Canonical explicit Bäcklund transformations with spectrality for constrained flows of soliton hierarchies

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Abstract It is shown that explicit Bäcklund transformations (BTs) for the high-order constrained flows of soliton hierarchy can be constructed via their Darboux transformations and Lax representation, and these BTs are canonical transformations including Bäcklund parameter $\eta$ and possess a spectrality property with respect to $\eta$ and the 'conjugated' variable $\mu$ for which the pair $(\eta, \mu)$ lies on the spectral curve. As model we present the canonical explicit BTs with the spectrality for high-order constrained flows of the Kaup-Newell hierarchy and the KdV hierarchy.

Keywords Bäcklund transformation, canonical transformation, spectrality, constrained flows of soliton equation, Lax representation.

1 Introduction

Bäcklund transformations (BTs) are an important tool in the studies of integrable systems [1]. In recent years some new properties of BTs for finite-dimensional integrable Hamiltonian systems have attracted some attention (see [2]-[7]). These explicit BTs are shown to be canonical transformations including Bäcklund parameter $\eta$ and to satisfy a spectrality property with respect to $\eta$ and the 'conjugated' variable $\mu$ for which the pair $(\eta, \mu)$ or $(\eta, f(\mu))$ for some function $f(\mu)$ lies on the spectral curve. The spectrality property of BTs is connected with the problem of separation of variables, in fact, the sequence of Bäcklund parameters $\eta_j$ together with the conjugate variables $\mu_j$ can be considered as the separation variables for the finite-dimensional integrable Hamiltonian
systems. These BTs are defined as symplectic integrable maps and can be viewed as time discretization of particular flows of Liouville integrable systems \[2, 3, 5, 6\].

The constrained flows of soliton equations which can be transformed into finite-dimensional Hamiltonian systems have been widely studied recently. The Lax representation for the high-order constrained flows can always be deduced from the adjoint representation for the soliton hierarchy \[8, 9\]. We pointed out in \[10\] that based on the results of Darboux transformations (DTs) for soliton hierarchy and the Lax representation for the constrained flows, one can construct explicit BTs including Bäcklund parameter \(\eta\) for the high-order constrained flows of soliton equations. We proceed further to develop the ideas with some new BTs and study the problem of constructing Bäcklund transformation for the high-order constrained flows of the soliton hierarchy. We show that these BTs be canonical transformations by presenting their generating functions and possess the spectrality property with respect to \(\eta\) and conjugate variable \(\mu\). In contrast with the few example of this kind of BTs presented in \[3\]-\[7\], this paper presents a way to find infinite number of explicit BTs with the properties mentioned above by means of DTs for the soliton hierarchy and the Lax representations for the high-order constrained flows. We illustrate the idea by the high-order constrained flows of the Kaup-Newell hierarchy and KdV hierarchy.

The paper is organized as follows. In section 2, we briefly describe the high-order constrained flows of the Kaup-Newell hierarchy. In section 3 we first present DTs for the constrained flows, then we construct infinite number of explicit BTs for the high-order constrained flows of the Kaup-Newell hierarchy and show them to be canonical transformations and to satisfy the spectrality property by using the first three high-order constrained flows as model. The infinite number of canonical explicit BTs with spectrality for the constrained flows of the KdV hierarchy are presented in section 4 and 5.

2 High-order constrained flows of the Kaup-Newell hierarchy

The Kaup-Newell hierarchy \[11\]

\[
\begin{align*}
\frac{du}{dt_n} &= \left( \begin{array}{c} q \\ r \end{array} \right) = \frac{\delta H_{2n-2}}{\delta u}, & n = 0, 1, \cdots
\end{align*}
\]

(2.1)
with
\[ H_{2n} = \frac{1}{2n} (4a_{2n+2} - qc_{2n+1} - rb_{2n+1}), \quad J = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \]
is associated with the following Kaup-Newell spectral problem
\[ \psi_x = U(u, \lambda) \psi \equiv \begin{pmatrix} -\lambda^2 & \lambda q \\ \lambda r & \lambda^2 \end{pmatrix} \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (2.2) \]
and the evolution of \( \psi \)
\[ \psi_{tn} = V^{(n)}(u, \lambda) \psi \equiv \sum_{i=0}^{n-1} \begin{pmatrix} a_{2i+1} \lambda^{2i+1} & b_{2i+1} \lambda^{2i} \\ c_{2i+1} \lambda^{2i+1} & -a_{2i} \lambda^{2i} \end{pmatrix} \psi, \quad (2.3) \]
where
\[ a_0 = 1, \quad b_0 = c_0 = a_1 = 0, \quad b_1 = -q, \quad c_1 = -r, \]
\[ a_2 = -\frac{1}{2} qr, \quad b_2 = c_2 = 0, \quad b_3 = \frac{1}{2} (q^2 r + q x), \quad c_3 = \frac{1}{2} (qr^2 - r^2 x), \ldots, \]
and in general, \( a_{2m+1} = b_{2m} = c_{2m} = 0, \)
\[ \begin{pmatrix} c_{2m+1} \\ b_{2m+1} \end{pmatrix} = L \begin{pmatrix} c_{2m-1} \\ b_{2m-1} \end{pmatrix} = -L^m \begin{pmatrix} r \\ q \end{pmatrix}, \quad a_{m,x} = qc_{m+1} - rb_{m+1}, \]
\[ L = \frac{1}{2} \begin{pmatrix} D - r D^{-1} q D & -r D^{-1} r D \\ -q D^{-1} q D & -D - q D^{-1} r D \end{pmatrix}, \quad D = \frac{\partial}{\partial x}, \quad DD^{-1} = D^{-1} D = 1. \]

We have
\[ \frac{\delta \lambda}{\delta q} = \phi_1^2, \quad \frac{\delta \lambda}{\delta r} = -\phi_1^2. \]

The high-order constrained flows of the Kaup-Newell hierarchy consist of the equations obtained from the spectral problem (2.2) for \( N \) distinct \( \lambda_j \) and the restriction of the variational derivatives for conserved quantities \( H_{2n} \) and \( \lambda_j \) (see \cite{8,9,12,13,14})
\[ \Phi_{1,x} = -\Lambda^2 \Phi_1 + q \Lambda \Phi_2, \quad \Phi_{2,x} = r \Lambda \Phi + \Lambda^2 \Phi_2, \quad (2.4a) \]
\[ \frac{\delta H_{2n}}{\delta u} - \frac{1}{2} \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} < \Phi_2, \Phi_2 > \\ -< \Phi_1, \Phi_1 > \end{pmatrix} = 0, \quad (2.4b) \]
where \( n = 0, 1, \ldots, \Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T, i = 1, 2, \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_N), < \cdot, \cdot > \) denotes the inner product. The Lax representation for the constrained flow (2.4) is given by \cite{8,9}
\[ M_x^{(n)} = [U, M^{(n)}], \quad (2.5) \]
with Lax matrix $M^{(n)}$

$$M^{(n)} = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & -A^{(n)} \end{pmatrix} = V^{(n+1)} + M_0 \quad (2.6)$$

$$M_0 = \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda^2 \lambda_j^2 \phi_{1j} \phi_{2j} & -\lambda \lambda_j^3 \phi_{1j}^2 \\ \lambda \lambda_j^3 \phi_{2j}^2 & -\lambda^2 \lambda_j^3 \phi_{1j} \phi_{2j} \end{pmatrix},$$

and the Lax pair for (2.4)

$$\psi_x = U(u, \lambda) \psi, \quad M^{(n)}(u, \lambda) \psi = \mu \psi. \quad (2.7)$$

The spectral curve $\Gamma$,

$$\Gamma : W(\lambda, \mu; \{P_i\}) = det(M^{(n)}(u, \lambda) - \mu) = 0$$

is

$$W(\lambda, \mu; \{P_i\}) = \mu^2 - (A^{(n)}(\lambda))^2 - B^{(n)}(\lambda)C^{(n)}(\lambda) = 0, \quad (2.8)$$

where $\{P_i\}$ are integrals of motion for (2.4).

We present the first three high-order constrained flows as follows.

1. For $n = 0$, (2.4) becomes a finite-dimensional integrable Hamiltonian system (FDIHS)

$$\Phi_{1,x} = \frac{\partial \tilde{H}_0}{\partial \Phi_2}, \quad \Phi_{2,x} = -\frac{\partial \tilde{H}_0}{\partial \Phi_1}, \quad (2.9)$$

$$\tilde{H}_0 = - < \Lambda^2 \Phi_1, \Phi_2 > + \frac{1}{4} < \Lambda \Phi_1, \Phi_1 > < \Lambda \Phi_2, \Phi_2 >,$$

with Lax matrix $M^{(0)}$

$$A^{(0)} = \lambda^2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda^2 \lambda_j^2}{\lambda^2 - \lambda_j^2} \phi_{1j} \phi_{2j}, \quad B^{(0)} = -\frac{1}{2} \lambda < \Lambda \Phi_1, \Phi_1 > - \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda \lambda_j^3}{\lambda^2 - \lambda_j^2} \phi_{1j}^2,$$

$$C^{(0)} = \frac{1}{2} \lambda < \Lambda \Phi_2, \Phi_2 > + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda \lambda_j^3}{\lambda^2 - \lambda_j^2} \phi_{2j}^2. \quad (2.10)$$

The spectral curve $\Gamma$ is

$$W(\lambda, \mu; \{P_i\}) = \mu^2 - \lambda^4 - \lambda^2 P_{-1} - P_0 - \sum_{j=1}^{N} \frac{P_j}{\lambda^2 - \lambda_j^2} = 0, \quad (2.11)$$
with \( P_{-1} = -\tilde{H}_0 \),

\[
P_0 = < \Lambda^4 \Phi_1, \Phi_2 > - \frac{1}{4} < \Lambda^3 \Phi_1, \Phi_1 > < \Lambda \Phi_2, \Phi_2 > \\
- \frac{1}{4} < \Lambda \Phi_1, \Phi_1 > < \Lambda^3 \Phi_2, \Phi_2 > + \frac{1}{4} < \Lambda^2 \Phi_1, \Phi_2 >^2 ,
\]

\[
P_j = \lambda^6_j \phi_{1j} \phi_{2j} - \frac{1}{4} \lambda^5_j \phi^2_{1j} - \frac{1}{4} \lambda^5_j \phi^2_{1j} + \frac{1}{2} \lambda^4_j \phi_{1j} \phi_{2j} < \Lambda^2 \Phi_1, \Phi_2 > - \frac{1}{4} \lambda^3_j \phi^2_{1j} < \Lambda^3 \Phi_2, \Phi_2 > \\
+ \frac{1}{4} \sum_{k \neq j} \frac{1}{\lambda^2_j - \lambda^2_k} (2 \lambda^4_j \lambda^6_k \phi_{1j} \phi_{2j} \phi_{1k} \phi_{2k} - \lambda^5_j \lambda^6_k \phi^2_{1j} \phi_{2k} - \lambda^3_j \lambda^6_k \phi^2_{1j} \phi^2_{2k} ), \quad j = 1, ..., N.
\]

\( P_1, ..., P_N \) are \( N \) independent integrals of motion in involution for FDIHS (2.9).

(2) For \( n = 1 \), (2.4) can be written as a FDIHS

\[
Q_x = \frac{\partial \tilde{H}_1}{\partial P}, \quad P_x = - \frac{\partial \tilde{H}_1}{\partial Q}, \quad (2.12)
\]

with

\[
\tilde{H}_1 = - < \Lambda^2 \Phi_1, \Phi_2 > - \frac{p_1}{\sqrt{2}} < \Lambda \Phi_1, \Phi_1 > + \frac{q_1}{\sqrt{2}} < \Lambda \Phi_2, \Phi_2 > - q_1^2 p_1^2 ,
\]

\[
Q = (\phi_{11}, ..., \phi_{1N}, q_1)^T, \quad P = (\phi_{21}, ..., \phi_{2N}, p_1)^T, \quad q_1 = \frac{q}{\sqrt{2}}, \quad p_1 = \frac{r}{\sqrt{2}}.
\]

and Lax matrix \( M^{(1)} \)

\[
A^{(1)} = \lambda^4 - q_1 p_1 \lambda^2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda^2 \lambda^2_j}{\lambda^2 - \lambda^2_j} \phi_{1j} \phi_{2j} ,
\]

\[
B^{(1)} = - \sqrt{2} q_1 \lambda^3 - \frac{1}{2} \lambda < \Lambda \Phi_1, \Phi_1 > - \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda \lambda^3_j}{\lambda^2 - \lambda^2_j} \phi^2_{1j} ,
\]

\[
C^{(1)} = - \sqrt{2} p_1 \lambda^4 + \frac{1}{2} \lambda < \Lambda \Phi_2, \Phi_2 > + \frac{1}{2} \sum_{j=1}^{N} \frac{\lambda \lambda^3_j}{\lambda^2 - \lambda^2_j} \phi^2_{2j} . \quad (2.13)
\]

(3) For \( n = 2 \), by introducing the following Jacobi-Ostrogradsky coordinates

\[
Q = (\phi_{11}, ..., \phi_{1N}, q_1, q_2)^T, \quad P = (\phi_{21}, ..., \phi_{2N}, p_1, p_2)^T ,
\]

\[
q_1 = q, \quad q_2 = r, \quad p_1 = - \frac{3}{16} q r^2 + \frac{1}{4} r_x, \quad p_2 = \frac{3}{16} q^2 r + \frac{1}{4} q_x,
\]

(2.4) can be transformed into a FDIHS

\[
Q_x = \frac{\partial \tilde{H}_2}{\partial P}, \quad P_x = - \frac{\partial \tilde{H}_2}{\partial Q} , \quad (2.14)
\]
with
\[
\tilde{H}_2 = -<\Lambda^2\Phi_1, \Phi_2> - \frac{1}{2}q_2 <\Lambda\Phi_1, \Phi_1> + \frac{1}{2}q_1 <\Lambda\Phi_2, \Phi_2> \\
\quad - \frac{1}{64}q_1^2q_2^2 - 4p_1p_2 - \frac{3}{4}q_1^2q_2p_1 + \frac{3}{4}q_1q_2^2p_2,
\]
and Lax matrix \(M^{(2)}\)
\[
A^{(2)} = \lambda^6 - \frac{1}{2}q_1q_2\lambda^4 + (p_2q_2 - p_1q_1)\lambda^2 + \frac{1}{2}\sum_{j=1}^{N} \frac{\lambda^2\lambda_j^2}{\lambda^2 - \lambda_j^2}\phi_{1j}\phi_{2j},
\]
\[
B^{(2)} = -q_1\lambda^5 + \left(\frac{1}{8}q_1^2q_2 + 2p_2\right)\lambda^3 - \frac{1}{2}\lambda <\Lambda\Phi_1, \Phi_1> - \frac{1}{2}\sum_{j=1}^{N} \frac{\lambda\lambda_j^3}{\lambda^2 - \lambda_j^2}\phi_{1j}^2,
\]
\[
C^{(2)} = -q_2\lambda^5 + \left(\frac{1}{8}q_1q_2^2 - 2p_1\right)\lambda^3 + \frac{1}{2}\lambda <\Lambda\Phi_2, \Phi_2> + \frac{1}{2}\sum_{j=1}^{N} \frac{\lambda\lambda_j^3}{\lambda^2 - \lambda_j^2}\phi_{2j}^2.
\]

3 BTs for high-order constrained flows of the Kaup-Newell hierarchy

Let \(\psi(x, \eta)\) be a solution of (2.2) and (2.3) with \(\lambda = \eta\). It is known [1, 15, 16] that the Darboux transformation (DT) for the Kuap-Newell hierarchy is given by
\[
\bar{\psi} = T\psi, \quad T = \begin{pmatrix} f\lambda & -\eta \\ -\eta & f^{-1}\lambda \end{pmatrix}, \quad f = \frac{\psi_2(x, \eta)}{\psi_1(x, \eta)},
\]
and
\[
\bar{q} = -2\eta f + qf^2, \quad \bar{\bar{r}} = \frac{2}{f}\eta + \frac{r}{f^2},
\]
namely, \(\bar{\psi}, \bar{q}\) and \(\bar{\bar{r}}\) satisfy (2.2), (2.3) and the equation (2.1).

Now let \(\psi(x, \eta)\) be a solution of (2.4) with \(\lambda = \eta\). Motivated by the DT for the Kaup-Newell hierarchy, it can be shown that the DT for the constrained flows (2.4) consists of (3.1), (3.2) and
\[
\bar{\phi}_{1j} = \frac{1}{\sqrt{\lambda_j^2 - \eta^2}}(\lambda_j\phi_{1j}f - \eta\phi_{2j}),
\]
\[
\bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j^2 - \eta^2}}(f^{-1}\lambda_j\phi_{2j} - \eta\phi_{1j}), \quad j = 1, \ldots, N,
\]
namely, under the transformation (3.1), (3.2) and (3.3), $\bar{q}, \bar{r}, \bar{\psi}$ and $\bar{\phi}_{1j}, \bar{\phi}_{2j}$ satisfy (2.4) and (2.7). We have

$$M^{(n)}(\bar{u}, \bar{\Psi}_1, \bar{\Psi}_2, \lambda)^T = TM^{(n)}(u, \Psi_1, \Psi_2, \lambda).$$  \tag{3.4}$$

It follows from (2.7)

$$f = \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)} = \frac{C^{(n)}(\eta)}{\mu + A^{(n)}(\eta)}. \tag{3.5}$$

By substituting (3.5) into (3.2) and (3.3), we obtain infinite number ($n = 0, 1,...$) of explicit BT $B_\eta$ for the constrained flows (2.4) as follows

$$\bar{q} = q \left( \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)} \right)^2 - 2\eta \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)}, \tag{3.6a}$$

$$\bar{r} = r \left( \frac{B^{(n)}(\eta)}{\mu - A^{(n)}(\eta)} \right)^2 + 2\eta \frac{B^{(n)}(\eta)}{\mu - A^{(n)}(\eta)}, \tag{3.6b}$$

$$\bar{\phi}_{1j} = \frac{1}{\sqrt{\lambda_j^2 - \eta^2}} \left( \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)} - \lambda_j \phi_{1j} - \eta \phi_{2j} \right), \tag{3.6c}$$

$$\bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j^2 - \eta^2}} \left( \frac{B^{(n)}(\eta)}{\mu - A^{(n)}(\eta)} - \lambda_j \phi_{2j} - \eta \phi_{1j} \right), \quad j = 1,...,N. \tag{3.6d}$$

It is found from (3.1) and (3.4)

$$(\lambda^2 - \eta^2)A^{(n)}(\lambda) = (\lambda^2 + \eta^2)A^{(n)}(\lambda) + f \lambda \eta B^{(n)}(\lambda) - \frac{\lambda \eta}{f} C^{(n)}(\lambda), \tag{3.7a}$$

$$(\lambda^2 - \eta^2)B^{(n)}(\lambda) = 2f \lambda \eta A^{(n)}(\lambda) + f^2 \lambda^2 B^{(n)}(\lambda) - \eta^2 C^{(n)}(\lambda) \tag{3.7b}$$

$$(\lambda^2 - \eta^2)C^{(n)}(\lambda) = -\frac{2\eta \lambda}{f} A^{(n)}(\lambda) - \eta^2 B^{(n)}(\lambda) + \frac{\lambda^2}{f^2} C^{(n)}(\lambda). \tag{3.7c}$$

Using the first constrained flows as model, we now show the BTs (3.6) to be canonical transformations by presenting their generating functions and check spectrality property with respect to the Bäcklund parameter $\eta$ and the `conjugated' variable $\mu.$
(1) For the first constrained flow, the FDIHS (2.9), by comparing the coefficients of $\lambda^3$ in (3.7b) after substituting (2.10), one gets
\[
f = \frac{2\eta + \sqrt{4\eta^2 + \langle \Lambda \Phi_1, \Phi_1 \rangle \langle \Lambda \bar{\Phi}_1, \bar{\Phi}_1 \rangle}}{\langle \Lambda \Phi_1, \Phi_1 \rangle}.
\] (3.8)

Then we have from (3.3)
\[
\phi_{2j} = -\frac{\sqrt{\lambda_j^2 - \eta^2 \phi_{1j}}}{\eta} + \frac{2\lambda_j \phi_{1j}}{\langle \Lambda \Phi_1, \Phi_1 \rangle} = \frac{\partial S^{(0)}}{\partial \phi_{1j}}, \tag{3.9a}
\]
\[
\phi_{2j} = \frac{\sqrt{\lambda_j^2 - \eta^2 \phi_{1j}}}{\eta} - \frac{2\lambda_j \phi_{1j}}{\langle \Lambda \Phi_1, \Phi_1 \rangle} = -\frac{\partial S^{(0)}}{\partial \phi_{2j}}, \tag{3.9b}
\]
where the generating function \( S^{(0)} \) for the canonical transformation (3.6) is given by
\[
S^{(0)}_\eta = 2\ln \left( 2\eta - \sqrt{4\eta^2 + \langle \Lambda \Phi_1, \Phi_1 \rangle \langle \Lambda \bar{\Phi}_1, \bar{\Phi}_1 \rangle} \right) - 2\ln \langle \Lambda \bar{\Phi}_1, \bar{\Phi}_1 \rangle
\]
\[
+ \frac{1}{\eta} \sqrt{4\eta^2 + \langle \Lambda \Phi_1, \Phi_1 \rangle \langle \Lambda \bar{\Phi}_1, \bar{\Phi}_1 \rangle} - \langle \Delta \Phi_1, \bar{\Phi}_1 \rangle,
\] (3.10)
where \( \Delta = \text{diag} \left( \frac{1}{\eta} \sqrt{\lambda_1^2 - \eta^2}, ..., \frac{1}{\eta} \sqrt{\lambda_N^2 - \eta^2} \right) \).

Furthermore, it is found from (2.10), (3.3) and (3.5)
\[
\tilde{\mu} \equiv \frac{\partial S^{(0)}}{\partial \eta} = -\frac{1}{\eta^2} \sqrt{4\eta^2 + \langle \Lambda \Phi_1, \Phi_1 \rangle \langle \Lambda \bar{\Phi}_1, \bar{\Phi}_1 \rangle} + \frac{1}{\eta^2} \sum_{j=1}^{N} \frac{1}{\sqrt{\lambda_j^2 - \eta^2}} \lambda_j^2 \phi_{1j} \phi_{1j}
\]
\[
= \frac{2}{\eta^3} \left( A^{(0)}(\eta) + fB^{(0)}(\eta) \right) = \frac{2}{\eta^3} \tilde{\mu},
\] (3.11)
which implies that the pair \( (\eta, \frac{1}{2} \eta^3 \tilde{\mu}) \) lies on the spectral curve \( \Gamma \)
\[
W(\eta, \frac{1}{2} \eta^3 \tilde{\mu}; \{P_i\}) = 0,
\]
namely satisfies the spectrality property [3]. Consider the composition \( B_{\eta_N} \circ ... \circ B_{\eta_1} \) of the Bäcklund transformation \( B_\eta \). Then the corresponding generating function
\( S^{(0)}_{\eta_1...\eta_N} \) becomes the generating function of the canonical transformation from \((\Phi_1, \Phi_2)\) to \\{\((\eta_i, \tilde{\mu}_i)\)\} given by the equations

\[
\phi_{2j} = \frac{\partial S^{(0)}_{\eta_1...\eta_N}}{\partial \phi_{1j}}, \quad \tilde{\mu}_j = \frac{\partial S^{(0)}_{\eta_1...\eta_N}}{\partial \eta_j}.
\]

The separation variables \\{\((\eta_i, \tilde{\mu}_i)\)\} satisfy the separation equations given by the spectral curve \((2.11)\)

\[
\tilde{\mu}_i^2 = \frac{4}{\eta^6} [\eta_i^4 + \eta_i^2 P_{-1} + P_0 + \sum_{j=1}^{N} \frac{P_j}{\eta_i^2 - \lambda_j^2}], \quad i = 1, ..., N.
\]

(2) For the second constrained flow, the FDIHS \((2.12)\), by comparing the coefficients of \(\lambda_5\) and \(\lambda_3\) in \((3.7b)\) after substituting \((2.13)\), one gets

\[
f = \frac{\eta + \sqrt{\eta^2 + 2q_1\bar{q}_1}}{\sqrt{2}q_1}, \quad (3.12a)
\]

\[
p_1 = \frac{1}{2\sqrt{2}\eta(\eta - \sqrt{2}f q_1)} \left( f^2 < \Lambda \Phi_1, \Phi_1 > - < \Lambda \Phi_1, \Phi_1 > + 2\sqrt{2}f^2 \bar{q}_1 \right). \quad (3.12b)
\]

Then by substituting \((3.12)\), we have from \((3.3)\) and \((3.2)\)

\[
\phi_{2j} = -\frac{\sqrt{\lambda_j^2 - \eta^2 \bar{\phi}_{1j}}}{\eta} + \frac{\lambda_j \phi_{1j}}{\sqrt{2q_1}} + \frac{\lambda_j \phi_{1j} \sqrt{\eta^2 + 2q_1\bar{q}_1}}{\sqrt{2}\eta q_1} = \frac{\partial S^{(1)}}{\partial \phi_{1j}}, \quad (3.13a)
\]

\[
\bar{\phi}_{2j} = \frac{\sqrt{\lambda_j^2 - \eta^2 \phi_{1j}}}{\eta} - \frac{\sqrt{2}\lambda_j q_1 \phi_{1j}}{\eta(\eta + \sqrt{\eta^2 + 2q_1\bar{q}_1})} = -\frac{\partial S^{(1)}}{\partial \phi_{1j}}, \quad (3.13b)
\]

\[
p_1 = \frac{-(\eta + \sqrt{\eta^2 + 2q_1\bar{q}_1})^2}{4\sqrt{2}\eta q_1 \sqrt{\eta^2 + 2q_1\bar{q}_1}} < \Lambda \Phi_1, \Phi_1 > - \frac{\eta \bar{q}_1}{\sqrt{\eta^2 + 2q_1\bar{q}_1}} \]

\[+ \frac{1}{2\sqrt{2}\eta \sqrt{\eta^2 + 2q_1\bar{q}_1}} < \Lambda \Phi_1, \Phi_1 > = \frac{\partial S^{(1)}}{\partial q_1}, \quad (3.13c)
\]

\[
\bar{p}_1 = \frac{\sqrt{2}q_1^2}{2\eta \sqrt{\eta^2 + 2q_1\bar{q}_1}(\eta + \sqrt{\eta^2 + 2q_1\bar{q}_1})^2} < \Lambda \bar{\Phi}_1, \bar{\Phi}_1 > + \frac{\eta q_1}{\sqrt{\eta^2 + 2q_1\bar{q}_1}} \]

\[- \frac{1}{2\sqrt{2}\eta \sqrt{\eta^2 + 2q_1\bar{q}_1}} < \Lambda \Phi_1, \Phi_1 > = -\frac{\partial S^{(1)}}{\partial \bar{q}_1}, \quad (3.13d)
\]
where the generating function $S^{(1)}$ for the canonical transformation (3.6) is given by

$$S^{(1)} = \frac{\eta + \sqrt{\eta^2 + 2q_1 \bar{q}_1}}{2\sqrt{2}\eta q_1} < \Lambda\Phi_1, \Phi_1 > - \eta \sqrt{\eta^2 + 2q_1 \bar{q}_1}$$

$$+ \frac{q_1}{\sqrt{2}\eta(q + \sqrt{\eta^2 + 2q_1 \bar{q}_1})} < \Lambda\bar{\Phi}_1, \bar{\Phi}_1 >$$

where $\Delta = \text{diag} \left( \frac{1}{\eta} \sqrt{\lambda_1^2 - \eta^2}, \ldots, \frac{1}{\eta} \sqrt{\lambda_N^2 - \eta^2} \right)$.

By a direct calculation, one can check the spectrality property by means of (3.13) and (2.13)

$$\frac{\partial S^{(1)}}{\partial \eta} = -\frac{\bar{q}_1}{\sqrt{2}\eta^2 \sqrt{\eta^2 + 2q_1 \bar{q}_1}} < \Lambda\Phi_1, \Phi_1 > - \frac{q_1}{\sqrt{2}\eta^2 \sqrt{\eta^2 + 2q_1 \bar{q}_1}} < \Lambda\bar{\Phi}_1, \bar{\Phi}_1 >$$

$$+ \frac{1}{\eta^2} \sum_{j=1}^{N} \frac{1}{\sqrt{\lambda_j^2 - \eta^2}} \lambda_j^2 \phi_{1j} \bar{\phi}_{1j} = \frac{2\eta^2 + 2q_1 \bar{q}_1}{\sqrt{\eta^2 + 2q_1 \bar{q}_1}} = \frac{2}{\eta^3} [A^{(1)}(\eta) + fB^{(1)}(\eta)] = \frac{2}{\eta^3} \mu. \tag{3.15}$$

(3) For the third constrained flow, the FDIHS (2.14), by comparing the coefficient of $\lambda^7$ in (3.7b) after substituting (2.15), one gets

$$f = \frac{\eta + \sqrt{\eta^2 + q_1 \bar{q}_1}}{q_1}. \tag{3.16}$$

Then we have from (3.3), (3.7b) and (3.7c)

$$\phi_{2j} = \frac{\lambda_j f \phi_{1j}}{\eta} + \sqrt{\lambda_j^2 - \eta^2} \phi_{1j} = \frac{\partial S^{(2)}}{\partial \phi_{1j}}, \tag{3.17a}$$

$$\bar{\phi}_{2j} = -\frac{\lambda_j \bar{\phi}_{1j}}{\eta f} + \sqrt{\lambda_j^2 - \eta^2} \phi_{1j} = -\frac{\partial S^{(1)}}{\partial \bar{\phi}_{1j}}, \tag{3.17b}$$

$$q_2 = \frac{4(f^2 p_2 - \bar{p}_2)}{\eta \sqrt{\eta^2 + q_1 \bar{q}_1}} = \frac{\bar{q}_1}{2\sqrt{\eta^2 + q_1 \bar{q}_1}} = -\frac{\partial S^{(2)}}{\partial p_2}, \tag{3.17c}$$

$$\bar{q}_2 = \frac{4(f^2 p_2 - \bar{p}_2)}{\eta f^2 \sqrt{\eta^2 + q_1 \bar{q}_1}} = \frac{q_1}{2\sqrt{\eta^2 + q_1 \bar{q}_1}} = \frac{\partial S^{(2)}}{\partial \bar{p}_2}. \tag{3.17d}$$
By a straightforward calculation, one can check the spectrality property by means of (3.17) and (2.15) where the generating function $S^{(2)}$ for the canonical transformation (3.6) is given by

\[
S^{(2)} = \frac{1}{\eta(\sqrt{\eta^2 + q_1 q_1})} \left[ -\frac{9}{32} \eta^6 - 2f^2 p_2^2 - 2f^{-2} \bar{p}_2^2 + \frac{1}{2} \eta q_1 p_2 (\sqrt{\eta^2 + q_1 q_1} + 3\eta) \\
- \frac{1}{2} \eta q_1 \bar{p}_2 (\sqrt{\eta^2 + q_1 q_1} - 3\eta) + 4p_2 \bar{p}_2 \right] - \frac{1}{32} \eta(\sqrt{\eta^2 + q_1 q_1})^3 - \frac{3}{16} \eta^2 \sqrt{\eta^2 + q_1 q_1} \\
+ \frac{1}{2f} \langle \Lambda \Phi, \Phi_1 \rangle > + \frac{f}{2\eta} \langle \Delta \Phi, \Phi_1 \rangle > - \langle \Delta \Phi, \Phi_1 \rangle >.
\]

By a straightforward calculation, one can check the spectrality property by means of (3.17) and (2.15)

\[
\frac{\partial S^{(2)}}{\partial \eta} = \frac{2}{\eta^3} [A^{(2)}(\eta) + fB^{(2)}(\eta)] = \frac{2}{\eta^3} \mu.
\]

In the exactly same way we can construct infinite number of explicit BTs for the high-order constrained flows (2.4) ($n = 3, 4, ...$) with the properties mentioned above.

4 High-order constrained flows of the KdV hierarchy

Consider the Schrödinger eigenvalue problem [17]

\[
\phi_{xx} + (\lambda + u)\phi = 0,
\]

\[(4.1)\]
which can be rewritten as in the matrix form
\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}_x = U \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}, \quad U = \begin{pmatrix}
0 & 1 \\
-\lambda - u & 0
\end{pmatrix}.
\] (4.2)

Take
\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}_{tn} = V^{(n)}(u, \lambda) \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}, \quad (4.3)
\]
where
\[
V^{(n)} = \sum_{i=0}^{n+1} \begin{pmatrix}
a_i & b_i \\
c_i & -a_i
\end{pmatrix} \lambda^{n+1-i} + \begin{pmatrix}
0 & 0 \\
b_{n+2} & 0
\end{pmatrix},
\]
a_0 = b_0 = 0, \quad c_0 = -1, \quad a_1 = 0, \quad b_1 = 1, \quad c_1 = -\frac{1}{2} u,
\]
a_2 = \frac{1}{4} u_x, \quad b_2 = -\frac{1}{2} u, \quad c_2 = \frac{1}{2} (u_{xx} + u^2), \ldots,
and in general for \( k = 1, 2, \ldots, \)
\[
a_k = -\frac{1}{2} b_{k,x}, \quad b_{k+1} = L b_k = -\frac{1}{2} L^{k-1} u, \quad c_k = -\frac{1}{2} b_{k,xx} - b_{k+1} - b_k u, \quad (4.4)
\]
with
\[
L = -\frac{1}{4} \partial^2 - u + \frac{1}{2} \partial^{-1} u_x, \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \partial^{-1} \partial = 1.
\]

Then the compatibility condition of Eqs (4.2) and (4.3) gives rise to the KdV hierarchy
\[
u_{tn} = \partial \frac{\delta H_n}{\delta u} = -2b_{n+2,x}, \quad n = 0, 1, \ldots, \quad (4.5)
\]
where \( H_n = \frac{4b_{n+3}}{2n+3}. \) We have
\[
\frac{\delta \lambda}{\delta u} = \phi_1^2. \quad (4.6)
\]

The high-order constrained flows of the KdV hierarchy consist of the equations obtained from the spectral problem (4.2) for \( N \) distinct \( \lambda_j \) and the restriction of the variational derivatives for the conserved quantities \( H_n \) and \( \lambda_j \) \( [12, 18, 19] \)

\[
\Phi_{1,x} = \Phi_2, \quad \Phi_{2,x} = -(\Lambda + u) \Phi_1, \quad (4.7a)
\]
\[
D \left[ \frac{\delta H_n}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} \right] \equiv D \left[ -2b_{n+2} - \sum_{j=1}^{N} \phi_j^2 \right] = 0, \quad (4.7b)
\]

The auxiliary linear problems associated with (4.7) are
\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = M^{(n)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]
(4.8)
with
\[
M^{(n)} = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & -A^{(n)} \end{pmatrix} = \sum_{k=0}^{n+1} \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix} \lambda^{n+1-k} + \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j} \phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j} \phi_{2j} \end{pmatrix}.
\]
The spectral curve \( \Gamma \) is given by the formula (2.8). We present the first three high-order constrained flows as follows.

1. For \( n = 0 \), (4.7) becomes a FDIHS
\[
\Phi_{1,x} = \frac{\partial \tilde{H}_0}{\partial \Phi_2}, \quad \Phi_{2,x} = -\frac{\partial \tilde{H}_0}{\partial \Phi_1},
\]
(4.9)
with Lax matrix \( M^{(0)} \)
\[
A^{(0)} = \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j} \phi_{2j}}{\lambda - \lambda_j}, \quad B^{(0)} = 1 - \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j},
\]
\[
C^{(0)} = -\lambda - \frac{1}{2} < \Phi_1, \Phi_1 > + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{2j}^2}{\lambda - \lambda_j}.
\]
(4.10)
The spectral curve \( \Gamma \) is
\[
W(\lambda, \mu; \{P_i\}) = \mu^2 + \lambda - \sum_{j=1}^{N} \frac{P_j}{\lambda - \lambda_j} = 0,
\]
(4.11)
with
\[
P_j = \frac{1}{2} (\phi_{2j}^2 + \lambda_j \phi_{1j}^2 + \frac{1}{2} < \Phi_1, \Phi_1 > \phi_{1j}^2) + \frac{1}{2} \sum_{k \neq j} \frac{\phi_{1j} \phi_{2j} \phi_{1k} \phi_{2k} - \phi_{1k}^2 \phi_{2k}^2}{\lambda_j - \lambda_k}, \quad j = 1, ..., N.
\]
\( P_1, ..., P_N \) are \( N \) independent integrals of motion in involution for FDIHS (4.9).
(2) For \( n = 1 \), (4.7) can be written as a FDIHS

\[ Q_x = \frac{\partial \tilde{H}_1}{\partial P}, \quad P_x = -\frac{\partial \tilde{H}_1}{\partial Q}, \]  

(4.12)

with

\[ \tilde{H}_1 = \frac{1}{2} < \Lambda \Phi_1, \Phi_1 > + \frac{1}{2} < \Lambda \Phi_2, \Phi_2 > + \frac{1}{2} q_1 < \Phi_1, \Phi_1 > + \frac{1}{8} q_1^3 + 4 p_1^2, \]

\[ Q = (\phi_{11}, ..., \phi_{1N}, q_1)^T, \quad P = (\phi_{21}, ..., \phi_{2N}, p_1)^T, \quad q_1 = u, \quad p_1 = \frac{1}{8} u_x \]

and Lax matrix \( M^{(1)} \)

\[ A^{(1)} = 2 p_1 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j} \phi_{2j}}{\lambda - \lambda_j}, \quad B^{(1)} = \lambda - \frac{1}{2} q_1 - \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{ij}^2}{\lambda - \lambda_j}, \]

\[ C^{(1)} = -\lambda^2 - \frac{1}{2} q_1 \lambda - \frac{1}{2} < \Phi_1, \Phi_1 > + \frac{1}{4} q_1^2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{2j}^2}{\lambda - \lambda_j}. \]  

(4.13)

(3) For \( n = 2 \), by introducing the following Jacobi-Ostrogradsky coordinates

\[ Q = (\phi_{11}, ..., \phi_{1N}, q_1, q_2)^T, \quad P = (\phi_{21}, ..., \phi_{2N}, p_1, p_2)^T, \]

\[ q_1 = u, \quad q_2 = u_{xx} + 5 u^2, \quad p_1 = -\frac{1}{32} u_{xxx}, \quad p_2 = -\frac{1}{32} u_x, \]

(4.7) can be transformed into a FDIHS

\[ Q_x = \frac{\partial \tilde{H}_2}{\partial P}, \quad P_x = -\frac{\partial \tilde{H}_2}{\partial Q}, \]  

(4.14)

with

\[ \tilde{H}_2 = \frac{1}{2} < \Phi_2, \Phi_2 > + \frac{1}{2} < \Lambda \Phi_1, \Phi_1 > + \frac{1}{2} q_1 < \Phi_1, \Phi_1 > - \frac{5}{32} q_1^2 q_2 - 32 p_1 p_2 + \frac{5}{16} q_1^4 - 160 q_1 p_2^2 + \frac{1}{64} q_2^2, \]

and Lax matrix \( M^{(2)} \)

\[ A^{(2)} = -8 p_2 \lambda + 2 p_1 + 12 q_1 p_2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j} \phi_{2j}}{\lambda - \lambda_j}, \]

\[ B^{(2)} = \lambda^2 - \frac{1}{2} q_1 \lambda + \frac{1}{8} q_2 - \frac{1}{4} q_1^2 - \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{1j}^2}{\lambda - \lambda_j}, \]

\[ C^{(2)} = -\lambda^3 - \frac{1}{2} q_1 \lambda^2 + \left( \frac{1}{8} q_2 - \frac{1}{2} q_1^2 \right) \lambda - \frac{1}{2} < \Phi_1, \Phi_1 > + \frac{1}{8} q_1 q_2 \]

\[ - \frac{3}{8} q_1^3 - 64 p_2^2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\phi_{2j}^2}{\lambda - \lambda_j}. \]  

(4.15)
5 BTs for high-order constrained flows of the KdV hierarchy

Let $\psi(x, \eta)$ be a solution of (4.2) and (4.3) with $\lambda = \eta$. It is known \[1, 20\] that there is the DT for the KdV hierarchy given by

$$\bar{\psi} = T \psi, \quad T = \begin{pmatrix} -f & 1 \\ -\lambda + f^2 & -f \end{pmatrix}, \quad f = \frac{\psi_2(x, \eta)}{\psi_1(x, \eta)},$$

and

$$\bar{u} = u + 2\ln \psi_1 = -u - 2\eta - 2f^2,$$

namely under the transformation (5.1) and (5.2), $\bar{\psi}, \bar{u}$ satisfy (4.2), (4.3) and (4.5).

We now consider the DTs for high-order constrained flows (4.7). Let $\psi(x, \eta)$ be a solution of (4.8) with $\lambda = \eta$. Motivated by the DT for the KdV hierarchy, we find that the DT for the constrained flows (4.7) consists of (5.1), (5.2) and

$$\bar{\phi}_{1j} = \frac{1}{\sqrt{\lambda_j - \eta}} (\phi_{2j} - f\phi_{1j}), \quad \bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j - \eta}} [(\eta - \lambda + f^2)\phi_{1j} - f\phi_{2j}], \quad j = 1, ..., N,$$

namely under the transformation (5.1), (5.2) and (5.3), $\bar{\psi}, \bar{u}$ and $\bar{\phi}_{1j}, \bar{\phi}_{2j}$ satisfy (4.7) and (4.8).

By substituting (3.5) into (5.2) and (5.3), we obtain infinite number ($n = 0, 1, ...$) of the explicit BT $B_\eta$ for the constrained flows (4.7) as follows

$$\bar{u} = -u - 2\eta - 2\left(\frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)}\right)^2,$$

$$\bar{\phi}_{1j} = \frac{1}{\sqrt{\lambda_j - \eta}} \left(-\frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)}\phi_{1j} + \phi_{2j}\right),$$

$$\bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j - \eta}} \left[(\eta - \lambda_j + \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)})^2 \phi_{1j} - \frac{\mu - A^{(n)}(\eta)}{B^{(n)}(\eta)}\phi_{2j}\right], \quad j = 1, ..., N.$$

It is found from (3.4)

$$(\lambda - \eta)\bar{A}^{(n)}(\lambda) = (-\lambda + \eta + 2f^2)A^{(n)}(\lambda) + f(f^2 + \eta - \lambda)B^{(n)}(\lambda) - fC^{(n)}(\lambda),$$

(5.5a)
\[(\lambda - \eta)B^{(n)}(\lambda) = 2fA^{(n)}(\lambda) + f^2B^{(n)}(\lambda) - C^{(n)}(\lambda), \quad (5.5b)\]

\[(\lambda - \eta)C^{(n)}(\lambda) = -2f(f^2 + \eta - \lambda)A^{(n)}(\lambda) - (f^2 + \eta - \lambda)^2B^{(n)}(\lambda) + f^2C^{(n)}(\lambda). \quad (5.5c)\]

(1) For the first constrained flow, the FDIHS (4.9), using (4.10) and (5.5b), one gets

\[f = \sqrt{-\eta - \frac{1}{2} < \Phi_1, \Phi_1 > - \frac{1}{2} < \Phi_1, \Phi_1 >} = \frac{\partial S^{(0)}}{\partial \phi_{1j}}, \quad (5.6)\]

Then we have from (5.3)

\[\phi_{2j} = \sqrt{\lambda_j - \eta \phi_{1j}} + \phi_{1j} \sqrt{-\eta - \frac{1}{2} < \Phi_1, \Phi_1 > - \frac{1}{2} < \Phi_1, \Phi_1 >} = \frac{\partial S^{(0)}}{\partial \phi_{1j}}, \quad (5.7a)\]

\[\bar{\phi}_{2j} = -\sqrt{\lambda_j - \eta \phi_{1j}} - \bar{\phi}_{1j} \sqrt{-\eta - \frac{1}{2} < \Phi_1, \Phi_1 > - \frac{1}{2} < \Phi_1, \Phi_1 >} = -\frac{\partial S^{(0)}}{\partial \phi_{1j}}, \quad (5.7b)\]

where the generating function \(S^{(0)}\) for the canonical transformation (5.4) is given by

\[S^{(0)} = -\frac{2}{3} \left( -\eta - \frac{1}{2} < \Phi_1, \Phi_1 > - \frac{1}{2} < \Phi_1, \Phi_1 > \right)^{\frac{3}{2}} + \sum_{j=1}^{N} \sqrt{\lambda_j - \eta \phi_{1j}\bar{\phi}_{1j}}.\]

Furthermore, it is found

\[\frac{\partial S^{(0)}}{\partial \eta} = 2 \sum_{j=1}^{N} \frac{1}{\eta - \lambda_j} \phi_{1j} \phi_{2j} - \frac{1}{\eta - \lambda_j} \phi_{1j}^2 + \frac{f}{\eta - \lambda_j} \phi_{1j} + f\]

\[= A^{(0)}(\eta) + f B^{(0)}(\eta) = \mu, \quad (5.8)\]

which implies that the pair \((\eta, \frac{\partial S^{(0)}}{\partial \eta})\) lies on the spectral curve \(\Gamma\) (4.11), namely the canonical transformation satisfies the spectrality property. (5.7) and (5.8) were also given in [5] in a little different way. Consider the composition \(B_{\eta_1...\eta_N} = B_{\eta_1} \circ ... \circ B_{\eta_N}\) of the Bäcklund transformation \(B_{\eta_1}\). Then the corresponding generating function \(S_{\eta_1...\eta_N}^{(0)}\) becomes the generating function of the canonical transformation from \((\Phi_1, \Phi_2)\) to \((\eta, \mu)\) given by the equations

\[\phi_{2j} = \frac{\partial S_{\eta_1...\eta_N}^{(0)}}{\partial \phi_{1j}}, \quad \mu_j = \frac{\partial S_{\eta_1...\eta_N}^{(0)}}{\partial \eta_j}.\]
The separation variables \( \{ (\eta_i, \mu_i) \} \) satisfy the separation equations given by the spectral curve \([4.11]\)

\[
\mu_i^2 = -\eta_i + \sum_{j=1}^{N} \frac{P_j}{\eta_i - \lambda_j}, \quad i = 1, ..., N.
\]

(2) For the second constrained flow, the FDIHS \([4.12]\), by comparing the coefficients of \( \lambda \) and \( \lambda^0 \) in \([5.5\text{b}]\) after substituting \([4.13]\), one gets

\[
f = \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}, \quad (5.9a)
\]

\[
p_1 = \frac{1}{8f}(\eta q_1 + f^2q_1 - \frac{1}{2}q_1^2 - <\Phi_1, \Phi_1> - <\bar{\Phi}_1, \bar{\Phi}_1>). \quad (5.9b)
\]

Then by substituting \([5.9a]\), we have from \([5.2]\), \([5.3]\) and \([5.9b]\)

\[
\phi_{2j} = \sqrt{\lambda_j - \eta \bar{\phi}_{1j}} + \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}\phi_{1j} = \frac{\partial S^{(1)}}{\partial \phi_{1j}}, \quad (5.10a)
\]

\[
\bar{\phi}_{2j} = -\sqrt{\lambda_j - \eta \phi_{1j}} - \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}\bar{\phi}_{1j} = -\frac{\partial S^{(1)}}{\partial \phi_{1j}}, \quad (5.10b)
\]

\[
p_1 = -\frac{2q_1^2 + 2\eta q_1 - 2\eta \bar{q}_1 + q_1 \bar{q}_1 + 2 <\Phi_1, \Phi_1> + 2 <\bar{\Phi}_1, \bar{\Phi}_1>}{16\sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}} = \frac{\partial S^{(1)}}{\partial q_1}, \quad (5.10c)
\]

\[
\bar{p}_1 = \frac{2\bar{q}_1^2 - 2\eta(q_1 - \bar{q}_1) + q_1 \bar{q}_1 + 2 <\Phi_1, \Phi_1> + 2 <\bar{\Phi}_1, \bar{\Phi}_1>}{16\sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}} = -\frac{\partial S^{(1)}}{\partial \bar{q}_1}, \quad (5.10d)
\]

where the generating function \( S^{(1)} \) for the canonical transformation \([5.4]\) is given by

\[
S^{(1)} = \sum_{j=1}^{N} \sqrt{\lambda_j - \eta \phi_{1j} \bar{\phi}_{1j}} + \frac{1}{4}q_1^2 \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)} + \frac{1}{2} \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)} <\Phi_1, \Phi_1> + \frac{1}{2}q_1 \left( \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)} \right)^3 \\
+ \frac{2}{5} \left( \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)} \right)^5 - \frac{1}{2} \eta \bar{q}_1 \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)}.
\]
By a direct calculation, one can find the spectrality property:

\[
\frac{\partial S^{(1)}}{\partial \eta} = -\frac{1}{2} \sum_{j=1}^{N} \phi_{1j} \bar{\phi}_{1j} + \frac{1}{2} f \sum_{j=1}^{N} \phi_{1j}^2 - \frac{1}{4} \eta f (\langle \Phi_1, \Phi_1 \rangle + \langle \bar{\Phi}_1, \bar{\Phi}_1 \rangle)
+ \frac{1}{8f} [q_1 \bar{q}_1 - 8\eta^2 - 2\eta(q_1 + \bar{q}_1)] = A^{(1)}(\eta) + fB^{(1)}(\eta) = \mu.
\]

(3) For the third constrained flow, the FDIHS (5.14), by comparing the coefficient of \(\lambda^2\) in (5.13b) after substituting (4.15), one gets

\[
f = \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1)},
\]

Then by substituting (5.13), we have from (5.2), (5.3) and (5.3)

\[
\phi_{2j} = \sqrt{\lambda_j - \eta \phi_{1j}} + \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1) \phi_{1j}} = \frac{\partial S^{(2)}}{\partial \phi_{1j}},
\]

\[
\bar{\phi}_{2j} = -\sqrt{\lambda_j - \eta \phi_{1j}} + \sqrt{-\eta - \frac{1}{2}(q_1 + \bar{q}_1) \phi_{1j}} = -\frac{\partial S^{(2)}}{\partial \bar{\phi}_{1j}},
\]

\[
q_2 = -\bar{q}_2 + 8f^4 + 24\eta f^2 + 4q_1 f^2 - 128p_2 f + 16\eta^2 + 12\eta q_1 + 6q_1^2 = -\frac{\partial S^{(2)}}{\partial p_2},
\]

\[
p_1 = -\frac{1}{4} f^5 + \frac{1}{8} q_1^3 - \frac{3}{4} \eta f^3 + 4p_2 f^2 - \frac{1}{2} \eta^2 f + \frac{3}{8} \eta q_1 f + \frac{1}{32} \bar{q}_2 f - 10q_1 p_2 + \frac{1}{4} f \frac{3}{8} q_1^3
\]

\[
+ \frac{3}{2} \eta q_1^2 + 2\eta^2 q_1 - \frac{1}{8} q_1 \bar{q}_2 - 64p_2^2 - \frac{1}{8} \eta \bar{q}_2 + \frac{1}{4} q_1^2 - \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle + \frac{1}{2} \langle \bar{\Phi}_1, \bar{\Phi}_1 \rangle = \frac{\partial S^{(1)}}{\partial q_1},
\]

\[
\bar{p}_2 = -p_2 - \frac{1}{8} \eta f - \frac{1}{8} q_1 f - \frac{1}{8} f^3 = -\frac{\partial S^{(2)}}{\partial q_2},
\]

\[
\bar{p}_1 = -\frac{9}{4} f^5 - \frac{25}{8} q_1 f^3 - \frac{19}{4} \eta f^3 - 8p_2 f^2 - \frac{5}{2} \eta^2 f - \frac{31}{8} \eta q_1 f + \frac{3}{32} \bar{q}_2 f - \frac{9}{8} q_1^2 f
\]

\[
- 2q_1 p_2 - 12q_2 \eta - \frac{1}{4} f \left[ \frac{3}{8} q_1^3 + \frac{3}{2} \eta q_1^2 + 2\eta^2 q_1 - \frac{1}{8} q_1 \bar{q}_2 \right]
\]

\[
- 64p_2^2 - \frac{1}{8} \eta \bar{q}_2 + \frac{1}{4} \eta \bar{q}_1^2 - \frac{1}{2} \langle \Phi_1, \Phi_1 \rangle - \frac{1}{2} \langle \bar{\Phi}_1, \bar{\Phi}_1 \rangle = \frac{\partial S^{(1)}}{\partial \bar{q}_1},
\]
where the generating function $S^{(2)}$ for the canonical transformation (5.4) is given by

$$S^{(2)} = \sum_{j=1}^{N} \sqrt{\lambda_j - \eta \phi_{1j} \bar{\phi}_{1j} + \frac{1}{2} f(\Phi_1, \bar{\Phi}_1) + \frac{1}{8} f^3 \bar{q}_2}$$

$$+ \frac{f}{4} \left[ \frac{1}{2} \eta \bar{q}_2 + \frac{1}{2} q_1 \bar{q}_2 + 256p_2^2 - \frac{5}{14} q_1^3 - \frac{1}{14} q_1^2 q_1 - \frac{1}{7} \eta q_1^2 - \frac{4}{7} q_1 q_1^2 + \frac{5}{7} \eta q_1 \bar{q}_1 - \frac{2}{7} \eta^2 q_1 \right]$$

$$+ \frac{1}{14} q_1^3 - \frac{1}{7} \eta q_1^2 + \frac{2}{7} \eta q_1 \bar{q}_1 - \frac{2}{7} \eta q_1 \bar{q}_2 + 4 \eta p_2 \bar{q}_1 - 6 \eta p_2 q_1^2 - 2 \eta p_2 q_1 - 4 \eta q_1 - 2 \eta p_2 q_1^2.$$

(5.15)

By a direct calculation, one can find the spectrality property

$$\frac{\partial S^{(2)}}{\partial \eta} = A^{(2)}(\eta) + f B^{(2)}(\eta) = \mu.$$

Similarly we can construct infinite number of explicit BTs for the high-order constrained flows (4.7) $(n = 3, 4, ...)$ with the properties mentioned above.

6 Conclusion

In contrast with few examples of Bäcklund transformations with the properties described above presented in [2, 3, 4, 5, 6], we propose a way to construct infinite number of explicit Bäcklund transformations for high-order constrained flows of soliton hierarchy by means of the Darboux transformations for soliton equations and the Lax representation for the high-order constrained flows. By constructing the generating functions, it is shown that these BTs are canonical transformations including Bäcklund parameter $\eta$ and a spectrality property holds with respect to the Bäcklund parameter $\eta$ and the conjugate variable $\mu$ with the pair $(\eta, \mu)$ lying on the spectral curve.

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