Quantum-improved phase estimation with a displacement-assisted SU(1,1) interferometer

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By performing two local displacement operations (LDOs) inside an SU(1,1) interferometer, called as the displacement-assisted SU(1,1) [DSU(1,1)], both the phase sensitivity based on homodyne detection and quantum Fisher information (QFI) with and without photon losses are investigated in this paper. In this DSU(1,1) interferometer, we focus our attention on the extent to which the introduced LDO affects the phase sensitivity and the QFI, even in the realistic scenario. Our analyses show that the estimation performance of DSU(1,1) interferometer is always better than that of SU(1,1) interferometer without the LDO, especially the phase precision of the former in the ideal scenario gradually approaching to the Heisenberg limit via the increase of the LDO strength. More significantly, different from the latter, the robustness of the former can be enhanced markedly by regulating and controlling the LDO. Our findings would open an useful view for quantum-improved phase estimation of optical interferometers.
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I. INTRODUCTION

Quantum metrology is an excellent candidate of parameter estimation theory to serve as the high-precision requirement of various quantum information tasks [1-6], such as quantum sensor [1,2] and quantum imaging [3,4]. Thus, how to achieve the higher precision of quantum metrology has become a general consensus among scientists. To this end, the optical interferometers, e.g., Mach-Zehnder interferometer (MZI) [7-13] and SU(1,1) interferometer [14-21], are often used for understanding the subtle phase variations thoroughly. Generally, in optical-interferometer systems, it is possible to obtain the higher precision of phase estimation using three probe strategies: generation, modification and readout [22]. In the probe generation stage, nonclassical quantum resources as the inputs of the MZI have been proven to more effectively enhance the precision measurement than its classical counterpart [23-27]. In particular, when using the NOON states [25,28], the two-mode squeezed vacuum states (TMSVS) [25] and the twin Fock states [27], the standard quantum limit (SQL) [13] that is not exceeded by only exploiting the classical resources can be easily beaten, even infinitely reaching at the famed Heisenberg limit (HL) [18,26]. These states, however, are extremely sensitive to noisy environments [29], so that non-Gaussian resources [12,30-33] as an alternative that can be produced by taking advantage of non-Gaussian operations on an arbitrary initial state play an important role in improving the estimation performance of the MZI, even in the presence of noisy scenarios [12,33]. Apart from the generation stage, many efforts have devoted to conceiving the probe modification by replacing the conventional beam splitters in the conventional MZI with the optical parametric amplifiers (OPAs) [14-18,21,22,34], which is also called as SU(1,1) interferometer proposed first by Yurke [16]. In this SU(1,1) interferometer with two OPAs, the first OPA (denoted as OPA₁) is used not only to obtain the entangled resources but also to eliminate amplified noise; while the usage of the second OPA (denoted as OPA₂) can result in the signal enhancement [21,22], which paves a feasible way to achieve the higher precision of phase estimation. Taking advantage of these features, an SU(1,1) interferometer scheme with the phase shift induced by a Kerr medium was suggested by Chang [22], pointing out that the significant improvement of both the phase sensitivity and quantum Fisher information can be achieved even in the presence of photon losses. In addition, the noiseless quantum amplification of parameter-dependent processes was used to SU(1,1) interferometer, indicating how this process results in the HL [35]. More interestingly, by using the non-Gaussian operations inside the SU(1,1) interferometer, both the phase sensitivity and the robustness of this interferometer system against the photon losses can be further enhanced [36]. From works [12,23,33,36], we also notice that the usage of non-Gaussian operations can significantly improve the estimation performance of the optical interferometers, but at the expense of the high cost of implementing these operations.

To solve the above problem, the local operations containing the local squeezing operation (LSO) [37-39] and the local displacement operation (LDO) [40] are one of the most promising choices. In particular, J. Sahota and D. F. V. James suggested a quantum-enhanced phase estimation scheme by applying the LSO into the MZI [39]. However, it should be mentioned that the LSO plays a key role in quantum metrology [39], quantum key distribution [38] and entanglement distillation [37], but the
degree of the LSO is not infinite, e.g., its maximum attainable degree for the TMSVS about 1.19 (10.7 dB) [41]. For this reason, here we suggest a quantum-improved phase estimation of the SU(1,1) interferometer based on the LDO, which can be called as the displacement-assisted SU(1,1) [DSU(1,1)] interferometer. Under the framework of this DSU(1,1) interferometer, we not only derive its explicit forms of both the quantum Fisher information (QFI) and the phase sensitivity based on homodyne detection, but also consider the effects of photon losses on both the QFI and the phase sensitivity, even in the presence of photon losses. In particular, this increasing LDO can narrow the gap for the phase sensitivity between with and without photon losses. This implies that the usage of the sufficiently large LDO can make the SU(1,1) interferometer systems more robust against photon losses.

The remainder of this paper is arranged as follows. In section II, we first describe the theoretical model of DSU(1,1) interferometer, and then give the relationship between the output and input operators for this interferometer. In sections III, for the ideal scenario, we analyze and discuss both the QFI and the phase sensitivity based on homodyne detection in DSU(1,1) interferometer, before making a comparison about phase sensitivities containing the SQL, the HL and the DSU(1,1) interferometer scheme in section VI. Subsequently, we also consider the effects of photon losses on both the QFI and the phase sensitivity of DSU(1,1) interferometer in section V. Finally, our main conclusions are drawn in the last section.

II. THE DSU(1,1) INTERFEROMETER AND ITS RELATIONSHIP BETWEEN THE OUTPUT AND INPUT OPERATORS

Now, let us begin with introducing the theoretical model of DSU(1,1) interferometer, whose structure is comprised of two OPAs, two LDOs and a linear phase shift, as depicted in Fig. 1. For simplicity, here we only consider both a squeezed vacuum state $|\xi\rangle_a$ and a coherent state $|\beta\rangle_b$ as the inputs of DSU(1,1) interferometer in paths $a$ and $b$ respectively. After these input states pass through the OPA1, paths $a$ and $b$ respectively experience the same LDO process, denoted as $\hat{D}_a(\gamma) = e^{i\gamma a^\dagger - \gamma^* a}$ and $\hat{D}_b(\gamma) = e^{i\gamma b^\dagger - \gamma^* b}$ with $\gamma = |\gamma| e^{i\theta_1}$, so that the probe state $|\psi_r\rangle$ can be achieved. Then, we also assume that path $a$ serves as the reference path, while path $b$ undergoes a linear phase shifter for producing a phase shift $\phi$ to be estimated. Finally, after paths $a$ and $b$ recombine in the OPA2, we can extract the phase information about the value of $\phi$ by implementing the homodyne detection in path $a$. Indeed, the relationship between the output and input operators for DSU(1,1) interferometer can be given by

$$\hat{a}_2 = W_1 + Y \hat{a}_0 - Z \hat{b}_0,$$
$$\hat{b}_2 = W_2 + e^{i\phi} (Y \hat{b}_0 - Z \hat{a}_0),$$
(1)

where $W_1$ and $W_2$ are caused by the LDO process, and

$$Y = \cosh g_1 \cosh g_2 + e^{i(\theta_2 - \theta_1 - \phi)} \sinh g_1 \sinh g_2,$$
$$Z = e^{i\theta_1} \sinh g_1 \cosh g_2 + e^{i(\theta_2 - \phi)} \cosh g_1 \sinh g_2,$$
$$W_1 = \gamma \cosh g_2 - \gamma^* e^{i(\theta_2 - \phi)} \sinh g_2,$$
$$W_2 = \gamma e^{i\phi} \cosh g_2 - \gamma^* e^{i\theta_2} \sinh g_2,$$
(2)

with $g_1 (g_2)$ and $\theta_1 (\theta_2)$ respectively representing the gain factor and the phase shift in the OPA1 (OPA2). According to Eq. (1), one can further derive the explicit form of phase sensitivity, which is a prerequisite for our analysis and discussion about the estimation performance of DSU(1,1) interferometer in the following sections.
III. THE QFI AND PHASE SENSITIVITY OF DSU(1,1) INTERFEROMETER IN AN IDEAL SCENARIO

So far, we have described the schematic of DSU(1,1) interferometer in detail. In this section, we shall present and analyze the estimation performance of DSU(1,1) interferometer from the perspective of both quantum Fisher information and phase sensitivity in an ideal scenario. Moreover, for the sake of discussion, in the following sections, we also assume that the DSU(1,1) interferometer is in the balanced case, i.e., $\theta_2 - \theta_1 = \pi$ and $g_1 = g_2 = g$ (set $\theta_1 = 0$ and $\theta_2 = \pi/2$ for simplicity).

A. The QFI

To directly assess the estimation performance of a unknown phase parameter without any detection strategies, it is an enormous success for utilizing the QFI of the probe state since the quantum Cramér-Rao bound (QCRB) $\Delta \phi_{QCRB}$ representing the ultimate precision is in inverse proportion to the QFI (denoted as $F$). In addition, the increased value of the QFI indicates that the estimation precision becomes more excellent. In this context, the QFI for an arbitrary pure state in the ideal scenario can be expressed as

$$F = 4(\langle \psi'_\phi | \psi'_\phi \rangle - | \langle \psi'_\phi | \psi_\phi \rangle |^2),$$  \hspace{1cm} (3)

where $| \psi_\phi \rangle = e^{i \phi \hat{b} \dagger} | \psi_\gamma \rangle$ is the state vector prior to the OPA$_2$ and $\psi'_\phi = \partial | \psi_\phi \rangle / \partial \phi$. Thus, if the probe state is obtained, Eq. (3) can be rewritten as

$$F = 4[\langle \psi_\gamma | \hat{n}^2 | \psi_\gamma \rangle - \langle \psi_\gamma | \hat{n} | \psi_\gamma \rangle^2],$$  \hspace{1cm} (4)

where $\hat{n} = \hat{b}^\dagger \hat{b}$ is the photon number operator of path $b$. As a consequence, based on Eq. (4), when inputting the state $| \psi_m \rangle = | \xi \rangle_a \otimes | \beta \rangle_\gamma$, one can obtain the explicit form of the QFI of DSU(1,1) interferometer (see Appendix A for more details), i.e.,

$$F = 4(\Gamma_2 + \Gamma_1 - \Gamma_1^2),$$  \hspace{1cm} (5)

where $\Gamma_m$ ($m = 1, 2$) are the average value of operators $\hat{b}^\dagger \hat{b}^m$ with respect to the probe state. By using Eq. (5), one also can obtain the QCRB providing the ultimate phase precision of DSU(1,1) interferometer regardless of detection schemes [43,44], i.e.,

$$\Delta \phi_{QCRB} = \frac{1}{\sqrt{\nu F}},$$  \hspace{1cm} (6)

with the number of trials $\nu$ (for simplicity, set $\nu = 1$). From Eq. (6), it is obvious that, the larger the value of $F$, the smaller the $\Delta \phi_{QCRB}$, which implies the attainability of the higher phase sensitivity. In order to see this point, Fig. 2 shows both the QFI and the QCRB changing with the gain factor $g$ and the LDO strength $| \gamma |$. As we can see from Fig. 2(a), compared to SU(1,1) interferometer without the LDO (the black solid line), with
the increase of |γ| = 1, 2, 3, the QFI can be increased well, which would straightway lead to the decrease of the QCRB [see Fig. 2(c)], meaning the enhanced phase sensitivity. These improvement effects of both the QFI and the QCRB, however, can decrease with the increase of g. To show the performance of both the QFI and the QCRB in the sufficiently large case of g, e.g., g = 2, as respectively depicted in Figs. 2(b) and 2(d), for several different |β| = 1, 2, 3, we thus find that both the QFI and the QCRB are still improved by increasing the value of |γ|. These results fully indicate that the performance of the DSU(1,1) interferometer for the ideal scenario performs better than that of the SU(1,1) interferometer without the LDO in terms of the QFI and the QCRB.

B. The phase sensitivity via the homodyne detection

In this subsection, we mainly focus on analyzing the phase sensitivity of DSU(1,1) interferometer. For this purpose, it is unavoidable to choose a specific detection scheme, such as homodyne detection [22, 34, 45], intensity detection [15, 19, 46], and parity detection [12, 25, 28, 47], for reading the phase information. In addition, compared with both intensity and parity detections, the homodyne detection is computationally convenient and compatible with existing experimental technology, thereby playing potential applications in quantum communication [48–52]. For this reason, the phase parameter φ can be estimated by exploiting the homodyne detection (see Fig. 1), whose detected variable can be treated as the amplitude quadrature \(^\hat{X}\), i.e.,

\[
\hat{X} = \hat{a}_2 + \hat{a}_2^\dagger \div \sqrt{2}
\]  

(7)

Using Eq. (7) and the error propagation formula, the phase sensitivity of DSU(1,1) interferometer can be thus given by

\[
\Delta \phi = \frac{\sqrt{\Delta^2 \hat{X}}}{\partial (\hat{X}) / \partial \phi},
\]  

(8)

with \(\Delta^2 \hat{X} = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2\). It is clearly seen from Eq. (8) that, for an arbitrary value of φ, the corresponding phase sensitivity can be analytically derived, which can refer to Appendix B for more details.

To find the optimal point φ corresponding to the minimum of phase sensitivity, we plot the phase sensitivity \(\Delta \phi\) as a function of φ for several different |γ| = 0, 1, 2, 3, as shown in Fig. 3(a). Significantly, for all given values |γ| = 0, 1, 2, 3, the minimum of phase sensitivity can be always found at the optimal point \(\phi = 0\). In addition, it is more interesting that, when fixed the optimal point \(\phi = 0\), as the LDO strength |γ| increases, the corresponding value of phase sensitivity can further decrease, which fully demonstrates that the usage of the LDO in SU(1,1) interferometer systems is more beneficial for improving the phase sensitivity than that without the LDO. The reason for these phenomena is that the increase of the LDO strength |γ| contributes to increase the slope \(\partial \langle \hat{X} \rangle / \partial \phi\) of the output signal \(\langle \hat{X} \rangle\), thereby giving rise to the increase of the denominator in Eq. (8), as pictured in Fig 3(b). After achieving the optimal point \(\phi = 0\), we can further obtain the phase sensitivity \(\Delta \phi\) of DSU(1,1) interferometer around this optimal point, i.e.,

\[
\Delta \phi|_{\phi=0} = \frac{e^{-r}}{2 |\beta| \sinh g \cosh g + 2 |\gamma| \sinh g},
\]  

(9)

where the second term of the denominator results from the LDO. In particular, when \(|\gamma| = 0\) corresponding to the SU(1,1) interferometer without the LDO, its phase sensitivity \(\Delta \phi\) is consistent with the previous work [17]. To visually compare the phase sensitivity of SU(1,1) interferometers with and without the LDO, Fig. 4 shows the phase sensitivity changing with \(g\), |β| and |γ|. In Figs. 4(a) and 4(b), the black solid line indicates the phase sensitivity of SU(1,1) interferometer without the LDO, which can be exceeded by that of DSU(1,1) interferometer (|γ| ≠ 0), and the phase sensitivity of SU(1,1) interferometer systems can be further enhanced via the increase of the LDO strength |γ|. The reason may be that the QFI can be effectively increased by increasing the values of |γ|, which can be seen in Figs. 2(a) and 2(b).
Moreover, we also find that, for given a LDO strength $|\gamma|$, the corresponding phase sensitivity can be further improved by increasing the values of $g$ and $|\beta|$, but for sufficiently large $g$ and $|\beta|$, the improvement effects cannot be clearly distinguished. For this reason, when given $g = 2$ and $|\beta| = 3$, we also respectively plot the $\log_{10} \Delta \phi$ versus $|\gamma|$, as shown in Figs. 4(c) and 4(d). Obviously, with the increase of $|\gamma|$, the improved effects of phase sensitivity are still evident. For instance, at a fixed $g = 2$, the corresponding phase sensitivity for $|\beta| = 1$ can be further improved by about 57% when $|\gamma|$ ranges from 0 to 5, see the black dashed line in Fig. 4(c).

IV. PHASE SENSITIVITY OF DSU(1,1) INTERFEROMETER WITH THE SQL AND THE HL

After presenting both the QFI and the phase sensitivity of DSU(1,1) interferometer in an ideal scenario, we shall make a comparison about phase sensitivities including the SQL, the HL and the DSU(1,1) interferometer scheme. For this purpose, the total mean photon number inside DSU(1,1) interferometer should be introduced, which can be defined as

$$N_{Total} = \langle \psi_\gamma | \hat{a} \hat{a}^\dagger + \hat{b} \hat{b}^\dagger | \psi_\gamma \rangle,$$

where $|\psi_\gamma \rangle = D_a(\gamma) D_b(\gamma) \hat{U}_{OPA1} |\psi_{in}\rangle$ is the probe state just after the LDO with both $\hat{U}_{OPA1} = \exp(g_1 e^{-i \theta_1} \hat{\alpha} \hat{b} - g_1 e^{i \theta_1} \hat{\alpha}^\dagger \hat{b}^\dagger)$ being the OPA1 process and $|\psi_{in}\rangle = |\xi\rangle_a \otimes |\beta\rangle_b$.

FIG. 4: (Color online) Phase sensitivity changing with (a) $g$, (b) $|\beta|$ and (c)-(d) $|\gamma|$. Among them, in (a), the fixed values are $|\beta| = r = 1$ for different $|\gamma| = 0, 1, 2, 3$; in (b), the fixed values are $g = r = 1$ for different $|\gamma| = 0, 1, 2, 3$; in (c), the fixed values are $g = 2, r = 1$ for different $|\beta| = 1, 2, 3$; in (d), the fixed values are $|\beta| = 3, r = 1$ for different $g = 0.5, 1, 2$. Other parameters are as following: $\phi = \theta_\xi = 0$ and $\theta_\beta = \theta_\gamma = \pi/2$.

FIG. 5: (Color online) Comparison about precision limits involving the SQL, the HL and the SU(1,1) interferometer with the LDOs. (a) and (b) respectively corresponds to phase sensitivity $\Delta \phi$ changing with $g$ and $|\gamma|$, when fixed $|\gamma| = |\beta| = r = 1$ and $|\beta| = g = r = 1$. 
being the input state of DSU(1,1) interferometer. It is worth noting that the total mean photon number $N_{\text{Total}}$ inside DSU(1,1) interferometer is different from the total mean photon number $N_{in}$, which in our scheme can be given by

$$N_{\text{Total}} = N_{in} \cosh 2g + 2 \sinh^2 g + 2 |\gamma|^2,$$

where the first two terms result from the amplification process of $N_{in}$ and the spontaneous process prior to the implementation of the LDO, and the last two terms stem from the LDO process. According to Eq. (11), one can respectively derive the SQL and the HL, i.e.,

$$\Delta \phi_{\text{SQL}} = \frac{1}{\sqrt{N_{\text{Total}}}},$$

$$\Delta \phi_{\text{HL}} = \frac{1}{N_{\text{Total}}}. \quad (12)$$

To see if the phase precision of DSU(1,1) interferometer can surpass the SQL, even closing to the HL, we make a comparison about the phase sensitivity changing with $g$ and $|\gamma|$, as shown in Fig. 5(a) and 5(b). It is clearly seen from Fig. 5(a) that, at fixed values of $|\beta| = |\gamma| = r = 1$, for a large range of $g$ (i.e., $g > 0.24$), the phase precision of DSU(1,1) interferometer scheme can easily break through the SQL (solid green line), even gradually approaching to HL (solid red line) with the increase of $g$. In addition, when given $g = |\beta| = r = 1$, it is found in Fig. 5(b) that the phase precision of DSU(1,1) interferometer scheme is always superior to the SQL, and can gradually approach to HL when increasing the value of $|\gamma|$, which implies that the applications of the LDO into the SU(1,1) interferometer systems are salutary aspects to the enhancement of phase precision, making it close to the HL.

V. THE QFI AND PHASE SENSITIVITY OF THE PHOTON-LOSS DSU(1,1) INTERFEROMETER

For the realistic environment, the photon losses are of vital importance in restricting the precision of quantum metrology. In particular, how the photon losses affect the quantum-noise cancellation in SU(1,1) interferometer was discussed in detail both theoretically and experimentally [53]. As a result, in this section, we shall analyze and discuss the behaviors of both the QFI and the phase sensitivity in the photon-loss DSU(1,1) interferometer.

A. The effects of photon losses on the QFI

Due to the existence of photon losses, it is not suitable for deriving the QFI via the conventional method given in Eq. (4). To solve this problem, a novel variational method was proposed by Escher [54], which has been used in the photon-loss single-(or multi-)parameter estimation systems [22, 55, 56]. Inspired by this lightspot, below we would derive the explicit form of the QFI in the photon-loss DSU(1,1) interferometer, with the help of the variational method.

Originally, we first denote the probe state as $|\psi_{\gamma}\rangle \equiv |\psi_{\gamma}\rangle_S$ where $|\psi_{\gamma}\rangle_S$ is an initial probe state of DSU(1,1) interferometer system $S$. Because of the photon losses, the encoding process of the probe state $|\psi_{\gamma}\rangle$ to an unknown phase $\phi$ is no longer the unitary evolution, so that the system $S$ is expanded into the enlarged one $S + E$ ($E$ represents the photon-loss environment system). Under such circumstances, the initial probe state $|\psi_{\gamma}\rangle_S$ in the enlarged systems $S + E$ experiences the unitary phase encoding process $U_{S+E}(\phi)$, which can be described as [54]

$$|\psi\rangle_{S+E} = U_{S+E}(\phi)|\psi_{\gamma}\rangle_S|0\rangle_E = \sum_{j=0}^{\infty} \hat{K}_j(\phi)|\psi_{\gamma}\rangle_S|j\rangle_E, \quad (13)$$

where $|0\rangle_E$ is the initial state of the photon-loss system $E$, $|j\rangle_E$ is the orthogonal basis of the $|0\rangle_E$, and $\hat{K}_j(\phi)$ is the Kraus operator only working on the $|\psi_{\gamma}\rangle_S$, whose expression can be given by [54]

$$\hat{K}_j(\phi) = \sqrt{\left(\frac{1 - \eta}{j!}\right)} e^{\phi b^\dagger b - \eta j} \eta^{b^\dagger b/2} b_j,$$ \quad (14)

with the variational parameter $\lambda$ and the strength of the photon losses $\eta$ ($\eta = 0$ and $\eta = 1$ respectively denote the complete absorption and lossless cases). In this situation, the QFI for the DSU(1,1) interferometer with the photon losses can be given by [54, 55, 57]

$$F_L = \min_{\{\hat{K}_j(\phi)\}} C_Q[|\psi_{\gamma}\rangle_S, \hat{K}_j(\phi)], \quad (15)$$

with the upper bound of the QFI in the photon-losses systems [54]

$$C_Q[|\psi_{\gamma}\rangle_S, \hat{K}_j(\phi)] = 4 \left[ S_{S+E} \langle \psi'|\psi\rangle_{S+E} - S_{S+E} \langle \psi'|\psi\rangle_{S+E}^2 \right]. \quad (16)$$

Upon substituting Eqs. (13) into (16), so that

$$C_Q[|\psi_{\gamma}\rangle_S, \hat{K}_j(\phi)] = 4 \left[ \langle \hat{H}_1(\phi) \rangle^2_S - \langle \hat{H}_1(\phi) \rangle^2_S \right], \quad (17)$$

where the symbol of $\langle \cdot \rangle$ is the inner product with respect to the initial probe state $|\psi_{\gamma}\rangle_S$ and

$$\hat{H}_1(\phi) = \sum_{j=0}^{\infty} \frac{d\hat{K}_j(\phi)}{d\phi} \frac{d\hat{K}_j(\phi)}{d\phi},$$

$$\hat{H}_2(\phi) = \frac{d}{d\phi} \sum_{j=0}^{\infty} \frac{d\hat{K}_j(\phi)}{d\phi} \frac{d\hat{K}_j(\phi)}{d\phi}. \quad (18)$$
Thus, combining Eqs. (14) and (17), Eq. (16) can be rewritten as

$$C_Q[|\psi_\gamma\rangle_S, \hat{K}_j(\phi)] = 4(\eta + \eta \lambda - \lambda)^2 \langle \Delta^2 \hat{n} \rangle + 4\eta(1 - \eta)(1 + \lambda)^2 \langle \hat{n} \rangle, \quad (19)$$

with the symbol of $\langle \Delta^2 \hat{n} \rangle$ representing the variance of the $|\psi_\gamma\rangle_S$. From Eq. (19), when minimizing the upper bound of the QFI, the optimal value of $\lambda$ is calculated as

$$\lambda_{\text{opt}} = \frac{\langle \Delta^2 \hat{n} \rangle}{(1 - \eta) \langle \Delta^2 \hat{n} \rangle + \eta \langle \hat{n} \rangle} - 1, \quad (20)$$

so that according to Eqs. (15) and (20), the explicit form of the QFI with the photon losses can be finally derived as

$$F_L = \frac{4\eta F \langle \hat{n} \rangle}{(1 - \eta) F + 4\eta \langle \hat{n} \rangle}, \quad (21)$$
where $F$ corresponds to the lossless case given in Eq. (5). For our scheme, when considering the $|\xi\rangle_b \otimes |\beta\rangle_b$ as the inputs of DSU(1,1) interferometer, Eq. (21) can be rewritten as

$$F_L = \frac{4\eta F T_1}{(1-\eta)F + 4\eta T_1}. \quad (22)$$

It is clearly seen from Eq. (22) that when $\eta = 1$, one can obtain $F_L = F$ corresponding to the ideal case. To intuitively elaborate the effects of the photon losses on the estimation performance of DSU(1,1) interferometer, we show the behaviors of both the QFI log$_{10}F_L$ and the QCRB log$_{10}\Delta_f$ changing with $\eta$ and $|\gamma|$, as depicted in Fig. 6. As can be seen from Figs. 6(a), when given a certain value of $|\gamma|$, the value of the QFI decreases with the decrease of $\eta$, which means that the QFI is heavily influenced by the strength of the photon losses $\eta$. Even so, we also can find that, at a fixed $\eta$, e.g., $\eta = 0.6$, the QFI can be further improved by increasing the value of $|\gamma|$, implying that the usage of the LDO is conducive to resisting the photon losses, thereby achieving the higher phase sensitivity of SU(1,1) interferometer systems, which can be seen in Fig. 6(b). These results are also true for the QCRB, as shown in Figs. 6(c) and 6(d). More interestingly, at fixed $\eta = 0.6$ and $|\gamma| = 1$, it is also possible to further enhance both the QFI and the QCRB via the increasing parameters of $g$ and $|\beta|$, as seen in Fig. 7.

Finally, to show the advantages of exploiting the LDO into the SU(1,1) interferometer, we take the difference between the QCRB with the photon losses and the one without both the LDOs and the photon losses, i.e., $\Delta = \log_{10} \Delta \phi_{L-\text{QCRB}} - \log_{10} \Delta \phi_{\text{QCRB}}$. If the condition of $\Delta < 0$ is true, then exploitation of the LDO can effectively improve the robustness of SU(1,1) interferometer systems. To see this point, we contourplot the difference $\Delta$ as a function of $\eta$ and $|\gamma|$, as shown in Fig. 8. It is evident that, with the increase of $|\gamma|$, the improved region of $\Delta < 0$ increases and more photon losses can be tolerated.

B. The effects of photon losses on the phase sensitivity

Now, let us examine the effects of photon losses on the phase sensitivity of DSU(1,1) interferometer. For this purpose, we assume that the same photon losses occur at between the phase shift and the OPA$_2$, as shown in Fig. 9. In general, under the photon losses process, the lossy channel can be simulated by inserting the fictitious beam splitter (FBS) with a transmissivity $T$. It is worth mentioning that, the smaller the values of $T$, the more serious the photon losses.

For the state vector prior to the OPA$_2$, $|\psi_f\rangle$, after going through the photon-loss channel, the output state $|\psi_{\text{out}}\rangle$ in the enlarged systems $S + E$ can be expressed as $|\psi_{\text{out}}\rangle = U_{BS}^{\dagger} U_{BS} |\psi_f\rangle |0\rangle_a |0\rangle_b$. Thus, when passing through the OPA$_2$, the final output state $|\psi_f\rangle$ can be given by

$$|\psi_f\rangle = U_{OPA2} \tilde{U}_{BS}^{\dagger} U_{BS} |\psi_f\rangle |0\rangle_{a,b}, \quad (23)$$

where $|0\rangle_{a,b} = |0\rangle_a \otimes |0\rangle_b$ is the vacuum noise, $U_{OPA2}$ is the OPA$_2$ process, and $U_{BS}^{\dagger} U_{BS} = \exp[\arccos \sqrt{T} \hat{\Theta} \cdot \hat{\Theta}^\dagger b]$, $\hat{\Theta} \in \{a, b\}$, represent the FBS operators acting on mode $\hat{\Theta}$, with $\hat{\Theta}_v$ being the vacuum noise operators. Further, by utilizing the transformations of the FBS, e.g., $(U_{BS}^{\dagger} \hat{\Theta} U_{BS}^\dagger) \hat{\Theta} U_{BS} = \sqrt{T} \hat{\Theta} + \sqrt{1-T} \hat{\Theta}_v$, one can derive the phase sensitivity $\Delta \phi_L$ with the photon losses, i.e.,

$$\Delta \phi_L = \sqrt{(\Delta \phi)^2 + \frac{(1-T) \cosh 2g}{4T(\Lambda_1 + \Lambda_2)^2}}, \quad (24)$$

where $\Delta \phi$ can be given in the Eq. (B5) of Appendix B, and

$$\Lambda_1 = |\beta| \sinh g \cosh \sin (\phi + \theta_\beta),$$
$$\Lambda_2 = |\gamma| \sinh g \sin (\phi + \theta_\gamma). \quad (25)$$
In order to explore whether the photon losses have an effect on the optimal point \( \phi \) corresponding to the minimum of phase sensitivity, we illustrate the phase sensitivity \( \Delta \phi \) with \( T = 0.6 \) (dashed lines) as a function of \( \phi \) for several different values \(|\gamma|\) = 0, 1, 2, as shown in Fig. 10(a). As a comparison, the solid lines represent the ideal case. As we can easily see, the minimum of phase sensitivity is always found at the optimal point \( \phi = 0 \), whether there is photon loss or not. More significantly, with the increase of \(|\gamma|\) = 0, 1, 2, the gap of the phase sensitivity \( \Delta \phi \) between with and without the photon losses can be further reduced around the optimal point \( \phi \). These phenomena result from that the increase of the LDO strength \(|\gamma|\) can still increase the slope \( \partial \langle \hat{X} \rangle / \partial \phi \) of the output signal \( \langle \hat{X} \rangle \) even in the presence of photon losses, which can be shown in Fig. 10(b).

In this context, therefore, the phase sensitivity with the photon losses at the optimal point \( \phi = 0 \) can be calculated as

\[
\Delta \phi_{L, \phi=0} = \left[ \frac{(1 - T) \cosh 2g}{4T (|\beta| \sinh g \cosh g + |\gamma| \sinh g)^2} + (\Delta \phi_{|\phi=0})^2 \right]^{1/2},
\]

where the second term of the square root derives from the photon losses, and \( \Delta \phi_{|\phi=0} \) is the phase sensitivity without the photon losses given in Eq. (25). In particular, when \( T = 1 \), the corresponding phase sensitivity becomes the ideal case.

In Fig. 11(a), we show the advantage of SU(1,1) interferometer robust against the photon losses. Obviously, with the decrease of \( T \), the phase sensitivity of DSU(1,1) interferometer systems would fade away, but this decline can be further slowed by increasing \(|\gamma|\) = 0, 1, 2, 3. To some extent, this phenomenon reveals that the usage of the LDO can make the whole SU(1,1) interferometer systems more robust against the photon losses, when comparing to the case without the LDO. To visualize this point, we make a comparison about the phase sensitivity between with (dashed lines) and without (solid lines) photon losses, as shown in Fig. 11(b). Surprisingly, at the same accessible parameters, the gap for the phase sensitivity between with and without photon losses narrows down with the increase of \(|\gamma|\) = 0, 1, 2, 3, even showing that the phase sensitivity with photon losses for \(|\gamma|\) = 2 (3) at certain small range of \( g < 0.436 \) (0.699) performs better than that without both the photon losses and the LDO (black solid line). In addition, the aforementioned gap can be further reduced by increasing the value of \( g \).
VI. CONCLUSIONS

In summary, we have presented the positive contribution of the LDO for improving the estimation performance of SU(1,1) interferometer in terms of both the QFI and the phase sensitivity based on homodyne detection. The numerical results show that the increase of the LDO strength is conducive to the enhancement of the QFI and the phase sensitivity. In particular, for the ideal case, the phase sensitivity of DSU(1,1) interferometer scheme can gradually approach to the HL. From a realistic point of view, we further investigate both the QFI and phase sensitivity in the presence of photon losses. Our analyses indicate that, when given the same parameters, the DSU(1,1) interferometer scheme can also obtain the higher QFI and the better phase sensitivity than the SU(1,1) interferometer without the LDO under the photon-loss case. More interestingly, the sufficiently large LDO can strengthen the robustness of SU(1,1) interferometer systems against the photon losses.

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Appendix A: The QFI for DSU(1,1) interferometer in the ideal scenario

For DSU(1,1) interferometer with the input state \(|\psi_{in}\rangle = |\xi\rangle_a \otimes |\beta\rangle_b\), the QFI in the ideal scenario can be given by Eq. (5) with \(C_{m} = \langle \psi_{\gamma} | \hat{b}^m \hat{b}^m | \psi_{\gamma} \rangle\), \((m = 1, 2)\) to be calculated. For this reason, here we need to introduce the characteristic function (CF), in which for any probe state \(|\psi_{\gamma}\rangle\) of DSU(1,1) interferometer, its CF can be expressed as

\[
C_{W}(\alpha_1, \alpha_2) = \text{Tr}[\hat{\rho}_{\gamma} \hat{D}(\alpha_1) \hat{D}(\alpha_2)],
\]

with \(\hat{\rho}_{\gamma} = |\psi_{\gamma}\rangle \langle \psi_{\gamma}|\) being the density operator of the probe state and \(\hat{D}(\alpha_1) = \exp(\alpha_1 \hat{a}^\dagger - \alpha_1^* \hat{a})\) being the displacement operator.

Thus, the average value \(\Gamma_m = \langle \psi_{\gamma} | \hat{b}^m \hat{b}^m | \psi_{\gamma} \rangle\) can be derived as

\[
\Gamma_m = \Omega_m C_N(0, \alpha_2),
\]

where \(\Omega_m = \frac{d^{2m}}{d\alpha_2^m d(-\alpha_2)^m \cdots |_{\alpha_2 = \alpha_2 = 0}}\) is the partial differential operator and \(C_N(0, \alpha_2) = e^{\alpha_2^2 / 2} C_W(0, \alpha_2)\) is the normal ordering form of the CF. For DSU(1,1) interferometer with the input state \(|\psi_{in}\rangle = |\xi\rangle_a \otimes |\beta\rangle_b\), the corresponding probe state can be given by \(|\psi_{\gamma}\rangle = D_{\alpha}(\gamma) \hat{D}_{\beta}(\gamma) \hat{U}_{OPA1} \psi_{in}\rangle\), so that according to Eq. (A2), one can obtain

\[
\Gamma_m = \Omega_m \exp \left[-\Delta_1 |z_2|^2 + \Delta_2^2 z_2 - \Delta_2 z_2^2\right] - \Delta_3 (e^{-i\theta \xi} z_2^2 + e^{i\theta \xi} z_2^2),
\]

where

\[
\Delta_1 = \cosh^2 r \sinh^2 g,
\]

\[
\Delta_2 = 2\gamma_2 + \beta \cosh g,
\]

\[
\Delta_3 = \frac{1}{4} \sinh 2r \sinh^2 g.
\]

Therefore, substituting Eq. (A3) into Eq. (5), one can obtain the explicit expression of the QFI for DSU(1,1) interferometer in the ideal case.

Appendix B: Phase sensitivity via homodyne detection

Combining Eqs. (11) and (7), one can derive the variance \(\Delta^2 \hat{X}\) as

\[
\Delta^2 \hat{X} = \frac{|U|^2 (\cosh 2r - \sinh 2r \cos \Delta) + |V|^2}{2},
\]

where

\[
U = |U| e^{i\theta U} = \cosh^2 g - e^{-i\phi} \sinh^2 g,
\]

\[
V = (1 - e^{-i\phi}) \sinh g \cosh g,
\]

\[
\Delta = \theta_e + 2\theta U,
\]

and the derivative of the variance is given by

\[
\frac{\partial \langle \hat{X} \rangle}{\partial \phi} = \sqrt{2} \left(|\beta| \cosh g \sin \Theta_1 + |\gamma| \sin \Theta_2\right) \sinh g,
\]

with

\[
\Theta_1 = \phi + \theta_\beta,
\]

\[
\Theta_2 = \phi + \theta_\gamma.
\]

Substituting Eqs. (B1) and (B3) into the error propagation formula shown in Eq. (8), the explicit expression of the phase sensitivity of DSU(1,1) interferometer in the ideal case can be given by

\[
\Delta \phi = \sqrt{|U|^2 + |V|^2 (\cosh 2r - \sinh 2r \cos \Delta) + \frac{2}{\sinh g (|\beta| \cosh g \sin \Theta_1 + |\gamma| \sin \Theta_2)}}.
\]

In particular, when \(\phi = \theta_e = 0\), the variance \(\Delta^2 \hat{X} = e^{-2r}/2\). Moreover, by utilizing the results given in Eq. (B3) at the optimal phase point \(\phi = 0\), one can find the absolute value of the derivative of the variance

\[
\left| \frac{\partial \langle \hat{X} \rangle}{\partial \phi} \right| = \sqrt{2} \sinh g (|\beta| \cosh g \sin \theta_\beta + |\gamma| \sin \theta_\gamma).
\]

Finally, after achieving \(\sin \theta_\beta = \sin \theta_\gamma = 1\) by taking \(\theta_\beta = \theta_\gamma = \pi/2\), one can obtain Eq. (9).
[1] T. J. Proctor, P. A. Knott, and J. A. Dunningham, Multiparameter Estimation in Networked Quantum Sensors, Phys. Rev. Lett. 120, 080501 (2018).

[2] Y. Maleki, and M. S. Zubairy, Distributed phase estimation and networked quantum sensors with W-type quantum probes, Phys. Rev. A 105, 032428 (2022).

[3] G. Sorelli, M. Gessner, M. Walschaers, and N. Treps, Optimal Observables and Estimators for Practical Superresolution Imaging, Phys. Rev. Lett. 127, 123604 (2021).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photon. 5, 222 (2011).

[5] S. Y. Lee, Y. Jo, T. Jeong, J. Kim, D. H. Kim, D. Kim, D. Y. Kim, Y. S. Ihn, and Z. Kim, Observable bound for Gaussian illumination, Phys. Rev. A 105, 042412 (2022).

[6] J. Aasi, J. Abadie, B. Abbott, R. Abbott, T. Abbott, M. Abernathy, C. Adams, T. Adams, P. Addesso, R. Adhikari et al., Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light, Nat. Photonics 7, 613 (2013).

[7] C. M. Caves, Quantum-mechanical noise in an interferometer, Phys. Rev. D 23, 1693 (1981).

[8] W. Ge, K. Jacobs, Z. Eldredge, A. V. Gorshkov, and M. Foss Feig. Distributed Quantum Metrology with Linear Networks and Separable Inputs, Phys. Rev. Lett. 121, 043604 (2018).

[9] S. Pang and A. N. Jordan, Optimal adaptive control for quantum metrology with time-dependent Hamiltonians, Nat. Commun. 8, 14695 (2017).

[10] J. Y. Wu, N. Toda, and H. F. Hofmann, Quantum enhancement of sensitivity achieved by photon-number-resolving detection in the dark port of a two-path interferometer operating at high intensities, Phys. Rev. A 100, 013814 (2019).

[11] Y. B. Aryeh, Phase estimation by photon counting measurements in the output of a linear Mach–Zehnder interferometer, J. Opt. Soc. Am. B 29, 2754 (2012).

[12] H. Zhang, W. Ye, C. P. Wei, C. J. Liu, Z. Y. Liao, and L. Y. Hu, Improving phase estimation using number conserving operations, Phys. Rev. A 103, 052602 (2021).

[13] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, Science 306, 1330 (2004).

[14] W. Du, J. Kong, G. Z. Bao, P. Y. Yang, J. Jia, S. Ming, C. H. Yuan, J. F. Chen, Z. Y. Ou, M. W. Mitchell, and W. P. Zhang, SU(2)-in-SU(1,1) Nested Interferometer for High Sensitivity, Loss-Tolerant Quantum Metrology, Phys. Rev. Lett. 128, 033601 (2022).

[15] L. L. Guo, Y. F. Yu, and Z. M. Zhang, Improving the phase sensitivity of an SU(1,1) interferometer with photon added squeezed vacuum light, Opt. Express 26, 29099 (2018).

[16] B. Yurke, S. L. McCall, and J. R. Klauder, SU(2) and SU(1,1) interferometers, Phys. Rev. A 33, 4033 (1986).

[17] D. Li, C. H. Yuan, Z. Y. Ou, W. P. Zhang, The phase sensitivity of an SU(1,1) interferometer with coherent and squeezed-vacuum light, New J. Phys. 16, 073020 (2014).

[18] D. Li, B. T. Gard, Y. Gao, C. H. Yuan, W. P. Zhang, H. Lee, J. P. Dowling, Phase sensitivity at the Heisenberg limit in an SU(1,1) interferometer via parity detection, Phys. Rev. A 94, 063840 (2016).

[19] W. N. Plick, J. P. Dowling, G. S. Agarwal, Coherent-light-boosted, sub-shot noise, quantum interferometry, New J. Phys. 12, 083014 (2010).

[20] O. Seth, X. F. Li, H. N. Xiong, J. Y. Luo and Y. X. Huang, Improving the phase sensitivity of an SU(1,1) interferometer via a nonlinear phase encoding, J. Phys. B: At. Mol. Opt. Phys. 53, 205503 (2020).

[21] S. K. Chang, C. P. Wei, H. Zhang, Y. Xia, W. Ye, and L. Y. Hu, Enhanced phase sensitivity with a nonconventional interferometer and nonlinear phase shifter, Phys. Lett. A 384, 126755 (2020).

[22] S. K. Chang, W. Ye, H. Zhang, L. Y. Hu, J. H. Huang, and S. Q. Liu, Improvement of phase sensitivity in an SU(1,1) interferometer via a phase shift induced by a Kerr medium, Phys. Rev. A 105, 033704 (2022).

[23] R. Carranza and C. C. Gerry, Photon-subtracted two-mode squeezed vacuum states and applications to quantum optical interferometry, J. Opt. Soc. Am. B 29, 2581 (2012).

[24] H. Kwon, K. C. Tan, T. Volkoff, and H. Jeong, Nonclassicality as a Quantifiable Resource for Quantum Metrology, Phys. Rev. Lett. 122, 040503 (2019).

[25] J. P. Dowling, Quantum optical metrology—the lowdown on high-N00N states, Contemp. Phys. 49, 125 (2008).

[26] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, and S. D. Huver, Quantum Metrology with Two-Mode Squeezed Vacuum: Parity Detection Beats the Heisenberg Limit, Phys. Rev. Lett. 104, 103602 (2010).

[27] R. A. Campos, C. C. Gerry, and A. Benmoussa, Optical interferometry at the Heisenberg limit with twin Fock states and parity measurements, Phys. Rev. A 68, 023810 (2003).

[28] I. Afek, O. Ambar, and Y. Silberberg, High-N00N states by mixing quantum and classical light, Science 328, 879 (2010).

[29] T. Ono and H. F. Hofmann, Effects of photon losses on phase estimation near the Heisenberg limit using coherent light and squeezed vacuum, Phys. Rev. A 81, 033819 (2010).

[30] W. Ye, H. Zhong, Q. Liao, D. Huang, L. Y. Hu, and Y. Guo, Improvement of self-referenced continuous-variable quantum key distribution with quantum photon catalysis, Opt. Express 27, 17186 (2019).

[31] D. Braun, P. Jian, O. Pinel, and N. Treps, Precision measurements with photon-subtracted or photon-added Gaussian states, Phys. Rev. A 90, 013821 (2014).

[32] Y. Ouyang, S. Wang, and L. J. Zhang, Quantum optical interferometry via the photon-added two-mode squeezed vacuum states, J. Opt. Soc. Am. B 33, 1373 (2016).

[33] H. Zhang, W. Ye, C. P. Wei, Y. Xia, S. K. Chang, Z. Y. Liao, and L. Y. Hu, Improved phase sensitivity in a quantum optical interferometer based on multiphoton catalytic two-mode squeezed vacuum states, Phys. Rev. A 103, 013705 (2021).

[34] X. Y. Hu, C. P. Wei, Y. F. Yu, and Z. M. Zhang, Enhanced phase sensitivity of an SU(1,1) interferometer with displaced squeezed vacuum light, Front. Phys. 11, 114203 (2016).

[35] G. S. Agarwal and L. Davidovich, Quantifying quantum-amplified metrology via Fisher information, Phys. Rev. Res. 4, L012014 (2022).

[36] J. Xin, Phase sensitivity enhancement for the SU(1,1) interferometer using photon level operations, Opt. Express
29, 43970-43984 (2021).

[37] S. L. Zhang and P. V. Loock, Local Gaussian operations can enhance continuous-variable entanglement distillation, Phys. Rev. A 84, 062309 (2011).

[38] V. C. Usenko and F. Grosshans, Unidimensional continuous-variable quantum key distribution, Phys. Rev. A 92, 062337 (2015).

[39] J. Sahota and D. F. V. James, Quantum-enhanced phase estimation with an amplified Bell state, Phys. Rev. A 88, 063820 (2013).

[40] J. Fiurasek, Improving entanglement concentration of Gaussian states by local displacements, Phys. Rev. A 84, 012335 (2011).

[41] Y. J. Wang, W. H. Zhang, R. X. Li, L. Tian, and Y. H. Zheng, Generation of -10.7 dB unbiased entangled states of light, Appl. Phys. Lett. 118, 134001 (2021).

[42] Y. M. Zhang, X. W. Li, W. Yang, and G. R. Jin, Quantum Fisher information of entangled coherent states in the presence of photon loss, Phys. Rev. A 88, 043832 (2013).

[43] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. 72, 3439 (1994).

[44] S. Luo, Wigner-Yanase Skew Information and Uncertainty Relations, Phys. Rev. Lett. 91, 180403 (2003).

[45] S. Ataman, Optimal Mach-Zehnder phase sensitivity with Gaussian states, Phys. Rev. A 100, 063821 (2019).

[46] S. Ataman, Phase sensitivity of a Mach-Zehnder interferometer with single-intensity and difference-intensity detection, Phys. Rev. A 98, 043856 (2018).

[47] S. Adhikari, N. Bhusal, C. You, H. Lee, and J. P. Dowling, Phase estimation in an SU(1,1) interferometer with displaced squeezed states, Opt. Express 1, 438 (2018).

[48] C. Weedbrook, Continuous-variable quantum key distribution with entanglement in the middle, Phys. Rev. A 87, 022308 (2013).

[49] F. Grosshans and P. Grangier, Continuous Variable Quantum Cryptography using Coherent States, Phys. Rev. Lett. 88, 057902 (2002).

[50] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. 74, 145 (2002).

[51] F. Dell’Anno, S. De Siena, L. Albano, and F. Illuminati, Continuous-variable quantum teleportation with non-Gaussian resources, Phys. Rev. A 76, 022301 (2007).

[52] S. L. Braunstein and H. J. Kimble, Teleportation of Continuous Quantum Variables, Phys. Rev. Lett. 80, 869 (1998).

[53] J. Xin, H. L. Wang, and J. T. Jing, The effect of losses on the quantum-noise cancellation in the SU(1,1) interferometer, Appl. Phys. Lett. 109, 051107 (2016).

[54] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, Nat. Phys. 7, 406 (2011).

[55] J. D. Yue, Y. R. Zhang, and H. Fan, Quantum-enhanced metrology for multiple phase estimation with noise, Sci. Rep. 4, 5933 (2014).

[56] F. Albarelli, M. Mazelanik, M. Lipka, A. Streltsov, M. Pariniak, and R. Demkowicz-Dobrzański, Quantum Asymmetry and Noisy Multimode Interferometry, Phys. Rev. Lett. 128, 240504 (2022).

[57] M. Zwierz, C. A. Perez-Delgado, and P. Kok, General Optimality of the Heisenberg Limit for Quantum Metrology, Phys. Rev. Lett. 105, 180402 (2010).