Noncommutative quantum analogs of constant curvature spaces

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Abstract

The quantum $N$-dimensional orthogonal vector Cayley-Klein spaces with different combinations of quantum structure and Cayley-Klein scheme of contractions and analytical continuations are described for multipliers, which include the first and the second powers of contraction parameters in the transformation of deformation parameter. The noncommutative analogs of constant curvature spaces are introduced. The low dimensional spaces with $N = 3, 4$ are discussed in detail and all quantum analogs of the fibered spaces corresponding to nilpotent values of contraction parameters are given. As a result the wide variety of the quantum deformations are obtained.

Keywords: quantum spaces; contractions; spaces of constant curvature; fiber spaces

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1 Introduction

Spaces of constant curvature are among all spaces most symmetric ones. Such $N$-dimensional space has maximal motion group depending on $N(N + 1)/2$ group parameters. Due to this reason they have many fields of application both in mathematics and physics. The uniform axiomatics of all constant curvature spaces of arbitrary dimension was elaborated by R.I.Pimenov [1], where the set of parameters taking real, imaginary and nilpotent values was introduced. Practically these spaces can be obtained from the spherical space by contractions and analytical continuations known as a Cayley-Klein scheme [2], where instead of zero tending contraction parameters of [3] are used nilpotent valued parameters. Accordingly their motion groups are obtained from orthogonal group by Cayley-Klein contractions and analytical continuations.
Quantization theory of simple Lie groups was suggested by L.D. Faddeev and his school [4]. Mathematically quantum groups are noncommutative and noncocommutative deformations of Hopf algebras. With the quantum orthogonal groups are connected quantum orthogonal vector spaces (or quantum Euclidean spaces), which are defined as an algebra functions with the set of generators satisfying certain commutation relations. Quantum orthogonal sphere of arbitrary dimension $S^N_q$ have been suggested in this paper [4]. The standard Podles quantum sphere [5] is connected with quantum unitary group and can be viewed as a quotient $SU_q(2)/U(1)$. The example of a quantum 4-sphere motivated by Poisson structure was presented in [6]. The twisted deformations which quantize the semiclassical structure defined by a generic element of the Cartan subalgebra results in twisted spheres [7]. The discrete family of Podles quantum spheres can be thought of as the family of q-deformed fuzzy spheres [8]. So there are different approaches to a noncommutative spaces.

It is naturally to construct quantum analogs of Cayley-Klein groups and spaces with the same contractions and analytical continuations scheme which holds in commutative case starting from the quantum orthogonal groups and quantum vector spaces. Two new aspects are added in noncommutative quantum case as compared with the commutative one. First of all a transformation of the deformation parameter $z = Jv$ under contraction need be added for quantum group [9] or space. Secondly two mathematical structure (quantum deformation and Cayley-Klein scheme) need be combined. This structures can be combined in a different ways [10]. Different combinations of quantum structure and Cayley-Klein scheme are described with the help of permutations $\sigma$.

In the papers [11]–[15], quantum orthogonal Cayley-Klein groups in Cartesian basis and the noncommutative analogs of the possible commutative kinematics [16] was constructed starting from the mathematical theory of quantum groups and quantum vector spaces [4]. But analysis in the previous papers was confined to the minimal multiplier $J$ in transformation of deformation parameter, which has the first power multiplication of contraction parameters. This restriction imply that for certain combinations of quantum structure and Cayley-Klein scheme some contractions do not exist. In order to all Cayley-Klein contractions was possible for all permutations $\sigma$ it is necessary to regard non-minimal multiplier $J$, which include the first and the second powers of contraction parameters. In this paper we find non-minimal multipliers for the general quantum orthogonal Cayley-Klein spaces (Sec. 3)
and describe in detail noncommutative quantum analogs of constant curvature spaces for $N = 3$ (Sec. 4) and $N = 4$ (Sec. 5). In Sec. 2 we briefly remind the definition of the commutative Cayley-Klein spaces.

## 2 Spaces of constant curvature

The axiomatic description of the most symmetric spaces, namely constant curvature spaces was given by R.I. Pimenov [1]. All $3^N \ N$-dimensional constant curvature spaces can be realized [2] on the spheres

$$S^N(j) = \{\xi_1^2 + j_2^2\xi_2^2 + \ldots + (1, N + 1)^2\xi_{N+1}^2 = 1\},$$

in $(N + 1)$-dimensional vector space $O^{N+1}(j)$, where

$$(i, k) = \prod_{l=\min(i,k)}^{\max(i,k)-1} j_l, \quad (k, k) \equiv 1,$$

and each of parameters $j_k$ takes three values $1, t_k, i, \ k = 1, \ldots, N$. Here $t_k$ are nilpotent generators $t_k^2 = 0$, with commutative law of multiplication $t_kt_m = t_m t_k \neq 0, \ k \neq m$. We shall demand that the following heuristic rules be fulfilled: for a real or complex $a$ the expression $a/t_k$ is defined only for $a = 0$, the expression $t_m/t_k$ is defined only for $m = k$, then $t_k/t_k = 1$.

The intrinsic Beltrami coordinates $r_k = \xi_{k+1}\xi_1^{-1}, \ k = 1, 2, \ldots, N$ present the coordinate system for constant curvature space $S^N(j)$, which coordinate lines $r_k = \text{const}$ are geodesic. $S^N(j)$ has positive curvature for $j_1 = 1$, negative for $j_1 = i$ and it is flat for $j_1 = t_1$. For a flat space the Beltrami coordinates coincide with the Cartesian ones. Nilpotent value of the contraction parameter $j_k = t_k, \ k > 1$ correspond to a fiber space with $(k - 1)$-dimensional base $S_b = \{r_1, \ldots, r_{k-1}\}$ and $(N - k + 1)$-dimensional fiber $S_f = \{r_k, \ldots, r_N\}$. In order to avoid terminological misunderstanding let us stress that we have in view locally trivial fibering, which is defined by the projection $pr : S^N(t_k) \to S_b$. This fibering constitute foundation of the semi-Riemannian geometry [17]–[22], [23] and has nothing to do with the principal bundle. Imaginary value of parameter $j_k = i$ correspond to pseudo-Riemannian space. All nine constant curvature planes (or Cayley-Klein planes) are represented on Fig.1.
| $j_1$ | 1 | $\iota_1$ | i |
|------|----|----------|----|
| 1    | ![Spherical](image) | ![Euclid](image) | ![Lobachevsky](image) |
| $\iota_1$ | ![Newton(+)](image) | ![Galilei](image) | ![Newton(-)](image) |
| i    | ![anti de Sitter](image) | ![Minkowski](image) | ![de Sitter](image) |

Fig.1. Nine constant curvature planes. Fibers are represented by thick lines. Light cones in $(1 + 1)$ kinematics are drawn by dotted lines.

Part of constant curvature spaces $S^4(j)$ for $j_1 = 1, \iota_1, i, j_2 = \iota_2, i, j_3 = j_4 = 1$ can be regarded as $(1 + 3)$ space-time models or kinematics [16], if one interprets $r_1$ as the time axis and the rest as the space ones [2, 15].
3 Quantum orthogonal groups and quantum Cayley-Klein spaces

According to FRT theory [4], the algebra function on quantum orthogonal group $\text{Fun}(SO_q(N))$ (or simply quantum orthogonal group $SO_q(N)$) is the algebra of noncommutative polynomials of $n^2$ variables $t_{ij}, i, j = 1, \ldots, n$, which are subject of commutation relations

$$R_qT_1T_2 = T_2T_1R_q$$

(1)

and additional relations of $q$-orthogonality

$$TCT^t = C, \quad T^tC^{-1}T = C^{-1}. \tag{2}$$

Here $T_1 = T \otimes I$, $T_2 = I \otimes T \in M_{n^2}(\mathbb{C}(t_{ij}))$, $T = (t_{ij})_{i,j=1}^N \in M_n(\mathbb{C}(t_{ij}))$, $I$ is unit matrix in $M_n(\mathbb{C})$, $C = C_0q^\rho$, $\rho = \text{diag}(\rho_1, \ldots, \rho_N)$, $(C_0)_{ij} = \delta_{ij}$, $i' = N + 1 - i$, $i, j = 1, \ldots, N$, that is $(C)_{ij} = q^{\rho_i} \delta_{ij}$ and $C^{-1} = C$.

$$\rho_1, \ldots, \rho_N = \begin{cases} (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}), N = 2n + 1 \\ (n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1), N = 2n. \end{cases} \tag{3}$$

The numerical matrix $R_q$ is the well-known solution [4] of Yang-Baxter equation and its elements fulfils the role of the structure constant of quantum group generators.

Let us remind the definition of the quantum vector space [4].

**Definition 1.** An algebra $O_N^q(\mathbb{C})$ with generators $x_1, \ldots, x_N$ and commutation relations

$$\hat{R}_q(x \otimes x) = qx \otimes x - \frac{q - q^{-1}}{1 + q^{N-2}} x'C x W_q, \tag{4}$$

where $\hat{R}_q = PR_q$, $Pu \otimes v = v \otimes u$, $\forall u, v \in C^n$, $W_q = \sum_{i=1}^N q^{\rho_i} e_i \otimes e_{i'}$,

$$x'C x = \sum_{i,j=1}^N x_i C_{ij} x_j = \epsilon x_{n+1}^2 + \sum_{k=1}^n \left(q^{-\rho_k} x_k x_{k'} + q^{\rho_k} x_{k'} x_k\right), \tag{5}$$

$\epsilon = 1$ for $N = 2n+1$, $\epsilon = 0$ for $N = 2n$ and vector $(e_i)_k = \delta_{ik}$, $i, k = 1, \ldots, N$ is called the algebra of functions on $N$-dimensional quantum Euclidean space (or simply the quantum Euclidean space).
The coaction of the quantum group $SO_q(N)$ on the noncommutative vector space $O^N_q(C)$ is given by
\[
\delta(x) = T \hat{\otimes} x, \quad \delta(x_i) = \sum_{k=1}^{n} t_{ik} \otimes x_k, \quad i = 1, \ldots, n
\]
and quadratic form (5) is invariant $x^t C x = \text{inv}$ with respect to this coaction.

The matrix $C$ has non-zero elements only on the secondary diagonal. They are equal to unit in the commutative limit $q = 1$. Therefore the quantum group $SO_q(N)$ and the quantum vector space $O^N_q(C)$ are described by equations (1)–(5) in a skew-symmetric basis, where for $q = 1$ the invariant form $x^t C_0 x = \text{inv}$ is given by the matrix $C_0$ with the only non-zero elements on the secondary diagonal which are all equal to real units.

New generators $y = D^{-1} x$ of the quantum Euclidean space $O^N_q(C)$ in arbitrary basis are obtained [11],[12] with the help of non-degenerate matrix $D \in M_N$ and they are subject of the commutation relations
\[
\hat{R}(y \otimes y) = qy \otimes y - \frac{\lambda}{1 + q^{N-2}} y^t C' y W,
\]
where $\hat{R} = (D \otimes D)^{-1} \hat{R}_q(D \otimes D)$, $W = (D \otimes D)^{-1} W_q$, $C' = D^t C D$. The corresponding quantum group $SO_q(N)$ is generated in arbitrary basis by $U = (u_{ij})_{i,j=1}^N$, where $U = D^{-1} T D$. The commutation relations of the new generators are
\[
\tilde{R} U_1 U_2 = U_2 U_1 \tilde{R}
\]
and $q$-orthogonality relations look as follows
\[
U \tilde{C} U^t = \tilde{C}, \quad U^t (\tilde{C})^{-1} U = (\tilde{C})^{-1},
\]
where $\tilde{R} = (D \otimes D)^{-1} R_q(D \otimes D)$, $\tilde{C} = D^{-1} C (D^{-1})^t$.

In many cases the most natural basis is the Cartesian basis, where the invariant form $\text{inv} = y^t y$ is given by the unit matrix $I$. The transformation from the skew-symmetric basis $x$ to the Cartesian basis $y$ is described by the matrix $D$, which is a solution of the following equation
\[
D^t C_0 D = I.
\]
This equation has many solutions. Take one of these, namely
\[
D = \frac{1}{\sqrt{2}} \begin{pmatrix}
I & 0 & -i\tilde{C}_0 \\
0 & \sqrt{2} & 0 \\
\tilde{C}_0 & 0 & iI
\end{pmatrix}, \quad N = 2n + 1.
\]
where \( \tilde{C}_0 \) is the \( n \times n \) matrix with real units on the secondary diagonal. For \( N = 2n \) the matrix \( D \) is given by (7) without the middle column and row. The matrix (7) provides one of the possible combinations of the quantum group structure and the Cayley-Klein scheme of group contractions. All other similar combinations are given by the matrices \( D_\sigma = D V_\sigma \), obtained from (7) by the right multiplication on the matrix \( V_\sigma \in M_N \) with elements \((V_\sigma)_{ik} = \delta_{\sigma, k}\), where \( \sigma \in S(N) \) is a permutation of the \( N \)-th order. The matrices \( D_\sigma \) are solutions of equation (6).

We derive the quantum Cayley-Klein spaces with the same transformation of the Cartesian generators \( y = \psi \xi, \ \psi = \text{diag}(1, (1, 2), \ldots, (1, N)) \in M_N \), as in commutative case [2], [12]. The transformation \( z = Jv \) of the deformation parameter \( q = e^z \) should be added in quantum case.

\textbf{Definition 2.} Algebra \( O^N_v(j; \sigma; C) \) with Cartesian generators \( \xi_1, \ldots, \xi_N \) and commutation relations

\[
\hat{R}_\sigma(j) \xi \otimes \xi = e^{Jv} \xi \otimes \xi - \frac{2 \sinh Jv}{1 + e^{Jv(N-2)}} \xi^t C_\sigma(j) \xi W_\sigma(j),
\]

where

\[
\hat{R}_\sigma(j) = \Psi^{-1} \hat{R}_\sigma \Psi, \quad W_\sigma(j) = \Psi^{-1} W_\sigma, \\
C_\sigma(j) = \psi D^t \sigma CD \sigma \psi = \psi V^t \sigma D \sigma \psi, \quad \Psi = \psi \otimes \psi,
\]

is called the \( N \)-dimensional quantum Cayley-Klein vector space.

In explicit form commutation relations (8) are

\[
\xi_{\sigma_k} \xi_{\sigma_m} = \xi_{\sigma_m} \xi_{\sigma_k} \cosh Jv - i \xi_{\sigma_m} \xi_{\sigma_k} \frac{(1, \sigma_{k'})}{(1, \sigma_k)} \sinh Jv, \ k < m < k', \ k \neq m',
\]

\[
\xi_{\sigma_k} \xi_{\sigma_m} = \xi_{\sigma_m} \xi_{\sigma_k} \cosh Jv - i \xi_{\sigma_m} \xi_{\sigma_k} \frac{(1, \sigma_{m'})}{(1, \sigma_m)} \sinh Jv, \ m' < k < m, \ k \neq m',
\]

\[
[\xi_{\sigma_k}, \xi_{\sigma_{k'}}] = 2i \epsilon \sinh \left( \frac{Jv}{2} \right) \left( \cosh Jv \right)^{n-k} \xi_{\sigma_{n+1}} \frac{(1, \sigma_{n+1})^2}{(1, \sigma_k)(1, \sigma_{k'})} +
\]

\[
+ i \frac{\sinh(Jv)}{(\cosh Jv)^{k+1}(1, \sigma_k)(1, \sigma_{k'})} \sum_{m=k+1}^{n} \left( \cosh Jv \right)^m ((1, \sigma_m)^2 \xi_{\sigma_m}^2 + (1, \sigma_{m'})^2 \xi_{\sigma_{m'}}^2),
\]

where \( k, m = 1, 2, \ldots, n \), \( N = 2n + 1 \) or \( N = 2n \), \( k' = N + 1 - k \), permutation \( \sigma = (\sigma_1, \ldots, \sigma_N) \) describes definite combination of the quantum group structure and Cayley-Klein scheme of group contraction. The invariant form
under the coaction of the corresponding quantum orthogonal group on the quantum Cayley-Klein space $O_N^v(j; \sigma; C)$ is written as

$$\text{inv}(j; \sigma) = \left( \epsilon(1, \sigma_{n+1})^2 \xi^2_{\sigma_{n+1}} \frac{(\cosh Ju)^n}{\cosh(Ju/2)} + \right.$$

$$+ \sum_{k=1}^n \left( (1, \sigma_k)^2 \xi^2_{\sigma_k} + (1, \sigma_{k'})^2 \xi^2_{\sigma_{k'}} \right) (\cosh Ju)^{k-1} \cosh(Ju \rho_k). \right) \quad (11)$$

**Definition 3.** The quantum Euclidean space $O_N^q(C)$ with the antiinvolution $x^* = C^tx$ (or in components $x_k^* = q^{pk} x_{k'}$, $k = 1, \ldots, N$) is called the quantum real Euclidean space $O_N^q(R)$ \[^4\].

Similar definition is hold for quantum Cayley-Klein space.

**Definition 4.** The quantum Cayley-Klein vector space $O_N^v(j; \sigma; C)$ with the antiinvolution

$$\xi^*_\sigma_k = \xi_\sigma_k \cosh Ju \rho_k + i \xi_{\sigma_{k'}} \frac{(1, \sigma_{k'})}{(1, \sigma_k)} \sinh Ju \rho_k, \quad \xi^*_{\sigma_{n+1}} = \xi_{\sigma_{n+1}},$$

$$\xi^*_{\sigma_{k'}} = \xi_{\sigma_{k'}} \cosh Ju \rho_{k'} - i \xi_{\sigma_k} \frac{(1, \sigma_{k'})}{(1, \sigma_k)} \sinh Ju \rho_{k}, \quad k = 1, \ldots, N \quad (12)$$

is called the quantum real Cayley-Klein vector space $O_N^v(j; \sigma)$.

The multiplier $J$ in the transformation $z = Ju$ of the deformation parameter need be chosen in such a way that all indefinite relations in commutators, which appear under nilpotent values of the contraction parameters are canceled.

**Theorem.** The quantum $N$-dimensional Cayley-Klein vector space $O_N^v(j; \sigma)$ exist for all possible contractions $j_k = i_k, k = 1, \ldots, N - 1$, if multiplier $J$ in transformation of deformation parameter $z = Ju$ is taken in the form

$$J = J_0 \bigcup J_1 = J_0 \bigcup \bigcup_{k=1}^{(k)} J^{(k)}_1, \quad (13)$$

where $J_0, J^{(k)}_1, J_1$ are given by \[^14\],\[^17\],\[^18\].

**Proof.** As far as multipliers $(1, \sigma_k)$ and $(1, \sigma_{k'})$ enter symmetrically into the commutators \[^19\],\[^20\], we can put $\sigma_k < \sigma_{k'}$ without loss of generality. Then indefinite relations in commutators \[^19\] take the form $(1, \sigma_k)(1, \sigma_{k'})^{-1} = (\sigma_k, \sigma_{k'})$, where $k = 1, 2, \ldots, n$ for $N = 2n + 1$ and $k = 1, 2, \ldots, n - 1$ for $N = 2n$. They are eliminated by the multiplier

$$J_0 = \bigcup_k (\sigma_k, \sigma_{k'}). \quad (14)$$
It has the first power multiplication of contraction parameters and is the minimal multiplier, which guarantees the existence of the Hopf algebra structure for the associated quantum group $SO_v(N; j; \sigma)$. The analysis in the paper [15] was confined to this minimal case.

**Definition 5.** The “union” of two multipliers is understood as the multiplication of all parameters $j_k$, which occur at least in one multiplier and the power of $j_k$ in the “union” is equal to its maximal power in both multipliers, for example, $(j_1j_2^2) \cup (j_2j_3) = j_1j_2^2j_3$.

If we take into account indefinite relations in commutators (10), then we come to the non-minimal multiplier $J$, which consist of contraction parameters in the first and the second powers. The indefinite relations in commutators (10) have the form

\[
\sum_{m=k+1}^n [(1, \sigma_m)^2 + (1, \sigma_{m'})^2] = \sum_{m=k+1}^n (1, \sigma_m)^2 \\
(1, \sigma_k) (1, \sigma_k')
\]

for even $N = 2n$ and

\[
(1, \sigma_{n+1}) + \sum_{m=k+1}^n [(1, \sigma_m)^2 + (1, \sigma_{m'})^2] = (1, \sigma_{n+1}) + \sum_{m=k+1}^n (1, \sigma_m)^2 \\
(1, \sigma_k) (1, \sigma_k')
\]

for odd $N = 2n + 1$. Let us introduce numbers

\[i_k = \min\{\sigma_{k+1}, \ldots, \sigma_n\},\]

then $k$th expression in (15) or (16) is equal to

\[
\frac{(1, i_k)^2}{(1, \sigma_k) (1, \sigma_k')} = \begin{cases} 
(1, \sigma_k)^{-2}(\sigma_k, \sigma_k')^{-1}, & i_k < \sigma_k \\
(\sigma_k, i_k)(i_k, \sigma_k')^{-1}, & \sigma_k < i_k < \sigma_k' \\
(\sigma_k, \sigma_k')(\sigma_k', i_k)^2, & i_k > \sigma_k'
\end{cases}
\]

and compensative multiplier for this expression is as follows

\[J_1^{(k)} = \begin{cases} 
(i_k, \sigma_k)^2(\sigma_k, \sigma_k'), & i_k < \sigma_k \\
(i_k, \sigma_k'), & \sigma_k < i_k < \sigma_k' \\
1, & i_k > \sigma_k'.
\end{cases}\]

For all expressions in (15) or (16) compensative multiplier $J_1$ is obtained by the union

\[J_1 = \bigcup_k J_1^{(k)}\]
Therefore the non-minimal multiplier $J$ in the transformation $z = Jv$ of the deformation parameter is given by (13) and include the first and the second powers of contraction parameters. △

**Definition 6.** The quotient algebra $S_{q}^{N-1}$ of the algebra $O_{q}^{N}(R)$ by the relation $x^* x = x^* C x = 1$ is called the $(N-1)$-dimensional quantum orthogonal sphere [4].

We define quantum orthogonal Cayley-Klein sphere in a similar way.

**Definition 7.** The quotient algebra $S_{v}^{N-1}(j; \sigma)$ of the algebra $O_{v}^{N}(j; \sigma)$ by the relation $\text{inv}(j; \sigma) = 1$ (11) is called the $(N-1)$-dimensional quantum orthogonal Cayley-Klein sphere.

The quantum analogs of the intrinsic Beltrami coordinates on this quantum sphere are given by the sets of independent right or left generators

$$r_{\sigma,-1} = \xi_{\sigma_{1}}\xi_{1}^{-1}, \quad \tilde{r}_{\sigma,-1} = \xi_{1}^{-1}\xi_{\sigma_{i}}, \quad i = 1, \ldots, N, \quad i \neq k, \quad \sigma_{k} = 1.$$ 

The reason for definition of right and left generators is the simplification of expressions for commutation relations. It is possible to use only, say, right generators, but its commutators are cumbersome, when all contraction parameters are not nilpotent.

### 4 Quantum Cayley-Klein vector spaces $O_{v}^{3}(j; \sigma)$ and orthogonal spheres $S_{v}^{2}(j; \sigma)$

The 3-dimensional quantum Cayley-Klein vector spaces $O_{v}^{3}(j; \sigma), j = (j_{1}, j_{2})$ are generated by $\xi_{\sigma_{1}}, \xi_{\sigma_{2}}, \xi_{\sigma_{3}}$ with commutation relations (see (9), (10))

$$\xi_{\sigma_{1}}\xi_{\sigma_{2}} = \xi_{\sigma_{2}}\xi_{\sigma_{1}} \cosh(Jv) - i\xi_{\sigma_{2}}\xi_{\sigma_{3}} \frac{(1, \sigma_{3})(1, \sigma_{1})}{(1, \sigma_{3})} \sinh(Jv),$$

$$\xi_{\sigma_{2}}\xi_{\sigma_{3}} = \xi_{\sigma_{3}}\xi_{\sigma_{2}} \cosh(Jv) - i\xi_{\sigma_{1}}\xi_{\sigma_{2}} \frac{(1, \sigma_{1})(1, \sigma_{3})}{(1, \sigma_{3})} \sinh(Jv),$$

$$[\xi_{\sigma_{1}}, \xi_{\sigma_{3}}] = 2i\xi_{\sigma_{2}} \frac{(1, \sigma_{2})^{2}}{(1, \sigma_{1})(1, \sigma_{3})} \sinh(Jv/2)$$

and has invariant form (11)

$$\text{inv}(j; \sigma) = \left( (1, \sigma_{1})^{2}\xi_{\sigma_{2}}^{2} + (1, \sigma_{3})^{2}\xi_{\sigma_{3}}^{2} \right) \cosh(Jv/2) + (1, \sigma_{2})^{2}\xi_{\sigma_{2}}^{2} \cosh(Jv).$$
The antiinvolution \((12)\) of the Cartesian generators is written as

\[
\xi_{\sigma_1}^* = \xi_{\sigma_1} \cosh J v \rho_1 + i \xi_{\sigma_3} \frac{(1, \sigma_3)}{(1, \sigma_1)} \sinh J v \rho_1, \quad \xi_{\sigma_2}^* = \xi_{\sigma_2} \\
\xi_{\sigma_3}^* = \xi_{\sigma_3} \cosh J v \rho_1 - i \xi_{\sigma_1} \frac{(1, \sigma_1)}{(1, \sigma_3)} \sinh J v \rho_1, \quad \rho_1 = \frac{1}{2}, \rho_2 = 0.
\] (21)

By the analysis of the multiplier \((13)\) for \(N = 3\) and commutation relations \((19)\) of the quantum space generators we have find three permutations with a different multipliers \(J\), namely \(J = j_1 j_2\), for \(\sigma_0 = (1, 2, 3)\), \(J = j_1\) for \(\sigma' = (1, 3, 2)\) and \(J = j_1^2 j_2\) for \(\hat{\sigma} = (2, 1, 3)\).

Orthogonal quantum 2-spheres \(S^2_v(j; \sigma) = O^3_v(j; \sigma)/\{\text{inv}(j; \sigma) = 1\}\) are characterized by the right \(r_k\) or left \(\hat{r}_k\) quantum analogs of the intrinsic Beltrami coordinates, whose commutation relations can be obtained only for fixed permutation \(\sigma\). Let us discuss these cases of different permutations separately.

### 4.1 Permutation \(\sigma_0 = (1, 2, 3)\), multiplier \(J = j_1 j_2\)

According to \((19), (21)\) the corresponding quantum Cayley-Klein vector space is characterized by the following commutation relations of generators and their involutions

\[
O^3_v(j; \sigma_0) = \left\{ \begin{array}{l}
\xi_1 \xi_2 = \xi_2 \xi_1 \cosh j_1 j_2 v - i \xi_2 \xi_3 j_1 j_2 \sinh j_1 j_2 v, \\
\xi_2 \xi_3 = \xi_3 \xi_2 \cosh j_1 j_2 v - i \xi_1 \xi_2 \frac{1}{j_1 j_2} \sinh j_1 j_2 v, \\
[\xi_1, \xi_3] = 2 i \xi_2^2 j_1^2 \sinh j_1 j_2 \frac{v}{2}, \\
\xi_1^* = \xi_1 \cosh j_1 j_2 \frac{v}{2} + i \xi_3 j_1 j_2 \sinh j_1 j_2 \frac{v}{2}, \\
\xi_3^* = \xi_3 \cosh j_1 j_2 \frac{v}{2} - i \xi_1 \frac{1}{j_1 j_2} \sinh j_1 j_2 \frac{v}{2}, \quad \xi_2^* = \xi_2 \end{array} \right\}.
\] (22)

In the commutative case \((v = 0)\) the nilpotent value of the first contraction parameter \(j_1 = \iota_1\) and \(j_2 = 1\) gives the semi-Euclidean space with one dimensional base \(\{\xi_1\}\) and two dimensional fiber \(\{\xi_2, \xi_3\}\). The noncommutative deformations of this fiber semi-Euclidean space is obtained by putting
\[ j_1 = \nu_1 \text{ in (22), namely} \]

\[ O^2_3(\nu_1; \sigma_0) = \left\{ [\xi_1, \xi_2] = [\xi_1, \xi_3] = 0, [\xi_3, \xi_2] = iv_1\xi_2, \xi_1^* = \xi_1, \xi_2^* = \xi_2, \xi_3^* = \xi_3-iv_1\nu/2 \right\}. \]  

(23)

For \( v = 0 \) contraction \( j_1 = 1, j_2 = \nu_2 \) transforms Euclidean space \( E_3 \) to the space with two dimensional base \( \{\xi_1, \xi_2\} \) and one dimensional fiber \( \{\xi_3\} \). Its quantum analogs is obtained by this contraction of commutators \( (22) \)

\[ O^3_3(\nu_2; \sigma_0) = \left\{ [\xi_1, \xi_2] = 0, [\xi_3, \xi_1] = iv_2\xi_2, [\xi_3, \xi_2] = iv_1\xi_1, \xi_1^* = \xi_1, \xi_2^* = \xi_2, \xi_3^* = \xi_3-iv_1\nu/2 \right\}. \]  

(24)

The involutive generators \( \xi^*_k, k = 1, 2, 3 \) are the same as in \( (23) \).

The invariant form for permutation \( \sigma_0 \) is obtained from \( (20) \)

\[ \text{inv}(j; \sigma_0) = (\xi_1^2 + j_1^2j_2^2u_2^2) \cosh j_1j_2v + j_1^2u_2^2 \cosh j_1j_2v. \]  

(25)

Orthogonal quantum 2-sphere \( S^2_v(j; \sigma_0) = O^2_v(j; \sigma_0) \setminus \{ \text{inv}(j; \sigma_0) = 1 \} \) is described by the quantum analogs of Beltrami coordinates with commutation relations

\[ S^2_v(j; \sigma_0) = \left\{ r_1 = \hat{r}_1(\cosh j_1j_2v - iv_2j_1j_2 \sinh j_1j_2v), \right\} \]

\[ r_2 - \hat{r}_2 = 2iv_1r_1 \sinh j_1j_2v, \hat{r}_1r_2 = (\hat{r}_2 \cosh j_1j_2v - \frac{1}{j_1j_2} \sinh j_1j_2v)r_1. \]  

(26)

For \( j_1 = \nu_1, j_2 = 1 \) we obtain from \( (26) \) that the left generators are equal to the right \( \hat{r}_1 = r_1, \hat{r}_2 = r_2 \) and the orthogonal quantum plane has the following commutation relations

\[ S^2_v(\nu_1; \sigma_0) = \left\{ [r_2, r_1] = ivr_1 \right\}. \]  

(27)

For \( j_2 = \nu_2, j_1 = 1 \) we obtain from \( (26) \) the quantum analog of the cylinder with \( r_1 = \hat{r}_1 \) being its cyclic generatrix and \( \hat{r}_2 = r_2 - iv_1^2r_1^2 \). If \( j_1 = i \), then cylinder has hyperbolic generatrix. The Beltrami generators of noncommutative cylinder has the following commutation relations

\[ S^2_v(\nu_2; \sigma_0) = \left\{ [r_2, r_1] = ivr_1(1 + j_1^2r_1^2) \right\}. \]  

(28)

For \( j_1 = \nu_1, j_2 = \nu_2 \) the quantum Galilei plane is given by \( (27) \).
4.2 Permutation $\sigma' = (1, 3, 2)$, multiplier $J = j_1$

As it follows from \([19], [21]\) for permutation $\sigma'$ the commutation relations of generators and their involutions of the quantum Cayley-Klein vector space $O_3^v(j; \sigma')$ is described by

$$O_3^v(j; \sigma') = \left\{ \xi_1 \xi_3 = \xi_3 \xi_1 \cosh j_1 v - i \xi_3 \xi_2 j_1 \sinh j_1 v, \right.$$

$$\xi_3 \xi_2 = \xi_2 \xi_3 \cosh j_1 v - i \xi_1 \xi_3 \frac{1}{j_1} \sinh j_1 v, \quad [\xi_1, \xi_2] = 2i \xi_3^2 j_2 \frac{j_1^2 j_2^2}{\eta_1} \sinh j_1 \frac{v}{2},$$

$$\xi_1^* = \xi_1 \cosh (j_1 \frac{v}{2} + i \xi_3 j_1 \sinh j_1 \frac{v}{2}), \quad \xi_2^* = \xi_2 \cosh j_1 \frac{v}{2} - i \xi_1 \xi_3 \frac{1}{j_1} \sinh j_1 \frac{v}{2}, \quad \xi_3^* = \xi_3 \right\}, \quad (29)$$

For $j_1 = \iota_1$ the quantum semi-Euclidean space $O_3^v(\iota_1; \sigma')$ is connected with the space $O_3^v(\iota_1; \sigma_0)$ \([23]\) by the replacement $\xi_2 \rightarrow \xi_3$ and vice-versa, i.e. by the renumbering of the fiber generators. So it can not be regarded as the independent nonequivalent deformation of the fiber space. For $j_1 = 1, j_2 = \iota_2$ in \((29)\) we obtain the noncommutative deformation of the fiber space with 2-dimensional commutative base $\{\xi_1, \xi_2\}$ and 1-dimensional fiber $\{\xi_3\}$

$$O_3^v(\iota_2; \sigma') = \left\{ \xi_1 \xi_3 = \xi_3 \xi_1 \cosh v - i \xi_3 \xi_2 \sinh v, \quad \xi_3 \xi_2 = \xi_2 \xi_3 \cosh v - i \xi_1 \xi_3 \sinh v, \right.$$

$$[\xi_1, \xi_2] = 0, \quad \xi_3^* = \xi_3, \quad \xi_1^* = \xi_1 \cosh \frac{v}{2} + i \xi_2 \sinh \frac{v}{2}, \quad \xi_2^* = \xi_2 \cosh \frac{v}{2} - i \xi_1 \sinh \frac{v}{2} \right\}. \quad (30)$$

The invariant forms for permutation $\sigma'$ is obtained from \((20)\)

$$\text{inv}(j; \sigma') = (\xi_1^2 + j_1^2 \xi_2^2) \cosh j_1 \frac{v}{2} + j_1^2 j_2^2 \xi_3^2 \cosh j_1 v. \quad (31)$$

Orthogonal quantum 2-sphere $S_2^v(j; \sigma') = O_3^v(j; \sigma')/\{\text{inv}(j; \sigma') = 1\}$ has two Beltrami generators with commutation relations

$$S_2^v(j; \sigma') = \left\{ \hat{r}_2 = (\cosh j_1 v + i \hat{r}_1 j_1 \sinh j_1 v) r_2, \right\}.$$
quantum plane $S$ spaces with 1-dimensional base and can be regarded as noncommutative deformation of the semi-Riemanian with $r$ quantum plane. It can not be regarded as the nonequivalent deformation of the orthogonal quantum plane.

For $j_1 = \ell_1$ in (32) the quantum plane $S^2_v(\ell_1; \sigma')$ is connected with the quantum plane $S^2_v(\ell_1; \sigma_0)$ (27) by the replacement $r_1 \to r_2$ and vice-versa, so it can not be regarded as the nonequivalent deformation of the orthogonal quantum plane.

For $j_2 = \ell_2, j_1 = 1$ we obtain from (32) the quantum analog of the cylinder with $r_1 = \hat{r}_1$ being its cyclic ($j_1 = 1$) or hyperbolic ($j_1 = i$) generatrix and $\hat{r}_2 = (\cosh j_1 v + i r_1 j_1 \sinh j_1 v) r_2$. This quantum cylinder is described by

$$S^2_v(\ell_2; \sigma') = \left\{ [r_1, r_2] = i (r_2 + j^2_1 r_1 r_2) \frac{1}{j_1} \tanh j_1 v \right\}$$

and can be regarded as noncommutative deformation of the semi-Riemanian spaces with 1-dimensional base $\{r_1\}$ and 1-dimensional fiber $\{r_2\}$. Spaces (30) and (33) give an example of contraction, when deformation parameter remain unchanged. Physically these spaces can be interpreted as quantum analogs of the $(1 + 1)$ nonrelativistic Newton kinematics with constant curvature.

4.3 Permutation $\hat{\sigma} = (2, 1, 3)$, multiplier $J = j^2_1 j_2$

The commutation relations of generators and their involutions of the quantum Cayley-Klein vector space $O^3_v(j; \hat{\sigma})$ follow from (10), (21)

$$O^3_v(j; \hat{\sigma}) = \left\{ \xi_2 \xi_1 = \xi_1 \xi_2 \cosh(j^2_1 j_2 v) - i \xi_1 \xi_3 j_2 \sinh(j^2_1 j_2 v), \right.\]

$$\left. \xi_1 \xi_3 = \xi_3 \xi_1 \cosh(j^2_1 j_2 v) - i \xi_2 \xi_1 \frac{1}{j_2} \sinh(j^2_1 j_2 v), \right.\]

$$\left. \left[ \xi_2, \xi_3 \right] = 2i \xi_2 \frac{1}{j_2} \sinh(j^2_1 j_2 v/2), \xi_2 = \xi_2 \cosh(j^2_1 j_2 v/2) + i \xi_3 j_2 \sinh(j^2_1 j_2 v/2), \right.\]

$$\left. \xi_3 = \xi_3 \cosh(j^2_1 j_2 v/2) - i \xi_2 \frac{1}{j_2} \sinh(j^2_1 j_2 v/2), \xi^* = \xi_1 \right\}. \quad (34)$$

The nilpotent value of the first contraction parameter $j_1 = \ell_1$ and $j_2 = 1$ in (34) gives the new quantum semi-Euclidean space

$$O^3_v(\ell_1; \hat{\sigma}) = \left\{ [\xi_1, \xi_2] = [\xi_1, \xi_3] = 0, \left[ \xi_2, \xi_3 \right] = i v \xi^2_1, \xi^*_k = \xi_k, k = 1, 2, 3 \right\}. \quad (35)$$
which is not isomorphic to \((23)\).

For \(j_1 = 1, j_2 = \nu_2\) quantum space \(O^3_v(\nu_2; \hat{\sigma})\) is transformed to \(O^3_v(\nu_2; \sigma_0)\) by the replacement of the base generators \(\xi_1 \to \xi_2\) and vice-versa, therefore it does not present new noncommutative deformation.

For \(j_1 = \nu_1, j_2 = \nu_2\) commutation relations of generators are given by \((35)\).

The invariant form for permutation \(\hat{\sigma}\) is given by \((20)\)

\[
\text{inv}(j; \hat{\sigma}) = j_1^2(j_2^2 + j_2^3) \cosh j_1 j_2 \frac{v}{2} + j_1^2 \cosh j_1 j_2 v. \tag{36}
\]

Orthogonal quantum 2-sphere \(S^2_v(j; \hat{\sigma}) = O^3_v(j; \hat{\sigma})/\{\text{inv}(j; \hat{\sigma}) = 1\}\) is characterized by the commutation relations

\[
S^2_v(j; \hat{\sigma}) = \left\{ \begin{array}{l}
\hat{r}_1 = r_1 \cosh j_1 j_2 v - i r_2 j_2 \sinh j_1 j_2 v, \\
r_2 = \hat{r}_2 \cosh j_1 j_2 v - i r_1 \frac{1}{j_2} \sinh j_1 j_2 v, \end{array} \right\}. \tag{37}
\]

For \(j_1 = \nu_1, j_2 = 1\) we obtain from \((37)\) that the left generators are equal to the right \(\hat{r}_1 = r_1, \hat{r}_2 = r_2\) and orthogonal quantum plane is as follows

\[
S^2_v(\nu_1; \hat{\sigma}) = \left\{ [r_1, r_2] = iv \right\}. \tag{38}
\]

This is the simplest deformation of the Euclid plane because commutator is proportional to the number \(iv\), instead of operator in \((27)\).

For \(j_2 = \nu_2, j_1 = 1\) the quantum cylinder \(S^2_v(\nu_2; \hat{\sigma})\) is given by

\[
S^2_v(\nu_2; \hat{\sigma}) = \left\{ [r_1, r_2] = iv(1 + j_1^2 r_1^2) \right\}. \tag{39}
\]

For \(j_1 = \nu_1, j_2 = \nu_2\) the simplest quantum deformation of the Galilei plane is given by \((38)\).

5 Quantum spaces \(O_4^4(j; \sigma)\) and \(S_3^3(j; \sigma)\)

Quantum vector spaces \(O_4^4(j; \sigma), j = (j_1, j_2, j_3)\) are generated by \(\xi_{\sigma_l}, l = 1, \ldots, 4\) with commutation relations \((k = 2, 3)\)

\[
O_4^4(j; \sigma) = \left\{ \xi_{\sigma_1} \xi_{\sigma_k} - \xi_{\sigma_k} \xi_{\sigma_1} \cosh(Jv) - i \xi_{\sigma_k} \xi_{\sigma_1} \frac{(1, \sigma_1)}{(1, \sigma_1)} \sinh(Jv) \right\},
\]
\[ \xi_{\sigma_k} \xi_{\sigma'} = \xi_{\sigma_k} \xi_{\sigma'} \cosh(Jv) - i \xi_{\sigma_k} \xi_{\sigma'} \left( \frac{1}{1, \sigma_1} \right) \sinh(Jv), \quad [\xi_{\sigma_2}, \xi_{\sigma'}] = 0, \]

\[ [\xi_{\sigma_1}, \xi_{\sigma'}] = i \left( \xi_{\sigma_2} (1, \sigma_2)^2 + \xi_{\sigma'} (1, \sigma_2')^2 \right) \left( \frac{\sinh(Jv)}{(1, \sigma_1)(1, \sigma_1')} \right) \],

where \( \sigma_1 = \sigma_4, \sigma_2 = \sigma_3 \). The antiinvolution \( [12] \) of the Cartesian generators is written as

\[ \xi_{\sigma_1}^* = \xi_{\sigma_1} \cosh(Jv) + i \xi_{\sigma_4} \left( \frac{1}{1, \sigma_1} \right) \sinh(Jv), \quad \xi_{\sigma_2}^* = \xi_{\sigma_2}, \]

\[ \xi_{\sigma_4}^* = \xi_{\sigma_4} \cosh(Jv) - i \xi_{\sigma_1} \left( \frac{1}{1, \sigma_4} \right) \sinh(Jv), \quad \xi_{\sigma_3}^* = \xi_{\sigma_3}, \]

since according with \( [3] \) \( \rho_1 = 1, \rho_2 = 0 \).

By the analysis of the multiplier \( [13] \) for \( N = 4 \) and commutation relations \( [10] \) of the quantum space generators we have find minimal multiplier \( J = (\sigma_1, \sigma_1) \), which takes three values \( J_0 = (1, 1') = j_1 j_2 j_3 \) for permutation \( \sigma_0 = (1, 2, 3, 4), \ J_I = (1, 2') = j_1 j_2 \) for \( \sigma_I = (1, 2, 4, 3), \ J_{II} = (1, 2) = j_1 \) for \( \sigma_{II} = (1, 3, 4, 2) \), i.e. for permutations with \( \sigma_1 = 1 \) and three non-minimal multipliers \( J = (1, \sigma_1)(1, \sigma_1') \), namely \( J_{III} = (1, 2')(1, 1') = j_1^2 j_2 j_3 \) for \( \sigma_{III} = (3, 1, 2, 4) \), \( J_{IV} = (1, 2)(1, 1') = j_1 j_2 j_3 \) for \( \sigma_{IV} = (2, 1, 3, 4) \), \( J_{V} = (1, 2)(1, 2') = j_1^2 j_2 \) for \( \sigma_{V} = (2, 1, 4, 3) \), i.e. for permutations with \( \sigma_1 \neq 1 \).

### 5.1 Quantum fibered spaces \( O^4_v(j; \sigma) \)

We shall not describe all six combinations of Cayley-Klein and quantum structures in full details but shall concentrate our attention on fibered spaces, which correspond to nilpotent values of the contraction parameters. The careful analysis of the commutation relations \( [10] \) for the above mentioned permutations and nilpotent value of the first contraction parameter \( j_1 = \xi_1, j_2 = j_3 = 1 \) gives two nonisomorphic quantum fibered spaces with 1-dimensional base \( \{\xi_1\} \) and 3-dimensional fiber \( \{\xi_2, \xi_3, \xi_4\} \). These quantum fibered spaces are obtained for permutations \( \sigma_0, \sigma_{III} \) and are characterized by the following nonzero commutation relations

\[ O^4_v(\xi_1; \sigma_0) = \left\{ [\xi_4, \xi_k] = iv\xi_1 \xi_k, \ k = 2, 3 \right\}, \quad O^4_v(\xi_1; \sigma_{III}) = \left\{ [\xi_3, \xi_4] = iv\xi_1^2 \right\}. \]

(42)
In both cases base generator commute with all fiber generators and the last ones are not closed under commutation relations. The same properties are hold for \( j_1 = \iota_1, j_2 = \iota_2, j_3 = 1, j_1 = \iota_1, j_3 = \iota_3, \) and \( j_2 = \iota_2, j_3 = \iota_3, \) i.e. for successive enclosed projections or repeatedly fibered spaces.

When the second contraction parameter takes nilpotent value \( j_2 = \iota_2, j_1 = j_3 = 1, \) then the fibered commutative space with 2-dimensional base \( \{ \xi_1, \xi_2 \} \) and 2-dimensional fiber \( \{ \xi_3, \xi_4 \} \) is obtained. There are three its nonisomorphic noncommutative analogs, which are given by \(^{40}\) for permutations \( \sigma_0, \sigma_{II}, \sigma_{III}. \) Their nonzero commutators are as follows

\[
O^4_v(\iota_2; \sigma_0) = \left\{ \xi_4, \xi_k = iv\xi_1\xi_k, \ k = 2, 3 \right\}, \quad O^4_v(\iota_2; \sigma_{II}) = \left\{ \xi_3, \xi_4 = iv\xi_1^2 \right\},
\]

\[
O^4_v(\iota_2; \sigma_{III}) = \left\{ \xi_1\xi_k = \xi_k(\xi_1 \cosh v - i\xi_2 \sinh v), \xi_k\xi_2 = (\xi_2 \cosh v - i\xi_1 \sinh v)\xi_k, \ k = 3, 4 \right\}.
\]

The base generators commute for all permutations. The fiber generators commute only for permutation \( \sigma_{II}. \) The base generators do not commute with the fiber generators for all permutations. The fiber generators are not closed with respect commutation relations for \( \sigma_0 \) and \( \sigma_{III}. \) The same properties are hold for \( j_1 = 1, j_2 = \iota_2, j_3 = \iota_3. \)

The fibered commutative space with 3-dimensional base \( \{ \xi_1, \xi_2, \xi_3 \} \) and 1-dimensional fiber \( \{ \xi_4 \} \) is obtained for nilpotent value of the third parameter \( j_3 = \iota_3, j_1 = j_2 = 1. \) We have find two nonisomorphic quantum fibered spaces for this values of contraction parameters, which are given by \(^{40}\) for permutations \( \sigma_0, \sigma_{II} \) and are characterized by the nonzero commutation relations

\[
O^4_v(\iota_3; \sigma_0) = \left\{ \xi_4, \xi_k = iv\xi_1\xi_k, \ k = 2, 3, \ [\xi_1, \xi_4] = iv(\xi_2^2 + \xi_3^2) \right\},
\]

\[
O^4_v(\iota_3; \sigma_{II}) = \left\{ \xi_1\xi_k = \xi_k(\xi_1 \cosh v - i\xi_2 \sinh v), \xi_k\xi_2 = (\xi_2 \cosh v - i\xi_1 \sinh v)\xi_k, \ k = 3, 4, \ [\xi_1, \xi_2] = i\xi_3^2 \sinh v \right\}.
\]
The base generators commute for permutation $\sigma_0$, but do not commute for permutation $\sigma_{II}$. In the last case they are closed with respect commutation relations.

In general quantum spaces $O^4_v(j; \sigma)$ have commutative base for all permutations, when fibering is defined by $j_1 = \iota_1$ or $j_2 = \iota_2$. When fibering is defined by $j_3 = \iota_3$ the 3-dimensional base is also commutative for permutation $\sigma_0$. The only exception is the quantum space $O^4_v(\iota_3; \sigma_{II})$, where 3-dimensional base is noncommutative, but closed with respect commutation relations. For all permutations and all nilpotent values of contraction parameters the fibers are noncommutative and nonclosed except for $O^4_v(\iota_2; \sigma_{II})$, where both 2-base and 2-fiber are commutative.

The antiinvolution of generators is easily obtained from general expressions (11). For $O^4_v(\iota_1; \sigma_0)$, $O^4_v(\iota_2; \sigma_0)$, $O^4_v(\iota_3; \sigma_0)$ we have $\xi^*_m = \xi_m$, $m = 1, 2, 3$, $\xi^*_4 = \xi_m - i\nu \xi_1$. For $O^4_v(\iota_1; \sigma_{II})$, $O^4_v(\iota_2; \sigma_{II})$ antiinvolution looks very simple $\xi^*_k = \xi_k$, $k = 1, 2, 3, 4$. The most complicate antiinvolution

$$
\xi^*_1 = \xi_1 \cosh j_1 v + i\xi_2 j_1 \sinh j_1 v,
$$

$$
\xi^*_2 = \xi_2 \cosh j_1 v - i\xi_1 \frac{1}{j_1} \sinh j_1 v,
$$

has quantum spaces $O^4_v(\iota_2; \sigma_{II}), O^4_v(\iota_3; \sigma_{II})$.

### 5.2 Quantum deformations of constant curvature spaces $S^3_v(j; \sigma)$

The invariant form of $O^4_v(j; \sigma)$ is given by (11) for $N = 4$

$$
\text{inv}(j; \sigma) = [(1, \sigma_1)^2 \xi^2_{\sigma_1} + (1, \sigma_1)^2 \xi^2_{\sigma_2} + ((1, \sigma_2)^2 \xi^2_{\sigma_2} + (1, \sigma_3)^2 \xi^2_{\sigma_3}) \cosh J v] \cosh J v.
$$

The 3-dimensional quantum orthogonal sphere $S^3_v(j; \sigma)$ is obtained as the quotient of $O^4_v(j; \sigma)$ by $\text{inv}(j; \sigma) = 1$. It is described by the noncommutative sets of right and left space generators $r_k = \xi_{k+1} \xi^{-1}$, $\tilde{r}_k = \xi^{-1} \xi_{k+1}$, $k = 1, 2, 3$. For different permutations $\sigma_0, \sigma_I, \ldots, \sigma_V$ these spheres are

$$
S^3_v(j; \sigma_0) = \left\{ r_1 r_2 = r_2 r_1, \tilde{r}_m r_3 = \left( \tilde{r}_3 \cosh J_0 v - i \frac{1}{J_0} \sinh J_0 v \right) r_m, m = 1, 2, \right\},
$$

where

$$
\tilde{r}_m = (\cosh J_0 v + i\tilde{r}_3 J_0 \sinh J_0 v) r_m, m = 1, 2,
$$

(45)
Commutation relations for nonminimal multipliers are more simple

\[ r_3 - \hat{r}_3 = ij_1^2 \left( \hat{r}_1 r_1 + j_2^2 \hat{r}_2 r_2 \right) \frac{1}{J_0} \sinh J_0 v, \quad J_0 = j_1 j_2 j_3. \]

\[
S^3_v(j; \sigma_I) = \left\{ r_1 r_3 = r_3 r_1, \quad \hat{r}_m r_2 = \left( \hat{r}_2 \cosh J_I v - i \frac{1}{J_I} \sinh J_I v \right) r_m, \quad m = 1, 3 \right\},
\]

where

\[
\hat{r}_m = (\cosh J_I v + i \hat{r}_2 J_I \sinh J_I v) r_m, \quad m = 1, 3,
\]

\[
r_2 - \hat{r}_2 = ij_1^2 \left( \hat{r}_1 r_1 + j_2^2 \hat{r}_2 r_2 \right) \frac{1}{J_I} \sinh J_I v, \quad J_I = j_1 j_2.
\]

\[
S^3_v(j; \sigma_{II}) = \left\{ r_2 r_3 = r_3 r_2, \quad \hat{r}_m r_1 = \left( \hat{r}_1 \cosh J_{II} v - i \frac{1}{J_{II}} \sinh J_{II} v \right) r_m, \quad m = 2, 3 \right\},
\]

where

\[
\hat{r}_m = (\cosh J_{II} v + i \hat{r}_1 J_{II} \sinh J_{II} v) r_m, \quad m = 2, 3,
\]

\[
r_1 - \hat{r}_1 = ij_1^2 j_2^2 \left( \hat{r}_2 r_2 + j_3^2 \hat{r}_3 r_3 \right) \frac{1}{J_{II}} \sinh J_{II} v, \quad J_{II} = j_1.
\]

Commutation relations for nonminimal multipliers are more simple

\[
S^3_v(j; \sigma_{III}) = \left\{ [r_1, r_2] = [r_1, r_3] = 0, \quad [r_2, r_3] = i \left( 1 + j_1^2 \mathbf{r}(j) \right) \frac{1}{J_{III}} \tanh J_{III} v \right\},
\]

\[
S^3_v(j; \sigma_{IV}) = \left\{ [r_1, r_2] = [r_2, r_3] = 0, \quad [r_1, r_3] = i \left( 1 + j_1^2 \mathbf{r}(j) \right) \frac{1}{J_{IV}} \tanh J_{IV} v \right\},
\]

\[
S^3_v(j; \sigma_{V}) = \left\{ [r_1, r_3] = [r_2, r_3] = 0, \quad [r_1, r_2] = i \left( 1 + j_1^2 \mathbf{r}(j) \right) \frac{1}{J_{V}} \tanh J_{V} v \right\},
\]

where \( \mathbf{r}(j) = r_1^2 + j_2^2 r_2^2 + j_3^2 r_3^2 \), \( J_{III} = j_1^2 j_2 j_3 \), \( J_{IV} = j_1^2 j_2 j_3 \), \( J_{V} = j_1^2 j_2 j_3 \).

All quantum orthogonal spheres \( S^3_v(j; \sigma) \) can be divided into two classes relative to their properties under nilpotent values of contraction parameters. These properties depend on transformation of deformation parameter and are different for minimal first order multipliers \( J_0, J_I, J_{II} \) and for non-minimal multipliers \( J_{III}, J_{IV}, J_{V} \). Let us regard these two classes separately.

For minimal multipliers all quantum analogs of 3-dimensional space with zero curvature \( j_1 = \iota_1 \) are isomorphic and can be obtained from

\[
S^3_v(\iota_1; \sigma_0) = \left\{ [r_1, r_2] = 0, \quad [r_3, r_1] = i \nu r_1, \quad [r_3, r_2] = i \nu r_2 \right\}
\]
by permutations of generators \( r_k, \ k = 1, 2, 3 \).

For \( j_2 = \iota_2 \) the commutative space has 1-dimensional base \( \{ r_1 \} \) and 2-dimensional fiber \( \{ r_2, r_3 \} \). Corresponding quantum space

\[
S^3_\nu(\iota_2; \sigma_0) = \left\{ [r_1, r_2] = 0, [r_3, r_m] = ivr_m(1 + j_1^2 r_1^2), m = 1, 2 \right\}
\]

is transformed to \( S^3_\nu(\iota_2; \sigma_I) \) by the substitution \( 2 \to 3 \) and vice versa. Both spaces have noncommutative fiber. New quantum deformation with commutative fiber is given by

\[
S^3_\nu(\iota_2; \sigma_{II}) = \left\{ [r_2, r_3] = 0, [r_1, r_m] = i\nu(1 + j_1^2 r_1^2 + j_2^2 r_2^2), m = 2, 3 \right\}
\]

When \( j_3 = \iota_3 \) there are three nonisomorphic quantum spaces: one with commutative base \( \{ r_1, r_2 \} \)

\[
S^3_\nu(\iota_3; \sigma_0) = \left\{ [r_1, r_2] = 0, [r_3, r_m] = ivr_m(1 + j_1^2 r_1^2 + j_2^2 r_2^2), m = 1, 2 \right\}
\]

and two with noncommutative base: \( S^3_\nu(\iota_3; \sigma_I) \), which has commutation relations \((46)\), where

\[
r_2 - \hat{r}_2 = ij_1^2 \hat{r}_1 r_1 \frac{1}{J_I} \sinh J_I v
\]

and \( S^3_\nu(\iota_3; \sigma_{II}) \), which has commutation relations \((47)\), where

\[
r_1 - \hat{r}_1 = ij_1^2 j_2^2 \hat{r}_2 r_2 \frac{1}{J_{II}} \sinh J_{II} v.
\]

In the case of nonminimal multipliers all quantum deformations of the Euclid space are isomorphic to

\[
S^3_\nu(\iota_1; \sigma_V) = \left\{ [r_1, r_3] = [r_2, r_3] = 0, [r_1, r_2] = iv \right\}
\]

and are the simplest ones.

For \( j_2 = \iota_2 \) two quantum spaces with commutative fiber are isomorphic

\[
S^3_\nu(\iota_2; \sigma_V) = \left\{ [r_1, r_3] = [r_2, r_3] = 0, [r_1, r_2] = iv(1 + j_1^2 r_1^2) \right\} \cong S^3_\nu(\iota_2; \sigma_{IV}),
\]
but the quantum space with noncommutative fiber

\[ S^3_v(\iota_2; \sigma_{III}) = \left\{ [r_1, r_2] = [r_1, r_3] = 0, \ [r_2, r_3] = iv(1 + j_1^2j^2_2) \right\} \quad (55) \]

presents new quantum deformation. In spite of the same commutation relations (54) and (55) the quantum spaces \( S^3_v(\iota_2; \sigma_{V}) \) and \( S^3_v(\iota_2; \sigma_{III}) \) need be regarded as different one because of their different fibering.

For \( j_3 = \iota_3 \) on the contrary two quantum spaces with commutative base are isomorphic

\[ S^3_v(\iota_3; \sigma_{III}) = \left\{ [r_1, r_2] = [r_2, r_3] = 0, \ [r_1, r_3] = iv \left( 1 + j_1^2(r_1^2 + j_2^2r_2^2) \right) \right\} \cong S^3_v(\iota_3; \sigma_{IV}). \quad (56) \]

New quantum deformation with noncommutative base is given by

\[ S^3_v(\iota_3; \sigma_{V}) = \left\{ [r_1, r_2] = i \left( 1 + j_1^2(r_1^2 + j_2^2r_2^2) \right) \frac{1}{j_1^2j_2^2} \tanh j_1^2j_2^2v, \ [r_1, r_3] = [r_2, r_3] = 0 \right\}. \quad (57) \]

Let us stress that deformation parameter remain untouched under this last contraction (57).

Physically quantum spaces (50), (51), (54), (55) with \( j_2 = \iota_2 \) can be interpreted as quantum analogs of the \((1 + 2)\) nonrelativistic Newton kinematics of constant curvature (zero curvature or Galilei kinematics, when \( j_1 = \iota_1 \)).

### 6 Conclusion

The quantum Cayley-Klein spaces of constant curvature \( O^N_v(j; \sigma) \) are uniformly obtained from the quantum Euclidean space \( O^N_q \) in Cartesian coordinates by the standard trick with real, complex, and nilpotent numbers, using a q-analog of Beltrami coordinates. The transformation of the quantum deformation parameter \( Z = Jv \) under contraction is the important ingredient of the noncommutative quantum groups and noncommutative quantum spaces. Unlike previous papers on this subject contraction parameters of the second power are included in the multiplier \( J \), what make all contractions admissible. The different combinations of quantum structure and Cayley-Klein scheme of contractions and analytical continuations are described with the help of permutations \( \sigma \). As a result the quantum orthogonal deformations of
three and four dimensional Caley-Klein vector spaces as well as those of two and three dimensional constant curvature spaces are obtained.

For three dimensional Caley-Klein vector spaces we have found two nonisomorphic quantum deformations (23), (35) of semi-Euclidean space with one dimensional base and two dimensional fiber, as well as two different deformations (24), (30) of semi-Euclidean space with two dimensional base and one dimensional fiber. Four dimensional Caley-Klein vector spaces have two quantum deformations (42) of commutative fibered space with one dimensional base and three dimensional fiber, three nonisomorphic deformations (43) of semi-Euclidean space with two dimensional base and two dimensional fiber, as well as two deformations (44) of fibered space with three dimensional base and one dimensional fiber.

Concerning spaces of constant curvature there are two nonequivalent deformations (27), (38) of Euclidean plane and three deformations (28), (33), (39) of cylinder or Newton plane. In the dimensional three we have found two quantum deformations (49), (53) of Euclidean space, four different deformations (50), (51), (54), (55) of semi-Riemannian space with one dimensional base and two dimensional fiber and five nonisomorphic quantum deformations (46), (47), (52), (50), (57) of semi-Riemannian space with two dimensional base and one dimensional fiber.

This demonstrate a wide variety of the quantum deformations of of the fibered semi-Riemannian spaces. One of their remarkable property is that for some of them commutation relations of generators are proportional to a numbers (38), (53) instead of generators, i.e. the simplest possible deformations are realized. The unique quantum deformation of the rigid algebraic structure of simple Lie groups and Lie algebras [4] is transformed into the spectrum of nonisomorphic deformations of the more flexible contracted structure of non-semisimple Lie groups and associated noncommutative spaces.

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