First Boundary Value Problem for Cordes-Type Semilinear Parabolic Equation with Discontinuous Coefficients

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1. Introduction

Let $E_n$ be an $n$-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$ and $\Omega$ be a bounded domain in $E_n$ with boundary $\partial \Omega$ of the class $C^2$ or simply a convex domain. Set $Q_T = \Omega \times (0, T)$ and $\Gamma (Q_T) = \partial Q_T \setminus [t = T]$. Consider in $Q_T$ the Dirichlet problem:

$$
\sum_{i,j=1}^{n} a_{ij} (t, x) u_{x_i x_j} - u_t + g(t, x, u) = f(t, x), \quad (t, x) \in Q_T,
$$

(1)

$$
u|_{\Gamma (Q_T)} = 0.
$$

(2)

It is assumed that the coefficients $a_{ij} (t, x), i, j = 1, 2, \ldots, n$, of the operator

$$
L = \sum_{i,j=1}^{n} a_{ij} (t, x) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t},
$$

(3)

are bounded measurable functions satisfying the uniform parabolicity condition

$$\gamma |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij} (t, x) \xi_i \xi_j \leq \gamma^{-1} |\xi|^2,$$

(4)

for $\gamma \in (0, 1), \forall (t, x) \in Q_T, \forall \xi \in E_n$, and the Cordes-type condition

$$
\left( \sum_{i,j=1}^{n} a_{ij} (t, x) \right)^2 \leq \frac{1}{n - \mu^2 - \delta}.
$$

(5)

Here, $\mu = \left( \text{ess inf} \sum_{i=1}^{n} a_{ii} (t, x) \right)/\left( \text{ess sup} \sum_{i=1}^{n} a_{ii} (t, x) \right)$, and the number $\delta \in (0, (1/(n+1)))$. The nonlinear term $g(t, x, u): Q_T \rightarrow E_1$, satisfies the Caratheodory condition, that is, $g$ is a measurable function with respect to variables $(t, x) \in \Omega$, and for almost all $(t, x) \in Q_T$ continuously depend on the variable $u \in E_1$. Also, the growth condition

$$|g(t, x, u)| \leq b_0 |u|^q, \quad b_0 > 0,$$

(6)

is satisfied.

The space $W^{2,1}_p (Q_T), p > 1$, is a closure of function class $u \in C^\infty (\overline{Q_T} \cap C (\overline{Q_T}), u|_{\Gamma (Q_T)} = 0$ with respect to norm...
\[ \|u\|_{W^{2,1}_p(Q_T)} = \|u\|_{L_p(Q_T)} + \sum_{i=1}^n \|\partial_{x_i} u\|_{L_p(Q_T)} + \|\partial_t u\|_{L_p(Q_T)} + \sum_{i=1}^n \|\partial_{x_i}^2 u\|_{L_p(Q_T)}. \]  

(7)

Here, \( u_t, u_{ij} \) and \( u_{x_i} \) denote the weak derivatives \( u_{x_i}, u_{ij} \), and \( u_{x_i} \), respectively, \( i, j = 1, \ldots, n \). The conjugate number is denoted by \( p' \), i.e., \( 1 < p < \infty \), \((1/p) + (1/p') = 1 \). By the same letter \( C \), we denote different positive constants, and the value of \( C \) is not essential for purposes of this study.

For \( p \in [1, \infty] \), we denote by \( \|v\|_{L^p(Q_T)} \) or simply \( \|v\|_p \), the norm of \( v \) in a Banach space \( L^p[0, T; L^p(\Omega)] \) defined as

\[ \|g\|_p = (\int_0^T \|g(t, \cdot)\|_{L^p(\Omega)}^{p'})^{1/p}. \]

A function \( u(t, x) \in W^{2,1}_p(Q_T) \) is called the strong solution (almost everywhere) of problems (1) and (2) if it satisfies equation (1), a.e., in \( Q_T \).

In this study, we will make essential use of the existence results given in Theorem 1.1 of [1] (see, also [2]) for Cordes-type parabolic equations satisfying (5). In [1], the estimate

\[ \|u\|_{W^{2,1}_p(Q_T)} \leq C\|u\|_{L^2(Q_T)}, \]  

(8)

was proved for all \( u \in W^{2,1}_p(Q_T) \), and when \( T < T_0 \) with \( T_0 = T_0(n, L, \Omega) \) to be sufficiently small and positive constant \( C \) depends on \( n, L, \Omega \).

In the stationary case, i.e., the solution does not depend on the time variable (the elliptic equation), from examples ([3], p. 48), it is followed that the equation \( Lu = f \) is solvable in \( W^{2,1}_p(Q_T) \) for no \( p > 1 \) (see [3–8]) if the coefficients are discontinuous. In the absence of \( g(t, x, u) \), the strong solvability of the Dirichlet problem for quasi-linear parabolic equations under more restrictive then (5) conditions see, e.g., [9, 10].

If the trace of matrix \( [a_{ij}(t, x)] \) is constant, condition (5) is exactly Cordes condition (see, e.g., [7, 11–13]):

\[ \sum_{i,j=1}^n a_{ij}^2(t, x) \leq \frac{1}{n-1} - \delta. \]  

(9)

For the strong solvability problem in \( W^{2,1}_p(\Omega) \) for any \( p > 1 \) for parabolic equations with discontinuous coefficients, we refer [8, 14, 15], where the leading coefficients are taken from the VMO class. We refer [16] on exact growth conditions for strong solvability of nonlinear elliptic equations \( \Delta u = g(x, u, u_x) \) in \( W^{2,1}_p(\Omega) \) whenever \( p > n \).

The aim pursued in this paper is to prove the strong solvability of Dirichlet problems (1) and (2) in the space \( W^{2,1}_p(Q_T) \) for \( T \) to be sufficiently small, \( \|f(t, x)\|_{L^1(Q_T)} \) norm to be sufficiently small, and the coefficients to satisfy (5).

2. Main Result

In order to carry out the proof of main Theorem 1, we need the following assertion from [1].

**Lemma 1.** Let \( u(t, x) \in W^{2,1}_p(Q_T) \) function in \( Q_T = \Omega \times [0, T] \) and conditions (2), (4), and (5) be fulfilled for \( u(t, x) \) and coefficients of the operator \( L \); the domain \( \Omega \) is of \( C^2 \) class or simply convex. Then, there exists sufficiently small \( T_0 \) depending on \( L, n, \Omega \) such that, for \( T < T_0 \), estimate (8) holds with the constant \( C \) depending on \( L, n, \Omega \).

The following assertion is the main result of this paper.

**Theorem 1.** Let \( n > 4, 0 < q < (n + 1/n - 1) \), and conditions (4)–(6) be fulfilled, and \( \partial \Omega \in C^2 \). Let \( T_0 \) be a number in Lemma 1 and \( T \leq T_0 \). Then, problems (1) and (2) have at least one strong solution in the space \( W^{2,1}_2(Q_T) \) for any \( f(t, x) \in L_2(Q_T) \) satisfying

\[ \|f\|_{L_2(Q_T)} \leq CB_0^{-1/q-1} \text{mes}_{n+1} Q_T ((q(p-1)/(p+1))^{-1}(1/2(q-1))). \]

(10)

**Proof.** In order to get the solvability of problem (1) and (2), we apply the Schauder fixed point theorem on completely continuous mappings of a compact subset in the Banach space (see, e.g. [4], p. 257, or [17]).

Set \( L^q(Q_T) \) as a basic Banach space. In this space, we define the set \( V_2 = \{ u \in W^{2,1}_2(Q_T) \} \), \( \|u\|_{W^{2,1}_2(Q_T)} \leq K \), where the number \( K \) will be chosen later. Show that \( V_2 \), is compact in \( L^q(Q_T) \). By using the condition \( 2q < 2(n + 1)/(n - 1) \) and Sobolev–Kondrachov’s compact embedding theorem, the space \( W^{2,1}_2(Q_T) \) is imbedded into \( L^q(Q_T) \) compactly. On the contrary, \( W^{2,1}_2(Q_T) \) is compact. Therefore, \( L^q(Q_T) \) is compact.

Show \( V_2 \) is convex. For any \( u_1, u_2 \in V_2 \) and \( t \in [0, 1] \), it holds \( u = tu_1 + (1-t)u_2 \in V_2 \):

\[ \|u\|_{W^{2,1}_2(Q_T)} \leq t\|u_1\|_{W^{2,1}_2(Q_T)} + (1-t)\|u_2\|_{W^{2,1}_2(Q_T)} \leq K. \]

(11)

For \( u(t, x) \in V_2 \), denote \( \nu(t, x) \in W^{2,1}_2(Q_T) \) the solution of the Dirichlet problem:

\[ L\nu + g(t, x, u) = f(t, x), \quad (t, x) \in Q_T, \]

\[ \nu|_{\Gamma} = 0. \]

(12)

(13)

For fixed \( u(t, x) \in V_2 \) and \( f \in L_2(Q_T) \), problems (12) and (13) are uniquely solvable in the space \( W^{2,1}_2(Q_T) \); because of the assumptions on domain and \( q \), we get the Dirichlet problem for equation (1) (for its solvability, we refer \([1, 2, 9, 10] \)):

\[ L\nu = F(t, x), \quad (t, x) \in Q_T, \quad u|_{\Gamma} = 0, \]

(14)

where \( F = f(t, x) - g(t, x) \in L_2(Q_T) \).

We have
\[ \|F\|_{L^2(Q_T)} \leq \|f\|_{L^2(Q_T)} + \|g\|_{L^2(Q_T)} \leq \|f\|_{L^2(Q_T)} + \|u\|_{L^2(Q_T)} + b_0 \|u\|_{L^2(Q_T)}. \]  

(15)

By using the chain of imbeddings, \( W^{2,1}_2(Q_T) \subseteq L^2(Q_T) \) and \( u \in W^{2,1}_2(Q_T) \), the norm \( \|u\|_{L^2(Q_T)} \) is finite.

Insert an operator \( A : u \rightarrow \nabla \) acting on \( L^2(Q_T) \), where \( \nabla \) is a solution of problems (12) and (13):

\[ Au = \nabla. \]  

(16)

Show that operator \( A \) is completely continuous in \( L^2(Q_T) \). Let \( \{u_m\} \) be a convergence sequence in \( L^2(Q_T) \) with \( u_m \rightarrow u_0 \). Show that its image is convergent in \( L^2(Q_T) \) with \( v_m \rightarrow v_0 \), where \( v_0 = Au_0, v_m = Au_m \).

Then,

\[ L\nu_m = -g(t, x, u_m) + f, \]

\[ L\nu_0 = -g(t, x, u_0) + f. \]  

(17)

We have

\[ L(v_m - v_0) = -g(t, x, u_m) - g(t, x, u_0). \]  

(18)

Set \( g_m = g(t, x, u_m) \) and \( g = g(t, x, u) \), and show that

\[ \|g_m - g\|_{L^2(Q_T)} \rightarrow 0 \]  

for \( m \rightarrow \infty \).  

(19)

For that, from \( u_m \rightarrow u_0 \) in \( L^2(Q_T) \) follows the convergence in measure in \( Q_T \). This and the Carathéodory condition imply that the convergence in measure \( (g_m - g_0)^2 \rightarrow 0 \). To prove (19), it remains to show the equicontinuity of \( \{g_m\} \), which follows from equicontinuity of \( |u_m|^2 \). The convergence \( u_m \rightarrow u_0 \) in \( L^2(Q_T) \) implies equicontinuity of \( |u_m|^2 \).

Applying Vitali’s theorem, we get

\[ \|g_m - g\|_{L^2(Q_T)} \rightarrow 0 \]  

as \( m \rightarrow \infty \).  

(20)

To show \( v_m \rightarrow v_0 \) in \( L^2(Q_T) \), we use the estimate from Lemma 1 for sufficiently small \( T' \) with \( T \leq T' \).

\[ \|v_m - v_0\|_{W^{2,1}_2(Q_T)} \leq C\|L(v_m - v_0)\|_{L^2(Q_T)} = C\|g_m - g\|_{L^2(Q_T)} \rightarrow 0. \]  

(21)

By virtue of \( W^{2,1}_2(Q_T) \subseteq L^2(Q_T) \), it follows that

\[ \|v_n - v_0\|_{L^2(Q_T)} \rightarrow 0 \]  

as \( n \rightarrow \infty \).  

(22)

The complete continuity of operator \( A \) in \( L^2(Q_T) \) has been shown.

Now, we have to show \( u \in V_2 \) implies \( \nabla = Au \in V_2 \). For this, applying Lemma 1, it follows that

\[ \|\nabla\|_{W^{2,1}_2(Q_T)} \leq C\|F\|_{L^2(Q_T)} \leq C(\delta, \gamma, n) \|u\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)}. \]  

(23)

Using Holder’s inequality and the imbedding chain

\[ W^{2,1}_2(Q_T) \rightarrow W^{1,2}_2(Q_T) \rightarrow L^2(Q_T), \]  

(24)

it follows that

\[ \|g\|_{L^2(Q_T)} \leq \left( \int_{Q_T} b_0^2 |u|^{2q} dx dt \right)^{1/2} = b_0 \|u\|^q_{L^q(Q_T)} \]

\[ \leq Cb_0 \|u\|_{L^q(Q_T)}^{(1/2)-(q(n-1)/2(n+1))} \leq Cb_0 \|u\|_{W^{2,1}_2(Q_T)}^{(1/2)-(q(n-1)/2(n+1))} \leq C_2 b_0 \|u\|_{L^2(Q_T)} \|\nabla u\|_{W^{1,2}_2(Q_T)} \]  

(25)

Using Lemma 1, this is exceeded:

\[ C_1 b_0 \|\nabla u\|_{W^{1,2}_2(Q_T)}^{(1/2)-(q(n-1)/2(n+1))} \leq \|u\|_{L^2(Q_T)}. \]  

(26)

Using estimate (26) in (23), we get

\[ \|\nabla\|_{W^{2,1}_2(Q_T)} \leq C_2 \|u\|_{L^2(Q_T)} \]  

(27)

Let \( K \) be such that

\[ C_{2,5} \|\nabla u\|_{L^2(Q_T)}^{(1/2)-(q(n-1)/2(n+1))} \leq K. \]  

(28)

For such number \( K \) to exist, condition (10) is sufficient. To prove it, set the notation

\[ a = b_0 \|\nabla u\|_{L^2(Q_T)}^{(1/2)-(q(n-1)/2(n+1))}, \]

\[ b = \|\nabla u\|_{L^2(Q_T)}. \]  

(29)

Inequality (28) takes the form

\[ aK^q + b \leq K, \]

\[ aK^q - K + b \leq 0, \]  

(30)

\[ K > 0. \]

The function \( f(K) = aK^q - K, K \geq 0 \), takes its minimal in \( K_0 = (1/qa)^{1/(q-1)} \). Indeed, \( df/dK = aqK^{q-1} - 1 \); then, for \( K^{q-1} = (1/qa), df/dK(K_0) = 0; (d^2 f/dK^2)(K_0) > 0 \). Therefore, for \( b \leq f(K_0) \), inequality (30) is solvable with respect to \( K \). To finish the proof, it remains to set sufficiently small \( T' \) so that condition (10) is satisfied. It is possible since \( \text{mes}_{n+1}Q_T = \text{mes}_{n}Q_T \); the power on \( \text{mes}_{n+1}Q_T \) is positive, i.e., \( (1/2) - (q(n-1)/2(n+1)) > 0 \).

This completes the proof of Theorem 1. \( \square \)

3. Conclusion

In this paper, the strong solvability problem for a class of second-order semilinear parabolic equations is studied. For the strong solvability of the first boundary value problem for a class of parabolic equations having a nonlinear term, a sufficient condition is found for the power growth condition. In the proof, the Schauder fixed point theorem in the Banach space is used. Also, some a priori estimates are shown in order to realize the legitimate.
Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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