ON RELIABILITY FUNCTION OF GAUSSIAN CHANNEL WITH NOISY FEEDBACK: ZERO TRANSMISSION RATE

For information transmission a discrete time channel with independent additive Gaussian noise is used. There is also feedback channel with independent additive Gaussian noise, and the transmitter observes without delay all outputs of the forward channel via that feedback channel. Transmission of nonexponential number of messages is considered and the achievable decoding error exponent for such a combination of channels is investigated. It is shown that for any finite noise in the feedback channel the achievable error exponent is better than similar error exponent of the no-feedback channel. Method of transmission/decoding used in the paper strengthens the earlier method used by authors for BSC. In particular, for small feedback noise, it allows to get the gain of 23.6% (instead of 14.3% earlier for BSC).

§ 1. Introduction and main results

We consider the discrete time channel with independent additive Gaussian noise, i.e. if \( \mathbf{x} = (x_1, \ldots, x_n) \) is the input codeword then the received block \( \mathbf{y} = (y_1, \ldots, y_n) \) is

\[
y_i = x_i + \xi_i, \quad i = 1, \ldots, n,
\]

where \( \mathbf{\xi} = (\xi_1, \ldots, \xi_n) \) are independent \( \mathcal{N}(0,1) \)-Gaussian random variables, i.e. \( E\xi_i = 0, \ E\xi_i^2 = 1 \). There is also a noisy feedback channel which allows to the transmitter to observe (without delay) all outputs of the forward channel

\[
z_i = y_i + \sigma \eta_i, \quad i = 1, \ldots, n,
\]

where \( \mathbf{\eta} = (\eta_1, \ldots, \eta_n) \) are independent (and independent of \( \mathbf{\xi} \)) \( \mathcal{N}(0,1) \)-Gaussian random variables, i.e. \( E\eta_i = 0, \ E\eta_i^2 = 1 \). The value \( \sigma > 0 \), characterizing the feedback channel noise intensity, is given. No coding is used in the feedback channel (i.e. the receiver simply re-transmits all received outputs to the transmitter). In other words, the feedback channel is “passive” (see Fig. 1).

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We assume that the input block $\mathbf{x}$ satisfies the constraint
\[ \sum_{i=1}^{n} x_i^2 \leq nA, \]
where $A$ is a given constant. We denote by AWGN($A$) the channel (1) without feedback, and by AWGN($A, \sigma$) that channel with noisy feedback (2).

Since Shannon’s paper [1] it has been known that even noiseless feedback does not increase the capacity of the Gaussian channel (or any other memoryless channel). However, feedback allows to improve the decoding error probability (or simplify the effective transmission methods). In the case of noiseless feedback possibility of such improvement of the decoding error probability with respect to no-feedback channel was shown for a number of channels in [2–9].

We consider the case when the overall transmission time $n$ and $M = e^{o(n)}$ equiprobable messages $\{\theta_1, \ldots, \theta_M\}$ are given. After the moment $n$, the receiver makes a decision $\hat{\theta}$ on the message transmitted. We are interested in the best possible decoding error exponent (and whether it exceeds the similar exponent of the channel without feedback).

Such problem (for $R = 0$) was first considered in [10, 11] for a binary symmetrical channel. Later in [12, 13], the case of positive rates (i.e. $R > 0$) was also investigated. The paper aim is to get similar (in fact, much stronger) results for a Gaussian channel.

Some results for channels with noiseless feedback can be found in [2–9], and for the case of noisy feedback – in [14, 15] (see also discussion in [11]).

In order to compare with this paper results, we remind briefly earlier results from [10–13]. There the binary symmetrical channel BSC($p$) with similar feedback channel BSC($p_1$) was considered. It was shown in [10–13] that there exists a certain critical value $p_{\text{crit}}(p, R) > 0$, such that if $p_1 < p_{\text{crit}}(p, R)$, then it is possible to improve the decoding error exponent of the no-feedback channel. If, in particular, both $R$ and $p_1$ are small then the gain is 14.3%. In order to get such improvement the transmission/decoding method with one “switching” moment was developed and investigated.

The method of papers [10, 11] was applied to Gaussian channel AWGN($A, \sigma$) in [16] with similar to papers [10, 11] results (in particular, with the same asymptotic gain 14.3%).

**Remark 1.** The transmission method used in [10–13], reduces the problem to testing of two most probable (at some fixed moment) messages. It was mentioned in [11] Remark 1 and [13] Remark 3] that such method is not optimal even for one switching moment.
In the paper, still using one switching moment, we essentially improve the transmission/decoding method of [10]–[13]. We show that for any noise intensity \( \sigma^2 < \infty \) it is possible to improve the best decoding error exponent \( E(M, A) \) of AWGN(A) channel without feedback.

The transmission/decoding method with one switching moment, giving such improvement is described in §§ 2-3. It strengthens the method introduced by authors earlier in [10]–[13]. Of course, if \( \sigma \) is not small then the gain is small, but it is strongly positive. In other words, in the problem considered there is no any critical level \( \sigma_{\text{crit}} \), beyond which it is not possible to improve the exponent \( E(M, A) \).

Remark 2. The paper methods can be applied for BSC as well, strengthening the results of [10]–[13]. In particular, for BSC there is no critical level \( p_{\text{crit}}(p) < 1/2 \), beyond which it is not possible to improve the exponent \( E(M, p) \).

Remark 3. We consider the case when feedback noise intensity \( \sigma^2 > 0 \) is fixed and does not depend on the number of messages \( M \). The case when the value \( \sigma^2 M \) is small, corresponds, in a sense, to the noiseless feedback case (cf. [16]).

For \( x, y \in \mathbb{R}^n \) denote

\[
(x, y) = \sum_{i=1}^{n} x_i y_i, \quad \|x\|^2 = (x, x), \quad d(x, y) = \|x - y\|^2.
\]

A subset \( C = \{x_1, \ldots, x_M\} \) with \( \|x_i\|^2 = An, \) \( i = 1, \ldots, M \) is called a \((M, A, n)\)-code of length \( n \).

For a code \( C = \{x_i\} \) denote by \( P_e(C) \) the minimal possible decoding error probability

\[
P_e(C) = \min_{i} \max P(e|x_i),
\]

where \( P(e|x_i) \) – conditional decoding error probability provided \( x_i \) was transmitted, and minimum is taken over all decoding methods (it will be convenient for us to denote the transmitted message both \( \theta_i \) and \( x_i \)).

For \( M \) messages and AWGN(A) channel denote by \( P_e(M, A, n) \) the minimal possible decoding error probability for the best \((M, A, n)\)-code. We are interested in the best exponent (in \( n \)) of that function

\[
E(M, A) = \limsup_{n \to \infty} \frac{1}{n} \ln \frac{1}{P_e(M, A, n)}.
\]

Similarly, for AWGN(A, \( \sigma \)) channel denote by \( P_e(M, A, \sigma, n) \) the minimal possible decoding error probability and introduce the function

\[
F(M, A, \sigma) = \limsup_{n \to \infty} \frac{1}{n} \ln \frac{1}{P_e(M, A, \sigma, n)}.
\]

In the paper we consider the case when \( M \) is a fixed number of messages, or \( M = M_n \to \infty \) as \( n \to \infty \), but \( M_n = e^{o(n)} \) (it corresponds to zero-rate of transmission).
It is known that \( E(M, A) \) is attained for a simplex code \([17, 6]\)
\[
E(M, A) = \frac{AM}{4(M - 1)}. \tag{4}
\]

It is also known that if \( \sigma = 0 \) (i.e. in the case of noiseless feedback) then for a fixed \( M \)
\[
F(M, A, 0) = \frac{A}{2}. \tag{6}
\]

For AWGN\((A, \sigma)\) channel denote by \( F_1(M, A, \sigma) \) the best error exponent for the transmission method with one switching moment, described in §§2–3. Then \( F_1(M, A, \sigma) \leq F(M, A, \sigma) \) for all \( M, A, \sigma \).

One of two the paper main results is as follows.

**Theorem 1.** Let \( \ln M = o(n) \), \( n \to \infty \).

\( a) \) If \( \sigma \to 0 \) then the formula holds
\[
F_1(M, A, \sigma) \geq \frac{AM}{4(M - 1)} \left[ 1 + \frac{1}{2 + \sqrt{5}} - \frac{1}{2M} + o(1) \right]. \tag{5}
\]

Since \( 1/(2 + \sqrt{5}) \approx 0.236 \), then for large \( M \) the formula \([5]\) gives 23.6% of improvement with respect to no-feedback channel.

\( b) \) If \( \sigma \to \infty \) then the formula holds
\[
F_1(M, A, \sigma) \geq \frac{AM}{4(M - 1)} \left[ 1 + \frac{1}{56\sigma^2} + O(\sigma^{-4}) \right] > E(M, A) = \frac{AM}{4(M - 1)}. \tag{6}
\]

In §3 the second paper main results – a more general theorem 2, valid for any \( \sigma^2 < \infty \), is proved. Theorem 1 follows from it.

In a standard way reliability functions \( E(R, A) \) and \( F(R, A, \sigma) \) of no-feedback channel and AWGN\((A, \sigma)\) channel with noisy feedback can be defined. Then from theorem 1 we get

**Corollary.** \( a) \) For \( \sigma \to 0 \) and \( R = 0 \) the formula holds
\[
F(0, A, \sigma) \geq F_1(0, A, \sigma) \geq \frac{A}{4} \left[ 1 + \frac{1}{2 + \sqrt{5}} + o(1) \right]. \tag{7}
\]

\( b) \) For \( \sigma \to \infty \) and \( R = 0 \) the inequality holds
\[
F(0, A, \sigma) \geq F_1(0, A, \sigma) \geq \frac{A}{4} \left[ 1 + \frac{1}{56\sigma^2} + O(\sigma^{-4}) \right] > E(0, A) = \frac{A}{4}. \tag{8}
\]

In order to simplify formulas we will pay attention only to exponential (in \( n \)) terms, omitting power factors. Moreover, \( f \sim g \) means that \( n^{-1} \ln f = n^{-1} \ln g + o(1), n \to \infty \). Similarly \( f \lesssim g \), etc. is meant. Greek letters \( \xi, \eta, \zeta, \xi_1, \ldots \) designate \( \mathcal{N}(0, 1) \)–Gaussian random variables.
In § 2 the transmission method with one switching moment and in § 3 its decoding are described. In § 4 that method is investigated and general theorem 2 is proved. Using theorem 2 in § 5 theorem 1 is proved.

Some preliminary (and simplified) version of the paper results (without detailed proofs) were published in [18].

§ 2. Improved transmission/decoding method

We use the transmission strategy with one fixed switching moment at which the coding function will be changed. The transmission method used earlier in [10]–[13] (and in [16]) reduced the problem to testing of two most probable (at some fixed moment) messages. We improve that strategy in both transmission and decoding stages.

In order to simplify formulas we start with case \( M \leq \frac{(n+2)}{2} \). We partition the total transmission time \([1,n]\) on two phases: \([1,M-1]\) (phase I) and \([M,2M-2]\) (phase II). Thus the total length of the code used is \(2M-2\). The remaining time \([2M-1,n]\) is not used.

After moment \(2M-2\) the receiver makes a decision in favor of the most probable message \(\theta_i\) (based on all received on \([1,2M-2]\) signals).

Each of \(M\) codewords \(\{x_i\}\) of length \(2M-2\) have the form \(x_i = (x'_i, x''_i)\), where \(x'_i\) has length \(M-1\) (to be used on phase I) and \(x''_i\) has length \(M-1\) (to be used on phase II). Similarly, the received block \(y\) has the form \(y = (y', y'')\), where \(y'\) is the block received on phase I and \(y''\) is the block received on phase II. Denote by \(z'\) the received (by the transmitter) block on phase I. The codewords first parts \(\{x'_i\}\) are fixed, while the second parts \(\{x''_i\}\) will depend on the block \(z'\) received by the transmitter on phase I.

We set two positive constants \(A_1, A_2\) such that

\[ A_1 + A_2 = nA, \]  

and denote

\[ \beta = \frac{A_2}{A_1}, \quad A_3 = \frac{MA_1}{M-1}, \quad A_4 = \frac{MA_2}{M-1}, \quad \mu = A_2 \frac{M-1}{A_3}. \]  

Then \(A = (1+\beta)A_1/n\).

Denoting

\[ d_i = d(x'_i, y') = \|y' - x'_i\|^2, \]

arrange the distances \(\{d_i, i = 1, \ldots, M\}\) for the receiver after phase I in the increasing order, and denote

\[ d^{(1)} = \min_i d(x'_i, y') \leq d^{(2)} \leq \ldots \leq d^{(M)} = \max_i d(x'_i, y') \]

(case of tie has zero probability). Let also \(x^{(1)}, \ldots, x^{(M)}\) be the corresponding ranking of codewords \(\{x'\}\) after phase I for the receiver, i.e \(x^{(1)}\) is the closest to \(y'\) codeword, etc.

Similarly, denoting

\[ d_i^{(t)} = d(x'_i, z') = \|y' - z'_i\|^2, \]
arrange the distances \( \{d_i^{(t)}, i = 1, \ldots, M\} \) for the transmitter after phase I in the increasing order, and denote

\[
d^{(1)t} = \min_i d_i^{(t)} \leq d^{(2)t} \leq \ldots \leq d^{(M)t} = \max_i d_i^{(t)}.
\]

Let also \( x^{(1)t}, \ldots, x^{(M)t} \) be the corresponding ranking of codewords \( \{x^i\} \) after phase I for the transmitter, i.e. \( x^{(1)} \) is the closest to \( z' \) codeword, etc.

**Transmission.** On phase I the transmitter uses a simplex code of \( M \) codewords \( \{x'_i\} \) of length \( M - 1 \) such that \( \|x'_i\|^2 = A_1 \).

For phase II we set a number \( \tau_0 > 0 \). Based on the received block \( z' \) the transmitter selects three most probable codewords \( x^{(1)t}, x^{(2)t}, x^{(3)t} \) and calculates for them the value

\[
d_{23}^{(t)} = d^{(3)t} - d^{(2)t} = \tau A_3 \geq 0.
\]

The code \( \{x''_k\} \) with \( \|x''_k\|^2 = A_2, k = 1, \ldots, M \) used by the transmitter on phase II depends on codewords \( x^{(1)t}, x^{(2)t}, x^{(3)t} \) and the value \( \tau \) as follows.

C a s e 1. If after phase I

\[
d_{23}^{(t)} = \tau A_3 \leq \tau_0 A_3,
\]

then on phase II the transmitter uses the same simplex code of \( M \) codewords \( \{x'_i\} \) of length \( M - 1 \), such that \( \|x''_k\|^2 = A_2 \).

C a s e 2. If after phase I

\[
d_{23}^{(t)} = \tau A_3 > \tau_0 A_3,
\]

then on phase II the transmitter uses another code \( \{x''_k\} \) with \( \|x''_k\|^2 = A_2, k = 1, \ldots, M \):

a) two most probable messages \( \theta_i, \theta_j \) have opposite codewords \( x''_i = -x''_j \) which have nonzero coordinates only at moment \( M - 2 \);

b) remaining \( M - 2 \) messages \( \{\theta_k\} \) use a simplex code of \( M - 2 \) codewords \( \{x''_k\} \) of length \( M - 3 \) trailed by 0 at moment \( M - 2 \). All those codewords \( \{x''_k\} \) are orthogonal to the first two codewords \( \{x'_1, x'_2\} \).

This transmission method strengthens the method used in \([10]–[13]\). The code used in case I helps in the case when after phase I three most probable codewords \( x^{(1)t}, x^{(2)t}, x^{(3)t} \) are approximately equiprobable.

**Decoding.** Due to noise in the feedback channel the receiver does not know exactly codewords \( x^{(1)t}, x^{(2)t}, x^{(3)t} \) and the value \( \tau \) for them, and therefore it does not know the code used on phase II. But it may evaluate probabilities of all possible codewords \( x^{(1)t}, x^{(2)t}, x^{(3)t} \) and the value \( \tau \) for them, and so find the probabilities with which any code was used.

It allows to the receiver, based on the received block \( y \), to find posterior probabilities \( \{p(y|x_i)\} \) and make decision in favor of most probable message \( \theta_i \). Such full decoding is described in details below.

### § 3. Full decoding and error probability \( P_e \)

Note that

\[
\ln \frac{p(y|x_2)}{p(y|x_1)} = (x_2 - x_1, y) - \frac{1}{2} (\|x_2\|^2 - \|x_1\|^2).
\]
If \( x_{\text{true}} \) is the true codeword then \( y = x_{\text{true}} + \xi \) and \( \xi = (\xi', \xi'') = (\xi_1, \ldots, \xi_n) \), where all \( \{\xi_i\} \) are independent \( \mathcal{N}(0,1) \)-Gaussian random variables. If \( x_{\text{true}} = x_1 \), then

\[
\ln \frac{p(y|x_2)}{p(y|x_1)} = (x_2 - x_1, \xi) - \frac{1}{2}||x_2 - x_1||^2,
\]

where \( (x, \xi) \) is \( \mathcal{N}(0, ||x||^2) \)-Gaussian random variable.

The receiver makes decision after moment \( n \) using all received block \( y \). If after phase I the difference \( d^{(3)} - d^{(2)} \) is rather close to \( \tau_0 A_3 \) (see \((11)\) and \((12)\)) then due to noise in the feedback link the receiver can not be sure which code was used by the transmitter on phase II (since lists \( \{x^{(1)}_i, x^{(2)}_i\} \) and \( \{x^{(1)}', x^{(2)}'\} \) may turn out to be different). But based on \( y' \) the receiver knows the probability distribution of the code used by the transmitter on phase II. Then in the decoding it should take into account that distribution.

Note that if \( \theta_{\text{true}} = \theta_1 \) then

\[
d_i - d_l = 2A_3 + 2(x'_i - x'_l, \xi'), \quad i = 2, \ldots, M.
\]

If \( \theta_{\text{true}} = \theta_1 \), then for decoding error probability \( P_e \) we have

\[
P_e = \mathbf{P} \left\{ \max_{t \geq 2} \ln \frac{p(y|\theta_t)}{p(y|\theta_1)} \geq 0|\theta_1 \right\} \leq M \mathbf{P} \left\{ \ln \frac{p(y|\theta_2)}{p(y|\theta_1)} \geq 0|\theta_1 \right\} = M \mathbf{P} \left\{ X + Y \geq 0|\theta_1 \right\},
\]

where

\[
X = \ln \frac{p(y'|\theta_2)}{p(y'|\theta_1)}, \quad Y = \ln \frac{p(y''|\theta_2)}{p(y''|\theta_1)}, \quad i = 2, \ldots, M.
\]

In order to investigate random variable \( Y \) introduce the following sets of random events (conditions):

\[
Z_1 = \left\{ z' : d^{(t)}_{23} \leq \tau_0 A_3 \right\},
\]

\[
Z_2 = \left\{ z' : d^{(t)}_{23} > \tau_0 A_3, \{x'_1, x'_2\} = \{x^{(1)}_i, x^{(2)}_i\} \right\},
\]

\[
Z_3 = \left\{ z' : d^{(t)}_{23} > \tau_0 A_3, \left| \{x'_1, x'_2\} \cap \{x^{(1)}_i, x^{(2)}_i\} \right| = 1 \right\},
\]

\[
Z_4 = \left\{ z' : d^{(t)}_{23} > \tau_0 A_3, \{x'_1, x'_2\} \cap \{x^{(1)}_i, x^{(2)}_i\} = \emptyset \right\}.
\]

We assume that the true message is \( \theta_1 \). Then using sets \( Z_2, Z_3, Z_4 \) it will be possible to describe all possible relations between pairs \( \{x^{(1)}_i, x^{(2)}_i\} \) and \( \{x^{(1)}', x^{(2)'}\} \) of most probable messages for the receiver and the transmitter, respectively.

Denote

\[
p_k = \mathbf{P}(Z_k|y', x'_1), \quad k = 1, \ldots, 4.
\]
We have
\[
\frac{p(y''|y', \theta_2)}{p(y''|y', \theta_1)} = \mathbb{E}_{z'}p(y''|z', x''_1) = \mathbb{E}_{z'}\left[ y'' \sum_{k=1}^{4} p_k e^{(y'' - x''_2 - x''_1)} \right],
\]
where blocks $x''_1, x''_2$ depend on $k$ (via $z_k$). If $\theta_{true} = \theta_1$, then $y'' = x''_1 + \xi''$, and
\[
e^{-A_2} \sum_{k=1}^{4} p_k e^{(x''_1, x''_2) + (\xi'' - x''_2)} = \sum_{k=1}^{4} p_k e^{(\xi'' - x''_2)}.
\]
Therefore
\[
P_e \leq M \mathbb{E}_{y''} \mathbb{P} \{ X + Y = 0 \mid \theta_1 \} = M \mathbb{E}_{y''} \mathbb{P} \{ X + Y = 0 \mid y', \theta_1 \} \leq M \mathbb{E}_{y''} \mathbb{P} \{ X + \ln 4 - A_2 + \max_k \{ (x''_1, x''_2) + (\xi'' - x''_2) + \ln p_k \} \geq 0 \mid y', \theta_1 \} \leq M \sum_{k=1}^{4} \mathbb{E}_{y''} \mathbb{P} \{ X + \ln 4 - A_2 + (x''_1, x''_2) + (\xi'' - x''_2) + \ln p_k \geq 0 \mid y', \theta_1 \} \leq \frac{1}{2} M \sum_{k=1}^{4} \mathbb{E} \xi_e \exp \left\{ -\frac{(A_2 - X - (x''_1, x''_2) - \ln p_k)_{+}^2}{2 ||x''_2 - x''_1||^2} \right\} = \frac{1}{2} M \sum_{k=1}^{4} e^{-B_k},
\]
where
\[
B_k = -\ln \mathbb{E} \xi_e e^{-b_k}, \quad k = 1, \ldots, 4,
\]
\[
b_k = \frac{[A_2 + A_3 - (x''_1 - x'_1, \xi') - (x''_1, x''_2) - \ln p_k]_{+}^2}{2 ||x''_2 - x''_1||^2}.
\]
Here $(x''_1 - x'_1, \xi') = \sqrt{2A_3} \xi_i$, where $\xi \sim \mathcal{N}(0, 1)$, for all $k$, and $x''_1, x''_2$ depend on $k$. In particular,
\[
b_1 = \frac{1}{4A_4} [A_3 + A_4 - (x''_1 - x'_1, \xi') - \ln p_1]_{+}^2,
\]
\[
b_2 = \frac{1}{8A_2} [A_3 + 2A_2 - (x''_1 - x'_1, \xi') - \ln p_2]_{+}^2,
\]
\[
b_3 = \frac{1}{4A_2} [A_3 + A_2 - (x''_1 - x'_1, \xi') - \ln p_3]_{+}^2,
\]
\[
b_4 = \frac{(M - 3)}{4A_2(M - 2)} \left[ A_3 + \frac{(M - 2)A_2}{M - 3} - (x''_1 - x'_1, \xi') - \ln p_4 \right]_{+}^2.
\]
Then
\[
F_1(M, A, \sigma) \geq \min_{k=1, \ldots, 4} B_k. \quad (14)
\]

8
We should find values $B_k, p_k, k = 1, \ldots, 4$ and choose optimal parameters $\beta, \tau_0$. We show below that in interesting for us cases probabilities $p_1, p_3, p_4$ are small, and therefore the probability $p_2$ is close to 1. Moreover, we omit estimates for values $p_4, B_4$, since clearly $p_4 < p_3$ and $B_4 \geq B_3$.

We start with the simplest term $B_2$. Note that if $\xi \sim N(0, 1)$, then

$$E e^{-a(b-\xi)^2/2} = \frac{e^{-ab^2/(2+2a)}}{\sqrt{1+a}}, \quad a > -1.$$ 

Neglecting $\ln p_k$, we get (as $n \to \infty$)

$$E \xi' e^{-b_2} \leq P\{b_2 = 0\} + E \exp \left\{ -\frac{1}{8A_2} [A_3 + 2A_2 - (x'_2 - x'_1, \xi')]^2 \right\} =$$

$$= P \left\{ \sqrt{2A_3} \xi \geq A_3 + 2A_2 \right\} + E \exp \left\{ -\frac{A_3 + 2A_2}{4A_2} \left[ \frac{A_3 + 2A_2}{\sqrt{2A_3}} - \xi \right]^2 \right\} \leq$$

$$\leq \Phi \left( \frac{-A_3 + 2A_2}{\sqrt{2A_3}} \right) + \exp \left\{ -\frac{A_3 + 2A_2}{4} \right\} \leq$$

$$\leq 2 \exp \left\{ -\frac{A_3 + 2A_2}{4} \right\} = 2 \exp \left\{ -\frac{MA_n(1+2\mu)}{4(M-1)(1+\beta)} \right\},$$

where we used simple inequality

$$\Phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du \leq \frac{1}{2} e^{-z^2/2}, \quad z \geq 0. \quad (15)$$

Inequality (15) will be regularly used in the paper. Therefore

$$B_2 \geq \frac{MA(1+2\mu)}{4(M-1)(1+\beta)} - \frac{1}{n}. \quad (16)$$

Calculation of values $B_1, B_3$ will demand more efforts. It is done in the next section.

**§ 4. Probabilities $p_1, p_3$ and values $B_1, B_3$. Theorem 2**

It will be convenient to use the following technical result, which allows instead of a simplex code to consider an orthogonal code. Let $\{z_i \in \mathbb{R}^n, i = 1, \ldots, M\}$ - a simplex code with $\|z_i\|^2 = A$ and $M \leq n$. Since $\sum_{k=1}^{M} z_k = 0$, then denoting $r = A/(M-1)$, we have

$$\|z_i - z_j\|^2 = 2rM; \quad (z_i, z_j) = -r, \quad i \neq j; \quad (z_3 - z_1, z_3 - z_2) = rM.$$ 

Set an arbitrary vector $u_0 \in \mathbb{R}^n$, such that $u_0 \perp \{z_1, \ldots, z_M\}$ and $\|u_0\|^2 = r$, and consider vectors $u_i = z_i + u_0, i = 1, \ldots, M$. Then $(u_i, u_j) = 0, i \neq j$ and $\|z_i - z_j\| = \|u_i - u_j\|$ for any $i, j$. In particular, we have $u_0 = M^{-1} \sum_{j=1}^{M} u_j$. This result can be formulated as follows.
Proposition 1. Let \( \{z_i \in \mathbb{R}^n, i = 1, \ldots, M\} \) be a simplex (i.e. equidistant) code with \( \|z_i\|^2 = A \). Then it can be represented as

\[
  z_i = u_i - u_0, \quad i = 1, \ldots, M, \quad u_0 = \frac{1}{M} \sum_{j=1}^{M} u_j, \tag{17}
\]

where \( \{u_i, i = 1, \ldots, M\} \) are mutually orthogonal (i.e. \( (u_i, u_j) = 0 \) for \( i \neq j \)) vectors with \( \|u_i\|^2 = AM/(M-1) \).

Using Proposition 1 we replace vectors \( \{x'_i\} \) by orthogonal vectors \( \{u_i = x'_i + u_0\} \) such that \( \|u_i\|^2 = A_3 \) and \( (u_i, u_j) = 0, i \neq j \). Then \( x'_1 - x'_i = u_1 - u_i \). Denote

\[
  (u_i, \xi'_i) = \sqrt{A_3} \xi_i = u_i A_3, \quad (u_i, \eta'_i) = \sqrt{A_3} \eta_i = v_i A_3, \quad i = 1, \ldots, M.
\]

Note that if we would omit \( \{\ln p_i\} \) from \( \{b_i\} \), then, for example, we have

\[
  B_1 = -\ln E \mathbf{e}^{b_1} \sim \min_{u_1, u_2} \left\{ \frac{1}{4A_4} \left[ A_3 + A_4 + u_1 \sqrt{A_3} - u_2 \sqrt{A_3} \right]^2 + u_1^2 + u_2^2 \right\} = \frac{A_3(1 + \beta)}{4},
\]

which corresponds to no-feedback case. Similar estimates would hold for \( B_3 \) as well. Therefore for given \( y' = \{u_i\} \) we should evaluate and take into account conditional probabilities \( p_k = P(Z_k | y', x'_1) \).

Note also that for large \( M \) values \( b_1, b_3, b_4 \) are approximately equal. Then it would be sufficient to evaluate how close is \( p_2 \) to 1.

In order to evaluate \( p_1 \), introduce events

\[
  A_{12} = \left\{ \max \{d_1^{(t)}, d_2^{(t)}\} \leq \min_{i \geq 3} d_i^{(t)} \leq \max \{d_1^{(t)}, d_2^{(t)}\} + \tau_0 A_3 \right\} = \left\{ (d_1^{(t)} - d_2^{(t)})_+ \leq \min_{i \geq 3} d_i^{(t)} - d_1^{(t)} \leq (d_2^{(t)} - d_1^{(t)})_+ + \tau_0 A_3 \right\},
\]

\[
  A_{13} = \left\{ \max \{d_1^{(t)}, d_3^{(t)}\} \leq d_2^{(t)} \leq \max \{d_1^{(t)}, d_3^{(t)}\} + \tau_0 A_3 \right\},
\]

\[
  A_{23} = \left\{ \max \{d_2^{(t)}, d_3^{(t)}\} \leq d_1^{(t)} \leq \max \{d_2^{(t)}, d_3^{(t)}\} + \tau_0 A_3 \right\}.
\]

Then

\[
  p_1 \leq (M - 1)^2 [P \{A_{12} | y', x'_1\} + P \{A_{23} | y', x'_1\} + P \{A_{13} | y', x'_1\}] \leq 3(M - 1)^2 P \{A_{12} | y', x'_1\},
\]

since \( P \{A_{23} | y', x'_1\} \leq \min \{P \{A_{12} | y', x'_1\}, P \{A_{13} | y', x'_1\}\} \), and due to symmetry

\[
  P \{A_{12} | y', x'_1\} \sim P \{A_{13} | y', x'_1\}. \]

Therefore it is sufficient to evaluate the probability

\[
  P \{A_{12} | y', x'_1\}.
\]

If \( x'_1 \) is the true \( x'_1 \), then for \( i \geq 2 \)

\[
  d_i^{(t)} - d_1^{(t)} = 2(u_1 - u_i, \xi' + \sigma \eta') + 2A_3 = 2A_3(1 + u_1 - u_i) + 2\sigma \sqrt{A_3} (\eta_1 - \eta_i).
\]
Denote
\[ w_i\sigma = (1 + u_1 - u_i)\sqrt{A_3}, \quad i = 2, \ldots, M, \quad s\sigma = \tau_0\sqrt{A_3}/2. \] (19)

Then \( \{\eta_i\} \) are independent \( \mathcal{N}(0,1) \)-Gaussian random variables.

\[
P\{A_{12}|y', x'_1\} = P \left\{ 0 \leq \eta_1 - (w_2 + \eta_1 - \eta_2)_+ + \min_{i \geq 3} \{w_i - \eta_i\} \leq s|y'\right\} = 
\]
\[ = P \left\{ 0 \leq \eta_1 + \min_{i \geq 3} \{w_i - \eta_i\} \leq s, w_2 + \eta_1 - \eta_2 < 0|y'\right\} + 
\]
\[ + P \left\{ 0 \leq \eta_2 - w_2 + \min_{i \geq 3} \{w_i - \eta_i\} \leq s, w_2 + \eta_1 - \eta_2 \geq 0|y'\right\} \leq 
\]
\[ \leq (M - 2) \left[ P \{w_3 + \eta_1 - \eta_3 \leq s\} + P \{w_3 - w_2 + \eta_2 - \eta_3 \leq s\} \right] \leq 
\]
\[ \leq 2(M - 2)e^{-[w_3-(w_2)+-s]^2/4}, \]

where on the last step the inequality (15) was used.

Using (18), for \( p_1 \) we get
\[
\ln p_1 \leq -[w_3 - (w_2)_+-s]^2/4 + \ln(6M^3). \] (20)

Consider values \( b_1 \) and \( B_1 \). Below in brackets, for short, we omit relatively small term \( \ln(6M^3) \), but it will be taken into account in the final result. Using (20) we have
\[
b_1 \geq \frac{1}{4A_4} \left[ A_3 + A_4 - (u_2 - u_1, \xi') + \frac{1}{4}(w_3 - (w_2)_+-s)^2 \right]_+ = 
\]
\[ = \frac{A_3}{4\beta} \left[ 1 + \beta + u_1 - u_2 + \frac{1}{4A_3}(w_3 - (w_2)_+-s)^2 \right]_+ = 
\]
\[ = \frac{A_3}{4\beta} \left[ \beta + y_2 + \gamma(y_3 - (y_2)_+-\tau_0/2)^2 \right]_+^2, \] (21)

where we denoted
\[
\gamma = 1/(4\sigma^2), \quad y_i = \sigma u_i/\sqrt{A_3} = 1 + u_1 - u_i. \] (22)

Therefore (if integration limits are not pointed out then it is done over all possible area)
\[
\left(\frac{2\pi}{A_3}\right)^{3/2} E_{\xi} e^{-b_1} = \iiint \exp \left\{ -b_1(y_2, y_3) - \frac{A_3(u_1^2 + u_2^2 + u_3^2)}{2} \right\} du_1 du_2 du_3 = 
\]
\[ = \iiint \exp \left\{ -b_1(y_2, y_3) - \frac{A_3}{2} \left[ u_1^2 + (1 + u_1 - y_2)^2 + (1 + u_1 - y_3)^2 \right] \right\} du_1 dy_2 dy_3 = 
\]
\[ = \sqrt{\frac{2\pi}{3A_3}} \int \int e^{-b_1(y_2, y_3) - A_3 g(y_2, y_3)/3} = \sqrt{\frac{2\pi}{3A_3}} \int \int e^{-A_3 f_1(y_2, y_3)/(12\beta)} dy_2 dy_3, \] (23)
where
\[ g(u,v) = 1 + u^2 + v^2 - uv - u - v = \left( v - \frac{1 + u}{2} \right)^2 + \frac{3(1 - u)^2}{4}, \]
(24)

\[ f_1(y_2, y_3) = 3 \left[ \beta + y_2 + \gamma(y_3 - (y_2) + - \tau_0/2)^2_+ + 4\beta g(y_2, y_3). \right. \]
Represent the last integral in the right-hand side of (23) as follows
\[ \int \int e^{-A_3 f_1(y_2, y_3)/(12\beta)} dy_2 dy_3 = I_1 + I_2 + I_3, \]
(25)

where
\[ I_i = \int \int_{V_i} e^{-A_3 f_1(y_2, y_3)/(12\beta)} dy_2 dy_3, \quad i = 1, 2, 3, \]
(26)
and
\[ V_1 = \{ \beta + y_2 \leq 0 \}, \quad V_2 = \{-\beta \leq y_2 \leq 0 \}, \quad V_3 = \{ y_2 \geq 0 \}. \]
(27)

We evaluate consecutively integrals \( I_1, I_2, I_3. \) For \( I_1 \) we have
\[ I_1 \leq \int \int_{V_1} e^{-A_3 g(y_2, y_3)/3} dy_2 dy_3 = \]
\[ = \int_{y_2 \leq -\beta}^{\infty} \int_{-\infty}^{e^{-A_3(y_2-1)^2/4} e^{-A_3(2y_3-y_2-1)^2/12} dy_3 dy_2} \sim e^{-A_3(1+\beta)^2/4}. \]
(28)

Consider the integral \( I_2. \) Denoting \( z = \gamma(y_3 - \tau_0/2)^2_+, \) we have
\[ f_1(y_2, y_3) = 3(y_2 + \beta + z)^2 + 4\beta g(y_2, y_3) = \]
\[ = (3 + 4\beta) \left[ y_2 + \frac{3z + \beta(1 - 2y_3)}{3 + 4\beta} \right]^2 + 3\beta(1 + \beta) + f_3(y_3), \]
\[ f_3(y_3) = \frac{3\beta}{(3 + 4\beta)} \left\{ (1 + \beta)(1 - 2y_3)^2 + 4z(z + 1 + 2\beta + y_3) \right\}. \]
Therefore
\[ I_2 \leq \int \int_{y_2 \geq -\beta} e^{-A_3 f_1(y_2, y_3)/(12\beta)} dy_2 dy_3 \lesssim e^{-A_3(1+\beta)^2/4} \int_{-\infty}^{\infty} e^{-A_3 f_3(y_3)/(12\beta)} dy_3. \]
If \( y_3 \leq \tau_0/2, \) then \( z = 0. \) Set some level \( \tau_0/2 < u < 1/2. \) Then denoting \( z_0 = \gamma(u - \tau_0/2)^2, \)
we have
\[\int_{-\infty}^{\infty} e^{-A_3 f_3(y_3)/12\beta} dy_3 \leq \int_{-\infty}^{u} \exp \left\{ -\frac{A_3(1+\beta)}{4(3+4\beta)} (1-2y_3)^2 \right\} dy_3 + \int_{u}^{\infty} \exp \left\{ -\frac{A_3(1+\beta)(1-2y_3)^2}{4(3+4\beta)} - \frac{A_3 z_0(z_0 + 1 + 2\beta + u)}{3+4\beta} \right\} dy_3 \lesssim \exp \left\{ -\frac{A_3(1+\beta)}{4(3+4\beta)} (1-2u)^2 \right\} + \exp \left\{ -\frac{A_3 z_0(z_0 + 1 + 2\beta + u)}{3+4\beta} \right\}.
\]

Set \( u \), such that \((1+\beta)(1-2u)^2 = 4z_0(1+2\beta) \), i.e. set
\[2u = \frac{\sqrt{1+\beta} + \tau_0 \sqrt{\gamma(1+2\beta)}}{\sqrt{1+\beta} + \sqrt{\gamma(1+2\beta)}}.
\]

Then we get
\[\int_{-\infty}^{\infty} e^{-A_3 f_3(y_3)/12\beta} dy_3 \lesssim \exp \left\{ -\frac{A_3(1+\beta)(1-\tau_0)^2}{4(3+4\beta)(1+2\sigma)^2} \right\}
\]
and therefore
\[I_2 \lesssim \exp \left\{ -\frac{A_3(1+\beta)}{4} \left[ 1 + \frac{(1-\tau_0)^2}{(3+4\beta)(1+2\sigma)^2} \right] \right\}. \tag{29}
\]

Consider the integral \( I_3 \) from (28). Represent it as follows
\[I_3 = I_{31} + I_{32},
\]
\[I_{31} = \iint_{y_2 \geq 0, y_3 \leq y_2 + \tau_0/2} e^{-A_3 f_1(y_2,y_3)/(12\beta)} dy_2 dy_3,
\]
\[I_{32} = \iint_{y_2 \geq 0, y_3 \geq y_2 + \tau_0/2} e^{-A_3 f_1(y_2,y_3)/(12\beta)} dy_2 dy_3. \tag{30}
\]

If \( y_3 \leq y_2 + \tau_0/2 \), then
\[f_1(y_2,y_3) = \beta (2y_3 - 1 - y_2)^2 + 3(1 + \beta)(\beta + y_2^2).
\]

Integrating first over \( y_3 \), and then over \( y_2 \), we get
\[I_{31} \lesssim e^{-A_3(1+\beta)/4} \int_{0}^{\infty} \exp \left\{ -\frac{A_3}{12} (1-y_2 - \tau_0)^2 - \frac{A_3}{4\beta} (1+\beta) y_2^2 \right\} dy_2 \leq \]
\[\leq e^{-A_3(1+\beta)/4} \int_{0}^{\infty} \exp \left\{ -\frac{A_3 (3+4\beta)}{12\beta} \left[ y_2 - \frac{\beta(1-\tau_0)}{3+4\beta} \right]^2 - \frac{A_3 (1+\beta)(1-\tau_0)^2}{4(3+4\beta)} \right\} dy_2 + \]
\[+ e^{-A_3(1+\beta)/4} \int_{1-\tau_0}^{\infty} e^{-A_3(1+\beta) y_2^2/(4\beta)} dy_2 \lesssim \exp \left\{ -\frac{A_3(1+\beta)}{4} \left[ 1 + \frac{(1-\tau_0)^2}{3+4\beta} \right] \right\}.
\]
Consider the integral \( I_{32} \). Denoting \( u = y_3 - y_2 - \tau_0 / 2 \), we have

\[
f_1(y_2, y_3) \geq f_4(y_2, u) = 3(\beta + y_2)^2 + 6\gamma(\beta + y_2)u^2 + 4\beta \left[ 1 + y_2^2 + (u + \tau_0 / 2)^2 + y_2(u + \tau_0 / 2) - 2y_2 - u - \tau_0 / 2 \right] =
\]

\[
u^2[6\gamma(\beta + y_2) + 4\beta] - 4\beta u(1 - \tau_0 - y_2) + 3(\beta + y_2)^2 + \beta(4 + 4y_2^2 + \tau_0^2 + 2y_2\tau_0 - 8y_2 - 2\tau_0).
\]

First we integrate over \( u \) and then over \( y_2 \). Since

\[
f_4(y_2, 0) = \left[ y_2\sqrt{3 + 4\beta} - \frac{\beta(1 - \tau_0)}{\sqrt{3 + 4\beta}} \right]^2 - \frac{\beta^2(1 - \tau_0)^2}{3 + 4\beta} + 3\beta^2 + \beta(4 - 2\tau_0 + \tau_0^2),
\]

we have

\[
I_{32} \leq \int \int e^{-A_3 f_4(y_2, u)/(12\beta)} dudy_2 \lesssim e^{A_3(1 - \tau_0)^2/[6(3\gamma + 2)]} \int e^{-A_3 f_4(y_2, 0)/(12\beta)} dy_2 \lesssim \exp \left\{ -\frac{A_3(1 + \beta)}{4} \left[ 1 + \frac{(1 - \tau_0)^2}{3 + 4\beta} - \frac{2(1 - \tau_0)^2}{3(1 + \beta)(3\gamma + 2)} \right] \right\}
\]

then

\[
I_3 \lesssim \exp \left\{ -\frac{A_3(1 + \beta)}{4} \left[ 1 + \frac{(1 - \tau_0)^2}{3 + 4\beta} - \frac{2(1 - \tau_0)^2}{3(1 + \beta)(3\gamma + 2)} \right] \right\} \tag{32}
\]

That estimate is applicable for all \( \gamma > 0 \). If \( \gamma \) is small, then \( \beta \) should be chosen such that

\[
\beta < 9\gamma / (2 - 9\gamma).
\]

As a result, from (23), (25), (28), (29) and (32) we get

\[
E\xi e^{-b_1} \lesssim e^{-A_3(1 + \beta)^2 / 4} + \exp \left\{ -\frac{A_3(1 + \beta)}{4} \left[ 1 + \frac{(1 - \tau_0)^2}{(3 + 4\beta)(1 + 2\sigma)^2} \right] \right\} + \exp \left\{ -\frac{A_3(1 + \beta)}{4} \left[ 1 + \frac{(1 - \tau_0)^2}{3 + 4\beta} - \frac{2(1 - \tau_0)^2}{3(1 + \beta)(3\gamma + 2)} \right] \right\}
\]

and then

\[
B_1 \geq \frac{A_3(1 + \beta)}{4n} \times \left[ 1 + \min \left\{ \beta, \frac{(1 - \tau_0)^2}{(3 + 4\beta)(1 + 2\sigma)^2}, \frac{(1 - \tau_0)^2}{3 + 4\beta} - \frac{2(1 - \tau_0)^2}{3(1 + \beta)(3\gamma + 2)} \right\} \right] - \frac{3 \ln M}{n}, \tag{34}
\]

where the last term in the right-hand side of (34) takes into account the term omitted in (21).
Values $p_3$ and $B_3$ are evaluated similarly to values $p_1, B_1$. Introduce sets
\[ Z_{31} = \{ z' : d_{23}^{(t)} > \tau_0 A_3, \{ x''(1)^t, x''(2)^t \} = \{ x_1', x_3' \} \}, \]
\[ Z_{32} = \{ z' : d_{23}^{(t)} > \tau_0 A_3, \{ x''(1)^t, x''(2)^t \} = \{ x_2', x_3' \} \]
and consider conditional probabilities $P_{31} = P \{ Z_{31} | y', x_1' \}$ and $P_{32} = P \{ Z_{32} | y', x_1' \}$. Then
\[ p_3 \leq (M - 2) (P_{31} + P_{32}) \leq 2(M - 2) p_{31}, \]
since $P_{31} \geq P_{32}$. Then it is sufficient to evaluate $P_{31}$. Using notations (19), we have
\[ P_{31} = P \left\{ \min_{i=2,4,5,...} d_i^{(t)} > \max \{ d_1^{(t)}, d_3^{(t)} \} + \tau_0 A_3 | y', x_1' \right\} = \]
\[ = P \left\{ \min_{i=2,4,5,...} [w_i + 1 - \eta_1] > (w_3 + 1 - \eta_3) + s | y' \right\} \leq \]
\[ \leq P \left\{ w_2 + 1 - 2 > (w_3 + 1 - 3) + s | y' \right\} = q_1 + q_2, \]
where
\[ q_1 = P \{ \eta_1 - \eta_2 > s - w_2, \eta_3 - \eta_1 > w_3 \}, \]
\[ q_2 = P \{ \eta_3 - \eta_2 > s - w_2 + w_3, \eta_1 - \eta_3 > -w_3 \}. \]
Here, for example, $\eta_1 - \eta_2 \sim N(0, 1)$. For probabilities $q_1, q_2$ we use simple estimates (see (19))
\[ \ln q_1 \leq -\frac{1}{4} \left[ \max \{ s - w_2, w_3 \} \right]_+^2, \quad \ln q_2 \leq -\frac{1}{4} \left[ \max \{ s - w_2 + w_3, -w_3 \} \right]_+^2. \]
Those estimates turn out to be sufficiently accurate, although it is possible to strengthen them using dependence among random variables. Then
\[ \ln(P_{31}/2) \leq -\frac{1}{4} \left[ \min \left\{ \max \{ s - w_2, w_3 \}, \max \{ s - w_2 + w_3, -w_3 \} \right\} \right]_+^2. \]
Using notations (22), after standard analysis we get
\[ \ln p_3 - \ln(4M) \leq -\gamma A_3 r_2^2(y_2, y_3, \tau_0), \]
where
\[ r(y_2, y_3, \tau_0) = \begin{cases} \min \{ \tau_0/2 - y_2, -y_3 \}, & y_2 \leq \tau_0/2, y_3 \leq 0, \\ y_3, & y_2 \leq \tau_0/2, y_3 \geq 0, \\ 0, & y_2 \geq \tau_0/2, y_3 \leq y_2 - \tau_0/2, \\ \tau_0/2 - y_2 + y_3, & y_2 \geq \tau_0/2, y_3 \geq y_2 - \tau_0/2. \end{cases} \]
We have
\[ b_3 = \frac{1}{4A_2} \left[ A_3 + A_2 - (u_2' - u_1', \xi') - \ln p_3 \right]_+^2 \geq \frac{A_3}{4\mu} \left[ \mu + y_2 + \gamma r_2^2(y_2, y_3, \tau_0) \right]_+^2, \]
where \((y_2, y_3) \in \mathbb{R}^2\). In order to simplify the right-hand side of (37), first we evaluate contribution to \(B_3\) of points \((y_2, y_3) \in D_0\), where

\[
D_0 = \{y_2, y_3 : y_2 \leq -\mu\}.
\]

Using simple inequality \(f_3(y_2, y_3) \geq 4\mu g(y_2, y_3)\), and integrating first over \(y_2\), and then over \(y_3\), we get

\[
J_0 = \int \int_{D_0} e^{-A_3g(y_2, y_3)/3} dy_2 dy_3 \lesssim \int_{-\infty}^{\infty} e^{-A_3(2y_3 - 1 + \mu)^2/12 - A_3(1 + \mu)^2/4} dy_3 \sim e^{-A_3(1 + \mu)^2/4}. \tag{38}
\]

For remaining points \((y_2, y_3) \in \mathbb{R}^2 \setminus D_0 = \{y_2, y_3 : y_2 > -\mu\} = \mathcal{D}\) we have

\[
b_3 \geq \frac{A_3}{4\mu} \left[\mu + y_2 + \gamma r^2(y_2, y_3, \tau_0)\right]^2. \tag{39}
\]

In order to use the formula (36) it is convenient to partition the remaining integration area \(\mathcal{D}\) on four parts

\[
\mathcal{D} = \sum_{i=1}^{4} D_i, \tag{40}
\]

where

\[
D_1 = \{-\mu \leq y_2 \leq \tau_0/2, y_3 \leq 0\}, \\
D_2 = \{-\mu \leq y_2 \leq \tau_0/2, y_3 \geq 0\}, \\
D_3 = \{y_2 \geq \tau_0/2, y_3 \leq y_2 - \tau_0/2\}, \\
D_4 = \{y_2 \geq \tau_0/2, y_3 \geq y_2 - \tau_0/2\}. \tag{41}
\]

Then similarly to (23) we have

\[
\frac{2\pi \sqrt{3}}{A_3} E_{\xi} e^{-b_3} \leq J_0 + \sum_{i=0}^{4} J_i, \tag{42}
\]

where

\[
J_i = \int \int_{D_i} e^{-A_3f_3(y_2, y_3)/(12\mu)} dy_2 dy_3, \quad i = 1, \ldots, 4, \tag{43}
\]

\[
f_3(y_2, y_3) = 3 \left[\mu + y_2 + \gamma r^2(y_2, y_3, \tau_0)\right]^2 + 4\mu g(y_2, y_3),
\]

and \(g(u, v)\) is defined in (24).

We evaluate consecutively integrals \(J_1, \ldots, J_4\), starting with \(J_1\). For \((y_2, y_3) \in D_1\) we have

\[
f_3(y_2, y_3) \geq 3(\mu + y_2)^2 + \mu (2y_3 - 1 - y_2)^2 + 3\mu(1 - y_2)^2.
\]


Integrating first over \( y_3 \leq 0 \), and then over all \( y_3 \), we get

\[
J_1 = \int\int_{D_1} e^{-A_3 f_3(y_2, y_3)/(12\mu)} dy_3 dy_2 \lesssim \int \exp \left\{ -\frac{A_3}{12\mu} \left[ 3(\mu + y_2)^2 + \mu(1 + y_2)^2 + 3\mu(1 - y_2)^2 \right] \right\} dy_2 \lesssim \int \exp \left\{ -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{1}{3 + 4\mu} \right] \right\}.
\]

(44)

Consider the integral \( J_2 \). Then

\[
f_3(y_2, y_3) = 3(\mu + y_2 + \gamma y_3^2)^2 + \mu(2y_2 - 1 - y_3)^2 + 3\mu(1 - y_3)^2 = \left[ \sqrt{3 + 4\mu y_2 + \frac{\mu - 2\mu y_3 + 3\gamma y_3^2}{\sqrt{3 + 4\mu}}} \right]^2 + f_{31}(y_3),
\]

where

\[
f_{31}(y_3) = \frac{12\mu}{(3 + 4\mu)} \left\{ (1 + \mu)^2 + \gamma^2 y_3^4 + \gamma y_3^2(1 + 2\mu + y_3) + (1 + \mu)(y_3^2 - y_3) \right\} \geq \frac{12\mu(1 + \mu)}{(3 + 4\mu)} \left\{ 1 + \mu + (\gamma + 1) \left[ y_3 - \frac{1}{2(\gamma + 1)} \right]^2 - \frac{1}{4(\gamma + 1)} \right\}.
\]

Therefore integrating first over all \( y_2 \), and then over all \( y_3 \), we get

\[
J_2 \lesssim \exp \left\{ -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{1}{3 + 4\mu} - \frac{1}{(3 + 4\mu)(\gamma + 1)} \right] \right\}.
\]

(45)

For the integral \( J_3 \) similarly to (44) we get

\[
J_3 \lesssim \exp \left\{ -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{1}{3 + 4\mu} \right] \right\}.
\]

(46)

Consider the integral \( J_4 \). Denoting \( z = \tau_0/2 - y_2 + y_3 \), we have

\[
f_3(y_2, z) = (3 + 4\mu) \left[ y_2 + \frac{3\gamma z^2 + \mu(2z - \tau_0 - 1)}{3 + 4\mu} \right]^2 + 3\mu(1 + \mu) + f_3(z),
\]

\[
f_3(z) = \frac{3\mu}{(3 + 4\mu)} \left\{ 4\gamma^2 z^4 + 2\gamma z^2(4 + 4\mu + \tau_0 - 2z) + (1 + \mu)(2z - \tau_0 - 1)^2 \right\}.
\]

Note that

\[
3\gamma z^2 + \mu(2z - \tau_0 - 1) \geq 0, \quad z \geq z_0 = \frac{1 + \tau_0}{1 + \sqrt{1 + 3\gamma(1 + \tau_0)/\mu}}.
\]

17
Therefore for $z \leq z_0$ we integrate over all $y_2 \geq 0$

$$\int_{0 \leq z \leq z_0, y_2 \geq 0} e^{-A_3 f_3(y_2, y_3)/(12\mu)} dy_2 dy_3 \lesssim e^{-A_3(1+\mu)/4} \exp \left\{ -\frac{A_3}{12\mu} \min_{0 \leq z \leq z_0} f_3(z) \right\} \lesssim$$

$$\lesssim \exp \left\{ -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{(1 + \tau_0)^2}{(3 + 4\mu)} \left[ 1 - \frac{2}{1 + \sqrt{1 + 3\gamma(1 + \tau_0)/\mu}} \right] \right] \right\},$$

since

$$\min_{0 \leq z \leq z_0} f_3(z) \geq \frac{3\mu(1 + \mu)(1 + \tau_0)^2}{(3 + 4\mu)} \left[ 1 - \frac{2}{1 + \sqrt{1 + 3\gamma(1 + \tau_0)/\mu}} \right]^2.$$

If $z \geq z_0$ then minimum of the function $f_3(y_2, z)$ is attained for $y_2 = 0$. Also

$$f_3(0, z) \geq \frac{3\mu(1 + \mu) + 2\mu}{\sqrt{3\gamma + 2z}} - \frac{1 + \tau_0}{\sqrt{3\gamma + 2}} \geq \frac{3\mu\gamma(1 + \tau_0)^2}{3\gamma + 2}.$$

Therefore

$$\int_{z \geq z_0, y_2 \geq 0} e^{-A_3 f_3(y_2, z)/(12\mu)} dy_2 dz \leq \int_{z \geq z_0} e^{-A_3 f_3(0, z)/(12\mu)} dz \lesssim$$

$$\lesssim \exp \left\{ -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{\gamma(1 + \tau_0)^2}{(3\gamma + 2)(1 + \mu)} \right] \right\},$$

which gives

$$\ln J_4 \lesssim -\frac{A_3(1 + \mu)}{4} \left[ 1 + \frac{(1 + \tau_0)^2}{(3 + 4\mu)} \min \left\{ \frac{\gamma}{1 + \gamma}, \left[ 1 - \frac{2}{1 + \sqrt{1 + 3\gamma(1 + \tau_0)/\mu}} \right]^2 \right\} \right]. \quad (47)$$

Then from (42), (38) and (44)–(47) we get

$$B_3 \lesssim \frac{A_3(1 + \mu)}{4n} \left[ 1 + \min \left\{ \mu, \frac{\gamma}{(3 + 4\mu)(1 + \gamma)} \left[ 1 - \frac{2}{1 + \sqrt{1 + 3\gamma(1 + \tau_0)/\mu}} \right]^2 \right\} \right]. \quad (48)$$

As a result, from (14) we get a general result for any $\sigma < \infty$.

Theorem 2. Let $\ln M = o(n)$, $n \to \infty$. Then for any $\sigma < \infty$ the inequality holds

$$F(M, A, \sigma) \geq F_1(M, A, \sigma) \geq \max_{\beta, \tau_0} \min_{k = 1, 2, 3} B_k + o(1) > E(M, A), \quad n \to \infty, \quad (49)$$

where values $B_1, B_2, B_3$ are defined in (34), (16) and (18), respectively.

The relation (49) has been proved provided $M \leq (n + 2)/2$. In fact, the formula (49) remains valid for any $M$ such that $M = e^{o(n)}$, $n \to \infty$. Indeed, note that instead of simplex codes $\{x'_i\}$ or $\{x''_i\}$ on phases I–II we may use “almost” equidistant codes, for which, for
example, $\|x'_i - x'_j\|^2 = 2An(1 + o(1)), n \to \infty, i \neq j$. All calculations then remain essentially the same. Such codes do exist due to the following result.

Denote by $S_n$ the unit sphere in $\mathbb{R}^n$ centered at $0$.

Proposition 2. For any $\rho \in (0, 1)$ and $n \geq 3$ there exists a code $C = \{x_1, \ldots, x_M\} \subset S_n$ with $|(x_i, x_j)| \leq \rho$, $i \neq j$, such that

$$M \geq \rho e^{n\rho^2/2}. \quad (50)$$

Proof. Denote by $\Omega(\theta)$ the area of the “cap” cut out from $S_n$ by the cone of half-angle $\theta$. In particular, the area of $S_n$ equals $\Omega(\pi)$. Then for any $0 < \theta < \pi/2$ there exists a code $C = \{x_1, \ldots, x_M\} \subset S_n$ with $|(x_i, x_j)| \leq \cos \theta$, $i \neq j$, such that

$$M \geq \frac{\Omega(\pi)}{2\Omega(\theta)}.$$

For the ratio $\Omega(\theta)/\Omega(\pi)$ the following estimate is known \cite{17, formula (27)}

$$\frac{\Omega(\theta)}{\Omega(\pi)} \leq \frac{\Gamma \left( \frac{n}{2} + 1 \right) \sin^{n-1} \theta}{n \Gamma \left( \frac{n+1}{2} \right) \sqrt{\pi} \cos \theta}.$$

Using that estimate for $\rho = \cos \theta$ we get

$$M \geq \frac{\Omega(\pi)}{2\Omega(\theta)} \geq \frac{n\Gamma \left( \frac{n+1}{2} \right) \sqrt{n} \rho}{2\Gamma \left( \frac{n}{2} + 1 \right) \left( 1 - \rho^2 \right)^{n\rho/2}} \geq \frac{n\Gamma \left( \frac{n+1}{2} \right) \sqrt{n} \rho e^{n\rho^2/2}}{2\Gamma \left( \frac{n}{2} + 1 \right) \sqrt{\pi}}.$$

From the last inequality for $n \geq 3$ the estimate (50) follows. $\triangle$

§ 5. Proof of Theorem 1

We need to investigate asymptotics of values $B_1, B_2$ and $B_3$ when $\sigma \to 0$ and $\sigma \to \infty$.

a) If $\sigma \to 0$ then set $\tau_0 > 0$ such that $\tau_0 \to 0$. Then for $B_1$ from (34) we have as $n \to \infty, \sigma \to 0$

$$B_1 \geq \frac{AM}{4(M-1)} \left[ 1 + \min \left\{ \beta, \frac{1}{3+4\beta} \right\} \right] + o(1).$$

For $B_2$ from (16) we have

$$B_2 \geq \frac{AM}{4(M-1)} \left[ 1 + \frac{\beta}{1+\beta} - \frac{2\beta}{(1+\beta)M} \right] + o(1).$$

For $B_3$ from (18) we have

$$B_3 \geq \frac{AM(1+\mu)}{4(M-1)(1+\beta)} \left[ 1 + \min \left\{ \mu, \frac{1}{3+4\mu} \right\} \right] \geq \frac{AM}{4(M-1)} \left[ 1 + \min \left\{ \beta, \frac{1}{3+4\beta} \right\} - \frac{\beta}{M(1+\beta)} \right] + o(1).$$
Then we get as \( \sigma \to 0 \) and \( n \to \infty \)

\[
\min_{k=1,2,3} B_k \geq \frac{AM}{4(M-1)} \left[ 1 + \min \left\{ \frac{\beta}{1+\beta}, \frac{1}{3+4\beta} \right\} - \frac{2\beta}{(1+\beta)M} + o(1) \right].
\]

We set \( \beta \) such that both terms under minimization become equal, i.e. set

\[
\beta = \left( \sqrt{5} - 1 \right)/4 \approx 0.3090.
\]

Then we get

\[
\min_{k=1,2,3} B_k \geq \frac{AM}{4(M-1)} \left[ 1 + \frac{1}{2+\sqrt{5}} - \frac{1}{2M} + o(1) \right],
\]

from which the formula (5) follows.

b) In that case \( \sigma \to \infty \), i.e. \( \gamma \to 0 \). Set \( \tau_0 > 0 \) such that \( \tau_0 \to 0 \) and choose \( \beta = \gamma/7, 1 \). Then after simple calculations we get

\[
\min_{k=1,2,3} B_k \geq \frac{AM}{4(M-1)} \left[ 1 + \frac{\gamma}{14} + o(\gamma^2) \right] - \frac{3 \ln M}{n}, \quad \gamma \to 0,
\]

from which the formula (6) follows.

Note that in both extreme case as \( \sigma \to 0 \) or \( \sigma \to \infty \) the value \( \tau_0 \) was chosen such that \( \tau_0 > 0 \), but \( \tau_0 \to 0 \). For intermediate values of \( \sigma \) optimal \( \tau_0 \neq 0 \).

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