FINITE ELEMENT APPROXIMATIONS
FOR A LINEAR CAHN-HILLIARD-COOK EQUATION
DRIVEN BY THE SPACE DERIVATIVE OF A SPACE-TIME WHITE NOISE

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Abstract. We consider an initial- and Dirichlet boundary- value problem for a linear Cahn-Hilliard-
Cook equation, in one space dimension, forced by the space derivative of a space-time white noise. First,
we propose an approximate regularized stochastic parabolic problem discretizing the noise using linear
splines. Then fully-discrete approximations to the solution of the regularized problem are constructed
using, for the discretization in space, a Galerkin finite element method based on $H^2$-piecewise polyno-
mials, and, for time-stepping, the Backward Euler method. Finally, we derive strong a priori estimates
for the modeling error and for the numerical approximation error to the solution of the regularized
problem.

1. Introduction

Let $T > 0$, $D = (0,1)$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Then we consider the following
model initial- and Dirichlet boundary- value problem for a linear Cahn-Hilliard-Cook equation: find a
stochastic function $u : [0,T] \times D \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
\partial_t u + \partial_x^4 u + \mu \partial_x^2 u &= \partial_x \dot{W}(t,x) \quad \forall (t,x) \in (0,T) \times D, \\
\partial_x^{2m} u(t,\cdot) \big|_{\partial D} &= 0 \quad \forall t \in (0,T), \ m = 0,1, \\
u(0,x) &= 0 \quad \forall x \in D,
\end{aligned}
$$

(1.1)
a.s. in $\Omega$, where $\dot{W}$ denotes a space-time white noise on $[0,T] \times D$ (see, e.g., [23], [11]) and $\mu$ is a real
constant for which there exists $\kappa \in \mathbb{N}$ such that

$$
(\kappa - 1)^2 \pi^2 \leq \mu < \kappa^2 \pi^2,
$$

(1.2)
where $\mathbb{N}$ is the set of all positive integers. The above stochastic partial differential equation combines two
independent characteristics. On the one hand it corresponds to the linearization of the Cahn-Hilliard-
Cook equation around a homogeneous initial state, in the spinodal region, that governs the dynamics
of spinodal decomposition in metal alloys; see e.g. [4], and references therein. On the other hand the
forcing noise is a derivative of a space-time white noise that physically arises in generalized Cahn-Hilliard
equations, which are equations of conservative type describing the evolution of an order parameter in
phase transitions (see [10]; cf. [12], [2], [19]).

The mild solution of the problem above (cf. [9]) is given by the formula

$$
u(t,x) = \int_0^t \int_D \Psi(t-s;x,y) dW(s,y),
$$

(1.3)
where

$$
\Psi(t;x,y) = -\sum_{k=1}^{\infty} e^{-\lambda_k^2 (\xi_k^2 - \mu) t} \varepsilon_k(x) \varepsilon_k^*(y) \quad \forall (t,x,y) \in (0,T] \times D \times D,
$$

(1.4)

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with \( \lambda_k := k \pi \) for \( k \in \mathbb{N} \), and \( \varepsilon_k(z) := \sqrt{2} \sin(\lambda_k z) \) for \( z \in \overline{D} \) and \( k \in \mathbb{N} \). Observe that \( \Psi(t; x, y) = -\partial_x G(t; x, y) \), where \( G(t; x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 (x - y)^2} \varepsilon_k(x) \varepsilon_k(y) \) for all \((t, x, y) \in (0, T) \times \overline{D} \times \overline{D} \), is the space-time Green kernel of the corresponding deterministic parabolic problem: find a deterministic function \( w : [0, T] \times \overline{D} \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
\partial_t w + \partial_x^2 w + \mu \partial_x^2 w &= 0 \quad \forall (t, x) \in (0, T) \times D, \\
\partial_x^2 w(t, \cdot) \big|_{\partial D} &= 0 \quad \forall t \in (0, T], \quad m = 0, 1, \\
w(0, x) &= w_0(x) \quad \forall x \in D.
\end{align*}
\]

(1.5)

The goal of the paper at hand is to propose and analyze a methodology of constructing finite element approximations to \( u \).

1.1. The regularized problem. Our first step is to construct below an approximate to (1.1) regularized problem getting inspiration from the work [1] for the stochastic heat equation with additive space-time white noise (cf. [13, 15]).

Let \( N_* \in \mathbb{N} \), \( \Delta t := \frac{T}{N}, J_\ast \in \mathbb{N} \) and \( \Delta x := \frac{x}{J} \). Then, consider a partition of the interval \([0, T]\) with nodes \((t_n)_{n=0}^{N} = \frac{n \Delta t}{N} \) and a partition of \( \overline{D} \) with nodes \((x_j)_{j=0}^{J} \) given by \( t_n := n \Delta t \) for \( n = 0, \ldots, N_* \) and \( x_j := j \Delta x \) for \( j = 0, \ldots, J_* \). Also, set \( T_n := (t_{n-1}, t_n) \) for \( n = 1, \ldots, N_* \), and \( D_j := (x_{j-1}, x_j) \) for \( j = 1, \ldots, J_* \).

First, let \( S_* \) be the space of functions which are continuous on \( \overline{D} \) and piecewise linear over the above specified partition of \( D \), i.e.,

\[
S_* := \left\{ s \in C(\overline{D}; \mathbb{R}) : s \big|_{D_j} \in \mathbb{P}^1(D_j) \quad \text{for} \quad j = 1, \ldots, J_* \right\} \subset H^1(D).
\]

It is well-known that \( \text{dim}(S_*) = J_* + 1 \) and that the functions \( (\psi_i)_{i=1}^{J_*+1} \subset S_* \) defined by:

\[
\psi_1(x) := \frac{1}{\Delta x} (x_1 - x)^+, \quad \psi_{i+1}(x) := \frac{1}{\Delta x} (x - x_{i-1})^+, \\
\psi_i(x) := \frac{1}{\Delta x} \left[ (x - x_{i-2}) \chi_{e_{i-2},e_{i-1}} + (x - x_i) \chi_{e_{i-1},e_i} \right], \quad i = 2, \ldots, J_*
\]

consist the well-known hat functions basis of \( S_* \), where, for any \( A \subset \mathbb{R} \), by \( \chi_A \) we denote the indicator function of \( A \). Next, consider the fourth-order linear stochastic parabolic problem:

\[
\begin{align*}
\partial_t \tilde{u} + \partial_x^4 \tilde{u} + \mu \partial_x^2 \tilde{u} &= \partial_x \tilde{W} \quad \text{in} \quad (0, T] \times D, \\
\partial_x^2 \tilde{u}(t, \cdot) \big|_{\partial D} &= 0 \quad \forall t \in (0, T], \quad m = 0, 1, \\
\tilde{u}(0, x) &= 0 \quad \forall x \in D,
\end{align*}
\]

(1.6)
a.e. in \( \Omega \), where:

\[
\tilde{W}(t, x) := \frac{1}{\Delta x} \sum_{n=1}^{N_*} \sum_{i=1}^{J_*+1} \chi_{T_n}(t) \left[ \sum_{\ell=1}^{J_*+1} \left( \sum_{m=1}^{J_*+1} G_{\ell,m}^{-1} R_{n,m} \right) \psi_\ell(x) \right], \quad \forall (t, x) \in [0, T] \times \overline{D},
\]

\( G \) is a real, \((J_*+1) \times (J_*+1)\), symmetric and positive definite matrix with

\[
G_{i,j} := (\psi_j, \psi_i)_{0,D}, \quad i, j = 1, \ldots, J_* + 1,
\]

and

\[
R_{n,i} := \int_{T_n} \int_D \psi_i(x) \, dW(t,x), \quad i = 1, \ldots, J_* + 1, \quad n = 1, \ldots, N_*
\]

The solution of the problem (1.6), has the integral representation (see, e.g., [17])

\[
\tilde{u}(x, t) = \int_{0}^{t} \int_{D} G(t - s; x, y) \partial_y \tilde{W}(s, y) \, ds \, dy \\
= \int_{0}^{t} \int_{D} \Psi(t - s; x, y) \tilde{W}(s, y) \, ds \, dy,
\]

(1.7)
Remark 1.1. A simple computation verifies that $G$ is a tridiagonal matrix with $G_{1,1} = G_{J+1,J+1} = \frac{\Delta x}{3}$, $G_{i,i} = \frac{2\Delta x}{3}$ for $i = 2, \ldots, J$, and $G_{i,i+1} = \frac{-\Delta x}{6}$ for $i = 1, \ldots, J+1$. Since $G$ is symmetric we have in addition that $G_{i-1,i} = \frac{-\Delta x}{6}$ for $i = 2, \ldots, J+1$.

Remark 1.2. Let $\mathcal{I} = \{(n,i) : n = 1, \ldots, N, i = 1, \ldots, J+1\}$. Using the properties of the stochastic integral (see, e.g., [23]), we conclude that $R_{n,i} \sim N(0, \Delta t G_{i,i})$ for all $(n,i) \in \mathcal{I}$. Also, we observe that $\mathbb{E}[R_{n,i} R_{n',j}] = 0$ for $(n,i), (n',j) \in \mathcal{I}$ with $n \neq n'$, and hence they are independent since they are Gaussian. In addition, we have that $\mathbb{E}[R_{n,i} R_{n',j}] = \Delta t G_{i,j}$ for $(n,i), (n,j) \in \mathcal{I}$. Thus, for a given $n$ the random variables $(R_{n,i})_{i=1}^{J+1}$ are Gaussian and correlated, with correlation matrix $\Delta t G$.

1.2. The numerical method. Our second step is to construct finite element approximations of the solution $\hat{u}$ to the regularized problem.

Let $M \in \mathbb{N}$, $\Delta t := T/M$, $\tau_m := m \Delta t$ for $m = 0, \ldots, M$, and $\Delta m := (\tau_{m-1}, \tau_m)$ for $m = 1, \ldots, M$. Also, let $r \in \{2,3\}$, and $M_h^r \subset H^2(D) \cap H^1_0(D)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most $r$ over a partition of $D$ in intervals with maximum mesh-length $h$. Then, computable fully-discrete approximations of $\hat{u}$ are constructed by using the Backward Euler finite element method, which first sets

$$
\hat{U}_h^0 := 0
$$

and then, for $m = 1, \ldots, M$, finds $\hat{U}_h^m \in M_h^r$ such that

$$
(\hat{U}_h^m - \hat{U}_h^{m-1}, \chi)_{0,D} + \Delta t \left[ \left( \left( \hat{U}_h^m \right)' \hat{U}_h^m, \chi \right)_{0,D} + \mu \left( \left( \hat{U}_h^m \right)', \chi \right)_{0,D} \right] = \int_{\Delta m} \left( \partial_x \hat{W}, \chi \right)_{0,D} d\tau
$$

for all $\chi \in M_h^r$, where $(\cdot, \cdot)_{0,D}$ is the usual $L^2(D)$-inner product.

1.3. An overview of the paper and related references. Our analysis first focus on the estimation of the modeling error, i.e. the difference $u - \hat{u}$, in terms of the discretization parameters $\Delta t$ and $\Delta x$. Indeed, working with the integral representation of $u$ and $\hat{u}$, we obtain (see Theorem 3.1)

$$
\max_{t \in [0,T]} \left\{ \int_D \left| u(t,x) - \hat{u}(t,x) \right|^2 dx \right\}^{\frac{1}{2}} \leq C_{me} \left( \epsilon^{-\frac{1}{2}} \Delta x^{\frac{1}{2} - \epsilon_1} + \Delta t^{\frac{1}{2} - \epsilon_2} \right), \quad \forall \epsilon \in (0, \frac{1}{2}],
$$

where $C_{me}$ is a positive constant that is independent of $\Delta x$, $\Delta t$ and $\epsilon$. Next target in our analysis, is to provide the fully discrete approximations of $\hat{u}$ defined in Section 1.2 with a convergence result, which is achieved by proving the following strong error estimate (see Theorem 5.3)

$$
\max_{0 \leq m \leq M} \left\{ \int_D \left( \int_D \left| \hat{U}_h^m(x) - \hat{u}(\tau_m,x) \right|^2 dx \right) d\tau \right\}^{\frac{1}{2}} \leq C_{ne} \left( \epsilon_1^{-\frac{1}{2}} \Delta t^{-\epsilon_1} + \epsilon_2^{-\frac{1}{2}} \Delta x^{-\epsilon_2} \right),
$$

for all $\epsilon_1 \in (0, \frac{1}{8}]$ and $\epsilon_2 \in (0, \nu(r))$ with $\nu(2) = \frac{1}{3}$ and $\nu(3) = \frac{1}{2}$, where $C_{ne}$ is a positive constant independent of $\epsilon_1$, $\epsilon_2$, $\Delta t$, $h$, $\Delta x$ and $\Delta t$. To get the error estimate (1.1), we use as an auxilliary tool the Backward-Euler time-discrete approximations of $\hat{u}$ which are defined in Section 4. Thus, we can see the numerical approximation error as a sum of two types of error: the time-discretization error and the space-discretization error. The time-discretization error is the approximation error of the Backward Euler time-discrete approximations which is estimated in Theorem 1.2 while the space-discretization error is the error of approximating the Backward Euler time-discrete approximations by the Backward Euler finite element approximations, which is estimated in Proposition 5.2.

Let us expose some related bibliography. The work [18] contains a general convergence analysis for a class of time-discrete approximations to the solution of stochastic parabolic problems, the assumptions of which may cover problem (1.1). However, the approach we adopt here is different since first we introduce a space-time discretization of the noise and then we analyze time-discrete approximations to the solution. We would like to note that we are not aware of another work providing a rigorous convergence analysis for fully discrete finite element approximations to a stochastic parabolic equation forced by the space derivative of a space-time white noise. We refer the reader to our previous work [14], [15] and to [16] for the construction and the convergence analysis of Backward Euler finite element approximations of the solution to the problem (1.1) when $\mu = 0$ and an additive space-time white noise $\hat{W}$ is forced instead of
\[ \frac{\partial}{\partial t} \hat{W}. \] Finally, we refer the reader to [8], [1], [13], [3], [22] and [24] for the analysis of the finite element method for second order stochastic parabolic problems forced by an additive space-time white noise.

We close the section by an overview of the paper. Section 2 introduces notation, and recalls or proves several results often used in the paper. Section 3 is dedicated to the estimation of the modeling error. Section 4 defines the Backward Euler time-discrete approximations of \( \hat{u} \) and analyzes its convergence. Section 5 contains the error analysis for the Backward Euler fully-discrete approximations of \( \hat{u} \).

### 2. Notation and Preliminaries

#### 2.1. Function spaces and operators.

Let \( I \subset \mathbb{R} \) be a bounded interval. We denote by \( L^2(I) \) the space of the Lebesgue measurable functions which are square integrable on \( I \) with respect to Lebesgue’s measure \( dx \), provided with the standard norm \( \| g \|_{0,I} := \left( \int_I |g(x)|^2 \, dx \right)^{1/2} \) for \( g \in L^2(I) \). The standard inner product in \( L^2(I) \) that produces the norm \( \| \cdot \|_{0,I} \) is written as \( ( \cdot, \cdot )_{0,I} \), i.e., \( (g_1, g_2)_{0,I} := \int_I g_1(x) g_2(x) \, dx \) for \( g_1, g_2 \in L^2(I) \). Let \( \mathbb{N}_0 \) be the set of the nonnegative integers. For \( s \in \mathbb{N}_0 \), \( H^s(I) \) will be the Sobolev space of functions having generalized derivatives up to order \( s \) in the space \( L^2(I) \), and \( (\cdot, \cdot)_{s,I} \) its usual norm, i.e. \( \| g \|_{s,I} := \left( \sum_{\ell=0}^{s} \| \partial^{\ell} g \|_{0,I}^2 \right)^{1/2} \) for \( g \in H^s(I) \). Also, by \( H_0^1(I) \) we denote the subspace of \( H^1(I) \) consisting of functions which vanish at the endpoints of \( I \) in the sense of trace. We note that in \( H_0^1(I) \) the, well-known, Poincaré-Friedrich inequality holds, i.e., there exists a nonegative constant \( C_{PF} \) such that

\[
\| g \|_{0,I} \leq C_{PF} \| \partial g \|_{0,I} \quad \forall g \in H_0^1(I).
\]

The sequence of pairs \( ( (\lambda_k^2, \varepsilon_k) )_{k=1}^{\infty} \) is a solution to the eigenvalue/eigenfunction problem: find nonzero \( \varphi \in H^2(D) \cap H_0^1(D) \) and \( \sigma \in \mathbb{R} \) such that \( -\partial^2 \varphi = \sigma \varphi \) in \( D \). Since \( (\varepsilon_k)_{k=1}^{\infty} \) is a complete \((\cdot, \cdot)_{0,D}-\)orthonormal system in \( L^2(D) \), for \( s \in \mathbb{R} \), a subspace \( V^s(D) \) of \( L^2(D) \) is defined by

\[
V^s(D) := \left\{ v \in L^2(D) : \sum_{k=1}^{\infty} \lambda_k^2 (v, \varphi_k)_{0,D}^2 < \infty \right\}
\]

which is provided with the norm \( \| v \|_{V^s} := \left( \sum_{k=1}^{\infty} \lambda_k^2 (v, \varphi_k)_{0,D}^2 \right)^{1/2} \) \( \forall v \in V^s(D) \). For \( s \geq 0 \), the pair \((V^s(D), \| \cdot \|_{V^s})\) is a complete subspace of \( L^2(D) \) and we set \((\hat{V}^s(D), \| \cdot \|_{\hat{V}^s}) := (V^s(D), \| \cdot \|_{V^s})\). For \( s < 0 \), we define \((\hat{V}^s(D), \| \cdot \|_{\hat{V}^s})\) as the completion of \((V^s(D), \| \cdot \|_{V^s})\), or, equivalently, as the dual of \((\hat{V}^s(D), \| \cdot \|_{\hat{V}^s})\). Let \( m \in \mathbb{N}_0 \). It is well-known (see [21]) that

\[
\hat{H}^m(D) = \left\{ v \in H^m(D) : \partial^2 v \big|_{\partial D} = 0 \quad \text{if} \quad 0 \leq i < \frac{m}{2} \right\}
\]

and there exist positive constants \( C_{m,a} \) and \( C_{m,b} \) such that

\[
C_{m,a} \| v \|_{m,a} \leq \| v \|_{H^m} \leq C_{m,b} \| v \|_{m,b}, \quad \forall v \in \hat{H}^m(D).
\]

Also, we define on \( L^2(D) \) the negative norm \( \| \cdot \|_{a,D} \) by

\[
\| v \|_{-m,D} := \sup \left\{ \frac{(v, \varphi)_{0,D}}{\| \varphi \|_{m,D}} : \varphi \in \hat{H}^m(D) \quad \text{and} \quad \varphi \neq 0 \right\}, \quad \forall v \in L^2(D),
\]

for which, using (2.3), it is easy to conclude that there exists a constant \( C_{-m} > 0 \) such that

\[
\| v \|_{-m,D} \leq C_{-m} \| v \|_{H^{-m}}, \quad \forall v \in L^2(D).
\]

Let \( L_2 = (L^2(D), (\cdot, \cdot)_{0,D}) \) and \( L(L_2) \) be the space of linear, bounded operators from \( L_2 \) to \( L_2 \). We say that, an operator \( \Gamma \in L(L_2) \) is Hilbert-Schmidt, when \( \| \Gamma \|_{HS} := \left( \sum_{k=1}^{\infty} \| \Gamma (\varepsilon_k) \|_{0,D}^2 \right)^{1/2} < +\infty \), where \( \| \Gamma \|_{HS} \) is the so called Hilbert-Schmidt norm of \( \Gamma \). We note that the quantity \( \| \Gamma \|_{HS} \) does not change when we replace \( (\varepsilon_k)_{k=1}^{\infty} \) by another complete orthonormal system of \( L_2 \), as it is the sequence \( (\varphi_k)_{k=0}^{\infty} \) with \( \varphi_0(z) := 1 \) and \( \varphi_k(x) := \sqrt{2} \cos(k \pi z) \) for \( k \in \mathbb{N} \) and \( z \in \mathbb{T} \). It is well known (see, e.g., [7]) that an operator \( \Gamma \in L(L_2) \) is Hilbert-Schmidt iff there exists a measurable function \( g : D \times D \to \mathbb{R} \) such that \( (\Gamma(v))(\cdot) = \int_D g(\cdot, y) v(y) \, dy \) for \( v \in L^2(D) \), and then, it holds that

\[
\| \Gamma \|_{HS} = \left( \int_D \int_D g^2(x, y) \, dxdy \right)^{1/4}.
\]
Let $\mathcal{L}_{\text{hs}}(\mathbb{L}_2)$ be the set of Hilbert Schmidt operators of $\mathcal{L}(\mathbb{L}^2)$ and $\Phi : [0, T] \rightarrow \mathcal{L}_{\text{hs}}(\mathbb{L}_2)$. Also, for a random variable $X$, let $E[X]$ be its expected value, i.e., $E[X] := \int_\Omega X dP$. Then, the Itô isometry property for stochastic integrals, which we will use often in the paper, reads

$$
(2.6) \quad E\left[\left\| \int_0^T \Phi(t) dW \right\|_{\mathbf{H}_D}^2\right] = \int_0^T \|\Phi(t)\|_{\mathbf{H}_D}^2 dt.
$$

Let $\hat{\Pi} : L^2((0, T) \times D) \rightarrow L^2((0, T) \times D)$ be a projection operator defined by

$$
(2.7) \quad \hat{\Pi} g(t, x) := \frac{1}{N^4} \sum_{n=1}^N \sum_{i=1}^{J_n-1} \sum_{\ell=1}^{J_{\ell+1}} G_{i,\ell}^{-1} \int_{T_n \times D} g(s, y) \psi_{\ell}(y) dsdy \psi_i(x), \quad \forall \ (t, x) \in T_n \times D,
$$

for $n = 1, \ldots, N_*$ and for $g \in L^2((0, T) \times D)$, for which holds that

$$
(2.8) \quad \left( \int_0^T \int_D (\hat{\Pi} g)^2 dx dt \right)^{\frac{1}{2}} \leq \left( \int_0^T \int_D g^2 dx dt \right)^{\frac{1}{2}}, \quad \forall \ g \in L^2((0, T) \times D).
$$

Now, in the lemma below, we relate the stochastic integral of the projection $\hat{\Pi}$ of a deterministic function to its space-time $L^2-$inner product with the discrete space-time white noise kernel $\hat{W}$ defined in Section 1.1 (cf. Lemma 2.1 in [14]).

**Lemma 2.1.** For $g \in L^2((0, T) \times D)$, it holds that

$$
(2.9) \quad \int_0^T \int_D \hat{\Pi} g(t, x) dW(t, x) = \int_0^T \int_D \hat{W}(s, y) g(s, y) dsdy.
$$

**Proof.** To obtain (2.9) we work, using (2.7) and the properties of the stochastic integral, as follows:

$$
\begin{align*}
\int_0^T \int_D \hat{\Pi} g(t, x) dW(t, x) & = \int_0^T \int_D \left( \hat{\Pi} g \right)^2 dW(t, x) \\
& = \frac{1}{N^4} \sum_{n=1}^N \sum_{i=1}^{J_n-1} \sum_{\ell=1}^{J_{\ell+1}} G_{i,\ell}^{-1} \int_{T_n \times D} \left( \sum_{i=1}^{J_n-1} \sum_{\ell=1}^{J_{\ell+1}} G_{i,\ell}^{-1} \psi_i(x) \psi_{\ell}(y) \right) dsdy \\
& = \frac{1}{N^4} \sum_{n=1}^N \int_{T_n \times D} \hat{\Pi} g(s, y) \left( \sum_{i=1}^{J_n-1} \sum_{\ell=1}^{J_{\ell+1}} G_{i,\ell}^{-1} \psi_i(x) \psi_{\ell}(y) \right) dsdy \\
& = \int_0^T \int_D \hat{W}(s, y) g(s, y) dsdy.
\end{align*}
$$

We close this section by observing that: if $c_* > 0$, then

$$
(2.10) \quad \sum_{k=1}^{\infty} \lambda_k^{-1} (1+c_* \epsilon) \leq \left( \frac{1+2c_*}{c_* \epsilon^2} \right) \frac{1}{\epsilon}, \quad \forall \epsilon \in (0, 2],
$$

and if $(\mathcal{H}, (\cdot, \cdot)_\mathcal{H})$ is a real inner product space, then

$$
(2.11) \quad (g - v, g)_\mathcal{H} \geq \frac{1}{2} \left[ (g, g)_\mathcal{H} - (v, v)_\mathcal{H} \right], \quad \forall \ g, v \in \mathcal{H}.
$$

**2.2. Linear elliptic and parabolic operators.** Let us define the elliptic differential operators $\Lambda_B, \tilde{\Lambda}_B : \dot{\mathbf{H}}^4(D) \rightarrow L^2(D)$ by $\Lambda_B v := \partial^4 v + \mu \partial^2 v$ and $\tilde{\Lambda}_B v := \Lambda_B v + \mu^2 v$ for $v \in \dot{\mathbf{H}}^4(D)$, and consider the corresponding Dirichlet fourth-order two-point boundary value problems: given $f \in L^2(D)$ find $v_B, \tilde{v}_B \in \dot{\mathbf{H}}^4(D)$ such that

$$
(2.12) \quad \Lambda_B v_B = f \quad \text{in} \ D
$$

and

$$
(2.13) \quad \tilde{\Lambda}_B \tilde{v}_B = f \quad \text{in} \ D.
$$
Assumption (2.12) yields that when \( \kappa = 1 \) or \( \kappa \geq 2 \) and \( \mu \neq \lambda_{s-1}^2 \), the operator \( \Lambda_B \) is invertible and thus the problem (2.13) is well-posed. However, the problem (2.13) is always well-posed. Letting \( T_B, \tilde{T}_B : L^2(D) \to \hat{H}^2(D) \) be the solution operator of (2.12) and (2.13), respectively, i.e. \( T_B f := \Lambda_B^{-1} f = v_B \) and \( \tilde{T}_B f := \tilde{\Lambda}_B^{-1} f = \tilde{v}_B \), it is easy to verify that

\[
T_B f = \sum_{k=1}^{\infty} \frac{(\varepsilon_k f)_{m,r}}{\lambda_k^2(\lambda_k^2 - \mu)} \cdot \varepsilon_k \quad \text{and} \quad \tilde{T}_B f = \sum_{k=1}^{\infty} \frac{(\varepsilon_k f)_{m,r}}{\lambda_k^2(\lambda_k^2 - \mu) + \mu^2} \cdot \varepsilon_k, \quad \forall f \in L^2(D),
\]

and

\[
\|T_B f\|_{m,D} + \|\tilde{T}_B f\|_{m,D} \leq C_{r,m} \|f\|_{m-4,D}, \quad \forall f \in H^{m+4,m-4}(D), \quad \forall m \in \mathbb{N}_0,
\]

where \( C_{r,m} \) is a positive constant which depends on \( f \) but depends on the \( D \) and \( m \). Observing that

\[
(\tilde{T}_B v_1, v_2)_{0,D} = (v_1, \tilde{T}_B v_2)_{0,D}, \quad \forall v_1, v_2 \in L^2(D),
\]

and in view (2.14), the map \( \tilde{\gamma}_B : L^2(D) \times L^2(D) \to \mathbb{R} \) defined by

\[
\tilde{\gamma}_B(v,w) = (\tilde{T}_B v, w)_{0,D} \quad \forall v, w \in L^2(D),
\]

is an inner product on \( L^2(D) \).

Let \( (S(t)w_0)_{t \in [0, T]} \) be the standard semigroup notation for the solution \( w \) of (1.3). Then, the following a priori bounds hold (see Appendix A): for \( t \in \mathbb{N}_0 \), \( \beta \geq 0 \) and \( p \geq 0 \), there exists a constant \( C_{\beta,\ell,p,\mu_T} > 0 \) such that:

\[
\int_t^{t_b} (\tau - t_a)^{\beta} \|\partial_t^i S(\tau)w_0\|^2_{H^p} d\tau \leq C_{\beta,\ell,p,\mu_T} \|w_0\|^2_{H^{p+4\ell-2\beta-2}} \quad \forall w_0 \in H^{p+4\ell-2\beta}(D) \quad \text{and} \quad t_a, t_b \in [0, T] \quad \text{with} \quad t_b > t_a.
\]

2.3. Discrete spaces and operators. For \( r \in \{2, 3\} \), let \( M_h^r \subset H^1_0(D) \cap H^r(D) \) be a finite element space consisting of functions which are piecewise polynomials of degree at most \( r \) over a partition of \( D \) in intervals with maximum mesh-length \( h \). It is well-known (cf., e.g., [5]) that the following approximation property holds:

\[
\inf_{\chi \in M_h^r} \|v - \chi\|_{2,D} \leq C_{FM,r} h^{s-1}\|v\|_{s+1,D}, \quad \forall v \in H^{s+1}(D) \cap H^1(D), \quad \forall s \in \{2, 3\},
\]

where \( C_{FM,r} \) is a positive constant that depends on \( r \) and is independent of \( h \) and \( v \). Then, we define the discrete elliptic operators \( \Lambda_{B,h}, \tilde{\Lambda}_{B,h} : M_h^r \to M_h^r \) by

\[
(\Lambda_{B,h}^\varphi, \chi)_{0,D} := (\partial^2 \varphi, \partial^2 \chi)_{0,D} + \mu (\partial^2 \varphi, \chi)_{0,D}, \quad \forall \varphi, \chi \in M_h^r,
\]

and

\[
\tilde{\Lambda}_{B,h}^\varphi := \Lambda_{B,h}^\varphi + \mu^2 \varphi, \quad \forall \varphi \in M_h^r.
\]

Also, let \( P_h : L^2(D) \to M_h^r \) be the usual \( L^2(D) \)-projection operator onto \( M_h^r \) for which it holds that

\[
(P_h f, \chi)_{0,D} = (f, \chi)_{0,D}, \quad \forall \chi \in M_h^r, \quad \forall f \in L^2(D),
\]

A finite element approximation \( \tilde{v}_{B,h} \in M_h^r \) of the solution \( \tilde{v}_B \) of (2.13) is defined by the requirement

\[
\tilde{\Lambda}_{B,h} \tilde{v}_{B,h} = P_h f,
\]

where the operator \( \tilde{\Lambda}_{B,h} \) is invertible since

\[
(\tilde{\Lambda}_{B,h} \chi, \chi)_{0,D} \geq \frac{r^2}{2} \left( \|\partial^2 \chi\|^2_{0,D} + \mu^2 \|\chi\|^2_{0,D} \right), \quad \forall \chi \in M_h^r.
\]

Thus, we denote by \( \tilde{T}_{B,h} : L^2(D) \to M_h^r \) the solution operator of (2.20), i.e.

\[
\tilde{T}_{B,h} f := \tilde{v}_{B,h} = \tilde{\Lambda}_{B,h}^{-1} P_h f, \quad \forall f \in L^2(D).
\]

Next, we derive an \( L^2(D) \) error estimate for the finite element method (2.20).
Proposition 2.1. Let \( r \in \{2, 3\} \). Then we have

\[
\| \widetilde{T}_B f - \widetilde{T}_{B,h} f \|_{0,D} \leq C \left\{ \begin{array}{ll}
h^4 \| f \|_{0,D}, & r = 3, \\
h^3 \| f \|_{-1,D}, & r = 3, \\
h^2 \| f \|_{-1,D}, & r = 2,
\end{array} \right.
\tag{2.22}
\]

where \( C \) is a positive constant independent of \( h \) and \( f \).

Proof. Let \( f \in L^2(D) \), \( e = \widetilde{T}_B f - \widetilde{T}_{B,h} f \) and \( \widetilde{v} = \widetilde{T}_B e \). To simplify the notation we define \( B : H^2(D) \times H^2(D) \rightarrow \mathbb{R} \) by \( B(v,w) := \langle \partial^2 v, \partial^2 w \rangle_{o,D} + \mu (\partial^2 v, w)_{o,D} + \mu^2 (v, w)_{o,D} \) for \( v, w \in H^2(D) \). It is easily seen that

\[
B(v,w) \leq \sqrt{2} (1 + \mu) \left( \| \partial^2 v \|_{0,D}^2 + \mu^2 \| v \|_{0,D}^2 \right)^{\frac{1}{2}} \| w \|_{2,D} \quad \forall v, w \in H^2(D),
\tag{2.23}
\]

Later in the proof we shall use the symbol \( C \) for a generic constant that is independent of \( h \) and \( f \), and may changes value from one line to the other.

First, we observe that \( \| e \|_{0,D}^2 = B(e, \widetilde{v}) \). Then, we use the Galerkin orthogonality to get

\[
\| e \|_{0,D}^2 = B(e, \widetilde{v} - \chi), \quad \forall \chi \in M^r_h,
\]

which, along with (2.23), leads to

\[
\| e \|_{0,D}^2 \leq C \left( \| \partial^2 e \|_{0,D}^2 + \mu^2 \| e \|_{0,D}^2 \right)^{\frac{1}{2}} \inf_{\chi \in M^r_h} \| \widetilde{v} - \chi \|_{2,D}.
\tag{2.24}
\]

Using again (2.23) and the Galerkin orthogonality, we obtain

\[
\| \partial^2 e \|_{0,D}^2 + \mu^2 \| e \|_{0,D}^2 \leq 2 B(e, e)
\leq 2 B(e, \widetilde{T}_B f - \chi)
\leq C \left( \| \partial^2 e \|_{0,D}^2 + \mu^2 \| e \|_{0,D}^2 \right)^{\frac{1}{2}} \| \widetilde{T}_B f - \chi \|_{2,D}, \quad \forall \chi \in M^r_h,
\]

which yields that

\[
\| e \|_{0,D}^2 \leq C \inf_{\chi \in M^r_h} \| \widetilde{T}_B f - \chi \|_{2,D}.
\tag{2.25}
\]

Combining (2.24), (2.25) and (2.17), we arrive at

\[
\| e \|_{0,D}^2 \leq C \inf_{\chi \in M^r_h} \| \widetilde{T}_B f - \chi \|_{2,D} \inf_{\chi \in M^r_h} \| \widetilde{v} - \chi \|_{2,D}
\leq C h^{s+s'-2} \| \widetilde{T}_B f \|_{s+1,D} \| \widetilde{T}_B e \|_{s'+1,D}, \quad \forall s, s' \in \{2, r\}.
\tag{2.26}
\]

Let \( r = 2 \). We use (2.26) and (2.15) to get

\[
\| e \|_{0,D}^2 \leq C h^2 \| \widetilde{T}_B f \|_{3,D} \| \widetilde{T}_B e \|_{3,D}
\leq C h^2 \| f \|_{-1,D} \| e \|_{-1,D}
\leq C h^2 \| f \|_{-1,D} \| e \|_{0,D},
\]

from which we conclude (2.22) for \( r = 2 \).

Let \( r = 3 \). We use (2.26) with \( s' = 3 \) and (2.15) to obtain

\[
\| e \|_{0,D}^2 \leq C h^{s+1} \| \widetilde{T}_B f \|_{s+1,D} \| \widetilde{T}_B e \|_{4,D}
\leq C h^{s+1} \| f \|_{-3,D} \| e \|_{0,D}, \quad s = 2, 3,
\]

from which we conclude (2.22) for \( r = 3 \).
Let $\tilde{\gamma}_{b,h} : L^2(D) \times L^2(D) \to \mathbb{R}$ be defined by

$$\tilde{\gamma}_{b,h}(f,g) = \langle \tilde{T}_{b,h}f, g \rangle_{0,D} \quad \forall f, g \in L^2(D).$$

Then, as a simple consequence of (2.21), the following inequality holds

$$(2.27) \quad \tilde{\gamma}_{b,h}(f,f) \geq \frac{1}{2} \left( \| \partial^2 \tilde{T}_{b,h}f \|^2_{0,D} + \mu^2 \| \tilde{T}_{b,h}f \|^2_{0,D} \right), \quad \forall f \in L^2(D).$$

Thus, observing that

$$(2.28) \quad \tilde{\gamma}_{b,h}(f,f) \leq C \| f \|^2_{-2,D}, \quad \forall f \in L^2(D).$$

**Proof.** Let $f \in L^2(D)$, $\psi = \tilde{T}_{b}f$ and $\psi_h = \tilde{T}_{b,h}f$. Then, we have

$$(2.29) \quad (\tilde{T}_{b,h}f, f)_{0,D} = (\tilde{\Lambda}_b \psi, \psi)_h$$

$$= (\tilde{T}_{b,h}f, f)_{0,D} + \mu (\tilde{T}_{b,h}f, \psi_h)_{0,D} + \mu^2 \langle \psi_h, \psi \rangle_h$$

$$\leq \frac{1}{2} \left( \| \partial^2 \tilde{T}_{b,h}f \|^2_{0,D} + \mu^2 \| \tilde{T}_{b,h}f \|^2_{0,D} \right) + \varepsilon \left( \| \partial^2 \psi_h \|^2_{0,D} + \mu^2 \| \psi_h \|^2_{0,D} \right), \quad \forall \varepsilon > 0.$$

Setting $\varepsilon = \frac{1}{4}$ in (2.24) and then combining it with (2.27), we obtain

$$(2.30) \quad \| \partial^2 \psi_h \|^2_{0,D} + \mu^2 \| \psi_h \|^2_{0,D} \leq 16 \left( \| \partial^2 \psi \|^2_{0,D} + \mu^2 \| \psi \|^2_{0,D} \right).$$

Finally, (2.29) with $\varepsilon = \frac{1}{2}$, (2.30) and (2.15) yield

$$\tilde{\gamma}_{b,h}(f,f) \leq 8 \left( \| \partial^2 \psi \|^2_{0,D} + \mu^2 \| \psi \|^2_{0,D} \right)$$

$$\leq 8 \left( 1 + \mu^2 \right) \| \tilde{T}_{b}f \|^2_{2,D}$$

$$\leq 8 \left( 1 + \mu^2 \right) C_{R,D} \| f \|^2_{-2,D}.$$

Thus, we arrived at (2.28). \hfill \Box

### 3. An Estimate for the Modeling Error

In this section, we estimate the modeling error in terms of $\Delta t$ and $\Delta x$ (cf. Theorem 3.1 in [14]).

**Theorem 3.1.** Let $u$ be the solution of (1.1) and $\tilde{u}$ be the solution of (1.6). Then, there exists a real constant $C > 0$, independent of $\Delta t$ and $\Delta x$, such that

$$(3.1) \quad \max_{[0,T]} \mathbb{E} \left[ \left\| u - \tilde{u} \right\|^2_{0,D} \right] \leq C \left( \omega_0(\Delta t) \Delta t \frac{1}{2} + e^{-\frac{1}{2} \Delta x} \right), \quad \forall \epsilon \in (0, \frac{1}{2}],$$

where $\omega_0(\Delta t) := \sqrt{1 + \Delta t^2}$.

**Proof.** Using (1.3), (1.7) and Lemma 2.1, we conclude that

$$(3.2) \quad u(t,x) - \tilde{u}(t,x) = \int_0^T \int_D \left[ X_{(t,x)}(s) \Psi(t; s; x, y) - \tilde{\Psi}(t; x, y) \right] dW(s, y), \quad \forall (t, x) \in [0, T] \times \bar{D},$$

where $\tilde{\Psi} : (0, T) \times D \to L^2((0, T) \times D)$ is given by

$$\tilde{\Psi}(t; x, y) := \frac{1}{\Delta T} \int_{T_n} \int_{T_n} X_{(t,x)}(s') \left[ \sum_{i=1}^{j_n} \psi_i(y) \sum_{t=1}^{j_n} G_{t,k}^{-1} \int_D \Psi(t; s'; x, y') \psi_i(y') dy' \right] ds', \quad \forall (s, y) \in T_n \times D,$$

for $n = 1, \ldots, N_*$. Let $\Theta := \mathbb{E} \left[ \left\| u - \tilde{u} \right\|^2_{0,D} \right]^{\frac{1}{2}}$ and $t \in (0, T)$. Using (3.2) and Itô isometry (2.6), we obtain

$$\Theta(t) = \left\{ \int_0^T \int_D \left[ X_{(t,x)}(s) \Psi(t; s; x, y) - \tilde{\Psi}(t; x, y) \right]^2 dx dy ds \right\}^{\frac{1}{2}}.$$
Now, we introduce the splitting
\begin{equation}
\Theta(t) \leq \Theta_A(t) + \Theta_B(t),
\end{equation}
where
\begin{align*}
\Theta_A(t) := & \left\{ \sum_{n=1}^{N_1} \int_D \int_D \int_{T_n} \left[ \frac{1}{\Delta t} \int_{T_n} \mathcal{X}_{(0,t)}(s') \Psi(t-s'; x, y) ds' - \Psi(t; x, y) \right] dx dy ds \right\}^{\frac{1}{2}} \\
\Theta_B(t) := & \left\{ \sum_{n=1}^{N_1} \int_D \int_D \int_{T_n} \mathcal{X}_{(0,t)}(s) \Psi(t-s; x, y) - \frac{1}{\Delta t} \int_{T_n} \mathcal{X}_{(0,t)}(s') \Psi(t-s'; x, y) ds' \right\}^{\frac{1}{2}} dx dy ds.
\end{align*}

Also, to simplify the notation in the rest of the proof, we set \( \mu_k := \lambda^2_k (\lambda^2_k - \mu) \) for \( k \in \mathbb{N} \), and use the symbol \( C \) to denote a generic constant that is independent of \( \Delta t \) and \( \Delta x \) and may change value from one line to the other.

- **Estimation of \( \Theta_A(t) \):** Using (1.4) and the \((\cdot, \cdot)_{0,D} - \)orthogonality of \((\varepsilon_k)_{k=1}^\infty\), we have
\begin{align*}
\Theta_A^2(t) &= \frac{1}{\Delta t} \sum_{n=1}^{N_1} \int_D \int_D \int_{T_n} \mathcal{X}_{(0,t)}(s') \left[ \Psi(t-s'; x, y) - \sum_{\ell, i=1}^{j+1} G_{i,\ell}^{-1} (\Psi(t-s'; x, \cdot), \psi_{\ell}(\cdot))_{0,D} \psi_i(y) \right] ds' \right) ^2 dy dx \\
&= \frac{1}{\Delta t} \sum_{n=1}^{N_1} \sum_{k=1}^\infty \left( \int_D \mathcal{X}_{(0,t)}(s) e^{-\mu_k (t-s)} ds \right)^2 \int_D \left( \varepsilon_k(y) - \sum_{\ell, i=1}^{j+1} G_{i,\ell}^{-1} (\varepsilon_{\ell}'(x), \psi_{\ell}(x))_{0,D} \psi_i(y) \right)^2 dy
\end{align*}
from which, using the Cauchy-Schwarz inequality, follows that
\begin{equation}
\Theta_A^2(t) \leq \sum_{k=1}^\kappa A_k(t) B_k + \sum_{k=\kappa+1}^\infty A_k(t) B_k,
\end{equation}
where
\begin{align*}
A_k(t) &:= 2 \lambda^2_k \int_0^t e^{-2\mu_k (t-s')} ds', \\
B_k &:= \int_D \left( \varphi_k(y) - \sum_{\ell, i=1}^{j+1} G_{i,\ell}^{-1} (\varphi_{\ell}(x), \psi_{\ell}(x))_{0,D} \psi_i(y) \right)^2 dy.
\end{align*}

First, we observe that
\begin{align*}
\sqrt{B_k} &\leq \max_{1 \leq j \leq j'} \sup_{x, y \in D_j} | \varphi_k(x) - \varphi_k(y) | \\
&\leq \min \{ 1, \lambda_k \Delta x \} \\
&\leq \min \left\{ 1, (\sqrt{2} \lambda_k \Delta x)^\theta \right\}, \quad \forall \theta \in [0, 1], \quad \forall k \in \mathbb{N}.
\end{align*}

Next, we use (1.2), to obtain
\begin{align*}
A_k(t) &\leq \frac{1-e^{-2\mu_k t}}{\lambda^2_k - \mu} \\
&\leq \frac{(\kappa+1)^2}{1+2\kappa} \frac{1}{\lambda^2_k}, \quad \forall k \geq \kappa + 1.
\end{align*}
Thus, from (3.4), (3.5) and (3.6), we conclude that
\begin{equation}
\Theta_A^2(t) \leq C \left( (\Delta x)^2 \sum_{k=1}^\kappa \lambda^2_k + (\Delta x)^2 \sum_{k=\kappa+1}^\infty \frac{1}{\lambda^2_k} \right)^{\frac{1}{2}}
\end{equation}
which yields
\begin{equation}
\Theta_A(t) \leq C (\Delta x) \theta \left( \sum_{k=1}^\infty \frac{1}{\lambda^2_k} \right)^{\frac{1}{2}}, \quad \forall \theta \in [0, \frac{1}{2}).
\end{equation}
• Estimation of $\Theta_\beta(t)$: For $t \in (0, T]$, let $\hat{N}(t) := \min \{ \ell \in \mathbb{N} : 1 \leq \ell \leq N_{\ast} \text{ and } t \leq t_\ell \}$ and

$$\hat{T}_n(t) := T_n \cap (0, t) = \begin{cases} T_n, & \text{if } n < \hat{N}(t), \\ (t_{\hat{N}(t)-1}, t), & \text{if } n = \hat{N}(t), \end{cases} n = 1, \ldots, \hat{N}(t).$$

Thus, using (1.4) and the $(\cdot, \cdot)_{a, D}$-orthogonality of $(\varepsilon_k)_{k=1}^\infty$ and $(\varphi_k)_{k=1}^\infty$ as follows

$$\Theta^2_n(t) = \frac{1}{(\Delta t)^2} \sum_{n=1}^{N_{\ast}} \int_D \int_D \int_{T_n} \left[ \int_{T_n} \left[ X_{(0,t)}(s) \Psi(t-s; x, y) \right. \right. - \left. \left. \int_{T_n} X_{(0,t)}(s') e^{-\mu_k(t-s')} \right] ds' \left. \right] dx dy ds \right]^2 dxdyds$$

$$\Theta^2_n(t) = \frac{1}{(\Delta t)^2} \sum_{n=1}^{N_{\ast}} \int_D \int_D \int_{T_n} \left[ \int_{T_n} \left[ X_{(0,t)}(s) \right. \right. - \left. \left. \int_{T_n} X_{(0,t)}(s') e^{-\mu_k(t-s')} \right] ds' \right] dx dy ds \right]^2 dxdyds$$

we conclude that

$$\Theta^2_n(t) \leq \sum_{k=1}^{\infty} \lambda^2_k \left( \frac{1}{(\Delta t)^2} \sum_{n=1}^{\hat{N}(t)} \Psi^2_k(t) \right),$$

where

$$\Psi^k_n(t) := \int_{T_n} \left[ \int_{T_n} \left[ X_{(0,t)}(s) e^{-\mu_k(t-s')} \right. \right. - \left. \left. \int_{T_n} X_{(0,t)}(s') e^{-\mu_k(t-s')} \right] ds' \right] dx dy ds \right]^2 ds.$$

Let $k \in \mathbb{N}$ and $n \in \{1, \ldots, \hat{N}(t) - 1\}$. Then, we have

$$\Psi^k_n(t) = \int_{T_n} \left( \int_{T_n} \int_{s'} \mu_k e^{-\mu_k(t-s)} d\tau ds' \right)^2 ds$$

$$\leq \int_{T_n} \left( \int_{T_n} \int_{t_{n-1}}^{\max\{s', s\}} \mu_k e^{-\mu_k(t-s)} d\tau ds' \right)^2 ds$$

$$\leq 2 \int_{T_n} \left( \int_{T_n} \int_{t_{n-1}}^{s'} \mu_k e^{-\mu_k(t-s)} d\tau ds' \right)^2 ds + 2 \int_{T_n} \left( \int_{T_n} \int_{t_{n-1}}^{s} \mu_k e^{-\mu_k(t-s)} d\tau ds' \right)^2 ds$$

$$\leq 2 \Delta t \left( \int_{T_n} \int_{t_{n-1}}^{s'} \mu_k e^{-\mu_k(t-s)} d\tau ds' \right)^2 + 2 (\Delta t)^2 \int_{T_n} \left( \int_{t_{n-1}}^{s} \mu_k e^{-\mu_k(t-s)} d\tau \right)^2 ds,$$

from which, after using the Cauchy-Schwarz inequality, we arrive at

$$\Psi^k_n(t) \leq 4 (\Delta t)^2 \int_{T_n} \left( \int_{t_{n-1}}^{s} \mu_k e^{-\mu_k(t-s)} d\tau \right)^2 ds.$$
Considering, now, the case $n = \hat{N}(t)$, we have

\begin{equation}
(3.13) \quad \Psi_{\hat{N}(t)}^k(t) = \Psi_A^k(t) + \Psi_B^k(t)
\end{equation}

with

\begin{align*}
\Psi_A^k(t) & := \int_{\hat{N}(t)-1}^t \left( \int_{\hat{N}(t)-1}^s \mu_k e^{-\mu_k(t-\tau)} d\tau ds' + \int_{t}^{\hat{N}(t)} e^{-\mu_k(t-s')} ds' \right)^2 ds, \\
\Psi_B^k(t) & := \int_{\hat{N}(t)}^t \left( \int_{\hat{N}(t)-1}^t e^{-\mu_k(t-s')} ds' \right)^2 ds.
\end{align*}

For $k \leq \kappa$, we obtain

\begin{equation}
(3.14) \quad \frac{1}{(\Delta t)^2} \Psi_{\hat{N}(t)}^k(t) \leq C \Delta t.
\end{equation}

For $k \geq \kappa + 1$, we have

\begin{align*}
\Psi_B^k(t) & \leq \frac{\Delta t}{\mu_k} \left[ 1 - e^{-\mu_k(t-\hat{N}(t)-1)} \right]^2 \\
& \leq \frac{\Delta t}{\mu_k} (1 - e^{-\mu_k \Delta t})^2
\end{align*}

and

\begin{align*}
\Psi_A^k(t) & \leq \int_{\hat{N}(t)-1}^t \left[ \int_{\hat{N}(t)-1}^s \mu_k e^{-\mu_k(t-\tau)} d\tau ds' + \Delta t e^{-\mu_k(t-s)} \right]^2 ds \\
& \leq 2 \int_{\hat{N}(t)-1}^t \left[ \int_{\hat{N}(t)-1}^s \mu_k e^{-\mu_k(t-\tau)} d\tau ds' \right]^2 ds + \frac{(\Delta t)^2}{\mu_k} \left[ 1 - e^{-2\mu_k(t-\hat{N}(t)-1)} \right] \\
& \leq 2 \int_{\hat{N}(t)-1}^t \left[ \int_{\hat{N}(t)-1}^{\max\{s,s'\}} \mu_k e^{-\mu_k(t-\tau)} d\tau ds' \right]^2 ds + \frac{(\Delta t)^2}{\mu_k} (1 - e^{-2\mu_k \Delta t}) \\
& \leq 8 (\Delta t)^2 \int_{\hat{N}(t)-1}^t \left[ \int_{\hat{N}(t)-1}^s \mu_k e^{-\mu_k(t-\tau)} d\tau \right]^2 ds + \frac{(\Delta t)^2}{\mu_k} (1 - e^{-2\mu_k \Delta t}) \\
& \leq 8 (\Delta t)^2 \int_{N(t)-1}^t \left[ e^{-\mu_k(t-s)} - e^{-\mu_k(t-\hat{N}(t)-1)} \right]^2 ds + \frac{(\Delta t)^2}{\mu_k} (1 - e^{-2\mu_k \Delta t}),
\end{align*}

which, along with (3.13), gives

\[ \Psi_{\hat{N}(t)}^k(t) \leq \frac{5(\Delta t)^2}{\mu_k} (1 - e^{-2\mu_k \Delta t}) + \frac{\Delta t}{\mu_k} (1 - e^{-\mu_k \Delta t})^2. \]

Since the mean value theorem yields: $1 - e^{-\mu_k \Delta t} \leq \mu_k \Delta t$, the above inequality takes the form

\begin{equation}
(3.15) \quad \frac{1}{(\Delta t)^2} \Psi_{\hat{N}(t)}^k(t) \leq 6 \frac{1-e^{-2\mu_k \Delta t}}{\mu_k}.
\end{equation}

Combining (3.8), (3.12), (3.14) and (3.15) we obtain

\begin{align*}
\Theta_B^2(t) & \leq C \left[ \Delta t + \sum_{k=\kappa+1}^{\infty} \lambda_k^2 \frac{1-e^{-2\mu_k \Delta t}}{\mu_k} \right] \\
& \leq C \left[ \Delta t + \sum_{k=1}^{\infty} \frac{1-e^{-\mu_k \Delta t}}{\lambda_k^2} \right],
\end{align*}

(3.16)
with \( c_0 = \frac{2(1+2\kappa)}{(\kappa+1)^2} \). To get a convergence estimate we have to exploit the way the series depends on \( \Delta t \) in the above relation:

\[
\sum_{k=1}^{\infty} \frac{1-e^{-c_0 \lambda_k^2 \Delta t}}{\lambda_k^2} \leq \frac{1-e^{-c_0 \pi^4 \Delta t}}{\pi^4} + \int_1^{\infty} \frac{1-e^{-c_0 x^4 \Delta t}}{x^4} \, dx
\]

\[
\leq C \left( (1-e^{-c_0 \pi^4 \Delta t}) + \Delta t \int_1^{\infty} x^2 \, e^{-c_0 x^4 \Delta t} \, dx \right)
\]

\[
\leq C \left[ (\Delta t)^{\frac{3}{2}} + 1 \right] (\Delta t)^{\frac{1}{2}}.
\]

Using the bounds (3.10) and (3.17) we conclude that

\[
\Theta_B(t) \leq C \left[ (\Delta t)^{\frac{3}{4}} + 1 \right] (\Delta t)^{\frac{1}{4}}.
\]

The error bound (3.11) follows by observing that \( \Theta(0) = 0 \) and combining the bounds (3.1), (3.7), (3.17) and (2.10). \( \square \)

### 4. Time-Discrete Approximations

The Backward Euler time-stepping method for problem (1.6) specifies an approximation \( \hat{U}^m \) of \( \hat{u}(\tau_m, \cdot) \) starting by setting

\[
\hat{U}^0 := 0,
\]

and then, for \( m = 1, \ldots, M \), by finding \( \hat{U}^m \in \hat{H}^1(D) \) such that

\[
\hat{U}^m - \hat{U}^{m-1} + \Delta \tau \Lambda_B \hat{U}^m = \int_{\Delta_m} \partial_x \hat{W} \, ds \quad \text{a} \text{.s}..\]

The method is well-defined when the differential operator \( Q_{B, \Delta \tau} := I + \Delta \tau \Lambda_B : \hat{H}^1(D) \to L^2(D) \) is invertible. It is easily seen that \( Q_{B, \Delta \tau} \) is invertible when \( 1 + \Delta \tau \lambda_k^2 (\lambda_k^2 - \mu) \neq 0 \) for \( k \in \mathbb{N} \), or equivalently when: \( \kappa = 1 \) or \( \kappa \geq 2 \) and \( \Delta \tau \max_{1 \leq k \leq \kappa-1} \lambda_k^2 (\mu - \lambda_k^2) \neq 1 \). If \( \kappa \geq 2 \), then it is easily seen that \( \max_{1 \leq k \leq \kappa-1} \lambda_k^2 (\mu - \lambda_k^2) \leq \frac{4}{\kappa^2} \), so the condition \( \Delta \tau \frac{4}{\kappa^2} < 1 \) is a sufficient condition for the invertibility of \( Q_{B, \Delta \tau} \).

#### 4.1. The Deterministic Case.

The Backward Euler time-discrete approximations of the solution \( w \) to the deterministic problem (1.5) are defined as follows: first we set

\[
W^0 := w_0,
\]

and then, for \( m = 1, \ldots, M \), we find \( W^m \in \hat{H}^1(D) \) such that

\[
W^m - W^{m-1} + \Delta \tau \Lambda_B W^m = 0.
\]

Obviously, the Backward Euler time-discrete approximations are well-defined when \( Q_{B, \Delta \tau} \) is invertible. Our next step, is to derive an error estimate in a discrete in time \( L^2(D) \) norm, taking into account that, in contrast to the case \( \mu = 0 \) considered in [14], the operator \( \Lambda_B \) is not always invertible.

**Proposition 4.1.** Let \( (W^m)_{m=0}^M \) be the Backward Euler time-discrete approximations of the solution \( w \) of the problem (1.5) defined in (1.3) - (1.4). Also, we assume that \( \kappa = 1 \), or \( \kappa \geq 2 \) and \( \Delta \tau \mu^2 < \frac{1}{4} \). Then, there exists a constant \( C > 0 \), independent of \( \Delta \tau \), such that

\[
\left( \sum_{m=1}^{M} (\Delta \tau \| W^m - w(\tau_m, \cdot) \|_{3/2}^{2}) \right)^{\frac{1}{2}} \leq C (\Delta \tau)^{\theta} \| w_0 \|_{3/2}^{\theta - 2}, \quad \forall w_0 \in \hat{H}^2(D), \quad \forall \theta \in [0, 1].
\]
Proof. The estimate (4.5) will be established by interpolation, after proving it for $\theta = 1$ and $\theta = 0$.

Let $w_0 \in \dot{H}^2(D)$. According to the discussion in the beginning of this section, when $\kappa = 1$ or $\kappa \geq 2$ and $\Delta \tau \mu^2 < \frac{1}{4}$, the existence and uniqueness of the time-discrete approximations $(W^m)^{m=0}$ is secured. We omit the case $\kappa = 1$ since then the operator $\Lambda_B$ is invertible and the proof of (4.3) follows moving along the lines of the proof of Proposition 1.1 in [13], or alternatively moving along the lines of the proof below using the operator $T_B$ instead of $\bar{T}_B$. Here, we will proceed with the proof of (4.3) under the assumption $\Delta \tau \mu^2 < \frac{1}{4}$, without using somewhere a possible invertibility of $\Lambda_B$. In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$ and may changes value from one line to the other.

Let $E^m(\cdot) := w(\tau_m, \cdot) - V^m(\cdot)$ for $m = 0, \ldots, M$ and $\sigma_m := \int_{\Delta_m} [w(\tau_m, \cdot) - w(\tau, \cdot)] d\tau$ for $m = 1, \ldots, M$. Then, combining (4.9) and (4.4), we conclude that

$$\tilde{\gamma}_B(E^m - E^{m-1}) + \Delta \tau E^m = \Delta \tau \mu^2 \bar{T}_B E^m + \left( \sigma_m - \mu^2 \bar{T}_B \sigma_m \right), \quad m = 1, \ldots, M.$$  \hspace{1cm} (4.6)

Now, take the $L^2(D)$—inner product with $E^m$ of both sides of (4.6), to obtain

$$\tilde{\gamma}_B(E^m - E^{m-1}, E^m) + \Delta \tau \|E^m\|^2_{0,D} = \Delta \tau \mu^2 E^m - \gamma_B(E^m, E^m)$$

$$+ (\sigma_m - \mu^2 \bar{T}_B \sigma_m, E^m)_{0,D}, \quad m = 1, \ldots, M.$$ \hspace{1cm} (4.7)

Using (2.11), (4.7) and (2.15), we arrive at

$$\tilde{\gamma}_B(E^m, E^m) - \tilde{\gamma}_B(E^{m-1}, E^{m-1}) + \Delta \tau \|E^m\|^2_{0,D} \leq 2 \Delta \tau \mu^2 \tilde{\gamma}_B(E^m, E^m)$$

$$+ C \Delta \tau^{-1} \|\sigma_m\|^2_{0,D}, \quad m = 1, \ldots, M.$$ \hspace{1cm} (4.8)

Since $2 \Delta \tau \mu^2 < 1$, (4.8) yields

$$\tilde{\gamma}_B(E^m, E^m) \leq C \Delta \tau^{-1} \sum_{\ell = 1}^{m} \|\sigma\|_{0,D}^2 \frac{1}{(1 - 2 \Delta \tau \mu^2)^{\ell-1}},$$

$$\leq C e^{4\Delta \tau^2 \Delta \tau^{-1} \sum_{\ell = 1}^{m} \|\sigma\|_{0,D}^2}, \quad m = 1, \ldots, M.$$ \hspace{1cm} (4.9)

Next, we use the Cauchy-Schwarz inequality to bound $\sigma_m$ as follows:

$$\|\sigma_m\|_{0,D}^2 \leq C \int_B \left( \int_{\Delta_m} \int_{\Delta} |\partial_{x,x} w(s, x)| ds d\tau \right)^2 dx$$

$$\leq C \Delta \tau^{-1} \int_{\Delta} \|\partial_{x} w(s, \cdot)\|^2_{0,D} ds, \quad m = 1, \ldots, M.$$ \hspace{1cm} (4.10)

Thus, (4.10) and (4.9) yield

$$\tilde{\gamma}_B(E^m, E^m) \leq C \Delta \tau^2 \int_{0}^{T} \|\partial_{x} w(s, \cdot)\|^2_{0,D} ds, \quad m = 1, \ldots, M.$$ \hspace{1cm} (4.11)

Combining (4.8), (4.11) and (4.10), we have

$$\tilde{\gamma}_B(E^m, E^m) - \tilde{\gamma}_B(E^{m-1}, E^{m-1}) + \Delta \tau \|E^m\|^2_{0,D} \leq C \Delta \tau^2 \int_{0}^{T} \|\partial_{x} w(s, \cdot)\|^2_{0,D} ds$$

$$+ C \Delta \tau^3 \int_{0}^{T} \|\partial_{x} w(s, \cdot)\|^2_{0,D} ds$$ \hspace{1cm} (4.12)

for $m = 1, \ldots, M$. Summing with respect to $m$ from 1 up to $M$ and using the fact that $E^0 = 0$, (4.12) yields

$$\tilde{\gamma}_B(E^M, E^M) + \sum_{m=1}^{M} \Delta \tau \|E^m\|^2_{0,D} \leq C \Delta \tau^2 \int_{0}^{T} \|\partial_{x} w(s, \cdot)\|^2_{0,D} ds.$$ \hspace{1cm} (4.13)
Finally, use (4.13) and (2.16) (with $\beta = 0$, $\ell = 1$, $p = 0$) to obtain

$$
(4.14) \quad \left( \sum_{m=1}^{M} \Delta \tau \| E^m \|_{\omega,D}^2 \right)^{1/2} \leq C \Delta \tau \| w_0 \|_{H^2},
$$

which establishes (4.5) for $\theta = 1$.

First, we observe that (4.4) is written equivalently as

$$
\tilde{T}_B(W^m - W^{m-1}) + \Delta \tau W^m = \Delta \tau \mu^2 \tilde{T}_B W^m, \quad m = 1, \ldots, M,
$$

from which, after taking the $L^2(D)$--inner product with $W^m$, we obtain

$$
\tilde{\gamma}_B(W^m - W^{m-1}, W^m)_{\omega,D} + \Delta \tau \| W^m \|^2_{\omega,D} = \Delta \tau \mu^2 \tilde{\gamma}_B(W^m W^m), \quad m = 1, \ldots, M.
$$

Then, we combine (2.11) and (4.15) to have

$$
(4.16) \quad (1 - 2 \Delta \tau \mu^2) \tilde{\gamma}_B(W^m, W^m) + 2 \Delta \tau \| W^m \|^2_{\omega,D} \leq \tilde{\gamma}_B(W^{m-1}, W^{m-1}), \quad m = 1, \ldots, M.
$$

Since $4 \mu^2 \Delta \tau < 1$, (4.16) yields that

$$
\tilde{\gamma}_B(W^m, W^m) \leq \frac{1}{1 - 2 \mu^2 \Delta \tau} \tilde{\gamma}_B(W^{m-1}, W^{m-1})
\leq e^{4\mu^2 \Delta \tau} \tilde{\gamma}_B(W^{m-1}, W^{m-1}), \quad m = 1, \ldots, M,
$$

from which, applying a simple induction argument, we conclude that

$$
(4.17) \quad \max_{0 \leq m \leq M} \tilde{\gamma}_B(W^m, W^m) \leq C \tilde{\gamma}_B(w_0, w_0).
$$

Now, summing with respect to $m$ from 1 up to $M$, and using (4.17), (4.16) yields

$$
(4.18) \quad \sum_{m=1}^{M} \Delta \tau \| W^m \|^2_{\omega,D} \leq C \| \tilde{T}_B w_0, w_0 \|_{\omega,D}
\leq \| w_0 \|_{-2,D} \| \tilde{T}_B w_0 \|_{2,D}.
$$

Thus, using (4.18), (2.16) and (2.24), we obtain

$$
(4.19) \quad \left( \sum_{m=1}^{M} \Delta \tau \| W^m \|^2_{\omega,D} \right)^{1/2} \leq C \| w_0 \|_{-2,D}
\leq C \| w_0 \|_{H^{-2}}.
$$

In addition we have

$$
\sum_{m=1}^{M} \Delta \tau \| w(\tau_m, \cdot) \|^2_{\omega,D} \leq \sum_{m=1}^{M} \int_{\Omega} \left( \int_{\Delta_m} \partial_\tau \left[ (\tau - \tau_{m-1}) w^2(\tau, x) \right] d\tau \right) dx
\leq \sum_{m=1}^{M} \int_{\Omega} \left[ w^2(\tau, x) + 2 (\tau - \tau_{m-1}) w_\tau(\tau, x) w(\tau, x) \right] dx
\leq \sum_{m=1}^{M} \int_{\Delta_m} \left[ 2 \| w(\tau, \cdot) \|^2_{\omega,D} + (\tau - \tau_{m-1})^2 \| w_\tau(\tau, \cdot) \|^2_{\omega,D} \right] d\tau
\leq 2 \int_{0}^{T} \left[ \| w(\tau, \cdot) \|^2_{\omega,D} + \tau^2 \| w_\tau(\tau, \cdot) \|^2_{\omega,D} \right] d\tau,
$$

which, along with (2.16) (taking $(\beta, \ell, p) = (0, 0, 0)$ and $(\beta, \ell, p) = (2, 1, 0)$) and (2.24), yields

$$
(4.20) \quad \left( \sum_{m=1}^{M} \Delta \tau \| w(\tau_m, \cdot) \|^2_{\omega,D} \right)^{1/2} \leq C \| w_0 \|_{H^{-2}}.
$$

Thus, the estimate (4.5) for $\theta = 0$ follows easily combining (4.19) and (4.20). □
4.2. The Stochastic Case. Next theorem combines the convergence result of Proposition 4.1 with a discrete Duhamel’s principle in order to prove a discrete in time $L^\infty_t(L^2_x)$ convergence estimate for the time discrete approximations of $\hat{u}$ (cf. [14], [22]).

**Theorem 4.2.** Let $\hat{u}$ be the solution of (1.5) and $(\hat{U}^m)^*_m=0$ be the time-discrete approximations defined by (4.1) - (4.2). Also, we assume that $\kappa = 1$, or $\kappa \geq 2$ and $\Delta \tau \mu^2 < \frac{1}{4}$. Then, there exists a constant $C > 0$, independent of $\Delta t$, $\Delta x$ and $\Delta \tau$, such that

$$(4.21) \quad \max_{1 \leq m \leq M} \mathbb{E} \left[ \left\| \hat{U}^m - \hat{u}(\tau, \cdot) \right\|_{0, D}^2 \right]^{\frac{1}{2}} \leq C \omega_1(\Delta \tau, \epsilon) \Delta \tau^{-\epsilon}, \quad \forall \epsilon \in (0, \frac{1}{8}],$$

where $\omega_1(\Delta \tau, \epsilon) := \epsilon^{-\frac{1}{4}} + (\Delta \tau)^{c} (1 + (\Delta \tau)^{\frac{1}{4}} + (\Delta \tau)^{\frac{3}{4}})^{\frac{1}{4}}$.

**Proof.** Let $I : L^2(D) \rightarrow L^2(D)$ be the identity operator, $\Lambda : L^2(D) \rightarrow \hat{H}^4(D)$ be the inverse elliptic operator $\Lambda := (I + \Delta \tau \Lambda_0)^{-1}$ which has Green function $G_{\Lambda}(x, y) = \sum_{k=1}^{\infty} \frac{\gamma_k(x)\gamma_k(y)}{1 + \Delta \tau \lambda_k^2}$, i.e. $\Lambda f(x) = \int_D G_{\Lambda}(x, y)f(y) dy$ for $x \in D$ and $f \in L^2(D)$. Also, we set $G_{\phi}(x, y) := -\partial_y G_{\Lambda}(x, y) = -\sum_{k=1}^{\infty} \frac{\gamma_k(x)\gamma_k(y)}{1 + \Delta \tau \lambda_k^2}$, and define $\Phi : L^2(D) \rightarrow \hat{H}^4(D)$ by $\Phi(f)(x) := \int_D G_{\Lambda}(x, y)f(y) dy$ for $f \in L^2(D)$. Also, for $m \in \mathbb{N}$, we denote by $G_{\Lambda, \phi, m}$ the Green function of the operator $\Lambda^{-1}\Phi$. In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$, $\Delta \tau$ and $\Delta x$, and may changes value from one line to the other.

Using (4.2) and a simple induction argument, we conclude that

$$\hat{U}^m = \sum_{j=1}^{m} \int_{\Delta_j} \Lambda^{m-j} \Phi \hat{W}(\tau, \cdot) d\tau, \quad m = 1, \ldots, M,$$

which is written, equivalently, as follows:

$$(4.22) \quad \hat{U}^m(x) = \int_0^{\tau_m} \int_D \hat{K}_m(\tau; x, y) \hat{W}(\tau, y) dyd\tau, \quad \forall x \in D, \quad m = 1, \ldots, M,$$

where $\hat{K}_m(\tau; x, y) := \sum_{j=1}^{m} \chi_{\Delta_j}(\tau) G_{\lambda, \phi, m-j+1}(x, y)$, $\forall \tau \in (0, T)$, $\forall x, y \in D$.

Let $m \in \{1, \ldots, M\}$ and $E^m := \mathbb{E} \left[ \|\hat{U}^m - \hat{u}(\tau_m, \cdot)\|_{0, D}^2 \right]$. First, we use (1.5), (1.4), (4.1), (4.2) and (2.9) to obtain

$$E^m = \mathbb{E} \left[ \int_D \left( \int_0^{\tau_m} \int_D \chi_{(0, \tau_m]}(\tau) \left[ \hat{K}_m(\tau; x, y) - \Psi(\tau_m - \tau; x, y) \right] \hat{W}(\tau, y) dyd\tau \right)^2 dx \right]$$

$$\leq \int_0^{\tau_m} \left( \int_D \left( \int_D \left[ \hat{K}_m(\tau; x, y) - \Psi(\tau_m - \tau; x, y) \right]^2 dy \right) dx \right) d\tau$$

$$\leq \sum_{\ell=1}^{m} \int_{\Delta_\ell} \left( \int_D \left[ G_{\lambda, \phi, m-\ell+1}(x, y) - \Psi(\tau_m - \tau; x, y) \right]^2 dy \right) dx d\tau.$$

Now, we introduce the splitting

$$(4.23) \quad \sqrt{E^m} \leq \sqrt{B^m_1} + \sqrt{B^m_2},$$

where

$$B^m_1 := \sum_{\ell=1}^{m} \int_{\Delta_\ell} \left( \int_D \left[ G_{\lambda, \phi, m-\ell+1}(x, y) - \Psi(\tau_m - \tau_{\ell-1}; x, y) \right]^2 dy \right) dx d\tau,$$

$$B^m_2 := \sum_{\ell=1}^{m} \int_{\Delta_\ell} \left( \int_D \left[ \Psi(\tau_m - \tau_\ell; x, y) - \Psi(\tau_m - \tau; x, y) \right]^2 dy \right) dx d\tau.$$
By the definition of the Hilbert-Schmidt norm, we have

\[
B_1^m \leq \Delta \sum_{k=1}^m \sum_{\ell=1}^\infty \int_D \left( \int_D \left[ G_{\Lambda \Phi, m-\ell+1}(x, y) \varphi_k(y) \, dy \right] \Psi(\tau_m - \tau_{\ell-1}; x, y) \varphi_k(y) \, dy \right)^2 \, dx 
\]

\[
\leq \sum_{k=1}^m \sum_{\ell=1}^\infty \Delta \| A^{\ell} \Phi \varphi_k - S(\tau_m - \tau_{\ell-1}) \varphi_k' \|_{0, D}^2
\]

\[
\leq \sum_{k=1}^m \lambda_k^2 \sum_{\ell=1}^\infty \Delta \| A^{\ell} \varepsilon_k - S(\tau_{\ell}) \varepsilon_k \|_{0, D}^2
\]

Let \( \theta \in [0, \frac{1}{4}) \). Using the deterministic error estimate (4.5) and (2.10), we obtain

\[
\sqrt{B_1^m} \leq C (\Delta \tau)^\theta \left( \sum_{k=1}^m \lambda_k^2 \| \varepsilon_k \|_{L^{2+\theta-2}}^2 \right)^{\frac{1}{2}}
\]

(4.24)

\[
\leq C (\Delta \tau)^\theta \left( \sum_{k=1}^m \frac{1}{\lambda_k^{1+8(\frac{1}{4}-\theta)}} \right)^{\frac{1}{2}}
\]

\[
\leq C \frac{1}{\sqrt{\tau}} (\Delta \tau)^\theta.
\]

Using, again, the definition of the Hilbert-Schmidt norm we have

\[
B_2^m = \sum_{k=1}^\infty \sum_{\ell=1}^m \int_{\Delta \tau} \| S(\tau_m - \tau_{\ell-1}) \varphi_k' - S(\tau_m - \tau) \varphi_k' \|_{0, D}^2 \, d\tau
\]

(4.25)

\[
= \sum_{k=1}^\infty \lambda_k^2 \sum_{\ell=1}^m \int_{\Delta \tau} \| S(\tau_m - \tau_{\ell-1}) \varepsilon_k - S(\tau_m - \tau) \varepsilon_k \|_{0, D}^2 \, d\tau
\]

Observing that \( S(t) \varepsilon_k = e^{-\lambda_k^2 (\lambda_k^2 - \mu)^t} \varepsilon_k \) for \( t \geq 0 \), (4.25) yields

\[
B_2^m = \sum_{k=1}^\infty \lambda_k^2 \sum_{\ell=1}^m \int_{\Delta \tau} \left( \int_D \left[ e^{-(\lambda_k^2 + \mu \lambda_k^2)(\tau_m - \tau_{\ell-1})} - e^{-(\lambda_k^2 - \mu \lambda_k^2)(\tau_m - \tau)} \right] \varepsilon_k^2(x) \, dx \right) \, d\tau
\]

(4.26)

\[
= \sum_{k=1}^\infty \lambda_k^2 \sum_{\ell=1}^m \int_{\Delta \tau} e^{-2(\lambda_k^2 - \mu \lambda_k^2)(\tau_m - \tau)} \left[ 1 - e^{-(\lambda_k^2 - \mu \lambda_k^2)(\tau - \tau_{\ell-1})} \right]^2 \, d\tau
\]

\[
\leq B_{2,1}^m + B_{2,2}^m,
\]

where

\[
B_{2,1}^m := \sum_{k=1}^K \lambda_k^2 \sum_{\ell=1}^m \int_{\Delta \tau} e^{-2\lambda_k^2(\lambda_k^2 - \mu)(\tau_m - \tau)} \left[ 1 - e^{-(\lambda_k^2 - \mu \lambda_k^2)(\tau - \tau_{\ell-1})} \right]^2 \, d\tau,
\]

\[
B_{2,2}^m := \sum_{k=K+1}^\infty \lambda_k^2 \sum_{\ell=1}^m \int_{\Delta \tau} e^{-2\lambda_k^2(\lambda_k^2 - \mu)(\tau_m - \tau)} \left[ 1 - e^{-(\lambda_k^2 - \mu \lambda_k^2)(\tau - \tau_{\ell-1})} \right]^2 \, d\tau.
\]
First, we estimate $B_{2,1}^m$ and $B_{2,2}^m$ as follows

\begin{align}
B_{2,1}^m & \leq \sum_{k=\kappa+1}^{\infty} \lambda_k^2 \left( 1 - e^{-\lambda_k^2 (\lambda_k^2 - \mu) \Delta \tau} \right)^2 \left[ \int_0^{\tau_m} e^{-2(\lambda_k^2 - \mu) (\tau_m - \tau)} \, d\tau \right] \\
& \leq \frac{1}{2} \sum_{k=\kappa+1}^{\infty} \frac{1 - e^{-2(\lambda_k^2 - \mu) \Delta \tau}}{\lambda_k^2 - \mu} \\
& \leq \frac{(\kappa + 1)^2}{2(1 + 2\kappa)} \sum_{k=\kappa+1}^{\infty} \frac{1 - e^{-2(\lambda_k^2 - \mu) \Delta \tau}}{\lambda_k^2} \\
& \leq C \sum_{k=1}^{\infty} \frac{1 - e^{-c_0 \lambda_k^4 \Delta \tau}}{\lambda_k}
\end{align}

with $c_0 = \frac{2(1 + 2\kappa)}{(\kappa + 1)^2}$, and

\begin{align}
B_{2,2}^m & \leq C \sum_{k=1}^{\kappa} \sum_{\ell=1}^{m} \int_{2\kappa}^{2\kappa + 1} \left[ 1 - e^{-((\lambda_k^2 - \mu) \lambda_k^2) (\tau - \tau_\ell)} \right]^2 \, d\tau \\
& \leq C \sum_{k=1}^{\kappa} \sum_{\ell=1}^{m} \int_{2\kappa}^{2\kappa + 1} \left[ (\lambda_k^2 - \mu \lambda_k^2) (\tau - \tau_\ell) \right]^2 \, d\tau \\
& \leq C \left( \Delta \tau \right)^2.
\end{align}

Finally, we combine (4.28), (4.27), (4.26) and (3.17), to obtain

\begin{align}
\sqrt{B_2^m} & \leq C \left( 1 + (\Delta \tau)^{\frac{3}{2}} + (\Delta \tau)^{\frac{3}{2}} \right)^{\frac{1}{2}} (\Delta \tau)^{\frac{1}{2}}.
\end{align}

The estimate (4.24) follows by (4.23), (4.21) and (4.29).

5. Convergence of the Fully-Discrete Approximations

To get an error estimate for the fully-discrete approximations of $\hat{u}$ defined by (1.12)-(1.13), we proceed by comparing them with their time-discrete approximations defined by (4.1)-(4.2) and using a discrete Duhamel principle (cf. [14, 22]).

5.1. The Deterministic Case. The Backward Euler finite element approximations of the solution to (1.5) are defined as follows: first, set

\begin{align}
W_0^h & := P_h w_0,
\end{align}

and then, for $m = 1, \ldots, M$, find $W_h^m \in M_h^r$ such that

\begin{align}
W_h^m - W_h^{m-1} + \Delta \tau \Lambda_{B,h} W_h^m = 0,
\end{align}

which is possible when $\mu^2 \Delta \tau < 4$.

Next, we derive a discrete in time $L_r^2(L_{\theta}^2)$ estimate for the error approximating the Backward Euler time-discrete approximations of the solution to (1.5) defined in (4.3)-(4.4), by the Backward Euler finite element approximations defined in (5.1)-(5.2). The main difference with the case $\mu = 0$, which has been considered in [14], is that, our assumption (1.2) on $\mu$, can not ensure the coerciveness of the discrete elliptic operator $\Lambda_{B,h}$.

Theorem 5.1. Let $r = 2$ or $3$, $w$ be the solution to the problem (1.5), $(W^m)_m=0$ be the time-discrete approximations of $w$ defined in (4.3)-(4.4), and $(W_h^m)_m=0 \subset M_h^r$ be the fully-discrete approximations of $w$ defined in (5.1)-(5.2). Also, we assume that $\mu^2 \Delta \tau < \frac{1}{4}$. If $w_0 \in \tilde{H}^2(D)$, then, there exists a nonnegative constant $\tilde{c}_1$, independent of $h$ and $\Delta \tau$, such that

\begin{align}
\left( \sum_{m=1}^{M} \Delta \tau \|W^m - W_h^m\|_{0,D}^2 \right)^{\frac{1}{2}} \leq \tilde{c}_1 \lambda^{\ell_i(r)} \|w_0\|_{\tilde{H}^r}, \quad \forall \theta \in [0,1],
\end{align}
where

\begin{equation}
\ell_\ast(r) := \begin{cases} 
2 & \text{if } r = 2 \\
4 & \text{if } r = 3
\end{cases} \quad \text{and} \quad \xi_\ast(r, \theta) := (r + 1) \theta - 2.
\end{equation}

**Proof.** The error estimate \((5.3)\) follows by interpolation, after showing that holds for \(\theta = 0\) and \(\theta = 1\). In the sequel, we will use the symbol \(C\) to denote a generic constant that is independent of \(\Delta \tau\) and \(h\), and may change value from one line to the other.

Let \(E^m := W_h^m - W^m\) for \(m = 0, \ldots, M\). First, use \((5.2)\) and \((5.4)\) to obtain

\begin{equation}
W_h^m - W_h^{m-1} + \Delta \tau \bar{\Lambda}_{B,h} W_h^m = \Delta \tau \mu^2 W_h^m,
\end{equation}

\begin{equation}
W^m - W^{m-1} + \Delta \tau \bar{\Lambda}_h W^m = \Delta \tau \mu^2 W^m
\end{equation}

for \(m = 1, \ldots, M\). Then, combine \((5.5)\) and \((5.6)\), to get the following error equation

\begin{equation}
\bar{T}_{B,h}(E^m - E^{m-1}) + \Delta \tau E^m = \Delta \tau \mu^2 \bar{T}_{B,h} E^m - \Delta \tau (\bar{T}_{B} - \bar{T}_{B,h}) \bar{\Lambda}_h W^m, \quad m = 1, \ldots, M.
\end{equation}

Taking the \(L^2(D)\)--inner product with \(E^m\) of both sides of \((5.7)\), it follows that

\begin{equation}
\bar{\gamma}_{B,h}(E^m - E^{m-1}, E^m) + \Delta \tau \|E^m\|_{0,D}^2 = \Delta \tau \mu^2 \bar{\gamma}_{B,h}(E^m, E^m)
\end{equation}

for \(m = 1, \ldots, M\). Applying a simple induction argument based on \((5.8)\) and then using that \(4 \Delta \tau \mu^2 < 1\), we get

\begin{equation}
\max_{0 \leq m \leq M} \bar{\gamma}_{B,h}(E^m, E^m) \leq C \left[ \bar{\gamma}_{B,h}(E^0, E^0) + \Delta \tau \sum_{\ell = 1}^{M} \| (\bar{T}_{B} - \bar{T}_{B,h}) \bar{\Lambda}_h W^\ell \|_{0,D}^2 \right].
\end{equation}

Summing with respect to \(m\) from 1 up to \(M\), using \((5.10)\) and observing that \(\bar{T}_{B,h} E^0 = 0\), \((5.8)\) gives

\begin{equation}
\sum_{m=1}^{M} \Delta \tau \|E^m\|_{0,D}^2 \leq C \sum_{m=1}^{M} \Delta \tau \| (\bar{T}_{B} - \bar{T}_{B,h}) \bar{\Lambda}_h W^m \|_{0,D}^2.
\end{equation}

Let \(r = 3\). Then, by \((5.13)\), \((5.11)\) and the Poincaré-Friedrich inequality, we obtain

\begin{equation}
\left( \sum_{m=1}^{M} \Delta \tau \|E^m\|_{0,D}^2 \right)^{\frac{1}{2}} \leq C h^4 \left( \sum_{m=1}^{M} \Delta \tau \|\bar{\Lambda}_h W^m\|_{0,D}^2 \right)^{\frac{1}{2}}
\end{equation}

\begin{equation}
\leq C h^4 \left[ \sum_{m=1}^{M} \Delta \tau \left( \|\partial^2 W^m\|_{0,D}^2 + \|\partial^2 W^m\|_{0,D}^2 + \|\partial^2 W^m\|_{0,D}^2 \right) \right]^{\frac{1}{2}}.
\end{equation}

Taking the \(L^2(D)\)--inner product of \((5.4)\) with \(\partial^4 W^m\) and then integrating by parts, we obtain

\begin{equation}
(\partial^2 W^m - \partial^2 W^{m-1}, \partial^2 W^m)_{0,D} + \Delta \tau \|\partial^4 W^m\|_{0,D}^2 + \mu \Delta \tau \|\partial^2 W^m, \partial^4 W^m\|_{0,D} = 0, \quad m = 1, \ldots, M.
\end{equation}

Using \((5.11)\), \((5.13)\) and the Cauchy-Schwarz inequality we obtain

\begin{equation}
\|\partial^2 W^m\|_{0,D}^2 + 2 \Delta \tau \|\partial^4 W^m\|_{0,D}^2 \leq \|\partial^2 W^{m-1}\|_{0,D}^2 + 2 \mu \Delta \tau \|\partial^2 W^{m-1}\|_{0,D} \|\partial^4 W^m\|_{0,D}, \quad m = 1, \ldots, M,
\end{equation}

which, after using the geometric mean inequality, yields

\begin{equation}
\|\partial^2 W^m\|_{0,D}^2 + \Delta \tau \|\partial^4 W^m\|_{0,D}^2 \leq \|\partial^2 W^{m-1}\|_{0,D}^2 + \Delta \tau \mu^2 \|\partial^2 W^m\|_{0,D}^2, \quad m = 1, \ldots, M.
\end{equation}
Since $2\mu^2 \Delta r < 1$, from (5.14) follows that
\[
\frac{\partial^2 W^m}{\partial^2 W^m} \leq \frac{1}{1 - \mu^2 \Delta r} \frac{\partial^2 W^{m-1}}{\partial^2 W^m}, \quad m = 1, \ldots, M,
\]
from which, applying a simple induction argument, we conclude that
\[
(5.15) \quad \max_{0 \leq m \leq M} \frac{\partial^2 W^m}{\partial^2 W^m} \leq C \| w_0 \|_{L^2(D)}^2.
\]
Next, sum both sides of (5.14) with respect to $m$, from 1 up to $M$, and use (5.15) to conclude that
\[
(5.16) \quad \sum_{m=1}^{M} \Delta \| \partial^4 W^m \|_{0, D}^2 \leq C \| w_0 \|_{L^2(D)}^2.
\]
Taking the $L^2(D)$–inner product of (5.1) with $\partial^2 W^m$, and then integrating by parts, it follows that
\[
(5.17) \quad (\partial W^m - \partial W^{m-1}, \partial W^m) + \Delta \| \partial^3 W^m \|_{0, D}^2 + \mu \Delta r (\partial W^m, \partial^3 W^m) = 0, \quad m = 1, \ldots, M.
\]
Using (2.11), (5.17), the Cauchy-Schwarz inequality and the geometric mean inequality, we obtain
\[
(5.18) \quad \| \partial W^m \|_{0, D}^2 + \Delta \| \partial^3 W^m \|_{0, D}^2 \leq \| \partial W^{m-1} \|_{0, D}^2 + \mu \Delta r \| \partial W^m \|_{0, D}^2, \quad m = 1, \ldots, M.
\]
Since $2 \mu^2 \Delta r < 1$, proceeding as in obtaining (5.15) and (5.16) from (5.14), we arrive at
\[
(5.19) \quad \left( \sum_{m=1}^{M} \Delta \| E^m \|_{0, D}^2 \right)^{1/2} \leq C \mu \| w_0 \|_{\mu^2}.
\]
Let $r = 2$. Then, by (2.22), (5.11) and the Poincaré-Friedrich inequality, we obtain
\[
(5.20) \quad \left( \sum_{m=1}^{M} \Delta \| E^m \|_{0, D}^2 \right)^{1/2} \leq C \mu^2 \left( \sum_{m=1}^{M} \Delta \| \bar{\Lambda}_b W^m \|_{1, D}^2 \right)^{1/2}
\leq C \mu^2 \left[ \sum_{m=1}^{M} \Delta \left( \| \partial^3 W^m \|_{0, D}^2 + \| \partial W^m \|_{0, D}^2 \right) \right]^{1/2}.
\]
Combining, now, (5.20), (5.18) and (2.3), we obtain
\[
(5.21) \quad \left( \sum_{m=1}^{M} \Delta \| E^m \|_{0, D}^2 \right)^{1/2} \leq C \mu^2 \| w_0 \|_{\mu^4}.
\]
Thus, relations (5.19) and (5.21) yield (5.3) and (5.4) for $\theta = 1$.

Since $\mu^2 \Delta r < 1$, using (5.15), we have
\[
\bar{\gamma}_{b,h}(W^m_h - W^{m-1}_h) + \Delta r W^m_h = \Delta r \mu^2 \bar{\gamma}_{b,h} W^m_h, \quad m = 1, \ldots, M,
\]
from which, after taking the $L^2(D)$–inner product with $W^m_h$, we obtain
\[
(5.22) \quad \bar{\gamma}_{b,h}(W^m_h - W^{m-1}_h, W^m_h) + \Delta r \| W^m_h \|_{0, D}^2 = \Delta r \mu^2 \bar{\gamma}_{b,h}(W^m_h, W^m_h), \quad m = 1, \ldots, M.
\]
Then we combine (5.22) with (2.11) to have
\[
(5.23) \quad (1 - 2 \Delta r \mu^2) \bar{\gamma}_{b,h}(W^m_h, W^m_h) + 2 \Delta r \| W^m_h \|_{0, D}^2 \leq \bar{\gamma}_{b,h}(W^{m-1}_h, W^{m-1}_h), \quad m = 1, \ldots, M.
\]
Since $4 \mu^2 \Delta r < 1$, (5.22) yields that
\[
\bar{\gamma}_{b,h}(W^m_h, W^m_h) \leq \frac{1}{1 - 2 \mu^2 \Delta r} \bar{\gamma}_{b,h}(W^{m-1}_h, W^{m-1}_h)
\leq e^{4 \mu^2 \Delta r} \bar{\gamma}_{b,h}(W^{m-1}_h, W^{m-1}_h), \quad m = 1, \ldots, M,
\]
from which, applying a simple induction argument, we conclude that
\[
\max_{0\leq m\leq M} \tilde{\gamma}_{B,\delta}(W_h^m, W_h^m) \leq C \tilde{\gamma}_{B,\delta}(W_h^0, W_h^0).
\]
Summing with respect to \( m \) from 1 up to \( M \), and using (5.24), (5.23) gives
\[
\Delta \tau \sum_{m=1}^M \|W_h^m\|_{0,D}^2 \leq C (T_{B,\delta} w_0, w_0)_{0,D}
\]
Finally, using (5.25), (2.28) and (2.4) we obtain
\[
\tau \Delta \sum_{m=1}^M \|W_h^m\|_{0,D}^2 \leq C (T_{B,\delta} w_0, w_0)_{0,D}
\]
Finally, combining (5.26) with (5.19) to get
\[
\left( \sum_{m=1}^M \Delta \tau \|W_h^m-W_h^m\|_{0,D}^2 \right) \frac{1}{4} \leq C \|w_0\|_{H^2}^2,
\]
which yields (5.3) and (5.4) for \( \theta = 0 \).

5.2. The Stochastic Case. Our first step is to show the existence of a Green function for the solution operator of a discrete elliptic problem.

**Lemma 5.1.** Let \( r = 2 \) or \( 3 \), \( \epsilon > 0 \) with \( \mu^2 \epsilon < 4 \), \( f \in L^2(D) \) and \( \psi_h \in M_h^r \) such that
\[
\psi_h + \epsilon \Lambda_{B,\delta} \psi_h = P_h f.
\]
Then there exists a function \( A_{\epsilon,h} \in H^2(D \times D) \) such that \( A_{\epsilon,h} \theta(D \times D) = 0 \) and
\[
\psi_h(x) = \int_D A_{\epsilon,h}(x,y) f(y) dy \quad \forall x \in \overline{D}
\]
and \( \psi_h(x,y) = A_{\epsilon,h}(x,y) \) for \( x,y \in \overline{D} \).

**Proof.** Let \( \delta_{\epsilon,h} : M_h^r \times M_h^r \rightarrow \mathbb{R} \) be the inner product on \( M_h^r \) given by
\[
\delta_{\epsilon,h}(\phi, \chi) := \epsilon (\Lambda_{B,\delta} \phi, \chi)_{0,D} + (\phi, \chi)_{0,D}
\]
We can construct a basis \( (\chi_i)_{i=1}^{n_h} \) of \( M_h^r \) which is \( L^2(D) \)-orthonormal, i.e., \( (\chi_i, \chi_j)_{0,D} = \delta_{ij} \) for \( i, j = 1, \ldots, n_h \), and \( \delta_{\epsilon,h} \)-orthogonal, i.e., there exist \( (\lambda_{\epsilon,h,i})_{i=1}^{n_h} \subset (0, +\infty) \) such that \( \delta_{\epsilon,h}(\chi_i, \chi_j) = \lambda_{\epsilon,h,i} \delta_{ij} \) for \( i, j = 1, \ldots, n_h \) (see Section 8.7 in [9]). Thus, there are \( (\mu_j)_{j=1}^{n_h} \subset \mathbb{R} \) such that \( \psi_h = \sum_{j=1}^{n_h} \mu_j \chi_j \), and (5.27) is equivalent to \( \mu_i = \frac{1}{\lambda_{\epsilon,h,i}} (f, \chi_i)_{0,D} \) for \( i = 1, \ldots, n_h \). Finally, we obtain (5.28) with \( A_{\epsilon,h}(x,y) = \sum_{j=1}^{n_h} \chi_j(x) \chi_j(y) / \lambda_{\epsilon,h,j} \).

Our second step is to compare, in a discrete in time \( L^\infty(T^\infty)(L^2(T^2)) \) norm, the Backward Euler time-discrete approximations of \( \tilde{u} \) with the Backward Euler finite element approximations of \( \tilde{u} \).

**Proposition 5.2.** Let \( r = 2 \) or \( 3 \), \( \tilde{u} \) be the solution of the problem (1.4), \( (\tilde{U}_h^m)_{m=0}^\infty \) be the Backward Euler finite element approximations of \( \tilde{u} \) defined in (1.8), (1.9), and \( (\tilde{U}_h^m)_{m=0}^\infty \) be the Backward Euler time-discrete approximations of \( \tilde{u} \) defined in (1.1)–(1.2). Also, we assume that \( \mu^2 \Delta \tau \leq \frac{1}{4} \). Then, there exists a nonnegative constant \( \tilde{c}_2 \), independent of \( \Delta x, \Delta t, h \) and \( \Delta \tau \), such that
\[
\max_{0\leq m\leq M} \left( \mathbb{E} \left[ \|\tilde{U}_h^m - \tilde{U}_h^m\|_{0,D}^2 \right] \right)^{1/2} \leq \tilde{c}_2 \epsilon \left( \|\tilde{U}_h^m\|_{1,D} \right)^{-\epsilon}, \quad \forall \epsilon \in (0, \nu(r)],
\]
where

\[ \nu(r) := \begin{cases} \frac{1}{2} & \text{if } r = 2, \\ \frac{1}{4} & \text{if } r = 3. \end{cases} \]

**Proof.** Let \( I : L^2(D) \to L^2(D) \) be the identity operator and \( \Lambda_h : L^2(D) \to M_N^2 \) be the inverse discrete elliptic operator given by \( \Lambda_h := (I + \Delta h \Lambda_{A,h})^{-1} P_h \), having a Green function \( G_{\Lambda_h} = A_h^{-1} \) according to Lemma 5.1 and taking into account that \( \mu^2 \Delta h < 4 \). Also, we define an operator \( \Phi_h : L^2(D) \to M_N^2 \) by \( (\Phi_h f)(x) := \int_D G_{\Lambda_h}(x,y) f(y) \, dy \) for \( f \in L^2(D) \) and \( x \in D \), where \( G_{\Lambda_h}(x,y) = -\partial_y A_h(x,y) \). Then, we have that \( \Lambda_h f = \Phi_h f \) for all \( f \in H^1(D) \). Also, for \( \ell \in \mathbb{N} \), we denote by \( \Lambda_{h\ell} \) the Green function of \( \Lambda_{h\ell} \).

In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta t, \Delta x, h \) and \( \Delta \tau \), and may changes value from one line to the other.

Applying, an induction argument, from (1.9) we conclude that

\[ \hat{U}^m_h = \sum_{j=1}^m \int_{\Delta_j} \Lambda_h^{m-j} \Phi_h \hat{W}^m(\tau, \cdot) \, d\tau, \quad m = 1, \ldots, M, \]

which is written, equivalently, as follows:

\[ \hat{U}^m_h(x) = \int_0^{\tau_m} \int_D \hat{D}_{h,m}(\tau; x, y) \hat{W}^m(\tau, y) \, dyd\tau \quad \forall x \in D, \quad m = 1, \ldots, M, \]

where \( \hat{D}_{h,m}(\tau; x, y) := \sum_{j=1}^m \Lambda_{h,\ell}(\tau) G_{\Lambda_{h,\ell}}(x,y) \) \( \forall \tau \in [0, T], \forall x, y \in D \). Using (4.22), (5.31), the Itô-isometry property of the stochastic integral, (2.5) and the Cauchy-Schwarz inequality, we get

\[
\begin{align*}
\mathbb{E} \left[ \| \hat{U}^m - \hat{U}^m_h \|_{0,D}^2 \right] & \leq \int_0^{\tau_m} \left( \int_D \left[ \hat{D}_{m}(\tau; x, y) - \hat{D}_{h,m}(\tau; x, y) \right]^2 \, dy dx \right) \, d\tau \\
& \leq \sum_{j=1}^m \int_{\Delta_j} \| \Lambda_h^{m-j} \Phi - \Lambda_h^{m-j} \Phi_h \|_{H^2}^2 \, d\tau, \quad m = 1, \ldots, M,
\end{align*}
\]

where \( \Lambda \) and \( \Phi \) are the operators defined in the proof of Theorem 4.2. Now, we use the definition of the Hilbert-Schmidt norm and the deterministic error estimate (5.3), to obtain

\[
\begin{align*}
\mathbb{E} \left[ \| \hat{U}^m - \hat{U}^m_h \|_{0,D}^2 \right] & \leq \sum_{j=1}^m \Delta \tau \left[ \sum_{k=1}^\infty \| \Lambda_h^{m-j} \Phi \varphi_k - \Lambda_h^{m-j} \Phi_h \varphi_k \|_{0,D}^2 \right] \\
& \leq \sum_{k=1}^\infty \Delta \tau \left[ \sum_{\ell=1}^m \| \Lambda^\ell \varphi_k - \Lambda_h^\ell \varphi_k \|_{0,D}^2 \right] \\
& \leq \sum_{k=1}^\infty \lambda_k^2 \left[ \sum_{\ell=1}^\infty \Delta \tau \| \Lambda^\ell \varepsilon_k - \Lambda_h^\ell \varepsilon_k \|_{0,D}^2 \right] \\
& \leq C h^2 \ell_*(\theta) \sum_{k=1}^\infty \lambda_k^2 \| \varepsilon_k \|_{H^2(\ell_*(\theta))}^2, \quad m = 1, \ldots, M, \quad \forall \theta \in [0, 1].
\end{align*}
\]

Thus, we arrive at

\[ \max_{1 \leq m \leq M} \left( \mathbb{E} \left[ \| \hat{U}^m - \hat{U}^m_h \|_{0,D}^2 \right] \right)^{\frac{1}{2}} \leq C h^2 \ell_*(\theta) \left( \sum_{k=1}^\infty \lambda_k^2 \left[ \frac{1}{2} + \frac{2\ell_*(\theta)}{\ell_*(\theta) + \ell_*(\theta)} \right] \right)^{\frac{1}{2}}, \quad \forall \theta \in [0, 1]. \]

It is easily seen that the series in the right hand side of (5.32) convergences iff \( \nu(r) > \ell_*(r) \). Thus, setting \( \epsilon = \nu(r) - \ell_*(r) \theta \), requiring \( \epsilon \in (0, \nu(r)] \), and combining (5.32) and (2.10), we arrive at the estimate (5.29). \( \Box \)

The available error estimates allow us to conclude a discrete in time \( L^\infty_\tau (L^2_D(L^2_x)) \) convergence of the Backward Euler fully-discrete approximations of \( \hat{u} \).
Theorem 5.3. Let \( r = 2 \) or \( 3 \), \( \nu(r) \) be defined by (5.30), \( \hat{u} \) be the solution of problem (1.6), and \( \hat{\hat{U}}_m^{(m)} \) be the Backward Euler finite element approximations of \( \hat{u} \) constructed by (1.8)-(1.9). Then, there exists a nonnegative constant \( C \), independent of \( h \), \( \Delta t \), \( \Delta \tau \) and \( \Delta x \), such that: if \( \mu^2 \Delta \tau \leq \frac{1}{4} \), then

\[
\max_{0 \leq m \leq M} \left\{ \mathbb{E} \left[ \| \hat{\hat{U}}_m^{(m)} - \hat{u}(\tau_m, \cdot) \|_{0,s}^2 \right] \right\}^{\frac{1}{2}} \leq C \left[ \omega_s(\Delta \tau, \epsilon_1) \Delta \tau^{\frac{3}{8} - \epsilon_1} + c_2^{-\frac{1}{2}} h^{\nu(r) - c_2} \right]
\]

forall \( \epsilon_1 \in (0, \frac{1}{4}] \) and \( c_2 \in (0, \nu(r)] \), where \( \omega_s(\Delta \tau, \epsilon_1) := c_1^{-\frac{1}{2}} + (\Delta \tau)^{\epsilon_1} (1 + (\Delta \tau)^{\frac{3}{4}} (\Delta \tau)^{\frac{3}{4}}) \).

Proof. The estimate is a simple consequence of the error bounds (5.29) and (1.21). \( \Box \)

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REFERENCES

[1] E.J. Allen, S.J. Novosel and Z. Zhang. Finite element and difference approximation of some linear stochastic partial differential equations. Stochastics Stochastics Rep., vol. 64, pp. 117–142, 1998.
[2] A. Are, M.A. Katsoulakis and A. Szepessy. Coarse-Grained Langevin Approximations and Spatiotemporal Acceleration for Kinetic Monte Carlo Simulations of Diffusion of Interacting Particles. Chin. Ann. Math., vol. 30B(6), pp. 653–682, 2009.
[3] L. Bin. Numerical method for a parabolic stochastic partial differential equation. Master Thesis 2004-03, Chalmers University of Technology, Göteborg, Sweden, June 2004.
[4] D. Blömker, S. Maier-Paape and T. Wanner. Second phase spinodal decomposition for the Cahn-Hilliard-Cook equation. Transactions of the AMS, vol. 360, pp. 449–489, 2008.
[5] J.H. Bramble and S.R. Hilbert. Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation. SIAM J. Numer. Anal., vol. 7 (1970), pp. 112-124.
[6] A. Debussche and L. Zambotti. Conservative Stochastic Cahn-Hilliard equation with reflection. Annals of Probability, vol. 35, pp. 1706-1739, 2007.
[7] N. Dunford and J.T. Schwartz. Linear Operators. Part II. Spectral Theory. Self Adjoint Operators in Hilbert Space. Reprint of the 1963 original. Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.
[8] W. Grecksch and P.E. Kloeden. Time-discretised Galerkin approximations of parabolic stochastic PDEs. Bull. Austral. Math. Soc., vol. 54, pp. 79–85, 1996.
[9] G. H. Golub and C. F. Van Loan. Matrix Computations. Second Edition, The John Hopkins University Press, Baltimore, 1989.
[10] P. C. Hohenberg and B.I. Halperin. Theory of dynamic critical phenomena. J. Rev. Mod. Phys. vol. 49, pp. 435–479, 1977.
[11] G. Kallianpur and J. Xiong. Stochastic Differential Equations in Infinite Dimensional Spaces. Institute of Mathematical Statistics, Lecture Notes-Monograph Series vol. 26, Hayward, California, 1995.
[12] M.A Katsoulakis and D.G. Vlachos. Coarse-grained stochastic processes and kinetic Monte Carlo simulators for the diffusion of interacting particles. J. Chem. Phys., vol. 119, pp. 9412–9427, 2003.
[13] P.E. Kloeden and S. Shot. Linear-implicit strong schemes for Itô-Galerkin approximations of stochastic PDEs. Journal of Applied Mathematics and Stochastic Analysis., vol. 14, pp. 47–53, 2001.
[14] G.T. Kossioris and G.E. Zouraris, Fully-discrete finite element approximations for a fourth-order linear stochastic parabolic equation with additive space-time white noise, Mathematical Modelling and Numerical Analysis 44, 289-322 (2010).
[15] G.T. Kossioris and G.E. Zouraris, Finite element approximations for a linear fourth-order parabolic SPDE in two and three space dimensions with additive space-time white noise, http://dx.doi.org/doi:10.1016/j.apnum.2012.01.003, Applied Numerical Mathematics (to appear).
[16] S. Larsson and A. Mesforush, Finite element approximation of the linearized Cahn-Hilliard-Cook equation,IMA J. Numer. Anal. 31, 1315-1333 (2011).
[17] J.L. Lions and E. Magenes. Non-Homogeneous Boundary Value Problems and Applications. Vol. I. Springer–Verlag, Berlin - Heidelberg, 1972.
[18] J. Printems. On the discretization in time of parabolic stochastic partial differential equations. Mathematical Modelling and Numerical Analysis, vol. 35, pp. 1055–1078, 2001.
[19] T.M. Rogers, K.R. Elder and R.C. Desai. Numerical study of the late stages of spinodal decomposition. Physical Review B, vol. 37, pp. 9638-9651, 1988.
Appendix A.

Let \( t > 0 \) and \( \mu_k := \lambda_k^2 (\lambda_k^2 - \mu) \) for \( k \in \mathbb{N} \) First, we recall that \( \mathcal{S}(t)w_0 = \sum_{k=1}^{\infty} e^{-\mu_k t} (w_0, \varepsilon_k)_{0,D} \varepsilon_k \) for \( t \geq 0 \), and set \( \tilde{\mathcal{S}}(t)w_0 = e^{-\mu^2 t} \mathcal{S}(t)w_0 \) for \( t \geq 0 \). Next, follow Chapter 3 in [21], to obtain

\[
\| \partial_t^\ell \tilde{\mathcal{S}}(t)w_0 \|_{\text{L}^2_{\text{HP}}} \leq \tilde{C}_{\mu,\ell} \sum_{k=1}^{\infty} \lambda_k^{2(p+4\ell)} e^{-\lambda_k^2 t} (w_0, \varepsilon_k)_{0,D}^2,
\]

where \( \tilde{C}_{\mu,\ell} := \left( 1 + \frac{4\mu^2}{\pi^2} + \frac{4\mu^2}{\pi^2} \right)^{2\ell} \). Now, use (A.1), to have

\[
\int_{t_a}^{t_b} (\tau - t_a)^\beta \left\| \partial_t^\ell \tilde{\mathcal{S}}(\tau)w_0 \right\|_{\text{L}^2_{\text{HP}}}^2 d\tau \leq \tilde{C}_{\mu,\ell} \sum_{k=1}^{\infty} \lambda_k^{2(p+4\ell+2\beta)} \left( \int_{t_a}^{t_b} [\lambda_k^2 (\tau - t_a)]^\beta e^{-\lambda_k^2 \tau} d\tau \right) (w_0, \varepsilon_k)_{0,D}^2
\]

\[
\leq \tilde{C}_{\mu,\ell} \sum_{k=1}^{\infty} \lambda_k^{2(p+4\ell+2\beta-2)} \left( \int_{0}^{\infty} \rho^\beta e^{-(\rho+\lambda_k^2 t_a)} d\rho \right) (w_0, \varepsilon_k)_{0,D}^2
\]

\[
\leq \tilde{C}_{\mu,\ell} \left( \int_{0}^{\infty} \rho^\beta e^{-\rho} d\rho \right) \sum_{k=1}^{\infty} \lambda_k^{2(p+4\ell+2\beta-2)} (w_0, \varepsilon_k)_{0,D}^2,
\]

which yields

\[
\int_{t_a}^{t_b} (\tau - t_a)^\beta \left\| \partial_t^\ell \tilde{\mathcal{S}}(\tau)w_0 \right\|_{\text{L}^2_{\text{HP}}}^2 d\tau \leq \tilde{C}_{\beta,\ell,\mu} \| w_0 \|_{\text{H}^{p+4\ell+2\beta-2}}^2,
\]

where \( \tilde{C}_{\beta,\ell,\mu} = \tilde{C}_{\mu,\ell} \int_{0}^{\infty} x^\beta e^{-x} dx \). Observing that \( \partial_t^\ell \mathcal{S}(t)w_0 = e^{\mu^2 t} \sum_{m=0}^{\ell} \mu_m (\ell - m) \partial_t^m \mathcal{S}(t)w_0 \), and using (A.2), we conclude that

\[
\int_{t_a}^{t_b} (\tau - t_a)^\beta \left\| \partial_t^\ell \mathcal{S}(\tau)w_0 \right\|_{\text{L}^2_{\text{HP}}}^2 d\tau \leq e^{2\mu^2 T} C_{\beta,\ell,\mu} \sum_{m=0}^{\ell} \| w_0 \|_{\text{H}^{p+4m+2\beta-2}}^2
\]

which yields (2.16) with \( C_{\beta,\ell,\mu,T} = C_{\beta,\ell,\mu} e^{2\mu^2 T} \ell \). \( \square \)