KINETIC MODELS OF CONSERVATIVE ECONOMIES WITH NEED-BASED TRANSFERS AS WELFARE

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Abstract. Kinetic exchange models of markets utilize Boltzmann-like kinetic equations to describe the macroscopic evolution of a community wealth distribution corresponding to microscopic binary interaction rules. We develop such models to study a form of welfare called need-based transfer (NBT). In contrast to conventional centrally organized wealth redistribution, NBTs feature a welfare threshold and binary donations in which above-threshold individuals give from their surplus wealth to directly meet the needs of below-threshold individuals. This structure is motivated by examples such as the gifting of cattle practiced by East African Maasai herders or food sharing among vampire bats, and has been studied using agent-based simulation. From the regressive to progressive kinetic NBT models developed here, moment evolution equations and simulation are used to describe the evolution of the community wealth distribution in terms of efficiency, shape, and inequality.

1. Introduction.

1.1. Kinetic exchange models of markets. Kinetic exchange models of markets utilize Boltzmann-like equations in which individuals exchanging money in a trade are considered analogous to gas particles changing velocities after a collision [11, 17, 20]. Commonly, the microscopic trade rules considered involve individuals exchanging random fractional amounts of each other’s wealths [4, 8, 10]. Where

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and \( w \) are two individuals’ respective wealths before a trade, the individuals’ post-trade wealths are given by

\[
v^* = p_1v + q_1w, \quad w^* = q_2v + p_2w.
\]

The coefficients \( p_i, q_i \) are the non-negative random variables whose laws determine the shape of the steady state distribution [12].

The relative density of individuals with wealth \( w \geq 0 \) at time \( t \geq 0 \) is then given by \( f(w, t) \), which evolves according to the following Boltzmann-like equation

\[
\partial_t f = Q_-(f, f) + Q_+(f, f),
\]

where \( Q_-(f, f) \) is the collisional loss operator that gives the losses at wealth \( w \) resulting from interactions/trades, \( Q_+ \) gives the gains at wealth \( w \) resulting from trades, and \( f(w, 0) = f_0(w) \) is the initial wealth distribution. For models fitting this description, a characteristic function has been defined which can classify the tail of the steady state distribution according to the given coefficients \( p_i, q_i \) [12]. Of particular interest is whether the tail of the wealth distribution satisfies a Pareto power law, in which case the Pareto index informs of the concentration of community wealth among the super rich [9, 12, 18].

Such kinetic exchange models of markets have been able to recover realistic wealth distributions from simple microscopic interaction rules, which may lead one to wonder if wealth inequality is simply a natural consequence of many individuals exchanging money in binary trades [7]. Beyond providing insight into natural wealth evolution in an economy, taxation and wealth redistribution have also been considered, where portions of individuals’ wealths are extracted via a tax on transfers, and a redistribution operator governs how the taxed wealth gets distributed back to the population [4, 5, 22].

In addition to global redistribution of wealth, redistribution has been modeled at the microscopic level via interaction with a background [21]. In contrast to [21] the fundamental concept of need-based transfers, discussed in Section 1.2 does not deal with any background or policy maker but achieves microscopic welfare redistribution through binary interactions between agents of different wealth.

Optimal control at the microscopic level has recently been utilized with a cost functional related to wealth inequality [13] which complements our time- and need-efficient redistribution relative to a viability threshold discussed in Section 4.

In this paper we use kinetic equations to model a different form of wealth redistribution called need-based transfer in which tax money is not centrally collected and redistributed; rather, it is within binary interactions that individuals with surplus give directly to individuals with deficit as determined by a fixed welfare threshold.

1.2. Need-based transfers. Need-based transfers (NBTs) are purpose-driven binary exchanges of wealth, instituted as a form of risk pooling or welfare in order to support community members’ viable participation in an economy [1, 15, 16]. This concept is motivated by the reciprocal gifting of cattle - called osotua - practiced among the Maasai herders of East Africa [2]. Relative to an established viability threshold, NBTs involve donations where an above-threshold individual gifts, e.g. cattle, from their surplus to a below-threshold individual in order to bring the recipient’s wealth up to threshold and so preserve their ability to viably participate in the economy.

Such a binary welfare system is also used by vampire bats who gift food to starving roost mates via a mouth-to-mouth transfer [6, 24, 25]. NBTs are interesting
in that the mechanism of risk pooling and welfare is not centrally organized but conducted on an individual-to-individual basis in binary interactions. This is often because the economic environment makes central redistribution unfeasible. For example, centrally collecting cattle devastates food and water resource. Similarly, emergency help after major catastrophic events like earthquakes or hurricanes typically has a large component of altruistic binary help since communication and transport failure may make centrally organized help unfeasible.

A microscopic description of NBTs with a welfare threshold \( \theta \in \mathbb{R} \) and pre-trade wealths \( v, w \in \mathbb{R} \) is the following:

\[
\begin{align*}
    v^* &= v + H(v + w - 2\theta)(\theta - v)H(\theta - v) - (\theta - w)H(\theta - w) \\
    w^* &= w + H(v + w - 2\theta)(\theta - w)H(\theta - w) - (\theta - v)H(\theta - v),
\end{align*}
\]

where \( H \) is the standard Heaviside step function. Essentially, a transfer will occur only if the sum of the two individuals’ wealths are such that both individuals can be at or above threshold after a transfer. Then, if one individual has a deficit, the entire amount will be gifted by the one with surplus. Thus transfers only and always occur when a below-threshold individual interacts with an above-threshold individual whose surplus is greater than their deficit; then the entire deficit will be gifted to bring the recipient up to the viability/welfare threshold.

Gifting only part of an individual’s deficit is not considered here for simplicity. This may be contextually defensible when transaction costs, e.g. time and energy required to regurgitate blood or track down cattle from multiple nomadic herds, are high. Also, as partially meeting an individual’s need does not guarantee their viability, partial gifts may not be given as they put the donor at risk without assuring a recipient’s viability and potential for future reciprocation.

The economics of such NBTs has been studied using agent-based simulations, and in isolated dyads has been found to result in greater survival than no exchanges or other welfare efforts \([2]\). Also, when considering networks of individuals who practice NBTs across edges, survival increases with the size of the network and network mean degree, though with diminishing returns \([15]\); lower variance in network degrees is also found to be preferable \([16]\). It has also been found that regressive wealth redistribution via prioritizing donations from the individuals with the smallest surpluses acts as a cutting stock optimization heuristic, most efficiently matching deficits to surpluses and resulting in the highest short-term survival. On the other hand, progressive transfers that prioritize the richest individuals giving first encourages a more stable wealth distribution that leads to higher long-term survival \([16]\).

As NBTs consist of binary wealth redistribution, it is natural to consider modeling the evolution of an NBT economy using kinetic exchange models of markets. Moreover, there are multiple advantages of kinetic integro-differential equation (IDE) models over agent-based simulation. In particular, the analytical structure of IDEs allows for proving some results, and even if results are still numerical, the computation time with the IDE model is tremendously reduced compared to agent-based simulations. Also, the concept of the evolution of the wealth distribution is better captured in the kinetic framework.

Thus, in this paper we develop and examine kinetic models of NBT economies. These models are conservative in that total wealth in the economy is conserved. In Section 2, we develop a random interaction model with diffusion as a natural
wealth evolution approximation and utilize moment evolution equations to characterize the shape of the wealth distribution. In Section 3 we develop a regressive to progressive NBT policy model and utilize simulation to compare efficiency of transfers and resulting wealth inequality with observations of the agent-based simulation study [16], while also contributing new conceptual insights. In Section 4 we present and interpret an instantaneous control policy, which seeks to maximize the rate of successful transfers.

2. Random interaction kinetic NBT model. If we assume that collisions or interactions as described in Equation (1) occur at rate 1, and consider \( f(w, t) \) as the relative density of individuals with wealth \( w \in \mathbb{R} \) at time \( t \geq 0 \), the Boltzmann-like wealth distribution evolution equation corresponding to the microscopic transfers described in Equation (1) is:

\[
\partial_t f(w, t) = Q(f, f)(w, t) := \int_{-\infty}^{\theta} \int_{2\theta - u}^{\infty} \left[ -\delta(w - u) - \delta(w - v) + \delta(w - \theta) + \delta(w - u - v + \theta) \right] \\
\times f(v, t) f(u, t) \, dv \, du.
\]

Note that wealth is defined on the whole real line. Thus the only important wealth value is the threshold: agents with wealth above threshold are potential donors, agents with wealth below threshold are potential receivers of need based transfers. In particular, there is no special meaning associated with \( w = 0 \) and thus we will in the following set the threshold to \( \theta = 0 \).

Equation (2) can be understood as \( u \) denoting the wealth of a below-threshold individual and \( v \) denoting the wealth of a donor (individual with enough surplus to cover the deficit \( \theta - u \) and still be above threshold, i.e. \( v > 2\theta - u \)). Thus, when these individuals interact, there are density losses (sinks) at their pre-trade wealths \( u, v \) and density gains (sources) at their post-trade wealths, \( u^* = \theta \) and \( v^* = v - (\theta - u) \). In Equation (2), the interactions occur at rates proportional to the relative densities of individuals with wealths \( u \) and \( v \), i.e. one can think of individuals being selected uniformly at random from the wealth distribution and paired. Hence, this is a model for the RA ← RG policy of [16] where no preference is considered in how below- and above-threshold individuals are paired. Interactions in this kinetic model are thus naturally similar to colliding gas particles in that interaction is random rather than organized or intelligent.

From Equation (2), it is apparent that steady state \( f_\infty \) takes one of the following forms:

(i) No individuals with surplus: \( \text{supp}(f_\infty) \subseteq (-\infty, \theta] \).

(ii) No individuals with deficit: \( \text{supp}(f_\infty) \subseteq [\theta, \infty) \).

(iii) Individuals with deficit and surplus but no individuals with enough surplus to match the deficits of below-threshold individuals: \( \max\{\text{supp}(f_\infty)\} \in (\theta, \infty) \) and \( (\theta - \max\{\text{supp}(f_\infty) \cap (-\infty, \theta)\}) > \{\max\{\text{supp}(f_\infty)\} - \theta\} \).

If there are no individuals with surplus, or if there are no individuals with deficit, no transfers described by Equation (1) can occur. Contextually, case (iii) says that if the richest individual’s surplus is less in magnitude than the least poor individual’s deficit then no one is eligible for a binary transfer and thus the wealth distribution ceases to change. Figure 1 illustrates simple examples of these different types of steady state. These illustrations are conceptual rather than naturally observed steady states; for example, case (i) can only occur when \( f_\infty(w) = f_0(w) \) has initial
Figure 1. Simple examples of the different cases of steady states for Equation (2) where $\theta = 0$; the area under the probability density curve is shaded for visibility.

support $(-\infty, \theta]$. However, Figure 2(b) shows how steady states of type (ii) and (iii) may be observed as steady states for different initial conditions.

Integrating (2) with a well-behaved test function $\phi(w)$ gives the weak form of the Boltzmann equation:

$$
\int_{-\infty}^{\infty} \phi(w) \partial_t f(w, t) \, dw = \int_{-\infty}^{\theta} \int_{2\theta - u}^{\infty} \left[ -\phi(u) - \phi(v) + \phi(\theta) + \phi(u + v - \theta) \right] \times f(v, t) f(u, t) \, dv \, du.
$$

In particular, we are interested in the evolution of the $n$-th moments, which are obtained by integrating (2) with $\phi(w) = (w - \theta)^n$:

$$
M_n(t) := \int_{-\infty}^{\infty} (w - \theta)^n f(w, t) \, dw.
$$

These moments are not central moments in that $\theta$ is not necessarily the mean wealth; however, since $\theta$ is central in determining the shape of our wealth distributions, it will be advantageous to examine moments about the threshold $\theta$ rather than the mean wealth.

For $f(w, t)$ satisfying (2), Equations (3) and (4) give that the $n$-th moment of $f$ evolves as follows:

$$
\frac{d}{dt} M_n(t) = \int_{-\infty}^{0} \int_{-\infty}^{\infty} \left[ -y^n - x^n + 0^n + (x + y)^n \right] f(x + \theta, t) f(y + \theta, t) dx \, dy
$$

Significantly, if $f(w, 0) = f_0(w)$ is a probability density, then (5) gives the following:

(i) $M_0(t) = 1$ is constant as $\frac{d}{dt} M_0(t) = 0$, and $M_0(0) = 1$.
(ii) $M_1(t) = M_1(0)$ is constant as $\frac{d}{dt} M_1(t) = 0$.
(iii) $\frac{d}{dt} M_n(t) \leq 0$ for $n \geq 2$, as the integrand in (5) is negative for $y < 0, x > -y$, see Appendix A.

$M_0(t) = 1$ means that $f(w, t)$ remains a probability density, and when combined with the fact that the mean wealth, $M_1(t)$ is constant, we have that the total wealth of the community is conserved, i.e. (2) defines a conservative kinetic exchange model. Observation (iii) finds the second moment to be non-increasing, which intuitively makes sense as transfers naturally bring wealths more central, i.e. the distance between two agents’ wealths decreases in a transfer with all wealths coming closer to $\theta$. $\frac{d}{dt} M_3(t) \leq 0$ in this case corresponds to individuals with very large surpluses coming closer to threshold more rapidly than individuals with very large deficits.
This is also intuitive as individuals to the far left have very few who can help them though individuals to the far right can help many.

Figure 2 illustrates the moment evolutions and steady state distributions for three different initial conditions that evolve according to (2). Note that two of the steady state cases previously described are shown: case (ii) where there are no deficits remaining for the Normal initial condition, and case (iii) where some deficits remain which are too large to match with remaining surpluses for the Uniform and Gamma initial distributions.

By observing that $\frac{d}{dt} M_0(t) = \frac{d}{dt} M_1(t) = 0$, we equivalently notice that $\phi(w) = 1$ and $\phi(w) = w$ are collision invariants, i.e. these test functions result in the collisional term on the right hand side of Equation (3) being equal to 0. Actually, linear functions of the form $\phi(w) = aw + b$, where $a, b \in \mathbb{R}$, are the only collision invariants for (3). Taking the derivative of both sides of $[ \phi(u) - \phi(v) + \phi(\theta) + \phi(u + v - \theta) ] = 0$ with respect to $u$ gives that $\phi'(u) = \phi'(u + v - \theta)$. Since $v$ can vary in the argument of $\phi'(u + v - \theta)$, it must be that $\phi'$ is constant and therefore $\phi$ is linear.

It is important to note that Equation (2) only models the welfare wealth redistribution; there is no natural wealth evolution considered, i.e. non-welfare influences like commerce, investments, consumption, etc. are not modeled in (2). How to model the natural evolution of wealth is an important challenge that depends on
the community being modeled. In the area of econophysics, much effort is given
to constructing models of modern national economies and the growth of firms [7].
However, the economies of the Maasai herders and vampire bats are driven by much
different natural processes.

As an early effort to examine how NBTs function in a context of natural wealth
evolution, we presently consider that individual wealths evolve according to an
additive random walk, i.e. wealths change by random amounts rather than random
percentages. Such changes in wealth may reasonably describe the process of wealth
evolution for a majority of citizens in a modern economy where income and random
expenses primarily determine wealth rather than investments [19]. To incorporate
this random additive walk into the model (2), we simply add a diffusion term. For
\( Q(f, f)(w, t) \) the NBT collision term from Equation (2), and
\( D > 0 \) the kinetic NBT with diffusion model is the following:

\[
\partial_t f(w, t) = Q(f, f)(w, t) + D\partial_w^2 f(w, t).
\] (6)

Using the same approach as before to investigate moment evolution (Equations
(3) and (4)), and assuming \( f \) decays at least quadratically, we have the following:

(i) \( M_0(t) = 1 \) is constant.
(ii) \( M_1(t) = M_1(0) \) is constant.
(iii) \( \frac{d}{dt} M_2(t) = S_2(f, f)(t) + 2D \) where \( S_2 \) is non-positive.
(iv) \( \frac{d}{dt} M_3(t) = S_3(f, f)(t) + 6D(M_1 - \theta) \), where \( S_3 \) is non-positive.

\( S_2(f, f)(t) = \int_{-\infty}^{0} \int_{0}^{\infty} 2xyf(x + \theta, t)f(y + \theta, t)dx dy \leq 0 \) is equal to \( \frac{d}{dt} M_2(t) \) for
the kinetic NBT model without diffusion. Similarly \( S_3(f, f)(t) = \int_{-\infty}^{0} \int_{0}^{\infty} 3xy(x + y)f(x + \theta, t)f(y + \theta, t)dx dy \leq 0 \) is equal to \( \frac{d}{dt} M_3(t) \) for the kinetic model without
diffusion. Thus, \( S_2 \) and \( S_3 \) describe the reduction in variance and decrease in
skew respectively due to the welfare redistribution, but we have a linear increase in
variance and a linear increase in skew (if \( M_1 > \theta \)) due to the diffusion. Figure 3
shows the evolution of the wealth distribution for three different initial conditions
as well as the evolution of the second and third moments for this kinetic NBT with
diffusion model.

It is interesting that for a wide range of initial conditions (Fig. 3(a)), there
appears to be a fast convergence to some sort of metastable manifold as in Fig.
3(b); then the density functions very slowly flatten. The NBT redistribution acts
on the different initial wealth distributions similarly and the diffusion smooths the
curves. From Figure 3(c), in the long-term the impact of diffusion and redistribution
somewhat balance and then gradually diffusion continues to linearly increase the
variance in the distribution. Figure 3(d) shows long-term linear decay in the skew.

Essentially, diffusion pushes individuals to the left and right (increases density
of very poor and very rich individuals), and the NBT redistribution brings wealths
closer to threshold, but in an asymmetric way. In a community where a large portion
of individuals have moderate surplus or deficit and a minority have extreme wealth
or debt, a very wealthy individual can give to the large portion of individuals with
moderate deficits, but a very poor individual can only receive help from the rare
extremely wealthy. This causes the number of individuals or mass at the right of
the distribution to come central faster than mass to the left. The diffusion and
NBT redistribution cause increasingly negative skew (left skew), allowing mass to
escape to the left while preserving the mean wealth; thus steady state will be flat.
3. Kinetic NBT policies. In [16], regressive to progressive NBT redistribution policies were examined. The concept is that in response to natural events generating a set of surpluses and a set of deficits, when restricting donations to be binary it matters who gets to request help first and also who is prioritized to give help.

It is shown that the choice of the distribution scheme not only impacts immediate viability; it also impacts the distribution of wealth of the survivors. In particular, for a large range of initial wealth distributions, regressive redistribution acts as a cutting stock optimization heuristic, most efficiently matching deficits to surpluses, i.e. the fewest individuals below threshold after transfers. However, efficiently matching deficits results in a more at-risk community with lower long-term survival rates in agent simulations. Progressive redistribution results in a more stable wealth distribution in the long-term. Here, in Section 3, we develop corresponding kinetic models of regressive to progressive NBT redistribution and recover observations from the agent-based study as well as contribute new observations.

The random interaction kinetic NBT model (2) treats individuals somewhat analogously to gas particles in that individuals interact uniformly at random. To attribute preference in whether to participate in a transfer or not - and therefore establish redistribution policies - we first define an exact-match kinetic NBT model. To do so we choose in addition to a recipient threshold (welfare threshold) $\theta \in \mathbb{R}$
also a donor threshold, \( \theta + \epsilon_0 \) (\( \epsilon_0 \geq 0 \) not necessarily small). A transfer will occur only if the exchange will cause an individual with wealth below \( \theta \) to end with wealth \( \theta \) and an individual with wealth above \( \theta + \epsilon_0 \) to end with exactly \( \theta + \epsilon_0 \) as a result of the transfer. Hence, the microscopic description of the exact-match kinetic NBT model, given \( \theta, \epsilon_0 \), is

\[
\begin{align*}
v^* &= v + \delta (v + w - 2\theta - \epsilon_0) \left[ (\theta - v)H(\theta - v) - (\theta - w)H(\theta - w) \right] \\
w^* &= w + \delta (w + v - 2\theta - \epsilon_0) \left[ (\theta - w)H(\theta - w) - (\theta - v)H(\theta - v) \right],
\end{align*}
\]

and the macroscopic kinetic equation is

\[
\begin{align*}
\partial_t f(w, t) &= \int_{-\infty}^{\theta} \left[ -\delta(w - u) - \delta(w + u - \epsilon_0 - \theta) + \delta(w - \theta) + \delta(w - \theta - \epsilon_0) \right] \\
\times f(\epsilon_0 + \theta - u, t) f(u, t) \, du.
\end{align*}
\] (7)

The steady state of Equation (7), given initial condition \( f_0(w) = f(w, 0) \), is the following:

\[
\begin{align*}
f_\infty(w) &= f_0(w) - \left[ (H(\theta - w) + H(w - \theta - \epsilon_0)) \min\{f_0(w), f_0(2\theta + \epsilon_0 - w)\} \\
&\quad + \delta(w - \theta) \int_{-\infty}^{\theta} \min\{f_0(v), f_0(2\theta + \epsilon_0 - v)\} dv \\
&\quad + \delta(w - \theta - \epsilon_0) \int_{\theta + \epsilon_0}^{\infty} \min\{f_0(v), f_0(2\theta + \epsilon_0 - v)\} dv. \right]
\] (8)

Since the model restricts transfers to exact symmetric matches across respective recipient donor thresholds, \( f \) simply evolves by equally decreasing matching densities and sending that mass to the thresholds; steady state is reached when one of the matching densities reaches 0 in the limit. Figure 4 provides an example of how an initial wealth distribution evolves according to exact match redistribution (7), and illustrates the steady state.
Choosing a donor threshold \( \theta + \epsilon_0 \) automatically guarantees that no individuals with wealth below that threshold will be able to give. Thus, a higher donor threshold (larger \( \epsilon_0 \)) corresponds to only wealthier individuals being able to give. In this sense, exact-match transfers with large donor thresholds can be considered as describing more progressive wealth redistribution than policies with smaller donor thresholds.

Certainly the exact match policy of Eq. (7) by itself is not a viable NBT strategy. However, it allows to assign a policy consideration \( p(\epsilon_0) \) that describes the probability of a donor to engage in a NBT, if, after the transfer the donor will end up with a wealth of \( \theta + \epsilon_0 \). To do this we encode donor preference by a probability density function \( p \) such that \( p(\epsilon_0) d\epsilon \) is the probability a donor threshold will be selected between \( \theta + \epsilon_0 \) and \( \theta + \epsilon_0 + d\epsilon \).

The policies we will focus on are then defined by their respective \( \alpha \) values:

(a) **progressive** policy: \( \alpha = 0.05 \)
(b) **flat** policy: \( \alpha = 0 \)
(c) **regressive** policy: \( \alpha = -0.05 \).

\( p_\alpha \) for these policies is illustrated in Figure 5, where \( L = 100 \).

What Equation (9) essentially describes is that an individual will have varying comfort with donations depending on what their post-transfer wealth will be, e.g. for the regressive policy, since there is greater density for smaller \( \epsilon \), an individual will be more likely to give if the recipient needs only a small amount. A progressive policy might be understood as a donor reasoning that since less wealthy donors may be able to meet smaller needs, they should reserve their large surplus for an individual who has a larger deficit. When needs are met in a binary way, this can be quite sensible [16].

The flat policy kinetic equation (Equations (9), (10)) is equivalent to the random interaction kinetic NBT model (2), but with interaction rate \( \frac{1}{L} \). As a proof of concept, Figure 6 shows a comparison of the numerical steady state solution of the flat policy kinetic IDE and the wealth distribution resulting from agent-based simulation using the microscopic description of equation (1); the initial wealth distribution was chosen to be gamma, which is considered to be qualitatively realistic for natural wealth distributions [3, 7].

For numerical experiments of the kinetic NBT redistribution policies (Figures 7 and 8), \( \theta = 0 \) is considered and two different initial conditions are used: (i) gamma distribution and (ii) uniform distribution. Again, the gamma distribution is chosen as qualitatively representative of naturally observed wealth distributions; the uniform distribution is chosen because it allows for comparability of effectiveness of each policy in meeting the needs of below-threshold individuals.

For all results in Figures 7 and 8, ‘steady state’ is considered to have been reached at time \( T \) such that \( ||f(w,T) - f(w,T-\Delta t)||_2 < 10^{-5} \); in the tables, \( T \) is rescaled by
Figure 5. Probability densities for probability of choosing donor threshold \( \theta + \epsilon \) for regressive, flat, and progressive policies with \( \theta = 0 \) and maximal wealth \( L = 100 \). The equation for these parameterized donor threshold probability distributions is given in Equation 10.

Figure 6. Flat policy comparison with agent-based simulation. A gamma initial condition is used for \( f_0(w) \) and \( 10^4 \) agents are sampled from this distribution as well. Equation (9) is used with \( \alpha = 0 \) to find the steady state solution of the Boltzmann-like equation; for the agents, interactions are randomly generated and transfers are conducted according to the microscopic description of equation (1) until all \( 10^4 \) agents are at or above threshold.

the minimum \( T \) value for comparability. \( \Delta t = 1 \) is used for simulations. Population below threshold is found as \( \int_{-\infty}^{0} f(w, T) \, dw \) for Figure 8 and is considered essentially 0 for Figure 7. Gini index \( G \) for the viable population (individuals with wealth above threshold) is calculated as \( G = 1 - 2A \), where \( A \) is the area under the Lorenz curve \((x(r), y(r))\) [14]. \( x(r) \) and \( y(r) \) are defined as follows:

\[
x(r) = \frac{\int_{0}^{r} f(s) \, ds}{\int_{0}^{\infty} f(s) \, ds}, \quad y(r) = \frac{\int_{0}^{r} (s - \theta) f(s) \, ds}{\int_{0}^{\infty} (s - \theta) f(s) \, ds}
\]
as \( r \) takes values from \( \theta \) to the maximal wealth.

Recalling that decreasing \( \alpha \) corresponds to making the policy more regressive, the following numerical observations are inspired by Figures 7 and 8: As \( \alpha \) decreases,

(i) The Gini index increases (more inequality)
(ii) The convergence rate increases
(iii) The fraction of the population below threshold at steady state decreases

In Figure 7 and Figure 8, the steady state distributions of each policy are qualitatively, and in terms of inequality, predictable or reinforce the regressive/progressive natures of the policies. As expected, a more regressive policy (lower \( \alpha \)) results in greater inequality (higher Gini index).

Interestingly, the rate of convergence for the regressive policy is greater than for the progressive policy. For example, with the Gamma initial condition (Figure 7), the progressive policy took 71 times as long as the regressive policy to reach a state where the change in \( f \) was less than \( 10^{-5} \). This is a new observation with respect to the work of [1, 2, 15], but intuitively makes sense as the higher donor thresholds preferred in the progressive model reduce the number of potential donors, making it harder for below-threshold individuals to find a donor.

Figure 8 echoes an observation made in [16], where regressive transfers were found to be a sort of cutting-stock optimization heuristic [23] for best matching all of the deficits to surpluses. We see that here also the regressive policy results in more individuals above threshold in steady state than the other policies. Essentially, by allowing individuals with small surplus to give more frequently, the regressive

| Policy     | \( T \) | Gini index |
|------------|--------|------------|
| Regressive | 1      | .785       |
| Flat       | 3.7    | .732       |
| Progressive| 71.7   | .685       |

Figure 7. Steady state distributions and data for parameterized kinetic NBT policies with initial condition \( f_0(w) \sim \text{Gamma} \).
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4. Control of NBTs. In the kinetic NBT policy model (9), $p(\epsilon)$ may be considered not to be a fixed distribution of donor thresholds, but rather a time-varying prioritization of donor thresholds, $p(\epsilon, t)$ that could be considered as a control. As such, we define a natural optimal control problem to consider:

- Minimize $\frac{d}{dt} \int_{-\infty}^{\theta} f(w, t) dw$,
- Subject to $f$ evolving according to (9),
- With $f_0(w) = f(w, 0)$ initial wealth distribution,
- Where $p(\epsilon, t)$ is the control.

Essentially the solution to this problem seeks at every time $t$ to identify $p(\epsilon, t)$ that reduces the number of individuals below threshold as quickly as possible.

For a simple form of $p$, namely

$$p_c(\epsilon, t) = \delta(\epsilon - \epsilon_0(t)), \quad (11)$$

we have from Equation (9) that $\frac{d}{dt} \int_{-\infty}^{\theta} f(w, t) dw = - \int_{-\infty}^{\theta} f(w, t) f(2\theta + \epsilon(t) - w, t) dw$. Thus, the solution can be stated by defining $\epsilon_0(t)$ as follows:

$$\epsilon_0(t) = \arg\min_{\epsilon \geq 0} \left\{ - \int_{-\infty}^{\theta} f(w, t) f(2\theta + \epsilon - w, t) dw \right\}. \quad (12)$$

Note that $\epsilon_0(t)$ is a function that acts as a parameter for the parameterized probability density function $p_c(\epsilon, t)$. Thus for a wealth distribution $f(w, t)$ at time
Figure 9. Wealth distributions at $t = 200$ for various policies. The notation used in the legend is such that $fb_p$: 0.2918 means that for the progressive policy (p), the fraction of the population below threshold ($fb_p$) is equal to 0.2918. The initial condition is chosen to be a Gamma distribution. $fb_o$ identifies the optimal policy corresponding to (11) and (12).

For a simple illustration, consider $f(w)$ to be defined as illustrated in Figure 10, and let $\theta = 0$. One can see that choosing $\epsilon = 1$ will lead to

$$
\frac{d}{dt} \int_{-\infty}^{0} f(w) \, dw = - \int_{-\infty}^{0} f(w)f(2\theta + \epsilon - w) \, dw \\
= - \int_{-1}^{0} f(w)f(1 - w) \, dw \\
= - \int_{-1}^{0} (.4)(.3) \, dw \\
= -.12
$$
Figure 10. A simple example density $f(w)$ to illustrate why the optimal control policy leads to a uniform distribution of surpluses. Choosing a donor threshold of 1 maximizes the product of matching donor-recipient densities.

whereas choosing $\epsilon = 2$ would lead to $\frac{d}{dt} \int_{-\infty}^\theta f(w) \, dw = -0.04$, and $\epsilon = 0$ leads to $\frac{d}{dt} \int_{-\infty}^\theta f(w) \, dw = -0.08$. To most quickly reduce the fraction of individuals below threshold, the interval of surpluses with the largest corresponding density would be targeted as donors.

A control problem similar to most quickly reducing the fraction of individuals below threshold - yet different - would be to most rapidly reduce the total deficit, i.e. minimize $\frac{d}{dt} \int_{-\infty}^\theta w f(w, t) \, dw$. Again considering $p_c(\epsilon, t) = \delta(\epsilon - \epsilon_0(t))$, the solution to this instantaneous control problem is given by

$$\epsilon_0(t) = \arg\min_{\epsilon \geq 0} \left\{ -\int_{-\infty}^\theta w f(w, t) f(2\theta + \epsilon - w, t) \, dw \right\}.$$

5. Conclusions. In this paper, we introduce conservative kinetic exchange models to describe a binary welfare system called need-based transfers, which is inspired by individual-to-individual gift-giving relative to a viability threshold like the exchanges of Maasai herders or vampire bats. With random interactions determining welfare exchanges and random additive changes modeling natural wealth evolution, we use moment evolution equations to describe the evolution of the shape of an NBT community wealth distribution. While increasing variance and decreasing skew cause mass to escape to the left and the distribution to flatten in this model, there is apparent convergence in the short-term to some attractor manifold. Expanding the model to consider regressive to progressive wealth redistribution policies, the kinetic model recovers observations made from an agent-based study related to inequality and fraction of individuals above the viability threshold in steady state. Additionally, it is found that regressive policies are more time-efficient at pairing donors and recipients and thus have a faster convergence rate. Finally, a control policy is identified which optimizes instantaneous reduction in fraction of individuals below threshold. These models and observations support understanding of an interesting welfare system where central redistribution is unfeasible and welfare efforts are restricted to binary donations.
Our model shares a fundamental issue with most other kinetic models of economic interactions: The wealth variable $w$ is unbounded, i.e. we assume there are enough instances of very high wealth that allows to define a probability density $f(w)$ for very high wealth. In addition, we also allow the wealth variable to become infinitely negative, i.e. we have a probability density for individuals whose wealth falls very far below threshold. While the upper limit for wealth is a moving boundary, a lower limit $B$ may represent a bankruptcy level. An interesting model for future work will be to study the influence of wealth transfers, where individuals with $w < B$ will receive NBT from a random collection of donors such that their wealth goes up to the threshold after the transfer.

Appendix A. Proof of non-increasing moments. It is claimed in Section 2 that the moments for the random interaction kinetic NBT model of Equation 2 are non-increasing. Consider the integrand in the moment evolution equation (5), and let $a = -y > 0$, and $b = x + y > 0$. The integrand becomes the following:

$$(x + y)^n - (x^n + y^n) = b^n - [(a + b)^n + (-a)^n] =: P(a, b; n).$$

If $n$ is odd, then $(-a)^n = -a^n$, and

$$P(a, b; n) = (a^n + b^n) - (a + b)^n = - \left[ \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k} b^k \right] < 0.$$

If $n$ is even, then $(-a)^n = a^n$, and

$$P(a, b; n) = b^n - (a + b)^n - a^n = - \left[ \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^k \right] - a^n < 0.$$

In conclusion, $\frac{d}{dt} M_n(t) \leq 0$ for all $n \in \mathbb{N}$.

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