SINGLE-POINT BLOW-UP FOR A MULTI-COMPONENT REACTION-DIFFUSION SYSTEM

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Abstract. In this work, we prove single-point blow-up for any positive, radially decreasing, classical and blowing-up solution of a system of $m \geq 3$ heat equations in a ball of $\mathbb{R}^n$, which are coupled cyclically by superlinear monomial reaction terms. We also obtain lower pointwise estimates for the blow-up profiles.

1. Introduction. In this paper, we consider the following reaction-diffusion system:

$$
\begin{align*}
\frac{\partial u_i}{\partial t} - \delta_i \Delta u_i &= u_i^{p_i} - 1, \quad i = 1, \ldots, m, \quad u_{m+1} = u_1, \quad t > 0, \quad x \in \Omega, \\
u_i &= 0, \quad i = 1, \ldots, m, \quad t > 0, \quad x \in \partial \Omega, \\
\end{align*}
$$

where $u_i = u_i(t, x)$ and $\Omega = B(0, R) = \{ x \in \mathbb{R}^n \mid |x| < R \}$ with $n \in \mathbb{N}^*$ and $R > 0$. We assume throughout this paper that

$$
m \geq 3, \quad \Omega = B(0, R), \quad p_i > 1, \quad \delta_i > 0,
$$

and

$$
u_{i, 0} \in L^\infty(\Omega), \quad u_{i, 0} \geq 0, \quad \text{radially symmetric, radially nonincreasing}, \quad i = 1, \ldots, m.
$$

It is known that under the assumptions (2)–(3), the system (1) has a unique nonnegative, radially symmetric and radially nonincreasing maximal solution $(u_1, \ldots, u_m)$, classical for $t > 0$. This fact follows by standard contraction mapping and maximum principle arguments. The maximal existence time of $(u_1, \ldots, u_m)$ is denoted by $T^* \in (0, \infty]$. If, moreover, $T^* < \infty$, then

$$
\limsup_{t \to T^*} \sum_{i=1}^m \|u_i(t)\|_{\infty} = \infty.
$$

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and we say that the solution blows up in finite time with blow-up time $T^*$. Also, without risk of confusion, we shall denote $\rho = |x|$, $u_i = u_i(t, \rho)$, $i = 1, ..., m$. So we have

$$\partial_\rho u_i := \frac{\partial u_i}{\partial \rho} \leq 0, \quad i = 1, ..., m, \quad \text{on} \quad (0, T^*) \times \overline{\Omega}. \quad (4)$$

In the papers [12, 13], Renclawowicz was shown that there exist blowing-up solutions (i.e. $T^* < \infty$) of (1) for some values of the parameters $p_i$, $i = 1, ..., m$. Moreover, under the equidiffusivity assumption (i.e. $\delta_1 = \delta_2 = ... = \delta_m$), we know that monotone in time solutions i.e:

$$\partial_t u_i := \frac{\partial u_i}{\partial t} \geq 0, \quad i = 1, ..., m, \quad \text{on} \quad (0, T^*) \times \overline{\Omega}, \quad (5)$$

satisfy the following blow-up rate estimates:

$$c(T^* - t)^{-\alpha_i} \leq \|u_i(t)\|_\infty \leq C(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m, \quad \text{for all} \quad t \in (0, T^*), \quad (6)$$

for some positive constants $c$ and $C$, where

$$\alpha_i = \frac{1 + p_i + \sum_{l=i+1}^{m+i-2} p_i p_{i+1} ... p_l}{p_1 ... p_m - 1}, \quad i = 1, ..., m, \quad (7)$$

$p_{m+k} = p_k$ for all integer $k$, the standard scaling exponents of system (1) (see [2, 15]). We note also that (6) is still available without the equidiffusivity assumption for a very particular class of radial decreasing solutions (see [12]).

The first major topic of this paper is concerned with single-point blow-up for nonnegative, radially symmetric, radially nonincreasing, classical and blowing-up solution of (1). The blow-up set of solutions of nonlinear equations and systems has drawn wide interest since the eighties, when Weissler ([16]) has proved that blow-up occurs only at the point $x = 0$ for the scalar equation (i.e. $m = 1$) with $p_1$ is a large number and for a particular initial data. See [4, 10] for further results in the scalar case. As an extension of the method in [4], Friedman and Giga have proved single-point blow-up for radially decreasing solutions of (1) only for $n = 1$, $m = 2$, $\delta_1 = \delta_2$ and $p_1 = p_2$. More recently, Souplet ([14]) proved that single-point blow-up occurs for a large class of radial decreasing solutions, in a ball or in the whole space for $n \geq 2$ and without the restrictive condition $p_1 = p_2$. However, the equidiffusivity assumption $\delta_1 = \delta_2$ is still needed and, in addition, it is required that the solution satisfies the upper type I blow-up rate estimates (6). Recently, the author, Souplet and Tayachi ([8]) removed the previously made extra assumptions. More precisely, they improved the results in two directions:

(i) without assuming the type I blow-up rate estimate (6);
(ii) without assuming equidiffusivity, i.e. for any $\delta_1, \delta_2 > 0$.

The previous manuscript concerns a more general type of nonlinearity. As far as we know, the question of single-point blow-up has not been considered so far for parabolic systems with more than two components. We shall concentrate, for simplicity, on the model case (1). We obtain the following Theorem:

**Theorem 1.1.** Let $(u_1, ..., u_m)$ be a blowing-up solution of (1) under the assumptions given by (2) and (3). Then blow-up occurs only at the origin, i.e.

$$\sup_{0 \leq t < T^*} \sum_{i=1}^{m} u_i(t, \rho) < \infty, \quad \text{for all} \quad \rho \in (0, R). \quad (8)$$
The proof of Theorem 1.1 is long and technical and requires many intermediate steps. As mentioned above, we follow the general strategy developed in [8] for systems of two equations. However, in order to treat systems with an arbitrary number of components, some nontrivial modifications are required. In particular, Steps 5-6 of the proof of Proposition 3 lead to a weaker estimate on solutions than the one which was obtained in [8] in (essentially) the same steps of the same proposition for a two-component system. To exclude blow-up, an additional Proposition 4 is needed. There are a couple of additional differences between [8] and the present manuscript: the proof of (22) by induction and the use of (29) to obtain the upper bound required by Proposition 2.

Remark 1. The result of Theorem 1.1 remains true for the Cauchy problem (that is, (1) with $R = \infty$ and $\partial \Omega = \emptyset$) provided $u_{1,0}, \ldots, u_{m,0}$ are not all constant. This follows from straightforward modifications of the proof.

Finally, in the case of monotone in time solutions, we extend the lower pointwise estimates on the final blow-up profiles from [14] to the system (1).

Theorem 1.2. Let $(u_1, \ldots, u_m)$ be a blowing-up solution of (1) under the assumptions given by (2), (3) and (5). Then there exist constants $\varepsilon_0, \varepsilon_1 > 0$, such that

$$|x|^{2\alpha_i} u_i(T^*, x) \geq \varepsilon_0, \quad 0 < |x| < \varepsilon_1, \quad i = 1, \ldots, m,$$

where $\alpha_i, i = 1, \ldots, m$, are given by (7).

Note that the existence of a nonnegative, radially symmetric, radially nonincreasing and classical solution of (1) such that $T^* < \infty$ and $\partial_t u_i \geq 0, i = 1, \ldots, m$, can be obtained for initial data $(\lambda u_{1,0}, \ldots, \lambda u_{m,0})$, with $\lambda > 0$ large enough, whenever $(\lambda u_{1,0}, \ldots, \lambda u_{m,0})$ satisfies (3) and

$$
\begin{align*}
\delta_i \Delta u_{i,0} + u_{i+1,0}^{p_i} & \geq 0, & & \text{on } \Omega,
\end{align*}
$$

for all $i = 1, \ldots, m$.

The layout of the remaining parts of this paper is as follows. In Section 2, we prove Theorem 1.1 by a contradiction argument and assuming asymptotic comparison properties of all two components in a row near blow-up points (Proposition 1). Sections 3-6 are next devoted to proving these properties. In Section 3, we establish upper blow-up estimates away from the origin (Proposition 2). To do so, we take advantage of some useful linear algebraic properties related with the $m$ system (1), obtained in [2, 15]. In Section 4, we prove a local criterion for excluding blow-up (Proposition 3). The proof of this criterion presents additional difficulties as compared with the case $m = 2$ in [8]. Namely, we need to combine the analysis in similarity variables with a weaker, preliminary nondegeneracy property (Proposition 4) in original variables. In Section 5, we show the ODE behavior for rescaled solutions and the local interdependence of the components for the ODE system. In Section 6, we then prove the asymptotic comparison properties by using a contradiction argument and the results of Sections 3-5. Finally, in Section 7, we establish the pointwise lower bounds on the blow-up profiles (Theorem 1.2).

2. Proof of Theorem 1.1 assuming asymptotic comparison properties.

To prove the Theorem 1.1, we argue by contradiction, we assume that there exists $\rho_0 \in (0, R)$ such that
Under (9), we have the following asymptotic comparison properties:

**Proposition 1.** Under the hypotheses of Theorem 1.1. If there exists \( \rho_0 \in (0, R) \) such that (9) holds, then for all \([\rho_1, \rho_2]\) a compact subinterval of \((0, \rho_0)\), there exist constants \(C_1, C_2 > 0\) (possibly depending on the solution \((u_1, ..., u_m)\) and on \(\rho_0, \rho_1, \rho_2\)), such that for all \(i = 1, ..., m\)

\[
C_1 \leq (T^* - t)^{\alpha_i} u_i(t, \rho) \leq C_2 \quad \text{on} \quad [T^*/2, T^*) \times [\rho_1, \rho_2].
\]

In particular, there exist \(C_1', C_2' > 0\) such that for all \(i = 1, ..., m\)

\[
C_1' \leq \frac{u_i^{q_{i+1}}(t, \rho)}{u_{i+1}^q(t, \rho)} \leq C_2' \quad \text{on} \quad [T^*/2, T^*) \times [\rho_1, \rho_2],
\]

where

\[
q_i = 1 + p_i + \sum_{i=1+1}^{m+i-2} p_i, \quad p_i = \frac{1}{\left(\frac{\rho_2}{\rho_1} - 1\right)}.
\]

The Proposition 1 will be proved in Sections 3-6.

With Proposition 1 at hand, as in extension of the maximum principle approach developed in [8, 14, 7, 3, 4], we define the auxiliary functions \(J_i, i = 1, ..., m\), as follows:

\[
J_i(t, \rho) = \partial_{\rho} u_i + \varepsilon c_i(\rho) u_i^{\gamma_i},
\]

with

\[
c_i(\rho) = q_i \sin^2 \left(\frac{\pi (\rho - \rho_1)}{\rho_2 - \rho_1}\right), \quad \rho_1 \leq \rho \leq \rho_2,
\]

where \(q_i, i = 1, ..., m\), are given by (12), and \(\gamma_i > 1, \varepsilon > 0\) and \(\rho_2 > \rho_1 > 0\) are to be fixed later. Our goal is to show that the auxiliary functions \(J_i, i = 1, ..., m\), satisfying a system of parabolic inequalities (see Lemma 2.1 below). From a maximum principle, we deduce that \(J_i \leq 0, i = 1, ..., m\), on \([T_1, T^*) \times [\rho_0/4, \rho_0/2]\) for some \(T_1 \in (0, T^*)\). By integrating these inequalities, we obtain upper bounds on \(u_i, i = 1, ..., m\), away from \(\rho_0/4\). The fact that \((u_1, ..., u_m)\) is nonnegative, radially symmetric and radially nonincreasing solution of (1) rules out blow-up at \(\rho_0\). We note that \(J_i \in C((0, T^*) \times [0, R]) \cap W_{Loc}^{1,2,q}((0, T^*) \times [0, R])\), for all \(1 < q < \infty\) and \(i \in \{1, ..., m\}\), by parabolic \(L^q\)-regularity.

**Lemma 2.1.** Under the hypotheses of Theorem 1.1, let \(\rho_1 = \rho_0/4\) and \(\rho_2 = \rho_0/2\) where \(\rho_0\) is as (9). Then there exist \(\gamma_i > 1, i = 1, ..., m\), and \(T_i \in (0, T^*)\), such that, for any \(\varepsilon \in (0, 1]\), the functions \(J_i, i = 1, ..., m\), defined in (13)–(14) satisfy

\[
\partial_t J_i - \delta_i \partial_{\rho p} J_i - \delta_i \frac{n-1}{\rho} \partial_{\rho} J_i + \delta_i \frac{n-1}{\rho^2} J_i \leq p_i u_i^{p_i-1} J_i - 2\varepsilon \delta_i \gamma_i c_i^{\beta_i} u_i^{\gamma_i-1} J_i,
\]

for all \(i = 1, ..., m\) and \((t, \rho) \in [T_i, T^*) \times (\rho_1, \rho_2)\).

**Proof.** Let \(F_i) = u_i^{\gamma_i}\). By differentiation of (13), we obtain

\[
\partial_t J_i - \delta_i \partial_{\rho p} J_i = \partial_t (\partial_{\rho} u_i) + \varepsilon c_i \partial_t F_i - \delta_i \partial_{\rho p} u_i - \delta_i \varepsilon c_i^{\beta_i} F_i - 2\varepsilon \delta_i \gamma_i c_i^{\beta_i} \partial_{\rho} F_i - \delta_i \varepsilon c_i \partial_{\rho p} F_i = \partial_{\rho} (\partial_t u_i - \delta_i \partial_{\rho p} u_i) + \varepsilon \left(c_i (\partial_t F_i - \delta_i \partial_{\rho p} F_i) - 2\delta_i \gamma_i c_i^{\beta_i} \partial_{\rho} F_i - \delta_i c_i^{\beta_i} F_i\right).
\]
Using
\[
\partial_p (\partial_t u_i - \delta \partial_{\rho p} u_i) = \partial_p \left( \delta \frac{n-1}{\rho} \partial_p u_i + u_{i+1}^{p_i} \right) = \delta \frac{n-1}{\rho} \partial_{\rho p} u_i - \frac{n-1}{\rho^2} \partial_p u_i + p_i u_{i+1}^{p_i-1} \partial_p u_{i+1},
\]
\[
\partial_t F_i - \delta \partial_{\rho p} F_i = \gamma_i u_i^{\gamma_i-1} \partial_t u_i - \delta \gamma_i (\gamma_i - 1) u_i^{\gamma_i-2} (\partial_{\rho} u_i)^2 - \delta \gamma_i u_i^{\gamma_i-1} \partial_{\rho p} u_i \leq \gamma_i u_i^{\gamma_i-1} (\partial_t u_i - \delta \partial_{\rho p} u_i) = \gamma_i u_i^{\gamma_i-1} \left( \delta \frac{n-1}{\rho} \partial_p u_i + u_{i+1}^{p_i} \right).
\]
\[
\partial_p u_i = J_i - \varepsilon_c u_i^{\gamma_i} \quad \text{and} \quad \partial_{\rho p} u_{i+1} = J_{i+1} - \varepsilon_c u_{i+1}^{\gamma_i+1}, \quad \text{we obtain}
\]
\[
\partial_t J_i - \delta \partial_{\rho p} J_i \leq \delta \frac{n-1}{\rho} \partial_p (J_i - \varepsilon_c u_i^{\gamma_i}) - \delta \frac{n-1}{\rho^2} (J_i - \varepsilon_c u_i^{\gamma_i}) + p_i u_{i+1}^{p_i-1} (J_{i+1} - \varepsilon_c u_{i+1}^{\gamma_i+1}) + \varepsilon u_i^{\gamma_i-1} \left[ \gamma_i c_i \left( \delta \frac{n-1}{\rho} \partial_p u_i + u_{i+1}^{p_i} \right) - 2 \gamma_i \delta \partial_{\rho p} c_i u_i - \delta \varepsilon c_i^{\gamma_i} \partial_p u_i - \delta \frac{n-1}{\rho^2} J_i \right]
\]
\[
= \delta \frac{n-1}{\rho} \partial_p J_i - \delta \varepsilon c_i^{\gamma_i} \partial_p J_i - \delta \varepsilon c_i^{\gamma_i} \partial_{\rho p} J_i - \delta \varepsilon c_i^{\gamma_i} \partial_{\rho p} u_i - \delta \frac{n-1}{\rho^2} J_i + \delta \varepsilon c_i^{\gamma_i} \partial_{\rho p} u_i + p_i u_{i+1}^{p_i-1} (J_{i+1} - \varepsilon c_i^{\gamma_i+1}) + \varepsilon u_i^{\gamma_i-1} \left[ \gamma_i c_i \left( \delta \frac{n-1}{\rho} \partial_p u_i + u_{i+1}^{p_i} \right) - 2 \delta \gamma_i c_i^{\gamma_i} (J_i - \varepsilon_c u_i^{\gamma_i}) - \delta c_i^{\gamma_i} u_i \right].
\]
Consequently,
\[
\partial_t J_i - \delta \partial_{\rho p} J_i - \delta \frac{n-1}{\rho} \partial_p J_i + \delta \frac{n-1}{\rho^2} J_i \leq p_i u_{i+1}^{p_i-1} J_{i+1} - 2 \varepsilon c_i^{\gamma_i+1} u_i^{\gamma_i+1} J_i + \varepsilon H_i,
\]
with
\[
H_i := - \rho c_i^{\gamma_i+1} u_i^{\gamma_i+1} - u_i^{\gamma_i-1} \left[ \gamma_i c_i u_i^{p_i} + 2 \delta \varepsilon c_i^{\gamma_i} c_i^{\gamma_i} + \delta u_i \left( \frac{n-1}{\rho} \left( \frac{c_i}{c_i'} - c_i^{\gamma_i} \right) \right) \right].
\]
We have
\[
\tilde{H}_i := \frac{H_i}{c_i u_i^{\gamma_i-1} u_{i+1}^{p_i}} = - \rho c_i^{\gamma_i+1} u_i^{\gamma_i+1} + \gamma_i + 2 \delta \gamma_i c_i^{\gamma_i} u_i^{\gamma_i-1} + \delta \xi_i(\rho) \frac{u_i}{u_{i+1}^{p_i}},
\]
on \( (0, T^*) \times (\rho_1, \rho_2) \), with
\[
\xi_i(\rho) = \frac{n-1}{\rho} \left( 1 - \frac{c_i'}{c_i'} \right) - \frac{c_i'}{c_i'}, \quad \rho_1 < \rho < \rho_2.
\]
As in [8, pp. 1907–1908], it follows that there exists \( C_3 = C_3(n, \rho_0) > 0 \) such that
\[
\xi_i(\rho) \leq C_3, \quad \text{for all } \rho \in (\rho_1, \rho_2).
\]
By (17) and (18), for some \( C_4 = C_4(n, \rho_0) > 0 \), we have
\[
\tilde{H}_i \leq - \rho c_i^{\gamma_i+1} u_i^{\gamma_i+1} + \gamma_i + \delta \gamma_i c_i^{\gamma_i} u_i^{\gamma_i-1} + \delta \xi_i(\rho) \frac{u_i}{u_{i+1}^{p_i}}.
\]
We choose \( \gamma_1 \) such that

\[
1 < \gamma_1 < p_i \frac{q_i}{q_i-1}
\]

and set

\[
\gamma_{i+1} = 1 + \frac{q_i}{q_i+1} (\gamma_i - 1), \quad i = 1, \ldots, m,
\]

where \( q_i, i = 1, \ldots, m, \) are given by (12), which, in turn, guarantees

\[
1 < \gamma_i < p_i \frac{q_i+1}{q_i-1}, \quad i = 1, \ldots, m.
\]

Indeed, by recurrence if \( 1 < \gamma_i < p_i \frac{q_i+1}{q_i-1}, \) then by (21), (7) and the fact that \( \alpha_i + 1 = p_i \alpha_{i+1}, \) it follows that

\[
1 < \gamma_{i+1} = 1 + \frac{q_i}{q_i+1} (\gamma_i - 1) < 1 + \frac{q_i}{q_i+1} \left( \frac{p_i}{q_i-1} - 1 \right) = 1 + p_i - \frac{q_i}{q_i+1} = \frac{\alpha_i + 1 - p_i \alpha_{i+1} - \alpha_i}{\alpha_{i+1}} = \frac{\alpha_{i+1} + 1 + p_i \alpha_{i+1} - \alpha_i}{\alpha_{i+1}} = p_i + \frac{\alpha_{i+1} + 1}{\alpha_{i+1}} = p_i + \frac{q_i+2}{q_i+1}.
\]

Let the constants \( C_1, C_2 > 0 \) be given by Proposition 1. By (10), (21) and (7), we have

\[
\left( C_1 C_2 \frac{q_i}{q_i+1} \right)^{\gamma_i - 1} \leq C_1^{\gamma_i - 1} u_i^{\gamma_i - 1} \leq C_2^{\gamma_i - 1} u_i^{\gamma_i - 1} \leq \left( C_2 C_1 \frac{q_i}{q_i+1} \right)^{\gamma_i - 1}
\]

on \( [T^*/2, T^*) \times (\rho_1, \rho_2) \).

Taking \( \gamma_i > 1, i = 1, \ldots, m, \) close enough to 1, we deduce from (21)–(23) and (12), that there exist \( \eta > 0 \) and \( T_0 \in (T^*/2, T^*) \) such that

\[
-\frac{p_i q_i}{q_i+1} \frac{u_i^{q_i+1}}{u_i^{q_i-1}} + \gamma_i \leq -\eta,
\]

for all \( T_0 < t < T^* \) and \( \rho_1 < \rho < \rho_2 \). Moreover, by (11), (21) and (22), it follows that there exists \( T_1 \in (T_0, T^*) \) such that

\[
C_4 \delta_i \gamma_i \frac{u_i^{q_i}}{u_i^{q_i+1}} + \delta_i C_3 \frac{u_i}{u_i^{q_i+1}} \leq \eta,
\]

for all \( T_1 < t < T^* \) and \( \rho_1 < \rho < \rho_2 \). Therefore, by (19), (24) and (25), we deduce that

\[
\tilde{H}_2 \leq 0, \quad i = 1, \ldots, m \quad \text{on} \ [T_1, T^*) \times (\rho_1, \rho_2)
\]

and the lemma follows from (16).

With Proposition 1 and Lemma 2.1 at hand, we can now conclude the proof of Theorem 1.1.
Proof of Theorem 1.1. Let \((u_1, \ldots, u_m)\) be a solution of system (1) satisfying the hypotheses of Theorem 1.1 and assume for contradiction that there exists \(\rho_0 \in (0, R)\) such that

\[
\limsup_{t \to T^*} \sum_{i=1}^{m} u_i(t, \rho_0) = \infty. \tag{26}
\]

Also, since \((u_1, \ldots, u_m) \not\equiv (0, \ldots, 0)\), it is easy to see that \(u_1, \ldots, u_m > 0\) in \((0, T^*) \times (0, R)\), hence \(\partial_{\rho} u_i(t, \cdot) \not\equiv 0\), for \(i = 1, \ldots, m\), for each \(t \in (0, T^*)\). Next, let \(i \in \{1; \ldots; m\}\), we have \(\partial_t u_i - \delta_i \partial_{\rho_i} u_i - \delta \frac{\alpha_i + 1}{\rho} \partial_{\rho} u_i = f_i(t, \rho)\) on \((0, T^*) \times (0, R)\), with \(f_i(t, \rho) = u_i^{\rho_i + 1}\). Since \(\partial_{\rho} u_i \leq 0\), a strong maximum principle guarantees

\[
\partial_{\rho} u_i < 0, \quad i = 1, \ldots, m \quad \text{on} \quad (0, T^*) \times (0, R). \tag{27}
\]

Set \(\rho_1 = \rho_0/4, \rho_2 = \rho_0/2\) and let \(J_i, T_1\) be given by Lemma 2.1. Since \(c_i(\rho_1) = c_i(\rho_2) = 0\), we have \(J_i \leq 0\), \(i = 1, \ldots, m\), on \(((T_1, T^*) \times \{\rho_1\}) \cup ((T_1, T^*) \times \{\rho_2\})\).

Taking \(\varepsilon > 0\) sufficiently small and using (27) we see that \(J_i \leq 0\) on \(\{T_1\} \times [\rho_1, \rho_2]\). Then, by a maximum principle, we obtain \(J_i \leq 0\) on \((T_1, T^*) \times [\rho_1, \rho_2]\).

Consequently,

\[-\partial_{\rho} u_i \geq \varepsilon c_i(\rho) u_i^{\gamma_i}, \quad i = 1, \ldots, m \quad \text{on} \quad (T_1, T^*) \times [\rho_1, \rho_2].\]

By integration, we obtain

\[u_i^{1-\gamma_i}(t, \rho_2) \geq q_i(\gamma_i - 1)\varepsilon \int_{\rho_1}^{\rho_2} \sin^2 \left(\frac{\pi(\rho - \rho_1)}{\rho_2 - \rho_1}\right) d\rho > 0, \quad i = 1, \ldots, m,\]

for all \(T_1 \leq t < T^*\). It follows that \(u_i(t, \rho_2)\) is bounded for \(i = 1, \ldots, m\) and \(T_1 \leq t < T^*\). Since \(\partial_{\rho} u_i \leq 0\), for all \(i = 1, \ldots, m\), this leads to a contradiction with (26) and proves the theorem. \(\square\)

3. Upper type I estimates away from the origin. In this section, we prove the upper type I estimates away from the origin for nonnegative, radially symmetric, nonlinearly decreasing and classical solutions of (1).

Proposition 2. Under the hypotheses of Theorem 1.1. Then, there exists a constant \(C_0 > 0\) (depending only on \(n, \rho_i, \delta_i, R, T^*\)) such that

\[u_i(t, \rho) \leq C_0 \rho^{-\alpha_i} (T^* - t)^{-\alpha_i}, \quad i = 1, \ldots, m, \tag{28}\]

for all \(t \in (0, T^*)\) and \(0 < \rho \leq R\).

Proof. Since our domain is bounded, then we denote by \(\lambda_1\) the first eigenvalue of \(-\Delta\) in \(H^1_0(B(0, R))\) and \(\varphi_1\) the corresponding eigenfunction such that \(\varphi_1 > 0\) and \(\int_{B(0, R)} \varphi_1(x) dx = 1\). Let \(t \in (0, T^*)\) and \(i \in \{1; \ldots; m\}\), multiplying the equation \(i\) in (1) by \(\varphi_1\) and integrating by parts, we obtain

\[
\frac{d}{dt} \int_{B(0, R)} u_i(t, x) \varphi_1(x) dx = \int_{B(0, R)} u_i^{\rho_i + 1}(t, x) \varphi_1(x) dx - \delta_i \lambda_1 \int_{B(0, R)} u_i(t, x) \varphi_1(x) dx.
\]

Let \(y_i(t) = \int_{B(0, R)} u_i(t, x) \varphi_1(x) dx, \quad i = 1, \ldots, m\). Using Jensen’s inequality, we obtain

\[y_i'(t) \geq y_i^{\rho_i + 1}(t) - \delta_i \lambda_1 y_i(t), \quad i = 1, \ldots, m.\]
We put $Y_i(t) = e^{\delta_i t} y_i(t)$, $i = 1, ..., m$. Then, there exists $M > 0$ such that
\[ Y'_i(t) \geq MY_{i+1}^p(t), \quad i = 1, ..., m, \quad Y_{m+1} = Y_1. \]
Here and in the rest of the proof, $M$ denotes a positive constant depending only on $T^*$, $\delta_i$, $p_i$, $n$, $R$ and which may vary from line to line. By [15, Lemma 4, p. 579], there exists $M$ such that
\[ (Y_1 \ldots Y_m)'(t) \geq M(Y_1 \ldots Y_m)^r(t), \quad \text{where } r = \frac{1 + \alpha_1 + \ldots + \alpha_m}{\alpha_1 + \ldots + \alpha_m}. \]
(29)
Since $(Y_1 \ldots Y_m)(t) = \infty$, integrating (29) from $t$ to $T^*$ yields
\[ (Y_1 \ldots Y_m)(t) \leq M(T^* - t)^{-1/(r-1)} = M(T^* - t)^{-(\alpha_1 + \ldots + \alpha_m)}. \]
(30)
Using (30) and similarly as in [15, pp. 580–582], we obtain
\[ y_i(t) \leq M(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m \quad \text{on } [0, T^*), \]
where $\alpha_i$ are given by (7). Therefore,
\[ y_i(t) \leq M(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m \quad \text{on } [0, T^*). \]
For $0 < \rho < R/2$, since $u_i$, $i = 1, ..., m$, are radially symmetric and radially nonincreasing, we deduce that
\[ \rho^n u_i(t, \rho) \leq M \int_{B(0, R/2)} u(t, |x|)dx \leq M \int_{B(0, R/2)} u_i(t, |x|) \varphi_1(x)dx \leq M(T^* - t)^{-\alpha_i}. \]
The case when $R/2 < \rho < R$ then follows from the radial nonincreasing property. This completes the proof. \( \square \)

4. A local criterion for excluding blow-up. In this section, we prove a sufficient, local smallness condition, at any given time sufficiently close to $T^*$, for excluding blow-up at a given point $a_0 \neq 0$.

**Proposition 3.** Under the hypotheses of Theorem 1.1. Let $a_0, a_1$ be such that $0 < a_1 < a_0 < R$. There exist $\eta, \tau_0 > 0$ (depend only on $p_i, \delta_i, a_0, a_1, n, R, T^*$) such that if, for some $t_1 \in [T^* - \tau_0, T^*)$, we have
\[ (T^* - t_1)^{\alpha_i} u_i(t_1, a_1) \leq \eta, \quad i = 1, ..., m, \]
then $a_0$ is not a blow-up point of $(u_1, ..., u_m)$, i.e. $(u_1, ..., u_m)$ is uniformly bounded in the neighborhood of $(T^*, a_0)$.

The proposition will be proved in Subsection 4.2. As in [8], the proof uses similarity variables, delayed smoothing effects (see Subsection 4.1 below), and upper type I estimates away from the origin (Proposition 3). However, a new difficulty arises, caused by the fact that some algebraic properties for the system of 2–equations may fail for general systems. For this reason, we prove a preliminary result (Proposition 4 below) adapted from [11, Theorem 25.3, p. 195], which gives a sufficient smallness condition on nonnegative solution of (1) at a neighborhood of $(T^*, a)$ for excluding blow-up at a given point $a \in \mathbb{R}^n$. 
Proposition 4. Let \( a \in \mathbb{R}^n, \rho, T > 0 \), let \((u_1, ..., u_m)\) be a classical solution of (1) in \( Q = (T - \rho^2, T) \times B(a, \rho) \). There exists \( \varepsilon(n, p_1, ..., p_m) > 0 \) such that if \((u_1, ..., u_m)\) satisfies
\[
(T - t)^{\alpha_i}u_i(t, x) \leq \varepsilon, \quad i = 1, ..., m, \quad (t, x) \in Q, \tag{32}
\]
then \((u_1, ..., u_m)\) is uniformly bounded in the neighborhood of \((T, a)\).

Proof. Let \( \varepsilon \in (0, 1) \). By a space-time translation, we may assume that \( a = 0 \) and \( T = \rho^2 \). By scaling, we may also assume that \( \rho = 1 \). Indeed \( \bar{u}_i(t, x) := \rho^{2\alpha_i}u_i(\rho^2t, \rho x), i = 1, ..., m \), solve (1) in \((0, 1) \times B(0, 1)\) and (32) is equivalent to
\[
\bar{u}_i(t, x) \leq \varepsilon(1 - t)^{-\alpha_i}, i = 1, ..., m, \quad \text{for all} \quad (t, x) \in (0, 1) \times B(0, 1).
\]
Let \( \alpha = \max(\alpha_1, ..., \alpha_m) \). We put
\[
v_i = u_i^{\frac{1}{\alpha}}, \quad i = 1, ..., m.
\]
Then, for all \( i = 1, ..., m \), we have
\[
\partial_t v_i - \delta_i \Delta v_i = \frac{\alpha_i}{\alpha} u_i^{\frac{1}{\alpha} - 1} (\partial_t u_i - \delta_i \Delta u_i) - \delta_i \frac{\alpha_i}{\alpha} - 1 u_i^{\frac{1}{\alpha} - 2} |\nabla u_i|^2. \tag{33}
\]
Since \( \frac{\alpha_i}{\alpha} - 1 \geq 0 \), and using (1), we obtain
\[
\partial_t v_i - \delta_i \Delta v_i \leq \frac{\alpha_i}{\alpha} u_i^{\frac{1}{\alpha} - 1} v_i^{p_i + 1}, \quad i = 1, ..., m. \tag{34}
\]
If \( p_i \geq \frac{\alpha_i}{\alpha_i + 1} \), by (32) and using the fact that \( \alpha_i + 1 = p_i \alpha_i + 1 \), we obtain
\[
\partial_t v_i - \delta_i \Delta v_i \leq \frac{\alpha_i}{\alpha} v_i^{\frac{1}{\alpha} - 1 + p_i - \frac{\alpha_i}{\alpha_i + 1}} (1 - t)^{-1} u_i^{\frac{\alpha_i}{\alpha_i + 1}}.
\]
If \( p_i < \frac{\alpha_i}{\alpha_i + 1} \), then \( \alpha - \alpha_i - 1 = \alpha - p_i \alpha_i + 1 > 0 \) and (34) can be see as the following inequality
\[
\partial_t v_i - \delta_i \Delta v_i \leq \frac{\alpha_i}{\alpha} v_i^{\frac{1}{\alpha} - 1} u_i^{\frac{\alpha_i}{\alpha} - 1} u_i^{p_i + 1}, \quad i = 1, ..., m. \tag{35}
\]
By (32) and Young’s inequality, we obtain
\[
\partial_t v_i - \delta_i \Delta v_i \leq \frac{\alpha_i}{\alpha} v_i^{\frac{1}{\alpha} - 1} (1 - t)^{-1} \left( \frac{\alpha - \alpha_i - 1}{\alpha} u_i^{\frac{\alpha_i}{\alpha}} + \frac{\alpha_i + 1}{\alpha} u_i^{p_i + 1} \right).
\]

Therefore, it follows that
\[
\partial_t v_i - \delta_i \Delta v_i \leq K \varepsilon^{\gamma_i} (1 - t)^{-1} \left( u_i^{\frac{\alpha_i}{\alpha}} + u_i^{\frac{\alpha}{4\alpha}} \right), \quad i = 1, ..., m, \tag{36}
\]
where \( K \) denotes a positive constant which may vary from line to line and
\[
\gamma_i = \begin{cases} \frac{\alpha_i}{\alpha} + 1 - p_i - \frac{\alpha_i}{\alpha_i + 1}, & \text{if} \quad p_i \geq \frac{\alpha_i}{\alpha_i + 1}, \\ \frac{1}{\alpha_i}, & \text{if} \quad p_i < \frac{\alpha_i}{\alpha_i + 1}. \end{cases}
\]

It is easy to see that \( \gamma_i > 0 \). Taking \( \gamma = \min_i \gamma_i > 0 \).

Then, we put
\[
\beta = \min \left( \frac{1}{2}, \frac{1}{4\alpha} \right). \tag{37}
\]
We recall that for each $R > 0$, we may find $\phi \in C^2(\mathbb{R}^n)$ such that
\begin{equation}
\phi(x) = 0 \quad \text{for } |x| \geq R/\sqrt{2}, \quad \phi(x) \geq 1 \quad \text{for } |x| \leq R/2
\end{equation}
and
\begin{equation}
|\nabla \phi|^2 + |\Delta \phi|^2 \leq C(R, n)\phi^{2(1-\beta)}, \quad \forall \, x \in \mathbb{R}^n.
\end{equation}
See [11, p. 196].

Choose $R = 1$ and put
\begin{equation*}
W_i = v_i^2\phi^2, \quad i = 1, \ldots, m.
\end{equation*}
Assuming $0 < \varepsilon < 1$, by (36), we have
\begin{align*}
\partial_t W_i - \delta_i \Delta W_i & \leq K\varepsilon^\gamma (1-t)^{-1} (v_i^2 \phi^2 + v_i \phi v_{i+1} \phi) - 2\delta_i \phi^2 |\nabla v_i|^2 - 8\delta_i v_i \phi \nabla v_i \nabla \phi - \delta_i v_i^2 \Delta \phi^2, \\
& \leq K\varepsilon^\gamma (1-t)^{-1} (v_i^2 \phi^2 + v_i^2 \phi^2) - 2\delta_i \phi^2 |\nabla v_i|^2 - 8\delta_i v_i \phi \nabla v_i \nabla \phi - \delta_i v_i^2 \Delta \phi^2, \\
& = K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) - 2\delta_i \phi^2 |\nabla v_i|^2 - 8\delta_i v_i \phi \nabla v_i \nabla \phi - \delta_i v_i^2 \Delta \phi^2,
\end{align*}
for all $(t, x) \in (0, 1) \times B(0, 1)$ and $i = 1, \ldots, m$. Since $4|v_i \phi \nabla v_i \phi| \leq \phi^2 |\nabla v_i|^2 + 4v_i^2 |\nabla \phi|^2$, it follows that
\begin{align*}
\partial_t W_i - \delta_i \Delta W_i & \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + 8\delta_i v_i^2 (|\nabla \phi|^2 + |\Delta \phi|^2) \\
& \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + 8\delta_i v_i^2 \phi^{2(1-\beta)} \\
& \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + 8\delta_i v_i^2 \phi^{2(1-\beta)} \\
& \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + 8\delta_i C v_i^2 \phi^{1-\beta} \\
& \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + 8\delta_i C v_i^2 \phi^{1+\beta}.
\end{align*}
Then by (32) and (37), it follows that
\begin{align*}
\partial_t W_i - \delta_i \Delta W_i & \leq K\varepsilon^\gamma (1-t)^{-1} (W_i + W_{i+1}) + K\varepsilon^{2\alpha/\gamma} (1-t)^{-1/2} + K\varepsilon^{2\alpha/\gamma} (1-t)^{-1/2} W_i.
\end{align*}

Since $W_i = 0$ on $(0, 1) \times \partial B(0, 1)$, from the variation of constants formula, we prove that
\begin{align*}
\|W_i(t)\|_\infty & \leq \|W_i(1/2)\|_\infty + K\varepsilon^\gamma \int_{1/2}^t (1-s)^{-1} \left( \|W_i(s)\|_\infty + \|W_{i+1}(s)\|_\infty \right) ds \\
& + K\varepsilon^{2\alpha/\gamma} \int_{1/2}^t (1-s)^{-1/2} ds + K\varepsilon^{2\alpha/\gamma} \int_{1/2}^t (1-s)^{-1/2} \|W_i(s)\|_\infty ds.
\end{align*}
We put
\begin{equation*}
W = \sum_{i=1}^m W_i.
\end{equation*}
We obtain
\begin{align*}
\|W(t)\|_\infty & \leq \|W(1/2)\|_\infty + K\varepsilon^\gamma \int_{1/2}^t (1-s)^{-1} \|W(s)\|_\infty ds \\
& + K\varepsilon^{2\beta} \int_{1/2}^t (1-s)^{-1/2} ds + K\varepsilon^{2\beta} \int_{1/2}^t (1-s)^{-1/2} \|W(s)\|_\infty ds \\
& \leq K + K\varepsilon^\lambda \int_{1/2}^t (1-s)^{-1} \|W(s)\|_\infty ds,
\end{align*}
where \( \lambda = \min(\gamma, 2\beta) > 0 \). From Gronwall’s lemma, we deduce that
\[
\|W(t)\|_\infty \leq K \exp \left( K \varepsilon^\lambda \int_{1/2}^t (1 - s)^{-1} \, ds \right) \leq K (1 - t)^{-K \varepsilon^\lambda},
\]
for all \( 1/2 < t < 1 \). \( \square \)

Then, it follows that for all \( i = 1, \ldots, m, \)
\[
u_i(t, x) \leq K (1 - t)^{-K \varepsilon^\lambda \frac{\alpha}{\alpha}} \quad \text{for all } |x| < 1/2 \text{ and } 1/2 < t < 1.
\]

Taking \( R = 1/2 \) instead of \( R = 1 \) in (38), by similar calculations as for (36) but with (42) instead of (32), it follows that
\[
\partial_t v_i - \delta_i \Delta v_i \leq K (1 - t)^{-K \varepsilon^\lambda \frac{\alpha}{\alpha}} \left( \frac{\alpha}{\alpha} u_i + u_{i+1} \right), \quad i = 1, \ldots, m.
\]

As before, we obtain
\[
\|W(t)\|_\infty \leq K \exp \left( K \int_{1/2}^t (1 - s)^{-K \varepsilon^\lambda \frac{\alpha}{\alpha}} \, ds \right), \quad \text{for all } 1/2 < t < 1.
\]

Taking a smaller \( \varepsilon > 0 \) such that \( K \varepsilon^\lambda \frac{\alpha}{\alpha} < \frac{1}{2} \), it follows that
\[
\|W(t)\|_\infty \leq K \exp \left( K \int_{1/2}^t (1 - s)^{-1/2} \, ds \right) \leq K \exp \left( -2K (1 - t)^{1/2} \right),
\]
for all \( 1/2 < t < 1 \). We see that \( u_i, i = 1, \ldots, m \) are bounded in a neighborhood of \((1, 0)\).

4.1. **Similarity variables and delayed smoothing effects.** In this subsection, for any given \( a \in \mathbb{R} \), we define the (one-dimensional) similarity variables around \((T^*, a)\), associated with \((t, \rho) \in (0, T^*) \times \mathbb{R} \), by:
\[
\sigma = -\log(T^* - t) \in [\hat{\sigma}, \infty), \quad \theta = \frac{\rho - a}{\sqrt{T^* - t}} = e^{\sigma/2}(\rho - a) \in \mathbb{R},
\]
where \( \hat{\sigma} = -\log T^* \) (cf. [5]). For given \( \delta > 0 \), let \( U \) be a (strong) solution of
\[
\partial_t U - \delta \partial_{\rho\rho} U = H(t, \rho), \quad 0 < t < T^*, \quad \rho \in \mathbb{R},
\]
where \( H \in L^\infty_{loc}([0, T), L^\infty(\mathbb{R})) \) is a given function. Then
\[
W = W_a(\sigma, \theta) = (T^* - t)^{\sigma} U(t, y) = e^{-\sigma \theta} U(T^* - \sigma, a + \theta e^{-\sigma/2})
\]
is a solution of
\[
W_\sigma - \mathcal{L}_\delta W + \alpha W = e^{-(\alpha+1)\sigma} H(T^* - \sigma, a + \theta e^{-\sigma/2}), \quad \sigma > \hat{\sigma}, \quad \theta \in \mathbb{R},
\]
where
\[
\mathcal{L}_\delta = \delta \frac{\partial^2}{\partial \theta^2} - \frac{\theta}{2} \partial_\theta = \delta K_\delta^{-1} \partial_\theta (K_\delta \partial_\theta), \quad K_\delta(\theta) = (4\pi\delta)^{-1/2} e^{-\theta^2/4\delta}.
\]
We denote by \((T_\delta(\sigma))_{\sigma \geq 0}\) the semigroup associated with \( \mathcal{L}_\delta \). More precisely, for each \( \phi \in L^\infty(\mathbb{R}) \), we set \( T_\delta(\sigma) \phi := w(\sigma, \cdot) \), where \( w \) is the unique solution of
\[
\begin{cases}
  w_\sigma = \mathcal{L}_\delta w, & \theta \in \mathbb{R}, \quad \sigma > 0, \\
  w(0, \theta) = \phi(\theta), & \theta \in \mathbb{R}.
\end{cases}
\]
For any \( \phi \in L^\infty(\mathbb{R}) \), we put
\[
\|\phi\|_{L^q_{K_\delta}} = \left( \int_{\mathbb{R}} |\phi(\theta)|^q K_\delta(\theta) \, d\theta \right)^{1/q}, \quad 1 \leq q < \infty.
\]
Let $1 \leq r < q < \infty$ and $\delta > 0$, then, by Jensen’s inequality,
\[
\|\phi\|_{L^q_{\rho_k}} \leq \|\phi\|_{L^q_{\rho_k}}^\delta, \quad 1 \leq r < q < \infty.
\] (47)

The semigroups $(T_\delta(\sigma))_{\sigma \geq 0}$ have the following properties (cf. [8]), which will be useful when dealing with system (1) with unequal diffusivities:

**Lemma 4.1.** 1. (Contraction) For any $1 \leq q < \infty$, we have
\[
\|T_\delta(\sigma)\phi\|_{L^q_{\rho_k}} \leq \|\phi\|_{L^q_{\rho_k}}, \quad \text{for all } \delta > 0, \quad \sigma \geq 0, \quad \phi \in L^\infty(\mathbb{R}).
\] (48)
Moreover, for all $0 < \delta \leq \lambda < \infty$, we have
\[
\|T_\delta(\sigma)\phi\|_{L^q_{\rho_k}} \leq \left(\frac{\lambda}{\delta}\right)^{1/2} \|\phi\|_{L^q_{\rho_k}}, \quad \text{for all } \sigma \geq 0, \quad \phi \in L^\infty(\mathbb{R}).
\] (49)

2. (Delayed regularizing effect) For any $1 \leq r < q < \infty$, there exist $\hat{C}, \sigma^* > 0$ such that
\[
\|T_\delta(\sigma)\phi\|_{L^q_{\rho_k}} \leq \hat{C}\|\phi\|_{L^q_{\rho_k}}, \quad \text{for all } \delta > 0, \quad \sigma \geq \sigma^*, \quad \phi \in L^\infty(\mathbb{R}).
\] (50)
Moreover, for all $0 < \delta \leq \lambda < \infty$, we have
\[
\|T_\delta(\sigma)\phi\|_{L^q_{\rho_k}} \leq \hat{C}\left(\frac{\lambda}{\delta}\right)^{1/2} \|\phi\|_{L^q_{\rho_k}}, \quad \text{for all } \sigma \geq \sigma^*, \quad \phi \in L^\infty(\mathbb{R}).
\] (51)

4.2. **Proof of Proposition 3.** As in [8], we split the proof in several steps. Without loss of generality, we assume $a_1 \geq a_i, \ i = 1, ..., m.$

**Step 1. Definition of suitably modified solutions.** By (28) in Proposition 2, we know that
\[
(T^* - t)^{\alpha_i}u_i(t, y) \leq N_0, \quad i = 1, ..., m, \quad 0 \leq t < T^*, \quad a_1 \leq y < R,
\] (52)
with $N_0 = C_0 a_1^{-n}$. We shall thus truncate the radial domain and consider suitably controlled extensions of the solution to the real line. We first define the following extensions $\bar{u}_i \geq 0$ of $u_i, \ i = 1, ..., m,$ by setting:
\[
\bar{u}_i(t, y) := \begin{cases} u_i(t, y), & y \in [a_1, R], \\ 0, & y \in \mathbb{R} \setminus [a_1, R] \end{cases}, \quad \text{for any } t \in [0, T^*). \] (53)

Next, let $M \geq N_0$ to be chosen below. For given $t_0 \in [0, T^*)$, let $(\bar{u}_1, ..., \bar{u}_m) = (\bar{u}_1(t_0, \cdot), ..., \bar{u}_m(t_0, \cdot))$ be the solution of the following auxiliary problem:
\[
\begin{align*}
\partial_t \bar{u}_i - \bar{\delta}_{ij} \partial_{yy} \bar{u}_i &= \bar{u}_{i+1}^1, \quad i = 1, ..., m, \quad \bar{u}_{m+1} = u_1 \quad t_0 < t < T^*, \quad y \geq a_1, \\
\bar{u}_i(t, a_1) &= M(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m, \quad t_0 < t < T^*, \\
\bar{u}_i(t_0, y) &= \bar{u}_i(t_0, y), \quad i = 1, ..., m, \quad y \geq a_1.
\end{align*}
\] (54)

It is clear that $\bar{u}_1, ..., \bar{u}_m \geq 0$ exist on $[t_0, T^*) \times [a_1, \infty).$ Also, using (52) and $M \geq N_0$, we deduce from the maximum principle that
\[
\bar{u}_i \leq \bar{u}_i, \quad i = 1, ..., m \quad \text{on } [t_0, T^*) \times [a_1, \infty). \] (55)

Now choosing
\[
M = \max\left(N_0, \alpha_1^{-1}N_0^{p_1}, ..., \alpha_m^{-1}N_0^{p_m}\right). \] (56)

We may thus use $(M(T^* - t)^{-\alpha_i}, ..., M(T^* - t)^{-\alpha_m})$ as a supersolution of the inhomogeneous, linear heat equation in (54), verified by $\bar{u}_i, \ i = 1, ..., m,$ on $[t_0, T^*) \times [a_1, \infty),$ and infer from the maximum principle that
\[
0 \leq \bar{u}_i \leq M(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m, \quad \text{on } [t_0, T^*) \times [a_1, \infty). \] (57)
We next extend \((\overline{w}_1, ..., \overline{w}_m)\) by odd reflection for \(y < a_1\), i.e., we set:
\[
\overline{w}_i(t, a_1 - y) = 2M(T^* - t)^{-\alpha_i} - \overline{w}_i(t, a_1 + y), \quad i = 1, ..., m, \quad t_0 \leq t < T^*, \quad y > 0.
\]
(58)
From (57), along with (55) and (53), we have
\[
0 \leq \overline{w}_i \leq 2M(T^* - t)^{-\alpha_i}, \quad i = 1, ..., m, \quad \text{on } [t_0, T^*) \times \mathbb{R}
\]
and
\[
\overline{u}_i \leq \overline{w}_i, \quad i = 1, ..., m, \quad \text{on } [t_0, T^*) \times \mathbb{R}.
\]
(59)
(60)
On the other hand, by (54) and (58), we see that \((\overline{u}_1, ..., \overline{u}_m)\) is a solution of
\[
\partial_t \overline{u}_i - \delta_i \partial_{yy} \overline{u}_i = F_i(t, y), \quad i = 1, ..., m, \quad t_0 < t < T^*, \quad y \in \mathbb{R},
\]
(61)
where
\[
F_i(t, y) := \begin{cases}
2\alpha_i M(T^* - t)^{-\alpha_i} - \overline{w}_{i+1}^p(t, 2a_1 - y), & y < a_1, \\
\overline{w}_{i+1}^p(t, y), & y \geq a_1.
\end{cases}
\]
(62)
Since \(F_i \in L^q_t((t_0, T^*) \times \mathbb{R})\), for all \(1 < q < \infty\) and \(i \in \{1, ..., m\}\), by parabolic \(L^q\)-regularity, it follows that \((\overline{u}_1, ..., \overline{u}_m)\) is a strong solution of (61) (Not a classical solution as mentioned in [8] for \(m = 2\)).

**Step 2.** Self-similar rescaling of modified solutions. We now fix \(a \in (a_1, a_0)\) (say, \(a = (a_0 + a_1)/2\)) and pass to self-similar variables \((\sigma, \theta)\) around \((T^*, a)\), cf. (44). In these variables, we first define the rescaled solution \((\tilde{w}_1, ..., \tilde{w}_m) = (\tilde{w}_{1,a}, ..., \tilde{w}_{m,a})\), associated with the extended solution \((\overline{u}_1, ..., \overline{u}_m)\), namely,
\[
\tilde{w}_i(\sigma, \theta) = (T^* - t)^{\alpha_i} \overline{w}_i(t, y), \quad i = 1, ..., m, \quad \hat{\sigma} \leq \sigma < \infty, \quad \theta \in \mathbb{R},
\]
(63)
where \(\hat{\sigma} = -\log T^*\). For given \(t_0 \in [0, T^*)\) (cf. Step 1), we also define \((\overline{w}_1, ..., \overline{w}_m) = (\overline{w}_{1,a}(\sigma_0; \cdot, \cdot), ..., \overline{w}_{m,a}(\sigma_0; \cdot, \cdot))\), associated with the modified solution \((\overline{u}_1, ..., \overline{u}_m)\) on \([t_0, \infty)\), where \((\tilde{w}_1, ..., \tilde{w}_m)\) does not. Actually, in Step 3, the \((\overline{w}_1, ..., \overline{w}_m)\) will be used as auxiliary functions in order to establish suitable estimates on \((\tilde{w}_1, ..., \tilde{w}_m)\) itself.

Set \(\ell = a - a_1 > 0\). Owing to (59), (60), we have
\[
\tilde{w}_i \leq \overline{w}_i \leq 2M, \quad i = 1, ..., m, \quad \text{on } [\sigma_0, \infty) \times \mathbb{R}
\]
(65)
and, for all \(\sigma \geq \hat{\sigma}\),
\[
\theta \rightarrow \tilde{w}_i(\sigma, \theta), \quad i = 1, ..., m, \quad \text{are nonincreasing for } \theta \in [-\ell e^{\sigma/2}, \infty),
\]
(66)
due to (4). Then, using (45), (62) and \(\alpha_i + 1 = p_i \alpha_{i+1}\), we see that \((\overline{w}_1, ..., \overline{w}_m)\) is a solution of
\[
\partial_\sigma \overline{w} - L_\delta \overline{w} + \alpha_i \overline{w}_i = G_i(\sigma, \theta), \quad i = 1, ..., m, \quad \sigma_0 < \sigma < \infty, \quad \theta \in \mathbb{R},
\]
(67)
where
\[
G_i(\sigma, \theta) := e^{-(\alpha_i + 1)\sigma} F_i(T^* - e^{-\sigma}, a + \theta e^{-\sigma/2}) \leq \tilde{w}_{i+1}^p(\sigma) + 2\alpha_i M \chi_{\{\theta < -\ell e^{\sigma/2}\}}.
\]
(68)
Also, using the last two conditions in (54), along with (63), (64) and (65), we see that
\[
\overline{w}_i(\sigma_0) \leq \tilde{w}_i(\sigma_0) + 2M \chi_{\{\theta < -\ell e^{\sigma_0/2}\}}, \quad i = 1, ..., m.
\]
(69)
In the next steps, we shall estimate \((\tilde{w}_1, \ldots, \tilde{w}_m)\) by using semigroup and delayed smoothing arguments.

**Step 3. First semigroup estimates for \((\tilde{w}_1, \ldots, \tilde{w}_m)\).** We claim that, for all \(\sigma_0 \geq \hat{\sigma}\) and \(\sigma > 0\), we have

\[
\tilde{w}_1(\sigma_0 + \sigma) \\
\leq e^{-\alpha_1 \sigma} T_{\hat{\sigma}}(\sigma) \left[ \tilde{w}_1(\sigma_0) + 2M \chi_{\{ \theta < -\varepsilon e^{\sigma_0/2} \}} \right] + \int_0^\sigma e^{-\alpha_i(\sigma-\tau)} T_{\hat{\sigma}}(\sigma-\tau) \tilde{w}_{i+1}(\sigma_0 + \tau) \, d\tau \\
+ 2\alpha_i M \int_0^\sigma e^{-\alpha_i(\sigma-\tau)} T_{\hat{\sigma}}(\sigma-\tau) \chi_{\{ \theta < -\varepsilon e^{(\sigma_0+\tau)/2} \}} \, d\tau \tag{70}
\]

and that, moreover,

\[
\sum_{i=1}^m \tilde{w}_i(\sigma_0 + \sigma) \leq e^{M_1 \sigma} S(\sigma) \left[ \sum_{i=1}^m \tilde{w}_i(\sigma_0) + 2mM \chi_{\{ \theta < -\varepsilon e^{\sigma_0/2} \}} \right] + 2\alpha_1 M \int_0^\sigma e^{M_1(\sigma-\tau)} S(\sigma-\tau) \chi_{\{ \theta < -\varepsilon e^{(\sigma_0+\tau)/2} \}} \, d\tau, \tag{71}
\]

where

\[
(S(\sigma))_{\sigma \geq 0} = \sum_{i=1}^m T_{\hat{\sigma}}(\sigma) \text{ and } M_1 = \max((2M)^{p_1-1}, \ldots, (2M)^{p_m-1}).
\]

Let us first verify (70). We fix \(\sigma_0 \geq \hat{\sigma}\) and consider \(\bar{w}_i = \bar{w}_{i,a}(\sigma_0; \cdot, \cdot), i = 1, \ldots, m, \)

defined in (64) with \(\sigma_0 = -\log(T^* - t_0)\). We use (67) and the variation of constants formula to write

\[
\bar{w}_i(\sigma_0 + \sigma) = e^{-\alpha_1 \sigma} T_{\hat{\sigma}}(\sigma) \bar{w}_i(\sigma_0) + \int_0^\sigma e^{-\alpha_i(\sigma-\tau)} T_{\hat{\sigma}}(\sigma-\tau) G_i(\sigma_0 + \tau, \cdot) \, d\tau
\]

for all \(\sigma > 0\), hence, by (68),

\[
\bar{w}_i(\sigma_0 + \sigma) \leq e^{-\alpha_1 \sigma} T_{\hat{\sigma}}(\sigma) \bar{w}_i(\sigma_0) + 2\alpha_i M \int_0^\sigma e^{-\alpha_i(\sigma-\tau)} T_{\hat{\sigma}}(\sigma-\tau) \chi_{\{ \theta < -\varepsilon e^{(\sigma_0+\tau)/2} \}} \, d\tau \\
+ \int_0^\sigma e^{-\alpha_i(\sigma-\tau)} T_{\hat{\sigma}}(\sigma-\tau) \bar{w}_{i+1}(\sigma_0 + \tau) \, d\tau. \tag{72}
\]

Inequality (70) then follows from (72), (69) and (65).

To verify (71), we set \(H := \sum_{i=1}^m \bar{w}_i\). Adding up (72), and recalling \(\alpha_1 \geq \alpha_i\), we easily get

\[
H(\sigma_0 + \sigma) \leq S(\sigma) H(\sigma_0) + \int_0^\sigma S(\sigma - \tau) \left[ M_1 H(\sigma_0 + \tau) + D(\tau) \right] \, d\tau, \quad \sigma \geq 0, \tag{73}
\]

where

\[
D(\tau, \cdot) = 2\alpha_1 M \chi_{\{ \theta < -\varepsilon e^{(\sigma_0+\tau)/2} \}}, \quad \tau \geq 0.
\]

Set

\[
\tilde{H}(\sigma_0 + \sigma) := e^{M_1 \sigma} S(\sigma) H(\sigma_0) + \int_0^\sigma e^{M_1(\sigma-\tau)} S(\sigma-\tau) D(\tau) \, d\tau, \quad \sigma \geq 0. \tag{74}
\]
By direct computation, using the semigroup properties of \((S(\sigma))_{\sigma \geq 0}\) and Fubini’s theorem, we see that
\[
\hat{H}(\sigma_0 + \sigma) = S(\sigma)H(\sigma_0) + \int_0^\sigma S(\sigma - \tau) [M_1 \hat{H}(\sigma_0 + \tau) + D(\tau)] \, d\tau, \quad \sigma > 0. \tag{75}
\]
Combining (73), (75) and using the positivity-preserving property of \((S(\sigma))_{\sigma \geq 0}\), we obtain
\[
[H - \hat{H}]_+(\sigma_0 + \sigma) \leq M_1 \int_0^\sigma S(\sigma - \tau)[H - \hat{H}]_+(\sigma_0 + \tau) \, d\tau, \quad \sigma > 0. \tag{76}
\]
Letting now \(\delta = \max(\delta_1, \ldots, \delta_m, 1)\) and \(K = K_\delta\), we deduce from (49) in Lemma 4.1 that
\[
\|S(\sigma)\phi\|_{L^q_K} \leq \tilde{C}\|\phi\|_{L^q_K}, \quad \sigma \geq 0, \quad \phi \in L^\infty(\mathbb{R}), \quad 1 \leq q < \infty, \tag{77}
\]
with \(\tilde{C} = \tilde{C}(\delta_1, \ldots, \delta_m) \geq 1\). Therefore, it follows from (76) that
\[
\|H - \hat{H}\|_+(\sigma_0 + \sigma) \leq \tilde{C}M_1 \int_0^\sigma \|H - \hat{H}\|_+(\sigma_0 + \tau) \, d\tau, \quad \sigma > 0,
\]
and we infer from Gronwall’s Lemma that \(H(\sigma_0 + \sigma) \leq \hat{H}(\sigma_0 + \sigma)\) for all \(\sigma \geq 0\). Inequality (71) then follows from (65) and (69).

**Step 4. Small time estimate of rescaled solutions.** At this point, we set, as before, \(\delta = \max(\delta_1, \ldots, \delta_m, 1)\) and \(K = K_\delta\), and we fix
\[
q > \max(p_1, \ldots, p_m) \tag{78}
\]
and let \(\sigma^*\) be given by Lemma 4.1(2), with \(r = 1\). We note that, by Lemma 4.1, we have
\[
\|S(\sigma)\phi\|_{L^q_K} \leq \tilde{C}_0\|\phi\|_{L^q_K}, \quad \sigma \geq \sigma^*, \quad \phi \in L^\infty(\mathbb{R}), \tag{79}
\]
with \(\tilde{C}_0 = \tilde{C}_0(p_i, \delta_i) \geq 1\). Also, by (77), we have
\[
\|S(\sigma)\chi_{\{\theta < -A\}}\|_{L^q_K} \leq \tilde{C}\|\chi_{\{\theta < -A\}}\|_{L^q_K} = \tilde{C} \left((4\pi)^{-1/2} \int_{-\infty}^{-A} \exp\left(-\frac{\theta^2}{4\delta}\right) \, d\theta\right)^{1/r} \leq \tilde{C}C_0 \exp(-(8r\delta)^{-1}A^2), \quad \text{for all } A > 0 \text{ and } 1 \leq r \leq q, \tag{80}
\]
with \(C_0 = C_0(p_i, \delta) \geq 1\).

Let \(\eta > 0\). We claim that there exists \(\tau_1 \in (0, T^*)\), depending only on \(\eta\) and on the parameters
\[
p_i, \delta_i, m, a_0, a_1, n, R, T^*, \tag{81}
\]
such that:

For any \(t_1 \in [T^* - \tau_1, T^*)\) satisfying (31) and \(\sigma_1 = -\log(T^* - t_1)\), we have
\[
\sum_{i=1}^{m} \|\tilde{w}_i(\sigma_1 + \sigma)\|_{L^q_K} \leq \tilde{C}_1 \eta, \quad 0 < \sigma \leq \sigma^*, \tag{82}
\]
with \(\tilde{C}_1 = (m + 1)\tilde{C}e^{M_1}\sigma^* > 0\).

To prove the claim, we choose \(\sigma_0 = \sigma_1\) in (71). Observe that, by assumption (31) and owing to (4), we have \(\tilde{w}_i(\sigma_1, \cdot) \leq \eta\) on \(\mathbb{R}\), hence
\[
\sum_{i=1}^{m} \|\tilde{w}_i(\sigma_1)\|_{L^q_K} \leq m\eta. \tag{83}
\]
Using (71), (77), (80), (83), \( e^{\sigma_1} = (T^* - t_1)^{-1} \geq \tau_1^{-1} \) and assuming \( \tau_1 < 1 \), we deduce that, for \( 0 \leq \sigma \leq \sigma^* \),

\[
\sum_{i=1}^{m} \| \tilde{w}_i(\sigma + \sigma^* + \tau) \|_{L^k_h} \leq m \tilde{C} e^{M_1 \sigma^*} \eta + 2 m \tilde{C} C_0 M e^{M_1 \sigma^*} \exp(-8 \delta \tau_1)^{-1} \ell^2 \\ + 2 \alpha_1 C_0 \tilde{C} \sigma^* Me^{M_1 \sigma^*} \exp(-8 \delta \tau_1)^{-1} \ell^2 \\ \leq \tilde{C} e^{M_1 \sigma^*} \left[ m \eta + 2 C_0 M (\alpha_1 \sigma^* + m) \exp(-8 \delta \tau_1)^{-1} \ell^2 \right].
\]

For \( \tau_1 \in (0, T^*) \) sufficiently small, depending only on \( \eta \) and on the parameters in (81), we finally get (82) with \( \tilde{C}_1 = (m + 1) \tilde{C} e^{M_1 \sigma^*} \).

**Step 5. Large time estimate of rescaled solutions.** We claim that there exists \( \eta_0 > 0 \) depending only on the parameters in (81), such that: for any \( \eta \in (0, \eta_0) \), there exists \( \tau_0 \in (0, \tau_1(\eta)) \), such that for any \( t_1 \in [T^* - \tau_0, T^*) \) satisfying (31), we have

\[
\mathcal{A}_{\eta, t_1} = (0, \infty),
\]

where \( \sigma_1 = -\log(T^* - t_1) \) and

\[
\mathcal{A}_{\eta, t_1} = \left\{ \sigma > 0 : \sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \sigma^* + \tau) \|_{L^k_h} \leq 2 \tilde{C} \tilde{C}_1 \eta, \; \tau \in [0, \sigma] \right\}.
\]

First observe that \( \mathcal{A}_{\eta, t_1} \neq \emptyset \), due to (82) and the continuity of the function \( \sigma \mapsto \sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \sigma^* + \sigma) \|_{L^k_h} \). We denote

\[
\mathcal{T} = \sup \mathcal{A}_{\eta, t_1} \in (0, \infty].
\]

Assume for contradiction that \( \mathcal{T} < \infty \). Then by (82), we have

\[
\sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \sigma^* + \sigma) \|_{L^k_h} \leq 2 \tilde{C} \tilde{C}_1 \eta, \quad \sigma^* \leq \sigma \leq \mathcal{T}.
\]

For \( 0 \leq \tau \leq \mathcal{T} \), we apply (71) with \( \sigma_0 = \sigma_1 + \tau \) and \( \sigma = \sigma^* \). Using (77), (79), (80), (83), (85), \( e^{\sigma_1} = (T^* - t_1)^{-1} \geq \tau_0^{-1} \) and assuming \( \tau_0 < 1 \), we get

\[
\sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \sigma^* + \tau) \|_{L^k_h} \leq m \tilde{C}_0 e^{M_1 \sigma^*} \left( \sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \tau) \|_{L^k_h} \right) \\ + 2 m^2 \tilde{C}_0 M e^{M_1 \sigma^*} \exp(-8 \delta \tau_0)^{-1} \ell^2 e^\tau \\ + 2 m \alpha_1 M \tilde{C}_0 C M e^{M_1 \sigma^*} \exp(-8 \delta \tau_0)^{-1} \ell^2 e^\tau \\ \leq 2 m \tilde{C}_1 \tilde{C}_0 \tilde{C} e^{M_1 \sigma^*} \eta + 2 m \tilde{C}_0 \tilde{C}(m + \alpha_1 \sigma^*) M e^{M_1 \sigma^*} \exp(-8 \delta \tau_0)^{-1} \ell^2 e^\tau.
\]

Put \( \tilde{C}_2 = (2m + 1) \tilde{C}_1 \tilde{C}_0 \tilde{C} e^{M_1 \sigma^*} \). For \( \tau_0 \in (0, \tau_1(\eta)) \) sufficiently small, depending only on \( \eta \) and on the parameters in (81), it follows that

\[
\sum_{i=1}^{m} \| \tilde{w}_i(\sigma_1 + \sigma^* + \tau) \|_{L^k_h} \leq \tilde{C}_2 \eta, \quad 0 \leq \tau \leq \mathcal{T}.
\]
Next let $0 < \sigma \leq \bar{T}$. Now using (70) with $\sigma_0 = \sigma_1 + \sigma^*$, (77), (80), $T_\delta_i(\sigma) \leq S(\sigma)$ and $e^{\alpha_i} \geq \tau_0^{-1}$, we obtain
\[
\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} \leq \|T_\delta_i(\sigma)\|_{L_K^1} + 2M\|T_\delta_i(\sigma)\chi_{\{\theta < -\ell e^{(\sigma_1 + \sigma^*)/2}\}}\|_{L_K^1} + \int_0^\sigma e^{\alpha_i(\tau - \sigma)}\|T_\delta_i(\sigma - \tau)\|_{L_K^1} d\tau + 2\alpha_i M \int_0^\sigma e^{\alpha_i(\tau - \sigma)}\|T_\delta_i(\sigma - \tau)\chi_{\{\theta < -\ell e^{(\sigma_1 + \sigma^* + \tau)/2}\}}\|_{L_K^1} d\tau,
\]
hence,
\[
\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} \leq \tilde{C}\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} + 2\tilde{C}_0 M \exp\left((-8\delta \tau_0^{-1}\ell^2\right) + \tilde{C} \int_0^\sigma e^{\alpha_i(\tau - \sigma)}\|w_{i+1}(\sigma_1 + \sigma^* + \tau)\|_{L_K^1} d\tau + 2\alpha_i \tilde{C} C_0 M \int_0^\sigma \exp\left((-8\delta \tau_0^{-1}\ell^2 e^\tau\right) d\tau.
\]
By taking $\tau_0$ possibly smaller (dependence as above), we may ensure that
\[
2\tilde{C} C_0 M \exp\left((-8\delta \tau_0^{-1}\ell^2\right) + 2\alpha_i \tilde{C} C_0 M \int_0^\sigma \exp\left((-8\delta \tau_0^{-1}\ell^2 e^\tau\right) d\tau \leq \eta^2,
\]
hence,
\[
\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} \leq \tilde{C}\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} + \eta^2 + \tilde{C} \int_0^\sigma e^{\alpha_i(\tau - \sigma)}\|w_{i+1}(\sigma_1 + \sigma^* + \tau)\|_{L_K^1} d\tau.
\]
Then by (78), (47) and (86), we obtain
\[
\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} \leq \tilde{C}\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} + \eta^2 + \tilde{C}(\tilde{C}_2 \eta)^{p_i} \int_0^\sigma e^{\alpha_i(\tau - \sigma)} d\tau.
\]
Therefore
\[
\|w_i(\sigma_1 + \sigma^*)\|_{L_K^1} \leq \tilde{C}\|w(\sigma_1 + \sigma^*)\|_{L_K^1} + \eta^2 + \tilde{C}\tilde{C}_2 \eta^{p_i} \omega_i.
\]
Finally, for $\sigma = \bar{T}$ in (88), by definition of $\bar{T}$ and by using (82) with $\sigma = \sigma^*$, we obtain
\[
2\tilde{C} \bar{T}_1 \eta = \sum_{i=1}^m \|w_i(\sigma_1 + \sigma^* + \bar{T})\|_{L_K^1} \leq \tilde{C} \sum_{i=1}^m \|w(\sigma_1 + \sigma^*)\|_{L_K^1} + m\eta^2 + \sum_{i=1}^m \tilde{C} \tilde{C}_2 \eta^{p_i} \omega_i + C_4 \left[\eta^2 + \sum_{i=1}^m \eta^{p_i}\right],
\]
hence
\[
\tilde{C} \bar{T}_1 \eta \leq C_4 \left[\eta + \sum_{i=1}^m \eta^{p_i-1}\right],
\]
where $C_4 > 0$ depends on the parameters in (81).

Since $p_i > 1$ for all $i = 1, \ldots, m$, for $\eta \in (0, \eta_0]$ with $\eta_0$ sufficiently small, we reach a contradiction. Consequently, $\bar{T} = \infty$ and the claim is proved.

**Step 6. Conclusion.** Let $\eta_0$ be as in Step 5. Let $\eta \in (0, \eta_0)$, we know that there exists $\tau_0 \in (0, \tau_1(\eta))$, such that for any $t_1 \in [T^* - \tau_0, T^*)$ satisfying (31), we have
\[ A_{\eta, t_1} = (0, \infty). \] It follows from the definition of \( A_{\eta, t_1} \) that
\[ \| \tilde{w}_i(\sigma) \|_{L^1_K} \leq C\eta, \quad \text{for all } 1 \leq i \leq m. \] (89)

Set \( L := \int_{-1}^{0} K(\theta) \, d\theta > 0. \) For all \( t \in (T^* - \ell^2, T^*), \) recalling (44), we have \( \ell e^{\sigma/2} \geq 1, \) hence
\[ \tilde{w}_i(\sigma, 0) \leq L^{-1} \int_{-1}^{0} \tilde{w}_i(\sigma, \theta)K(\theta) \, d\theta, \quad 1 \leq i \leq m. \] (90)

to (66). Let then \( \hat{t}_1 = T^* - \min(t^2, e^{-(\sigma_i + \sigma^*)}). \) It follows from (53), (63), (89), (90) that, for all \( t \in [\hat{t}_1, T^*), \)
\[ \sum_{i=1}^{m} (T^* - t)^{\alpha_i} u_i(t, a) = \sum_{i=1}^{m} \tilde{w}_i(\sigma, 0) \leq 2L^{-1} \sum_{i=1}^{m} \| \tilde{w}_i(\sigma) \|_{L^1_K} \leq 2mCL^{-1}\eta. \]

Therefore, it follows that
\[ u_i(t, a) \leq C\eta(T^* - t)^{-\alpha_i}, \quad \text{for all } t \in [\hat{t}_1, T^*). \]

Taking \( \eta \) small in such a way as to apply Proposition 4, it follows that \( u_i, i = 1, \ldots, m, \) are uniformly bounded in the neighborhood of \((a_0, T^*). \)

5. Convergence of rescaled solutions to solutions of a system of ordinary differential equalities. We note that Proposition 3 is not yet sufficient to establish the lower type I estimates of Proposition 1. Indeed, Proposition 3 shows no blow-up at a given point, assuming that all components violate the type I estimate. Therefore, in order to improve from Proposition 3 to Proposition 1, we need to show a suitable interdependence of the components (under the assumption of blow-up at a point different from the origin). In order to do so, for given \( \rho_1 \in (0, R), \) we again switch to similarity variables around \((T^*, \rho_1), \) already used in the previous section. Namely, we set:

\[ \sigma = -\log(T^* - t), \quad \theta = \frac{\rho - \rho_1}{\sqrt{T^* - t}} = e^{\sigma/2}(\rho - \rho_1), \] (91)

and consider the rescaled solution \((W_1, \ldots, W_m) = (W_{1, \rho_1}, \ldots, W_{m, \rho_1})\) associated with \((u_1, \ldots, u_m):\)

\[ W_i(\sigma, \theta) = (T^* - t)^{\alpha_i} u_i(t, \rho), \] (92)

defined for \( \sigma \in [\hat{\sigma}, \infty) \) with \( \hat{\sigma} = -\log T^* \) and \( \theta \in (-\rho_1 e^{\sigma/2}, (R - \rho_1)e^{\sigma/2}). \)

The goal of this section is to show that any such rescaled solution \((W_1, \ldots, W_m)\) behaves, in a suitable sense as \( \sigma \to \infty \) and \( \theta \to \infty, \) like a (distribution) solution of the following system of ordinary differential equalities:

\[ \phi_i' + \alpha_i \phi_i = \phi_{i+1}^{\rho_i}, \quad i = 1, \ldots, m, \quad \phi_{m+1} = \phi_1. \] (93)

on the whole real line \((-\infty, \infty)\) (however, we shall eventually only use the fact that \((\phi_1, \ldots, \phi_m)\) solves (93) on some bounded open interval). Moreover, we single out a simple but crucial property of local interdependence of components for solutions of (93).

**Proposition 5.** Under the hypotheses of Theorem 1.1. Let \( \rho_1 \in (0, R) \) and let \((W_1, \ldots, W_m)\) be defined by (91)-(92).

(i) Then, for all sequence \( \sigma_j \to \infty, \) there exists a subsequence (not relabeled) such that, for each \( \sigma \in \mathbb{R}, \)

\[ \phi_i(\sigma) = \lim_{\theta \to \infty} \left( \lim_{j \to \infty} W_i(\sigma + \sigma_j, \theta) \right), \] (94)
exists and is finite for all \(i = 1, \ldots, m\), where the limits in \(j\) are uniform for \((\sigma, \theta)\) in bounded subsets of \(\mathbb{R} \times \mathbb{R}\), and the limits in \(\theta\) are monotone nonincreasing.

(ii) The functions \(\phi_i, i = 1, \ldots, m\), defined in (94) belong to \(BC(\mathbb{R})\) and \((\phi_1, \ldots, \phi_m)\) is a nonnegative solution in \(D'(\mathbb{R})\) of system (93).

(iii) If \(\phi_i(0) = 0\) for some \(i \in \{1; \ldots; m\}\), then \(\phi_i(0) = \ldots = \phi_m(0) = 0\).

Proof. (i) Let \(A = \min(p_1/2, R - p_1) > 0\). By (28), we have that

\[
(W_1, \ldots, W_m) \text{ is bounded on the set } D = \{(\sigma, \theta) \in \mathbb{R} \times \mathbb{R}, \sigma > \sigma_0, |\theta| \leq Ae^{\sigma'/2}\}
\]

and \((W_1, \ldots, W_m)\) solves the system

\[
\partial_\sigma W_i - \delta_i \partial_\theta W_i + \left[ \frac{\bar{w}}{2} - \delta_i \left( \frac{\left( n-1 \right)e^{-\sigma/2}}{\rho_i + \theta e^{-\sigma/2}} \right) \right] \partial_\theta W_i + \alpha_i W_i = W_i^{p_i} \quad \text{on } D,
\]

\[
(96)
\]

\(i = 1, \ldots, m\) and \(W_{m+1} = W_1\).

For each compact \(Q\) of \(\mathbb{R} \times \mathbb{R}\), the sequences \((W_i^Q)\) for all \(i = 1, \ldots, m\) are defined on \(Q\) for \(j\) large enough and, owing to (95), they are bounded in \(L^\vartheta(Q)\) for each \(q \in (1, \infty)\). Therefore, by (96) and parabolic estimates (see, e.g. [11, p.438]), the sequences \((W_i^Q)\) are bounded in \(W^{1;2;\vartheta}(Q)\) for each compact \(Q\) of \(\mathbb{R} \times \mathbb{R}\) and each \(q \in (1, \infty)\). Fixing \(\alpha \in (0, 1)\) and using the compact embeddings \(W^{1;2;\vartheta}(Q) \subset C^{\alpha,1+\alpha/2}(Q)\) for \(q\) large, we deduce that, for some subsequence (not relabeled), \((W_1^Q, \ldots, W_m^Q)\) converges, in \(C^{\alpha,1+\alpha/2}(Q)\) for each compact \(Q\) of \(\mathbb{R} \times \mathbb{R}\), to some \(i = 1, \ldots, m\) for each \(q \in (1, \infty)\).

Moreover, since \(\partial_\mu u_i \leq 0\) by (4), we have \(\partial_\theta W_{i,j} \leq 0\) on \(D\) and therefore, for each \(\sigma \in \mathbb{R}\),

\[
\mathbb{R} \ni \theta \mapsto w_i(\sigma, \theta), i = 1, \ldots, m, \text{ are nonincreasing.}
\]

Since \(w_i, i = 1, \ldots, m, \text{ are bounded and nonincreasing, we may define}

\[
\phi_i(\sigma) = \lim_{\theta \to +\infty} w_i(\sigma, \theta),
\]

which proves assertion (i).

(ii) We first observe that the properties of the sequence obtained in the previous paragraph allow us to pass to the limit in the distribution sense in (96), it follows in particular that \((w_1, \ldots, w_m)\) is a (continuous bounded) solution of

\[
\partial_\sigma w_i - \delta_i \partial_\theta w_i + \frac{\bar{w}}{2} \partial_\theta w_i + \alpha_i w_i = w_i^{p_i}, \quad i = 1, \ldots, m, \quad w_{m+1} = w_1 \quad \text{on } D'(\mathbb{R}^2).
\]

\[
(98)
\]

We can then obtain (93) by a standard argument based on multiplication by test functions. However, in order to avoid dealing with the terms \(\frac{\bar{w}}{2} \partial_\theta w_i, i = 1, \ldots, m\), in the passage to the limit, it is convenient not to work in the current similarity variables. Thus put

\[
U_i(t, x) := (T^* - t)^{-\alpha_i} w_i \left( -\log(T^* - t), \frac{\sqrt{T^* - t - 1}}{T^* - t - 1} \right), \quad i = 1, \ldots, m,
\]

\[
(99)
\]

for \(x \in \mathbb{R}\) and \(-\infty < t < T^*\). We observe that

\[
U_i^* := \lim_{x \to \infty} U_i(t, x) = (T^* - t)^{-\alpha_i} \phi_i(-\log(T^* - t)).
\]

Moreover, \((U_1, \ldots, U_m)\) solves the system

\[
\partial_t U_i - \delta_i \partial_x U_i = U_i^{p_i}, \quad i = 1, \ldots, m, \quad x \in \mathbb{R}, \quad -\infty < t < T^*.
\]

\[
(101)
\]
Fix $\chi \in \mathcal{D}(\mathbb{R})$ and $\xi \in \mathcal{D}((-, T^*))$ with $\int_{\mathbb{R}} \chi = 1$. For $j \in \mathbb{N}$, replacing $x$ by $x + j$ in (101) and testing with $\xi(t)\chi(x)$, we obtain
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} U^j_{i+1}(t, x + j) \xi(t) \chi(x) \, dx \, dt = \left\langle U^j_{i+1}(\cdot, \cdot + j), \xi \otimes \chi \right\rangle = \left\langle \partial_t U_i - \delta_i \partial_{xx} U_i (\cdot, \cdot + j), \xi \otimes \chi \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} (-\xi(t)\chi(x) - \delta_i \xi(t)\chi_{xx}(x)) \, U_i(t, x + j) \, dx \, dt.
\]
Due to the boundedness of $w_i$ and (100), we may therefore apply the dominated convergence theorem on the first and last terms of (102). Taking $\int_{\mathbb{R}} \chi = 1$ and $\int_{\mathbb{R}} \chi_{xx} = 0$ into account, we thus obtain
\[
\int_{\mathbb{R}} \left( U^j_{i+1} \right)^{p_i}(t) \xi(t) \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( U^j_{i+1} \right)^{p_i}(t) \chi(x) \xi(t) \, dx \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} (-\xi(t)\chi(x) - \delta_i \xi(t)\chi_{xx}(x)) \left( U^j_{i+1} \right)^{p_i}(t) \, dx \, dt = \int_{\mathbb{R}} -\xi(t)\left( U^j_{i+1} \right)^{p_i}(t) \, dt.
\]
It follows that $U^+_i = \left( U^j_{i+1} \right)^{p_i}$ for all $i = 1, \ldots, m$, in the distribution sense. Converting back to $\phi_i$, the conclusion follows.

(iii) Let $i \in \{1; \ldots; m\}$. Assume for contradiction that, for instance, $\phi_i(0) = 0$ and $\phi_{i+1}(0) > 0$. Then, by continuity, there exists $\eta > 0$ such that $|\phi_{i+1}^{p_i} - \alpha_i \phi_i|(\sigma) \geq \eta$ on $(\eta, \eta) \subset I$. Consequently $\phi_i' \geq \eta$ in $\mathcal{D}'((\eta, \eta))$. It is well known that this guarantees
\[
\phi_i(y) - \phi_i(x) \geq \int_x^y \eta \, d\sigma = \eta(y - x) \quad \text{for } -\eta < x < y < \eta.
\]
In particular $\phi_i(x) \leq \phi(0) + \eta x = \eta x < 0$ for all $x \in (-\eta, 0)$: a contradiction. By recurrence, we prove $\phi_i(0) = 0$ if and only if $\phi_{i+1}(0) = \cdots = \phi_{m+1-i}(0) = 0$.

6. Completion of proof of Proposition 1. We now turn to complete the proof of Proposition 1. It is based on a contradiction argument and the results of Sections 3-5.

Proof of Proposition 1. The upper estimate in (10) follows from (28) in Proposition 2. To prove the lower estimates, since $\partial_x u_i \leq 0$ and since $u_i > 0$ on $[T^*/2, T^*) \times [0, R]$ by the strong maximum principle, it suffices to show that, for each $\rho_1 \in (0, \rho_0),$
\[
\liminf_{t \to T^*} (T^* - t)^{\alpha_i} u_i(t, \rho_1) > 0, \quad i = 1, \ldots, m.
\]
We argue by contradiction and assume for instance that there exist $i \in \{1; \ldots; m\}, \rho_1 \in (0, \rho_0)$ and a sequence $t_j \to T^*$ such that
\[
\lim_{j \to \infty} (T^* - t_j)^{\alpha_i} u_i(t_j, \rho_1) = 0.
\]
Set $\sigma_j := -\log(T^* - t_j) \to \infty$, let $(W_1, \ldots, W_m)$ be defined by (91)-(92) and
let \((\phi_1, ..., \phi_m)\) be given by Proposition 5(i). Since \(W_i(\sigma, \theta) \leq W_i(\sigma, 0)\) for all \(\theta \in [0, (R - \rho_1)e^{\gamma/2}]\) due to (4), it follows from (94) that
\[
\phi_i(0) = \lim_{\theta \to \infty} \left( \lim_{j \to \infty} W_i(\sigma_j, \theta) \right) \leq \lim_{j \to \infty} W_i(\sigma_j, 0) = \lim_{j \to \infty} (T^* - t_j)^{\alpha_i} u_i(t_j, \rho_1) = 0.
\]
By Proposition 5(ii) and (iii), it follows that \(\phi_i(0) = \phi_{i+1}(0) = \cdots = \phi_{m+i-1}(0) = 0\).
Therefore, with \(\eta\) given by Proposition 3, we deduce from (94) that there exists \(\theta_0 > 0\) such that
\[
\lim_{j \to \infty} W_i(\sigma_j, \theta_0) \leq \eta/2, \quad i = 1, ..., m.
\]
Then, for all \(j\) sufficiently large, we have
\[W_i(\sigma_j, \theta_0) \leq \eta, \quad i = 1, ..., m,
\]
hence, in view of (91)-(92),
\[
(T^* - t_j)^{\alpha_i} u_i(t_j, \rho_1 + \theta_0 \sqrt{T^* - t_j}) \leq \eta.
\]
Taking \(j\) large enough so that \(\rho_1 + \theta_0 \sqrt{T^* - t_j} < (\rho_0 + \rho_1)/2\) and \(T^* - t_j \leq \tau_0\), we conclude from Proposition 3 that \(\rho_0\) is not a blow-up point: a contradiction. \(\square\)

7. **Proof of Theorem 1.2.** Let \((u_1, ..., u_m)\) be a solution of system (1) satisfying the hypotheses of Theorem 1.2 and \(i \in \{1; ..., m\}\). Since \(\partial_i u_i \geq 0\) and \(\partial_\rho u_i \leq 0\) then,
\[
\frac{\partial}{\partial \rho} \left( \frac{1}{2} \delta_i (\partial_\rho u_i)^2 + u_i u_{i+1}^{p_i} \right) = \left( \delta_i \partial_{\rho\rho} u_i + u_i^{p_i} \right) \partial_\rho u_i + p_i u_i u_{i+1}^{p_i-1} \partial_\rho u_{i+1}
\]
\[
= \left( \partial_i u_i - \delta_i \frac{n-1}{p_i} \partial_\rho u_i \right) \partial_\rho u_i + p_i u_i u_{i+1}^{p_i-1} \partial_\rho u_{i+1} \leq 0.
\]
Then
\[
\left( \frac{1}{2} \delta_i (\partial_\rho u_i)^2 + u_i u_{i+1}^{p_i} \right)(t, \rho) \leq \left( \frac{1}{2} \delta_i (\partial_\rho u_i)^2 + u_i u_{i+1}^{p_i} \right)(t, 0) = u_i u_{i+1}^{p_i}(t, 0).
\]
On the other hand, by (6), there exists \(C > 0\) such that
\[
u_{i+1}(t, 0) \leq C(T^* - t)^{-\alpha_{i+1}} = C \left( (T^* - t)^{-\alpha_i} \right)^{\alpha_{i+1}/\alpha_i}
\]
\[
\leq C u_i^{\alpha_{i+1}/\alpha_i}(t, 0), \quad \text{for all } t \in (T^*/2, T^*).
\]
Therefore, we obtain
\[
\frac{1}{2} \delta_i (\partial_\rho u_i)^2(t, \rho) \leq C u_i^{p_i \alpha_{i+1}/\alpha_i-1}(t, 0), \quad \text{for all } t \in (T^*/2, T^*) \text{ and } \rho \in [0, R].
\]
Then
\[
\|\partial_\rho u_i(t)\|_\infty \leq \sqrt{\frac{2C}{\delta_i}} u_i^{(q+1)/2}(t, 0), \quad \text{for all } t \in (T^*/2, T^*),
\]
with \(q = p_i \alpha_{i+1}/\alpha_i = (\alpha_i + 1)/\alpha_i\). Arguing as in [14, p. 187], it follows that there exists \(\varepsilon_0, \varepsilon_1 > 0\) such that
\[
u_i(T^*, |x|) \geq \varepsilon_0 |x|^{-2\alpha_i}, \quad \text{for all } x \in (0, \varepsilon_1).
\]
REFERENCES

[1] X. Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, J. Differ. Equations, 78 (1989), 160–190.

[2] F. Fila and P. Quittner, The blow-up rate for a semilinear parabolic system, J. Math. Anal. Appl., 238 (1999), 468–476.

[3] A. Friedman and Y. Giga, A single point blow-up for solutions of semilinear parabolic systems, J. Fac. Sci. Univ. Tokyo Sec. IA. Math., 34 (1987), 65–79.

[4] A. Friedman and B. Mcleod, Blow-up of positive solution of semilinear heat equations, Indiana Univ. Math. J., 34 (1985), 425–447.

[5] Y. Giga and R. V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math., 38 (1985), 297–319.

[6] M. A. Herrero and J. J. A. Velázquez, A blow up result for semilinear heat equations in the supercritical case, Preprint.

[7] N. Mahmoudi, Single-point blow-up for a semilinear reaction-diffusion system, Differ. Equ. Appl., 6 (2014), 563–591.

[8] N. Mahmoudi, Ph. Souplet and S. Tayachi, Improved conditions for single-point blow-up in reaction-diffusion systems, J. Differ. Equations, 259 (2015), 1494–1541.

[9] H. Matano and F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation, Comm. Pure Appl. Math., 57 (2004), 1494–1541.

[10] C. E. Mueller and F. B. Weissler, Single point blow-up for a general semilinear heat equation, Indiana Univ. Math. J., 34 (1985), 881–913.

[11] P. Quittner and Ph. Souplet, Superlinear Parabolic Problems Blow-Up, Global Existence and Steady States, Birkhäuser Verlag AG, Basel Boston Berlin, 2007.

[12] J. Renclawowicz, Blow-up, global existence and growth rate estimates in nonlinear parabolic systems, Colloq. Math., 86 (2000), 43–66.

[13] J. Renclawowicz, Global existence and blow-up of solutions for a completely coupled Fujita type system, Appl. Math., 27 (2000), 203–218.

[14] Ph. Souplet, Single-point blow-up for a semilinear parabolic system, J. Eur. Math. Soc., 11 (2009), 169–188.

[15] M. Wang, Blow-up rate for a semilinear reaction diffusion system, Comput. Math. Appl., 44 (2002), 573–585.

[16] F. B. Weissler, Single point blow-up for a semilinear initial value problem, J. Differ. Equations, 55 (1984), 204–224.

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