Multiplication of distributions in any dimension: applications to \( \delta \)-function and its derivatives

F. Bagarello

Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, I - 90128 Palermo, Italy
E-mail: bagarell@unipa.it
web page: www.unipa.it/~bagarell

Abstract

In two previous papers the author introduced a multiplication of distributions in one dimension and he proved that two one-dimensional Dirac delta functions and their derivatives can be multiplied, at least under certain conditions. Here, mainly motivated by some engineering applications in the analysis of the structures, we propose a different definition of multiplication of distributions which can be easily extended to any spatial dimension. In particular we prove that with this new definition delta functions and their derivatives can still be multiplied.
1 Introduction

In the literature several examples of multiplication of distributions exist, more or less interesting and more or less useful for concrete applications. In [2] and [3] we have proposed our own definition of multiplication in one spatial dimension, $d = 1$, and we have proved that two or more delta functions can be multiplied and produce, as a result, a linear functional over $\mathcal{D}(\mathbb{R})$. However, our definition does not admit a natural extension to $d > 1$. This is a strong limitation, both mathematically and for concrete applications: for instance, it is known to the civil engineers community that the problem of fracture mechanics may be analyzed considering beams showing discontinuities along the beam span, [4, 5]. The classical approach for finding solutions of beams with discontinuities consists in looking for continuous solutions between discontinuities and imposing continuity conditions at the fractures. In [4, 5] a different strategy has been successfully proposed, modeling the flexural stiffness and the slope discontinuities as delta functions. However, in this approach, the problem of multiplying two delta functions naturally arises, and this was solved using the definition given in [2]. This application proved to be useful not only to get a solution in a closed form but also to get numerical results in a reasonable simple way, when compared with the older existing approaches. These very promising results, however, have been discussed only in $d = 1$ since the mathematical framework, which is behind the concrete application, only existed in one dimension, [2, 3]. It is not surprising, therefore, that extensions of our procedure to higher spatial dimensions has been strongly urged in order to consider the same problem for more general physical systems, i.e. for physical systems in higher dimensions like two or three-dimensional beams, in which the fractures can be schematized as two or three-dimensional delta functions.

In a completely different field of research the same necessity appeared: for instance, in the analysis of stochastic processes the need for what is called a renormalization procedure for the powers of white noise, where the white noise is essentially a delta function, has produced many results, see [1]. Also, applications to electric circuits exist which are surely less mathematically oriented, [6], and again are based on the possibility of giving a meaning to $\delta^2(x)$. With these motivations we have considered the problem of defining the multiplication of two distributions in more than one spatial dimension. However, our original definition cannot be easily extended to $d > 1$, because the regularization proposed in [7] and used in [2, 3] does not work without major changes in this case. For this reason we propose here a different definition of multiplication, which works perfectly in any spatial dimension. Moreover, this new approach returns results which are very close to those in [2, 3], for $d = 1$.

The paper is organized as follows:

in the next Section we briefly recall the main definitions and results of [2] and [3];

in Section 3 we propose a different definition of multiplication in one dimension and we show that results which are not significantly different from those of Section 2 are recovered;

in Section 4 we extend the definition to an arbitrary spatial dimension and prove that with
this new definition two delta functions can be multiplied.

2 A brief resume of our previous results

In [2, 3] we have introduced a (family of) multiplications of distributions making use of two different regularization discussed in the literature and adapted to our purposes. Here, for completeness’ sake, we briefly recall our strategy without going into too many details.

The first ingredient was first introduced in [7], where it was proven that, given a distribution $T$ with compact support, the function
\begin{equation}
\mathbf{T}^0(z) \equiv \frac{1}{2\pi i} T \cdot (x - z)^{-1}
\end{equation}
exists and is holomorphic in $z$ in the whole $z$-plane minus the support of $T$. Moreover, if $T(x)$ is a continuous function with compact support, then $T_{\text{red}}(x, \epsilon) \equiv \mathbf{T}^0(x + i\epsilon) - \mathbf{T}^0(x - i\epsilon)$ converges uniformly to $T(x)$ on the whole real axis for $\epsilon \to 0^+$. Finally, if $T$ is a distribution in $\mathcal{D}'(\mathbb{R})$ with compact support then $T_{\text{red}}(x, \epsilon)$ converges to $T$ in the following sense
\begin{equation}
T(\Psi) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} T_{\text{red}}(x, \epsilon) \Psi(x) \, dx
\end{equation}
for every test function $\Psi \in \mathcal{D}(\mathbb{R})$.

As discussed in [7], this definition can be extended to a larger class of one-dimensional distributions with support not necessarily compact, $\mathcal{V}'(\mathbb{R})$, while it is much harder to extend the same definition to more than one spatial dimension.

The second ingredient is the so-called method of sequential completion, which is discussed for instance in [8], and it follows essentially from very well known results on the regularity of the convolution of distributions and test functions. Let $\Phi \in \mathcal{D}(\mathbb{R})$ be a given function with \text{supp} $\Phi \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \Phi(x) \, dx = 1$. We call $\delta$--sequence the sequence $\delta_n$, $n \in \mathbb{N}$, defined by $\delta_n(x) \equiv n \Phi(nx)$. Then, $\forall T \in \mathcal{D}'(\mathbb{R})$, the convolution $T_n \equiv T \ast \delta_n$ is a $C^\infty$--function for any fixed $n \in \mathbb{N}$. The sequence $\{T_n\}$ converges to $T$ in the topology of $\mathcal{D}'$, when $n \to \infty$. Moreover, if $T(x)$ is a continuous function with compact support, then $T_n(x)$ converges uniformly to $T(x)$.

We are now ready to recall our original definition: for any couple of distributions $T, S \in \mathcal{D}'(\mathbb{R})$, $\forall \alpha, \beta > 0$ and $\forall \Psi \in \mathcal{D}(\mathbb{R})$ we start defining the following quantity:
\begin{equation}
(S \otimes T)^{(\alpha, \beta)}_n(\Psi) \equiv \frac{1}{2} \int_{-\infty}^{\infty} [S_n^{(\beta)}(x) T_{\text{red}}(x, \frac{1}{n^\alpha}) + T_n^{(\beta)}(x) S_{\text{red}}(x, \frac{1}{n^\alpha})] \Psi(x) \, dx
\end{equation}
where
\begin{equation}
S_n^{(\beta)}(x) \equiv (S \ast \delta_n^{(\beta)})(x),
\end{equation}
with $\delta_n^{(\beta)}(x) \equiv n^\beta \Phi(n^\beta x)$, which is surely well defined for any choice of $\alpha, \beta, T, S$ and $\Psi$. Moreover, if the limit of the sequence $\{(S \otimes T)^{(\alpha, \beta)}_n(\Psi)\}$ exists for all $\Psi(x) \in \mathcal{D}(\mathbb{R})$, we define $(S \otimes T)^{(\alpha, \beta)}(\Psi)$ as:
\begin{equation}
(S \otimes T)^{(\alpha, \beta)}(\Psi) \equiv \lim_{n \to \infty} (S \otimes T)^{(\alpha, \beta)}_n(\Psi)
\end{equation}
Of course, as already remarked in [2], the definition (2.4) really defines many multiplications of distributions. In order to obtain one definite product we have to fix the positive values of $\alpha$ and $\beta$ and the particular function $\Phi$ which is used to construct the $\delta$-sequence. Moreover, if $T(x)$ and $S(x)$ are two continuous functions with compact supports, and if $\alpha$ and $\beta$ are any pair of positive real numbers, then: (i) $T^2_S (x) S_{red} (x, \frac{1}{m})$ converges uniformly to $S(x) T(x)$; (ii) $\forall \Psi(x) \in D(\mathbb{R}) \Rightarrow (T \otimes S)(\alpha, \beta)(\Psi) = \int_{-\infty}^{\infty} T(x) S(x) \Psi(x) \, dx$.

It is furthermore very easy to see that the product $(S \otimes T)(\alpha, \beta)$ is a linear functional on $D(\mathbb{R})$. The continuity of such a functional is, on the contrary, not obvious at all, but, as formulas (2.6)-(2.11) show, is a free benefit of our procedure.

We now recall few results obtained in [2, 3].

If we assume $\Phi$ to be of the form

$$\Phi(x) = \begin{cases} \frac{m}{N_m} \cdot \exp\{ \frac{1}{x^2-1} \}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

where $m$ is an even natural number and $N_m$ is a normalization constant which gives $\int_{-1}^{1} \Phi(x) \, dx = 1$, and we put $A_j \equiv \int_{-\infty}^{\infty} \Phi(x) x^j \, dx$, whenever it exists, we have proved that:

if $m > 1$ then

$$(\delta \otimes \delta)(\alpha, \beta) = \begin{cases} \frac{1}{\pi} A_2 \delta, & \alpha = 2\beta \\ 0, & \alpha > 2\beta; \end{cases}$$

(2.6)

if $m > 2$ then

$$(\delta \otimes \delta')(\alpha, \beta) = 0 \quad \forall \alpha \geq 3\beta;$$

(2.7)

if $m > 3$ then

$$(\delta' \otimes \delta')(\alpha, \beta) = \begin{cases} \frac{-6}{\pi} A_4 \delta, & \alpha = 4\beta \\ 0, & \alpha > 4\beta. \end{cases}$$

(2.8)

Also, if $m > 3$ then

$$(\delta \otimes \delta'')(\alpha, \beta) = \begin{cases} \frac{6}{\pi} A_4 \delta, & \alpha = 4\beta \\ 0, & \alpha > 4\beta. \end{cases}$$

(2.9)

If $m > 4$ then

$$(\delta' \otimes \delta'')(\alpha, \beta) = 0 \quad \forall \alpha \geq 5\beta. \quad (2.10)$$

Finally, if $m > 5$ then

$$(\delta'' \otimes \delta'')(\alpha, \beta) = \begin{cases} \frac{120}{\pi} A_6 \delta, & \alpha = 6\beta \\ 0, & \alpha > 6\beta. \end{cases}$$

(2.11)

It is worth stressing that formula (2.6), for $\alpha > 2\beta$, coincides with the result given by the neutrix product discussed by Zhi and Fisher, see [9]. Also, because of our technique which strongly relies on the Lebesgue dominated convergence theorem, LDCT, the above formulas only give sufficient conditions for the product between different distributions to exist. In other words:
formula \((2.6)\) does not necessarily implies that \((\delta \otimes \delta)_{(\alpha,\beta)}\) does not exist for \(\alpha < 2\beta\). In this case, however, different techniques should be used to check the existence or the non-existence of \((\delta \otimes \delta)_{(\alpha,\beta)}\).

More remarks on this approach can be found in [2] where, in particular the above results are extended to the product between two distributions like \(\delta^{(l)}\) and \(\delta^{(k)}\) for generic \(l, k = 0, 1, 2, \ldots\). In [3] we have further discussed the extension of the definition of our multiplication to more distributions and to the case of non commuting distributions, which is relevant in quantum field theory.

3 A different definition in \(d = 1\)

The idea behind the definition we introduce in this section is very simple: since the regularization \(T \rightarrow T_{\text{red}}\) cannot be easily generalized to higher spatial dimensions, \(d > 1\), we use twice the convolution procedure in \((2.3)\), \(T \rightarrow T_n^{(\alpha)} = T \ast \delta^{(\alpha)}_n\), with different values of \(\alpha\) as we will see.

Let \(\Phi(x) \in \mathcal{D}(\mathbb{R})\) be a given non negative function, with support in \([-1, 1]\) and such that \(\int_{\mathbb{R}} \Phi(x) \, dx = 1\). In the rest of this section, as in [2] [3], we will essentially consider the following particular choice of \(\Phi(x)\),

\[
\Phi(x) = \begin{cases} 
\frac{n^m}{N_m} \cdot \exp\left\{ \frac{-1}{2x^2} \right\}, & |x| < 1 \\
0, & |x| \geq 1.
\end{cases}
\]  

where \(m\) is some fixed even natural number and \(N_m\) is a normalization constant fixed by the condition \(\int_{-1}^1 \Phi(x) \, dx = 1\). As we have discussed in [2] the sequence \(\delta_n^{(\alpha)}(x) = n^\alpha \Phi(n^\alpha x)\) is a delta-sequence for any choice of \(\alpha > 0\). This means that: (1) \(\delta_n^{(\alpha)}(x) \to \delta(x)\) in \(\mathcal{D}'(\mathbb{R})\) for any \(\alpha > 0\); (2) for any \(n \in \mathbb{N}\) and for all \(\alpha > 0\) if \(T(x)\) is a continuous function with compact support then the convolution \(T_n^{(\alpha)}(x) = (T \ast \delta_n^{(\alpha)})(x)\) converges to \(T(x)\) uniformly for all \(\alpha > 0\); (3) if \(T(x) \in \mathcal{D}(\mathbb{R})\) then \(T_n^{(\alpha)}(x)\) converges to \(T(x)\) in the topology \(\tau_\mathcal{D}\) of \(\mathcal{D}(\mathbb{R})\); (4) if \(T \in \mathcal{D}'(\mathbb{R})\) then \(T_n^{(\alpha)}(x)\) is a \(C^\infty\) function and it converges to \(T\) in \(\mathcal{D}'(\mathbb{R})\) as \(n\) diverges independently of \(\alpha > 0\).

We remark here that all these results can be naturally extended to larger spatial dimensions, and this will be useful in the next section.

Our next step is to replace definition \((2.4)\) with our alternative multiplication. To begin with, let us consider two distributions \(T, S \in \mathcal{D}'(\mathbb{R})\), and let us compute their convolutions \(T_n^{(\alpha)}(x) = (T \ast \delta_n^{(\alpha)}) (x)\) and \(S_n^{(\beta)}(x) = (T \ast \delta_n^{(\beta)}) (x)\) with \(\delta_n^{(\alpha)}(x) = n^\alpha \Phi(n^\alpha x)\), for \(\alpha, \beta > 0\) to be fixed in the following. \(T_n^{(\alpha)}(x)\) and \(S_n^{(\beta)}(x)\) are both \(C^\infty\) functions, so that for any \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) and for each fixed \(n \in \mathbb{N}\), the following integrals surely exist:

\[
(S \ast T)^{\alpha,\beta}_n(\Psi) \equiv \frac{1}{2} \int_{\mathbb{R}} \left[ S_n^{(\alpha)}(x) T_n^{(\beta)}(x) + S_n^{(\beta)}(x) T_n^{(\alpha)}(x) \right] \Psi(x) \, dx,
\]  

\[
(S \ast_d T)^{\alpha,\beta}_n(\Psi) \equiv \int_{\mathbb{R}} S_n^{(\alpha)}(x) T_n^{(\beta)}(x) \Psi(x) \, dx,
\]  

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\[(S \circ_{\text{ex}} T)^{(\alpha,\beta)}_n(\Psi) \equiv \int_{\mathbb{R}} S^{(\beta)}_n(x) T^{(\alpha)}_n(x) \Psi(x) \, dx. \tag{3.4}\]

The suffix \(d\) stands for \textit{direct}, while \(\text{ex}\) stands for \textit{exchange}. This is because the two related integrals remind us the direct and the exchange contributions in an energy Hartree-Fock computation, typical of quantum many-body systems. It is clear that if \(S \equiv T\) then the three integrals above coincide: \(\left( S \circ S \right)^{(\alpha,\beta)}_n(\Psi) = (S \circ_d S)^{(\alpha,\beta)}_n(\Psi) = (S \circ_{\text{ex}} S)^{(\alpha,\beta)}_n(\Psi)\), for all \(\alpha, \beta, \Psi, n\). In general, however, they are different and we will discuss an example in which they really produce different results when \(n \to \infty\). We say that the distributions \(S\) and \(T\) are \(\circ\)-multiplicable, and we indicate with \((S \circ T)^{(\alpha,\beta)}\) their product, if the following limit exists for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\):

\[(S \circ T)^{(\alpha,\beta)}(\Psi) = \lim_{n \to \infty} (S \circ T)^{(\alpha,\beta)}_n(\Psi). \tag{3.5}\]

Analogously we put, when the following limits exist,

\[(S \circ_d T)^{(\alpha,\beta)}(\Psi) = \lim_{n \to \infty} (S \circ_d T)^{(\alpha,\beta)}_n(\Psi). \tag{3.6}\]

and

\[(S \circ_{\text{ex}} T)^{(\alpha,\beta)}(\Psi) = \lim_{n \to \infty} (S \circ_{\text{ex}} T)^{(\alpha,\beta)}_n(\Psi). \tag{3.7}\]

Again, it is clear that, whenever they exist, \((S \circ S)^{(\alpha,\beta)}(\Psi) = (S \circ S)^{(\beta,\alpha)}(\Psi) = (S \circ_d S)^{(\alpha,\beta)}(\Psi) = (S \circ_{\text{ex}} S)^{(\alpha,\beta)}(\Psi)\), for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) while they are different, in general, if \(S \neq T\). Of course, the existence of these limits in general will depend on the values of \(\alpha\) and \(\beta\) and on the particular choice of \(\Phi(x)\). For this reason, as in [2, 3], we are really defining a \textit{class of multiplications} of distributions and not just a single one. We have already discussed in [2] a simple example which shows how these different multiplications may have a physical interpretation in a simple quantum mechanical system. We will discuss further physical applications of our procedure in a forthcoming paper.

We now list a set of properties which follow directly from the definitions:

1. if \(S(x)\) and \(T(x)\) are continuous functions with compact support then

\[(S \circ T)^{(\alpha,\beta)}(\Psi) = (S \circ_d T)^{(\alpha,\beta)}(\Psi) = (S \circ_{\text{ex}} T)^{(\alpha,\beta)}(\Psi) = \int_{\mathbb{R}} S(x) T(x) \Psi(x) \, dx, \tag{3.8}\]

for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) and for all \(\alpha, \beta > 0\).

2. for all fixed \(n\), for all \(\alpha, \beta > 0\) and for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) we have

\[(S \circ_d T)^{(\alpha,\beta)}_n(\Psi) = (S \circ_{\text{ex}} T)^{(\alpha,\beta)}_n(\Psi), \tag{3.9}\]

3. for all \(n, \alpha, \beta > 0\) and for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) we have

\[(S \circ_d T)^{(\alpha,\beta)}_n(\Psi) = (T \circ_d S)^{(\beta,\alpha)}_n(\Psi), \quad \text{and} \quad (S \circ_{\text{ex}} T)^{(\alpha,\beta)}_n(\Psi) = (T \circ_{\text{ex}} S)^{(\beta,\alpha)}_n(\Psi). \tag{3.10}\]
4. given \( S, T \in \mathcal{D}'(\mathbb{R}) \) such that the following quantities exist we have

\[
(S \odot T)_{(\alpha,\beta)}(\Psi) = \frac{1}{2} \left\{ (S \odot_d T)_{(\alpha,\beta)}(\Psi) + (S \odot_{ex} T)_{(\alpha,\beta)}(\Psi) \right\}
\]  

(3.11)

We will now discuss a few examples of these definitions, beginning with maybe the most relevant for concrete applications, i.e. the multiplication of two delta functions.

**Example nr.1:** \((\delta \odot \delta)_{(\alpha,\beta)}\)

First of all we remind that, in this case, the \(\odot, \odot_d\) and \(\odot_{ex}\) multiplications all coincide. If the following limit exists for some \(\alpha, \beta > 0\), we have

\[
(\delta \odot \delta)_{(\alpha,\beta)}(\Psi) = \lim_{n \to \infty} \int_{\mathbb{R}} \delta_n^{(\alpha)}(x) \delta_n^{(\beta)}(x) \Psi(x) \, dx = \lim_{n \to \infty} n^{\alpha+\beta} \int_{\mathbb{R}} \Phi(n^\alpha x) \Phi(n^\beta x) \Psi(x) \, dx,
\]

for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\). It is an easy exercise to check that this limit does not exist, if \(\Psi(0) \neq 0\) and \(\alpha = \beta\). Therefore we consider, first of all, the case \(\alpha > \beta\). In this case we can write

\[
(\delta \odot \delta)_{(\alpha,\beta)}(\Psi) = n^\beta \int_{-1}^{1} \Phi(x) \Phi(xn^\alpha - \alpha) \Psi(xn^{-\alpha}) \, dx,
\]

and the existence of its limit can be proved, as in [2, 3], using the LDCT. Choosing \(\Phi(x)\) as in (3.1), and defining \(B_m = \frac{1}{eN_m} \int_{-1}^{1} x^m \Phi(x) \, dx\), which is surely well defined and strictly positive for all fixed even \(m\), it is quite easy to deduce that

\[
(\delta \odot \delta)_{(\alpha,\beta)}(\Psi) = \begin{cases} 
B_m \Psi(0) = B_m \delta(\Psi), & \alpha = \beta \left(1 + \frac{1}{m}\right) \\
0, & \alpha > \beta \left(1 + \frac{1}{m}\right)
\end{cases}
\]

(3.12)

As we can see, this result is quite close to the one in (2.6). Analogously to what already stressed in Section 2, formula (3.12) does not imply that \((\delta \odot \delta)_{(\alpha,\beta)}(\Psi)\) does not exist if \(\alpha < \beta \left(1 + \frac{1}{m}\right)\) because using the LDCT we only find sufficient conditions for \((\delta \odot \delta)_{(\alpha,\beta)}(\Psi)\) to exist. However, with respect to our results in [2], here we can say a bit more, because we have \((\delta \odot \delta)_{(\alpha,\beta)}(\Psi) = (\delta \odot \delta)_{(\beta,\alpha)}(\Psi)\), which was not true in general for the multiplication \(\odot_{(\alpha,\beta)}\) introduced in [2]. Therefore we find that

\[
(\delta \odot \delta)_{(\alpha,\beta)}(\Psi) = \begin{cases} 
B_m \delta(\Psi), & \alpha = \beta \left(1 + \frac{1}{m}\right)^{-1}, \text{ or } \alpha = \beta \left(1 + \frac{1}{m}\right) \\
0, & \alpha < \beta \left(1 + \frac{1}{m}\right)^{-1} \text{ or } \alpha > \beta \left(1 + \frac{1}{m}\right)
\end{cases}
\]

(3.13)

while nothing can be said in general if \(\alpha \in \left[\beta \left(1 + \frac{1}{m}\right)^{-1}, \beta \left(1 + \frac{1}{m}\right)\right]\).

**Example nr.2:** \((\delta \odot \delta')_{(\alpha,\beta)}\)

In this case the three multiplications \(\odot, \odot_d\) and \(\odot_{ex}\) do not need to coincide. Indeed we will find serious differences between the three, as expected.
First of all we concentrate on the computation of \((\delta \odot_d \delta')(\alpha,\beta)\). Because of (3.9) this will also produce \((\delta \odot_{ex} \delta')(\beta,\alpha)\). We have

\[
(\delta \odot_d \delta')_n(\alpha,\beta)(\Psi) = n^{\alpha+2\beta} \int_{\mathbb{R}} \Phi(n^\alpha x) \Phi'(n^\beta x) \Psi(x) dx,
\]

where \(\Phi'\) is the derivative of \(\Phi\). Again, it is easy to check that the limit of this sequence does not exist, if \(\alpha = \beta\), for all \(\Psi(x) \in \mathcal{D}(\mathbb{R})\) but only for those \(\Psi(x)\) which go to zero fast enough when \(x \to 0\). Let us then consider \((\delta \odot_d \delta')_n(\alpha,\beta)(\Psi)\) for \(\alpha > \beta\). In this case we can write

\[
(\delta \odot_d \delta')_n(\alpha,\beta)(\Psi) = n^{2\beta} \int_{-1}^{1} \Phi(x) \Phi'(xn^\beta - \alpha) \Psi(xn^{-\alpha}) dx,
\]

and by the LDCT we deduce that

\[
(\delta \odot_d \delta')(\alpha,\beta)(\Psi) = \begin{cases} 
K_m \delta(\Psi), & \alpha = \beta \frac{m+1}{m-1} \\
0, & \alpha > \beta \frac{m+1}{m-1}, 
\end{cases}
\]

where \(K_m = \frac{m}{N_m e} \int_{-1}^{1} x^{m-1} \Phi(x) dx\). We see that, contrarily to (2.7), we can obtain a non trivial result with the \(\odot_d\) multiplication. It is clear therefore that also the \((\delta \odot_{ex} \delta')\) can be non trivial, as remarked above.

The situation is completely different for the \((\delta \odot \delta')(\beta,\alpha)\). Indeed, it is not difficult to understand that the LDCT cannot be used to prove its existence. The reason is quite general and is the following:

suppose that for two distributions \(T\) and \(S\) in \(\mathcal{D}'(\mathbb{R})\) \((T \odot_d S)_{(\alpha,\beta)}\) exists for \(\alpha\) and \(\beta\) such that \(\alpha > \gamma\beta\), where \(\gamma\) is some constant larger than 1 appearing because of the LDCT. For instance here \(\gamma = \frac{m+1}{m-1}\), while in Example nr.1 we had \(\gamma = 1 + \frac{1}{m}\). Therefore, since if \((S \odot_{ex} T)_{(\beta,\alpha)}\) and \((S \odot_d T)_{(\alpha,\beta)}\) both exist and coincide, using (3.11) we have

\[
(S \odot T)_{(\alpha,\beta)}(\Psi) = \frac{1}{2} \{ (S \odot_d T)_{(\alpha,\beta)}(\Psi) + (S \odot_d T)_{(\beta,\alpha)}(\Psi) \}
\]

(3.15)

Of course, \((S \odot T)_{(\alpha,\beta)}(\Psi)\) exists if \((S \odot_d T)_{(\alpha,\beta)}(\Psi)\) and \((S \odot_d T)_{(\beta,\alpha)}(\Psi)\) both exist, which in turn implies that \(\alpha > \gamma\beta\) and, at the same time, that \(\beta > \gamma\alpha\). These are clearly incompatible. Therefore in order to check whether \((S \odot T)_{(\alpha,\beta)}(\Psi)\) exists or not it is impossible to use the LDCT which only gives sufficient conditions for the multiplication to be defined: some different strategy should be considered.

**Example nr.3:** \((\delta' \odot \delta')(\alpha,\beta)\)

As for Example nr.1 we remark that here the \(\odot, \odot_d\) and \(\odot_{ex}\) multiplications all coincide. If the following limit exists for some \(\alpha, \beta > 0\), we have

\[
(\delta' \odot \delta')(\alpha,\beta)(\Psi) = (\delta' \odot \delta')(\beta,\alpha)(\Psi) = \lim_{n \to \infty} \int_{\mathbb{R}} \delta_n'(\alpha)(x) \delta_n'(\beta)(x) \Psi(x) dx =
\]
\[ \lim_{n \to \infty} n^{2\alpha+2\beta} \int_{\mathbb{R}} \Phi'(n^{\alpha} x) \Phi'(n^{\beta} x) \Psi(x) \, dx, \]
for all \( \Psi(x) \in \mathcal{D}(\mathbb{R}) \). As before, it is quite easy to check that this limit does not exist, if \( \Psi(0) \neq 0 \), when \( \alpha = \beta \). Therefore we start considering the case \( \alpha > \beta \). In this case we can write

\[ (\delta' \circ \delta')^{(\alpha,\beta)}_n(\Psi) = n^{\alpha+2\beta} \int_{-1}^{1} \Phi'(x) \Phi'(xn^{\beta-\alpha}) \Psi(xn^{-\alpha}) \, dx, \]

and again the existence of its limit can be proved using the LDCT. Choosing \( \Phi(x) \) as in (3.1) and defining \( \tilde{B}_m = \frac{m}{eN_m} \int_{-1}^{1} x^{m-1} \Phi'(x) \, dx \), which surely exists for all fixed even \( m \), we deduce that

\[ (\delta' \circ \delta')^{(\alpha,\beta)}(\Psi) = \begin{cases} \tilde{B}_m \delta(\Psi), & \alpha = \beta \frac{m+1}{m-2} \\ 0, & \alpha > \beta \frac{m+1}{m-2}. \end{cases} \] (3.16)

However this is not the end of the story, because we still can use the symmetry \( (\delta' \circ \delta')^{(\alpha,\beta)}(\Psi) = (\delta' \circ \delta')^{(\beta,\alpha)}(\Psi) \) discussed before. We find

\[ (\delta' \circ \delta')^{(\alpha,\beta)}(\Psi) = \begin{cases} \tilde{B}_m \delta(\Psi), & \alpha = \beta \frac{m-2}{m+1}, \text{ or } \alpha = \beta \frac{m+1}{m-2} \\ 0, & \alpha < \beta \frac{m-2}{m+1}, \text{ or } \alpha > \beta \frac{m+1}{m-2}. \end{cases} \] (3.17)

while nothing can be said in general if \( \alpha \in \left[ \beta \frac{m-2}{m+1}, \beta \frac{m+1}{m-2} \right] \). Needless to say, we need here to restrict to the following values of \( m \): \( m = 4, 6, 8, \ldots \).

Summarizing we find that results which are very close to those in [2] can be recovered with each one of the definitions in (3.5), (3.6) or (3.7). The main differences essentially arise from the lack of symmetries of these two last definitions compared to definition (3.5) and the one in [2].

### 4 More spatial dimensions and conclusions

While the definition of the multiplication given in [2], as we have stressed before, cannot be extended easily to \( \mathbb{R}^d, d > 1 \), definitions (3.5), (3.6) or (3.7) admit a natural extension to any spatial dimensions. We concentrate here only on the symmetric definition, (3.5), since it is the most relevant one for the application we are interested in here. Of course no particular differences appear in the attempt of extending \( \circ_d \) and \( \circ_{ex} \) to \( d > 1 \).

The starting point is again a given non negative function \( \Phi(\underline{x}) \in \mathcal{D}(\mathbb{R}^d) \) with support in \( I_1 := [-1,1] \times \cdots \times [-1,1] \), and such that \( \int_{I_1} \Phi(\underline{x}) \, d\underline{x} = 1 \). In this case the delta-sequence

\[ d \text{ times} \]

is \( \delta^{(\alpha)}_n(\underline{x}) = n^{d\alpha} \Phi(n^{\alpha} \underline{x}) \), for any choice of \( \alpha > 0 \). The same results listed in the previous section again hold in this more general situation. For instance, if \( T \in \mathcal{D}'(\mathbb{R}^d) \) then \( \{ T^{(\alpha)}_n(\underline{x}) \} = (T * \delta^{(\alpha)}_n)(\underline{x}) \) is a sequence of \( C^\infty \) functions and it converges to \( T \) in \( \mathcal{D}'(\mathbb{R}^d) \) as \( n \) diverges, independently of \( \alpha > 0 \).
Therefore, let us consider two distributions $T, S \in \mathcal{D}'(\mathbb{R}^d)$, and let us consider their convolutions $T_n^{(\alpha)}(x) = (T * \delta_n^{(\alpha)})(x)$ and $S_n^{(\beta)}(x) = (S * \delta_n^{(\beta)})(x)$ with $\delta_n^{(\alpha)}(x) = n^{\alpha} \Phi(n^\alpha x)$, for $\alpha, \beta > 0$. As usual, $T_n^{(\alpha)}(x)$ and $S_n^{(\beta)}(x)$ are both $C^\infty$ functions, so that the following integral surely exists:

$$
(S \circ T)_n^{(\alpha,\beta)}(x) = \frac{1}{2} \int_{\mathbb{R}^d} \left[ S_n^{(\alpha)}(x) T_n^{(\beta)}(x) + S_n^{(\beta)}(x) T_n^{(\alpha)}(x) \right] \Psi(x) \, dx,
$$

where $\forall \Psi(x) \in \mathcal{D}(\mathbb{R}^d)$. As before the two distributions $S$ and $T$ are $\odot$-multiplicable if the following limit exists independently of $\Psi(x) \in \mathcal{D}(\mathbb{R}^d)$:

$$
(S \circ T)_{(\alpha,\beta)}(x) = \lim_{n \to \infty} (S \circ T)_n^{(\alpha,\beta)}(x).
$$

We consider in the following the $\odot$-multiplication of two delta functions, considering two different choices for the function $\Phi(x)$ both extending the one-dimensional case.

The starting point of our computation is the usual one: if it exists, $(\delta \circ \delta)_{(\alpha,\beta)}(\Psi)$ must be such that

$$
(\delta \circ \delta)_{(\alpha,\beta)}(\Psi) = \lim_{n \to \infty} n^{\alpha+\beta} \int_{\mathbb{R}^d} \Phi(n^\alpha x) \Phi(n^\beta x) \Psi(x) \, dx.
$$

Again, this limit does not exist if $\alpha = \beta$, but for very peculiar functions $\Psi(x)$. If we consider what happens for $\alpha > \beta$ then the limit exists under certain extra conditions.

For instance, if we take $\Phi(x) = \prod_{j=1}^d \Phi(x_j)$, where $\Phi(x_j)$ is the one in (3.1), the computation factorizes and the final result, considering also the symmetry of the multiplication, is a simple extension of the one in (3.13):

$$
(\delta \circ \delta)_{(\alpha,\beta)}(\Psi) = \begin{cases} 
B_m^d \delta(\Psi), & \alpha = \beta \left(1 + \frac{1}{m} \right)^{-1}, \text{ or } \alpha = \beta \left(1 + \frac{1}{m} \right) \\
0, & \alpha < \beta \left(1 + \frac{1}{m} \right)^{-1} \text{ or } \alpha > \beta \left(1 + \frac{1}{m} \right)
\end{cases}
$$

(4.3)

A different choice of $\Phi(x)$, again related to the one in (3.1), is the following:

$$
\Phi(x) = \begin{cases} 
\frac{\|x\|^m}{N_m^c} \exp\left\{-\frac{1}{\|x\|^2-1}\right\}, & \|x\| < 1 \\
0, & \|x\| \geq 1
\end{cases}
$$

(4.4)

where $N_m^c$ is a normalization constant and $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$. With this choice, defining $C_m = \frac{1}{N_m^c} \int_{\mathbb{R}^d} \|x\| \Phi(x) \, dx$ and using the symmetry property of $\odot_{(\alpha,\beta)}$, we find

$$
(\delta \circ \delta)_{(\alpha,\beta)}(\Psi) = \begin{cases} 
C_m \delta(\Psi), & \alpha = \beta \left(1 + \frac{d}{m} \right)^{-1}, \text{ or } \alpha = \beta \left(1 + \frac{d}{m} \right) \\
0, & \alpha < \beta \left(1 + \frac{d}{m} \right)^{-1} \text{ or } \alpha > \beta \left(1 + \frac{d}{m} \right).
\end{cases}
$$

(4.5)

Therefore the limit defining the product of two delta can be defined, at least under certain conditions, also with this choice of $\Phi(x)$ . The main differences between the above results are the values of the constants and the fact that $d$ explicitly appears in the result in (4.3), while it only appears in the condition relating $\alpha$ and $\beta$ in (4.5).
The conclusion of this short paper is that the use of sequential completion, properly adapted for our interests, is much simpler and convenient. The next step of our analysis will be to use our results in applications to three-dimensional engineering structures, trying to extend the results in [4, 5].

Acknowledgments

This work has been partially supported by M.U.R.S.T.

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