On total edge and total vertex irregularity strength of pentagon cactus chain graph with pendant vertices

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Abstract. Let \( G(V, E) \) be a graph with a non-empty set of vertices \( V \) and a set of edges \( E \). A total \( k \)-labeling \( f : V \cup E \to \{1, 2, ..., k\} \) is called an edge irregular total \( k \)-labeling if the weight of each edge is distinct, where the weight of an edge \( e = xy \) is \( wt(e) = f(xy) + f(x) + f(y) \). Whereas, \( f \) is a vertex irregular total \( k \)-labeling if the weight \( wt(x) \neq wt(y) \) for two distinct vertices \( x, y \) of \( V(G) \) where the weight \( wt(x) = f(x) + \sum_{ux \in E(G)} f(u) \). The minimum \( k \) for which \( G \) has an edge irregular total \( k \)-labeling is called the total edge irregularity strength (tes) of \( G \). Further, the total vertex irregularity strength (tvs) of \( G \) is the minimum \( k \) for which \( G \) has a vertex irregular total \( k \)-labeling. In this paper, we find tes and tvs of heptagon cactus chain graph with pendants and get the results as follows:

\[
tes(C(P_t^n)) = \left\lceil \frac{6t + 2}{3} \right\rceil \text{ and } tvs(C(P_t^n)) = \left\lceil \frac{6t + 2}{4} \right\rceil.
\]

1. Introduction

Given an undirected and simple graph \( G = (V, E) \) where \( |V(G)| \) is finite. A labeling \( f \) on \( G \) is an assigning from elements of \( G \) (vertices or edges or both of them) into a set of positive integers. If \( f \) assigns the vertex set \( V(G) \) into a set of positive integers, then \( f \) is called a vertex labeling. If its domain is the edge set \( E(G) \), then \( f \) is an edge labeling. Further, if its domain \( V(G) \cup E(G) \), then \( f \) is a total labeling [1].

Baca et al. was first introduced an irregular total labeling of graph [2]. The function \( f \) from \( V(G) \cup E(G) \) into a set \( \{1, 2, 3, ..., k\} \) is called an edge irregular total \( k \)-labeling if the weight \( wt(e_1) \neq wt(e_2) \) for two distinct edges \( e_1, e_2 \) of \( E(G) \) where \( wt(e = uv) = f(u) + f(uv) + f(v) \). The symbol tes(\( G \)) stands for a total edge irregularity strength of \( G \) that is the minimum number \( k \) in which \( G \) has an edge irregular total \( k \)-labeling. Moreover, the function \( f \) is named a vertex irregular total \( k \)-labeling if the weight \( wt(x) \neq wt(y) \) for two distinct vertices \( x, y \) of \( V(G) \) where the weight \( wt(x) = f(x) + \sum_{ux \in E(G)} f(u) \). Moreover, the symbol tvs(\( G \)) denote a total vertex irregularity strength of \( G \), that is the minimum value \( k \) such that \( G \) has a vertex irregular total \( k \)-labeling.

The bounds for tes of any graph was given by Baca as follows:

\[
\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq \text{tes}(G) \leq |E(G)|
\]

(1)

Furthermore, Baca et al. proposed the lower and upper bounds for tvs of any graphs as follows [2]:

\[
\frac{(p + \delta)}{\Delta + 1} \leq \text{tvs}(G) \leq p + \Delta - 2\delta + 1,
\]
where \( p = |V(G)| \), \( \delta = \min\{\text{degree } d(v)|v \in V(G)\} \), and \( \Delta = \max\{d(v)|v \in V(G)\} \). Anholcer, et al. provided another bounds in [2]. Meanwhile, Nurdin et al. [3] showed a property of tvs for any connected graph with \( n_i \) vertices of degrees \( i \) for \( i = \delta, \delta + 1, \delta + 2, \ldots, \Delta \):

\[
tv(G) \geq \max \left\{ \frac{\delta(G) + n_\delta(G)}{\delta(G) + 1}, \frac{\delta(G) + n_{\delta+1}(G)}{\delta(G) + 2}, \ldots, \frac{\delta(G) + \sum_{i=\Delta(G)} n_i}{\Delta(G) + 1} \right\}
\]

(2)

A block of the graph \( G = (V, E) \) is a subgraph \( H \) of \( G \) that is maximal and \( H \) does not have cut-vertex. A block-cut vertex of \( G \) is a bipartite graph \( (A, B) \) such that \( A \) contains cut vertices of \( G \) and \( B \) contains all blocks of \( G \). Two vertices \( v \) and \( b_i \) in the block-cut vertex \( (A, B) \) are adjacent if the cut vertex \( v \) is incident to an edge of block \( b_i \) [4]. A cactus graph is a connected graph that consists of some blocks and each block is an edge or a cycle. A pentagon cactus is a cactus graph whose all blocks are cycle \( C_5 \). Borishevich and Doslic [4] offered the cactus chain graph in 2015. In this research, we focus on a pentagon cactus chain graph that is a pentagon cactus in which its block cut vertex graph is a path. Whereas, a pentagon cactus chain graph with pendant vertices is given on Definition 1.

**Definition 1.** A pentagon cactus chain graph with 3 pendant vertices, represented by \( C(P_t^3) \), is a pentagon cactus chain graph that consists of \( r \) blocks cycle \( C_5 \) with 3 pendant vertices on each block. The vertex set is \( V(P_t^3) = \{a_i, b_i, c_i, d_i, a_{i+1}, b_i', c_i', d_i'\} \) and the edge set is \( E(C(P_t^3)) = \{a_i b_i, b_i a_{i+1}, a_{i+1} c_i, c_i d_i, a_i d_i, b_i b_i', c_i c_i', d_i d_i'\} \). The pendant vertices are \( b_i' \) connected to \( b_i \), \( c_i' \) connected to \( c_i \), and \( d_i' \) connected to \( d_i \), for \( i = 1, 2, 3, \ldots, r \).

Many results on tes or tvs of any graph classes have been found by researchers [2]. Ivanco dan Jendrol verified tes of trees [2]. Arockiamary [5] established tes of diamond snake graph that is a class of cycle \( C_4 \) cactus chain graph. Indriati et al. found tes of generalized helm graphs and prisms with outer pendant edges [7]. For more results on tes and tvs of some graph classes, readers can find in [2],[8],[13],[14], and [15]. Recently, Rosyida and Indriati found tes of cycle \( C_3 \) and cycle \( C_4 \) cactus chain graphs with pendant vertices [9]. Also, tvs of cycle \( C_4 \) cactus chain graph with one pendant vertex on each block (tadpole chain graph \( T_r(4,1) \)) was proved in [10]. In this research, we find an exact value of tes and tvs of pentagon cactus chain graph with 3 pendant vertices on each block.

**2. Methods**
The method used in this research is as follows:

(i) defining the pentagon cactus chain graph with 3 pendant vertices, denoted by \( C(P_t^3) \) (Definition 1);

(ii) determining the lower bound for tes and tvs of \( C(P_t^3) \) through the formulas given in [2] and [3]:

\[
tes(C(P_t^3)) \geq \left\lceil \frac{5r+3}{3} \right\rceil \geq \left\lceil \frac{8r+2}{3} \right\rceil \geq \left\lceil \frac{6r+3}{3} \right\rceil; \quad tvs(C(P_t^3)) \geq \left\lceil \frac{6r+3}{3} \right\rceil.
\]

put \( k = \left\lceil \frac{8r+2}{3} \right\rceil \) and \( l = \left\lceil \frac{6r+3}{3} \right\rceil \);

(iii) showing that the upper bound for tes and tvs are \( tes(C(P_t^3)) \leq k \) and \( tvs(C(P_t^3)) \leq l \) through trial and error process beginning from pentagon cactus chain graph with length \( r = 2, r = 3 \), and so forth until we get a fixed pattern of edge irregular total \( k \)-labeling or vertex irregular total \( l \)-labeling;

(iv) constructing formulas for vertex and edge labels;
(v) showing that the weights of edges or vertices are diverse under the labeling $f$;
(vi) obtaining exact values $tes = k$ and $tvs = l$.

3. Results and Discussion
There are only a few papers which discussed labeling on chain graphs, among others [11], [12], [9], and [10]. In order to continue previous work in [9] and [10], we verify $tes$ and $tvs$ of pentagon cactus chain graph with 3 pendant vertices $C(P_t^3)$. These problems have not been investigated until now and this is the novelty of this research.

3.1. Tes of pentagon cactus chain graph with 3 pendant vertices $C(P_t^3)$
The result of $tes$ of pentagon cactus chain graph $C(P_t^3)$ is presented in Theorem 3.1.

**Theorem 3.1.** If $C(P_t^3)$ is a pentagon cactus chain graph with 3 pendant vertices on each block and the length is $r$ ($r \geq 2$), then

$$tes(C(P_t^3)) = \left[\frac{8r+2}{3}\right].$$

**Proof.** According to Inequality (1.), we get the lower bound

$$tes(C(P_t^3)) \geq \left\lceil\frac{|E(C(P_t^3))| + 2}{3}\right\rceil = \left\lceil\frac{8r+2}{3}\right\rceil$$

for $r \geq 2$. Let $k = \left\lceil\frac{8r+2}{3}\right\rceil$. We verify that $k$ is an upper bound for $tes(C(P_t^3))$ by constructing the function $\lambda: V \cup E \rightarrow \{1, 2, \ldots, k\}$ where vertex and edge labels are defined in Table 1 and Table 2.

| Case 1: for $i \equiv 1 \ mod \ 3$ | Case 2: for $i \equiv 2 \ mod \ 3$ | Case 3: for $i \equiv 0 \ mod \ 3$ |
|---------------------------------|---------------------------------|---------------------------------|
| $\lambda(a_i) = \frac{1}{3}(8i - 5)$ | $\lambda(a_i) = \frac{1}{3}(8i - 4)$ | $\lambda(a_i) = \frac{1}{3}(8i - 6)$ |
| $\lambda(b_i) = \frac{1}{3}(8i - 5)$ | $\lambda(b_i) = \frac{1}{3}(8i - 1)$ | $\lambda(b_i) = \frac{1}{3}(8i - 3)$ |
| $\lambda(b'_i) = \frac{1}{3}(8i - 5)$ | $\lambda(b'_i) = \frac{1}{3}(8i - 4)$ | $\lambda(b'_i) = \frac{1}{3}(8i - 6)$ |
| $\lambda(c_i) = \frac{2}{3}(8i + 2)$ | $\lambda(c_i) = \frac{1}{3}(8i + 2)$ | $\lambda(c_i) = \frac{8i + 2}{3}$ |
| $\lambda(c'_i) = \frac{1}{3}(8i + 1)$ | $\lambda(c'_i) = \frac{1}{3}(8i + 2)$ | $\lambda(c'_i) = \frac{8i + 2}{3}$ |
| $\lambda(d_i) = \frac{2}{3}(8i - 5)$ | $\lambda(d_i) = \frac{1}{3}(8i - 4)$ | $\lambda(d_i) = \frac{8i - 6}{3}$ |
| $\lambda(d'_i) = \frac{2}{3}(8i + 1)$ | $\lambda(d'_i) = \frac{1}{3}(8i + 2)$ | $\lambda(d'_i) = \frac{8i + 2}{3}$ |

Under the labeling $\lambda$, we get the weight of each edge as follows:

$w(a_i b_i) = 8i - 5; w(c_i c'_i) = 8i; w(b_i a_{i+1}) = 8i + 1; w(d_i d'_i) = 8i - 1$;

$w(a_{i+1} b_i) = 8i - 3; w(c_i d_i) = 8i - 2; w(c_{i+1} a_i) = 8i + 2; w(d_i a_i) = 8i - 4$.

for $i = 1, 2, \ldots, r$. It is obvious that all labels on the graph $C(P_t^3)$ in Table 1 are not more than $k$ and the edge weights are different, i.e., $8i - 5, 8i - 4, \ldots, 8i, 8i + 1, 8i + 2$. Therefore, the labeling $\lambda$ is an edge irregular total $k$-labeling. Thus, $tes(C(P_t^3)) = \left[\frac{8r+2}{3}\right]$. □
Table 2. Labels of edges under total labeling $\lambda$

| Case 1: for $i \equiv 1 \text{ mod } 3$ | Case 2: for $i \equiv 2 \text{ mod } 3$ | Case 3: for $i \equiv 0 \text{ mod } 3$ |
|---------------------------------|---------------------------------|---------------------------------|
| $\lambda(a_ib_i) = \frac{1}{3}(8i - 5)$ | $\lambda(a_ib_i) = \frac{1}{3}(8i - 10)$ | $\lambda(a_ib_i) = \frac{1}{3}(8i - 6)$ |
| $\lambda(b_ib_i') = \frac{1}{3}(8i + 1)$ | $\lambda(b_ib_i') = \frac{1}{3}(8i - 4)$ | $\lambda(b_ib_i') = \frac{1}{3}(8i)$ |
| $\lambda(b_i a_{i+1}) = \frac{1}{3}(8i + 2)$ | $\lambda(b_i a_{i+1}) = \frac{1}{3}(8i + 2)$ | $\lambda(b_i a_{i+1}) = \frac{8i + 2}{3}$ |
| $\lambda(a_{i+1}c_i) = \frac{1}{3}(8i - 2)$ | $\lambda(a_{i+1}c_i) = \frac{1}{3}(8i + 2)$ | $\lambda(a_{i+1}c_i) = \frac{1}{3}(8i)$ |
| $\lambda(c_ic_i') = \frac{1}{3}(8i - 5)$ | $\lambda(c_ic_i') = \frac{1}{3}(8i - 4)$ | $\lambda(c_ic_i') = \frac{1}{3}(8i - 6)$ |
| $\lambda(c_id_i) = \frac{1}{3}(8i - 5)$ | $\lambda(c_id_i) = \frac{1}{3}(8i - 4)$ | $\lambda(c_id_i) = \frac{1}{3}(8i - 3)$ |
| $\lambda(d_id_i') = \frac{1}{3}(8i + 1)$ | $\lambda(d_id_i') = \frac{1}{3}(8i - 1)$ | $\lambda(d_id_i') = \frac{8i}{3}$ |
| $\lambda(d_ia_i) = \frac{1}{3}(8i - 2)$ | $\lambda(d_ia_i) = \frac{1}{3}(8i - 4)$ | $\lambda(d_ia_i) = \frac{1}{3}(8i)$ |

An example of edge irregular total 12–labeling on pentagon cactus chain graph $C(P_t^4)$ is given in Figure 1. Let $f$ be a total labeling on the chain graph in Figure 1. Vertex and edge labels on the graph are marked with black color numbers and the weights of edges are marked with red color numbers. We can see that the labels are at most 12 and the edge weights are all diverse. Therefore, $f$ is an edge irregular total 12–labeling and tes($C(P_t^4)$) = 12.

Figure 1. Edge irregular total 12–labeling of $C(P_t^4)$

3.2. tvs of pentagon cactus chain graph with 3 pendant vertices $C(P_t^3)$

We present formula for tvs of pentagon cactus chain graph $C(P_t^3)$ in Theorem 3.2.

Theorem 3.2. If $C(P_t^3)$ is a pentagon cactus chain graph with 3 pendant vertices on each block and the length is $r$ ($r \geq 2$), then

$$\text{tvs}(C(P_t^3)) = \left\lceil \frac{6r+3}{4} \right\rceil.$$
Proof. According to Inequality (2. ), we get the lower bound as follows:

$$tvsl(C(Pr^n_r)) \geq \max \left\{ \frac{1+3r}{2}, \frac{3r+3}{3}, \frac{6r+3}{4}, \frac{6r+5}{5} \right\} = \frac{6r+3}{4} \quad (for \ r \geq 2).$$

Let $k = \left\lfloor \frac{8r+2}{3} \right\rfloor$. We will prove that $k$ is an upper bound for $tvsl(C(Pr^n_r))$. Let us consider a $k$-total labeling $f: V \cup E \rightarrow \{1,2,\ldots,k\}$ with labels of vertices and edges are presented in Table 3 and Table 4.

**Table 3. Labels of vertices under total labeling $f$**

| Case 1: for $r$ is odd | Case 2: for $r$ is even |
|------------------------|------------------------|
| $f(a_1) = i, i = 1,2,\ldots,r$; $f(a_{r+1}) = 1$ | $f(a_1) = f(a_2) = 2$ |
| $f(b_1) = 2, i = 1,2,\ldots,r$ | $f(a_{i+1}) = i + 3, i = 1,2,\ldots,(r-1)$ |
| $f(b'_1) = 1, i = 1,2,\ldots,r$ | $f(b_1) = 2, i = 1,2,\ldots,r$ |
| $f(c_i) = r, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor$ | $f(c_i) = r + 1, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor$ |
| $f(c'_i) = r + i, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor$ | $f(c'_i) = i, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor$ |
| $f(c''_i) = 2i - 2r + \left\lfloor \frac{6r+3}{4} \right\rfloor$ | $f(c''_i) = r + i, i = 1,2,\ldots,\left\lfloor \frac{6r+3}{4} \right\rfloor + 1$ |
| $f(d_i) = r, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor + 1$ | $f(d_i) = r + 1, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor + 1$ |
| $f(d'_i) = i, i = 1,2,\ldots,2r - \left\lfloor \frac{6r+3}{4} \right\rfloor$ | $f(d'_i) = 2i - 2r + \left\lfloor \frac{6r+3}{4} \right\rfloor$ |
| $f(d''_i) = 2i - 2r + \left\lfloor \frac{6r+3}{4} \right\rfloor - 1, i = 2,3,\ldots,\left\lfloor \frac{6r+3}{4} \right\rfloor$ | $f(d''_i) = r + i, i = 2,3,\ldots,\left\lfloor \frac{6r+3}{4} \right\rfloor$ |

Let us consider a

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

for $r \geq 2$.
Table 4. Labels of edges under total labeling $f$

| Case 1: for $r$ is odd | Case 2: for $r$ is even |
|------------------------|------------------------|
| $f(a,b_i) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r$ | $f(a,b_i) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r$ |
| $f(b_i b_{i}^{'})) = i, i = 1, 2, ..., r$ | $f(b_i a_{i+1}) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r$ |
| $f(b_i a_{i+1}) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r$ | $f(a_{i+1} c_i) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r - 1$ |
| $f(a_{i+1} c_i) = \left[\frac{6r^{i}+3}{4}\right] - 1, i = 1, 2, ..., r - 1$ | $f(a_{r+1} c_i) = \left[\frac{6r^{i}+3}{4}\right] - 1$ |
| $f(c_i c_i^{'})) = r + i + 1, i = 1, 2, ..., 2r - \left[\frac{6r^{i}+3}{4}\right] - 1$ | $f(c_i c_i^{'})) = r + i + 1, i = 1, 2, ..., 2r - \left[\frac{6r^{i}+3}{4}\right] - 1$ |

The weights of vertices are displayed in Table 5.

Table 5. The weights of vertices under total labeling $f$

| Case 1: for $r$ is odd | Case 2: for $r$ is even |
|------------------------|------------------------|
| $w(b_i) = 2\left[\frac{6r^{i}+3}{4}\right] + i, i = 1, 2, ..., r$ | $w(b_i) = i + 1, i = 1, 2, ..., r$ |
| $w(d_i) = 2\left[\frac{6r^{i}+3}{4}\right] + r + 2i - 1, i = 1, 2, ..., r$ | $w(d_i) = 2\left[\frac{6r^{i}+3}{4}\right] + r + 2i, i = 1, 2, ..., r$ |
| $w(c_i) = 2\left[\frac{6r^{i}+3}{4}\right] + r + 2i, i = 1, 2, ..., r$ | $w(c_i) = 2\left[\frac{6r^{i}+3}{4}\right] + r + 2i + 1, i = 1, 2, ..., r$ |
| $w(a_i) = 2\left[\frac{6r^{i}+3}{4}\right] - 1; w(a_{r+1}) = 2\left[\frac{6r^{i}+3}{4}\right]$ | $w(a_i) = 2\left[\frac{6r^{i}+3}{4}\right]; w(a_{r+1}) = 2\left[\frac{6r^{i}+3}{4}\right] + 1$ |
| $w(a_{i+1}) = 4\left[\frac{6r^{i}+3}{4}\right] + i - 3, i = 1, 2, ..., r - 1$ | $w(a_{i+1}) = 4\left[\frac{6r^{i}+3}{4}\right] + i - 1, i = 1, 2, ..., r - 1$ |
It is shown that vertex and edge labels of $C(P_t^3)$ are at most $k$ and the weights of vertices are different. Hence, the labeling $f$ is a vertex irregular total $k$-labeling. Thus, $tvs(C(P_t^3)) = \left\lceil \frac{6r+3}{4} \right\rceil$ and the proof is complete.

An illustration of vertex irregular total 9–labeling on pentagon cactus chain graph $C(P_t^3)$ is shown in Figure 2. Let $f$ be a total labeling on the chain graph in Figure 2. Vertex and edge labels on the graph are indicated with black color numbers and the weights of vertices are indicated with red color numbers. It is clear that the biggest number of labels of vertices and edges is 9 and the weights of vertices are all distinct. Therefore, $f$ is a vertex irregular total 9-labeling and $tvs(C(P_t^3)) = 9$.

![Figure 2](image-url)

**Figure 2.** Vertex irregular total 9–labelling of $C(P_t^3)$

4. Conclusion
In this paper, we have obtained formulas for tes and tvs of pentagon chain graph $C(P_t^3)$ as follows:

$$tes(C(P_t^3)) = \left\lceil \frac{8r+2}{3} \right\rceil$$
$$tvs(C(P_t^3)) = \left\lceil \frac{6r+3}{4} \right\rceil$$

As further research, we will determine formulas for tes and tvs of generalized cyclic chain graph with pendant vertices and develop an algorithm.

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