Positive Solutions for a Singular Superlinear Fourth-Order Equation with Nonlinear Boundary Conditions

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Abstract

We show the existence of positive solutions for a singular superlinear fourth-order equation with nonlinear boundary conditions. Consider the singular superlinear fourth-order nonlinear boundary value problem

\[
\begin{cases}
  u^{(4)}(x) = \lambda h(x)f(u(x)), & x \in (0,1), \\
  u(0) = u'(0) = 0, \\
  u''(1) = 0, \\
  u''(1) + c(u(1))u(1) = 0,
\end{cases}
\]

where \( \lambda > 0 \) is a small positive parameter, \( f : (0, \infty) \to \mathbb{R} \) is continuous, superlinear at 0, and is allowed to be singular at 0, and \( h : [0,1] \to [0, \infty) \) is continuous. Our approach is based on the fixed-point result of Krasnoselskii type in a Banach space.

1. Introduction

Consider the singular superlinear fourth-order nonlinear boundary value problem

\[
\begin{cases}
  u^{(4)}(x) = \lambda h(x)f(u(x)), & x \in (0,1), \\
  u(0) = u'(0) = 0, \\
  u''(1) = 0, \\
  u''(1) + c(u(1))u(1) = 0,
\end{cases}
\]

where \( \lambda > 0 \) is a small positive parameter, \( f : (0, \infty) \to \mathbb{R} \) is continuous, superlinear at 0, and is allowed to be singular at 0, and \( h : [0,1] \to [0, \infty) \) is continuous. If \( f(0) > 0 \), then (1) is called a positone problem; if \( f(0) < 0 \), then (1) is called a semipositone problem.

The study of semipositone problems was formally introduced by Castro and Shivaji [1]. From an application viewpoint, one is usually interested in the existence of positive solutions for semipositone problems. Significant processes on second-order semipositone problems have taken place in the last 10 years, see [1–5] and the references therein.

Fourth-order boundary value problems modeling bending equilibria of elastic beams have been considered in several papers [6–9]. Most of them are concerned with nonlinear equations with null boundary conditions. When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities. See for instance, [10–12] and the references therein.

For instance, Cabrera et al. [10] studied the existence of positive solutions for the fourth-order positone problem

\[
\begin{cases}
  u^{(4)}(x) = f(t, u(x), (Hu)(x)), & x \in (0,1), \\
  u(0) = u'(0) = 0, \\
  u''(1) = 0, \\
  u''(1) + \tau(u(1)) = 0,
\end{cases}
\]

where \( f(t, x, y) : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is increasing in \( x \) and decreasing in \( y \), for fixed \( t \in [0, +\infty) \). For convenience, we denote \( \tau(s) = c(s)s \) in this article. In [10–12], the authors studied positive solutions of fourth-order nonlinear boundary value problems in the positone case based on a mixed monotone operator method and a well-known fixed-point theorem in cones.

It should be noted that nonlinear part \( f \) is either bounded or positive states in [1, 10–12]. In this paper, we prove the existence of positive solution to (1) by assuming that \( f : (0, \infty) \to \mathbb{R} \) is continuous and is allowed to be singular at 0; in other words, \( f \) may be unbounded from below and satisfies the superlinear condition. Moreover, we prove a useful lemma (Lemma 3) in this paper which plays a key role...
to guarantee the positivity of solution. It can be obtained by concavity and convexity of solution or calculation for the second-order boundary value problem, but for the fourth-order boundary value problem, this becomes complicated.

In addition, we will replace \( h(x)f(u(x)) \) by \( m(x) \) and \( y \) by \( c(s) \), and we perform a study of the sign of Green’s function \( G(x, s) \), and we perform a study of the sign of Green’s function \( G(x, s) \), and we perform a study of the sign of Green’s function \( G(x, s) \). We shall make the following assumptions:

\[
\begin{align*}
\text{(H1): } & h: [0, 1] \to [0, \infty) \text{ is continuous.} \\
\text{(H2): } & c: [0, \infty) \to [0, \infty) \text{ is continuous.} \\
\text{(H3): } & \text{there exists a constant } \beta \text{ with } \beta < (1/2) \text{ such that}
\end{align*}
\]

\[
\limsup_{t \to 0^+} t^\beta |f(t)| < +\infty.
\]

\[
\text{(H4): } f: (0, \infty) \to \mathbb{R} \text{ is continuous and}
\lim_{s \to +\infty} f(s) s^{-1} = 0.
\]

By a positive solution of (1), we mean function \( u \in C^4[0, 1] \cap C^2(0, 1) \) with \( u > 0 \) on \((0, 1)\) and satisfying (1).

Our main result is as follows.

**Theorem 1.** Let \((H1)-(H4)\) hold. Then, there exists a constant \( \lambda_0 > 0 \) such that (1) has a positive solution \( u_0 \) for \( \lambda < \lambda_0 \) with \( u_0 \to \infty \) as \( \lambda \to 0^+ \) uniformly on compact subsets of \((0, 1)\). The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we prove the main result that the existence of a large positive solution to (1) for \( \lambda \) small.

### 2. Notation and Preliminaries

Suppose that \( E \) is a real Banach space which is partially ordered by a cone \( P \subset E \), i.e., \( x \leq y \) if and only if \( y - x \in P \). If \( x \leq y \) and \( x \neq y \), then we denote \( x < y \) or \( y > x \). By \( \theta \), we denote the zero element of \( E \). Recall that a nonempty closed convex set \( P \subset E \) is a cone if it satisfies (i) \( x \in P, \lambda \geq 0 \implies \lambda x \in P \) and (ii) \( x \in P, -x \in P \implies x = \theta \).

We first recall the following fixed-point result of Krasnoselskii type in a Banach space (see e.g., [1], Theorem 12.3).

**Lemma 1.** Let \( P \) be a Banach space and \( T: P \to P \) be a completely continuous operator. Suppose there exist \( \xi \in E, \xi_0 \neq 0 \) and positive constants \( r, R \) with \( r \neq R \) such that

\[
\begin{align*}
(a) & \text{ If } y \in P \text{ satisfies } y = \xi Ty, \; \xi \in [0, 1], \text{ then } \|y\| \neq r \\
(b) & \text{ If } y \in P \text{ satisfies } y = Ty + \xi_0 h, \; \xi \geq 0, \text{ then } \|y\| \neq R
\end{align*}
\]

Then, \( T \) has a fixed point \( y \in P \) with \( \min\{r, R\} < \|y\| < \max\{r, R\} \).

In the sequel, we describe the Banach space where the fixed points are found, as well as some notations which are used along the paper. Let us consider the Banach space \( \mathcal{B} = C(I), I = [0, 1] \) coupled with the norm

\[
\|u\|_\infty = \max_{t \in I} |u(t)|,
\]

and the Banach space \( X = L^1(0, 1) \) equipped with the norm \( \|\phi\|_X = \int_0^1 |\phi(t)| dt \).

**Lemma 2.** Let \( w \) satisfy

\[
\begin{align*}
\text{(a)} & \quad w''(x) = m(x), \quad x \in (0, 1), \\
\text{(b)} & \quad w(0) = w'(0) = 0, \\
\text{(c)} & \quad w''(1) = 0, \quad w'''(1) + yw(1) = 0,
\end{align*}
\]

where \( m \in L^1(0, 1), m(t) \geq 0 \) for a.e. \( t \in (0, 1), \gamma \in [0, 2] \). Then,

\[
w(x) \geq \frac{3 - \gamma}{9 - \gamma} \|w\|_\infty q(x), \quad x \in (0, 1),
\]

where

\[
q(x) = \min(x^2, x) = x^2.
\]

**Proof of Lemma 2.** It is easy to verify that

\[
w(x) = \int_0^1 G(x, s)m(s) ds = \mathcal{L}w(x).
\]

Green’s function \( G(x, s) \) is computed as follows:

\[
G(x, s) = \frac{1}{6}
\]

\[
\begin{align*}
s^3(3x - s) + \frac{\gamma}{6 - 2\gamma} x^2 s^2(x - 3)(s - 3), & \quad 0 \leq s \leq x \leq 1, \\
x^2(3s - x) + \frac{\gamma}{6 - 2\gamma} x^2 s^2(x - 3)(s - 3), & \quad 0 \leq x \leq s \leq 1.
\end{align*}
\]
Obviously, \( G(x, s) \) is positive for any \( y \in [0, 2] \), while \( G(x, s) \) changes sign for \( y \) is large enough, and we can compute the value of \( G(x, s) \) when \( y = 10000 \) by Mathematic 9.0:

\[
G(0.587022479, 0.999) = -0.0000059476,
g(0.5852750296, 0.5852750296) = 0.00979528106.
\]

(11)

\[
G(x, s) = \frac{x^2 (3s - x) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3)}{x_0^2 (3s - x_0) + (y/(6 - 2y))x_0^2 s^2 (x_0 - 3)(s - 3)} \geq \frac{x^2 (3 - (x/s))}{x_0^2 (3 - (x_0/s))} \geq \frac{2x^2}{3s^2 + (y/(6 - 2y))s^2 (s - 3)} \geq \frac{3 - y}{9 - y^2}.
\]

(13)

If \( x \leq s \leq x_0 \),

\[
G(x, s) = \frac{x^2 (3s - x) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3)}{s^2 (3x_0 - x) + (y/(6 - 2y))x_0^2 s^2 (x_0 - 3)(s - 3)} \geq \frac{x^2 (3 - (x/s))}{s (3x_0) + (y/(6 - 2y))x_0^2 s (x_0 - 3)(s - 3)} \geq \frac{(3x - s)}{3x_0 + (y/(6 - 2y))(s - 3)} \geq \frac{2x}{3 + (y/(6 - 2y))} \geq \frac{3 - y}{9 - y^2}.
\]

(14)

If \( s \leq x \leq x_0 \),

\[
G(x, s) = \frac{s^2 (3s - x) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3)}{s^2 (3x_0 - x) + (y/(6 - 2y))x_0^2 s^2 (x_0 - 3)(s - 3)} \geq \frac{s^2 (3 - x)}{s (3x_0 - x) + (y/(6 - 2y))x_0^2 s (x_0 - 3)(s - 3)} \geq \frac{3x - s}{3x_0 + (y/(6 - 2y))} \geq \frac{2x}{3 + (y/(6 - 2y))} \geq \frac{3 - y}{9 - y^2}.
\]

(15)

If \( x_0 \leq s \leq x \),

\[
G(x, s) = \frac{s^2 (3s - x) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3)}{x^2 (3s - x_0) + (y/(6 - 2y))x_0^2 s^2 (x_0 - 3)(s - 3)} \geq \frac{s^2 (3s - x)}{x^2 (3s - x_0) + (y/(6 - 2y))x_0^2 s (x_0 - 3)(s - 3)} \geq \frac{3x - s}{3s + (y/(6 - 2y))} \geq \frac{2x}{3 + (y/(6 - 2y))} \geq \frac{3 - y}{9 - y^2}.
\]

(16)

Thus,

\[
w(x) = \int_0^1 G(x, s) G(x_0, s) w(s) ds \geq 3 - \frac{y}{9 - y^2} \|w\|_\infty q(x), \quad x \in (r, 1).
\]

(17)

**Lemma 3.** Let \( k \in L^1(0, 1) \) with \( k \geq 0 \), and let \( u \in C^4[0, 1] \cap C^3(0, 1) \) satisfy

\[
\left\{
\begin{array}{ll}
u^{\prime\prime}(x) & \geq -k, \quad x \in (0, 1),
u(0) & \geq 0, \quad u^{\prime}(0) \geq 0, 
u^{\prime\prime}(1) & \geq 0, \quad u^{\prime\prime}(1) + yu(1) \geq 0.
\end{array}
\right.
\]

(18)

Suppose \( \|u\|_\infty > 2 ((9 - y)/(3 - y))^2 \|k\|_X \). Then, \( u(x) \geq 0 \)

\[
\|w\|_\infty = \inf \left\{ \|w\|_\infty : \int_0^1 w(x) dx = 0, \quad w(0) = 1, \quad w^{\prime}(0) = 0, \quad w^{\prime\prime}(1) + yw(1) = 0 \right\}.
\]

(19)

**Proof.** Let \( w_0(x) \in \mathcal{B} \) be the unique solution of the problem

\[
\left\{
\begin{array}{ll}
w^{\prime\prime}(x) & = -k, \quad x \in (0, 1),
w(0) & = 0, \quad w^{\prime}(0) = 0, 
w^{\prime\prime}(1) & = 0, \quad w^{\prime\prime}(1) + yw(1) = 0.
\end{array}
\right.
\]

(20)

Then,

\[
-\omega_0(x) = \frac{1}{6} \int_0^1 (s^2 (3s - s) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3))k(s) ds
\]

\[
+ \int_0^1 (s^2 (3s - x) + (y/(6 - 2y))x^2 s^2 (x - 3)(s - 3))k(s) ds,
\]

(21)
and accordingly,
\[-w_0(x)\]
\[\leq \int_0^x \left( 3x^2 s + \frac{y}{6-2y} x^2 s^2 (x-3)(s-3) \right) k(s) ds \]
\[+ \int_x^1 \left( 3x^2 s + \frac{y}{6-2y} x^2 s^2 (x-3)(s-3) \right) k(s) ds \]
\[\leq \int_0^1 \left( 3x^2 s + \frac{y}{6-2y} x^2 s(x-3)(s-3) \right) k(s) ds \]
\[\leq x^2 \int_0^1 \left( 3s + \frac{y}{6-2y} (1-3)^2 s \right) k(s) ds \leq \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)}. \tag{22} \]

Set \( y(x) = u(x) - w_0(x), (y''(1) + y y(1) - y(y(0) + y'(0))/(6+y)) \leq y''(1) \leq (y''(1) + y y(1) - y(y(0) + y'(0)), y(1) = \| \gamma \|_{\vDash q(x)}. \)

Then,
\[
\begin{align*}
  y''(x) &\geq 0, \quad x \in (0, 1), \\
  y(0) &\geq 0, \quad y'(0) \geq 0, \\
  y''(1) &\geq 0, \quad y''(1) + y y(1) \geq 0,
\end{align*}
\tag{23}
\]

and by Lemma 2, we can get
\[
y(x) \geq \frac{3-\gamma}{9-\gamma} y\|\|_{\vDash q(x)}, \quad x \in (r, 1). \tag{24} \]

It follows from (22) and (24) and the fact that \( y(x) = u(x) - w_0(x) \) that
\[
u(x) = y(x) + w_0(x)
\[\geq \frac{3-\gamma}{9-\gamma} y\|\|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \]
\[= \frac{3-\gamma}{9-\gamma} \| u - w_0 \|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \]
\[\geq \frac{3-\gamma}{9-\gamma} \| u \|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| w_0 \|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \]
\[\geq \frac{3-\gamma}{9-\gamma} \| u \|_{\vDash q(x)} + \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} + \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \]
\[\geq \frac{3-\gamma}{9-\gamma} \| u \|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \]
\[= \left( \frac{3-\gamma}{9-\gamma} \| u \|_{\vDash q(x)} - \frac{9-\gamma}{3-\gamma} \| \alpha \|_{\vDash q(x)} \right) q(x), \tag{25} \]

for \( x \in (r, 1) \). This completes the Proof of Lemma 3.

3. Proof of the Main Results

Proof of Theorem 1. Let \( \lambda > 0 \). For \( v \in \vDash \), define \( \mathcal{L}_\lambda v = u \), where \( u \) is the solution of
\[
\begin{align*}
u''(x) &= \lambda h(x)f(\bar{v}), \quad x \in (0, 1), \\
u(0) &= 0, \quad u'(0) = 0, \\
u''(1) &= 0, \quad u''(1) + \gamma u(1) = 0,
\end{align*}
\tag{26}
\]

where
\[
\bar{v}(x) = \max \{ v(x), q(x) \}, \tag{27}
\]

\( y_v = c(\| v(1) \|) \), and \( q(x) \) is defined in Lemma 2. Then,
\[
u(x) = \lambda \int_0^x G_v(x, s) h(s) f(\bar{v}(s)) ds,
\]
where Green’s function \( G_v(x, s) \) is given by
\[
\begin{align*}
G_v(x, s) &= \frac{1}{6} \left\{ \begin{array}{ll}
\frac{s^2 (3-x) + \frac{y}{6-2y} s^2 (x-3)(s-3)}{6}, & 0 \leq s \leq x \leq 1, \\
\frac{x^2 (3-x) + \frac{y}{6-2y} x^2 (x-3)(s-3)}{6}, & 0 \leq x \leq s \leq 1,
\end{array} \right.
\tag{28}
\end{align*}
\]

Note that \( G_v(x, s) \leq 1 \) for all \( x, s \). By (H3), there exists a constant \( M > 0 \) depending on \( \| v \|_{\vDash q(x)} \) such that
\[
| f(\bar{v}(x)) | \leq \frac{M}{s^2} \leq \frac{M}{q^2} \in L^1 (0, 1). \tag{29}
\]

It follows from the Lebesgue dominated convergence theorem that
\[
u(x) \in \vDash \text{ i.e. } \mathcal{L}_\lambda : \vDash \rightarrow \vDash. \tag{31}
\]

Define the cone \( P \) in \( \vDash \) by
\[
P = \left\{ u \in \vDash : \| u \|_{\vDash} \geq 0, \quad u(x) \geq \frac{3-\gamma}{9-\gamma} \| u \|_{\vDash q(x)}, \quad x \in (r, 1) \right\}. \tag{32}
\]

For \( u \in P \), it follows that, from (H3) and Lemma 3, we can get \( \mathcal{L}_\lambda (P) \subset P \).

We next show that \( \mathcal{L}_\lambda : P \rightarrow P \) is a completely continuous operator.

Now, we show \( \mathcal{L}_\lambda : P \rightarrow P \) is continuous. To this end, let \( \{ v_n \} \in C[0, 1] \) be such that \( v_n \rightarrow v \) in \( P \) and let
\[
u_n = \mathcal{L}_\lambda v_n, \quad u = \mathcal{L}_\lambda v. \tag{33}
\]

Fix \( x, z \in (0, 1) \), and define
\[
H(z) = x^2 (3-x) + \frac{z}{6-2z} x^2 z^2 (x-3)(s-3), \quad 0 \leq z \leq 2, \tag{34}
\]

Then, \( H'(z) = (1/(6-2z^2)) x^2 z^2 (x-3)(s-3) \), and there exists a constant \( N \) such that \( |H'(z)| \leq N \) for \( 0 \leq z \leq 2 \), and the mean value theorem gives
\[
\left| G_v_n(x, s) - G_v(x, s) \right| \leq N | v_n' - v'_n |. \tag{35}
\]

Notice that
\( \bar{\sigma}_n(s) = |f(\bar{y}(s)) - f(\bar{y}(s))| \longrightarrow 0, \ n \longrightarrow \infty \) for \( s \in (0, 1) \),
\( \bar{y}(s) = \bar{y}(s) \)
and the fact
\[
|f(\bar{y}(x))| \leq \frac{M}{q^p} \in L^1(0, 1). \tag{37}
\]

Now, these together with the Lebesgue dominated convergence theorem guarantee that
\[
\| \mathcal{L}_A \bar{v}_n - \mathcal{L}_A \bar{v} \| \leq \lambda \left( \int_0^1 G_{n_{\lambda}}(x, s) - G_{n}(x, s) |h(s)| f(\bar{v}_n) \right) + \int_0^1 G_{n_{\lambda}}(x, s) h(s) f(\bar{v}_n) ds + \int_0^1 G_{n_{\lambda}}(x, s) \bar{\sigma}_n(s) ds \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{38}
\]

Hence, \( \mathcal{L}_A : P \longrightarrow P \) is continuous.

Finally, we prove that \( \mathcal{L}_A : P \longrightarrow P \) is completely continuous. In fact, for \( x \in P \),
\[
\| \mathcal{L}_A \bar{v} \| \leq \int_0^1 G_{n_{\lambda}}(x, s) h(s) \frac{M}{q^p} \, ds, \tag{39}
\]
and for \( x, x' \in [0, 1] \), we have
\[
\| \mathcal{L}_A \bar{v}(x) - \mathcal{L}_A \bar{v}(x') \| \leq \int_0^1 G_{n_{\lambda}}(x, s) - G_{n_{\lambda}}(x', s) |h(s)| \frac{M}{q^p} \, ds. \tag{40}
\]

Now, the Arzela–Ascoli theorem guarantees that \( \mathcal{L}_A : P \longrightarrow P \) is compact.

Let \( a > 1 \) be such that \( f(z) > 0 \) for \( z \geq a \). Since \( \lim_{x \to + \infty} x^q f(x) < \infty \), there exists a constant \( b > 0 \) such that
\[
|f(z)| \leq \frac{b}{z^p}, \quad z \in (0, a). \tag{41}
\]
Hence,
\[
f(z) \geq - \frac{b}{z^p}, \tag{42}
\]
and
\[
|f(z)| \leq \frac{b}{z^p} + \bar{f}(\max(z, a)) \tag{43}
\]
for all \( z > 0 \), where \( \bar{f}(x) = \sup_{z \geq x} f(z) \) for \( z \geq a \). Note that \( f \) is nondecreasing.

Suppose \( \lambda < (a/(2(\lambda + c_2 \bar{f}(a))) \) where \( c_1 = \frac{b}{z^p} \int_0^1 h(s)/q^p \, ds \) and \( c_2 = \int_0^1 h(s) \, ds \). Next, we shall verify the assumptions of Lemma 1 hold for \( \mathcal{L}_A \).

(a) There exists \( r_1 > 0 \) such that if \( u \in P \) satisfy \( u = \theta \mathcal{L}_A u, \theta \in (0, 1) \), then \( \|u\|_{\infty} = \|u\|_{\infty} \neq r_1 \).

Indeed, let \( u \in P \) satisfy \( u = \theta \mathcal{L}_A u \) for some \( \theta \in (0, 1] \). Then, \( u/\theta = \mathcal{L}_A u \), and therefore, \( u \) satisfies
\[
\|u\|_{\infty} \leq \lambda \left( \int_0^1 G_{n_{\lambda}}(x, s) \bar{\sigma}_n(s) \right) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

(b) There exists \( R_3 > r_1 \) such that if \( u = \mathcal{L}_A u + \xi, \xi \geq 0 \), then \( \|u\|_{\infty} \neq R_3 \).

Let \( u \in P \) satisfy \( u = \mathcal{L}_A u + \xi \) for some \( \xi \geq 0 \). Then, \( u - \xi = \mathcal{L}_A u \), and so,
\[
\begin{align*}
    u(x) - \xi &= \lambda \int_{0}^{1} G_u(x, s) h(s) f(\bar{u}) ds. \\
    \text{Let} \\
    k(x) &= \frac{b h(x)}{q^p(x)} x \in (0, 1]. \\
    \text{Then, } k \in X. \text{ Since } u \text{ satisfies} \\
    \begin{cases}
        u''' = \lambda h(x) f(\bar{u}), & 0 < x < 1, \\
        u(0) = \xi \geq 0, & u'(0) = u''(1) = 0, \\
        u''(1) + \lambda, u(1) = \lambda, \xi \geq 0,
    \end{cases} \\
    \text{and by (41) and (42),} \\
    h(x) f(\bar{u}(x)) \geq -\frac{b h(x)}{q^p(x)} \geq -\frac{b h(x)}{q^p(x)} = -k(x),
\end{align*}
\]

\[\begin{align*}
    u(x) &\geq \left( \frac{3 - \gamma}{9 - \gamma} \left( \frac{9 - \gamma}{3 - \gamma} \right)^{\frac{q}{q}} \right) q(x), \quad x \in (r, 1). \\
    \text{Suppose } \|u\|_{\infty} > \max \left\{ \left( (9 - \gamma)/(3 - \gamma) \right)^{\frac{q}{q}}, 2/q(\alpha) \right\}. \\
    \text{Then,} \\
    \|u\|_{\infty} \geq \frac{2(9 - \gamma)^2 - (9 - \gamma)^2}{2(9 - \gamma)^2} \|u\|_{\infty} q(\sigma) \geq \frac{q(\sigma)}{2} \|u\|_{\infty}, \\
    s \in [\tau + \sigma, 1 - \sigma] = \Lambda.
\end{align*}\]

\[\begin{align*}
    G_u(x, s) \geq \frac{1}{6} \begin{cases}
    2x^2 s^2, & \text{if } x \leq x, \quad \frac{1}{\lambda} \geq \frac{1}{256}, \\
    2x^2 s^2, & \text{if } x \leq s.
    \end{cases}
\end{align*}\]
for \( s, x \in \Lambda, \) it follows from (52), (54), and (55) that

\[\begin{align*}
    u(x) &\geq \lambda \left( \int_{\Lambda} G_u(x, s) h(s) f(\bar{u}) ds + \int_{\Lambda} G_u(x, s) h(s) f(\bar{u}) ds \right) \\
    &\geq \lambda \left( (1/256) f((q(\sigma)\|u\|_{\infty})/2) \int_{\Lambda} h(s) ds - \|k\|_{X} \right), \\
    \text{for } x \in \Lambda, \text{ where } \bar{f}(x) = \inf_{z \geq x} f(z). \text{ Consequently,} \\
    (1/256) f((q(\sigma)\|u\|_{\infty})/2) \int_{\Lambda} h(s) ds - \|k\|_{X} \leq \frac{1}{\lambda} \|u\|_{\infty}
\end{align*}\]

Since the left side of this inequality goes to \( \infty \) as \( \|u\|_{\infty} \to \infty, \) it follows that \( \|u\|_{\infty} < R_1 \) for \( R_1 \gg 1, \) which establishes (b).

By Lemma 1, \( \mathcal{A} \) has a fixed point \( u_1 \) with \( \|u_1\|_{\infty} > r_1. \) Since (19) holds with \( \xi = 0 \) and \( r_1 \to \infty \) as \( \lambda \to 0, \) it follows that \( u_1 \) is a positive solution of (1) if \( \lambda \) is sufficiently small and \( u_1(x) \to \infty \) as \( \lambda \to 0 \) uniformly for \( x \) in compact subsets of \( (0, 1). \) This completes the proof of Theorem 1.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Authors’ Contributions**

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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