Topological Expansion and Exponential Asymptotics in 1D Quantum Mechanics

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Abstract

Borel summable semiclassical expansions in 1D quantum mechanics are considered. These are the Borel summable expansions of fundamental solutions and of quantities constructed with their help. An expansion, called topological, is constructed for the corresponding Borel functions. Its main property is to order the singularity structure of the Borel plane in a hierarchical way by an increasing complexity of this structure starting from the analytic one. This allows us to study the Borel plane singularity structure in a systematic way. Examples of such structures are considered for linear, harmonic and anharmonic potentials. Together with the best approximation provided by the semiclassical series the exponentially small contribution completing the approximation are considered. A natural method of constructing such an exponential asymptotics relied on the Borel plane singularity structures provided by the topological expansion is developed. The method is used to form the semiclassical series including exponential contributions for the energy levels of the anharmonic oscillator.

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1 Introduction

In this paper we continue our investigations to represent basic quantities of the quantum mechanics in the form of Balian - Bloch representation i.e. in the form of the Laplace-Borel transforms in which the conjugate variables are an action and the Planck constant $\hbar$ [1]. The key results, which the paper is relied on, has been published earlier [2]. The present work develops these key ideas and using the explicit form of the fundamental solutions [11, 12] to the 1D Schrödinger equation expresses the Balian-Bloch representation in the form of what we call a topological expansion. We describe also the way of using the representation to construct extended JWKB approximations in the form of so called exponential asymptotics (sometimes called also the hyperasymptotics, see [23, 24, 25, 26, 27] and the references cited there) and we consider some particular applications of the Balian-Bloch method as well.

For simplicity the potentials considered in this paper are assumed to be polynomial but its main results are valid for more general meromorphic potentials as well.

Being Borel summable the fundamental solutions define their corresponding Borel functions with the help of which they can be represented in the form of the Borel transformation from the Borel plane of the action variable to the complex plane of the $\hbar^{-1}$-variable. For the polynomial potentials these Borel functions are in fact all the same despite the fact that they are defined by different fundamental solutions [3]. This means of course that the fundamental solutions themselves are in close relations to each other being in fact a mutual analytical continuation of each other in the $\hbar^{-1}$ plane [2, 3].

Therefore, to get any of the fundamental solutions it is necessary only to choose properly an integration path in the Borel plane. However, to do it a detailed knowledge of singularity distribution of the Borel function in the Borel plane is necessary. It is the aim of this paper to provide us with an effective tool for studying these singularities. Namely, we develop an expansion for the Borel function called topological in which an expansion parameter is the complexity of the Borel plane corresponding to successive terms of the expansion.

With the help of the fundamental solutions we can solve most of the 1D quantum mechanical problems so that the corresponding quantities involved in the problems considered depend on different pieces of the fundamental solutions used. These quantities themselves can have then semiclassical representations which can be Borel summable and can serve as a source of their semiclassical approximations as well. It is clear that the corresponding Borel plane singularities of these quantities are defined then by the pieces of the fundamental solutions constructing them. Therefore the topological expansion method can be applied also to determine the approximate singularity structure for these quantities as well.

The semiclassical expansions used as a source of approximations are considered as insufficient providing us with unavoidable nonvanishing errors. It is well known that the reasons for these errors are immanent due to the divergence of the semiclassical series so that the latter as asymptotic neglect the exponentially small contributions. Nevertheless, since the series are Borel summable they have to contain the full information about such exponential contributions. A common goal of many approaches was just to recover these contributions leading to a formulation of so called resurgent theory [24, 25, 26, 27].

Let us note, however, that the exponentially small contributions is of its own importance since in many cases of quantities considered these contributions are dominant. Among the latter cases the most well known one is the difference between the energy levels of different parities in the symmetric double well [30]. But these are also the cases of transition probabilities in the tunnelling phenomena [30] or their adiabatic limits in the time dependent problem.
of transitions between two (or more) energy levels (see \[32, 33\] and references cited there) or the exponential decaying of resonances in the week electric field (see \[34, 35\] and references cited there).

In an approach of our paper we make full use of the Borel summability of the quantities considered as well as of the corresponding topological expansions to construct the relevant exponential asymptotics.

However, as a necessary step of our formulation is the knowledge of the Borel plane singularity structure of any considered quantity. It is just the topological expansion which allows us to build this knowledge step by step.

The topological expansion is constructed directly from the Fröman and Fröman representation of the fundamental solutions which themselves are given in the forms of functional series \([2, 3, 4]\). Therefore, we shall start in the next section with the detailed presentation of the series.

In Section 3 the topological series representation for the Borel functions is introduced and its convergence is proved. This representation provides us with an algorithm for approximate calculations of Borel functions being alternative to the ones relied on the Padé approximants \([5, 8, 9, 14]\), continued fractions \([14]\) or conformal transformations \([14]\).

In Section 4 singularity structures of the topological series expansion are analyzed and their hierarchic form (which gives rise to the name of the series) is established. We consider there as the simplest examples the 'first sheet' singularity structures of the linear and harmonic potentials. In particular we describe completely the singularity structure of the Borel plane of the harmonic oscillator Joos function found first by Voros by a different method \([19]\).

The results obtained in Sections 3 and 4 are applied in the next section where the solution of the so called connection problem within the framework of the Balian-Bloch representation is discussed.

In Section 6 we discuss the problem of the exponential asymptotics \([23, 24, 25, 26, 27]\). We show that this problem has a natural solution in the framework of the Balian-Bloch representation and gets a natural support from the topological expansion method.

In Section 7 the energy levels of the single-well anharmonic potential are considered in order to show how to use the topological expansion to construct their extended exponential asymptotics.

Finally, in Section 8 we summarise our results.

2 Fundamental solutions

Let us remind basic notions of our considerations (see \([2]\), for details).

The fundamental solutions satisfy the Schrödinger equation:

$$\psi''(x, \lambda, E) - \lambda^2 q(x, E) \psi(x, \lambda, E) = 0 \quad (2.1)$$

where: \(q(x, E) = V(x) - E\), \(\lambda = \sqrt{2m\hbar}^{-1}\). Both \(\lambda\) and \(E\) can take on complex values. \(V(x)\) is assumed to be a polynomial of any degree \(n \geq 1\). A Stokes line pattern (see \([11, 12]\) for necessary definitions) relevant for our considerations is shown in Fig. 1. (A total number \(p\) of sectors equals to \(n + 2\) in this case.)

The following fundamental solution \(\psi^1(x, \lambda, E)\) to \((2.1)\) can be attached to the sector \(S_1\):

$$\psi^1(x, \lambda, E) = q^{-\frac{1}{4}}(x, E)e^{\frac{1}{4} \lambda \int_{x_0}^{x} q(y, E) dy} \chi^1(x, \lambda, E)$$
\[ \Re \left[ \sigma \int_{x_0}^{x} q_2^2(y, E) dy \right] < 0, \quad x \in S_1, \quad \sigma = \pm 1 \quad (2.2) \]

\[ q(x_0, E) = 0 \]

with the "amplitude factor" \( \chi_{\sigma}^1(x, \lambda, E) \) given by the following functional series:

\[ \chi_{\sigma}^1(x, \lambda, E) = 1 + \sum_{n \geq 1} \left( \frac{\sigma_k}{2\lambda} \right)^n \int_{\gamma_\sigma^0(x)} \int_{\gamma_\sigma^0(y_n-1)} dy_1 \cdots dy_n \omega(y_1) \cdots \omega(y_n) \cdot \left[ 1 - e^{2\lambda \xi(y_1,x)} \right] \cdots \left[ 1 - e^{2\lambda \xi(y_n,y_n-1)} \right] \quad (2.3) \]

where:

\[ \omega(y) = \frac{1}{4} \left[ \frac{\tilde{q}''(y)}{\tilde{q}_2(y)} - \frac{5}{4} \frac{\tilde{q}''(y)}{\tilde{q}_2^2(y)} \right] = -q^{-\frac{1}{4}}(y) \left( q^{-\frac{1}{4}}(y) \right)'' \quad (2.4) \]

\[ \xi(x_0, x) = -\sigma \int_{x_0}^{x} q_2^\frac{1}{2}(y, E) dy \]

and where an obvious dependence of \( \omega, q, \xi, \) etc. on \( E \) has been dropped. We shall also put \( \sigma = -1 \) in (2.2)-(2.4) assuming that in (2.2) the corresponding inequality is satisfied in this case.

Fig. 1. The Stokes graph for a general polynomial potential

In the sector \( S_1 \) the following semiclassical expansion for \( \chi_1(x, \lambda) \) takes place:

\[ \chi_1(x, \lambda) \sim \chi_{1s}^1(x, \lambda) = 1 + \sum_{n \geq 1} \frac{\kappa_{1,s}(x)}{2\lambda^n} \]
\[ \kappa_{1,n}(x) = \int_{\infty_k}^{x} d\xi_n q^{-\frac{1}{4}}(\xi_k) \left( q^{-\frac{1}{4}}(\xi_n) \int_{\infty_k}^{\xi_n} d\xi_{n-1} q^{-\frac{1}{4}}(\xi_{n-1}) \right) \] (2.5)

\[ \cdot \left( \ldots q^{-\frac{1}{4}}(\xi_2) \int_{\infty_k}^{\xi_2} d\xi_1 q^{-\frac{1}{4}}(\xi_1) \left( q^{-\frac{1}{4}}(\xi_1) \right)'' \ldots \right)'' , \quad k = 1, 2, \ldots \]

As it has been shown in \[2\] if \( x \) stays in \( S_1 \) of Fig. 1 then we can define for \( \Re s < 0 \) the following Laplace transformation of the amplitude factor \( \chi_1(x, \lambda) \):

\[ \tilde{\chi}_1(x, s) = \frac{1}{2\pi i} \int_C e^{-2\lambda s} \frac{\chi_1(x, \lambda)}{\lambda} d\lambda \] (2.6)

with the integration contour \( C \) shown in Fig. 2. (The factor 2 in the exponential function in \( (2.5) \) is introduced for convenience). By the form \( (2.6) \) \( \tilde{\chi}_1(x, s) \) is defined holomorphically in the half-plane \( \Re s < \Re \xi(x_0, x) \) and since \( \Re \xi(x_0, x) \) is positive \( \tilde{\chi}_1(x, s) \) appears to be, in fact, the Borel transform of \( \chi_1(x, \lambda) \). The contour \( C \) in \( (2.6) \) can be chosen as a circle with its radius \( \lambda \) to be large enough to substitute \( \chi_1(x, \lambda) \) by its semiclassical series \( (2.5) \). Then for \( |s| < |\xi(x_0, x)| \) the LHS of \( (2.6) \) can be integrated to give the following Borel series:

\[ \tilde{\chi}_1(x, s) = 1 + \sum_{n \geq 1} \kappa_{1,n}(x) \frac{(-s)^n}{n!} \] (2.7)

convergent in the circle \( |s| < |\Re \xi(x_0, x)| \). The point \( s_0(x) = \xi(x_0, x) \) is a singularity for \( \tilde{\chi}_1(x, s) \) closest to the origin.

\[ \Psi_1(x, \lambda), \Psi_1(x, \lambda) \]

Fig. 2. The cut \( \lambda \)-plane corresponding to the global solution \( \Psi(x, \lambda) \)

The transformations \( (2.3) \) can be inverted to give:

\[ \chi_1(x, \lambda) = 2\lambda \int_C e^{2\lambda s} \tilde{\chi}_1(x, s) ds \] (2.8)
where the contour $\tilde{C}$ starts at the infinity $\Re(\lambda s) = -\infty$ and ends at $s = 0$. Since $\tilde{C}$ can be freely deformed in the half plane $\Re s \leq 0$ the formula (2.8) define $\chi_1(x, \lambda)$ in the whole sheet shown in Fig. 2 excluding the points of the negative real half-axis.

The following is worth to note.

The formula (2.8) is certainly valid for $\Re \lambda > 0$ when the contour $\tilde{C}$ stays in the half plane $\Re s < 0$. It can be continued, however, to other domains of the Riemann $\lambda$-surface corresponding to $\chi_1(x, \lambda)$ if accompanied with suitable changes of the variable $x$. Thus, for example, when continuing $x$ to the sector $S_2$ and deforming the contour $C$ in Fig. 2 into $C_1$ the formula \( (2.3) \) will then define $\tilde{\chi}_1(x, s)$ in the half plane $\Re s > 0$. On the other hand the inverse formula (2.8) defines then $\chi_1(x, \lambda)$ in the half plane $\Re \lambda < 0$ with the contour $C$ in the formula deformed (anticlockwise) from its position in the left half-plane to its new position in the right half of the $s$-plane. The function $\chi_1(x, \lambda)$ fulfils then for $\lambda > 0$ the condition: $\chi_1(x, -\lambda) \equiv \chi_2(x, \lambda)$. Possible singularities of $\tilde{\chi}_1(x, s)$ existing in the corresponding half-plane $\Re s > 0$ when $x$ stays in the sector $S_1$ move to the half-planes $\Re s < 0$ when $x$ moves to the sector $S_2$ (see also Section 5 for a relevant discussion).

3 Topological expansion of Borel function $\tilde{\chi}_1(x, s)$

As it follows from the definition of $\tilde{\chi}_1(x, s)$ if we want to learn something about it we have to analyze $\chi_1(x, \lambda)$ as given by \( (2.3) \). We shall show below that if $x$ stays in $S_1$ (see Fig. 2) then it is possible to represent each term of the series in \( (2.3) \) in the form of the Laplace inverse formula \( (2.8) \) defines then $\chi_1(x, \lambda)$ in the half plane $\Re \lambda > 0$ which the $\xi$-Riemann surface of which is three sheeted surface (see Fig. 3 and Appendix 2). On this surface $\tilde{\omega}(\xi(x))$ becomes additionally infinitely periodic with its complex periods acting however between different sheets of the surface. As a result of this an image of each root of $q(x)$ proliferates infinitely on the $\xi$-Riemann surface with every such a copy giving rise to still new branch point and sheet. The only exception of the latter rule is the linear potential case the $\xi$-Riemann surface of which is three sheeted with a single root branch point of the third degree.
Opening the brackets in (3.1) we get:

\[
Y_n(\xi, \lambda) = \sum_{0 \leq r_1 < \ldots \leq n} \sum_{0 \leq 2q \leq n} Y_{n; r_1 \ldots r_{2q}}(\xi, \lambda)(-1)^{r_1-r_2+r_3-\ldots+r_{2q-1}-r_{2q}} \quad (3.2)
\]

where

\[
Y_{n; r_1}(\xi) = \int_{\tilde{\gamma}_1(\xi)} d\xi_1 \ldots \int_{\tilde{\gamma}_1(\xi_{n-1})} d\xi_n \tilde{\omega}(\xi_1) \ldots \tilde{\omega}(\xi_n) = \frac{\Omega_n(\xi)}{n!} \quad (3.3)
\]

with

\[
\Omega(\xi) = \int_{\tilde{\gamma}_1(\xi)} d\eta \tilde{\omega}(\eta) \quad (3.4)
\]

and

\[
Y_{n; r_1 \ldots r_{2q}}(\xi, \lambda) = \int_{\tilde{\gamma}_1(\xi)} d\xi_1 \ldots \int_{\tilde{\gamma}_1(\xi_{n-1})} d\xi_n \tilde{\omega}(\xi_1) \ldots \tilde{\omega}(\xi_n) e^{2\lambda(\xi_{r_1} - \xi_{r_2} + \xi_{r_3} - \ldots + \xi_{r_{2q-1}} - \xi_{r_{2q}})} \quad (3.5)
\]

where

\[
\xi_0 \equiv \xi, \quad q = 1, 2, 3, \ldots
\]
Note that all the integrals in (3.5) are absolutely convergent. Therefore, it should be now obvious that to each integral in (3.5) the following Laplace transformation form can be given:

\[ Y_{r_1\ldots r_2 q}(\xi, \lambda) = \int_{C} dse^{2\lambda s} \tilde{Y}_{r_1\ldots r_2 q}(\xi, s) \]  

(3.6)

where the integration contour \( \tilde{C} \) starts at \( \Re s = -\infty \) and ends at \( s = 0 \) and the Laplace transform \( Y_{r_1\ldots r_2 q}(\xi, s) \) is to be determined. We do it in Appendix 1. An important observation done there is that it is possible to rearrange the order of terms in the series (2.3) in such a way to sum it in an accordance with the increasing \( q \) rather than \( n \) - the number of the integrations in (3.5). (All these are still possible since the series (2.3) is absolutely convergent). As a result of such reordering \( \chi_1(\xi, \lambda) \) can be represented as the following sum:

\[ \chi_1(\xi, \lambda) = 2\lambda \sum_{q \geq 0} \chi_{1}^{(q)}(\xi, \lambda) \]  

(3.7)

\( \xi \in S_1, \quad |\arg \lambda| < \pi \)

where

\[ \chi_{1}^{(q)}(\xi, \lambda) = \int_{\tilde{C}} dse^{2\lambda s} \tilde{\Phi}_{1}^{(q)}(\xi, s) \]  

(3.8)

with \( \tilde{\Phi}_{1}^{(q)}(\xi, s) \), \( q \geq 0 \) given by formulae (A1.12) of Appendix 1 and with the contour \( \tilde{C} \) shown in Fig. 5. Of course, since the series (3.7) is absolutely and uniformly convergent we have also:

\[ \chi_1(\xi, \lambda) = 2\lambda \int_{C} dse^{2\lambda s} \tilde{\Phi}_1(\xi, s) \]  

(3.9)

with \( \tilde{\Phi}_1(\xi, s) \) given by (A1.11) so that the corresponding Laplace transform \( \chi_1(\xi, s) \) defined by (2.6) can be identified as:

\[ \chi_1(\xi, s) \equiv \tilde{\Phi}_1(\xi, s) \]  

(3.10)

The expansions (3.7) and (A1.10) shall be called further topological expansions for the following two reasons:

1. the higher term of the series in (A1.12), the more complicated is its Riemann surface;
2. the Riemann surface \( R_q \) corresponding to the term \( \tilde{\Phi}_{1}^{(q)}(\xi, s) \) in (A1.11) can be reduced to some \( R_{q'} \) with \( q' < q \) when deprived of some singular points of \( \tilde{\Phi}_{1}^{(q)}(\xi, s) \) i.e. a set \( S_q \) of all singularities of \( \tilde{\Phi}_{1}^{(q)}(\xi, s) \) includes a set \( S_{q'} \) corresponding to \( \tilde{\Phi}_{1}^{(q')}(\xi, s) \) (see the next section).

3.2. Analytic properties of \( \tilde{\chi}_1(\xi, s) \)

The analytic properties of \( \tilde{\chi}_1(\xi, s) \) have been established in Section 3 of Appendix 1. As it follows from Appendix A1.3 the Laplace transform \( \tilde{\chi}_1(\xi, s) \) is holomorphic in some vicinity of
the point \( s = 0 \) for \( \xi \in \mathbb{R}(d^*) \) i.e. it is the Borel function \( \chi_1(\xi, \lambda) \) corresponding to \( \chi_1(\xi, \lambda) \). For \( \Re \xi > 0 \), however, \( \tilde{\chi}_1(\xi, s) \) is holomorphic in the half plane \( \Re s < 0 \). Therefore, the asymptotic series constructed for \( \chi_1(\xi, \lambda) \) when \( \lambda \to \infty \) is Borel summable to the function itself - a result which is in a full accordance with the corresponding one obtained in \([2]\) and mentioned in Section 2.

4 Singularity structure of \( \tilde{\chi}_1(\xi, s) \)

Because of (3.10) this is the singularity structure of \( \tilde{\Phi}_1(\xi, s) \) and the latter structure is determined by the corresponding singularity structures of \( \tilde{\Phi}_1^{(q)}(\xi, s) \) due to (A1.10). These structures can be investigated by the analytic continuation procedure of the formulae (A1.11) - (A1.12) with respect to \( s \) and \( \xi \) and are, on its own, determined completely by the corresponding singularity structures of \( \tilde{\omega}(\xi) \) and the integrations present in (A1.11) and (A1.12) (see Appendix 1). These integrations can give rise to singularities due to the following two mechanisms \([13]\):

1. moving singularity of the integrand approaches a fixed limit of the integration or, inversely, a moving limit of an integration approaches a fixed singularity of the integrand (so called end point (EP-) singularities).

2. moving singularity of the integrand approaches some another singularity pinching unavoidably in that way the integration contour (so called pinch (P-) singularities).

In the convolution integrals of the formula (A1.11) only the functions \( \tilde{\omega}(\xi) \) and \( \Omega(\xi) \) can give rise to both the (EP- and P-) singularity mechanisms since a dependence of the integrals on the remaining partners of the convolutions are holomorphic.

From the defining formulae (A1.12) and from the \( \xi \)-Riemann surface structure on which \( \tilde{\omega}(\xi) \) and \( \Omega(\xi) \) are defined (this structure was sketched in the previous section) it follows also that even for the simplest cases of first few \( \tilde{\Phi}_1^{(q)}(\xi, s) \)'s their global \( (\xi, s) \)-Riemann surface structures are too complicated to be fully handled and only some crude descriptions of them are possible limited to a few first sheets and a few singularities.

However, in making the corresponding analysis by limiting ourselves to first few \( q \)'s we are free in deforming the integration contours in (A1.12) i.e. the limitation of \( \tilde{\gamma}_1(\xi) \) to the canonical choices is no longer valid. This observation is very important and proves that the Borel function \( \tilde{\chi}_1(\xi, s) \) constructing initially for the fundamental solution of the sector \( S_1 \) is universal i.e. each Borel summable solution to the Schrödinger equation (2.1) can be obtained by the Borel transformation of \( \tilde{\chi}_1(\xi, s) \) with a properly chosen integration path in the Borel plane. A discussion of the latter property of the Borel summable solutions and some of its consequences is postponed however to another paper \([3]\).

Having in mind the incredible (in general) complexity of the \( (\xi, s) \)-Riemann surface structure of \( \tilde{\Phi}_1(\xi, s) \) we shall describe first a general procedure of getting this structure for first few \( \tilde{\Phi}_1^{(q)}(\xi, s) \)'s taking into account also a few singularities of \( \tilde{\omega}(\xi) \) and \( \Omega(\xi) \) and next we try to give as full as possible a description of such structures for the linear and harmonic potentials.

\[ q = 0 \]
It is seen from (A1.12) that $\tilde{\Phi}_1^{(0)}(\xi, s)$ is an entire function of $s$ for any $\xi$ not coinciding with singularities of $\tilde{\omega}(\xi)$. Its singularities in the $\xi$-variable coincide therefore with those of $\Omega(\xi)$ and consequently with those of $\tilde{\omega}(\xi)$ as the EP- singularities shown in Fig. 3.

$q = 1$

This is the Riemann surface structure of $\tilde{\Phi}_1^{(q)}(\xi, s)$ as defined by (A1.12) for $q = 1$.

\[
\tilde{\Phi}_1^{(1)}(\xi, s) = \int_{\tilde{C}(s)} d\eta \tilde{\omega}(\xi - \eta)(2s - 2\eta) \frac{I_1 \left( 8(s - \eta)\Omega(\xi - \eta) - 4(s - \eta)\Omega(\xi) \right)^{\frac{1}{2}}}{\left[ 8(s - \eta)\Omega(\xi - \eta) - 4(s - \eta)\Omega(\xi) \right]^\frac{1}{2}} \tag{4.1}
\]

It follows from (4.1) that singularities of the subintegral function are essential singularities coinciding with the branch points of $\Omega(\xi)$ and $\Omega(\xi - \eta)$. Because of the single $\eta$-integration in (4.1) only the EP-mechanism can generate singularities in the 's-plane' since all the $\eta$-singularities are the moving ones (depending linearly on $\xi$) so that the positions of all the (essential) singularities of $\tilde{\Phi}_1^{(1)}(\xi, s)$ coincide again with those of $\Omega(\xi - s)$ and $\Omega(\xi)$. Therefore, these positions on the $\xi, s$-Riemann surface are the following:

\[
\xi = \zeta_k, \quad \xi - s = \zeta_k, \quad k = 1, 2, \ldots, \text{ etc.} \tag{4.2}
\]

The nature of all these singularities is not altered by the integrations i.e. all they are branch points. Therefore, the resulting pattern of cuts on the corresponding Riemann surface which follows from Fig. 3 is sketched in the figures 4-5.

$q = 2$

This is the Riemann surface structure of $\tilde{\Phi}_1^{(2)}(\xi, s)$ as defined by (A1.11) for $q = 2$. From (A1.11) we have:

\[
\tilde{\Phi}_1^{(2)}(\xi, s) = \int_{\tilde{C}(s)} d\eta \int_{\tilde{\gamma}(\xi)} d\xi_1 \tilde{\omega}(\xi_1 - \eta)\tilde{\omega}(\xi_1)(2s - 2\eta) \frac{I_2(z_{\frac{1}{2}})}{z} \tag{4.3}
\]

\[z = 4(s - \eta) \left( \Omega(\xi) - 2\Omega(\xi_1) + 2\Omega(\xi_1 - \eta) \right) \]

Note that the $\xi$-integration in (4.3) runs across a sheet of the $\xi$-Riemann surface shown in Fig. 4 (where the $s$ variable is to be substituted by the $\eta$ one). However, contrary to the close correspondence between the distributions of sectors and turning points on the Stokes graph of Fig. 1 and of sheets and the corresponding cuts on Fig. 3 such a correspondence is lost in the case of Fig. 4 i.e. we are left only with some properly arranged system of branch points and cuts.

Since the half of the cuts in Fig. 4 are moving then except of the EP-singularities the P-singularities are also generated by both the $\xi$- and $\eta$-integrations in (4.3).
Consider first results of the $\xi$-integration in (4.3).

The EP-singularities which follow from this integration coincide (with the corresponding substitution $s$ by $\eta$) with those in the figures 4 and 5 are given again by (4.2).

A generation of P-singularities can be performed by moving singularities depending on $\eta$ (see Fig. 4). For example, moving clockwise the singularity $\eta + \zeta_1$ around the end point $\xi$
of $\tilde{\gamma}_1(\xi)$ and next pinching $\tilde{\gamma}_1(\xi)$ against $\zeta_1$ we generate a singularity of (4.3) at $\eta = 0$ in the $\eta$-Riemann surface. It is placed however on another sheet of the surface since to achieve it we had to go around the branch point singularity $\xi - \zeta_1$, shown in Fig. 5, in the clockwise direction.

To obtain all other $\eta$-plane singularities generated by the $\xi$-integration in (4.3) we proceed in the same way as described above. All these singularities lie on sheets which can be reached by going around the two branch points (in any direction - clockwise or anticlockwise) shown in Fig. 5. Therefore all these singularities are shared by the actual positions of the branch points cuts of Fig. 5. They can become visible by cutting the $\eta$-plane in a different way or moving appropriately both the branch points to the left.

![Diagram](image.png)

Fig. 6. The $\xi$-plane singularities corresponding to subintegral function in (4.3)

Choosing for example the last possibility and moving $\xi$ toward Sector 3 we shall arrive at the situation shown in Fig. 6. If $\xi$ and $\eta$ are moved so that $\Re \xi < \Re \zeta_2 = \Re (\eta + \zeta_1)$ then a further motion of $\eta + \zeta_1$ upwards to the point $\zeta_2$ pinches the path $\tilde{\gamma}_1(\xi)$ producing in that way a singularity at $\eta = \zeta_{21} \equiv \zeta_2 - \zeta_1$. It lies to the right from the cut at $\xi - \zeta_1$ in the '\eta-plane' and is screened therefore by the cut just mentioned when $\Re \xi > \Re \zeta_2$ (see Fig. 7).

By the identical analyses applied to each pair $\eta - \zeta_i$, $\zeta_j$ of the singularities lying on the sheet in Fig. 4 the singularities at $s = \zeta_{ij}$ or at $s = \zeta_{ji} = -\zeta_{ij}$ can be produced being screened by cuts at $s = \xi - \zeta_j$ or at $s = \xi - \zeta_i$, correspondingly. All the singularities produced in this way are branch points.

According to (4.3) the second, final integration is performed over the $\eta$-plane providing $\tilde{\Phi}^{(2)}_1(\xi,s)$ with all its $\xi$- and $s$-plane singularities. This integration transforms all the $\eta$-singularities obtained by the first ($\xi$-)integration into the corresponding $s$-ones (by the EP-mechanism) and provides us with additional $\xi$-singularities by the pinch mechanism. Pinching for example the singularity $\xi - \zeta_2$ against $\zeta_{21}$ we obtain the $\xi$-singularity at $\xi = \zeta_2 + \zeta_{21}$ lying...
on a sheet of the $\xi$-Riemann surface originated by the branch point at $\zeta_2$ on Fig. 4. This branch point is screened, of course, by the cut at $\xi = s + \zeta_2$ when $\Re s > \Re \zeta_{ij}$ (see Fig. 8). Therefore, figures 7-8 show the complete singularity structure of $\tilde{\Phi}_1^{(2)}(\xi, s)$ when continued in $\xi$ in the way shown in Fig. 6.

Fig. 7. The $s$-plane singularities corresponding to $\tilde{\Phi}_1^{(2)}(\xi, s)$

4.1. The analytic structure of the Borel function for the linear potential

We can put for this case $q(x, E) \equiv x$ and $\xi = x^{3/2}$ with the corresponding Stokes graph shown in Fig. 9. and we shall consider $\tilde{\Phi}_1(\xi, s)$ as the Borel function defined by the fundamental solution $\Psi_1(x, \lambda)$.

Fig. 8. The $\xi$-plane singularities corresponding to $\tilde{\Phi}_1^{(0)}(\xi, s)$
At the first glance the corresponding analysis seems to be simple because of the simplicity of the relevant functions $\tilde{\omega}(\xi) = -\frac{5}{16} \frac{1}{\xi^2}$ and $\Omega(\xi) = \frac{5}{16} \frac{1}{\xi}$ as a result of which the three sheeted Riemann surface branching at $\xi = 0$ (the surface being the image of the two sheeted $x$-plane by the transformation $\xi = x^{3/2}$) decouples into three independent sheets. The unity of the surface is recovered however by the solution $\Psi_1(x,\lambda)$ which being holomorphic at $x = 0$ branches at this point as $\xi^{2/3}$ when considered as a function of $\xi$. However, since we are interested in the properties of the Borel function $\tilde{\Phi}_1(\xi, s)$ determined rather by $\chi_1(\xi, \lambda)/2\lambda$ it is the latter the $(\xi, \lambda)$-dependence of which is most important.

Fig. 9. The Stokes graph for the linear potential

The latter dependence can be established to some extent noticing that continuing analytically the solution $\Psi(\xi, \lambda) = \xi^{-\frac{1}{6}} e^{-\lambda \xi} \chi_1(\xi, \lambda)$ in the $\lambda$-plane (whilst $\xi$ is fixed) by rotating $\lambda$ by the angle $\pm6\pi$ we come back with the beginning of the integration path $\gamma(\xi)$ in $\chi_1(\xi, \lambda)$ to the infinity of the first sector. Of course, this path is by the above continuation deformed from the initial canonical one into the one surrounding the point $\xi = 0$ twice (in the direction suitable to the sign) to end eventually at the point $\xi$. This is because continuing $\chi_1(\xi, \lambda)$ in $\lambda$ in the above way we have to shift the infinite end of the path to the neighbour sectors each time when $\lambda$ changes by $\pm\pi$ (this operation keeps the factor $e^{-\lambda \xi}$ always vanishing in the infinities of the passed sectors).

However, the above $\lambda$-continuation of $\chi_1(\xi, \lambda)$ is equivalent to its continuation to the same point $\xi$ along the deformed path $\tilde{\gamma}(\xi)$ starting from its initial canonical form. Since by this latter continuation the argument of $\xi$ changes also by $\pm6\pi$ then the factor $\xi^{-\frac{1}{6}}$ of $\Psi(\xi, \lambda)$ acquires minus by this continuation so does the factor $\chi_1(\xi, \lambda)$ since by this continuation $\Psi(\xi, \lambda)$ can not change because it branches at $\xi = 0$ as $\xi^{2/3}$. It follows therefore that $\chi_1(\xi, \lambda)$ branches at $\xi = 0$ as $\xi^{1/6}$.

From the latter observation it follows further directly that the Borel function $\tilde{\Phi}_1(\xi, s)(\equiv \tilde{\chi}_1(\xi, s))$ branches at the infinity point of its $s$-plane also as $s^{1/6}$. This can be seen noticing that to recover the factor $\chi_1(\xi, \lambda)$ by the Borel transformation of $\tilde{\Phi}_1(\xi, s)$ we have to change successively the integration path in the transformation from the negative real halfaxis to the positive one (and vice versa) according to which the sector the infinite end of the deformed
path $\tilde{\gamma}(\xi)$ is actually in. These Borel transformation paths are again the deformations of each other obtained by moving the infinite end of them along the circle of infinite radius i.e. all the singularities of $\tilde{\Phi}_1(\xi, s)$ are avoided by these deformations. Since after six such changes the Borel transformation of $\tilde{\Phi}_1(\xi, s)$ has to change its sign in comparison with its initial value so $\tilde{\Phi}_1(\xi, s)$ itself has to do it.

Therefore, we conclude that for fixed $\xi$ the s-Riemann surface of $\tilde{\Phi}_1(\xi, s)$ is built of six sheets.

The above situation is however not so simple when the formulae (A1.12)-(A1.14) defining $\tilde{\Phi}_1(\xi, s)$ are considered. The Bessel functions in these formulae convert the simple pole of $\Omega(\xi)$ at $\xi = 0$ into a corresponding root (of the forth order) branch points accompanied by essential singularities (see Appendix 3.1). Also the successive $\xi$- and $\eta$-integrations in these formulae have to generate unavoidably the branch points at $\xi = 0$, $s = 0$ and $\xi = s$ of the logarithmitic type. This is of course because the representation of $\tilde{\Phi}_1(\xi, s)$ given by (A1.12)-(A1.14) is singular providing us with the correct positions of singularities but not necessarily with their nature. The above example of the linear oscillator shows that the proper behaviour of $\tilde{\Phi}_1(\xi, s)$ close to its singularities is obtained only by the full resummation of these series. Nevertheless, in more complicated cases of potentials an information the series provide us are certainly very useful. Also in the case just considered.

Namely, taking into account the recurrent relations (A1.14) we can establish inductively that $\tilde{\Phi}_1(\xi, s)$ being defined on its six sheeted $(\xi, s)$-Riemann surface has on its first two sheets singularities shown in the figures 10a,b (see Appendix 3.1 for details). The point $s = 0$ on the sheet of Fig. 10b is regular for $\tilde{\Phi}_1(\xi, s)$, according to general results of App. 1. According to this analysis the points $\xi - s = 0$ are the four order branch points of $\tilde{\Phi}_1(\xi, s)$ and simultaneously its essential singularities but we should have in mind that the last two properties can be incorrect.

The same property concerns the points $\xi = 0$ and $s = 0$ the latter being on the second and further sheets of Fig. 10b. All these points arrange themselves to build in the considered approximation of $\tilde{\Phi}_1(\xi, s)$ infinitely sheeted Riemann surface. However, even for this simple case the topology of the surface except its first two sheets is too complicated to be fully described.

Fig. 10. The $\xi$- and $s$-plane singularities corresponding to $\tilde{\Phi}_1^{(0)}(\xi, s)$
Nevertheless, one general conclusion valid at least for all the polynomial potentials can be drawn from the above consideration. Namely, if for a general polynomial potential we consider any pair of neighbour sectors joined by the analytic continuation in \( \lambda \) when \( \lambda \to e^{\pm i\pi} \lambda \) and we continue a fundamental solution defined in one of the sectors to the second along the canonical path then the corresponding 's-plane' singularity structure of the first sheet of the respective Borel function \( \Phi_1(\xi, s) \) is exactly the same as for the 'simplest' case of the linear potential described above.

4.2. An alternative non-standard Borel representation for the linear potential wave function

In the previous subsection we have made a disappointed note that even in such a simple case as the linear potential one the corresponding Borel function properties which follow from the topological expansion are quite complicated. We have however shown also that the actual structure of the linear potential Borel plane should be rather simple. Below, we want to show that indeed this complication is apparent and changing a little bit the definition of the Borel function one can simplified the latter enormously for the case considered. Namely, let us replace the definition (2.7) of the Borel function by the following one

\[
\tilde{\chi}_{alt}^1(\xi, \sigma) = \sum_{n \geq 0} \frac{(-\sigma)^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})} \kappa_{1,n}(\xi) \tag{4.4}
\]

which corresponds to the following representation of \( \tilde{\chi}_{alt}^1(\xi, \sigma) \) by the Laplace transformation

\[
\tilde{\chi}_{alt}^1(\xi, \sigma) = \frac{1}{\pi i} \int_{-i\infty + \lambda_0}^{+i\infty + \lambda_0} e^{-2\lambda \sigma} \frac{\chi_1(\xi, \lambda)}{(2\lambda)^{\frac{1}{2}}} d\lambda \tag{4.5}
\]

so that the invers Borel transformation is given by

\[
\chi_1(\xi, \lambda) = (2\lambda)^{\frac{1}{2}} \int_{-\infty}^{0} e^{2\lambda \sigma} \tilde{\chi}_{alt}^1(\xi, \sigma) d\sigma \tag{4.6}
\]

Let us now make use of the fact that the fundamental solution \( \Psi_1(x, \lambda) \) can be given the following integral representation (see [30], Mathematical appendix)

\[
\Psi_1(x, \lambda) = \frac{i}{\sqrt{\pi}} (2\lambda)^{\frac{3}{2}} \int_C e^{\lambda(x y - \frac{y^3}{3})} dy \tag{4.7}
\]

where we put \( x \) real and positive and the contour \( C \) is shown in Fig. 9.

Changing in (4.7) the integration variable \( y \) into \( x^{-1/4} y \) and next putting \( 2\sigma = x^{3/4} y - x^{-3/4} y^3/3 + 2x^{2/3}/3 \) we can bring the integral to the following form
\[
\Psi_1(x, \lambda) = x^{-\frac{3}{4}}e^{-\frac{4}{3}x^{\frac{1}{2}}} \sqrt{\frac{3}{\pi}} (2\lambda)^{\frac{1}{2}} \int_{-\infty}^{0} e^{2\lambda \sigma} d\sigma \tag{4.8}
\]

\[
\left[ (-3(\sigma - \frac{1}{3}x^{\frac{3}{2}})x^{\frac{3}{4}} + 3x^{\frac{3}{4}}\sqrt{\sigma(\sigma - \frac{2}{3}x^{\frac{3}{2}})})^{\frac{1}{2}} - (-3(\sigma - \frac{1}{3}x^{\frac{3}{2}})x^{\frac{3}{4}} - 3x^{\frac{3}{4}}\sqrt{\sigma(\sigma - \frac{2}{3}x^{\frac{3}{2}})})^{\frac{1}{2}} \right] d\sigma
\]

Hence for \( \tilde{\chi}^{alt}_1(\xi, \sigma) \) we get finally

\[
\tilde{\chi}^{alt}_1(\xi, \sigma) = \sqrt{\frac{3}{\pi}} \left[ (-3(\sigma - \frac{1}{3}\xi)(\frac{2}{3}\xi)^{\frac{1}{2}} + 3(\frac{2}{3}\xi)^{\frac{1}{2}}\sqrt{\sigma(\sigma - \xi)})^{\frac{1}{2}} - (-3(\sigma - \frac{1}{3}\xi)(\frac{2}{3}\xi)^{\frac{1}{2}} - 3(\frac{2}{3}\xi)^{\frac{1}{2}}\sqrt{\sigma(\sigma - \xi)})^{\frac{1}{2}} \right] \tag{4.9}
\]

It follows from (4.9) that \( \tilde{\chi}^{alt}_1(\xi, \sigma) \) is defined on the two sheeted Riemann surface having the branch points \( \sigma = 0 \) and \( \sigma = \xi \) as its unique singularities.

The non standard representation (4.5)-(4.6) of the Borel function considered above shows that the complicated form (2.7) of the standard one depends on the representation itself and it can be simplified greatly by the proper choice of such a representation.

### 4.3. The singularity structure of the Borel function for the harmonic oscillator

Making, if necessary, a suitable rescaling we can put in this case \( q(x) = x^2 + 1 \) (assuming the energy to be negative). The corresponding Stokes graph is then shown in Fig. 11 and we choose as usually the sector 1 to provide us with the fundamental solution \( \Psi_1(x, \lambda) \) and its Borel function \( \tilde{\Phi}_1(\xi, s) \). Because of the last conclusion of Section 4.1 we consider now a case of the Riemann surface structure corresponding to \( \tilde{\Phi}_1(\xi, s) \) when \( \xi(= \int_{-i}^{x} \sqrt{y^2 + 1} dy) \) is continued to the sector 3 of Fig. 11 (along a canonical path). Then the first sheets of \( \tilde{\Phi}_1^{(1)}(\xi, s) \) and \( \tilde{\Phi}_1^{(2)}(\xi, s) \) are shown in the figures 12a,b and 13a,b respectively.

Using again the formulae (A1.14) we can show inductively that the first sheets of \( \tilde{\Phi}_1^{(2q)}(\xi, s) \) and \( \tilde{\Phi}_1^{(2q+1)}(\xi, s) \) look as in the figures 14 and 15. All the detailed considerations establishing these can be found in Appendix 3.2.

### 4.4 Borel plane structure of harmonic oscillator Joos function

When in the consideration of the previous subsection we shall push \( R\xi \) to minus infinity (this corresponds to push \( x \) to the infinite point \( \infty_3 \) of the sector 3 of Fig. 11) than we get the 'Borel plane' singularity structure of the so called Joos function for the harmonic oscillator. The last name is given to the coefficient \( \chi_{1-\infty_3}(\lambda) \equiv \lim_{\xi \to \infty_3} \chi_1(\xi, \lambda) \) \( \tag{19} \) so that the energy spectrum of the harmonic oscillator is given by \( \chi_{1-\infty_3}(\lambda) = 0 \). Note that in the limit \( \xi \to \infty_3 \) all the functions \( \tilde{\Phi}_1^{(2q+1)}(\xi, s) \) vanish so that the corresponding limiting functions \( \tilde{\Phi}_1^{(2q)}(s) \) contribute only to \( \tilde{\chi}_{1-\infty_3}(s) \).
As it follows from the considerations of the previous subsection the singularity structure of the latter function is determined by the branch points distributed along the imaginary axes of the $s$-Riemann surface. This distribution can be described completely if instead of the Borel function $\tilde{\chi}_{1\to3}(s)$ we shall consider the one corresponding to $\log \chi_{1\to3}(\lambda)$. To this end let us note that as it follows from Fig. 11a the normal sector of $\chi_{1\to3}(\lambda)$ (i.e. the one where $\chi_{1\to3}(\lambda)$ is holomorphic and can be expanded into the semiclassical series (2.5)) is defined by $|\arg \lambda| < \pi$. One can easily find also (by analytic continuation in $\lambda$) that

$$\chi_{2\to4}(\lambda) = \chi_{1\to3}(\lambda e^{i\sigma \pi}), \quad 0 < |\arg \lambda| < \pi$$

(4.10)

where $\sigma = \arg \lambda/|\arg \lambda|$ and $\chi_{1\to3}(\lambda)$ are the canonical coefficients corresponding to the graphs of Fig. 11b and 11c respectively. Despite (4.10) these two canonical coefficients obey the following two other relations

Fig. 11. The Stokes graphs corresponding to the harmonic oscillator

```
\[ \chi_{1\to3}(\lambda) \chi^\pm_{2\to4}(\lambda) = 1 + e^{\pm i\lambda}, \quad 0 < \arg \lambda < \pi \]  
(4.11)  
\[ \chi_{1\to3}(\lambda) \chi^\pm_{2\to4}(\lambda) = 1 + e^{-\pm i\lambda}, \quad -\pi < \arg \lambda < 0 \]  

in which the fact that \( 2 \int_{i}^{i} \sqrt{y^2 + 1} dy = \pi i \) has been used.

The relations (4.11) follows as a result of an identity which the four fundamental solutions \( \Psi_k, \ k = 1, ..., 4 \), corresponding to the Stokes graphs of Fig. 11 have to satisfy since only two of them can be linearly independent.

Using (4.11) we get from (4.11)

\[ \chi_{1\to3}(\lambda) \chi_{1\to3}(\lambda e^{-i\sigma \pi}) = 1 + e^{\pi i \sigma \lambda} \]  
(4.12)

where \( \sigma = \arg \lambda/|\arg \lambda| \) and \( 0 < |\arg \lambda| < \pi \).

The formula (2.6) can now be used directly to define the Laplace transform \( \tilde{\chi}_{1\to3}(s) \) of the Joos function \( \chi_{1\to3}(\lambda) \) with the integration contour \( C_{13} \) running around the negative half of the real axis of the \( \lambda \)-plane. In this way \( \tilde{\chi}_{1\to3}(s) \) is defined by (2.6) as the holomorphic function in the half-plane \( \Re s < 0 \).

![Fig. 12. The "first sheets" singularities of \( \tilde{\Phi}^{(1)}(\xi, s) \) for the harmonic potential](image)

The analytic structure of \( \tilde{\chi}_{1\to3}(s) \) can, however, be best handled if we consider \( \log^* \tilde{\chi}_{1\to3}(s) \) rather than the function itself \( [13] \). Namely, we have:

\[ \log^* \tilde{\chi}_{1\to3}(s) = \frac{1}{2\pi i} \int_{C_{13}} e^{-2\lambda s} \log \chi_{1\to3}(\lambda) d\lambda \]  
(4.13)

and we can use (4.12) to calculate (4.13) exactly. (Note, that \( \chi_{1\to3}(\lambda) \) does not vanish in the \( \lambda \)-plane cut along the negative half of the real axis). Using (4.12) we have:
\[
\log^* \tilde{\chi}_{1\to3}(s) = \frac{1}{2\pi i} \int_{C^u(\lambda_0)} e^{-2\lambda s} \log(1 + e^{\pi i\lambda}) d\lambda + \frac{1}{2\pi i} \int_{C^d(\lambda_0)} e^{-2\lambda s} \log(1 + e^{-\pi i\lambda}) d\lambda + \frac{1}{2\pi i} \int_{C_{13}(\lambda_0)} e^{-2\lambda s} \log(1 + e^{\pi i\lambda}) d\lambda
\]

where \( C_{13}(\lambda) \) is one of the contours \( C_{13} \) crossing the real axis at \( \lambda_0 > 0 \) and \( C^u(\lambda_0), C^d(\lambda_0) \) are parts of it lying above and below of the real axis correspondingly.

Performing the integrations in the first two integrals in (4.14) (by expanding the logarithms) and changing \( \lambda \) into \(-\lambda\) in the third one we get:

\[
\log^* \tilde{\chi}_{1\to3}(s) = \frac{1}{4\pi i} \sum_{n \leq 1} \frac{(-1)^{n+1}}{n} \left( \frac{e^{\pi i n \lambda_0}}{s - \frac{\pi n}{2}} - \frac{e^{-2\pi i n \lambda_0}}{s + \frac{\pi n}{2}} \right) e^{-2\lambda_0 s} + \frac{1}{2\pi i} \int_{C'(\lambda_0)} e^{2\lambda s} \log(\chi_{1\to3}(\lambda)) d\lambda
\]

where \( C'(\lambda_0) \) is the contour encircling (anticlockwise) the point \( \lambda = 0 \) and starting and ending at the point \( \lambda = -\lambda_0 \) of the real axis. Since the right hand side of (4.15) is independent of \( \lambda_0 \) it can be calculated at \( \lambda_0 \to 0 \). It can be shown (see Appendix 5) that the integral in (4.15) vanishes in this limit and therefore we finally get:

\[
\log^* \tilde{\chi}_{1\to3}(s) = \frac{1}{2} \sum_{n \leq 1} \frac{(-1)^{n+1}}{s^2 + \frac{n^2\pi^2}{4}} = \frac{1}{2is} \left( \frac{1}{2is} - \frac{1}{\sin(2is)} \right)
\]
The result (4.16) was established essentially by Voros [19] but here it is obtained directly by the definition (4.13) of the Laplace transform for \( \log \chi_{1 \rightarrow 3}(\lambda) \). The inverse transformation can be also performed to give the known expression for \( \chi_{1 \rightarrow 3}(\lambda) \) [19].

\[
\xi \gamma (\xi) \zeta - (2 - 1) \zeta - \zeta^2 \zeta^2 (\zeta - 1) \zeta
\]

Fig. 14. The "first sheets" singularities of \( \tilde{\Phi}_{1}^{(2q)}(\xi, s) \) for the harmonic potential

Summarizing the above analyses one can see that despite the clear way of obtaining corresponding singularity patterns and the underlying structures of the Riemann surfaces both they become still more and more complicated with increasing \( q \). The following main observations follow, however, from this analyses:

1. The set \( S_{q+1} \) of singularities corresponding to \( \tilde{\Phi}_{1}^{(q+1)}(\xi, s) \) contains the set \( S_{q} \) of these corresponding to \( \tilde{\Phi}_{1}^{(q)}(\xi, s) \).

2. The new singularities which belong to \( S_{q+1} \setminus S_{q} \) are generated on the sheets originated by the singularities of \( S_{q} \); the latter is true both on the \( \xi \)- and on the \( s \)-planes.
The following comment concerning the positions of the singularities themselves and their relation to the Feynman path integral is in order. From the above discussion it is seen that these positions are determined by the values of the classical action the latter takes on along suitable classical paths corresponding to the case considered. The paths are real as well as complex (i.e. they are real or complex solutions to the classical equation of motion). They contribute to calculated quantities \( \tilde{\Phi}_1^{(2q+1)}(\xi, s) \), \( q \geq 0 \), in a hierarchical way described above so that a path with greater absolute value of the real part of the corresponding action contributes to the later term \( \tilde{\Phi}_1^{(q)}(\xi, s) \) of the topological expansion. In this way the latter expansion reflects its close relation to the semiclassical expansion based on the Feynman path integral and the saddle-point technique as well as it confirms the role of complex classical paths in such calculations [15, 16, 17, 18].

5 An application: the connection problem

The connection problem is an old problem of the JWKB theory which in the context of the Balian-Bloch representation was considered first by Voros [19]. We shall discuss again this problem within the framework of our formalism to show the equivalence of the solution it provides with the corresponding method used in our earlier papers (see, for example, [12, 22, 28]).

The main question is how the JWKB formula, being a good approximation to a given solution in some domain of the \( x \)-plane, should be changed (in order to remain still the good approximation of the solution) when the solution is continued analytically to another domain of the \( x \)-plane. The problem can be solved in many different ways depending on the type of the considered solutions (see, for example, [20, 21, 24, 25]). In particular, it can be solved
with the aid of fundamental solutions (see [12] for the relevant procedure in an application to matrix element evaluations in JWKB approximation).

In the framework of the Balian-Bloch representation the solution of the problem is the following. Consider the fundamental solution to (2.1) given by (2.2)-(2.4) and continued along a path $\gamma_1$ to sector 2 (see Fig. 2). As it follows from the previous section analysis continuing analytically along the considered path we can not meet singularities above the corresponding path $\tilde{\gamma}_1(\xi)$ in the $\xi$-plane and, therefore, the singularity pattern of $\tilde{\chi}_1(\xi, s)$ in the $s$-plane looks like in Fig. 16a i.e. there are only two cuts on the relevant sheet. Since the integration along $\tilde{C}$ is limited only to lie in the left half-plane we can deform it freely in this half-plane to the position $\tilde{C}_3$, for example (see Fig. 16c). But doing this we have to integrate also along the cuts starting at the points $s = \xi - \zeta_1$ and $s = \xi - \zeta_1$, respectively.

Fig. 16. The Borel plane singularities corresponding to $\tilde{\chi}_1(\xi, s)$
Thus, $\psi_1(\xi, \lambda)$ is represented in this way as the following sum:

$$
\psi_1(\xi, \lambda) = 2q^{-\frac{1}{2}}(\xi)e^{-\lambda \xi} \int_C e^{2\lambda s} \chi_1(\xi, s) ds \quad (5.1)
$$

$$
2q^{-\frac{1}{2}}(\xi)e^{-\lambda \xi} \left( \int_{C_1} + \int_{C_2} + \int_{C_3} \right) e^{2\lambda s} \chi_1(\xi, s) ds
$$

where $\xi = \int_{x_0}^x q^{\frac{1}{2}} dy$. It should be noticed now that each term of the sum in (5.1) is a solution to the Schrödinger equation (2.1) (see, for example, [3]). It is also not difficult to see that each of the solutions generated by the integrations along $\tilde{C}_1$ and $\tilde{C}_2$ is proportional to the fundamental solution defined in the sector 2 of Fig. 2, whilst the remaining third solutions generated by the integration along $\tilde{C}_3$ - to $\psi_3(\xi, \lambda)$ - the fundamental solution defined in the sector 3. (An easy way to establish these facts is to investigate the behaviour of these solutions when $\xi$ goes to $\infty_2$ and $\infty_3$ correspondingly ($\infty_k$ being the infinity point in the sector $k$). In this way a linear combination of the fundamental solutions $\psi_2$ and $\psi_3$ to form the solution $\psi_1(\xi, \lambda)$ is realized simply by moving the contour $\tilde{C}$ in the $s$-plane.

The connection problem arises when $\psi_1(\xi, \lambda)$ is continued to the sector 3 by crossing the sector 2 i.e. along some non-canonical path $\gamma_1$" in Fig. 2. At the end of such a continuation the dominant character of the JWKB factor $q^{-1/4} \exp(-\lambda \xi)$ is lost in favour of the amplitude factor $\chi_1(\xi, \lambda)$ but the series (2.3) does not give then an easy answer to what actually happens when $\Re \xi \to \infty$ along such a path. (In fact $\chi_1(\xi, \lambda)$ behaves then as $\exp(2\lambda \xi)$).

In the $s$-plane the analytic continuation just described results in a deformation of the contour $\tilde{C}$ to the form shown in Fig. 16d (broken line). It follows obviously from the figure that the dominant contribution to $\psi_1(\xi, \lambda)$ comes now from the integration along $\tilde{C}_3$ i.e. from the solution proportional to $\psi_2(\xi, \lambda)$ and this is the way by which the connection problem is solved within the framework of the Balian-Bloch representation. (Note, that both the solutions defined by the integrations along $\tilde{C}_1$ and $\tilde{C}_3$ are subdominant when $\lambda \to \infty$ and $\xi$ stays in the sector 3 or when $\Re \xi \to \infty_3$ and $\lambda$ is fixed).

It is easy to see further, that the linear combination in the RHS of (5.1) can be explicitly reconstructed with the aid of the canonical coefficients $\alpha_{i/j \to k}$ ($\alpha_{i/j \to k} = \lim_{x \to \infty_k} \psi_i(x)/\psi_j(x)$, see [2]) as follows:

$$
\psi_1(\xi, \lambda) = \alpha_{1/2 \to p} \psi_2(\xi, \lambda) + \alpha_{1/p \to 2} \psi_3(\xi, \lambda) = \alpha_{1/2 \to p} \psi_2(\xi, \lambda) + \alpha_{1/p \to 2} \alpha_{p/2 \to 3} \psi_2(\xi, \lambda) + \alpha_{1/p \to 2} \alpha_{p/3 \to 2} \psi_3(\xi, \lambda) \quad (5.2)
$$

where the sequence of terms of the last sum in (5.2) corresponds strictly to the sequence of integrations along $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ in (5.1). The first linear combination appears in (5.2) when the contour $\tilde{C}_1$ is deformed to the position $\tilde{C}_p$ shown in Fig. 16b.

It is also worthwhile to note that the formula (5.2) giving us the continuation of $\psi_1(\xi, \lambda)$ to the sector 3 along the noncanonical path $\gamma_1$" can be also used to obtain in a simple way the improved connection formula of Silverstone [24] (see also the recent work of Fröman and Fröman [28]) with $\psi_3$ playing the role of the subdominant contribution. It is enough to this end to substitute each term in the sums in (5.2) by its corresponding JWKB approximation (i.e. none cumbersome Borel resummation used by Silverstone is necessary).
6 Exponential asymptotics

The problem of the semiclassical expansions for physical quantities is strictly related to the problem of so called exponentially small contributions absent (by definition) when only the bare semiclassical expansions of these quantities are considered \[23, 24, 25, 26, 27\]. The exponentially small contributions become important if the accuracy of the best semiclassical approximation is considered to be insufficient. There are however two aspects of this problem a difference between which was, according to our knowledge, not discussed properly.

The first one appears when the corresponding semiclassical series if Borel summed does not reproduce correctly the quantity considered. A discrepancy has to be of course exponentially small not contributing to the semiclassical limit. It can however be incorporated into the resurgent quantity by choosing another integration path in the Borel 'plane' i.e. the original path has had to be chosen incorrectly if the quantity considered was to be Borel summable (see \[3\] for the corresponding discussion). If it is the case then the Borel summation along the correct path has to incorporate all the exponentially small contributions to the quantity considered.

The second aspect appears when the Borel transform reproduces completely the quantity considered and the semiclassical series is used as a source of the best approximation. As it is well known (see for example \[29\]) the latter is obtained in this case by abbreviating the series on its least term (since the series is divergent) the order of which is proportional to actual value of \(\lambda(h^{-1})\) (in fact \(n\) should be equal to the integer part of \(\lambda|s_0|\) where \(s_0\) is a singularity of the Borel function closest to the origin). The remainder (i.e. the difference between the quantity and its approximation) is then exponentially small quantity.

To improve this approximation using still the semiclassical tools we have to be able to identify the exponential factor of the remainder and to multiply the last factor again by some optimal abbreviation of a new semiclassical expansion of the remainder. Next, we should be able to repeat this procedure to the remainder of the remainder constructing in this way still more and more accurate semiclassical approximation which includes as many exponentially small contributions as we need to make the approximation as good as we wish. In Appendix 4 we show how to do it. According to the beginning of this discussion the exponentially small contributions obtained in this way have to be determind by the singularity structure of the corresponding Borel functions provided for example by the topological expansions. The results of Appendix 4 confirm these expectations.

7 Exponential asymptotics of energy levels

Using the approximation scheme which follows obviously from the topological expansion and from the results of App. 4 we shall determine in this section the way of obtaining the semiclassical exponential asymptotic for energy levels of the anharmonic oscillillator corresponding to the potential \(V(x) = x^2 + x^4\) with the Stokes graph shown in Fig. 17 and drawn for \(E > 0\). Taking into account the symmetry of the potential we can write the quantization condition for the energy levels in the form

\[
\exp\left(\frac{\lambda}{2} \oint_K \sqrt{V(x) - E(\lambda)} dx \pm i\frac{\pi}{2}\right) = \chi_1 \rightarrow 3(E^\pm(\lambda), \lambda)
\] (7.1)
where the contour $K$ is shown in Fig. 17 and "±" in (7.1) correspond to the even and odd parities of the levels respectively.

Making the complex conjugation of both the sides of (7.1) we get an alternative condition for the energy level quantization. Both the versions are important since they determine the normal sector of $E(\lambda)$ to be $0 < |\arg \lambda| < 3\pi/2$ for $\lambda$ sufficiently large [2]. Because of this the semiclassical series of $E(\lambda)$ is, as we have shown in our earlier paper [2] (see also [3, 4, 15, 23]), Borel summable to $E(\lambda)$ itself and the singularity structure of $\tilde{E}(s)$ on its Borel plane is also determined by (7.1) and its complex conjugation. As it follows from (7.1) this structure is symmetric with respect to the real axis of the $s$-plane and $E(\lambda)$ can be recovered by integrating $\tilde{E}(s)$ along the negative halfaxis. Of course to apply to the last integral the procedure of App. 4 we have to know a detailed distribution of singularities of $\tilde{E}(s)$ on its Borel plane. But instead of this we can use (7.1) directly to establish the respective exponentially small contributions to $E(\lambda)$. Namely ordering these contributions according to their exponential smallness we can treat each such a contribution as a correction to its predecessors and use the Taylor series expansion to take into account the corresponding contribution. So we can write

$$E(\lambda) = E_0(\lambda) + E_1(\lambda) \ldots $$

(7.2)

with $E_1(\lambda)$ being polynomially dependent on $\lambda^{-1}$ and with the contributions $E_1(\lambda), E_2(\lambda), \ldots$, being each exponentially small with respect to their predecessors. Of course, $E_0(\lambda)$ is constructed in the standard way (see [1] and App. 4) and, for a given $\lambda$, it has some well defined numerical value.

The corresponding Taylor expansion of $\chi_{1 \rightarrow 3}(E(\lambda), \lambda)$ with respect to the dependence of the latter on the energy is the following

$$\chi_{1 \rightarrow 3}(E(\lambda), \lambda) = \chi_{1 \rightarrow 3}(E_0(\lambda), \lambda) + \frac{\partial \chi_{1 \rightarrow 3}(E_0(\lambda), \lambda)}{\partial E}(E_1(\lambda) + E_2(\lambda) + \ldots) + \ldots $$

(7.3)
Since the quantities in (7.2) are real we shall write the corresponding quantization condition in its real form, too, to get

\[
\sin \left( \frac{\lambda}{2i} \oint_K \sqrt{V(x) - E_0^\pm(\lambda)} \, dx - \frac{\lambda}{4i} (E_1^\pm(\lambda) + E_2^\pm(\lambda) + \ldots) \oint_K \frac{dx}{\sqrt{V(x) - E_0^\pm(\lambda)}} + \ldots \right) = \mp \Re \left( \chi_{1\to3}(E^\pm(\lambda), \lambda) + (E_1^\pm(\lambda) + E_2^\pm(\lambda) + \ldots) \frac{\partial \chi_{1\to3}(E_0^\pm(\lambda), \lambda)}{\partial E} + \ldots \right)
\]

(7.4)

It is now clear that we can apply to the coefficient \( \chi_{1\to3}(E_0(\lambda), \lambda) \) and its derivatives the procedure of App. 4 considering \( E_0(\lambda) \) as having well defined value so that the singularity structure of the corresponding Borel function \( \tilde{\chi}_{1\to3}(E, s) \) is determined just by the value of \( E \) equal to \( E_0(\lambda) \).

Assuming the exponential asymptotics for \( \chi_{1\to3}(E_0(\lambda), \lambda) \) to be ordered in a way analogous to (7.2) we get for the first two terms of (7.2)

\[
\sin \left( \frac{\lambda}{2i} \oint_K \sqrt{V(x) - E_0^\pm(\lambda)} \, dx \right) = \mp \Re \left( \chi_{1\to3}^{(0)}(E_0^\pm(\lambda), \lambda) \right)
\]

(7.5)

and

\[
E_1^\pm(\lambda) = \frac{\pm \Re \left( \chi_{1\to3}^{(1)}(E_0^\pm(\lambda), \lambda) \right)}{\Re \oint_K \frac{dx}{\sqrt{V(x) - E_0^\pm(\lambda)}} \cos \left( \frac{\lambda}{2i} \oint_K \frac{dx}{\sqrt{V(x) - E_0^\pm(\lambda)}} \right) + \Re \left( \frac{\partial \chi_{1\to3}^{(0)}(E_0^\pm(\lambda), \lambda)}{\partial E} \right)}
\]

(7.6)

where \( \chi_{1\to3}^{(0)}(E_0(\lambda), \lambda) \) (we shall suppress the parity indices as unimportant for our further considerations) is given by the respective number of the first terms of the series (2.5) (i.e. abreviated at its corresponding least term; note also that the integrations in (2.5) go from \( \infty_1 \) to \( \infty_3 \) along the canonical path) whilst the exponential contribution \( \chi_{1\to3}^{(1)}(E_0(\lambda), \lambda) \) is determined according to App. 4 by the singularities of \( \tilde{\chi}_{1\to3}(E_0(\lambda), s) \) in its Borel plane closest to the origin.

Applying now the approximations following from the topological expansion (A1.11) for \( \tilde{\chi}_{1\to3}(E_0(\lambda), s) \) we can write (keeping only the first two terms of this expansion)

\[
\tilde{\chi}_{1\to3}(E_0(\lambda), s) = \tilde{\Phi}_{1\to3}^{(0)}(E_0(\lambda), s) + \tilde{\Phi}_{1\to3}^{(2)}(E_0(\lambda), s)
\]

(7.7)

where

\[
\tilde{\Phi}_{1\to3}^{(0)}(E_0(\lambda), s) = I_0 \left( \sqrt{\int_{\infty_1}^{\infty_3} \tilde{\omega}(\xi, E_0(\lambda)) \, d\xi} \right)
\]

\[
\tilde{\Phi}_{1\to3}^{(2)}(E_0(\lambda), s) = \int_s^0 d\eta \int_{\infty_1}^{\infty_3} d\xi \tilde{\omega}(\xi + \eta, E_0(\lambda)) \tilde{\omega}(\xi, E_0(\lambda)) I_2 \left( \frac{\sqrt{\eta}}{\xi} \right)
\]

(7.8)
\[ z = 4(s - \eta) \int_{\infty}^{\infty} \tilde{\omega}(\zeta, E_0(\lambda))d\zeta + 8(s - \eta) \left( \int_{\infty}^{\xi} - \int_{\infty}^{\xi+\eta} \right) \tilde{\omega}(\zeta, E_0(\lambda))d\zeta \]

Assuming the same order of approximation for \( \chi_1(x, E_0(\lambda), \lambda) \) we can see that when \( x \) stays in the sector 3 as it is shown in Fig. 18a then its Borel plane looks as in Fig. 18b on which \( C_1 \) is the path of the Borel integration to recover \( \chi_1(x, E_0(\lambda), \lambda) \). The distribution of the singularities on the figure follows now from (7.7). Fig. 18c shows the Borel plane for \( \tilde{\chi}_{1\to3}(E_0(\lambda), s) \) i.e. when \( x \to \infty \). The singular points are \( \zeta_C = \int_{B(E_0(\lambda))} \sqrt{V(x) - E_0(\lambda)}dx \),

\[ -\zeta_C \rightleftharpoons \zeta_C - \zeta_A = \int_{A(E_0(\lambda))} \sqrt{V(x) - E_0(\lambda)}dx \] and \( \zeta_A - \zeta_C \). Therefore, \( \chi_{1\to3}(E_0(\lambda), \lambda) \) can be given as

\[ \chi_{1\to3}(E_0(\lambda), \lambda) = 2\lambda \int_{C} e^{2\lambda s} \tilde{\chi}_{1\to3}(E_0(\lambda), s)ds \quad (7.9) \]

where the integration path \( C \) is shown in Fig.18c.
The singularity patterns of \(\chi_1(x, E_0(\lambda), \lambda)\) for the anharmonic oscillator (figure (a)) and \(\tilde{\chi}_1, (x, E_0(\lambda), s)\) (figure (b)). Figure (c) shows the latter pattern for \(x(\xi) \to \infty\).

The path \(C\) differs from the one considered in App. 4 but this does not prevent us from applying the procedure of this appendix. Therefore according to the approximation (7.7) we have

\[
\chi_{1\to3}(E_0(\lambda), \lambda) = 2\lambda \int_C e^{2\lambda s} \left( \Phi_1^{(0)}(E_0(\lambda), s) + \Phi_1^{(2)}(E_0(\lambda), s) \right) ds
\]

so that

\[
\chi_{1\to3}^{(0)}(E_0(\lambda), \lambda) = \sum_{k=0}^{n_0} \frac{(-1)^k}{(2\lambda)^k} \frac{\partial^k}{\partial s^k} \left( \Phi_1^{(0)}(E_0(\lambda), s) + \Phi_1^{(2)}(E_0(\lambda), s) \right) |_{s=0}
\]

where \(n_0 = |\lambda\zeta_C|\).

Using the formulae (A4.5) and (A4.6) of App. 4 for \(\chi_{1\to3}^{(1)}(E_0(\lambda), \lambda)\) we get

\[
\chi_{1\to3}^{(0)}(E_0(\lambda), \lambda) = -\sum_{j=C, -C, A-C, C-A} \frac{(n_0 + 1)!}{(2\lambda)^{n_0} \zeta_j^{m_0}} \sum_{m=0}^{n_1} \frac{(-1)^m \kappa_j^{(m)}(E_0(\lambda), 0)}{(2\lambda)^{m+1}}
\]

where \(\zeta_j = -\zeta_C, \zeta_A - C = \zeta_C - C, \zeta_A - C = \zeta_C - \zeta_A, n_1 = |\lambda\zeta_A|\) (with \(\zeta_A\) determining the common distance of singularities of \(\kappa_j(E_0(\lambda), s)\) closest to the origin) and \(\kappa_j's\) are given by

\[
\kappa_j(E_0(\lambda), s) = \frac{1}{2\pi i} \int_{K_j} dt \frac{\Phi_1^{(2)}(E_0(\lambda), \zeta_j + t)}{(1 + \frac{t}{\zeta_j})^{n_0}}
\]

where the contours \(K_j\) surround the cuts originated by the singularities of \(\Phi_1^{(2)}(E_0(\lambda), s)\) at \(\zeta_j's\) each shifted to the origin \(s = 0\) of the Borel plane.
8 Summary

In this paper we have found the representation for the Borel functions of the quantities relevant for the 1D quantum mechanics. The representation takes the form of the topological expansion. This expansion provides us with an algorithm determining in a systematic way the singularity structure of the Borel plane for the relevant quantities and orders the appearing of the Borel plane singularity structures in a hierarchical way allowing for the formulation of the approximation scheme of the semiclassical calculations alternative to the other ones [8, 9, 10, 11, 12].

We have also remarked limitations of our method in the description of the proper nature of singularities of the quantities represented by the expansion.

We have formulated also the scheme of the semiclassical approximations including the exponentially small contributions to the desired order of accuracy. It makes use of the Borel plane singularity structure in the most natural and effective way, particularly, if it is accompanied by the topological expansion method of approximations of the Borel functions.

We have demonstrated the action of both the methods considering some simple (but not quite trivial) examples of their applications in Sections 4-5 and 7. However, it was not our aim in this paper to perform some numerical tests of the method presented. Rather we have limited ourselves to test both the methods as theoretical tools for better understanding of the mutual relations between the semiclassical expansions, Borel plane singularity structure and the exponential asymptotics. For the latter goal both the expansions (i.e. the topological and the exponential ones) appeared to be very useful. Nevertheless, their test as a practical method of extended semiclassical approximations is certainly desired.

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Appendix 1

A1.1. The Laplace transforms \( \tilde{Y}_{n;r_1,...,r_q}(\xi, s) \)

We shall determine below the Laplace transforms \( \tilde{Y}_{n;r_1,...,r_q}(\xi, s) \), \( n \geq q \geq 0 \), as defined by (3.5). To begin with consider first the case \( r_1 = 0 \) and \( q = 1 \). We have:

\[
Y_{n;r}^{(1)}(\xi, \lambda) = \int_{\gamma_1(\xi)} d\xi_1 \cdots \int_{\gamma_{n-1}(\xi_{n-1})} d\xi_n \tilde{\omega}(\xi_1) \cdots \tilde{\omega}(\xi_r) \cdots \tilde{\omega}(\xi_n) e^{2\lambda(\xi - \xi_r)}\quad (A1.1)
\]

\( r = 1, ..., n, \quad n = 1, 2, ... \) etc.

The multiple integral in (A1.1) can be rewritten further as follows:
\[ Y_{n,r}^{(1)}(\xi, \lambda) = \int_{\tilde{\gamma}(\xi)} d\xi e^{2\lambda(\xi - \xi_r)} \tilde{\omega}(\xi_r) \Omega_{n-1}(\xi, \xi_r) Y_{n-r}^{(0)}(\xi_r) \]  
(A1.2) 

\[ r = 1, \ldots, n, \quad n = 1, 2, \ldots \text{ etc.} \]

where \( Y_{n-r}^{(0)}(\xi_r) \equiv Y_{n-r}(\xi_r) \) is defined by (3.3) and

\[ \Omega_{n-1}(\xi, \xi_r) = ((r - 1)!)^{-1}(\Omega(\xi) - \Omega(\xi_r))^{r-1} \]  
(A1.3)

Making further in (A1.2) a change \( \xi_r \to \xi_r - s \) of the integration variable we get finally:

\[ Y_{n,r}^{(1)}(\xi, \lambda) = \int_{\tilde{C}} dse^{2\lambda s} \tilde{Y}_{n,r}^{(1)}(\xi_r) \]  
(A1.4) 

\[ r = 1, \ldots, n, \quad n = 1, 2, \ldots \text{ etc.} \]

where

\[ \tilde{Y}_{n,r}^{(1)}(\xi, s) = -\tilde{\omega}(\xi - s)\Omega_{r-1}(\xi, \xi - s)Y_{n-r}^{(0)}(\xi - s) \]  
(A1.5)

and the contour \( \tilde{C} \) runs from the infinity \( \Re s = -\infty \) to the origin \( s = 0 \). Note, that because of \( \Re \xi > 0 \) (by assumption) the contour \( \tilde{C} \) is independent of \( r \) and \( n \) and also, as it follows from (A1.5), \( \tilde{Y}_{n,r}^{(1)}(\xi, s) \) is holomorphic for \( \Re s < \Re \xi \).

Reasoning in the completely similar way we get for \( \tilde{Y}_{n,r_1,...,r_2}^{(2)}(\xi, s) \):

\[ \tilde{Y}_{n,r_1,...,r_2}^{(2)}(\xi, s) = -\int_{\tilde{\gamma}(\xi-s)} d\xi_1 \tilde{\omega}(\xi_1+s)\tilde{\omega}(\xi_1)\Omega_{r_1-1}(\xi, \xi_1+s)\Omega_{r_2-r_1-1}(\xi_1+s, \xi_1)Y_{n-r_2}^{(0)}(\xi_1) \]  
(A1.6)

The remaining Laplace transforms \( \tilde{Y}_{n,r_1,...,r_2}^{(2q+1)}(\xi, s) \) and \( \tilde{Y}_{n,r_1,...,r_2}^{(2q)}(\xi, s) \), \( q = 1, 2, \ldots \text{ etc.} \) can be defined recurrently as follows:

\[ \tilde{Y}_{n,r_1,...,r_2}^{(2q)}(\xi, s) = -\int_{\tilde{C}(s)} d\eta \int_{\tilde{\gamma}(\xi-s)} d\xi_1 \tilde{\omega}(\xi_1+s)\tilde{\omega}(\xi_1+\eta)\Omega_{r_1-1}(\xi, \xi_1+s) \]  
(A1.7) 

\[ \Omega_{r_2-r_1-1}(\xi_1+s, \xi_1+\eta)\tilde{Y}_{n-r_2+r_3-2,...,r_2-}^{(2q-2)}(\xi_1+\eta, \eta), q = 2, 3, \ldots \]

and

\[ \tilde{Y}_{n,r_1,...,r_2}^{(2q+1)}(\xi, s) = -\int_{\tilde{C}(s)} d\eta \tilde{\omega}(\xi-s+\eta)\Omega_{r_1-1}(\xi, \xi_1-s+\eta) \]  
(A1.8) 

\[ \tilde{Y}_{n-r_1+r_2-1,...,r_2}^{(2q)}(\xi_1-s+\eta, \eta), \quad q = 1, 2, \ldots \]
Every of them is holomorphic in $\xi$ and $s$ for $\Re \xi > 0$ and $\Re s < \Re \xi$. The contour $\tilde{C}(s)$ in (A1.7)-(A1.8) starts at the point $s$ with $\Re s < \Re \xi$ and ends at $s = 0$.

A1.2. Topological expansion

A further step we can do is to fix $q$ and to take sums with respect to $n$, $r_1, \ldots, r_{2q}$. It can be done as follows. First, we consider rather $\chi_1(\xi, \lambda)/(2\lambda)$ than $\chi_1(\xi, \lambda)$ itself. Next we note that to each term $-(-2\lambda)^{-n-1}\tilde{Y}_n(\xi, \eta_1, \ldots, \eta_q)$ there corresponds the following Laplace transform:

$$
\frac{1}{n!} s^n \ast \tilde{Y}_n(\xi, \eta_1, \ldots, \eta_q)
$$

where the star means the convolution of the factors.

The sums we are now looking for are the following:

$$
\tilde{\Phi}_1(q)(\xi, s) = \sum_{1 \leq r_1 < \ldots < r_q \leq n} \frac{(-1)^{r_1 + r_2 + \ldots + r_q}}{n!} s^n \ast \tilde{Y}_n(\xi, \eta_1, \ldots, \eta_q)
$$

(A1.10)

so that the series:

$$
\tilde{\Phi}_1(\xi, s) = \sum_{q \geq 0} \tilde{\Phi}_1(q)(\xi, s)
$$

(A1.11)

(its convergence is discussed below) represents a function $\tilde{\Phi}_1(\xi, s)$ such that $\partial\tilde{\Phi}_1(\xi, s)/\partial s$ is the Laplace transform of $\chi_1(\xi, \lambda)$.

The sums in (A1.10) can be performed explicitly to give:

$$
\tilde{\Phi}_1^{(0)}(\xi, s) = I_0 \left( \sqrt{4s\Omega(\xi)} \right)
$$

$$
\tilde{\Phi}_1^{(2q)}(\xi, s) = \int_{\tilde{C}(s)} d\eta_1 \int_{\tilde{C}(\eta_1)} d\eta_2 \ldots \int_{\tilde{C}(\eta_{q-1})} d\eta_q \int_{\infty}^{\xi} d\xi_1 \int_{\infty}^{\xi} d\xi_2 \ldots \int_{\infty}^{\xi} d\xi_q
$$

$$
\tilde{\omega}(\xi_1 + \eta_1) \tilde{\omega}(\xi_1 + \eta_2) \cdots \tilde{\omega}(\xi_q + \eta_q) \tilde{\omega}(\xi_q)(2s - 2\eta_1)^{2q} \frac{I_{2q}(\tilde{z}_{2q})}{\tilde{z}_{2q}}
$$

$$
\tilde{z}_{2q} = 4(s - \eta_1)\Omega(\xi) + 8(s - \eta_1) \sum_{p=1}^{q} (\Omega(\xi_p + \eta_{p+1}) - \Omega(\xi_p + \eta_p)),
$$

$$
\eta_{q+1} \equiv 0, \quad q = 1, 2, \ldots
$$

(A1.12)

$$
\tilde{\Phi}_1^{(2q+1)}(\xi, s) = \int_{\tilde{C}(s)} d\eta_1 \ldots \int_{\tilde{C}(\eta_{q})} d\eta_{q+1} \tilde{\omega}(\xi - \eta_1) \tilde{\omega}(\xi - \eta_2) \cdots \tilde{\omega}(\xi - \eta_q) \int_{\infty}^{\xi} d\xi_1 \int_{\infty}^{\xi} d\xi_2 \ldots \int_{\infty}^{\xi} d\xi_q
$$
\[ \tilde{\omega}(\xi_1 + \eta_2)\tilde{\omega}(\xi_1 + \eta_3) \cdots \tilde{\omega}(\xi_q + \eta_{q+1})\tilde{\omega}(\xi_q)(2s - 2\eta_j)^{2q+1} \frac{I_{2q+1}(z_{2q+1})}{z_{2q+1}} \]

\[ z_{2q+1} = 4(s - \eta_1)\Omega(\xi) + 8(s - \eta_1)\sum_{p=0}^{q}(\Omega(\xi_p + \eta_{p+2}) - \Omega(\xi_p + \eta_{p+1})) , \]

\[ \xi_0 \equiv \xi, \quad \eta_{q+2} \equiv 0, \quad q = 0, 1, 2, \ldots \]

The functions \( I_q(x), q \geq 0 \), in (A1.12) are the modified Bessel functions (of the first kind, see [31] p.5, formula (12)). The results (A1.12) have been obtained from (A1.10) by using repeatedly the following sum rule [31]:

\[ \sum_{k \geq 0} \frac{1}{k!} \frac{(t/2)^k}{z^{-\nu + k}I_{\nu + k}(z^{1/2})} = (z + t)^{-\nu}I_\nu((z + t)^{1/2}) \quad (A1.13) \]

valid for any \( \nu \).

The formulae (A1.12) provide us with the general forms of \( \tilde{\Phi}^{(q)}_1(\xi,s) \)'s. However, for the singularity analysis of the latter the more convenient representation for them is the following recurrent one:

\[ \tilde{\Phi}^{(2q+2)}_1(\xi, s) = - \int_{\tilde{C}(s)} \frac{d\eta}{\tilde{C}(\eta)} \int \frac{d\eta'}{\tilde{C}(\eta')} \int d\eta_{12} \tilde{\omega}(\xi_{12})\tilde{\omega}(\eta_{12})(2s - 2\eta) \]

\[ \tilde{\Phi}^{(2q)}_1(\xi_1 - \eta', \eta - \eta') = \frac{I_1 \left( \sqrt{4(s - \eta)(\Omega(\xi) - 2\Omega(\xi_1) + \Omega(\xi_1 - \eta'))} \right)}{\sqrt{4(s - \eta)(\Omega(\xi) - 2\Omega(\xi_1) + \Omega(\xi_1 - \eta'))}} \]

\[ \tilde{\Phi}^{(2q+1)}_1(\xi, s) = - \int_{\tilde{C}(s)} \frac{d\eta}{\tilde{C}(\eta)} \int d\eta' \tilde{\omega}(\eta - \eta') \quad (A1.14) \]

\[ \tilde{\Phi}^{(2q)}_1(\xi - \eta', \eta - \eta')I_0 \left( \sqrt{-4(s - \eta)(\Omega(\xi) - \Omega(\xi - \eta'))} \right) \quad q = 0, 1, 2, \ldots \]

where \( \tilde{\Phi}^{(0)}_1(\xi, s) \) is given by (A1.12).

Note that (A1.14) can be obtained from (A1.12) and vice versa by applying the following relations:

\[ \int_0^1 dx I_m(\sqrt{\alpha x})I_m(\sqrt{\beta(1 - x)})(\alpha x)^{1/2}\beta(1 - x))^{1/2}n = 2\alpha^m\beta^n I_{m+n+1}\left( \sqrt{\alpha + \beta} \right)^{m+n+1} \quad (A1.15) \]

\[ \frac{(s - \eta)^n}{n!} = \frac{1}{(k-1)!(n-k)!} \int_{\eta}^{s} d\eta' (s - \eta')^{k-1}(\eta' - \eta)^{n-k} \]

**A1.3. Analytic properties of the functions \( \tilde{\Phi}^{(0)}_1(\xi, s) \)**
Since each of the functions $I_q(z^{1/2})/z_q^{1/2}$, $q \geq 0$, is an entire function of its argument then it follows from (A1.12) that possible singularities of $\tilde{\Phi}_1^{(q)}(\xi, s)$ are generated by the (known) singularities of the functions $\tilde{\omega}(\eta)$ and $\Omega(\eta)$ and their integrations present in (A1.12). However, it can be easily checked that the conditions:

$$\Re \xi > 0 \quad \text{and} \quad \Re s < \Re \xi$$

(A1.16)

determine the domain where the integrands in (A1.12) are holomorphic. Therefore, this is also the domain of holomorphicity of $\tilde{\Phi}_1^{(q)}(\xi, s)$ since all the integration paths in (A1.1), (A1.12) can be chosen to lie completely in this domain.

Let us note, however, that as it follows from (A1.12) each $\tilde{\Phi}_1^{(q)}(\xi, s)$, $q \geq 1$ can be continued analytically from the domain (A1.16) to any point $\xi$ of the $\xi$-Riemann surface of $\tilde{\omega}(\eta)$ if the distribution of branch points of $\tilde{\omega}(\eta)$ along a path of the corresponding analytical continuation is such that the distance of any of them from the path is greater than $|s|$. This statement is the direct conclusion from the corresponding formulae in (A1.12) since all the integrations on the $\xi$-Riemann surface present there are performed inside a strip no wider than $|s|$. Let us note further that for the polynomial potentials the branch points of $\tilde{\omega}(\eta)$ are isolated and on each sheet of the $\xi$-Riemann surface of $\tilde{\omega}(\eta)$ their numbers are finite. The distances between them on each sheet are nothing but the corresponding distances between turning points measured by the action. Therefore, there is the smallest distance $d$ among them. If we take, therefore, $s$ in (A1.12) such that $|s| < d' < d/2$ then we can penetrate by paths of the analytical continuations the whole $\xi$-Riemann surface of $\tilde{\omega}(\xi)$ if the former is deprived all the circular vicinities of radius $d''$, $d' < d'' < d/2$, centered at each branch point of the surface. We shall denote the corresponding part of the $\xi$-Riemann surface as $R(d'')$.

Consider now a question of convergence of the series in (A1.11). We shall show below that the series is convergent absolutely and uniformly in the domain $R(d'')$. It means that the series (A1.11) determines $\Phi_1(\xi, s)$ as the holomorphic function in these domains.

To this end let us note that if $|s|$ is chosen to satisfy the condition $|s| < d' < d''$ all the integration paths $\tilde{\gamma}(\xi - \eta)$ can be deformed then to lie inside an infinite strip $S(\xi, s)$ bounded by the paths $\tilde{\gamma}(\xi)$ and $\tilde{\gamma}(\xi - s)$ so having the width $|s|$ with the one end of the strip being placed at the infinity $\infty_1$ and the other one being a segment $(\xi, \xi - s)$. The latter bound can be chosen as such because the path $\tilde{C}(s)$ can be deformed to a segment (with its ends anchored at the origin and at $s$). Introducing now the following functions:

$$|\tilde{\omega}|(\xi_r, \eta_1) = \limsup_{\eta \in \tilde{C}(\eta_1)} |\tilde{\omega}(\xi_r + \eta_1)|$$

(A1.17)

$$|\tilde{\rho}|(\xi, \eta_1) = \int_{\tilde{\gamma}_1(\xi)} |d\xi_r||\tilde{\omega}|(\xi_r, \eta)$$

we have:

$$|\Omega(\xi_r + \eta)| < \tilde{\rho}(\xi, \eta_1), \quad \eta \in \tilde{C}(\eta_1), \quad \xi_r \in \tilde{\gamma}_1(\xi)$$

(A1.18)
and for $q$ large enough:

$$|z_q| < 8(q + 1)|s - \eta_1|\tilde{\rho}(\xi, \eta_1)$$  \hspace{1cm} (A1.19)$$

$$|2^q z_q^{-\frac{1}{2}} I_q| < \frac{1}{q!} \exp(2|s - \eta_1|\tilde{\rho}(\xi, \eta_1))$$

so that:

$$|\tilde{\Phi}_1^{(2q)}(\xi, s)| < ((2q)!q!(q - 1)!)^{-1} \int_0^{|s|} dx x^{q-1}(|s| - x)^{2q}$$

$$\cdot \left( \int_{\tilde{\gamma}_1(\xi - m)} |d\eta| |\tilde{\omega}|^2(\eta, \eta_1) \right)^q \exp(2(|s| - x)\tilde{\rho}(\xi, \eta_1))$$  \hspace{1cm} (A1.20)$$

$$|\tilde{\Phi}_1^{(2q+1)}(\xi, s)| < ((2q + 1)!(q!)^2)^{-1} \int_0^{|s|} dx x^q(|s| - x)^{2q+1} |\tilde{\omega}|(\eta, \eta_1)$$

$$\cdot \left( \int_{\tilde{\gamma}_1(\xi - m)} |d\eta| |\tilde{\omega}|^2(\eta, \eta_1) \right)^q \exp(2(|s| - x)\tilde{\rho}(\xi, \eta_1))$$

where $x = |\eta_1|$. Introducing yet:

$$|\omega|(\xi, s) = \limsup_{\eta_1 \in \tilde{C}(s)} |\tilde{\omega}|(\xi, \eta_1)$$

$$\tilde{\rho}(\xi, s) = \limsup_{\eta_1 \in \tilde{C}(s)} \tilde{\rho}(\xi, \eta_1)$$  \hspace{1cm} (A1.21)$$

$$Q(\xi, s) = \limsup_{\eta_1 \in \tilde{C}(s)} \int_{\tilde{\gamma}_1(\xi - m)} |d\eta| |\tilde{\omega}|(\eta, \eta_1)$$

we obtain finally for $q \to \infty$:

$$|\tilde{\Phi}_1^{(2q)}(\xi, s)| < \frac{|s|^{3q}}{(3q)!q!} Q^{2q}(\xi, s) e^{2|s|\tilde{\rho}(\xi, s)}$$  \hspace{1cm} (A1.22)$$

$$|\tilde{\Phi}_1^{(2q+1)}(\xi, s)| < \frac{|s|^{3q+2}}{(3q + 2)!q!} Q^{2q}(\xi, s) |\omega|(\xi, s) e^{2|s|\tilde{\rho}(\xi, s)}$$

The bounds (A1.22) show clearly that the series (A1.11) is convergent in the assumed domain $\mathbf{R}(d'')$ since $Q(\xi, s), \tilde{\rho}(\xi, s)$ and $|\omega|(\eta, s)$ are finite there.
Appendix 2

If \( x_p \) is a simple zero of \( q(x) \) then the point \( \xi_p = \xi(x_0, x_p) \) is the branch point for the function \( \tilde{\omega}(\xi) \) defined by (2.3) which can be expounded around the point \( \xi_p \) into the following series:

\[
\tilde{\omega}(\xi) = \sum_{k \geq -3} \tilde{\omega}_k(\xi_p)(\xi - \xi_p)^{2k/3} \quad (A2.1)
\]

The coefficients \( \tilde{\omega}_k(\xi_p) \) in (A2.1) are defined by the identity: \( \tilde{\omega}(\xi(x_0, x)) \equiv \omega(x)q^{-\frac{1}{2}}(x) \) and the following expansions of \( \xi(x_0, x) \) and \( \omega(x)q^{-\frac{1}{2}}(x) \) (see (2.4)) around \( x_p \):

\[
\xi(x_0, x) - \xi_p = \sum_{k \geq 0} \xi_k(x_p)(x - x_p)^{k + \frac{3}{2}}
\]

and

\[
\omega(x)q^{-\frac{1}{2}}(x) = \sum_{k \geq 0} \omega_k(x_p)(x - x_p)^{k - 3}
\]

In particular, the coefficient at the most singular term in (A2.1) \( \tilde{\omega}_{-3} = -5/36 \) i.e. it is potential independent. It depends, however, on the multiplicity of zero of \( q(x) \) at \( x_p \), namely, \( \tilde{\omega}_{-3} = -n(n + 4)/[4(n + 2)^2] \) for the \( n \)-fold zero.

Appendix 3

We establish here the Riemann surface structure of \( \tilde{\Phi}^{(q)}_1(\xi, s) \) for the linear and harmonic potentials.

3.1 The linear potential

According to Section 4.1 the Riemann surface structure of \( \tilde{\Phi}^{(1)}_1(\xi, s) \) for this case is determined by

\[
\tilde{\Phi}^{(1)}_1(\xi, s) = -\frac{5}{8} \int_{C(s)} d\eta \frac{s - \eta}{(\xi - \eta)^2} \frac{I_1 \left( \left( \frac{\eta - \frac{s}{2}}{\xi - \eta} - \frac{\frac{s}{4} - \eta}{\xi} \right)^{\frac{1}{2}} \right)}{\left( \frac{\eta - \frac{s}{2}}{\xi - \eta} - \frac{\frac{s}{4} - \eta}{\xi} \right)^{\frac{1}{2}}} \quad (A3.1)
\]

From (A3.1) it follows that its subintegral function is singular at \( \xi = \eta \) and at \( \xi = 0 \) where it behaves as \( e^{\pm(\xi-\eta)^{-\frac{1}{2}}}(\xi-\eta)^{-\frac{3}{2}} \) and \( e^{\pm\xi^{-\frac{3}{2}}\xi^{\frac{1}{2}}} \) respectively. The \( \eta \)-integration generates only a singularity at \( s = \xi \) (by the EP-mechanism) leaving the singularity at \( \xi = 0 \) and its character unchanged. Therefore, assuming \( \xi \) to be continued to the sector 2 (along the canonical path) the corresponding first sheets of the Riemann surface look as in Fig. 10a,b.

Consider now \( \tilde{\Phi}^{(2)}_1(\xi, s) \). Its Riemann surface structure is defined by:
\[
\tilde{\Phi}^{(2)}_1(\xi, s) = -\frac{25}{64} \int_{\tilde{C}(s)} d\eta \int d\xi_1 \frac{(s - \eta)^2}{(\xi - \eta)^2 \xi_1} \frac{I_2(z^{\frac{1}{2}})}{z} \tag{A3.2}
\]

\[z = \frac{5}{4}(s - \eta) \left( \frac{1}{\xi} - \frac{2}{\xi_1} + \frac{2}{\xi_1 - \eta} \right)\]

As it follows from (A3.2) the subintegral function is singular at \(\xi_1 = \eta, \xi = 0\) and at \(\xi_1 = 0\) behaving there as \(e^{\pm(\xi_1 - \eta)^{-\frac{3}{2}}} (\xi_1 - \eta)^{-\frac{5}{4}}\), \(e^{\pm \xi^{-\frac{3}{2}} \xi_1^{\frac{1}{2}}}\) and \(e^{\pm (\xi - \eta)^{-\frac{3}{2}} \xi_1^{\frac{1}{2}}}\) respectively.

The \(\xi_1\)-integration in (A3.2) generates the EP-singularities at \(\xi = \eta\) and at \(\xi = 0\) but also the P-singularity at \(\eta = 0\) (when the singularity at \(\xi_1 = \eta\) move around the end point of \(\tilde{\gamma}(\xi)\) clockwise pinching the latter against the singular point \(\xi_1 = 0\)).

The final \(\eta\)-integration in (A3.2) generates the EP-singularity at \(s = \xi\) and \(s = 0\) and the P-singularity at \(\xi = 0\). Therefore, the 'closest' singularities of \(\tilde{\Phi}^{(2)}_1(\xi, s)\) are the following:

\[\xi = 0, \; \xi = \eta, \; s = 0 \tag{A3.3}\]

Let us make a general note that the EP-mechanism repeats the distribution of the branch points and cuts whilst the P-one generates new branch points on the Riemann surfaces obtained by the EP-mechanism always however enforcing specific ways of moving around the singularities generated by the EP-mechanism.

All the singularities in (A3.3) are the root branch points (of the forth order) accompanied by essential singularities as we have mentioned above. The singularity at \(s = 0\), however, to be reached needs to round the branch point at \(s = \xi\) moving clockwise i.e. it lies on the sheet opened by the latter branch point. This singularity is therefore a consequence of the singularity at \(\xi = 0\).

A similar note concerns the singularity at \(\xi = 0\). In fact there are two such singularities the one on the sheet shown in Fig. 10a (arising by the EP-mechanism) and the second one at the sheet opened by the branch point at \(\xi = \eta\) i.e. to reach it one needs to round this point clockwise.

One can conclude therefore that the P-mechanism applied once has generated a singularity at \(s = 0\) and applied twice has generated a new singularity at \(\xi = 0\) (on a different sheet) from the old one. It is clear that this mechanism will proliferate the last singularity on all the sheets of the Riemann surface of \(\tilde{\Phi}_1(\xi, s)\) except the sheet we have started with on which the point \(s = 0\) is regular for \(\tilde{\Phi}_1(\xi, s)\).

Now we can use the formulae (A1.14) to prove the form of the first sheet of the Riemann surface as shown on Fig. 10a,b. Namely, assuming for \(\tilde{\Phi}^{(2q)}_1(\xi, s)\) the form of this sheet shown in the last figure we deduce that it remains unchanged for \(\tilde{\Phi}^{(2q+2)}_1(\xi, s)\) whilst it is deprived of the singularity at \(s = 0\) for \(\tilde{\Phi}^{(2q+1)}_1(\xi, s)\).

Indeed, consider the subintegral function in the first of the formulae (A1.14) defining \(\tilde{\Phi}^{(2q+2)}_1(\xi, s)\). It has singularities at the following points:

\[\xi = 0, \; \xi_1 = 0, \; \xi_1 - \eta' = 0, \; \xi_1 - \eta = 0, \; \eta - \eta' = 0 \tag{A3.4}\]
shown on Fig. 19a for the $\xi_1$-Riemann surface (when the rest of the variables are fixed).

Fig. 19. The "first sheets" singularity pattern of the subintegral function defining $\Phi^{(2q+1)}_1(\xi,s)$ for the linear potential before the $\xi_1$-integration (a) and after it (b).

The $\xi_1$-integration provides us with the EP-singularities at $\xi = 0$, $\xi - \eta = 0$ and $\xi - \eta' = 0$ and with the P-ones at $\eta = 0$, $\eta' = 0$ (when the point $\xi$ is rounded by $\eta$ and $\eta'$ clockwise to touch the point $\xi = 0$ by the latter) and at $\eta = \eta'$ (when the point $\eta(\eta')$ rounds $\xi$ clockwise (anticlockwise) to touch $\eta'(\eta)$). We get in this way the singularity pattern before the $\eta'$-integration shown in Fig. 19b where the singularities at $\eta' = 0$ and at $\eta = \eta'$ are screened by the $\xi$-cut and to get them one has to go around $\xi$ clockwise or anticlockwise respectively.

The $\eta'$-integration therefore does not do much now providing us with the singularity at $\eta = \xi$ and at $\eta = 0$ (by the EP-mechanism) and at $\xi = 0$ (by the P-one) with the latter singularity placed on a sheet opened by the singularity at $\xi = \eta$.

The final $\eta$-integration repeat only the singularity pattern described just above so we are left with the distribution of the singularities as shown in Fig. 10a,b.

Consider now the second formula (A1.14). There is no the $\xi_1$-integration and therefore the singularity at $\eta' = 0$ is not generated and the other singularities at $\xi = 0$, $\eta = 0$ and $s = 0$ can not be generated as well by the further $\eta'$- and $\eta$-integrations. Besides the generation of the singularities at $\xi = s$ goes exactly in the same way so that the final picture of the corresponding Riemann surface is the same as in Fig. 10a,b except of missing of the respective singularities at $\xi = 0$ and $s = 0$ on the lower sheets.

3.2 The harmonic potential

We assume here $\bar{\gamma}_1(\xi)$ to be continued canonically to the sector 3 of Fig. 11 and we put $\xi(i) = \int_{-1}^1 \sqrt{x^2 + 1} dx = \zeta$. Nor $\bar{\omega}(\xi)$ nor $\Omega(\xi)$ are now simple functions of $\xi$. $\bar{\omega}(\xi)$ is periodic (with its period $2\zeta$ acting between different sheets of the infinitely sheeted Riemann surface on which this function is defined) whilst $\Omega(\xi)$ is not.

37
Consider however again $\tilde{\Phi}_1^{(1)}(\xi, s)$ as given by (4.1). The closest singularities of the subintegral function are shown in Fig. 20a,b i.e. they are:

$$\xi = 0, \quad \xi - \zeta = 0, \quad \xi - \eta = 0, \quad \xi - \eta - \zeta = 0$$

(A3.5)

Therefore the $\eta$-integration in (4.1) provides us with the following singularities of $\tilde{\Phi}_1^{(1)}(\xi, s)$ shown in Fig. 12a,b:

$$\xi = 0, \quad \xi - \zeta = 0, \quad \xi - s = 0, \quad \xi - \zeta - s = 0$$

(A3.6)

i.e. none the P-singularity is generated.

Consider next $\tilde{\Phi}_1^{(2)}(\xi, s)$. According to (4.3) singularities of the subintegral function in this formula are now the following:

$$\xi = 0, \quad \xi - \zeta = 0, \quad \xi_1 = 0, \quad \xi_1 - \zeta = 0 \quad \xi_1 - \eta, \quad \xi_1 - \zeta - \eta = 0$$

(A3.7)

and we can use Fig. 20 to show the corresponding situation situation for the $\xi_1$- and $\eta$-dependence by making the substitution $\xi \rightarrow \xi_1$ in the figure.

The $\xi_1$-integration generates the following singularities:

$$\xi = 0, \quad \xi - \zeta = 0, \quad \xi - \eta = 0, \quad \xi - \zeta - \eta = 0$$

(A3.8)
by the EP-mechanism and

\[ \eta = 0, \quad \eta - \zeta = 0, \quad \eta + \zeta = 0 \quad (A3.9) \]

by the P-mechanism. The latter singularities lie on the 'lower' sheets of the \( \eta \)-Riemann surface.

Finally, integrating in (4.3) over \( \eta \) we generate singularities of \( \tilde{\Phi}_1^{(2)}(\xi, s) \) at

\[ \xi - s = 0, \quad \xi - \zeta - s = 0, \quad s - \zeta = 0, \quad s + \zeta = 0 \quad (A3.10) \]

by the EP-mechanism and at

\[ \xi + \zeta = 0, \quad \xi = 0, \quad \xi - \zeta = 0, \quad \xi - 2\zeta = 0 \quad (A3.11) \]

by the P-mechanism. The proper distribution of these singularities is shown on Fig. 13.

Now we can proceed inductively assuming for \( \tilde{\Phi}_1^{(2q)}(\xi, s) \) the singularity pattern shown in Fig.14a,b where \( \tilde{\gamma}(\xi) \) is the integration path in the formulae (A1.12)-(A1.14) and \( C \) the corresponding path to recover from \( \chi_1^{(2q)}(\xi, \lambda) \) by the Borel transformation (at \( s = 0 \) \( \tilde{\Phi}_1^{(2q)}(\xi, s) \) is then regular).

Taking into account the second of the formulae (A1.14) we see that the singularities of the subintegral function are determined mostly by its factor \( \tilde{\Phi}_1^{(2q)}(\xi - \eta', \eta - \eta') \) according to which and Fig. 21a these singularity are at the points

\[ \xi - \eta = 0, \quad \xi - \eta - \zeta = 0, \quad \xi - \eta' - k\zeta = 0, \quad k = -(2q - 1), ..., 2q \quad (A3.12) \]

\[ \eta' - \eta + k\zeta = 0, \quad k = -(2q - 1), ..., (2q - 1), \quad k \neq 0 \]

shown for the case of the corresponding \( \eta' \)-Riemann surface on Fig. 21b.
Therefore, making the \( \eta' \)-integration we obtain the '\( \eta \)-plane' singularity pattern shown in Fig. 21c on which the \( \xi \)-dependent singularities are created by the EP-mechanism whilst the two fixed ones on the imaginary axis by the P-mechanism. Other singularities generated in the last way appear on the lower sheets originated by the two singularities at \( \eta = \zeta \) and \( \eta = \zeta' \).

The successive \( \eta \)-integration does change nothing in the \( s \)-variable singularity pattern (in comparison with this on Fig. 21c) so providing us finally with its form shown in Fig. 15b but seriously changes the original pattern of Fig.14a. Namely, the EP-mechanism generates the \( \xi \)-singularities at the points \( \xi = s - (2q - 1)\zeta, s - (2q - 2)\zeta, ..., s - \zeta, s + \zeta, ..., s + 2q\zeta \) and by the P-mechanism at the points \( \xi = -2q\zeta, ..., -\zeta, 0, \zeta, ..., (2q + 1)\zeta \). As the final result we have for \( \bar{\Phi}^{(2q+1)}(\xi, s) \) the picture of Fig.15 for its both types of singularities.

The corresponding analysis of the case \( \bar{\Phi}^{(2q+2)}(\xi, s) \) is still a little bit more tedious but nevertheless direct due to the first of the formulae (A1.14). The valid singularities of the subintegral function in this case are

\[
\begin{align*}
\xi_1 &= 0, \quad \xi_1 - \eta = 0, \\
\xi_1 - \eta = 0, \quad \xi_1 - \eta - \zeta = 0, \quad \xi_1 - \eta' - k\zeta = 0, \quad k = -(2q - 1), ..., 2q \\
\eta' - \eta + k\zeta &= 0, \quad k = -(2q - 1), ..., (2q - 1), \quad k \neq 0
\end{align*}
\]

The first \( \xi_1 \)-integration is performed on the sheet shown in Fig. 22a. By the EP- and P-mechanism it generates \( \eta \)- and \( \eta' \)-singularities. Limiting ourselves to collect only these singularities which appear on the \( \eta' \)-sheet on which the result of this \( \xi_1 \)-integration is regular at \( \eta' = 0 \) we arrive at the pattern shown in Fig. 22b.
The successive $\eta'$-integration leads us to the '\eta-plane' pattern shown in Fig. 22c where the two $\xi$-dependent singularities were produced by the EP-mechanism whilst the fixed ones by the EP- and P-mechanisms simultaneously with the exception of the highest two ones at $\eta = -(2q+1)\zeta$ and $\eta = (2q+1)\zeta$ which are generated by the P-mechanism only.

The final $\eta$-integration in (A1.14) provide us with a pattern analogous to the one of Fig. 14a by the EP- and P-mechanism and with the pattern of Fig. 14b by the EP-mechanism when $q$ is substituted by $q+1$.

**Appendix 4**

We describe here a procedure allowing us to construct in a systematic way the optimum semiclassical representation for the Borel summable quantity including both the main contribution coming from the semiclassical series abbreviated at its least term and the corresponding exponential contributions of an arbitrary order. In its finite form the procedure
provides us with the exact formula for the quantity considered. However, if continued infinitely the procedure give rise to the question of convergence of the infinite functional series we get by it.

To this goal we shall consider the basic quantity given by the formula (2.8). Integrating in it by parts we get

$$
\chi_1(\xi, \lambda) = 2\lambda \left( \sum_{k=0}^{n} \frac{(-1)^k}{(2\lambda)^{k+1}} \chi_1^{(k)}(\xi, 0) + \frac{(-1)^{n+1}}{(2\lambda)^{n+1}} \int_{\mathcal{C}} e^{2\lambda s} \tilde{\chi}_1^{(n+1)}(\xi, s) ds \right) 
$$

(A4.1)

According to the well known prescription (which can be easily justified by the analysis similar to the one performed below) we should put

$$n = n_0 = \left\lfloor \lambda |s_0| \right\rfloor$$

in (A4.1) where \(\lfloor x \rfloor\) means the integer part of \(x\) and \(s_0\) is a singularity of \(\tilde{\chi}_1(\xi, s)\) closest to the origin. Next we should extract from the integral the exponentially small factor and finally continue the procedure to the remaining Borel integral. This could be done in the following way.

First the \(n_0 + 1\)th derivative of \(\tilde{\chi}_1(\xi, s)\) can be given the form

$$
\tilde{\chi}_1^{(n_0+1)}(\xi, s) = \frac{(-1)^{n_0+1}(n_0+1)!}{(2\pi i)^2} \int_{K} \tilde{\chi}_1(\xi, s + t) t^{-n_0-2} dt
$$

(A4.2)

with the integration contour \(K\) in (A4.2) surrounding anticlockwise the negative half axis of the Borel plane (see Fig. 5).

As it follows from Fig. 2 \(s_0 = \xi - \zeta_1 = \xi\) (since \(\zeta_1 = 0\), see Fig. 4). Deforming the contour \(K\) to surround the cuts generated by the points \(\xi - \zeta_1\) and \(\xi - \zeta_2\) of Fig. 5 (for \(\xi\) chosen as in the figure they are the unique cuts visible in these positions) and shifting the integration variable in the corresponding integral we get

$$
\tilde{\chi}_1^{(n+1)}(\xi, s) = \frac{(-1)^{n_0+1}(n_0+1)!}{(2\pi i)^2} \sum_{j=1}^{2} \int_{K_j} \tilde{\chi}_1(\xi - \zeta_j + t)(\xi - \zeta_j - s + t)^{-n_0-2} dt
$$

(A4.3)

where the contours \(K_j\) surround (anticlockwise) the cuts whose origins are at the point \(t = 0\).

Further substituting (A4.3) to (A4.1) and changing both the order of integrations in (A4.3) and the integration variables themselves we get as a result of these calculations

$$
\chi_1(\xi, \lambda) = \sum_{k=0}^{n_0} \frac{(-1)^k}{(2\lambda)^k} \chi_1^{(k)}(\xi, 0) - \sum_{j=1}^{2} \frac{(n_0+1)!}{(2\lambda)^{n_0}(\xi - \zeta_j)^{n_0}} \int_{\mathcal{C}} e^{2\lambda s} \kappa_j(\xi, s) ds
$$

(A4.4)

where

$$
\kappa_j(\xi, s) = \frac{1}{2\pi i} \int_{K_j} dt \frac{\tilde{\chi}_1(\xi, \xi - \zeta_j + t)}{(1 + \frac{t}{\xi - \zeta_j})^{n_0}}
$$

$$
\frac{1}{(t + \xi - \zeta_j + \frac{n_0}{\lambda} \ln(1 + \frac{t}{\xi - \zeta_j}) - s) \left( t + \xi - \zeta_j + \frac{n_0}{\lambda} \ln(1 + \frac{t}{\xi - \zeta_j}) - s \right)}
$$

$$
j = 1, 2
$$

(A4.5)
and where the contours $K_j$ run again around the cuts anchored at $t = 0$.

The form (A4.5) for $\kappa$’s allows us to continue the procedure of getting the asymptotic series expansions for the integrals in (A4.4) and next to abbreviate the series at their least terms. The latter are to be determined by the singularities generated by the $t$-integrals in the $s$-plane (as a result of the pinch mechanism) closest to the origin of the plane. It is easy to see that among possible candidates for the latter are the singularities at $s = \xi + \zeta_0$ for $\kappa_1(\xi, s)$ and the ones at $s = \xi - \zeta_2, \xi + \zeta_0/\lambda - \zeta_2$ for $\kappa_2(\xi, s)$ (all the singularities are generated by the P-mechanism at $t = 0$). However, the integrations in (A4.5) along the corresponding cuts open possibilities for new singularities to appear generated by the $t$-singularities shared by the cuts. These possibilities still enrich the variety of singularities which have to be taken into account in choosing the one closest to the origin of the $s$-plane.

Therefore, to construct the representation (A4.1) for each of the two integrals in (A4.4) we have to choose from the singularities corresponding to each $\kappa$ the ones which are closest to the origin. When these choices are done the procedure described above can be repeated.

Let us call $\kappa_j, j = 1, 2$, defined by (A4.5) the first generation family considering $\kappa_0(\xi, s) \equiv \tilde{\chi}_1(\xi, s)$ as the zeroth generation one. It is clear that the general form of the optimum semiclassical representation for $\chi_1(\xi, \lambda)$ is the following

$$
\chi_1(\xi, \lambda) = 2\lambda \sum_{m=0}^{n_0} \frac{(-1)^m \chi_1(\xi, 0)}{(2\lambda)^{m+1}} \left(e^{2\lambda s} \kappa_{j_1, \ldots, j_k}(\xi, s)ight) ds
$$

where $j_0 \equiv 0$ and $\kappa_{j_1, \ldots, j_p}$’s constitute the $p + 1$th generation family. The latter is constructed from the $p$th one (with $j_p$ as its singular points and with $n_{j_p}/\lambda = |\zeta_{j_p}|$ being the singularity closest to the origin) according to the formulae (A4.2)-(A4.5).

It is important to stress that (A4.6) is exact and its RHS becomes an approximation to the left one only when the last sum of the RHS containing the integrals is rejected.

**Appendix 5**

We shall show below that the last term on the right hand side sum in (4.15) has to vanish when $\lambda_0 \to 0$. To this end let us note that we can rewrite the integral present in this term in the following way:

$$
\int_{C'} \exp(2\lambda s) \log \chi_1 \to 3(\lambda) d\lambda = \int_{C_1} \left[\exp(2\lambda s) \log \chi_1 \to 3(\lambda) - \exp(-2\lambda s) \log \chi_1 \to 3(\lambda) d\lambda\right]
$$

(A5.1)
\[
+ \int_{C^d} \exp(-2\lambda s)[1 + \exp(-2\pi i\lambda)]d\lambda + \int_{C^{du}} \exp(-2\lambda s)[1 + \exp(2\pi i\lambda)]d\lambda
\]

where \(C_\frac{1}{2}\) is the half-circle of radius \(\lambda_0\) lying in the right half of the \(\lambda\)-plane, and \(C^{du}\) and \(C^d\) are the corresponding upper and lower halves of \(C_\frac{1}{2}\). We have also made use of the relations (4.11) to obtain the final form of (A3.1). It follows now from (4.10) that we have:

\[
\lim_{\lambda \to 0} \chi_1 \to 3(\lambda) = \sqrt{2} \text{for } |\arg\lambda| < \pi
\]

(A5.2)

Therefore, we can conclude that both \(\log|\chi_1\to 3(\lambda)|\) and \(\arg\chi_1\to 3(\lambda)\) are bounded in the halfplane \(\Re\lambda \geq 0\). The vanishing of all the integrals in (A5.1) when \(\lambda_0 \to 0\) follows now directly from the last conclusion.

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