THE $d$-DIMENSIONAL BOOTSTRAP PERCOLATION MODELS WITH THRESHOLD AT LEAST DOUBLE EXPONENTIAL

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Abstract. Consider a $p$-random subset $A$ of initially infected vertices in the discrete cube $[L]^d$, and assume that the neighbourhood of each vertex consists of the $a_i$ nearest neighbours in the $±e_i$-directions for each $i \in \{1, 2, \ldots, d\}$, where $a_1 \leq a_2 \leq \ldots \leq a_d$. Suppose we infect any healthy vertex $v \in [L]^d$ already having $r$ infected neighbours, and that infected sites remain infected forever. In this paper we determine the $(d-1)$-times iterated logarithm of the critical length for percolation up to a constant factor, for all $d$-tuples $(a_1, \ldots, a_d)$ and all $r \in \{a_2 + \cdots + a_d + 1, \ldots, a_1 + a_2 + \cdots + a_d\}$.

Moreover, we reduce the problem of determining this (coarse) threshold for all $d \geq 3$ and all $r \in \{a_d + 1, \ldots, a_1 + a_2 + \cdots + a_d\}$, to that of determining the threshold for all $d \geq 3$ and all $r \in \{a_d + 1, \ldots, a_{d-1} + a_d\}$.

1. Introduction

The study of bootstrap processes on graphs was initiated in 1979 by Chalupa, Leath and Reich [10], and is motivated by problems arising from statistical physics, such as the Glauber dynamics of the zero-temperature Ising model, and kinetically constrained spin models of the liquid-glass transition (see, e.g., [5][15][18–20]). The $r$-neighbour bootstrap process on a locally finite graph $G$ is a monotone cellular automata on the configuration space $\{0, 1\}^V(G)$, (we call vertices in state 1 “infected”), evolving in discrete time in the following way: 0 becomes 1 when it has at least $r$ neighbours in state 1, and infected vertices remain infected forever. Throughout this paper, $A$ denotes the initially infected set, and we write $[A] = G$ if the state of each vertex is eventually 1.

We will focus on anisotropic bootstrap models, which are $d$-dimensional analogues of a family of (two-dimensional) processes studied by Duminil-Copin, van Enter and Hulshof [11][12][14]. In these models the graph $G$ has vertex set $[L]^d$, and the neighbourhood of each vertex consists of the $a_i$ nearest neighbours in the $−e_i$ and $e_i$-directions for each $i \in [d]$, where $a_1 \leq \cdots \leq a_d$ and $e_i \in \mathbb{Z}^d$ denotes the $i$-th canonical unit vector. In other words, $u, v \in [L]^d$ are neighbours if (see Figure 1 for $d = 3$)

$$u - v \in N_{a_1, \ldots, a_d} := \{±e_1, \ldots, ±a_1e_1\} \cup \cdots \cup \{±e_d, \ldots, ±a_de_d\}. \quad (1)$$

We also call this process the $N_{a_1, \ldots, a_d}^{r}$-model. Our initially infected set $A$ is chosen according to the Bernoulli product measure $\mathbb{P}_p = \bigotimes_{v \in [L]^d} \text{Ber}(p)$, and we are interested in the so-called critical length for percolation, for small values of $p$

$$L_c(N_{r}^{a_1, \ldots, a_d}, p) := \min \{L \in \mathbb{N} : \mathbb{P}_p([A] = [L]^d) \geq 1/2\}. \quad (2)$$
The analysis of these bootstrap processes for \( a_1 = \cdots = a_d = 1 \) was initiated by Aizenman and Lebowitz \cite{1} in 1988, who determined the magnitude of the critical length up to a constant factor in the exponent for the \( N_2^{1,\cdots,1} \)-model (in other words, they determined the ‘metastability threshold’ for percolation). In the case \( d = 2 \), Holroyd \cite{16} determined (asymptotically, as \( p \to 0 \)) the constant in the exponent (this is usually called a sharp metastability threshold).

For the general \( N_1^{a_1,\cdots,1} \)-model with \( 2 \leq r \leq d \), the threshold was determined by Cerf and Cirillo \cite{8} and Cerf and Manzo \cite{9}, and the sharp threshold by Balogh, Bollobás and Morris \cite{3} and Balogh, Bollobás, Duminil-Copin and Morris \cite{2}; for all \( d \geq r \geq 2 \) there exists a computable constant \( \lambda(d, r) \) such that, as \( p \to 0 \),

\[
L_c(N_1^{a_1,\cdots,1}, p) = \exp((r-1)\left(\frac{\lambda(d, r) + o(1)}{p^{1/(d-r+1)}}\right)).
\]

The \( N_1^{a_1,a_2} \)-model is called isotropic when \( a_1 = a_2 \) and anisotropic when \( a_1 < a_2 \). Hulshof and van Enter \cite{14} determined the threshold for the first interesting anisotropic model given by the family \( N_3^{1,2} \), and the corresponding sharp threshold was determined by Duminil-Copin and van Enter \cite{11}.

The threshold was also determined in the general case \( r = a_1 + a_2 \) by van Enter and Fey \cite{13} and the proof can be extended to all \( a_2 + 1 \leq r \leq a_1 + a_2 \): as \( p \to 0 \),

\[
L_c(N_r^{a_1,a_2}, p) = \exp(\Theta(\lambda_{r-a_2}(p))),
\]

where for each \( i \in [a_1] \),

\[
\lambda_i(p) = \lambda_i(p, a_1, a_2) = \begin{cases} 
  p^{-i} & \text{if } a_2 = a_1, \\
  p^{-i} (\log p)^2 & \text{if } a_2 > a_1.
\end{cases}
\]

1.1. Anisotropic bootstrap percolation on \([L]^d\). In this paper we consider the \( d \)-dimensional analogue of the anisotropic bootstrap process studied by Duminil-Copin, van Enter and Hulshof. In dimension \( d = 3 \), we write \( a_1 = a, a_2 = b \) and \( a_3 = c \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The neighbourhood \( N_{a_1,a_2,a_3} \) with \( a_1 = 1, a_2 = 2 \) and \( a_3 = 4 \). The \( e_1 \)-axis is towards the reader, the \( e_2 \)-axis is vertical, and the \( e_3 \)-axis is horizontal.}
\end{figure}

These models were studied by van Enter and Fey \cite{13} for \( r = a + b + c \); they determined the following bounds on the critical length, as \( p \to 0 \),

\[
\log \log L_c(N_r^{a,b,c}, p) = \Theta(\lambda_a(p))
\]

Note that, by (5) the critical length is doubly exponential in \( p \) when \( r = a + b + c \). It is not difficult to show that the critical length is polynomial in \( p \) if \( r \leq c \).
On the other hand, we have shown in [4] that the critical length is singly exponential in the case \( r \in \{c + 1, \ldots, c + \delta\} \): as \( p \to 0 \),
\[
\Omega \left( p^{-1/2} \right) \leq \log L_c \left( \mathcal{N}^{a,b,c}_r, p \right) \leq O \left( p^{-b}(\log \frac{1}{p})^2 \right).
\]
(6)

We moreover determined the magnitude of the critical length up to a constant factor in the exponent in the cases \( r \in \{c + 1, c + 2\} \), for all triples \((a, b, c)\), except for \( r = c + 2 \) when \( c = a + b - 1 \) (see Section 6 in [4]): set \( s := r - c \in \{1, 2\} \), then, as \( p \to 0 \),
\[
\log L_c \left( \mathcal{N}^{a,b,c}_r, p \right) = \begin{cases}
\Theta \left( p^{-s/2} \right) & \text{if } c = b = a, \\
\Theta \left( p^{-s/2}(\log \frac{1}{p})^{1/2} \right) & \text{if } c = b > a, \\
\Theta \left( p^{-s/2}(\log \frac{1}{p})^{3/2} \right) & \text{if } c \in \{b + 1, \ldots, a + b - s\}, \\
\Theta \left( p^{-s} \right) & \text{if } c = a + b, \\
\Theta \left( p^{-s}(\log \frac{1}{p})^2 \right) & \text{if } c > a + b.
\end{cases}
\]
(7)

While we conjecture that \( \log L_c \left( \mathcal{N}^{a,b,a+b-1}_{a+b+1}, p \right) = \Theta \left( p^{-1}(\log \frac{1}{p})^2 \right) \).

In this paper we generalize (5) by showing that the critical length is doubly exponential in \( p \) for each \( r \in \{c+b+1, \ldots, c+b+a\} \). Indeed, we determine \( \log_{((d-1))} \left( L_c \left( \mathcal{N}^{a_1, \ldots, a_d}_{a_d+\ldots+a_2+i}, p \right) \right) \) up to a constant factor, for all dimensions \( d \geq 3 \) and every \( i \in [a_1] \).

The following is our main result.

**Theorem 1.1.** For each \( d \geq 3 \) and \( i \in [a_1] \), as \( p \to 0 \),
\[
L_c \left( \mathcal{N}^{a_1, \ldots, a_d}_{a_d+\ldots+a_2+i}, p \right) = \exp_{(d-1)}(\Theta_i(p)).
\]
(8)

The techniques in this paper can be used to reduce the general problem of determining \( L_c \left( \mathcal{N}^{a_1, \ldots, a_d}_r, p \right) \) (coarse threshold) for all \( d \geq 3 \) and all \( r \in \{a_d + 1, \ldots, a_1 + a_2 + \cdots + a_d\} \), to that of determining \( L_c \left( \mathcal{N}^{a_1, \ldots, a_d}_r, p \right) \) for all \( d \geq 3 \) and all \( r \in \{a_d + 1, \ldots, a_1 + a_2 \} \) (the 2-critical families only, see Definition 1.5 and Section 4 below).

**Corollary 1.2.** For every \( m \in \{2, \ldots, d\} \) and \( i \in [a_{m-1}] \), as \( p \to 0 \), the following holds: if \( L_c \left( \mathcal{N}^{a_1, \ldots, a_m}_{a_m+i}, p \right) = \exp \Theta \left( \xi_i(p) \right) \), for some function \( \xi_i(p) = \xi_i(p, a_1, \ldots, a_m) \) then
\[
L_c \left( \mathcal{N}^{a_1, \ldots, a_d}_{a_d+\ldots+a_m+i}, p \right) = \exp_{(d-m+1)}(\Theta_i(p)).
\]

Note that in this corollary, it is an open problem to determine the functions \( \xi_i(p) \) for all \( m \geq 4 \) and \( i \in [a_{m-1}] \). While for \( m = 3 \), we only know \( \xi_i(p) \) for \( i = 1, 2 \) (except for \( i = 2 \) when \( a_3 = a_1 + a_2 - 1 \)) by [7], and it is unknown for \( i \in \{3, \ldots, a_2\} \).

1.2. The BSU model. The model we study here is a special case of the following extremely general class of \( d \)-dimensional monotone cellular automata, which were introduced by Bollobás, Smith and Uzzell [7].

Let \( \mathcal{U} = \{X_1, \ldots, X_m\} \) be an arbitrary finite family of finite subsets of \( \mathbb{Z}^d \setminus \{0\} \). We call \( \mathcal{U} \) the *update family*, each \( X \in \mathcal{U} \) an *update rule*, and the process itself *BSU-bootstrap percolation*. Let \( \Lambda \) be either \( \mathbb{Z}^d \) or \([L]^d \) or \( \mathbb{Z}_L^d \) (the \( d \)-dimensional torus of sidelength \( L \)).

Given a set \( A \subset \Lambda \) of initially *infected* sites, set \( A_0 = A \), and define for each \( t \geq 0 \),
\[
A_{t+1} = A_t \cup \{x \in \Lambda : x + X \subset A_t \} \text{ for some } X \in \mathcal{U}.
\]
The set of eventually infected sites is the closure of $A$, denoted by $[A]_U = \bigcup_{t \geq 0} A_t$, and we say that there is percolation when $[A]_U = \Lambda$.

For instance, our $A_{r^a_1 \cdots a_d}$-model is the same as $A_{r^a_1 \cdots a_d}$-bootstrap percolation, where $A_{r^a_1 \cdots a_d}$ is the family consisting of all subsets of size $r$ of the neighbourhood $N_{a_1 \cdots a_d}$ in $[1]$, and we denote $[A] = [A]_{A_{r^a_1 \cdots a_d}}$.

Let $S^{d-1}$ be the unit $(d-1)$-sphere and denote the discrete half space orthogonal to $u \in S^{d-1}$ as $H_u := \{ x \in \mathbb{Z}^d : \langle x, u \rangle < 0 \}$. The stable set $S = S(U)$ is the set of all $u \in S^{d-1}$ such that no rule $X \in U$ is contained in $H_u$. Let $\mu$ denote the Lebesgue measure on $S^{d-1}$. The following classification of families was proposed in [1] for $d = 2$ and extended to all dimensions in [0]: A family $U$ is

- **subcritical** if for every hemisphere $H \subset S^{d-1}$ we have $\mu(H \cap S) > 0$.
- **critical** if there exists a hemisphere $H \subset S^{d-1}$ such that $\mu(H \cap S) = 0$, and every open hemisphere in $S^{d-1}$ has non-empty intersection with $S$;
- **supercritical** otherwise.

For dimension $d = 2$, Bollobás, Duminil-Copin, Morris and Smith proved a universality result in [0], determining the critical length (with $A = \bigotimes_{v \in \mathbb{Z}_L^2} \text{Ber}(p)$)

$$L_c(U, p) := \min \{ L \in \mathbb{N} : \mathbb{P}_p([A]_U = \mathbb{Z}_L^2) \geq 1/2 \},$$

up to a constant factor in the exponent for all two-dimensional critical families $U$, which we can briefly state as follows.

**Theorem 1.3 (Universality).** Let $U$ be a critical two-dimensional family. There exists a computable positive integer $\alpha = \alpha(U)$ such that, as $p \to 0$, either

$$\log L_c(U, p) = \Theta(p^{-\alpha}),$$

or

$$\log L_c(U, p) = \Theta(p^{-\alpha}(\log \frac{1}{p})^2).$$

Proving a universality result of this kind for higher dimensions is a challenging open problem. However, there is a weaker conjecture about all critical families and all $d \geq 3$, stated by the authors in [0].

**Conjecture 1.4.** Let $U$ be a critical $d$-dimensional family. There exists $r \in \{2, \ldots, d\}$ such that, as $p \to 0$

$$\log (r-1) L_c(U, p) = p^{-\Theta(1)},$$

**Definition 1.5.** We say that a $d$-dimensional update family $U$ is $r$-critical if it satisfies condition (11) (so, roughly speaking, $U$ behaves like the classical $r$-neighbour model).

Observe that the family $N_{r^a_1 \cdots a_d}$ is critical if and only if

$$r \in \{a_d + 1, \ldots, a_1 + \cdots + a_d\}.$$

As an illustration, let us verify this for $d = 3$: If $r > a + b + c$ then every $u \in S^2$ is in the stable set, since there is no rule of $N_{r^a b^c}$ contained in $\mathbb{H}_u^3$. Thus $S(N_{r^a b^c}) = S^2$, and the model is subcritical. For each $i = 1, 2, 3$, let us denote by

$$S^1_i := \{(u_1, u_2, u_3) \in S^2 : u_i = 0\}$$
the unit circle contained in $S^2$ that is orthogonal to the vector $e_i$. When $r \leq c$, for every $u \notin S^3_1$ either $\{r'e_3 : r' \in [r]\}$ or $\{-r'e_3 : -r' \in [r]\}$ is contained in $\mathbb{H}_u^3$, so $u$ is not in the stable set. Therefore $S(N_r^{a,b,c}) \subset S^3_1$, so the hemisphere $H_3$ pointing in the $e_3$-direction satisfies $H_3 \cap S = \emptyset$ and $N_r^{a,b,c}$ is supercritical.

Finally, when $r \in \{c + 1, \ldots, a + b + c\}$, every canonical unit vector is in the stable set since $r > c \geq b \geq a$, so every open hemisphere in $S^2$ intersects $S(N_r^{a,b,c})$. Moreover, for each $u \notin S^1_1 \cup S^1_2 \cup S^1_3$, $\mathbb{H}_u^3$ intersects all three coordinate axis (see Figure 2), hence there is a rule contained in $\mathbb{H}_u^3$ since $r \leq a + b + c$. It follows that $S(N_r^{a,b,c}) \subset S^1_1 \cup S^1_2 \cup S^1_3$ and every hemisphere $H \subset S^2$ satisfies $\mu(H \cap S) = 0$, so $N_r^{a,b,c}$ is critical, as claimed.

Indeed, a careful analysis would lead us to all possibilities for the stable set of the family $N_r^{a_1, \ldots, a_d}$ in dimensions $d \geq 3$. Some cases are:

$$S(N_r^{a_1, \ldots, a_d}) = \begin{cases} 
\{ \pm e_1, \ldots, \pm e_d \} & \text{for } a_d < r \leq a_1 + a_2, \\
S^1_{1,2} \cup \{ \pm e_3, \ldots, \pm e_d \} & \text{for } a_1 + a_2 < r \leq a_1 + a_3, \\
S^1_{1,2} \cup S^1_{1,3} \cup \{ \pm e_4, \ldots, \pm e_d \} & \text{for } a_1 + a_3 < r \leq a_2 + a_3, \\
\vdots & \\
S^1_{1,2} \cup S^1_{2,3} \cup \cdots \cup S^1_{d-2} \cup S^1_{d-2} & \text{for } a_2 + \cdots + a_d < r \leq a_1 + a_2 + \cdots + a_d,
\end{cases}$$

where, $S^1_{i,k}$ is the unit circle contained in $S^d-1$ that contains vectors $e_i, e_k$, while $S^1_{d-2} \subset S^d-1$ is the $(d-2)$-sphere orthogonal to vector $e_i$.

For instance, if $d = 3$ Note that by (6), the family $N_r^{a,b,c}$ is 2-critical for all $r \in \{c + 1, \ldots, c + b\}$ (first 3 cases in (12)). On the other hand, Theorem 1.1 implies that $N_r^{a,b,c}$ is 3-critical for all $r \in \{c + b + 1, \ldots, c + b + a\}$ (last case in (12)).
2. Upper bounds

To prove upper bounds, it is enough to give one possible way of growing from $A$ step by step until we fill the whole of $[L]^d$.

**Definition 2.1.** A rectangular block is a set of the form $R = [l_1] \times \cdots \times [l_d] \subset \mathbb{Z}^d$. We say that a rectangular block $R$ is internally filled if $R \subset [A \cap R]$, and denote this event by $I^\bullet(R)$.

Given $d \geq 2$ and $a_2 \leq \ldots \leq a_d$, let us denote

$$s_d := a_2 + a_3 + \cdots + a_d.$$

As usual in bootstrap percolation, we actually prove a stronger proposition.

**Proposition 2.2.** Given $d \geq 3$, fix $i \in [a_1]$ and consider $\mathcal{N}_{s_d+i}^{a_1,\ldots,a_d}$-bootstrap percolation. There exists a constant $\Gamma = \Gamma(d,a_d) > 0$ such that, if

$$L = \exp(d^{-1}) (\Gamma \lambda_i(p)),$$

then $\mathbb{P}_p (I^\bullet([L]^d)) \to 1$, as $p \to 0$.

One key step in the proof of this proposition is to refine the upper bounds in (3) for all dimensions, which can be done by using standard renormalization techniques.

**Lemma 2.3** (Renormalization). Given $d \geq 2$, fix $i \in [a_1]$ and consider $\mathcal{N}_{s_d+i}^{a_1,\ldots,a_d}$-bootstrap percolation. There exists a constant $N_0 = N_0(d,a_d) > 0$ such that,

$$\mathbb{P}_p ([A] = [N]^d) \geq 1 - \exp (-\Omega(N)),$$

for all $p$ small enough and $N \geq N_0$.

**(Proof.** For $d = 2$, it follows from (3) and renormalization techniques (see e.g. [21]). For $d \geq 3$ it follows by induction on $d \geq 3$, meaning, Proposition $2.2$ with $d$ implies Lemma $2.3$ with $d$, while Lemma $2.3$ with $d-1$ implies Proposition $2.2$ with $d$ (see the proof of Proposition $2.2$ below). \)

Now, we are ready to show the upper bound for $L_0(\mathcal{N}_{s_d+i}^{a_1,\ldots,a_d},p)$.

**(Proof of Proposition 2.2.** We use induction on $d \geq 3$. Assume that the proposition holds for all dimensions $2, 3, \ldots, d - 1$. Set $L = \exp(d^{-1}) (\Gamma \lambda_i(p))$, where $\Gamma > 0$ is a large constant to be chosen. Let $C$ be another large constant ($\Gamma$ will depend on $C$), $N = \exp(d^{-2})(C \lambda_i(p))$, and consider the rectangular block

$$R := [N]^{d-1} \times [a_d] \subset [L]^d,$$

and the events $F_L := \{ \exists$ a copy of $R$ contained in $A \}$, and $G_L := \{ [A \cup R] = [L]^d \}$. Note that $\mathbb{P}_p (I^\bullet([L]^d)) \geq \mathbb{P}_p (F_L) \mathbb{P}_p (G_L | R \subset A)$, so we need to show that $\mathbb{P}_p (F_L) \to 1$ and $\mathbb{P}_p (G_L | R \subset A) \to 1$, as $p \to 0$.

Indeed, there are roughly $L^d/|R|$ disjoint (therefore independent) copies of $R$ (which we label $Q_1, \ldots, Q_{L^d/|R|}$), and $|R| \leq \exp(d^{-2}) (p^{-2i})$, so

$$\mathbb{P}_p (F_L) \leq \mathbb{P}_p \left( \bigcap_i (Q_i \not\subset A) \right) \leq [1 - \mathbb{P}_p (R \subset A)]^{L^d/|R|} \leq \exp \left( - p |R| L^d / |R| \right) \leq \exp \left( - \exp \left( d \Gamma \lambda_i(p) - c e^2 C \lambda_i(p) \log \frac{1}{p} - p^{-2i} \right) \right) \leq \exp \left( - \exp \left( d \Gamma \lambda_i(p) - c e^3 C \lambda_i(p) \right) \right).$$
By taking $\Gamma \geq 3C$ we conclude $\mathbb{P}_p(F_L) \to 1$, as $p \to 0$.

Next, set $M = \exp_{d-2}(p^{-2a_2})$, and consider the rectangular block

$$R' := [N]^{d-1} \times [M] \supset R.$$  

In order to prove that $\mathbb{P}_p(G_L | R \subset A) \to 1$, as $p \to 0$ it is enough to verify that

$$\mathbb{P}_p(\mathcal{I}^*(R') | R \subset A) \to 1, \text{ as } p \to 0,$$

then $R'$ will grow with high probability to fill the whole of $[L]^d$, since each of its $(d-1)$-faces is of supercritical size for the corresponding induced $(d-1)$-dimensional bootstrap process on that face. More precisely, on the face orthogonal to the (easiest grow) $e_i$-direction with volume $N^{d-1} \geq \exp_{d-2}(2C\lambda_i(p))$, by induction hypothesis the corresponding critical length is $L_c(\mathcal{N}_{s_{d-1}+1}^{a_1,a_2,...,a_d-1}, p) = \exp_{d-2}(\Theta(\lambda_i(p)) \leq N^{d-1}$ if $C$ is large; on the face orthogonal to the (second hardest) $e_2$-direction with volume $MN^{d-2} \geq \exp_{d-2}(2\lambda_2 - 2a_2)$ (and shape such that it is much larger than a critical droplet in all $d - 1$ directions) the corresponding critical length is $L_c(\mathcal{N}_{s_{d-1}+1}^{a_1,a_2,...,a_d}, p) = \exp_{d-2}(\Omega(\lambda_2(p)) \leq MN^{d-2}$, and on the face orthogonal to the (hardest) $e_{c,d}$-direction with volume $MN^{d-2}$ as well the corresponding critical length is $L_c(\mathcal{N}_{s_{d-1}+1}^{a_1,a_2,...,a_d}, p) = \exp_{d-2}(\Omega(\lambda_2 - a_1 + 1(p)) \leq MN^{d-2}$.

Finally, by Lemma 2.3 (applied with $d - 1$),

$$\mathbb{P}_p(\mathcal{I}^*(R') | R \subset A) \geq (1 - e^{-\Omega(N)})^M \geq \exp(-2Me^{-\Omega(N)}) \to 1,$$

as $p \to 0$, and (14) follows.  

\[\square\]

3. Lower bounds

In this section we will prove the lower bounds, and the proof is an application of the Cerf-Cirillo method (see Section 3.3) and the components process (see Definition 3.5 below), a variant of an algorithm introduced Bollobás, Duminil-Copin, Morris, and Smith. [6] We will prove the following.

**Proposition 3.1.** Given $d \geq 3$, fix $i \in [a_1]$ and consider $\mathcal{N}_{s_{d+1}}^{a_1,...,a_d}$-bootstrap percolation. There exists a constant $\gamma = \gamma(d,a_1) > 0$ such that, if

\[L \leq \exp_{d-1}(\gamma\lambda_i(p)),\]

then $\mathbb{P}_p(\mathcal{I}^*(|[L]^d)|) \to 0$, as $p \to 0$.

In order to show this proposition, we need to introduce a notion about rectangular blocks which is an approximation to being internally filled, and this notion requires a strong concept of connectedness; we define both concepts in the following.

**Definition 3.2.** For $d \geq 1$, let $G^d = (V,E)$ be the graph with vertex set $[L]^d$ and edge set given by $E = \{uv : \|u - v\|_\infty \leq 2a_d\}$. We say that a set $S \subset [L]^d$ is $d$-strongly connected if it is connected in the graph $G^d$.

**Definition 3.3.** We say that the rectangular block $R \subset [L]^d$ is internally spanned by $A$, if there exists a strongly connected set $S \subset [A \cap R]$ such that $R$ is the smallest rectangular block containing $S$. We denote this event by $I^*(R)$.

Note that when a rectangular block is internally filled then it is also internally spanned. Now, given $T \subset [L]^d$, let us denote by $\text{long}(T)$ the largest sidelength of the smallest
rectangle containing $T$, and let
\[
\text{diam}(T) := \max\{\text{long}(S) : S \subset T, S \text{ strongly connected}\}.
\]
Since $I^\ast([L]^d)$ is an increasing event, Proposition 3.1 is a consequence of the following result.

**Proposition 3.4.** Given $d \geq 3$, fix $i \in [a_1]$ and consider $\mathcal{N}_{a_{d+i}}$-bootstrap percolation. There exists a constant $\gamma = \gamma(d, a_d) > 0$ such that, if
\[
L = \exp\left( (d-1) \left( \gamma \lambda_i(p) \right) \right),
\]
then, as $p \to 0$,
\[
\mathbb{P}\left( \text{diam}([A]) \geq \log L \right) \leq L^{-1}.
\]

The rest of this paper is devoted to the proof of this result.

### 3.1. The components process

The following is an adaptation of the spanning algorithm in [6, Section 6.2]. We will use it to show an Aizenman-Lebowitz-type lemma, which says that when a rectangular block is internally spanned, then it contains internally spanned rectangular blocks of all intermediate sizes (see Lemmas 3.8 and 3.9 below).

**Definition 3.5 (The components $d$-process).** Consider $\mathcal{N}_{a_1 \cdots a_d}$-bootstrap percolation on $[L]^d$ with $r > a_d$. Let $A = \{v_1, \ldots, v_{|A|}\} \subset [L]^d$. Set $\mathcal{R} := \{S_1, \ldots, S_{|A|}\}$, where $S_i = \{v_i\}$ for each $i = 1, \ldots, |A|$. Then repeat the following steps until STOP:

1. If there exist distinct sets $S_1, S_2 \in \mathcal{R}$ such that $S_1 \cup S_2$ is strongly connected, then remove them from $\mathcal{R}$, and replace by $[S_1 \cup S_2]$.
2. If there do not exist such sets in $\mathcal{R}$, then STOP.

**Remark 3.6.** We highlight that the condition $r > a_d$ (equivalent to $\mathcal{N}_{a_1 \cdots a_d}$ is not supercritical) guarantees that at any stage of the component process, if $S = [S_1 \cup S_2]$ is added to the collection $\mathcal{R}$, then the smallest rectangular block (which is finite) containing $S$ is internally spanned.

Since $G^d$ is finite, the process stops in finite time; so that we can consider the final collection $\mathcal{R}'$ and set $V(\mathcal{R}') = \bigcup_{S \in \mathcal{R}'} S$.

**Lemma 3.7.** $V(\mathcal{R}') = [A]$.

**Proof.** See Lemma 3.10 in [4]. \qed

The following is a variant of the Aizenman-Lebowitz Lemma in [1].

**Lemma 3.8.** Consider $\mathcal{N}_{r}^{a_1 \cdots a_d}$-bootstrap percolation with $r \geq a_d + 1$. For every $k \leq \text{diam}([A])$, there exists an internally spanned rectangular block $R \subset [L]^d$ satisfying
\[
k \leq \text{diam}(R) \leq 2a_d k.
\]

**Proof.** Let $S$ be the first set that appears in the components process such that $\text{diam}(S) \geq k$, and let $R$ be the smallest block containing $S$. Since $\text{diam}(S) = \text{diam}(R)$, it only remains
to show that \( \text{diam}(S) \) at most \( 2adk \). In fact, we know that \( S = [S_1 \cup S_2] \) for some sets \( S_t \) such that, \( \text{diam}(S_t) \leq k - 1 \) for each \( t = 1, 2 \). Since \( S \) is strongly connected, we conclude
\[
\text{diam}(S) \leq \text{diam}(S_1) + \text{diam}(S_2) + 2ad \leq 2adk.
\]

\[ \square \]

Basically, the same proof of this lemma (by using the components \((d - 1)\)-process) allows us to conclude the following.

**Lemma 3.9.** Consider \( \mathcal{N}^{a_1,\ldots,a_d-1}_r \)-bootstrap percolation with \( r \geq a_{d-1} + 1 \). For every \( k, l \leq \text{diam}([A]) \), there exists an internally spanned copy of the rectangular block \( W \times [h] \), with \( W \subset [L]^{d-2} \), satisfying \( \text{diam}(W) \leq 2ad-l, h \leq 2ad-k \) and either 
\[
\text{diam}(W) \geq l \text{ or } h \geq k.
\]

### 3.2. Anisotropic bootstrap percolation on \([N]^{d-1}\) with subcritical sizes.

Let us fix \( d \geq 3 \), \( i \in [a_1] \) and consider \( \mathcal{N}^{a_1,\ldots,a_d-1}_{s_{d-1}+i} \)-bootstrap percolation on \([N]^{d-1}\), where
\[
N \leq \exp_{(d-2)}(\gamma \lambda_i(p))
\]
and \( \gamma = \gamma(d,a_{d-1}) > 0 \) is a small constant (so that percolation is unlikely). Note that for \( d = 3 \), \( \exp(\gamma \lambda_i(p)) \approx L_c(\mathcal{N}^{a_1,a_2}_{a_3+i},p) \) by [3], while we will deduce that \( \exp_{(d-2)}(\gamma \lambda_i(p)) \approx L_c(\mathcal{N}^{a_1,\ldots,a_d-1}_{s_{d-1}+i},p) \) by induction on \( d \).

**Definition 3.10.** We define the component (or cluster) at \( ([N/2], \ldots, [N/2]) \in [N]^{d-1} \) as the \((d-1)\)-strongly connected component containing \( ([N/2], \ldots, [N/2]) \) in the graph induced by \([A \cap [N]^{d-1}]\), and we denote it by \( K = \mathcal{K}(A, i, a_1, \ldots, a_{d-1}) \subset [N]^{d-1} \).

The following results are standard in bootstrap percolation.

**Proposition 3.11.** Consider \( \mathcal{N}^{a_1,\ldots,a_d-1}_r \)-bootstrap percolation. For any \( \varepsilon > 0 \), there exists \( \gamma = \gamma(d,a_{d-1}) > 0 \) such that if \( N \leq \exp_{(d-2)}(\gamma \lambda_i(p)) \), as \( p \to 0 \),
\[
\begin{align*}
(a) & \, \mathbb{P}_p(\text{diam}(K) \geq p^{-i-\varepsilon}) \leq N^{-\varepsilon}, \text{ when } d = 3, \\
(b) & \, \mathbb{P}_p(\text{diam}(K) \geq \exp_{(d-3)}(\lambda_i(p))) \leq N^{-\varepsilon}, \text{ when } d \geq 4.
\end{align*}
\]

The proof of this proposition goes by induction on \( d \) (like the proof of Lemma 2.3), by combining it with Proposition 3.4. The base case is given by (a), thus, this is the only case that we will prove. Moreover, when \( d = 3 \), the proof in the isotropic case \( a_1 = a_2 \) follows from usual application of the the Aizenman-Lebowitz Lemma (see for instance, the paragraph after Theorem 7.1 of [6] with \( \alpha = i \)). While the proof in the case \( a_1 < a_2 \) is basically the same as that of Theorem 8.1 of [6], with some minor modifications; for completeness, we will prove this case only.

**Proof of Proposition 3.11 (a).** Assume that \( a_1 < a_2 \) and let \( \delta = \delta(\varepsilon) > 0 \) be small. If \( \text{diam}(K) \geq p^{-i-\varepsilon} \), then by Lemma 3.3 there exists a rectangle \( R = [w] \times [h] \) such that \( w \leq p^{-i-\varepsilon}, h \leq \delta p^{-1}\log \frac{1}{p} \) and either, \( w \geq \Omega(p^{-i-\varepsilon}) \) or \( h \geq \Omega(\delta p^{-1}\log \frac{1}{p}) \).

If \( w \geq \Omega(p^{-i-\varepsilon}) \), since \( R \) is internally spanned, every copy of the slab \([2a_2^3] \times [h] \) must contain \( i \) vertices of \( A \) within constant distance, so for \( \delta \) small,
\[
\mathbb{P}_p(I^X(R)) \leq (1 - e^{-\Omega(\delta p^{-1}\log \frac{1}{p})})^{\Omega(p^{-i-\varepsilon})} \leq \exp(-p^{C\delta}p^{-i-\varepsilon}) \leq \exp(-p^{-i-\varepsilon/2}).
\]
Analogously, if \( h \geq \Omega(\delta p^{-i} \log \frac{1}{p}) \), every copy of the slab \([w] \times [2a_2^2]\) must contain \( a_2 + i - a_1 \) vertices of \( A \) within constant distance, so

\[
\mathbb{P}_p(I^\times(R)) \leq (1 - e^{-\Omega(p^{a_2-i+a_1} \cdot 3a_2)})^{\Omega(\delta p^{-i} \log \frac{1}{p})} \leq O(p^{a_2-i-a_1+1})^{\Omega(\delta p^{-i} \log \frac{1}{p})} \leq \exp(-\delta^2 p^{-i}(\log p)^2).
\]

Since there are at most \( N^3 \) copies of the rectangle \( R \) in \([N]^2\), then

\[
\mathbb{P}_p(\text{diam}(K) \geq p^{-i-\varepsilon}) \leq N^3 \exp(-\delta^3 p^{-i}(\log p)^2) \leq \exp(3\gamma f_i(p) - \delta^3 p^{-i}(\log p)^2) \leq N^{-\varepsilon},
\]

for \( \gamma \ll \delta^3 \).

**Proposition 3.12.** Consider \( N_{a_{d-1}+1}^{a_1-\ldots-a_d-1}\)-bootstrap percolation. As \( p \to 0 \),

(a) \( \mathbb{E}_p(|K|) \leq \sqrt{p} \), given that \( \text{diam}(K) \leq p^{-i-\varepsilon} \), when \( d = 3 \).

(b) \( \mathbb{E}_p(|K|) \leq o(1) \), given that \( \text{diam}(K) \leq \exp_{(d-3)}(\lambda_i(p)) \), when \( d \geq 4 \).

Again, the proof is by induction on \( d \), as that of Proposition 3.11. The base case is (a) and we will focus on that again, whose a straightforward application of Aizenman-Lebowitz Lemma (for the isotropic case, see for instance, (3.30) in [8]). We will prove Proposition 3.12(a) in the anisotropic case, and the proof is similar to that of Lemma 5.4 in [13] (with \( i = a \) only), which does not seem to be complete.

**Proof of Proposition 3.12(a).** It is enough to consider two cases. If \( 1 \leq \text{diam}(K) \leq 6a_2^2 \) then there is a vertex in \( A \) within constant distance of “the origin” \([N/2], [N/2]\). On the other hand, if \( \text{diam}(K) > 6a_2^2 \), by Lemma 3.8 there exists an internally spanned rectangular block \( R = [w] \times [h] \subset [N]^2 \) with \( 3a_2 \leq \text{diam}(R) \leq 6a_2^2 \). In particular, \( w, h \leq 6a_2^2 \) and either \( w > 3a_2 \) or \( h > 3a_2 \). So we have two subcases:

If \( w > 3a_2 \), then

\[
\mathbb{P}_p(I^\times(R)) \leq (1 - e^{-\Omega(p^{i-6a_2^2})})^{3a_2} \leq O(p^{i-3a_2}).
\]

And, for \( h > 3a_2 \),

\[
\mathbb{P}_p(I^\times(R)) \leq (1 - e^{-\Omega(p^{i+1-a_1+6a_2^2})})^{3a_2} \leq O(p^{3a_2(i-a_1+1)}) \leq O(p^{3a_2i}).
\]

Finally, there are at most \( O(N^2) \) possible choices for the rectangular block \( R \), thus

\[
\mathbb{E}_p(|K|) \leq O(6a_2^2p) + O(N^2 \cdot N^2 \cdot p^{3a_2i}) \leq O(p) + O(p^{-4i-4a+3a_2i}) \leq p^{1/2},
\]

for \( \varepsilon > 0 \) small, since \( 3a_2 \geq 6 \). \(\square\)

### 3.3. The proof via Cerf-Cirillo method.

In this section we reproduce a result that was proved in [3] (and used again in [2]), which is an adaptation of some ideas from [8] and [17]. Then, we use it to prove Proposition 3.4.

Let us consider two-colored graphs, that is, simple graphs with two types of edges, which we will label “good” and “bad”.

**Definition 3.13.** We say that a two-colored graph is **admissible** if it either contains at least one bad edge, or if every component is a clique (i.e., a complete graph).

For any set \( S \), we let

\[
\Lambda(S) := \{ \text{admissible two-colored graphs with vertex set } S \times [2] \}.
\]

And, for each \( m \in \mathbb{N} \) we let

\[
\Omega(S, m) := \{ \mathcal{P} = (G_1, \ldots, G_m) : G_t \in \Lambda(S) \text{ for each } t \in [m] \},
\]
be the set of sequences of two-colored admissible graphs on $S \times [2]$ of length $m$. We shall sometimes think of $G_t$ as a two-colored graph on $S \times [2t - 1, 2t]$, and trust that this will cause no confusion.

Now, for each $P \in \Omega(S, m)$, let $G_P$ denote the graph with vertex set $V(G_P) = S \times [2m]$, and the following edge set $E(G_P)$:

(i) $G_P[S \times \{2y - 1, 2y\}] = G_y$,
(ii) $\{(x, y), (x', y + 1)\} \in E(G_P) \iff x = x'$,
(iii) $\{(x, y), (x', y')\} \notin E(G_P)$ if $|y - y'| \geq 2$.

Edges in $G_P$ of type (i) are labelled good and bad in the obvious way, to match the label of the corresponding edge in $G_y$. Thus $G_P$ has three types of edge: good, bad, and unlabelled.

Given $G \in \Lambda(S)$, let $E^g(G)$ denote the set of good edges, and $E^b(G)$ denote the bad edges, so that $E(G) = E^g(G) \cup E^b(G)$. If $uv \in E^g(G)$, then we shall write $u \sim v$. For each vertex $v = (x, y) \in V(G_P)$, let

$$\Gamma_P(v) := \{u \in V(G_P) : u \sim v \text{ and } u \neq v\},$$

and let $d_P(v) = |\Gamma_P(v)|$. Note that $d_P(v)$ is the number of good edges incident with $v$.

Finally, let $X(P)$ denote the event that there is a connected path across $G_P$ (i.e., a path from the set $S \times \{1\}$ to the set $S \times \{2m\}$). The following lemma was first stated in [3], then in [2], but the proof is due to Cerf and Cirillo [8].

Lemma 3.14 (Cerf and Cirillo [8], see Lemma 35 of [3]). For each $0 < \alpha < 1/2$ and $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds for all $m \in \mathbb{N}$ and all finite sets $S$ with $\alpha^4|S|\varepsilon \geq 1$.

Let $P = (G_1, \ldots, G_m)$ be a random sequence of admissible two-coloured graphs on $S \times [2]$, chosen according to some probability distribution $f_\Omega$ on $\Omega(S, m)$. Suppose $f_\Omega$ satisfies the following conditions:

(a) Independence: $G_i$ and $G_j$ are independent if $i \neq j$.
(b) BK condition: For each $t \in [m]$, $r \in \mathbb{N}$, and each $x_1, y_1, \ldots, x_r, y_r \in V(G_t)$,

$$\mathbb{P}\left(\bigcap_{j=1}^r (x_j \sim y_j) \cap \bigcap_{j \neq j'} (x_j \neq x_{j'}) \cap (E^b(G_t) = \emptyset)\right) \leq \prod_{j=1}^r \mathbb{P}(x_j \sim y_j),$$

and for each $t \in [m]$ and $v \in V(G_P)$,

(c) Bad edge condition: $\mathbb{P}\left(E^b(G_t) \neq \emptyset\right) \leq |S|^{-\varepsilon}$,
(d) Good edge condition: $\mathbb{E}(d_P(v)) \leq \delta$.

Then

$$\mathbb{P}(X(P)) \leq \alpha \alpha^m|S|.$$

We are ready to prove the lower bound.

Proof of Proposition 3.4. We use induction on $d \geq 3$. Assume that the proposition holds for all dimensions $2, 3, \ldots, d - 1$. In particular, Propositions 3.11 and 3.12 hold for dimension $d$. Fix $i \in [n_1]$ and consider $\mathcal{N}_{a_1, \ldots, a_d}$-bootstrap percolation. Fix a small constant $\varepsilon > 0$ and let $\gamma > 0$ be the constant given by Proposition 3.11 then take $L = \exp(d_1 - 1)(\gamma \lambda_1(p))$. Let us show that $\mathbb{P}(\text{diam}(A)) \geq \log L \leq L^{-1}$, as $p \to 0$. 


Suppose that \( \text{diam}([A]) \geq \log L \), then by Lemma 3.8, there exists an internally spanned rectangular block \( R \subset [L]^d \) satisfying
\[
\frac{\log L - 1}{2a_d} \leq \text{diam}(R) \leq \log L - 1.
\]
Let \( N = \text{diam}(R) \), then we can assume for simplicity that \( R \subset [N]^d \). Moreover, there is a strongly connected path \( X \) in \([A \cap R]\) joining two opposite \((d - 1)\)-faces of \([N]^d\), and we can assume that this happens along the \((e_d)\)-direction, so \( X \) goes from the set \( \{(x_1, \ldots, x_d) \in [N]^d : x_d = 1\} \) to the set \( \{(x_1, \ldots, x_d) \in [N]^d : x_d = N\} \).

Now, let \( m = \lfloor N/4a_d \rfloor \) and partition \([N]^d\) into blocks \( B_1, \ldots, B_{2m} \), each of size \([N]^{d-1} \times [2a_d] \) (for simplicity, assume that \( N \) is a multiple of \( 4a_d \)). So, \( B_j = \{(x_1, \ldots, x_d) \in [N]^d : x_d \in [2a_d(j - 1) + 1, 2a_dj]\} \), for each \( j \in [2m] \).

For each \( j \), let us consider a \((d - 1)\)-dimensional bootstrap process on \( B'_j := [N]^{d-1} \times \{j\} \) as follows: Take the initially infected set \( A' \sim \bigotimes_{v \in [N]^d} \text{Ber}(2a_d p) \) and then run the \( \mathcal{N}^n_{d_{d-1}+i} \)-bootstrap process, independently on each \( B'_j \). Note that this defines a concatenated process on \([N]^{d-1} \times [2m]\) consisting of \( 2m \) independent \((d - 1)\)-dimensional processes, and couples our original \( \mathcal{N}^n_{d_d} \)-process in the following way:

The probability of having a vertex in \( A \cap \{(d - 1)\} \times \{2a_d, 2a_dj\} \subset B_j \) is at most \( 2a_d p \), which is the initial density (for \( A' \)) in \( B'_j \). Also, each vertex in \( B_j \) has at most \( a_d \) neighbors in \([N]^d \setminus B_j \). Thus, the projection of components of \([A \cap B_j]\) onto the \((d - 1)\)-plane orthogonal to \( e_d \) is coupled by the components in \([A' \cap B'_j]\). In particular, the existence of \( X \) implies the existence of a strongly connected path \( X' \subset \bigcup\{A' \cap B'_j\} \) from the set \( \{(x_1, \ldots, x_d) \in [N]^d : x_d = 1\} \) to the set \( \{(x_1, \ldots, x_d) \in [N]^d : x_d = 2m\} \).

Next, set \( S = [N]^{d-1} \), and for each \( j \in [2m] \), let \( [A](j) := [A' \cap B'_j] \), and define a two-colored graph \( G_j \) on \( S \times [2] \) by
\[
uv \in E(G_j) \iff u', v' \text{ are in the same strong component of } [A](j),
\]
where \( u' \) is the element of \([N]^{d-1} \times \{2j - 1, 2j\}\) corresponding to \( u \) in the natural isomorphism, and define “good” edges by
\[
u \sim v \iff \text{there exists an internally filled strongly connected component }
X \subset [A](j) \text{ such that } u, v \in X \text{ and } \text{diam}(X) \leq (\log N)^{1+\varepsilon}.
\]
Note that \( G_j \) is admissible (see for instance, the proofs of the lower bounds for Theorem 1 in [2] and [3]). Therefore, it is enough to check that the sequence \( \mathcal{P} = (G_1, \ldots, G_m) \in \Omega(S, m) \) satisfies the conditions of Lemma 3.14.

In fact, condition (a) follows by construction, while condition (b) follows from the van Berg-Kesten Lemma (again, see the proof of Theorem 1 in [3]).

Now, since \( N \leq \log L = \exp\{d-2\}(\gamma \lambda^i(p)) \), by Proposition 3.11, we conclude, as \( p \to 0 \),
\[
\mathbb{P}_p(\text{diam}([A](j)) > (\log N)^{1+\varepsilon}) \leq N^{-\varepsilon},
\]
and by Proposition 3.12 for \( p \) small,
\[
\mathbb{E}_p(dp(v)) \leq O(\mathbb{E}_p(|K|)) \leq O(\sqrt{p}) = o(1).
\]
Finally, by Lemma 3.14 we conclude that for \( \alpha > 0 \) small,
\[
\mathbb{P}_p(X(\mathcal{P})) \leq \alpha^{\lfloor N/4a_d \rfloor N^{d-1}} \leq 1/L^{2d},
\]
then, summing over all possible choices of $R \subset [L]^d$ we get
\[ P_p(\text{diam}([A]) \geq \log L) \leq 1/L, \]
and we are done. \qed

Note that in the above proof for $d \geq 4$, when defining “good” edges $u \sim v$, we could replace the size $(\log N)^{1+\epsilon}$ by $O(\log N)$. That refinement would improve the lower bound for the $(d−1)$-times iterated logarithm of the threshold by just a constant factor.

4. Future work

In dimension $d = 3$, a problem which remains open is the determination of the threshold for $a_3 + 3 \leq r \leq a_2 + a_3$ (the 2-critical families). We believe that the techniques used in [4] can be adapted to cover these cases (though significant technical obstacles remain); recall that in this case, by (6), the critical length is singly exponential.

For dimensions $d \geq 4$, by using the techniques in [9], it can be shown that as $p \to 0$, $\log L_c(\mathcal{N}_{a_d+1}^{a_d}, p) \geq \Omega\left( p^{-1/(d-1)} \right)$, so that $\log L_c(\mathcal{N}_r^{a_1,\ldots,a_d}, p) \geq \Omega\left( p^{-1/(d-1)} \right)$ for $r \geq a_d + 1$.

On the other hand, as it was shown in the appendix of [4], by using Lemma 2.3 and decomposing $[L]^d$ as $L^{d-2}$ disjoint copies of $[L]^2$ all of them parallel to the $e_{d-1}$ and $e_d$-directions, we can see that for $r \in \{a_d + 1, \ldots, a_d + a_{d-1}\}$,
\[ \log L_c(\mathcal{N}_r^{a_1,\ldots,a_d}, p) \leq O\left( \log L_c(\mathcal{N}_r^{a_d-1,a_d}, p) \right) = O\left( p^{-r-a_d}(\log p)^2 \right). \]

So, it follows that the critical length is singly exponential in the cases
\[ r \in \{a_d + 1, \ldots, a_d + a_{d-1}\}, \]
and the family is 2-critical, by Definition 1.5. It is an interesting open problem to find the critical length for all critical anisotropic models in all dimensions, and by Corollary 1.2 we need to do it for the 2-critical families only.

Problem 4.1. Determine the critical length $L_c(\mathcal{N}_r^{a_1,\ldots,a_d}, p)$ for all $d \geq 3$, $a_1 \leq \cdots \leq a_d$ and all $r \in \{a_d + 1, \ldots, a_d + a_{d-1}\}$.

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