Abstract. Various algebraic properties of Heilbronn’s exponential sum can be deduced through the use of supercharacter theory, a novel extension of classical character theory due to Diaconis-Isaacs and André. This perspective yields a variety of formulas and provides a method for computing the number of solutions to Fermat-type congruences that is much faster than the naïve method. It also reproduces Heath-Brown’s well-known bound for Heilbronn’s sum.

1. Introduction

The theory of supercharacters, of which classical character theory is a special case, was recently introduced by P. Diaconis and I.M. Isaacs in 2008 [7], generalizing the basic characters studied by C. André [1–3]. The original aim of supercharacter theory was to provide new tools for studying groups, such as the unipotent matrix groups $U_n(q)$, that had proven intractable from the perspective of classical character theory. However, recent work indicates that supercharacters on abelian groups are intimately tied to various exponential sums arising in the theory of numbers [5, 6, 8, 9]. Our aim here is to explore another such connection, demonstrating that many properties of Heilbronn’s exponential sum can be deduced through supercharacter theory.

We adopt the standard notation $e(x) = \exp(2\pi ix)$, so that the function $e(x)$ is periodic with period 1. The letter $p$ will always denote an odd prime number and $g$ a primitive root modulo $p^2$. The constant implicit in our frequent use of $\ll$ is understood to be independent of $p$. A Heilbronn sum is an exponential sum of the form

$$H_p(a) = \sum_{\ell=1}^{p-1} e\left(\frac{a\ell p}{p^2}\right).$$

The sums $H_p(a)$ are real-valued and obey the obvious bound $|H_p(a)| \leq p - 1$.

Answering a long-standing open problem raised by Heilbronn in the 1940s, Heath-Brown used a version of Stepanov’s method to establish that $H_p(a) \ll$
$p^{11/12}$ uniformly for $a$ coprime to $p$ \cite{11}. This upper bound was later improved by Heath-Brown and Konyagin to $p^{7/8}$ \cite{13} and by Shkredov to $p^{59/68} \log^{5/34} p$ \cite{17} and then to $p^{31/36} \log^{1/6} p$ \cite{18}.

In this note, we show that Heilbronn sums arise as the values of supercharacters on $\mathbb{Z}/p^2\mathbb{Z}$ induced by the action of a certain subgroup of the unit group $(\mathbb{Z}/p^2\mathbb{Z})^\times$. This observation, coupled with the general techniques from \cite{5,9}, permit us to derive a variety of identities involving Heilbronn sums. A brief review of basic facts about supercharacters on abelian groups is undertaken in Section 2, after which we construct the relevant supercharacter theory for Heilbronn sums in Section 3. An exact formula involving Heilbronn sums for computing the number of solutions to Fermat-type congruences $ax^p + by^p \equiv cz^p \pmod{p^2}$ is given in Section 4. In fact, our method turns out to be much faster than the naïve method (a detailed justification of this assertion is provided in Appendix A). We conclude in Section 5 with an exact formula for quartic sums involving Heilbronn sums, that recovers Heath-Brown’s bound $H_p(a) \ll p^{11/12}$ \cite{11}.

2. Supercharacters on abelian groups

Before proceeding, we recall a few basic facts about supercharacters on abelian groups. Since complete details can be found in \cite{5,2}, we content ourselves with a quick overview of the relevant facts required in our particular case.

Let $A$ be a subgroup of $GL_d(\mathbb{Z}/n\mathbb{Z})$ that is closed under the transpose operation and let $X_1, X_2, \ldots, X_N$ denote the orbits in $G = (\mathbb{Z}/n\mathbb{Z})^d$ under the action of $A$. The functions

$$\sigma_i(y) = \sum_{x \in X_i} e\left(\frac{x \cdot y}{n}\right),$$

where $x \cdot y$ denotes the formal dot product of two elements of $(\mathbb{Z}/n\mathbb{Z})^d$, are called supercharacters on $(\mathbb{Z}/n\mathbb{Z})^d$ and the sets $X_i$ are referred to as superclasses. It turns out that supercharacters are constant on superclasses, and hence we may employ the notation $\sigma_i(X_j)$ without confusion. The $N \times N$ matrix

$$U = \frac{1}{\sqrt{n^d}} \left[ \sigma_i(X_j) \frac{1}{\sqrt{|X_i|}} \right]_{i,j=1}^{N}$$

is symmetric (i.e., $U = U^T$) and unitary. In fact, the matrix $U$ encodes an analogue of discrete Fourier transform (DFT) on the space of all superclass functions (i.e., functions $f : (\mathbb{Z}/n\mathbb{Z})^d \to \mathbb{C}$ that are constant on each superclass) and satisfies many of the standard properties of the DFT \cite{5}.

It turns out that a variety of exponential sums that are relevant to the theory of numbers can be realized as supercharacters on abelian groups in the manner described above. This approach was first undertaken to study Ramanujan sums \cite{9} and, a short while later, Gaussian periods \cite{8}. The general theory is developed in \cite{5}, where a number of such examples (see Table 1) are discussed. A novel and visually compelling class of exponential sums is considered from the supercharacter perspective in \cite{6}.
Table 1. Gaussian periods, Ramanujan sums, Kloosterman sums, and Heilbronn sums appear as supercharacters arising from the action of a group $A$ of automorphisms on an abelian group $G$. Here $p$ denotes an odd prime number.

The main result we require is the following, which identifies the set of all matrices that are diagonalized by the unitary matrix (3) as the span of a certain family of matrices containing combinatorial information about the superclasses. A complete proof and further details can be found in [5] (see also [9]).

**Lemma 1.** Let $A = A^T$ be a subgroup of $GL_d(\mathbb{Z}/n\mathbb{Z})$, let $X = \{X_1, X_2, \ldots, X_N\}$ denote the set of superclasses induced by the action of $A$ on $(\mathbb{Z}/n\mathbb{Z})^d$, and let $\sigma_1, \sigma_2, \ldots, \sigma_N$ denote the corresponding supercharacters. For each fixed $z$ in $X_k$, let $c_{i,j,k}$ denote the number of solutions $(x_i, y_j) \in X_i \times X_j$ to the equation $x + y = z$.

1. $c_{i,j,k}$ is independent of the representative $z$ in $X_k$ which is chosen,
2. The identity
   \[ \sigma_i(X_\ell)\sigma_j(X_\ell) = \sum_{k=1}^{N} c_{i,j,k} \sigma_k(X_\ell) \]  
   holds for $1 \leq i, j, k, \ell \leq N$.
3. The matrices $T_1, T_2, \ldots, T_N$, whose entries are given by
   \[ [T_i]_{j,k} = \frac{c_{i,j,k} \sqrt{|X_k|}}{\sqrt{|X_j|}}. \]
   each satisfy
   \[ T_i U = U D_i, \]
   where
   \[ D_i = \text{diag}(\sigma_i(X_1), \sigma_i(X_2), \ldots, \sigma_i(X_N)). \]
   In particular, the $T_i$ are simultaneously unitarily diagonalizable.
4. Each $T_i$ is a normal matrix (i.e., $T_i^* T_i = T_i T_i^*$) and the set $\{T_1, T_2, \ldots, T_N\}$ forms a basis for the algebra of all $N \times N$ matrices $T$ such that $U^* T U$ is diagonal.
3. A supercharacter theory for Heilbronn sums

We are now in a position to represent Heilbronn sums as the values of certain supercharacters on \( \mathbb{Z}/p^2\mathbb{Z} \). Following the general outline described in Section 2, we first require a group of automorphisms \( A \) to act upon \( G = \mathbb{Z}/p^2\mathbb{Z} \). To this end, we need the following elementary lemma, whose proof is omitted.

**Lemma 2.** If \( p \) is an odd prime, then \( x^p \equiv y^p \) (mod \( p^2 \)) if and only if \( x \equiv y \) (mod \( p \)).

It follows that

\[
A = \{1^p, 2^p, \ldots, (p-1)^p\}
\]

is a subgroup of \( (\mathbb{Z}/p^2\mathbb{Z})^\times \) of order \( p-1 \). Letting \( A \) act upon \( \mathbb{Z}/p^2\mathbb{Z} \) by multiplication, we obtain the orbits

\[
\begin{align*}
X_1 &= gA, \\
X_2 &= g^2A, \\
& \vdots \\
X_{p-1} &= g^{p-1}A, \\
X_p &= A, \\
X_{p+1} &= \{p, 2p, \ldots, (p-1)p\}, \\
X_{p+2} &= \{0\},
\end{align*}
\]

where \( g \) denotes a primitive root modulo \( p^2 \) that will remain fixed throughout this paper. We have adopted this somewhat unusual labeling scheme in order to simplify the structure of certain matrices and streamline a number of formulas which appear later. For \( 1 \leq i, j \leq p \), we find that

\[
\sigma_j(X_i) = \sum_{\ell=1}^{p-1} c \left( \frac{g^{ij} \ell p^2}{p^2} \right) = \sum_{\ell=1}^{p-1} c \left( \frac{g^{ij+\ell p}}{p^2} \right) = H_p(g^{ij}).
\]

We pause to make the following simple observation.

**Lemma 3.** The value of \( H_p(g^k) \) depends only upon \( k \) (mod \( p \)).

Upon performing some additional elementary computations to evaluate the remaining values of \( \sigma_j(X_i) \), we obtain the supercharacter table corresponding to the supercharacter theory on \( \mathbb{Z}/p^2\mathbb{Z} \) arising from the action of \( A \) (see Table 2). Also of relevance is the unitary matrix \( U \) defined by (3), which is given by

\[
U = \frac{1}{p} \begin{bmatrix}
H_p(1) & H_p(g) & H_p(g^2) & \cdots & H_p(g^{p-1}) \\
H_p(g) & H_p(g^2) & H_p(g^3) & \cdots & H_p(g^{p}) \\
H_p(g^2) & H_p(g^3) & H_p(g^4) & \cdots & H_p(g^p) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_p(g^{p-1}) & H_p(1) & H_p(g) & \cdots & H_p(g^{p-2}) \\
-1 & -1 & -1 & \cdots & -1 \\
\sqrt{p-1} & \sqrt{p-1} & \sqrt{p-1} & \cdots & \sqrt{p-1}
\end{bmatrix}.
\]
Proof. It suffices to establish that
\[\sigma \] since each eigenvalue
\[b \] we have
\[c \] we conclude that
\[g \] Lemma 4. If
\[z \] the number of solutions
\[X_1 \times X_j \] to
\[x + y \equiv z \pmod{p^2}, \] (13)
where \(z\) is a fixed element of \(X_k\), recalling that the value of \(c_{i,j,k}\) is independent of the representative \(z\) of \(X_k\). Lemma 4 ensures that \(U\) simultaneously diagonalizes the matrices \(T_1, T_2, \ldots, T_{p+2}\) whose entries are given by (5). To be more specific, we have \(T_1 U = U D_1\), where
\[D_i = \text{diag} (\sigma_i (X_1), \sigma_i (X_2), \ldots, \sigma_i (X_{p+2})). \] (14)
Since each eigenvalue \(\sigma_i (X_k)\) is real and \(U\) is unitary, it follows that each \(T_i\) is real and symmetric. In order to describe the matrices \(T_1, T_2, \ldots, T_p\), we first require a few elementary facts about the \(c_{i,j,k}\).

Lemma 4. If \(1 \leq i, j, k \leq p\), then \(c_{i,j,k} = c_{\pi(i,j,k)}\) for any permutation \(\pi(i, j, k)\).

Proof. It suffices to establish that \(c_{i,j,k} = c_{j,i,k}\) and \(c_{i,j,k} = c_{i,k,j}\). Since \(c_{i,j,k}\) denotes the number of solutions \((a^p, b^p)\) in \(A \times A\) to
\[a^p g^i + b^p g^j \equiv g^k \pmod{p^2}, \] (15)
it follows immediately that \(c_{i,j,k} = c_{j,i,k}\). On the other hand, if we let \(b'\) denote the inverse of \(b\) modulo \(p\), then (15) is equivalent to \((-ab')^p g^i + (b')^p g^k \equiv g^j \pmod{p^2}\) by Lemma 2. Since the coefficients of \(g^i\) and \(g^j\) in the preceding range over \(A \times A\), we conclude that \(c_{i,j,k} = c_{i,k,j}\). \(\square\)
Lemma 5. If $1 \leq i \leq p$ and $j \neq i$, then

$$\sum_{k=1}^{p+2} c_{i,j,k} = p - 1. \quad (16)$$

Proof. We first note that if $1 \leq i \leq p$ and $j \neq i$, then $c_{i,j,p+2} = 0$, since $a^p g^i + b^p g^j \equiv 0 \pmod{p^2}$ has no solutions when $i \neq j$. Indeed, the preceding is equivalent to $a^p g^{i-j} \equiv (-b)^p \pmod{p^2}$, which is inconsistent because $g^{i-j} A \cap A = \emptyset$. Since $c_{i,j,p+2} = 0$ and $(\mathbb{Z}/p^2\mathbb{Z}) \setminus \{0\} = X_1 \cup X_2 \cup \cdots \cup X_{p+1}$, it follows that any sum of the form $x + y$, where $x$ and $y$ belong to $X_i$ and $X_j$, respectively, also belongs to $(\mathbb{Z}/p^2\mathbb{Z}) \setminus \{0\}$. For $1 \leq k \leq p - 1$, the superclass $X_k$ has precisely $p - 1$ distinct representatives whence $x + y$ belongs to $X_k$ for precisely $(p - 1) c_{i,j,k}$ pairs $(x,y)$ in $X_i \times X_j$. Thus $(p - 1)^2 = |X_i \times X_j| = \sum_{k=1}^{p+1} (p - 1) c_{i,j,k}$, which implies (16). \(\square\)

We now have all of the information required to describe the general structure of $T_1, T_2, \ldots, T_p$.

Lemma 6. If $1 \leq i \leq p$, then

$$T_i = \begin{bmatrix} 0 & \cdots & 0 & C_i & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sqrt{p-1} & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad (17)$$

where $C_i = [c_{i,j,k}]_{j,k=1}^p$.

Proof. Suppose that $1 \leq i \leq p$. Since $T_i$ is real and symmetric, it suffices to establish that the final two columns of $T_i$ have the desired form. In what follows, $a$ and $b$ denote units modulo $p$.

We first show that the upper-right $p \times 2$ submatrix is of the form claimed. Let us consider the coefficients $c_{i,j,p+1}$ for $j \neq i$. Since

$$a^p g^i + b^p g^j \equiv p \pmod{p^2} \iff a g^i + b g^j \equiv 0 \pmod{p},$$

for each fixed $a$ we may let $b \equiv -a g^{i-j} \pmod{p}$ to obtain a solution to the preceding congruences. In particular, this implies that $c_{i,j,p+1} \geq 1$ for $j \neq i$. However, Lemmas 4 and 5 tell us that $\sum_{j=1}^{p+2} c_{i,j,p+1} = p - 1$, from which it follows that

$$c_{i,j,p+1} = \begin{cases} 1 & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases} \quad (18)$$

as claimed. Turning our attention to the final column of $T_i$, we note that the proof of Lemma 5 tells us that $c_{i,j,p+2} = 0$ for $j \neq i$. Moreover, $c_{i,j,p+2} = p - 1$ for

$$0 \leq j \leq$$

and

$$p \leq j \leq p - 1.$$
1 \leq i \leq p \text{ since}\n\begin{align*}
a^p g^i + b^p g^i & \equiv 0 \pmod{p^2} \\
g^i(a^p + b^p) & \equiv 0 \pmod{p^2}
\end{align*}

has exactly $p - 1$ solutions $\{(a, -a) : 1 \leq a \leq p - 1\}$. In other words,

$$c_{i,j,p+2} = \begin{cases} 
0 & \text{if } j \neq i, \\
p - 1 & \text{if } j = i, 
\end{cases} \quad (19)$$

That the lower-right $2 \times 2$ submatrix of $T_i$ is identically zero follows easily from the fact that $a^p g^i$ is a unit modulo $p^2$. \hfill \Box

4. The third moment and Fermat’s Last Theorem

Although it is not obvious from their definition, Heilbronn’s exponential sums are intimately related to a certain family of congruences connected to Fermat’s Last Theorem. Since the details and history of Fermat’s Last Theorem are well-known, we make no attempt to discuss the topic in depth, recalling only that this famous conjecture (proved by Andrew Wiles [20]), asserts that the equation $x^n + y^n = z^n$ has no integral solutions $x, y, z \geq 1$ if $n \geq 3$. Moreover, the general case can be easily reduced to the consideration of odd prime exponents.

**Theorem 1.** If $p \nmid abc$, then the number of solutions $(x, y, z)$ in $(\mathbb{Z}/p^2\mathbb{Z})^3$ to the generalized Fermat congruence

$$ax^p + by^p \equiv cz^p \pmod{p^2} \quad (20)$$

which satisfy $p \nmid xyz$ is precisely

$$p^3(p - 1)F(p; a, b, c), \quad (21)$$

where $F(p; a, b, c)$ denotes the nonnegative integer

$$F(p; a, b, c) = 1 - \frac{2}{p} + \frac{1}{p^2} \sum_{\ell=1}^{p} H_p(ag^\ell)H_p(bg^\ell)H_p(cg^\ell), \quad (22)$$

where $g$ denotes a primitive root modulo $p^2$. In particular, the equation

$$ax^p + by^p \equiv cz^p$$

has no solutions in integers with $p \nmid xyz$ whenever $F(p; a, b, c) = 0$.

**Proof.** If $p \nmid abc$ then $a, b, c$ are congruent modulo $p^2$ to some powers $g^i, g^j, g^k$ of $g$. We may assume without loss of generality that $1 \leq i, j, k \leq p$ since $g^{i+j+k} \equiv g^i(g^jx)^p \equiv g^i \pmod{p^2}$ and so forth. Computing the $(j, k)$ entry of the matrix identity $T_i = UD_iU$ reveals that

$$c_{i,j,k} = \frac{\sqrt{|X_i|}}{p^2 \sqrt{|X_k|}} \sum_{\ell=1}^{p+2} \sigma_i(X_\ell)\sigma_j(X_\ell)\sigma_k(X_\ell)$$

$$= \frac{1}{p^3} \sum_{\ell=1}^{p+2} \sigma_i(X_\ell)\sigma_j(X_\ell)\sigma_k(X_\ell)$$

\[ \frac{1}{p^2 |X_k|} \sum_{\ell = 1}^{p + 2} |X_\ell| \sigma_{\ell}(X_\ell) \sigma_i(X_\ell) \sigma_k(X_\ell), \]

the last line following from the fact that \( U = U^T \). Thus

\[ p^2(p - 1)c_{i,j,k} = (p - 1) \sum_{\ell = 1}^{p} H_p(\alpha g^\ell) H_p(\beta g^\ell) H_p(\gamma g^\ell) + (p - 1)(-1)^3 + (p - 1)^3, \]

which yields

\[ \sum_{\ell = 1}^{p} H_p(\alpha g^\ell) H_p(\beta g^\ell) H_p(\gamma g^\ell) = p^2(c_{i,j,k} - 1) + 2p. \tag{23} \]

Now recall that \( c_{i,j,k} \) denotes the number of solutions to the congruence

\[ g^i x^p + g^j y^p \equiv g^k \pmod{p^2} \]

with \( 1 \leq x, y \leq p - 1 \). Since there are \( p - 1 \) different representatives of the superclass \( X_k = g^k A \), there are \( (p - 1)c_{i,j,k} \) solutions to \( \text{20} \) with \( 1 \leq x, y, z \leq p - 1 \). By considering \( (x + rp, y + sp, z + tp) \) for \( 0 \leq r, s, t \leq p - 1 \) we obtain \( p^3(p - 1)c_{i,j,k} \) distinct solutions to \( \text{20} \). Solving \( \text{23} \) for \( c_{i,j,k} \) immediately provides us with the desired formulas \( \text{21} \) and \( \text{22} \). \( \square \)

As the preceding theorem illustrates, cubic sums of Heilbronn sums control, in a precise manner, whether the generalized Fermat congruence \( \text{20} \) possesses any nontrivial solutions. Indeed, we consider a solution satisfying \( p | xyz \) trivial since if, say \( p | x \), the congruence reduces to \( by^p \equiv cz^p \pmod{p^2} \), which has no solutions if \( b = g^i \) and \( c = g^j \) for \( i \not\equiv j \pmod{p} \) and has only the \( p(p - 1) \) obvious solutions otherwise.

We remark that \( \text{22} \) can be used to efficiently evaluate \( F(p; a, b, c) \) for many triples \( (a, b, c) \) in succession. In fact, one can compute \( F(p; a, b, c) \) for all triples \( (a, b, c) \) simultaneously by taking advantage of the identity \( T_i = U D_i U \) and fast matrix multiplication. These algorithms are much faster than the naive approach of counting the number of solutions to \( \text{20} \). Appendix A contains a detailed justification of these claims.

We present in Table 3 numerical values of the function \( F(p) = F(p; 1,1,1) \), which corresponds to the classical Fermat congruence \( x^p + y^p \equiv z^p \pmod{p^2} \). In particular, \( F(p) = 0 \) occurs precisely when the Fermat equation \( x^p + y^p = z^p \) has no solutions in integers satisfying \( p | xyz \).

At this point it is worth mentioning Kummer’s proof of Fermat’s Last Theorem for regular primes. Recall that a prime \( p \) is called *regular* if \( p \) does not divide the class number of the cyclotomic field \( \mathbb{Q}(\zeta) \) where \( \zeta = e^{\frac{2\pi i}{p}} \). It is well-known that \( p \) is regular if and only if \( p \) does not divide the numerator of the Bernoulli numbers \( B_2, B_4, \ldots, B_{p-3} \) [19, p. 198]. Although Kummer himself believed that there are infinitely many regular primes, this conjecture remains open. On the other hand, Jensen proved that there are infinitely many irregular primes (i.e., primes which are not regular), the first few of which are

37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, 307, 311, 347, 353, 379, 389, 401, 409, 421, 433, 461, 463, 467, 491, 523,
The first major step in Kummer’s approach is establishing that if \( p \) is an odd regular prime, then \( x^p + y^p = z^p \) has no integral solutions with \( p \nmid xyz \) (in the terminology of [4], this is referred to as the first case of Fermat’s Last Theorem).

A glance at Table 3 reveals we have actually established that the Fermat equation \( x^p + y^p = z^p \) has no integral solutions \( x, y, z \geq 1 \) with \( p \nmid xyz \) if \( p \) is one of the irregular primes highlighted in boldface above.

| \( p \) | \( F(p) \) | \( p \) | \( F(p) \) | \( p \) | \( F(p) \) | \( p \) | \( F(p) \) | \( p \) | \( F(p) \) |
|---|---|---|---|---|---|---|---|---|---|
| 3  | 0  | 127 | 2  | 283 | 0  | 461 | 0  | 647 | 0  | 853 | 2  |
| 5  | 0  | 131 | 0  | 283 | 2  | 463 | 2  | 653 | 0  | 857 | 6  |
| 7  | 2  | 137 | 0  | 293 | 0  | 467 | 0  | 659 | 0  | 859 | 2  |
| 11 | 0  | 139 | 2  | 307 | 2  | 479 | 0  | 661 | 2  | 863 | 0  |
| 13 | 2  | 149 | 0  | 311 | 0  | 487 | 2  | 673 | 2  | 877 | 2  |
| 17 | 0  | 151 | 2  | 313 | 2  | 491 | 0  | 677 | 0  | 881 | 0  |
| 19 | 2  | 157 | 2  | 317 | 0  | 499 | 2  | 683 | 0  | 883 | 2  |
| 23 | 0  | 163 | 2  | 331 | 2  | 503 | 0  | 691 | 8  | 887 | 6  |
| 29 | 0  | 167 | 0  | 337 | 8  | 509 | 0  | 701 | 12 | 907 | 8  |
| 31 | 2  | 173 | 0  | 347 | 0  | 521 | 0  | 709 | 2  | 911 | 6  |
| 37 | 2  | 179 | 6  | 349 | 2  | 523 | 2  | 719 | 0  | 919 | 2  |
| 41 | 0  | 181 | 2  | 353 | 0  | 541 | 2  | 727 | 2  | 929 | 6  |
| 43 | 2  | 191 | 0  | 359 | 0  | 547 | 8  | 733 | 2  | 937 | 2  |
| 47 | 0  | 193 | 8  | 367 | 2  | 557 | 0  | 739 | 2  | 941 | 0  |
| 53 | 0  | 197 | 0  | 373 | 2  | 563 | 0  | 743 | 0  | 947 | 0  |
| 59 | 12 | 199 | 2  | 379 | 2  | 569 | 0  | 751 | 2  | 953 | 0  |
| 61 | 2  | 211 | 2  | 383 | 0  | 571 | 2  | 757 | 8  | 967 | 2  |
| 67 | 2  | 223 | 2  | 389 | 0  | 577 | 2  | 761 | 0  | 971 | 6  |
| 71 | 0  | 227 | 6  | 397 | 2  | 587 | 0  | 769 | 2  | 977 | 6  |
| 73 | 2  | 229 | 2  | 401 | 0  | 593 | 0  | 773 | 0  | 983 | 0  |
| 79 | 8  | 233 | 0  | 409 | 2  | 599 | 0  | 787 | 8  | 991 | 2  |
| 83 | 6  | 239 | 0  | 419 | 6  | 601 | 8  | 797 | 0  | 997 | 2  |
| 89 | 0  | 241 | 2  | 421 | 8  | 607 | 2  | 809 | 0  | 1009 | 2  |
| 97 | 2  | 251 | 0  | 431 | 0  | 613 | 2  | 811 | 2  | 1013 | 0  |
| 101 | 0 | 257 | 0  | 433 | 2  | 617 | 0  | 821 | 0  | 1019 | 0  |
| 103 | 2 | 263 | 0  | 439 | 2  | 619 | 8  | 823 | 2  | 1021 | 2  |
| 107 | 0 | 269 | 0  | 443 | 6  | 631 | 2  | 827 | 0  | 1031 | 0  |
| 109 | 2 | 271 | 2  | 449 | 0  | 641 | 0  | 829 | 2  | 1033 | 2  |
| 113 | 0 | 277 | 2  | 457 | 8  | 643 | 2  | 839 | 0  | 1039 | 8  |

Table 3. Values of \( F(p) = F(p; 1, 1, 1) \) as \( p \) ranges over the first 174 odd primes. Primes \( p \) for which \( F(p) = 0 \) satisfy the property that the corresponding Fermat equation \( x^p + y^p = z^p \) has no solutions in integers with \( p \nmid xyz \).

5. The fourth moment and Heath-Brown’s bound

Using the fact that the matrices \( T_i \) are simultaneously unitarily diagonalizable, we can obtain a variety of quartic formulas involving Heilbronn sums. For
instance, a special case of the following formula can be used to obtain HeathBrown’s bound \( H_p(u) \ll p^{11/12} \) for \( p \mid u \).

**Theorem 2.** Letting \( g \) denote a primitive root modulo \( p^2 \), for \( 1 \leq i, j, k, \ell \leq p \) we have

\[
p^2 \sum_{r=1}^{p} c_{i,k,r}c_{j,\ell,r} = \sum_{r=1}^{p} H_p(g^{i+r}) H_p(g^{j+r}) H_p(g^{k+r}) H_p(g^{\ell+r})
\]

\[
+ \begin{cases} 
  -2p^2 + 3p & \text{if } i = k \text{ and } j = \ell, \\
  p^3 - 4p^2 + 3p & \text{if } i \neq k \text{ and } j \neq \ell, \\
  0 & \text{otherwise}.
\end{cases}
\]

In particular,

\[
\sum_{\ell=1}^{p} H_p^4(g^\ell) = p^2 \sum_{\ell=1}^{p} c_{i,i,\ell}^2 + 2p^2 - 3p.
\] (24)

**Proof.** Since \( U = U^* \) it follows from Lemma \([\text{I}]) that \( T_iT_j = UD_jD_iU \) for \( 1 \leq i, j \leq p + 2 \). Letting \( 1 \leq i, j, k, \ell \leq p \) and recalling that \( |X_i| = p - 1 \) in this range, it follows from the symmetry of \( T_j \), \([\text{I}9]) \) and \([\text{I}9]) \) that

\[
[T_iT_j]_{k,\ell} = \frac{1}{p - 1} \sum_{r=1}^{p+2} c_{i,k,r}c_{j,\ell,r}
\]

\[
= \frac{1}{p - 1} \sum_{r=1}^{p+2} \frac{c_{i,k,r}\sqrt{|X_r|}}{|X_k|} \cdot \frac{c_{j,\ell,r}\sqrt{|X_r|}}{|X_\ell|}
\]

\[
= \frac{1}{p - 1} \sum_{r=1}^{p+2} |X_r| c_{i,k,r}c_{j,\ell,r} + \begin{cases} 
  p - 1 & \text{if } i = k \text{ and } j = \ell, \\
  1 & \text{if } i \neq k \text{ and } j \neq \ell, \\
  0 & \text{otherwise}.
\end{cases}
\]

On the other hand, using the fact that \( U = U^T \) we have

\[
[UD_jD_iU]_{k,\ell} = \frac{1}{p^2} \sum_{r=1}^{p+2} \frac{\sigma_k(X_r)\sqrt{|X_r|}}{|X_k|} \cdot \sigma_i(X_r)\sigma_j(X_r) \cdot \frac{\sigma_r(X_\ell)\sqrt{|X_\ell|}}{|X_r|}
\]

\[
= \frac{1}{p^2} \sum_{r=1}^{p+2} \frac{|X_r|\sigma_i(X_r)\sigma_j(X_r)\sigma_k(X_r)\sigma_\ell(X_\ell)}{p - 1}
\]

\[
= \frac{1}{p^2} \left( \sum_{r=1}^{p} \sigma_i(X_r)\sigma_j(X_r)\sigma_k(X_r)\sigma_\ell(X_\ell) + 1 + (p - 1)^3 \right)
\]

\[
= \frac{1}{p^2} \left( \sum_{r=1}^{p} H_p(g^{i+r}) H_p(g^{j+r}) H_p(g^{k+r}) H_p(g^{\ell+r}) + 1 + (p - 1)^3 \right).
\]

Equating our expressions for the matrix entries \([T_iT_j]_{k,\ell}\) and \([UD_jD_iU]_{k,\ell}\) yields the desired result. \(\square\)
As a direct consequence of the preceding, we obtain for \( p \nmid u \) the estimate
\[
H_p(u) \ll p^{\frac{1}{2}} \left( \sum_{k=1}^{p} c_{i,j,k}^2 \right)^{\frac{1}{4}},
\]
thereby recovering Heath-Brown’s observation [12, Lem. 1]. Building upon the preceding, the following result should be of interest to those seeking to obtain more precise bounds on Heilbronn sums. In particular, it suggests that dramatic improvements in the estimation of the quantities \( c_{i,j,k} \) should lead directly to improved bounds on Heilbronn sums.

**Theorem 3.** If \( 1 \leq i \leq p \), then
\[
\max_{1 \leq k \leq p} c_{i,j,k} \ll p^{\beta} \quad \Rightarrow \quad H_p(u) \ll p^{\frac{3}{4} + \beta}
\]
whenever \( p \nmid u \).

We defer the proof of Theorem 3 until the end of this section. Instead, we remark that the preceding theorem recovers Heath-Brown’s nontrivial estimate [11]. Indeed, it follows from a theorem of Mit’kin [15] that we may take \( \beta = \frac{2}{3} \) in (26). See Lemma 10 in Appendix B for a detailed explanation.

**Corollary 1** (Heath-Brown). If \( p \nmid u \), then \( H_p(u) \ll p^{\frac{11}{12}} \).

Before proving Theorem 3, we require the following simple lemma.

**Lemma 7.** For \( 1 \leq i \leq p \),
\[
\sum_{k=1}^{p} c_{i,j,k} = p - 2.
\]

For \( 0 < \alpha < 1 \) the number of \( k \) for which \( c_{i,j,k} \geq p^{\alpha} \) and \( 1 \leq k \leq p \) is less than \( p^{1-\alpha} \).

**Proof.** First observe that there are exactly \( (p - 1)^2 \) pairs \((a, b)\) with \( 1 \leq a, b \leq p - 1 \). Since \( c_{i,j,k} \) is independent of the representative from \( X_k \) which is chosen, it follows that as \((x, y)\) ranges over \( X_1 \times X_2 \), the sum \( x + y \) assumes values in \( X_k \) exactly \( |X_k| c_{i,j,k} \) times. In light of (18) and (19), we obtain
\[
(p - 1)^2 = \sum_{k=1}^{p+2} |X_k| c_{i,j,k} = \sum_{k=1}^{p} (p - 1) c_{i,j,k} + 0 + (p - 1),
\]
which implies (27). Letting \( M \) denote the number of \( k \) for which \( c_{i,j,k} \geq p^{\alpha} \) and \( 1 \leq k \leq p \), we see that \( p > \sum_{j=1}^{p} c_{i,j,k} \geq M p^{\alpha} \), whence \( M \leq p^{1-\alpha} \), as claimed. \( \square \)

**Pf. of Theorem 3.** Using the fact that \( c_{i,j,k} \geq 0 \) for all \( k \), repeated applications of the Cauchy-Schwarz inequality yield
\[
\sum_{k=1}^{p} c_{i,j,k}^2 = \sum_{k=1}^{p} c_{i,j,k} c_{k,j,k}^{\frac{1}{2}} \leq \left( \sum_{k=1}^{p} c_{i,j,k} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{p} c_{i,j,k}^3 \right)^{\frac{1}{2}}.
\]
\[ < p^{1/2} \left( \sum_{k=1}^{p} c_{i,j,k}^{2} \right)^{1/2} \]

by (27)

\[ \leq p^{1/2} \left( \sum_{k=1}^{p} c_{i,j,k}^{4} \right)^{1/4} \]

by (27).

Let \( 0 < \alpha < 1 \) and continue in the preceding fashion to obtain for each natural number \( n \) the inequality

\[ \sum_{k=1}^{p} c_{i,j,k}^{2} < p^{1/2} \left( \sum_{k=1}^{p} c_{i,j,k}^{2n+1} \right)^{1/p} \]

\[ = p^{1-1/p} \left( \sum_{k=1}^{p} c_{i,j,k}^{2n+1} \right)^{1/p} \]

\[ = p^{1-1/p} \left( \sum_{c_{i,j,k} < p^{n}} c_{i,j,k}^{2n+1} + \sum_{c_{i,j,k} \geq p^{n}} c_{i,j,k}^{2n+1} \right)^{1/p} \]

\[ \ll p^{1-1/p} \left( p \cdot p^{\alpha(2n+1)} + p^{1-\alpha} \cdot p^{\beta(2n+1)} \right)^{1/p} \]

by Lem. 7

\[ = p^{1-1/p} \left( p^{1+\alpha(2n+1)} + p^{1-\alpha+\beta(2n+1)} \right)^{1/p} \]  

(28)

Setting

\[ 1 + \alpha(2n+1) = 1 - \alpha + \beta(2n+1) \]

and solving for \( \alpha \) we find that

\[ \alpha = \frac{\beta(2n+1)}{2^n + 2}. \]

Plugging this value of \( \alpha \) into (28) we obtain

\[ \sum_{k=1}^{p} c_{i,j,k}^{2} \ll p^{1-1/p} \left( 2p^{1+\frac{\beta(2n+1)^2}{2^n+2}} \right)^{1/p} \]

\[ = 2^{1/p} p^{1-1/p} p^{\frac{\beta(2n+1)}{2^n+2}} \]

\[ = 2^{1/p} p^{1+\frac{\beta(2n+1)}{2^n+2}}. \]

Letting \( n \) tend to infinity we obtain the inequality

\[ \sum_{k=1}^{p} c_{i,j,k}^{2} \ll p^{1+\beta}. \]

Plugging this into (25) we obtain (26). \[ \square \]
Appendix A. The computational cost of finding $c_{i,j,k}$

Building upon the material in Section 4, we consider the computational cost of finding $c_{i,j,k} = F(p; g^i, g^j, g^k)$ and, subsequently, how many computations are needed to evaluate $c_{i,j,k}$ for all $i, j, k$ simultaneously. In both instances, the method of Theorem 1 is faster than the naïve method. In particular, computing the number of solutions to a generalized Fermat congruence (20) can be done much more rapidly using the method of Heilbronn sums than by brute force.

Recall that $g(x) = \Theta(f(x))$ if there exist constants $c_1, c_2$ so that $c_1 f(x) \leq g(x) \leq c_2 f(x)$ holds for sufficiently large $x$. Since $\log_2(d) = \Theta(\log d)$, we shall ignore subscripts in logarithms. A flop is one of the following floating point operations: addition, subtraction, multiplication, division, and cosine. In order to characterize the computation cost associated with a flop, it is necessary to break each operation down into its component parts. A bit operation is an addition or multiplication of two bits.

- Addition and subtraction of two $d$ bit numbers can be carried out using the usual bit carry algorithm with $\Theta(d)$ bit operations.
- Currently, the fastest algorithm (asymptotically) for floating point multiplication is Fürer’s algorithm [10], which multiplies $d$ bit numbers using $\Theta(d \log(d) \exp(O(\log^*(d))))$ operations. Here $\log^*(d)$ denotes the iterated logarithm of $d$, which is the number of times the logarithm must be applied until the result is at most 1. In particular, $\exp(O(\log^*(d)))$ is also $O(\log(\log(d)))$.
- A division $x/y$ can be computed by using Newton’s method to approximate $1/y$ and then multiplying $x(1/y)$. Newton’s method is quadratic in convergence and requires a multiplication and a subtraction at each step. To obtain accuracy to $d$ bits requires $\Theta(\log(d))$ multiplications.
- Evaluating cosine to $d$ bits also uses $O(\log(d))$ multiplications by the Arithmetic-Geometric Mean approach [16, Table 5.3, p. 100]. Since each multiplication requires only $O(\log(d) \log(\log(d)))$ operations, finding cosine to $d$ bits uses only $O(d \log^2(d) \log(\log(d)))$ bit operations.

These results are collected in Table 4.

| Floating Point Operation | Number of bit operations |
|--------------------------|--------------------------|
| Addition/Subtraction     | $\Theta(d)$              |
| Multiplication           | $O(d \log(d) \log(\log(d)))$ |
| Division/Cosine          | $O(d \log^2(d) \log(\log(d)))$ |

Table 4. Number of bit operations per flop.

**Proposition 1.** The number of bit operations to compute $x^p \pmod{p^2}$ is

$$O(\log^2(p) \log(\log(p)) \log(\log(\log(p)))).$$
Proof. Write $p$ in binary so $p = a_0 + a_12 + a_22^2 + \cdots + a_k2^k$, where $k = \lfloor \log(p) \rfloor + 1$ and $a_i \in \{0, 1\}$. Next observe that
\[
x^p = ((\cdots (x^{a_k})^2 \cdots )^2 \cdot x^{a_1})^2 \cdot x^{a_0}.
\]
For instance, 13 is 1101 base 2, and $x^{13} = ((x^1)^2 \cdot x^1)^2 \cdot x^1$. This expression for $x^p$ has $O(\log(p))$ multiplications, each of which is done modulo $p^2$ and so with $O(\log(p))$ bits. The result immediately follows. \qed

Proposition 2. The brute force approach to counting the number of solutions to $ax^p + by^p \equiv cz^p \pmod{p^2}$ requires $O(p^3 \log(p))$ bit operations.

Proof. Since $x^p \equiv y^p \pmod{p^2}$ if and only if $x \equiv y \pmod{p}$ (see Lemma 2), it is only necessary to evaluate the left and right hand sides for $p^3$ triples $(x, y, z)$.

It takes $O(p^2 \log(p) \log(\log(p)))$ bit operations to precompute $w^p$ for all values of $w$ in $(\mathbb{Z}/p^2\mathbb{Z})^3$; this is dominated by the $O(p^3)$ number of triples. Each evaluation requires one addition, and each number in $\{0, 1, \ldots, p^2 - 1\}$ requires $O(\log(p))$ bits to represent, hence the evaluations can be done in $O(p^3 \log(p))$ bit operations. \qed

The method of Heilbronn sums given by Theorem 1 is much faster.

Proposition 3. The Heilbronn sums approach can calculate $F = F(p, s^i, t^j, r^k)$ exactly using
\[
O(p^2 \log^2(p) \log(\log(p)))
\]
bit operations.

Proof. As with the naïve approach, start by precomputing the values of $\ell^p \pmod{p^2}$ for $\ell \in \{1, 2, \ldots, p - 1\}$, taking $O(p \log(p) \log(\log(p)))$ bit operations.

To calculate each value of $H_p(a)$, the $p - 1$ values of $\ell^p$ must be multiplied by $a$, then multiplied by $2\pi/p^2$ (which only needs to be computed once). Since Heilbronn sums are real, only $\cos(2\pi \ell^p/p^2)$ needs to be computed for each $\ell$. The resulting $(p - 1)/2$ values are summed to obtain $H_p(a)$, requiring one division and $\Theta(p)$ flops. Since each computation of $H_p$ uses $\Theta(p)$ flops, the overall computation of $c_{i,j,k}$ uses $\Theta(p^2)$ flops.

However, the naïve method requires only integer multiplication, while the Heilbronn sums approach requires floating point arithmetic. To truly compare the methods, it is necessary to understand how error propagates and how many bits of precision are needed.

Let $\tilde{H}_p$ denote the floating number approximation to $H_p$. Suppose $|\tilde{H}_p - H_p| < \epsilon_1 < 1$, and let
\[
\tilde{F} = 1 - \frac{2}{p} + \frac{1}{p^2} \sum_{\ell=1}^{p} \tilde{H}_p(\ell g^\ell)\tilde{H}_p(b g^\ell)\tilde{H}_p(c g^\ell)
\]
be our approximation to the true value of $F$. For any six numbers $x, y, z$ and $\tilde{x}, \tilde{y}, \tilde{z}$ with $\max\{|x - \tilde{x}|, |y - \tilde{y}|, |z - \tilde{z}|\} \leq \epsilon \leq 1$,
\[
|\tilde{x}\tilde{y}\tilde{z} - xyz| \leq |xyz||(1 + \epsilon)^3 - 1| \leq 7\epsilon|xyz|.
\]
Using the trivial bound $|H_p(u)| \leq p - 1$ for $p \nmid u$, it follows that

$$|\tilde{H}_p(a\ell^p)\tilde{H}_p(b\ell^p)\tilde{H}_p(c\ell^p) - H_p(a\ell^p)H_p(b\ell^p)H_p(c\ell^p)| \leq 7(p - 1)^3\varepsilon_1.$$ 

Using the above bound together with a floating point representation for $1 - 2/p$ with error at most $\varepsilon_1$ yields

$$|\tilde{F} - F| \leq \varepsilon_1 + 7(p - 1)^3/p^2\varepsilon_1 = 7p\varepsilon_1.$$

If $\varepsilon_1 < (14p)^{-1}$ then rounding $F$ to the nearest integer returns $F$.

Now consider the error in $H_p(a)$. Each of the individual $a\ell^p$ are computed exactly mod $p^2$. Suppose $2\pi/p^2$ is computed to within an additive error of $\varepsilon_2$. Since the $a\ell^p$ are mod $p^2$, and hence at most $p^2 - 1$, the additive error in $a\ell^p(2\pi/p^2)$ is at most $p^2\varepsilon_2$. The derivative of cosine is absolutely bounded by 1, so the absolute error in each $e(\cdot)$ is also at most $p^2\varepsilon_2$. Adding $p$ of these terms together to form $H_p(a)$ makes the absolute error at most $p^3\varepsilon_2$. Hence the goal is to make

$$p^3\varepsilon_2 < (14p)^{-1} \implies \varepsilon_2 < (14p^4)^{-1}.$$ 

Evaluating $2\pi/p^2$ to $4\log_2 p + \log_2 14$ bits achieves this error bound. So the floating point operations and exponentiations can use only $O(\log(p))$ bits, and the result follows.

Now let us consider the computation of $c_{i,j,k}$ for all possible values of $a = g^i$, $b = g^j$, and $c = g^k$ in (20) simultaneously.

**Proposition 4.** The naïve method for finding $c_{i,j,k}$ for all $i,j,k$ uses $O(p^4\log(p)\log(\log(p)))$ bit operations.

**Proof.** Since the $c_{i,j,k}$ can be viewed as the number of solutions $(x,y)$ with $1 \leq x,y \leq p - 1$ to $x^p g^i + y^p g^j \equiv g^k \pmod{p^2}$, the naïve method here involves testing this equation for $(x,y,i,j) \in \{1,2,\ldots,p - 1\}^4$, and so requires $\Theta(p^4)$ multiplications and additions, each involving numbers that can be represented with $O(\log(p))$ bits. The result immediately follows. (Note that it is not necessary to test the equation for all $k$ as the values of $g^k$ can be precomputed in $\Theta(p^2)$ flops to make finding the inverse function a constant time operation.)

**Proposition 5.** Finding all the values of $c_{i,j,k}$ can be accomplished using Lemma 6 in $O(p^{3.3727}\log^2(p)\log(\log(p)))$ bit operations.

**Proof.** Finding all the values of $c_{i,j,k}$ is equivalent to calculating $T_i = UD_iU^*$ for all values of $i$. Here $D_i$ is the diagonal matrix in (7), so this is equivalent to finding the entries of $U$, $D_i$ doing the $\Theta(p^2)$ operations needed to find $UD_i$, and then the $\Theta(f_{MM}(p + 2))$ operations needed to find $UD_iU^*$, where $f_{MM}(n)$ denotes the number of operations for a matrix multiplication of two $n \times n$ matrices. Currently, the best result for matrix multiplication is $f_{MM}(n) = O(n^{2.3727})$ due to Williams [21]. To compute $U$ requires $H_p(g^{4\ell})$ for $\ell \in \{1,\ldots,p - 2\}$ together with the square root of $p - 1$ (as shown earlier, each $H_p(a)$ can be found with $\Theta(p)$ operations). So the final matrix multiplication dominates the number of operations. Using $\Theta(f_{MM}(p))$ for $p$ values of $i$ gives $\Theta(p^{3.3727})$ operations.
This gives the number of floating point operations; finding the number of bit operations requires analyzing the error propagation. Again let \( \hat{H}_P(a) \) is the floating point approximation to the true value \( H_P(a) \), where \(|\hat{H}_P(a) - H_P(a)| \leq \epsilon_1 \). (Similarly, there is \( P^{1/2} \) which is the floating point approximation to \( p^{1/2} \).)

The entries of \( UDU^* \) are the sum of \( p \) products of three \( H_P(a) \). As earlier, the difference between these products formed from the \( \hat{H}_P(a) \) versus the true \( H_P(a) \) is at most \( 7p^3 \epsilon_1 \) for \( \epsilon_1 \leq 1 \). Adding together \( p \) of these products increases the error to \( 7p^4 \epsilon_1 \). Hence the entries of \( UDU^* \) differ from their true values by \( 7p^4 \epsilon_1 \).

As earlier, if \( 2\pi/p^2 \) is represented to within an additive error of \( \epsilon_2 \), then \( \epsilon_1 \leq 2p \epsilon_2 \). Hence for the final error to be smaller than \( 1/2 \) (so that the simple rounding allows the exact \( c_{i,j,k} \) values to be found) it suffices to have

\[
7p^4 \epsilon_1 < 1/2 \iff 14p^6 \epsilon_2 < 1 \iff p < (\epsilon_2/14)^{1/6}.
\]

Using \( 6 \log_2(p) + \log_2(14) \) bits to approximate each \( \hat{H}_P(a) \) value makes the final error at most \( 1/2 \), then simple rounding gives the correct value of \( c_{i,j,k} \).

Table 5 summarizes these results.

| Method                              | Number of steps                          |
|-------------------------------------|-----------------------------------------|
| Naïve method                        | \( O(p^4 \log(p) \log(\log(p))) \)     |
| Heilbronn sum and matrix multiplication | \( O(p^{3.3727} \log^2(p) \log(\log(p))) \) |

**Table 5. Operations to compute \( c_{i,j,k} \) for all \( i, j, \) and \( k \).**

Note that if the simple \( \Theta(p^3) \) method is used to compute \( UDU^* \), the \( p^{3.3727} \) factor becomes \( p^4 \) and the improvement is lost.

**APPENDIX B. TRUNCATED LOGARITHMS AND MIT’KIN’S BOUND**

In what follows, the truncated logarithm

\[
L_p(u) = 
\left(
\frac{u^2}{2} + \cdot \cdot \cdot + \frac{u^{p-1}}{p-1}
\right)
\]

plays an important rôle. We include this material for the sake of readers more familiar with supercharacter theory than with analytic number theory. A detailed proof of the following can be found in the lecture notes of E. Kowalski [14].

**Lemma 8.** If \( u \not\equiv 0, 1 \pmod{p} \), then

\[
1 - (1 - u)^p \equiv u^p + pL_p(u) \pmod{p^2}.
\] (29)

If \( u \not\equiv 0 \pmod{p} \), then

\[
L_p(u) = -u^pL_p(u^{-1}) \pmod{p}.
\] (30)

The following important lemma was discovered independently by Mit’kin [15] and Heath-Brown [11]. The version stated below is from [14], which provides a detailed proof and an explicit constant based upon Stepanov’s method.
Lemma 9 (Mit’kin, Heath-Brown). The quantity
\[ \mathcal{N}_r = \{ 2 \leq x \leq p : L_p(x) \equiv r \, (\text{mod } p) \} \]
satisfies \(|\mathcal{N}_r| \leq 44p^{\frac{2}{3}}\) for all \(r\).

We require the preceding lemma in order to bound the \(c_{i,j,k}\) for \(1 \leq i, k \leq p\), since these are precisely the terms which appear in the expression \(25\).

Lemma 10. For \(1 \leq i, k \leq p\), we have \(c_{i,j,k} \ll 44p^{\frac{2}{3}}\)

Proof. The constant \(c_{i,j,k}\) denotes the number of solutions to the congruence \(x + y \equiv g^k \, (\text{mod } p^2)\) where both \(x\) and \(y\) belong to \(X_i\). Letting \(x = a^p g^i\) and \(y = (-b)^p g^i\), we find that
\[
\begin{align*}
c_{i,j,k} &= \# \{ (a, b) : 1 \leq a, b \leq p - 1, \quad a^p - b^p \equiv g^{k-i} \, (\text{mod } p^2) \} \\
&= \frac{1}{p-1} \# \{ (a, b, c) : 1 \leq a, b, c \leq p - 1, \quad a^p - b^p \equiv c^p g^{k-i} \, (\text{mod } p^2) \}.
\end{align*}
\]
Since \((a, a, c)\) is never a solution to the preceding congruence, we may perform the substitution \(b = a - au^{-1}\) where \(2 \leq u \leq p - 1\) to obtain
\[
\begin{align*}
c^p g^{k-i} &\equiv a^p - (a - au^{-1})^p \, (\text{mod } p^2) \\
&\equiv a^p (1 - (1 - u^{-1})^p) \, (\text{mod } p^2) \\
&\equiv a^p (1 - u^{-p} + pL_p(u^{-1})) \, (\text{mod } p^2) \\
&\equiv a^p (u^{-p} - u^{-p}pL_p(u)) \, (\text{mod } p^2) \\
&\equiv u^{-p}a^p (1 - pL_p(u)) \, (\text{mod } p^2).
\end{align*}
\]
Rewriting this as
\[
1 - pL_p(u) \equiv (uc)^p g^{k-i} \, (\text{mod } p^2),
\]
we see that for each of the \(p - 1\) possible values of \(uc\), the preceding congruence has at most \(44p^{2/3}\) corresponding \(u\). Therefore
\[
c_{i,j,k} \leq \frac{1}{p-1} \cdot (p-1)44p^{\frac{2}{3}} = 44p^{\frac{2}{3}}. \quad \square
\]

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