The spin-one DKP Equation with a nonminimal vector interaction in the presence of minimal uncertainty in momentum

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(Dated: February 18, 2021)

In this work, we consider the relativistic Duffin-Kemmer-Petiau equation for spin-one particles with a nonminimal vector interaction in the presence of minimal uncertainty in momentum. By using the position space representation we exactly determine the bound-states spectrum and the corresponding eigenfunctions. We discuss the effects of the deformation and nonminimal vector coupling parameters on the energy spectrum analytically and numerically.

PACS numbers: 03.65.Pm, 03.65.Ge
Keywords: Duffin–Kemmer–Petiau equation, spin-1 particles, nonminimal vector coupling, minimal uncertainty in momentum, bound state solutions.
I. INTRODUCTION

In a series of studies Duffin, Kemmer, and Petiau put forward a first order differential equation, hereafter (DKPe), to describe the dynamics mesons [22, 26, 40, 62]. Although the DKPe presents similarities to the Dirac’s equation, its matrices obey a different and more complex commutation rule schemes. Until the 1970s, DKP formalism developed [50] with increasing doubts over the DKPe’s relationships with the Klein-Gordon (KG) and Proca equations [27]. Between 1970 and 1980, interest in the DKPe waned, believing that DKPe was equivalent to the KG and Proca equations [14]. After showing that the equivalence is valid only in the special case where symmetry exists [52], interest in the solutions of the DKPe has increased in the last decades.

The DKPe with different types of couplings is used in a wide area of physics. For example in the modelling of the: meson scattering by nuclei [22], large and short distance interactions in quantum chromodynamics [30], boson dynamics in curved space-time [13, 42], covariant Hamilton dynamics [44], non-inertial effect of rotating frames [19], Galilei covariance [24], the Aharonov-Bohm phenomenon [4, 12], dynamics of vector bosons in the expanding universe [67], commutative and noncommutative spaces [11, 28], thermodynamic properties of bosons [5, 13], Bose-Einstein condensation [1, 17] and etc...

It is a well-known fact that the nonminimal vector couplings to the KG and Proca equations produce results that contradict the predictions of the non-relativistic quantum mechanics [16, 25, 31, 69]. With a nonminimal vector coupling, one refers to a sort of charge conjugate invariant coupling which transforms like a vector under a Lorentz transformation. If the nonminimal vector potential is invariant under charge conjugation, then, one can vector coupling, one refers to a sort of charge conjugate invariant coupling which transforms like a vector under a Lorentz transformation. If the nonminimal vector potential is invariant under charge conjugation, then, one can not discriminate the particle from its antiparticle [20]. Since the DKPe, unlike from KG and Proca equations, allows a Lorentz transformation. If the nonminimal vector potential is invariant under charge conjugation, then, one can vector coupling, one refers to a sort of charge conjugate invariant coupling which transforms like a vector under a Lorentz transformation. If the nonminimal vector potential is invariant under charge conjugation, then, one can vector coupling, one refers to a sort of charge conjugate invariant coupling which transforms like a vector under a Lorentz transformation.

On the other hand, various researches regarding the quantum gravity [3, 68] and cosmology [10], string theory [64, 65], noncommutative geometry [54], black hole physics [70] and thermodynamics [63] show that a minimal observable length should exist. This minimal length (ML) may be introduced as an additional uncertainty in position measurements $\Delta x_{min}$, which leads to a generalization of Heisenberg’s uncertainty principle. Kempf with his collaborators [41, 47-49] showed that a ML can be obtained out of the generalized Heisenberg algebra with the form

$$[X_i, P_j] = i\hbar \delta_{ij} (1 + \alpha P^2); \quad \ [X_i, X_j] = i\hbar \alpha J_{ij},$$  \hspace{1cm} (1)$$

where $\alpha$ is the parameter of deformation. It is worth mentioning that the deformed algebra leading to quantized space time was introduced for first time by Snyder [66].

It is well known that the curvature of space-time becomes important at great distances. On a general curved space-time, there is no concept of a plane wave. This implies that there is a finite lower bound to the precision with which the corresponding momentum can be described. This can be represented with a nonzero minimal uncertainty in momentum (MUM) measurement. It has been argued in [9] that in the presence of a cosmological constant the Heisenberg uncertainty principle receives a correction term due to the background curvature, which is known as the "extended uncertainty principle" (EUP),

$$\langle \Delta X_i \rangle \langle \Delta P_j \rangle \geq \frac{\hbar \delta_{ij}}{2} \left(1 + \alpha (\Delta X_i)^2\right),$$  \hspace{1cm} (2)$$

where the deformation parameter is proportional to the cosmological constant. It is obvious that Eq. (2) yields a nonzero minimal uncertainty in momentum as $\langle \Delta P \rangle_{min} = \frac{\hbar \sqrt{\alpha}}{2}$. Recently, Mignemi showed that one can derive Eq. (2) from the definition of the quantum mechanics on the anti-de Sitter spacetime [57]. Moreover, in that case, he stated that the modified Heisenberg algebra corresponding to the EUP as follows:

$$[X_i, P_j] = i\hbar (\delta_{ij} + \alpha X_i X_j); \quad \ [X_i, X_j] = 0; \quad [P_i, P_j] = i\hbar \alpha L_{ij}.$$  \hspace{1cm} (3)$$

Here, $L_{ij}$ is the angular momenta while $i, j = 1, 2, 3$. In the position space, one particular explicit representation of the position and momentum operators that obeys Eq. (3) is given with

$$X_i = \frac{x_i}{\sqrt{1 - \alpha r^2}}; \quad \quad (4a)$$
$$P_i = -i\hbar \sqrt{1 - \alpha r^2} \frac{\partial}{\partial x_i}; \quad \quad (4b)$$

It is worth noting that as a consequence of the deformation of the usual algebra the conventional inner product definition is needed to be modified with [33]

$$\langle \psi | \phi \rangle = \int \frac{d^3r}{\sqrt{1 - \alpha r^2}} \psi^* \phi.$$  \hspace{1cm} (5)$$
In the last decades by considering modifications to the momentum and position operators the extension of the Heisenberg algebra is being examined extensively \cite{8,21,29,32,33,43,53,56,61}. However, to the best of our knowledge, the rich structure of the DKP due to non-minimum vector coupling has not been studied with a ML. Our purpose in this study is to consider the spin-one DKP with a nonminimal vector interaction in the presence of the MUM. The structure of the manuscript is constructed as follows: In section II we briefly introduce the DKP formalism and discuss the nonminimal coupling vector interaction. In section III we study the effect of the MUM on the spectrum of spin-one particles in the presence of a nonminimal vector linear potential in \((3 + 1)\) dimensional. In the final section, we give our conclusion.

II. THE DKP EQUATION

In the case of non-interacting scalar and vector bosons the DKP is defined with the natural units, \((\hbar = c = 1)\), as follows:

\[
[i\beta^\sigma \partial_\sigma - \mu] \Psi(\vec{r}, t) = 0, \quad \text{where} \quad \sigma = 0, 1, 2, 3.
\]

Here, \(m\) is the mass of the spin-one particle. The DKP matrices, \(\beta^\sigma\), satisfy the following DKP algebra

\[
\beta^\alpha \beta^\sigma \beta^\lambda + \beta^\lambda \beta^\sigma \beta^\alpha = g^{\alpha \kappa} \beta^\lambda + g^{\kappa \lambda} \beta^\sigma.
\]

where \(g^{\alpha \kappa} = \text{diag}(1, -1, -1, -1)\) is the metric tensor of the Minkowski space-time while \((g^{\alpha \kappa})^2 = 1\). In the spin-one sector, the irreducible DKP matrices are given with \(10 \times 10\) matrix sets. In this manuscript, we employ the following DKP matrices:

\[
\beta^0 = \begin{pmatrix} 0 & \hat{0}_{1 \times 3} & \hat{0}_{1 \times 3} & \hat{0}_{1 \times 3} \\
0 & \sigma_3 & \sigma_3 & \sigma_3 \\
0 & \sigma_3 & \sigma_3 & \sigma_3 \\
0 & \sigma_3 & \sigma_3 & \sigma_3 \end{pmatrix}, \quad \beta^k = \begin{pmatrix} 0 & \hat{0}_{1 \times 3} & \hat{u}^j_{1 \times 3} & \hat{0}_{1 \times 3} \\
0 & \sigma_3 & \sigma_3 & -iS^j_{3 \times 3} \\
0 & \sigma_3 & -iS^j_{3 \times 3} & \sigma_3 \\
0 & \sigma_3 & \sigma_3 & \sigma_3 \end{pmatrix},
\]

where \(j = 1, 2, 3\),

\[
\hat{0}_{1 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad \hat{I}_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix},
\]

\[
\hat{u}^1_{1 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad \hat{u}^2_{1 \times 3} = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0 \end{pmatrix}, \quad \hat{u}^3_{1 \times 3} = \begin{pmatrix} 0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix},
\]

\(S^j_{3 \times 3}\) are the usual spin-one matrices as given

\[
S^1_{3 \times 3} = i \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \end{pmatrix}, \quad S^2_{3 \times 3} = i \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0 \end{pmatrix}, \quad S^3_{3 \times 3} = i \begin{pmatrix} 0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix},
\]

When the interactions are taken into account the DKP is defined with

\[
[i\beta^\sigma \partial_\sigma - \mu - U] \Psi(\vec{r}, t) = 0, \quad \text{where} \quad \sigma = 0, 1, 2, 3.
\]

with the natural units. Here, \(U\) is the general potential energy matrix that can be expressed with 100 irreducible matrices in the spin-one sector. In this case, the four-current, \(J^\mu\), satisfies

\[
\partial_\mu J^\mu + \frac{i\nabla}{2} (U - \eta^0 U^1 \eta^0) \Psi = 0,
\]

where \(\nabla = \Psi^1 \eta^0\). Note that, the four-current is conserved when \(U\) is Hermitian with respect to \(\eta^0\), \cite{20}. In the spin-one sector the potential energy matrices can be constructed by well-defined Lorentz structures such as two-vector, two-scalar, two pseudo-vector, a pseudo-scalar, and eight tensor terms. However, in applications tensor terms are discarded since they issue non-causal effects \cite{60}. In this manuscript we consider a non-minimal vector interaction in the form of

\[
U = i [P, \beta^\mu] A_\mu.
\]
Here, $P$ denotes the projection operator, thus, $P^2 = P$ and $P^\dagger = P$. It is worth noting that the considered potential energy matrices behaves as a vector under the Lorentz transformation [20]. By choosing the potential energy matrix in this way, it is shown that four currents are conserved [20].

In this manuscript, we consider a time-independent potential energy, therefore we assume the spin-one wave function can be expressed in the form of

$$\Psi(\vec{r}, t) = \psi(\vec{r}) e^{-iEt}.$$ 

Here, $E$ is the energy of the spin-one boson particle. Then, Eq. (12) reduces to

$$\left[ \beta^\dagger E + \beta^0 \cdot \vec{P} - m - i [P, \beta^\mu] A^\mu \right] \psi(\vec{r}) = 0.$$ 

(15)

III. DKP EQUATION IN THE PRESENCE OF MINIMUM UNCERTAINTY IN MOMENTUM

In this section, we examine the dynamics of a vector boson by solving the DKPe in the presence of ML by considering a nonminimal vector coupling. We take the wave function in the form of [36]

$$\psi(\vec{r}) = \begin{pmatrix} \phi_F \\ F \\ G \\ H \end{pmatrix},$$ 

(16)

where $\phi$ is a scalar function, and

$$\vec{F} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \vec{G} = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{pmatrix}.$$ 

(17)

We consider the parity operator as $P = \beta^\mu \beta_\mu - 2$. By using the chosen representation, which is given in Eq. (8), we obtain the parity operator matrix as

$$P = \text{diag} \left( 1 1 1 1 0 0 0 0 0 0 \right).$$ 

(18)

We employ the position and momentum operators that are given in Eq. (4) by considering the assumption of the presence of the ML. Then, we derive a compact form of the time-independent DKPe out of Eq. (15)

$$i \sqrt{1 - \alpha r^2} \left( \vec{A} - \frac{\vec{A}}{\sqrt{1 - \alpha r^2}} \right) \times \vec{F} = m \vec{H},$$ 

(19a)

$$\sqrt{1 - \alpha r^2} \left( \vec{A} + \frac{\vec{A}}{\sqrt{1 - \alpha r^2}} \right) \vec{G} = m \phi,$$ 

(19b)

$$i \sqrt{1 - \alpha r^2} \left( \vec{A} + \frac{\vec{A}}{\sqrt{1 - \alpha r^2}} \right) \times \vec{H} = m \vec{F} - (E - iA_0) \vec{G},$$ 

(19c)

$$\sqrt{1 - \alpha r^2} \left( \vec{A} - \frac{\vec{A}}{\sqrt{1 - \alpha r^2}} \right) \phi = m \vec{G} - (E + iA_0) \vec{F}.$$ 

(19d)

In order to solve these coupled equations we follow [60], and assume that wave function components have the form of

$$\phi = \varphi(r) Y_{JM}(\theta, \phi)$$ 

(20a)

$$\vec{F} = \sum_L F_{nLL}(r) Y_{rL1}(\theta, \phi)$$ 

(20b)

$$\vec{G} = \sum_L G_{nLL}(r) Y_{rL1}(\theta, \phi)$$ 

(20c)

$$\vec{H} = \sum_L H_{nLL}(r) Y_{rL1}(\theta, \phi),$$ 

(20d)
where $Y_{JM}(\theta, \phi)$ is the spherical harmonics of order $J$, $Y_{JM}^M(\theta, \phi)$ are the vector spherical harmonics, and $\Phi_{nJ}(r)$, $F_{nJL}(r)$, $G_{nJL}(r)$, and $H_{nJL}(r)$ are unnormalized radial wave functions. In this manuscript, we examine a spherical symmetric vector potential in the form of

$$\vec{A} = \frac{A(r)}{r} \hat{r}.$$  \hspace{1cm} (21)

Then, by using the properties of vector spherical harmonics \[2, 7, 23\], we obtain the following radial differential equations:

$$\sqrt{1 - \alpha r^2} \zeta_j \left( \frac{d}{dr} + \frac{j + 1}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) F_0 = -mH_+(22)$$

$$\sqrt{1 - \alpha r^2} \zeta_j \left( \frac{d}{dr} + \frac{j}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) F_0 = -mH_-(23)$$

$$\sqrt{1 - \alpha r^2} \left[ \zeta_j \left( \frac{d}{dr} + \frac{j + 1}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) F_{+1} + \zeta_j \left( \frac{d}{dr} + \frac{j}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) F_{-1} \right] = -mH_0, (24)$$

$$\sqrt{1 - \alpha r^2} \left[ -\xi_j \left( \frac{d}{dr} + \frac{j + 1}{r} + \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) H_{+1} + \xi_j \left( \frac{d}{dr} - \frac{j}{r} + \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) H_{-1} \right] + (E - iA_0) G_{+1} = mF_{+1}, \hspace{1cm} (26)$$

$$\sqrt{1 - \alpha r^2} \left[ -\zeta_j \left( \frac{d}{dr} + \frac{j + 1}{r} + \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) H_0 + (E - iA_0) G_{-1} = mF_{-1}, \hspace{1cm} (27)$$

$$\sqrt{1 - \alpha r^2} \left[ \zeta_j \left( \frac{d}{dr} + \frac{j + 1}{r} + \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) H_{+1} + \zeta_j \left( \frac{d}{dr} - \frac{j}{r} + \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) H_{-1} \right] + (E - iA_0) G_0 = mF_0, \hspace{1cm} (28)$$

$$\sqrt{1 - \alpha r^2} \left[ \xi_j \left( \frac{d}{dr} - \frac{j - 1}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) \varphi + (E + iA_0) F_{+1} = mG_{+1}, \hspace{1cm} (30)$$

$$\sqrt{1 - \alpha r^2} \zeta_j \left( \frac{d}{dr} - \frac{j}{r} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) \varphi + (E + iA_0) F_{-1} = mG_{-1}. \hspace{1cm} (31)$$

where

$$\xi_j = \sqrt{\frac{j + 1}{2J + 1}}, \hspace{1cm} \zeta_j = \sqrt{\frac{j}{2J + 1}}.$$

Nedjadi et al., in \[60\], presented a procedure to decouple ten-coupled differential equations into two classes of coupled differential equation sets by taking the parity into account. We follow their procedure and consider Eqs. \[22, 23, 25\], and \[29\] for the natural parity states which relates $F_0, G_0, H_{+1}$ and $H_{-1}$ functions. We take $\varphi, H_0, F_{+1}, F_{-1}, G_{+1},$ and $G_{-1}$ functions as zero. On the other hand, for the unnatural parity states, we examine Eqs. \[24, 25, 29, 27, 31\], and \[31\] that associate $F_{+1}, F_{-1}, G_{+1}, G_{-1}, H_0$, and $\varphi$ functions, while we assume $F_0, G_0, H_{+1}$ and $H_{-1}$, thus, Eqs. \[22, 23, 25\], and \[29\] are zero.

A. $(-1)^J$ parity states

In this subsection we obtain the energy eigenvalue function for the natural parity states. From Eqs. \[22, 23, \] and \[29\] we find $H_{+1}, H_{-1}, G_0$ in terms of $F_0$. Then, we employ them in Eq. \[28\]. After a little algebra we find

$$\left[ (1 - \alpha r^2) \left( \frac{d^2}{dr^2} - \frac{j(j + 1)}{r^2} - \frac{A_r}{r \sqrt{1 - \alpha r^2}} - \frac{A_r^2}{\sqrt{1 - \alpha r^2}} \right) - \alpha r \left( \frac{d}{dr} - \frac{A_r}{\sqrt{1 - \alpha r^2}} \right) + (E^2 + A_0^2 - m^2) \right] F_0 = 0. \hspace{1cm} (32)$$

Then, we consider a vector potential energy with the following components:

$$A_0 = \lambda_0 \frac{r}{\sqrt{1 - \alpha r^2}}, \hspace{1cm} (33)$$

$$A_r = \lambda_r \frac{r}{\sqrt{1 - \alpha r^2}}. \hspace{1cm} (34)$$
Then, we reach

\[ \left[ \left( \sqrt{1 - \alpha r^2} \frac{d}{dr} \right)^2 - \left(1 - \alpha r^2 \right) \frac{J(J+1)}{r^2} - \frac{\lambda r}{1 - \alpha r^2} - \frac{(\lambda_r^2 - \lambda_0^2) r^2}{1 - \alpha r^2} + E^2 - m^2 \right] F_0 = 0. \]  

(35)

Next, we introduce a new variable \( \rho \) via the coordinate transformation \( r = \alpha \rho \). We obtain

\[ \left[ (1 - \rho) \frac{d^2}{d\rho^2} + \left( \frac{1}{2} - \rho \right) \frac{d}{d\rho} - \frac{J(J+1)}{4 \rho} - \frac{1}{4} \left( \frac{\lambda_r^2}{\alpha^2} - \frac{\lambda_0^2}{\alpha^2} \right) \right] F_0 = 0. \]  

(36)

For the general solution, we follow an Ansatz as

\[ F_0 = \rho^\nu (1 - \rho)^\mu \Xi(\rho). \]  

(37)

Then, we reach

\[ \left[ (1 - \rho) \frac{d^2}{d\rho^2} + \left( \frac{1}{2} + 2a - (1 + 2a + 2b) \right) \frac{d}{d\rho} + \frac{v_1}{\rho} + \frac{v_2}{1 - \rho} - u \right] \Xi(\rho) = 0, \]  

(38)

where

\[ v_1 = a \left( \frac{a - 1}{2} \right) - \frac{J(J+1)}{4}, \]  

(39)

\[ v_2 = b \left( \frac{b - 1}{2} \right) - \frac{1}{4} \left( \frac{\lambda_r^2}{\alpha^2} + \frac{\lambda_0^2}{\alpha^2} - \frac{\lambda_r^2}{\alpha^2} \right), \]  

(40)

\[ u = (a + b)^2 - \frac{E^2 - m^2}{4 \alpha} - \frac{J(J+1)}{4} - \frac{\lambda_r^2 - \lambda_0^2}{4 \alpha^2}. \]  

(41)

For the roots

\[ a = \frac{J + 1}{2}, \]  

(42a)

\[ b = \frac{1}{4} + \frac{1}{4} \sqrt{1 + 4 \left( \lambda_r \frac{\lambda_0 - 1}{\alpha + 1} - \frac{\lambda_r^2}{\alpha^2} \right)}, \]  

(42b)

\( v_1 \) and \( v_2 \) vanish. Therefore, Eq. (38) reduces to the hypergeometric differential equation

\[ \left[ \rho (1 - \rho) \frac{d^2}{d\rho^2} + \left( C - (1 + A + B) \rho \right) \frac{d}{d\rho} - A B \right] \Xi(\rho) = 0, \]  

where

\[ \Xi = N_1 \times _2F_1 (A; B; C; \rho). \]  

Here, \( N_1 \) is the normalization constant, and

\[ A = a + b + \sqrt{\frac{E^2 - m^2}{4 \alpha} + \frac{J(J+1)}{4} + \frac{\lambda_r^2 - \lambda_0^2}{4 \alpha^2}}, \]  

(43a)

\[ B = a + b - \sqrt{\frac{E^2 - m^2}{4 \alpha} + \frac{J(J+1)}{4} + \frac{\lambda_r^2 - \lambda_0^2}{4 \alpha^2}}, \]  

(43b)

\[ C = \frac{1}{2} + 2a. \]  

(43c)

We want to emphasize that to obtain a nonsingular solution, we do not consider the second solution of the hypergeometric differential equation by equating its normalization constant to zero. After that for the quantization, we use the well-known condition

\[ B = -n. \]  

(44)
which yields to

\[ E_{n;J} = \pm \sqrt{m^2 + 4\alpha \left( n + \frac{2J + 3}{4} + \frac{1}{4} \sqrt{1 + 4\left( \frac{\lambda_r}{\alpha} \left( \frac{\lambda_r}{\alpha} + 1 \right) - \frac{\lambda_0^2}{\alpha^2} \right)} \right)^2 - \alpha J (J + 1) - \frac{(\lambda_r^2 - \lambda_0^2)}{\alpha}}. \] (45)

We see that the energy eigenvalue expression contains an additional correction term which depends on the deformation \( \alpha \). It is worth noting that the presence of a correction term proportional to \( n^2 \) indicates the appearance of hard confinement due to the deformation. This is similar to the energy eigenvalue function of a particle in a square well potential whose boundaries are placed at \( \pm \frac{\pi \sqrt{\alpha}}{2} \). The second correction term is proportional to \( J (J + 1) \), so, it mimics a kind of rotational energy and removes the degeneracy of the usual spectrum according to \( J \) number. In addition, in the limit of \( \alpha \to 0 \), the energy level for the spin-one DKPe with a nonminimal vector interaction reduces to

\[ E_{n;J} = \sqrt{m^2 + \lambda_r + (4n + J + 3) \sqrt{\lambda_r^2 - \lambda_0^2}}. \] (46)

which is the same result of ordinary case \[20\]. We demonstrate these results graphically by assigning some numerical values to the deformation and nonminimal vector coupling parameter \( \lambda_0 \). Note that in all graphs of this manuscript we assume \( m = 1 \) and \( \lambda_r = 1 \). In fig. 1 for \( \alpha = J = 0 \), we present the behavior of the energy eigenvalue function versus \( n \) with four different values of \( \lambda_0 \). We see a constant energy value for \( \lambda_0 = \lambda_r \) as foreseen in Eq. (46). On the other hand, we observe that for decreasing values of \( \lambda \), the energy increases faster for small \( n \) quantum numbers. We observe that the increments of these increases gradually slow down as the quantum numbers become larger.

![Graph showing energy levels versus quantum number, n, for different values of \( \lambda_0 \).](image)

FIG. 1. Energy levels versus the quantum number, \( n \), for different values of \( \lambda_0 \).

Next, we assign a fixed value for the vector coupling parameter, \( \lambda_0 = 0.5 \), with \( J = 0 \) and examine the behaviour of the energy eigenvalue function versus \( n \) with four different values of the deformation parameter in fig. 2. When \( \alpha \) is equal to zero, we observe an increase in the energy function as predicted in Eq. (46). For non-zero alpha values, we see that the increase in energy function varies linearly in accordance with Eq. (45) with respect to the quantum number \( n \). The increase is greater in larger deformation parameters.

Another interesting property arises for the energy level spacing which is defined by \( \Delta E_{n;J} = E_{n+1;J} - E_{n;J} \). We observe that for large \( n \),

\[ \lim_{n \to \infty} |\Delta E_{n;J}| = 2\sqrt{\alpha}, \] (47)
It is worth noting that the energy spacing tends to zero in the absence of the MUM. For the graphical illustration, we take $\lambda_0 = 0.5$ and plot the energy level spacing versus quantum number $n$ for $J = 0$ with different values of deformation parameter in fig. 3.

We observe that for small $n$, the changes in between the energy levels are not constant. For higher quantum numbers, the energy level spacings become constant and it is proportional to the deformation parameter as predicted in Eq. (47).
B. \((-1)^{J+1}\) parity states

In this subsection we investigate the unnatural parity states and derive the energy eigenvalue function. In the general case, where \(A_0\) and \(A_r\) have non zero values as given in Eqs. \(53\) and \(54\). Castro et al. \cite{20} stated that a decoupling process can not be executed successfully for the Eqs. \(24\), \(25\), \(26\), \(27\), \(28\), \(29\), and \(31\). Instead, for \(\lambda_0 = 0\), thus, \(A_0 = 0\), Eqs. \(20\), \(21\), \(30\), \(31\), and \(33\) reduce to the following forms, respectively.

\[
-\sqrt{1-\alpha r^2} \zeta_J \left( \frac{d}{dr} - \frac{J+1}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}} \right) H_0 = m F_{+1} - E G_{+1},
\]

\[
-\sqrt{1-\alpha r^2} \zeta_J \left( \frac{d}{dr} + \frac{J}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}} \right) H_0 = m F_{-1} - E G_{-1},
\]

\[
-\sqrt{1-\alpha r^2} \zeta_J \left( \frac{d}{dr} - \frac{J-1}{r} - \frac{A_r}{\sqrt{1-\alpha r^2}} \right) \varphi = m G_{+1} - E F_{+1},
\]

\[
-\sqrt{1-\alpha r^2} \zeta_J \left( \frac{d}{dr} + \frac{J}{r} - \frac{A_r}{\sqrt{1-\alpha r^2}} \right) \varphi = m g_{-1} - E F_{-1}.
\]

After a little algebra, we express these four equations in the form of

\[
\begin{pmatrix}
F_{+1} \\
G_{+1}
\end{pmatrix} = \sqrt{1-\alpha r^2} \frac{E}{E^2 - m^2} \begin{pmatrix}
E \zeta_J & m \zeta_J \\
 m \zeta_J & E \zeta_J
\end{pmatrix} \begin{pmatrix}
\frac{d}{dr} - \frac{J+1}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}} \\
\frac{d}{dr} + \frac{J+1}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}}
\end{pmatrix} \begin{pmatrix}
\varphi \\
H_0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
f_{-1} \\
g_{-1}
\end{pmatrix} = \sqrt{1-\alpha r^2} \frac{E}{E^2 - m^2} \begin{pmatrix}
-E \zeta_J & m \zeta_J \\
-m \zeta_J & E \zeta_J
\end{pmatrix} \begin{pmatrix}
\frac{d}{dr} + \frac{J}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}} \\
\frac{d}{dr} + \frac{J}{r} + \frac{A_r}{\sqrt{1-\alpha r^2}}
\end{pmatrix} \begin{pmatrix}
\varphi \\
h_0
\end{pmatrix}
\]

We take \(g_{+1}\) and \(g_{-1}\) from Eqs. \(62\), \(63\) and employ in Eq. \(29\). We find

\[
\left[ (1-\alpha) \left( \frac{d^2}{dr^2} - \alpha \frac{d}{dr} - \frac{J(J+1)}{r^2} - \frac{A r}{(1-\alpha^2)} \right) + \tilde{k}_1 \right] H_0 + \frac{\alpha E}{m} \sqrt{J(J+1)} \varphi = 0,
\]

where \(\tilde{k}_1 = E^2 - m^2 + \lambda_r + \alpha J(J+1)\). Alike, we draw \(f_{+1}\) and \(f_{-1}\) from Eqs. \(62\), \(63\) and use in Eq. \(29\). We get

\[
\left[ (1-\alpha) \left( \frac{d^2}{dr^2} - \alpha \frac{d}{dr} - \frac{J(J+1)}{r^2} - \frac{A r}{(1-\alpha^2)} \right) + \tilde{k}_2 \right] \varphi + \frac{\alpha E}{m} \sqrt{J(J+1)} H_0 = 0,
\]

while \(\tilde{k}_2 = E^2 - m^2 - (3\lambda_r - \alpha) + \alpha J(J+1)\). For simplicity, we set \(J = 0\), and examine a particular solution among the general solution. It is worth noting that under this choice, \(H_0\) and \(\varphi\) decouple from each others in Eqs. \(55\) and \(54\) as

\[
\left[ (1-\alpha) \frac{d^2}{dr^2} - \alpha \frac{d}{dr} - A r \left( \frac{\lambda r}{\alpha} - 1 \right) \frac{1}{r^2} + E^2 - m^2 - (3\lambda_r - \alpha) \right] \varphi = 0,
\]

\[
\left[ (1-\alpha) \frac{d^2}{dr^2} - \alpha \frac{d}{dr} - A r \left( \frac{\lambda r}{\alpha} - 1 \right) \frac{1}{r^2} + E^2 - m^2 + \lambda_r \right] H_0 = 0.
\]

At the next step, we introduce a variable change \(\rho = \alpha r^2\). We obtain

\[
\left[ (1-\rho) \rho \frac{d^2}{d\rho^2} + \left( \frac{1}{2} - \rho \right) \frac{d}{d\rho} + \frac{E^2 - m^2}{4 \alpha} + \frac{1}{4} - \frac{A r \left( \frac{\lambda r}{\alpha} + 1 \right)}{4(1-\rho)} + \frac{\lambda_r}{4 \alpha} \left( \frac{\lambda r}{\alpha} - 2 \right) \right] \varphi = 0,
\]

\[
\left[ (1-\rho) \rho \frac{d^2}{d\rho^2} + \left( \frac{1}{2} - \rho \right) \frac{d}{d\rho} + \frac{E^2 - m^2}{4 \alpha} + \frac{\lambda^2}{4 \alpha^2} - \frac{A r \left( \frac{\lambda r}{\alpha} - 1 \right)}{4(1-\rho)} \right] H_0 = 0.
\]

Since Eqs. \(58\) and \(59\) are similar to Eq. \(56\), they can be solved exactly in the same manner. We follow the recipe written in between Eqs. \(37\) and \(44\) and obtain the energy spectra as

\[
E_\varphi = \pm \sqrt{m^2 + 4 \lambda_r + 4 \alpha \left( n + \frac{1}{2} \right) \left( n + \frac{3}{2} + \frac{\lambda r}{\alpha} \right)},
\]

\[
E_{H_0} = \pm \sqrt{m^2 + 4 \alpha \left( n + \frac{1}{2} \right) \left( n + \frac{1}{2} + \frac{\lambda r}{\alpha} \right)}.
\]
Finally, we take $J = 0$ and $\lambda_0 = 0.5$ and plot the energy functions $E_\varphi$ and $E_{H_0}$ versus $n$ for two nonzero deformation parameter values in fig. [4]. We observe that for all quantum numbers $E_\varphi$ is greater than $E_{H_0}$. The change of the deformation parameter in small quantum numbers does not cause much difference in the energy values. As the quantum numbers grow, the change of the deformation parameter has a greater effect on the energy values. For $E_\varphi$ and $E_{H_0}$, these effects are similar.

![Graph of energy functions](image)

**FIG. 4.** Variation of DKP energies with $n$ for different values of the deformation parameters $\alpha$.

### IV. CONCLUSION

In this paper, we discussed the various consequences of considering a non-minimal vector interaction in the presence of minimal uncertainty in momentum in the Duffin-Kemmer-Petiau (DKP) formalism. We obtained eigenfunctions in terms of the hypergeometric function, analytically. We exposed an explicit calculation of the energy eigenvalue function for the bound states of the spin-one DKP equation in three-dimension spaces by using the quantization condition. Since the energy eigenfunctions depend on the nonminimal coupling constants and the deformation parameters, we revealed the effects of them on the energy values analytically. We strengthened our findings by presenting these effects in several figures. Moreover, we found that the energy eigenvalues depend on the quantum number $n^2$ like square well problem. For large $n$, we found that the energy level spacings become a constant which is proportional to the deformation parameter. Our finding predicted a discontinuity in the energy levels.

### ACKNOWLEDGMENT

One of the author, B.C. Lütfüoğlu, was partially supported by the Turkish Science and Research Council (TÜBİTAK).

[1] L.M. Abreu, Alexandre L. Gadelha, B.M. Pimentel, and E.S. Santos. Galilean dkp theory and bose–einstein condensation. *Physica A*, 419:612 – 621, 2015.

[2] S. Ait-Tahar, J.S. Al-Khalili, and Y. Nedjadi. A relativistic model for alpha-nucleus elastic scattering. *Nucl. Phys. A*, 589(2):307 – 319, 1995.
[3] D. Amati, M. Ciafaloni, and G. Veneziano. Superstring collisions at planckian energies. *Phys. Lett. B.*, 197(1):81 – 88, 1987.

[4] H. Aounallah and A. Boumali. Solutions of the duffin–kemmer equation in non-commutative space of cosmic string and magnetic monopole with allowance for the aharonov–bohm and coulomb potentials. *Phys. Part. Nuclei Lett.*, 16:195, 2019.

[5] H. Aounallah, B.C. Lütfiçoğlu, and J. Kříž. Thermal properties of a two-dimensional duffin-kemmer-petiau oscillator under an external magnetic field in the presence of a minimal length. *Mod. Phys. Lett. A.*, 35:2050278, 2020.

[6] M.K. Bahar and F. Yasuk. Relativistic solutions for the spin-1 particles in the two-dimensional smorodinsky–winternitz potential. *Ann. Phys.*, 344:105 – 117, 2014.

[7] Roger C Barrett and Youcef Nedjadi. Meson-nuclear interactions in the duffin-kemmer-petiau formalism. *Nucl. Phys. A.*, 585(1):311 – 312, 1995. Hypermuclear and Strange Particle Physics.

[8] H. Benzair, T. Boudjedaa, and M. Merad. Propagator of Dirac oscillator in 2D with the presence of the minimal length uncertainty. *Eur. Phys. J. Plus.*, 132:94, 2017.

[9] Brett Bolen and Marco Cavaglià. (Anti-)de Sitter black hole thermodynamics and the generalized uncertainty principle. *Gen. Relativ. Gravit.*, 37(7):1255–1262, 2005.

[10] Pasquale Bosso and Octavio Obregón. Minimal length effects on quantum cosmology and quantum black hole models. *Class. Quantum Gravity*, 37(4):045003, 2020.

[11] A. Boumali and H. Aounallah. Exact solutions of scalar bosons in the presence of the aharonov-bohm and coulomb potentials in the gravitational field of topological defects. *Adv. High Energy Phys.*, 2018:1031763, 2018.

[12] A. Boumali and H. Aounallah. Exact solutions of vector bosons in the presence of the aharonov-bohm and coulomb potentials in the gravitational field of topological defects in non-commutative space-time. *Rev. Mex. Fis.*, 66:192, 2020.

[13] A. Boumali and H. Hassanabadi. The thermal properties of a two-dimensional dirac oscillator under an external magnetic field. *Eur. Phys. J. Plas.*, 128:124, 2013.

[14] Abdelmalek Boumali, Lyazid Chetouani, and Hassan Hassanabadi. Two-dimensional duffin–kemmer–petiau oscillator under an external magnetic field. *Can. J. Phys.*, 91(1):1–11, 2013.

[15] T R Cardoso, L B Castro, and A S de Castro. On the nonminimal vector coupling in the duffin-kemmer-petiau theory and the confinement of massive bosons by a linear potential. *J. Phys. A: Math. Theor.*, 43(5):055306, 2010.

[16] Tatiana R Cardoso and Antonio S. de Castro. Estados estacionários de partículas sem spin em potenciais quadrados. *Rev. Bras. Ensino Fís.*, 30:2306.1–2306.10, 2008.

[17] R. Casana, V.Ya. Fainberg, B.M. Pimentel, and J.S. Valverde. Bose–einstein condensation and free dkp field. *Phys. Lett. A.*, 316(1):33 – 43, 2003.

[18] L.B Castro. Quantum dynamics of scalar bosons in a cosmic string background. *Eur. Phys. J. C.*, 75:287, 2015.

[19] L.B Castro. Non-inertial effects on the quantum dynamics of scalar bosons. *Eur. Phys. J. C.*, 76:61, 2016.

[20] L.B Castro and L.P. de Oliveira. Remarks on the spin-one duffin-kemmer-petiau equation in the presence of nonminimal vector interactions in (3 + 1) dimensions. *Adv. High Energy Phys.*, 2014:784072, 2014.

[21] W. S. Chung and H. Hassanabadi. A new higher order gup: one dimensional quantum system. *Eur. Phys. J. C.*, 79:213, 2019.

[22] B.C. Clark, R.J. Furnstahl, L.Kurth Kerr, John Rusnak, and S. Hama. Pion-nucleus scattering at medium energies with densities from chiral effective field theories. *Phys. Lett. B.*, 427(3):231 – 234, 1998.

[23] A. S. de Castro. On duffin–kemmer–petiau particles with a mixed minimal-nonminimal vector coupling and the nondegenerate bound-states for the one-dimensional inversely linear background. *J. Math. Phys.*, 51(10):102302, 2010.

[24] M de Montigny, F C Khanna, A E Santana, E S Santos, and J D M Viana. Galilean covariance and the duffin-kemmer-petiau equation. *J. Phys. A: Math. Theor.*, 33(31):L273–L278, 2000.

[25] Luiz P. de Oliveira. Quantum dynamics of relativistic bosons through nonminimal vector square potentials. *Ann. Phys.*, 372:320 – 328, 2016.

[26] R. J. Duffin. On the characteristic matrices of covariant systems. *Ann. Phys.*, 11:114–1144, 1938.

[27] V. Ya. Fainberg and B.M. Pimentel. On Equivalence of Duffin–Kemmer–Petiau and Klein–Gordon equations. *Prog. Theor. Phys.*, 30(2):275–281, 2000.

[28] M Falek and M Merad. DKP oscillator in a noncommutative space. *Commun. Theor. Phys.*, 50(3):587–592, 2008.

[29] Nasrin Farahani, Hassan Hassanabadi, Jan Kříž, Won Sang Chung, and Saber Zarrinkamar. DSR-GUP Black Hole based on COW experiment and Einstein–Bohr’s photon box. *Eur. Phys. J. C.*, 80(8):696, 2020.

[30] V. Gribov. QCD at large and short distances (annotated version). *Eur. Phys. J. C.*, 10:71–90, 1999.

[31] Ralph F. Guertin and Thomas L. Wilson. Noncausal propagation in spin-0 theories with external field interactions. *Phys. Rev. D.*, 15:1518–1531, 1977.

[32] B. Hamil and M. Merad. Dirac and Klein-Gordon oscillators on anti-de Sitter space. *Eur. Phys. J. Plus*, 133(5):174, 2018.

[33] B. Hamil, M. Merad, and T. Birklandan. The duffin-kemmer-petiau oscillator in the presence of minimal uncertainty in momentum. *Phys. Scr.*, 95:075309, 2020.

[34] H. Hassanabadi, N. Farahani, W. S. Chung, and B. C. Lütfiçoğlu. Investigation of unruh temperature of black holes by using the egup formalism. *EPL*, 130(4):40001, 2020.

[35] H. Hassanabadi, E. Maghsoodi, W. S. Chung, and M. de Montigny. Thermodynamics of the Schwarzschild and Reissner–Nordström black holes under the Snyder–de Sitter model. *Eur. Phys. J. C.*, 79:936, 2019.

[36] H. Hassanabadi, Z. Molaee, M. Ghominejad, and S. Zarrinkamar. Spin-one dkp equation in the presence of coulomb and harmonic oscillator interactions in (1 + 3)-dimension. *Adv. High Energy Phys.*, 2012:489641, 2012.

[37] H. Hassanabadi, Z. Molaee, M. Ghominejad, and S. Zarrinkamar. Duffin–Kemmer–Petiau Equation with a Hyperbolic Potential in (2+1) Dimensions for Spin-One Particles. *Few-Body Syst.*, 54(11):1765–1772, 2013.
[38] H. Hassanabadi, Z. Molaee, and S. Zarrinkamar. Relativistic vector bosons under pÔschl–teller double-ring-shaped coulomb potential. *Mod. Phys. Lett. A.*, 27(39):1250228, 2012.

[39] H. Hassanabadi, S. Zarrinkamar, and E. Maghsoodi. Minimal length dirac equation revisited. *Eur. Phys. J. Plus*, 128:25, 2013.

[40] S. Hassanabadi, A. A. Rajabi, S. Zarrinkamar, and H. Hassanabadi. DKP equation under a vector Yukawa-type potential. *Phys. Part. Nucl. Lett.*, 10(1):28–32, 2013.

[41] Haye Hinrichsen and Achim Kempf. Maximal localization in the presence of minimal uncertainties in positions and in momenta. *J. Math. Phys.*, 37(5):2121–2137, 1996.

[42] A.N. Ikot, H.P. Obong, and H. Hassanabadi. Minimal Length Quantum Mechanics of Dirac Particles in Noncommutative Space. *Chin. Phys. Lett.*, 32-030201, 2015.

[43] V. Kanatchikov. On the duffin-kemmer-petiau formulation of the covariant hamiltonian dynamics in field theory. *Rep. Math. Phys.*, 46(1):107 – 112, 2000. Proceeding of the bdXXXI Symposium on Mathematical Physics.

[44] Nicholas Kemmer. The particle aspect of meson theory. *Proc. R. Soc. A.*, 173(952):91–116, 1939.

[45] Achim Kempf. Uncertainty relation in quantum mechanics with quantum group symmetry. *J. Math. Phys.*, 35(9):4483–4496, 1994.

[46] Achim Kempf. Non-pointlike particles in harmonic oscillators. *J. Phys. A: Math. Theor.*, 30(6):2093–2101, 1997.

[47] Achim Kempf, Gianpietro Mangano, and Robert B. Mann. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev. D*, 52:1108–1118, 1995.

[48] Toichiro Kinoshita. On the Interaction of Mesons with the Electromagnetic Field. I. *Prog. Theor. Phys.*, 5(3):473–488, 1950.

[49] R. E. Kozack, B. C. Clark, S. Hama, V. K. Mishra, R. L. Mercer, and L. Ray. Spin-one kemmer-duffin-petiau equations and intermediate-energy deuteron-nucleus scattering. *Phys. Rev. C*, 40:2181–2194, 1989.

[50] R. A. Krajeck and Michael Martin Nieto. Historical development of the bhabha first-order relativistic wave equations for arbitrary spin. *Am. J. Phys.*, 45(9):818–822, 1977.

[51] L. J. Kurth, B. C. Clark, E. D. Cooper, S. Hama, R. L. Mercer, and G. W. Hoffmann. Relativistic impulse approximation treatment of the elastic scattering of 400 mev \(\pi^{\pm}\) on \(^{28}\text{Si}\). *Phys. Rev. C*, 50:2624–2626, 1994.

[52] Michele Maggiore. The algebraic structure of the generalized uncertainty principle. *Phys. Lett. B*, 319(1):83 – 86, 1993.

[53] E. Maghsoodi, H. Hassanabadi, and W. S. Chung. Effect of the new extended uncertainty principle on black hole thermodynamics. *EPL*, 129(5):50001, 2020.

[54] M. Merad, F. Zeronou, and M. Falek. Relativistic particle in electromagnetic fields with a generalized uncertainty principle. *Mod. Phys. Lett. A*, 27(15):1250080, 2012.

[55] S. Mignemi. Extended uncertainty principle and the geometry of (anti)-de sitter space. *Mod. Phys. Lett. A*, 25(20):1697–1703, 2010.

[56] Z. Molaee, M. K. Bahar, F. Yasuk, and H. Hassanabadi. Minimal length Quantum Mechanics of Dirac Particles in Noncommutative Space. *Chin. Phys. Lett.*, 27(15):1250080, 2012.

[57] A. Moradzadeh and H. Hassanabadi. Quasi-maxwell equation for spin-1 particles. *Int. J. Mod. Phys. E*, 23(02):1450007, 2014.

[58] Y. Nedjadi and R. C. Barrett. Solution of the central field problem for a duffin–kemmer–petiau vector boson. *J. Math. Phys.*, 35(9):4517–4533, 1994.

[59] Pouria Pedram. New approach to nonperturbative quantum mechanics with minimal length uncertainty. *Phys. Rev. D*, 85:024016, 2012.

[60] G. Petiau. Ph.d. thesis. *Acad. Roy. Belg., A. Sci. Mem. Collect.*, 16:2, 1936.

[61] S. Saghafigar and Kourosh Nozari. Black hole thermodynamics in snyder phase space. *Int. J. Mod. Methods Mod. Phys.*, 14(11):1750164, 2017.

[62] Fabio Scardigli. Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment. *Phys. Lett. B*, 452(1):39 – 44, 1999.

[63] Fabio Scardigli and Roberto Casadio. Generalized uncertainty principle, extra dimensions and holography. *Class. Quantum Gravity*, 20(18):3915–3926, 2003.

[64] Hartland S. Snyder. Quantized space-time. *Phys. Rev.*, 71:38–41, 1947.

[65] Y. Sucu and N. ¨Unal. Vector bosons in the expanding universe. *Eur. Phys. J. C*, 44(2):287–291, 2005.

[66] G. Veneziano. A stringy nature needs just two constants. *Europhysics Letters (EPL)*, 2(3):199–204, 1986.

[67] B Vijayalakshmi, M Scetharaman, and P M Mathews. Consistency of spin-1 theories in external electromagnetic fields. *J. Phys. A: Math. Theor.*, 12(5):665–677, 1979.

[68] Li Xiang, Yi Ling, You-Gen Shen, Cheng-Zhou Liu, Hong-Sheng He, and Lan-Fang Xu. Generalized uncertainty principles, effective newton constant and the regular black hole. *Ann. Phys.*, 396:334 – 350, 2018.