Δm_d, Δm_s/Δm_d AND ε_K IN QUENCHED QCD

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I present quenched, lattice QCD calculations of the hadronic matrix elements relevant for
B^0_{d(s)} - \bar{B}^0_{d(s)} and K^0 - \bar{K}^0 mixing and briefly review the status of lattice predictions.

1 Introduction

Neutral meson mixing is a rich source of information on the Standard Model (SM). For instance, the
frequencies with which B_d and B_s mesons oscillate into their anti-particles yield constraints on the
Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{td} which determines the most poorly known
side of the unitarity triangle. K^0 - \bar{K}^0 mixing, on the other hand, through its measured contribution
to indirect CP violation in K \to ππ decays, provides a constraint on the triangle’s summit. These
constraints require quantification of the non-perturbative QCD dynamics which modify the simple,
underlying quark processes. The uncertainties in this quantification must be reduced to allow for as
stringent a test of the SM as possible with the triangle’s angles soon to be measured at the B-Factories,
HERA, the Tevatron and the LHC. Lattice QCD provides a first principle tool which can help achieve
this goal. In what follows, I present results of lattice calculations performed with C.J. David Lin and
the UKQCD Collaboration as well as a summary of lattice predictions.

2 B^0_{d(s)} - \bar{B}^0_{d(s)} mixing

B^0_q and \bar{B}^0_q (q=d, s) are not eigenstates of the weak hamiltonian and can therefore oscillate into
one another with a frequency given by the mass difference, Δm_q, of the eigenstates of the full SM
hamiltonian. In the SM, the dominant contribution to this mass difference is given by

Δm_q ≃ \frac{G_F^2}{8\pi^2} M_W^2 |V_{td}|^2 S_0 (x_t) \frac{|⟨\bar{B}_q | O^B=2 (μ) | B_q⟩|^2}{2 M_{B_q}},

O^B=2 = [\bar{b} γ^μ (1-γ^5) q] [\bar{b} γ_μ (1-γ^5) q].

1 Invited talk given at the XXXIVth Rencontres de Moriond: Electroweak Interactions and Unified Theories, Les Arcs,
France, 13-20 March 1999
2 On leave from Centre de Physique Théorique, Case 907, CNRS Luminy, F-13288 Marseille Cedex 9, France.
3 For other recent lattice reviews, please see [1].
\[ x_q \equiv (m_q^2/M_W^2) \] and \( \eta_B, S_0 \) and \( C_B \) are short-distance quantities, calculated perturbatively. Therefore, \(|V_{td}|\) can be determined from a measurement of \( \Delta m_q \) once the non-perturbative matrix element \( \langle B_q | O_q^{\Delta B = 2} | B_q \rangle \) is quantified. This is where the lattice enters.

While the measurement of \( \Delta m_d \) provides a direct determination of \(|V_{td}|\), one may also consider the ratio

\[
\frac{\Delta m_s}{\Delta m_d} \equiv \frac{|V_{ts}|^2}{|V_{td}|^2} \frac{M_{B_d}}{M_{B_s}} \frac{\langle \bar{B}_q | O_q^{\Delta B = 2} | B_q \rangle}{\langle B_d | O_d^{\Delta B = 2} | B_d \rangle} = \frac{|V_{ts}|^2}{|V_{td}|^2} \frac{M_{B_d}}{M_{B_s}} \ell_{sd} \equiv \frac{|V_{ts}|^2}{|V_{td}|^2} \frac{M_{B_q}}{M_{B_d}} \xi^2 .
\]

This ratio gives another possible constraint on \(|V_{td}|\) since with three generations, \(|V_{ts}| \simeq |V_{cb}|\). It further has the advantage that many common factors and uncertainties in the evaluation of the matrix elements cancel. Measuring \( \Delta m_s \), however, remains an experimental challenge, as the neutral \( B \) mesons oscillate rapidly: \( \Delta m_s \geq 12.4 \text{ ps}^{-1} \) at 95\% CL versus \( \Delta m_d = 0.477(17) \text{ ps}^{-1} \) for \( B_d \) mesons. Nevertheless, even this lower bound on \( \Delta m_s \) provides a significant constraint on \(|V_{td}|\) as can be seen in Figure 1.

Traditionally, the matrix element \( \langle B_q | O_q^{\Delta B = 2} | B_q \rangle \) is normalized by its vacuum saturation value:

\[
B_{B_q}(\mu) \equiv \frac{\langle \bar{B}_q | O_q^{\Delta B = 2} (\mu) | B_q \rangle}{\langle \bar{B}_q | O_q^{\Delta B = 2} | B_q \rangle_{\text{VSA}}} = \frac{\langle \bar{B}_q | O_q^{\Delta B = 2} (\mu) | B_q \rangle}{(8/3)M_{B_q}^2 f_{B_q}^2} .
\]

While one can actually determine the matrix element itself on the lattice, \( B \)-parameters are obtained from ratios of correlation functions in which many statistical and systematic uncertainties are expected to cancel. Furthermore, the matrix element has mass dimension four and therefore suffers very strongly from the uncertainty associated with the determination of the lattice cutoff which is of order 10\% in present day quenched calculations. As we shall also see later, it is advantageous to get the matrix element from an independent determination of \( B_{B_q} \) and \( f_{B_q} \) and the experimental value of \( M_{B_q} \).

### 2.1 A parte on decay constants

Because the leptonic decay constants of \( B \) mesons are required, I briefly digress to comment on their values. Many lattice groups have calculated these constants over the years. A recent compilation can be found in where the following summary numbers, which include uncertainties due to quenching, are given:

\[ f_{B_d} = 175 \pm 35 \text{ MeV} \text{ , } \quad f_{B_s} = 200 \pm 35 \text{ MeV} \text{ and } f_{B_s}/f_{B_d} = 1.14 \pm 0.08 \text{ ,} \]

\( ^d \text{Assuming three-generation unitarity and present day constraints on CKM parameters, } |V_{tb}| = 1 \text{ to high accuracy.} \)
Table 1: Results for B-meson B-parameters obtained with “relativistic” heavy quarks. \( \beta \) is the coupling at which the calculations were performed. \( \beta = \infty \) corresponds to results extrapolated to the continuum limit. \( \mu \) is the matching scale used. The numbers in italics are derived from the published results. Running is performed at two-loops using the procedure of J. Flynn et al. which assumes \( m_b = 5 \) GeV. \( \hat{B}_{B_d}^{nlo} \) is the RG-invariant B-parameter at NLO.

| action        | \( \beta [\text{GeV}] \) | \( B_{B_d}(\mu) \) | \( B_{B_d}(m_b) \) | \( \hat{B}_{B_d}^{nlo} \) | \( B_{B_s}/B_{B_d} \) |
|---------------|---------------------------|---------------------|---------------------|---------------------|---------------------|
| UKQCD(MFI SW) | (preliminary) 6.2 2.6     | 0.95(3)             | 0.91(3)             | 1.45(5)             | 0.99(3)             |
| BBS98        | Wilson \( \infty \) 2     | 0.94(5)             | 0.89(4)             | 1.41(7)             | 1.03(3)             |
| JLQCD96      | Wilson 6.3                | 1.02(13)            | 0.96(12)            | 1.53(19)            | \( \sim 1 \)        |
| BS96         | Wilson \( \infty \) 2     | 0.96(6)(4)          | 0.90(6)(4)          | 1.44(9)(6)          | 1.01(4)             |
| ELC92        | Wilson 6.4 3.7            | 0.86(5)             | 0.84(5)             | 1.34(8)             | \( \sim 0.99 \)     |
| BDHS88       | Wilson 6.1 2              | 1.01(15)            | 0.95(14)            | 1.51(22)            |                     |

in a normalization where \( f_\pi = 131 \) MeV. While the effects of quenching in \( (f_{B_s}/f_{B_d}) \) appear to be small in simulations,\( ^{11} \), Quenched \( \chi PT \) (Q\( ^{\chi PT} \)) indicates that they could be significant.\( ^{12} \)

### 2.2 \( \Delta B = 2 \) matrix elements and B-parameters

In Table 1, I present our results for \( B_{B_d} \) and \( B_{B_s}/B_{B_d} \) along with a compilation of results obtained by other groups who use, as we do, “relativistic” heavy quarks, as opposed to NRQCD or static quarks.\( ^{13} \) Quenching errors for these B-parameters and B-parameter ratios have been studied with the help of Q\( ^{\chi PT} \)\( ^{14} \) and have been found to be small. Combining this information with the results of Table 1, I give the following estimates:

\[
\hat{B}_{B_d}^{nlo} = 1.4(1) \quad \text{and} \quad \frac{B_{B_s}}{B_{B_d}} = 1.00(3) \ . \tag{5}
\]

In order to use these results to extract \( |V_{td}| \) from a measurement of \( \Delta m_d \), we need to combine them with a determination of \( f_{B_d} \). Using the estimate given in Section 2.1, I quote:

\[
f_{B_d} \sqrt{\hat{B}_{B_d}^{nlo}} = 207(42) \text{ MeV} \ . \tag{6}
\]

This prediction can be compared to the value obtained from an overconstrained, unitarity-triangle fit with \( f_{B_d} \sqrt{\hat{B}_{B_d}^{nlo}} \) left as a fit parameter:\( ^{15} \) \( f_{B_d} \sqrt{\hat{B}_{B_d}^{nlo}} = 223(13) \) MeV. This fit incorporates lattice predictions for \( r_{sd} \) and \( B_K \) consistent with the ones given below. Agreement is excellent, indicating a general consistency of the SM and lattice calculations. The central value and error bars, of course, reflect the choices made by the authors for the various inputs they use.

### 2.3 \( SU(3) \) breaking in \( \frac{\Delta m_s}{\Delta m_d} \)

There are at least two possible ways of obtaining \( r_{sd} \) from the lattice:

a) taking the product

\[
r_{sd}^{(a)} = \left( \frac{M_{B_s}}{M_{B_d}} \right)^2 \left( \frac{f_{B_s}}{f_{B_d}} \right)^2 \left( \frac{B_{B_s}}{B_{B_d}} \right) \, , \tag{7}
\]

with \( (f_{B_s}/f_{B_d}) \) and \( (B_{B_s}/B_{B_d}) \) determined on the lattice and \( (M_{B_s}/M_{B_d}) \) measured experimentally;
Table 2: Results for $r_{sd}$ as obtained using methods a) and b) with “relativistic” heavy quarks.

| action | $\beta$ | $r_{sd}^{(a)}$ | $r_{sd}^{(b)}$ |
|--------|--------|----------------|----------------|
| UKQCD (preliminary) | 6.2 | 1.37(13) | 1.70(28) |
| BBS98 Wilson | $\infty$ | 1.42(5)(28) | 1.76(10)(42) |

w/ results of Eqs. (4) and (5)

| | $r_{sd}^{(b)}$ |
|----------------|----------------|
| 8 | 1.34(19) |

b) from a direct determination of the ratio

$$r_{sd}^{(b)} = \frac{\langle \bar{B}_s | O_{\Delta S=2} | B_s \rangle}{\langle \bar{B}_d | O_{\Delta B=2} | B_d \rangle}.$$  

Our results for $r_{sd}$, together with the results of other groups who use “relativistic” heavy quarks are summarized in Table 2. Comparison of $r_{sd}^{(a)}$ at our two values of the lattice spacing ($\beta = 6.2$ and 6.0) suggests that discretization errors are small. Furthermore, we find that $r_{sd}^{(a)}$ and $r_{sd}^{(b)}$ are compatible, though the latter is less accurate and less reliable: its heavy-quark and light-quark-mass dependences are stronger and the corresponding extrapolations are less well controlled.

On the basis of these results and the comments on quenching in Sections 2.1 and 2.2, I quote as summary values:

$$r_{sd} = 1.4(2) \quad \text{or} \quad \xi = \sqrt{r_{sd}} \left( \frac{M_{B_d}}{M_{B_s}} \right) = 1.16(8).$$  

3 K$^0$–$\bar{K}^0$ Mixing

K$^0$–$\bar{K}^0$ mixing induces indirect CP violation in $K \rightarrow \pi\pi$ decays, quantified by the parameter $\epsilon_K$:

$$\epsilon_K e^{-i\xi} \simeq C_\epsilon C_K(\mu) B_K(\mu) A^2 \lambda^{10} \eta \left[ (1 - \bar{\rho}) A^2 S_0(x_t) + P_0(x_t, x_c, \ldots) \right] = (2.280 \pm 0.013) \times 10^{-3},$$  

with

$$\langle \bar{K}^0 | O_{\Delta S=2}(\mu) | K^0 \rangle = \frac{8}{3} M_K^2 f_K^2 \times B_K(\mu) \quad \text{and} \quad O_{\Delta S=2} = \left[ \bar{s} \gamma_\mu (1 - \gamma^5) d \right] \left[ \bar{s} \gamma^\mu (1 - \gamma^5) d \right].$$

This in turn leads to a hyperbolic constraint on the summit ($\bar{\rho}, \bar{\eta}$) of the unitarity triangle, once the $B$-parameter $B_K$ is determined (see Figure 1). Here, $C_\epsilon$ is obtained from well measured quantities, $A$ and $\lambda$ are Wolfenstein parameters and $\eta_2$, $C_K$, $S_0$ and $P_0$ incorporate perturbative, short-distance physics ($P_0$ also contains CKM factors). We calculate $B_K$ on the same lattices as the $\Delta B = 2$ matrix elements.

3.1 Chiral subtractions

Even though the basic ingredients, such as the operator mixing alluded to in the Appendix, are very similar to those used to calculate the $\Delta B = 2$ matrix elements, the physics here is very different, as it is governed by chiral symmetry. In the continuum, $O_{\Delta S=2}$ is in the $(27,1)$ representation of $SU(3)_L \times SU(3)_R$. On the lattice, however, the explicit breaking of chiral symmetry implies the following chiral expansion:

$$\langle \bar{K}^0(q) | O_{\Delta S=2}(\bar{p}) | K^0(\bar{p}) \rangle_{\text{lat}} = \alpha_K + \beta_K \bar{M}_K^2 + \gamma_K (p \cdot q) + \cdots,$$  

\[12\]
Table 3: Results for $B_K^{(NDR)}(2\text{ GeV})$. The results in the second half of the table were obtained with discretizations of the quark action which maintain a partial or full chiral symmetry, obviating the need for chiral subtractions.

| Ref.           | action  | $\beta$ | $B_K^{(NDR)}(2\text{ GeV})$ |
|----------------|---------|---------|-----------------------------|
| UKQCD98        | MFI SW  | 6.2     | 0.72^{+8}_{-6}              |
| APE98          | SW      | 6.0, 6.2| 0.68(21)^{(a)}              |
| GBS97          | Wilson  | 6.0     | 0.74(4)(5)                  |
| JLQCD99        | Wilson  | $\infty$| 0.69(7)                     |
| JLQCD98        | Staggered| $\infty$| 0.628(42)                   |
| KGS98          | Staggered| $\infty$| 0.62(2)(2)                  |
| BS97           | Domain Wall | $\infty$| 0.628(47)^{(b)}           |

(a) matched to NDR; (b) matched at tree level

where $\alpha_K$ and $\beta_K$ are pure lattice artefacts, while $\gamma_K(p \cdot q)$ and higher-order terms contain the physical contributions. In our calculation, where we match onto the continuum at one loop, the artefacts $\alpha_K$ and $\beta_K$ are proportional to $\alpha_s^2$ and $a\alpha_s$. The problem is that even though these factors are small, the physical contributions are chirally suppressed compared to $\alpha_K$.

To quantify and subtract the unphysical contributions, we study the chiral behavior of the $\Delta S = 2$ matrix element as a function of $M^2_K$ and $p \cdot q$. At $\beta = 6.2$ we find that artefacts such as $\alpha_K$ and $\beta_K$ are small and consistent with zero for all matching scales in the range $1/a \to \pi/a$. We have checked that our results are robust to procedure by normalizing the $\Delta S = 2$ matrix element in a variety of ways and using different mass and recoil variables for the chiral expansion. The determination of $B_K$ from the corresponding physical expansion terms should thus be reliable. At $\beta = 6.0$, the lattice artefacts are around 2 standard deviations away from zero and the results are less robust to procedure. Our findings, together with results obtained with less improved actions, suggest that discretization errors represent an important part of the traditionally observed residual chiral violations.

3.2 Results for $B_K$

We take our $\beta = 6.2$ result as our best estimate. We run it to 2 GeV at two-loops with $n_f = 0$ in the $\overline{\text{MS}}$-NDR scheme (small running). Our results, together with those of other groups are summarized in Table 3. On the basis of $\chi$PT and preliminary unquenched results,\(^{22}\) Sharpe estimates that $SU(3)$-breaking corrections\(^{23}\) and unquenching may lead to an $\mathcal{O}(10\%)$ increase in $B_K$ and ascribes an $\mathcal{O}(15\%)$ error to $B_K$ to account for the uncertainties in this estimate.\(^{24}\) Bijnens et al.\(^{25}\) reach similar conclusions. I choose to include these effects as a contribution to the error but not to the central value.

On the basis of these conclusions and the results given in Table 3, I quote:

$$B_K^{(NDR)}(2\text{ GeV}) = 0.65(10) \quad \text{or} \quad \hat{B}_K^{\text{nlo}} = 0.89(14),$$

where $\hat{B}_K^{\text{nlo}}$ is the two-loop RG-invariant $B$-parameter obtained from $B_K^{(NDR)}(2\text{ GeV})$ with $n_f = 3$ and $\alpha_s(2\text{ GeV}) = 0.3$. Again, this result compares very favorably to the SM fit of\(^{15}\), but this time with $\hat{B}_K^{\text{nlo}}$ as a fit parameter instead of $f_B s \sqrt{\hat{B}_K^{\text{nlo}}}$. $\hat{B}_K^{\text{nlo}} = 0.87^{+0.34}_{-0.20}$.

\(^{e}\)Calculations are performed with degenerate or nearly degenerate $s$ and $d$ quarks.
Table 4: Parameters of our calculations. “# cfs” is the number of gauge-field configurations on which the various matrix elements are computed (i.e. our statistics). $c_{SW}$ is the mean-field-improved coefficient of the SW term. $a^{-1}(m_{\rho})$ is the inverse lattice spacing as determined from a calculation of the $\rho$-meson mass.

| $\beta$ | lattice size | # cfs | $c_{SW}$ | $a^{-1}(m_{\rho})$ |
|---------|--------------|-------|----------|-------------------|
| 6.2     | $24^3 \times 48$ | 188   | 1.442    | 2.57(8) GeV       |
| 6.0     | $16^3 \times 48$ | 498   | 1.479    | 1.96(5) GeV       |

4 Conclusions

The lattice provides a means for calculating $\Delta B = 2$ and $\Delta S = 2$ matrix elements from first principles. A reliable determination of these matrix elements will be crucial for testing the SM with the forthcoming experiments on CP violating $B$ decays. Moreover, the hadronic matrix elements which appear in supersymmetric extensions of the SM can also be considered\[4]. In the next few years, more and more unquenched calculations will be performed, enabling a better quantification of quenching effects and eventually yielding fully unquenched results.

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Appendix: details of the calculations

We describe quarks with a mean-field-improved, Sheikholeslami-Wohlert (MFI SW) action. Compared to the standard Wilson action, the leading discretization errors are formally reduced by a factor of order $\alpha_s(a)$, and the mean-field-improvement may give additional numerical suppression. We perform calculations at two values of the cutoff, corresponding to couplings $\beta \equiv 3/(2\pi\alpha_s(a)) = 6.0$ (coarser lattice) and 6.2 (finer lattice). This enables us to quantify discretization errors. The parameters of the simulations are summarized in Table 4. Note that our simulations have high statistics. Unfortunately, because of the very high numerical cost of including the feedback of quarks on the gauge-fields, both calculations are performed in the quenched approximation.

Because a physical pion would feel the boundaries of the box in which we work and because the algorithms we use slow down rapidly for lighter quarks, we are restricted to work with quarks with masses on the order of $m_s/2$ or more. Thus, to obtain results at the physical values of the $u$, $d$ and $s$ masses, we perform all calculations for three values of the light-quark mass roughly in the range $m_s/2 \rightarrow m_s$ and extrapolate or interpolate to the physical mass values.

Furthermore, the graininess of our lattice forbids us from working with quarks whose masses are much larger than $m_c$. Thus, we perform all calculations for five values of the heavy-quark mass around that of the charm and extrapolate to the physical $b$-quark mass.

Finally, because Wilson fermions break chiral symmetry explicitly, the left-left operator, $O_{\Delta F=2}$, mixes with four-quark operators of different chirality. We subtract these wrong chirality contributions and match to the $\overline{\text{MS}}$-NDR scheme at one loop. To estimate the systematic error associated with our procedure, we vary the matching scale in the range $1/a \rightarrow \pi/a$. For the $B$-parameters discussed here, dependence on this scale is very small.
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