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Fixed-Time Output-Constrained Synchronization of Unknown Chaotic Financial Systems Using Neural Learning

Qijia Yao 1, Hadi Jahanshahi 2,*, Larissa M. Batrancea 3,*, Naif D. Alotaibi 4 and Mircea-Iosif Rus 5

1 School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China
2 Department of Mechanical Engineering, University of Manitoba, Winnipeg, MB R3T 5V6, Canada
3 Department of Business, Babeș-Bolyai University, 7 Horea Street, 400174 Cluj-Napoca, Romania
4 Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia
5 National Institute for Research and Development in Constructions, Urbanism and Sustainable Spatial Development “URBAN INCERC”, 117 Calea Floresti, 400524 Cluj-Napoca, Romania

* Correspondence: jahanshahi.hadi90@gmail.com (H.J.); larissa.batrancea@ubbcluj.ro (L.M.B.)

Abstract: This article addresses the challenging problem of fixed-time output-constrained synchronization for master–slave chaotic financial systems with unknown parameters and perturbations. A fixed-time neural adaptive control approach is originally proposed with the aid of the barrier Lyapunov function (BLF) and neural network (NN) identification. The BLF is introduced to preserve the synchronization errors always within the predefined output constraints. The NN is adopted to identify the compound unknown item in the synchronization error system. Unlike the conventional NN identification, the concept of indirect NN identification is employed, and only a single adaptive learning parameter is required to be adjusted online. According to the stability argument, the proposed controller can ensure that all error variables in the closed-loop system regulate to the minor residual sets around zero in fixed time. Finally, simulations and comparisons are conducted to verify the efficiency and benefits of the proposed control strategy. It can be concluded from the simulation results that the proposed fixed-time neural adaptive controller is capable of achieving better synchronization performance than the compared linear feedback controller.

Keywords: synchronization; chaotic financial system; fixed-time control; neural network; output constraint

MSC: 34C28; 37D45; 92B20; 68T01; 93C40

1. Introduction

It is recognized that the financial system is a type of extremely complex nonlinear system. Investigating the complex dynamical behavior of the financial system is a significant topic in both macroeconomics and microeconomics. It is well known that the financial system possesses both periodic and chaotic characteristics. When a chaotic phenomenon occurs, the financial system exhibits irregular behavior and becomes difficult to forecast. Moreover, the chaotic financial system is unavoidably affected by unknown parameters and perturbations. As a result, the above issues bring great difficulty to the control and synchronization of chaotic financial systems.

Recently, a large range of advanced control algorithms was put forward for the control and synchronization of chaotic financial systems. In [1,2], a class of nonlinear financial systems was proposed, and its complicated characteristics were studied. Through numerical modeling, Chen et al. [3] examined the dynamics of a financial system with time-delayed feedback. The chaos dynamics of a fractional financial system with time delay was analyzed in [4]. For the synchronization of finance systems, a special matrix structure-based controller, a hybrid feedback controller, and an active controller were presented in [5]. Yu et al. [6] developed speed feedback and linear feedback controllers for the stabilization...
of a new hyperchaotic finance system. In [7], the complex features of a financial and economic system where three parameters change simultaneously was investigated. An adaptive control scheme was proposed by Vargas et al. [8] for the synchronization of hyperchaotic financial systems with unknown parameters. An adaptive control approach was carried out in [9] for the function projective synchronization of financial systems. Zheng and Du [10] designed a unique matrix structure-based control law for the projective synchronization of hyperchaotic financial systems. A linear feedback controller was developed by Ding and Xu [11] to deal with the mixed synchronization of chaotic financial systems. In [12], a simple straightforward controller was presented for the hybrid synchronization of a class of chaotic financial systems.

Jajarmi et al. [13] developed efficient optimal and adaptive controllers for the hyperchaotic suppression and synchronization of a financial model. Moreover, a fractional form of this financial model was also hyper-chaotically suppressed and synchronized by using an active linear state feedback controller. For the synchronization and anti-synchronization of fractional financial systems, an active control strategy was presented by Huang and Cai [14]. A linear feedback control scheme was proposed in [15] to synchronize a time-delayed fractional-order chaotic financial system. An adaptive sliding mode controller was designed by Hajipour et al. [16] for the stabilization of a fractional hyperchaotic financial system. By utilizing an adaptive control method, Gong et al. [17] addressed the chaotic synchronization of fractional financial systems. In [18], the chaos and complexity of a fractional financial system with time delays were systematically studied. From the perspective of fractional calculus with circuit realization, Chen et al. [19] examined the effects of market confidence on a financial system. A nonlinear model predictive controller was carried out by Jahanshahi et al. [20] to stabilize a variable-order fractional hyperchaotic economic system. In [21], a proportional–integral–derivative (PID) controller based on whale optimization was designed for the control and synchronization of two identical fractional-order financial chaotic systems.

Note that the above controllers are mainly designed based on the nominal model information of the financial system. However, the parameters of the financial system may be completely unknown in some extreme cases. Under this situation, the above controllers are no longer applicable. The intelligent control is a good solution to this problem, owing to the powerful universal identification capability of the neural network (NN) and fuzzy logic system. With the use of NN and fuzzy identifications, the intelligent control is model-free with strong robustness against unknown parameters and perturbations. An integrated terminal sliding mode controller based on a fuzzy disturbance observer was developed in [22] for the synchronization of hyperchaotic financial systems. Bekiros et al. [23] proposed a fuzzy mixed $H_2/H_{\infty}$ control approach for the stabilization of a hyperchaotic financial system. In [24], a finite-time terminal sliding mode control scheme with deep recurrent NN was carried out for a fractional-order chaotic financial system with market confidence. A reinforcement learning-based control algorithm was presented by Ding et al. [25] for a class of nonlinear macroeconomic systems. Nevertheless, it should be emphasized that the conventional NN and fuzzy identifications require a large amount of adaptive parameters learning online and thus, the intelligent control unavoidably suffers from the heavy computational burden.

Additionally, the above control approaches can only achieve the asymptotic stability or, at best, exponential stability of the resulting closed-loop system. Alternatively, the finite-time control can guarantee that the synchronization errors of the master–slave chaotic financial systems regulate to zero or the minor residual sets around zero in finite time. A robust finite-time controller was designed by Xin and Zhang [26] to stabilize a fractional financial system with market confidence. By using an active finite-time fault-tolerant control strategy, the tracking control and stabilization of a fractional financial risk system were studied in [27]. However, the settling time of the finite-time control is heavily dependent on the initial states of the system. To overcome this weakness, the concept of fixed-time control was provided [28–31]. The fixed-time control can be regarded as a typical class of
finite-time control, whose settling time is bounded and the upper bound of the settling time is unaffected by initial system states. For the chaotic suppression of the power system, a fixed-time fast nonsingular terminal sliding mode control scheme was proposed by Ni et al. [32]. A fixed-time adaptive backstepping control algorithm was presented to stabilize a Lorenz chaotic system [33]. Yao [34] developed a fixed-time adaptive integral sliding mode controller for the synchronization of second-order chaotic systems.

It is important to take the output constraints into account for the control and synchronization of chaotic financial systems. On the one hand, the output-constrained control is an effective approach to balance the steady-state and transient response performance of a controller. On the other hand, keeping the system outputs always within the predefined output constraints can also ensure the safe operation of the system for practical applications. The barrier Lyapunov function (BLF) is a commonly used method to handle output constraints [35–37]. The BLF is a special class of Lyapunov function that grows to infinity when system outputs approach certain bounds. Accordingly, the predefined output constraints can never be exceeded by keeping the BLF bounded during the whole synchronization process. A neural adaptive state feedback controller was designed in [38] for the synchronization of second-order chaotic systems with output constraints. Moreover, a neural adaptive output feedback controller was also developed by integrating with a high-gain observer. However, the above controllers can merely guarantee the uniform ultimate boundedness of the resulting closed-loop system, meaning that synchronization errors can only regulate to the minor residual sets around zero within the infinite settling time.

Inspired by the above observations, this article addresses the challenging problem of fixed-time output-constrained synchronization for master–slave chaotic financial systems with unknown parameters and perturbations. A fixed-time neural adaptive control approach is originally proposed under the fixed-time control framework and by the aid of BLF and NN identification. Actually, the main novelties and contributions of this article can be summarized as the following three aspects. (1) Benefiting from the utilization of BLF, the proposed controller can preserve synchronization errors always within the predefined output constraints. (2) Unlike the conventional NN identification, the concept of indirect NN identification is employed, and only a single adaptive learning parameter is required to be adjusted online. This way, the computational burden of the proposed controller is significantly decreased. (3) According to the stability argument, the proposed controller can ensure all error variables in the closed-loop system regulate to the minor residual sets around zero in fixed time. To the best of the authors’ knowledge, there are very limited existing controllers that can achieve such excellent synchronization performance under identical circumstances.

The rest of this article is structured as follows. Section 2 describes the problem and gives some mathematical preparations. Section 3 provides the control design and stability argument. Section 4 presents the simulations and comparisons. Finally, Section 5 concludes this research.

2. Problem Description and Mathematical Preparations

2.1. Problem Description

Consider the synchronization of master–slave financial systems. Without loss of generality, the master chaotic financial system is described as

$$\dot{x} = h(x),$$

(1)

where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) is the master system state vector, \( n \) is the dimension of the chaotic financial system, and \( h(\cdot) \) is an unknown nonlinear vector field. Moreover, the slave system is described as

$$\dot{y} = h(y) + u + d,$$

(2)
where \( y = [y_1, y_2, \ldots, y_n]^T \in \mathbb{R}^n \) is the slave system state vector, \( u \) is the control input vector, and \( d \) is the perturbation vector. Define \( e = y - x \) as the synchronization error vector. Then, the synchronization error can be deduced as
\[
\dot{e} = h(y) - h(x) + u + d = u + \delta,
\]
where \( \delta = h(y) - h(x) + d \) denotes the compound unknown item.

The objective of this study is constructing a suitable controller \( u \) to guarantee that the synchronization errors \( e \) can regulate to the minor residual sets around zero in fixed time. Moreover, during the whole synchronization process, the synchronization errors \( e \) can never exceed the following output constraints:
\[
-k_a(t) < e(t) < k_a(t),
\]
where \( k_a(t) = [k_{a1}, k_{a2}, \ldots, k_{an}]^T \in \mathbb{R}^n \) is the predefined time-varying bound vector with \( k_{ai} > 0 \) (\( i = 1, 2, \ldots, n \)). Obviously, the initial synchronization errors must meet the condition \(-k_a(0) < e(0) < k_a(0)\).

2.2. Mathematical Preparations

**Lemma 1** [31]. Consider the nonlinear system \( x(t) = f(x(t)), x(t) \in \mathbb{R}^n \), where \( f(\cdot) \) is a continuous vector field. If there exists a positive definite function \( V(x) \) such that
\[
\dot{V}(x) \leq -\kappa_1 V^p(x) - \kappa_2 V^q(x) + \varphi,
\]
where \( \kappa_1 > 0, \kappa_2 > 0, 0 < p < 1, q > 1, \) and \( \varphi > 0 \), then the nonlinear system is practically fixed-time stable and \( V(x) \) can regulate to the minor residual set around zero in fixed time as follows:
\[
\Phi = \left\{ V(x) \in \mathbb{R} \mid V(x) \leq \min \left\{ \left( \frac{\varphi}{\kappa_1 (1 - \tau)} \right)^\frac{1}{p}, \left( \frac{\varphi}{\kappa_2 (1 - \tau)} \right)^\frac{1}{q} \right\} \right\},
\]
where \( 0 < \tau < 1 \), and the settling time is bounded with \( T_s \leq \frac{1}{\kappa_1 (1 - \tau) p} + \frac{1}{\kappa_2 (1 - \tau) q} \).

**Lemma 2** [39]. Consider a continuous nonlinear function \( f(Z), Z \in \mathbb{R}^n \); it can be identified by a radial basis function NN (RBFNN) as
\[
f(Z) = W^T \Theta(Z) + \epsilon(Z),
\]
where \( W \in \mathbb{R}^N \) is the ideal NN weight, \( \Theta(Z) = [\theta_1(Z), \theta_2(Z), \ldots, \theta_N(Z)]^T \in \mathbb{R}^N \) is the basis function vector, \( \epsilon(Z) \) is the identification error with \( |\epsilon(Z)| \leq \varepsilon \), and \( N \) is the neuron number. Generally, \( s_i(Z) \) is chosen as the following Gaussian function:
\[
\theta_i(Z) = \exp \left( -\|Z - \zeta_i\|^2 / w_i^2 \right), \quad i = 1, 2, \ldots, N,
\]
where \( \zeta_i = [\zeta_{i1}, \zeta_{i2}, \ldots, \zeta_{in}]^T \in \mathbb{R}^n \) is the center of the Gaussian function and \( w_i \) is the width.

**Lemma 3** [36]. For \( |x| < k_b \), we have the following inequality:
\[
\ln \left( \frac{k_b^2}{k_b^2 - x^2} \right) \leq \frac{x^2}{k_b^2 - x^2}.
\]

**Lemma 4** [40]. For \( x \in \mathbb{R}, y \in \mathbb{R}, p > 0, q > 0, \) and \( r > 0 \), we have the following inequality:
\[
|x|^p |y|^q \leq \frac{p}{p + q} r |x|^{p+q} + \frac{q}{p + q} r^{-\frac{q}{p}} |y|^{p+q}.
\]
Lemma 5 [40]. For \( x_i \in \mathbb{R} \) \( (i = 1, 2, \ldots, n) \) and \( 0 < p \leq 1 \), we have the following inequality:
\[
\left( \sum_{i=1}^{n} |x_i| \right)^p \leq \sum_{i=1}^{n} |x_i|^p \leq n^{1-p} \left( \sum_{i=1}^{n} |x_i| \right)^p.
\] (10)

3. Control Design and Stability Argument

3.1. Control Design

Define the variable \( Z = [x^T, y^T]^T \). By Lemma 2, the compound uncertain item \( \delta \) can be identified by an RBFNN as
\[
\delta = W^* \Theta(Z) + \varepsilon(Z),
\] (11)
where \( W^* \in \mathbb{R}^N \) is the ideal NN weight, \( \Theta(Z) \in \mathbb{R}^N \) is the Gaussian basis function vector, \( \varepsilon(Z) \) is the identification error with \( \|\varepsilon(Z)\| \leq \bar{\varepsilon} \), and \( N \) is the neuron number. Consider the inequality \( \|W^*\| \leq \bar{W} \). Substituting it into (11), we have
\[
\|\delta\| \leq \|W^*\| \|\Theta(Z)\| + \|\varepsilon(Z)\| \leq B_1 \Delta,
\] (12)
where \( B_1 = \max\{\bar{W}, \bar{\varepsilon}\} \) is an unknown constant and \( \Delta = \|\Theta(Z)\| + 1 \) is a known function. Then, the fixed-time neural adaptive controller is designed as
\[
\begin{align*}
\dot{\hat{B}}_s & = -\dot{\mu}_1 \hat{B}_s + \frac{\Delta^2 \|\varepsilon\|^2}{2\eta^2},
\end{align*}
\] (13)
where \( k_1 > 0, k_2 > 0, 0 < p < 1, q > 1, \eta > 0, \) and \( \hat{B}_s \) is the identification of \( B_s = B^2 \). Moreover, the parametric adaptive learning law is designed as
\[
\dot{\hat{B}}_s = -\mu_1 \hat{B}_s + \frac{\Delta^2 \|\varepsilon\|^2}{2\eta^2},
\] (14)
where \( \mu_1 > 0 \) and \( \mu_2 > 0 \).

3.2. Stability Argument

After the above control design, we can present the main theorem of this work as follows.

Theorem 1. Consider the master–slave chaotic financial systems. If the fixed-time neural adaptive controller (13) with the parametric adaptive learning law (14) is employed, then the resulting closed-loop system is practically fixed-time stable, and error variables \( e \) and \( \hat{B}_s \) can regulate to the minor residual sets around zero in fixed time. Moreover, during the whole synchronization process, the synchronization errors \( e \) can always stay within the predefined output constraints.

Proof. The following BLF is employed:
\[
V = \frac{1}{2^2} \sum_{i=1}^{n} \ln \frac{k_i^2}{k_i^2 - \bar{\varepsilon}_i^2} + \frac{1}{2\mu_2} \hat{B}_s^2,
\] (15)
where \( \hat{B}_s = B_s - \hat{B}_s \) denotes the identification error of \( B_s \). Calculating the time derivative of \( V \) gives

\[
\dot{V} = \sum_{i=1}^{n} e_i \left( \frac{c_i^2}{k_{ai}^2 - c_i^2} \right) + \frac{1}{\mu_1} \hat{B}_s \tilde{\mu}.
\]

Substituting the fixed-time neural adaptive controller (13) with the parametric adaptive learning law (14) into (16), we have

\[
\dot{V} = -k_1 \sum_{i=1}^{n} \left( \frac{c_i^2}{k_{ai}^2 - c_i^2} \right) + \frac{1}{\mu_1} \hat{B}_s \tilde{\mu} + \eta \| \delta \| - \frac{1}{\mu_1} \hat{B}_s B_s + \frac{1}{2\mu_1} \hat{B}_s B_s.
\]

Consider the following inequalities:

\[
\| e \| \leq \frac{B_s^2 \| e \|}{2\eta} + \frac{\eta^2}{2} = \frac{B_s^2 \| e \|}{2\eta} + \frac{\eta^2}{2},
\]

where \( \phi \) is defined as

\[
\phi = \left( \frac{\mu_1}{4\mu_2} \right)^{\frac{\mu_1}{\mu_2}} + \left( \frac{\mu_1}{4\mu_2} \right)^{\frac{\mu_1}{\mu_2}} - \frac{\mu_1}{2\mu_2} B_s^2 + \frac{\mu_1}{4\mu_2} \tilde{B}_s^2.
\]
where \( \bar{\nu} = \frac{\nu + 1}{\nu} \). Suppose there exists a compact set \( \Psi \) and a positive constant \( \psi \) that satisfies \( \Psi = \{ \bar{B}_s | B_s \leq \psi \} \). Then, it follows from (22) and (23) that

\[
\left( \frac{\mu_1}{4\mu_2} \right)^{\frac{\nu + 1}{\nu}} + \left( \frac{\mu_1}{4\mu_2} \right)^{\frac{\nu + 1}{\nu}} - \frac{\mu_1}{2\mu_2} B_s^2 \leq \alpha, \tag{24}
\]

where \( \alpha \) is defined as

\[
\alpha = \begin{cases} 
(1 - \bar{\nu}) \bar{\nu}^{\frac{\nu}{1+\nu}}, & \psi < 2 \sqrt{\frac{\mu_2}{\nu}}, \\
\left( \frac{\mu_1}{4\mu_2} \psi^2 \right)^{\frac{\nu + 1}{\nu}} - \frac{\mu_1}{4\mu_2} \psi^2, & \psi \geq 2 \sqrt{\frac{\mu_2}{\nu}}.
\end{cases} \tag{25}
\]

Substituting (25) into (20) and employing Lemma 5, we further have

\[
\dot{V} \leq -\kappa_1 V^{\frac{\nu + 1}{\nu}} - \kappa_2 V^{\frac{\nu + 1}{\nu}} + \bar{\nu}, \tag{26}
\]

where \( \kappa_1, \kappa_2, \) and \( \bar{\nu} \) are defined as

\[
\kappa_1 = \min \left\{ 2^{\frac{\nu + 1}{\nu}} k_1, \left( \frac{\mu_1}{2} \right)^{\frac{\nu + 1}{\nu}} \right\}, \tag{27}
\]

\[
\kappa_2 = n^{\frac{\nu - 1}{\nu}} \min \left\{ 2^{\frac{\nu + 1}{\nu}} k_1, \left( \frac{\mu_1}{2} \right)^{\frac{\nu + 1}{\nu}} \right\}, \tag{28}
\]

\[
\bar{\nu} = \alpha + \frac{\mu_1}{2\mu_2} B_s^2 + \frac{\eta^2}{2}. \tag{29}
\]

By Lemma 1, the resulting closed-loop system is practically fixed-time stable and \( V \) can regulate to the minor residual set around zero in fixed time as follows:

\[
\Phi = \left\{ V \bigg| V \leq \min \left\{ \left( \frac{\bar{\nu}}{\kappa_1 (1 - \tau)} \right)^{\frac{\nu + 1}{\nu}}, \left( \frac{\bar{\nu}}{\kappa_2 (1 - \tau)} \right)^{\frac{\nu + 1}{\nu}} \right\} \right\}. \tag{30}
\]

where \( 0 < \tau < 1 \) and the settling time is bounded with \( T_s \leq \frac{2}{\kappa_1 \tau (1 - p)} + \frac{2}{\kappa_2 \tau (\psi - 1)} \). Define the variable \( \sigma = 2 \min \left\{ \left( \frac{\bar{\nu}}{\kappa_1 (1 - \tau)} \right)^{\frac{\nu + 1}{\nu}}, \left( \frac{\bar{\nu}}{\kappa_2 (1 - \tau)} \right)^{\frac{\nu + 1}{\nu}} \right\} \). Combining with the definition of \( V \), we have

\[
\ln \frac{k_{ai}^2}{k_{ai}^2 - e_i} \leq \sigma, \quad i = 1, 2, \ldots, n, \tag{31}
\]

\[
B_s^2 \leq \mu_2 \sigma. \tag{32}
\]

Rearranging (31) gives

\[
|e_i| \leq k_{ai} \sqrt{1 - e^{-\sigma}}, \quad i = 1, 2, \ldots, n. \tag{33}
\]

It follows from (32) and (33) that the error variables \( e \) and \( \bar{B}_s \) can regulate to the minor residual sets around zero in fixed time. Moreover, from (33), we further have that during the whole synchronization process, the synchronization errors \( e \) can always stay within the predefined output constraints \( -k_u(t) < e(t) < k_u(t) \) if the initial condition \( -k_u(0) < e(0) < k_u(0) \) is met. Hence, the proof is finished. \( \square \)

**Remark 1.** The NN is adopted to identify the compound unknown item in the synchronization error system. It should be pointed out that the conventional NN identification requires a total number
of 3\( N \) adaptive parameters to be learned online. Alternatively, the indirect NN identification (12) is employed in this article and requires only a single adaptive learning parameter. Benefiting from the indirect NN identification, the proposed controller is computationally simple. This distinctive feature makes the proposed controller suitable for practical implementations.

**Remark 2.** For the convenience of the readers to have a better understanding about our work, the structure of the proposed fixed-time neural adaptive control approach is depicted in Figure 1.

![Figure 1. Structure of the proposed fixed-time neural adaptive control approach.](image1)

4. Simulations and Comparisons

In this section, simulations and comparisons are conducted in the MATLAB/Simulink environment to demonstrate the proposed control approach. The ode45 integrator is utilized for numerical integration with the time interval \( t_s = 0.01 \) s. According to [1,2], the following chaotic financial system is considered:

\[
\begin{align*}
\dot{x}_1 &= x_3 + (x_2 - a)x_1, \\
\dot{x}_2 &= 1 - bx_2 - x_1^2, \\
\dot{x}_3 &= -x_1 - cx_3,
\end{align*}
\]  

(34)

where \( x_1 \) is the interest rate, \( x_2 \) is the investment demand, \( x_3 \) is the price index, \( a \) is the savings, \( b \) is the investment cost, and \( c \) is the commodity demand elasticity. When we select a group of typical parameters as \( a = 0.9, b = 0.2, \) and \( c = 1.2 \), the chaotic financial system (34) exhibits chaotic behaviors. The phase diagram of system (34) with the initial states \( x(0) = [0.62, 0.83, 0.72]^T \) is given in Figure 2.

![Figure 2. Cont.](image2)
The proposed fixed-time neural adaptive controller is designed as

\[
\begin{align*}
\dot{y}_1 &= y_3 + (y_2 - a) y_1 + u_1 + d_1, \\
\dot{y}_2 &= 1 - b y_2 - y_1^2 + u_2 + d_2, \\
\dot{y}_3 &= -y_1 - c y_3 + u_3 + d_3. 
\end{align*}
\] (35)

For the synchronization of master–slave financial systems, system (34) is chosen as the master system. Moreover, the slave system is chosen as

\[
\begin{align*}
\dot{x}_1 &= x_3 + (x_2 - a) x_1 + u_1 + d_1, \\
\dot{x}_2 &= 1 - b x_2 - x_1^2 + u_2 + d_2, \\
\dot{x}_3 &= -x_1 - c x_3 + u_3 + d_3. 
\end{align*}
\]

In the simulations, we set the parameters as \(a = 0.9\) and \(c = 1.2\), and the perturbations as \(d_1 = 0.1 \sin(0.8 t)\), \(d_2 = 0.1 \sin(0.6 t)\), and \(d_3 = 0.1 \sin(0.5 t)\), respectively. It should be noticed that \(a\), \(b\), \(c\), \(d_1\), \(d_2\), and \(d_3\) are all assumed to be unknown in the control design. Moreover, the initial master and slave system states are chosen as \(x(0) = [0.8, 1.5, -1]^T\) and \(y(0) = [1.2, 1, -0.4]^T\). The time-varying bounds are predefined as \(k_{a1} = k_{a2} = k_{a3} = 0.9999 \exp(-0.4 t) + 0.0001\).

Define \(e = y - x\) as the synchronization error vector. According to (13) and (14), the proposed fixed-time neural adaptive controller is designed as

\[
\begin{align*}
\dot{u} &= \begin{bmatrix}
-k_1 e_1 \\ -2 k_2 e_2 \\ -k_3 e_3
\end{bmatrix} + \begin{bmatrix}
k_1 \Delta e_1 \\ k_2 \Delta e_2 \\ k_3 \Delta e_3
\end{bmatrix}
- \frac{\beta_s \Delta^2 e}{2 \eta^2}.
\end{align*}
\] (36)

Moreover, the parametric adaptive learning law is designed as

\[
\dot{\beta}_s = -\mu_1 \beta_s + \mu_2 \frac{\Delta^2 \| e \|^2}{2 \eta^2}. 
\] (37)

In the simulations, the design gains of the proposed controller (36) are selected as \(k_1 = 8\), \(k_2 = 2\), \(p = 3/5\), \(q = 7/5\), \(\mu_1 = 1\), \(\mu_2 = 1\), and \(\eta = 1\). Seven neurons are suggested to construct the RBFNN. The center of the Gaussian function is set as \(\zeta_i = [-3, -2, -1, 0, 1, 2, 3]^T\), and the width is chosen as \(\omega_i = 6\).

Besides the proposed controller, the conventional linear feedback controller is also utilized for comparisons. The linear feedback controller is designed as

\[
u = -ke, 
\] (38)
where \( k > 0 \). A common feature of the proposed and compared controllers is that they are both model-free. In the simulations, the design gain of the compared controller (38) is selected as \( k = 8 \).

The simulations results are provided in Figures 3–6. Figure 3 shows the time response of the master and slave system states. It is clearly seen that the slave system states can synchronize to the master system states rapidly and exactly under both the proposed and compared controllers. Specifically, the time response of the synchronization errors is given in Figure 4. It is not difficult to find that the steady-state synchronization errors under the proposed controller are much smaller than those under the linear feedback controller. Specifically, the steady-state synchronization errors under the proposed controller are within the range of \([-0.0072, 0.0072]\). By contrast, the steady-state synchronization errors under the linear feedback controller are within the range of \([-0.015, 0.015]\). Moreover, the convergence speed of the proposed controller is also much faster than that of the linear feedback controller.

![Figure 3. Time response of the master and slave system states.](image1)

![Figure 4. Time response of the synchronization errors.](image2)
Figure 5. Time response of the control inputs. 

Figure 6. Time response of the adaptive learning parameter.

Figure 5 presents the time response of the control inputs. The time response of the adaptive learning parameter is presented in Figure 6. It is obvious that the adaptive learning parameter is changing smoothly with time. Moreover, only a single adaptive learning parameter is required to be adjusted online and thus the computational burden of the proposed controller is significantly decreased. It can be concluded from the simulation results that the proposed controller can realize superior synchronization performance compared to the linear feedback controller associated with higher steady-state synchronization accuracy and faster convergence speed. This is mainly benefiting from the indirect NN identification. By adopting the NN identification to compensate for the compound unknown parameters, the proposed controller is not only highly insensitive to unknown parameters, but also strongly robust against perturbations. 

Furthermore, the integrated absolute errors (IAEs) and integrated time absolute errors (ITAEs) of the proposed controller and the linear feedback controller are listed in Table 1.
to quantitatively compare their synchronization performance. The IAEs are defined as
\[ IAE_{e_1} = \int_0^t |e_1(\tau)|d\tau, \]
\[ IAE_{e_2} = \int_0^t |e_2(\tau)|d\tau, \]
and \[ IAE_{e_3} = \int_0^t |e_3(\tau)|d\tau, \]
which corresponds to the steady-state performance of the controller. Moreover, the ITAEs are defined as
\[ ITAE_{e_1} = \int_0^t \tau |e_1(\tau)|d\tau, \]
\[ ITAE_{e_2} = \int_0^t \tau |e_2(\tau)|d\tau, \]
and \[ ITAE_{e_3} = \int_0^t \tau |e_3(\tau)|d\tau, \]
which corresponds to the transient performance of the controller. As shown in Table 1, the IAEs of the linear feedback controller are 400% larger than the proposed controller. Moreover, the ITAEs of the linear feedback controller are 300% larger than the proposed controller. This means that the proposed controller can realize the superior steady-state and transient performance compared to the linear feedback controller.

Table 1. IAEs and ITAEs of the controllers.

|          | \( IAE_{e_1} \) | \( IAE_{e_2} \) | \( IAE_{e_3} \) | \( ITAE_{e_1} \) | \( ITAE_{e_2} \) | \( ITAE_{e_3} \) |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Proposed | 0.0194          | 0.0237          | 0.0249          | 0.0603          | 0.0583          | 0.0669          |
| Compared | 0.0708          | 0.0689          | 0.0621          | 0.2206          | 0.1932          | 0.1923          |

To summarize, it can be concluded from the simulation results that the proposed fixed-time neural adaptive controller is capable of achieving better synchronization performance and stronger robustness against unknown parameters and perturbations than the compared linear feedback controller in terms of superior steady-state and transient performance.

5. Conclusions

In this article, a fixed-time neural adaptive control approach is originally proposed for the synchronization of unknown chaotic financial systems with output constraints. The proposed controller is developed by introducing the BLF to handle the output constraints and adopting the NN to identify the unknown part of the synchronization error system. Particularly, the concept of indirect NN identification is employed, which makes the proposed controller computation simple. The stability argument shows that the proposed controller can ensure all error variables in the closed-loop system regulate to the minor residual sets around zero in fixed time. Finally, simulations and comparisons indicate that the proposed controller can realize better synchronization than the compared linear feedback controller, owing to the utilization of NN identification. Future investigations of this work are to further improve the proposed control design by taking time delays into consideration.

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