SU(N) Wigner-Racah algebra for the matrix of second moments of embedded Gaussian unitary ensemble of random matrices

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Abstract

Recently Pluhar and Weidenmüller [Ann. Phys. (N.Y.) 297, 344 (2002)] showed that the eigenvectors of the matrix of second moments of embedded Gaussian unitary ensemble of random matrices generated by $k$-body interactions (EGUE($k$)) for $m$ fermions in $N$ single particle states are $SU(N)$ Wigner coefficients and derived also an expression for the eigenvalues. Going beyond this work, we will show that the eigenvalues of this matrix are square of a $SU(N)$ Racah coefficient and thus the matrix of second moments of EGUE($k$) is solved completely by $SU(N)$ Wigner-Racah algebra.
I. INTRODUCTION

Interacting finite quantum systems such as nuclei, atoms, quantum dots, nanometer-scale metallic grains etc. are governed by Hamiltonians of low (compared to the number of particles in the system) particle rank. Therefore, for many purposes, the random matrix models appropriate for these systems are embedded random matrix ensembles of $k$-body interactions originally introduced, via nuclear shell model calculations, by French and Wong and Bohigas and Flores [1]. For a system of $m$ spinless fermions in $N$ single particle states (we will use fermions throughout this paper and turn briefly to bosons at the end) the embedded Gaussian unitary ensemble of random matrices of $k$-body interactions $\text{EGUE}(k)$ is generated by defining the Hamiltonian $H$, which is given to be $k$-body, to be GUE in the $k$-particle spaces and then propagating it to the $m$ particle spaces by using the geometry (direct product structure) of the $m$-particle spaces. Just as $\text{EGUE}(k)$, the $\text{EGOE}(k)$ and other embedded ensembles are defined [2]. With $m$ particle space dimension given by $N_m = \binom{N}{m}$, one has the unitary groups $SU(N)$, $U(N_k)$ and $U(N_m)$ with $\text{EGUE}(k)$ invariant under $U(N_k)$ and the embedding in $m$-particle spaces is defined by $SU(N)$; note that a GUE in $m$ particle spaces is invariant under $U(N_m)$ but not the $\text{EGUE}(k)$, $k < m$. Very early, using the so called binary correlation approximation, Mon and French [3] and later French et al [4] derived some analytical properties of embedded ensembles valid in the dilute limit (defined by $(N, m, k) \to \infty$, $m/N \to 0$ and $k/m \to 0$). However only recently rigorous analytical results, valid for any $(N, m, k)$ are derived for these ensembles by Benet et al. [5] and very soon Pluhar and Weidenmüller (hereafter called PW) demonstrated [6] that these results indeed follow from considerations based on the $SU(N)$ embedding algebra. With all the $m$-particle matrix elements being linear combinations of the $k$-particle matrix elements (see Eq. (9) ahead), the
joint distribution for the matrix elements will be a multivariate Gaussian. Thus all
the information about EGUE\((k)\) is in the covariance matrix or the matrix of second
moments (Eq. (10) ahead). PW have shown that the eigenvectors of this matrix are
\(SU(N)\) Wigner (or Clebsch-Gordon (CG)) coefficients and derived the expression for
their eigenvalues using a duality relation for EGUE\((k)\). The purpose of this paper is
to show that the eigenvalues can be written as \(SU(N)\) Racah coefficients and thus
the matrix of second moments is solved completely by \(SU(N)\) Wigner-Racah algebra.
To this end results for \(SU(N)\) Racah coefficients given in [7] and [8,9] are used. We
will start with some basic results given in PW paper.

II. BASIC DEFINITIONS AND RESULTS

Let us begin with \(m\) particles in \(N\) single particle states (unfortunately in PW
\(\ell\) is used in place of \(N\) but to keep the notations same as in our earlier papers
[2,10], we use \(N\)). The single particle (sp) creation operator \(a_i^\dagger\) for any \(i\)-th sp state
transforms as the irreducible representation \(\{1\}\) of \(U(N)\) and similarly a product of \(r\)
creation operators transform, as we have fermions, as the irrep \(\{1^r\}\) in Young tableaux
notation. Let us add that a \(U(N)\) irrep \(\{\lambda_1, \lambda_2, \ldots, \lambda_N\}\) defines the corresponding
\(SU(N)\) irrep as \(\{\lambda_1 - \lambda_N, \lambda_2 - \lambda_N, \ldots, \lambda_{N-1} - \lambda_N\}\) with \(N - 1\) rows (there are
also other equivalent ways of defining \(SU(N)\) irreps given a \(U(N)\) irrep [6]). This
\(U(N) \leftrightarrow SU(N)\) correspondence is used throughout and therefore we use \(U(N)\) and
\(SU(N)\) interchangeably. In PW, \(\{1^r\}\) is denoted by \(f_r\) and we will follow this notation
from now on. With \(v_{1^r}\) denoting irreps (and other multiplicity labels) of the groups in
a subgroup chain of \(U(N)\) that supply the labels needed for a complete specification
of any \(m\)-particle state (for the purpose of the present paper the subgroup chain need
not be specified), the operator \(\prod_{i=1}^r a_i^\dagger\) and a normalized \(r\)-particle creation operator
\( A^f(v_r) \) behave as the \( SU(N) \) tensors \( T^{f,v_r} \) and \( \frac{1}{\sqrt{r!}} T^{f,v_r} \) respectively. Using the composition formula,

\[
T^{f_m v_m} = \sum_{v_k, v_s} C^{f_m v_m}_{f_k v_k} T^{f_k v_k} T^{f_s v_s}, \quad s = m - k
\]

(1)

where \( C^{f_m v_m}_{f_k v_k} \) is a \( SU(N) \) CG coefficient, a \( m \)-particle state \( |f_m v_m\rangle = A^f(f_m v_m) |0\rangle \) can be written as a product of \( k \) and \( s = m - k \) particle states as,

\[
|f_m v_m\rangle = (\frac{m}{k})^{-\frac{1}{2}} \sum_{v_k, v_s} A^f(f_k v_k) |f_s v_s\rangle C^{f_m v_m}_{f_k v_k}
\]

(2)

Some properties of the CG coefficients, used later in simplifications, are

\[
C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b} = (-1)^{\phi(f_a, f_b; f_{ab})} C^{f_{ab} v_{ab}}_{f_b v_b f_a v_a}, \quad C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b} = C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b}
\]

\[
C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b} = (-1)^{\phi(f_a, f_b; f_{ab})} \sqrt{\frac{d(f_{ab})}{d(f_a)}} C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b}
\]

\[
C^{f_{a 0 0}}_{f_a v_a} = 1, \quad C^{f_{a 0 0}}_{f_a v_a} = \frac{1}{d(f_a)}, \quad (C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b})^* = C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b}
\]

\[
\sum_{v_a, v_b} \left( C^{f_{ab} v_{ab}}_{f_a v_a f_b v_b} \right)^* C^{f'_{ab} v'_{ab}}_{f_a v_a f_b v_b} = \delta_{f_a f'_a} \delta_{f_b f'_b} \delta_{v_a v'_a} \delta_{v_b v'_b}
\]

(3)

In (3), \( \phi \) is a function that defines the phase for the \( 1 \leftrightarrow 2 \) interchange in the CG coefficients, \( d(f) \) is the dimension of the irrep \( f \) and \( \overline{f} \) is the irrep conjugate to \( f \). For \( f_r = \{1^r\}, \overline{f_r} = \{1^{N-r}\} \) and it also contains a phase factor as given in Eq. (53) of [7] (this is also seen easily by comparing the second and third equalities in Eq. (3) with the corresponding relations for the standard CG coefficients for angular momentum [11]). Similarly with \( f_a = f_k, f_b = \overline{f_k} \), one has

\[
f_{ab} = g_{\nu} = \{2^{\nu} 1^{N-2\nu}\}, \quad \nu = 0, 1, \ldots, k.
\]

Note that \( g_0 = \{0\} \) for \( SU(N) \) and also \( g_{\nu} = \overline{g_{\nu}} \). The function \( \phi \) in Eq. (3) is of the form \( \phi(\lambda_1, \lambda_2, \lambda_3) = F(\lambda_1) + F(\lambda_2) + F(\lambda_3) \) where \( F \) is some function with \( F(\lambda) \) an integer and \( F(\lambda) = F(\overline{\lambda}) \); for \( SU(N) \) irreps that appear in this paper these results
are valid as can be seen from Eq. (60) of [7]) except that there can be an overall $N$ dependent factor which will not change any of the final results.

$SU(N)$ irreducible tensors $B_k(g_\nu \omega_\nu)$ constructed out of $A^\dagger(f_k v_k)A(f_k v'_k)$ are defined by,

$$B_k(g_\nu \omega_\nu) = \sum_{v_k,v'_k} A^\dagger(f_k v_k)A(f_k v'_k) C^{g_\nu \omega_\nu}_{f_k v_k f_k v'_k}$$  \hspace{1cm} (4)

It is useful to note that the tensors $B$'s in (4) multiplied by $k!$ are, to within a phase factor, same as the tensors defined in Eq. (48) of [7]. Wigner-Eckart theorem decomposes the matrix elements of $B_k(g_\nu \omega_\nu)$ in $m$-particle spaces into a reduced matrix element $\langle || || \rangle$ and a CG coefficient,

$$\langle m_m | B_k(g_\nu \omega_\nu) | m'_m \rangle = \langle m_m || B_k(g_\nu) || m_m \rangle C^{g_\nu \omega_\nu}_{m_m m_m}$$  \hspace{1cm} (5)

Two important properties of $B_k(g_\nu \omega_\nu)$ are,

$$\langle \langle B_k(g_\nu \omega_\nu) \rangle \rangle^k = \sqrt{N_k} \delta_{g_\nu,\{0\}}$$  \hspace{1cm} (6)

$$\langle \langle B_k(g_\nu \omega_\nu)B_k(g_\mu \omega_\mu) \rangle \rangle^k = \delta_{g_\nu,g_\mu} \delta_{\omega_\nu,\omega_\mu}$$

In (6), $\langle \langle \rangle \rangle^k$ denotes trace over the $k$-particle spaces. The first equality in Eq. (6) easily follows from the fact that here $g_\nu = \{0\}$ as traces are scalars with respect to $SU(N)$ and then applying the fifth equality in Eq. (3). Similarly the second equality in Eq. (6) follows from the fact that only $g_\nu = g_\mu$ will give a scalar. Now we will turn to EGOE($k$).

Consider a $k$-body hamiltonian

$$H(k) = \sum_{v_a,v_b} V_{v_a v_b}(k)A^\dagger(f_k v_a)A(f_k v_b)$$  \hspace{1cm} (7)

where $V_{v_a v_b}(k)$ are matrix elements of $H(k)$ in $k$-particle space and form a GUE, i.e. $V_{v_a v_b}(k)$ are independent Gaussian variables with zero center and variance given by
\[ \overline{V_{v_a v_{d}}(k)} \overline{V_{v_c v_{d}}(k)} = \frac{\lambda^2}{N_k} \delta_{v_a v_d} \delta_{v_b v_c} \quad (8) \]

In Eq. (8) the overline indicates ensemble average and \( \lambda^2 \) is ensemble averaged variance of \( H(k) \) in \( k \)-particle space. The \( m \)-particle matrix elements of \( H(k) \) are, with \( s = m - k \),

\[ H_{v_1 v_2}(k) = \langle f_m v_{m}^1 | H(k) | f_m v_{m}^2 \rangle = (m) \sum_{v_a v_b v_s} (C_{f_m v_{a} f_s v_s})^* C_{f_m v_{b} f_s v_s} \overline{V_{v_a v_b}}(k) \quad (9) \]

Eq. (9) is obtained easily by substituting the definition (7) for \( H(k) \), then inserting complete set of states between \( A^\dagger \) and \( A \) operators and applying Eq. (2). The EGUE(\( k \)) in \( m \geq k \) spaces is defined by Eqs. (7)-(9). Now it is clear that for any analysis of EGUE(\( k \)) all one need to know is the covariance between any two \( m \)-particle matrix elements \( H_{v_1 v_2}(k) \) and this defines the matrix of second moments,

\[ A_{v_1 v_2 : v_3 v_4} = \overline{H_{v_1 v_2}(k) H_{v_3 v_4}(k)} \quad (10) \]

As stressed by PW, most important step in EGUE(\( k \)) analysis is to derive a "generalized eigenvalue expansion" of \( A \) defined by \( A_{ij} = \sum_k C_{ik} E_k C_{jk} \) with \( E_k \) the eigenvalues and \( C_{jk} \) the eigenvectors such that \( E_k \) are positive and \( C \)'s hermitian. To this end, it is useful to consider the unitary decomposition of \( H(k) \) in terms of the \( SU(N) \) tensors \( B_k(g_\nu \omega_\nu) \),

\[ H(k) = \sum_{g_\nu \omega_\nu} B_k(g_\nu \omega_\nu) W_{g_\nu \omega_\nu}(k) \quad (11) \]

The expansion coefficients \( W_{g_\nu \omega_\nu}(k) \) are easily given by

\[ W_{g_\nu \omega_\nu}(k) = \langle \langle H(k) B_k(g_\nu \omega_\nu) \rangle \rangle_k \quad (12) \]

and this follows by using the definition (11) and Eq. (6). Most significant property of the \( W \) coefficients is that they are independent Gaussian variables with zero center
and variance given by (derived using Eqs. (11), (4), (7) and (8) in that order and using the orthonormal properties of the CG coefficients),

$$ W_{g_1 \omega_1}(k)W_{g_2 \omega_2}(k) = \frac{\lambda^2}{N_k} \delta_{g_1 g_2} \delta_{\omega_1 \omega_2} \tag{13} $$

### III. MATRIX OF SECOND MOMENTS

Firstly we will derive an expression for the covariance $H_{v_1 m_1 v_2 m_2}(k)H_{v_3 m_3 v_4 m_4}(k)$ in terms of $SU(N)$ CG coefficients and Racah coefficients and then turn to the eigenvalues and eigenvectors of $A$, the matrix of second moments. Applying Eqs. (10), (11) and (5) in that order gives,

$$ H_{v_1 m_1 v_2 m_2}(k)H_{v_3 m_3 v_4 m_4}(k) = \sum_{g_1 \omega_1, g_2 \omega_2} \langle f_m v_1^2 | B_k(g_1 \omega_1) W_{g_1 \omega_1}(k) | f_m v_2^2 \rangle \langle f_m v_3^2 | B_k(g_2 \omega_2) W_{g_2 \omega_2}(k) | f_m v_4^2 \rangle 
= \sum_{g_1 \omega_1, g_2 \omega_2} W_{g_1 \omega_1}(k)W_{g_2 \omega_2}(k) \langle f_m v_1^2 | B_k(g_1 \omega_1) | f_m v_2^2 \rangle \langle f_m v_3^2 | B_k(g_2 \omega_2) | f_m v_4^2 \rangle 
= \frac{\lambda^2}{N_k} \sum_{g \omega} \langle f_m v_1^2 | B_k(g \omega) | f_m v_2^2 \rangle \langle f_m v_3^2 | B_k(g \omega) | f_m v_4^2 \rangle 
= \frac{\lambda^2}{N_k} \sum_{g \omega} |\langle f_m | B_k(g \omega) | f_m \rangle|^2 C_{f_m v_1^2 f_m v_2^2}^{g \omega} C_{f_m v_3^2 f_m v_4^2}^{g \omega} \tag{14} $$

Eqs. (55) and (56) of [7] together with Eqs. (3) (see also the remark just after Eq. (4)) allows one to write the reduced matrix element in Eq. (14) as a $SU(N)$ Racah or $U$-coefficient,

$$ |\langle f_m | B_k(g \omega) | f_m \rangle|^2 = \frac{(N_m)^2 (k)^2}{d(g \omega) (N_{m-k})} [U(f_m f_{N-k} f_m f_k; f_m-k \omega) + \text{c.c.}] \tag{15} $$

With $\{2^{\nu}1^{N-2\nu}\} = \{1^{\nu}\} \otimes \{1^{N-\nu}\} - \{1^{\nu-1}\} \otimes \{1^{N-\nu+1}\}$ where $\otimes$ denotes Kronecker product, the dimension $d(g \omega)$ is given by
\[ d(g_\nu) = d(\nu) = (N_\nu)^2 - (N_{\nu-1})^2 = \frac{(N!)^2(N + 1)(N - 2\nu + 1)}{(\nu!)^2(N - \nu + 1)^2} \]  

(16)

Before going further let us define, for a given \((m, N)\), a function \(\Lambda^\nu(k)\),

\[ \Lambda^\nu(k) = \binom{m - \nu}{k} \binom{N - m + k - \nu}{k} \]  

(17)

Now, using Eq. (16) and substituting the expression given by Eq. (6.1) of [7] for the \(U\)-coefficient in (15), it is seen that

\[ |\langle f_m | B_k(g_\nu) | f_m \rangle|^2 = \Lambda^\nu(m - k), \quad \nu = 0, 1, 2, \ldots, k \]  

(18)

Combining (14) with (18) yield an expression for the covariance between \(H\) matrix elements in \(m\)-particle spaces,

\[ \langle H_{v_m^1 v_m^2}(k) H_{v_m^3 v_m^4}(k) \rangle = \frac{\lambda^2}{N_k} \sum_{\nu=0,1,\ldots,k; \omega_\nu} \frac{(N_m)^2 (m_k)^2}{d(g_\nu)(N_{m-k})} [U(f_m f_{N-k} f_m f_k; f_{m-k} g_\nu)]^2 C_{f_m v_m^1 f_m v_m^2}^{g_\nu \omega_\nu} C_{f_m v_m^3 f_m v_m^4}^{g_\nu \omega_\nu} \]

\[ = \frac{\lambda^2}{N_k} \sum_{\nu=0,1,\ldots,k; \omega_\nu} \{\Lambda^\nu(m - k)\} C_{f_m v_m^1 f_m v_m^2}^{g_\nu \omega_\nu} C_{f_m v_m^3 f_m v_m^4}^{g_\nu \omega_\nu} \]

(19)

Eq. (19) will be useful in deriving expressions for the moments of \(H\) spectrum, i.e. \(\langle H^p \rangle^m\). However the disadvantage of (19) is that it is not in a proper form to give the eigenvalues and eigenvectors of the matrix \(A\) in Eq. (10). In order obtain them, the CG coefficients in Eq. (14) should be changed to \(C_{f_m v_m^1 f_m v_m^2}^{g_\nu \omega_\nu} C_{f_m v_m^3 f_m v_m^4}^{g_\nu \omega_\nu}\) and this can be accomplished by a \(SU(N)\) Racah transform. Using Eq. (3.2.17) of [9] one has for example,

\[ \sum_{\omega_\nu} C_{f_m v_m^1 f_m v_m^2}^{g_\nu \omega_\nu} C_{f_m v_m^3 f_m v_m^4}^{g_\nu \omega_\nu} = \sum_{g_\mu g_\nu} U(f_m f_{m1} f_m g_\mu; g_\nu g_\mu) C_{f_m v_m^1 f_m v_m^2}^{g_\mu \omega_\mu} C_{f_m v_m^3 f_m v_m^4}^{g_\nu \omega_\nu} \]  

(20)

Then Eqs. (20) and (3) will give,
\[
\sum_{\omega \nu} C_{f_m v_m}^{g_\nu \omega \nu} C_{f_m v_m}^{g_\nu \omega \nu} = \sum_{g \mu \omega \mu} \sqrt{\frac{d(g_\nu)}{d(g_\mu)}} U(f_m f_m f_m; g_\nu g_\mu) C_{f_m v_m}^{g_\mu \omega \mu} C_{f_m v_m}^{g_\mu \omega \mu} \left\{ \lambda^2 \frac{(N_m)^2 (m)^2}{N_k (N_m-k)} \right\}
\]

Finally, Eqs. (21) and (15) combined with (14) produce the generalized eigenvalue expansion of the matrix of second moments \(A\),

\[
\begin{aligned}
A_{v_i v_3} v^2 = & \sum_{g_\nu} \frac{1}{d(g_\nu) d(g_\mu)} \left[ U(f_m f_m; f_m f_m; g_\nu g_\mu) \right]^2 \left\{ \lambda^2 \frac{(N_m)^2 (m)^2}{N_k (N_m-k)} \right\} \\
& \end{aligned}
\]

(22)

Obviously the quantity in the curly brackets in Eq. (22) gives the eigenvalues of \(A\) and the \(C\)'s are eigenvectors.

**IV. EIGENVALUES AS SU(N) RACAH COEFFICIENTS AND THEIR APPLICATIONS**

In order to proceed further, it is useful to consider \(6j\) symbols of \(SU(N)\) and they are defined by (see Eq. (3.2.18) of [9]),

\[
U(\lambda_1 \lambda_2 \lambda_3; \lambda_{12} \lambda_{23}) = \sqrt{d(\lambda_{12}) d(\lambda_{23})} (-1)^{\phi(\lambda_2, \lambda_3, 0) + \phi(\lambda_{12}, \lambda_{3}, \lambda) + \phi(\lambda_1, \lambda_2, \lambda_{12})} \left\{ \begin{array}{ccc}
\lambda_1 & \lambda_{23} & \lambda \\
\lambda_3 & \lambda_{12} & \lambda_2 \\
\end{array} \right\}
\]

(23)

In (23) \(\lambda\)'s are \(SU(N)\) irreps and the four couplings involved in the \(U\)-coefficient are assumed to be multiplicity free (for the applications in the present paper this assumption is always valid). Symmetry properties of the \(6j\)-symbol appearing on the r.h.s of (23) are well known [8,9]. In the present analysis, the Biedenharn-Elliott sum rule extended to \(SU(N)\) [8,9] plays a central role. This sum rule relates a product
of three Racah coefficients (weighted appropriately by dimension factors and phase factors with the irreps in the Racah coefficients appearing in some particular order) with sum over a common irrep label to a product of two Racah coefficients. After converting the Racah coefficients in (22) into $6j$ symbols of $SU(N)$ using Eq. (23) and then applying the symmetry properties of the $6j$ symbols, it is seen that the sum in the square brackets in Eq. (22) is exactly in the required form. Applying the Biedenharn-Elliott sum rule, the sum then simplifies to

$$\frac{N_{m-k}}{N_k d(g_\mu)} U^2(f_m f_{N-m+k} f_m f_{m-k} : f_k g_\mu)$$

Here $\mu = 0, 1, \ldots, m - k$. Now the eigenvalues of the matrix $A$, in terms of the $U(N)$ Racah coefficients is given by

$$E_\mu = \frac{\lambda^2}{N_k d(g_\mu)(N_k)} \left[U(f_m f_{N-m+k} f_m f_{m-k}; f_k g_\mu)\right]^2 ; \quad \mu = 0, 1, \ldots, m - k$$

with degeneracy $d(g_\mu)$ (see Eq. (16)). Eq. (24) is the central result of this paper. With this, the matrix $A$ is completely specified by the $U(N)$ Wigner and Racah coefficients. Now substituting the formula (Eq. (61)) of [7]) for the $U$-coefficients in Eq. (24) produces the result of PW,

$$A_{v_1 v_2 v_3, v_4 v_5 v_6} = H_{v_1 v_2 v_3, v_4 v_5 v_6}^2(k) H_{v_1 v_2 v_3, v_4 v_5 v_6}^2(k) = \sum_{g_{\mu \omega \mu}} C_{g_{\mu \omega \mu}}^{g_{\mu \omega \mu}} C_{f_m f_m f_m f_m}^{g_{\mu \omega \mu}} E_\mu$$

where,

$$E_\mu = \frac{\lambda^2}{N_k} \Lambda^\mu(k) ; \quad \mu = 0, 1, \ldots, m - k.$$ 

Note that the function $\Lambda^\mu(k)$ is defined by Eq. (17).

Information about EGUE($k$) is contained in the ensemble averaged moments $M_p = \langle H^p \rangle^m$ and the bivariate moments $\Sigma_{pq} = \langle H^p \rangle^m \langle H^q \rangle^m$. In deriving the formulas for the lower order moments, we will show the usefulness of Eq. (19). Obviously, ensemble averaged centroid is zero and the variance is
\[
\langle H^2 \rangle^m = \frac{1}{N_m} \sum_{v_i^1, v_i^m, v_i^m} H_{v_i^1} H_{v_i^m} v_i^1 v_i^m
\]
\[
= \frac{1}{N_m} \sum_{g_{\mu, \omega_{\mu}}} \Lambda^\nu(k) \sum_{v_i^1, v_i^m} C_{g_{\mu, \omega_{\mu}}}^{\nu(o)} f_{m_{v_i^1} m_{v_i^m}} C_{g_{\mu, \omega_{\mu}}}^{\nu(o)} f_{m_{v_i^1} m_{v_i^m}}
\]
\[
= \Lambda^0(k)
\]

The second equality follows from (25) and the final result follows by applying (3); note that \(\sum v_{\mu} C_{g_{\mu, \omega_{\mu}}}^{\nu(o)} f_{m_{v_i^1} m_{v_i^m}} = \sqrt{N_m} \delta_{\mu,0}\). The variance in (27) is in \(\lambda^2/N_k\) units and this factor is dropped as all the quantities we consider from now on are all scaled with respect to \(\{\langle H^2 \rangle^m\}^{1/2}\). As the third moment is zero, we will turn to the fourth moment,

\[
\langle H^4 \rangle^m = \frac{1}{N_m} \sum_{v_i^1, v_i^m, v_i^m, v_i^m, v_i^m} H_{v_i^1} H_{v_i^m} H_{v_i^m} H_{v_i^m} v_i^1 v_i^m v_i^m v_i^m
\]
\[
= \frac{1}{N_m} \sum_{g_{\mu, \omega_{\mu}}} \left\{ 2 \left[ \sum_{g_{\nu, \omega_{\nu}}} \langle f_m v_i^1 | B_k(g_{\nu, \omega_{\nu}}) | f_m v_i^m \rangle \langle f_m v_i^m | B_k(g_{\nu, \omega_{\nu}}) | f_m v_i^m \rangle \right] \times \right.
\]
\[
\left[ \sum_{g_{\mu, \omega_{\mu}}} \langle f_m v_i^1 | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \langle f_m v_i^m | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \right] + \left[ \sum_{g_{\mu, \omega_{\mu}}} \langle f_m v_i^1 | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \langle f_m v_i^m | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \right] \times \right.
\]
\[
\left[ \sum_{g_{\mu, \omega_{\mu}}} \langle f_m v_i^1 | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \langle f_m v_i^m | B_k(g_{\mu, \omega_{\mu}}) | f_m v_i^m \rangle \right] \right\}
\]
\[
= 2 [\Lambda^0(k)]^2 + \frac{1}{N_m} \sum_{v_i^1, v_i^m, v_i^m, v_i^m, v_i^m} \left\{ \sum_{\nu=0,1,\ldots,k;\omega_{\nu}} \left\{ \Lambda^\nu(m-k) \right\} C_{g_{\mu, \omega_{\nu}}}^{\nu(o)} f_{m_{v_i^1} m_{v_i^m}} \right\} \times \right.
\]
\[
\left\{ \sum_{\mu=0,1,\ldots,m-k;\omega_{\mu}} \left\{ \Lambda^\nu(k) \right\} C_{g_{\mu, \omega_{\nu}}}^{\nu(o)} f_{m_{v_i^1} m_{v_i^m}} \right\}
\]
\[
= 2 [\Lambda^0(k)]^2 + \frac{1}{N_m} \sum_{\nu=0}^{\min\{k,m-k\}} \Lambda^\nu(m-k) \Lambda^\nu(k) d(\nu)
\]

(28)

The second equality in Eq. (28) follows by applying Eqs. (11) and (13). In the third equality, it is easy to recognize the first term. The second term follows by applying Eq. (19) and (25) to the two pieces in the corresponding term in the second equality.
The final result follows by applying the orthonormality of the CG coefficients. Eqs. (27,28) will give the excess ($\gamma_2$) parameter of the density of eigenvalues of EGUE($k$),

$$
\gamma_2 = \frac{\langle H^4 \rangle}{\langle H^2 \rangle^2} = 3 - \left[ \frac{1}{N_m} \min\{k,m-k\} \sum_{\nu=0}^{\Lambda^\nu(m-k) \Lambda^\nu(k) d(\nu)} \right] - 1 \quad (29)
$$

Turning now to the lowest bivariate moment $\Sigma_{11}$, it is easily seen that

$$
\Sigma_{11} = \langle H^m \rangle \langle H^m \rangle = \frac{1}{(N_m)^2} \sum_{v_i^m,v_j^m} H_{v_i^m,v_j^m} H_{v_i^m,v_j^m} = \frac{1}{N_m} \Lambda^0(m-k) \quad (30)
$$

Applying (19) and recognizing that only $\nu = 0$ will contribute to the traces give immediately Eq. (30). However Eq. (25) generates a different formula for $\Sigma_{11}$ and equating it to (30) gives the identity (derived in PW using the duality transformation),

$$
\frac{1}{N_m} \sum_{\nu=0}^{m-k} \Lambda^\nu(k) d(\nu) = \Lambda^0(m-k). \quad (31)
$$

Finally $\Sigma_{22}$ is given by

$$
\Sigma_{22} = \langle H^2 \rangle^m \langle H^2 \rangle^m = \frac{1}{(N_m)^2} \sum_{v_i^m,v_j^m,v_i'^m,v_j'^m} \left| H_{v_i^m,v_j^m} \right|^2 \left| H_{v_i'^m,v_j'^m} \right|^2
$$

$$
= \frac{1}{(N_m)^2} \sum_{v_i^m,v_j^m,v_i'^m,v_j'^m} \left| H_{v_i^m,v_j^m} \right|^2 \left| H_{v_i'^m,v_j'^m} \right|^2 + \frac{2}{(N_m)^2} \sum_{v_i^m,v_j^m,v_i'^m,v_j'^m} \left| H_{v_i^m,v_i'^m} H_{v_j^m,v_j'^m} \right|^2 (32)
$$

Here in the second equality used is the property $\bar{x^2 y^2} = (\bar{x^2}) (\bar{y^2}) + 2[\bar{xy}]^2$ of Gaussian variables $x$ and $y$. Similarly the final result follows by applying (25) to the second term in the second equality and simplifying the CG coefficients using Eq. (3). Now, the variance of the distribution of the variances of the $H$ spectra over the EGUE($k$) ensemble is

$$
\hat{\Sigma}_{22} = \frac{\Sigma_{22}}{\langle H^2 \rangle^m} - 1 = \frac{2}{(N_m)^2} \sum_{\nu=0}^{m-k} \left[ \frac{\Lambda^\nu(k)}{\Lambda^0(k)} \right]^2 d(\nu) \quad (33)
$$
Eqs. (27, 29, 31, 33) are also given by Benet et al [5]; this paper neither gives details of the derivations nor uses $SU(N)$ Racah coefficients. Also in this work, $\hat{\Sigma}_{11}$, $\hat{\Sigma}_{22}$ and $\gamma_2 + 1$ are denoted by $S$, $R$ and $Q$ respectively while we have followed Ref. [3].

V. CONCLUSIONS

Going beyond PW, matrix elements of the matrix of second moments are written explicitly in terms of $SU(N)$ Wigner and Racah coefficients and this result is obtained by recognizing that the reduced matrix elements of $B_k(g_\nu)$ are $SU(N)$ Racah coefficients. With this one has Eq. (19) and this is converted into the generalized eigenvalue expansion form by first applying a $SU(N)$ Racah transform and then applying the Biedenharn-Elliott sum rule extended to $SU(N)$. This gives the eigenvalues of the matrix of second moments explicitly in terms of $SU(N)$ Racah coefficients (Eq. (24)). The two different forms given by Eqs. (19, 25) for the covariances of $m$-particle $H$ matrix elements, give in a simple manner the formulas for the low order moments $M_p$ that define the state density and the bivariate moments $\Sigma_{pq}$ that give information about fluctuations.

Although EGUE($k$) for only fermions is considered in this paper, all the results in fact translate to those of EGUE($k$) for bosons by using the well known $N \to -N$ symmetry [12,13], i.e. in the fermion results replace $N$ by $-N$ and then take the absolute value of the final result. For example, the $m$ boson space dimension is $d(m) = \left| \left( \begin{array}{c} -N \\ m \end{array} \right) \right| = \{(N - m + 1)_m\}$. More importantly the eigenvalues of the matrix of the second moments are

$$\Lambda^\nu_B(k) \rightarrow \left| \left( \begin{array}{c} m - \nu \\ k \end{array} \right) \left( \begin{array}{c} -N - m + k - \nu \\ k \end{array} \right) \right| = \left( \begin{array}{c} m - \nu \\ k \end{array} \right) \left( \begin{array}{c} N + m + \nu - 1 \\ k \end{array} \right)$$

This result was explicitly derived in [14]; see Eq. (14) of this paper. Moreover for bosons, $\{k\} \otimes \{k^{N-1}\} \to g_\nu = \{2\nu, \nu^{N-2}\}$, $\nu = 0, 1, \cdots, k$. Also, the $N \to -N$
symmetry and Eq. (16) give \( d(\nu) = \{(N + \nu - 1)_{\nu}\}^2 - \{(N + \nu - 2)_{\nu-1}\}^2 \) and this is same as Eq. (15) of [14]. Similarly Eqs. (27,29,31,33) extend directly to the boson EGUE\((k)\) with \( \Lambda^\nu(k) \) replaced by \( \Lambda^\nu_B(k) \) defined in Eq. (34). In addition, for fermions to bosons there is also a \( m \leftrightarrow N \) symmetry and this connects fermion results (say for \( M_p \) and \( \Sigma_{pq} \)) in dilute limit to boson results in dense limit [13].

Recently there is considerable interest in mesoscopic physics to study EGUE\((k)\) for fermions with spin [15] and here the embedding algebra is \( U(2N) \supset U(N) \otimes SU(2) \) with \( SU(2) \) generating spin. The approach presented in Sections II-IV is being applied to this system; some useful results for the \( U(2N) \supset U(N) \otimes SU(2) \) Wigner-Racah algebra are available in Refs. [16,17]. Finally, Wigner-Racah algebra analysis of embedded ensembles with more general group symmetries (see [2,6,17,18] for examples) should be possible in future, thus opening up a new direction in random matrix theory.

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