Determination of $|V_{cb}|$ and $m_b$ from Inclusive $B \rightarrow X_c\ell\nu$ and $B \rightarrow X_s\gamma$ Decays at Belle

K. Abe,9 K. Abe,49 I. Adachi,9 H. Aihara,51 D. Anipko,1 K. Aoki,25 T. Arakawa,32 K. Arinstein,1 Y. Asano,56 T. Aso,55 V. Aulchenko,1 T. Aushev,21 T. Aziz,47 S. Bahinipati,4 A. M. Bakich,46 V. Balagura,15 Y. Ban,37 S. Banerjee,47 E. Barberio,24 M. Barbero,8 A. Bay,21 I. Bedny,1 K. Belous,14 U. Bitenc,16 I. Bizjak,16 S. Blyth,27 A. Bondar,1 A. Bozek,30 M. Bračko,23,16 J. Brodzicka,9,30 T. E. Browder,8 M.-C. Chang,50 P. Chang,29 Y. Chao,29 A. Chen,27 K.-F. Chen,29 W. T. Chen,27 B. G. Cheon,3 R. Chistov,15 J. H. Choi,18 S.-K. Choi,7 Y. Choi,45 Y. K. Choi,45 A. Chuvikov,39 S. Cole,46 J. Dalseno,24 M. Danilov,15 M. Dash,57 R. Dowd,24 J. Dragic,9 A. Drutskoy,4 S. Eidelman,1 Y. Enari,25 D. Epifanov,1 S. Fratina,16 H. Fujii,9 M. Fujikawa,26 N. Gabyshev,1 A. Garmash,39 T. Gershon,9 A. Go,27 V. Gokhroo,47 P. Goldenzweig,4 B. Golob,22,16 A. Gorišek,16 M. Grosse Perdekamp,11,40 H. Haba,18 J. Haba,9 K. Harada,25 T. Harada,35 Y. Hasegawa,44 N. C. Hastings,51 K. Hayasaka,25 H. Hayashii,26 M. Hazumi,9 D. Heffernan,35 T. Higuchi,9 L. Hinz,21 T. Hokuue,25 Y. Hoshi,49 K. Hoshina,54 S. Hou,27 W.-S. Hou,29 Y. B. Hsiung,29 Y. Igarashi,9 T. Iijima,25 K. Ikado,25 A. Imoto,26 K. Inami,25 A. Ishikawa,51 H. Ishino,52 K. Itoh,51 R. Itoh,9 M. Iwasaki,6 M. Iwasaki,51 Y. Iwasaki,9 C. Jacoby,21 M. Jones,8 H. Kakuno,51 J. H. Kang,58 J. S. Kang,18 P. Kapusta,30 S. U. Kataoka,26 N. Katayama,9 H. Kawai,2 T. Kawasakai,32 H. R. Khan,52 A. Kibayashi,52 H. Kichimi,9 N. Kikuchi,50 H. J. Kim,20 H. O. Kim,45 J. H. Kim,45 S. K. Kim,43 T. H. Kim,58 Y. J. Kim,6 K. Kinoshita,4 N. Kishimoto,25 S. Korpar,23,16 Y. Kozakai,25 P. Križan,22,16 P. Krokovny,9 T. Kubota,25 R. Kulasiri,4 R. Kumar,36 C. C. Kuo,27 E. Kurihara,2 A. Kusaka,51 A. Kuzmin,1 Y.-J. Kwon,58 J. S. Lange,5 G. Leder,13 J. Lee,43 S. E. Lee,43 Y.-J. Lee,29 T. Lesiak,30 J. Li,8 A. Limosani,9 C. Y. Lin,29 S.-W. Lin,29 Y. Liu,6 D. Liventsev,15 J. MacNaughton,13 G. Majumder,47 F. Mandl,13 D. Marlow,39 T. Matsumoto,53 A. Matyja,30 S. McOnie,46 T. Medvedeva,15 Y. Mikami,50 W. Mitaroff,13 K. Miyabayashi,26 H. Miyake,25 H. Miyata,32 Y. Miyazaki,25 R. Mizuk,15 D. Mohapatra,57 G. R. Moloney,24 T. Mori,52 J. Mueller,38 A. Murakami,41 T. Nagamine,50 Y. Nagasaka,10
T. Nakagawa, Y. Nakahama, I. Nakamura, E. Nakano, M. Nakao, H. Nakazawa, Z. Natkaniec, K. Neichi, S. Nishida, K. Nishimura, O. Nitoh, S. Noguchi, M. Nakao, H. Nakazawa, Z. Natkaniec, K. Neichi, S. Nishida, K. Nishimura, O. Nitoh, S. Noguchi, T. Nozaki, A. Ogawa, S. Ogawa, T. Ohshima, T. Okabe, S. Okuno, S. L. Olsen, S. Ono, W. Ostrowicz, H. Ozaki, P. Pakhlov, G. Pakhlova, H. Palka, C. W. Park, H. Park, K. S. Park, N. Parslow, L. S. Peak, M. Pernicka, R. Pestotnik, M. Peters, L. E. Piilonen, A. Poluektov, F. J. Ronga, N. Root, J. Rorie, M. Rozanska, H. Sahoo, S. Saitoh, Y. Sakai, H. Sakamoto, H. Sakaue, T. R. Sarangi, N. Sato, N. Satoyama, K. Sayeed, T. Schietinger, O. Schneider, P. Schönmeyer, J. Schümann, C. Schwanda, A. J. Schwartz, R. Seidl, T. Seki, K. Senyo, M. E. Sevior, M. Shapkin, Y.-T. Shen, H. Shibuya, B. Shwartz, V. Sidorov, J. B. Singh, A. Sokolov, A. Somov, N. Soni, R. Stamen, S. Stanić, M. Starić, H. Stoeck, A. Sugiyama, K. Sumisawa, T. Sumiyoshi, S. Suzuki, S. Y. Suzuki, O. Tajima, N. Takada, F. Takasaki, K. Tamai, N. Tamura, K. Tanabe, M. Tanaka, G. N. Taylor, Y. Teramoto, X. C. Tian, I. Tikhomirov, K. Trabelsi, Y. T. Tsai, Y. F. Tse, T. Tsuboyama, T. Tsukamoto, K. Uchida, Y. Uchida, S. Uehara, T. Uglov, K. Ueno, Y. Unno, S. Uno, P. Urquijo, Y. Ushiroda, Y. Usov, G. Varner, K. E. Varvell, S. Villa, C. C. Wang, C. H. Wang, M.-Z. Wang, M. Watanabe, Y. Watanabe, J. Wicht, L. Widhalm, J. Wiechczynski, E. Won, C.-H. Wu, Q. L. Xie, B. D. Yabsley, A. Yamaguchi, H. Yamamoto, S. Yamamoto, Y. Yamashita, M. Yamauchi, Heyoung Yang, L. Zhang, Z. P. Zhang, V. Zhilich, T. Ziegler, A. Zupanc, and D. Zürcher

(The Belle Collaboration)

1Budker Institute of Nuclear Physics, Novosibirsk
2Chiba University, Chiba
3Chonnam National University, Kwangju
4University of Cincinnati, Cincinnati, Ohio 45221
5University of Frankfurt, Frankfurt
6The Graduate University for Advanced Studies, Hayama
7Gyeongsang National University, Chinju
8 University of Hawaii, Honolulu, Hawaii 96822
9 High Energy Accelerator Research Organization (KEK), Tsukuba
10 Hiroshima Institute of Technology, Hiroshima
11 University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
12 Institute of High Energy Physics, Chinese Academy of Sciences, Beijing
13 Institute of High Energy Physics, Vienna
14 Institute of High Energy Physics, Protvino
15 Institute for Theoretical and Experimental Physics, Moscow
16 J. Stefan Institute, Ljubljana
17 Kanagawa University, Yokohama
18 Korea University, Seoul
19 Kyoto University, Kyoto
20 Kyungpook National University, Taegu
21 Swiss Federal Institute of Technology of Lausanne, EPFL, Lausanne
22 University of Ljubljana, Ljubljana
23 University of Maribor, Maribor
24 University of Melbourne, Victoria
25 Nagoya University, Nagoya
26 Nara Women's University, Nara
27 National Central University, Chung-li
28 National United University, Miao Li
29 Department of Physics, National Taiwan University, Taipei
30 H. Niewodniczanski Institute of Nuclear Physics, Krakow
31 Nippon Dental University, Niigata
32 Niigata University, Niigata
33 University of Nova Gorica, Nova Gorica
34 Osaka City University, Osaka
35 Osaka University, Osaka
36 Panjab University, Chandigarh
37 Peking University, Beijing
38 University of Pittsburgh, Pittsburgh, Pennsylvania 15260
Abstract

We present an analysis of the Belle measured moments of the lepton energy and hadronic mass spectra in $B \to X_c \ell \nu$ decays and the photon energy spectrum in $B \to X_s \gamma$ decays using theoretical expressions derived in the 1S and kinetic schemes. The magnitude of the Cabibbo-Kobayashi-Maskawa matrix element $V_{cb}$, the $b$-quark mass and other non-perturbative parameters are extracted. In the 1S scheme analysis we find $|V_{cb}| = (41.49 \pm 0.52(\text{fit}) \pm 0.20(\tau_B)) \times 10^{-3}$ and $m_b^{1S} = (4.729 \pm 0.048)$ GeV. In the kinetic scheme, we obtain $|V_{cb}| = (41.93 \pm 0.65(\text{fit}) \pm 0.07(\alpha_s) \pm 0.63(\text{th})) \times 10^{-3}$ and $m_b^{\text{kin}} = (4.564 \pm 0.076(\text{fit}) \pm 0.003(\alpha_s))$ GeV.
I. INTRODUCTION

The most precise determinations of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{cb}|$ are obtained using combined fits to inclusive $B$ decay distributions [2, 3]. These analyses are based on calculations of the semileptonic decay rate in the frameworks of the Operator Product Expansion (OPE) [4] and the Heavy Quark Effective Theory (HQET) [2, 5] which predict this quantity in terms of $|V_{cb}|$, the $b$-quark mass $m_b$, and a few non-perturbative matrix elements that enter at the order $1/m_b^2$.

Several studies have shown that the spectator model decay rate, in which bound state effects are neglected, is the leading term in a well-defined expansion controlled by the parameter $\Lambda_{\text{QCD}}/m_b$ [5, 6, 7, 8]. Non-perturbative corrections to this leading approximation arise only to order $1/m_b^2$. The key issue in this approach is the ability to separate non-perturbative corrections (expressed as a series in powers of $1/m_b$), and perturbative corrections (expressed in powers of $\alpha_s$). There are various different methods to handle the energy scale $\mu$ used to separate long-distance from short-distance physics.

The coefficients of the $1/m_b$ power terms are expectation values of operators that include non-perturbative physics. In this framework, non-perturbative corrections are parameterised by quark masses and matrix elements of higher dimensional operators which are presently poorly known. The experimental accuracy already achieved, and that expected from larger data sets recorded by the $B$-factories, make the ensuing theory uncertainty a major limiting factor. The extraction of the non-perturbative parameters describing the heavy quark masses, kinetic energy of the $b$ quark and the $1/m_b^2$ corrections directly from the data has therefore become a key issue.

The non-calculable, non-perturbative quantities are parametrised in terms of expectation values of hadronic matrix elements, which can be related to the shape (characterised by moments) of inclusive decay spectra [2, 8, 9]. High precision comparison of theory and experiment requires a precise determination of the heavy quark masses, as well as the nonperturbative matrix elements that enter the expansion. These are $\lambda_{1,2}$ or $\mu_{x,G}$ which parameterise the nonperturbative corrections to inclusive observables at $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_b^2)$. At $\mathcal{O}(\Lambda_{\text{QCD}}^3/m_b^3)$, more matrix elements occur, denoted by $\rho_{1,2}$ and $\tau_{1-4}$ or $\rho_{D,LS}$.

In this paper we make use of the Heavy Quark Expansions (HQEs) that express the semileptonic decay width $\Gamma_{s,1}$, moments of the lepton energy and hadronic mass spectra in
$B \to X_c \ell \nu$ decays and the photon energy spectrum in $B \to X_s \gamma$ decays in terms of the running kinetic quark masses $m_b^{\text{kin}}$ and $m_c^{\text{kin}}$ as well as the 1S $b$-quark mass $m_b^{1S}$. Further details of these two schemes are discussed in later sections of the paper. These schemes should ultimately yield consistent results for $|V_{cb}|$. The precision of the $b$-quark mass is also important for the determination of $|V_{ub}|$, the least well understood element in the CKM matrix, and a limiting factor in the uncertainty on the unitarity triangle.

II. EXPERIMENTAL INPUT

Belle has measured the partial branching fractions $B(B \to X_c \ell \nu)_{E_\ell > E_{\text{min}}}$ and the first, second, third and fourth moments of the truncated electron energy spectrum in $B \to X_c e \nu$, $\langle E_\ell \rangle_{E_\ell > E_{\text{min}}}$, $\langle (E_\ell - \langle E_\ell \rangle)^2 \rangle_{E_\ell > E_{\text{min}}}$, $\langle (E_\ell - \langle E_\ell \rangle)^3 \rangle_{E_\ell > E_{\text{min}}}$ and $\langle (E_\ell - \langle E_\ell \rangle)^4 \rangle_{E_\ell > E_{\text{min}}}$, for nine different electron energy thresholds ($E_{\text{min}} = 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8$ and $2.0$ GeV) \[10\].

We have measured the first, second central and second non-central moments of the hadron invariant mass squared ($M_X^2$) spectrum in $B \to X_c \ell \nu$, $\langle M_X^2 \rangle_{E_\ell > E_{\text{min}}}$, $\langle (M_X^2 - \langle M_X^2 \rangle)^2 \rangle_{E_\ell > E_{\text{min}}}$ and $\langle M_X^4 \rangle_{E_\ell > E_{\text{min}}}$ for seven different lepton energy thresholds ($E_{\text{min}} = 0.7, 0.9, 1.1, 1.3, 1.5, 1.7$ and $1.9$ GeV) \[11\].

For $B \to X_s \gamma$ we have measured the first and second moments of the truncated photon energy spectrum, $\langle E_\gamma \rangle_{E_\gamma > E_{\text{min}}}$ and $\langle (E_\gamma - \langle E_\gamma \rangle)^2 \rangle_{E_\gamma > E_{\text{min}}}$. These measurements are available for six minimum photon energies ($E_{\text{min}} = 1.8, 1.9, 2.0, 2.1, 2.2$ and $2.3$ GeV) \[12\].

Hence, there are a total of 71 Belle measurements of inclusive spectra available for use in the fits \[13\]. The measurements used in the 1S and kinetic scheme fit analyses are shown in Table \[1\]. We have excluded measurements that do not have corresponding theoretical predictions. Measurements with higher cutoff energies (i.e. electron energy and hadron mass moments with $E_{\text{min}} > 1.5$ GeV and photon energy moments with $E_{\text{min}} > 2$ GeV) are not used to determine the HQE parameters, as theoretical predictions are not considered reliable in this region. Finally, we have also excluded points where correlations with neighbouring points are too high.

The value of $|V_{cb}|$ is dependent on the $B$ meson lifetimes. The measured semileptonic ratios can be written as $B_{s,l} = \tau_{\text{eff}} \Gamma_{s,l}$ in terms of an effective lifetime, $\tau_{\text{eff}} = f_+ \tau_+ + f_0 \tau_0$. Using the most recent world average values for the lifetimes and the $b$-hadron fractions, we obtain $\tau_{\text{eff}} = (1.585 \pm 0.006)$ ps \[14\]. For simplicity we refer to this quantity simply as $\tau_B$. 


TABLE I: Experimental input used in the 1S and kinetic scheme analyses. The values of $E_{\text{min}}$ are given in GeV. The 1S (kinetic) scheme analysis uses a total of 24 (31) measurements.

### III. 1S SCHEME ANALYSIS

#### A. Theoretical input

The inclusive spectral moments of $B \to X_c \ell \nu$ decays have been derived in the 1S scheme up to $\mathcal{O}(1/m_b^3)$ [2]. The theoretical expressions for the truncated moments have the following form (where $\langle X \rangle_{E_{\text{min}}}$ represents any of the experimental observables):

$$
\langle X \rangle_{E_{\text{min}}} = X^{(1)} + X^{(2)} \Lambda + X^{(3)} \Lambda^2 + X^{(4)} \Lambda^3 + X^{(5)} \lambda_1 + X^{(6)} \lambda \lambda_1 + X^{(7)} \lambda_2 + X^{(8)} \lambda \lambda_2 + X^{(9)} \rho_1 \\
+ X^{(10)} \rho_2 + X^{(11)} \tau_1 + X^{(12)} \tau_2 + X^{(13)} \tau_3 + X^{(14)} \tau_4 + X^{(15)} \varepsilon + X^{(16)} \varepsilon_{\text{BLM}}^2 + X^{(17)} \varepsilon \Lambda .
$$

(1)

The coefficients $X^{(k)}$, determined by theory, are functions of $E_{\text{min}}$. The non-perturbative corrections are parametrized by $\Lambda$ ($\mathcal{O}(m_b)$), $\lambda_1$ and $\lambda_2$ ($\mathcal{O}(1/m_b^2)$), and $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$, $\rho_1$ and $\rho_2$ ($\mathcal{O}(1/m_b^2)$).

Predictions for the partial branching fractions are obtained using the following expression,

$$
B(B \to X_c \ell \nu)_{E_{\text{min}}} = \langle X \rangle_{B,E_{\text{min}}} \frac{\eta_{\text{QED}} |V_{cb}|^2 G_F^2 m^5}{192 \pi^3 \tau_B} ,
$$

(2)

where $\langle X \rangle_{B,E_{\text{min}}}$ is an expression of the form of Eq. 1, $m$ is the 1S reference mass, $m = m_{\Upsilon(1S)}/2$, $\eta_{\text{QED}} = 1.007$, and $G_F^2 m^5/(192 \pi^3) = 5.4 \times 10^{-11}$. 


In this analysis, we determine a total of seven parameters: $|V_{cb}|$, $\Lambda$, $\lambda_1$, $\tau_1$, $\tau_2$, $\tau_3$ and $\rho_1$. One of the higher order parameters, $\tau_4$, is set to zero, and from available constraints, e.g. $B^*-B$ mass splitting, the remaining parameters in Eq. 1 are set to: $\lambda_2 = 0.1227 - 0.0145 \lambda_1$ and $\rho_2 = 0.1361 + \tau_2$, following advice from Ref. [2]. The parameter $\Lambda$ is the difference between the $b$-quark mass and the reference value about which it is expanded, i.e., $\Lambda = m_{\Upsilon(1S)/2} - m_b^{1S}$. We will present our results in terms of $m_b^{1S}$ in place of $\Lambda$.

B. The $\chi^2$ function

The fit takes into account both experimental and theoretical uncertainties. Following the approach in Ref. [2], an element of the combined experimental and theoretical error matrix is given by

$$\sigma_{ij}^2 = \sigma_i \sigma_j c_{ij},$$

where $i$ and $j$ denote the observables and $c_{ij}$ is the experimental correlation matrix element. The total error on the observable $i$ is defined as

$$\sigma_i = \sqrt{(\sigma_i^{\text{exp}})^2 + (Af_n m_{Bn}^2)^2 + (B_i/2)^2}$$

for the $n$th hadron moment ,

$$\sigma_i = \sqrt{(\sigma_i^{\text{exp}})^2 + (Af_n (m_B/2)^n)^2 + (B_i/2)^2}$$

for the $n$th lepton moment ,

$$\sigma_i = \sqrt{(\sigma_i^{\text{exp}})^2 + (Af_n (m_B/2)^n)^2 + (B_i/2)^2}$$

for the $n$th photon moment ,

and $f_0 = f_1 = 1$, $f_2 = 1/4$ and $f_3 = 1/(6\sqrt{3})$. Here, $\sigma_i^{\text{exp}}$ are the experimental errors, $B_i = X^{(16)}$ are the coefficients of the last computed terms in the perturbation series (providing the error on the uncalculated higher order perturbative terms). The dimensionless parameter $A$ contains various theoretical errors (uncalculated power corrections, uncalculated effects of order $(\alpha_s/4\pi)\Lambda_{\text{QCD}}^2/m_b^2$, and effects not included in the OPE, i.e., duality violation), and is multiplied by dimensionful quantities. We fix $A = 0.001$ as in Ref. [2]. For $B \rightarrow X_s\gamma$, the accessible phase space is limited, and the theoretical extraction of $m_b$ is affected by shape function effects. So, $A$ is multiplied by the ratio of the difference from the end point relative to $E_{\text{min}} = 1.8$ GeV.

As the fit does not provide strong constraints on the $O(1/m_b^2)$ parameters, it is necessary to provide constraints to ensure their convergence to sensible values. We achieve this by
introducing extra terms in the $\chi^2$ function of the fit,

$$\chi_{\text{param}}^2(m_\chi, M_\chi) = \begin{cases} 
0 & |O| \leq m_\chi^3, \\
(|\langle O \rangle| - m_\chi^3)^2 / M_\chi^6 & |O| > m_\chi^3,
\end{cases} \quad (7)$$

where $(m_\chi, M_\chi)$ are both quantities of order $\Lambda_{\text{QCD}}$, and $\langle O \rangle$ represents any $O(1/m_b^3)$ parameters. In the fit, we take $M_\chi = m_\chi = 500$ MeV after Ref. [2]. The parameter $m_\chi$ may have a value anywhere between 500 MeV and 1 GeV.

The overall form of the $\chi^2$ function used in the 1S fit is

$$\chi^2 = \sum_{i,j} (\langle X \rangle_i^{\text{meas}} - \langle X \rangle_i^{1S}) \text{cov}_{ij}^{-1} (\langle X \rangle_j^{\text{meas}} - \langle X \rangle_j^{1S}) + \sum_{i=1}^2 \chi_{\rho_i}^2 + \sum_{i=1}^3 \chi_{\tau_i}^2, \quad (8)$$

where $\langle X \rangle_i^{\text{meas}}$ are the measured moments and $\langle X \rangle_i^{1S}$ are the corresponding 1S scheme predictions.

C. Fit results and discussion

Minimizing the $\chi^2$ function in Eq. (8) using MINUIT [15], we find the following results for the fit parameters,

$$|V_{cb}| = (41.49 \pm 0.52_{\text{fit}} \pm 0.20_{\tau_B}) \times 10^{-3} ,$$

$$m_b^{1S} = (4.729 \pm 0.048) \text{ GeV} , \text{ and}$$

$$\lambda_1 = (-0.30 \pm 0.04) \text{ GeV}^2 .$$

The first error is the uncertainty from the fit including experimental and theory errors, and the second error (on $|V_{cb}|$ only) is due to the uncertainty on the average $B$ lifetime. The correlations between these fit parameters are provided in Table III. Using the measurement of the partial branching fraction at $E_{\text{min}} = 0.6$ GeV, we obtain for the semileptonic branching ratio (over the full lepton energy range),

$$B(B \to X_c \ell \nu) = (10.62 \pm 0.25)\% .$$

The measured moments compared to the 1S scheme predictions are shown in Figs. 1 and 2.

We assess the stability of the fit in two ways (Table III): by repeating the fit only to $B \to X_c \ell \nu$ data (21 measurements), (a) to (c); and by releasing the $m_\chi$ constraint on
TABLE II: Correlation coefficients of the parameters in the 1S fit.

| $|V_{cb}|$ | $m_b^{1S}$ | $\lambda_1$ |
|----------|------------|---------|
| 1.000    | -0.539     | -0.330  |
| $m_b^{1S}$ | 1.000     | 0.871   |
| $\lambda_1$ |          | 1.000   |

FIG. 1: Fit results to the electron energy spectrum moments. The yellow band represents the fit error, and the red band gives the theory and fit errors combined. Filled circles represent data points used in the fit, and open circles are points not used in the fit.
FIG. 2: Fit results to the hadron invariant mass squared and photon energy spectrum moments. The yellow band represents the fit error, and the red band gives the theory and fit errors combined. Filled circles represent data points used in the fit, and open circles are points not used in the fit.

To study the effect of the estimated theoretical uncertainties, we repeat the fit with all theoretical uncertainties set to zero; (b) and (e). All studies give consistent results with acceptable values of $\chi^2/\text{ndf}$. Figure 3 shows the contour plots for the fits corresponding to Table III(a) ($B \rightarrow X_c \ell \nu$ data only) and III(d) (full fit).
| Data used | $m_\chi$ ($\text{GeV}$) | $\sigma^2_{\text{theory}}$ | $\chi^2/\text{ndf}$ | $|V_{cb}| \times 10^3$ | $m_b^{1S}$ ($\text{GeV}$) | $\lambda_1$ ($\text{GeV}^2$) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| (a) $E_{e(0,1,2,3)}$, $M_{X(1,2)}^2$ | 0.5 | yes | 4.9/13 | 41.52 ± 0.77 | 4.723 ± 0.103 | −0.307 ± 0.083 |
| (b) $E_{e(0,1,2,3)}$, $M_{X(1,2)}^2$ | 0.5 | no | 9.4/13 | 41.64 ± 0.58 | 4.655 ± 0.075 | −0.348 ± 0.052 |
| (c) $E_{e(0,1,2,3)}$, $M_{X(1,2)}^2$ | 0.8 | yes | 1.9/13 | 42.82 ± 1.09 | 4.588 ± 0.118 | −0.505 ± 0.271 |
| (d) $E_{e(0,1,2,3), E\gamma(1,2)}$, $M_{X(1,2)}^2$ | 0.5 | yes | 5.7/17 | 41.49 ± 0.52 | 4.729 ± 0.048 | −0.302 ± 0.043 |
| (e) $E_{e(0,1,2,3), E\gamma(1,2)}$, $M_{X(1,2)}^2$ | 0.5 | no | 10.8/17 | 41.42 ± 0.45 | 4.695 ± 0.046 | −0.321 ± 0.035 |
| (f) $E_{e(0,1,2,3), E\gamma(1,2)}$, $M_{X(1,2)}^2$ | 0.8 | yes | 3.9/17 | 42.19 ± 0.73 | 4.709 ± 0.066 | −0.489 ± 0.087 |

TABLE III: Stability of the 1S fit result. $\sigma^2_{\text{theory}}$ refers to whether or not the theory error is included in the fit.

FIG. 3: Fit results for $m_b^{1S}$ and $\lambda_1$ on the left, and $m_b^{1S}$ and $|V_{cb}|$ on the right. The fit to the $B \to X_c \ell \nu$ data only ($B \to X_c \ell \nu$ and $B \to X_s \gamma$ data combined) is shown by a dashed blue line (solid red line). The regions correspond to $\Delta \chi^2 = 1$.

IV. KINETIC SCHEME ANALYSIS

A. Theoretical input

Spectral moments of $B \to X_c \ell \nu$ decays have been derived up to $\mathcal{O}(1/m_b^2)$ in the kinetic scheme [8]. Compared to the original paper, the theoretical expressions used in the fit contain an improved calculation of the perturbative corrections to the lepton energy moments [16] and account for the $E_{\text{min}}$ dependence of the perturbative corrections to the hadronic mass...
moments \[17\]. For the $B \rightarrow X_s \gamma$ moments, the (biased) OPE prediction and the bias correction have been calculated \[9\].

All these expressions depend on the $b$- and $c$-quark masses $m_b(\mu)$ and $m_c(\mu)$, the non-perturbative parameters $\mu^2_\pi(\mu)$ and $\mu^2_G(\mu)$ ($\mathcal{O}(1/m^2_b)$), $\tilde{\rho}^3_D(\mu)$ and $\tilde{\rho}^3_{LS}(\mu)$ ($\mathcal{O}(1/m^3_b)$), and $\alpha_s$ \[18\]. The theoretical uncertainties can be separated into two categories: non-perturbative (related to the expansion in $1/m_b$) and perturbative (related to the expansion in $\alpha_s$).

Following the recipe in Ref. \[8\], the non-perturbative uncertainties are evaluated by varying $\mu^2_\pi$ and $\mu^2_G$ ($\tilde{\rho}^3_D$ and $\tilde{\rho}^3_{LS}$) by $\pm 20\%$ ($\pm 30\%$) around their “nominal” values $m_b = 4.6$ GeV, $m_c = 1.18$ GeV, $\mu^2_\pi = 0.4$ GeV$^2$, $\tilde{\rho}^3_D = 0.1$ GeV$^3$, $\mu^2_G = 0.35$ GeV$^2$ and $\tilde{\rho}^3_{LS} = -0.15$ GeV$^3$, corresponding to the uncertainty of the respective Wilson coefficient. All these variations are considered uncorrelated for a given moment. The theoretical covariance matrix is then constructed by treating these errors as fully correlated for a given moment with different $E_{\text{min}}$ while they are treated as uncorrelated between moments of different order.

For the moments of the photon energy spectrum, we take 30\% of the absolute value of the bias correction as its uncertainty. This additional theoretical error is considered uncorrelated for moments with different $E_{\text{min}}$ and different order.

The theoretical uncertainties mentioned so far (non-perturbative, bias correction) are used to construct a theoretical covariance matrix for the measurements and are thus included in the fit. The perturbative uncertainties are estimated by repeating the fit, setting $\alpha_s$ to a different value. For lepton and photon energy (hadronic mass) moments, we vary $\alpha_s$ within $\pm 0.04$ ($\pm 0.1$) around the central value 0.22 (0.3). These ranges of variation follow the recommendations in Ref. \[8\]. The different treatment of the hadron mass moments is due to the fact that the calculation of the perturbative corrections to these moments is less complete.

### B. The $\chi^2$ function

We use a $\chi^2$ function with seven free parameters: the semileptonic $b \rightarrow c$ branching fraction $\mathcal{B}(B \rightarrow X_c(\ell \nu))$, $m_b$, $m_c$, $\mu^2_\pi$, $\tilde{\rho}^3_D$, $\mu^2_G$ and $\tilde{\rho}^3_{LS}$,

$$\chi^2 = \sum_{i,j} (\langle X^\text{meas}_i \rangle - \langle X^\text{kin}_i \rangle)^{\text{cov}^{-1}}_{ij} (\langle X^\text{meas}_j \rangle - \langle X^\text{kin}_j \rangle). \tag{9}$$
Here, $\langle X \rangle_i^{\text{meas}}$ are the measured moments. $\langle X \rangle_i^{\text{kin}}$ are the corresponding kinetic scheme predictions that depend on these free parameters. The covariance matrix is the sum of the experimental and theoretical error matrices.

We determine the CKM element $|V_{cb}|$ by treating it as an eighth free parameter in the fit. $|V_{cb}|$ is related to the semileptonic width $\Gamma(B \to X_c \ell \nu)$ by

$$\frac{|V_{cb}|}{0.0417} = \left( \frac{\Gamma(B \to X_c \ell \nu)}{1.55 \text{ ps}} \right)^{1/2} \times (1 - 0.0018) \times (1 + 0.30(\alpha_s - 0.22)) \times (1 - 0.66(m_b - 4.6 \text{ GeV}) + 0.39(m_c - 1.15 \text{ GeV})
\begin{align*}
&+ 0.013(\mu^2 - 0.4 \text{ GeV}^2) + 0.09(\rho^3_D - 0.1 \text{ GeV}^3) \\
&+ 0.05(\mu^2 - 0.35 \text{ GeV}^2) - 0.01(\rho^3_{LS} + 0.15 \text{ GeV}^3).\end{align*}
$$

Using this expression, we calculate $\Gamma(B \to X_c \ell \nu)$ from $|V_{cb}|$ and add the following term to the $\chi^2$ function,

$$\chi''^2 = \chi^2 + \left( \frac{B_{X_c \ell \nu}}{\Gamma(B \to X_c \ell \nu) - \tau_B} \right)^2 / \sigma_{\tau_B}^2. \quad (11)$$

As $\mu^2_G$ and $\rho^3_{LS}$ are determined from $B^* - B$ mass splitting and heavy quark sum rules and because the expressions depend only weakly on these parameters, we fix $\mu^2_G$ and $\rho^3_{LS}$ to $0.35 \pm 0.07 \text{ GeV}^2$ and $-0.15 \pm 0.1 \text{ GeV}^3$, respectively, by adding the following terms to the $\chi^2$ function,

$$\chi''''^2 = \chi''^2 + (\mu^2_G - 0.35 \text{ GeV}^2)^2/(0.07 \text{ GeV}^2)^2 + (\rho^3_{LS} + 0.15 \text{ GeV}^3)^2/(0.1 \text{ GeV}^3)^2. \quad (12)$$

The minimization of the $\chi''''^2$ is performed using MINUIT [15].

C. Fit results and discussion

The result of the kinetic scheme analysis is shown in Table [V] and in Figs. [4] and [5]. The value of the $\chi^2$ function at the minimum is 17.76, compared to $(31 - 7)$ degrees of freedom. All results are preliminary.

To assess the stability of the fit, we have repeated the analysis using lepton energy moments only, hadron mass moments only and photon energy moments only. The result is shown in Fig. [6]. In general, changes are well covered by the fit uncertainty though the $B \to X_s \gamma$ data seems to prefer lower values of $m_b$.

Finally, the result for $|V_{cb}|$ reads

$$|V_{cb}| = (41.93 \pm 0.65(\text{fit}) \pm 0.07(\alpha_s) \pm 0.63(\text{th})) \times 10^{-3}.$$
TABLE IV: Results of the kinetic scheme fit. The error from the fit contains the uncertainties related to the experiment, the non-perturbative corrections and the bias correction. $\sigma(\alpha_s)$ is the uncertainty related to the perturbative corrections. In the lower part of the table, the correlation matrix of the parameters is shown.

The first error is due to all uncertainties taken into account in the fit (experimental error in the moment measurements, non-perturbative corrections and bias correction to the moments, uncertainty in $\tau_B$). The second error is obtained by varying $\alpha_s$ in the expressions for the moments and in Eq. 10. In Eq. 10 we vary $\alpha_s$ by $\pm 0.008$ around the central value of 0.22 [19]. The last error is a 1.5% uncertainty due to the limited accuracy of the theoretical expression for the semileptonic width, assessed in Ref. [5].

V. SUMMARY

We have performed a fit to the Belle measured spectral moments of the lepton energy and hadronic mass spectrum in charmed semileptonic $B$ decays, and the photon energy spectrum of inclusive radiative $B$ decays using expressions for the moments in terms of HQE parameters in the $1S$ mass and kinetic mass schemes. The fits produce values of $|V_{cb}|$
FIG. 4: Partial branching fractions and lepton energy moments, compared to the kinetic scheme fit result. The yellow bands show the theoretical uncertainty included in the fit (non-perturbative corrections, bias correction). The open symbols correspond to measurements not used in the fit.

that are consistent between the two schemes. In the 1S scheme analysis we find $|V_{cb}| = (41.49 \pm 0.52(\text{fit}) \pm 0.20(\tau_B)) \times 10^{-3}$, and in the kinetic scheme we obtain $|V_{cb}| = (41.93 \pm 0.65(\text{fit}) \pm 0.07(\alpha_s) \pm 0.63(\text{th})) \times 10^{-3}$. The heavy quark parameters, $m_b^{\text{kin,1S}}$ and $\lambda_5^{\mu^2}$, have been extracted with values that are consistent with previous determinations $[2, 3]$. 

Constant feedback between theory and experiment should further confirm the understanding of the OPE in all measurable regions of phase space. The accuracy achieved by the Belle measurements is unprecedented by any other experiment and the uncertainty on the heavy quark parameters and $|V_{cb}|$ reflect these improvements.
FIG. 6: Stability of the kinetic scheme fit. The fit is repeated using lepton energy moments only, hadron mass moments only and photon energy moments only. The ellipses are $\Delta \chi^2 = 1$.

VI. ACKNOWLEDGMENTS

We thank the theorists working on the $1S$ scheme: C.W. Bauer, Z. Ligeti, M. Luke, A.V. Manohar and M. Trott, and those working on the kinetic scheme: P. Gambino, N. Uraltsev and I. Bigi for providing the Mathematica and Fortran code that describe the respective calculations. We thank the KEKB group for the excellent operation of the accelerator, the KEK cryogenics group for the efficient operation of the solenoid, and the KEK computer group and the National Institute of Informatics for valuable computing and Super-SINET network support. We acknowledge support from the Ministry of Education, Culture, Sports, Science, and Technology of Japan and the Japan Society for the Promotion of Science; the Australian Research Council and the Australian Department of Education, Science and Training; the National Science Foundation of China and the Knowledge Innovation Program of the Chinese Academy of Sciences under contract No. 10575109 and IHEP-U-503; the Department of Science and Technology of India; the BK21 program of the Ministry of Education of Korea, the CHEP SRC program and Basic Research program (grant No. R01-2005-000-10089-0) of the Korea Science and Engineering Foundation, and the Pure Basic
Research Group program of the Korea Research Foundation; the Polish State Committee for Scientific Research; the Ministry of Science and Technology of the Russian Federation; the Slovenian Research Agency; the Swiss National Science Foundation; the National Science Council and the Ministry of Education of Taiwan; and the U.S. Department of Energy.

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