Ramanujan circulant graphs and the conjecture of Hardy-Littlewood and Bateman-Horn

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December 10, 2013

Abstract

In this paper, we determine the bound of the valency of the odd circulant graphs which guarantees to be Ramanujan for each fixed number of vertices. In almost of the cases, the bound coincides with the trivial bound, which comes from the trivial estimate of the largest non-trivial eigenvalue of the circulant graph. As exceptional cases, the bound in fact exceeds the trivial one by two. We then prove that such exceptionals occur only in the cases where the number of vertices has at most two prime factors and is represented by a quadratic polynomial in a finite family and, moreover, under the conjecture of Hardy-Littlewood and Bateman-Horn, exist infinitely many.

2010 Mathematics Subject Classification: Primary 11M41, Secondary 05C25, 05C75, 11N32.

Key words and phrases: Ramanujan graph, circulant graph, Hardy-Littlewood and Bateman-Horn conjecture, prime number, almost prime number.

1 Introduction

Let X be a regular graph with standard assumptions, that is, finite, undirected, connected and simple. Spectral analysis on X is an important topic in several interest of mathematics, such as combinatorics, group theory, differential geometry, and number theory. Especially, the topics around Ramanujan and expander graphs are focused; these are related each other and have common interest in the second eigenvalue (or the spectral gap) of the adjacency operator on X (cf. [HLW, Lu]).

The notion of Ramanujan graph was defined in [LPS]: The graph X is called Ramanujan if its largest non-trivial eigenvalue (in the sense of absolute value) is not greater than the Ramanujan bound $2\sqrt{k - 1}$, where $k$ is the valency (or degree) of X. In view of the theory of zeta functions, the Ramanujan property means that the associated Ihara zeta function satisfies the “Riemann hypothesis”. Here the Ihara zeta functions are regarded as a graph analogue of the Selberg zeta functions on locally symmetric spaces. Similarly to the case of the usual prime number (or geodesic) theorem, one has a good estimate for the number of the prime cycles in X if it is Ramanujan (cf. [T2]). Therefore, for a given graph, we want to examine whether it is Ramanujan or not in easy way.

As one can be seen from the estimation of the isoperimetric constant, the Ramanujan graphs are very much connected in some sense. The complete graph $K_m$ with $m$-vertices which is the densest graph with the eigenvalues \{ $m - 1, -1, \ldots, -1$ \} is in fact Ramanujan. Also, some

∗Partially supported by Grant-in-Aid for Scientific Research (C) No. 24540022.
†Partially supported by Grant-in-Aid for Young Scientists (B) No. 24740018.
neighbors of $K_m$ are expected to be Ramanujan (cf. [AR]). Now, we want to estimate the precise boundary of the number of removable edges from the complete graph preserving the Ramanujan property.

We formulate our problem in the general setting. Let $G$ be the set of all (isomorphic classes of) graphs with standard assumptions and $G_{m,k}$ the subset of $G$ consisting of the graphs with $m$-vertices and $k$-valency. Similarly, let $R$ and $R_{m,k}$ be the set of all Ramanujan graphs in $G$ and $G_{m,k}$, respectively. For a given subset $X$ of $G$, we put $X_{m,k} = X \cap G_{m,k}$ and decide the set

$$\Gamma_m = \{ k \in V_m \mid X_{m,k} \subset R_{m,k} \}$$

for each $m$, where $V_m = \{ k \mid X_{m,k} \neq \emptyset \}$. If $K_m$ can be realized in $X$, we have $m - 1 \in \Gamma_m$ because $G_{m,m-1} = R_{m,m-1} = \{K_m\}$.

In this paper, we take as $X$ the easiest family of the Cayley graphs, that is, the set of odd circulant graphs (i.e., the Cayley graphs of cyclic groups $\mathbb{Z}_m$ of odd order $m$), and try to decide

$$\hat{l}_m = \max \{ l \in \mathbb{N} \mid m - l \in V_m, [m - l, m - 1] \cap V_m \subset \Gamma_m \}, \quad m \in 2\mathbb{N} + 1.$$

Here $\hat{l}_m$ means the maximum number of edge-removal preserving the Ramanujan property from the odd complete graph $K_m = \text{Cay}(\mathbb{Z}_m, \mathbb{Z}_m \setminus \{0\}) \in X$. Our main result is the following theorem which says our problem associates a classical problem in analytic number theory.

**Theorem 1.1.** Let $X$ be the family of circulant graphs of odd order. Then, for $m \geq 15$, we have

$$\hat{l}_m = l_{0,m} + \varepsilon_m,$$

where $l_{0,m} = 2\lfloor \sqrt{m - \frac{3}{2}} \rfloor + 1$ and $\varepsilon_m \in \{0, 2\}$. Here $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$. Moreover, the case $\varepsilon_m = 2$ occurs only if $m$ is represented by one of the quadratic polynomials $k^2 + 5k + c$ for some $c \in \{\pm 1, \pm 3, \pm 5\}$ and is either a prime or a product of two distinct primes $p, q$ with $p < q < 4p$.

We notice that $l_{0,m}$ comes from a trivial estimate of the largest non-trivial eigenvalue. We also remark that the above condition is not sufficient; if $q$ is very close to $4p$, then one can in fact observe that $\varepsilon_m = 0$ even if $m = pq$ can be represented by one of the above quadratic polynomials (cf. §4). Let us call $m$ ordinary (resp. exceptional) if $\varepsilon_m = 0$ (resp. $\varepsilon_m = 2$).

Our result suggests that the existence of infinitely many exceptions for our $X$ is related to the well-known conjecture of Hardy-Littlewood [HL] and Bateman-Horn [BH] on primes represented by polynomials, or to the Iwaniec’s important result [I] (see also the recent result [La]) on the almost-primes represented by a quadratic polynomial.

We also consider the case of odd abelian family in the final section. For the case of even circulant family, we need a slightly different formulation coming from the symmetricity of generating sets for the groups. We treat this case in another paper [K]. Moreover, we will discuss the case of the simplest non-abelian family, that is, the dihedral family, in forthcoming paper [HKY].

2 Preliminaries

2.1 Cayley graphs and their eigenvalues

Let $X$ be a $k$-regular graph with $m$-vertices which is finite, undirected, connected, and simple. The adjacency matrix $A_X$ of $X$ is the symmetric matrix of size $m$ whose entry is 1 if the
corresponding pair of vertices are connected by an edge and 0 otherwise. We call the eigenvalues of $A_X$ the eigenvalues of $X$. The set $\Lambda(X)$ of all eigenvalues of $X$ is given as

$$\Lambda(X) = \{ \lambda_i \mid k = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{m-1} \geq -k \}.$$  

Remark that $-k \in \Lambda(X)$ if and only if $X$ is bipartite (cf. [DSV]). Let $\mu(X)$ be the largest non-trivial eigenvalue of $X$ in the sense of absolute value, that is,

$$\mu(X) = \max\{ |\lambda| \mid \lambda \in \Lambda(X), |\lambda| \neq k \}.$$  

Then, $X$ is called Ramanujan if the inequality $\mu(X) \leq 2\sqrt{k-1}$ holds. Here the constant $2\sqrt{k-1}$ in the right hand side of this inequality is often called the Ramanujan bound for $X$ and is denoted by $RB(X)$.

Let $G$ be a finite group with the identity element $e$ and $S$ a Cayley subset of $G$, that is, a symmetric set of generators for $G$ satisfying $e \notin S$. Then, the Cayley graph $X(G, S)$ is the $|S|$-regular graph with vertex set $G$ and the edge set $\{(x, y) \in G \times G \mid x^{-1}y \in S\}$, which is undirected, connected, and simple. The adjacency matrix of $X(G, S)$ is described in terms of the right regular representation of $G$ (cf. [TT]). In particular, if $G$ is a finite abelian group, then we have

$$\Lambda(X(G, S)) = \left\{ \lambda_{\chi} = \sum_{s \in S} \chi(s) \mid \chi \in \hat{G} \right\}.$$  

Here $\hat{G}$ is the dual group of $G$.

### 2.2 A problem for Ramanujan circulants

Fix $m$ a positive integer and let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order $m$. Moreover, put $S$ the set of all Cayley subsets of $\mathbb{Z}_m$. We call a Cayley graph $X(S) = X(\mathbb{Z}_m, S)$ with $S \in S$ a circulant graph of order $m$. Since the dual group of $\mathbb{Z}_m$ consists of the characters $\chi_j(a) = e^{2\pi j a/m}$ ($0 \leq j \leq m - 1$), the set of all eigenvalues of $X(S)$ is given by

$$\Lambda(X(S)) = \{ \mu_j(S) \mid 0 \leq j \leq m - 1 \},$$

where $\mu_0(S) = |S|$ and

$$\mu_j(S) = \sum_{a \in S} e^{2\pi j a/m} = -\sum_{b \in \mathbb{Z}_m \setminus S} e^{2\pi jb/m}, \quad 1 \leq j \leq m - 1. \tag{2.1}$$

Besides the valency $|S|$ of a circulant graph $X(S)$, we call $l(S) = |\mathbb{Z}_m \setminus S| = m - |S|$ the covalency of $X(S)$. Now we divide the set $S$ of Cayley subsets of $\mathbb{Z}_m$ by the covalency as

$$S = \bigsqcup_{l \in \mathcal{L}} S_l, \quad S_l = \{ S \in S \mid l(S) = l \}.$$  

Here $\mathcal{L} = \{ l(S) \mid S \in S \}$ is the set of values of covalency. For example, we have $\mathbb{Z}_m \setminus \{0\} \in S_1$ and $\{ \pm 1 \} \in S_{m-2}$ for which the attached graphs are the complete graph $K_m = X(\mathbb{Z}_m \setminus \{0\})$ and the cycle graph $C_m = X(\{ \pm 1 \})$ of order $m$, respectively. These Cayley subsets give the non-trivial eigenvalues

$$\mu_j(\mathbb{Z}_m \setminus \{0\}) = -1, \quad \mu_j(\{ \pm 1 \}) = 2\cos \frac{2\pi j}{m}, \quad 1 \leq j \leq m - 1.$$
Moreover, if we put

\begin{equation}
S^{(l)} = \mathbb{Z}_m \setminus \{0, \pm 1, \pm 2, \ldots, \pm \frac{l-1}{2}\}
\end{equation}

for an odd integer \( l \) with \( 1 \leq l \leq m - 2 \), then one sees that \( S^{(l)} \) is an element of \( S_l \) with the non-trivial eigenvalues

\begin{equation}
\mu_j(S^{(l)}) = -\sum_{b=-\frac{l-1}{2}}^{\frac{l-1}{2}} e^{\frac{2\pi jb}{m}} = -\frac{\sin \frac{\pi jl}{m}}{\sin \frac{\pi}{m}}, \quad 1 \leq j \leq m - 1.
\end{equation}

The Cayley subset \( S^{(l)} \) often appears in our discussion.

From the definition, the circulant graph \( X(S) \) is Ramanujan if and only if \( \mu(S) \leq \text{RB}(S) \) where \( \mu(S) = \mu(X(S)) \) and \( \text{RB}(S) = \text{RB}(X(S)) \). Observe that the Ramanujan bound \( \text{RB}(S) = 2\sqrt{m - l - 1} \) is depend only on the covalency \( l = l(S) \) of \( S \in S_l \). Moreover, we remark that \( S_1 = \{\mathbb{Z}_m \setminus \{0\}\} \) and \( X(\mathbb{Z}_m \setminus \{0\}) \) is a Ramanujan circulant because \( \mu(\mathbb{Z}_m \setminus \{0\}) = | - 1 | \leq 2\sqrt{m - 2} = \text{RB}(\mathbb{Z}_m \setminus \{0\}) \). These observations naturally lead us to evaluate the bound

\[ \hat{l} = \max \left\{ l \in \mathcal{L} \mid X(S) \text{ is Ramanujan for all } S \in \bigsqcup_{1 \leq k \leq l} S_k \right\}, \]

which means the maximal number of edge-removal from the complete graph \( K_m \) preserving the Ramanujan property. In particular, \( \hat{l} = m - 2 \) is equivalent to say that \( X(S) \) is Ramanujan for all \( S \in S \).

In this paper, we treat only the case of odd \( m \). Then, each \( S \in S \) has even number of elements because of symmetry, and hence \( \mathcal{L} = \{1, 3, \ldots, m - 2\} \) consists of odd integers. Moreover, \( -|S| \) does not appear in \( \Lambda(X(S)) \) because \( X(S) \) has odd vertices and thus is not bipartite. (It is known that \( X(S) \) is bipartite if and only if \( m \) is even and all the elements of \( S \) are odd. See, e.g., [He].) Therefore, we have \( \mu(S) = \max \{|\mu_j(S)| \mid 1 \leq j \leq m - 1\} \).

### 3 Initial results

The following lemma says that, on the determination of \( \hat{l} \), we may assume that \( m \geq 15 \).

**Lemma 3.1.** \( \hat{l} = m - 2 \) if and only if \( 3 \leq m \leq 13 \).

**Proof.** Remark that the cycle graph \( C_m = X(\{\pm 1\}) \) is Ramanujan, whence \( X(S) \) is whenever \( |S| = 2 \). Therefore, to prove the “only if” part, it suffices to show that there exists \( S \in S_{m-4} \) such that \( X(S) \) is not Ramanujan for \( m \geq 15 \). Actually, let \( S = \{\pm \frac{m-1}{2}, \pm \frac{m-3}{2}\} \in S_{m-4} \). From (2.1), we have \( |\mu_1(S)| = 4 \cos \frac{\pi}{m} \cos \frac{2\pi}{m} \), which is monotonic increasing. Hence, for \( m \geq 15 \),

\[ \mu(S) \geq |\mu_1(S)| = 4 \cos \frac{\pi}{m} \cos \frac{2\pi}{m} \geq 4 \cos \frac{\pi}{15} \cos \frac{2\pi}{15} = 3.57 \ldots > 2\sqrt{3} = \text{RB}(S). \]

The converse is direct. \( \square \)
3.1 Trivial bound

We first show that there exists a lower bound of $\hat{l}$.

**Lemma 3.2.** We have $\hat{l} \geq l_0$, where

$$l_0 = 2\lfloor \sqrt{m} - \frac{3}{2} \rfloor + 1.$$ 

Here $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$.

**Proof.** From (2.1), for any $S \in S_l$ with $1 \leq l < m/2$, we have $|\mu_j(S)| \leq \min\{ \sum_{a \in S} 1, \sum_{b \in \mathbb{Z}_m \setminus S} 1 \} \leq \min\{|S|, l(S)| = l \}$ for all $1 \leq j \leq m-1$ and hence $\mu(S) \leq l$. Therefore, if $l \leq \text{RB}(S) = 2\sqrt{m} - l - 1$, equivalently $l \leq 2(\sqrt{m} - 1)$, then $X(S)$ is Ramanujan. Now, the claim follows because $l_0$ coincides with the maximum odd integer satisfying $l < m/2$ and $l \leq 2(\sqrt{m} - 1)$.

We call $l_0$ the *trivial bound* of $\hat{l}$. Note that, since $x - 1 < \lfloor x \rfloor \leq x - 1$, we have

$$2(\sqrt{m} - 2) < l_0 \leq 2(\sqrt{m} - 1),$$

and hence $l_0 \sim 2\sqrt{m}$ as $m \to +\infty$. See the table below for the explicit value of $l_0$ and $\hat{l}$ for small $m$ (remark that $\hat{l} = m - 2$ for $3 \leq m \leq 13$ from Lemma 3.1).

| $m$  | 3   | 5   | 7   | 9   | 11  | 13  | 15  | 17  | 19  | 21  | 23  | 25  | 27  | 29  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $l_0$ | 1   | 3   | 5   | 5   | 5   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   | 7   |
| $\hat{l}$ | 1   | 3   | 5   | 7   | 9   | 11  | 7   | 7   | 7   | 7   | 9   | 9   | 7   | 9   |
| $m$  | 31  | 33  | 35  | 37  | 39  | 41  | 43  | 45  | 47  | 49  | 51  | 53  | 55  |
| $l_0$ | 9   | 9   | 9   | 9   | 9   | 11  | 11  | 11  | 11  | 11  | 11  | 11  | 11  |
| $\hat{l}$ | 9   | 9   | 11  | 11  | 11  | 11  | 11  | 11  | 11  | 13  | 13  | 13  | 13  |

Table 1: $l_0$ and $\hat{l}$ for small $m$.

3.2 Beyond the trivial bound

As you find from Table 1, we can indeed prove the following theorem.

**Theorem 3.3.** There exists $\varepsilon \in \{0, 2\}$ such that $\hat{l} = l_0 + \varepsilon$ for $m \geq 15$.

To prove the theorem, it is sufficient to show that there exists $S \in S_{l_0+4}$ such that $X(S)$ is not Ramanujan. Actually, for large $m$, we claim that $X(S^{(l_0+4)})$ is not Ramanujan where $S^{(l)}$ is defined in (2.2). More strongly, we show the following

**Lemma 3.4.** $X(S^{(l_0+2h)})$ is not Ramanujan if $m \geq 39$ and $2 \leq h \leq \lfloor \frac{1}{4}(\sqrt{m} - 2)^2 \rfloor$.

**Proof.** Assume that $\lfloor \frac{1}{4}(\sqrt{m} - 2)^2 \rfloor \geq 2$. Using (3.1), we have $2(\sqrt{m} - 2 - h) < l_0 + 2h < \frac{m}{2}$. Then the expression (2.3) together with the inequality above leads us to the evaluation

$$|\mu_1(S^{(l_0+2h)})| - \text{RB}(S^{(l_0+2h)}) > \frac{2\pi(\sqrt{m} - (2-h))}{\sin \frac{m}{2}} - 2(\sqrt{m} - 1)$$

$$= 2(h - 1) - \frac{4\pi^2}{3} \frac{1}{\sqrt{m}} + O(m^{-1}).$$
This shows that \( \mu(S^{(l_0+2h)}) \geq |\mu_1(S^{(l_0+2h)})| > RB(S^{(l_0+2h)}) \) for \( m \gg 0 \). In fact, one can check that the right hand side of (3.2) is positive whenever \( m \geq 39 \).

**Proof of Theorem 3.3.** From Lemma 3.4 we know that \( X(S^{(l_0+4)}) \) is not Ramanujan for \( m \geq 39 \). Moreover, one can see that the situations for \( 15 \leq m \leq 37 \) are the same as above by checking \( |\mu_1(S^{(l_0+4)})| > RB(S^{(l_0+4)}) \) individually.

We remark that the above discussion does not work for the case \( h = 1 \), that is, \( l = l_0 + 2 \).

### 3.3 A criterion for ordinary \( m \)

From Theorem 3.3, our task is to determine the number \( \varepsilon \in \{0, 2\} \) satisfying \( \hat{l} = l_0 + \varepsilon \) for a given \( m \). Let us call \( m \) ordinary if \( \varepsilon = 0 \) and exceptional otherwise. This is based on the numerical fact that there are much more \( m \) of the former type rather than the latter. The aim of this subsection is to give a criterion for ordinary \( m \).

Let \( k \in \mathbb{Z}_{>0} \) and put \( I_k = \{ x \in \mathbb{R} | \lfloor \sqrt{x} - \frac{3}{2} \rfloor = k \} = [k^2 + 3k + \frac{9}{4}, k^2 + 5k + \frac{25}{4}) \). We now study an interpolation function \( d(x) \) for the difference between \( |\mu_1(S^{(l_0+2)})| \) and \( RB(S^{(l_0+2)}) \) on \( m \in I_k \cap (2\mathbb{Z} + 1) \), that is,

\[
d(x) = \frac{\sin \frac{\pi(2k+3)}{x}}{\sin \frac{\pi}{x}} - 2\sqrt{x - 2k - 4}, \quad x \in I_k.
\]

Notice that \( d(m) > 0 \) for \( m \in I_k \cap (2\mathbb{Z} + 1) \) implies that \( X(S^{(l_0+2)}) \) is not Ramanujan and hence \( m \) is ordinary. Therefore, we are interested in the sign of the values of \( d(x) \) on \( I_k \cap (2\mathbb{Z} + 1) \). The following lemma is crucial in our study.

**Lemma 3.5.** Let \( m \in I_k \cap (2\mathbb{Z} + 1) \).

1. \( d(m) < 0 \) for all \( m \in I_k \cap (2\mathbb{Z} + 1) \) when \( k = 1, 2, 3 \).
2. \( d(m) < 0 \) if and only if \( m \in [k^2 + 5k - c, k^2 + 5k + 5] \) with

\[
c = \begin{cases} 
3 & 4 \leq k \leq 18, \\
5 & k \geq 19.
\end{cases}
\]

**Proof.** The assertions for \( k \leq 8 \) are direct. Let \( k \geq 9 \). We first claim that \( d(x) \) is monotone decreasing on \( I_k \). Actually, using the inequalities \( x - \frac{4^3}{6} < \sin x < x \) and \( 1 - \frac{4^4}{25x^3} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} \), we have

\[
d'(x) = -\frac{1}{\sqrt{x - 2k - 4}} + \frac{\pi}{x^2(\sin \frac{\pi}{x})^2}
\left(-2(2k + 3) \cos \frac{\pi(2k+3)}{x} \sin \frac{\pi}{x} + \sin \frac{\pi(2k+3)}{x} \cos \frac{\pi}{x}\right)
\]

\[
< -\frac{1}{\sqrt{x}} + \frac{\pi^2(2k+3)(3(2k+3)^2 - 2)}{6x^3(1 - \frac{4}{15}(\frac{\pi}{x})^2)^2}
\left(1 - \frac{\pi^2(2k+3)^2 - 1}{4x^2(3(2k+3)^2 - 2)}\right)
\]

\[
< -\frac{1}{\sqrt{x}} + \frac{\pi^2(2k+3)^3}{2x^3(1 - \frac{4}{15}(\frac{\pi}{x})^2)^2}.
\]

Here we have clearly \( 2k + 3 < 2(k + 3) \), \( x > \pi \) and \( k(k + 3) < x < (k + 3)^2 \) for \( x \in I_k \). Therefore,

\[
d'(x) < -\frac{1}{\sqrt{x}} + \frac{144\pi^2(k+3)^3}{25x^3} < -\frac{1}{k+3} + \frac{144\pi^2}{25k^3} < 0
\]
for \( k \geq 9 \). This shows the assertion. Next we investigate the value \( D(k) = D(k, c) = d(k^2 + 5k + c) \) where \( c \leq 6 \) is an integer not depending on \( k \). It is easy to see that

\[
D(k) = \frac{3c' - 16\pi^2}{12} k^{-1} + O(k^{-2}),
\]

where \( c' = 25 - 4c \). This shows that \( D(k) < 0 \) for \( k \gg 0 \) if the leading coefficient is negative, that is, \( c > \frac{75 - 16\pi^2}{12} = -6.90 \ldots \). Actually, for \( k \geq 49 \), one can see that \( D(k, -7) > 0 \) and \( D(k, -6) < 0 \), whence, together with the monotoneness of \( d(x) \), we obtain the desired claim for \( k \geq 49 \). The rest of assertions, that is, for \( 9 \leq k \leq 48 \), are also checked individually. This completes the proof because \( k^2 + 5k + c \) is odd if and only if \( c \) is.

From this lemma, one can obtain a criterion for ordinary \( m \).

**Theorem 3.6.** Let \( m \geq 15 \) be an odd integer. Put

\[
J = \{ 2n + 1 \mid 7 \leq n \leq 14 \} \cup \bigcup_{c \in \{ \pm 1, \pm 3, \pm 5 \}} J_c,
\]

where

\[
J_c = \begin{cases} 
\{ k^2 + 5k + c \mid k \geq 4 \} & c \in \{ \pm 1, \pm 3, 5 \}, \\
\{ k^2 + 5k - 5 \mid k \geq 19 \} & c = -5.
\end{cases}
\]

Then, \( m \) is ordinary if \( m \notin J \).

**Proof.** Suppose that \( m \) is not in \( J \). Then, from Lemma 3.5, one sees that \( d(m) > 0 \), in other words, \( |\mu_1(S^{(l_0+2)})| > RB(S^{(l_0+2)}) \). This shows that \( m \) is ordinary.

Theorem 3.6 implies that from now on we may concentrate only on \( m \) with \( m \in J \) and leads us to imagine that the quadratic polynomials

\[
f_c(k) = k^2 + 5k + c, \quad c \in \{ \pm 1, \pm 3, \pm 5 \},
\]

play important roles in our study. We remark that the constant

\[
c' = 25 - 4c > 0,
\]

which was in the proof of Lemma 3.5 is nothing but the discriminant of \( f_c(k) \) and will often appear in several arguments.

### 4 Spectral consideration

In the subsequent discussion, we only consider the case where \( m \in J \), that is, \( m \) can be written as \( m = f_c(k) = k^2 + 5k + c \) for some \( k \in \mathbb{Z}_{\geq 0} \) and \( c \in \{ \pm 1, \pm 3, \pm 5 \} \). For such \( m \), we clarify when exceptionals occur. Hence, from now on, we concentrate on the circulant graphs \( X(S) \) with \( S \in S_{l_0+2} \). In this section, we use the notations \( RB = 2\sqrt{m - (l_0 + 2) - 1} \) and

\[
\hat{\mu} = \max_{S \in S_{l_0+2}} \mu(S),
\]

for simplicity. From the definition, \( m \) is exceptional if and only if \( \hat{\mu} \leq RB \). Therefore, we have to decide \( \hat{\mu} \) for a given \( m \).
4.1 A necessary condition for exceptionals

The aim of this subsection is to obtain the following necessary condition for exceptionals, which we can relatively easily reach the conclusion.

Proposition 4.1. Let $m \geq 15$ be an odd integer. If $m$ is exceptional, then $m \in J$ which is in either of the following three types;

(I) $m = p$ is an odd prime.

(II) $m = pq$ is a product of two odd primes $p$ and $q$ satisfying $p < q < 4p$.

(III) $m = 25, 49$.

Proof. From Theorem 3.6, it is enough to consider only the case where $m \in J$.

Assume that $m$ is a composite. One can easily see that there are finitely many $m \in J$ such that $m = p^2$, that is, $m = 25, 49$. It is directly checked that these are all exceptional. For the other cases, let $p$ be the minimum prime factor of $m$ and write $m = pt$ with $3 \leq p < t$. If one can take $S \in S_{l_0 + 2}$ as $\mathbb{Z}_m \setminus S \subset \{0, \pm p, \pm 2p, \ldots, \pm \frac{t-1}{2}p\}$, then $m$ is ordinary because

$$|\hat{\mu}(S)| = \left| \sum_{b \in \mathbb{Z}_m \setminus S} e^{4\pi ib/m} \right| = l_0 + 2 \geq RB$$

from the definition of $l_0$. Such $S$ can be in fact taken if and only if $l_0 + 2 = \#(\mathbb{Z}_m \setminus S) \leq \#\{0, \pm p, \pm 2p, \ldots, \pm \frac{t-1}{2}p\} = t$, that is, $2\sqrt{pt} - \frac{3}{2} + 3 \leq t$, equivalently $t \geq 4p - 3$. Therefore, if $t$ is either composite or odd prime with $t \geq 4p - 3$, then $m$ is ordinary. This shows the claim.

Remark 4.2. As we have stated in the above proof, the necessary condition $p < q < 4p$ in (II) can be actually reduced to $p < q \leq 4p - 5$. We also remark that there are finitely many $m \in J$ of the form of both $m = p(4p - 1)$ and $m = p(4p - 3)$; $m = 33$ and $m = 27, 85, 451$, respectively. These are of course ordinary.

4.2 Exceptionals of type (I)

It is easy to see that $m \in J$ of type (I) is actually exceptional.

Theorem 4.3. Every odd prime $m = p \in J$ are exceptional.

Proof. Let $m = p \in J$ be a prime. Then, one can easily see that $\hat{\mu} = |\mu_1(S^{(l_0+2)})|$ because the map $\mathbb{Z}_m \to \mathbb{Z}_m$ defined by $x \mapsto jx$ is bijective for all $1 \leq j \leq m - 1$. Hence, from Lemma 3.5, $m$ is exceptional if and only if $m \in J$.

4.3 Exceptionals of type (II)

In this subsection, we assume that $m = f_c(k) \in J$ is of type (II). Namely, there exists odd distinct primes $p$ and $q$ with $p < q < 4p$ such that $m = pq$. From Proposition 4.1, our task is clear up whether or not such $m$ is in fact exceptional. We at first show that one can narrow down the candidates of $\hat{\mu}$ as follows.
Lemma 4.4. We have $\hat{\mu} = \max\{\mu(0), \mu(1), \mu(2)\}$ where

\begin{align}
\mu(0) &= \sin \frac{\pi (l_0 + 2)}{pq} \sin \frac{2\pi}{p}, \\
\mu(1) &= q + (l_0 - 2 - q) \cos \frac{2\pi}{p}, \\
\mu(2) &= \begin{cases}
p + (l_0 + 2 - p) \cos \frac{2\pi}{q} \\
p + 2p \cos \frac{2\pi}{q} + (l_0 + 2 - 3p) \cos \frac{4\pi}{q}
\end{cases} \quad (p < q < \frac{(3p + 2)^2}{4p}).
\end{align}

Proof. From the definition, we have

$$\hat{\mu} = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq j \leq pq - 1} |\mu_j(S)| \right\} = \max\{\mu(0), \mu(1), \mu(2)\},$$

where

$$\mu(0) = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq j \leq pq - 1} |\mu_j(S)| \right\},$$

$$\mu(1) = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq j \leq p - 1} |\mu_j(S)| \right\},$$

$$\mu(2) = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq j \leq p - 1} |\mu_j(S)| \right\}.$$

Hence, it is enough to show that $\mu(0), \mu(1), \mu(2)$ are equal to (4.1), (4.2), (4.3), respectively. The expression (4.1), that is, $\mu(0) = |\mu_1(S_{l_0+2})|$, can be seen similarly as in the proof of Theorem 4.1. Therefore, it suffices to consider the other two cases.

For $\mu(1)$, we have

$$\mu(1) = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq l \leq pq - 1} \sum_{t = 0}^{p-1} \# \{ b \in \mathbb{Z}_{pq} \mid b \not\in S, b \equiv t \pmod{p} \} e^{\frac{2\pi i t s}{pq}} \right\}.$$

Now, we introduce the notation

$$T_h(a, b) = \begin{cases}
\{0, \pm a, \pm 2a, \ldots, \pm \frac{b-1}{2}a\} & (h = 0), \\
\{\pm h, \pm a \pm h, \pm 2a \pm h, \ldots, \pm \frac{b-1}{2}a \pm h\} & (h \geq 1)
\end{cases}$$

for odd $a, b \in \mathbb{Z}_{>0}$. Note that $\#T_h(a, b) = b$ if $h = 0$ and $2b$ otherwise. We may assume that $p < q \leq 4p - 5$ (see Remark 4.2). This condition implies that we can not take $S \in \mathcal{S}_{l_0+2}$ as $\mathbb{Z}_{pq} \setminus S \subset T_0(p, q)$ but can as $\mathbb{Z}_{pq} \setminus S \subset T_0(p, q) \cup T_1(p, q)$. This together with (4.1) shows the expression (4.2).

Similarly, we have

$$\mu(2) = \max_{S \in \mathcal{S}_{l_0+2}} \left\{ \max_{1 \leq l \leq q - 1} \sum_{t = 0}^{q-1} \# \{ b \in \mathbb{Z}_{pq} \mid b \not\in S, b \equiv t \pmod{q} \} e^{\frac{2\pi i t s}{pq}} \right\}.$$

The condition $p < q$ implies that we can not take $S \in \mathcal{S}_{l_0+2}$ as $\mathbb{Z}_{pq} \setminus S \subset T_0(q, p)$. However, if $l_0 + 2 \leq 3p$, that is, $2\sqrt{pq} - \frac{3}{2} + 3 \leq 3p$, equivalently $(p, q) = (3, 5), (3, 7)$ or $p < q < \frac{(3p + 2)^2}{4p}$ if $p \geq 5$, then we can take $S \in \mathcal{S}_{l_0+2}$ as $\mathbb{Z}_{pq} \setminus S \subset T_0(q, p) \cup T_1(q, p)$ and hence, together with (4.2), $\mu(2) = p + (l_0 + 2 - p) \cos \frac{2\pi}{q}$. If $\frac{(3p + 2)^2}{4p} \leq q < 4p$, then we can take $S \in \mathcal{S}_{l_0+2}$ as $\mathbb{Z}_{pq} \setminus S \subset T_0(q, p) \cup T_1(q, p)$ but can as $\mathbb{Z}_{pq} \setminus S \subset T_0(q, p) \cup T_1(q, p) \cup T_2(q, p)$, whence $\mu(2) = p + 2p \cos \frac{2\pi}{q} + (l_0 + 2 - 3p) \cos \frac{4\pi}{q}$. These show the expression (4.3). \qed
We next analytically evaluate the difference between $\mu^{(i)}$ and $\text{RB}$ on $J \cap I_k$ for each $i \in \{0, 1, 2\}$. Before that, we notice that when $m = f_c(k) = pq \in J \cap I_k$ is of type (II), we have $l_0 + 2 = 2k + 3$. Moreover, if we put $x = \sqrt{\frac{2}{p}}$, then $1 < x < 2$ and $p = \sqrt{f_c(k)}$ and $q = \sqrt{f_c(k)x}$. Based on these facts, we study the following functions

$$D_c^{(i)}(k; x) = M_c^{(i)}(k; x) - A, \quad k \in \mathbb{Z}_{>0}, \ 1 < x < 2,$$

where $M_c^{(i)}(k; x)$ is defined by

$$M_c^{(0)}(k; x) = \frac{\sin \frac{\pi C}{B}}{\sin \frac{\pi x}{B}},$$
$$M_c^{(1)}(k; x) = Bx + \left( C - Bx \right) \cos \frac{2\pi x}{B},$$
$$M_c^{(2)}(k; x) = \begin{cases} 
\frac{B}{x} + \left( C - Bx \right) \cos \frac{2\pi x}{Bx} & (1 < x < \frac{3}{4}), \\
\frac{B}{x} + \frac{2B}{x} \cos \frac{2\pi x}{Bx} + \left( C - \frac{3B}{x} \right) \cos \frac{4\pi x}{Bx} & (\frac{3}{4} < x < 2),
\end{cases}$$

with

$$A = 2\sqrt{f_c(k)} - (l_0 + 2) - 1 = 2\sqrt{k^2 + 3k + c - 4},$$
$$B = \sqrt{f_c(k)} = \sqrt{k^2 + 5k + c},$$
$$C = l_0 + 2 = 2k + 3.$$

**Lemma 4.5.** For a fixed $x$, we have

\begin{align*}
(4.6) \quad & M_c^{(0)}(k; x) = 2k + 3 - \frac{4\pi^2}{3} k^{-1} + O(k^{-2}), \\
(4.7) \quad & M_c^{(1)}(k; x) = 2k + 3 - 2\pi x^2 (2 - x) k^{-1} + O(k^{-2}), \\
(4.8) \quad & M_c^{(2)}(k; x) = \begin{cases} 
2k + 3 - \frac{2\pi^2 x^2 (2x - 1)}{x^3} k^{-1} + O(k^{-2}) & (1 < x < \frac{3}{4}), \\
2k + 3 - \frac{4\pi^2 x^2 (4x - 5)}{x^3} k^{-1} + O(k^{-2}) & (\frac{3}{4} < x < 2),
\end{cases}
\end{align*}

and

\begin{align*}
(4.9) \quad & A = 2k + 3 - \frac{c'}{4} k^{-1} + O(k^{-2})
\end{align*}

as $k \to \infty$.

**Proof.** These are direct. \qed

We first show that one does not have to take account of both $D_c^{(0)}$ and $D_c^{(2)}$ in our discussion.

**Lemma 4.6.** We have $D_c^{(0)}(k; x) < 0$ and $D_c^{(2)}(k; x) < 0$ on $1 < x < 2$ for any $c \in \{\pm 1, \pm 3, \pm 5\}$ and $k \geq k_0$ with a sufficiently large $k_0 \in \mathbb{N}$.

**Proof.** Notice that $D_c^{(0)}(k; x) = D(k)$ where $D(k) = D(k, c)$ is defined in the proof of Lemma 3.6. Therefore, as we have seen in the lemma, $D_c^{(0)}(k; x) = D(k) < 0$ because $c \in \{\pm 1, \pm 3, \pm 5\}$ if $k$ is sufficiently large.
Similarly, from (4.8) and (4.9), we have
\[
D_c^{(2)}(k; x) = \begin{cases} 
\frac{c' x^3 - 8\pi^2 (2x - 1)}{4x^3} k^{-1} + O(k^{-2}) & (1 < x < \frac{3}{2}), \\
\frac{c' x^3 - 16\pi^2 (4x - 5)}{4x^3} k^{-1} + O(k^{-2}) & (\frac{3}{2} < x < 2).
\end{cases}
\]
Hence the result follows from the fact that the coefficient of \(k^{-1}\) is negative for any \(1 < x < 2\).

For the function \(D_c^{(1)}\), we have the asymptotic expansion
\[
D_c^{(1)}(k; x) = \frac{c' - 8\pi^2 x(2 - x)}{4} k^{-1} + O(k^{-2})
\]
from (4.7) and (4.9), and thus \(D_c^{(1)}\) becomes negative only if \(x\) is close to 2, in other words, \(q\) is close to \(4p\). More precisely, we obtain the following

**Lemma 4.7.** There exists constants \(x_1\) and \(x_2\) with \(1 < x_1 < x_2 < 2\) such that

1. \(D_c^{(1)}(k; x) < 0\) on \(1 < x < x_1\),
2. \(D_c^{(1)}(k; x) > 0\) on \(x_2 < x < 2\),

for any \(c \in \{\pm1, \pm3, \pm5\}\) and \(k \geq k_0\) with a sufficiently large \(k_0 \in \mathbb{N}\).

**Proof.** We write \(D_c^{(1)}(k; x) = B(1 - \cos \frac{2\pi x}{B})(x - X(x; k, c))\) with
\[
X(x; k, c) = \frac{C}{B} - \frac{C - A}{B(1 - \cos \frac{2\pi x}{B})}.
\]
Since \(B(1 - \cos \frac{2\pi x}{B}) > 0\), \(D_c^{(1)}(k; x) < 0\) if and only if \(x < X(x; k, c)\). Hence, noticing that \(C > A\) and \(\cos \frac{2\pi x}{B}\) is monotone decreasing for \(1 < x < 2\) when \(k \geq 3\), one finds that if \(1 < x < X(1; k, c)\) (resp. \(X(2; k, c) < x < 2\)), then \(D_c^{(1)}(k; x) < 0\) (resp. \(D_c^{(1)}(k; x) > 0\)). Here, from the expansion
\[
X(x; k, c) = 2 - \frac{c'}{8\pi^2 x^2} + O(k^{-1}),
\]
for a given \(\varepsilon > 0\), there exists \(k(\varepsilon; x, c) \in \mathbb{N}\) such that for any \(k \geq k(\varepsilon; x, c)\) we have \(2 - \frac{c'}{8\pi^2 x^2} - \varepsilon < X(x; k, c) < 2 - \frac{c'}{8\pi^2 x^2} + \varepsilon\). This implies that for any \(k \geq \max\{k(\varepsilon; 1, c), k(\varepsilon; 2, c)\}\) we have \(2 - \frac{c'}{8\pi^2} - \varepsilon < X(1; k, c)\) and \(X(2; k, c) < 2 - \frac{c'}{32\pi^2} + \varepsilon\). Therefore, we can take \(x_1\) and \(x_2\) in the assertion as
\[
x_1 = \min_{c \in \{\pm1, \pm3, \pm5\}} \mathfrak{p}_1(c) = \mathfrak{p}_1(-5) = 1.4300 \cdots,
\]
\[
x_2 = \max_{c \in \{\pm1, \pm3, \pm5\}} \mathfrak{p}_2(c) = \mathfrak{p}_2(5) = 1.9841 \cdots,
\]
where \(\mathfrak{p}_1(c) = 2 - \frac{c'}{8\pi^2}\) and \(\mathfrak{p}_2(c) = 2 - \frac{c'}{32\pi^2}\).

Now we state the main result in this subsection, which follows immediately from Lemma 4.7.

**Theorem 4.8.** There exists constants \(\xi_1\) and \(\xi_2\) with \(1 < \xi_1 < \xi_2 < 4\) such that, for sufficiently large \(m = pq\) of type (II),

1. \(m\) is exceptional if \(1 < \frac{4}{p} < \xi_1\).
(2) $m$ is ordinary if $\xi_2 < \frac{8}{5} < 4$. \hfill\Box

Remark 4.9. From Lemma 4.5 one can find the asymptotic order of $\mu^{(0)}$, $\mu^{(1)}$, $\mu^{(2)}$ and $\text{RB}$. Actually, the expansions (4.6), (4.7), (4.8) and (4.9) assert in the proof of Lemma 4.7.

$$\begin{align*}
\mu^{(1)} < \mu^{(2)} < \mu^{(0)} < \text{RB}_{l_0+2} & \quad (1 < x < \gamma_1), \\
\mu^{(1)} < \mu^{(0)} < \mu^{(2)} < \text{RB}_{l_0+2} & \quad (\gamma_1 < x < \gamma_2), \\
\mu^{(1)} < \mu^{(2)} < \mu^{(0)} < \text{RB}_{l_0+2} & \quad (\gamma_2 < x < \gamma_3), \\
\mu^{(2)} < \mu^{(1)} < \mu^{(0)} < \text{RB}_{l_0+2} & \quad (\gamma_3 < x < \gamma_4), \\
\mu^{(2)} < \mu^{(0)} < \mu^{(1)} < \text{RB}_{l_0+2} & \quad (\gamma_4 < x < \gamma_5(c)), \\
\mu^{(2)} < \mu^{(0)} < \text{RB}_{l_0+2} < \mu^{(1)} & \quad (\gamma_5(c) < x < 2),
\end{align*}$$

for sufficiently large $k > 0$. Here $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ and $\gamma_5(c)$ are the real roots in the interval $(1, 2)$ of the equations $2x^3 - 6x + 3 = 0$, $x^3 - 12x + 15 = 0$, $x^6 - 2x^5 + 8x - 10 = 0$, $3x^3 - 6x^2 + 2 = 0$ and $8\pi^2x^3 - 16\pi^2x^2 + c' = 0$, respectively. Remark that one can numerically check the inequality $\overline{x}_1(c) < \gamma_5(c) < \underline{x}_2(c)$ as the table below, where $\overline{x}_1(c) = 2 - \frac{c'}{8\pi^2}$ and $\underline{x}_2(c) = 2 - \frac{c'}{32\pi^2}$ are defined in the proof of Lemma 4.7.

| $c$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\overline{x}_1(c)$ | $\gamma_5(c)$ | $\underline{x}_2(c)$ |
|-----|-------------|-------------|-------------|-------------|------------------|-------------|------------------|
| $-5$ | $1.3843\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.4300\ldots$ | $1.8297\ldots$ | $1.8575\ldots$ |
| $-3$ | $1.5234\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.5313\ldots$ | $1.8653\ldots$ | $1.8828\ldots$ |
| $-1$ | $1.3843\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.6327\ldots$ | $1.8980\ldots$ | $1.9081\ldots$ |
| $1$ | $1.3843\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.7340\ldots$ | $1.9284\ldots$ | $1.9335\ldots$ |
| $3$ | $1.3843\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.8353\ldots$ | $1.9570\ldots$ | $1.9588\ldots$ |
| $5$ | $1.3843\ldots$ | $1.5765\ldots$ | $1.7579\ldots$ | $1.7925\ldots$ | $1.9366\ldots$ | $1.9839\ldots$ | $1.9841\ldots$ |

Table 2: The explicit values of $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$, $\overline{x}_1(c)$, $\gamma_5(c)$ and $\underline{x}_2(c)$.

See Figure 1-6 which show actual values of $\mu^{(0)}$, $\mu^{(1)}$, $\mu^{(2)}$ and $\text{RB}$ for $m = f_x(k) = pq \in J$ with $k = 10^4$ for each $c \in \{ \pm 1, \pm 3, \pm 5 \}$, where the horizontal axis shows $x = \sqrt{\frac{4}{p}}$ and the left and right vertical dashed lines describe $\overline{x}_1(c)$ and $\underline{x}_2(c)$, respectively. As we have seen in (4.10), the inequality $\mu^{(1)} > \text{RB}$ holds when $x$ is very close to 2.
5 Arithmetic consideration

Let $m \geq 15$. Then, $m$ is one of the followings: type (I), (II) and the others. Remark that, from Theorem 5.6 and Proposition 4.1 except for 25 and 49, exceptionals belong to the set $J$ with both of type (I) and (II). In this section, we investigate the existence of infinitely many ordinaries and exceptionals of each type.

We first show the following assertion on ordinaries outside of $J$.

**Theorem 5.1.** In each type of (I) and (II), there exists ordinary $m \notin J$ infinitely many.

**Proof.** It is easy to see that $p \mathbb{Z} \cap J = \emptyset$ if and only if $(\frac{c'}{p}) = -1$ for all $c \in \{\pm 1, \pm 3, \pm 5\}$ where $c' = 25 - 4c$ and $(\frac{c}{p})$ is the Legendre symbol. Since $(\frac{c}{p}) = -1$ if and only if

$$p = \begin{cases} 
\pm 2 \pmod{5} & (c = \pm 5), \\
\pm 2, \pm 5, \pm 6, \pm 8, \pm 13, \pm 14, \pm 15, \pm 17, \pm 18 \pmod{37} & (c = -3), \\
\pm 2, \pm 3, \pm 8, \pm 10, \pm 11, \pm 12, \pm 14 \pmod{29} & (c = 1), \\
\pm 2, \pm 8, \pm 10 \pmod{21} & (c = 3), \\
\pm 2, \pm 5, \pm 6 \pmod{13} & \end{cases}$$

this is equivalent to say that $p$ is of the form $p = at + b$ where $a = 5 \cdot 13 \cdot 21 \cdot 29 \cdot 37 = 1464645$, $b \in \{2, 8, 32, 97, 128, 242, \ldots, 1464637, 1464643\}$ and $t \in \mathbb{Z}$ from the Chinese reminder theorem. The Dirichlet theorem of arithmetic progression tells us there exists infinitely many primes of such forms (the first few are given by 97, 577, 827, 853, 947, \ldots) and hence we have the assertion of type (I). Moreover, for each prime $p$ satisfying the above condition, one can take a prime $q$ satisfying $p < q < 2p$ because of the Bertrand-Chebyshev theorem, and then $pq$ is not in $J$. This shows the assertion of type (II).

Next, we discuss about infinitely many existence of both ordinaries and exceptionals inside of $J$ (we remark that, from Theorem 4.3 there are no ordinaries in $J$ of type (I)). To state our results, we recall the well-known conjecture of Hardy-Littlewood [HL] and Bateman-Horn [BH].


| $k$ | $c = -5$ | $c = -3$ | $c = -1$ | $c = 1$ | $c = 3$ | $c = 5$ |
|-----|---------|---------|---------|---------|---------|---------|
| 4   | -       | 33      | 35      | 37      | 39      | 41      |
| 5   | -       | 47      | 49      | 51      | 53      | 55      |
| 6   | -       | 63      | 65      | 67      | 69      | 71      |
| 7   | -       | 81      | 83      | 85      | 87      | 89      |
| 8   | -       | 101     | 103     | 105     | 107     | 109     |
| 9   | -       | 123     | 125     | 127     | 129     | 131     |
| 10  | -       | 147     | 149     | 151     | 153     | 155     |
| 11  | -       | 173     | 175     | 177     | 179     | 181     |
| 12  | -       | 201     | 203     | 205     | 207     | 209     |
| 13  | -       | 231     | 233     | 235     | 237     | 239     |
| 14  | -       | 263     | 265     | 267     | 269     | 271     |
| 15  | -       | 297     | 299     | 301     | 303     | 305     |
| 16  | -       | 333     | 335     | 337     | 339     | 341     |
| 17  | -       | 371     | 373     | 375     | 377     | 379     |
| 18  | -       | 411     | 413     | 415     | 417     | 419     |
| 19  | -       | 451     | 453     | 455     | 457     | 459     | 461     |
| 20  | -       | 495     | 497     | 499     | 501     | 503     | 505     |
| 21  | -       | 541     | 543     | 545     | 547     | 549     | 551     |
| 22  | -       | 589     | 591     | 593     | 595     | 597     | 599     |
| 23  | -       | 639     | 641     | 643     | 645     | 647     | 649     |
| 24  | -       | 691     | 693     | 695     | 697     | 699     | 701     |
| 25  | -       | 745     | 747     | 749     | 751     | 753     | 755     |
| 26  | -       | 801     | 803     | 805     | 807     | 809     | 811     |
| 27  | -       | 859     | 861     | 863     | 865     | 867     | 869     |
| 28  | -       | 919     | 921     | 923     | 925     | 927     | 929     |
| 29  | -       | 981     | 983     | 985     | 987     | 989     | 991     |
| 30  | -       | 1045    | 1047    | 1049    | 1051    | 1053    | 1055    |
| 31  | -       | 1111    | 1113    | 1115    | 1117    | 1119    | 1121    |
| 32  | -       | 1179    | 1181    | 1183    | 1185    | 1187    | 1189    |
| 33  | -       | 1249    | 1251    | 1253    | 1255    | 1257    | 1259    |
| 34  | -       | 1321    | 1323    | 1325    | 1327    | 1329    | 1331    |
| 35  | -       | 1395    | 1397    | 1399    | 1401    | 1403    | 1405    |
| 36  | -       | 1471    | 1473    | 1475    | 1477    | 1479    | 1481    |
| 37  | -       | 1549    | 1551    | 1553    | 1555    | 1557    | 1559    |
| 38  | -       | 1629    | 1631    | 1633    | 1635    | 1637    | 1639    |
| 39  | -       | 1711    | 1713    | 1715    | 1717    | 1719    | 1721    |
| 40  | -       | 1795    | 1797    | 1799    | 1801    | 1803    | 1805    |
| 41  | -       | 1881    | 1883    | 1885    | 1887    | 1889    | 1891    |
| 42  | -       | 1969    | 1971    | 1973    | 1975    | 1977    | 1979    |
| 43  | -       | 2059    | 2061    | 2063    | 2065    | 2067    | 2069    |
| 44  | -       | 2151    | 2153    | 2155    | 2157    | 2159    | 2161    |
| 45  | -       | 2245    | 2247    | 2249    | 2251    | 2253    | 2255    |
| 46  | -       | 2341    | 2343    | 2345    | 2347    | 2349    | 2351    |
| 47  | -       | 2439    | 2441    | 2443    | 2445    | 2447    | 2449    |
| 48  | -       | 2539    | 2541    | 2543    | 2545    | 2547    | 2549    |
| 49  | -       | 2641    | 2643    | 2645    | 2647    | 2649    | 2651    |
| 50  | -       | 2745    | 2747    | 2749    | 2751    | 2753    | 2755    |

Table 3: List of small exceptionals $m = p \in J$ of type (I) (blue bold numbers) and $m = pq \in J$ with $p < q < 4p$ of type (II) (red bold numbers).
Conjecture 5.2. Let \( f_1(x), \ldots, f_r(x) \in \mathbb{Z}[x] \) and \( f(x) = f_1(x) \cdots f_r(x) \). Suppose that \( f_1(x), \ldots, f_r(x) \) satisfy the following conditions:

(i) \( f_1(x), \ldots, f_r(x) \) are distinct.

(ii) \( f_1(x), \ldots, f_r(x) \) are irreducible in \( \mathbb{Z}[x] \).

(iii) The leading coefficients of \( f_1(x), \ldots, f_r(x) \) are positive.

(iv) There is no prime \( \ell \) so that \( \ell | f(n) \) for all \( n \in \mathbb{Z}_{>0} \).

Then, we have

\[
\pi(f_1, \ldots, f_r; x) = \#\{ n \leq x \mid f_1(n), \ldots, f_r(n) \text{ are all prime} \}
\approx \frac{1}{(\deg f_1) \cdots (\deg f_r)} C(f_1, \ldots, f_r) \frac{x}{(\log x)^r},
\]

where \( C(f_1, \ldots, f_r) \) is the Hardy-Littlewood constant defined by

\[
C(f_1, \ldots, f_r) = \prod_p \left( 1 - \frac{\nu_f(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-r}
\]

with \( \nu_f(p) \) being the number of solutions \( n \) in \( \mathbb{Z}_p \) of the congruence \( f(n) \equiv 0 \pmod{p} \).

Now, we can state our results.

Theorem 5.3. Under Conjecture 5.2

(1) there exists exceptional \( m \) infinitely many both of types (I) and (II).

(2) there exists ordinary \( m \) infinitely many of type (II).

To prove the assertion, we use the following lemma.

Lemma 5.4. For \( a, y \in \mathbb{Z}_{>0} \) and \( c \in \mathbb{Z} \), let

\[
p = p(a, y) = a^2(2a + 1)^2 y^2 - a(2a + 1)(8a + 5)y + (4c - 9)a^2 + (4c - 5)a + c,
\]

\[
q = q(a, y) = 16a^4 y^2 - 8a^2(8a + 1)y + 4(4c - 9)a^2 + 16a + 1,
\]

\[
k = k(a, y) = 4a^3(2a + 1)^2 y^2 - a(32a^2 + 20a + 1)y + 2(4c - 9)a^2 + (4c - 1)a.
\]

(1) The identity \( pq = k^2 + 5k + c \) holds. Moreover, \( p < q < 4p \) for \( y \gg 0 \) with

\[
\lim_{y \to \infty} \frac{q}{p} = \left( 2 - \frac{2}{2a + 1} \right)^2 < 4, \quad \lim_{a \to \infty} \lim_{y \to \infty} \frac{q}{p} = 4.
\]

(2) If we consider \( p \) and \( q \) as polynomials in \( \mathbb{Z}[y] \), then each of them satisfies the four conditions in Conjecture 5.2 for any \( c \in \{ \pm 1, \pm 3, \pm 5 \} \) when \( a \equiv 1, 4, 7, 13 \pmod{15} \).

Proof. The identity \( pq = k^2 + 5k + c \) and the above limit formulas for \( \frac{q}{p} \) can be checked directly. Moreover, since the coefficients of \( y^2 \) in both \( q - p \) and \( 4p - q \) are positive for all \( a > 0 \), one sees that \( p < q < 4p \) for \( y \gg 0 \). Now, let us write \( p = p(y) \) and \( q = q(y) \). For all \( a > 0 \), \( p(y) \) and \( q(y) \) satisfy the conditions (i) and (iii) obviously. Moreover, one sees that (ii) is also true for all
a > 0 and c ∈ {±1, ±3, ±5} since p(y) and q(y) have the non-square discriminants c′a^2(2a + 1)^4 and 2^8c′a^6, respectively.

Put d_p (resp. d_q) the greatest common divisor of the coefficients of p(y) (resp. q(y)). Under the primitive situation of p and q, that is, d_p = d_q = 1, it is sufficient to check the condition (iv) only for the case ℓ = 2, 3 because deg p = deg q = 2.

At first, it is easy to see that d_p = ((a, 5)a, c) and d_q = 1. Therefore, for all a > 0 and c ∈ {±1, ±3, ±5} with (a, c) = 1, the polynomials p(y) and q(y) are both primitive. When ℓ = 2, the condition is obvious for all a > 0 and c ∈ {±1, ±3, ±5} because p(0) ≡ q(0) ≡ 1 (mod 2). The values of p(y) at y = 0, 1, 2 are congruent modulo ℓ = 3 to ca^2 + (c + 1)a + c, (c + 2)a^2 + (c + 2)a + c, (c + 2)a^2 + (c + 2)a + c, respectively. Also, we have ca^2 + a + 1, (c + 2)a^2 + 1, ca^2 + 2a + 1 for the values q(y) at y = 0, 1, 2, respectively. Thus, except for the case (a, c) ≡ (0, 0), (2, 0) (mod 3), p(y)q(y) is not congruent to the zero polynomial modulo 3 for all a > 0 and c ∈ {±1, ±3, ±5}.

Summing up the above discussion, the polynomials p(y) and q(y) satisfy the condition (iv) if a ≠ 0 (mod 5) when c = ±5, a ≡ 1 (mod 3) when c = ±3 and for all a when c = ±1. Therefore, solving the congruences a ≠ 0 (mod 5) and a ≡ 1 (mod 3), we obtain the second assertion.

Proof of Theorem 5.3. Since each of the six polynomial f_c(x) with c ∈ {±1, ±3, ±5} satisfies the four conditions in Conjecture 5.2 together with Theorem 4.3 one obtains the assertion for exceptionals of type (I).

Now, let ξ_1 = x_1^2 = 2.0451 ... ξ_2 = x_2^2 = 3.9365 ... be the constants obtained in Theorem 4.8. Take a ∈ ℤ_{>0} satisfying a ≡ 1, 4, 7, 13 (mod 15) and (2 − 2^{−2a+1})^2 < ξ_1, that is, a = 1. Then, under Conjecture 5.2, the corresponding p(y) and q(y) in Lemma 6.4 represent infinitely many primes at the same time. Moreover, if both p(y) and q(y) are prime, then m = p(y)q(y) = f_c(k(y)) ∈ J and, form Theorem 4.8, it is exceptional. This shows the assertion for exceptionals of type (II). Furthermore, if we take a ∈ ℤ_{>0} satisfying a ≡ 1, 4, 7, 13 (mod 15) and (2 − 2^{−2a+1})^2 > ξ_2 (notice that the smallest such a is 64), under Conjecture 5.2 from Theorem 4.8 again, one similarly proves the assertion for ordinals of type (II). This completes the proof.

Example 5.5. Consider the case where a = 1 and c = −5, that is,

p = 9y^2 − 39y − 59,  q = 16y^2 − 72y − 99.

Then, as we have seen above, m = pq is exceptional if both p and q are prime for sufficiently large y ≫ 0. Notice that, since 1 < (2 − 2^{−2a+1}) = 1.3333 ... < γ_1 = 1.3843 ... where γ_1 is defined in Remark 4.9 the inequality μ(1) < μ(2) < μ(0) = ̄μ < RB holds for such m. The first few of such p and q are given in Table 4.

On the other hand, if we replace c with −7, that is,

p = 9y^2 − 39y − 77,  q = 16y^2 − 72y − 131.

Then, m = pq ∉ J and hence m is ordinary from Theorem 3.6. Actually, as one finds from Table 5, the inequality RB < μ(0) = ̄μ holds.

Example 5.6. Consider the case where a = 64 and c = 5, that is,

p = 68161536y^2 − 4268352y + 46021,  q = 268435456y^2 − 16809984y + 181249.
Table 4: Differences between $\mu^{(i)}$ and RB for $m = pq$ with $a = 1$, $c = -5$.

| $y$ | $p$ | $q$ | $\frac{a}{p}$ | $\mu^{(0)} - \text{RB}$ | $\mu^{(1)} - \text{RB}$ | $\mu^{(2)} - \text{RB}$ |
|-----|-----|-----|---------------|-----------------|-----------------|-----------------|
| 7   | 109 | 181 | 1.660...      | $-1.11 \times 10^{-2}$ | $-8.21 \times 10^{-2}$ | $-2.17 \times 10^{-2}$ |
| 17  | 1879| 3301| 1.756...      | $-7.58 \times 10^{-4}$ | $-4.86 \times 10^{-3}$ | $-1.09 \times 10^{-3}$ |
| 25  | 4591| 8101| 1.764...      | $-3.11 \times 10^{-4}$ | $-1.98 \times 10^{-3}$ | $-4.42 \times 10^{-4}$ |
| 35  | 9601| 16981| 1.768...     | $-1.49 \times 10^{-4}$ | $-9.50 \times 10^{-4}$ | $-2.09 \times 10^{-4}$ |
| 40  | 12781| 22621| 1.768...    | $-1.12 \times 10^{-4}$ | $-7.13 \times 10^{-4}$ | $-1.57 \times 10^{-4}$ |
| 62  | 32119| 56941| 1.772...    | $-4.46 \times 10^{-5}$ | $-2.83 \times 10^{-4}$ | $-6.20 \times 10^{-5}$ |
| 82  | 57259| 101581| 1.774...  | $-2.50 \times 10^{-5}$ | $-1.59 \times 10^{-4}$ | $-3.47 \times 10^{-5}$ |
| 104 | 93229| 165469| 1.774...  | $-1.53 \times 10^{-5}$ | $-9.77 \times 10^{-5}$ | $-2.12 \times 10^{-5}$ |

Table 5: Differences between $\mu^{(i)}$ and RB for $m = pq$ with $a = 1$, $c = -7$.

| $y$ | $p$ | $q$ | $\frac{a}{p}$ | $\mu^{(0)} - \text{RB}$ | $\mu^{(1)} - \text{RB}$ | $\mu^{(2)} - \text{RB}$ |
|-----|-----|-----|---------------|-----------------|-----------------|-----------------|
| 13  | 937 | 1637| 1.747...      | $1.07 \times 10^{-4}$ | $-8.13 \times 10^{-4}$ | $-6.21 \times 10^{-4}$ |
| 43  | 14887| 26357| 1.770...     | $4.70 \times 10^{-6}$ | $-5.11 \times 10^{-4}$ | $-3.36 \times 10^{-5}$ |
| 60  | 29983| 53149| 1.772...     | $2.30 \times 10^{-6}$ | $-2.54 \times 10^{-4}$ | $-1.64 \times 10^{-5}$ |
| 81  | 55813| 99013| 1.774...     | $1.22 \times 10^{-6}$ | $-1.36 \times 10^{-4}$ | $-8.73 \times 10^{-6}$ |
| 158 | 218437| 387917| 1.775...    | $3.11 \times 10^{-7}$ | $-3.48 \times 10^{-5}$ | $-2.19 \times 10^{-6}$ |
| 211 | 392383| 697013| 1.776...    | $1.73 \times 10^{-7}$ | $-1.93 \times 10^{-5}$ | $-1.21 \times 10^{-6}$ |
| 225 | 440773| 793669| 1.776...    | $1.52 \times 10^{-5}$ | $-1.70 \times 10^{-5}$ | $-1.06 \times 10^{-6}$ |
| 249 | 548221| 973957| 1.776...    | $1.23 \times 10^{-7}$ | $-1.38 \times 10^{-5}$ | $-8.69 \times 10^{-7}$ |

Table 6: Differences between $\mu^{(i)}$ and RB for $m = pq$ with $a = 64$, $c = 5$.

| $y$ | $p$ | $q$ | $\frac{a}{p}$ | $\mu^{(0)} - \text{RB}$ | $\mu^{(1)} - \text{RB}$ | $\mu^{(2)} - \text{RB}$ |
|-----|-----|-----|---------------|-----------------|-----------------|-----------------|
| 39  | 103507276549| 407634920449| 3.938...      | $-5.79 \times 10^{-11}$ | $2.17 \times 10^{-11}$ | $-6.61 \times 10^{-11}$ |
| 134 | 1223336627269| 4817774691329| 3.938...      | $-4.90 \times 10^{-12}$ | $1.84 \times 10^{-14}$ | $-5.59 \times 10^{-12}$ |
| 165 | 1854993585541| 7305381823489| 3.938...      | $-3.23 \times 10^{-12}$ | $1.21 \times 10^{-14}$ | $-3.69 \times 10^{-12}$ |
| 178 | 2158870385989| 8562116992001| 3.938...      | $-2.77 \times 10^{-12}$ | $1.04 \times 10^{-14}$ | $-3.17 \times 10^{-12}$ |
| 279 | 5304571299589| 20890594526209| 3.938...      | $-1.13 \times 10^{-12}$ | $4.25 \times 10^{-15}$ | $-1.29 \times 10^{-12}$ |
| 433 | 12777090072709| 50321416668161| 3.938...      | $-4.69 \times 10^{-13}$ | $1.76 \times 10^{-15}$ | $-5.35 \times 10^{-13}$ |
| 468 | 14927014718149| 58785940423681| 3.938...      | $-4.02 \times 10^{-13}$ | $1.51 \times 10^{-15}$ | $-4.58 \times 10^{-13}$ |
| 499 | 16970160763909| 66832308978689| 3.938...      | $-3.53 \times 10^{-13}$ | $1.32 \times 10^{-15}$ | $-4.03 \times 10^{-13}$ |
In this case, \( m = pq \) is ordinary if both \( p \) and \( q \) are prime for sufficiently large \( y \gg 0 \). Notice that, since \( \gamma_5(5) = 1.9839 \ldots < (2 - \frac{2}{2a+1}) = 1.9845 \ldots < 2 \) where \( \gamma_5(5) \) is also defined in Remark 4.3, the inequality \( \mu^{(2)} < \mu^{(0)} < RB < \mu^{(1)} = \tilde{\mu} \) holds for such \( m \). See Table 6.

**Remark 5.7.** Let us denote the fractional part of a real number \( x \) by \( \{x\} \). If the sequence \( \{\sqrt{p} - \frac{3}{2}\} \) \( p \) prime is included in a closed interval, then one easily sees that there can not be infinitely many exceptional primes. In this sense, this phenomena on the existence of exceptional primes is also related to \( \{\sqrt{p}\} \) which distributes uniformly in the interval \([0, 1)\) (cf. [DW], [DL]).

6 Numerical consideration

Let \( \rho_E(x) \) be the number of exceptionals \( m \leq x \). It is now natural to ask how \( \rho_E(x) \) behaves as \( x \) tends to infinity. The aim of this section is to consider this question by giving some conjectures which are obtained by numerical studies. Notice that to investigate \( \rho_E(x) \) it is enough to know \( \pi_E(c; x) \) for \( c \in \{\pm 1, \pm 3, \pm 5\} \) where \( \pi_E(c; x) \) is the number of \( k \leq x \) such that \( f_c(k) = k^2 + 5k + c \) is exceptional, since \( \rho_E(x) \sim \sum_{c \in \{\pm 1, \pm 3, \pm 5\}} \pi_E(c; \sqrt{x}) \) from Theorem 5.6. Moreover, it is sufficient to investigate \( \pi_E^{(1)}(c; x) \) and \( \pi_E^{(II)}(c; x) \), the number of \( k \leq x \) such that \( f_c(k) \) is exceptional of type (I) and (II), respectively, because of the identity

\[
\pi_E(c; x) \sim \pi_E^{(1)}(c; x) + \pi_E^{(II)}(c; x),
\]

which is immediate from Proposition 4.4.

6.1 Distribution of exceptionals of type (I)

From Theorem 4.3 we have \( \pi_E^{(1)}(c; x) = \pi(f_c; x) \), where \( \pi(f; x) \) with \( f \in \mathbb{Z}[x] \) is defined in Conjecture 5.2. Hence, from the conjecture of Hardy-Littlewood and Bateman-Horn, the asymptotic behavior of \( \pi_E^{(1)}(c; x) \) is expected as follows.

**Conjecture 6.1.** It holds that

\[
\pi_E^{(1)}(c; x) \sim C^{(1)}(c) \frac{x}{\log x},
\]

where \( C^{(1)}(c) \)

\[
C^{(1)}(c) = \frac{C(f_c)}{2} = \prod_{p \geq 3} \left(1 - \frac{(\zeta')_p}{p-1}\right) = \begin{cases} 
1.18219 \ldots & (c = -5), \\
1.18219 \ldots & (c = -3), \\
1.12674 \ldots & (c = -1), \\
0.927881 \ldots & (c = 1), \\
0.807233 \ldots & (c = 3), \\
1.77328 \ldots & (c = 5),
\end{cases}
\]

with \( \zeta' = 25 - 4c \).

6.2 Distribution of exceptionals of type (II)

For \( a > 1 \), let \( P_2(a) \) be the set of all \( pq \) where \( p \) and \( q \) are distinct primes satisfying \( p < q < ap \). Moreover, for a polynomial \( f \in \mathbb{Z}[x] \), let \( \pi_2(f, a; x) \) be the number of \( k \leq x \) such that \( f(k) \in P_2(a) \). From Theorem 4.8 and the observation in Remark 4.9, one may expect that
\( \pi_E^{(\Pi)}(c; x) \) asymptotically behaves as \( \pi_2(f; \gamma_5(c)^2; x) \), where \( \gamma_5(c) \) is a constant also defined in Remark 4.9. We here notice that Conjecture 5.2 with \( r = 1 \) asserts that \( \pi(f; x) \asymp \pi(x) \) for any \( f \in \mathbb{Z}[x] \) satisfying the conditions in Conjecture 5.2 where \( \pi(x) \sim \frac{x}{\log x} \) is the number of primes \( p \leq x \). Based on this observation, we may expect the same situation for \( \pi_2(f; a; x) \), that is, \( \pi_2(f; a; x) \asymp \pi_2(a; x) \) where \( \pi_2(a; x) \) is the number of \( m \leq x \) such that \( m \in P_2(a) \). For \( \pi_2(a; x) \), we can say the following (for more precise discussion, see [DM], [Ha]).

Lemma 6.2. It holds that

\[
\pi_2(a; x) \asymp \frac{x}{(\log x)^2}.
\]

Proof. Fix any prime number \( p_0 \). For \( x \geq a p_0^2 \), by the prime number theorem, we have

\[
\pi^2(x; a) = \sum_{p \leq \sqrt{x}} \sum_{p < q < ap} 1 + \sum_{\sqrt{x} < p \leq x} \sum_{p < q \leq \frac{x}{p}} 1
\]

\[
= \sum_{p \leq \sqrt{x}} \left( \frac{ap}{\log ap} - \frac{p}{\log p} + O(1) \right) + \sum_{\sqrt{x} < p \leq x} \left( \frac{\frac{x}{p}}{\log \frac{x}{p}} - \frac{p}{\log p} + O(1) \right)
\]

\[
= \sum_{p \leq \sqrt{x}} \left( \frac{ap}{\log ap} - \frac{p}{\log p} \right) + \sum_{\sqrt{x} < p \leq x} \left( \frac{\frac{x}{p}}{\log \frac{x}{p}} - \frac{p}{\log p} \right) + O(\sqrt{x} / \log x).
\]

Let us write the first and the second sums of the rightmost hand side as \( A \) and \( B \), respectively.

Since \( x \geq a p_0^2 \), we have

\[
\frac{a}{c(a) + 1} \frac{p}{\log p} < \frac{ap}{\log ap} < \frac{a p}{\log p} \quad \text{where} \quad c(a) = \frac{\log a}{\log p_0}.
\]

This shows that

\[
(6.1) \quad \frac{2}{a} \left( \frac{a}{c(a) + 1} - 1 \right) \frac{x}{(\log x)^2} \ll A \ll \frac{2}{a} \left( a - 1 \right) \frac{x}{(\log x)^2}.
\]

Here, we have used the formula

\[
(6.2) \quad \sum_{p \leq x} \frac{p}{\log p} \sim \frac{1}{2} \left( \frac{x}{\log x} \right)^2,
\]

which follows from the Abel summation formula with the fact that there exists a constant \( c > 0 \) such that \( \vartheta(x) = \sum_{p \leq x} \log p = x + O(x \exp(-c \sqrt{\log x})) \) (see, e.g., [MV]).

On the other hand, we have

\[
\log a \frac{x}{(\log x)^2} \sim \frac{x}{\log x} \sum_{\frac{x}{p} < p \leq \sqrt{x}} \frac{1}{p} < \sum_{\frac{x}{p} < p \leq \sqrt{x}} \frac{\frac{x}{p}}{\log \frac{x}{p}} < \frac{2x}{\log x} \sum_{\frac{x}{p} < p \leq \sqrt{x}} \frac{1}{p} \sim 2 \log a \frac{x}{(\log x)^2},
\]

where we have used the fact that there exists constants \( b \) and \( c' > 0 \) such that \( \sum_{p \leq x} \frac{1}{p} = \log \log x + b + O(\exp(-c' \sqrt{\log x})) \) (see [MV] again). This together with (6.2) implies

\[
(6.3) \quad \left( \log a - 2 + \frac{2}{a} \right) \frac{x}{(\log x)^2} \ll B \ll \left( 2 \log a - 2 + \frac{2}{a} \right) \frac{x}{(\log x)^2}.
\]

Combining (6.1) and (6.3), we have

\[
\left( \frac{2 \log p_0}{\log p_0 + \log a} + \log a - 2 \right) \frac{x}{(\log x)^2} \ll A + B \ll 2 \log a \frac{x}{(\log x)^2}.
\]

Notice that if we take \( p_0 \geq 11 \), then the coefficients of the leftmost hand side is positive for all \( a > 1 \). This completes the proof. \( \square \)
These observations lead us to expect the following.

**Conjecture 6.3.** There exists a constant $C^{(II)}(c)$ such that

$$\pi_E^{(II)}(c; x) \sim C^{(II)}(c) \frac{x}{(\log x)^2}.$$  

Here we give a numerical computation for the values $\pi_E^{(II)}(c; x)/x(\log x)^2$ with $x \leq 5 \times 10^7$ for each $c \in \{\pm 1, \pm 3, \pm 5\}$ in Figure 7-12.

**Remark 6.4.** Under Conjecture 5.2, one can show the relation $\pi_E^{(II)}(c; x) \gg \frac{x}{(\log x)^2}$. Actually, from our construction of exceptionals in the proof of Theorem 5.3 we have

$$\pi_E^{(II)}(x; c) \gg \# \{ y \leq x \mid p(1, y) \text{ and } q(1, y) \text{ are both primes} \} \sim \frac{x}{(\log x)^2}.$$  

Here,

$$p(1, y) = 9y^2 - 39y + 9c - 14, \quad q(1, y) = 16y^2 - 72y + 16c - 19.$$  

**6.3 Some Remarks**

**Remark 6.5.** Let $P_2$ be the set of all $pq$ where $p$ and $q$ are distinct primes with $p < q$. Moreover, for $f \in \mathbb{Z}[x]$, let $\pi_2(f; x)$ be the number of $k \leq x$ such that $f(k) \in P_2$. Similar to the discussion in §6.2, one may expect that $\pi_2(f; x)$ is asymptotically equal to a constant multiple of $\pi_2(x)$ if $f$ satisfies suitable conditions, that is,

$$\pi_2(f; x) \asymp \pi_2(x) \sim \frac{x \log \log x}{\log x}.$$  

Here, $\pi_2(x)$ is the number of $m \leq x$ such that $m \in P_2$. Notice that the second equality relation in (6.4) was obtained by Landau [La] (see also [HW]).
A positive integer having at most two distinct prime factors is called an *almost prime*. When \( f(x) \) is a quadratic polynomial, it is shown by Iwaniec \([I]\) and Lemke-Oliver \([Le]\) that there are infinitely many \( k \) such that \( f(k) \) is almost prime. More precisely, they prove that

\[
\pi(f; x) + \pi_2(f; x) \gg \frac{x}{\log x}
\]

if \( f \) satisfies suitable conditions. Of course, the expectation (6.4) is more stronger than the result (6.5). In Figure 13, we give a numerical computation of \( \pi_2(f; x)/(\frac{x}{\log x} \log \log x) \) for \( x \leq 5 \times 10^4 \) with \( f(k) = k^2 + 1 \), which is studied in \([I]\).

Figure 13: The asymptotic of \( \pi_2(f; x)/(\frac{x}{\log x} \log \log x) \) with \( f(k) = k^2 + 1 \) for \( x \leq 5 \times 10^4 \).

**Remark 6.6.** If one can prove that there exists infinitely many exceptions in the framework of graph theory, then, from Theorem 3.6 and Proposition 4.1, one may obtain a theorem of Iwaniec \([I]\) and Lemke-Oliver \([Le]\) type, for at least one of \( f_c \). Much more stronger, if one can prove the existence of infinitely many exceptional primes in such a framework, then we can say that the conjecture of Hardy-Littlewood and Bateman-Horn is true for at least one of \( f_c \).

## 7 Ramanujan abelian graphs of odd order

Our problem can be discussed more general situation. Namely, we can determine \( \hat{l} \) for any finite abelian group \( G \) of odd order \( m \), instead of \( \mathbb{Z}_m \). Let \( \hat{G} \) be the dual group of \( G \) and \( \mathcal{S} \) the set of all Cayley subset of \( G \). Notice that, since \( m \) is odd, there is no element in \( G \) whose order is two. This means that \( |\mathcal{S}| \) is even for any \( S \in \mathcal{S} \) and hence \( l(S) = m - |\mathcal{S}| \) is always odd. We denote by \( X(S) \) the Cayley graph of \( G \) attached to \( S \in \mathcal{S} \) and \( \Lambda(S) \) the set of all eigenvalues of \( X(S) \). As we have explained in Section 2.1, it can be written as \( \Lambda(S) = \{\lambda_1| \chi \in \hat{G}\} \) where \( \lambda_1_G = |\mathcal{S}| \) with \( 1_G \) being the trivial character of \( G \) and

\[
\lambda_\chi = \sum_{a \in S} \chi(a) = -\sum_{b \in G \setminus S} \chi(b), \quad \chi \neq 1_G.
\]

From the same discussion as in the proof of Lemma 5.2, it is immediate to see that \( \hat{l} \geq l_0 = 2[\sqrt{m} - \frac{3}{2}] + 1 \). Let us also call \( G \) ordinary (resp. exceptional) if \( \hat{l} = l_0 \) (resp. \( \hat{l} \geq l_0 + 2 \)).

From the fundamental theorem of finite abelian groups, we may assume that \( G \) is a direct sum of finite number of cyclic groups. We remark that we here do not consider \( G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) because it can be checked that all the Cayley graphs of \( G \) are Ramanujan. The following theorem says that there are only finitely many exceptionals \( G \) which are not cyclic.
Theorem 7.1. Let $G$ be a finite abelian group of odd order which is not cyclic. Then, $G$ is
ordinary except for the cases $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ where $p$ is odd prime with $3 \leq p \leq 17$. In
the exceptional cases, we have

$$
\hat{i} = \begin{cases} 
  l_0 + 2 & p = 7, 11, 13, 17, \\
  l_0 + 4 & p = 5.
\end{cases}
$$

Before giving a proof of the theorem, it is convenient to prove the following lemma.

Lemma 7.2. Let $G_1, G_2$ be finite abelian groups with $|G_1| = n_1, |G_2| = n_2$, respectively, where
$n_1, n_2$ are odd integer with $n_1 \leq n_2$. If $n_2 \geq 4n_1 - 3$, then $G = G_1 \oplus G_2$ is ordinary.

Proof. Take a non-trivial character $\chi = \eta \otimes \chi_{G_2} \in \hat{G}$ with $\eta$ being a non-trivial character of
$G_1$. As in the proof of Proposition 4.1, the condition $n_2 \geq 4n_1 - 3$ implies that one can take
$S \in S_{l_0+2}$ as $G \setminus S \subset \{(0, g_2) \in G | g_2 \in G_2\}$. This shows that

$$
|\lambda_{\chi_0}| = \sum_{(g_1, g_2) \in G_1 \setminus S} \chi_0(g_1, g_2) = l_0 + 2 \geq \text{RB}
$$

and hence asserts that $X(S)$ is not Ramanujan.

Proof of Theorem 7.1. From the fundamental theorem of finite abelian groups, we may assume
that $G$ is of the form $G = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ where $m_1, m_2, \ldots, m_r$ are odd integers with
$m_1 | m_2 | \cdots | m_r$. Moreover, because $G$ is not cyclic, we may further assume that $r \geq 2$. Let
$G_1 = \mathbb{Z}_{m_1}$ and $G_2 = \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_r}$ and $n_1 = 1$ and $n_2 = m_2 \cdots m_r$, respectively. If $r \geq 3$, then one easily sees that $n_2 \geq 4n_1 - 3$ and hence, from Lemma 7.2, $G$ is ordinary. Therefore, it
is sufficient to study only the case $r = 2$.

Suppose that at least one of $m_1$ and $m_2$ has two prime factors. Then, since $m_1 | m_2$, it can
be written as $m_1 = p^{t_1} t_1$ and $m_2 = p^{t_2} t_2$ for some odd prime $p$ and odd integers $t_1, t_2$ with
$(p, t_1) = (p, t_2) = 1$. Here, at least one of $t_1$ and $t_2$ are greater than one. Now, from the Chinese
reminder theorem, we have $G = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \cong \mathbb{Z}_{p^{t_1}} \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{p^{t_2}} \oplus \mathbb{Z}_{t_2}$, whence, from Lemma 7.2,
again, $G$ is ordinary. Therefore, we may assume that $m_1$ and $m_2$ can be respectively written as
$m_1 = p^s$ and $m_2 = p^t$ for some $p$ and $s \leq t$. Moreover, we see that $n_2 \geq 4n_1 - 3$ if $t \geq 2$ or
$t = s + 1$ with $s = 1$ when $p = 3$ or $t = s + 1$ when $p \geq 5$. Hence, it is enough to consider only the cases $(m_1, m_2) = (3^s, \ast)$ with $s \geq 2$ or $(m_1, m_2) = (p^s, p^t)$ with $s \geq 1$ for $p \geq 5$.

Assume that $G$ is the former, that is, $G = \mathbb{Z}_{3^s} \oplus \mathbb{Z}_{3^{s+1}}$ with $s \geq 2$. In this case, we can take
$S \in S_{l_0+2}$ as $G \setminus S \subset \{(g_1, g_2) \in G | g_1 \neq 1 \wedge g_2 \leq 3^{s+1}\}$ because $l_0 + 2 = 2 \lfloor 3^{s \sqrt{3}} - \frac{1}{2} \rfloor + 3 < 3^{2s}$. Then, for such $S$, we have $|\lambda_{\chi_0}| = l_0 + 2$ where $\chi_0(g_1, g_2) = e^{\frac{2\pi ig_1}{3}}$. This shows that $G$ is ordinary.

We next consider the latter, that is, $G = \mathbb{Z}_{p^{s}} \oplus \mathbb{Z}_{p^{t}}$ with $s \geq 1$. At first, let $s \geq 2$. Then, we can similarly take $S \in S_{l_0+2}$ as $G \setminus S \subset \{(g_1, g_2) \in G | g_1 \neq 1 \wedge g_2 \leq p^t\}$ because $l_0 + 2 = 2p^s - 1 < p^{2s-1}$ and hence, by the same reason as above, $G$ is ordinary.

Now, only the cases $G = \mathbb{Z}_{p^{s}} \oplus \mathbb{Z}_{p^{t}}$ with $p \geq 5$ are left. Let $h \geq 1$. If $p \geq 2h - 3$, then, we can take
$S \in S_{l_0+2h}$ as $G \setminus S \subset \{(0, g_2) \in G | 1 \leq g_2 \leq p\} \cup \{(\pm 1, \pm g_2) \in G | 1 \leq g_2 \leq \lfloor \frac{p}{2} \rfloor \} \cup \{(\pm 1, 0)\}$ because $l_0 + 2h(= 2p - 3 + 2h) \leq 3p$. For such $S$, we have $|\lambda_{\chi_0}| = p + (p - 3 + 2h) \cos \frac{2\pi}{p}$ where
$\chi_0(g_1, g_2) = e^{\frac{2\pi ig_1}{p}}$. We notice that this is the largest, that is, $|\lambda_{\chi_0}| = \mu_{l_0+2h} = \max_{S \in S_{l_0+2h}} \mu(S)$.

Since $l_0 \geq 2h$ is equivalent to $\mu_{l_0+2h} \leq \mu_{l_0+2h}$, this implies that $p + (p - 3 + 2h) \cos \frac{2\pi}{p} \leq 2 \sqrt{p^2 - 2p + 2}$. Let

$$
d(p, h) = p + (p - 3 + 2h) \cos \frac{2\pi}{p} - 2 \sqrt{p^2 - 2p + 2}.
$$
Then, one can see that $d(p, 1) > 0$ if and only if $p \geq 19$, $d(p, 2) > 0$ if and only if $p \geq 7$ and $d(p, 3) > 0$ for all $p \geq 5$. This completes the proof.

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