COMMENTS ON BOSONISATION AND BIPRODUCTS

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Abstract We clarify the relation between the ‘bosonisation’ construction (due to the author) which can be used to turn a Hopf algebra $B$ in $H\mathcal{M}$ or $H^\mathcal{M}$ into an equivalent ordinary Hopf algebra, and a version of Radford’s theorem (also due in this form to the author) which does the same for $B$ in $H\mathcal{M}$. We also comment on reconstruction from the category of $B$-comodules.

Recently there appeared an interesting paper by Fischman and Montgomery [FM] in which some abstract constructions due to the author for Hopf algebras in braided categories [M6] [M7] [M8] [M9] were applied to obtain a Schur’s double centraliser theorem for color Lie algebras. Unfortunately, in the introduction and preliminaries of their paper, some incorrect statements are made which could cause confusion and which we would like to correct here in this note. The most serious is the identification $H\mathcal{M} = H^\mathcal{M}$ which we cover in Section 8. The main results about color-Lie algebras, etc., in [FM] are, fortunately, not affected. Finally, we add some related remarks about a recent preprint of Pareigis [P2].

Most of these comments were made to Susan Montgomery at the LMS meeting on Noncommutative Rings, Durham, UK in July 1992, where we were shown the first draft of [FM] and pointed out [M6] [M7] [M8] [M9]. We work over a ground field $k$ and use the usual notations $S, \epsilon$ for the antipode and counit, and the notation $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ for the coproduct applied to an element $h \in H$. We use the symbols $\triangleleft$, etc., for smash products, $\triangleright$ for smash coproducts and $\triangleright\triangleright$ when both are made simultaneously.

1. Dual quasitriangular structures. Let $H$ be a dual quasitriangular (or coquasitriangular) Hopf
algebra, i.e. equipped with a convolution-invertible linear map $\mathcal{R} : H \otimes H \to k$ obeying

$$
\mathcal{R}(h \otimes gf) = \sum \mathcal{R}(h(1) \otimes f)\mathcal{R}(h(2), g), \quad \mathcal{R}(fg \otimes h) = \sum \mathcal{R}(f \otimes h(1))\mathcal{R}(g \otimes h(2))
$$

$$
\sum g(1)h(1)\mathcal{R}(h(2) \otimes g(2)) = \sum \mathcal{R}(h(1) \otimes g(1))h(2)g(2)
$$

These axioms are nothing other than the axioms of a quasitriangular structure introduced by Drinfeld[1], with product replaced by convolution product of $\mathcal{R}$ as a linear functional. They were formulated by the author in 1989[2, below Thm 2.1],[3, Thm 4.1] cf.[2, Sec. 7.4-7.4] as equivalent under Tannaka-Krein reconstruction to the category $\mathcal{M}_H$ of right comodules becoming braided in the sense of [4]. The result was generalised in [3] to the dual quasitriangular quasi-Hopf algebra setting in the Fall of 1990. One can use left comodules $\mathcal{M}_H$ just as well.

The basic properties of dual quasitriangular Hopf algebras as algebraic objects are due to the author in [3, Appendix], again from the Fall of 1990. For example

**Proposition 1** [3, Prop A.5] Let $H$ be a dual quasitriangular Hopf algebra. Then the square of the antipode is inner in the convolution algebra $H \to H$ and hence the antipode is bijective.

This is nothing more than the dual of standard results (due ultimately to V.G. Drinfeld[4]) for quasitriangular Hopf algebras. From such basic results in [3] we see that the assumption that the antipode of $H$ is bijective in the paper of Fischman and Montgomery (e.g. in [5, Thm 2.15]) should be deleted as superfluous.

The matrix bialgebras $A(R)$ of [4] associated to a matrix $R$ are dual quasitriangular whenever $R$ obeys the so-called quantum Yang-Baxter equations or Artin braid relations. This result is due to the author in 1989 in [1, p. 141] as $\mathcal{R} : A(R) \to A(R)^*$ and was elaborated further as $\mathcal{R} : A(R) \otimes A(R) \to k$ by Larson and Towber [6] under another name.

2. **Braided groups.** In case the attribution in [5, p. 596] is not clear, we would like to stress that the theory of ‘braided groups’ or bialgebras and Hopf algebras $B$ living in a braided category is due to the present author in 1989 and V. Lyubashenko in 1990, and arose in both cases in connection with conformal field theory and generalised Tannaka-Krein reconstruction theorems. See [1] and [3] for a review. The theory is more subtle than the theory of Hopf algebras in symmetric categories, which have an older history such as [2]. If $A, B$ are two algebras in the braided category, one checks first that $A \otimes B$ has a braided tensor product algebra structure [3,4] with product morphism $(\cdot_A \otimes \cdot_B) \circ \Psi_{B,A}$ where $\Psi_{B,A} : B \otimes A \to A \otimes B$ is the braiding
morphism between any two objects in the category. Note that objects in a category need not be sets with elements. A bialgebra in a braided category is an algebra $B$ which is also a coalgebra with coproduct and counit $\Delta_B, \epsilon_B$ algebra homomorphisms. Here $\Delta_B : B \rightarrow B \otimes B$ to the braided tensor product algebra above. The morphisms $B \rightarrow B$ form an algebra under convolution and an antipode is defined as usual as an inverse for the identity. Basic lemmas about Hopf algebras in braided categories or braided groups are due to the author. For example,

**Proposition 2** [M10, Fig. 2] The antipode $S_B$ of a braided group $B$ is braided-antimultiplicative in the sense

$$S_B \circ \cdot_B = \cdot_B \circ \Psi_{B,B} \circ (S_B \otimes S_B), \quad \Delta_B \circ S_B = (S_B \otimes S_B) \circ \Psi_{B,B} \circ \Delta_B.$$ 

The proof [M10] in the braided case is non trivial and makes use of the coherence theorem for braided categories [JS] in the form of a diagrammatic notation in which algebraic information ‘flows’ along braids and tangles. Further basic properties such tensor products of modules and comodules in the category, smash products and coproducts in the category, etc, can be found in [M10, M9] where we give diagrammatic proofs for the module versions. The comodule versions are the same with all diagrams turned up-side-down. There is also a theory of quasitriangular structures and dual quasitriangular structures for such categorical braided groups.

3. **Braided groups in $M^H$.** In case it is not clear from [FM, Remark 1.6(e)], the theory and construction of bialgebras and Hopf algebras in the specific braided category of comodules of a dual quasitriangular Hopf $H$ is due to the author in [M5, M6]. It was presented at St Petersburg in September, 1990 and at the AMS Special Session on Hopf Algebras, San Francisco, January 1991, where it was presented to the ‘Hopf algebras community’ including Susan Montgomery. This was new even for the symmetric (unbraided) case of primary interest in [FM]. We use the notation $\Delta_B b = \sum b^{(i)} \otimes b^{(2)}$ for the coproduct of $B$ and $\beta(b) = \sum b^{(i)} \otimes b^{(2)}$ for a left or right coaction on it.

Basic properties are now $\Delta_B$ an algebra homomorphism to the braided tensor product algebra $B \underline{\otimes} B$ (the underscore is to distinguish it from the usual tensor product algebra) and that $S_B$ is braided-antimultiplicative (deduced from Proposition 2),

$$\Delta_B(bc) = \sum b^{(1)}_1 c^{(i)}_1 \otimes b^{(2)}_2 c^{(i)}_2 \mathcal{R}(b^{(2)}_2 \otimes c^{(2)}_2)$$

$$S_B(bc) = \sum (S_Bc^{(i)})(S_Bb^{(i)})\mathcal{R}(b^{(2)} \otimes c^{(2)}_2)$$

(2)
in the case of $B \in \mathcal{M}^H$, and similarly
\[
\Delta_B(bc) = \sum \mathcal{R}(e^{(1)}_{(1)} \otimes b^{(1)}_{(2)})b^{(1)}_{(1)} e^{(2)}_{(1)} \otimes b^{(2)}_{(2)} c_{(2)}
\]
\[
S_B(bc) = \sum \mathcal{R}(e^{(1)} \otimes b^{(1)}) (S_{BC}^{(2)}) (S_{B}^{(2)} b^{(2)})
\]
(3)
in the case of $H \mathcal{M}$. Here we would like to clarify [FM, eqn. (1.9)]: the property for $\Delta_B$ is part of the definition while the property for $S_B$ is derived from Proposition 2 due to the author. As well as studying such braided groups, the paper [M6] introduced a ‘transmutation’ procedure $B(\cdot, \cdot)$ for their construction by means of a generalised Tannaka-Krein reconstruction theorem. The property (2) is verified directly in [M3, Prop. A.5] for the example $B(H,H) \in \mathcal{M}^H$ associated canonically to $H$ itself.

4. **Dualisation.** We recall that it is one of the standard ideas behind the theory of ordinary Hopf algebras that for every construction in the category of $k$-modules based on hypotheses and proofs by commutative diagrams, there is a dual one by reversing all arrows. The axioms of a Hopf algebra are self-dual in this respect, while the axioms of a comodule are dual to those of a module, the construction of smash products dual to that of smash coproducts, etc. For constructions which can be expressed as commuting diagrams it is therefore redundant to give both module and comodule versions and one could not really be considered mathematically new once the other is known. The constructions to be considered here are all clearly of this diagrammatic type. Note that it is not any specific Hopf algebra which is being dualised here. This is analogous to the convention that a result for left modules is not new when reworked for right modules, etc., the symmetry in this case being reversal of $\otimes$ in the category of $k$-modules.

5. **Bosonisation.** One of the main results to date about braided groups is the ‘bosonisation theorem’ introduced by the author in [M9]. This generalises the Jordan-Wigner bosonisation transform for $\mathbb{Z}_2$-graded systems in physics, and associates to every Hopf algebra $B$ in the braided category of representations of $H$ an equivalent ordinary Hopf algebra $B \bowtie H$. The paper [M9] focused on the case where $B$ is in $H \mathcal{M}$ where $H$ is quasitriangular.

**Proposition 3** [M9, Thm 4.1] Let $H$ be quasitriangular. If $B \in H \mathcal{M}$ is a braided group then $B \bowtie H$ defined as the smash product by the canonical action $\bowtie$ of $H$ (by which $B$ is an object) and coproduct $\Delta b = \sum b_{(1)} \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} b_{(2)}$ for all $b \in B$ is a Hopf algebra over $k$. Here $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ is the quasitriangular structure.

As emphasised later in [M8, Thm 4.2], the coproduct is also a smash coproduct, namely by the induced coaction $\beta(b) = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} b$ introduced by the author in [M7, Thm 3.1]. Hopf
algebras which are both smash products and smash coproducts by the same (co)acting Hopf algebra have been called ‘biproducts’ by Radford\[R\]. The observation that bosonisations $B\triangleright\triangleleft H$ are examples of this type is due to the author in [M8]. This is not quite clear from [FM, Remark 1.17].

If $H$ is a dual quasitriangular Hopf algebra then it is a trivial matter to dualise the above construction in the sense of Section 4 above. If $B$ is a braided group in $\mathcal{M}^H$ we obviously make a smash coproduct $H\triangleright\triangleleft B$ by the canonical coaction of $H$ and smash product by the induced (right) action $b\triangleright h = \sum b^{(1)} R(b^{(2)} \otimes h)$. Likewise if $B$ in $H\mathcal{M}$ as in [FM] we make a left handed smash coproduct $B\triangleright\triangleleft H$ by the canonical coaction and smash product by the induced left action $h\triangleright b = \sum R(b^{(1)} \otimes h) b^{(2)}$. These are clearly the variants of the bosonisation theorem for Hopf algebras in $\mathcal{M}^H$ and $H\mathcal{M}$. Their further properties along the lines of [M9] are obtained by turning the diagrammatic proofs of these properties up-side-down or reversing arrows. For example:

**Proposition 4** cf[M9, Thm 4.2] Let $H$ be a dual quasitriangular Hopf algebra and $B$ a Hopf algebra in $\mathcal{M}^H$. The $B$-comodules in $\mathcal{M}^H$ can be identified canonically with $H\triangleright\triangleleft B$-comodules as monoidal categories over $k\mathcal{M}$.

**Proof** Cf[M9] a $B$-comodule in the category means a vector space which is an $H$-comodule and a $B$-comodule which intertwines the $H$-coaction. The corresponding coaction of $H\triangleright\triangleleft B$ consists of the $B$-coaction followed by the $H$-coaction. Using the properties of a dual quasitriangular structure one finds that this is an identification of monoidal categories (i.e. that the tensor product of comodules is respected). \(\Box\)

Likewise, left comodules of $B \in H\mathcal{M}$ in the category can be identified with left comodules of $B\triangleright\triangleleft H$. The more conceptual proof of Proposition 4 along the lines in [M9] is that

$$B(H\triangleright\triangleleft B, H) \cong B(H, H) \triangleright\triangleleft B$$

where $B(, )$ is the transmutation procedure from [M6, Thm 4.2] mentioned in Section 3, and the right hand side is the categorical smash coproduct of Hopf algebras in $\mathcal{M}^H$ with braided tensor product algebra structure. The corresponding categorical smash products were introduced in [M9, Thm 2.4] for just this purpose in the module setting.

We consider these results to be variants of a single bosonisation theorem and did not explicitly elaborate all (four) natural cases in [M9]. This differs from the presentation in [FM, Remark 1.17]
where the historical order also appears to us unclear (from the published left module version of the bosonisation theorem one at once derives the right module and the comodule versions). In any case, the first draft of [FM] shown to the author at the LMS meeting in July 1992 did not contain exactly the left or right comodule bosonisation formulae above, but a version appropriate to a more unnatural set-up involving $H^{\text{cop}}$-modules, see [FM, Remark 1.16]. It was explained to Susan Montgomery at the time that the correct dualisation of [M9] did not need such a formulation with $H^{\text{cop}}$. We would like to stress that this is as true for the right comodule bosonisation and biproducts as for the left comodule bosonisation and biproducts. Otherwise [FM, Remark 1.16] could lead to some confusion.

6. Duality and bosonisation. When $B, H$ are finite-dimensional then not only are the module and comodule bosonisation constructions dual, the specific examples are dual as ordinary Hopf algebras as well. One has [M11, Lem 4.3]

\[ H^* \bowtie B^* \cong (B \bowtie H)^* \]

where $B^* = B^{\text{op/cop}}$ is the natural dual of $B \in H\mathcal{M}$ where $H$ is quasitriangular. This is the new algebraic result in [M11] (which otherwise develops an application). Note that the most natural identification of categories is $H\mathcal{M} = \mathcal{M}^{H^*}$ by evaluation of a right coaction to give a left action of the dual. This is why we use the right comodule bosonisation in (5) and [M11].

7. Biproducts. In case it is not clear from [FM] we would like to stress that the picture for general biproducts $B \bowtie H$ (denoted $B \rtimes H$ in [FM, Prop. 1.15]) in terms of Hopf algebras in braided categories is due to the author in [M8, Prop. A.2] [M7]. We no longer assume that $H$ is quasitriangular or dual quasitriangular etc. Radford [R] elaborated the conditions for a smash product algebra and smash coproduct coalgebra $B \bowtie H$ to be a Hopf algebra, and characterised the Hopf algebras obtained in this way as those equipped with a Hopf algebra projection.

Let $H$ be a Hopf algebra with bijective antipode. The braided category of left crossed $H$-modules $H\mathcal{M}$ consists of objects $V$ which are both modules and comodules compatible in a natural way

\[ \sum h_{(1)} v^{(i)} \otimes h_{(2)} \triangleright v^{(2)} = \sum (h_{(1)} \triangleright v)^{(i)} h_{(2)} \otimes (h_{(1)} \triangleright v)^{(2)}. \]

These were studied by Yetter in [Y] as a generalisation of Whitehead’s crossed $G$-modules [W] and shortly thereafter by the author in [M7], where we observed that this category is nothing other than a reformulation appropriate to the case of $H$ infinite-dimensional of the well-known
braided category $D(H)\mathcal{M}$, where $D(H)$ is the quantum double construction of V.G. Drinfeld\cite{D1}.

One just views a module of $H^{\text{op}} \subset D(H)$ as an $H$-comodule in the usual way.

Now consider $H$ acting and coacting on an algebra and coalgebra $B$ as in the setting of \cite{R}.

It was observed in \cite{M7} (below Cor. 2.3 there) that $B \in H^H\mathcal{M}$ according to (\ref{eqn1}) is precisely one of Radford’s principal conditions for $B \bowtie H$ to be a bialgebra. The supplementary conditions that $B$ is an $H$-module coalgebra and an $H$-comodule algebra in \cite{R} were omitted from \cite{M7, Cor 2.3} but should be understood. In \cite{M8} (see proof of Proposition A.2) we noted further that the second of Radford’s principal conditions also has a categorical meaning, namely that $B$ is a bialgebra in $H^H\mathcal{M}$. Radford’s condition for an antipode corresponds to $B$ a Hopf algebra in $H^H\mathcal{M}$.

Hence

**Proposition 5** \cite{M3, Prop. A.2} Let $H$ be a Hopf algebra with bijective antipode. If $B$ is a Hopf algebra in the braided category $H^H\mathcal{M}$ then $B \bowtie H$ is an ordinary Hopf algebra with projection $B \bowtie H \to H$. Conversely, every Hopf algebra with projection to $H$ is of the form $B \bowtie H$ for $B$ a Hopf algebra in $H^H\mathcal{M}$.

As in \cite{M7}, we emphasised the $D(H)\mathcal{M}$ point of view but also explained that the same result holds for $H^H\mathcal{M}$ in the infinite-dimensional case. The work \cite{M8} was circulated in January 1992 and its original form is archived on ftp.kurims.kyoto-u.ac.jp as kyoto-net/92-02-07-majid. The right handed version for $B \in \mathcal{M}^H_H$ works just as well and gives an ordinary Hopf algebra $H \bowtie B$ by right handed smash product and coproduct. The right-handed version of (\ref{eqn1}) is

$$\sum v^{(i)} \bowtie h_{(1)} \otimes v^{(2)} h_{(2)} = \sum (v \bowtie h_{(2)})^{(i)} \otimes h_{(1)} (v \bowtie h_{(2)})^{(2)}$$

and when $H$ is finite-dimensional one has $H^H_\ast \mathcal{M} = \mathcal{M}^H_H$ by the standard identifications.

**8. Relating bosonisation and biproducts.** The biproduct point of view on our bosonisation constructions was emphasised in \cite{FM} based on an identification $H^H_\ast \mathcal{M} = H^H_\ast \mathcal{M}$ which is stated several times \cite[p.594, eqn (1.16), below Prop 1.15, Remark 1.16]{FM}. Indeed, the introduction of this paper does not discuss the author’s bosonisation work at all on this basis. Unfortunately, such an identification is incorrect and rather misleading because bosonisation and biproducts are not the same that one can eliminate one and replace it by the other. This is the principal confusion in \cite{FM} which we clarify in this note. It is important because bosonisations have many remarkable properties (such as Proposition 4 above) which are not true for general biproducts.

The functor which connects bosonisation to biproducts is in any case due to the author. In \cite[Prop. 3.1]{M7} we showed that if $H$ is quasitriangular then there is a functor of braided
monoidal categories

\[ H \mathcal{M} \leftrightarrow H_H \mathcal{M}, \quad (V, \triangleright) \mapsto (V, \triangleright, R(\triangleright)), \quad R(\triangleright)v = \sum R^{(2)} \otimes R^{(1)}_\triangleright v. \]  

(8)

This adds to an action \( \triangleright \) the induced coaction as used in Proposition 3. One verifies that the two fit together to form a crossed module, and that this identification respects tensor products. This means that an algebra, coalgebra, Hopf algebra etc in \( H \mathcal{M} \) can be viewed in \( H_H \mathcal{M} \). It is clear that \( B \triangleright H \) constructed by bosonisation can also be viewed as an example of a biproduct as already noted above and in [M8]. This functor in [M7] was the first of its kind. As an application we showed that the quantum double \( D(H) \) was itself a biproduct (in fact, a bosonisation).

The version when \( H \) is dual quasitriangular is obviously the functor

\[ \mathcal{M}H \leftrightarrow \mathcal{M}_H^H, \quad (V, \beta) \mapsto (V, \beta, \triangleright), \quad \triangleright h = v^{(1)}R(v^{(2)} \otimes h) \]  

(9)

for the right comodule version, or

\[ H \mathcal{M} \leftrightarrow H_H^H \mathcal{M}, \quad (V, \beta) \mapsto (V, \beta, \triangleright), \quad h \triangleright v = R(v^{(1)} \otimes h)v^{(2)} \]  

(10)

for the left comodule version. Here \( \triangleright \) is the induced action as in Section 5 and one can check directly from (8) that this identification respects tensor products. This is the variant or our result from [M7] which was used in the final version of [FM]. We note in passing that the category \( H_H \mathcal{M} \) is an example of a more general construction of the ‘Pontryagin dual’ [M12] or ‘double’ (in the sense of V.G. Drinfeld) of any monoidal category. The above functors likewise have a categorical origin [M13, Prop 2.5].

**Proposition 6** Let \( H \) be a quasitriangular Hopf algebra with \( H \neq k \). Then the functor \( H \mathcal{M} \rightarrow H_H^H \mathcal{M} \) introduced in [M7] is never an isomorphism. Likewise, let \( H \) be a dual quasitriangular Hopf algebra with \( H \neq k \). Then \( H \mathcal{M} \rightarrow H_H \mathcal{M} \) is never an isomorphism.

**Proof** This is clear in the finite-dimensional case from the construction in [M7], where this functor was introduced as pull back along a Hopf algebra projection \( D(H) \rightarrow H \). Since \( D(H) \) as a vector space is \( H^* \otimes H \), this can never be isomorphism. An isomorphism of categories would, by Tannaka-Krein reconstruction, require such an isomorphism. This is the conceptual reason. For a formal proof which includes the infinite-dimensional case, consider \( H \in H_H \mathcal{M} \) by the left regular coaction \( \Delta \) and left adjoint action. If in the image of the first functor (with \( H \) quasitriangular), then \( \sum h^{(1)} \otimes h^{(2)} = \sum R^{(2)} \otimes R^{(1)}(1)hSR^{(1)}(2) \) for all \( h \) in \( H \). Applying \( \epsilon \) to the second factor tells us that \( h = \epsilon(h) \) for all \( h \), i.e. \( H = k \). This object in \( H_H \mathcal{M} \) can be
in the image of the second functor, but this is iff the dual quasitriangular structure is trivial and \( H \) commutative. On other hand, consider \( H \in \mathcal{H} \) by the left regular action and left adjoint coaction. If in the image of the second functor (with \( H \) dual quasitriangular) then
\[
hg = \mathcal{R}(g(1)Sg(3) \otimes h)g(2).
\]
Setting \( g = 1 \) tells us that \( h = \epsilon(h) \) again, hence \( H = k \). This object can be in the image of the first functor, but this is iff the quasitriangular structure is trivial and \( H \) cocommutative.

This means in turn that general ‘biproduts’ associated to \( B \in \mathcal{H} \) are much more general than the Hopf algebras obtained by bosonisation when \( B \in \mathcal{H} \) for \( H \) quasitriangular or \( B \in \mathcal{H} \) for \( H \) dual quasitriangular, both of the latter being viewable as examples of biproducts.

9. Symmetric braiding. We would like to stress that inspite of the term ‘symmetric braiding’ used in \( \text{FM} \) for the CT structure of a cotriangular (CT) Hopf algebra, there is no canonical braid group action in this setting. This can be misleading. For example, the colour-Lie algebras etc. and their universal enveloping algebra studied in \( \text{FM} \) are defined in the obvious way with transposition replaced by a representation of the symmetric group. There is a theory of truly braided-Lie algebras and their enveloping algebras introduced in \( \text{M14} \) but the definition of the Jacobi identity is rather more complicated in the braided case.

10. Reconstruction. In the 1990 paper \( \text{M6, Thm 2.2} \) we introduced and proved a very general reconstruction theorem which yields a Hopf algebra \( \text{Aut}(\mathcal{C}, \omega, \mathcal{V}) \in \mathcal{V} \) associated to a monoidal functor \( \omega : \mathcal{C} \rightarrow \mathcal{V} \) between a monoidal category \( \mathcal{C} \) and a braided monoidal category \( \mathcal{V} \). This is the automorphism (or endomorphism) braided group of a functor and is due to the author as a significant generalisation of usual Tannaka-Krein ideas. It is constructed in \( \text{M6} \) as representing object for \( \text{Nat}(\omega, \omega(\ )). \) As an application of it we obtained in \( \text{M6} \) the transmutation construction \( B(\ , H) \) mentioned above, which turns any ordinary Hopf algebra mapping to \( H \) into a braided group in \( \mathcal{M}^H \). Here \( H \) is dual quasitriangular. So far, this transmutation remains one of the main constructions for Hopf algebras in braided categories such as \( \mathcal{M}^H \).

Recently there appeared an interesting preprint \( \text{P2} \) in the introduction of which the following question is posed: Let \( H \) be dual quasitriangular, \( B \) a Hopf algebra in \( \mathcal{M}^H \) and \( \omega \) the forgetful functor from \( B \)-comodules in \( \mathcal{M}^H \) to \( \mathcal{M}^H \). What is the automorphism braided group \( \text{Aut}(\omega) \) in \( \mathcal{M}^H \) reconstructed in this case? Our generalised reconstruction work is recalled explicitly in \( \text{P2, Sec. 3.4.1} \) and a partial answer (for the coalgebra structure when \( B \) is only a coalgebra) is obtained \( \text{P2, Cor 5.7} \) as a main result of the paper.
Here we point out that this question was already answered (the full braided group structure of \( \text{Aut}(\omega) \)) in the course of our 1991 paper on bosonisation [M9]. Some refinements of the problem to the case of ‘limited reconstruction’ over a control category in [P2] remain interesting and will not be addressed here.

**Proposition 7** Let \( H \) be dual quasitriangular and \( B \) a Hopf algebra in \( \mathcal{M}^H \). Then the forgetful functor \( \omega \) from \( B \)-comodules in \( \mathcal{M}^H \) to \( \mathcal{M}^H \) has as automorphism braided group the Hopf algebra \( B(H, H) \triangleright B \) in \( \mathcal{M}^H \) mentioned in (4). It has the smash coproduct coalgebra and braided tensor product algebra, and is a transmutation of the bosonisation \( H \triangleright B \) of \( B \).

**Proof** Under the equivalence in Proposition 4, the forgetful functor \( \omega \) becomes the functor induced by push-out along the canonical Hopf algebra map \( H \triangleright B \to H \) defined by the counit of \( B \). But the automorphism braided group of a functor induced by push out is exactly the definition of the transmutation construction \( B(\ , H) \). So the answer is exactly the transmutation \( B(H \triangleright B, H) \). But the abstract definition of bosonisation \( H \triangleright B \) is (4) i.e., exactly such that its transmutation is \( B(H, H) \triangleright B \). These are exactly the conceptual steps (in comodule form) which led to the author’s bosonisation theory [M9] in the first place. \( \square \)

This demonstrates how one may use the bosonisation theory of [M9]: we convert our problem for the braided group \( B \) to one for its equivalent ordinary Hopf algebra \( H \triangleright B \). Explicitly, the braided group \( B(H, H) \) associated to \( H \) was introduced as the automorphism braided group of the identity functor from \( \mathcal{M}^H \) to itself [M9] and corresponds to \( B = k \). Its structure is \( H \) as a coalgebra, with the right adjoint coaction and modified product [M6]

\[
\sum h^{(1)} \otimes h^{(2)} = \sum h_{(2)} \otimes (Sh_{(1)})h_{(3)}, \quad h \cdot g = \sum h_{(2)}g_{(2)}\mathcal{R}((Sh_{(1)})h_{(3)} \otimes Sg_{(1)})
\] (11)

in terms of the structure of \( H \). This \( B(H, H) \) coacts on any \( B \) by the same map \( \beta \) by which \( H \) coacts on \( B \) as an object (the tautological coaction). The fact that one can then make a (braided) smash coproduct by this and still obtain a Hopf algebra in the braided category with the braided tensor product algebra structure reflects the fact that \( B(H, H) \) is braided-commutative with respect to \( B \) in a certain (unobvious) sense introduced in [M9]. This was the key idea behind the construction in [M9, Sec. 2]. This \( B(H, H) \triangleright B \) has product and coproduct
defined diagrammatically cf. [M9, Sec. 2]
as generator of the category SuperVec of superspaces with their $\mathbb{Z}_2$-graded transposition. This novel application of quasitriangular Hopf algebras was generalised in 1991 in [M15] to generate

the braided category of anyonic or $\mathbb{Z}_n$-graded vector spaces introduced there. Unfortunately, in all these examples the adjoint coaction of $H$ is trivial and $B(H,H) = H$ is viewed trivially in $\mathcal{M}^H$. Hence the algebra structure of $B(H,H)$ is the usual tensor product one (and its smash coproduct the usual one as well). The result is a Hopf algebra in $\mathcal{M}^H$ just because $B$ is.

The same applies for Hopf algebras in the braided category of $\mathbb{Z}_n$-graded vector spaces.

To give a more non-trivial example, let $q \in k^*$ and $H = GL_q(2)$ defined as

\[ k\langle \alpha, \beta, \gamma, \delta, C \rangle \]

modulo the relations

\[ \alpha\beta = q^{-1}\beta\alpha, \quad \alpha\gamma = q^{-1}\gamma\alpha, \quad \beta\delta = q^{-1}\delta\beta, \quad \gamma\delta = q^{-1}\delta\gamma \]

\[ \beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = (q^{-1} - q)\beta\gamma, \quad C = \alpha\delta - q^{-1}\beta\gamma \]

essentially as in [D1][FRT] for $SU_q(2)$. We equip it now with dual quasitriangular structure determined by the associated solution $R$ of the quantum Yang-Baxter equations. More precisely (for our application) we take $R$ with a non-standard normalisation as explained in [M19], fixed instead by $R(C \otimes C) = q^6$. Note that one cannot set $C = 1$ as one would for the usual dual quasitriangular Hopf algebra $SU_q(2)$.

The braided group $B(GL_q(2), GL_q(2)) = BGL_q(2)$ is likewise a variant of the braided group $BSU_q(2)$ introduced by the author in [M5][M17]. We define it as $k\langle a, b, c, d, D \rangle$ modulo the relations

\[ ba = q^2ab, \quad ca = q^{-2}ac, \quad da = ad, \quad bc = cb + (1 - q^{-2})a(d - a) \]

\[ db = bd + (1 - q^{-2})ab, \quad cd = dc + (1 - q^{-2})ca, \quad D = ad - q^2cb \]

It has ‘matrix’ coproduct $\Delta u = u \otimes u$ and $\Delta D = D \otimes D$ when we regard the generators as a matrix $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The braided group antipode for $u$ is as for $BSU_q(2)$ in [M5][M17] times $D^{-1}$. The braiding $\Psi$ between the generators is also as listed for $BSU_q(2)$ in [M5][M17]. This $BGL_q(2)$ lives in $\mathcal{M}^{GL_q(2)}$ with a coaction which has the same ‘matrix conjugation’ form on the generators $u$ as the right adjoint coaction of $GL_q(2)$.

Let $B = k^2_q = k(x, y)/(yx - qxy)$ the $q$-deformed plane with right coaction of $GL_q(2)$ given by transformation of the $(x, y)$ as a row vector by the $GL_q(2)$ generators as a matrix, i.e. $\beta(x) = x \otimes \alpha + y \otimes \gamma$ and $\beta(y) = x \otimes \beta + y \otimes \delta$. One of the first applications of braided groups to physics was to show that this ‘quantum-braided plane’ $k^2_q$ is a Hopf algebra in $\mathcal{M}^{GL_q(2)}$ with
This result is due to the author in [M18], where $GL_q(2)$ above is formulated as $SU_q(2)$, the ‘dilatonic’ central extension.

We use the same matrix transformation for the braided coaction of $BGL_q(2)$ on $A_q^2$. Under this, $A_q^2$ becomes a right comodule algebra in the braided category [M19, Prop. 3.7].

Example 8 The automorphism braided group $BGL_q(2)\bowtie A_q^2$ in $M^{GL_q}(2)$ is generated by $BGL_q(2)$ and the quantum-braided plane $A_q^2$ as subalgebras with the cross relations

$$xa = ax, \quad ya = bx(q - q^{-1}) + ay, \quad xb = q^{-1}bx, \quad yb = qby$$

$$xc = qcx, \quad yc = (1 - q^{-2})(d - a)x + q^{-1}cy, \quad xd = dx, \quad yd = dy - q^{-2}(q - q^{-1})bx$$

It has the matrix coproduct of $BGL_q(2)$ and

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta y = y \otimes 1 + 1 \otimes y$$

extended as a braided group in $M^{GL_q}(2)$.

Proof The cross relations are exactly the braided tensor product algebra as in Section 2, computed for the present setting in terms of $R$ in [M19, Lem. 3.4]. This gives the relations shown. For the coproduct we know that we have the same form as the smash coproduct by the coaction of $GL_q(2)$ on $A_q^2$ but viewed now as a coaction of $BGL_q(2)$. To extend the coproduct to products of the generators we use its braided-multiplicativity, with $\Psi$ determined from the coaction. This was computed in terms of $R$ in [M19, Prop. 3.2] and in our case is

$$\Psi(a \otimes x) = x \otimes a + (1 - q^2)y \otimes c, \quad \Psi(b \otimes x) = q^{-1}x \otimes b + (q - q^{-1})y \otimes (a - d), \quad \Psi(c \otimes x) = qx \otimes c$$

$$\Psi(d \otimes x) = x \otimes d + (1 - q^{-2})y \otimes c, \quad \Psi\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \otimes y) = y \otimes \left(\begin{array}{cc} a & qb \\ q^{-1}c & d \end{array}\right)$$

while the braiding $\Psi(x \otimes a) = a \otimes x$ etc., has just the same form as the cross relations already given. It is enough to specify the coproduct and braiding on the generators since the braiding $\Psi$ itself extends ‘multiplicatively’ by functoriality and the Hexagon coherence identities, as explained in [M17]. □
The construction of linear braided groups such as the quantum-braided plane works for general quantum planes associated to suitable matrix data [M18]. Another example is the 1-dimensional case $B = A_q = k[x]$, the braided line [M16]. Such 'linear braided groups' have been very extensively studied since [M18] as the true foundation for $q$-deformed geometry. See [M20] for a review. Their bosonisations were used in [M18] to define inhomogeneous quantum groups and are also extensively studied since then.

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