Reachability in Infinite Dimensional Unital Open Quantum Systems with Switchable GKS-Lindblad Generators

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Abstract. In quantum systems theory one of the fundamental problems boils down to: given an initial state, which final states can be reached by the dynamic system in question. Here we consider infinite dimensional open quantum dynamical systems following a unital Kossakowski-Lindblad master equation extended by controls. More precisely, their time evolution shall be governed by an inevitable potentially unbounded Hamiltonian drift term $H_0$, finitely many bounded control Hamiltonians $H_j$ allowing for (at least) piecewise constant control amplitudes $u_j(t) \in \mathbb{R}$ plus a bang-bang (i.e. on-off) switchable noise term $\Gamma_V$ in Kossakowski-Lindblad form. Generalizing standard majorization results from finite to infinite dimensions, we show that such bilinear quantum control systems allow to approximately reach any target state majorized by the initial one as up to now only has been known in finite dimensional analogues.—The proof of the result is currently limited to the control Hamiltonians $H_j$ being bounded and noise terms $\Gamma_V$ with compact normal $V$. 
1. Introduction and Overview

1.1. Markovian Bilinear Quantum Control

The Kossakowski-Lindblad equation\[^{25,26,30,21}\] plays a central role in quantum dynamics since it characterizes the infinitesimal generators of the semigroup of all (invertible\(^{1a}\)) Markovian quantum maps.

As in\[^{46}\] a quantum map (c\(_{\text{ptp}}\) map\(^{1b}\)) is called (time-dependent) Markovian, if it is the solution of a (time-dependent) Markovian master equation

\[
\dot{F}(t) = -(iH(t) + \Gamma(t))F(t), \quad F(0) = 1,
\]

where Markovianity and c\(_{\text{ptp}}\) property are guaranteed by the Kossakowski-Lindblad form of \(\Gamma(t)\) in Eq. (3). Then for finite dimensional Hamiltonians and noise terms one can show that those Markovian quantum maps (including both, time-dependent and time-independent ones) are infinitesimal divisible into products of exponentials of Kossakowski-Lindblad generators\[^{46}\] hence leading to Lie semigroup structure\[^{14}\]. In contrast, non-Markovian quantum maps (existing even arbitrarily close to the identity map) are Kraus maps\[^{27}\] that are not solutions of a Kossakowski-Lindblad master equation and hence the set of all invertible quantum maps (including Markovian and non-Markovian ones) has no Lie-semigroup structure (details in\[^{46,14,39}\]).

For most of the work, we focus on the corresponding induced system \(\Sigma\) acting on the state space \(\mathbb{D}(\mathcal{H})\) of all density operators \(\rho\) and following the time-dependent Kossakowski-Lindblad master equation of the form

\[
\dot{\rho}(t) = -(iH(t) + \Gamma(t))(\rho(t)),
\]

where Markovianity and c\(_{\text{ptp}}\) property are guaranteed by the Kossakowski-Lindblad form of \(\Gamma(t)\) in Eq. (3). Then for finite dimensional Hamiltonians and noise terms one can show that those Markovian quantum maps (including both, time-dependent and time-independent ones) are infinitesimal divisible into products of exponentials of Kossakowski-Lindblad generators\[^{46}\] hence leading to Lie semigroup structure\[^{14}\]. In contrast, non-Markovian quantum maps (existing even arbitrarily close to the identity map) are Kraus maps\[^{27}\] that are not solutions of a Kossakowski-Lindblad master equation and hence the set of all invertible quantum maps (including Markovian and non-Markovian ones) has no Lie-semigroup structure (details in\[^{46,14,39}\]).

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\[
\dot{\rho}(t) = -(iH(t) + \Gamma(t))(\rho(t)),
\]

where \(H(t)\) denotes the adjoint action of some time-dependent Hamiltonian \(H(t)\) on \(\mathbb{D}(\mathcal{H})\), i.e. \(H(t)(\rho) := \text{ad}_{H(t)}(\rho) = [H(t), \rho]\). For Markovianity take the noise term \(\Gamma(t)\) in the usual Kossakowski-Lindblad form

\[
\Gamma(t) := \sum_k \gamma_k(t) \Gamma_{V_k} \quad \text{with} \quad \Gamma_{V_k}(\rho) := \frac{1}{2}(V_k^\dagger V_k \rho + \rho V_k^\dagger V_k) - V_k \rho V_k^\dagger.
\]

The time dependence of \(H(t)\) is brought about by adding to the (usually inevitable, possibly unbounded) system Hamiltonian \(H_0\), bounded control Hamiltonians of the type \(u_j(t)H_j\) to give \(H(t) := H_0 + \sum_{j=1}^m u_j(t)H_j\), where the control amplitudes \(u_j(t) \in \mathbb{R}\) are typically modulated in a manner at least allowing for piecewise constant controls.

In the finite-dimensional case, an unambiguous separation of the dissipative part and the coherent part results by choosing the \(V_k\) traceless—as described by Kossakowski, Gorini and Sudarshan in the celebrated work of\[^{21}\].

\(^{1a}\)Here invertibility only means invertible as linear map, not necessarily as quantum map.

\(^{1b}\)C\(_{\text{ptp}}\) maps are linear completely positive and trace-preserving.
In infinite dimensions, this separation is a bit delicate yet not crucial for the sequel. More important is the restriction to compact noise terms $V_k$. In our case of interest, the noise terms can be individually switched on and off (as ‘bang-bang controls’), so it suffices to study a single noise term\footnote{Clearly, collectively switched noise in the sense of $\gamma(t) = \gamma_k(t)$ for all $k$ is more subtle.}.

\[ \Gamma(t)(\rho) := \gamma(t)\Gamma_V \]  

with $\gamma(t) \in \{0, \gamma_*\}$ and $\gamma_* > 0$—w.l.o.g. we always assume $\gamma_* = 1$. With these stipulations, we refer to the master equation (2) as GKSL-equation henceforth.

In the limiting case of $\gamma(t) = 0$ for all times, the control system of Eq. (2) turns into a closed Hamiltonian system referred to as $\Sigma_0$, while for switchable noise with a single $V$-term the system will be labelled $\Sigma_V$.

Note that with the identifications $A := iH_0$, $B_j := iH_j$, $B_0 := \Gamma_V$ and $X(t) = \rho(t)$ one formally gets a standard bilinear control system \[\begin{align*}
\dot{X}(t) &= -(A + \sum_{j=0}^{m} u_j(t)B_j)X(t) \quad \text{with} \quad X(0) = X_0,
\end{align*}\]

also identifying $u_0(t) = \gamma(t)$. This covers a broad class of quantum control problems including coherent and incoherent feedback \cite{33, 17, 38, 22}. Accessibility of such bilinear Markovian quantum systems (in finite dimensions) was analysed i.a. in terms of symmetries in previous work \cite{39}.

In the following, we are interested in characterising the reachable sets of $\Sigma_V$ which take the form of a semigroup orbit

\[ \text{reach}_{\Sigma_V}(\rho_0) := \mathcal{S}_{\Sigma_V} \cdot \rho_0 := \{ F(\rho_0) \mid F \in \mathcal{S}_{\Sigma_V} \} , \]  

where $\rho_0 \in \mathcal{D}(H)$ denotes an arbitrary initial density operator and $\mathcal{S}_{\Sigma_V}$ is the semigroup generated by the one-parameter semigroups

\[ e^{-t(i\text{ad}H_0 + i\sum_{j=1}^{m} u_j\text{ad}H_j + u_0\Gamma_V)} \quad \text{with} \quad u_1, \ldots, u_m \in \mathbb{R}, \quad u_0 \in \{0,1\}. \]

If all involved operators are bounded, $e^{-t(i\text{ad}H_0 + i\sum_{j=1}^{m} u_j\text{ad}H_j + u_0\Gamma_V)}$ is given by the exponential series and $\text{reach}_{\Sigma_V}(\rho_0)$ can alternatively be defined as the collection of all endpoints $\rho(T)$, $T \geq 0$ of trajectories of (2) for piecewise constant controls and initial value $\rho(0) = \rho_0$. For the general case ($H_0$ unbounded), defining $\text{reach}_{\Sigma_V}(\rho_0)$ via trajectories is problematic since Eq. (2) allows classical solutions only on a dense domain of initial states. Yet, Eq. (5) also works for unbounded $H_0$, as $-i\text{ad}H_0 + i\sum_{j=1}^{m} u_j\text{ad}H_j + u_0\Gamma_V$ does generate a unique strongly continuous semigroup (details in Appendix D).
We start the discussion by $\Sigma_0$, assuming for the moment that the noise is switched off, i.e. $\gamma(t) = 0$. In finite dimensions such a system is **fully unitarily controllable** if it satisfies the Lie-algebra rank condition \[ \langle iH_0, iH_j | j = 1, 2, \ldots, m \rangle_{\text{Lie}} = \text{su}(\mathcal{H}) \quad \text{(or } \text{u}(\mathcal{H})) \]. (6)

Then reachable sets are unitary group orbits of the respective initial states \[ \text{reach}_{\Sigma_0}(\rho_0) = \{ U\rho_0 U^\dagger | U \in \mathcal{U}(\mathcal{H}) \} . \]

If the Lie closure $\mathfrak{k} := \langle iH_0, iH_j | j = 1, 2, \ldots, m \rangle_{\text{Lie}}$ in Eq. (6) is but a proper compact subalgebra $\mathfrak{k} \subsetneq \text{su}(\mathcal{H})$, one likewise gets a subgroup orbit now by limiting $U$ to elements of $K := \exp \mathfrak{k} \subsetneq \mathcal{U}(\mathcal{H})$, see, e.g., [13, 39].

Yet already in open finite dimensional quantum systems $\Sigma_V$, it is more intricate to characterise reachable sets: In the unital case, i.e. for $\Gamma(t)(1) = 0$, one finds by the seminal work of [43, 1] and [2] on majorization the inclusion \[ \text{reach}_{\Sigma}(\rho_0) \subseteq \{ \rho \in \mathbb{D}(\mathcal{H}) | \rho \prec \rho_0 \} \]
as used in [44]. Henceforth, $\text{reach}_{\Sigma}(\rho_0)$ denotes the closure\footnote{In both, finite and infinite dimensions, there is a canonical choice for the topology on $\mathbb{D}(\mathcal{H})$—we will come back to this point later.} of $\text{reach}_{\Sigma}(\rho_0)$. In the special case $\Gamma(t) = \gamma(t)\Gamma_V \neq 0$ (where unitality of $\Gamma(t)$ boils down to normality of $V$) one can obtain equality even for unswitchable noise if there are no bounds on the coherent controls $u_k(t)$ and already the control Hamiltonians (without the drift $iH_0$) satisfy $\langle iH_j | j = 1, 2, \ldots, m \rangle_{\text{Lie}} = \text{su}(\mathcal{H})$, a scenario we called **Hamiltonian controllable** (fully $H$-controllable) [14, 35].

For many experiments this is hopelessly idealising unless one can switch off the noise—a scenario studied below—because then one is allowed to “use” also the drift Hamiltonian $H_0$ for controlling the system in the course of noise-free evolution. However, for all physical scenarios (requiring the drift Hamiltonian $H_0$ for full controllability of its Hamiltonian part) with sizeable constant noise, the above inclusion is far from being tight and—even worse—the overestimation of the reachable set increases with system size. In these cases, **Lie-semigroup** techniques help to estimate the reachable set [14, 35].

Yet there are indeed instances of unitarily controllable systems of the type $\Sigma_V$ in which the noise can be switched as bang-bang control. An important experimental incarnation are superconducting qubits coupled to an open transmission line [10]. Then, for normal $V$, one can saturate the above inclusion to get $\text{reach}(\rho_0) = \{ \rho \in \mathbb{D}(\mathcal{H}) | \rho \prec \rho_0 \}$ as shown in [4, 39]. In another extreme, $\Gamma_V$ models coupling the system to a bath of temperature zero (entailing $V$ is the nilpotent matrix $\sigma_-$). In this case $\text{reach}(\rho_0) = \mathbb{D}(\mathbb{C}^n)$ [16].

Here the goal is to transfer the former result (with normal $V$) from finite to infinite-dimensional systems on separable complex Hilbert spaces $\mathcal{H}$. 
1.2. Main Result

In infinite dimensions, establishing unitary controllability for $\Sigma_0$ is more intricate. One of the most general results currently known is the following [24]:

Let $H_0, ..., H_m$ be selfadjoint operators on a separable Hilbert space $\mathcal{H}$. Further assume that

1. $H_0$ is bounded or unbounded, but has only pure point spectrum. The eigenvalues $x_k, k \in \mathbb{N}$ are non-degenerate and rationally independent.
2. The operators $H_1, ..., H_m$ are bounded and the set $\{H_1, ..., H_m\}$ is connected with respect to the complete set of eigenvectors $\phi_k \in \mathcal{H}, k \in \mathbb{N}$ of $H_0$.

Then the unitary system

$$\dot{U}(t) = -iH(t)U(t) \quad \text{with} \quad U(0) = 1,$$

is strongly approximately operator controllable in the following sense:

**Definition 1.** The unitary control system [7] is called strongly approximately operator controllable, if the strong closure (in $U(\mathcal{H})$) of the reachable set $\text{reach}(1)$ coincides with $U(\mathcal{H})$.

The result can be generalized to eigenvalues $x_k, k \in \mathbb{N}$ with finite multiplicities, but this requires more technical conditions on the control Hamiltonians: We have to ensure that trace-free finite rank operators commuting with all eigenprojectors of $H_0$ are contained in the strong closure of the Lie algebra generated by the $H_j, j = 1, \ldots, m$. More challenging are drift Hamiltonians with rationally dependent eigenvalues. However, they can be studied in terms of certain non-Abelian von Neumann algebras; cf. [24] for details. Similar results were derived earlier in terms of Galerkin approximations in [5] and were refined more recently in [8].

If all Hamiltonians (including $H_0$) are bounded, we use approximate versions of the Lie algebra rank condition, the most straightforward one being

$$\langle iH_0, iH_j, i1 | j = 1, 2, \ldots, m \rangle_{\text{Lie}} = u(\mathcal{H}).$$

Using the continuity of the exponential map in the strong topology [24] it is easy to see that this condition is sufficient for strong operator controllability of (7). Our conjecture is that it is not necessary, but counter examples are not known (their construction is subject of current research). Stronger types

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\textsuperscript{1}This means that the associated graph (which roughly speaking indicates where a transition from energy level $k$ to $l$ is possible) has to be connected, cf. [24].
of convergence can be achieved if all the Hamiltonians $H_j, j = 0,\ldots,m$ are even compact. Since the strong closure of the algebra $\mathcal{K}(\mathcal{H})$ of compact operators is $\mathcal{B}(\mathcal{H})$, it is clear that Eq. (5) is implied by
\[\langle iH_0, iH_j | j = 1, 2,\ldots,m \rangle_{\text{Lie}} = u(\mathcal{K}(\mathcal{H})) ,\]
where the closure is taken in the uniform (operator norm) topology. If the other implication also holds is still unclear, but unlikely. Note that a compact operator can be the strong limit of a sequence in $\mathcal{B}(\mathcal{H})$ without being the uniform limit.

Let us fix some final notations with regard to the GKS-equation: $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}^1(\mathcal{H})$ denote the spaces of all bounded and trace-class operators on $\mathcal{H}$, respectively. Thus $\mathbb{D}(\mathcal{H}) \subset \mathcal{B}^1(\mathcal{H})$ is precisely the set of all positive semi-definite (selfadjoint) trace-class operators with trace 1. Moreover, $\| \cdot \|_1$ stands for the trace norm on $\mathcal{B}^1(\mathcal{H})$ (see Appendix A for more detail on the trace class). To begin with, all Hamiltonians $H_0, H_j$ are assumed to be taken from $\mathcal{B}(\mathcal{H})$, while later $H_0$ may be any unbounded selfadjoint operator.

In this setting, the operator solutions of (1) are globally well-defined (with respect to $t \in \mathbb{R}$) for arbitrary piecewise continuous controls (even more irregular controls are admissible) and for each fixed $t \in \mathbb{R}^+$ the corresponding map is ultraweakly continuous (cf. footnote 4a) and cptp. In particular for constant controls they form uniformly continuous semigroups of ultraweakly continuous cptp-maps, [30, Thm. 1 & 2].

With these notions and notations and taking majorization from finite to infinite dimensions by way of sequence spaces as introduced by Gohberg and Markus [20] (see Sec. 2.1.), our main result for $\Gamma(t) = \gamma(t)\Gamma_V(t)$ reads:

**Theorem 1.** Given the Markovian control system $\Sigma_V$
\[
\dot{\rho}(t) = -i\left[H_0 + \sum_{j=1}^m u_j(t)H_j, \rho\right] - \gamma(t)\left(\frac{1}{2}(V^\dagger V\rho + \rho V^\dagger V) - V\rho V^\dagger\right),
\]
where

1. the drift $H_0$ is selfadjoint and the controls $H_1,\ldots,H_m$ are selfadjoint and bounded,
2. the Hamiltonian part $\Sigma_0$ is strongly (approximately) operator controllable in the sense of Def. 4,
3. the noise term $V \in \mathcal{K}(\mathcal{H}) \setminus \{0\}$ is compact, normal and switchable by $\gamma(t) \in \{0,1\}$.

Then the $\| \cdot \|_1$-closure of the reachable set of any initial state $\rho(0) = \rho_0 \in \mathbb{D}(\mathcal{H})$ under the system $\Sigma_V$ exhausts all states majorized by the initial state $\rho_0$
\[
\overline{\text{reach}_{\Sigma_V}(\rho_0)} = \{ \rho \in \mathbb{D}(\mathcal{H}) | \rho \prec \rho_0 \}.
\]
In order to arrive at this result, the paper is organised as follows: Section 2 first takes majorization from finite to infinite dimensions in 2.1 before combining ideas of von Neumann with $C$-numerical ranges for majorization in infinite dimensions 2.2. Section 3 then presents the idea of the main theorem, the proof details themselves being relegated to the Appendix. Appendix A contains technical basics, while Appendix B gives the proofs to Section 2.2. Finally Appendix C provides the proof of the main theorem for bounded $H_0$, while Appendix D relaxes it to unbounded $H_0$.

2. From Majorization via $C$-Numerical Range to Reachability

2.1. Majorization in Finite and Infinite Dimensions

Generalizing majorization to infinite dimensions is somewhat delicate. Following [20], one may define majorization first on the space of all real null sequences $c_0(\mathbb{N})$ and then on the space of all absolutely summable sequences $\ell^1(\mathbb{N})$. As we need a concept of majorization on density operators, for our purposes it suffices to introduce majorization solely on the summable sequences of non-negative numbers $\ell^1_+(\mathbb{N})$, which is rather intuitive.

In the notation of [2, 31], take a real vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and let $x^\downarrow = (x^\downarrow_1, \ldots, x^\downarrow_n)^T$ denote its decreasing re-arrangement $x^\downarrow_1 \geq x^\downarrow_2 \geq \ldots \geq x^\downarrow_n$. For two vectors $x, y \in \mathbb{R}^n$, we say $x$ is majorized by $y$ (written $x \prec y$) if

$$\sum_{j=1}^k x^\downarrow_j \leq \sum_{j=1}^k y^\downarrow_j \text{ for all } k \in \mathbb{N},$$

and

$$\sum_{j=1}^\infty x_j = \sum_{j=1}^\infty y_j.$$ 

By definition $x \prec y$ depends only on the entries of $x$ and $y$ but not on their initial arrangement, so $\prec$ is permutation invariant.

Now for sequences $x \in \ell^1_+(\mathbb{N})$, this re-arrangement procedure works just the same way, and all the non-zero entries of $x$ are again contained within the rearranged sequence $x^\downarrow$. However, be aware that $x$ and $x^\downarrow$ may differ in the number of their zero entries.

Definition 2. Consider $x, y \in \ell^1_+(\mathbb{N})$ and $\rho, \omega \in \mathbb{D}(\mathcal{H})$.

(a) We say that $x$ is majorized by $y$, denoted by $x \prec y$, if the sum inequalities $\sum_{j=1}^k x^\downarrow_j \leq \sum_{j=1}^k y^\downarrow_j$ hold for all $k \in \mathbb{N}$, and $\sum_{j=1}^\infty x_j = \sum_{j=1}^\infty y_j$.

(b) $\omega$ majorizes $\rho$, denoted by $\rho \prec \omega$, if $\lambda^\downarrow(\rho) \prec \lambda^\downarrow(\omega)$ where $\lambda^\downarrow(\cdot) \in \ell^1_+(\mathbb{N})$ denotes the (non-modified) eigenvalue sequence\(^{2a}\) of the respective state.

\(^{2a}\) Usually, the eigenvalue sequence $\lambda^\downarrow(T)$ of a compact operator $T$ on $\mathcal{H}$ is obtained by arranging its non-zero eigenvalues in the decreasing order of their magnitudes and each eigenvalue is repeated as many times as its (necessarily finite) algebraic multiplicity. If the spectrum of $T$ is finite itself, then the sequence is filled with zeros, cf. [32, Ch. 15]. However, in order to get the result of Lemma 8 with respect to an orthonormal basis (and not just an orthonormal system), and also to properly define the $C$-spectrum of $T$ later on,
Remark 1. In Definition 2 (b) it does not matter whether one considers the usual (non-modified) or the modified eigenvalue sequence (for the purpose of this remark denoted by $\lambda^\downarrow$ and $\lambda_m^\downarrow$ respectively). More precisely, these sequences by construction share the same non-zero entries so $\lambda^\downarrow = \lambda_m^\downarrow$.

As in finite dimensions, majorization in infinite dimensions has a number of different characterizations, the following two being particularly advantageous for our purposes.

Lemma 1 ([29], Thm. 3.3). For $\rho, \omega \in \mathbb{D}(\mathcal{H})$ the following are equivalent:

(a) $\rho \prec \omega$.
(b) There exists a bi-stochastic quantum map $T \in \mathcal{S}(\mathcal{H})$ (cf. Def. 2 in Appendix A) such that $T(\omega) = \rho$.

Proposition 1. Let $x, y \in \ell_1^+ \mathbb{N}$ be non-increasing and let $(e_n)_{n \in \mathbb{N}}$ be some orthonormal basis of $\mathcal{H}$. Then the following statements are equivalent:

(a) $x \prec y$.
(b) There exists a selfadjoint $A \in \mathcal{B}^1(\mathcal{H})$ with diagonal entries $(x_n)_{n \in \mathbb{N}}$ and eigenvalues $(y_n)_{n \in \mathbb{N}}$.
(c) There exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U \text{diag}(y)U^\dagger$ has diagonal entries $(x_n)_{n \in \mathbb{N}}$.

Here, “diagonal” always refers to the orthonormal basis $(e_n)_{n \in \mathbb{N}}$, i.e. the map $\text{diag} : \ell_1^+ \mathbb{N} \to \mathcal{B}^1(\mathcal{H})$ is given by $x \mapsto \sum_{n=1}^{\infty} x_n (e_n, \cdot e_n)$.

Proof. “(a) $\Rightarrow$ (b)”: Assume $x \prec y$. By [20] Prop. IV there exist orthonormal bases $(\phi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that $H = \sum_{n=1}^{\infty} y_n (\psi_n, \cdot \psi_n) \phi_n$ satisfies $\langle \phi_n, H\phi_n \rangle = x_n$ for all $n \in \mathbb{N}$. Consider the unitary operator $U \in \mathcal{B}(\mathcal{H})$ which transforms $(\phi_n)_{n \in \mathbb{N}}$ into $(e_n)_{n \in \mathbb{N}}$, then $A := UHU^\dagger \in \mathcal{B}^1(\mathcal{H})$ does the job.

“(b) $\Rightarrow$ (a)” follows from [19]. “(b) $\Leftrightarrow$ (c)”: The statement is obvious.

We conclude with a classical result on sub-majorization (without proof) which will be needed in the following subsection.

Lemma 2 ([31], 3.H.3.b). Let $x, y \in \mathbb{R}^n$ such that $\sum_{j=1}^{k} x_j^+ \leq \sum_{j=1}^{k} y_j^+$ for all $k = 1, \ldots, n$. Then for arbitrary $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$ one has $\sum_{j=1}^{n} c_j x_j^+ \leq \sum_{j=1}^{n} c_j y_j^+$.

If the range of $T$ is infinite-dimensional and the kernel of $T$ finite-dimensional then put $\dim(\ker T)$ zeros at the beginning of the eigenvalue sequence of $T$. If the range and the kernel of $T$ are infinite-dimensional, mix infinitely many zeros into the eigenvalue sequence of $T$ (since for the $C$-spectrum arbitrary permutations will be applied to the modified eigenvalue sequence, we need not specify this mixing procedure further). If the range of $T$ is finite-dimensional, leave the eigenvalue sequence of $T$ unchanged.
2.2. Combining a von Neumann Idea with \(C\)-Numerical Ranges

In finite dimensions, Ando [2, Thm. 7.4] has shown that majorization can be characterized in an elegant way via the \(C\)-numerical range [28] [11]

\[
W_C(T) := \{ \text{tr}(CU^\dagger TU) \mid U \in U(\mathcal{H}) \}
\]

with \(C \in \mathcal{B}^1(\mathcal{H}), T \in \mathcal{B}(\mathcal{H}) \) and \(U(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \) being the unitary group on \(\mathcal{H}\). Here, we generalize his approach to infinite dimensions (Prop. 2 below) using a recent result in [15]. Later on, this characterization will greatly simplify handling continuity properties of majorization (cf. Lemma 5).

For our purpose, we need a relation connecting the \(C\)-numerical range and \(C\)-spectrum of a compact operator \(T\) given by

\[
P_C(T) := \left\{ \sum_{j=1}^{\infty} \lambda_j(C)\lambda_{\sigma(j)}(T) \mid \sigma : \mathbb{N} \to \mathbb{N} \text{ is a permutation} \right\}
\]

on one hand-side with \((\lambda_j(C))_{j \in \mathbb{N}}\) and \((\lambda_j(T))_{j \in \mathbb{N}}\) being the modified eigenvalue sequences (cf. footnote 2a) of \(C\) and \(T\) on the other. Note that each element in \(W_C(T)\) and \(P_C(T)\) is bounded by \(\|C\|_1\|T\|_{\text{op}}\) —thus the closures of \(W_C(T)\) and \(P_C(T)\) constitute compact subsets of \(\mathbb{C}\).

If \(C, T\) are normal, one has the inclusion \(\overline{W_C(T)} \subseteq \text{conv}(\overline{P_C(T)})\). Yet under further assumptions on the operators one can even achieve equality.

**Lemma 3** ([15], Coro. 3.1). Let \(C \in \mathcal{B}^1(\mathcal{H})\) and \(T \in \mathcal{K}(\mathcal{H})\) both be normal, such that the eigenvalues of \(C\) are collinear, i.e. the eigenvalues all lie on a common line. Then \(\overline{W_C(T)} = \text{conv}(\overline{P_C(T)})\).

In fact, Lemma 3 induces a von Neumann-type of trace (in-)equality [15] for compact, selfadjoint operators. Its proof is in Appendix B.

**Corollary 1.** Let \(C \in \mathcal{B}^1(\mathcal{H})\) and \(T \in \mathcal{K}(\mathcal{H})\) both be selfadjoint. Then

\[
\sup_{U \in U(\mathcal{H})} \text{tr}(CU^\dagger TU) = \sum_{j=1}^{\infty} \lambda_j^+(C)\lambda_j^+(T^+) + \sum_{j=1}^{\infty} \lambda_j^-(C)\lambda_j^-(T^-),
\]

where \(\lambda_j^+(C), \lambda_j^+(T^+), \lambda_j^-(C), \lambda_j^-(T^-)\) denote the decreasing eigenvalue sequences of the positive semi-definite operators \(C^+, T^+\) and \(C^-, T^-,\) respectively, where \(C = C^+ - C^-\) and \(T = T^+ - T^-\) as usual.

To simplify notation, we use the following abbreviation.

**Definition 3.** Let \(C \in \mathcal{B}^1(\mathcal{H}), T \in \mathcal{B}(\mathcal{H})\) both be selfadjoint. We define \(K_C(T) := \sup_{U \in U(\mathcal{H})} \text{tr}(CU^\dagger TU) \in \mathbb{R}\) or, equivalently, \(K_C(T) := \sup W_C(T) = \max W_C(T)\).
Note that if $C$ and $T$ are positive semi-definite, then $K_C(T)$ turns into the $C$-numerical radius $r_C(T)$ of $T$. Now this definition gives rise to the following result, whose finite-dimensional analogue can be found, e.g., in [2, Thm. 7.4].

**Proposition 2.** For $\rho, \omega \in \mathbb{D}(\mathcal{H})$ the following statements are equivalent.

(a) $\rho \prec \omega$

(b) $K_\rho(T) \leq K_\omega(T)$ for all selfadjoint $T \in \mathcal{K}(\mathcal{H})$.

**Proof.** “(a) ⇒ (b)”: Keeping in mind that $\rho, \omega \geq 0$, Coro. I yields

$$K_\rho(T) = \max W_\rho(T) = \sum_{j=1}^{\infty} \lambda_1^j(\rho) \lambda_1^j(T^+)$$

and similarly for $K_\omega(T)$. Moreover, Lemma 2 implies

$$\sum_{j=1}^{n} \lambda_1^j(\rho) \lambda_1^j(T^+) \leq \sum_{j=1}^{n} \lambda_1^j(\omega) \lambda_1^j(T^+)$$

for all $n \in \mathbb{N}$ and thus it follows $K_\rho(T) \leq K_\omega(T)$ for all selfadjoint $T \in \mathcal{K}(\mathcal{H})$.

“(b) ⇒ (a)”: Let $k \in \mathbb{N}$ and let $(e_n)_{n \in \mathbb{N}}$ be any orthonormal basis of $\mathcal{H}$. Consider the (finite-rank) projection $\Pi_k = \sum_{j=1}^{k} \langle e_j, \cdot \rangle e_j$. As $\Pi_k$ is compact and selfadjoint with eigenvalues 1 (of multiplicity $k$) and 0 (of infinite multiplicity), Coro. I yields $K_\rho(\Pi_k) = \sum_{j=1}^{k} \lambda_1^j(\rho)$ and $K_\omega(\Pi_k) = \sum_{j=1}^{k} \lambda_1^j(\omega)$. Now by assumption, one has

$$\sum_{j=1}^{k} \lambda_1^j(\rho) = K_\rho(\Pi_k) \leq K_\omega(\Pi_k) = \sum_{j=1}^{k} \lambda_1^j(\omega)$$

for all $k \in \mathbb{N}$ which shows $\rho \prec \omega$ and thus concludes this proof.

3. Idea behind the Main Result

Below we sketch the proof of our main result Thm. I. A full proof will be given in Appendix C. Here, we sketch central ideas and key lemmas, the proofs of which are either straightforward or postponed to Appendices C and D. For convenience, let us first recall the precise statement of Thm. I.

**Theorem**. Given the Markovian control system $\Sigma_V$

$$\dot{\rho}(t) = -i \left[ H_0 + \sum_{j=1}^{m} u_j(t) H_j, \rho \right] - \gamma(t) \left( \frac{1}{2} (V^\dagger V \rho + \rho V^\dagger V) - V \rho V^\dagger \right),$$

where

(1) the drift $H_0$ is selfadjoint and the controls $H_1, \ldots, H_m$ are selfadjoint and bounded,
(2) the Hamiltonian part $\Sigma_0$ is strongly (approximately) operator controllable in the sense of Def. \[1\].

(3) the noise term $V \in \mathcal{K}(\mathcal{H}) \setminus \{0\}$ is compact, normal and switchable by $\gamma(t) \in \{0, 1\}$.

Then the $\|\cdot\|_1$-closure of the reachable set of any initial state $\rho(0) = \rho_0 \in \mathbb{D}(\mathcal{H})$ under the system $\Sigma_V$ exhausts all states majorized by the initial state $\rho_0$

$$\text{reach}_{\Sigma_V}(\rho_0) = \{\rho \in \mathbb{D}(\mathcal{H}) \mid \rho \prec \rho_0\}.$$ 

The following lemmas play a crucial role in the proof of Theorem \[1\]. The first one reveals a beautiful eigenspace structure of the noise generators $\Gamma_V$ whenever $V$ is normal and compact, and it follows by direct computation.

**Lemma 4.** Let $V \in \mathcal{K}(\mathcal{H})$ be normal, $(f_j)_{j \in \mathbb{N}}$ its orthonormal eigenbasis and $(\nu_j)_{j \in \mathbb{N}}$ its modified eigenvalue sequence, hence $V = \sum_{j=1}^{\infty} \nu_j \langle f_j, \cdot \rangle f_j$ (cf. Appendix \[A\]). Then for all $X \in \mathcal{B}(\mathcal{H})$, the noise operator $\Gamma_V$ given by Eq. (4) acts like

$$\langle f_j, \Gamma_V(X)f_k \rangle = \left( \frac{1}{2} |\nu_j - \nu_k|^2 - i \text{Im}(\nu_j \overline{\nu_k}) \right) \langle f_j, Xf_k \rangle$$

for all $j, k \in \mathbb{N}$. In particular, each rank-1 operator of the form $\langle f_k, \cdot \rangle f_j$ is an eigenvector of $\Gamma_V$ to the eigenvalue $\frac{1}{2} |\nu_j - \nu_k|^2 - i \text{Im}(\nu_j \overline{\nu_k})$ and the kernel of $\Gamma_V$ contains span$\{(f_j, \cdot)f_j \mid j \in \mathbb{N}\}$. Moreover, it follows

$$\exp(-t \Gamma_V)(\langle f_k, \cdot \rangle f_j) = \exp\left(-\frac{t}{2} |\nu_j - \nu_k|^2 \right) \exp(it \text{Im}(\nu_j \overline{\nu_k})) \langle f_k, \cdot \rangle f_j$$

for all $t \in \mathbb{R}$ and $j, k \in \mathbb{N}$.

The following lemmas provide two crucial approximation results.

**Lemma 5.** For all $\rho_0 \in \mathbb{D}(\mathcal{H})$ the set $\{\rho \in \mathbb{D}(\mathcal{H}) \mid \rho \prec \rho_0\}$ is closed w.r.t. the trace norm $\|\cdot\|_1$.

**Lemma 6** (Unitary channel approximation). Consider a subset $R \subseteq \mathcal{U}(\mathcal{H})$ of the unitary group of $\mathcal{H}$ such that $\overline{R} = \mathcal{U}(\mathcal{H})$, i.e. its strong closure relative to $\mathcal{U}(\mathcal{H})$ yields the full group. Furthermore, let $\rho \in \mathbb{D}(\mathcal{H})$ and $U \in \mathcal{U}(\mathcal{H})$. Then for all $\varepsilon > 0$ one can find $\tilde{U} \in R$ such that $\|U \rho U^\dagger - \tilde{U} \rho \tilde{U}^\dagger\|_1 < \varepsilon$.

Now in our control setting we do not have direct access to the “pure” noise generator $-\Gamma_V$. However, we may use the Lie-Trotter product formula (cf. [36] Thm. VIII.29) to approximate the noise dynamics $\exp(-t \Gamma_V)_{t \in \mathbb{R}^+}$:
Lemma 7 (Trotter trick). For $u_1(t) = \ldots = u_m(t) = 0$ and $\gamma(t) = 1$ (i.e. noise only) the operator solution of (2) reads $(\exp(-itH_0 - t\Gamma_V))_{t \in \mathbb{R}_0^+}$. Then for $t \geq 0$ (uniformly on bounded intervals) one has
\[
\lim_{n \to \infty} \left\| \left( \exp \left( \frac{itH_0}{n} \right) \exp \left( -\frac{itH_0 - t\Gamma_V}{n} \right) \right)^n - \exp(-t\Gamma_V) \right\|_{op} = 0.
\]
Thus given a time $t \geq 0$ and precision $\varepsilon > 0$, to “simulate” $\exp(-t\Gamma_V)$ within this precision it suffices to apply the noisy evolution as well as the unitary channel $\exp(itH_0/N)$ to the system—in an alternating manner, $N$ times (for sufficiently large $N \in \mathbb{N}$). — Now we are ready to outline the proof of Thm. 1.

Sketch of the proof of Theorem 1. “⊆”: As $V$ is assumed to be normal one has $(iH(t) + \gamma(t)\Gamma_V)(1) = 0$ at all times so the operator solution of Eq. (2) is in $\mathcal{S}(\mathcal{H})$, i.e. a bi-stochastic quantum map and one can never leave the set of states majorized by $\rho_0$ (cf. Lemma 1). By Lemma 5 the $\parallel \cdot \parallel_1$-closure yields
\[
\overline{\text{reach}_{\Sigma_V}(\rho_0)} \subseteq \overline{\{ \rho \in \mathcal{D}(\mathcal{H}) \mid \rho < \rho_0 \}} = \{ \rho \in \mathcal{D}(\mathcal{H}) \mid \rho < \rho_0 \}.
\]
“⊇”: As $V \in \mathcal{K}(\mathcal{H})$ is normal we can diagonalize it (see Appendix A) with orthonormal eigenbasis $(e_j)_{j \in \mathbb{N}}$. Now, let $\varepsilon > 0$ and $\rho \in \mathcal{D}(\mathcal{H})$ with $\rho < \rho_0$ be given. We have to find $\rho_F \in \text{reach}_{\Sigma_V}(\rho_0)$ such that $\| \rho - \rho_F \|_1 < \varepsilon$. By assumption there exist $x,y \in \ell^1_+(\mathbb{N})$, $x,y \neq 0$ as well as $W_1,W_2 \in \mathcal{U}(\mathcal{H})$ such that $\rho = W_1 \text{diag}(x)W_1^\dagger$, $\rho_0 = W_2 \text{diag}(y)W_2^\dagger$ with $x \prec y$. Here, diag refers to the above eigenbasis of $V$. Applying Prop. 1 to $x,y$ gives us unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U \text{diag}(y)U^\dagger$ has diagonal entries $(x_n)_{n \in \mathbb{N}}$. The proof roughly consists of three steps shown here:
\[
\rho_0 = W_2 \text{diag}(y)W_2^\dagger \xrightarrow{\text{Step } 1} U \text{diag}(y)U^\dagger \xrightarrow{\text{Step } 2} \text{diag}(x) \xrightarrow{\text{Step } 3} W_1 \text{diag}(x)W_1^\dagger = \rho. \tag{10}
\]
Step 1 and 3 merely apply a unitary channel; assuming strong operator controllability, we may use unitary channels giving the target state with arbitrary precision (cf. Lemma 6). Step 2 is about getting rid of all off-diagonal elements of $U \text{diag}(y)U^\dagger$ to reach $\text{diag}(x)$ by applying pure noise $\exp(-t\Gamma_V)$ in the limit $t \to \infty$ (cf. Lemma 1). As expected there are a few delicate issues:

- We have no access to pure noise, as in our setting we cannot switch off $H_0$. Yet by a Trotter-type argument we can approximate the desired noise with arbitrary precision, cf. Lemma 7 and Lemma 14.
- If the eigenvalues of $V$ are not pairwise different, there are some “matrix” elements left untouched by the noise as a consequence of (9). So one may need permutation channels (which in particular are unitary) to rearrange those elements into “spots” where the noise affects them.
As in Step 1 and 3 we have to approximate these permutation channels. Here we use the approximation property of the trace class (cf. Lemma 9), i.e., we invoke decoherence on a sufficiently large but finite “block” of the density operator so we only need finitely many permutations.

Applying Prop. 1 requires that \( \rho_0, \rho \) are unitarily diagonalized so that the original and the modified eigenvalue sequences of these states coincide (which either means the states are finite-rank or have trivial kernel)—else the zeros that have to be added for the modified eigenvalue sequence prevent this. In the latter case we can proceed to states \( \rho', \rho'_0 \) which satisfy the assumptions of Prop. 1 and which are close (in trace norm) to the original states, and execute the scheme of Eq. (10).

Altogether this is enough to perform the scheme suggested in Eq. (10) with arbitrary precision. So \( \rho \prec \rho_0 \) is in the \( \| \cdot \|_1 \)-closure of the reachable set. The full proof with all detail is in Appendices C and D.

### 4. Conclusions and Outlook

For the first time, here we have derived sufficient conditions under which a quantum dynamical system can actually reach (in the closure) all quantum states majorized by the respective initial state in an infinite dimensional quantum system following a controlled Markovian master equation. To this end, we have extended the standard unital GKSL master equation to an infinite dimensional bilinear control system \( \Sigma_V \) the unitary part of which has to be operator controllable and the dissipative part (generated by a single normal compact noise term \( V \)) has to be bang-bang switchable. This takes recent results on finite dimensional systems [4, 39] to infinite dimensions. — While the generalization from a single such \( V \) to several commuting compact noise terms \( V_k \) is obvious, a generalization beyond compact \( V \) seems challenging. One may also relax considerations to weak-* continuity of the semigroup, which goes beyond the standard GKSL-equation, as pursued by Carbone, Fagnola [9] and more recently by Siemon, Holevo and Werner [40].

For applying the results to broader classes of physical systems, one may think of further generalizations. The current setup restricts us to (possibly unbounded) system Hamiltonians \( H_0 \) with discrete spectrum such as bound systems where particles are trapped within an unbounded potential (e.g., harmonic oscillators). To look at more interesting setups where processes like ionization, tunneling and evaporation play a role, we have to use operators with continuous spectrum. However, in this area even coherent control is not understood well enough (if at all). Closing this gap is therefore an obvious (yet non-trivial!) next step.

Thus the spirit of Sudarshan still promises insightful results to come.
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Appendix A: Notation and Basics

For a comprehensive introduction to infinite-dimensional separable Hilbert spaces and Schatten-class operators we refer to, e.g., [3, 31, 36]. As we will encounter compact normal operators repeatedly, let us first recap the well-known diagonalization result.

Lemma 8 ([3], Thm. VIII.4.6). Let $T \in \mathcal{K}(\mathcal{H})$ be normal, i.e. $T^\dagger T = TT^\dagger$. Then there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that

$$T = \sum_{j=1}^{\infty} \tau_j \langle e_j, \cdot \rangle e_j$$

where $(\tau_j)_{j \in \mathbb{N}}$ is the modified eigenvalue sequence (cf. footnote 2a) of $T$.

Moreover, recall that the set of trace-class operators is given by

$$\mathcal{B}^1(\mathcal{H}) := \{ C \in \mathcal{B}(\mathcal{H}) \mid \|C\|_1 := \text{tr} (\sqrt{C^\dagger C}) < \infty \} \subseteq \mathcal{K}(\mathcal{H}),$$

which forms a Banach space under the trace norm $\| \cdot \|_1$ and constitutes a two-sided ideal in the $C^*$-algebra of all bounded operators $\mathcal{B}(\mathcal{H})$. Important properties of the trace are

$$\text{tr} (\langle (x, \cdot )y \rangle T) = \langle x, Ty \rangle \quad \text{and} \quad |\text{tr}(CT)| \leq \|C\|_1 \|T\|_{\text{op}} \quad (11)$$

for all $x, y \in \mathcal{H}$, $C \in \mathcal{B}^1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$. Furthermore, the trace class has the approximation property:

Lemma 9 ([15], Lemma 3.2). Let $C \in \mathcal{B}^1(\mathcal{H})$ and let $(e_n)_{n \in \mathbb{N}}$ be any orthonormal basis of $\mathcal{H}$. For arbitrary $k \in \mathbb{N}$, let $\Pi_k := \sum_{j=1}^{k} \langle e_j, \cdot \rangle e_j$ denote the orthogonal projection onto $\text{span}\{e_1, \ldots, e_k\}$. Then the sequence of “block approximations” $(\Pi_n C \Pi_n)_{n \in \mathbb{N}}$ converges (in trace norm) to $C$, i.e.

$$\lim_{n \to \infty} \|C - \Pi_n C \Pi_n\|_1 = 0.$$ 

Definition 4. (a) A linear map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G})$ is trace-preserving if $T(\mathcal{B}^1(\mathcal{H})) \subseteq \mathcal{B}^1(\mathcal{G})$ with $\text{tr}(T(A)) = \text{tr}(A)$ for all $A \in \mathcal{B}^1(\mathcal{H})$. 
(b) A bi-stochastic quantum map is a linear, ultraweakly continuous, completely positive, unital (identity-preserving) and trace-preserving map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G})$. We define

$$S(\mathcal{H}, \mathcal{G}) := \{ T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G}) \mid T \text{ is a bi-stochastic quantum map} \}$$

and $S(\mathcal{H}) := S(\mathcal{H}, \mathcal{H})$.

Thus using the terminology of [44, Def. 2], a bi-stochastic quantum map is a Heisenberg quantum channel which also is trace-preserving and its restriction to the trace class is a Schrödinger quantum channel. Using [44, Prop. 2] this directly implies the following.

**Lemma 10.** Let $T \in S(\mathcal{H}, \mathcal{G})$ and consider the restricted (well-defined) map $T_{\mathcal{B}^1} : \mathcal{B}^1(\mathcal{H}) \to \mathcal{B}^1(\mathcal{G})$. Then $\|T\|_{\text{op}} = \|T_{\mathcal{B}^1}\|_{\text{op}} = 1$ so

$$\|T(B)\|_{\text{op}} \leq \|B\|_{\text{op}} \quad \text{and} \quad \|T(A)\|_1 \leq \|A\|_1$$

for all $B \in \mathcal{B}(\mathcal{H})$, $A \in \mathcal{B}^1(\mathcal{H})$.

**Appendix B: Proof of von Neumann Type of Trace Inequality**

The following von Neumann type of trace (in-)equality was used in Sec. 2.2.

**Corollary** [34]. Let $C \in \mathcal{B}^1(\mathcal{H})$ and $T \in \mathcal{K}(\mathcal{H})$ both be selfadjoint. Then

$$\sup_{U \in \mathcal{U}(\mathcal{H})} \text{tr}(CU^TU) = \sum_{j=1}^{\infty} \lambda_j^+(C^+)\lambda_j^+(T^+) + \sum_{j=1}^{\infty} \lambda_j^-(C^-)\lambda_j^-(T^-),$$

where $\lambda_j^+(C^+)$, $\lambda_j^+(T^+)$ and $\lambda_j^-(C^-)$, $\lambda_j^-(T^-)$ denote the decreasing eigenvalue sequences of the positive semi-definite operators $C^+$, $T^+$ and $C^-$, $T^-$, respectively, where $C = C^+ - C^-$ and $T = T^+ - T^-$ as usual.

Below, we provide a proof of the above statement, which to the best of our knowledge is new. To this end, we need the notion of set convergence using the Hausdorff metric on compact subsets (of $\mathbb{C}$) and the associated notion of convergence, see, e.g., [34]. The distance between $z \in \mathbb{C}$ and any non-empty compact subset $A \subseteq \mathbb{C}$ is defined by

$$d(z, A) := \min_{w \in A} d(z, w) = \min_{w \in A} |z - w| .$$

The ultraweak topology is the weak-* topology on $\mathcal{B}(\mathcal{H})$ inherited by the isometrically isomorphic map $\psi : \mathcal{B}(\mathcal{H}) \to (\mathcal{B}^1(\mathcal{H}))'$, $B \mapsto \text{tr}(B(\cdot))$. 

4a The ultraweak topology is the weak-* topology on $\mathcal{B}(\mathcal{H})$ inherited by the isometrically isomorphic map $\psi : \mathcal{B}(\mathcal{H}) \to (\mathcal{B}^1(\mathcal{H}))'$, $B \mapsto \text{tr}(B(\cdot))$. 

4b This ultraweak topology is the weak-* topology on $\mathcal{B}(\mathcal{H})$ inherited by the isometrically isomorphic map $\psi : \mathcal{B}(\mathcal{H}) \to (\mathcal{B}^1(\mathcal{H}))'$, $B \mapsto \text{tr}(B(\cdot))$. 

Based on (12) the Hausdorff metric $\Delta$ on the set of all non-empty compact subsets of $\mathbb{C}$ is given by

$$\Delta(A, B) := \max \left\{ \max_{z \in A} d(z, B), \max_{z \in B} d(z, A) \right\}.$$ 

The following characterization of the Hausdorff metric is readily verified.

**Lemma 11.** Let $A, B \subset \mathbb{C}$ be two non-empty compact sets and let $\varepsilon > 0$. Then $\Delta(A, B) \leq \varepsilon$ if and only if for all $z \in A$, there exists $w \in B$ with $d(z, w) \leq \varepsilon$ and vice versa.

With this metric one can introduce the notion of convergence for sequences $(A_n)_{n \in \mathbb{N}}$ of non-empty compact subsets of $\mathbb{C}$ such that the maximum-operator is continuous in the following sense.

**Lemma 12.** Let $(A_n)_{n \in \mathbb{N}}$ be a bounded sequence of non-empty, compact subsets of $\mathbb{R}$ which converges to $A \subset \mathbb{R}$. Then the sequence of real numbers $(\max A_n)_{n \in \mathbb{N}}$ is convergent with

$$\lim_{n \to \infty} (\max A_n) = \max \left( \lim_{n \to \infty} A_n \right) = \max A.$$

**Proof.** Let $\varepsilon > 0$. By assumption, there exists $N \in \mathbb{N}$ such that $\Delta(A_n, A) < \varepsilon$ for all $n \geq N$. Hence, by Lemma 11 there exists $a_n \in A_n$ such that $|\max A - a_n| < \varepsilon$ and thus

$$\max A < a_n + \varepsilon < \max A_n + \varepsilon.$$

Similarly, there exists $a \in A$ such that $|\max A_n - a| < \varepsilon$ and thus

$$\max A_n < a + \varepsilon < \max A + \varepsilon.$$

Combining both estimates, we get $|\max A - \max A_n| < \varepsilon$. \hfill $\square$

Just like [15] Thm. 3.1] one can show the following:

**Lemma 13.** Let $C \in \mathcal{B}^1(\mathcal{H}), T \in \mathcal{B}(\mathcal{H})$ and $(C_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{B}^1(\mathcal{H})$ which converges to $C$ w.r.t. $\| \cdot \|_1$. Then

$$\lim_{n \to \infty} W_{C_n}(T) = W_C(T).$$

Moreover, if $T$ is compact as well, then

$$\lim_{k \to \infty} W_C(\Pi_k T \Pi_k) = W_C(T),$$

where $\Pi_k$ is the orthogonal projection onto the span of the first $k$ elements of an arbitrary orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$. 

Proof of Corollary 1. Let $C \in B^{1}(\mathcal{H})$ and $T \in \mathcal{K}(\mathcal{H})$ both be selfadjoint and let us first assume that $T$ has at most $k \in \mathbb{N}$ non-zero eigenvalues. Then

$$\max_{\mathcal{U} \subseteq \mathcal{H}} \text{tr}(CU^\dagger TU) = \max_{j=1}^{\infty} W_{\omega}(T) = \max_{j=1}^{\infty} W_{\omega}(\Pi_{k}T\Pi_{k})$$

where we used the identity $(\Pi_{k}T\Pi_{k})^{\pm} = \Pi_{k}T^{\pm}\Pi_{k}$.

On the other hand, due to Prop. 2 and $\rho_{n} < \omega$ for all $n \in \mathbb{N}$, one has

$$K_{\omega}(T) = \lim_{n \to \infty} K_{\rho_{n}}(T) \leq K_{\rho_{0}}(T)$$

which again by Prop. 2 (as $T$ was chosen arbitrarily) implies $\omega < \rho_{0}$. \hfill $\square$
Lemma 6 (Unitary channel approximation). Consider a subset $R \subseteq \mathcal{U}(H)$ of the unitary group on $\mathcal{H}$ such that $\overline{R} = \mathcal{U}(\mathcal{H})$, i.e. its strong closure relative to $\mathcal{U}(\mathcal{H})$ yields the full group. Furthermore let $\rho \in \mathbb{D}(\mathcal{H})$ and $U \in \mathcal{U}(\mathcal{H})$. Then for all $\varepsilon > 0$ one can find $\tilde{U} \in R$ such that $\|U\rho U^\dagger - \tilde{U}\rho \tilde{U}^\dagger\|_1 < \varepsilon$.

Proof. Due to Lemma 8 there exists a modified eigenvalue sequence $(r_n)_{n \in \mathbb{N}} \in \ell_1^+(\mathbb{N})$ and an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that $\rho = \sum_{j=1}^{\infty} r_j \langle e_j, \cdot \rangle e_j$. Then one also finds $N \in \mathbb{N}$ such that the “tail” of $\rho$ is sufficiently small, i.e.

$$\sum_{j=N+1}^{\infty} r_j < \frac{\varepsilon}{6} \quad \text{and} \quad \sum_{j=1}^{N} r_j > 0. \quad (14)$$

By assumption there is $\tilde{U} \in R \subseteq \mathcal{U}(\mathcal{H})$ such that

$$\|U e_j - \tilde{U} e_j\|_H < \varepsilon / (6 \sum_{j=1}^{N} r_j) \quad \text{for all} \ j = 1, \ldots, N. \quad (15)$$

Moreover, the triangle inequality, non-negativity of $r_n$ and the trace norm identity $\|\langle x, \cdot \rangle y\|_1 = \|x\| \|y\|$ for all $x, y \in \mathcal{H}$ imply

$$\|U \rho U^\dagger - \tilde{U} \rho \tilde{U}^\dagger\|_1 \leq \|U \rho U^\dagger - U \rho \tilde{U}^\dagger\|_1 + \|U \rho \tilde{U}^\dagger - \tilde{U} \rho \tilde{U}^\dagger\|_1$$

$$= \left\| \sum_{j=1}^{\infty} r_j \langle U e_j - \tilde{U} e_j, \cdot \rangle U e_j \right\|_1 + \left\| \sum_{j=1}^{\infty} r_j \langle \tilde{U} e_j, \cdot \rangle (U e_j - \tilde{U} e_j) \right\|_1$$

$$\leq \sum_{j=1}^{\infty} r_j \|U e_j - \tilde{U} e_j\| \left( \|U e_j\| + \|\tilde{U} e_j\| \right).$$

Splitting the sum at $N$ and using Eq. (14) and (15) finally yields the estimate

$$\|U \rho U^\dagger - \tilde{U} \rho \tilde{U}^\dagger\|_1 \leq 2 \sum_{j=1}^{N} r_j \|U e_j - \tilde{U} e_j\| + 4 \sum_{j=N+1}^{\infty} r_j < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square$$

Next, let us refine Lemma 7 in terms of precision as follows:

Lemma 14. Let $V \in \mathcal{K}(\mathcal{H}) \setminus \{0\}$ be normal, $H_0 \in \mathcal{B}(\mathcal{H})$ be selfadjoint, $\rho \in \mathbb{D}(\mathcal{H})$ be arbitrary and $[0, T] \subset \mathbb{R}_0^+$ be given. Furthermore let $R \subseteq \mathcal{U}(\mathcal{H})$ with $\overline{R} = \mathcal{U}(\mathcal{H})$, where the closure is taken in $\mathcal{U}(\mathcal{H})$. Then for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ and $U_1, \ldots, U_m \in R$ such that for all $s \in [0, T]$

$$\left\| \exp(-s\Gamma_V)(\rho) - \prod_{j=1}^{m} \left( \text{Ad}_{U_j} \circ \exp \left( \frac{-isH_0 - s\Gamma_V}{m} \right) \right)(\rho) \right\|_1 < \varepsilon$$

Proof. By Lemma 7 there exists $m \in \mathbb{N}$ with

$$\left\| \exp(-s\Gamma_V) - \left( \exp \left( \frac{isH_0}{m} \right) \circ \exp \left( \frac{-isH_0 - s\Gamma_V}{m} \right) \right)^m \right\|_{\text{op}} < \frac{\varepsilon}{2}, \quad (16)$$
for all $s \in [0,T]$, where $F$ is a unitary channel and $G$ is unital (because $V$ is normal) and reflects the operator solution of (2) with $u_1(s) = \cdots = u_m(s) = 0$ and $\gamma(s) = 1$, i.e. the noisy but uncontrolled evolution of the system.

For convenience define $\rho_j := (G \circ (F \circ G)^{m-j})(\rho)$ for $j = 1, \ldots, m$. Then, Lemma 6 yields $U_j \in R \subseteq U(\mathcal{H})$ with $\|F(\rho_j) - U_j \rho_j U_j^\dagger\|_1 < \frac{\varepsilon}{2m}$ for all $s \in [0,T]$. Finally, Lemma 11 and Lemma 15 (below) imply

$$\|\exp(-s \Gamma_V)(\rho) - \prod_{j=1}^m (\text{Ad}_{U_j} \circ g)(\rho)\|_1$$

$$\leq \|\exp(-s \Gamma_V) - (F \circ G)^m\|_{op} \|\rho\|_1 + \|(F \circ G)^m(\rho) - \prod_{j=1}^m (\text{Ad}_{U_j} \circ G)(\rho)\|_1$$

$$< \frac{\varepsilon}{2} + \sum_{j=1}^m \left( \prod_{k=1}^{j-1} (\text{Ad}_{U_k} \circ G) \circ (F - \text{Ad}_{U_j}) \circ G \circ (F \circ G)^{m-j}(\rho) \right)_{\rho_j}$$

$$\leq \frac{\varepsilon}{2} + \sum_{j=1}^m \left( \prod_{k=1}^{j-1} \|\text{Ad}_{U_k}\|_{op} \|G\|_{op} \right) \|F(\rho_j) - U_j \rho_j U_j^\dagger\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for all $s \in [0,T]$.

A simple and readily verified induction argument shows:

**Lemma 15.** Let $m \in \mathbb{N}$ and $A_1, \ldots, A_m$, $B_1, \ldots, B_m : D \to D$ be arbitrary maps acting on some common domain $D$ be given. Then

$$\prod_{j=1}^m A_j - \prod_{j=1}^m B_j = \sum_{j=1}^m \left( \prod_{k=1}^{j-1} A_k \circ (A_j - B_j) \circ \prod_{k=j+1}^m B_j \right).$$

Here and henceforth, the order of the “product” $\prod_{j=1}^m A_j$ shall be fixed by $A_1 \circ \cdots \circ A_m$.

**Proof of Theorem 1.**

“$\subseteq$”: Obviously, $\rho_0 \in \{\rho \in \mathbb{D}(\mathcal{H}) \mid \rho \prec \rho_0\}$ and by assumption of $V$ being normal, $\Gamma(1) = -VV^\dagger + \frac{1}{2}(V^\dagger V + V^4) = 0$. Thus the operator solution of (2) remains in $\mathbb{S}(\mathcal{H})$ for $t \geq 0$, and by Lemma 11 the set $\{\rho \in \mathbb{D}(\mathcal{H}) \mid \rho \prec \rho_0\}$ is

---

Note that $\rho_j = \rho_j(s)$ depends on $s \in [0,T]$ as $F$ and $G$ do so. Moreover, the set $\{\rho_j(s) \mid s \in [0,T]\}$ is compact as $F$ and $G$ are continuous in $s$ and hence the proof of Lemma 6 can be easily modified to obtain the desired result.
forward invariant, i.e. solutions of the given control problem can never leave the set of states majorized by $\rho_0$. Taking the $\|\cdot\|_1$-closure by Lemma[3] yields

$$\text{reach}_{\Sigma_V}(\rho_0) \subseteq \{ \rho \in \mathbb{D}(\mathcal{H}) \mid \rho \prec \rho_0 \} = \{ \rho \in \mathbb{D}(\mathcal{H}) \mid \rho < \rho_0 \}.$$  

"$\supseteq$": As $V \in \mathcal{K}(\mathcal{H})$ is normal, by Lemma[5] there exists an orthonormal basis $(f_j)_{j \in \mathbb{N}}$ of $\mathcal{H}$ such that $V = \sum_{j=1}^{\infty} v_j \langle f_j, \cdot \rangle f_j$ with modified eigenvalue sequence $(v_j)_{j \in \mathbb{N}}$. Whenever we use the term “diagonal” or “diag” in the following, it always refers to $(f_j)_{j \in \mathbb{N}}$.

Let $\varepsilon > 0$ and $\rho \in \mathbb{D}(\mathcal{H})$ with $\rho < \rho_0$ be given. We now have to find $\rho_F \in \text{reach}_{\Sigma_V}(\rho_0)$ such that $\|\rho - \rho_F\|_1 < \varepsilon$. As seen before there exist $x, y \in \ell_1^1(\mathbb{N})$, $x, y \neq 0$ as well as unitary $W_1, W_2 \in \mathcal{B}(\mathcal{H})$ such that

$$\rho = W_1 \text{diag}(x)W_1^\dagger$$ and $$\rho_0 = W_2 \text{diag}(y)W_2^\dagger$$

with $x < y$ (so $x$ and $y$ denote the modified eigenvalue sequence of $\rho$ and $\rho_0$, respectively, see also Remark[1]).

First assume that the original and the modified eigenvalue sequence of $\rho$ as well as $\rho_0$ coincide, i.e. $x = x^\dagger$, $y = y^\dagger$ from the start (necessary to apply Prop.[1]). The subsequent steps of the proof were sketched in the main text on page[12] where Step 1 & 3 are the mere application of a suitable unitary channel whereas Step 2 is about (approximately) getting rid of all “off-diagonal” elements $\langle f_j, U W_2^\dagger \rho_0 W_2 U^\dagger f_k \rangle$ of $U W_2^\dagger \rho_0 W_2 U^\dagger = U \text{diag}(y)U^\dagger$.

**Step 1:** By assumption $\Sigma_0$ is strongly operator controllable so we have the unitary orbit of $\rho_0$ in the closure of $\text{reach}_{\Sigma_V}(\rho_0)$. Although we may not have access to $\text{Ad}_{U W_2^\dagger} = U W_2^\dagger(\cdot) W_2 U^\dagger$ directly, by Lemma[6] we find $\tilde{U} \in \mathcal{B}(\mathcal{H})$ unitary such that $\tilde{U} \rho_0 \tilde{U}^\dagger \in \text{reach}_{\Sigma_V}(\rho_0)$ with

$$\|U \text{diag}(y)U^\dagger - \tilde{U} \rho_0 \tilde{U}^\dagger\|_1 = \|U W_2^\dagger \rho_0 W_2 U^\dagger - \tilde{U} \rho_0 \tilde{U}^\dagger\|_1 < \varepsilon/3.$$  

**Step 2:** By Lemma[4] the pure noise generator $\Gamma_V$ acts like

$$|\langle f_j, \exp(-it\Gamma_V)(X)f_k \rangle| = |\exp \left( - \frac{t |v_j - v_k|^2}{2} \right) \exp(it \text{Im}(v_j \overline{v_k})) \langle f_j, X f_k \rangle|$$

$$= \exp \left( - \frac{t |v_j - v_k|^2}{2} \right) |\langle f_j, X f_k \rangle| \leq |\langle f_j, X f_k \rangle|$$

on arbitrary $X \in \mathcal{B}(\mathcal{H})$ for all $j, k \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. Evidently,

$$\lim_{t \to \infty} \langle f_j, \exp(-it\Gamma_V)(X)f_k \rangle = \begin{cases} 0 & \text{if } v_j \neq v_k, \\ \langle f_j, X f_k \rangle & \text{else}. \end{cases}$$
If we assume \( v_j \neq v_k \) for all \( j \neq k \), all the off-diagonal terms of \( X \) vanish in the limit \( t \to \infty \) and one is left with \( \sum_{j=1}^{\infty} \langle f_j, X f_j \rangle \langle f_j, \cdot \rangle f_j =: P(X) \). Note that this projection map has Kraus operators \((\langle f_j, \cdot \rangle f_j)_{j=1}^{\infty} \) so \( P \in S\mathcal{H} \).

Since we want to approximate a density operator in the trace norm, we only have to care about a sufficiently large upper left block of the matrix representation \((\langle f_j, X f_k \rangle)_{j,k \in \mathbb{N}} \) as the rest is “already small” in the trace norm. More formally, by Lemma 9 there exists \( N_1 \in \mathbb{N} \) such that

\[
\| \tilde{U} \rho_0 \tilde{U}^\dagger - \Pi_n \tilde{U} \rho_0 \tilde{U}^\dagger \Pi_n \|_1 < \frac{\varepsilon}{24}
\]

for all \( n \geq N_1 \), where \( \Pi_n := \sum_{j=1}^{n} \langle f_j, \cdot \rangle f_j \) for all \( n \in \mathbb{N} \).

Of course, there is no reason for the eigenvalues of \( V \) to be pairwise different. Therefore we have to make sure that the upper left block is large enough such that it corresponds to at least two different eigenvalues of \( V \)—thus we have access to partial decoherence, which we then may spread anywhere needed via permutation channels.

Due to \( V \neq 0 \) and \( v_j \to 0 \) as \( j \to \infty \) (compactness of \( V \)), there exists \( M \in \mathbb{N} \) such that \( v_1 \neq v_M \). On the other hand \((\ref{eq:19})\) still holds so we define \( N := \max\{N_1, M\} \). Then, by construction and \((\ref{eq:18})\), we know that \( \langle f_1, X f_M \rangle \) (and \( \langle f_M, X f_1 \rangle \)) tend to zero when pure noise is applied.

Thus we find \( \alpha \in \mathbb{N}_0 \), \( \alpha \leq N(N-1)/2 \) (number of matrix elements above the diagonal), permutation operators \( \sigma_1, \ldots, \sigma_\alpha \in \mathcal{U}(\mathcal{H}) \) (in abuse of notation we write \( f_j \mapsto \sigma_l f_j = f_{\sigma_l^{-1}(j)} \)), yet the explicit form of \( \sigma_j \) is not that important) and relaxation times \( s_1, \ldots, s_\alpha \in \mathbb{R}_0^+ \) such that

- the permutations only operate non-trivially on the \( N \times N \)-block, i.e. for all \( l = 1, \ldots, \alpha \) and \( k > N \) one has \( \sigma_l f_k = f_k \).
- for every matrix element \( \langle f_k, f_j \rangle \) with \( j, k = 1, \ldots, N \), \( j \neq k \) there exists a permutation \( \sigma_l \) with \( 1 \leq l \leq \alpha \) such that \( \langle f_k, f_j \rangle \) sits in the “relaxation” spot (i.e. \( \langle f_1, f_M \rangle \) or \( \langle f_M, f_1 \rangle \)). More precisely,

\[
\| \text{Ad}_{\sigma_l} \circ \exp(-s_l \Gamma_V) \circ \text{Ad}_{\sigma_l} (\langle f_k, \cdot \rangle f_j) \|_1 \leq \frac{\varepsilon}{12N^2}.
\]

- after having successively applied all operations from \((\ref{eq:20})\), every matrix element \( \langle f_k, f_j \rangle \) is in its original spot because all \( \langle f_k, f_j \rangle \) are eigenvectors of \( \exp(-s_l \Gamma_V) \).
Now, using linearity of the involved maps, the estimate in question reads

\[
\|P(\tilde{U} \rho_0 \tilde{U}^\dagger) - \prod_{m=1}^{\alpha} (\text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\tilde{U} \rho_0 \tilde{U}^\dagger)\|_1 \leq \\
(\|P\|_{\text{op}} + \prod_{m=1}^{\alpha} \|\text{Ad}_{\sigma_m}\|_{2\text{op}} \|\exp(-s_m \Gamma_V)\|_{\text{op}})\|\tilde{U} \rho_0 \tilde{U}^\dagger - \Pi_N \tilde{U} \rho_0 \tilde{U}^\dagger \Pi_N\|_1 \\
+ \|\sum_{j,k=1}^{N} \langle f_j, \tilde{U} \rho_0 \tilde{U}^\dagger f_k \rangle (P - \prod_{m=1}^{\alpha} \text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\langle f_k, \cdot \rangle f_j)\|_1
\]

The first summand is smaller than \(2 \cdot \frac{\epsilon}{12} = \frac{\epsilon}{12}\) by Lemma 10 and (19). For the second one notice that

\[
(P - \prod_{m=1}^{\alpha} \text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\langle f_k, \cdot \rangle f_j) = 0
\]

for all \(j \in \mathbb{N}\). Now, \(P(\langle f_k, \cdot \rangle f_j) = 0\) whenever \(j \neq k\). Moreover,

\[
\|\prod_{m=1}^{\alpha} (\text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\langle f_k, \cdot \rangle f_j)\|_1 \leq \frac{\epsilon}{12N^2}.
\]

by (18) and (20). Putting together gives the estimate

\[
\|P(\tilde{U} \rho_0 \tilde{U}^\dagger) - \prod_{m=1}^{\alpha} (\text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\tilde{U} \rho_0 \tilde{U}^\dagger)\|_1 \\
\leq \frac{\epsilon}{12} + \sum_{j,k=1}^{N} \langle f_j, \tilde{U} \rho_0 \tilde{U}^\dagger f_k \rangle \|\prod_{m=1}^{\alpha} (\text{Ad}_{\sigma_m} \circ \exp(-s_m \Gamma_V) \circ \text{Ad}_{\sigma_m})(\langle f_k, \cdot \rangle f_j)\|_1 \\
\leq \frac{\epsilon}{12} + \sum_{j,k=1, j \neq k}^{N} \frac{\epsilon}{12N^2} \leq \frac{\epsilon}{6}.
\]

This leaves us with two problems:

1. We have to approximate all permutation channels.
2. We do not have access to pure noise \((\exp(-t \Gamma_V))_{t \in \mathbb{R}_0^+}\) within the given control problem.

For solving the first problem we exploit that we can strongly approximate every unitary channel. First, to simplify the upcoming computations, let
us assume w.l.o.g. that $\sigma_{\alpha}$ is the identity and let us introduce the notation $\pi_l := \sigma_l \circ \sigma_{l-1}^\dagger$ for $l \in \{2, \ldots, \alpha\}$ and $\pi_1 := \sigma_1$. Moreover, define

$$\omega_l := (\exp(-s_l \Gamma_V) \circ \prod_{m=l+1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ \exp(-s_m \Gamma_V)))(\check{U}\rho_0 \check{U}^\dagger) \in \mathbb{D}(\mathcal{H})$$

for every $l \in \{1, \ldots, \alpha\}$ as then by Lemma 3 we find $\tilde{\pi}_l \in \mathcal{U}(\mathcal{H})$ which we have access to within $\text{reach}_{\Sigma_0}$ (and thus $\text{reach}_{\Sigma_V}$) such that

$$\|\pi_l^\dagger \omega_l \pi_l - \pi_l^\dagger \omega_l \pi_l\|_1 < \frac{\varepsilon}{12\alpha}.$$ (21)

Then a telescope argument (cf. Lemma 15) yields the estimate

$$\left\|\left( \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ \exp(-s_m \Gamma_V)) - \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ \exp(-s_m \Gamma_V)) \right)(\check{U}\rho_0 \check{U}^\dagger)\right\|_1 \leq \sum_{m=1}^{\alpha} \left\|\left( \prod_{l=1}^{m-1} (\text{Ad}_{\tilde{\pi}_l^\dagger} \circ \exp(-s_l \Gamma_V)) \circ (\text{Ad}_{\tilde{\pi}_m^\dagger} - \text{Ad}_{\pi_m^\dagger})\right)(\omega_l)\right\|_1 \leq \sum_{m=1}^{\alpha} \left( \prod_{l=1}^{m-1} \|\text{Ad}_{\tilde{\pi}_l^\dagger} \|_{\text{op}} \|\exp(-s_l \Gamma_V)\|_{\text{op}} \right)\|\pi_l^\dagger \omega_l \pi_l - \pi_l^\dagger \omega_l \pi_l\|_1 < \frac{\varepsilon}{12}$$

where in the last step we once again used Lemma 10.

For the second problem we luckily may approximate the pure noise as precisely as needed using Lemma 14. For every $l = 1, \ldots, \alpha$ define

$$\rho_l := \prod_{m=l+1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ \exp(-s_m \Gamma_V))(\check{U}\rho_0 \check{U}^\dagger) \in \mathbb{D}(\mathcal{H}).$$

Then by Lemma 14 there exists a CPTP map $F_l$ which we have access to such that $\|\exp(-s_l \Gamma_V)(\rho_l) - F_l(\rho_l)\|_1 < \frac{\varepsilon}{12\alpha}$. Just as before

$$\left\|\left( \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ \exp(-s_m \Gamma_V)) - \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ F_m) \right)(\check{U}\rho_0 \check{U}^\dagger)\right\|_1 < \frac{\varepsilon}{12}.$$ (22)

**Step 3:** The current state $\check{\rho} := \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ F_m)(\check{U}\rho_0 \check{U}^\dagger)$ of the system is “close to diag($x$)” in the trace distance as we saw before. Now we want to apply the unitary channel generated by $W_1$ so again by Lemma 3 one finds unitary $\tilde{W} \in \mathcal{B}(\mathcal{H})$ such that $\|W_1 \check{\rho} W_1^\dagger - \tilde{W} \check{\rho} \tilde{W}^\dagger\|_1 < \frac{\varepsilon}{3}$. Then one has $\rho_F = \text{Ad}_{\tilde{W}^\dagger} \circ \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m^\dagger} \circ F_m)(\check{U}\rho_0 \check{U}^\dagger) \in \text{reach}_{\Sigma_V}(\rho_0)$ and by (17)

$$\|\rho - \rho_F\|_1 \leq \|W_1 P(UW_1^\dagger \check{\rho} W_2 U^\dagger) W_1^\dagger - W_1 P(\check{U}\rho_0 \check{U}^\dagger) W_1^\dagger\|_1 + \|W_1 P(\check{U}\rho_0 \check{U}^\dagger) W_1^\dagger - \check{W}_1 \check{\rho} \check{W}_1^\dagger\|_1 + \|\check{W}_1 \check{\rho} \check{W}_1^\dagger - \rho_F\|_1.$$
As all channels involved are in $S(\mathcal{H})$, by Lemma 10 we ultimately obtain
\[
\|\rho - \rho_F\|_1 \leq \|\text{Ad}_{W_1}\|_{\text{op}}\|P\|_{\text{op}}\|UW_2^\dagger \rho_0 W_2 U^\dagger - \tilde{U}\rho_0 \tilde{U}^\dagger\|_1 \\
+ \|\text{Ad}_{W_1}\|_{\text{op}}\|P(\tilde{U}\rho_0 \tilde{U}^\dagger) - \prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m} \circ \exp(-s_m \Gamma_V))(\tilde{U}\rho_0 \tilde{U}^\dagger)\|_1 \\
+ \|\text{Ad}_{W_1}\|_{\text{op}}\|\prod_{m=1}^{\alpha} (\text{Ad}_{\pi_m} \circ \exp(-s_m \Gamma_V))(\tilde{U}\rho_0 \tilde{U}^\dagger)\|_1 \\
+ \|W_1 \tilde{\rho} W_1^\dagger - \rho_F\|_1 < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12} + \frac{\varepsilon}{3} = \varepsilon.
\]

Now what happens if we cannot apply Prop. 1 directly, i.e. if the original and the modified eigenvalue sequence of $\rho = W_1 \text{diag}(x)W_1^\dagger$ or $\rho_0 = W_2 \text{diag}(y)W_2^\dagger$ do not coincide? Given $\varepsilon > 0$, we first of all find $N \in \mathbb{N}$ such that
\[
\sum_{j=N+1}^{\infty} x_j^\dagger < \frac{\varepsilon}{12} \quad \sum_{j=N+1}^{\infty} y_j^\dagger < \frac{\varepsilon}{12}.
\tag{22}
\]

Take unitaries $X, Y \in B(\mathcal{H})$ so that $X\rho X^\dagger = \text{diag}(x_1^\dagger, \ldots, x_N^\dagger, *, *, \ldots)$ and $Y\rho_0 Y^\dagger = \text{diag}(y_1^\dagger, \ldots, y_N^\dagger, *, *, \ldots)$ where the diagonal entries differ from the original only by a permutation on a finite block. As the tail of these new diagonals is “already small” we may change these elements within the realm of approximation. Given $\sum_{j=1}^{N} x_j^\dagger \leq \sum_{j=1}^{N} y_j^\dagger$ (because $\rho < \rho_0$) where this inequality may or may not be strict, we want to fill up $X\rho X^\dagger$ with small entries such that the traces match. Define $\varphi := \sum_{j=1}^{N} (y_j^\dagger - x_j^\dagger)$ where $0 \leq \varphi < \frac{\varepsilon}{12}$ due to (22) and $\rho \geq 0$, as well as $m := \lceil \varphi/x_k^\dagger \rceil \in \mathbb{N}$. Here $k \in \{1, \ldots, N\}$ is chosen such that $x_k^\dagger$ is the smallest non-zero entry of $(x_1^\dagger, \ldots, x_N^\dagger)$. The new (eigenvalue) sequences then are $\hat{x} := (x_1^\dagger, \ldots, x_k^\dagger, \frac{\varphi}{m}, \ldots, \frac{\varphi}{m}, 0, 0, \ldots)$ (where $\varphi/m$ occurs $m$ times) and $\hat{y} := (y_1^\dagger, \ldots, y_N^\dagger, 0, 0, \ldots)$. These sequences satisfy $\hat{x}^\dagger = \hat{x}, \hat{y}^\dagger = \hat{y}$ and $\hat{x} \prec \hat{y}$ (for this note that if $k < N$ then majorization forces $\sum_{j=1}^{k} x_j^\dagger = \sum_{j=1}^{k} y_j^\dagger = 1$ and thus $\varphi = 0$) so we could apply Prop. 1 to them. Now to
\[
\omega := \frac{\text{diag}(\hat{x})}{\sum_{j=1}^{N} y_j^\dagger} \quad \text{and} \quad \omega_0 := \frac{\text{diag}(\hat{y})}{\sum_{j=1}^{N} y_j^\dagger},
\]
which are both in $\mathbb{D}(\mathcal{H})$, we can apply the original scheme which yields a CPTP map $f$ on $\mathcal{H}$ such that $f(\omega_0) = \omega_F \in \text{reach}_{\Sigma_V}(\omega_0)$ and $\|\omega - \omega_F\|_1 < \frac{\varepsilon}{6}$.
Of course linearity implies \( \| \text{diag}(\hat{x}) - f(\text{diag}(\hat{y})) \|_1 < \frac{\xi}{6} \). The final scheme goes as follows:

\[
\rho_0 \xrightarrow{Y} Y\rho_0 Y^\dagger \approx \text{diag}(\hat{y}) \xrightarrow{f} \text{diag}(\hat{x}) \approx X\rho^\dagger X^\dagger \rightarrow \rho
\]

More precisely, by Lemma 6 we find unitaries \( \tilde{X}, \tilde{Y} \in B(\mathcal{H}) \) such that

\[
\| Y\rho_0 Y^\dagger - \tilde{Y}\rho_0 \tilde{Y}^\dagger \|_1 < \frac{\xi}{6}, \quad \| \tilde{X}^\dagger (f \circ \text{Ad}_{\tilde{Y}})(\rho_0) \tilde{X} - X^\dagger (f \circ \text{Ad}_Y)(\rho_0) X \|_1 < \frac{\xi}{6}
\]

and \( \rho_F := (\text{Ad}_{\tilde{X}} \circ f \circ \text{Ad}_Y)(\rho_0) \in \text{reach}_{\Sigma_V}(\rho_0) \). Putting things together,

\[
\| \rho - \rho_F \|_1 \leq \| \rho - X^\dagger \text{diag}(\hat{x}) X \|_1 + \| X^\dagger \text{diag}(\hat{x}) X - X^\dagger f(\text{diag}(\hat{y})) X \|_1 + \| X^\dagger f(\text{diag}(\hat{y})) X - \rho_F \|_1
\]

\[
< \frac{\xi}{6} + \| \text{Ad}_{X^\dagger} \|_{\text{op}} \frac{\xi}{6} + \| (\text{Ad}_{X^\dagger} \circ f)(\text{diag}(\hat{y})) - \rho_F \|_1
\]

\[
\leq \frac{\xi}{3} + \| \text{Ad}_{X^\dagger} \|_{\text{op}} \| f \|_{\text{op}} \| \text{diag}(\hat{y}) - Y\rho_0 Y^\dagger \|_1 + \| \text{Ad}_{X^\dagger} \|_{\text{op}} \| f \|_{\text{op}} \| Y\rho_0 Y^\dagger - \tilde{Y}\rho_0 \tilde{Y}^\dagger \|_1 + \| (\text{Ad}_{X^\dagger} \circ f \circ \text{Ad}_Y)(\rho_0) - \rho_F \|_1 < \varepsilon
\]

so \( \rho \in \text{reach}_{\Sigma_V}(\rho_0) \), which concludes the proof.

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**Appendix D: Theorem 1 for Unbounded Drift \( H_0 \)**

The physically relevant case of an unbounded system Hamiltonian \( H_0 \) is a bit more intricate. To show that Eq. (2) is well-defined in this general setting, we have to resort to some basic results from the theory of strongly continuous one-parameter semigroups as presented in [12, Ch. 1.9 & Ch. 5.5].

To begin with, for selfadjoint \( H_0, \ldots, H_m \in B(\mathcal{H}) \) and \( u_1, \ldots, u_m \in \mathbb{R} \) and setting \( H := H_0 + \sum_{j=0}^m u_j H_j \) and \( \mathbf{H} := \text{ad}_H \), the solution of

\[
\dot{\rho}(t) = -i\mathbf{H}(\rho(t)) \quad \rho(0) = \rho_0 \in \mathbb{D}(\mathcal{H}), \quad (23)
\]

is obviously given by applying the corresponding unitary channel

\[
\rho(t) = e^{-it\mathbf{H}} \rho_0 e^{it\mathbf{H}}
\]

for all \( t \in \mathbb{R}_+ \) (even for all \( t \in \mathbb{R} \)). The control case \( H(t) \) with piecewise constant control amplitudes \( u_j(t) \) is solved by compositions of such solutions.

Now let us assume that \( H_0 \) is unbounded and defined on some dense domain \( D(H_0) \subset \mathcal{H} \). Then \( H := H_0 + \sum_{j=0}^m u_j H_j \) is selfadjoint with dense
domain $D(H) = D(H_0)$ and Stone’s Theorem implies that $-iH$ is the infinitesimal generator of a strongly continuous group $U(t) = e^{-itH}$ of unitary operators. The corresponding one-parameter group $U(t) := \text{Ad}_{U(t)} := e^{-itH(\cdot)} e^{itH}$ of unitary channels (isometries!) is strongly continuous on the trace class $B^1(\mathcal{H})$ and hence it is generated via the densely defined, closed operator $-iH$. More precisely, one has the following result\footnote{NB: Davies \cite{Davies2012} proved the sequel on the Banach space of all selfadjoint trace-class operators. As selfadjointness is neither used nor necessary, we extend the results to $B^1(\mathcal{H})$.} which allows us to identify $-iH$ with $-i \text{ad}_H$.

**Lemma 16** \cite{Davies2012}, Ch. 5, Lemma 5.1. The domain $D(H)$ of $H$ is the set of all $\rho \in B^1(\mathcal{H})$ such that $\rho(D(H)) \subset D(H)$ and such that the operator $H\rho - \rho H$ on $D(H)$ is norm bounded with an extension to a trace class operator on $\mathcal{H}$. Moreover, one has the identity $H(\rho) = H\rho - \rho H =: \text{ad}_H(\rho)$.

In the sequel, the explicit form of $D(H)$ is irrelevant as is the explicit construction of $U_t$, e.g., via the Post-Widder inversion formula

$$U(t)(\rho) = \lim_{n \to \infty} \left( 1 - \frac{-itH}{n} \right)^{-n}(\rho)$$

for all $t \geq 0$ (even for all $t \in \mathbb{R}$) and $\rho \in B^1(\mathcal{H})$. Recall that the ODE \footnote{NB: Davies \cite{Davies2012} proved the sequel on the Banach space of all selfadjoint trace-class operators. As selfadjointness is neither used nor necessary, we extend the results to $B^1(\mathcal{H})$.} has a classical solution only for a dense set of initial values, namely for $\rho_0 \in D(H) \subset B^1(\mathcal{H})$. Nevertheless, we will write $U(t) = e^{-itH}$ although this only holds formally.

**Lemma 17.** Let any $V \in B(\mathcal{H})$. The operator $(-iH - \Gamma V)$ on $B^1(\mathcal{H})$ with domain equal to that of $D(H)$ is the generator of a strongly continuous, positive, trace-preserving semigroup on $B^1(\mathcal{H})$, formally denoted by $(e^{-itH - t\Gamma V})_{t \in \mathbb{R}^+}$.

**Proof.** Apply \cite{Davies2012} Thm. 5.2] with $\mathcal{J} := V(\cdot)V^\dagger$ so $\mathcal{J}^*(1) = V^\dagger V$.\hfill\qed

The corresponding semigroup is completely positive by its GKS form, yet positivity and trace-preservation would suffice for what follows.

Thus even if $H_0$ is unbounded, for every choice of constant controls Eq. (2) gives rise to a strongly continuous one-parameter semigroup $(e^{-i(H - t\Gamma)V})_{t \in \mathbb{R}^+}$ and therefore we obtain a well-defined reachable sets $\text{reach}_{\Sigma_V}(\rho_0)$ in the sense of Eq. \footnote{NB: Davies \cite{Davies2012} proved the sequel on the Banach space of all selfadjoint trace-class operators. As selfadjointness is neither used nor necessary, we extend the results to $B^1(\mathcal{H})$.} for all initial values $\rho_0 \in \mathbb{D}(\mathcal{H})$. The fact, that the ODE \footnote{NB: Davies \cite{Davies2012} proved the sequel on the Banach space of all selfadjoint trace-class operators. As selfadjointness is neither used nor necessary, we extend the results to $B^1(\mathcal{H})$.} allows classical solutions only on a dense domain of initial values, may be neglect when specifying $\text{reach}_{\Sigma_V}(\rho_0)$.

With the stage being set, we only need the Trotter product formula for contraction semigroups on Banach spaces before we can highlight how the proof of Theorem \footnote{NB: Davies \cite{Davies2012} proved the sequel on the Banach space of all selfadjoint trace-class operators. As selfadjointness is neither used nor necessary, we extend the results to $B^1(\mathcal{H})$.} changes to the new frame.
Lemma 18 ([37], Thm. X.51). Let $A_1$ and $A_2$ be generators of contraction semigroups on $\mathcal{B}^1(\mathcal{H})$, i.e. strongly continuous semigroups of operator norm less or equal one for all $t \in \mathbb{R}_+$. Suppose that the closure of $(A_1+A_2)$ generates a contraction semigroup on $\mathcal{B}^1(\mathcal{H})$ and denote by $(A_1+A_2)$ its closure. Then for all $\rho \in \mathcal{B}^1(\mathcal{H})$ and all (fixed) $t \geq 0$

$$\lim_{n \to \infty} \| (e^{-tA_1/n}e^{-tA_2/n})^n(\rho) - e^{-t(A_1+A_2)}(\rho) \|_1 = 0.$$ 

With all these ingredients we are prepared to

Generalizing the proof of Thm. 1 to unbounded $H_0$, $\subseteq$: As $V$ is assumed to be normal, one has $\Gamma_V(1) = 0$ and so the corresponding one-parameter semigroup is in $\mathbb{S}(\mathcal{H})$, i.e. it consists of bi-stochastic quantum maps. To see that $e^{-itH-it\Gamma_V}(\rho) \prec \rho$ for all $\rho \in \mathbb{D}(\mathcal{H})$ and $t \in \mathbb{R}_+$ we note

- $e^{-itH}(\rho) \prec \rho$ as unitary channels do not change the eigenvalues. Thus, majorization cannot increase if the noise $\Gamma_V$ is switched off.
- $e^{-it\Gamma_V}(\rho) \prec \rho$ by Lemma 1.

Therefore and since $\prec$ is a preorder (so in particular transitive), one has

$$(e^{-itH/n}e^{-it\Gamma_V/n})^n(\rho) \in \{ \omega \in \mathbb{D}(\mathcal{H}) | \omega \prec \rho \}$$

(24)

for all $n \in \mathbb{N}_0$. Now apply Lemma 18 to conclude that $(e^{-itH/n}e^{-it\Gamma_V/n})^n(\rho)$ converges to $e^{-itH-it\Gamma_V}(\rho)$ in trace norm for all $t \in \mathbb{R}_+$ and $n \to \infty$. Then, by Eq. (24) in combination with Lemma 5 (the set of majorized states is trace-norm closed), we conclude $e^{-itH-it\Gamma_V}(\rho) \prec \rho$.

We saw earlier that $(e^{-itH})_{t \in \mathbb{R}_+}$ is a contractive semigroup. The same holds for $(e^{-it\Gamma_V})_{t \in \mathbb{R}_+}$ by Lemma 10 as well as $(e^{-itH-it\Gamma_V})_{t \in \mathbb{R}_+}$ by Lemma 17 & 18 Prop. 2 [4d]. The respective generators are all densely defined (on at least $D(\mathcal{H})$) and closed, so Lemma 18 yields

$$\lim_{n \to \infty} \| (e^{-itH/n}e^{-it\Gamma_V/n})^n(\rho) - e^{-itH-it\Gamma_V}(\rho) \|_1 = 0$$

for all $\rho \in \mathcal{B}^1(\mathcal{H})$, which shows the inclusion in question.

$\supseteq$: Generalizing this inclusion to unbounded $H_0$ is easier as we only have to modify Lemma 7. By the same line of reasoning, Lemma 18 gives

$$\lim_{n \to \infty} \| (e^{itH/n}e^{-(itH-it\Gamma_V)/n})^n(\rho) - e^{-it\Gamma_V}(\rho) \|_1 = 0$$

for all $\rho \in \mathbb{D}(\mathcal{H})$ and $t \geq 0$. With this, the original proof of $\supseteq$ holds without further changes—since we have never used the Trotter product formula explicitly in the uniform or norm topology, but only in the strong topology, i.e. when applied to some density or trace-class operator, see also Lemma 14.

Thereby all instances of Theorem 1 are finally proven. $\Box$

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[4d] The proof only uses positivity and trace-preservation, so we apply the respective result.
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