Superconductor-to-Spin-Density-Wave Transition in Quasi-One-Dimensional Metals with Ising Anisotropy

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Abstract

We study a mechanism for superconductivity in quasi-one-dimensional materials with Ising anisotropy. In an isolated chain Ising anisotropy opens a spin gap; if inter-chain coupling is sufficiently weak, single particle hopping is suppressed and the physics of coupled chains is controlled by a competition between pair hopping and exchange interaction. Spin density wave and triplet superconductivity phases are found separated by a first order phase transition. For particular parameter values a second order transition described by SO(4) symmetry is found.

I. INTRODUCTION

This paper study a mechanism of superconductivity for quasi-one-dimensional (Q1D) metals. Well known examples of Q1D materials exhibiting superconductivity are organic compounds $(\text{TMTSF})_2X$ and $(\text{TMTTF})_2X$. At this moment there is no consensus on the origin of the superconductivity this metals. Two possibilities are discussed in the literature. One is BCS theory with the attraction mediated by either phonons or magnons \cite{1,2}. The second approach developed by early workers in the field tries to accommodate the ‘gology’ picture of a one-dimensional conductor to the materials of interest \cite{3}. The crystal structure of the materials allows one to treat them as an array of the weakly coupled one-dimensional conductors formed by chains of flat molecules. It is assumed further that at high temperature every conductor can be approximately described as a Luttinger liquid (LL) characterized by dimensionless parameters $K_c, s$. Depending on those parameters, at least one of four susceptibilities, spin-density, charge-density, singlet Cooper pair or triplet Cooper pair, diverges at low temperature. In a three-dimensional crystal a phase with a broken symmetry corresponding to the most divergent susceptibility will be formed at low temperature. Thus, the efforts were invested into the study of the LL parameters as a function of the temperature and the pressure with the hope of finding conditions under which the Cooper pair susceptibility becomes divergent. Due to Coulomb repulsion, however, it was proven to be difficult to move the system into the region of divergent Cooper pair susceptibility \cite{4}.
The inability of the ‘g-ology’ picture to offer an explanation for the superconducting phase, by contrast to its success in describing the spin-density wave (SDW) phase [5,6] prompted many researchers to think in terms of a hybrid approach [1]. It is known that in the majority of Q1D materials SDW and Pierls-SDW phases are stable at ambient pressure. In order to stabilize superconductivity external pressure has to be applied. The pressure acts to increase the transverse coupling between chains driving the system closer to an anisotropic 3D Fermi liquid. It was suggested that this 3D Fermi liquid is a crucial requirement for the superconductivity while the SDW origin is essentially one-dimensional. With this agreed upon one can use BCS theory and its strong-coupling modification to derive different experimentally relevant quantities.

In this paper we will not follow this path. Instead, it will be demonstrated that both SDW and superconductivity can be derived within the framework of weakly coupled one-dimensional conductors. Our proposal is very similar in spirit, although different in detail to that of Carlson et. al. [7] who studied consequences of stripe formation in models of the cuprate superconductors.

In its current state the mechanism discussed in this paper cannot be applied directly to the organic superconductors because it requires substantial Ising anisotropy whereas in the organic materials deviations from the rotational symmetry are very small. Yet, it may be a useful step toward constructing viable mechanism for this materials. We discuss possible modifications at the end of the paper. It is also can be considered as a guide in the search for new superconducting materials. In addition to that it is interesting from purely theoretical point of view.

There are two parts to the proposed mechanism. At high energies inter-chain couplings can be neglected and the physics is purely one-dimensional. Unlike the ‘g-ology’ picture, however, it is accepted that chains are not simple Luttinger liquids in high energy regime. Instead, we assume that an interaction which is anisotropic in spin space opens a gap $\Delta_s$ in the spin sector.

Opening of the gap dramatically changes the low energy behavior of the system. For reasonable values of $K_c$ ($1/2 < K_c < 1$) there are now two relevant operators [8]. One is longitudinal SDW ($SDW_z$), the other is triplet superconductivity (TSC) dual to SDW. Provided that inter-chain transversal coupling is smaller then $\Delta_s$ an effective low energy hamiltonian can be easily derived. It describes the competition of this two order parameters. The intra-chain electron-electron repulsion makes $K_c$ smaller then unity. In this situation SDW is more relevant then TSC. Yet, under pressure, the influence of the next-to-nearest-neighbor coupling increases. This increases the effective Josephson coupling between the chains and decreases the effective exchange coupling. Therefore, it is possible to observe superconductivity under pressure.

An essential feature of the approach is its many-body character. Its main ingredient is severe spin-charge separation. Unlike the usual BCS mechanism, this route to superconductivity does not require attractive interaction.

Another interesting property of the mechanism is that for some parameter values the symmetry group of the low energy effective hamiltonian is enhanced. Generically, the symmetry is $U(1) \otimes U(1)$, one copy of $U(1)$ for the gauge transformation and another one for translation along the chains. If parameters of the hamiltonian are fine-tuned the symmetry group is enlarged to $SO(4)$. An extra symmetry is particle-hole rotation connecting SDW...
and TSC ground states.

The rest of the paper is organized as follows. In Section II we introduce high energy single-chain hamiltonian and discuss its spectrum and correlation functions. We introduce our low-energy hamiltonian in Section III. The phase diagram is calculated in Section IV. The effect of long range Coulomb interaction is considered in Section V. The SO(4) symmetry is derived in Section VI. Finally, Section VII contains discussion.

II. HIGH ENERGY DESCRIPTION OF THE SYSTEM

It is usually agreed upon that high energy physics of Q1D materials is purely one-dimensional. The hamiltonian of a single chain can be written as follows:

\[ H = \int_{-L}^{L} dx \mathcal{H}, \]

\[ \mathcal{H} = \mathcal{H}_c + \mathcal{H}_s \]

\[ \mathcal{H}_c = \frac{v_c}{2} \left( K_c^{-1} (\nabla \Phi_c)^2 + K_c (\nabla \Theta_c)^2 \right) \]

\[ \mathcal{H}_s = \frac{v_s}{2} \left( K_s^{-1} (\nabla \Phi_s)^2 + K_s (\nabla \Theta_s)^2 \right) + \frac{v_s g_{bs}}{\pi a^2} \eta_{L\uparrow} \eta_{R\uparrow} \eta_{R\downarrow} \eta_{L\downarrow} \cos \sqrt{8\pi} \Phi_s, \]

where we used the Abelian bosonization prescription [9]:

\[ \psi_{p\sigma}^\dagger (x) = (2\pi a)^{-1/2} \eta_{p\sigma} e^{i\sqrt{2\pi} \rho_{p\sigma} (x)} = (2\pi a)^{-1/2} \eta_{p\sigma} e^{i\sqrt{\pi/2} \left( \Theta_{c}(x) + p \Phi_{c}(x) + \sigma (\Theta_{s}(x) + p \Phi_{s}(x)) \right)}, \]

Equation (3) implies that the spectrum of charge excitations is gapless. The presence or the absence of the gap in the spin sector is a subtler matter, and depends on the relation between \( g_{bs} \) and \( K_s \). A renormalization group (RG) analysis of (4) leads to [9,10]:

\[ \frac{dK_s}{d\ell} = -2g_{bs}^2, \]

\[ \frac{dg_{bs}}{d\ell} = -2g_{bs} (K_s - 1). \]

The infrared properties in the spin sector are defined by the value of the RG flow invariant:

\[ I = g_{bs}^2 - (K_s - 1)^2. \]

For positive \( I \) the cosine term is relevant and the spin bosons spectrum has a gap. For \( I \leq 0 \) the cosine term scales down to zero and the spectrum is gapless. Physically, the value of \( I \) corresponds to the spin rotational symmetry of the interaction. If the interaction is easy axis (plane) then \( I > 0 \) (\( I < 0 \)).

Let us now assume that the interaction has an easy axis. In this case there is a gap.
\[ \Delta_s \propto \Lambda \exp \left( -\frac{\pi}{2} I^{-1/2} \right) \]  

in the spin sector. Here \( \Lambda \) is the ultraviolet cut-off energy.

Opening of the spin gap does not imply long-range ordering. All expectation values disallowed by symmetries of the Hamiltonian \( \mathcal{H} \) remain zero. For example, the operator density corresponding to SDW order takes the form:

\[ S_z(2k_F) = \frac{1}{2} \left( \psi_{L\uparrow}^\dagger \psi_{R\uparrow} - \psi_{L\downarrow}^\dagger \psi_{R\downarrow} \right) \propto e^{i\sqrt{2\pi} \Phi_c} \sin \sqrt{2\pi} \Phi_s. \]  

The non-vanishing spin gap fixes the field \( \sqrt{2\pi} \Phi_s \) near \( \pm \pi/2 \) so the expectation value \( \langle \sin \sqrt{2\pi} \Phi_s \rangle \) is non-zero. The SDW order parameter, however, is zero because charge boson field \( \Phi_c \) fluctuates:

\[ \langle S_z \rangle \propto \langle e^{i\sqrt{2\pi} \Phi_c} \rangle = 0. \]

The correlation function \( \langle S_z S_z \rangle \) decays algebraically with the exponent \( K_c \) for large separations.

The state with the spin gap was proposed and described first in Ref. [8].

III. INTERCHAIN COUPLING FOR \( \omega < \Delta_S \)

In this section we determine the effect of inter-chain coupling on the assumption \( \omega < \Delta_s \). The Hamiltonian density which incorporates such interactions is:

\[ \mathcal{H} = \sum_i \mathcal{H}_i + \sum_{ij} \mathcal{H}_{ij} \]

\[ \mathcal{H}_{ij} = t_{ij} \sum_{\sigma} \left( \psi_{L\sigma i}^\dagger \psi_{L\sigma j} + \psi_{R\sigma i}^\dagger \psi_{R\sigma j} \right) + \text{h.c.}, \]

where \( i \) and \( j \) labels chains and the single chain Hamiltonian density \( \mathcal{H}_i \) is \( \mathcal{H}_c \). The single particle inter-chain hopping is represented by the Hamiltonian density \( \mathcal{H}_{ij} \). It will be demonstrated below that the other physically important interaction the inter-chain density-density repulsion is irrelevant.

An addition of a single electron or hole to a chain with a spin gap creates a state with a spin boson soliton. The energy of such soliton is at least \( \Delta_s \). Yet, it is possible to add a pair of particles or a particle and a hole in such a way that the soliton is not created and the spin bosons are left undisturbed. This implies that while the single particle hopping in our situation is greatly diminished, the particle-particle (Cooper pair) hopping and the particle-hole (exchange) hopping survive the presence of the spin gap [7,11]. Technically, when transverse hopping is smaller then \( \Delta_s \) the fast modes (those, whose energy exceeds the spin gap) can be easily ‘integrated out’. The slow degrees of freedom are charge bosons with the energy less then the spin gap. The effective dynamics of these modes are given by the following Hamiltonian density:

\[ \mathcal{H}_{\text{eff}} = \frac{v_c}{2} \sum_j \left( K_{i,j}^{-1} (\nabla \Phi_{cj})^2 + K_c (\nabla \Theta_{cj})^2 \right) \]

\[ + 2a_c^{-1} \sum_{ij} \left( J_{i,j}^{dw} (\eta_{L\uparrow i}^j \eta_{R\uparrow j}^i + \eta_{L\downarrow i}^j \eta_{R\downarrow j}^i) \cos \sqrt{2\pi} (\Phi_{ci} - \Phi_{cj}) \right. \]

\[ \left. - J_{i,j}^{ac} (\eta_{L\uparrow i}^j \eta_{R\downarrow j}^i + \eta_{L\downarrow i}^j \eta_{R\uparrow j}^i) \cos \sqrt{2\pi} (\Theta_{ci} - \Theta_{cj}) \right). \]
Here $J^{sdw}$ is the effective exchange coupling and $J^{sc}$ is the effective Josephson coupling. The hamiltonian itself is derived in the Appendix. The ultraviolet cut-off for this effective theory is equal to $a_c^{-1} = \Delta_s/v_c$.

It is necessary to explain the physical meaning of different terms of (13). The origin of the first term is obvious. This term is responsible for the intra-chain dynamics of the charge bosons. The second term acts to order $\Phi_c$. As one can infer from (10) the field $\Phi_c$ should be viewed as the phase of the SDW. When $\exp(i\sqrt{2}\pi\Phi_s)$ acquires finite expectation value the ground state becomes SDW$_z$. One has to remember that $\langle \sin \sqrt{2}\pi\Phi_s \rangle$ is non-zero. Unlike $S_z$, the superconducting order parameter in SDW phase is zero. It is proportional to $\exp\left(i\sqrt{2}\pi\Theta_c\right)$. Since $\Theta_c$ is dual to $\Phi_c$ the ordering of the latter implies strong fluctuations of $\Theta_c$ and, therefore, $\langle \exp\left(i\sqrt{2}\pi\Theta_c\right) \rangle = 0$.

The last term of (13) describes the Josephson coupling between pairs of the chains. If the field $\Theta_c$ is ordered the ground state becomes the longitudinal triplet superconductor. The operator density

$$\Delta_{\pi 0} = \psi_{L\uparrow}^\dagger \psi_{R\downarrow}^\dagger - \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \propto e^{i\sqrt{2}\pi\Theta_c} \sin \sqrt{2}\pi\Phi_s \tag{14}$$

acquires anomalous non-zero expectation value. We see that the field $\Theta_c$ plays a role of the superconducting order parameter phase. The SDW order parameter has zero expectation value in the superconducting phase.

In a purely one-dimensional system a ground state with a broken symmetry cannot exist. However, due to inter-chain coupling in (13) the problem at hand becomes three-dimensional where the broken symmetry phase can be stabilized at low temperature. In our situation the longitudinal triplet superconductivity and SDW$_z$ are the only types of broken symmetry which are allowed to exist. Other possibilities, such as charge-density wave, transversal SDW (SDW$_{x,y}$), longitudinal TSC or singlet superconductivity are incompatible with the spin gap and (4). Which of the two allowed ground states, TSC or SDW, has lower energy at $T = 0$ depends on the parameters of the effective hamiltonian.

There is a relative minus sign between the second and the third term of (13). This has important consequence. If the ground state is a superconductor the order parameter is uniform over the chain array. In case of SDW, however, the order parameter has the opposite sign on the neighboring chains. Therefore, next to nearest neighbor coupling acts to stabilize the superconductivity and to destabilize SDW. This circumstance will be used when we will map out the phase diagram of (13) in the next section.

Before we continue with the phase diagram we want to explain why the inter-chain density-density repulsion can be neglected. We have mentioned above that the charge-density expectation value is zero due to the spin gap. Therefore, in the first order of the inter-chain density-density coupling $V_\perp$ the contribution of this term is exactly zero. In the next order the contribution is non-zero. It is proportional to $\exp\left(\pm\sqrt{8}\pi\left(\Phi_{ci} - \Phi_{cj}\right)\right)$. This operator is irrelevant and can be neglected.

**IV. PHASE DIAGRAM**

In this section we will determine the phase diagram of the system as a function of $J$’s and temperature. Before we start constructing the diagram let us discuss possible constraints on
the values of the model parameters. There are several parameters our hamiltonian (13) is characterized by: $a_c, K_c, J_{ij}$. We assume that the exchange constants $J_{sdw}$ are equal to the Josephson coupling constants $J^{sc}$. Why this is a reasonable approximation is discussed in the Appendix. We will put all $J_{ij}$ equal to zero unless $i$ and $j$ are nearest neighbors or next-to-nearest neighbors. It is accepted that the next-to-nearest neighbor coupling constant $J_2$ is smaller then the nearest neighbor constant $J_1$: $J_1 > J_2$. When the interaction is repulsive $K_c$ is bigger then one half. For the Hubbard model it has been found that $K_c$ is smaller then two. We will confine $K_c$ to this interval:

$$1/2 < K_c < 1. \quad (15)$$

Outside of this interval the proposed mechanism cannot work: for $K_c < 1/2$ the Josephson coupling is irrelevant.

To map out the phase diagram we will use mean field theory. One constructs the mean-field hamiltonian for the coupled chain problem by re-writing the inter-chain interaction term:

$$4 \sum_{ij} J_{ij} a_c^{-1} \cos \sqrt{2\pi} \left( \Theta_{ci} - \Theta_{cj} \right) = \mathcal{H}_{MF} + \Delta \mathcal{H}, \quad (16)$$

$$\mathcal{H}_{MF} = -\frac{2}{a_c} \sum_{ij} J_{ij} \left( \cos \sqrt{2\pi} \Theta_{ci} \left( \cos \sqrt{2\pi} \Theta_{cj} \right) + \cos \sqrt{2\pi} \Theta_{cj} \left( \cos \sqrt{2\pi} \Theta_{ci} \right) \right), \quad (17)$$

$$\Delta \mathcal{H} = -\frac{2}{a_c} \sum_{ij} J_{ij} \left( 2 \cos \sqrt{2\pi} \left( \Theta_{ci} - \Theta_{cj} \right) - \cos \sqrt{2\pi} \Theta_{ci} \left( \cos \sqrt{2\pi} \Theta_{cj} \right) - \cos \sqrt{2\pi} \Theta_{cj} \left( \cos \sqrt{2\pi} \Theta_{ci} \right) \right) \quad (18)$$

and neglecting $\Delta \mathcal{H}$. The mean-field single-chain hamiltonian density is:

$$\mathcal{H}_{MF} = \sum_i \mathcal{H}_{ci} - \tilde{J}_+ a_c^{-1} \cos \sqrt{2\pi} \Phi_{ci} - \tilde{J}_- a_c^{-1} \cos \sqrt{2\pi} \Theta_{ci}, \quad (19)$$

$$\tilde{J}_+ = J_+ \langle \cos \sqrt{2\pi} \Phi_c \rangle, \quad (20)$$

$$\tilde{J}_- = J_- \langle \cos \sqrt{2\pi} \Theta_c \rangle. \quad (21)$$

The expression for $\mathcal{H}_c$ is given by (9). Both cosines are relevant interactions. Therefore, they produce gap $\Delta_c$ in the excitation spectrum of the charge boson. But the nature of the ground state, superconducting or magnetic, depends on the values of $K_c$ and the effective coupling constants (‘+’ for TSC and ‘-’ for SDW):

$$J_\pm = 2J_1 \left( \cos \delta^z_y + \cos \delta^z_z \right) \pm 2J_2 \left( \cos \left( \delta^z_y + \delta^z_z \right) + \cos \left( \delta^z_y - \delta^z_z \right) + \cos 2\delta^z_y + \cos 2\delta^z_z \right). \quad (22)$$

The quantities $\delta^z_{y,z}$ are differences of the order parameter phase on neighboring chains. When $J_2$ is zero all $\delta$’s are zero, too. Otherwise, they have to be determined to maximize the effective coupling constants $J_\pm$. Simple calculation shows that $\delta^z_{y,z}$ deviate from zero when $|J_2|$ grows bigger then $J_1/6$:

$$J_+ = \begin{cases} 4J_1 + 8J_2 & \text{if } J_2 > -J_1/2, \\ -4J_2 - J_1^2/(3J_2) & \text{if } J_2 < -J_1/2, \end{cases} \quad J_- = \begin{cases} 4J_1 - 8J_2 & \text{if } J_2 < J_1/2, \\ 4J_2 + J_1^2/(3J_2) & \text{if } J_2 > J_1/2. \end{cases} \quad (23)$$
Of course, $J_{1,2} > 0$ in our system. We will consider the case of negative $J_2$ for completeness of the presentation.

We did not put the products of the Klein factors in (19). This is permissible as long as the superconductivity and SDW do not co-exist. Then it is possible to use the following representation of the Klein factors:

$$\eta_{L\uparrow}\eta_{R\uparrow} = i, \quad \eta_{L\downarrow}\eta_{R\downarrow} = i, \quad \eta_{L\uparrow}\eta_{R\downarrow} = i, \quad \eta_{R\uparrow}\eta_{L\downarrow} = i. \quad (24)$$

The averages of cosine of $\Phi_c$ and $\Theta_c$ have to be found self-consistently. From the dimensional analysis:

$$\langle \cos \sqrt{2\pi}\Theta_c \rangle \propto \left(\frac{\Delta_c}{\Delta_s}\right)^{1/(2\kappa_c)}. \quad (25)$$

The required self-consistency condition reads:

$$\tilde{J}_+ \left(\frac{\Delta_s}{\Delta_c}\right)^{2-1/(2\kappa_c)} \propto \Delta_s. \quad (26)$$

These two equations give the following expression for the gap:

$$\Delta_c \propto \Delta_s \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-1/\kappa_c}}. \quad (27)$$

This is the equation for the gap due to the formation of the superconducting ground state. The equation for the SDW gap is:

$$\Delta_c \propto \Delta_s \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-\kappa_c}}. \quad (28)$$

The zero-temperature transition between the superconductivity and SDW occurs when these two gaps are the same:

$$\Delta_s \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-1/\kappa_c}} \propto \Delta_s \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-\kappa_c}}. \quad (29)$$

The critical temperature $T_c$ for a particular set of parameters equals to the biggest gap:

$$T_c = \Delta_s \max \left\{ \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-1/\kappa_c}}, \left(\frac{J_+}{\Delta_s}\right)^{\frac{1}{2-\kappa_c}} \right\}. \quad (30)$$

The equation (29) also gives the position of the tricritical point where the superconductivity, SDW and the phase without the charge gap co-exist.

Now it is possible to construct the phase diagram. We will plot it on ($J_2$, $T$) plane. It will be explained later how such diagram can be related to the ($p$, $T$) diagram measured experimentally. One has to fix the value of $\kappa_c$ in the interval (15) and $r = J_1/\Delta_s$ in the interval $0 < r \ll 1$. The value of $J_2$ is constrained to:

$$- J_1 < J_2 < J_1. \quad (31)$$
The phase boundaries are determined by (30). The tricritical point is given by (29). It is always located at right-hand side ($J_2 > 0$) part of the diagram. This is a consequence of (14). The point of $T = 0$ transition is located right beneath the tricritical point. The diagram itself is presented on fig.1. The phase with no broken symmetry is denoted ‘Spin gap’.

Usually, experimental results are presented on the $(p, T)$ plane. In order to establish connection between fig.1 and the experiment we make three assumptions: (i) the coupling constants $J_{1,2}$ are decreasing functions of the transverse lattice constant $b_\perp$; (ii) $J_2$ is more sensitive to change in $b_\perp$ than $J_1$ in the sense that $J_2/J_1$ is a decreasing function of $b_\perp$; (iii) both pressure and temperature affect $J_{1,2}$ through variation of $b_\perp$; all other effects of temperature and pressure are unimportant. The supposition (i) immediately implies that $J_{1,2}$ are increasing functions of the pressure and decreasing functions of the temperature. If we neglect the dependence of $J_1$ on $b_\perp = b_\perp(p, T)$ at all then (ii) is trivially satisfied. In such a case by applying pressure at zero temperature we move the system from SDW at low temperature and pressure are unimportant. The supposition (i) immediately implies that the tricritical point moves to the right. This effect is commonly referred to as re-entrance. It is a well-established feature of the experimental phase diagram of Q1D organic compounds [6].

Fig.1 suggests that the critical temperature of TSC grows with the pressure. This contradicts the experimental data for the organic superconductors [1] which show that $p > p_{TCP}$ where $p_{TCP}$ is the pressure at the tricritical point. We will address this issue in Section VII.

Let us study the validity of the mean field approach. We will evaluate the energy correction introduced by the operator $\Delta H$. This correction has to be smaller then the mean field energy $E^{(0)} \propto LN_\perp \Delta_s^2/v_c$ ($L$ is the length of chains, $N_\perp$ is the number of chains). The first order correction is zero: $\langle \Delta H \rangle_{MF} = 0$. Second order contribution is given by the formula:

$$
E^{(2)} = L \int dx d\tau \langle \Delta H(0,0)\Delta H(x,\tau) \rangle_{MF},
$$

$$
\Delta H(x,\tau) = e^{-\tau} \int dx' \Delta H(x)e^{\tau} \int dx' \Delta H(x').
$$

The problem at hand is reduced to the estimation of the bosonic correlation functions. When $x/v_c$ or $\tau$ exceed $\Delta_c^{-1}$ the integrand of (32) vanishes exponentially fast. Thus, the integration can be restricted to $\Delta_s^{-1} < \sqrt{\tau^2 + x^2/v_c^2} < \Delta_c^{-1}$. At such small distances the effect of the charge gap is negligible. This simplifies our task even further: the bosons may be considered to be free. The evaluation of integral (32) gives:

$$
E^{(2)} \propto (L/v_c) \left( \frac{\Delta_s}{\Delta_c} \right)^{2-2/K_c} \sum_{ij} J_{ij}^2 e^{2iq \cdot (R_i-R_j)} \propto LN_\perp \frac{\Delta_c}{\bar{z} v_c},
$$

$$
\bar{z} = \left( \frac{\sum_{ij} J_{ij} e^{iq \cdot (R_i-R_j)}}{N_\perp \sum_{ij} J_{ij}^2 e^{2iq \cdot (R_i-R_j)}} \right)^2,
$$

Fig.1
\[ q = \begin{cases} 0 & \text{for TSC} \\ b_{\perp}^{-1} (\pi + \delta_y, \pi + \delta_z) & \text{for SDW} \end{cases} \]  

(36)

Two-dimensional vector \( \mathbf{R}_i \) shows the position of \( i \)th chain. The quantity \( \bar{z} \) can be considered as an effective co-ordination number of a chain. For models with the nearest neighbor interaction only and \( q = 0 \) the value of \( \bar{z} \) coincides with the co-ordination number of a chain. In the above expression for \( E^{(2)} \) a constant independent of \( \Delta_c \) has been omitted. We see now that the condition for the mean field ground state to be a good approximation is

\[ \bar{z} \gg 1. \]  

(37)

This condition is satisfied if the number of transverse directions is big or if the transverse interaction is sufficiently long ranged. Inequality (37) is consistent with findings of [11] where corrections for the inter-chain mean field theory were also evaluated.

V. EFFECT OF LONG RANGE COULOMB INTERACTION

The phase diagram on fig.1 is a mean-field result. It ignores the contribution of the Goldstone mode to the free energy. Such contribution is unimportant at \( T = 0 \). At \( T > 0 \) those modes get excited and increase the entropy of the system. Due to the entropy of these modes the first order phase transition line separating TSC and SDW phases bends to the right creating re-entrance region even on \((J_2/J_1, T)\) diagram.

The reason behind the re-entrant behavior is difference between the Goldstone mode contribution to the entropy of TSC and SDW phases. The long-range Coulomb interaction opens a gap in the spectrum of the Bogoliubov mode (Goldstone mode in the superconducting phase) but leaves some SDW sliding modes without such gap. Therefore, even if at a given value of \( J_{1,2} \) the ground state energy of the superconducting phase is bigger then that of SDW the entropic contribution of the sliding mode may trigger the transition into SDW phase as the temperature grows. On \((p, T)\) phase diagram both thermal expansion discussed in the previous section and the Goldstone modes work together forming the re-entrance region.

To study the effect of the long-range Coulomb interaction the following term has to be added to the total hamiltonian:

\[ H_C = \frac{v_c g_C}{4\pi} \sum_{ij} \int dx dx' \frac{\nabla \Phi_i(x) \nabla \Phi_j(x')}{\sqrt{(x - x')^2 + (\mathbf{R}_i - \mathbf{R}_j)^2}} \approx v_c g_C \int \frac{dk_\parallel}{2\pi} \frac{d^2k_\perp}{(2\pi)^2} \frac{k_\parallel^2}{k_\parallel^2 + k_\perp^2} \Phi_k \Phi_k^\dagger. \]  

(38)

Here \( g_C \propto e^2/(v_c \varepsilon) \) is the interaction constant. Simple power counting suggests that (38) is a marginal operator. Provided that \( g_C \ll 1 \) the effect of the long-range Coulomb repulsion on the mean-field gap (27) and (28) is negligible. However, it changes the dynamics of the Goldstone modes significantly. In SDW phase the long-wavelength hamiltonian of the sliding mode is:

\[ H_{sdw}^{sdw} = \int \frac{dk_\parallel}{2\pi} \frac{d^2k_\perp}{(2\pi)^2} b_{\perp}^2 H_{k}^{sdw}, \]  

(39)

\[ H_{k}^{sdw} = \frac{1}{2} \left( v_c \mathcal{K}_c \Pi_k \Pi_k^\dagger + \left( v_c \mathcal{K}_c^{-1} k_\parallel^2 + \frac{v_{sdw}^2}{k_c v_c} k_\perp^2 + 2v_c g_C b_{\perp}^{-2} \frac{k_\parallel^2}{k_\parallel^2 + k_\perp^2} \right) \Phi_k \Phi_k^\dagger \right). \]  

(40)
\[ \Pi_k = k_{||}\Theta_k, \quad (41) \]
\[ v_{sdw}^2 \propto J_-\Delta b_{||}^2 \left< \cos \sqrt{2\pi\Phi} \right>^2 \propto \Delta^2 b_{||}^2. \quad (42) \]

Here we used an approximation \( \cos \Phi \approx -\left(\Phi^2/2\right) \left< \cos \Phi \right> \) to simplify the transversal coupling. The meaning of the constant \( v_{sdw} \) is the velocity of the sliding mode in the direction normal to the chains. From this expression the dispersion of the Goldstone bosons is found to be:

\[ \omega^2 = v_c^2k_{||}^2 + v_{sdw}^2k_{||}^2 + 2K_c v_c^2g_b b_{||}^2 \frac{k_{||}^2}{k_{||}^2 + k_{\perp}^2} \quad (43) \]

The structure of this formula becomes more clear if we re-parameterize it:

\[ \omega^2 = \frac{v_c^2 + v_{sdw}^2}{2} + \frac{v_c^2 - v_{sdw}^2}{2} \cos 2\phi \left( k_{||}^2 + 2K_c v_c^2g_b b_{\perp}^2 \cos^2 \phi, \right) \]
\[ k_{||}^2 = k_{||}^2 + k_{\perp}^2, \quad \tan \phi = k_{\perp}/k_{||}. \quad (44) \]

We see that the gap in the sliding mode spectrum closes at \( \phi = \pi/2 \), that is, for mode momenta normal to the chain direction.

Similar procedure for the Bogoliubov mode gives:

\[ \omega^2 = \left( \frac{v_c^2 + v_{sc}^2}{2} + \frac{v_c^2 - v_{sc}^2}{2} \cos 2\phi \right) \left( k_{||}^2 + 2K_c v_c^2g_b b_{\perp}^2 \right), \quad (46) \]
\[ v_{sc}^2 \propto J_-\Delta b_{\perp}^2 \left< \cos \sqrt{2\pi\Theta} \right>^2 \propto \Delta^2 b_{\perp}^2. \quad (47) \]

Unlike SDW sliding mode, the Bogoliubov mode has a gap for any direction of its momentum. The difference stems from the fact that for TSC case the long-range Coulomb interaction is expressed not in terms of the field variable but rather in terms of the conjugate momentum. The gap is lowest for \( \phi = \pi/2 \). Its value is \( \Delta g^{1/2}_b \).

**VI. SO(4) TRANSFORMATION**

In this section we will show that at some value of the model parameters the symmetry group of the hamiltonian (13) is enhanced. One can observe straightforwardly that (13) is invariant under the global shifts

\[ \Theta_{ci}(x) \rightarrow \Theta_{ci}(x) + c, \quad (48) \]
\[ \Phi_{ci}(x) \rightarrow \Phi_{ci}(x) + c. \]

The first of these equations corresponds to the gauge transformation, the second corresponds to the translation along the chains. Thus, the symmetry group of the hamiltonian is \( U(1) \otimes U(1) \). We now show that if the additional conditions:

\[ K_c = 1, \quad J_{ij}^{sdw} = -(-1)^{i-j} J_{ij}^{sc}, \quad (49) \]

are imposed (where \( i \) and \( j \) are sites in the 2D bipartite lattice formed by the chains and \((-1)^{i-j} = 1\) if \( i \) and \( j \) are on the same sublattice and \(-1\) if on opposite sublattices) then the
The hamiltonian symmetry group is SO(4). To demonstrate this we construct an explicit set of SO(4) transformations which leave the hamiltonian invariant. As a first step it is convenient to re-fermionize the hamiltonian using (5) with $a_c$ instead of $a$. We do this first for the inter-chain interaction term:

$$H_{\text{eff} \perp} = 8\pi^2 a_c \sum_{i,j} \left[ J_{ij}^{\text{sdw}} \left( \psi_{L \uparrow}^\dagger \psi_{R \uparrow} \psi_{R \uparrow}^\dagger \psi_{L \uparrow}^\dagger \psi_{L \downarrow}^\dagger \psi_{L \downarrow}^\dagger \psi_{R \downarrow}^\dagger \psi_{R \downarrow}^\dagger \right) + J_{ij}^{\text{sc}} \left( \psi_{L \uparrow}^\dagger \psi_{R \downarrow} \psi_{R \downarrow}^\dagger \psi_{L \uparrow}^\dagger \psi_{L \downarrow}^\dagger \psi_{R \downarrow} \psi_{R \uparrow} \psi_{L \downarrow}^\dagger \psi_{R \downarrow}^\dagger \right) \right] e^{-i\sqrt{2}\pi(\Phi_s - \Phi_{s'})} + \text{h.c.}$$

(50)

To show the SO(4) structure of this operator we define a four component spinor

$$\Psi^\dagger = \left( \begin{array}{c} \psi_L^\dagger \psi_R^\dagger \\ \psi_{L \downarrow}^\dagger \psi_{R \downarrow}^\dagger \end{array} \right)$$

and 2×2 matrices:

$$\hat{g} = \Psi_L \Psi_R^\dagger = \left( \begin{array}{cccc} \psi_{L \uparrow}^\dagger & \psi_{R \uparrow}^\dagger \\ \psi_{L \downarrow}^\dagger & \psi_{R \downarrow}^\dagger \end{array} \right).$$

(52)

With this notation the hamiltonian density $H_{\text{eff} \perp}$ can be re-written as:

$$H_{\text{eff} \perp} = 4\pi^2 a_c \sum_{i,j} \left[ \left( J_{ij}^{\text{sdw}} + J_{ij}^{\text{sc}} \right) \text{tr}\hat{g}_i \sigma_z \sigma_j + \left( J_{ij}^{\text{sdw}} - J_{ij}^{\text{sc}} \right) \text{tr}\hat{g}_i \hat{g}_j \right] e^{-i\sqrt{2}\pi(\Phi_s - \Phi_{s'})} + \text{h.c.}$$

(53)

We now define a set of transformations which (i) leaves the spin boson part of (53) invariant and (ii) acts as SO(4) on matrices $\hat{g}$. To do this we introduce matrices $\sigma_{\mu\nu}$ which act on the four-component spinor (51). These matrices are determined by:

$$\sigma_{\mu\nu} = (-i/2) \left[ \gamma_{\mu}, \gamma_{\nu} \right] ,$$

$$\sigma_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} \left( \begin{array}{cc} \sigma_k & 0 \\ 0 & \sigma_k \end{array} \right) , \quad \sigma_{0i} = \left( \begin{array}{cc} \sigma_i & 0 \\ 0 & -\sigma_i \end{array} \right) .$$

(54)

(55)

with $\gamma$-matrices of Dirac theory [12]:

$$\gamma_i = \left( \begin{array}{cc} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{array} \right) , \quad i = 1, 2, 3, \quad \gamma_0 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) , \quad \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) .$$

(56)

Consider the following transformation group of the spinor $\Psi$:

$$\Psi' = \exp \left( i \sum_{\mu,\nu=0}^{3} \omega_{\mu\nu} \sigma_{\mu\nu} \right) \Psi$$

(57)

with $\omega_{\mu\nu}$ being c-numbers. The form of (54) and (55) shows that $\Psi_L$ and $\Psi_R$ transform independently under (57):

$$\Psi'_L = U \Psi_L$$

$$\Psi'_R = V \Psi_R$$

$$\exp \left( i \sum_{\mu,\nu=0}^{3} \omega_{\mu\nu} \sigma_{\mu\nu} \right) = \left( \begin{array}{cc} U & 0 \\ 0 & V \end{array} \right) , \quad U, V - \text{unitary.}$$

(58)

(59)

(60)
Because the two L components of $\Psi$ have the same spin boson part (i.e. same factors of $\exp(\pm i\sqrt{\pi/2}\Phi_s)$ and $\exp(\pm i\sqrt{\pi/2}\Theta_s)$) as do the two R components and the transformation (57) does not mix L and R, the spin boson is left invariant under (57). Therefore, spin boson operators commute with this transformation, as (i) claims.

It is routinely proven in introductory courses on the quantum electrodynamics that the object $R_\mu = \Psi^\dagger \gamma_\mu \Psi$ transforms as a four-dimensional vector under (57). Therefore, the transformations (57) comprise SO(4) group. Unlike the theory of Dirac equation we do not need to deal with $R_\mu$. To prove (ii) it is required to know the action of these transformation on $\hat{g}$. As a consequence of (58) and (59) this action is given by:

$$\hat{g}' = U \hat{g} V^\dagger.$$  

(61)

On a bipartite lattice a global SO(4) rotation can be defined by specifying matrices $U_A$ and $V_A$ on sublattice $A$ and assigning $U_B$ and $V_B$ according to:

$$U_B = \sigma_z U_A \sigma_z, \quad V_B = \sigma_z V_A \sigma_z.$$  

(62)

If equation (49) is obeyed then in (53) the term proportional to $(J_{sdw}^{ij} + J_{sc}^{ij})$ is not vanishing only for $i$ and $j$ on different sublattices and the term proportional to $(J_{sdw}^{ij} - J_{sc}^{ij})$ only for $i$ and $j$ on the same sublattice. So the total expression (53) is invariant under (62).

The same re-fermionization procedure applied to $H_c$ gives:

$$H_c = \sum_i \frac{1}{2} (K_c + 1/K_c) v_c \sum_\sigma \left( \psi^\dagger_{Li} \partial_x \psi_{Li} - \psi^\dagger_{Ri} \partial_x \psi_{Ri} \right) - 4\pi \left( K_c - 1/K_c \right) (n_{Lti} + n_{L,ti}) (n_{Rti} + n_{R,ti}) - (K_c + 1/K_c) v_s \left( (\nabla \Phi_{si})^2 + (\nabla \Theta_{si})^2 \right)$$

$$= \sum_i \frac{(-1)}{2} (K_c + 1/K_c) v_c \left( \Psi^\dagger_i \sigma_z \partial_x \Psi_i \right) + 4\pi \left( K_c - 1/K_c \right) \text{tr} \hat{g} \sigma_z \hat{g}^\dagger \sigma_z$$

$$- (K_c + 1/K_c) v_s \left( (\nabla \Phi_{si})^2 + (\nabla \Theta_{si})^2 \right).$$

The first and the last terms are invariant under SO(4) rotations. The second term is not invariant, but vanishes if (49) holds.

To summarize, for generic parameter values the symmetry group of the hamiltonian (13) is $U(1) \otimes U(1)$ but if extra conditions (49) apply then the symmetry of the low energy hamiltonian is SO(4). The symmetry transformations connect smoothly the superconducting ground state and the SDW ground state. A similar situation takes place in $U < 0$ Hubbard model on a square lattice [13] and in a certain Q1D system near charge density wave-singlet superconductivity transition [14]. In the latter paper the symmetry was investigated using non-Abelian bosonization technique and found to be SU(2). On phenomenological level our model is an analog of SO(5) theory of high-$T_c$ superconductors proposed by S.-C. Zhang [14]. The long-range Coulomb interaction destroys this symmetry (as seen, for example, from (58)).

VII. DISCUSSION

In this paper we proposed a many-body theory of the transition from the spin-density wave to the superconducting state in Q1D materials. The essential component of the proposed mechanism is the existence of the high energy spin gap on a single chain. This gap
is a many-body phenomena unaccessible from the mean field theory. At the temperature lower than $\Delta_s$ all intra-chain interactions and particle hopping are modified by the presence of the spin gap. Only the exchange interaction and the Josephson tunneling survive. The competition of these two determines the low-temperature phase of the system. Our picture allows us to produce a phase diagram which is qualitatively similar to that of the organic superconductors. In particular, we were able to explain the re-entrant region of the phase diagram.

In our approach the superconducting phase is deeply connected to SDW. This connection manifests itself through the existence of the quantum symmetry between the superconducting ground state and SDW ground state. In any real system this symmetry is violated. Depending on the violation we have either TSC or SDW as a ground state. As we demonstrated, the physical long-range Coulomb interaction further damages the symmetry by modifying the Goldstone spectrum of TSC and SDW in different ways. If this symmetry can be observed experimentally is unclear.

It is also important to note that the superconducting phase is stabilized in a system with purely repulsive interactions. The total energy is reduced because the energy of transverse hopping is smaller in the superconducting phase. This is in obvious contrast with BCS theory where the optimization of total energy is achieved by lowering the potential energy of the electron-electron interaction. Even more interesting, in BCS superconductivity the pairing operator is only marginally relevant while in our system it is always relevant. Thus, $T_{c,\text{BCS}} \propto \exp(-1/g)$ where $g$ is the coupling constant. In our case $g \propto J_+ / \Delta_s$ and $T_c$ is proportional to some power of $g$. In this sense, the proposed mechanism is ‘high-temperature’ superconductivity.

A crucial prerequisite for experimental implementation of the mechanism is a strong Ising anisotropy of the spin-spin interaction. Although, in the organic superconductors this interaction does possess some anisotropy, the latter seems to be quite weak [13]. The situation becomes even more aggravated if one realize that the size of the spin gap $\langle \rangle$ is exponentially small at weak anisotropy. There are two possible ways out of this problem. First, one can try to improve the estimate $\langle \rangle$. Second, one may consider the system with the small value of $\Delta_s$. We believe, that the latter is a correct route toward the realistic description of the Q1D organic metals. An indication that the regime with small $\Delta_s$ is experimentally relevant comes from dependence $T_c = T_c(p)$. Experiments show that the critical temperature is a decreasing function of the pressure. The situation can be explained qualitatively if one assume that at high pressure the system becomes 3D anisotropic Fermi liquid. Since Fermi liquid with repulsive interaction is stable against transition into the superconducting phase the superconducting properties deteriorate under pressure. The spin gap also has to close before the system can become Fermi liquid. Thus, the compound inevitably enters the regime where the spin gap is comparable or smaller then the transversal interactions and the transversal hopping. Capturing this regime is a significant theoretical challenge.

The major reason why the system with small $\Delta_s$ is difficult to describe is the presence of the unquenched operator of the single-electron transversal hopping. When expressed in terms of the bosonic fields $\Theta_{s,c}$ and $\Phi_{s,c}$ this operator has distinct non-local structure: it creates kinks in both $\Theta_{s,c}$ and $\Phi_{s,c}$ fields. This makes it impossible to apply directly our method to such systems. However, an important conclusion can be drawn. Q1D metal without well-developed spin gap on every chain is close to an array of transversally coupled...
LL. This puts us in a situation studied before with ‘g-ology’ approach [3]. We already mentioned in Introduction about its failure to describe the superconducting phase. Our study suggests that the breaking of the spin-rotational invariance can help recover this phase. Indeed, the scaling dimension of the spin-density and charge-density wave susceptibilities is $d_{dw} = K_c + K_s$. That of the Cooper pair susceptibility (longitudinal triplet and singlet) is $d_{sc} = 1/K_c + K_s$. In case of full rotational invariance ($K_s = 1$) only one of these two is smaller then 2. However, if $K_s < 1$ both can be less then 2 simultaneously. In this situation there are several relevant operators in the system. As we have seen, at the ambient pressure the ground state is likely to be a density wave since the exchange interaction is more relevant then the Josephson coupling operator ($\varepsilon_{dw} < \varepsilon_{sc}$ if $K_c < 1$). At the elevated pressure the superconducting state benefits from next-to-nearest-neighbor processes and may become the ground state.

Despite the appeal of this discussion we have to issue a warning. The arguments we made in the previous passage are based on the knowledge of LL scaling dimensions of different symmetry-breaking operators. In the presence of the unquenched single-electron hopping those dimensions may change in some unknown manner. That is why in this paper we chose to study the case of well-developed spin gap where the single-electron hopping is irrelevant.

To conclude, we propose a mechanism of SDW and superconductivity in Q1D materials. This mechanism stabilizes the triplet superconducting state without any attraction between the electrons. The effective hamiltonian possesses approximate SO(4) symmetry which connects TSC and SDW ground states. Depending on how this symmetry is broken the low-temperature phase is either TSC or SDW.

VIII. APPENDIX

In this Appendix we will derive the effective low-energy hamiltonian (13). First we recognize the fact that in the presence of the well defined spin gap $\Delta_s$ the transverse single-electron hopping is ineffective: when a electron is added to or subtracted from a chain it creates a soliton in the field $\Phi_s$ whose energy is at least $\Delta_s$. Yet, correlated hopping of two electrons can survive. For example, an addition of a Cooper pair to a chain does not induce a kink. This can be easily seen from (14): the operator $\Delta_{0}^{\pi}$ does not contain exponentials of $\Theta_s$ which create kinks. The same is true for an addition of a particle-hole pair given by operator (11). In fact, these two possibilities are the most relevant inter-chain processes. Their competition dominates the low-energy properties of our system.

To obtain the low-energy description we have to eliminate all states whose energy is higher then $\Delta_s$. A particular version of the elimination procedure we will use here is very similar to the Schrieffer-Wolf transformation known in the theory of Kondo effect. The idea is to split the Hilbert spaces of a single chain into two subspaces. The charge subspace $W_{cj}$ for the chain $j$ is spanned by the vectors of the form $|\psi_{cj}\rangle |0_{sj}\rangle$ where $|0_{sj}\rangle$ is the ground state of the spin boson. The effective hamiltonian density on the subspace $W_c = \sum_i W_{ci}$ can be defined by its action on $|\psi\rangle \in W_c$:

$$H_{\text{eff}} |\psi\rangle = \mathcal{P}_c \left( \sum_i H_i - H_{\perp} \mathcal{P}_c^\perp \left( \sum_j H_{sj} + H_{cj} - E \right)^{-1} \mathcal{P}_c^\perp H_{\perp} \right) \mathcal{P}_c |\psi\rangle = E |\psi\rangle,$$

(64)
Here $\mathcal{P}_c$ is the orthogonal projector on $\mathcal{W}_c$ and $\mathcal{P}_c^\perp = 1 - \mathcal{P}_c$.

Unlike usual hamiltonian (64) depends on the eigenvalue $E$. We will demonstrate that this dependence is weak and for our purposes (64) can be treated as an ordinary hamiltonian. The expression (64) is quite general. It is possible to simplify it by noting that the partial matrix element $\langle 0_{s j} | H_\perp | 0_{s j} \rangle$ is zero. This allows to drop the projectors from formula (64):

$$H_{\text{eff}} |\psi_c\rangle = \left( \sum_i H_{ci} - \left\langle H_\perp \left( \sum_j H_{sj} + H_{cj} - E \right) \right\rangle^{-1} H_\perp \right) |\psi_c\rangle = E |\psi_c\rangle,$$

(65)

where $\langle \ldots \rangle_s$ denotes the partial matrix element of an operator with respect to the spin boson ground state. We will calculate this matrix element below.

It is convenient to re-write the inverse operator as follows:

$$\left\langle H_\perp \left( \sum_j H_{sj} + H_{cj} - E \right) \right\rangle^{-1} = \int_0^\infty du \left\langle H_\perp e^{-u \left( \sum_j H_{sj} + H_{cj} - E \right)} H_\perp \right\rangle_s \approx \int_0^\infty du \left\langle H_\perp (0) H_\perp (u) \right\rangle_s,$$

(66)

where we took into account that, to the lowest order in $t_\perp$, $\left( \sum_j H_{cj} - E \right) |\psi_c\rangle \approx 0$ and as well as that $\langle H_\perp (0) H_\perp (u) \rangle_s$ vanishes exponentially for $u > \Delta_s^{-1}$. The calculation of the partial matrix element $\langle H_\perp (0) H_\perp (u) \rangle_s$ can be reduced to the calculation of the Green’s function for the spin bosons:

$$\langle H_\perp (0) H_\perp (u) \rangle_s = \int dxdx' \sum_{ij} \langle H_{ij} (x, 0) H_{ij} (x', u) \rangle_s,$$

(67)

$$= \frac{1}{(2\pi a)^2} \int dxdx' \sum_{ij} t_{ij}^2 \sum_{p\sigma p'\sigma'} \eta_{p\sigma}^i \eta_{p'\sigma'}^j \langle x, 0, e^{i\sqrt{2\pi} \phi_p (x, 0)} e^{-i\sqrt{2\pi} \phi_{p'\sigma'} (x, 0)} e^{i\sqrt{2\pi} \phi_p (x', u)} e^{-i\sqrt{2\pi} \phi_{p'\sigma'} (x', u)} \rangle_s,$$

$$\times \langle x, 0, e^{i\sqrt{2\pi} \phi_p (x, 0)} e^{-i\sqrt{2\pi} \phi_{p'\sigma'} (x, 0)} e^{i\sqrt{2\pi} \phi_p (x', u)} e^{-i\sqrt{2\pi} \phi_{p'\sigma'} (x', u)} \rangle_s + \text{h.c.}.$$

The operator $\phi_{p\sigma}$ is defined by (3). The first term in the brackets corresponds to a tunneling event in which an electron and a hole hop from chain $i$ to chain $j$. The second term corresponds to tunneling of a Cooper pair (two electrons) from $j$ to $i$. For every pair of chains $i$ and $j$ there are 64 terms in the expression (67). Yet, only a fraction of them has non-zero partial matrix element $\langle 0_s \ldots | 0_s \rangle$. For this matrix element to be non-zero the term must be ‘neutral’ with respect to kink creation operator $\exp (i \sqrt{\pi / 2\Theta_s})$. If at $u = 0$ a kink (anti-kink) at $i$th chain is created then for the term to be ‘neutral’ an anti-kink (kink) has to be created at $u > 0$. This condition immediately reduces the number of possible terms. The surviving terms have the form $\langle T_{ij} T_{ij}^\dagger \rangle_s$ where $T$ is one of the following:

$$\psi_{L_1}^\dagger (x, 0) \psi_{R_1}^\dagger (x', u), \psi_{L_1}^\dagger (x, 0) \psi_{R_1} (x', u), \psi_{R_1}^\dagger (x, 0) \psi_{L_1} (x', u);$$

(68)

$$\psi_{L_1}^\dagger (x, 0) \psi_{R_1}^\dagger (x', u), \psi_{R_1}^\dagger (x, 0) \psi_{L_1} (x', u);$$

(69)

$$\psi_{L_1}^\dagger (x, 0) \psi_{L_1} (x', u), \psi_{R_1}^\dagger (x, 0) \psi_{R_1} (x', u);$$

(70)
The correlation function

\[ \psi_{Lt}^+(x, 0) \psi_{Lt}^+(x', 0) \psi_{Rt}^+(x, 0) \psi_{Rt}^+(x', 0); \]  
\[ \psi_{Lt}^+(x, 0) \psi_{Lt}^+(x', 0) \psi_{Rt}^+(x, 0) \psi_{Rt}^+(x', 0). \]

(71)

(72)

On this list only two first lines are of interest to us since they are the most relevant operators. Let us calculate the matrix element

\[
\langle \psi_{L_{tj}}^+(x, 0) \psi_{R_{tj}}^+(x', u) \psi_{R_{tj}}^+(x', u) \rangle_{s}
= \langle \psi_{L_{tj}}^+(x, 0) \psi_{R_{tj}}^+(x', u) \rangle_{s} \langle \psi_{R_{tj}}^+(x', u) \rangle_{s}
- \frac{1}{4\pi^2a^2} \eta_{L^j}^i \eta_{R^j}^i \eta_{R^j}^{i'} \eta_{L^j}^{i'} \int dx du e^{i\sqrt{2\pi}2(\Theta_{ci}(x,0)+\Phi_{ci}(x,0))} e^{-i\pi/2(\Theta_{ci}(x',u)+\Phi_{ci}(x',u))} \int dx' du |G_s(x-x', u)|^2,
\]

\[G_s = \left\langle 0_s \right| e^{i\sqrt{2\pi}(\Theta_{ci}(x,0)+\Phi_{ci}(x,0))} e^{-i\pi/2(\Theta_{ci}(x',u)+\Phi_{ci}(x',u))} |0_s \rangle.
\]

(73)

The correlation function \(G_s\) decays exponentially for \(|x-x'|/v_s\) and \(|u|\) above \(\Delta_s^{-1}\). Unfortunately, the operator (73) is quite complicated due to its non-local structure. An obvious way to simplify it is to approximate \(G_s\) by a delta-function. This step is acceptable for the low-momentum boson whose wave length is bigger then \(v_c/\Delta_s\). Yet, the presence of high-momentum bosons render this procedure unsatisfactory. This obstacle can be overcome by eliminating high-momentum bosons from the effective theory. The procedure itself is very similar to the elimination of the spin bosons: in addition to averaging over the spin boson ground state it is required to average over the ground states of high-momentum charge bosons. After this procedure (73) is reduced to:

\[
\int dx dx' du \left\langle \psi_{L_{tj}}^+(x, 0) \psi_{R_{tj}}^+(x', 0) \psi_{R_{tj}}^+(x', u) \psi_{R_{tj}}^+(x', u) \right\rangle_{s,c} = \int dx dx' du \left\langle \psi_{L_{tj}}^+(x, 0) \psi_{R_{tj}}^+(x', 0) \psi_{L_{tj}}^+(x', u) \psi_{L_{tj}}^+(x', u) \right\rangle_{s,c}
= \int dx dx' du \left\langle \psi_{L_{tj}}^+(x, 0) \psi_{R_{tj}}^+(x', 0) \psi_{R_{tj}}^+(x', u) \psi_{R_{tj}}^+(x', u) \right\rangle_{s,c}
\]

(75)

(76)

In this formula the superscript ‘\(\rangle\)’ is used to denote the quantities associated with the high-momentum charge bosons. Putting the expression (75) back into (60) and performing the integration over \(u\) and \(x-x'\) one gets an inter-chain interaction term of the form:

\[
J_{sdw} e^{i\sqrt{2\pi}(\Phi_{ci}(x,0)-\Phi_{cj}(x,0))} \int dx du |G_s(x, u)|^2
\]

\[J_{sdw} = \frac{a_c}{4\pi^2a^2} \int dx du |G_s(x, u)|^2
\]

(77)

(78)

The evaluation of this integral requires knowledge of \(G_s\) at the intermediate values of \(r\):

\[\Delta_s^{-1} < r < \Delta_s^{-1}.\]

This knowledge is unavailable. To circumvent this problem we neglect any anomalous dimension to \(G_s\) and approximate it by \((\Lambda r)^{-1/2}\). This gives:
\[ J_{sdw} \propto \frac{t^2}{\Delta_s} \left( \frac{\Delta_s}{\Lambda} \right)^{(\kappa_c + 1/\kappa_c)/2 - 1}. \] 

(79)

Other terms can be processed in a similar fashion. The resultant inter-chain interaction is:

\[ H_{\text{eff}, \perp} = \sum_{ij} J_{ij}^{sdw} \left( \eta^i_{L\uparrow} \eta^i_{R\uparrow} \eta^j_{R\downarrow} \eta^j_{L\downarrow} + \eta^i_{L\downarrow} \eta^i_{R\downarrow} \eta^j_{R\uparrow} \eta^j_{L\uparrow} \right) \int dx a_c^{-1} e^{i\sqrt{2\pi}(\Phi_{ci} - \Phi_{cj})} \]

(80)

\[ -J_{sc} \left( \eta^i_{L\uparrow} \eta^i_{R\uparrow} \eta^j_{R\downarrow} \eta^j_{L\downarrow} + \eta^i_{L\downarrow} \eta^i_{R\downarrow} \eta^j_{R\uparrow} \eta^j_{L\uparrow} \right) \int dx a_c^{-1} e^{i\sqrt{2\pi}(\Theta_{ci} - \Theta_{cj})} + \text{h.c.} \]

The value of \( J_{sc} \) can be obtained without any calculation if one takes the duality transformation of the charge hamiltonian: \( \Theta_c \leftrightarrow \Phi_c, \kappa_c \leftrightarrow 1/\kappa_c \). The quantity dual to \( J_{sdw} \) is \( J_{sc} \).

A consequence of (78) and the duality is the following estimate:

\[ J_{sc} \propto \frac{t^2}{\Delta_s} \left( \frac{\Delta_s}{\Lambda} \right)^{(\kappa_c + 1/\kappa_c)/2 - 1} \propto J_{sdw}. \] 

(81)

Thanks to the duality, we can make a more accurate statement about these two coupling constants: \( J_{sdw} = F(\kappa_c) J_{sc} \), where the function \( F(\kappa_c) \) is unity when \( \kappa_c \) is unity. Away from this point, \( J_{sdw} / J_{sc} = O(1) \).

The cut-off of this effective theory of the charge bosons is now equal to \( a_c^{-1} \).

The products of Klein factors do not commute with each other. Thus, in general, they cannot be dropped. Yet, within an inter-chain mean field approximation the contribution of the Josephson tunneling term in SDW phase is zero and, conversely, the contribution of the exchange term in the superconducting phase is also zero. In such a case, it is permissible to ignore Klein factors.
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FIG. 1. Mean-field phase diagram of our model. Solid lines mark the location of the second-order phase transition into SDW or TSC. Dash line corresponds to the first order phase transition between SDW and TSC. Label ‘Spin gap’ denotes the phase in which the charge bosons are disordered. The gap in the charge sector (equations (27) and (28)) are shown by dotted lines. The unphysical region $J_2 < 0$ is shaded.