Super-Horizon Scale Dynamics of Multi-Scalar Inflation

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We consider the dynamics of a multi-component scalar field on super-horizon scales in the context of inflationary cosmology. We present a method to solve the perturbation equations on super-horizon scales, i.e., in the long wavelength limit, by using only the knowledge of spatially homogeneous background solutions. In doing so, we clarify the relation between the perturbation equations in the long wavelength limit and the background equations. Then as a natural extension of our formalism, we provide a strategy to study super-horizon scale perturbations beyond the standard linear perturbation theory. Namely we reformulate our method so as to take into account the nonlinear dynamics of the scalar field.

§1. Introduction

Study of multi-scalar inflation is a current topic in cosmology, motivated mainly by the fact that supergravity theories suggest the existence of many flat directions in the scalar field potential. Observationally we will be able to determine the accurate spectrum of primordial density fluctuations in the near future, e.g., by the next-generation MAP and PLANCK satellites. At that time, to test a variety of models of inflation, we will need a systematic method to evaluate the spectrum for a wide class of inflaton models.

In this paper, we consider perturbations of a multi-component scalar field in the long wavelength limit. That is, we consider modes whose wavelengths exceed the Hubble horizon scale. In the case of single-scalar inflation, the evolution of a perturbation in the long wavelength limit is very simple. The behavior of the adiabatic growing mode is specified by $R_c = \text{constant}$, where $R_c$ is the spatial curvature perturbation of the comoving hypersurface. In the case of multi-scalar inflation, the same is not true. Even if the wavelength of a perturbation exceeds the horizon scale, $R_c$ changes in time. The evolution of super-horizon scale perturbations for multi-scalar inflation has been investigated in some particular models that allow analytical treatments. However, since the feasibility of analysis of a model does not imply the viability of the model, it is necessary to develop a method that has a wide range of applicability. In this context, a rather general framework to study this issue has been given by Sasaki and Stewart under the assumption of the slow rolling evolution of a multi-component scalar field. However, it may well happen that some components of the scalar field do not satisfy the slow rolling condition during inflation. Furthermore, the slow rolling condition will be eventually violated toward the end of inflation when the reheating (or preheating) commences. Hence it is much more desirable to exclude the slow rolling assumption. In this connection, Taruya and Nambu have recently discussed a method to obtain the general solution for long wavelength perturbations. Unfortunately, however, they have not clarified several delicate issues associated with super-horizon scale perturbations that are characteristic of general relativity. Here we develop a general framework to study super-horizon scale perturbations without assuming slow rolling. Note that, although we use the terminology of ‘multi-scalar inflation’, our framework will be valid for any stage of the universe as long as the energy momentum tensor is dominated by a multi-component scalar field. Then we extend our formalism so as to include the effect of nonlinearity of the scalar field potential.

Suppose we are given a set of equations that governs the evolution of a system. If we know a complete set of solutions to this set of equations, we have no difficulty in constructing a general solution of the perturbation around a fixed background solution. By definition, a complete set of solutions contains a sufficient number of constants of motion that parametrize the different solutions. If we take a derivative...
with respect to one of such constants of motion, a solution to the perturbation equations can be obtained. The same is true for the cosmological perturbations. If we could obtain a full set of solutions to the Einstein equations coupled with the equations of matter fields, it would be trivially easy to spell out the perturbation around a fixed background universe. But it is hopeless to expect so. Nevertheless, if we restrict our attention to spatially homogeneous and isotropic universes, we may be able to obtain a full set of solutions. Then one may expect that the perturbation in the long wavelength limit is obtained from the knowledge of this restricted class of solutions. In fact, although restricted by the slow rolling assumption, the formula for the curvature perturbation on the comoving hypersurface derived in Ref. [4] is a good example of such a case.

Let us consider an \( n \)-component scalar field whose action is given by

\[
S_{\text{matter}} = -\int d^4x \sqrt{-g} \left( g^\mu\nu \partial_\mu \phi \cdot \partial_\nu \phi + U(\phi) \right),
\]

where we have used vector notation to represent the \( n \)-component scalar field. That is, \( \phi \cdot \phi = h_{pq} \phi^p \phi^q \), where \( h_{pq} \) is the metric in the scalar field space. For simplicity we assume \( h_{pq} = \delta_{pq} \) in the discussions below, but the generalization to a non-trivial \( h_{pq} \) is straightforward, which we shall discuss later. The equations for the spatially flat, homogeneous and isotropic configuration are given by

\[
\begin{align*}
\ddot{\phi}^p + 3H \dot{\phi}^p + U^{\parallel p} &= 0, \\
H^2 &= \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + U \right),
\end{align*}
\]

where dot means a derivative with respect to the cosmological time \( t \). We adopt the units \( 8\pi G = c = 1 \), \( H := \dot{a}/a \) is the Hubble parameter with \( a \) being the cosmic scale factor, and \( U^{\parallel p} = h^{pq} \partial U/\partial \phi^q \). A solution to these equations gives a background solution in the context of cosmological perturbation theory. Hence we refer to these equations as the background equations.

Substituting Eq. (1.3) into Eq. (1.2), we have \( n \) coupled second order differential equations. Now, suppose that we know a complete set of solutions for these equations. The complete set of solutions will contain \( 2n \) parameters, \( \lambda^\alpha (\alpha = 1, 2, \ldots, 2n) \) which distinguish the various solutions. One of them, say \( \lambda^1 \), just represents the change in the origin of the time coordinate, \( t_0 \). So let us denote the general solution by \( \phi(t + \lambda^1, \lambda^\alpha) \), where \( \alpha \) runs from 2 to \( 2n \). Then a naive anticipation is that we would obtain the \( 2n \) solutions for the perturbation in the long wavelength limit by \( \partial \phi(t + \lambda^1, \lambda^\alpha)/\partial \lambda^\alpha \), or more concisely by \( \partial \phi(t, \lambda^\alpha)/\partial \lambda^\alpha \) where \( \lambda^1 = t \).

However, things are not so simple. First of all, it is not guaranteed if we can restrict our consideration to spatially homogeneous and isotropic configurations from the beginning to obtain the perturbation in the long wavelength limit. Furthermore, it is well known that, when dealing with cosmological perturbations particularly on super-horizon scales, it is essential to make a clear statement about the choice of gauge before one talks about the behavior of the perturbation. In the above discussion, it was not clarified in what gauge the solutions were given, even if they actually would describe the long wavelength perturbation. One of the purposes of this paper is to clarify these points.

The paper is organized as follows. In §2, we consider the linear perturbation equations in the long wavelength limit and clarify its relation to the background equations. In §§2.1, we introduce the basic notation and list the basic equations for the perturbation of a multi-component scalar field. In §§2.2, we discuss the long wavelength perturbation in the \( B = H_L = 0 \) gauge, where \( B \) is the shift vector perturbation and \( H_L \) is the trace part of the spatial metric perturbation (see Eq. (2.21) below). In this gauge we find the e-folding number \( N := \log a \) is unperturbed even under the presence of the perturbation, hence it can be regarded as the natural time coordinate instead of the cosmological time, \( t \). Then we show that there exists a simple relation between the perturbation equations in the long wavelength limit and the derivative of the background equations with respect to the parameters \( \{ \lambda^\alpha \} := \{ N, \lambda^\alpha \} \). This result indicates there is a unique correspondence between the choice of the time coordinate for the background equations and that of gauge for the perturbation equations. To strengthen our assertion, in Appendix we consider the perturbation equations in a couple of other choices of time coordinates and show that this is indeed the case. Then using this relation, we give a formula to calculate the scalar field perturbation in the flat slicing from the solutions of the background equations. Very recently the same formula has been obtained independently by Kodama and Hamazaki[6] by a different approach. In §§2.3, we use this formula to evaluate the curvature perturbation on the comoving hypersurface at the end of inflation. Then we take the slow rolling limit and recover the result of Sasaki and Stewart[4]. In §3, we consider an
extension of our formalism beyond the limitation of the standard linear perturbation theory. There we only assume the smallness of the spatial metric perturbation and the non scalar-type perturbation but the other components of perturbations are not supposed to be small. We find almost everything goes parallel to the linear case. The summary of this paper is given in §4.

Throughout this paper, we follow the notation and sign convention used in Kodama and Sasaki[5].

§2. Linear Perturbation

2.1. Basic equations

First we write down a set of well known equations for cosmological perturbations. We consider only the scalar-type metric perturbation and write the perturbed metric as

\[ ds^2 = a^2 \left[ -(1 + 2AY) d\eta^2 - 2BY \, dy dx^j + \left( (1 + 2H_L Y) \delta_{ij} + 2H_T \chi_{ij} \right) dx^i dx^j \right], \]  

where \( Y \) is the spatial scalar harmonic with the eigenvalue \( k^2 \), \( Y_j = -k^{-1} \nabla_j Y \), and \( \chi_{ij} = k^{-2} \nabla_i \nabla_j Y + \frac{1}{2} \delta_{ij} Y \). We note the background equations (1.2) and (1.3) can be rewritten in terms of the conformal time, \( \eta \), as

\[ \phi'' + 2H \phi' + k^2 U = 0, \]  

\[ H^2 = \frac{1}{3} \left( \frac{1}{2} \phi'^2 + a^2 U \right), \]

where the prime represents a derivative with respect to \( \eta \) and \( H := a'/a \).

The perturbation of the scalar field equation becomes

\[ \chi'' + 2H \chi' + k^2 \chi + a^2 U \chi' - 2(\phi'' + 2H \phi') A - \phi' A' + k^2 \left( 3R' - k \sigma_g \right) = 0, \]

where \( \chi = \delta \phi \), and we have introduced \( k \sigma_g := H_T' - k B \) and \( R := H_L + \frac{1}{3} H_T \); the former represents the shear of the \( \eta = \) constant hypersurface and the latter the spatial curvature perturbation.

The \( (0) \)-component of the perturbed Einstein equations is given by

\[ 2 \left[ 3H^2 A - (3R' - k \sigma_g) - k^2 \mathcal{R} \right] = -\phi' \cdot \chi' - a^2 U_{ij} \chi + A \phi'^2. \]

The \( (i) \)-component is given by

\[ 2 \left[ H A - \mathcal{R}_i \right] = \phi' \cdot \chi. \]

The trace part of the \( (j) \)-component becomes

\[ 2 \left[ a^2 U A + HA' - \left[ \frac{d}{d\eta} + 2H \right] \left( \mathcal{R}' - \frac{1}{3} k \sigma_g \right) - \frac{k^2}{3} (A + \mathcal{R}) \right] = \phi' \cdot \chi' - a^2 U_{ij} \chi. \]

Finally, the traceless part of the \( (j) \)-component is

\[ k \sigma'_g + 2H k \sigma_g - k^2 (A + \mathcal{R}) = 0. \]

The long wavelength limit of the perturbation, i.e., the limit \( k^2 / H^2 \to 0 \), is described by taking \( k^2 \to 0 \) in the above equations.

An important geometrical quantity which plays a central role in the following discussion is the \( e \)-folding number of cosmic expansion. The perturbed \( e \)-folding number of expansion is defined by

\[ \bar{N} = \int_{t_0}^T \bar{H} dt, \]

where \( \bar{H} \) is the perturbed Hubble parameter given by the 1/3 of the expansion \( \bar{\theta} \) of the \( \eta = \) constant hypersurface;

\[ \bar{H} = \frac{1}{3} \bar{\theta} = H(1 + K_g Y); \]

\[ K_g := -A + \frac{1}{H} (H'_L + \frac{1}{3} k B) = -A + \frac{1}{H} (\mathcal{R}' - \frac{1}{3} k \sigma_g), \]

(2.10)
and $\tau$ is the proper time along the integral curve of the vector normal to the $\eta = \text{constant}$ hypersurfaces;

$$d\tau = (1 + AY) a \, d\eta.$$  \hspace{1cm} (2.11)

One then readily sees that

$$\dot{N} = \int_{\eta_0}^{\eta} \left( H + (R' - \frac{1}{3} k \sigma_g) Y \right) \, d\eta.$$  \hspace{1cm} (2.12)

Thus the $e$-folding number will be unperturbed if we take the gauge $H' = B = 0$, which implies $R' = k \sigma_g / 3$. One can then expect that the equations obtained by perturbing the background equations by taking $N = \ln a$ as the time coordinate will closely resemble the perturbation equations in this gauge.

2.2. $N$ as a time coordinate: the $H' = B = 0$ gauge

If we take $N$ as the time coordinate, the background equations can be written as

$$H \frac{d}{dN} \phi_N^p + 3 H^2 \phi_N^p + U^{ij} = 0,$$  \hspace{1cm} (2.13)

$$H^2 \left( 1 - \frac{1}{6} \phi_N^2 \right) = \frac{1}{3} U,$$  \hspace{1cm} (2.14)

where $\phi_N \equiv \frac{d\phi}{dN}$. We note that Eq. (2.14) is the $(0^0)$-component of the background Einstein equations. Together with this equation, the trace part of the $(ij)$-component of the Einstein equations gives

$$\frac{1}{H} \frac{dH}{dN} = - \frac{1}{2} \phi_N^2,$$  \hspace{1cm} (2.15)

which may be also obtained by substituting Eq. (2.13) into the $N$-derivative of Eq. (2.14).

Let us consider the perturbation equations in the long wavelength limit in the $B = H' = 0$ gauge. As we noted, in terms of the geometrical variables $R$ and $k \sigma_g$, this gauge condition implies

$$R' - \frac{1}{3} k \sigma_g = 0.$$  \hspace{1cm} (2.16)

Thus among the metric variables, the perturbed field equation contains only $A$. We also note that $A$ in this gauge represents the perturbation of the Hubble parameter, $\delta H / H = -AY$, as seen from Eq. (2.10).

From the $(0^0)$-component of the perturbed Einstein equations (2.3), one can see that $A$ is expressed in terms of $\chi$ as

$$2 U A = - H^2 \phi_N \cdot \chi_N - U_{ij} \chi^i \chi^j + 2 U^{ij} A - H^2 \phi_N^p A_N = 0.$$  \hspace{1cm} (2.17)

Together with Eq. (2.4), this gives

$$A_N = \phi_N \cdot \chi_N.$$  \hspace{1cm} (2.18)

This equation is not independent of Eqs. (2.17) and (2.13) because of the contracted Bianchi identities. In fact, Eq. (2.18) can be directly verified by taking the $N$-derivative of Eq. (2.17) and using Eqs. (2.13), (2.14) and (2.17). An equivalent, and perhaps a simpler way to obtain the closed equation for $\chi$ is to eliminate $A$ and $A_N$ from Eq. (2.17) by using Eqs. (2.18) and (2.20).

An important fact is that the closed equation for $\chi$ is obtained without using the $(0^0)$-component or the traceless part of the $(ij)$-component of the Einstein equations, both of which are absent in the
background equations. We also see that Eq. (2.20) exactly corresponds to the perturbation of Eq. (2.15). This indicates that we are on the right track. If we obtain a complete set of solutions of Eq. (2.17) (supplemented by Eqs. (2.18) and (2.20)), the only remaining task is to solve for \( R \) and \( k\sigma_g \). Let us denote the general solution of Eq. (2.17) by

\[
\chi = c^\alpha \chi^{(\alpha)}, \tag{2.21}
\]

where \( \alpha \) runs from 1 to \( 2n \), \( \chi^{(\alpha)} \) are the \( 2n \) independent solutions and \( c^\alpha \) are arbitrary constants.

From the long wavelength limit of the traceless \((ij)\)-component, Eq. (2.8), we have

\[
k\sigma_g \propto 1/a^2. \tag{2.22}
\]

Under the present gauge condition (2.16), this implies

\[
R_N = k\sigma_g \propto 1/a^3 H. \tag{2.23}
\]

On the other hand, the \((0i)\)-component of the perturbed Einstein equations, Eq. (2.6), is rewritten as

\[
R_N = A - \frac{1}{2}\phi_N \cdot \chi. \tag{2.24}
\]

Using Eqs. (2.13), (2.14) and (2.18), \( R_N \) can be expressed in terms of \( \chi \) as

\[
R_N = \frac{H^2}{2U} \left( \frac{d\phi_N}{dN} \cdot \chi - \phi_N \cdot \frac{d\chi}{dN} \right). \tag{2.25}
\]

Hence from Eq. (2.23), we find

\[
W[\chi] := \frac{a^3 H^3}{2U} \left( \frac{d\phi_N}{dN} \cdot \chi - \phi_N \cdot \frac{d\chi}{dN} \right) = a^3 H R_N \tag{2.26}
\]

should be a constant, which can be directly proved by using Eq. (2.17) with the aid of Eq. (2.18).

Now, we can calculate \( R \) and \( k\sigma_g \) for each independent solution of the scalar field perturbation. We obtain

\[
R_N = W_{(\alpha)} \left( \int_{N_b}^N dN \right) / a^3 H, \quad k\sigma_g = \frac{3W_{(\alpha)}}{a^2 c}, \tag{2.27}
\]

where \( W_{(\alpha)} \) is a constant given by

\[
W_{(\alpha)} = W[\chi^{(\alpha)}], \tag{2.28}
\]

and \( N_b \) can be arbitrary chosen. We leave it unspecified here.

The arbitrariness of the choice of \( N_b \) is the reflection of a residual gauge degree of freedom in the present gauge. Note that there is a trivial solution \( \chi = 0 \) for which \( R = c^0 = \text{constant} \). The addition of this solution corresponds to a variation of \( N_b \). Hence we have \( 2n + 1 \) integration constants for the whole set of the perturbation equations in this gauge. This is not a contradiction because there is a residual gauge degree of freedom in the gauge \( B = H_L = 0 \). Under an infinitesimal transformation of the time coordinate,

\[
N \rightarrow N - \delta N, \tag{2.29}
\]

\( H_L \) transforms as

\[
H_L \rightarrow H_L + \delta N. \tag{2.30}
\]

Hence the gauge condition \( H_L = 0 \) allows a further gauge transformation given by \( \delta N = c = \text{constant} \), which corresponds to an infinitesimal time translation mode. Applying this gauge transformation to the null perturbation, we obtain a pure gauge mode,

\[
\chi = c\phi_N, \quad R = c. \tag{2.31}
\]

This fact implies that one of the \( 2n \) solutions of Eq. (2.17) should be proportional to this time translation mode. In fact, we can verify that \( \chi = \phi_N \) is a solution of Eq. (2.17) by direct substitution. Hence we may set

\[
\chi^{(1)} = \phi_N. \tag{2.32}
\]
Then we also find \(W^{(1)} = 0\).

The issue of the number of physical degrees of freedom becomes transparent by constructing a gauge invariant quantity. A convenient choice is the scalar field perturbation \(\chi_F\) on the flat hypersurface (defined by \(\mathcal{R} = 0\)). It is given by

\[
\chi_F = \chi - \phi_N \mathcal{R}.
\]  

(2.33)

Then the general solution for \(\chi_F\) is given by

\[
\chi_F = c^\alpha \left( \chi^{(\alpha)} - W^{(\alpha)} \psi \right),
\]  

(2.34)

where \(W^{(1)} = 0\) and we have redefined \(c^1\) by the replacement; \(c^1 \rightarrow c^1\), and

\[
\psi := \phi_N \int_{N_0}^{N} \frac{dN}{e^{3N} H}.
\]  

(2.35)

One sees that the change of \(N_0\) is always absorbed in the redefinition of \(c^1\). Hence, only the 2n integration constants remain.

Since there exists at least one solution that has a non-vanishing \(W\), we may assume \(W^{(2)} \neq 0\) without any loss of generality. Then it is worthwhile to mention that it is always possible to set \(W^{(\alpha)} = 0\) for \(\alpha \neq 2\) by redefinition of the complete set of solutions. In fact, \(W\) vanishes for the linear combination of the solutions defined by \(\chi^{(\alpha)} - \frac{W^{(\alpha)}}{W^{(2)}} \chi^{(2)}\) for \(\alpha \geq 3\). It may be also useful to note that \(c^\alpha W^{(\alpha)}\) describes the amplitude of the adiabatic decaying mode, while the adiabatic growing mode amplitude is given by \(c^1\).

Let us now consider the perturbation of the background equations (2.13) and (2.14). Let us assume that the general solution for the background equations is known. Except for the trivial time translation, the general solution contains \(2n - 1\) integration constants, \(\lambda^\alpha\), which distinguish the various solutions. Each solution gives a curve in the phase space of the scalar field parametrized by some time coordinate \(\lambda^1\). Let \(\chi^A = (\lambda^\rho, \lambda^q)\) \((A = 1, 2, \ldots, 2n)\) be the phase space coordinates where \(\pi^q\) are the momentum variables. Then the general solution is expressed as \(\bar{\chi}^A = \tilde{\chi}^A(\lambda^\beta)\). We may regard this as a coordinate transformation of the phase space coordinates. We see that the perturbation of a given background solution is given by the Jacobian \(\partial \bar{\chi}^A / \partial \lambda^\beta\), including the time translation mode. It is then easy to convince ourselves that a necessary condition for these background solutions to describe the solutions of the perturbation in the long wavelength limit is the non-disturbance of the time coordinate under the presence of a perturbation, since the commutative property of the partial derivatives; \(\partial^2 / \partial \lambda^\alpha \partial \lambda^\beta = \partial^2 / \partial \lambda^\beta \partial \lambda^\alpha\), should be maintained. As we have seen, the \(\epsilon\)-folding number \(N = \ln \alpha\) is indeed such a coordinate for the gauge \(B = H' = 0\). Thus we take \(N\) as the time coordinate in the background equations and set \(\lambda^\alpha = (N, \lambda^\alpha)\). Note that the background equations (2.13) and (2.14) have no explicit \(N\)-dependence. Hence \(\phi_N\) is a solution to the perturbed background equations, which corresponds to the time translation mode.

Taking the derivative of Eqs. (2.13) and (2.14) with respect to \(\lambda^\alpha\), we obtain

\[
H \frac{d}{dN} H \frac{d}{dN} \phi^\rho + 3H^2 \phi^\rho + U |p|^q \phi^q - 2U |p| H = 0,
\]  

(2.36)

\[
2U |p| = H^2 \phi_N \frac{d}{dN} \phi^\rho + U |p| \phi^\rho,
\]  

(2.37)

where the suffix \(\lambda\) represents \(\partial / \partial \lambda^\alpha\). We find these equations are equivalent to those for the perturbation in the long wavelength limit in the \(B = H' = 0\) gauge, Eqs. (2.17) and (2.18), respectively, with the identifications \(\phi^\rho = \chi\) and \(H/3 = -A\). Further, as we have already seen, the \(\lambda^\alpha\)-derivative of Eq. (2.15) is equivalent to Eq. (2.20). Thus we conclude that a complete set of solutions in the long wavelength limit, \(\chi^{(\alpha)}\), can be constructed from the solutions of the background equations; \(\chi^{(\alpha)} = \partial \phi / \partial \lambda^\alpha\). Once \(\chi^{(\alpha)}\) are obtained, the corresponding set of gauge-invariant quantities \(\chi_F^{(\alpha)}\) is readily obtained as

\[
\chi_F^{(\alpha)} = \chi^{(\alpha)} - W^{(\alpha)} \psi,
\]  

(2.38)

where \(W^{(\alpha)}\) is given by Eq. (2.28).
Before closing this subsection, we mention the generalization of the scalar field metric $h_{pq}$ to a non-trivial one. Except for the perturbed field equation (2.17) or (2.36), the only modification is to replace all the derivatives in the equations with the covariant ones with respect to $h_{pq}$. For example,

$$\frac{d}{dN} \phi_{\lambda}^p \rightarrow \frac{D}{dN} \phi_{\lambda}^p = \frac{d}{dN} \phi_{\lambda}^p + \Gamma^p_{qr} \phi_{\lambda}^q \phi_{\lambda}^r,$$

(2.39)

where $\Gamma^p_{qr}$ is the connection of $h_{pq}$. As for the perturbed field equation, we have to add the curvature term in addition to the covariantization of the derivatives. That is, Eq. (2.17) is modified as

$$H \frac{D}{dN} (H \chi_N^p) + 3H^2 \chi_N^p + H^2 R^p_{qr} \phi_N^q \chi^q + U^{|p|q} \chi^q + 2U^{|p} A - H^2 \phi_N^p A_N = 0,$$

(2.40)

and similarly for Eq. (2.36).

2.3. Curvature perturbation on the comoving hypersurface

Among the various geometrical quantities, one of the most convenient representations of the perturbation amplitude is the curvature perturbation of the comoving hypersurface, $\mathcal{R}_c$. For the adiabatic growing mode perturbation, it is known to stay constant on super-horizon scales. Here, we relate $\mathcal{R}_c$ with the scalar field perturbation and present a formula that can be used to evaluate $\mathcal{R}_c$ at the end of inflation in terms of the initial data of the scalar field perturbation. Then we take the slow rolling limit and show how our formula reduces to the one obtained in Ref. [1]. To obtain $\mathcal{R}_c$ that is directly relevant to observational quantities, one must solve the evolution of $\mathcal{R}_c$ during the reheating stage after inflation. One may also have to evaluate the possible contribution of the isocurvature perturbation. We defer these issues to future studies and derive here a formula valid up to a time $N = N_c$ before the reheating commences. It may be noted, however, that there are situations in which one can neglect the time variation of $\mathcal{R}_c$ during reheating. For example, if all the trajectories in the phase space of the scalar field converge to one path before reheating, the evolution during reheating is essentially the same as the case of single-scalar inflation and there will be no additional change in the amplitude of $\mathcal{R}_c$.

As we have seen in the previous section, the spacetime configuration of the scalar field on super-horizon scales is completely determined by the background solutions and the parameters $\alpha^\lambda$ characterizing the background solutions play the role of phase space coordinates. This implies the value of the scalar field depends on the spacetime coordinates $\{x^\mu\}$ only through the phase space coordinates for the spatially homogeneous configurations; $\phi(x^\mu) = \phi(\lambda^\alpha(x^\mu))$. Thus it will be useful to express $\mathcal{R}_c$ in the language of the phase space of the background scalar field.

The comoving hypersurface is determined by the condition, $T_i^0 = 0$, i.e.,

$$\phi_N \cdot d\phi = 0,$$

(2.41)

In general, we cannot expect this condition to determine a surface in the phase space of the scalar field since it is not integrable. However, in the case of linear perturbation, this condition reduces to

$$\bar{\phi}_N(N) \cdot (\phi - \bar{\phi}(N)) = 0,$$

(2.42)

where the barred quantities represent the background values. Now it is manifest that this condition determines a surface in the phase space. For each $\lambda^\alpha$, the value of $N$ at which the comoving surface in the phase space is crossed is different. We denote this difference by $\Delta N$. Namely,

$$\bar{\phi}_N(N) \cdot (\phi(N + \Delta N, \lambda^\alpha) - \bar{\phi}(N)) = 0,$$

(2.43)

where $\Delta N$ is a function of $\lambda^\alpha = (N, \lambda^\alpha)$. From the spacetime point of view, through the dependence of $\lambda^\alpha$ on $x^\mu$, $\Delta N$ is a function of $x^\mu$. Hence we may consider the infinitesimal coordinate transformation $N \rightarrow N - \Delta N$ and move from the $H'_L = B = 0$ gauge to the comoving gauge. Then we have

$$\mathcal{R}_c(N) = \mathcal{R}(N) + \Delta N.$$

(2.44)

On the other hand, the comoving condition (2.43) is rewritten as

$$\phi_N(N) \cdot (\chi(N) + \phi_N(N)\Delta N) = 0.$$

(2.45)
Thus we have $\Delta N = -(\phi_N \cdot \chi)/\phi_N^2$. Therefore we obtain

$$R_c = e^\alpha \frac{\phi_N \cdot \chi_{(\alpha)}}{\phi_N^2} + c^\alpha W(\alpha) \int_{N_0}^N \frac{dN}{a^3 H}, \quad (2.46)$$

where we remind that $\chi_{(\alpha)} = \partial \phi/\partial \lambda^\alpha$. Given the background solutions, this gives the evolution of $R_c$ up to $N = N_c$.

In the above, we have derived the expression for $R_c$ in the language of the phase space for the background solutions. But, of course, it can be derived in the language of the standard linear perturbation theory. In the defining equation for $\chi_F$, Eq. (2.33), we evaluate the right hand side on the comoving hypersurface to obtain

$$\chi_F = \chi_c - \phi_N R_c, \quad (2.47)$$

where $\chi_c$ is the scalar field perturbation on the comoving hypersurface. Then taking the inner product of the above equation with $\phi_N$ and using the comoving hypersurface condition $\phi_N \cdot \chi_c = 0$, we find

$$R_c = -\frac{\phi_N \cdot \chi_F}{\phi_N^2}. \quad (2.48)$$

Inserting the expression (2.34) for $\chi_F$ to this equation gives Eq. (2.46).

The remaining task is to determine $c^\alpha$. In the inflationary universe scenario, scalar field perturbations are generally induced by quantum vacuum fluctuations. The evaluation of the vacuum fluctuations is most conveniently done with respect to $\chi_F$ since there exists a closed action for $\chi_F$ that gives a complete description of the scalar-type perturbation [2]. Hence we assume that $\chi_F$ is given at some initial epoch when the wavelength exceeds the horizon scale.

Suppose that the initial condition of $\chi_F$ and $D\chi_F/dN$ is given at a time, $N = N_0$. The constant $c^\alpha W(\alpha)$ can be readily evaluated by inserting $\chi_F$ into the operator $W$:

$$W[\chi_F] = c^\alpha W(\alpha) \left(1 + \frac{H^2}{2U} \phi_N^2\right) = \frac{3H^2}{U} c^\alpha W(\alpha). \quad (2.49)$$

Hence,

$$c^\alpha W(\alpha) = \left[\frac{a^3 H}{6} \left(\frac{D\phi_N}{dN} \cdot \chi_F - \phi_N \cdot \frac{D\chi_F}{dN}\right)\right]_{N_0}. \quad (2.50)$$

Then adopting the notation $\hat{\chi}^A := \chi^A$ for $A \leq n$ and $\hat{\chi}^A := \chi^A_{A-n}$ for $A > n$, $c^\alpha$ is evaluated from Eq. (2.34) as

$$c^\alpha = \left[(\hat{\chi}^{-1})_A^{(\alpha)} \left\{\hat{\chi}_F + (c^\beta W(\beta))\hat{\psi}^A\right\}\right]_{N_0}, \quad (2.51)$$

where $\hat{\chi}_F$ and $\hat{\psi}$ are defined in the same way as $\hat{\chi}$, and $(\hat{\chi}^{-1})_A^{(\alpha)}$ is the inverse matrix of $\hat{\chi}_{(\alpha)}$, i.e.,

$$(\hat{\chi}^{-1})_A^{(\alpha)} = \frac{\partial \lambda^\alpha}{\partial \hat{\chi}^A}. \quad (2.52)$$

So far our discussion has been completely general. Now we consider the slow rolling limit. We assume all the components of the scalar field satisfy the slow rolling condition:

$$\phi_N^2 \ll 1, \quad \left|\frac{D\phi_N}{dN}\right| \ll |\phi_N|. \quad (2.53)$$

In this limit, the background equations reduce to

$$\phi_N^p = -\frac{U^p}{3H^2} = -(\ln U)^p, \quad H^2 = \frac{1}{3} U. \quad (2.54)$$

Thus the momentum variables cease to be dynamical and the $2n$-dimensional phase space reduces to the $n$-dimensional configuration space of the scalar field. Correspondingly the number of the parameters $\lambda^\alpha$ reduces to $n$. The perturbed field equation is very simple:

$$\frac{D}{dN} \phi^p_A = -(\ln U)^p |q| \phi^q_A, \quad (2.55)$$
where we have assumed the scalar space curvature is small; \( |R_{prqs}^p| \ll 1 \). From this equation, we easily see that \( W[\phi] = 0 \) for all the \( n \)-independent solutions. Thus the slow rolling assumption kills the adiabatic decaying mode as in the case of single-scalar inflation. This implies that \( \chi_F \) is directly given by the background solutions:

\[
\chi_F = c^a \chi(\alpha) = c^a \frac{\partial \phi}{\partial \lambda^a},
\]

where \( \alpha \) now runs from 1 to \( n \). In other words, the flat slicing does not perturb the \( e \)-folding number \( N \) in the slow rolling limit.

The formula (2.51) for \( c^a \) reduces to

\[
c^a = \left( \chi^{-1}_p(a) \chi_F^p \right)_{N_0} = \left[ \frac{\partial \lambda^a}{\partial \phi^p} \chi_F^p \right]_{N_0}.
\]

For definiteness, we set \( \chi(1) = \phi_N \) (i.e., \( \lambda^1 = N \)) as before. The curvature perturbation \( R_c \) is then given by

\[
R_c = -c^a \frac{\phi_N \cdot \chi(\alpha)}{\phi_N^2}
\]

\[
= - \left\{ \left[ \frac{\partial N}{\partial \phi^p} \chi_F^p \right]_{N_0} + \left[ \frac{\partial \lambda^a}{\partial \phi^p} \chi_F^p \right]_{N_0} \phi_N \cdot \phi_a \right\},
\]

where \( a = 2, 3, \ldots, n \) and \( \phi_a = \partial \phi / \partial \lambda^a \).

If we synchronize the time to be \( N = N_e \) at the end of inflation for all the background scalar field trajectories, we have \( \partial U / \partial \lambda^a = 0 \) at \( N = N_e \). Then using the slow rolling equation of motion (2.54) we find

\[
0 = \frac{\partial U}{\partial \lambda^a} = \frac{\partial U}{\partial \phi^p} \frac{\partial \phi^p}{\partial \lambda^a} = -U \phi_N \cdot \phi_a,
\]

at \( N = N_e \). Thus the evaluation of Eq. (2.58) at \( N = N_e \) gives

\[
R_c(N_e) = - \left[ \frac{\partial N}{\partial \phi^p} \chi_F^p \right]_{N_0}.
\]

This formula coincides with the one derived by Sasaki and Stewart.

\section{Quasi-Nonlinear Perturbation}

An advantage of the formalism developed in the previous section is that it can be extended so as to take into account the non-linearity of the scalar field dynamics. The basic idea is to consider the linearization with respect only to inhomogeneities of the spatial metric. For simplicity, we again assume the trivial metric for the scalar field space, \( h_{pq} = \delta_{pq} \).

We consider the following metric form,

\[
ds^2 = -\frac{1}{H^2} dN^2 + e^{2N} \left( (1 + 2H_L) \delta_{ij} + 2H_T_{ij} \right) dx^i dx^j,
\]

where \( \tilde{H} \) depends both on \( N \) and \( x^i \). Here and in what follows the tilde is attached to a non-linearly perturbed quantity. We consider \( H_L \) and \( H_T_{ij} \) to be small of \( O(\epsilon) \). In accordance with the linear case, we assume the existence of a potential function \( H_T \) for \( H_T_{ij} \) such that

\[
- \nabla^2 H_T_{ij} = \left[ \nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2 \right] H_T.
\]

\(^{\ast}\) It seems this condition is necessary for the slow rolling approximation to be consistent. However, we are unable to give a proof.

\(^{\ast\ast}\) In Ref. [4], the minus sign on the right hand side of this formula is absent. The reason is that \( N \) there is defined as the \( e \)-folding number counted backwards from the time \( N = N_e \) for convenience of the spectrum evaluation.
We also assume the spatial derivatives of $\tilde{\dot{H}}$ and $\tilde{\dot{\phi}}$ are small of $O(\epsilon)$. This assumption corresponds to taking the long wavelength limit. Furthermore, we assume the gauge condition $dH_L/dN = O(\epsilon^2)$ can be imposed consistently. We then linearize the scalar field and Einstein equations with respect to $\epsilon$ while keeping the other nonlinear terms as they are.

First let us consider the scalar field equation. Keeping the terms linear in $\epsilon$, the field equation reduces to

$$0 = \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ g^{\mu\nu} \sqrt{-g} \partial_{\nu} \tilde{\phi} \right]$$

$$= \frac{\tilde{H}}{e^{3N}} (1 - 3H_L) \frac{d}{dN} \tilde{H} e^{3N} (1 + 3H_L) \frac{d}{dN} \tilde{\dot{\phi}} + U(\tilde{\phi})^p$$

$$= \tilde{H} \frac{d}{dN} \tilde{H} \frac{d}{dN} \tilde{\dot{\phi}} + 3\tilde{H}^2 \frac{d}{dN} \tilde{\dot{\phi}} + U(\tilde{\phi})^p.$$  

(3.3)

This is the same form as the background equation (2.13).

Next let us consider the $(0)_i$-component of the Einstein equations. To the linear order in $\epsilon$, we have

$$G^0_0 = -3\tilde{H}^2,$$

$$T^0_0 = -\frac{1}{2} \tilde{H}^2 \tilde{\phi}_N - U(\tilde{\phi}).$$  

(3.4)

Therefore we find

$$\tilde{H}^2 \left( 1 - \frac{1}{6} \tilde{\phi}_N^2 \right) = \frac{1}{3} U(\tilde{\phi}),$$  

(3.5)

which is also the same form as the background equation (2.14).

Thus we find that, in the long wavelength limit, solving the equations for the perturbed scalar field in the $dH_L/dN = 0$ gauge is completely equivalent to finding the spatially homogeneous background solutions.

The remaining task is to determine $R := H_L + \frac{1}{3} H_T$ and $k\sigma_g := e^N \tilde{H} dH_T/dN$ as before (note that $k\sigma_g$ is not $k$ times $\sigma_g$ but regarded as a single symbol here). The traceless part of the $(i)_i$-component of the Einstein equations is essentially the same as Eq. (2.8) also in the present case. This tells us that we have

$$k\sigma_g = 3e^N \tilde{H} R_N \propto \frac{1}{e^{2N}}.$$  

(3.6)

The $(0)_i$-component of the Einstein tensor becomes

$$G^0_i = \nabla_i \tilde{H}^2 + 2\tilde{H}^2 \nabla_i R_N,$$  

(3.7)

while that of the energy momentum tensor is given by

$$T^0_i = -\tilde{H}^2 \tilde{\phi}_N \cdot \nabla_i \tilde{\phi}.$$  

(3.8)

Hence we have

$$e^{3N} \tilde{H} \nabla_i R_N = -e^{3N} \left( \nabla_i \tilde{H} + \frac{1}{2} \tilde{H} \tilde{\phi}_N \cdot \nabla_i \tilde{\phi} \right),$$  

(3.9)

which is expected to be constant in time because of Eq. (3.4). In fact, we find

$$\frac{d}{dN} (e^{3N} \tilde{H} \nabla_i R_N) = -\frac{d}{dN} \left[ e^{3N} \left( \nabla_i \tilde{H} + \frac{1}{2} \tilde{H} \tilde{\phi}_N \cdot \nabla_i \tilde{\phi} \right) \right]$$

$$= \frac{e^{3N}}{2H} \left( -\frac{2U(\tilde{\phi})}{\tilde{H}} \nabla_i \tilde{H} + \tilde{H}^2 \tilde{\phi}_N \nabla_i \tilde{\phi}_N + \nabla_i U(\tilde{\phi}) \right)$$

$$= 0,$$  

(3.10)

where Eqs. (3.3) and (3.5) are used in the second equality and the spatial derivative of Eq. (3.4) in the last equality. This verifies the consistency of our assumptions, in particular the gauge condition $dH_L/dN = O(\epsilon^2)$.

Let us reinterpret the discussions given in §§2.3 in the present context. For simplicity we assume that the linear perturbation is valid at $N = N_0$. Otherwise, the evaluation of the initial perturbation becomes too difficult. As in §§2.3, we give the initial data in terms of $\chi_F$ and $d\chi_F/dN$. 


Then from Eq. (3.6), the evolution of $R$ is given by

$$R = C_R \int_{N_0}^{N} \frac{dN}{e^{3N\tilde{H}}},$$  
(3.11)

where the coefficient $C_R$ is expressed in terms of the initial values of $\chi_F$ and $d\chi_F/dN$ as

$$C_R = \left[ \frac{e^{3N\tilde{H}}}{6} \left( \frac{d\phi_N}{dN} \cdot \chi_F - \phi_N \cdot d\chi_F/dN \right) \right]_{N_0},$$  
(3.12)

which corresponds to Eq. (2.50) for the linear case. In the above, we have chosen $R(N_0) = 0$ by setting $N_b = N_0$ for convenience.

Using Eq. (2.34), the initial data for the scalar field are given by

$$\tilde{\phi}(N_0) = \phi(N_0) + \chi_F(N_0),$$  
$$\phi_N(N_0) = \left( 1 - \frac{C_R}{e^{3N_0\tilde{H}_0}} \right) \phi_N(N_0) + \chi_{F N}(N_0),$$  
(3.13)

where $\tilde{H}_0 = \tilde{H}(N_0)$ and $X_{F N} = dX_F/dN$. The evolution of $\tilde{\phi}$ is determined by solving the nonlinear background field equation (3.3) supplemented with Eq. (3.5).

In §2.3, to evaluate the amplitude of the perturbation at the end of inflation, it was necessary to move to the comoving hypersurface. In terms of the background solutions, this amounts to finding a comoving surface in the phase space. However, as noted there, the comoving condition (2.41) does not specify a surface in the phase space in the nonlinear case. To circumvent this difficulty, here we propose to use the constant Hubble hypersurface in substitution of the comoving one. It is apparent from Eq. (3.5) that there exists a surface in the phase space corresponding to the constant Hubble hypersurface. From the fact that the right hand side of Eq. (3.5) is constant in time, we find

$$\nabla_i \ln \tilde{H} = -\frac{1}{2} \tilde{\phi} \cdot \nabla_i \tilde{\phi} + \frac{d_i}{e^{3N\tilde{H}}},$$  
(3.14)

where $d_i$ is a time-independent vector. At a later epoch, the second term in the right hand side can be neglected and the difference between the comoving gauge and the constant Hubble gauge becomes negligibly small.

Then as a function of $\chi_F(N_0)$ and $X_{F N}(N_0)$, we define $\Delta N_H$ as the difference of the time to cross the constant Hubble surface in the phase space, namely, by the condition,

$$\tilde{H}(\phi(N + \Delta N_H), \phi_N(N + \Delta N_H)) = \text{independent of } \lambda^a.$$  
(3.15)

Finally, we find the curvature perturbation on the constant Hubble hypersurface $\mathcal{R}_H$ as

$$\mathcal{R}_c \simeq \mathcal{R}_H = \mathcal{R} + \Delta N_H.$$  
(3.16)

In the case the slow rolling condition (2.53) is satisfied at the initial time, the first term in the right hand side of Eq. (3.16) can be totally neglected and we simply have

$$\mathcal{R}_c \simeq \mathcal{R}_H = \Delta N_H,$$  
(3.17)

where $\Delta N_H$ will now be a function of $\chi_F(N_0)$ alone.

§4. Summary

In this paper, we have investigated the dynamics of a multi-component scalar field on super-horizon scales, i.e., in the long wavelength limit. We have shown that there is a simple relation between the perturbation equations in the long wavelength limit and the background equations. That is, the derivative of the general solution of the background equations with respect to a parameter that characterizes different solutions satisfies the same equation as the perturbation in the long wavelength limit does. However, we have also found that the explicit form of the relation depends on the choice of gauge, and the choice of gauge corresponds to that of a time coordinate in the phase space of the background scalar field. We have
found that the simplest form of the relation is obtained in the gauge in which the $e$-folding number $N$ of cosmic expansion is unperturbed even under the presence of the perturbation. Then using this result, we have given a method to calculate the amplitude of the spatial curvature perturbation on the comoving hypersurface, $R_c$, from the knowledge of the background solutions alone.

As a natural extension of our approach, we have considered to take into account the nonlinearity of the scalar field dynamics in the perturbation. We have found that this can be actually done under several reasonable assumptions. The result provides a powerful tool to evaluate the effect of nonlinearity of the scalar field potential during the inflationary stage. In particular, the effect of non-gaussian statistics of the perturbation, given a gaussian distribution of the initial perturbation due to quantum vacuum fluctuations, can be evaluated by studying the background solutions alone.

The present nonlinear extension is similar to the so-called anti-Newtonian approximation 9) or the gradient expansion method 10). It may be worthwhile to clarify how our result is related to these methods.

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**Appendix A**

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**Other Choices of Gauge**

In this appendix, we compare the long wavelength perturbation equations and the background equations in a couple of other choices of gauge. We find the choice of the time coordinate in the scalar field phase space determines the corresponding gauge condition for the perturbation as in the case of the $H'_L = B = 0$ gauge. However, the resulting equations turn out to be less tractable.

### A.1. $t$ as a time coordinate: the synchronous gauge

In the synchronous gauge, $A = B = 0$, the cosmological time $t$ remains to be the proper time along curves normal to the constant time hypersurfaces. Hence we expect $t$ to be the relevant time coordinate to parametrize the background solutions.

The perturbation equations (2.4) and (2.5) in the long wavelength limit in the synchronous gauge become

$$\ddot{\chi}^p + 3H\dot{\chi}^p - \dot{\phi}^p \left( \frac{k\sigma_g}{a} - 3\dot{R} \right) + U|^p|_q \chi^q = 0.$$  \hfill(A.1)

$$2H \left( \dot{3}\dot{R} - \frac{k\sigma_g}{a} \right) = \dot{\phi} \cdot \dot{\chi} + U|^p|_q \chi^q.$$  \hfill(A.2)

Eliminating $\left( \frac{3\dot{R} - k\sigma_g}{a} \right)$ from these equations, we obtain an equation written in terms of $\chi$ alone.

Note that $\left( \dot{R} - \frac{k\sigma_g}{3a} \right)$ in this gauge represents the perturbation of the Hubble parameter as seen from Eq. (2.10).

Once we know the solutions for $\chi$, we can evaluate the spatial curvature perturbation with the aid of the $(0)$-component of the Einstein equations, Eq. (2.6), which now reduces to

$$\dot{\mathcal{R}} = -\frac{\dot{\phi} \cdot \chi}{2}.$$  \hfill(A.3)

Integrating this equation, $\mathcal{R}$ can be evaluated. But different from the $H'_L = B = 0$ gauge, we have no knowledge of the explicit time dependence of the integrand at all. So the integration must be done for each mode separately.

On the other hand, the time dependence of $k\sigma_g$ can be found from the traceless part of the $(i)$-component, Eq. (2.8), as

$$k\sigma_g \propto \frac{1}{a^2}.$$  \hfill(A.4)
Using Eqs. (A-2) and (A-3), this implies
\[ k\sigma_g a^2 = \frac{a^3}{2H} (\ddot{\phi} \cdot \chi - \dot{\phi} \cdot \dot{\chi}) = \text{constant} . \] (A-5)

As before, the general solution of Eq. (A-1) with (A-2) contains \(2n\) integration constants. When we evaluate \(\mathcal{R}\), there appears an additional integration constant. Hence, we have \(2n+1\) integration constants in total. This situation is exactly parallel to the case of the \(H'_L = B = 0\) gauge. Namely, one mode is responsible for a gauge degree of freedom, since the synchronous gauge condition allows an additional gauge transformation given by \(t \rightarrow t + c\), where \(c\) is a constant. Then the gauge mode is found to be
\[ \chi = c \dot{\phi}, \quad \mathcal{R} = c H. \] (A-6)

Also parallel to the \(H'_L = B = 0\) gauge, there exists a trivial solution:
\[ \chi = 0, \quad \mathcal{R} = \text{constant}. \] (A-7)

As before, this gives one of the solutions for the gauge-invariant scalar field perturbation \(\chi_F\) as
\[ \chi_F = \chi - \frac{\dot{\phi}}{H} \mathcal{R} \propto \phi_N . \] (A-8)

Now let us turn to the comparison of the perturbation equations with the background equations. As we have mentioned in the beginning, since the cosmological time is unperturbed in the synchronous gauge, the relevant time coordinate will be \(t\). Thus taking the derivatives of Eqs. (1-2) and (1-3) with respect to \(\lambda = (t, \lambda^a)\), we have
\[ \dddot{\phi}_\lambda + 3H_\lambda \dot{\phi}^p + 3H \phi^p_\lambda + U^{|p}|_q \phi^q_\lambda = 0 , \] (A-9)
and
\[ 2HH_\lambda = \frac{1}{3} \left( \phi \cdot \dot{\phi}^p_\lambda + U^{|p}|_q \phi^q_\lambda \right) , \] (A-10)
where the suffix \(\lambda\) again represents the partial derivative with respect to \(\lambda^a\). It is readily seen that these are equivalent to Eqs. (A-1) and (A-2), with the identifications,
\[ \phi_\lambda = \chi, \quad H_\lambda = \dot{\mathcal{R}} - \frac{1}{3} \frac{k\sigma_g}{a} . \] (A-11)

**A.2. \(H\) as a time coordinate: the constant Hubble gauge**

As another example, we consider to take the Hubble parameter \(H\) as a time coordinate. Then the relevant choice of gauge will be to take the constant Hubble gauge in which \(H\) is unperturbed under the presence of the perturbation.

From the expression for the perturbed Hubble parameter, Eq. (2-10), we find the condition for the constant Hubble gauge as
\[ \mathcal{H} A - \mathcal{R}' + \frac{1}{3} k\sigma_g = 0 . \] (A-12)

In this gauge, the perturbation equations (2-4) and (2-5), in the long wavelength limit, reduce to
\[ \chi^{ppp} + 2H \chi^{pp} - \{2\phi^{ppp} + H \phi^{pp}\} A - \phi^{ppp} A' + a^2 U^{|p}|q \chi^q = 0 , \] (A-13)
\[ A = \frac{1}{\phi^{pp}} \left( \phi' \cdot \chi' + a^2 U^{|p}|q \chi^q \right) . \] (A-14)

From Eq. (2-8), we find \(k\sigma_g \propto 1/a^2\), and from Eq. (2-6), we have \(-(2/3)k\sigma_g = \phi' \cdot \chi\). Hence, we find
\[ k\sigma_g a^2 = -\frac{3a^2}{2} \phi' \cdot \chi = \text{constant} . \] (A-15)

Taking the time derivative of this equation, we find the constraint,
\[ \phi' \cdot \chi' = a^2 U^{|p}|q \chi^q . \] (A-16)
This means that the initial condition for the scalar field perturbation cannot be chosen arbitrarily. This is because the constant Hubble gauge condition completely fixes the time slicing and hence the constant time surfaces in the phase space of the scalar field. So the perturbation in the direction normal to the constant time surfaces is not allowed.

Hence, in the present gauge, the general solution for scalar field perturbations has $2n - 1$ integration constants. The curvature perturbation can be evaluated by integrating

$$\mathcal{R}' = \dot{H} - \frac{1}{2} \phi' \cdot \chi = 2 \dot{H} \frac{\phi' \cdot \chi'}{\phi'^2} - \frac{1}{2} \phi' \cdot \chi.$$  \hfill (A.17)

Again, the integration of this equation is non-trivial.

As in the previous two cases, we have a trivial solution

$$\chi = 0, \quad \mathcal{R} = \text{constant}. \hfill (A.18)$$

This again tells us that $\phi_N$ is a solution for $\chi_F$.

Now let us consider the background equations. Using $H$ as the time coordinate, the background equations are written as

$$\frac{4}{3} \frac{d}{dH} \left( \frac{\dot{\phi}_H}{\phi_H} \right) - 6 \frac{H}{\phi_H} \phi_H^p + U^{|p} = 0,$$

$$H^2 = \frac{1}{3} \left( \frac{2}{\phi_H^2} + U \right), \hfill (A.19)$$

where subscript $H$ represents the derivative with respect to $H$. The latter equation is the equation that constrains the scalar field in its phase space. Different from the previous two cases, these equations do not have the invariance with respect to the time translation because they contain $H$ explicitly. Hence, $\phi_H$ is not a solution of the perturbation equations.

Taking the derivative of Eqs. (A.19) with respect to $\lambda = \lambda^a$, we obtain

$$\phi_{\lambda''} + 2 \dot{\mathcal{H}} \phi_{\lambda'} - \left( 2 \phi_{p''} + \mathcal{H} \phi_{p'} \right) \frac{2 \phi' \cdot \phi_{\lambda'}}{\phi'^2} - \phi_{p''} \left( \frac{2 \phi' \cdot \phi_{\lambda'}}{\phi'^2} \right)' + a^2 U^{|p} q \phi_{\lambda}^q = 0,$$

$$\phi' \cdot \phi_{\lambda} = a^2 U_{|p} \phi_{\lambda}^{|p}. \hfill (A.20)$$

These are equivalent to Eqs. (A.13) and (A.14) supplemented by the constraint (A.16).

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