Abstract Let $X \neq \emptyset$ an arbitrary set and $U \subset 2^X$ a non-empty set of subsets. The function $\mu : U \to \{0,1\}$ is called binary set function. If $\mu$ is countably additive, then it is called a measure. The paper gives some definitions and properties of these functions, its purpose being that of suggesting the reconstruction of the measure theory within this frame, by analogy with [1], [2].

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1. Set Rings and Function Rings

1.1 We note with $B_2$ the set $\{0,1\}$, called the binary Boole (or Boolean) algebra, together with the discrete topology, the order $0 \leq 1$ and the laws: the logical complement ‘‘−’’, the reunion ‘‘∪’’, the product ‘‘⋅’’, the modulo 2 sum ‘‘⊕’’, the coincidence ‘‘⊗’’:

$$
\begin{array}{c|c|c|c|c|c}
- & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\begin{array}{c|c|c|c|c|c}
\cup & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\begin{array}{c|c|c|c|c|c}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\begin{array}{c|c|c|c|c|c}
\oplus & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\begin{array}{c|c|c|c|c|c}
\otimes & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
$$
a) b) c) d) e)
table (1)

1.2 Let $X \neq \emptyset$ be an arbitrary set, that we shall call the total set. In the set $2^X$ of the subsets of $X$, the order is given by the inclusion and the laws are: the complementary relative to $X$ ‘‘−’’, the reunion ‘‘∨’’, the difference ‘‘−’’, the intersection ‘‘∧’’, the symmetrical difference ‘‘Δ’’ and the coincidence ‘‘Θ’’ that is defined like this:

$$
A \Theta B = A \overline{\Delta} B
$$

1.3 Theorem Let $U \subset 2^X$ a set of subsets of $X$. The next statements are equivalent:

a) $A, B \in U \Rightarrow A \vee B, A - B \in U$

b) $A, B \in U \Rightarrow A \Delta B, A \wedge B \in U$

and the next statements are equivalent too:

c) $A, B \in U \Rightarrow A \wedge B, \overline{B} - A \in U$

d) $A, B \in U \Rightarrow A \Theta B, A \vee B \in U$
1.4 **Remark** In the previous theorem, the conditions a), c); b), d) are dual.

1.5 a) The set $U$ that fulfills one of 1.3 a), b) is called set ring, or ring of subsets of $X$ (on $X$). N. Bourbaki calls such a set clan.
   b) Similarly, if $U$ fulfills one of 1.3 c), d), it is called set ring, or ring of subsets of $X$ (on $X$), the dual structure of the structure from a).

1.6 **Remark** a) $(U, \Delta, \wedge)$ is really a non-unitary, commutative ring. Its neuter element is $\emptyset$.
   b) $(U, \Theta, \lor)$ is itself a non-unitary, commutative ring. Its neuter element is $X$.

1.7 a) If $X$ belongs to the ring $(U, \Delta, \wedge)$, then $(U, \Delta, \wedge)$ is called a set algebra.
   b) If $\emptyset$ belongs to the ring $(U, \Theta, \lor)$, then $(U, \Theta, \lor)$ is called a set algebra too.

1.8 **Remark** a) The condition that $(U, \Delta, \wedge)$ is a set algebra ($(U, \Theta, \lor)$ is a set algebra) implies the one that $U$ is a unitary set ring, because if $X \in U$ (if $\emptyset \in U$), then it is the unit of the ring.
   b) Generally speaking, the unit, if it exists, is given by $\lor_{A \in U} A$ (by $\wedge_{A \in U} A$).

1.9 **Remark** The set algebras are not what is usually meant by the $F$-algebra structures, where $F$ is a field.

1.10 Let $f : X \rightarrow B_2$ a function. Its support is by definition the set:

\[
\text{supp } f = \{ x \mid x \in X, f(x) = 1 \}
\]  

(1)

1.11 If

\[
\text{supp } f = A
\]  

(1)

$f$ will be noted sometimes with $\chi_A$. This function is called the characteristic function of the set $A \subset X$.

1.12 Let us define for the set ring $(U, \Delta, \wedge)$, respectively for the set ring $(U, \Theta, \lor)$, the set

\[
U' = \{ f \mid f : X \rightarrow B_2, \text{supp } f \in U \}
\]  

(1)

1.13 $(U', \Theta, \cdot, \cdot)$ and $(U', \Theta, \cup, \cup)$ are $B_2$-algebras, where '$\cdot$' is the symbol of two laws: the product of the functions and the product of the functions with scalars (both induced from $B_2$), while '$\cup$' is the dual of '$\cdot$'.

1.14 The associations

\[
U \ni A \leftrightarrow \chi_A \in U'
\]

are ring isomorphisms. They allow us many times to identify the set rings $U \subset 2^X$ and the function rings $U' \subset B_2^X$. 
2. Additive and Countably Additive Set Functions

2.1 Theorem Let \( U \subset 2^X \) a non-empty family of subsets of \( X \) and \( \mu : U \to B_2 \) a function.

a) If \((U, \Delta, \wedge)\) is a set ring, then the next statements are equivalent:
   a.1) \( \forall A, B \in U, A \wedge B = \emptyset \Rightarrow \mu(A \vee B) = \mu(A) \oplus \mu(B) \) \( (1) \)
   a.2) \( \forall A, B \in U, \mu(A \Delta B) = \mu(A) \oplus \mu(B) \) \( (2) \)

b) If \((U, \Theta, \vee)\) is a set ring, then the next statements are equivalent:
   b.1) \( \forall A, B \in U, A \vee B = X \Rightarrow \mu(A \wedge B) = \mu(A) \otimes \mu(B) \) \( (3) \)
   b.2) \( \forall A, B \in U, \mu(A \Theta B) = \mu(A) \otimes \mu(B) \) \( (4) \)

2.2 a) Let \((U, \Delta, \wedge)\) be a set ring. A function \( \mu : U \to B_2 \) that fulfills one of the equivalent conditions 2.1 a.1), a.2) is called additive, or finitely additive.

b) In a dual manner, let \((U, \Theta, \vee)\) be a set ring. A function \( \mu : U \to B_2 \) that fulfills one of the equivalent conditions 2.1 b.1), b.2) is called additive*, or finitely additive*.

2.3 The sets of functions \( U \to B_2 \) which are additive, respectively additive* are noted with \( Ad(U) \), respectively \( Ad^*(U) \). They are naturally organized as \( B_2 \)-linear spaces.

2.4 Theorem a) Let \( \mu \in Ad(U) \). For \( A, B \in U \), we have:

   a.1) \( \mu(\emptyset) = 0 \) \( (1) \)
   a.2) \( \mu(A - B) = \mu(A) \oplus \mu(A \wedge B) \) \( (2) \)
   a.3) \( \mu(A \vee B) \otimes \mu(A \wedge B) \otimes \mu(A \Delta B) = 0 \) \( (3) \)

b) If \( \mu \in Ad^*(U) \), then the next properties are true:

   b.1) \( \mu(X) = 1 \) \( (4) \)
   b.2) \( \mu(B - A) = \mu(A) \otimes \mu(A \vee B) \) \( (5) \)
   b.3) \( \mu(A \wedge B) \otimes \mu(A \vee B) \otimes \mu(A \Theta B) = 1 \) \( (6) \)

where \( A, B \in U \).

2.5 Let \( a : N \to B_2 \),

\[ a_n = a(n), n \in N \] \( (1) \)
a binary sequence. If the support of \( a : \{n \mid n \in N, a_n = 1\} \) is a finite set, then the summation modulo 2 has sense:

\[ \sum_{n \in N} a_n = \begin{cases} 1, & \text{supp } a \text{ is odd} \\ 0, & \text{supp } a \text{ is even} \end{cases} \] \( (2) \)

where we have noted with \(| \cdot |\) the number of elements of a finite set and where, by definition:

\[ |\emptyset| = 0 \] \( (3) \)
is even. If the support of \( a \) is not finite, then the symbol \( \sum_{n \in N} a_n \) refers to a divergent series.

2.6 Let \( A : N \to B_2 \),

\[ A_n = A(n), n \in N \] \( (1) \)
a sequence of sets. If for any \( x \in X \) the set \( \{ n \mid n \in N, x \in A_n \} \) is finite, then the symmetrical difference has sense:

\[
\Delta_n A_n = \{ x \mid x \in X, \{ n \mid n \in N, x \in A_n \} \text{ is odd} \}
\]

and if not, the symbol \( \Delta_n A_n \) refers to a divergent series of sets.

2.7 **Theorem** Let \((U, \Delta, \land, \lor) \subset 2^X\) be a set ring and \( \mu : U \to B_2 \) a function. The following statements are equivalent:

a) For any sequence of sets \( A_n \in U, n \in N \), the conditions

a.1) \( n \neq m \Rightarrow A_n \land A_m = \emptyset \)

and

a.2) \( \lor_{n \in N} A_n \in U \)

imply

a.3) \( \{ n \mid n \in N, \mu(A_n) = 1 \} \) is finite

and

a.4) \( \mu(\lor_{n \in N} A_n) = \sum_{n \in N} \mu(A_n) \)

b) For any sequence of sets \( A_n \in U, n \in N \), the conditions

b.1) \( \forall x \in X, \{ n \mid n \in N, x \in A_n \} \) is finite

and

b.2) \( \Delta_n A_n \in U \)

imply

b.3) \( \{ n \mid n \in N, \mu(A_n) = 1 \} \) is finite

and

b.4) \( \mu(\Delta_n A_n) = \sum_{n \in N} \mu(A_n) \)

**Proof**

a) \( \Rightarrow \) b) Let \( A_n \in U, n \in N \) so that b.1), b.2) are true under the form:

\[
\forall x \in X, \{ n \mid n \in N, x \in A_n \} \in \{0,1\}
\]

\[
\Delta_n A_n = \lor_{n \in N} A_n \in U
\]

a.1), a.2) being fulfilled, a.3), a.4) are also fulfilled, thus b.3), b.4) are fulfilled.

b) \( \Rightarrow \) a) If \( A_n \in U, n \in N \) satisfies a.1), a.2), then b.1), b.2) are true, thus b.3), b.4) are true resulting that a.3), a.4) are fulfilled.

2.8 a) A function \( \mu : U \to B_2 \) that satisfies one of the equivalent conditions 2.7 a), b) is called countably additive, or measure.

b) We take in consideration the duals of 2.5, 2.6, 2.7. A function \( \mu : U \to B_2 \) that fulfills one of the duals of the previous equivalent conditions is called countably additive* or measure*.

2.9 The sets of countably additive, respectively countably additive* \( U \to B_2 \) functions are noted with \( Ad_c(U) \), respectively with \( Ad^*_c(U) \).

These sets are \( B_2 \)-linear spaces.
2.10 The inclusions $\text{Ad}_c(U) \subset \text{Ad}(U)$, $\text{Ad}_c^*(U) \subset \text{Ad}^*(U)$ are easily shown.

2.11 The terminology of additive function, countably additive function and measure is the same if the domain of the function is a $B_2$-algebra $U'$ included in $B_2^X$, via the identification from 1.14.

### 3. Examples

3.1 Let $X \neq \emptyset$ and $U \subset 2^X$ a set ring. The null function $0 : U \rightarrow B_2$ is a measure; it is the null element of the linear space $\text{Ad}_c(U)$.

3.2 Suppose that $\mu : U \rightarrow B_2$ is a measure and $A \in U$. The function $\mu_1 : U \rightarrow B_2$ that is defined by:

$$\mu_1(B) = \mu(A \wedge B), \quad B \in U$$

is a measure, called the restriction of $\mu$ at $A$.

**Proof** Let $A_n \in U$, $n \in N$ be disjoint two by two with $\bigvee_{n \in N} A_n \in U$, resulting that the sets $A \wedge A_n \in U$, $n \in N$ are disjoint two by two with

$$\bigvee_{n \in N} (A \wedge A_n) = A \wedge \bigvee_{n \in N} A_n \in U \quad (2)$$

Because $\mu$ is a measure, the set $\{n \mid n \in N, \mu(A \wedge A_n) = 1\}$ is finite and it is true:

$$\mu_1(\bigvee_{n \in N} A_n) = \mu(A \wedge \bigvee_{n \in N} A_n) = \mu(A \wedge (A \wedge A_n)) =$$

$$= \sum_{n \in N} \mu_1(A_n) \mu(A \wedge A_n) = \sum_{n \in N} \mu_1(A_n) \quad (3)$$

3.3 We fix $x_0 \in X$. The function $\chi^{(x_0)} : U \rightarrow B_2$ defined by:

$$\chi^{(x_0)}(A) = \chi_A(x_0), \quad A \in U$$

is a measure. More general, the sum of these functions is a measure too and this means that to each finite set $H \subset X$ it is associated a function $\chi^H : U \rightarrow B_2$ defined in the following way:

$$\chi^H(A) = \sum_{x \in H} \chi_A(x), \quad A \in U \quad (2)$$

When $H$ is the empty set, we find the example 3.1.

3.4 $(S_2, \oplus, \cdot, \cdot)$ is the $B_2$-algebra of the binary sequences $x_n \in B_2, n \in N$, where the sum of the sequences '$\oplus$', the product of the sequences '$\cdot$' and the product of the sequences with scalars '$\cdot$' is made coordinatewise. We mention here that the families of sequences $(x^p_n)_n \in S_2, p \in N$ that are disjoint two by two are these that satisfy:

$$p \neq p' \Rightarrow \forall n, x^p_n \cdot x_{n'}^{p'} = 0 \quad (1)$$

Let $k \in N$ and we define $\mu_k : S_2 \rightarrow B_2$ by:

$$\mu_k((x_n)) = x_k \cdot (x_n) \in S_2 \quad (2)$$
- the projection of the vector \((x_n)\) of \(S_2\) on the \(k\)-th coordinate. More general, if \(H \subset N\) is a finite set

\[
H = \{k_1, \ldots, k_p\}
\]

then we have the sum of functions \(\mu_H : S_2 \to B_2\),

\[
\mu_H = \mu_{k_1} \oplus \ldots \oplus \mu_{k_p}
\]

\(\mu_k\) and \(\mu_H\) are countably additive; if \(H\) is empty, then \(\mu_H\) is by definition the null function.

3.5 a) We say that the sequence \(x_n \in B_2, n \in N\) converges to \(x^0 \in B_2\) if

\[
\exists N \in N, \forall n \geq N, x_n = x^0
\]

If so, the unique \(x^0\) with this property (because \(x\) is a function) is called the limit of \((x_n)\). If the previous statement is made under the weaker form: the sequence \((x_n)\) is convergent, this means that such an \(x^0\) like at (1) (uniquely) exists. The limit of the sequence \((x_n)\) has the usual notation \(\lim_{n \to \infty} x_n\).

b) \((S_2^0, \oplus, \cdot, \cdot)\) is the \(B_2\)-algebra of the binary sequences \(x_n \in B_2, n \in N\) that converge to 0. We define the measure \(\mu : S_2^0 \to B_2\) by

\[
\mu((x_n)) = \bigoplus_{n \in N} x_n, (x_n) \in S_2^0
\]

3.6 \((S_2^c, \oplus, \cdot, \cdot)\) is the \(B_2\)-algebra of the convergent binary sequences \(x_n \in B_2, n \in N\) and we define \(\mu : S_2^c \to B_2\) by:

\[
\mu((x_n)) = \lim_{n \to \infty} x_n, (x_n) \in S_2^c
\]

\(\mu\) is additive, but it is not countably additive. In order to see this, we give the example of the sequence of convergent sequences (the canonical base of \(S_2^c\)):

\[
\varepsilon^n : N \to B_2, \varepsilon^n(m) = \begin{cases} 1, & n = m \\ 0, & \text{else} \end{cases}, m, n \in N
\]

\((\varepsilon^n)_n\) are disjoint two by two, their reunion is the constant 1 sequence that is convergent and on the other hand

\[
\mu\left( \bigcup_{n \in N} \varepsilon^n \right) = 1 \neq 0 = \bigoplus_{n \in N} \mu(\varepsilon^n)
\]

3.7 A variant of 3.4 is obtained if we take instead of \(S_2 = B_2^N\) an arbitrary function \(B_2\)-algebra \(U \subset B_2^X\). Let \(x_0 \in X\); the function \(\mu_{x_0} : U \to B_2\) defined like this:

\[
\mu_{x_0}(f) = f(x_0), f \in U
\]

is a measure. More general, if \(H \subset X\) is a finite set, then the function \(\mu_H : U \to B_2\) defined in the following manner:
\[
\mu_H(f) = \sum_{x \in H} f(x), \quad f \in U
\]

is a measure. If \( H \) is empty, then by definition \( \mu_H \) is the null function.

We mention the fact that \( f^p \in U, \quad p \in N \) are disjoint two by two if

\[ p \neq p' \Rightarrow \forall x \in X, f^p(x) \cdot f^{p'}(x) = 0 \]

(3)

3.8 We note with \( R_f(X) \) the ring - relative to \( \Delta, \wedge \) - of the finite subsets of \( X \). The function \( \mu_f^X : R_f(X) \to B_2 \),

\[
\mu_f^X(A) = \begin{cases} 1, & |A| \text{ is odd} \\ 0, & |A| \text{ is even} \end{cases}, \quad A \in R_f(X)
\]

is a measure, called the \textit{finite Boolean measure}.

3.9 We note with \( \text{Inf}_f \) the ring of the \textit{inferiorly finite sets} \( A \subset R \), i.e. the sets with the following property:

\[ \forall \alpha \in R, (-\infty, \alpha) \wedge A \text{ is finite} \]

We fix some \( \alpha \in R \) and we define \( \mu_\alpha : \text{Inf}_f \to B_2 \) by:

\[
\mu_\alpha(A) = \mu_f((-\infty, \alpha) \wedge A), \quad A \in \text{Inf}_f
\]

(1)

\( \mu_\alpha \) is countably additive: for any family \( A_n \in \text{Inf}_f, n \in N \) of two by two disjoint sets so that

\[ \bigvee_{n \in N} A_n \in \text{Inf}_f, \text{ only a finite number of sets } A_n \text{ fulfill } (\neg \infty, \alpha) \wedge A_n \neq \emptyset \text{ etc.} \]

3.10 a) Let \( X \subset R \) and \( t \in R \) a point so that

\[ \forall t' < t, (t', t) \wedge X \text{ is infinite} \]

b) We say that the function \( f : X \to B_2 \) has a \textit{left limit} in \( t \), noted with

\[ f(t - 0) \in B_2, \text{ if the next property is true:} \]

\[
\exists t' < t, \forall t' \in (t', t) \wedge X, f(t') = f(t - 0)
\]

(1)

c) We note with \( \text{Lim}_X^-(t) \) the \( B_2 \)-algebra of the \( X \to B_2 \) functions that have a left limit in \( t \).

d) The function \( \mu : \text{Lim}_X^-(t) \to B_2 \):

\[
\mu(f) = f(t - 0), \quad f \in \text{Lim}_X^-(t)
\]

(1)

is a measure, this example being analogue to 3.7.

e) Other examples of measures of the same type with this one may be given.

3.11 a) For \( a, b \in R \setminus \{\infty\} \), the \textit{symmetrical interval} \([(a, b)) \) is defined by:

\[
[(a, b)) = \begin{cases} (a, b), & a < b \\ \emptyset, & b = a \end{cases}
\]

(1)

b) We note with \( \text{Sym}^- \) the set ring - relative to \( \Delta, \wedge \) - generated by the symmetrical intervals \([(a, b)) \).

c) We define \( \mu : \text{Sym}^- \to B_2 \) by:
\[
\mu(A) = \begin{cases} 
1, & \text{if } \sup A = \infty \\
0, & \text{else}
\end{cases}
\] (2)

where \( A \in \text{Sym}^- \). Because in a sequence of sets \( A_n \in \text{Sym}^-, n \in N \) that are disjoint two by two with \( \vee_{n \in N} A_n \in \text{Sym}^- \) at most one satisfies the condition \( \sup A_n = \infty \), it may be shown that \( \mu \) is a measure.

3.12 a) We define the next \( B_2 \)-algebras of functions \( f : R \rightarrow B_2 \):

\[
I_{(a,b)} = \{ f \mid [(a,b)] \land \text{supp } f \text{ is finite}, a,b \in R \lor \{\infty\} \} \quad (1)
\]

\[
I_\infty = \{ f \mid \text{supp } f \text{ is finite} \} \quad (2)
\]

and the integrals

\[
\int_a^b f = \sum_{t \in [(a,b)]} f(t), f \in I_{(a,b)}) \quad (3)
\]

\[
\int_{-\infty}^\infty f = \sum_{t \in R} f(t), f \in I_\infty \quad (4)
\]

b) The next \( I_{(a,b)} \rightarrow B_2, I_\infty \rightarrow B_2 \) functions:

\[
\mu(f) = \int_a^b f, f \in I_{(a,b)} \quad (5)
\]

\[
\mu(f) = \int_{-\infty}^\infty f, f \in I_\infty \quad (6)
\]

are measures.

3.13 a) The set \( S \subset 2^R \) defined in the next way:

\[
S = \{ (a_1,b_1) \Delta \ldots \Delta (a_p,b_p) \Delta \{c_1,\ldots,c_n\} \mid a_1,b_1, \ldots, a_p,b_p, c_1,\ldots,c_n \in R, p,n \in N \} \quad (1)
\]

is a ring of subsets of \( R \) and we have supposed that

\[
p = 0 \Rightarrow (a_1,b_1) \Delta \ldots \Delta (a_p,b_p) = \emptyset \quad (2)
\]

\[
n = 0 \Rightarrow \{c_1,\ldots,c_n\} = \emptyset \quad (3)
\]

b) The function \( \mu : S \rightarrow B_2 \) given by:

\[
\mu((a_1,b_1) \Delta \ldots \Delta (a_p,b_p) \Delta \{c_1,\ldots,c_n\}) = \pi(p + n) \quad (4)
\]

where \( \pi : N \rightarrow B_2 \) is the parity function:

\[
\pi(m) = \begin{cases} 
1, & \text{if } m \text{ is odd} \\
0, & \text{if } m \text{ is even}
\end{cases}, m \in N
\] (5)

- is additive, but it is not countably additive. In order to see this fact, we take the sequence

\[
\left[ \frac{1}{n + 2}, \frac{1}{n + 1} \right) = \left( \frac{1}{n + 2}, \frac{1}{n + 1} \right) \Delta \left( -\frac{1}{n + 2}, \frac{1}{n + 1} \right) \in S, n \in N
\] (6)

of sets that are disjoint two by two, satisfying

\[
\vee_{n \in N} \left[ \frac{1}{n + 2}, \frac{1}{n + 1} \right) = (0,1) \in S
\] (7)
\[ \{ n \mid \mu\left(\frac{1}{n+2}, \frac{1}{n+1}\right) = 1 \} = \emptyset \] (8)

\[ \mu((0,1)) = 1 \neq 0 = \sum_{n \in N} \mu\left(\frac{1}{n+2}, \frac{1}{n+1}\right) \] (9)

### 3.14

a) We note with
\[ R^*_f(X) = \{ H \mid H \subset X, \bar{H} \text{ is finite} \} \] (1)

This set is a set ring relative to \( \Theta, \lor \) and it is the dual structure of \( R_f(X) \).

b) A typical example of measure* is given by the function \( \mu^*_f : R^*_f(X) \rightarrow B_2 \) that is defined in the next manner:
\[ \mu^*_f(H) = \begin{cases} 0, & |H| \text{ is odd} \\ 1, & |H| \text{ is even} \end{cases} \] (2)

(In the equations (1), (2) the superior bar notes two things: the complementary of a set and the logical complement.)

Let the sequence of sets \( A_n \in R^*_f(X), n \in N \) that are disjoint* two by two:
\[ n, m \in N, n \neq m \Rightarrow A_n \lor A_m = X \ (i.e. \ A_n \land A_m = \emptyset) \] (3)

so that \( \bigcap_{n \in N} A_n \in R^*_f(X) \). Because from the definition of \( R^*_f(X) \), the set
\[ \bigcap_{n \in N} A_n = \lor_{n \in N} \bar{A}_n \] (4)

is finite, there results the existence of a rank \( N \) with the property that \( \bar{A}_n \) are empty for \( n > N \). We have:
\[ \mu^*_f\left(\bigcap_{n \in N} A_n\right) = \mu^*_f\left(\bigcap_{n \in N} \bar{A}_n\right) = \mu^*_f\left(\lor_{n \in N} A_n\right) = \mu^*_f\left(\lor_{n \in N} \bar{A}_n\right) = \mu^*_f\left(A_0 \lor A_1 \lor \ldots \lor A_N\right) = \mu^*_f\left(A_0\right) \oplus \mu^*_f\left(A_1\right) \oplus \ldots \oplus \mu^*_f\left(A_N\right) = \sum_{n \in N} \mu^*_f\left(A_n\right) = \sum_{n \in N} \mu^*_f\left(\bar{A}_n\right) = \bigoplus_{n \in N} \mu^*_f\left(A_n\right) \] (5)

### 4. The Behavior of the Measures Relative to the Monotonous Sequences of Sets

#### 4.1

a) The family \( A_n \subset X, n \in N \) is called ascending sequence of sets if
\[ A_0 \subset A_1 \subset A_2 \subset \ldots \] (1)

In this case, the reunion \( \bigvee_{n \in N} A_n \) is called the limit of the sequence and is noted sometimes with \( \lim_{n \to \infty} A_n \).

b) The family \( A_n \subset X, n \in N \) is called descending sequence of sets if
\[ A_0 \supset A_1 \supset A_2 \supset \ldots \] (2)
The intersection $\bigwedge_n A_n$ is called the limit of the sequence and is noted sometimes with $\lim_{n \to \infty} A_n$.

c) If the sequence $A_n \subset X, n \in \mathbb{N}$ is either ascending, or descending, then we say that it is monotonous.

4.2 **Theorem** Let $U \subset 2^X$ a set ring and the function $\mu : U \to B_2$.

a) Let $A_n \in U, n \in \mathbb{N}$ an arbitrary ascending sequence of sets satisfying the property that the set

$$A = \bigvee_{n \in \mathbb{N}} A_n$$

belongs to $U$. If $\mu$ is a measure, then the binary sequence $(\mu(A_n))_n$ is convergent (see 3.5 a)) and it is true:

$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$

b) Suppose that $\mu$ is additive and it satisfies the property: for any ascending sequence $A_n \in U, n \in \mathbb{N}$ of sets so that its reunion $A$ belongs to $U$, the binary sequence $(\mu(A_n))_n$ is convergent and the relation (2) takes place. Then $\mu$ is a measure.

**Proof**
a) We have the disjoint reunion:

$$A = A_0 \cup (A_1 - A_0) \cup \ldots \cup (A_n - A_{n-1}) \cup \ldots$$

Because $\mu$ is a measure, it results that there exists $N \in \mathbb{N}$ so that

$$n > N \Rightarrow \mu(A_{n+1} - A_n) = 0$$

thus

$$\mu(A) = \mu(A_0) \oplus \mu(A_1) \oplus \ldots \oplus \mu(A_N) = \mu(A_0) \oplus \mu(A_1) \oplus \ldots \oplus \mu(A_N) = \mu(A_{N+1})$$

(4) is equivalent with the convergence of the sequence $(\mu(A_n))_n$, as it can be rewritten under the form:

$$n > N \Rightarrow \mu(A_{n+1}) = \mu(A_n)$$

and (5) is equivalent in this situation with (2). In the last equations, we have used 2.4 a.2) under the form:

$$\mu(A_{n+1} - A_n) = \mu(A_{n+1}) \oplus \mu(A_n) = \mu(A_{n+1}) \oplus \mu(A_n), n \in \mathbb{N}$$

b) Let $A_n \in U, n \in \mathbb{N}$ a sequence of sets that are disjoint two by two and let us suppose that their reunion

$$A = \bigvee_{n \in \mathbb{N}} A_n$$

belongs to $U$. We define the sequence $A_n \in U, n \in \mathbb{N}$ by:

$$A_n = A_0 \cup A_1 \cup \ldots \cup A_n, n \in \mathbb{N}$$

and it is remarked that it is ascending and (1) is satisfied. The hypothesis states the convergence of the sequence with the general term

$$\mu(A_n) = \mu(A_0) \oplus \mu(A_1) \oplus \ldots \oplus \mu(A_n)$$

in other words there exists an $N \in \mathbb{N}$ for which the implication

$$n > N \Rightarrow \mu(A_n) = 0$$
is true. The relation (2) becomes
\[ \mu(A) = \mu(A_0') \oplus \mu(A_1') \oplus ... \oplus \mu(A_N') = \bigoplus_{n \in N} \mu(A_n') \] (12)
i.e. \( \mu \) is a measure.

4.3 Theorem

It is considered the set ring \( U \) and the function \( \mu : U \to B_2 \).

a) We suppose that \( A_n \in U, n \in N \) is an arbitrary descending sequence of sets whose intersection
\[ A = \bigwedge_{n \in N} A_n \] (1)
belongs to \( U \) and that \( \mu \) is a measure. Then the binary sequence \( (\mu(A_n))_n \) is convergent and it is true:
\[ \mu(A) = \lim_{n \to \infty} \mu(A_n) \] (2)

b) Let us suppose that \( \mu \) is additive and the next property is satisfied: for any descending sequence \( A_n \in U, n \in N \) of sets so that its intersection \( A \) belongs to \( U \), the binary sequence \( (\mu(A_n))_n \) is convergent and the relation (2) is true. In these circumstances \( \mu \) is a measure.

Proof

a) Let us remark for the beginning that the set
\[ \bigvee_{n \in N} (A_0 - A_n) = A_0 - \bigwedge_{n \in N} A_n = A_0 - A \] (3)
belongs to \( U \) and the sequence of sets
\[ A_0 - A_0 \subset A_0 - A_1 \subset A_0 - A_2 \subset ... \] (4)
is ascending. We apply 4.2 a) resulting that the binary sequence \( (\mu(A_0 - A_n))_n \) is convergent and that it takes place
\[ \mu(A_0 - A) = \lim_{n \to \infty} \mu(A_0 - A_n) \] (5)
From (5) it results that
\[ \mu(A_0) \oplus \mu(A) = \mu(A_0) \oplus \lim_{n \to \infty} \mu(A_n) \] (6)
and we have the validity of (2).

b) Let \( A_n' \in U, n \in N \) a sequence of sets that are disjoint two by two with the property that their reunion
\[ A' = \bigvee_{n \in N} A_n' \] (7)
belongs to \( U \). We define the sequence of sets from \( U \):
\[ A_n = A' - (A_0' \vee A_1' \vee ... \vee A_n') = (A' - A_0') \wedge (A' - A_1') \wedge ... \wedge (A' - A_n') \] (8)
where \( n \in N \) that proves to be descending and its meet
\[ A = \bigwedge_{n \in N} A_n = \bigwedge_{n \in N} (A' - A_k') = A' - \bigvee_{n \in N} A_k' = A' - A' = \emptyset \] (9)
belongs to \( U \). The hypothesis states that the binary sequence \( (\mu(A_n))_n \) is convergent and the relation (2) becomes:
\[ 0 = \mu(A) = \lim_{n \to \infty} \mu(A_n) \] (10)
There exists a rank \( N \in N \) so that for any \( n > N \) we have:
We have that $(\mu(A'_n))_n$ converges to 0 and if $k > N$ then
\[
\mu(\bigvee_{n \in N} A'_n) = \mu(A') = \sum_{n=0}^{k} \mu(A'_n) = \sum_{n \in N} \mu(A'_n)
\]

5. Derivable Measures

5.1 In this paragraph we shall consider that the total space $X$ is equal with $\mathbb{R}^n$, $n \geq 1$. The elements $x \in X$ will be consequently $n$-tuples $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

5.2 We define the family
\[
U_n = \{A \mid A \subset \mathbb{R}^n, A \text{ is bounded}\}
\]
It is a set ring (relative to $\Delta$ and $\wedge$).

5.3 Let $A \in U_n$ be a bounded set. Its diameter is defined to be the real non-negative number
\[
d(A) = \sup_{x, y \in A} \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}
\]

5.4 We define the locally finite sets from $\mathbb{R}^n$ to be these sets $H \subset \mathbb{R}^n$ with the property that
\[
\forall A \in U_n, A \wedge H \text{ is finite}
\]

5.5 The set of the locally finite sets from $\mathbb{R}^n$ is noted with $\text{Loc}^{(n)}$ and it is a set ring.

5.6 Proposition Let us take a set $H \in \text{Loc}^{(n)}_f$. The function $\mu_H : U_n \rightarrow B_2$ defined by:
\[
\mu_H(A) = \pi(|A \wedge H|), A \in U_n
\]
is a measure (the function $\pi$ was defined at 3.13 (5)).

Proof Let $A_p \in U_n$, $p \in \mathbb{N}$ a family of sets that are disjoint two by two with the property that $\bigvee_{p \in \mathbb{N}} A_p \in U_n$. Because $\bigvee_{p \in \mathbb{N}} A_p \wedge H$ is a finite set, there exists a number $N \in \mathbb{N}$ with:
\[
p > N \Rightarrow A_p \wedge H = \emptyset
\]
We infer that
\[
\{p \mid \mu_H(A_p) = 1\} \subset \{0, 1, \ldots, N\}
\]
\[
\mu_H(\bigvee_{p \in \mathbb{N}} A_p) = \pi(|\bigvee_{p \in \mathbb{N}} A_p \wedge H|) = \pi(|\bigvee_{p \in \mathbb{N}} (A_p \wedge H)|) = \pi(|A_0 \wedge H| + |A_1 \wedge H| + \ldots + |A_N \wedge H|) = \pi(|A_0 \wedge H|) \oplus \pi(|A_1 \wedge H|) \oplus \ldots \oplus \pi(|A_N \wedge H|) =
\]
\[
= \sum_{p \in N} \pi(|A_p \wedge H|) = \sum_{p \in N} \mu_H(A_p)
\]

5.7 **Proposition** The function \( \mu_H \in Ad_c(U_n) \) that was previously defined fulfills the property that for any \( A \in U_n \) and \( x \in A \):

\[
\exists \varepsilon > 0, \exists a \in B_2, \forall B \in U_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu_H(B) = a
\]  

(1)

**Proof** We define the real positive number

\[
\varepsilon = \begin{cases} 
\min \{ (x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 \mid y \in H - \{x\}, H - \{x\} \neq \emptyset \} & , H - \{x\} = \emptyset \\
1 & 
\end{cases}
\]

(2)

Such an \( \varepsilon \) exists, if not there would exist a sphere \( S_x \) with the center in \( x \) and the property that \( S_x \wedge H \) is infinite and this is a contradiction with the hypothesis \( H \in Loc_f^{(n)} \).

Any bounded set \( B \in U_n \) with the properties that \( x \in B \) and \( d(B) < \varepsilon \) fulfills the relations:

\[
(B - \{x\}) \wedge H = \emptyset
\]

(3)

\[
\mu_H((B - \{x\}) \vee \{x\}) = \mu_H(B - \{x\}) \oplus \mu_H(\{x\}) = \mu_H(\{x\}) = \pi(\{x\} \wedge H) = \begin{cases} 
1, x \in H \\
0, x \notin H 
\end{cases}
\]

(4)

(5)

5.8 Let now \( \mu : U_n \to B_2 \) be a measure.

a) We say that it is *derivable* in \( x \in A \), where \( A \subseteq U_n \), if

\[
\exists \varepsilon > 0, \exists a \in B_2, \forall B \in U_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu(B) = a
\]

(1)

b) In the case that the property of derivability of \( \mu \) takes place in any \( x \in A \), we say that \( \mu \) is *derivable on* \( A \).

c) If \( \mu \) is derivable on any set \( A \subseteq U_n \), then it is called *derivable*.

5.9 The number \( a \in B_2 \) depending on \( x \in A \) and the function \( A \ni x \mapsto a \in B_2 \) whose existence is stated in 5.8 are called the *derivative of* \( \mu \) *in* \( x \), respectively the *derivative function of* \( \mu \) *in* \( x \).

5.10 The derivative of \( \mu \) in \( x \) and the derivative function of \( \mu \) in \( x \) are noted with \( d\mu(x) \).

Other notations are:

- \( \frac{d\mu}{dt}(x) \), if \( n = 1 \)
- \( \frac{d\mu}{ds}(x) \), if \( n = 2 \)
- \( \frac{d\mu}{dv}(x) \), if \( n = 3 \)

5.11 **Remark** The set \( B \subseteq U_n \) formed by one element, \( x \in A \)

\[
B = \{x\}
\]

(1)

has the property that for any \( \varepsilon > 0 \),

\[
x \in B \text{ and } d(B) = 0 < \varepsilon
\]

(2)
from where it is inferred that, if $\mu$ is derivable in $x$, then $d\mu(x)$, that generally does not depend on $B$, is given by:

$$d\mu(x) = \mu(\{x\})$$  \hfill (3)

5.12 a) We suppose that $\mu$ is a derivable measure on $A$. The set

$$\text{supp}_A d\mu = \{x \mid x \in A, d\mu(x) = 1\} = \{x \mid x \in A, \mu(\{x\}) = 1\}$$  \hfill (1)

is called the \textit{support of $d\mu$ on $A$}.

b) If $\mu$ is derivable (on any set $A \in U_n$), then by definition the set

$$\text{supp } d\mu = \{x \mid x \in \mathbb{R}^n, d\mu(x) = 1\} = \{x \mid x \in \mathbb{R}^n, \mu(\{x\}) = 1\}$$  \hfill (2)

is called the \textit{support of $d\mu$ (on $\mathbb{R}^n$)}.

5.13 \textbf{Theorem} We consider the derivable measure $\mu : U_n \to \mathbb{B}_2$ on the closed set $A \in U_n$ ($A$ is compact). Then the set $\text{supp}_A d\mu$ is finite.

\textbf{Proof} Let us suppose that $\text{supp}_A d\mu$ is infinite, in contradiction with the conclusion of the theorem. Because $A$ is bounded, there exists (Cesaro) a convergent sequence $x^p \in \text{supp}_A d\mu$, $p \in N$ and the fact that $A$ is closed implies that

$$x = \lim_{p \to \infty} x^p$$  \hfill (1)

belongs to $A$. We apply the hypothesis of derivability of $\mu$ in $x$:

$$\exists \varepsilon > 0, \forall B \in U_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu(B) = \mu(\{x\})$$  \hfill (2)

We fix $\varepsilon$, $B$ like above so that for some $x^p \neq x$ it is true in addition $x^p \in B$. The set $B - \{x^p\}$ satisfies the same hypothesis like $B$, that is:

$$x \in B - \{x^p\} \text{ and } d(B - \{x^p\}) \leq d(B) < \varepsilon$$  \hfill (3)

and the conclusion must be the same:

$$\mu(B - \{x^p\}) = \mu(\{x\})$$  \hfill (4)

It is inferred that:

$$\mu(\{x\}) = \mu(B) = \mu((B - \{x^p\}) \cup \{x^p\}) =$$

$$= \mu(B - \{x^p\}) \oplus \mu(\{x^p\}) = \mu(\{x\}) \oplus 1$$  \hfill (5)

The last equation is a contradiction, having its origin in our supposition that $\text{supp}_A d\mu$ is infinite.

5.14 \textbf{Corollary} Let the measure $\mu : U_n \to \mathbb{B}_2$.

a) If $\mu$ is derivable on the topological closure $\overline{A}$ of the set $A \in U_n$, then:

\textbf{a.1)} the set $\text{supp}_A d\mu$ is finite

\textbf{a.2)} $\forall x \in A, \exists \varepsilon > 0, \forall B \in U_n, (x \in B \text{ and } d(B) < \varepsilon) \Rightarrow \mu(B - \{x\}) = 0$  \hfill (1)

\textbf{a.3)} $\mu(A) = \sum_{x \in A} \mu(\{x\})$  \hfill (2)

\textbf{a.4)} For any partition $A_i \subset A, i \in I$, we have

$$\mu(A) = \sum_{i \in I} \mu(A_i)$$  \hfill (3)

b) If the measure $\mu$ is derivable, then the set $\text{supp } d\mu$ is locally finite.
Proof a.3) If \( \text{supp}_A \ d\mu \) is empty, then for any \( x \in A \) we have that \( x \notin \text{supp}_A \ d\mu \) and by replacing in 5.8 (1) \( B \) with \( A \) and \( a \) with \( \mu(\{x\}) \), it results
\[
\mu(A) = \mu(\{x\}) = 0
\]
making the statement of the theorem obvious.

We suppose now that
\[
\text{supp}_A \ d\mu = \{x^1, \ldots, x^n\}, \ p \geq 1
\]
There exists a partition \( A_1, \ldots, A_p \in U_n \) of \( A \) with the property that \( x^i \in A_i, \ i = 1, p \) and moreover
\[
\mu(A) = \mu(\bigvee A_i) = \sum_{i=1}^{p} \mu(A_i) = \sum_{i=1}^{p} \mu(\{x^i\}) = \sum_{x \in A} \mu(\{x\}) = \pi(p)
\]

b) \( \mu \) is derivable on the compacts \( \overline{A} \in U_n \) and from a.1) all the sets
\[
A \land \text{supp} \ d\mu = \text{supp}_A \ d\mu
\]
are finite.

5.15 Let us suppose that \( \mu : U_n \to B_2 \) is derivable on the topological closure \( \overline{A} \) of \( A \in U_n \). The binary number
\[
\mu(A) = \sum_{x \in A} \mu(\{x\}) = \sum_{x \in A} d\mu(x) = \sum_{x \in R^n} f(x) \cdot d\mu(x)
\]
is noted with \( \int d\mu \), \( \int f \cdot d\mu \) or \( \int f \cdot d\mu \) and is called the integral of \( f : R^n \to B_2 \) relative to \( \mu \), where the relation between \( f \) and \( A \) is by definition the following:
\[
A = \{x \mid x \in R^n, f(x) = 1\} = \text{supp} \ f
\]

5.16 We note with
\[
I_{\text{Loc}}^{(n)} = \{f \mid f : R^n \to B_2, \text{supp} \ f \in \text{Loc}_{(n)}_f \}
\]
the \( B_2 \)-algebra of the functions with locally finite support, that are called locally integrable functions.

5.17 Theorem a) The function \( g \in I_{\text{Loc}}^{(n)} \) defines a derivable measure \( \mu^g : U_n \to B_2 \) by the formula:
\[
\mu^g(A) = \pi(\{A \land \text{supp} \ g \}), \ A \in U_n
\]
It is true the relation
\[
d\mu^g(x) = g(x), \ x \in R^n
\]
b) Conversely, if \( \mu : U_n \to B_2 \) is a derivable measure, then there exists in a unique manner the function \( g \in I_{\text{Loc}}^{(n)} \) so that
\[
\mu(A) = \pi(\{A \land \text{supp} \ g \}), \ A \in U_n
\]
being also true the relation
\[
d\mu(x) = g(x), \ x \in R^n
\]
Proof a) The fact that \( \mu^g \) is a measure was already proved at 5.6, if we put
\[
\mu^g (A) = \mu_{\text{supp } g} (A), \ A \in U_n
\]
and (2) results from
\[
d\mu^g (x) = \mu^g (\{x\}) = \pi(\{x\} \wedge \text{supp } g ) = \begin{cases} 
\pi(1) = 1, & \text{if } x \in \text{supp } g \\
\pi(0) = 0, & \text{if } x \not\in \text{supp } g 
\end{cases} = g(x)
\]

b) If \( \mu \) is derivable, then \( \text{supp } d\mu \in L_{oc}^{(n)} \) from 5.14 b) and the function 
\[
g : R^n \to B_2 \text{ defined by:}
\]
\[
g(x) = d\mu(x) = \mu(\{x\}), \ x \in R^n
\]
is locally integrable. As (4) was proved at (7), we prove (3) by taking into account 5.14 a.3): 
\[
\mu(A) = \sum_{x \in A} \mu(\{x\}) = \pi(\{A \wedge \text{supp } d\mu \}) = \pi(\{A \wedge \text{supp } g \}), \ A \in U_n
\]

5.18 Corollary For \( g \in I_{Loc}^{(n)} \) and \( A \in U_n \) it is defined the integral 
\[
\int_A g = \int_A d\mu^g = \mu^g (A) = \pi(\{A \wedge \text{supp } g \})
\]

6. The Lebesgue-Stieltjes Measure

6.1 We say about the function \( f : R \to B_2 \) that 

a) it has a left limit in any point \( t \in R \cup \{\infty\} \), if (see 3.10) 
\[
\forall t \in R \cup \{\infty\}, \exists t' < t, \exists f(t - 0) \in B_2, \forall \xi \in (t',t), \ f(\xi) = f(t - 0)
\]

b) it is left continuous in any point \( t \in R \), if a) is true in any \( t \in R \) and moreover: 
\[
\forall t \in R, \ f(t) = f(t - 0)
\]

6.2 We fix a function \( f \) satisfying the properties from 6.1. We prolong \( f \) to \( R \cup \{\infty\} \) by left continuity in the point \( \infty \): 
\[
f(\infty) = f(\infty - 0)
\]
and we note this new function with \( f \) too.

6.3 The relation 
\[
\mu([[a_1,b_1)]) \Delta ... \Delta [[a_n,b_n)]) = f(a_1) \oplus f(b_1) \oplus ... \oplus f(a_n) \oplus f(b_n)
\]
where \( a_1,..,a_n,b_1,..,b_n \in R \cup \{\infty\} \) obviously defines an additive function \( \mu : Sym^- \to B_2 \) (see 3.11 for the definition of the symmetrical intervals and of \( Sym^- \)). Our purpose is that of proving the next:

6.4 Theorem \( \mu \) is a measure.

Proof Let \( A_n \in Sym^-, n \in N \) a sequence of sets that are disjoint two by two with the property that the reunion 
\[
A = \bigvee_{n \in N} A_n
\]
belongs to \( Sym^- \). We can suppose without loss that all the sets \( A_n \) are non-empty.

Case a) \( (t_n)_n \) is a real strictly increasing sequence that converges to \( b \) and we have:
\[ -\infty < a = t_0 < t_1 < t_2 < \ldots < b \leq \infty \]  
\[ A_n = [t_n, t_{n+1}), \; n \in \mathbb{N} \]  
\[ A = [a, b) \]  

There exists an \( N \in \mathbb{N} \) with 
\[ n > N \Rightarrow \mu(A_n) = f(t_n) \oplus f(t_{n+1}) = f(b - 0) \oplus f(b - 0) = 0 \]  
and we can write that 
\[ \bigoplus_{n \in \mathbb{N}} \mu(A_n) = \sum_{n=0}^{N} \mu(A_n) = f(t_0) \oplus f(t_1) \oplus f(t_1) \oplus f(t_2) \oplus \ldots \oplus f(t_N) \oplus f(t_{N+1}) \]  
\[ = f(t_0) \oplus f(t_{N+1}) = f(a) \oplus f(b - 0) = f(a) \oplus f(b) = \mu(A) \]  

Case b) \( A \) is of the general form 
\[ A = [a_1, b_1) \lor [a_2, b_2) \lor \ldots \lor [a_k, b_k) \]  
where 
\[ -\infty < a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_k < b_k \leq \infty \]  

We note 
\[ A_{n,i} = A_n \land [a_i, b_i) \]  
where \( n \in \mathbb{N} \) and \( i = 1, \ldots, k \); we have: 
\[ \bigoplus_{n \in \mathbb{N}} \mu(A_n) = \bigoplus_{n \in \mathbb{N}} (\mu(A_{n,1} \lor A_{n,2} \lor \ldots \lor A_{n,k})) = \]  
\[ = \bigoplus_{n \in \mathbb{N}} (\mu(A_{n,1} \oplus \mu(A_{n,2}) \oplus \ldots \oplus \mu(A_{n,k})) = \]  
\[ = \bigoplus_{n \in \mathbb{N}} \mu(A_{n,1}) \oplus \bigoplus_{n \in \mathbb{N}} \mu(A_{n,2}) \oplus \ldots \bigoplus_{n \in \mathbb{N}} \mu(A_{n,k}) = \]  
\[ = \mu(\biglor_{n \in \mathbb{N}} A_{n,1}) \oplus \mu(\biglor_{n \in \mathbb{N}} A_{n,2}) \oplus \ldots \oplus \mu(\biglor_{n \in \mathbb{N}} A_{n,k}) = \]  
\[ = \mu([a_1, b_1)) \lor [a_2, b_2)) \lor \ldots \lor \mu([a_k, b_k)) = \]  
\[ = \mu([a_1, b_1) \lor [a_2, b_2) \lor \ldots \lor [a_k, b_k)) = \mu(A) \]  

6.5 The measure \( \mu \) that was defined at 6.3 is called the \( (\text{left}) \) Lebesgue-Stieltjes measure associated to \( f \).

6.6 The right dual construction is made starting from an \( R \to B_2 \) function (see 6.1) with a right limit in any \( t \in (-\infty) \lor R \), right continuous in any \( t \in R \) that is prolonged (see 6.2) to \( (-\infty) \lor R \) by right continuity in the point \(-\infty\). It is defined then (see 6.3) a measure 
\[ Sym^+ \to B_2, \]  
where \( Sym^+ \), the dual of \( Sym^- \), is the set ring generated by the symmetrical intervals 
\[ ((a,b)) = \begin{cases} 
(a,b], \ a < b \\
(b,a], \ b < a \\
\emptyset, \ a = b 
\end{cases} \]  
where \( a, b \in (-\infty) \lor R \).

6.7 **Theorem** Let \( \mu_1 : Sym^- \to B_2 \) an arbitrary measure.  

a) The function 
\[ g(t) = \mu_1([(a,t)]) \]
where $a, t \in \mathbb{R} \lor \{\infty\}$ is left continuous on $\mathbb{R} \lor \{\infty\}$.

b) $\mu_1$ is the left Lebesgue-Stieltjes measure associated to $g$.

**Proof**

a) It is considered the sequence $(t_n)_n$

$$-\infty < a = t_0 < t_1 < t_2 < ... < t \leq \infty$$

that is strictly increasing and convergent to $t$. The sets

$$A_n = [t_n, t_{n+1}), n \in \mathbb{N}$$

belong to $\text{Sym}^-$ and are disjoint two by two and their reunion

$$\forall \ n \in \mathbb{N}, A_n = [a, t)$$

is an element from $\text{Sym}^-$ too. It results that there exists $N \in \mathbb{N}$ with

$$n > N \Rightarrow \mu_1(A_n) = \mu_1([t_n, t_{n+1})) = \mu_1([a, t_n)) \oplus \mu_1([a, t_{n+1}]) = g(t_n) \oplus g(t_{n+1}) = 0$$

showing the existence of $g(t-0)$.

b) We have that

$$\mu_1([(a_1, b_1)) \Delta ... \Delta [(a_n, b_n)]) = \mu_1([(a_1, b_1)) \Delta ... \Delta [(a_n, b_n)]) = \mu_1([(a_1, b_1)) \oplus ... \oplus \mu_1([(a_n, b_n)]) = g(a_1) \oplus g(b_1) \oplus ... \oplus g(a_n) \oplus g(b_n)$$

is true for any $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R} \lor \{\infty\}$.

### 7. Measurable Spaces and Measurable Functions. The Integration of the Binary Functions Relative to a Measure

7.1 It is called *measurable space* a pair $(X, U)$ where $X$ is a set and $U \subset 2^X$ is a ring of subsets of $X$. The sets $A \in U$ are called *measurable*.

7.2 We say that we have defined a *measurable*, or *integrable function* $f : (X, U) \rightarrow B_2$ where $(X, U)$ is a measurable space, if it is given the function $f : X \rightarrow B_2$ with $\text{supp } f \in U$, i.e. the support of $f$ is a measurable set.

7.3 Recall that the set of the binary functions

$$U' = \{ f \mid f : (X, U) \rightarrow B_2, f \text{ is measurable} \}$$

(see 1.13, 1.14) is a $B_2$-algebra relative to the obvious laws. As ring, it is isomorphic with $U$.

7.4 Let $(X, U)$ a measurable space and $M \subset X$. Because the set
\[ U \land M = \{A \land M \mid A \in U\} \] (1)
is a set ring, the pair \((M, U \land M)\) is a measurable space, called measurable subspace of \((X, U)\).

7.5 **Proposition** a) If \( f : (X, U) \to B_2 \) is measurable, then its restriction \( f_{|M} : (M, U \land M) \to B_2 \) is measurable.

b) If \( g : (M, U \land M) \to B_2 \) is measurable, then it can be prolonged to a measurable function \( f : (X, U) \to B_2 \).

7.6 Let us suppose that \((X, U)\) is a measurable space, \( f : (X, U) \to B_2 \) is a measurable function and \( \mu : U \to B_2 \) is a measure. The number \( \mu(supp \ f) \) is called the integral of \( f \) relative to \( \mu \) and is noted with \( \int f \cdot d\mu \).

7.7 Let \( f_n, f \in U', n \in N \).

a) If
\[
\forall n \in N \quad supp \ f_0 \subseteq supp \ f_1 \subseteq supp \ f_2 \subseteq \ldots
\]
then we say that \( f_n \) converges, or tends increasingly to \( f \) and this fact is noted with \( f_n \uparrow f \).

b) If
\[
\forall n \in N \quad supp \ f_0 \supseteq supp \ f_1 \supseteq supp \ f_2 \supseteq \ldots
\]
then we say that \( f_n \) converges, or tends decreasingly to \( f \) and this fact is noted with \( f_n \downarrow f \).

c) In one of the situations from a), b) we say that \( f_n \) converges, or tends monotonously to \( f \) and the notation is \( f_n \downarrow f \).

7.8 Let us suppose that \( f, g : (X, U) \to B_2 \) are measurable and that \( \mu : U \to B_2 \) is a measure. We say that \( f \) and \( g \) are equal almost everywhere and we write this fact with \( f = g \text{ a.e.} \) if
\[
\mu(\{x \mid f(x) \neq g(x)\}) = \mu(supp \ f \Delta supp \ g) = 0
\]
or, in an equivalent manner, if
\[
\mu(supp \ f) = \mu(supp \ g)
\]
7.9 **Proposition** The function \( U' \ni f \mapsto \int f \cdot d\mu \in B_2 \) satisfies the following properties:

a) it is linear
b) \( f_n \downarrow f \Rightarrow \int f_n \cdot d\mu \to \int f \cdot d\mu \)
c) \( f = g \text{ a.e.} \Leftrightarrow \int f \cdot d\mu = \int g \cdot d\mu \)

where \( f_n, f, g \in U', n \in N \).

**Proof** b) is a restatement of 4.2 a) and 4.3 a).
7.10 **Corollary** If \( f_n \in U', \ n \in N \) converges to 0 decreasingly, then
\[
\int f_n \cdot d\mu \to 0
\] (1)

7.11 Let \( f_n, f : (X, U) \to B_2, \ n \in N \) measurable and \( \mu : U \to B_2 \) a measure. We say that \( f_n \) tends to \( f \) in measure and we note this property with \( f_n \to f \) if
\[
\int f_n \cdot d\mu \to \int f \cdot d\mu
\] (2)

7.12 Let \( f, \chi_A : X \to B_2 \) two functions, where \( f \) is arbitrary and \( \chi_A \) is the characteristic function of the set \( A \subset X \). If \( A \wedge \text{supp} \ f \in U \) - condition that is called of integrability- then the number
\[
\int_A f \cdot d\mu = \int (\chi_A \cdot f) \cdot d\mu
\] (1)
is called the integral of \( f \), on \( A \), relative to \( \mu \).

7.13 The function \( f : U \to B_2 \) defined by
\[
(f \cdot \mu)(A) = \int_A f \cdot d\mu
\] (1)
where \( A, \text{supp} \ f \in U \) is a measure, that coincides with the restriction of \( \mu \) at \( \text{supp} \ f \).

8. **Riemann Integrals**

8.1 We end the paper with a short paragraph that introduces the Riemann integrals of the \( f : R \to B_2 \) functions (generalizations are possible to \( f : R^n \to B_2 \) functions). The main feature for this type of integral is considering the set ring \( R_f(R) \) and the finite Boolean measure (see 3.8) \( \mu^f_R : R_f(R) \to B_2 \).

8.2 For the set \( A \subset R \), the property \( A \wedge \text{supp} \ f \in R_f(R) \) (see 7.12) is called the condition of Riemann integrability of \( f \) on \( A \). If it is fulfilled, we say that \( f \) is Riemann integrable, or integrable in the sense of Riemann, on \( A \).

8.3 **Special cases** for 8.2. a) \( [(a, b)) \wedge \text{supp} \ f \in R_f(R) \) (see 3.12 (1) for the definition of \( I_{[(a, b))]}, \ a, b \in R \lor [\infty] \). These functions are called left integrable (in the sense of Riemann) from \( a \) to \( b \).

\( R \wedge \text{supp} \ f \in R_f(R) \) (see 3.12 (2) for the definition of \( I_\infty \)). These functions are called integrable (in the sense of Riemann).

\( \forall a, b \in R, (a, b) \wedge \text{supp} \ f \in R_f(R) \) (see 5.4, 5.5, 5.16 for the definition of \( I^{(1)}_{\text{loc}} \)). These functions are called locally integrable (in the sense of Riemann) and they have a locally finite support.

\( \forall a, b \in R \lor [\infty], (a, b) \wedge \text{supp} \ f \in R_f(R) \) defines the \( B_2 \)-algebra of functions \( I_{\text{Sup}} \). We say about these functions that they are left integrable (in the sense of Riemann) and
that they have the support superiorly finite, dual notion to that of inferiorly finite set that was defined at 3.9.

8.4 If \( f \) is Riemann integrable on \( A \), then the number (see 7.12 (1))
\[
\int_A f \cdot d\mu_f^R = \mu_f^R (A \land \text{supp } f) = \sum_{x \in A} f(x)
\]
is called the integral, in the sense of Riemann, of \( f \), on \( A \).

8.5 Special cases for 8.4
a) \( f \in I_{[(a,b))}, a,b \in \mathbb{R} \lor \{\infty\} \); the integral \( \int_a^b f \cdot d\mu_f^R \) is noted with \( \int_a^b f \) and is called the left integral (in the sense of Riemann) of \( f \) from \( a \) to \( b \).

b) \( f \in I_{\infty} \); the integral \( \int_{\mathbb{R}} f \cdot d\mu_f^R \) is usually noted with \( \int_{-\infty}^{\infty} f \) and is called the integral (in the sense of Riemann) of \( f \).

8.6 The cases 8.3 a) and 8.5 a) have right duals, that refer to symmetrical intervals of the form \( ((a,b]), a,b \in \{\infty\} \lor \mathbb{R} \) (see 6.6).

8.7 We define the subring of sets \( \text{Sym}' \subset \text{Sym}^- \) to be the one that is generated by the symmetrical intervals \( [[a,b)), a,b \in \mathbb{R} \) (at \( \text{Sym}^- \) we had \( [[a,b)), a,b \in \mathbb{R} \lor \{\infty\} \)).

8.8 a) Let us suppose that \( f \in I_{(1)}^{\text{Loc}} \). Then the measure \( f \cdot \mu_f^R : \text{Sym}' \to B_2 \) (see 7.13) is called the indefinite integral of \( f \).

b) The function \( F^- : \mathbb{R} \to B_2 \), which is defined in the next manner:
\[
F^-(t) = f \cdot \mu_f^R (\{[a,t]\}), t \in \mathbb{R}
\]
where \( a \in \mathbb{R} \) is a parameter is called the left primitive of \( f \).

c) The left primitive \( F^-(t) \) has a left limit and it is left continuous in any \( t \in \mathbb{R} \).

8.9 If at 8.8 \( f \in I_{\text{Sup}} \) (where \( I_{\text{Sup}} \subset I_{(1)}^{\text{Loc}} \)), then \( f \cdot \mu_f^R \) is extended to \( \text{Sym}^- \) and \( F^- \) is extended to \( \mathbb{R} \lor \{\infty\} \), by left continuity in the point \( \infty \). \( f \cdot \mu_f^R \) is in this situation the left Lebesgue-Stieltjes measure associated to \( F^- \) (see 6.3).

8.10 Together with the duals in the left-right sense that have appeared having their origin in the order of \( \mathbb{R} \), the previous notions have also another type of duality, so called in the algebraical sense, resulting by the replacement of \( 0 \) with \( 1 \) and viceversa, to be compared, from the table 1.1, the laws ‘⊕’ and ‘⊗’. For example, the algebraical dual of \( \int_a^b f \) is defined like this:
\[ \int_{a}^{b} f = \bigotimes_{x \in ([a,b))} f(x) \]  

(1)

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