Photonic crystal slab laplace operator for image differentiation: supplementary material

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This document provides supplementary information to “Photonic crystal slab laplace operator for image differentiation,” https://doi.org/10.1364/OPTICA.5.000251. This supplementary document consists of six sections. In Section 1, we derive the effective Hamiltonian near the Γ point, when the states are two-fold degenerate at the Γ point. In Section 2 we provide numerical validation of the effective Hamiltonian. In Section 3, we calculate the transmission for an arbitrarily polarized state from the transmission response for S and P polarized light. In Section 4, we plot the S matrix of the device. In Section 5, we plot the transmittances |t_o|, |t_p|, and |t_u| in a larger wavevector region for the device configuration that operates as a differentiator. Such information on transmittance is used in demonstrating the performance of the differentiator. In Section 6, we demonstrate the performance of the device for linearly polarized light.

1. DERIVATION OF THE EFFECTIVE HAMILTONIAN

In this section, we derive the $2 \times 2$ effective Hamiltonian (Equation (11) in the primary text) near the Γ point. We assume that the system has $C_{4v}$ symmetry, and at the Γ point the system supports a pair of doubly degenerate states, denoted as $|x\rangle$ and $|y\rangle$, respectively. With these states as bases, the $2 \times 2$ Hamiltonian in the vicinity of Γ has the following general form:

$$\hat{H}(k) = \hat{A}(k) - i\hat{B}(k)$$ (S1)

where $\hat{A}(k)$ and $\hat{B}(k)$ are both Hermitian and

$$\hat{A}(k) = f(k)\hat{\sigma}_+ + f^*(k)\hat{\sigma}_- + g(k)\hat{\sigma}_z + h(k)\hat{I} + \omega_0\hat{I}$$
$$\hat{B}(k) = r(k)\hat{\sigma}_+ + r^*(k)\hat{\sigma}_- + s(k)\hat{\sigma}_z + t(k)\hat{I} + \gamma_0\hat{I}$$ (S2)

We are interested in the lowest-order non-vanishing terms in the functions $f$, $g$, $h$, $r$, $s$ and $t$. By virtue of the two-fold degeneracy, we have

$$f(0) = g(0) = h(0) = r(0) = s(0) = t(0) = 0$$ (S3)

Also, $g$, $h$, $s$ and $t$ are real since $\hat{A}(k)$ and $\hat{B}(k)$ are Hermitian.

In general, the Hamiltonian $\hat{H}(k)$ is constrained by the symmetric group $G$ of the system:

$$\forall g \in G, \quad \hat{D}(g)\hat{H}(k)\hat{D}^{-1}(g) = \hat{H}(D(g)k)$$ (S4)

where $\hat{D}(g)$ and $D(g)$ are the representations of $g$ in the Hilbert space and the $k$ space, respectively.

For our system, $G = C_{4v} = \{E, 2C_4, C_2, 2C_2, 2C_2\}$. As we choose $|x\rangle$ and $|y\rangle$ as bases for the Hilbert space and $k_x$ and $k_y$ for the $k$ space, the representations $\hat{D}(g)$ and $D(g)$ have the same matrix forms. Now, we consider the constraint on $\hat{H}(k)$ from each symmetry element in $G$.

1. Inversion symmetry $C_2$.

$$\hat{D}(C_2) = D(C_2) \equiv \Pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ (S5)
From Equation (S4) we have:

\[ f(k) = f(-k), \quad g(k) = g(-k), \quad h(k) = h(-k) \]
\[ r(k) = r(-k), \quad s(k) = s(-k), \quad t(k) = t(-k) \]  
(S6)

Thus, the lowest order terms in \( f, g, h, r, s \) and \( t \) must all be quadratic.

2. Four-fold rotational symmetry \( C_4 \).

\[ \hat{D}(C_4) = D(C_4) \equiv R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  
(S7)

From Equation (S4) we have:

\[ f^*(k) = -f(Rk), \quad g(k) = -g(Rk), \quad h(k) = h(Rk) \]
\[ r^*(k) = -r(Rk), \quad s(k) = -s(Rk), \quad t(k) = t(Rk) \]  
(S8)

3. Mirror symmetry \( \sigma_y \).

\[ \hat{D}(\sigma_y) = D(\sigma_y) \equiv M_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  
(S9)

Here we consider the mirror plane perpendicular to the \( y \)-axis. From Equation (S4) we have:

\[ f(k) = -f(M_y k), \quad g(k) = g(M_y k), \quad h(k) = h(M_y k) \]
\[ r(k) = -r(M_y k), \quad s(k) = s(M_y k), \quad t(k) = t(M_y k) \]  
(S10)

The other mirror symmetry \( \sigma_x \) doesn’t provide new constraints as it can be realized with a combination of \( \sigma_y \) and \( C_{4v} \).

Combining all the symmetry requirements, we determine the forms of the lowest order terms in all the functions. As an example, we consider the function \( f(k) \), which can be expanded as

\[ f(k) = C_x k_x^2 + C_y k_y^2 + C_{xy} k_x k_y \]  
(S11)

where all the \( C \) coefficients are in general complex. Since \( f(k) = -f^*(Rk) = -f(M_y k) \), we have \( C_x = C_y = 0, C_{xy} = C_{xy}^* \equiv C \). Thus \( f(k) = C k_y k_y \), where \( C \) is real. Repeating the procedures for all the functions, we summarize the results below:

\[
\begin{align*}
   f(k) &= C k_y k_y, \\
   g(k) &= B(k_x^2 - k_y^2), \\
   h(k) &= A(k_x^2 + k_y^2), \\
   r(k) &= C k_x k_y, \\
   s(k) &= B'(k_x^2 - k_y^2), \\
   t(k) &= A'(k_x^2 + k_y^2)
\end{align*}
\]  
(S12)

where all the coefficients are real.

Therefore, we obtain the Hamiltonian in Equation (11) of the main text:

\[ \hat{H}(k) = (\omega_0 - i\gamma_0 + a|k|^2)\mathbf{1} + b(k_x^2 - k_y^2)\hat{\sigma}_z + c k_x k_y \hat{\sigma}_x \]  
(S13)

where \( a = A - iA', \ b = B - iB', \ c = C - iC' \) are the complex coefficients.

2. VALIDATION OF THE EFFECTIVE HAMILTONIAN

In this section we provide quantitative validation of the effective Hamiltonian. Figure (S1) plots the band structure of the photonic crystal slab near frequency \( \omega_0 = 0.38749 \times 2\pi c / a \). Figure (S1) (a-d) exhibit contour plots of \((\omega - \omega_0)\) and \((\gamma - \gamma_0)\) for the two bands numerically calculated using guided-mode expansion method, while (e-h) show the corresponding analytical results from the effective Hamiltonian (Equation (S13)). The complex coefficients \( a = 0.33 - 0.16 i, b = 0.24 - 0.16 i, c = 8.78 - 0.08 i \) are fitted from the band dispersions along the \( \Gamma-X \) and \( \Gamma-M \) directions. Figure (S2) plots the numerical and analytical results of the band structure of the same photonic crystal slab near frequency \( \omega_0 = 0.47656 \times 2\pi c / a \). The fitted complex coefficients are \( a = 0.68 - 0.14 i, b = 1.11 - 0.12 i, c = 2.24 + 0.15 i \). In both cases, the numerical results agree with the analytical expressions excellently, which provide the validations of the effective Hamiltonian.
Fig. S1. Band structure of the photonic crystal slab near frequency $\omega_0 = 0.38749 \times 2\pi c / a$. The structure has the dielectric constant $\epsilon = 12$, the thickness $d = 0.55a$, and the radius $r = 0.111a$ of the holes. (a-d) Contour plots of the band structure numerically calculated using guided-mode expansion method. (a) $(\omega - \omega_0)$ vs. $(k_x,k_y)$ for the lower band. (b) $(\omega - \omega_0)$ vs. $(k_x,k_y)$ for the upper band. (c) $(\gamma - \gamma_0)$ vs. $(k_x,k_y)$ for the lower band. (d) $(\gamma - \gamma_0)$ vs. $(k_x,k_y)$ for the upper band. $\gamma_0 = 2.0 \times 10^{-4} \times 2\pi c / a$. The $(\omega - \omega_0)$ and $(\gamma - \gamma_0)$ contours for both bands are anisotropic. (e-h) Corresponding contour plots of the band structure analytically calculated from the effective Hamiltonian (Equation (S13)). The complex coefficients $a = 0.33 - 0.16i$, $b = 0.24 - 0.16i$, $c = 8.78 - 0.08i$ are fitted from the band dispersions along the $\Gamma$-X and $\Gamma$-M directions. The numerical results agree with the analytical expressions excellently. In all the plots $(\omega - \omega_0)$ and $(\gamma - \gamma_0)$ are in the units of $10^{-4} \times 2\pi c / a$.

Fig. S2. Band structure of the photonic crystal slab near frequency $\omega_0 = 0.47656 \times 2\pi c / a$. The structure has the dielectric constant $\epsilon = 12$, the thickness $d = 0.55a$, and the radius $r = 0.111a$ of the holes. (a-d) Contour plots of the band structure numerically calculated using guided-mode expansion method. (a) $(\omega - \omega_0)$ vs. $(k_x,k_y)$ for the lower band. (b) $(\omega - \omega_0)$ vs. $(k_x,k_y)$ for the upper band. (c) $(\gamma - \gamma_0)$ vs. $(k_x,k_y)$ for the lower band. (d) $(\gamma - \gamma_0)$ vs. $(k_x,k_y)$ for the upper band. $\gamma_0 = 1.3 \times 10^{-4} \times 2\pi c / a$. The $(\omega - \omega_0)$ contours for both bands are almost completely circular, while the $(\gamma - \gamma_0)$ contours are anisotropic. Nonetheless, $(\gamma - \gamma_0)$ are much smaller than $(\omega - \omega_0)$ and thus don’t affect the isotropy of the band structure much. (e-h) Corresponding contour plots of the band structure analytically calculated from the effective Hamiltonian (Equation (S13)). The complex coefficients $a = 0.68 - 0.14i$, $b = 1.11 - 0.12i$, $c = 2.24 + 0.15i$ are fitted from the band dispersions along the $\Gamma$-X and $\Gamma$-M directions. The numerical results agree with the analytical expressions excellently. In all the plots $(\omega - \omega_0)$ and $(\gamma - \gamma_0)$ are in the units of $10^{-4} \times 2\pi c / a$. 
3. TRANSMITTANCE OF ARBITRARILY POLARIZED STATES

In this section we compute the transmission of an arbitrarily polarized state, given the transmission response of the system for $S$ and $P$ polarized light.

The transmission response of the system is described by the $S$ matrix:

$$ S = \begin{pmatrix} t_{ss} & t_{sp} \\ t_{ps} & t_{pp} \end{pmatrix} $$  \hspace{1cm} (S14)

where $t_{ss}, t_{sp}, t_{ps}$ and $t_{pp}$ are complex. Here we adopt $|s\rangle$ and $|p\rangle$ as the basis states. For a general input state $|e\rangle$, the output state is $S |e\rangle$.

Consider an input beam with its polarization described by a polarization density matrix $\rho$, normalized such that $I = \text{tr} \rho$ is the incident light intensity. The transmitted state is then described by $\rho' = S \rho S^\dagger$, with the transmission defined as:

$$ T = \frac{\text{tr} \rho' \rho}{\text{tr} \rho} $$  \hspace{1cm} (S15)

Now let’s consider some specific examples of incident states:

1. **$S$ polarized light**

$$ \rho = \hat{I} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \hspace{1cm} \rho' = \hat{I} \begin{pmatrix} |s|^2 t_{ss}^* & t_{ss} t_{sp}^* \\ t_{ps} t_{ss}^* & |ps|^2 \end{pmatrix} $$

$$ T \equiv |t_s|^2 = |t_{ss}|^2 + |t_{ps}|^2 $$  \hspace{1cm} (S16)

Similarly, for $P$ polarized light,

$$ |t_p|^2 = |t_{sp}|^2 + |t_{pp}|^2 $$  \hspace{1cm} (S17)

2. **Unpolarized light**

$$ \rho = \hat{I} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \hspace{1cm} \rho' = \hat{I} \begin{pmatrix} |s|^2 + |p|^2 & t_{ss} t_{sp}^* + t_{sp} t_{pp}^* \\ t_{sp} t_{ss}^* + t_{pp} t_{ps}^* & |ps|^2 + |pp|^2 \end{pmatrix} $$

$$ T \equiv |t_u|^2 = \frac{1}{2} (|t_{ss}|^2 + |t_{ps}|^2 + |t_{sp}|^2 + |t_{pp}|^2) $$  \hspace{1cm} (S18)

From Equation (S16-S18) we have:

$$ |t_u| = \sqrt{\frac{|t_s|^2 + |t_p|^2}{2}} $$  \hspace{1cm} (S19)

which is Equation (19) in the main text.

3. **Left circularly polarized light**

$$ \rho = \hat{I} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \hspace{1cm} \rho' = \hat{I} \begin{pmatrix} |s|^2 + |p|^2 + i(t_{sp} t_{ss}^* - t_{ss} t_{sp}^*) & t_{ss} t_{ps}^* + t_{sp} t_{pp}^* + i(t_{sp} t_{ps}^* - t_{ss} t_{pp}^*) \\ t_{ps} t_{ss}^* + t_{pp} t_{sp}^* - i(t_{sp} t_{ps}^* - t_{ss} t_{pp}^*) & |ps|^2 + |pp|^2 - i(t_{sp} t_{ps}^* - t_{ss} t_{pp}^*) \end{pmatrix} $$

$$ T \equiv |t_l|^2 = \frac{1}{2} (|t_{ss}|^2 + |t_{sp}|^2 + |t_{ps}|^2 + |t_{pp}|^2 + i(t_{sp} t_{ss}^* - t_{ss} t_{sp}^* + t_{ps} t_{pp}^* - t_{pp} t_{ps}^*)) $$  \hspace{1cm} (S20)

which differs from Equation (S18) only in the interference term.

Note that for our system with an isotropic band structure, $S$ is diagonal due to the single-band excitation effect:

$$ S = \begin{pmatrix} t_{ss} & 0 \\ 0 & t_{pp} \end{pmatrix} $$  \hspace{1cm} (S21)

Then the interference term in Equation (S20) disappears and we have $|t_l| = |t_u|$. So the differentiation performance will be the same for unpolarized or circularly polarized light.
Fig. S3. S matrix of the system. (a) $|t_{ss}|(k_x, k_y)$, (b) $|t_{sp}|(k_x, k_y)$, (c) $|t_{ps}|(k_x, k_y)$, (d) $|t_{pp}|(k_x, k_y)$. Due to the effect of single-band excitation, $t_{sp} = t_{ps} = 0$, thus the $S$ and $P$ polarizations are decoupled. On the other hand, $|t_{ss}|$ and $|t_{pp}| \propto (k_x^2 + k_y^2)$ near the $\Gamma$ point, which is required by Laplacian.

Fig. S4. Transmittance $|t|$ as a function of $k_x$ and $k_y$ at frequency $\omega_0 = 0.47656 \times 2\pi c/a$ for (a) $S$, (b) $P$ and (c) unpolarized light, which are all isotropic near $\Gamma$, which is the major relevant wavevector region for image processing, but anisotropic at larger $k$.

4. S MATRIX

In this section we demonstrate the $S$ matrix of our device. As we show in Equation (2) of the main text, the $S$ matrix of the system is

$$S(k_x, k_y) = \begin{pmatrix} a_s(k_x^2 + k_y^2) & 0 \\ 0 & a_p(k_x^2 + k_y^2) \end{pmatrix}$$  \hspace{1cm} (S22)

where $a_s \neq a_p$.

As a numerical demonstration, Figure (S3) plots the $S$ matrix of the system computed using the Rigorous Coupled Wave Analysis. Figure (S3) (a-d) show $|t_{ss}|$, $|t_{sp}|$, $|t_{ps}|$ and $|t_{pp}|$, respectively, as functions of $(k_x, k_y)$. It can be seen that $|t_{sp}| = |t_{ps}| = 0$, while $|t_{ss}|, |t_{pp}| \propto |k|^2$ near the $\Gamma$ point. All the numerical results agree with the theoretical analysis above (Equation (S22)).

5. TRANSMITTANCE AT LARGER WAVEVECTORS

Figure (S4) plots the transmittance $|t|$ as a function of $k_x$ and $k_y$ for $S$, $P$ and unpolarized light in a larger wavevector region as compared to Figure 4 of the primary text. All $|t|$ are isotropic near the $\Gamma$ point. They become anisotropic at larger wavevector $k$. The background transmittances are all 1. The occurrences of transmittance dips at larger $k$ are due to other guided resonance modes.

We calculate the performance of our differentiator based on the transmittance $|t_u|$ shown in Figure (S4)(c), which covers all the relevant wavevector regions for the incident images that we assumed.
Fig. S5. (a) Transmittance $|t|$ as a function of $k_x$ and $k_y$ at frequency $\omega_0 = 0.47656 \times 2\pi c / a$ for y-polarized light, which is anisotropic near $\Gamma$. (b) Incident Stanford emblem with a size of $2610a \times 1729a$. (c) Calculated transmitted image with $y$-polarized light, which clearly shows the edges with different orientations.

6. TRANSMITTANCE OF LINEARLY POLARIZED LIGHT

In this section we demonstrate the performance of our device for linearly polarized light. For concreteness, we consider incident light that is linearly polarized along $y$ direction.

In the case where the incident light is linearly polarized, a device described by Equation (S22) does not perform an ideal Laplacian operation since the decomposition of linearly polarized light into the $S$ and $P$ polarization is wavevector-dependent. As shown in Figure (S5)(a), the transmittance $|t|$ is anisotropic even for wavevector $k$ near the $\Gamma$ point. Nevertheless, in practice such a device can still perform detection for edges with arbitrary orientations with linearly polarized incident light. To demonstrate this, Figure (S5)(c) shows the calculated transmitted image of the incident image (Figure (S5)(b)) with linearly polarized light, which clearly shows all the edges, even though the intensity and resolution vary for edges with different orientations.