Universal Rank Inference via Residual Subsampling with Application to Large Networks *

Xiao Han¹, Qing Yang² and Yingying Fan¹
University of Southern California¹ and Purdue University²

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Abstract

Determining the precise rank is an important problem in many large-scale applications with matrix data exploiting low-rank plus noise models. In this paper, we suggest a universal approach to rank inference via residual subsampling (RIRS) for testing and estimating rank in a wide family of models, including many popularly used network models such as the degree corrected mixed membership model as a special case. Our procedure constructs a test statistic via subsampling entries of the residual matrix after extracting the spiked components. The test statistic converges in distribution to the standard normal under the null hypothesis, and diverges to infinity with asymptotic probability one under the alternative hypothesis. The effectiveness of RIRS procedure is justified theoretically, utilizing the asymptotic expansions of eigenvectors and eigenvalues for large random matrices recently developed in [8] and [9]. The advantages of the newly suggested procedure are demonstrated through several simulation and real data examples.

* Xiao Han is Postdoctoral Scholar, Data Sciences and Operations Department, Marshall School of Business, University of Southern California, Los Angeles, CA 90089 (E-mail: xhan011@e.ntu.edu.sg). Qing Yang is Visiting Assistant Professor, Department of Statistics, Purdue University, West Lafayette, IN, 47907 (E-mail: qyang1@e.ntu.edu.sg). Yingying Fan is Professor of Data Sciences and Operations, Dean’s Associate Professor in Business Administration, Data Sciences and Operations Department, Marshall School of Business, University of Southern California, Los Angeles, CA 90089 (E-mail: fanying@marshall.usc.edu). This work was partly supported by NIH Grant 1R01GM131407 and NSF CAREER Award DMS-1150318.
1 Introduction

Matrix data have been popularly encountered in various big data applications. For example, many science and social applications consist of individuals with complicated interaction systems. Such system can often be modeled using a network with the nodes representing the $n$ individuals and the edges representing the connectivity among individuals. The overall connectivity can thus be recorded in an $n \times n$ adjacency matrix whose entries are zeros and nonzeros, representing the corresponding pair of nodes unconnected or connected, respectively. Examples include the friendship network, the citation network, the predator-prey interaction network, and many others.

There has been a large literature on statistical methods and theory proposed for analyzing matrix data. In the network setting, the observed adjacency matrix is frequently modeled as the summation of a deterministic low rank mean matrix and a random noise matrix, where the former stores all useful information in the data and is often the interest. One popular assumption made in most studies is that the rank $K$ of the latent mean matrix is known. However, in practice, such $K$ is generally unknown and needs to be estimated. This paper focuses on estimation and inference on the low rank $K$ in a general model setting including many popularly used network models as special cases.

In our model, the data matrix $X$ can be “roughly” decomposed as a low rank mean matrix $H$ with $K$ spiked eigenvalues and a noise matrix $W$ whose components are mostly independent. Here, $K$ is assumed to be fixed but unknown. To infer $K$ with quantified statistical uncertainty, we propose a universal approach for Rank Inference by Residual Subsampling (RIRS). Specifically, we consider the hypothesis test

$$H_0 : K = K_0 \text{ vs. } H_1 : K > K_0$$

(1)

with $K_0$ some pre-specified positive integer. The spiked mean matrix with rank $K_0$ can be estimated by eigen decomposition, subtracting which from the observed data matrix yields the residual matrix. Then by appropriately subsampling the entries of the residual matrix, we can construct a test statistic. We prove that under the null hypothesis, the test statistic converges in distribution to the standard normal, and under the alternative hypothesis, some spiked structure remains in the residual matrix and the constructed test statistic behaves very differently. Thus, the hypothesis test in (1) can be successfully conducted. Then by sequentially testing the hypothesis (1) for $K_0 = 1, \cdots, K_{\text{max}}$ with $K_{\text{max}}$ some large enough positive integer, we
can estimate $K$ as the first integer that makes our test fails to reject. We provide theoretical justifications on the effectiveness of our procedure.

A key for RIRS to work well is the carefully designed subsampling scheme. Although the noise matrix $W$ has mostly independent components, the residual matrix is only an estimate of $W$ and has correlated components. Intuitively speaking, if too many entries of the residual matrix are sampled, the accumulated estimation error and the correlation among sampled entries would be too large, rendering the asymptotic normality invalid. We provide both theoretical and empirical guidance on how many entries to subsample. In the special case where the diagonals of the data matrix $X$ are nonzero independent random variables (which corresponds to selfloops in network models), a special deterministic sampling scheme can also be used and the RIRS test takes a simpler form.

The structure of low rank mean matrix plus noise matrix is very general and includes many popularly used network models such as the Stochastic Block Model (SBM, [11, 20, 1]), Degree Corrected SBM (DCSBM, [13]), Mixed Membership (MM) Model, and Degree Corrected Mixed Membership (DCMM) Model [3] as special cases. RIRS test is applicable to all these network models, and in fact, goes beyond them.

Substantial efforts have been made in the literature on estimating $K$ in some specific network models, where $K$ is referred to as the number of communities. For example, [18] proposed an MCMC algorithm based on the allocation sampler to cluster the nodes in SBM and simultaneously estimate $K$. [3] developed a general variational inference algorithm to estimate the parameters in MM model with $K$ chosen according to some BIC criterion. [12] considered testing (1) with $K_0 = 1$ and proposed a signed polygon statistic which can accommodate the degree heterogeneity in the DCMM model. [10] proposed EZ statistics constructed by “frequencies of three-node subgraphs” to test (1) with $K_0 = 1$ in the setting of DCSBM. [4] introduced a linear spectral statistic to test $H_0 : K = 1$ vs. $H_1 : K = 2$ under the SBM. Compared to these works, we consider more general model and general positive integer $K_0$ that can be larger than 1.

There is also a popular line of work uses likelihood based methods to estimate $K$. For example, [7], [15], [19], and [21], among others. [6] proposed a network cross-validation method for estimating $K$ and proved the consistency of the estimator under SBM. [16] proposed to estimate $K$ using the spectral properties of two graph operators – the non-backtracking matrix and the Bethe Hessian matrix. [22] proposed to sequentially extract one community at a time by optimizing some extraction criterion, based on which they proposed a hypothesis test for testing the number of commu-
ities empirically via permutation method. [5] proposed a new test based on the asymptotic distribution of the largest eigenvalue of the appropriately rescaled adjacency matrix for testing whether a network is Erdös Rényi or not, and suggested a recursive bipartition algorithm for estimating $K$. [17] generalized the test in [5] for testing whether a network is SBM with some specific $K_0$, and proposed a sequential testing idea to estimate the true number of communities.

Among the existing literature reviewed above, the works by [5] and [17] are most closely related to ours. The main idea in both papers is that under the null hypothesis which assumes that the matrix data follows SBM with $K_0$ communities, the model parameters can be estimated and then the residual matrix can be rescaled. The rescaled residual matrix will be close to a generalized Wigner matrix whose extreme eigenvalues (after recentering and rescaling) converge in distribution to the Tracy-Widom distribution. However, under the alternative hypothesis, the extreme eigenvalues behave very differently. At a high level, this idea is related to ours in the sense that our proposal is also based on the residual matrix.

RIRS test differs from the literature in the way of using the residual matrix. Instead of investigating the spectral distribution of the residual matrix, we construct RIRS test by subsampling just a fraction of the entries in residual matrix. The subsampling idea ensures that the noise accumulation caused by estimating the mean matrix does not dominate the signal which guarantees the nice performance of our test. Compared to the existing literature, RIRS test behaves more like a nonparametric one in the sense that we do not assume any specific structure of the low rank mean matrix. Yet, it is also simple and fast to implement by nature. Our asymptotic theory is also new to the literature. It is built on the recent developments on random matrix theory in [8], which establishes the asymptotic expansions of the eigenvectors for a very general class of random matrices. This powerful result allows us to establish the sampling properties of RIRS test in equally general setting.

The remaining of the paper is organized as follows: Section 2 presents the model setting and motivation for RIRS. We introduce our new approach and establish its asymptotic theoretical results in Section 3. Simulations under various models are conducted to justify the performance of RIRS in Section 4. We further apply RIRS to a real data example in Section 5. All proofs are relegated to the Appendix and the Supplementary Material.
1.1 Notations

We introduce some notations that will be used throughout the paper. We use $a \ll b$ to represent $a/b \to 0$ and write $a \lesssim b$ if there exists a positive constant $c$ such that $0 \leq a \leq cb$. We say that an event $\mathcal{E}_n$ holds with high probability if $\mathbb{P}(\mathcal{E}_n) = 1 - O(n^{-l})$ for some positive constant $l$ and sufficiently large $n$. For a matrix $A$, we use $\lambda_j(A)$ to denote the $j$-th largest eigenvalue, and $\|A\|_F$, $\|A\|$, and $\|A\|_\infty$ to denote the Frobenius norm, the spectral norm, and the maximum elementwise infinity norm, respectively. In addition, denote by $A(k)$ the $k$th row of the matrix $A$. For a unit vector $x = (x_1, \cdots, x_n)^T$, let $d_x = \|x\|_\infty = \max |x_i|$ represent the vector infinity norm.

2 Model setting and motivation

2.1 Model setting

Consider an $n \times n$ symmetric random matrix $\tilde{X}$ which admits the following decomposition

$$\tilde{X} = H + W,$$

(2)

where $H = \mathbb{E}(\tilde{X})$ is the mean matrix with some fixed but unknown rank $K \ll n$ and $W$ is the noise matrix with bounded and independent entries on and above the diagonals. As mentioned in the introduction, model (2) includes popularly used network models as special cases. In such applications, the observed matrix $X$ is the adjacency matrix and can be either $\tilde{X}$ or $\tilde{X} - \text{diag}(\tilde{X})$, with the former corresponding to networks with self-loops and the latter corresponding to networks without self-loops, respectively. An important and interesting question is inferring the unknown rank $K$, which corresponds to the number of communities in network models. We address the problem by testing the hypotheses (1) under the universal model (2).

We note that with some transformation, model (2) can accommodate nonsymmetric matrices. In fact, for any matrix $\tilde{X}$ that can be written as the summation of a rank $K$ mean matrix and a noise matrix of independent components, we can define a new matrix as

$$
\begin{pmatrix}
0 & \tilde{X} \\
\tilde{X}^T & 0
\end{pmatrix}.
$$

It is seen that this new matrix has the same structure as in (2) with rank $2K$, and our new method and theory both apply. For simplicity of presentation, hereafter we assume the symmetric matrix structure for $\tilde{X}$ and $X$. 

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Write the eigen-decomposition of $H$ as $VDV^T$, where $D = \text{diag}(d_1, \ldots, d_K)$ collects the nonzero eigenvalues of $H$ in decreasing magnitude and $V = (v_1, \cdots, v_K)$ is the matrix collecting the corresponding eigenvectors. Denote by $d_1, \cdots, d_n$ the eigenvalues of $X$ in decreasing magnitude and $\hat{v}_1, \cdots, \hat{v}_n$ the corresponding eigenvectors. We next discuss the motivation of RIRS.

2.2 Motivation

To gain insights, consider the simple case when the observed data matrix $X = \tilde{X}$ and follows model (2). Then $E\tilde{W} = 0$. Thus intuitively, as $n \to \infty$, the normalized statistic $\sum_{i=1}^{n} w_{ii} / \sqrt{\sum_{i=1}^{n} Ew_{ii}^2}$ converges in distribution to standard normal. Meanwhile, we expect $\sum_{i=1}^{n} Ew_{ii}^2 / \sum_{i=1}^{n} w_{ii}^2$ to converge to 1 in probability as $n \to \infty$. The above two results entail that

$$\frac{\sum_{i=1}^{n} w_{ii}}{\sqrt{\sum_{i=1}^{n} w_{ii}^2}}$$

is asymptotically normal as the matrix size $n \to \infty$.

In the ideal case where the eigenvalues $d_1, \cdots, d_K$ and eigenvectors $v_1, \cdots, v_K$ are known, a test of the form (3) can be constructed by replacing $w_{ii}$ with $\tilde{w}_{ii}$ where $\tilde{W} = (\tilde{w}_{ij}) = X - \sum_{k=1}^{K_0} d_k v_k v_k^T$. Under the null hypothesis, $\tilde{W} = W$ and the corresponding test statistic (constructed in the same way as (3)) is asymptotically normal. However, under the alternative hypothesis, $\tilde{W}$ still contains some information from the $K - K_0$ smallest spiked eigenvalues and the corresponding eigenvectors and the test statistic is expected to exhibit different asymptotic behavior. Thus, the hypotheses in (1) can be successfully tested by using this statistic.

In practice, the eigenvalues and eigenvectors of $H$ are unavailable and need to be estimated. A natural estimate of $\tilde{W}$ takes the form

$$\hat{W} = (\hat{w}_{ij}) = X - \sum_{k=1}^{K_0} \hat{d}_k \hat{v}_k \hat{v}_k^T.$$  \hfill (4)

Under $H_0$, the residual matrix $\hat{W}$ is expected to be close to $W$, which motivates us to consider test of the form

$$\tilde{T}_n = \frac{\sum_{i=1}^{n} \hat{w}_{ii}}{\sqrt{\sum_{i=1}^{n} \hat{w}_{ii}^2}}$$ \hfill (5)
Intuitively, the asymptotic behavior of the above statistic is expected to be close to the one in (3). Thus, by examining the asymptotic behavior of $T_n$ we can test the desired hypotheses. In fact, it will be made clear later that one form of RIRS test is based on this intuition.

The statistic in (3) only uses the diagonals of $\mathbf{W}$. In theory, the asymptotically normality remains true if we aggregate any and all entries of the matrix $\mathbf{W}$ (instead of just the diagonals) and normalize properly, thanks to the independence of the entries on and above the diagonals of $\mathbf{W}$. However, this does not translate into the asymptotic normality of the test based on $\hat{\mathbf{W}}$ for at least two reasons: First, in applications absence of selfloops, the observed data matrix $\mathbf{X}$ takes the form $\hat{\mathbf{X}} - \operatorname{diag}(\hat{\mathbf{X}})$ and thus $\hat{\mathbf{W}}$ estimates $\mathbf{W} - \operatorname{diag}(\mathbf{X})$ which has nonrandom diagonals. Consequently, test constructed using diagonals of $\hat{\mathbf{W}}$ becomes invalid. Second, the entries of $\hat{\mathbf{W}}$ are all correlated and have errors coming from estimating the corresponding entries of $\mathbf{W}$. Aggregating too many entries of $\hat{\mathbf{W}}$ will cause too much noise accumulation. This together with the correlations among $\hat{w}_{ij}$ makes the asymptotic normality of the corresponding test statistic invalid. This heuristic argument is formalized in a later Section 3.5. Thus to overcome these difficulties, we need to carefully choose which and how many entries to aggregate. These issues are formally addressed in the next section.

3 Rank inference via residual subsampling

3.1 A universal RIRS test

The key ingredient of RIRS is subsampling the entries of $\hat{\mathbf{W}}$. Specifically, define i.i.d. Bernoulli random variables $Y_{ij}$ with $P(Y_{ij} = 1) = \frac{1}{m}$ for $1 \leq i < j \leq n$, where $m$ is some positive integer diverging with $n$ at a rate that will be specified later. In addition, set $Y_{ji} = Y_{ij}$ for $i < j$. A universal RIRS test that works under the broad model (2) takes the following form

$$T_n = \frac{\sqrt{m} \sum_{i \neq j} \hat{w}_{ij} Y_{ij}}{\sqrt{2 \sum_{i \neq j} \hat{w}_{ij}^2}}. \quad (6)$$

The effect of $m$ is to control on average how many entries of the residual matrix to aggregate for calculating the test statistic. It will be made clear in a moment that $m$ needs to grow to infinity in order for the central limit theorem to kick in. However, the growth rate cannot be too fast because otherwise the noise accumulation and the correlation in $\hat{w}_{ij}$ will make the asymptotic normality invalid.
The following conditions will be used in our theoretical analysis.

**Condition 1.** $W$ is a symmetric matrix with independent and bounded upper triangular entries (including the diagonals) and $Ew_{ij} = 0$ for $i \neq j$.

**Condition 2.** There exists a positive constant $c_0$ such that $\frac{|d_i|}{|d_j|} \geq 1 + c_0$ for all $1 \leq i < j \leq K, d_i \neq -d_j$.

**Condition 3.** There exists a positive sequence $\theta_n$, which may tend to 0 as $n \to \infty$, such that $|d_i| |d_j| \geq 1 + c_0$ for all $1 \leq i < j \leq K, d_i \neq -d_j$.

**Condition 3.** There exists a positive sequence $\theta_n$, which may tend to 0 as $n \to \infty$, such that $\sigma_{ij}^2 = \text{var}(w_{ij}) \leq \theta_n$ and $\max_{1 \leq i \leq n} |h_{ii}| \lesssim \theta_n$, where $h_{ii}$’s are the diagonal entries of matrix $H$. In addition, $\alpha_n^2 = \max_i \sum_{j=1}^n \sigma_{ij}^2 \to \infty$ as $n \to \infty$, $|d_K| \gtrsim \alpha_n^2$ and $|d_K| \gtrsim n^\epsilon$ for some positive constant $\epsilon$.

**Condition 4.** $\|V\|_\infty \lesssim \frac{1}{\sqrt{n}}$.

**Condition 5.** It holds that $\sum_{i \neq j} \alpha_n^2 \gg m$ and $\sum_{i \neq j} \sigma_{ij}^2 \gtrsim n^\epsilon \left( \frac{\sum_{k=1}^K (Y^T V_k)^2}{m} + \frac{n^2 \alpha_n^2}{m \sigma_K^2} \right)$ for some positive constant $\epsilon$.

Conditions 1-2 are also imposed in [8], where asymptotic expansions of spiked eigenvectors are established. The results therein serve as the theoretical foundation of RIRS. Random matrix satisfying Condition 1 is often termed as generalized Wigner matrix in the literature. Conditions 2 and 3 restrict the spiked eigenvalues of the low rank mean matrix. The constraint $|d_K| \gtrsim \alpha_n^2$ in Condition 3 is a technical condition for controlling the noise accumulation in our test caused by estimating $w_{ij}$. It can be easily satisfied by many network models with low rank structure. To see this, note that if $w_{ij}, j \geq i \geq 1$ follows Bernoulli distribution and $\sigma_{ij}^2 \sim \theta_n$, then $\alpha_n^2 \sim n \theta_n$. Since $h_{ij}$’s and $\sigma_{ij}^2$’s are the means and variances of Bernoulli random variables, respectively, we have $h_{ij} \sim \sigma_{ij}^2 \sim \theta_n$ and $\|H\|_F = \left\{ \sum_{i,j} h_{ij}^2 \right\}^{1/2} \sim n \theta_n$. Note also that $\|H\|_F = \left\{ \sum_{i=1}^K d_i^2 \right\}^{1/2}$ and $K$ is finite. These together with $\alpha_n^2 \sim n \theta_n$ derived earlier ensure that $|d_K| \gtrsim \alpha_n^2$ is not hard to be satisfied. In fact, if in addition $d_1 \sim d_K$ and $\theta_n \lesssim 1$, we have $|d_K| \gtrsim \alpha_n^2$ satisfied. Condition 4 is a technical condition needed to prove the key Lemmas 2-3. Condition 5 characterizes what kind of $m$ can make RIRS succeed. More detailed discussion on the choice of $m$ will be given in a later section.

**Theorem 3.1.** Assume Conditions 1-5. Under null hypothesis in (1) we have

$$T_n \xrightarrow{d} N(0, 1), \text{ as } n \to \infty. \quad (7)$$
Theorem 3.2. Assume Conditions 1-5 and the alternative hypothesis in (1). If \( \sum_{i \neq j} (\sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j))^2 \ll \sum_{i \neq j} \sigma_{ij}^2 \), then as \( n \to \infty \),

\[
\sqrt{m} \left( \sum_{i \neq j} \hat{w}_{ij} Y_{ij} - \sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i) v_k(j) Y_{ij} \right) \xrightarrow{d} N(0, 1). \tag{8}
\]

If instead,

\[
\left| \sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i) v_k(j) \right| \gg \sqrt{m} \left( \sqrt{\sum_{i \neq j} \sigma_{ij}^2} + \sum_{k=K_0+1}^{K} |d_k| \right), \tag{9}
\]

we have

\[
P(|T_n| > C) \to 1, \text{ as } n \to \infty \tag{10}
\]

for arbitrarily large positive constant \( C \).

By Theorems 3.1 and 3.2, we have the following Corollary about the size and power of RIRS.

**Corollary 1.** Under the conditions of Theorem 3.1, we have

\[
\lim_{n \to \infty} P(|T_n| \geq \Phi^{-1}(1 - \alpha/2)|H_0) = \alpha,
\]

where \( \Phi^{-1}(t) \) is the inverse of the standard normal distribution function, and \( \alpha \) is the pre-specified significance level. Alternatively under the same conditions for ensuring (10), we have

\[
\lim_{n \to \infty} P(|T_n| \geq \Phi^{-1}(1 - \alpha/2)|H_1) = 1.
\]

We remark that the above theoretical results hold even under extreme degree heterogeneity in network models. In fact, in degree corrected membership model the mean matrix takes the form

\[
H = \Theta \Pi \Pi^T \Theta,
\tag{11}
\]

where \( \mathbf{B} \) is a \( K \times K \) nonsingular matrix with all entries taking values in \([0, 1] \), \( \Theta = \text{diag}(\vartheta_1, \ldots, \vartheta_n) \) with \( \vartheta_i > 0 \) is the degree heterogeneity matrix, and \( \Pi = (\pi_1, \ldots, \pi_n)^T \) is a \( n \times K \) matrix of probability mass vectors. Since we do not have any direct constraints on the smallest variance of \( w_{ij} \), all our theoretical results remain to hold even when \( \max_j \vartheta_j / \min_j \vartheta_j \to \infty \).
3.2 Choice of \( m \)

It is seen from the previous two theorems that the tuning parameter \( m \) plays a crucial role for RIRS to achieve the desired size with high power. Condition 5 provides general conditions on the choice of \( m \) for ensuring the null and alternative distributions in (7) and (8). For (10) to hold, we also need the additional assumption (9). In some special cases, these conditions boil down to simpler forms which can provide us more specific guideline on the choice of \( m \).

As an example, we consider a special case that

\[
\min_{i \neq j} \sigma_{ij}^2 \sim \max_{i \neq j} \sigma_{ij}^2, \quad |d_1| \lesssim n\theta_n, \quad \text{and for } K_0 < K,
\]

(12)

\[
\sum_{i \neq j} \sigma_{ij}^2 \lesssim \left| \sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i)v_k(j) \right|.
\]

For SBM with \( K \) communities, there are at most \( K(K+1)/2 \) different variances in the entries of the adjacency matrix and hence the first condition in (12) is not hard to be satisfied. For other network models, this condition may also be satisfied with additional assumptions. Note that \( \|H\|_F = \left\{ \sum_{i,j} h_{ij}^2 \right\}^{1/2} = \left\{ \sum_{i=1}^{K} d_i^2 \right\}^{1/2} \). If the entries of \( \tilde{X} \) follow Bernoulli distribution then \( h_{ij} \sim \sigma_{ij}^2 \), and thus the second condition in (12) is satisfied in view of Condition 3. To understand the intuition of the third condition, note that under the alternative hypothesis we have the following decomposition

\[
\tilde{X} = \sum_{k=1}^{K_0} d_k v_k v_k^T + \sum_{k=K_0+1}^{K} d_k v_k v_k^T + W.
\]

The second term on the right hand side corresponds to the signal missed by the null hypothesis, and the third term corresponds to the noise. Thus, the third condition in (12) intuitively says that under the alternative hypothesis, the cumulative missed signal \( \sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i)v_k(j) \) cannot be dominated by the noise accumulation.

The next theorem specifies what kind of \( m \) satisfies the two inequalities in Condition 5 and (9).

**Theorem 3.3.** Set \( \theta_n = \max_{i \neq j} \sigma_{ij}^2 \). Assume (12). Then \( m \) satisfying the following condition

\[
\frac{n^e}{\theta_n} \log n + n^{e-1} \theta_n^{-2} (\log n)^2 \ll m \ll n^2 \theta_n (\log n)^{-2}
\]

(13)
makes Condition 5 and inequality (9) hold. Consequently, (7) and (10) hold under Conditions 1–4. Moreover, a sufficient condition for (13) is $n^{1-\epsilon} \ll m \ll n^{1+2\epsilon}(\log n)^{-1}$ under Conditions 1–4.

It is seen that Theorem 3.3 allows for a wide range of values for $m$. In theory, any $m$ satisfying (13) guarantees the asymptotic size and power of our test. In implementation, we found smaller $m$ in this range yields better empirical size.

It is also seen from (13) that RIRS works with very sparse networks. In fact, the only sparsity condition imposed by (13) is that $\theta_n = \max_{i \neq j} \sigma_{ij}^2 \gg n^{-1+\epsilon/2}$, where $\epsilon$ is a constant that can be arbitrarily small. In SBM, this corresponds to the very sparse setting with average degree of order $n^{-1+\epsilon/2}$. Our sparsity condition is significantly weaker than the ones in related work in the literature. In particular, both [5] and [17] considered dense SBM with $\theta_n$ bounded below by some constant.

We remark that sparser models have been considered in the network literature, though mostly in estimation instead of inference problems. For example, [21] proposed a model selection criterion for estimating $K$ under the very sparse setting of SBM with $n\theta_n/\log n \to \infty$. [16] established the consistency of their method for estimating $K$ under the setting $n\theta_n = O(1)$.

We need slightly stronger assumption on the sparsity level because we consider the statistical inference problem of hypothesis testing, which involves more delicate analyses for establishing the asymptotic distributions of the test statistic.

3.3 A special case: networks with self-loops

We formalize the heuristic arguments in Section 2.2 about the ratio statistic $\tilde{T}_n$ in (5) when the network admits self-loops. In such case, the general test (6) still works. However, the simpler one $\tilde{T}_n$ can enjoy similar asymptotic properties.

**Theorem 3.4.** Suppose that Conditions 1-4 hold, the network contains self-loops and $\sqrt{\sum_{i=1}^{n} \sigma_{ii}^2} \gg n^\epsilon$ for some positive constant $\epsilon$.

(i) Under null hypothesis we have

$$\tilde{T}_n \overset{d}{\to} N(0, 1), \text{ as } n \to \infty. \quad (14)$$

(ii) Under alternative hypothesis, if further $\sum_{i=1}^{n} (\sum_{k=K_0+1}^{K} d_k v(i)^2)^2 \ll$
\[ \sum_{i=1}^n \sigma_{ii}^2, \text{ we have} \]
\[ \frac{\sum_{i=1}^n \hat{w}_{ii} - \sum_{k=K_0+1}^K d_k}{\sqrt{\sum_{i=1}^n \hat{w}_{ii}^2}} \xrightarrow{d} N(0, 1), \text{ as } n \to \infty. \tag{15} \]

If instead, \(|\sum_{k=K_0+1}^K d_k|^2 \gg \sum_{i=1}^n \sigma_{ii}^2 + \sum_{i=1}^n \left(\sum_{k=K_0+1}^K d_k v_k^2(i)\right)^2\), then
\[ \mathbb{P}(|\tilde{T}_n| > C) \to 1, \tag{16} \]
for arbitrarily large positive constant \(C\).

It is seen that with the same critical value \(\Phi^{-1}(1 - \alpha/2)\), \(\tilde{T}_n\) enjoys the same properties on size and power as \(T_n\). In addition, since the construction of \(\tilde{T}_n\) does not depend on any tuning parameter, the implementation is much easier.

### 3.4 Estimation of \(K\)

RIRS naturally suggests a simple method for estimating the rank \(K\). The idea is similar to the one in [17]. That is, we sequentially test the following hypotheses
\[ H_0 : K = K_0 \quad \text{vs.} \quad H_1 : K > K_0, \]
for \(K_0 = 1, 2, ..., K_{\text{max}}\) at the significant level \(\alpha\) using RIRS. Here, \(K_{\text{max}}\) is some prespecified positive integer. Once RIRS fails to reject a value of \(K_0\), we stop and use it as the estimate of the rank. Since we assume the true value of \(K\) is finite, it is easy to see from Corollary 1 that with asymptotical probability \(1 - \alpha\) we can identify the true value of the rank.

### 3.5 Networks without self-loops: why subsampling?

In this section, we formalize the heuristic arguments given in Section 2.2 on why sub-sampling is necessary. We theoretically show that \(T_n\) is no longer a valid test for \(H_0\) without the ingredient of subsampling.

We start with introducing some additional notations that will be used in this subsection. For any matrices \(M_1\) and \(M_2\) of appropriate dimensions, let
\[ \mathcal{R}(M_1, M_2, t) = - \sum_{l=0, l \neq 1}^L \frac{M_1^T E W^l M_2}{t^{l+1}}, \quad \mathcal{P}(M_1, M_2, t) = t \mathcal{R}(M_1, M_2, t), \]
where $L$ is a positive integer such that
\[
\frac{\alpha_n^{L+1}(\log n)^{L+1}}{|d_K|^{L-2}} \to 0.
\]

By Lemma 1 and Theorem 1 of [8], there exists a unique $t_k$ such that \(t_k \to 1, \quad 1 \leq k \leq K\) and \(d_k - t_k = v_k^T W v_k + O_p(\frac{\alpha_n}{|d_k|})\). Define
\[
b_{e_i,k,t}^T = e_i^T - R(e_i, V_{-k}, t)\left((D_{-k})^{-1} + R(V_{-k}, V_{-k}, t)\right)^{-1} v_{-k}^T,
\]
\[
s_{k,i} = b_{e_i,k,t} - e_i^T v_k, \quad s_k = \sum_{i=1}^n s_{k,i}, \quad s_k(i) = e_i^T s_k,
\]
and \(r_k = V_{-k}(t_k D_{-k}^{-1} - I)^{-1} V_{-k}^T W^2 v_k\), where $V_{-k}$ is the submatrix of $V$ by removing the $k$-th column, and we slightly abuse the notation and use $D_{-k}$ to denote the submatrix of $D$ by removing the $k$th diagonal entry.

Further define $a_k = \sum_{i=1}^n v_k(i), \quad k = 1, \ldots, K$ and
\[
R(K) = 2 \sum_{k=1}^K \frac{1}{t_k} \frac{v_k^T W^2 v_k a_k}{d_k} + 2 \sum_{k=1}^K a_k^2 v_k^T W^2 v_k + \sum_{k=1}^K v_k^T \text{diag}(W) v_k a_k^2 + 2 \sum_{k=1}^K a_k s_k^T \text{diag}(W) v_k + 2 \sum_{k=1}^K a_k \frac{1}{t_k} r_k. \tag{17}
\]

We have the following theorems.

**Theorem 3.5.** Suppose that Conditions 1–4 hold and
\[
\sum_{i<j} \sigma_{ij}^2 \left(1 - \sum_{k=1}^{K_0} a_k^2 v_k(i) v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j) s_k(i) + v_k(i) s_k(j))\right) \geq n^{r_1} \left(1 + \frac{n^2 \alpha^2}{d_{K_0}^2}\right), \tag{18}
\]
for some positive constant $\epsilon_1$. Under null hypothesis we have, as $n \to \infty$,
\[
\sqrt{n} \sum_{i \neq j} \tilde{w}_{ij} + R(K_0) \quad \xrightarrow{d} N(0,1). \quad 2 \sqrt{\sum_{i<j} \sigma_{ij}^2 \left(1 - \sum_{k=1}^{K_0} a_k^2 v_k(i) v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j) s_k(i) + v_k(i) s_k(j))\right)} \to N(0,1).
\]

13
Theorem 3.6. Suppose that Conditions 1–4 hold. In addition, assume
\[(18)\]
holds with \(K_0\) and \(d_{K_0}\) replaced with \(K\) and \(d_K\), respectively. Under alternative hypothesis we have, as \(n \to \infty\),
\[
\frac{\sum_{i \neq j} \hat{w}_{ij} + R(K) - \sum_{k=K_0+1}^{K} d_k a_k^2}{2 \sqrt{\sum_{i<j} \sigma_{ij}^2 \left(1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))\right)}} \xrightarrow{d} N(0,1).
\]

It is seen from Theorems 3.5 and 3.6 that aggregating all entries of the residual matrix leads to a statistic with bias and variance taking very complicated forms under both null and alternative hypotheses. The complicated forms of bias and variance limit the practical usage of the above results. In addition, and more importantly, these results may even fail to hold in some cases.

To understand this, note that the variance of \(\sum_{i \neq j} \hat{w}_{ij} + R(K)\) in Theorem 3.5 is approximately equal to
\[
4 \sum_{i<j} \sigma_{ij}^2 \left(1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))\right)^2.
\]
Condition (18) is imposed to put a lower bound on the variance. Without this condition, the asymptotic normality in Theorem 3.5 will no longer hold. However, we next give an example where inequality (18) fails to hold.

Consider networks with eigenvector taking the form \(v_1 = \frac{1}{\sqrt{n}} 1\). Then \(a_1 = \sqrt{n}\). Since \(v_k, k \geq 2\) are orthogonal to \(v_1\), we have \(a_k = 0, k \geq 2\). By Condition 4 and Theorem C.1 in the Supplementary file, we have \(\max_i |s_1(i)| \lesssim \frac{\alpha_n^2}{\sqrt{nd_1}}\). Combining this with Condition 3 and using the fact \(v_1 = \frac{1}{\sqrt{n}} 1\), we have
\[
\sum_{i<j} \sigma_{ij}^2 \left(1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))\right)^2 \leq \frac{\alpha_n^4}{nd_1^2} \sum_{i<j} \sigma_{ij}^2 \lesssim \left(1 + \frac{n^2 \alpha_n^2}{d_{K_0}^2}\right).
\]
where in the last line we have used \( \sum_{i<j} \sigma_{ij}^2 \leq n \alpha_n^2 \) and \( \alpha_n \lesssim n \). This contradicts to (18)! Therefore, in this case the central limit theorem fails to hold under the null hypothesis. In fact, by checking the proof of Theorem 3.5, we see that the intrinsic problem is when aggregating too many terms from the residual matrix, the noise accumulation is no longer negligible, canceling the first order term \( \sum_{i \neq j} \sigma_{ij}^2 \), and consequentially makes the central limit theorem fail. Similar phenomenon happens under the alternative hypothesis as well. This justifies the necessity of the subsampling step.

4 Simulation studies

In this section, we use simulations to justify the performance of RIRS in testing and estimating \( K \), where Section 4.1 considers the network model and Section 4.2 considers more general low rank plus noise matrices. The nominal level is fixed to be \( \alpha = 0.05 \) in all settings.

4.1 Network models

Consider the DCMM model (11). We simulate two types of nodes: pure node with \( \pi_i \) chosen from the set of unit vectors

\[
\text{PN}(K) = \{e_1, \ldots, e_K\},
\]

and the mixed membership node with \( \pi_i \) chosen from

\[
\text{MM}(K, x) = \left\{ \left(x, 1-x, 0, \ldots, 0\right), \left(1-x, x, 0, \ldots, 0\right), \left(\frac{1}{K}, \ldots, \frac{1}{K}\right) \right\}
\]

where \( x \in (0, 1) \). Note that DCMM (11) includes SBM, DCSBM, and MM models as special cases.

1) SBM

When all rows of \( \Pi \) are chosen from the pure node set \( \text{PN}(K) \) and the degree heterogeneity matrix \( \Theta = r I_n \), the DCMM (11) reduces to the SBM with the following mean matrix structure

\[
H = r \Pi \Pi^T, \quad r \in (0, 1), \quad \pi_i \in \text{PN}(K), \quad i = 1, \ldots, n.
\] (20)

We generate 200 independent adjacency matrices each with \( n = 1000 \) nodes and \( K \) equal-sized communities from the above SBM (20). We set
\( \mathbf{B} = (B_{ij})_{K \times K} \) with \( B_{ij} = \rho^{|i-j|}, \ i \neq j \) and \( B_{ii} = (K + 1 - i)/K \). We experiment with \( \rho = 0.1 \) and 0.9. The value of \( r \) ranges from 0.1 to 0.9, with smaller \( r \) corresponding to sparser network model. For all values of \( K \), we choose \( m = \sqrt{n} \) in calculating the RIRS test statistics \( T_n \) and \( \tilde{T}_n \) for networks without and with self-loops, respectively.

The performance of RIRS is compared with the methods in [17], where two versions of test – one with and one without bootstrap correction – were proposed when the network is absent of self-loops (i.e. \( X_{ii} = 0, i = 1, \ldots, n \)). The empirical sizes and powers of both methods when \( \rho = 0.1 \) are reported in Tables 1 and 2 for \( K = 2 \) and 3, respectively. The corresponding computation times are reported in Table 4. We also compare the performance of \( T_n \) and \( \tilde{T}_n \) when \( \rho = 0.1 \) and 0.9, receptively, in Table 3 in the existence of selfloops.

From Tables 1 and 2, we observe that the performance of RIRS is relatively robust to the sparsity level \( r \), with size close to the nominal level and power close to 1 in almost all settings. On contrary, the method in [17] without bootstrap has much worse performance when the sparsity level is high or when the number of communities is large. In fact, when \( K = 2 \), the method in [17] without bootstrap correction suffers from size distortion for smaller \( r \) (sparser setting). This phenomenon becomes even more severe when \( K = 3 \), where the sizes are equal or close to one at all sparsity levels. With such distorted size, it is no longer meaningful to compare the power. Therefore we omit its power in Table 2. With bootstrap correction, the method in [17] performs much better and is comparable to RIRS except for the setting of \( r = 0.1 \) and \( K = 3 \), where the size is severely distorted. However, from Table 4 we see that the computational cost for the bootstrap method in [17] is much higher than that of RIRS. Table 3 suggests that when \( \rho \) is large, that is, denser connections between communities, \( \tilde{T}_n \) performances better than \( T_n \), and vice versa.

Finally, we present in Figure 1 the histogram plots as well as the fitted density curves of our test statistics from 1000 repetitions when \( K = 2, \rho = 0.1, \) and \( r = 0.7 \) under the null hypothesis. The standard normal density curves are also plotted as reference. It visually confirms that the asymptotic null distribution is standard normal.

2) DCMM

Next consider the general DCMM model (11). The number of repetition is still 200. We simulate the node degree parameters \( \vartheta_j \)'s independently from the uniform distribution over \([0.5, 1]\). The vectors \( \pi_i \) are chosen
Table 1: Empirical size and power under SBM with $K = 2$ and $\rho = 0.1$.

| $r$ | No selfloops | Selfloops | No selfloops | Selfloops | No selfloops | Selfloops |
|-----|---------------|-----------|---------------|-----------|---------------|-----------|
|     | RIRS ($T_n$)  | Lei (no bootstrap) | Lei (bootstrap) | RIRS ($\tilde{T}_n$) | size | power | size | power | size | power |
| 0.1 | 0.025 | 0.025 | 1 | 0.025 | 0.025 | 1 | 0.035 | 0.035 | 1 | 0.085 | 0.085 | 1 | 0.815 | 0.815 |
| 0.3 | 0.075 | 0.075 | 1 | 0.075 | 0.075 | 1 | 0.02 | 0.02 | 1 | 0.06 | 0.06 | 1 | 0.625 | 0.625 |
| 0.5 | 0.045 | 0.045 | 1 | 0.045 | 0.045 | 1 | 0.02 | 0.02 | 1 | 0.065 | 0.065 | 1 | 0.94 | 0.94 |
| 0.7 | 0.065 | 0.065 | 1 | 0.065 | 0.065 | 1 | 0.055 | 0.055 | 1 | 0.05 | 0.05 | 1 | 0.945 | 0.945 |
| 0.9 | 0.04 | 0.04 | 1 | 0.04 | 0.04 | 1 | 0.065 | 0.065 | 1 | 0.075 | 0.075 | 1 | 0.72 | 0.72 |

Table 2: Empirical size and power under SBM for $K = 3$ and $\rho = 0.1$.

| $r$ | No selfloops | Selfloops | No selfloops | Selfloops | No selfloops | Selfloops |
|-----|---------------|-----------|---------------|-----------|---------------|-----------|
|     | RIRS ($T_n$)  | Lei (no bootstrap) | Lei (bootstrap) | RIRS ($\tilde{T}_n$) | size | power | size | power | size | power | size | power |
| 0.1 | 0.065 | 0.065 | 1 | 0.065 | 0.065 | 1 | 0.095 | 0.095 | 1 | 0.08 | 0.08 | 1 | 0.98 | 0.98 |
| 0.3 | 0.075 | 0.075 | 1 | 0.075 | 0.075 | 1 | 0.046 | 0.046 | 1 | 0.07 | 0.07 | 1 | 0.99 | 0.99 |
| 0.5 | 0.04 | 0.04 | 1 | 0.04 | 0.04 | 1 | 0.015 | 0.015 | 1 | 0.03 | 0.03 | 1 | 0.955 | 0.955 |
| 0.7 | 0.05 | 0.05 | 1 | 0.05 | 0.05 | 1 | 0.045 | 0.045 | 1 | 0.065 | 0.065 | 1 | 1 | 1 |
| 0.9 | 0.06 | 0.06 | 1 | 0.06 | 0.06 | 1 | 0.075 | 0.075 | 1 | 0.095 | 0.095 | 1 | 1 | 1 |

Table 3: Size and power of $T_n$ and $\tilde{T}_n$ under SBM with selfloops when $K = 3$, $\rho = 0.1$ or $\rho = 0.9$.

| $r$ | $\rho = 0.1$ | $\rho = 0.9$ | $\rho = 0.1$ | $\rho = 0.9$ |
|-----|--------------|--------------|--------------|--------------|
|     | RIRS ($T_n$) | RIRS($\tilde{T}_n$) | RIRS($T_n$) | RIRS($\tilde{T}_n$) |
| 0.1 | 0.045 | 0.025 | 0.1 | 0.1 | 0.075 | 0.025 | 0.72 | 0.72 |
| 0.3 | 0.04 | 0.04 | 0.33 | 0.33 | 0.4 | 0.4 | 0.99 | 0.99 |
| 0.5 | 0.06 | 0.05 | 0.69 | 0.69 | 0.04 | 0.04 | 0.97 | 0.97 |
| 0.7 | 0.05 | 0.04 | 0.945 | 0.945 | 0.035 | 0.035 | 1 | 1 |
| 0.9 | 0.07 | 0.07 | 0.965 | 0.965 | 0.03 | 0.03 | 0.945 | 0.945 |

| $r$ | $\rho = 0.9$ | $\rho = 0.9$ | $\rho = 0.9$ | $\rho = 0.9$ |
|-----|--------------|--------------|--------------|--------------|
| 0.1 | 0.025 | 0.025 | 0.8 | 0.15 | 0.1 | 0.075 | 0.075 | 0.66 | 0.72 |
| 0.3 | 0.04 | 0.05 | 0.345 | 0.345 | 0.28 | 0.28 | 1 | 1 |
| 0.5 | 0.05 | 0.065 | 0.645 | 0.645 | 0.535 | 0.535 | 1 | 1 |
| 0.7 | 0.045 | 0.045 | 0.765 | 0.765 | 0.77 | 0.77 | 1 | 1 |
| 0.9 | 0.055 | 0.055 | 0.915 | 0.915 | 0.895 | 0.895 | 1 | 1 |
Null density without loop

Null density with loop

Figure 1: Histogram plots and the estimated densities (red curves) of RIRS test statistic when $K = 2$ and $r = 0.7$. Left: $T_n$ when no selfloop; Right: $\tilde{T}_n$ when selfloops exist.

Table 4: Average computation time (in seconds) for test statistics in Table 1 and Table 6 in one replication under SBM with no selfloops, $K = 2$ and $r = 0.5$.

|             | RIRS ($T_n$) | Lei (no bootstrap) | Lei (bootstrap) |
|-------------|--------------|--------------------|-----------------|
| Size        | 0.504        | 0.906              | 0.432           |
| $K$         | 0.906        | 2.88               | 14.410          |
| Time        | 147.142      |                    |                 |

from $PN(K) \cup MM(K, 0.2)$, with $n_0$ pure nodes from each community and $(n - Kn_0)/3$ nodes from each mixed membership probability mass vector in $MM(K, 0.2)$. We select $n_0 = 0.35n$ when $K = 2$ and $n_0 = 0.25n$ when $K = 3$. The matrix $B$ is chosen to be the same as in the SBM with $\rho = 0.1$. The network size $n$ ranges from 800 to 2000. The empirical sizes and powers are summarized in Table 5.

Since [17] only considers SBM, the tests therein are no longer applicable in this setting. RIRS performs well and similarly to the SBM setting. Figure 2 presents the histogram plots as well as the fitted density curves of RIRS under the null hypothesis from 1000 repetitions when $K = 3$ and $n = 1500$. These results well justify our theoretical findings.

3). Estimating the Number of Communities

We use the method discussed in Section 3.4 to estimate the number of communities $K$. Since the approaches in [17] are not applicable to the DCMM model, we only compare the performance of RIRS with [17] in SBM setting in the absence of selfloops. The proportions of correctly estimated
Table 5: Empirical size and power of RIRS under DCMM model.

| K = 2 | No Selfloop ($T_n$) | Selfloop ($\tilde{T}_n$) | K = 3 | No Selfloop ($T_n$) | Selfloop ($\tilde{T}_n$) |
|-------|---------------------|--------------------------|-------|---------------------|--------------------------|
|       | Size | Power ($K_0=1$) | Size | Power ($K_0=1$) | Size | Power ($K_0=2$) | Size | Power ($K_0=1$) | Size | Power ($K_0=2$) |
| n     |      |                |      |                |      |                |      |                |      |                |
| 800   | 0.045 | 1 | 0.08 | 1 | 0.05 | 1 | 0.58 | 1 | 0.08 | 1 | 0.845 |
| 1000  | 0.04  | 1 | 0.05  | 1 | 0.025 | 1 | 0.68 | 1 | 0.06 | 1 | 0.92 |
| 1200  | 0.065 | 1 | 0.05  | 1 | 0.045 | 1 | 0.77 | 1 | 0.07 | 1 | 0.92 |
| 1500  | 0.045 | 1 | 0.03  | 1 | 0.075 | 1 | 0.9 | 0.055 | 1 | 0.98 |
| 1800  | 0.075 | 1 | 0.055 | 1 | 0.045 | 1 | 0.98 | 0.065 | 1 | 0.995 |
| 2000  | 0.075 | 1 | 0.065 | 1 | 0.05 | 1 | 0.965 | 0.045 | 1 | 1 |

Figure 2: DCMM. Histogram plots and the estimated densities (red curves) of RIRS when $K = 3$ and $n = 1500$. Left: $T_n$ when no selfloop; Right: $\tilde{T}_n$ when selfloops exist.

$K$ are calculated over 200 replications and tabulated in Table 6 for SBM and in Table 7 for DCMM model.

Table 6 shows that RIRS generally has comparable estimation accuracy with Lei’s method under the SBM. While for DCMM model (Table 7), RIRS can also estimate the number of communities with high accuracy. In particular, the estimation accuracy gets closer and closer to the expected value of 95% as $n$ increases, which is consistent with our theory.

4.2 Low rank data matrix

RIRS can be applied to other low rank data matrices beyond the network model. In this section, we generate $n \times n$ data matrix $X$ from the following model

$$X = H + W = VDV^T + W.$$
Table 6: Proportion of correctly estimated \( K \) under SBM.

| \( r \) | \( K = 2 \) | \( K = 3 \) |
|---|---|---|
| \( T_n \) (no bootstrap) | \( T_n \) (bootstrap) | \( T_n \) (no bootstrap) | \( T_n \) (bootstrap) |
| \( RIRS \) | \( Lei \) | \( Lei \) | \( RIRS \) | \( Lei \) | \( Lei \) | \( RIRS \) | \( Lei \) |
| 0.1 | 0.93 | 0 | 0.97 | 0.815 | 0.285 | 0 | 0.165 | 0.105 |
| 0.3 | 0.94 | 0.795 | 0.955 | 0.96 | 0.745 | 0 | 0.935 | 0.645 |
| 0.5 | 0.945 | 0.925 | 0.925 | 0.95 | 0.95 | 0.005 | 0.98 | 0.895 |
| 0.7 | 0.97 | 0.915 | 0.945 | 0.96 | 0.955 | 0.065 | 0.995 | 0.955 |
| 0.9 | 0.94 | 0.94 | 0.935 | 0.955 | 0.955 | 0.275 | 0.975 | 0.945 |

Table 7: Proportion of correctly estimated \( K \) under DCMM.

| \( n \) | \( K = 2 \) | \( K = 3 \) |
|---|---|---|
| \( T_n \) (no bootstrap) | \( T_n \) (bootstrap) |
| \( RIRS \) | \( Lei \) | \( Lei \) | \( RIRS \) | \( Lei \) | \( Lei \) | \( RIRS \) | \( Lei \) |
| 800 | 0.935 | 0.935 | 0.93 | 0.965 | 0.935 | 0.935 | 0.505 | 0.625 | 0.805 | 0.865 | 0.915 | 0.935 |
| 1000 | 0.94 | 0.94 | 0.945 | 0.945 | 0.945 | 0.945 | 0.79 | 0.85 | 0.93 | 0.895 | 0.935 | 0.965 |
| 1200 | 0.95 | 0.95 | 0.955 | 0.955 | 0.955 | 0.955 | 0.79 | 0.85 | 0.93 | 0.895 | 0.935 | 0.965 |
| 1500 | 0.96 | 0.96 | 0.965 | 0.965 | 0.965 | 0.965 | 0.79 | 0.85 | 0.93 | 0.895 | 0.935 | 0.965 |
| 1800 | 0.97 | 0.97 | 0.975 | 0.975 | 0.975 | 0.975 | 0.79 | 0.85 | 0.93 | 0.895 | 0.935 | 0.965 |
| 2000 | 0.98 | 0.98 | 0.985 | 0.985 | 0.985 | 0.985 | 0.79 | 0.85 | 0.93 | 0.895 | 0.935 | 0.965 |

where the residual matrix \( \mathbf{W} \) is symmetric with upper triangle entries (including the diagonal ones) i.i.d from uniform distribution over \((-1,1)\). Let \( \mathbf{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix} \), where \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are \( n_1 \times K \) and \((n - n_1) \times K \) matrices respectively. We randomly generate an \( n_1 \times n_1 \) Wigner matrix and collect its \( K \) eigenvectors corresponding to the largest \( K \) eigenvalues to form \( \mathbf{V}_1 \). We set \( \mathbf{V}_2 = \frac{\sqrt{K}}{\sqrt{n-n_1}} \mathbf{\Pi} \) with \( \mathbf{\Pi} = (\pi_1, ..., \pi_{n-n_1})^T \), where \( \pi_i \in \Pi \) and the number of rows taking each distinct value from \( \Pi \) is the same. The diagonal matrix \( \mathbf{D} = n \times \text{diag}(K, K - 1, ..., 1) \). The multiplier \( n \) in the construction of \( \mathbf{D} \) is to make the norm of each column of \( \mathbf{V} \) one. We set \( n_1 = n/2 \) and range the value of \( n \) from 100 to 500. When \( K = 2 \), the empirical sizes and powers as well as the proportions of correctly estimated \( K \) over 500 repetitions are recorded in Table 8. It is seen that both \( T_n \) and \( \tilde{T}_n \) performs well, with \( \tilde{T}_n \) having slightly higher power. This higher power further translates into better estimation accuracy (closer to 95%) of estimated \( K \).

5 Real data analysis

We consider a popularly studied network of political blogs assembled by [2]. The nodes are blogs over the period of two months before the 2004 U.S. Presidential Election. The edges are the web links between the blogs. These
blogs have known political divisions and were labeled into two communities ($K = 2$) by [2] – the liberal and conservative communities. This blog data has been frequently used in the literature, see [14], [23] and [17] among others. It is widely believed to follow a degree corrected block model. For the readers’ convenience, we cite a graph (Figure 3) from [14], which modeled the data using the degree corrected block model. Following the literature, we ignore the directions and study only the largest connected component, which has $n = 1222$ nodes. Consider the following two hypothesis tests:

\[
\begin{align*}
(HT1) & : \ H_0 : K = 1 \ vs \ H_1 : K > 1. \\
(HT2) & : \ H_0 : K = 2 \ vs \ H_1 : K > 2.
\end{align*}
\]

[17] considered (HT2) and obtained test statistic values 1172.3 and 491.5, corresponding to the test without bootstrap and with bootstrap, respectively. Both are much larger than the critical value (about 1.454) from the Tracy-Widom distribution, and thus the null hypothesis in (HT2) was strongly rejected. This is not surprising because the testing procedure in [17] is based on the SBM. It is possible that the model is misspecified when applying the tests therein.

RIRS does not depend on any specific network model structure and is expected to be more robust to model misspecification. Since most of the diagonal entries of $X$ are zero, we use the test statistic $T_n$. Noticing that the observed data matrix $X$ is non-symmetric, we consider two simple transformations:

\[
\begin{align*}
\text{Method 1 : } \tilde{X}_1 &= X + X^T; \\
\text{Method 2 : } \tilde{X}_2 &= \left( \begin{array}{cc} 0 & X \\ X^T & 0 \end{array} \right)_{2n \times 2n}.
\end{align*}
\]

The transformation in Method 2 is general and can be applied to even non-square data matrix $X$. After the transformations, $\text{rank}(\mathbb{E}(\tilde{X}_1)) = K$ and $\text{rank}(\mathbb{E}(\tilde{X}_2)) = 2K$. The results of applying $T_n$ to the two hypothesis test problems (HT1) and (HT2), together with the estimated number of communities by the sequential testing procedure are reported in Table 9.
see that for both transformations, RIRS consistently estimated the number of communities to be 2, which is consistent with the common belief in the literature.

Table 9: Hypothesis testing and estimation results for the political blog data.

|                | Method 1 |       | Method 2 |       | Decision                  |
|----------------|----------|-------|----------|-------|---------------------------|
|                | Test Statistic | P-value | Test Statistic | P-value |                         |
| (HT1)          | 3.3527               | 0.0008               | 2.7131               | 0.0067 | Reject $H_0$ in (HT1)    |
| (HT2)          | -1.2424              | 0.2141              | -0.8936              | 0.3716 | Accept $H_0$ in (HT2)    |
| Estimate       | 2                    |       | 2                    |       | $K = 2$                  |

Figure 3: (FIG.2. in [14]). Divisions of the blog network data using the degree corrected block model. The node colors reflect community labels.
A Proof of the main results

We introduce a definition that will be used frequently in the proof.

**Definition 1.** Let $\xi_n$ and $\zeta_n$ be some random (or deterministic) variables. We say $\xi_n \prec \zeta_n$ or equivalently $\xi_n = O_\prec(\zeta_n)$, if for any pair of positive constants $(\epsilon, D)$, there exists some positive integer $n_0(\epsilon, D)$ depending only on $\epsilon$ and $D$ such that for all $n \geq n_0(\epsilon, D)$ we have $\mathbb{P}[|\xi_n| > n^\epsilon|\zeta_n|] \leq n^{-D}$.

A.1 Outline of The Proof

The proof of our main results highly depends on Lemmas 2 and 3 in the Supplement, which are the asymptotic expansions of the eigenvectors $\hat{v}_k(i)$ and eigenvalues $\hat{d}_k$. Briefly speaking, Lemma 1 in Section B.1 of the Supplement establishes the relation between $\hat{w}_{ij}$ and $(\hat{v}_k(i), \hat{d}_k)$, from which we can obtain the asymptotic expansion of $\hat{w}_{ij}$. Substituting this asymptotic expansion into the proposed test statistics, we are able to prove our main theorems by further careful analysis and calculations. In the main paper we only provide the proofs of Theorems 3.1 and 3.2. All the other proofs are relegated to the supplement.

A.2 Proof of Theorem 3.1

The result in Theorem 3.1 can be obtained by combing the following two results.

\[
\text{CLT : } \frac{\sqrt{m} \sum_{i \neq j} \hat{w}_{ij} Y_{ij}}{\sqrt{2 \sum_{i \neq j} \mathbb{E} w_{ij}^2}} \overset{d}{\rightarrow} N(0, 1), \quad (A.1)
\]

\[
\text{Consistency: } \frac{\sum_{i \neq j} \hat{w}_{ij}^2}{\sum_{i \neq j} \mathbb{E} w_{ij}^2} = 1 + o_p(1). \quad (A.2)
\]

We next proceed with proving (A.1) and (A.2).

We first verify the central limit theorem (A.1). By (B.35) we have
\[
\sum_{i \neq j} w_{ij}Y_{ij} = \sum_{i \neq j} w_{ij}Y_{ij} - \sum_{k=1}^{K_0} \sum_{i \neq j} (v_k(i)v_k(j)Y_{ij})v_k^T Wv_k 
\tag{A.3}
\]

\[- \sum_{k=1}^{K_0} \sum_{i \neq j} Y_{ij} \frac{e_i^T \mathbb{E}W^2v_kv_k(j) + e_j^T \mathbb{E}W^2v_kv_k(i)}{t_k}
\]

\[- \sum_{k=1}^{K_0} \sum_{i \neq j} Y_{ij} \frac{r_k(i)v_k + r_k(j)v_k(i)}{t_k} + 2 \sum_{k=1}^{K_0} \frac{v_k^T \mathbb{E}W^2v_k}{d_k} \sum_{i \neq j} (v_k(i)v_k(j)Y_{ij})
\]

\[- 2 \sum_{k=1}^{K_0} \sum_{i \neq j} Y_{ij} v_k(i)s_{k,j}^T Wv_k - \sum_{k=1}^{K_0} \sum_{i \neq j} \frac{d_k e_i^T \mathbb{E}Wv_k e_j^T \mathbb{E}Wv_k Y_{ij}}{t_k^2}
\]

\[+ \sum_{i \neq j} (Y_{ij} \mathcal{O}_\prec (\alpha n_{nd_k})).\]

Recall that \(\mathbb{E}Y_{ij} = \frac{1}{m}, a_k = \sum_{i=1}^{n} v_k(i)\) and \(|a_k| \leq \sqrt{n}\). Our aim is to bound all terms on the right hand side of equation (A.3) except for the first term \(\sum_{i \neq j} w_{ij}Y_{ij}\). We begin with splitting the term

\[\sum_{i \neq j} (v_k(i)v_k(j)Y_{ij})v_k^T Wv_k\]

into two parts:

\[\frac{1}{m} \sum_{i \neq j} (v_k(i)v_k(j))v_k^T Wv_k \quad \text{and} \quad \sum_{i \neq j} (Y_{ij} - \frac{1}{m})(v_k(i)v_k(j))v_k^T Wv_k.\]

For the first part, first note that since \(|w_{ij}| \leq C\), we have \(|v_k^T \mathbb{E}Wv_k| \lesssim 1\).

Then by Theorem C.1 in the supplementary material and Condition 1 we have

\[\frac{1}{m} \sum_{i \neq j} (v_k(i)v_k(j))v_k^T Wv_k = \frac{1}{m} (a_k^2 - 1)v_k^T Wv_k \]

\[= \frac{1}{m} (a_k^2 - 1)(v_k^T (W - \mathbb{E}W) v_k + v_k^T \mathbb{E}Wv_k) = (a_k^2 + 1)(O_\prec (\alpha n_{m \sqrt{n}}) + O(\frac{1}{m})).\]

For the second part, first note that \(\mathbb{E}(v_k^T Wv_k)^2 = \text{var}(v_k^T Wv_k) + \mathbb{E}^2(v_k^T Wv_k) \lesssim \alpha_n^2/n + 1\). Since \(Y_{ij}, i \leq j\) are i.i.d with \(\mathbb{E}Y_{ij} = \frac{1}{m}\), Theorem C.1 and
Condition 1 ensure that

\[
\text{var}\left(\sum_{i \neq j} (Y_{ij} - \frac{1}{m})(v_k(i)v_k(j))v_k^T Wv_k\right)
\]

\[
= \mathbb{E}\left[\text{var}\left(\sum_{i \neq j} (Y_{ij} - \frac{1}{m})(v_k(i)v_k(j))v_k^T Wv_k|W\right)\right]
\]

\[
\lesssim \frac{1}{m} \sum_{i \neq j} (v_k(i)v_k(j))^2 \mathbb{E}(v_k^T Wv_k)^2 \lesssim \frac{\alpha_n^2}{mn} + \frac{1}{m},
\]

then

\[
\sum_{i \neq j} (Y_{ij} - \frac{1}{m})(v_k(i)v_k(j))v_k^T Wv_k = O_p\left(\frac{\alpha_n}{\sqrt{mn}} + \frac{1}{\sqrt{m}}\right).
\]  \hspace{1cm} (A.5)

Therefore,

\[
\sum_{i \neq j} (v_k(i)v_k(j)Y_{ij})v_k^T Wv_k = (a_k^2 + 1)O_n\left(\frac{\alpha_n}{m\sqrt{n}}\right) + O\left(\frac{a_k^2 + 1}{m}\right) + O_p\left(\frac{\alpha_n}{\sqrt{mn}}\right) + O_p\left(\frac{1}{\sqrt{m}}\right).
\]  \hspace{1cm} (A.6)

Similar to (A.6), we get

\[
\frac{v_k^T EW^2 v_k}{d_k} \sum_{i \neq j} (v_k(i)v_k(j)Y_{ij}) \lesssim \frac{a_k^2 + 1}{m} + \frac{1}{\sqrt{m}},
\]

and

\[
\sum_{i \neq j} Y_{ij} \frac{e_k^T EW^2 v_k v_k(j) + e_k^T EW^2 v_k v_k(i)}{t_k} \lesssim \frac{|a_k|\sqrt{n}}{m} + \frac{1}{\sqrt{m}}.
\]  \hspace{1cm} (A.7)

Next we split the term \(\sum_{i \neq j} Y_{ij} \frac{d_k s_{k,j}^T Wv_k v_k(i)}{t_k}\) into the following two parts

\[
\frac{1}{m} \sum_{i \neq j} \frac{d_k s_{k,j}^T Wv_k v_k(i)}{t_k} \quad \text{and} \quad \sum_{i \neq j} \frac{Y_{ij} - \frac{1}{m}}{d_k s_{k,j}^T Wv_k v_k(i)}.
\]

By (C.25), we have

\[
\|\mathcal{R}(1, V_{-k}, t)\left( (D_{-k})^{-1} + \mathcal{R}(V_{-k}, V_{-k}, t_k) \right)^{-1} V_{-k}^T \|
\]

\[
= \left\| \sum_i \mathcal{R}(e_i, V_{-k}, t)\left( (D_{-k})^{-1} + \mathcal{R}(V_{-k}, V_{-k}, t_k) \right)^{-1} V_{-k}^T \right\| \lesssim \sqrt{n}.
\]
In light of (B.37), (C.25), Theorem C.1, Corollary 2 in the supplementary material and Condition 4, the following three results hold:

\[
\frac{d_k(s_k - 1)^TW_kv_k}{t_k} = -\frac{d_k a_k R(1, V_{-k}, t)((D_{-k})^{-1} + R(V_{-k}, V_{-k}, t))^{-1}V_{-k}^TW_kv_k}{t_k} - \frac{d_k a_k^2 v_k^TW_k}{t_k} + O_\prec(\alpha_n + |a_k|\sqrt{n}) + O_\prec(|a_k|\alpha_n),
\]

\[
\frac{1}{m} \sum_{i \neq j} \frac{d_k s_{k,i}^TW_kv_k(i)}{t_k} = \frac{1}{m} d_k 1^TW_kE_Wv_k + O_\prec\left(\frac{|a_k|}{m}\alpha_n\right)
\]

\[
= O_\prec\left(\frac{|a_k|}{m}\alpha_n\right)
\]

and

\[
\sum_{i \neq j} (Y_{ij} - \frac{1}{m} d_k s_{k,i}^TW_kv_k(i)) t_k = O_p\left(\frac{\alpha_n}{\sqrt{m}}\right),
\]

where the calculation of the variance of the second part \(\sum_{i \neq j} (Y_{ij} - \frac{1}{m} d_k s_{k,i}^TW_kv_k(i)) t_k\) is similar to that of (A.5). Therefore,

\[
\sum_{i \neq j} Y_{ij} d_k s_{k,i}^TW_kv_k(i) t_k = O_\prec\left(\frac{|a_k| + 1}{m}\alpha_n + |a_k|\sqrt{n}\right) + O_p\left(\frac{\alpha_n}{\sqrt{m}}\right). \quad (A.8)
\]

For the term \(\sum_{i \neq j} Y_{ij} r_k(i)v_k(j) + r_k(j)v_k(i) t_k\), we write

\[
\sum_{i \neq j} Y_{ij} r_k(i)v_k(j) + r_k(j)v_k(i) t_k = \frac{1}{m} \sum_{i \neq j} r_k(i)v_k(j) + r_k(j)v_k(i) t_k + \sum_{i \neq j} (Y_{ij} - \frac{1}{m}) r_k(i)v_k(j) + r_k(j)v_k(i) t_k.
\]

(A.9)

It follows from Conditions 2–4, \(\frac{t_k}{a_k} \to 1\) in Section 3.5 and Corollary 2 that

\[
|1^TR_k| = |1^TV_{-k}(t_kD_{-k}^{-1} - I)^{-1}V_{-k}^TW^2v_k| \lesssim \sqrt{n}\alpha_n^2,
\]

26
\[ |r_k(i)| = |e_i^T V_{-k} (t_k D_{-k}^{-1} - I)^{-1} V_{-k}^T \mathbb{E} W^2 v_k| \lesssim \frac{\alpha^2}{\sqrt{n}}, \]

and thus

\[
\frac{1}{m} \sum_{i \neq j} r_k(i)v_k(j) + r_k(j)v_k(i) t_k = \frac{2 a_k 1^T r_k}{t_k m} - \frac{2}{m} \sum_{i=1}^n r_k(i)v_k(i) t_k = O\left( \frac{|a_k| \sqrt{n} + 1}{m} \right).
\]

As for (A.5), calculating the variance of the second term on the right hand side of (A.9) yields

\[
\sum_{i \neq j} (Y_{ij} - \frac{1}{m}) r_k(i)v_k(j) + r_k(j)v_k(i) t_k = O_p\left( \frac{1}{\sqrt{m}} \right).
\]

Therefore,

\[
\sum_{i \neq j} Y_{ij} r_k(i)v_k(j) + r_k(j)v_k(i) t_k = O_\prec\left( \frac{|a_k| \sqrt{n}}{m} \right) + O_p\left( \frac{1}{\sqrt{m}} \right). \tag{A.10}
\]

Now for the term \( \sum_i d_i e_i^T W v_k e_j^T W v_k Y_{ij} \), similarly we write

\[
\sum_i d_i e_i^T W v_k e_j^T W v_k Y_{ij} = \sum_{i \neq j} d_i e_i^T W v_k e_j^T W v_k t_k^2 + \sum_{i \neq j} d_i e_i^T W v_k e_j^T W v_k (Y_{ij} - \frac{1}{m}).
\]

By Corollary 2, the first part has order

\[
\sum_i d_i e_i^T W v_k e_j^T W v_k t_k^2 = \frac{d_k 1^T W v_k 1^T W v_k}{m t_k^2} - \frac{n}{m} \frac{d_k (e_i^T W v_k)^2}{m t_k^2} = O_\prec\left( \frac{\alpha^2}{m|d_k|} \right) = O_\prec\left( \frac{1}{m} \right).
\]

Moreover, it follows from Corollary 2 and Theorem C.1 that

\[
\text{var}\left( \sum_{i \neq j} d_i e_i^T W v_k e_j^T W v_k (Y_{ij} - \frac{1}{m}) \right) \leq \sum_{i \neq j} \text{var}\left( \frac{d_i e_i^T W v_k e_j^T W v_k}{t_k^2} \right) \frac{4 \mathbb{E}(e_i^T W v_k)^2 \mathbb{E}(e_j^T W v_k)^2}{m d_k^2} \leq \sum_{i \neq j} \mathbb{E}(e_i^T W v_k)^2 \mathbb{E}(e_j^T W v_k)^2 \leq \sum_{i \neq j} \mathbb{E}(e_i^T W v_k)^2 \mathbb{E}(e_j^T W v_k)^2 \leq \frac{\alpha^2}{m d_k^2}.
\]

27
\[ \sum_{i \neq j} \sqrt{E(e_i^T Wv_k - Ee_i^T Wv_k)^4 + (Ee_j^T Wv_k)^4} \frac{E(e_i^T Wv_k - Ee_i^T Wv_k)^4}{md_k^2} \]

\[ \leq \frac{\alpha^4}{md_k^2} \leq \frac{1}{m}. \]

Therefore

\[ \sum_{i \neq j} \frac{d_k e_i^T Wv_k e_j^T Wv_k Y_{ij}}{t_k^2} = O_p\left(\frac{1}{m}\right) + O_p\left(\frac{1}{\sqrt{m}}\right). \quad (A.11) \]

Finally, consider the residual term

\[ \sum_{i \neq j} (Y_{ij}O_\prec(\frac{\alpha_n}{nd_k})) = \sum_{i \neq j} O_\prec(\frac{\alpha_n}{nd_k})(Y_{ij} - \frac{1}{m}) + \frac{1}{m} \sum_{i \neq j} O_\prec(\frac{\alpha_n}{nd_k}). \]

Note that \( Y_{ij} \) is independent of \( O_\prec(\frac{\alpha_n}{nd_k}) \). Calculating the variance of \( \sum_{i \neq j} O_\prec(\frac{\alpha_n}{nd_k})(Y_{ij} - \frac{1}{m}) \) gives us

\[ \sum_{i \neq j} O_\prec(\frac{\alpha_n}{nd_k})(Y_{ij} - \frac{1}{m}) = O_p\left(\frac{1}{\sqrt{m}}\right) \times O_\prec\left(\frac{n\alpha_n}{|d_k|}\right). \]

The “mean” of the residual term should be

\[ \frac{1}{m} \sum_{i \neq j} O_\prec(\frac{\alpha_n}{nd_k}) = O_\prec\left(\frac{n\alpha_n}{md_k}\right). \]

Therefore we have

\[ \sum_{i \neq j} (Y_{ij}O_\prec(\frac{\alpha_n}{nd_k})) = O_\prec\left(\frac{n\alpha_n}{md_k}\right) + O_p\left(\frac{1}{\sqrt{m}}\right) \times O_\prec\left(\frac{\alpha_n}{|d_k|}\right). \quad (A.12) \]

So far we have found the orders of all other terms on the right hand side of equation (A.3) except for \( \sum_{i \neq j} w_{ij} Y_{ij} \). Note that

\[ \text{var}(\sum_{i \neq j} w_{ij} Y_{ij}) = \frac{2}{m} \sum_{i \neq j} Ew_{ij}^2. \quad (A.13) \]

According to the orders (A.6), (A.7), (A.8), (A.10), (A.11) and (A.12), we can conclude that as long as

\[ \frac{2}{m} \sum_{i \neq j} Ew_{ij}^2 \geq n^t \left( \frac{\sum_{k=1}^{K_0} a_k^2 n}{m^2} + \frac{\alpha_n^2}{m} + \frac{n^2 \alpha_n^2}{m^2 d_K^2} \right), \]
the term \( \sum_{i \neq j} w_{ij} Y_{ij} \) dominates all other terms on the right hand side of (A.3). Moreover, by the condition \( \sum_{i \neq j} E w_{ij}^2 \gg m \) and the independence between \( Y_{ij} \) and \( w_{ij} \) we have

\[
\frac{m^2}{(\sum_{i \neq j} E w_{ij}^2)^2} \sum_{i \neq j} E w_{ij}^4 Y_{ij}^4 \lesssim \frac{m \sum_{i \neq j} E w_{ij}^2}{(\sum_{i \neq j} E w_{ij}^2)^2} \lesssim \frac{m}{(\sum_{i \neq j} E w_{ij}^2)} \to 0.
\]

Therefore, the central limit theorem (A.1) holds by Lyapunov CLT.

We now show the consistency of \( \sum_{i \neq j} \hat{w}_{ij}^2 \) in (A.2). By (B.22), we have

\[
\hat{w}_{ij}^2 = w_{ij}^2 + 2 w_{ij} O_p \left( \frac{1}{n} \right) - 2 \sum_{k=1}^{K_0} \frac{d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))}{t_k}
+ O_p \left( \frac{1}{n^2} \right) + (\sum_{k=1}^{K_0} \frac{d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))}{t_k})^2
- 2 O_p \left( \frac{1}{n} \right) \sum_{k=1}^{K_0} d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))
\]

and

\[
\sum_{i \neq j} \hat{w}_{ij}^2 = \sum_{i \neq j} w_{ij}^2 + 2 \sum_{i \neq j} \left( w_{ij} O_p \left( \frac{1}{n} \right) \right) - 2 \sum_{k=1}^{K_0} \sum_{i \neq j} \frac{d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))}{t_k}
+ O_p (1) + (\sum_{i \neq j} \frac{d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))}{t_k})^2
- 2 \sum_{i \neq j} O_p \left( \frac{1}{n} \right) \sum_{k=1}^{K_0} d_k (e_i^T Wv_k v_k(j) + e_j^T Wv_k v_k(i))
\]

Combing the fact \( \text{var}(\sum_{i \neq j} w_{ij}^2) \leq \sum_{i \neq j} E w_{ij}^4 \lesssim \sum_{i \neq j} E w_{ij}^2 \) with Condition 5

\[
\sum_{i \neq j} E w_{ij}^2 \gg n^\epsilon,
\]

we have

\[
\frac{\sum_{i \neq j}^n w_{ij}^2}{\sum_{i \neq j}^n E w_{ij}^2} = 1 + o_p(1).
\]
Then to prove (A.2), it suffices to show
\[
\sum_{i \neq j} \frac{w_{ij}^2}{\sum_{i \neq j} w_{ij}^2} = 1 + o_p(1). \tag{A.17}
\]
We now check the other terms on the right hand side of (A.14) to verify (A.17). First of all,
\[
|\sum_{i \neq j} w_{ij} O_\prec(\frac{1}{n})| \leq O_\prec(\frac{1}{n}) \sum_{i \neq j} |w_{ij}| \leq O_\prec(1) \times \sqrt{\sum_{i \neq j} w_{ij}^2} = O_\prec(\sqrt{\sum_{i \neq j} \mathbb{E}w_{ij}^2}).
\]
Condition (A.15) further implies that
\[
|\sum_{i \neq j} w_{ij} O_\prec(\frac{1}{n})| = (\sum_{i \neq j} \mathbb{E}w_{ij}^2) \times O_p(1). \tag{A.18}
\]
Now consider the term \(\sum_{i \neq j} w_{ij} \frac{d_k e_i^T W_k v_k(j) + e_j^T W_k v_k(i)}{t_k}\). We will only provide detail for proving \(\sum_{i \neq j} w_{ij} \frac{d_k e_i^T W_k v_k(j)}{t_k}\) because the other part can be proved similarly. Write
\[
\sum_{i \neq j} w_{ij} \frac{d_k e_i^T W_k v_k(j)}{t_k} = \sum_{i \neq j} \frac{d_k w_{ij}^2 v_k^2(j)}{t_k} + \sum_{i \neq j, l \neq j} \frac{d_k w_{ij} w_{il} v_k(j) v_k(l)}{t_k}.
\]
Direct calculations yield
\[
\mathbb{E} \left| \sum_{i \neq j} \frac{d_k w_{ij}^2 v_k^2(j)}{t_k} \right| \lesssim \frac{1}{n} \sum_{i \neq j} \mathbb{E}w_{ij}^2, \quad \sum_{i \neq j, l \neq j} \mathbb{E} \frac{d_k w_{ij} w_{il} v_k(j) v_k(l)}{t_k} = 0, \quad \text{and}
\]
\[
\text{var}(\sum_{i \neq j, l \neq j} \frac{d_k w_{ij} w_{il} v_k(j) v_k(l)}{t_k}) \lesssim \sum_{i \neq j, l \neq j} \mathbb{E}w_{ij}^2 \mathbb{E}w_{il}^2 v_k^2(j) v_k^2(l) \lesssim \frac{\alpha_n^2}{n^2} \sum_{i \neq j} \mathbb{E}w_{ij}^2.
\]
Thus, \(\sum_{i \neq j} w_{ij} \frac{d_k e_i^T W_k v_k(j)}{t_k} = o_p(1) \times \sum_{i \neq j} \mathbb{E}w_{ij}^2\). And consequently,
\[
\sum_{i \neq j} w_{ij} \frac{d_k(e_i^T W_k v_k(j) + e_j^T W_k v_k(i))}{t_k} = o_p(1) \times \sum_{i \neq j} \mathbb{E}w_{ij}^2. \tag{A.19}
\]
Next by Theorem C.1 and Condition 5, we have
\[
\sum_{i \neq j} O_\prec \left(\frac{1}{n} \sum_{k=1}^{K_0} \frac{d_k(e_i^T W_k v_k(j) + e_j^T W_k v_k(i))}{t_k}\right) = O_\prec(\alpha_n) = O_\prec(\alpha_n^2)
\]
\[
= O_\prec(n^{-\epsilon/2} \sum_{i \neq j} \mathbb{E}w_{ij}^2). \tag{A.20}
\]
Finally, similar to (A.20), it holds by Condition 5 that

\[
\sum_{i \neq j} \left( \sum_{k=1}^{K_0} \frac{d_k (e_i^T W v_k(j) + e_j^T W v_k(i))}{t_k} \right)^2 \leq \sum_{k=1}^{K_0} \sum_{i \neq j} \frac{d_k^2 (e_i^T W v_k(j) + e_j^T W v_k(i))^2}{t_k^2} = O_\prec \left( \alpha_n^2 \right) = O_\prec \left( n^{-\epsilon/2} \sum_{i \neq j} \mathbb{E} w_{ij}^2 \right).
\]

Substituting the arguments (A.16), (A.18), (A.19), (A.20) and (A.21) into equation (A.14), we complete the proof of (A.17). Thus, (A.2) is proved and the results in the theorem follow automatically.

### A.3 Proof of Theorem 3.2

According to Lemma 1, under the alternative hypothesis \((K > K_0)\), we have the following expansion

\[
\hat{w}_{ij} - \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) = w_{ij} - \sum_{k=1}^{K_0} \left[ \Delta(d_k) \hat{v}_k(i) \hat{v}_k(j) + d_k \Delta(v_k(i)) v_k(j) \right. \\
+ \left. d_k \Delta(v_k(j)) v_k(i) + d_k \Delta(v_k(j)) \Delta(v_k(i)) \right].
\]

(A.22)

The only difference from the expression of \(\hat{w}_{ij}\) under the null hypothesis is the extra term \(\sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j)\) on the left hand side of (A.22). Notice that this term is non random. Define

\[
\bar{w}_{ij} = \hat{w}_{ij} - \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j).
\]

Then the central limit theorem for \(\sum_{i \neq j} \bar{w}_{ij}\) can be obtained by using the same proof as that for Theorem 3.1. Thus,

\[
\left( T_n - \frac{\sqrt{n} \sum_{i \neq j} \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) Y_{ij}}{\sqrt{2 \sum_{i \neq j} \hat{w}_{ij}^2}} \right) \sqrt{\frac{\sum_{i \neq j} \hat{w}_{ij}^2}{\sum_{i \neq j} \bar{w}_{ij}^2}} \to N(0,1).
\]

(A.23)
Note that $\hat{w}_{ij}$ is the residual term under the alternative hypothesis. Similarly to (A.2), we also have

$$\frac{\sum_{i \neq j} \hat{w}_{ij}^2}{\sum_{i \neq j} \mathbb{E} w_{ij}^2} = 1 + o_p(1).$$

(A.24)

Moreover, direct calculations show that

$$\sum_{i \neq j} \hat{w}_{ij}^2 = \sum_{i \neq j} \hat{w}_{ij}^2 + 2 \sum_{i \neq j} \hat{w}_{ij} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right) + \sum_{i \neq j} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)^2.$$

(A.25)

We first prove (8) in Theorem 3.2. It follows from Cauchy-Schwarz inequality, (A.24) and the condition $\sum_{i \neq j} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)^2 \ll \sum_{i \neq j} \mathbb{E} w_{ij}^2$ that

$$|\sum_{i \neq j} \hat{w}_{ij} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)| \leq \sqrt{\sum_{i \neq j} \hat{w}_{ij}^2 \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)^2} = o_p(\sum_{i \neq j} \mathbb{E} w_{ij}^2).$$

(A.26)

Combining (A.25) with (A.26) and in view of (A.24) we arrive at

$$\frac{\sum_{i \neq j} \hat{w}_{ij}^2}{\sum_{i \neq j} \mathbb{E} w_{ij}^2} = 1 + o_p(1).$$

(A.27)

This together with (A.23) and (A.24) completes the proof of (8) in Theorem 3.2.

Next consider the case when condition (9) is true. The definition of $\tilde{w}_{ij}$ entails that

$$\sum_{i \neq j} \tilde{w}_{ij}^2 = \sum_{i \neq j} \tilde{w}_{ij}^2 + \sum_{i \neq j} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)^2$$

(A.28)

$$\leq 2 \left( \sum_{i \neq j} \tilde{w}_{ij}^2 + \sum_{i \neq j} \left( \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) \right)^2 \right)$$

$$\leq 2 \left( \sum_{i \neq j} \tilde{w}_{ij}^2 + (K - K_0) \sum_{k=K_0+1}^{K} d_k^2 \sum_{i \neq j} v_k^2(i) v_k^2(j) \right)$$

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\[
\begin{align*}
&\leq 2\left(\sum_{i \neq j} \tilde{w}_{ij}^2 + (K - K_0) \sum_{k=K_0+1}^K d_k^2\right) \\
&\leq 2 \sum_{i \neq j} \tilde{w}_{ij}^2 + 2(K - K_0)\left( \sum_{k=K_0+1}^K |d_k|^2 \right) \lesssim \left( \sum_{i \neq j} \tilde{w}_{ij}^2 + \sum_{k=K_0+1}^K |d_k|^2 \right)^2.
\end{align*}
\]

By (A.23), (A.24) and (A.28), in order to obtain (10), it suffices to show that
\[
\frac{\sqrt{m}\left| \sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j) \mathbf{Y}_{ij} \right|}{\sqrt{\sum_{i \neq j} \tilde{w}_{ij}^2 + \sum_{k=K_0+1}^K |d_k|^2}} \rightarrow_{i.p.} \infty. \tag{A.29}
\]

For the term \(\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j) \mathbf{Y}_{ij}\) in the numerator, we calculate its expectation and variance respectively as follows:
\[
E \\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j) \mathbf{Y}_{ij} = \frac{\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j)}{m}
\]
and
\[
\text{var} \left( \sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j) \mathbf{Y}_{ij} \right) \lesssim \frac{(\sum_{k=K_0+1}^K d_k)^2 \sum_{i \neq j} \mathbf{v}_k^2(i) \mathbf{v}_k^2(j)}{m} \leq \frac{(\sum_{k=K_0+1}^K d_k)^2}{m}.
\]

Therefore we have
\[
\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j) \mathbf{Y}_{ij} = \frac{\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j)}{m} + O_P\left( \frac{\sum_{k=K_0+1}^K |d_k|}{\sqrt{m}} \right). \tag{A.30}
\]

Combining (A.30) with condition (9) that
\[
\frac{|\sum_{k=K_0+1}^K d_k \sum_{i \neq j} \mathbf{v}_k(i) \mathbf{v}_k(j)|}{\sqrt{m}\left( \sum_{i \neq j} \tilde{w}_{ij}^2 + \sum_{k=K_0+1}^K |d_k|^2 \right)^{1/2}} \gg 1
\]
as well as (A.27) for the denominator, we conclude (A.29) and thus complete the proof.
Supplementary material for “Universal Rank Inference via Residual Subsampling with Application to Large Networks”

XIAO HAN, QING YANG AND YINGYING FAN

This Supplementary Material contains some main theorems, key lemmas and their proofs as well as additional technical details.

B Proof of other main results

B.1 Lemma 1 and its proof

Recall the definition of the residual matrix $\hat{W}$ in (4). In this section we connect its entries $\hat{w}_{ij}$ with $(\hat{v}_k(i), \hat{d}_k)$, which is important for analyzing the asymptotic properties of $\hat{w}_{ij}$.

**Lemma 1.** Let $\Delta(d_k) = \hat{d}_k - d_k$ and $\Delta(v_k(i)) = \hat{v}_k(i) - v_k(i)$. For $K \geq K_0$, we have

$$\hat{w}_{ij} = w_{ij} + \delta(K > K_0) \sum_{k=K_0+1}^{K} d_k v_k(i) v_k(j) - \sum_{k=1}^{K_0} [\Delta(d_k) \hat{v}_k(i) \hat{v}_k(j)] \quad (B.1)$$

where $\delta(K > K_0) = \begin{cases} 1, & K > K_0, \\ 0, & otherwise. \end{cases}$

**Proof.** By the definition of $\hat{W}$, we have

$$\hat{W} = X - \sum_{k=1}^{K_0} \hat{d}_k \hat{v}_k \hat{v}_k^T = W + \sum_{k=1}^{K} d_k v_k v_k^T - \sum_{k=1}^{K_0} \hat{d}_k \hat{v}_k \hat{v}_k^T. \quad (B.2)$$

Equation (B.1) follows directly from (B.2) by considering each entry separately. \qed
B.2 Proof of Theorem 3.3

We start with analyzing the first inequality $\sum_{i \neq j} \sigma_{ij}^2 \gg m$ in Condition 5. Combing the assumptions $\theta_n = \max_{i \neq j} \sigma_{ij}^2$, $\min_{i \neq j} \sigma_{ij}^2 \gtrsim \theta_n (\log n)^{-1}$ with Condition 3, we can see that

$$n \theta_n (\log n)^{-1} \lesssim \alpha_n^2 \lesssim n \theta_n \quad \text{and} \quad \sum_{i \neq j} \sigma_{ij}^2 \gtrsim n^2 \theta_n (\log n)^{-1}. \quad (B.3)$$

Then the first inequality $\sum_{i \neq j} \sigma_{ij}^2 \gg m$ is satisfied if

$$n^2 \theta_n (\log n)^{-2} \gg m. \quad (B.4)$$

By Conditions 3 and (B.3), we have

$$\frac{|d_K|}{\alpha_n} \gtrsim \alpha_n \gtrsim \sqrt{n \theta_n (\log n)^{-1}}. \quad (B.5)$$

Moreover, the assumption $|d_1| \lesssim n \theta_n$ yields

$$\frac{|d_K|}{\alpha_n} \gtrsim \frac{|d_1|}{\alpha_n} \lesssim \sqrt{n \theta_n}. \quad (B.6)$$

It follows from (B.5) and (B.6) that

$$\theta_n \gtrsim n^{2\epsilon - 1} \log n \quad (B.7)$$

is a sufficient condition for ensuring $\frac{|d_K|}{\alpha_n} \gtrsim n^\epsilon$ in Condition 3. In view of (B.4) and (B.7), any $m$ satisfying

$$m \ll n^{1 + 2\epsilon} (\log n)^{-1} \quad (B.8)$$

makes the first inequality in Condition 5 hold.

Next, we discuss the second inequality in Condition 5, i.e.

$$\sum_{i \neq j} \sigma_{ij}^2 \gtrsim n^\epsilon \left( \frac{n \sum_{k=1}^{K_0} (1^T v_k)^2}{m} + \alpha_n^2 + \frac{n^2 \alpha_n^2}{md^2_K} \right).$$

By assumption (12) and $\theta_n = \max_{i \neq j} \sigma_{ij}^2$ and $\min_{i \neq j} \sigma_{ij} \gtrsim \theta_n (\log n)^{-1}$, it suffices to have

$$\sum_{i \neq j} \sigma_{ij}^2 \gtrsim n^2 \theta_n (\log n)^{-1} \gg \frac{n^{1+\epsilon} \sum_{k=1}^{K_0} (1^T v_k)^2}{m} + n^\epsilon \alpha_n^2 + \frac{n^{2+\epsilon} \alpha_n^2}{md^2_K}. \quad (B.9)$$
Now we compare the three terms on the very right hand side of (B.9) with $n^2\theta_n(\log n)^{-1}$ one by one. Note that by (B.5), the second term $n^4\theta_n^2 \lesssim n^{1+\epsilon}\theta_n \ll n^2\theta_n$, making no contribution to the choice of $m$. For the third term, it is easy to see from (B.5) that $n\theta_n \lesssim d_K^2\alpha_n^{-2}\log n$, which guarantees that

$$\frac{n^2\alpha_n^2}{md_K^2} \lesssim \frac{n^{1+\epsilon}\log n}{m\theta_n}.$$  

Therefore, any $m$ satisfying

$$m \gg n^{\epsilon-1}\theta_n^{-2}(\log n)^2$$  

ensures that $\frac{n^2\alpha_n^2}{md_K^2} \ll n^2\theta_n(\log n)^{-1}$. Finally, for the first term, it suffices to have

$$m \gg n^{\epsilon-1}\theta_n^{-1}(\log n)\sum_{k=1}^{K_0}(1^Tv_k)^2. \quad (B.11)$$

By Cauchy-Schwarz inequality we know that $\sum_{k=1}^{K_0}(1^Tv_k)^2 \leq nK_0$. The above two results entail that (B.11) is satisfied as long as

$$m \gg \frac{n^\epsilon}{\theta_n} \log n. \quad (B.12)$$

Combining (B.9), (B.10) and (B.12), it is easy to see that any $m$ satisfying

$$m \gg \frac{n^\epsilon}{\theta_n} \log n + n^{\epsilon-1}\theta_n^{-2}(\log n)^2$$  

(B.13)

can make the second inequality in Condition 5 hold. Moreover, it follows from (B.7) that

$$m \gg n^{1-\epsilon}$$  

(B.14)

is a sufficient condition for (B.13).

Summarizing the arguments above tells that (13) in Theorem 3.3 is sufficient for Condition 5 (see (B.4) and (B.13)), and $n^{1-\epsilon} \ll m \ll n^{1+2\epsilon}(\log n)^{-1}$ is a sufficient condition to ensure (13) (see (B.8) and (B.14)).

To complete the proof of Theorem 3.3, it remains to verify inequality (9) in Theorem 3.2 under condition (13). By (B.5) and (B.6) we have $n\theta_n(\log n)^{-1} \lesssim d_k \lesssim n\theta_n$ for all $k = 1, \cdots, K$. This together with (B.3) entails that

$$n\sqrt{\theta_n(\log n)^{-1}} \lesssim \sqrt{\sum_{i\neq j}\sigma_{ij}^2} + \sum_{k=K_0+1}^{K}|d_k| \lesssim n\sqrt{\theta_n}.$$
This implies that the inequality
\[ \sqrt{m} \ll \frac{\sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i)v_k(j)}{n\sqrt{\theta_n}} \]  
(B.15)
is sufficient for (9). Furthermore, it follows from the third condition in (12) that
\[ \frac{\sum_{k=K_0+1}^{K} d_k \sum_{i \neq j} v_k(i)v_k(j)}{n\sqrt{\theta_n}} \geq \frac{\sum_{i \neq j} \sigma^2_{ij}}{n\sqrt{\theta_n}} \geq n\sqrt{\theta_n}(\log n)^{-1}. \]
Thus, any \( m \ll n^2\theta_n(\log n)^{-2} \) is sufficient for (9). This completes our proof.

**B.3 Proof of (14) in Theorem 3.4**
The asymptotic distribution (14) under the null hypothesis in Theorem 3.4 can be concluded from the following two results:
\[ \frac{\sum_{i=1}^{n} \hat{w}_{ii}}{\sqrt{\sum_{i=1}^{n} Ew_{ii}^2}} \xrightarrow{d} N(0,1) \quad \text{and} \quad \frac{\sum_{i=1}^{n} \hat{w}_{ii}^2}{\sum_{i=1}^{n} Ew_{ii}^2} = 1 + o_p(1). \]  
(B.16)
(B.17)

We first prove (B.16). Under the null hypothesis \( K = K_0 \), Lemma 1 ensures that
\[ \hat{w}_{ij} = w_{ij} - \sum_{k=1}^{K_0} \left[ \Delta(d_k)\hat{v}_k(i)\hat{v}_k(j) + d_k\Delta(v_k(i))v_k(j) + d_k\Delta(v_k(j))v_k(i) + O(\frac{n^2}{nd_k}) \right]. \]  
(B.18)

By (C.21) in Lemma 3, Theorem C.1 and Corollary 2 as well as Condition 4, we have
\[ \Delta(d_k) = \hat{d}_k - d_k = \frac{v_k^T E\hat{W}^2v_k}{d_k} + v_k^T E\hat{W}v_k + v_k^T (W - E\hat{W})v_k + O(\frac{\alpha_n^2}{\sqrt{nd_k}}) \]
\[ = O(\frac{\alpha_n^2}{d_k}) + O(\min\{1, \frac{\alpha_n}{\sqrt{n}}\}) + O(\frac{1}{\sqrt{n}}) \]
\[ = O(1) + O(\sqrt{\theta_n}), \]  
(B.19)
where in the second step we have used $v_k^T E W v_k = 0$ because the network has self loops. Recall the definition of $r_k$ in Section 3.5. Note that the following results can be proved by Theorem C.1, Corollary 2 and (C.24):

$$|r_k(i)| = |e_i^T V_{-k}(t_k D_{-k}^{-1} - I)^{-1} V_k^T E W^2 v_k| \lesssim \frac{1}{\sqrt{n}} \cdot \alpha_n^2, \quad (B.20)$$

$$|s_{k,i}^T W v_k - e_i^T W v_k|$$

$$= |\mathcal{R}(e_i, V_{-k}, t_k) (D_{-k})^{-1} + \mathcal{R}(V_{-k}, V_{-k}, t_k))^{-1} V_k^T W v_k + e_i^T v_k v_k^T W v_k|$$

$$\lesssim \left\| \mathcal{R}(e_i, V_{-k}, t_k) (D_{-k})^{-1} + \mathcal{R}(V_{-k}, V_{-k}, t_k))^{-1} \right\| \left\| V_k^T W v_k \right\|$$

$$+ \left| e_i^T v_k v_k^T W v_k \right|$$

$$= \frac{1}{\sqrt{n} t_k} \cdot O_\prec(1) + \frac{1}{\sqrt{n}} \cdot O_\prec(1).$$

By (C.20) in Lemma 3 and noting that $d_k \sim t_k$ for all $k = 1, \cdots, K$, we have

$$\Delta(v_k(i)) = \hat{v}_k(i) - v_k(i)$$

$$= \frac{e_i^T W v_k}{t_k} + \frac{r_k(i)}{t_k^2} + \frac{e_i^T W^2 v_k}{t_k^2} - v_k(i) \frac{3 v_k^T E W^2 v_k}{2 t_k^2} + \frac{s_{k,i}^T W v_k - e_i^T W v_k}{t_k}$$

$$+ O_\prec \left( \frac{\alpha_n}{\sqrt{\alpha_n^2}} \right)$$

$$= \frac{e_i^T W v_k}{t_k} + O(\alpha_n^2) + \frac{O(\alpha_n^2/\sqrt{n}) + O(\alpha_n^2/\sqrt{n})}{t_k^2} + O_\prec \left( \frac{1}{\sqrt{n} \cdot |t_k|} \right) + \frac{O_\prec(\alpha_n^2)}{\sqrt{\alpha_n^2}}$$

$$= \frac{e_i^T W v_k}{t_k} + O_\prec \left( \frac{1}{\sqrt{n} |d_k|} \right),$$

uniformly for all $i = 1, \cdots, n$, where the penultimate step uses Theorem C.1, Corollary 2 and (B.20). Condition 4 together with (B.21) and Corollary 2 ensures $\|\hat{v}_k\|_\infty \lesssim \frac{1}{\sqrt{n}} + O_\prec(\frac{\alpha_n}{\sqrt{n} |d_k|})$. Using this and substituting (B.19) and (B.21) into (B.18) gives us

$$\hat{w}_{ij} = w_{ij} + O_\prec \left( \frac{1}{n} \right) - \sum_{k=1}^{K_0} \frac{d_k (e_i^T W v_k v_k(j) + e_j^T W v_k v_k(i))}{t_k}.$$  

Therefore,

$$\frac{1}{n} \sum_{i=1}^{n} \hat{w}_{ii} = \frac{1}{n} \sum_{i=1}^{n} w_{ii} + O_\prec \left( \frac{1}{n} \right) - 2 \sum_{k=1}^{K_0} \frac{d_k v_k^T W v_k}{n t_k}. \quad (B.23)$$
Since $X$ contains selfloops, $\mathbb{E} v_k^T W v_k = 0$. Together with Corollary 2 and the fact that $\|v_k\|_\infty \lesssim \frac{1}{\sqrt{n}}$, we have

$$v_k^T W v_k = O_\prec \left( \frac{\alpha_n}{\sqrt{n}} \right).$$

Then it follows from $\alpha_n \lesssim \sqrt{n}$ that (B.23) can be simplified to

$$\frac{1}{n} \sum_{i=1}^n \hat{w}_{ii} = \frac{1}{n} \sum_{i=1}^n w_{ii} + O_\prec \left( \frac{1}{n} \right). \quad (B.24)$$

It is easy to see that $\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^n w_{ii} \right) = 0$ and $\sqrt{\mathbb{V} \mathbb{A}r \left( \frac{1}{n} \sum_{i=1}^n w_{ii} \right)} = \frac{1}{n} \sqrt{\sum_{i=1}^n \mathbb{E} w_{ii}^2} \gg n^{\epsilon^{-1}}$ by the condition of Theorem 3.4. Therefore (B.24) can be rewritten as

$$\frac{1}{n} \sum_{i=1}^n \hat{w}_{ii} = \frac{1}{n} \sum_{i=1}^n w_{ii} + o_p \left( \sqrt{\mathbb{V} \mathbb{A}r \left( \frac{1}{n} \sum_{i=1}^n w_{ii} \right)} \right). \quad (B.25)$$

Moreover, according to Condition 1 it holds that

$$\frac{\sum_{i=1}^n \mathbb{E} w_{ii}^4}{\left( \sum_{i=1}^n \mathbb{E} w_{ii}^2 \right)^2} \lesssim \frac{\sum_{i=1}^n \mathbb{E} w_{ii}^2}{\left( \sum_{i=1}^n \mathbb{E} w_{ii}^2 \right)^2} \to 0,$$

satisfying Lyapunov’s condition. Thus, $\sum_{i=1}^n w_{ii} / \sqrt{\sum_{i=1}^n \mathbb{E} w_{ii}^2} \overset{d}{\to} N(0, 1)$. This together with (B.25) proves (B.16).

Next we prove (B.17). Condition 1 yields

$$\mathbb{V} \mathbb{A}r \left( \sum_{i=1}^n w_{ii}^2 \right) \leq \sum_{i=1}^n \mathbb{E} w_{ii}^4 \lesssim \sum_{i=1}^n \mathbb{E} w_{ii}^2,$$

which implies that

$$\mathbb{V} \mathbb{A}r \left( \frac{\sum_{i=1}^n w_{ii}^2}{\sum_{i=1}^n \mathbb{E} w_{ii}^2} \right) \lesssim \frac{1}{\sum_{i=1}^n \mathbb{E} w_{ii}^2} \to 0.$$

Therefore we have

$$\frac{\sum_{i=1}^n w_{ii}^2}{\sum_{i=1}^n \mathbb{E} w_{ii}^2} = 1 + o_p(1). \quad (B.26)$$

Then to verify (B.17), it suffices to show that

$$\frac{\sum_{i=1}^n w_{ii}^2}{\sum_{i=1}^n \mathbb{E} w_{ii}^2} = 1 + o_p(1).$$
By equation (B.26) and the condition $\sum_{i=1}^{n} E w_{ii}^2 \geq n^{2\epsilon}$ in Theorem 3.4, we only need to show

\[
\sum_{i=1}^{n} \hat{w}_{ii}^2 = \sum_{i=1}^{n} w_{ii}^2 + O_p(1 + o_p(1)) + O_p(1). \tag{B.27}
\]

In view of equation (B.22), it holds uniformly over all $i = 1, \ldots, n$ that

\[
\hat{w}_{ii}^2 = w_{ii}^2 + 2 w_{ii} O_{\prec} \left( \frac{1}{n} \right) - 4 \sum_{k=1}^{K_0} d_k e_i^T W v_k v_k(i) \left( \frac{1}{n^2} \right) \tag{B.28}
\]

and thus

\[
\sum_{i=1}^{n} \hat{w}_{ii}^2 = \sum_{i=1}^{n} w_{ii}^2 + 2 \sum_{i=1}^{n} (w_{ii} O_{\prec} \left( \frac{1}{n} \right)) - 4 \sum_{k=1}^{K_0} \sum_{i=1}^{n} (w_{ii} d_k e_i^T W v_k v_k(i)) + O_p(1) \tag{B.29}
\]

To prove (B.27), we study the terms on the right hand side of (B.29). To begin with, we know that

\[
\left| \sum_{i=1}^{n} w_{ii} O_{\prec} \left( \frac{1}{n} \right) \right| \leq \sum_{i=1}^{n} w_{ii} |O_{\prec} \left( \frac{1}{n} \right)| = O_p(1). \tag{B.30}
\]

Next, we write

\[
\sum_{i=1}^{n} \frac{d_k e_i^T W v_k v_k(i)}{t_k} = \sum_{i=1}^{n} \frac{d_k w_{ii} v_k^2(i)}{t_k} + \sum_{1 \leq l \neq i \leq n} \frac{d_k w_{ii} w_{il} v_k(i) v_k(l)}{t_k}. \tag{B.31}
\]

For the first term on the right hand side of (B.31), it follows from $E \left| \sum_{i=1}^{n} \frac{d_k w_{ii}^2 v_k^2(i)}{t_k} \right| \lesssim \frac{1}{n} \sum_{i=1}^{n} E w_{ii}^2 \lesssim 1$ that

\[
\sum_{i=1}^{n} \frac{d_k w_{ii}^2 v_k^2(i)}{t_k} = O_p(1).
\]
For the second term, it follows from the calculations

\[ E\left( \sum_{1 \leq i \neq l \leq n} \frac{d_kw_{il}w_{il}v_k(l)}{t_k} \right) = 0 \quad \text{and} \]

\[ \text{var}\left( \sum_{1 \leq i \neq l \leq n} \frac{d_kw_{il}w_{il}v_k(l)}{t_k} \right) \lesssim \sum_{1 \leq i \neq l \leq n} Ew_{ii}^2 Ew_{il}^2 v_k^2(l) \]

\[ \lesssim \frac{\alpha_n^2}{n^2} \sum_{i=1}^n Ew_{ii}^2 \lesssim \frac{\sum_{i=1}^n Ew_{ii}^2}{n} \lesssim 1 \]

that

\[ \sum_{1 \leq i \neq l \leq n} \frac{d_kw_{il}w_{il}v_k(l)}{t_k} = O_p(1). \]

Therefore, \[ \sum_{i=1}^n w_{ii} \frac{d_ke_i^T Wv_k(i)}{t_k} = O_p(1). \] (B.32)

Since \( \mathbb{E} W = 0 \), it follows from \( d_k \sim t_k \), Condition 4 and Corollary 2 that

\[ \frac{d_ke_i^T Wv_k(i)}{t_k} = O_{\prec}(\alpha_n/n) = O_{\prec}(\frac{1}{\sqrt{n}}). \]

Thus, by Condition 4, and Corollary 2 we can see that

\[ \sum_{i=1}^n \left( \sum_{k=1}^{K_0} \frac{d_ke_i^T Wv_k(i)}{t_k} \right)^2 \leq K_0 \sum_{i=1}^n \left( \sum_{k=1}^{K_0} \frac{d_k^2(e_i^T Wv_k(i))^2}{t_k^2} \right) \] (B.33)

\[ = O_{\prec}(\frac{\alpha_n^2}{n}) = O_{\prec}(1). \]

Therefore, we have

\[ \left| \sum_{i=1}^n \sum_{k=1}^{K_0} O_{\prec}(\frac{1}{n}) \frac{d_ke_i^T Wv_k(i)}{t_k} \right| = \sum_{i=1}^n \sum_{k=1}^{K_0} O_{\prec}(\frac{1}{n}) O_{\prec}(\frac{1}{\sqrt{n}}) = O_{\prec}(\frac{1}{\sqrt{n}}). \] (B.34)

Then (B.27) is concluded from combining the arguments (B.29), (B.30), (B.32), (B.33) and (B.34). Therefore, equation (B.17) holds and the proof of (14) in Theorem 3.4 is completed.
B.4 Proof of Theorem 3.5

Recall that Theorem 3.5 assumes the null hypothesis $K = K_0$. Plugging the expansions of $\Delta(d_k)$ and $\Delta(v_k(i))$ in Lemma 3 into the expression in Lemma 1, and using results in Theorem C.1, (C.3) in Corollary 2 and (B.20) we arrive at

$$
\hat{w}_{ij} = w_{ij} - \sum_{k=1}^{K_0} v_k(i)v_k(j)(-2 \frac{v_k^T E W^2 v_k}{d_k} + v_k^T W v_k)
$$

(B.35)

$$
- \sum_{k=1}^{K_0} \frac{e_i^T W^2 v_k(j) + e_j^T W^2 v_k(i)}{t_k}
- \sum_{k=1}^{K_0} \frac{r_k(i) v_k(j) + r_k(j) v_k(i)}{t_k}
- \sum_{k=1}^{K_0} \frac{d_k(s_{k,i}^T W v_k(j) + s_{k,j}^T W v_k(i))}{t_k}
- \sum_{k=1}^{K_0} \frac{d_k e_i^T W v_k e_j^T W v_k}{t_k^2} + O_{\prec}(\frac{\alpha n}{n |d| K_0}),
$$

uniformly over all $i, j$. Summing up (B.35) over subscripts $i$ and $j$ yields

$$
\sum_{i \neq j} \hat{w}_{ij} = \sum_{i \neq j} w_{ij} - \sum_{k=1}^{K_0} v_k^T W v_k \sum_{i \neq j} (v_k(i)v_k(j))
$$

(B.36)

$$
- \sum_{k=1}^{K_0} \frac{e_i^T W^2 v_k(j) + e_j^T W^2 v_k(i)}{t_k}
- \sum_{k=1}^{K_0} \frac{r_k(i) v_k(j) + r_k(j) v_k(i)}{t_k}
- \sum_{k=1}^{K_0} \frac{d_k(s_{k,i}^T W v_k(j) + s_{k,j}^T W v_k(i))}{t_k}
+ 2 \sum_{k=1}^{K_0} \frac{v_k^T E W^2 v_k}{d_k} \sum_{i \neq j} (v_k(i)v_k(j)) - 2 \sum_{k=1}^{K_0} s_{k,i} v_k \sum_{i=1}^n v_k(i) + O_{\prec}(\frac{n\alpha n}{d K_0}),
$$

where we have made use of the following relationship, which is a direct consequence of Theorem C.1 and Corollary 2

$$
\sum_{k=1}^{K_0} \sum_{i \neq j} d_k e_i^T W v_k e_j^T W v_k = \sum_{k=1}^{K_0} \frac{d_k 1^T W v_k 1^T W v_k}{t_k^2} - \sum_{k=1}^{K_0} \sum_{i=1}^n \frac{d_k(e_i^T W v_k)^2}{t_k^2}
= O_{\prec}(\frac{\alpha^2 n}{d K_0}).
$$

Denote by $a_k = \sum_{i=1}^n v_k(i)$. By Cauchy–Schwarz inequality, $|a_k| \leq \sqrt{n}$. Furthermore, Corollary 2 ensures that

$$
1^T (W^2 - E W^2) v_k = O_{\prec}(\alpha_n^2).
$$
Therefore by Theorem C.1 and Corollary 2 we have

\[
\sum_{k=1}^{K_0} \sum_{i \neq j} e_i^T W^2 v_k(j) + e_j^T W^2 v_k(i) t_k \\
= \sum_{k=1}^{K_0} e_i^T W^2 v_k(j) + e_j^T W^2 v_k(i) t_k \\
+ \sum_{k=1}^{K_0} \sum_{i \neq j} e_i^T (W^2 - E W^2) v_k(j) + e_j^T (W^2 - E W^2) v_k(i) t_k \\
= 2 \sum_{k=1}^{K_0} \frac{1T E W^2 v_k a_k}{t_k} + 2 \sum_{k=1}^{K_0} \frac{1T (W^2 - E W^2) v_k a_k}{t_k} \\
- 2 \sum_{k=1}^{K_0} \sum_{i=1}^{n} e_i^T W^2 v_k(i) t_k \\
= 2 \sum_{k=1}^{K_0} \frac{1T E W^2 v_k a_k}{t_k} + O_{\prec}\left(\frac{n \alpha_n}{d_{K_0}}\right) + O_{\prec}\left(\sqrt{n \alpha_n}\right),
\]

where we have used the simple inequality that \( \alpha_n \lesssim \sqrt{n} \). Then (B.36) can be written as

\[
\sum_{i \neq j} \hat{w}_{ij} = \sum_{i \neq j} w_{ij} (1 - \sum_{k=1}^{K_0} a_k^2 v_k(i) v_k(j)) - \sum_{k=1}^{K_0} a_k^2 v_k^T \text{diag}(W) v_k - 2 \sum_{k=1}^{K_0} \frac{1T E W^2 v_k a_k}{t_k} - 2 \sum_{k=1}^{K_0} a_k \frac{1T r_k}{t_k} - 2 \sum_{k=1}^{K_0} \sum_{i=1}^{n} e_i^T W^2 v_k(i) t_k \\
- \sum_{k=1}^{K_0} \sum_{i=1}^{n} e_i^T \text{diag}(W) v_k - 2 \sum_{k=1}^{K_0} \frac{1T E W^2 v_k a_k}{t_k} - 2 \sum_{k=1}^{K_0} a_k \frac{1T r_k}{t_k} \\
- \sum_{k=1}^{K_0} \frac{v_k^T E W^2 v_k}{d_k} a_k^2 - 2 \sum_{k=1}^{K_0} a_k s_k^T \text{diag}(W) v_k + O_{\prec}\left(\frac{n \alpha_n}{d_{K_0}}\right) + 1
\]
\[
\begin{align*}
    &= 2 \sum_{i<j} w_{ij} (1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))) \\
&\quad - R(K_0) + O_s\left(\frac{n\alpha_n}{|d_{K_0}|} + 1\right).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\sum_{i\neq j} \hat{w}_{ij} + R(K_0)
&= 2 \sum_{i<j} w_{ij} (1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))) \\
&\quad + O_s\left(\frac{n\alpha_n}{|d_{K_0}|} + 1\right),
\end{align*}
\]

where \( S_n \) is the sum of independent random variables and its variance equals to

\[
\text{var}(S_n) = 4 \sum_{i<j} \sigma_{ij}^2 \left( 1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j)) \right)^2.
\]

In addition, by (C.25) and the definition of \( s_k(j) \), we have

\[
\|s_k(j) - e_j\| \lesssim \frac{1}{\sqrt{n}}.
\]

This together with \(|a_k| \leq \sqrt{n}\) implies that

\[
\max_{1 \leq i<j \leq n} |1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))| \lesssim 1.
\]

Combining the above result with the condition \( \max_{1 \leq i<j \leq n} |w_{i,j}| \leq C \) and (18) we have

\[
[\text{var}(S_n)]^{-2} \times \left[ 2 \sum_{i<j} Ew_{ij}^4 (1 - \sum_{k=1}^{K_0} a_k^2 v_k(i)v_k(j) - \sum_{k=1}^{K_0} a_k (v_k(j)s_k(i) + v_k(i)s_k(j))^4 \right]
\]

\[
\lesssim [\text{var}(S_n)]^{-1} \rightarrow 0.
\]
Combining (18) with Lyapunov’s condition, we can conclude that \( \sum_{i \neq j} \hat{w}_{ij} + R(K_0) \) converges weakly to the standard normal distribution after centralization and normalization, i.e.

\[
\sum_{i \neq j} \hat{w}_{ij} + R(K_0) \xrightarrow{d} N(0,1).
\]

This completes the proof of the theorem.

B.5 Proof of (15) and (16) in Theorem 3.4

Recall (A.22). The proof of (15) and (16) in Theorem 3.4 is similar to that of Theorem 3.2 and thus we omit the details.

B.6 Proof of Theorem 3.6

The proof of Theorem 3.6 is almost the same as Theorem 3.2 by defining \( \tilde{w}_{ij} = \hat{w}_{ij} - \sum_{k=K_0+1}^{K} d_k v_k(i)v_k(j) \) and thus we omit the details.

C Some Lemmas and Their Proofs

Before presenting the keys Lemmas, we first state the following Corollary, which is a direct consequence of Theorem 7 and Corollary 1 in [9]. This lemma is used throughout the proofs of this paper. In fact, all of our order small terms \( O_\prec(\cdot) \) are followed by this Corollary.

**Corollary 2.** Under Conditions 1-4, it holds that for any positive integers \( l \) and \( r \), and unit vectors \( x \) and \( y \),

\[
\mathbb{E} \left[ x^T(W^l - \mathbb{E}W^l)y \right]^{2r} \leq C_r(\min\{\alpha_n^{l-1}, d_x\alpha_n^l, d_y\alpha_n^l\})^{2r},
\]

\[
x^T(W^l - \mathbb{E}W^l)y = O_\prec(\min\{\alpha_n^{l-1}, d_y\alpha_n^l, d_x\alpha_n^l\}),
\]

where \( d_x = \|x\|_\infty \), \( d_y = \|y\|_\infty \) and \( C_r \) is some positive constant determined only by \( r \). Moreover, for any positive constants \( a \) and \( b \), there exists \( n_0(a,b) > 0 \) such that for any positive integer \( l \)

\[
\sup_{\|x\|=\|y\|=1} \mathbb{P}(|x^T(W^l - \mathbb{E}W^l)y| \geq n^a \min\{\alpha_n^{l-1}, d_y\alpha_n^l, d_x\alpha_n^l\}) \leq n^{-b},
\]

for any \( n \geq n_0(a,b) \).
The following Theorem C.1 is Theorem 6 in [9], which we include here for easier reference.

**Theorem C.1.** For any unit vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\mathbb{E}\mathbf{x}^T\mathbf{W}^l\mathbf{y} \lesssim \alpha_n^l,$$  \hspace{1cm} (C.4)

where $l$ is a positive integer. Furthermore, if the number of non zero entries of $\mathbf{x}$ is bounded, then for any positive integer $l$,

$$\mathbb{E}\mathbf{x}^T\mathbf{W}^l\mathbf{y} \lesssim \alpha_n^l\alpha_n,$$  \hspace{1cm} (C.5)

Now we are ready to proceed to the key lemmas as well as their proofs.

### C.1 Lemma 2
We will need the following notations for the proof of the lemma.

$$A_{\mathbf{x},k,t} = \mathcal{P}(\mathbf{x}, \mathbf{v}_k, t) - \mathcal{P}(\mathbf{x}, \mathbf{V}_{-k}, t) \left[ t(D_{-k})^{-1} + \mathcal{P}(\mathbf{V}_{-k}, \mathbf{V}_{-k}, t) \right]^{-1} \mathcal{P}(\mathbf{V}_{-k}, \mathbf{v}_k, t),$$

$$\tilde{b}_{k,t} = \left[ z^2(A_{\mathbf{v}_k,k,t}/2t) \right]^{-1},$$

where $\mathcal{P}(\cdot)$ is defined at the beginning of Section 3.5, $\mathbf{x}$ is a vector of appropriate dimension and the derivative in $(\cdot)'$ is with respect to $t$.

**Lemma 2.** Under Conditions 1-4, we have the following expansion

$$\mathbf{e}_i^T\mathbf{V}_k\mathbf{v}_k^T \mathbf{v}_k = \left( \tilde{P}_{k,t_k} - 2t_k^{-1}\tilde{P}_{k,t_k}^2 \mathbf{v}_k^T\mathbf{W}\mathbf{v}_k + O_{\prec}(\frac{\alpha_n^2}{\sqrt{nt_k^2}}) \right).$$

$$\times \left[ A_{\mathbf{e}_i,k,t} - t_k^{-1}\mathbf{b}_{\mathbf{e}_i,k,t}^T \mathbf{W}\mathbf{v}_k - \mathbf{e}_i^T(W^2 - \mathbb{E}W^2)v_k + O_{\prec}(\frac{\alpha_n^3}{\sqrt{nt_k^3}}) \right].$$

$$\times \left[ A_{\mathbf{v}_k,k,t} - t_k^{-1}\mathbf{b}_{\mathbf{v}_k,k,t}^T \mathbf{W}\mathbf{v}_k + O_{\prec}(\frac{\alpha_n^2}{\sqrt{nt_k^2}}) \right].$$  \hspace{1cm} (C.6)

In addition, we have

$$\tilde{d}_k = t_k + \mathbf{v}_k^T\mathbf{W}\mathbf{v}_k + O_{\prec}(\frac{1}{\sqrt{n}}).$$  \hspace{1cm} (C.7)

**Proof.** Proof of this lemma is similar to the one of Lemma 6 in [9]. Recall the conditions required in the Lemma 6 of [9]. Condition 1 therein is our Condition 2. And according to their proof, Conditions 2 and 4 therein are needed just for the sake of following two statements:
1. Lemma 7 of [9] holds, that is \( \max_{1 \leq k \leq K} \|v_k\|_\infty = \|V\|_\infty \lesssim \frac{1}{\sqrt{n}} \). This is Condition 4 of our paper.

2. \( \frac{d_k}{\alpha_{\infty}} \geq n^\epsilon \) for some positive constant \( \epsilon \). This is actually ensured by Condition 3 of our paper.

Therefore, Lemma 6 in [9] also holds under the conditions of our paper, which directly implies (C.7) and the following expansion

\[
\mathbf{u}^T \mathbf{v}_k \mathbf{v}_k^T = \frac{\tilde{d}_k^2 u^T [\mathbf{G}(\tilde{d}_k) - \mathbf{F}_k(\tilde{d}_k)] v_k v_k^T [\mathbf{G}(\tilde{d}_k) - \mathbf{F}_k(\tilde{d}_k)] v_k}{\tilde{d}_k^2 v_k^T [\mathbf{G}'(\tilde{d}_k) - \mathbf{F}'_k(\tilde{d}_k)] v_k}
\]

\[
= \left[ \tilde{p}_{k,t_k} - 2 t_k^{-1} \tilde{p}_{k,t_k}^2 v_k^T \mathbf{W} v_k + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k}} \right) \right]
\]

\[
\times \left[ A_{u,k,t_k} - t_k^{-1} b_{u,k,t_k} \mathbf{W} v_k + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k}} \right) \right]
\]

\[
\times \left[ A_{v,k,t_k} - t_k^{-1} b_{v,k,t_k} \mathbf{W} v_k + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k}} \right) \right],
\]  

(C.8)

for \( \mathbf{u} = \mathbf{e}_i \) or \( \mathbf{v}_k \), where

\[
\mathbf{G}(z) = (\mathbf{W} - z \mathbf{I})^{-1}
\]  

(C.9)

and

\[
\mathbf{F}_k(z) = \mathbf{G}(z) \mathbf{V}_k \mathbf{D}^{-1}_k + \mathbf{V}_k^T \mathbf{G}(z) \mathbf{V}_k^{-1} \mathbf{V}_k^T \mathbf{G}(z).
\]  

(C.10)

Comparing (C.8) with Lemma 2, we see that to prove Lemma 2 we only need to show when \( \mathbf{u} = \mathbf{e}_i \), the term \( A_{e_i,k,t_k} - t_k^{-1} b_{e_i,k,t_k} \mathbf{W} v_k + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k}} \right) \) in (C.8) can be further expanded as \( A_{e_i,k,t_k} - t_k^{-1} b_{e_i,k,t_k} \mathbf{W} v_k - c_k(z) - \frac{e_i^T (\mathbf{W}^2 - z \mathbf{W}) v_k}{t_k^2} + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k^3}} \right) \). In fact, by comparing these two terms we see that Lemma 2 indeed provides higher order expansion of the remainder term \( O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n t_k^3}} \right) \) in (C.8). We next discuss how to obtain this higher order expansion.
By Theorem C.1 and Corollary 2, we have

\[
x^T G(z) y = -z^{-1} x^T y - z^{-2} x^T W y - \sum_{l=2}^{L} z^{-(l+1)} x^T \mathbb{E} W^l y \quad \text{(C.11)}
\]

\[
- \sum_{l=L+1}^{\infty} z^{-(l+1)} x^T W^l y - \sum_{l=2}^{L} z^{-(l+1)} x^T (W^l - \mathbb{E} W^l) y
\]

\[
= -z^{-1} x^T y - z^{-2} x^T W y - z^{-3} x^T W^2 y - \sum_{l=2}^{L} z^{-(l+1)} x^T \mathbb{E} W^l y + O_{\prec} \left( \frac{\alpha_n^3}{\sqrt{n} |z|^4} \right),
\]

for all \( z \sim t_k \) and \( \min\{\|x\|_{\infty}, \|y\|_{\infty}\} \lesssim \frac{1}{\sqrt{n}} \). Similar to (A.22)-(A.26) in [9], we have the following higher order expansions

\[
e_i^T G(z) v_k = -z^{-1} e_i^T v_k - z^{-2} e_i^T W v_k - z^{-3} e_i^T W^2 v_k - \sum_{l=3}^{L} z^{-(l+1)} e_i^T \mathbb{E} W^l v_k + O_{\prec} \left( \frac{\alpha_n^3}{\sqrt{n} |z|^4} \right),
\]

\[
v_k^T G(z) v_k = -z^{-1} - z^{-2} v_k^T W v_k - \sum_{l=2}^{L} z^{-(l+1)} v_k^T \mathbb{E} W^l v_k + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n} |z|^3} \right),
\]

\[
v_k^T G(z) V_{-k} = -z^{-2} v_k^T W V_{-k} - \sum_{l=2}^{L} z^{-(l+1)} v_k^T \mathbb{E} W^l V_{-k} + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n} |z|^3} \right),
\]

\[
e_i^T G(z) V_{-k} = -z^{-1} e_i^T V_{-k} - z^{-2} e_i^T W V_{-k} - z^{-3} e_i^T W^2 V_{-k} - \sum_{l=3}^{L} z^{-(l+1)} e_i^T \mathbb{E} W^l V_{-k} + O_{\prec} \left( \frac{\alpha_n^3}{\sqrt{n} |z|^4} \right),
\]

\[
V_{-k}^T G(z) V_{-k} = -z^{-1} - z^{-2} V_{-k}^T W V_{-k} - \sum_{l=2}^{L} z^{-(l+1)} V_{-k}^T \mathbb{E} W^l V_{-k} + O_{\prec} \left( \frac{\alpha_n^2}{\sqrt{n} |z|^3} \right).
\]
It follows from (C.12)-(C.15) that
\[
e_T F_k(z)v_k = R(e_i, V_{-k}, z) \left[ D^{-1}_k + R(V_{-k}, V_{-k}, z) \right]^{-1} R(V_{-k}, v_k, z)
- z^{-2} R(e_i, V_{-k}, z) \left[ D^{-1}_k + R(V_{-k}, V_{-k}, z) \right]^{-1} V_k^T Wv_k + O_\prec \left( \frac{\alpha_n}{\sqrt{n|z^4}} \right).
\]

(C.17)

Moreover, according to the proof of Lemma 6 in [9], the term \[ A_{u_k,t_k} - t_k^{-1} b_{u_k,t_k}^T Wv_k + O_\prec \left( \frac{\alpha_n^2}{\sqrt{n|t_k^3}} \right) \] is the expansion of \[ \hat{t}_k u^T \left[ G(\hat{t}_k) - F_k(\hat{t}_k) \right] v_k. \]

i.e.
\[
\hat{t}_k u^T \left[ G(\hat{t}_k) - F_k(\hat{t}_k) \right] v_k = A_{u_k,t_k} - t_k^{-1} b_{u_k,t_k}^T Wv_k + O_\prec \left( \frac{\alpha_n^2}{\sqrt{n|t_k^3}} \right).
\]

(C.18)

Therefore by (C.7), (C.12) and (C.17) we have
\[
(C.19)
\]

\[
\hat{t}_k e_T \left[ G(\hat{t}_k) - F_k(\hat{t}_k) \right] v_k
= P(e_i, v_k, \hat{t}_k) - P(e_i, V_{-k}, \hat{t}_k) \left[ t_k D^{-1}_k + P(V_{-k}, V_{-k}, \hat{t}_k) \right]^{-1}
\times P(V_{-k}, v_k, \hat{t}_k) - \hat{t}_k^{-1} e_i^T Wv_k - \hat{t}_k^{-2} e_i^T (W^2 - EW^2) v_k + \hat{t}_k^{-1} R(e_i, V_{-k}, \hat{t}_k)
\times \left[ D^{-1}_k + R(V_{-k}, V_{-k}, \hat{t}_k) \right]^{-1} V_k^T Wv_k + O_\prec \left( \frac{\alpha_n^3}{\sqrt{n|t_k^3}} \right)
\]

\[
= P(e_i, v_k, t_k) - P(e_i, V_{-k}, t_k) \left[ t_k D^{-1}_k + P(V_{-k}, V_{-k}, t_k) \right]^{-1}
\times P(V_{-k}, v_k, t_k) - t_k^{-1} e_i^T Wv_k - t_k^{-2} e_i^T (W^2 - EW^2) v_k + t_k^{-1} R(e_i, V_{-k}, t_k)
\times \left[ D^{-1}_k + R(V_{-k}, V_{-k}, t_k) \right]^{-1} V_k^T Wv_k + O_\prec \left( \frac{\alpha_n^3}{\sqrt{n|t_k^3}} \right)
\]

\[
= A_{e_i,k,t_k} - t_k^{-1} b_{e_i,k,t_k}^T Wv_k - t_k^{-2} e_i^T (W^2 - EW^2) v_k + O_\prec \left( \frac{\alpha_n^3}{\sqrt{n|t_k^3}} \right).
\]

That is to say, we can replace (C.19) by (C.18) and this completes the proof.

\[ \square \]

C.2 Lemma 3

Lemma 3. Under Conditions 1-4, fixing the direction \( \hat{v}_k \) such that \( \hat{v}_k^T v_k \geq 0 \), we have the following expansion
\[
\hat{v}_k(i) = v_k(i) + \frac{r_k(i)}{t_k^2} + \frac{e_T^2 W^2 v_k}{t_k^2} - v_k(i) \frac{3v_k^T W^2 v_k}{2t_k^2} + \frac{s_{k,i}^T Wv_k}{t_k} + O_\prec \left( \frac{\alpha_n}{\sqrt{n|t_k^2}} \right).
\]

(C.20)
where \( r_k \) and \( s_{k,i} \) are defined in Section 3.5. Moreover,

\[
d_k - d_k = \frac{v_k^T E W^2 v_k}{d_k} + v_k^T W v_k + O\left(\frac{\alpha_n^2}{\sqrt{n}d_k}\right). \tag{C.21}
\]

**Proof.** We prove (C.20) first. By (A.35) and (A.36) of [9], we have

\[
t_k R(v_k, v_k, t_k) = \mathcal{P}(v_k, v_k, t_k) = -1 + O\left(\frac{\alpha_n^2}{t_k^4}\right) \tag{C.22}
\]

and

\[
\|t_k R(v_k, V_{-k}, t_k)\| = \|\mathcal{P}(v_k, V_{-k}, t_k)\| \lesssim \frac{\alpha_n^2}{t_k^2}. \tag{C.23}
\]

By Theorem C.1 and Corollary 2, we have the following inequalities

\[
R(e_i, v_k, t_k) \lesssim \frac{1}{\sqrt{n} |t_k|}, \tag{C.24}
\]

\[
\|b_{e_i, k, t_k} - e_i\| = \|R(e_i, V_{-k}, t)\left((D_{-k})^{-1} + R(V_{-k}, V_{-k}, t_k)\right)^{-1} V_{-k}^T \| \lesssim \frac{1}{\sqrt{n}}, \tag{C.25}
\]

\[
\|b_{v_k, k, t_k} - v_k\| = \left\|\frac{R(v_k, V_{-k}, t)}{t_k} \left((D_{-k})^{-1} + R(V_{-k}, V_{-k}, t_k)\right)^{-1} V_{-k}^T \right\| \lesssim \frac{\alpha_n^2}{t_k^2}, \tag{C.26}
\]

\[
\|R_{v_k, k, t_k} + 1 + \frac{v_k^T E W^2 v_k}{t_k^2} \| \leq \left\|\mathcal{P}(v_k, v_k, t_k) + 1 + \frac{v_k^T E W^2 v_k}{t_k^2}\right\| \tag{C.27}
\]

\[
= \left\|\sum_{l=2}^{L} \frac{1}{t_k} v_k^T E W^l v_k\right\| + O\left(\frac{\alpha_n^3}{t_k^4}\right) \lesssim \frac{\alpha_n^3}{|t_k|^3},
\]

\[
t_k^2 \tilde{R}(v_k, v_k, t_k) = (1 + \frac{2}{t_k^2} v_k^T E W^2 v_k + \sum_{l=3}^{L} \frac{1}{t_k} v_k^T E W^l v_k)^{-1} \tag{C.28}
\]

\[
= (1 + \frac{2}{t_k} v_k^T E W^2 v_k + O\left(\frac{\alpha_n^3}{|t_k|^3}\right))^{-1} = 1 - \frac{2}{t_k} v_k^T E W^2 v_k + O\left(\frac{\alpha_n^3}{|t_k|^3}\right).
\]

Similarly we have

\[
t_k^2 \tilde{R}(v_k, V_{-k}, t_k) = \left(\sum_{l=2}^{L} \frac{1}{t_k} v_k^T E W^l v_k\right)^{-1} \lesssim \frac{\alpha_n^2}{|t_k|^2}. \tag{C.29}
\]
By (83) and (A.16) of [8], we have
\[
\left\| \left( D_{-k}^{-1} + R(V_{-k}, V_{-k}, t_k) \right)^{-1} \right\| \lesssim 1, \tag{C.30}
\]
and
\[
\left\| D_{-k}^{-1} + R(V_{-k}, V_{-k}, t_k) \right\| \lesssim |t_k|. \tag{C.31}
\]
By (86) of [8], (C.22), (C.22), (C.28), (C.29), (C.30) and (C.31), we conclude that
\[
\frac{1}{t_k^2} \bar{P}_{k,t_k} = \left( \frac{A_{v_k,k,t_k}}{t_k} \right)' = R'(v_k, v_k, t_k) - 2R'(v_k, V_{-k}, t_k) \left( D_{-k}^{-1} + R(V_{-k}, V_{-k}, t_k) \right)^{-1} R(V_{-k}, v_k, t_k)
\]
\[
= \frac{1}{t_k^2} - \frac{2}{t_k} v_k^T E W^2 v_k + O(\alpha_n^3/|t_k|^3). \tag{C.32}
\]
Therefore
\[
\bar{P}_{k,t_k} = 1 + \frac{2}{t_k} v_k^T E W^2 v_k + O(\alpha_n^3/|t_k|^3). \tag{C.33}
\]

Recalling the definition of \( r_k \) in Section 3.5, we have
\[
\left\| A_{e_i,k,t_k} + v_k(i) + \frac{1}{t_k^2} e_i^T E W^2 v_k + e_i^T r_k \right\| \leq \left\| P(e_i, v_k, t_k) + v_k(i) + \frac{1}{t_k^2} e_i^T E W^2 v_k \right\|
\]
\[
+ \left\| P(e_i, V_{-k}, t_k) \left( t_k (D_{-k})^{-1} + P(V_{-k}, V_{-k}, t_k) \right)^{-1} P(V_{-k}, v_k, t_k) + e_i^T r_k \right\|
\]
\[
= \left\| \sum_{l=2}^L \frac{1}{t_k^2} e_i^T E W^l v_k \right\| + O(\frac{\alpha_n^3}{\sqrt{n}|t_k|^3}) \lesssim \frac{\alpha_n^2}{\sqrt{n}|t_k|^3}. \tag{C.34}
\]

By Corollary 2 we have
\[
|b_{e_i,k,t_k}^T (W - EW)v_k| + |v_k^T (W - EW)v_k| = O(\frac{\alpha_n}{\sqrt{n}}).
\]

It follows from (C.25) and (C.26) that
\[
E b_{e_i,k,t_k}^T W v_k \lesssim \frac{\theta_n}{\sqrt{n}}, \tag{C.35}
\]
By the expressions from (C.22)-(C.36), we have

\[
\begin{align*}
\mathbb{E}b^T_{\hat{v}_k,k,t_k} Wv_k - \mathbb{E}v^T_{k} Wv_k & \lesssim \frac{\alpha_n^2 t_k}{t_k^2}. \quad (C.36)
\end{align*}
\]

Choosing \(u = v_k\), by Lemma 6 of [9] we have

\[
\begin{align*}
\mathbf{e}^T_{k} \hat{v}_k & \mathbf{v}^T_{k} v_k = \left[ \bar{P}_{k,t_k} - 2t_k^{-1} \bar{P}^2_{k,t_k} \mathbf{v}^T_k Wv_k + O_\prec \left( \frac{\alpha_n}{\sqrt{nt_k}} \right) \right] \\
& \times \left[ A_{v_k,k,t_k} - t_k^{-1} b_{v_k,k,t_k}^T Wv_k + O_\prec \left( \frac{\alpha_n^2}{\sqrt{nt_k}} \right) \right] \\
& = \bar{P}_{k,t_k} A_{v_k,k,t_k} \mathbf{v}^T_{k} Wv_k - \bar{P}_{k,t_k} A_{v_k,k,t_k} \mathbf{e}^T_{k} (W^2 - \mathbb{E}W^2) \mathbf{v}_k + O_\prec \left( \frac{\alpha_n^3}{\sqrt{nt_k}} \right).
\end{align*}
\]

Choosing \(u = v_k\), by Lemma 6 of [9] we have

\[
\begin{align*}
(\mathbf{v}^T_{k} \hat{v}_k)^2 = \frac{\bar{P}^2_{k,t_k} \left[ \mathbf{G}(\hat{t}_k) - \mathbf{F}(\hat{t}_k) \right] \mathbf{v}_k \mathbf{v}^T_{k} \left[ \mathbf{G}(\hat{t}_k) - \mathbf{F}(\hat{t}_k) \right] \mathbf{v}_k}{\bar{P}^2_{k,t_k} \mathbf{v}^T_{k} \left[ \mathbf{G}(\hat{t}_k) - \mathbf{F}(\hat{t}_k) \right] \mathbf{v}_k} \\
= \left[ \bar{P}_{k,t_k} - 2t_k^{-1} \bar{P}^2_{k,t_k} \mathbf{v}^T_k Wv_k + O_\prec \left( \frac{\alpha_n}{\sqrt{nt_k}} \right) \right] \\
& \times \left[ A_{v_k,k,t_k} - t_k^{-1} b_{v_k,k,t_k}^T Wv_k + O_\prec \left( \frac{\alpha_n^2}{\sqrt{nt_k}} \right) \right]^2 \\
= \bar{P}_{k,t_k} A_{v_k,k,t_k}^2 + O_\prec \left( \frac{\alpha_n^2}{\sqrt{nt_k}} \right).
\end{align*}
\]

Since we fix the direction of \(\hat{v}_k\) such that \(\mathbf{v}^T_{k} \hat{v}_k \geq 0\), we can obtain the expansion of \(\mathbf{v}^T_{k} \hat{v}_k\) as follows

\[
\mathbf{v}^T_{k} \hat{v}_k = -\sqrt{\bar{P}_{k,t_k} A_{v_k,k,t_k}} + O_\prec \left( \frac{\alpha_n}{\sqrt{nt_k}} \right).
\]

Divide (C.37) by \(\mathbf{v}^T_{k} \hat{v}_k\). According to (C.27) and (C.33) we can expand the estimator \(\hat{v}_k(i)\) to higher order as follows

\[
\hat{v}_k(i) = v_k(i) + \frac{\mathbf{r}_k(i)}{t_k^2} + \frac{\mathbf{e}^T_{k} W^2 v_k}{t_k^2} - v_k(i) \frac{3\mathbf{v}^T_{k} W^2 v_k}{2t_k} + \frac{s^T_{k,t_k} Wv_k}{t_k} + O_\prec \left( \frac{\alpha_n}{\sqrt{nd_k}} \right), \quad (C.39)
\]
where we have used the inequality that $\alpha_n^2 \lesssim |d_k|$ by Conditions 3-4. This completes the proof of (C.20).

Now we focus on the proof of (C.21). From (C.7) we have

$$\hat{d}_k = t_k + v_k^T W v_k + O_\prec(\frac{1}{\sqrt{n}}).$$

Combining with the definition of $t_k$ and

$$1 + d_k (R(v_k, v_k, z_0) - R(v_k, V_{-k}, z_0)) (D_{-k}^{-1} + R(V_{-k}, V_{-k}, z))^{-1} R(V_{-k}, v_k, z_0) \lesssim \frac{\alpha_n^3}{d_k},$$

(C.40)

$$z_0 = d_k + \frac{v_k^T E W^2 v_k}{d_k},$$

we conclude that

$$t_k = z_0 + O(\frac{\alpha_n}{|d_k|}).$$

Hence we have

$$\hat{d}_k - d_k = \frac{v_k^T E W^2 v_k}{d_k} + v_k^T W v_k + O_\prec(\frac{1}{\sqrt{n}}).$$

This proves (C.21), and thus concludes the proof of the lemma. \qed

References

[1] Abbe, E. (2017). Community detection and stochastic block models: recent developments. *Journal of Machine Learning Research* 18, 177:1–177:86.

[2] Adamic, L. A. and N. Glance (2005). The political blogosphere and the 2004 u.s. election: Divided they blog. In *Proceedings of the 3rd International Workshop on Link Discovery*, New York, USA, pp. 36–43. ACM.

[3] Airoldi, E. M., D. M. Blei, S. E. Fienberg, and E. P. Xing (2008). Mixed membership stochastic blockmodels. *Journal of machine learning research* 9, 1981–2014.

[4] Banerjee, D. and Z. Ma (2017). Optimal hypothesis testing for stochastic block models with growing degrees. *arXiv preprint arXiv:1705.05305*.

[5] Bickel, P. J. and P. Sarkar (2016). Hypothesis testing for automated community detection in networks. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 78(1), 253–273.
[6] Chen, K. and J. Lei (2018). Network cross-validation for determining the number of communities in network data. *Journal of the American Statistical Association* 113(521), 241–251.

[7] Daudin, J.-J., F. Picard, and S. Robin (2008). A mixture model for random graphs. *Statistics and Computing* 18(2), 173–183.

[8] Fan, J., Y. Fan, X. Han, and J. Lv (2019a). Asymptotic theory of eigenvectors for large random matrices. *arXiv preprint arXiv:1902.06846*.

[9] Fan, J., Y. Fan, X. Han, and J. Lv (2019b). Simple: Statistical inference on membership profiles in large networks. *arXiv preprint arXiv:1910.01734*.

[10] Gao, C. and J. Lafferty (2017). Testing for global network structure using small subgraph statistics. *arXiv preprint arXiv:1710.00862*.

[11] Holland, P. W., K. B. Laskey, and S. Leinhardt (1983). Stochastic blockmodels: First steps. *Social Networks* 5, 109–137.

[12] Jin, J., Z. T. Ke, and S. Luo (2019). Optimal Adaptivity of Signed-Polygon Statistics for Network Testing. *arXiv e-prints*, arXiv:1904.09532.

[13] Karrer, B. and M. E. J. Newman (2011a). Stochastic blockmodels and community structure in networks. *Phys. Rev. E* 83, 016107.

[14] Karrer, B. and M. E. J. Newman (2011b). Stochastic blockmodels and community structure in networks. *Phys. Rev. E* 83, 016107.

[15] Latouche, P., E. Birmelé, and C. Ambroise (2012). Variational bayesian inference and complexity control for stochastic block models. *Statistical Modelling* 12(1), 93–115.

[16] Le, C. M. and E. Levina (2015). Estimating the number of communities in networks by spectral methods. *arXiv e-prints*, arXiv:1507.00827.

[17] Lei, J. (2016). A goodness-of-fit test for stochastic block models. *The Annals of Statistics* 44, 401–424.

[18] McDaid, A. F., T. B. Murphy, N. Friel, and N. J. Hurley (2013). Improved bayesian inference for the stochastic block model with application to large networks. *Computational Statistics & Data Analysis* 60, 12–31.

[19] Saldana, D., Y. Yu, and Y. Feng (2017). How many communities are there? *Journal of Computational and Graphical Statistics* 26, 171–181.
[20] Wang, Y. J. and G. Y. Wong (1987). Stochastic blockmodels for directed graphs. *Journal of the American Statistical Association* 82, 8–19.

[21] Wang, Y. X. R. and P. J. Bickel (2017). Likelihood-based model selection for stochastic block models. *Ann. Statist.* 45(2), 500–528.

[22] Zhao, Y., E. Levina, and J. Zhu (2011). Community extraction for social networks. *Proceedings of the National Academy of Sciences* 108(18), 7321–7326.

[23] Zhao, Y., E. Levina, and J. Zhu (2012). Consistency of community detection in networks under degree-corrected stochastic block models. *Ann. Statist.* 40(4), 2266–2292.