Abstract

Within the classical approximation we calculate the static $Q\bar{Q}$ potential via the AdS/CFT relation for nonzero temperature and arbitrary internal orientation of the quarks. We use a higher order curvature corrected target space background. For timelike Wilson loops there arises a critical line in the orientation-distance plane which is shifted to larger distances relative to the calculation with uncorrected background. Beyond that line there is no $Q\bar{Q}$-force. The overall vanishing of the force for antipodal orientation known from zero temperature remains valid. The spacelike Wilson loops yield a string tension for a (2+1)-dimensional gauge theory, independent of the relative internal orientation, but sensitive to the background correction.
1 Introduction

Part of the recent interest in relations between gauge field theories and string theory/supergravity on certain nontrivial backgrounds [1, 3, 2] is due to a conjecture expressing the Wilson loops in the gauge theory by the partition function of a string fulfilling boundary conditions set by the geometry of the Wilson loop [4, 3, 5]. This has been used to calculate the heavy quark-antiquark potential for $\mathcal{N} = 4$ super Yang-Mills in 3+1 dimensions in the t'Hooft limit and large effective coupling from the classical string action in $\text{AdS}_5 \times S^5$ [4, 5]. The procedure has been extended to general dimensions and to geometries arising as some limit of near extremal D-brane configurations [4, 6, 7, 8]. These background configurations with compactified Euclidean time coordinate are related to equilibrium $T > 0$ field theory. Since the periodicity condition breaks supersymmetry one gains access to the quark-antiquark potential in non-supersymmetric gauge theories (hopefully QCD) in one dimension lower [3, 5, 8].

The present paper will be concerned mainly with two aspects of this kind of calculations. In the $\text{AdS}_5 \times S^5$ calculation of ref. [4] a Coulombic potential arises, which is switched off for quark-antiquarks oriented antipodally with respect to the internal $S^5$. This is in agreement with an argument telling that the corresponding configuration is of BPS type [9, 4]. We would like to know what happens in this antipodal case for $T > 0$, where supersymmetry is broken.

The classical approximation used on the string side of the duality is justified in the t'Hooft limit for large effective coupling, corresponding to small curvature of $\text{AdS}_5 \times S^5$ or of the relevant region of the near extremal D-brane configuration for $T > 0$. A complete treatment has to include both quantum corrections and corrections of the target space background, which after all has to be a solution for the effective string action exact in $\alpha'$.

As a first step in this direction we perform the classical calculation using the next order in $\alpha'$ corrected background of ref. [11, 12]. At $T > 0$ the quark-antiquark potential vanishes beyond some critical distance [7, 8]. This total screening is expected to be weakened if the corrections just mentioned are taken into account [10]. Therefore, our focus will be on the behaviour of the critical line in the ($\Delta \Theta, L$)-plane (relative $S^5$ orientation and distance), if the correction of [11, 12] is switched on.

Our 10D metric for non-zero temperature $T > 0$ is given by

$$
\begin{align*}
\text{d}s^2 &= \alpha' \left\{ \frac{U_T^2}{R^2} (1 - \frac{U_T^4}{U^4}) e^{\gamma_A(U_T)} (\text{d}x^0)^2 + \frac{U_T^2}{R^2} e^{\gamma_C(U_T)} (\text{d}x^i)^2 \\
&+ \frac{R^2}{U^2 (1 - \frac{U_T^4}{U^4})} e^{\gamma_B(U_T)} (\text{d}U)^2 + R^2 e^{\gamma_D(U_T)} (d\Omega_5)^2 \right\} = G_{MN} \text{d}x^M \text{d}x^N,
\end{align*}
$$

with [8]

$$
U_T = \pi R^2 T (1 + 15\gamma)^{-1},
$$

The dimensionless parameter $\frac{1}{U_T}$ controls the curvature and is related to Yang-Mills quantities via $R^2 = \sqrt{2g_{YM}^2 N}$, which is equal to the effective coupling in the t'Hooft limit.
\[ A(\rho) = -75\rho^4 - \frac{1225}{16}\rho^8 + \frac{695}{16}\rho^{12}, \]
\[ B(\rho) = 75\rho^4 + \frac{1175}{16}\rho^8 - \frac{4585}{16}\rho^{12}, \]
\[ C(\rho) = -\frac{25}{16}\rho^8 - \frac{25}{16}\rho^{12}, \]
\[ D(\rho) = \frac{15}{16}\rho^8 + \frac{15}{16}\rho^{12}. \]

(3)

We consider
\[ \gamma = 0 \quad \text{or} \quad \gamma = \frac{1}{8}\zeta(3) R^{-6} \]

(4)
to describe either the near extremal D-3 brane metric used in refs. [7, 8] or the corresponding first order higher curvature corrected metric studied in ref. [11, 12]. In these target space metrics we calculate the Nambu-Goto action
\[ S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{\det(G_{MN}\partial_{\mu}x^{M}\partial_{\nu}x^{N})} \]

(5)

for stationary (\(\delta S = 0\)) string world surfaces fulfilling certain boundary conditions at \(U \to \infty\). In a first case, called timelike, the boundary of the string world sheet at \(U \to \infty\) consists out of two lines running at \(\vec{\Theta} = \vec{\Theta}_Q\) or \(\vec{\Theta}_{\bar{Q}}\) parallel to the \(x^0\)-axis one times around the compactified \(x^0\), with compactification length \(1/T\). In the second case, called spacelike, the boundary lines at \(U \to \infty\) run parallel to the \(x^2\)-axis at
\[ x^1 = \pm\frac{L}{2}, \quad x^2 = x^3 = 0, \quad \vec{\Theta} = \vec{\Theta}_Q \quad \text{or} \quad \vec{\Theta}_{\bar{Q}}. \]

(6)

Via the CFT/AdS conjecture specified for Wilson loops in [4, 5, 6] one has in classical approximation for case 1
\[ S_{\text{case 1}} = \frac{1}{T} \cdot V_1(L, \Delta \Theta, T) \]

(8)

and for case 2 (\(Y\) measures the large \(x^2\) interval.)
\[ S_{\text{case 2}} = Y \cdot V_2(L, \Delta \Theta), \]

(9)

where \(V_1(L, \Delta \Theta, T)\) is the static quark-antiquark potential in 3-dimensional space at temperature \(T\) for separation \(L\) and an angle \(\Delta \Theta\) between the internal orientation \(\vec{\Theta}_Q\) and \(\vec{\Theta}_{\bar{Q}}\) in \(S_5\). \(V_2(L, \Delta \Theta)\) for \(Y, L \gg \frac{1}{T}\) is the potential in 3-dimensional space-time (i.e. 2-dim. space) at zero temperature [3].

\(^3\Theta\) with \(|\vec{\Theta}| = 1\) parametrises \(S^5\).
2 Timelike Wilson lines

In this section it is convenient to choose the gauge

\[\tau = x^0, \quad \sigma = x^1.\]  

Furthermore, due to the symmetry of the problem, it is sufficient to look for world surfaces with \(\partial_\tau x^M = \delta_0^M, \partial_\sigma x^2 = \partial_\sigma x^3 = 0,\) only. Then the action becomes \(\text{(' denotes derivatives with respect to } \sigma, \Theta \text{ is an angle variable on the great circle on } S^5 \text{ passing } \vec{\Theta}_Q \text{ and } \vec{\Theta}^-_Q.\)

\[
S = \frac{1}{2\pi T} \int_{-\frac{L}{2}}^{+\frac{L}{2}} d\sigma \ e^{\frac{\gamma}{2} A(U')_M} \sqrt{\frac{U'^4 - U_T^4}{R^4}} e^{\gamma C(U'_T)} + U'^2 e^{\gamma B(U'_T)} + \Theta'^2 \frac{U'^4 - U^4}{U^2} e^{\gamma D(U'_T)}. 
\]

The wanted solutions \(U(\sigma), \Theta(\sigma)\) are most easily found by making use of the conservation laws derived from the Lagrangian \(L\) \((S = \frac{1}{2\pi T} \int d\sigma L)\)

\[
C_1 = \frac{\partial L}{\partial U'} U' + \frac{\partial L}{\partial \Theta'} \Theta' - L, \\
C_2 = \frac{\partial L}{\partial \Theta'}. 
\]

To be as close as possible to ref. [4] we parametrise the conserved quantities \(C_1\) and \(C_2\) by \(U_0 = U(0)\) and a parameter \(l\) via

\[
C_2 = U_0 l, \\
C_1 = -\frac{U_0^2 \ e^{\frac{\gamma}{2} \{C(\frac{\delta}{\gamma}) - D(\frac{\delta}{\gamma})\}}}{R^2} \sqrt{(1 - \delta^4) e^{\gamma \{A(\frac{\delta}{\gamma}) + D(\frac{\delta}{\gamma})\}} - l^2}, \quad \text{(13)}
\]

with

\[
\delta = \frac{U_T}{U_0}. \quad \text{(14)}
\]

Then using the notation

\[
f_\gamma(y, \delta) = (y^4 - \delta^4) e^{\gamma \{A(\frac{\delta}{\gamma}) + C(\frac{\delta}{\gamma}) - D(\frac{\delta}{\gamma})\}} - l^2 y^2 e^{\gamma \{C(\frac{\delta}{\gamma}) - D(\frac{\delta}{\gamma})\}} \quad \text{(15)}
\]

straightforward integration yields

\[
\sigma(U) = \frac{R^2}{U_T} \delta \sqrt{f_\gamma(1, \delta)} \int_1^{U_0} \frac{y \ e^{\frac{\gamma}{2} \{B(\frac{\delta}{\gamma}) - C(\frac{\delta}{\gamma})\}}}{\sqrt{y^4 - \delta^4} \sqrt{f_\gamma(y, \delta) - f_\gamma(1, \delta)}} dy, \quad \text{(16)}
\]

\[
\Theta(U) = l \ e^{\frac{\gamma}{2} \{D(\frac{\delta}{\gamma}) - C(\frac{\delta}{\gamma})\}} \int_1^{U_0} \frac{y^2 \ e^{\frac{\gamma}{2} \{B(\frac{\delta}{\gamma}) + C(\frac{\delta}{\gamma}) - 2D(\frac{\delta}{\gamma})\}}}{\sqrt{y^4 - \delta^4} \sqrt{f_\gamma(y, \delta) - f_\gamma(1, \delta)}} dy, \quad \text{(17)}
\]

\(^4\text{Making use of } U'(0) = 0.\)
and in particular, using (2), for \( L = 2\sigma(U \to \infty) \), \( \Delta \Theta = 2\Theta(U \to \infty) \)

\[
L \cdot T = \frac{2}{\pi} (1 + 15\gamma) \delta \sqrt{f_\gamma(1, \delta)} \int_1^\infty \frac{e^{\frac{1}{2}\{R(\frac{y}{\delta})-C(\frac{y}{\delta})\}}}{\sqrt{y^4 - \delta^4}} \sqrt{f_\gamma(y, \delta) - f_\gamma(1, \delta)} dy ,
\]

(18)

\[
\Delta \Theta = 2l e^{\frac{1}{2}\{D(\delta)-C(\delta)\}} \int_1^\infty \frac{y^2 e^{\frac{1}{2}\{B(\frac{y}{\delta})+C(\frac{y}{\delta})+D(\frac{y}{\delta})\}}}{\sqrt{y^4 - \delta^4}} \sqrt{f_\gamma(y, \delta) - f_\gamma(1, \delta)} dy .
\]

(19)

Inserting the solution into the action functional (13) and making use of the conservation laws (12), (13) again, we get via (8) the static \( Q\bar{Q} \) potential. Due to the behaviour of the integrand at large \( U \) it is divergent. With a preliminary cutoff \( \Lambda \) it looks like

\[
V_1(\Lambda) = \frac{U_T}{\pi} \delta^{-1} \int_1^\Lambda \frac{y^4 - \delta^4}{\sqrt{y^4 - \delta^4}} e^{\frac{1}{2}\{2A(\frac{y}{\delta})+B(\frac{y}{\delta})+C(\frac{y}{\delta})+D(\frac{y}{\delta})\}} dy .
\]

(20)

Defining a renormalised potential along the lines of \([4, 7, 8]\) requires a comment. The potential \( V_1(\Lambda) \) is linearly divergent. Subtracting the divergence yields a renormalised \( V_1 \), fixed up to a finite additive constant which per se is completely irrelevant. From the experience with the \( \Delta \Theta = 0, \gamma = 0 \) case \([4, 8]\), we expect that beyond some critical distance the configuration of two strings stretching straightly from \( U = \Lambda \) to the horizon at \( U = U_T \) at fixed values \( x_1 = \pm \frac{L}{2}, \Theta = \pm \frac{\Delta \Theta}{2} \) is energetically favoured in comparison to our smooth U-shaped configuration (16), (17). Choosing \( 2\cdot \frac{\Lambda-U_T}{2\pi} \), i.e. twice the energy stored in a string stretching between \( U_T \) and \( \Lambda \) as the subtraction term, the competing piecewise straight string configuration has been normalised to zero in \([4, 8]\). As a consequence, in the \( L \)-region with \( V_1 \geq 0 \) the \( Q\bar{Q} \) force vanishes.

To extend this framework to our modified background (1), we first of all have to check, whether the competing piecewise straight string configuration (pssc) \([\bar{4}, \bar{3}]\) still has an action (5) independent of \( L \). Since also with (1) we have \( G_{00} = 0 \) at \( U = U_T \), the determinant of the induced metric vanishes for the part of the pssc extending along the horizon. Only the pieces of the pssc running off the horizon contribute:

\[
V^{(\Lambda)}_{\text{pssc}} = \frac{U_T}{\pi} \int_1^{U_T} e^{\frac{1}{2}\{(A(y^{-1})+B(y))\}} dy .
\]

(21)

For \( \gamma = 0 \) this reduces to \( \frac{\Lambda-U_T}{2\pi} \).

Now we choose \( V^{(\Lambda)}_{\text{pssc}} \) as the subtraction constant to get a renormalised \( V_1 \). Then the potential of the competing pssc is zero again, and our U-shaped configuration has a potential

\[
V_1 = \lim_{\Lambda \to \infty} \left( V_1^{(\Lambda)} - V^{(\Lambda)}_{\text{pssc}} \right) ,
\]

\footnote{At \( x^1 = -\frac{L}{2}, \Theta = -\frac{\Delta \Theta}{2} \) from \( U = \Lambda \) to \( U = U_T \), then along the horizon to \( x^1 = +\frac{L}{2}, \Theta = +\frac{\Delta \Theta}{2} \), and then at fixed \( x^1 \), \( \Theta \) back to \( U = \Lambda \).}
leading with (2) to

\[
\frac{V_1}{R^2T} = (1 + 15\gamma)^{-1}\left\{ \delta^{-1} \int_1^\infty \left( \frac{\sqrt{y^4 - \delta^4} e^{\frac{2}{\gamma} (2A(y^2) + B(y^2))} + C(\delta) + D(\delta) - C(\delta))}{\sqrt{f_\gamma(y, \delta) - f_\gamma(1, \delta)}} - 1 \right) dy \\
+ 1 - \delta^{-1} + \int_1^\infty \left( 1 - e^{\frac{2}{\gamma} (A(y^{-1}) + B(y^{-1}))} \right) dy \right\}.
\]

(22)

The potential \( V_1 \) as a function of \( L \) and \( \Delta \Theta \) is obtained by inserting in (22) the inversion of (18), (19) i.e. \( \delta = \delta(L, \Delta \Theta) \) and \( l = l(L, \Delta \Theta) \). The qualitative features of this inversion problem are best understood by looking at the contour lines of \( L, \Delta \Theta \) and \( V_1 \) in the \((L^2, \delta^4)\)-plane. Since we are discussing string configurations with \( U_T \leq U_0 < \infty \) we have \( 0 \leq \delta \leq 1 \). In addition, the \( \Delta \Theta \)-formula requires \( l^2 \geq 0 \), while the \( L \)-formula needs \( l^2 \leq (1 - \delta^4) e^{\gamma \{A(\delta) + D(\delta)\}} \). Altogether our set of formulae is applicable in the region

\[
\mathcal{G} = \left\{ (l^2, \delta^4) \mid 0 \leq l^2 \leq (1 - \delta^4) e^{\gamma \{A(\delta) + D(\delta)\}}, \delta^4 \geq 0 \right\}.
\]

A generic feature of these integrals is their monotonous increase with \( l^2 \) at fixed \( \delta \). As a consequence, \( V_1 \) and \( \Delta \Theta \) are monotonously increasing functions of \( l^2 \). Due to the inverse behaviour of the square root prefactor it is difficult to make a definite statement concerning \( L \).

The following discussion of the contour line maps is based on explicit analysis of the \( \gamma = 0 \) case. However, the qualitative features, forced by the singularities of the functions under discussion, are not changed by switching on \( \gamma \). We start with the \( \Delta \Theta \) lines in \( 0 \leq \delta^4 \leq 1 \). Here \((l^2, \delta^4) = (0, 1)\) is a singular point\(^6\). Across this point run all contour lines from \( \Delta \Theta = 0 \) up to \( \Delta \Theta = \infty \). The \( \Delta \Theta = 0 \) line goes straight down to the point \((0, 0)\), the \( \Delta \Theta = \infty \) line first runs parallel to the \( l^2 \) axis to \((2, 1)\) and then straight down to \((2, 0)\). The \( \Delta \Theta = \pi \) line goes through \((l^2, \delta^4) = (1, 0)\).

Let us turn to the \( L \)-lines. Again \((0, 1)\) is a singular point. Here the \( L = \infty \) line at \( \delta^4 = 1 \) meets the straight zero-line connecting \((1, 0)\) with \((0, 1)\). The approach of all lines with \( 0 \leq L < \infty \) to the singular point is possible due to the balance between the logarithmic divergent integral for \( \delta^4 \to 1 \) and the vanishing square root prefactor. Therefore, \textit{all} lines with \( L > 0 \) start tangent to the \( L = \infty \) line into the region \( l^2 < 0 \). Those with small enough \( L \) must turn to the \( l^2 > 0 \) region, since they must stay close to the zero line also away from the singular point. Altogether \( L \)-lines up to some maximal value \( L_{\text{max}} \) cross the \( \delta^4 \)-axis twice. For \( l^2 = 0 \) the inversion of (18) exists only for \( L \leq L_{\text{max}} \), for the \( \gamma = 0 \) case see also \textsuperscript{4, 8}. From the point where the \( L_{\text{max}} \)-line touches the \( \delta^4 \)-axis to the point \((l^2, \delta^4) = (1, 0)\) extends a ridge of decreasing height. For a given \( 0 \leq l^2 < 1 \) the inverse of (18) exists for values of \( L \) smaller than the corresponding ridge height, only.

\textsuperscript{6}Using the fact that \( A(\rho) + D(\rho) \) as well as \( C(\rho) - D(\rho) \) in \( 0 \leq \rho \leq 1 \) are negative and monotonously decreasing one can convince oneself that the integrals in (18),(19) and (22) are well defined inside \( \mathcal{G} \) for arbitrary \( \gamma \). They exist even in a larger region, for the simplest case \( \gamma = 0 \) in \( 0 \leq \delta < 1 \), \( -\infty < l^2 < 2 \).

\textsuperscript{7}Vanishing of the factor \( l \) versus logarithmic divergence of the integral for \( \delta^4 \to 1 \).
Fig. 1 Critical lines $V_1 = 0$ as functions of $\frac{1}{\pi} \cdot \Delta\Theta$ (horizontal) and $\frac{\pi}{2} T \cdot L$ (vertical). The lowest line is for $\gamma = 0$, the others for $\gamma = 0.01$, 0.02, and 0.05 respectively.

Finally, the $V_1$ map is governed by a singularity with crossing contour lines at $(l^2, \delta^4) = (1, 0)$. On the $l^2$-axis $V_1$ tends to $\infty(-\infty)$ for $l^2 > 1(< 1)$. Also $l^2 = 2, \ 0 \leq \delta^4 < 1$ is a line of $V_1 = \infty$. On the contrary, $V_1$ takes finite values on $\delta^4 = 1, \ l^2 < 2$. In particular one finds $V_1 = 0$ at $(0, 1)$. The zero line of $V_1$ from $(0, 1)$ to $(1, 0)$ is the critical line we are after. Beyond this critical line there is no longer any force between the $Q\bar{Q}$-pair. From the analysis of [7, 8] and a theorem in [13] we know that, for $\gamma = 0$, the zero line coming from $(0, 1)$ at first must go to the $l^2 < 0$ region to cross the $\delta^4$-axis at a value below the $L$-ridge. We have no general proof that the $V_1 = 0$ line is below the $L$-ridge in whole $G$, but there is numerical evidence for this scenario for not too large $\gamma$.

After this qualitative discussion of the behaviour of the potential we present the results of a numerical evaluation. In fig.1 the critical lines, $(V_1 = 0)$, in the $(\Delta\Theta, L)$-plane, obtained for various values of $\gamma$ are shown. Beyond the critical line there is no $Q\bar{Q}$ force. Although for $\gamma = 0$, i.e. the zeroth order metric used in [7, 8], the potential is $R$-dependent, the $V_1 = 0$ contour line in the $(\Delta\Theta, L)$-plane is independent of $R$. With the first order corrected metric of [11, 12] the $R$-dependence originates from the $\gamma$-dependence via (4). Obviously, the correction acts in the right direction, the critical line for the onset of total screening is shifted to larger distances over the whole range of $\Delta\Theta$. The chosen values of $\gamma = 0.01, 0.02$ and 0.05 correspond to $R^2 = 2.468, 1.959$ and $1.443$ respectively.
Fig. 2 \( \frac{1}{RT} \cdot V_1 \) as a function of \( \frac{\pi}{2} T \cdot L \) at \( \gamma = 0 \) for various values of \( \frac{1}{\pi} \Delta \Theta \), from left to right: 0.8, 0.5, 0.

To get an impression of the influence of \( \Delta \Theta \) on the potential we show in fig. 2 \( \frac{1}{RT} \cdot V_1 \) as a function of \( \frac{\pi}{2} T \cdot L \) for various values of \( \frac{1}{\pi} \Delta \Theta \) in the case of zeroth order metrics, i.e. \( \gamma = 0 \). The \( \bar{Q}Q \) force is zero for distances larger than the value of the zero of the drawn potential. For illustration we also followed the potential a little bit for positive values. The cusp with the accompanying backward movement indicates that one goes beyond the L-ridge discussed above.

For a better understanding of the mechanism switching off the potential completely for \( \Delta \Theta \rightarrow \pi \) we show (for \( \gamma = 0 \)) in fig. 3 the \((l^2, \delta^4)\)-plane with the region \( \mathcal{G} \), the contour line \( V_1 = 0 \) and contour lines of \( \Delta \Theta \). Any \( \Delta \Theta \)-line below \( \pi \) enters \( \mathcal{G} \) and the region of nonvanishing force for small enough values of \( \delta^4 \). The \( \Delta \Theta = \pi \)-line touches \( \mathcal{G} \) at the point \((1, 0)\) only. This figure also illustrates some statements made in the qualitative discussion above.

Finally, fig. 4 shows, again for \( \gamma = 0 \), some contour lines of \( \frac{\pi}{2} T \cdot L \) in the \((l^2, \delta^4)\)-plane. The numerical curves illustrate the general analytic arguments given above. The L-ridge is visible clearly. The L-line touching the \( \delta^4 \) axis corresponds to \( L_{\text{max}} = 0.435 \).
Fig. 3 \((l^2, \delta^4)\) plane with region \(G\) for \(\gamma = 0\) (shadowed). Contour lines of \(\frac{1}{\pi} \cdot \Delta \Theta\) are shown for 0.5, 0.8, 1, from left to right. The dashed line is the \(V_1 = 0\) contour line.

Fig. 4 \((l^2, \delta^4)\) plane with region \(G\) for \(\gamma = 0\) (shadowed). Contour lines of \(\frac{\pi}{2} T \cdot L\) are shown for 0.5, 0.435, 0.3, 0.2, 0.1, from left to right. The dashed line is the \(V_1 = 0\) contour line.
Closing this section we present the results of an expansion of (18), (19), (22) for small $\delta$. It is related to either $T \to 0$ at fixed distance or to small distances at fixed temperature. Within this expansion one encounters the following $l^2$-dependent integrals, expressible by the hypergeometric function $F$, the elliptic integrals $K$ and $E$ as ($n \geq 0$)

$$
\begin{align*}
    b_{2n}(l) &= \int_1^\infty \frac{dy}{y^{2n} \sqrt{(y^2 - 1)(y^2 + 1 - l^2)}} = \frac{1}{2} B(n + \frac{1}{2}, \frac{1}{2}) F(n + \frac{1}{2}, \frac{1}{2}, n + 1, l^2 - 1), \\
    b_{-2}(l) &= \int_1^\infty \left( \frac{y^2}{\sqrt{(y^2 - 1)(y^2 + 1 - l^2)}} - 1 \right) dy \\
    &= 1 + \frac{\pi}{2} \sqrt{2 - l^2} F(\frac{1}{2}, \frac{1}{2}, 1, 1 - l^2) - \frac{\pi}{2} \sqrt{2 - l^2} F(-\frac{1}{2}, \frac{1}{2}, 1, 1 - l^2). \quad (24)
\end{align*}
$$

All $b(l)$ are monotonously increasing with $l^2$. Special values are

$$
\begin{align*}
    b_{2n}(0) &= \frac{1}{4} B(\frac{n}{2} + \frac{1}{4}, \frac{1}{2}), \\
    b_{-2}(0) &= 1 + \frac{1}{\sqrt{2}} \left( K(\frac{\sqrt{2}}{2}) - 2 E(\frac{\sqrt{2}}{2}) \right) \approx 0.401, \\
    b_{2n}(1) &= \frac{1}{2} B(n + \frac{1}{2}, \frac{1}{2}), \\
    b_{-2}(1) &= 1. \quad (25)
\end{align*}
$$

Then we get ($c(\gamma) = \int_1^\infty (1 - e^{\pm(A(\frac{1}{2}) + B(\frac{1}{2}))})\, dx$)

$$
\frac{V_1}{R^2 T} = \frac{1}{1 + 15\gamma} \left( \frac{b_{-2}(l) - 1}{\delta} + 1 + c(\gamma) - \frac{1 + 75\gamma}{2} b_2(l) \, \delta^3 + O(\delta^7) \right), \quad (26)
$$

$$
L \cdot T = \frac{2}{\pi} (1 + 15\gamma) \sqrt{1 - l^2 - (1 + 75\gamma) \delta^4 + O(\delta^8)} \\
\cdot \left( b_2(l) \, \delta + \frac{1 + 75\gamma}{2} b_6(l) \, \delta^5 + O(\delta^9) \right), \quad (27)
$$

$$
\Delta \Theta = 2l \left( b_0(l) + \frac{1 + 75\gamma}{2} b_4(l) \, \delta^4 + O(\delta^8) \right). \quad (28)
$$

The insertion of inversed (27), (28) into (26) is straightforward and will not be discussed here.

### 3 Spacelike Wilson loops

To describe a string world sheet satisfying the boundary condition (7) it is convenient to choose the gauge

$$
\tau = x^2, \quad \sigma = x^1. \quad (29)
$$
The leading terms in the expansion of all the integrals in (33)-(35) for subtraction as for
then we find with the notation
singularity of the integral at
\( \delta > 0 \).

The conserved quantities \( \tilde{C}_1 \) and \( \tilde{C}_2 \) defined analogously to (12) will be parametrised by
\[
\tilde{C}_2 = U_0 \ell \\
\tilde{C}_1 = -\frac{U_0^2}{R^2} e^{\gamma C(\delta)} \sqrt{1 - I^2 e^{-\gamma (C(\delta) + D(\delta))}} .
\]

Then we find with the notation
\[
g_\gamma(y, \delta) = y^4 e^{2\gamma (C(y) - C(\delta))} - y^2 \int e^{\gamma (C(y) - D(\delta) - 2C(\delta))} dy,
\]
repeating the steps in section 2,
\[
L \cdot T = 2 \pi (1 + 15\gamma) \sqrt{g_\gamma(1, \delta)} \delta \int_1^\infty \frac{e^{\frac{2}{\gamma} (B(y) - C(y))}}{\sqrt{y^4 - \delta^4}} \sqrt{g_\gamma(y, \delta) - g_\gamma(1, \delta)} dy,
\]
\[
\Delta \Theta = 2l \int_1^\infty \frac{y^2 e^{\frac{2}{\gamma} (B(y) + C(y) - 2D(\delta) - 2C(\delta))}}{\sqrt{y^4 - \delta^4}} \sqrt{g_\gamma(y, \delta) - g_\gamma(1, \delta)} dy,
\]
\[
\frac{V_2}{R^2 T} = (1 + 15\gamma)^{-1} \left\{ \delta^{-1} \int_1^\infty \left( \frac{y^4 e^{\frac{2}{\gamma} (B(y) + 3C(y) - 2C(\delta))}}{\sqrt{y^4 - \delta^4}} \sqrt{g_\gamma(y, \delta) - g_\gamma(1, \delta)} - 1 \right) + 1 - \delta^{-1} \right\} .
\]

To define the renormalised potential \( V_2 \) we have chosen as the subtraction \( \frac{\Delta U_T}{\pi} \). Contrary to the previous section the pssc configuration has a \( L \)-dependent action. This is due to different behaviour on the horizon, \( G_{22} \neq 0 \) versus \( G_{00} = 0 \). With the same subtraction as for \( V_2 \) one gets, e.g. for \( \Delta \Theta = 0 \),
\[
V_{2, \text{ppsc}} = \frac{U_T^2}{2\pi R^2} e^{\gamma C(1)} \cdot L + \frac{U_T}{\pi} \int_1^\infty \left( \frac{y^2 e^{\frac{2}{\gamma} (B(y) + C(y))}}{\sqrt{y^4 - 1}} - 1 \right) dy .
\]

We expect this to be energetically disfavoured over the whole range of \( L \).

The \( L - \delta \) relation differs qualitatively from that in section 2. Due to the absence of \( \delta \) in the square root prefactor we can reach arbitrary large values of \( L \) by approaching the singularity of the integral at \( \delta = 1 \). Furthermore, \( L \) is monotonously increasing with \( \delta^4 \).\footnote{This is obviously for \( \gamma = 0 \). It can be demonstrated by simple estimates also for small enough values \( \gamma > 0 \).}
due to the equality of the leading term of the expansion of the respective integrands for $y \to 1$. Denoting with a fixed $\epsilon > 0$

$$I(\epsilon, l) = \int_1^{1+\epsilon} \frac{dy}{\sqrt{y^4 - \delta^4} \sqrt{g_\gamma(y, \delta) - g_\gamma(1, \delta)}},$$

we get

$$L \cdot T = \frac{2}{\pi} (1 + 15\gamma) \sqrt{g_\gamma(1, 1)} e^{\frac{2}{2}(B(1)-C(1))} (I(\epsilon, l) + O(1)),$$

$$\Delta \Theta = 2l e^{B(1)-C(1)-2D(1)} (I(\epsilon, l) + O(1)),$$

$$\frac{V_2}{R^2 T} = (1 + 15\gamma)^{-1} e^{B(1)+C(1)} (I(\epsilon, l) + O(1)).$$

(37)

Using the first line of (37) to eliminate $I$ in the last line one arrives at

$$V_2 = \frac{\pi R^2 T^2}{2} (1 - \frac{265}{8} \gamma) \cdot \frac{L}{\sqrt{1 - l^2 e^{4\gamma}}} + O(1).$$

(38)

At the very end we want to know the potential for fixed $\Delta \Theta$. According to the second line of (37) at $L \to \infty$ this requires $l \to 0$. Using the first line of (37) to eliminate $I$ in the second line one gets $l = \frac{\Delta \Theta}{\pi L T} e^{(15+D(1))} + o(\frac{1}{L})$ and altogether up to linear terms in $\gamma$

$$V_2(\Delta \Theta, L) = \frac{\pi R^2 T^2}{2} (1 - \frac{265}{8} \gamma) \cdot L + \frac{1}{4\pi} R^2 (\Delta \Theta)^2 (1 + \frac{15}{8} \gamma) \cdot \frac{1}{L} + O(1).$$

(39)

The terms which we neglected and indicated as $O(1)$ are, up to eq. (38), more precisely $\propto L e^{-\pi TL}$, as follows by repeating the estimates of ref. [14]. There the case $\Delta \Theta = 0$ was studied and the absence of any Lüscher term $\propto 1/L$ in the large $L$ expansion was pointed out to be an obstacle for the potential, derived in the approach of [5, 6], to be identified with a QCD potential. We find it interesting that for $\Delta \Theta \neq 0$ an $1/L$ term arises. However, due to the wrong sign and a coupling dependence it cannot be interpreted as a Lüscher term. One the other side one should keep in mind that this term is also not Coulombic since in 2+1 dimensions the Coulomb term is $\propto \log L$.

4 Conclusions

We showed that within the classical approximation the static quark-antiquark potential is completely switched off for antipodal $Q\bar{Q}$ orientation in the internal $S^5$ also for $T > 0$. In contrast to the $T = 0$ case this is, due to the broken SUSY, not required by any BPS argument. The effect is present both for the zeroth order background metrics of refs. [7, 8] as well as for the higher curvature corrected background of ref. [11, 12]. One should add, that certainly this is an effect of the backgrounds used. In other circumstances, with BPS

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9We thank P. Olesen for a comment on this fact.
arguments set out of work, nonvanishing $QQ$ forces may be possible for all orientations in internal space. Here, for example, we have in mind a $\mathcal{N} = 1, \ T = 0$ case based on a simple orbifold construction over $S^5$ [15].

The replacement of the lowest order background by the corrected ones shifts the critical line in the orientation-distance plane to larger distances for all values of the relative orientation in internal space. This shift acts in the right direction, if one follows the arguments of ref. [10] stating the vanishing of the total screening as soon as all corrections are taken into account.

From the study of spacelike Wilson loops we got the string tension for a temperature zero gauge theory in (2+1) dimensions. This tension is independent of the relative internal orientation of the quarks. However, the tension is sensible to switching on the string target space background corrections. The subleading terms in a large distance expansion of the potential are dependent on the relative internal orientation. In particular we found a nextleading $1/L$ term, absent for parallel orientation.

In this picture $T$ plays the role of a regularization parameter which, to make contact with (2+1)-dimensional non-supersymmetric QCD, has to be sent to infinity. To keep a finite tension, $R^2$ has to go to zero. Obviously, our approximation is not suited to study this limit. However, our formula (39) suggests that the $\Delta \Theta$ dependent $1/L$ term drops out in this limit. This indeed should happen since there is no place in QCD for the internal $S^5$ orientation parameter which is a remnant of $\mathcal{N} = 4$ SUSY.

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