A unique Fock quantization for fields in non-stationary spacetimes

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In curved spacetimes, the lack of criteria for the construction of a unique quantization is a fundamental problem undermining the significance of the predictions of quantum field theory. Inequivalent quantizations lead to different physics. Recently, however, some uniqueness results have been obtained for fields in non-stationary settings. In particular, for vacua that are invariant under the background symmetries, a unitary implementation of the classical evolution suffices to pick up a unique Fock quantization in the case of Klein-Gordon fields with time-dependent mass, propagating in a static spacetime whose spatial sections are three-spheres. In fact, the field equation can be reinterpreted as describing the propagation in a Friedmann-Robertson-Walker spacetime after a suitable scaling of the field by a function of time. For this class of fields, we prove here an even stronger result about the Fock quantization: the uniqueness persists when one allows for linear time-dependent transformations of the field in order to account for a scaling by background functions. In total, paying attention to the dynamics, there exists a preferred choice of quantum field, and only one SO(4)-invariant Fock representation for it that respects the standard probabilistic interpretation along the evolution. The result has relevant implications e.g. in cosmology.

1. INTRODUCTION

In inhomogeneous cosmology, the study of quantum phenomena is often affected by two related types of ambiguities which influence the physical predictions, and which appear owing to the infinite number of degrees of freedom that are present in the inhomogeneities. The first of these ambiguities concerns the choice of variables (usually scalar or tensor fields) which are employed to describe, or parameterize, the local degrees of freedom of the system. The second ambiguity is generic in quantum field theory, and concerns the selection of an appropriate quantum representation for the field theory obtained once a choice of field variables is made.

The freedom in the choice of fields has in fact potentially non-trivial consequences. Namely, in non-stationary and spatially homogeneous scenarios, like those provided by the typical kind of background solutions encountered in cosmology, different choices of “fundamental” fields arise naturally from time-dependent transformations, each of them leading in general to different dynamics. Normally, these transformations consist in a scaling of the fields by time-dependent functions, so that the linearity of the field equations and of the structures of the system are maintained. But, while in theories with a finite number of degrees of freedom these linear transformations can be promoted generally to quantum unitary operators, the situation changes drastically in quantum field theory. In this latter case, linear transformations do not always admit a unitary implementation and distinct dynamics usually call for inequivalent representations. Therefore, it is clear that the final quantum theory depends intimately on which fields (and hence on which dynamics) are viewed as the fundamental ones.

An important class of systems for which this kind of considerations has a major relevance are quantum matter fields propagating in a classical non-stationary spacetime, like e.g. the case of matter fields in inflationary backgrounds. Another large class is found in the quantization of local gravitational degrees of freedom (in contraposition to the quantization of matter in gravitational backgrounds). This includes inhomogeneous models with certain types of symmetries, such as the so-called Gowdy models [1], as well as the quantization of gravitational perturbations around homogeneous, cosmological backgrounds. For this latter class, the parametrization of the metric components by means of appropriate fields is only constrained by symmetry arguments, usually leaving an ample freedom in the choice that can be reinterpreted in terms of a time-dependent scaling of the fields. In addition, in

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the case of matter fields, it is often convenient to scale them by a suitable combination of background functions, typically related to the scale factor of the background spacetime, but whose specific form depends on the particular system under discussion.\footnote{As an example, let us mention the case of the gauge-invariant energy density perturbation amplitude in perfect fluid cosmologies; see e.g. [3].}

For this kind of non-stationary settings, the commented freedom in the choice of fields often allows one to select –as fundamental one– fields which effectively live in an auxiliary static background, although generally subject to time-dependent potentials. This permits one to simplify the corresponding dynamics at least partially. For instance, let us mention again the case of the Gowdy cosmologies [1], which are spacetimes with two spacelike Killing isometries and a compact spatial topology. In the particular case of a three-torus topology and a content of linearly polarized gravitational waves, the local gravitational degrees of freedom of these cosmologies can be described by a single scalar field defined on the circle. The field evolves with respect to a quadratic and explicitly time-dependent Hamiltonian which, for a particular choice of field parametrization, can be seen as the Hamiltonian of a free particle on the circle with a time-dependent mass [3]. Similar descriptions can be obtained for the Gowdy models with the other possible spatial topologies, namely the three-sphere and the three-handle [1], with the difference that in those cases the field is effectively defined on the sphere, \(S^3\).

Also in the case of (inhomogeneous) cosmological perturbations, and more straightforwardly for quantum test fields in a non-stationary gravitational background, one can map the theory to a field model in a static background. For instance, for scalar matter fields coupled to Friedmann-Robertson-Walker (FRW) spacetimes (with applications e.g. to inflation [4–6]), a Klein-Gordon equation for the matter field becomes, after a re-scaling in terms of the scale factor, a linear wave equation with a time-dependent mass, but now in a static auxiliary background with the same spatial topology and dimension as the FRW cosmology. On general grounds, the dynamics of a massive scalar field \(\varphi\) in an FRW spacetime \(ds^2 = a^2(t)\left(-dt^2 + \gamma_{ij}dx^idx^j\right)\) –where \(\gamma_{ij}\) \((i = 1, 2, 3)\) is the standard Riemannian metric of either a three-sphere, a three-dimensional flat space, or a three-dimensional hyperboloid– is governed by the equation

\[
\dot{\varphi} + 2\frac{\dot{a}}{a}\varphi - \Delta \varphi + m^2 a^2 \varphi = 0. \tag{1}
\]

The dot denotes time derivative with respect to the conformal time \(t\), and \(\Delta\) is the Laplace-Beltrami operator associated with the spatial metric \(\gamma_{ij}\). It is then straightforward to check that, if one introduces the time-dependent scaling \(\phi = a\varphi\), the dynamics in the new field description is dictated by

\[
\dot{\phi} - \Delta \phi + s(t)\phi = 0, \tag{2}
\]

where \(s(t) = m^2 a^2 - \left(\dot{a}/a\right)\). The system can now be treated as a Klein-Gordon field propagating in a static background \(ds^2 = -dt^2 + \gamma_{ij}dx^idx^j\) but in the presence of a time-varying potential \(V(\phi) = s(t)\phi^2/2\). Notice that \(s(t)\) can be interpreted as a nonnegative time-dependent mass \(m(t) = s^{1/2}(t)\), provided that \(s(t) \geq 0\). The same arguments can be applied when the scalar field coupled to the FRW spacetime is also subject to a time-dependent potential quadratic in the field.

On the other hand, as we have remarked, even if the field parametrization of the system is specified, there exists a second type of ambiguity that affects the physical predictions. This concerns the selection of a quantum representation for the field theory in hand: in the case of fields, with local degrees of freedom, the system admits infinite non-equivalent representations of the canonical commutation relations (CCR’s) [7] and there is no general procedure to select a preferred quantum description.

The usual strategy to pick up a distinguished representation in a given field theory is to exploit the classical symmetries. For instance, invariance under the Poincaré group, adapted to the theory under consideration, is the criterion imposed to arrive at a unique representation in ordinary quantum field theory. In particular, for a scalar field theory, this imposition of Poincaré invariance selects a complex structure [7], which is the mathematical object that encodes the ambiguity in the quantization and determines the vacuum state of the Fock representation.\footnote{Poincaré invariance selects in fact a continuous family of representations, characterized by a mass parameter. A unique element of that family is obtained when the Poincaré group is properly adapted to the dynamics of the system, since one of the generators of the group is a time-like Killing vector which provides a natural decomposition in positive and negative frequencies.}

For systems with time translation invariance, this symmetry is also exploited in order to formulate the so-called energy criterion and then choose a preferred complex structure [8]. But when the symmetries are restricted, as it is the case for non-stationary spacetimes or for manifestly time-dependent systems, extra requirements must be imposed to complete the quantization process. E.g., for a free scalar field in \(1 + 1\) de Sitter spacetime, a unique de Sitter invariant Fock vacuum is selected by looking for an invariant Gaussian solution to a regulated Schrödinger equation [9].

Recently, the unitary implementation of the dynamics has been successfully employed as an additional criterion to specify a unique Fock quantization for the linearly polarized Gowdy model with three-torus topology. In particular, it has been shown that there is essentially one field parametrization and one Fock representation which is invariant under the symmetries of the (auxiliary) background and allows a unitary implementation of the corresponding field dynamics [10,11]. Thus, both ambiguities have been resolved in that case. Furthermore, by demanding a unitary dynamics and invariance under the classical symmetries,
unique Fock representations have been specified as well for free scalar fields with generic time-dependent mass terms defined on the circle, the two-sphere, or the three-sphere [12], thus endorsing the unitarity criterion. As we have pointed out, the case of the two-sphere covers the other two possible topologies of the Gowdy cosmologies. On the other hand, a scalar field with time-dependent mass on the three-sphere not only describes test fields in an FRW spacetime, but in addition finds applications in the treatment of cosmological perturbations [4, 13]. In this perturbative framework realistic scenarios are explored by considering small inhomogeneous departures from FRW spacetimes, and one frequently has to deal with linear wave equations with a quadratic time-dependent potential (for a brief account, see e.g. [12]). Thus, as far as this entire class of cosmological systems is concerned, and restricting the discussion to the specific field parametrization considered, our previous results assert that there is a unique Fock representation such that the vacuum state has the symmetries of the background and the evolution is unitarily implementable.

In this article we go beyond those results by analyzing the remaining freedom in the choice of fields. As we have already mentioned, the freedom that we are interested in considering amounts to time-dependent scalings of the field, which respect the linearity of the field equations and hence of the space of solutions. We will therefore study the most general compatible linear canonical transformation, which consists of a time-dependent scaling of the configuration variable, the inverse scaling of its canonical momentum, and the possible addition to this momentum of a time-dependent contribution which is also linear in the configuration variable. We will show that, unless the transformation is trivial, the dynamics of the transformed field is such that it is impossible to find a Fock representation which is invariant under the background symmetries and allows a unitary implementation of the evolution. In this respect, the inclusion in the transformed momentum of a contribution linear in the configuration variable is important because, in practice, one usually starts with a field description other than the one with the privileged dynamics. Then, in general, the latter can be reached only if such a contribution is taken into account.3

2. CLASSICAL CONSIDERATIONS

Before we turn to the proof of our statement about the uniqueness of the choice of field under time-dependent scalings in the case of the three-sphere, let us describe the classical set-up of our theory.

We consider a real scalar field on $S^3$, with field equation

$$\ddot{\phi} - \Delta \phi + s(t) \phi = 0,$$

where the dot stands for the time derivative, $\Delta$ is now the Laplace-Beltrami operator corresponding to the standard metric on $S^3$ and $s(t)$ is an arbitrary time function (apart from some extremely mild conditions about its derivative, stated in [12]). After a decomposition of the field in terms of (hyper)spherical harmonics (see e.g. [14]), the degrees of freedom are encoded in a discrete set of modes $q_{n\ell m}$, where $n$ is any non-negative integer, and the other two integers $\ell$ and $m$ vary from zero to $n$ and from $-\ell$ to $\ell$, respectively. The modes obey decoupled equations of motion

$$\ddot{q}_{n\ell m} + [\omega_n^2 + s(t)]q_{n\ell m} = 0,$$

where $\omega_n^2 = n(n + 2)$ are the eigenvalues of the Laplace-Beltrami operator. These equations are independent of the labels $\ell$ and $m$, and therefore, for each $n$, there is a degeneracy of $g_n = (n + 1)^2$ modes with the same dynamics.

We restrict our analysis to the sector of non-zero modes from now on. Obviously, the inclusion or not of the zero mode does not affect the properties related to the presence of an infinite number of degrees of freedom. Let us then introduce the annihilation-like variables

$$a_{n\ell m} = \frac{1}{\sqrt{2\omega_n}} (\omega_n q_{n\ell m} + i p_{n\ell m})$$

(5)

which, together with their complex conjugates (the creation-like variables $a^*_{n\ell m}$), provide a complete set of kinematical variables, or coordinates, for the phase space of the considered sector. Here, $p_{n\ell m} = q_{n\ell m}$ is the canonically conjugate momentum of the configuration variable $q_{n\ell m}$. The evolution of these variables can be expressed as a linear transformation which is block-diagonal (i.e., respects the harmonic labels) and has the form

$$a_{n\ell m}(t) = a_n(t)a_{n\ell m}(t_0) + \beta_n(t)a^*_{n\ell m}(t_0).$$

(6)

The time functions $\alpha_n(t)$ and $\beta_n(t)$, which depend as well on the reference time $t_0$, completely characterize the classical evolution operator (see [12] for details).

3 Related to this fact, this kind of contribution is necessary if one wants to consider time-dependent transformations between field descriptions whose quadratic Hamiltonian is free of undesirable terms containing the product of the configuration and momentum field variables (see e.g. the discussion in [13]).
According to our previous discussion, we now consider a time-dependent canonical transformation of the form

\[ \varphi := f(t)\phi, \quad P_\varphi := \frac{P_\phi}{f(t)} + g(t) \sqrt{f(t)} \phi, \]

where \( h \) is the determinant of the metric on \( S^3 \). Our aim is to investigate the unitary implementability of the dynamics of the new canonical fields \( \varphi, P_\varphi \) obtained with this transformation. It is shown in [11] that there is no loss of generality in arbitrarily fixing the transformation at the reference time \( t_0 \) so that \( f(t_0) = 1 \) and \( g(t_0) = 0 \), conditions that we assume in the following. Furthermore, we require that the functions \( f \) and \( g \) be real and differentiable, and that \( f(t) \) vanish nowhere, so that the transformation does not spoil the differential formulation of the field theory, nor introduces singularities.

The dynamics of the transformed fields admits a description like (6), but with new coefficients \( \tilde{\alpha}_n(t) \) and \( \tilde{\beta}_n(t) \), related to the old ones by:

\[ \tilde{\alpha}_n(t) = f_+(t)\alpha_n(t) + f_-(t)\beta_n^*(t) + \frac{i}{2} \frac{g(t)}{\omega_n}[\alpha_n(t) + \beta_n^*(t)], \]

\[ \tilde{\beta}_n(t) = f_+(t)\beta_n(t) + f_-(t)\alpha_n^*(t) + \frac{i}{2} \frac{g(t)}{\omega_n}[\alpha_n^*(t) + \beta_n(t)], \]

where \( 2f_+(t) := f(t) \pm 1/f(t) \).

### 3. UNIQUENESS OF THE QUANTIZATION

#### 1. Symmetries and unitarity condition

The possible Fock quantizations of the system are effectively determined by the different complex structures that can be defined on phase space. Strictly speaking, a complex structure \( J \) is a real linear transformation on phase space which is compatible with its canonical symplectic structure, and whose square is minus the identity, \( J^2 = -1 \). Together with the symplectic structure, a complex structure defines a state, usually called the vacuum, and hence a representation of the CCR’s.\(^5\) The corresponding positive and negative frequency components are obtained, respectively, with the projections \( (1 - iJ)/2 \) and \( (1 + iJ)/2 \).

For the systems under consideration, as we have seen, one can coordinate the phase space by the creation and annihilation-like variables \( \{a_n, a_n^\dagger\} \), associated with the harmonic modes. Using these variables, it was shown in [12] that the complex structures that remain invariant under the group \( SO(4) \) of rotations on the three-sphere can be parameterized by a sequence of complex pairs \( (\kappa_n, \lambda_n) \), with \( |\kappa_n|^2 - |\lambda_n|^2 = 1 \) for every integer \( n > 0 \). Given the relation between Fock quantizations and complex structures, it follows that the quantizations whose vacuum state is invariant under the symmetry of the field equations, \( SO(4) \), are again characterized by the sequence of pairs \( (\kappa_n, \lambda_n) \) [12].

Among the set of those \( SO(4) \)-invariant Fock representations, there still exist infinitely many non-equivalent ones. We then impose the unitary implementation of the dynamics as an additional criterion to select a Fock representation among them. Unitary implementation of a given canonical transformation essentially means that one is able to define in a consistent way a quantum version of the transformation by means of the action of a unitary operator. In the present case, we require this for all the canonical transformations generated by the dynamics, i.e. for the entire set of transformations [6] that describe the evolution to all possible final times \( t \).\(^6\) Let us then suppose that we have a certain Fock quantization, i.e. we are given a Hilbert space and operators \( \hat{a}_{ntm} \) and \( \hat{a}_{ntm}^\dagger \) corresponding to our classical observables. When the transformation [6] is applied to our quantum operators, we obtain new operators:

\[ \hat{a}_{ntm}(t) := \alpha_n(t)\hat{a}_{ntm} + \beta_n(t)\hat{a}_{ntm}^\dagger. \]

By construction, the set of operators \( \hat{a}_{ntm}(t) \), together with their adjoints, gives us a new representation of the CCR’s. We say that the classical transformation [6] admits a unitary implementation when this new representation is unitarily equivalent to the original one. If this is the case, there indeed exist unitary operators \( \hat{U}(t) \) corresponding to the classical transformation, i.e., such

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\(^4\) Namely, \( J \) must be a symplectomorphism and, if \( \Omega(\cdot, \cdot) \) is the symplectic form, the bilinear map \( \Omega(J \cdot, \cdot) \) must be positive definite. This requirement guarantees that \( \Omega(J^2 \cdot, \cdot) = \Omega(J \cdot, \cdot) \) provides an inner product.

\(^5\) To be precise, a complex structure determines a state of the Weyl algebra generated by the configuration and momentum field variables, from which a representation of the CCR’s can be defined (see e.g. [13] for a general introduction).

\(^6\) We emphasize that the functions appearing in [6] depend on the field parametrization. After the transformation [7], the general form of those functions is given in [6] and [8].
that $\tilde{a}_{nlm}(t) = \tilde{U}^{-1}(t)\tilde{a}_{nlm}\tilde{U}(t)$. In standard quantum mechanics the condition for unitary implementation is trivially satisfied but, as mentioned in the Introduction, this is not generally so when an infinite number of degrees of freedom are present.

As far as Fock quantizations are concerned, the unitary implementation of a linear canonical transformation essentially depends on the effect of the transformation on the vacuum of the considered representation. In fact, each Fock representation comes with its own set of creation and annihilation operators, and the application of a linear canonical transformation leads to new ones, like e.g. in (10). One can then ask whether the formal state in the kernel of all the new annihilation operators, interpretable as with its own set of creation and annihilation operators, and the application of a linear canonical transformation leads to new ones, a finite particle production. For a linear transformation of the form (10) and assuming that the original vacuum is annihilated by all the operators $a_{nlm}$, for instance, the condition for a unitary implementation becomes then just the summability over all modes $(n, l, m)$ of the square complex norm of the functions $b_n(t)$ corresponding to this transformation.

It is worth noticing that the criterion that we are using here is just the unitary implementation of the transformations which correspond to the evolution of the system in finite intervals of time, in opposition to the technically more involved condition of the existence of a well-defined self-adjoint quantum Hamiltonian (which in the cases under study is necessarily time dependent). For a discussion on the existence of a quantum Hamiltonian for linear theories see e.g. [17]. On a different context, the review article [18] also provides a modern approach to unitarity issues in field theory, starting from the time-dependent quantum harmonic oscillator.

2. Uniqueness result

We are now in adequate conditions to show that, after applying a transformation of the considered type (7), the dynamics becomes such that one cannot attain a unitary implementation of the corresponding evolution with respect to any $SO(4)$-invariant Fock representation.

Let us first make the unitary implementability condition fully explicit in our case. Suppose that we are given a $SO(4)$-invariant Fock representation of the CCR’s, determined by a sequence of pairs $(\kappa_n, \lambda_n)$ as explained above. It is shown in detail in [12] (see also [11]) that the dynamics associated with the transformed canonical pair $(\varphi, P_e)$ is unitarily implementable in the considered $SO(4)$-invariant Fock quantization if and only if the sequences $\{ \sqrt{\omega^2_n} \}$ are square summable over $n$ for all possible values of $t$, where

$$\tilde{b}_n^l(t) := (\kappa_n^2)^2 \tilde{b}_n^l(t) - \lambda_n^2 \tilde{b}_n^l(t) + 2i\kappa_n^2 \lambda_n \text{Im}[\tilde{a}_n(t)].$$

(11)

We will now see that this summability condition can only be fulfilled if the transformation (7) is in fact the identity transformation, i.e., if $f(t)$ is the identity function and $g(t)$ vanishes. The arguments, whose complete technical details will appear in [19], are analogous to those presented in [11].

So, let us assume that $\{ \sqrt{\omega^2_n(t)} \}$ is square summable at all times. This implies that, for every $t$, the terms of this sequence must tend to zero in the limit of infinite $n$. Then, the same must occur with $\tilde{b}_n^l(t)/(\kappa_n^2)^2$, since both $g_n$ and $|\kappa_n|$ are larger than 1. By substituting in the expression of $\tilde{b}_n^l(t)$ the asymptotic limits of $\alpha_n(t)$ and $\beta_n(t)$ (which were analyzed in [12]), as well as that of $\omega_n$, we conclude that the sequences given by

$$e^{\alpha_n^+\tau} - \frac{\lambda_n^2}{(\kappa_n^2)^2} e^{-\alpha_n^+\tau} \frac{f_n(t)}{2} - \frac{\lambda_n}{\kappa_n} \sin [(n + 1)\tau] f_n(t)$$

(12)

tend to zero for all values of $t$ when $n \to \infty$. Here, $\tau := t - t_0$.

We are now in a situation completely similar to that studied in [11]. Applying the type of arguments presented in Appendix A of that reference one can prove that, if (12) tends indeed to zero at all times, and hence the same happens with its imaginary part, then the sequences with terms $1 - \text{Re} \left[ \frac{\lambda_n^2}{(\kappa_n^2)^2} \right]$ and $\text{Im} \left[ \frac{\lambda_n^2}{(\kappa_n^2)^2} \right]$ cannot tend simultaneously to zero on any (infinite) subsequence of the natural numbers.

Let us now restrict our attention to expression (12) for the particular set of values $\tau = 2\pi q/p$, where $q$ and $p$ are arbitrary positive integers subject only to the condition that $t = \tau + t_0$ belongs to the allowed domain for this time parameter. For each fixed value of $p$, we then consider the subsequence of natural numbers of the form $n = mp - 1$, where $m$ can take any positive integer value. Since the real and imaginary parts of (12) must tend to zero on all subsequences, it follows that both

7 In rigorous terms, a canonical transformation $T$ is unitarily implementable in a Fock representation defined by a complex structure $J$ if and only if the operator $T + JTJ$ is of the Hilbert-Schmidt type, on the one-particle Hilbert space defined by $J$ [16].
\( \left(1 - \text{Re}\left[\frac{\lambda^2}{\omega^2}(\kappa^2)\right]\right) f. (t_0 + 2\pi q / p) \) and \( \text{Im}\left[\frac{\lambda^2}{\omega^2}(\kappa^2)\right] f. (t_0 + 2\pi q / p) \) must approach a vanishing limit as \( m \) goes to infinity, for every possible value of \( p \) and \( q \). However, we know that the time-independent coefficients in these expressions cannot tend simultaneously to zero on any subsequence of the natural numbers. Therefore, our conditions can only be fulfilled if \( f. (t_0 + 2\pi q / p) \) vanishes for all the possible values of \( p \) and \( q \) or, equivalently, if

\[
f^2 \left( t_0 + \frac{2\pi q}{p} \right) = 1. \tag{13}
\]

But, given that the set \( \{ t_0 + 2\pi q / p \} \) is dense and that \( f(t) \) is a continuous function with \( f(t_0) = 1 \), it follows that \( f(t) \) must be the unit function.

In order to show that the function \( g(t) \) in (17) vanishes, we go back and consider the sequences \( \sqrt{\pi} \beta_n(t) / (\kappa_n) \), but specialized now to the case \( f(t) = 1 \). We recall that the terms of these sequences must tend to zero at infinite \( n \), because of the square summability of \( \sqrt{\pi} \beta_n(t) \). Then, using again the asymptotic limits of \( \alpha_n(t) \) and \( \beta_n(t) \), we conclude that the sequences given by

\[
g(t) - 4 \frac{\lambda_n}{\kappa_n} \omega_n \sin [(n + 1)\tau] e^{-i(n+1)\tau} \tag{14}
\]

must have a vanishing limit as well. In obtaining this last expression, we have employed the fact that the constants \( \lambda_n \) must tend to zero when \( n \to \infty \), as it follows from the unitarity of the dynamics once \( f(t) = 1 \) has been established (see [11]).

Finally, it can be proved [19] that the real and imaginary parts of (14) cannot both tend to zero for all possible values of \( t \) unless the function \( g(t) \) vanishes. As in Appendix A of [11], the crucial argument involves Lebesgue dominated convergence (see e.g. [20]), which guarantees that a sequence with terms of the type \( \sin^n (nt) \) cannot tend to zero for all values of time in a given interval.

This concludes the proof that asserts and generalizes previous results involving uniqueness theorems for Fock quantizations in non-stationary settings [10–12]. In this case, a unitary implementation of the dynamics using \( SO(4) \)-invariant complex structures not only selects a unique quantum representation of the CCR’s, but also fixes the choice of field, picking up a preferred dynamics.

4. DISCUSSION AND CONCLUSIONS

It is not difficult to extend our result to other compact spatial manifolds of dimension \( d \leq 3 \), for which the analysis of [12] already supports the uniqueness of the Fock representation for fields satisfying a Klein-Gordon equation with time-dependent mass in an (“inertially” foliated) static and homogenous background. We call again \( \omega_n \) the eigenvalues of the Laplace-Beltrami operator, forming an increasing sequence, and \( g_n \) the dimension of the corresponding eigenspace. In addition, we suppose that there exists a characterization of the complex structures that are invariant under the symmetries of the field equations similar to that discussed for \( SO(4) \). Then, one can try and repeat the proof along the lines explained above for the three-sphere. One can easily realize that a key point to elucidate whether the function \( g(t) \) may differ from zero is the square summability of the sequence \( \sqrt{\pi} / \omega_n \). For \( d \)-spheres, this is the case only for the circle [11]. Nonetheless, even in this case, the dynamics of the new field variables (obtained with \( g(t) \neq 0 \)) can be implemented unitarily with an invariant complex structure if and only if the same happens for the original field variables, so that no new invariant representation with unitary evolution is permitted by changing the momentum [11].

As we have pointed out, a discipline where the proved uniqueness has deep consequences is in cosmology. Our result immediately eliminates the ambiguity in the quantization of fields in such cosmological scenarios, selecting a preferred quantization and confirming the robustness of its physical predictions. Moreover, it does so guaranteeing that unitarity is not lost in the field dynamics, in spite of the fact that the quantum field theory is realized in a non-stationary background.

In particular, this applies to inflationary cosmological models consisting of a massive Klein-Gordon scalar field propagating in an FRW spacetime. In addition to inflation, our results are relevant for the study of perturbations around non-stationary homogeneous solutions of the Einstein equations. For instance, this happens in the case of perfect fluids in FRW cosmologies (for which the perturbations of the energy-momentum tensor are isotropic) when the perturbations are adiabatic as well. It can be shown that the dynamics of the gauge-invariant energy density perturbation amplitude, or equivalently of the Bardeen potential, is dictated then by a wave equation with a time-varying mass [2, 4], in a static and homogeneous spacetime with the spatial topology of the FRW background. Another important example is the propagation of gravitational waves in FRW spacetimes with the topology of a three-sphere, treated as tensor perturbations [2]. Again, for isotropic perturbations of the energy-momentum tensor, these gravitational waves satisfy a field equation which belongs to the considered class. Conceptually, there should be no obstruction to apply our uniqueness theorem to these tensor quantities, selecting in this way fundamental fields (with the appropriate dynamics) as well as preferred Fock representations for them. Finally, we also mention the perturbations of an FRW universe with \( S^3 \)-topology whose matter content is a massive scalar field. These perturbations were studied in [21]. The coefficients of the expansion of the matter perturbations in harmonics, in a suitable gauge and with an appropriate scaling.
satisfy an equation whose solutions reproduce those for a free field with a time-dependent mass (in a static background), up to asymptotic corrections for large harmonic numbers.

In summary, we have seen that there exists not only a unique representation for the CCR’s that implements unitarily the dynamics using SO(4)-invariant vacuum states [12], but we have also reached a deeper conclusion: there is a unique choice of fundamental field such that the theory is compatible with both requests –dynamics and symmetries. The possible ambiguities of the quantization are completely removed. In one hand, the strong results showed here have an immediate application in physics, particularly in cosmology. On the other hand, they are a basis to extend powerful uniqueness theorems in standard quantum mechanics to quantum field theory.

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