Topological Susceptibility of Monte-Carlo Generated Projected Vortices

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We determine the topological susceptibility from center projected vortices and demonstrate that the topological properties of the \(SU(2)\) Yang-Mills vacuum can be extracted from the vortex content. We eliminate spurious ultraviolet fluctuations by two different smoothing procedures. The extracted susceptibility is comparable to that obtained from full field configurations.

One of the main aims of present day lattice investigations is to obtain a consistent picture of the QCD vacuum. During the last years, the center vortex model turned out to be a good candidate for that. The vortex model was invented at the end of the seventies \([1]\). Due to the lack of an identification method for vortices, almost no numerical investigations were done for 25 years. Maximal center gauge \([2]\) and center projection gave us a means to identify vortices and led to new investigations about the predictive power of the vortex model. The vortex model explains the confinement properties of the QCD vacuum \([1,2,3,4,5]\). By the idea that monopoles are located on vortices \([6]\), it is even strongly related to the dual superconductor picture of confinement. As recently \([7]\) suggested, the vortex picture is also related to the topological properties of the QCD vacuum. We report on numerical calculations which support this statement.

We extract center vortices from lattice configurations by direct maximal center gauge \([8]\), i.e. by a maximization of the gauge fixing functional

\[
\max_G \sum_i |\text{tr} U_i^G|^2
\]

under gauge transformations \(G\), where the \(U_i\) are the link variables. This functional biases link variables toward the center of the gauge group, e.g. in the \(SU(2)\) case considered henceforth, toward the elements \(\pm 1\). Physically, the idea is to transform as much physical information as possible to the center part of the configuration. In a second step, center projection, the magnetic flux is concentrated in tubes (P-vortices) which propagate in time and form closed two-dimensional surfaces in dual space. Since the Pontryagin index \(Q\), a four-dimensional integral over the scalar product of electric and magnetic field strengths, is a topological quantity, one can hope that it is effectively unchanged by a compression of electric and magnetic fluxes.

For thin center vortices in the continuum, the topological charge \(Q\) is given by their self-intersection number \([7]\)

\[
Q = -\frac{1}{16} \varepsilon_{\mu\nu\rho\sigma} \int S d^2 \sigma \int S d^2 \sigma' \delta^4 (\vec{x}(\sigma) - \vec{x}(\sigma'))
\]

where \(\vec{x}(\sigma)\) parametrizes the two-dimensional vortex surface \(S\) in four-dimensional space-time. This expression is only well defined if the orientation of \(S\) is specified. Lattice calculations show that, in the QCD vacuum, projected vortices have a random structure and are in general non-orientable \([8]\). This demonstrates that vortex surfaces consist of patches of alternating orientation which are separated by closed lines where the orientation switches. These flips of orientation of P-vortices are related to the color structure of the original thick vortices. In an Abelian description, these lines are world-lines of Abelian monopoles with the two halves of the magnetic monopole flux going in opposite directions. The positions of the monopoles depend on the color structure of the vortices and the choice of the \(U(1)\) subgroup.

According to Eq. (2), in the continuum an intersection point contributes \(\pm 1/2\) to \(Q\). On the lattice, an intersection point joins \(4 \times 4\) plaquettes, see Fig. [9]: therefore, every joined pair of electric and magnetic plaquettes contributes \(\pm 1/32\). There are further contributions to \(Q\) from “writhing” points. These are lat-
tice sites where electric and magnetic plaquettes share a point of the four-dimensional lattice and belong to the same surface region. In spite of these contributions, $Q$ remains quantized.

Using the above prescription, we will determine $Q$ for Monte-Carlo generated gauge field configurations. However, first we have to solve three problems. Due to their transverse fluctuations inside the original thick vortices, P-vortices are plagued by spurious ultraviolet fluctuations. We remove them by the elementary cube transformations (a)-(d) in Fig. 2 or by blocking the gauge field configuration such as to arrive at a two or three times coarser lattice. A further problem is that Abelian monopole lines determined by Abelian projection are not always on vortices; some 3% of monopole cubes are not pierced by P-vortices [9]. Instead of trying to adjust monopole trajectories to vortices, we randomly assign orientations to the vortex plaquettes, with two different choices of bias allowing us to explore the extreme cases of either maximizing or minimizing the monopole line density. In our measurements, the monopole line density varies between the two extremes by a factor of around ten. The third problem is that, in contrast to the continuum, on the lattice P-vortices in general do not intersect at points, they intersect along lines, and some monopole lines coincide with vortex intersections and writhing points. We remove these artefacts by transferring the P-vortices to finer lattices of $1/3$ or, if necessary, $1/9$ the lattice spacing. Then we apply elementary cube transformations once at each lattice site whenever this allows us to remove an instance of the above coarse graining problems.

Fig. 1 depicts the results of the numerical measurements of the topological susceptibility $\chi$ for the coupling $\beta = 2.5$ on a $16^4$ lattice (filled squares), where 1156 samples were taken, and also for $\beta = 2.3$ both on a $16^4$ lattice (crosses, 1183 samples) and on a $12^4$ lattice (open squares, 4622 samples). The displayed results were obtained using the maximal monopole density. Despite the monopole line density varying by a factor of around ten when instead using the minimal monopole density, $\chi$ only differs by at most 1%. This observation agrees with the random vortex surface model [10,11]. The vertical error bars in Fig. 3 are compounded from the statistical uncertainty of the susceptibility and the statistical uncertainty of the string tension measurements. The latter uncertainty in addition leads to the horizontal error bars in the bottom panel in Fig. 3, since the evaluation of $\sqrt{\sigma}$ was also used to determine the lattice spacing $a$ by equating $\sqrt{\sigma} = 440$ MeV. Since smoothing step (d), cf. Fig. 2, leads already to a lowering of the string tension $\sigma$, we expect the physical $\chi$ between the (a)-(c) and (a)-(d) values:

\[(166 \text{MeV})^4 \leq \chi_{\text{phys}} \leq (230 \text{MeV})^4\]

in the upper part of Fig. 3. The lattice spacing which
should be reached by blocking is related to the thickness of the physical vortices and roughly lies between 0.4 fm and 0.6 fm. From the bottom panel in Fig. 3 we obtain therefore the estimate:

$$174 \text{ MeV}^4 \leq \chi_{\text{phys}} \leq 224 \text{ MeV}^4.$$  

(4)

These ranges for the topological susceptibility correspond well with values extracted from the full SU(2) lattice Yang-Mills ensemble.

It is interesting to consider the contribution of the intersection points alone to the topological susceptibility [12]. It is suppressed by a factor of around $2^4$. Therefore, the value of $\chi$ is dominated by the contributions from the writhing points. This result agrees with the invariance of $\chi$ under strong changes of the monopole configurations, which can only influence the contributions from intersection points [12]. In addition, even the contribution of the intersection points alone varies only by about 5% with the monopole density. Thus, it seems that the minimal monopole density required by the non-orientability of the vortices is sufficient to randomize the relative surface orientations at the intersection points.

The results which we obtain for the topological susceptibility support the vortex picture of the QCD vacuum, i.e. thick random vortices with color structure. These vortices can be located in an appropriate gauge like the maximal center gauge. Center projection of the field configurations corresponds to a compression of the magnetic flux into quantized tubes. Thick vortices can explain the string tension between color charges and its Casimir scaling properties, and, as shown within this work, the topological properties of the QCD vacuum.

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