Toric-friendly groups
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Let $G$ be a connected linear algebraic group over a field $k$. We say that $G$ is toric-friendly if for any field extension $K/k$ and any maximal $K$-torus $T$ in $G$ the group $G(K)$ acts transitively on $(G/T)(K)$. Our main result is a classification of semisimple (and under certain assumptions on $k$, of connected) toric-friendly groups.

Introduction

Let $k$ be a field and $X$ be a homogeneous space of a connected linear algebraic group $G$ defined over $k$. The first question one usually asks about $X$ is whether or not it has a $k$-point. If the answer is “yes”, then one often wants to know whether or not the set $X(k)$ of $k$-points of $X$ forms a single orbit under the group $G(k)$.

In this paper we shall focus on the case where the geometric stabilizers for the $G$-action on $X$ are maximal tori of $G_k := G \times_k \overline{k}$ (here $\overline{k}$ stands for a fixed algebraic closure of $k$). Such homogeneous spaces arise, in particular, in the study of the adjoint action of a connected reductive group $G$ on its Lie algebra or of the conjugation action of $G$ on itself; see [Colliot-Thélène et al. 2011]. It is shown in Corollary 4.6 of the same reference (see also [Kottwitz 1982, Lemma 2.1]) that every homogeneous space $X$ of this type has a $k$-point, assuming that $G$ is split and $\text{char}(k) = 0$. Therefore it is natural to ask if this point is unique up to translations by $G(k)$.

Definition 0.1. Let $k$ be a field. We say that a connected linear $k$-group $G$ is toric-friendly if for every field extension $K/k$ the following condition is satisfied:

(*) For every maximal $K$-torus $T$ of $G_k := G \times_k K$, the group $G(K)$ has only one orbit in $(G_K/T)(K)$; equivalently, the natural map $\pi : G(K) \to (G_K/T)(K)$ is surjective.
Examining the cohomology exact sequence associated to the $K$-subgroup $T$ of $G_K$ [Serre 1994, I.5.4, Proposition 36], we see that $G$ is toric-friendly if and only if $\ker[H^1(K, T) \to H^1(K, G)] = 1$ for every field extension $K/k$ and every maximal $K$-torus $T$ of $G_K$.

Observe that $G$ is toric-friendly if and only if condition $(\ast)$ of Definition 0.1 is satisfied for all finitely generated extensions $K/k$.

We are interested in classifying toric-friendly groups. In Section 1 we partially reduce this problem to the case where the group is semisimple. The rest of this paper will be devoted to proving the following classification theorem for semisimple toric-friendly groups.

**Main Theorem 0.2.** Let $k$ be a field. A connected semisimple $k$-group $G$ is toric-friendly if and only if $G$ is isomorphic to a direct product $\prod_i R_{F_i/k}G'_i$, where each $F_i$ is a finite separable extension of $k$ and each $G'_i$ is an inner form of $\text{PGL}_{n_i,F_i}$ for some integer $n_i$.

**Notation.** Unless otherwise specified, $k$ will denote an arbitrary field. For any field $K$ we denote by $K_s$ a separable closure of $K$.

By a $k$-group we mean an affine algebraic group scheme over $k$, not necessarily smooth or connected. However, when talking of a reductive or semisimple $k$-group, we implicitly assume smoothness and connectedness.

Let $S$ be a $k$-group. We denote by $H^i(k, S)$ the $i$-th flat cohomology set for $i = 0, 1$ [Waterhouse 1979, 17.6]. If $S$ is abelian, we denote by $H^i(k, S)$ the $i$-th flat cohomology group for $i \geq 0$ [Berhuy et al. 2007, Appendix B]. There are exact sequences for flat cohomology similar to those for Galois cohomology, [Waterhouse 1979, 18.1; Berhuy et al. 2007, Appendix B]. When $S$ is smooth, the flat cohomology $H^i(k, S)$ can be identified with Galois cohomology.

### 1. First reductions

**Lemma 1.1.** Let $1 \to U \to G \to G' \to 1$ be an exact sequence of smooth connected $k$-groups, where $U$ is unipotent. We assume that $U$ is $k$-split, that is, has a composition series over $k$ whose successive quotients are isomorphic to $\mathbb{G}_{a,k}$. Then $G$ is toric-friendly if and only if $G'$ is toric-friendly.

**Proof.** Choose a field extension $K/k$ and a maximal $K$-torus $T \subset G_K$. Set $T' = \varphi(T) \subset G'_K$, then $T'$ is a maximal torus of $G'_K$. The map $\varphi^T : T \to T'$ is an isomorphism, because $T \cap U_K = 1$ (as $U_K$ is unipotent). Conversely, let us start from a maximal torus $T'$ of $G'_K$. The preimage

$$H = \varphi^{-1}(T') \subset G_K$$
of $T'$ is smooth and connected, so any maximal torus $T$ of $H$ maps isomorphically onto $T'$ and therefore it is maximal in $G_K$.

Now we have a commutative diagram

$$
\begin{array}{ccc}
H^1(K, T) & \longrightarrow & H^1(K, G) \\
\varphi^T_* & \downarrow & \varphi_* \\
H^1(K, T') & \longrightarrow & H^1(K, G')
\end{array}
$$

Since $\varphi^T : T \to T'$ is an isomorphism of tori, the left vertical arrow $\varphi^T_*$ is an isomorphism of abelian groups. On the other hand, by [Sansuc 1981, Lemma 1.13], the right vertical arrow $\varphi_*$ is a bijective map. We see that the top horizontal arrow in the diagram is injective if and only if the bottom horizontal arrow is injective, which proves the lemma. □

Let $k$ be a perfect field and $G$ be a connected $k$-group. Recall that over a perfect field the unipotent radical of $G$ makes sense; that is, the “geometric” unipotent radical over an algebraic closure is defined over $k$, by Galois descent. We denote the unipotent radical of $G$ by $R_u(G)$.

**Corollary 1.2.** Let $k$ be a perfect field, $G$ be a connected $k$-group, and $R_u(G)$ be its unipotent radical. Then $G$ is toric-friendly if and only if the associated reductive $k$-group $G/R_u(G)$ is toric-friendly.

**Proof.** Since $k$ is perfect, the smooth connected unipotent $k$-group $R_u(G)$ is $k$-split [Borel 1991, Theorem 15.4], and the corollary follows from Lemma 1.1. □

Let $k$ be a field. We recall that a $k$-group $G$ is called special if $H^1(K, G) = 1$ for every field extension $K/k$. This notion was introduced by J.-P. Serre [1958]. Semisimple special groups over an algebraically closed field were classified by A. Grothendieck [1958]; we shall use his classification later on.

Recall that a $k$-torus $T$ is called quasitrivial, if its character group $\chi(T)$ is a permutation Galois module. Split tori and, more general, quasitrivial tori are special.

**Proposition 1.3.** Let $1 \to C \to G \xrightarrow{\varphi} G' \to 1$ be an exact sequence of $k$-groups, where $G$ and $G'$ are reductive, and $C \subset G$ is central, hence of multiplicative type (not necessarily connected or smooth).

(a) If $G$ is toric-friendly, so is $G'$.

(b) If $C$ is a special $k$-torus, then $G$ is toric-friendly if and only if $G'$ is toric-friendly.

**Proof.** Let $K/k$ be a field extension. The map $T \mapsto T' := \varphi(T)$ is a bijection between the set of maximal $K$-tori $T \subset G_K$ and the set of maximal $K$-tori $T' \subset G'_K$. 

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For such $T$ and $T' = \varphi(T)$ we have commutative diagrams

\[
\begin{array}{ccc}
G_K & \xrightarrow{\varphi} & G'_K \\
\downarrow{\pi} & & \downarrow{\pi'} \\
G_K/T & \xrightarrow{\varphi_*} & G'_K/T'
\end{array}
\quad \begin{array}{ccc}
G(K) & \xrightarrow{\varphi} & G'(K) \\
\downarrow{\pi} & & \downarrow{\pi'} \\
(G_K/T)(K) & \xrightarrow{\varphi_*} & (G'_K/T')(K)
\end{array}
\]

where $\varphi_* : G_K/T \sim G'_K/T'$ is an isomorphism of $K$-varieties, and the induced map on $K$-points $\varphi_* : (G_K/T)(K) \rightarrow (G'_K/T')(K)$ is a bijection. Now, if $G$ is toric-friendly, then the map $\pi : G(K) \rightarrow (G_K/T)(K)$ is surjective, and we see from the right-hand diagram that then the map $\pi' : G'(K) \rightarrow (G'_K/T')(K)$ is surjective as well. This shows that $G'$ is toric-friendly, thus proving (a).

To prove (b), assume that $G'$ is toric-friendly and $C$ is a special $k$-torus. Then the map $\pi' : G'(K) \rightarrow (G'_K/T')(K)$ is surjective (because $G'$ is toric-friendly) and the map $\varphi : G(K) \rightarrow G'(K)$ is surjective (because $C$ is special). We see from the right-hand diagram that the map $\pi : G(K) \rightarrow (G_K/T)(K)$ is surjective as well. Hence $G$ is toric-friendly. \hfill $\square$

We record the following immediate corollary of Proposition 1.3(b).

**Corollary 1.4.** Let $G$ be a reductive $k$-group. Suppose that the radical $R(G)$ is a special $k$-torus (in particular, this condition is satisfied if $R(G)$ is a quasitrivial $k$-torus). Then $G$ is toric-friendly if and only if the semisimple group $G/R(G)$ is toric-friendly. \hfill $\square$

The next result follows from Corollaries 1.2 and 1.4. It partially reduces the problem of classifying toric-friendly groups $G$ to the case where $G$ is semisimple.

**Corollary 1.5.** Let $k$ be a perfect field. Let $G$ be a connected $k$-group containing a split maximal torus. Then $G$ is toric-friendly if and only if the semisimple group $G/R(G)$ is toric-friendly. \hfill $\square$

The following two lemmas will be used to reduce the problem of classifying adjoint semisimple toric-friendly groups $G$ to the case where $G$ is an absolutely simple adjoint $k$-group.

**Lemma 1.6.** A direct product $G = G' \times_k G''$ of connected $k$-groups is toric-friendly if and only if both $G'$ and $G''$ are toric-friendly.

**Proof.** Let $K/k$ be a field extension. Let $T' \subset G'_K$ and $T'' \subset G''_K$ be maximal $K$-tori, then $T := T' \times_K T'' \subset G_K$ is a maximal $K$-torus, and every maximal $K$-torus
in $G_K$ is of this form. The commutative diagram

$$
\begin{array}{ccc}
G(K) & \longrightarrow & G'(K) \times G''(K) \\
\downarrow & & \downarrow \\
(G_K/T)(K) & \longrightarrow & (G'_K/T')(K) \times (G''_K/T'')(K)
\end{array}
$$

shows that every $K$-point of $G_K/T$ lifts to $G$ if and only if every $K$-point of $G'_K/T'$ lifts to $G'$ and every $K$-point of $G''_K/T''$ lifts to $G''$.

**Lemma 1.7.** Let $l/k$ be a finite separable field extension, $G'$ a connected $l$-group, and $G = R_{l/k}G'$. Then $G$ is toric-friendly if and only if $G'$ is toric-friendly.

**Proof.** Let $K/k$ be a field extension. Then $l \otimes_k K = L_1 \times \cdots \times L_r$, where $L_i$ are finite separable extensions of $K$. It follows that $G_K = \prod_i R_{L_i/K}G_{L_i}$. Let $T \subset G_K$ be a maximal $K$-torus, then $T = \prod_i R_{L_i/K}T_i'$, where $T_i'$ is a maximal $L_i$-torus of $G_{L_i}$ for each $i$. We have

$$G(K) = G_K(K) = \left( \prod_i R_{L_i/K}G_{L_i}' \right)(K) = \prod_i G_{L_i}'(L_i) = \prod_i G'(L_i)$$

and similarly $(G_K/T)(K) = \prod_i (G'_{L_i}/T'_i)(L_i)$, yielding a commutative diagram

$$
\begin{array}{ccc}
G(K) & \longrightarrow & \prod_i G'(L_i) \\
\downarrow & & \downarrow \\
(G_K/T)(K) & \longrightarrow & \prod_i (G'_{L_i}/T'_i)(L_i)
\end{array}
$$

If $G'$ is toric-friendly, then the right vertical arrow in the diagram is surjective, hence the left vertical arrow is surjective and $G$ is toric-friendly.

Conversely, assume that $G$ is toric-friendly. Let $l/k$ be a field extension and $T' \subset G'_L$ a maximal $l$-torus. Set $K := l$ and $T := T'$ in the diagram above. Then we can identify $L$ with one of $L_i$ in the decomposition $l \otimes_k K = L_1 \times \cdots \times L_r$, say with $L_1$. In this way we identify $G'_L$ with $G_{L_1}'$ and $G'_L/T'$ with $G'_{L_1}/T'_1$. Since $G$ is toric-friendly, the left vertical arrow in the diagram is surjective, hence the right vertical arrow is also surjective. This means that the map $G'(L_i) \rightarrow (G'_{L_1}/T'_1)(L_i)$ is surjective for each $i$ and in particular, for $i = 1$. Consequently, the map $G'(L) \rightarrow (G'_{L}/T')(L)$ is surjective, and $G'$ is toric-friendly, as desired. \qed

**2. The elementary obstruction**

**2.1.** Let $K$ be a field and $X$ be a smooth geometrically integral $K$-variety. Write $g = \text{Gal}(K_s/K)$, where $K_s$ is a fixed separable closure of $K$. Recall from [Colliot-Thélène and Sansuc 1987, Definition 2.2.1] that the **elementary obstruction** $\text{ob}(X)$...
is the class in \( \text{Ext}^1_g(K_s(X)^*/K_s^*, K_s^*) \) of the extension
\[
1 \to K_s^* \to K_s(X)^* \to K_s(X)^*/K_s^* \to 1.
\]
In particular, \( \text{ob}(X) = 0 \) if and only if this extension of \( g \)-modules splits. If \( X \) has a \( K \)-point, then \( \text{ob}(X) = 0 \) [Colliot-Thélène and Sansuc 1987, Proposition 2.2.2(a)]. Conversely, if \( Y \) is a \( T \)-torsor over \( K \) for some \( K \)-torus \( T \), and \( \text{ob}(Y) = 0 \), then \( Y \) has a \( K \)-point, by Lemma 2.1(iv) of [Borovoi et al. 2008]. However, if \( X \) is an \( H \)-torsor over \( K \) for some simply connected semisimple \( K \)-group \( H \), then \( \text{ob}(X) = 0 \) even when \( X \) has no \( K \)-points; see Lemma 2.2(viii) of that same reference. (The standing assumption in [Borovoi et al. 2008] is that \( \text{char}(K) = 0 \); however, the proofs of Lemmas 2.1(iv) and 2.2(viii) go through in arbitrary characteristic.)

The following key lemma was suggested to us by J.-L. Colliot-Thélène.

**Lemma 2.2.** Let \( K \) be a field, \( T \) be a \( K \)-torus, \( H \) be a simply connected semisimple \( K \)-group, \( X \) be a \( H \)-torsor over \( K \) and \( Y \) be a \( T \)-torsor over \( K \). If \( Y \) has an \( F \)-point over the function field \( F = K(X) \) of \( X \), then \( Y \) has a \( K \)-point.

**Proof.** Since \( H \) is simply connected, \( \text{ob}(X) = 0 \); see Section 2.1 above. Suppose \( Y \) has an \( F \)-point. This means that there exist a \( K \)-rational map \( X \to Y \). By [Wittenberg 2008, Lemma 3.1.2], if we have a \( K \)-rational map \( X \to Y \) between smooth geometrically integral \( K \)-varieties, then \( \text{ob}(X) = 0 \) implies \( \text{ob}(Y) = 0 \). Since \( T \) is a \( K \)-torus, if \( \text{ob}(Y) = 0 \), then \( Y(K) \neq \emptyset \); see Section 2.1 above. Thus in our situation \( Y \) has a \( K \)-point, as claimed. \( \square \)

**Lemma 2.3.** Let \( k \) be a field. Assume we have a commutative diagram of \( k \)-groups

\[
\begin{array}{ccc}
S & \to & T \\
\downarrow & & \downarrow \\
H & \to & G
\end{array}
\]

where \( G \) is a smooth connected \( k \)-group, the vertical map \( T \to G \) is the inclusion of a maximal \( k \)-torus \( T \) into \( G \), and \( H \) is semisimple and simply connected. If there exists a field extension \( K/k \) such that the map
\[
H^1(K, S) \to H^1(K, T)
\]
is nontrivial, then \( G \) is not toric-friendly.

**Proof.** Choose \( K \) and \( s \in H^1(K, S) \) such that the image \( t \in H^1(K, T) \) of \( s \) in \( H^1(K, T) \) is nontrivial. Let \( h \in H^1(K, H) \) be the image of \( s \in H^1(K, S) \) in \( H^1(K, H) \), and let \( g \in H^1(K, G) \) be the image of \( t \) (and of \( h \)) in \( H^1(K, G) \), as
shown in the commutative diagram below:

\[
\begin{array}{ccc}
H^1(K, S) & \longrightarrow & H^1(K, T) \\
\downarrow & & \downarrow \\
H^1(K, H) & \longrightarrow & H^1(K, G)
\end{array}
\]

Let \( X \) be an \( H \)-torsor over \( K \) representing \( h \) and let \( F = K(X) \) be the function field of \( X \). We denote by \( h_F \) the image of \( h \) in \( H^1(F, H) \), and similarly we define \( s_F, t_F, \) and \( g_F \). Clearly \( X \) has an \( F \)-point, hence \( h_F = 1 \) in \( H^1(F, H) \) and therefore \( g_F = 1 \) in \( H^1(F, G) \). On the other hand, by Lemma 2.2, \( t_F \neq 1 \). We conclude that the kernel of the natural map \( H^1(F, T) \to H^1(F, G) \) contains \( t_F \neq 1 \) and hence, is nontrivial. This implies that \( G \) is not toric-friendly. \( \square \)

2.4. Let \( G \) be a reductive \( k \)-group. Let \( G^{ss} \) be the derived group of \( G \) (it is semisimple), and let \( G^{sc} \) be the universal cover of \( G^{ss} \) (it is semisimple and simply connected). Consider the composed homomorphism \( f : G^{sc} \to G \).

Let \( K/k \) be a field extension. There is a canonical bijective correspondence \( T \leftrightarrow T^{sc} \) between the set of maximal \( K \)-tori \( T \subset G_K \) and the set of maximal \( K \)-tori \( T^{sc} \subset G^{sc} \). Starting from a maximal \( K \)-torus \( T \subset G_K \), we define a maximal \( K \)-torus \( T^{sc} := f^{-1}(T) \subset G^{sc}_K \). Conversely, starting from a maximal \( K \)-torus \( T^{sc} \subset G^{sc}_K \), we define a maximal \( K \)-torus \( T := f(T^{sc}) \cdot R(G)_K \subset G_K \), where \( R(G) \) is the radical of \( G \).

Proposition 2.5. Let \( G \) be a reductive \( k \)-group. Let \( G^{sc} \) and \( f : G^{sc} \to G \) be as in Section 2.4 above. Let \( K/k \) be a field extension, \( T \subset G_K \) be a maximal \( K \)-torus of \( G_K \), and set \( T^{sc} = f^{-1}(T) \subset G^{sc}_K \) as above. If the natural map \( H^1(K, T^{sc}) \to H^1(K, T) \) is nontrivial, then \( G \) is not toric-friendly.

Proof. Immediate from Lemma 2.3. \( \square \)

Proposition 2.6. Let \( G \) be a semisimple \( k \)-group, \( f : G^{sc} \to G \) be the universal covering and \( C := \ker(f) \). Then the following conditions are equivalent:

(a) \( G \) is toric-friendly.

(b) The map \( H^1(K, T^{sc}) \to H^1(K, T) \) is trivial (identically zero) for every field extension \( K/k \) and every maximal \( K \)-torus \( T^{sc} \) of \( G^{sc} \). Here \( T := f(T^{sc}) \).

(c) The map \( H^1(K, C) \to H^1(K, T^{sc}) \) is surjective for every field extension \( K/k \) and every maximal \( K \)-torus \( T^{sc} \) of \( G^{sc} \).

(d) The connecting homomorphism \( \partial_T : H^1(K, T) \to H^2(K, C) \) is injective for every field extension \( K/k \) and every maximal \( K \)-torus \( T \) of \( G \).

(e) The natural map \( H^1(K, T) \to H^1(K, G) \) is injective for every field extension \( K/k \) and every maximal \( K \)-torus \( T \) of \( G \).
Proof. (a) ⇒ (b) by Proposition 2.5. Examining the cohomology sequence

\[ H^1(K, C) \rightarrow H^1(K, T^{sc}) \rightarrow H^1(K, T) \rightarrow H^2(K, C) \]

associated to the exact sequence \( 1 \rightarrow C \rightarrow T^{sc} \rightarrow T \rightarrow 1 \) of \( k \)-groups, we see that (b), (c) and (d) are equivalent.

(d) ⇒ (e): The diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & C & \rightarrow & T^{sc} & \rightarrow & T & \rightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & C & \rightarrow & G^{sc} & \rightarrow & G & \rightarrow & 1
\end{array}
\]

of \( K \)-groups induces compatible connecting morphisms

\[
\begin{array}{ccc}
H^1(K, T) & \rightarrow & H^2(K, C) \\
\downarrow & & \downarrow \\
H^1(K, G) & \rightarrow & H^2(K, C)
\end{array}
\]

Suppose \( \alpha, \beta \in H^1(K, T) \) map to the same element in \( H^1(K, G) \). Then the diagram above shows that \( \partial_T(\alpha) = \partial_T(\beta) \) in \( H^2(K, C) \). Part (d) now tells us that \( \alpha = \beta \).

(e) ⇒ (a) is obvious, since (a) is equivalent to the assertion that \( H^1(K, T) \rightarrow H^1(K, G) \) has trivial kernel for every \( K \) and \( T \); see Definition 0.1. □

Corollary 2.7. With the assumptions and notation of Proposition 2.6, if \( G \) is toric-friendly and quasisplit, then

(a) the map \( H^1(K, G^{sc}) \rightarrow H^1(K, G) \) is trivial for every \( K/k \),
(b) the map \( H^1(K, C) \rightarrow H^1(K, G^{sc}) \) is surjective for every \( K/k \),
(c) the connecting map \( \partial_G : H^1(K, G) \rightarrow H^2(K, C) \) has trivial kernel for every \( K/k \).

Proof. Examining the cohomology sequence

\[ H^1(K, C) \rightarrow H^1(K, G^{sc}) \rightarrow H^1(K, G) \rightarrow H^2(K, C) \]

associated to the exact sequence \( 1 \rightarrow C \rightarrow G^{sc} \rightarrow G \rightarrow 1 \), we see that (a), (b) and (c) are equivalent.

To prove (a), recall that since \( G_K \) is quasisplit, by a theorem of Steinberg [1965, Theorem 1.8] every \( x^{sc} \in H^1(K, G^{sc}) \) lies in the image of the map \( H^1(K, T^{sc}) \rightarrow \)
$H^1(K, G^{sc})$ for some maximal $K$-torus $T^{sc}$ of $G^{sc}$. Since $G$ is toric-friendly, by Proposition 2.6 the map $H^1(K, T^{sc}) \to H^1(K, T)$ is trivial. The commutative diagram

$$
\begin{array}{ccc}
H^1(K, T^{sc}) & \longrightarrow & H^1(K, T) \\
\downarrow & & \downarrow \\
H^1(K, G^{sc}) & \longrightarrow & H^1(K, G)
\end{array}
$$

now shows that the image of $x^{sc}$ in $H^1(K, G)$ is 1. Thus the map $H^1(K, G^{sc}) \to H^1(K, G)$ is trivial. □

**Theorem 2.8.** Let $G$ be a split semisimple $k$-group and $f : G^{sc} \to G$ be its universal covering map. If $G$ is toric-friendly, then $G^{sc}$ is special.

**Proof.** Let $T^{sc}$ be a split maximal torus of $G^{sc}$. Recall that $T^{sc}$ is special (as is any split torus). Set $C = \ker f$, then $C \subset T^{sc}$. For any field extension $K/k$, the map $H^1(K, C) \to H^1(K, G^{sc})$ factors through $H^1(K, T^{sc}) = 1$ and hence is trivial. By Corollary 2.7(b) this map is also surjective. This shows that $H^1(K, G^{sc}) = 1$ for every $K/k$, that is, $G^{sc}$ is special. □

**Remark 2.9.** Our proof of Theorem 2.8 goes through for any (not necessarily split) semisimple $k$-group $G$, as long as $G^{sc}$ contains a special maximal $k$-torus $T^{sc}$. In particular, Theorem 2.8 remains valid for any quasisplit semisimple $k$-group $G$, in view of Lemma 2.10 below. This lemma is a special case of [Colliot-Thélène et al. 2004, Lemma 5.6]; however, for the sake of completeness we supply a short self-contained proof.

**Lemma 2.10.** Let $G$ be a semisimple, simply connected, quasisplit $k$-group over a field $k$. Let $B \subset G$ be a Borel subgroup defined over $k$, and let $T \subset B \subset G$ be a maximal $k$-torus of $G$ contained in $B$. Then $T$ is a quasitrivial $k$-torus.

**Proof.** We write $\bar{k}$ for a fixed algebraic closure of $k$. Let $X^\vee(T)$ denote the group of cocharacters of $T$. Let $R^\vee = R^\vee(G_{\bar{k}}, T_{\bar{k}}) \subset X^\vee(T)$ denote the coroot system of $G_{\bar{k}}$ with respect to $T_{\bar{k}}$, and let $\Pi^\vee \subset R^\vee$ denote the basis of $R^\vee$ corresponding to $B$. The Galois group $\text{Gal}(k_s/k)$ acts on $X^\vee(T)$. Since $T$, $G$, and $B$ are defined over $k$, the subsets $R^\vee$ and $\Pi^\vee$ of $X^\vee(T)$ are invariant under this action. Since $G$ is simply connected, $\Pi^\vee$ is a $\mathbb{Z}$-basis of $X^\vee(T)$. Thus $\text{Gal}(k_s/k)$ permutes the $\mathbb{Z}$-basis $\Pi^\vee$ of $X^\vee(T)$; in other words, $T$ is a quasitrivial torus. □

**Remark 2.11.** A similar assertion for adjoint quasisplit groups was proved by G. Prasad [1989, Proof of Lemma 2.0].

3. **Examples in type $A$**

Let $k$ be a field and $A$ a central simple $k$-algebra of dimension $n^2$. We write $GL_{1,A}$ for the $k$-group with $GL_{1,A}(R) = (A \otimes_k R)^*$ for any unital commutative $k$-algebra.
Let $K$ be a field. Recall that an $n$-dimensional commutative étale $K$-algebra is a finite product $E = \prod_i L_i$, where each $L_i$ is a finite separable field extension of $K$ and $\sum_i [L_i : K] = n$. For such $E = \prod_i L_i$ we define a $K$-torus $R_{E/K G_m, E} := \prod_i R_{L_i/K G_m, L_i}$, then $(R_{E/K G_m, E})(K) = E^*$. Clearly the $K$-torus $R_{E/K G_m, E}$ is quasitrivial.

**Proposition 3.1.** Let $k$ be a field, and let $A/k$ be a central simple $k$-algebra of dimension $n^2$. 

(a) The $k$-group $G = \text{GL}_{1,A}$ is toric-friendly. 

(b) The $k$-group $\text{PGL}_{1,A} := \text{GL}_{1,A} / \mathbb{G}_m,k$ is toric-friendly. 

(c) In particular, $\text{GL}_{n,k}$ and $\text{PGL}_{n,k}$ are toric-friendly.

**Proof.** (a) Let $K/k$ be a field extension and let $T \subset G_K = \text{GL}_{1,A \otimes k K}$ be a maximal $K$-torus. Let $E$ be the centralizer of $T$ in $A \otimes k K$. An easy calculation over a separable closure $K_s$ of $K$ shows that $E$ is an $n$-dimensional commutative étale $K$-subalgebra of $A \otimes k K$ and that $T = R_{E/K G_m, E}$. It follows that $T$ is quasitrivial, hence special. Since all maximal $K$-tori $T \subset G_K$ are special, $G$ is toric-friendly.

(b) follows from (a) and Corollary 1.4. To deduce (c) from (a) and (b), set $A = M_n(k)$ (the matrix algebra). □

We now come to the main result of this section, which asserts that a toric-friendly semisimple groups of type $A$ is necessarily an adjoint group.

**Proposition 3.2.** Let $k$ be a field. Consider a $k$-group $G = (\text{SL}_{n_1} \times \cdots \times \text{SL}_{n_r}) / C$, where $C \subset \mu := \mu_{n_1} \times \cdots \times \mu_{n_r}$ is a central subgroup of $G^{\text{sc}} = \text{SL}_{n_1} \times \cdots \times \text{SL}_{n_r}$, not necessarily smooth. If $C \neq \mu$, then $G$ is not toric-friendly.

Before proceeding with the proof, we fix some notation. Let $L/K$ be a finite separable field extension of degree $n$. Set

$$R^1_{L/K}(\mathbb{G}_m) := \ker[ N_{L/K} : R_{L/K \mathbb{G}_m, L} \rightarrow \mathbb{G}_m,K ],$$

where $N_{L/K}$ is the norm map. Clearly $R^1_{L/K}(\mathbb{G}_m)$ can be embedded into $\text{SL}_{n,K}$ as a maximal $K$-torus. The embedding $K \hookrightarrow L$ induces an embedding $\mu_{n,K} \hookrightarrow R^1_{L/K \mathbb{G}_m}$, where $n = [L : K]$.

The following two lemmas are undoubtedly known. We include short proofs below because we have not been able to find appropriate references.
Lemma 3.3. There is a commutative diagram

\[ \begin{array}{c}
K^* / K^{*n} \xrightarrow{\cong} H^1(K, \mu_n) \\
\downarrow \\
K^* / N_{L/K}(L^*) \xrightarrow{\cong} H^1(K, R_{L/K}^1 \mathbb{G}_m) 
\end{array} \] (1)

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the embedding \( \mu_n \hookrightarrow R_{L/K}^1 \mathbb{G}_m \), and the left vertical arrow is the natural projection.

Proof. Apply the flat cohomology functor to the commutative diagram of commutative \( K \)-groups

\[ \begin{array}{c}
1 \rightarrow \mu_n, K \rightarrow \mathbb{G}_{m, K} \xrightarrow{n} \mathbb{G}_{m, K} \rightarrow 1 \\
\downarrow \\
1 \rightarrow R_{L/K}^1 \mathbb{G}_m \rightarrow R_{L/K}^1 \mathbb{G}_m \xrightarrow{N_{L/K}} \mathbb{G}_{m, K} \rightarrow 1 
\end{array} \]

and use Hilbert’s Theorem 90. \( \square \)

Lemma 3.4. Suppose \( r \mid n \). Then there is a commutative diagram

\[ \begin{array}{c}
K^* / K^{*n} \xrightarrow{\cong} H^1(K, \mu_n) \\
\downarrow^{(n/r)_*} \\
K^* / K^{*r} \xrightarrow{\cong} H^1(K, \mu_r) ,
\end{array} \]

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the homomorphism \( \mu_n \xrightarrow{n/r} \mu_r \) given by \( x \mapsto x^{n/r} \), and the left vertical arrow is the natural projection.

Proof. Similar to that of Lemma 3.3, using the commutative diagram

\[ \begin{array}{c}
1 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \\
\downarrow^{n/r} \downarrow^{n/r} \downarrow \text{id} \\
1 \rightarrow \mu_r \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \\
\end{array} \]

Example 3.5. The group \( G = \text{SL}_{n,k} \) (\( n \geq 2 \)) is not toric-friendly.

Proof. Since \( \text{SL}_n \) is special, it suffices to construct an extension \( K/k \) and a maximal \( K \)-torus \( T := R_{L/K}^1 \mathbb{G}_m \) such that \( H^1(K, T) \neq 1 \). In view of Lemma 3.3 it suffices to show that \( N_{L/K}(L^*) \neq K^* \) for some field extension \( K/k \) and some finite
separable field extension \( L/K \) of degree \( n \). This is well known; see for example the proof of [Rowen 1980, Proposition 3.1.46]. We include a short proof below as a way of motivating a related but more complicated argument at the end of the proof of Proposition 3.2.

Let \( L := k(x_1, \ldots, x_n) \), where \( x_1, \ldots, x_n \) are independent variables, and \( K := L^{\Gamma} \), where \( \Gamma \) is the cyclic group of order \( n \) that acts on \( L \) by cyclically permuting \( x_1, \ldots, x_n \). For \( 0 \neq a \in k[x_1, \ldots, x_n] \), let \( \deg(a) \in \mathbb{N} \) denote the degree of \( a \) as a polynomial in \( x_1, \ldots, x_n \). If \( a \in k(x_1, \ldots, x_n) \) is of the form \( a = b/c \) with nonzero \( b, c \in k[x_1, \ldots, x_n] \), then we define \( \deg(a) = \deg(b) - \deg(c) \). This yields the usual degree homomorphism \( \deg : L^* \to \mathbb{Z} \). Since \( N_{L/K}(a) = \prod_{\gamma \in \Gamma} \gamma(a) \), we see that \( \deg(N_{L/K}(a)) = n \deg(a) \) is divisible by \( n \), for every \( a \in L^* \). On the other hand, \( s_1 = x_1 + \cdots + x_n \in K \) has degree 1. This shows that \( N_{L/K}(L^*) \neq L^* \), as claimed.

\[ \square \]

**3.6. Proof of Proposition 3.2.** Let \( K/k \) be a field extension. For each \( i = 1, \ldots, r \), let \( L_i \) be a separable field extension of degree \( n_i \) over \( K \), and let \( T = T_1 \times \cdots \times T_r \) be a maximal \( K \)-torus of \( G^\times \), where \( T_i := R_{L_i/K}(G_m) \). By Proposition 2.6 it suffices to show that the composition

\[
H^1(K, C) \to H^1(K, \mu) \to H^1(K, T)
\]

is not surjective for some choice of extensions \( K/k \) and \( L_i/K_i \). Since \( C \subseteq \mu \), there exist a prime \( p \) and a nontrivial character \( \chi : \mu \to \mu_p \) such that \( \chi(C) = 1 \). By Proposition 1.3(a) we may assume that \( C = \ker(\chi) \). For notational simplicity, let us suppose that \( n_1, \ldots, n_s \) are divisible by \( p \) and \( n_{s+1}, \ldots, n_r \) are not, for some \( 0 \leq s \leq r \). Then it is easy to see that \( \chi \) is of the form

\[
\chi(c_1, \ldots, c_r) = c_1^{d_1 n_1/p} \cdots c_s^{d_s n_s/p}
\]

for some integers \( d_1, \ldots, d_s \). Since \( \chi \) is nontrivial on \( \mu \), we have \( s \geq 1 \) and \( d_i \) is not divisible by \( p \) for some \( i = 1, \ldots, s \), say for \( i = 1 \). That is, we may assume that \( d_1 \) is not divisible by \( p \).

Lemma 3.3 gives a concrete description of the second map in (2). To determine the image of the map \( H^1(K, C) \to H^1(K, \mu) \), we examine the cohomology exact sequence

\[
\begin{array}{c}
H^1(K, C) \xrightarrow{} H^1(K, \mu) \xrightarrow{\chi_*} H^1(K, \mu_p) \\
\prod_{i=1}^r K^*/K^{*n_i} \xrightarrow{\chi_*} K/K^{*p}
\end{array}
\]

induced by the exact sequence \( 1 \to C \to \mu \xrightarrow{\chi} \mu_p \to 1 \). The image of \( H^1(K, C) \) in \( H^1(K, \mu) \) is the kernel of \( \chi_* \). By Lemma 3.4, \( \chi_* \) maps the class of \((a_1, \ldots, a_r)\)
in $H^1(K, \mu) = \prod_{i=1}^r K^*/K^{s_i}$ to the class of $a_1^{d_1} \cdots a_s^{d_s}$ in $H^1(K, \mu_p) = K/K^p$.

In other words, the image of $H^1(K, C)$ in $H^1(K, \mu)$ is the subgroup of classes of $r$-tuples $(a_1, \ldots, a_r)$ in $H^1(K, \mu) = \prod_{i=1}^r K^*/K^{s_i}$ such that $a_1^{d_1} \cdots a_s^{d_s} \in K^p$.

Hence, the image of $H^1(K, C)$ in $H^1(K, T) = \prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ consists of classes of $r$-tuples $(a_1, \ldots, a_r)$ such that $a_1^{d_1} \cdots a_s^{d_s} \in K^p$.

It remains to construct a field extension $K/k$, separable field extensions $L_i/K$ of degree $n_i$ for $i = 1, \ldots, r$, and an element $\alpha \in H^1(K, T) = \prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$, which cannot be represented by $(a_1, \ldots, a_r) \in (K^*)^r$ such that $a_1^{d_1} \cdots a_s^{d_s} \in K^p$.

This will show that the map $H^1(K, C) \to H^1(K, T)$ is not surjective, as claimed.

Set $L := k(x_1, \ldots, x_n)$, where $n = n_1 + \cdots + n_r$ and $x_1, \ldots, x_n$ are independent variables. The symmetric group $S_n$ acts on $L$ by permuting these variables; we embed $S_{n_1} \times \cdots \times S_{n_r}$ into $S_n$ in the natural way, by letting $S_{n_1}$ permute the first $n_1$ variables, $S_{n_2}$ permute the next $n_2$ variables, etc. Set $K := L^{S_{n_1} \times \cdots \times S_{n_r}}$, $s_1 := x_1 + \cdots + x_n \in K$ and

$$L_1 := K(x_1), \; L_2 := K(x_{n_1+1}), \ldots \; L_r := K(x_{n_1+\cdots+n_{r-1}+1}).$$

Clearly $[L_i : K] = n_i$. We claim the class of $(s_1, 1, \ldots, 1)$ in $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ cannot be represented by any $(a_1, \ldots, a_r) \in (K^*)^r$ with $a_1^{d_1} \cdots a_s^{d_s} \in K^p$.

Let $\deg : L^* \to \mathbb{Z}$ be the degree map, as in Example 3.5. Arguing as we did there, we see that $\deg(N_{L_i/K}(a))$ is divisible by $n_i$ for every $i = 1, \ldots, r$ and every $a \in L_i^*$. In particular, $(a_1, \ldots, a_r) \mapsto \deg(a_i) + n_i \mathbb{Z}$ is a well-defined function $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*) \to \mathbb{Z}/n_i \mathbb{Z}$, and consequently,

$$f(a_1, \ldots, a_n) := d_1 \deg(a_1) + \cdots + d_s \deg(a_s) + p \mathbb{Z}$$

is a well-defined function $H^1(K, T) \to \mathbb{Z}/p \mathbb{Z}$. We have

$$f(a_1, \ldots, a_n) = \deg(a_1^{d_1} \cdots a_s^{d_s}).$$

If $a_1^{d_1} \cdots a_s^{d_s} \in K^p$, then $f(a_1, \ldots, a_r) = 0$ in $\mathbb{Z}/p \mathbb{Z}$. On the other hand, since $\deg(1) = 0$, $\deg(s_1) = 1$ and $d_1$ is not divisible by $p$, we conclude that $f(s_1, 1, \ldots, 1)$ is nonzero in $\mathbb{Z}/p \mathbb{Z}$. This proves the claim and the proposition.

4. Groups of type $C_n$ and outer forms of $A_n$

**Proposition 4.1.** No absolutely simple $k$-group of type $C_n$ ($n \geq 2$) is toric-friendly.

**Proof.** Clearly we may assume that $k$ is algebraically closed. We may also assume that $G$ is adjoint, see Proposition 1.3(a). We see that $G = \text{PSp}_{2n}$ and $G^{sc} = \text{Sp}_{2n}$. By Example 3.5, $\text{SL}_2$ is not toric-friendly. This means that there exist a field extension $K/k$, a maximal $K$-torus $S \subset \text{SL}_{2, K}$, and a cohomology class $a_S \in H^1(K, S)$ such
that $a_\mathcal{S} \neq 1$. We consider the standard embedding

$$ (\text{SL}_2)^n = (\text{Sp}_2)^n \hookrightarrow \text{Sp}_{2n}, \quad n \geq 2. $$

Set $T^{sc} = S^n \subset (\text{Sp}_2)^n \subset \text{Sp}_{2n} = G^{sc}$. Let $\iota : S \hookrightarrow T^{sc} = S^n$ be the embedding as the first factor. Set $a^{sc} = \iota_*(a_S) \in H^1(K, T^{sc})$. Let $T$ be the image of $T^{sc}$ in $G = \text{PSp}_{2n}$, and let $a$ be the image of $a^{sc}$ in $H^1(K, T)$.

Now observe that the homomorphism

$$ \chi : T^{sc} = S^n \rightarrow S, \quad (x_1, \ldots, x_n) \mapsto x_1x_2^{-1}, $$

factors through $T$ (recall that $n \geq 2$). Since $\chi \circ \iota = \text{id}_S$, we see that $a \neq 1$. On the other hand, the image of $a^{sc}$ in $H^1(K, G^{sc})$ is 1 (because $G^{sc} = \text{Sp}_{2n}$ is special), hence $a \in \ker[H^1(K, T) \rightarrow H^1(K, G)]$, and we see that $G = \text{PSp}_{2n}$ is not toric-friendly. \hfill $\square$

**Proposition 4.2.** No absolutely simple $k$-group of outer type $A_n$ ($n \geq 2$) is toric-friendly.

**Lemma 4.3.** Let $k$ be a field, $K/k$ a separable quadratic extension, and $D/K$ a central division algebra of dimension $r^2$ over $K$ with an involution $\sigma$ of the second kind (i.e., $\sigma$ acts nontrivially on $K$ and trivially on $k$). Then there exists a finite separable field extension $F/k$ such that $K_F := K \otimes_k F$ is a field and $D \otimes_K K_F$ is split, that is, $K_F$-isomorphic to the matrix algebra $M_r(K_F)$.

**Proof of the lemma.** Since there are no nontrivial central division algebras over finite fields, we may assume that $k$ and $K$ are infinite. Let

$$ H = \{ x \in D \mid x^\sigma = x \} $$

denote the $k$-space of Hermitian elements of $D$. Consider the embedding $D \hookrightarrow M_r(K_s)$ induced by an isomorphism $D \otimes_K K_s \cong M_r(K_s)$, where $K_s$ is a separable closure of $K$. An element $x$ of $D$ is called semisimple regular if its image in $D \otimes_K K_s \cong M_r(K_s)$ is a semisimple matrix with $r$ distinct eigenvalues. A standard argument using an isomorphism $D \otimes_k K_s \cong M_r(K_s) \times M_r(K_s)$ shows that there is a dense open subvariety $H_{\text{reg}}$ in the space $H$, consisting of semisimple regular elements. Clearly $H_{\text{reg}}$ is defined over $k$ and contains $k$-points.

Let $x \in H_{\text{reg}}(k) \subset D$ be a semisimple regular Hermitian element. Let $L$ be the centralizer of $x$ in $D$. Since $x$ is Hermitian ($\sigma$-invariant), the $k$-algebra $L$ is $\sigma$-invariant. Since $x$ is semisimple and regular, the algebra $L$ is a commutative étale $K$-subalgebra of $D$ of dimension $r$ over $K$, as is easily seen by passing to $K_s$. Clearly $L$ is a field, $[L : K] = r$, and $L$ is separable over $k$. Since $L \subset D$ and $[L : K] = r$, the field $L$ is a splitting field for $D$; see, for example, [Pierce 1982, Corollary 13.3].
Since \( L \supset K \), we see that \( \sigma \) acts nontrivially on \( L \). Let \( F = L^{(\sigma)} \) denote the subfield of \( L \) consisting of elements fixed by \( \sigma \). Then \([L : F] = 2\) and \([F : k] = r\). Clearly \( F \) is separable over \( k \). Since \( F \cap K = k \) and \( FK = L \), we conclude that \( L = K \otimes_k F := K_F \). This completes the proof of the lemma.

4.4. Proof of Proposition 4.2. By Proposition 1.3(a) we may assume that \( G \) is adjoint. By Lemma 4.3 there is a finite separable field extension \( F/k \) such that \( G_F \cong \text{PSU}(L^{n+1}, h) \), where \( L/F \) is a separable quadratic extension and \( h \) is a Hermitian form on \( L^{n+1} \). It suffices to prove that \( G_F = \text{PSU}(L^{n+1}, h) \) is not toric-friendly.

Set \( S = \mathbb{R}^1_{L/K} \mathbb{G}_m \). We set \( G_F^{sc} = \text{SU}(L^{n+1}, h) \). We may assume that \( h \) is a diagonal form [Knus 1991, Proposition 6.2.4(1); Scharlau 1985, Theorem 7.6.3]. Consider the diagonal torus \( S^{n+1} \subset \text{U}(L^{n+1}, h) \) and set \( T^{sc} = S^{n+1} \cap \text{SU}(L^{n+1}, h) \).

We claim that there exists a field extension \( K/F \) such that \( H^1(K, S) \neq 1 \). Indeed, take \( K = F((t)) \), the field of formal Laurent series over \( F \). Then by [Serre 1968, Proposition V.2.3(c)], \( H^1(K, S) \cong H^1(F, S) \times \mathbb{Z}/2\mathbb{Z} \neq 1 \).

Now let \( a_S \in H^1(K, S) \), \( a_S \neq 1 \), and consider the embedding

\[ \iota : S \hookrightarrow T^{sc} \subset S^{n+1}, \quad x \mapsto (x, x^{-1}, 1, \ldots, 1). \]

Set \( a^{sc} = \iota_*(a_S) \in H^1(K, T^{sc}) \). Let \( T \) be the image of \( T^{sc} \) in \( G_F = \text{PSU}(L^{n+1}, h) \) and \( a \) be the image of \( a^{sc} \) in \( H^1(K, T) \).

Note that the homomorphism

\[ \chi : T^{sc} \rightarrow S, \quad (x_1, \ldots, x_n, x_{n+1}) \mapsto x_1x_3^{-1}, \]

factors through \( T \) (recall that \( n \geq 2 \)). Since \( \chi \circ \iota = \text{id}_S \), we see that \( a \neq 1 \). Now by Proposition 2.5, \( G_F \) and hence \( G \) are not toric-friendly.

5. Classification of semisimple toric-friendly groups

Lemma 5.1. Let \( k \) be an algebraically closed field. If a semisimple \( k \)-group \( G \) is toric-friendly, then it is adjoint of type \( A \), that is, \( G \cong \prod_i \text{PGL}_{n_i} \) for some integers \( n_i \geq 2 \).

Proof. First assume that \( G \) is simple. By Theorem 2.8 the simply connected cover \( G^{sc} \) of \( G \) is special. By a theorem of Grothendieck [1958, Theorem 3], \( G^{sc} \) is special if and only if \( G \) is of type \( A_n, n \geq 1 \) or \( C_n, n \geq 2 \). Proposition 4.1 rules out the second possibility. Thus \( G \) is of type \( A \).

Now let \( G \) be semisimple. By Proposition 1.3(a), \( G^{ad} \) is toric-friendly. Write \( G^{ad} = \prod_i G_i \), where each \( G_i \) is an adjoint simple group, then by Lemma 1.6 each \( G_i \) is toric-friendly. As we have seen, this implies that each \( G_i \) is of type \( A \), that is, isomorphic to \( \text{PGL}_{n_i} \) for some \( n_i \). By Proposition 3.2, \( G \) is adjoint, that is, \( G = G^{ad} = \prod_i \text{PGL}_{n_i} \).
5.2. **Proof of the Main Theorem 0.2.** If $G$ is toric-friendly, then clearly $G_\mathbb{K}$ is toric-friendly, where $\mathbb{K}$ is an algebraic closure of $k$. By Lemma 5.1, $G$ is adjoint of type $A$. Write $G = \prod_i R_{F_i/k} G'_i$, where each $F_i/k$ is a finite separable extension and $G'_i$ is a form of $\text{PGL}_{n_i,F_i}$. By Lemmas 1.6 and 1.7, each $G'_i$ is toric-friendly, and by Proposition 4.2, $G'_i$ is an inner form of $\text{PGL}_{n_i,F_i}$.

Conversely, by Proposition 3.1 an inner form $G'_i$ of $\text{PGL}_{n_i,F_i}$ is toric-friendly. By Lemmas 1.6 and 1.7, the product $G = \prod_i R_{F_i/k} G'_i$ is toric-friendly. □

**Corollary 5.3.** Let $G$ be a nontrivial semisimple $k$-group. Then there exist a field extension $K/k$ and a maximal $K$-torus $T \subset G$ that is not special. Equivalently, there exist a field extension $K/k$ and a maximal $K$-torus $T$ of $G$ such that $H^1(K, T) \neq 1$.

*Proof.* Assume the contrary, that is, that for any field extension $K/k$, any maximal $K$-torus $T \subset G_K$ is special. We may and shall assume that $G$ is split. Recall that for a (quasi)split group, by [Steinberg 1965, Theorem 11.1], every element of $H^1(K, G)$ lies in the image of the map $H^1(K, T) \to H^1(K, G)$ for some maximal $K$-torus $T$ of $G$. Thus, under our assumption we have $H^1(K, G) = 1$ for every field extension $K/k$, that is, $G$ is special. By [Grothendieck 1958, Theorem 3], this is only possible if $G$ is simply connected and has components only of types $A$ and $C$. On the other hand, $G$ is clearly toric-friendly (see Definition 0.1), and by the Main Theorem 0.2 no nontrivial simply connected semisimple group can be toric-friendly, a contradiction. □

The next result follows immediately from the Main Theorem 0.2 and Corollary 1.4.

**Corollary 5.4.** Let $G$ be a split reductive $k$-group. The group $G$ is toric-friendly if and only if it satisfies these two conditions:

(a) the center $Z(G)$ of $G$ is a $k$-torus, and

(b) the adjoint group $G^\text{ad} := G/Z(G)$ is a direct product of simple adjoint groups of type $A$. □

Note that in condition (a) we allow the trivial $k$-torus $\{1\}$.

By Corollary 1.4 if $G$ is a reductive $k$-group such that $G/R(G)$ is toric-friendly and $R(G)$ is special, then $G$ is toric-friendly. The example below shows that when $G/R(G)$ is toric-friendly but $R(G)$ is not special, $G$ need not be toric-friendly.

**Example 5.5.** Let $k = \mathbb{R}$, $G = \mathbb{U}_2$, the unitary group in two complex variables. Then $Z(G)$ is the group of scalar matrices in $G$, it is connected, hence $R(G) = Z(G)$ and $G/R(G) = G^\text{ad} = \text{PSU}_2$. Since $\text{PSU}_2$ is an inner form of $\text{PGL}_{2,\mathbb{R}}$, by the Main Theorem 0.2 it is toric-friendly. However, the group $G = \mathbb{U}_2$ is not toric-friendly. This does not contradict Corollary 1.4, because $R(G) = Z(G)$ is not special: $H^1(\mathbb{R}, Z(G)) = \mathbb{R}^*/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$. 


To show that $G = U_2$ is not toric-friendly, set $S = R^1_{C/R} G_m$. Let $T$ be the diagonal maximal $R$-torus of $U_2$. Set $G^{sc} = SU_2$, $T^{sc} = T \cap SU_2$, then $T^{sc} \cong S$.

Let $a^{sc} \in H^1(R, T^{sc})$ be the cohomology class of the cocycle given by the element $-1 \in T^{sc}(R)$ of order 2. Let $a \in H^1(R, T)$ be the image of $a^{sc}$ in $H^1(R, T)$. Clearly $a \neq 1$. By Proposition 2.5, $G$ is not toric-friendly.

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