An $Sp(4,R)$ geometrization on the quantum dynamics of two-component Bose-Einstein condensate

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The method of geometrization arises as an important tool in understanding the entanglement of quantum fields and the behavior of the many-body system. The symplectic structure of the boson operators provides a natural way to geometrize the quantum dynamics of the bosonic systems of quadratic Hamiltonians, by recognizing that the time evolution operator corresponds to a real symplectic matrix in $Sp(4,R)$ group. We apply this geometrization scheme to study the quantum dynamics of the spinor Bose-Einstein condensate systems, demonstrating that the quantum dynamics of this system can be represented by trajectories in a six dimensional manifold. It is found that the trajectory is quasi-periodic for coupled bosons. The expectation value of the observables can also be naturally calculated through this approach.

I. INTRODUCTION

The methods of geometry have played a central role in the modern physics theory such as general relativity or Yang-Mills theory [1, 2] as well as the AdS/CFT correspondence [3, 4]. The later one reveals a duality between a strongly correlated quantum field and a weakly interacted gravity, which further motivates the study on the connection between geometry and quantum entanglement [5–7].

However, in these geometrization examples, either it is semi-classical or strongly correlated fields are present. This motivates us to wonder what kind of geometry can be revealed in the weakly interacting quantum system. Recently, it is proposed that the ground states of a weakly interacting boson system with Bose-Einstein condensation (BEC) can be mapped to a Poincare disk with $SU(1,1) \simeq Sp(2,R)$ symmetry [8]. This opens up an interesting possibility to understand the dynamics of a weak coupled system from a geometric point of view.

In this paper, we consider a more general two-component spinor Bose-Einstein condensate (BEC) system [9–15]. Since the boson operators naturally realize the symplectic Lie algebra, the ground state of this system can be identified as the coherent state of the non-compact real symplectic Lie group $Sp(4,R)$, which is defined as the $4 \times 4$ complex matrices set $\{ \begin{pmatrix} U & V \\ \bar{V}^T & \bar{U}^T \end{pmatrix} : UU^T - VV^T = I \}$, where $U, V$ are $2 \times 2$ complex matrices [16, 17]. The ground state can be parameterized as a point in a more complicated 6-dimensional space $Sp(4,R)/U(2)$, which can be shown to reduce to the direct product of two Poincare disk in certain limit. Given its high dimensions, however, this space is very easily to visualize with the aid of polar coordinate defined in this paper. Meanwhile, within this approach, the calculation of the time evolution operator is converted to calculating the exponential of a $4 \times 4$ matrix. As a result, the time evolution of any coherent state, as the ground state of some Hamiltonian, can be mapped to a trajectory in this parameter manifold. Although the quantum dynamics of BEC system has been studied by many approaches [18–21], this geometrizing method has its own advantages, including its geometric intuitiveness, easy visualization, more importantly, its generality and versatility, i.e. it can deal with the most general form of quadratic Hamiltonian of two-component bosonic system, meanwhile, applies to time independent and time dependent dynamics in a uniform way. Furthermore, this approach can also be easily generalized to N-component boson systems.

Comparing to the previous work of the $SU(1,1)$ geometrization of the one component BEC system, some new features emerge in our $Sp(4,R)$ geometrization scheme. Due to the coupling of the two components of bosons, their trajectory in the parameter manifold will display some quasi-periodic behaviors, which is very similar to two coupled classical harmonic oscillators with incommensurate frequencies. Although these evolution trajectories is not chaotic, it is possible that some non-linear perturbation may lead to some chaotic dynamics [22]. In this regards, this geometrization may provide an interesting relation with the behavior of out-of-time ordering correlators (OTOC) [23–25]. Since there are two eigen modes, we can have a mixed situation with one stable and one unstable mode [26]. In this case, only one coordinate of the trajectory will approaches to the boundary of the ground state manifold, corresponding to the number of the unstable mode. Meanwhile, in this case, the number operator expectation value of one species of boson grow exponentially in a much higher rate than the other species. These features demonstrate the richness of the higher dimensional geometrization method.

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The rest of this paper is organized as follows. In section II, we introduce the model Hamiltonian of two-component BEC. Making use of its \(Sp(4, R)\) symmetry, we identify its ground state as the \(Sp(4, R)\) coherent state, and establish its correspondence to a point in the six-dimensional space \(SP(4, R)/U(2)\), which is dubbed as ground state parameter (GSP) manifold. In section III, we reveal that the quantum dynamic evolutions can be understand as \(Sp(4, R)\) group actions on the GSP manifold. Different types of quantum dynamical evolution, stable or unstable, are represented by different types of curves in the GSP manifold. In section IV, the GSP manifold is described by a polar coordinate. Based on this coordinate, we can specify the boundary and metric of this space. The evolutions of expectation of operators can be understand by a similar group action method and was discussed in section V. Finally, we briefly conclude in section VI.

II. HAMILTONIAN AND GROUND STATE PARAMETER MANIFOLD

We consider the Hamiltonian of two-component Bose-Hubbard model with synthetic spin-orbit coupling, which can be written as

\[
H = \sum_{i,j,k} \epsilon_{ij}(k) a_{i,k}^\dagger a_{j,k} + \frac{1}{2V} \sum_{i,j,k,k',q} U_{ij} a_{i,k-q}^\dagger a_{j,k'} a_{j,k'} a_{i,k},
\]

where \(i, j = 1, 2\) label the two species of bosons. After taking the mean field approximation, we find that the two-component BEC Hamiltonian can be expressed in a matrix form as

\[
H_k = \Psi_k^\dagger H \Psi_k, \quad H_k = \begin{pmatrix} \xi(k) & \eta(k) \\ \eta^*(k) & \xi^*(k) \end{pmatrix},
\]

where \(\Psi_k = (a_{1,k}, a_{2,k}, a_{1,-k}^\dagger, a_{2,-k}^\dagger)^T\). Here \(\xi(k)\) is a 2 \(\times\) 2 Hermitian matrix given by

\[
\xi(k) = \begin{pmatrix} \epsilon_{11}(k) + 2 \sum_j U_{1j} |\psi_j|^2 & \epsilon_{12}(k) \\ \epsilon_{21}(k) & \epsilon_{22}(k) + 2 \sum_j U_{2j} |\psi_j|^2 \end{pmatrix}
\]

and \(\eta(k)\) is a complex symmetric matrix given by

\[
\eta = \begin{pmatrix} U_{11} \psi_1^2 & U_{12} \psi_1 \psi_2 \\ U_{21} \psi_1 \psi_2 & U_{22} \psi_2^2 \end{pmatrix}
\]

Here \(\psi_i = \sqrt{N_i/V} e^{i \theta_i}\) is the condensate wavefunction. In this paper, we only consider uniform condensate. In this case, \(\eta\) is always a constant matrix.

In the mean field level, the Hamiltonian for different \(k\) is decoupled. Hence, in the following, we will always consider a generic choice of \(k\) and make the \(k\) dependence implicit in our convention for simplicity. By diagonalizing \(gH_k\), where \(g = diag(1, 1, -1, -1)\), we can get the quasiparticles, whose annihilation operators are given by the Bogoliubov transformation \(\alpha_{i,k} = \sum_{j=1}^2 u_{ij} a_{j,-k} + v_{ij} a_{j,k}^\dagger\), where \(u, v\) are general 2 \(\times\) 2 complex matrices, if the eigen-energies of these two excited states both are real. And the Bogoliubov transformation has the property \([16, 17]\)

\[
u u^T - u v^T = I, \quad u v^T = v u^T,
\]

where \(I\) is the 2 \(\times\) 2 identity matrix. Hence, the ground state of the Hamiltonian can be defined as \(\alpha_i(G) = 0\).

On the other hand, if we define

\[
X_{ij} = a_{i,k}^\dagger a_{j,-k} + a_{i,k} a_{j,-k}^\dagger, \quad \Lambda_{ij}^k = a_{i,k}^\dagger a_{i,k} + a_{i,-k} a_{i,-k}^\dagger
\]

where \(i, j, k, l = 1, 2\), with ten of them independent, they satisfy the commutation relations of the Lie algebra of the real symplectic group \(Sp(4, R)\) \([16, 17]\), i.e.

\[
[X_{ij}, X_{kl}] = [X_{ij}, \Lambda_{kl}^i] = [\Lambda_{ij}^k, \Lambda_{kl}^j] = 0,
\]

\[
[X_{ij}, \Lambda_{kl}^i] = \Lambda_{kl}^i \delta_{jk}^i + X_{ij}^l \delta_{jk}^l + X_{ij}^k \delta_{jl}^i + \Lambda_{ij}^l \delta_{jl}^k,
\]

\[
[X_{ij}, \Lambda_{kl}^j] = \Lambda_{kl}^j \delta_{ik}^j + X_{ij}^l \delta_{ik}^l + X_{ij}^k \delta_{il}^j + \Lambda_{ij}^l \delta_{il}^k,
\]

\[
[X_{ij}, \Lambda_{kl}^l] = -\Lambda_{ik}^l \delta_{jk}^l - \Lambda_{jk}^l \delta_{ik}^l + \Lambda_{il}^j \delta_{jk}^i + \Lambda_{jk}^i \delta_{il}^j,
\]

\[
[X_{ij}, \Lambda_{kl}^k] = \Lambda_{ik}^l \delta_{jk}^l - \Lambda_{jk}^l \delta_{ik}^l + \Lambda_{il}^j \delta_{jk}^i - \Lambda_{jk}^i \delta_{il}^j.
\]

We can write the Hamiltonian \(H_k\) in terms of these generators as

\[
H_k = \xi_k \Lambda_{ij}^j + \eta_k (1 + \frac{1}{2} \eta(k) - \frac{1}{2} \eta(k) \Lambda_{ij}^j)
\]

where we have adopted the Einstein summation and \(i\) is the row index, \(j\) the column index for the matrices \(\xi(k), \eta(k)\). Hence, this Hamiltonian gives a representation of the Lie algebra \(sp(4, R)\), and the time evolution operator \(e^{-iHt}\) gives a unitary representation of the noncompact real symplectic group \(Sp(4, R)\).

Similar to the one-component BEC \([27]\), the ground state of the two-component BEC Hamiltonian \(H_k\) can be given by the coherent state \([15]\)

\[
|Z\rangle = N e^{\sum_{ij} Z_{ij} a_{i,k}^\dagger a_{j,-k}} |0\rangle = N e^{-\frac{1}{2} Z_{ij} \Lambda_{ij}^j} |0\rangle
\]

where \(i, j = 1, 2, |0\rangle\) is the vacuum, and \(N = det(I - ZZ^T)^{1/2}\) is the normalization factor (see Appendix A). Here, we have adopted the Einstein summation convention. The inner product of two coherent states is given by

\[
(Z'|Z) = \frac{det(I - Z'Z'^T)^{1/2} det(I - ZZ^T)^{1/2}}{det(I - ZZ^T)}.
\]

The detailed derivation of above equation is in Appendix A.

Substituting the expression of \(|Z\rangle\) to the defining equation of the ground state \(\alpha_i(G) = 0\), we see that this equation is satisfied.
if $Z = u^{-1}v$ with matrices $u, v$ determined by the Hamiltonian. According to Eq.(3), we have $I - ZZ^\dagger = (u^\dagger u)^{-1}$, indicating that $1 - ZZ^\dagger$ is positive definite, which can be denoted as $I - ZZ^\dagger > 0$. Meanwhile, we also have $(u^{-1}v)^T = u^{-1}v$, meaning that the matrix $Z$ is symmetric. Hence, the ground state for each $k$ corresponding to a point of the space of $2 \times 2$ symmetric complex matrices satisfying a constraint. This space can be denoted as

$$B = \{ Z : Z^T = Z, I - ZZ^\dagger > 0 \},$$

which is parameterized by 3 complex number $Z_{11}, Z_{22}, Z_{12}$ and is 6-dimension in real parameters. In mathematical literature, this space belongs to one type of the so called Cartan’s classification on the Lie groups \[16, 28\], and is isometric to the quotient space $Sp(4, R)/U(2)$, where $U(2)$ is the unitary group of degree two. In the following, we will call the space $B$ as ground state parameter (GSP) manifold.

### III. GROUP ACTION AND QUANTUM DYNAMICS

In this section, we will study the quantum dynamics of the ground state. In other words, we want to calculate $e^{-iH_k t} |Z \rangle$, where $|Z \rangle$ is a general coherent state which could be a ground state of $H_k$ or not. Before calculating this general case, we will start from calculating the time evolution of the vacuum, i.e. $e^{-iH_k t} |0 \rangle$.

Because the generators of $sp(4, R)$ are highly non-commuting operators, the expansion of exponential will be quite complicated. To proceed, it is very helpful to decompose the time evolution operator into a normal ordered form as [16]

$$U(t) \equiv e^{-iH_k t} = e^{-\frac{i}{2} \mu_{ij} X^{ij}} e^{\zeta_{ij} \Psi_k^i \Psi_k^j} e^{-\frac{i}{2} \nu_{ij} X^{ij}}, \quad (11)$$

where Einstein summation convention has been adopted. This decomposition is the generalization of the normal order decomposition for $SU(2)$ or $SU(1, 1)$ [29]. Here we also assume that the $2 \times 2$ matrices $\mu, \zeta, \nu$ implicitly depend on $t$. With this decomposition, the time evolution of the vacuum is easy to find

$$e^{-iH t} |0 \rangle = \mathcal{N} e^{-\frac{i}{2} \mu_{ij} X^{ij}} |0 \rangle = |Z = \mu \rangle,$$

which is a coherent state.

By the virtue of the representation theory, such a decomposition should also hold true for the group element corresponding to the time evolution operator. Hence, we can covert the decomposition of the time evolution operator to the decomposition of the group element of $Sp(4, R)$.

In order to find the group element corresponding the time evolution operator, we need to rewrite the Lie algebra generators in the matrix form. Note that the generators can also be expressed as

$$X_{ij} = \Psi_k^i \beta_{ij} \Psi_k^j, \quad X^{ij} = \Psi_k^i \beta^{ij} \Psi_k^j, \quad \xi_{ij} = \Psi_k^j \beta_{ij} \Psi_k^i$$

$$(\beta_{ij})_{ab} = \delta_{a,2+i} \delta_{b,j} + \delta_{a,2+j} \delta_{b,i},$$

$$(\beta^{ij})_{ab} = \delta_{a,1+i} \delta_{b,j} + \delta_{a,1+j} \delta_{b,i},$$

$$(\beta^{ij})_{ab} = \delta_{a,2} \delta_{b,1} \delta_{2+a,b+1,a}.$$

Also it is easy to verify that for any two $4 \times 4$ matrices $A$ and $B$, we have

$$[\Psi_k^i A \Psi_k^j, \Psi_k^k B \Psi_k^l] = \Psi_k^i g[a, b] g[b, c] \Psi_k^l,$$ \quad (12)

where $g$ is a diagonal matrix given by $g = \text{diag}(1, 1, -1, -1)$. Therefore, we can introduce the following matrices

$$Y_{ij} = g \beta_{ij}, \quad Y^{ij} = g \beta^{ij}, \quad Y_{ij} = g \beta_{ij}$$ \quad (13)

which satisfies the commutation relations of the $sp(4, R)$ Lie algebra of Eq.(7). Actually, the way we choose this set of matrix generators is similar to that we can choose $\{ \sigma_z, \sigma^+, \sigma^- \}$ as the generators of $su(2)$ Lie algebra. As a result, we find that the symplectic matrix corresponding to the time evolution operator is $e^{-i(t/2)I_{ij} + \frac{i}{2} \delta_{ij} (k \epsilon + \frac{1}{2} \delta_{ij} (k \epsilon^*)} = e^{-igH_k t}$, which belongs to the $Sp(4, R)$ group. Therefore, the decomposition in the matrix form corresponding to Eq.(11) is

$$e^{-igH_k t} = e^{-\frac{i}{2} \mu_{ij} Y^{ij}} e^{\zeta_{ij} Y_{ij}} e^{-\frac{i}{2} \nu_{ij} Y^{ij}}.$$ \quad (14)

Since $e^{-igH_k t} \in Sp(4, R)$, it can be expressed as

$$e^{-igH_k t} = \begin{pmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{V}^* & \mathcal{U}^* \end{pmatrix},$$ \quad (15)

where $\mathcal{U}, \mathcal{V}$ are $2 \times 2$ complex matrices satisfying [16, 17]

$$\mathcal{U} \mathcal{U}^T - \mathcal{V} \mathcal{V}^T = I, \quad \mathcal{U} \mathcal{V}^T = \mathcal{V} \mathcal{U}^T.$$ \quad (16)

On the other hand, through direct calculation, we find that the creation and annihilation operators will give rise to matrices with upper and lower triangular form as

$$e^{-\frac{i}{2} \mu_{ij} Y^{ij}} = \begin{pmatrix} I & \mu \\ 0 & I \end{pmatrix}, \quad e^{-\frac{i}{2} \nu_{ij} Y^{ij}} = \begin{pmatrix} I & 0 \\ \nu & I \end{pmatrix},$$ \quad (17)

Meanwhile, the operator $Y_{ij}$ will give a blocked diagonal matrix as

$$e^{\xi_{ij} Y_{ij}} = \begin{pmatrix} (O^{-1})^T & 0 \\ 0 & O \end{pmatrix},$$ \quad (18)

where $O \in \mathcal{U}(2)$. Collect all the above results, the decomposition of Eq.(14) can be rewritten as

$$\begin{pmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{V}^* & \mathcal{U}^* \end{pmatrix} = \begin{pmatrix} I & \mu \\ 0 & I \end{pmatrix} \begin{pmatrix} (O^{-1})^T & 0 \\ 0 & O \end{pmatrix} \begin{pmatrix} I & 0 \\ \nu & I \end{pmatrix}.$$ \quad (19)

Here the 2 by 2 matrices $\mu, \nu$ and $O$ are still unknown. By using Eq.(16), it can be directly verified that the above equation has the following solution [30]

$$\mu = \mathcal{V}(\mathcal{U}^*)^{-1}, \quad \nu = (\mathcal{U}^*)^{-1} \mathcal{V}, \quad O = \mathcal{U}^*.$$ \quad (20)
Thus, the decomposition of Eq.(14) is satisfied. Then the time evolution of the vacuum is just
\[ e^{-iHt}|0\rangle = |\mathcal{V}(\mathcal{U}^*)^{-1}\rangle \] (21)
with matrices \( \mathcal{U}, \mathcal{V} \) determined by the parameters of the Hamiltonian.

Using the same method, we can study the time evolution of a generic state in the GSP manifold. To this end, we also seek a normal ordered decomposition similar to the form of Eq.(11) for the operator \( e^{-iHt}e^{-\frac{i}{2}Z_{ij}X^{ij}} \). By the virtue of representation theory, such a decomposition can be converted to a decomposition in the matrix level, i.e.
\[
\begin{pmatrix}
\mathcal{U} & \mathcal{V} \\
\mathcal{V}^* & \mathcal{U}^*
\end{pmatrix}
\begin{pmatrix}
I & Z \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
I & Z' \\
0 & I
\end{pmatrix}
\begin{pmatrix}
(O'^{-1})^T & 0 \\
0 & O'
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
\nu' & I
\end{pmatrix}.
\] (22)
And the solution is given by
\[
Z' = (\mathcal{V}Z + \mathcal{V})(\mathcal{V}Z + \mathcal{U}^*)^{-1},
\]
\[
O' = \mathcal{V}^*Z + \mathcal{U}^*,
\]
\[
\nu' = (\mathcal{V}^*Z + \mathcal{U}^*)^{-1}\mathcal{V}^*.
\] (23)
Hence it is easy to see that \( e^{-iHt}|Z\rangle = |Z'\rangle \). Therefore the time evolution will map a coherent state to another coherent state.

In summary, we have converted the problem of calculating of the time evolution of a generic state \( |Z\rangle \) in the GSP manifold into the calculation of the exponential of a matrix, i.e. \( e^{-ig\mathcal{H}t} \), which can be easily calculated through numerical method. Furthermore, this method also works for the case where the Hamiltonian is time dependent. In this case, the group element corresponding to the time evolution operator is
\[
U(t) = \mathcal{T} e^{-ig \int \mathcal{H}_k(t) dt} \equiv \lim_{N \to \infty} \prod_{n=1}^{N} e^{-ig \mathcal{H}_k(\frac{\pi}{2})} \] (24)
Here \( \mathcal{T} \) means the time order. Numerically, this can be approximated by taking \( N \) as a large finite number.

In Fig. 1, we show the numerical results of trajectories of \( Z(t) \) in complex plane evolving under the Hamiltonians with real eigen-energies, which are all stable modes. In Fig. 1 (a), the system starts from the vacuum \( |0\rangle \), and evolving according to the Hamiltonian \( \mathcal{H}_1 \) in which the two species of bosons are totally decoupled. The parameters of \( \mathcal{H}_1 \) is given by
\[
\xi = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{2}
\end{pmatrix} U_{11}, \quad \eta = \begin{pmatrix}
2 & 0 \\
3 & 0
\end{pmatrix} U_{11}.
\] (25)
The two real eigen-energies of \( \mathcal{H}_1 \) are \( E_1 \approx 0.43U_{11}, \quad E_2 \approx 0.75U_{11} \). In this case, we can see that the off-diagonal terms are zero, and the diagonal terms \( Z_{11}, Z_{22} \) form circles in the complex plane, and the evolution is periodic for the two decouple parts with different periods generally. In Fig. 1 (b), the system starts from a non-vacuum state \( |Z_0\rangle \), \( Z_0 = \begin{pmatrix}
0.2 & 0.25 \\
0.25 & 0.2i
\end{pmatrix} \), and evolves under the same decoupled Hamiltonian \( \mathcal{H}_1 \). In this case, the evolution is no longer periodic and \( Z_{12} \) becomes non-zero.

In Fig. 1 (c-d), the systems evolve under the same Hamiltonian \( \mathcal{H}_2 \) with two species of bosons coupled, whose two real eigen-energies are \( E_1 \approx 0.28U_{11}, \quad E_2 \approx 0.62U_{11} \). The parameters of \( \mathcal{H}_2 \) is given by
\[
\xi = \begin{pmatrix}
1 & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{2}
\end{pmatrix} U_{11}, \quad \eta = \begin{pmatrix}
2 & 1 \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} U_{11}.
\] (26)
Again, the trajectory in panel (c) starts from the vacuum \( |0\rangle \), the one in panel (d) starts from non-vacuum state \( |Z_0\rangle \). We can see in both cases, the evolution is not periodic any more and is also quite complicated, and the trajectory is very similar to the behavior of two coupled classical harmonic oscillators with incommensurate frequencies. The trajectory is still confined inside a finite volume of the GSP manifold.
IV. THE GEOMETRY OF THE GSP MANIFOLD

The above method not only works for the evolving Hamiltonian with both eigen-energies real, but also works for one with imaginary eigen-energy. Similar to the $SU(1, 1)$ case, the trajectories will also approach to the boundary of the GSP manifold, which reveals their unstable nature. To proceed, we need first define the boundary of the GSP manifold by introducing the polar coordinate.

A. Polar Coordinate and Boundary

According to Autonne-Takagi factorization [31], every complex symmetric matrix can be decomposed as $u\Lambda u^T$, where $u$ is a unitary matrix, and $\Lambda$ is a real diagonal with non-negative entries. This gives us the polar coordinate of the complex symmetric matrix [32]. Hence, in our case, $Z$ can be decomposed as $Z = u\Lambda u^T$, where $u \in U(2)$ and a diagonal matrix $\Lambda = \text{diag}(r_1, r_2)$ with two eigenvalues satisfying $r_1 \geq r_2$. Note that every elements in the group $U(2)$ can be expressed as

$$ u = e^{i\delta} e^{-i\frac{\pi}{2}\sigma_z} e^{-i\frac{\pi}{2}\sigma_y} e^{-i\frac{\pi}{2}\sigma_z}. \quad (27) $$

We can absorb the phase factor $e^{i\delta} e^{-i\frac{\pi}{2}\sigma_z}$ in $u$ to $\Lambda$, and write our new coordinate as

$$ Z = u(\varphi, \theta) \begin{pmatrix} r_1 e^{i\tau_1} & 0 \\ 0 & r_2 e^{i\tau_2} \end{pmatrix} u(\varphi, \theta)^T, \quad (28) $$

where $u(\varphi, \theta) = e^{i\frac{\pi}{2}\sigma_y} e^{-i\frac{\pi}{2}\sigma_y}$, and $r_1, r_2, \varphi \in [-\pi, \pi]$, $\theta \in [0, \pi]$, and $\sigma_{y,z}$ are Pauli matrices. In the following, we will still call this coordinate as polar coordinate. In this coordinate, the eigenvalues of $I - ZZ^\dagger$ are $1 - r_1^2, 1 - r_2^2$. Hence the condition $I - ZZ^\dagger > 0$ is equivalent to $1 - r_1^2 > 0$, and $1 - r_2^2 > 0$. And the the boundary of the GSP manifold is naturally given by $Z(r_1, r_2, \tau_1, \tau_2, \varphi, \theta) : r_1 = 1$, which is five-dimensional. To visualize the trajectories, we define $\lambda_1 = r_1 e^{i\tau_1}, \lambda_2 = r_2 e^{i\tau_2}, \lambda_3 = \frac{1}{2} e^{i\varphi}$, which are within the unit disk in the complex plane.

In (c), we show the trajectories in the polar coordinate system we defined above. In panel (a) and (b), the time-evolution is under a coupled Hamiltonian $\mathcal{H}_3$ with the following parameters

$$ \xi = \begin{pmatrix} 1 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} U_{11}, \ \eta = \begin{pmatrix} 2 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} U_{11}. \quad (29) $$

and eigen-energies $E_1 \simeq 0.33 U_{11}, \ E_2 \simeq 1.8 U_{11}$. The imaginary eigen-energy is the one unstable mode. The initial state is the vacuum in (a) and non-vacuum in (b). In both cases, only coordinate $\lambda_1$ approaches to the boundary corresponding one unstable mode.

In panels (c) and (d), the time-evolution is under a coupled Hamiltonian $\mathcal{H}_4$ with the parameters

$$ \xi = \begin{pmatrix} 1 & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} U_{11}, \ \eta = \begin{pmatrix} 2 & 1 \\ 1 & 3/2 \end{pmatrix} U_{11}. \quad (30) $$

and eigen-energies $E_1 \simeq 0.6 i U_{11}, \ E_2 \simeq 2.53 i U_{11}$. Again, the initial state is the vacuum in (c) and non-vacuum in (d). One can see for the Hamiltonian with two imaginary eigen-energies, both $\lambda_1$ and $\lambda_2$ approach to the boundary of GSP manifold. In summary, the number of coordinates approaching to the boundary equals to the number of imaginary eigen-energies.

B. Fidelity and Metric

Now we turn to the relations between the fidelity $F_{Z,Z+dZ} = |\langle Z + dZ | Z \rangle|^2$ and the metric of the GSP manifold. Actually, the inner product between two coherent states is given by Eq.(10), whose modular square can be rewritten as

$$ |\langle Z' | Z \rangle|^2 = \frac{\det(I - Z' Z^\dagger) \det(I - ZZ^\dagger)}{\det(I - Z' Z^\dagger) \det(I - ZZ^\dagger)}. \quad (31) $$

We can consider the fidelity of two neighboring states by taking $Z' = Z + dZ$. In order to make a connection with a metric, we take the logarithm of the fidelity and expand it to the second order of $dZ$. The result is

$$ \ln(F_{Z,Z+dZ}) = -\text{Tr}[dZ^\dagger A^{-1} dZ(I + Z^\dagger A^{-1} Z)], \quad (32) $$
where $A = 1 - ZZ^\dagger$. Making use of the identity $I + ZZ^\dagger A^{-1}Z = (I - ZZ^\dagger)^{-1}$, we can rewrite the above equation as

$$\ln(F_{Z,Z+dz}) = -\text{Tr}[dZ^\dagger(1 - ZZ^\dagger)^{-1}dZ(1 - ZZ^\dagger)^{-1}].$$

On the other hand, it is known that the metric of the GSP manifold is given by $ds^2 = \frac{1}{2} \text{Tr}[dZ^\dagger(1 - ZZ^\dagger)^{-1}dZ(1 - ZZ^\dagger)^{-1}]$. Furthermore, by plug in Eq. (28), we can write the metric in the polar coordinate,

$$ds^2 = \frac{dr^2 + r^2d\theta^2}{(1 - r_1^2)^2} + \frac{dr^2 + r^2d\theta^2}{(1 - r_2^2)^2} + \frac{r^2 + 2r_1r_2\cos(\tau_1 - \tau_2)d\phi^2 + \frac{2r_1r_2\cos(2\theta)d\phi^2 + \frac{2r_2\cos(2\theta)}{1 - r_2^2}dr_2d\phi}{(1 - r_2^2)^2} + \frac{r_2\cos(2\theta)}{1 - r_2^2}dr_2d\phi}{(1 - r_2^2)^2} + \frac{2(r_2^2 - r_2^2)^2\cos^2\theta}{(1 - r_2^2)^2}d\phi^2 + \frac{r_1r_2\sin(\tau_1 - \tau_2)\ln\theta}{(1 - r_1^2)(1 - r_2^2)}d\phi^2. \tag{34}$$

We notice that the first term in $ds^2$ is just the metric of a Poincaré disk. Hence, when $d\theta = 0$ and $d\phi = 0$, $ds^2$ is just the sum of the metrics of two Poincaré disks. We know that the geodesic connecting the origin and a point $z$ in the Poincaré disk is just radius through $z$. Hence, we can see that, in our case, the geodesic connecting the $0$ and $Z(r_1, r_2, \tau_1, \tau_2, \theta, \phi)$ should be given by two radii in the two Poincaré disks, whose parameter equation can be written $\{Z_t : Z_t = Z_t(f_{r_1}(t), f_{r_2}(t), \tau_1, \tau_2, \theta, \phi), 0 \leq t \leq t_1\}$, where $t$ is a parameter. And $f_{r_i}(t)$'s represent the two radii in the two Poincaré disks, and satisfy $0 \leq f_{r_1}(t) < 1, f_{r_2}(0) = 0, f_{r_2}(t_1) = r_i, i = 1, 2.$

It would be very interesting to find a Hamiltonian $H'$ that makes $\{Z_t = e^{-iH't}|0\}$ along the geodesic between the origin $0$ and $Z = Z(r_1, r_2, \tau_1, \tau_2, \theta, \phi)$, satisfying $Z_0 = Z$. Inspired by the similar problem in $SU(1, 1)$ case [8], we expect that this Hamiltonian is of the form $H'/U_{11} = \begin{pmatrix} 0 & iZ' \\ -iZ'^* & 0 \end{pmatrix}$, where $Z'$ is a $2 \times 2$ complex symmetric matrix in the GSP space, and its polar coordinate is $Z'(r_1', r_2', \tau_1', \tau_2', \theta', \phi')$. It can be numerically verified that under this Hamiltonian, $Z_t$ is given by $Z_t = Z_t(\text{tanh}(r_1'tU_{11}), \text{tanh}(r_2'tU_{11}), \tau_1', \tau_2', \theta', \phi')$, see Fig(3). Hence, if we let

$$r_1' = \frac{\text{arctanh}(r_1)}{t_1U_{11}}, \quad r_2' = \frac{\text{arctanh}(r_2)}{t_1U_{11}}, \quad (\tau_1', \tau_2', \theta', \phi') = (\tau_1, \tau_2, \theta, \phi), \tag{35}$$

$Z_t$ will go along the geodesic.

The geodesic connecting any two points $Z_1$ and $Z_2$ can be gotten by first mapping $Z_1$ to $0$, $Z_2$ to $Z_2'$ by $Z' = (UZ + V)(V^*Z + U^*)^{-1}$, and getting the geodesic connecting $0$ and $Z_2'$. Then mapping this geodesic back to get the geodesic connecting $Z_1$ and $Z_2$.

$$Z_t \text{ through the Cartan-Killing form at the origin, then mapping to other place through the group action [33]. Hence, we have established the relation }$$

$$ds^2 = -\frac{1}{2} \ln(F_{Z,Z+dz}). \tag{33}$$

V. DYNAMICS OF OBSERVABLE OVER COHERENT STATE

In this section, we will study the time evolution of a kind of physical observables $O$ which can be written as the linear combination of the Lie algebra generators of $\mathfrak{sp}(4, R)$. However, there is a problem with the $\mathfrak{sp}(4, R)$ generators Eq.(13), i.e. they are not orthogonal to each other, given the definition of the inner product of between two Lie algebra generators $A, B$ as $\text{Tr}A^\dagger B$ [33]. We need a set of $\mathfrak{sp}(4, R)$ generators which are orthogonal to each other. The matrix form of these generators can be defined.

Figure 3. Geodesic $Z_t = Z_t(f_{r_1}(t), f_{r_2}(t), \tau_1, \tau_2, \theta, \phi)$ connecting $0$ and $Z = \begin{pmatrix} 0.2 + 0.2i & 0.3i \\ 0.3i & 0.4 \end{pmatrix}$. (a) The light blue dots represent $0$, and the red dots represent $Z$. The geodesic is along the radium direction, the angle variables in the polar coordinate do not change. (b) Comparison between $f_{r_1}(t)$, $f_{r_2}(t)$ and $\text{tanh}(r_1'tU_{11}), \text{tanh}(r_2'tU_{11})$, with $r_1', r_2'$ defined as Eq.(35).
as
\[
\gamma_0 = \frac{1}{2} I \otimes I, \quad \gamma_1 = \frac{1}{2} I \otimes \sigma_y, \quad \gamma_2 = \frac{1}{2} \sigma_z \otimes I, \quad (36)
\]
\[
\gamma_3 = \frac{1}{2} \sigma_z \otimes \sigma_z, \quad \gamma_4 = \frac{1}{2} \sigma_z \otimes \sigma_z, \quad \gamma_5 = \frac{1}{2} i \sigma_x \otimes I, \quad (37)
\]
\[
\gamma_6 = \frac{1}{2} i \sigma_x \otimes \sigma_x, \quad \gamma_7 = \frac{1}{2} i \sigma_x \otimes \sigma_x, \quad \gamma_8 = \frac{1}{2} i \sigma_y \otimes I, \quad (38)
\]
\[
\gamma_9 = \frac{1}{2} i \sigma_y \otimes \sigma_x, \quad \gamma_{10} = \frac{1}{2} i \sigma_y \otimes \sigma_z,
\]
where $\sigma_{x,y,z}$ are Pauli matrices, and $I$ is $2 \times 2$ identity matrix. Notice that the way we choose this set of matrix generators is similar to that we can choose $\{\sigma_x, \sigma_y, \sigma_z/2\}$ as the generators of $\mathfrak{su}(2)$ Lie algebra. These matrix generators satisfy $\text{Tr}(\gamma_i \gamma_j) = \delta_{ij}$. And the corresponding operator representation of these matrices is given by $K_i = \Psi_k^\dagger \gamma_i \Psi_k$, which are all Hermitian. Meanwhile, $K_0$ is the Casimir operator. As a result, the operator $\hat{O}$ can be written as $\hat{O} = \sum_i C_i K_i$. The time evolution of the operator $\hat{O}$ is $\hat{O}(t) = e^{i H_k t} \hat{O} e^{-i H_k t} = \sum_i C_i e^{i H_k t} K_i e^{-i H_k t}$. Since $e^{-i H_k t}$ gives a representation of the $Sp(4, R)$ group, the time evolution of any $\mathfrak{sp}(4, R)$ generators $K_i$ is also a linear combination of $\mathfrak{sp}(4, R)$ generators,

\[
e^{i H_k t} K_i e^{-i H_k t} = \sum_j d_{ij}(t) K_j. \quad (39)
\]
Using representation theory, the factors $d_{ij}(t)$ can also be calculated by calculating the time evolution of the corresponding matrix $\gamma_i$ of the generators $K_i$,

\[
\gamma_i(t) = e^{i g H_k t} \gamma_i e^{-i g H_k t} = \sum_j d_{ij}(t) \gamma_j, \quad (40)
\]
where the $d_{ij}(t)$ are given by $d_{ij}(t) = \text{Tr}(\gamma_j^\dagger \cdot \gamma_i(t))$. As a result, the expectation value of $\hat{O}(t)$ over the vacuum is $\langle \hat{O}(t) \rangle = \sum_{ij} C_i d_{ij}(t) \langle K_j \rangle$.

If we want to calculate the expectation value of $\hat{O}(t)$ over a generic coherent state $\langle Z(\hat{O}(t)) | Z \rangle$, we need to find a Hamiltonian which gives $e^{-i H_k t} | 0 \rangle = | Z \rangle$. Actually, we don’t need to find $H_k$. Instead, we only need to find the group element corresponding to $e^{-i H_k t}$, which is of the form $e^{-i g H_0} = \left( \begin{array}{ccc} \mathcal{U} & \mathcal{V} \\ \mathcal{V}^* & \mathcal{U}^* \end{array} \right)$, where $\mathcal{U}, \mathcal{V}$ are $2 \times 2$ complex matrices satisfying $\mathcal{U} \mathcal{U}^\dagger = I$, $\mathcal{U} \mathcal{V}^\dagger = \mathcal{V} \mathcal{U}^\dagger$, and $\mathcal{V}(\mathcal{U}^\dagger)^{-1} = Z$. As a result, $\mathcal{U} \mathcal{U}^\dagger = (I - Z^\dagger Z)^{-1}$. Since $\mathcal{U} \mathcal{U}^\dagger$ is Hermitian, $(I - Z^\dagger Z)^{-1}$ can be diagonalized as $u \mathcal{D} u^\dagger$, where $D$ is diagonal matrix, and $u$ is unitary matrix. Hence, we can take $\mathcal{U} = u \sqrt{D}$, and $\mathcal{V} = Z \mathcal{U}^\dagger$.

Then the corresponding time evolution equation of $\gamma_i$ is

\[
\gamma_i'(t) = e^{i g H_0 t} e^{i g H_k t} \gamma_i e^{-i g H_k t} e^{-i g H_0 t} = \sum_j d'_{ij}(t) \gamma_j, \quad (41)
\]
where $d'_{ij}(t) = \text{Tr}(\gamma_j^\dagger \cdot \gamma_i(t))$. As a result, $\langle Z(\hat{O}(t)) | Z \rangle = \sum_{ij} C_i d'_{ij}(t) \langle K_j \rangle$. Furthermore, this method can also be used to calculate the expectation value of operators with four boson operators, which can be written as the quadratic form of the operator generators of the $\mathfrak{sp}(4, R)$ algebra. Hence, this method can also be used to calculate OTOC [23–25].

In Fig.4, we show our numerical results for the dynamics of the expectation value of number operator $n_{i, \pm k} = a_{i, \pm k}^\dagger a_{i, \pm k}$ using above method. In all these figures, the system starts from the vacuum state. In Fig.4(a), the system evolves under the decoupled Hamiltonian $H_1$. The number operator expectation value of each kind of boson oscillates according to their own period. In Fig.4(b), the system evolves under the decoupled Hamiltonian $H_2$ with two species of bosons with growth rates close to each other. In addition, in all the cases, the number operator grows exponentially for both species of bosons. But $\langle n_{i, \pm k} \rangle$ grows faster than $\langle n_{2, \pm k} \rangle$. In Fig.4(d), the system evolves according to a Hamiltonian with two imaginary eigen-energies, and the expectation value of number operator grows exponentially for both species of bosons with growth rates close to each other. In addition, in all the cases, the number operator grows exponentially for both species of bosons. But $\langle n_{i, \pm k} \rangle$ grows faster than $\langle n_{2, \pm k} \rangle$. We also compared the results for the number operator by using this method and by using the Taylor expansion method in Appendix B.
VI. CONCLUSION

In this paper, we provide a general method to study the ground state and the quantum dynamics of the two-component Bose-Einstein condensate system. We first demonstrated that the ground state of the two-component BEC for a generic momentum $k$ is given by a coherent state of $Sp(4, R)$, and can be parameterized as a point in a six dimensional manifold $Sp(4, R)/U(2)$. We then showed that the quantum dynamics of the system corresponds to trajectory in this six dimensional manifold by using the group action on this manifold. And finally, the group action on the operator also provides us a tool to calculate the expectation value of physical observables. In summary, we convert the calculation of time evolution operator to calculating the exponential of a matrix.

Throughout this paper, we demonstrate our formalism of quantum dynamics of two-component BEC for a single generic momentum. It is also very interesting to study the collective behavior in the momentum space, as well as the Floquet dynamics by periodic driven. We will address these investigations in future works. Since our geometrization method is complete general for two-component bosons, its applications are not restricted to BEC systems. Other two-component boson problem from high energy physics [34], quantum optics or quantum information [35] can also be studied by this method. Furthermore, our method can also be generalized to N-component bosonic system, the corresponding group of which is $Sp(2N, R)$.

ACKNOWLEDGMENTS

C.Y.W. thanks Tin-Lun Ho for valuable discussions. Y. H. was supported by the Natural Science Foundation of China under Grant No. 11874272 and Science Specialty Program of Sichuan University under Grant No. 2020SCUNL210.

Appendix A: calculation of the overlap between two coherent states

In this section, we will calculate the overlap between two coherent states $\langle Z'|Z \rangle = N'N\langle 0|e^{-\frac{1}{2}Z'^\dagger X^k}e^{-\frac{1}{2}Z_j X^j}|0 \rangle$. We seek to the decomposition

$$e^{-\frac{1}{2}Z'^\dagger X^k}e^{-\frac{1}{2}Z_j X^j} = e^{-\frac{1}{2}\mu_{ij} X^i}e^{\delta_{k}^i X^k}e^{-\frac{1}{2}\nu_{ij} X^j}.$$ (A1)

If such a decomposition exists, then we have $\langle Z'|Z \rangle = N'N\langle 0|e^{\delta_{k}^i X^k}|0 \rangle$. Furthermore, since $e^{\delta_{k}^i X^k} = e^{\sum_{k, i} \delta_{k}^i (a_{k, k} a_{k, k} + a_{k, -k} a_{-k, -k} + \delta_{k}^i)}$, we have $\langle Z'|Z \rangle = N'N\langle 0|e^{\mu_{ij} X^i}|0 \rangle$.

On the other hand, we also have the decomposition in the matrix level corresponding to Eq.(A1),

$$\begin{pmatrix} I & 0 \\ -Z^\dagger I \end{pmatrix} \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & \mu \\ 0 & I \end{pmatrix} \begin{pmatrix} (O^{-1})^T & 0 \\ 0 & O \end{pmatrix} \begin{pmatrix} I & 0 \\ \nu & I \end{pmatrix}. \quad (A2)$$

Here, the solution can be given by $O = I - Z^\dagger Z, \mu = ZO^{-1}, \nu = -O^{-1}Z^\dagger$. Meanwhile we have $(O^{-1})^T = e^\zeta$. Hence,

$$e^{\text{Tr} \zeta} = \det e^\zeta = \det[(O^{-1})^T] = (\det O)^{-1}. \quad (A3)$$

As a result, the overlap is

$$\langle Z'|Z \rangle = N'N \det(I - ZZ^\dagger)^{-1}. \quad (A4)$$

Taking $Z' = Z$, we have the normalization factor $N = \det(I - ZZ^\dagger)^{\frac{1}{2}}$. 
where we have defined that for non-negative integer $m$, function, and we have used its property that for non-negative integer $H$ evolving Hamiltonian is $H$.

**Appendix B: Taylor expansion of the coherent state**

The Taylor expansion of the coherent state is

$$|Z\rangle = N e^{\sum_{ij} Z_{ij} \phi^i \phi^j}$$

$$= N \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{1}{2} Z_{ij} \phi^i \phi^j)^k |0\rangle$$

$$= N \sum_{k=0}^{\infty} \sum_{m,n=0}^{k} \sqrt{m!(k-m)!n!(k-n)!} \sum_{q=0}^{\min(m,n)} \frac{Z_1^{m-q} Z_2^{n-q} Z_{12}^{k-m-n+q}}{q!(m-q)!(n-q)! \Gamma(k-m-n+q+1)} |m, k-m; n, k-n\rangle$$

$$= N \sum_{k=0}^{\infty} \sum_{m,n=0}^{k} \sqrt{(k-m)!(k-n)!} \frac{2F_1(-m,-n;k-m-n+1,z)}{m!n! \Gamma(k-m-n+1)} Z_1^m Z_2^n Z_{12}^{k-m-n} |m, k-m; n, k-n\rangle$$

where we have defined $z = \frac{Z_{11} Z_{22}}{Z_{12} Z_{21}}$, and the basis $|m, k-m; n, k-n\rangle = \frac{a_{1m}^{k-m}}{\sqrt{m!}} \frac{a_{2n}^{k-n}}{\sqrt{(k-m)!}} |0\rangle$, $\Gamma(n)$ is the gamma function, and we have used its property that for non-negative integer $n$, $\Gamma(n+1) = n!$, $\Gamma(-n) = \infty$. $2F_1(a; b; c; z)$ is the hypergeometric function [36]

$$2F_1(a; b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(q)_n = \{ 1, q(q+1) \cdots (q+n-1) \}$ and $n > 0$. And we also used the following property of the hypergeometric function that for non-negative integer $m$,

$$\lim_{c \rightarrow -m} \frac{2F_1(a; b; c; z)}{\Gamma(c)} = \frac{(a)_{m+1} (b)_{m+1}}{(m+1)!} z^{m+1} 2F_1(a + m + 1, b + m + 1; m + 1; z).$$

In Fig. 5, we compared the results of expectation value of the number operator by using the method in Sec.V and by using Taylor expansion method. In the Taylor expansion method, we take a cutoff of $k = 100$. We can see that these two methods agree very well with each other when the expectation value is small. However, at large value, the results using Taylor expansion method deviate from the method in Sec.V, due to the difficulties in accurately calculating the factorial of large numbers.

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