Lattice charge overlap and the elastic limit

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The results of a lattice simulation of time-separated charge overlap for the charged pion are discussed. The expected result \( \sim \exp[-(E_q - m_\pi)t] \) for large charge density overlap time separations, \( t \), is clearly visible in the Fourier transform, indicating that the elastic limit can be achieved at low to medium momentum values on present-sized lattices. The implications of this result for direct lattice simulations of hadron structure functions are discussed and a brief presentation of the lattice formalism is given.

1. INTRODUCTION

In trying to better understand the internal structure of hadrons in lattice QCD, charge overlap techniques are playing an increasingly important role. These methods are based upon the simulation of the hadronic matrix elements \( < h(0)|T[J_d^\mu(r,t)J_u^\nu(0)]|h(0)> \), where \( J_d^\mu \) and \( J_u^\nu \) are \( d, u \) flavor current densities. In their original form\[1,2\] these measurements, using \( t = 0 \) between the currents, allowed qualitative studies of lattice hadrons to be carried out. However, it has been pointed out that large Euclidean time separations of the currents allows one to extract form factor data from this matrix element in the so-called elastic limit\[3\]. This is interesting in its own right, but it is also clear that by replacing the flavor currents in this matrix element with the full electromagnetic ones, we are then studying the basic matrix element needed for hadron structure functions\[4\]. Thus, in moving from the qualitative to the quantitative stage, the elastic limit of these matrix elements forms an important bridge, leading to both elastic and inelastic properties of hadrons. The question of how readily this limit may be implemented in lattice simulations is the subject of the present investigation.

2. THEORY

Consider the lattice Euclidean time-separated charge overlap distribution for zero momentum pions:

\[
Q_{00}^{du}(q^2, t) = \sum_r e^{i\mathbf{q} \cdot \mathbf{r}} P_{00}^{du}(r, t),
\]

where

\[
\begin{align*}
Q_{00}^{du}(q^2, t) &\equiv \\
&= \sum_{\mathbf{x}} \langle \pi^+(0)|T[-\rho^d(\mathbf{r} + \mathbf{x}, t)\rho^u(\mathbf{x}, 0)]|\pi^+(0)\rangle e^{-(E_q - m_\pi)t},
\end{align*}
\]

For large Euclidean times the sum reduces to a single term and we find in the continuum limit that

\[
Q_{00}^{du}(q^2, t) \xrightarrow{t \gg 1} \frac{(E_q + m_\pi)^2}{4E_qm_\pi} \frac{F_\pi^2(q^2)}{\pi} e^{-(E_q - m_\pi)t},
\]

when the SU(2) flavor symmetry is present.

Figure 1. Symbolic representation of the \( d, u \) charge overlap measurement.

Thus, by separating the currents in time it is in principle possible to damp out the intermediate
state contributions and to measure form factors at arbitrary lattice momenta. On a finite space-time lattice, however, achieving this elastic limit is problematical. The exponential damping factor, \( e^{-\left(E_q - m_\pi\right) t} \), is not large and the fixed time positions of the initial and final pion interpolating fields limit the possible charge density separation times. We now turn to a numerical investigation of this limit.

3. SIMULATION

The present discussion is based on the numerical results found in Ref. [4], where more calculational details may be found. This study was conducted on 12 quenched \( \beta = 6.0 \) SU(3) configurations \((16^3 \times 24)\) at \( \kappa = .154 \) (the largest \( \kappa \) value studied in Ref. [5]). The pion interpolating fields were located at time slices 4 and 21 and the current densities were positioned as symmetrically possible in time between these sources. Two Wilson quark propagators per configuration, with origins at the positions of the interpolating sources, were necessary to extract the relative overlap function \( \mathcal{P}_{0u0}(r, t) \) in Eq. (2). \( \mathcal{P}_{0u0}(r, t) \) is actually the large Euclidean time limit of a similarly denoted quantity in Ref. [4].

The pion interpolating field at time slice 21 was constructed with quark propagators which were smeared over the entire spatial volume using the lattice Coulomb gauge. Both point and smeared pion fields were used at time slice 4. These propagators produce four-point functions which project exactly on zero pion momentum. Charge density self-contractions were neglected in forming these quantities.

Fig. 2 represents a \( \log_{10} \) plot of the Fourier transform, \( \mathcal{Q}_{0u0}^{du}(q^2, t) \), of the spatial density charge correlation function as a function of relative time separation, \( t \), between charge density operators. (We also assume the continuum relation \( E_q = \left(m_\pi^2 + q^2\right)^{1/2} \).) Actually shown in this figure are results for both point-to-smeared (✸) as well as smeared-to-smeared (✷) correlation functions. In all cases, the expected functional dependence \( \sim e^{-\left(E_q - m_\pi\right) t} \) is present, indicating that by time step 7 or 8 single exponential behavior has emerged. This is similar to the number of time steps needed in hadron spectrum calculations. This behavior is remarkable because although we are damping out intermediate states as \( t \) increases, we are also moving closer to possible contaminations from the fixed interpolating fields at either time end. In fact, we do not see any indications of such contamination in the data. These correlation functions are also unusual because the asymptotic approach is from below, indicating damping of negative terms in the \( d, u \) correlation function. Ref. [6] indicates that in the continuum limit, these are primarily positive G-parity states.

Although the point-to-smeared and smeared-to-smeared results exhibit essentially overlapping error bars, the smeared-to-smeared values are
systematically low compared to the point-to-
smear results, indicating a slight dependence
on the form of the interpolating field used. How-
evertheless, the results of Ref. [3] indicate that such a
dependence decreases as \( \kappa \to \kappa_{cr} \), that is, as the
physical regime is approached.

By fitting time steps 7 through 10 of Fig. 2 with
a single exponential of known slope and removing
the kinematical factor in Eq. (4), a value of \( F_{\pi}(q^2) \)
can be obtained. The results for the two calcu-
lated \( q^2/m_{\pi}^2 \) values using point-to-smeared corre-
lation functions are shown in Fig. 3 (\( \bullet \)). The val-
ues found are consistent within errors with vector
dominance, shown as the solid line. Also shown
in this figure are the results (\( \Box \)) from a previous
three-point-function simulation of the pion form
factor[8]. Comparison shows that the error bars
of these different techniques are of the same or-
der order of magnitude for similar numbers of configu-
trations. (Note the different \( \beta \) values and lattice
sizes of the two simulations, however.)

4. STRUCTURE FUNCTIONS

In order to begin to understand the significance
of these results for structure functions, let us re-
call the basic form of these quantities (continuum
Minkowski expression):

\[
W_{\alpha\beta}(q^2, \nu) = \frac{1}{2\pi} \int d^4x e^{-iq \cdot x} \cdot \frac{1}{2m}(\pi^+(0)|[J_{\alpha}(x), J_{\beta}(0)]|\pi^+(0)).
\]  

We adopt the lab frame where the external pion
momentums are zero and use the full electromag-
netic currents. In this frame \( \nu \equiv -p \cdot q/m = q_0 \).
In order to make contact with lattice expressions,
we imagine discretizing space, but keeping the
time variable continuous. Using the correspond-
ences,

\[
\int d^3x \rightarrow a^3 \sum_x,
\]

\[
|\pi^+(p)) \rightarrow [N_s a^3 2E_p]|\pi^+(p)>,
\]

\[
J_{\alpha}^{cont.} \rightarrow a^{-3} J_{\alpha}^{latt.},
\]

where \( N_s \) is the number of space points in the
lattice, this gives (all quantities on the lattice)

\[
W_{\alpha\beta}(q^2, \nu) = \frac{N_s}{2\pi} \int_{-\infty}^{\infty} dt e^{i\nu t} \sum_x e^{-iq \cdot x} \cdot <\pi^+(0)|[J_{\alpha}(x), J_{\beta}(0)]|\pi^+(0)>. \]  

By a standard set of manipulations, this can be
shown to result in the expression:

\[
W_{\alpha\beta}(q^2, \nu) = N_s^2 \sum_x \delta(\nu - E_x + m_\pi)
\cdot <\pi^+(0)|J_{\alpha}(0)|X(q) > < X(q)|J_{\beta}(0)|\pi^+(0) >. \]  

How is this quantity to be measured on the
lattice? Let us consider the generalization of
Eqs. (3) and (4) to arbitrary components of the
full electromagnetic current. Define in Euclidean
space

\[
Q_{\alpha\beta}(q^2, t) \equiv \sum_{r,x} e^{-iq \cdot r} \cdot <\pi^+(0)|T[J_{\alpha}(x + r, t)J_{\beta}(x, 0)]|\pi^+(0) >. \]  

Figure 3. Extracted pion form factor, \( F_\pi \), as a
function of \( q^2/m_{\pi}^2 \). The solid line is vector domi-
nance.
Again assuming $t > 0$, this results in
\[ Q_{\alpha\beta}(q^2, t) = N^2_x \sum_x e^{-(E_x - m_x) t} \cdot X(q) \]
\[ \cdot < \pi^+(0)|J_\alpha(0)|X(q) > < X(q)|J_\beta(0)|\pi^+(0) > . \] (9)

We notice that the connection between Eq. (8) and (9) is formally given by the inverse Laplace transform,
\[ W_{\alpha\beta}(q^2, \nu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt e^{\nu t} Q_{\alpha\beta}(q^2, t), \] (10)
where $c > 0$ and the contour is closed in the left-half complex $t$-plane. Since we are at fixed spatial momentum $q$, the lattice data extracted will actually cut a parabola-like path given by $x = q^2/(2m\sqrt{q^2 - q_0^2})$ in the $x$, $q^2$ plane ($x \equiv q^2/(2mv)$ and $q^2 \equiv q_\pi^2 - q_0^2$). In the lab frame the range of $\nu$ when $0 < \nu < 1$ is $|q| < \nu < (E_\pi - m_\pi)$. One can assure these kinematical constraints in the lattice model by restricting the assumed forms of $Q_{\alpha\beta}(q^2, t)$ to assure contributions to $W_{\alpha\beta}(q^2, \nu)$ only within the continuum range of $\nu$. We also need to make a quasi-continuum assumption about the lattice data: quantities like $W_{00}(q^2, \nu)$ should be very poorly fit by a positively weighted sum of exponentials. This assumption implies that $Q_{00}(q^2, t)$ is given by products of exponentials and inverse powers of $t$. (In this case Eq. (8) becomes purely formal but Eq. (10) continues to hold.) This whole discussion has been carried out for the pion, but it is clear that there is no barrier to applying these techniques to the phenomenologically more interesting proton case as well.

5. DISCUSSION

There are now two workable techniques for extracting form factor data from lattice simulations: direct current insertion and elastic charge overlap. The elastic limit also makes it possible to perform direct simulations of hadron structure functions. Attention so far has been focused on the moments of such functions, which are given by the operator product expansion. These expansions are based upon separation of the short-distance physics, calculated perturbatively, from the long-distance part, which can be evaluated on the lattice in the form of certain operator expectation values. Because of the perturbative assumption these methods work best at large $q^2$; direct lattice simulations must use low dimensionless $(qa)^2$, so these two techniques should be complementary. The question of whether the direct method can reach the scaling regime is still open, but note that the chiral limit (elastic) $q^2$ range in Ref. 8 was $0.6 \leq q^2 \leq 1.9 GeV^2$. Assuming the elastic limit of all necessary flavor-diagonal and non-diagonal four-point functions can be demonstrated, direct simulations of structure functions should be feasible with current computer technology.

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7. Note that an alternate, momentum-smeared (no $x$ sum in Eq. (8)) $d, u$ correlation function discussed in Ref. 8 never exhibits the crucial exponential behavior $e^{-(E_q - m_q) t}$ as the charge densities are separated in time $t$; it is no longer considered a viable continuum measurement by the present author.
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