1 Introduction

The existence of Kähler-Einstein metrics on a compact Kähler manifold has been the subject of intensive study over the last few decades, following Yau’s solution to Calabi’s conjecture (see \[Ya2\], \[Au\], \[Ti1\], \[Ti2\]). The Ricci flow, introduced by Richard Hamilton in \[Ha1\], \[Ha2\], has become one of the most powerful tools in geometric analysis. The Ricci flow preserves the Kählerian property, so it provides a natural flow in Kähler geometry, referred as the Kähler-Ricci flow. Using the Kähler-Ricci flow, Cao \[Ca\] gave an alternative proof of the existence of Kähler-Einstein metrics on a compact Kähler manifold with trivial or negative first Chern class. In early 90’s, Hamilton and Chow also used the Ricci flow to give another proof of
classical uniformization for Riemann surfaces (see [Ha2], [Ch]). Recently Perelman [Pe] has made a major breakthrough in studying the Ricci flow. The convergence of the Kähler-Ricci flow on Kähler-Einstein Fano manifolds was claimed by Perelman and it has been generalized to any Kähler manifolds admitting a Kähler-Ricci soliton by the second named author and Zhu [TiZh]. Previously, in [ChTi], Chen and the second named author proved that the Kähler-Ricci flow converges to a Kähler-Einstein metric if the initial metric is of non-negative bisectional curvature.

However, most projective manifolds do not have a definite or trivial first Chern class. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or on varieties canonically associated to them, i.e., canonical models. Tsuji [Ts] used the Kähler-Ricci flow to prove the existence of a canonical singular Kähler-Einstein metric on a minimal projective varieties canonically associated to them, i.e., canonical models. In this paper, we propose a program of finding canonical metrics on manifolds of general type. In its sequel, we will extend our results here to higher dimensional varieties. To do it, we will study the Kähler-Ricci flow starting from any Kähler metrics and show its limiting behavior at time infinity.

Let $X$ be an $n$-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form $\omega$ on $X$. In local coordinates $z_1, ..., z_n$, we can write $\omega$ as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dz_i \wedge d\bar{z}_j,$$

where $\{g_{ij}\}$ is a positive definite hermitian matrix function. Consider the Kähler-Ricci flow

$$\frac{\partial \omega(t, \cdot)}{\partial t} = -Ric(\omega(t, \cdot)) - \omega(t, \cdot), \quad \omega(0, \cdot) = \omega_0,$$  \hspace{1cm} (1.1)

where $\omega(t, \cdot)$ is a family of Kähler metrics on $X$ and $Ric(\omega(t, \cdot))$ denotes the Ricci curvature of $\omega(t, \cdot)$ and $\omega_0$ is a given Kähler metrics. If the canonical class $K_X$ of $X$ is ample and $\omega_0$ represents $K_X$, Cao proved in [Ca] that (1.1) has a global solution $\omega(t, \cdot)$ for all $t > 0$ and $\omega(t, \cdot)$ converges to a Kähler-Einstein metric on $X$. If $K_X$ is numerically effective (i.e. nef), Tsuji proved in [Ts] under additional assumption $[\omega_0] > K_X$ that (1.1) has a global solution $\omega(t, \cdot)$. This additional assumption was removed in [TiZh], moreover, if $K_X$ is also big, $\omega(t, \cdot)$ converges to a singular Kähler-Einstein metric with locally bounded Kähler potentials as $t$ tends to $\infty$ (see [TiZh]). Our problem is to show how $\omega(t, \cdot)$ behaves at time infinity.

If $X$ is a minimal Kähler surface of non-negative Kodaira dimension, then $K_X$ is numerically effective. The Kodaira dimension $\nu(X)$ of $X$ is equal to 0, 1, 2. If $\nu(X) = 0$, then a finite cover of $X$ is either a K3 surface or a complex torus, then after an appropriate scaling, $\omega(t, \cdot)$ converges to the unique Calabi-Yau metric in the Kähler class $[\omega_0]$ (cf. [Ca]). If $\nu(X) = 2$, i.e., $X$ is of general type, then $\omega(t, \cdot)$ converges to the unique Kähler-Einstein orbifold metric on its canonical model as $t$ tends to $\infty$ (see [TiZh]). If $\nu(X) = 1$, then $X$ is a minimal elliptic surface and does not admit any Kähler-Einstein current which is smooth outside a subvariety. Hence, one does not expect that $\omega(t, \cdot)$ converges to a smooth metric outside a subvariety of $X$.

In this paper, we study the limiting behavior of $\omega(t, \cdot)$ as $t$ tends to $\infty$ in the case that $X$ is a minimal elliptic surface. In its sequel, we will extend our results here to higher dimensional manifolds, that is, we will study the limiting behavior of (1.1) when $X$ is a $n$-dimensional variety of Kodaira dimension in $(0, n)$ and with numerically positive $K_X$. Hence, our first question is to identify limiting candidates. Since $X$ is a minimal elliptic surface, there is a holomorphic map $f : X \rightarrow \Sigma$ such that $K_X = \pi^*L$ for some ample line bundle $L$ over the curve $\Sigma$. Let $\Sigma_{reg}$ consist of all $s \in \Sigma$ such that $f^{-1}(s)$ is smooth and let $X_{reg} = f^{-1}(\Sigma_{reg})$. For any $s \in \Sigma_{reg}$, $f^{-1}(s)$ is
an elliptic curve. Thus the $L^2$-metric on the moduli of elliptic curves induces a metric $\omega_{WP}$ on $\Sigma_{reg}$. We call a metric $\omega$ canonical if it is smooth on $\Sigma_{reg}$ and extends appropriately to $\Sigma$ and satisfies

$$\text{Ric}(\omega) = -\omega + \omega_{WP}, \quad \text{on } \Sigma_{reg}.$$  

Such a metric exists and is unique in a suitable sense.  

**Theorem 1.1** Let $f : X \to \Sigma$ be a minimal elliptic surface of $\nu(X) = 1$ with singular fibres $X_{s_1} = m_1F_1$, $\ldots$, $X_{s_k} = m_kF_k$ with multiplicity $m_i \in \mathbb{N}$, $i = 1, \ldots, k$. Then for any initial Kähler metric, the Kähler-Ricci flow (1.1) has a global solution $\omega(t, \cdot)$ for all time $t \in [0, \infty)$ satisfying:

1. $\omega(t, \cdot)$ converges to $f^*\omega_\infty \in -2\pi c_1(X)$ as currents for a positive current $\omega_\infty$ on $\Sigma$;
2. $\omega_\infty$ is smooth on $\Sigma_{reg}$ and satisfies as currents on $\Sigma$

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP} + \sum_{i=1}^{k} \frac{m_k - 1}{m_k} [s_i],$$  

where $\omega_{WP}$ is the induced Weil-Petersson metric and $[s_i]$ is the current of integration associated to the divisor $s_i$ on $\Sigma$, in particular, $\omega_\infty$ is a generalized Kähler-Einstein metric on $\Sigma_{reg}$;

3. for any compact subset $K \in X_{reg}$, there is a constant $C_K$ such that

$$||\omega(t, \cdot) - f^*\omega_\infty||_{L^\infty(K)} + e^t \sup_{s \in K} ||\omega(t, \cdot)|_{f^{-1}(s)}||_{L^\infty} \leq C_K,$$  

moreover, the scalar curvature of $\omega(t, \cdot)$ is uniformly bounded on any compact set of $X_{reg}$.

**Remark 1.1** We conjecture that $\omega(t, \cdot)$ converges to $f^*\omega_\infty$ in the Gromov-Hausdorff topology and in the $C^\infty$-topology outside singular fibers.

An elliptic surface $f : X \to \Sigma$ is an elliptic fibre bundle if it does not admit any singular fibre.

**Corollary 1.1** Let $f : X \to \Sigma$ be an elliptic fibre bundle over a curve $\Sigma$ of genus greater one. Then the Kähler-Ricci flow (1.1) has a global solution with any initial Kähler metric. Furthermore, $\omega(t, \cdot)$ converges with uniformly bounded scalar curvature to the pullback of the Kähler-Einstein metric on $\Sigma$.

This theorem seems to be the first general convergence result on collapsing of the Kähler-Ricci flow. Combining the results in [1][2][3], we give a metric classification for Kähler surfaces with an nef canonical line bundle by the Kähler-Ricci flow.

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1Such canonical metrics can be also defined for higher dimensional manifolds. We refer the readers to section 5 for more details.
2 Preliminaries

2.1 Reduction of the Kähler-Ricci flow

In this section, we will reduce the Kähler-Ricci flow (1.1) to a parabolic equation on Kähler potentials for any compact Kähler manifold $X$ with its canonical line bundle $K_X \geq 0$. Let $X$ be an $n$-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form $\omega$ on $X$. In local coordinates $z_1, ..., z_n$, $\omega$ can be written in the form of

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dz_i \wedge d\bar{z}_j,$$

where $\{g_{ij}\}$ is a positive definite hermitian matrix function. The curvature tensor for $g$ is locally given by

$$R_{ijkl} = -\frac{\partial^2 g_{ij}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{pq} \frac{\partial g_{ip}}{\partial z_k} \frac{\partial g_{jq}}{\partial \bar{z}_l}, \quad i,j,k=1,2,\ldots,n.$$

And the Ricci curvature is given by

$$R_{ij} = -\frac{\partial^2 \log \det(g_{kl})}{\partial z_i \partial \bar{z}_j}, \quad i,j=1,2,\ldots,n.$$

So its Ricci curvature form is

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{ij} dz_i \wedge d\bar{z}_j = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{kl}).$$

Let $Ka(X)$ denote the Kähler cone of $X$, that is,

$$Ka(X) = \{ [\omega] \in H^{1,1}(X, \mathbb{R}) \mid [\omega] > 0 \}.$$

Suppose that $\omega(t, \cdot)$ is a solution of (1.1) on $[0, T)$. Then its induced equation on Kähler classes in $Ka(X)$ is given by the following ordinary differential equation

$$\left\{ \begin{array}{l}
\frac{\partial [\omega]}{\partial t} = -2\pi c_1(X) - [\omega] \\
[\omega]_{t=0} = [\omega_0].
\end{array} \right. \quad (2.1)$$

It follows

$$[\omega(t, \cdot)] = -2\pi c_1(X) + e^{-t}([\omega_0] + 2\pi c_1(X)).$$

Now if we assume that $X$ has semi-positive canonical bundle $K_X$, then there is a large integer $\mu$ such that any basis of $H^0(X, \mu K_X)$ gives rise to an embedding $f$ into a projective space. Recall the Kodaira dimension $\nu(X)$ of $X$ is defined to be the dimension of the image of this embedding. This dimension is in fact independent of choices of the basis and embedding. Moreover, using this embedding, one can see easily that there is a positive $(1,1)$-form $\chi$ such that $f^* \chi$ represents $-2\pi c_1(X)$. Choose the reference Kähler metric $\omega_t$ by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi). \quad (2.2)$$

Here we abuse the notation by identifying $\chi$ and $f^* \chi$ for simplicity. Then the solution of (1.1) can be written as

$$\omega = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi.$$
We can always choose a smooth volume form $\Omega$ on $X$ such that $\text{Ric}(\Omega) = \chi$. Then the evolution for the Kähler potential $\varphi$ is given by the following initial value problem:

$$\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\nu(X))t}(\omega_0 + \sqrt{-1}\partial \bar{\partial} \varphi)^2}{\Omega} - \varphi$$

$$\varphi|_{t=0} = 0.$$  \hfill (2.3)

The following was proved in [TiZha]. When $\omega_0 > \chi$ and $K_X \geq 0$, it was proved by Tsuji in [TS]. It was also proved in [CaLa] under a stronger assumption.

**Theorem 2.1** The Kähler-Ricci flow (2.3) has a global solution for all time $t \in [0, \infty)$ if $K_X$ is nef.

The evolution equation for the scalar curvature $R$ is given by

$$\frac{\partial R}{\partial t} = \Delta R + |\text{Ric}|^2 + R.$$  \hfill (2.4)

Then the following proposition is an immediate conclusion from the maximum principle for the parabolic equation (2.4).

**Proposition 2.1** The scalar curvature along the Kähler-Ricci flow (1.1) is uniformly bounded from below.

**2.2 Minimal surfaces with positive Kodaira dimension**

An elliptic fibration of a surface $X$ is a proper, connected holomorphic map $f : X \to \Sigma$ from $X$ to a curve $\Sigma$ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface $X$ of $\nu(X) = 1$ must be an elliptic surface. Such an elliptic surfaces is sometimes called a properly elliptic surface. Since we assume that $X$ is minimal, all fibres are free of $(-1)$-curves. A very simple example is the product of two curves, one elliptic and the other of genus $\geq 2$.

Let $f : X \to \Sigma$ be an elliptic surface. The differential $df$ can be viewed as an injection of sheaves $f^*(K_\Sigma) \to \Omega^1_X$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then $df$ vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of $f$ we have the following exact sequence

$$0 \to f^*(K_\Sigma) \to \Omega^1_X \to \Omega_{X/\Sigma} \to 0$$

including an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of $f$ on $X$. The following theorem is well-known (cf. [BaHuPeVa]).

**Theorem 2.2** Let $f : X \to \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1 F_1$, ..., $X_{s_k} = m_k F_k$. Then

$$K_X = f^*(K_\Sigma \otimes (f_{s_1} \mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(\sum (m_i - 1)F_i),$$  \hfill (2.5)

or

$$K_X = f^*(L \otimes \mathcal{O}_X(\sum (m_i - 1)F_i),$$  \hfill (2.6)

where $L$ is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on $\Sigma$.  


Note that \( \deg(f_*\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0 \) and the equality holds if and only if \( f \) is locally trivial. The following invariant
\[
\delta(f) = \chi(\mathcal{O}_X) + \left( 2g(\Sigma) - 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{m_i} \right) \right)
\]
determines the Kodaira dimension of \( X \).

**Proposition 2.2** (cf. [BaHuPeVa]) Let \( f : X \to \Sigma \) be a relatively minimal elliptic fibration and \( X \) be compact. Then \( \nu(X) = 1 \) if and only if \( \delta(f) > 0 \).

Kodaira classified all possible singular fibres for \( f \). A fibre \( X_s \) is stable if
\begin{enumerate}
\item \( X_s \) is reduced,
\item \( X_s \) contains no \((-1)\)-curves,
\item \( X_s \) has only node singularities.
\end{enumerate}

The only stable singular fibres are of type \( I_b \) for \( b > 0 \), therefore such singular fibres are particularly interesting. Let \( \mathcal{S}_1/\Gamma_1 \cong \mathbb{C} \) be the period domain, where \( \mathcal{S}_1 = \{ z \in \mathbb{C} \mid \text{Im}z > 0 \} \) is the upper half plane and \( \Gamma_1 = \text{SL}(2, \mathbb{Z})/\{\pm 1\} \) is the modular group acting by \( z \to \frac{az+b}{cz+d} \). The \( j \)-function gives an isomorphism \( \mathcal{S}_1/\Gamma_1 \to \mathbb{C} \) with
\begin{enumerate}
\item \( j(z) = 0 \) if \( z = e^{\frac{\pi}{\sqrt{-1}}t} \) modular \( \Gamma_1 \),
\item \( j(z) = 1 \) if \( z = \sqrt{-1} \) modular \( \Gamma_1 \).
\end{enumerate}

Thus any elliptic surface \( f : X \to \Sigma \) gives a period map \( p : \Sigma_{\text{reg}} \to \mathcal{S}_1/\Gamma_1 \). Set \( J : \Sigma_{\text{reg}} \to \mathbb{C} \) by \( J(s) = j(p(s)) \). For a stable fibre \( X_s \) of type \( I_b \), the functional invariant \( J \) has a pole of order \( b \) at \( s \) and the monodromy is given by \( \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \). Now choose a semi-positive \((1,1)\)-form \( \chi \in -2\pi c_1(X) \) to be the pullback of a Kähler form \( \chi \) on the base \( \Sigma \) and \( \chi = f^*\chi \) might vanish somewhere due to the presence of singular fibres. Then as we discussed in the previous subsection, one can reduce the Kähler-Ricci flow \( (1.1) \) to an evolution equation on Kähler potentials. Hence, the Kähler-Ricci flow \( (1.1) \) has a global solution and provides a canonical way of deforming any given Kähler metric to a canonical metric on the canonical model of minimal elliptic surfaces of positive Kodaira dimension. As described in Theorem \( (1.1) \) this canonical metric on \( \Sigma \) satisfies the curvature equation
\[
\text{Ric}(g_\infty) = -g_\infty + g_{WP} + \sum_{i=1}^{k} \left( \frac{m_i-1}{m_i} \right) [s_i].
\]

It corresponds to \( (2.6) \), where the pullback of the Weil-Petersson metric \( g_{WP} \) by the period map \( p \) is the curvature of the dualizing sheaf \( f_*\Omega_{X/\Sigma} \) and the current \( \sum_{i=1}^{k} \left( \frac{m_i-1}{m_i} \right) [s_i] \) corresponds to the residues from the multiple fibres.
3 A parabolic Schwarz lemma

In this section we will establish a parabolic Schwarz lemma for compact Kähler manifolds. This is inspired by \[Ya1\]. This will lead us to identify and estimate the collapsing on the vertical direction for properly minimal elliptic surfaces and in general certain fibre spaces. It also plays a key role in estimating the scalar curvature along the Kähler-Ricci flow. Let \( f : X \to Y \) be a non-constant holomorphic mapping between two Kähler manifolds. Suppose \( \dim X = n \) and the Kähler metric \( \omega(t, \cdot) \) on \( X \) is deformed by the following Kähler-Ricci flow (1.1). Then we have the following parabolic Schwarz lemma for metrics.

**Theorem 3.1** If the holomorphic bisectional curvature of \( Y \) with respect to a fixed Kähler metric \( h_{\alpha\overline{\beta}} \) is bounded from above by a negative constant \( -K \) and the Kähler-Ricci flow (1.1) exists for all \( t \in [0, T) \), then

\[
f^*h \leq \frac{C(t)}{K} \omega(t, \cdot),
\]

where \( C(t) \) is a positive function in \( t \) dependent on the initial metric \( \omega_0 \) and \( \lim_{t \to \infty} C(t) = 1 \) if \( T = \infty \).

**Proof** Choose normal coordinate systems for \( g = \omega(t, \cdot) \) on \( X \) and \( h \) on \( Y \) respectively. Let \( u = \text{tr}_g(h) = g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^\alpha h_{\alpha\overline{\beta}} \) and we will calculate the evolution of \( u \).

\[
\Delta u = g^{i\overline{j}} \partial_i \partial_{\overline{j}} (g^{k\overline{l}} f_i^\alpha f_{\overline{l}}^\beta h_{\alpha\overline{\beta}}) = g^{i\overline{j}} g^{k\overline{l}} R_{k\overline{l}i} f_i^\alpha f_{\overline{j}}^\beta h_{\alpha\overline{\beta}} + g^{i\overline{j}} g^{k\overline{l}} f_i f_{\overline{k}} f_{\overline{l}} h_{\alpha\overline{\beta}} - g^{i\overline{j}} g^{k\overline{l}} S_{\alpha\overline{\beta}k\overline{l}i} f_i^\alpha f_{\overline{k}}^\beta f_{\overline{l}}^\gamma,
\]

where \( S_{\alpha\overline{\beta}k\overline{l}i} \) is the curvature tensor of \( h_{\alpha\overline{\beta}} \) and the Laplacian \( \Delta \) acts on functions \( \phi \) by

\[
\Delta \phi = g^{i\overline{j}} \partial_i \partial_{\overline{j}} \phi.
\]

By the definition of \( u \) we have

\[
\Delta u \geq g^{i\overline{j}} g^{k\overline{l}} R_{k\overline{l}i} f_i^\alpha f_{\overline{j}}^\beta h_{\alpha\overline{\beta}} + Ku^2.
\]

Now

\[
\frac{\partial u}{\partial t} = -g^{i\overline{j}} g^{k\overline{l}} \partial_k \partial_{\overline{l}} f_i^\alpha f_{\overline{j}}^\beta h_{\alpha\overline{\beta}} = g^{i\overline{j}} g^{k\overline{l}} (R_{k\overline{l}i} + g_{k\overline{l}}) f_i^\alpha f_{\overline{j}}^\beta h_{\alpha\overline{\beta}} = g^{i\overline{j}} g^{k\overline{l}} R_{k\overline{l}i} f_i^\alpha f_{\overline{j}}^\beta h_{\alpha\overline{\beta}} + u,
\]

therefore

\[
\frac{d}{dt} - \Delta)u \leq u - Ku^2.
\]

Applying the maximal principle, we have

\[
\frac{d}{dt} u_{\text{max}} \leq u_{\text{max}} - Ku_{\text{max}}^2.
\]

Thus

\[
u_{\text{max}}(t) \leq \frac{1}{K - Ce^{-t}}
\]

and it proves the theorem. \( \square \)
By similar argument as in the proof of Theorem 3.1, one can also derive the following Schwarz lemma for volume forms with weaker curvature bounds on the target manifold.

**Theorem 3.2** Suppose $\dim X = n \geq \dim Y = \kappa$. Let $\chi$ be the Kähler form on $Y$ with respect to the Kähler metric $h_{\alpha\overline{\beta}}$. If $\text{Ric}(h) \leq -K h$ for some $K > 0$ and the Kähler-Ricci flow \((1.1)\) exists for all $t \in [0, T)$, then there exists a constant $C > 0$ dependent on the initial metric $\omega_0$ such that

$$\frac{\omega^{n-k} \wedge f^* \chi^k}{\omega^n} \leq C.$$  \hspace{1cm} (3.3)

Suppose $2\pi c_1(X) = -[f^* \chi]$ for a Kähler form $\chi$ on $Y$, i.e. $K_X$ is a semi-positive line bundle pulled back from an ample line bundle from $Y$. Then we can remove the curvature assumption on $Y$. From now on, for convenience, we will write $f^* \chi$ as $\chi$. Since $c_1(X) \leq 0$, by Theorem 2.11 the Kähler-Ricci flow has long time existence.

**Theorem 3.3** Let $f : X \to Y$ be a holomorphic fibration such that $2\pi c_1(X) = -[f^* \chi]$ for some Kähler form $\chi$ on $Y$. Then the Kähler-Ricci flow \((1.1)\) exists for all $t \in [0, \infty)$ and there exist constants $A, C > 0$ such that for all $(t, z)$,

$$f^* \chi(t, z) \leq C \omega(t, z) e^{A\phi(t, z)} \max_{[0,t] \times X} \{\log \frac{\Omega}{e^{(n-k)\chi(s, \cdot)} e^{-A\phi}}\},$$  \hspace{1cm} (3.4)

where $\Omega$ is the volume form on $X$ such that $\text{Ric}(\Omega) = f^* \chi$.

**Proof** Let $u = g^{k\overline{l}} f_i^\alpha f_j^\beta \chi_{\alpha\beta}$ and choose normal coordinates for $g$ and $\chi$. We will calculate the evolution for $\log u$. Note that $\Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u}$ and

$$\Delta u = g^{k\overline{l}} g^{j\overline{k}} R_{kj} f_i^\alpha f_j^\beta + g^{k\overline{l}} g^{j\overline{k}} f_{i,k}^\alpha f_j^\beta \chi_{\alpha\beta} - g^{j\overline{k}} g^{i\alpha} S_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_l^\gamma f_k^\delta.$$  \hspace{1cm} (3.5)

Applying the Cauchy-Schwartz inequality, we have

$$|\nabla u|^2_g = \sum_{i,j,k,\alpha,\beta} f_i^\alpha f_j^\beta f_{i,k}^\alpha f_{j,k}^\beta \leq \sum_{i,j,\alpha,\beta} \left| f_i^\alpha f_j^\beta \right|^2 \left( \sum_k \left| f_{i,k}^\alpha \right|^2 \right)^{1/2} \left( \sum_l \left| f_{j,k}^\beta \right|^2 \right)^{1/2}$$

$$= \left( \sum_{i,\alpha} \left| f_i^\alpha \right|^2 \right) \left( \sum_{j,\beta} \left| f_j^\beta \right|^2 \right)^{1/2}$$

$$\leq \left( \sum_{j,\beta} \left| f_j^\beta \right|^2 \right) \left( \sum_{i,k,\alpha} \left| f_{i,k}^\alpha \right|^2 \right).$$

Let $C$ be a constant satisfying $S_{\alpha\beta\gamma\delta} \leq C \chi_{\alpha\beta} \chi_{\gamma\delta}$. Then we have

$$\frac{\frac{d}{dt} - \Delta} u \log u = \frac{1}{u} \left( -g^{k\overline{l}} g^{j\overline{k}} f_{i,k}^\alpha f_{j,l}^\beta \chi_{\alpha\beta} + g^{k\overline{l}} g^{j\overline{k}} S_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_l^\gamma f_k^\delta + \frac{|\nabla u|^2}{u} \right) + 1$$

$$\leq \frac{-1}{u} g^{j\overline{k}} g^{i\alpha} S_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_l^\gamma f_k^\delta + 1$$

$$\leq -Cu + 1.$$
On the other hand,

\[
(\frac{d}{dt} - \Delta)\varphi = -\text{tr}(\sqrt{-1}\partial\bar{\partial}\varphi) + \frac{\partial\varphi}{\partial t} = -\text{tr}(\omega - \omega_t) + \frac{\partial\varphi}{\partial t} = \text{tr}(\omega_t) + \frac{\partial\varphi}{\partial t} - n.
\]

Notice that \(\varphi\) is uniformly bounded from above from the equation (2.3) by the maximum principle. Combining the above estimates we have

\[
(\frac{\partial}{\partial t} - \Delta)(\log u - 2A\varphi) \\
\leq -2(A - C)u - 2A\log \frac{\Omega}{e^{(n-\kappa)n_0\omega^n}} + 2A\varphi + 2nA + 1 \\
\leq -Au + 2A\log \frac{\Omega}{e^{(n-\kappa)n_0\omega^n}} + 2A\varphi + 2nA + 1 \\
\leq -Au + 2A\log \frac{\Omega}{e^{(n-\kappa)n_0\omega^n}} + CA
\]

if we choose \(A\) sufficiently large.

Suppose on each time interval \([0, t]\), the maximum of \(\log u - A\varphi\) is achieved at \((t_0, z_0)\), by the maximum principle we have

\[
u(t_0, z_0) \leq 2\log \frac{\Omega}{e^{(n-\kappa)n_0\omega^n}}(t_0, z_0)
\]

and

\[
u(t, z) \leq u(t_0, z_0)e^{2A\varphi(z,t) - 2A\varphi(t_0, z_0)}.
\]

This completes the proof. \(\Box\)

4 Estimates

In this section, we prove the uniform zeroth order and second order estimate of the potential \(\varphi\) along the Kähler-Ricci flow. A gradient estimate is also derived and it gives a uniform bound of the scalar curvature. We assume that \(f : X \to \Sigma\) is a properly minimal surface over a curve \(\Sigma\) with singular fibres over \(\Delta = \{s_1, \ldots, s_k\} \subset \Sigma\). Let \(X_{s_i} = f^{-1}(s_i)\) be the corresponding singular fibre for \(i = 1, \ldots, k\) and \([S] = \sum_{i=1}^{k} [X_{s_i}] = f^*(\sum_{i=1}^{k} [s_i])\) be the divisor containing all the singular fibres. We can always find a hermitian metric \(h\) on the line bundle induced by \([S]\) such that \(\text{Ric}(h)\) is a multiple of \(\chi\).

4.1 The zeroth order and volume estimates

We will derive the zeroth order estimates for \(\varphi\) and \(\frac{d\varphi}{dt}\).

Lemma 4.1 Let \(\varphi\) be a solution of the Kähler-Ricci flow (2.3). There exists a positive constant \(C\) depending only on the initial data such that \(\varphi \leq C\).
Proof This is a straightforward application of the maximum principle. Let \( \varphi_{\max}(t) = \max_X \varphi(t, \cdot) \).

Applying the maximum principle, we have

\[
\frac{\partial \varphi_{\max}}{\partial t} \leq \log \frac{e^t \omega^2}{\Omega} - \varphi_{\max} \\
\leq \log 2 \wedge (\omega_0 - \chi) + e^{-t}(\omega_0 - \chi)^2 - \varphi_{\max} \\
\leq C - \varphi_{\max}.
\]

This gives a uniform upper bound for \( \varphi \).

\[\Box\]

Lemma 4.2 There exists a positive constant \( C \) depending only on the initial data such that

\[
\frac{\partial \varphi}{\partial t} \leq C. \tag{4.1}
\]

Proof Differentiating on both sides of (2.3) we obtain

\[
\frac{\partial}{\partial t} \left( e^t \frac{\partial \varphi}{\partial t} \right) = \Delta \frac{\partial \varphi}{\partial t} + 1 - e^{-t}tr_\omega(\omega_0 - \chi) - \frac{\partial \varphi}{\partial t}, \tag{4.2}
\]

where \( \Delta \) is the Laplacian operator of the metric \( g \). It can be rewritten as

\[
\frac{\partial}{\partial t} (e^t \frac{\partial \varphi}{\partial t}) = \Delta (e^t \frac{\partial \varphi}{\partial t}) + e^t - tr_\omega(\omega_0 - \chi),
\]

and

\[
\frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} + \varphi \right) = \Delta \left( \frac{\partial \varphi}{\partial t} + \varphi \right) + tr_\omega(\chi) - 1. \tag{4.3}
\]

So

\[
\frac{\partial}{\partial t} (e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t) = \Delta (e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t) - tr_\omega(\omega_0).
\]

Applying the maximum principle, we have

\[
e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t \leq C'
\]

for some uniform constant \( C' \) only depending on the initial data. Hence

\[
\frac{\partial \varphi}{\partial t} \leq \frac{e^{-t} \varphi}{1 - e^{-t}} + C' \leq C.
\]

\[\Box\]

Lemma 4.3 There exists a positive constant \( C \) depending only on the initial data such that

\[
|\varphi| \leq C. \tag{4.4}
\]

Proof It suffices to derive the lower bound for \( \varphi \). Consider \( u(t, z) = \max_X \varphi(t, \cdot) - \varphi(t, z) \geq 0 \).

Fix \( \delta > 0 \). For any \( p > 1 \), since both \( \varphi \) and \( \frac{\partial \varphi}{\partial t} \) are bounded from above, using equation (2.3), we have

\[
\int_X e^{p\delta u}(\omega^2 - \omega_t^2) \leq \int_X e^{p\delta u} \omega^2 \leq Ce^{-t} \int_X e^{p\delta u} \omega_0^2. \tag{4.5}
\]
Calculate
\[
\int_X e^{\delta u}(\omega^2 - \omega_t^2)
= \sqrt{-1} \int_X e^{\delta u} \partial \bar{\partial}(-u) \wedge (\omega + \omega_t)
= \frac{2\sqrt{-1}}{p\delta} \int_X \partial e^{\frac{\delta u}{2}} \wedge \bar{\partial} e^{\frac{\delta u}{2}} \wedge (\omega + \omega_t)
\geq \frac{2\sqrt{-1}}{p\delta} \int_X \partial e^{\frac{\delta u}{2}} \wedge \bar{\partial} e^{\frac{\delta u}{2}} \wedge \omega_t
\geq \frac{\sqrt{-1}C}{p\delta} e^{-t} \int_X \partial e^{\frac{\delta u}{2}} \wedge \bar{\partial} e^{\frac{\delta u}{2}} \wedge \omega_0.
\] (4.6)

Combining (4.5) and (4.6) we obtain
\[
\int_X |\nabla e^{\frac{\delta u}{2}}|^2 \omega_0^2 \leq Cp \int_X e^{\delta u} \omega_0^2.
\]
The Sobolev inequality \(|f|_{L^4}^4 \leq C||f||_{H^1}^2\) implies that for all \(p > 1\)
\[
||e^{\delta u}||_{L^{2p}} \leq C \delta p ||e^{\delta u}||_{L^p}.
\]
Now we can apply Moser’s iteration by successively replacing \(p\) by \(2^k\) and let \(k \to \infty\). Then the standard argument shows that
\[
||e^{\delta u}||_{L^\infty} \leq C||e^{\delta u}||_{L^1}.
\]
Then we only need to bound the quantity \(||e^{\delta u}||_{L^1}\). Note that \(A\omega_0 - \sqrt{-1}\partial \bar{\partial}u \geq \chi + e^{-t}(\omega_0 - \chi) + \sqrt{-1}\partial \bar{\partial} \varphi > 0\) if we choose \(A > 0\) sufficiently large. The lemma is proved if we apply the following proposition. It is proved by the second named named author in [TI1] based on a result in [HI].

**Proposition 4.1** There exists \(\delta > 0\) and \(C\) depending only on \((X, \omega_0)\) such that
\[
\int_X e^{-\delta \varphi} \omega_0^2 \leq C,
\] (4.7)
for all \(\varphi \in C^2(X)\) satisfying \(\omega_0 + \sqrt{-1}\partial \bar{\partial} \varphi > 0\) and \(\sup_X \varphi = 0\).

This completes the proof.

Since \(e^t \omega^2 = e^{\frac{\delta u}{2}} + \varphi \Omega\) and \(||\varphi||_{C^0}\) is uniformly bounded, from the uniform upper bound for \(\frac{\partial \varphi}{\partial t}\) we conclude that the normalized volume form \(e^t \omega^2\) is uniformly bounded above and a lower bound for it will also give a lower bound for \(\frac{\partial \varphi}{\partial t}\).

**Lemma 4.4** There exist constants \(\lambda_1 > 0\) and \(C > 0\) such that for all \((t, z) \in [0, \infty) \times X\) we have the following volume estimate
\[
\frac{1}{C} |S|^2 \omega_h^2 \leq e^t \omega^2 \Omega \leq C.
\]
Here \(h\) is a fixed hermitian metric equipped on the line bundle induced by the divisor \([S]\) such that \(Ric(h) > 0\) is a multiple of \(\chi\).
\textbf{Proof} It suffices to prove the lower bound of the volume form $e^t \omega^2$. Notice that $\log \frac{e^t \omega^2}{\Omega} = \frac{\partial \varphi}{\partial t} + \varphi$ and hence the evolutions for $\log \frac{e^t \omega^2}{\Omega}$ and $\varphi$ are prescribed by

$$
(\frac{\partial}{\partial t} - \Delta) \log \frac{e^t \omega^2}{\Omega} = tr_\omega(\chi) - 1 \quad \text{and} \quad (\frac{\partial}{\partial t} - \Delta) \varphi = tr_\omega(\omega_t) + \log \frac{e^t \omega^2}{\Omega} - \varphi. 
$$

Combining the above equations, at any point $(t, z)$ we have for $\lambda > 0$

$$
(\frac{\partial}{\partial t} - \Delta)(\log \frac{e^t \omega^2}{\Omega} + 2A\varphi - \lambda \log |S|^2_h) \geq 2Atr_\omega(\omega_t) + 2A\log \frac{e^t \omega^2}{\Omega} - 2A\varphi - (4A + 1) 
$$

if we choose $A$ sufficiently large. Suppose on each time interval $[0, T]$, the minimum of $\log \frac{e^t \omega^2}{\Omega} + 2A\varphi - \lambda \log |S|^2_h$ is achieved at $(t_0, z_0)$, then by the maximum principle at $(t_0, z_0)$ we have

$$
tr_\omega(\omega_t)(t_0, z_0) \leq 2 \log \frac{\Omega}{e^t \omega^2}(t_0, z_0) + C. 
$$

But for some $\lambda > 0$ we have at $(t_0, z_0)$

$$
C + 2 \log \frac{\Omega}{e^t \omega^2} \geq tr_\omega(\omega_t) \geq \frac{\omega_t^2}{\omega^2} \geq \left( \frac{\Omega}{e^t \omega^2} \right)^{\frac{\lambda}{2}} \left( \frac{\partial \Omega}{\partial t} \right)^{\frac{\lambda}{2}} \geq C(|S|^{2\lambda}_h \frac{\Omega}{e^t \omega^2})^{\frac{\lambda}{2}}.
$$

For each $\delta > 0$, there is the following elementary inequality

$$
\log x \leq x^\delta + C_\delta \quad \text{for all} \quad x > 0.
$$

It follows that at $(t_0, z_0)$, we have for some small $\delta < \frac{1}{2}$

$$
(|S|^{2\lambda}_h \frac{\Omega}{e^t \omega^2})^{\frac{\lambda}{2}} \leq C((\frac{\Omega}{e^t \omega^2})^{\delta} + 1)
$$

and by multiplying $|S|^{2\lambda_1}_h$,

$$
(|S|^{2\lambda_1}_h \frac{\Omega}{e^t \omega^2})^{\frac{\lambda}{2}} \leq C((|S|^{2\lambda_1}_h \frac{\Omega}{e^t \omega^2})^{\delta} + 1).
$$

We have $2\lambda + 4\delta \lambda_1 = 2\lambda$ if $\lambda_1$ is chosen by $\lambda_1 = \frac{\lambda}{1 - 2\delta}$. Therefore $|S|^{2\lambda_1}_h \frac{\Omega}{e^t \omega^2}(t_0, z_0) \leq C$ and

$$
\frac{e^{t \omega^2}}{|S|^{2\lambda_1}_h \Omega} e^\varphi(t, z) \geq \frac{e^{t_0 \omega^2}}{|S|^{2\lambda_1}_h \Omega} e^\varphi(t_0, z_0).
$$

Both $\varphi$ and $\frac{e^{t_0 \omega^2}}{|S|^{2\lambda_1}_h \Omega}(t_0, z_0)$ are uniformly bounded from below, hence the lemma is proved. \qed
This also shows that there is a uniform lower bound for \( \frac{\partial \varphi}{\partial t} \) with at worse log poles near the singular fibres.

**Lemma 4.5** There exists a constant \( C > 0 \) such that

\[
\frac{\partial \varphi}{\partial t} \geq C(\lambda_1 \log |S|^2_h - 1). \tag{4.11}
\]

**Proof** We only have to show \( \frac{\partial \varphi}{\partial t} \) is uniformly bounded from below. This is obtained by the previous lemma and

\[
\frac{\partial \varphi}{\partial t} = \log \frac{e^t \omega^2}{\Omega} - \varphi.
\]

\[\square\]

### 4.2 A partial second order estimate

In this section, we slightly modify the proof of the parabolic Schwarz lemma to derive a partial second order estimate. This will imply that along the Kähler-Ricci flow \((1.1)\) the metric collapses along the fiber direction exponentially fast outside the singular fibres.

**Lemma 4.6** (The partial second order estimate) For any \( \delta > 0 \) there exists a constant \( C > 0 \) depending on \( \delta \) such that

\[
\text{tr} \omega(\chi) \leq C\frac{|S|^2_h}{\delta} \tag{4.12}
\]

**Proof** By Lemma 4.5 for any \( \delta > 0 \)

\[
|S|^{2\delta} \frac{\partial \varphi}{\partial t} \leq C.
\]

Let \( u = g^7 f^a \int f^2 \chi_{\alpha \beta} \). Following the similar calculation in Section 3 we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\log |S|^{2\delta} u - 3A\varphi) \leq -2Au - 3A\frac{\partial \varphi}{\partial t} + \delta \text{tr}_\omega(Ric(h)) + CA \leq -Au - 3A\frac{\partial \varphi}{\partial t} + CA
\]

for sufficiently large \( A \). Suppose on each time interval \([0, T]\), the maximum of \( \log |S|^{2\delta} u - A\varphi \) is achieved at \((t_0, z_0)\), by the maximum principle we have

\[
(|S|^{2\delta} u)(t_0, z_0) \leq -3(|S|^{2\delta} \frac{\partial \varphi}{\partial t})(t_0, z_0) + C \leq C.
\]

Combining with the uniform bound of \( |\varphi| \), we can conclude that \( |S|^{2\delta} u \) is uniformly bounded and the theorem is proved. \[\square\]

**Corollary 4.1** Let \( X_s \) be a non-singular fibre for any \( s \in \Sigma_{\text{reg}} \). Then along the Kähler-Ricci flow \((1.1)\), \( \omega \) decays exponentially fast on \( X_s \). Furthermore if \( \Delta_s \) is the Laplacian on \( X_s \) with respect to \( \omega_0|_{X_s} \), then there exist constants \( \lambda_2 > 0 \) and \( C > 0 \) such that

\[
-e^{-t} \leq \Delta_s \varphi \leq \frac{Ce^{-t}}{|S|^{2\lambda_2}(s)}. \tag{4.13}
\]
Proof Applying the partial second order estimate, we have

$$0 < e^{-t} + \Delta_s \varphi = \frac{\omega|_{X_s}}{\omega_0|_{X_s}} = \frac{\omega \wedge \chi}{\omega_0 \wedge \chi} = \frac{\omega \wedge \chi - \omega^2}{\omega_0 \wedge \chi} \leq tr_\omega(\chi) \frac{\omega^2}{\omega_0 \wedge \chi} \leq Ce^{-t}$$

for some uniform constants $C$ and $\lambda_2$. This proves the corollary.

The partial second-order estimate enables us to derive the following strong partial $C^0$-estimate.

**Corollary 4.2** There exist constants $\lambda_3 > 0$ and $C > 0$ such that for all $s \in \Sigma_{reg}$

$$|\sup_{X_s} \varphi - \inf_{X_s} \varphi| \leq \frac{Ce^{-t}}{|S|^{2\lambda_3}(s)}.$$

**Proof** Let $\theta(s)$ be the smooth family of standard flat metrics on the elliptic fibres over $\Sigma_{reg}$ such that $\int_{X_s} \theta(s) = \int_{X_s} \omega_0$ for all $s \in \Sigma_{reg}$. Let $\Delta_{\theta(s)}$ be the Laplacian of $\theta(s)$ on each smooth fibre $X_s$. By Green’s formula, we have

$$\varphi - \frac{1}{\int_{X_s} \theta(s)} \int_{X_s} \varphi \theta(s) = \int_{X_s} \Delta_{\theta(s)} \varphi(y)(G_s(x, y) + A_s) \theta(s),$$

where $G_s(\cdot, \cdot)$ is Green’s function with respect to $\theta(s)$ and $A_s = \inf_{X_s \times X_s} G_s(\cdot, \cdot)$. Since $(X_s, \theta(s))$ is a flat torus, one can easily show that Green’s function $G_s(\cdot, \cdot)$ is uniformly bounded below by a multiple of $\text{Diam}^2(X_s, \theta(s))$. However the diameter $\text{diam}(X_s, \theta(s))$ might blow up near the singular fibres and actually there exist $\lambda > 0$ such that

$$\text{diam}(X_s, \theta(s)) \leq \frac{C}{|S(s)|^\lambda}.$$

Therefore $A_s \geq -\frac{C}{|S(s)|^\lambda}$ for some constant $C$ and we have on each smooth fibre $X_s$,

$$|\sup_{X_s} \varphi - \inf_{X_s} \varphi| \leq C \sup_{X_s} |\Delta_{\theta(s)} \varphi||S(s)|^{2\lambda}. $$

But for some $\mu > 0$ and $C > 0$ we have

$$|\Delta_{\theta(s)} \varphi| = |\Delta_s \varphi| \frac{\omega_0|_{X_s}}{\omega_0} = |\Delta_s \varphi| \frac{\omega \wedge \chi}{\omega_0 \wedge \chi} \leq \frac{Ce^{-t}}{|S(s)|^\mu},$$

where the last inequality follows from Corollary 4.1 and Lemma 5.4. This completes the proof of the corollary.

### 4.3 Gradient estimates

In this section we will adapt the arguments in [ChYa] to obtain a uniform bound for $|\nabla \frac{\partial \varphi}{\partial t}|_g$ and the scalar curvature $R$. Let $u = \frac{\partial \varphi}{\partial t} + \varphi = \log \frac{e^{\varphi}}{\mu}$. The evolution equation for $u$ is given by

$$\frac{\partial u}{\partial t} = \Delta u + tr_\omega(\chi). \quad (4.14)$$

We will obtain a gradient estimate for $u$, which will help us bound the scalar curvature from below. Note that $u$ is uniformly bounded from above, so we can find a constant $A > 0$ such that $A - u \geq 1$. 

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Theorem 4.1 There exist positive integers $\lambda_1$, $\lambda_5$ and a uniform constant $C > 0$ such that

(i) $|S_h^{2\lambda_1} |\nabla u|^2 \leq C(A - u)$,

(ii) $-|S_h^{2\lambda_5} \Delta u \leq C(A - u)$,

where $\nabla$ is the gradient operator with respect to the metric $g$.

Proof Standard computation gives the following evolution equations for $|\nabla u|^2$ and $\Delta u$:

$$
(\frac{\partial}{\partial t} - \Delta)|\nabla u|^2 = |\nabla u|^2 + (\nabla tr(\chi) \cdot \nabla u + \nabla tr(\chi) \cdot \nabla u) - |\nabla \nabla u|^2 - |\nabla \nabla u|^2 \tag{4.15}
$$

$$
(\frac{\partial}{\partial t} - \Delta)\Delta u = \Delta u + g^{ij} g^{kl} R_{kli} u_j + \Delta tr(\chi). \tag{4.16}
$$

On the other hand, $\nabla_i \nabla_j u = -R_{ij} - \chi_{ij}$, hence

$$
(\frac{\partial}{\partial t} - \Delta)\Delta u = \Delta u - |\nabla \nabla u|^2 - g^{ij} g^{kl} \chi_{ij} u_k u_l + \Delta tr(\chi).
$$

We shall now prove the first inequality. Let

$$
H = |S_h^{2\lambda_1} \frac{|\nabla u|^2}{A-u} + tr(\chi)).
$$

The evolution equation for $H$ is given by

$$
|S_h^{2\lambda_1} \frac{|\nabla u|^2}{A-u} - |\nabla \nabla u|^2 - |\nabla \nabla u|^2| + (\nabla tr(\chi) \cdot \nabla u + \nabla tr(\chi) \cdot \nabla u) \tag{4.15}
$$

$$
(\frac{\partial}{\partial t} - \Delta)|S_h^{2\lambda_4} tr(\chi)

= |S_h^{2\lambda_4} \frac{\partial}{\partial t} - \Delta)tr(\chi) - \Delta |S_h^{2\lambda_4} tr(\chi) - (\nabla |S_h^{2\lambda_4} tr(\chi) + \nabla |S_h^{2\lambda_4} tr(\chi)

\nabla H = (|S_h^{2\lambda_4} \frac{|\nabla u|^2}{A-u} - |S_h^{2\lambda_4} \frac{|\nabla u|^2}{(A-u)^2} + \nabla |S_h^{2\lambda_4} \frac{|\nabla u|^2}{A-u} + \nabla |S_h^{2\lambda_4} tr(\chi) + |S_h^{2\lambda_4} tr(\chi).

Since $tr(\chi) \leq C$ and $|S_h^{2\lambda_5}$ can be considered as functions pulled back from the base, one can easily show that

$$
|\nabla |S_h^{2\lambda_4}|^2 \leq C|S_h^{4\lambda_4 - 2} tr(\chi)

and

$$
|\Delta |S_h^{2\lambda_4}| \leq C|S_h^{2\lambda_4 - 2} tr(\chi).$$

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Also note that $|S|_{h}^{2\lambda}u$ is a bounded function on $X$ for any $\delta > 0$. Calculate
\[
(\frac{\partial}{\partial t} - \Delta)H
\leq |S|_{h}^{2\lambda - 2}(C(\epsilon)(\frac{\|\nabla u\|^2}{A - u} + \|\nabla u\|^2) + \epsilon\|\nabla tr_{\omega}(\chi)\|^2) + C(\epsilon)|S|_{h}^{2\lambda - 1}\frac{\|\nabla u\|^3}{(A - u)^2}.
\]
\[+(\frac{\partial}{\partial t} - \Delta)|S|_{h}^{2\lambda}tr_{\omega}(\chi) + \epsilon|S|_{h}^{2\lambda}|\nabla tr_{\omega}(\chi)|^2 + C(\epsilon)
\]
\[+(1 - \epsilon)(\nabla H \cdot \frac{\nabla u}{A - u} + \nabla H \cdot \frac{\nabla u}{A - u})
\]
\[+\frac{1}{2}|S|_{h}^{2\lambda}\frac{\|\nabla u\|^2 + \|\nabla u\|^2}{A - u} - \epsilon|S|_{h}^{2\lambda}\frac{\|\nabla u\|^4}{(A - u)^3}
\]
for small $\epsilon > 0$. Similar calculation as in the proof of the Schwarz lemma shows that
\[
(\frac{\partial}{\partial t} - \Delta)|S|_{h}^{2\lambda}tr_{\omega}(\chi) + \epsilon|S|_{h}^{2\lambda}|\nabla tr_{\omega}(\chi)|^2 \leq C.
\]
At any point $(t_0, z_0)$ $H$ achieves its maximum, by the maximum principle one has $\nabla H = 0$ and
\[0 \leq -\frac{\epsilon}{2}|S|_{h}^{2\lambda}\frac{\|\nabla u\|^4}{(A - u)^3}(t_0, z_0) + C.
\]
Therefore $H(t_0, z_0) \leq C$ and
\[H \leq C.
\]
Now we can prove the second inequality by making use of the first one. Let $K = |S|_{h}^{2\lambda - 1}\frac{\Delta u}{A - u}$ with $\Delta u \leq A - 1$. Then max $K$ is uniformly bounded below from zero. By standard calculation we have
\[
(\frac{\partial}{\partial t} - \Delta)K
\]
\[= |S|_{h}^{2\lambda}\frac{\|\nabla u\|^2 - \Delta u + g^j g^k \Gamma^l_{jk} u_l \Delta tr_{\omega}(\chi)}{A - u}
\]
\[+ \Delta u|S|_{h}^{2\lambda}\frac{\Delta u}{A - u} + [\nabla|S|_{h}^{2\lambda}] \cdot \nabla(\frac{\Delta u}{A - u}) + (\nabla|S|_{h}^{2\lambda}) \cdot \nabla(\frac{\Delta u}{A - u})]
\]
\[+ |S|_{h}^{2\lambda}\frac{\nabla(\Delta u) \cdot \nabla u}{(A - u)^2} + 2|S|_{h}^{2\lambda}\frac{\Delta u|\nabla u|^2}{(A - u)^3} + (\frac{\partial}{\partial t} - \Delta)|S|_{h}^{2\lambda}\frac{A - u}{A - u}.
\]
and
\[
(\frac{\partial}{\partial t} - \Delta)(K + 3H)
\]
\[\leq \nabla(K + 3H) \cdot (\frac{\nabla u}{A - u} + \frac{\nabla|S|_{h}^{2\lambda}}{|S|_{h}^{2\lambda}}) + \nabla(K + 3H) \cdot (\frac{\nabla u}{A - u} + \frac{\nabla|S|_{h}^{2\lambda}}{|S|_{h}^{2\lambda}})
\]
\[+ |S|_{h}^{2\lambda}\frac{\|\nabla u\|^2 - \Delta u + g^j g^k \Gamma^l_{jk} u_l \Delta tr_{\omega}(\chi)}{A - u}
\]
\[+ C|S|_{h}^{2\lambda - 2}\frac{\Delta u}{A - u} - 2|S|_{h}^{2\lambda}\frac{\|\nabla u\|^2}{A - u} + C
\]
\[\leq \nabla(K + 3H) \cdot (\frac{\nabla u}{A - u} + \frac{\nabla|S|_{h}^{2\lambda}}{|S|_{h}^{2\lambda}}) + \nabla(K + 3H) \cdot (\frac{\nabla u}{A - u} + \frac{\nabla|S|_{h}^{2\lambda}}{|S|_{h}^{2\lambda}})
\]
\[- C|S|_{h}^{2\lambda - 2}\frac{\Delta u}{A - u} - \frac{3}{2}|S|_{h}^{2\lambda}\frac{\|\nabla u\|^2 + \|\nabla u\|^2}{A - u}.
\]
Here we make use of the fact that
\[-\Delta tr_\omega(\chi) \leq R_{ij} \chi_{ij} + Ctr_\omega^2(\chi) = -u_{ij} \chi_{ij} + \chi_{ij}^2 + C \leq C(|\nabla \nabla u| + 1) .\]

At any point \((t_0, z_0)\) where \(H\) achieves its maximum, by the maximum principle one has \(\nabla(K + 3H) = 0\) and
\[|S|^2h \frac{|\nabla \nabla u|^2 + |\nabla \nabla u|^2}{A - u}(t_0, z_0) \leq C.\]

Hence \(K(t_0, z_0) \leq C\). Since \(H\) is always bounded, one has \(K(t, z) \leq C\) for any \((t, z)\).

By the volume estimate we have the following immediate corollary.

**Corollary 4.3** For any \(\delta > 0\), there exists \(C > 0\) depending on \(\delta\) such that

1. \( |S|^2h_{h \delta} |\nabla \nabla u|^2 \leq C, \)
2. \(-|S|^2h_{h \delta} \Delta u \leq C. \)

Now we are in the position to prove a uniform bound for the scalar curvature. The following corollary tells that the Kähler-Ricci will collapse with bounded scalar curvature away from the singular fibres.

**Corollary 4.4** Along the Kähler-Ricci flow \(1.1\) the scalar curvature \(R\) is uniformly bounded on any compact set of \(X_{\text{reg}}\). More precisely, there exist constants \(\lambda_6 > 0\) and \(C\) such that

\[-C \leq R \leq \frac{C}{|S|^2h_{h \delta}}. \] (4.17)

**Proof** It suffices to give an upper bound for \(R\) by Proposition 2.1. Notice that \(R_{ij} = -u_{ij} - \chi_{ij}\) and then
\[R = -\Delta u - tr_\omega(\chi). \]

By Corollary 4.3 and the partial second order estimate, we have
\[R \leq \frac{C}{|S|^2h_{h \delta}}. \]

It will be interesting to know if sectional curvatures are uniformly bounded on any compact set of \(X_{\text{reg}}\). It is not expected to be true for higher dimension. For example, we can choose \(X = X_1 \times X_2\) where \(X_1\) is a Calabi-Yau manifold and \(X_2\) is a compact Kähler manifold of \(c_1(X_2) < 0\). We can also choose the initial metric \(\omega_0(x_1, x_2) = \omega_1(x_1) + \omega_2(x_2)\) where \(Ric(\omega_1) = 0\) and \(Ric(\omega_2) = -\omega_2\). Then along the Kähler-Ricci flow \(1.1\), the solution \(\omega(t, \cdot)\) is given by
\[\omega(t, x_1, x_2) = e^{-t} \omega_1(x_1) + \omega_2(x_2). \]

The bisectional curvature of \(\omega_t\) will blow up along time if the bisectional curvature of \(\omega_1\) on \(X_1\) does not vanish.
4.4 The second order estimates

In this section, we prove a second order estimate for the potential \( \phi \) along the Kähler-Ricci flow. First we will prove a formula which allows us to commute the \( \partial \bar{\partial} \) operator and the push-forward operator for smooth functions on \( X \). Integrating along each fibre with respect to the initial metric \( \omega_0 \), we get a function on \( \Sigma \):

\[
\phi = \frac{1}{\operatorname{vol}(X_s)} \int_{X_s} \phi \omega_0.
\]

This can be considered as a push-forward of \( \phi \).

**Lemma 4.7** Let \( \phi \) be a smooth function defined on \( X \), then we have

\[
\partial \bar{\partial} \int_{X_s} \phi \omega_0 = \int_{X_s} \partial \bar{\partial} \phi \wedge \omega_0.
\]  

**(4.18)**

**Proof** It suffices to prove that the push forward and \( \partial \bar{\partial} \) are commutative. Let \( \pi : M \rightarrow B \) be an analytic deformation of a complex manifold \( M_0 = \pi^{-1}(0) \). Choose a sufficiently small neighborhood \( \Delta \subset B \) such that \( M_\Delta = \pi^{-1}(\Delta) = \bigcup \Delta \times U_i \) with local coordinates \( (z^1, \ldots, z^n, t) \), where \( z^i \) is the coordinate on \( U_i \) and \( t \) on \( \Delta \). Now choose any test function \( \zeta \) on \( B \) with \( \operatorname{supp} \zeta \subset \Delta \) and a partition of unity \( \rho_i \) with \( \operatorname{supp} \rho_i \subset \Delta \times U_i \). Let \( \varphi_i = \rho_i \phi \). We calculate

\[
\int_{M_\Delta} \zeta \partial \bar{\partial} \phi \wedge \omega = \sum_i \int_{\Delta \times U_i} \partial \bar{\partial} \zeta \wedge \varphi_i \omega = \int_{\Delta} \partial \bar{\partial} \zeta \left( \sum_i \int_{U_i} \varphi_i \omega \right)
\]

\[
= \int_{\Delta} \partial \bar{\partial} \zeta \left( \int_{M_t} \varphi \omega \right) = \int_{\Delta} f \partial \bar{\partial} \left( \int_{M_t} \varphi \omega \right).
\]

On the other hand we have

\[
\int_{M_\Delta} \zeta \partial \bar{\partial} \phi \wedge \omega = \int_{M_\Delta} \zeta \left( \sum_i \partial \bar{\partial} \varphi_i \wedge \omega \right) = \sum_i \int_{\Delta \times U_i} \zeta \partial \bar{\partial} \varphi_i \omega
\]

\[
= \int_{\Delta} \zeta \left( \sum_i \int_{U_i} \partial \bar{\partial} \varphi_i \wedge \omega \right) = \int_{\Delta} \zeta \left( \int_{M_t} \partial \bar{\partial} \varphi \wedge \omega \right).
\]

Therefore

\[
\int_{\Delta} \zeta \partial \bar{\partial} \left( \int_{M_t} \varphi \omega \right) = \int_{\Delta} \zeta \left( \int_{M_t} \partial \bar{\partial} \varphi \wedge \omega \right)
\]

for any testing function \( f \) and hence

\[
\partial \bar{\partial} \int_{M_t} \varphi \omega = \int_{M_t} \partial \bar{\partial} \varphi \wedge \omega.
\]

\[\square\]

**Lemma 4.8** There exists a constant \( C > 0 \) such that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \operatorname{tr} \omega_0(\omega) \leq C(\operatorname{tr} \omega(\omega_0) + 1).
\]  

**(4.19)**

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Proof Choose a normal coordinate system for $g_0$ such that $g$ is diagonalized. By straightforward calculation we have
\[
(\frac{\partial}{\partial t} - \Delta) \text{tr}_{\omega_0}(\omega) \leq -\text{tr}_{\omega_0}(\omega) - g^i_\alpha g^k_\beta g_i^j \text{tr}_{\omega_0}(\omega) + C \text{tr}_{\omega_0}(\omega) \text{tr}_{\omega}(\omega_0).
\] (4.20)

It can also be shown that
\[
|\nabla \text{tr}_{\omega_0}(\omega)|^2 = \sum_{i,j,k} g^{k\alpha} g_{i\alpha,k} g_{j\beta,k}^j \leq \sum_{i,j} \left( \sum_k |g_{i\alpha}^k|^2 |g_{j\beta,k}^j|^2 \right)^{\frac{1}{2}} \leq \left( \sum_i \left( \sum_k |g_{i\alpha}^k|^2 \right)^{\frac{1}{2}} \right)^2 \leq \left( \sum_i \left( \sum_k |g_{i\alpha}^k|^2 \text{tr}_{\omega_0}(\omega) \right) g_{i\alpha}^k g_{j\beta,k}^j \right)^2 \leq \text{tr}_{\omega_0}(\omega) g_{i\alpha}^k g_{j\beta,k}^j g_{i\alpha,k}^j g_{j\beta,k}^j.
\] (4.21)

Combined with the above inequalities, the lemma follows by calculating $(\frac{\partial}{\partial t} - \Delta) \log \text{tr}_{\omega_0}(\omega)$. □

Lemma 4.9
\[
\Delta(e^t(\varphi - \overline{\varphi})) \leq -\text{tr}_{\omega}(\omega) + \frac{1}{\text{Vol}(X_s)} \text{tr}_{\omega}(\int_{X_s} \omega_0^2) + 2e^t.
\] (4.22)

Proof Applying equation (4.18), we have
\[
\Delta(\varphi - \overline{\varphi}) = \text{tr}_{\omega}((\omega - \omega_t) - \text{tr}_{\omega}(\frac{1}{\text{Vol}(X_s)} \int_{X_s} \partial \theta \wedge \omega_0)) = 2 - \text{tr}_{\omega}(\omega_t) - \frac{1}{\text{Vol}(X_s)} \text{tr}_{\omega}(\int_{X_s} \omega \wedge \omega_0 - \int_{X_s} \omega_t \wedge \omega_0) \leq -e^{-t} \text{tr}_{\omega}(\omega_0) + \frac{e^{-t}}{\text{Vol}(X_s)} \text{tr}_{\omega}(\int_{X_s} \omega_0^2) + 2.
\] □

Theorem 4.2 (Second order estimates) There exist constants $\lambda_7, A$ and $C > 0$ such that
\[
\text{tr}_{\omega_0}(\omega)(t, z) \leq Ce^{Ae^t(\varphi - \overline{\varphi})(t, z)} \text{sup}_{|S|_{\lambda_7}^2(\varphi - \overline{\varphi})(t, z)} \text{inf}_{X \times [0, T]} (|S|_{\lambda_7}^2(\varphi - \overline{\varphi})) + C.
\] (4.23)

Proof Put $H = |S|_{\lambda_7}^2(\log \text{tr}_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi}))$. We will apply the maximum principle on the
evolution of $H$. There exists a constant $C > 0$ such that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) H
\]
\[
= |S|^{2\lambda_7} \left( \frac{\partial}{\partial t} - \Delta \right) \log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi}) - (\nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}} + \nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}})
\]
\[
+ |\nabla |S|^{2\lambda_7}|^2
\]
\[
\leq C |S|^{2\lambda_7} tr_{\omega}(\omega_0) - A |S|^{2\lambda_7} tr_{\omega}(\omega_0) - (\nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}} + \nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}})
\]
\[
+ |S|^{2\lambda_7} [-Ae^t(\varphi - \overline{\varphi}) - Ae^t \frac{\partial(\varphi - \overline{\varphi})}{\partial t} + \frac{1}{\text{Vol}(X_s)} \int \nabla \omega^2] + Ae^t
\]
\[
+ |\nabla |S|^{2\lambda_7}|^2
\]
\[
\frac{1}{\text{Vol}(X_s)} \int \nabla \omega^2 \leq tr_{\omega}(\chi) \sup_{X_s} \frac{\omega^2}{\omega SF \wedge \chi}.
\]
If we assume Lemma 5.4 then $e^t |S|^{2\lambda_7}(\varphi - \overline{\varphi})$, $|S|^{2\lambda_7} \frac{\partial(\varphi - \overline{\varphi})}{\partial t}$ and $|S|^{2\lambda_7} tr_{\omega}(\int \nabla \omega^2)$ are uniformly bounded if $\lambda_7$ is chosen to be sufficiently large. Also we have
\[
\Delta |S|^{2\lambda_7} \leq C |S|^{2\lambda_7 - 2} tr_{\omega}(\chi)
\]
and
\[
\frac{|\nabla |S|^{2\lambda_7}|^2}{|S|^{2\lambda_7}} \leq C |S|^{2\lambda_7 - 2} tr_{\omega}(\chi)
\]
for a uniform constant $C > 0$. Therefore we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) H
\]
\[
\leq C |S|^{2\lambda_7} tr_{\omega}(\omega_0) - A |S|^{2\lambda_7} tr_{\omega}(\omega_0) - (\nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}} + \nabla H \cdot \frac{\nabla |S|^{2\lambda_7}}{|S|^{2\lambda_7}}) + Ae^t.
\]
Assume $H$ achieves its maximum at $(t_0, z_0)$ on $[0, T] \times X$. Applying the maximum principle, we have $\nabla H(t_0, z_0) = 0$ and then
\[
\{|S|^{2\lambda_7} tr_{\omega}(\omega_0)\}(t_0, z_0) \leq Ce^{t_0}.
\]
This implies
\[
\{|S|^{2\lambda_7} tr_{\omega}(\omega)\}(t_0, z_0) \leq C.
\]
The theorem is proved then by comparing $H$ at any point $(t, z) \in [0, T] \times X$ and $(t_0, z_0)$. \hfill \Box

5 Generalized Kähler-Einstein metrics and the Kähler-Ricci flow

5.1 Limiting metrics on canonical models and Weil-Petersson metrics

Let $X$ be a smooth projective manifold of dim $X = n$. Suppose that $\mu K_X$ is base point free for $\mu >> 1$ and $\nu(X) = \kappa$ with $0 < \kappa \leq n$. Then
\[
|\mu K_X| : X \rightarrow X_\mu \subset \mathbb{CP}^{N_\nu}
\]
is a holomorphic map for \( \mu >> 1 \). Fix such \( \mu \), we have a holomorphic fibration \( f : X \to X_{can} \) such that \( \mu K_X = f^*O(1) \), where \( X_{can} \) is the canonical model of \( X \) and coincides with \( X_{\mu} \) for \( \mu \) sufficiently large. The abundance conjecture claims that there is such a holomorphic map \( f \) whenever \( K_X \) is nef. If \( K_X \) is also big, it was proved by Kawamata. If \( \kappa = n \), \( X \) is a minimal model of general type with \( K_X \) big and nef., the Kähler-Ricci flow deforms any Kähler metric onto a unique singular Kähler-Einstein metric on \( X \) (see \([15], [17]\)). If \( 0 < \kappa < n \), then for a generic fibre \( X_s \), one has \( \mu K_{X_s} = f^*O(1) \) for some \( \mu \in \mathbb{N} \), thus \( \mu K_{X_s} \) is trivial and \( X_s \) is a Calabi-Yau. We can choose \( \chi \) to be a multiple of the Fubini-Study metric of \( CP^N \) restricted on \( X_{can} \) such that \( f^*\chi \in -2\pi c_1(X) \). Notice that \( f^*\chi \) is a smooth semi-positive \((1,1)\)-form on \( X \).

Remark 5.2 The function \( \psi(z,s) = \psi_s(z) \) is well-defined as a smooth function on \( X_{reg} \).

Lemma 5.1 For any Kähler class \([\omega]\) on \( X \), there is a smooth function \( \psi \) such that \( \omega_{SF} := \omega + \sqrt{-1}\partial\bar{\partial}\psi \) is a closed semi-flat \((1,1)\)-form in the following sense: the restriction of \( \omega_{SF} \) to each smooth \( X_s \subset X_{reg} \) is Ricci flat.

**Proof** On each smooth fiber \( X_s \), let \( \omega_s \) be the restriction of \( \omega \) to \( X_s \) and \( \partial_V \) and \( \bar{\partial}_V \) be the restriction of \( \partial \) and \( \bar{\partial} \) to \( X_s \). Then by the Hodge theory, there is a unique function \( h_s \) on \( X_s \) defined by

\[
\begin{align*}
\partial_V \bar{\partial}_V h_s &= -\partial_V \bar{\partial}_V \log \omega_s^{n-\kappa} \\
\int_{X_s} e^{h_s} \omega_s &= \int_{X_s} \omega_s.
\end{align*}
\]

(5.1)

By Yau’s solution to the Calabi conjecture, there is a unique \( \psi_s \) to the following Monge-Ampère equation

\[
\begin{align*}
\frac{(\omega_s + \sqrt{-1}\partial_V \bar{\partial}_V \psi_s)^{n-\kappa}}{\omega_s^{n-\kappa}} &= e^{h_s} \\
\int_{X_s} \psi_s \omega_s^{n-\kappa} &= 0.
\end{align*}
\]

(5.2)

Since \( f \) is holomorphic, \( \psi(z,s) = \psi_s(z) \) is well-defined as a smooth function on \( X_{reg} \).

By the Hodge theory, there exists a volume form on \( X \) such that \( \sqrt{-1}\partial\bar{\partial}\log \Omega = \chi \). Define

\[
F = \frac{\Omega}{\left( \frac{n}{\kappa} \right)^{\omega_s^{n-\kappa} \wedge \chi^\kappa}}.
\]

(5.3)

We will show properties of \( F \) and examine how \( F \) behaves near singular fibers. Define \( h(z,s) = h_s(z) \), then

\[
F = \frac{\Omega}{\left( \frac{n}{\kappa} \right)^{\omega_s^{n-\kappa} \wedge \chi^\kappa}} e^{-h} \geq 0.
\]

(5.4)

It follows that \( F \) has at most poles along \( X_{can} \setminus X_{can}^0 \).

Lemma 5.2 \( F \) is the pullback of a function on \( X_{can} \). Furthermore, there exists \( \epsilon > 0 \) such that

\[
F \in L^{1+\epsilon}(X_{can}).
\]

(5.5)
Proof Since $\chi$ is the pullback from $X_{\text{Can}}$, we have
\[ \sqrt{-1} \partial_V \bar{\partial}_V \log \Omega = \sqrt{-1} \partial_V \bar{\partial}_V \log \omega_{SF}^{n-\kappa} \wedge \chi^\kappa = 0 \]
on each smooth fibre $X_s$. Thus $F$ is constant along each smooth fibre $X_s$ and so it is the pullback of a function from $X_{\text{Can}}$. Now we prove the second statement.

\[ \int_{X_{\text{Can}}} F^{1+\epsilon} \chi^\kappa = \frac{1}{\int_{X_s} \omega_{SF}^{n-\kappa}} \int_X F^{1+\epsilon} \chi^\kappa \wedge \omega_{SF}^{n-\kappa} \]
\[ = \frac{1}{\binom{n}{\kappa}} \int_{X_s} \omega_{SF}^{n-\kappa} \int_X F^\epsilon \Omega \leq C. \]

The last inequality holds for sufficiently small $\epsilon > 0$ because $F$ can have at worst pole singularities. \qed

There is a canonical hermitian metric on the push-forward of the dualizing sheaf $f_*(\Omega_{X/X_{\text{Can}}}^{n-\kappa}) = (f_1 \mathcal{O}_X)^\vee$ over $X_{\text{Can}}^0$.

**Definition 5.1** Let $X$ be a projective manifold of complex dimension $n$. Suppose its canonical line bundle $K_X$ is semi-positive and $0 < \kappa = \nu(X) < n$. Let $X_{\text{Can}}$ be the canonical model of $X$ by pluricanonical maps. We define a canonical hermitian metric $h_{\text{can}}$ on $f_*(\Omega_{X/X_{\text{Can}}}^{n-\kappa})$ in the way that for any smooth $(n-\kappa,0)$-form $\eta$ on a smooth fiber $X_s$,

\[ |\eta|^2_{h_{\text{can}}} = \frac{\eta \wedge \bar{\eta} \wedge \chi^\kappa}{\omega_{SF}^{n-\kappa} \wedge \chi^\kappa} = \frac{\int_{X_s} \eta \wedge \bar{\eta}}{\int_{X_s} \omega_{SF}^{n-\kappa}}. \]

(5.6)

Now let us recall some facts on the Weil-Petersson metric on the moduli space $M$ of polarized Calabi-Yau manifolds of dimension $n-\kappa$. Let $\mathcal{X} \to \mathcal{M}$ be a universal family of Calabi-Yau manifolds. Let $(U; t_1, \ldots, t_m)$ be a local holomorphic coordinate chart of $\mathcal{M}$, where $m = \dim \mathcal{M}$. Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $i(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira-Spencer map $i$. The Weil-Petersson metric is defined by the $L^2$-inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi-Yau manifolds, we can express it as follows: Let $\Psi$ be a nonzero holomorphic $(n-\kappa,0)$-form on the fibre $\mathcal{X}_t$ and $\Psi \cdot i(\frac{\partial}{\partial t_i})$ be the contraction of $\Psi$ and $\frac{\partial}{\partial t_i}$. Then the Weil-Petersson metric is given by

\[ \left( \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \right)_{\omega_{WP}} = -\frac{\int_{\mathcal{X}_t} \Psi \cdot i(\frac{\partial}{\partial t_i}) \wedge \Psi \cdot i(\frac{\partial}{\partial t_j})}{\int_{\mathcal{X}_t} \Psi \wedge \Psi}. \]

(5.7)

One can also represent $\omega_{WP}$ as the curvature form of the first Hodge bundle $f_* \Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$. Let $\Psi$ be a nonzero local holomorphic section of $f_* \Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$ and one can define the hermitian metric $h_{WP}$ on $f_* \Omega_{\mathcal{X}/\mathcal{M}}^{n-\kappa}$ by

\[ |\Psi_t|^2_{h_{WP}} = \int_{\mathcal{X}_t} \Psi_t \wedge \overline{\Psi_t}. \]

(5.8)

Then the Weil-Petersson metric is given by

\[ \omega_{WP} = \text{Ric}(h_{WP}) \]

(5.9)
Lemma 5.3

\[ \text{Ric}(h_{\text{can}}) = \omega_{WP}. \]  

(5.10)

**Proof** Let \( u = \frac{\Psi \wedge \overline{\Psi}}{i \omega_{SF}} \). Notice \( \Psi \) restricted on each fibre \( X_t \) is a holomorphic \((n - \kappa, 0)\)-form and \( \Psi \wedge \overline{\Psi} \) is a Calabi-Yau volume form, therefore \( u \) is constant along each fibre and can be considered as the pullback of a function on \( M \). Then by definition

\[
\omega_{WP} = -\sqrt{-1} \partial \overline{\partial} \log \int_{X_t} \omega_{SF}^{n-\kappa} = -\sqrt{-1} \partial \overline{\partial} \log u,
\]

where the last equality makes use of the fact that \( \int_{X_t} \omega_{SF}^{n-\kappa} = \text{constant} \). At the same time

\[
\text{Ric}(h_{\text{can}}) = -\sqrt{-1} \partial \overline{\partial} \log \frac{\Psi \wedge \overline{\Psi} \wedge \chi^\kappa}{\omega_{SF}^{n-\kappa} \wedge \chi^\kappa} = -\sqrt{-1} \partial \overline{\partial} \log u.
\]

This proves the lemma. \( \square \)

**Definition 5.2 (Canonical metrics on \( X_{\text{can}} \))** We define the generalized Kähler-Einstein metric \( \omega \) in the class of \(-2\pi f_* c_1(X)\) on \( X_{\text{can}}^0 \) with respect to the fibration \( f : X \rightarrow X_{\text{can}} \) by

\[
\text{Ric}(\omega) = -\omega + \omega_{WP}.
\]

(5.11)

In general if \( X \rightarrow \Sigma \) is a Calabi-Yau fibration, we can define a generalized Kähler-Einstein metric \( \lambda \omega \in 2\pi c_1(\Sigma) + 2\pi c_1(f_* \Omega_X^{n-\kappa}) \) by

\[
\text{Ric}(\omega) = \lambda \omega + \omega_{WP},
\]

(5.12)

where \( \lambda = -1, 0, 1 \).

The following theorem is the main result of this section and its proof is essentially due to the work of Kolodziej [Ko1, Ko2].

**Theorem 5.1** Suppose that \( X_{\text{can}} \) is smooth (or has at worst orbifold singularities), then there is a unique solution \( \varphi_\infty \in \text{PSH}(\chi) \cap C^0(X_{\text{can}}) \) of the following equation on \( X_{\text{can}} \)

\[
(\chi + \sqrt{-1} \partial \overline{\partial} \varphi)^\kappa = F e^\varphi \chi^\kappa.
\]

(5.13)

Furthermore, \( \omega = \chi + \sqrt{-1} \partial \overline{\partial} \varphi_\infty \) is a positive closed current on \( X_{\text{can}} \). If \( \omega \) is smooth on \( X_{\text{can}}^0 \), then the Ricci curvature of \( \omega \) on \( X_{\text{can}}^0 \) is given by

\[
\text{Ric}(\omega) = -\chi_\omega + \omega_{WP}.
\]

(5.14)

In fact, these canonical metrics belong to a class of Kähler metrics which generalize Calabi’s extremal metrics. Let \( Y \) be a Kähler manifold of complex dimension \( n \) together with a fixed closed \((1,1)\)-form \( \theta \). Fix a Kähler class \([\omega]\), denote by \( \mathcal{K}_{[\omega]} \) the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form \( \omega_\varphi = \omega + \sqrt{-1} \partial \overline{\partial} \varphi \). One may consider the following equation:

\[
\partial V_\varphi = 0,
\]

(5.15)

where \( V_\varphi \) is defined by

\[
\omega_\varphi(V_\varphi, \cdot) = \partial (S(\omega_\varphi) - \text{tr}_{\omega_\varphi}(\theta)).
\]

(5.16)
Clearly, when $\theta = 0$, (5.15) is exactly the equation for Calabi’s extremal metrics. For this reason, we call a solution of (5.15) a generalized extremal metric. If $Y$ does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric $\omega_\varphi$ satisfies

$$S(\omega_\varphi) - tr_{\omega_\varphi}(\theta) = \mu,$$

where $\mu$ is the constant given by

$$\mu = \frac{n(2\pi c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $2\pi c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$Ric(\omega_\varphi) = \lambda \omega_\varphi + \theta,$$

that is, $\omega_\varphi$ is a generalized Kähler-Einstein metric. This can be proved by an easy application of the Hodge theory. More interestingly, if we take $\theta$ to be the pull-back of $\omega_{WP}$ by $f : X^0_{\text{can}} \to \mathcal{M}_{CY}$, then we get back those generalized Kähler-Einstein metrics which arise from limits of the Kähler-Ricci flow.

### 5.2 Minimal surfaces of general type

We start with minimal surfaces of general type. Let $X_{\text{can}}$ be the canonical model of a minimal surface of general type from the contraction map $f : X \to X_{\text{can}}$. Possibly, $X_{\text{can}}$ has rational singularities of $A-D-E$-type by contracting the (-2)-curves. Since $K_{\text{can}}$ is ample and $f^*K_{\text{can}} = K_X$, we can assume the smooth closed (1,1)-form $\chi = f^*\chi \in -2\pi c_1(X)$ and $\chi$ is a Kähler from on $X_{\text{can}}$. It is shown in [TiZha] that the Kähler-Ricci flow (1.1) converges to the canonical metric $g_{KE}$ on $X$, which is the pullback of the smooth orbifold Kähler-Einstein metric on the canonical model $X_{\text{can}}$, although $g_{KE}$ might vanish along those (-2)-curves.

### 5.3 Minimal elliptic surfaces of Kodaira dimension one

Now consider minimal elliptic surfaces. From Lemma 5.1, we know that there exists a closed semi-flat (1,1)-form $\omega_{SF}$ in $[\omega_0]$.

**Lemma 5.4** Let $F$ be the function on $\Sigma$ defined by $F = \frac{\Omega}{2\omega_{SF} \wedge \chi}$ as (5.3). Let $B \subset \Sigma$ be a small disk with center 0 such that all fibres $X_s$, $s \neq 0$, are smooth. There exists a constant $C > 0$ such that

1. If $X_0$ is of type $mI_0$, then

   $$\frac{1}{C}|s|^{-2(m-1)}m \leq F|_B \leq C|s|^{-2(m-1)}m;$$

   \hspace{1cm} (5.17)

2. If $X_0$ is of type $mI_b$ or $I^*_b$, $b > 0$, then

   $$-\frac{1}{C}|s|^{-2(m-1)}m \log |s|^2 \leq F|_B \leq -C|s|^{-2(m-1)}m \log |s|^2;$$

   \hspace{1cm} (5.18)

3. If $X_0$ is of any other type, then

   $$\frac{1}{C} \leq F|_B \leq C.$$

   \hspace{1cm} (5.19)
Proof Let $Y$ be the fibration of $f$ over $B$.

1. If $X_0$ is of type $mI_0$, we start with a fibration $\tilde{Y} = C \times \tilde{B}/L$, where $L = Z + Z \cdot z(w)$ is a holomorphic family of lattices with $z$ being a holomorphic function on $\tilde{B}$ satisfying: $z(w) = z(0) + const \cdot w^{m/2}$, $w$ is the coordinate on $\tilde{B}$, $h \in \mathbb{N}$. The automorphism of $C \times B$ given by $(c, w) \to (c + \frac{1}{m}, e^{\frac{2\pi\sqrt{-1}}{m}}w)$ descends to $\tilde{Y}$ and generates a group action without fixed points. We can assume that $\tilde{Y}$ is the quotient of $\tilde{Y}$ by the group action. Therefore $\omega_{SF}$ is a smooth family of Ricci-flat metrics over $B$. Choose a local coordinate $s$ on $B$ centered around 0, and a covering $\{\mathcal{U}_\alpha\}$ of a neighborhood $U$ of $X_0$ in $X$ by small polydiscs. Since the function $f^s$ vanishes to order $m$ along $X_0$, we can in each $U_\alpha$ choose a holomorphic function $w_\alpha$ on $U_\alpha$ as the $m$th root of $f^s$, with

$$w_\alpha^m = f^s$$

and on $U_\alpha \cap U_\beta$

$$w_\alpha = e^{-\frac{2\pi\sqrt{-1}k_{\alpha\beta}}{m}}w_\beta$$

for $k_{\alpha\beta} \in \{0, 1, ..., m - 1\}$. On each $U_\alpha$, $ds \wedge d\bar{s} = m^2 |s|^{\frac{2(m-1)}{m}}dw_\alpha \wedge d\bar{w}_\alpha$. Then $|s|^{\frac{2(m-1)}{m}}F$ is smooth and bounded away from zero on $Y$. Thus (5.17) is proved.

2. If $X_0$ is of type $I_b$, $b > 0$, we can assume $Y = C \times B/L$, where

$$L = Z + Z \frac{b}{2\pi\sqrt{-1}} \log s.$$ 

Let $\gamma_0$ be an arc passing through 0 in $B$ and $\gamma$ be an arc on $X$ transverse to $X_0$ with $f \cdot \gamma = \gamma_0$. We also assume that $\gamma$ does not pass through any double point of $X_0$. $\Omega = F \omega_{SF} \wedge \chi$ is smooth and non-degenerate and so is $\chi$ along $\gamma$. Since $F = \frac{\Omega}{\omega_0 \wedge \chi} \wedge SF$, it suffices to estimate the function $\frac{\omega_0}{\omega_{SF}}|X_s$ restricted to $\gamma$ near $X_0$. Along $\gamma$, $\omega_0|X_s$ pulls back to a metric uniformly equivalent to a fixed flat metric $\omega_C$, so it suffices to estimate $\frac{\omega_C}{\omega_{SF}}|X_s$. But

$$\frac{\omega_C}{\omega_{SF}}|X_s = \frac{\int_{X_s} \omega_C}{\int_{X_s} \omega_{SF}} = \frac{b \log |s|}{2\pi \int_{X_s} \omega_{SF}}$$

and $\text{Vol}(X_s) = \int_{X_s} \omega_{SF}$ is a constant independent of $s$. Therefore there exists a constant $C$ such that

$$-\frac{1}{C} \log |s|^2 \leq F \leq -C \log |s|^2.$$

If $X_0$ is of type $mI_b$, $b > 0$, we start with a fibration $f : Y = C \times \tilde{B}/L$ and $L = Z + Z \frac{mb}{2\pi\sqrt{-1}} \log w$ and $w$ is the coordinate function of $\tilde{B}$. So $Y_0 = C_1 + C_2 + ... + C_{mb}$ is of type $I_{mb}$. The automorphism $(c, w) \to (c, e^{\frac{2\pi\sqrt{-1}}{m}}w)$ of $C \times \tilde{B}$ induces a fibre-preserving automorphism of order $m$ on $Y$. Such an automorphism generates a group action on $Y$ without fixed points and the quotient of $Y$ has a singular fibre of type $mI_b$. Then by using the same arguments for singular fibres of type $mI_0$, we can prove (5.18). A fibration of type $I_b^0$ ($b > 0$) is obtained by taking a quotient of a fibration of type $I_{2b}$ after resolving the $A_1$-singularities. The lattices can be locally written as $L = s^2 Z + Z s^2 \frac{b}{2\pi\sqrt{-1}} \log s$. Then the above argument gives the required estimate for $F$. 

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3. If $X_0$ is not of type $mI_b$, $b \geq 0$ or $I_b^*$, $b > 0$, it must be of type $I_0^*$, $II$, $III$, $IV$, $IV^*$, $III^*$ or $II^*$. Such a singular fibre is not a stable fiber. By the table of Kodaira (cf. [Ko]), the functional invariant $J(s)$ is bounded near 0 and $J(0) = 0$ or 1. One can write down the table of local lattices of periods and the periods are bounded near the singular fibre. For example, if $X_0$ is of type $II$, then $X_0$ is a cuspidal rational curve with $J(s) = s^{3m+1}$, $m \in \mathbb{N} \cup \{0\}$ in the local normal representation. On each fibre $X_s$ the above fixed flat metric $\omega_C$ on $C$ has uniformly bounded area, therefore

$$0 < \frac{1}{C} \leq \frac{\omega_C}{\omega_S} \mid_{X_s} = \frac{\int_{X_s} \omega_C}{\int_{X_s} \omega_S} \leq C.$$ 

The estimate is then proved by the same argument as that in the previous case. □

Lemma 5.5 There is a unique solution $\varphi_\infty$ of the following equation on $\Sigma$

$$\chi + \sqrt{-1} \partial \bar{\partial} \varphi = F e^\varphi \chi$$

satisfying $\sup_{\Sigma} |\varphi| \leq C$. Furthermore, we have $\varphi_\infty \in C^0(\Sigma_{\text{reg}}) \cap C^\infty(\Sigma_{\text{reg}})$.

Proof This is a corollary of Theorem 5.2, but still we give an elementary proof for the sake of completeness. Rewrite equation (5.20) as

$$\Delta \varphi = F e^\varphi - 1,$$

where $\Delta$ is the Laplacian operator with respect to $\chi$. We will apply the method of continuity to find the solutions of the following equation parameterized by $t \in [0,1]$: 

$$\Delta \varphi = e^\varphi \left( \frac{2\omega_S \wedge \chi}{\Omega} + t \right)^{-1} - 1.$$  

(5.22)

Obviously equation (5.22) is solvable for all $t \in (0, 1]$. To solve for $t = 0$ we need to derive the uniform $C^0$-estimate for $\varphi_t$. By the maximum principle,

$$\sup_{\Sigma \times (0, 1]} \varphi_t \leq \sup_\Sigma \log \left( \frac{2\omega_S \wedge \chi}{\Omega} \right) \leq C.$$

By Lemma 5.4 $||F||_{L^p}$ is bounded for some $p > 1$, then the standard $L^p$ estimate gives

$$||\varphi_t||_{L^p} \leq C(||F||_{L^p} + 1) \leq C.$$

The Sobolev embedding theorem implies

$$||\varphi_t||_{L^\infty} \leq C$$

for $t \in (0, 1]$. With the $C^0$ estimate, we can derive the uniform $C^k$-estimate for $\varphi_t$ by the local estimates of the standard theory of linear elliptic PDE due to the fact that $\Delta$ has uniformly bounded coefficients. Therefore there exists $\varphi_\infty \in C^\infty(\Sigma_{\text{reg}})$ satisfying equation (5.20).

Now we prove the uniqueness. Suppose there is another solution $\psi$ solving (5.13) with

sup $\Sigma |\psi| \leq C$. We define

$$\varphi_\epsilon = \varphi + \epsilon \log |S|^2_h$$

such that $|S|^2_h \leq 1$. Then

$$\sqrt{-1} \partial \bar{\partial} (\varphi_\epsilon - \psi) = \left( \frac{1}{|S|^2_h} e^{\varphi_\epsilon - \psi} - 1 \right) (\chi + \sqrt{-1} \partial \bar{\partial} \psi) - \epsilon \chi.$$
Note that \( \{ \varphi_\infty - \psi \geq 0 \} \cap (\cup_i X_{p_i}) = \emptyset \), then

\[
- \int_{\{ \varphi_\infty - \psi \geq 0 \}} |\nabla \psi (\varphi_\infty - \psi)|^2 (\chi + \sqrt{-1} \partial \bar{\partial} \psi) \\
= \int_{\{ \varphi_\infty - \psi \geq 0 \}} (\varphi_\infty - \psi) \sqrt{-1} \partial \bar{\partial} (\varphi_\infty - \psi) \\
= \int_{\{ \varphi_\infty - \psi \geq 0 \}} (\varphi_\infty - \psi) \left( \frac{1}{|S|_{\text{het}}^2} e^{\varphi_\infty - \psi} - 1 \right) (\chi + \sqrt{-1} \partial \bar{\partial} \psi) - \int_{\{ \varphi_\infty - \psi \geq 0 \}} \epsilon \chi \\
\geq -C \epsilon.
\]

Let \( \epsilon \to 0 \), by Fatou’s lemma,

\[
\int_{\{ \varphi_\infty - \psi \geq 0 \}} |\nabla \psi (\varphi_\infty - \psi)|^2 (\chi + \sqrt{-1} \partial \bar{\partial} \psi) = 0,
\]

therefore \( \varphi_\infty - \psi = 0 \) on each path connected component of \( \{ \varphi_\infty - \psi \geq 0 \} \). On the other hand, we can apply the same argument for \( \psi_\epsilon \) and it infers that

\[ \varphi_\infty - \psi = 0. \]

\[ \square \]

**Corollary 5.1** Let \( f : X \to \Sigma \) be a minimal elliptic surface of \( \nu(X) = 1 \) with singular fibres \( X_{s_1} = m_1 F_1, \ldots, X_{s_k} = m_k F_k \) with multiplicity \( m_i \in \mathbb{N}, i = 1, \ldots, k \). If \( \varphi_\infty \) is the solution in Lemma 5.5, then \( \omega_\infty = \chi + \sqrt{-1} \partial \bar{\partial} \varphi_\infty \) is a Kähler form on \( \Sigma \) and Ricci curvature of \( \omega_\infty \) is given by the following formula

\[
\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{SF} + \sum_{i=1}^{k} \frac{m_k - 1}{m_k} [s_i],
\]

where \( \omega_{WP} \) is the induced Weil-Petersson metric and \([s_i]\) is the current of integration associated to the divisor \( s_i \) on \( \Sigma \). In particular, if \( f : X \to \Sigma \) has only singular fibres of type \( m I_0 \), then \( \chi_\infty \) is a hyperbolic cone metric on \( X \) given by

\[
\text{Ric}(\chi_\infty) = -\omega_\infty. \tag{5.24}
\]

This tells us that \( \omega_\infty \) is canonical in the sense that it is more or less a hyperbolic metric with a correction term \( \omega_{SF} + \sum_{i=1}^{k} \frac{m_k - 1}{m_k} [s_i] \) inherited from the elliptic fibration structure of \( X \). Also we notice that the residues only come from multiple fibres.

### 6 Convergence

#### 6.1 Sequential convergence

In this section we will prove a sequential convergence of the Kähler-Ricci flow to a canonical metric on the base \( \Sigma \).

**Lemma 6.1** There exist constants \( \lambda \) and \( C \) such that

\[
\left| \nabla g_0 \frac{\partial \varphi}{\partial t} \right|^2 \leq C e^{\frac{C}{|S|_{\text{het}}^2}}. \tag{6.1}
\]
Proof Put \( u = \frac{\partial \varphi}{\partial t} + \varphi \) and calculate

\[
\frac{1}{2} \left| \nabla_{g_0} \frac{\partial \varphi}{\partial t} \right|^2 - |\nabla_{g_0} \varphi|^2 \\
\leq \left| \nabla_{g_0} (\frac{\partial \varphi}{\partial t} + \varphi) \right|^2 = \frac{\sqrt{-1} \partial u \wedge \bar{\partial} u \wedge \omega_0}{\omega_0} = |\nabla u|^2 \frac{\omega \wedge \omega_0}{\omega_0}
\]

\[
\leq |\nabla u|^2 tr_{\omega_0}(\omega).
\]

The lemma follows from Theorem 4.2 (the second order estimate), Corollary 4.2 and Theorem 4.1 (the gradient estimate) since \( |\nabla_{g_0} \varphi| \) can be bounded by \( tr_{\omega_0}(\omega) \). \( \square \)

We first prove a weak convergence for \( \frac{\partial \varphi}{\partial t} \) by picking a sequence.

**Lemma 6.2** There exists a sequence \( \{t_j\}_{j=1}^\infty \) such that \( \frac{\partial \varphi}{\partial t}(t_j, \cdot) \) converges to 0 weakly. Thus by Lemma 6.1 \( \frac{\partial \varphi}{\partial t}(t_j, \cdot) \) converges to 0 in \( C^{0,\alpha} \) for any \( 0 < \alpha < 1 \) on any compact set of \( X_{reg} \).

**Proof** For any \( T_2 > T_1 > 0 \) and \( z \in X \) we have

\[
\int_{T_1}^{T_2} \frac{\partial \varphi}{\partial t}(s, z) ds = \varphi(T_2, z) - \varphi(T_1, z).
\]

Since \( |\varphi|_{C^0} \) is uniformly bounded on \( X \), we have

\[
\left| \int_{T_1}^{T_2} \frac{\partial \varphi}{\partial t}(s, z) ds \right| \leq C.
\]

Choose a countable basis \( \{U_\alpha\}_{\alpha \in A} \) for the topology of \( X \). Then on each \( U_\alpha \), \( \int_{T_1}^{T_2} (\int_{U_\alpha} \frac{\partial \varphi}{\partial t} \omega_0^2) ds \) is uniformly bounded independent of the choice of \( T_1 \) and \( T_2 \). Therefore by applying the mean value theorem with \( T_1, T_2, T_2 - T_1 \to \infty \), we can show that by passing to a sequence \( \frac{\partial \varphi}{\partial t}(t_k, \cdot, \cdot) \)

\[
\lim_{k \to \infty} \int_{U_\alpha} \frac{\partial \varphi}{\partial t}(t_k, \cdot, \cdot) = 0.
\]

By taking the diagonal sequence \( t_j \) of \( t_{k,\alpha} \), one has for each \( \alpha \in A \)

\[
\lim_{j \to \infty} \int_{U_\alpha} \frac{\partial \varphi}{\partial t}(t_j, \cdot, \cdot) \omega_0^2 = 0.
\]

Therefore \( \frac{\partial \varphi}{\partial t}(t_j, \cdot, \cdot) \) converges weakly to 0 on \( X \). \( \square \)

**Theorem 6.1** There exists a sequence \( \{t_j\}_{j=1}^\infty \) with \( t_j \to \infty \) such that \( \varphi(t_j, \cdot) \) converges to the pullback of a function \( \varphi_\infty \) given by equation (5.27) uniformly on any compact set of \( X_{reg} \) in the sense of \( C^{1,1} \).

**Proof** Consider any test function \( \zeta \in C^\infty_0(\Sigma_{reg}) \). We calculate

\[
\int_X \zeta e^t \omega^2 = \int_X \zeta \left[ 2(\chi + \sqrt{-1} \partial \bar{\partial} \varphi) \wedge \omega_0 + 2\sqrt{-1} e^t \partial \bar{\partial}(\varphi - \bar{\varphi}) \wedge (\chi + \sqrt{-1} \partial \bar{\partial} \varphi) + e^{-t}(\omega_0 - \chi + \sqrt{-1} e^t \partial \bar{\partial}(\varphi - \bar{\varphi}))^2 \right].
\]
Notice that \( \int_X \zeta \partial \overline{\partial} (\varphi - \bar{\varphi}) \wedge (\chi + \sqrt{-1} \partial \overline{\partial} \varphi) = \int_X (\varphi - \bar{\varphi}) \partial \overline{\partial} \zeta \wedge (\chi + \sqrt{-1} \partial \overline{\partial} \varphi) = 0 \) and
\[
\int_X \zeta e^{-t} (\omega_0 - \chi + \sqrt{-1} e^t \partial \overline{\partial} (\varphi - \bar{\varphi}))^2 = \int_X \zeta [e^{-t} (\omega_0 - \chi)^2 + 2 \sqrt{-1} \partial \overline{\partial} (\varphi - \bar{\varphi}) \wedge (\omega_0 - \chi) + \sqrt{-1} e^t \partial \overline{\partial} (\varphi - \bar{\varphi}) \wedge \sqrt{-1} \partial \overline{\partial} (\varphi - \bar{\varphi})] = \int_X [2 (\varphi - \bar{\varphi}) \sqrt{-1} \partial \overline{\partial} \zeta \wedge (\omega_0 - \chi) + e^t (\varphi - \bar{\varphi}) \partial \overline{\partial} \zeta \wedge \sqrt{-1} \partial \overline{\partial} (\varphi - \bar{\varphi})] + O(e^{-t}) = O(e^{-t}).
\]
Therefore
\[
\int_X 2 \zeta (\chi + \sqrt{-1} \partial \overline{\partial} \varphi) \wedge \omega_0 = \int_X \zeta e^t \omega^2 + O(e^{-t}) = \int_X \zeta e^t \frac{\partial \varphi}{\partial t} \Omega + O(e^{-t}).
\]

Since \( \chi + \sqrt{-1} \partial \overline{\partial} \varphi \) sits on the base \( \Sigma \) and the volume along each smooth fibre given by \( \omega_{SF} \) is the same as that of \( \omega_0 \), we have
\[
\int_X 2 \zeta (\chi + \sqrt{-1} \partial \overline{\partial} \varphi) \wedge \omega_{SF} = \int_X \zeta e^t \frac{\partial \varphi}{\partial t} \frac{\Omega}{\chi \wedge \omega_{SF}} \chi \wedge \omega_{SF} + O(e^{-t}).
\]
So for all \( \zeta \in C^\infty_0 (\Sigma_{reg}) \), we have
\[
\lim_{t_j \to \infty} \int_X 2 \zeta (\chi + \sqrt{-1} \partial \overline{\partial} \varphi) \wedge \omega_{SF} = \int_X \zeta e^t \frac{\partial \varphi}{\partial t} \frac{\Omega}{\chi \wedge \omega_{SF}} \chi \wedge \omega_{SF}.
\]

since \( \frac{\partial \varphi}{\partial t}(t_j, \cdot) \to 0 \). By taking the convergent subsequence we have
\[
\frac{\chi + \sqrt{-1} \partial \overline{\partial} \varphi}{\chi} = e^{\varphi - \varphi_{\infty}} \frac{\Omega}{2 \chi \wedge \omega_{SF}}.
\]

Notice \( \varphi_{\infty} \) is uniformly \( C^0 \) bounded since \( \varphi \) is uniformly bounded in \( C^0(X) \) along the Kähler-Ricci flow. Also by the uniqueness of the solution for the equation above, we have proved the theorem. \( \square \)

### 6.2 Uniform convergence

In this section we will prove a uniform convergence of the Kähler-Ricci flow. Since \( \varphi \) and \( \varphi_{\infty} \) are both uniformly bounded on \( X \). Therefore for any \( \epsilon > 0 \), there exists \( r_\epsilon > 0 \) with \( \lim_{\epsilon \to 0} r_\epsilon = 0 \), such that for any \( z \in \cup_{t=1}^{\mu} B_{r_\epsilon} (p_t) \) and \( t > 0 \) we have
\[
(\varphi - \varphi_{\infty} + \epsilon \log |S|_{h_t}^2)(t, z) < -1 \quad \text{and} \quad (\varphi - \varphi_{\infty} - \epsilon \log |S|_{h_t}^2)(t, z) > 1,
\]

where \( B_{r_\epsilon}(p_t) \) is a geodesic tube centered at the singular fibre \( X_{p_t} \) with radius \( r_\epsilon \) with respect to the metric \( \omega_0 \). Suppose the semi-flat closed form is given by \( \omega_{SF} = \omega_0 + \sqrt{-1} \partial \overline{\partial} \rho_{SF} \) and \( \rho_{SF} \) blows up near the singular fibres. We can always find an approximation \( \rho_\epsilon \) for \( \rho_{SF} \) such that \( \rho_\epsilon \) is smooth on \( X \) and on \( X \setminus \cup_{t=0}^{\mu} B_{r_\epsilon} (p_t) \)
\[
\rho_\epsilon = \rho_{SF}.
\]

We also define \( \omega_{SF, \epsilon} = \omega_0 + \sqrt{-1} \partial \overline{\partial} \rho_\epsilon \). Now we define the twisted difference of \( \varphi \) and \( \varphi_{\infty} \) by
\[
\psi^-_\epsilon = \varphi - \varphi_{\infty} - e^{-t} \rho_\epsilon + \epsilon \log |S|_{h_t}^2 \quad \text{and} \quad (6.2)
\]
\[
\psi^+_\epsilon = \varphi - \varphi_{\infty} - e^{-t} \rho_\epsilon - \epsilon \log |S|_{h_t}^2 \quad \text{and} \quad (6.3)
\]

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Lemma 6.3 There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there exists $T_\epsilon > 0$ such that for and $z \in X$ and $t > T_\epsilon$ we have

\[ \psi_\epsilon^-(t, z) \leq 4\epsilon \quad \text{and} \quad \psi_\epsilon^+(t, z) \geq -4\epsilon. \]  
(6.4)

\( \psi_\epsilon \)

Proof The evolution for $\psi_\epsilon^-$ is given by

\[
\frac{\partial \psi_\epsilon^-}{\partial t} = \log \frac{e^t (\chi_\infty + \epsilon \chi + e^{-t} \omega_{SF, \epsilon} + \sqrt{-1 i \partial \bar{\partial} \psi_\epsilon^-})^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2.
\]  
(6.6)

Since $\rho_\epsilon$ is bounded on $X$, we can always choose $T_1 > 0$ sufficiently large such that for $t > T_1$ we have $\psi_\epsilon^-(t, z) < -\frac{1}{2}$ on $\cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ and $e^{-t} \frac{\omega^2_{SF}}{\chi_\infty \wedge \omega_{SF}} \leq \epsilon$ on $X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$. We will discuss in two cases for $t > T_1$.

1. If $\psi_{\epsilon, \max}(t) = \max_X \psi_\epsilon^-(t, \cdot) = \psi_\epsilon^-(t, z_{\max, t}) > 0$ for all $t > T_1$. Then $z_{\max, t} \in X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ for all $t > T_1$. Applying the maximum principle at $z_{\max, t}$ we have

\[
\frac{\partial \psi_\epsilon^-}{\partial t}(t, z_{\max, t}) \leq \{ \log \frac{e^t (\chi_\infty + \epsilon \chi + e^{-t} \omega_{SF, \epsilon})^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \}(t, z_{\max, t})
\]

\[
= \{ \log \frac{(\chi_\infty + \epsilon \chi) \wedge \omega_{SF, \epsilon} + e^{-t} \frac{\omega^2_{SF}}{\omega_{SF}}}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \}(t, z_{\max, t})
\]

\[
= \{ \log \frac{(\chi_\infty + \epsilon \chi) \wedge \omega_{SF} + e^{-t} \frac{\omega^2_{SF}}{\omega_{SF}}}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2 \}(t, z_{\max, t})
\]

\[
\leq -\psi_\epsilon^-(t, z_{\max, t}) + \log(1 + 2\epsilon) + \epsilon.
\]

Applying the maximum principle again, we have

\[
\psi_\epsilon^- \leq 4\epsilon + O(e^{-t}).
\]  
(6.7)

2. If there exists $t_0 \geq T_1$ such that $\max_{z \in X} \psi_\epsilon^-(t_0, z) = \psi_\epsilon^-(t_0, z_0) < 0$ for some $z_0 \in X$. Assume $t_1$ is the first time when $\max_{z \in X, t \leq t_1} \psi_\epsilon^-(t, z) = \psi_\epsilon^-(t_1, z_1) > 4\epsilon$. Then $z_1 \in X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ and applying the maximum principle we have

\[
\psi_\epsilon^-(t_1, z_1) \leq \{ \log \frac{(\chi_\infty + \epsilon \chi) \wedge \omega_{SF} + e^{-t_1} \frac{\omega^2_{SF}}{\omega_{SF}}}{\chi_\infty \wedge \omega_{SF}} + \epsilon \log |S|_h^2 \}(t_1, z_1)
\]

\[
\leq \log(1 + 2\epsilon) + \epsilon < 4\epsilon.
\]

which contradicts the assumption that $\psi_\epsilon^-(t_1, z_1) > 4\epsilon$. Hence we have

\[
\psi_\epsilon^- \leq 4\epsilon.
\]  
(6.8)

By the same argument we have

\[
\psi_\epsilon^+ \geq -4\epsilon.
\]  
(6.9)

This completes the proof. \( \square \)

Lemma 6.4 We have the point-wise convergence of $\varphi$ on $X_{reg}$. Namely, for any $z \in X_{reg}$ we have

\[
\lim_{t \to \infty} \varphi(t, z) = \varphi_\infty(z).
\]  
(6.10)
Proof By lemma \ref{lemma}, we have for $t > T_\varepsilon$

$$\varphi_\infty(t, z) + \varepsilon \log |S|^2_{(t, z)} - 4\varepsilon \leq \varphi(t, z) \leq \varphi_\infty(t, z) - \varepsilon \log |S|^2_{(t, z)} + 4\varepsilon. \quad (6.11)$$

Then the lemma is proved by letting $\varepsilon \to 0$.

Since we have the uniform zeroth and second order estimates for $\varphi$ away from the singular fibres, we derive our main theorem.

Theorem 6.2 $\varphi$ converges to the pullback of a function $\varphi_\infty$ given by equation \ref{equation} on $\Sigma$ uniformly on any compact set of $X_{reg}$ in the sense of $C^{1,1}$.

7 An alternative deformation and large complex structure limits

Mirror symmetry and the SYZ conjecture make predictions for Calabi-Yau manifolds with "large complex structure limit point" (cf. \cite{SYZ}). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov-Hausdorff sense to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. \cite{GW}) in the case of K3 surfaces.

The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Ooguri-Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of K3 surfaces is McLean's metric.

In this section, we will apply a deformation for a family of Calabi-Yau metrics and derive Mclean's metric \cite{Mc} without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a K3 surface $X$ can be identified as the mirror to the large Kähler limit of $X$ as shown in \cite{GW}, so we can fix the complex structure on $X$ and deform the Kähler class to infinity. Let $f : X \to \mathbb{C}P^1$ be an elliptic K3 surface. Let $\chi \geq 0$ be the pullback of a Kähler form on $\mathbb{C}P^1$ and $\omega_0$ be a Kähler form on $X$. We construct a reference Kähler metric $\omega_t = \chi + t\omega_1$ and $[\omega_t]$ tends to $[\chi]$ as $t \to 0$. We can always scale $\omega_1$ so that the volume of each fibre of $f$ with respect to $\omega_t$ is $t$. Suppose $\Omega$ is a Ricci-flat volume form on $X$ with $\partial\bar{\partial}\log \Omega = 0$. Then Yau's proof \cite{Ya2} of Calabi's conjecture yields a unique solution $\varphi_t$ to the following Monge-Ampère equation for $t \in (0, 1]$

$$\left\{ \begin{array}{l}
\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^2}{\Omega} = C_t \\
\int_X \varphi_t \Omega = 0,
\end{array} \right. \quad (7.1)$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t, \cdot) = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$.

The following theorem is the main result of this section.

Theorem 7.1 Let $f : X \to \mathbb{C}P^1$ be an elliptically fibred K3 surface with 24 singular fibres of type $I_1$. Then the Ricci-flat metrics $\omega(t, \cdot)$ converges to the pullback of a Kähler metric $\tilde{\omega}$ on $\mathbb{C}P^1$ in any compact set of $X_{reg}$ in $C^{1,1}$ as $t \to 0$. The Kähler metric $\tilde{\omega}$ on $\mathbb{C}P^1$ satisfies the equation

$$Ric(\tilde{\omega}) = \omega_{WP}. \quad (7.2)$$

Proof All the estimates can be obtained by the same argument in Section 4 with little modification. It is relatively easy compared to the Kähler-Ricci flow because there is no such a term.
We apply similar argument Section 6.1 to prove the weak convergence. It is not difficult to show that for any test function \( \zeta \in C_0^\infty(\mathbb{CP}_1^{reg}) \) we have
\[
\int_X 2\zeta(\chi + \sqrt{-1} \partial \bar{\partial} \varphi_t) \wedge \omega_{SF} = \int_X \zeta \Omega + O(t).
\]
By taking the convergent subsequent we have
\[
\frac{\chi + \sqrt{-1} \partial \bar{\partial} \varphi_0}{\chi} = \frac{\Omega}{2 \chi \wedge \omega_{SF}}.
\]
Since such \( \varphi_0 \) with bounded \( \| \varphi_0 \|_{L^\infty} \) is unique and we have the 2nd order estimates for \( \varphi_t \), the convergence is then uniform and this completes our proof. \( \square \)

This limit metric \( \tilde{\omega} \) coincides with McLean’s metric as obtained by Gross and Wilson \cite{GrWi}. Their construction is certainly much more delicate and gives an accurate approximation near the singular fibres by the Ooguri-Vafa metrics. Also Mclean’s metric is an example of the generalized Kähler-Einstein metric defined in Definition 5.2 satisfying
\[
\text{Ric}(\omega) = -\lambda \omega + \omega_{WP}
\]
when \( \lambda = 0 \).

8 Generalizations and problems

8.1 A metric classification for surfaces of non-negative Kodaira dimension

In this section we will give a metric classification for surfaces of non-negative Kodaira dimension. Any surface \( X \) with \( K_X \geq 0 \) must be minimal and \( \nu(X) > 0 \).

1. When \( \nu(X) = 2 \), \( X \) is a minimal surface of general type and we have the following theorem.

**Theorem 8.1** \cite{TiZha} If \( X \) is a minimal complex surface of general type, then the global solution of the Kähler-Ricci flow converges to a positive current \( \omega_\infty \) which descends to the Kähler-Einstein orbifold metric on its canonical model. In particular, \( \omega_\infty \) is smooth outside finitely many rational curves and has local continuous potential.

2. When \( \nu(X) = 1 \), \( X \) is a minimal elliptic surface. Theorem 1.1 shows that the Kähler-Ricci flow deforms any Kähler metric to a unique generalized Kähler-Einstein metric \( \omega \) on its canonical model \( X_{can} \).

3. When \( \nu(X) = 0 \), \( X \) is a Calabi-Yau surface. The Kähler-Ricci defined in \cite{Ca} deforms any Kähler metric to a Calabi-Yau metric in the same Kähler class.

When \( X \) is not minimal, the Kähler-Ricci flow \cite{Li2} will develop singularities at finite time. Let \( \omega_0 \) be the initial Kähler metric and \( T \) be the first time such that \( e^{-t}[\omega_0] - (1 - e^{-t})2\pi c_1(X) \) fails to be a Kähler class. The Kähler-Ricci flow has a smooth solution \( \omega(t, \cdot) \) on \([0, T)\) converging to a degenerate metric as \( t \) tends to \( T \) (cf. \cite{TiZha}, also see \cite{CaLa}). This degenerate metric is actually smooth outside a subvariety \( C \). Such a \( C \) is characterized by the condition that \( e^{-T}[\omega_0] - (1 - e^{-T})2\pi c_1(X) \) vanishes along \( C \). This implies that \( C \) is a disjoint union of finitely many rational curves with self-intersection \(-1\). Then we can blow down these \((-1)\)-curves and
obtain a complex surface \( X' \) and \( e^{-T}[\omega_0] - (1 - e^{-T})2\pi c_1(X) \) descends to a Kähler class on \( X' \). Choose a Kähler metric \( \omega_T \) representing this class and then \( (1 - e^{-T})^{-1}(\omega_T - e^{-T}\omega_0) \) represents \(-2\pi c_1(X)\). Define 
\[
\omega_t(\cdot) = \frac{e^{-t} - e^{-T}}{1 - e^{-T}}\omega_0 + \frac{1 - e^{-t}}{1 - e^{-T}}\omega_T
\]
and write \( \omega(t, \cdot) = \omega_t(\cdot) + \sqrt{-1}\partial \bar{\partial} \varphi(t, \cdot) \), then \( \varphi \) solves (2.8) on \([0, T)\), moreover, \( \varphi \) converges to a bounded function \( \varphi(T, \cdot) \) as \( t \) tends to \( T \) and \( \varphi(T) \) is smooth outside \( C \). Applying the above vanishing property of \( C \) (cf. [TiZha]), one can show that \( \varphi(T, \cdot) \) descends to a continuous function on \( X' \). We believe that \( \varphi(T, \cdot) \) is actually \( C^{1,1} \). Since \( \omega_T \) is a smooth metric on \( X' \), we can consider the flow (2.8) on \( X' \) with \( \varphi(T, \cdot) \) as the initial potential. We expect that (2.8) still has a unique solution on \((0, T')\) where \( T' \) is the first time such that \( e^{-t}[\omega_T] - (1 - e^{-t})2\pi c_1(X') \) fails to be a Kähler class, hence, [14] has a smooth solution \( \omega(t, \cdot) \) on \( X' \times (0, T') \). Either \( T' \) is \( \infty \) or we can repeat the previous procedure and continue the flow [14]. After finitely many times, we will get a minimal complex surface with non-negative Kodaira dimension. Then the flow has a global solution which falls into one of the cases described above. In order to complete this, one needs to do the following steps: 1. prove an optimal estimate for \( \varphi(T, \cdot) \); 2. prove that the flow (2.8) has a unique solution under weaker assumptions on smoothness of initial data \( \varphi(T, \cdot) \). Of course, if one can get sufficient regularity in step 1, step 2 follows from the standard theory of parabolic equations. We will address these problems in a forthcoming paper.

### 8.2 Higher dimension

In this section, we discuss possible generalizations of Theorem [14] for higher dimension. First, as we assumed in Section 5.1, let \( X \) be a non-singular variety of \( \dim X = n \) such that \( \mu K_X \) is base point free for \( \mu \) sufficiently large. Then the pluricanonical map defines a holomorphic fibration \( f: X \to X_{can} \) by the linear system \( |\mu K_X| \), where \( X_{can} \) is the canonical model of \( X \).

1. If \( \nu(X) = n \), \( K_X \) is big and nef. Hence \( X \) is a minimal variety of general type. The Kähler-Ricci flow will deform any Kähler metric to a canonical Kähler-Einstein metric on \( X \) (cf. [TiZha], [Ts]).

2. If \( \nu(X) = 1 \), \( X_{can} \) is a curve. With little modification of the proof, Theorem [14] can be generalized and the Kähler-Ricci flow will converge.

3. If \( 1 < \mu(X) < n \), the fibration structure of \( f \) can be very complicated. A large number of the calculations can be carried out as in this paper and we expect the Kähler-Ricci flow will converge appropriately to the pullback of a canonical metric \( \omega_\infty \) on \( X_{can} \) such that \( Ric(\omega_\infty) = -\omega_\infty + \omega_{WP} \) on \( X_{can} \).

When \( K_X \) is nef, the Kähler-Ricci flow has long time existence, yet it does not necessarily converge, although the abundance conjecture predicts that \( \mu K_X \) is globally generated for \( \mu \) sufficiently large. Hence, the problem of convergence of the Kähler-Ricci flow for nef \( K_X \) can be considered as the analytic version of the abundance conjecture. If \( K_X \) is not nef, the flow will develop finite time singularities. Let \( \omega_0 \) be the initial Kähler metric and \( T \) the first time such that \( e^{-t}[\omega_0] - (1 - e^{-t})2\pi c_1(X) \) fails to be a Kähler class. The potential \( \varphi(T, \cdot) \) is bounded and smooth outside an analytic set of \( X \) (cf. [TiZha]). Let \( X_T \) be the metric completion of \( \omega(T, \cdot) \). We conjecture that this is an analytic variety. It might be a flip of \( X \), or a variety obtained by certain standard algebraic procedure. Of course \( X_T \) might have singularities and it is not clear at all how to develop the notion of a weak Ricci flow on a singular variety. Suppose such
a procedure can be achieved and the Kähler-Ricci flow can continue on $X_1$, then after applying the above procedure finitely many times on $X_1$, $X_2$, ..., $X_N$, we might have $K_{X_N} \geq 0$ and get the minimal model of $X$.

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**References**

[Au] Aubin, T. *Equations du type Monge-Ampère sur les variétés Kähleriennes compacts*, Bull. Sc. Math. 102 (1976), 119-121.

[BaMu] Bando, S. and Mabuchi, T., *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[BaHuPeVa] Barth, W., Hulek, K., Peters, C., and Van De Ven, A., *Compact complex surfaces*, 2003, Springer.

[Ca] Cao, H., *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985), no. 2, 359–372.

[CaLa] Cascini, P. and La Nave, P., *Kähler-Ricci Flow and the Minimal Model Program for Projective Varieties*, preprint.

[Ch] Chow, B., *The Ricci flow on the 2-sphere*, J. Differential Geom. 33 (1991), no. 2, 325–334.

[ChTi] Chen, X.X. and Tian, G., *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. 147 (2002), no. 3, 487–544.

[ChYa] Cheng, S. Y. and Yau, S. T., *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), no. 3, 333–354.

[CiKoMo] Clemens, H., Kollar, J. and Mori, S., *Higher-dimensional complex geometry*, Astérisque No. 166 (1988), 144 pp. (1989).

[Do] Donaldson, S. K., *Scalar curvature and projective embeddings, I.*, J. Differential Geom. 59 (2001), no. 3, 479-522.

[GrWi] Gross, M. and Wilson, P. M. H., *Large complex structure limits of K3 surfaces*, J. Differential Geom. 55 (2000), no. 3, 475–546.

[Ha1] Hamilton, R., *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255–306.

[Ha2] Hamilton, R., *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.
[Hö] Hörmander, L., *An introduction to complex analysis in several variables*, Van Nostrand, Princeton 1973.

[Ko] Kōdaira, K., On compact complex analytic surfaces I, Ann. of Math. (2) 71 (1960), 111–152. II, Ann. of Math. (2) 77 (1963), 563–626. III, Ann. of Math. (2) 78 (1963), 1–40.

[Kol1] Kolodziej, S., *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69–117.

[Kol2] Kolodziej, S., *Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds*, preprint.

[KoSo] Kontsevich, M. and Soibelman, Y., *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publishing, River Edge, NJ, 2001.

[LiYa] Li, P. and Yau, S.T., *Estimates of eigenvalues of a compact Riemannian manifold*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 205–239, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[Mi] Miranda, R., *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research] ETS Editrice, Pisa, 1989. vi+108 pp.

[Mc] McLean, R., *Deformations of calibrated submanifolds*, Comm. Anal. Geom. 6 (1998), no. 4, 705–747.

[OoVa] Ooguri, H. and Vafa, C., *Summing up Dirichlet instantons*, Phys. Rev. Lett. 77 (1996), no. 16, 3296–3298.

[Pe] Perelman, P., *The entropy formula for the Ricci flow and its geometric applications*, preprint math.DG/0211159.

[PhSt] Phong, D. H. and Sturm, J., *On stability and the convergence of the Kähler-Ricci flow*, preprint math.DG/0412185.

[SeTi] Sesum, N. and Tian, G., *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman)*, lecture note.

[Si] Siu, Y-T. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser Verlag, Basel 1987.

[SoWe] Song, J. and Weinkove, B., *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, preprint math.DG/0410418.

[StYaZa] Strominger, A., Yau, S.T. and Zaslow, E., *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

[Ti1] Tian, G., *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. 89 (1987), no. 2, 225–246.

[Ti2] Tian, G., *On Calabi’s conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101, no. 1 (1990), 101-172.
[TiZha] Tian, G. and Zhang, Z., *A note on the Kähler-Ricci flow on projective manifolds of general type*, preprint.

[TiZhu] Tian, G. and Zhu, X., *Convergence of Kähler Ricci flow*, preprint.

[Ts] Tsuji, H. *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), 123-133.

[Ya1] Yau, S.T., *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. 100 (1978), no. 1, 197–203.

[Ya2] Yau, S.T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. 31 (1978), 339-411.

[Ya3] Yau, S.T., *Open problems in geometry*, Proc. Symposia Pure Math. 54 (1993), 1-28.

[Zh] Zhang, Z., Thesis at MIT.