The phenomenology of neutrinos with Majorana mass terms and standard-model interactions derived in the charge-parity basis

R. Plaga
Federal Office for Information Security (BSI), D-53175 Bonn, Germany
rainer.plaga@bsi.bund.de

March 25, 2022

Abstract

The physical mechanisms that make a neutrino with standard-model (SM) weak interactions (“standard-model-interaction (SMI) neutrino”) a “lepton-number conservation (LNC) violating” neutrino such as the Majorana neutrino are analysed in a basis of two Majorana states that have opposite charge-parity (“charge-parity basis”).

It is necessary to assume that Majorana neutrinos interact with a certain weak-interaction Hamiltonian “$H_{PE}$” to prove that they have the same phenomenology (to first order) as the SMI-neutrino in the limit $m \to 0$. But $H_{PE}$ violates lepton-number conservation and is therefore qualitatively different from the SM Hamiltonian “$H_{SM}$”. Because even a Majorana neutrino is nowadays believed to interact with SM-interactions, $H_{PE}$ is excluded. This means that the above necessary assumption for the proof of the “Dirac-Majorana confusion theorem” can no longer be made.

Non standard-model Majorana mass terms modify the equation of motion of the neutrino by being different in sign and/or value for the two components of the charge-parity basis. A small Majorana mass that is larger than any Dirac mass makes the neutrino not a Majorana but a “pseudo-Majorana” particle that has no definite chirality and therefore has a different phenomenology than the physical neutrino. A combination of a large Majorana and Dirac mass of nearly equal value makes the neutrino a Majorana neutrino. However if this Majorana neutrino has SM interactions, its weak transition amplitudes are a factor $\sqrt{2}$ smaller than the ones observed for the physical neutrino. Only with a small Dirac mass that is larger than any Majorana mass (and in the massless case), the physical neutrino’s phenomenology is correctly predicted by the SM. Such a mass combination makes the neutrino a Dirac- or (the most likely possibility for the physical neutrino) Pontecorvo’s pseudo-Dirac particle which features neutrino-antineutrino oscillations, that violate LNC. Pseudo-Dirac neutrinos enable a completely negligible rate for neutrinoless double-beta...
decay if there is no Majorana-mass independent decay mechanism. Off-diagonal components of the mass matrix in the charge-parity basis make the neutrino a mixture of Dirac field with a different particle and anti-particle mass (i.e. a mass that violates $CPT$ invariance) and a pseudo-Dirac field. Such a neutrino leads to a phenomenology similar to the one with additional generations of sterile neutrinos.

1 Introduction

The question whether the physical neutrino is a Majorana particle, is one of the major open problems in particle physics[19]. The standard-model of particle physics[25] is not believed to be a complete theory, but to be correct to very good approximation. Therefore, while it is possible that the neutrino has extra properties that are not contained in the standard model (like e.g. Majorana mass terms), it is generally believed that its weak interaction is described, at least to good approximation, by the standard model. Let us call a neutrino with this property “standard-model-interaction (SMI) neutrino”.

This paper analyses the phenomenology of the SMI-neutrino with a Majorana mass term for the first time systematically in the “charge-parity basis”. The elements of this basis are two Majorana states, and therefore it seems well adapted to study this question. The aim is to fully understand the physical mechanisms that can “make” the SMI-neutrino a lepton-number conservation (LNC) violating neutrino such as the Majorana neutrino. The analysis is performed only for a single flavor because flavour mixing is not its topic. The full second quantised field theory is used throughout, because a first quantised treatment of Majorana neutrinos is impossible.

In section 2 I critically review the definition of a “Majorana neutrino”, introduce the charge-parity basis and formulate the definition in this basis. The physical mechanisms that can induce a SMI neutrino to violate LNC are analysed in section 3. In section 4 I will work out the phenomenology of neutrinos with mass terms that couple the components of opposite charge parity. Section 5 summarises the novel theoretical results and section 6 explains what they mean for the “physical”, i.e. really existing, neutrino.

1.1 Preview: which widely held beliefs are put into question and where these contradictions are resolved

The results of the paper contradict two widely held beliefs. The first is that a “Majorana-Dirac confusion theorem” applies to the physical neutrino. I will explain why the confusion theorem can only be proved under the assumption that Majorana neutrinos have an exotic, non-SM weak interaction in section 3.1.2. This assumption was only tenable until it was generally accepted that the SM describes the neutrino’s weak interaction to good approximation, and therefore the confusion theorem must not necessarily hold any more. This insight clears the way to not reject out of hand a straightforward demonstration
that the physical neutrino is no Majorana neutrino (in section 3.2.3), “because it contradicts the confusion theorem”.

The second belief is that a small Majorana mass makes neutrinos Majorana particles that are not their own charge conjugate and that in the basis of Majorana states the mass matrix has Majorana masses on the diagonal and Dirac masses as the off-diagonal elements. I will demonstrate in section 3.2.1:
- this belief is absolutely correct, however not for the Majorana neutrino but rather for states that I christen “pseudo-Majorana” and the basis formed with them
- the physical neutrino is not a pseudo-Majorana neutrino.

In the charge-parity basis both Dirac and Majorana masses remain on the diagonal of the mass matrix and its off-diagonal elements violate CP T invariance (section 4).

2 The definition of Majorana neutrino fields and the charge-parity basis

A Dirac neutrino field $\nu_D$ at a position $x, t$ in space-time can be written in compact form as [29, 5]:

$$\nu_D(x, t) = \sum_k b_k^\dagger v_k + \sum_k d_k^\dagger u_k$$  \hspace{1cm} (1)

Here the sum extends over all momentum and spin states (“modes”) the field can be in. The ($+$) symbolises the modes with positive energy and ($-$) the modes with negative energy, i.e. modes that are reversed in time in the sense that instead of a particle or antiparticle “moving into” mode $k$ (being created, symbolised by the $+$), a particle or antiparticle is “removed from” mode $k$ (being annihilated, symbolised by the $-$). $b_k^\dagger$ is the creation operator for a particle (symbolised by the $b$) with spinor $v_k$ in mode $k$, and $d_k$ is the annihilation operator for an antiparticle (symbolised by the $d$) with spinor $u_k$ in mode $k$. Below “$\nu_x$” symbolises a neutrino field in a state $x$ and $|\rangle$ a state vector that contains the amplitudes of the states a system can be in. |$\nu_x\rangle$ is a shorthand for a system that has amplitude 1 for being in the neutrino state $\nu_x$.

In the Majorana representation of the $\gamma$ matrices[12], which I will use throughout this paper (see appendix 7.1 for explicit $\gamma$ matrices I chose), the operation of “charge conjugation” is defined as taking the Hermitian conjugate of the field[5]:

$$\nu(x, t)^c = \nu(x, t)^\dagger T$$ \hspace{1cm} (Majorana representation)  \hspace{1cm} (2)

The transposition operator $T$ is to be applied only to the spinors ($u, v$), but not to the creation and annihilation operators $[27]$. Therefore charge conjugation turns the spinor column $u$ to a spinor column $v$, and e.g. a creation operator $b_k^\dagger$ into an annihilation operator $b_k^\dagger$. Applying eq.(2) to eq.(1) one obtains:

$$\nu_D^c(x, t) = \sum_k (-) b_k u_k + \sum_k (+) d_k^\dagger v_k$$  \hspace{1cm} (3)
Comparing eq. (1) and eq. (3) one finds that charge conjugation is fully characterised by: “replace $b$ with $d$ and $b^\dagger$ with $d^\dagger$”. For the correct description of charged fermions, like electrons, it is necessary to make the fundamental assumption that the operators $b^\dagger$ and $d^\dagger$ are qualitatively different. In other words, one has to assume that there are two types of fundamental fields “particles” and “antiparticles”. Majorana asked[18]: might for neutrinos $b^\dagger \equiv d^\dagger$? A neutrino with this property, a “Majorana neutrino” $\nu_M$, is self-charge conjugate i.e. it fulfils the “Majorana condition”[4]:

$$\nu_M(x,t)^c = e^{i\alpha} \nu_M(x,t)$$

(4)

Here $e^{i\alpha}$ is a phase factor which is called “the charge-parity of the field”. There are two widespread misconceptions about this definition.

The first misconception is that charge conjugation flips chirality, i.e. a left-chiral neutrino becomes a right-chiral antineutrino under charge conjugation[1]. It is in principle well known[2] that charge conjugation has no effect whatsoever on the spatial and spin modes $k$. In particular a field creating a (massless) neutrino with chirality=-1 (left-chiral neutrino) is transformed by charge conjugation into a field creating an antineutrino with unchanged chirality = -1. There is confusion about this fact in the literature, presumably because a spinor $$(1-\gamma_5)\frac{1}{2}v(x) = v_L$$

is indeed turned into a spinor state $$(1+\gamma_5)\frac{1}{2}u(x) = u_R$$ with opposite chirality by charge conjugation[3]. However, if the left-chiral neutrino field state:

$$\nu_{DL}(x,t) = \sum_k^{(+)} b_k^\dagger v_kL + \sum_k^{(-)} d_k u_kR$$

(5)

is charge conjugated, both the spinor and the creation operator and annihilation are charge conjugated and one obtains:

$$\nu_{DL}(x,t)^c = \sum_k^{(-)} b_k^\dagger u_kR(x) + \sum_k^{(+)} d_k^\dagger v_kL(x)$$

(6)

i.e. indeed a left-chiral neutrino field is charge-conjugated to a left-chiral antineutrino field. Charge conjugation only affects the “particle-antiparticle” character of the field.

The second misconception is a claim[13, 15] that “dressed” (a fancy expression for “weakly interacting”) neutrinos cannot be self-charge conjugate i.e. cannot fulfil eq. (4) “because the weak interaction is not invariant under charge conjugation”. The weak interaction of a left-chiral Majorana neutrino $\nu_{ML}$ is not invariant under charge-conjugation because e.g. the virtual reaction $\nu_{ML} \rightarrow e^-$

---

1The author was under the spell of this misconception himself. I am very much indebted to P. Pal for explaining me this point in detail (see also Ref. [21]).

2See e.g. fig.3.5 in the classic textbook of Perkins[23], that shows that the neutrino helicity does not change under charge conjugation. But in the massless limit helicity and chirality are identical (see e.g. [21] for a proof), so that neither changes chirality under charge conjugation.

3For a spinor in the Majorana representation chosen here, charge conjugation is complex conjugation. $\gamma_5$ is purely imaginary in the Majorana representation. Therefore charge conjugation turns $1-\gamma_5$ to $1+\gamma_5$. 

---
W^{+} takes place but its charge conjugate $\nu_{ML} \rightarrow e^{+} W^{-}$ does not. The fact that the field $\nu_{ML}$ does not change under charge conjugation does not invalidate the fact that its weak reactions are not charge-conjugation invariant. Therefore the non-invariance of the reaction under charge-conjugation is not in contradiction to a self-charge conjugacy of the neutrino. See appendix B.2 for a detailed refutation of a published proof for the above claim and further comments.

I now introduce the charge-parity basis consisting of two Majorana states $\nu_{+}$ and $\nu_{-}$. Below I will often drop the arguments $x, t$ and often also a subindex $R, L$ indicating the chirality of the neutrino state, because the set of states of greatest interest for the present analysis has just the elements of $\nu$ (eq. (1)) and $\nu^c$ (eq. (3)). $\nu_{+}$ is defined as:

$$\nu_{+} = \frac{1}{\sqrt{2}}(\nu_{D} + \nu_{D}^c)$$

(7)

which fulfils eq. (4) with a phase (or charge-parity) $e^{i\alpha} = +1$:

$$\nu_{+}^c = \nu_{+}.$$  

(8)

The other component is:

$$\nu_{-} = \frac{1}{\sqrt{2}}(\nu_{D} - \nu_{D}^c),$$

(9)

which possesses a charge-parity of $-1$:

$$\nu_{-}^c = -\nu_{-}.$$  

(10)

Obviously $\nu_{+}$ and $\nu_{-}$ are both Majorana states $\nu_{M}$ because they fulfil eq. (4). We can then describe the Dirac neutrino as

$$\nu_{D} = \frac{1}{\sqrt{2}}(\nu_{+} + \nu_{-}),$$

(11)

and the Dirac antineutrino as

$$\nu_{D}^c = \frac{1}{\sqrt{2}}(\nu_{+} + \nu_{-})^c = \frac{1}{\sqrt{2}}(\nu_{+} - \nu_{-}) \neq \nu_{D}.$$  

(12)

In order to formulate the definition of the Majorana neutrino in the charge-parity basis we need to permanently restrict the set of states available to the neutrino to a subset of fields that fulfils the Majorana condition eq. (4). The Majorana condition eq. (4), i.e. definition of the Majorana neutrino evidently always holds if the

- **Majorana restriction**:

The neutrino state $e^{i\alpha} \nu_{-}$ (or equivalently $e^{i\alpha} \nu_{+}$) is permanently excluded from the physically accessible Hilbert space.

is guaranteed by some physical mechanism (for one angle $\alpha$). If the Majorana restriction holds in general, the neutrino “is” a Majorana field.
3 Mechanisms that “make” the neutrino a Majorana neutrino, or another LNC violating neutrino

Originally it was simply assumed that the neutrino might “be” a Majorana neutrino in the sense that the statement “the state $\nu^-$ does not exist” (i.e. the Majorana restriction) is a law of nature. However, as we discuss in detail below, in the unmodified standard model the neutrino is a Dirac neutrino, and according to eq.(11) then both states $\nu_+$ and $\nu_-$ do exist. Therefore, to make the neutrino a Majorana particle, some mechanism beyond the SM needs to implement the Majorana restriction. Which mechanisms can exclude components of the charge parity basis from the accessible Hilbert space without violating the above assumption that the SM describes the weak interaction to good approximation? In principle there are two possibilities:

1. Non-SM weak interactions of the field
2. Non-SM masses of the field

I will now analyse both possibilities in turn.

3.1 Possibility 1: Weak interaction Hamiltonians that make Majorana neutrinos

3.1.1 SMI-neutrinos cannot be made Majorana neutrinos by the weak interaction

Can the weak interaction create a massless (or only Dirac-massive) Majorana neutrino? As discussed in the Introduction, today there is a broad consensus that the neutrino’s charge current Hamiltonian is the one described by the standard model i.e.

$$H_{SM} = \frac{g}{\sqrt{2}} (W^- \bar{\nu} \gamma_\mu (1 - \gamma_5) \nu_D + W^+ \bar{\nu}_D \gamma_\mu (1 + \gamma_5) \nu).$$  (13)

Clearly this interaction creates a massless (or only Dirac massive) neutrino as $\nu_D$, i.e. as a Dirac particle. But if a different charged current Hamiltonian, let us call it “$H_{PE}$”, would create e.g. only the field state $\nu_+$, then the field state $\nu_-$ would be permanently excluded (i.e the Majorana restriction would be fulfilled for $\alpha=0$) and a massless neutrino would be created as a Majorana neutrino. The charged-current interaction term that achieves this is:

$$H_{PE} = \frac{g}{2\sqrt{2}} (W^- (\bar{\nu} \gamma_\mu (1 - \gamma_5) \nu_+) + W^+ (\bar{\nu}_D \gamma_\mu (1 + \gamma_5) \nu)).$$  (14)

It was first discussed by Pauli[22] (his eq.(22)) and Enz[7]. In eq.(14) I replaced a hadronic current in their expression for $H_{PE}$ by the W-boson for simplicity. I propose to call $H_{PE}$ “Pauli-Enz”(PE) interaction.

Using eq.(7) $H_{PE}$ can be rewritten as:

$$H_{PE} = \frac{1}{\sqrt{2}} H_{SM} + \frac{g}{4} (W^- (\bar{\nu} \gamma_\mu (1 - \gamma_5) \nu_D) + W^+ (\bar{\nu}_D \gamma_\mu (1 + \gamma_5) \nu)).$$  (15)
The SM necessarily does conserve lepton number\[^{17}\] by mapping a positron in a neutrino and an electron into an antineutrino with a SU(2) transformation. We see that $H_{PE}$ does not merely add a non-SM LNC violating second term but that it changes the first SM term by a factor $\frac{1}{\sqrt{2}}$. Moreover the LNC violating second term is not small, but of equal magnitude as the first term. Therefore, if neutrinos would interact with $H_{PE}$, the SM would be wrong. In appendix\[^{7,4}\] explicitly demonstrate LNC violation in a weak reaction with $H_{PE}$. Summarising I formulate the following:

- **Fact**

  *If the weak interaction would implement the Majorana restriction (“make the neutrino Majorana”) - in the sense that it would produce even a massless neutrino as Majorana particle - the SM would not describe weak interactions to good approximation.*

Therefore - if the physical neutrino is a Majorana particle - it must be because of the action of non-SM Majorana mass term (see section\[^{3.2}\] where we will discuss how this mechanism keeps the SM intact).

Let us examine the argument of this subsection again, from a somewhat different angle. Some SM bosons happen to be “a priori” self-charge conjugate, namely the photon, the $Z_0$ and the Higgs boson (even though they are not called Majorana fields, because this designation is reserved for fermions). The neutrino fields happens not to be self-charge conjugate in the SM, i.e. the SM weak interaction produces only Dirac neutrinos and antineutrinos if they are massless or only Dirac-massive. One can consider a theory of weak interactions with a charged-current Hamiltonian “$H_{PE}$” in which massless neutrinos are produced as self-charge conjugate fields (i.e. are self-charge conjugate “a priori”) but such a theory cannot be the SM in which massless neutrinos are Dirac particles because they are obtained by a SU(2) transformation from electrons, which are necessarily Dirac particles because they are electrically charged.

### 3.1.2 SM and LNC violating weak interactions of Majorana neutrinos: the confusion theorem

In the late 1950s, long before the SM was established, the phenomenology of parity-violating Majorana neutrinos was clarified in a flurry of papers\[^{22, 7, 6, 26}\]. At that time the possibility that the weak interaction charged current Hamiltonian is the LNV violating $H_{PE}$ was entirely reasonable. The title of Enz’ Ref.\[^{7}\] is: *Fermi Interaction with Non-Conservation of “Lepton Charge” and of Parity*. The standard case of a weak-interaction Hamiltonian $H_{SM}$ was equally reasonable. (At the time, the expression for $H_{PE}$ and $H_{SM}$ contained a hadronic current instead of the still unknown W-boson, I will use the modern notation below.)

In the previous subsection\[^{3.1.1}\] I excluded the possibility that $H_{PE}$ is the Hamiltonian of the physical neutrino. In this subsection I review the consequences of not excluding it, in order to understand the origin of the “confusion theorem” and why one of the assumptions needed for its proof can no longer be made.
In the 1950s there was no way to decide a priori if $H_{PE}$ and $H_{SM}$ is the correct Hamiltonian and therefore all authors [6, 26] implicitly made the

- **Assumption PE**
  1. If neutrinos are Majorana particles they interact with $H_{PE}$ and
  2. If they are Dirac neutrinos they interact with a Hamiltonian $H_{SM}$.

The following theorem can be proved if, and only if, assumption PE is true.

- **Confusion theorem**
  Dirac and Majorana neutrinos have the same phenomenology in the limit $m \to 0$.

This statement was originally called “equivalence theorem” by Radicati and Touschek [26] and reappeared as “confusion theorem” in the more modern literature [15]. Here is its proof:

If the neutrino is massless, it is either perfectly right- or left-chiral. From eqs. (14,13) the transition elements from a positron and $W^+ \to$ a massless right-chiral Majorana neutrino interacting via $H_{PE}$ and to a massless Dirac neutrino interacting via $H_{SM}$ (the reactions eqs. (53,54) discussed in appendix 7.4) is equal to:

$$\langle (e^+W^-)_k | H_{SM} | \bar{\nu}_{DR} \rangle = \langle (e^+W^-)_k | H_{PE} | \bar{\nu}_{+R} \rangle = c \neq 0 \quad (16)$$

Here the index $k$ symbolises a well defined spatial and spin state of $W^-e^+$. The transition from an electron and $W^+ \to$ a right-chiral neutrino is 0 for vanishing mass in both cases:

$$\langle (e^-W^+)_k | H_{SM} | \bar{\nu}_{DR} \rangle = \langle (e^-W^+)_k | H_{PE} | \bar{\nu}_{+R} \rangle = 0 \quad (17)$$

The equality of these generic transition amplitudes of a Majorana neutrino interacting with $H_{PE}$ and a Dirac neutrino interacting with $H_{SM}$ (an analogous argument for the weak neutral currents) prove that Dirac and Majorana neutrino have the same weak interaction phenomenology if, and only if, assumption PE is true.

$\langle (e^-W^+)_k | H_{PE} | \bar{\nu}_{+R} \rangle$ does not vanish for massive Majorana neutrinos, because massive neutrinos cannot be perfectly left-chiral. But $\langle (e^-W^+)_k | H_{SM} | \bar{\nu}_{DR} \rangle$ does vanish even for massive Dirac neutrinos due to lepton-number conservation. Therefore the confusion theorem only holds in the massless limit. End of proof.

To the best of my knowledge it was never pointed out explicitly that the confusion theorem is only valid if assumption PE is. In the 1950s no such mention was necessary, because, at the time, PE might well have been correct and under this circumstance massless Dirac and Majorana neutrinos could not be discriminated. However, nowadays it is believed that even a Majorana neutrino would have SM weak interactions to good approximation, i.e. that PE.1 can no longer be made. Summarising I formulate the following
Fact

If neutrinos are assumed to interact with SM interactions to good approximation, assumption PE.1, that is necessary for the proof of the confusion theorem, can no longer be made. This does not mean that in principle Dirac and Majorana neutrino cannot have the same phenomenology in the massless limit. Rather it only means that one cannot conclude from the confusion theorem that they have. In section 3.2.3 I will discuss how the SMI-neutrino can be made a light Majorana neutrino by assigning certain values for the Majorana and Dirac masses. Based on the above Fact, it will not be possible to simply conclude that such a Majorana neutrino has the weak-interaction phenomenology of the Dirac neutrino in the limit $m \to 0$ “because of the confusion theorem”. Rather we will have to calculate its weak-interaction transition amplitude to a $W^-$ and $e^+$ and to compare it with the transition amplitude in eq. (16).

3.2 Possibility 2: Mass terms that make pseudo-Majorana, pseudo-Dirac and Majorana neutrinos

How can a Majorana mass term make the neutrino self-charge conjugate, i.e. a Majorana neutrino? In other words, how can a mass restrict the set of possible field states, i.e. implement the Majorana restriction of section 2? In order to answer this question, I will first derive the equation of motion of a free massive neutrino in the charge-parity base. Let us write eq. (11) in vectorial form:

$$\nu_D = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_+ \\ \nu_- \end{pmatrix}$$

(18)

so that the upper component has charge parity +1 and the lower charge parity = –1. We now search for the mass matrix in this basis.

I consider a slightly extended standard-model in which neutrinos can possess “Dirac masses”. These masses are assumed to arise in exactly the same way as the masses of the charged leptons via coupling to the Higgs field. The introduction of such masses has been aptly called “correcting an oversight” of the original SM formulation[25], because there is no known reason that the coupling of the neutrino to the SM Higgs field must vanish exactly. The Dirac mass term in the neutrino Lagrangian is then:

$$L_{m_{\text{Dirac}}} = m_D (\bar{\nu}_D \nu_D + \nu_D \bar{\nu}_D)$$

(19)

Inserting eqs. (11,12) into the mass term eq. (19) one gets in the charge-parity basis:

$$L_{m_{\text{Dirac}}} = m_D (\bar{\nu}_+ \nu_+ + \bar{\nu}_+ \nu_- + \bar{\nu}_- \nu_+ + \bar{\nu}_- \nu_-)$$

(20)

From the eqs. (8,10) $\bar{\nu}_+ = \nu_+ \gamma_0$ and $\nu_- = -\nu_- \gamma_0$ so that the second and third term cancel each other and we are left with:

$$L_{m_{\text{Dirac}}} = m_D (\bar{\nu}_+ \nu_+ + \bar{\nu}_- \nu_-) = (\bar{\nu}_+ \nu_-) M_D \begin{pmatrix} \nu_+ \\ \nu_- \end{pmatrix}$$

(21)
with

\[ M_D = \begin{pmatrix} m_D & 0 \\ 0 & m_D \end{pmatrix}. \]  

(22)

The Dirac mass term of the SMI neutrino is seen to be mathematically equivalent to mass terms of the two component Majorana fields which have the same sign and value.

There are plausible theoretical reasons to suspect that non-normalisable non-standard-model Majorana mass terms of the neutrino exist\(^{30}\). In the particle-antiparticle basis they are:

\[ L_{\text{Majorana}}^{M} = m_M (\bar{\nu}_D \nu_D + \bar{\nu}_c \nu_c) \]  

(23)

with \( m_M \approx 10^{-5} - 10^{-1} \text{ eV} \). Inserting eqs. (11,12) into eq.(23) yields in the charge parity basis:

\[ L_{\text{Majorana}}^{M} = m_M (\bar{\nu}_+ \nu_+ - \bar{\nu}_- \nu_-) = (\bar{\nu}_+ \nu_-) M_M \begin{pmatrix} \nu_+ \\ \nu_- \end{pmatrix} \]  

(24)

with

\[ M_M = \begin{pmatrix} m_M & 0 \\ 0 & -m_M \end{pmatrix}. \]  

(25)

From this equation and eq.(21) the total mass matrix with both Dirac and Majorana mass terms is:

\[ M_t = M_D + M_M = \begin{pmatrix} m_+ & 0 \\ 0 & m_- \end{pmatrix} = \begin{pmatrix} m_D + m_M & 0 \\ 0 & m_D - m_M \end{pmatrix} \]  

(26)

From the Lagrangian of the free neutrino field

\[ L = \bar{\nu}_D i \gamma^\mu \frac{\partial}{\partial x^\mu} \nu_D + \bar{\nu}_c M_i \nu_D \]  

(27)

the equations of motion for \( \nu_+ \) and \( \nu_- \) are:

\[ i \gamma^\mu \frac{\partial}{\partial p^\mu} \nu_+(x,t) + (m_D + m_M) \nu_+(x,t) = 0 \]  

(28)

and

\[ i \gamma^\mu \frac{\partial}{\partial p^\mu} \nu_-(x,t) + (m_D - m_M) \nu_-(x,t) = 0 \]  

(29)

It is now clear in principle how finite Majorana masses can restrict (or enlarge) the state space of the neutrino: by making the mass term of the \( \nu_+ \) and \( \nu_- \) component different in value or even sign. This requires no modification in the weak interactions and together with the form of eq.(26) this means that:

- **Fact**
  
  The Majorana mass terms are a mere non-SM addition to the SM Lagrangian. They leave the weak-interaction and Higgs-mass part of the SM completely intact.
To study the effect of this on the neutrino’s state space we have to discriminate between the following cases:

1. \( m_D \) and \( m_M \) are “small” (with respect to the smallest energies in laboratory neutrino reactions, i.e. \( m_D, M \ll \text{keV} \)) and
   (1.1) \( m_M > m_D \) (section 3.2.1) (opposite sign mass term in eqs. (28,29))
   (1.2) \( m_M \leq m_D \) (section 3.2.2) (like sign mass term in eqs. (28,29))

2. \( m_D \) and \( m_M \) are “large” with respect to the largest energies occurring in laboratory neutrino reactions i.e. \( m_D, M \gg 100 \text{ GeV} \) but equal to each other to within a difference smaller than the masses of the physical neutrino (section 3.2.3).

Case 1.1 includes the simplest LNC violating case of a single small Majorana mass term without a Dirac mass term and will therefore be discussed first. Contrary to expectation, it does not make the neutrino a Majorana neutrino. Case 1.2 is well known to correspond to a pseudo-Dirac neutrino if \( m_D \gg m_M \), we will find that this is true also for \( m_D \geq m_M \). Only Case 2. enforces the Majorana restriction of section 2 and makes the SMI neutrino a Majorana particle.

### 3.2.1 Small \( m_D < m_M \): pseudo-Majorana neutrinos

If \( m_M \) is “small” (in the above sense) and larger than \( m_D \) the mass terms in eq. (28) and eq. (29) have an opposite sign. It is well known that the sign of the mass term in the Dirac equation is reversed by multiplying the field by \( \gamma_5 \), because \( \gamma^\mu \) and \( \gamma_5 \) anticommute\(^{[28]} \). If the sign of the mass terms in eq. (28) and eq. (29) the same (as it is for case 1.2), it has no effect on observable physics, because \( \gamma_5 \) is then just an overall phase factor of both \( \nu_+ \) and \( \nu_- \). However, if the sign is different, as assumed for this subsection, \( \nu_+ \) and \( \nu_- \) obtain a relative phase of \( \gamma_5 \) which has observable effects. Independent of the special form of the spatial state (which is of no concern for our discussion) the joint solutions of the system of equations eq. (28) and eq. (29) can have either have the following “particle”-solution form:

\[
\nu_{pM+} = \frac{1}{\sqrt{2}} (\gamma_5 \nu_+ + \nu_-) = \frac{1}{2} (1 + \gamma_5) \nu - (1 - \gamma_5) \nu^c
\]  

(30)

or the corresponding “antiparticle”-solution form:

\[
\nu_{pM-} = \frac{1}{\sqrt{2}} (\gamma_5 \nu_+ - \nu_-) = \frac{1}{2} (1 + \gamma_5) \nu^c - (1 - \gamma_5) \nu
\]

(31)

The right-hand form of the equation is obtained by inserting eqs. (7,9). Taking into account that charge conjugation does not flip chirality (see discussion in section 2) one immediately concludes from the right-hand side that charge conjugation transforms \( \nu_{pM+} \) to \( \nu_{pM-} \). Therefore \( \nu_{pM} \) does not fulfill eq. (4) and is no Majorana field, in spite of containing neutrino and antineutrino components
of equal size. The reason is that the restriction is to a field where a phase factor $\gamma_5$ appears where a phase factor $i$ would have been necessary to effect the Majorana restriction of section 2 with $\alpha = e^{i\frac{\pi}{4}}$. The limit $m_M \to 0$ is not smooth, because the phase shift of $\gamma_5$ appears for arbitrarily small $m_M$ but not for $m_M = 0$.

Because $\nu_{pM+}$ contains both neutrino and antineutrino components, just like a Majorana neutrino does (eq. (11)), I christen it “pseudo-Majorana” neutrino. $\nu_{pM+}$ is composed of the components that can both interact weakly according to $H_{SM}$, i.e. it is the “active component”, whereas $\nu_{pM-}$ is sterile. From eqs. (30,31) the pseudo-Majorana neutrino has no definite chirality because it has components of left- and right chirality (remember from section 2 that the chirality of a field does not depend on whether it is particle or antiparticle). In the massless limit chirality = helicity, i.e. the pseudo-Majorana neutrino has no definite helicity in the massless limit. But the physical neutrino was of course experimentally determined to have a helicity = 1 in a state that is massless to good approximation under the sole assumption that angular momentum is conserved\[10\]. Summarising:

- **Fact**
  
  *The physical neutrino is definitely no pseudo-Majorana neutrino (that has no definite chirality), because it is experimentally observed to have helicity = 1 in the massless limit and therefore has a chirality = 1 to good approximation.*

The pseudo-Majorana neutrino possesses two other properties that were occasionally erroneously assigned to Majorana neutrinos. Below I will explain them. Firstly pseudo-Majorana neutrinos are self-$CPT$ conjugate in the following sense: the $CPT$ transforms\[5\] of $\nu_{pM+}$:

$$\nu'_{pM+} = CPT \nu_{pM+} (CPT)^{-1} = \frac{i}{\sqrt{2}} (\nu_+ - \gamma_5 \nu_-)$$

(32)

would also interact weakly with SM interactions. Therefore a counterfactual, “physical” pseudo-Majorana neutrino would be in state of incoherent mixture with the density matrix:

$$\rho(\nu_{pM}) = \frac{1}{\sqrt{2}} \left( |\nu_{pM+}\rangle \langle \nu_{pM+}| + |\nu'_{pM+}\rangle \langle \nu'_{pM+}| \right).$$

(33)

which is $CPT$ self-conjugate. The $CPT$ self-conjugacy is seen to be the result of the simple fact that it makes no difference whether the phase shift $\gamma_5$ is applied to $\nu_+$ or $\nu_-$. Secondly in the “active-sterile” basis with the components $\nu'_{pM+}$ and $\nu'_{pM-}$

$$\nu_{pM} = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu'_{pM-} \\ \nu'_{pM+} \end{pmatrix}.$$  

(34)

\[4\gamma_5 “comes close”, it is purely imaginary in the Majorana representation.]
(but not the charge parity basis) the neutrino has a mass matrix with $m_M$ on the diagonal and $m_D$ on the off-diagonal. Inserting eqs. (30,31) it is easy to verify that:

$$\langle \bar{\nu}_p M - \bar{\nu}_p M^+ \rangle M^p M \left( \begin{array}{c} \nu_{pM}^- \\ \nu_{pM}^+ \end{array} \right) = L^\text{Dirac}_m + L^\text{Majorana}_m$$

(35)

with

$$M^p M = \left( \begin{array}{cc} m_M & m_D \\ m_D & m_M \end{array} \right)$$

(36)

for $L^\text{Dirac}_m$ and $L^\text{Majorana}_m$ from eqs. (24,25). This mass matrix (or a similar one, typically with the upper diagonal element set to 0) is often presented in the literature with the claim that its mass eigenstates for $m_D \ll m_M$, that are $\nu_{pM}^+$ and $\nu_{pM}^-$ to good approximation, are Majorana states. The origin of this misconception lies in the first misconception about charge conjugation, discussed in section 2. It is evident e.g. from eq. (30) that $\nu_{pM}^+$ would be self-charge conjugate if charge conjugation would flip chirality, but this is not the case. The fact that both components of $\nu_{pM}^+$ have SM weak interaction might have led to a confusion that it conforms to the confusion theorem (section 3.1.2). Summarising three bases have been discussed: “active-sterile” basis of pseudo-Majorana states $\nu_{pM}^+$ and $\nu_{pM}^-$, the “particle-antiparticle” basis of Dirac states $\nu_D$ and $\nu_D^\text{c}$ and the “charge-parity” basis of Majorana state $\nu_+$ and $\nu_-$. All of them can be used to study the structure of the neutrino field. But they must not be confused.

### 3.2.2 Small $m_M \leq m_D$: pseudo-Dirac neutrinos

This case is similar to the one discussed in the previous subsection 3.2.1, but there is no relative phase between $\nu_+$ and $\nu_-$ because the sign of the mass terms in eq. (28) and eq. (29) is the same:

$$\nu_{pD} = \frac{1}{\sqrt{2}}(\nu_+ + \nu_-) \neq \nu_{pD}^c.$$ 

(37)

$\nu_{pD}$ the well known pseudo-Dirac state[31], proposed by Pontecorvo in the very first paper on neutrino oscillations[24].

Upon production pseudo-Dirac neutrinos are Dirac neutrinos. But because $m_+ \neq m_-$ the $\nu_+$ and $\nu_-$ components of a pseudo-Dirac neutrino have different momenta and develop a relative phase of $e^{i\beta}$ with $\beta = \pi \Delta m^2/(2 E)$ after propagating a distance $D$ (with $\Delta m^2 = m_+^2 - m_-^2$). The phase angle is $\beta = \pi$ when:

$$D(\beta = \pi) = L = 2 \frac{E}{\Delta m^2} = \frac{E}{2m_D m_M}.$$ 

(38)

The pseudo-neutrino has then oscillated into a pseudo-antineutrino

$$\nu_{pD}(\beta = \pi) = \frac{1}{\sqrt{2}}(\nu_- + e^{i\pi} \nu_+) = \frac{1}{\sqrt{2}}(\nu_+ - \nu_-) = \nu_{Dc}.$$ 

(39)
These oscillations violate LNC, a lepton oscillates into an anti-lepton that can then effect a SM weak reaction with a right-chiral amplitude suppressed by $a = (m_{\nu} - p_{\beta})^2$. Such an anti-lepton is sterile to first approximation but violates LNC with a typically very small reaction amplitude $a$.

Neutrinoless double $\beta$-decay is strongly suppressed for pseudo-Dirac neutrinos because here the initial weakly produced neutrino strictly conserves lepton number. It then oscillates and violates LNC with a wavelength $L$ (eq.(38)) that is very large compared to nuclear dimensions $r_N$. A very rough estimate assumes that an emitted neutrino oscillates over a distance $D = r_N$ developing a small antineutrino amplitude that then annihilates with another emitted neutrino. Approximating $r_N \approx 0.1/E$ (for energies on the order of a few MeV), it yields a suppression factor relative to the decay rate induced by a Majorana neutrinos of $1/(r_N)^4 = 10^{-15} (\frac{\Delta m^2}{E})^8$. Assuming $\Delta m^2 < 10^{-5}$ and $m_\nu = 1$ eV, this yields a completely negligible lower limit on the pseudo-Dirac neutrino induced neutrinoless double-beta decay half life of $^{76}$Ge (assuming a Majorana-neutrino induced half life of $10^{24}$ years for $m_M = 1$ eV[15]) on the order of $10^{70}$ years. If the physical neutrino were a pseudo-Dirac neutrino an observation of neutrinoless double-beta decay would be evidence for new physics beyond the SM and beyond the mere addition of Majorana mass terms.

3.2.3 Large $m_D \approx m_M$: the Majorana field

In essence Majorana’s insight was that the two components $\nu_+ \text{ and } \nu_-$ of a Dirac field are dynamically independent, i.e. they are both solutions to the Dirac equation. Therefore if one of the two components would have a mass so large that its particles are not produced in experiments and the other had a small mass a Dirac field would only create Majorana particles and thus effectively be a Majorana field.

Such a scenario is realised if $m_D$ and $m_M$ are both larger than the reaction energy of a given experiment so that particles of the $\nu_+$ field have a too large mass $m_+ = m_D + m_M$ (eq.(26)) to appear in the laboratory and - at the same time - $m_M$ is equal to $m_D$ within to better than an eV or so, so that the physical Majorana neutrino $\nu_-$ has a small mass $m_- = m_D - m_M < eV$, that does not violate experimental bounds on neutrino masses. Such a scenario seems contrived, because there is no known reason why the Majorana mass should be on the weak scale and nearly exactly equal to the Dirac mass but it cannot be excluded that there is some theoretical reason for such a fine tuning.

If a SMI-neutrino would be such a Majorana neutrino, what would be its interactions? We had concluded in section 3.1.2 that the historic confusion theorem no longer tells us that they must be the same as the ones for a Dirac neutrino for $m \to 0$ because this theorem rests on the assumption PE.1 that states that a Majorana neutrino does not interact with SM-model weak interactions but the exotic lepton number violating Pauli-Enz interaction. Therefore we have to

---

5 Another possibility would be Majorana masses that depend on charge-parity, so that $m_+ \gg eV > m_-$. 14
calculate the transition element from a positron and \( W^- \) state to an an approximately right-chiral Majorana neutrino as in eq.\( (16) \) with SM interactions:

\[
\langle (e^+ W^-)_k | H_{SM} | \bar{\nu}_{+R} \rangle = \frac{1}{\sqrt{2}} \left( \langle (e^+ W^-)_k | H_{SM} | \bar{\nu}_{DR} \rangle + \langle (e^+ W^-)_k | H_{SM} | \bar{\nu}_{cDR} \rangle \right) = c/\sqrt{2} \quad (40)
\]

Here \( \bar{\nu}_+ \) was decomposed according to eq.\( (7) \), the value \( c \) of \( \langle (e^+ W^-)_k | H_{SM} | \bar{\nu}_{DR} \rangle \) was taken from eq.\( (16) \) and the antineutrino component

\[
\langle (e^+ W^-)_k | H_{SM} | \bar{\nu}_{cDR} \rangle = 0 \quad (41)
\]

from eq.\( (13) \). The transition amplitude is a factor \( 1/\sqrt{2} \) smaller than the one of eq.\( (16) \).

- **Fact**
  
  A Majorana neutrino with SM-interactions would be produced with a cross section \( 1/2 \) times smaller than expected for SMI-Dirac neutrinos. Therefore the hypothesis that the physical neutrino is a Majorana neutrino is in quantitative contradiction to experimental evidence.

The reduction would apply in an analogous manner also to neutral current reactions. This excludes that the physical neutrino is in the state \( \nu_- \) (or \( \nu_+ \)).

This effective “reduction” in weak transition amplitude due to a mass occurs also in other familiar circumstances. In an analogous manner, a non-relativistic neutrino (or electron) produced at rest has half the weak transition amplitude squared compared to one produced at ultra-relativistic energies because it has a sterile, right-chiral component with the same amplitude as the active left-chiral amplitude. A Dirac antineutrino at rest and a massless Majorana neutrino emitted in a \( \beta \)-decay together with an electron, both have sterile left-chiral and an active right-chiral amplitude of value \( 1/\sqrt{2} \).

Summarising, Majorana neutrinos with SM interactions need a fine-tuned combination of Dirac and Majorana mass terms and have a phenomenology that quantitatively differs from the one of the observed neutrino in having an effective weak transition amplitude \( \sqrt{2} \) smaller than the observed one.

### 3.3 Summary of the restriction mechanisms, comparison with experiment

Table \( \text{II} \) summarises the diverse mechanisms that restrict or enlarge the set of states that a SMI-neutrino can be in, thus creating various LNC violating neutrino types. Pseudo-Dirac neutrinos exclude no states but include states that the SMI-neutrino can normally not be in, namely the right(left)-chiral (anti)neutrino.

Experimental evidence about physical SMI-neutrinos was found to agree only with Dirac or pseudo-Dirac neutrinos: pseudo-Majorana neutrinos have a qualitatively and Majorana neutrinos a quantitatively different phenomenology. Based on this insight I formulate the following
Field: Excl.: Incl.: Reason for exclusion or inclusion:
No Maj. mass: - - -
Interaction-ind. Maj.: \(\nu_-\) - \(\nu_-\) not part of weak-interaction theory
Mass-ind. Majorana: \(\nu_-\) - \(\langle \nu_- | \nu_p \rangle = 0\) for \(E(\nu) \ll 100\) GeV
Pseudo Majorana: \(\nu_D, \nu_M, \nu_L, \nu_R\) - Phase diff. \(\gamma_5\) btw. \(\nu_+\) and \(\nu_-\)
Pseudo-Dirac: - \(\nu_{\text{DR}}, \bar{\nu}_{\text{DL}}\) Different momentum of \(\nu_+\) and \(\nu_-\)

Table 1: Weak-interaction induced Majorana (second row, section 3.1), mass-induced Majorana (third row, section 3.2.3), pseudo-Majorana (fourth row, section 3.2.1) and pseudo-Dirac neutrino (fifth row, section 3.2.2) have different mechanisms that exclude or include the states from the set of states the original SMI-neutrino (first row) can be in. “Excl.” stands for excluded states, “Incl.” for included states relative to the original Majorana-massless SMI-neutrino. \(\nu_p\) is the state of the physical neutrino.

• Conjecture

The physical neutrino is probably a pseudo-Dirac neutrino because it seems very likely that small Majorana masses are induced by some see-saw mechanism[30] and a larger Dirac mass is then needed to predict the observed phenomenology for a SMI-neutrino.

4 Off-axis \(CPT\)-violating masses

In the charge-parity basis (eq.(18)), the mass matrix remained diagonal even with Majorana masses (eq.(26)). It is interesting to ask what happens when we add off-diagonal terms \(m_O\) so that:

\[
M = \begin{pmatrix}
m_+ & m_O \\
m_O & m_-
\end{pmatrix}
\]  

(42)

The mass eigenstates of this matrix are no longer the Majorana states \(\nu_{+,-}\). In order to find the eigenstates \(\nu' = U
\nu_D\) the matrix \(M\) must be diagonalised with a unitary matrix \(U = \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)
\end{pmatrix}\) such that \(M' = U M U^{-1}\) is a diagonal matrix:

\[
M' \nu' = \begin{pmatrix}
m_+ & 0 \\
0 & m_-
\end{pmatrix} \begin{pmatrix}
\cos(\theta) \nu_+ + \sin(\theta) \nu_- \\
-\sin(\theta) \nu_+ - \cos(\theta) \nu_-
\end{pmatrix}.
\]  

(43)

One finds the following expression for the mixing angle

\[
\theta = \arctan \left( \frac{2m_0}{m_+ - m_-} \right) = \arctan \left( \frac{m_0}{m_M} \right).
\]  

(44)

and for the eigenmasses

\[
m'_{+,-} = \frac{1}{2} \left( m_+ + m_- \pm \sqrt{(m_+ - m_-)^2 - 4(m_+ m_- - m_0^2)} \right)
\]  

(45)
m’ is positive if $m_+ m_- \geq m_0^2$. Up to this section I considered the case $m_0 = 0$ that yields $\theta = 0$. If $m_0 \neq 0$ there is maximal mixing i.e. $\theta = 45^\circ$ if $m_+ \rightarrow m_-$ i.e. $m_M \rightarrow 0$. According to eqs.(11,12) in this case $\nu' = \begin{pmatrix} \nu_D \\ \nu_D \end{pmatrix}$, i.e. the mass eigenstates are the Dirac particle and anti-particle state. This means that particle and antiparticle have different masses and thereby the mass violates $CPT$ invariance. For intermediate angles of $\theta$ between 0 and $\pi/4$ the mass-eigenstates are mixtures between Dirac and pseudo states (pseudo-Dirac if $m_\geq 0$ and pseudo-Majorana if $m_- < 0$), i.e. $CPT$ is still violated. Therefore one might exclude the existence of non-vanishing masses off-diagonal in the charge-parity basis because they are incompatible with basic assumptions on which quantum field theory is built[20]. However, the possibility of $CPT$ violation in the neutrino sector has been widely discussed[2, 8, 9], and it is interesting to characterise the precise meaning of the mixing angle $\theta$. With finite $\theta$ SM interactions could produce a neutrino together with a positron as a mixture of Dirac/pseudo Dirac neutrino:

$$\nu_{D-pD} = \cos(\pi/4 - \theta)\nu_D + \sin(\pi/4 - \theta)\nu_{pD}$$  \hfill (46)$$

The pseudo-Dirac amplitude following from this expression is $A_{pD} = \sin^2(\pi/4 - \theta)$. Thereby $\theta$ determines the pseudo-Dirac amplitude with which a neutrino with Majorana and off-axis mass terms oscillates into a sterile state. This mechanism is therefore an interesting - if exotic - alternative to new sterile neutrino generations that mix with active generations to interpret the tentative experimental evidence for sterile neutrinos[16].

Such a scenario is predictive. If it were possible to determine $m_+$, $m_-$ and $m_0$ by measuring the mass of neutrino and antineutrino (eq.(45)) and the mixing angle of sterile neutrinos (eq.(44)) then from $m_M = (m_+ - m_-)/2$ and $m_D = (m_+ + m_-)/2$, the wavelength of the neutrino - sterile neutrino oscillation could be predicted via eq.(38).

5 Summary of the novel theoretical results

If the standard model describes the weak interaction correctly to good approximation, the only way to make the neutrino a Majorana neutrino is by way of mass terms (section 3.1). If there is a small Majorana mass term that is larger than the Dirac mass term, the neutrino field consists of a neutrino and antineutrino component with opposite chirality (pseudo-Majorana neutrino, section 3.2.1). If there is a large Dirac mass term and a large Majorana mass term of nearly equal value the neutrino is a Majorana neutrino. Such a neutrino has a SM weak interaction transition amplitude that is half as large as the experimentally observed one (section 3.2.3). The neutrino is a pseudo-Dirac neutrino not only if a small Dirac mass is much larger than a finite Majorana mass, but

---

6 Only the pseudo-Dirac case is studied below, because pseudo-Majorana states do not have the observed phenomenology (section 3.2.1), i.e. we assume that $m_+ m_- \geq m_0^2$. 

17
also if it is just larger or of equal value. (section 3.2.2). Mass terms that couple field states with opposite charge parity violate $\text{CPT}$ invariance (with different neutrino and antineutrino masses) (section 4).

6 Conclusions about the physical neutrino

Physical neutrinos have a definite helicity to good approximation and can therefore not be pseudo-Majorana neutrinos. The precise quantitative agreement of the neutrino’s phenomenology with the properties expected in the standard model[8] makes it impossible that any of the three neutrino field is a Majorana field for which it would be expected that the squared weak transition amplitude squared is half as large as expected in the SM (section 3.2.3). Under the assumption that the physical neutrino is a SMI-neutrino to good approximation (which is universally made in high-energy physics) it is therefore certain that:

- $m_D > m_M$,
- the physical neutrino is either a Dirac or pseudo-Dirac particle,
- neutrinoless double-$\beta$ decay with a non negligible rate cannot be due only to a Majorana mass term.

Moreover because of the observational fact that neutrinos oscillate among 3 generations[8], at least two neutrino generations must have a finite Dirac mass. It is very likely that neutrinos are pseudo-Dirac particles because there is no plausible reason why the Majorana mass of the neutrino should vanish exactly. This is good news, because with $\text{CPT}$ invariance they oscillate from neutrino to antineutrino either with maximal amplitude, or, if $\text{CPT}$ invariance is broken (which would be even more interesting), with a smaller amplitude. Pseudo-Dirac neutrinos offer excellent prospects for a definite discovery of even extremely small Majorana mass terms because even extremely large oscillation lengths in principle can be probed with current technology[3]. Non-negligible neutrinoless double-beta decay assumes a role exclusively as a channel to search for new physics beyond the mere addition of Majorana mass terms (e.g. [11]).

Acknowledgments

I thank Palash Pal and Carlo Giunti for recent inspiring and helpful discussions on the subject of Majorana neutrinos and Silvia Pezzoni for convincing me that 18 years ago - contrary to my then firm conviction - I did not understand the nature of Majorana neutrinos at all.

References

[1] F.T. Avignone III et al., Rev.Mod.Phys. 80,481 (2008).
[2] G. Barenboim et al. Phys. Lett. B 537, 227 (2002).
[3] J.F. Beacom et al., Phys.Rev.Lett. 92, 011101 (2004).
[4] S. Bilenky, Ann. Fond. Louis de Broglie, 31, 139 (2006).
[5] J.D. Bjorken, S.D. Drell, Relativistic Quantum Fields, Mc Graw Hill, New York (1965).
[6] K.M. Case, Phys.Rev.107, 307 (1957);
J.A. McLennan, Phys.Rev.106, 821 (1957);
J. Serpe, Nucl.Phys. 4, 183 (1957).
[7] C. Enz, Nuovo Cimento, Ser. 10 6, 250 (1957).
[8] C. Giunti, C.W. Kim, Fundamentals of Neutrino Physics and Astrophysics,
Oxford University Press, 2008.
[9] C. Giunti, M. Laveder, Phys.Rev. D82, 113009 (2010);
A. A. Aguilar-Arevalo et al (MiniBooNE Collaboration), arXiv:1109.3480,
subm. to Phys.Rev.Lett.
[10] M. Goldhaber, L. Grodzins & A. W. Sunyar, Phys. Rev. 109, 1015 (1958).
[11] P. Gu, Neutrinoless double beta decay with pseudo Dirac neutrinos,
arXiv:1101.5106, (2011).
[12] C. Itzykson, J.-B. Zuber, Quantum-Field Theory, Mc Graw Hill, New York
(1985).
[13] B. Kayser, Phys. Rev. D26, 1662 (1982).
[14] B. Kayser, A.S. Goldhaber, Phys.Rev. D28, 2341 (1983).
[15] B. Kayser, F. Gibrat-Debu and F. Perrier, The Physics of Massive Neutrinos
(World Scientific, Singapore, 1989).
[16] J. Kopp et al., Phys.Rev.Lett.107, 091801 (2011).
[17] E. Kraus, Ann. Phys. 262, 155 (1998).
[18] E. Majorana, Nuovo Cimento, Ser.8 14, 171 (1937).
[19] R.N. Mohapatra et al., Rept. Prog. Phys. 70, 1757-1867 (2007)
[20] L.B. Okun, C, P, T are Broken. Why Not CPT?, arxiv.org/abs/hep-ph/0210052 (2002).
[21] P.B. Pal, Dirac, Majorana and Weyl fermions, arXiv: 1006:1718v2 (2010).
[22] W. Pauli, Nuovo Cimento, Ser.10 6, 204 (1957).
[23] D.H. Perkins, Introduction to High Energy Physics, Addison Wesley, London (1982).
7 Appendices

7.1 The Majorana representation of $\gamma$ matrices

I choose the following basis of $\gamma$ matrices:

$$
\gamma^0 = \begin{pmatrix}
0 & \sigma^2 \\
\sigma^2 & 0
\end{pmatrix}
$$

$$
\gamma^1 = \begin{pmatrix}
0 & \sigma^1 \\
\sigma^1 & 0
\end{pmatrix}
$$

$$
\gamma^2 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
$$

$$
\gamma^3 = \begin{pmatrix}
0 & \sigma^3 \\
\sigma^3 & 0
\end{pmatrix}
$$

(47)

Itzykson & Zuber\[12]\ choose a slightly different form in their appendix, with which the definition of charge conjugation (my eq. (2)) is:

$$
\nu(x,t)^c = -i \nu(x,t)^T \quad \text{(alternative Majorana representation)}
$$

(48)

7.2 Can a weakly interacting neutrino be self-charge ($C$) conjugate?

It has been argued\[13, 15]\ that “dressed”, i.e. weakly interacting neutrinos cannot be self-charge conjugate i.e. cannot fulfil eq. (4). This “argument C” runs like this\[15]:

“A predominantly left-chiral Majorana neutrino can virtually decay into a $e^-$ $W^+$ pair but not a $e^+$ $W^-$ pair due to the weak interaction’s charge-parity violation. $e^- W^+$ is not an eigenstate of $C$ and therefore the Majorana neutrino is
neither, because it has maximally $C$-violating interactions.”

The final decay state $e^- W^+$ is indeed not an eigenstate of $C$ and therefore has no definite charge-parity. But from this fact one can only infer the charge-parity of the initial Majorana neutrino if the interaction were charge-parity conserving.

Let us consider a different, charge-parity conserving reaction where a similar inference can be drawn correctly: the decay $\pi_0 \rightarrow 2 \gamma$. The charge parity of the final state is +1, because the photon has a charge parity of -1 (a fact derived from the structure of an electromagnetic quantum field[5]). Because the decay proceeds via the electromagnetic interaction which conserved charge-parity, one can infer that the $\pi_0$ has a charge parity of +1.

On the contrary, the decay $\nu_M \rightarrow e^+ W^-$ proceeds via the weak interaction which does not conserve charge parity so no conclusion about the charge parity of the neutrino can be drawn from the charge parity of the final state, i.e. the above “argument C” is wrong.

The inference of the charge parity of a field from the one of another in decay processes needs always to be rooted in the determination of the charge parity of some field from the structure of the quantum field (as was done for the photon[5] in the above example). This rooting is done in eqs. (8) to (11) for the case of the Majorana fields that make up a Dirac quantum field. The fact that for an only weakly interacting field, like the neutrino, one can draw no conclusions about the charge parity of decay products, does not invalidate this rooting analysis.

Another angle on this problem is the following. Is a weakly interacting Dirac neutrino in a self-parity ($\mathcal{P}$) conjugate state? For a massless neutrino the answer is “no, because the weak interaction violates parity”. The parity conjugate of a left-handed neutrino is a right-handed neutrino, i.e. it is not self-parity conjugate. We saw in section 3.1 that a massless SMI-neutrino is also not self-charge conjugate.

But masses can change these conclusions. In the limit $m_D \rightarrow E$, where E is the reaction energy, Dirac mass terms can “make” the neutrino self-parity conjugate (see section 7.3 for a detailed discussion), even though it is produced in a reaction that still violates $\mathcal{P}$-parity (as indicated by the properties of other particles taking part in the reaction).

In a similar manner Majorana and Dirac masses make the neutrino self-charge conjugate (as outlined in section 3.2.3) even if it was produced in a reaction that violates charge parity. Therefore one cannot conclude from the noninvariance of the reaction-Hamiltonian with respect to $\mathcal{C}$ or $\mathcal{P}$ that some fermion fields taking part in the reaction are not made self-charge or self-parity invariant by special mass terms.

Nor can one conclude from the invariance of a reaction-Hamiltonian to $\mathcal{C}$ that the fermion fields of the reaction products are self-charge conjugate, consider e.g. the electromagnetic reaction $\gamma \rightarrow e^+ + e^-$. The concepts “$\mathcal{C}$-invariance of reaction-Hamiltonian” and “$\mathcal{C}$-invariance of a field taking part in the reaction” are logically independent concepts.

Summarising, weakly interacting, neutral fields with spin 1/2 can in principle be self-charge ($\mathcal{C}$) conjugate.
7.3 How a Dirac mass term can make a neutrino field self-parity conjugate

A massless neutrino that was produced by $H_{SM}$

$$\nu_{DL} = \int_{k}^{(+)} b_{k}^{\dagger} v_{L}(x) + \int_{k}^{(-)} d_{k} u_{R}(x)$$ (49)

is not self-parity conjugate because the parity conjugation operation $P$ flips chirality, i.e. $\gamma_{0} v_{L}(-x) = -v_{R}(x)$, $\gamma_{0} u_{L}(-x) = u_{R}(x)$ but does not change $b,d^{\dagger}$ and only changes the sign of $d,b^{\dagger}$[5], so that:

$$P\nu_{DL}P^{-1} = \int_{k}^{(+)} b_{k}^{\dagger} v_{R}(x) + \int_{k}^{(-)} d_{k} u_{L}(x) \neq \nu_{DL}$$ (50)

However, it is well known (e.g.[21]), that if the neutrino has a Dirac mass $m_{D} \rightarrow E$, i.e. the decay is non-relativistic (NR), the two chiralities are produced and annihilated with equal amplitude i.e.

$$\nu_{NR-D} = \int_{k}^{(+)} b_{k}^{\dagger} (v_{L}(x) + v_{R}(x)) + \int_{k}^{(-)} d_{k} (u_{L} + u_{R}(x))$$ (51)

and clearly:

$$P\nu_{NR-D}P^{-1} = \nu_{NR-D}$$ (52)

The field was made self-parity conjugate by the Dirac mass in the limit $m_{D} \rightarrow E$, because it excluded states with a definite chirality of 1 or -1.

7.4 Weak reactions with $H_{PE}$ vs. $H_{SM}$

If $H_{PE}$ would be the weak-interaction Hamiltonian weak reactions would violate LNC. We can illustrate a LNC reaction that produces neutrinos:

$$e^{+}W^{-} \rightarrow \bar{\nu}_{D}$$ (53)

and

$$e^{+}W^{-} \rightarrow \nu_{+}.$$ (54)

The left-hand side of both equations has a lepton number of -1 (of the positron). The right-hand side of eq. (53) (a Dirac neutrino) also has a lepton number of $\ell = -1$ because

$$\mathcal{L}\bar{\nu}_{D} = -\bar{\nu}_{D}$$ (55)

where $\mathcal{L}$ is the lepton-number operator. Therefore lepton number is conserved in eq. (53). However, the right-hand side of eq. (54) (a Majorana neutrino) has no well defined lepton number because from eq. (55),

$$\mathcal{L}\nu_{D} = \nu_{D}$$ (56)

22
and eq. (7) it follows that the Majorana neutrino is no eigenstate of the lepton-number operator $\mathcal{L}$:

$$\mathcal{L}\nu_+ = \frac{1}{\sqrt{2}} (\nu - \nu^c) \neq \ell\nu_+.$$ (57)

The lepton number $\ell$ of $\nu_+$ is undefined, rather then being close to -1, which would be necessary if the SM described the interaction to good approximation. LNC in eq. (54). Therefore the reaction eq. (54) does not occur for a massless (or Dirac-massive) neutrino with $H_{SM}$. 

23