Huygens’ principle in classical electrodynamics:
a distributional approach

Gerald Kaiser
Signals & Waves, Austin, TX
http://www.wavelets.com

June 23, 2009

Abstract

By combining Pauli algebra with distribution theory, we give a very compact and conceptually simple formulation of Huygens’ principle in classical electrodynamics. The Stratton-Chu and Kottler-Franz representations of an electromagnetic field as surface integrals are derived with minimal effort and maximal clarity. They are then generalized by allowing the integration surfaces to move freely, so that charge distributions in arbitrary motion are represented.

1 Electrodynamics with the Pauli algebra

The Pauli algebra is a generalization of vector analysis in $\mathbb{R}^3$. One defines the products of two vectors $A, A'$ by

$$AA' = A \cdot A' + iA \times A'. \quad (1)$$

Thus $AA'$ consists of a scalar part and a (pseudo-) vector part, the latter represented by an imaginary axial vector, which is actually the oriented area spanned by $A$ and $A'$:

$$\langle AA' \rangle_s = A \cdot A', \quad \langle AA' \rangle_v = iA \times A'.$$

The two common bilinear expressions are thus united into a single complex entity. A general element of the algebra, here called a Pauli number, is represented by a complex scalar plus a complex vector

$$A = A_0 + A, \quad A_0 \in \mathbb{C}, \quad A \in \mathbb{C}^3.$$
and the product of two such elements is

\[ A A' = A_0 A'_0 + A \cdot A' + A_0 A'_0 + i A \times A' \]  

(2)

A concrete representation of the algebra is given in terms of 2 \( \times \) 2 matrices by the correspondence

\[ A \leftrightarrow \begin{bmatrix} A_0 + A_3 & A_1 + i A_2 \\ A_1 - i A_2 & A_0 - A_3 \end{bmatrix}, \quad A_0, A_1, A_2, A_3 \in \mathbb{C}, \]  

(3)

where \( A A' \) is represented by the matrix product. Define the spacetime differential operators

\[ D = \partial_t + \nabla \quad \text{and} \quad \overline{D} = \partial_t - \nabla, \]

which act on a Pauli-valued field \( A(x) = A(x, t) \) by

\[ D A = (\partial_t + \nabla)(A_0 + A) = (\dot{A}_0 + \nabla \cdot A) + (\dot{A} + \nabla A_0 + i \nabla \times A) \]  

\[ \overline{D} A = (\partial_t - \nabla)(A_0 + A) = (\dot{A}_0 - \nabla \cdot A) + (\dot{A} - \nabla A_0 - i \nabla \times A), \]

where \( \dot{A}_0 = \partial_t A_0, \dot{A} = \partial_t A \), and whose product is the scalar wave operator:

\[ \overline{D} D = D \overline{D} = \partial_t^2 - \nabla^2 = \Box. \]

Now consider the scalar wave equation

\[ \Box f(x) = g(x) \]  

(5)

where \( g(x) \) is a given source function which, for convenience, is assumed to be a distribution of compact support. The wave radiated by \( g \) is the unique causal solution

\[ f(x) = \int_{\mathbb{R}^4} d^4 x' P(x - x') g(x') = P * g(x), \]  

(6)

\[ ^1 \text{Along with complex numbers and quaternions, the Pauli algebra is one of the simplest examples of Clifford algebra. Although its first application to physics was in quantum mechanics, it has also turned out to be useful in other fields, especially classical electrodynamics [B99]. The relation to Pauli matrices is found by choosing } A_0 = 0 \text{ and } A_k = \nabla x_k \text{ to be the unit vector in the direction of } x_k. \text{ The matrix representing } A_k \text{ is then the Pauli matrix } \sigma_k. \]

\[ ^2 \text{In this context, causality simply means that } f \text{ is supported in the future region of } g. \text{ If } g \text{ vanishes at } t = -\infty \text{ as assumed here, it suffices to take the 'initial condition' } f(x, -\infty) = 0. \]
where $\ast$ denotes spacetime convolution and $P$ is the retarded propagator, which is the wave radiated by $g(x) = \delta(x) \equiv \delta(x)\delta(t)$:

$$P(x) = P(x, t) = \frac{\delta(t - |x|)}{4\pi|x|}, \quad \Box P(x) = \delta(x). \quad (7)$$

Hence the wave operator is invertible on the space of such fields, with

$$\Box^{-1} = P \ast .$$

Since $\Box$ is a scalar operator, it operates on Pauli fields $\mathcal{A}(x)$, expressed in Cartesian coordinates, by

$$\Box \mathcal{A}(x) = \Box A_0(x) + \Box A(x).$$

Thus we may extend the wave equation (5) to Pauli fields as

$$\Box \mathcal{F}(x) = \mathcal{G}(x) \quad (8)$$

where $\mathcal{G}(x)$ is a Pauli-valued distribution with compact support. The unique causal solution is

$$\mathcal{F}(x) = P \ast \mathcal{G}(x) = \int_{\mathbb{R}^4} d^4x' P(x - x') \mathcal{G}(x'). \quad (9)$$

We now apply the Pauli algebra to classical electrodynamics, more or less following [B99]. To minimize the appearance of unnecessary parameters, we use natural Lorentz-Heaviside units, where $\varepsilon_0 = \mu_0 = c = 1$. An electromagnetic field in free space consists of two vector fields $E(x), H(x)$ satisfying Maxwell’s equations

$$\dot{E} - \nabla \times H = -J \quad \nabla \cdot E = \rho \quad (10)$$
$$\dot{H} + \nabla \times E = 0 \quad \nabla \cdot H = 0 \quad (11)$$

where $(\rho, J)$ is a given charge-current density. The obvious symmetry of these equations suggest combining the two fields into a single complex field

$$\mathcal{F}(x) = E(x) + iH(x), \quad (12)$$

for which Maxwell’s equations reduce to

$$\dot{\mathcal{F}} + i\nabla \times \mathcal{F} = -J \quad \nabla \cdot \mathcal{F} = \rho. \quad (13)$$
Now interpret $\mathbf{F}(x)$ as a Pauli field with vanishing scalar component. Then (4) shows that (13) further reduces to the single equation

$$\mathbb{D}\mathbf{F} = \rho - \mathbf{J} \equiv \mathbf{J}.$$ \hspace{1cm} (14)

The homogeneous equations (11) state that the source $\mathbf{J}$ is real, but it will be useful to allow $\mathbf{J}$ to be complex:

$$\mathbf{J} = \mathbf{J}_e + i\mathbf{J}_m \quad \mathbf{J}_e = \rho_e - \mathbf{J} \quad \mathbf{J}_m = \rho_m - \mathbf{J}_m$$ \hspace{1cm} (15)

where $\mathbf{J}_e$ and $\mathbf{J}_m$ represent electric and magnetic sources, respectively. Although Maxwell’s equations require $\mathbf{J}_m = 0$, virtual magnetic sources will be needed in the formulation of the general Huygens principle.

To solve (14) for $\mathbf{F}$, apply $\mathbb{D}$:

$$\Box \mathbf{F} = \mathbb{D} \mathbb{D} \mathbf{F} = \mathbb{D} \mathbf{J}$$ \hspace{1cm} (16)

and note that

$$\mathbb{D} \mathbf{J} = (\partial_t - \nabla)(\rho - \mathbf{J}) = (\dot{\rho} + \nabla \cdot \mathbf{J}) + (i\nabla \times \mathbf{J} - \nabla \rho - \dot{\mathbf{J}}).$$ \hspace{1cm} (17)

Since the left side of (16) is a pure vector field, the scalar component of the right side of (17) must vanish. This gives the continuity equation

$$\langle \mathbb{D} \mathbf{J} \rangle_s = \dot{\rho} + \nabla \cdot \mathbf{J} = 0,$$ \hspace{1cm} (18)

whose real and imaginary parts state that electric and magnetic charge are conserved. Assuming the initial condition $\mathbf{F}(x, -\infty) = 0$, we obtain the unique causal solution

$$\mathbf{F} = P \ast (\mathbb{D} \mathbf{J}) = \mathbb{D} (P \ast \mathbf{J}),$$ \hspace{1cm} (19)

where the last equality follows because $\Box$ commutes with $\mathbb{D}$ and $\mathbb{D}$. Thus

$$\mathbf{F}(x) = \mathbb{D} \mathbf{A}(x)$$ \hspace{1cm} (20)

where the Pauli field

$$\mathbf{A} = P \ast \mathbf{J} = \Phi - \mathbf{A} \quad \text{with} \quad \Phi = P \ast \rho \quad \text{and} \quad \mathbf{A} = P \ast \mathbf{J},$$ \hspace{1cm} (21)

representing the 4-potential, is the causal solution of the wave equation

$$\Box \mathbf{A} = \mathbb{D} \mathbb{D} \mathbf{A} = \mathbb{D} \mathbf{F} = \mathbf{J}.$$ \hspace{1cm} (22)
In fact,

$$\mathbf{F} = \mathcal{D} \mathbf{A} = (\dot{\Phi} + \nabla \cdot \mathbf{A}) - \nabla \Phi - \dot{\mathbf{A}} + i \nabla \times \mathbf{A}$$

(23)

shows that $\mathbf{A}$ satisfies the Lorenz gauge condition$^3$

$$\dot{\Phi} + \nabla \cdot \mathbf{A} = 0.$$  

(24)

If $\mathcal{J}$ is complex as in (15), then so is $\mathbf{A}$:

$$\mathbf{A} = \mathbf{A}_e + i \mathbf{A}_m$$

$$\mathbf{A}_e = \Phi_e - \mathbf{A}_e$$

$$\mathbf{A}_m = \Phi_m - \mathbf{A}_m.$$  

(25)

The free Maxwell field is then given by

$$\mathbf{E} = \text{Re} \mathbf{F} = -\nabla \Phi_e - \dot{\mathbf{A}}_e - \nabla \times \mathbf{A}_m$$

$$\mathbf{H} = \text{Im} \mathbf{F} = -\nabla \Phi_m - \dot{\mathbf{A}}_m + \nabla \times \mathbf{A}_e.$$  

(26)

Of course, the homogeneous Maxwell equations require $\mathcal{J}_m = \mathbf{A}_m = 0$. But the expressions (26) with a virtual magnetic 4-potential $\mathbf{A}_m$ will be used to formulate Huygens’ principle.

### 2 Electromagnetic Huygens principle

The assumption that $\mathcal{J}(x)$ is compactly supported was made for convenience and can be relaxed. While it is reasonable to assume that the spatial support of $\mathcal{J}$ is bounded at any time, we want to allow sources persisting in time, for example a set of charged particles following world lines or extended charged systems evolving in time. This includes, among other things, time-harmonic systems. The above results remain valid provided the integrals converge.

Let the sources be spatially bounded. To simplify the analysis, assume that the spatial support of $\mathcal{J}(x,t)$ is contained in the interior of a closed surface $S \subset \mathbb{R}^3$ at all times $t$.\footnote{This will be generalized to sources in arbitrary motion in Section 3 by allowing $\lambda$ to depend on time. Here we assume a fixed surface $S$, as is commonly done in the derivation of Huygens’ principle.} We assume that $S$ is a smooth manifold, at least of class $C^2$. Denote the exterior of $S$ by $E$ and its interior by $E'$. Let $\lambda(x)$ be

$^3$Evidently, the Lorenz gauge is selected by causality.
a $C^2$ function such that\(^5\)

\[
x \in E \Rightarrow \lambda(x) > 0
\]
\[
x \in S \Rightarrow \lambda = 0 \quad \text{and} \quad |\nabla \lambda| = 1
\]
\[
x \in E' \Rightarrow \lambda(x) < 0.
\]

The characteristic functions $\chi$ and $\chi'$ of $E$ and $E'$ may be written in terms of the Heaviside step function $H$ as \(^6\)

\[
\chi(x) = H(\lambda(x)) = \begin{cases} 1, & x \in E \\ 0, & x \in E' \end{cases}
\]
\[
\chi'(x) = H(-\lambda(x)) = \begin{cases} 0, & x \in E \\ 1, & x \in E' \end{cases}
\]

Then

\[
N(x) \equiv \nabla \chi(x) = -\nabla \chi'(x) = \delta_S(x)n(x)
\]

where

\[
n(x) = \nabla \lambda(x) \quad \text{and} \quad \delta_S(x) = H'(\lambda(x)) = \delta(\lambda(x)).
\]

Thus $n$ is the outward unit normal on $S$ and $d^3x \delta_S(x)$ is the 2D area measure on $S$:

\[
dl x \delta_S(x) = dS(x).
\]

**Remark:** Since the characteristic function $\chi(x)$ does not depend on the choice of $\lambda$, neither does the distributional field $N = \nabla \chi$. The introduction of $\lambda$ is thus seen to be merely a convenience which makes the concepts easier to understand by using the relation $H' = \delta$. Similar remarks apply when $\lambda(x, t)$ is time-dependent, allowing for moving boundaries.

Let $F'$ be an interior field whose source

\[
\mathcal{J}' \equiv \mathcal{D} F'
\]

\(^5\)An example of a function with these properties is

\[
\lambda(x) = \begin{cases} d(x), & x \in E \\ 0, & x \in S \\ -d(x), & x \in E' \end{cases}
\]

where $d(x)$ is the shortest distance from $x$ to $S$.

\(^6\)For $x \in S$, we define $\chi(x) = \chi'(x) = 1/2$; but this singular case will not be needed in the sequel.
is spatially supported in the exterior region $E$. (This includes the case $J' = 0$, where $F'$ is globally sourceless.) Since the spatial support of $J'$ is by definition closed, it must actually be contained in some closed set $V \subset E$. Hence $F'$ is defined and sourceless in an open neighborhood of $S$ as well as in its interior $E'$. Thus both $F$ and $F'$ are defined and sourceless on a neighborhood of $S$.

We shall construct a field $F^s$ whose sources are concentrated on $S$ at all times and which coincides with the given field $F$ in $E$ and with $F'$ in $E'$. The two partial fields are ‘glued’ into a single field defined by

$$F^s(x) = \chi(x)F(x) + \chi'(x)F'(x), \quad x = (x,t) \in \mathbb{R}^4,$$

and the source of $F^s$ is defined by applying $\mathbb{D}$ in a distributional sense:

$$J^s = \rho^s - J^s \equiv \mathbb{D}F^s.$$  \hspace{1cm} (32)

Note that $\mathbb{D}\chi = \nabla \chi = \mathbf{N}$ and

$$\mathbb{D}(\chi F) = (\nabla \chi)F + \chi \mathbb{D}F = \mathbf{N}F + \chi \mathbb{D}F.$$  \hspace{1cm} (31)

Similarly, since $\mathbb{D}\chi' = \nabla \chi' = -\nabla \chi = -\mathbf{N}$,

$$\mathbb{D}(\chi' F') = -\mathbf{N}F + \chi' \mathbb{D}F'.$$

Therefore

$$J^s = \mathbf{N}F^j + \chi J + \chi' J'$$  \hspace{1cm} (33)

where

$$F^j = F - F' = E^j + iH^j \quad E^j = E - E', \quad H^j = H - H'$$  \hspace{1cm} (34)

is the jump field across $S$. Since $J$ is supported in $E'$ and $J'$ is supported in $E$, we have the global identities

$$\chi J \equiv 0 \quad \text{and} \quad \chi' J' \equiv 0.$$  \hspace{1cm} (35)

Hence

$$J^s = \mathbf{N}F^j = \mathbf{N} \cdot F^j + i\mathbf{N} \times F^j = \delta_S(n \cdot F^j + in \times F^j)$$

is a distributional (4D) charge-current density supported on $S$, with a surface charge-current density $(\sigma, K)$ given by

$$\rho^s = \delta_S \sigma \quad \sigma = n \cdot F^j$$

$$J^s = \delta_S K \quad K = -in \times F^j.$$  \hspace{1cm} (36)
Like $J$, $J^S$ satisfies the distributional continuity equation

$$\langle \vec{D} J^S \rangle_s = \dot{\rho}^S + \nabla \cdot J^S = 0,$$

which states that charge, now restricted to flow on $S$, is conserved.

Note that even though $J$ is real, $J^S$ is in general complex, consisting of electric and magnetic sources on $S$:

$$J^S = J^S_e + i J^S_m \quad J^S_e = \rho^S_e - J^S_e \quad J^S_m = \rho^S_m - J^S_m$$

with

$$\rho^S_e = \delta_S \sigma_e, \quad \sigma_e = n \cdot E^j \quad J^S_e = \delta_S K_e, \quad K_e = n \times H^j$$

$$\rho^S_m = \delta_S \sigma_m, \quad \sigma_m = n \cdot H^j \quad J^S_m = \delta_S K_m, \quad K_m = -n \times E^j.$$ (39)

If we wish to construct a physically realizable surface source $J^S$, then the absence of magnetic monopoles requires it to be real:

$$J^S_m = 0 \iff n \cdot H^j = 0 \text{ and } n \times E^j = 0 \text{ on } S.$$ (41)

That is, the normal component of $H^S$ and tangential components of $E^S$ must be continuous across $S$. It can be shown that the scalar condition follows from the vector condition and Maxwell’s homogeneous vector equation. Since we are free to choose any sourceless interior field $F'$, (39) and (41) can be viewed as a set of boundary conditions for $(E', H')$ with $(E, H)$ given. Thus we look for an interior field $F' = E' + i H'$ such that

$$\dot{F}' + i \nabla \times F' = 0 \text{ in } E'$$

$$n \times E' = n \times E \text{ and } n \cdot H' = n \cdot H \text{ on } S.$$ (42)

(Recall that $F'$ actually extends as a sourceless field to a neighborhood of $S$.) This boundary-value problem has a unique solution if $F$ is continuous in an open neighborhood of $S$, which will be the case if $J$ is continuous in time. (Recall that we have also assumed $S$ to be of class $C^2$.) For this unique interior field, (39) and (41) are the jump conditions on the interface between the interior and exterior regions [J99, pp 16–18].

If the interior field does not satisfy (42), then the Huygens representations we are developing, while useful mathematically for expressing the given ‘real’

---

7This follows in the frequency domain from Equation (6.38) in [CK92].

8This is a sufficient but not necessary condition, as follows from the properties of the propagator (7). Due to the factor $\delta(t-r)$, the spread of $J$ in both time and space tends to smooth $F$. 

8
field $F$ (i.e., with $\mathbb{J}_m = 0$) in terms of surface integrals, cannot be realized physically by actual surface sources. This is what was meant by saying that the magnetic sources $\mathbb{J}^m$ are ‘virtual.’ In either case, we now derive the Huygens representations.

The 4-vector potential for $F^S$ is given by (21) and (29) as

$$
\Phi^S(x) = P \ast \rho^S(x) = \int_{\mathbb{R}^4} d^4x' \delta_S(x') P(x - x') n(x') \cdot F^j(x') \quad (43)
$$

$$
\Phi^S(x) = \frac{1}{4\pi} \int_S dS(x') r^{-1} [n \cdot F^j]
$$

$$
A^S(x) = P \ast J^S(x) = -i \int_{\mathbb{R}^4} d^4x' \delta_S(x') P(x - x') n(x') \times F^j(x')
$$

$$
A^S(x) = \frac{1}{4\pi i} \int_S dS(x') r^{-1} [n \times F^j],
$$

where $r = |x - x'|$ and

$$[u] = u(x', t - r), \quad [v] = v(x', t - r)$$

denote retarded expressions. Hence the electric and magnetic 4-potentials are

$$
\Phi^S_e(x) = \frac{1}{4\pi} \int dS(x') r^{-1} [n \cdot E^j] \quad (44)
$$

$$
\Phi^S_m(x) = \frac{1}{4\pi} \int dS(x') r^{-1} [n \cdot H^j]
$$

$$
A^S_e(x) = \frac{1}{4\pi} \int dS(x') r^{-1} [n \times H^j]
$$

$$
A^S_m(x) = -\frac{1}{4\pi} \int dS(x') r^{-1} [n \times E^j],
$$

Substituting these into (26) gives

$$
\begin{align*}
E^S &= -\nabla \Phi^S_e - \dot{A}^S_e - \nabla \times A^S_m \\
H^S &= -\nabla \Phi^S_m - \dot{A}^S_m + \nabla \times A^S_e.
\end{align*}
$$

If we choose $F' = 0$ and assume $x \in E$, then $F^j = F$ and (45) reduce to the Stratton-Chu equations [HY99, page 32]. Since $F' = 0$ does not satisfy (42) except in the trivial case $F = 0$, the Stratton-Chu formulation of Huygens’ principle requires virtual magnetic sources.

However, if $F'$ is chosen to be the unique solution of (42), the Stratton-Chu equations reduce to the simpler expressions

$$
E^S = -\nabla \Phi^S_e - \dot{A}^S_e \qquad H^S = \nabla \times A^S_e. \quad (46)
$$
Returning to the general case (44) and (45), note that \( A^S \) involves only the tangential components of \( F^j \) on \( S \) while \( \Phi^S \) involves only the normal components. The latter can be eliminated as follows. Begin with

\[
\dot{F}^S = -i\nabla \times F^S - J^S = -i\nabla \times (i\nabla \times A^S - \nabla \Phi^S - \dot{A}^S) - J^S \\
= \nabla \times \nabla \times A^S + iN \times F^j.
\]

This involves only the tangential component \( n \times F^j \) of \( F^j \) on \( S \), and it can be integrated using the initial condition \( A^S(x, -\infty) = 0 \) to obtain

\[
F^S = \nabla \times \nabla \times \partial_t^{-1} A^S + i\nabla \times A^S + iN \times \partial_t^{-1} F^j.
\] (47)

where \( \partial_t^{-1} A^S(x, t) = \int_{-\infty}^t dt' A^S(x, t') \).

If we choose \( F' = 0 \) and assume \( x \in E \), then (47) reduce to the Kottler-Franz equations [HY99, page 34]:

\[
E^S = \nabla \times \nabla \times \partial_t^{-1} A^S_e - \nabla \times A^S_m, \quad H^S = \nabla \times \nabla \times \partial_t^{-1} A^S_m + \nabla \times A^S_e.
\] (48)

However, note that (47) are global, remaining valid when \( x \in E' \) (where \( F^S = F' \) ) and, in a distributional sense, even when \( x \in S \) as indicated by the last term.

Like the Stratton-Chu equations, (47) involve virtual magnetic sources on \( S \). If we assume that the interior field satisfies the physical boundary conditions (42), then (48) simplify to

\[
E^S = \nabla \times \nabla \times \partial_t^{-1} A^S_e, \quad H^S = \nabla \times A^S_e.
\] (49)

### 3 Moving sources

The above can be generalized to sources in arbitrary motion simply by letting \( \lambda \) depend on time, so that the 2D surface

\[ S_t = \{ x : \lambda(x, t) = 0 \} \subset \mathbb{R}^3 \]

enclosing the source \( J(x, t) \) at time \( t \) is time-dependent. The 3D hypersurface

\[ \tilde{S} = \{ (x, t) : \lambda(x, t) = 0 \} \subset \mathbb{R}^4 \]
now represents the history of $S_t$. It is the oriented boundary separating the exterior and interior spacetime regions:

$$\tilde{E} = \{ x : \lambda(x) > 0 \}, \quad \tilde{E}' = \{ x : \lambda(x) < 0 \}, \quad \tilde{S} = \partial \tilde{E}' = -\partial \tilde{E}.$$ 

Define the Huygens field

$$\tilde{F}(x) = \chi(x) F(x) + \chi'(x) F'(x) \quad (50)$$

where

$$\chi(x) = H(\lambda(x)) \quad \text{and} \quad \chi'(x) = H(-\lambda(x)) = 1 - \chi(x)$$

are the characteristic functions of $\tilde{E}$ and $\tilde{E}'$. Then the same arguments as above give

$$\tilde{J}(\tilde{S}) \equiv \tilde{D} \tilde{F}(\tilde{S}) = \{ \frac{d}{dt} \tilde{\lambda} \tilde{F}_j + \tilde{n} \times \tilde{F}_j \} \quad (51)$$

with charge- and current distributions

$$\rho(\tilde{S}) = \delta(\tilde{S}) \tilde{n} \cdot \tilde{F}_j, \quad \tilde{J}(\tilde{S}) = -\delta(\tilde{S}) \frac{d}{dt} \tilde{\lambda} \tilde{F}_j + \tilde{n} \times \tilde{F}_j \quad (52)$$

where

$$\delta(\tilde{S}) = \delta(\lambda(x)) \quad (53)$$

is a distribution supported on $\tilde{S}$, whose interpretation is given by the measure

$$d^4x \delta(\tilde{S}) = dt \, d^3x \, \delta(\lambda(x), t) = dt \, dS_t(x).$$

The term $-\dot{\lambda} \tilde{F}_j$ is a drag current on generated by its motion. Since

$$-\dot{\lambda} \tilde{F}_j = \dot{\lambda} n \times (n \times \tilde{F}_j) - \dot{n} \tilde{n} \cdot \tilde{F}_j,$$

it has both normal and tangential component. Equations (43) generalize to

$$\Phi(\tilde{x}) = \int_{\mathbb{R}^4} d^4x' \, \delta(\tilde{S}) \tilde{P}(x - x') \tilde{n}(x') \cdot \tilde{F}_j(x') \quad (54)$$

$$A(\tilde{x}) = -\int_{\mathbb{R}^4} d^4x' \, \delta(\tilde{S}) \tilde{P}(x - x') \left\{ \dot{\lambda}(x') \tilde{F}_j(x') + \dot{n}(x') \times \tilde{F}_j(x') \right\}$$

The electric and magnetic 4-potentials are the real and imaginary parts, and the Stratton-Chu equations for a moving surface are obtained exactly as in (45). Enforcing the homogeneous Maxwell equations on $\tilde{S}$ gives the boundary conditions

$$\tilde{n} \cdot \tilde{H}_j = 0 \quad \text{and} \quad \dot{\tilde{\lambda}} \tilde{H}_j + \tilde{n} \times \tilde{E}_j = 0 \quad (55)$$

Note that the vector condition implies the scalar condition if $\dot{\tilde{\lambda}} \neq 0$. (As stated earlier, this is also true when $\dot{\tilde{\lambda}} = 0$, though not as obviously.) It can then be used to determine the interior field.
4 Local partitions and nonlinearity

The expression (50) defines a global field $F^S$ using a partition of spacetime into the exterior and interior regions separated by the interface $\tilde{S}$:

$$\mathbb{R}^4 = \tilde{E} \cup \tilde{S} \cup \tilde{E}'.$$

This can be generalized to a finite (or even infinite) sum of partial fields $F_k$ defined on regions $\tilde{E}_k$ with characteristic functions $\chi_k$:

$$F(x) = \sum_k \chi_k(x)F_k(x), \quad (56)$$

where the superscript $\tilde{S}$ has been dropped. We assume that $F_k$ is a source-less field in an open spacetime region $O_k$ containing the closure of $\tilde{E}_k$, thus extending beyond its boundary. Since

$$\chi_k \mathbb{D}F_k = 0 \quad \forall k,$$

the source of $F$ is the distribution

$$\mathbb{J} = \mathbb{D}F = \sum_k (\mathbb{D}\chi_k)F_k = \sum_k \{\dot{\chi}_k F_k + \nabla \chi_k \cdot F_k + i\nabla \chi_k \times F_k\} \quad (57)$$

$$\Rightarrow \rho = \sum_k \nabla \chi_k \cdot F_k \quad \text{and} \quad J = -\sum_k \{\dot{\chi}_k F_k + i\nabla \chi_k \times F_k\}.$$

Since $\mathbb{D}\chi_k$ is supported on $\partial \tilde{E}_k$, $\mathbb{J}$ is supported on the ‘cellular’ boundary

$$\tilde{S} = \bigcup_k \partial \tilde{E}_k. \quad (58)$$

Furthermore, since

$$x \in \partial \tilde{E}_k \cap \partial \tilde{E}_l \Rightarrow \mathbb{D}(\chi_k + \chi_l) = 0, \quad (59)$$

$\mathbb{J}$ depends only on the jump fields

$$F_{k,l}^j = F_k - F_l$$

---

9 Since $\partial \tilde{E}_k$ is oriented by the unit normal pointing into its interior $\tilde{E}_k$, each interface $\partial \tilde{E}_k \cap \partial \tilde{E}_l$ between adjoining regions occurs twice in (58), with opposite orientations. Hence the oriented sum (chain) $\sum_k \partial \tilde{E}_k$ vanishes but the set-theoretic union $\tilde{S}$ does not.

10 It makes no sense to say that $\mathbb{D}\chi_k = -\mathbb{D}\chi_l$ since $\mathbb{D}\chi_k$ and $\mathbb{D}\chi_l$ are both infinite on $\partial \tilde{E}_k \cap \partial \tilde{E}_l$. On the other hand, $\chi_k + \chi_l$ is the characteristic function of $\tilde{E}_k \cup \tilde{E}_l \cup (\partial \tilde{E}_k \cap \partial \tilde{E}_l)$, hence (59) makes sense.
across the interfaces between adjoining regions $\tilde{E}_k, \tilde{E}_l$. Thus

$$\mathbf{F} = \tilde{\mathcal{D}} \mathbf{A} \quad \text{where} \quad \mathbf{A} = P \ast \mathbb{J}$$  \hspace{1cm} (60)

gives a generalized Huygens representation of $\mathbf{F}$ in terms of sources supported on $\tilde{S}$. Furthermore, the projection property \(^{11}\)

$$\chi_l(x) \chi_m(x) = \delta_{lm} \chi_l(x)$$  \hspace{1cm} (61)

implies that nonlinear expressions in $\mathbf{F}$ have similar partitions. For example, the scalar Lorentz invariant

$$\mathbf{F}^2 = \mathbf{F} \cdot \mathbf{F} = \mathbf{E}^2 - \mathbf{H}^2 + 2i \mathbf{E} \cdot \mathbf{H}$$

has the local partition

$$\mathbf{F}^2 = \sum_k \chi_k \mathbf{F}_k^2,$$

and the electromagnetic energy-momentum density

$$\mathcal{S} \equiv \frac{1}{2} \mathbf{F} \mathbf{F}^* = \frac{1}{2} \left\{ \mathbf{F} \cdot \mathbf{F}^* + i \mathbf{F} \times \mathbf{F}^* \right\} = \mathbf{U} + \mathcal{S}$$  \hspace{1cm} (62)

has the local partition

$$\mathcal{S} = \sum_k \chi_k \mathcal{S}_k \quad \mathbf{U} = \sum_k \chi_k \mathbf{U}_k \quad \mathbf{S} = \sum_k \chi_k \mathbf{S}_k.$$

Hence the local power density (rate of increase of energy density) is

$$\langle \mathcal{D} \mathcal{S} \rangle_s = \dot{\mathbf{U}} + \nabla \cdot \mathbf{S} = \sum_k \left\{ \chi_k \dot{\mathbf{U}}_k + \nabla \chi_k \cdot \mathbf{S}_k \right\} + \sum_k \chi_k \left\{ \dot{\mathbf{U}}_k + \nabla \cdot \mathbf{S}_k \right\}.$$

Since $\mathbf{F}_k$ is sourceless in $\mathcal{O}_k$, it follows from Poynting’s theorem that

$$\dot{\mathbf{U}}_k + \nabla \cdot \mathbf{S}_k = 0 \quad \text{in} \quad \mathcal{O}_k$$

\(^{11}\)Equation (61) fails numerically on $\partial \tilde{E}_k \cap \partial \tilde{E}_l$ where $\chi_k(x) = \chi_l(x) = 1/2$, but it holds weakly, in the sense of distributions, i.e.,

$$\int d^4x \chi_l(x) \chi_m(x) f(x) = \delta_{lm} \int d^4x \chi_l(x) f(x)$$

for any continuous function $f$ with compact support or rapid decay (needed when $\tilde{E}_k$ or $\tilde{E}_l$ are unbounded).
and thus

$$\dot{U} + \nabla \cdot S = \sum_k \{ \dot{\chi}_k U_k + \nabla \chi_k \cdot S_k \}. \tag{63}$$

Here $\dot{\chi}_k U_k$ is the rate of increase in the energy density coming into $\tilde{E}_k$ due the motion of the boundary $\partial \tilde{E}_k$, and $\nabla \chi_k \cdot S_k$ is that due to the incoming momentum flowing through $\partial \tilde{E}_k$. Due to (59), the right side of (63) involves only the differences

$$U_{kl}^j = U_k - U_l \quad \text{and} \quad S_{kl}^j = S_k - S_l \quad \text{on} \quad \partial \tilde{E}_k \cap \partial \tilde{E}_l.$$

Since the general partition (56) allows arbitrary choices of sourceless fields $F_k$ in domains $\mathcal{O}_k$ containing the closure of $\tilde{E}_k$, these differences need not vanish. By enforcing boundary conditions on any interface $\partial \tilde{E}_k \cap \partial \tilde{E}_l$, the corresponding terms can be made to vanish. But then that interface can be removed, thus merging the two cells into one.

It is instructive to confirm (63) using the expression (57) for the surface current $\bar{J}$. Recall that $\bar{J}$ is generally complex, including magnetic as well as electric sources:

$$\bar{J} = \bar{J}_e + i \bar{J}_m.$$

The generalized Poynting theorem for a complex surface current density is derived by applying the distributional Maxwell equations

$$\tilde{F} + i \nabla \times F = -J \quad \tilde{F}^* - i \nabla \times F^* = -J^*$$

to

$$\dot{U} + \nabla \cdot S = \frac{1}{2} \left\{ \tilde{F} \cdot F^* + F \cdot \tilde{F}^* + i \nabla \times F \cdot F^* - i F \cdot \nabla \times F^* \right\},$$

which gives

$$\dot{U} + \nabla \cdot S = -\frac{1}{2} (J \cdot F^* + F \cdot J^*) = -J_e \cdot E - J_m \cdot H. \tag{64}$$

The right side is, like $J$, a distribution supported on $\tilde{S}$. The partitions

$$-J = \sum_k \{ \dot{\chi}_k F_k + i \nabla \chi_k \times F_k \} \quad \text{and} \quad F = \sum_l \chi_l F_l$$

14
\( -J \cdot F^* = \sum_{kl} \{\chi_l \dot{\chi}_k F_k \cdot F^*_l + i\chi_l \nabla \chi_k \times F_k \cdot F^*_l\}. \)

Using \( \nabla \chi_k \times F_k \cdot F^*_l = \nabla \chi_k \cdot F_k \times F^*_l \), (64) gives

\[ \dot{U} + \nabla \cdot S = \frac{1}{2} \sum_{kl} \{ (\chi_l \dot{\chi}_k + \chi_k \dot{\chi}_l) F_k \cdot F^*_l + i(\chi_l \nabla \chi_k + \chi_k \nabla \chi_l) \cdot F_k \times F^*_l \}. \]

But the projection property (61) implies the distributional identities

\[ \chi_l \dot{\chi}_k + \chi_k \dot{\chi}_l = \delta_{kl} \dot{\chi}_k \quad \text{and} \quad \chi_l \nabla \chi_k + \chi_k \nabla \chi_l = \delta_{kl} \nabla \chi_k, \]

therefore

\[ \dot{U} + \nabla \cdot S = \sum_k \{ \dot{\chi}_k U_k + \nabla \chi_k \cdot S_k \} \]

in agreement with (63). This confirms the consistency of our computations involving bilinear distributional expressions.\(^\text{12}\) It gives reason to hope that partition methods could be useful in the synthesis of global solutions to nonlinear equations from local solutions.

### Acknowledgements

I thank David Colton and Thorkild Hansen for helpful discussions, and Arje Nachman for his sustained support of this work, most recently through AFOSR Grant #FA9550-08-1-0144.

### References

[B99] W E Baylis, *Electrodynamics: A Modern Geometric Approach*. Birkhäuser Progress in Mathematical Physics vol 17, Boston, 1999

[CK92] D Colton and R Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer, Berlin, 1992

\(^{12}\)Quadratic expressions in singular distributions such as \( \delta(x) \) do not make sense, but evidently products such as \( \chi_i \dot{\chi}_k \) and \( \chi_i \nabla \chi_k \) do, due to the mild nature of the singularity of \( \chi_i \) (i.e., its finite jump discontinuity).
[H66] D Hestenes, *Space-Time Algebra*. Gordon and Breach, New York, 1966

[HY99] T B Hansen and A Yaghjian, *Plane-Wave Theory of Time-Domain Fields: Near-Field Scanning Applications*. IEEE Press, 1999

[J99] J D Jackson, *Classical Electrodynamics*, third edition. John Wiley & Sons, New York, 1999