A MIRROR THEOREM FOR GENUS TWO GROMOV–WITTEN INVARIANTS OF QUINTIC THREEFOLDS

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Abstract. We derive a closed formula for the generating function of genus two Gromov–
Witten invariants of quintic 3-folds and verify the corresponding mirror symmetry conjecture
of Bershadsky, Cecotti, Ooguri and Vafa.

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1. Introduction

The computation of the Gromov–Witten (GW) theory of compact Calabi–Yau 3-folds is a central problem in geometry and physics where mirror symmetry plays a key role. In the early 90’s, the physicists Candelas and his collaborators [2] surprised the mathematical community to use the mirror symmetry to derive a conjectural formula of a certain generating function (the \( J \)-function, see Section 3 for its definition) of genus zero Gromov–Witten invariants of

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a quintic 3-fold in terms of the period integral or the $I$-function of its B-model mirror. The effort to prove the formula directly leads to the birth of mirror symmetry as a mathematical subject. Its eventual resolution by Givental \cite{Givental-1993} and Liu–Lian–Yau \cite{Liu-Lian-Yau-1995} was considered to be a major event in mathematics during the 90’s. Unfortunately, the computation of higher genus GW invariants of compact Calabi–Yau manifolds (such as quintic 3-folds) turns out to be a very difficult problem. For the last twenty years, many techniques have been developed. These techniques have been very successful for so-called semisimple cases such as Fano or toric Calabi–Yau 3-folds. In fact, they were understood thoroughly in several different ways. But these techniques have little effect on our original problem on compact Calabi–Yau 3-folds. For example, using B-model techniques, Bershadsky, Cecotti, Ooguri and Vafa (BCOV) have already proposed a conjectural formula for genus one and two Gromov–Witten invariants of quintic 3-folds as early as 1993 \cite{Bershadsky-Cecotti-Ooguri-Vafa-1993} (see also \cite{Givental-1993b}). It took another ten years for Zinger to prove BCOV’s conjecture in genus one \cite{Zinger-2003}. During the last ten years, an effort has been made to push Zinger’s technique to higher genus without success. Nevertheless, the problem inspires many developments in the subject such as the modularity problem \cite{Batyrev-2005, Donagi-2005} and algebraic mathematical GLSM theory \cite{Borisov-Carlos-2013b}. It was considered as one of guiding problems in the subject. The main purpose of this article is to prove BCOV’s conjecture in genus two. To describe the conjecture explicitly, let us consider the so called $I$-function of the quintic 3-fold

$$I(q, z) = z^2 \sum_{d \geq 0} q^d \prod_{k=1}^{5d} (5H + kz) \prod_{k=1}^{d} (H + kz)^5,$$

where $H$ is a formal variable satisfying $H^4 = 0$. The $I$-function satisfies the following Picard–Fuchs equation

$$\left( D_H^4 - 5q \prod_{k=1}^{4} (5H + kz) \right) I(q, z) = 0,$$

where $D_H := zq \frac{dq}{dq} + H$. We separate $I(q, z)$ into components:

$$I(q, z) = zI_0(q) + I_1(q)H + z^{-1}I_2(q)H^2 + z^{-2}I_3(q)H^3.$$

The genus zero mirror symmetry conjecture of quintic 3-fold can be phrased as a relation between the $J$- and the $I$-function

$$J(Q) = \frac{I(q)}{I_0(q)}$$

up to a mirror map $Q = q e^{\tau_Q(q)}$, where

$$\tau_Q(q) = \frac{I_1(q)}{I_0(q)}.$$

The leading terms of $I_0$ and $\tau_Q$ are

$$I_0(q) = 1 + 120q + 113400q^2 + 168168000q^3 + O(q^4),$$

$$\tau_Q(q) = 770q + 717825q^2 + \frac{3225308000}{3} q^3 + O(q^4).$$

Now we introduce the following degree $k$ “basic” generators

$$\mathcal{X}_k := \frac{d^k}{du^k} \left( \log \frac{I_0}{L} \right), \quad \mathcal{Y}_k := \frac{d^k}{du^k} \left( \log \frac{I_0I_{1,1}}{L^2} \right), \quad \mathcal{Z}_k := \frac{d^k}{du^k} \left( \log (q^\frac{1}{5} L) \right),$$
where $I_{1,1} := 1 + qq^{-\frac{1}{2}}$. Let $L := (1 - 5^5 q)^{-\frac{1}{5}}$ and $du := L dq$. Some numerical data for these generators are given in Remark 3.4.

Let $F_{g}^{GW}(Q)$ be the generating function of genus $g$ Gromov–Witten invariants of a quintic 3-fold. The following is an equivalent formulation of the genus two mirror conjecture (Conjecture 3.10) of [42, 27] (see Section 3.5 for the argument).

**Conjecture 1.1.** The genus 2 GW generating function $F_{2}^{GW}(Q)$ for the quintic threefold is given by

$$F_{2}^{GW}(Q) = \frac{I_{0}^{2}}{L^{2}} \left( \frac{70 X_{3}}{9} + \frac{575 X_{2}^{2}}{18} + \frac{5 Y X_{2}}{6} + \frac{557 X^{3}}{72} - \frac{629 Y X^{2}}{72} - \frac{23 Y^{2} X}{24} - \frac{Y^{3}}{24} \\ + \frac{625 Z_{2}^{2}}{36} - \frac{175 Z Y X}{9} + \frac{1441 Z_{2}^{2} X}{48} - \frac{25 Z(X^{2} + Y^{2})}{24} - \frac{3125 Z^{2}(X + Y)}{288} \\ + \frac{41 Z_{2}^{3}}{144} - \frac{625 Z^{3}}{128} + \frac{2233 Z Z_{2}}{72} + \frac{547 Z_{3}}{72} \right),$$

where $Q = qe^{\tau_{Q}(q)}$.

In particular, the leading terms of $F_{2}^{GW}$ are

$$F_{2}^{GW}(Q) = -\frac{5}{144} + \frac{575}{48} Q + \frac{5125}{2} Q^{2} + \frac{7930375}{6} Q^{3} + O(Q^{4}).$$

The following is our Main Theorem:

**Theorem 1.2.** The above genus two mirror conjecture of quintic 3-fold holds.

**Remark 1.3.** A consequence of above conjecture is that $(L/I_{0})^{2}(Q)F_{2}^{GW}(Q)$ (more generally, $(L/I_{0})^{2g-2}(Q)F_{g}^{GW}(Q)$) is a homogeneous polynomial of the generators $X_{k}, Y_{k}, Z_{k}$. We refer to this property as finite generation. On the other hand, the original conjecture (Conjecture 3.10) is an inhomogeneous polynomial of five generators. We found it easier to work with a homogeneous polynomial than an inhomogeneous polynomial. Since the Taylor expansions of the generators are known, we can easily compute numerical genus two GW-invariants for any degree.

As we mentioned previously, it has been ten years since Zinger proved the genus one BCOV mirror conjecture. A key new advancement during last ten years was the understanding of global mirror symmetry which was in the physics literature in the beginning but somehow lost in its translation into mathematics in the early 90’s. The idea of the global mirror symmetry [2, 1, 27, 7] is to view GW theory as a particular limit (large complex structure limit) of the global B-model theory. Physicists use the results about the other limits such as the Gepner limit and conifold limit to yield the computation of the large complex structure limit/GW theory. Interestingly, one of the key pieces of information they used is the regularity of the Gepner limit. The latter can be interpreted as the existence of FJRW theory. A natural consequence of the above global mirror symmetry perspective is a prediction that the GW/FJRW generating functions are quasi-modular forms in some sense and hence are polynomials of certain finitely many canonical generators [1, 27, 42]. This imposes a strong structure for GW theory and we refer it as the finite generation property. For anyone with experience on the complexity of numerical GW invariants, it is not difficult to appreciate how amazing the finite generation property is! In fact, it immediately reduces an infinite computation for all degree to a finite computation of the coefficients of a polynomial. Therefore, it should be considered as one of the fundamental problems in the subject of higher genus GW theory.
The above global mirror symmetry framework was successfully carried out by the third author and his collaborators \[32, 36, 28\] for certain maximal quotients of Calabi–Yau manifolds. These examples are interesting in their own right. Unfortunately, we understand very little about the relation between the GW-theory of a Calabi–Yau manifold and its quotient. Hence, the success on its quotient has only a limited impact on our original problem. Several years ago, an algebraic-geometric curve-counting theory was constructed by Fan–Jarvis–Ruan for so called \textit{gauged linear sigma model (GLSM)} (see \[41, 26\] for its physical origin). One application of mathematical GLSM theory is to interpret the above global mirror symmetry as wall-crossing problem for a certain stability parameter $\epsilon$ of the GLSM-theory. It leads to a complete new approach to attack the problem without considering B-model at all. The first part of the new approach is to vary the stability parameter from $\epsilon = \infty$ (stable map theory) to $\epsilon = 0^+$ (quasimap \[9\] or stable quotient \[34\] theory) and has been successfully carried out recently by several authors \[8, 11, 12, 44\]. Suppose that $F^S_Q(q)$ is the the genus $g$ generating function of stable quotient theory. Then, the above authors have proved

$$F^G_W(q) = I^2_{0-2g}(q)F^S_Q(q),$$

which explains the appearance of $I^2_0$ at the right hand side of the conjecture. In the current paper, we take the next step to calculate the genus two generating function in quasimap theory and verify the conjectural formula in \[1, 42\].

The current paper relies on certain geometric input from \[5\] (see also \[6\]) which we now describe. Recall that the virtual cycle of the GLSM (stable map with $p$-field in this case) moduli space was constructed using cosection localization on an open moduli space \[30\]. The cosection is not $\mathbb{C}^*$-invariant which prevents us from applying the localization technique. A naive idea is to construct a compactification of the GLSM moduli space such that the cosection localized class can be identified with the virtual cycle of the compactified moduli space. Hopefully, the latter carries $\mathbb{C}^*$-action and the localization formula can be applied. Unfortunately, it is not easy to make naive idea work due to the difficulty of extending the cosection to the compactified moduli space. Working with Qile Chen, the last two authors solved the problem by introducing a certain “reduced virtual cycle” on an appropriate log compactification of the GLSM moduli space. In a sense, the current article and \[5\] belong together. Of course, \[5\] provides a general tool with applications beyond the current article. We will briefly describe \[5\] (specialized to a quintic 3-fold) in Section 2.

Taking the localization formula of \[5\] as an input, we can express genus $g$ Gromov–Witten invariants of a quintic 3-fold as a graph sum of twisted equivariant Gromov–Witten invariants of $\mathbb{P}^4$ and certain \textit{effective invariants}. When $g = 2, 3$, the effective invariants can be computed from known degree zero Gromov–Witten invariants. The main content of the current article is to extract a closed formula for the generating functions. This is of course difficult in general. We solve it using Givental formalism. A subtle and yet interesting phenomenon is the choice of twisted theory. The general twisted theory naturally depends on six equivariant parameters, five for the base $\mathbb{P}^4$ and one for the twist. It is complicated to study the general twisted theory, and therefore Zagier–Zinger \[43\] specialize the equivariant parameters of the base to scalar multiples of $(1, \xi, \xi^2, \xi^3, \xi^4, 0)$, where $\xi$ is a primitive fifth root of unity. Under this specialization, they show that the twisted theory is generated by the five generators predicted by physicists. Unfortunately, in our work we cannot set the equivariant parameter for the twist to zero. As a consequence, we have to introduce four \textit{extra generators}. It was a miracle to us that the terms involving the four extra generators cancel and we have our
The appearance of four extra generators has a direct impact to our future work for \( g \geq 3 \). For example, while no additional geometric input is needed to apply our method to genus 3 to prove the conjectural formula of Klemm-Katz-Vafa [29], and we could proceed with the methods developed in this paper by brutal force, the resulting proof would not be very illuminating. Recall that there is a conjectural formula up to genus 51 by A. Klemm and his collaborators [27]. Our eventual goal is to reach genus 51 and go beyond. To do so, we have to understand better the cancellation of terms involving extra generators. We will leave this to a future research [23].

The paper is organized as follows. In Section 2 we will summarize the relevant compactified moduli space and its localization formula from [5]. The detailed analysis of contributions of the localization graphs and their closed formulae in terms of generators are stated in Section 3. The main theorem directly follows from these closed formulae. In Section 4 we derive important results about the twisted theory, which we then apply in Section 5 to yield a proof of the closed formulae. Finally, we would like to mention several independent approaches to higher genus problem by Maulik and Pandharipande [35], Chang–Li–Li–Liu [4] and Guo–Ross [24, 25].

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2. A localization formula

The main geometric input is a formula computing GW-invariants of a quintic 3-fold in terms of a twisted theory and a certain “effective” theory. This formula is obtained by localization on a compactified moduli space of stable maps with a \( p \)-field. The proof of the formula and details about the moduli space can be found in [5] (see also [6]).

In Section 2.1 we give an overview of the definition of the relevant moduli spaces, and in Section 2.2 we explain the localization formula in the general case. We then, in Section 2.3 specialize to genus two. Finally, in Section 2.4 we illustrate the formula by directly computing the genus two, degree one invariant.

2.1. Moduli space. Let \( \overline{M}_{g,0}(Q_5, d) \) denote the moduli space of genus-\( g \), unpointed, degree-\( d \) stable maps \( f: C \to Q_5 \) to the quintic threefold \( Q_5 \subset \mathbb{P}^4 \). This moduli space admits a perfect obstruction theory and hence a virtual class \( [\overline{M}_{g,0}(Q_5, d)]^{\text{vir}} \) of virtual dimension zero, whose degree is defined to be the Gromov–Witten invariant \( N_{g,d} \).

For the computations in this paper, it will be much more convenient to work with the moduli space \( \overline{Q}_{g,0}(Q_5, d) \) of genus-\( g \), unpointed, degree-\( d \) stable quotients (or quasimaps) to the quintic threefold \( Q_5 \subset \mathbb{P}^4 \) instead. We refer to [34, 9] for the definition of this moduli space, mentioning here just that it parameterizes prestable curves \( C \) together with a line bundle \( L \) and sections \( s \in H^0(L^{\otimes 5}) \) satisfying the equation of \( Q_5 \) and a stability condition. The moduli space \( \overline{Q}_{g,0}(Q_5, d) \) also admits a perfect obstruction theory and virtual class \( [\overline{Q}_{g,0}(Q_5, d)]^{\text{vir}} \), which after integration defines the stable quotient invariant \( N_{g,d}^{SQ} \). By the wall-crossing formula [8, 11], the information of the \( N_{g,d}^{SQ} \) is equivalent to the information of
the $N_{g,d}$. In genus $g \geq 2$, the wall-crossing formula says that, if

$$F_{g}^{GW}(q) := \sum_{d=0}^{\infty} q^{d} N_{g,d}, \quad F_{g}^{SQ}(q) := \sum_{d=0}^{\infty} q^{d} N_{g,d}^{SQ}$$

are the generating series of stable map and stable quotient invariants, then

$$F_{g}^{GW}(Q) = F_{0}^{2g-2} F_{g}^{SQ}(q).$$

It is difficult to directly access the stable quotient (or stable map) theory of $Q_{5}$, in part because $Q_{5}$ is a “non-linear” object. By results of Chang–Li \cite{3} we can instead work on the more linear moduli space $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$ of stable quotients with a $p$-field, that is the cone $\pi_{*}(\omega_{\pi} \otimes L^{-\otimes 5})$ over $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$. Here, $\pi$ denotes the universal curve and $L$ denotes the universal line bundle $L$. The moduli space $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$ also has a perfect obstruction theory and hence a virtual class. However, because $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$ is in general not compact, this virtual class cannot directly be used to define invariants. To circumvent this problem, Chang–Li introduce a cosection $\sigma$ of the obstruction sheaf, and show that

$$N_{g,d}^{SQ} = \int_{[\overline{Q}_{g,0}(\mathbb{P}^{4}, d)]^{vir}} (-1)^{1-g+5d},$$

where $[\overline{Q}_{g,0}(\mathbb{P}^{4}, d)]^{vir}$ is the cosection localized virtual class of $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$, which is supported on the compact moduli space $\overline{Q}_{g,0}(Q_{5}, d)$.

The main new idea of \cite{5} is to define a modular compactification $X_{g,d}$ of $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$ with a perfect obstruction theory extending the one of $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$ such that the cosection $\sigma$ extends without acquiring additional degeneracy loci. It is then easy to see that

$$\int_{[\overline{Q}_{g,0}(\mathbb{P}^{4}, d)]^{vir}} (-1)^{1-g+5d} = \int_{[X_{g,d}]^{vir}} (-1)^{1-g+5d}.$$ 

If $X_{g,d}$ furthermore admits a non-trivial torus action (and the perfect obstruction theory is equivariant), we can apply virtual localization to the right hand side to express it in terms of hopefully simpler fixed loci.

A suitable compactification $X_{g,d}$ is given by a space of stable quotients $(C, L, s)$ together with a log-section $\eta: C \to \mathbb{P} := \mathbb{P}(\omega_{C} \otimes L^{-\otimes 5} \oplus \mathcal{O}_{C})$ where the target is equipped with the divisorial log-structure corresponding to the infinity section. The open locus where the log-section $\eta$ does not touch the infinity section is clearly isomorphic to $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$, and the canonical perfect obstruction theory of $X_{g,n,d}$ extends the one of $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$. Unfortunately, the cosection $\sigma$ becomes singular on the complement of $\overline{Q}_{g,0}(\mathbb{P}^{4}, d)$, Still, it can be extended to a homomorphism to a non-trivial line bundle $L_{N}^{vir}$, which comes with a canonical section $\mathcal{O} \to L_{N}^{vir}$, and we can define a new (reduced) perfect obstruction theory by “removing” (taking the cone) under the induced map from the obstruction theory to the complex $[\mathcal{O} \to L_{N}^{vir}]$. With this reduced perfect obstruction theory, $X_{g,d}$ satisfies all desired properties.

**Remark 2.1.** The construction of the modular compactification and its reduced perfect obstruction theory can also be carried out in the setting of stable maps, and it satisfies all of the analogous properties.

\footnote{To be precise, we use the result of Chang–Li to rewrite Gromov–Witten invariants of the quintic in terms of stable maps with $p$-fields. After that, we apply the wall-crossing \cite{12} to move to stable quotients with a $p$-field.}
The vertices $v$ accordingly. Furthermore, let graphs $\Gamma$ with $n$ either the zero or infinity section. We define the bivalent structure $V_{\Gamma}$ of corresponding equivariant parameter. The fixed loci of $C_{\Gamma}$ Localization.

2.2. Localization. We now consider virtual localization \cite{22} of the reduced perfect obstruction theory of $X_{g,d}$ with respect to the $C^*$-action of $X_{g,d}$ that scales $\eta$. Let $t$ be the corresponding equivariant parameter. The fixed loci of the $C^*$-action are indexed by bivalent graphs $\Gamma$ with $n$ legs. We let $V(\Gamma)$, $E(\Gamma)$ be the corresponding sets of vertices and edges. The vertices $v \in V(\Gamma)$ correspond to components (or isolated points) of the curve $C$ sent to either the zero or infinity section. We define the bivalent structure $V(\Gamma) = V_0(\Gamma) \sqcup V_\infty(\Gamma)$ accordingly. Furthermore, let $g(v)$ (respectively, $d(v)$) be the genus of such a component (respectively, the degree of $L$ on this component). A vertex $v$ is unstable if and only if it corresponds to an isolated point of $C$, that is when $g(v) = 0$ and either $n(v) = 1$ or $n(v) = 2$ and $d(v) = 0$. The edges $e \in E(\Gamma)$ correspond to the remaining components of $C$, which are rational, each contracted to a point in $\mathbb{P}^4$, and mapped via a degree-$\delta(e)$ Galois cover to the corresponding component of $\mathbb{P}$ (with a possible base point at the zero section).

The fixed locus corresponding to a dual graph $\Gamma$ is (up to a finite map) isomorphic to

$$M_\Gamma := \prod_{v \in V_0(\Gamma)} \overline{Q}_{g(v),n(v)}(\mathbb{P}^4, d(v)) \times_{(\mathbb{P}^4)^{E(\Gamma)}} \prod_{v \in V_\infty(\Gamma)} \overline{Q}_{g(v),n(v)}(\mathbb{P}, d(v), \mu(v))^\sim,$$

where unstable moduli spaces $\overline{Q}_{g(v),n(v)}(\mathbb{P}^4, d(v))$ are defined to be a copy of $\mathbb{P}^4$, and the rubber moduli space $\overline{Q}_{g(v),n(v)}(\mathbb{P}, d(v), \mu(v))^\sim$ generically parameterizes stable quotients $(C, L, s)$ together with a nonzero holomorphic section of $\omega \otimes L^{-\otimes 5}$ up to scaling with zeros prescribed by $\mu(v)$. Here, $\mu(v)$ is the $n(v)$-tuple of integers consisting of 0 for each leg, and $\delta(e) - 1$ for each edge $e$ at $v$. In analogy with \cite{39}, we will also refer to the rubber moduli space as the “effective” moduli space.

In order for a fixed locus corresponding to a graph to be non-empty, there are many constraints on the decorated dual graph. First, we must have

$$g = \sum_{v \in V(\Gamma)} g(v) + h^1(\Gamma), \quad d = \sum_{v \in V(\Gamma)} d(v).$$

Second, there is a stability condition which says that for any vertex $v \in V_0(\Gamma)$ of genus zero and valence one, the corresponding unique edge $e$ needs to satisfy $\delta(e) > 1 + 5d(v)$. Third, for every $v \in V_\infty(\Gamma)$, the partition $\mu(v)$ must have size $2g(v) - 2 - 5d(v) \geq 0$. Note that the third condition implies that $g(v) \geq 1$ for each $v \in V_\infty(\Gamma)$. A localization graph satisfying all these conditions is depicted in Figure 1.

![Figure 1. A localization graph in genus $g_1 + g_2 + 9$ and degree $d_1 + d_2 + 2$. The bottom vertices lie in $V_0$, the top vertices lie in $V_\infty$. The pair $(g(v), d(v))$ is specified at each vertex. An edge $e$ is thick if $\delta(e) = 2$. Otherwise, $\delta(e) = 1.$](image-url)
The contribution of a decorated graph $\Gamma$ can then be written as

$$\frac{1}{|\text{Aut}|} \int_{M_\Gamma} \Delta^! \left( \prod_{v \in V_0(\Gamma)} \frac{e(-R\pi_{v,*}(\omega_{\pi_v} \otimes L^{-\otimes 5}) \otimes [1])}{\prod_{e \in E\text{ at } v} (t - 5e_v^*H)(\frac{t-5e_v^*H}{\delta(e)} - \psi_e)} \cap [\mathcal{Q}_{g,v,n}(\mathbb{P}^4, d(v))]^{\text{vir}} \right) \times \prod_{v \in V_\infty(\Gamma)} -t + 5H - \psi_0 \cap [\mathcal{Q}_{g,v,n}(\mathbb{P}, d(v), \mu(v))^{\text{red}}] \times \prod_{e \in E(\Gamma)} \frac{1}{\delta(e) \prod_{i=1}^{\delta(e)-1} \frac{1}{t - 5e_v^*H(\delta(e))}}.$$ 

with the notation:

- $\Delta: ([\mathbb{P}^4]|E(\Gamma)| \rightarrow ([\mathbb{P}^4]|E(\Gamma)| \times ([\mathbb{P}^4]|E(\Gamma)|):$ diagonal map
- $\pi_v$: universal curve corresponding to $v \in V_0(\Gamma)$
- $e_v^*(H)$: pullback of $H$ via any of the two evaluation maps corresponding to $e$
- $[1]$: line bundle with Chern class $t$
- $5H - \psi_0$: a universal divisor class on the effective moduli space
- $[\mathcal{Q}_{g,v,n}(\mathbb{P}, d(v), \mu(v))^\text{red}]$, a reduced virtual class, which is discussed below

In the unstable case that $v \in V_0(\Gamma)$ has valence 2 and is connected to two edges $e_1$ and $e_2$, we define

$$\frac{e(-R\pi_{v,*}(\omega_{\pi_v} \otimes L^{-\otimes 5}) \otimes [1])}{\prod_{e \in E\text{ at } v} (t - 5e_v^*H)(\frac{t-5e_v^*H}{\delta(e)} - \psi_e)} \cap [\mathcal{Q}_{g,v,n}(\mathbb{P}^4, d(v))]^{\text{vir}}$$

to be

$$\frac{1}{(t - 5H) \left( \frac{t-5H}{\delta(e_1)} + \frac{t-5H}{\delta(e_2)} \right)}.$$ 

and when $v$ has genus zero and is connected to a single edge $e$, we define (2) as

$$(t - 5H)^{5d(v)+1}(5d(v)+1)! \delta(e)^{5d(v)+1}.$$ 

Most parts of (1) are effectively computable, such as the integrals over the moduli spaces for $v \in V_0(\Gamma)$, which are twisted $\mathbb{P}^4$ invariants of $\mathbb{P}^4$. The most difficult part of the formula is related to the effective moduli spaces. Fortunately, these integrals are highly constrained.

The reduced virtual class

$$[\mathcal{Q}_{g,n}(\mathbb{P}, d, \mu)^{\text{red}}]$$

has dimension

$$n - (2g - 2 - 5d),$$

which is one more than the naive virtual dimension. Together with pullback properties of the reduced virtual class, this implies that the virtual class vanishes unless all parts of $\mu$ are 0 or 1. Furthermore, there is a dilaton equation which implies that

$$\int_{[\mathcal{Q}_{g,n}(\mathbb{P}, d, \mu)^{\text{red}}]} \psi_0^{n-2g+2} = \frac{(2g - 2 + n)!}{(4g - 4 - 5d)!} \int_{[\mathcal{Q}_{g,2g-2}(\mathbb{P}, d, (1, \ldots, 1))^\text{red}}} 1$$

as long as $g \geq 2$. When $g = 1$, we need to have $d = 0$, and $\mathcal{Q}_{1,n}(\mathbb{P}, 0, (0, \ldots, 0))^\sim \cong \mathcal{Q}_{1,n} \times \mathbb{P}^4$.

The virtual class is also explicit, and given by

$$[\mathcal{Q}_{1,n}(\mathbb{P}, 0, (0, \ldots, 0))^\text{red}] = e((T_{\mathbb{P}^4} + \mathcal{O} - \mathcal{O}_{\mathbb{P}^4}(5)) \otimes \mathcal{E}^\vee) = 205H^4 + 40H^3\lambda_1,$$
where $E$ denotes the Hodge (line) bundle, and $\lambda_1$ its first Chern class. There are also explicit formulae for $5H - \psi_0$ in this case, for instance, when $n = 1$, we have $5H - \psi_0 = 5H - \lambda_1 = 5H - \psi_1$, and, when $n = 2$, we have $5H - \psi_0 = 5H - \psi_1 = 5H - \psi_2$.

All in all, (1) gives an explicit computation of any Gromov–Witten invariant of the quintic, up to the determination of the constants $c_{g,d} := \int_{[Q_{g,2g-2}(F,d,(1,...,1))]^{red}} 1 \in \mathbb{Q}$, which are defined for every $g \geq 2, d \geq 0$ such that $d \leq \frac{2g-2}{5}$. We call these constants “effective invariants”. Note that for any particular genus, only finitely many of these invariants are needed.

**Remark 2.2.** All of the discussion of this section can also be carried out in the stable maps setting, with only the following essential differences: Vertices $v \in V_0(\Gamma)$ with $(g(v), n(v)) = (0,1)$ are stable unless $d(v) = 0$. Therefore, for such vertices $v$, the corresponding edge $e$ only needs to have $\delta(e) > 1$. Accordingly, we also need to replace $5d(v) + 1$ by 1 in (3).

Because of this, there are many more localization graphs in the stable maps setting than in the stable quotient setting. In fact, while in any case, there are only finitely many types of localization graphs for any fixed $g$ and $d$, in the stable quotient setting, for fixed $g$, the number of localization graphs does not depend on $d$ (as long as $d$ is large enough). This is the main technical advantage of the stable quotient theory for our purpose.

**2.3. Formula in genus two.** We now specialize the localization formula to genus two, and apply it to the computation of the generating series

$$F_{2}^{SQ}(q) = \sum_{d=0}^{\infty} (-1)^{1-g+5d} q^d \int_{[X_{2,d}]^{vir}} 1.$$ 

There are 5 localization graphs, which are shown in Figure 2. Note that the fifth graph can only occur when $d = 0$, and that its contribution is given by the constant $-c_{2,0}$. We label the first four graphs by $\Gamma^2$, $\Gamma^1$, $\Gamma^0_b$ and $\Gamma^0_a$, respectively.
We introduce the bracket notation
\[
\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n}^{t,\text{SQ}} = \sum_{d=0}^{\infty} q^d \int \mathbb{P}^4 \mathbb{P}^4 \text{vir} e(R\pi_* L_*^{\otimes 5} \otimes [1]) \prod_{i=1}^{n} \text{ev}_i^*(\alpha_i)
\]

\[
= \sum_{d=0}^{\infty} (-1)^{1-g+5d} \int \mathbb{P}^4 \mathbb{P}^4 \text{vir} e(-R\pi_* (\mu \otimes L_*^{\otimes 5}) \otimes [1]) \prod_{i=1}^{n} \text{ev}_i^*(\alpha_i).
\]

The contribution of \(\Gamma^2\) is then simply given by
\[
\text{Cont}_{\Gamma^2} := \langle \rangle_{2,0}^{t,\text{SQ}}.
\]
For the contribution of \(\Gamma^1\), we first need the computation
\[
\int_{\mathcal{M}_{1,1} \times \mathbb{P}^4} \frac{t}{-t + 5H - \psi_0} (205H^4 + 40H^3 \lambda_1) = -\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}.
\]
Therefore,
\[
\text{Cont}_{\Gamma^1} := -\left\langle \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi)} \right\rangle_{1,1}^{t,\text{SQ}},
\]
and we can also directly compute:
\[
\text{Cont}_{\Gamma^0} := \left\langle \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi_1)}, \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi_2)} \right\rangle_{0,2}^{t,\text{SQ}}.
\]
Finally, for the contribution of \(\Gamma_0^1\), we need to know
\[
\int_{\mathcal{M}_{1,2} \times \mathbb{P}^4} \frac{t}{-t + 5H - \psi_0} (205H^4 + 40H^3 \lambda_1) = \frac{5}{3} H^3 t^{-1} + \frac{65}{8} H^4 t^{-2}.
\]
Thus,
\[
\text{Cont}_{\Gamma_0^1} := \left\langle \frac{\frac{5}{3} (H^3 \otimes H^4 + H^1 \otimes H^3) t^{-1} + \frac{65}{8} H^4 \otimes H^4 t^{-2}}{(t - 5H)(t - 5H - \psi_1)(t - 5H)(t - 5H - \psi_2)} \right\rangle_{0,2}^{t,\text{SQ}}.
\]
Summing all contributions gives
\[
F_2^{\text{SQ}}(q) = \text{Cont}_{\Gamma^2} + \text{Cont}_{\Gamma^1} + \frac{1}{2} \text{Cont}_{\Gamma_0^1} + \frac{1}{2} \text{Cont}_{\Gamma_0^0} - c_{2,0}.
\]

2.4. Example. To illustrate our genus two formula, we compute explicitly the degree one invariant, that is the coefficient of \(q^1\) in \(F_2^{\text{SQ}}\) defined by (4), and match it with the value given by Conjecture [4] that is the \(q^1\)-coefficient of
\[
F_2^{\text{SQ}}(q) = (I_0(q))^{-2} F_2^{GW}(Q) = -\frac{5}{144} + \frac{325}{16} q + \frac{366875}{24} q^2 + \frac{1030721125}{48} q^3 + O(q^4).
\]
Note that the value of the constant \(c_{2,0}\) is irrelevant for this computation. To compute the \(q^1\)-coefficients of the other terms on the right hand side of (4), we use localization for an additional diagonal \((\mathbb{C}^*)^2\)-action on the base \(\mathbb{P}^4\) with corresponding localization parameters \(\lambda_i\). We refer to [31, Section 7] and [22] for the enumeration of fixed points and identification of localization contributions that we use below. It will be convenient for the computation to assume that \(\lambda_i = \xi \lambda\) where \(\xi\) is a primitive fifth root of unity. We index the fixed points of \(\mathbb{P}^4\) by \(i \in \{0, \ldots, 4\}\). Note that the Euler class of the Poincaré dual of fixed point \(i\) is given by \(\prod_{j \neq i}(\lambda_i - \lambda_j)\).
2.4.1. Genus two contribution. We begin with the computation of $\text{Cont}_{12}$ via localization. First, note that there are two types of fixed loci for localization on $\mathcal{Q}_{2,0}(\mathbb{P}^4, 1)$.

The first type of fixed locus is where, except for an order one base point, the entire source curve is contracted to a fixed point $i$. Such a fixed locus is isomorphic to $\mathcal{M}_{2,1}$. The contribution of the virtual class of this fixed locus is

\[ \prod_{j \neq i} \left( \frac{1}{(\lambda_i - \lambda_j)^2 - \lambda_1(\lambda_i - \lambda_j) + \lambda_2} \right) \],

where, by abuse of notation, $\lambda_1$ and $\lambda_2$ denote the Chern classes of the Hodge bundle. The contribution of the twisting by $e(R\pi_* \mathcal{L}^{\otimes 5} \otimes [1])$ is

\[ \frac{(t + 5\lambda_i - 5\psi_1) \cdots (t + 5\lambda_i - \psi_1)(t + 5\lambda_i)}{(t + 5\lambda_i)^2 - \lambda_1(t + 5\lambda_i) + \lambda_2} \].

Summing the product of (5) and (6) over all five fixed loci, taking the coefficient of $t^0$ and integrating, gives the total resulting contribution of these 5 fixed loci

\[
\int_{\mathcal{M}_{2,1}} 1370\psi_1^4 - 3075\psi_1^2\lambda_1^2 + 2100\psi_1\lambda_1^3 + 75\psi_1^2\lambda_2 + 2925\psi_1\lambda_1\lambda_2
\]

\[ = 1370 \cdot \frac{1}{1152} - 3075 \cdot \frac{7}{2880} + 2100 \cdot \frac{1}{1440} + 75 \cdot \frac{7}{5760} + 2925 \cdot \frac{1}{2880} = -\frac{4285}{1152} , \]

where we used a few well-known intersection numbers on $\mathcal{M}_{2,1}$.

The second type of fixed locus corresponds to the locus of two genus one components contracted to fixed points $i \neq j$, and which are connected by a degree one cover of the torus fixed curve connecting $i$ and $j$. This fixed locus is isomorphic to $\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$ (up to a $\mathbb{Z}/2\mathbb{Z}$-automorphism group that we will address later). The contribution of the virtual class to this fixed locus is

\[ \prod_{k \neq i} \frac{\lambda_i - \lambda_k - \lambda_{1a}}{\lambda_i - \lambda_k} \prod_{k \neq j} \frac{\lambda_j - \lambda_k - \lambda_{1b}}{\lambda_j - \lambda_k} \frac{1}{(\lambda_i - \lambda_j - \psi_{1a})(\lambda_j - \lambda_i - \psi_{1b})} , \]

where $\psi_{1a}$, $\psi_{1b}$, $\lambda_{1a}$ and $\lambda_{1b}$ denote the cotangent and Hodge classes on each $\mathcal{M}_{1,1}$-factor. The contribution of the twisting is

\[ \frac{(t + 5\lambda_i) \cdots (t + 5\lambda_j)}{(t + 5\lambda_i - \lambda_{1a})(t + 5\lambda_j - \lambda_{1b})} . \]

Summing up the contribution from all of the 20 fixed loci gives

\[ \frac{2965}{288} . \]

2.4.2. Genus one contribution. We now compute $\text{Cont}_{11}$. There are also two types of fixed loci for $\mathcal{Q}_{1,1}(\mathbb{P}^4, 1)$.

The first type of fixed locus is where, except for an order one base point, the entire source curve is contracted to a fixed point $i$. Such a fixed locus is isomorphic to $\mathcal{M}_{1,2}$. The contribution of the virtual class of this fixed locus is

\[ \prod_{j \neq i} \frac{\lambda_i - \lambda_j - \lambda_1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_j - \psi_2)} , \]
and the contribution of the twisting is
\[
\frac{(t + 5\lambda_i - 5\psi_2) \cdots (t + 5\lambda_i - \psi_2)(t + 5\lambda_i)}{t + 5\lambda_i - \lambda_1}.
\]
In addition, we need to consider the insertion. For this, note that \(\text{ev}_1^*(H) = \lambda_i\), and that the descendent \(\psi\)-class is \(\psi_1\). Thus, the insertion gives a factor of
\[
\frac{-\frac{5}{3}\lambda_i^3 + \frac{5}{24}\lambda_i^4 t^{-1}}{(t - 5\lambda_i)(t - 5\lambda_i - \psi_1)}.
\]
Summing over \(i\) gives the contribution of
\[
\frac{975}{64}.
\]

The second type of fixed locus is where there is a genus one curve contracted to a fixed point \(i\) which is connected via a node to a rational component mapping isomorphically to the fixed line connecting fixed point \(i\) to another fixed point \(j\) such that the preimage of fixed point \(j\) is the marking. This locus is isomorphic to \(\overline{\mathcal{M}}_{1,1}\). The contribution of the virtual class is
\[
\prod_{k \neq i} (\lambda_i - \lambda_k - \lambda_1)^{-1} \prod_{k \neq i} (\lambda_j - \lambda_k) \lambda_i - \lambda_j - \psi_1,
\]
and the contribution of the twisting is
\[
\frac{(t + 5\lambda_i)(t + 4\lambda_i + \lambda_j) \cdots (t + 5\lambda_j)}{t + 5\lambda_i - \lambda_1}.
\]
Now, \(\text{ev}_1^*(H) = \lambda_j\), and the descendent \(\psi\)-class is given by \(\lambda_i - \lambda_j\). So, the insertion gives a factor of
\[
\frac{-\frac{5}{3}\lambda_j^3 + \frac{5}{24}\lambda_j^4 t^{-1}}{(t - \lambda_j)(t - 4\lambda_j - \lambda_i)}.
\]
Summing over all \(i \neq j\) gives
\[
-\frac{3425}{288}.
\]

2.4.3. Genus zero contributions. We finally compute \(\text{Cont}_{\Omega}^{\text{a}} + \text{Cont}_{\Omega}^{\text{b}}\). Note that we can combine the two insertions:
\[
\left( -\frac{5}{3}H^3 + \frac{5}{24}H^4 t^{-1}\right)^2 + \frac{5}{3}(H^3 \otimes H^4 + H^4 \otimes H^3) t^{-1} + \frac{65}{8} H^4 \otimes H^4 t^{-2} = \frac{25}{9} H^3 \otimes H^3 + \frac{95}{72} (H^3 \otimes H^4 + H^4 \otimes H^3) t^{-1} + \frac{4705}{576} H^4 \otimes H^4 t^{-2}
\]
There are also two types of fixed loci for \(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^4, 1)\). The first type of fixed locus is where except for an order one base point the entire source curve is contracted to a fixed point \(i\). Such a fixed locus is isomorphic to a point.

The contribution of the virtual class for this fixed locus is given by
\[
1 \prod_{j \neq i} (\lambda_i - \lambda_j)^2,
\]
and the contribution of the twisting is
\[
(t + 5\lambda_i)^6.
\]
On this fixed locus, the descendent classes vanish, and we have $ev_1^*(H) = ev_2^*(H) = \lambda_i$. Thus, the insertion gives a factor of
\[
\frac{25 \lambda_i^6 + 95 \lambda_i^7 t^{-1} + 4705 \lambda_i^8 t^{-2}}{(t - 5\lambda_i)^4}
\]
In total, the contribution of the fixed loci of first type is
\[
\frac{1967}{192}.
\]
A fixed locus of second type is where the map is a degree one cover of a fixed line of $\mathbb{P}^4$ such that marking 1 is mapped to fixed point $i$ and marking 2 is mapped to fixed point $j$. Such a locus is again just a point. The contribution of the virtual class is then given by
\[
\frac{1}{\prod_{k \neq i}(\lambda_i - \lambda_k) \prod_{k \neq j}(\lambda_j - \lambda_k)}
\]
and the contribution of the twisting is
\[(t + 5\lambda_i)(t + 4\lambda_i + \lambda_j) \cdots (t + 5\lambda_j).
\]
We have $ev_1^*(H) = \lambda_i, ev_2^*(H) = \lambda_j$, the descendent class at marking one is $\lambda_j - \lambda_i$, while the other descendent class is $\lambda_i - \lambda_j$. So, the insertion gives a factor of
\[
\frac{25 \lambda_i^3 \lambda_j^3 + 95 \lambda_i^3 \lambda_j^4 t^{-1} + 95 \lambda_i^4 \lambda_j^3 t^{-1} + 4705 \lambda_i^4 \lambda_j^4 t^{-2}}{(t - 5\lambda_i)(t - 5\lambda_j)(t - 4\lambda_i - \lambda_j)(t - \lambda_i - 4\lambda_j)}
\]
Thus, the total contribution is
\[
\frac{3001}{144}.
\]
2.4.4. Final result. Collecting all the contributions gives
\[
-\frac{4285}{1152} + \frac{12965}{288} + \frac{975}{64} - \frac{3425}{288} + \frac{1}{2} \left( \frac{1967}{192} + \frac{3001}{144} \right) = \frac{325}{16},
\]
which is the expected coefficient of $q^1$ in $F_{SQ}^2(q)$.

3. Proof of the Main Theorem

In this section, we provide a list of closed formulae for the contribution of each graph in Section 2.3. Based on these closed formulae, we prove the main theorem. One subtlety are the extra generators appearing in the twisted theory. They mysteriously cancel each other when we sum up the contributions. We will come back to these cancellations in higher genus in [23]. The proof of these formulae will be presented in Section 5.

3.1. Genus zero mirror theorem for the twisted theory of $\mathbb{P}^4$. Let $I(t, q, z)$ be the $I$-function of the twisted invariants, that is explicitly
\[
I(t, q, z) = z \sum_{d \geq 0} q^d \prod_{j=1}^{5d} (5H + jz - t) \prod_{k=1}^d (H + kz)^5,
\]
which satisfies the following Picard–Fuchs equation
\[
\left( D_H^5 - q \prod_{k=1}^5 (5D_H + kz - t) \right) I(t, q, z) = 0,
\]
where \( D_H := D + H := z q \frac{d}{dq} + H \). The \( I \)-function has the following form when expanded in \( z^{-1} \):

\[
I(t, q, z) = z I_0(q) + I_1(t, q) + z^{-1} I_2(t, q) + z^{-2} I_3(t, q) + \cdots.
\]

The genus zero mirror theorem \cite{LIN} relates \( I(t, q, z) \) to the \( J \)-function, defined by

\[
J(t, z) := -tz + t(-z) + \sum_i \varphi^i \left( \left\langle \left\langle \frac{\varphi_i}{z - \psi} \right\rangle \right\rangle_{0,1} (t(\psi)),
\]

where we define the double bracket for the twisted Gromov–Witten invariants by

\[
\left\langle \left\langle \gamma_1(\psi), \ldots, \gamma_m(\psi) \right\rangle \right\rangle^t_{g,m} (t(\psi)) = \sum_n \frac{1}{n!} \left\langle \gamma_1(\psi), \ldots, \gamma_m(\psi), t(\psi), \ldots, t(\psi) \right\rangle^t_{g,m+n}.
\]

\[
= \sum_{n,d} \frac{q^d}{n!} \int_{[M_{g,m+n}(P^4,d)]^{vir}} e(R^* \pi_* f^* \mathcal{O}(5) \otimes [1]) \prod_{j=1}^m \gamma_j(\psi_j) \prod_{k=m+1} t(\psi_k),
\]

where we recall that [1] is a line bundle with first Chern class \( t \), and where \( \{\varphi_i\} \) is any basis of \( H^*(P^4) \) with dual basis \( \{\varphi^i\} \) under the inner product

\[
(a, b)^t := \int_{P^4} a \cup b \cup (5H - t).
\]

To state the precise relationship, we write

\[
I_1(t, q) = I_1(q) H + I_{1:a}(q)t,
\]

and define the mirror map by

\[
\tau(q) = \frac{I_1(t, q)}{I_0(q)} = H \frac{I_1(q)}{I_0(q)} + t \frac{I_{1:a}(q)}{I_0(q)}.
\]

Then Givental’s mirror theorem states that the \( J \)-function of the twisted invariants can be computed from the \( I \)-function by

\[
J(\tau(q), z) = \frac{I(t, q, z)}{I_0(q)}.
\]

### 3.2. Quantum product and “extra” generators

We consider the quantum product in the twisted theory, which is defined for any point \( t \in H^*_{\mathbb{C}^*}(P^4) \) by

\[
a \ast_t b := \sum_i \varphi^i \left\langle \left\langle a, b, \varphi_i \right\rangle \right\rangle^t_{0,3} (t) = \sum_i \varphi^i \langle a, b, \varphi_i \rangle^t_{0,3}.
\]

In the basis \( \{H^k\} \), the quantum product \( \hat{\ast}_\tau \), in which

\[
\hat{\tau} = H + q \frac{d}{dq} \tau,
\]

can be identified with a \( 5 \times 5 \)-matrix. It is not hard to see that it has the following form

\[
\hat{\ast}_\tau = A := \begin{pmatrix}
I_{1,1:a}t & I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} & I_{1,5} \\
I_{2,2,a}t^2 & I_{2,2:a}t & I_{2,3} & I_{2,4} & I_{2,5} \\
I_{3,3,c}t^3 & I_{3,3:b}t^2 & I_{3,3:a}t & I_{3,4} & I_{3,5} \\
I_{4,4,a}t^4 & I_{4,4,c}t^3 & I_{4,4:b}t^2 & I_{4,4:a}t & I_{4,4} \\
I_{5,5,c}t^5 & I_{5,5:a}t^4 & I_{5,5:b}t^3 & I_{5,5:a}t^2 & I_{5,5:a}t
\end{pmatrix}^t,
\]
where the $I_{i,i}$ and $I_{i,i*}$ are certain power series in $q$. Recall that by the basic theory of Frobenius manifolds, the $S$-matrix $S(t, z)$, which is defined by

$$S(t, z)\varphi^i = \varphi^i + \sum_j \varphi^j \left< \varphi_j, \frac{\varphi_i}{z - \psi} \right> (t),$$

is a solution of the quantum differential equation

$$dS(t, z) = dt * t S(t, z),$$

where the differential $d$ acts on the coordinates $t \in H^*_C(\mathbb{P}^4)$ (but not on the Novikov variable $q$). By using the divisor equation and the matrix introduced above, we can write down the explicit form of the quantum differential equation at $t = \tau(q)$:

$$D_H S(\tau(q), z) = A(t, q) \cdot S(\tau(q), z).$$

Since the $S$-matrix can be obtained from the derivatives of the $I$-function by Birkhoff factorization (see e.g. [13] and see also Proposition 4.1 below), we see that all the entries in this matrix $A$ can be written explicitly in terms of the derivatives of $I_k$. In particular, we have

$$I_{1,1} = 1 + q \frac{d}{dq} \left( \frac{I_1}{I_0} \right), \quad I_{1,1,a} = q \frac{d}{dq} \left( \frac{I_{1,a}}{I_0} \right).$$

**Remark 3.1.** One can check that our $I_{p,p}$ coincide with the $I_p$ in Theorem 1 of Zagier–Zinger’s paper [43].

Recall that we have introduced the following degree $k$ “basic” generators

$$X_k := \frac{d^k}{du^k} \left( \log \frac{I_0}{L} \right), \quad Y_k := \frac{d^k}{du^k} \left( \log \frac{I_0 I_{1,1}}{L^2} \right), \quad Z_k := \frac{d^k}{du^k} \left( \log (q^k L) \right),$$

By using the entries in the quantum product matrix, we define the following four “extra” generators

$$Q = \frac{1}{L} \left( I_{1,1,a} - \frac{1}{5} \right), \quad \mathcal{P} = \frac{I_{1,1} I_{2,2,b}}{L^2} + Q^2 + \frac{L^4}{2} Q,$$

$$\tilde{Q} := \frac{d}{du} Q + (X - Y) Q, \quad \tilde{\mathcal{P}} := \frac{d}{du} \mathcal{P} + (X + Y) \mathcal{P}.$$

Their degrees are defined by

$$\deg Q := 1, \quad \deg \mathcal{P} := 2, \quad \deg \frac{d}{du} := 1,$$

so that we have

$$\deg \tilde{Q} = 2, \quad \deg \tilde{\mathcal{P}} = 3.$$

**Remark 3.2.** We will see that, in the genus 2 case, only linear terms of the following two extra generators are involved: $\tilde{\mathcal{P}}$ and $\tilde{Q}$.

**Remark 3.3.** For any $k$, the generator $Z_k$ can be written as a (non-homogeneous) polynomial of $L$. This is because $Z_1$ is a polynomial of $L$ and

$$\frac{d}{du} L = \frac{1}{5} (L^5 - 1).$$
For example, the first several of them are
\[ Z_1 = \frac{1}{L} \left( \frac{d}{dq} (\log L) + \frac{1}{5} \right) = \frac{L^4}{5}, \]
\[ Z_2 = \frac{4}{25} L^3 (L^5 - 1), \]
\[ Z_3 = \frac{4}{125} L^2 (L^5 - 1)(8L^5 - 3). \]

**Remark 3.4.** Some leading terms of the basic generators are given by
\[ X = -\frac{505}{2} q - \frac{1425100}{100} q^2 - \frac{4155623250}{250} q^3 + O(q^4) \]
\[ Y = -\frac{360}{2} q - \frac{1190450}{100} q^2 - \frac{3759611500}{250} q^3 + O(q^4) \]
\[ Z_1 = \frac{505}{2} q - \frac{2534575}{250} q^2 - \frac{10290963500}{250} q^3 + O(q^4) \]
\[ Z_2 = \frac{4}{25} L^3 (L^5 - 1), \]
\[ Z_3 = \frac{4}{125} L^2 (L^5 - 1)(8L^5 - 3). \]

and some leading terms of the extra generators are given by
\[ Q = -\frac{1}{5} - 149 q - 271030q^2 - 591997100q^3 + O(q^4) \]
\[ P = -\frac{3}{50} - \frac{399}{10} q - 12732q^2 + 131454705q^3 + O(q^4) \]
\[ \tilde{Q} = -120q - 473525q^2 - 1622526750q^3 + O(q^4) \]
\[ \tilde{P} = 12q + \frac{331965}{2} q^2 + 984651825q^3 + O(q^4). \]

In particular, from this data one can see the degree zero genus two Gromov–Witten invariant should be equal to the coefficient of the \( Z_3 \) in the homogenous polynomial \( 5^3 \frac{L^2}{I_0} F_2 \) (see Conjecture [1,1]), which is \(-\frac{5}{114}\).

### 3.3. A list of closed formulae for the contribution of localization graphs.

We now give a list of closed formulae for the contribution of each of the localization graphs in Section 2.3. We also rewrite each contribution in terms of Gromov–Witten double brackets using the wall-crossing formula \(^2\):

\[ \langle \rangle_{g,n}^{t, SQ} = I_0^{2g-2} \langle \rangle_{g,n}^{t} (\tau(q)) \]

The first proposition concerns the purely twisted contribution \( \text{Cont}_{\Gamma^2} = L^{-2} \text{Cont}'_{\Gamma^2} \), where

\[ \text{Cont}'_{\Gamma^2} := (L/I_0)^2 \cdot (\langle \rangle_{2,2}^t (\tau(q))). \]

**Proposition 3.5.** The contribution of \( \Gamma^2 \) is a degree 3 homogeneous polynomial in the basic and extra generators, to be precise

\[
\text{Cont}'_{\Gamma^2} = -\frac{5}{1152} \tilde{P} - \left( \frac{31X}{576} + \frac{1}{48} Y + \frac{205Z_1}{2304} \right) \cdot \tilde{Q} \\
- \frac{19X_3 + 67X^3}{1152} - \frac{Y(X_2 + 5XY)}{48} - \frac{25X_2(X + Z_1)}{288} - \frac{101YX^2}{1152} - \frac{Y^3}{24} - \frac{107Z_1X^2}{384} \\
- \frac{19Z_1YX}{64} - \frac{3Z_1Y^2}{16} - \frac{65Z_2X}{1536} - \frac{715Z_1^2X}{1152} - \frac{45Z_1^2Y}{128} - \frac{829Z_1Z_2}{1728} + \frac{349Z_3}{13824}
\]

\(^2\)While no proof exactly applies to our situation, the proofs [10], [8], and [12] can be easily adapted.
The leading terms of the genus two graph contribution are

\[ \text{Cont}_{\Gamma^2} = L^{-2} \text{Cont}'_{\Gamma^2} = \frac{1645}{1152} q + \frac{1842665}{576} q^2 + \frac{2419134175}{288} q^3 + O(q^4). \]

Note that the degree 1 term coincides with the one in the localization computation in Section 2.4:

\[ \frac{1645}{1152} = -\frac{4285}{1152} + \frac{1}{2} \cdot \frac{2965}{288}. \]

We next consider the contribution from the graph \( \Gamma^1 \), which is a genus-one twisted theory with a special insertion. It can be rewritten as \( \text{Cont}_{\Gamma^1} = L^{-2} \text{Cont}'_{\Gamma^1} \), where

\[ \text{Cont}'_{\Gamma^1} := \frac{L^2}{I_0} \left\langle \left\langle \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi)} \right\rangle \right\rangle_{0,1}^t (\tau(q)). \]

**Proposition 3.6.** The contribution of \( \Gamma^1 \) is a degree 3 homogeneous polynomial in the basic and extra generators, to be precise

\[ -\text{Cont}'_{\Gamma^1} = \frac{473}{576} \hat{P} + (\frac{1}{48} \mathcal{Y} - \frac{25}{96} \mathcal{X} + \frac{2093}{1152} \mathcal{Z}_1) \cdot \hat{Q} \]

\[ + \frac{25}{72} (\mathcal{X}_3 + \mathcal{X}^3) + \frac{41}{48} \mathcal{Z}_1 \mathcal{Y} - \frac{1871}{576} \mathcal{X}_1 \mathcal{X} \mathcal{Y}^2 - \frac{1271}{192} \mathcal{Z}_1 \mathcal{Y}^2 \mathcal{X} + \frac{779}{56} \mathcal{Z}_1 \mathcal{Y}^2 \mathcal{X}^2 + \frac{4945}{864} \mathcal{Z}_1 \mathcal{Z}_2 - \frac{155}{3456} \mathcal{Z}_3. \]

The leading terms of the genus one graph contribution are

\[ -\text{Cont}_{\Gamma^1} = -L^{-2} \text{Cont}'_{\Gamma^1} = \frac{1925}{576} q - \frac{2344025}{288} q^2 - \frac{4831529575}{144} q^3 + O(q^4). \]

Note that the degree 1 term coincides with the one in the localization computation in Section 2.4:

\[ \frac{1925}{576} = \frac{975}{64} - \frac{3425}{288}. \]

The remaining two (non-trivial) graphs involve a genus-zero two-pointed twisted theory. We rewrite them as \( \text{Cont}_{\Gamma^0_a} = L^{-2} \text{Cont}'_{\Gamma^0_a} \) and \( \text{Cont}_{\Gamma^0_b} = L^{-2} \text{Cont}'_{\Gamma^0_b} \), where

\[ \text{Cont}'_{\Gamma^0_a} := L^2 \left\langle \left\langle \frac{-\frac{5}{3} H^3 \otimes H^4 t^{-1} + \frac{5}{3} H^4 \otimes H^3 t^{-1} + \frac{65}{8} H^4 \otimes H^4 t^{-2}}{(t - 5H)(t - 5H - \psi_1)(t - 5H)(t - 5H - \psi_2)} \right\rangle \right\rangle_{0,2}^t (\tau(q)), \]

\[ \text{Cont}'_{\Gamma^0_b} := L^2 \left\langle \left\langle \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi_1), \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi_2)}} \right\rangle \right\rangle_{0,2}^t (\tau(q)). \]
Proposition 3.7. The contributions of \( \Gamma^0_a \) and \( \Gamma^0_b \) are both degree 3 homogeneous polynomials in the basic and extra generators, to be precise

\[
\text{Cont}_{\Gamma^0_a} = - \frac{13}{8} \mathcal{P} - \left( \frac{1}{12} \mathcal{X} + \frac{199}{48} \mathcal{Z}_1 \right) \cdot \mathcal{Q}
+ \frac{41}{24} \mathcal{X}^3 + \mathcal{X}^2 \mathcal{Y} + \mathcal{Z}_1^2 \mathcal{Y} + \frac{77}{24} \mathcal{Z}_1 \mathcal{X}^2 + \frac{41}{12} \mathcal{Z}_1 \mathcal{Y} \mathcal{X}
+ \frac{677}{96} \mathcal{Z}_2 \mathcal{X} - \frac{277}{24} \mathcal{Z}_1^2 \mathcal{X} - \frac{599}{72} \mathcal{Z}_1 \mathcal{Z}_2 + \frac{805}{288} \mathcal{Z}_3,
\]

\[
\text{Cont}_{\Gamma^0_b} = - \frac{5}{576} \mathcal{P} + \left( \frac{205}{288} \mathcal{X} \mathcal{Y} + \frac{265}{384} \mathcal{Z}_1 \right) \cdot \mathcal{Q}
+ \frac{7595}{576} \left( \mathcal{X}_3 + \mathcal{X}^3 \right) + \frac{15595}{288} \mathcal{X} \mathcal{X}_2 - \frac{8405}{576} \mathcal{Y} \mathcal{X} + 2 \mathcal{Z}_1 \mathcal{X} + \mathcal{Z}_1^2
+ \left( \frac{250}{9} \mathcal{X}_2 + \frac{215}{576} \mathcal{X}^2 \right) \mathcal{Z}_1 + \frac{117215}{2304} \mathcal{Z}_2 \mathcal{X} - \frac{2405}{192} \mathcal{Z}_1^2 \mathcal{X} + \frac{1585}{64} \mathcal{Z}_1 \mathcal{Z}_2 + \frac{30325}{2304} \mathcal{Z}_3.
\]

Corollary 3.8. The summation of the two contributions is

\[
\text{Cont}_{\Gamma^0} = \text{Cont}_{\Gamma^0_a} + \text{Cont}_{\Gamma^0_b} = L^{-2} \text{Cont}_{\Gamma^0} = \frac{17905}{576} q + \frac{11650385}{288} q^2 + \frac{13428251725}{144} q^3 + O(q^4).
\]

Note that the degree 1 term coincides with the one in the localization computation in Section 2.4

\[
\frac{17905}{576} = \frac{1967}{192} + \frac{3001}{144}
\]

3.4. Proof of the Main Theorem. Our main result now follows from the propositions in the last subsection.

Theorem 3.9. The genus two Gromov–Witten free energy of a quintic Calabi–Yau threefold is given by

\[
F^{GW}_2(Q) = \frac{I^2}{L^2} \left( \frac{70}{9} \mathcal{X}_3 + \frac{575}{18} \mathcal{X} \mathcal{X}_2 + \frac{5 \mathcal{Y} \mathcal{X}^2}{6} + \frac{557}{72} \mathcal{X}^3 - \frac{629}{72} \mathcal{Y} \mathcal{X}^2 - \frac{23}{24} \mathcal{Y}^2 \mathcal{X} - \frac{24}{24} \mathcal{Y}^3 \right)
+ \frac{625}{36} \mathcal{Z} \mathcal{X}_2 - \frac{175}{9} \mathcal{Z} \mathcal{Y} \mathcal{X} + \frac{14141}{48} \mathcal{Z}_2 \mathcal{X} - \frac{25}{24} \mathcal{Z} (\mathcal{X}^2 + \mathcal{Y}^2) - \frac{3125}{288} \mathcal{Z}^2 (\mathcal{X} + \mathcal{Y})
+ \frac{41}{48} \mathcal{Z}_2 \mathcal{Y}^3 - \frac{265}{144} \mathcal{Z}_3 \mathcal{Z}_2 + \frac{2233}{128} \mathcal{Z}_2 \mathcal{Z}_3 - \frac{547}{72} \mathcal{Z}_3.
\]

Proof. By dimension considerations, the \( \text{Cont}_{\Gamma^i} \) have no constant term in \( q \). Therefore, the value of \( c_{2,0} = -N_{2,0} \) can be read off from (see \[37\])

\[
N_{2,0} = \frac{1}{2} \int_{Q_5} (c_3(Q_5) - c_1(Q_5) c_2(Q_5)) \cdot \int_{\mathcal{M}_{2,0}} \lambda_1^3 = \frac{1}{2} \cdot (-200) \cdot \frac{|B_4|}{4} \cdot \frac{|B_2|}{2} \cdot \frac{1}{2!} = -\frac{5}{144}.
\]
The rest is just a direct consequence of the following formula

\[
L^2 \cdot F_2^{SQ}(q) = -c_{2,0} L^2 + \frac{1}{2} \text{Cont}'_{\Gamma^0} - \text{Cont}'_{\Gamma^1} + \text{Cont}'_{\Gamma^2},
\]

the wall-crossing formula

\[
F_2^{GW}(Q) = I_0(q)^2 \cdot F_2^{SQ}(q),
\]

and the formulae for \(\text{Cont}_i\) in Proposition 3.5, Proposition 3.6 and Corollary 3.8 \(\square\)

3.5. **Equivalence between our Main Theorem and the physicists’ conjecture.** The closed formula for the genus two Gromov–Witten potential was first proposed by BCOV [1] and further clarified by Yamaguchi–Yau [42]. Later, Klemm–Huang–Quackenbush [27] extended the result to genus 51. We will follow the notation of [27].

In [42, 27], the authors introduce the following basic generators

\[
A_p := \frac{(-q \frac{d}{dq})^p(qI_{1,1}(q))}{qI_{1,1}(q)}, \quad B_p := \frac{(-q \frac{d}{dq})^p I_0(q)}{I_0(q)}, \quad X := \frac{-5q}{1-5q}
\]

and the change of variables

\[
B_1 = u, \quad A_1 = v_1 - 1 - 2u \quad B_2 = v_2 + uv_1, \quad B_3 = v_3 - uv_2 + uv_1X - \frac{2}{5}uX.
\]

The following is the original physical conjecture.

**Conjecture 3.10.** Let \(F_g\) be the genus \(g\) Gromov–Witten potential and

\[
P_g(q) := \left(\frac{-5}{I_0(1-5q)}\right)^{g-1} F_g(Q).
\]

When \(g = 2\), we have the following explicit formula

\[
P_2 = \frac{25}{144} - \frac{625 v_1}{288} + \frac{25 v_1^2}{24} - \frac{5 v_1^3}{24} - \frac{625 v_2}{36} + \frac{25 v_1 v_2}{6} + \frac{350 v_3}{9} - \frac{5759 X}{3600} - \frac{167 v_1 X}{720} + \frac{v_1^2 X}{6} - \frac{475 v_2 X}{12} + \frac{41 X^2}{3600} - \frac{13 v_1 X^2}{288} + \frac{X^3}{240}.
\]

**Proposition 3.11.** Both Conjecture 3.10 and Conjecture 1.1 are equivalent to

\[
F_2^{GW}(Q) = \left(\frac{L^2}{L_0^2}\right) \left(\frac{70 X_3}{9} + \frac{575 X X_2}{18} + \frac{557 X^3}{72} + \frac{5 Y X_2}{6} - \frac{629 Y X^2}{72} - \frac{23 Y^2 X}{24} - \frac{Y^3}{24}\right)
\]

\[
+ \frac{125 X_2 L^4}{36} - \frac{5 X^2 L^4}{24} - \frac{35 Y X L^4}{9} - \frac{5 Y^2 L^4}{24} - \frac{1441 X L^3}{24} - \frac{141 Y L^3}{300} - \frac{41 Y^2 L^3}{300}
\]

\[
+ \frac{31459 X L^8}{7200} - \frac{2141 Y L^8}{7200} + \frac{29621 L^{12}}{12000} - \frac{116369 L^7}{36000} + \frac{547 L^2}{750}.
\]

**Proof.** First notice that

\[
u_1 = A_1 + 1 + 2B_1, \quad v_2 = -A_1 B_1 - 2B_1^2 - B_1 + B_2,
\]

\[
u_3 = -A_1 B_1^2 - B_1 A_1 X - 2B_1^3 - 2B_1^2 X - B_1^2 + B_1 B_2 - \frac{3}{5} B_1 X + B_3.
\]
Therefore, Conjecture 3.10 is equivalent to

\[
P_2 = \frac{385}{36} A_1 B_1 - \frac{1045}{18} A_1 B_1^2 + \frac{5923}{360} B_1 X + \frac{37}{18} B_1^2 X + \frac{425}{9} B_1 B_2 + \frac{565}{48} B_1 - \frac{205}{288} A_1 - \frac{13 X^2 B}{144} - \frac{65}{12} A_1^2 B_1 + \frac{25}{6} A_1 B_2 - \frac{13 X^2 A_1}{288} + \frac{49}{36} B_1 X A_1 - \frac{865}{9} B_1^3 + \frac{115}{6} B_1^2 + \frac{350}{9} B_3 - \frac{333}{200} X - \frac{475}{36} B_2 - \frac{335}{720}.
\]

Next we define

\[
\tilde{A}_p := \left( -q \frac{d}{dq} \right)^p \log(q^\frac{1}{2} I_0), \quad \tilde{B}_p := \left( -q \frac{d}{dq} \right)^p \log(q^\frac{1}{5} I_{1,1}),
\]

so that

\[
A_1 = -\frac{4}{5} + \tilde{A}_1, \quad B_1 = \frac{1}{5} + \tilde{B}_1, \quad B_2 = \frac{1}{25} + \frac{2}{5} \tilde{B}_1 + (\tilde{B}_2 + \tilde{B}_1^2), \quad B_3 = \frac{1}{125} + \frac{3}{25} \tilde{B}_1 + \frac{3}{5} (\tilde{B}_2 + \tilde{B}_1^2) + (\tilde{B}_3 + 3 \tilde{B}_1 \tilde{B}_2 + \tilde{B}_3).
\]

On the other hand, by definition of $X_p$, $Y_p$ and $L$, we have

\[
\tilde{A} = -L(Y + Z - X), \quad \tilde{B}_1 = -L(X + Z), \quad \tilde{B}_2 = L^2(X_2 + Z_2) - \frac{L}{5} X(X + Z), \quad \tilde{B}_3 = -L^3(X_3 + Z_3) + \frac{3}{5} L^2 X(X_2 + Z_2) - \frac{L}{25} (6X^2 - 5X)(X + Z).
\]

Also $X = 1 - L^3$ and the $Z_k$ are all polynomials of $L$ (see Remark 3.3). Finally, a few direct computations show that, after the above change of variables, both conjectures are equivalent to equation (9).

\[
4. \text{Structures of the twisted invariants}
\]

The twisted theory of $\mathbb{P}^4$ is semisimple, and can be computed by the Givental–Teleman formula using $R$-matrices. In this section, we write down the basic data and relations for the twisted invariants, and then we derive closed formulae for the entries of the $R$-matrix up to $z^3$, which is all we need for the computation in genus two.

4.1. $S$-matrix. Recall that the $S$-matrix is a fundamental solution of the quantum differential equation

\[
D_H S(\tau(q), z) = A(t, q) \cdot S(\tau(q), z),
\]

where we recall that $D := z q \frac{d}{dq}$ and $D_H := D + H$. Moreover, the $S$-matrix can be obtained from the derivatives of the $I$-function by Birkhoff factorization. We start from $I$-function. In the rest of this and next sections, we will fix the base point

\[
(t = \tau(q) = H \frac{I_1(q)}{I_0(q)} + t \frac{I_{1,n}(q)}{I_0(q)}).
\]

As a convention, we will omit the base point $t$ in the double bracket $\langle\langle-\rangle\rangle_{q,t}^{g,n}(t)$ and the $S$-matrix $S(\tau(q), z)$ when $t = \tau$.

The following proposition is an application of Birkhoff factorization.
Proposition 4.1. We have the following formulae for the $(\cdot, \cdot)^t$-adjoint of the $S$-matrix

\[
S^*(z)(1) = \frac{I(z)}{zI_0}
\]

\[
S^*(z)(H) = \left( \frac{D_H - I_{1,1}}{I_{1,1}} \right) \frac{I(z)}{zI_0}
\]

\[
S^*(z)(H^2) = \det \left( \begin{bmatrix} I_{3,3} & 0 & 0 & 0 \\ I_{3,3} & I_{2,2} & 0 & 0 \\ 0 & I_{2,2} & I_{1,1} & 0 \\ 0 & 0 & 0 & I_{1,1} \end{bmatrix} \right) \frac{I(z)}{zI_0}
\]

\[
S^*(z)(H^3) = \det \left( \begin{bmatrix} I_{4,4} & 0 & 0 & 0 \\ I_{4,4} & I_{3,3} & 0 & 0 \\ 0 & I_{3,3} & I_{2,2} & 0 \\ 0 & 0 & 0 & I_{2,2} \end{bmatrix} \right) \frac{I(z)}{zI_0}
\]

\[
S^*(z)(H^4) = \det \left( \begin{bmatrix} I_{5,5} & 0 & 0 & 0 \\ I_{5,5} & I_{4,4} & 0 & 0 \\ 0 & I_{4,4} & I_{3,3} & 0 \\ 0 & 0 & 0 & I_{3,3} \end{bmatrix} \right) \frac{I(z)}{zI_0}
\]

Note that there is a differential operator $D_H$ in the determinants. We define the differential operation from top to bottom.

Proof. Noting that the $S$-matrix is a solution of equation (10), and that $S(z)\varphi^i = \varphi^i + O(z^{-1})$, this proposition follows from a direct computation:

\[
S^*(z)(1) = \frac{I(z)}{zI_0}
\]

\[
S^*(z)(H) = \frac{1}{I_{1,1}} \left( D_H - I_{1,1}^t \right) S^*(z)(1)
\]

\[
S^*(z)(H^2) = \frac{1}{I_{2,2}} \left( D_H - I_{2,2}^t \right) S^*(z)(H) - \frac{I_{2,2}^t}{I_{2,2}} t^2 S^*(z)(1)
\]

\[
S^*(z)(H^3) = \frac{1}{I_{3,3}} \left( (D_H - I_{3,3}^t) S^*(z)(H^2) - I_{3,3}^t t^2 S^*(z)(H) - I_{3,3}^t t^3 S^*(z)(1) \right)
\]

\[
S^*(z)(H^4) = \frac{1}{I_{4,4}} \left( (D_H - I_{4,4}^t) S^*(z)(H^3) - I_{4,4}^t t^2 S^*(z)(H^2) - I_{4,4}^t t^3 S^*(z)(H) - I_{4,4}^t t^4 S^*(z)(1) \right)
\]

\[
\square
\]

Now we can write down all the entries of $S$-matrix. However, it soon becomes too complicated to write down all the explicit formulae for further computations. We want to establish an equation satisfied by

\[
S^*(z) \left( \frac{1}{t} + 5 \frac{H}{t^2} + 25 \frac{H^2}{t^3} + 125 \frac{H^3}{t^4} + 625 \frac{H^4}{t^5} \right).
\]

This equation will help us to simplify some computations involving the $S$-matrix (for example the computation of the modified $V$-matrix in Section 5.1). Also by using this equation we can deduce closed formulae for some special entries of the $R$-matrix and prove some important identities.
Lemma 4.2. Define
\[ \tilde{H}_4 := \frac{1}{t} + 5 \frac{H}{t^2} + 25 \frac{H^2}{t^3} + 125 \frac{H^3}{t^4} + 625 \frac{H^4}{t^5}. \]

Then, we have
\[ (D_H - \frac{1}{5} t) S^*(z)(\tilde{H}_4) + \frac{1}{5} I(z) = 0 \]

Furthermore,
\[ S^*(t - 5H)(\tilde{H}_4) = S^* \left( \frac{1 - 5H}{2} \right) (\tilde{H}_4) = \tilde{H}_4. \]

Remark 4.3. The \( \tilde{H}_4 \) in this lemma can be viewed an element of the dual basis, which satisfies
\[ (\tilde{H}_4, 1) = (\tilde{H}_4, H) = (\tilde{H}_4, H^2) = (\tilde{H}_4, H^3) = 0. \]

Proof. We define formally
\[ S^*_5 = \text{det} \begin{pmatrix} D_H - I_{5,5:a} t & -I_{5,5:b} t^2 & -I_{5,5:c} t^3 & -I_{5,5:d} t^4 & -I_{5,5:e} t^5 \\ -1 & D_H - I_{4,4:a} t & -I_{4,4:b} t^2 & -I_{4,4:c} t^3 & -I_{4,4:d} t^4 \\ -1 & -1 & D_H - I_{3,3:a} t & -I_{3,3:b} t^2 & -I_{3,3:c} t^3 \\ -1 & -1 & -1 & D_H - I_{2,2:a} t & -I_{2,2:b} t^2 \\ -1 & -1 & -1 & -1 & D_H - I_{1,1:a} t \end{pmatrix} \frac{I(z)}{zI_0} \]

Since \( H^5 = 0 \), we have
\[ S^*_5 = S^*(z)(H^5) = 0. \]

On the other hand, by symmetry of the quantum product (see Section 4.4.1 for more details):
\begin{align*}
I_{5,5:a} &= \frac{1}{5} (1 - I_{4,4}) \\
I_{5,5:b} &= \frac{1}{25} (1 - I_{3,3}) - \frac{1}{5} I_{4,4:a} \\
I_{5,5:c} &= \frac{1}{125} (1 - I_{2,2}) - \frac{1}{25} I_{3,3:a} - \frac{1}{5} I_{4,4:b} \\
I_{5,5:d} &= \frac{1}{625} (1 - I_{1,1}) - \frac{1}{125} I_{2,2:a} - \frac{1}{25} I_{3,3:b} - \frac{1}{5} I_{4,4:c} \\
I_{5,5:e} &= \frac{1}{3125} (1 - I_0) - \frac{1}{625} I_{1,1:a} - \frac{1}{125} I_{2,2:b} - \frac{1}{25} I_{3,3:c} - \frac{1}{5} I_{4,4:d}. 
\end{align*}

Hence by replacing the first row of the matrix with the first row plus the sum of all the \((k + 1)\)-th row multiplied by \( \frac{1}{5^k} I_{5-k,5-k} \) for \( k = 1, 2, 3, 4 \), we have
\[ S^*_5 = \text{det} \begin{pmatrix} D_H - \frac{1}{5} t & \frac{4}{25} D_H - \frac{t^2}{25} & \frac{1}{125} D_H - \frac{t^3}{125} & \frac{1}{625} D_H - \frac{t^4}{625} & \frac{1}{3125} D_H - \frac{t^5}{3125} (1 - I_0) \\ -1 & D_H - I_{4,4:a} t & -I_{4,4:b} t^2 & -I_{4,4:c} t^3 & -I_{4,4:d} t^4 \\ -1 & -1 & D_H - I_{3,3:a} t & -I_{3,3:b} t^2 & -I_{3,3:c} t^3 \\ -1 & -1 & -1 & D_H - I_{2,2:a} t & -I_{2,2:b} t^2 \\ -1 & -1 & -1 & -1 & D_H - I_{1,1:a} t \end{pmatrix} \frac{I(z)}{zI_0} \]
Note that, in general, the determinant will change when we perform a row transformation for a matrix containing the operator $D_H$. However in this case, it does not since there are only constant terms $-1$ under the diagonal. By Proposition 4.1 we obtain

$$0 = S_5^* = \left( D_H - \frac{1}{5} t \right) \left( S_4^* + \frac{t}{5} S_3^* + \frac{t^2}{25} S_2^* + \frac{t^3}{125} S_1^* + \frac{t^4}{625} S_0^* \right) + \frac{t^5}{3125} \frac{I(z)}{z} = 0$$

where $S_k := S(z)^*(H^k)$ for $k = 0, 1, 2, 3, 4$. This proves the first statement of the lemma.

In particular, letting $z = t - 5H$, we have $D_H = H + (t - 5H)q \frac{d}{dq}$, and the equation becomes

$$(t - 5H) \left( -\frac{1}{5} + q \frac{d}{dq} \right) S^*(t - 5H)(\tilde{H}_4) + \frac{1}{5} = 0.$$ 

We can solve this equation using the initial condition

$$S^*(z)(H^k)|_{q=0} = H^k.$$ 

It implies the second statement of the lemma. The third can be proved similarly. \hfill \Box

4.2. $\Psi$-matrix and $R$-matrix: computations of $\Psi 1$ and $R^* 1$. The twisted theory of $\mathbb{P}^4$ is semisimple, in the sense that there exist idempotents $e_\alpha$ with respect to the quantum product (at $t = \tau(q)$):

$$e_\alpha \ast_{\tau} e_\beta = \delta_{\alpha\beta} e_\alpha$$

In addition, we recall the definition of the normalized canonical basis

$$\tilde{e}_\alpha := \Delta^{-\frac{1}{2}}_\alpha e_\alpha, \quad \Delta^{-1}_\alpha := (e_\alpha, e_\alpha)^t.$$

By results of Dubrovin and Givental [14, 19, 21], there exists an asymptotic fundamental solution of the quantum differential equation (10) which has the following form

$$(12) \quad \tilde{S}(t, z) = \Psi^{-1}(t) R(t, z) e^{U(t)/z},$$

where $\Psi^{-1}$ is the change of basis from a flat basis to the normalized canonical basis, $U$ is a diagonal matrix with entries the canonical coordinates

$$U = \text{diag}(u^0, u^1, \ldots, u^4)$$

and $R(t, z) = 1 + O(z)$ is a matrix of formal power series in $z$. While there is no direct relation between $\tilde{S}$ and the $S$-matrix defined by two point correlators in Section 3, in the proof of Lemma 4.6 we will discuss a relation between their fully equivariant generalizations.

We rewrite the fundamental solution in coordinates as

$$(13) \quad \tilde{S}_{i\alpha}(z) = R_{i\alpha}(z) e^{w^\alpha/z} = \sum_{\beta} \Psi_{i\beta} R_{\beta\alpha}(z) e^{w^\alpha/z}.$$ 

where, viewing $\tilde{S}$ as a linear transformation from $H^\ast_{\mathcal{C}^*}(\mathbb{P}^4)$ with basis $\{\tilde{e}_\alpha\}$ to $H^\ast_{\mathcal{C}^*}(\mathbb{P}^4)$ with a flat basis, we write $\tilde{S}_{i\alpha}(z) := (H^i, \tilde{S}(z) \tilde{e}_\alpha)$, and where, viewing $R$ as a linear transformation written in the normalized canonical basis, we set

$$\Psi_{i\beta} := (H^i, \tilde{e}_\beta)^t, \quad R_{i\alpha\beta}(z) := (\tilde{e}_\alpha, R(z) \tilde{e}_\beta)^t, \quad R_{i\alpha}(z) := (H^i, R(z) \tilde{e}_\alpha)^t.$$
Proposition 4.4. Let \( R(z) = 1 + R_1 z + R_2 z^2 + \cdots \), and 
\[
(R_k)_i^\alpha := (H^i, R_k e^\alpha)^t.
\]

Then, we have 
\[
\Psi_{0\bar{\alpha}} = 1 + \frac{q_\alpha}{I_0 \cdot q_\alpha^2} \cdot \left( \frac{-t}{5} \right)^{-\frac{2}{5}}
\]
where \( q_\alpha = -5\xi_\alpha q^\frac{1}{5} \) and 
\[
(14) \quad (R_1)_0^\alpha = \frac{1 - \frac{1}{12} q_\alpha}{t \cdot q_\alpha}, \quad (R_2)_0^\alpha = \frac{1 - \frac{13}{12} q_\alpha - \frac{287}{288} q_\alpha^2}{t^2 \cdot q_\alpha^2}, \quad (R_3)_0^\alpha = \frac{2}{5} - \frac{293}{660} q_\alpha + \frac{5447}{1440} q_\alpha^2 + \frac{5039}{10368} q_\alpha^3.
\]

Proof. By the following Lemma 4.6, the functions 
\[
(15) \quad \tilde{I}_\alpha(q, z) = e^{u^\alpha/z} I_0 R_{0\bar{\alpha}}(z) \quad \forall \alpha
\]
are solutions of the Picard–Fuchs equation 
\[
(D^5 - q \prod_{k=1}^{5} (5D + kz - t)) I(q, z) = 0.
\]

The proposition then follows from the following asymptotic expansion of \( \tilde{I}_\alpha \) (Lemma 4.5 and 4.6).

Lemma 4.5. There exist constants \( C_\alpha, c_{1\alpha}, c_{2\alpha}, \ldots \) such that 
\[
\tilde{I}_\alpha(z) = C_\alpha \cdot \frac{1 + q_\alpha}{q_\alpha^2} \left( 1 + \frac{1 + c_{1\alpha} q_\alpha}{q_\alpha} \cdot t^{-1} z + \frac{1 + (c_{1\alpha} - 1) q_\alpha + c_{2\alpha} q_\alpha^2}{q_\alpha^2} \cdot t^{-2} z^2 \right.
\]
\[
+ \frac{2}{5} + (c_{1\alpha} - \frac{24}{5}) q_\alpha + (c_{2\alpha} - c_{1\alpha} - \frac{14}{5}) q_\alpha^2 + c_{3\alpha} q_\alpha^3 \cdot t^{-3} z^3 + \ldots \Big) e^{u^\alpha/z}
\]
where \( u^\alpha \) satisfies 
\[
q \frac{d}{dq} u^\alpha = L_\alpha := \frac{\frac{q_\alpha}{1 + q_\alpha}}{1 + q_\alpha}.
\]

Proof. By the arguments in the above proposition, applying the Picard–Fuchs equation to (15), we can see that \( R_{0\bar{\alpha}}(z) \) satisfies the following equation 
\[
(16) \quad \left( D_\alpha^5 - q \prod_{k=1}^{5} (5D_\alpha + kz - t) \right) I_0 R_{0\bar{\alpha}}(z) = 0,
\]
where \( D_\alpha := D + q \frac{d}{dq} u^\alpha \). After writing down this equation carefully, we can first solve \( q \frac{d}{dq} u^\alpha \) by looking at the coefficient of \( z^0 \) of (16). After that, we can determine the coefficients of \( z^k \) in \( R_{0\bar{\alpha}}(z) \) one by one. The coefficient of \( z^1 \) in (16) determines the coefficient of \( z^0 \) in \( R_{0\bar{\alpha}}(z) \), and hence \( \Psi_{0\bar{\alpha}} \) up to a constant \( C_\alpha \). At each step, we need to introduce a new undetermined constant \( c_{k\alpha} \).

We cannot directly fix the constant terms of the \( R \)-matrix because of the poles of the entries of the \( R \)-matrix. The idea to solve this problem is to consider a more general equivariant theory first, and then take the limit. We will do so in the following lemma.
Lemma 4.6. The functions $\tilde{I}_\alpha = e^{u_\alpha/z} I_0 R_\alpha(z)$ are solutions of Picard–Fuchs equation
\[
\left( D^5 - q \prod_{k=1}^{5} (5D + k z - t) \right) I(q, z) = 0.
\]
Furthermore, the constants $C_\alpha$ and $c_{\alpha i}$ for $i = 1, 2, 3$ in Lemma 4.5 are independent of $\alpha$, and have the following values
\[
C_\alpha = \left( -\frac{t}{5} \right)^{-\frac{4}{5}}, \quad c_{1\alpha} = -\frac{1}{12}, \quad c_{2\alpha} = -\frac{287}{288}, \quad c_{3\alpha} = \frac{5039}{10368}.
\]

Proof. There is a standard method of fixing the constants of the $R$-matrix in equivariant Gromov–Witten theory using an explicit formula for the $R$-matrix when the Novikov parameters are sent to zero. Unfortunately, it does not directly apply to the twisted theory that we are considering since we do not work equivariantly on $\mathbb{P}^4$, and the theory hence becomes non-semisimple at $q = 0$. To solve this problem, we first introduce a more general equivariant theory, and then take a limit to recover the original theory.

We introduce the $(\mathbb{C}^*)^5$-action which acts diagonally on the base $\mathbb{P}^4$, and we denote by $\lambda_i$ the corresponding equivariant parameters. To simplify the computation, we set $\lambda_i = \xi_i \lambda$.

We consider the corresponding twisted theory of $\mathbb{P}^4$, which has the $I$-function
\[
I(t, \lambda, q, z) = z \sum_{d \geq 0} q^d \prod_{j=1}^{5d} \frac{(5H + jz - t)}{\prod_{k=1}^{d} ((H + kz)^5 - \lambda^5)},
\]
which satisfies the Picard–Fuchs equation
\[
\left( D^5_H - \lambda^5 - q \prod_{k=1}^{5} (5D_H + k z - t) \right) I(t, \lambda, q, z) = 0.
\]

Similarly to the previous discussion, we introduce the mirror map $\tau(q)$, the $S$- and $R$-matrix
\[
S(t, \lambda, z), \quad R(t, \lambda, z)
\]
at point $\tau(q)$, the normalized canonical basis, . . . . It is clear that by taking $\lambda \to 0$ we recover the twisted theory considered in this paper.

One main advantage of the more general twisted theory is that it stays semisimple at $q = 0$ because the classical equivariant cohomology of $\mathbb{P}^4$ has a basis of idempotents:
\[
e_{\alpha} |_{q=0} = \prod_{\beta \neq \alpha} \frac{(H - \xi^\beta \lambda)}{(\xi^\alpha \lambda - \xi^\beta \lambda)} = \prod_{\beta \neq \alpha} \frac{(H - \xi^\beta \lambda)}{5\xi^{4\alpha} \lambda^4}
\]
Hence, the normalized canonical basis stays well-defined at $q = 0$, and we have
\[
\bar{e}_{\alpha} |_{q=0} = \frac{\prod_{\beta \neq \alpha} (H - \xi^\beta \lambda)}{\sqrt{5(5\lambda^5 - t\xi^{4\alpha} \lambda^4)}},
\]
and
\[
\Psi_{\alpha} |_{q=0} = \frac{-t + 5\xi^\alpha \lambda}{\sqrt{5(5\lambda^5 - t\xi^{4\alpha} \lambda^4)}}.
\]

By uniqueness of fundamental solutions of the quantum differential equation (see also [21, 13]), we have
\[
S(t, \lambda, z) (\Psi(t, \lambda)^{-1} |_{q=0}) \Gamma^{-1}(t, z) C^{-1}(\lambda, z) = \Psi(t, \lambda)^{-1} R(t, \lambda, z) e^{U(t, \lambda)/z}
\]
for constant matrices (with respect to \( q \)) given by
\[
C(\lambda, z) = \text{diag}\left( \{ e^{\sum_{k>0, j \neq i} \frac{B_{2k}}{2k(2k-1)} (\lambda_i - \lambda_j)^{2k-1}} \}_{i=0,1,2,3,4} \right),
\]
(20)
\[
\Gamma(t, z) = e^{\sum_{k>0} \frac{B_{2k}}{2k(2k-1)} (1 - 5t)^{2k-1}}.
\]
Indeed, these constant matrices together give exactly the constant term of the \( R \)-matrix.

Notice that the \( \tilde{e}_\alpha \mid_{q=0} \) are eigenvectors for the eigenvalues \( h_\alpha \) of the classical multiplication by \( H \). Hence, by the Picard–Fuchs equation \([17]\) for \( I(t, \lambda, q, z) = zI_0(q) \cdot S(t, \lambda, z)^* 1 \), the identity \([19]\), and the fact that \( \Gamma(t, z) \) and \( C(\lambda, z) \) are diagonal, we see that
\[
\tilde{I}_\alpha(t, \lambda, q, z) = e^{u_\alpha(q)/z} I_0(q) R_{0\bar{\alpha}}(z)
\]
satisfies the Picard–Fuchs equation
\[
\left( (D + h_\alpha)^5 - \lambda^5 - q \prod_{k=1}^5 (5D + 5h_\alpha + kz - t) \right) \tilde{I}_\alpha(t, \lambda, q, z) = 0.
\]
(21)
At the limit \( t = 0 \), this proves the first part of the lemma.

Now we can use a similar method as in the proof of Lemma \([4,5]\) to compute the \( R \)-matrix. By the definition of \( \tilde{I}_\alpha(t, \lambda, q, z) \), we have the following Picard–Fuchs equation
\[
\left( (D + L_\alpha)^5 - \lambda^5 - q \prod_{k=1}^5 (5D + 5L_\alpha + kz - t) \right) I_0(q) R_{0\bar{\alpha}}(t, q, z) = 0.
\]
where \( L_\alpha = h_\alpha + q \frac{d}{dq} u_\alpha \). The coefficient of \( z^0 \) gives us an equation for \( L_\alpha \):
\[
q(5L_\alpha - t)^5 + (L_\alpha^5 - \lambda^5) = 0.
\]
(22)
We can choose the basis \( \tilde{e}_\alpha \mid_{q=0} \) such that the solution \( L_\alpha \) satisfies the following
\[
L_\alpha = \xi^\alpha \lambda + O(q), \quad \text{for} \quad \alpha = 0, 1, 2, 3, 4,
\]
and from here we see that \( h_\alpha = \xi^\alpha \lambda \). Then, the coefficient of \( z^1 \) of the Picard–Fuchs equation and the initial condition \([18]\) imply that
\[
I_0(q) \cdot \Psi_{0\bar{\alpha}} = \frac{5L_\alpha - t}{\sqrt{5(5\lambda^5 - L_\alpha^4 t)}}
\]
Next, we look at the coefficients of \( z^2 \), \( z^3 \) and \( z^4 \) of the equation. By equation \([22]\) we have
\[
q \frac{d}{dq} L_\alpha = -\frac{(5L_\alpha - t)(L_\alpha^5 - \lambda^5)}{5L_\alpha^4 t - 25\lambda^5}.
\]
Using this relation, we can solve \( R_1, R_2, R_3 \) inductively as rational functions of \( L_\alpha \). The explicit formulae are:
\[
(R_1)_\alpha^0 = \frac{1}{(L_\alpha^4 t - 5\lambda^5)^2} \left( 13L_\alpha^{12}t^2 + 5L_\alpha^8 - 19L_\alpha^6 + 11L_\alpha^4 t - \frac{111L_\alpha^6}{15} \right) \lambda^5 + \frac{75L_\alpha^4}{4} + \frac{10L_\alpha^3 t}{3} - \frac{3t^2 L_\alpha^2}{2} \lambda^{10}
\]
\[
(R_2)_\alpha^0 = \frac{1}{(L_\alpha^4 t - 5\lambda^5)^6} \left( -\frac{13L_\alpha^{12}t^6}{25} + \frac{5L_\alpha^8 t^4}{60} + \frac{5L_\alpha^6 t^2}{288} + \frac{313L_\alpha^{14}t^4}{150} + \frac{47L_\alpha^{10}t^2}{12} + \frac{871L_\alpha^{8}t^2}{72} - \frac{1181L_\alpha^{10}t^2}{12} + \frac{915L_\alpha^{12}t^2}{4} \right) \lambda^5
\]
\[
+ \frac{517L_\alpha^{12}t^6}{1800} - \frac{67L_\alpha^{10}t^4}{48} + \frac{6343L_\alpha^{14}t^4}{24} + \frac{37255L_\alpha^{12}t^2}{36} + \frac{22855L_\alpha^{10}t^2}{16} - \frac{9375L_\alpha^{12}t}{4} \lambda^{10}
\]
\[
+ \frac{209L_\alpha^8 t^4}{18} + \frac{3379L_\alpha^{10}t^4}{24} + \frac{22855L_\alpha^{10}t^2}{36} - \frac{9625L_\alpha^{10}t^2}{36} - \frac{9375L_\alpha^{12}t^2}{4} \lambda^{15}
\]
\[
+ \frac{205L_\alpha^{10}t^8}{8} - \frac{180L_\alpha^8 t^6}{72} + \frac{4675L_\alpha^{12}t^6}{2} + \frac{5625L_\alpha^{10}t^4}{32} - \frac{25L_\alpha^2 t^2}{2} + \frac{1375L_\alpha^{12}t^2}{2} \lambda^{20}
\]
Proposition 4.7. We have the following formula for the $\Psi$-matrix:

\[
\Psi_{\alpha \delta} = \Delta_{\alpha}^{-\frac{1}{2}} = \frac{1 + q_{\alpha}}{I_0 \cdot q_{\alpha}^2} \cdot \left( \frac{-t}{5} \right)^{-\frac{1}{2}}
\]

\[
\Psi_{1\delta} = \frac{1}{I_{1,1}} \left( L_{\alpha} - I_{1,1} a t \right) \frac{1 + q_{\alpha}}{I_0 \cdot q_{\alpha}^2} \left( \frac{-t}{5} \right)^{-\frac{1}{2}} = \det \left( \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}} \right) \Psi_{0\delta}
\]

\[
\Psi_{2\delta} = \det \begin{pmatrix}
\frac{L_{\alpha} - I_{2,2,2} a t}{I_{2,2}} & -\frac{I_{2,2,2} a t}{I_0 - I_{1,1} a t} \\
-1 & \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}}
\end{pmatrix} \Psi_{0\delta}
\]

\[
\Psi_{3\delta} = \det \begin{pmatrix}
\frac{L_{\alpha} - I_{3,3} a t}{I_{3,3}} & -\frac{I_{3,3} a t}{I_0 - I_{1,1} a t} \\
-1 & \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}}
\end{pmatrix} \Psi_{0\delta}
\]

Notice that each time we have a constant to fix, and we fix these constants by $[20]$. To be precise, the constant terms of the above $(R_k)_0^\alpha$ are fixed the following initial condition

\[
R(z)^0_{|L_\alpha = \xi^\alpha} = R(z)^0_{|q = 0} = e^{-\frac{1}{\overline{\alpha}} \left( \frac{1}{1-\xi^\alpha} + \frac{1}{\overline{\alpha}} \right) z + \frac{1}{\overline{\alpha^2}} \left( \frac{1}{1-\xi^\alpha} \right)^2} + O(z^4)
\]

where we have used

\[
\sum_{\alpha = 1,2,3,4} \frac{1}{1-\xi^\alpha} = 2, \quad \sum_{\alpha = 1,2,3,4} \frac{1}{(1-\xi^\alpha)^2} = -1.
\]

Finally, by taking $\lambda = 0$ in the above formulae of $R_k(t, \lambda)$ (note that in this limit $L_\alpha$ becomes $\frac{t q_{\alpha}}{I_0 + q_{\alpha}}$ as in Lemma 4.5), we recover the results of Lemma 4.5 and obtain the constant terms as well. \qed

4.3. $\Psi$-matrix and $R$-matrix: computations of the remaining entries. From Proposition 4.4 we are able to compute the $\Psi$-matrix and $R$-matrix using the asymptotic expansion of Proposition 4.4.

Proposition 4.7. The following formula for the $\Psi$-matrix:

\[
\Psi_{0\delta} = \Delta_{\alpha}^{-\frac{1}{2}} = \frac{1 + q_{\alpha}}{I_0 \cdot q_{\alpha}^2} \cdot \left( \frac{-t}{5} \right)^{-\frac{1}{2}}
\]

\[
\Psi_{1\delta} = \frac{1}{I_{1,1}} \left( L_{\alpha} - I_{1,1} a t \right) \frac{1 + q_{\alpha}}{I_0 \cdot q_{\alpha}^2} \left( \frac{-t}{5} \right)^{-\frac{1}{2}} = \det \left( \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}} \right) \Psi_{0\delta}
\]

\[
\Psi_{2\delta} = \det \begin{pmatrix}
\frac{L_{\alpha} - I_{2,2,2} a t}{I_{2,2}} & -\frac{I_{2,2,2} a t}{I_0 - I_{1,1} a t} \\
-1 & \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}}
\end{pmatrix} \Psi_{0\delta}
\]

\[
\Psi_{3\delta} = \det \begin{pmatrix}
\frac{L_{\alpha} - I_{3,3} a t}{I_{3,3}} & -\frac{I_{3,3} a t}{I_0 - I_{1,1} a t} \\
-1 & \frac{L_{\alpha} - I_{1,1} a t}{I_{1,1}}
\end{pmatrix} \Psi_{0\delta}
\]
Proof. Define $\Psi_i^\beta := (H_i, e^\beta)^t$. Then since
$$1 = \sum_{\alpha} c_{\alpha},$$
we have
$$\Psi_0^\beta = 1 = \Psi_0^\beta \Delta_{\beta}^{-\frac{1}{2}}$$
i.e. $\Delta_{\beta}^{-\frac{1}{2}} = \Psi_0^\beta$. Recall that $\Psi_0^\beta$ was computed in Proposition 4.4. Using Proposition 4.1 we get
$$\Psi_{0\bar{\alpha}} = 1 + \frac{q_\alpha}{I_0 \cdot q_\alpha^2} \left( -\frac{t}{5} \right)^{-\frac{1}{2}}$$

$$\Psi_{1\bar{\alpha}} = \frac{1}{I_{1,1}} \left( L_\alpha - I_{1,1; 1} t \right) + \frac{q_\alpha}{I_0 \cdot q_\alpha^2} \cdot \left( -\frac{t}{5} \right)^{-\frac{1}{2}} = \det \left( \frac{L_\alpha - I_{1,1; 1}}{I_{1,1}} \right) \Psi_{0\bar{\alpha}}$$

$$\Psi_{2\bar{\alpha}} = \det \left( \frac{L_\alpha - I_{2,2; 1} t}{I_{2,2}} - 1 \right) \frac{I_2+I_3}{I_{1,1}} \Psi_{0\bar{\alpha}}$$

$$\Psi_{3\bar{\alpha}} = \det \left( \frac{L_\alpha - I_{3,3; 1} t}{I_{3,3}} - 1 \right) \frac{I_2+I_3}{I_{1,1}} \Psi_{0\bar{\alpha}}$$

The relations
$$\sum_{\alpha} \Psi_{j\bar{\alpha}} \Psi_{k\bar{\alpha}} = (H^j, H^k)^t$$
provide a consistency check of the constants $C_\alpha$ fixed in Lemma 4.6.

We have already computed $R_{0\bar{\alpha}}$. Next we compute the remaining entries of the $R$-matrix. By applying Proposition 4.1 to Equation (15), we have the following inductive formula:

$$R(z)^{1\alpha}_{11} = \frac{1}{I_{1,1}} \left( \Delta_\alpha^2 \frac{dq}{dz} \Delta_\alpha^\frac{1}{2} R(z)^{0\alpha}_{00} + L_\alpha - I_{1,1; 1} t R(z)^{0\alpha}_{00} \right)$$

$$R(z)^{2\alpha}_{22} = \frac{1}{I_{2,2}} \left( \Delta_\alpha^2 \frac{dq}{dz} \Delta_\alpha^\frac{1}{2} R(z)^{1\alpha}_{11} + \left( L_\alpha - I_{2,2; 1} t \right) R(z)^{1\alpha}_{11} - I_{2,2; 1} t^2 R(z)^{0\alpha}_{00} \right)$$

$$R(z)^{3\alpha}_{33} = \frac{1}{I_{3,3}} \left( \Delta_\alpha^2 \frac{dq}{dz} \Delta_\alpha^\frac{1}{2} R(z)^{2\alpha}_{22} + \left( L_\alpha - I_{3,3; 1} t \right) R(z)^{2\alpha}_{22} - I_{3,3; 1} t^2 R(z)^{1\alpha}_{11} - I_{3,3; 1} t^3 R(z)^{0\alpha}_{00} \right)$$

$$R(z)^{4\alpha}_{44} = \frac{1}{I_{4,4}} \left( \Delta_\alpha^2 \frac{dq}{dz} \Delta_\alpha^\frac{1}{2} R(z)^{3\alpha}_{33} + \left( L_\alpha - I_{4,4; 1} t \right) R(z)^{3\alpha}_{33} - I_{4,4; 1} t^2 R(z)^{2\alpha}_{22} - I_{4,4; 1} t^3 R(z)^{1\alpha}_{11} - I_{4,4; 1} t^3 R(z)^{0\alpha}_{00} \right)$$

Together with Proposition 4.4 (more precisely Equation (14)), we can then write down all the necessary entries of the $R$-matrix.

We will now compute some very explicit entries of the $\Psi$- and $R$-matrix. For this, let us introduce the notation

$$R^1_{4\alpha}(z) := (\tilde{H}_4, R(z)e^\alpha)^t, \quad (R_k^4)^\alpha := (\tilde{H}_4, R_k e^\alpha)^t.$$
Then, we can use this equation to compute the following entries of the Ψ- and R-matrix.

4.3.1. The entries $\Psi^{4\alpha}$. Note that
$$\Psi^{4\alpha} := (\tilde{H}_4, e^\alpha) = (R_0)^{4\alpha}$$
since $R_0$ is the identity matrix in any basis.

Consider the coefficient of $z^0$ of Equation (23). Since $L_\alpha - \frac{t}{5} = \frac{1}{1+q_\alpha}$, we obtain
$$\Psi^{4\bar{\alpha}} = -\frac{1 + q_\alpha}{t} I_0 \Psi^{0\bar{\alpha}}.$$

Recall that $\Delta^{\frac{1}{2}} = \frac{I_0 q_\alpha^2}{1+q_\alpha} \cdot (\frac{t}{5})^{-\frac{2}{5}} = (\Psi^{0\bar{\alpha}})^{-1}$, so that we have
$$\Psi^{4\alpha} = \frac{1 + q_\alpha}{t} I_0, \quad \Psi^{4\bar{\alpha}} = \frac{(1 + q_\alpha)^2}{q_\alpha^2 t} \left( -\frac{t}{5} \right)^{\frac{2}{5}}.$$

4.3.2. The entries $(R_k)^{4\alpha}$. Considering the coefficient of $z^1$ of equation (23), we obtain
$$q \frac{d}{dq} \Psi^{4\bar{\alpha}} + \left( L_\alpha - \frac{t}{5} \right) (R_1)^{4\bar{\alpha}} + \frac{I_0}{5} (R_1)_{0\alpha} = 0$$

Since $q \frac{d}{dq} \frac{(1+q_\alpha)^2}{q_\alpha^2} = -\frac{2(1+q_\alpha)}{5q_\alpha^2}$, we have
$$-I_0 \frac{2}{5t} - \frac{1}{5} \frac{1}{1+q_\alpha} (R_1)^{4\alpha} + \frac{I_0}{5} (R_1)_{0\alpha} = 0$$

Hence, by (14), we have
$$(R_1)^{4\alpha} = \frac{(1 + q_\alpha)(12 - 25q_\alpha)}{12 t^2 \cdot q_\alpha} I_0.$$

Furthermore, by considering the coefficients of $z^2$ and $z^3$ of equation (23) one after another, we deduce
$$(R_2)^{4\alpha} = \frac{(1 + q_\alpha)(288 - 1176q_\alpha + 625q_\alpha^2)}{288 t^3 \cdot q_\alpha^3} I_0,$$
$$(R_3)^{4\alpha} = \frac{(1 + q_\alpha)(20736 - 460512q_\alpha + 338868q_\alpha^2 + 11875q_\alpha^3)}{51840 t^4 \cdot q_\alpha^3} I_0.$$

4.4. Generators and relations. Next, we derive some basic relations in order to be able to write down the closed formula for the $S$ and $R$-matrices in terms of a minimal number of generators.

4.4.1. Quantum product and relations between $I$-functions. Recall that in the flat basis, the quantum product $\hat{\tau}^* x$ can be written in the following form
$$\hat{\tau}^* x H^{k-1} = I_{k,k} H^k + I_{k,k;\alpha} H^{k-1} t + I_{k,k;\beta} H^{k-2} t^2 + \cdots.$$

By [43], the functions $I_{k,k}$ have the following properties
$$(24) \quad I_{0,0} I_{1,1} \cdots I_{4,4} = (1 - 5^5 q)^{-1}, \quad I_{p,p} = I_{4-p,4-p}.$$

Recall $A$ is the matrix for $\hat{\tau}^*$ in the flat basis $\{H^k\}$.
Lemma 4.8. The characteristic polynomial of $A$ is given by
\[
\det(x - A) = \frac{x^5 - q(5x - t)^5}{1 - 5^5 q}.
\]
In particular, we have
\[
(\dot{\tau} *)^5 - q(5\dot{\tau} * - t)^5 = 0.
\]
Proof. Note that the eigenvalues of the matrix $A$ are just $du^\alpha$. Recall that by Lemma 4.5,
\[
du^\alpha = L^\alpha dq, \quad L^\alpha = -\frac{t\xi^\alpha q^\frac{1}{5}}{1 - 5^5 q^\frac{1}{5}},
\]
so that the characteristic polynomial of $A$ is
\[
\prod_{\alpha} (x - L^\alpha) = \frac{x^5 - q(5x - t)^5}{1 - 5^5 q}.
\]
In fact, one can see that the numerator is just the coefficient of $z^0$ of (16) with $q \frac{dq}{du^\alpha}$ replaced by $x$. □

The characteristic polynomial gives us many relations between the entries $\{I_{k,k}; a, I_{k,k}; b, \cdots\}$ in the matrix $A$. However, these do not cover all relations. For additional relations, we can use the symmetry of the quantum product
\[
(\dot{\tau} *_r H^k, H^j)^t = (\dot{\tau} *_r H^j, H^k)^t.
\]
Example 4.9. By taking $(k, j) = (0, 3)$ and $(1, 2)$ in (27), we obtain the following relations between $I_{k,k; a}$:
\[
I_{1,1} - 5I_{1,1; a} = I_{4,4} - 5I_{4,4; a}, \quad I_{2,2} - 5I_{2,2; a} = I_{3,3} - 5I_{3,3; a}
\]
Furthermore, we have
\[
-5I_{5,5; a} = I_0 - 1, \quad \sum_{k=1}^5 I_{k,k; a} = t^{-1} \text{Tr} A = -\frac{5^5 q}{1 - 5^5 q}.
\]
Together with the symmetry of $I_{k,k}$: $I_{3,3} = I_{1,1}$ and $I_{4,4} = I_0$, we deduce
\[
I_{1,1; a} + I_{2,2; a} = \frac{I_{2,2} - 1}{10} - \frac{1}{2} \cdot \frac{5^5 q}{1 - 5^5 q}.
\]
In the end of this subsection, we conclude that by using the characteristic polynomial of $A$ and the symmetry of quantum product, there are indeed only two independent functions in $\{I_{k,k; a}, I_{k,k; b}, \cdots\}$:

Lemma 4.10. Denoting the entries in $A$ by $a_{i,j}$, we have
\[
a_{i,j} \in I_{1,1}^{-2} I_{2,2}^{-1} \mathbb{Q}[I_0, I_{1,1}, I_{2,2}, L, \mathcal{P}, \mathcal{Q}], \quad \forall i, j = 0, 1, \cdots, 4.
\]
Proof. Let us list all the relations mentioned in this subsection.

First, the basic relations (Zagier–Zinger’s relations) are
\[
I_{k,k} = I_{4-k,4-k}, \quad \text{for } k = 0, 1, 2, 3, 4,
\]
and
\[
R_{0,0}^2 R_{1,1}^2 I_{2,2} = X := \frac{1}{1 - 5^5 q}.
\]
Next, we can use the symmetry (27) and the characteristic polynomial (23). For the extra $I_{5,5,*}$-functions, we have

\[ I_{5,5:a} = \frac{1}{5} (1 - I_{0,0}), \quad I_{5,5:b} = \frac{1}{625} (25 - 25I_{3,3} - 125I_{4,4:a}), \]

\[ I_{5,5:c} = \frac{1}{625} (5 - 5I_{2,2} - 25I_{3,3;a} - 125I_{4,4:b}), \]

\[ I_{5,5:d} = \frac{1}{625} (1 - I_{1,1} - 5I_{2,2:a} - 25I_{3,3;b} - 125I_{4,4:c}), \]

\[ I_{5,5:e} = \frac{1}{625} (-I_{1,1:a} - 5I_{2,2:b} - 25I_{3,3:c} - 125I_{4,4:d} + \frac{1}{5} (1 - I_{0,0})). \]

For the extra $I_{4,4,*}$-functions, we have

\[ I_{4,4:a} = \frac{1}{5} (I_{0,0} + (5I_{1,1:a} + 5I_{2,2:a} - I_{2,2} - 5I_{3,3:a})), \]

\[ I_{4,4:b} = \frac{1}{5} (-I_{2,2:a} + 5I_{2,2:b} + I_{4,4:a}), \]

\[ I_{4,4:c} = \frac{1}{5} (-I_{3,3:b} + 5I_{3,3:c} + I_{4,4:b}). \]

For the extra $I_{3,3,*}$, $I_{2,2,*}$-functions, we have

\[ I_{3,3:a} = \frac{1}{5} (I_{1,1} + (5I_{2,2,a} - I_{2,2})), \quad I_{2,2:a} = -I_{1,1:a} - \frac{1}{10} (1 - I_{2,2}) - \frac{1}{2} (X - 1) \]

Moreover, for the other extra $I$-functions, we have

\[ I_{3,3:b} = \frac{1}{100 I_{2,2}} (25 L^{10} - 40 L^5 - 200 L^2 P + 4 I_{1,1} I_{2,2} - I_{2,2}^2) \]

\[ I_{3,3:c} = -\frac{10 L^5 - I_{1,1}^2 I_{2,2} - 2 I_{0,0} I_{1,1}^2 I_{2,2} - I_{1,1}^2 I_{2,2}}{250 I_{1,1} I_{2,2}} + \frac{20 I_{1,1} I_{2,2}^2}{20 I_{1,1} I_{2,2}} + \frac{L^2 Q^2}{10 I_{1,1} - I_{1,1}^2 I_{2,2}} \]

\[ I_{4,4:d} = \frac{10 I_{1,1} L^5 - 10 L^5 + 2 I_{0,0} I_{1,1}^2 I_{2,2} - I_{1,1}^2 I_{2,2}}{1250 I_{1,1}^2 I_{2,2}} + \frac{L^5 (-5 I_{1,1} L^5 + I_{1,1} I_{2,2} + 8 I_{1,1} - 8)}{100 I_{1,1}^2 I_{2,2}} \]

\[ + \frac{L^2 \left( 25 L^{10} - 40 L^5 + 2 I_{1,1} I_{2,2} \right) Q^2 + P \left( \frac{L^2 (5 L^5 - I_{2,2})}{50 I_{1,1} I_{2,2}} - \frac{L^3 (5 L^5 - 2 I_{1,1})}{5 I_{1,1}^2 I_{2,2}} \right) Q}{I_{1,1}^2 I_{2,2}} \]

\[ - \frac{2 L^4}{I_{1,1}^2 I_{2,2}} + \frac{L^4 P^2}{I_{1,1}^2 I_{2,2}} \]

The lemma follows from a careful examination of all these formulae. \[\Box\]

4.4.2. Identities between derivatives of basic and extra generators.

**Lemma 4.11.** We have the following identities between the basic generators and their derivatives:

\[ \gamma_2 = -3 \chi_2 - \gamma^2 - \chi^2 - \frac{15}{4} Z_2 \]

\[ \gamma_4 = -4 \chi \chi_3 - 3 \chi_2 - 6 \chi^2 \chi_2 - \chi^4 - \frac{15}{4} (Z_2 \chi^2 + Z_2 \chi_2 + Z_2 \chi_3) \]

\[ - \frac{23}{24} Z_4 + \frac{29}{9} Z_1^2 Z_2 - \frac{65}{72} Z_1 Z_3 - \frac{3}{4} Z_2^2 \]

**Proof.** This is just a restatement of the identities in [42]. \[\Box\]
Lemma 4.12. We have the following identities for the extra generators

\begin{align}
\frac{d^2}{du^2} Q &= -2 \mathcal{X} \frac{d}{du} Q - (4 \mathcal{X}_2 + 2 \mathcal{X}^2 + \frac{15}{4} \mathcal{Z}_2) Q \\
& \quad - \frac{5}{2} (2 \mathcal{Z}_1 \mathcal{X}_2 + \mathcal{Z}_1 \mathcal{X}^2 + \mathcal{Z}_2 \mathcal{X}) - \frac{125 \mathcal{Z}_1 \mathcal{Z}_2}{24} - \frac{5 \mathcal{Z}_3}{24}, \\
\mathcal{P} &= - (3 \mathcal{X} + \mathcal{Y}) \frac{d}{du} \mathcal{P} + (2 \mathcal{X}_2 - \mathcal{X}^2 - 2 \mathcal{Y} \mathcal{X} + \mathcal{Y}^2 + \frac{15}{4} \mathcal{Z}_2) \mathcal{P} \\
& \quad - \frac{1}{2} \left( \left( \frac{d}{du} + \mathcal{X} - \mathcal{Y} \right) Q \left( \frac{d}{du} + \mathcal{X} - \mathcal{Y} \right) (2 \mathcal{Q} + 5 \mathcal{Z}_1) \right) - \frac{15 \mathcal{Z}_1^2 \mathcal{X}^2}{4} - \frac{55 \mathcal{Z}_1 \mathcal{Z}_2 \mathcal{X}}{4} \\
& \quad + \frac{5}{2} \left( \mathcal{Z}_2 (\mathcal{X}_2 + \mathcal{X}^2) + \mathcal{Z}_1^2 \mathcal{X}_2 + \mathcal{Z}_3 \mathcal{X} \right) + \frac{305 \mathcal{Z}_2^2}{64} - \frac{35 \mathcal{Z}_1^2 \mathcal{Z}_2}{8} + 10 \mathcal{Z}_1^4.
\end{align}

Proof. Recall that in Lemma 4.2 we have proved the following identity for two types of special \( S^* \)-matrices

\begin{align}
S^* (t - 5H)(\tilde{H}_4) &= \tilde{H}_4, \\
S^* \left( \frac{1}{2} (t - 5H) \right) (\tilde{H}_4) &= \tilde{H}_4
\end{align}

On the other hand, we have the explicit formula for \( S^*1 \):

\[ S^* (t - 5H) 1 = S^* \left( \frac{t - 5H}{2} \right) 1 = I_0^{-1} \]

and all the other columns of the special \( S^* \)-matrices can be computed by Birkhoff factorization starting from \( S^*1 \) (see Proposition 4.11). Comparing with (33) and (34), we will get many identities between basic generators, extra generators \( \mathcal{P}, \mathcal{Q} \) and their derivatives.

Let us introduce

\[ \mathcal{P}_k := \frac{d^{k-2}}{du^{k-2}} \mathcal{P}, \quad \mathcal{Q}_k := \frac{d^{k-1}}{du^{k-1}} \mathcal{Q}. \]

In particular, we have \( \mathcal{Q} = \mathcal{Q}_1, \quad \mathcal{P} = \mathcal{P}_2 \). By definition, the degrees of \( \mathcal{P}_k \) and \( \mathcal{Q}_k \) are both \( k \). By Lemma 4.11 and (33), we can solve \( \mathcal{Q}_1 \) and \( \mathcal{P}_4 \) as rational functions of the basic generators and \( \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3 \). Furthermore by using (34), we can solve \( \mathcal{Q}_3 \) as rational functions of the basic generators and \( \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}_1, \mathcal{Q}_2 \). Finally, by applying the formula for \( \mathcal{Q}_3 \) to the formula for \( \mathcal{P}_4 \), we also write down \( \mathcal{P}_1 \) as rational functions of the basic generators and the four extra generators \( \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}_1, \mathcal{Q}_2 \). Since the details of solving these equations are tedious, we omit them. The resulting two formulae for \( \mathcal{P}_4 \) and \( \mathcal{Q}_3 \) are precisely what we claim. \qed

Corollary 4.13. The ring generated by all basic generators, extra generators and their derivatives is isomorphic to

\[ \mathbb{Q}[\mathcal{X}, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, \mathcal{Q}, \tilde{Q}, \mathcal{P}, \tilde{P}, L]. \]

Proof. This follows from the preceding relations and Remark 3.3. \qed

Remark 4.14. Notice that here \( \mathbb{Q}[\mathcal{X}, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, \mathcal{Q}, \tilde{Q}, \mathcal{P}, \tilde{P}, L] \) is defined as the ring generated by these nine generators. The corollary does not imply that there are no other relations between the generators.

5. Proof of key propositions

Recall that the key propositions 3.7, 3.6 and 3.5 directly imply the Main Theorem. In this section, we will finish their proofs.
5.1. Proof of Proposition 3.7. We want to compute the contributions of graphs with a genus 0 quasimap vertex. Recall that there are two graphs, with contributions given by the following correlators

\[
\begin{align*}
\text{Cont}'_{10} := L^2 \left< \left< \frac{\tilde{H}^3}{t-5H} + \frac{5}{3} \frac{H^4 t^{-1}}{(t-5H)(t-5H-\psi)} + \frac{65}{5} \frac{H^4 t^{-1}}{(t-5H)(t-5H-\psi)} \right>_{0,2} \right>, \\
\text{Cont}'_{10} := L^2 \left< \left< \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t-5H)(t-5H-\psi)} \right>_{0,2} \right>,
\end{align*}
\]

Definition 5.1. We define the following modified S-matrix

\[
\bar{S}'(t)(\gamma) := \gamma + \sum_j e_j \left< \frac{\gamma}{t-5H-\psi} \right>_{0,2}^j,
\]

and a modified V-matrix

\[
\bar{V}(t)(\gamma_1 \otimes \gamma_2) := \left( \gamma_1 \otimes \gamma_2, \frac{1}{2t-5(1 \otimes H + H \otimes 1)} \right)^t + \left< \left< \frac{\gamma_1}{t-5H-\psi} \right>_{0,2} \right>,
\]

By definition, we can write the modified matrix as a specialization of the original matrix, for example

\[
\bar{S}'(t) = \text{Res}_{z=t-5H} S^*(z) \frac{1}{t-5H-z} = S^*(z)_{|z=t-5H}.
\]

Example 5.2. As pointed out in the proof of Lemma 4.12, by the definition of the I-function for the twisted theory (7), we have

\[
\bar{S}'(t)(1) = \frac{1}{I_0}.
\]

By definition of \(\bar{V}\), the genus 0 contribution is just the modified V-matrix with given insertions. Since (see e.g. [19])

\[
V(t, z, w)(\gamma_1 \otimes \gamma_2) = \frac{1}{z+w} \left( S(t, z) \gamma_1, S(t, w) \gamma_2 \right)^t
\]

the modified V-matrix can be computed as follows

\[
\bar{V}(t)(\gamma_1 \otimes \gamma_2) = \left( \gamma_1 \otimes \gamma_2, \sum_\alpha \bar{S}'(e_\alpha) \otimes \bar{S}'(e^\alpha) \right)^t
\]

\[
= \left( \gamma_1 \otimes \gamma_2, \sum_{j=0,1,2,3} \bar{S}'(H^j) \otimes \bar{S}'(H^{3-j}) - \frac{65}{625} \bar{S}'(\tilde{H}_4) \otimes \bar{S}'(\tilde{H}_4) \right)^t
\]

(35)

where we have used \((\tilde{H}_4, \tilde{H}_4)^t = -\frac{625}{625}\).

Denote by

\[
\bar{S}_k := \bar{S}'(t) H^k, \quad \bar{S}^i_k := (\tilde{H}_i, \bar{S}_k)^t,
\]

where \(\{\tilde{H}_i\}\) is the dual basis of \(\{H^k\}\).
Lemma 5.3. For $i = 0, 1, 2, 3, 4; k = 0, 1, 2, 3$, the entries of the modified $S$-matrix satisfy

$$S_k^i \in \frac{t^{k-i}}{I_0 \tau_{\mathcal{I}_1} \cdots \tau_{\mathcal{I}_k}} Q[\mathcal{X}, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, L, \mathcal{P}, \mathcal{Q}, \mathcal{P}_3, \mathcal{Q}_2, \mathcal{I}, \mathcal{I}_2]$$

where $\mathcal{I}_k := \frac{I_0 + k}{L}$ for $k = 1, 2, 3$ (notice that $\mathcal{I}_3 = \mathcal{I}_1$). Moreover, if we define

$$\deg \mathcal{I}_k = 1 \quad \text{for } k = 1, 2, 3$$

and rewrite $L^j$ ($j = 2, 3, 4, 7, 8, 12$) in $S_k^i$ as polynomials of $\mathcal{I}_k$ ($k = 1, 2, 3$) (see Remark [3.3]), then for all $i = 0, 1, \cdots, 4$ and $k = 0, 1, \cdots, 3$

$$(i^{-k} I_0 \tau_{\mathcal{I}_1} \cdots \tau_{\mathcal{I}_k}) \cdot S_k^i \in Q[\mathcal{X}, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, \mathcal{Z}, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{P}, \mathcal{Q}, \mathcal{P}, \mathcal{Q}, \mathcal{I}_1, \mathcal{I}_2]$$

are homogeneous polynomials of degree $k$.

Proof. This lemma is a direct consequence of Proposition [1.1] and the fact $\mathcal{S}_0 = I_0^{-1}$. To be more explicit, by a direct Birkhoff factorization computation we have

$$S_1 = \frac{1}{I_0 \tau_{\mathcal{I}_1}} \left( (L^3 + 5 \mathcal{X}) \cdot H - \left( \frac{t^4}{8} + \mathcal{Q} \cdot \mathcal{X} \right) \cdot t \right)$$

$$S_2 = \frac{1}{I_0 \tau_{\mathcal{I}_2}} \left( \left( 4 L^3 - 3 L^8 + 5 L^4 \mathcal{X} \mathcal{Y} + 25 \mathcal{X} \mathcal{Y} - 25 \mathcal{X}_2 \right) \cdot H^2 
= \frac{-16 L^3 + 17 L^8}{10} + \frac{L^4}{2} \mathcal{X} \mathcal{Y} + 5 \mathcal{Q} \cdot \mathcal{X} \mathcal{Y} + 10 \mathcal{X}_2 - \frac{t^2}{10} (L^4 + 5 \mathcal{X}) \cdot t H 
\right) + \frac{8 L^3 - 11 L^8}{50} + \frac{L^4}{2} (-3 \mathcal{X} + 2 \mathcal{Y}) - \mathcal{P} \cdot \mathcal{Q} + \mathcal{X} \mathcal{Y} - \mathcal{X}_2 + \frac{t^2}{50} (L^4 + 5 \mathcal{X} + 5 \mathcal{X}_2)
$$

$$S_3 = \frac{1}{I_0 \tau_{\mathcal{I}_2}} \left( \left( 36 L^1^2 - 47 L^7 + 12 L^2 + (60 L^8 - 55 L^3) \mathcal{X} + 25 L^4 \mathcal{X}^2 + \mathcal{X}_2 + 125 (\mathcal{X}^3 + 3 \mathcal{X} \mathcal{X}_2 + \mathcal{X}_3) \right) \mathcal{X} \mathcal{Y} + 25 \mathcal{X} \mathcal{Y} - 25 \mathcal{X}_2 \right) \cdot H^3 
= \frac{211 L^1^2 - 282 L^7 + 67 L^2}{10} + \left( 33 L^3 - 31 L^8 \right) \mathcal{X}^2 \mathcal{Y} - \frac{5}{2} L^4 \mathcal{X} - 15 L^4 \mathcal{X}_2 - 75 \mathcal{X}^3 + 3 \mathcal{X} \mathcal{X}_2 + \mathcal{X}_3 + 5 \mathcal{Q} (L^4 + 5 \mathcal{X}) 
+ (4 L^3 - 3 L^8) \mathcal{Q} + 5 L^4 \mathcal{X} \mathcal{Y} + 25 \mathcal{X} \mathcal{Y} - \mathcal{X}_2 + \frac{t^2}{5} (3 L^8 - 4 L^3 - 5 L^4 \mathcal{X} \mathcal{Y} + 25 \mathcal{X}_2 - 2 \mathcal{X} \mathcal{Y})) \right) \cdot t H^2 
= \frac{103 L^1^2 - 131 L^7 + 31 L^2}{25} + \frac{26 L^8 - 23 L^3 \mathcal{X}^2}{5} + \frac{L^4 (3 \mathcal{X}_2 - 2 \mathcal{X} \mathcal{Y}) + 15 (\mathcal{X}^3 + 3 \mathcal{X} \mathcal{X}_2 + \mathcal{X}_3)}{5}
$$

where we have used the relations between $\{ I_{k,k_1}, I_{k,k_2}, \cdots \}$ presented in the proof of Lemma [4.10] and the differential equations in Lemma [4.12].

Finally, by applying the formulae in the proof of Lemma [5.1] and the formula

$$\mathcal{S}^t(t) (\mathcal{H}_4) = \mathcal{H}_4$$

in Lemma [4.2] to equation [35], with

$$\gamma_1 \otimes \gamma_2 = \frac{1}{1 - t^{-1} \otimes 5H - 5H \otimes t^{-1} + 25H \otimes H} \left( \frac{5 H^3 \otimes H^4 t^{-1} + \frac{5}{3} H^4 \otimes H^3 t^{-1} + \frac{65}{3} H^4 \otimes H^4 t^{-2}}{t - 5H} \right)$$

or

$$\gamma_1 \otimes \gamma_2 = \frac{5 H^3 \otimes H^4 t^{-1}}{t - 5H} \otimes \frac{5 H^3 \otimes H^4 t^{-1}}{t - 5H}.$$
we can write down the explicit formula of $L^2 \cdot \mathcal{V}(t)$ as a homogeneous polynomial of degree 3 in 
\[ \mathbb{Q}[\mathcal{X}, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, \mathcal{Z}, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{P}, \mathcal{Q}, \mathcal{P}, \mathcal{Q}]. \]

Note that here we have used 
\[ I_0^2 T_1^2 T_2 = L^2, \quad \deg L^2 = 3. \]

The explicit computations exactly give us the formulae presented in Proposition 3.7

5.2. Proof of Proposition 3.6. There is only one graph with a genus 1 quasimap vertex, and it has the contribution
\[ \text{Cont}'_{1,1} := \frac{L^2}{I_0} \left\langle \left( \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi)} \right) \right\rangle_{1,1}. \]

This type of twisted invariants can be computed by the following two steps. First, by using the modified $S$-matrix $\overline{S}(t)$, we can write the modified descendent correlators in terms of the $\overline{S}$ action on the ancestor genus one invariants (see [31] and [20]). Explicitly, for any $\alpha \in H^*_C(\mathbb{P}^4)$, we have
\[ \left\langle \left( \frac{\alpha}{t - 5H - \psi} \right) \right\rangle_{1,1} = \left\langle \left( \frac{S(\bar{\psi}) \alpha}{t - 5H - \psi} \right)_{+} \right\rangle_{1,1} = \left\langle \left( \frac{\overline{S}(t) \alpha}{t - 5H - \psi} \right) \right\rangle_{1,1}, \]

where $\bar{\psi}$ is the pullback psi class from $\overline{M}_{1,1}$, we expand $\frac{1}{t - 5H - \psi}$ in terms of positive powers of $H$ and $\psi$ with $S$ acting on the left, and the bracket $[\cdot]_+$ picks out the terms with a non-negative power of $\psi$.

Next, we evaluate the ancestor invariants of the following form
\[ \left\langle \gamma(\bar{\psi}) \right\rangle_{1,1} = \int_{\overline{M}_{1,1}} \Omega_{1,1}(\gamma(\psi_1)) \]

by using Givental–Teleman’s reconstruction theorem [20,40]. We recall here only the general shape of the reconstruction theorem and refer the reader to [38] for more details. For $2g - 2 + n > 0$, the theorem says that
\[ \Omega_{g,n}(\gamma_1, \cdots, \gamma_n) = RT \omega_{g,n}(\gamma_1, \cdots, \gamma_n) = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} (\iota_{\Gamma})_* T \omega_{\Gamma}(\gamma_1, \cdots, \gamma_n) \]

where $G_{g,n}$ is the set of genus $g$, $n$ marked point stable graphs, $\iota_{\Gamma}$ is the canonical morphism $\iota_{\Gamma}: \overline{M}_{\Gamma} \to \overline{M}_{g,n}$, the translation action $T$ is given by
\[ T \omega_{g,n}(-) := \sum_{k=0}^{\infty} \frac{1}{k!} (\mathcal{P}_k)_* \omega_{g,n+k}(-, T(\psi)^k) \]

for the cohomological valued formal series
\[ T(z) = T_1 z^2 + T_2 z^3 + T_3 z^4 + \cdots := z (1 - R^{-1}(z) \mathbf{1}), \]

and below we will define $\omega_{\Gamma}(\gamma_1, \cdots, \gamma_n)$ for each graph under discussion.

In our case, the only insertion $\gamma$ is given by
\[ \gamma(\psi) := \overline{S}(t) \frac{-\frac{5}{3} H^3 + \frac{5}{24} H^4 t^{-1}}{(t - 5H)(t - 5H - \psi)}. \]
The contribution can be written as a summation over two stable graphs:

\[ \text{Cont}_{\Gamma_1} = \frac{L^2}{I_0} \int_{\mathcal{M}_{1,1}} \Omega_{1,1}(\gamma(\psi)) = \frac{L^2}{I_0} \left( \text{Cont}_{\Gamma_a} + \text{Cont}_{\Gamma_b} \right) \]

where \( \Gamma_a \) and \( \Gamma_b \) are the following stable graphs:

\[ \Gamma_a := \gamma \quad \Gamma_b := \gamma \quad g=0. \]

We now compute the contribution of each graph separately.

5.2.1. Contribution of \( \Gamma_a \). We have

\[ T\omega_{\Gamma_a}(\gamma) := T\omega_{1,1}(R^{-1}(\psi) \cdot \gamma) = \omega_{1,1}(\gamma) - \psi_1 \omega_{1,1}(R_1 \gamma) + \psi_1 \omega_{1,2}(\gamma, T_1) = \sum_\alpha \left( e^\alpha, \gamma \right)^t + \psi_1 \left( - (e^\alpha, R_1 \gamma)^t + (e^\alpha, \gamma)^t (e^\alpha, T_1)^t \right) \]

where we have used

\[ \omega_{g,n}(\gamma_1, \cdots, \gamma_n) = \Delta^{g-1} \prod_{i=1}^n (e^\alpha, \gamma_i)^t. \]

In particular, we have

\[ T\omega_{\Gamma_a}(e_\alpha) = 1 + \psi_1 \left( (R_1^* e_\alpha, \sum e^\beta)^t + (R_1)_0^\alpha \right). \]

Note that the formula for \( (R_1)_0^\alpha \) is in equation \( \text{(14)} \). After a direct computation using the formula for \( (R_1)_0^\alpha \) and the inductive formulæ for the \( \Psi \)- and \( R \)-matrix of Section \( \text{4.3} \) we obtain

\[ (R_1^* e_\alpha, \sum e^\beta)^t = \frac{5}{12t} + \frac{5}{t q_\alpha} + \frac{25}{2t q_\alpha^2 L^2} (12(L^5 - 1)L^2 + 60L^3 \mathcal{X} + 75L^4 \mathcal{Q} + 250\mathcal{P}) + \frac{625}{t q_\alpha^3 L^3} (2(L^5 - 1)L^3 + L^4(8\mathcal{X} + \mathcal{Y}) + 50L\mathcal{P} + (15L^5 - 5)\mathcal{Q}) + \frac{625}{2t q_\alpha^4 L^4} (8(L^5 - 1)L^4 + 10L^5(2\mathcal{X} + \mathcal{Y}) + 10\mathcal{X} + 250L^2\mathcal{P} + 25L(3L^5 - 2)\mathcal{Q}) \]

5.2.2. Contribution of \( \Gamma_b \). Let

\[ W(w, z) := \sum_\alpha e_\alpha \otimes e^\alpha - R^{-1}(z)e_\alpha \otimes R^{-1}(w)e^\alpha \]

then we have \( W(w, z) = W_0 + W_1 + W_2 + \cdots \) where

\[ W_0 = \sum_\alpha R_1 e_\alpha \otimes e^\alpha, \quad W_1 = \sum_\alpha \left( - R_2 z - R_2^* w \right) e_\alpha \otimes e^\alpha \]

\[ W_2 = \sum_\alpha \left( R_3 z^2 + (R_1 R_2 - R_3) w z + R_3^* w^2 \right) e_\alpha \otimes e^\alpha \]
The contribution from the loop type graph is given by

\[ T\omega_{11}(\gamma) := T\omega_{0,3}(W(\psi), R^{-1}(\psi) \cdot \gamma) = \omega_{0,3}(W_0, \gamma) + \sum_\alpha \Delta_\alpha^{-1}(e^\alpha, R_1 e^\alpha, \gamma) \]

\[ = \sum_\alpha (\alpha, \gamma^t) = \sum_\alpha (\alpha, \gamma^t) (R_1)_{\alpha\alpha} \]

In particular,

\[ T\omega_{11}(\epsilon^\alpha) = (R_1)_{\alpha\alpha}. \]

Again after a direct computation using the formula for \((R_1)_{0,0}\) and using the formulae from Section 13, we obtain

\[
(R_1)_{\alpha\alpha} = \frac{5^5}{12t} + \frac{5}{t_\alpha q_\alpha} + \frac{25}{2t q_\alpha^3 L^2} (4(L^5 - 1)L^2 + 20L^3 \chi + 25L^4 \bar{Q} + 50 \bar{P}) \\
+ \frac{125}{t q_\alpha^3 L^3} (2(L^5 - 1)L^2 + 10L^4 \chi + 50L \bar{P} + (25L^5 - 10) \bar{Q}) \\
+ \frac{625}{2t q_\alpha^4 L^4} (6\chi + 2\psi + 50L^2 \bar{P} + 5L(4L^5 - 3) \bar{Q})
\]

To summarize, given

\[ \gamma(\psi) = \gamma_0 + \gamma_1 \psi + \cdots \]

we have

\[ \text{Cont}_{11} = \int_{\mathcal{M}_{1,1}} T\omega_{11}(\gamma) = \frac{1}{24} \sum_\alpha ((\gamma_1, e^\alpha)^t + (\gamma_0, e^\alpha)^t \left( (R_1^* e_\alpha, \sum e^\beta)^t + (R_1)_{0,0} \right)), \]

\[ \text{Cont}_{12} = \frac{1}{2} \int_{\mathcal{M}_{0,3}} T\omega_{11}(\gamma) = \frac{1}{2} \sum_\alpha (\gamma_0, e^\alpha)^t (R_1)_{\alpha\alpha}. \]

Finally, by equation (13), (37) and (38), together with the explicit formula for the insertion

\( (\gamma_0, e^\alpha) = \frac{t I_0}{6000L^2} \left( 5381L^{12} - 7274L^7 + 1893L^2 + L^8(800\chi + 410Y) + 25L^4(80\chi_2 - 43\chi^2 + 82\chi Y) \\
- 8850L^3 \chi + 20250(\chi^3 + 3\chi \chi_2 + \chi_3) - (1975L^4 + 10250\chi) \bar{Q} + 250 \bar{P} \right) - \frac{t q_\alpha I_0}{15000} \left( 3L^5 + KL^2 \\
+ 15L \chi + 39 \right) - \frac{t q_\alpha^3 L_0}{75000} \left( 2L^5 + KL^2 + 10L \chi + 39 \right) - \frac{t q_\alpha^4 L_0}{375000} \left( L^4 + KL + 5 \chi \right) - \frac{t q_\alpha^6 L_0^2 K}{1875000}. \)

\( (\gamma_1, e^\alpha) = \frac{I_0}{6000L^2} \left( 5381L^{12} - 7274L^7 + 1893L^2 + L^8(800\chi + 410Y) + 25L^4(80\chi_2 - 43\chi^2 + 82\chi Y) \\
- 8850L^3 \chi + 20250(\chi^3 + 3\chi \chi_2 + \chi_3) - (1975L^4 + 10250\chi) \bar{Q} + 250 \bar{P} \right) - \frac{t q_\alpha I_0}{15000} \left( 3L^5 + KL^2 \\
+ 15L \chi + 39 \right) - \frac{t q_\alpha^3 L_0}{75000} \left( 2L^5 + KL^2 + 10L \chi + 39 \right) - \frac{t q_\alpha^4 L_0}{375000} \left( L^4 + KL + 5 \chi \right) - \frac{t q_\alpha^6 L_0^2 K}{1875000}. \)

where \( K := 123L^8 - 164L^3 - 205L^4(\chi + Y) + 1025(\chi_2 - \chi Y) + 25 \bar{Q}, \) and where (*) denote some \( \alpha \)-independent functions which we have omitted because they are irrelevant for the further computations, we arrive at the following formulae.
Lemma 5.4. Both \( \frac{L^2}{I_0} \text{Cont}_{\Gamma_1^3} \) and \( \frac{L^2}{I_0} \text{Cont}_{\Gamma_1^1} \) are degree 3 homogeneous polynomials in the basic and extra generators, to be precise

\[
\frac{L^2}{I_0} \text{Cont}_{\Gamma_1^3} = \frac{5}{576} (\mathcal{X} + 2Z_1) \hat{Q} + \frac{205}{576} (\mathcal{X}^3 + 4\mathcal{X}^2 + \mathcal{X}^3 - 3Z_1\mathcal{Y}) + \frac{305 Z_1X_2}{288} + \frac{5 Z_1\mathcal{X}^2}{384} + \frac{3055 Z_2\mathcal{X}}{2304} - \frac{95 Z_1^2\mathcal{X}}{144} - \frac{205 Z_1^2\mathcal{Y}}{288} + \frac{9275 Z_1Z_2}{13824} + \frac{5285 Z_3}{13824}
\]

\[
\frac{L^2}{I_0} \text{Cont}_{\Gamma_1^1} = -\frac{473}{576} \hat{P} + \left( \frac{145 \mathcal{X}^2}{576} - \frac{\mathcal{Y}}{48} - \frac{2113 Z_1}{1152} \right) \hat{Q} - \frac{45}{64} (\mathcal{X}^3 + \mathcal{X}^3) - \frac{299 \mathcal{X}^2}{64}
\]

Again using Givental–Teleman’s theorem, we see that we need to compute

\[
\langle\langle\rangle\rangle_{2,0} = \int_{\mathcal{M}_{2,0}} \Omega_{2,0} = \sum_{\Gamma \in G_{2,0}} \frac{1}{|\text{Aut}(\Gamma)|} \text{Cont}_\Gamma.
\]

There are 7 stable graphs in \( G_{2,0} \):

\[
\Gamma_2^1 := g=2 \quad \Gamma_2^2 := g=1 \quad \Gamma_2^3 := g=1 \quad \Gamma_2^4 := g=1 \quad \Gamma_2^5 := g=0 \quad \Gamma_2^6 := g=0 \quad \Gamma_2^7 := g=0
\]

Hence we can write the contribution as summation over contributions of the stable graphs:

\[
\langle\langle\rangle\rangle_{2,0}^t = \left( \text{Cont}_{\Gamma_2^1}^t + \frac{\text{Cont}_{\Gamma_2^2}^t}{2} + \frac{\text{Cont}_{\Gamma_2^3}^t}{2} + \frac{\text{Cont}_{\Gamma_2^4}^t}{8} + \frac{\text{Cont}_{\Gamma_2^5}^t}{8} + \frac{\text{Cont}_{\Gamma_2^6}^t}{12} \right)
\]

5.3. Proof of Proposition 3.5. Finally, we deal with the remaining graph with a single genus two quasimap vertex. It has the contribution

\[
\text{Cont}_{\Gamma_2^1}^t := \frac{L^2}{I_0} \cdot \langle\langle\rangle\rangle_{2,0}^t.
\]

Again using Givental–Teleman’s theorem, we see that we need to compute

\[
\langle\langle\rangle\rangle_{2,0}^t = \int_{\mathcal{M}_{2,0}} \Omega_{2,0} = \sum_{\Gamma \in G_{2,0}} \frac{1}{|\text{Aut}(\Gamma)|} \text{Cont}_\Gamma.
\]

In the remaining part of this section, we compute each of these contributions.

5.3.1. Trivial graph. The first graph \( \Gamma_2^1 \) is a single genus-two vertex. We directly compute its contribution:

\[
\text{Cont}_{\Gamma_2^1} = \int_{\mathcal{M}_{2,0}} T \omega_{2,0}
\]

\[
= \int_{\mathcal{M}_{2,0}} \left( \omega_{2,0} + (\pi_1)_* \omega_{2,1}(T(\psi)) + \frac{1}{2!} (\pi_2)_* \omega_{2,2}(T(\psi)^2) + \frac{1}{3!} (\pi_3)_* \omega_{2,3}(T(\psi)^3) \right)
\]

\[
= \omega_{2,1}(T_3) \int_{\mathcal{M}_{2,1}} \psi_1^4 + \frac{2}{2} \omega_{2,2}(T_1 T_2) \int_{\mathcal{M}_{2,2}} \psi_1^2 \psi_2^2 + \frac{1}{6} \omega_{2,3}(T_3^3) \int_{\mathcal{M}_{2,3}} \psi_1^2 \psi_2^3 \psi_3^3
\]

\[
= \sum_{\beta} \Delta_{\beta} \left( \frac{1}{1152} (e^\beta, T_3)^t + \frac{29}{5760} (e^\beta, T_1)(e^\beta, T_2)^t + \frac{7}{6 \cdot 240} \left( (e^\beta, T_1)^t \right)^3 \right)
\]
where we have used some known intersection numbers in $\overline{M}_{g,n}$.

5.3.2. **One-edge graphs.** There are two graphs with one edge:

\[ \Gamma_2^2 := \begin{array}{c}
\circ \\
g = 1
\end{array} \quad \Gamma_3^2 := \begin{array}{c}
\circ \\
g = 1
\end{array} \]

For the computation of $\text{Cont}_{\Gamma_2^2}$, it is useful to notice that

\[ T\omega_{1,1}(e_\beta) = 1 + (e_\beta, T_1)^t \cdot \psi. \]

With this, we can compute:

\[
\text{Cont}_{\Gamma_2^2} = \int_{\overline{M}_{1,1} \times \overline{M}_{1,1}} (T\omega_{1,1} \otimes T\omega_{1,1})(W(\psi_1, \psi_2))
\]

\[
= \frac{1}{24^2} \sum_{\alpha, \beta} \left( (e_\beta, T_1)^t (e_\alpha, R_1 e_\beta)^t (e_\alpha, T_1)^t - 2(e_\alpha, (R_2^*) e_\beta)^t (e_\beta, T_1)^t + (e_\alpha, (R_1 R_2 - R_3) e_\beta)^t \right)
\]

We next compute $\text{Cont}'(\Gamma_3^2)$ using

\[ T\omega_{1,2}(-) = \omega_{1,2}(-) + (\pi_1)_* \omega_{1,3}(-, T(\psi)) + \frac{1}{2!} (\pi_2)_* \omega_{1,4}(-, T(\psi)^2) \]

and $R_2 + R_2^* = R_1^2$:

\[
\text{Cont}_{\Gamma_3^2} = \int_{\overline{M}_{1,1}} T\omega_{1,2}(W(\psi_1, \psi_2))
\]

\[
= \langle \omega_{1,2}(W_2(\psi_1, \psi_2)) \rangle_{1,2} + \langle \omega_{1,3}(W_1(\psi_1, \psi_2), T_1), \psi_3^2 \rangle_{1,3}
\]

\[
+ \omega_{1,3}(W_0, T_2) \langle \psi_3^2 \rangle_{1,3} + \frac{1}{2!} \omega_{1,4}(W_0, T_1^2) \langle \psi_3^2 \psi_4^2 \rangle_{1,4}
\]

\[
= \sum_\beta \Delta_\beta \left( \frac{1}{24} (e_\beta, (R_3^* + R_1 R_2) e_\beta)^t - \frac{1}{12} (e_\beta, (R_2^* + R_2) e_\beta)^t (e_\beta, T_1)^t + \frac{1}{24} (e_\beta, R_1 e_\beta)^t (e_\beta, T_2)^t + \frac{1}{24} (e_\beta, R_1 e_\beta)^t \left( (e_\beta, T_1)^t \right)^2 \right)
\]

5.3.3. **Two-edge graphs.** There are two graphs with two edges:

\[ \Gamma_4^2 := \begin{array}{c}
\circ \\
g = 1
\end{array} \begin{array}{c}
\circ \\
g = 0
\end{array} \quad \Gamma_5^2 := \begin{array}{c}
\circ \\
g = 0
\end{array} \begin{array}{c}
\circ \\
g = 0
\end{array} \]

We compute $\Gamma_4^2$ as follows:

\[
\text{Cont}_{\Gamma_4^2} = \int_{\overline{M}_{1,1}} (T\omega_{1,1} \otimes \omega_{0,3})(W_0 + W_1, W_0)
\]

\[
= \frac{1}{24} \left( \sum_{\alpha, \beta} (e_\beta, T_1)^t (e_\alpha, R_1 e_\alpha)^t (e_\alpha, T_1)^t + (e_\beta, -R_2^* e_\alpha)^t (e_\alpha, R_1 e_\alpha)^t \right)
\]

\[
= \frac{1}{24} \sum_{\alpha, \beta} \Delta_{\alpha}^{1/2} \Delta_{\beta}^{1/2} \left( (e_\beta, T_1)^t (R_1)_{\alpha \beta} (R_1)_{\alpha \alpha} - (R_1)_{\alpha \alpha} (R_2^*)_{\beta \alpha} \right)
\]

For $\Gamma_5^2$, recalling

\[ T\omega_{0,4}(-) = \omega_{0,4}(-) + \omega_{0,5}(-, T_1) \cdot \psi_1, \]
we can write

\[
\text{Cont}_{\Gamma_5^2} = \int_{\mathcal{M}_{0,4}} T\omega_{0,4}(W \otimes W)
= \omega_{0,5}(W_0^2, T_1) - 2 \sum_{\alpha} \omega_{0,4}(W_0, (R_2 + R_2^*)e_{\alpha}, e_{\alpha})
= \sum_{\beta} \Delta_{\beta}\left(\left((e_{\beta}, R_1 e_{\beta})^t\right)^2 (e_{\beta}, T_1)^t - 2(e_{\beta}, R_1 e_{\beta})^t(e_{\beta}, (R_2 + R_2^*)e_{\beta})^t\right)
\]

5.3.4. Three-edge graphs. There are also two graphs with three edges:

\[
\Gamma_6^2 := \begin{array}{c}
\bullet \\
g = 0
\end{array} \quad \Gamma_7^2 := \begin{array}{c}
\bullet \\
g = 0
\end{array}
\]

The computation of their contributions is similar to the previous ones:

\[
\text{Cont}_{\Gamma_6^2} = (\omega_{0,3} \otimes \omega_{0,3})(W_0 \otimes W_0 \otimes W_0)
= \sum_{\alpha, \beta} (R_1 e_{\beta}^\alpha, e_{\alpha}^\beta)^t (e_{\alpha}^\beta, R_1 e_{\beta}^\alpha)^t
= \sum_{\alpha, \beta} (\Delta_{\alpha} \Delta_{\beta})^{1/2} (R_1)_{\alpha\alpha} (R_1)_{\beta\beta} (R_1)_{\alpha\beta}
\]

\[
\text{Cont}_{\Gamma_7^2} = \sum_{\alpha, \beta} (\Delta_{\alpha} \Delta_{\beta})^{1/2} (R_1)_{\alpha\beta}^3
\]

5.3.5. Conclusion. For \(k = 1, 2, \cdots, 7\), we introduce

\[
\text{Cont}_{\Gamma_k^2} := \frac{L^2}{I_6^2} \text{Cont}_{\Gamma_k^2}
\]

After a long but direct computation using the \(R\)-matrix formulae in Section 4.3 we arrive at the following proposition.

**Proposition 5.5.** Both \(\text{Cont}_{\Gamma_1^2}\) and \(\text{Cont}_{\Gamma_2^2}\) are degree 3 homogeneous polynomials in the basic generators:

\[
\text{Cont}_{\Gamma_1^2} = \frac{1}{2^{15}3^6} (24475 Z_3 - 17565 Z_1 Z_2)
\]

\[
\text{Cont}_{\Gamma_2^2} = \frac{5}{576} (X_3 + 4X_2^2 + X^3 - YX^2) + \frac{5}{144} (Z_1 X_2 - Z_2 Y X - Z_1^2 (X + Y))
+ \frac{25}{768} Z_2 X + \frac{1}{2^{14}3^6} (139157 Z_3 - 12531 Z_1 Z_2)
\]
Both $\text{Cont}_{13}^t$ and $\text{Cont}_{14}^t$ are degree 3 homogeneous polynomials in the basic and extra generators:

$$\text{Cont}_{13}^t = \frac{289}{6912} \tilde{P} + \frac{1345}{13824} Z_1 \tilde{Q} - \frac{1}{24} (X_3 + 2X X_2 + X^2 + 3 + Y X^2) - \frac{1}{6} Z_1 (X^2 + Y X) - \frac{3347}{13824} Z_2 X - 682 Z_2 Y - 1630 Z_1^2 Y + \frac{21573 Z_1 Z_2 - 68155 Z_3}{2^{12} 3^5}$$

$$\text{Cont}_{14}^t = -\frac{289}{6912} \tilde{P} + \left(\frac{5 X}{288} - \frac{865 Z_1}{13824}\right) \tilde{Q} + \frac{1}{24} (3X + Y) (Y X - X_2) - \frac{5 Z_1 X_2}{24} + \frac{53 Z_1 X^2}{192} + \frac{1}{12} Z_1 Y^2 + \frac{151 Z_1 Y X}{288} - \frac{205 Z_2 X}{13824} + \frac{1055 Z_1^2 X}{2304} - \frac{235 Z_2 Y}{6912} + 3455 Z_1 X Y - \frac{161 Z_2 X}{2304} + \frac{325 Z_1^2 X}{384} - \frac{605 Z_1 X Y}{1152} + \frac{5225 Z_3}{82944}$$

The sum $\text{Cont}_{13}^t + \text{Cont}_{14}^t + \text{Cont}_{12}^t$ is a degree 3 homogeneous polynomial in the basic and extra generators:

$$\text{Cont}_{5,6,7}^t := \frac{1}{8} \text{Cont}_{13}^t + \frac{1}{8} \text{Cont}_{14}^t + \frac{1}{12} \text{Cont}_{12}^t$$

$$= -\frac{5}{1152} \tilde{P} - \left(\frac{3 X + Y}{48} + \frac{245 Z_1}{2304}\right) \tilde{Q} - \frac{1}{24} (X^3 + 3 X Y X^2 + 3 Y^2 X + Y^3 + 8 Z_1 X^2 + 11 Z_1 (Y X + \frac{Y^2}{2}) + Z_2 Y) + \frac{161 Z_2 X}{2304} - \frac{325 Z_1^2 X}{384} - \frac{605 Z_1 X Y}{1152} - \frac{15449 Z_1 Z_2}{27648} + \frac{5225 Z_3}{82944}$$

Proposition 3.5 now follows directly from (39) and Proposition 5.5.

For reference, we also provide individual formulae for $\text{Cont}_{13}^t$, $\text{Cont}_{14}^t$ and $\text{Cont}_{12}^t$. They are rational functions in the basic and extra generators.

**Lemma 5.6.** Introduce the following inhomogeneous polynomial

$$\mathcal{E} := \left(4 \tilde{Q}^2 + 2 (3 X + Y) \tilde{P}\right) L - 20 \tilde{Q} \tilde{P} L^2 + 25 \tilde{P}^2 L^3 + \frac{1}{5} \tilde{Q} (3 X + Y) L^5 - 10 \tilde{Q}^2 L^5 + 25 \tilde{Q} \tilde{P} L^7 + \frac{1}{25} (3 X + Y)^2 L^9 + \frac{25}{4} \tilde{Q}^2 L^{11}.$$
We have the following polynomiality for $\text{Cont}'_{\Gamma_5}$, $\text{Cont}'_{\Gamma_6}$ and $\text{Cont}'_{\Gamma_7}$:

$$\text{Cont}'_{\Gamma_5} = -\frac{3E}{L^5-1} + \frac{1183}{90}\tilde{P} - \left(\frac{79X}{6} + \frac{71Y}{10} - \frac{3221Z_1}{360}\right)\tilde{Q}$$

$$+ \frac{1}{5}\left(\frac{6X^3 - 2YX^2 - 8Y^2X}{5} + \frac{101Z_1X^2}{4} + \frac{431Z_1YX}{30} + \frac{43Z_1Y^2}{20}\right)$$

$$- \frac{119Z_2X}{72} + \frac{61Z_2Y}{36} + \frac{879Z_1^2X}{20} + \frac{557Z_1^2Y}{180} + \frac{176539Z_1Z_2}{6912} - \frac{17389Z_3}{6912}$$

$$\text{Cont}'_{\Gamma_6} = \frac{E}{L^5-1} - \frac{869}{240}\tilde{P} + \left(\frac{21X}{5} + \frac{11Y}{5} - \frac{439Z_1}{288}\right)\tilde{Q}$$

$$- \frac{1}{5}\left(\frac{7X^3 + 9YX^2 + 3Y^2X + Y^3}{5}\right) - \frac{877Z_1X^2}{60} - \frac{359Z_1YX}{30} - \frac{51Z_1Y^2}{20}$$

$$- \frac{33Z_2X}{32} - \frac{31Z_2Y}{48} - \frac{967Z_1^2X}{144} - \frac{1087Z_1^2Y}{6912} - \frac{76795Z_1Z_2}{6912} + \frac{6565Z_3}{6912}$$

$$\text{Cont}'_{\Gamma_7} = \frac{3E}{L^5-1} - \frac{1147}{80}\tilde{P} + \left(\frac{127X}{10} + \frac{71Y}{10} - \frac{5957Z_1}{480}\right)\tilde{Q}$$

$$+ \frac{1}{5}\left(\frac{-X^3 + 9YX^2 + 9Y^2X - Y^3}{5}\right) - \frac{399Z_1X^2}{20} - \frac{91Z_1YX}{10} - \frac{43Z_1Y^2}{20}$$

$$+ \frac{467Z_2X}{96} - \frac{67Z_2Y}{48} - \frac{3669Z_1^2X}{240} + \frac{91Z_1^2Y}{2304} - \frac{65321Z_1Z_2}{21461Z_3} + \frac{6912}{6912}$$

The geometric meaning of this inhomogeneous polynomial is unclear. It has the following expansion

$$\frac{E}{L^5-1} = 45q + 227400q^2 + 1195603370q^3 + 5913833272300q^4 + O(q^5).$$

All coefficients are integers.

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