Ricci flow smoothing for locally collapsing manifolds

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Abstract
We show that for certain locally collapsing initial data with Ricci curvature bounded below, one could start the Ricci flow for a definite period of time. This provides a Ricci flow smoothing tool, with which we find topological conditions that detect the collapsing infranil fiber bundles over controlled Riemannian orbifolds among those locally collapsing regions with Ricci curvature bounded below. In the appendix, we also provide a local distance distortion estimate for certain Ricci flows with collapsing initial data.

Mathematics Subject Classification 53C21 · 53C23 · 53E20

1 Introduction
The Ricci flow with initial data $(M, g)$, first introduced by Hamilton [22] on 3-manifolds to deform a given Riemannian metric $g$ with positive Ricci curvature to a canonical one, is a smooth family of Riemannian metrics $g(t)$ on $M$ solving the following initial value problem for $t \geq 0$:

$$\begin{align*}
\frac{\partial}{\partial t} g(t) &= -2\text{Rc}_{g(t)}; \\
g(0) &= g.
\end{align*}$$

(1.1)

Hamilton [22] shows that if $M$ is a closed manifold, this initial value problem is always solvable up to a certain time, depending on the initial data $(M, g)$. In harmonic coordinates, the Ricci flow becomes a non-linear heat-type equation for the metric tensor, and by the nature of the heat flows, notably Shi’s estimates [43], a key effect of running Ricci flow is that the evolved metric has much improved regularity:
Here the constants \( C_l \) depend on the dimension of \( M \), as well as \( \| Rm_g \|_{L^\infty(M, g)} \). In fact, the finiteness of \( \| Rm_g \|_{L^\infty(M, g)} \) guarantees the Ricci flow solution to (1.1) to exist for a definite amount of time determined by its value, even if \((M, g)\) is complete but non-compact.

In view of Shi’s estimates (1.2), the Ricci flow also becomes a useful tool to smooth a given Riemannian metric by replacing the initially given metric \( g \) with the evolved metric \( g(t) \), whose regularity is controlled by (1.2). In order to uniformly control the evolved metric according to (1.2), a uniform lower bound of the Ricci flow existence time then becomes crucial.

When the initial data has bounded sectional curvature, notable applications of the Ricci flow smoothing method include [9,16,17,40], where the regularity of the smoothing metric plays a key role in understanding the fine structures of the collapsing geometry with bounded curvature. Imposing uniform two-sided bounds on the sectional curvature is a rather strong requirement, and a more general (and natural) situation is to only assume a uniform Ricci curvature lower bound. Imposing uniform two-sided bounds on the sectional curvature is a rather strong requirement, and a more general (and natural) situation is to only assume a uniform Ricci curvature lower bound. In this case, the Ricci flow smoothing technique usually enables one to obtain Cheeger–Gromov (smooth) convergence to the regular part of the non-collapsing Gromov–Hausdorff (rough) limit spaces. Applications of such technique have been illustrated by the work [46] on almost Einstein manifolds, by the work [37] on the Gromov–Hausdorff limits of Kähler manifolds with Ricci curvature bounded below, and by the work [44] where the Gromov–Hausdorff limits of 3-manifolds with Ricci curvature bounded below is shown to be topological manifolds, confirming the 3-dimensional case of a conjecture due to Cheeger and Colding [7].

In all the above mentioned results, the existence time lower bound of Ricci flows—which is critical for the smoothing purpose as we have discussed—depends on the uniform volume ratio lower bound of the initial metric: when the initial volume ratio at some point becomes smaller, the existence time of the Ricci flow becomes shorter. However, in many natural situations, especially for the purpose of smoothing a given initial metric by running the Ricci flow, there may be no uniform volume ratio lower bound to be assumed; and the purpose of the current paper, sequential to the previous work [29], is then to show that in certain cases when the initial data have a uniform Ricci curvature lower bound but without any uniform volume ratio lower bound, the Ricci flow could still be started locally for a definite period of time. Throughout the paper, for a compact subset \( K \subset M \) and \( R > 0 \), we will let \( B_g(K, R) \) denote the geodesic \( R \)-neighborhood of \( K \), i.e. \( B_g(K, R) := \{ x \in M : d_g(x, K) < R \} \).

Our first result concerns starting Ricci flows with possibly collapsing initial data locally modeled on Euclidean spaces:

**Theorem 1.1** For any \( \alpha \in (0, 10^{-1}) \) and \( R \in (0, 100) \), there are positive constants \( \delta_{\varepsilon}(m, R, \alpha) < 1 \) and \( \varepsilon_{\delta}(m, R, \alpha) < 1 \) to the following effect: let \((M^m, g)\) be an \( m \)-dimensional Riemannian manifold with \( Rc_g \geq -(m-1)g \), if a connected subset \( K \) of \( M \) satisfies \( B_g(K, 2R) \subseteq M \), and if for some \( \delta \leq \delta_{\varepsilon} \), any \( p \in B_g(K, R) \) satisfies

1. \( d_{GH}(B(p, 10^{-1}R), B_{\delta}^{\varepsilon}(10^{-1}R)) < \delta \), and
2. rank \( \Gamma_g(p) = m - k \),

then there is a Ricci flow solution with initial data \((B_g(K, \frac{R}{4}), g)\), existing for a period no shorter than \( \varepsilon_{\delta}^2 \), and with curvature control

\[
\forall t \in [0, \varepsilon_{\delta}^2], \quad \sup_{B_g(K, \frac{R}{4})} |Rm_{g(t)}|_{g(t)} \leq \alpha t^{-1} + \varepsilon_{\delta}^2.
\]
Here $\tilde{\Gamma}_\delta(p)$ denotes the pseudo-local fundamental group, and is defined for each $\delta \in (0, 1)$ as

$$\tilde{\Gamma}_\delta(p) := \text{Image}\{\pi_1(B_g(p, \delta), p) \to \pi_1(B_g(K, R), p)\}.$$  

This concept is introduced in [32] and is originated from the concept of the fibered fundamental group $\Gamma(p) := \text{Image}\{\pi_1(B_g(p, \delta), p) \to \pi_1(B_g(p, 2), p)\}$, which captures all those loops based at $p$ and contained in $B_g(p, \delta)$, but are allowed to be deformed (with fixed base point) within $B_g(p, 2)$. The fibered fundamental group has been a crucial concept for our understanding of the local structure of manifolds with Ricci curvature bounded below. The important work of Kapovitch and Wilking [34] has shown that the fibered fundamental group is almost nilpotent, and so is the pseudo-local fundamental group, according to the work of Naber and Zhang [38]; see also [32, Lemma 2.2]. In fact, the almost nilpotency of groups of these kinds has been a key property to investigate even for manifolds with sectional curvature bounded below; see also the previous works [18,33] for some remarkable results.

Our previous result [32, Theorem 1.4] tells that if a closed manifold $(M, g)$ with $\text{Rc}_g \geq -(m - 1)g$ is sufficiently Gromov–Hausdorff close to some closed manifold $(N, h)$ with bounded geometry, if $b_1(M) - b_1(N) = \dim M - \dim N$, then the Ricci flow with initial data $(M, g)$ exists for a definite amount of time, independent of the volume $|M|_g$. This theorem is global in nature, and sees limited applications in general settings—similar issues arise for the work [14], where the Ricci flow smoothing is applied to closed manifolds with bounded Ricci curvature for the first time.

In contrast, an important feature of Theorem 1.1 is that it is purely local. Since the assumed Ricci curvature lower bound could be directly obtained via rescaling, this theorem lends itself as an agile smoothing tool in various contexts. Also notice that the Assumption (1) in Theorem 1.1 can be replaced as $B_g(K, R)$ is $\delta$-Gromov–Hausdorff close to a given lower dimensional smooth manifold (not necessarily complete) with a uniform lower bound on the $C^{1, \frac{1}{2}}$ harmonic radius. Moreover, this theorem can be generalized to the setting where the initial data may locally collapse with only scalar curvature bounded below: see Theorem 2.5.

If the collapsing limit is a controlled Riemannian orbifold, there are still certain conditions that guarantee the Ricci flow to exist for a definite period of time. Our second result concerns starting the Ricci flow on possibly collapsing initial data locally modeled on flat orbifolds:

**Theorem 1.2** For any $\alpha \in (0, 10^{-1})$, $m, l \in \mathbb{N}$ and $R \in (0, 100)$, there are positive constants $\delta_0(m, l, R, \alpha) < 1$ and $\varepsilon_O(m, \alpha) < 1$ to the following effect: let $(M^m, g)$ be an $m$-dimensional Riemannian manifold with $\text{Rc}_g \geq -(m - 1)g$, if a connected subset $K$ of $M$ satisfies $B_g(K, 2R) \Subset M$, and if for some $k \leq m$ and $\delta \leq \delta_0$ the following assumptions are satisfied by any $p \in B_g(K, R)$:

1. there is a finite group $G_p < O(k)$ with $|G_p| \leq l$ and a surjective group homomorphism $\phi_p : \pi_1(B_g(K, R), p) \to G_p$,
2. $d_{GH}(B_g(p, 4^{-1}R), \mathbb{R}^k(4^{-1}R)/G_p) < \delta$, and
3. rank $\Gamma_\delta(p) = m - k$,

then there is a Ricci flow solution with initial data $(B_g(K, \frac{R}{4}), g)$, existing for a period no shorter than $\varepsilon_O^2$, and with curvature control

$$\forall t \in (0, \varepsilon_O^2), \quad \sup_{B_g(K, \frac{R}{4})} |\text{Rm}_{g(t)}|_{g(t)} \leq \alpha t^{-1} + \varepsilon_O^{-2}.$$

Again, here we may say that $B_g(K, R)$ is $\delta$-Gromov–Hausdorff close to a $k$-dimensional controlled orbifold, in the sense that each point has its orbifold group of size bounded by $l$...
Definition 1.3 We say that a metric space \((Z, d_Z)\) is a locally \((l, \delta, \bar{\iota})\)-controlled \(k\)-dimensional Riemannian orbifold if

(a.) \((Z, d_Z)\) is a \(k\)-dimensional Riemannian orbifold;
(b.) \(\forall \bar{z} \in Z\), there is a finite group \(G_{\bar{z}} < O(k)\) of order not exceeding \(l\), such that

\[
d_{GH}(B_{d_Z}(\bar{z}, \bar{\iota}), \mathbb{B}^k(\bar{\iota})/G_{\bar{z}}) < \delta.
\]

Again, here we notice that each \(G_{\bar{z}} < O(k)\) may well have a non-empty fixed point set in \(S^{k-1}\). With such definition, we have

Theorem 1.4 Given \(m, l \in \mathbb{N}\) and \(\bar{\iota} \in (0, 1)\), there is a positive constant \(\delta_F(m, l, \bar{\iota}) > 0\) to the following effect: let \((M^m, g)\) be an \(m\)-dimensional Riemannian manifold with \(\text{Rec}_g \geq -(m - 1)g\), if a connected subset \(K \subseteq M\) satisfies \(B_g(K, 8\bar{\iota}) \subseteq M\), and if for some \(k \leq m\) and \(\delta \leq \delta_F\) it satisfies the following assumptions:

1. there is a locally \((l, \delta, \bar{\iota})\)-controlled \(k\)-dimensional Riemannian orbifold \((Z^k, d_Z)\) such that \(d_{GH}(B_g(K, 4\bar{\iota}), Z) < \delta_F(m, l, \bar{\iota})\),
2. with \(\Phi : B_g(K, 4\bar{\iota}) \to Z\) denoting a \(\delta\)-Gromov–Hausdorff approximation and \(G_{\Phi(p)}\) denoting the orbifold group at \(\Phi(p) \in Z\), there exists a surjective group homomorphism

\[
\phi_p : \pi_1(B_g(K, R), p) \to G_{\Phi(p)},
\]

3. for any \(p \in B_g(K, 2\bar{\iota})\), rank \(\tilde{\Gamma}_\delta(p) = m - k\),

then there is an open subset \(U \subseteq M\) with \(K \subseteq U \subseteq B_g(K, \bar{\iota})\), such that \(U\) is an infranil fiber bundle over \(Z_{4\bar{\iota}} := \{z \in Z : d_Z(z, \delta Z) > 4\bar{\iota}\}\).

The emphasis of this theorem is of course the collapsing setting, i.e. the case \(k < m\). When \(k = m\) it can be seen that there cannot be a surjective group homomorphism from the fundamental group to the orbifold group. Since otherwise, there would be a sequence of \(m\)-dimensional Riemannian manifolds with uniformly bounded curvature converging to an \(m\)-dimensional orbifold in the pointed Gromov–Hausdorff sense, and this is impossible.

Theorem 1.4 generalizes [38, Proposition 6.6] in two fronts: it removes the Ricci curvature upper bound assumption when \(|G_{\Phi(z)}| = 1\), and it allows the collapsing limit to be any...
singular orbifold when $|G_{\Phi(z)}| > 1$. While the removal of the assumed Ricci curvature upper bound in [38, Proposition 6.6] is expected among experts (see [38, Remark 6.5] and Remark 8), perhaps the more substantial contribution of Theorem 1.4 is that it copes with singular collapsing limits (see Remark 7), giving a topological condition that detects the nilpotent Killing structure \textit{a la} Cheeger, Fukaya and Gromov [9, Definition 1.5] in the context of Ricci curvature bounded from below.

From the perspective of applications, Theorem 1.4 will provide a technical tool for the possible generalization of our rigidity theorem on the first Betti number ([32, Theorem 1.1]) of Ricci curvature bounded from below.0.4–0.6] and the third last paragraph on [28, Page 2] for potential applications of Theorem 1.4 in the study of the locally bounded Ricci covering geometry (see e.g. [31])—notice that the maximal nilpotency rank assumption on the fibered fundamental groups guarantees the local universal covering to be uniformly non-collapsing, as shown in [38].

2 Initial data locally collapsing to Euclidean model spaces

In this section we prove our first Ricci flow local existence theorem. We fix a complete Riemannian manifold $(M, g)$ with Ricci curvature bounded below as $\text{Rc}_g \geq -(m - 1)g$. Given a compact subset $K \subset M$, the neighborhood $B_g(K, R) := \{ x \in M : d_g(x, K) < R \}$ has compact closure in $M$ for any $R > 0$, thanks to the completeness of $(M, g)$.

Fixing any $R > 0$ and any compact subset $K \subset M$, we could already start a Ricci flow with initial data $(B_g(K, R), g)$, based on the conformal transformation technique due to Hochard [25, §6] (see also [24,35,44] for other applications and refinements of this technique)—we will blow the boundary to infinity using a well-known conformal factor, and apply Shi’s short-time existence theorem to the newly obtained metric:

**Theorem 2.1** (Shi’s short-time existence) There are positive constants $C_S(m)$ and $T_S(m, K)$ such that if $(M^m, g)$ is an $m$-dimensional complete Riemannian manifold with sectional curvature uniformly bounded by $K > 0$ in absolute value, then there exists a complete Ricci flow solution $g(t)$ defined on $M \times [0, T_S]$, satisfying

$$\forall t \in [0, T_S], \sup_{M} |\text{Rm}_{g(t)}|_{g(t)} \leq C_{St}^{-1}. \quad (2.1)$$

We now prove the following short-time existence result:

**Lemma 2.2** (Locally starting the Ricci flow) Let $K \subset M$ be a compact subset of a complete Riemannian manifold $(M^m, g)$ with $\text{Rc}_g \geq -(m - 1)g$. For any $R > 0$ there is a smooth family of Riemannian metrics $g(t)$ on $B_g(K, \frac{R}{4})$ satisfying

$$\begin{cases}
\partial_t g(t) = -2\text{Rc}_{g(t)} & \text{on } B_g(K, \frac{R}{4}) \times [0, T], \\
g(0) = g & \text{on } B_g(K, \frac{R}{4}),
\end{cases}$$

defined up to some time $T > 0$ such that

$$\forall t \in (0, T], \sup_{B_g(K, \frac{R}{4})} |\text{Rm}_{g(t)}|_{g(t)} \leq C_{T}^{-1},$$

where the positive constants $C$ and $T$ depend on $g$, $K$ and $R$.

**Proof** Let $\{x_j\} \subset B_g(K, \frac{R}{4})$ be a maximal collection of points such that $B_g(x_j, \frac{R}{4}) \cap B_g(x_j', \frac{R}{4}) = \emptyset$ whenever $j \neq j'$. By volume comparison it is clear that $\{B_g(x_j, \frac{R}{4})\}$
covers $B_g(K, \frac{R}{4})$ and the multiplicity of this covering is bounded above, at each point of $B_g(K, \frac{R}{4})$, by $V^m_1(\frac{5R}{8})V^m_1(\frac{R}{8})^{-1}$, where $V^m_r(r)$ denotes, for any $r > 0$, the volume of a geodesic $r$-ball in the space form of sectional curvature equal to $-1$.

We now let $\psi_j : M \to [0, 1]$ be the cut-off function constructed in [6], supported in $B_g(x_j, \frac{K}{4})$ with $\psi_j|_{B_g(x_j, \frac{K}{2})} \equiv 1$, and satisfying the control $R||\nabla \psi_j||_{C^0(M)} + R^2||\Delta_g \psi_j||_{C^0(M)} \leq C(m)$. We then have $\psi := \sum_j \psi_j$ supported in $B_g(K, R)$, $\psi(x) \geq 1$ whenever $x \in B_g(K, \frac{R}{4})$, and

$$R||\nabla \psi||_{C^0(M)} + R^2||\Delta_g \psi||_{C^0(M)} \leq \frac{C(m)V^m_1(\frac{5R}{8})}{V^m_1(\frac{R}{8})} =: C_0(m, R).$$

We could find a smooth cut-off function $u : [0, \infty) \to [0, 1]$, such that $u|_{[1, \infty)} \equiv 0$, $u(0) = 1$, $-10 < u' < 0$ and $|u''| < 10$. We then define the function $\rho := u \circ \psi : B_g(K, \frac{3R}{4}) \to [0, 1]$, which clearly satisfies the gradient bound

$$R||\nabla \rho||_{C^0(B_g(K, \frac{3R}{4}))} + R^2||\Delta_g \rho||_{C^0(B_g(K, \frac{3R}{4}))} \leq 100C_0(m, R). \quad (2.2)$$

For any $\theta \in (0, \frac{1}{2})$ to be fixed later, we define the function $w_\theta : [0, 1) \to [0, \infty)$ as

$$w_\theta(s) := \begin{cases} 0 & \text{for } s \in [0, 1 - \theta); \\ -\ln(1 - (s - 1 + \theta)^2\theta^-2) & \text{for } s \in [1 - \theta, 1). \end{cases} \quad (2.3)$$

Notice that for each $s \in (1 - \frac{3}{2}\theta, 1)$, with $\zeta := \zeta(\theta, s) := \frac{1}{4}\theta(1 - s) > 0$, we have

$$0 < s - \zeta < s + \zeta < 1, \quad e^{w_\theta(s + \zeta) - w_\theta(s - \zeta)} \leq 1 + 2\theta, \quad \text{and} \quad 10^{-2}\theta^2 \leq \zeta e^{w_\theta(s - \zeta)} \leq 1. \quad (2.4)$$

In fact, we could perturb $w_\theta$ slightly to make it smooth, still vanishing identically on $[0, 1 - \frac{3}{2}\theta)$, while keeping (2.4) and the following derivative control true for $s \in [1 - \frac{3}{2}\theta, 1)$:

$$0 < w_\theta'(s) \leq \frac{2\theta}{\theta^2 - (s - 1 + \theta)^2} \quad \text{and} \quad 0 < w_\theta''(s) \leq \frac{4\theta^2}{(\theta^2 - (s - 1 + \theta)^2)^2}. \quad (2.5)$$

Now we define the conformal factor $f_\theta = w_\theta \circ \rho : B_g(K, \frac{3R}{4}) \to [0, \infty)$, and consider the new metric $h_\theta = e^{2f_\theta}g$, defined on $B_g(K, \frac{3R}{4})$. The function $f_\theta$ blows to infinity the boundary points of $\partial B_g(K, \frac{3R}{4})$, and $h$ becomes a complete Riemannian metric on $B_g(K, \frac{3R}{4})$. Since $f_\theta(x) = 0$ whenever $x \in B_g(K, \frac{R}{4})$, we have $h_\theta \equiv g$ on $B_g(K, \frac{R}{4})$, which contains $K$. Moreover, since $B_g(K, \frac{3R}{4})$ is compact in the complete Riemannian manifold $M$, we have

$$\sup_{\wedge^2 TB_g(K, \frac{3R}{4})} |K_g| =: \kappa(K, g, R) < \infty,$$

with $K_g$ denoting the sectional curvature evaluated at some tangent plane, viewed as a point in $\wedge^2 TB_g(K, \frac{3R}{4})$. We could now compute the corresponding sectional curvature of $h$ under an orthonormal frame $\{e_a\}$ as following:

$$(K_{h_\theta})_{ab} = e^{-2f_\theta}\left((K_g)_{ab} - \sum_{c \neq a, b} |\nabla_c f_\theta|^2 + \nabla^2_{aa} f_\theta + \nabla^2_{bb} f_\theta\right).$$
By the estimates (2.2) on $\rho$ and (2.5) on $u_\theta$, we have the bounds
\[
e^{-f_0} \| f_0 \|_{C^1(B_g(K, \frac{3R}{4}))} \leq 10^7 k C_0(m, R)^2 \theta^{-1} R^{-2},
\]
\[
e^{-2f_0} \| f_0 \|_{C^2(B_g(K, \frac{3R}{4}))} \leq 10^9 k^2 C_0(m, R)^2 \theta^{-2} R^{-2} + 10^3 k \| \psi \|_{C^2(M)} \theta^{-3} < \infty,
\]
and since $\kappa(g) < \infty$, while $f_0(x) \to \infty$ as $x \to \partial B_g(K, \frac{3R}{4})$, we see that
\[
\sup_{\wedge^2 T B_g(K, \frac{3R}{4})} |K_h| \leq 10^9 k^2 \left( C_0(m, R)^2 R^{-2} + \kappa(K, g, R) + \| \psi \|_{C^2(M)} \right) \theta^{-4} < \infty.
\]

(2.6)

(2.7)

We could therefore appeal to Shi’s existence theorem (Theorem 2.1) to start a Ricci flow $h(t)$ on the complete non-compact Riemannian manifold $(B_g(K, \frac{3R}{4}), h)$, which has a global curvature bound depending on $\kappa(K, g, R)$, $\| \psi \|_{C^2(M)}$ and $\theta$. The maximal existence time of the Ricci flow $h(t)$ is therefore bounded below by some $T > 0$ determined by $m$, $k$, $\theta$, $\kappa(K, g, R)$ and $\| \psi \|_{C^2(M)}$. Shi’s existence theorem also provides the following curvature bound:

\[
\forall t \in (0, T], \quad \sup_{B_g(K, \frac{R}{4})} \left| R_{h(t)} \right| \leq C_{St} T^{-1},
\]

where $C_S > 0$ depends on $m$, $k$, $\kappa(g)$ and $\| \psi \|_{C^2(M)}$. Now restricting the flow $h(t)$ to $K$, where $g \equiv h = h(0)$, we obtain the desired Ricci flow with initial data $(K, g)$ and curvature bound.

\[
\square
\]

Remark 2 In fact, we can also let $\frac{9R^2}{16} (1 - \psi)$ be obtained by slightly smoothing the square of the distance to $B_g(K, \frac{R}{4})$. The key Laplacian upper bound of this cut-off function is then a consequence of the Laplacian comparison: $\Delta_g d^2 \leq 2n$. Such cut-off function has been discussed in [42].

Although this lemma enables us to start a Ricci flow locally, we have no uniform control on the flow, as both the maximal existence time and the curvature bound in (2.1) depend on the specific geometric structure of the space in consideration, encoded in $\kappa(K, g, R)$ and $\| \psi \|_{C^2(M)}$ as shown by (2.7). To prove Theorem 1.1, we will need to reduce their dependence to only on the Ricci curvature.

A typical approach to obtain a uniform lower bound on the maximal existence time of a Ricci flow is to invoke Perelman’s pseudo-locality theorem (see [39, Theorem 10.1], or another version [46, Proposition 3.1]). In the complete non-compact setting, a detailed proof of the pseudo-locality theorem could be found in [5, Theorem 8.1]. We will let $I_g(U)$ denote the isoperimetric constant of the domain $U$ equipped with the metric $g$, while letting $I_m = m^m \omega_m$ denote the $m$-dimensional Euclidean isoperimetric constant, and state the pseudo-locality theorem as the following:

Theorem 2.3 (Perelman’s pseudo-locality for Ricci flows) For any $\alpha \in (0, 10^{-1})$, there are positive constants $\delta_P (m, \alpha) < 1$ and $\epsilon_P (m, \alpha) < 1$, such that for any $m$-dimensional complete Ricci flow solution $(M, g(t))$ defined on $t \in [0, T)$, if each time slice has bounded curvature, then for any $x \in M$ satisfying

\[
\inf_{B_g(0)} R_g(0) \geq -1,
\]

and

\[
I_g(0) \left( B_g(0)(x, 1) \right) \geq (1 - \delta_P) I_m.
\]

(2.8)

(2.9)
we have the following curvature bound for any \( t \in [0, T) \cap (0, \varepsilon_p^2] \):
\[
|\text{Rm}_{g(t)}|_{g(t)}(x_0) \leq \alpha t^{-1} + \varepsilon_p^2. \tag{2.10}
\]

For those initial data satisfying the assumptions of Theorem 2.3 at every point, we could obtain the uniform existence time lower bound by a contradiction argument: if the existence time \( T \) of the Ricci flow is shorter than \( \varepsilon_p^2 \), then for some sequence \( t_i \nearrow T \) we could observe points \( x_i \in M \) such that \( \lim_{t_i \to T} |\text{Rm}_{g(t_i)}|_{g(t_i)}(x_i) = \infty \); especially, we will get
\[
|\text{Rm}_{g(t_i)}|_{g(t_i)}(x_i) > 2\alpha T^{-1} + \varepsilon_p^2 \text{ for all } i \text{ large enough, contradicting the conclusion (2.10)}
\]

In the setting of Theorem 1.1, however, we could not directly apply the pseudo-locality theorem to the Ricci flow obtained from Lemma 2.2, since the almost Euclidean isoperimetric constant assumption (2.9) fails drastically for the initial data in our consideration. In order to overcome this difficulty, we will pull the conformally transformed initial metric back to the local universal covering space, which, under the maximal rank assumption of Theorem 1.1 (Item (2)), is non-collapsing (see [38, Proposition 5.9]). By the known connection between the isoperimetric constant and the volume ratio lower bound in the setting of Ricci curvature bounded below (see e.g. [20]), we then expect to improve the isoperimetric constant lower bound on the covering space. We begin with the following

**Lemma 2.4** (Almost Euclidean condition for local normal covering spaces) For any sufficiently small \( \varepsilon > 0 \) fixed, there are positive constants \( \delta_{AE} \leq 1 \) and \( r_{AE} \leq 1 \), solely determined by \( \varepsilon \) and \( m \), to the following effect: let \( B_g(p, 10) \) be a geodesic ball in a complete Riemannian manifold \((M^m, g)\) with \( \text{Rc}_g \geq -(m-1)g \), let \( \pi : Y \to B_g(p, 10) \) be any normal covering with deck transformation group \( G \); suppose for some \( \delta \leq \delta_{AE} \) it holds

1. \( d_{GH}(B_g(p, 10), \mathbb{B}^k(10)) < \delta \), and
2. the group \( \widehat{G}_{\delta}(p) := \{ \gamma \in G : d_{\pi^*g}(\gamma \cdot \tilde{p}, \tilde{p}) < 2\delta \} \) has nilpotency rank equal to \( m - k \),

then for any \( r \in (0, r_{AE}] \) and any \( \bar{x} \in \pi^{-1}(B_g(p, 7)) \subset Y \) we have
\[
|B_{\pi^*g}(\bar{x}, r)|_{\pi^*g} \geq (1 - \varepsilon)\omega_mr^m, \tag{2.11}
\]
and
\[
I_{\pi^*g}(B_{\pi^*g}(\bar{x}, r)) \geq (1 - \varepsilon)I_m. \tag{2.12}
\]

**Remark 3** We will rely on [38, Lemma 5.3 (ii)], the “non-localness” property of the nilpotency rank, whose proof is based on [38, Theorem 2.26] and [38, Lemma 5.2]. Here we notice that [38, Lemma 5.2] applies to any discrete isometric group action, and if we replace [38, Theorem 2.26] by [38, Theorem 4.25], then the conclusion of [38, Lemma 5.3 (ii)] holds for any normal covering: if \( \pi : X \to B_g(p, 10) \) is a normal covering with deck transformation group \( G \), then for any \( \varepsilon > 0 \) sufficiently small, there is a constant \( \Psi_{NZ}(\varepsilon/m) \in (0, \varepsilon) \), such that
\[
\forall x \in B_g(p, 7), \quad \text{rank} \, \widehat{G}_\varepsilon(x) \geq \text{rank} \, \widehat{G}_{\Psi_{NZ}(\varepsilon/m)}(p), \tag{2.13}
\]
where we recall that \( \widehat{G}_{\delta}(x) = \{ \gamma \in G : d_{\pi^*g}(\gamma \cdot \tilde{x}, \tilde{x}) < 2\delta \} \), for any \( \tilde{x} \in \pi^{-1}(x) \subset X \).

**Proof** By [4, Theorem 1.1], we obtain dimensional constants \( C_{0,m} > 0, \tilde{\delta}_{0,m} > 0, \tilde{\varepsilon}_{0,m} > 0 \) and \( \tilde{\eta}_{0,m} > 0 \). We require that \( \varepsilon \leq \min \{ \delta_{0,m}, \tilde{\delta}_{0,m}, \tilde{\eta}_{0,m}, 10^{-1} \} \) and put \( \varepsilon' := \frac{\varepsilon}{16} \min \{ 1, C_{0,m} \} \) for all such \( \varepsilon \). We also let \( r_0 = r_0(\varepsilon') \in (0, 1) \) be the constant such that
\[
\forall r \in (0, r_0], \quad (1 - \varepsilon')\omega_mr^m \leq V^m_{r_0}(r) \leq (1 + \varepsilon')\omega_mr^m. \tag{2.14}
\]
where \( V^m_{\lambda_1}(r) \) is the volume of geodesic \( r \)-ball in the space form of sectional curvature equal to \(-1\).

By Colding’s volume continuity theorem, [12, Main Lemma 2.1], we obtain for \( \varepsilon' \) the corresponding positive constants \( \delta_C = \delta_C(\varepsilon') < 1 \), \( \Lambda_C = \Lambda_C(\varepsilon') < 1 \) and \( R_C = R_C(\varepsilon') > 1 \), such that if the Ricci curvature is bounded below by \(- (m - 1) \Lambda_C\), the Gromov–Hausdorff distance between a geodesic \( R_C \)-ball (in the universal covering space) and \( \mathbb{B}^m(R_C) \) is less than \( \delta_C \), then the concentric unit geodesic ball has volume no less than \( (1 - \varepsilon') \omega_m \). We then put \( \varepsilon'' := r_0 \Lambda_C \delta_C R_C^{-1} \sqrt{\varepsilon'/((m - 1))} \) in [38, Proposition 5.4] to obtain positive constants \( \delta_{NZ} = \delta_{NZ}(\varepsilon'') < 1 \) and \( r' := r_{NZ}(\varepsilon'') \in (\delta_{NZ}(\varepsilon''), 1) \), so that given the maximal rank assumption, when a geodesic 2-ball (in the collapsing manifold) is \( \delta_{NZ} \)-close to \( \mathbb{B}^k(2) \) in the pointed Gromov–Hausdorff sense, the corresponding geodesic \( r' \)-ball in the universal covering space is \( \varepsilon'' \)-close to \( \mathbb{B}^m(r') \) in the pointed Gromov–Hausdorff topology. Finally we put \( \delta_{AE}(\varepsilon) := \Psi_{NZ}(\delta_{NZ}|m) \leq \delta_{NZ}(\varepsilon'') \) —here \( \Psi_{NZ} > 0 \) is the uniform constant obtained in [38, Lemma 5.3] (see also Remark 3)—we point out that \( \Psi_{NZ}(\delta|m) \) is monotone increasing in \( \delta \), as readily checked from the proof of [38, Lemma 5.3].

Now suppose \( B_{\hat{g}}(p, 10) \subset M \) satisfies the assumptions (1) and (2) with \( \delta < \delta_{AE} \). For any \( x \in B_{\hat{g}}(p, 7) \), since \( B_{\hat{g}}(x, 2) \subset B_{\hat{g}}(x, 10) \), by \( d_{GH}(B_{\hat{g}}(p, 10), \mathbb{B}^k(10)) < \delta \) we have

\[
d_{GH}(B_{\hat{g}}(x, 2), \mathbb{B}^k(2)) < \delta. \tag{2.15}
\]

On the other hand, the given normal covering \( \pi : X \to B_{\hat{g}}(p, 10) \) restricts to a normal covering \( \pi^{-1}(B_{\hat{g}}(x, 2)) \to B_{\hat{g}}(x, 2) \) with deck transformation group \( G \). Therefore, with some \( \tilde{x} \in \pi^{-1}(x) \subset X \) fixed, we have

\[
\widehat{\varrho}_{\delta_{NZ}(x)} := \{ \gamma \in G : d_{\pi^*g}(\gamma \cdot \tilde{x}, \tilde{x}) < 2\delta_{NZ} \}.
\]

By [38, Lemma 5.3 (ii)], especially (2.13), we have \( \text{rank} \widehat{\varrho}_{\delta_{NZ}(x)} \geq \text{rank} \widehat{\varrho}_{\delta_{NZ}(\delta_{NZ}|m)}(p) \). Moreover, since \( \delta < \delta_{AE} = \Psi_{NZ}(\delta_{NZ}|m) \), we have \( \widehat{\varrho}_{\delta}(p) \leq \widehat{\varrho}_{\delta_{AE}}(p) \), whence the lower bound

\[
\text{rank} \widehat{\varrho}_{\delta_{NZ}(x)} \geq \text{rank} \widehat{\varrho}_{\delta}(p) = m - k. \tag{2.16}
\]

Recalling that \( \delta_{NZ} = \delta_{NZ}(\varepsilon'') \), we could apply [38, Proposition 5.4] with (2.15) and (2.16) to see

\[
d_{GH}(B_{\pi^*\hat{g}}(\tilde{x}, r'), \mathbb{B}^m(r')) \leq \varepsilon'' r'.
\]

Now consider the rescaled metric \( \tilde{g} := \lambda^{-2} \pi^*g \) with

\[
\lambda(\varepsilon) := \min \left\{ r_0, \Lambda_C, \sqrt{\varepsilon'/((m - 1))}, r' R_C^{-1} \right\}, \tag{2.17}
\]

we have for any \( \tilde{x} \in \pi^{-1}(B_{\hat{g}}(p, 7)) \) that

\[
d_{GH}(B_{\tilde{g}}(\tilde{x}, R_C), \mathbb{B}^m(R_C)) < \delta_C,
\]

and we have the Ricci curvature lower bound

\[
\text{Rc}_{\tilde{g}} \geq - \min \left\{ (m - 1) \Lambda_C^2, \varepsilon' \right\} \tilde{g}. \tag{2.18}
\]

Consequently, applying [12, Main Lemma 2.1] we have for any \( \tilde{x} \in \pi^{-1}(B_{\hat{g}}(p, 7)) \),

\[
\left| B_{\tilde{g}}(\tilde{x}, 1) \right|_{\tilde{g}} \geq (1 - \varepsilon') \omega_m. \tag{2.19}
\]
By the volume ratio comparison (2.14) we have
\[ \forall r \in (0, 1), \forall \tilde{x} \in \pi^{-1}(B_{\bar{g}}(p, 7)), \quad |B_{\bar{g}}(\tilde{x}, r)|_{\bar{g}} \geq (1 - \varepsilon)\omega_m r^m. \quad (2.20) \]

On the other hand, by (2.18), (2.19) and the choice of \( \varepsilon' \), we could apply [4, Theorem 1.1] to see
\[ \forall r \in (0, \varepsilon'), \quad I_{\bar{g}}(B_{\bar{g}}(\tilde{x}, r)) \geq (1 - \varepsilon)I_m. \quad (2.21) \]
for any \( \tilde{x} \in \pi^{-1}(B_{\bar{g}}(p, 7)) \). Notice the scaling invariance of these estimates.

Now we scale back to the original metric \( \pi^*g \) and the estimates (2.20) and (2.21) remain valid for geodesic balls centered anywhere in \( X \), with radii not exceeding \( r_{AE} := \varepsilon'_1\lambda(\varepsilon) \). By (2.17) and the bound of \( r' \in (\delta'_1, 1) \) in [38, Proposition 5.8], we have the following bound on \( r_{AE} \), solely determined by \( m \) and \( \varepsilon' \):
\[ \min \left\{ r_0(\varepsilon'), \Lambda_C(\varepsilon'), \sqrt{\varepsilon'/(m - 1)}, \delta_{AE}(\varepsilon'')R_C(\varepsilon')^{-1} \right\} \varepsilon' \leq r_{AE} \leq \varepsilon', \quad (2.22) \]
with \( \varepsilon' = \frac{\varepsilon}{10} \min \left\{ 1, C_{0,m}^{-1} \right\} \) and \( \varepsilon'' \) determined by \( \varepsilon' \) via Colding’s theorem. \( \square \)

With the almost Euclidean isoperimetric constant estimate on the local universal covering space, we could apply the pseudo-locality theorem on the covering space and write down the details of proving Theorem 1.1.

**Proof of Theorem 1.1** The short-time existence of the Ricci flow is already shown in Lemma 2.2, and here we only need to bound the existence time from below by a constant only depending on \( m \). We fix an \( \alpha \in (0, 10^{-1}) \) to begin our discussion.

Recall that the conformal factor \( f_\theta \) we used in Lemma 2.2 involves an extra parameter \( \theta \in (0, \frac{1}{2}) \), which we now fix. Let \( \varepsilon' := \frac{\delta_p}{4} \), where \( \delta_p = \delta_p(m, \alpha) > 0 \) is the dimensional constant provided by Theorem 2.3. We then define (assuming \( \delta_p \leq 1 \))
\[ \theta := \frac{1}{2} \left( \frac{4 - \delta_p}{4 - 2\delta_p} \right)^{-\frac{1}{m-1}} - \frac{1}{2} \quad (2.23) \]
This dimensional constant is thus defined so that \( (1 - 2\theta)^{-m}(4 - \delta_p) = (4 + \delta_p) \). Once \( \theta \) is chosen, we will drop this subscript in our writing of \( f = f_\theta \) and \( w = w_\theta \), given in Lemma 2.2.

We set \( \delta_E(m, R, \alpha) := 10^{-2}\delta_{AE}(\varepsilon')R \).

Letting \( \pi : X \rightarrow B_{\bar{g}}(K, R) \) denote the universal covering map, we now pull the metric \( g \) back to \( X \). Clearly \( \text{Rc}_{\pi^*g} \geq -(m - 1)\pi^*g \) on \( X \). We notice that the conformal change of the pull-back metric is exactly the pull-back of the conformal change:
\[ \pi^*h = \pi^*(e^{2f}g) = e^{2f_{\pi^*}}\pi^*g, \]
and the conformal factor is \( e^{f_{\pi^*}} =: e_{\bar{f}}^\pi \). We also denote \( \bar{\rho} := \pi^*\rho \). Equipping \( X_0 := \pi^{-1}(B_{\bar{g}}(K, \frac{3R}{4})) \) with the covering metric \( \pi^*h \), we have made it a complete Riemannian manifold. Moreover, the bounds on the sectional curvature of \( \pi^*h \) remain the same as that of \( h \): by (2.7) we have
\[ \sup_{\wedge^2TX_0} |K_{\pi^*h}| \leq 10^9k^2 \left( C_0(m, R)^2R^{-2} + \kappa(K, g, R) + \|\psi\|_{C^2(M)} \right) \theta^{-4} < \infty. \quad (2.24) \]
Therefore, we could start Ricci flow \( (X_0, \tilde{h}(t)) \) from the initial data \( (X_0, \pi^*h) \) by Theorem 2.1. Notice that the Ricci flow \( \tilde{h}(t) \) is invariant under the action of \( \pi_1(B_{\bar{g}}(K, R)) \),

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as does the initial data \( h \); therefore \((X_0, \tilde{h}(t))\) covers \((B_g(K, \frac{3R}{4}), h(t))\), the original flow obtained in Lemma 2.2, and consequently, the existence time of \( h(t) \) is the same as the existence time of \( \tilde{h}(t) \), which we will bound from below via the pseudo-locality theorem (Theorem 2.3).

To apply Theorem 2.3 to the Ricci flow \((X_0, \tilde{h}(t))\), we start with checking the relevant properties satisfied by \((B_g(K, \frac{3R}{4}), g)\). By our assumption that \( \delta_E = 10^{-2}\delta_{AE} R \), we have for any \( \tilde{x} \in X_0 \),

\[
d_{GH} \left( B_{\pi^*g}(\pi(\tilde{x}), 10^{-1} R), \mathbb{B}^m(10^{-1} R) \right) < 10^{-2}\delta_{AE} R,
\]

and that \( \text{rank} \ G_{\delta_E}(\pi(\tilde{x})) = \text{rank} \ G_{\delta_E}(\pi(\tilde{x})) = m - k \); therefore, after rescaling \( \pi^*g \mapsto 10^4 R^{-2}\pi^*g \) and applying Lemma 2.4, we see that

\[
\forall \tilde{x} \in X_0, \; \forall r \in (0, 10^{-2}r_{AE} R], \; I_{\pi^*g} \left( B_{\pi^*g}(\tilde{x}, r) \right) \geq (1 - \varepsilon') I_m.
\]

(2.25)

Now we consider the corresponding bounds on the metric \( \pi^*h \). We first see that the scalar curvature is uniformly bounded from below. By the standard formula, we have

\[
R_{\pi^*h} = e^{-2f} \left( R_{\pi^*g} - \frac{4(m - 1)}{m - 2} e^{-m-2f} \Delta_{\pi^*g} e^{m-2f} \right)
= e^{-2f} \left[ R_{\pi^*g} - (m - 1) \left( (2w'' + (m - 2)(w')^2) |\nabla_{\pi^*g} \tilde{\rho}|_{\pi^*g}^2 + 2w' \Delta_{\pi^*g} \tilde{\rho} \right) \right].
\]

(2.26)

Since \( \pi \) is a covering map, by (2.2) we have

\[
R \|\nabla_{\pi^*g} \tilde{\rho}\|_{C^0(X)} + R^2 \|\Delta_{\pi^*g} \tilde{\rho}\|_{C^0(X)} \leq 100C_0(m, R).
\]

Consequently, as \( R_{\pi^*g} \geq -(m - 1)\pi^*g \), we have the scalar curvature lower bound for \( \pi^*h \) as

\[
R_{\pi^*h} \geq -10^2m(m - 1)C_0(m, R)^2 R^{-2}\theta^{-4} =: C_1(m, R, \theta).
\]

(2.27)

Notice that here \( \theta \) is already determined by \( \alpha \in (0, 10^{-1}) \).

Moreover, we need to control the isoperimetric constant of \( \pi^*h \) around any given point in \( X_0 \), and we only need to focus on the region \( U := \tilde{\rho}^{-1}((1 - \frac{3}{2}\theta, 1)) \subset X_0 \), since \( \pi^*g(\tilde{x}) = \pi^*h(\tilde{x}) \) whenever \( \tilde{\rho}(\tilde{x}) \leq 1 - \frac{3}{2}\theta \). Fixing any \( \tilde{x} \in U \), let us denote \( s = \tilde{\rho}(\tilde{x}) \) and define \( U_s := \tilde{\rho}^{-1}(s - \zeta, s + \zeta) \) for the moment, with \( \zeta = \frac{1}{3}\theta(1 - s) \). The key feature for the points in \( U_s \) is the following metric equivalence:

\[
\forall \tilde{y} \in U_s, \; e^{2w(s-\zeta)}\pi^*g(\tilde{y}) \leq \pi^*h(\tilde{y}) \leq e^{2w(s+\zeta)}\pi^*g(\tilde{y}).
\]

(2.28)

As we have \( 1 - \frac{3}{2}\theta < s < 1 \), and as indicated in (2.4), we could consider all radii \( r > 0 \) bounded as

\[
r < \text{min} \left\{ \frac{\theta^2 R}{10^6 C_0(m, R)}, \frac{10^{-2}r_{AE}(\varepsilon') R}{10^4 C_0(m, R)}, \frac{10^{-2}r_0(\varepsilon') R}{10^4 C_0(m, R)} \right\},
\]

(2.29)

where \( \zeta = \frac{1}{3}\theta(1 - s) \) and \( r_0(\varepsilon') \) is defined as in (2.14).

We now claim that \( B_{\pi^*h}(\tilde{x}, r) \subset U_s \); suppose otherwise, there is some \( \tilde{y} \in B_{\pi^*h}(\tilde{x}, r) \setminus U_s \), we could then let \( \gamma : [0, 1] \to X_0 \) be a minimal \( \pi^*h \)-geodesic connecting \( \tilde{x} = \gamma(0) \) to \( \tilde{y} = \gamma(1) \); since \( \tilde{x} \in U_s \), there must be a first time \( t_0 \in (0, 1) \) such that \( \gamma(t_0) \in \partial U_s \) and
\( \gamma([0, t_0]) \subset U; \) by the continuity of \( \rho, \) we know that \(|s - \tilde{\rho}(\gamma(t_0))| = \xi; \) but by (2.4) and (2.28) we have
\[
|\tilde{\rho}(\tilde{x}) - \tilde{\rho}(\gamma(t_0))| \leq \int_0^{t_0} |\pi^*g(\nabla_{\pi^*g} \tilde{\rho}(\gamma(t)), \gamma'(t))| \, dt
\]
\[
\leq \|u'|_{C^0([0,1])} \|\nabla_g \psi\|_{C^0(B_2(\kappa,R))} d_{\pi^*g}(\tilde{x}, \gamma(t_0))
\]
\[
\leq 10^3 C_0(m, R) R^{-1} d_{\pi^*g}(\tilde{x}, \gamma(t_0))
\]
\[
\leq 10^3 C_0(m, R) R^{-1} e^{-w(s-\xi)} r
\]
\[
\leq \frac{\xi}{10};
\]
this provides a contradiction, and the claim is proven.

Consequently, on \( B_{\pi^*h}(\tilde{x}, r) \) we also have the uniform metric equivalence (2.28), and so we could easily compare the volume of subsets in the ball, measured in the two metrics. Now for any region \( \Omega \subset B_{\pi^*h}(\tilde{x}, r), \) we could estimate
\[
|\Omega|_{\pi^*h} \leq e^{m(w(s+\xi))}|\Omega|_{\pi^*g} \quad \text{and} \quad |\partial \Omega|_{\pi^*h} \geq e^{(m-1)w(s-\xi)}|\partial \Omega|_{\pi^*g},
\]
and by (2.25), we get the scaling-invariant estimate
\[
|\partial \Omega|_{\pi^*h}^m \geq \frac{(1 - \epsilon') I_m}{(1 + 2\theta)^{m-1}} |\Omega|_{\pi^*h}^{m-1}.
\]
Since \( \Omega \subset B_{\pi^*h}(\tilde{x}, r) \) is arbitrarily chosen, by our choice of the constants \( \epsilon' \) and \( \theta, \) we get the control of the isoperimetric constant:
\[
I_{\pi^*h}(B_{\pi^*h}(\tilde{x}, r)) \geq (1 - \delta_p) I_m
\]
for any \( \tilde{x} \in X_0 \) and any \( r > 0 \) in the range specified by (2.29). Notice that the estimate (2.31) remains unchanged under rescaling of the metric.

Now by (2.27), (2.25) for the region where \( \pi^*h = \pi^*g \) and (2.31) for the region \( U, \) we could apply the pseudo-locality theorem to obtain a lower bound of the existence time: Consider the dimensional constant
\[
\mu(m, R, \alpha) := \min \left\{ C_1(m, R, \theta), \frac{\theta^2 R}{10^6 C_0(m, R)}, 10^{-2} R A E R \right\},
\]
and rescale the the flow \( (X_0, \tilde{h}(t)) \) to \( (X_0, \bar{h}(s)), \) with \( s = \mu^{-2} t \) and \( \bar{h}(s) := \mu^{-2} \tilde{h}(s). \) Now we have
\[
R_{\tilde{h}(0)} \geq -1 \quad \text{and} \quad \forall \tilde{x} \in X_0, \quad I_{\tilde{h}(0)} \left(B_{\tilde{h}(0)}(\tilde{x}, 1)\right) \geq (1 - \delta_p) I_m.
\]
Applying Theorem 2.3, we see that the the existence time of the rescaled flow \( (X_0, \tilde{h}(s)) \) is bounded below by \( \epsilon^2_p > 0, \) as previously discussed. Now scaling back, we see that the existence time \( T \) for the flow \( (X_0, \tilde{h}(t)) \) is bounded below as
\[
T \geq \mu^2 \epsilon^2_p =: \epsilon^2_E(m, R, \alpha),
\]
which is a constant only depending on \( m \) and \( \alpha, \) whence the desired lower bound of the existence time for the original Ricci flow. Moreover, we have the curvature estimate
\[
\forall s \in (0, \epsilon^2_p], \quad \sup_{X_0} \left| R_{\tilde{h}(s)} \right|_{\tilde{h}(s)} \leq \alpha s^{-1} + \epsilon^{-2}.
\]
Rescaling back and restricting our attention to \( B_{\delta}(K, \frac{R}{4}) \), which is unaffected by the conformal transformation, we get the desired curvature control.

**Remark 4** The application of the pseudo-locality theorem in proving the existence of Ricci flows seems to appear in the work [47] of Topping for the first time. Here we notice that checking the completeness of the pull-back metric on the local universal covering is necessary for the application of Theorem 2.3. An example where the pseudo-locality theorem fails for incomplete Ricci flows is given by Topping; see e.g. [28, Example 0.3].

**Remark 5** It seems that the initially assumed Ricci lower bound allows us to apply the version of the pseudo-locality theorem due to Tian and the second-named author ([46, Proposition 3.1]) directly, as is done in our previous result [32, Theorem 1.4]. But the conformal transformation involved here prevents us from doing so—we do not have a uniform \( C^2 \) control of the cut-off function \( \psi \) in the definition of the conformal factor: with \( \tilde{f} = \pi^*(w \circ u \circ \psi) \), the lower bound of the conformally transformed Ricci curvature

\[
\text{Re}_{\pi^*h} = \text{Re}_{\pi^*g} - (m-2)\left( \nabla^2 \tilde{f} - \nabla \tilde{f} \otimes \nabla \tilde{f} \right) - \left( \Delta \tilde{f} + (m-2)|\nabla \tilde{f}|^2 \right) \pi^*g
\]

depends on the full Hessian bound \( \| \nabla^2 \tilde{f} \|_{C^0(B_{\delta}(K, R))} \), which has no uniform *apriori* estimate. In contrast, the scalar curvature of \( \pi^*h \) only depends on \( \| \Delta \psi \|_{C^0(B_{\delta}(K, R))} \), which is indeed uniformly bounded and leaves a chance to apply Perelman’s original version of the pseudo-locality theorem.

Extracting the technical essence involved in the proof of Theorem 1.1, we can generalize this theorem to the setting where only the initial *scalar curvature* lower bound is known.

**Theorem 2.5** For any \( \alpha \in (0, 10^{-1}) \), \( C > 0 \) and \( R > 0 \) there are constants \( \delta_{SC}, \epsilon_{SC} \in (0, 1) \) solely determined by \( m, C, R \) and \( \alpha \), such that if \( K \) is a compact subset of an \( m \)-dimensional Riemannian manifold \((M, g)\) with scalar curvature bounded below by \(-1\), satisfying

1. the universal covering space of \( B_{\delta}(K, R) \) has isoperimetric constant no less than \((1 - \delta_{SC})I_m \) everywhere at scale \( 1 \), and
2. there is a cut-off function \( \rho \) supported in \( B_{\delta}(K, \frac{3R}{4}) \) such that

\[
0 \leq \rho \leq 1, \quad \rho|_{B_{\delta}(K, \frac{R}{4})} \equiv 1 \quad \text{and} \quad R \| \rho \|_{C^1(M)} + R^2 \max_M \Delta \rho \leq C,
\]

then there is a Ricci flow solution with initial data \((B_{\delta}(K, \frac{3R}{4}), g)\), existing at least up to \( \epsilon_{SC}^2 \), and satisfying

\[
\forall t \in (0, \epsilon_{SC}^2], \quad \sup_{B_{\delta}(K, \frac{R}{4})} \left| Rm_{g(t)} \right|_{g(t)} \leq \alpha t^{-1} + \epsilon_{SC}^{-2}.
\]

**Sketch of proof** To indicate the proof, while Lemma 2.2 directly enables us to start a Ricci flow with initial data \((K, g)\), we notice that the conformally transformed metric using \( \tilde{\rho} \) on the universal covering of \( B_{\delta}(K, R) \) has scalar curvature lower bound given by (2.26), and Assumption (2) provides a uniform lower bound of this scalar curvature, just as (2.27); on the other hand, Assumption (1) together with the \( \| \rho \|_{C^1(M)} \) bound ensure that the isoperimetric constant of the conformally transformed metric (on the universal covering) is sufficiently close to the Euclidean isoperimetric constant, verifying (2.31)—we have all the ingredients ready to apply the pseudo-locality theorem (Theorem 2.3) and obtain the desired existence time and curvature bounds of the Ricci flow.

Similar to Theorem 1.1, this theorem may serve as a smoothing tool in the study of the uniform behavior of Riemannian manifolds whose scalar curvature are locally bounded below, a program initiated by Gromov [21]; see also [2,45].
3 Initial data locally collapsing to orbifold model spaces

This section is devoted to the proof of Theorem 1.2, based on the discussion in the last section. We will prove the theorem by a contradiction argument. Now suppose for some \( R > 0 \) fixed we have a sequence of data \( \{ K_i \subset (M_i, g_i) \} \) satisfying the assumptions of the theorem, but with the Ricci flow existence time on \( B_{g_i}(K_i, \frac{R}{4}) \) decaying to zero. Notice that by Lemma 2.2, as long as the ambient manifold has Ricci curvature uniformly bounded below, we could start the Ricci flow regardless of the specific geometry of the region \( B_{g_i}(K_i, \frac{R}{4}) \).

Now letting \( \pi_i : X_i \rightarrow B_{g_i}(K_i, R) \) denote the universal covering equipped with the covering metric \( \pi_i^* g_i \) and setting \( X_{i,0} = \pi_i^{-1}(B_{g_i}(K_i, \frac{3R}{4})) \), if \( X_{i,0} \) were everywhere almost Euclidean at a fixed scale, i.e. (2.25) holds in the setting of Theorem 1.2, then applying Theorem 2.3 to the conformally transformed metric, we could uniformly bound the Ricci flow existence time from below. Therefore, by the contradiction hypothesis, we could find \( p_i \in B_{g_i}(K_i, \frac{3R}{4}) \) such that geodesic balls centered at \( \tilde{p}_i \in \pi_i^{-1}(p_i) \subset X_{i,0} \) with a fixed size cannot be locally almost Euclidean.

More specifically, we may assume that there is some \( G < O(k) \) with \( |G| \leq l \) such that

\[
d_{GH}\left(B_{g_i}(p_i, 4^{-1}R), \mathbb{B}^k(4^{-1}R)/G\right) = \delta_i \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]

however, for \( \epsilon' = \frac{\delta_0(\alpha)}{4} \) and any \( r \in (0, \frac{R}{10}) \) fixed, the isoperimetric constant satisfies

\[
I_{\pi^* g_i} \left( B_{\pi^* g_i}(\tilde{p}_i, r) \right) \leq (1 - \epsilon') I_m. \tag{3.1}
\]

On the other hand, from Assumption (1) of Theorem 1.2, we have surjective group homomorphisms \( \phi_i : \pi_1(B_{g_i}(K_i, R), p_i) \rightarrow G \), and consequently we have \( \pi_1(B_{g_i}(K_i, R), p_i)/\ker \phi_i = H_i \cong G \). To facilitate our argument, we will let \( H \) denote all \( H_i \) as they are all isomorphic to one another, and let \( \phi : H \overset{\approx}{\rightarrow} G \) denote the group isomorphism induced by \( \phi_i \) for \( i \) large enough (after possibly passing to a sub-sequence); we will also perform the rescaling \( g \mapsto 10^5 R^{-2}g =: g' \).

Since \( \ker \phi_i \cong \pi_1(B_{g_i}(K_i, R), p_i) \), the covering map \( \pi_i \), when restricted to the \( g'_i \)-metric closure \( \pi_i^{-1}(B_{g'_i}(p_i, 50)) \), induces a normal covering \( \pi_{i,0} : \pi_i^{-1}(B_{g'_i}(p_i, 50)) \rightarrow Y_i := \pi_i^{-1}(B_{g'_i}(p_i, 50))/\ker \phi_i \), equipped with the quotient metric \( \bar{g}_i \) of \( \pi_i^* g'_i \). Clearly, the deck transformation group of this covering is \( \ker \phi_i \). Moreover, the deck transformation group of the covering \( \bar{\pi}_i : Y_i \rightarrow B_{g'_i}(p_i, 50) \equiv Y_i/H \) is nothing but \( H \), which is isomorphic to the local orbifold group \( G \). Clearly, \( H \leq Isom(Y_i, \bar{g}_i) \).

To produce a contradiction, we will show that for \( i \) large enough, \( B_{g_i}(\tilde{p}_i, 10) \subset Y_i \) is sufficiently Gromov–Hausdorff close to \( \mathbb{B}^k(10) \), for some \( \tilde{p}_i \in \pi_i^{-1}(p_i) \subset Y_i \) fixed. Applying Lemma 2.4 to the normal covering \( \pi_{i,0} \), we could then find an estimate contradicting (3.1) for some \( r > 0 \) fixed. To prove such Gromov–Hausdorff proximity we rely on yet another contradiction argument: assuming

\[
\liminf_{i \rightarrow \infty} d_{GH}\left( B_{\bar{g}_i}(\tilde{p}_i, 10), \mathbb{B}^k(10)\right) \geq \delta_{AE} \left( 2^{-1} \epsilon' \right), \tag{3.2}
\]

\[\square\] Springer
we will deduce a contradiction from the situation summarized in the following diagram:

\[
\begin{array}{c}
\pi_i^{-1}(B_{\tilde{g}_i}(p_i, 50)), \pi_i^*(\tilde{g}_i) \xrightarrow{\pi_i} (Y_i, \tilde{g}_i) \xrightarrow{\delta_{AE} = \text{GH apart}} (\mathbb{B}^k(50), \tilde{\text{g}_{\text{Euc}}}) \\
\end{array}
\]

\[
\begin{array}{c}
\pi_i \\
\end{array}
\]

\[
\begin{array}{c}
(B_{\tilde{g}_i}(p_i, 50), \tilde{g}_i) \xleftarrow{GH \text{ close}} (\tilde{Z}, d_Z) \\
\end{array}
\]

(3.3)

Since $\text{Rec}_{\tilde{g}_i} \geq -(m - 1)\tilde{g}_i$, we have $\{(Y_i, \tilde{g}_i, \tilde{p}_i)\}$ sub-converges to some $(Y_\infty, d_\infty, \tilde{p}_\infty)$ in the pointed and $H$ equivariant Gromov–Hausdorff topology. Especially, since $B_{\tilde{g}_i}(p_i, 50) \to Z := \overline{\mathbb{B}^k(50)}/G$ in the pointed Gromov–Hausdorff topology, we see that $Y_\infty/H \equiv Z$. We will identify $Y_\infty$ with its completion under $d_\infty$, making it a compact and connected metric space.

We now aim at proving the following

**Lemma 3.1** $B_{d_\infty}(\tilde{p}_\infty, 10) \equiv \mathbb{B}^k(10)$.

It seems not obvious how to re-construct $B_{d_\infty}(\tilde{p}_\infty, 10)$ directly out of the quotient and the group action, and we need to rely on the metric measure property of $Y_\infty$, appealing to the theory of $\text{RCD}(0, k)$ space. Eventually, we will show that $B_{d_\infty}(\tilde{p}_\infty, 20)$ is a non-collapsing $\text{RCD}(0, k)$ space, but as its $k$-dimensional Hausdorff measure is equal to the volume of $\mathbb{B}^k(20)$, by the recent work [15] we can conclude that $B_{d_\infty}(\tilde{p}_\infty, 10)$ is isometric to the $k$-Euclidean 10-ball.

**Proof** Notice that if $G < O(k)$ fixes no point on $\mathbb{S}^{k-1}$, then $\mathbb{S}^{k-1}/G$ is a manifold and $Y_\infty \equiv \overline{\mathbb{B}^k(50)}$. The situation is more complicated when $G$ does not act on $\mathbb{S}^{k-1}$ freely; see Remark 1. In this case, since $G < O(k)$, the fixed point set of any element of $G$ is a vector sub-space of $\mathbb{R}^k$. Especially, the singular part $\Sigma$ of $Z$ is cut out by sub-spaces of $\mathbb{R}^k$. Consequently, every $x \in Z$ has a small radius $r_x > 0$ such that $B_{d_Z}(x, r_x)$ is the geodesic $r_x$-ball centered at $x$ in a metric cone with vertex $x$.

On the other hand, for the finite quotient $\tilde{\pi}_\infty : Y_\infty \to Z$ we may decompose $Y_\infty$ into a union of the orbit of a fundamental domain $\Lambda$, i.e. $Y = \cup_{\gamma \in H\gamma\Lambda}$, and for any regular point $y \in Y_\infty$ there is a unique $\gamma \in H$ such that $\gamma.y \in \Lambda$—here we put $\mathcal{R} := \{\gamma \in Y_\infty : \forall y \in H \gamma \gamma, \gamma.y \neq y\}$ as the regular part of $Y_\infty$, and the singular part is $S := Y_\infty \setminus \mathcal{R}$. Notice that for $\gamma \in H\setminus\{I\} \gamma \Lambda$ may intersect $\Lambda$ non-trivially, and the intersection is contained in $S$. Moreover, $\tilde{\pi}_\infty|_{\Lambda}$ is a bijective local isometry onto $Z$, and we notice that $\tilde{\pi}_\infty$ maps $\mathcal{R}$ and $S$ respectively to the regular and singular parts of $Z$. It is also clear that $\mathcal{R}$ is an open subset.

We begin with the following

**Claim 3.2** $\tilde{p}_\infty$ is a fixed point under the action of $H \leq \text{Isom}(Y_\infty, \tilde{p}_\infty)$.

**Proof of the claim** We begin with considering any $\gamma \in H$ such that $\gamma.\Lambda \cap \overline{\Lambda} \neq \emptyset$—by the connectedness of $Y_\infty$ we can always find such a $\gamma$. If $\gamma.\tilde{p}_\infty \neq \tilde{p}_\infty$, then $\gamma.\tilde{p}_\infty \in \partial \Lambda \setminus \tilde{p}_\infty$, as $\text{int}(\Lambda)$ consists of regular points. Since $\tilde{\pi}_\infty(\tilde{p}_\infty) = \tilde{\pi}_\infty(\gamma.\tilde{p}_\infty) = z \in \Sigma$, we must have $\gamma.\tilde{p}_\infty \in S$. As the fixed point set, $S$ must be totally geodesic, and thus the minimal geodesic segment $\sigma$ connecting $\tilde{p}_\infty$ to $\gamma.\tilde{p}_\infty$ has its interior entirely lying in $S$. Since $\tilde{\pi}_\infty(S) = \Sigma$ and $\tilde{\pi}_\infty(\tilde{p}_\infty) = \tilde{\pi}_\infty(\gamma.\tilde{p}_\infty) = z$, under the local isometry $\tilde{\pi}_\infty$ the geodesic segment $\sigma$ becomes a smooth geodesic loop $\tilde{\pi}_\infty \circ \sigma$ based at $z \in Z$, whose interior lying in $\Sigma$. But this is impossible because any geodesic segment of $\Sigma$ emanating from the vertex is a straight line segment, and in particular no geodesic loop in $\Sigma$ can be based at $z \in Z$.
Setting $H_1 = \{ y \in H : \gamma \cdot \Lambda \cap \Lambda \neq \emptyset \}$, the same argument shows that if $\gamma \cdot \Lambda \cap \bigcup_{y' \in H_1} \gamma' \cdot \Lambda \neq \emptyset$, then $\gamma \cdot \bar{p}_\infty = \bar{p}_\infty$. For each $j \geq 1$, setting $H_{j+1} = \{ y \in H : \exists y' \in \bigcup_{i=1}^{j} H_i, \gamma \cdot \Lambda \cap \gamma' \cdot \Lambda \neq \emptyset \}$, then we could inductively show that $\gamma \cdot \bar{p}_\infty = \bar{p}_\infty$ if $\gamma \in H_{j+1}$. Since $Y = \bigcup_{y \in H} \gamma \cdot \Lambda$ and $Y$ is connected, we have $\bigcup_{j=1}^{l'} H_j = H$ for some $l' \leq l$, and thus we have shown that $\gamma \cdot \bar{p}_\infty = \bar{p}_\infty$ for any $\gamma \in H$.

From this claim, we see that $H$ acts on each geodesic ball centered at $\bar{p}_\infty$, and $Y_\infty = B_{d_\infty}(\bar{p}_\infty, 50)$. Moreover, $H$ sends a minimal geodesic emanating from $\bar{p}_\infty$ to another minimal geodesic emanating from $\bar{p}_\infty$, and since $Z$ is a metric cone over $S^{k-1}/G$, $Y_\infty$ is also a geodesic ball in a metric cone centered at the vertex $\bar{p}_\infty$.

In the same vein, if $\bar{x} \in S$, then its isotropy group $H_{\bar{x}} := \{ y \in H : \gamma \cdot \bar{x} = \bar{x} \}$ acts on a small ball around $\bar{x}$: since $|H| \leq l$ and $d_\infty(\gamma \cdot \bar{x}, \bar{x}) > 0$ for any $\gamma \neq H_{\bar{x}}$, we can find $\bar{r}_\infty \in (0, r_{\infty}(\bar{x}))$ such that $\gamma \cdot B_{d_\infty}(\bar{x}, \bar{r}_\infty) \cap B_{d_\infty}(\bar{p}_\infty, \bar{r}_\infty) = \emptyset$ whenever $\gamma \in H \setminus H_{\bar{x}}$, but as $H_{\bar{x}} \leq Isom(Y_\infty, d_\infty)$, we know that $H_{\bar{x}}$ acts on $B_{d_\infty}(\bar{x}, \bar{r}_\infty)$ by isometries. Moreover, the collection $\{ \gamma \cdot B_{\bar{x}} : \gamma \in H \}$ is in bijective correspondence with the left coset of $H_{\bar{x}}$. Since $H_{\bar{x}} \gamma^{-1}$ acts isometrically on $\gamma \cdot B_{\bar{x}}$, and $B_{\bar{x}} / H_{\bar{x}} \equiv (\gamma \cdot B_{\bar{x}}) / (\gamma H_{\bar{x}} \gamma^{-1})$ for any $\gamma \in H$, we have $B_{\bar{x}} / H_{\bar{x}} \equiv \bar{\pi}_\infty(B_{\bar{x}}) \subset Z$; in fact we also have $B_{\bar{x}} / H_{\bar{x}} \equiv B_{d_{\bar{x}}}(\bar{\pi}_\infty(\bar{x}), \bar{r}_\infty)$ as $B_{\bar{x}}$ is a geodesic ball and $\bar{\pi}_\infty$ is taking quotient by isometries. We further notice that $B_{d_{\bar{x}}}(\bar{\pi}_\infty(\bar{x}), \bar{r}_\infty)$, as a geodesic ball in a metric cone $Z$, is itself the geodesic $\bar{r}_\infty$-ball in a metric cone with vertex $\bar{\pi}_\infty(\bar{x})$, and consequently we know that $B_{\bar{x}}$ is the geodesic $\bar{r}_\infty$-ball centered at $\bar{x}$ in a metric cone with vertex $\bar{x} \in Y_\infty$. Especially, if $\bar{x} \in S$ has a tangent cone isometric to $\mathbb{R}^k$, then we must have $B_{\bar{x}} \equiv \mathbb{R}^k(\bar{r}_\infty)$. On the other hand, it is clear that for any $\bar{x} \in R$, $B_{d_{\bar{x}}}(\bar{x}, \bar{r}_\infty) \equiv \mathbb{R}^k(\bar{r}_\infty)$ with $\bar{r}_\infty = d_{\infty}(\bar{x}, S)$. We will also let $B_{\bar{x}}$ denote $B_{d_{\bar{x}}}(\bar{x}, \bar{r}_\infty)$ when $\bar{x} \in R$.

On the other hand, notice that every point in the regular part $R$ has (any of) its tangent cone isometric to $\mathbb{R}^k$. According to the work of Colding and Naber [13], the regular part of $Y_\infty$ is very well connected. Letting $H^k$ denote the $k$-dimensional Hausdorff measure, we have the following

**Claim 3.3** For $H^k \times H^k$-a.e. pair of points $(x, y) \in R_{20} \times R_{20}$, there is a minimal geodesic in $B_{d_{\infty}}(\bar{p}_\infty, 40)$ with all of whose interior points having the unique tangent cone $\mathbb{R}^k$. Here we employ the notation $R_r := R \cap B_{d_{\infty}}(\bar{p}_\infty, r)$ for any $r \in (0, 50)$.

**Proof of the claim** Notice that the underlying manifolds $(Y_i, \bar{g}_i)$ are complete with boundary, and by [13, Theorem 1.1] we know that as long as an limiting minimal geodesic is contained in $B_{d_{\infty}}(\bar{p}_\infty, 40)$, connecting two regular points whose tangent cones are $\mathbb{R}^k$, and can be extended slightly towards both ends, then the tangent cones centered at its interior are all isometric to $\mathbb{R}^k$. Here we point out that Colding and Naber’s estimates are uniform (independent of $\{Y_i\}$) as long as the limit geodesic in consideration is contained within $B_{d_{\infty}}(\bar{p}_\infty, 40)$, since the Li-Yau gradient estimate [36, Theorem 1.2] and Harnack inequality [36, Theorem 2.1] for manifolds with boundary are uniform as long as we stay a definite distance away from the boundary points. In fact, by [13, Theorem 1.20], for $\nu \times \nu$-a.e. pair of points $(x, y) \in R_{20} \times R_{20}$, since the minimal geodesic connecting them is entirely contained in $B_{d_{\infty}}(\bar{p}_\infty, 40)$, every interior point of this geodesic has tangent cone isometric to $\mathbb{R}^k$. Here $\nu$ is the renormalized measure, defined for any geodesic ball $B_{d_{\infty}}(x, r) \subset Y_\infty$ by $\nu(B_{d_{\infty}}(x, r)) = \lim_{i \to \infty} \nu(B_{\bar{g}_i}(\bar{p}_i, 1))^{-1} B_{\bar{g}_i}(x_i, r)$, where $B_{\bar{g}_i}(x_i, r) \xrightarrow{pGH} B_{d_{\infty}}(x, r)$ as $i \to \infty$. By the volume comparison it is clear that $\nu(B_{d_{\infty}}(x, r)) > 0$ for any $x \in R_{20}$ and $r \in (0, 1)$, implying that for $H^k \times H^k$-a.e. pair of points in $R_{20} \times R_{20}$, the minimal geodesic connecting these two points has tangent cone $\mathbb{R}^k$ at all of its interior points.

\(\square\)
Notice that if there is a minimal geodesic connecting two points in $\mathcal{R}$ and passing through $\tilde{p}_\infty$, then the tangent cone at $\tilde{p}_\infty$ is isometric to $\mathbb{R}^k$, but since $B_{d_\infty}(\tilde{p}_\infty, 20)$ is a geodesic ball in a metric cone centered at the vertex $\tilde{p}_\infty$, we must have $B_{d_\infty}(\tilde{p}_\infty, 20) \equiv \mathbb{B}^k(20)$, and the lemma is proven.

In general, there may be no minimal geodesic connecting regular points and passing through $\tilde{p}_\infty$, but the same reasoning shows that when a unit-speed minimal geodesic $\sigma: [0, r] \to B_{d_\infty}(\tilde{p}_\infty, 40)$ connects two points in $\mathcal{R}_{20}$, then it has a small tubular neighborhood locally diffeomorphic to $(-\varepsilon, \varepsilon) \times \mathbb{B}^{k-1}(\varepsilon)$ for some $\varepsilon > 0$: since every interior point of $\sigma$ is the center of a small geodesic ball isometric to one centered at the cone vertex in a metric cone, and the regular-convexity ensures that the tangent cone at each interior point is isometric to $\mathbb{R}^k$, we see that each interior point $\sigma(t)$ has a small radius $\tilde{r}_\sigma(t) > 0$ such that $B_{d_\infty}(\sigma(t), \tilde{r}_\sigma(t)) \equiv \mathbb{B}^k(\tilde{r}_\sigma(t))$; by the compactness of $\sigma([0, r])$ we can find a minimal radius $\tilde{r}_\sigma > 0$ such that the $\tilde{r}_\sigma$-tubular neighborhood of $\sigma([0, r])$ is locally diffeomorphic to the product mentioned above. In fact, since $\sigma$ is a distance minimizer when restricted in each $B_{d_\infty}(\sigma(t), \tilde{r}_\sigma(t))$, it satisfies the Euclidean geodesic equation, and thus becomes a straight line segment, i.e. we can identify the $\tilde{r}_\sigma$-tubular neighborhood of $\sigma([0, r])$ isometrically as

$$B_{d_\infty}(\sigma([0, r]), \tilde{r}_\sigma) \equiv \{ \tilde{v} \in \mathbb{R}^k : \exists t \in [0, r], |\tilde{v} - t\tilde{e}| < \tilde{r}_\sigma \}, \quad (3.4)$$

where $\tilde{e} \in \mathbb{R}^k$ is a unit vector.

Intuitively speaking, such tubular neighborhoods provide sufficiently regular “tunnels” that connect regular points in different fundamental domains of $B_{d_\infty}(\tilde{p}_\infty, 20)$, and such tunnels exist in abundance as $\mathcal{R}$ is of full Hausdorff measure in each fundamental domain. The fact that $\mathcal{R}_{20}$ is very well connected enables us to prove the following segment inequality.

**Claim 3.4** For any $u \in L^1(B_{d_\infty}(\tilde{p}_\infty, 40))$ and any $\tilde{x}_0, \tilde{y}_0 \in \mathcal{R}$ with $r = d_\infty(\tilde{x}_0, \tilde{y}_0)$, we have

$$\int_{B_{d_\infty}(\tilde{x}_0, r)} \int_{B_{d_\infty}(\tilde{y}_0, r)} \mathcal{F}_u(\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y}) \, d\mathcal{H}^k(\tilde{x}) \leq 2^{k+3} r \mathcal{H}^k(B_{d_\infty}(\tilde{y}_0, 4r))$$

$$\int_{B_{d_\infty}(\tilde{y}_0, 4r)} u \, d\mathcal{H}^k,$$

(3.5)

where for any $\tilde{x}, \tilde{y} \in \mathcal{R}$ we define $\mathcal{F}_u(\tilde{x}, \tilde{y}) := \inf_{\sigma_{\tilde{x}\tilde{y}}} \int_{\sigma_{\tilde{x}\tilde{y}}} u$, and the infimum is taken over all minimal geodesics $\sigma_{\tilde{x}\tilde{y}}$ connecting $\tilde{x}$ and $\tilde{y}$ and entirely contained in $\mathcal{R}_{40}$.

**Remark 6** We need to reprove such an inequality, rather than directly applying the original one due to Cheeger and Colding [8], because (3.5) is considered with respect to the $k$-dimensional Hausdorff measure, rather than the renormalized measure $\nu$.

**Proof of the claim** We put $\mathcal{F}_u(\tilde{x}, \tilde{y}) = \mathcal{F}_u^+(\tilde{x}, \tilde{y}) + \mathcal{F}_u^-(\tilde{x}, \tilde{y})$, where

$$\mathcal{F}_u^+(\tilde{x}, \tilde{y}) := \inf_{\sigma_{\tilde{x}\tilde{y}}} \int_{\sigma_{\tilde{x}\tilde{y}}}^d(\tilde{x}, \tilde{y}) u(\sigma_{\tilde{x}\tilde{y}}(t)) \, dt,$$

and

$$\mathcal{F}_u^-(\tilde{x}, \tilde{y}) := \inf_{\sigma_{\tilde{x}\tilde{y}}} \int_{0}^{d(\tilde{x}, \tilde{y})} u(\sigma_{\tilde{x}\tilde{y}}(t)) \, dt.$$

Since $\mathcal{F}_u^+(\tilde{x}, \tilde{y}) = \mathcal{F}_u^-((\tilde{x}, \tilde{y})$, by Fubini’s theorem we have

$$\int_{B_{d_\infty}(\tilde{x}_0, 4r)} \int_{B_{d_\infty}(\tilde{y}_0, 4r)} \mathcal{F}_u^+((\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y}) \, d\mathcal{H}^k(\tilde{x})$$

$$= \int_{B_{d_\infty}(\tilde{x}_0, 4r)} \int_{B_{d_\infty}(\tilde{y}_0, 4r)} \mathcal{F}_u^-((\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y}) \, d\mathcal{H}^k(\tilde{x}),$$

and we only need to establish the estimate for $\mathcal{F}_u^+$. 

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We now fix any \( \tilde{x} \in B_{d_{\infty}}(\tilde{x}_0, r) \cap \mathcal{R} \). By the triangle inequality, the unit-speed minimal geodesic \( \sigma_{\tilde{x}\tilde{y}} \) connecting \( \tilde{x} \) to any \( \tilde{y} \in B_{d_{\infty}}(\tilde{y}_0, r) \cap \mathcal{R} \) is entirely contained in \( B_{d_{\infty}}(\tilde{y}, 4r) \). Moreover, as we have discussed before in (3.4), \( \sigma_{\tilde{x}\tilde{y}} \) is a straight line segment which has a Euclidean tubular neighborhood. Even though it is not possible to obtain a uniform size of the neighborhood, the key observation is that however small the neighborhood is, infinitesimally

\[
\tilde{\theta} \text{ neighborhood, the key observation is that however small the neighborhood is, infinitesimally}
\]

we can decompose the volume form \( d\mathcal{H}^k \) at \( \sigma_{\tilde{x}\tilde{y}}(s) \) into polar coordinate as \( s^{k-1}d\theta(\tilde{v})ds \), where \( d\theta \) is the volume form on \( S^{k-1} \), the standard \( k-1 \) sphere, and \( \tilde{v} \in S^{k-1} \) is essentially \( \tilde{\theta}(0) \)—since \( B_{\tilde{x}} \equiv B^k(\tilde{r}_x) \), all unit-speed minimal geodesics emanating from \( \tilde{x} \) is parametrized by \( S^{k-1} \). Adapting to the polar structure at \( \tilde{x} \), we will let \( \sigma_{\tilde{x}\tilde{v}} \) denote the minimal geodesic emanating from \( \tilde{x} \) with initial direction \( \tilde{v} \in S^{k-1} \). Consequently, we have

\[
\mathcal{F}^+_u \left( \sigma_{\tilde{x}\tilde{v}} \left( \frac{t}{2}, \sigma_{\tilde{x}\tilde{v}}(t) \right) \right) d\mathcal{H}^k(\sigma_{\tilde{x}\tilde{v}}(t)) \leq \left( \int_{\frac{t}{2}}^t u(\sigma_{\tilde{x}\tilde{v}}(s)) \, ds \right) \, t^{k-1}d\theta(\tilde{v})dt \leq 2^{k-1} \left( \int_{\frac{t}{2}}^t u(\sigma_{\tilde{x}\tilde{v}}(s))s^{k-1}d\theta(\tilde{v})ds \right) dt.
\]

Moreover, we will let \( \tilde{r}_x(\tilde{x}, \tilde{v}) := \sup \{ t > 0 : \sigma_{\tilde{x}\tilde{v}}(t) \in \mathcal{R} \cap B_{d_{\infty}}(\tilde{y}_0, 4r) \} \)—notice that \( \sigma_{\tilde{x}\tilde{v}} \) is always defined at least up to \( t = \tilde{r}_x \) and \( \tilde{x} \in B_{d_{\infty}}(\tilde{y}_0, 4r) \), so \( \tilde{r}_x(\tilde{x}, \tilde{v}) \geq 0 \). On the other hand, by the triangle inequality, it is clear that \( \tilde{r}_x(\tilde{x}, \tilde{v}) \leq 6r \) for any \( \tilde{v} \in S^{k-1} \). Now since \( \tilde{v} \in S^{k-1} \) exhausts all possible directions of minimal geodesics emanating from \( \tilde{x} \) and reaching to \( \tilde{y} \in B_{d_{\infty}}(\tilde{y}_0, r) \cap \mathcal{R} \), we can integrate the above inequality with respect to \( \tilde{y} \in B_{d_{\infty}}(\tilde{y}_0, r) \cap \mathcal{R} \) and apply Fubini’s theorem to see that

\[
\int_{B_{d_{\infty}}(\tilde{y}_0, r)} \mathcal{F}^+_u(\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y}) \leq \int_{S^{k-1}} \int_{0}^{\tilde{r}_x(\tilde{x}, \tilde{v})} \mathcal{F}^+_u \left( \sigma_{\tilde{x}\tilde{v}} \left( \frac{t}{2}, \sigma_{\tilde{x}\tilde{v}}(t) \right) \right) d\mathcal{H}^k(\sigma_{\tilde{x}\tilde{v}}(t)) \leq 2^{k-1} \int_{S^{k-1}} \int_{0}^{\tilde{r}_x(\tilde{x}, \tilde{v})} u(\sigma_{\tilde{x}\tilde{v}}(s))s^{k-1}d\theta(\tilde{v})ds \, dt \leq 2^{k+2}r B_{d_{\infty}}(\tilde{y}_0, 4r) u \, d\mathcal{H}^k.
\]

Here on the left-hand side we are essentially integrating regular \( \tilde{y} \), but as \( \mathcal{R} \) is of full Hausdorff measure, it is the same as integrating \( \tilde{y} \) all over \( B_{d_{\infty}}(\tilde{y}_0, r) \). Integrating the last inequality with respect to \( \tilde{x} \in B_{d_{\infty}}(\tilde{x}_0, r) \cap \mathcal{R} \), we have

\[
\int_{B_{d_{\infty}}(\tilde{x}_0, r)} \int_{B_{d_{\infty}}(\tilde{y}_0, r)} \mathcal{F}^+_u(\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y})d\mathcal{H}^k(\tilde{x}) \leq 2^{k+2}r \mathcal{H}^k \left( B_{d_{\infty}}(\tilde{y}_0, 4r) \right) \int_{B_{d_{\infty}}(\tilde{y}_0, 4r)} u \, d\mathcal{H}^k,
\]
as \( B_{d_{\infty}}(\tilde{x}_0, r) \subset B_{d_{\infty}}(\tilde{y}_0, 4r) \). Adding its symmetric part we obtain

\[
\int_{B_{d_{\infty}}(\tilde{x}_0, r)} \int_{B_{d_{\infty}}(\tilde{y}_0, r)} \mathcal{F}^-_u(\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y})d\mathcal{H}^k(\tilde{x}) \leq 2^{k+3}r \mathcal{H}^k \left( B_{d_{\infty}}(\tilde{y}_0, 4r) \right) \int_{B_{d_{\infty}}(\tilde{y}_0, 4r)} u \, d\mathcal{H}^k,
\]
which is the desired inequality. 

In order to show that \( B_{d_{\infty}}(\tilde{p}_{\infty}, 10) \equiv B^n(10) \), we will rely on the recent developments in the theory of RCD spaces, notably [15], together with a volume consideration. To this end, we equip \( W_\infty := B_{d_{\infty}}(\tilde{p}_{\infty}, 20) \) with the \( k \)-dimensional Hausdorff measure \( \mathcal{H}^k \), and prove the following

**Claim 3.5** The metric measure space \((W_\infty, d_{\infty}, \mathcal{H}^k)\) is a compact, connected and non-collapsing RCD(0, \( k \)) space.
Although the metric measure space \((W_\infty, d_\infty, \nu)\), as the metric-measure limit of the closed geodesic balls \((\bar{B}_{\tilde{g}_i}(\tilde{p}_i, 20), \tilde{g}_i, |B_{\tilde{g}_i}(\tilde{p}_i, 1)|^{-1} \, \d V_{\tilde{g}_i})\), is already an \(RCD(m - 1, m)\) space by the standard theory, it is not immediate that the metric limit \((W_\infty, d_\infty)\), when equipped with the \(k\)-dimensional Hausdorff measure \(\mathcal{H}^k\), is an \(RCD(0, k)\) space. Before proving the claim, let us briefly recall the definition of \(RCD(0, k)\) spaces proposed in \cite{1}; see also \cite{19,26}. For a complete, Hausdorff and separable metric measure space \((X, d_X, m)\) the Cheeger energy is defined for any \(f \in L^2(X)\) as

\[
Ch(f) := \inf \left\{ \liminf_{j \to \infty} \frac{1}{2} \int_X (\text{lip } f_j)^2 \, \d m : f_j \in \text{Lip}_b(X) \cap L^2(X), \lim_{j \to \infty} \| f_j - f \|_{L^2(X)} = 0 \right\},
\]

where \(\text{Lip}_b(X)\) denote the collection of locally bounded Lipschitz functions on \(X\), and for any such \(f\), \(\text{lip } f\) is defined at any \(x \in X\) as \(\text{lip } f(x) := \liminf_{r \to 0} \sup_{y \in B_d(x, r)} |f(x) - f(y)| d_X(x, y)^{-1}\). The Banach space \(W^{1,2}(X)\) is defined as \(\{ f \in L^2(X) : Ch(f) < \infty \}\).

**Definition 3.6** We say that a complete, Hausdorff and separable metric measure space \((X, d_X, m)\) an \(RCD(0, k)\) space if:

1. there is a carré du champ \(\Gamma : W^{1,2}(X) \times W^{1,2}(X) \to L^1(X)\) such that \(\forall f \in W^{1,2}(X)\), if \(\Gamma(f, f) \leq 1\) m.a.e. on \(X\), then it has 1-Lipschitz representative;
2. Infinitesimally Hilbertian: the Cheeger energy can be written as \(2Ch(f) = \int_X \Gamma(f, f) \, \d m\) for any \(f \in W^{1,2}(X)\);
3. there exists \(X\) such that \(m(B_{d_X}(x, r)) \leq Ce^{Cr}\) for some fixed \(C > 1\) and any \(r > 0\);
4. Bakry-Émery inequality: \(\forall f \in \text{Dom}(\Delta)\) with \(\Delta f \in W^{1,2}(X)\), we have

\[
\int_X \Gamma(f, f) \Delta \varphi \, \d m \geq 2 \int_X \left( \frac{1}{k} (\Delta f)^2 + \Gamma(f, \Delta f) \right) \varphi \, \d m,
\]

whenever \(\varphi \in \text{Dom}(\Delta) \cap L^\infty(X)\) satisfies \(\varphi \geq 0\) and \(\Delta \varphi \in L^\infty(X)\).

Moreover, if \(m = \mathcal{H}^k\), the \(k\)-dimensional Hausdorff measure, then we say that \((X, d_X, m)\) is a non-collapsing \(RCD(0, K)\) space, denoted by \(\text{nc}RCD(0, K)\) space; see \cite[Definition 1.1]{15}.

Here we notice that \(\Delta\) denotes the Laplace operator on \(X\): for some \(f \in \text{Dom}(\Delta) \subset W^{1,2}(X)\), we say that \(u \in L^2(X)\) satisfies \(\Delta f = u\) if

\[
\forall \varphi \in W^{1,2}(X), \quad \int_X \Gamma(f, \varphi) \, \d m = -\int_X u \varphi \, \d m.
\]

Notice that here \(f\) is a critical point of the functional \(E_u(f) := \int_X \frac{1}{2} \Gamma(f, f) + f \d \mathcal{H}^k\). When \(X\) has a boundary in a given context, we require the energy functional \(E_u\) to only consider \(W^{1,2}(W_\infty)\) functions subject to a given boundary condition \(f|_{\partial X}\). In this situation, we require the test function \(\varphi\) to vanish on \(\partial X\), both in the definition of \(\Delta\) and in Condition (4) of Definition 3.6.

**Proof of Claim 3.5** If \(f \in \text{Lip}_b(W_\infty)\), then clearly \(\nabla f\) is defined \(\mathcal{H}^k\)-a.e. in \(\mathcal{R}_{20}\). Consequently, we can define the carré du champ \(\Gamma\) for \(f_1, f_2 \in \text{Lip}_b(W_\infty)\) as \(\Gamma(f_1, f_2) := \langle \nabla f_1, \nabla f_2 \rangle\) for \(\mathcal{H}^k\)-a.e. point in \(W_\infty\). Here \((\langle \cdot, \cdot \rangle)\) denotes the Euclidean inner product. Notice that here the orbifold nature of \(Z\) is crucial—any regular point of \(W_\infty\) is the center of a small enough geodesic ball isometric to a \(k\)-dimensional Euclidean ball.
Now for any \( f_1, f_2 \in W^{1,2}(W_\infty) \), there is a sequence \{fa_j\} \subset Lip_b(W_\infty) such that for \( a = 1, 2 \),
\[
\lim_{j \to \infty} \|fa_j - fa\|_{L^2(W_\infty)} = 0 \quad \text{and} \quad \lim_{j \to \infty} \|\nabla fa_j\|_{L^2(W_\infty)}^2 = 2Ch(fa) < \infty.
\]

Passing to sub-sequences, we can assume that \( fa_j \xrightarrow{\mathcal{H}^k-a.e.} fa \) for \( a = 1, 2 \), and thus we can define \( \Gamma(f_1, f_2) := \lim_{j \to \infty} \|\nabla f_{1j}, \nabla f_{2j}\| \) for \( \mathcal{H}^k \)-a.e. point in \( W_\infty \). (Here we can truncate \( |\nabla fa_j| \) by \( n \in \mathbb{N} \), define a limit for each \( n \) and take a diagonal limit as \( n \to \infty \).) Clear, \( \Gamma \) maps into \( L^1(W_\infty) \) by the dominated convergence theorem: we have \( |\Gamma(f_{1j}, f_{2j})| \leq |\nabla f_{1j}| |\nabla f_{2j}| \in L^1(W_\infty) \) by Hölder’s inequality, and consequently
\[
\int_{W_\infty} |\Gamma(f_1, f_2)| \, d\mathcal{H}^k = \int_{W_\infty} \lim_{j \to \infty} |\Gamma(f_{1j}, f_{2j})| \, d\mathcal{H}^k
\leq \liminf_{j \to \infty} \|\nabla f_{1j}\|_{L^2(W_\infty)} \|\nabla f_{2j}\|_{L^2(W_\infty)}
\leq 2\sqrt{Ch(f_1)Ch(f_2)} < \infty.
\]

By the same reasoning, \( \Gamma \) makes the Cheeger energy a quadratic form: if \( f_j \xrightarrow{L^2} f \in W^{1,2}(W_\infty) \) and \( \|\nabla f_j\|_{L^2(W_\infty)}^2 \to 2Ch(f) \), then \( \|\nabla f_j\|_{L^2(W_\infty)} \leq 2Ch(f)^{\frac{1}{2}} < \infty \) for all \( j \) sufficiently large, and after possibly passing to a sub-sequence, by the dominated convergence theorem we have
\[
2Ch(f) = \lim_{j \to \infty} \|\nabla f_j\|_{L^2(W_\infty)}^2
= \int_{W_\infty} \lim_{j \to \infty} \langle \nabla f_j, \nabla f_j \rangle \, d\mathcal{H}^k
= \int_{W_\infty} \Gamma(f, f) \, d\mathcal{H}^k.
\]

While we have defined the carré du champ \( \Gamma \) and checked the infinitesimally Hilbertian property (Condition (2)), given \( f \in W^{1,2}(W_\infty) \) with \( \Gamma(f, f) \leq 1 \), we are yet to find a 1-Lipschitz function \( \tilde{f} \) such that \( \tilde{f} = f \) for \( \mathcal{H}^k \)-a.e. point in \( W_\infty \). We now assume to have a sequence \( \{f_j\} \subset Lip_b(W_\infty) \) such that \( f_j \xrightarrow{\mathcal{H}^k-a.e.} f \) and \( \|\nabla f_j\|_{L^2(W_\infty)}^2 \to 2Ch(f) \), then
\[
\frac{1}{2} \|\nabla f_j\|_{L^2(W_\infty)}^2 \xrightarrow{\mathcal{H}^k-a.e.} \Gamma(f, f) \leq 1 \text{ after possibly passing to a sub-sequence, still denoted by } \{f_j\}.
\]
In order to find a Lipschitz representative of \( f \), we need to check the uniform equicontinuity of \( \{f_j\} \)—notice that \( |\nabla f_j| \) only converges to a bounded function \( \mathcal{H}^k \)-almost everywhere, and we do not have a uniform bound for \( \|\nabla f_j\|_{L^\infty(W_\infty)} \). Therefore, we will need to rely on the segment inequality (3.5) which controls the modulus of continuity by \( \|\nabla f_j\|_{L^2_{loc}(W_\infty)} \) (the local \( L^2 \)-average).

Now for any \( \tilde{p}, \tilde{q} \in W_\infty \) with \( d_\infty(\tilde{p}, \tilde{q}) = r \leq 1 \), for any \( \varepsilon > 0 \) there are \( \tilde{p}_\varepsilon, \tilde{q}_\varepsilon \in R_{20} \) such that \( d_\infty(\tilde{p}, \tilde{p}_\varepsilon) + d_\infty(\tilde{q}, \tilde{q}_\varepsilon) < \varepsilon r \). Fixing a minimal geodesic \( \sigma_\varepsilon : [0, 1] \to M \) connecting \( \tilde{p}_\varepsilon \) and \( \tilde{q}_\varepsilon \), we have a positive radius \( \tilde{r}_{\sigma_\varepsilon} \) such that the \( \tilde{r}_{\sigma_\varepsilon} \)-tubular neighborhood of \( \sigma_\varepsilon \) is isometric to an Euclidean domain; see (3.4). We now fix \( n \in \mathbb{N} \) large enough so that...
$10n^{-1} r < \tilde{r}_{\sigma}$. For $\tilde{x}_s := \sigma(s)$ with $s = 1, \ldots, n$, we define

$$A^n_{s-1} := \left\{ \tilde{x} \in B_{d_{\infty}}(\tilde{x}_{s-1}, n^{-1} r) \cap \mathcal{R} : \int_{B_{d_{\infty}}(\tilde{x}_s, n^{-1} r)} |\nabla f_j| (\tilde{x}, \tilde{y}) \, d\mathcal{H}^k(\tilde{y}) \leq 32^{k+1} r \int_{B_{d_{\infty}}(\tilde{x}_s, 4n^{-1} r)} |\nabla f_j| \right\}.$$

and for any $\tilde{x} \in A^n_{s-1}$, we define

$$B^n_s(\tilde{x}) := \left\{ \tilde{y} \in B_{d_{\infty}}(\tilde{x}_s, n^{-1} r) \cap \mathcal{R} : |\nabla f_j|(\tilde{x}, \tilde{y}) \leq 32^{k+2} r \int_{B_{d_{\infty}}(\tilde{x}_s, 4n^{-1} r)} |\nabla f_j| \right\}.$$

We also define $A^n_s := B_{d_{\infty}}(\tilde{x}_s, n^{-1} r) \cap \mathcal{R}$ and $B^n_s(\tilde{x}) := B_{d_{\infty}}(\tilde{x}_s, n^{-1} r) \cap \mathcal{R}$ for any $\tilde{x} \in A^n_0$. Now for each $s = 1, \ldots, n$, we apply Claim 3.4 to $|\nabla f_j|$ on $B_{d_{\infty}}(\tilde{x}_{s-1}, n^{-1} r) \cup B_{d_{\infty}}(\tilde{x}_{s-1}, n^{-1} r)$ and see that

$$\int_{B_{d_{\infty}}(\tilde{x}_{s-1}, n^{-1} r)} \int_{B_{d_{\infty}}(\tilde{x}_s, n^{-1} r)} |\nabla f_j| \, d\mathcal{H}^k \, d\mathcal{H}^k \leq 2^{5k+3} n^{-1} r \int_{B_{d_{\infty}}(\tilde{x}_s, 4n^{-1} r)} |\nabla f_j| \, d\mathcal{H}^k.$$

The right-hand side constant is obtained as $B_{d_{\infty}}(\tilde{x}_s, 10n^{-1} r) \subset B_{d_{\infty}}(\sigma([0, 1]), \tilde{r}_{\sigma})$, and thus

$$\mathcal{H}^k(B_{d_{\infty}}(\tilde{x}_s, n^{-1} r)) = \omega_k(n^{-1} r)^k \quad \text{and} \quad \mathcal{H}^k(B_{d_{\infty}}(\sigma(\tilde{x}_s), 4n^{-1} r)) = \omega_k(4n^{-1} r)^k$$

for each $s = 0, 1, \ldots, n$. Now by Chebyshev’s inequality we see for each $s = 1, \ldots, n$ that

$$\mathcal{H}^k(A^n_{s-1}) \geq \frac{3}{4} \mathcal{H}^k(B_{d_{\infty}}(\tilde{x}_{s-1}, n^{-1} r)), \quad \text{and} \quad \mathcal{H}^k(B^n_s(\tilde{x})) \geq \frac{31}{32} \mathcal{H}^k(B_{d_{\infty}}(\tilde{x}_s, n^{-1} r))$$

whenever $\tilde{x} \in A^n_{s-1}$. Especially, $A^n_s \cap B^n_s(\tilde{x}) \neq \emptyset$ for any $s = 1, \ldots, n$ and any $\tilde{x} \in A^n_{s-1}$.

On the other hand, by the assumption that $|\nabla f_j| \xrightarrow{\mathcal{H}^k-a.e.} \Gamma(f, f) \leq 1$ when $j \to \infty$ we see for all $j$ large enough and any $s = 1, \ldots, n$ that

$$\int_{B_{d_{\infty}}(\tilde{x}_s, 4n^{-1} r)} |\nabla f_j|^2 \, d\mathcal{H}^k \leq 2.$$

Combining this inequality with (3.6) and starting with any $\tilde{x}_s^0 \in A^n_0$ fixed, we could then inductively find minimal geodesic segments $\sigma_s$ connecting $\tilde{x}_s^0 \in A^n_s \cap B^n_s(\tilde{x}_{s-1}^0)$, such that

$$\forall s = 1, \ldots, n, \quad |f_j(\tilde{x}_s^0) - f_j(\tilde{x}_{s-1}^0)| = \int_{\sigma_s} |\nabla f_j| = 32^{k+4} n^{-1} r.$$

Here we notice that each $\sigma_s$ is contained in $\mathcal{R}_{40}$ since $\sigma_s$ maps into $B_{d_{\infty}}(\tilde{x}_s, 8n^{-1} r) \subset B_{d_{\infty}}(\tilde{p}_\infty, 23)$ for $n \geq 8$. Adding these inequalities up with respect to $s$, we see that

$$|f_j(\tilde{x}_n^0) - f_j(\tilde{x}_0^0)| \leq 32^{k+4} r. \quad (3.7)$$

Since $d_{\infty}(\tilde{p}, \tilde{x}_0^0) + d_{\infty}(\tilde{q}, \tilde{x}_n^0) < (2n^{-1} + \varepsilon)r$, by the continuity of $f_j$ and the independence of the estimate (3.7) on $n$ and $\varepsilon$, we can let $n \to \infty$, then send $\varepsilon \to 0$, and see that

$$|f_j(\tilde{p}) - f_j(\tilde{q})| \leq 32^{k+4} d_{\infty}(\tilde{p}, \tilde{q}).$$
As the uniform constant $32^{k+4}$ is clearly independent of $j$, this proves the equicontinuity of the family $\{f_j\}$, and the Arzelà-Ascoli theorem allows us to extract a continuous limit function $\tilde{f}$. Clearly, $\tilde{f} = f$ holds $\mathcal{H}^k$-a.e.; on the other hand, since $\Gamma'(f, f) \leq 1$, the infinitesimally Hilbertian property ensures that the upper gradient $|Df|$ of $f$ is $\mathcal{H}^k$-a.e. bounded above by 1, and by the locality of the upper gradient (see [19, (2.7)]), we see that $|D\tilde{f}| \leq 1$ holds $\mathcal{H}^k$ almost everywhere, implying that $\tilde{f}$ is Lipschitz with $lip \tilde{f} \leq 1$. We have now finished verifying Condition (1).

By the Claim 3.2, we see that Condition (3) is readily checked as

$$\mathcal{H}^k(B_{d_{\infty}}(\tilde{p}_{\infty}, r)) = |H| B_{d_2}(z, r) = \omega_2 r^2 < \omega_k e^r.$$  

We now check the Bakry-Émery inequality for $f \in Dom(\Delta)$, i.e. Condition (4). By the infinitesimally Hilbertian property, we know that $\Delta$ is a linear operator, and that it is a local operator. Consequently, for any $\tilde{x} \in \mathcal{R}_{20}$, let $B_{\tilde{x}}$ be the $k$-Euclidean neighborhood of $\tilde{x}$, then $(\Delta f)|_{B_{\tilde{x}}} = \Delta(f|_{B_{\tilde{x}}}) \mathcal{H}^k$-a.e. whenever $\Delta f \in W^{1,2}(W_{\infty})$. But in this case, we can handle the Laplace operator as the Euclidean one: as $(\Delta f)|_{B_{\tilde{x}}} \in W^{1,2}(B_{\tilde{x}})$, by the elliptic regularity in the $k$-Euclidean space we see that $\|f\|_{C^2(\frac{1}{2}B_{\tilde{x}})} < \infty$ where $\frac{1}{2}B_{\tilde{x}}$ is the concentric ball as $B_{\tilde{x}}$ but with half the radius. Consequently, as $\Delta f \in W^{1,2}(W_{\infty})$, we can estimate at $\mathcal{H}^k$-a.e. regular point $\tilde{x}$ that

$$\Delta \Gamma(f, f)(\tilde{x}) = 2|Hess_f|^2(\tilde{x}) + 2\Gamma(f, \Delta f)(\tilde{x})$$

$$\geq \frac{2}{k}(\Delta f)^2(\tilde{x}) + 2\Gamma(f, \Delta f)(\tilde{x}). \quad (3.8)$$

Especially, we see that $\Delta \Gamma(f, f)$ is well-defined $\mathcal{H}^k$ almost everywhere on $\mathcal{R}_{20}$. On the other hand, since $\mathcal{R}_{20}$ is an open set, and especially $\tilde{r}_{\tilde{x}} = d_{\infty}(\tilde{x}, S)$ for any $\tilde{x} \in \mathcal{R}_{20}$, we could mollify $f$ to find $f_j \in C^2(\mathcal{R}_{20})$ so that $f_j|_S = 0$ and $f_j \xrightarrow{\mathcal{H}^k-a.e.} f$ as $j \to \infty$.

Consequently, this also ensures

$$\frac{1}{k}(\Delta f_j)^2 + \Gamma(f_j, \Delta f_j) \xrightarrow{j \to \infty} \frac{1}{k}(\Delta f)^2 + \Gamma(f, \Delta f) \quad \mathcal{H}^k\text{-a.e. as } j \to \infty, \quad (3.9)$$

since by (3.8) the limit is $\mathcal{H}^k$-a.e. well defined on $\mathcal{R}_{20}$; and similarly

$$\Gamma(f_j, f_j) \xrightarrow{j \to \infty} \Gamma(f, f) \quad \mathcal{H}^k\text{-a.e. as } j \to \infty. \quad (3.10)$$

Now for any non-negative $\varphi \in Dom(\Delta) \cap L^\infty(W_{\infty})$ with $\Delta \varphi \in L^\infty(W_{\infty})$ such that $\varphi = 0$ on $\partial W_{\infty} = \{\tilde{x} \in Y : d_{\infty}(\tilde{x}, \tilde{p}_{\infty}) = 20\}$, we have

$$\int_{W_{\infty}} \Gamma(f, f) \Delta \varphi \, d\mathcal{H}^k = \lim_{j \to \infty} \int_{W_{\infty}} \Gamma(f_j, f_j) \Delta \varphi \, d\mathcal{H}^k$$

$$= \lim_{j \to \infty} \int_{W_{\infty}} \Delta \Gamma(f_j, f_j) \varphi \, d\mathcal{H}^k$$

$$\geq \lim_{j \to \infty} 2 \int_{W_{\infty}} \left(\frac{1}{k}(\Delta f_j)^2 + \Gamma(f_j, \Delta f_j)\right) \varphi \, d\mathcal{H}^k$$

$$\geq 2 \int_{W_{\infty}} \left(\frac{1}{k}(\Delta f)^2 + \Gamma(f, \Delta f)\right) \varphi \, d\mathcal{H}^k,$$

where the first equality holds because of (3.10) and the dominated convergence theorem, while the last inequality holds due to (3.9) and Fatou’s lemma. Condition (4) is now verified.
With all conditions in Definition 3.6 checked, and since the measure $\mathcal{H}^k$ is nothing but the $k$-dimensional Hausdorff measure, we conclude that $(Y_\infty, d_\infty, \mathcal{H}^k)$ is an $nc\text{RC}D(0, k)$ space. □

Moreover, since $B_{d_\infty}(\tilde{p}_\infty, 20)/H = W_\infty/H \equiv B_{d_Z}(z, 20) \subset Z$, we see that

$$\mathcal{H}^k \left( B_{d_\infty}(\tilde{p}_\infty, 20) \right) = \left| H \right| \left| B_{d_Z}(z, 20) \right| = \left| \mathbb{B}^k(20) \right|.$$ 

Therefore, applying [15, Theorem 1.5] to the $nc\text{RC}D(0, k)$ space $(W_\infty, d_\infty, \mathcal{H}^k)$, we have

$$B_{d_\infty}(\tilde{p}_\infty, 10) \equiv \mathbb{B}^k(10).$$ 

This concludes the proof of the lemma. □

Now since $\lim_{i \to \infty} d_{GH}(B_{g_i}(\tilde{p}_i), 10)$, $B_{d_\infty}(\tilde{p}_\infty, 10) = 0$, Lemma 3.1 produces a contradiction to (3.2) as $i \to \infty$, and this contradiction shows that that for any $i$ sufficiently large,

$$10R^{-1}\delta_i' := d_{GH}(B_{g_i}(\tilde{p}_i), \mathbb{B}^k(10)) < \delta_{AE}(2^{-1}\varepsilon').$$

Without loss of generality, we may assume that $\delta_i' \geq \delta_i$.

On the other hand, since for each $i$ the map $\pi_{i, 0} : \pi_i^1(B_{g_i}(p_i, 25)) \to Y_i$ is a normal covering with deck transformation group $\ker \phi_i$, so is its restriction to $\pi_i^1(\tilde{B}_{g_i}(\tilde{p}_i, 10) \to B_{\tilde{g}_i}(\tilde{p}_i, 10)$.

Since the pseudo-local fundamental group $\tilde{\Gamma}_{\delta_i'}(p_i) \leq \pi_1(B_{g_i}(K_i, R), p_i)$ is characterized by

$$\tilde{\Gamma}_{\delta_i'}(p_i) \equiv \left\{ \gamma \in \pi_1(B_{g_i}(K_i, R), p_i) : d_{\pi^*_g_i}(\gamma, \tilde{p}_i) < 2\delta_i' \right\},$$

we have the following identity of almost nilpotent groups:

$$\tilde{G}_{\delta_i'}(\tilde{p}_i) = \left\{ \gamma \in \ker \phi_i : d_{\pi^*_g_i}(\gamma, \tilde{p}_i) < 20R^{-1}\delta_i' \right\} = \tilde{\Gamma}_{\delta_i'}(p_i) \cap \ker \phi_i.$$

Moreover, since

$$\tilde{G}_{\delta_i'}(\tilde{p}_i) = \tilde{\Gamma}_{\delta_i'}(p_i) \cap \ker \phi_i \subseteq \tilde{\Gamma}_{\delta_i'}(p_i)$$

and $[\pi_1(B_{g_i}(K_i, R), p_i) : \ker \phi_i] = |G| < \infty$,

we must have $\left| \tilde{\Gamma}_{\delta_i'}(p_i) : \tilde{G}_{\delta_i'}(\tilde{p}_i) \right| \leq 1$, and thus rank $\tilde{G}_{\delta_i'}(\tilde{p}_i) = \text{rank} \tilde{\Gamma}_{\delta_i'}(p_i)$—the almost nilpotency of $\tilde{G}_{\delta_i'}(\tilde{p}_i)$ is guaranteed by its normality within $\tilde{\Gamma}_{\delta_i'}(p_i)$, which is almost nilpotent for $\delta_i$ sufficiently small (see [34,38]).

Since $\delta_i \leq \delta_i'$, we have $\tilde{\Gamma}_{\delta_i}(p_i) \leq \tilde{\Gamma}_{\delta_i'}(p_i)$ by the definition of the pseudo-local fundamental group, and the Assumption (3), i.e. rank $\tilde{\Gamma}_{\delta_i}(p_i) = m - k$, then implies that rank $\tilde{\Gamma}_{\delta_i'}(p_i) \geq m - k$. Consequently, we have

$$\text{rank} \tilde{G}_{\delta_i'}(\tilde{p}_i) \geq m - k.$$  

Now applying Lemma 2.4 to the normal covering $\pi_i^1(\tilde{B}_{g_i}(\tilde{p}_i, 10))) \to B_{\tilde{g}_i}(\tilde{p}_i, 10)$ with deck transformation group $\ker \phi_i$, by (3.11), (3.12) and (3.13), we can find some scale $\tilde{r} := 10^{-1}r_{AE}R \in (0, \frac{R}{10})$, such that for all $i$ large enough, the isoperimetric constant has lower bound

$$I_{\pi^*_g_i}(B_{\pi^*_g_i}(\tilde{p}_i, \tilde{r})) > (1 - 2^{-1}\varepsilon')I_m,$$

contradicting (3.1). As discussed at the very beginning of the section, this contradiction establishes Theorem 1.2. Notice that the a priori dependence of $\delta_O$ is on $Z$, but as $Z \equiv \mathcal{S}$
\(\mathbb{B}^k(25)/G\), and \(G < O(k)\) has no more than \(l\) elements, there are only finitely many such finite groups, and the dependence of \(\delta_O\) is then reduced to \(l\), the order of the local orbifold group (besides the dimension \(k \leq m\)).

## 4 Infranil fiber bundles over controlled orbifolds

In this section, we apply the local Ricci flow smoothing result to prove Theorem 1.4, which characterizes the situation, via topological data, when a collapsing domain with Ricci curvature lower bound is actually an infranil fiber bundle over an orbifold. In order to apply Theorem 1.2 for the smoothing purpose, we need to keep track of the distance change after locally running the Ricci flow. We have the following distance distortion estimate according to [28, Lemma 1.11]:

**Lemma 4.1** (Distance distortion) For any \(\alpha \in (0, 1)\), there is a positive quantity \(\Psi_D(\alpha|m)\) with \(\lim_{m \to 0} \Psi_D(\alpha|m) = 0\), such that under the assumption of Theorem 1.1, for any \(x, y \in B_g(p, 2)\) and any \(t \in (0, \varepsilon^2_p(m, \alpha))\), if \(d_g(x, y) \leq \sqrt{t}\), then we have

\[
|d_{\theta(t)}(x, y) - d_g(x, y)| \leq \Psi_D(\alpha|m)\sqrt{t}.
\]

**Proof** As in the proof of Theorem 1.1, we consider the universal covering \(\tilde{\pi} : X \to B_g(p, 6)\) and we have a Ricci flow solution \(\tilde{h}(t)\) on \(X\), with \(\tilde{h}(0) = \pi^*h\) and in particular on \(\pi^{-1}(B_g(p, 4))\) we have \(\tilde{h}(0) = \pi^*g\). Notice that for each \(t \in [0, \varepsilon^2_p]\), the fundamental group \(\Gamma := \pi_1(B_g(p, 6))\) acts on \((X, \tilde{h}(t))\) by discrete isometries and the Ricci flow \(g(t)\) on \(B_g(p, 4)\) is the quotient flow \((\pi^{-1}(B_g(p, 4)), \tilde{h}(t))/\Gamma\).

Recall that \((X, \pi^*h)\) satisfies the assumption of Theorem 2.3 at every point, then we could apply [28, Lemma 1.11] to find some positive quantity \(\Psi_D(\alpha|m)\) with \(\lim_{m \to 0} \Psi_D(\alpha|m) = 0\), such that for any \(\tilde{x}, \tilde{y} \in X\) with \(d_{\pi^*g}(\tilde{x}, \tilde{y}) \leq \sqrt{t}\), we have the estimate

\[
|d_{\tilde{h}(t)}(\tilde{x}, \tilde{y}) - d_{\pi^*g}(\tilde{x}, \tilde{y})| \leq \Psi_D(\alpha|m)\sqrt{t}.
\]

By the discussion above, the isometric action of \(\Gamma\) on \((\pi^{-1}(B_g(p, 4)), \tilde{h}(t))\) for any \(t \in [0, \varepsilon^2_p]\) ensure that the above distance comparison descends to \(B_g(p, 2)\), since any minimal geodesic \(\gamma\) realizing the \((g\text{- or } g(t)\text{-})\)distance between two points in \(B_g(p, 2)\) lie entirely within \(B_g(p, 4)\), and is lifted to a minimal \((\pi^*g\text{- or } \tilde{h}(t)\text{-})\)geodesic \(\tilde{\gamma}\) with the same length. This gives the desired distance distortion estimate (4.1).

We are now ready to prove the infranil fiber bundle theorem:

**Proof of Theorem 1.4** Let us recall that in [9, Theorem 2.1] there is dimensional constant \(\delta(m) > 0\) which we denote \(\lambda_{CFG}\). Let us fix the largest possible \(\alpha_F \in (0, 10^{-1})\) so that \(\Psi_D(\alpha_F|m) \leq 10^{-2}\lambda_{CFG}^2\), and we can consider \(\alpha_F\) as a constant only determined by \(m\). Now there are constants \(\delta_O(\alpha_F) > 0\) and \(\varepsilon_O(\alpha_F) > 0\) obtained from Theorem 2.3. Now we put

\[
\delta_F(m, l, \tilde{l}, \alpha_F) := 10^{-2} \min \{\varepsilon_O(m, l, \tilde{l}, \alpha_F)\lambda_{CFG}^2, \delta_O(m, l, \tilde{l}, \alpha_F)\}.
\]

For any \(\delta < \delta_F\) and any \(p \in B_g(K, 4\tilde{l})\), we have

\[
d_{GH}\left(B_g(p, \tilde{l}), \mathbb{B}^k(\tilde{l})/\Gamma_{\Phi(p)}\right) \leq d_{GH}\left(B_g(p, \tilde{l}), B_dz(\Phi(p), \tilde{l})\right)
\]

\[
+ d_{GH}\left(B_dz(\Phi(p), \tilde{l}), \mathbb{B}^k(\tilde{l})/\Gamma_{\Phi(p)}\right)
\]

\[
< \delta_O.
\]

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and Theorem 1.2 applies to start the Ricci flow over $\bar{B}_g(K, \bar{\iota})$ for a period no shorter than $T_F := \varepsilon_F^2$, satisfying the curvature bound

$$\sup_{B_g(K, \bar{\iota})} \left| \mathbf{Rm} \right|_{g(T_F)} \leq 2T_F^{-1}.$$  

Moreover, Shi’s estimate in [43] provides finite constants $C_n$ ($n \geq 1$) such that

$$\sup_{B_g(K, \bar{\iota})} \left| \nabla^n \mathbf{Rm} \right|_{g(T_F)} \leq C_n(m, T_F, \alpha_F).$$

Putting $c := \min \{ 2T_F, \bar{\iota}^2 \}$, the rescaled metric $g(T_F)' := c^{-1} g(T_F)$ then satisfies, with $C_0 = 1$, the regularity estimates

$$\forall n \in \mathbb{N}, \sup_{B_g(K, \bar{\iota})} \left| \nabla^n \mathbf{Rm} \right|_{g(T_F)'} \leq C_n'(m, T_F, \alpha_F).$$

By the scaling invariant estimate in Lemma 4.1, we know that the identity map on $B_g(K, \bar{\iota})$ provides a $(1, \Psi_D(\alpha_F))$-Gromov–Hausdorff approximation between the metrics $c^{-3} g$ and $g(T_F)'$—here for any $\varepsilon > 0$, we define a $(1, \varepsilon)$-Gromov–Hausdorff approximation between metric spaces $(A, d_A)$ and $(B, d_B)$ as an $\varepsilon$-dense map $\varphi : A \to B$ such that $\forall a \in A$, $\varphi|_{B_d(a, \varepsilon)} : B_d(a, \varepsilon) \to B_d(\varphi(a), 1 + \varepsilon)$ is an $\varepsilon$-Gromov–Hausdorff approximation; see also [28, §2] for the definition.

Therefore, the region $(B_g(K, \bar{\iota}), g(T_F)')$ is a $(1, \Psi_D(\alpha_F))$-Gromov–Hausdorff close to a $(k, l, 1)$-controlled orbifold with regular metric by (4.2), and combining the localization and the frame bundle argument (see [9, 17]), it is easily seen that $K$ has an open neighborhood $U \subset B_g(K, \bar{\iota})$ which fibers over $\mathbb{Z}_{\bar{\iota}}$ by infranil fibers. In fact, by [9, Theorem 1.7], there is a nilpotent Killing structure over $U$. Here we notice that the only technical alternation is replacing [9, Theorem 2.6] with [28, Theorem 2.2], which says that the $(1, \Psi_D)$-Gromov–Hausdorff collapsing suffices to produce the same conclusions of [9, Theorem 2.6].

Remark 7 In [27], Huang proved a fiber bundle theorem for manifolds with uniform Ricci curvature and local rewinding volume lower bounds, that collapse to lower dimensional manifolds with bounded geometry; see [27, Theorem 1.3]. The proof relies on applying the canonical Reifenberg method, developed in [7,10], to the non-collapsing local universal covering space. If the collapsing limit is instead a singular orbifold, as in the setting of Theorem 1.4, an attempt to apply the canonical Reifenberg method would have to be performed on the finite covering space $Y$ of $B_{g'}(p, 25)$, which has a geodesic 10-ball Gromov–Hausdorff close to $\mathbb{R}^k(10)$—see the proof of Theorem 1.2 in §3. However, the harmonic almost splitting map obtained from the canonical Reifenberg method is not necessarily equivariant (with respect to the $H$ action on $Y$ and the $G \cong H$ action on $\mathbb{R}^k(20)$)—it is not obvious that such a harmonic map defines a topological fiber bundle structure on $B_{g'}(p, 10)$ over the orbifold neighborhood. On the contrary, the Ricci flow is a natural regularization that respects isometric group actions, particularly adaptive to the collapsing setting. While it is possible that an equivariant Reifenberg method appears in future works, the Ricci flow smoothing method (Theorems 1.1 and 1.2) remains necessary at the current stage.

Remark 8 We also point out that when $Z$ is (a bounded open subset of) a $k$-dimensional manifold with locally bounded geometry, Theorem 1.4 provides a localization of [28, Theorem B]; see [31, Remark 2]. But our emphasis here is the extra orbifold singularity which may very well occur in the collapsing geometry; see Remark 1.
In fact, if \( f : B_g(p, 20) \to \mathbb{B}(20)/G_O \) is an infranil fiber bundle over an orbifold neighborhood with \( G_O < O(k), |G_O| < \infty \), and \( \text{diam}_g(f^{-1}(z)) < |G_O|^{-1} \delta \) for any \( z \in \mathbb{B}(20)/G_O \), we also have the structure described by the assumptions of Theorem 1.4, presenting a local inverse to this theorem. More detailed descriptions about the infranil fiber bundle structure over an orbifold chart, as well as the locally unwrapped neighborhood, could be found in [17, §7] and [30, §3].

We now explain the above mentioned point, following the notations in [17, Definition 7-3]. By the fact that both \( \mathbb{B}(20) \times F \) and \( B_g(p, 20) \) are smooth manifolds, it implies that the action of the finite orbifold group \( G_O \) is free of fixed points, implying that \( \theta : G_O \to \text{Aff}(F) \) is injective. Here we notice that the fiber \( F \) is diffeomorphic to an infranil manifold, and \( \text{Aff}(F) \) is the affine transformation group of \( F \); see e.g. [17, (0-3-2)]. On the other hand, let \( \pi : X \to \mathbb{B}(20) \times F \) denote the universal covering. We clearly see that \( X \cong \mathbb{B}(20) \times \tilde{F} \), with \( \tilde{F} \) denoting the universal covering of \( F \). Consequently, \( \pi_1(\mathbb{B}(20) \times F) = \pi_1(F) \). Notice that the map \( q \circ \pi : X \to B_g(p, 20) \), with \( q : \mathbb{B}(20) \times F \to B_g(p, 20) \) denoting the quotient map of the \( G_O \) action, is actually a covering map, thanks to the discrete and free action of \( G_O \). Then the simple connectedness of \( X \) makes it the universal covering space of \( B_g(p, 20) \). Since \( \pi \) is a normal covering, we have \( \pi_1(F) \leftarrow \pi_1(B_g(p, 20)) \), with the quotient \( H := \pi_1(B_g(p, 20))/\pi_1(F) \) acting on \( \mathbb{B}(20) \times F \) so that \( \mathbb{B}(20) \times F/H = B_g(p, 20) \). By the correspondence between the covering spaces of \( B_g(p, 20) \) and subgroups of \( \pi_1(B_g(p, 20)) \), we know that \( H \cong G_O \), whence a surjective group homomorphism

\[
\pi_1(B_g(p, 20)) \to \pi_1(B_g(p, 20))/\pi_1(F) \cong G_O.
\]

To see that \( \text{rank } \tilde{\Gamma}_\delta(p) = m - k \), we notice that \( \text{diam}_g q^{-1}(f^{-1}(Z)) \leq \delta \) for all \( z \in \mathbb{B}(20)/G_O \), and thus \( \tilde{\Gamma}_\delta(\pi(\tilde{p})) = \pi_1(\mathbb{B}(20) \times F) = \pi_1(F) \), for any \( \tilde{p} \in \pi^{-1}(p) \subset X \). Since \( F \) is diffeomorphic to an infranil manifold, \( \pi_1(F) \) is almost nilpotent with \( \text{rank } \pi_1(F) = \text{dim } F = m - k \). The following splitting short exact sequence of groups

\[
0 \to \pi_1(F) \to \pi_1(B_g(p, 20)) \xrightarrow{\theta} G_O \to 0,
\]

then tells that \( \tilde{\Gamma}_\delta(p) = \tilde{\Gamma}_\delta(\pi(\tilde{p})) \rtimes G_O \), implying that \( \text{rank } \tilde{\Gamma}_\delta(p) = \text{rank } \pi_1(F) = m - k \).

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**Appendix A: Local distance distortion estimates for Ricci flows with collapsing initial data**

The distance distortion estimates along Ricci flows is a crucial issue in view of its many natural applications (besides Lemma 4.1 in the proof of Theorem 1.4, see also e.g. [11] for a survey). In [29] a uniform distance distortion estimate for Ricci flows with collapsing initial data has been obtained. That estimate is for compact Ricci flow solutions and only compares the distance functions on nearby positive time slices. Here we present a new distance distortion estimate which can be seen both as an extension (to \( t = 0 \)) and a localization of the estimate in [29, Theorem 1.1] under some extra assumption on the Ricci curvature:

**Theorem A.1** Given a positive integer \( m \), positive constants \( \bar{C}_0, C_R, T \leq 1 \) and \( \alpha \in (0, \frac{1}{20m-1}) \), there are constants \( C_D(\bar{C}_0, C_R, m) \geq 1 \) and \( T_D(\bar{C}_0, C_R, m) \in (0, T) \) such that
for an m-dimensional complete Ricci flow \((M, g(t))\) with bounded curvature for \(t \in [0, T]\), if for some \(x_0 \in M\) and any \(t \in [0, T]\) we have

\[
R_{g(t)} \geq -C_R \text{ in } B_{g(t)}(x_0, 10),
\]

and the initial metric has a uniform bound \(\tilde{C}_0\) on the doubling and Poincaré constant for the geodesic ball \(B_{g(0)}(x_0, 10)\), then for any \(x, y \in B_{g(t)}(x_0, \sqrt{tD})\) and \(t \in [0, T]\), we have

\[
C_D^{-1} d_{g(0)}(x, y)^{1+4(m-1)\alpha} d_{g(0)}(x, y) \leq d_{g(t)}(x, y) \leq C_D d_{g(0)}(x, y)^{1-4(m-1)\alpha}.
\]

The proof of the lemma is based on the theory of local entropy developed in [48,49], as well as the local entropy lower bound by volume ratio obtained in [29]. It is inspired by the corresponding results for non-collapsing initial data in [28], and relies on a ball containment argument as surveyed in [11, §3].

**Remark 9** Here we will rely on [41, Theorem 2.1], and we point out that the assumptions [41, (1) and (2)] are only needed locally, i.e. the same conclusion of [41, Theorem 2.1] holds even if these conditions on the doubling and Poincaré constants are assumed only within the geodesic ball \(B_{g(0)}(x_0, 10)\): notice that the only global result needed in the proof of this theorem is [41, Theorem 2.2], in which the constants involved only depend on the dimension of the manifold; whereas all other arguments leading to the application of [41, Theorem 2.2] only depend on the doubling and Poincaré constants within the geodesic ball in question.

**Proof** By taking \(r_0 = \sqrt{t}\) in [39, Lemma 8.3] (see also [23, §17]), the Ricci curvature upper bound in (A.2) tells that if \(x, y \in B_{g(t)}(x_0, 10)\) whenever \(t \leq T\), then

\[
d_{g(0)}(x, y) \leq d_{g(t)}(x, y) + 3(m - 1)\sqrt{t}.
\]

To control the distance expansion, we first establish the following *a priori* estimate

**Claim A.2** There exists a uniform constant \(\tilde{C}_1(\tilde{C}_0, C_R, m) > 0\) such that if \(x, y \in B_{g(t)}(x_0, 10)\) for any \(t \leq t_0 := d_0^2\) with \(d_0 := d_{g(0)}(x, y) \leq \sqrt{t}\), then \(d_{g(t_0)}(x, y) \leq \tilde{C}_1 d_0\).

**Proof of the claim** To see this, we let \(\sigma : [0, 1] \to B_{g(0)}(x, d_0)\) be a minimal \(g(0)\)-geodesic realizing \(d_0\), and without loss of generality we may assume that \(d_{g(t_0)}(x, y) \geq 10d_0\). Now let \(\{\sigma(s_i)\}_{i=1}^N\) be a maximal collection of points on the image of \(\sigma\) so that \(d_{g(t_0)}(\sigma(s_i), \sigma(s_j)) \geq 2d_0\) when \(i \neq j\). By the maximality of \(\{\sigma(s_i)\}\), we see that \(B_{g(t_0)}(\sigma(s_i), d_0) \cap B_{g(t_0)}(\sigma(s_j), d_0) = \emptyset\), and it is also clear that \(\{B_{g(t_0)}(\sigma(s_i), 2d_0)\}\) cover \(\text{Image}(\sigma)\). Especially, we have

\[
d_{g(t_0)}(x, y) \leq \sum_{i=1}^{N-1} d_{g(t_0)}(\sigma(s_i), \sigma(s_{i+1})) \leq 4N d_0.
\]

We therefore only need to bound \(N\) uniformly from above. Moreover, by (A.4) we see that for each \(i = 1, \ldots, N\),

\[
\forall y' \in B_{g(t_0)}(\sigma(s_i), d_0), \quad d_{g(t_0)}(\sigma(s_i), y') \leq d_{g(t_0)}(\sigma(s_i), y') + \alpha \leq (1 + 8(m - 1)\sqrt{\alpha})d_0.
\]
Therefore, it is easily seen that each \( B_{g(t)}(\sigma(s_i), d_0) \subset B_{g(0)}(\sigma(s_i), (1 + 8(m - 1)\sqrt{\alpha})d_0) \), and consequently, as each \( \sigma(s_i) \in B_{g(0)}(x, d_0) \), we have
\[
\bigcup_{i=0}^{k} B_{g(t)}(\sigma(s_i), d_0) \subset B_{g(0)}(x, (2 + 8(m - 1)\sqrt{\alpha})d_0).
\] (A.5)

We now study the local entropy associated to the various metric balls. We first recall that the uniform bound \( C_0 \) on the doubling and Poincaré constants gives a uniform bound on the Sobolev constant \( C_S = C_S(m, \tilde{C}_0) \), according to [41, Theorem 2.1] (see also Remark 9 and [29, Proposition 2.1]). Therefore, following the argument in [29, §3.1], we get to [29, (3.1)], and by (A.1) we have
\[
\mathcal{W}\left( B_{g(0)}(x, 8md_0), g(0), v^2, \tau \right) \
\geq \log |B_{g(0)}(x, 8md_0)| d_0^{-m} - (C_R + (8md_0)^{-2}) \tau - \frac{m}{2} \log (512C_{Sem}^3 \pi).
\]
for any \( \tau > 0 \) and any \( v \in W^{1,2}_0(B_{g(0)}(x, 8md_0)) \) with \( \int_M v^2 dV_g = 1 \). Here due to the selection of \( v \), we have the Perelman’s \( \mathcal{W} \)-entropy equal to the local entropy \( \mathcal{W}(B_{g(0)}(x, 8md_0), g(0), v^2, \tau) \), as defined in [48, §2]. Taking the infimum of \( \mathcal{W}(B_{g(0)}(x, 8md_0), g(0), v^2, \tau) \) among all admissible \( v \) described above, we see that for any \( \tau > 0 \),
\[
\mu\left( B_{g(0)}(x, 8md_0), g(0), \tau \right) \
\geq \log |B_{g(0)}(x, 8md_0)| d_0^{-m} - (C_R + (8md_0)^{-2}) \tau - \frac{m}{2} \log (512C_{Sem}^3 \pi).
\]
Consequently, for \( v(B_{g(0)}(x, 8md_0), g(0), \tau) = \inf_{s \in (0, \tau]} \mu\left( B_{g(0)}(x, 8md_0), g(0), s \right) \) we have
\[
v(B_{g(0)}(x, 8md_0), g(0), \tau) \
\geq \log |B_{g(0)}(x, 8md_0)| d_0^{-m} - (C_R + (8md_0)^{-2}) \tau - \frac{m}{2} \log (512C_{Sem}^3 \pi). \quad (A.6)
\]
On the other hand, by (A.2) we have \( |R_{g(t)}| \leq 2m(m - 1)\alpha \tau^{-1} \) for \( \tau > 0 \), and applying [48, Theorem 3.6], we can bound the local entropy from above by volume ratio:
\[
v\left( B_{g(t)}(\sigma(s_i), d_0), g(t), d_0^2 \right) \leq \log \frac{|B_{g(t)}(\sigma(s_i), d_0)|}{\omega_m d_0^m} + (2^{m+7} + 2m(m - 1)\alpha \tau^{-1} d_0^2); \quad (A.7)
\]
while by (A.2) and the effective monotonicity of the local entropy ([48, Theorem 5.4]), we see that
\[
v\left( B_{g(t)}(\sigma(s_i), d_0), g(t), d_0^2 \right) \geq v\left( B_{g(0)}(\sigma(s_i), 3d_0), g(0), d_0^2 + t \right) - 1. \quad (A.8)
\]
Moreover, since \( B_{g(0)}(\sigma(s_i), 3d_0) \subset B_{g(0)}(x, 8md_0) \) for each \( i = 1, \ldots, N \), applying [48, Proposition 2.1] and (A.6) we see that
\[
v\left( B_{g(0)}(\sigma(s_i), 3d_0), g(0), \tau \right) \
\geq \log |B_{g(0)}(x, 8md_0)| d_0^{-m} - (C_R + (8md_0)^{-2}) \tau - \frac{m}{2} \log (512C_{Sem}^3 \pi). \quad (A.9)
\]
We now obtain a uniform bound on \( N \). Denoting \( v_i,s(r, \tau) := v(B_{g(s)}(\sigma(s_i), r), g(s), \tau) \) for each \( i = 1, \ldots, N, r \leq 8md_0, s \leq t_0 \) and \( \tau \leq d_0^2 + t_0 \), we consecutively apply (A.7),
(A.8) and (A.9) with $t_0 = d_0^2$ to see that

\[
|B_{g(t_0)}(\sigma(s), d_0)| d_0^{-m} \geq \omega_m e^{v_{i,t_0}(d_0,d_0^2) - 2m^2 - 2m(M-1)\alpha - 2} \geq \omega_m e^{2m^2 - 2C_R d_0^2 - 2m(M-1)\alpha - 2} \geq \frac{\omega_m e^{2m^2 - 2C_R d_0^2 - 2m(M-1)\alpha - 2}}{(512C_S em^3 \pi)^{\frac{m}{2}}} |B_{g(t_0)}(x, 8md_0)| d_0^{-m}.
\]

Therefore, by the mutual disjointness of $\{B_{g(t_0)}(\sigma(s), d_0)\}$ and (A.5) we have

\[
|B_{g(0)}(x, 8md_0)|_{g(0)} \geq \sum_{i=1}^{N} |B_{g(t_0)}(\sigma(s), d_0)|_{g(t_0)} \geq \sum_{i=1}^{N} |B_{g(t_0)}(\sigma(s), d_0)|_{g(t)} \geq \sum_{i=1}^{N} C |B_{g(0)}(x, 8md_0)|_{g(0)},
\]

where $C = \omega_m e^{2m^2 - 2C_R d_0^2 - 2m(M-1)\alpha - 2} (512C_S em^3 \pi)^{\frac{m}{2}}$. Consequently, we easily see that

\[
N \leq \frac{\omega_m^{-1} e^{2m^2 + C_R + 2m^2(M-1)^2 + 2(88C_S em^3 \pi)^{\frac{m}{2}}} + 10}{4 \tilde{C}_1(\tilde{C}_0, C_R, m)},
\]

since $\alpha < 1$, and we emphasize that $C_S$ is solely determined by $\tilde{C}_0$ and $m$. \hfill $\Box$

Now let $T_0 \leq T$ be the first time when some point $y \in B_{g(0)}(x_0, \sqrt{T_0})$ sees $d_{g(t_0)}(x_0, y) \geq 10$, then applying the estimate established by the claim with $x_0$, $y$, we have $T_0 \geq 100 \tilde{C}_1^{-2} =: T_D(\tilde{C}_0, C_R, m)$. Especially, for any $x, y \in B_{g(0)}(x_0, \sqrt{T_D})$, we have (A.4) holds without any extra assumption, and for any such points,

\[
\forall 0 < s < t \leq T_D, \quad \left(\frac{s}{t}\right)^{4(M-1)\alpha} \leq \frac{d_{g(t)}(x, y)}{d_{g(s)}(x, y)} \leq \left(\frac{t}{s}\right)^{4(M-1)\alpha}.
\]

Therefore, reasoning in the same way as in [28, Appendix], we have

\[
\forall x, y \in B_{g(0)}(x_0, \sqrt{T_D}), \quad d_{g(0)}(x, y) \leq 3md_{g(t)}(x, y) \leq \frac{3m}{4(M-1)\alpha + 1}.
\]

We now show the other side of the estimate following the same argument as in [28, Appendix]. Given $x, y \in B_{g(0)}(x_0, \sqrt{T_D})$ satisfying $d_0 = d_{g(0)}(x, y) \leq \sqrt{T_D}$ and given $t \leq T_D$, we begin with setting $N := \left[d_0 t^{-\frac{1}{2}}\right] + 1$ so that $d_0 < N \sqrt{t} \leq 2 \sqrt{T} \leq 2$. Dividing a minimal $(g(0))$-geodesic $\sigma$ that realizes $d_0$ into $N$ pieces of equal length, i.e. $|\sigma|_{[s_i, s_{i+1}]} = N^{-1} d_0$ for $0 = s_0 < \cdots < s_N = 1$, by the claim above we see that

\[
\forall i = 0, 1, \ldots, N - 1, \quad d_{g(N^{-2}d_0^2)}(\sigma(s_i), \sigma(s_{i+1})) \leq \tilde{C}_1 N^{-1} d_0;
\]

moreover, since $t > N^{-2} d_0^2$, by (A.13) we have

\[
d_{g(N^{-2}d_0^2)}(x, y) \geq (N^{-2} d_0^2 t^{-1})^{2(M-1)\alpha} d_{g(t)}(x, y).
\]
Now adding through $i = 0, 1, \ldots, N - 1$, by these inequalities we have

$$d_0 = \sum_{i=0}^{N-1} |\sigma|_{[s_i, s_{i+1}]},$$

and consequently, we have

$$d_g(x, y) \leq 16^m \bar{C}_1 d_0^{1-4(m-1)\alpha}.$$  

Combining this with \((A.14)\) we have the desired estimate

$$\forall x, y \in B_{g(0)}(x_0, \sqrt{T_D}), \forall t \leq T_D, \quad C_D^{-1} d_{g(0)}(x, y)^{1+4(m-1)\alpha} \leq d_{g(t)}(x, y) \leq C_D d_{g(0)}(x, y)^{1-4(m-1)\alpha},$$

where $C_D := \max \{ (8m)^{4m}, 16^m \bar{C}_1 \}$, only depending on $\bar{C}_0$ and $C_R$. \hfill \Box

Arguing in the same way, if \((A.2)\) can be assumed globally on $M$, then we have the following

**Corollary A.3** With the same assumptions as in Theorem A.1, but with \((A.2)\) replaced by

$$\forall t \leq T, \quad \sup_M |\text{Re}_{g(t)}|_{g(t)} \leq 2(m - 1)\alpha t^{-1}, \quad (A.15)$$

then we have for any $t \leq T$ and $x, y \in B_{g(0)}(x_0, 5)$ satisfying $d_{g(0)}(x, y) \leq 1$,

$$C_D^{-1} d_{g(0)}(x, y)^{1+4(m-1)\alpha} \leq d_{g(t)}(x, y) \leq C_D d_{g(0)}(x, y)^{1-4(m-1)\alpha}. \quad (A.16)$$

**Proof** By \((A.15)\), we see that \((A.4)\) holds for any $x, y \in B_{g(0)}(x_0, 5)$, without any extra assumption, and thus Claim A.2 holds for all such points. Therefore, the rest of the arguments follow without needing to confine ourselves in a smaller geodesic ball. \hfill \Box

**Remark 10** The referee kindly pointed out to us that a similar distance distortion should hold under curvature bound and scaling invariant injectivity radius lower bond $r_{inj} \geq ct^\frac{1}{2}$ using a similar argument in the proof.

**References**

1. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded below. Duke Math. J. 163(7), 1405–1490 (2014)
2. Bamler, R.: A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature. Math. Res. Lett. 23(2), 325–337 (2016)
3. Bando, S., Kasue, A., Nakajima, H.: On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth. Invent. Math. 97(2), 313–349 (1989)
4. Cavalletti, F., Mondino, A.: Almost Euclidean isoperimetric inequalities in spaces satisfying local Ricci curvature lower bounds. Int. Math. Res. Not. (2018). https://doi.org/10.1093/imrn/rny070
5. Chau, A., Tam, L.-F., Yu, C.: Pseudolocality for the Ricci flow and applications. Can. J. Math. 63(1), 55–85 (2011)
6. Cheeger, J., Colding, T.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. Math. 144(1), 189–237 (1996)
7. Cheeger, J., Colding, T.: On the structure of spaces with Ricci curvature bounded below: I. J. Differ. Geom. 46(3), 406–480 (1997)
8. Cheeger, J., Colding, T.: On the structure of spaces with Ricci curvature bounded below. III. J. Differ. Geom. 54, 37–74 (2000)
9. Cheeger, J., Fukaya, K., Gromov, M.: Nilpotent structures and invariant metrics on collapsed manifolds. J. Am. Math. Soc. 5(2), 327–372 (1992)
10. Cheeger, J., Jiang, W., Naber, A.: Rectifiability of singular sets in noncollapsed spaces with Ricci curvature bounded below. Ann. Math. (2) 193(2), 407–538 (2021)
11. Chen, X., Wang, B.: Remarks of weak-compactness along Kähler Ricci flow. In: Proceedings of the Seventh International Congress of Chinese Mathematicians, Advanced Lectures in Mathematics (ALM) 44, vol. II, pp. 203–233. International Press, Somerville (2019)
12. Colding, T.H.: Ricci curvature and volume convergence. Ann. Math. (2) 145(3), 477–501 (1997)
13. Colding, T.H., Naber, A.C.: Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. Ann. Math. (2) 176(2), 1173–1229 (2012)
14. Dai, X., Wei, G., Ye, R.: Smoothing Riemannian metrics with Ricci curvature bounded below. Manuscr. Math. 90(1), 49–61 (1996)
15. De Philippis, G., Gigli, N.: Non-collapsed spaces with Ricci curvature bounded below. J. École Polytech. Math. 5, 613–650 (2018)
16. Fukaya, K.: A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. J. Differ. Geom. 28, 1–21 (1988)
17. Fukaya, K.: Collapsing Riemannian manifolds to ones with lower dimension II. J. Math. Soc. Jpn. 41(2), 333–356 (1989)
18. Fukaya, K., Yamaguchi, T.: The fundamental groups of almost non-negatively curved manifolds. Ann. Math. (2) 136(2), 253–333 (1992)
19. Gigli, N.: Lecture notes on differential calculus on RCD spaces. Publ. Res. Inst. Math. Sci. 54(4), 855–918 (2018)
20. Gromov, M.: Paul Levi’s isoperimetric inequality. Preprint, IHÉS
21. Gromov, M.: Metric inequalities with scalar curvature. Geom. Funct. Anal. 28(3), 645–726 (2018)
22. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. 17(2), 255–306 (1982)
23. Hamilton, R.: The formation of singularities in Ricci flow. In: Hsiung, C.C., Yau, S.T. (eds.) Surveys in Differential Geometry (Cambridge, MA, 1993), vol. II, pp. 7–136. International Press, Cambridge (1995)
24. He, F.: Existence and applications of Ricci flows via pseudolocality. Preprint, arXiv:1610.01735
25. Hochard, R.: Short-time existence of the Ricci flow on complete, non-collapsed 3-manifolds with Ricci curvature bounded from below. Preprint, arXiv:1603.08726
26. Honda, S.: Collapsed Ricci limit spaces as non-collapsed RCD spaces. SIGMA Symmetry Integr. Geom. Methods Appl. 16, 021 (2020)
27. Huang, H.: Fibrations and stability of compact group actions on manifolds with local bounded Ricci covering geometry. Front. Math. China 15(1), 69–89 (2020)
28. Huang, H., Kong, L., Rong, X., Shicheng, X.: Collapsed manifolds with Ricci bounded covering geometry. Trans. Am. Math. Soc. 373(11), 8039–8057 (2020)
29. Huang, S.: Notes on Ricci flows with collapsing initial data (I): distance distortion. Trans. Am. Math. Soc. 373(6), 4389–4414 (2020)
30. Huang, S.: On the long-time behavior of immortal Ricci flows. Calc. Var. Partial Differ. Equ. 60(2), 78 (2021)
31. Huang, S., Rong, X., Wang, B.: Collapsing geometry with Ricci curvature bounded below and Ricci flow smoothing. SIGMA Symmetry Integr. Geom. Methods Appl. 16, 123 (2020)
32. Huang, S., Wang, B.: Rigidity of the first Betti number via Ricci flow smoothing. Preprint, arXiv:2004.09762
33. Kapovitch, V., Petrunin, A., Tuschmann, W.: Nilpotency, almost nonnegative curvature and the gradient flow on Alexandrov spaces. Ann. Math. (2) 171(1), 343–373 (2010)
34. Kapovitch, V., Wilking, B.: Structure of fundamental groups of manifolds with Ricci curvature bounded below. Preprint, arXiv:1105.5955
35. Lai, Y.: Ricci flow under local almost non-negative curvature conditions. Adv. Math. 343, 353–392 (2019)
36. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. 156(3–4), 153–201 (1986)
37. Liu, G., Székelyhidi, G.: Gromov–Hausdorff limits of Kähler manifolds with Ricci curvature bounded below II. Commun. Pure Appl. Math. 74(5), 909–931 (2021)
38. Naber, A., Zhang, R.: Topology and $\epsilon$-regularity theorems on collapsed manifolds with Ricci curvature bounds. Geom. Topol. 20(5), 2575–2664 (2016)
39. Perelman, G.: The entropy formula for the Ricci flow and its applications. Preprint, arXiv:math/0211159
40. Ruh, E.A.: Almost flat manifolds. J. Differ. Geom. 17(1), 1–14 (1982)
41. Saloff-Coste, L.: A note on Poincaré, Sobolev, and Harnack inequalities. Int. Math. Res. Not. 2, 27–38 (1992)
42. Schoen, R., Yau, S.-T.: Lectures on differential geometry. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y. Cheng. With a preface translated from the Chinese by Kaising Tso. Conference Proceedings and Lecture Notes in Geometry and Topology, I., pp. v+235. International Press, Cambridge (1994). ISBN: 1-57146-012-8
43. Shi, W.-X.: Deforming the metric on complete Riemannian manifolds. J. Differ. Geom. 30(1), 223–301 (1989)
44. Simon, M., Topping, P.: Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces. Geom. Topol. 25(2), 913–948 (2021)
45. Sormani, C.: Scalar curvature and intrinsic flat convergence. In: Gigli, N. (ed.) Measure Theory in Non-smooth Spaces, Partial Differential Equations and Measurement Theory, pp. 288–338. De Gruyter Open, Warsaw (2017)
46. Tian, G., Wang, B.: On the structure of almost Einstein manifolds. J. Am. Math. Soc. 28(4), 1169–1209 (2015)
47. Topping, P.: Ricci flow compactness via pseudolocality, and flows with incomplete metrics. J. Eur. Math. Soc. (JEMS) 12(6), 1429–1451 (2010)
48. Wang, B.: The local entropy along Ricci flow part A: the no-local-collapsing theorems. Camb. J. Math. 6(3), 267–346 (2018)
49. Wang, B.: The local entropy along Ricci flow part B: the pseudo-locality theorems. Preprint, arXiv:2010.09981

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