Phase transition on the edge of the $\nu = \frac{5}{2}$ Pfaffian and anti-Pfaffian quantum Hall state

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Starting from the edge reconstructed Pfaffian state and the anti-Pfaffian state for the filling fraction $\nu = \frac{5}{2}$ fractional quantum Hall (FQH) state with the filled Landau levels included, we find that interactions between counterpropagating edge modes can induce phase transitions on the edge. In the new ‘Majorana-gapped’ phases, a pair of counter propagating neutral Majorana modes becomes gapped. The quasiparticle tunneling exponent changes from $g = 1/4$ to $g = 1/2$ for the edge reconstructed Pfaffian state, and changes from $g = 1/2$ to $g = 0.55 - 0.75$ for the anti-Pfaffian state, in the new Majorana-gapped phases. The new phases are candidate states for the observed $\nu = \frac{5}{2}$ state. Furthermore, Majorana-gapped phases provide examples that non-trivial quantum phase transitions can happen on the edge of a FQH state without any change in bulk topological order.

I. INTRODUCTION

With the prospect of non-abelian anyons and the potential application of topologically protected qubits, there has been a renewed interest in exotic fractional quantum Hall (FQH) states, especially the experimentally easiest accessible state at filling fraction $\nu = \frac{5}{2}$.

In 1991, two different non-Abelian FQH states were proposed both at filling fraction $\nu = \frac{5}{2}$ (or $\nu = \frac{2}{5}$). This has raised an very interesting possibility that the observed $\nu = \frac{5}{2}$ state may be a non-Abelian state. One of the proposed states, the Moore-Read Pfaffian wavefunction, has received a lot of attention recently, both in experiment and in theory.

However, it is currently still unclear what the true nature of the $\nu = \frac{5}{2}$ bulk state is; many candidate states, or trial wavefunctions, exist, some of which predict non-abelian statistics, others predict less exotic abelian statistics. We briefly review these candidates.

Experimentally the bulk state can be probed through transport measurements on the edge, for instance through tunneling between FQH edges induced by a quantum point contact (QPC). The different candidate states make specific predictions for the charge $e^*$ and exponent $g$ of the tunneling quasiparticles, and these can in principal be measured in experiment. In the regime of weak tunneling at sharp edges, chiral Luttinger liquid (cLL) theory describes the form of e.g. the differential tunneling conductance and the tunneling current noise as a function of temperature and applied bias, with $e^*$ and $g$ as continuously-varying fitting parameters.

Very recently two experimental groups reported results on tunneling across a QPC in the $\nu = \frac{5}{2}$ state; Ref. 17 focuses on scaling of the differential conductance, Ref. 18 measures shot-noise. The results are most consistent with a quasiparticle charge $e^* = 1/4$ and exponent $g = 1/2$.

In this paper we introduce two other candidates for the $\nu = \frac{5}{2}$ FQH edge. First, starting from the existing so-called ‘anti-Pfaffian’ bulk state, we consider interactions between the counterpropagating edge modes. As we change the interaction strength, we find that there is a transition to a new phase on the edge, with different values for $e^*$ and $g$. Note that this is really a phase transition on the edge, since the bulk state does not change. By appropriately tuning the edge interactions one should be able to observe this quantum phase transition through e.g. a change in $e^*$ and $g$.

We call this new phase the ‘Majorana-gapped’ phase, as the anti-Pfaffian Majorana mode becomes gapped. The Majorana-gapped phase has 2 and 1/2 right-moving edge branches and 1 left-moving edge branches, while the standard edge phase for the anti-Pfaffian state has 3 right-moving edge branches and 1 and 1/2 left-moving edge branches. During the transition from the standard edge phase to the Majorana-gapped phase, half a left-moving edge branch (a Majorana fermion mode) pairs up with half a right-moving edge branch and this opens up an energy gap.

Second, we start with the edge reconstructed Pfaffian state, which has 2 and 1/2 branches of right movers and 1 branch of left movers. Such an edge can also undergo a phase transition into a Majorana-gapped phase which has 2 branches of right movers and 1/2 branch of left movers. The values of $e^*$ and $g$ can be changed by the phase transition.

The above result is for the clean edge. In the presence of impurities, the picture is different. In that case, as we change the interaction strength between different edge branches beyond a threshold, a right-moving Majorana fermion mode pairs up with a left-moving Majorana fermion mode and they become localized. If we assume that the localized modes do not contribute to tunneling between the edges, then we can treat those localized modes as if they are gapped. Under this assumption, the clean edge and dirty edge behave similarly.

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1 Here, with a change of $e^*$ we mean that another quasiparticle with a different charge becomes the most dominant quasiparticle which is observed in experiments. The phase transition does not change the fixed charge $e^*$ of a given quasiparticle; it can change the exponent $g$ for all quasiparticles.
or the localization phase transition, it is necessary to include the supposedly completely filled lowest Landau level in the framework, or to include the additional edge branches from the edge reconstruction. The new phases require that the different edge modes have substantial interactions with each other.

Numerical simulations for small-size closed systems which by construction ignore edge effects, suggested that the Moore-Read Pfaffian trial wavefunction is the most likely candidate for the actual \( \nu = \frac{5}{2} \) FQH bulk state. To compare with actual experiments, which obviously have an edge, it is necessary, as our examples in this paper show, to include the edge-aspect as well. This is emphasized as well in a very recent numerical study, Ref. [22], which considers a disc-geometry with an edge, and a varying confining potential; for a sharp edge the Pfaffian is found to be favored, but for a smooth edge the groundstate is a different state which bears the marks of some form of edge reconstruction.

The ‘Majorana-gapping’ transition is a quantum phase transition on the edge only which does not affect the bulk state. Such a kind of edge-only quantum phase transition has been studied in Ref. [23]. Here we find a new type of edge-only quantum phase transition where we lose (gain) a fractional branch of right-movers (right-movers) through the transition.

Our paper is organized as follows. We review the different candidates for the \( \nu = \frac{5}{2} \) state in Sec. II. Section III is the core of our paper. Here we show that for the anti-Pfaffian state there exists an operator which for certain density-density interactions becomes relevant and can drive a phase transition. In the new phase a pair of counterpropagating Majorana modes becomes gapped. For the new phase we determine the quasiparticle spectrum and which of these quasiparticles is the most relevant. In Sec. IV we apply the same formalism to the edge reconstructed Pfaffian state. We discuss and summarize our results in Sec. V.

II. LIST OF CANDIDATE STATES FOR \( \nu = \frac{5}{2} \) (\( \nu = \frac{1}{2} \)) FQH STATE.

The Majorana-gapped phase at the edge of the anti-Pfaffian state is just one of many possible edge states at filling fraction \( \nu = \frac{5}{2} \). Therefore, in this section, we will review some known edge states for filling fraction \( \nu = \frac{5}{2} \). Or, to be more precise, for filling fraction \( \nu = \frac{1}{2} \) modulo completely filled Landau levels.

It is well known that, at a given filling fraction, FQH states may have many different internal structures – topological orders. \(^{2,24,25,26,27,28,29,30}\) So it is not clear which topological order describes a particular experimentally observed \( \nu = \frac{1}{2} \) (\( \nu = \frac{5}{2} \)) FQH state. However, the following five topological orders are simple and are more likely to describe the observed \( \nu = \frac{1}{2} \) (\( \nu = \frac{5}{2} \)) FQH states. The five topological orders are:

| state | \# of branches | \( e^* \) | \( g \) | \( g_e \) |
|-------|----------------|--------|-------|--------|
| Pfaffian | \( 2R + 2R \) | 1/4 | 0.25 | 3 |
| 331  | \( 2R + 2R \) | 1/4 | 0.375 | 3 |
| \( U(1) \times SU(2)_2 \) | \( 2R + 2R \) | 1/4 | 0.5 | 3 |
| anti-Pfaffian | \( 1R + \frac{1}{2}L \) | 1/2 | 0.5 | 3 |
| Majorana-gapped Pfaffian | \( 2R + \frac{1}{2}L + 2R \) | 1/4 | 0.5 | 3 |
| Majorana-gapped anti-Pfaffian | \( \frac{1}{2}L + \frac{1}{2}R \) | 1/4 | 0.5-0.75 | 1.8-2.0 |

(A) The electrons first pair into charge \( 2e \) bosons and the charge \( 2e \) bosons then condense into the Laughlin state described by the following wave function:

\[
\prod_{i<j}(Z_i - Z_j)^{g} e^{-\frac{1}{2} \sum |Z_i|^2}.
\]

The effective theory of this state has a form \(^{2,29,30}\)

\[
\mathcal{L} = \sum_{i,j} K_{ij} \frac{1}{4\pi} a_{ij} \partial_{\nu} a_{ij} e^{i\nu \lambda}
\]

with \( K \) a \( 1 \times 1 \) matrix \( K = 8 \).

(B) The charge \( 2e/3 \) quasiparticles on top of the \( \nu = 1/3 \) Laughlin state condense into a second level hierarchical FQH state.\(^{2,31,32}\) The effective theory of such a state is given by \(^{11}\) with \( K = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix} \). Since \( \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = W \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix} W^T \) with \( W = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \), such a state has the same topological order as the 331 double layer state.\(^{30}\)

(C) The FQH state proposed in Ref. [\text{??}] and described by the Pfaffian wave function

\[
\Psi_{pd}\{\{z_i\}\} = A\left(\frac{1}{z_1 - z_2} \frac{1}{z_2 - z_3} \cdots \right) \prod_{i<j}(z_i - z_j)^2 e^{-\frac{1}{2} \sum |z_i|^2}
\]

where \( A \) is the antisymmetrization operator.

(D) The anti-Pfaffian state, which is the particle-hole conjugate of the Pfaffian state.\(^{10,11}\)

(E) The FQH state proposed in Refs. [\text{??}] and described by the wave function

\[
\Psi\{\{z_i\}\} = [\chi_2(\{z_i\})]^{2} \prod_{i<j}(z_i - z_j)e^{-\frac{1}{4} \sum |z_i|^2}
\]
where $\chi_2(\{z_i\})$ is the fermion wave function of two filled Landau levels. In the Appendix we provide a more detailed description of the edge theory of this state.

Other topological orders at $\nu = \frac{1}{2}$ have more complicated internal structures and are unlikely to appear. For convenience, we will use $K = 8$, 331, Pfaffian, anti-Pfaffian, and $U(1) \times SU_2(2)$ to denote the above five topological orders respectively.

The $K = 8$ and the 331 states are abelian FQH states, whose quasiparticles all have abelian statistics. The bulk low energy effective theories for the two FQH state are given by $U(1)$ Chern-Simons (CS) theory, Eq. \ref{eq:CS}. The edge excitations of the $K = 8$ state are described by a single density mode (or more precisely, a $U(1)_R$ Kac-Moody (KM) algebra, where the subscript $R$ indicates that the excitations are right moving). The number of low energy edge excitations for the $K = 8$ state is the same as one filled Landau level, as measured by the low temperature specific heat. Thus we say that the $K = 8$ state has one branch of edge excitations. The edge excitations of the 331 state are described by two density modes (which form a $U(1)_R \times U(1)_R$ KM algebra). Using the similar definition in terms of specific heat, the 331 state has two branches of edge excitations.

The Pfaffian, anti-Pfaffian, and $U(1) \times SU_2(2)$ states are non-abelian states. Some of their quasiparticles have non-abelian statistics. The edge excitations of the Pfaffian state are described by a density mode (the $U(1)_R$ KM algebra) and a free chiral Majorana fermion (the Ising$_R$ conformal field theory), or in other word, by a $U(1)_R \times Ising_R$ conformal field theory (CFT). Such an edge state has one and a half branches of right-moving edge excitations as measured by specific heat. The edge excitations of the anti-Pfaffian state are described by $U(1)_R \times U(1)_L \times Ising_L$ CFT. The edge excitations for the anti-Pfaffian state have one and a half branches of left-moving edge excitations and one branch of right-moving edge excitations. For the $U(1) \times SU_2(2)$ state, the bulk effective theory is a $U(1) \times SU_2(2)$ CS theory and the edge excitations are described by $U(1)_R \times SU_2(2)_R$ KM algebra. The edge state has two and a half branches of right-moving excitations.

The theory of edge excitations for both abelian and non-abelian FQH states were well developed\cite{9511228,9511224,9511235,9511236}. In Table \ref{table:edge_conformal} we list the relevant results. Here $e^*$ is the quasiparticles charge and $g$ is the exponent in the correspond- ing quasiparticle Green’s function: $\langle \psi_{qp}^\dagger \psi_{qp}\rangle \sim 1/t^g$. In terms of scaling dimensions $\Delta$ the exponent $g$ is twice the scaling dimension of the quasiparticle operator.

The results we find in this paper for the Majorana-gapped edge phases of the anti-Pfaffian and edge-reconstructed Pfaffian states are also included in the table. Note that the anti-Pfaffian edge state and its Majorana-gapped edge state are two edge phases of the same anti-Pfaffian bulk FQH state (and similarly for the Pfaffian edge states). For the Majorana-gapped anti-Pfaffian phase we find that the exponent of the quasiparticle Green’s function is non-universal; the exact value of $g$ depends on the interaction. Nevertheless there are two dominant quasiparticles, one with $e^* = 1/4$ and the exponent $g$ in the range $g \in [0.55 - 0.75]$, and one with $e^* = 1/2$ and $g \in [0.5 - 0.7]$.

### III. MAJORANA-GAPPED PHASE OF THE ANTI-PFAFFIAN

This section hold the main results of our paper. We show in detail how to calculate scaling dimensions of quasiparticle operators for the anti-Pfaffian state in the presence of density-density interactions. We identify a charge-transfer operator that can be relevant. This operator is a product of a left-moving Majorana fermion and a right-moving complex fermion. Condensation of this operator gaps the left-moving Majorana mode and half of the right-moving fermionic mode.

In the new phase, dubbed ‘Majorana-gapped’ phase, several quasiparticle operators have become gapped as well, and we determine the spectrum of ungapped quasiparticles. Next, we find the quasiparticle with the lowest scaling dimension which is expected to dominate tunneling. Finally, we consider the effect of impurities.

#### A. Non-universality for non-chiral edges

Fractional quantum Hall states which are described by fully chiral edge theories are called ‘universal’, because correlation function exponents are independent of the the exact microscopic details of e.g. the interaction between different edge branches.

This situation is no longer the case for FQH state described by an edge theory with branches moving in opposite directions, i.e., a non-chiral edge. In this case the scaling dimensions of operators depend on the exact form of the interaction between the different edge branches, and a more detailed analysis is required to predict the fate of e.g. tunneling exponents.

In some cases (i.e., for some regions in the space of all possible interactions) the result is that the tunneling exponents are indeed non-universal, in other cases the properties of the system are dominated by a certain fixed point for which the tunneling exponents do acquire universal values. In this sense one can construct a phase diagram in ‘interaction-space’.

Such an analysis typically focusses on two types of interactions: density-density-interactions, which determine the scaling dimensions of all quasiparticle operators, and charge-transfer operators. Charge-transfer operators move charge between the different edge branches, and as such it is also the mechanism through which different edge branches equilibrate. Charge-transfer operators violate no symmetry and are allowed to appear in the action. If a charge-transfer operator has a relevant scaling dimension, the condensation of this operator can lead to a different phase.
The action for this system is given by

$$S = \frac{1}{4\pi} \int dxdt \left[ iK_{ij} \partial_x \phi_i \partial_t \phi_j + V_{ij} \partial_x \phi_i \partial_x \phi_j \right. $$

$$\left. + \lambda \left( v_\lambda \partial_x - \partial_t \right) \lambda \right].$$

The generic quasiparticle operator has a form of a bosonic vertex operator $e^{i\phi}$ times a Majorana operator. Majorana (CFT primary field) operators are the identity operator $\mathbb{1}_\lambda$, the Majorana fermion operator $\lambda$ and the spin operator $\sigma_\lambda$. The charge and the bosonic contribution to the (mutual) statistics of such a quasiparticle operator can be determined from the inverse of the $K$-matrix:

$$\theta = \pi \mathbb{1}_\lambda K^{-1}, \quad q = \mathbb{1}_\lambda K^{-1}, \quad \theta_{jk} = \pi \mathbb{1}_j K^{-1} \mathbb{1}_k,$$

where $\theta$ is the statistical phase, $q$ is the charge, $\mathbb{1} = (1, 1, \ldots, 1)$ is the so-called charge vector and unit of charge is $e = 1$. The Majorana branch is charge-neutral and commutes with the bosonic branches. Its contribution to mutual statistics is

$$\frac{1}{\pi} \theta_{\lambda\lambda} = \pm 1, \quad \frac{1}{\pi} \theta_{\lambda\sigma} = \pm \frac{1}{2},$$

where we fix the sign to be $+1$ for right-moving branches and $-1$ for left-moving ones.

The quasiparticle spectrum is obtained by first identifying physical electron operators, which have charge $e$ and fermionic statistics. For the anti-Pfaffian state we are considering here, the physical electron operators are $e_1 = e^{i\phi_1}, e_2 = e^{i\phi_2}, e_3 = e^{i\phi_3}, e_4 = \lambda e^{-2i\phi_4}$, and any combination of these $e_i$ with total charge $e$.

The remainder of the quasiparticle spectrum is formed by those quasiparticle operators that are local with respect to all these electron operators, i.e., the phase induced by moving a quasiparticle around any electron operators should be a multiple of $2\pi$. The allowed quasiparticles can straightforwardly be found from these rules and are listed for convenience in Table II.

### Table II: Allowed quasiparticles in the $\nu = \frac{2}{3}$ anti-Pfaffian

| $\lambda$-sector | $q$ | $\phi_4$ |
|-------------------|-----|---------|
| $\mathbb{1}_\lambda$ | $m_1 + m_2 + m_3 - \frac{m_4}{2}$ | $e^{im_4\phi_4}$ |
| $\lambda$ | $m_1 + m_2 + m_3 - \frac{m_4}{2}$ | $e^{im_4\phi_4}$ |
| $\sigma_\lambda$ | $\frac{1}{4} + m_1 + m_2 + m_3 - \frac{m_4}{2}$ | $e^{im_4 - \frac{1}{2}i\phi_4}$ |

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### C. Calculating scaling dimension of quasiparticle operators, boost parameters

For the matrices $K$ and $V$ in the action, Eq. (5), there exist a (non-orthogonal) basis $\tilde{\phi}$ such that $K$ is a pseudo-identity and $V$ is diagonal. In such a basis, the scaling dimension $\Delta$ of a quasiparticle operator $e^{i\tilde{\phi}}$ would be given by $\Delta = \frac{1}{2} |V|$. In general, with $K$ given and fixed, the scaling dimension of quasiparticle operators thus depends on the precise form of the 4-by-4 matrix $V$ in the basis $\phi$.

A parametrization of the most generic density-density interaction $V$ requires ten real parameters. With a suitable choice of parameters the scaling dimension depends on only three of these parameters. This goes as follows.

Let $M_1$ be the matrix that brings $K$ into pseudo-identity form,

$$\tilde{K} = K^T M_1^T M_1 = -\mathbb{1}_{N^-} \oplus \mathbb{1}_{N^+},$$

where $N^\pm$ is the number of positive/negative eigenvalues of $K$, in our case $N^- = 1, N^+ = 3$. Next we diagonalize $V$ with a matrix $M_2 \in SO(N^-, N^+)$,

$$\tilde{V} = M_2^T M_1^T V M_1 M_2 = \text{Diag}(v_4, v_1, v_2, v_3).$$

Note that $M_2$ leaves $\tilde{K}$ invariant. Furthermore, $M_2$ can be decomposed in a pure boost $B$ and a pure rotation $R \in SO(N^-) \times SO(N^+)$, $M_2 = BR$.  

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2 For purposes of finding the quasiparticle spectrum we do not need to consider mutual statistics of two $\sigma$ operators.
In our case we use\footnote{The extra rotation incorporated in our \(M_1\) is added for later convenience.}

\[
M_1 = \begin{pmatrix}
\sqrt{\frac{1}{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{pmatrix},
\] (10)

and \(B \in SO(1,3)\) is the familiar pure boost from the Lorentz-group,

\[
B = \begin{pmatrix}
\gamma & \frac{\beta_1 \gamma}{\sqrt{1+\beta_1^2}} + 1 & \frac{\beta_2 \gamma}{\sqrt{1+\beta_2^2}} + 1 & \frac{\beta_3 \gamma}{\sqrt{1+\beta_3^2}} + 1 \\
\frac{\beta_1 \gamma}{\sqrt{1+\beta_1^2}} + 1 & \frac{\beta_1 \beta_2 \gamma^2}{\sqrt{1+\beta_1^2+\beta_2^2}} + 1 & \frac{\beta_2 \beta_3 \gamma^2}{\sqrt{1+\beta_2^2+\beta_3^2}} + 1 & \frac{\beta_1 \beta_3 \gamma^2}{\sqrt{1+\beta_1^2+\beta_3^2}} + 1 \\
\frac{\beta_2 \gamma}{\sqrt{1+\beta_2^2}} + 1 & \frac{\beta_2 \beta_3 \gamma^2}{\sqrt{1+\beta_2^2+\beta_3^2}} + 1 & \frac{\beta_2 \gamma^2}{\sqrt{1+\beta_2^2}} + 1 & \frac{\beta_2 \beta_3 \gamma^2}{\sqrt{1+\beta_2^2+\beta_3^2}} + 1 \\
\frac{\beta_3 \gamma}{\sqrt{1+\beta_3^2}} + 1 & \frac{\beta_1 \beta_3 \gamma^2}{\sqrt{1+\beta_1^2+\beta_3^2}} + 1 & \frac{\beta_1 \beta_3 \gamma^2}{\sqrt{1+\beta_1^2+\beta_3^2}} + 1 & \frac{\beta_3 \gamma^2}{\sqrt{1+\beta_3^2}} + 1
\end{pmatrix},
\] (11)

with \(\gamma = 1/\sqrt{1-(\beta_1^2 + \beta_2^2 + \beta_3^2)}\). An explicit specification of \(R\) is not required at this point.

What is important from this decomposition is that the scaling dimension of operator \(e^{i\vec{\beta} \cdot \vec{\phi}} = e^{i\vec{\beta} \cdot \vec{\phi}}\) is now given by \(\Delta = \frac{1}{2} \vec{\beta} \cdot \vec{\beta}\), where

\[
\tilde{\Delta} = M_1 B^2 M_1^T.
\] (12)

In our case, \(V\) is parametrized by four (eigenvalues \(v_j\)) plus three (rotation \(R \in SO(3)\)) plus three (boost parameters \(\beta_j\)) equalling a total of ten parameters. But the scaling dimension depends only on the three boost parameters \(\beta_j\); one parameter for each pair of counter-propagating edge modes. The scaling dimension in the bosonic sector of any quasiparticle operator thus becomes a function of a vector \(\beta = (\beta_1, \beta_2, \beta_3)\) inside the unit 3D-ball \(\beta^2 < 1\) (cf. \(|v| < c\)).

There is also a contribution to the scaling dimension from the Majorana sector,

\[
\Delta_\lambda = \frac{1}{2}, \quad \Delta_\sigma = \frac{1}{16},
\] (13)

which simply needs to be added to the bosonic scaling dimension of a quasiparticle operator to obtain the total scaling dimension. Here we note that the density-density interactions between the Majorana sector and the boson sectors are always irrelevant and we ignore those interactions in our calculations of scaling dimensions. We will consider other interactions between the Majorana sector and the boson sectors below.

D. Majorana mode becomes gapped through ‘null’ charge-transfer operator

Now that we know how to calculate scaling dimensions, we can probe \(\vec{\beta}\)-space for charge-transfer operators with low scaling dimension. Charge-transfer operators are total charge zero operators, which typically move electrons between the different branches. Since they violate no symmetry or conservation, charge-transfer operators are in principle allowed to appear in the action, Eq. (5), and if relevant (in RG sense) can cause a transition to another phase.

In \(\vec{\beta}\)-parameter-space surfaces of constant scaling dimension typically are of ellipsoidal shape, for example

\[
\Delta_{+\epsilon_2-\epsilon_3} = 1 + \frac{2 \beta_1^2}{1 - \beta_2^2}, \quad \Delta_{+\epsilon_1-\epsilon_4} = 2 \left(\frac{\beta_2 + \frac{\beta_3}{\sqrt{2}} + \sqrt{3}}{3(1 - \beta^2)}\right)^2,
\] (14)

where \(+\epsilon_i - \epsilon_j\) stands for the combination of an \(\epsilon_i\) creation and an \(\epsilon_j\) destruction operator, which transfers charge \(e\) between branches \(i\) and \(j\).

Regions inside parameter space where charge-transfer operators are relevant (\(\Delta < 2\)) are thus ellipsoids inside the unit ball \(\beta^2 < 1\).

One charge transfer operator we are particularly interested in is \(\lambda \epsilon_i (\epsilon_i + \phi_1 + \phi_2 + \phi_3) \equiv \hat{n}\), that is, the operator which simultaneously destroys \(\epsilon_1\) and \(\epsilon_4\) and creates \(\epsilon_2\) and \(\epsilon_3\). Its scaling dimension is

\[
\Delta_{-\epsilon_1-\epsilon_4+\epsilon_2+\epsilon_3} = 1 + \frac{3(\sqrt{\frac{2}{3}} - \beta_3)^2}{1 - \beta^2}.
\] (15)

There is a whole disc in \(\vec{\beta}\)-space for which the scaling dimension of \(\hat{n}\) is identically 1, namely when \(\beta_3 = \sqrt{2/3}\) (with \(\beta_1^2 + \beta_2^2 < 1/3\)).

On this disc the bosonic part of the operator \(\hat{n}\) has the scaling dimensions and statistics of a charge-neutral right-moving complex fermion, which we write as a combination of two Majorana fermions \(\eta\) and \(\zeta\), such that \(\hat{n} = \lambda(\eta + i\zeta)\). Note that \(\hat{n}\) resembles the ‘neutral null vector’ from the \(\nu = \frac{5}{2}\) FQH case, as in it is a zero-charge operator with equal left and right conformal dimensions, \(h = h\).

We are now approaching the step where the Majorana mode acquires a gap. Clearly this is a key ingredient in our procedure. But the argument of gaining itself is almost trivial: consider a system with two counter-propagating fermions \(\psi_1\) and \(\psi_2\), with dispersion relation \(E_{1/2}(k) = \pm vk\), which has gapless excitations at zero energy; adding a coupling \(\sum_x \Gamma \psi_1(x)\psi_2(x) + \text{H.c.}\) changes the dispersion to \(E_k(x) = \sqrt{\Gamma^2 + (vk)^2}\) and opens up a gap at zero energy.

Let us assume the interaction \(V\) is such that the operator \(\hat{n}\) is relevant, and include \(\hat{n}\) and its Hermitian conjugate in the action Eq. (5) with a constant coupling \(\Gamma\), i.e., we are not considering disorder at this point. Then the effect of this term... \(\Gamma(\hat{n} + \hat{n}^\dagger) = 2\Gamma \lambda n\), is that the counterpropagating Majorana modes \(\lambda\) and \(\eta\) become gapped whereas \(\zeta\) is left untouched. In other words the left-moving Majorana mode and a right-moving bosonic mode disappear and a right-moving Majorana mode emerges.
FIG. 1: Different representations of the gapless branches in the anti-Pfaffian and Majorana-gapped phases. (a) In the basis \( \hat{\phi} \) the anti-Pfaffian has three right-moving bosonic branches (solid lines) and two left-moving modes, one bosonic and one Majorana mode (dotted line). (b) After a basistransformation to basis \( \hat{\phi} \) the right-moving charge-neutral mode \( \hat{\phi}_3 \) can be expressed as two right-moving Majorana modes \( \eta \) and \( \phi \). (c) In the Majorana-gapped phase, the modes \( \lambda \) and \( \eta \) acquire a gap and are dropped. (d) An additional basistransformation explicitly separates the charge mode, \( \tilde{\phi}_\nu \) (double solid line), from the neutral modes (\( \phi_{\sigma, L/R} \)).

![Figure 1](image)

FIG. 2: Shown is the cross-section, at \( \beta_4 = 0 \), of the parameter space unit ball \( |\beta|^2 < 1 \). The scaling dimension of the ‘null’ operator \( \hat{\eta} \), the charge-transfer operator which induces the transition to the Majorana-gapped phase, is identically one at the disc \( \beta_3 = \sqrt{2/3} \approx 0.82 \) (indicated by thick black line). This null operator \( \hat{\eta} \) is relevant in a substantial volume of \( \beta \) parameter space; the grey area indicates the ellipsoidal volume where \( \hat{\eta} \) had scaling dimension \( \Delta \leq 3/2 \), i.e., the region where \( \hat{\eta} \) is relevant even in the presence of disorder.

Figure 2 shows the volume of parameter space in which \( \hat{\eta} \) is a relevant operator. A schematic representation of the branches and different bases before and after Majorana-gapping is given in Fig. 1.

Before we continue to determine the quasiparticle spectrum in the Majorana-gapped system we would like to note that the operator \( \hat{\eta} \) is not unique; due to the permutation symmetry between branches \( \hat{\phi}_1, \hat{\phi}_2 \) and \( \hat{\phi}_3 \) there is a total of three such operators \( \hat{\eta} \).

E. Quasiparticle spectrum in gapped system

With the gapping of the \( \lambda \) Majorana fermion, some of the quasiparticle operators in the original spectrum, Table II, have likely developed a gap as well and have disappeared from the low-energy spectrum. For the Majorana-gapped phase we would like to find out (i) which quasiparticles have survived the gapping and (ii) which of these survivors have the lowest scaling dimension and are thus expected to dominate e.g. tunneling processes.

To obtain the quasiparticle spectrum we follow the same procedure as for any other \( \chi \)LL FQH edge system: we identify the physical electron operators and determine those quasiparticles which are single-valued with respect to the electron operators. The non-standard part is how to remove the degree of freedom associated with the now-gapped \( \lambda \) Majorana fermion and insert the now-emerged \( \zeta \) Majorana fermion.

Setting \( \beta_3 = \sqrt{2/3} \), the following steps will find the quasiparticle spectrum for arbitrary \( \beta_1 \) and \( \beta_2 \) \((\beta_1^2 + \beta_2^2 < 1/3)\). To illustrate the procedure we will use the example where \( \beta_1 = 0 = \beta_2 \), for which the intermediate basis-dependent values are relatively simple.

We transform the basis from \( \hat{\phi} \) to \( \hat{\tilde{\phi}} \) such that \( \tilde{K} \) is the pseudo-identity, \( \tilde{V} \) is diagonal, and \( \tilde{n} = \lambda e^{i\tilde{\phi}_3} \), which is achieved by \( M_1 \) and \( B \). Eqs. (11) and (11), and \( R = \text{Diag}(1,1,-1,-1) \). As far as the electron operators go, \( e_4 \) contains a \( \lambda \) operator and hence becomes gapped, so we drop \( e_4 \) from the spectrum. The three remaining physical electron operators have the following form in the basis \( \hat{\tilde{\phi}} \),

\[
e_1 = e^{i(-\sqrt{2} \hat{\tilde{\phi}}_1 + \sqrt{2} \hat{\tilde{\phi}}_2 - \hat{\tilde{\phi}}_3)},
\]
\[
e_2 = e^{i(-\sqrt{2} \hat{\tilde{\phi}}_1 + \sqrt{2} \hat{\tilde{\phi}}_3 - \hat{\tilde{\phi}}_2)},
\]
\[
e_3 = e^{i(-\sqrt{2} \hat{\tilde{\phi}}_2 + \sqrt{2} \hat{\tilde{\phi}}_3 - \hat{\tilde{\phi}}_1)}.
\]

For the operator \( e^{i\tilde{\phi}_3} = \eta + i\zeta \) we expect that gapping will get rid of the \( \eta \)-part and effectively leave a Majorana operator \( \zeta \) times some overall phase: \( e^{i\tilde{\phi}_3} \rightarrow \zeta \). Note that this includes all three electron operators for which we thus make the identification

\[
e_1 \simeq \zeta e^{i(-\sqrt{2} \hat{\tilde{\phi}}_1 + \sqrt{2} \hat{\tilde{\phi}}_2)},
\]
\[
e_2 \simeq \zeta e^{i(-\sqrt{2} \hat{\tilde{\phi}}_1 - \sqrt{2} \hat{\tilde{\phi}}_3 - \hat{\tilde{\phi}}_2)},
\]
\[
e_3 \simeq \zeta e^{i(-\sqrt{2} \hat{\tilde{\phi}}_2 - \sqrt{2} \hat{\tilde{\phi}}_3 - \hat{\tilde{\phi}}_1)}.
\]
TABLE III: Quasiparticle spectrum in the Majorana-gapped phase. Quasiparticles are identified by three integers \( m_1, m_2, \) and \( m_3 \) and a \( \zeta \)-Majorana sector. The corresponding vertex operator are easiest expressed in the ungapped basis \( \phi \), where the \( \phi_1, \phi_2 \) and \( \phi_3 \) contributions are still \( e^{i(m_1\phi_1+m_2\phi_2+m_3\phi_3)} \). The three \( m_i \) and the Majorana sector fix the coefficient of \( \phi_3 \), as shown in the table. The quasiparticle charge \( q \) is listed, as well as a correction to the scaling dimension, \( \Delta_{\text{cor}} \), as explained in the text.

| \( \zeta \)-sector | \( \phi_3 \) | \( q \) | \( \Delta_{\text{cor}} \) |
|---|---|---|---|
| \( I_\zeta \) | \( e^{i(-m_1+m_2+m_3)\phi_3} \) | \( \frac{m_1+\frac{m_2}{2}+\frac{m_3}{2}}{2} \) | 0 |
| \( \zeta \) | \( e^{i(-m_1+m_2+m_3-\frac{1}{2})\phi_3} \) | \( \frac{1}{2}+\frac{m_1}{2}+\frac{m_2}{2}+\frac{m_3}{2} \) | 0 |
| \( \sigma_\zeta \) | \( e^{i(-m_1+m_2+m_3-\frac{1}{4})\phi_3} \) | \( \frac{1}{4}+\frac{m_1}{2}+\frac{m_2}{2}+\frac{m_3}{2} \) | \( \frac{1}{16} \) |

Next, we look for quasiparticles which are single-valued with respect to these three electron operators, with the generic form \( e^{i(\tilde{l}_1\phi_1+\tilde{l}_2\phi_2+\tilde{l}_3\phi_3)} \) times a \( \zeta \)-sector Majorana operator \( I_\zeta, \zeta, \) or \( \sigma_\zeta \).

Solving for allowed \( \tilde{l}_1, \tilde{l}_2 \) and \( \tilde{l}_3 \) now is a computationally trivial task of solving a set of three linear equations given by the mutual statistics equations, Eqs. (6) and (7).

The resulting expressions for the \( \tilde{l}_i \) will involve various square roots, which tend to become more ugly for generic values for \( \beta_1 \) and \( \beta_2 \). However, the mutual statistics equations are basis-invariant, and hence can be solved in any basis. As it turns out, these equations become really simple in the original \( \phi \) basis.

Making the identifications \( \zeta \simeq e^{i\frac{\pi}{4}\phi_3} \) and \( \sigma_\zeta \simeq e^{i\frac{3\pi}{4}\phi_3} \) we can transform back to the original basis \( \phi \). In this basis, the generic quasiparticle operator has the form \( e^{i(l_1\phi_1+l_2\phi_2+l_3\phi_1)} \). Single-valuedness with the three electron operators forces \( l_1, l_2, \) and \( l_3 \) to be integers \( m_i \); \( l_4 \) is determined by the constraint

\[
-l_4 - l_1 + l_2 + l_3 = \begin{cases} 
0 & \text{for } I_\zeta \text{-sector}, \\
1 & \text{for } \zeta \text{-sector}, \\
\frac{1}{2} & \text{for } \sigma_\zeta \text{-sector},
\end{cases}
\]

which will assure the appropriate coefficient of \( \phi_3 \) for each Majorana sector in the \( \phi \) basis.

So the result is that the quasiparticle spectrum in the Majorana-gapped phase can be labelled by three integers \( m_1, m_2, \) and \( m_3, \) and a Majorana \( \zeta \)-sector, as shown in Table III. Note that these expressions are independent of \( \beta_1 \) and \( \beta_2 \). The charge is given by \( q = l_1 + l_2 + l_3 - \frac{l_4}{4} \).

Even the scaling dimension can be calculated in the \( \phi \) basis for all \( \beta_1 \) and \( \beta_2 \), however, in the \( \sigma_\zeta \) sector the calculation \( \frac{\Delta}{2} \) would assign a scaling dimension of \( \frac{3}{2} \) to the \( \sigma \)-operator; we know this should be \( \frac{1}{4} \) for a Majorana \( \sigma \) operator, and so scaling dimension calculations need to be corrected for this. This stems from the identification \( \sigma_\zeta \simeq e^{i\frac{\pi}{4}\phi_3} \), which gives the correct statistics and is valid for the purpose of enumerating the quasiparticle spectrum, but may not be true as operator equality.

F. Dominant quasiparticles in gapped system, charge separation

Having determined the quasiparticle spectrum, we now look for the most dominant quasiparticles.

As far as (non-)universality goes, the gapping of the pair of Majorana modes has removed one pair of counterpropagating bosonic modes from the system, and with it one boost-parameter. Two counterpropagating pairs remain with corresponding boost parameters \( \beta_1 \) and \( \beta_2 \).

And so in principle we now have to repeat our procedure of looking for dominant charge-transfer operators on the disc \( \beta_1^2 + \beta_2^2 < 1/3 \). However, so far we have considered the most general density-density interaction \( V \). We expect the interaction to show traces of the underlying Coulomb interaction; especially, we expect that there will be a single charge mode which will separate itself from the other (neutral) modes.

Here we will consider the limit where the charged mode is completely separated from the neutral modes. This decouples one of the right-moving bosonic modes from the left-moving one and eliminates one boost parameter. The condition for charge-separation is \( \beta_2 = (\sqrt{2}\beta_1 - \sqrt{3})/4 = -1/(4\sqrt{3}) \). The one remaining boost parameter is \( \beta_1 \), with \( |\beta_1| < \sqrt{5}/4 = \sqrt{1 - \beta_2^2 - \beta_3^2} \).

So we continue our analysis of scaling dimensions of operators on the line \( \beta_1 \). A plot of scaling dimensions for several quasiparticle operators is given in Fig 3. Upon closer inspection though, there is some regularity in the spectrum. For instance, charge-transfer operators can have a minimal scaling dimension of one, which is obtained for \( \beta_1 = 0, \pm \frac{1}{4}, \pm \frac{5}{12}, \pm \frac{1}{7}, \pm \frac{15}{28}, \ldots \) which appears to form an on-going series, and in between such points the same ‘spectrum’ of scaling dimensions is repeated.

So it seems we only need to consider the interval \( 0 \leq \beta_1 \leq 1/4 \). At \( \beta_1 = 0 \) the most dominant quasiparticle operator is a charge \( q = 1/4 \) \( \sigma_\zeta \)-sector operator, with scaling dimension \( \Delta = 0.275 \). Upon increasing \( \beta_1 \) the scaling dimension increases monotonically to a value of \( \Delta = 0.375 \) at \( \beta_1 = 1/4 \). At \( \beta_1 = 1/4 \) the quasiparticle operator with the lowest scaling dimension is a charge \( q = 1/2 \) \( I_\zeta \)-sector operator with \( \Delta = 0.25 \). Its scaling dimension increases monotonically in the opposite direction, reaching a maximum at \( \beta_1 = 0 \) of \( \Delta = 0.35 \).

It is tempting to suggest that the charge-transfer operator with the smallest scaling dimension will dominate and fix the system to be either at the \( \beta_1 = 0 \) or at the \( \beta_1 = 1/4 \) point. However, since both charge-transfer operators have scaling dimension between 1 and 3/2 on the interval, they are both relevant, and it depends on the strength of the coefficient if one dominates over the other. Similarly, in our analysis we cannot single out a most dominant quasiparticle, it is simply too close to call. In that sense we find the Majorana-gapped charge-separated phase to be non-universal: there are two dominant quasiparticles, with charges of 1/4 and 1/2 and scaling dimensions ranging between 0.25 and 0.375.
parameter particles in the Majorana-gapped phase as function of boost parameter \(\beta_1\). Plotted are charge-transfer operators with scaling dimension \(\Delta < 3/2\) (dashed blue lines), electron operator with lowest scaling dimension (solid green line), and two operators with lowest scaling dimension of all operators: a charge \(q = 1/4\) quasiparticle (dashed-dotted red line) and a charge \(q = 1/2\) quasiparticle (solid blue line). Notice that there is a pattern which repeats itself for \(\beta_1 > 0.25\), turning into a series with shrinking width as \(\beta_1\) approaches its maximum allowed value \(\beta_1 = \sqrt{5}/4 \approx 0.56\). Note that there are many quasiparticle operators with scaling dimensions smaller than 1.5 that we did not include on this graph.

### G. Only strong interaction leads to Majorana-gapped phase

Having identified the Majorana-gapped phase, we can ask what kind of interaction will lead to such a phase. In the Majorana-gapped charge-separated phase the interaction is characterized by 5 remaining parameters: if we pick \(\phi_2\) as the charged mode these are the three \(V\) eigenvalues \(v_4, v_1\) and \(v_3\), an angle \(\alpha\) for rotations between branches \(\phi_1\) and \(\phi_3\), and the boost parameters \(\beta_1\).

A crucial ingredient for the Majorana-gapped phase is to include the two filled (lowest) Landau level modes. If these two modes are spatially well-separated on the edge from the inner two modes one would expect the interactions to be small between these blocks of edge branches. We find that the interaction required for the Majorana-gapped phase is such that this kind of separation of two modes is not possible: all right-moving branches have to interact with the left-moving branch with similar strength.

### H. Disorder: localization instead of gapping

The assumption we made so far was that the charge-transfer coupling strength \(\Gamma\) was uniform along the edge. A more realistic assumption would be to consider \(\Gamma = \Gamma(x)\) to be fluctuating with position due to random disorder. Also, with disorder we do not need to worry about momentum mismatch between different edge modes.

With disorder present, we expect instead of the gapping of the pair of left and right moving Majorana modes, \(\lambda\) and \(\eta\), that they will become localized. Here we will assume that the localized modes do not contribute to the tunneling between edges. In particular, they do not affect the value of exponent \(g\). So as long as the calculation of \(g\) is concerned, we treat the localized modes as if they are gapped. Thus the above calculation of \(g\) also applies to the disordered edge with localization.

### IV. MAJORANA-GAPPED PHASE OF THE EDGE-RECONSTRUCTED PFAFFIAN STATE

We now apply the same mechanism of Majorana gapping on a different state: the edge-reconstructed Pfaffian state. By itself the Pfaffian state is fully chiral and gapping of pairs of counterpropagating modes cannot occur. However, the edge might be unstable towards edge reconstruction. Edge reconstruction effectively adds pairs of counterpropagating charged bosonic modes to the edge. Here we will analyze the state in which edge reconstruction has introduced one such pair of edge modes to the Pfaffian state.

In the edge-reconstructed Pfaffian there are three bosonic modes \(\phi_1, \phi_2\) and \(\phi_3\), and one neutral Majorana mode \(\lambda\). The left-moving branch is \(\phi_1\), the other branches are right-moving. The \(K\)-matrix is \(K = \text{Diag}(-1, 1, 2)\). Electron operators are \(e_1 = e^{i\phi_1}, e_2 = e^{i\phi_2}\) and \(e_3 = \lambda e^{-2i\phi_3}\).

The ‘null’ operator \(\lambda e^{i(2\phi_1 + \phi_2 + 2\phi_3)}\) is a charge-transfer operator with equal left and right conformal dimensions \(h = \tilde{h} = \frac{1}{4}\). Introducing boost parameters \(\tilde{\beta} = (\beta_1, \beta_2)\), similar to Eq. (11), we can parametrize scaling dimensions of quasiparticle operators. The scaling dimension of \(\lambda e^{i(2\phi_1 + \phi_2 + 2\phi_3)}\) becomes one at the point \(\beta_1 = -1/2, \beta_2 = -1/\sqrt{2}\).

We perform a basis transformation from \(\phi\) to \(\tilde{\phi}\); in this basis \(K = \text{Diag}(-1, 1, 1)\), \(\lambda e^{i(2\phi_1 + \phi_2 + 2\phi_3)} = \lambda e^{i\phi_1}\), and the \(\tilde{\phi}_2\) branch carries all the charge. If the null operator is relevant it will gap the right-moving Majorana mode \(\lambda\) and half of the left-moving bosonic mode \(\phi_1\) leaving a left-moving Majorana mode \(\zeta\).

In the Majorana-gapped phase, the gapless physical electron operators are \(e_1 = \zeta e^{i(\sqrt{2}\tilde{\phi}_2)}\) and \(e_2 = e^{i(\sqrt{2}\tilde{\phi}_2 + \tilde{\phi}_3)}\); \(e_3\) acquires a gap. The quasiparticle spectrum can be labelled by two integers \(m_1\) and \(m_2\) and the \(\zeta\) Majorana sector, as follows, with charge and scaling dimension included:

\[
\mathbb{1}_\zeta\text{-sector: } e^{i\frac{m_1}{\sqrt{2}}\tilde{\phi}_2 + (m_2 - m_1)\tilde{\phi}_3}, \quad q = \frac{m_1}{2}, \quad \Delta = \frac{m_1^2}{4} + \frac{(m_2 - m_1)^2}{2}, \quad (23)
\]
\[ \zeta \text{-sector: } \zeta e^{i \frac{m_1}{4} \phi_2 + (m_2 - m_1) \phi_3}, \quad q = \frac{m_1}{2}, \quad \Delta = \frac{m_1^2}{4} + \frac{(m_2 - m_1)^2}{2} + \frac{1}{2}, \quad (24) \]

\[ \sigma \zeta \text{-sector: } \sigma \zeta e^{i \frac{m_1 + 1}{4} \phi_2 + (m_2 - m_1 + \frac{1}{2}) \phi_3}, \quad q = \frac{m_1 + 1}{2}, \quad \Delta = \frac{(m_1 + 1)^2}{4} + \frac{(m_2 - m_1 + \frac{1}{2})^2}{2}. \quad (25) \]

Note that here the Majorana-gapping effectively removes all pairs of counterpropagating bosonic modes that existed before. Hence the scaling dimensions of all operators becomes fixed. In other words, there is no remaining boost parameter degree of freedom.

In the Majorana-gapped edge-reconstructed there are three operators with smallest scaling dimension \( \Delta = \frac{1}{2} \): one charge \( q = 1/2 \) operator \((m_1 = m_2 = 1 \text{ in } \zeta \text{-sector})\), and two charge \( q = 1/4 \) operators \((m_1 = 0, m_2 = 0, 1 \text{ in the } \sigma \zeta \text{-sector})\). The electron operator with smallest scaling dimensions has \( \Delta = \frac{1}{2} \).

V. DISCUSSION

A. Tunneling through bulk in new edge phase

To detect the phase transition to the Majorana-gapped phase on the edge of the anti-Pfaffian state one would have to observe a change in quasiparticle tunneling exponent \( g \). This presents a dilemma: even though \( g \) itself is an intrinsic property of the edge, a measurement of \( g \) requires the quasiparticle to tunnel through the bulk. But in the bulk the phase transition does not occur, so is it even possible for the quasiparticle to tunnel? We assume that edge quasiparticles in the Majorana-gapped phase can indeed tunnel through the bulk and we do not run into obvious inconsistencies (of e.g. having a quasiparticle charge on the edge which does not exist in the bulk). Whether or not this assumption is fully justified is not yet understood.

B. Charge transfer in the bulk

We like to stress that that operator \( \lambda e^{i (2 \phi_4 - \phi_1 + \phi_2 + \phi_3)} \) not only appears in the edge effective Hamiltonian, the corresponding operator also appears in the 2D bulk effective Hamiltonian for the 2D anti-Pfaffian state. Such a bulk operator transfers charges between different condensates (note that the anti-Pfaffian state is formed by several condensates: the spin-down electrons in the first Landau level, the spin-up electrons in the fast Landau level, the spin-up electrons in the second Landau level, etc.). With such an operator present in the bulk Hamiltonian, one naturally expects that the 2D anti-Pfaffian state has \( 1_L + (5/2)_R \) branches of the edge excitations. The \((3/2)_L + 3_R \) branches of the edge excitations proposed in Refs. [10,11] can be viewed as a result of edge reconstruction of the \( 1_L + (5/2)_R \) edge.

C. Effects of spin conservation

So far we have ignored the effect of spin conservation. In the presence of magnetic field, the \( z \)-component of spin \( S_z \) is still conserved. By examine the spin quantum number of the charge transfer operator \( \lambda e^{i (2 \phi_4 - \phi_1 + \phi_2 + \phi_3)} \) in the anti-Pfaffian state, we find that it carries \( S_z = 1 \). Therefore, the \( S_z \) conservation prevents \( \lambda e^{i (2 \phi_4 - \phi_1 + \phi_2 + \phi_3)} \) from appearing in the edge Hamiltonian. In this case the Majorana-gapped phase for the anti-Pfaffian state cannot appear. Thus to have the Majorana-gapped phase for the anti-Pfaffian state we either need to break the \( S_z \) conservation, or to consider the \( \nu = 9/2 \) anti-Pfaffian state where there exists a charge transfer operator which carries \( S_z = 0 \).

The charge transfer operator for the edge reconstructed Pfaffian state has \( S_z = 0 \). Thus the \( S_z \) conservation will not prevent the appearance of the Majorana-gapped phase. The Majorana-gapped phase is more likely to appear for edge reconstructed Pfaffian state.

D. Determining the true nature of the \( \nu = \frac{5}{2} \) state

Glancing at Table [1] we have to conclude that the quest to determine the nature of the observed \( \nu = \frac{5}{2} \) FQH state is far from over. The first experimental results\cite{17,18} suggest that likely \( e^* = 1/4 \) and \( g = 0.5 \). If these are confirmed to be the correct values we can scratch a few candidates off the list, but we would not be able to distinguish between the anti-Pfaffian, \( U(1) \times SU_2(2) \) and Majorana-gapped edge-reconstructed Pfaffian states. Electron tunneling is expected to be the same for these three states as well.

Additional measurements which would probe the number of left- and right-moving edge branches would be required to settle this issue. For instance a thermal Hall conductance\cite{19} measurement distinguishes between the three states with \( e^* = 1/4 \) and \( g = 0.5 \). As far as the presence of non-abelian statistics goes the prospect is somewhat brighter, as five out of the seven candidate states are non-abelian. Furthermore the non-abelian statistics is carried by similar Ising spin fields in all these cases, hence experimental setups based on interference should give qualitatively similar results.

E. Summary

In the paper, we studied the effect of a charge transfer process described by neutral bosonic operators in the \( \nu = \frac{5}{2} \) anti-Pfaffian state and edge reconstructed Pfaffian state. On the edge, such operators have a form \( \lambda e^{i (2 \phi_4 - \phi_1 + \phi_2 + \phi_3)} \) or \( \lambda e^{i (2 \phi_1 + \phi_2 + 2 \phi_3)} \). Such operators transfer charges between edge branches and create/annihilate a Majorana fermion \( \lambda \). The operator respects all the symmetries and is local with respect to all the electron operators. Thus such an operator is allowed.
in the effective edge Hamiltonian. We find that, for a certain range of interactions between the edge branches, the operators \( \lambda e^{i(2\phi_1-\phi_3+\phi_1+\phi_3)} \) or \( \lambda e^{i(2\phi_1+\phi_2+2\phi_3)} \) represent relevant perturbations. The effect of such a relevant perturbation opens up a gap for a pair of left and right moving Majorana fermion modes.

For the anti-Pfaffian state, before the 1D gapping transition at the edge, the state has 3/2 branches of left-movers and 3 branches of right-movers. After the gapping transition, the same 2D anti-Pfaffian state has 1 branch of left-movers and 2 and 1/2 branches of right-movers. For the edge reconstructed Pfaffian state, before the 1D gapping transition at the edge, the state has 1 branches of left-movers and 2 and 1/2 branches of right-movers. After the gapping transition, the same state has 1/2 branch of left-movers and 2 branches of right-movers.

The phase transition changes the scaling dimension of quasiparticle operators on the edge, which can in principle be observed in experiment. For FQH edge states with counterpropagating edge modes it was known that interactions between the edge branches have to be taken into account to determine the phase of the edge. It was previously shown that under certain conditions a full left- and right-moving branch could pair up and open up a gap. The research is partially supported by NSF Grant No. DMR-0706078.

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APPENDIX: THE \( U(1) \times SU_2(2) \) EDGE STATE

The edge excitations of the \( U(1) \times SU_2(2) \) state is described by a charge density mode \( \rho(x) \), an \( S^z \) density mode \( \bar{\rho}(x) \), plus a Majorana fermion \( \lambda(x) \):

\[
[\rho_{k}, \bar{\rho}_{k'}] = \frac{\nu}{2\pi} k\delta_{k+k'}, \quad \nu = \frac{1}{2}
\]

\[
[\bar{\rho}_{k}, \bar{\rho}_{k'}] = \frac{1}{2\pi} k\delta_{k+k'},
\]

\[
\{\lambda_{k}, \lambda_{k'}\} = \delta_{k+k'}, \quad \lambda_{k}^{\dagger} = \lambda_{-k}
\]

\[
H = 2\pi \sum_{k>0} [V \rho_{k} \bar{\rho}_{k} + \bar{V} \bar{\rho}_{k} \rho_{k} + \sum_{k>0} V_{\lambda} k \lambda_{-k} \lambda_{k}]
\]

There are three electron operators given by

\[
\Psi_{e,3}(x) = \lambda(x)e^{2i\phi(x)}
\]

\[
\Psi_{e,1}(x) = i\Psi_{e,2}(x) = e^{\pm i\phi(x)}e^{2i\phi(x)}
\]

The \( e/4 \) quasiparticle operators are given by

\[
\psi_{q,1} = \sigma(x)e^{i\frac{1}{2}\phi(x)}e^{\frac{1}{4}\phi(x)}
\]

\[
\psi_{q,2} = \sigma(x)e^{-i\frac{1}{2}\phi(x)}e^{\frac{1}{4}\phi(x)}
\]

We find that

\[
\langle \psi^{\dagger}(x,t)\psi(x',t') \rangle \sim (z-z')^{-g}
\]

with

\[
g = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}.
\]
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