ON THE DERIVED FUNCTORS OF DESTABILIZATION AND OF ITERATED LOOP FUNCTORS

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Abstract. These notes explain how to construct small functorial chain complexes which calculate the derived functors of destabilization (respectively iterated loop functors) in the theory of modules over the mod 2 Steenrod algebra; this shows how to unify results of Singer and of Lannes and Zarati.

1. Introduction

These lecture notes examine the interface between unstable modules \( \mathcal{U} \) over the Steenrod algebra \( A \) and \( M \), the category of \( A \)-modules, in particular the structure of the left derived functors of destabilization \( D : \mathcal{M} \to \mathcal{U} \) (which is the left adjoint to the inclusion \( \mathcal{U} \subset \mathcal{M} \)) and of the family of iterated loop functors, \( \Omega_t : \mathcal{U} \to \mathcal{U} \) (the left adjoint to the suspension functor \( \Sigma^{-t} : \mathcal{U} \to \mathcal{U} \)), for \( t \in \mathbb{N} \). Throughout, the prime 2 is fixed and the underlying field \( F \) denotes the prime field \( \mathbb{F}_2 \). (There are analogous results for odd primes, which are not presented in detail here.)

The two basic ingredients which are used are the Singer functors \( R_s \), \( s \in \mathbb{N} \), which are defined for all \( A \)-modules, and the Singer residue map \( F[u^{\pm 1}] \to \Sigma^{-1}F \), which is \( A \)-linear and induces differentials. Part of the interest of the current approach is that it provides a clear explanation of the relationship between the methods of Lannes and Zarati [LZ87] and those of Singer ([Sin81] etc.).

The notes explain how to construct a natural chain complex \( D_M \), for \( M \) an \( A \)-module, with homology calculating the derived functors of destabilization, and, for \( t \in \mathbb{N} \) and \( N \) an unstable module, a chain complex \( C_t N \), with homology calculating the derived functors of \( \Omega_t^0 \). The existence of such a chain complex goes back to the work of Singer [Sin78, Sin80], but the construction given here is new.

The complex \( C_t N \) is given as a quotient of \( D(\Sigma^{-t}N) \) and the projection

\[
D(\Sigma^{-t}N) \twoheadrightarrow C_t N
\]

induces the natural transformation \( D_s(\Sigma^{-t}N) \to \Omega_t^s N \) between left derived functors in homology.

The chain complex \( D_M \) is also related to the chain complex \( \Gamma^+ M \) introduced by Singer [Sin83] and Hung and Sum [HS95] (who work at odd primes), to calculate the homology of \( M \) over the Steenrod algebra. Namely, there is a natural inclusion

\[
D_M \hookrightarrow \Gamma^+ M
\]

which, in homology, induces the Lannes-Zarati homomorphism (up to dualizing) [LZ87], the derived form of:

\[
DM \to F \otimes_{ad} M,
\]

thus giving rise to \( D_s M \to \text{Tor}_s^F(F,M) \). (A word of warning: \( DM \) is concentrated in degrees \( \geq 0 \), hence the map to \( F \otimes_{ad} M \) is not in general surjective.)
This morphism is of interest, since it is intimately related to the mod-2 Hurewicz morphism.

A number of exercises, of varying levels of difficulty, are included, reflecting the origin of this text as lecture notes. The reader is encouraged to attempt them all, since they are essential to the understanding of the subject.

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2. Background

Throughout, the prime $p$ is taken to be 2 and the ground field $\mathbb{F}$ is the field with two elements. All the results introduced have analogues for odd primes, although the arguments are slightly more complicated in the odd primary situation.

A general reference for the theory of (unstable) modules over the Steenrod algebra is the book by Schwartz [Sch94] and, for $A$-modules, that of Margolis [Mar83]. References for the individual results stated can be found for example in the author’s papers [Pow10a, Pow10b, Pow12]; many go back to Massey and Peterson and the work of Singer.

2.1. The Steenrod algebra as a quadratic algebra. The mod-2 Steenrod algebra $A$ is, by definition, the algebra of stable cohomology operations for mod-2 singular cohomology. Hence the Steenrod algebra can be identified with the cohomology $\text{H}^*(\text{H}F_2)$ (here $\text{H}^*(-)$ always denote cohomology with mod-2 coefficients) of the Eilenberg-MacLane spectrum, $\text{H}F_2$, which represents mod-2 cohomology.

The algebra $A$ is a (non-homogeneous) quadratic algebra, as explained below. Let $\tilde{A}$ be the (homogeneous) quadratic algebra

$$\tilde{A} := T(Sq^i | i \geq 0) / \sim$$

where $Sq^i$ has degree $i$ and $\sim$ corresponds to the Adem relations:

$$Sq^a Sq^b = \sum_j \binom{b - j - 1}{a - 2j} Sq^{a+b-j} Sq^j,$$

where $Sq^0$ is considered as an independent generator. Since the relations are homogeneous of length 2, the algebra $\tilde{A}$ is a homogeneous quadratic algebra and, in particular, has a length grading in addition to the internal grading coming from the degrees of the generators.

There is a surjection of algebras $\tilde{A} \twoheadrightarrow A$ which corresponds to imposing the relation $Sq^0 = 1$; the algebra $A$ inherits a length filtration from $\tilde{A}$ (no longer
a grading). The relations defining $\mathcal{A}$ have length $\leq 2$, which means that $\mathcal{A}$ is quadratic.

The graded $\mathcal{A}$ associated to the length filtration can also be described as a quotient of $\mathcal{A}$, namely $\mathcal{A} = \tilde{\mathcal{A}} / \langle Sq^0 \rangle$.

This is again a homogeneous quadratic algebra. Moreover, it has the important property that it is Koszul. This notion, introduced by Priddy [Pri70], is at the origin of the existence of small resolutions for calculating the homology of the Steenrod algebra; the Koszul dual is the (big) Lambda algebra.

The complexes introduced here are related to the quadratic Koszul nature of $\mathcal{A}$ and to the relation between the Steenrod algebra and invariant theory; many of the ideas developed go back to the work of Singer [Sin78, Sin80, Sin83] etc.

Remark 2.1. The odd primary analogues depend upon the work of M`ui [M`ui86, M`ui75], which describes the (more-complicated) relationship between invariant theory and the Steenrod algebra.

See for example the work of Hung and Sum [HS95] generalizing Singer’s invariant-theoretic description of the Lambda-algebra to odd primes and Zarati’s generalization [Zar84] of his work with Lannes [LZ87] - and the author’s paper [Pow10b].

2.2. The category of $\mathcal{A}$-modules. Let $\mathcal{M}$ denote the category of (left) $\mathcal{A}$-modules. This is an abelian category with additional structure; namely, the fact that $\mathcal{A}$ is a Hopf algebra implies that the tensor product (as graded vector spaces) of two $\mathcal{A}$-modules has a natural $\mathcal{A}$-module structure. Explicitly, the Steenrod squares act via:

\[
Sq^n(x \otimes y) = \sum_{i+j=n} Sq^i(x) \otimes Sq^j(y);
\]

this corresponds to the fact that the diagonal $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is determined by $\Delta Sq^n = \sum_{i+j=n} Sq^i \otimes Sq^j$. Since $\mathcal{A}$ is a connected algebra (concentrated in non-negative degrees, with $\mathcal{A}^0 = F$) the Hopf algebra conjugation $\chi : \mathcal{A}^\circ \to \mathcal{A}$ is determined by the diagonal [MM65] and is an isomorphism of algebras, where $\mathcal{A}^\circ$ is $\mathcal{A}$ equipped with the opposite algebra structure ($\chi$ is an anti-automorphism of $\mathcal{A}$).

Via $\chi$, the category of left $\mathcal{A}$-modules is equivalent to the category of right $\mathcal{A}$-modules: a right $\mathcal{A}$-module can be considered as a left $\mathcal{A}$-module by setting $am := m\chi(a)$.

Hence $\mathcal{M}$ has a duality functor:

\[
(-)\vee : \mathcal{M}^{\text{op}} \to \mathcal{M}
\]

\[
M \mapsto M\vee := \text{Hom}_F(M, F),
\]

where the usual right $\mathcal{A}$-module structure on $M\vee$ is regarded as a left structure via $\chi$.

Notation 2.2. For $n \in \mathbb{Z}$, let $\Sigma^n F$ denote the $\mathcal{A}$-module $F$ in degree $n$.

Remark 2.3. Since $\mathcal{A}$ is connected, $\{\Sigma^n F | n \in \mathbb{Z}\}$ gives a set of representatives of isomorphism classes of the simple objects of $\mathcal{M}$.

Example 2.4. Duality gives $(\Sigma^n F)^\vee = \Sigma^{-n} F$.

Definition 2.5. For $n \in \mathbb{Z}$, the $n$th suspension functor $\Sigma^n : \mathcal{M} \to \mathcal{M}$ is $\Sigma^n F \otimes -$.

Example 2.6. For $n \in \mathbb{Z}$, the free $\mathcal{A}$-module $\Sigma^n \mathcal{A}$ is a projective object of $\mathcal{M}$ (which is the projective cover of $\Sigma^n F$).

Proposition 2.7. The category $\mathcal{M}$ has enough projectives, with set of projective generators $\{\Sigma^n \mathcal{A} | n \in \mathbb{Z}\}$. 
Proposition 2.8. For \( n \in \mathbb{Z} \), \( \Sigma^n : \mathcal{M} \to \mathcal{M} \) is an exact functor which is an equivalence of categories, with inverse \( \Sigma^{-n} : \mathcal{M} \to \mathcal{M} \). In particular, \( \Sigma^n \) preserves projectives.

Proof. Exercise. □

2.3. Unstable modules and destabilization. Whereas the cohomology of a spectrum (object from stable homotopy theory which represents a cohomology theory) is simply an \( \mathcal{A} \)-module, the cohomology of a space has further structure; it is an algebra (via the cup product) and the underlying \( \mathcal{A} \)-module is unstable.

Definition 2.9. An \( \mathcal{A} \)-module \( M \) is unstable if \( Sq^i x = 0 \), \( \forall i > |x| \). The full subcategory of unstable modules is denoted \( \mathcal{U} \subset \mathcal{M} \).

For later use, the following definition is recalled, which uses the tensor product of \( \mathcal{U} \).

Definition 2.10. An algebra in \( \mathcal{M} \) is a graded algebra such that the structure morphisms are \( \mathcal{A} \)-linear. An unstable algebra \( K \) is an unstable module which is a commutative algebra in \( \mathcal{M} \) (and hence in \( \mathcal{U} \)) such that the Cartan condition holds: \( Sq^{|x|} x = x^2 \), \( \forall x \in K \). Unstable algebras form a category \( \mathcal{K} \), with morphisms the algebra morphisms which are \( \mathcal{A} \)-linear. Forgetting the algebra structure yields a functor \( \mathcal{K} \to \mathcal{U} \).

In a few places the terminology nilpotent, reduced, nil-closed will be used; for convenience the definition is recalled (see [Sch94] for further details).

Definition 2.11. An unstable module \( N \) is nilpotent if the operation \( Sq_0 \) (where \( Sq_0(x) = Sq^{|x|}(x) \)) acts locally nilpotently; a key example of a nilpotent unstable module is \( \Sigma M \), for any unstable module \( M \).

An unstable module \( M \) is reduced if \( \text{Hom}_\mathcal{U}(N, M) = 0 \) for any nilpotent module \( N \) and nil-closed if, in addition, \( \text{Ext}^1_\mathcal{U}(N, M) = 0 \) for all nilpotents \( N \).

Proposition 2.12. The category \( \mathcal{U} \) is an abelian subcategory of \( \mathcal{M} \) and is closed under the tensor product \( \otimes \) of \( \mathcal{M} \).

Proof. Exercise. □

Remark 2.13. The duality functor \((\mathcal{M})^\vee : \mathcal{M}^\vee \to \mathcal{M}\) does not preserve \( \mathcal{U} \), since the relation \( Sq^0 = 1 \) implies that an unstable module is concentrated in degrees \( \geq 0 \). The dual \( \mathcal{M}^\vee \) of a module \( M \) concentrated in degrees \( \geq 0 \) is concentrated in degrees \( \geq 0 \) if and only if \( M = M^0 \); for example, the dual of \( \Sigma \mathbb{F} \) is not unstable.

Example 2.14. For \( n \in \mathbb{N} \), the suspension functor \( \Sigma^n : \mathcal{M} \to \mathcal{M} \) restricts to an exact functor \( \Sigma^n : \mathcal{U} \to \mathcal{U} \) (given by \( \Sigma^n \mathbb{F} \otimes - \)). However, this is not an equivalence of categories if \( n > 0 \).

The notion of destabilization arises naturally through topological considerations, for example when passing from stable homotopy theory (spectra) to unstable homotopy theory (spaces).

Definition 2.15. Let \( D : \mathcal{M} \to \mathcal{U} \) be the left adjoint to the (exact) inclusion functor \( \mathcal{U} \hookrightarrow \mathcal{M} \).

Exercise 2.16. For \( M \) an \( \mathcal{A} \)-module, show that the linear subspace \( BM := \langle Sq^i(x) | i > |x| \rangle \), as \( x \) ranges over elements of \( M \), is a sub \( \mathcal{A} \)-module. Deduce that \( DM \cong M/BM \) as an \( \mathcal{A} \)-module (which is unstable, by construction).
Namely, from the explicit construction, if \( f : M \to N \) is a morphism of \( \mathcal{A} \)-modules with \( N \) unstable, there is a natural factorization:

\[
\begin{array}{c}
M \xrightarrow{f} N \\
\downarrow \quad \downarrow \\
M/\text{BM}.
\end{array}
\]

\textbf{Notation 2.17.} For \( n \in \mathbb{Z} \), let \( F(n) \) denote \( D(\Sigma^n \mathcal{A}) \). (The unstable module \( F(n) \) is the free unstable module on a generator of degree \( n \).)

\textbf{Exercise 2.18.} Show that

1. \( F(n) = 0 \) for \( n < 0 \) and \( F(0) = F \);
2. \( F(n) \) is a projective object of \( \mathcal{U} \), \( \forall n \in \mathbb{Z} \).

\textbf{Proposition 2.19.} The category \( \mathcal{U} \) has enough projectives and the set \( \{ F(n) \, | \, n \in \mathbb{N} \} \) forms a set of projective generators.

\textit{Proof.} Exercise. \( \square \)

\textbf{Proposition 2.20.} The functor \( D : \mathcal{M} \to \mathcal{U} \) is right exact (but not exact) and preserves projectives.

\textit{Proof.} Exercise. \( \square \)

The functor \( D \) can be used to define division functors. The most important examples considered here are the (iterated) loop functors.

\textbf{Definition 2.21.} For \( n \in \mathbb{N} \), let \( \Omega^n : \mathcal{U} \to \mathcal{U} \) denote the composite functor \( D \circ \Sigma^{-n} \), where \( \Sigma^{-n} : \mathcal{M} \to \mathcal{M} \) is restricted to a functor \( \mathcal{U} \to \mathcal{U} \).

\textbf{Proposition 2.22.} For \( n \in \mathbb{N} \), the functor \( \Omega^n : \mathcal{U} \to \mathcal{U} \) is left adjoint to \( \Sigma^n : \mathcal{U} \to \mathcal{U} \); it is right exact (but not exact for \( n > 0 \)) and preserves projectives.

\textit{Proof.} Exercise. \( \square \)

\textbf{Proposition 2.23.} For \( n \in \mathbb{N} \) there is a natural equivalence of functors

\[
\Omega^n D \cong D \Sigma^{-n} : \mathcal{M} \to \mathcal{U}.
\]

\textit{Proof.} Exercise. \( \square \)

\textbf{Example 2.24.} Another important division functor which can be constructed by using destabilization is Lannes’ \( T \)-functor:

\[
TM := D \left( M \otimes H^* (B\mathbb{Z}/2)^{\vee} \right),
\]

where \( B\mathbb{Z}/2 \) is the classifying space of the group \( \mathbb{Z}/2 \), which has the homotopy type of \( \mathbb{R}P^\infty \). Exercise: verify that \( T \) is left adjoint to the functor \( H^* (B\mathbb{Z}/2) \otimes - : \mathcal{U} \to \mathcal{U} \).

\textbf{Exercise 2.25.} Let \( M \) be an unstable module which is of finite type (ie \( \dim(M^n) \) is finite \( \forall n \)). Show that the functor \( D(\cdot \otimes M^n) \) is left adjoint to \( M \otimes - \). (This left adjoint is usually referred to as the division functor by \( M \) and written \( (\cdot : M) \); see [Lan92] for general considerations on such functors.)
2.4. Derived functors. The abelian categories $\mathcal{A}$ and $\mathcal{U}$ both have enough projectives, hence one can do homological algebra in them. Recall that a projective resolution $P_\bullet$ of an object $M$ of an abelian category is a complex of projectives

$$\cdots \to P_s \to P_{s+1} \to \cdots \to P_1 \to P_0,$$

with $P_s$ in homological degree $s$, and which has homology concentrated in degree zero with $H_0(P_s) \cong M$. This will frequently be denoted by $P_\bullet : M \to \cdots$. Note that the arrow corresponds to the surjection $P_0 \to M$.

**Remark 2.26.** If $M$ is an unstable module, there are two possible notions of projective resolution: a projective resolution in $\mathcal{U}$, $P_\bullet : M \to \cdots$ (that is, by projectives in $\mathcal{U}$), or a resolution in $\mathcal{A}$, $F_\bullet : M \to \cdots$, by free $\mathcal{A}$-modules.

**Definition 2.27.** For $s, n \in \mathbb{N}$, let

1. $D_s : \mathcal{A} \to \mathcal{U}$ denote the $s$th left derived functor of $D : \mathcal{A} \to \mathcal{U}$;
2. $\Omega^n : \mathcal{U} \to \mathcal{U}$ denote the $n$th left derived functor of $\Omega : \mathcal{U} \to \mathcal{U}$.

Explicitly, if $F_\bullet : M \to \cdots$ is a free resolution in $\mathcal{A}$ of an $\mathcal{A}$-module $M$, then $D_sM$ is the $s$th homology of the complex $DF_\bullet$. Note that $DF_\bullet$ is a complex with each object $DF_s$ projective in $\mathcal{U}$; it is a projective resolution of $DM$ if and only if all the higher derived functors $D_sM$ vanish.

**Exercise 2.28.** Let $M$ be an $\mathcal{A}$-module and suppose that there exists $t \in \mathbb{N}$ such that $\Sigma^tM$ is unstable (such a $t$ does not exist in general). Show that, for all $s \in \mathbb{N}$, there exists a natural morphism $D_sM \to \Omega^s\Sigma^tM$. This exhibits the close relationship between derived functors of destabilization and of iterated loop functors.

**Example 2.29.** Derived functors of destabilization are highly non-trivial. For example we consider a lower bound for $D_1(\Sigma^{-1}F)$ as follows.

Recall that $H^*(B\mathbb{Z}/2) \cong \mathbb{F}[u]$, where $|u| = 1$; this is an unstable algebra, and this fact determines its structure as an $\mathcal{A}$-module. (Explicitly, the total Steenrod power $S^q = \sum_{i \in \mathbb{N}} Sq^i$ on $u$ is $S^q(u) = u(1+u)$, via the Cartan condition and instability and this determines the structure via the Cartan formula for cup products, which implies that $S^q$ is multiplicative.)

One can form the localization $\mathbb{F}[u^{-1}]$, so that $u^{-1}$ is a class of degree $-1$. This has an $\mathcal{A}$-algebra structure (not unstable!), which is determined by the total Steenrod power of $u^{-1}$. This can be calculated by using the multiplicity of $S^q$:

$$1 = S^q(1) = S^q(u^{-1}u) = S^q(u^{-1})S^q(u),$$

giving $S^q(u^{-1}) = \frac{u^{-1}}{1 + u}$, which translates as

$$S^{q+n}(u^{-1}) = u^n$$

for all $n \in \mathbb{N}$.

Let $\hat{P}$ denote the sub $\mathbb{F}[u]$-module of $\mathbb{F}[u^{\pm 1}]$ generated by $u^{-1}$, so that there is a (non-split) short exact sequence of $\mathcal{A}$-modules:

$$0 \to \mathbb{F}[u] \to \hat{P} \to \Sigma^{-1}\mathbb{F} \to 0.$$

It is straightforward to see that $D\hat{P} = 0$ and $D\Sigma^{-1}\mathbb{F} = 0$. Hence the long exact sequence for derived functors

$$\cdots \to D_1\hat{P} \to D_1(\Sigma^{-1}\mathbb{F}) \to D\mathbb{F}[u] \to D\hat{P} \to D\Sigma^{-1}\mathbb{F} \to 0$$

shows that $D_1(\Sigma^{-1}\mathbb{F}) \to D\mathbb{F}[u] \cong \mathbb{F}[u]$ is surjective. (As we shall see, it is an isomorphism, by Corollary [1.18] as in Lannes and Zarati [LZ97].) Thus $D_1(\Sigma^{-1}\mathbb{F})$ is infinite, even though $\Sigma^{-1}\mathbb{F}$ has total dimension one.
Remark 2.30. The $A$-module $\hat{P}$ is bounded below; however $\Sigma^t \hat{P}$ is never an unstable module.

2.5. Motivation for studying derived functors of destabilization and of iterated loop functors. The functors $D_s : \mathcal{M} \rightarrow \mathcal{U}$ arise naturally in algebraic topology.

Example 2.31. There is a Grothendieck spectral sequence calculating $\text{Ext}^*_A(M, N)$ in terms of $\text{Ext}^*_U$ when $N$ is an unstable module. This is the spectral sequence derived from considering $\text{Hom}_A(\neg, N)$ as the composite functor $\text{Hom}_U(D(\neg), N)$.

The spectral sequence has the form

$$\text{Ext}^p_U(D^q M, N) \Rightarrow \text{Ext}^{p+q}_A(M, N).$$

When $N$ is injective in $\mathcal{U}$ (for example $N = \mathbb{F}$ or $N = H^*(BV)$, for $V$ an elementary abelian 2-group), the spectral sequence degenerates to the isomorphism

$$\text{Ext}^q_A(M, N) \cong \text{Hom}_U(D_q M, N).$$

Such Ext-groups are important for calculating the $E^2$-term of the Adams spectral sequence - for example, this motivated Lannes and Zarati’s work on the derived functors of destabilization [LZ87] and is intimately related to an approach to the Segal conjecture for elementary abelian $p$-groups.

Example 2.32. Derived functors of destabilization occur in studying the relationship between the cohomology of a spectrum $E$ and the cohomology of the infinite loop space $\Omega^\infty E$ associated to $E$.

Recall that there is an adjunction counit $\Sigma^\infty \Omega^\infty E \rightarrow E$, where $\Sigma^\infty$ is the suspension spectrum functor. This gives rise to a commutative diagram in $\mathcal{M}$:

$$H^*(E) \xrightarrow{DH^*(E)} H^*(\Sigma^\infty \Omega^\infty E) \cong H^*(\Omega^\infty E)$$

where the factorization exists since $H^*(\Omega^\infty E)$ is an unstable algebra and, in particular, an unstable module.

Recall that $\mathcal{K}$ denotes the category of unstable algebras and that the Steenrod-Epstein enveloping algebra functor $U : \mathcal{U} \rightarrow \mathcal{K}$ is left adjoint to the forgetful functor $\mathcal{K} \rightarrow \mathcal{U}$. This is given explicitly by

$$UM := S^*(M)/\langle Sq^m m = m^2 \rangle,$$

the quotient of the free commutative algebra on $M$ given by imposing the Cartan condition.

Hence, the above induces a morphism of unstable algebras:

$$U(DH^*(E)) \rightarrow H^*(\Omega^\infty E).$$

When $E = \Sigma^n H\mathbb{F}_2$ is a suspension of the mod-2 Eilenberg-MacLane spectrum, this is an isomorphism (exercise). However, in general it is far from being an isomorphism (examples can be given by considering suspension spectra $E = \Sigma^\infty X$).

Kuhn and McCarty [KM13] have shown that there exists an algebraic approximation to $H^*(\Omega^\infty E)$ which is expressed in terms of the derived functors of destabilization. This generalizes earlier work of Lannes and Zarati [LZ84] for suspension spectra.
Example 2.33. Similar considerations arise in giving an algebraic approximation to $H^*(\Omega^n X)$ in terms of $H^*(X)$, when $X$ is a pointed space. As a first approximation, one shows (exercise) that there is a natural morphism of unstable algebras

$$U(\Omega^n QH^*(X)) \to H^*(\Omega^n X),$$

where $Q : \mathcal{X}_a \to \mathcal{U}$ is the ‘indecomposables’ functor, defined on the category $\mathcal{X}_a$ of augmented unstable algebras by $QK := \overline{K}/K^2$, where $\overline{K}$ is the augmentation ideal. (Here the base point of $X$ provides the augmentation of $H^*(X)$.) This can be shown to be an isomorphism for Eilenberg-MacLane spaces but, in general, is far from being an isomorphism.

Under suitable hypotheses on the space $X$, in particular supposing that the cohomology of $X$ is of the form $UM$ for some unstable module $M$, Harper and Miller [HM89] gave an algebraic approximation to $H^*(\Omega^n X)$, which is expressed in terms of the derived functors of certain iterated loop functors. It is expected that their results can be generalized.

Remark 2.34. Note that the topological based loop space functor $\Omega$ is right adjoint to the reduced suspension functor $\Sigma$ and the infinite loop space functor $\Omega^\infty$ is right adjoint to the suspension spectrum functor $\Sigma^\infty$ (at the level of homotopy categories). The suspension functor $\Sigma$ commutes with cohomology; however, since $H^*(-)$ is contravariant, the algebraic approximations to these functors are left adjoints.

3. First results on derived functors of $D$ and $\Omega^n$

In this section, elementary results on the derived functors of $D$ and $\Omega^n$ are considered, as a warm-up to constructing the chain complexes which compute the respective derived functors.

3.1. Derived functors of $\Omega$. There is a simple chain complex which calculates the derived functors of $\Omega$; to define it requires the introduction of the doubling (or Frobenius) functor $\Phi : \mathcal{M} \to \mathcal{M}$.

Definition 3.1. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be the functor which associates to $M$ the module $\Phi M$ concentrated in even degrees with $(\Phi M)^{2k} = M^k$ and action of the Steenrod algebra determined by $Sq^{2i}(\Phi x) = \Phi(Sq^{i}x)$.

Let $\lambda_M : \Phi M \to M$ be the natural morphism (of graded vector spaces) defined by $\lambda_M(\Phi x) = Sq^{i=1}(x)$.

Remark 3.2. The functor $\Phi$ and the linear transformation $\lambda$ are defined for all $\mathcal{A}$-modules.

Proposition 3.3. If $M$ is an unstable module, $\lambda_M : \Phi M \to M$ is $\mathcal{A}$-linear; hence $\lambda$ induces a natural transformation $\lambda : \Phi|_{\mathcal{A}} \to 1_{\mathcal{A}}$.

Proof. Exercise.

Remark 3.4. For $M$ an unstable module, $\lambda_M$ is injective if and only if $M$ is a reduced unstable module. (Exercise: verify this assertion.)

Proposition 3.5. The functor $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies the following properties:

1. $\Phi$ is exact;
2. $\Phi$ commutes with tensor products, in particular $\Phi(\Sigma M) \cong \Sigma^2 \Phi M$.

Proof. Exercise.

Remark 3.6. The odd primary version of $\Phi$ does not commute with tensor products; behaviour of $\Phi \Sigma$ is complicated, whereas $\Phi \Sigma^2 \cong \Sigma^{2p} \Phi$. 

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**Exercise 3.7.** For $K$ an unstable algebra, show that $\lambda_K : \Phi K \to K$ is a morphism of unstable algebras. If $K$ is reduced (equivalently has no nilpotent elements as a commutative algebra), show that $\Phi K$ identifies with the subalgebra of $K$ generated by the squares of elements of $K$. For example $\Phi \mathbb{F}[u] \cong \mathbb{F}[u^2] \subset \mathbb{F}[u]$.

**Proposition 3.8.** For an unstable module $M$, the higher derived functors $\Omega_s$, $s > 1$ of $\Omega$ are trivial, (i.e. $\Omega_s = 0$ for $s > 1$) and there is a natural exact sequence in $\mathcal{U}$

$$0 \to \Sigma \Omega_1 M \to \Phi M \xrightarrow{\lambda_M} M \to \Sigma \Omega M \to 0.$$ 

In particular, the complex in $\mathcal{H}$:

$$\Sigma^{-1} \Phi M \xrightarrow{\Sigma^{-1} \lambda_M} \Sigma^{-1} M$$

has homology $\Omega M$ and $\Omega_1 M$ in homological degrees 0, 1 respectively.

**Proof.** By definition, $\Omega M$ is the destabilization of $\Sigma^{-1} M$. Hence (using the fact that $M$ is unstable),

$$\Sigma \Omega M \cong M/\langle Sq|x\rangle,$$

which is precisely the cokernel of $\lambda_M$.

It is a standard fact that the free unstable modules $F(n)$ are reduced (exercise: verify this), hence $\lambda_P$ is a monomorphism if $P$ is a projective unstable module.

Consider a projective resolution $P_\bullet \to M$ in $\mathcal{U}$. By the above property, the natural transformation $\lambda$ gives rise to a short exact sequence of complexes:

$$0 \to \Phi P_\bullet \to P_\bullet \to \Sigma \Omega P_\bullet \to 0.$$ 

The functors $\Phi$ and $\Sigma$ are exact, hence the homology of $P_\bullet$ and $\Phi P_\bullet$ is concentrated in homological degree zero, where it is respectively $M$ and $\Phi M$, whereas the homology of $\Sigma \Omega P_\bullet$ is isomorphic to $\Sigma \Omega M$ in homological degree $s$, by construction of the derived functors.

The long exact sequence in homology in low degrees gives the exact sequence

$$0 = H_1(\Phi P_\bullet) \to H_1(\Sigma \Omega P_\bullet) \to H_0(\Phi P_\bullet) = \Phi M \xrightarrow{\lambda_M} M = H_0(P_\bullet) \to H_0(\Sigma \Omega P_\bullet),$$

which shows that the kernel of $\lambda_M$ is isomorphic to $\Sigma \Omega_1 M$, as required.

Considering higher homological degree, it follows immediately that $\Omega_s M = 0$ for $s > 1$. The final statement is clear. \hfill $\square$

**Corollary 3.9.** For $p = 2$, an unstable module $M$ is reduced if and only if $\Omega_1 M = 0$.

**Proof.** Exercise. \hfill $\square$

**Exercise 3.10.**

1. Zarati [Zar90] showed that an unstable module $M$ (for $p = 2$) is nil-closed if and only if $M$ and $\Omega M$ are both reduced. Prove this. (Hint: first show that $\Omega_1 N$ is a nilpotent unstable module if $N$ is nilpotent.)
2. Assuming Zarati’s result, show that $M$ is nilclosed if and only if $\Omega_s^t M = 0$ for $s > 0$ and $t \leq 2$.
3. Give an example of a nilpotent unstable module $N$ such that $\Omega N$ is reduced(!).
3.2. Applications of $\Omega$ and $\Omega_1$.

**Proposition 3.11.** For $C_\bullet$ a chain complex of reduced unstable modules, $\Omega C_\bullet$ has homology which fits into a natural short exact sequence:

$$0 \to \Omega H_s(C_\bullet) \to H_s(\Omega C_\bullet) \to \Omega_1 H_{s-1}(C_\bullet) \to 0.$$  

*Proof.* Since each $C_\bullet$ is reduced, the natural transformation $\lambda$ induces a short exact sequence of complexes

$$0 \to \Phi C_\bullet \to C_\bullet \to \Sigma C_\bullet \to 0.$$  

Using the exactness of $\Phi$ and $\Sigma$ together with the naturality of $\lambda$, the associated long exact sequence in homology is

$$\ldots \to \Phi H_s(C_\bullet) \xrightarrow{\lambda} H_s(C_\bullet) \to \Sigma H_s(\Omega C_\bullet) \to \Phi H_{s-1}(C_\bullet) \to \ldots .$$

By Proposition 3.11, the cokernel of $\lambda H_s(C_\bullet)$ is $\Sigma H_s(\Omega C_\bullet)$ and the kernel is $\Omega_1 H_{s-1}(C_\bullet)$. Applying these identifications for $s$ and $s-1$ respectively gives the stated short exact sequence.  

**Corollary 3.12.** For $s,t \in \mathbb{N}$ and $M$ an unstable module, there is a natural short exact sequence

$$0 \to \Omega^t_s M \to \Omega^{s+1}_t M \to \Omega_1 \Omega^{t-1}_s M \to 0.$$  

In particular

$$\Omega^t_s = \begin{cases} 0 & s > t \\ (\Omega_1)^t & s = t. \end{cases}$$

*Proof.* Let $P_\bullet \to M$ be a projective resolution of $M$ in $\mathcal{A}$ and take $C_\bullet := \Omega^t P_\bullet$, which is a complex of projective unstable modules (which are reduced). The homology of $C_\bullet$ is, by definition, $H_s(C_\bullet) = \Omega^t_s M$, whereas $H_s(\Omega C_\bullet) = \Omega^{t+1}_s M$. The required short exact sequence is furnished by Proposition 3.11.

The final statement is proved by a straightforward induction upon $t$.  

**Remark 3.13.** More generally, for natural numbers $t_1,t_2$ and an unstable module $M$, there is a Grothendieck spectral sequence

$$\Omega^t_1 \Omega^t_2 \Rightarrow \Omega^{t_1+t_2}.$$  

The short exact sequence corresponds to the case $t_1 = 1$.

**Definition 3.14.** For $M$ an $\mathcal{A}$-module, the connectivity of $M$, $\text{conn}(M) \in \mathbb{Z} \cup \{-\infty, \infty\}$, is

$$\text{conn}(M) := \sup\{i | M_j = 0, \forall j \leq i\}.$$  

**Lemma 3.15.** For $M$ an $\mathcal{A}$-module, $\text{conn}(\Phi M) = 2\text{conn}(M) + 1$.

*Proof.* Exercise.

**Proposition 3.16.** For $s,k \in \mathbb{N}$ and $M$ an unstable module:

$$\text{conn}(\Omega^{s+k}_1 M) \geq 2^s (\text{conn}(M) - k).$$

*Proof.* It is clear that $\text{conn}(\Omega M) \geq \text{conn}(M) - 1$ and, by Lemma 3.15, $\text{conn}(\Omega_1 M) \geq 2\text{conn}(M)$. The general result is proved by induction upon $s$, using the Grothendieck short exact sequence of Corollary 3.12 for the inductive step. (Exercise: fill in the details.)

**Remark 3.17.** Since an unstable module $M$ is always concentrated in non-negative degrees, $\text{conn}(M) \geq -1$, hence it is clear that the previous statement is not optimal.
3.3. **Interactions between loops and destabilization.** Recall from Proposition 2.23 that, for $t \in \mathbb{N}$, there is a natural isomorphism between $\Omega^t D, D \Sigma^{-t} : \mathcal{M} \to \mathcal{M}$. The following result is another application of Proposition 3.11:

**Corollary 3.18.** For $M$ an $\mathcal{A}$-module, there is a natural short exact sequence:

$$0 \to \Omega(D_s M) \to D_s(\Sigma^{-1} M) \to \Omega_1(D_{s-1} M) \to 0.$$  

**Proof.** Let $F_\bullet \to M$ be a free resolution of $M$ (in $\mathcal{M}$) and take $C_\bullet = D F_\bullet$, which is a complex of projective unstable modules.

Proposition 2.23 implies that $\Omega C_\bullet$ is naturally isomorphic to $D(\Sigma^{-1} F_\bullet); \Sigma^{-1} F_\bullet$ is a projective resolution of $\Sigma^{-1} M$, hence the homology of $\Omega C_\bullet$ calculates the derived functors $D_s(\Sigma^{-1} M)$, whereas the homology of $C_\bullet$ calculates the derived functors $D_s M$. The result follows immediately from Proposition 3.11.\hfill \Box

**Remark 3.19.** The module $\Omega_1(D_{s-1} M)$ is the obstruction to $\Omega(D_s M) \to D_s(\Sigma^{-1} M)$ being an isomorphism. This is zero if and only if $D_{s-1} M$ is reduced, by Corollary 3.9.

**Remark 3.20.** More generally, for $m \in \mathbb{N}$, there is a Grothendieck spectral sequence $\Omega^m p D q \Rightarrow D_{p+q} \Sigma^{-m}$. The short exact sequence corresponds to the case $m = 1$.

3.4. **Connectivity for $D_s$.** The explicit identification of the functor $D$ via $D M = M/BM$ leads to the following result:

**Lemma 3.21.** For $M$ an $\mathcal{A}$-module, the natural surjection $M \twoheadrightarrow D M$ is an isomorphism in degrees $\leq 2(\text{conn} M + 1)$.

**Proof.** The lowest degree element (if it exists - i.e. if $\text{conn} M$ is finite) of $M$ has degree $\text{conn} M + 1$, hence the lowest degree element of $BM$ has degree at least $2(\text{conn} M + 1) + 1$. The result follows.\hfill \Box

The following statement is a general result for connected algebras, stated here for the Steenrod algebra.

**Lemma 3.22.** An $\mathcal{A}$-module $M$ has a free resolution $F_\bullet \to M$ in $\mathcal{M}$ with $\text{conn}(F_s) \geq \text{conn}(M) + s$.

**Proof.** Exercise.\hfill \Box

The following weak result is sufficient for the initial applications; a much stronger result holds (combine Lemma 5.6 with Theorem 5.8).

**Proposition 3.23.** For $0 < s \in \mathbb{N}$ and $M$ an $\mathcal{A}$-module

$$\text{conn}(D_s M) \geq 2(\text{conn} M + s).$$

**Proof.** It is sufficient to treat the case $\text{conn} M$ finite (the other cases are clear), hence we may take a free resolution $F_\bullet \to M$ as in Lemma 3.22. Consider the natural surjection of complexes $F_\bullet \to D F_\bullet$. For $s > 0$, the portion pertinent to $H_s$ is

$$
\begin{array}{cccc}
F_{s+1} & \to & F_s & \to & F_{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
D F_{s+1} & \to & D F_s & \to & D F_{s-1},
\end{array}
$$

where the top row is exact and the homology of the bottom row (in degree $s$) is $D_s M$, by definition. The vertical morphisms are all isomorphisms in degrees $\leq 2(\text{conn} M + s)$, by the hypothesis on $F_\bullet$ together with Lemma 5.24 the result follows.\hfill \Box
Proposition 3.23 implies that

\[ \text{conn} \]

Proof. Consider the long exact sequence for the derived functors \( D_s \) associated to the short exact sequence (1):

\[ \ldots \rightarrow D_s(M^{>c}) \rightarrow D_s(M) \rightarrow D_s(M^{>c}) \rightarrow D_{s-1}(M^{>c}) \rightarrow \ldots . \]

Proposition 3.25 implies that \( \text{conn}(D_s(M^{>c})) \geq 2(c+s) \) and \( \text{conn}(D_{s-1}(M^{>c})) \geq 2(c+s-1) \). The first statement follows immediately, implying the second. \( \square \)

Remark 3.26. This result shows that, to study the derived functors \( D_sM \), it would suffice to consider modules which are \( \text{bounded above} \) (i.e., such that \( M^{>c} = 0 \) for \( c \gg 0 \)).

3.5. Comparing \( D_s \) and \( \Omega'_t \). This section establishes a precise relationship between the derived functors of destabilization and of iterated loop functors. (This material is slightly technical and is not required for the subsequent results, hence can be skipped on first reading.)

Throughout, \( M \) is an iterated desuspension of an unstable module, so that there exists \( T \in \mathbb{N} \) such that \( \Sigma^TM \) is unstable \( \forall t \geq T \). If \( M \neq 0 \), \( \text{conn}(M) \) is finite; by Lemma 3.22, there exists a free resolution of \( M \) in \( \mathcal{M} \), \( F_* \rightarrow M \), with \( \text{conn}(F_s) \geq \text{conn}(M) + s \). Consider the free resolution \( \Sigma^tF_* \) of \( \Sigma^tM \), for \( t \geq T \). Then, by construction, \( D(\Sigma^tF_*) \) is a complex of projective unstable modules which has homology \( H_s(D(\Sigma^tF_*)) \cong D_s(\Sigma^tM) \) and, in particular, \( H_0(D(\Sigma^tF_*)) = \Sigma^tM \); Proposition 3.25 implies that, for \( s > 0 \), \( \text{conn}(H_s(D(\Sigma^tF_*))) = \text{conn}(D_s(\Sigma^tM)) \geq 2(\text{conn}M + s + t) \geq 2(\text{conn}M + t + 1) \).

Remark 3.27. The hypothesis upon \( T \) implies that \( \text{conn}M + T + 1 \geq 0 \).

The complex \( D(\Sigma^tF_*) \) can be seen as an approximation to a projective resolution (in unstable modules) of the unstable module \( \Sigma^tM \). More precisely, one has the following

Lemma 3.28. There is a short exact sequence of complexes of projectives in \( \mathcal{U} \):

\[ 0 \rightarrow D(\Sigma^tF_*) \rightarrow P_* \rightarrow Q_* \rightarrow 0 \]

such that

(1) \( P_* \) is a projective resolution of \( \Sigma^tM \) in \( \mathcal{U} \);
(2) \( D(\Sigma^tF_*) \rightarrow P_* \) induces an isomorphism on \( H_0 \);
(3) \( Q_0 = 0 \) and, \( \forall s \), \( \text{conn}(Q_s) \geq 2(\text{conn}M + t + 1) \).
Proof. Exercise (this is a general argument in homological algebra). □

**Proposition 3.29.** For $M$ an $A$-module and $t \in \mathbb{N}$ such that $\Sigma^t M$ is unstable, then for all $s \in \mathbb{N}$, the natural morphism

$$D_s M \to \Omega^s_1 \Sigma^t M$$

is an isomorphism in degrees $\leq 2(\text{conn} M + 1) + t$.

**Proof.** Consider $F_\bullet \to M$ as above and the short exact sequence (2) of Lemma 3.28. Applying the functor $\Omega^t$ and using the natural isomorphism $\Omega^t D(\Sigma^t F_\bullet) \cong DF_\bullet$ given by Proposition 2.23, this yields a short exact sequence of complexes

$$0 \to DF_\bullet \to \Omega^t P_\bullet \to \Omega^t Q_\bullet \to 0,$$

(exercise: why is this a short exact sequence, even though $\Omega$ is not exact?), where the first morphism induces $D_s M \to \Omega^t s \Sigma^t M$ is homology.

The connectivity condition on $Q_\bullet$ implies that $\text{conn}(\Omega^t Q_\bullet) \geq 2(\text{conn} M + 1) + t$. The result follows from the long exact sequence in homology. □

**Corollary 3.30.** For $M$ an $A$-module and $T \in \mathbb{N}$ such that $\Sigma^T M$ is unstable, there is a natural isomorphism

$$D_s M \cong \lim_{T \leq t \to \infty} \Omega^t_1 \Sigma^t M$$

and the inverse system stabilizes locally for $t \gg 0$ (i.e. in any given degree).

**Proof.** Exercise. □

**Exercise 3.31.** Let $M \in \mathcal{M}$ be a module which is bounded below ($M^n = 0$ for $n \ll 0$). Show that, for fixed $s, d \in \mathbb{N}$, there exist $c, T \in \mathbb{N}$ such that $\Sigma^T(M/M > c)$ is unstable and, for all $t \geq T$,

$$(D_s M)^d \cong \left( \Omega^t_1 \Sigma^t(M/M > c) \right)^d.$$ 

4. Singer functors

Singer introduced a series of functors which are indispensable in understanding the derived functors of iterated loops and destabilization. This section recalls the definition of variants of these.

4.1. The unstable Singer functors $R_s$. Following Lannes and Zarati [LZ87], we recall the construction of the unstable Singer functors $R_s$, for $s \in \mathbb{N}$; by convention $R_0$ is the identity functor $R_0 : \mathcal{U} \to \mathcal{U}$.

**Notation 4.1.** For $s \in \mathbb{N}$, let $D(s)$ denote the Dickson algebra of Krull dimension $s$, which is defined as the algebra of invariants

$$D(s) := H^*(BV_s)^{GL_s},$$

where the action of the general linear group on the cohomology of the classifying space $BV_s$ is induced by the natural action on $V_s = \mathbb{F}^\oplus s$.

**Exercise 4.2.** Show that $D(s)$ is a sub unstable algebra of $H^*(BV_s)$.

The Dickson algebra $D(s)$ has underlying algebra the polynomial algebra $\mathbb{F} [\omega_{s,i} | 0 \leq i \leq s - 1]$, where $\omega_{s,i}$ is the Dickson invariant of degree $2^s - 2^i$ (for example, the top Dickson invariant, $\omega_{s,0}$, is the product of all non-zero classes in $H^1(BV_s)$).

**Remark 4.3.** The algebra $D(s)$ must not be confused with the destabilization functors considered here.

**Notation 4.4.** For $K$ an unstable algebra, let $K-\mathcal{U}$ denote the category of $K$-modules in $\mathcal{U}$; forgetting the module structure defines a functor $K-\mathcal{U} \to \mathcal{U}$.
An object of $K\mathcal{U}$ is an unstable module $M$ equipped with a $K$-module structure such that the structure map $K \otimes M \to M$ is $\mathcal{A}$-linear.

**Proposition 4.5.** For $K$ an unstable algebra, the category $K\mathcal{U}$ has a unique abelian structure such that $K\mathcal{U} \to \mathcal{U}$ is exact. Moreover the tensor product of $K$-modules $\otimes_K$ defines a tensor structure on $K\mathcal{U}$, with unit $K$ (i.e $K\mathcal{U}$ is a symmetric monoidal category $(K\mathcal{U}, \otimes_K, K)$).

**Proof.** Exercise. \qed

Recall that $H^*( BV_1) \cong F[u]$, with $|u| = 1$, has a canonical unstable algebra structure.

**Definition 4.6.** For $M$ an unstable module, let

1. $St_1 : \Phi M \to F[u] \otimes M$ denote the linear map (not $\mathcal{A}$-linear) defined by $St_1(x) := \sum u^{|x|-1} \otimes Sq^i(x)$;

2. $R_1 M$ denote the sub $F[u]$-module of $F[u] \otimes M$ generated by $\{St_1(x) | x \in M\}$.

**Proposition 4.7.** [LZ87] The sub $F[u]$-module $R_1 M \subset F[u] \otimes M$ is stable under the $\mathcal{A}$-action, hence this defines a functor $R_1 : \mathcal{U} \to F[u]\mathcal{U}$.

**Proof.** Exercise (Remark 4.11 below provides some hints). \qed

**Remark 4.8.** Forgetting the $F[u]$-module structure, $R_1$ is frequently considered as a functor $R_1 : \mathcal{U} \to \mathcal{U}$. However, the $F[u]$-module structure is important when considering iterated loop functors in Section 5.2.

The functor $R_1$ has many remarkable properties, such as the following. (Their nature is exhibited by considering the nil-localization; for this, see [Pow12].)

**Proposition 4.9.** (Cf. [LZ87].)

1. The functor $R_1 : \mathcal{U} \to F[u]\mathcal{U}$ is exact; more precisely, the underlying $F[u]$-module is isomorphic to $F[u] \otimes \Phi M$.

2. There is a natural isomorphism of unstable modules

$$F \otimes_{F[u]} R_1 M \cong \Phi M;$$

the canonical surjection is written $\rho_1 : R_1 M \to \Phi M$ and there is a natural short exact sequence in $F[u]\mathcal{U}$:

$$0 \to u R_1 M \to R_1 M \to \Phi M \to 0$$

3. The functor $R_1$ preserves tensor products: there is a natural isomorphism $R_1 (M \otimes N) \cong R_1 M \otimes_{F[u]} R_1 N$.

**Proof.** Exercise. \qed

**Exercise 4.10.**

1. Show that the projection $\rho_1 : R_1 M \to \Phi M$ is compatible with $\lambda_M$, namely the following diagram commutes:

$$\begin{array}{ccc}
R_1 M & \longrightarrow & F[u] \otimes M \\
\rho_1 \downarrow & & \downarrow \varepsilon \\
\Phi M & \longrightarrow & \lambda_M M,
\end{array}$$

where $\varepsilon$ is induced by the augmentation of $F[u]$. 
(2) Show that the total Steenrod power $St_1$ is multiplicative, when $K$ is an unstable algebra. Namely

$$St_1(xy) = St_1(x)St_1(y)$$

where the product on the right hand side is formed in the unstable algebra $\mathbb{F}[u] \otimes K$.

(3) For $K$ an unstable algebra, show that $R_1K$ is naturally an unstable algebra, equipped with a natural inclusion $\mathbb{F}[u] \hookrightarrow R_1K$, so that $R_1$ defines a functor $R_1 : \mathcal{X} \to \mathbb{F}[u] \downarrow \mathcal{X}$ to the category of $\mathbb{F}[u]$-algebras in $\mathcal{X}$.

(4) For $K$ an unstable algebra, show that $R_1$ induces a functor $R_1 : \mathcal{K} \to R_1\mathcal{K}$.

(5) Determine the structure of $R_1\mathbb{F}[u_2] \subset \mathbb{F}[u_1, u_2]$ and identify it as the ring of invariants for the action of the upper triangular subgroup $B_2 \subset GL_2$.

Remark 4.11. Lannes and Zarati [LZ87] showed that $R_1$ is intimately related to destabilization. Namely, the short exact sequence (see Example 2.29)

$$0 \to \mathbb{F}[u] \to \hat{P} \to \Sigma^{-1}\mathbb{F} \to 0$$

defines a non-trivial class $e_1 \in \text{Ext}^1_{\mathcal{K}}(\Sigma^{-1}\mathbb{F}, \mathbb{F}[u])$. For an unstable module $M$, tensoring gives a short exact sequence

$$0 \to \mathbb{F}[u] \otimes M \to \hat{P} \otimes M \to \Sigma^{-1}M \to 0$$

and the long exact sequence for derived functors of destabilization induces a morphism

$$\alpha_M : D_1(\Sigma^{-1}M) \to D(\mathbb{F}[u] \otimes M) = \mathbb{F}[u] \otimes M.$$ Considering the case $M = \Sigma N$, Lannes and Zarati observed that $\alpha_{\Sigma N}$ induces a surjection

$$D_1N \to \Sigma R_1N \subset \Sigma\mathbb{F}[u] \otimes N.$$ (Exercise: prove this, by using the fact that $N$ is unstable to show that $B(\Sigma \hat{P} \otimes M) \subset \Sigma P \otimes N$ - see [LZ87].)

In the case $N = \mathbb{F}$, Lannes and Zarati proved moreover that $D_1\mathbb{F} \cong \Sigma R_1\mathbb{F} \cong \Sigma\mathbb{F}[u]$ (this follows directly from the chain complex constructed in Section 5.1 below). Proposition 3.11 shows that $D_1(\Sigma^{-1}\mathbb{F}) \cong \Omega D_1\mathbb{F}$, which is therefore isomorphic to $\mathbb{F}[u]$.

Remark 4.12. The functor $R_1$ has topological significance: let $X$ be a pointed topological space and write $EZ/2$ for the universal cover of $BZ/2$, which is an acyclic space equipped with a free $Z/2$-action. (An explicit model is given by $S^\infty = \text{colim}_{n \to \infty} S^n$, with projection $S^\infty \to \mathbb{R}P^\infty$ induced by the $Z/2$-Galois coverings $S^n \to \mathbb{R}P^n$.)

The diagonal of $X$ induces a $Z/2$-equivariant map $EZ/2_+ \wedge X \to EZ/2_+ \wedge X \wedge X$ (here $(-)_+$ denotes the addition of a disjoint basepoint) and passage to the quotient by the $Z/2$-action gives:

$$\Delta_2 : BZ/2_+ \wedge X \to \mathcal{G}_2X := EZ/2_+ \wedge_{Z/2} (X \wedge X)$$

In reduced mod-2 cohomology, this induces

$$\Delta_2^* : \hat{H}^*(\mathcal{G}_2X) \to H^*(BZ/2) \otimes \hat{H}^*(X)$$

and the image of $\Delta_2^*$ is $R_1\hat{H}^*(X)$. This is related to the construction of the Steenrod reduced power operations.

The Singer functors can be iterated. For example, $R_1R_1 : \mathcal{K} \to R_1\mathbb{F}[u]-\mathcal{K}$ (see Exercise 3.11) and $R_1R_1M$ is the sub $R_1\mathbb{F}[u]$-module of $\mathbb{F}[u_1, u_2] \otimes M$ which is generated by $St_2(x) := St_1(St_1(x))$. 


Notation 4.13. For \( s \in \mathbb{N} \) and a fixed basis of \( V_s \cong \mathbb{F}^{\oplus s} \), define \( St_s \) as a linear map

\[
St_s : \Phi^s M \to H^*(BV_s) \otimes M
\]

inductively by \( St_s = St_1 \circ St_{s-1} \).

Remark 4.14. Here, for precision, one should indicate the basis element used in the definition of each \( St_1 \) (cf. \textbf{[LZ87]}). However, this issue is resolved by the following result.

Lemma 4.15. \textbf{[LZ87]} The linear map \( St_s \) takes values in \( D(s) \otimes M \subset H^*(BV_s) \otimes M \), hence is independent of the choice of basis of \( V_s \) used in the definition.

By construction, the iterated Singer functor \( R^a_1 \) comes equipped with a natural inclusion \( R^a_1 M \hookrightarrow H^*(BV_s) \otimes M \), which depends upon the basis used in the construction. This dependency is removed by the following definition:

Definition 4.16. For \( s \in \mathbb{N} \), let \( R_s : \mathcal{A} \to D(s)-\mathcal{A} \) be the functor defined by:

\[
R_s(M) := (D(s) \otimes M) \cap R^a_1 M.
\]

Remark 4.17. The functor can be defined explicitly by taking \( R_s M \) to be the sub \( D(s) \)-module of \( D(s) \otimes M \) generated by \( St_s(x) \), \( \forall x \in M \). The advantage of the previous construction is that it implies immediately that this submodule is stable under \( \mathcal{A} \).

Remark 4.18. The quadratic nature of the construction is exhibited by the identity:

\[
R_s = \bigcap_{a+b+2=s} R^a_1 R^b_2 R^1_1^b.
\]

This shows that the functors \( R_s \) are determined by the generating functor \( R_1 \) and the relation \( R_2 \hookrightarrow R_1 R_1 \).

Exercise 4.19. Make the previous statement precise and prove it (hint: consider generators for \( GL_s \)).

Exercise 4.20. For \( 0 < s \in \mathbb{N} \) and any inclusion \( i_s : V_{s-1} \subset V_s \), show that the canonical inclusions of the Dickson invariants fit into a commutative diagram in \( \mathcal{K} \):

\[
\begin{array}{ccc}
D(s) & \to & H^*(BV_s) \\
\Phi D(s-1) & \to & D(s-1) & \to & H^*(BV_{s-1})
\end{array}
\]

(Use Exercise 4.17 for the first inclusion of the bottom row.) In particular, there is a canonical surjection of unstable algebras \( D(s) \to \Phi D(s-1) \).

Explicitly, show that \( i_s^* \) maps \( \omega_{s,0} \) to zero and \( \omega_{s,i} \mapsto \omega_{s-1,i-1}^2 \) for \( i > 0 \).

Exercise 4.21. For \( M \) in \( D(s-1)-\mathcal{A} \), show that \( \Phi M \) lies naturally in \( \Phi D(s-1)-\mathcal{A} \) and hence, via the surjection \( D(s) \to \Phi D(s-1) \), in \( D(s)-\mathcal{A} \). (An analogous result holds replacing the module categories such as \( D(s)-\mathcal{A} \) by the category \( D(s)-\mathcal{K} \) of \( D(s) \)-modules in \( \mathcal{K} \).)

Proposition 4.22. For \( s \in \mathbb{N} \),

1. \( R_s : \mathcal{A} \to D(s)-\mathcal{A} \) is exact and commutes with tensor products: \( R_s(M \otimes N) \cong R_s M \otimes D(s) R_s N \).

2. The natural transformation \( \rho_s \) induces a natural surjection \( \rho_s : R_s \to \Phi R_{s-1} \) via the inclusion \( R_s \hookrightarrow R_s R_{s-1} \) composed with \((\rho_1)_{R_{s-1}}\), which fits into a short exact sequence in \( D(s)-\mathcal{A} \):

\[
0 \to \omega_{s,0} R_s M \to R_s M \to \Phi R_{s-1} M \to 0.
\]
Proof. Exercise. (Hint: for $M = F$, the short exact sequence corresponds to the natural projection $D(s) \to \Phi D(s - 1).$) □

Remark 4.23. The class $c_1 \in \text{Ext}_{\mathcal{A}}(\Sigma^{-1}F, F[u])$ gives rise, via Yoneda product, to the class $\alpha_s \in \text{Ext}_{\mathcal{D}}(\Sigma^{-s}F, H^*(BV_s))$ and it is a fundamental result of Singer’s that this class is invariant under the action of $GL_s$ (see [LZS7], for example).

Standard methods of homological algebra (it is easier to think in terms of derived categories) show that the functor $D$ induces a natural morphism (for $t \in \mathbb{N}$)

$$\text{Ext}_{\mathcal{D}}^t(M, N) \to \text{Hom}_{\mathcal{D}}(D_tM, D_{t-s}N).$$

Thus, the class $e_s$ induces a linear morphism (natural in $M \in \mathcal{M}$)

$$\alpha^M_s : D_s(\Sigma^{-s}M) \to D(H^*(BV_s) \otimes M).$$

If $M$ is unstable, the right hand side is $H^*(BV_s) \otimes M$ and Lannes and Zarati show that $\alpha^M_s$ induces a map

$$\alpha^M_s : D_s(\Sigma^{-s}M) \to R_sM$$

(again this follows from the results of the next section). This exhibits the relationship between the Singer functor $R_s$ and the derived functor of destabilization $D_s$.

4.2. Singer functors for $\mathcal{M}$. The unstable Singer functors $R_s : \mathcal{U} \to D(s)\mathcal{U}$ forget $\mathcal{U}$ generalize to

$$\mathcal{M} \xrightarrow{R_1} D(s)\mathcal{M} \xrightarrow{\text{forget}} \mathcal{M},$$

where $D(s)\mathcal{M}$ is the category of $D(s)$-modules in $\mathcal{M}$.

Recall that localization gives an inclusion $F[u] \leftarrow F[u \pm 1]$ of $\mathcal{A}$-algebras. If $M$ is not unstable, then the Steenrod total power $St_1$ will not take values in $F[u] \otimes M$; if $M$ is bounded above it takes values in $F[u \pm 1] \otimes M$ but, in the general case, it is necessary to use a large tensor product (half-completed tensor product - see [Pow10], for example) so that $St_1$ is a linear map

$$St_1 : \Phi M \to F[u \pm 1] \otimes M.$$ 

With this modification, $R_1$ is defined as in the unstable case, so that $R_1M$ comes equipped with a canonical inclusion $R_1M \hookrightarrow F[u \pm 1] \otimes M$. Many of the good properties of $R_1$ pass to this setting, in particular:

Proposition 4.24. The functor $R_1 : \mathcal{M} \to F[u] \mathcal{M}$ is exact.

Proof. Exercise. □

The higher functors $R_s$ are constructed as before; the large tensor product leads to some technical issues.

Localization inverting the top Dickson invariant gives a commutative diagram of $\mathcal{A}$-algebras:

$$\begin{array}{ccc}
D(s) & \xrightarrow{=} & H^*(BV_s) \\
\downarrow & & \downarrow \\
D(s)[\omega^{-1}_{s,0}] & \xrightarrow{=} & H^*(BV_s)[\omega^{-1}_{s,0}] \\
\end{array}$$

which, in the case $s = 1$, corresponds to $F[u] \hookrightarrow F[u \pm 1]$. The localized Dickson algebra $D(s)[\omega^{-1}_{s,0}]$ is the appropriate generalization of $F[u \pm 1]$.

The general Singer functors $R_s : \mathcal{M} \to D(s)\mathcal{M}$, come equipped with a natural embedding

$$R_sM \hookrightarrow D(s)[\omega^{-1}_{s,0}] \otimes M,$$
and are exact. Moreover, they can be constructed from iterates of $R_1$ by imposing the quadratic relation $R_2$.

**Remark 4.25.** Care must be taken in considering the composition because of the large tensor product; see [Pow10b](#) (which is written for the case $p$ odd, but the methods apply to $p = 2$.)

As in the unstable case, one has the following fundamental short exact sequence:

**Proposition 4.26.** For $0 < s \in \mathbb{N}$ and $M$ an $\mathcal{A}$-module, there is a natural short exact sequence in $D(s)\mathcal{A}$:

$$0 \to \Sigma^{-1} R_s \Sigma M \to R_s M \to \Phi R_{s-1} M \to 0.$$

**Proof.** Cf. [Pow10b]. □

4.3. **The Singer differential.** There is a new phenomenon when considering the Singer functors defined on $\mathcal{A}$, corresponding to the Singer differential.

**Proposition 4.27.** The residue, namely the unique non-trivial map of graded vector spaces:

$$\partial : F[u^{\pm 1}] \to \Sigma^{-1} F,$$

is $\mathcal{A}$-linear. (Equivalently, $u^{-1}$ is not in the image of $Sq^i$, $\forall i > 0$).

**Proof.** Essential exercise. □

**Definition 4.28.** For $M$ an $\mathcal{A}$-module, let $d_M : R_1 M \to \Sigma^{-1} M$ denote the composite natural transformation:

$$R_1 M \hookrightarrow F[u^{\pm 1}] \otimes M \overset{\partial \otimes M}{\Rightarrow} \Sigma^{-1} M.$$

**Exercise 4.29.** Show that, if $M$ is unstable, then $d_M : R_1 M \to \Sigma^{-1} M$ is trivial.

The following result is the basis for building the chain complex calculating the derived functors of destabilization.

**Proposition 4.30.** For $M$ an $\mathcal{A}$-module, the cokernel of $\Sigma d_M : \Sigma R_1 M \to M$ is $DM$.

**Proof.** Essential exercise. □

5. **Constructing chain complexes**

Recall from Section 4.2 that $R_\ast : \mathcal{A} \to D(s)\mathcal{A}$ is an exact functor and that there is a natural differential $d_M : R_1 M \to \Sigma^{-1} M$ (see Definition 4.28). Moreover, there is a natural inclusion $R_s \hookrightarrow R_{s-1} R_1$. These are the key ingredients to constructing the chain complexes which calculate the derived functors of destabilization and of iterated loop functors.

5.1. **Destabilization.**

**Definition 5.1.** For $M$ an $\mathcal{A}$-module and $1 \leq s \in \mathbb{Z}$, let $d_{s,M} : R_s M \to R_{s-1} (\Sigma^{-1} M)$ denote the natural morphism given as the composite:

$$R_s M \hookrightarrow R_{s-1} R_1 M \overset{R_{s-1} d_M}{\Rightarrow} R_{s-1} (\Sigma^{-1} M),$$

so that $d_{1,M}$ identifies with $d_M$.

**Proposition 5.2.** For $M$ an $\mathcal{A}$-module and $s \in \mathbb{N}$, the composite

$$R_{s+2}(M) \overset{d_{s+2,M}}{\Rightarrow} R_{s+1} (\Sigma^{-1} M) \overset{d_{s+1,M}}{\Rightarrow} R_s (\Sigma^{-2} M)$$

is trivial.
The connecting morphism \( \lambda \) applies to \( M \). Moreover, in homology this induces a long exact sequence in homology.

**Proof.** (Indications - see [Pow10b] for a proof in the case \( p \) odd - the method also applies to \( p = 2 \)). Using the quadratic nature of the functors \( R_s \), it is straightforward to reduced to the case \( s = 0 \) (exercise).

This case is proved using the relationship between the Steenrod algebra and invariant theory, as in the work of Singer [Sin83]; one method is to embed the diagram in the \((p = 2)\)-analogue of the chain complex \( \Gamma_s M \) considered by Hung and Sum [HS95] (their arguments adapt to the case \( p = 2 \)). \( \square \)

Recall that the category of chain complexes for an abelian category is abelian (or prove this as an exercise!).

**Corollary 5.3.** There is an exact functor \( \mathcal{D} : \mathcal{M} \to \text{Ch}(\mathcal{M}) \) with values in \( \mathbb{N} \)-graded chain complexes defined by

\[
\mathcal{D}_n M := \Sigma R_n(\Sigma^{s-1}M)
\]

\[
d_n : \mathcal{D}_n M \to \mathcal{D}_{n-1} M := \Sigma d_{s, \Sigma^{s-1}} M.
\]

**Proof.** Proposition 5.2 implies that \( \mathcal{D} M \) is a chain complex and the construction is functorial. Since \( R_s : \mathcal{M} \to \mathcal{M} \) is an exact functor (forgetting the action of \( D(s) \)) and \( \Sigma \) is exact, the functor \( \mathcal{D} \) is exact. \( \square \)

As shown by the work of Singer on the derived functors of iterated loop functors [Sin80], a key input to the proof is to have a short exact sequence of complexes which gives rise to the long exact sequence of derived functors of destabilization.

**Notation 5.4.** For \( M \) an \( \mathcal{A} \)-module and \( s \in \mathbb{N} \), let \( \mathbb{D}_s M \) denote \( H_d(\mathbb{D}_s M) \), so that \( \mathbb{D}_0 M = DM \), by Proposition 4.30.

**Proposition 5.5.** For \( M \) an \( \mathcal{A} \)-module, there is a natural short exact sequence of chain complexes:

\[
0 \to \Sigma^{-1} \mathcal{D}_*(\Sigma M) \to \mathcal{D}_* M \to \Sigma^{-1} \Sigma \mathcal{D}_{s-1}(\Sigma M) \to 0.
\]

Moreover, in homology this induces a long exact sequence in \( \mathcal{M} \):

\[
\ldots \to \Sigma^{-1} \mathbb{D}_s(\Sigma M) \to \mathbb{D}_s M \to \Sigma^{-1} \Sigma \mathbb{D}_{s-1}(\Sigma M) \xrightarrow{\lambda} \Sigma^{-1} \Sigma \mathbb{D}_{s-1}(\Sigma M) \to \ldots .
\]

The connecting morphism \( \lambda_0 \) identifies with \( \Sigma^{-1} \lambda_{DM} \), using the identification \( \mathbb{D}_0 M = DM \).

**Proof.** (Indications. Cf. [Pow10b], which treats the case \( p \) odd.) The first statement follows from the naturality of the construction of the chain complex of Proposition 4.26 and of the differential.

For the final statement, the long exact sequence is the long exact sequence in homology, using the exactness of the functors \( \Sigma \) and \( \Phi \). The identification of \( \lambda_0 \) is a straightforward exercise. \( \square \)

**Lemma 5.6.** For \( M \) an \( \mathcal{A} \)-module and \( s \in \mathbb{N} \), \( \text{conn}(\mathbb{D}_s M) \geq 2^s (\text{conn} M + s) \).

**Proof.** Exercise. \( \square \)

**Proposition 5.7.** \( \mathbb{D}_s(\Sigma^{t+1} \mathcal{A}) = 0 \) \( \forall t \in \mathbb{Z} \) and \( 0 < s \in \mathbb{N} \).

**Proof.** For \( s = 1 \) and \( t \in \mathbb{Z} \) recall that \( D(\Sigma^{t+1} \mathcal{A}) = F(t + 1) \), so that the long exact sequence of Proposition 5.3 is of the following form:

\[
\ldots \to \Sigma^{-1} \mathbb{D}_1(\Sigma^{t+1} \mathcal{A}) \xrightarrow{\alpha_{t+1}} \mathbb{D}_1(\Sigma^{t} \mathcal{A}) \to \Sigma^{-1} \Phi F(t + 1) \xrightarrow{\Sigma^{-1} \lambda} F(t + 1) \to \ldots .
\]

The morphism \( \lambda \) is injective, since \( F(t + 1) \) is reduced, hence the morphism \( \alpha_{t+1} \) is surjective. Since \( \text{conn}(\mathbb{D}_1(\Sigma^{t+1} \mathcal{A})) \to \infty \) as \( t \to \infty \), by Lemma 5.6 it follows that \( \mathbb{D}_1(\Sigma^{t} \mathcal{A}) = 0 \) \( \forall t \in \mathbb{Z} \). (Exercise: provide the details of this argument.)

This forms the initial step of an induction upon \( s \); the inductive step is similar (but easier). \( \square \)
Theorem 5.8. For $M$ an $\mathcal{A}$-module, there is a natural isomorphism

$$H_s(\mathcal{D}M) \cong D_sM.$$ 

Proof. This follows by standard arguments of homological algebra, since $D_0M = DM$ by Proposition 4.30 and $D_s$ vanishes for $s > 0$ on the projectives of $\mathcal{M}$, by Proposition 5.7. □

Remark 5.9. A priori, from the construction, it is not clear that the homology of the complex is unstable.

From this result, one recovers immediately one of the main results of Lannes and Zarati [LZ87]:

Corollary 5.10. For $M$ an unstable module and $s \in \mathbb{N}$, there is a natural isomorphism

$$D_s(\Sigma^{1-s}M) \cong \Sigma R_sM$$

and a short exact sequence of unstable modules

$$0 \to R_sM \to D_s(\Sigma^{-s}M) \to \Omega_1D_{s-1}(\Sigma^{1-s}M) \to 0.$$ 

In particular, if $M$ is reduced, then $D_s(\Sigma^{-s}M) \cong R_sM$.

Proof. The first statement is a consequence of the vanishing of the relevant differentials in the chain complex $\mathcal{D}_*M$ under the given hypotheses and the second follows from the short exact sequence of Corollary 4.18. Finally it is clear that $R_sM$ is reduced if $M$ is reduced (exercise). Hence, by induction on $s$, one sees that $D_s(\Sigma^{-s}M)$ is reduced and the $\Omega_1$ term vanishes. □

Remark 5.11. Kuhn and McCarty [KM13], who work in homology, give a geometric construction of the analogous chain complex, notably giving a geometric construction of the Singer functors and the differential. The reader should compare their approach, which shows the relationship with the Dyer-Lashof operations.

Exercise 5.12. Show that, if $M$ is a finite $\mathcal{A}$-module (of finite total dimension), then the derived functors $D_sM$ are all non-trivial for $s \gg 0$.

5.2. Iterated loops. Fix $t \in \mathbb{N}$, which corresponds to the number of loops $\Omega^t$.

Notation 5.13. For $t \in \mathbb{N}$, let $R_{1/t} : \mathcal{M} \to \mathbb{F}[u].-\mathcal{M}$ denote the functor

$$R_{1/t}M := \mathbb{F}[u]/(u^t) \otimes_{\mathbb{F}[u]} R_1M$$

equipped with the natural projection $R_1M \to R_{1/t}M$ in $\mathbb{F}[u].-\mathcal{M}$.

Example 5.14. For $M$ an $\mathcal{A}$-module, $R_{1/0}M = 0$ and there is a natural isomorphism $R_{1/1}M \cong \Phi M$.

Exercise 5.15. Show that

1. $R_{1/t}$ is exact;
2. $R_{1/t}$ induces a functor $\mathcal{X} \to \mathbb{F}[u]/(u^t) \downarrow \mathcal{X}$.

Lemma 5.16. For $N$ an unstable module, the differential $d_{\Sigma^{-t}N}$ induces a commutative diagram

$$R_1(\Sigma^{-t}N) \xrightarrow{d_{\Sigma^{-t}N}} \Sigma^{-t-1}N \xrightarrow{d_{\Sigma^{-t}N}} R_{1/t}(\Sigma^{-t}N)$$

in particular the morphism $d_{1/t,N} : R_{1/t}(\Sigma^{-t}N) \to \Sigma^{-t-1}N$ is $\mathcal{A}$-linear.
Definition 5.19. For integers $0 \leq s \leq t$, let $R_{s/t} : \mathcal{M} \to D(s)\mathcal{M}$ denote the functor defined by

$$
R_{s/t}M := \text{image}[R_sM \hookrightarrow (R_1)^{\otimes s}M \twoheadrightarrow R_{1/t} \circ R_{1/(t-1)} \circ \ldots \circ R_{1/(t-s+1)}M],
$$

equipped with the canonical surjection $R_1M \twoheadrightarrow R_{s/t}M$ in $D(s)\mathcal{M}$.

Remark 5.20. The functor $R_{s/t}$ is zero if $s > t$, since $R_{1/0} = 0$.

Exercise 5.21. For an integer $0 \leq s \leq t$, and an unstable module $N$, there is an induced morphism in $\mathcal{M}$, which fits into the commutative diagram:

$$
\begin{array}{ccc}
R_{s/t}M & \rightarrow & R_{s-1}R_1M \\
\downarrow & & \downarrow \\
R_{s-1/(t-s+1)}M & \rightarrow & R_{s-1/(t-s+1)}R_{1/(t-s+1)}M. \\
\end{array}
$$

Hence, as in the construction of $d_{1/t}/N$, there is an induced morphism in $\mathcal{M}$, which is given for $N \in \mathcal{M}$ as the composite:

$$
R_{s/t}(\Sigma^{-(t-s+1)}N) \longrightarrow R_{s-1/t}R_{1/(t-s+1)}(\Sigma^{-(t-s+1)}N) \twoheadrightarrow R_{s-1/(t-s+1)}N.
$$

Lemma 5.23. For integers $1 \leq s \leq t$ and an unstable module $N$, the following diagram commutes:

$$
\begin{array}{ccc}
R_s(\Sigma^{-(t-s+1)}N) & \rightarrow & R_{s-1}R_1(\Sigma^{-(t-s+1)}N) \\
\downarrow & & \downarrow \\
R_{s/t}(\Sigma^{-(t-s+1)}N) & \rightarrow & R_{s-1/t}(\Sigma^{-(t-s+1)}N).
\end{array}
$$

Proof. Exercise. □
Remark 5.24. The hypothesis that $N$ is unstable is essential for this compatibility, as in Lemma 5.16.

There is also an analogue of the short exact sequence of Proposition 4.26 based on the observation that, for $t \geq 1$, the natural surjection $\rho_1 : R_1M \twoheadrightarrow \Phi M$ factorizes naturally across a surjection $\rho_1 : R_{1/t}M \twoheadrightarrow \Phi M$.

**Proposition 5.25.** For integers $1 \leq s \leq t$, the morphism $\rho_1$ induces a short exact sequence for the functors $R_{s/t}$ which forms the bottom row of the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Sigma^{-1}R_s \Sigma M & \rightarrow & R_s M & \rightarrow & \rho_s \Phi R_{s-1} M & \rightarrow & 0 \\
0 & \rightarrow & \Sigma^{-1}R_{s/t-1} \Sigma M & \rightarrow & R_{s/t} M & \rightarrow & \rho_s \Phi R_{s-1/t-1} M & \rightarrow & 0,
\end{array}
\]

where the top row is provided by Proposition 4.26 and the vertical morphisms are the canonical surjections.

**Proof.** (Indications.) The only non-trivial point is to identify the kernel in the bottom row; this is clear in the case $s = 1$ and the higher cases are treated by induction. □

**Exercise 5.26.** For $M$ a finite $A$-module (ie. the total dimension is finite) and integers $1 \leq s \leq t$,

1. show that the total dimension of $R_{s/t}M$ is finite;
2. calculate $\text{conn}(R_{s/t}M)$ in terms of $\text{conn}M$ and $s, t$;
3. calculate the top dimension of $R_{s/t}M$ in terms of the top dimension of $M$ and $s, t$.

(Hint: use information on the algebra $R_{s/t}F$, which can be obtained inductively using Proposition 5.26)

**Definition 5.27.** Let $\mathcal{C}_s^t : \mathcal{U} \rightarrow \text{Ch}(\mathcal{M})$ denote the exact functor defined on an unstable module $N$ by $\mathcal{C}_s^t N := \Sigma R_{s/t}(\Sigma^{-(t-s+1)}N)$ and with differential $\Sigma d_{s/t,N}$.

**Remark 5.28.** The fact that $\mathcal{C}_s^t$ is a chain complex (namely $d^2 = 0$) is a consequence of the corresponding result for $\mathcal{D}$, which follows from Proposition 5.2.

**Exercise 5.29.** Show that the chain complex $\mathcal{C}_s^t N$ is bounded for any unstable module, $N$. Namely, $\mathcal{C}_s^t N = 0$ for $s > t$.

**Proposition 5.30.** For $N$ an unstable module, there is a natural surjection of chain complexes:

\[\mathcal{D}(\Sigma^{-t}N) \twoheadrightarrow \mathcal{C}_s^t(N).\]

Moreover, the short exact sequences of Proposition 4.26 induce a short exact sequence of chain complexes which fits into the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Sigma^{-1}\mathcal{D}_s(\Sigma^{-(t+1)}N) & \rightarrow & \mathcal{D}_s(\Sigma^{-t}N) & \rightarrow & \Sigma^{-1}\Phi \mathcal{D}_{s-1}(\Sigma^{-t+1}N) & \rightarrow & 0 \\
0 & \rightarrow & \Sigma^{-1}\mathcal{C}_{s-1}^{t+1}(N) & \rightarrow & \mathcal{C}_s^t(N) & \rightarrow & \Sigma^{-1}\Phi \mathcal{C}_{s-1}^t(N) & \rightarrow & 0.
\end{array}
\]

**Exercise 5.31.** Show that $\mathcal{C}_s^t N = (\Sigma^{-1}\Phi N \rightarrow \Sigma^{-1}N)$. 


Remark 5.32. The chain complex $C_s \cdot N$ has the same formal properties as that constructed by Singer in [Sin80] (they are believed to be naturally isomorphic).

The current presentation, being based upon quotients of the Singer functors and the Singer differential, makes explicit the relationship between $D \Sigma^{-t}$ and $C'$. Theorem 5.33. For $N$ an unstable module and $s, t \in \mathbb{N}$, there is a natural isomorphism

$$\Omega_s^t N \cong H_s(C_s \cdot N).$$

Moreover, the surjection of chain complexes $D(A) \rightarrow C(A)$ of Proposition 5.30 induces the natural transformations $D_s(A) \rightarrow \Omega_s^t N$ in homology.

Proof. The proof of the first point is formally similar to that of Theorem 5.8 but the inductive step is easier, since a double induction on $t$ and $s$ can be used. This argument is identical to that used in Singer [Sin80], which only requires the formal properties of the chain complex.

The final statement follows by general arguments of homological algebra. □

The following result is analogous to Corollary 5.10 (and is implicit in [Sin80]).

Corollary 5.34. For $N$ an unstable module, $s, t \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $k \geq t - s + 1$, there is a natural isomorphism:

$$\Omega_s^t(\Sigma^k N) \cong \Sigma R_s/t(\Sigma^{k-1+1}) N$$

of unstable modules.

Under these hypotheses, there is a short exact sequence of unstable modules:

$$0 \rightarrow P_s/t-1(\Sigma^{k-1+1}) N \rightarrow \Omega_s^t(\Sigma^{k-1} N) \rightarrow \Omega_s^{t-1}(\Sigma^{k-1} N) \rightarrow 0.$$

Proof. Exercise. □

Remark 5.35. Unlike the functor $R_1$, the functor $R_{s/t}$ restricted to $\mathcal{M}$ does not send reduced unstable modules to reduced objects if $t > 1$; in particular, $R_{s/t} F \cong F |u|/u!$ is not reduced for $t > 1$.

Example 5.36. When $t = 1$, the above result gives

1. for $s = 0$ and $k \geq 2$, $\Omega(\Sigma^k N) \cong \Sigma \Sigma^{k-2} N \cong \Sigma^{k-1} N$, as expected;
2. for $s = 1$, we require $k \geq 1$ and get $\Omega_1(\Sigma^k N) \cong \Sigma R_{1/1}(\Sigma^{k-1} N) \cong \Sigma^{-1} \Phi \Sigma^k N$, using the identification $R_{1/1} \cong \Phi$ and $\Phi \Sigma \cong \Sigma^2 \Phi$.

5.3. The Lannes-Zarati homomorphism. Singer [Sin83] constructed a chain complex $\Gamma_p M$ which computes the homology of $M$ over the Steenrod algebra as a sub-complex of a larger complex $\Gamma_p M$. Using the method of Hung and Sun [HS95] (adapted to the prime 2) it is possible to show the following:

Proposition 5.37. There is a natural inclusion of chain complexes

$$D(A) \rightarrow \Gamma_p^+ M.$$

In homology this induces the Lannes-Zarati homomorphism of [LZ87].

References
[HM89] J. R. Harper and H. R. Miller, Looping Massey-Peterson towers, Advances in homotopy theory (Cortona, 1988), London Math. Soc. Lecture Note Ser., vol. 139, Cambridge Univ. Press, Cambridge, 1989, pp. 69–86. MR 1055889 (91c:55032)
[HS95] Nguyễn H. V. Hung and Nguyễn Sum, On Singer’s invariant-theoretic description of the lambda algebra: a mod $p$ analogue, J. Pure Appl. Algebra 99 (1995), no. 3, 297–329. MR 1332903 (96c:55024)
[KM13] Nicholas Kuhn and Jason McCarty, The mod 2 homology of infinite loopspaces, Algebr. Geom. Topol. 13 (2013), no. 2, 687–745. MR 3044591
Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d’un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman. MR 1179079 (93j:55019)

Jean Lannes and Saïd Zarati, *Invariants de Hopf d’ordre supérieur et suite spectrale d’Adams*, Preprint, 1984.

Jean Lannes and Saïd Zarati, *Sur les foncteurs dérivés de la déstabilisation*, Math. Z. 194 (1987), no. 1, 25–59. MR 0871217 (88j:55014)

H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983, Modules over the Steenrod algebra and the stable homotopy category. MR 738773 (86j:55001)

John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) 81 (1965), 211–264. MR 0174052 (30 #4259)

Huỳnh Mùi, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 3, 319–369. MR 0422451 (54 #10440)

Huỳnh Mùi, *Cohomology operations derived from modular invariants*, Math. Z. 193 (1986), no. 1, 151–163. MR 852916 (88j:55015)

Geoffrey M. L. Powell, *Module structures and the derived functors of iterated loop functors on unstable modules over the Steenrod algebra*, J. Pure Appl. Algebra 214 (2010), no. 8, 1435–1449. MR 2593673

Geoffrey M. L. Powell, *On the derived functors of destabilization at odd primes*, arXiv:1101.0226 45 pages; [version accepted Acta Mathematica Vietnamica (31 pages), to appear (2013)], 2010.

Geoffrey M. L. Powell, *On unstable modules over the Dickson algebras, the Singer functors Rs and the functors Fixs*, Algebr. Geom. Topol. 12 (2012), 2451–2491 (electronic).

Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970), 39–60. MR 0265437 (42 #346)

Lionel Schwartz, *Unstable modules over the Steenrod algebra and Sullivan’s fixed point set conjecture*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994. MR 1282727 (95d:55017)

William M. Singer, *Iterated loop functors and the homology of the Steenrod algebra. II. A chain complex for Ω^n M*, J. Pure Appl. Algebra 16 (1980), no. 1, 85–97. MR 0549706 (81k:55049)

William M. Singer, *Iterated loop functors and the homology of the Steenrod algebra. I. An invariant theory and the lambda algebra*, Trans. Amer. Math. Soc. 280 (1983), no. 2, 673–693. MR 0716844 (85e:55029)

William M. Singer, *Iterated loop functors and the homology of the Steenrod algebra*, J. Pure Appl. Algebra 11 (1977/78), no. 1–3, 83–101. MR 0478155 (57 #17644)

Saïd Zarati, *Dérivés du foncteur de stabilisation en caractéristique impaire et applications*, Thèse d’État, Université Paris Sud, 1984.

Saïd Zarati, *Derived functors of the destabilization and the Adams spectral sequence*, Astérisque (1990), no. 191, 8, 285–298. International Conference on Homotopy Theory (Marseille-Luminy, 1988). MR 1098976 (92c:55020)

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