Calderón Zygmund decompositions on amenable groups

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Abstract

We propose a simple abstract version of Calderón–Zygmund theory, which is applicable to spaces with exponential volume growth, and then show that amenable Lie groups can be treated within this framework.

1 Abstract Calderón–Zygmund theory

Definition 1.1 We say that the space $M$ with metric $d$ and Borel measure $\mu$ has the Calderón–Zygmund property if there exists a constant $C$ such that for every $f$ in $L^1$ and for every $\lambda > C\frac{\|f\|_{L^1}}{\mu(M)}$ ($\lambda > 0$ if $\mu(M) = \infty$) we have a decomposition $f = \sum f_i + g$, such that there exist sets $Q_i$, numbers $r_i$, and points $x_i$ satisfying:

- $f_i = 0$ outside $Q_i$,
- $\int f_i \, d\mu = 0$,
- $Q_i \subset B(x_i, Cr_i)$,
- $\sum \mu(Q^*_i) \leq C\frac{\|f\|_{L^1}}{\lambda}$, where $Q^*_i = \{x : d(x, Q_i) < r_i\}$,
- $\sum \|f_i\|_{L^1} \leq C\|f\|_{L^1}$,
- $\|g\| \leq C\lambda$.

Since $g = f - \sum f_i$, we have $\|g\|_{L^1} \leq C'\|f\|_{L^1}$, hence $\|g\|_{L^2}^2 \leq C''\lambda\|f\|_{L^1}$.

Theorem 1.2 If $G$ is a connected amenable Lie group, $d$ is a right-invariant optimal control metric on $G$ then $G$ (with $d$ and left Haar measure) satisfies Calderón–Zygmund property.

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From now to the end of the section we consider a fixed separable metric space \(M\) with a metric \(d\) and measure \(\mu\). We assume that all balls have finite measure. All operations on sets are meant as operations in the lattice of measurable sets on \(M\).

We say that a set \(Q\) is a doubling set with a constant \(C\) iff \(\mu(\{x : d(x, Q) \leq \frac{1}{C}\text{diam}(Q)\}) \leq C\mu(Q)\).

We say that a family \(A\) of sets is a doubling family with a constant \(C\) if for each \(Q \in A\) it satisfies the following two conditions:

- the set
  \[
  \tilde{Q} = \bigcup_{R \in A, R \cap Q \neq \emptyset, \mu(R) \leq 2\mu(Q)} R
  \]
  is contained in a set \(S \in A\) such that \(\mu(S) \leq C\mu(Q)\).

- either \(\mu(M) \leq C\mu(Q)\) and \(M \in A\) or there is \(R \in A\) such that \(Q \subset R\) and \(2\mu(Q) \leq \mu(R) \leq C\mu(Q)\).

**Remark** We get equivalent definition of doubling family if in the definition of \(\tilde{Q}\) we replace \(\mu(R) \leq 2\mu(Q)\) by \(\mu(R) \leq \mu(Q)\). Namely, choose \(S \in A\) using second point of the definition so \(\tilde{Q} \subset S\) and \(\min(\mu(M), 2\mu(Q)) \leq \mu(S) \leq C\mu(Q)\).

Now, if \(R \cap Q \neq \emptyset\), then \(R \cap S \neq \emptyset\). Also, if \(\mu(R) \leq 2\mu(Q)\), then \(\mu(R) \leq \mu(S)\). So, \(\tilde{Q}_1 \subset \tilde{S}_2\), where \(\tilde{Q}_1\) uses first definition while \(\tilde{S}_2\) uses condition \(\mu(R) \leq \mu(Q)\). So, applying weaker version of first condition to \(S\) we get stronger version for \(Q\) with \(C\) replaced by \(C^2\).

We say that a family \(A\) of sets is dense if for a.e \(x\) and all \(r > 0\) there is a \(Q \in A\) such that \(x \in Q\) and \(\text{diam}(Q) \leq r\).

**Theorem 1.3** Assume \(A\) is a doubling family. The maximal function

\[
M_A f(x) = \sup_{Q \in A, x \in Q} \frac{1}{\mu(Q)} \int_Q |f|
\]

is of weak type \((1, 1)\).

If \(A\) is dense then for each \(f \in L^1(\mu)\)

\[
\lim_{r \to 0^+} \sup_{Q \in A, \text{diam}Q \leq r, x \in Q} \frac{1}{\mu(Q)} \int_Q |f| = |f|(x)
\]

except possibly for a set of \(\mu\) measure 0.

If \(A\) is dense and consists of doubling sets (with a common doubling constant \(C\)) then \(M\) satisfies Calderón–Zygmund property.

**Proof:** Let \(S_0\) be the set of all \(R \in A\) such that

\[
\int_R |f| > \lambda \mu(R).
\]
Let $v_0 = \sup_{R \in S_0} \mu(R)$. $v_0$ is finite since for all $R \in S_0$, $\mu(R) < \|f\|_{L^1}/\lambda$. We take as $R_0$ an element of $S_0$ such that $\mu(R_0) > v_0/2$ (we choose arbitrarily among all $R$ satisfying this bound). Now, we proceed inductively: $S_i$ is the set of all $R \in S_{i-1}$ such that $R$ is disjoint with all $\tilde{R}_j$, $j = 0, \ldots, i-1$. Put $v_i = \sup_{R \in S_i} \mu(R)$ and choose $R_i$ such that $\mu(R_i) > v_i/2$.

We choose as $Q_i$ element of $A$ such that $\tilde{R}_i \subset Q_i$ and $\mu(Q_i) \leq C\mu(R_i)$ (possible since our family is a doubling family).

$R_i$ are disjoint, so

$$\sum_i \mu(R_i) \leq \frac{1}{\lambda} \sum_i \int_{R_i} |f| \leq \frac{1}{\lambda} \|f\|_{L^1}$$

and

$$\sum_i \mu(\tilde{R}_i) \leq \sum_i \mu(Q_i) \leq C \sum_i \mu(R_i) \leq \frac{C}{\lambda} \|f\|_{L^1}.$$

If $\int_Q |f| > \lambda \mu(R)$ then by construction $R$ intersects some $R_i$ such that $\mu(R_i) \leq 2\mu(R_i)$, so $R \subset \tilde{R}_i \subset Q_i$. Hence, putting $E = \bigcup_i \tilde{R}_i$ we have $M_A f \leq \lambda$ outside $E$ and $\mu(E) \leq \frac{C}{\lambda} \|f\|_{L^1}$ which gives the first claim.

If the family $A$ is dense then the second claim is valid for continuous functions with bounded support. Since such functions are dense in $L^1(\mu)$ we get the second claim using weak type $(1,1)$ of $M_A$.

We put $U_0 = \tilde{R}_0$ and $U_i = \tilde{R}_i - \bigcup_{j<i} \tilde{R}_j$. Let $h_i(x) = f(x)$ on $U_i$ and 0 otherwise. We claim that

$$\int_{Q_i} |h_i| \leq C \lambda \mu(Q_i)$$

We need to consider three cases. In the first case $C \mu(Q_i) > \mu(M)$ and

$$\int_{Q_i} |f| > C \lambda \mu(Q_i).$$

Then

$$\|f\|_{L^1} > C \lambda \mu(Q_i) > \lambda \mu(M)$$

and there is no need to do Calderón-Zygmund decomposition ($\lambda$ is out of range). In the second case $\mu(Q_i) > 2\mu(R_i)$ and we put $Q = Q_i$. In the third case $\mu(Q_i) \leq 2\mu(R_i)$ we take as $Q$ an element of $A$ such that $Q_i \subset Q$, $\mu(Q) < C \mu(Q_i)$ and $\mu(Q) \geq 2\mu(Q_i) \geq 2\mu(R_i)$. If

$$\int_Q |f| \leq \lambda \mu(Q),$$

then

$$\int_{Q_i} |h_i| \leq \int_Q |f| \leq \lambda \mu(Q) \leq C \lambda \mu(Q_i).$$
If the inequality above does not hold, then
\[
\int_Q |f| > \lambda \mu(Q)
\]
and by our construction \( Q \) intersects some of \( R_j \) with \( j < i \) and \( 2\mu(R_j) > \mu(Q) \). So \( Q_i \subset Q \subset \tilde{R}_j \) and in this case \( h_i = 0 \) (since \( U_i = \emptyset \)).

Next, let \( \chi_{R_i} \) be the indicator (characteristic) function of \( R_i \). We put
\[
f_i = h_i - \frac{\int h_i}{\mu(R_i)} \chi_{R_i},
\]
\[
g = f - \sum f_i,
\]
\[
r_i = \frac{1}{C} \text{diam}(Q_i).
\]
For \( x_i \) we choose an arbitrary point from \( Q_i \), which finishes construction of Calderón-Zygmund decomposition.

We need to check that conditions of Calderón-Zygmund decomposition are satisfied. By construction \( f_i = 0 \) outside \( Q_i \) and \( \int f_i = 0 \). Also \( Q_i \subset \overline{B(x_i, Cr_i)} \). Since each \( Q_i \) is doubling (with constant \( C \)) we have
\[
\mu(Q_i^*) \leq C \mu(Q_i) \leq C^2 \mu(R_i)
\]
and
\[
\sum_i \mu(Q_i^*) \leq C^2 \mu(R_i) \leq \frac{C^2}{\lambda} \|f\|_{L^1}.
\]
Also
\[
\|f_i\|_{L^1} \leq 2\|h_i\|_{L^1} = 2 \int_{U_i} |f|
\]
so (noting that \( U_i \) are disjoint)
\[
\sum_i \|f_i\|_{L^1} \leq 2 \sum_i \int_{U_i} |f| \leq 2 \|f\|_{L^1}.
\]
For \( x \) outside \( E \) we have \( |g|(x) \leq |f|(x) \leq M_A f(x) \leq \lambda \) (the second inequality follows from second claim of our theorem). On \( E \), since \( R_i \) are disjoint, we have
\[
\sup_{x \in E} |g|(x) \leq \sup_i \frac{\int h_i}{\mu(R_i)} \leq \sup_i \frac{C \lambda \mu(Q_i)}{\mu(R_i)} \leq C^2 \lambda.
\]
2 Base case

In this section we will prove a special case of the main theorem.

**Theorem 2.1** Assume that $G$ is a connected and simply connected solvable Lie group, $N$ and $W$ are connected and simply connected nilpotent subgroups of $G$, $N$ is a normal subgroup, $G = WN$. Also assume that $G$ and $N$ are equipped with right invariant Riemannian metrics $d_G$ and $d_N$ such that for $x, y \in N$

$$\exp(C_1 d_G(x, y)) \leq 1 + d_N(x, y) \leq \exp(C_2(d_G(x, y) + 1))$$

Then $G$ contains dense doubling family of doubling sets and $G$ satisfies Calderón–Zygmund property.

Before we prove 2.1 we need some preparations. Let $W_0 = W/(W \cap N) = G/N$ and let $\pi : G \mapsto W_0$ be the quotient mapping. On $W_0$ we put quotient metric (from $W$): $d_{W_0}(x, y) = \inf_{v, z \in W, \pi(v) = x, \pi(z) = y} d_G(v, z)$.

The following lemma is known, but we provide proof for convenience.

**Lemma 2.2** Let $X$ and $Y$ be complete separable metric spaces, $Y$ compact, $R \subset X \times Y$ a compact subset. If for each $y \in Y$ the set $R_y = \{x : (x, y) \in R\}$ is nonempty, then there exist Borel measurable $\chi : Y \mapsto X$ such that $(\chi(y), y) \in R$ for all $y \in Y$.

**Proof:** Without loss of generality we may assume that $X = H = [0, 1]^\\infty$. Namely, it is well known that any complete separable metric space is homeomorphic to a subset of $H$, so we can treat $X$ as a subset of $H$. Then image of $R$ is a compact subset of $H \times Y$. Now, again without loss of generality we may assume that $X = [0, 1]$. Namely, it is well known that there is continuous function $f$ from $[0, 1]$ onto $H$. Putting $h(x, y) = (f(x), y)$ we get mapping from $[0, 1] \times Y$ onto $H \times Y$. $S = h^{-1}(R)$ is a closed (hence compact) subset of $[0, 1] \times Y$. Once we build $\chi$ for $S$ the composition $f \circ \chi$ gives result for $R$. So now $X = [0, 1]$. We claim that $\chi(y) = \inf(R_y)$ is Borel measurable and has required properties. First, since $R_y$ is nonempty $\chi$ is well defined. Next, since $R_y$ is compact, we have $\inf(R_y) \in R_y$ so indeed $(\chi(y), y) \in R$.

To show that $\chi$ is Borel measurable we need extra construction. For each rational $q \in [0, 1]$ consider set $T_q = R \cap ([0, q] \times Y)$ and let $F_q$ be projection of $T_q$ on $Y$. We put $\phi_q(y) = q$ for $y \in T_q$ and $\phi_q(y) = 1$ otherwise. Since $T_q$ and $F_q$ are compact $\phi_q$ is Borel measurable. We claim that $\chi(y) = \inf_q \phi_q(y)$. Clearly $(\chi(y), y) \in R$, so for any $q \geq \chi(y)$ we have $(\chi(y), y) \in T_q$, so $y \in F_q$ and $\phi_q(y) = q$. Since $\chi(y) = \inf(R_y)$, for $q < \chi(y)$ we have $[0, q] \cap R_y = \emptyset$, so $[0, q] \times \{y\} \cap R = \emptyset$. Hence $y \notin F_q$ and $\phi_q(y) = 1$. Together,

$$\inf_q \phi_q(y) = \inf_{q \geq \chi(y)} q = \chi(y).$$

Since infimum of countable family of Borel measurable functions is Borel measurable $\chi$ is Borel measurable. \(\diamondsuit\)
Lemma 2.3 There exists a Borel measurable map \( \chi \) from \( W_0 \) into \( W \) such that for all \( x \in W_0 \), \( \pi \circ \chi(x) = x \) and \( d_G(\chi(x), e) = d_{W_0}(x, e) \).

Proof: Consider the relation \( R(x, y) \Leftrightarrow (x = \pi(y) \wedge d_G(y, e) = d_{W_0}(x, e)) \). It is easy to see that \( R \) is closed, and that for each \( x \) the set \( \{ y : R(x, y) \} \) is a nonempty compact. Also, we may restrict \( x \) to stay in a compact set: we simply cover \( W_0 \) by a countable family of compact sets \( K_n \), build \( \chi_n \) in each \( K_n \) separately and then glue them in Borel measurable way (for example by taking smallest \( n \) such that \( x \) is in domain of \( \chi_n \)). Now the claim follows from \ref{lemma2.2}. \( \diamond \)

Lemma 2.4 \( B_G(e, r) \subset \chi(B_{W_0}(e, r))(B_G(e, 2r) \cap N) \).

Proof: If \( x \in B_G(e, r) \), then \( \pi(x) \in B_{W_0}(e, r) \). Let \( z = \chi(\pi(x)) \). We have \( x = z(z^{-1}x) \). Next \( \pi(z^{-1}x) = \pi(z)^{-1} \pi(x) = \pi(x)^{-1} \pi(x) = e \), so \( z^{-1}x \in N \). By \ref{lemma2.3} \( d_G(z, e) = d_G(x, e) \), so \( d_G(z^{-1}x, e) \leq d_G(z, e) + d_G(x, e) = 2d_G(x, e) \). \( \diamond \)

Lemma 2.5 We can normalize left invariant Haar measures \( \mu (\lambda, \nu) \) on \( G \) (\( W_0 \), \( N \) respectively) in such a way that the following holds: If \( \chi : W_0 \mapsto W \) is a Borel measurable function such that \( \pi \circ \chi(x) = x \), then

\[
\mu(\chi(Q)R) = \lambda(Q) \nu(R)
\]

for all Borel measurable \( Q \subset W_0 \), \( R \subset N \). The same normalization works for all such \( \chi \).

Proof: Fix \( \lambda \) and \( \nu \). Note that mapping \( (w, n) \mapsto \chi(w)n \) is a Borel measurable one-to-one mapping from \( W_0 \times N \) onto \( G \) with Borel measurable inverse. So formula \( \mu(\chi(Q)R) = \lambda(Q) \nu(R) \) defines measure \( \mu \) as a transport of \( \lambda \otimes \nu \). We need to check that resulting \( \mu \) is left invariant on \( G \) and does not depend on choice of \( \chi \). By definition and Fubini theorem, for nonnegative Borel measurable \( f \) on \( G \) we have

\[
\int_G f \, d\mu = \int_{W_0} \int_N f(\chi(w)n) \, d\nu(n) \, d\lambda(w)
\]

If \( h : W_0 \to N \) is Borel measurable by left invariance of \( \nu \) we have

\[
\int_{W_0} \int_N f(\chi(w)n) \, d\nu(n) \, d\lambda(w) = \int_{W_0} \int_N f(\chi(w)h(w)n) \, d\nu(n) \, d\lambda(w).
\]

If \( \chi_1 \) and \( \chi_2 \) are two different choices for \( \chi \) we have \( \chi_2(w) = \chi_1(w)h(w) \) with \( h(w) \in N \) so by the formula above \( \mu \) does not depend on choice of \( \chi \). Similarly, if \( z \in N \) putting \( h(w) = \chi(w)^{-1}z^{-1}\chi(w) \) we have \( \pi(h(w)) = \pi(\chi(w))^{-1}\pi(z^{-1})\pi(\chi(w)) = e \), so \( h(w) \in N \). Then \( z\chi(w)h(w)n = \chi(w)n \) and by the formula above

\[
\int_G f(zx) \, d\mu(x) = \int_{W_0} \int_N f(z\chi(w)n) \, d\nu(n) \, d\lambda(w)
\]
so measure $\mu$ is left invariant under action of $N$.

Finally, if $y \in W$ then $y\chi(w) = \chi(\pi(y)w)h(w)$. Again

$$\pi(h(w)) = (\pi(\chi(\pi(y)w)))^{-1}\pi(y\chi(w)) = (\pi(y)w)^{-1}\pi(y)w = e$$

so $h(w) \in N$ and by the formula above and left invariance of $\lambda$ we have

$$\int f(yx)d\mu(x) = \int_{W_0} \int_N f(\chi(\pi(y)w)h(w)n)d\nu(n)d\lambda(w)$$

$$= \int_{W_0} \int_N f(\chi(w)n)d\nu(n)d\lambda(w) = \int f(x)d\mu(x)$$

so $\mu$ is left invariant under action of $W$. Since $G = WN$ this means that $\mu$ is left invariant under action of $G$. \hfill \Box

**Lemma 2.6** Consider a nilpotent Lie group equipped with an optimal control metric $d$. The doubling property:

$$|B(x, 2r)| \leq C|B(x, r)|$$

holds with a constant $C$ which depends only on the dimension of the group. In particular, the constant is independent of $d$.

**Proof.** This follows via transference from free nilpotent group. Namely, let $G$ be a nilpotent Lie group of dimension $n$ and let $F$ be free nilpotent group of step $n$ on $n$ generators. Let $d$ be associated to vector fields $X_1, \ldots, X_n$. We map generators of $F$ to $X_1, \ldots, X_n$. Then we get homomorphism $\pi$ from $F$ to $G$, such that $B_G(r, e_G) = \pi(B_F(r, e_F))$ (where we use subscripts to distinguish objects in $F$ and $G$). We may consider $G$ as an $F$-space, where $x \in F$ acts on $G$ by left multiplication by $\pi(x)$. By Lemma 1.1 in [2] the following inequality holds, whenever locally compact group $F$ acts on locally compact space $G$ equipped with $F$ invariant measure and $A, B \subset F$,

$$|A||BY| \leq |BA||A^{-1}Y|.$$ 

Putting $A = A^{-1} = B_F(r, e_F)$, $B = B_F(2r, e_F)$, $Y = \{e_G\}$ we get:

$$|B_F(r, e_F)||B_G(2r, e_G)| = |A||BY| \leq |BA||A^{-1}Y| = |B(3r, e_F)||B_G(r, e_G)|$$

so

$$|B_G(2r, e_G)| \leq C|B_G(r, e_G)|$$

where $C = \frac{|B(3r, e_F)|}{|B_F(r, e_F)|}$. \hfill \Box

We say that a finite sequence of metrics $d_i, i = 0, \ldots, k$ on $N$ is a doubling chain of metrics iff

$$2d_{i+1}(x, y) \leq d_i(x, y) \leq 16d_{i+1}(x, y)$$

for all $x$ and $y$. 

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Lemma 2.7 Let $d$ and $\rho$ be two right-invariant Riemannian metrics on $N$. If there is $m \geq 2$ such that $m^2 \rho \geq d \geq m \rho$, then there exists doubling chain of metrics such that $d_0 = d$ and $d_k = \rho$.

Proof: Right-invariant Riemannian metric on a group is uniquely determined by corresponding scalar product at $e$. So it is enough to prove analogous lemma for quadratic forms. But two positive quadratic forms can be diagonalized simultaneously (we diagonalizes one form first, and then use orthogonal transformations with respect to this form to diagonalize the other one). For quadratic forms in diagonal form construction of required chain is straightforward. \hfill \Box

Lemma 2.8 If $x \in W$, and $d_1(n_1, n_2) = d_N(xn_1x^{-1}, xn_2x^{-1})$, then there is $C$ such that $d_1 \leq \exp(Cd_G(x,e))d_N$.

Proof: This is well known. \hfill \Box

The following theorem is due to M. Christ ([1] Theorem 11):

Theorem 2.9 Let $X$ be a space of homogeneous type. There exists a collection of open subsets $\{ Q_{\alpha, k} : k \in \mathbb{Z}, \alpha \in I_k \}$, and constants $\delta \in (0, 1)$, $a_0 > 0$ and $C < \infty$, such that

- $\forall k \mu(X - \bigcup_{\alpha} Q_{\alpha, k}) = 0$.
- If $l > k$, then either $Q_{\beta, l} \subset Q_{\alpha, k}$ or $Q_{\beta, l} \cap Q_{\alpha, k} = \emptyset$.
- For each $(k, \alpha)$ and each $l < k$, there is unique $\beta$ such that $Q_{\alpha, k} \subset Q_{\beta, l}$.
- $\text{diam}(Q_{\alpha, k}) \leq C\delta^k$.
- Each $Q_{\alpha, k}$ contains some ball $B(z_{\alpha, k}, a_0\delta^k)$.

We say that $\{ Q_{\alpha, k} \}$ is a family of dyadic cubes. For each $Q \in \{ Q_{\alpha, k} \}$ we choose a point $x_Q \in Q$ which we will call centerpoint of $Q$.

Remark First and second condition together mean that $\mu(Q_{\alpha, k} - \bigcup_{\beta \in J_\alpha} Q_{\beta, k+1}) = 0$ where $J_\alpha = \{ \beta : Q_{\beta, k+1} \subset Q_{\alpha, k} \}$.

In the sequel we will assume that if $Q_{\beta, l} \subset Q_{\alpha, k}$, then $\text{diam}(Q_{\alpha, k}) \geq 3^{l-k}\text{diam}(Q_{\beta, l})$ - to archive this it is enough to replace $Q_{\alpha, k}$ by $Q_{\alpha, mk}$ for $m$ large enough.

Lemma 2.10 Let $Q_{\alpha, k}$ be a family of dyadic cubes on $W_0$. There exists Borel measurable mapping $\iota : W_0 \mapsto W$ and constant $C_\iota$ such that $\pi \circ \iota = id_{W_0}$ and $\text{diam}(\iota(Q_{\alpha, k})) \leq C_\iota \text{diam}(Q_{\alpha, k})$.

Proof: It is enough to construct $\iota$ on $S = \bigcup_{\alpha} Q_{\alpha, l}$ for some $l$ in such a way that $\iota$ has continuous extension $\iota_{Q_{\alpha, l}}$ to the closure of each $Q_{\alpha, l}$. Namely, we can then order linearly the set $I_l$ and extend $\iota$ to whole $W_0$ by taking $\iota(x) = \iota_{\alpha}(x)$ where $\alpha$ is the smallest one such that $x$ belongs to the closure of $Q_{\alpha, l}$. Note that for given $x$ there
is only finitely many such $\alpha$, so $t$ is well defined and Borel measurable. Condition that $\pi(t(x)) = x$ is preserved when we take continuous extension or glue function from pieces. Also condition on diameters is preserved: if $x \in Q_{\beta,k}$ for $k < l$ and $x$ is in closure of $Q_{\alpha,l}$ then $Q_{\alpha,l} \subset Q_{\beta,k}$. So, in the following we will work on $S$ (replace $Q_{\alpha,k}$ by $S \cap Q_{\alpha,k}$) and consequently we will have $Q_{\alpha,k} = \bigcup_{Q_{\beta,k+1} \subset Q_{\alpha,k}} Q_{\beta,k+1}$.

Locally we can use smooth $t$. So we may assume that for some $l$ there is mapping $t_l$ such that $\pi \circ t_l = id_S$ and for all $k \geq l$ condition on diameters hold.

We choose a maximal family $K \subset I_l$ such that if $\alpha_1, \alpha_2 \in K$, $\alpha_1 \neq \alpha_2$, $Q_{\alpha_1,l} \subset Q_{\beta_1,k}$, $Q_{\alpha_2,l} \subset Q_{\beta_2,k}$, then $\beta_1 \neq \beta_2$. Next, for each $Q_{\alpha,k}$ with $k < l$ we choose $\gamma(\alpha, k)$ such that $Q_{\gamma(\alpha,k),k+1} \subset Q_{\alpha,k}$. If $\alpha_0 \in K$ and $Q_{\alpha_0,l} \subset Q_{\beta,k+1} \subset Q_{\alpha,k}$, then we choose $\beta$ as $\gamma(\alpha,k)$, otherwise we choose arbitrarily. Now, given $t_{k+1}$ we want to build $t_k$. We do this on each $Q_{\beta,k+1} \subset Q_{\alpha,k}$ separately. On $Q_{\gamma(\alpha,k),k+1}$ we put $t_k = t_{k+1}$. If $Q_{\beta,k+1} \subset Q_{\alpha,k}$ and $\beta \neq \gamma(\alpha,k)$ then put $x_0 = xQ_{\gamma(\alpha,k),k+1}$, $x_\beta = xQ_{\beta,k+1}$ and we choose $y_\beta \in W$ such that $d(y_\beta, t_{k+1}(x_0)) = d(x_\beta, x_0)$ and $\pi(y_\beta) = x_\beta$. Next, put $t_k(x) = t_{k+1}(x)(t_{k+1}(x_\beta))^{-1}y_\beta$.

Observe that sequence $t_k$ is convergen: because $K$ is maximal for each $Q_{\alpha,k}$ there exists $\beta_0 \in K$ and $Q_{\beta_1,m}$ such that $Q_{\beta_0,l} \subset Q_{\beta_1,m}$ and $Q_{\alpha,k} \subset Q_{\beta_1,m}$. By our choice of $\gamma(\alpha,k)$ we will have $t_n = t_m$ on $Q_{\beta_1,m}$ for all $n < m$.

Finally, we need to check diameter condition. By our definition

\[
\text{diam}(\pi(t_{\alpha,k})) \leq \max_{\beta_1,\beta_2 \in J_n} (d(y_{\beta_1}, y_{\beta_2}) + \text{diam}(\pi(t_{\beta_1,k+1}))) + \text{diam}(\pi(t_{\beta_2,k+1}))) \\
\leq 2 \max_{\beta \in J_n} (d(y_{\beta}, y_0) + \text{diam}(\pi(t_{\beta,k+1}))) \\
\leq 2 \text{diam}(Q_{\alpha,k}) + 2 \max_{\beta \in J_n} \text{diam}(\pi(t_{\beta,k+1})).
\]

Inductively for $k + j \leq l$,

\[
\text{diam}(\pi(t_{\alpha,k})) \leq \sum_{i=0}^{j-1} 2^i \max_{\beta:Q_{\beta,k+i} \subset Q_{\alpha,k}} \text{diam}(\pi(t_{\beta,k+i})) + 2^j \max_{\beta:Q_{\beta,k+j} \subset Q_{\alpha,k}} \text{diam}(\pi(t_{\beta,k+j})).
\]

Choosing $j$ such that $k + j = l$ and using estimate $\text{diam}(Q_{\beta,k+j}) \leq 3^{-j}\text{diam}(Q_{\alpha,k})$ we get

\[
\text{diam}(\pi(t_{\alpha,k})) = \text{diam}(\pi(t_{\beta,l})) \leq C \text{diam}(Q_{\beta,l}) \leq C 3^{k-l} \text{diam}(Q_{\alpha,k})
\]

so

\[
\text{diam}(t_{\pi(Q_{\alpha,k}))} \leq 2 \sum_{i=0}^{\infty} (2/3)^i \text{diam}(Q_{\alpha,k}) + C (2/3)^{k-l} \text{diam}(Q_{\alpha,k}) \leq C_1 \text{diam}(Q_{\alpha,k}).
\]

\[\diamondsuit\]

Proof of [2.7] We are going to construct a doubling family of doubling sets in $G$. We fix a family of dyadic cubes in $W_0$. 

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With each dyadic cube \( Q \) we associate metric \( d_{Q,0} \) on \( N \) by the formula:

\[
d_{Q,0}(n_1, n_2) = e^{-Mr_Q}d_N(y_Qn_1y_Q^{-1}, y_Qn_2y_Q^{-1})
\]

where \( r_Q \) is the diameter of \( Q \), \( y_Q = \iota(x_Q) \), \( x_Q \) is the centerpoint of \( Q \) and \( M \) is a large enough constant (to be specified later).

Next, for each \( Q \) with \( r_Q \geq 1 \) we choose a doubling chain of metrics \( d_{Q,j}, \ j = 0 \ldots k_Q \), such that \( d_{Q,0} \) is as above and \( d_{Q,k_Q} = d_{S,0} \) where \( S \) is the smallest dyadic cube strictly containing \( Q \). To use \( 2.7 \) we need to find \( m \geq 2 \) such that \( m^2d_{S,0} \geq d_{Q,0} \). We have:

\[
d_{S,0}(n_1, n_2) = e^{-Mr_S}d_N(y_Sn_1y_S^{-1}, y_Sn_2y_S^{-1}) = e^{-Mr_S}d_1(n_Qy_Q^{-1}, y_Qn_2y_Q^{-1})
\]

where \( d_1(n_1, n_2) = d_N(zn_1z^{-1}, zn_2z^{-1}) \) and \( z = y_Sy_Q^{-1} = \iota(y_S)(\iota(y_Q))^{-1} \). Since \( x_S, x_Q \in S \), \( d_W(z, e) = d_W(y_S, y_Q) \leq \text{diam}(\iota(S)) \leq C_1r_S \) and by \( 2.3 \)

\[
d_1(n_1, n_2) \leq \exp(C_2r_S)d_N(n_1, n_2),
\]

\[
d_{S,0}(n_1, n_2) \leq e^{-Mr_S}e^{C_3S}d_N(y_Qn_1y_Q^{-1}, y_Qn_2y_Q^{-1}) = e^{-Mr_S}e^{C_3S}e^{-Mr_Q}d_{Q,0}(n_1, n_2)
\]

so putting \( m = \exp(Mr_S - Mr_Q - C_2r_S) \) we have

\[
md_{S,0} \leq d_{Q,0}.
\]

Similarly, there is \( C_3 \) such that

\[
d_{Q,0} \leq e^{Mr_S}e^{C_3S}e^{-Mr_Q}d_{S,0}.
\]

Note that \( 1 \leq r_Q \leq r_S/3. \) Also \( C_2, C_3 \geq 0. \) If we choose \( M \geq (3/2)(2C_2 + C_3 + 1) \), then

\[
Mr_S - Mr_Q - C_2r_S \geq (2/3)Mr_S - C_2r_S \geq (2C_2 + C_3 + 1)r_S - C_2r_S
\]

\[
= (C_2 + C_3 + 1)r_S \geq 3,
\]

so \( m \geq \exp(3) \geq 2. \) Also

\[
Mr_S - Mr_Q \geq (2/3)Mr_S \geq (2C_2 + C_3)r_S
\]

so

\[
Mr_S - Mr_Q - 2C_2r_S - C_3r_S \geq 0
\]

and

\[
2(Mr_S - Mr_Q - C_2r_S) = Mr_S - Mr_Q + C_3r_S + (Mr_S - Mr_Q - 2C_2r_S - C_3r_S)
\]

\[
\geq Mr_S - Mr_Q + C_3r_S.
\]
Consequently
\[ d_{Q,0} \leq e^{M_r s} e^{C_3 r s} e^{-M_q} d_{S,0} \leq m^2 d_{S,0} \]
which ends verification of assumptions of 2.7.

Now, let \( \mathcal{A}_Q \) be a family of subsets of \( N \) defined as follows. If \( r_Q < 1 \), then \( \mathcal{A}_Q \) consists of all balls in \( N \) of radius \( r_Q \) with respect to \( d_{Q,0} \) metric. If \( r_Q \geq 1 \), then \( \mathcal{A}_Q \) consists of all balls in \( N \) with radius 1 with respect to some \( d_{Q,j}, j = 0, \ldots, k_Q \).

Finally, we define \( \mathcal{A} \) to be a family of all sets of form \( \iota(Q) R \) where \( R \in \mathcal{A}_Q \).

Note that for \( S \in \mathcal{A} \) diameter \( S \) is bounded by a constant times \( r_Q \). This is obvious if \( r_Q \leq 1 \), otherwise we compute:
\[
S y_Q^{-1} = \iota(Q) R y_Q^{-1} = \iota(Q) y_Q^{-1} y_Q R y_Q^{-1},
\]
\[
\iota(Q) y_Q^{-1} \subset B_G(e, C_1 r_Q),
\]
\[
y_Q R y_Q^{-1} \subset B_N(y_Q z y_Q^{-1}, e^{C_2 M r}) \subset B_G(y_Q z y_Q^{-1}, C_3 r)
\]
where \( R = B_{d_Q,j}(z, 1) \), so
\[
S y_Q^{-1} \subset B_G(e, C_1 r) B_G(y_Q z y_Q^{-1}, C_3 r) = B_G(y_Q z y_Q^{-1}, C_4 r)
\]

Also, if \( r_Q \leq 1 \) then \( \iota(Q) R \) is comparable to a ball, so it is a doubling set. Otherwise:
\[
\mu(S) = \lambda(Q) \nu(R),
\]
\[
B_G(e, r_Q) S \subset B_G(e, C r_Q) y_Q R \subset \chi(B_W(e, C r_Q)) B_N(e, e C r) y_Q R
\]
\[
\subset \chi(B_W(e, C r_Q)) y_Q B_{d_Q,j}(e, 1) B_{d_Q,j}(z, 1)
\]
\[
\subset \chi(B_W(e, C r_Q)) y_Q B_{d_Q,j}(z, 2)
\]
so
\[
\mu(B_G(e, r_Q) S) \leq \lambda(B_W(e, C r_Q) y_Q) \nu(B_{d_Q,j}(z, 2)) = \lambda(B_W(e, C r_Q)) \nu(B_{d_Q,j}(z, 2))
\]
\[
\leq C \lambda(Q) \nu(B_{d_Q,j}(z, 1)) = C \lambda(Q) \nu(R)
\]
so \( \mathcal{A} \) consists of doubling sets.

It remains to prove that \( \mathcal{A} \) is a doubling family. Note that second condition of the definition of doubling family holds, namely, if \( S = \iota(Q) R \) and the metric corresponding to \( S \) is not the smallest one in the chain, then we can take the next metric in the chain and obtain a ball \( T \) such that \( R \subset T, \iota(Q) T \in \mathcal{A}, 2 \nu(R) \leq \nu(T) \leq C \nu(R) \). If the metric corresponding to \( S \) is the smallest in the chain (in particular if \( r_Q \leq 1 \)), then we can replace the cube \( Q \) by bigger one. In both cases by 2.6 we have control of volume.

Again, first condition of the definition of doubling family for \( r_Q \leq 1 \) is easy. To get it for \( r_Q > 1 \), notice that it is enough to look at
\[
\tilde{S}_2 = \bigcup_{S_1 \in \mathcal{A}, S_1 \cap \tilde{S}_2 \neq \emptyset, \mu(S_1) \leq \mu(S_2)} S_1.
\]
Namely, according to the remark after definition we may enlarge $S_2$ to deduce original condition.

Consider $S_1, S_2 \in \mathcal{A}$ such that $S_1 \cap S_2 \neq \emptyset$. Then $Q_1 \cap Q_2 \neq \emptyset$ and since $Q_1$ and $Q_2$ are dyadic cubes either $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Since $\mu(S_1) \leq \mu(S_2)$, $Q_1 \subset Q_2$. Also, then the metric corresponding to $S_1$ is greater or equal to the metric corresponding to $S_2$; if $Q_1$ is strictly smaller than $Q_2$, then this follows form our construction, if $Q_1 = Q_2$, then bigger metric implies smaller volume of balls. Consequently, if $S_2 = \iota(Q_2)B_{d_{Q_2,j}}(z, 1)$$S_1 \subset \iota(Q_2)B_{d_{Q_2,j}}(z, 2)$

Like previously, if $d_{Q_2,j}$ is not last in the chain we can take $\tilde{S}_2 \subset T = \iota(Q_2)B_{d_{Q_2,j+1}}(z, 1)$, if $d_{Q_2,j}$ is last in the chain we enlarge $Q_2$ first and then take the next metric in the chain.

\[\Diamond\]

### 3 Reduction to the base case

**Definition 3.1** We say that the space $M$ with metric $d$ and Borel measure $\mu$ has the Calderón–Zygmund property on large scales if there is a decomposition like in 1.1, but $r_i > 1/2$ and $g$ may be unbounded and only satisfies:

$$\int_{B(x, 1)} |g|d\mu \leq C \lambda \mu(B(x, 1)).$$

**Definition 3.2** We say that the space $M$ with metric $d$ and Borel measure $\mu$ has the Calderón–Zygmund property on small scales if there is a decomposition like in 1.1, but only for $f$ which satisfies:

$$\int_{B(x, 1)} |f|d\mu \leq C \lambda \mu(B(x, 1)).$$

**Lemma 3.3** If $G$ is a connected Lie group, $d$ is a right-invariant optimal control metric on $G$ then $G$ (with $d$ and left Haar measure) satisfies Calderón–Zygmund property if and only if it satisfies Calderón–Zygmund property on large scales.

**Proof:** It is well known that optimal control metric satisfies doubling property for balls of bounded radius: for each $R$ there is a $C$ such that

$$\mu(B(x, 2r)) \leq C \mu(B(x, r))$$

for $r \leq R$. This doubling property implies Calderón–Zygmund property on small scales. Together with Calderón–Zygmund property on large scales we get (full) Calderón–Zygmund property.
To get the opposite implication, we fix $\lambda$ and apply Calderón–Zygmund property to $f$. We will denote by $C_{CZ}$ the constant from the Calderón–Zygmund property. Without loss of generality we may pretend that $C_{CZ} \geq 2$.

Decomposition given by Calderón–Zygmund property may fail conditions of Calderón–Zygmund property on large scales because some of $r_i$ may be smaller than or equal to $1/2$, so we want to correct this. Choose a maximal collection of non-intersecting balls $\{B(x, C_{CZ}/2)\}$ of radius $C_{CZ}/2$. If $r_i \leq 1/2$, then $Q_i \subset B(x, C_{CZ}/2)$. Since the collection $\{B(x, C_{CZ}/2)\}$ is maximal, there is an $\alpha$ such that $B(x, C_{CZ}/2)$ and $B(x, C_{CZ}/2)$ intersect, so

$$B(x, C_{CZ}/2) \subset B(x, C_{CZ}).$$

For each $i$ with $r_i \leq 1/2$ choose one $\alpha_i$ as above. Put

$$E_\beta = \bigcup_{\alpha_i = \beta} Q_i,$$

$$h_\beta = \sum_{\alpha_i = \beta} f_i$$

We have

$$\int h_\beta = 0,$$

$$E_\beta \subset B(x, C_{CZ}).$$

Let $A$ be set of all $\beta$ such that

$$\int |h_\beta| \leq C_{CZ} \lambda \mu(B(x, C_{CZ})).$$

Put

$$\tilde{g} = g + \sum_{\beta \notin A} h_\beta.$$ 

Now we construct a new family of functions taking those $f_i$ for which $r_i > 1/2$ and we add to it all $h_\beta$ with $\beta \notin A$. We associate set $E_\beta$ and radius 1 with each $h_\beta$ for $\beta \notin A$. We claim that this new family satisfies all conditions required by the Calderón–Zygmund property on large scales (with a new constant). Namely, we have:

$$f = \tilde{g} + \sum_{r_i > 1/2} f_i + \sum_{\beta \notin A} h_\beta,$$

$$\sum \|h_\beta\|_{L^1} \leq \sum \|f_i\|_{L^1} \leq C_{CZ} \|f\|_{L^1}.$$

Next

$$\mu(E_\beta^*) \leq \mu(B(x, C_{CZ} + 1)) \leq \mu(B(x, 2C_{CZ})) \leq C_1 \mu(B(x, C_{CZ}))$$
the last inequality due to doubling property for radii up to $C_{CZ}$. Then

$$\sum_{\beta \notin A} \mu(E_{\beta}^x) \leq C_1 \sum_{\beta \notin A} \mu(B(x_\beta, C_{CZ})) \leq \frac{C_1}{C_{CZ}} \sum_{\beta \notin A} \|h_\beta\|_{L^1} \leq C_1 \frac{\|f\|_{L^1}}{\lambda}$$

It remains to show that averages of $\tilde{g}$ over balls of radius 1 are bounded. Let $I_x = \{\beta \in A : B(x_\beta, C_{CZ}) \cap B(x, 1) \neq \emptyset\}$. The cardinality of $I_x$ is bounded by a constant $C_2$, since the balls $B(x_\beta, C_{CZ}/2)$ are disjoint and we have doubling property for radii up to $2C_{CZ}$. Now,

$$\int_{B(x, 1)} |\tilde{g}| \leq \int_{B(x, 1)} |g| + \sum_{\beta \in I_x} \int_{B(x, 1)} |h_\beta|$$

$$\leq C_{CZ} \lambda \mu(B(x, 1)) + \sum_{\beta \in I_x} C_{CZ} \lambda \mu(B(x_\beta, C_{CZ}))$$

$$\leq C_{CZ} \lambda \mu(B(x, 1)) + C_2 C_{CZ} \lambda \mu(B(x, 2C_{CZ}))$$

$$\leq C_3 \lambda \mu(B(x, 1)).$$

again by doubling for radii up to $2C_{CZ}$. \endproof

**Corollary 3.4** If a connected Lie group satisfies Calderón–Zygmund property for one optimal control metric, then is satisfies Calderón–Zygmund property for any other optimal control metric.

Namely, by 3.3 only behavior at large scales matters, and at large scales any two optimal control metric are equivalent. Hence, in the sequel we will usually say that a Lie group satisfies (or not) Calderón–Zygmund property, without mentioning metric.

**Lemma 3.5** Let $G_1$ and $G_2$ be connected Lie groups and let $K \subset G_1$ be a compact Lie group. Assume that $G_2 \subset G_1$ and $G_1 = KG_2$ or that $K$ is normal and $G_2 = G_1/K$. Then $G_1$ satisfies Calderón–Zygmund property if and only if $G_2$ satisfies Calderón–Zygmund property.

**Proof:** By 3.3 we may consider Calderón–Zygmund property on large scales. In more detail, consider case when $K$ is normal. Let $\pi$ be canonical projection from $G_1$ to the quotient. By choosing appropriate metrics we may assume that metric balls in $G_2$ are images of balls in $G_1$, that is $\pi(B(x, r)) = B(\pi(x), r)$. Consider $f \in L^1(G_2)$. Put $h = f \circ \pi$. Then $\|h\|_{L^1} = \|f\|_{L^1}$. We use Calderón–Zygmund property on large scales in $G_1$ and obtain $h = \sum h_i + s$. Let $f_i(x) = \int_{\pi \in K} h_i(y)$ and similarly $g(x) = \int_{\pi \in K} s(y)$. We have $f = \sum f_i + g$, $\int f_i = 0$, $\|f_i\|_{L^1} \leq \|h_i\|_{L^1}$. On $G_1$ we have $h_i = 0$ outside $Q_i$ so putting $R_i = \pi(Q_i)$ we have $f_i = 0$ outside $R_i$ and since $Q_i \subset B(x_i, C_{CZ})$ we have $R_i \subset \pi(B(x_i, C_{CZ})) = B(\pi(x_i), C_{CZ})$. We have $R_i^* = \pi(Q_i^*)$, and $\mu(R_i^*) = \mu(\pi^{-1}(R_i^*)) = \mu(KQ_i^*)$. In general $KQ_i^*$ has bigger
measure than $Q_i^*$, so this would cause trouble. However, $Q_i^* = \{ x : d(x, Q_i) < r_i \}$ and $KQ_i^* \subset \{ x : d(x, Q_i) < r_i + \text{diam}(K) \}$. By changing $r_i$ so that new $r_i + \text{diam}(K)$ is less or equal to old $r_i$, we get condition on measure of $R_i^*$. Such change is possible if old $r_i$ is bigger or equal to $\text{diam}(K)$, namely we divide $r_i$ by 2. Of course, to make sure that $R_i \subset B(x_i, Cr_i$) after such change of $r_i$ we need to multiply $C$ by 2 to compensate. When $r_i$ is bigger than $1/2$ and smaller than a fixed constant we may estimate measure of $R_i^*$ using doubling condition on measure of balls. Next,

$$\int_{B(x,1)} |g| \leq \int_{d(y,xK) < 1} |s(y)|.$$ 

\{ y : d(y,xK) < 1 \} can be covered by a bounded number of unit balls and by enlarging $C$ we get condition on $g$. So $G_2$ satisfies Calderón–Zygmund property on large scales.

When proving that $G_1$ has Calderón–Zygmund property on large scales using property of $G_2$ we have problem with defining functions $f_i$. To avoid this problem note that it is enough to build decomposition of nonnegative function: by combining decompositions of positive and negative parts we get decomposition of arbitrary function with slightly worse constants. When $f$ is nonnegative we put $h(x) = \int_{y \in xK} f(y)$ and apply decomposition to $h$ obtaining $h = \sum f_i + s$. We put $t(x) = f(x)/h(\pi(x))$ if $h(\pi(x)) > 0$ and $t(x) = 1$ otherwise. Now, take $f_i(x) = t(x)h_i(\pi(x))$ and $g(x) = t(x)s(\pi(x))$. When $h(\pi(x)) > 0$, then

$$\sum f_i(x) + g(x) = t(x)\left(\sum h_i(\pi(x)) + s(\pi(x))\right) = t(x)h(\pi(x)) = f(x).$$

When $h(\pi(x))$ in similar way $\sum f_i(x) + g(x) = 0$ and $f(x) = 0$ so again we get equality (modulo null sets). Note that $\int_{y \in xK} t(y) = 1$, so $\int f_i = \int h_i = 0$. Also, $t(y) \geq 0$, so $|g|(y) = t(y)|s(\pi(y))|$. Consequently

$$\int_{B(y,1)} |g|(y) \leq \int_{B(y,1)K} t(y)|s(\pi(y))| = \int_{B(\pi(y),1)} |s|$$

and

$$\|f_i\|_{L_1} = \|h_i\|_{L_1}$$

We get $Q_i$ in $G_1$ as counterimages of sets from $G_2$ and we take as $x_i$ arbitrary counterimage of corresponding point in $G_2$. The only nontrivial condition is $Q_i \subset B(x_i, Cr_i)$ since $Q_i$ in $G_1$ are bigger than corresponding sets in $G_2$. However, like in previous case we have $Q_i \subset B(x_i, Cr_i + \text{diam}(K))$. Since $r_i > 1/2$ by enlarging $C$ we can ensure that new $Cr_i$ is bigger than old $Cr_i + \text{diam}(K)$, so we obtained Calderón–Zygmund decomposition on large scales on $G_1$.

Case when $G_1 = K\overline{G_2}$ is similar, but more messy. To map functions from $G_1$ to $G_2$ we integrate on cosets of $K$. To map back we use measurable selector from $K/(K \cap G_2)$ into $K$ like in lemma 2.3. We transfer sets from $G_2$ to $G_1$ multiplying
them by $K$. We go back using measurable selector. Now, both ways we can enlarge sets so we need to compensate as above.

\[ \diamond \]

Remark. In similar way like lemma 3.5 we could transfer doubling family of doubling sets between $G_1$ and $G_2$. More precisely, we need to assume that sets in the family of diameter of order 1 are comparable to balls. We can compensate changing sizes like above using remark after definition of doubling family.

Lemma 3.6 Let $G$ be a simply connected solvable Lie groups with the Lie algebra $\mathfrak{g}$ and let $W \subset G$ be a Lie subgroup of $G$ corresponding to a Cartan subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ and $N$ be a Lie subgroup of $G$ corresponding to $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Then $W$ and $N$ are simply connected and $G = WN$.

Proof: Since $\mathfrak{w}$ is a Cartan subalgebra of $\mathfrak{g}$ we have

$$\mathfrak{g} = \mathfrak{w} + \mathfrak{n}.$$ 

Put $\mathfrak{w}_0 = \mathfrak{w} \cap \mathfrak{n}$. Let $v_1, \ldots, v_k \in \mathfrak{w}$ span complementary subspace to $\mathfrak{w}_0$. Put $v_i = \text{lin}(v_i, \ldots, v_k) + \mathfrak{n}$. Since $[\mathfrak{g}, \mathfrak{g}] \subset v$ each $v_i$ is an ideal in $\mathfrak{g}$ and consequently $v_{i+1}$ is an ideal in $v_i$. This means that $v_i$ is a semidirect product of one dimensional Lie algebra generated by $v_i$ and $v_{i+1}$. Let $V_i$ be simply connected Lie group with the Lie algebra $v_i$. Since $v_i$ is a semidirect product $V_i$ is also a semidirect product. Consequently, mapping $(t, g) \mapsto \exp(t v_i)g$ from $\mathbb{R} \times V_{i+1}$ into $V_i$ is one to one. Let $\tilde{N} = V_{k+1}$ be simply connected Lie group with the Lie algebra $\mathfrak{n} = v_{k+1}$. Simple induction shows that mapping $(t_1, \ldots, t_k, g) \mapsto \exp(t_1 v_1) \cdots \exp(t_k v_k)g$ from $\mathbb{R}^k \times \tilde{N}$ into $V_1 = G$ is one to one. Consequently $\tilde{N} = N$ and $N$ is simply connected. Similarly, we show that restriction of the mapping above to $\mathbb{R}^k \times \mathfrak{w}_0$ is one to one onto $W$, so also $W$ is simply connected. Finally, $\exp(t_i v_i) \in W$ so the expression above means that $G = WN$. \[ \diamond \]

Lemma 3.7 Let $G$ and $W$ be as in Lemma 3.6. Then center of $G$ is contained in $W$.

Proof: We will use notation form 3.6. First note that that action of $W$ on $G/W$ by inner authomorphisms of $G$ is equivalent to action on $N/(W \cap N)$. Let $z$ be in the center of $G$. Since $z$ is in the center it is invariant under inner authomorphisms and it leads to a fixpoint of the action of $W$ on $G/W$. So it is enough to show that the only fixpoint of action on $W$ on $G/W$ by inner authomorphisms of $G$ is $W$. This is equivalent to showing that only fixpoint of action of $W$ on $N/(W \cap N)$ is $(W \cap N)$. However, $N$ is a simply connected nilpotent Lie group, so exponential mapping from $\mathfrak{n}$ into $N$ is one to one. So it is enough to show that the only fixpoint of action of $W$ on $\mathfrak{n}/\mathfrak{w}_0$ is $\mathfrak{w}_0$. This is equivalent to showing that the only element of $\mathfrak{n}/\mathfrak{w}_0$ anihilated by $\mathfrak{w}$ is $\mathfrak{w}_0$. But this holds since $\mathfrak{w}$ is a Cartan subalgebra of $\mathfrak{g}$. \[ \diamond \]
Now we can reduce the general case to the base case.

**Step 1.** By the structural theory, amenable Lie group $G$ is a compact extension of a solvable Lie group $R$. Moreover, by Levy-Maltsev theorem there is a compact subgroup $K$ of $G$ such that $G = KR$. So by 3.5 the general case reduces to solvable one.

**Step 2.** $G$ is now solvable. Let $g$ be the complexification of the Lie algebra of $G$ and let $a$ be the Cartan subalgebra of $g$. Consider root space decomposition of $g$:

$$g = \bigoplus_{\alpha} g_{\alpha}$$

where $g_{\alpha} = \{ x \in g : \forall y \in a (\text{ad}(y) - \alpha(y))^n x = 0 \}$ and $n$ is the dimension of $g$. We have $g_0 = a$ and (by )

$$[g_{\alpha}, g_{\beta}] \subset g_{\alpha+\beta}.$$ 

Hence, for $y \in a$ linear mapping of $g$ which multiplies $x \in g_{\alpha}$ by imaginary part of $\alpha(y)$ is a derivation of $g$. Such derivations commute with complex conjugation on $g$, so they generate automorphisms of universal covering $\tilde{G}$ of $G$. Resulting automorphisms act as identity on $\exp g_0$ which by Lemma 3.7 contains center of $\tilde{G}$. Consequently we can pass to quotient and obtain automorphisms of $G$.

Obviously, the closure $T$ of all such automorphisms is a group isomorphic to a torus (and compact). It is easy to see that in the semidirect product $T \ltimes G$ we can find a subgroup $G_1$ such that all roots of $G_1$ are real. So, applying 3.5 twice, first to $G$ and $T \ltimes G$, then to $T \ltimes G_1$ and $G_1$ we reduce the solvable case to the case with all roots real.

**Step 3.** Now $G$ has all roots real. What matters for us, is that the Lie algebra $g$ of $G$ is exponential: the exponential mapping from $g$ to corresponding simply connected Lie group $\tilde{G}$ (which we will identify with universal covering of $G$) is a diffeomorphism. It follows that the center $Z$ of $\tilde{G}$ is an image of the center of $g$. Also, the kernel $N$ of the covering map from $\tilde{G}$ to $G$ is a central subgroup. So, we may identify $N$ with a lattice $L$ in the center of $g$. Let $K$ be a subgroup of $G$ corresponding to linear span $V$ of $L$. $K$ is isomorphic to $V/L$, so it is a normal torus. Again using 3.5 we may divide $G$ by $K$ reducing the problem to simply connected groups.

**Step 4.** Now $G$ is an exponential solvable Lie group. Like in step 2 we fix a Cartan subalgebra and root space decomposition of the Lie algebra $g$ of $G$:

$$g = \bigoplus_{\alpha} g_{\alpha}$$

(since after step 2 all roots are real we can skip complexification). Let $n$ be the subalgebra of $g$ generated by all $g_{\alpha}$ with $\alpha \neq 0$. Let $N$ be the subgroup of $G$ corresponding to $n$. $N \subset [G, G]$ is a normal divisor. Since $G$ is exponential $N$ is closed. Let $G_0$ be the subgroup of $G$ corresponding to $g_0$. $G_0$ is nilpotent, hence of polynomial growth. We need to get estimate on the distance in $N$. Let $d$ be optimal
control metric corresponding to basis $X_1, \ldots, X_m, Y_1, \ldots, Y_l$ of $g$ consisting of basis $X_1, \ldots, X_m$ of $n$ extended by vectors $Y_1, \ldots, Y_l$ from $g_0$ ($d$ is in fact a Riemannian distance). Consider a curve $\gamma : [0, 1] \mapsto G$ joining $e$ with $x \in N$. We have

$$\gamma'(s) = \sum_{i=1}^{m} a_i(s)X_i(\gamma(s)) + \sum_{i=1}^{l} a_{m+i}(s)Y_i((\gamma(s))$$

Let $\gamma_1$ be the solution of the differential equation

$$\gamma_1'(s) = \sum_{i=1}^{l} a_{m+i}(s)Y_i((\gamma_1(s))$$

with initial condition $\gamma_1(0) = e$. Consider images $\pi \circ \gamma$ and $\pi \circ \gamma_1$ of $\gamma$ and $\gamma_1$ under quotient projection $\pi : G \mapsto G/N$. Both $\pi \circ \gamma$ and $\pi \circ \gamma_1$ satisfy the same differential equation with the same initial condition, so

$$\pi \circ \gamma(s) = \pi \circ \gamma_1(s).$$

Put $\gamma_2(s) = \gamma_1^{-1}(s)\gamma(s)$. By the above $\pi(\gamma_2(s)) = e$ so $\gamma_2(s) \in N$. Next

$$\gamma_2'(s) = (\gamma_1^{-1}\gamma)'(s) = dL_{\gamma_1^{-1}(s)}\gamma'(s) + dR_{\gamma(s)}(\gamma_1^{-1})'$$

$$= \text{Ad}(\gamma_1^{-1}(s))dR_{\gamma_1^{-1}(s)}\gamma'(s) + dR_{\gamma(s)}(\gamma_1^{-1})'$$

On the other hand, since $\gamma_2 : [0, 1] \mapsto N$ there are (unique) $b_1, \ldots, b_m$ such that

$$\gamma_2'(s) = \sum_{i=1}^{m} b_i(s)X_i(\gamma_2(s))$$

Our vector fields are right invariant, so for each $s$

$$\sum_{i=1}^{m} b_i(s)X_i = \sum_{i=1}^{m} a_i(s)\text{Ad}(\gamma_1^{-1}(s))X_i + \sum_{i=1}^{l} a_{m+i}(s)(\text{Ad}(\gamma_1^{-1}(s)) - 1)Y_i$$

and

$$\sum_{i=1}^{m} b_i^2(s) \leq e^{Ct} \sum_{i=1}^{m} a_i^2(s)$$

where $t$ is length of $\gamma$. Hence, for $x \in N$, $d_G(x, e) < t$ implies $d_N(x, e) < te^{Ct}$, which in turn implies $1 + d_N(x) \leq \exp((C + 1)d_G(x))$.

**Step 5.** To obtain remaining estimate on distance we first prove that

$$B_N(e, e^t) \subset B_G(e, C(t + 1)) \cap N$$

for $C > 0$ large enough and $t > 0$. As in step 4 we may assume here that $X_1, \ldots, X_m$ is a basis of $\bigoplus_{\alpha \neq 0} g_\alpha$, such that each $X_i$ belongs to some $g_\alpha$ and that $Y_1, \ldots, Y_l$ is a basis of $g_0$ (and $d$ corresponds to fields $X_1, \ldots, X_m, Y_1, \ldots, Y_l$).

We need a lemma:
Lemma 3.8 Let $X_i, i = 1, \ldots, m, N$ and $d_N$ be as above. There exists a constant $k$ such that

$$B_N(e, r) \subset \left( \bigcup_{i=1}^{m} \{ \exp(sX_i) : |s| < r \} \right)^k$$

Proof: Note that we can replace $N$ by a free nilpotent group of the same step as our original $N$ (the claim is preserved when passing to a quotient). Hence, we may assume that $N$ has a one-parameter family of authomorphic dilations $\delta_s, s > 0$, such that

$$\delta_s(X_i) = sX_i$$

so

$$d(e, \delta_s(x)) = sd(e, x)$$

Since $X_i$ generate $N$ we can find $k$ such that the inclusion holds for $r = 1$. Applying $\delta_1$ to both sides we see that the inclusion holds for all $r > 0$.  

Fix $i$. There is $a$ such that $X_i \in g_a$. Fix $Z \in g_b$ such that $\alpha(Z) = 1$ and $b_1, \ldots, b_l$ such that $Z = \sum_{i=1}^{l} b_i Y_i$. Let $\gamma(s) = \exp(-3tsZ)$ for $0 \leq s \leq 1/3$, $\gamma(s) = \exp(3(s - 1/3))X_i \exp(-tZ)$ for $1/3 \leq s \leq 2/3$, $\gamma(s) = \exp(3(s - 2/3)Z) \exp(X_i) \exp(-tZ)$ for $2/3 \leq s \leq 1$. We have

$$\gamma(1) = \exp(tZ) \exp(X_i) \exp(-tZ) = \exp(\exp(tad(Z)X_i) = \exp(\exp(t)X_i)$$

$$\gamma'(s) = \sum_{i=1}^{m} a_i X_i(\gamma(s)) + \sum_{i=1}^{l} a_i(\gamma(s))Y_i(\gamma(s))$$

where $|a_i|$ are bounded by $C(t + 1)$ with $C$ large enough, so

$$\exp(\exp(r)X_i) \in B_G(e, C(r + 1)).$$

By the lemma

$$B_N(e, \exp(t)) \subset B_G(e, C_2(t + 1))$$

with (another) $C_2 > 0$ large enough. Now, let $1 < d_N(e, x) = e^t$. By the inclusion above $d_G(e, x) \leq C_2(t + 1)$, so $C_2^{-1}d_G(e, x) - 1 \leq t$, so

$$\exp(C_2^{-1}d_G(e, x) - 1) \leq e^t = d_N(e, x) \leq 1 + d_N(e, x).$$

When $d_G(e, x) \geq 2C_2^{-1}$ we have $C_2^{-1}d_G(e, x) - 1 \geq (2C_2)^{-1}d_G(e, x)$, so

$$\exp((2C_2)^{-1}d_G(e, x)) \leq 1 + d_N(e, x).$$

$\exp(3d_G(e, x))$ is locally Lischitz, so there is constant $C_3$ such that when $x \in N$ and $d_G(e, x) \leq 2C_2^{-1}$ we have $d_G(e, x) \leq C_3d_N(e, x)$ and

$$\exp(C_4d_G(e, x)) \leq 1 + d_N(e, x)$$

with $C_4 = (C_3 \exp(2C_2^{-1}))^{-1}$ which finishes our reduction of [12.2 to 2.1]
4 Necessary condition for Calderón–Zygmund property

We say that a set $A$ is $r$-doubling iff $|B(r)A| \leq 2|A|$.

**Theorem 4.1** Assume a group $M$ with a right-invariant metric $d$ and left-invariant measure satisfies Calderón–Zygmund property. If $M$ has infinite volume then $M$ contains $r$-doubling sets for arbitrarily large $r$.

**Proof:** It is enough to find sets $A(r_i)$ such that $|B(r_i)A(r_i)| \leq 2|A(r_i)|$ for a sequence of $r_i \to \infty$. We will suppose that the inverse inequality $|B(r)A| > 2|A|$ holds for all large $r$ and will derive contradiction with Calderón–Zygmund property.

If $M$ has infinite volume, we can take $\lambda$ arbitrarily small. Let $f(x) = 1/|B(1)|$ for $d(x, e) < 1$ and 0 otherwise. Of course, $\|f\|_{L^1} = 1$. Fix a Calderón–Zygmund decomposition of $f$ with small $\lambda$ (to be determined later). Put

$$E_k = \bigcup_{2^k < r_i \leq 2^{k+1}} Q_i,$$

$$F_k = B(1) \cup \bigcup_{j \geq k} E_j,$$

$$h_k = \sum_{2^k < r_i} f_i.$$

Let $\phi_k(x) = 1$ for $d(x, F_k) > 2^{k-1}$ and $\phi_k(x) = 2^{-k+1}d(x, F_k)$ otherwise.

Let $C$ be the constant from the definition of Calderón–Zygmund property. Fix $l > 4$ such that $2^{l-3} > C^2$. Assume that $|B(2^k+1)| \leq C/\lambda$ and $2^{k-1} > rl$. If $r$ is large enough, then there are no $r$ doubling sets and

$$|B(r)^j B(2^{k-1})F_k| \geq 2|B(r)^j B(2^{k-1})F_k|.$$

By induction

$$|B(2^{k-1})F_k| \leq 2^{-l} |B(r)^j B(2^{k-1})F_k|.$$

Next

$$|B(2^{k-1})F_k| \leq 2^{-l} |B(r)^j B(2^{k-1})F_k| \leq 2^{-l} |B(2^{k})F_k|$$

$$\leq 2^{-l}(|B(2^{k}+1)| + \sum \mu(Q^*_i)) \leq 2^{-l+1}C \|f\|_{L^1} / \lambda \leq 1/(4C\lambda)$$

so

$$\int_{B(2^{k-1})F_k} |g| \leq C\lambda |B(2^{k-1})F_k| \leq 1/4.$$

Also

$$\int g = \int f - \sum \int f_i = \int f = 1.$$
\[ \int \phi_k g \geq \int g - \int_{B(2^{k-1})F_k} |g| \geq 1 - 1/4 = 3/4. \]

If \( r_i \leq 2^{k-l} \) then
\[ \int \phi_k f_i \leq 2^{-k+1} r_i \| f_i \|_{L^1} \leq 2^{-l+1} \| f_1 \|_{L^1} \]
so
\[ \sum_{r_i \leq 2^{k-l}} \int \phi_k f_i \leq 2^{-l+1} \sum \| f_i \|_{L^1} \leq 2^{-l+1} C \| f_i \|_{L^1} \leq \| f \|_{L^1}/4 = 1/4. \]

Next
\[ \phi_k f = 0, \]
\[ \phi_k h_k = 0, \]
so
\[ - \sum_{2^{k-l} < r_i \leq 2^k} \int \phi_k f_i = \int \phi_k g - \int \phi_k f - \int \phi_k h_k - \sum_{r_i \leq 2^{k-l}} \int \phi_k f_i \]
\[ \geq 3/4 - 1/4 = 1/2 \]

Now, if \( m > 2C \), and \( \lambda \) is small enough so the inequalities above hold for \( k_j = k_0 + j l, j = 0, \ldots, m - 1 \) then
\[ \sum_{2^{k_0+(j-1)l} < r_i \leq 2^{k_0+jl}} \| f_i \|_{L^1} \geq \sum_{2^{k_0+(j-1)l} < r_i \leq 2^{k_0+jl}} \int \phi_k f_i \geq 1/2 \]
so
\[ \sum \| f_i \|_{L^1} \geq \sum_{j=0}^{m-1} \sum_{2^{k_0+(j-1)l} < r_i \leq 2^{k_0+jl}} \| f_i \|_{L^1} \geq \sum_{j=0}^{m-1} 1/2 = m/2 > C \| f \|_{L^1} \]
which gives a contradiction.

\[ \diamond \]

**Corollary 4.2** If a locally compact group \( G \) with a right-invariant geometric metric \( d \) and the left Haar measure satisfies Calderón–Zygmund property, then \( G \) is amenable.

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