TWO-DIMENSIONAL PAULI EQUATION IN NONCOMMUTATIVE PHASE-SPACE

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In this paper, we investigated the Pauli equation in a two-dimensional noncommutative phase-space by considering a constant magnetic field perpendicular to the plane. We mapped the noncommutative problem to the equivalent commutative one through a set of two-dimensional Bopp-shift transformation. The energy spectrum and the wave function of the two-dimensional noncommutative Pauli equation are found, where the problem in question has been mapped to the Landau problem. Further, within the classical limit, we have derived the noncommutative semi-classical partition function of the two-dimensional Pauli system of one-particle and N-particle systems. Consequently, we have studied its thermodynamic properties, i.e. the Helmholtz free energy, mean energy, specific heat and entropy in noncommutative and commutative phase-spaces. The impact of the phase-space noncommutativity on the Pauli system is successfully examined.

Keywords: Noncommutative phase-space, Pauli equation, Bopp-shift, semi-classical function partition, thermodynamic properties.

I. INTRODUCTION

In the last few years, there has been a growing interest in the study of two-dimensional systems, which have become an active area of research because of its implications in nanofabrication technology. Such as in graphene [1, 2] and other materials like Weyl semimetals [3], semiconductor quantum wells, quantum Hall and fractional Hall effects [4, 5], as well the Dirac relativistic oscillator [6], etc. However, despite their experimental success, it is very important to understand these systems from a theoretical point of view in which quantum mechanics plays a central role. Motivated by the efforts to understand string theory [7], black hole models and describe the quantum gravitation [8–10] using noncommutative geometry and by trying to have drawn considerable attention to the phenomenological implications, we concentrate on studying the problem of a non-relativistic spin-1/2 particle in the presence of an electromagnetic field within two-dimensional noncommutative phase-space. Besides, there are a large amount of articles has been devoted to the study physics within noncommutative geometry, particularly in quantum field theory [11, 12] and quantum mechanics [13, 14].

We present the essential formulas of noncommutative algebra we need in this manuscript. At very tiny scales (the string scale), the position coordinates do not commute with each other, neither do the momenta.

In the two-dimensional noncommutative phase-space, the operators of coordinates \( x_{j}^{nc} \) and momenta \( p_{j}^{nc} \) satisfy the following Heisenberg-like commutation relations

\[
\begin{align*}
[x_{j}^{nc}, x_{k}^{nc}] & = [x_{j}, x_{k}] = i\Theta_{jk} \\
[p_{j}^{nc}, p_{k}^{nc}] & = [p_{j}, p_{k}] = i\eta_{jk} \quad (j, k = 1, 2) \\
[x_{j}^{nc}, p_{k}^{nc}] & = [x_{j}, p_{k}] = i\hbar\delta_{jk}
\end{align*}
\]  

(1)

The noncommutative phase-space can be obtained using ordinary coordinates \( x_{j} \) and momenta \( p_{j} \) operators and with replacing the ordinary product by the Moyal product, which can be used as follows [15]

\[
\mathcal{F}(x^{nc}, p^{nc}) \mathcal{G}(x^{nc}, p^{nc}) = \mathcal{F}(x, p) \star \mathcal{G}(x, p) = e^{\frac{i}{2} \sqrt{\Theta_{ab} \partial_{a} \partial_{b} + \eta_{ab} \partial_{a} \partial_{b}}} \mathcal{F}(x_{a}, p_{a}) \mathcal{G}(x_{b}, p_{b}) ,
\]

(2)

where \( \mathcal{F}, \mathcal{G} \) are two functions vary in terms of \( x, p \) and assumed to be infinitely differentiable. The effective Planck constant (deformed Planck constant) is given by [16, 17]

\[
\hat{\hbar} = \hbar \left( 1 + \frac{\Theta\eta}{4\hbar^{2}} \right),
\]

(3)

where \( \frac{\Theta\eta}{4\hbar^{2}} \ll 1 \) is the condition of consistency in the usual commutative spacetime quantum mechanics, it is expected to be generally satisfied since the small parameters \( \Theta \) and \( \eta \) are of second order. \( \delta_{jk} \) is the identity matrix, \( \epsilon_{jk} \) is the Levi-Civita symbol, with \( \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0 \). And \( \Theta, \eta \) are the real-valued non-commutative parameters with the dimension of length\(^2\), momentum\(^2\) respectively, which are assumed to be extremely small. Note that experimental and theoretical investigations on noncommutative systems of the non-commutativity constants led to obtaining the following upper bound on the value of the noncommutative parameters [17]

\[
\Theta \leq 4.10^{-40} m^{2}; \quad \eta \leq 1.76.10^{-61} Kg^{2}m^{2}s^{-2}.
\]

(4)

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Besides, recent studies [18–20] revealed that the noncommutative parameters associated with different particles are not the same in noncommutative quantum mechanics.

The set of operators $x^{nc}_i, p^{nc}_i$ are related to the set $x_i, p_j$ in usual quantum mechanics by a non-canonical linear transformation referred to as Bopp-shift as follows [21]

$$
\begin{align*}
x^{nc}_i &= x - \frac{1}{2\hbar} \Theta_{ij} p_j, \\
y^{nc}_i &= y + \frac{1}{2\hbar} \Theta_{ij} p_j.
\end{align*}
$$

The quantum mechanical system will become merely the noncommutative one using equation (5) or (2). Let $H(x,p)$ be the Hamiltonian operator of the usual quantum system, then the corresponding noncommutative Schrödinger equation is given by

$$
H(x,p) \ast \psi(x,p) = H\left(x_i - \frac{\Theta_{ij}}{2\hbar} p_j, p_i + \frac{\eta_{ij}}{2\hbar} \right) \psi = E\psi.
$$

Noting that noncommutative term always can be treated as a perturbation in quantum mechanics.

In the ordinary two-dimensional commutative phase-space, the canonical variables $x_j$ and $p_i$ satisfy the following canonical commutation

$$
[x_j, x_k] = 0, \quad [p_j, p_k] = 0, \quad [x_j, p_k] = i\hbar \delta_{jk}.
$$

The paper is organized as follows. The formulation of the two-dimensional noncommutative geometry is briefly outlined in section I. The exact solution to the two-dimensional noncommutative Pauli equation is presented in section II. Section III is devoted to present the thermodynamic properties of the problem in question. Therefore, concluding with some remarks.

II. TWO-DIMENSIONAL NONCOMMUTATIVE PAULI EQUATION

The time-independent Pauli equation is given by [22]

$$
\frac{1}{2m_e} \left(p - \frac{e}{c} A\right)^2 \psi + \frac{e}{c} \phi \psi + \mu_B \sigma B \psi = E \psi,
$$

where $\psi = (\psi_1, \psi_2)^T$ is a two-component spinor, $p = i\hbar \nabla$ is the momentum operator, $m_e, e$ are the mass and charge of the electron, and $c$ is the speed of light. As well $\mu_B = \frac{e\hbar}{m_e c}$ is the Bohr magneton, $B$ is the applied magnetic field vector, with $A(r,t)$ is the vector potential and $\phi(r,t)$ is the Coulomb potential. $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

The time-independent Pauli equation in noncommutative phase-space is given by

$$
\left\{ \frac{1}{2m_e} \left(p^{nc} - \frac{e}{c} A^{\ast}\right)^2 + \frac{e}{c} \phi + \mu_B \sigma . B \right\} \bar{\psi} = E \bar{\psi},
$$

with $\bar{\psi}$ is the noncommutative spinor wave function. If the magnetic field $B$ oriented along the axis (Oz), which is often referred to as the Landau system, and based on the proposal that noncommutative observables corresponding to the commutative one [23], we have the following deduced noncommutative symmetric gauge

$$
A^{\ast} = (A^*_x, A^*_y, A^*_z) = B \left(-y^{nc}, x^{nc}, 0\right), \quad A^{\ast}_0 = e\phi^{\ast} = 0.
$$

Here the electron is unbound $\phi = 0$. Using equation (10), with $[p^{nc}_i, A^*_j] = 0$, equation (9) becomes

$$
\left\{ \frac{(p^{nc}_i)^2}{2m_e} + \frac{e^2}{2c^2} \left(A^{\ast}_i\right)^2 + \mu_B \sigma \cdot B \right\} \bar{\psi} = E \bar{\psi},
$$

where $\sigma_z = \pm 1$. It is easy to check that

$$
(p^{nc}_i)^2 = p^{2}_x + p^{2}_y - \frac{\eta}{\hbar} L_z + \frac{\eta^2}{4\hbar^2} \left(x^2 + y^2\right),
$$

$$
(A^{\ast}_i)^2 = \frac{B^2}{4} \left\{ x^2 + y^2 + \frac{\Theta_{ij}}{4\hbar^2} \left(p^{2}_x + p^{2}_y\right) - \frac{\Theta_{ij}}{\hbar} L_z \right\},
$$

$$
p^{nc}_i . A^{\ast}_j = -\frac{B}{2} \left\{ \frac{\Theta_{ij}}{2\hbar} \left(p^{2}_x + p^{2}_y\right) + \frac{\eta}{2\hbar} \left(y^2 + x^2\right) - \left(1 + \frac{\Theta_{ij}}{4\hbar^2}\right) L_z \right\},
$$

with

$$
L_z = (x_i \times p_i)_z = p_{iy} - p_{ix}.
$$

Using the three-equations (12-14) above, the Pauli equation reads

$$
\left\{ \frac{(p^{2}_x + p^{2}_y)}{2\tilde{m}} - \tilde{\omega} L_z + \frac{\tilde{m} \tilde{\omega}}{2} \left(x^2 + y^2\right) + \mu_B \sigma \cdot B \right\} \tilde{\psi} = E \tilde{\psi},
$$

with

$$
\tilde{m} = \frac{m_e}{(1 + \frac{\Theta_{ij}}{4\hbar^2})^2}, \quad \tilde{\omega} = \frac{eB + c\eta}{2c\hbar m_e (1 + \frac{\Theta_{ij}}{4\hbar^2})},
$$

$$
\frac{1}{2} \tilde{m} \tilde{\omega}^2 = \frac{m_e}{2c} \left(\frac{e B + c \eta}{2c\hbar m_e} \right)^2 + \frac{\eta}{2\hbar} \left(\frac{e^2 B^2}{c^2}\right).
$$

We assume that $\tilde{\omega}$ is the deformed cyclotron frequency, where in $\Theta \to 0, \eta \to 0$ limits, $\tilde{\omega}$ is reduced to $\frac{e B}{2c\hbar m_e}$.

On the other hand, in case of an atomic Hydrogen, the electron is bound to a proton by the Coulomb potential $A^*_0$, which is given by

$$
A^{\ast}_0 = \frac{e}{4\pi \epsilon_0} \sqrt{x^2 + y^2 + \frac{\Theta_{ij}}{4\hbar^2} \left(p^{2}_x + p^{2}_y\right) - \frac{\Theta_{ij}}{\hbar} L_z}.
$$

Our system looks like a two-dimensional harmonic oscillator with an additional interaction $(-\tilde{\omega} L_z + \mu_B \sigma \cdot B)$. This system corresponds to the Landau level problem; it corresponds to the motion of a charged particle in the $xy$ plane and subjected to a uniform magnetic field (in
the symmetric gauge) oriented along the axis (Oz), which means the particle is in interaction with its orbital and spin angular momentum. The Hamiltonian from equation (16) can be written as

\[ H_{\text{Pauli}}^{\text{nc}} = H_{nc}^{\text{ho}} - \tilde{\omega}L_z + \mu_B\sigma_zB. \]  

(19)

This problem will be solved simply by introducing operators of creation and annihilation of harmonic oscillator, thus we define

\[ a = \frac{1}{\sqrt{2}} \sqrt{\frac{\hbar}{m\omega}} (x - iy) + \frac{i}{\sqrt{2}} \sqrt{\frac{\hbar}{m\omega}} (px - ipy), \]

\[ a^\dagger = \frac{1}{\sqrt{2}} \sqrt{\frac{\hbar}{m\omega}} (x + iy) + \frac{i}{\sqrt{2}} \sqrt{\frac{\hbar}{m\omega}} (px + ipy), \]  

(20)

satisfying the following commutation relations

\[ [a, a^\dagger] = [b, b^\dagger] = 1. \]  

(21)

In terms of the ladder operators (20), our Hamiltonian terms can be re-written as

\[ H_{nc}^{\text{ho}} = \hbar\omega (a^\dagger a - b^\dagger b), \]  

(22)

\[ H_{nc}^{\text{ho}} = \hbar\omega (a^\dagger a + b^\dagger b + 1) - \hbar\tilde{\omega} (a^\dagger a - b^\dagger b) \]

\[ = \frac{\hbar\tilde{\omega}}{2\hbar} (b^\dagger b + 1). \]  

(23)

Eigenstates of our Hamiltonian are labeled by the number \( j \) of excitation quanta of the oscillator \( a \), and the number \( n \) of excitation quanta of the oscillator \( b \),

\[ a^\dagger a \mid n,j > = j \mid n,j > \quad \text{and} \quad b^\dagger b \mid n,j > = n \mid n,j >, \]  

(24)

where both \( n \) and \( j \) can take on any positive integer value. Therefore, our Pauli system becomes

\[ \{ \hbar\tilde{\omega} (3b^\dagger b - a^\dagger a + 1) + \mu_B\sigma_zB \} \mid n,j > = E \mid n,j >, \]  

(25)

with \( \pm 1 \) are the eigenvalue of \( \sigma_z \), therefore, our system energy spectrum (discretely quantised) reads

\[ E = \hbar\tilde{\omega} (3n - j + 1) \pm \mu_BB. \]  

(26)

The effect of the phase-space noncommutativity is reduced in \( \tilde{\omega} \). Thus, by using equation (17) we have

\[ E_{n,j} (\Theta, \eta) = \frac{\epsilon Bh + \eta}{2em_c} \left( 1 + \frac{\epsilon \Theta B}{4\hbar}\right) (3n - j + 1) \pm \mu_BB. \]  

(27)

The above spectrum is a bit different from that obtained in ref. [24] in the limit of \( \eta \to 0 \). However, the slight difference is because the authors considered the magnetic field term proportional to \( \frac{1}{2m_c} \). In the limits of \( \Theta \to 0 \) and \( \eta \to 0 \), the NC energy spectrum becomes commutative one, i.e. commutative Landau system [25].

After finding the energy spectrum, we now find the wave function. The time-independent Pauli equation (16) reads

\[ \left\{-\frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) - \tilde{\omega}L_z + \frac{\tilde{\omega}^2m}{2}\sigma_zB \right\} \psi = E\tilde{\psi}, \]

(28)

with \( r = \sqrt{x^2 + y^2} \). Now, we use cylindrical coordinates \((r, \Phi)\) to solve the corresponding Pauli equation. The wave function \( \tilde{\psi}(r, \Phi) \) can be given by

\[ \tilde{\psi}(r, \Phi) = R(r) e^{im\Phi}, \]  

(29)

knowing that \( m = 0, \pm 1, \pm 2, \pm 3, \ldots \) are the eigenvalues of the orbital angular momentum operator \( L_z \). By replacing equation (29) into equation (28), we obtain

\[ \left\{ -\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{d}{dr} + \frac{d^2}{dr^2} - \frac{m^2}{r^2} \right) - \tilde{\omega}m + \frac{\tilde{\omega}^2m^2}{2} - \mu_B\sigma_zB \right\} R = ER. \]  

(30)

We can now solve the two-dimensional Pauli equation by assuming a new functional form for \( R(r) \) which is

\[ R(r) = \frac{\gamma(r)}{\sqrt{r}}, \]  

(31)

and choosing the lower eigenvalue of \( \sigma_z \), thus the resulting equation for \( \gamma(r) \) is

\[ \left\{ -\frac{1}{2m} \frac{d^2}{dr^2} + \frac{m^2 - \frac{1}{3}}{2mr^2} - \tilde{\omega}m + \frac{\tilde{\omega}^2m^2}{2} - \mu_BB \right\} \gamma = E\gamma. \]  

(32)

We have used natural units with \( \hbar = c = e = 1 \) to simplify more in this part only. Let us find the corresponding wave functions. We can rewrite the left-hand side of the equation (32) in the form \( A^\dagger A \), with

\[ \begin{cases} A = \frac{d}{dr} - \left( \frac{|m| + \frac{3}{2}}{r} \right) + \tilde{\omega}r, \\ A^\dagger = -\frac{d}{dr} - \left( \frac{|m| - \frac{3}{2}}{r} \right) + \tilde{\omega}r, \end{cases} \]  

(33)

where the decomposition holds if \( |m| \leq 0 \), which leads to \( E_0 \geq 0 \). \( E_0 = 0 \) exists if and only if the solution of the equation \( A\psi_0 (r) = 0 \), thus

\[ \left( \frac{d}{dr} - \left( \frac{|m| + \frac{3}{2}}{r} \right) + \tilde{\omega}r \right) \psi_0 (r) = 0, \]  

(34)

hence

\[ \frac{d\psi_0 (r)}{\psi_0 (r)} = \left( \frac{|m| + \frac{3}{2}}{r} - \tilde{\omega}r \right) dr. \]  

(35)

We solve the above equation to find

\[ \psi_0 (r) = k_0 r^{|m| + \frac{3}{2}} \exp \left[ -\frac{1}{2} \tilde{\omega}r^2 \right], \]  

(36)

where \( k_0 \) is the normalization factor. In the limits of \( \Theta \to 0 \) and \( \eta \to 0 \), the above result reduces to the commutative one, which corresponds to that of ref. [26], and it is given by

\[ \psi_0 (r) = k_0 r^{|m| + \frac{3}{2}} \exp \left[ -\frac{B}{2r^2} \right]. \]  

(37)
III. THE SEMI-CLASSICAL PARTITION FUNCTION AND THERMODYNAMIC PROPERTIES IN NONCOMMUTATIVE PHASE-SPACE

In the language of the classical treatment, we investigate the thermodynamic properties of the two-dimensional noncommutative Pauli equation using the semi-classical partition function. We initially focus on the calculation of the semi-classical partition function \( Z \). Our studied system is semi-classical where the Hamiltonian is split as follows

\[
H^{2D}_{\text{Pauli}} = H_{\text{classic}} + H_{\text{ncl},\sigma},
\]

with \( H_{\text{ncl},\sigma} = \mu B \sigma_z B \). Therefore, the noncommutative partition function is separable into two independent parts as followed recently in our work in Ref. [27]

\[
Z = Z_{\text{cl}} Z_{\text{ncl}},
\]

where \( Z_{\text{ncl}} \) is the non-classical part of the partition function. To study our non-classical partition function, we assume that the passage between noncommutative classical mechanics and noncommutative quantum mechanics can be realized through the following generalized Dirac quantization condition [28, 29]

\[
\{f, g\} = \frac{1}{i\hbar} [F, G],
\]

where \( F, G \) stand for the operators associated with classical observables \( f, g \) and \( \{., \} \) stands for Poisson bracket. Using the condition above, we obtain from Eq.(1)

\[
\left\{x^{nc}_j, x^{nc}_k\right\} = \Theta_{jk},
\]

\[
\left\{p^{nc}_j, p^{nc}_k\right\} = \eta_{jk},
\]

\[
\left\{x^{nc}_j, p^{nc}_k\right\} = \delta_{jk} + \frac{\eta_{jk}}{4\hbar} = \delta_{jk}.
\]

It is important to mention that in terms of the classical limit, \( \frac{\Theta_{\eta}}{4\hbar^2} \ll 1 \) (check ref. [29]), thus \( \left\{x^{nc}_j, p^{nc}_k\right\} = \delta_{jk} \). Now based on the proposal that noncommutative observables \( F^{nc} \) corresponding to the commutative one \( F(x, p) \) can be defined by [23, 30, 31]

\[
F^{nc} = F(x^{nc}, p^{nc}),
\]

and for non-interacting particles, the classical partition function in noncommutative phase-space for \( N \) particles is written as follows [27, 28]

\[
Z_{\text{cl}} = \frac{1}{N!} \left(\frac{2\pi\hbar}{2\eta}\right)^{2N} \int e^{-\beta H_{\text{classic}}} d^2 x^{nc} d^2 p^{nc}.
\]

Using equation (39) we may derive the important thermodynamic quantities such as the Helmholtz free energy

\[
F = -\frac{1}{\beta} \ln Z,
\]

and the average energy

\[
U \equiv N \langle \varepsilon \rangle = -\frac{\partial}{\partial \beta} \ln Z,
\]

where \( \varepsilon \) is the mean energy, which is given by \(-\frac{\partial}{\partial \beta} \ln Z_1\). Also the specific heat (heat capacity)

\[
C_v = \frac{\partial}{\partial T} \langle \varepsilon \rangle,
\]

as well the entropy

\[
S = \frac{\partial F}{\partial T} - \frac{K_B \ln Z}{\beta} + \frac{1}{\beta} \frac{\partial}{\partial T} \ln Z.
\]

Now for a single particle, the noncommutative classical partition function is given by

\[
Z_{\text{cl},1} = \frac{1}{\hbar^2} \int e^{-\beta H_{\text{classic}}(x,p)} d^2 x^{nc} d^2 p^{nc},
\]

where \( d^2 \) is a shorthand notation serving as a reminder that the \( x \) and \( p \) are vectors in two-dimensional phase-space. The relation between equation (43) and (48) is given by the following formula

\[
Z_{\text{cl}} = \left(\frac{Z_{\text{cl},1}}{N!}\right)^N.
\]

From equation (5), we simply have

\[
d^2 x^{nc} d^2 p^{nc} = \left(1 - \frac{\Theta_{\eta}}{4\hbar^2}\right) d^2 x d^2 p,
\]

and we have \( \hbar \sim \Delta x^{nc} \Delta p^{nc} \), which is given by

\[
\hbar^2 = k^2 \left(1 + \frac{\Theta_{\eta}}{2\hbar^2}\right) + \mathcal{O}(\Theta^2\eta^2).
\]

Following now equation (48) we express the single particle noncommutative classical partition function as

\[
Z_{\text{cl},1} = \frac{1}{\hbar^2} \int e^{-\beta \left[\frac{x^2 + y^2}{2m} - \omega L_z + \frac{\omega^2}{2m} (x^2 + y^2)\right]} d^2 x^{nc} d^2 p^{nc}.
\]

We should mention again, as we emphasized in our previous work [27] that within the classical limit it is always possible to factorize our Hamiltonian into momentum and position terms. Thus, we have

\[
Z_{\text{cl},1} = \frac{1}{\hbar^2} \int e^{-\beta \left(\frac{x^2 + y^2}{2m}\right)} e^{-\beta \omega^2 (x^2 + y^2)} e^{\beta \omega L_z} d^2 p^{nc} d^2 x^{nc}.
\]
Using the same method used in our previous work [27], which depends on expanding exponentials containing \( \tilde{\omega} \), and by considering terms up to the second-order of \( \tilde{\omega} \), we find

\[
Z_{cl,1} = \frac{1}{N!} \int \frac{d^2 \xi}{\pi} \left( e^{-\frac{\beta}{h} \left[ \frac{p_x^2 + p_y^2}{m} \right]} \right) \left( 1 + \beta \tilde{\omega} x + \frac{1}{2} \beta^2 \tilde{\omega}^2 L_z^2 \right) \times \left( 1 - \beta \tilde{\omega}^2 d \frac{2}{\pi} \left( x^2 + y^2 \right) \right) d^2 p_{nc} d^2 x_{nc}.
\] (54)

Knowing that

\[
\left( 1 - \frac{\Theta N}{4\hbar^2} \right) \left( 1 - \frac{\Theta N}{2\hbar^2} \right) = 1 - \frac{3\Theta N}{4\hbar^2} + O \left( \Theta^2 + \eta^2 \right),
\] (55)

thus we have the convenient expression of \( Z_{cl,1} \)

\[
Z_{cl,1} = \frac{1}{N!} \int \frac{d^2 \xi}{\pi} \int e^{-\frac{\beta}{h} \left[ \frac{p_x^2 + p_y^2}{m} \right]} d^2 p d^2 x + \left( \frac{3\Theta N}{4\hbar^2} \right) \tilde{\omega} \int \frac{d^2 \xi}{\pi} \left[ \frac{p_x^2 + p_y^2}{m} \right] L_z^2 d^2 p d^2 x
\]

\[
+ \left( \frac{3\Theta N}{4\hbar^2} \right) \beta^2 \tilde{\omega}^2 \int \frac{d^2 \xi}{\pi} \left[ \frac{p_x^2 + p_y^2}{m} \right] \frac{1}{2} \left( x^2 + y^2 \right) d^2 p d^2 x.
\] (56)

In the right-hand side of the above equation, the second integral goes to zero, the third and fourth integrals cancel each other, then by using the known integral of Gaussian function \( \int e^{-ax^2} dx = \sqrt{\pi/a} \), we find

\[
Z_{cl,1} = \frac{1}{N!} \int \frac{d^2 \xi}{\pi} \int d^2 x e^{-\frac{\beta}{h} \left[ \frac{p_x^2 + p_y^2}{m} \right]} \frac{p_x^2 + p_y^2}{m} \frac{1}{2} \left( x^2 + y^2 \right) d^2 p d^2 x.
\] (57)

with \( \int d^2 x = \pi^2, A = \hbar (2\pi m_n K_B T)^{-1} \) are the area and the thermal de Broglie wavelength respectively.

We also propose another method based on the substitution of variables with the \( \text{Jacobi} \) matrix to compute the integral (52), explained in Appendix A, which gives the same results.

The non-classical partition function for a \( N \) particle is given by

\[
Z_{nc} = Z_{nc,1} \left( \sum_{\sigma_\mu = \pm 1} e^{\beta \mu B \sigma_z B} \right)^N = 2^N \cosh^N (\beta \mu B).
\] (58)

An important point to note is for a canonical ensemble that is classical and discrete, the canonical partition function is defined using sum as in the case of \( H_{nc,\sigma} \). But for a canonical ensemble that is classical and continuous, the canonical partition function is defined using integral.

Finally, the Pauli partition function (39) for a system of \( N \) particles in a two-dimensional noncommutative phase-space is

\[
Z = \frac{2^N N^2}{A^2 N!} \left( \frac{1 - \frac{3\Theta N}{4\hbar^2}}{1 + \frac{\Theta N}{8\hbar^2}} \right) \cosh^N (\beta \mu B).
\] (59)

In the vanishing limit of the noncommutativity, i.e. \( \Theta \to 0, \eta \to 0 \), the expression of \( Z \) reduces to that of the usual commutative phase-space, which is

\[
Z = \frac{2^N N^2}{A^2 N!} \cosh^N (\beta \mu B).
\] (60)

Following the relations (44, 45, 46, 47) and as a consequence of equation (59), we express the thermodynamic quantities in noncommutative phase-space, thus we have

\[
F^{nc} = \frac{N}{\beta} \ln \left[ \frac{2^2 \left( 1 - \frac{3\Theta N}{4\hbar^2} \right) \cosh (\beta \mu B)}{A^2 \left( 1 + \frac{\Theta N}{8\hbar^2} \right)} \right] + \frac{1}{\beta} \ln N!,
\] (61)

where \( \ln N! \approx N \ln N - N \).

\[
S^{nc} = K_B N \left( \frac{1}{\beta} - \mu B \tanh (\beta \mu B) \right) \left[ 1 + \frac{N \ln N}{\beta} + \beta \mu B \tanh (\beta \mu B) \right]
\] (62)

\[
U^{nc} = N \left[ \frac{1}{\beta} - \mu B \tanh (\beta \mu B) \right],
\] (63)

\[
\langle e^{nc} \rangle = \frac{1}{\beta} - \mu B \tanh (\beta \mu B),
\] (64)

\[
C_v^{nc} = -K_B \left[ \frac{1}{\beta^2} + \frac{(\mu B)^2}{\cosh^2 (\beta \mu B)} \right].
\] (65)

In the vanishing limit of the noncommutativity, the result of this paper will be reduced to that of commutative phase space. Such as

\[
F = \frac{N}{\beta} \ln \left[ \frac{2^2 \cosh (\beta \mu B)}{A^2 \left( 1 + \frac{\Theta N}{8\hbar^2} \right)} \right] + \frac{1}{\beta} \ln N!,
\] (66)

as well

\[
S = K_B \left[ \frac{1}{\beta} + \frac{N \ln N}{\beta} + \beta \mu B \tanh (\beta \mu B) \right] - \ln \left[ \frac{2^2 \cosh (\beta \mu B)}{A^2 \left( 1 + \frac{\Theta N}{8\hbar^2} \right)} \right].
\] (67)

Through further derivatives, we can go deeper and calculate the rest of the thermodynamic properties using the obtained partition function. Such as temperature \( T \), pressure \( P \), the magnetization \( \langle M \rangle \) and chemical potential \( \mu \).

IV. CONCLUSION

In this work, we have discussed the problem of a charged particle with a spin in interaction with an electromagnetic field moving in a two-dimensional noncommutative phase-space, by considering a constant magnetic field perpendicular to the plane. The approach
that we have took to map the noncommutative problem to the equivalent commutative one is the Bopp-shift transformation. We found the energy spectrum, which is discretely quantised and the wave function of the two-dimensional noncommutative Pauli equation. The effect of the noncommutative parameters on the energy spectrum and wave function is significant. In addition, the effect of the noncommutative parameters on the energy spectrum, which is represented by the equivalent commutative one is the Bopp-shift transformation. We found the energy spectrum, which is of the classical Maxwell-Boltzmann gas, as this happens upon the classical calculation of Landau problem. On the other hand, the quantum partition function for the Landau problem represents de Haas-van Alphen effect.

The results of the present work can be used to expand the study on a possible generalization to make the consideration of anyons, i.e., particles with arbitrary non-integer spin, which can exist in two-dimensional space.

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Appendix A: Integration using the substitution of multiple variables with Jacobian matrix

Here is a method based on the substitution of multiple variables with the determinant of the Jacobian matrix to compute the integral (52).

The substitution of multiple variables is as follows

\[
\begin{align*}
x &= x, \\
y &= y, \\
P_x &= p_x + \tilde{m}\omega y, \\
P_y &= p_y - \tilde{m}\omega x, \\
\end{align*}
\]

where the integral (52) is

\[
\text{Int} = \frac{1}{h^2} \int e^{-\beta \left[ \frac{p_x^2 + p_y^2}{2m} - \omega L_x + \frac{\tilde{m}\omega}{2} (x^2 + y^2) \right]} \, dp_x dp_y.
\]

The corresponding Jacobian matrix is

\[
J(x, y, P_x, P_y) = \begin{bmatrix} \frac{\partial x}{\partial p_x} & \frac{\partial x}{\partial p_y} & \frac{\partial x}{\partial P_x} & \frac{\partial x}{\partial P_y} \\
\frac{\partial y}{\partial p_x} & \frac{\partial y}{\partial p_y} & \frac{\partial y}{\partial P_x} & \frac{\partial y}{\partial P_y} \\
\frac{\partial P_x}{\partial p_x} & \frac{\partial P_x}{\partial p_y} & \frac{\partial P_x}{\partial P_x} & \frac{\partial P_x}{\partial P_y} \\
\frac{\partial P_y}{\partial p_x} & \frac{\partial P_y}{\partial p_y} & \frac{\partial P_y}{\partial P_x} & \frac{\partial P_y}{\partial P_y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \tilde{m}\omega & 1 & 0 \\
\tilde{m}\omega & 0 & 0 & 1 \end{bmatrix}.
\]

The determinant of the Jacobian matrix J is

\[
\text{Det}J(x, y, P_x, P_y) = \begin{vmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \tilde{m}\omega & 1 & 0 \\
\tilde{m}\omega & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\
0 & 1 \end{vmatrix} = 1.
\]

Therefore, the integral (A1) becomes

\[
= \frac{1}{h^2} \int e^{-\beta \left[ \frac{(p_x^2 + P_y^2)}{2m} \right]} \, dp_x dp_y \int dx dy
= \frac{1}{h^2 (1 + \frac{\tilde{m}\omega}{2})^2} \frac{2\pi m\omega}{\beta} \int dx dy,
\]

then by using the known integral of Gaussian function \(\int e^{-a(x^2+y^2)} \, dx = \frac{\pi}{a}\), with \(\int d^2 x = l^2\), \(A = \hbar (2\pi m_e K_B T)^{-\frac{1}{2}}\) we find

\[
\text{Int} = \frac{l^2 \left( 1 + \frac{3\tilde{m}\omega}{4}\right)}{\lambda^2 \left( 1 + \frac{6\tilde{m}\omega}{5}\right)^2}.
\]

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