Abstract—In this paper, we want to find out the determining factors of Chernoff information in distinguishing a set of Gaussian graphs. We find that Chernoff information of two Gaussian graphs can be determined by the generalized eigenvalues of their covariance matrices. We find that the unit generalized eigenvalue doesn’t affect Chernoff information and its corresponding dimension doesn’t provide information for classification purpose. In addition, we can provide a partial ordering using Chernoff information between a series of Gaussian trees connected by independent grafting operations. With the relationship between generalized eigenvalues and Chernoff information, we can do optimal linear dimension reduction with least loss of information for classification.

Key words: Gaussian graphs, generalized eigenvalue, Chernoff information, dimension reduction

I. INTRODUCTION

Gaussian graphical model is widely used in constructing the conditional independence of continuous random variables. It is used in many applications such as social network [1], economics [2], biology [3] and so on. Our work is about statistical inference problems related to Gaussian graphical model where we want to learn the error exponents for classification.

There are two different error events in this learning problem. If we want to classify the learnt distribution from the true one, we often use Kullback-Leibler (KL) distance as the error exponent [4], [5]. Otherwise, we use Chernoff information when we measure the average error exponent for the error probability in discerning M distributions based on a sequence of data drawn independently from one of M distributions. The minimum pair-wise Chernoff information among these distributions is the error exponent characterizing the performance of an M-ary hypothesis testing problem.

In algebraic analysis of hypothesis testing problem, we also use generalized eigenvalues of covariance matrices as a metric of the difference between them [6], [7]. Clearly, Chernoff information and generalized eigenvalues of covariance matrices are respectively probabilistic and algebraic ways to describe the difference between two Gaussian graphs. So there must be relation among topology, statistical distributions (Chernoff information), and algebra (generalized eigenvalues). This paper will study this relationship and show how Chernoff information can be determined by generalized eigenvalues. What’s more, we will show how topology differences affect generalized eigenvalues and thus Chernoff information.

More specifically, we find that two Gaussian graphs can be linearly and inversely transformed to two graphs whose covariance matrices are diagonal. And entries of the diagonal matrices are related to generalized eigenvalues. Thus we find that Chernoff information between two Gaussian graphs is an expression of generalized eigenvalues and a special parameter λ∗ which is also determined by generalized eigenvalues.

What’s more, we find that the unit generalized eigenvalue doesn’t affect Chernoff information and the corresponding dimension is not making contributions to differentiating two Gaussian graphs for classification problem.

Our former paper [8] dealt with the classification problem related to Gaussian trees. And we found that some special operations on Gaussian trees, namely, adding operation and division operation, don’t change Chernoff information between them. Now in this paper, we find that the two operations only add one extra unit generalized eigenvalue and don’t affect other generalized eigenvalues. So we can use generalized eigenvalues to prove the same probabilities. [8] also dealt with two Gaussian trees connected by one grafting operation and showed that Chernoff information between them is the same with that of two special 3-node trees whose weights are related to the underlying operation. In this paper, we extend this result to a Gaussian tree chain connected by independent grafting operations and provide a partial ordering of Chernoff information between these trees.

In practice, the collection and storage of data have cost and we may need to reduce the dimension of data. An good choice here is doing linear dimension reduction in collection stage. We linearly transform an N dimensional Gaussian vector x to an Nφ < N dimensional vector y = Ax, through an Nφ × N matrix A. An immediate question here is the optimal selection of A which can maximize the error exponent in the classification of two hypothesis. In our former paper [9], we only dealt with a simple, but non-trivial case with Nφ = 1.

In this work, we offer an optimal method to maximize the resulting Chernoff information after a linear transformation for an arbitrary Nφ ≥ 1.

Our major and novel results can be summarized as follows. We first provide the relationship between Chernoff information and generalized eigenvalues. According to the former result, we show that generalized eigenvalues which are equal to 1 make no contribution to Chernoff information. And we use this to explain why adding and division operations of [8]...
don’t affect Chernoff information between Gaussian trees. These results build a relationship between topology, statistical distribution and algebra. What’s more, we deal with Gaussian trees connected by more than one grafting operation and show a partial ordering inside the chain. At last, we provide an optimal linear dimension reduction method.

This paper is organized as below. In Section II we propose the model of our analysis. The relationship between topology, Chernoff information and generalized eigenvalues is shown in Section III. The partial ordering of Gaussian tree chain is presented in Section IV. Section V shows the optimal dimension reduction method. And in Section VI we conclude the paper.

II. SYSTEM MODEL

Consider a set of Gaussian graphs, namely, \( G_k(x), k = 1, 2, \ldots, M \), with their prior probabilities given by \( \pi_1, \pi_2, \ldots, \pi_M \). For simplification, we normalize the variance of all Gaussian variables to be 1 and the mean values to be 0. They share the same entropy, and thus the same determinant of their covariance matrices \( \Sigma_k = [\sigma_{ij}^{(k)}] \). The same entropy means the same amount of randomness. This assumption can let us compare these Gaussian graphs fairly. We want to do an \( M \)-ary hypothesis testing to find out from which Gaussian distribution the data sequence \( X = [x_1, \ldots, x_L] \) comes from. We define the average error probability of the hypothesis testing to be \( P_e \) and let \( E_c = \lim_{L \to \infty} -\frac{\ln P_e}{L} \) be the resulting error exponent, which depends on the smallest Chernoff information between the graphs [9], namely,

\[
E_c = \min_{1 \leq i \neq j \leq M} CI(\Sigma_i || \Sigma_j) \tag{1}
\]

where \( CI(\Sigma_i || \Sigma_j) \) is the Chernoff information between the \( i^{th} \) and \( j^{th} \) graphs.

For two \( 0 \)-mean Gaussian joint distributions, \( x_1 \sim N(0, \Sigma_1) \) and \( x_2 \sim N(0, \Sigma_2) \), their Kullback-Leibler divergence is as follows

\[
D(\Sigma_1 || \Sigma_2) = \frac{1}{2} \log \frac{\Sigma_2}{\Sigma_1} + \frac{1}{2} \text{tr}(\Sigma_2^{-1} \Sigma_1) - \frac{N}{2} \tag{2}
\]

where \( \text{tr}(X) = \sum_{i} x_{ii} \) is the trace of square matrix \( X \). We define a new distribution \( N(0, \Sigma_{\lambda}) \) in the exponential family of the \( N(0, \Sigma_1) \) and \( N(0, \Sigma_2) \), namely

\[
\Sigma_{\lambda}^{-1} = \Sigma_1^{-1} \lambda + \Sigma_2^{-1}(1-\lambda) \tag{3}
\]

So the Chernoff information is given below

\[
CI(\Sigma_1 || \Sigma_2) = D(\Sigma_{\lambda^*} || \Sigma_2) = D(\Sigma_{\lambda^*} || \Sigma_1) \tag{4}
\]

where \( \lambda^* \) is the point at which the latter equation is satisfied [10]. It is unique in interval \([0, 1]\).

We already know that the overall Chernoff information in an \( M \)-ary testing is bottle-necked by the minimum pair-wise difference [10], thus we next focus on the calculation of Chernoff information of pair-wise Gaussian graphs. Further research will show that pair-wise Chernoff information can be determined by the generalized eigenvalues of covariance matrices \( \Sigma_1, \Sigma_2 \) and \( \lambda^* \). And \( \lambda^* \) can also be determined by these generalized eigenvalues.

In addition to the full observation case, we will also study a dimension reduction case. For two \( N \) variables Gaussian graphs \( G_1, G_2 \), in the full observation case, we can have access to all \( N \) variables and only need to calculate Chernoff information \( CI(\Sigma_1 || \Sigma_2) \). But in the dimension reduction case, we can only observe an \( N_O \)-dim vector each time, namely, \( y = Ax \), where \( A \) is an \( N_O \times N \) matrix and \( x \in R^N, y \in R^{N_O} \).

The new variables follow joint distributions \( N(0, \Sigma_1) \) and \( N(0, \Sigma_2) \) where \( \Sigma_1 = A \Sigma A^T, i = 1, 2 \). For fixed \( N_O \), we want to find out the optimal \( A^* \) and its Chernoff information result \( CI(\Sigma_1^* || \Sigma_2^*) \), s.t.

\[
A^* = \arg \max_A CI(\Sigma_1^* || \Sigma_2^*) \tag{5}
\]

We only consider binary hypothesis testing in this dimension reduction problem.

III. GENERALIZED EIGENVALUES, CHERNOFF INFORMATION AND TOPOLOGY

Chernoff information is the measurement of the difference between statistical distributions. It is hard to be calculated directly and we rarely study its insights about the relationship between Chernoff information and structure characters. Chernoff information and generalized eigenvalues are both important parameters to describe Gaussian graphical models. We will expose the relationship between them.

A. Linear transformation to diagonal covariance matrix related to generalized eigenvalues

For two \( N \)-node 0-mean Gaussian graphs \( G_1 \) and \( G_2 \) on random variables \( x \), whose covariance matrices are \( \Sigma_1 \) and \( \Sigma_2 \), we can use an inverse linear transformation matrix \( P \) to transform them to \( x' = Px \) whose covariance matrices \( \Sigma_1' \) and \( \Sigma_2' \) are diagonal and related to the generalized eigenvalues of \( \Sigma_1 \) and \( \Sigma_2 \).

\( \Sigma_1 \) and \( \Sigma_2 \) are real symmetric positive definite matrices, so the eigenvalues of \( \Sigma_1 \Sigma_2^{-1} \) are all positive, as shown in Appendix A. The eigenvalue decomposition of \( \Sigma_1 \Sigma_2^{-1} \) is \( QAQ^{-1} \), where \( Q \) is an \( N \times N \) matrix and \( A = \text{Diag}(\lambda_i) \) is a diagonal matrix of eigenvalues, in which we put multiple eigenvalues adjacent. \( \lambda_i \) are the eigenvalues of \( \Sigma_1 \Sigma_2^{-1} \), namely, the generalized eigenvalues of \( \Sigma_1 \) and \( \Sigma_2 \). Note that \( Q \) may be non-orthogonal when \( \Sigma_1 \Sigma_2^{-1} \) isn’t symmetric.

**Proposition 1:** For two \( N \)-node 0-mean Gaussian graphs \( G_1 \) and \( G_2 \) whose covariance matrices are \( \Sigma_1 \) and \( \Sigma_2 \) respectively, we can construct a linear transformation matrix \( P = \left(Q^{-1} \Sigma_2 (Q^{-1})^T\right)^{-\frac{1}{2}} Q^{-1} \) and thus

\[
\Sigma_2' = P \Sigma_2 P^T = I_N \tag{6}
\]

\[
\Sigma_1' = P \Sigma_1 P^T = A \tag{7}
\]

where eigenvalue decomposition of \( \Sigma_1 \Sigma_2^{-1} \) is \( QAQ^{-1} \). The proof of Proposition 1 is shown in Appendix A.
We can treat $\Sigma'_1$ and $\Sigma'_2$ as two graphs $G'_1$ and $G'_2$ on $x'$ obtained from $G_1$ and $G_2$ by inverse linear transformation $P$. $G'_1$ and $G'_2$ are graphs with $N$ independent variables.

The distances of $G'_1$ and $G'_2$ are as follows. The distances between two Gaussian graphs can be determined by their covariance matrices $\Sigma_1$ and $\Sigma_2$.

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$D(\Sigma_1||\Sigma_2) = D(\Sigma'_1||\Sigma'_2) = \frac{1}{2} \sum_i (-\ln \lambda_i + \lambda_i - 1)$  \hspace{1cm} (8)

$D(\Sigma_2||\Sigma_1) = D(\Sigma'_2||\Sigma'_1) = \frac{1}{2} \sum_i (\ln \lambda_i + \frac{1}{\lambda_i} - 1)$  \hspace{1cm} (9)

$D(\Sigma_1||\Sigma_1) = D(\Sigma'_1||\Sigma'_1) = \frac{1}{2} \sum_i \left( \ln \lambda_i + \frac{1}{\lambda_i} - \frac{1}{\lambda_i + (1 - \lambda_i)\lambda_i} - 1 \right)$  \hspace{1cm} (10)

$D(\Sigma_2||\Sigma_2) = D(\Sigma'_2||\Sigma'_2)$

$CI(\Sigma_1||\Sigma_2) = CI(\Sigma'_1||\Sigma'_2) = D(\Sigma'_1||\Sigma'_1) = D(\Sigma'_2||\Sigma'_2)$  \hspace{1cm} (12)

where $\lambda_i^{-1} = \lambda_i + \lambda_i^{-1}$ and $\lambda_i = \lambda_i + \lambda_i^{-1}$.

In this way, we can conclude that the KL and CI divergences between two Gaussian graphs can be determined by their generalized eigenvalues.

B. Relationship between generalized eigenvalues and Chernoff information

Here we will show the relationship between Chernoff information of two Gaussian graphs and generalized eigenvalues of their covariance matrices $\Sigma_1$ and $\Sigma_2$ under the assumption of same entropy and normalized covariance matrix. Under this assumption, $\prod_{i=1}^N \lambda_i = |\Sigma_1||\Sigma_2| = 1$.

Proposition 2: For two $N$-node Gaussian distributions who have the same entropy and normalized covariance matrix $\Sigma_1$, $\Sigma_2$, their Chernoff information satisfies

$CI(\Sigma_1||\Sigma_2) = \frac{1}{2} \sum_i \left\{ \ln \left( \frac{1 - \lambda_i^*}{\lambda_i} \right) \sqrt{\lambda_i} + \frac{\lambda_i^*}{\sqrt{\lambda_i}} \right\}$  \hspace{1cm} (13)

where $\{\lambda_i\}$ are the generalized eigenvalues of $\Sigma_1$, $\Sigma_2$, namely eigenvalues of $\Sigma_1||\Sigma_2^{-1}$, and $\lambda^* \in [0, 1]$ is the unique result of

$\sum_i \lambda_i = 1, \lambda^* + (1 - \lambda^*)\lambda_i = N$  \hspace{1cm} (14)

We can prove this proposition from the equation (12), as shown in Appendix [A].

We find that generalized eigenvalues of covariance matrices $\Sigma_1$ and $\Sigma_2$ are the key parameters of Chernoff information. We can get Chernoff information with these $N$ generalized eigenvalues, so these $N$ parameters contain all the information about the difference between two Gaussian trees. The generalized eigenvalues are so important that we need more property about them.

C. Performance of unit generalized eigenvalues

In former equation, we find that unit generalized eigenvalues are very special to Chernoff information.

Proposition 3: Assuming that the generalized eigenvalues of $(N + 1)$-node $G'_1$ and $G'_2$ are the same with that of $N$-node $G_1$ and $G_2$ except a newly added unit generalized eigenvalue, the optimal parameter $\lambda^*$ of $(\Sigma'_1, \Sigma'_2)$ is the same with that of $(\Sigma_1, \Sigma_2)$ and $CI(\Sigma'_1||\Sigma'_2) = CI(\Sigma_1||\Sigma_2)$.

Proposition [A] can be proved from Proposition [A] as shown in Appendix [A].

Proposition 3 shows a possible way to do dimension reduction that we can reduce the dimension of Gaussian graphs from $N + 1$ to $N$ without changing their Chernoff information if we only remove an unit generalized eigenvalue in this procedure.

Gaussian trees are tree models whose definition can be seen in [8], represent the dependence of multiple Gaussian random variables by tree topologies. [8] dealt with the classification on Gaussian trees. In that paper, they defined two special operations on two Gaussian trees, namely adding operation and division operation, as shown in Fig. 1. We add the same leaf node or divide the same edge by adding or division operations. These two operations don’t change Chernoff information between two Gaussian trees. Next we will show how generalized eigenvalues change after adding or division operation.

Proposition 4: Assume that Gaussian trees $G'_1$ and $G'_2$ are obtained from $G_1$ and $G_2$ by adding operation or division operation. Their covariance matrices are $(\Sigma'_1, \Sigma'_2), (\Sigma_1, \Sigma_2)$ respectively. The generalized eigenvalues of $(\Sigma'_1, \Sigma'_2)$ are the same with that of $(\Sigma_1, \Sigma_2)$ except a newly added unit eigenvalue.

Proposition [B] is proved in Appendix [B]. From Proposition [B] and [C], we can conclude that adding and division operations don’t affect Chernoff information between two Gaussian trees.
This result has been proved in our former paper [8] and is one of key contribution of that paper.

IV. PARTIAL ORDERING IN INDEPENDENT GRAFTING CHAIN

Grafting operation is a topological operation by cutting down a subtree from another tree and pasting it to another location, as shown in Figure 2. It is defined in our former paper [8].

In that former paper, we have shown that two Gaussian trees connected by one grafting operation have the same Chernoff information with two special 3-node Gaussian trees whose weights are related to the underlying operation. Now we consider a more complex situation: two Gaussian trees connected by more than one grafting operation.

Gaussian trees connected by more grafting operations can’t be simplified to a fixed couple of small trees because the interaction of these grafting operations varies. Our initial expectation was that bigger difference in topology between two Gaussian trees leads to larger Chernoff information. But this may not be true for all situations.

Before we deal with a sequence of grafting operations, we need to constrain the interaction among them. So we will define the independence of grafting operations at first.

Definition 1: If all the grafting operations can be divided into different subtrees, as shown in Figure 3 then these grafting operations are independent. After regrouping all the nodes into small subtrees represented by circles, the whole tree has super star-shaped topology. The subtree in the center is unchanged during grafting operations. And grafting operations are involved in disjoint super leaf nodes of the tree.

In Figure 3 we show 4 independent grafting operations around the unchanged subtree. There are three types of grafting operations in the star-shaped topology. From left to right, the 1-st, 2-nd grafting operations belong to the first type, the 4-th one belongs to the second type and the 3-rd operation belongs to the third type. For the first type, we can cut a subtree, represented by a small circle with a number in it, from the super leaf node(subtree1,2) and paste it to another part of this super leaf node. In the second type, we can cut the unchanged subtree outside the super leaf node(subtree5) and paste it to another location in this super leaf node. But for the third type, we will cut a subtree, represented by a small circle with a number in it, from a super leaf node(subtree3) and paste it to another super leaf node(subtree4). The third kind of grafting operation involves two super leaf nodes of the star while the first and second kinds of operation only involve one.

If all the grafting operations are independent, then we can make the following conclusion.

Proposition 5: For two Gaussian trees connected by several independent grafting operations, \( \lambda^* = 1/2 \) holds.

The trees have the same number of nodes and the same entropy due to grafting operations. So \( \lambda^* \) satisfies \( tr(\Sigma_A, (\Sigma_A - \Sigma_B)) = 0 \), which can be transformed from the definition formulas of \( \lambda^* \). \( tr(\Sigma_A, (\Sigma_A - \Sigma_B)) \) is a summation formula with \( 4n \) term, where each 4 terms are related to one single grafting operation. So we can deal with the terms respectively and prove \( tr(\Sigma_A, (\Sigma_A - \Sigma_B)) = 0 \) eventually. More details can be found in Appendix F.

Considering a grafting chain \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \), intuition tells us that \( CI(T_1 || T_3) \) is likely larger than \( CI(T_1 || T_2) \) and \( CI(T_2 || T_3) \), because the difference between \( T_1 \) and \( T_3 \) is the accumulation of \( T_1 - T_2 \)'s difference and \( T_2 - T_3 \)'s difference. If the assumption is true, we only need to consider the adjacent pairs of Gaussian trees in the chain when looking for the minimum Chernoff information. Later results will tell us that it depends on the structure of the chain.

Proposition 6: For the grafting chain \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \leftrightarrow T_4 \), where all the grafting operations in the chain are independent, we can conclude that \( CI(T_i || T_j) \leq CI(T_p || T_q) \) if \( p \leq i \leq j \leq q \).

If we want to find out the minimum Chernoff information in this set, we only need to try \( n-1 \) pairs of \( T_i - T_j \), rather than all the \( \binom{n}{2} \) pairs. The number of candidates is significantly reduced.

It is a partial ordering because we can’t compare \( CI(T_1 || T_2) \) and \( CI(T_2 || T_3) \) even in a simplest chain \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \) without knowing the weights. We can only compare Chernoff information pairs \( CI(T_i || T_j), CI(T_p || T_q) \) when \( p \leq i \leq j \leq q \) ordering, and thus this result is a partial inequality, rather than a full ordering inequality.

In proposition 6 we constrain the grafting operations independent. So we may wonder whether the result suits for all the possible grafting chains \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \leftrightarrow \cdots \leftrightarrow T_n \) without independent assumption. Taking grafting chain \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \) in Figure 4 as an example, the two grafting operations are not independent. Some special cases is shown in Table I. In this table, we can find that \( CI(T_1 || T_3) < CI(T_1 || T_2) \) can hold in some special situations. This is a counter-intuitive result because more topological differences can’t lead to larger Chernoff information between Gaussian trees.
So we can only provide a partial ordering with independent grafting operations in this section. The problems of ordering is much more complex than what we have expected. For a grafting chain \( T_1 \leftrightarrow T_2 \leftrightarrow T_3 \), the Chernoff information between \( T_1 \) and \( T_2 \) may be larger than that between \( T_1 \) and \( T_3 \), even though the difference between \( T_1 \) and \( T_3 \) seems smaller. Here we only consider topological difference, rather than parameterized difference, between Gaussian trees. Topological difference is not the only contribution factor affecting such comparisons [5], [11].

### V. DIMENSION REDUCTION

The analysis in former section deals with Chernoff information in full-observation situation where we can get access to all the variables. But in practice, there is significant cost in collecting and storing data. So we can only use linearly transformed low-dimensional samples to do the classification. The linear transformation matrix should make sure that the reduced data have maximum information for classification.

Traditional dimension reduction methods, such as Principal Component Analysis (PCA) and other Representation Learning [12], aim to find the optimal features with maximum information. They only consider a distribution at one time and don’t care how we use the resulting low-dimensional data. But our method considers two hypotheses at the same time and want to keep the most information for classification. Some features may be important in PCA, but invalid in our method because these features in two hypothesis are similar. Our object function here is the Chernoff information between the resulting low-dimensional data. We choose the dimensions of \( x' \) corresponding to the first \( k \) rank and last \( N_O - k \) rank of \( \{\lambda_i\} \), where \( N_O + m - N \leq k \leq m \) and \( k \geq 0 \). We can choose a sub-matrix of \( \Sigma'_1 \) and \( \Sigma'_2 \) corresponding to the \( N_O \) chosen eigenvalues as the result of dimension reduction. The \( N_O \times N \) linear transformation matrix \( A_k \) is the corresponding \( N_O \) rows of \( P \) corresponding to the chosen eigenvalues.

**Proposition 7:** \( A' \) is the optimal \( N_O \times N \) linear transformation matrices to maximize the Chernoff information in transformed space, namely \( A' = \arg \max_{A \in \mathbb{A}_{N_O \times N}} CI(\Sigma'_1 || \Sigma'_2) \) where \( \Sigma'_i = A' \Sigma_i A'^T \) for \( i = 1, 2 \). So \( A' \in \{ A_k | N_O + m - N \leq k \leq m, k \geq 0 \} \).

Proof of proposition 7 can be seen in Appendix [5] and this proposition ensure the optimality of our method.

The observation is \( y = A' x \) and the covariance matrices of \( y \) in two hypothesis are

\[
\Sigma''_2 = A' \Sigma_2 A'^T = I_{N_O} \quad (15)
\]

\[
\Sigma''_1 = A' \Sigma_1 A'^T = \text{Diag}(\{\mu_i\}) \quad (16)
\]

where \( \Sigma''_1 \) and \( \Sigma''_2 \) are \( N_O \times N_O \) diagonal matrices and \( \{\mu_1, \mu_2, \ldots, \mu_{N_O}\} \) (including multiple eigenvalues) are \( N_O \) chosen eigenvalues.

\[
D(\Sigma''_1 || \Sigma''_1) = \frac{1}{2} \sum \left( \ln \left( \frac{\lambda + (1 - \lambda)\mu_i}{\lambda + (1 - \lambda)\mu_i} \right) - 1 \right) \quad (17)
\]

\[
D(\Sigma''_1 || \Sigma''_2) = \frac{1}{2} \sum \left( \ln \left( \frac{\lambda + (1 - \lambda)\mu_i}{\lambda + (1 - \lambda)\mu_i} \right) - 1 \right) \quad (18)
\]

\[
CI(\Sigma''_1 || \Sigma''_2) = \max_{\lambda \in [0, 1]} \min \{ D(\Sigma''_1 || \Sigma''_1), D(\Sigma''_1 || \Sigma''_2) \} \quad (19)
\]

Equation 19 is another definition of Chernoff information [10]. The distances are only related to the eigenvalues we have chosen.

### VI. CONCLUSION

In this paper, we show the relationship between topology, statistical distribution and algebra. Chernoff information between two Gaussian graphs can be determined by the generalized eigenvalues of their covariance matrices. Among them unit generalized eigenvalue is very special and doesn’t affect the Chernoff information. Adding and division operations on Gaussian trees only add a newly unit generalized eigenvalues and don’t change other generalized eigenvalues. Thus these operations keep the Chernoff information. We also extend
our former result about grafting operation to Gaussian trees connected by more than one independent grafting operation and provide a partial ordering among these trees. What’s more, we provide an optimal linear dimension reduction method with the metric of Chernoff information.

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APPENDIX A
PROOF OF PROPOSITION

Because $\Sigma_1$ and $\Sigma_2^{-1}$ are real symmetric positive definite matrices, $\Sigma_1 = L_1 L_1^T$ and $\Sigma_2^{-1} = L_2 L_2^T$ due to Cholesky decomposition where $L_1$ and $L_2$ are real non-singular triangular matrices.

The characteristic polynomial of $\Sigma_1 \Sigma_2^{-1}$ is

$$f(\Sigma_1 \Sigma_2^{-1}) = |\Lambda - \Sigma_1 \Sigma_2^{-1}|$$

$$= |\Lambda | - L_1 L_1^T L_2 L_2^T L_1 |$$

$$= |L_1 (\Lambda I - L_1^T L_2 L_2^T L_1) L_1^{-1}|$$

$$= |L_1 - L_1^T L_2 L_2^T L_1|$$

$$= |L_1 - (L_1^T L_2)(L_2^T L_2)^T|$$

So $\Sigma_1 \Sigma_2^{-1}$ has the same eigenvalues with $(L_1^T L_2)(L_2^T L_2)^T$, which is a real symmetric positive definite matrix. So the eigenvalues of $\Sigma_1 \Sigma_2^{-1}$ are all positive and $\Sigma_1 \Sigma_2^{-1}$ is a positive definite matrix.

The eigenvalue decomposition of $\Sigma_1 \Sigma_2^{-1}$ is $Q \Lambda Q^{-1}$, where $Q$ is an $N \times N$ matrix and $\Lambda = \text{Diag}(\lambda_i)$ is a diagonal matrix of eigenvalues, in which we put multiple eigenvalues adjacent. $\lambda_i$ are the eigenvalues of $\Sigma_1 \Sigma_2^{-1}$, namely, the generalized eigenvalues of $\Sigma_1$ and $\Sigma_2$.

And $\Sigma_1^{(1)} = Q^{-1} \Sigma_1 Q^{-1T} = [a_{i j}]$ and $\Sigma_2^{(1)} = Q^{-1} \Sigma_2 Q^{-1T} = [b_{i j}]$ satisfy $\Sigma_1^{(1)} = \Lambda \Sigma_2^{(1)}$.

$$a_{i j} = \lambda_i b_{i j}$$

(21)

$$a_{i j} = \lambda_j b_{i j} = \lambda_j b_{i j}$$

(22)

So for $\forall i \neq j$, $a_{i j} = b_{i j} = 0$ or $\lambda_i = \lambda_j$.

If the eigenvalues of $\Sigma_1 \Sigma_2^{-1}$ have $k$ different multiple eigenvalues $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ with multiplicity $n_1, n_2, \ldots, n_k$, so

$$\Sigma_2^{(1)} = \begin{bmatrix} I_1 & J_2 & \cdots & J_k \end{bmatrix}$$

(23)

$$\Sigma_1^{(1)} = \begin{bmatrix} \lambda^{(1)} J_1 & \lambda^{(2)} J_2 & \cdots & \lambda^{(k)} J_k \end{bmatrix}$$

(24)

where $J_i$ is $n_i \times n_i$ inverse matrix.

We can also get an inverse matrix $Q^{(1)} = \Sigma_2^{(1)}^{-0.5}$ so that

$$\Sigma_2^{(2)} = Q^{(1)} \Sigma_2^{(1)} Q^{(1)} = I_N$$

(25)

$$\Sigma_1^{(2)} = Q^{(1)} \Sigma_1^{(1)} Q^{(1)} = \Lambda$$

(26)

So we can construct a linear transformation matrix $P = Q^{(1)} Q^{-1} = (Q^{-1} \Sigma_2 (Q^{-1})^T)^{-1} Q^{-1}$ and thus

$$\Sigma_1 = P \Sigma_2 P^T = I_N$$

(27)

$$\Sigma_1 = P \Sigma_2 P^T = \Lambda$$

(28)
APPENDIX B
PROOF OF PROPOSITION
\[ \lambda^* \text{ satisfies } D(\Sigma^*_\lambda, || \Sigma'_i ||) = D(\Sigma'_\lambda, || \Sigma'_j ||), \text{ and thus} \]
\[ \frac{1}{2} \sum_i \left( \ln (\lambda^*(1 + (1 - \lambda^*)l_i) + \frac{1}{\lambda^* + (1 - \lambda^*)l_i} - 1 \right) \]
\[ = \frac{1}{2} \sum_i \left( \ln (\lambda^* + (1 - \lambda^*)l_i) + \frac{1}{\lambda^* + (1 - \lambda^*)l_i} - 1 \right) \]
\[ = \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = N \]
(29)

With knowing \( \prod_{i=1}^N \lambda_i = 1 \), we can conclude
\[ \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = N \]
(30)

We also know
\[ \lambda^* \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} + (1 - \lambda^*) \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = \]
\[ = \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} - N \]
(31)

So \( \lambda^* \in [0, 1] \) is the unique result of
\[ \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = N \]
(32)

And
\[ CI(\Sigma_1|| \Sigma_2) = D(\Sigma^*_\lambda, || \Sigma'_i ||) \]
\[ \frac{1}{2} \sum_i \left( \ln (\lambda^* + (1 - \lambda^*)l_i) + \frac{1}{\lambda^* + (1 - \lambda^*)l_i} - 1 \right) \]
\[ = \frac{1}{2} \sum_i \ln \left( (1 - \lambda^*) \sqrt{l_i} + \frac{\lambda^*}{\sqrt{l_i}} \right) \]
(33)

APPENDIX C
PROOF OF PROPOSITION
\[ \lambda^* \text{ is unique in the range of } [0, 1]. \text{ So if } \lambda^* \text{ satisfies equation (14), then} \]
\[ \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} + \frac{1}{\lambda^* + (1 - \lambda^*)l_i} = \]
\[ = \sum_i \frac{1}{\lambda^* + (1 - \lambda^*)l_i} + \frac{1}{\lambda^* + (1 - \lambda^*)l_i} \]
(34)

where \( \{ \lambda_i \} \) are the generalized eigenvalues of \( (\Sigma_1, \Sigma_2) \).

So the generalized eigenvalues \( \{ \lambda_i \} \cup \{ 1 \} \) of \( (\Sigma'_i, \Sigma'_j) \) and \( \lambda^* \) also satisfies equation (14). The optimal \( \lambda^* \) is the same. And
\[ CI(\Sigma'_i|| \Sigma'_j) = \frac{1}{2} N \sum_i \left\{ \ln \left( (1 - \lambda^*) \sqrt{l_i} + \frac{\lambda^*}{\sqrt{l_i}} \right) \right\} \]
\[ + \frac{1}{2} \left\{ \ln \left( (1 - \lambda^*) \sqrt{l_i} + \frac{\lambda^*}{\sqrt{l_i}} \right) \right\} \]
\[ = \frac{1}{2} N \sum_i \left\{ \ln \left( (1 - \lambda^*) \sqrt{l_i} + \frac{\lambda^*}{\sqrt{l_i}} \right) \right\} \]
\[ = CI(\Sigma_1|| \Sigma_2) \]
(35)

APPENDIX D
PROOF OF PROPOSITION
Assume that Gaussian trees \( G_1' \) and \( G_2' \) are obtained from \( G_1 \) and \( G_2 \) by adding operation or division operation. Their covariance matrices are \( (\Sigma_1', \Sigma_2') \) respectively. The generalized eigenvalues of \( (\Sigma_1', \Sigma_2') \) are the same with that of \( (\Sigma_1, \Sigma_2) \) except a new added unit eigenvalue.

We use the equation of block matrix to prove this proposition. We deal with adding operation and division operation in different subsections.

A. Proof on adding operation case
Adding operation is shown in Fig.1. Assume \( G_1 \) and \( G_2 \) have node 1, \ldots, \( N \), and their covariance matrices are
\[ G_1: \Sigma_1 = \begin{bmatrix} D & \alpha \cr \alpha^T & 1 \end{bmatrix} \]
\[ G_2: \Sigma_2, \Sigma_2^{-1} = \begin{bmatrix} A_2 & B_2 \\ B_2^T & a_2 \end{bmatrix} \]
Without loss of generality, the new edge is \( e_{N,N+1} \) with parameter \( w \). So new covariance matrices after adding operation are
\[ \Sigma_1' = \begin{bmatrix} D & \alpha \cr \alpha^T & w \cr w & 1 \end{bmatrix} \]
\[ \Sigma_2' = \begin{bmatrix} A_2 & B_2 \cr B_2^T & a_2 + \frac{w^2}{1-w} \cr 0 & 1 \end{bmatrix} \]
(36)

The generalized eigenvalues of \( \Sigma_1 \) and \( \Sigma_2 \) are the roots of
\[ |\lambda I - \Sigma'_1 \Sigma'_2^{-1} | = 0 \]
(37)

So new trees have an extra unit generalized eigenvalue without changing other eigenvalues.

B. Proof on division operation case
Division operation is shown in Fig.1. Assume that \( G_1 \) and \( G_2 \) have node 1, \ldots, \( N \), and the connecting points are \( p = 1, q = N \) without loss of generality.

If we set edge \( e_{1N} \) in \( G_2 \) to be 0, we can treat it as a new \( N \)-node tree, whose covariance matrix is \( A^{-1} \). We divide \( A \) as
\[ A = \begin{bmatrix} x & X^T & 0 \\ Z & Y & 0 \\ Y^T & y \end{bmatrix} \]
(38)

where \( X, Y \) are \((N-2)\times1\) matrices and \( Z \) is \((N-2)\times(N-2)\) matrix.
In $G_1$ and $G_1'$, we define the column node 1 to nodes 2 to $N - 1$ to be $\alpha_j$ (setting the respective rows to the nodes in $G_{T1}$, 0) and the column node $N$ to nodes 2 to $N - 1$ to be $\alpha_2$ (setting the respective rows to the nodes in $G_{T1}$, 0). Then in $G_1$ and $G_1'$, the column from node 1 to nodes 2 to $N - 1$ is $\alpha_1 + w_1 w_2 \alpha_2$, because node 1 can reach the nodes in $G_{T1}$, directly, but reach the nodes in $G_{T1}$ through node N with path weighted by $w_1 N = w_1 w_2$. And the column from node N to node 2 to $N - 1$ is $w_1 w_2 \alpha_1 + \alpha_2$ because node N can reach the nodes in $G_{T1}$, directly, but reach the nodes in $G_{T1}$ through node 1 with path weighted by $w_1 N = w_1 w_2$. In $G_1'$, the column from node N + 1 to nodes 2 to $N - 1$ is $w_1 \alpha_1 + w_2 \alpha_2$ because node N + 1 can reach the nodes in $G_{T1}$, through node 1 with path $c_{1,N+1}$ weighted by $w_1$, and reach the nodes in $G_{T1}$ through node N with path $c_{N,N+1}$ weighted by $w_2$.

The covariance matrices of these trees are as shown below:

$$G_1 : \Sigma_{11} = \begin{bmatrix} \alpha_1 + w_1 w_2 \alpha_2 & \alpha_1 T + w_1 w_2 \alpha_2 T & \alpha_2 T & \alpha_2 + w_1 w_2 \alpha_2 \end{bmatrix}$$

$$G_2 : \Sigma_{12} = \begin{bmatrix} w_1 w_2 & w_1 w_2 \alpha_1 & w_1 \alpha_1 + w_2 \alpha_2 \end{bmatrix}$$

$$G_1' : \Sigma_{21} = \begin{bmatrix} \alpha_1 + w_1 w_2 \alpha_2 & \alpha_1 T + w_1 w_2 \alpha_2 T & \alpha_2 T & \alpha_2 + w_1 w_2 \alpha_2 \end{bmatrix}$$

$$G_2' : \Sigma_{22} = \begin{bmatrix} w_1 w_2 & w_1 w_2 \alpha_1 & w_1 \alpha_1 + w_2 \alpha_2 \end{bmatrix}$$

The generalized eigenvalues of $\Sigma_1$ and $\Sigma_2$ are the roots of $|\lambda I - \Sigma_1 \Sigma_2^{-1}|$. The non-zero diagonal elements and terms respecting to one grafting operation. We can divide the polynomial into $n$ parts corresponding to each grafting operation, namely, $tr(\Sigma_1 \Sigma_2^{-1})$ is a sum of 4n terms, each 4 terms respecting to one grafting operation. We can divide the polynomial into $n$ parts corresponding to each grafting operation, namely, $tr(\Sigma_1 \Sigma_2^{-1}) = \sum k P_k$. Then we only need to prove each $P_k$ equals to 0.

As Fig. 3 shows, there are three different types of grafting operations here. After simplification by using adding and merging as shown in [8], the three types of subparts can be translated as shown in Fig. 5. In this figure, we label these relevant nodes as 1, 2, 3, … for simplification because it doesn’t change the result if we exchange labels of nodes.

Before we deal with $P_k$ in each case, we will show $\Sigma_1 \Sigma_2$ is positive definite matrix. $\Sigma_1$ and $\Sigma_2$ are positive definite matrices, namely $\Sigma_1 \Sigma_2^{\top} z > 0$ and $\Sigma_2 \Sigma_1^{\top} z > 0$ for any non-zero vector $z$. Then $\Sigma_1 \Sigma_2^{\top} z = \lambda \Sigma_1 \Sigma_2^{\top} z + \Sigma_1^{\top} (1 - \lambda) \Sigma_2^{\top} z > 0$ for any non-zero vector $z$. So $\Sigma_1 \Sigma_2$ is positive definite matrix and its order principal minors are invertible matrices. This result is useful in our later proof.

We define $f_i = \frac{1}{\lambda - 1} g_i$ and $g_i = \frac{w_1}{\lambda - 1}$. Then we prove $P_k = 0$ in different type of Fig. 5 later.

$$tr(A B) = \sum i a_{ij} b_{ij}$$ when $A, B$ are both symmetric matrices. Luckily, $A = \Sigma_0^2$, $B = \Sigma_1 \Sigma_2^{\top}$ are both symmetric matrices. What’s more, $B = \Sigma_1 \Sigma_2^{\top}$ only have 2n non-zero diagonal elements and 2n pairs of non-zero non-diagonal elements, where n is the number of grafting operations. For example, $b_{PP} = \frac{w_1}{\lambda - 1}$, $b_{QQ} = \frac{w_2}{\lambda - 1}$, $b_{PP} = \frac{w_3}{\lambda - 1}$, $b_{QQ} = \frac{w_4}{\lambda - 1}$ if we cut node i from node p with edge $c_{i,p}$ weighting w and paste it to node q, and there aren’t any other grafting operations involving these three nodes. So $tr(\Sigma_0 (\Sigma_1 \Sigma_2^{\top}))$ is a sum of 4n terms, each 4 terms respecting to one grafting operation. We can divide the polynomial into $n$ parts corresponding to each grafting operation, namely, $tr(\Sigma_1 \Sigma_2^{\top}) = \sum k P_k$. Then we only need to prove each $P_k$ equals to 0.

APPENDIX E

PROOF OF PROPOSITION 5

$\lambda^*$ satisfies $D(\Sigma_1^{\lambda}, |\Sigma_2|) = D(\Sigma_1, |\Sigma_1|)$. When $|\Sigma_1| = |\Sigma_2|$, it will become the unique root in $[0, 1]$ of $tr(\Sigma_1 (\Sigma_1^{\lambda} - \Sigma_2)) = 0$. We only need to prove $tr(\Sigma_0 (\Sigma_1^{\lambda} - \Sigma_2)) = 0$ before we get $\lambda^* = 1/2$. Fig. 5. Simplified separate grafting operation types
A. TYPE1

In this type, we focus on node 1 to 4, which is involved in this grafting operation. So

$$\Sigma_1^{-1} =
\begin{bmatrix}
1 + g_1 & f_1 \\
\frac{f_1}{1 + g_1 + g_2} & 1 + g_2 + g_3 + g_4 & f_3 \\
\frac{f_3}{1 + g_1 + g_3} & & 1 + g_3 \\
& & & V \\
& & & & K^{(1)}
\end{bmatrix}
$$

$$\Sigma_2^{-1} =
\begin{bmatrix}
1 + g_1 & f_1 \\
\frac{1 + g_2}{f_2} & 1 + g_2 + g_3 + g_4 & f_3 \\
\frac{f_3}{1 + g_1 + g_3} & & 1 + g_3 \\
& & & V \\
& & & & K^{(2)}
\end{bmatrix}
$$

where $V$ is a $4 \times (N - 4)$ matrix with all zero elements except $v_{3,1} = f_4$ and $K^{(3)}, K^{(4)}$ are covariance matrices of node 5 to $N$ in $G_1$ and $G_2$ respectively. And then

$$\Sigma_1^{-1} - \Sigma_2^{-1} =
\begin{bmatrix}
0 & f_1 & -f_1 \\
-1 & g_1 & 0 \\
f_1 & 0 & -1 \\
\end{bmatrix}
$$

$$\Sigma_{0.5}^{-1} = \frac{1}{2} \Sigma_1^{-1} + \frac{1}{2} \Sigma_2^{-1} =
\begin{bmatrix}
1 + \frac{1}{2} s_1 & \frac{1}{2} s_1 \\
\frac{1 + s_2}{2} & 1 + s_2 + s_3 + s_4 & f_3 \\
\frac{f_3}{1 + \frac{1}{2} s_1} & & 1 + \frac{1}{2} s_1 + s_3 \\
& & & V \\
& & & & K^{(3)}
\end{bmatrix}
$$

where $K^{(3)} = \frac{1}{2} K^{(1)} + \frac{1}{2} K^{(2)}$ is an invertible matrix because $\Sigma_{0.5}^{-1}$ is positive definite matrix. And we define

$$VK^{(3)^{-1}}V^T =
\begin{bmatrix}
0 & \frac{X}{|K^{(3)^{-1}}|} \\
0 & 0
\end{bmatrix}
$$

So $P_k$ related to this grafting operation is $g_1 m_{22} - g_1 m_{44} + 2 f_1 m_{12} - 2 f_1 m_{14}$ where $m_{ij}$ is the $ij$-th element of $\Sigma_{0.5}$.

$M_{ij}$ is the $(i, j)$ minor of $\Sigma_{0.5}^{-1}$, so

$$M_{22} = |K^{(3)}|(1 + g_1)(1 + g_3)(1 + g_2 + g_4 + \frac{1}{4} g_1)
+ \frac{1}{4} g_1 (g_2 + g_4) - (1 + g_1)(1 + g_3 + \frac{1}{4} g_1)X$$

$$M_{44} = |K^{(3)}|(1 + g_1)(1 + g_2)(1 + g_3 + g_4 + \frac{1}{4} g_1)
+ \frac{1}{4} g_1 (g_3 + g_4) - (1 + g_1)(1 + g_2 + \frac{1}{4} g_1)X$$

$$M_{12} = |K^{(3)}| \left( \frac{f_1}{2} (1 + g_3)(1 + g_2 + g_4 + \frac{1}{4} g_1)
+ \frac{1}{2} g_1 (g_2 + g_4) + \frac{1}{2} f_1 f_2 f_3 \right) - \frac{1}{2} f_1 (1 + g_3 + \frac{1}{2} g_1)X$$

$$M_{14} = |K^{(3)}| \left( \frac{f_1}{2} (1 + g_2)(1 + g_3 + g_4 + \frac{1}{2} g_1)
+ \frac{1}{2} g_1 (g_3 + g_4) + \frac{1}{2} f_1 f_2 f_3 \right) - \frac{1}{2} f_1 (1 + g_2 + \frac{1}{2} g_1)X$$

and

$$2 f_1 (-M_{12} + M_{14}) + g_1 (M_{22} - M_{44}) = 0$$

$$P_k = g_1 m_{22} - g_1 m_{44} + 2 f_1 m_{12} - 2 f_1 m_{14}
= (2 f_1 (-M_{12} + M_{14}) + g_1 (M_{22} - M_{44})) |\Sigma_{0.5}| = 0$$

B. TYPE2

In this type, we focus on node 1 to 3, which is involved in this grafting operation. So

$$\Sigma_1^{-1} =
\begin{bmatrix}
1 + g_1 & f_1 \\
\frac{f_1}{1 + g_1 + g_2} & 1 + g_1 + g_2 & f_2 \\
\frac{f_2}{1 + g_1 + g_3} & & 1 + g_3 \\
& & & V \\
& & & & K^{(1)}
\end{bmatrix}
$$

$$\Sigma_2^{-1} =
\begin{bmatrix}
1 + g_1 + g_2 & f_1 & f_2 \\
\frac{f_1}{1 + g_1} & 1 + g_1 & f_2 \\
\frac{f_2}{1 + g_1 + g_3} & & 1 + g_3 \\
& & & V \\
& & & & K^{(2)}
\end{bmatrix}
$$

where $V$ is a $3 \times (N - 3)$ matrix with all zero elements except $v_{3,1} = f_3$ and $K^{(3)}, K^{(4)}$ are covariance matrices of node 4 to $N$ in $G_1$ and $G_2$ respectively. And then we use the same progress and get

$$P_k = g_1 m_{22} - g_1 m_{44} + 2 f_1 m_{12} - 2 f_1 m_{14} = 0$$

where $m_{ij}$ is the $ij$-th element of $\Sigma_{0.5}$.

C. TYPE3

In this type, we focus on node 1 to 5, which is involved in this grafting operation. So
where $V$ is a $5 \times (N - 5)$ matrix with all zero elements except $v_{1,1} = f_5$, $v_{1,2} = f_6$, and $K(1)$, $K(2)$ are covariance matrices of node 6 to $N$ in $G_1$ and $G_2$ respectively. And then we use the same progress and get

\[ P_k = g_1 m_{22} - g_1 m_{33} + 2 f_1 m_{12} - 2 f_1 m_{13} = 0 \] (53)

where $m_{ij}$ is the $ij$-th element of $\Sigma_{q,i}$.

\[ P_k = 0 \] in all types of grafting operations if these operations are separate. And $tr(\Sigma_{q,5}(\Sigma_1^{-1} - \Sigma_2^{-1})) = \sum_k P_k = 0$.

So $\lambda^* = \frac{1}{2}$ if all the grafting operations in the chain are separate.

**APPENDIX F**

**PROOF OF PROPOSITION 5**

We prove this proposition by using adding and merging operation of $\Sigma$ repeatedly.

For $p \leq i \leq j \leq q$, there are $q - p$ grafting operations in the grafting chain $T_p \leftrightarrow \cdots \leftrightarrow T_q$. We divide these grafting operations into two sets: grafting operations from $T_i$ to $T_j$ and other grafting operations, namely, set 1 and set 2.

We can simplify tree pairs ($T_p, T_q$) and ($T_i, T_j$) into ($\tilde{T}_p, \tilde{T}_q$) and ($\tilde{T}_i, \tilde{T}_j$) respectively by using adding and merging operation of $\Sigma$ repeatedly. ($\tilde{T}_p, \tilde{T}_q$) have all the anchor nodes of all $q - p$ operations and the paths of backbone among these operations. ($\tilde{T}_i, \tilde{T}_j$) have all the anchor nodes of all set 1 operations and the paths of backbone among these operations. If all the grafting operations in the chain are independent, ($\tilde{T}_p, \tilde{T}_q$) have the same substructure with ($\tilde{T}_i, \tilde{T}_j$) after dropping the extra nodes. So $CI(T_p || T_q) \geq (T_i || T_j)$ holds by Proposition 2 of $\Sigma$. Then $CI(T_i || T_j) \leq CI(T_p || T_q)$.

Here we take a $T_1 \leftrightarrow T_2 \leftrightarrow T_3$ case as an example, as shown in Fig. 5. We can simplify the calculation of $CI(T_1 || T_3)$ and $CI(T_1 || T_2)$ as shown in Fig. 5. And then Proposition 2 of $\Sigma$ can tell us $CI(T_1 || T_3) \geq CI(T_1 || T_2)$. In the same way, we can conclude $CI(T_1 || T_3) \geq CI(T_2 || T_2)$.

**APPENDIX G**

**PROOF OF PROPOSITION 7**

We want to prove $CI(\Sigma_1 || \Sigma_2) \geq CI(\Sigma'_1 || \Sigma_2)$ where $\Sigma_i = A^* \Sigma_i A^* T$, $\Sigma_i = D \Sigma_i D^T$ for arbitrary $N_O \times N$ matrix $D$ and $i = 1, 2$. Due to equation (19), we only need to prove $D(\Sigma_1 || \Sigma_2) \geq D(\Sigma_1 || \Sigma_2)$ and $D(\Sigma_2 || \Sigma_1) \geq D(\Sigma_2 || \Sigma_1)$

where $\Sigma_1 = \lambda \Sigma_1^{-1} + (1 - \lambda) \Sigma_2^{-1}$ and $\Sigma_1^{-1} = \lambda \Sigma_1^{-1} + (1 - \lambda) \Sigma_2^{-1}$.

In the following part, I will prove $D(\Sigma_1 || \Sigma_1) \geq D(\Sigma_1 || \Sigma_2)$ and the proof of $D(\Sigma_2 || \Sigma_2) \geq D(\Sigma_2 || \Sigma_2)$ is similar.

\[ D(\Sigma_1 || \Sigma_1) = \frac{1}{2} \sum_i g(\nu_i) \] (54)

where $g(\nu_i) = ln(\lambda + 1 - \nu_i) + \frac{1}{1 + \lambda - \nu_i} - 1$ and $\{\nu_i\}$ are generalized eigenvalues of $\Sigma_1$ and $\Sigma_2$.

$g(1) = 0$, $g(\nu_i)$ is decreasing in $(0, 1)$ and increasing in $(1, +\infty)$. If $\nu_i^{(1)} \leq \nu_i^{(2)} \leq \nu_i^{(2)} \leq \cdots \leq \nu_i^{(N-1)} \leq \nu_i^{(1)}$, we can choose $\mu_i^{(1)} = \nu_i^{(1)}$ for $\nu_i^{(2)} \geq 1$ and $\mu_i^{(1)} = \nu_i^{(1)}$ for $\nu_i^{(2)} < 1$ so that $\{\nu_i^{(1)}\}$ is an $(N - 1)$ subset of $\{\nu_i^{(2)}\}$ and $\frac{1}{2} \sum_{i=1}^{N-1} g(\mu_i^{(1)}) \geq \frac{1}{2} \sum_{i=1}^{N-1} g(\nu_i^{(2)})$.

If $m$ elements of $\{\nu_i^{(2)}\}$ are greater than one, $\{\mu_i^{(1)}\}$ contains the first $k$ elements and last $N - 1 - k$ elements of $\{\nu_i^{(1)}\}$.

We first introduce a preparation of proof.

**A. The relationship of eigenvalues**

We have four covariance matrices $\Sigma_1, \Sigma_2, \Sigma'_1 = \Sigma_1 a_0^T a_0 \Sigma_2$ and $\Sigma'_2 = \Sigma_2 b_0^T b_0$. The eigenvalues of $\Sigma_1 \Sigma_2^{-1}$ are $\lambda_1(2) \leq \lambda_2(2) \leq \cdots \leq \lambda_N(2)$. The eigenvalues of $\Sigma_2^{-1} \Sigma_1$ are $\lambda_1(1) \leq \lambda_2(1) \leq \cdots \leq \lambda_N(1)$. For $\Sigma_1$ and $\Sigma_2$, we can find an inverse matrix $P$ where $\Sigma_1 = P \Sigma_1 P^T = diag(\lambda_1(2), \lambda_2(2), \cdots, \lambda_N(2))$ and $\Sigma_2 = P \Sigma_2 P^T = diag(\lambda_1(1), \lambda_2(1), \cdots, \lambda_N(1))$ for $P \Sigma_1 P^T = I$. So $P' = \begin{bmatrix} P' & b_0 \\ 0 & b_0 \end{bmatrix}$ satisfies $\Sigma_1' = P' \Sigma_1 P'^T = \begin{bmatrix} a_0^T & a_0^T b_0 \\ a_0 & a_0 b_0 \end{bmatrix}$. Therefore, $\Sigma_1' = P' \Sigma_1 P'^T$ and $\Sigma_2' = P' \Sigma_2 P'^T = \begin{bmatrix} I & b \& b \\ b^T & b_0 \end{bmatrix}$, where $a = (a_1, a_2, \cdots, a_N)^T$ and $b = (b_1, b_2, \cdots, b_{N-1})^T$. 
So \( \{\lambda_i^{(1)}\} \) are the roots of \( |\lambda \Sigma_2^{(1)} - \Sigma_1^{(1)}| = 0 \), which is equal to \( F(\lambda) = 0 \) where

\[
F(\lambda) = |\lambda \Sigma_2^{(1)} - \Sigma_1^{(1)}| = \begin{vmatrix} \lambda & \Sigma_1^{(1)} \\ \Sigma_2^{(1)^T} & a^T \end{vmatrix}
\]

\[
= \prod_{i=1}^{N-1} (\lambda - \lambda_i^{(2)}) \times \left\{ \lambda b_0 - a_0 - (\lambda b^T - a^T)(\lambda I - \Sigma_1^{(1)})^{-1}(\lambda b - a) \right\}
\]

\[
= \prod_{i=1}^{N-1} (\lambda - \lambda_i^{(2)}) \times \left\{ f_0(\lambda) - \sum_{i=1}^{N-1} f_i(\lambda) \right\}
\]

\[
= |\Sigma_2(1)| \lambda^N + \sum_{i=0}^{N-1} c_i \lambda^i
\]

and \( f_i(\lambda) = b_i \lambda - a_i \).

At first, we assume that there is no multiple eigenvalues for \( \{\lambda_i^{(2)}\} \) and \( f_i(\lambda_i^{(2)}) \neq 0 \) for all \( 1 \leq i \leq N - 1 \).

So

\[
(-1)^{N-k} F(\lambda_i^{(2)}) = (-1)^{N-k+1} f_k(\lambda_i^{(2)}) \prod_{i=1, i \neq k}^{N-1} (\lambda_i^{(2)} - \lambda_i^{(2)})
\]

\[
> 0
\]

And \( F(+\infty) = +\infty, F(-\infty) = (-1)^N \infty \) There is at least one root of \( F(\lambda) = 0 \) in \((-\infty, \lambda_1^{(2)}), (\lambda_N^{(2)}, +\infty) \) and \( (\lambda_i^{(2)}, \lambda_{i+1}^{(2)}) \) for \( 1 \leq i \leq N - 2 \).

So we can conclude that \( \lambda_1^{(1)} < \lambda_2^{(1)} < \lambda_2^{(2)} < \lambda_2^{(2)} < \cdots < \lambda_{N-1}^{(1)} < \lambda_{N-1}^{(2)} < \lambda_N^{(1)} \).

If \( \lambda_i^{(2)} \) has multiplicity \( n_i \), we can conclude that \( \lambda_i^{(2)} \) has multiplicity \( n_i - 1 \) in \( F(\lambda) = 0 \). If we put this exact multiple eigenvalues aside, other eigenvalues satisfy the ordering before.

If \( f_i(\lambda_i^{(2)}) = 0 \), we can conclude that \( \lambda_i^{(2)} \) is also the root of \( \lambda^{(2)}(\lambda) = 0 \). If we put this eigenvalue aside, other eigenvalues satisfy the ordering before.

So for all cases, \( \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_{N-1}^{(1)} \leq \lambda_{N-1}^{(2)} \leq \lambda_N^{(1)} \).

**B. Main proof**

For \( N \)-dimension graphs whose covariance matrices are \( A \) and \( B \), we can do inverse linear transformation by \( A' = KAK^T \) and \( B' = KBK^T \). Then we can do dimension-reduction from \( N \) to \( N - 1 \) by \( A'' = [I_{N-1}, 0]A'[I_{N-1}, 0]^T \) and \( B'' = [I_{N-1}, 0]B'[I_{N-1}, 0]^T \).

The generalized eigenvalues of \( A \) and \( B \) are \( \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \cdots \leq \lambda_N^{(1)} \), which are also generalized eigenvalues of \( A' \) and \( B' \). And the generalized eigenvalues of \( A'' \) and \( B'' \) are \( \lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_N^{(2)} \). So \( \lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_{N-1}^{(1)} \leq \lambda_{N-1}^{(2)} \leq \lambda_N^{(1)} \).

We can also do the procedure to reduce the dimension from \( N - 1 \) to \( N - 2 \) and get \( \lambda_1^{(2)} \leq \lambda_3^{(2)} \leq \lambda_2^{(2)} \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_{N-2}^{(1)} \leq \lambda_{N-2}^{(3)} \leq \lambda_{N-2}^{(2)} \) and \( \lambda_{N-2}^{(2)} \).

The generalized eigenvalues after \( k \) dimension reduction are \( \{\lambda_i^{(k+1)}\} \) of size \( N = k \).

In No. \((N - N_0)\) dimension reduction stage, we can always find an \( N_0 \) subset \( \{\mu_i^{(N-N_0)}\} \) of \( \{\lambda_i^{(N-N_0)}\} \) so that \( \frac{1}{2} \sum_{i=1}^{N_0} g(\mu_i^{(N-N_0)}) \geq \frac{1}{2} \sum_{i=1}^{N_0} g(\lambda_i^{(N-N_0)+1}) \) because \( \lambda_1^{(N-N_0)} \leq \lambda_2^{(N-N_0)+1} \leq \lambda_2^{(N-N_0)} \leq \lambda_2^{(N-N_0)+1} \leq \cdots \leq \lambda_{N_0}^{(N-N_0)} \leq \lambda_{N_0}^{(N-N_0)+1} \leq \lambda_{N_0}^{(N-N_0)} \).

We can always find \( N_0 \) subset \( \{\mu_i^{(N-N_0)-1}\} \) of \( \{\lambda_i^{(N-N_0)-1}\} \) and \( \{\mu_i^{(N-N_0)}\} \) of \( \{\lambda_i^{(N-N_0)}\} \), which satisfy \( \frac{1}{2} \sum_{i=1}^{N_0} g(\mu_i^{(N-N_0)-1}) \geq \frac{1}{2} \sum_{i=1}^{N_0} g(\mu_i^{(N-N_0)}) \geq \frac{1}{2} \sum_{i=1}^{N_0} g(\lambda_i^{(N-N_0)+1}) \) with the same reason.

With the same method, we can conclude that there exist an \( N_0 \) subset \( \{\lambda_i^{(1)}\} \) so that \( \frac{1}{2} \sum_{i=1}^{N_0} g(\lambda_i^{(1)}) \geq \frac{1}{2} \sum_{i=1}^{N_0} g(\lambda_i^{(N-N_0)+1}) \). If \( k \) elements of \( \{\lambda_i^{(N-N_0)+1}\} \) are greater than one, \( \{\lambda_i^{(1)}\} \) contains the first \( k \) elements and last \( N_0 - k \) elements of \( \{\lambda_i^{(1)}\} \).

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If \( k \) elements of \( \{\lambda_i^{(N-N_0)+1}\} \) are greater than one, \( \{\lambda_i^{(1)}\} \) contains the first \( k \) elements and last \( N_0 - k \) elements of \( \{\lambda_i^{(1)}\} \).

So \( \sum_{i=1}^{N} g(\Sigma^{(1)}) \geq \sum_{i=1}^{N} g(\Sigma^{(2)}) \) holds for arbitrary \( D \) and its corresponding \( A_k \).

In the same way, \( \sum_{i=1}^{N} g(\Sigma^{(2)}) \geq \sum_{i=1}^{N} g(\Sigma^{(3)}) \) holds for arbitrary \( D \) and its corresponding \( A_k \).

So \( C(I(\Sigma^{(1)}||\Sigma^{(2)}) \geq C(I(\Sigma^{(2)}||\Sigma^{(3)}) \) for for arbitrary \( D \) and \( A' \) is the optimal matrix in the set \( \{A_k|N_0 + m = N \leq k \leq m, k \geq 0\} \).