Zero rest-mass fields and the Newman-Penrose 
constants on flat space

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Abstract

Zero rest-mass fields of spin 1 (the electromagnetic field) and spin 2 propagating on flat space and their corresponding Newman-Penrose (NP) constants are studied near spatial infinity. The aim of this analysis is to clarify the correspondence between data for these fields on a spacelike hypersurface and the value of their corresponding NP constants at future and past null infinity. To do so, the framework of the cylinder at spatial infinity is employed to show that, expanding the initial data in terms spherical harmonics and powers of the geodesic spatial distance ρ to spatial infinity, the NP constants correspond to the data for the highest possible spherical harmonic at fixed order in ρ. In addition, it is shown that the NP constants at future and past null infinity, for both the Maxwell and spin-2 case, are related to each other as they arise from the same terms in the initial data. Moreover, it is shown that this observation is true for generic data (not necessarily time-symmetric). This identification is a consequence of both the evolution and constraint equations.

Keywords: Conformal methods, spinors, Newman-Penrose constants, cylinder at spatial infinity, soft-hair.

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1 Introduction

The concept of asymptotic simplicity is central for the understanding of isolated systems in general relativity. In this regard, Penrose’s proposal [18] is an attempt to characterise the fall-off behaviour of the gravitational field in a geometric manner —see also [10]. The essential mathematical idea behind for the Penrose proposal is that of a conformal transformation; given a spacetime (M, g) satisfying the Einstein field equations (the physical spacetime) one considers a 4-dimensional Lorentzian manifold M equipped with a metric g such that g and ̃g are conformal to each other, in other words

\[ g = \Xi^2 \tilde{g}, \]

where \( \Xi \) is the so-called conformal factor. The pair (M, g) can be called the unphysical spacetime.

The set of points where \( \Xi = 0 \) but d\( \Xi \neq 0 \) is called the null infinity and is denoted by \( \mathcal{I} \). If \( \tilde{g} \) satisfies the vacuum Einstein field equations (with vanishing Cosmological constant) near \( \mathcal{I} \), then the conformal boundary defines a smooth null hypersurface of M —see [10, 23]. One can identify two disjoint pieces of \( \mathcal{I} \): \( \mathcal{I}^- \) and \( \mathcal{I}^+ \) correspond to the past and future end points of

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null geodesics. If every null geodesic acquires two distinct endpoints at \( \mathcal{I} \), the spacetime \((\tilde{M}, \tilde{g})\) is said to be *asymptotically simple*—see [10, 23, 26] for precise definitions. The Minkowski spacetime, \((\mathbb{R}^4, \tilde{\eta})\) is the prototypical example of an asymptotically simple spacetime. In the standard conformal representation of the Minkowski spacetime, the unphysical spacetime can be identified with the Einstein cylinder \((M_\mathcal{E}, g_\mathcal{E})\) where \(M_\mathcal{E} \approx \mathbb{R} \times S^3\) and

\[
g_\mathcal{E} = dT \otimes dT - d\psi \otimes d\psi - \sin^2 \psi d\sigma, \quad \Xi = \cos(T) + \cos(\psi),
\]

where \( -\pi < T < \pi \), \( 0 < \psi < \pi \) and \( \sigma \) is the standard metric on \( S^2 \). In this conformal representation \( \mathcal{I}^\pm \) correspond to the sets of points on the Einstein cylinder, \( M_\mathcal{E} \equiv \mathbb{R} \times S^3 \), for which \( 0 < \psi < \pi \) and \( T = \pm(\pi - \psi) \). One can directly verify that \( \Xi|_{\mathcal{I}^\pm} = 0 \) while \( d\Xi|_{\mathcal{I}^\pm} \neq 0 \) —see [23]. Consequently, a distinguished region in the conformal structure of the Minkowski spacetime is *spatial infinity* \( i^0 \) for which both \( \Xi|_{i^0} \) and \( d\Xi|_{i^0} \) vanish. In this conformal representation, spatial infinity corresponds to a point in the Einstein cylinder with coordinates \( \psi = \pi \) and \( T = 0 \).

A natural problem to be considered is the existence of spacetimes whose conformal structure resembles that of the Minkowski spacetime. In this setting, the conformal Einstein field equations introduced originally in [6] provide a convenient framework for discussing global existence of *asymptotically simple* solutions to the Einstein field equations. An important application of these equations is the proof of the semi-global non-linear stability of the Minkowski spacetime given in [6]. In the latter work, the evolution of perturbed initial data close to exact Minkowski data is analysed. Nevertheless, the initial data is not prescribed on a Cauchy hypersurface \( \tilde{S} \) but in an hyperboloid \( \tilde{H} \) whose conformal extension in \( \tilde{M} \) intersects \( \mathcal{I} \). Therefore, an open problem in the framework of the conformal Einstein field equations is the analysis of the evolution of initial data prescribed on a Cauchy hypersurface \( S \) intersecting \( i^0 \) —see [4] for the proof of the global non-linear stability of the Minkowski spacetime employing different methods. One of the main difficulties in establishing a global result for the stability of the Minkowski spacetime using conformal methods lies on the fact that the initial data for the conformal Einstein field equations is not smooth at \( i^0 \). This is not unexpected since, as observed by Penrose—see [18, 19], the conformal structure of spacetimes with non-vanishing mass becomes singular at spatial infinity. A milestone in the resolution of this problem is the construction, originally introduced in [8], of a new representation of spatial infinity known as the *cylinder at spatial infinity*. In this representation, spatial infinity is not represented as a point but as set whose topology is that of a cylinder. This representation is well adapted to exploit the properties of curves with special conformal properties: *conformal geodesics*. In addition, it allows to formulate a regular finite initial value problem for the conformal Einstein field equations—other approaches for analysing the gravitational field near spatial infinity using different representations of spatial infinity have been also proposed in literature—see [21, 3, 2, 22].

The framework of the cylinder at spatial infinity and its connection with the conformal Einstein field equations have been exploited in an analysis of the *gravitational Newman-Penrose (NP) constants* in [11]. The NP constants, originally introduced in [17], are defined in terms of integrals over cuts \( \mathcal{C} \approx S^2 \) of \( \mathcal{I} \). The integrands in the expressions defining the NP constants are, however, written in a particular gauge adapted to \( \mathcal{I} \) (the so-called NP-gauge) while the natural gauge used in the framework of the cylinder at spatial infinity (the so-called F-gauge in [11]), is adapted to a congruence of conformal geodesics and hinged at a Cauchy hypersurface \( \tilde{S} \). This fact, which in first instance looks as an obstacle to analyse the NP constants, turns out to be advantageous since, once the relation between the NP-gauge and the F-gauge is clarified, one can relate the initial data prescribed on \( \tilde{S} \) with the gravitational NP constants at \( \mathcal{I} \).

In a recent work [15], the authors exploit the notion of these conserved quantities at \( \mathcal{I} \) to make inroads into the problem of the information paradox—see [14, 12, 13]. In the latter work, the concept of *soft hair* is motivated by means of an analysis of the conservation laws and symmetries of abelian gauge theories in Minkowski space. These conservation laws correspond
essentially to the electromagnetic version of the gravitational NP constants. With this motivation, in the present article zero rest-mass fields propagating on flat space and their corresponding NP constants are studied. Two physically relevant fields are analysed: the spin-1 and spin-2 zero rest-mass fields. The spin-1 field provides a description of the electromagnetic field while the spin-2 field on the Minkowski spacetime describes linearised gravity.

In this article it is shown how the framework of the cylinder at spatial infinity can be exploited to relate the corresponding NP constants with the initial data on a Cauchy hypersurface intersecting \( i^0 \) —see Propositions 5 and 6 for the spin-1 case and Proposition 7 and 8 for the spin-2 case. Additionally, it is shown that, for the class of initial data considered, the NP constants at \( \mathscr{I}^+ \) and \( \mathscr{I}^- \) coincide —see Theorems 1 and 2. We show that this identification arises from a delicate interplay between the evolution and constraint equations associated to these fields. In particular, our analysis highlights the connection between the smoothness of the fields at null infinity and the finiteness of the conserved quantities.

1.1 Outline of the paper

Section 2 contains a general discussion of the cylinder at spatial infinity and the F-gauge in the Minkowski spacetime. In Section 3, the Maxwell equations are written in the F-gauge and the initial data for the electromagnetic field on a spacelike hypersurface is discussed. In Section 4, the equations governing the massless spin-2 field are expressed in the F-gauge and the corresponding initial data is discussed. In Section 5, Bondi coordinates and a NP-frame for a conformal extension of the Minkowski spacetime is derived. Additionally, the relation between this frame and the one introduced in Section 2 is determined explicitly. In Section 6 the electromagnetic NP constants are introduced and written in the F-gauge. This construction is exploited to identify the electromagnetic NP constants with part of the initial data introduced in Section 3. In Section 7 a similar analysis is carried out for the spin-2 field; the corresponding NP constants are found and written in terms of the initial data. Section 8 provides with some concluding remarks. In addition, a general discussion of the connection on \( S^2 \) is given in Appendix A and a discussion of the \( \delta \) and \( \bar{\delta} \) operators of Newman and Penrose is provided in Appendix B.

1.2 Notations and Conventions

The signature convention for (Lorentzian) spacetime metrics will be \((+,−,−,−)\). In the rest of this article \( \{\alpha,\beta,\gamma,\ldots\} \) denote abstract tensor indices and \( \{a,b,c,\ldots\} \) will be used as spacetime frame indices taking the values 0,...,3. In this way, given a basis \( \{e_a\} \) a generic tensor is denoted by \( T_{ab} \) while its components in the given basis are denoted by \( T_{ab} \equiv T_{ab}{}^e e_a e_b \). Part of the analysis will require the use of spinors. In this respect, the notation and conventions of Penrose & Rindler [20] will be followed. In particular, capital Latin indices \( \{A,B,C,\ldots\} \) will denote abstract spinor indices while boldface capital Latin indices \( \{A,B,C,\ldots\} \) will denote frame spinorial indices with respect to a specified spin dyad \( \{\delta_A A\} \). The conventions for the curvature tensors are fixed by the relation

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) u^c \equiv R^c_{dab} u^d.
\]

2 The cylinder at spatial infinity and the F-Gauge

In this section a conformal representation of the Minkowski spacetime that is adapted to a congruence of conformal geodesics is discussed. This conformal representation, introduced originally in [8], is particularly suited for analysing the behaviour of fields near spatial infinity. In broad terms, in this representation spatial infinity \( i^0 \), which corresponds to a point in the standard compactification of the Minkowski spacetime, is blown up to a two-sphere \( S^2 \). In the subsequent discussion this representation will be referred as the cylinder at spatial infinity. The discussion of the cylinder at spatial infinity as presented in [8] is given in the language of fibre bundles. In particular,
the construction of the so-called extended bundle space is required —see [8, 1]. Nevertheless, a discussion which does not make use of this construction is presented in the following.

2.1 The cylinder at spatial infinity

Consider the Minkowski metric $\tilde{\eta}$ in Cartesian coordinates $\tilde{x}^\alpha = (\tilde{t}, \tilde{x}^i)$,

$$\tilde{\eta} = \eta_{\mu\nu} d\tilde{x}^\mu \otimes d\tilde{x}^\nu,$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Introducing polar coordinates defined by $\tilde{\rho} = \delta_{ij}\tilde{x}^i\tilde{x}^j$ where $\delta_{ij} = \text{diag}(1,1,1)$, and an arbitrary choice of coordinates on $S^2$, the metric $\tilde{\eta}$ can be written as

$$\tilde{\eta} = d\tilde{t} \otimes d\tilde{t} - d\tilde{\rho} \otimes d\tilde{\rho} - \tilde{\rho}^2 \sigma,$$

with $\tilde{t} \in (-\infty), \tilde{\rho} \in [0, \infty)$ and $\sigma$ denotes the standard metric on $S^2$. A common procedure to obtain a conformal representation of the Minkowski spacetime close to $i^0$ is to introduce inversion coordinates $x^\alpha = (t, x^i)$ defined by —see [23],

$$x^\mu = -\tilde{x}^\mu / X^2, \quad X^2 \equiv \tilde{\eta}_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu.$$

The inverse transformation is given by

$$\tilde{x}^\mu = -x^\mu / X^2, \quad X^2 = \eta_{\mu\nu} x^\mu x^\nu.$$

Using these coordinates one readily identifies the following conformal representation of the Minkowski spacetime

$$g_I = \Xi^2 \tilde{\eta},$$

where $g_I = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ and $\Xi = X^2$. Notice, additionally that, $X^2 = 1/\tilde{X}^2$. Introducing an unphysical polar coordinate defined as $\rho = \delta_{ij}x^i x^j$, one observes that the rescaled metric $g_I$ and conformal factor $\Xi$ read

$$g_I = dt \otimes dt - d\rho \otimes d\rho - \rho^2 \sigma, \quad \Xi = t^2 - \rho^2,$$

with $t \in (-\infty, \infty)$ and $\rho \in (0, \infty)$. In this conformal representation, spatial infinity $i^0$ corresponds to a point located at the origin. For future reference, observe that $\tilde{t}$ and $\tilde{\rho}$ are related to $t$ and $\rho$ via

$$\tilde{t} = - \frac{t}{t^2 - \rho^2}, \quad \tilde{\rho} = \frac{\rho}{t^2 - \rho^2}.$$ 

Then, one introduces a time coordinate $\tau$ defined via $t = \rho \tau$. In the coordinate system determined by $\tau$ and $\rho$ the metric $g_I$ is written as

$$g_I = \rho^2 d\tau \otimes d\tau - (1 - \tau^2) d\rho \otimes d\rho + \rho \tau d\rho \otimes d\tau + \rho \tau d\tau \otimes d\rho - \rho^2 \sigma.$$

The required conformal representation is obtained by considering the rescaled metric

$$g_C \equiv \frac{1}{\rho^2} g_I.$$ 

Introducing $\rho^* = -\ln \rho$ the metric $g_C$ explicitly reads

$$g_C = d\tau \otimes d\tau - (1 - \tau^2) d\rho^* \otimes d\rho^* - \tau d\tau \otimes d\rho^* - \tau d\rho^* \otimes d\tau - \sigma.$$

Observe that spatial infinity $i^0$, which is at infinity respect to the metric $g_C$, corresponds to a set which has the topology of $\mathbb{R} \times S^2$ —see [8, 1]. In what follows we continue using the coordinates
\((\tau, \rho)\) and call them the \textit{F-coordinates}. Following the conformal rescalings previously introduced one considers the conformal extension \((\mathcal{M}, g_C)\) where

\[
g_C = \Theta^2 \tilde{\eta}, \quad \Theta = \rho(1 - \tau^2),
\]

and

\[
\mathcal{M} \equiv \{ p \in \mathbb{R}^4 \mid -1 \leq \tau \leq 1, \ \rho(p) \geq 0 \}.
\]

In this representation future and past null infinity are located at

\[
\mathcal{I}^+ \equiv \{ p \in \mathcal{M} \mid \tau(p) = 1 \}, \quad \mathcal{I}^- \equiv \{ p \in \mathcal{M} \mid \tau(p) = -1 \},
\]

and the physical Minkowski spacetime can be identified with the region

\[
\tilde{\mathcal{M}} \equiv \{ p \in \mathcal{M} \mid -1 < \tau(p) < 1, \ \rho(p) > 0 \};
\]

In addition, the following sets will be distinguished:

\[
I \equiv \{ p \in \mathcal{M} \mid |\tau(p)| < 1, \ \rho(p) = 0 \}, \quad I^0 \equiv \{ p \in \mathcal{M} \mid \tau(p) = 0, \ \rho(p) = 0 \},
\]

\[
I^+ \equiv \{ p \in \mathcal{M} \mid \tau(p) = 1, \ \rho(p) = 0 \}, \quad I^- \equiv \{ p \in \mathcal{M} \mid \tau(p) = -1, \ \rho(p) = 0 \}.
\]

Notice that spatial infinity \(i^0\), which originally was a point in the \(g_I\)-representation, can be identified with the set \(I\) in the \(g_C\)-representation. In addition, one can intuitively think of the \textit{critical sets} \(I^+\) and \(I^-\) as the region where spatial infinity “touches” \(\mathcal{I}^+\) and \(\mathcal{I}^-\) respectively. Similarly, \(I^0\) represents the intersection of \(i^0\) and the initial hypersurface \(S \equiv \{ \tau = 0 \}\). See [8, 11] and [1] for further discussion of the framework of the cylinder at spatial infinity implemented for stationary spacetimes.

### 2.2 The F-gauge

In this section a brief discussion of the so-called F-gauge is provided —see [11, 1] for a discussion of the F-gauge in the language of fibre bundles. Following the philosophy of the previous section the discussion presented here will not make use of the extended bundle space —see [11, 1] for definitions. One of the motivations for the introduction of this gauge is that it exploits the properties of conformal geodesics. More precisely, in this framework, one introduces a null frame whose timelike leg corresponds to the tangent of a conformal geodesic starting from a fiduciary spacelike hypersurface \(S \equiv \{ \tau = 0 \}\). The notion of conformal geodesics, however, will not be discussed here —see [7, 9, 25, 26] for definitions and further discussion.

To start the discussion, consider the conformal extension \((\mathcal{M}, g_C)\) of the Minkowski spacetime and the F-coordinate system introduced in Section 2.1. Observe that the induced metric on the surface \(Q \equiv \{ \tau = \tau_*, \rho = \rho_* \}\), with \(\tau_*, \rho_*\) fixed, is the standard metric on \(S^2\). Consequently, one can introduce a complex null frame \(\{ \partial_+, \partial_- \}\) on \(Q\) as described in Appendix A. To propagate this frame off \(Q\) one requires that

\[
[\partial_\tau, \partial_\pm] = 0, \quad [\partial_\rho, \partial_\pm] = 0.
\]

Taking into account the above construction one writes, in spinorial notation, the following space-time frame

\[
e_{00'} = \frac{\sqrt{2}}{2} ((1 - \tau) \partial_\tau + \rho \partial_\rho), \quad e_{11'} = \frac{\sqrt{2}}{2} ((1 + \tau) \partial_\tau - \rho \partial_\rho),
\]

\[
e_{01'} = \frac{\sqrt{2}}{2} \partial_+, \quad e_{10'} = \frac{\sqrt{2}}{2} \partial_-.
\]
The corresponding dual coframe is given by
\[ \omega^0' = \frac{\sqrt{2}}{2} \left( d\tau - \frac{1}{\rho}(1 - \tau) d\rho \right), \quad \omega^1' = \frac{\sqrt{2}}{2} \left( d\tau + \frac{1}{\rho}(1 + \tau) d\rho \right), \]
\[ \omega^0' = \sqrt{2} \omega^+, \quad \omega^1' = \sqrt{2} \omega^- . \]

One can directly verify that
\[ g_C = \epsilon_{AB} \epsilon_{A'B'} \omega^{AA'} \omega^{BB'} . \]

The above construction and frame will be referred in the following discussion as the \( F \)-gauge. A direct computation using the Cartan structure equations shows that the only non-zero reduced connection coefficients are given by
\[ \Gamma^{00'}_{11} = \Gamma^{11'}_{11} = \frac{\sqrt{2}}{4}, \quad \Gamma^{00'}_{00} = \Gamma^{11'}_{00} = -\frac{\sqrt{2}}{4}, \]
\[ \Gamma^{10'}_{11} = \Gamma^{10'}_{00} = \frac{\sqrt{2}}{4} \omega, \quad \Gamma^{01'}_{00} = \Gamma^{01'}_{11} = \frac{\sqrt{2}}{4} \omega. \]

### 3 The electromagnetic field in the F-gauge

In this section the Maxwell equations on \( (\mathcal{M}, g_C) \) are discussed. After rewriting the equations in terms of the \( \mathfrak{d} \) and \( \overline{\mathfrak{d}} \) operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion, satisfy ordinary differential equations which can be explicitly solved in terms of special functions. The analysis given here is similar to the one for the Maxwell field on a Schwarzschild background in [29] and the gravitational field in [8]. Notice that, in contrast with the analysis presented in this section, in the latter references the equations and relevant structures are lifted to the extended bundle space. Additionally, the initial data considered in this analysis is generic and in particular is not assumed to be time symmetric.

#### 3.1 The spinorial Maxwell equations

The Maxwell equations in the 2-spinor formalism take the form of the spin-1 equation
\[ \nabla_A \phi_{AB} = 0 . \] (6)

Let \( \epsilon_A^A \) with \( \epsilon_0^A = \phi^A \) and \( \epsilon_1^A = \overline{\phi}^A \) denote a spin dyad adapted to the F-gauge so that \( \epsilon_{AA'} \epsilon_{A'B'} = \epsilon_A^A \epsilon_{A'B'} \), corresponds to the null frame introduced in Section 2.2. A direct computation shows that equation (6) implies a set of equations for the components of \( \phi_{AB} \) respect to \( \epsilon_A^A \): \( \phi_0 \equiv \phi_{AB} \overline{\phi}^A \overline{\phi}^B \), \( \phi_1 \equiv \phi_{AB} \phi^A \overline{\phi}^B \) and \( \phi_2 \equiv \phi_{AB} \overline{\phi}^A \overline{\phi}^B \), which can be split into a system of evolution equations
\[ (1 + \tau) \partial_+ \phi_0 - \rho \partial_{\rho} \phi_0 - \partial_+ \phi_1 = -\phi_0, \] (7a)
\[ \partial_+ \phi_1 - \frac{1}{2} \left( \partial_+ \phi_2 + \partial_- \phi_0 \right) = \frac{1}{2} (\overline{\phi} \phi_2 + \phi \phi_0), \] (7b)
\[ (1 - \tau) \partial_+ \phi_2 + \rho \partial_{\rho} \phi_2 - \partial_- \phi_1 = \phi_2, \] (7c)
and a constraint equation
\[ \tau \partial_+ \phi_1 - \rho \partial_{\rho} \phi_1 + \frac{1}{2} \left( \partial_- \phi_0 - \partial_+ \phi_2 \right) = \frac{1}{2} (\overline{\phi} \phi_2 - \phi \phi_0). \] (7d)

One can systematically solve the above equations decomposing the fields \( \phi_0, \phi_1, \phi_2 \) in spin-weighted spherical harmonics. To do so, one has to rewrite these equations in terms of the \( \mathfrak{d} \) and
\( \delta \) operators of Newman and Penrose. Using (105) of Appendix B and the fact that \( \phi_0, \phi_1 \) and \( \phi_2 \) have spin weights 1, 0 and -1, respectively, one finds that equations (7a)-(7d) can be rewritten as the following evolution equations

\[
(1 + \tau) \partial_\tau \phi_0 - \rho \partial_\rho \phi_0 + \delta \phi_1 = -\phi_0, \quad (8a)
\]

\[
\partial_\tau \phi_1 + \frac{1}{2} (\delta \phi_2 + \bar{\delta} \phi_0) = 0, \quad (8b)
\]

\[
(1 - \tau) \partial_\tau \phi_2 + \rho \partial_\rho \phi_2 + \bar{\delta} \phi_1 = \phi_2, \quad (8c)
\]

and the constraint equation

\[
\tau \partial_\tau \phi_1 - \rho \partial_\rho \phi_1 + \frac{1}{2} (\delta \phi_2 - \bar{\delta} \phi_0) = 0. \quad (8d)
\]

### 3.2 The transport equations for the electromagnetic field on the cylinder at spatial infinity

In order to analyse the behaviour of solutions of the Maxwell equations in a neighbourhood of the cylinder at spatial infinity we assume that \( \phi_0, \phi_1 \) and \( \phi_2 \) are smooth functions of \( \tau \) and \( \rho \). Moreover, taking into account equation (107) of Appendix B we make the Ansatz:

**Assumption 1.** The components of the Maxwell field admit a Taylor-like expansion around \( \rho = 0 \) of the form

\[
\phi_n = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p,\ell,m}(\tau) Y_{1-n,\ell-1,mp}, \quad (9)
\]

where \( a_{n,p,\ell,m} : \mathbb{R} \to \mathbb{C} \) and with \( n = 0, 1, 2 \).

**Remark 1.** Recalling that \( Y_{s';\ell';m'} = 0 \) for \( l' < |s'| \) then one notices that the lowest order in the expansion for \( \phi_0 \) is \( \mathcal{O}(\rho^2) \). This observation will play a role in Section 6 when the electromagnetic NP constants are computed in terms of the initial data. Expression (9) is not the most general Ansatz which is compatible with the Maxwell constraints. However, more general expansions, like

\[
\phi_n = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p,\ell,m}(\tau) Y_{1-n,\ell,mp},
\]

which follow from general multipolar expansions in electrostatics and magnetostatics and allow for higher harmonics at each order in \( p \) can be seen to have, in general, divergent Newman-Penrose constants.

To simplify the notation of the subsequent analysis let

\[
\phi_n^{(p)} = \left. \frac{\partial^p \phi_n}{\partial \rho^p} \right|_{\rho=0}, \quad (10)
\]

with \( n = 0, 1, 2 \). Formally differentiating equations (8a)-(8d) respect to \( \rho \) and evaluating at the cylinder \( I \) one obtains

\[
(1 + \tau) \dot{\phi}_0^{(p)} - (p - 1) \phi_0^{(p)} + \delta \phi_1^{(p)} = 0, \quad (11a)
\]

\[
\dot{\phi}_1^{(p)} + \frac{1}{2} (\delta \phi_2^{(p)} + \bar{\delta} \phi_0^{(p)}) = 0, \quad (11b)
\]

\[
(1 - \tau) \dot{\phi}_2^{(p)} + (p - 1) \phi_2^{(p)} + \bar{\delta} \phi_1^{(p)} = 0, \quad (11c)
\]

\[
\tau \dot{\phi}_1^{(p)} - p \phi_1^{(p)} + \frac{1}{2} (\delta \phi_2^{(p)} - \bar{\delta} \phi_0^{(p)}) = 0, \quad (11d)
\]

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where the dot denotes a derivative respect to \( \tau \). Using equations (108a)-(108b) of Appendix B and expansions encoded in equation (9) one obtains the following equations for \( a_{n,p,\ell,m} \)

\[
(1 + \tau)\dot{a}_{0,p,\ell,m} + \sqrt{\ell(\ell + 1)}a_{1,p,\ell,m} - (p - 1)a_{0,p,\ell,m} = 0, \quad (12)
\]

\[
\dot{a}_{1,p,\ell,m} + \frac{1}{2}\sqrt{\ell(\ell + 1)}(a_{2,p,\ell,m} - a_{0,p,\ell,m}) = 0, \quad (13)
\]

\[
(1 - \tau)\dot{a}_{2,p,\ell,m} - \sqrt{\ell(\ell + 1)}a_{1,p,\ell,m} + (p - 1)a_{2,p,\ell,m} = 0, \quad (14)
\]

\[
\tau\dot{a}_{1,p,\ell,m} - \frac{1}{2}\sqrt{\ell(\ell + 1)}(a_{2,p,\ell,m} + a_{0,p,\ell,m}) - pa_{1,p,\ell,m} = 0, \quad (15)
\]

for \( p \geq 1, 1 \leq \ell \leq p, -\ell \leq m \leq \ell \). Notice that equations (12)-(15) correspond, essentially, to the homogeneous part of the equations reported in [29]. Furthermore, \( a_{1,p,\ell,m} \) can be solved from (13) and (15) in terms of \( a_{0,p,\ell,m} \) and \( a_{2,p,\ell,m} \) to obtain

\[
a_{1,p,\ell,m} = \frac{\sqrt{\ell(\ell + 1)}}{2p}((1 - \tau)a_{2,p,\ell,m} + (1 + \tau)a_{0,p,\ell,m}). \quad (16)
\]

Substituting \( a_{1,p,\ell,m} \) as given in (16) into equations (12) and (14) one obtains

\[
(1 + \tau)a_{0,p,\ell,m} + \left( \frac{1}{2}\ell(\ell + 1)(1 + \tau) - (p - 1) \right)a_{0,p,\ell,m} + \frac{1}{2p}(\ell(\ell + 1)(1 - \tau)a_{2,p,\ell,m} = 0, \quad (17a)
\]

\[
(1 - \tau)a_{2,p,\ell,m} - \frac{1}{2p}(\ell(\ell + 1)(1 + \tau)a_{0,p,\ell,m} - \left( \frac{1}{2p}(\ell(\ell + 1)(1 - \tau) - (p - 1) \right)a_{2,p,\ell,m} = 0. \quad (17b)
\]

At this point one can proceed in analogous way as in [29] to obtain a fundamental matrix for the system (17a)-(17b): a direct computation shows that one can decouple the last system of first order equations and obtain the following second order equations

\[
(1 - \tau^2)\ddot{a}_{0,p,\ell,m} + 2(1 - (p - 1)\tau)a_{0,p,\ell,m} + (p + \ell)(\ell - p + 1)a_{0,p,\ell,m} = 0, \quad (18a)
\]

\[
(1 - \tau^2)\ddot{a}_{2,p,\ell,m} - 2(1 + (p - 1)\tau)a_{2,p,\ell,m} + (p + \ell)(\ell - p + 1)a_{2,p,\ell,m} = 0. \quad (18b)
\]

Dropping temporarily the subindices \( p, \ell, m \) observe that, if \( a_2(\tau) \) solves (18b) then \( a_2(-\tau) \) solves equation (18a). Equations (18a)-(18b) are particular examples of so-called Jacobi ordinary differential equations. Following the discussion of [29] one obtains the following:

**Proposition 1.** If \( p \geq 2, \ell < p, -\ell \leq m \leq \ell \) the solutions to the Jacobi equations (18a)-(18b) are polynomial in \( \tau \). For \( p \geq 2, \ell = p, -p \leq m \leq p \) one has

\[
a_{0,p,p,m}(\tau) = -\left( \frac{1 - \tau}{2} \right)^{p+1}\left( \frac{1 + \tau}{2} \right)^{-p-1}\left( C_{p,m} + C_{p,m}^\oplus \int_0^\tau ds \frac{ds}{(1+s)^p(1-s)^{p+2}} \right), \quad (19a)
\]

\[
a_{2,p,p,m}(\tau) = \left( \frac{1 + \tau}{2} \right)^{p+1}\left( \frac{1 - \tau}{2} \right)^{-p-1}\left( D_{p,m} + D_{p,m}^\oplus \int_0^\tau ds \frac{ds}{(1-s)^p(1+s)^{p+2}} \right). \quad (19b)
\]

where \( C_{p,m}^\oplus \) and \( D_{p,m}^\oplus \) are integration constants.

**Remark 2.** Observe that, for non-vanishing \( C_{p,m}^\oplus \) and \( D_{p,m}^\oplus \) the solutions \( a_{0,p,p,m}(\tau) \) and \( a_{2,p,p,m}(\tau) \) with \( p \geq 2, -p \leq m \leq p \), contain terms which diverge logarithmically near \( \tau = \pm 1 \).

**Remark 3.** The expressions of Proposition 1 are solutions to the Jacobi equations. To obtain a solution to the original system it is necessary to evaluate these expressions in the coupled system (17a)-(17b). In turn, this gives rise to restrictions on the integration constants.
3.3 Initial data for the Maxwell equations

Evaluating the constraint equation (8d) at $\tau = 0$ gives the following equation

$$\rho \partial_\rho \phi_1 - \frac{1}{2} (\bar{\partial} \phi_2 - \bar{\partial} \phi_0) = 0.$$  (20)

Consistent with the expressions encoded in equation (9) one considers on the initial hypersurface $S$ fields $\phi_n|_S$, with $n = 0, 1, 2$, which can be expanded as

$$\phi_n|_S = \sum_{p=|1-n|}^{\infty} \sum_{\ell=|1-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{0,p,\ell,m}(0) Y_{1-n;\ell-1-m} p^p,$$  (21)

Observe that once $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$ are given, $a_{1,p,\ell,m}(0)$ is already determined by virtue of equation (16) as

$$a_{1,p,\ell,m}(0) = \frac{\sqrt{\ell(\ell+1)}}{2p} (a_{2,p,\ell,m}(0) + a_{0,p,\ell,m}(0)).$$

In addition, observe that equations (17a)-(17b) are first order while equations (18a)-(18b) are second order. Consequently, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{2,p,\ell,m}(0)$ are determined, by virtue of equations (17a)-(17b) restricted to $S$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$.

The following remark plays an important role for the subsequent discussion of the electromagnetic NP constants:

**Remark 4.** For general $p, \ell$ and $m$, the free data is encoded in $a_{0,p,\ell,m}(0)$ and $a_{2,p,\ell,m}(0)$. Nevertheless, for $p = \ell$, a direct substitution of the solution (19a)-(19b) into equations (17a)-(17b) shows that $C_{0,m}^p = D_{0,m}^p = 0$ with $p \geq 2, -p \leq m \leq p$. Consequently, the potentially divergent terms in expressions (19a)-(19b) do not contribute to the electromagnetic field. Additionally, one has that

$$a_{0,p,m}(0) = a_{2,p,m}(0) = C_{p,m} = D_{p,m},$$  (22)

with $p \geq 2, -p \leq m \leq p$. Observe that the initial data considered is generic and the restriction (22) is a consequence of the interplay of the evolution and constraint equations. In other words, this condition does arise from restricting the class of initial data.

**Remark 5.** The convergence of the expansions encoded in (9) follows from the results of [30].

4 The massless spin-2 field equations in the F-gauge

In Section 3 the Maxwell equations (in the F-gauge) were discussed, these correspond in spinorial formalism to the spin-1 equations. In this section, a similar analysis is performed but now for a spin-2 field propagating on the Minkowski spacetime. As discussed in [27] the spin-2 equations where the background geometry is that of the Minkowski spacetime can be used to describe the linearised gravitational field. In [27] these equations were written in terms the lifts of the relevant structures to the extended bundle space. In this section, following the spirit of the present article, the equations will be discussed without making use of these structures. In a similar way as in the electromagnetic case studied in Section 3, after rewriting the equations in terms of the $\bar{\partial}$ and $\bar{\partial}$ operators, a general solution is obtained by expanding the fields in spin-weighted spherical harmonics. The resulting equations for the coefficients of the expansion satisfy ordinary differential equations which can be explicitly solved in terms of special functions.
4.1 The spin-2 equation

As discussed in [27], the linearised gravitational field over the Minkowski spacetime can be described with the so-called massless spin-2 field equation

$$\nabla_A^{\phantom{A}A} \phi_{ABCD} = 0. \quad (23)$$

Following an approach analogous to the one described in Section 3.1 for the electromagnetic field, it can be shown that equation (23) implies the following evolution equations for the components of the spinor $\phi_{ABCD}$

\begin{align*}
(1 + \tau) \partial_+ \phi_0 - \rho \partial_\mu \phi_0 - \partial_+ \phi_1 + \bar{\omega} \phi_1 &= -2 \phi_0, \quad (24a) \\
\partial_+ \phi_1 - \frac{1}{2} \partial_+ \phi_2 - \frac{1}{2} \partial_- \phi_0 - \bar{\omega} \phi_0 &= -\phi_1, \quad (24b) \\
\partial_+ \phi_2 - \frac{1}{2} \partial_- \phi_1 - \frac{1}{2} \partial_+ \phi_3 - \frac{1}{2} \bar{\omega} \phi_1 - \frac{1}{2} \bar{\omega} \phi_3 &= 0, \quad (24c) \\
\partial_+ \phi_3 - \frac{1}{2} \partial_+ \phi_4 - \frac{1}{2} \partial_- \phi_2 - \bar{\omega} \phi_4 &= \phi_3, \quad (24d) \\
(1 - \tau) \partial_+ \phi_4 + \rho \partial_\mu \phi_4 - \partial_- \phi_3 + \bar{\omega} \phi_3 &= 2 \phi_4, \quad (24e)
\end{align*}

and the constraint equations

\begin{align*}
\tau \partial_+ \phi_1 - \rho \partial_\mu \phi_1 - \frac{1}{2} \partial_+ \phi_2 + \frac{1}{2} \partial_- \phi_0 + \bar{\omega} \phi_0 &= 0, \quad (25a) \\
\tau \partial_+ \phi_2 - \rho \partial_\mu \phi_2 - \frac{1}{2} \partial_+ \phi_3 + \frac{1}{2} \partial_- \phi_1 - \frac{1}{2} \bar{\omega} \phi_3 + \frac{1}{2} \bar{\omega} \phi_1 &= 0, \quad (25b) \\
\tau \partial_+ \phi_3 - \rho \partial_\mu \phi_3 - \frac{1}{2} \partial_+ \phi_4 + \frac{1}{2} \partial_- \phi_2 - \bar{\omega} \phi_4 &= 0, \quad (25c)
\end{align*}

where the five components $\phi_0, \phi_1, \phi_2, \phi_3$ and $\phi_4$, given by

$$\begin{align*}
\phi_0 &= \phi_{ABCD}^{A} o^{B} o^{C} o^{D}, \\
\phi_1 &= \phi_{ABCD}^{A} o^{B} o^{C} t^{D}, \\
\phi_2 &= \phi_{ABCD}^{A} o^{B} t^{C} t^{D}, \\
\phi_3 &= \phi_{ABCD}^{A} t^{B} t^{C} t^{D}, \\
\phi_4 &= \phi_{ABCD}^{A} t^{B} t^{C} t^{D},
\end{align*}$$

have spin weight of 2, 1, 0, -1, -2 respectively. Taking into account this observation and equations (105) and (106) given in Appendix B one can rewrite (24a)-(25c) in terms of the $\bar{\sigma}$ and $\bar{\omega}$ as done for the electromagnetic case. A direct computation renders the following evolution equations

\begin{align*}
(1 + \tau) \partial_+ \phi_0 - \rho \partial_\mu \phi_0 + \bar{\omega} \phi_1 &= -2 \phi_0, \quad (26a) \\
\partial_+ \phi_1 + \frac{1}{2} \bar{\omega} \phi_0 + \frac{1}{2} \bar{\omega} \phi_2 &= -\phi_1, \quad (26b) \\
\partial_+ \phi_2 + \frac{1}{2} \bar{\omega} \phi_1 + \frac{1}{2} \bar{\omega} \phi_3 &= 0, \quad (26c) \\
\partial_+ \phi_3 + \frac{1}{2} \bar{\omega} \phi_2 + \frac{1}{2} \bar{\omega} \phi_4 &= \phi_3, \quad (26d) \\
(1 - \tau) \partial_+ \phi_4 + \rho \partial_\mu \phi_4 + \bar{\omega} \phi_3 &= 2 \phi_4, \quad (26e)
\end{align*}

and the constraint equations

\begin{align*}
\tau \partial_+ \phi_1 - \rho \partial_\mu \phi_1 + \frac{1}{2} \bar{\omega} \phi_2 - \frac{1}{2} \bar{\omega} \phi_0 &= 0, \quad (27a) \\
\tau \partial_+ \phi_2 - \rho \partial_\mu \phi_2 + \frac{1}{2} \bar{\omega} \phi_3 - \frac{1}{2} \bar{\omega} \phi_1 &= 0, \quad (27b) \\
\tau \partial_+ \phi_3 - \rho \partial_\mu \phi_3 + \frac{1}{2} \bar{\omega} \phi_4 - \frac{1}{2} \bar{\omega} \phi_2 &= 0. \quad (27c)
\end{align*}

With the equations already written in this way, one can follow the discussion of [27] for parametrising the solutions to equations (26a)-(27c).
4.2 The transport equations for the massless spin-2 field on the cylinder at spatial infinity

One proceeds in analogous way as in the electromagnetic case and assumes that the fields $\phi_n$ with $n = 0, 1, 2, 3, 4$, are smooth functions of $\tau$ and $\rho$. Taking into account equation (107) of Appendix B, we assume one can express the components the of the linearised gravitational field in a Taylor-like expansion around $\rho = 0$. More precisely, we make the Ansatz:

**Assumption 2.** In what follows we assume that the components of the spin-2 field have the expansions

$$
\phi_n = \sum_{p=|2-n|}^{\infty} \sum_{\ell=|2-n|}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p,\ell,m}(\tau) Y_{2-n;\ell-1,m} \rho^p
$$

where $a_{n,p,\ell,m} : \mathbb{R} \to \mathbb{C}$ and $n = 0, 1, 2, 3, 4$.

**Remark 6.** Recalling that $Y_{s';l',m'} = 0$ for $l' < |s'|$ then one notices that the lowest order in the expansion for $\phi_0$ is $\mathcal{O}(\rho^1)$. As in the case of the spin-1 field, one can consider more general expressions which are compatible with the spin-2 constraints which admit higher harmonics at every order. Some experimentation reveals, however, that these more general expansions lead to divergent NP constants —cf. Remark 1.

For the remaining part of this section, the $p$-th derivative respect to $\rho$ of the fields $\phi_n$ with $n = 0, 1, 2, 3, 4$ evaluated at the cylinder $I$, is denoted using the same notation as in equation (10). Then, by formally differentiating equations (26a)-(27c) respect to $\rho$ and evaluating at the cylinder $I$, one obtains the following equations

$$
(1 + \tau) \partial_\tau \phi_0^{(p)} + \partial_\tau \phi_1^{(p)} (p-2) \phi_0^{(p)} = 0,
(29a)
$$

$$
\partial_\tau \phi_1^{(p)} + \frac{1}{2} \tau \partial_\tau \phi_0^{(p)} + \frac{1}{2} \partial_\tau \phi_2^{(p)} + \phi_1^{(p)} = 0,
(29b)
$$

$$
\partial_\tau \phi_2^{(p)} + \frac{1}{2} \partial_\tau \phi_1^{(p)} + \frac{1}{2} \partial_\tau \phi_3^{(p)} = 0,
(29c)
$$

$$
\partial_\tau \phi_3^{(p)} + \frac{1}{2} \partial_\tau \phi_2^{(p)} + \frac{1}{2} \partial_\tau \phi_4^{(p)} + \phi_3^{(p)} = 0,
(29d)
$$

and

$$
(1 - \tau) \partial_\tau \phi_4^{(p)} + \partial_\tau \phi_3^{(p)} + (p-2) \phi_4^{(p)} = 0,
(29e)
$$

and

$$
\tau \partial_\tau \phi_1^{(p)} + \frac{1}{2} \partial_\tau \phi_0^{(p)} - \frac{1}{2} \partial_\tau \phi_0^{(p)} - p \phi_1^{(p)} = 0,
(30a)
$$

$$
\tau \partial_\tau \phi_2^{(p)} + \frac{1}{2} \partial_\tau \phi_1^{(p)} - \frac{1}{2} \partial_\tau \phi_1^{(p)} - p \phi_2^{(p)} = 0,
(30b)
$$

$$
\tau \partial_\tau \phi_3^{(p)} + \frac{1}{2} \partial_\tau \phi_2^{(p)} - \frac{1}{2} \partial_\tau \phi_2^{(p)} - p \phi_3^{(p)} = 0.
(30c)
$$

The last set of equations along with the expansion (28), in turn, imply the following equations for $a_{n,p,\ell,m}$ with $p \geq 2$ and $2 \leq \ell \leq p$:

$$
(1 + \tau) \dot{a}_0 + \lambda_1 a_1 - (p-2) a_0 = 0,
(31a)
$$

$$
\dot{a}_1 - \frac{1}{2} \lambda_1 a_0 + \frac{1}{2} \lambda_2 a_2 + a_1 = 0,
(31b)
$$

$$
\dot{a}_2 - \frac{1}{2} \lambda_2 a_0 + \frac{1}{2} \lambda_3 a_3 = 0,
(31c)
$$

$$
\dot{a}_3 - \frac{1}{2} \lambda_3 a_2 + \frac{1}{2} \lambda_4 a_4 - a_3 = 0.
(31d)$$
\[(1 - \tau)\dot{a}_4 - \lambda_1 a_3 + (p - 2)a_4 = 0, \quad (31e)\]

and
\[
\tau\dot{a}_1 + \frac{1}{2}\lambda_0 a_2 + \frac{1}{2}\lambda_1 a_0 - pa_1 = 0, \quad (32a)
\]
\[
\tau\dot{a}_2 + \frac{1}{2}\lambda_0 a_3 + \frac{1}{2}\lambda_0 a_1 - pa_2 = 0, \quad (32b)
\]
\[
\tau\dot{a}_3 + \frac{1}{2}\lambda_1 a_4 + \frac{1}{2}\lambda_0 a_2 - pa_3 = 0, \quad (32c)
\]

where \(\lambda_1 = \sqrt{(\ell - 1)(\ell + 2)}\) and \(\lambda_0 = \sqrt{\ell(\ell + 1)}\) and the labels \(p, \ell, m\) have been suppressed for conciseness. From equations (31b)-(31d) and (32a)-(32c) one obtains an algebraic system which can be written succinctly as
\[
\begin{pmatrix}
    p + \tau & -\frac{1}{2}(1 - \tau)\lambda_0 & 0 \\
    -\frac{1}{2}(1 + \tau)\lambda_0 & p & -\frac{1}{2}(1 - \tau)\lambda_0 \\
    0 & -\frac{1}{2}(1 + \tau)\lambda_0 & p - \tau
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    a_3
\end{pmatrix}
= \frac{1}{2}\lambda_1
\begin{pmatrix}
    (1 + \tau)a_0 \\
    0 \\
    (1 - \tau)a_4
\end{pmatrix}. \quad (33)
\]

Solving the above system and substituting \(a_0, a_1\) and \(a_3\) written in terms of \(a_0\) and \(a_4\) into equations (31a) and (31e) one obtains
\[
(1 + \tau)a_0 + (-(p - 2) + f(\tau, p, \ell))a_0 + g(\tau, p, \ell)a_4 = 0, \quad (34a)
\]
\[
(1 - \tau)\dot{a}_4 + (-(p - 2) + f(-\tau, p, \ell))a_4 + g(-\tau, p, \ell)a_0 = 0, \quad (34b)
\]

where
\[
f(\tau, p, \ell) = \frac{(1 + \tau)(\ell - 1)(\ell + 2)[4p^2 - 4p\tau + \ell(\ell + 1)(\tau^2 - 1)]}{4p(2p^2 - \ell(\ell + 1) + (\ell - 1)(\ell + 2)\tau^2)},
\]
\[
g(\tau, p, \ell) = \frac{(1 - \tau)^3(\ell + 1)(\ell - 1)(\ell + 2)}{4p(2p^2 - \ell(\ell + 1) + (\ell - 1)(\ell + 2)\tau^2)}.
\]

Together, the last equations entail the following decoupled equations
\[
(1 - \tau^2)\dot{a}_0 + (4 + 2(p - 1)\tau)a_0 + (p + \ell)(p - \ell + 1)a_0 = 0, \quad (35a)
\]
\[
(1 - \tau^2)\dot{a}_4 + (-4 + 2(p - 1)\tau)a_4 + (p + \ell)(p - \ell + 1)a_4 = 0. \quad (35b)
\]

It can be verified that if \(a_0(\tau)\) solves (35a) then \(a_0(-\tau)\) solves equation (35b). As in the electromagnetic case, these equations are Jacobi ordinary differential equations. For the solutions to these equations one has the following:

**Proposition 2.** For \(p \geq 3, p > \ell, -\ell \leq m \leq \ell\) the solutions to equations (35a)-(35b) are polynomial. For \(p \geq 3, p = \ell, -p \leq m \leq p\) one has
\[
a_{0,p,p,m}(\tau) = \left(\frac{1 - \tau}{2}\right)^{p+2}\left(\frac{1 + \tau}{2}\right)^{p-2}\left(C_{p,m} + C_{p,m}^\circ \int_0^\tau \frac{ds}{(1 + s)^{p-1}(1 - s)^{p+3}}\right), \quad (36a)
\]
\[
a_{4,p,p,m}(\tau) = \left(\frac{1 + \tau}{2}\right)^{p+2}\left(\frac{1 - \tau}{2}\right)^{p-2}\left(D_{p,m} + D_{p,m}^\circ \int_0^\tau \frac{ds}{(1 + s)^{p-1}(1 - s)^{p+3}}\right), \quad (36b)
\]

where \(C_{p,m}, C_{p,m}^\circ\) and \(D_{p,m}, D_{p,m}^\circ\) are integration constants.

**Remark 7.** Notice that for non-vanishing \(C_{p,m}^\circ\) and \(D_{p,m}^\circ\) diverges logarithmically near \(\tau = \pm 1\). The expressions of Proposition 2 are solutions to the Jacobi equations. To obtain a solution to the original system it is necessary to evaluate these expressions in the coupled system (34a)-(34b). In turn, this shows that the integration constants are not independent of each other.

**Remark 8.** As discussed in the next section, in contrast with the electromagnetic case, the singular terms in expressions (36a)-(36b) do represent genuine contributions to the spin-2 field.
4.3 Initial data for the spin-2 equations

Consistent with equations (28) one considers on the initial hypersurface $S$ fields $\phi_n|_S$, with $n = 0, 1, 2, 3, 4$ which can be expanded as

$$\phi_n|_S = \sum_{p=|2-n|}^{\infty} \sum_{\ell=2-n}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{n,p,\ell,m}(0) Y_{2-n;\ell-1m}^p.$$  \hspace{1cm} (37)

Observe that, by virtue of equation (33), the initial data $a_{1,p,\ell,m}(0)$, $a_{2,p,\ell,m}(0)$ and $a_{3,p,\ell,m}(0)$ is determined by $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. In addition, notice that, equations (34a)-(34b) are first order while equations (35a)-(35b) are second order. Therefore, the initial data $\dot{a}_{0,p,\ell,m}(0)$ and $\dot{a}_{4,p,\ell,m}(0)$ is determined, as a consequence of equations (34a)-(34b) restricted to $S$, by the initial data $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. The following remarks play an important role for the subsequent discussion of the spin-2 NP constants:

Remark 9. For general $p$, $\ell$ and $m$, the free initial data is encoded in $a_{0,p,\ell,m}(0)$ and $a_{4,p,\ell,m}(0)$. However, for $p = \ell$, a direct substitution of the solution (36a)-(36b) into equations (34a)-(34b) shows that $C_{p,m}^{\sigma} = D_{p,m}^{\sigma}$ and $C_{p,m} = D_{p,m}$. In other words, for $p \geq 3$, $-p \leq m \leq p$,

$$a_{0,p,p,m}(0) = a_{4,p,p,m}(0).$$  \hspace{1cm} (38)

Remark 10. In contrast with the electromagnetic case, in principle, initial data with $C_{p,m}^{\sigma} = D_{p,m}^{\sigma} \neq 0$, is admissible and consequently, for generic initial data the appearance of logarithmic singularities is expected. Nevertheless, for the computation of the NP constants $C_{p,m} = D_{p,m} = 0$ will be assumed —otherwise the expressions defining the NP constants diverge —see Section 7.

Remark 11. The solutions to the constraint equations correspond, in tensor frame notation to the equation

$$D^i \phi_{ij} = 0,$$

where $D_i$ denotes the covariant Levi-Civita derivative of the metric $h_{ij}$ intrinsic to the initial hypersurface $S$ and $\phi_{ij}$ corresponds to tensorial counterpart of the field $\phi_{ABCD}$. In the conformally flat setting, the solutions to these equations are known —see [5]. Moreover, in the latter reference, a general parametrisation to the solutions to this equation was given. Consequently, one could, in principle, rewrite the initial data considered in this section using this parametrisation. Nevertheless, this analysis will not be pursued here.

Remark 12. The convergence of the expansions (28) follows from the results given in [27].

5 The NP-gauge

In this section, an adapted frame satisfying the NP-gauge conditions and Bondi coordinates are constructed for the conformal extension $(\mathcal{M}, g_I)$ introduced in Section 2.1. For convenience of the reader, a general discussion of the NP-gauge conditions and the construction of Bondi coordinates is provided in the first part of this section.

5.1 The NP-gauge conditions and Bondi coordinates

This section provides a general discussion of the NP-gauge conditions and the construction of Bondi coordinates. A more comprehensive discussion of these gauge conditions and their consequences can be found in [23, 11, 26].

Let $(\mathcal{M}, \hat{g}, \Xi)$ denote a conformal extension of an asymptotically simple spacetime $(\mathcal{M}, \hat{g})$ where $\hat{g}$ satisfies the vacuum Einstein field equations with vanishing Cosmological constant. It is
a general result in the theory of asymptotics that for vacuum spacetimes with vanishing Cosmological constant the conformal boundary $\mathcal{I}$, with locus given by $\Xi = 0$, consists of two disjoint null hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$ each one having the topology of $\mathbb{R} \times S^2$—see [23, 26]. In this section the discussion will be particularised to $\mathcal{I}^+$. Nevertheless, the time dual results and constructions can be formulated for $\mathcal{I}^-$ in an analogous manner. To simplify the notation, the symbol $\simeq$ will be used to denote equality at $\mathcal{I}$, e.g. if $w$ is a scalar field on $\mathcal{M}$ that vanishes at $\mathcal{I}$ one writes $w \simeq 0$. Let $\{\hat{e}_{AA}\}$ denote a frame satisfying $g(\hat{e}_{AA}, \hat{e}_{BB}) = \epsilon_{AB} e_{AB}$ in a neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $\mathcal{I}^+$. Additionally, let $\Gamma_{AA}^{\phantom{AA}CD}$ denote the reduced connection coefficients of the Levi-Civita connection of $g$ defined respect to $\hat{e}_{AA}$. The frame $\hat{e}_{AA}$ is an adapted frame at $\mathcal{I}^+$ if the following conditions hold:

(i) The vector $\hat{e}_{11}$ is tangent to and parallly propagated along $\mathcal{I}^+$, i.e.,

$$\nabla_{11} \hat{e}_{11} \simeq 0.$$ 

(ii) On $\mathcal{U}$ there exists a smooth function $u$ inducing an affine parameter on the null generators of $\mathcal{I}^+$, namely $e_{11}(u) \simeq 1$. The vector $e_{00}$ is then defined as $e_{00} = g(\hat{e}_{u}, \cdot)$ so that it is tangent to the null generators of the hypersurfaces transverse to $\mathcal{I}$ defined by

$$\mathcal{N}_{u_o} \equiv \{ p \in \mathcal{U} \mid u(p) = u_o \},$$

with constant $u_o$.

(iii) The frame $\{\hat{e}_{AA}\}$ is tangent to the cuts $\mathcal{C}_{u_o} \equiv \mathcal{N}_{u_o} \cap \mathcal{I}^+ \simeq S^2$ and parallly propagated along $\mathcal{N}_{u_o}$, namely

$$\nabla_{00} \hat{e}_{AA} = 0 \quad \text{on} \quad \mathcal{N}_{u_o}.$$ Conditions (i)-(iii) can be encoded in the following requirements on the reduced connection coefficients $\Gamma_{AA}^{\phantom{AA}CD}$.

**Proposition 3 (adapted frame at $\mathcal{I}^+$).** Let $(\mathcal{M}, g, \Xi)$ be a conformal extension of an asymptotically simple spacetime $(\mathcal{M}, \hat{g})$ with vanishing Cosmological constant. On a neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $\mathcal{I}^+$ it is always possible to find a $g$-null frame $\{\hat{e}_{AA}\}$ for which

$$\Gamma_{10}^{11} \simeq 0, \quad \Gamma_{11}^{11} \simeq 0, \quad \Gamma_{1000} = \Gamma_{100}^{00} \simeq 0, \quad \Gamma_{1100} = \Gamma_{110}^{00} + \Gamma_{0101}, \quad \Gamma_{00}^{AB} = 0 \quad \text{on} \quad \mathcal{U}. \quad (39a)$$

The conformal freedom of the setting, i.e. the fact that instead of $(\mathcal{M}, g, \Xi)$ one can consider $(\mathcal{M}', g', \Xi')$ with

$$g' \mapsto \theta^2 g, \quad \Xi \mapsto \Xi' = \theta \Xi,$$

can be exploited to obtain an improved frame $e'_{AA}$ leading to further simplifications to the conditions given in Proposition 3. If in addition, one introduces an arbitrary function $\kappa$ constant along the generators of $\mathcal{I}^+$ and sets

$$e_{11}^\prime \simeq \theta^{-2} \kappa e_{11}, \quad \text{on} \quad \mathcal{I}^+, \quad (40)$$

one is lead to define an affine parameter $u'(u)$ such that $e_{11}$ is parallly propagated and $e_{11}(u') = 1$. This in turn implies $du'/du = \kappa^{-1} \theta^2$ which, integrating along the null generators of $\mathcal{I}^+$, renders

$$u'(u) = \frac{1}{\kappa} \int_{u_o}^{u} \theta^2(s)ds + u'_o, \quad (41)$$

where the integration constants $u_o$ and $u'_o$ identify a fiducial cut $\mathcal{C}_* \equiv C_{u_o}$. Observing that equation (40) also holds on $\mathcal{C}_*$, one prescribes the remaining part of the frame on $\mathcal{C}_*$ as

$$e_{00}^\prime = \kappa^{-1} \bar{e}_{00}, \quad e_{01}^\prime = \theta^{-1} \bar{e}_{01}, \quad e_{10}^\prime = \theta^{-1} \bar{e}_{10}, \quad \text{on} \quad \mathcal{C}_*. \quad (42)$$
Observe that, using equations (40) and (42), it can be verified that $g(e'_{AA'}, e_{BB'}) = \epsilon_{AA'}\epsilon_{BB'}$ on $C_*$. Using these expressions, one can exploit the freedom in choosing $\varpi$ and $\theta$ along with the conformal transformation laws for the relevant fields (connection coefficients and curvature spinors) and a general rotation of $e'_{01}$ and $e'_{10}$ of the form

$$e'_{01} \mapsto e^{ic}e'_{01}, \quad e'_{10} \mapsto e^{-ic}e'_{10},$$

where $c$ is a scalar function such that $c = 0$ at $C_*$, to obtain an improved frame $e'_{AA'}$ that satisfies the following conditions:

**Proposition 4 (NP-gauge conditions at $J^+$).** Let $(\hat{\mathcal{M}}, \hat{g})$ be an asymptotically simple spacetime. Locally, it is always possible to find a conformal extension $(\mathcal{M}', g', \Xi')$ for which there exist a $g'$-null frame $\{e'_{AA'}\}$ such that the reduced spin connection coefficients of the Levi-Civita connection of $g'$ respect to $e_{AA'}$ satisfy the gauge conditions:

$$\Gamma'_{00'BC} \simeq 0, \quad \Gamma'_{11'BC} \simeq 0, \quad (44a)$$

$$\Gamma'_{01'11} \simeq 0, \quad \Gamma'_{10'00} \simeq 0, \quad \Gamma'_{40'11} \simeq 0, \quad (44b)$$

$$\Gamma'_{10'01'} + \Gamma_{01'01} \simeq 0. \quad (44c)$$

Moreover, for the curvature one has

$$R' \simeq 0, \quad \Phi'_{12} \simeq 0, \quad \Phi'_{22} \simeq 0. \quad (44d)$$

where $R'$ and $\Phi'_{ab}$ are, respectively, the Ricci scalar and the components (respect to $e'_{AA'}$) of the trace-free Ricci spinor of the Levi-Civita connection of $g'$. Additionally, $e'_{00}(\Xi')$ is constant on $\mathcal{J}^+$. A frame $e'_{AA'}$ satisfying the conditions of Proposition 4 will be said to be a NP-frame. The proof of this Proposition can be found in [11] and [26]. The proof, in addition gives a procedure to determine $\theta$ and $\varpi$ by prescribing data on $C_*$, which is extended along $\mathcal{J}$ solving ordinary differential equations. Observing equations (40), (42) and (43) one concludes that in general, frames $e_{AA'}$ and $e'_{AA'}$ of Propositions 3 and 4 respectively, are related via a conformal transformation $g' = \theta^2 g$ and a Lorentz transformation encoded in $(\varpi, c)$ so that

$$e'_{11} \simeq \varpi^{-2}\varpi e_{11}, \quad e'_{00} \equiv \varpi e_{00}, \quad e'_{01} = e^{ic}\varpi^{-1}e_{01}, \quad e'_{10} = e^{-ic}\varpi^{-1}e_{10} \quad \text{on } U.$$

The function $\varpi$ corresponds a boost while $c$ encodes a spin.

In the discussion of the NP-gauge, is customary to complete the construction introducing *Bondi coordinates* as follows: choose an arbitrary coordinate system $\vartheta^a$ with $a = 2, 3$ on the cut $C_* \approx S^2$. Extend this coordinate system to $\mathcal{J}^+$ so that they remain constant along on its null generators. Recalling that $u'$, as defined in equation (41), corresponds to an affine parameter along the generators of $\mathcal{J}^+$ fixed by the condition $e'_{11}(u') = 1$, is then natural to use as an affine parameter on the hypersurfaces $\mathcal{N}_{\vartheta}^a$, transverse to $\mathcal{J}^+$, a parameter $r'$ fixed by the conditions $e'_{00}(r') = 1$ and $r' \simeq 0$. Using $r'$ and $u'$ defined as previously described, $(r', u', \vartheta^a)$ defines a Bondi coordinate system.

### 5.2 The NP frame and Bondi coordinates for the conformal extension $(\mathcal{M}, g_I)$

To implement the procedure described in Section 5.1 for the conformal extension $(\mathcal{M}, g_I)$ it is convenient to introduce null coordinates $u = t - \rho$ and $v = t + \rho$. Observe that the unphysical null coordinates $u$ and $v$ are related to the physical null coordinates $\tilde{u} = \tilde{t} - \tilde{\rho}$ and $\tilde{v} = \tilde{t} + \tilde{\rho}$ via $u = -1/\tilde{u}$ and $v = -1/\tilde{v}$. In these coordinates, the metric $g_I$ and conformal factor $\Xi$ read

$$g_I = \frac{1}{2}du \otimes dv + \frac{1}{2}dv \otimes du - \frac{1}{4}(v - u)^2 \sigma, \quad \Xi = uv.$$
In this representation, future null infinity $\mathcal{I}^+$ is located at $v = 0$ while past null infinity $\mathcal{I}^-$ is located at $u = 0$. Additionally, a $g_I$-null frame is given by

$$
e_{00}' = \sqrt{2} \partial_v, \quad e_{11}' = \sqrt{2} \partial_u, \quad e_{01}' = \frac{\sqrt{2}}{v-u} \partial_+, \quad e_{10}' = \frac{\sqrt{2}}{v-u} \partial_-.$$

A direct computation using the Cartan structure equations, one can verify that the only non-zero spin coefficients are

$$\Gamma_{10}^{10} = -\frac{\sqrt{2}}{2} \frac{\omega}{v-u}, \quad \Gamma_{01}^{10} = \frac{\sqrt{2}}{2} \frac{\omega}{v-u}, \quad \Gamma_{01}^{11} = \Gamma_{10}^{00} = -\frac{\sqrt{2}}{v-u}.$$

A direct inspection reveals that the frame $\{e_{\mathcal{A}\mathcal{A}'}\}$ does satisfy all the conditions of Proposition 4. In order to construct a frame satisfying the conditions defining the NP gauge one has to introduce a conformal rescaling a Lorentz transformation as follows: consider a conformal rescaling

$$g' = \theta^2 g_I,$$

and the following $g'$-null frame,

$$e_{00}' = \kappa^{-1} e_{00}, \quad e_{11}' = \kappa \theta^{-2} e_{11}, \quad e_{01}' = \theta^{-1} e_{01} \mathrm{i}, \quad e_{10}' = \theta^{-1} e_{10} \mathrm{i}.$$  

Some experimentation reveals that setting

$$\theta = \frac{2}{v-u}, \quad \kappa = \frac{4u^2}{(v-u)^2}, \quad c = 0,$$

one obtains the non-zero spin coefficients

$$\Gamma_{11}^{10} = \sqrt{2} \frac{u v}{u-v}, \quad \Gamma_{10}^{10} = -\frac{\sqrt{2}}{4} \omega, \quad \Gamma_{10}^{11} = \frac{\sqrt{2}}{4} \omega.$$  

In addition, observe that $\Gamma_{11}^{10} \simeq 0$. A further computation, using the NP-equations as given in [23] and equation (101) of Appendix A, shows that

$$R' = 0, \quad \Phi_{00}' = \Phi_{01}' = \Phi_{02}' = \Phi_{22}' = 0, \quad \Phi_{11}' = \frac{1}{2}.$$  

An inspection of the conditions of Proposition 4 shows that $e_{\mathcal{A}\mathcal{A}'}'$ constitutes a frame in the NP-gauge. To round up the discussion one can introduce Bondi coordinates $(r', u')$ fixed by the requirements

$$e_{00}'(r') = 1, \quad e_{11}'(u') = 1, \quad r' \simeq 0.$$  

A direct computation shows that

$$r' = \frac{4}{\sqrt{2}} \left( \frac{uv}{u-v} \right), \quad u' = -\frac{1}{\sqrt{2}} \frac{1}{u}.$$  

In these coordinates the frame $e_{\mathcal{A}\mathcal{A}'}'$ reads

$$e_{00}' = \partial_{r'}, \quad e_{11}' = -\frac{1}{2} r'^{2} \partial_{r'} + \partial_{u'}, \quad e_{01}' = \frac{\sqrt{2}}{2} \partial_+ , \quad e_{10}' = \frac{\sqrt{2}}{2} \partial_-.$$  

Observe that the Bondi coordinates $(r', u')$ are related to the physical coordinates $\tilde{\rho}$ and $\tilde{u}$, as introduced in Section 2, through

$$r' = -\frac{2}{\sqrt{2}} \frac{1}{\tilde{\rho}}, \quad u' = \frac{1}{\sqrt{2}} \tilde{u}.$$  

For future reference, notice that in the physical coordinates $\{\tilde{\rho}, \tilde{u}\}$ the NP-frame $e_{\mathcal{A}\mathcal{A}'}'$ is given by

$$e_{00}' = \sqrt{2} \tilde{\rho}^2 \partial_{\tilde{\rho}}, \quad e_{11}' = \sqrt{2} \partial_{\tilde{u}} - \sqrt{2} \partial_{\tilde{\rho}}, \quad e_{01}' = \frac{\sqrt{2}}{2} \partial_+, \quad e_{10}' = \frac{\sqrt{2}}{2} \partial_-.$$  

(50)
5.3 Relating the NP-gauge to the F-gauge

In general, a frame in the F-gauge and the NP-gauge will not coincide since, while the former is based on a Cauchy hypersurface, the latter is adapted to \( \mathcal{I} \). However, as \( g_C \) and \( g' \) are conformally related, \( g' = \kappa^2 g_C \), then the frames \( e_{AA'} \) and \( e'_{AA'} \) are related through a conformal rescaling and a Lorentz transformation

\[
e'_{AA'} = \kappa^{-1} \Lambda^B A \bar{\Lambda}^B A' e_{BB'}.
\]

To determine explicitly \( \kappa \) and \( \Lambda^A B \) observe that the frame \( \bar{e}_{AA'} \), introduced in Section 5.2, written in the F-coordinates, reads

\[
\bar{e}_{00'} = \frac{\sqrt{2}}{2\rho}(1-\tau)\partial_{\tau} + \rho \partial_{\rho}, \quad \bar{e}_{11'} = \frac{\sqrt{2}}{2\rho}(1+\tau)\partial_{\tau} - \rho \partial_{\rho}, \quad \bar{e}_{01'} = \frac{\sqrt{2}}{2\rho} \partial_{+}, \quad \bar{e}_{10'} = \frac{\sqrt{2}}{2\rho} \partial_{-}.
\]

In addition, one has

\[
\theta = \frac{1}{\rho}, \quad \varsigma = (1-\tau)^2.
\]

Then, from a direct comparison of equation (4) and (45) one concludes that

\[
g' = g_C.
\]

Moreover, using equations (46) and (52) the NP frame \( \{e'_{AA'}\} \) in the F-coordinates reads

\[
e'_{00'} = \frac{\sqrt{2}}{2} \frac{1}{\rho(1-\tau)^2} (1-\tau)\partial_{\tau} + \rho \partial_{\rho}, \quad e'_{11'} = \frac{\sqrt{2}}{2} \frac{1}{\rho(1+\tau)^2} (1+\tau)\partial_{\tau} - \rho \partial_{\rho},
\]

\[
e'_{01'} = \frac{\sqrt{2}}{2} \partial_{+}, \quad e'_{10'} = \frac{\sqrt{2}}{2} \partial_{-}.
\]

Comparing the last expressions for \( e'_{AA'} \) and \( e_{AA'} \) as given in equations (5a)-(5b) one concludes that

\[
\Lambda^0 0 = \frac{1}{\rho^{3/2}(1-\tau)}, \quad \Lambda^1 1 = \rho^{3/2}(1-\tau), \quad \kappa = 1.
\]

6 The electromagnetic NP constants

Consider the Minkowski spacetime \( (\bar{M}, \bar{\eta}) \) described through the physical coordinates \( (\bar{u}, \bar{\rho}) \) as defined in Section 2.1 and 5.2. In these coordinates one has

\[
\bar{\eta} = d\bar{u} \otimes d\bar{u} + d\bar{\rho} \otimes d\bar{\rho} + d\bar{\rho} \otimes d\bar{\rho} - \bar{\rho}^2 \sigma.
\]

From equations (1) and (45) one has that \( g' = \theta^2 \bar{\zeta}^2 \bar{\eta} \) and using equations (2), (3) and (52) one concludes that

\[
g' = \frac{1}{\bar{\rho}^2} \bar{\eta}.
\]

Let \( e'_A \), with \( e'_0 = o' A \) and \( e'_1 = \iota' A \), denote a spin dyad so that \( e'_{AA'} = e'_A e'_{A'} \) constitutes the NP-frame given in equation (50). Let \( \{\bar{o}^A, \bar{\iota}^A\} \) denote a spin dyad denoted by \( \bar{e}_A \) and defined via

\[
\theta^A = \rho^A, \quad \iota^A = \bar{i}^A.
\]

Notice that, by virtue of equation (55), the spin dyad \( \bar{e}_A \) is normalised respect to \( \bar{\eta} \). To introduce the electromagnetic NP constants as defined in [17] consider the physical Maxwell spinor \( \bar{\phi}_{AB} \) satisfying

\[
\nabla_{AA'} \bar{\phi}_{AB} = 0,
\]

where \( \nabla_{AA'} \) denotes the Levi-Civita connection respect to \( \bar{\eta} \). The components the physical Maxwell spinor respect to the spin dyad \( \bar{e}_A \) will be denoted, as usual, by \( \bar{\phi}_0 \equiv \bar{\phi}_{AB} \bar{\sigma}^A \bar{\sigma}^B \), \( \bar{\phi}_1 \equiv \bar{\phi}_{AB} \bar{\sigma}^A \bar{\tau}^B \), \( \bar{\phi}_2 \equiv \bar{\phi}_{AB} \bar{\tau}^A \bar{\tau}^B \).
Assumption 3. Following [17], the $\tilde{\phi}_0$ component is assumed to have an expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N} \tilde{\phi}_n^0 + O\left(\frac{1}{\tilde{\rho}^{3+N}}\right), \quad (57)$$

where the coefficients $\tilde{\phi}_n^0$ do not depend on $\tilde{\rho}$.

The electromagnetic NP constants are defined through the following integrals over cuts $C$ of null infinity:

$$F_{m,n}^k = \int_C \bar{Y}_{1;n+1,m} \tilde{\phi}_n^{n+1} dS,$$

where $n, m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n + 1$ and $dS$ denotes the area element respect to $\sigma$. In flat space, $F_{m,n}$ are absolutely conserved in the sense that their value is independent of the cut $C$ on which they are evaluated —see [17]. From these, only those given by $n = 0$ and $m = 0, 1$ are conserved in the general non-linear Einstein Maxwell theory —see [17].

6.1 Translation to the F-gauge

In view of equation (55), one has that, as a consequence of the standard conformal transformation law for the spin-1 equation —see [23], the spinor $\phi_{AB}'$, satisfying

$$\nabla_{A'} A' \phi_{AB}' = 0,$$

where $\nabla_{A'A'}$ is the Levi-Civita connection of $g'$, is related to $\tilde{\phi}_{AB}$ via

$$\phi_{AB}' = \tilde{\rho} \tilde{\phi}_{AB}. \quad (58)$$

Therefore, using equations (57), (56) and (58), one obtains

$$\phi_0 = \sum_{n=0}^{N} \tilde{\phi}_n^0 + O\left(\frac{1}{\tilde{\rho}^{N}}\right),$$

where $\phi_0^0 \equiv \phi_{AB}' d^A d^B$. From equation (50), one has that $e_{00}' = \sqrt{2} \tilde{\rho}^2 \partial_\rho$ and consequently

$$\frac{1}{\sqrt{2}} e_{00}'(\phi_0^0) = -\tilde{\rho}^1 + O(\tilde{\rho}^{-1}).$$

The repeated application of $e_{00}'$ to the above relation shows that in general

$$\frac{1}{2^{q/2}} e_{00}^{(q)}(\phi_0^0) = (-1)^q q! \tilde{\rho}_0^q + \sum_{i=q+1}^{N} (-1)^q \frac{(i+1)!}{(i-q+1)!} \tilde{\rho}_0^{i-q} + O\left(\frac{1}{\tilde{\rho}^{N-q}}\right),$$

where $e_{00}^{(q)}(\phi_0^0)$ denotes $q$ consecutive applications of $e_{00}'$ to $\phi_0$. Thus, the quantities $F_{m,n}$ can be written as

$$F_{m,n}^k = \int_C \bar{Y}_{1;n+1,m} e_{00}^{(n+1)}(\phi_0^0) dS. \quad (59)$$

Observe that the constants $F_{m,n}$ in the previous equation are expressed in terms of $g'$-associated quantities. In order to obtain a general expression for the electromagnetic NP quantities in the F-gauge one has to rewrite expression (59) in terms of $g_C$-related quantities. As discussed before, the frames $e_{AA'}$ and $e_{AA'}'$ are related through a conformal rescaling and a Lorentz transformation as given in equation (51). For the sake of generality, the first part of the discussion will be carried out for general $\kappa$ and $\Lambda_{AB}$. 

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6.1.1 Explicit computation of the first three constants

Let $\epsilon^{A}_A$, with $\omega^A = \omega^A$ and $\epsilon^A_1 = \epsilon^A$, denote a spin dyad normalised respect to $g_C$ as defined in Section 3. As a consequence of equation (51), the spin dyads $\epsilon^{A}_A$ and $\epsilon'^{A}_A$, giving rise to $e_{AA'}$ and $e'_{AA'}$, are related via

$$\epsilon'^{A}_A = \kappa^{-1/2} \Lambda^B A^B \epsilon^{A}_A.$$  \hspace{1cm} (60)

Additionally, the spinor field $\phi_{AB}$, satisfying

$$\nabla_{A'} \phi_{AB} = 0,$$

where $\nabla_{AA'}$ is the Levi-Civita connection respect to $g_C$, is related to $\phi'_{AB}$ via

$$\phi'_{AB} = \kappa^{-1} \phi_{AB}.$$  

Therefore, one has that

$$\phi'_{00} = \kappa^{-2} \Lambda^C \Lambda^D_0 \phi_{CD},$$

where $\phi_{CD} \equiv \epsilon^C e^D \phi_{CD}$. Using the Leibniz rule one obtains

$$e'_{00}(\phi') = \kappa^{-2} \left( \Lambda^C \Lambda^D_0 e_{00'}(\phi_{CD}) + 2 \phi_{CD} \Lambda^C \Lambda^D_0 e_{00'}(\Lambda^D_0) - 2 \kappa^{-1} \Lambda^C \Lambda^D_0 \phi_{CD} e_{00'}(\kappa) \right).$$  \hspace{1cm} (61)

Notice that, in the above expression, all the quantities except for the frame derivative $e'_{00}$ are $g_C$ related quantities, namely, given in the F-gauge and the F-coordinates. Using equation (51) one can expand expression (61). This leads to the following expression for the conserved quantities:

$$F^0_m = -\frac{1}{\sqrt{2}} \int_C Y_{1,1,m} \kappa^{-3} \left( \Lambda^C \Lambda^D_0 \Lambda^B_0 \Lambda^B \phi_{BB'} e_{00'}(\phi_{CD}) \right.$$  

$$+ 2 \kappa \phi_{CD} \Lambda^C \Lambda^D_0 e_{00'}(\Lambda^D_0) - 2 \kappa^{-1} \Lambda^C \Lambda^D_0 \phi_{CD} e_{00'}(\kappa) \right) dS.$$  \hspace{1cm} (62)

for $m = -1, 0, 1$. These correspond to the three electromagnetic NP quantities that remain conserved in the non-linear Einstein Maxwell theory. The last expression represent the electromagnetic counterpart of the gravitational NP quantities in the F-gauge as reported in [11] in equation (III.5). The last expression is general can be used, in principle, to find the electromagnetic NP constants in the F-gauge in the non-linear case. Nevertheless, particularising the discussion to the case analysed in this article simplifies the expressions considerably. To verify this, observe that, using the results of Section 5.3, equation (61) reduces to

$$e_{00'}(\phi') = (\Lambda^0_0)^{4}(e_{00'}(\phi_0)) + 2 \phi(\Lambda^0_0) e_{00'}(\Lambda^0_0).$$  \hspace{1cm} (63)

Using equations (5a) and (54) one observes that

$$e'_{00}(\Lambda^0_0) = \frac{\sqrt{2}}{4} \left( \Lambda^0_0 \right)^3,$$  \hspace{1cm} (64)

and more generally

$$e'_{00}(\Lambda^0_0) = \left( \frac{\sqrt{2}}{4} \right)^n (2n - 1)!!(\Lambda^0_0)^{2n+1}.$$  \hspace{1cm} (65)

Using equation (64) one gets

$$e_{00'}(\phi') = (\Lambda^0_0)^{4}(e_{00'}(\phi_0) + \frac{\sqrt{2}}{2} \phi_0).$$  \hspace{1cm} (66)
In order to write explicitly the first term of the last expression one uses equation (5a) and obtains

\[ e_{00}'(\phi_0) = \frac{1}{\sqrt{2}} \left( (1 - \tau) \partial_\tau \phi_0 + \rho \partial_\rho \phi_0 \right). \]  

(67)

Substituting equations (54) and (67) into equation (66) renders

\[ e_{00}'(\phi_0') = \frac{1}{\sqrt{2}} \rho^{-2} (1 - \tau)^{-4} \left( (1 - \tau) \partial_\tau \phi_0 + \rho \partial_\rho \phi_0 + \phi_0 \right). \]

Using the last expression, the quantities \( F_m^0 \) as determined in equation (59) are rewritten as

\[ F_m^0 = \lim_{\tau \to 1} \left( - \frac{1}{2} \int_{S^2} Y_{1,1,m} \rho^{-2} (1 - \tau)^{-4} \left( (1 - \tau) \partial_\tau \phi_0 + \rho \partial_\rho \phi_0 + \phi_0 \right) dS \right). \]  

(68)

Substituting the expansion (9) for \( \phi_0 \) into equation (68) and using the orthogonality relation

\[ \int_{S^2} Y_{s',\ell',m'} \bar{Y}_{s,\ell,m} = \delta_{\ell,\ell'} \delta_{m,m'}, \]  

(69)

one obtains

\[ F_m^0 = \lim_{\tau \to 1} \left( - \frac{1}{2 \times 2! (1 - \tau)^{-4}} \left( (1 - \tau) a_{0,2,2,m} + 3a_{0,2,2,m} \right) \right). \]  

(70)

Using the solution for \( a_{0,p,\ell,m} \) as given in equation (19a) and taking into account the discussion of the initial data of Section 3.3 showing that \( C^{(\sigma)}_{p,m} = 0 \), one gets

\[ F_m^0 = - \frac{1}{2 \times 2! \times 16} C_{2,m}. \]  

(71)

Remark 13. As discussed in Section 3.3 the condition \( C^{(\sigma)}_{p,m} = 0 \) with \( p \geq 2, -p \leq m \leq p \), does not represent a restriction on the class of initial data but arises as a necessary condition ensuring that the solutions (19a)-(19b) to the Jacobi equation correspond to a solution to the original equations (17a)-(17b). In the spin-2 case the analogous condition, in contrast, do represent a restriction on the class of initial data.

Proceeding in an analogous way, one can compute the next set of constants in the hierarchy, i.e., \( F_m^1 \). A direct computation using equations (66) and (64) renders

\[ e_{00}^{(2)}(\phi_0) = \left( \Lambda_0^0 \right)^6 \left( e_{00}'(\phi_0) + \frac{3\sqrt{2}}{2} e_{00}'(\phi_0) + \phi_0 \right). \]  

(72)

Using expression (5a) one has

\[ e_{00}^{(2)}(\phi_0) = \frac{1}{2} \left( (1 - \tau)^2 \partial_\tau^2 \phi_0 + 2\rho(1 - \tau) \partial_\tau \partial_\rho \phi_0 + \rho^2 \partial_\rho^2 \phi_0 - (1 - \tau) \partial_\tau \phi_0 + \rho \partial_\rho \phi_0 \right). \]

(73)

Substituting equations, (54), (67) and (73) into (72) renders

\[ e_{00}^{(2)}(\phi_0') = \frac{1}{2} \rho^{-3} (1 - \tau)^{-6} \left( (1 - \tau)^2 \partial_\tau^2 \phi_0 + 2\rho(1 - \tau) \partial_\tau \partial_\rho \phi_0 + \rho^2 \partial_\rho^2 \phi_0 \right) \]

\[ + 2(1 - \tau) \partial_\tau \phi_0 + 4\rho \partial_\rho \phi_0 + 2\phi_0. \]

Using the last expression the integral of equation (59) reads
\[ F^1_m = \lim_{\rho \to 0} \frac{1}{8} \int_{S^2} \tilde{Y}_{1:2,m} \rho^{-3} (1 - \tau)^{-6} \left( (1 - \tau)^2 \partial_\tau^2 \phi_0 + 2 \rho (1 - \tau) \partial_\rho \partial_\tau \phi_0 + \rho^2 \partial_\rho^2 \phi_0 + 2(1 - \tau) \partial_\tau \phi_0 + 4 \rho \partial_\rho \phi_0 + 2 \phi_0 \right) dS. \] (74)

Exploiting the orthogonality conditions (69) one gets
\[ F^1_m = \lim_{\tau \to 1} \left( \frac{1}{8 \times 3!} (1 - \tau)^{-6} \left( (1 - \tau)^2 \tilde{a}_{0,3:3,m} + 6(1 - \tau) \tilde{a}_{0,3:3,m} + 20a_{0,3:3,m} \right) \right). \] (75)

Consequently, using equation (19a) with \( C_{p,m}^\odot = 0 \) —see Remark 13, one obtains
\[ F^1_m = \frac{1}{8 \times 3! \times 32} C_{4,m}. \]

It is instructive to find explicitly one order more in this hierarchy — namely \( F^2_m \). A computation using equations (72) and (64) renders
\[ e^{(3)}_{00'}(\phi_0) = (\Lambda_0^0) \left( e^{(3)}_{00'}(\phi_0) + 3\sqrt{2} e^{(2)}_{00'}(\phi_0) + \frac{11}{2} e_{00'}(\phi_0) + \frac{3\sqrt{2}}{2} \phi_0 \right). \] (76)

Applying \( e_{00'} \) to equation (73) one obtains
\[ e^{(3)}_{00'}(\phi_0) = \frac{1}{2 \sqrt{2}} \left( (1 - \tau)^3 \partial_\tau^2 \phi_0 + 3 \rho^2 \partial_\rho^2 \phi_0 + 3 \rho^2 (1 - \tau)^2 \partial_\rho \partial_\tau \phi_0 + 3 \rho^2 (1 - \tau)^2 \partial_\rho^2 \phi_0 \right. \]
\[ \left. - 3(1 - \tau)^2 \partial_\tau^2 \phi_0 + 3 \rho^2 \partial_\rho^2 \phi_0 + (1 - \tau) \partial_\tau \phi_0 + \rho \partial_\rho \phi_0 \right). \] (77)

Using the last expression along with equations (54), (67), (73) one gets
\[ e^{(3)}_{00'}(\phi_0) = \frac{1}{2 \sqrt{2}} \rho^{-4} (1 - \tau)^{-8} \left( (1 - \tau)^3 \partial_\tau^2 \phi_0 + \rho^2 \partial_\rho^3 \phi_0 \right. \]
\[ + 3 \rho^2 (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 3 \rho^2 (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 9 \rho^2 \partial_\rho^2 \phi_0 + 3 (1 - \tau)^2 \phi_0 \]
\[ + 12 \rho (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 18 \rho \partial_\rho \phi_0 + 6 (1 - \tau) \partial_\tau \phi_0 + 6 \phi_0 \right). \]

Consequently, the quantities \( F^2_m \) as given in equation (59) read
\[ F^2_m = \lim_{\rho \to 0} \frac{1}{48} \int_{S^2} \tilde{Y}_{1:3,m} \rho^{-1} (1 - \tau)^{-8} \left( (1 - \tau)^3 \partial_\tau^2 \phi_0 + \rho^2 \partial_\rho^3 \phi_0 \right. \]
\[ + 3 \rho^2 (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 3 \rho^2 (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 9 \rho^2 \partial_\rho^2 \phi_0 + 3 (1 - \tau)^2 \phi_0 \]
\[ + 12 \rho (1 - \tau) \partial_\rho \partial_\tau \phi_0 + 18 \rho \partial_\rho \phi_0 + 6 (1 - \tau) \partial_\tau \phi_0 + 6 \phi_0 \right) dS. \] (78)

Exploiting the orthogonality condition (69) the last expression simplifies to
\[ F^2_m = \lim_{\tau \to 1} \left( \frac{1}{48 \times 4!} (1 - \tau)^{-8} \left( (1 - \tau)^3 \tilde{a}_{0,4:4,m} + 15 (1 - \tau)^2 \tilde{a}_{0,4:4,m} + 90 (1 - \tau) \tilde{a}_{0,4:4,m} + 210a_{0,4:4,m} \right) \right). \] (79)

Finally, using equation (19a) with \( C_{p,m}^\odot = 0 \) —see Remark 13, one obtains
\[ F^2_m = \frac{3}{48 \times 4! \times 128} C_{4,m}. \]
### 6.1.2 The general case

The previous discussion suggests that, in principle, it should be possible to obtain a general formula for \( F_{mn} \). Revisiting the calculation of \( F_{00}^n \), \( F_{11}^n \) and \( F_{22}^n \), one can obtain the following results concerning the overall structure of the electromagnetic NP constants in flat space:

**Lemma 1.** For any integer \( n \geq 1 \)

\[
\epsilon_{00}^{(n)}(\phi_0) = \left( \Lambda_0^0 \right)^{2(n+1)} \sum_{i=1}^{n} A_i \epsilon_{00}^{(i)}(\phi_0)
\]

for some coefficients \( A_i \) independent of \( \rho \) and \( \tau \).

**Proof.** To prove this result one proceeds by induction. Equations (66), (72) and (76) already then, applying \( e_0 \) as the basis of induction. Assume that

\[
\epsilon_{00}^{(n)}(\phi_0) = \left( \Lambda_0^0 \right)^{2(n+1)} \sum_{i=1}^{n} A_i \epsilon_{00}^{(i)}(\phi_0),
\]

then, applying \( e_{00}^{'} \) to the last expression one has

\[
\epsilon_{00}^{(n+1)}(\phi_0) = \left( \Lambda_0^0 \right)^{2(n+2)} \sum_{i=1}^{n} A_i \epsilon_{00}^{(i+1)}(\phi_0) + 2(n+1) \left( \Lambda_0^0 \right)^{2n+1} \left( \epsilon_{00}^{(0)} \Lambda_0^0 \right) \sum_{i=1}^{n} A_i \epsilon_{00}^{(i)}(\phi_0).
\]

Using equations (51), (54) and (64) one obtains

\[
\epsilon_{00}^{(n+1)}(\phi_0) = \left( \Lambda_0^0 \right)^{2(n+2)} \sum_{i=1}^{n+1} A_i \epsilon_{00}^{(i)}(\phi_0) + \frac{\sqrt{2}}{2} (n+1) \left( \Lambda_0^0 \right)^{2(n+2)} \sum_{i=1}^{n} A_i \epsilon_{00}^{(i)}(\phi_0).
\]

One can rearrange the last expression into

\[
\epsilon_{00}^{(n+1)}(\phi_0) = \left( \Lambda_0^0 \right)^{2(n+2)} \sum_{i=1}^{n+1} \tilde{A}_i \epsilon_{00}^{(i)}(\phi_0),
\]

where \( \tilde{A}_1 = A_1 \) and \( \tilde{A}_i = \frac{\sqrt{2}}{2} (n+1) A_i + A_{i-1} \) for \( i \geq 2 \). \( \square \)

**Lemma 2.** For any integer \( n \geq 1 \)

\[
\epsilon_{00}^{(n)}(\phi_0) = \sum_{i+j=k}^{i+j=k} B_{ij} \rho^i (1-\tau)^j \partial_\rho^{(i)} \partial_\tau^{(j)} \phi_0,
\]

for some coefficients \( B_{ij} \) independent of \( \rho \) and \( \tau \).

**Proof.** As in the proof of Lemma 1, one argues inductively. Equations (67), (73) and (77) serve as the basis of induction. Assume that

\[
\epsilon_{00}^{(n)}(\phi_0) = \sum_{i+j=k}^{i+j=k} B_{ij} \rho^i (1-\tau)^j \partial_\rho^{(i)} \partial_\tau^{(j)} \phi_0.
\]

then, applying \( e_{00}^{'} \) to the last expression renders

\[
\epsilon_{00}^{(n+1)}(\phi_0) = \sum_{i+j=k}^{i+j=k} B_{ij} \left( \rho^i (1-\tau)^j \epsilon_{00}^{'}(\partial_\rho^{(i)} \partial_\tau^{(j)} \phi_0) + (\partial_\rho^{(i)} \partial_\tau^{(j)} \phi_0) \epsilon_{00}^{'}(\rho^i (1-\tau)^j) \right).
\]
Using that
\[ e_{00}^r(\rho^i(1-\tau)^j) = \frac{1}{\sqrt{2}}(i+j)\rho^i(1-\tau)^j \]
and
\[ e_{00}^r(\partial^{(i)}_\rho \partial^{(j)}_\tau \phi_0) = \frac{1}{\sqrt{2}} \left( (1-\tau)\partial^{(i)}_\rho \partial^{(j+1)}_\tau \phi_0 + \rho \partial^{(j+1)}_\rho \partial^{(i)}_\tau \phi_0 \right), \]
on one obtains
\[ e_{00}^{(n+1)}(\phi_0) = \sum_{i+j=k}^{k=n} \frac{1}{\sqrt{2}} B_{ij} \left( \rho^i(1-\tau)^{j+1} \partial^{(i)}_\rho \partial^{(j+1)}_\tau \phi_0 + \rho^{j+1}(1-\tau)^j \partial^{(j+1)}_\rho \partial^{(i)}_\tau \phi_0 \right) + (i+j)\rho^i(1-\tau)^j \partial^{(i)}_\rho \partial^{(j)}_\tau \phi_0). \]
The last expression can be rearranged as
\[ e_{00}^{(n+1)}(\phi_0) = \sum_{i+j=k}^{k=n+1} B_{ij} \rho^i(1-\tau)^j \partial^{(i)}_\rho \partial^{(j)}_\tau \phi_0, \]
for some coefficients \( B_{ij} \) which depend only on \( B_{ij} \), \( i \) and \( j \).

**Proposition 5.** If the electromagnetic constants \( F^m_{n+} \) at \( \mathcal{S}^+ \) are finite, then \( F^m_{n+} \) depends only on the initial datum \( a_{0;n+2,n+2,m}(0) \) —that is, one has
\[ F^m_{n+} = Q^+(m,n)C_{n+2,m}, \]
where \( Q^+(m,n) \) is a numerical coefficient.

**Proof.** Using equation (54) and the results from Lemmas 1 and 2 one has that
\[ e_{00}^{(n)}(\phi_0') = \rho^{-(n+1)}(1-\tau)^{-2(n+1)} \sum_{q=1}^{n} \sum_{i+j=k}^{k=q} E_{ij} \rho^i(1-\tau)^j \partial^{(i)}_\rho \partial^{(j)}_\tau \phi_0, \]
for some coefficients \( E_{ij} \) independent of \( \rho \) and \( \tau \). Using the expansion for \( \phi_0 \) given in equation (9) one has
\[ e_{00}^{(n)}(\phi_0') = \rho^{-(n+1)}(1-\tau)^{-2(n+1)} \sum_{q=1}^{n} \sum_{i+j=k}^{k=q} \sum_{p=1}^{\infty} \sum_{p=1}^{\infty} \sum_{m'=1}^{m'} \sum_{m'=1}^{m'} \left\{ \frac{1}{p!} E_{ij} \rho^i(1-\tau)^j Y_{1;1',1',m'}, \partial^{(i)}_\rho \partial^{(j)}_\tau (\rho^p a_{0,p;1',1',m'}(\tau)) \right\}. \]
Noticing that
\[ \partial^{(i)}_\rho \partial^{(j)}_\tau (\rho^p a_{0,p;1',1',m'}(\tau)) = \partial^{(i)}_\rho (\rho^p) \partial^{(j)}_\tau (a_{0,p;1',1',m'}(\tau)), \]
and using that
\[ \partial^{(i)}_\rho \rho^p = \frac{(p+1)!}{(p-i+1)!} \rho^{p-i}, \]
one finds
\[ \epsilon_{00}^{(n)}(\phi_0') = (1 - \tau)^{-(n+1)} \sum_{q=1}^{n} \sum_{i+j=q}^{k=q} \sum_{p=1}^{\infty} \sum_{\ell'=-\ell=1}^{\infty} \left\{ E_{ijp} \rho^{-(n+1)} (1 - \tau)^{j} \right\} \]

where \( E_{ijp} = E_{ij}(p+1)/(p-i+1)! \). Notice that, the terms with \( p < n + 1 \) diverge when \( \rho \to 0 \) while the terms with \( p > n + 1 \) vanish when \( \rho \to 0 \). Integrating the last expression with \( Y_{1;n,m} \) and using the the orthogonality condition (69) one obtains

\[ \int_{S^2} \bar{Y}_{1;n,m} \epsilon_{00}^{(n)}(\phi_0') dS = (1 - \tau)^{-(n+1)} \sum_{q=1}^{n} \sum_{i+j=q}^{k=q} \sum_{p=1}^{\infty} \sum_{\ell'=-\ell=1}^{\infty} \left\{ E_{ijp} \rho^{-(n+1)} (1 - \tau)^{j} \right\} \delta_{\ell' - 1,n} \delta_{m',m} \theta_{\tau}^{(j)}(a_0,p,\ell',m'(\tau)) \]

Noticing that only the terms with \( \ell' = n + 1 \) and \( m = m' \) contribute to the sum and recalling that \( \ell' \leq p \) one realises that all the potentially diverging terms with \( p < n + 1 \) vanish. Taking this into account this observation one concludes that

\[ \lim_{\rho \to 0} \int_{S^2} \bar{Y}_{1;n,m} \epsilon_{00}^{(n)}(\phi_0') dS = (1 - \tau)^{-(n+1)} \sum_{q=1}^{n} \sum_{i+j=q}^{k=q} E_{ijn+1}(1 - \tau)^{j} \theta_{\tau}^{(j)}(a_0,n+1;n+1,m(\tau)). \]

Taking into account the expression for the electromagnetic NP quantities \( F_m^n \) in the F-gauge as given in equation (59), consistently with this definition, one replaces \( n \) with \( n + 1 \) to obtain

\[ F_m^n = \lim_{\tau \to 1} \left[ (1 - \tau)^{-(n+2)} \sum_{q=1}^{n+1} \sum_{i+j=q}^{k=q} E_{ijn+2}(1 - \tau)^{j} \theta_{\tau}^{(j)}(a_0,n+2;n+2,m(\tau)) \right]. \quad (82) \]

Therefore, if \( F_m^n \) is finite then it can only depend on the initial datum \( a_{0;n+2,n+2,m}(0) \). Moreover, since \( C_{n+2,m}^2 = 0 \). One concludes that

\[ F_m^n = Q^+(m,n)C_{n+2,m}, \]

where \( Q^+(m,n) \) is a numerical coefficient. In the last line the label + has been added to remind that the quantities correspond to the NP constants at \( \mathcal{S}^+ \).

**Remark 14.** Notice that to show that \( F_m^n \) is always finite then one would need to analyse the limit given in equation (82). This, however, requires a detailed analysis of the coefficients \( E_{ijp} \) which in addition would determine explicitly the numerical coefficient \( Q^+(m,n) \). The latter requires a lengthy computation which will not be pursued here.

### 6.2 The constants at \( \mathcal{S}^- \)

The analysis carried out in Sections 5 and 6 for the electromagnetic constants defined at \( \mathcal{S}^+ \), can be performed in a completely analogous way for \( \mathcal{S}^- \). To do so, consider a formal replacement \( \tau \to -\tau \) and consistently \( \theta_{\tau} \to -\theta_{\tau} \). Upon this formal replacement the roles of \( \ell = e_{00} \) and \( n = e_{11} \) as defined in (5a) and \( \phi_0 \) and \( \phi_2 \) are essentially interchanged. Then, following the discussion of Sections 5.3 and 6, one obtains *mutatis mutandis* the time dual of Proposition 5:
Proposition 6. If the electromagnetic constants $F_{m}^{n} -$ at $\mathcal{I}^-$ are finite, then $F_{m}^{n} -$ depends only on the initial datum $a_{2;n+2,n+2,m}(0)$. Moreover,

$$F_{m}^{n} = Q^-(m,n)D_{n+2,m},$$

where $Q^-(m,n)$ is a numerical coefficient.

Finally, recalling the results of propositions 5 and 6 and the discussion of the initial data given in Section 3.3 one obtains the following:

Theorem 1. If the electromagnetic NP constants $F_{m}^{n} +$ and $F_{m}^{n} -$ at $\mathcal{I}^+ \text{ and } \mathcal{I}^-$, are finite, then, up to a numerical factor $Q^+(m,n)/Q^-(m,n)$, coincide.

Remark 15. Observe that the conclusion of Theorem 1, which at first instance would seem to hold only for time-symmetric data, holds for generic initial data and is a consequence of the interplay between the evolution and constraint equations as discussed in Section 3.3.

Remark 16. The computations at order $n = 0, 1, 2$ given in Section 6.1.1 suggest that in fact $Q^+(m,n) = Q^-(m,n)$. Nevertheless, explicitly determining these factors require a lengthy computation which will not be pursued here.

7 The NP constants for the massless spin-2 field

In this section an analogous analysis to that given in Section 6 is performed for the case of the spin-2 massless field. The same notation as the one introduced in Section 6 will be used. In particular, the spin dyads $\tilde{\epsilon}^{A}_{A}$, $\epsilon^{A}_{A}$ and $\epsilon A^{A}$ associated to $\tilde{n}$, $g'$ and $g_C$ will be employed. To introduce the gravitational NP constants originally introduced in [17], let $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ denote the components of the spin-2 massless field $\phi_{ABCD}$ respect to $\tilde{\eta}^{A}_{A}$. The spin-2 equation reads

$$\tilde{\nabla}_{A'}^{A}\tilde{\phi}_{ABCD} = 0. \quad (83)$$

Assumption 4. Following [17], the component $\phi_0$ is assumed to have the expansion

$$\tilde{\phi}_0 = \sum_{n=0}^{N} \tilde{\tilde{\phi}}_{0}^{n} \frac{1}{\tilde{\rho}^{2+n}} + O\left(\frac{1}{\tilde{\rho}^{2+N}}\right), \quad (84)$$

where the coefficients $\tilde{\tilde{\phi}}_0^n$ do not depend on $\tilde{\rho}$.

As already mentioned, the field $\tilde{\phi}_{ABCD}$ provides a description of the linearised gravitational field over the Minkowski spacetime. In the full non-linear theory, the linear field $\tilde{\phi}_{ABCD}$ is replaced by the Weyl spinor $\Psi_{ABCD}$ and the analogue of equation (83) encodes the second Bianchi identity in vacuum —see [17]. The spin-2 NP quantities are defined through the following integrals over cuts $C$ of null infinity:

$$G_{m}^{n} \equiv \int_{C} \tilde{Y}_{2;n+2,m} \tilde{\phi}_{0}^{n+1} dS,$$

where $n, m \in \mathbb{Z}$ with $n \geq 0$, $|m| \leq n + 2$ and $dS$ denotes the area element respect to $\sigma$. The NP constants $G_{m}^{n}$ are absolutely conserved in the sense that their value is independent on the cut $C$ on which they are evaluated.

Remark 17. In particular, the constants $G_{m}^{0}$ are also conserved in the full non-linear case of the gravitational field where $\tilde{\phi}_0$ is replaced by the component $\Psi_0$ of the Weyl spinor $\Psi_{ABCD}$ —see [17]. These are the only constants of the hierarchy which are generically inherited in the non-linear case.
7.1 Translation to the F-gauge

An expression for the gravitational NP constants in the F-gauge has been given in Section III of [11]. In order to provide a self-contained discussion and for the ease of comparison with the analysis made in Section 6 the analogue of Formula (III.5) of [11] will be derived in accordance with the notation and conventions used in this article. In view of equation (55), one has that, as a consequence of the standard conformal transformation law for the spin-2 equation —see [23], the spinor $\phi'_{ABCD}$, satisfying

$$\nabla'_{A'} A' \phi'_{ABCD} = 0,$$

where $\nabla'_{AA'}$ is the Levi-Civita connection of $g'$, is related to $\phi_{ABCD}$ via

$$\phi'_{ABCD} = \tilde{\rho} \phi_{ABCD}.$$

(85)

Therefore, using equations (84), (56) and (85), one obtains

$$\phi'_{0} = \sum_{n=0}^{N} \tilde{\phi}_{0}^{n} + O\left(\frac{1}{\tilde{\rho}^{N}}\right),$$

where $\phi'_{0} \equiv \phi'_{ABCD}e'_{0}^{A} e'_{0}^{B} e'_{0}^{C} e'_{0}^{D}$. Using the last expansion and recalling that $e'_{00} = \sqrt{2}\rho^{2} \partial_{\tilde{\rho}}$ one obtains, after consecutive applications of $e'_{00}$, the expression

$$G_{m}^{n} = - \frac{(-1)^{n+1}}{(n+1)! 2^{(n+1)/2}} \int_{\mathcal{C}} \bar{Y}_{2:n+2,m} e'^{(n+1)}_{00}(\phi'_{0}) dS.$$

(86)

To derive an expression for the spin-2 NP constants in the F-gauge one recalls the relation between the $g'$ and $g_{C}$ representations and their associated spin dyads encoded in equation (60). Once again, as a consequence of the conformal transformation laws for the spin-2 equation one has that the spinor field $\phi_{ABCD}$ related to $\phi'_{ABCD}$ through

$$\phi'_{ABCD} = \kappa^{-1} \phi_{ABCD},$$

satisfies

$$\nabla_{A'} A' \phi_{ABCD} = 0,$$

where $\nabla_{AA'}$ represents the Levi-Civita connection respect to $g_{C}$. Additionally, one has that

$$\phi'_{0} = \kappa^{-3} \Lambda_{0}^{A} \Lambda_{0}^{B} \Lambda_{0}^{C} \Lambda_{0}^{D} \phi_{ABCD},$$

where $\phi_{ABCD} \equiv \epsilon_{A}^{A'} \epsilon_{B}^{B'} \epsilon_{C}^{C'} \epsilon_{D}^{D'} \phi_{ABCD}$. 

7.1.1 Explicit computation of the first constant

Using equation (86) and the Leibniz rule one obtains the analogue of Equation (III.5) of [11] written in accordance with the notation and conventions used in this article

$$G_{m}^{n} = - \frac{1}{\sqrt{2}} \int_{\mathcal{C}} \bar{Y}_{2:2,m} \kappa^{-4} \left( \Lambda_{0}^{A} \Lambda_{0}^{B} \Lambda_{0}^{C} \Lambda_{0}^{D} \phi_{ABCD} \right)$$

$$- 3 \phi_{ABCD} e'_{00}(\kappa) + 4\kappa \Lambda_{0}^{A} \Lambda_{0}^{B} \Lambda_{0}^{C} \phi_{ABCD} e'_{00}(\Lambda_{0}^{D}) dS.$$

(87)

Particularising the discussion to the case of the Minkowski spacetime, simplifies the expressions considerably. To see this, observe that, using the results of Section 5.3 and equation (64) one has that

$$e'_{00}(\phi'_{0}) = (\Lambda_{0}^{0})^{6} (e_{00}(\phi_{0}) + \sqrt{2}\phi_{0}).$$

(88)
A direct computation using equation (5a) and (66) renders
\[ e'_{00}'(\phi'_0) = \frac{\sqrt{2}}{\rho^3(1 - \tau)^6} \left( \frac{1}{2} (1 - \tau) \partial_{\tau} \phi_0 + \frac{1}{2} \rho \partial_{\rho} \phi_0 + \phi_0 \right). \]

Using the last expression, the quantities \( G^0_m \) as determined in equation (86) are rewritten as
\[ G^0_m = \lim_{\rho \to 0 \tau \to 1} \left( -\frac{1}{2} \int_{S^2} \tilde{Y}_{2,2,m} \rho^{-3}(1 - \tau)^{-6} (1 - \tau) \partial_{\tau} \phi_0 + \rho \partial_{\rho} \phi_0 + 2 \phi_0 \right) dS. \] (89)

Substituting the expansion for \( \phi_0 \) as succinctly encoded in (28)
\[ \phi_0 = \sum_{p=2}^{\infty} \sum_{\ell=2}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} a_{0,p,\ell,m}(\tau) Y_{2,\ell-m} \rho^p, \] (90)
into equation (68) renders
\[ G^0_m = \lim_{\rho \to 0 \tau \to 1} \left( -\frac{1}{2} (1 - \tau)^{-6} \sum_{p=2}^{\infty} \sum_{\ell=2}^{p} \sum_{m=-\ell}^{\ell} \frac{1}{p!} \rho^{n-3} \int_{S^2} \tilde{Y}_{2,2,m} (1 - \tau) a_{0,p,\ell,m} + (p + 2) a_{0,p,\ell,m} \right) dS. \]

Using the orthogonality relation (69) one obtains
\[ C^0_m = \lim_{\tau \to 1} \left( -\frac{1}{2 \times 3!} (1 - \tau)^{-6} \right) \left( 1 - \tau \right) a_{0,3,3,m} + 5 a_{0,3,3,m}. \]

Remark 18. The above expression is general and makes no assumption on the form of the initial data. An explicit calculation shows, however, that the limit will diverge unless one discards the logarithmic part of the solution in (36a). This observation brings to the forefront the close relation between the regularity at the conformal boundary (and in particular at \( \ell^0 \)) and the NP constants.

The previous remark motivates the following assumption:

Assumption 5. The initial data (37) is assumed to satisfy the regularity condition
\[ C^\oplus_{p,m} = D^\oplus_{p,m} = 0 \quad \text{for} \quad p \geq 3, \quad -p \leq m \leq p. \]
so that no logarithmic singularities arise in the solutions to the Jacobi equation (34a)-(34b).

Substituting the solution for \( a_{0,p,\ell,m} \) as given in equation (36a) for \( p = \ell = 3 \), taking into account the discussion of the initial data of Section 4.3 and setting \( C^\oplus_{3,m} = 0 \), consistent with Assumption 5, one obtains
\[ G^0_m = -\frac{1}{2 \times 3! \times 64} C_{3,m}. \] (91)

7.1.2 The general case

One can obtain in similar way to compute higher constants in the hierarchy \( G^n_m \). In order to obtain a general expression for the overall structure of \( G^n_m \), one proceeds inductively —in a similar way to the discussion of the electromagnetic NP constants \( F^n_m \).

Lemma 3. For any integer \( n \geq 1 \)
\[ e^{(n)}_{00}'(\phi'_0) = \left( \Lambda^0_0 \right)^{2(n+2)} \sum_{i=1}^{n} A_i e^{(i)}_{00}'(\phi_0) \]
for some coefficients \( A_i \) independent of \( \rho \) and \( \tau \).
Proof. As before, one argues by induction. Equation (88) for the case \( n = 0 \) constitutes the basis of induction. Assuming that
\[
e^{(n)}(\phi'_0) = \left(\Lambda_0^0\right)^{2(n+2)} \sum_{i=1}^{n} A_i e^{(i)}_0(\phi_0),
\]
and applying \( e^{(n)}_0 \) one obtains
\[
e^{(n+1)}(\phi'_0) = \left(\Lambda_0^0\right)^{2(n+2)} \sum_{i=1}^{n} A_i e^{(i)}_0(\phi_0) + 2(n+2) \left(\Lambda_0^0\right)^{2n+3} \sum_{i=1}^{n} A_i e^{(i)}_0(\phi_0).
\]
Making use of equations (51), (54) and (64) one gets
\[
e^{(n+1)}(\phi'_0) = \left(\Lambda_0^0\right)^{2(n+3)} \sum_{i=1}^{n} A_i e^{(i+1)}_0(\phi_0) + \sqrt{2} (n+1) \left(\Lambda_0^0\right)^{2(n+3)} \sum_{i=1}^{n} A_i e^{(i)}_0(\phi_0).
\]
One can rearrange the last expression into
\[
e^{(n+1)}(\phi'_0) = \left(\Lambda_0^0\right)^{2(n+3)} \sum_{i=1}^{n+1} \tilde{A}_i e^{(i)}_0(\phi_0),
\]
where \( \tilde{A}_1 = A_1 \) and \( \tilde{A}_i = \sqrt{2}(n+2)A_i + A_{i-1} \) for \( i \geq 2 \).

Remark 19. Observe that the conclusion of Lemma 2 in Section 6 is valid for any scalar field \( \phi \) on \( M \). Consequently, it can be applied without further change for the spin-2 case.

Proposition 7. If the NP constants \( G^m_n \) associated to a spin-2 field on the Minkowski spacetime at \( S^+ \) are finite, then \( G^m_n \) depends only on the initial datum \( a_{0;n+3,n+3,m}(0) \) —that is, one has
\[
G^m_n = Q^+(m,n)C_{n+3,m},
\]
where \( Q^+(m,n) \) is a numerical coefficient.

Proof. Taking into account Remark 19 and equation (54) one obtains using Lemmas 2 and 3 that
\[
e^{(n)}(\phi'_0) = \rho^{-(n+2)}(1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{i+j=k}^{\infty} \sum_{p=2}^{\infty} \sum_{\ell'=-\ell}^{\ell} \frac{1}{p!} E_{ij}\rho^j(1-\tau)^j \partial_p^{(i)} \partial_{\tau}^{(j)} \phi_0,
\]
for some coefficients \( E_{ij} \) independent of \( \rho \) and \( \tau \). Using the expansion for \( \phi_0 \) given in equation (90) one has
\[
e^{(n)}(\phi'_0) = \rho^{-(n+2)}(1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{i+j=k}^{\infty} \sum_{p=2}^{\infty} \sum_{\ell'=-\ell}^{\ell} \left\{ \frac{1}{p!} E_{ij}\rho^j(1-\tau)^j Y_{2,\ell'-1,m'} \partial_p^{(i)} \partial_{\tau}^{(j)} \phi_0 \right\}.
\]
Using equations (80) and (81) one finds
\[
e^{(n)}(\phi'_0) = (1-\tau)^{-2(n+2)} \sum_{q=1}^{n} \sum_{i+j=k}^{\infty} \sum_{p=2}^{\infty} \sum_{\ell'=-\ell}^{\ell} \left\{ E_{ij}\rho^{-p-(n+2)}(1-\tau)^j \partial_p^{(i)} \partial_{\tau}^{(j)} \phi_0 \right\}.
\]

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that the quantities correspond to the NP constants at $I$ where $Q$

Integrating the last expression with $\bar{G}$ if the appearance of logarithmic singularities. In accordance with Assumption 5 one concludes that, Observe that only the terms with $\ell$ obtained

At this point, a necessary condition for the above limit to be finite is to set $G_m^n$ to $0$ to avoid the appearance of logarithmic singularities. In accordance with Assumption 5 one concludes that, if $G_m^n$ is finite then it can only depend on the initial datum $a_{0;n+3,n+3,m}(0)$. Moreover,

\[ G_m^n = Q^+(m,n)C_{n+3,m}, \]

where $Q^+(m,n)$ is a numerical coefficient. In the last line the label $+$ has been added to remind that the quantities correspond to the NP constants at $\mathcal{F}^+$. □

Remark 20. Notice that Assumption 5 is a necessary condition if the full hierarchy of constants $G_m^n$ is required. Nevertheless, if one is only interested in a finite subset of these constants, say $G_m^{n'}$ at fixed order $n'$, then the restriction imposed by Assumption 5 to the initial data can be relaxed to $C_{p,m}^\oplus = D_{p,m}^\oplus = 0$ for $p = n' + 3, -p \leq m \leq p$.  

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7.2 The constants on $\mathcal{I}^-$

The time dual result can be obtained succinctly considering a formal replacement $\tau \rightarrow -\tau$ and consistently $\partial_\tau \rightarrow -\partial_\tau$. As previously discussed, upon this formal replacement the roles of $\ell = e_{00}$ and $n = e_{11}$ as defined in (5a) and $\phi_0$ and $\phi_4$ are essentially interchanged. Finally, one obtains \textit{mutatis mutandis} that the time dual of Proposition 7

**Proposition 8.** If the NP constants $G^m_n^-$ associated to a spin-2 field on the Minkowski spacetime at $\mathcal{I}^-$ are finite, then $G^m_n^-$ depends only on the initial datum $a_{4,n+3,n+3,m}(0)$ —that is, one has

$$G^m_n^- = Q^-(m,n)D_{n+3,m},$$

where $Q^-(m,n)$ is a numerical coefficient.

**Remark 21.** A necessary condition for $G^m_n^\pm$ to be finite is that the regularity condition of Assumption 5 is satisfied —see Remark 20. Nevertheless, to show that this condition is sufficient requires a detailed analysis of the coefficients $E_{ijn+3}$ in equation (92), which in addition would determine explicitly the numerical coefficient $Q^\pm(m,n)$. The latter requires a lengthy computation which will not be pursued here.

Recalling Propositions 7 and 8 and the discussion of the initial data given in Section 4.3 one obtains the following:

**Theorem 2.** If the spin-2 NP constants $G^m_n^+$ and $G^m_n^-$ at $\mathcal{I}^+$ and $\mathcal{I}^-$ in Minkowski spacetime, are finite, then, up to a numerical factor $Q^+(m,n)/Q^-(m,n)$, coincide.

**Remark 22.** This conclusion, which at first instance would seem to hold only for time-symmetric data, is a consequence of the field equations and, as discussed in Section 4.3, holds for generic initial data satisfying the regularity condition given in Assumption 5 —see also Remark 20.

**Remark 23.** A similar symmetric behaviour has been observed in the gravitational case in [28]. In that reference the Newman-Penrose constants at future and past null infinity of the spacetime arising from Bowen-York initial data have been computed.

7.3 The time symmetric case

It is of interest to analyse the case when the initial data is time-symmetric. An analysis of a spin-2 field on Minkowski spacetime with time-symmetric initial data in the framework of the cylinder at spatial infinity was given in [27]. In this reference it is shown that for time-symmetric initial data one has, for $p \geq 3$, $-p \leq m \leq p$, that

$$a_{0,p,p,m}(0) = a_{4,p,p,m}(0). \quad (93)$$

Nevertheless, as shown in Section 4.3 if the regularity condition $C^p_{p,m} = D^p_{p,m} = 0$ holds, then necessarily $a_{0,p,p,m}(0) = a_{4,p,p,m}(0)$. Combining this observation with the condition (93) valid for time-symmetric data, one concludes, for $p \geq 3$, $-p \leq m \leq m$, that

$$a_{0,p,p,m}(0) = a_{4,p,p,m}(0) = 0. \quad (94)$$

Therefore $C_{p,m} = D_{p,m} = 0$ and using Propositions 7 and 8 one concludes that all the constants in the hierarchy $G^m_n^\pm$ vanish.

**Proposition 9.** Given time-symmetric initial data for the spin-2 field on the Minkowski spacetime, if the regularity conditions (94) hold for $p \geq 3$, then the gravitational NP constants at $\mathcal{I}^+$ and $\mathcal{I}^-$, denoted by $G^m_n^+$ and $G^m_n^-$, are finite and vanish —that is, one has that

$$G^m_n^+ = G^m_n^- = 0.$$
8 Conclusions

In this article the correspondence between initial data given on a Cauchy hypersurface $\mathcal{S}$ intersecting $i^0$ on Minkowski spacetime for the spin-1 (electromagnetic) and spin-2 fields and their associated NP constants is analysed. This analysis has been done for the full hierarchy of NP constants $F^m_n$ and $G^m_n$ in the Minkowski spacetime.

For the electromagnetic case, it was shown that, once the initial data for the Maxwell spinor is written as an expansion of the form (21), the electromagnetic NP constants $F^m_\pm$ at $\mathfrak{I}^\pm$ can be identified with the initial datum $a_{0,p;\ell,m}(0)$ with $p = \ell = n + 2$. Since $1 \leq \ell \leq p$, one concludes that $F^m_\pm$ are in correspondence with the highest harmonic but are irrespective of the initial data for the lower modes $\ell < p$. In an analogous way, one can identify the electromagnetic NP constants $F^m_-$ at $\mathfrak{I}^-$ with the initial datum $a_{2,p;\ell,m}(0)$. Notice that the only restriction imposed on the initial data is to have the appropriate decay at infinity so that the electromagnetic NP constants can be defined. Apart from this requirement, the initial data encoded in (21) is completely general. As a by-product of the analysis of Section 6 and the discussion of the field equations given in Section 3.3 one concludes that the electromagnetic NP constants at $\mathfrak{I}^+$ and $\mathfrak{I}^-$ coincide —up to an irrelevant numerical factor. In this discussion, the field equations $\mathcal{S}$ played a dual role: on the one hand they allow to conclude that for $p = \ell$ one has that $C^\oplus_{p,m} = D^\oplus_{p,m} = 0$ so that the potentially singular part of the solutions (19a) and (19b) does not contribute to the electromagnetic field. On the other hand, the field equations further imply that $a_{0,p;\ell,m}(0) = a_{2,p;\ell,m}(0) = C_{p,m} = D_{p,m}$. This last observation is the one which ultimately relates the electromagnetic NP constants at $\mathfrak{I}^+$ and $\mathfrak{I}^-$. Observe that this result is irrespective of the initial data being time symmetric or not.

An analogous analysis was performed for a spin-2 field on a Minkowski background. In contrast with the electromagnetic case, for the spin-2 field the divergent terms at $\tau = \pm 1$ in expressions (36a)-(36b) are solutions which contribute to the field. In other words, $C^\otimes_{p,m} = D^\otimes_{p,m} \neq 0$ represents, in principle, admissible initial data. Consequently, for generic initial data, logarithmic singularities at null infinity arise. In such cases the spin-2 field does not have the appropriate decay and the associated NP constants are divergent. Thus, if the initial data for the field is written as an expansion of the form (37) and satisfy the regularity condition $C^\otimes_{p,m} = D^\otimes_{p,m} = 0$, then the spin-2 NP constants $G^m_\pm$ at $\mathfrak{I}^+$ and $\mathfrak{I}^-$ can be identified with the initial data $a_{0,n+3,n+3,m}(0)$ and $a_{4,n+3,n+3,m}(0)$, respectively. Moreover, as discussed in Section 4.3, if the regularity condition is satisfied, the field equations imply that $a_{0,p,p,m}(0) = a_{4,p,p,m}(0)$. Consequently, up to a numerical constant, $G^m_+ = G^m_-$ coincide. Notice that, this result is irrespective of the initial data being time symmetric or not. Furthermore, a direct consequence of this analysis is that, for time-symmetric data satisfying the regularity condition, the spin-2 NP constants vanish.

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A The connection on $S^2$

In this section expressions for the connection coefficients—of the Levi-Civita connection—respect to a complex null frame which do not make reference to any particular coordinate system on $S^2$ are obtained. To set up the notation, it is convenient to start the discussion writing the Cartan structure equations in accordance with the conventions used in this article:

$$d\omega^a = -\gamma^a_b \wedge \omega^b,$$

(95a)
\[ d \gamma^a_b = -\gamma^a_d \wedge \gamma^d_b + \Omega^a_b. \]  
(95b)

In the last expressions the curvature 2-form \( \Omega^a_b \) and connection 1-form \( \gamma^a_b \) are defined via

\[ \Omega^a_b = \frac{1}{2} R^a_{bcd} \omega^c \wedge \omega^d, \quad \gamma^a_b = \Gamma^a_{\alpha\beta} \omega^\alpha \wedge \omega^\beta. \]  
(96)

The connection coefficients \( \Gamma^a_{\alpha\beta} \) of the Levi-Civita connection \( \nabla \) with respect to a given frame \( e_a \) are defined as \( \Gamma^a_{\alpha\beta} = \langle \omega^a, \nabla_{e_a} e_{\beta} \rangle \).

In the remaining part of this appendix the discussion is particularised to the case of \( S^2 \). Let \( \{ \partial_+, \partial_- \} \) be a real null frame on \( S^2 \) with corresponding dual covectors \( \{ \omega^+, \omega^- \} \). Namely, one considers

\[ \sigma = 2(\omega^+ \otimes \omega^- + \omega^- \otimes \omega^+), \quad \sigma^b = \frac{1}{2}(\partial_+ \otimes \partial_- + \partial_- \otimes \partial_+), \]

where \( \sigma \) and \( \sigma^b \) denote the covariant and contravariant version of the standard metric on \( S^2 \). Furthermore, one assumes that

\[ \partial_+ = \overline{\partial}_-, \]  
(97)
and consequently \( \omega^+ = \overline{\omega^-} \). To start the discussion observe that \( [\partial_+, \partial_-] \) and its complex conjugate can be expressed as a linear combination of the basis vectors \( \partial_+ \) and \( \partial_- \). A direct inspection, taking into account the condition encoded in equation (97), reveals that

\[ [\partial_+, \partial_-] = \omega \partial_+ - \overline{\omega} \partial_-, \]  
(98)

where \( \omega \) is a scalar field over \( S^2 \). Using the no-torsion condition of the Levi-Civita connection \( \nabla \) on \( S^2 \) we get from equation (98) that

\[ \nabla_+ \partial_- - \nabla_- \partial_+ = \omega \partial_+ - \overline{\omega} \partial_-, \]  
(99)

where \( \nabla_+ \) and \( \nabla_- \) denote a covariant derivative in the direction of \( \partial_+ \) and \( \partial_- \) respectively. Using equation (99) and the metricity conditions \( \nabla_+ \sigma = 0, \nabla_- \sigma = 0 \), one finds that the only non-zero connection coefficients are all encoded in the scalar field \( \omega \):

\[ \Gamma^-_+ = \Gamma^+_+ = -\Gamma^-_- = -\Gamma^+_-_ = \omega. \]

The connection can be compactly encoded in the curvature 1-form \( \gamma^a_b \) as defined in equation (96). A direct computation renders

\[ \gamma^+_+ = \gamma^-_- = \overline{\omega} \omega^+ - \overline{\omega} \omega^-, \quad \gamma^-_+ = \gamma^+_-_ = 0. \]

Using the first Cartan structure equation encoded in (95a), one obtains

\[ d\omega^+ = -\overline{\omega} \omega^+ \wedge \omega^-, \quad d\omega^- = \overline{\omega} \omega^+ \wedge \omega^-. \]  
(100)

For completeness, using the above expressions and the second Cartan structure equation encoded in (95b), one can directly compute the curvature form \( \Omega^a_b \):

\[ \Omega^+_+ = \Omega^-_- = -2(|\omega|^2 + \frac{1}{2}(\partial_+ \omega + \partial_- \overline{\omega})) \omega^+ \wedge \omega^- \].

Notice that, in order to find further information about \( \omega \) one can exploit the fact that the Riemann curvature for maximally symmetric spaces \( (N, h) \) is given by

\[ R_{abcd} = \frac{1}{2} R(h_{ab}h_{bd} - h_{ad}h_{bc}), \]

where \( R \) is the Ricci scalar of the Levi-Civita connection of the metric \( h \) on \( N \). Since the Ricci scalar for \( S^2 \) is \( R = -2 \), using equation (96) one finds that

\[ \Omega^+_+ = \Omega^-_- = 2\omega^+ \wedge \omega^- \].

Consequently, one concludes that the scalar field \( \omega \) satisfies

\[ |\omega|^2 + \frac{1}{2}(\partial_+ \omega + \partial_- \overline{\omega}) = -1. \]  
(101)
B The \( \partial \) and \( \bar{\partial} \) operators

In this appendix, the operators \( \partial_+ \) and \( \partial_- \) are written in terms of the \( \partial \) and \( \bar{\partial} \) operators of Newman and Penrose. To fix the notation and conventions, let \( \partial_P \) and \( \bar{\partial}_P \) denote the \( \partial \) and \( \bar{\partial} \) operators [20] as defined in [23]. In the language of the NP-formalism [16, 20, 23], given a null frame represented by \( \{l, n, m, \bar{m}\} \) their corresponding covariant directional derivatives are denoted by \( \{D, \Delta, \delta, \bar{\delta}\} \). The operators \( \partial_P \) and \( \bar{\partial}_P \) acting on a quantity \( \eta \) with spin weight \( s \) can be written in terms of the \( \delta \) and \( \bar{\delta} \) derivatives as —see [23],

\[
\partial_P \eta = \delta\eta + s(\alpha - \beta)\eta, \quad \bar{\partial}_P \eta = \bar{\delta}\eta - s(\alpha - \bar{\beta})\eta, \quad (102)
\]

where \( \alpha \) and \( \beta \) denote the spin coefficients as defined in the NP formalism. The action of the directional derivatives \( \delta \) and \( \bar{\delta} \) on the vectors \( m \) and \( \bar{m} \), projected into the tangent space \( T(Q) \subset T(M) \) spanned by \( m \) and \( \bar{m} \), is encoded in

\[
\delta m^a = - (\bar{\alpha} - \beta)m^a, \quad \bar{\delta} \bar{m}^a = (\bar{\alpha} - \beta)\bar{m}^a \quad \text{on} \quad Q. \quad (103)
\]

The directional derivatives \( \nabla_+ \) and \( \nabla_- \) as defined on Appendix A are related to \( \delta \) and \( \bar{\delta} \) via

\[
\delta = \frac{1}{\sqrt{2}} \nabla_+, \quad \bar{\delta} = \frac{1}{\sqrt{2}} \nabla_. \quad (104)
\]

It follows from the discussion of Appendix A and equation (103) that

\[
\bar{\alpha} - \beta = - \frac{1}{\sqrt{2}} \bar{\nabla}, \quad \text{on} \quad Q \quad (104)
\]

Using equations (103) and (104) one obtains

\[
\nabla_+ \eta = \sqrt{2} \partial_P \eta + s\bar{\omega}\eta, \quad \nabla_- \eta = \sqrt{2} \bar{\partial}_P \eta - s\bar{\omega}\eta, \quad (105)
\]

To align the discussion with the conventions of [11, 29, 27] is convenient to define \( \bar{\partial} \) and \( \bar{\bar{\partial}} \) by rescaling \( \partial_P \) and \( \bar{\partial}_P \) as

\[
\bar{\partial} \equiv - \frac{1}{\sqrt{2}} \partial_P, \quad \bar{\bar{\partial}} \equiv - \frac{1}{\sqrt{2}} \bar{\partial}_P. \quad (106)
\]

The corresponding eigenfunctions \( Y_{s;\ell m} \) of the operator \( \bar{\partial} \bar{\partial} \), defining the spin-weighted spherical harmonics, will be assumed to be rescaled in accordance with equation (106). Exploiting that \( \{Y_{s;\ell m}\} \), with \( 0 \leq |s| \leq \ell \) and \( -\ell \leq m \leq \ell \), form a complete basis for functions of spin-weight \( s \) over \( S^2 \), given a scalar field \( \xi : Q \to \mathbb{R} \), with spin-weight \( s \), one can expand \( \xi \) as

\[
\xi = \sum_{\ell=s}^{\infty} \sum_{m=-\ell}^{\ell} C_{s\ell m} Y_{s;\ell m}. \quad (107)
\]

In addition, one has that

\[
\bar{\partial}(Y_{s;\ell m}) = \sqrt{(\ell - s)(\ell + s + 1)} Y_{s+1;\ell m}, \quad (108a)
\]

\[
\bar{\bar{\partial}}(Y_{s;\ell m}) = - \sqrt{(\ell + s)(\ell - s + 1)} Y_{s-1;\ell m}. \quad (108b)
\]

Notice that equation (107) as well as equations (108a)-(108b) do not depend on the specific choice of coordinates on \( Q \).
References

[1] A. Aceña & J. A. Valiente Kroon, *Conformal extensions for stationary spacetimes*, Class. Quantum Grav. **28**, 225023 (2011).

[2] R. Beig, *Integration of Einstein's equations near spatial infinity*, Proc. Roy. Soc. Lond. A **391**, 295 (1984).

[3] R. Beig & B. G. Schmidt, *Einstein's equation near spatial infinity*, Comm. Math. Phys. **87**, 65 (1982).

[4] D. Christodoulou & S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.

[5] S. Dain & H. Friedrich, *Asymptotically flat initial data with prescribed regularity at infinity*, Comm. Math. Phys. **222**, 569 (2001).

[6] H. Friedrich, *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. Roy. Soc. Lond. A **375**, 169 (1981).

[7] H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times*, J. Geom. Phys. **17**, 125 (1995).

[8] H. Friedrich, *Gravitational fields near space-like and null infinity*, J. Geom. Phys. **24**, 83 (1998).

[9] H. Friedrich, *Conformal geodesics on vacuum spacetimes*, Comm. Math. Phys. **235**, 513 (2003).

[10] H. Friedrich, *Smoothness at null infinity and the structure of initial data*, in 50 years of the Cauchy problem in general relativity, edited by P. T. Chruściel & H. Friedrich, Birkhauser, 2004.

[11] H. Friedrich & J. Kánnár, *Bondi-type systems near space-like infinity and the calculation of the NP-constants*, J. Math. Phys. **41**, 2195 (2000).

[12] S. W. Hawking, *Black hole explosions*, Nature **248**, 30 (1974).

[13] S. W. Hawking, *Particle creation by black holes*, Comm. Math. Phys. **43**, 199 (1975).

[14] S. W. Hawking, *Breakdown of predictability in gravitational collapse*, Phys. Rev. D **14**, 2460 (1976).

[15] S. W. Hawking, M. J. Perry, & A. Strominger, *Soft Hair on Black Holes*, Phys. Rev. Lett. **116**, 231301 (2016).

[16] E. T. Newman & R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3**, 566 (1962).

[17] E. T. Newman & R. Penrose, *New Conservation Laws for Zero Rest-Mass Fields in Asymptotically Flat Space-Time*, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **305**(1481), 175 (1968).

[18] R. Penrose, *Asymptotic properties of fields and space-times*, Phys. Rev. Lett. **10**, 66 (1963).

[19] R. Penrose, *Zero rest-mass fields including gravitation: asymptotic behaviour*, Proc. Roy. Soc. Lond. A **284**, 159 (1965).
[20] R. Penrose & W. Rindler, *Spinors and space-time. Volume 1. Two-spinor calculus and relativistic fields*, Cambridge University Press, 1984.

[21] B. G. Schmidt, *The decay of the gravitational field*, Comm. Math. Phys. **78**, 447 (1981).

[22] B. G. Schmidt, *Gravitational radiation near spatial and null infinity*, Proc. Roy. Soc. Lond. A **410**, 201 (1987).

[23] J. Stewart, *Advanced general relativity*, Cambridge University Press, 1991.

[24] G. Szegö, *Orthogonal polynomials*, volume 23 of *AMS Colloq. Pub.*, AMS, 1978.

[25] K. P. Tod, *Isotropic cosmological singularities*, in *The Conformal structure of space-time. Geometry, Analysis, Numerics*, edited by J. Frauendiener & H. Friedrich, Lect. Notes. Phys. **604**, page 123, 2002.

[26] J. A. Valiente Kroon, *Conformal methods in General Relativity*, Cambridge University Press, 2016.

[27] J. A. Valiente Kroon, *Polyhomogeneous expansions close to null and spatial infinity*, in *The Conformal Structure of Spacetimes: Geometry, Numerics, Analysis*, edited by J. Frauendiener & H. Friedrich, Lecture Notes in Physics, page 135, Springer, 2002.

[28] J. A. Valiente Kroon, *Asymptotic properties of the development of conformally flat data near spatial infinity*, Class. Quantum Grav. **24**, 3037 (2007)

[29] J. A. Valiente Kroon, *The Maxwell field on the Schwarzschild spacetime: behaviour near spatial infinity*, Proc. Roy. Soc. Lond. A **463**, 2609 (2007).

[30] J. A. Valiente Kroon, *Estimates for the Maxwell field near the spatial and null infinity of the Schwarzschild spacetime*, J. Hyp. Diff. Equns. **6**, 229 (2009).