A renormalisation group method.
II. Approximation by local polynomials

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Abstract

This paper is the second in a series devoted to the development of a rigorous renormalisation group method for lattice field theories involving boson fields, fermion fields, or both. The method is set within a normed algebra $\mathcal{N}$ of functionals of the fields. In this paper, we develop a general method—localisation—to approximate an element of $\mathcal{N}$ by a local polynomial in the fields. From the point of view of the renormalisation group, the construction of the local polynomial corresponding to $F \in \mathcal{N}$ amounts to the extraction of the relevant and marginal parts of $F$. We prove estimates relating $F$ and its corresponding local polynomial, in terms of the $T_\phi$ semi-norm introduced in part I of the series.

1 Introduction and main results

This paper is the second in a series devoted to the development of a rigorous renormalisation group method. In [6], we defined a normed algebra $\mathcal{N}$ of functionals of the fields. The fields can be bosonic, or fermionic, or both, and in most of this paper there is no distinction between these possibilities. The algebra $\mathcal{N}$ is equipped with the $T_\phi$ semi-norm, which is defined in terms of a normed space $\Phi$ of test functions. In the renormalisation group method, a sequence of test function spaces $\Phi_j$ is chosen, with corresponding normed algebras $\mathcal{N}_j$, and there is a dynamical system whose trajectories evolve through these normed algebras in the sequence $\mathcal{N}_0 \to \mathcal{N}_1 \to \mathcal{N}_2 \to \cdots$. The dimension of the dynamical system is unbounded, but a finite number of local polynomials in the fields represent the relevant (expanding) and marginal (neutral) directions for the dynamical system. These local polynomials play a central role in the renormalisation group approach.

In this paper, we develop a general method for the extraction from an element $F \in \mathcal{N}$ of a local polynomial $\text{Loc}_X F$, localised on a spatial region $X$, that captures the relevant and marginal parts of $F$. We also prove norm estimates which show that the norm of $\text{Loc}_X F$ is not much larger than the norm of $F$, while the norm of $F - \text{Loc}_X F$ is substantially smaller than the norm of $F$. The latter fact, which is crucial, indicates that $\text{Loc}_X F$ has encompassed the important part of $F$, leaving the irrelevant remainder $F - \text{Loc}_X F$. The method used in our construction of $\text{Loc}_X F$ bears some relation to ideas in [4].

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This paper is organised as follows. Section 1 contains the principal definitions and statements of results, as well as some of the simpler proofs. More substantial proofs are deferred to Section 2. Section 3 contains estimates for lattice Taylor expansions; these play an essential role in the proofs of Propositions 1.11–1.12, which provide the norm estimates on $\text{Loc}_X F$ and $F - \text{Loc}_X F$.

1.1 Fields and test functions

We recall some concepts and notation from [6].

Let $\Lambda = \mathbb{Z}^d / (R^m \mathbb{Z})$ denote the $d$-dimensional discrete torus of (large) side $R^m$, for integers $R \geq 2$ and $m \geq 1$. In [6], we have introduced an index set $\Lambda = \Lambda_b \sqcup \Lambda_f$. The set $\Lambda_b$ is itself a disjoint union of sets $\Lambda^{(i)}_b (i = 1, \ldots, s_b)$ corresponding to different species of boson fields. Each $\Lambda^{(i)}_b$ is either a finite disjoint union of copies of $\Lambda$, with each copy representing a distinct field component for that species, or is $\Lambda \sqcup \bar{\Lambda}$ when a complex field species is intended. The set $\Lambda_f$ has the same structure, with possibly a different number $s_f$ of fermion field species.

An element of $\mathbb{R}^{\Lambda_b}$ is called a boson field, and can be written as $\phi = (\phi_x)_{x \in \Lambda_b}$. Let $\mathcal{R}(\Lambda_b)$ denote the ring of functions from $\mathbb{R}^{\Lambda_b}$ to $\mathbb{C}$ having at least $p_N$ continuous derivatives, where $p_N$ is fixed. The fermion field $\psi = (\psi_y)_{y \in \Lambda_f}$ is a set of anticommuting generators for an algebra $\mathcal{N} = \mathcal{N}(\Lambda)$ over the ring $\mathcal{R}$. By definition, $\mathcal{N}$ consists of elements $F$ of the form

$$F = \sum_{y \in \Lambda^*_f} \frac{1}{y!} F_y \psi_y,$$

where each coefficient $F_y$ is an element of $\mathcal{R}$. We will use test functions $g : \Lambda^*_f \to \mathbb{C}$ as defined in [6]. Also, given a boson field $\phi$, we will use the pairing between elements of $\mathcal{N}$ and test functions defined in [6] and written as

$$\langle F, g \rangle_\phi = \sum_{x \in \Lambda^*_f} \frac{1}{z!} F_z (\phi) g_z.$$

For our present purposes, we distinguish between the boson and fermion fields only through the dependence of the pairing on the boson field $\phi$. When the distinction is unimportant, we use $\varphi$ to denote both kinds of fields, and identify $\bar{\Lambda}$ with $\Lambda \times \{1, 2, \ldots, p_\Lambda\}$, where $p_\Lambda$ is the number of copies of $\Lambda$ comprising $\bar{\Lambda}$. This $p_\Lambda$ is given by the sum, over all species, of the number of components within a species. Thus we can write the fields all evaluated at $x \in \Lambda$ as the sequence $\varphi(x) = (\varphi_1(x), \ldots, \varphi_{p_\Lambda}(x))$.

1.2 Local monomials and local polynomials

Let $e_1, \ldots, e_d$ denote the standard unit vectors in $\mathbb{Z}^d$, so that

$$\mathcal{U} = \{ \pm e_1, \ldots, \pm e_d \}$$

is the set of all $2d$ unit vectors. For $e \in \mathcal{U}$ and $f : \Lambda \to \mathbb{C}$, the difference operator is given by

$$\nabla^e f(x) = f(x + e) - f(x).$$
When \( e \) is one of the standard unit vectors \( \{ e_1, \ldots, e_d \} \), we refer to \( \nabla^e \) as a forward derivative. When \( e \) is the negative of a standard unit vector we refer to \( \nabla^e \) as a backward derivative, although it is the negative of a conventional backward derivative. We allow \( 2d \) directions in \( U \), rather than only \( d \), so as not to break lattice symmetries by favouring forward derivatives over backward derivatives. This introduces redundancy expressed by the identity

\[
\nabla^e + \nabla^{-e} = -\nabla^{-e} \nabla^e, \tag{1.5}
\]

which is straightforward to verify by evaluating both sides on a function \( f \). For \( \alpha \in \mathbb{N}_0^d \) with components \( \alpha(e) \in \mathbb{N}_0 \), we write

\[
\nabla^\alpha = \prod_{e \in U} \nabla^{\alpha(e)}, \quad \nabla^0 = \text{Id}, \tag{1.6}
\]

where the product is independent of the order of its factors.

A local monomial \( M \) is a finite product of fields and their derivatives, all to be evaluated at the same point in \( \Lambda \) (whose value we suppress). To be more precise, for \( m = (m_1, \ldots, m_p(m)) \) a finite sequence whose components \( m_k = (i_k, \alpha_k) \) are elements of \( \{1, \ldots, p_\Lambda\} \times \mathbb{N}_0^d \), we define

\[
M_m = \prod_{k=1}^{p(m)} \nabla^{\alpha_k} \varphi_{i_k} = \left( \nabla^{\alpha_1} \varphi_{i_1} \right) \cdots \left( \nabla^{\alpha_{p(m)}} \varphi_{i_{p(m)}} \right). \tag{1.7}
\]

The product in \( M_m \) is taken in the same order as the components \( i_k \) in \( m \). For example, if the sequence \( m \) is given by \( m = ((1, \alpha_1), (1, \alpha_1), (1, \alpha_2), (1, \alpha_2), (1, \alpha_2), (2, \alpha_3)) \) with \( \alpha_1 < \alpha_2 \), then

\[
M_m = (\nabla^{\alpha_1} \varphi_1)^2 (\nabla^{\alpha_2} \varphi_1)^3 \nabla^{\alpha_3} \varphi_2. \tag{1.8}
\]

It is convenient to denote the number of times \( m \) contains a given pair \( (i, \alpha) \) as \( n_{(i, \alpha)} = n_{(i, \alpha)}(m) \); in (1.7) we have \( n_{(1, \alpha_1)} = 2 \), \( n_{(1, \alpha_2)} = 3 \), \( n_{(2, \alpha_3)} = 1 \), and all other \( n_{(i, \alpha)} \) are zero. For a fermionic species \( i \), \( M_m = 0 \) when \( n_{(i, \alpha)} > 1 \). Permutations of the order of the components of \( m \) give plus or minus the same monomial. We will now define a subset \( m \) of sequences such that every non-zero monomial (1.8) is represented by exactly one \( m \in m \). First we fix an order \( \leq \) on the elements of \( \mathbb{N}_0^d \). Let \( m \) be the set whose elements are finite sequences as defined above and such that: (i) \( i_1 \leq \cdots \leq i_{p(m)} \); (ii) for \( i \) a fermionic species \( n_{(i, \alpha)} = 0, 1 \); (iii) for \( k < k' \) with \( i_k = i_{k'} \), \( \alpha_k \leq \alpha_{k'} \). Conditions (i) and (iii) together amount to imposing lexicographic order on the components of a sequence \( m \).

The degree of a local monomial \( M_m \) is the length \( p = p(m) \) of the sequence \( m \in m \). For \( m \) equal to the empty sequence \( \emptyset \) of length 0, we set \( M_\emptyset = 1 \). In addition, we specify a map which associates to each field species a value in \((0, +\infty]\) called the scaling dimension (also known as engineering dimension), which we abbreviate as the dimension of the field species. Following tradition, for \( i = 1, \ldots, p_\Lambda \), we denote the dimension of the species of the field \( \varphi_i \) by \([\varphi_i]\). This dimension does not depend on the value of the field, only on its species. Then we define the dimension of \( M_m \) by

\[
[M_m] = \sum_{k=1}^{p(m)} ([\varphi_{i_k}] + |\alpha_k|_1). \tag{1.9}
\]
with the degenerate case \([M_\varnothing] = [1] = 0\).

Let \(m_+\) denote the subset of \(m\) for which only forward derivatives occur. Given \(d_+ \geq 0\), let \(\mathcal{M}_+\) denote the set of monomials \(M_m\) with \(m \in m_+\), such that
\[
[M_m] \leq d_+. \tag{1.10}
\]

**Example 1.1.** Consider the case of a single real-valued boson field \(\varphi\) of dimension \([\varphi] = \frac{d-2}{2}\), with no fermion field. The space \(\mathcal{N}_j\) is reached after \(j\) renormalisation group steps have been completed. Each renormalisation group step integrates out a fluctuation field, with the remaining field increasingly smoother and smaller in magnitude. A basic principle is that there is an \(L > 0\) such that \(\varphi_x\) will typically have magnitude approximately \(L^{-j[\varphi]}\), and that moreover \(\varphi\) is roughly constant over distances of order \(L^j\). A block \(B\) in \(\mathbb{Z}^d\), of side \(L^j\), contains \(L^{dj}\) points, so the above assumptions lead to the rough correspondence
\[
\sum_{x \in B} |\varphi_x|^p \approx L^{(d-\rho[\varphi])j}. \tag{1.11}
\]
In the case of \(d = 4\), for which \([\varphi] = 1\), this scales down when \(p > 4\) and \(\varphi^p\) is said to be irrelevant. The power \(p = 4\) neither decays nor grows, and \(\varphi^4\) is called marginal. Powers \(p < 4\) grow with the scale, and \(\varphi^p\) is said to be relevant. The assumption that \(\varphi\) is roughly constant over distances of order \(L^j\) translates into an assumption that each spatial derivative of \(\varphi\) produces a factor \(L^{-j}\), so that, e.g.,
\[
\sum_{x \in B} |\nabla^\alpha \varphi_x|^p \approx L^{(d-\rho[\varphi]-\rho[\alpha])j}. \tag{1.11}
\]

Thus, in dimension \(d = 4\) with \(d_+ = 4\), \(\mathcal{M}_+\) consists of the relevant monomials
\[
1, \, \varphi, \, \varphi^2, \, \varphi^3, \, \nabla_i \varphi, \, \nabla_j \nabla_i \varphi, \, \varphi \nabla_i \varphi, \tag{1.12}
\]

as well as the marginal monomials
\[
\varphi^4, \, \nabla_k \nabla_j \nabla_i \varphi, \, \varphi \nabla_j \nabla_i \varphi, \, \varphi^2 \nabla_i \varphi, \tag{1.13}
\]

with each \(\nabla_l\) represents forward differentiation in the direction \(e_l \in \{+e_1, \ldots, +e_d\}\).

Let \(\mathcal{P}\) be the vector space over \(\mathbb{C}\) freely generated by all the monomials \((M_m)_{m \in m}\) of finite dimension. A polynomial \(P \in \mathcal{P}\) has a unique representation
\[
P = \sum_{m \in m} a_m M_m, \tag{1.14}
\]
where all but finitely many coefficients \(a_m \in \mathbb{C}\) are zero. Similarly, we define \(\mathcal{P}_+\) to be the vector subspace of \(\mathcal{P}\) freely generated by the monomials \((M_m)_{m \in m_+}\) of finite dimension. Given \(x \in \Lambda\), a polynomial \(P \in \mathcal{P}\) is mapped to an element \(P_x \in \mathcal{N}\) by evaluating the fields in \(P\) at \(x\). More generally, for any \(X \subset \Lambda\) and \(P \in \mathcal{P}\), we define an element of \(\mathcal{N}\) by
\[
P(X) = \sum_{x \in X} P_x. \tag{1.15}
\]

For a real number \(t\) we define \(\mathcal{P}_t\) to be the subspace of \(\mathcal{P}\) spanned by the monomials with \([M_m] \geq t\). Let
\[
v_+ = \{m \in m_+ : [M_m] \leq d_+\} = \{m \in m_+ : M_m \in \mathcal{M}_+\}, \tag{1.16}
\]
and let $V_+$ denote the vector subspace of $P_+$ generated by the monomials in $V_+$. By definition, the set $v_+$ is finite. The use of only forward derivatives to define $V_+$ breaks the Euclidean symmetry of $\Lambda$. We wish to replace $V_+$ by a symmetric family of polynomials, and this leads us to consider Euclidean symmetry in more detail.

Let $\Sigma$ be the group of permutations of $U$. Let $\Sigma_{\text{axes}}$ be the abelian subgroup of $\Sigma$ whose elements fix $\{e_i, -e_i\}$ for each $i = 1, \ldots, d$. In other words, elements of $\Sigma_{\text{axes}}$ act on $U$ by possibly reversing the signs of the unit vectors. Let $\Sigma_+$ be the subgroup of permutations that permute $\{e_1, \ldots, e_d\}$ onto itself and $\{-e_1, \ldots, -e_d\}$ onto itself. Then (i) $\Sigma_{\text{axes}}$ is a normal subgroup of $\Sigma$, (ii) every element of $\Sigma$ is the product of an element of $\Sigma_{\text{axes}}$ with an element of $\Sigma_+$, and (iii) the intersection of the two subgroups is the identity. Therefore, by definition, $\Sigma$ is the semidirect product $\Sigma = \Sigma_{\text{axes}} \rtimes \Sigma_+$.

An element $\Theta \in \Sigma$ acts on elements of $N_0^d$ via its action on components, as $(\Theta\alpha)(e) = \alpha(\Theta(e))$. The action of $\Theta$ on derivatives is then given by $\Theta\nabla^\alpha = \nabla^{\Theta\alpha}$. This allows us to define an action of the group $\Sigma$ on $P$ by linear transformations, determined by the action on any monomial in $P$ by possibly reversing the signs of the unit vectors. Let $\Theta \in \Sigma$.

The action of $\Theta$ on components is reversed by $\Theta$ is odd and otherwise $\Theta$ is even. The action of $\Theta$ on derivatives is then given by $\Theta\nabla^\alpha = \nabla^{\Theta\alpha}$. This allows us to define an action of the group $\Sigma$ on $P$ by linear transformations, determined by the action of $\Theta$ on components, as $(\Theta\alpha)(e) = \alpha(\Theta(e))$. The action of $\Theta$ on derivatives is then given by $\Theta\nabla^\alpha = \nabla^{\Theta\alpha}$. This allows us to define an action of the group $\Sigma$ on $P$ by linear transformations, determined by the action of $\Theta$ on components, as $(\Theta\alpha)(e) = \alpha(\Theta(e))$. The action of $\Theta$ on derivatives is then given by $\Theta\nabla^\alpha = \nabla^{\Theta\alpha}$.

As the notation indicates, the homomorphism can depend on $P$.

The polynomials in $V_+$ contain only forward derivatives and hence do not form an invariant subspace of $P$ under the action of $\Sigma$. We wish to replace $V_+$ by a suitable $\Sigma$-invariant subspace of $P$, which we will call $V$. As a first step in this process, we define a map that associates to a monomial $M \in M_+$ a polynomial $P(M) \in P$, by

$$P(M) = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta, M) \Theta M \quad (1.19)$$

where $\lambda(\Theta, M) = -1$ if the number of derivatives in $M$ that are reversed by $\Theta$ is odd and otherwise $\lambda(\Theta, M) = 1$. This is a homomorphism: for $\Theta, \Theta' \in \Sigma_{\text{axes}}$, $\lambda(\Theta\Theta', M) = \lambda(\Theta, M)\lambda(\Theta', M)$. Note that $P(M)$ consists of a linear combination of monomials whose degrees and dimensions are all equal to those of $M$. We claim that for any $M \in M_+$, the polynomial $P = P(M)$ of (1.19) obeys: $P(M)$ is $\Sigma_{\text{axes}}$-covariant; $M - P(M) \in P_t$ for some $t > \lceil M \rceil$ up to terms that vanish under the redundancy relation (1.5); and $P(\Theta M) = \Theta P(M)$ for $\Theta \in \Sigma$. The proof of this fact is deferred to Section 2.3.

To enable the use of the redundancy relation (1.5), let $R_1$ be the vector subspace of $P$ generated by the relation (1.5); this is defined more precisely as follows. First, $0 \in R_1$. Given nonzero $P \in P$, we recursively replace any occurrence of $\nabla^e \nabla^{-e}$ in any monomial in $P$ by the equivalent expression $-(\nabla^e + \nabla^{-e})$. This procedure produces monomials of lower dimension so eventually terminates. If the resulting polynomial is the zero polynomial, then $P \in R_1$, and otherwise $P \notin R_1$. The claim in the previous paragraph shows the existence of the polynomial $P$ of the next definition.
**Definition 1.2.** To each monomial $M \in \mathcal{M}_+$ we choose a polynomial $\hat{P}(M) \in \mathcal{P}$, which is a linear combination of monomials of the same degree and dimension as $M$, such that

(i) $\hat{P}(M)$ is $\Sigma_{\text{axes}}$-covariant,

(ii) $M - \hat{P}(M) \in \mathcal{P}_t + \mathcal{R}_1$ for some $t > |M|$,

(iii) $\Theta \hat{P}(M) = \hat{P}(\Theta M)$ for $\Theta \in \Sigma_+$.

Let $\mathcal{V}$ be the vector subspace of $\mathcal{P}$ spanned by the polynomials $\{\hat{P}(M) : M \in \mathcal{M}_+\}$. We also define $\mathcal{V}(X) = \{P(X) : P \in \mathcal{V}\}$.

Note that $\mathcal{V}$ depends on our choice of $\hat{P}(M)$ for each $M \in \mathcal{M}_+$, but is spanned by monomials of dimension at most $d_+$. The restriction of $\Theta$ to $\Sigma_+$ in item (iii) ensures that $\Theta M \in \mathcal{M}_+$ when $M \in \mathcal{M}_+$, so that $\hat{P}(\Theta M)$ makes sense.

**Example 1.3.** In practice, we may prefer to choose $\hat{P}$ satisfying the conditions of Definition 1.2 using a formula other than (1.19). For example, for $e \in \mathcal{U}$ let $M_e = \varphi \nabla^e \varphi$. The formula (1.19) gives

$$P(M_e) = (1/2) (\varphi \nabla^e \varphi + \varphi \nabla^{-e} \varphi),$$

but via (1.5) the simpler choice $\hat{P}(M_e) = \varphi \nabla^{-e} \varphi$ also satisfies the conditions of Definition 1.2.

**Proposition 1.4.** The subspace $\mathcal{V}$ is a $\Sigma$-invariant subspace of $\mathcal{P}$.

**Proof.** By Definition 1.2(iii), the set $\{\hat{P}(M) : M \in \mathcal{M}_+\}$ is mapped to itself by $\Sigma_+$. Since $\hat{P}(M)$ is $\Sigma_{\text{axes}}$-covariant, $\mathcal{V}$ is invariant under $\Sigma_+$ and $\Sigma_{\text{axes}}$. Thus, since $\Sigma = \Sigma_{\text{axes}} \rtimes \Sigma_+$, $\mathcal{V}$ is invariant under $\Sigma$. \hfill $\blacksquare$

### 1.3 The operator loc

A nonempty connected subset $\Lambda'$ of $\Lambda$ whose $l^\infty$ diameter is less than the period of the torus $\Lambda$ is called a *coordinate patch*. The diameter of $\Lambda'$ is defined using the distance in $\Lambda$ considered as a torus. For a coordinate patch $\Lambda'$ we define the *coordinate* $z = (x_1, \ldots, x_d)$ which maps $\Lambda'$ to $\mathbb{Z}^d$ so that its values at nearest-neighbour sites differ by a unit vector. For $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{N}^d$, we define the monomial $z^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. This is a function defined on $\Lambda'$.

We will define a class of test functions $\Pi = \Pi(\Lambda')$ which are polynomials in each argument by specifying the monomials which span $\Pi$. To a local monomial $M_m \in \mathcal{M}_+$ in *fields*, as in (1.7), we associate a monomial $p_m$ in $\Pi$ by replacing $\nabla^a_\alpha \varphi_{ik}$ by $z^\alpha_k$. Thus

$$p_m(z) = \prod_{k=1}^p z^\alpha_k,$$

which is a function of $z = (z_1, \ldots, z_{p(m)}) \in \prod_{k=1}^{p(m)} \Lambda'_k$. For example, we associate the monomial $z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} z_5^{\alpha_5} z_6^{\alpha_6}$ to the field monomial (1.8). However, we will also need the monomial $z_1^{\alpha_2} z_2^{\alpha_3} z_3^{\alpha_3} z_4^{\alpha_4} z_5^{\alpha_5} z_6^{\alpha_6}$ which cannot be obtained from $m \in \mathcal{m}_+$ because the condition (iii) below (1.8) requires $\alpha_2 \leq \alpha_3 \leq \alpha_1$, which is not the case in this example. Therefore we define $\mathcal{m}_+$ and $\mathcal{v}_+$ by dropping the order condition (iii) in $\mathcal{m}_+$ and $\mathcal{v}_+$. The space $\Pi$ is the span of $\{p_m : m \in \mathcal{v}_+\}$. 


Equivalently we can define the dimension of a polynomial on $\Lambda'_{i_1} \times \cdots \times \Lambda'_{i_p(m)}$ to be its polynomial degree plus $\sum_{k=1}^{p}([\varphi_{i_k}] + |\alpha_k|_1)$, consistent with (1.21). Then $\Pi$ consists of all polynomials whose dimension is at most $d$. In the following, we will also need the subspace $S\Pi$ of $\Pi$. This is the image of $\Pi$ under the symmetry operator $S$ defined in [6, Example 3.6].

Recall the definition from [6] that, given $X \subset \Lambda$, $N(X)$ consists of those $F \in \mathcal{N}$ such that $F_z(\phi) = 0$ for all $\phi$ whenever any component of $z$ lies outside of $X$. For nonempty $X \subset \Lambda$, we say $F \in \mathcal{N}_X$ if there exists a coordinate patch $\Lambda'$ such that $F \in \mathcal{N}(\Lambda')$ and $X \subset \Lambda'$. The condition $F \in \mathcal{N}_X$ guarantees that neither $X$ nor $F$ “wrap around” the torus.

**Proposition 1.5.** For nonempty $X \subset \Lambda$ and $F \in \mathcal{N}_X$, there is a unique $V \in \mathcal{V}$, depending on $F$ and $X$, such that
\[
\langle F, g \rangle_0 = \langle V(X), g \rangle_0 \quad \text{for all } g \in \Pi.
\] (1.22)
The polynomial $V$ does not depend on the choice of $\Lambda'$ implicit in the requirement $F \in \mathcal{N}_X$, as long as $X \subset \Lambda'$ and $F \in \mathcal{N}(\Lambda')$. Moreover, $\mathcal{V}(X)$ and $S\Pi$ are dual vector spaces under the pairing (1.2).

The proof of Proposition 1.5 is deferred to Section 2.1. It allows us to define our basic object of study in this paper, the map $\text{loc}_X$.

**Definition 1.6.** For nonempty $X \subset \Lambda$ we define $\text{loc}_X : \mathcal{N}_X \to \mathcal{V}(X)$ by $\text{loc}_X F = V(X)$, where $V$ is the unique element of $\mathcal{V}$ such that (1.22) holds. For $X = \emptyset$, we define $\text{loc}_\emptyset = 0$.

1.4 Properties of $\text{loc}$

By definition, for nonempty $X \subset \Lambda$ and $F \in \mathcal{N}_X$,
\[
\langle F, g \rangle_0 = \langle \text{loc}_X F, g \rangle_0 \quad \text{for all } g \in \Pi.
\] (1.23)

Also, if $F = V(X) \in \mathcal{V}(X)$ then trivially $\langle F, g \rangle_0 = \langle V(X), g \rangle_0$ and hence the uniqueness in Definition 1.6 implies that $\text{loc}_X F = V(X) = F$. Thus $\text{loc}_X$ acts as the identity on $\mathcal{V}(X)$. The following proposition shows that $\text{loc}$ behaves well under composition.

**Proposition 1.7.** For $X, X' \subset \Lambda$ and $F \in \mathcal{N}_{X \cup X'}$, excluding the case $X' = \emptyset \neq X$,
\[
\text{loc}_X \circ \text{loc}_{X'} = \text{loc}_X.
\] (1.24)

In particular, $\text{loc}_X \circ (\text{Id} - \text{loc}_X) = 0$ on $\mathcal{N}_X$.

**Proof.** If $X = \emptyset$ then both sides are zero, so suppose that $X, X' \neq \emptyset$. Let $g \in \Pi$. By (1.23),
\[
\langle \text{loc}_X \circ \text{loc}_{X'} F, g \rangle_0 = \langle \text{loc}_{X'} F, g \rangle_0 = \langle F, g \rangle_0 = \langle \text{loc}_X F, g \rangle_0.
\] (1.25)

Since $\text{loc}_X \circ \text{loc}_{X'} F$ and $\text{loc}_X F$ are both in $\mathcal{V}(X)$, their equality follows from the uniqueness in Definition 1.6.

The following proposition gives an additivity property of $\text{loc}$. 

**Proposition 1.8.** Let \( X \subset \Lambda \) and \( F \in \mathcal{N}_X \) for all \( x \in X \). Suppose that \( P \in \mathcal{V} \) obeys \( \text{loc}_{(x)} F_x = P_x \) for all \( x \in X \). Then \( \text{loc}_X F(X) = P(X) \), where \( F(X) = \sum_{x \in X} F_x \).

**Proof.** If \( X \) is empty then both sides are zero, so suppose that \( X \) is not empty. Let \( g \in \Pi \). It follows from (1.23), linearity of the pairing, and the assumption, that

\[
\langle \text{loc}_X F(X), g \rangle_0 = \langle F(X), g \rangle_0 = \sum_{x \in X} \langle F_x, g \rangle_0 = \sum_{x \in X} \langle \text{loc}_{(x)} F_x, g \rangle_0 = \sum_{x \in X} \langle P_x, g \rangle_0 = \langle P(X), g \rangle_0.
\]

(1.26)

Since \( \text{loc}_X F(X) \) and \( P(X) \) are both in \( \mathcal{V}(X) \), their equality follows from the uniqueness in Definition 1.6.

For nonempty \( X \subset \Lambda \), let \( \mathcal{E}(X) \) be the set of automorphisms of \( \Lambda \) which map \( X \) to itself. Here, an automorphism means an injective map from \( X \) to \( X \) under which nearest-neighbour points are mapped to nearest-neighbour points under both the map and its inverse. In particular, \( \mathcal{E}(\Lambda) \) is the set of automorphisms of \( \Lambda \). An automorphism \( E \in \mathcal{E}(\Lambda) \) defines a mapping of the boson field by \( (\phi_E)_x = \phi_{Ex} \). Then, for \( F = \sum_y \frac{1}{y!} F_y \psi^y \in \mathcal{N} \), we define \( E \) as a linear operator on \( \mathcal{N} \) by

\[
(EF)(\phi) = \sum_{y \in \Lambda^*_f} \frac{1}{y!} F_y (\phi_E) \psi^y = \sum_{y \in \Lambda^*_f} \frac{1}{y!} F_{E^{-1}y} (\phi_E) \psi^y,
\]

(1.28)

where in the second equality we have extended the action of \( E \) to component-wise action on \( \Lambda^*_f \), and we used the fact that summation over \( y \) is the same as summation over \( E^{-1}y \). The following proposition gives a Euclidean covariance property of \( \text{loc} \).

**Proposition 1.9.** For \( X \subset \Lambda \), \( F \in \mathcal{N}_X \) and \( E \in \mathcal{E}(\Lambda) \),

\[
E(\text{loc}_X F) = \text{loc}_{EX} (EF).
\]

(1.29)

**Proof.** We define \( E^* : \Phi \to \Phi \) by \( (E^* g)_x = g_{Ex} \). By (1.28), and by taking derivatives with respect to \( \phi_{x_i} \) for \( x_i \in \Lambda_b \), for \( x \in \Lambda^*_b \) we have

\[
(EF)_{x,y}(\phi) = F_{E^{-1}x,E^{-1}y}(\phi_E).
\]

(1.30)

Therefore,

\[
\langle EF, g \rangle_\phi = \sum_{z \in \Lambda^*_f} \frac{1}{z!} F_{E^{-1}z}(\phi_E) g_z = \sum_{z \in \Lambda^*_f} \frac{1}{z!} F_z(\phi_E) g_{Ez} = \langle F, E^* g \rangle_{\phi_E}.
\]

(1.31)

Since \( F \in \mathcal{N}_X \) there exists a coordinate patch \( \Lambda' \) containing \( X \) such that \( F \in \mathcal{N}(\Lambda') \). Let \( g \in \Pi_{EN'} \), and note that \( E^* \) maps test functions in \( \Pi_{EN'} \) to test functions in \( \Pi_{\Lambda'} \). By (1.23) and (1.31),

\[
\langle E \text{loc}_X F, g \rangle_0 = \langle \text{loc}_X F, E^* g \rangle_0 = \langle F, E^* g \rangle_0 = \langle EF, g \rangle_0 = \langle \text{loc}_{EX} EF, g \rangle_0.
\]

(1.32)

Since both \( E \text{loc}_X F \) and \( \text{loc}_{EX} EF \) are in \( \mathcal{V}(EX) \), their equality follows from the uniqueness in Proposition 1.5.
The subgroup of $\mathcal{E}(\Lambda)$ consisting of automorphisms that fix the origin is homomorphic to the group $\Sigma$, with the element $\Theta_E \in \Sigma$ determined from such an $E \in \mathcal{E}(\Lambda)$ by the action of $E$ on the set $U$ of unit vectors. Since $\mathcal{E}(\Lambda)$ is the semidirect product of the subgroup of translations and the subgroup that fixes the origin, we can use this homomorphism to associate to each element $E \in \mathcal{E}(\Lambda)$ a unique element $\Theta_E \in \Sigma$. The following proposition ensures that the polynomial $P \in \mathcal{V}$ determined by $\text{loc}_X F$ inherits symmetry properties of $X$ and $F$.

**Proposition 1.10.** For $X \subset \Lambda$ and $F \in \mathcal{N}_X$ such that $EF = F$ for all $E \in \mathcal{E}(X)$, the polynomial $P \in \mathcal{V}$ determined by $P(X) = \text{loc}_X F \in \mathcal{V}(X)$ obeys $\Theta_E P = P$ for all $E \in \mathcal{E}(X)$.

**Proof.** By Proposition 1.9 and by hypothesis, $EP(X) = \text{loc}_X EF = P(X)$. Therefore, for $g \in \Pi$,

$$\langle F, g \rangle_0 = \langle P(X), g \rangle_0 = \langle EP(X), g \rangle_0. \quad (1.33)$$

Since $EP(X) = (\Theta_E P)(X)$, this gives

$$\langle P(X), g \rangle_0 = \langle (\Theta_E P)(X), g \rangle_0, \quad (1.34)$$

and since $\Theta_E P \in \mathcal{V}$ by Proposition 1.4, the uniqueness in Proposition 1.5 implies that $\Theta_E P = P$, as required. \hfill \blacksquare

The next two propositions concern norm estimates, using the $T_\phi$ semi-norm defined in [6]. The $T_\phi$ semi-norm is itself defined in terms of a norm on test functions, and next we define the particular norm on test functions that we will use here.

The norm depends on a vector $h = (h_1, \ldots, h_{p_\Lambda})$ of positive real numbers, one for each field species and component, though in practice we take $h_k$ to depend only on the field species of the index $k$. Given $z = (z_1, \ldots, z_p) \in \Lambda^*$, we define $h^{-z} = \prod_{i=1}^p h_{k(z_i)}^{-1}$, where $k(z_i)$ denotes the copy of $\Lambda$ inhabited by $z_i \in \Lambda$. Given $p_\phi \geq 0$, the norm on test functions is defined by

$$\|g\|_{\Phi(h)} = \sup_{z \in \Lambda^*} \sup_{|\alpha| \leq p_\phi} h^{-z} |\nabla^\alpha_R g_z|, \quad (1.35)$$

where $\nabla^\alpha_R = R^{[\alpha]} \nabla^\alpha$. In terms of this norm, a semi-norm on $\mathcal{N}$ is defined by

$$\|F\|_{T_\phi} = \sup_{g \in B(\Phi)} |\langle F, g \rangle_\phi|, \quad (1.36)$$

where $B(\Phi)$ denotes the unit ball in $\Phi = \Phi(h)$. This $T_\phi$ semi-norm depends on the boson field $\phi$, via the pairing (1.2).

The next two propositions provide essential norm estimates on $\text{loc}$. Their proofs, which make use of the results in Section 3, are deferred to Section 2.2. Recall from [6] that a polymer is a union of blocks of side $R$ in a paving of $\Lambda$.

**Proposition 1.11.** Let $U \subset \Lambda$ be a polymer which is also a coordinate patch, and let $X$ be a polymer with $X \subset U$. For $F \in \mathcal{N}(U)$, there is a constant $\bar{C}'$, which depends only on $R^{-1} \text{diam}(U)$, such that

$$\|\text{loc}_X F\|_{T_\phi} \leq \bar{C}' \|F\|_{T_\phi}. \quad (1.37)$$
The next result, which is crucial, involves the $T_\phi$ semi-norm defined in terms of $\Phi(h)$, as well as the $T_\phi'$ semi-norm defined in terms of the $\Phi'(h')$ norm for which $R$ and $h$ of (1.35) are replaced by $R'$ and $h'$, with $R'$ chosen so that the side length of $\Lambda$ can be written as $(R')^{m'}$ for some integer $m'$. We define $L$ by $R' = LR$ and assume that $L > 1$; in practice we will choose $L$ to be large.

In addition, we assume that $h'$ and $h$ are chosen such that $h'_i/h_i \leq cL^{-[\varphi]}$ for each component $i$, where $c$ is a universal constant. Let

$$d'_+ = \min\{[M_m] : m \not\in \nu_+\},$$

(1.38)

where $\nu_+$ was defined in (1.16); thus $d'_+$ denotes the smallest dimension of a monomial not in the range of Loc. Let $[\varphi_{\min}] = \min\{[\varphi_i] : i = 1, \ldots, p_\Lambda\}$. Given a positive integer $A$, we define

$$\gamma = L^{-d'_+} + L^{-(A+1)[\varphi_{\min}]}.$$  

(1.39)

We use the term $R$-polymer to indicate a polymer constructed from blocks of side $R$ (as opposed to $R'$).

In anticipation of a hypothesis of Lemma 3.6, for the next proposition we impose the restriction that $p_\Phi \geq d'_+ - [\varphi_{\min}]$.

**Proposition 1.12.** Let $A < p_N$ be a positive integer, let $X$ be an $R$-polymer which is also a coordinate patch and let $Y \subset X$ be a nonempty $R$-polymer. For $i = 1, 2$, let $F_i \in \mathcal{N}(X)$. Then

$$\|F_1(1 - \text{loc}_Y)F_2\|_{T_\phi'} \leq \gamma \bar{C} \left(1 + \|\phi\|_{\\Phi'}^{A+d'/[\varphi_{\min}]+1} \sup_{0 \leq t \leq 1} (\|F_1F_2\|_{T_\phi} + \|F_1\|_{T_\phi} \|F_2\|_{T_\phi})\right),$$

(1.40)

where $\gamma$ is given by (1.39), and where $\bar{C}$ depends only on $R^{-1}\text{diam}(X)$.

For the special case with $F_1 = 1$, $F_2 = F$, and $\phi = 0$, Proposition 1.12 asserts that

$$\|F - \text{loc}_XF\|_{T_\phi'} \leq \gamma \bar{C}\|F\|_{T_\phi}.$$  

(1.41)

For the case of $d \geq 4$, $d_+ = d$, $[\varphi_{\min}] = \frac{d-2}{2}$, and with $A$ (and so $p_N$) chosen sufficiently large that $(A + 1)\frac{d-2}{2} \geq d + 1$, we have $d'_+ = d_+ + 1$ and $\gamma = O(L^{-d-1})$. This shows that, when measured in the $T_\phi'$ semi-norm, $F - \text{loc}_XF$ is substantially smaller than $F$ measured in the $T_\phi$ semi-norm.

### 1.5 An example

The following example is not needed elsewhere in this paper, but it serves to illustrate the evaluation of loc.

**Example 1.13.** Consider the case where there is a single complex boson field $\phi$, in dimension $d = 4$, with $[\varphi] = 1$, and with $d_+ = d = 4$. The list of relevant and marginal monomials is as in (1.12)–(1.13), but now each factor of $\varphi$ in those lists can be replaced by either $\phi$ or its conjugate $\bar{\phi}$. To define $\mathcal{V}$, for each monomial $M$ we choose $P(M)$ as in (1.19), except monomials which contain $\nabla^a\nabla^e$ for which we use $\nabla^{-a}\nabla^e$ as in Example 1.3 instead. Let $X \subset \Lambda$ be a coordinate patch and let $a, x \in X$.

(i) Simple examples are given by

$$\text{loc}_X|\phi_x|^6 = 0, \quad \text{loc}_{(a)}|\phi_x|^4 = |\phi_a|^4,$$

(1.42)
which hold since in both cases the pairing requirement of Definition 1.6 is obeyed by the right-hand
sides.

(ii) Let $\tau_x = \phi_x \bar{\phi}_x$, let $q : \Lambda \to \mathbb{C}$ have range strictly less than the period of the torus, and let

$$F = \sum_{x \in \Lambda, y \in \Lambda} q(x - y) \tau_y.$$  \hfill (1.43)

The assumption on the range of $q$ ensures that the coordinate patch condition in the definition of
loc is satisfied. We define

$$q^{(1)} = \sum_{x \in \Lambda} q(x), \quad q^{(**)} = \sum_{x \in \Lambda} q(x)x_1^2,$$  \hfill (1.44)

and assume that

$$\sum_{x \in \Lambda} q(x)x_i = 0, \quad \sum_{x \in \Lambda} q(x)x_ix_j = q^{(**)} \delta_{i,j} \quad i, j \in \{1, 2, \ldots, d\}.$$  \hfill (1.45)

We claim that

$$\text{loc}_X F = \sum_{x \in X} (q^{(1)} \tau_x + q^{(**)} \sigma_x),$$  \hfill (1.46)

where, with $\Delta = -\sum_{i=1}^d \nabla^{-e_i} \nabla^{e_i}$,

$$\sigma_x = \frac{1}{2} \left( \phi_x \Delta \bar{\phi}_x + \sum_{e \in \mathcal{U}} \nabla^e \phi_x \nabla^e \bar{\phi}_x + \Delta \phi_x \bar{\phi}_x \right).$$  \hfill (1.47)

To verify (1.46), we define

$$A = \sum_{y \in \Lambda} q(a - y) \tau_y.$$  \hfill (1.48)

By Proposition 1.8, it suffices to show that

$$\text{loc}_{(a)} A = q^{(1)} \sigma_a + q^{(**)} \sigma_a.$$  \hfill (1.49)

For this, it suffices to show that $A$ and $q^{(1)} \tau_a + q^{(**)} \sigma_a$ have the same zero-field pairing with test
functions $g \in \Pi$. By definition, $\langle A, g \rangle_0 = \sum_{y \in \Lambda} q(a - y)g_{a,y}$. Since the polynomial test function
$g = g_{y_1,y_2}$ is in $\Pi$, it is a quadratic polynomial in $y_1, y_2$ and we can write the coefficients of this
polynomial in terms of lattice derivatives of $g$ at the point $(a, a)$. For example the quadratic terms in $g$ are $(1/2) \sum_{i,j=1}^d (y_i - a_i)(y_j - a_j) \nabla^{e_i}_1 \nabla^{e_j}_2 g_{a,a}$. (The construction of lattice Taylor polynomials
is described below in (2.4).)

The constant term in $g$ is the zeroth derivative $g_{a,a}$. The linear terms vanish in the pairing
due to (1.45). For the quadratic terms with derivatives on both variables of $g$, the only non-
vanishing contribution to the pairing arises from $\frac{1}{2} \sum_{i=1}^d (y_i - a_i)^2 \nabla_i \nabla^{e_i}_1 \nabla^{e_i}_2 g_{a,a}$, due to (1.45), where the subscripts on the derivatives indicate on which argument they act. For the quadratic terms
with both derivatives on a single variable of $g$, by (1.45) we may assume that both derivatives are
in the same direction, and for those, we can replace the binomial coefficient \(\binom{y_i - a_i}{2}\) by \(\frac{1}{2}(y_i - a_i)^2\) due to the first assumption in (1.45), to see that the relevant terms for the pairing are

\[
\frac{1}{2} \sum_{i=1}^{d} (y_i - a_i)^2 \nabla_1^e \nabla_1^\epsilon g_{a,a} + \frac{1}{2} \sum_{i=1}^{d} (y_i - a_i)^2 \nabla_2^e \nabla_2^\epsilon g_{a,a}.
\]  

(1.50)

Since \(g\) is a polynomial of total degree at most 2, we can use (1.5) to replace derivatives \(\nabla\) by \(-\nabla\) in the above expressions involving two derivatives. Thus we obtain

\[
\langle A, g \rangle_0 = q^{(1)} g_{a,a} + q^{(**)1} \left( \Delta g_{a,a} + \sum_{e \in U} \nabla_1^e \nabla_2^e g_{a,a} + \Delta g_{a,a} \right).
\]  

(1.51)

By inspection, the right-hand side of (1.49) has the same pairing with \(g\) as \(A\), so (1.49) is verified.

(iii) Let

\[
F' = \sum_{x \in X, y \in \Lambda} q(x - y)(\tau_{xy} + \tau_{yx}).
\]  

(1.52)

By a similar analysis to that used in (ii),

\[
\text{loc}_X F' = \sum_{x \in X} \left(2q^{(1)} \tau_x + q^{(**)} \left( \phi_x \Delta \bar{\phi}_x + (\Delta \phi)_x \bar{\phi}_x \right) \right).
\]  

(1.53)

1.6 Supersymmetry and \(\text{loc}\)

For our application to self-avoiding walk in [1,2], we will use \(\text{loc}\) in the context of a supersymmetric field theory involving a complex boson field \(\phi\) with conjugate \(\bar{\phi}\), and a pair of conjugate fermion fields \(\psi, \bar{\psi}\), all of dimension \(d-2\). We now show that if \(F \in \mathcal{N}\) is supersymmetric then so is \(\text{Loc}_X F\).

The supersymmetry generator \(Q = d + \bar{z}\), which is discussed in [5, Section 6], has the following properties: (i) \(Q\) is an antiderivation that acts on \(\mathcal{N}\), (ii) \(Q^2\) is the generator of the gauge flow characterised by \(q \mapsto e^{-2\pi it}q\) for \(q = \phi_x, \psi_x\) and \(\bar{q} \mapsto e^{+2\pi it}\bar{q}\) for \(\bar{q} = \bar{\phi}_x, \bar{\psi}_x\), for all \(x \in \Lambda\). An element \(F \in \mathcal{N}\) is said to be gauge invariant if it is invariant under this flow and supersymmetric if \(QF = 0\). By property (ii), supersymmetric elements are gauge invariant. Let \(\hat{Q} = (2\pi i)^{-1/2}Q\). Then \(\hat{Q}\) is an antiderivation satisfying:

\[
\hat{Q}\phi = \psi, \quad \hat{Q}\psi = -\phi, \quad \hat{Q}\bar{\phi} = \bar{\psi}, \quad \hat{Q}\bar{\psi} = \bar{\phi}.
\]  

(1.54)

The gauge flow clearly maps \(\mathcal{V}\) to itself. Also, since the boson and fermion fields have the same dimension, \(Q\) also maps \(\mathcal{V}\) to itself. The following observation is a general one, but it has the specific consequences that if \(F\) is gauge invariant then so is \(\text{loc}_X F\), and if \(F\) is supersymmetric then \(Q\text{loc}_X F = \text{loc}_X QF = 0\) so \(\text{loc}_X F\) is supersymmetric. This provides a simplifying feature in the analysis applied in [8].

**Proposition 1.14.** The map \(Q : \mathcal{N} \to \mathcal{N}\) commutes with \(\text{loc}_X\).
Proof. Let $F \in \mathcal{N}$ and $g \in \Pi$. There is an explicitly computable map $Q^* : \Pi \to \Pi$ such that $\langle QF, g \rangle_0 = \langle F, Q^* g \rangle_0$. It then follows from (1.23) that
\[
\langle Q_{\text{loc}} X F, g \rangle_0 = \langle \text{loc}_X F, Q^* g \rangle_0 = \langle Q F, g \rangle_0 = \langle \text{loc}_X Q F, g \rangle_0.
\] (1.55)
Since $Q : \mathcal{V}(X) \to \mathcal{V}(X)$ by (1.54), it then follows from the uniqueness in Definition 1.6 that $Q_{\text{loc}} X F = \text{loc}_X Q F$.

1.7 Observables and the operator Loc

We now generalise the operator $\text{loc}$ in two ways: to modify the set onto which it localises, and to incorporate the effect of observable fields. The first of these is accomplished by the following definition.

**Definition 1.15.** For $Y \subset X \subset \Lambda$ and $F \in \mathcal{N}_X$, we define the linear operator $\text{loc}_{X,Y} : \mathcal{N} \to \mathcal{V}(Y)$ by
\[
\text{loc}_{X,Y} F = P_X(Y) \quad \text{with} \quad P_X \text{ determined by } P_X(X) = \text{Loc}_X F.
\] (1.56)

In other words, $\text{loc}_{X,Y} F$ evaluates the polynomial $\text{loc}_X F$ on the set $Y$ rather than on $X$. It is an immediate consequence of the definition that $\text{loc}_X = \text{loc}_{X,X}$, and that if $\{X_1, \ldots, X_m\}$ is a partition of $X$ then
\[
\text{loc}_X = \sum_{i=1}^m \text{loc}_{X,X_i}.
\] (1.57)
The following norm estimate for $\text{loc}_{X,Y}$ will be proved in Section 2.2.

**Proposition 1.16.** Let $U \subset \Lambda$ be a polymer which is also a coordinate patch, and let $X,Y$ be polymers with $Y \subset X \subset U$. There is a constant $\bar{C}'$, which depends only on $R^{-1}\text{diam}(U)$, such that for $F \in \mathcal{N}(U),$
\[
\|\text{loc}_{X,Y} F\|_{T_0} \leq \bar{C}'\|F\|_{T_0}.
\] (1.58)

Next, we incorporate the presence of an observable field, which is a species of complex boson field, denoted $\sigma, \bar{\sigma}$. The norm on test functions is now defined as in [6], with the previously chosen weights $w^{-1}_{\alpha, z_i} = b_i^{-z_i} R^{[\alpha]}$ for the non-observable fields. However, for the observable fields, we choose the weights differently, as follows. First, if $\alpha \neq 0$ then we choose $w_{\alpha, z_i} = 0$ when $i$ corresponds to the observable species. This eliminates test functions which are not constant in the observable variables. In addition, we set test functions equal to zero if their observable variables exceed one $\sigma$, one $\bar{\sigma}$, or one pair $\sigma \bar{\sigma}$. Therefore, modulo the ideal $\mathcal{I}$ of zero norm elements, a general element $F \in \mathcal{N}$ has the form
\[
F = F^{\varnothing} + F^a + F^b + F^{ab},
\] (1.59)
where $F^{\varnothing}$ is obtained from $F$ by setting $\sigma = \bar{\sigma} = 0$, while $F^a = F_\sigma \sigma$, $F^b = F_{\bar{\sigma}} \bar{\sigma}$, and $F^{ab} = F_{\sigma \bar{\sigma}} \sigma \bar{\sigma}$ with the derivatives evaluated at $\sigma = \bar{\sigma} = 0$. In the $T_\phi$ semi-norm we will always set $\sigma = \bar{\sigma} = 0$. We unite the above cases with the notation $F^\alpha = F_\alpha \sigma^{\alpha}$ for $\alpha \in \{\varnothing, a, b, ab\}$. This corresponds to a direct sum decomposition,
\[
\mathcal{N}/\mathcal{I} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab},
\] (1.60)
with canonical projections $\pi_\alpha : N/I \rightarrow N^\alpha$ defined by $\pi_\varnothing F = F_\varnothing$, $\pi_a F = F_a\sigma$, and so on. Note that

$$\|F\|_{T_0} = \sum_\alpha \|F_\alpha\sigma^\alpha\|_{T_0} = \sum_\alpha \|F_\alpha\|_{T_0}\|\sigma^\alpha\|_{T_0},$$

(1.61)

by definition. We use the same value $h_\sigma$ in the weight for both $\sigma$ and $\bar{\sigma}$. In particular, $h_\sigma = \|\sigma\|_{T_0} = \|\bar{\sigma}\|_{T_0}$.

On each of the subspaces on the right-hand side of (1.60), we choose a value for the parameter $d_+$ and construct corresponding spaces $V^\varnothing, V^a, V^b, V^{ab}$ as in Definition 1.2. We allow the freedom to choose different values for the parameter $d_+$ in each subspace, and in our application in [3, 7] we will make use of this freedom. Then we define

$$V = V^\varnothing \oplus V^a \oplus V^b \oplus V^{ab}. \quad (1.62)$$

The following definition extends the definition of the localisation operator by applying it in a graded fashion in the above direct sum decomposition.

**Definition 1.17.** Let $\Lambda'$ be a coordinate patch. Let $a, b \in \Lambda'$ be fixed. Let $X(\varnothing) = X$, $X(a) = X \cap \{a\}$, $X(b) = X \cap \{b\}$, and $X(ab) = X \cap \{a,b\}$. For $Y \subset X \subset \Lambda$ and $F \in N_X$, we define the linear operator $\operatorname{Loc}_{X,Y} : N_X \rightarrow \mathcal{V}(Y)$ by specifying its action on each subspace in (1.60) as

$$\operatorname{Loc}_{X,Y} F^\alpha = \sigma^\alpha \operatorname{loc}_{X,Y} F_\alpha, \quad (1.63)$$

and the linear map $\operatorname{Loc}_X : N_X \rightarrow \mathcal{V}(X)$ by

$$\operatorname{Loc}_X F = \operatorname{Loc}_{X,X} F = \operatorname{loc}_{X,X}^\varnothing F_\varnothing + \sigma \operatorname{loc}_{X,X \cap \{a\}}^a F_a + \bar{\sigma} \operatorname{loc}_{X,X \cap \{b\}}^b F_b + \sigma \bar{\sigma} \operatorname{loc}_{X,X \cap \{a,b\}}^{ab} F_{ab}. \quad (1.64)$$

The space $\mathcal{V}$ is defined by (1.62). Different choices of $d_+$ are permitted on each subspace, and the label $\alpha$ appearing on the operators $\operatorname{loc}$ on the right-hand side of (1.63)–(1.64) are present to reflect these choices.

It is immediate from the definition that

$$\pi_\alpha \operatorname{Loc}_{X,Y} = \operatorname{Loc}_{X,Y} \pi_\alpha \quad \text{for} \ \alpha = \varnothing, a, b, ab, \quad (1.65)$$

and from (1.57) that, for a partition $\{X_1, \ldots, X_m\}$ of $X$,

$$\operatorname{Loc}_X = \sum_{i=1}^m \operatorname{Loc}_{X,X_i}. \quad (1.66)$$

It is a consequence of Proposition 1.7 that

$$\operatorname{Loc}_{X'} \circ \operatorname{Loc}_X = \operatorname{Loc}_{X'} \quad \text{for} \ \ X' \subset X \subset \Lambda, \quad (1.67)$$

and therefore

$$\operatorname{Loc}_X \circ (\text{Id} - \operatorname{Loc}_X) = 0. \quad (1.68)$$

Also, by Proposition 1.9, for an automorphism $E \in \mathcal{E}(\Lambda)$,

$$E(\operatorname{Loc}_X F) = \operatorname{Loc}_{EX}(EF) \quad \text{if} \ F \in N_X^\varnothing. \quad (1.69)$$

Note that (1.69) fails in general for $F \in N_X \setminus N_X^\varnothing$, due to the fixed points $a, b$ in the definition of $\operatorname{Loc}_{X,Y} F$. The following two propositions extend the norm estimates for $\operatorname{loc}$ to $\operatorname{Loc}$.
Proposition 1.18. Let $U \subset \Lambda$ be a polymer which is also a coordinate patch, and let $X,Y$ be polymers with $Y \subset X \subset U$. There is a constant $\bar{C}'$, which depends only on $R^{-1}\text{diam}(U)$, such that for $F \in \mathcal{N}(U)$,

$$
\|\text{Loc}_{X,Y} F\|_{T_0} \leq \bar{C}' \|F\|_{T_0}.
$$

(1.70)

Note that the case $X = Y$ gives (1.70) for $\text{Loc}_X F$.

Proof. By definition, the triangle inequality, Proposition 1.16, and (1.61),

$$
\|\text{Loc}_{X,Y} F\|_{T_0} = \sum_{\alpha = \emptyset, a, b, ab} \|\sigma^\alpha \text{loc}_{X,Y} F_\alpha\|_{T_0} \leq \bar{C}' \sum_{\alpha = \emptyset, a, b, ab} \|\sigma^\alpha\|_{T_0} \|F_\alpha\|_{T_0} = \bar{C}' \|F\|_{T_0},
$$

(1.71)

where $\bar{C}' = \max_{\alpha} \bar{C}'_\alpha$, with $\bar{C}'_\alpha$ the constant arising in each of the four applications of Proposition 1.16. 

For the next proposition, which is applied in [7, Proposition 4.9], we write $d_\alpha$ for the choice of $d_+$, and $[\varphi_{\min}]$ for the common minimal field dimension on each space $\mathcal{N}^\alpha$ for $\alpha = \emptyset, a, b$ and $ab$. We choose the spaces $\Phi(h)$ and $\Phi'(h')$ as in Proposition 1.12. With $d_\alpha'$ defined as in (1.38), let

$$
\gamma_{\alpha,\beta} = (L^{-d_\alpha} + L^{-(A+1)[\varphi_{\min}]}) \left( \frac{b'_\sigma}{b_\sigma} \right)^{|\alpha \cup \beta|}.
$$

(1.72)

As in Proposition 1.12, for the next proposition we again require that $p_\Phi \geq d_+ - [\varphi_{\min}]$.

Proposition 1.19. Let $A < p_X$ be a positive integer, and let $\emptyset \neq Y \subset X \in \mathcal{P}$. Let $F_1 \in \mathcal{N}(X)$, and let $F_2 = \sum_\alpha F_2^\alpha \in \mathcal{N}(X)$ with $F_2^\emptyset = 0$ when $Y(\alpha) = \emptyset$. Let $F = F_1(1 - \text{loc}_Y) F_2$. Then

$$
\|F\|_{T_\Phi} \leq \bar{C} \sum_{\alpha, \beta = \emptyset, a, b, ab} \gamma_{\alpha,\beta} \left( 1 + \|\phi\|_{\Phi'} \right)^{A+d_\alpha/[\varphi_{\min}]+1} \times \sup_{0 \leq t \leq 1} \left( \|F_{1,\beta} F_{2,\alpha}\|_{T_{t\phi}} + \|F_{1,\beta}\|_{T_{0\phi}} \|F_{2,\alpha}\|_{T_0} \right) \|\sigma_{\alpha \cup \beta}\|_{T_0}.
$$

(1.73)

Proof. We use

$$
\|F\|_{T_\Phi} \leq \sum_{\alpha, \beta} \|\sigma_{\alpha \cup \beta}\|_{T_0} \|F_{1,\beta}(1 - \text{loc}_{Y(\alpha)}) F_{2,\alpha}\|_{T_\Phi}
$$

(1.74)

and apply Proposition 1.12 to each term. We also use

$$
\|\sigma_{\alpha \cup \beta}\|_{T_0} = (b'_\sigma)^{|\alpha \cup \beta|} = \|\sigma_{\alpha \cup \beta}\|_{T_0} \left( \frac{b'_\sigma}{b_\sigma} \right)^{|\alpha \cup \beta|}.
$$

(1.75)

The constant $\bar{C}$ is the largest of the four constants $\bar{C}_\alpha$ arising from Proposition 1.12.

\[\Box\]

2 The operator $\text{loc}$

In Section 2.1, we prove existence of the operator $\text{loc}$ and prove Proposition 1.5. In Section 2.2, we prove Propositions 1.11–1.12, using the results on Taylor polynomials proven in Section 3. Finally, in Section 2.3, we now prove the claim which guaranteed existence of the polynomials $\hat{P}$ used to define $\mathcal{V}$ in Definition 1.2.

Throughout this section, $\Lambda'$ is a coordinate patch in $\Lambda$, and we assume that $X \subset \Lambda'$ and $a \in \Lambda'$. The space of polynomial test functions is then $\Pi = \Pi_{\Lambda'}$. 

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2.1 Existence and uniqueness of loc: Proof of Proposition 1.5

Recall from [6, Proposition 3.5] that the pairing obeys

$$\langle F, g \rangle = \langle F, Sg \rangle$$

(2.1)

for all $F \in \mathcal{N}$, $g \in \Phi$, and for all boson fields $\phi$. The symmetry operator $S$ is defined in [6, Definition 3.4] (see also Section 3.2 below); it obeys $S^2 = S$. Let $m \in \mathfrak{m}$ have components $m_k = (i_k, \alpha_k)$ for $k = 1, \ldots, p(m)$, and, as discussed under (1.8), let $n_{(i, \alpha)}$ denote the number of times that $(i, \alpha)$ appears as a component of $m$. Recall from [6, Example 3.6] that, for any test function $g$,

$$\langle M_{m,a}, g \rangle_0 = \nabla^m (S g) \bar{a}, \quad \nabla^m = \prod_{k=1}^{p(m)} \nabla^{\alpha_k},$$

(2.2)

where on the right-hand side $\bar{a}$ indicates that each of the $p(m)$ arguments is evaluated at $a$, and $\nabla^{\alpha_k}$ acts on the variable $z_k$.

We specified a basis for $\Pi$ in (1.21), but now we require another basis. For $z = (x_1, \ldots, x_d)$ in $\Lambda'$ and $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{N}_0^d$, we define the binomial coefficient $(\alpha) = (\alpha_1!) \ldots (\alpha_d)!$. The new basis is obtained by replacing, in the definition (1.21) of $p_m$, the monomial $z_k^{\alpha_k}$ by the polynomial $(z_k^{\alpha_k})$. More generally, we can also move the origin. Thus for $m \in \mathfrak{m}_+$ and $a \in \Lambda'$ we define

$$b_{m,z}^{(a)} = \prod_{k=1}^{p} \left( z_k - a \right).$$

(2.3)

This is a polynomial function defined on $m \in \Lambda_{i_1}' \cdots \Lambda_{i_p(m)}'$. For $p(m) = 0$, we set $b_{m,z}^{(a)} = 1$. For any $a \in \Lambda'$, the set $\{ b_{m,z}^{(a)} : m \in \mathfrak{m}_+ \}$ is a basis for $\Pi$. For $g \in \Phi$, we define $\text{Tay}_a : \Phi \to \Pi$ by

$$(\text{Tay}_a g)_z = \sum_{m \in \mathfrak{m}_+} (\nabla^m g)_a b_{m,z}^{(a)}.$$

(2.4)

The following lemma shows that $\text{Tay}_a g$ is the lattice analogue of a Taylor polynomial approximation to $g$.

Lemma 2.1. (i) For $g \in \Phi$, $\text{Tay}_a g$ is the unique $p \in \Pi$ such that $\nabla^m (g - p)_z|_{z = \bar{a}} = 0$ for all $m \in \mathfrak{m}_+$. (ii) $\text{Tay}_a$ commutes with $S$. (iii) For $g \in \Pi$, $\text{Tay}_a g = g$.

For $m \in \mathfrak{m}_+$, let

$$f_{m}^{(a)} = N_m S b_{m}^{(a)},$$

(2.5)

where $N_m$ is a normalisation constant chosen so that case $m = m'$ holds in (2.6) below (its value is specified in (3.9)). The lexicographic ordering on $\mathfrak{m}_+$ implies that $f_{m}^{(a)} \neq f_{m'}^{(a)} \neq 0$ for $m \neq m'$. Since $\{ b_{m}^{(a)} \}_{m \in \mathfrak{m}_+}$ forms a basis of $\Pi$, the linearly independent set $\{ f_{m}^{(a)} \}_{m \in \mathfrak{m}_+}$ forms a basis of $\mathcal{S}\Pi$. The next lemma says that $\{ M_{m,a} \}_{m \in \mathfrak{m}_+}$ and $\{ f_{m}^{(a)} \}_{m' \in \mathfrak{m}_+}$ are dual bases of $\mathcal{V}_+$ and $\mathcal{S}\Pi$ with respect to the zero-field pairing.
Lemma 2.2. For \( m, m' \in \mathfrak{m}_+ \),
\[
\langle M_{m,a}, f_{m'}^{(a)} \rangle_0 = \delta_{m,m'},
\]
and for \( g \in \Phi \),
\[
(Tay_a S g) \equiv \sum_{m \in \mathfrak{v}_+} \langle M_{m,a}, g \rangle_0 f_{m,a}^{(a)}.
\]

Definition 2.3. Given \( a \in \Lambda' \), we define a linear map \( \text{loc}_{+,a} : \mathcal{N}_{\{a\}} \to \mathcal{V}_+(\{a\}) \) by
\[
\text{loc}_{+,a} F = \sum_{m \in \mathfrak{v}_+} \langle F, f_m^{(a)} \rangle_0 M_{m,a}.
\]

It is an immediate consequence of (2.8) and (2.6) that \( \text{loc}_{+,a} M_{m,a} = M_{m,a} \) for all \( m \in \mathfrak{v}_+ \). Since \( \mathcal{V}_+ \) is spanned by the monomials \( (M_m)_{m \in \mathfrak{v}_+} \), it follows that
\[
\text{loc}_{+,a} P_a = P_a \quad P \in \mathcal{V}_+.
\]

The following lemma shows that the map \( \text{loc}_{+,a} \) is dual to \( \text{Tay}_a \) with respect to the zero-field pairing of \( \mathcal{N} \) and \( \Phi \).

Lemma 2.4. For any \( a \in \Lambda \), \( F \in \mathcal{N}_{\{a\}} \), and \( g \in \Phi \),
\[
\langle \text{loc}_{+,a} F, g \rangle_0 = \langle F, \text{Tay}_a g \rangle_0.
\]

In particular, if \( g \in \Pi \), then
\[
\langle \text{loc}_{+,a} F, g \rangle_0 = \langle F, g \rangle_0.
\]

Proof. For (2.10), we use Definition 2.3, linearity of the pairing, (2.7), Lemma 2.1(ii) and (2.1) to obtain
\[
\langle \text{loc}_{+,a} F, g \rangle_0 = \sum_{m \in \mathfrak{v}_+} \langle F, f_m^{(a)} \rangle_0 \langle M_{m,a}, g \rangle_0 = \langle F, \text{Tay}_a S g \rangle_0
\]
\[
= \langle F, S \text{Tay}_a g \rangle_0 = \langle F, \text{Tay}_a g \rangle_0.
\]

For (2.11), we use (2.10) and the fact that \( \text{Tay}_a g = g \) for \( g \in \Pi \), by Lemma 2.1(iii).

Lemma 2.5. Given \( V_+ \in \mathcal{V}_+ \) and \( X \) such that \( \mathcal{N}(X) \subset \mathcal{N}_{\{a\}} \) there exists a unique \( V \in \mathcal{V} \) (depending on \( V_+ \) and \( X \)) such that
\[
\text{loc}_{+,a} V(X) = V_{+,a}.
\]

In particular, the map \( V_+ \mapsto V \) defines an isomorphism from \( \mathcal{V}_+ \) to \( \mathcal{V} \).

Proof. Fix \( V_+ = \sum_{m \in \mathfrak{v}_+} \alpha_m M_{m,a} \in \mathcal{V}_+(\{a\}) \); then \( \alpha_m = \langle V_+, f_m^{(a)} \rangle_0 \) by (2.6). Let \( \hat{P}_m = \hat{P}(M_m) \). We want to show that there is a unique \( V = \sum_{m' \in \mathfrak{v}_+} \beta_{m'} \hat{P}_{m'} \in \mathcal{V} \) such that
\[
\alpha_m = \sum_{m' \in \mathfrak{v}_+} \beta_{m'} \langle \hat{P}_{m'}(X), f_m^{(a)} \rangle_0 = \sum_{m' \in \mathfrak{v}_+} \beta_{m'} B_{m',m},
\]

where $B_{m',m} = \langle \hat{P}_{m'}(X), f_m^{(a)} \rangle_0$. Let $\hat{Q}_{m'} = \hat{P}_{m'} - M_{m'}$. According to Definition 1.2, $\hat{Q}_{m'} \in \mathcal{P}_t + \mathcal{R}_1$ for some $t > [M_{m'}]$. Since elements of $\mathcal{R}_1(X)$ annihilate test functions in pairings, it follows from (3.14)-(3.15) that, for $[M_{m'}] \geq [M_m]$,

$$B_{m',m} = \langle M_{m'}(X), f_m^{(a)} \rangle_0 + \langle \hat{Q}_{m'}(X), f_m^{(a)} \rangle_0 = |X|\delta_{m',m} + 0 = \delta_{m',m}. \quad (2.15)$$

Thus the matrix $B$ is triangular, with $|X|$ on the diagonal, and hence $B^{-1}$ exists. Then the row vector $\beta$ is given in terms of the row vector $\alpha$ by $\beta = \alpha B^{-1}$, and this solution is unique. Since $\mathcal{V}_+$ and $\mathcal{V}$ have the same finite dimension, the map $V_+ \mapsto V$ defines an isomorphism between these two spaces.

The following commutative diagram illustrates the construction of $\text{loc}_X$ in the next proof:

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\text{loc}_X} & \mathcal{V}(X) \\
\downarrow \text{loc}_{+,a} & & \downarrow \text{loc}_{+,a} = \mu_{X,a}^{-1} \\
\mathcal{V}_+(\{a\}) & & \\
\end{array}$$

**Proof of Proposition 1.5.** (i) *Existence of $V \in \mathcal{V}$.** Given $a \in X$, let $V(X) = (\mu_{X,a} \circ \text{loc}_{+,a})F$, where $\mu_{X,a} : \mathcal{V}_+(\{a\}) \rightarrow \mathcal{V}(X)$ denotes the map which associates the polynomial $V(X)$ to $V_{+,a}$ in Lemma 2.5. By (2.11) and Lemma 2.5, for all $g \in \Pi$,

$$\langle V(X), g \rangle_0 = \langle \text{loc}_{+,a}V(X), g \rangle_0 = \langle \text{loc}_{+,a}\mu_{X,a}\text{loc}_{+,a}F, g \rangle_0 = \langle \text{loc}_{+,a}F, g \rangle_0 = \langle F, g \rangle_0. \quad (2.16)$$

This establishes (1.22).

(ii) *Uniqueness.* Given two polynomials in $\mathcal{V}$ that satisfy (1.22), let $P$ be their difference. Then $P$ is a polynomial in $\mathcal{V}$ such that, for all $g \in \Pi$ and $a \in X$,

$$0 = \langle P(X), g \rangle_0 = \langle \text{loc}_{+,a}P(X), g \rangle_0, \quad (2.17)$$

where we used (2.11). By (2.6) $\text{loc}_{+,a}P(X) = 0$ is zero as an element of $\mathcal{V}_+(\{a\})$. By Lemma 2.5 $P = 0$. This proves uniqueness.

(iii) *Independence of $\mathcal{N}'$.** The polynomial $V$ does not depend on the choice of $\mathcal{N}'$ implicit in the requirement $F \in \mathcal{N}_X$, as long as $X \subset \Lambda'$ and $F \in \mathcal{N}(\Lambda')$ because if $\Lambda'$ and $\Lambda''$ are valid choices then so is $\Lambda' \cap \Lambda''$ and the resulting two constructions of $V$ satisfy (1.22) for all $g \in \Pi(\Lambda' \cap \Lambda'')$.

(iv) *Duality.* Namely, For $n \in \mathfrak{v}_+$, let $c_n$ be the vector $(c_n)_{n'} = B^{-1}_{n,n'}$, where $B$ is the matrix in the proof of Lemma 2.5. It follows from that proof that the pairing of $\sum_{n'}(c_n)_{n'}\hat{P}_{n'}(X)$ with $f_m^{(a)}$ is $\delta_{a,m}$. Thus the basis $(c_n)_{n \in \mathfrak{v}_+}$ is dual to the basis $(f_m^{(a)})_{m \in \mathfrak{v}_+}$ of $\Pi$. This completes the proof of Proposition 1.5.

It follows from (i) and (ii) above that, for any $a \in X$,

$$\text{loc}_XF = (\mu_{X,a} \circ \text{loc}_{+,a})F, \quad (2.18)$$
2.2 Proof of norm estimates for $\text{loc}$

We now prove Propositions 1.11, 1.12 and 1.16, using the following definition which we recall from [6, (3.37)]. Given $X \subset \Lambda$ and a test function $g \in \Phi$, we define

$$
\|g\|_{\Phi(X)} = \inf \{\|g - f\|_{\Phi} : f_z = 0 \text{ if all components of } z \text{ lie in } X\}. \tag{2.19}
$$

Let $f$ be as in (2.19). By definition, if $F \in \mathcal{N}(X)$ then $\langle F, g \rangle_{\phi} = \langle F, g - f \rangle_{\phi}$. Hence

$$
|\langle F, g \rangle_{\phi}| \leq \|F\|_{T_0} \|g - f\|_{\Phi},
$$

and by taking the infimum over $f$ we obtain

$$
|\langle F, g \rangle_{\phi}| \leq \|F\|_{T_0} \|g\|_{\Phi(X)} \quad F \in \mathcal{N}(X). \tag{2.20}
$$

**Proof of Propositions 1.11 and 1.16.** We use the notation in the proof of Lemma 2.5. By definition,

$$
\text{loc}_{+,a} F = \sum_{m' \in \mathbb{V}_+} \alpha_m M_{m',a} \quad \text{with} \quad \alpha_m = \langle F, f_m^{(a)} \rangle_0. \tag{2.21}
$$

Therefore, by (2.18) and the formula $\beta = \alpha B^{-1}$ of the proof of Lemma 2.5,

$$
\text{loc}_X F = \sum_{m \in \mathbb{V}_+} \beta_m \hat{P}_m(X) = \sum_{m, m' \in \mathbb{V}_+} \langle F, f_m^{(a)} \rangle_0 B_{m',m}^{-1} \hat{P}_m(X). \tag{2.22}
$$

By Definition 1.15, this implies that

$$
\|\text{loc}_{X,Y} F\|_{T_0} \leq \sum_{m, m' \in \mathbb{V}_+} |\langle F, f_m^{(a)} \rangle_0| \|B_{m',m}^{-1}\| \|\hat{P}_m(X)\|_{T_0}
$$

$$
\leq \frac{|Y|}{|X|} \sum_{m, m' \in \mathbb{V}_+} |\langle F, f_m^{(a)} \rangle_0| \|A_{m',m}^{-1}\| \|\hat{P}_m,0\|_{T_0}
$$

$$
\leq \|F\|_{T_0} \frac{|Y|}{|X|} \sum_{m, m' \in \mathbb{V}_+} |\langle f_m^{(a)} \rangle_{\Phi(U)}| \|A_{m',m}^{-1}\| \|\hat{P}_m,0\|_{T_0}, \tag{2.23}
$$

where we used (2.20) in the last inequality.

It is shown in Lemmas 3.2 and 3.4 that

$$
|\langle f_m^{(a)} \rangle_{\Phi(X)}| \leq C\hbar^{-m'} R^{[\alpha(m')]_1}, \quad \|\hat{P}_m,0\|_{T_0} \leq c R^{-[\alpha(m)]_1} \hbar^m, \tag{2.24}
$$

here $\hbar^m$ denotes the product over the components $(i_k, \alpha_k)$ of $m$ of $\hbar_{i_k}$. It therefore suffices to show that

$$
|A_{m',m}^{-1}| \leq C\hbar^m R^{-[\alpha(m')]_1} R^{[\alpha(m)]_1} \hbar^{-m}. \tag{2.25}
$$

The matrix elements $A_{m',m}$ can be computed using the formula

$$
A_{m',m}^{-1} = (I + (A - I))^{-1} = \sum_{j=0}^{[\mathbb{V}_+]_1} (-1)^j (A - I)^j, \tag{2.26}
$$

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where we have used the fact that the upper triangular matrix $A - I$ with zero diagonal is nilpotent. Consequently, $A_{m',m}^{-1}$ is bounded by a sum of products of factors of the form

$$|X|^{-1} |\langle \hat{P}_{m'}(X), f_m' \rangle_0| \leq \|\hat{P}_{m',0}\| T_0 \|f_m'\|_{\Phi(X)}, \quad (2.27)$$

where $\hat{X}$ is a polymer which extends $X$ in a minimal way to ensure that $P_{m'}(X) \in \mathcal{N}(\hat{X})$ for all $m' \in \nu_+$. Now repeated application of (2.24) gives rise to a telescoping product in which the powers of $R$ and $h$ exactly cancel, leading to an upper bound

$$\|\text{loc}_X Y F\|_{T_0} \leq \bar{C} \|F\|_{T_0}. \quad (2.28)$$

This proves Proposition 1.16, and the special case $Y = X$ then gives Proposition 1.11. $lacksquare$

For the proof of Proposition 1.12, we need some preliminaries. For a coordinate patch $X$, let $\Pi(X) \subset \Phi$ denote the set of test functions whose restriction to every argument in $X$ agrees with the restriction of an element of $\Pi$. Roughly speaking, it is convenient to decompose $\phi \in \Phi_j(X)$ into a “polynomial part” $f_1 \in \Pi_j(X)$ which is a good approximation to $\phi$ in $X$, plus a remainder $f_2$. More precisely, for $F \in \mathcal{N}(X)$, we define the semi-norm

$$\|F\|_{T_\phi(\Pi(X))} = \sup_{g \in \Pi(X) \cap B(\Phi)} |\langle F, g \rangle_\phi|. \quad (2.29)$$

This semi-norm based on $\Pi$ is admissible in [6, Definition 3.1] because it is equivalent to a choice of weight: by setting $u = 0$ on appropriate spatial derivatives only particular polynomials have finite norm. We also define, on $\Phi$, the semi-norm

$$\|g\|_{\tilde{\Phi}(X)} = \inf\{\|g - f\|_\phi : f \in \Pi(X)\}. \quad (2.30)$$

Lemma 2.6. Let $\epsilon > 0$, $X \subset \Lambda$, and $g \in \Phi$. Then there exists a decomposition $g = f + h$ with $f \in \Pi_X$, $\|g\|_{\tilde{\Phi}(X)} \leq \|h\|_{\Phi} \leq (1 + \epsilon)\|g\|_{\tilde{\Phi}(X)}$ and $\|f\|_{\Phi} \leq (2 + \epsilon)\|g\|_{\Phi}$.

Proof. By (2.30), we can choose $f \in \Pi(X)$ so that $h = g - f$ obeys $\|g\|_{\tilde{\Phi}(X)} \leq \|h\|_{\Phi} \leq (1 + \epsilon)\|g\|_{\tilde{\Phi}(X)}$, and then $\|f\|_{\Phi} \leq \|h\|_{\Phi} + \|g\|_{\Phi} \leq (2 + \epsilon)\|g\|_{\Phi}$. $lacksquare$

Proof of Proposition 1.12. We write $c$ for a generic constant and $\bar{c}$ for a generic constant that depends on $R^{-1}\text{diam}(X)$. Let $F \in \mathcal{N}(X)$ and $A < p_X$. We first apply [6, Proposition 3.11] to obtain

$$\|F\|_{T_\phi} \leq (1 + \|\phi\|_{\Phi})^{A+1} \left[\|F\|_{T_0} + \rho^{(A+1)} \sup_{0 \leq t \leq 1} \|F\|_{T_\phi}\right], \quad (2.31)$$

where, due to our choice of norm, $\rho^{(A+1)} \leq c L^{-A+1} [\tilde{\phi}_{\text{min}}]$. To estimate $\|F\|_{T_0}$, given a test function $g$, we choose $f \in \Pi(X)$ as in Lemma 2.6, and obtain

$$|\langle F, g \rangle_0| \leq |\langle F, f \rangle_0| + |\langle F, g - f \rangle_0|. \quad (2.32)$$

The first term on the right-hand side is at most $\|F\|_{T_0(\Pi(X))}\|f\|_{\Phi}$. Now we set $F = F_1(1 - \text{loc}_Y) F_2$. It follows from (1.23) that $\|(1 - \text{loc}_Y) F_2\|_{T_0(\Pi(X))} = 0$, and hence, by the product property of the
Proposition 1.11, this gives 
\[ Y_{\text{loc}} \leq Y \]
Therefore the first term on the right-hand side of (2.32) is zero. For the second term, we use
\[
|\langle F, g - f \rangle_0| \leq \|F\|_{T_0} \|g - f\|_{\mathcal{F}} \leq \|F\|_{T_0} (1 + \epsilon) \|g\|_{\mathcal{F}} \leq \|F\|_{T_0} (1 + \epsilon) c L^{-d_+} \|g\|_{\mathcal{F}'} ,
\]
where the final inequality is a consequence of Lemma 3.6. After taking the supremum over \( g \in B(\Phi') \), followed by the infimum over \( \epsilon > 0 \), we obtain \( \|F\|_{T_0} \leq c L^{-d_+} \|F\|_{T_0} \), and hence
\[
\|F\|_{T_0} \leq (1 + \|\phi\|_{\mathcal{F}'})^{A+1} \bar{c} \left( L^{-d_+} + L^{-(A+1)|\varphi_{\text{min}}|} \right) \sup_{0 \leq t \leq 1} \|F\|_{T_{t\phi}} .
\]
Next, we apply the triangle inequality and the product property of the \( T_{t\phi} \) semi-norm to obtain
\[
\|F\|_{T_{t\phi}} \leq \|F_1 F_2\|_{T_{t\phi}} + \|F_1\|_{T_{t\phi}} \|\text{loc}_Y F_2\|_{T_{t\phi}} .
\]
Since \( \text{loc}_Y F_2 \in \mathcal{V} \), it is a polynomial of dimension at most \( d_+ \), and hence of degree at most \( d_+ / |\varphi_{\text{min}}| \). It follows from [6, Proposition 3.10] that \( \|\text{loc}_Y F_2\|_{T_{t\phi}} \leq (1 + \|\phi\|_{\mathcal{F}'})^{d_+ / |\varphi_{\text{min}}|} \|\text{loc}_Y F_2\|_{T_0} \). With Proposition 1.11, this gives
\[
\|F\|_{T_{t\phi}} \leq \|F_1 F_2\|_{T_{t\phi}} + \bar{c}' (1 + \|\phi\|_{\mathcal{F}'})^{d_+ / |\varphi_{\text{min}}|} \|F_1\|_{T_{t\phi}} \|F_2\|_{T_0} .
\]
Since \( \|\phi\|_{\mathcal{F}} \leq c L^{-|\varphi_{\text{min}}|} \|\phi\|_{\mathcal{F}'} \leq c \|\phi\|_{\mathcal{F}'} \) due to our choice of norm, this gives
\[
\|F\|_{T_{t\phi}} \leq \|F_1 F_2\|_{T_{t\phi}} + \bar{c} (1 + \|\phi\|_{\mathcal{F}'})^{d_+ / |\varphi_{\text{min}}|} \|F_1\|_{T_{t\phi}} \|F_2\|_{T_0} .
\]
Substitution of (2.37) into (2.34) completes the proof. \( \blacksquare \)

### 2.3 The polynomials \( P(M) \)

We now prove the claim which guaranteed existence of the polynomials \( \hat{P} \) of Definition 1.2. These polynomials were used to define the \( \Sigma \)-invariant subspace \( \mathcal{V} \) of \( \mathcal{P} \).

**Lemma 2.7.** For any \( M \in \mathcal{M}_+ \), the polynomial \( P = P(M) \) of (1.19) obeys: (i) \( P(M) \) is \( \Sigma_{\text{axes}} \)-covariant, (ii) \( M - P(M) \in \mathcal{P}_t + \mathcal{R}_1 \) for some \( t > [M] \), and (iii) \( P(\Theta M) = \Theta P(M) \) for \( \Theta \in \Sigma \).

**Proof.** (i) For \( \Theta' \in \Sigma_{\text{axes}} \),
\[
\Theta' P = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta, M) \Theta' \Theta M = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta^{-1} \Theta, M) \Theta M = \lambda(\Theta^{-1}, M)|\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta, M) \Theta M = \lambda(\Theta', M) P ,
\]
as required.

(ii) Given \( M \in \mathcal{M}_+ \) and \( \Theta \in \Sigma_{\text{axes}} \), the monomial \( \Theta M \) is equal to \( M \) with derivatives switched from forward to backward in each coordinate where \( \Theta \) changes sign. Any derivative that was
switched can be restored to its original direction using (1.5), modulo a term in \( P_t + R_1 \). The use of (1.5) introduces a sign change for each restored derivative, with the effect that \( M \) is equal to \( \lambda(\Theta, M) \Theta M \) modulo \( P_t \). Therefore, \( M = P(M) \) is also in \( P_t + R_1 \).

(iii) Let \( M \in M_+ \), \( \Theta' \in \Sigma_3 \), and \( \Theta \in \Sigma_{\text{axes}} \). Since \( \Theta^{-1} \Theta \Theta' \in \Sigma_{\text{axes}} \), it makes sense to write \( \lambda(\Theta^{-1} \Theta \Theta', M) \). Also, since the number of derivatives that change direction in the transformation \( M \mapsto \Theta^{-1} \Theta \Theta' M \) is equal to the number that change direction in the transformation \( \Theta \Theta' M \mapsto \Theta \Theta' M \), it follows that \( \lambda(\Theta^{-1} \Theta \Theta', M) = \lambda(\Theta, \Theta' M) \), and hence

\[
\Theta' P(M) = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta, M) \Theta' \Theta M = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta^{-1} \Theta \Theta', M) \Theta \Theta' M = |\Sigma_{\text{axes}}|^{-1} \sum_{\Theta \in \Sigma_{\text{axes}}} \lambda(\Theta, \Theta' M) \Theta(\Theta' M) = P(\Theta' M),
\]

(2.39)

and the proof is complete.

\( \blacksquare \)

3 Lattice Taylor polynomials

Throughout this section we work in a coordinate patch \( \Lambda' \) as described above (2.3), but mainly keep this restriction tacit.

3.1 Taylor polynomials

Let \( a \in \Lambda' \). Recall the definition of the test functions \( b^{(a)}_{(m)} \) in (2.3), for \( m \in \mathbb{m}_+ \). We now prove Lemma 2.1.

Proof of Lemma 2.1. (i) Since \( \{ b^{(a)}_{(m)}, m \in \mathbb{m}_+ \} \) is a basis of \( \Pi \), any \( p \in \Pi \) is given by a unique linear combination of these basis elements. Thus it suffices to show that \( p = \text{Tay}_a g \) obeys the desired identity \( \nabla^m(g - p)|_{z = \bar{a}} = 0 \), and this assertion is implied by

\[
\nabla^m b^{(a)}_{(m', z)}|_{z = \bar{a}} = \delta_{m, m'}, \quad m, m' \in \mathbb{m}_+.
\]

(3.1)

To prove (3.1), it suffices to consider one species and the 1-dimensional case, since the derivatives and binomial coefficients all factor. For non-negative integers \( k, n \), it suffices to show that \( \nabla^n (x - a)^m_k |_{x = a} = \delta_{n,k} \), where we write \( \nabla_+ \) to emphasise that this is a forward derivative. We use induction on \( n \), noting first that when \( n = 0 \) we have \( \nabla^n (x - a)^m_k |_{x = a} = \binom{0}{k} = \delta_{0,k} = \delta_{n,k} \). To advance the induction, we assume that the identity holds for \( n - 1 \) (for all \( k \in \mathbb{N}_0 \)). Since \( \nabla_+(x - a)^m_k = (x - a)^{m+1}_k - (x - a)^m_k = (x - a)^{m-1}_{k-1} \) for all \( x \in \mathbb{Z} \), the induction hypothesis gives, as required,

\[
\nabla^n_+(x - a)^m_k |_{x = a} = \nabla^{n-1}_+(x - a)^{m-1}_{k-1} |_{x = a} = \delta_{n-1,k-1} = \delta_{n,k}.
\]

(3.2)

(ii) It follows from (2.4) that the Taylor expansion of \( g \) with permuted arguments is obtained by permuting the arguments of \( \text{Tay}_a g \), and from this it follows that \( \text{Tay}_a \) commutes with \( S \).

(iii) This follows from the uniqueness in (i).

\( \blacksquare \)
We also make note of a simple fact that we will use below. Suppose the components of \( m \in \mathfrak{m}_+ \) are \((i_k, \alpha_k)\) and the components of \( m' \in \mathfrak{m}_+ \) are \((i_k, \alpha'_k)\) where \( k \in \{1, \ldots, p\}\) and \( \alpha_k, \alpha'_k \in \mathbb{N}_0^d \). We say \( \alpha_k \geq \alpha'_k \) if each component of \( \alpha_k \) is at least as large as the corresponding component of \( \alpha'_k \).

By examining the proof of (3.1), we find that

\[
\nabla^m b^{(a)}_{m', z} = 0 \quad \text{if} \quad \alpha_k > \alpha'_k \quad \text{for some} \quad k = 1, \ldots, p, \tag{3.3}
\]

\[
\nabla^m b^{(a)}_{m, z} = 1. \tag{3.4}
\]

In other words, the condition \( z = \bar{a} \) is not needed in these cases.

### 3.2 Dual pairing

For \( m \in \mathfrak{m}_+ \) let \( \tilde{\Sigma}(m) \) be the set of permutations of \( 1, \ldots, p(m) \) that fix the species when they act on \( m \) by permuting components, i.e., \( \pi(i_k, \alpha_k) = (i_{\pi k}, \alpha_{\pi k}) \) with \( i_{\pi k} = i_k \). This is a group of order \( |\tilde{\Sigma}(m)| \).

There is also the subgroup \( \tilde{\Sigma}_0(m) \) of permutations that fix \( m \). It has order

\[
|\tilde{\Sigma}_0(m)| = \prod_{(i, \alpha)} n_{(i, \alpha)}(m)!, \tag{3.5}
\]

with \( n_{(i, \alpha)} \) as defined below (1.8): \( n_{(i, \alpha)} \) denotes the number of times that \((i, \alpha)\) appears as a component of \( m \).

For example, for \( m = ((1, \alpha_1), (1, \alpha_1), (1, \alpha_2), (1, \alpha_2), (1, \alpha_2), (2, \alpha_3)) \) with \( \alpha_1 < \alpha_2 \), we have \( |\tilde{\Sigma}(m)| = 5!! \) and \( |\tilde{\Sigma}_0(m)| = 2!3!! \). For this choice of \( m \),

\[
b^{(a)}_{m, z} = \begin{pmatrix} z_1 - a \\ \alpha_1 \\ \alpha_1 \\ \alpha_2 \\ \alpha_2 \\ \alpha_2 \\ \alpha_3 \\ \alpha_3 \end{pmatrix}. \tag{3.6}
\]

For this, or for any other \( m \in \mathfrak{m}_+ \), a permutation \( \pi \) in \( \tilde{\Sigma}(m) \) has an action on \( b^{(a)}_{m, z} \) either by mapping it to \( b^{(a)}_{\pi m, z} \) or to \( b^{(a)}_{m, \pi z} \), where \( \pi(z_1, \ldots, z_p) = (z_{\pi 1}, \ldots, z_{\pi p}) \). The two actions are related by \( b^{(a)}_{\pi m, z} = b^{(a)}_{m, \pi z} \). Therefore \( \tilde{\Sigma}_0(m) \) is the set of permutations that leave \( b^{(a)}_{m, z} \) invariant.

By the definition of the symmetry operator \( S : \Phi \to \Phi \) in \([6, \text{Definition 3.4}]\), for \( m \in \mathfrak{m}_+ \),

\[
(Sb^{(a)}_{m})_z = |\tilde{\Sigma}(m)|^{-1} \sum_{\sigma \in \tilde{\Sigma}(m)} \text{sgn}(\sigma_f) b^{(a)}_{\sigma m, z}, \tag{3.7}
\]

where \( \sigma_f \) denotes the restriction of \( \sigma \) to the fermion components of \( z \), and \( \text{sgn}(\sigma_f) \) denotes the sign of this permutation. In (2.5), we defined

\[
f^{(a)}_m = N_m Sb^{(a)}_m, \tag{3.8}
\]

and we now specify that

\[
N_m = \frac{|\tilde{\Sigma}(m)|}{|\tilde{\Sigma}_0(m)|}. \tag{3.9}
\]

We are now in a position to prove Lemma 2.2. Lemma 2.2(i) is subsumed by Lemma 3.1 and is proved in (3.13).
Proof of Lemma 2.2(ii). Let \( g \in \Pi \). By Lemma 2.1(ii), \( \text{Tay}_a S = \text{Tay}_a S^2 = S \text{Tay}_a S \). With (2.4) and (2.2), this gives
\[
(\text{Tay}_a S g)_z = \sum_{m \in \bar{v}_+} \langle M_{m,a}, g \rangle_0 S b_{m,z}^{(a)} = S \sum_{m \in \bar{v}_+} \langle M_{m,a}, g \rangle_0 b_{m,z}^{(a)}. \tag{3.10}
\]
Since \( \bar{\Sigma}_0(m) \) is the set of permutations that leave \( m \) invariant, the sum over \( \bar{v}_+ \) can be written as a sum over \( v_+ \), as
\[
S \sum_{m \in v_+} \langle M_{m,a}, g \rangle_0 b_{m,z}^{(a)} = \sum_{m \in v_+} \frac{1}{|\Sigma_0(m)|} \sum_{\sigma \in \Sigma(m)} \langle M_{\sigma m,a}, g \rangle_0 b_{\sigma m,z}^{(a)}. \tag{3.11}
\]
The anticommutativity of the fermions implies that \( \langle M_{\sigma m,a}, g \rangle_0 = \text{sgn}(\sigma_f) \langle M_{m,a}, g \rangle_0 \). Since \( b_{\sigma m,z}^{(a)} = b_{m,\sigma^{-1} z}^{(a)} \), it follows from (3.7)–(3.9) and the fact that \( S f_{m}^{(a)} = f_{m}^{(a)} \) that
\[
(\text{Tay}_a S g)_z = S \sum_{m \in v_+} \langle M_{m,a}, g \rangle_0 N_m S b_{m,z}^{(a)} = S \sum_{m \in v_+} \langle M_{m,a}, g \rangle_0 f_{m,z}^{(a)} = \sum_{m \in v_+} \langle M_{m,a}, g \rangle_0 f_{m,z}^{(a)}, \tag{3.12}
\]
and the proof is complete. \( \blacksquare \)

The next lemma provides statements concerning the duality of field monomials and test functions, for use in Section 2. In particular, (3.13) gives Lemma 2.2(i).

Lemma 3.1. The following identities hold, for \( a, x \in A' \):
\[
\langle M_{m,a}, f_{m'}^{(a)} \rangle_0 = \delta_{m,m'} \quad m, m' \in m_+; \tag{3.13}
\]
\[
\langle M_{m,x}, f_{m'}^{(a)} \rangle_0 = \delta_{m,m'} \quad m, m' \in m_+ \text{ with } [M_m] = [M_{m'}]; \tag{3.14}
\]
\[
\langle M_{m,x}, f_{m'}^{(a)} \rangle_0 = 0 \quad m \in m, m' \in m_+ \text{ with } [M_m] > [M_{m'}]. \tag{3.15}
\]

Proof. We begin with a preliminary observation. Let \( m \in m_+ \). It follows from (2.2), the identity \( S^2 = S \), and (3.7)–(3.9) that
\[
\langle M_{m,x}, f_{m'}^{(a)} \rangle_0 = \nabla^m (S f_{m'}^{(a)})|_{z=\bar{x}} = |\bar{\Sigma}_0(m')|^{-1} \sum_{\sigma \in \Sigma(m')} \text{sgn}(\sigma_f) \nabla^m b_{m',\sigma z}^{(a)}|_{z=\bar{x}} = |\bar{\Sigma}_0(m')|^{-1} \sum_{\sigma \in \Sigma(m')} \text{sgn}(\sigma_f) \nabla^m b_{\sigma m',z}^{(a)}|_{z=\bar{x}}, \tag{3.16}
\]
where in the last step we recalled that \( b_{z}^{(a)} = b_{m,\pi^{-1} z}^{(a)} \).

It is now easy to prove (3.13). Indeed, by (3.1) with \( x = a \), \( \nabla^m b_{\sigma m',z}^{(a)}|_{z=\bar{a}} = \delta_{m,\sigma m'} \). For \( m, m' \in m_+, m = \sigma m' \) holds if and only if \( m = m' \) and \( \sigma \in \bar{\Sigma}_0(m') \). Since \( n_{(i,a)} = 1 \) for fermion species \( i \), we have \( \text{sgn}(\sigma_f) = 1 \) for permutations that fix \( m \), and (3.13) follows.

For the proof of (3.14)–(3.15), we first observe that by the definition of the zero-field pairing, \( M_{m,x} \) has nonzero pairing only with test functions with the same number of variables as there are fields in \( M_{m,x} \). Therefore, we may assume that the number \( p(m) \) of fields in \( M_{m,x} \) is equal to the
number $p(m')$ of variables in $f_m^{(a)}$. Furthermore, the pairing only replaces the fields in $M_{m,x}$ with test functions whose arguments match the species of the fields. Thus, for $m, m' \in \mathfrak{m}$, the pairing $\langle M_{m,x}, f_m^{(a)} \rangle_0$ is zero unless $p(m) = p(m')$ and the components $(i_k, \alpha_k)$ of $m$ and the components $(i'_k, \alpha'_k)$ of $m'$ obey $i_k = i'_k$ for all $k = 1, \ldots, p(m)$. For (3.14), the condition that $[M_m] = [M_{m'}]$ therefore becomes the condition that $|\alpha|_1 = |\alpha'|_1$. Consider first the case where $\alpha_k \neq \alpha'_k$ for some $k$. Then, for some $k$, $\alpha_k > \alpha'_k$. Since $m, m'$ are elements of $\mathfrak{m}_+$ both the $\alpha_k$ and the $\alpha'_k$ are ordered within each species. Therefore it is also true that for any permutation $\sigma \in \hat{\Sigma}(m')$ there is some $k$ such that $\alpha_k > \alpha'_{\sigma k}$. By (3.3), in this case $\nabla^m f_{m'\sigma m,z}^{(a)} = 0$, so the right-hand side of (3.16) is zero. We are now reduced to the case $\alpha_k = \alpha'_{\sigma k}$ for all $k$. This means that $m = m'$ and we complete the proof of (3.14) as in the proof of (3.13), applying (3.4) rather than (3.1).

Finally, we prove (3.15). As in the proof of (3.14), the condition that $[M_m] > [M_{m'}]$ implies that for any $\sigma$ there is some $k$ such that $\alpha_k > \alpha'_{\sigma k}$. By (3.3), this implies that $\nabla^m f_{m'\sigma m,z}^{(a)} = 0$, and hence the right-hand side of (3.16) is zero, and (3.15) is proved.

3.3 Elementary norm estimates

Lemmas 3.2 and 3.4 are used in the proof of Proposition 1.11. Lemma 3.3 is used to prove Lemmas 3.4 and 3.6.

**Lemma 3.2.** For $m \in \mathfrak{v}_+$, let $\hat{P}_{m,x} = \hat{P}(M_{m,x})$, with $\hat{P}$ given by Definition 1.2. Then there is a constant $c$ such that

$$
\|\hat{P}_{m,x}\|_{T_0} \leq R^{-|\alpha(m)|_1} h^m, \tag{3.17}
$$

where $h^m$ denotes the product over the components $(i_k, \alpha_k)$ of $m$ of $h_{i_k}$.

**Proof.** By Definition 1.2, $\hat{P}_m$ is a sum of monomials of the same degree and dimension as $M_m$, so it suffices to prove (3.17) for a single such monomial $\bar{M}_m$. But for any test function $g$, by the definition of the $\Phi(h)$ norm in (1.35) we have

$$
|\langle \bar{M}_{m,x}, g \rangle_0| = |\nabla^{\bar{a}(m)}(Sg)_{\bar{z}}|_{z=\bar{z}}| \leq R^{-|\alpha(m)|_1} h^m \|Sg\|_{\Phi(h)} \leq R^{-|\alpha(m)|_1} h^m \|g\|_{\Phi(h)}, \tag{3.18}
$$

as required.

Let $X$ be a polymer constructed from unions of blocks of side $R$ in a paving of $\Lambda$. Given a block $B$, we denote its enlargement to a block of side $3R$, centred on $B$, by $\bar{B}$. Then we define the enlargement $\bar{X}$ of $X$ to be the union of $\bar{B}$ over the blocks $B$ in $X$. The following lemma shows that it is possible to estimate the $\Phi(X)$ norm of a test function $g$ using the values of $g$ only in the enlargement $\bar{X}$. In its statement, we write $z \in \bar{X}$ to mean that each component $z_i$ of $z$ lies in $\bar{X}$. Recall from (2.19) that the $\Phi(X)$ is defined in terms of the $\Phi = \Phi(h)$ norm of (1.35) by

$$
\|g\|_{\Phi(X)} = \inf\{\|g - f\|_\Phi : f_z = 0 \text{ if all components of } z \in \bar{A}^* \text{ are in } X\}. \tag{3.19}
$$

**Lemma 3.3.** There is a positive constant $c_1$, independent of $R$, such that for any $g \in \Phi$ and any polymer $X$ which is also a coordinate patch,

$$
\|g\|_{\Phi(X)} \leq c_1 \sup_{z \in \bar{X}} \sup_{|\beta|_1 \leq \rho_\Phi} h^{-\beta} |\nabla^\beta g_z|. \tag{3.20}
$$

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Proof. Let $Y$ be the subset of $\mathbb{R}^d$ obtained by taking the union of the closed unit blocks in $\mathbb{R}^d$ centred at the points $x \in X$, and let $\bar{Y}$ be defined similarly from $\bar{X}$. Let $Y_0 = R^{-1}Y$ and let $\bar{Y}_0 = R^{-1}\bar{Y}$. We fix a $C^\infty$ function $\chi_0 : \mathbb{R}^d \to [0, 1]$ with $\chi_0|_{Y_0} = 1$ and $\chi_0|_{\bar{Y}_0} = 0$. We can choose $\chi_0$ in such a way that its partial derivatives to any fixed order are bounded uniformly in $R$ and $X$. We define a test function $\chi_R \in \Phi$ by $\chi_{R,z} = \prod_{i=1}^{p(z)} \chi_0(z_i/R)$, where as usual $p(z)$ denotes the number of components of $z$.

Since $g\chi_R$ agrees with $g$ when evaluated on $X$, and is zero outside $\bar{X}$, it follows from the definition of the $\Phi(X)$ norm in (2.19) that

$$
\|g\|_{\Phi(X)} \leq \|g\chi_R\|_{\Phi} \leq \sup_{z \in X} h^{-z} \sup_{|\beta| \leq p_{\Phi}} \|\nabla_R^\beta (g\chi_R)\|.
$$

(3.21)

By the lattice product rule $\nabla_e(hf) = (T_e f)\nabla h + h\nabla f$, where $T_e$ is translation by the unit vector $e$. By the mean-value theorem, all forward and backward finite-difference derivatives $\nabla_R^\beta \chi_R$, up to order $|\beta|_1 \leq p_{\Phi}$, are uniformly bounded by a constant independent of $R$. Together, these give the desired estimate.

Lemma 3.4. Let $X$ be a polymer which is also a coordinate patch and let $a \in X$. There is a constant $C$, which depends on $m$ and the diameter of $R^{-1}X$, such that

$$
\|f_m^{(a)}\|_{\Phi(X)} \leq \bar{C} h^{-m} R^{|\alpha(m)|_1}.
$$

(3.22)

Proof. By the definition of $f_m^{(a)}$ in (3.8) and by Lemma 3.3, it suffices to show that for $z \in \bar{X}$ and for $|\beta|_1 \leq p_{\Phi}$,

$$
|\nabla_R^\beta f_m^{(a)}| \leq \bar{C} R^{a|_1},
$$

(3.23)

where $\bar{c}$ depends on $m$ and $R^{-1}X$. For this, we first note that if any component of $\beta$ exceeds the corresponding component of $\alpha = \alpha(m)$ then the left-hand side of (3.23) is equal to zero as in the proof of (3.15). Thus we may assume that each component of $\beta$ is at most the corresponding component of $\alpha$, and without loss of generality we may consider the 1-dimensional case. In this case, for $j = j_- + j_+ \leq k$, $|\nabla^{j_-}_- \nabla^{j_+}_+ (x-a)_k| = |(x-a-j_-)|$ and this is at most a multiple of $R^{k-j}$, with the multiple dependent on the ratio of the diameter of $X$ to $R$. This proves (3.23) and completes the proof of (3.22).

3.4 Taylor remainders and change of scale

The following Taylor remainder estimate is used to prove Lemma 3.6, which plays an important role in the proof of the crucial change of scale bound in Proposition 1.12. For its statement, given $a \in \mathbb{Z}^d$, $p \in \mathbb{N}$, $z = (z_1, \ldots, z_p)$ with $z_1, \ldots, z_p \in \mathbb{Z}^d$ and with $(z_i)_j \geq a_j$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, d$, and $t \in \mathbb{N}$, we define $S_t(a, z) = \{y = (y_1, \ldots, y_p) : y_i \in \mathbb{Z}^d : a_j - t \leq (y_i)_j \leq (z_i)_j\}$. We make use of the map $\text{Tail}_a : \Phi \to \Pi$ given by (2.4). It involves polynomials in the components of $z$ to maximal degree $s = d_+ - \sum_{k=1}^p |i(z_k)|$, where $i(z_k)$ denotes the field species corresponding to the component $z_k$. Also, given a test function $g \in \Phi^{(p)}$, we write $M_g = \sup_{y \in S_t(a, z)} \sup_{|\alpha|_1 = s+1} |\nabla^\alpha g_z|$ where the supremum over $\alpha$ is a supremum over only forward derivatives.
Lemma 3.5. For \( a \in \Lambda \), components of \( z = (z_1, \ldots, z_p) \) in \( \Lambda \), \( (z_i)_j \geq a_j \) for all \( i, j \), a coordinate patch \( \Lambda' \supset S_t(a, z) \), and \( |\beta|_1 = t \leq s \) (forward or backward derivatives), the remainder in the approximation of \( g \) by its Taylor polynomial obeys

\[
|\nabla^\beta(g - \text{Tay}_a g)_z| \leq M_g \left( \frac{|z - \vec{a}|_1}{s - t + 1} \right),
\]

with \( M_g \) and \( s \) as defined above.

Proof. Although our setup is such that the number of components \( dp \) of \( z \) is divisible by \( d \), this is artificial in the context of this proof and we can assume that \( d = 1 \), so that the increase from \( p \) to \( p + 1 \) increases the number of components of \( z \) by 1. This is important for the proof, which is by induction on the number of components. So without loss of generality, we set \( d = 1 \). Also without loss of generality, we assume that \( a = 0 \). Let \( f_z = \text{Tay}_a g = \text{Tay}_0 g_z \).

We first show that it suffices to establish (3.24) for the case \( |\beta|_1 = t = 0 \), namely

\[
|g_z - f_z| \leq M_g \left( \frac{|z|_1}{s + 1} \right),
\]

with the supremum defining \( M \) taken over \( S_0(z) \). In fact, for the case where \( \beta \) involves only forward derivatives, \( \nabla^\beta f \) is the degree \( s - t \) Taylor polynomial for \( \nabla^\beta g \), and it follows from (3.25) that

\[
|D^\beta(g - f)_z| \leq M_g \left( \frac{|z|_1}{s - t + 1} \right),
\]

which is better than (3.24). To allow also backward derivatives, we simply note that a single backward derivative is equal in absolute value to a forward derivative at a point translated backwards, and this translation is handled in our estimate by the extension of \( S_0(z) \) to \( S_t(z) \) in the definition of \( M_g \).

It remains to prove (3.25). The proof is by induction on \( p \) (with \( s \) held fixed). Consider first the case \( p = 1 \). For a function \( \phi \) on \( Z \), let \( (T\phi)_x = \phi_{x+1} \) and let \( D = T - I \). For \( m > 0 \), \( T^m = I + \sum_{n=1}^{m} (T - I)T^{m-1} \). Iteration of this formula \( s \) times gives

\[
T^m = I + \sum_{m \geq n_1 \geq 1} D^{n_1} + \sum_{m \geq n_1 > n_2 \geq 1} D^2T^{n_2-1} + \cdots = \sum_{\alpha = 0}^{s} \left( \frac{m!}{\alpha!} \right) D^{s+1} E,
\]

where

\[
E = \sum_{m \geq n_1 > n_2 > \cdots > n_{s+1} \geq 1} D^{s+1} T^{n_{s+1}-1}.
\]

We apply this operator identity to \( (T^{z_1} g)_0 \) and obtain, for \( p = 1 \),

\[
g_{z_1} = (T^{z_1} g)_0 = f_{z_1} + (Eg)_0.
\]

The remainder term obeys the estimate

\[
|(Eg)_0| \leq \sum_{m \geq n_1 > n_2 > \cdots > n_{s+1} \geq 1} \sup_{x \in S_0(z_1)} |D^{s+1} g_x| = \left( \frac{m}{s + 1} \right) \sup_{x \in S_0(z_1)} |D^{s+1} g_x|.
\]
This proves (3.25) for \( p = 1 \).

To advance the induction, we assume that (3.25) holds for \( p - 1 \). We write \( y = (z_1, \ldots, z_{p-1}) \) and \( z = (y, z_p) \), and apply the case \( p - 1 \) to \( g \) with the coordinate \( z_p \) regarded as a parameter. This gives

\[
g_z = \sum_{|\beta|_1 \leq s} \binom{y}{\beta} D^\beta g_{(0, z_p)} + \tilde{E},
\]

where by the induction hypothesis \(|\tilde{E}| \leq M(\frac{|y|_1}{s+1})\). We also apply the case \( p = 1 \) to obtain

\[
D^\beta g_{(0, z_p)} = \sum_{\alpha=0}^{s-|\beta|_1} \binom{z_p}{\alpha} D^\alpha D^\beta g_0 + E_1,
\]

with \(|E_1| \leq M(\frac{z_p}{s-|\beta|_1+1})\). The insertion of (3.32) into (3.31) yields

\[
g_z = \sum_{|\beta|_1 \leq s} \binom{y}{\beta} \sum_{\alpha=0}^{s-|\beta|_1} \binom{z_p}{\alpha} D^\alpha D^\beta g_0 + \sum_{|\beta|_1 \leq s} \binom{y}{\beta} E_1 + \tilde{E}.
\]

(3.33)

The first term on the right-hand side is just the Taylor polynomial \( f_z \) for \( g_z \). It therefore suffices to show that

\[
\sum_{|\beta|_1 \leq s} \binom{y}{\beta} \left( s - |\beta|_1 + 1 \right) + \left( \frac{|y|_1}{s+1} \right) \leq \left( \frac{|z|_1}{s+1} \right).
\]

(3.34)

However, (3.34) follows from a simple counting argument: the right-hand side counts the number of ways to choose \( s+1 \) objects from \(|z|_1\), while the left-hand side decomposes this into two terms in the first of which at least one object is chosen from the last coordinate of \( z \), and in the second of which no object is chosen from the last coordinate. This completes the proof of (3.25). 

The following lemma is used in this paper only in the proof of Proposition 1.12, and, for that purpose, only the second inequality on the right-hand side of (3.35) is needed. However, in [7, Lemma 1.2], we also need the first inequality of (3.35). The need for the first inequality of (3.35) leads us to apply Lemma 2.6 in the proof, rather than using the simpler inequality with \( h = g \) in (3.36).

**Lemma 3.6.** Fix \( L > 0 \). Let \( \Phi(h), \Phi'(h') \) be test function spaces defined via weights involving parameters \( R, h \) and \( R' = LR, h' \) respectively, and with \( p_\Phi \geq d'_+ - [c_{\min}] \). Suppose that \( h'_i / h_i \leq cL^{-[\phi_i]} \), where \( c \) is a universal constant, and where \( h \) and \( h' \) are vectors of length \( p_\Lambda \) whose components depend only on species. Let \( X \) be an \( R \)-polymer which is also a coordinate patch. There exists \( \tilde{C}_3 \), which is independent of \( L \) and depends on \( R \) only via \( R^{-1} \text{diam}(X) \), such that for any test function \( g \),

\[
\|g\|_{\Phi(X)} \leq \tilde{C}_3 L^{-d'_+} \|g\|_{\Phi'(X)} \leq \tilde{C}_3 L^{-d'_+} \|g\|_{\Phi'},
\]

(3.35)

with \( d'_+ \) given by (1.38).
Proof. We assume that $X$ is connected; if it is not then the following argument can be applied in a componentwise fashion. For connected $X$, let $a$ be the largest point which is lexicographically no larger than any point in $X$.

Given $g$, we use Lemma 2.6 to choose $f \in \Pi(X)$ such that $h = g - f$ obeys $\|h\|_{\Phi(X)} \leq 2\|g\|_{\Phi(X)}$. Then $g - (h - \text{Tay}_a h) \in \Pi(X)$, and hence

$$\|g\|_{\Phi(X)} = \|h - \text{Tay}_a h\|_{\Phi(X)} \leq \|h - \text{Tay}_a h\|_{\Phi(X)}.$$  \hspace{1cm} (3.36)

It suffices to prove that for every test function $h$,

$$\|h - \text{Tay}_a h\|_{\Phi(X)} \leq \frac{1}{2} C_3 L^{-d^*} \|h\|_{\Phi'(X)},$$  \hspace{1cm} (3.37)

since $\|h\|_{\Phi'(X)} \leq 2\|g\|_{\Phi'(X)} \leq 2\|g\|_{\Phi'}$.

The rest of the proof is concerned with proving (3.37). We write $a < b$ to denote $a \leq \text{const } b$ with a constant whose value is unimportant. Let $r = h - \text{Tay}_a h$. By Lemma 3.3,

$$\|r\|_{\Phi(X)} \prec \sup_{z \in X} \sup_{|\beta|_1 \leq p_\Phi} |\nabla^\beta R^z r_z|.$$  \hspace{1cm} (3.38)

By hypothesis, (3.38) implies that

$$\|r\|_{\Phi(X)} \prec \sup_{z \in X} \left( \sup_{|\beta|_1 \leq p_\Phi} L^{- \sum_{k \geq 1} |\beta|_1} |\nabla^\beta R^z r_z| \right) \hspace{1cm} \text{(3.39)}$$

where sum on the right-hand side is over the components present in $z$.

Consider first the case $\sum_{k \geq 1} |\beta|_1 > d_+$, for which $\nabla^\beta r_z = \nabla^\beta h_z$. By definition of $d'_+$ in (1.38), $\sum_{k \geq 1} |\beta|_1 \geq d'_+$. The contribution to the right-hand side of (3.39) due to this case is

$$\prec L^{-d'_+} \|h\|_{\Phi'(X)},$$  \hspace{1cm} (3.40)

as required (here there is no dependence on $R^{-1}\text{diam}(X)$ in the constant, and the hypothesis on $p_\Phi$ ensures that there are sufficiently many derivatives in the norm of $h$).

For the case $\sum_{k \geq 1} |\beta|_1 \leq d_+$, we write $t = |\beta|_1$ and $s = d_+ - \sum_{k \geq 1} |\beta_k| \geq t$. By Lemma 3.5, there exists $\tilde{c}$, depending on $R^{-1}\text{diam}(X)$, such that

$$|\nabla^\beta r_z| \leq \tilde{c} \sup_{|\alpha| = s + 1} R^{s-t+1} \sup_z |\nabla^\alpha h_z| \leq \tilde{c} R^{s-t+1} (R^t)^{s-1} (\beta')^z \|h\|_{\Phi'(X)},$$  \hspace{1cm} (3.41)

(the power of $R$ in the first line arises from the binomial coefficient in (3.24), and it is here that the constant develops its dependence on $R^{-1}\text{diam}(X)$) and hence

$$(\beta')^z |\nabla^\beta r_z| \leq \tilde{c} R^{s-t+1} (R^t)^{s-1} \|h\|_{\Phi'(X)} \prec \tilde{c} L^{s-1} \|h\|_{\Phi'(X)},$$  \hspace{1cm} (3.42)

Thus the contribution to (3.39) due to this case is

$$\prec \tilde{c} L^{-\sum_{k \geq 1} |\beta_k|} R^{s-t-1} \|h\|_{\Phi'(X)} = \tilde{c} L^{-d_+} \|h\|_{\Phi'(X)}.$$  \hspace{1cm} (3.43)

Since $d_+ + 1 \geq d'_+$ by the definition of $d'_+$, this completes the proof. 

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