A study of the Lorentz-Dirac equation in complex space-time for clues to emergent quantum mechanics

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Abstract. A hypothetical equation of motion is proposed for Kerr-Newman particles. Obtained by analytic continuation of the Lorentz-Dirac equation into complex space-time, the goal is to study the implications of this theory for emergent quantum mechanics. A new class of “runaway” solutions is found which bear similarity to the quantum phenomenon of Zitterbewegung. Other solutions incorporating external forces are also presented. The electromagnetic fields generated by these motions are studied. It is found that the retarded times are multi-sheeted functions of the field points. As a consequence, the solutions are not unique. A cloaking mechanism is found which inhibits electromagnetic radiation by combining the fields from several Riemann sheets for the retarded time. These results reinforce to some extent, but also pose additional conceptual questions for the idea that Kerr-Newman solutions provide insight into elementary particles and into emergent quantum mechanics.

1. Introduction
In the general relativity literature Kerr-Newman (KN) models for elementary particles have been studied [1, 2, 3, 4, 5, 6]. Their gyromagnetic g factor is exactly 2, the same as with the Dirac equation [3]. Complex manifold techniques have proved useful [7, 8, 9]. Emergent quantum mechanics at the Planck scale is now an active research area[10, 11, 12, 13, 14]. A dynamical theory for the KN particle could yield clues for this program [15]. The author has argued that this emergence might be cosmological in origin [16, 17]. The Lorentz-Dirac equation (LDE) is typically derived for exactly point-like classical charged particles. We study here a complexified version of it, and find new solutions which are similar to Zitterbewegung [18, 19, 20, 15].

2. A brief overview of the Kerr-Newman solution and complex Minkowski space
The KN particle is modeled as a point charge in a complex Minkowski space (CM4) [21, 22, 8, 9, 1, 2, 4, 6]. Gravity is ignorable for elementary particle scale parameters [2]. The Riemann-Silberstein complex vector field is used $W = E + iB$, or a complex Faraday tensor is introduced [22] $W^{\mu\nu} = F^{\mu\nu} + iF^{\mu\nu}$, $*W = -iW$, where $F$ is the Faraday tensor and $*F$ is its dual. $W$ is termed anti self-dual. The real part gives the physical Faraday tensor when evaluated on the real hyperspace $F_{\text{phys}}^{\mu\nu} = ReW^{\mu\nu}$. The static Kerr-Newman particle is modelled as a point charge located at a point in complex 3-space at $z_0 = x_0 + i b$, where $x_0$ and $b$ are real 3-vectors. One introduces a complex Coulomb potential $\Phi = q/\sqrt{(z - z_0)^2}$. $\Phi$ is holomorphic in each spatial coordinate and double-sheeted. It follows that $\nabla^2 \Phi = 0$ on the real hyperspace so long as $b \neq 0$. The prescription for making physical sense out of this complex displacement is
to take the Riemann-Silberstein vector as given by the analytic continuation of the electric field to complex values. Amazingly, this recipe yields the correct electromagnetic field of the Kerr-Newman solution \( \mathbf{W} = -\nabla \Phi \). As there is no charge on the real hyperspace, this leads to the so-called “sourceless” Kerr-Newman solution. The covariant version of this formula starts with a calculation of the analytic continuation of the real Faraday tensor to its value at the complex source point \( z_0 \), \( F^{\mu \nu} = \partial^\mu \Phi \delta^{(0)}(\omega) \), where \( \delta \) here denotes the Kronecker delta. This tensor will not be anti-self-dual automatically. In order to calculate the tensor \( W \) we must project out the anti-self-dual part. Then the real part will give the physical Faraday tensor. The metric tensor is found by solving the equations of general relativity by assuming that the metric can be expressed in Kerr-Schild coordinates in the form \( g_{\mu \nu} = \eta_{\mu \nu} - 2H k_\mu k_\nu \), where \( k \) is a null vector field, \( H \) a scalar field, and \( \eta \) is the Minkowski metric. Introducing a discontinuity on the Riemann cut leads to charges as sources for the fields [23]. When considering values for charge, mass, and angular momentum (\( q \), \( m \), and \( j \)) of elementary particles, the gravitational metric is well-approximated by a Minkowski metric except very near to the ring singularity [2]. They include the case where the mass is negative. The total electromagnetic energy stored in the static electric and magnetic fields of a Kerr-Newman solution is infinite [4]. Despite this, the mass is finite, and is arbitrary and unrelated to the electromagnetic energy. Consequently, we cannot interpret these solutions as purely electromagnetic particles. The Kerr-Newman solutions have event horizons whenever the mass is sufficiently large such that (in Planck units) \( m^2 > (j/m)^2 + q^2 \) [2, 3]. For the case of elementary particles, event horizons would not be present, and consequently the ring singularity would be naked.

3. The Lorentz-Dirac equation (LDE) and Runaway solutions

The LDE is [24, 25, 26, 27, 28] \( m_0 a^\mu = F^\mu_{\ ext} + m_0 \tau_0 (\dot{a}^\mu + \alpha^\lambda a_\lambda v^\mu) \) (metric signature is (+,−,−,−)). The notation and conventions are the same as [15] where \( F^\mu_{\ ext} \) is any external force and \( \tau_0 = \frac{2e^2}{3m_0 c^2} \). The vast literature on this equation is reasonably in agreement on two points. First, if we consider exactly point particles then the general consensus is that the LDE is correct in the sense that straightforward electromagnetic analysis leads to it. Second, this equation has one of two possible unphysical properties - either it has runaway solutions, or it has preacceleration. There is a tendency to replace it in practical calculations by an approximate equation developed first by Landau and Lifshitz [29, 30] which is suitable for non-point-like quasi-particles and free of pathologies. Since the Kerr-Newman solutions are predicated on the existence of an exactly point-like singularity in complex space-time, we are forced to consider the LDE and not the quasi-particle equation for describing them. The term \( m_0 \tau_0 \dot{a} \) is called the Schott term. Its origin can be understood from the following equation for the total momentum of the charged particle including electromagnetic momentum [31], \( p^\mu = mv^\mu - \frac{2}{3} e^2 a^\mu \). The term \( m_0 \tau_0 a^2 v \) is the force caused by emitted radiation. In the method developed in this paper, the runaway solutions lead to interesting behaviour, and therefore we study them in detail. The most famous runaway solution is [27] for motion in the 0-3 plane, \( v^\mu = (cosh(A e^{\tau/\tau_0} + B), 0, 0, sinh(A e^{\tau/\tau_0} + B))^\mu \), where \( A \) and \( B \) are constants. Call these type I runaway solutions. Here the focus will be on another type of runaway solution. Look for a solution which has vanishing acceleration squared in the absence of force, \( a^\lambda a_\lambda = 0 \), then the LDE becomes, \( a^\mu = \tau_0 \dot{a}^\mu \). The solution is therefore

\[
a^\mu(\tau) = e^{\tau/\tau_0} a^\mu(0), \quad a^\lambda(0)a_\lambda(0) = 0
\]

\[
v^\mu(\tau) = \tau_0 e^{\tau/\tau_0} a^\mu(0) + u^\mu, \quad z^\mu(\tau) = (\tau_0)^2 e^{\tau/\tau_0} a^\mu(0) + u^\mu \tau + \Omega^\mu
\]

where \( \Omega^\mu \) and \( u^\mu \) are constant 4-vectors. In complex space-time, there are new solutions of this type, and these shall be the focus of this paper. These solutions shall be referred to as type II. Note that the Laplace transform in \( \tau \) of a type II solution has a simple pole whereas the type
I solutions have an infinite number of poles. So the type II solutions are considerably simpler. Runaway solutions have been dismissed as unphysical for over 100 years and they have posed a profound challenge for classical electromagnetism. Note that both type I and II solutions are entire functions of \( \tau \) if it is considered as a complex variable. They are also entire functions of the constants of the motion \( a^\mu(0) \), \( \Omega^\mu \), and \( u^\mu \).

4. Analytic continuation of free particle equation and solutions

We let all variables become complex. Considering only \( \tau \), the position of the particle becomes not a curve but rather a two dimensional surface embedded in complex Minkowski space or \( \text{CM}^4 \). This is because, as a complex variable, \( \tau \) has both a real and imaginary part. We shall require that the particle moves along a curve in complex space-time parametrized by some real variable \( s \) so that the curve of the particle in complex Minkowski space is given by \( z^\mu(\tau(s)) \).

The analytic continuation of the LDE and its solutions to complex \( \tau \) are uniquely determined by their real hyperspace forms. We could interpret this as meaning that any analytic function \( \tau(s) \) yields a possible solution for the charged particle’s motion due to self-forces while moving in complex space-time. Despite this non-uniqueness, one particular analytic continuation will be studied in detail here because of its similarity to Zitterbewegung. Starting with all real values for the various initial parameters, set \( \tau(s) = s \) for \( s \) real and ranging from \(-\infty\) to \(+\infty\) and then analytically continue this curve to the curve \( \tau(s) = is \) along the imaginary \( \tau \) axis. An analytic transformation which accomplishes this is the following continuous function as \( \lambda \) vary from 0 to 1.

\[
\tau(\lambda, s) = s + (i - 1)\lambda s, \quad \lambda \in [0, 1].
\]

After the continuation in \( \tau \) is performed, the resulting real part of each coordinate is interpreted as the physical position of the particle in physical (ie. ordinary 4D) Minkowski space. Additional analytic continuations of the initial values \( \Omega^\mu \), \( u^\mu \), \( a(0)^\mu \) will also be made in order to obtain a physically sensible solution. The result is \( \tau = is \) and consequently the LDE becomes

\[
-m_0 \frac{d^2 z^\mu}{ds^2} = F^\mu_{\text{ext}} + im_0 \tau_0 \left( \frac{d^3 z^\mu}{ds^3} - \left( \frac{d^2 z}{ds^2} \right)^2 \frac{dz^\mu}{ds} \right)
\] (3)

We wish to interpret \( s \) as a new scaled proper time for the motion of the particle in the real hyperspace. Notice that then the inertial mass has the wrong sign. The Kerr-Newman metric has both positive and negative mass solutions though, so we therefore choose \( m_0 \) to be negative. We have then, in the absence of external force and defining \( \tau_0 \) positive

\[
m \frac{d^2 z^\mu}{ds^2} = im\tau_0 \left( \frac{d^3 z^\mu}{ds^3} - \left( \frac{d^2 z}{ds^2} \right)^2 \frac{dz^\mu}{ds} \right), \quad m = -m_0 > 0
\] (4)

Considering the type II runaway solutions we have

\[
a_s^\mu(s) = d^2 z^\mu/ds^2 = e^{-is/\tau_0}a_s^\mu(0), \quad a_s^\lambda(0)a_{s\lambda}(0) = 0
\] (5)

\[
u_s^\mu(s) = dz^\mu/ds = i\tau_0 e^{-is/\tau_0}a_s^\mu(0) + V^\mu
\] (6)

\[
z_s^\mu(s) = -\tau_0^2 e^{-is/\tau_0}a_s^\mu(0) + V^\mu s + \Omega^\mu
\] (7)

where \( s \) varies from \(-\infty\) to \( \infty \), and \( V \) and \( \Omega \) are constant 4-vectors, with \( V \) real and time-like.
\section{Circular rotating solution for free particles}

Consider analytic continuation of the constant 4-vector $a_\mu(0)$ to null vectors of the form $a_\mu(0) = \delta_\mu(0, 1, \pm i, 0)$ which have opposite chirality. Although $\delta_\mu$ can be a complex constant, by choice of the zero point for $s$ we can absorb its phase, and so we take it to be real and positive. We let $\Omega^\mu$ have a small imaginary part to give it the properties of a spinning particle as dictated by the Kerr-Newman static solution $\Omega^\mu = x^\mu_0 + ib^\mu$ where $b^\mu$ is real and space-like, and in the “average rest frame” where $V^\mu = (\gamma, 0, 0, 0)$, $b^\mu = (0, b, \ldots)$, and $\gamma$ is a real constant. In this example we have chosen to take $b$ along the 3-direction to maintain symmetry. It might be oriented in other directions too, but these shall not be considered here. The magnitude of $b$ is the angular momentum divided by mass for the Kerr-Newman particle. For an electron, it be oriented in other directions too, but these shall not be considered here. The magnitude of $b$ is

\begin{equation}
\gamma = \frac{\mu}{\sqrt{\mu^2 - 1}} \approx 1.0001 - \frac{999}{4\mu^2},
\end{equation}

and the angular frequency $\omega_z = \frac{2m_{obs}}{\hbar}$ yields $\gamma^2 = \frac{3\hbar}{4m_{obs}}$ as shown in [15]. Introducing the fine structure constant $\alpha = \frac{\hbar^2}{\pi}$, we have for a particle with charge $e$, $\gamma = \sqrt{\frac{3\alpha}{4\mu}} \approx 10.1379$. We can interpret the solution as a particle with an internal clock which has twice the frequency as mandated by quantum mechanics - the de Broglie clock which is the same as the result for the Dirac equation. In the average rest frame $Re z^\mu$ describes a circular orbit of radius $\gamma_0^2 \delta_\mu$ and angular frequency $\omega_z$. This is the projection onto the real hyperspace that is interpreted as the actual motion of the particle in physical space-time. The laboratory speed, determined by the real part of $z$ is $\gamma_0^2 \delta_\mu \omega_z$. If we accept the usual arguments regarding Zitterbewegung, then this would be the speed of light. This is only approximately true in our case. We find rather $\delta_\mu = \sqrt{\gamma^2 - 1}/\gamma \approx 1.0001$ so that the radius of oscillation is then approximately half of the (reduced) Compton wavelength $\lambda_C = 2\gamma_0$ and $r = \frac{\lambda_C \gamma}{2 \gamma^2 - 1}$, so that the speed is given by $r \omega_z = \sqrt{\gamma^2 - 1}/\gamma \approx 0.995c$. In most models for Zitterbewegung the speed is taken to be exactly $c$ and the radius is given by $r_S = \frac{b}{2m_{obs}} = \gamma_0$. This value, originally due
to Schroedinger, agrees with our radius to better than 1%. The angular momentum due to this circular motion alone may be calculated to be \( L = \frac{\hbar v^2}{2} \), which is slightly smaller than the spin of the electron by about 1%. The electromagnetic fields’ angular momentum might make up the deficit. Experimental proof of Zitterbewegung comes from the Darwin term [35] in hyperfine splitting of atomic lines in spectroscopy, and more directly by observing resonant behaviour in electron channeling [36].

6. Liénard-Wiechert potentials and the fields

The Liénard-Wiechert potentials provide a procedure for calculating the electromagnetic fields produced by a particle. In ordinary electromagnetism they are [25] \( A^\mu(x) = \frac{q v^\mu}{v(\tau - (x - z(\tau)))} \vert_{\tau = \tau_r} \). Note that \( \frac{dz^\mu(\tau)/d\tau}{dz(\tau)/ds - (x - z(\tau))} = \frac{dz^\mu(\tau(s))/ds}{dz(\tau(s))/ds - (x - z(\tau(s)))} \), and where \( \tau_r \) is the proper time which satisfies the null root equation \( (x - z(\tau_r)) \cdot (x - z(\tau_r)) = 0 \) or \( (x - z(\tau_r)) \cdot (x - z(\tau(s))) = 0 \). After analytic continuation the coordinates are expressed in terms of the new variable \( s \) and the complex valued potential \( A \) is used to calculate a complex Faraday tensor \( F \) and its dual \( *F \) from which the anti self-dual tensor \( W \) can be calculated from. If the particle trajectory is time-like in the (real valued) case of standard electromagnetic theory, then there are exactly two solutions to the null root equation, one in the past (retarded time) and one in the future (advanced time). When we analytically continue these equations into the complex plane, the null root equation becomes for complex \( z \) and real \( x \), \( (x - z(s))^\mu (x - z(s))_\mu = 0 \). This condition in the complex case is fundamentally different since both the real and imaginary parts must vanish. We are considering here the situation where \( z \) is parametrized along a curve by a real parameter \( s \), and the null root equation will quite likely have no solutions at all for real \( s \).

In general, the null root equation will have solutions which are not on the path of the particle. Nevertheless, these solutions are obtained by analytic continuation from ordinary solutions, and so they are valid ones. This is an important - indeed profound - difference as compared to the usual case. The set of complex solutions \( \{s_i\} \) can be quite large, even infinite. Some subset will be causal, but even this subset can be infinite. Why shouldn’t we include a superposition of several different roots at the same time? The solutions for the electromagnetic field are not unique and may be characterized by the number of roots included in the solution, by the particular values of these roots, and by their respective weights in the sum. For single-root solutions we have, letting \( v^\mu = dz^\mu/ds \), \( A^\mu(x) = \frac{q v^\mu}{v(s)(x - z(s))} \), \( s \in \{s_i\} \). If there are more than one retarded solution, then there is no reason to exclude multiple root solutions. For an \( N \) root solution we have \( A^\mu(x) = \sum_{j=1}^N w_j q v^\mu(s_j)/(x - z(s_j)) \), \( s_j \in \{s_i\} \), \( \sum_{j=1}^N w_j = 1 \). These 4-potentials are in general complex. The rule for calculating the complex Faraday tensor from them is \( W^{\mu\nu} = \partial^{[\mu} A^{\nu]} + i \* \partial^{[\mu} A^{\nu]} \), and the physical Faraday tensor on the real hyperspace \( x \) is the real part of this \( F_{\text{physical}} = Re W \). The weight factors \( w_j \) are arbitrary except that we would expect that they sum to one, and we may want to include only causal retarded times in the sum. For the time being we shall assume that these weighting factors are constants, but later on we shall consider the possibility that they are dependent on position. These multi-root solutions would look to an observer capable of measuring the electromagnetic fields produced by them like a particle made up of multiple constituents, but they are really analytic continuations associated with a single particle trajectory. So we see we have a whole family of possible solutions. Already in the literature there is evidence for this non-uniqueness. It is exploited in Wheeler-Feynman electrodynamics [37] by allowing a sum of advanced and retarded potentials to give solutions to Maxwell’s equation, although this is a fairly trivial case in comparison. The author has wondered whether these multi-root solutions might be related to the quark model for hadrons. They could look like a particle made up of multiple constituent point particles if probed electromagnetically. But these “particles” might be hard to separate because they are related
to a single particle trajectory in the complex space, with their positions linked together by the requirement that they must lie on different Riemann sheets of a single root equation, perhaps providing a mechanism for quark confinement. We can think of this analytic continuation by imagining we start with real-space Liénard-Wiechert potential for a single root, and write it in the following way:

$$A^{\mu}(x) = \sum_j w_j^{q_0 \mu(x_{\tau_j})} (x - z_{\tau_j})$$

(We haven’t done anything yet, since $\sum w_j = 1$).

But now we analytically continue each term in the sum to a different Riemann sheet, and thus obtain a different value for the retarded time for each term in the sum. This is a generalization of the usual analytic continuation for analytic functions in mathematics. The result is a field which is produced by complex source point charges at a countable set of different complex valued retarded times. The maximum number of source points is the number of Riemann sheets of the analytic function which describes the coordinates for the particle.

7. Calculation of retarded proper times for the type II runaway solutions

The Null root equation that must be satisfied is, for a real field point $x^\mu$, $(x + \tau_0^2 e^{-is/\tau_0} a_s(0) - V s - \Omega)^2 = 0$. The solution set $\{s_j\}$ will be implicit functions of $x$, $a_s(0)$, $u$, and $\Omega$, and in general will not be real and consequently $z(s_j)$ will not lie on the particle’s actual trajectory in the complex space. This is a consequence of relying on analytic continuation to guide us into the complex space. One might be tempted to reject this approach because of this development. We shall adopt a more liberal exploratory philosophy and accept this possibility in order to learn the consequences it leads to. We may rewrite this equation in the form

$$\begin{align*}
(s - C_A) (s - C_R) &= D e^{-is/\tau_0} \\
C_A &= \left( +2x^0 \gamma + \sqrt{(2x^0 \gamma)^2 - 4\gamma^2 R^2} \right) / 2\gamma^2, \\
C_R &= \left( +2x^0 \gamma - \sqrt{(2x^0 \gamma)^2 - 4\gamma^2 R^2} \right) / 2\gamma^2 \\
D_{\pm} &= -2\tau_0^2 a_{\pm}(0) \cdot x / \gamma^2 = 2\tau_0^2 (\delta_a r_{\perp} e^{\pm i\varphi}) / \gamma^2 \\
R^2 &= (x - \Omega)^2, \quad \Omega = (0, 0, 0, ib)
\end{align*}$$

(11-13)

It’s not possible to solve (11) in closed form, unless one uses a proposed generalization of the Lambert function [38]. We can proceed numerically. Start with $C_A$ and $C_R$ set to zero, so that the equation then becomes: $s^2 = D e^{-is/\tau_0}$. This can be solved in terms of Lambert functions [39].

$$s = -2i\tau_0 W_n(\pm i\sqrt{D}/(2\tau_0)), \ n \in \mathbb{Z}$$

(15)

Where $W_n$ is the nth branch of the Lambert function. There are an infinite number of Riemann sheets, and an infinite number of roots. Starting with this solution, we can analytically continue (11) to the correct values for $C_A$ and $C_R$ using numerical means. Thus we can conclude that there are likely to be an infinite number of solutions, unless the analytic continuation results in infinitely many of these different starting approximations coalescing to the same value. Numerical simulation of this analytic continuation suggests that this does not happen, and they indicate the existence of an infinite number of solutions. These solutions correspond to different Riemann sheets of the solution $s$ when viewed as a function of $x$. 


7.1. Calculation of the roots for the asymptotic field limit to study radiation

We assume that \( |z| \ll |x| \) and therefore we make the standard far-field approximation with \( x^\mu \) real

\[
(x - z(s))^2 = 0 \approx (t - z^0)^2 - (x^2 - 2x \cdot z), \quad x = r\hat{r}
\]

(16)

\[
t - z^0 = t - \gamma s \approx \pm (r - \hat{r} \cdot z) = \pm \left(r + \frac{1}{\tau_0} (a_\nu(0) \cdot x) e^{-is/\tau_0}/r - \hat{r} \cdot \Omega \right)
\]

(17)

For the retarded solution \( s \uparrow \) we choose the + sign and obtain

\[
\gamma s \uparrow -(t - r + \hat{r} \cdot \Omega) + \frac{1}{\tau_0} (a_\nu(0) \cdot \hat{r}) e^{-i\nu t/\tau_0} = 0
\]

(18)

\[
A \equiv (t - r + \hat{r} \cdot \Omega), \quad B \equiv \frac{1}{\tau_0} (a_\nu(0) \cdot \hat{r}), \quad \gamma s \uparrow - A + B e^{-i\nu t/\tau_0} = 0
\]

(19)

This equation can be solved for \( s \uparrow \) again in terms of Lambert functions

\[
s \uparrow = A/\gamma - i\tau_0 W_n \left( - \frac{i B e^{-iA/\gamma \tau_0}}{\gamma \tau_0} \right), \quad n \in \mathbb{Z}
\]

(20)

\[
s \uparrow_n = (t - r + \hat{r} \cdot \Omega)/\gamma - i\tau_0 W_n \left( - \frac{i \tau_0 (a_\nu(0) \cdot \hat{r}) e^{-i(t - r + \hat{r} \cdot \Omega)/\gamma \tau_0}}{\gamma \tau_0} \right)
\]

(21)

There are an infinite number of solutions as \( n \) is an arbitrary integer, and there are advanced solutions as well [15]. These null source points all appear to radiate [15], and so does any finite superposition of them. We view it as unlikely that even an infinite number of roots with fixed weighting factors could suppress radiation.

8. Suppressing or Cloaking radiation with weighting factors which are functions of \( x \)

In order to suppress the radiation, we consider allowing the weighting factors \( w_j \) to be functions of the field point \( x \). So long as these functions are analytic (or at least holomorphic) in the coordinates, the resulting fields will be too. It is convenient write \( A^\mu(x) = A_{\text{Coul}}^\mu(x) + A_{\text{Rad}}^\mu(x) \)

\[
A_{\text{Coul}}^\mu(x) = \sum_{j=1}^{N} w_j(x) \frac{q^\mu \delta_0^j}{v(s_j(x)) \cdot (x - z(s_j(x)))}, \quad \sum_{j=1}^{N} w_j(x) = 1
\]

(22)

\[
A_{\text{Rad}}^\mu(x) = \sum_{j=1}^{N} w_j(x) \frac{q^\mu \tau_0 e^{-is_j(x)/\tau_0} a_\nu^j(0)}{v(s_j(x)) \cdot (x - z(s_j(x)))}
\]

(23)

and where \( s_j(x) \) are the proper-time null roots. The potential \( A_{\text{Coul}} \) determines the Coulomb field and the potential \( A_{\text{Rad}} \) determines the radiation field. In order to calculate the radiation field, we must perform the following steps following our procedure \( W^{\mu\nu}(x) = \partial^\mu A^\nu + i(\partial^\mu A^\nu) \), where the real part of \( W \) gives the physical Faraday tensor. Note that \( \ast \partial^\mu A^\nu = i \partial^\mu A^\nu \Rightarrow W^{\mu\nu}(x) = 0 \). We have for large \( r \), \( \partial^\mu A^\nu = \partial^\mu A^\nu_{\text{Rad}} + o(1/r) \). If the \( A_{\text{Rad}} \) happens to be self-dual, then there will be no radiation. But the \( w_j(x) \) are now assumed to be arbitrary functions, and we can choose them such that this is the case. The following potential functions are self dual in this sense \( A_{SD}(x) = f(x^3 = x^0, x^1 = i x^2, s_\pm(0)) \) where \( f \) are arbitrary holomorphic functions of their arguments. So to cloak radiation for + or - chirality, we must choose the weighting functions to satisfy
\[ \sum_{j=1}^{N} w_j(x) \frac{qi\tau_0 e^{-is_j(x)/\tau_0}}{v(s_j(x)) \cdot (x - z(s_j(x)))} = f_\pm(x^3 \mp x^0, x^1 \pm ix^2); \quad \sum_{j=1}^{N} w_j(x) = 1 \]  

(24)

\( f_\pm = 0 \) is an allowed case. Since two equations must be satisfied, at least two roots must be included in the solution. Denote two such roots by index 1 and 2. The equations can be solved simply with the result

\[ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{C_1 - C_2} \begin{pmatrix} f_\pm - C_2 \\ -f_\pm + C_1 \end{pmatrix} \]

(25)

where the two \( C \) values in this formula are the functions multiplying the respective weight functions \( w \) in (24). To this solution might be added by superposition any combination of the weighting functions which satisfy the associated homogeneous equations. Thus, there are a large number of non-radiating solutions. When the cloaking conditions are satisfied, we can write \( W^{\mu\nu} = \partial^{[\mu} A^{\nu]}_{\text{Coul}} + i(\ast \partial^{[\mu} A^{\nu]}_{\text{Coul}}) \) since the radiation term vanishes. The asymptotic electric field will still vary as \( q\hat{r}/r^2 \), so the total charge is unchanged, but the charge near to the particle’s origin will appear to be non zero and smeared out as viewed by a remote observer. One question that naturally arises is what is the state of minimum electrostatic energy? This will depend on the number of roots included in the sum, assuming that there are some solutions for which the energy is not infinite. Relevant to the question of emergence, the quantum mechanical wave equations for free particles are closely related to non-radiating electromagnetic sources [40]. The assumption of \( x \) dependent weighting factors thus seems to be a requirement if this theory is to be taken seriously. Without it, we could not avoid radiation.

9. Motion with weak external forces applied

In an external electromagnetic field, we take the equation of motion to be

\[ m \frac{d^2 z^\mu}{ds^2} = qF^\mu_{\nu\text{ext}} \frac{dz^\nu}{ds} + im\tau_0 \left( \frac{d^3 z^\mu}{ds^3} - \left( \frac{d^2 z^\mu}{ds^2} \right)^2 \frac{dz^\mu}{ds} \right) \]

(26)

The external Faraday tensor \( F_{\text{ext}} \) here is real on the real hyperspace, but is generally complex in the rest of the complex space-time by analytic continuation. We assume that the fields are weak and slowly varying on the time scale of \( \tau_0 \). This reduces to simply the Lorentz equation if we ignore the Schott and radiation terms, which now have a factor of \( i \) in front of them. Consider the simplified case where we ignore the radiation term. The equation then becomes

\[ m \frac{d^2 z^\mu}{ds^2} = qF^\mu_{\nu\text{ext}} \frac{dz^\nu}{ds} + im\tau_0 \frac{d^3 z^\mu}{ds^3} \]

(27)

In this formula, the mass \( m \) is the observed mass divided by \( \gamma \), so \( m = m_e/\gamma = m_e/\sqrt{\frac{3}{4\alpha}} \).

10. The guiding center approximation

Assume that the particle moves in a highly localized oscillation at very high frequency, and the external electromagnetic fields are only very slowly varying in time and space on the scale of this oscillation. Then the weak field acts to weakly perturb the oscillating frequency and to guide the overall motion [15]. We start with a solution to the ordinary Lorentz equation for a trajectory of a time-like charged particle (the 4-position \( L \) is real)

\[ m \frac{d^2 L^\mu}{ds^2} = qF^\mu_{\nu} \frac{dL^\nu}{ds} \left( \frac{dL}{ds} \right)^2 = \gamma^2 \]

(28)
and then we add a complex correction term $\delta$ to it which represents the high frequency oscillation
\[ m \frac{d^2(L + \delta)}{ds^2} = qF_\nu \frac{d(L + \delta)}{ds} + im\tau_0 \frac{d^3(L + \delta)}{ds^3} \] (29)

Note that $s$ is the proper time scaled by $\gamma$ for the guiding center, and that the observed mass is $\gamma m$ in order to compensate for this. We make the following approximations: $F(z) \approx F(L)$, $d^3(L + \delta)/ds^3 \approx d^3\delta/\delta^3$, and $|qF_{\mu\nu}/m| \ll 1/\tau_0$ to obtain $m\frac{d^2\delta^\mu}{ds^2} = im\tau_0 \frac{d^3\delta^\mu}{ds^3}$. In general, these solutions will radiate energy. However, this equation will have multiple roots as did the Zitterbewegung solution, and so by using again the cloaking mechanism of position dependent weighting factors, the radiation rate should have a minimum and possibly be zero, and one would expect this to be a natural asymptotic state for the system after transients have died away.

10.1. Motion in constant fields

Look for solutions of the form $\delta = \delta(0)e^{i\omega s}$, so that $-m\omega^2\delta^\mu(0) - i\omega qF_{\mu\nu}\delta^\nu(0) - m\tau_0\omega^2\delta^\mu(0) = 0$. There are eight roots for $\omega$ [15]. The most general solution can be written $\delta^\mu(s) = \sum_{k=1}^{8} \delta_k^\mu(0)e^{i\omega_k s}$. The $\delta_k^\mu$ must be constructed to solve the four homogeneous equations.

10.2. Motion in a 3D harmonic oscillator central force

\[ m \frac{d^2z^j}{ds^2} = -kz^j + im\tau_0 \left( \frac{d^2z^j}{ds^3} - \frac{d^2z}{ds^2} \frac{dz^j}{ds} \right) \quad j = 1, 2, 3 \] (30)

There will be guided-center approximate solutions to this equation, and also exact solutions with $a = \alpha = 0$ as follows. If the radiation term vanishes, then the equation has solutions of the form $z^j(s) = z^j(0)e^{i\omega s}$ with $\mathbf{z}(0) \cdot \mathbf{z}(0) = 0$. We can take $\mathbf{z}(0)$ to be proportional to either of the null vectors $\mathbf{a}_\pm(0)$ 3-vectors for example for some arbitrary spin axis. Then the frequency equation becomes $\tau_0\omega^3 + \omega^2 - k/m = 0$. There are three roots to this which are approximately $\omega = \pm \sqrt{k/m}, -1/\tau_0$. A class of exact solutions are given by a linear superposition of these $z^j(s) = \mathbf{a}_\pm(0) \sum_{j=1}^{3} c_j e^{i\omega_j s}$ and $z^0(s) = \omega s$ for arbitrary complex constants $c_j$. These solutions are more complicated than the single-frequency Zitterbewegung solutions. Nevertheless, we expect that they will certainly have multiple null roots for the radiation calculation. Therefore, the same cloaking technique that was proposed to eliminate radiation in the Zitterbewegung case can be applied in the present circumstance to reduce or eliminate the radiation from these harmonic oscillator solutions as well.

10.3. Motion in a Coulomb field

Consider motion in a spherically symmetric Coulomb field of charge $Q$. On the real hyperspace we have $E_{ext} = \frac{Q}{r} \mathbf{r}$, $B_{ext} = 0$

\[ m \frac{d^2z^\mu}{ds^2} = qF_{0\nu} \frac{dz^\nu}{ds} + im\tau_0 \left( \frac{d^2z^\mu}{ds^3} - \left( \frac{d^2z}{ds^2} \right)^2 \frac{dz^\mu}{ds} \right) \] (31)

There will be guiding-center approximations to this equation, based on solutions to the Kepler problem. The simplest would be a circle. An exact analytic solution has not yet been found. There will undoubtedly be an infinite number of Riemann sheets for the retarded time function, and the radiation from these can be cloaked to the level of accuracy of the guiding-center approximation. Whether it is possible, using the cloaking method of $x$ dependent weighting functions, to suppress all radiation in some cases, as would be required for the ground state of the Hydrogen atom for example, is not clear at the present time, but it is a good possibility.
11. Interpretation and Conclusion

The picture that emerges from this model is of a free particle which has a peculiar internal oscillation, and whose electromagnetic fields are not unique, being dependent on a set of complex weighting functions. It eliminates runaway solutions and replaces them with oscillating solutions that look similar to Zitterbewegung and give a model for the de Broglie clock. They allow a cloaking mechanism for radiationless accelerated motion, both in the self-oscillation case, the central harmonic force case, and possibly the Coulomb force case. There might be solutions which do not have any singularities on the real physical hyperspace (ie. the physical Riemann sheet), and some of these might have finite electromagnetic energy. The non-uniqueness of the fields for a particle have a vague similarity to the wave-particle duality of quantum mechanics.

The particle would appear to an observer using an electromagnetic probe as possibly made up of a number of point-like constituents. In order to inhibit radiation, a new cloaking mechanism was invoked that relied on position dependent weighting functions for the different retarded times corresponding to different Riemann sheets superimposed. Some other possibilities and questions then naturally arise. For example, can the weighting functions change their values suddenly as the result of an observation being made? Thus giving a substantive explanation for wave-function collapse. Can different observers “see” different weighting functions for the same particle? Can we understand complementarity from this? What is a photon in this framework? How does the Riemann structure change when more than one particle is considered? What are the lowest energy states possible, assuming that there are solutions with finite electrostatic energy? What are the implications of this theory for non-abelian gauge theories? Could the multi-root solutions be related to the quark model for hadrons? Emergent quantum mechanics from Planck scale physics is a bottom-up approach whereas the model discussed here is more of a top-down one. It might thus be a useful stepping stone in the theoretical voyage from the Planck scale to the quantum atomic realm.

References

[1] Burinskii A 2007 arxiv.org 0712.0577
[2] Burinskii A 2008 Gravitation and Cosmology 14 109–122
[3] Carter B 1968 Phys. Rev. 174 1559
[4] Lynden-Bell D 2003 Stellar Astrophysical Fluid Dynamics p 309–375
[5] Newman E T 2002 Phys. Rev. D 65 104005
[6] Pekeris C L and Frankowski K 1987 Phys. Rev. A 36 5118
[7] Adamo T M, Kozameh C and Newman E T 2009 Living Reviews in Relativity (www.livingreviews.org/lrr-2009-6) 12
[8] Newman E T 1973 J. Math. Phys. 14 774
[9] Newman E T 1976 Gen. Relativ. Gravit. 7 107–111
[10] Adler S L 2004 Quantum theory as an emergent phenomenon (Cambridge University Press)
[11] Carroll R 2004 gr-qc/0406004
[12] Markopoulou F and Smolin L 2004 Phys. Rev. D 70 124029
[13] 't Hooft G 2000 arxiv.org hep-th/0003005
[14] 't Hooft G 2001 arxiv.org hep-th/0104219
[15] Davidson M 2011 arXiv:1109.4923
[16] Davidson M 2008 arxiv.org 0807.1990
[17] Davidson M 2010 Physica E 42 317–322
[18] Schroedinger E 1930 Sitzungberichte der Preussischen Akademie der Wissenschaften 24 418–428
[19] Hestenes D 1990 Found. Phys. 20 1213–1232
[20] Huang K 1952 Am. J. Phys. 20 479–484
[21] Newman E T, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 J. Math. Phys. 6 918
[22] Newman E T 1973 J. Math. Phys. 14 102
[23] Lynden-Bell D 2004 Phys. Rev. D 70 105017
[24] Dirac P A M 1938 P. Roy. Soc. Lond. A Mat. 167 148–169
[25] Jackson J D 1999 Classical electrodynamics, 3rd Ed (Wiley)
[26] Plass G N 1961 Rev. Mod. Phys. 33 37
[27] Rohrlich F 2007 *Classical charged particles* (World Scientific)
[28] Teitelboim C, Villarroel D and Weert C G 1980 *La Rivista del Nuovo Cimento* 3 1–64
[29] Landau L D and Lifshitz E M 1951 *The classical theory of fields* (Cambridge, MA: Addison-Wesley)
[30] Jackson J D 2007 *Am. J. Phys.* 75 844
[31] Teitelboim C 1970 *Phys. Rev. D* 1 1572
[32] Dirac P A M 1978 *The principles of quantum mechanics* (Clarendon Press)
[33] Barut A O and Bracken A J 1981 *Phys. Rev. D* 23 2454
[34] de Broglie L 1924 *Researches sur la théorie des quanta, P.H. D. Thesis* Ph.D. thesis
[35] Bjorken J D and Drell S D 1998 *Relativistic Quantum Mechanics* 1st ed (McGraw-Hill)
[36] Guoanere M, Spighel M, Cue N, Gaillard M J, Genre R, Kirsch R, Poizat J C, Remillieux J, Catillon P and Roussel L 2008 *Annales de la Fondation Louis de Broglie* 33 85–91
[37] Wheeler J A and Feynman R P 1945 *Rev. Mod. Phys.* 17 157
[38] Scott T C, Mann R and Martinez II R E 2006 *Appl. Algebr. Eng. Comm.* 17 41–47
[39] Corless R M, Gonnet G H, Hare D E G, Jerey D J and Knuth D E 1996 *Adv. Comput. Math.* 5 329—359
[40] Davidson M P 2007 *Ann. Phys.* 322 2195–2210