A PROBABILISTIC APPROACH TO SYSTEMS OF PARAMETERS AND
NOETHER NORMALIZATION

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Abstract. We study systems of parameters over finite fields from a probabilistic perspective, and use this to give the first effective Noether normalization result over a finite field. Our central technique is an adaptation of Poonen’s closed point sieve, where we sieve over higher dimensional subvarieties, and we express the desired probabilities via a zeta function-like power series that enumerates higher dimensional varieties instead of closed points. This also yields a new proof of a recent result of Gabber-Liu-Lorenzini and Chinburg-Moret-Bailly-Pappas-Taylor on Noether normalizations of projective families over the integers.

Given an \( n \)-dimensional projective scheme \( X \subseteq \mathbb{P}^r \) over a field, Noether normalization says that we can find homogeneous polynomials that induce a finite morphism \( X \to \mathbb{P}^n \). Such a morphism is determined by a system of parameters, namely by choosing homogeneous polynomials \( f_0, f_1, \ldots, f_n \) of degree \( d \) where \( X \cap V(f_0, f_1, \ldots, f_n) = \emptyset \). While over an infinite field any generic choice of linear polynomials will work, over a finite field we can ask:

**Question 1.1.** Let \( \mathbb{F}_q \) be a finite field and \( X \subseteq \mathbb{P}^r_{\mathbb{F}_q} \) be an \( n \)-dimensional closed subscheme.

1. What is the probability that a random choice of polynomials of degree \( d \) will yield a finite morphism \( X \to \mathbb{P}^n \)?
2. Can one effectively bound the degrees \( d \) for which such a finite morphism exists?

We answer these questions by studying the distribution of systems of parameters from both a geometric and probabilistic viewpoint. It is also useful to analyze partial systems of parameters, so for \( k \leq n \) we say that \( f_0, f_1, \ldots, f_k \) are parameters on \( X \) if

\[
\dim V(f_0, f_1, \ldots, f_k) \cap X = \dim X - (k + 1).
\]

By convention, the empty set has dimension \(-1\).

For the geometric side, we fix a field \( k \) and let \( S = k[x_0, \ldots, x_r] \) be the coordinate ring of \( \mathbb{P}^r_k \). We write \( S_d \) for the vector space of degree \( d \) polynomials in \( S \). In §4, we define a scheme \( \mathcal{D}_{k,d}(X) \) parametrizing collections that do not form parameters. The \( k \)-points of \( \mathcal{D}_{k,d}(X) \) are

\[
\mathcal{D}_{k,d}(X)(k) = \{(f_0, f_1, \ldots, f_k) \text{ that are not parameters on } X \} \subseteq S_d \times \cdots \times S_d \text{, } \text{ k+1 copies}
\]

We bound the codimension of these closed subschemes of the affine space \( S_d^{\oplus k+1} \).

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Theorem A. Let $X \subseteq \mathbb{P}_k^r$ be an $n$-dimensional closed subscheme. We have:

$$\text{codim } \mathcal{D}_{k,d}(X) = \begin{cases} \geq \binom{n-k+d}{n-k} & \text{if } k < n \\ 1 & \text{if } k = n. \end{cases}$$

This generalizes several results from the literature: the case $k = n$ is a classical result about Chow forms [GKZ08, 3.2.B]; the case $d = 1$ is a classical result about determinantal varieties [Mac94]; and the case $k = 0$ appears in [Ben11, Lemme 2.3]. If $k < n$, then the codimension grows as $d \to \infty$ and this factors into our asymptotic analysis over finite fields. It also leads to a uniform convergence result that allows us to go from a finite field to $\mathbb{Z}$.

For the probabilistic side, we work over a finite field $\mathbb{F}_q$ and compute the asymptotic probability that random polynomials $(f_0, f_1, \ldots, f_k)$ of degree $d$ are parameters on $X$. This is inspired by [Poo04], and it forms the heart of the paper. There is a bifurcation between the maximal and submaximal cases, reflecting Theorem A.

**Theorem B.** Let $X \subseteq \mathbb{P}_r^{r}_q$ be an $n$-dimensional closed subscheme. Then the asymptotic probability that random polynomials $(f_0, f_1, \ldots, f_k)$ of degree $d$ are parameters on $X$ is

$$\lim_{d \to \infty} \text{Prob}\left( (f_0, \ldots, f_k) \text{ of degree } d \text{ are parameters on } X \right) = \begin{cases} 1 & \text{if } k < n \\ \zeta_X(n+1)^{-1} & \text{if } k = n \end{cases}$$

where $\zeta_X(s)$ is the arithmetic zeta function of $X$.

The maximal case $k = n$ is due to Bucur and Kedlaya [BK12, Theorem 1.2], and is proven using Poonen’s closed point sieve. For submaximal cases where $k < n$, we adapt Poonen’s technique by sieving over closed subvarieties of dimension $n - k$. Just as the case $k = n$ depends on the zeta function of $X$, which counts points in $X$ of varying degrees, we show that each case $k < n$ depends on a power series that counts $(n - k)$-dimensional subvarieties of varying degrees. The full computation of these probabilities appears in Theorem 6.2, while the following corollary of Theorem 6.2 computes the first error term.

**Corollary 1.2.** Let $X \subseteq \mathbb{P}_r^{r}_q$ be a $n$-dimensional closed subscheme and let $k < n$. Then

$$\lim_{d \to \infty} \frac{\text{Prob}\left( (f_0, \ldots, f_k) \text{ of degree } d \text{ are not parameters on } X \right)}{q^{-(k+1)\binom{n-k+d}{n-k}}} = \# \left\{ (n-k)\text{-planes } L \subseteq \mathbb{P}_r^{r}_q \text{ such that } L \subseteq X \right\}.$$
Theorem C. Let \( X \subseteq \mathbb{P}^r_{F_q} \) where \( \dim X = n \). If \( \max \{ d, \frac{n}{d} \} \geq \deg(X) \) and
\[
d > \log_q \deg(X) + \log_q n + n \log_q d
\]
then there exist \( f_0, f_1, \ldots, f_n \) of degree \( d^{n+1} \) inducing a finite morphism \( \pi : X \to \mathbb{P}^n_{F_q} \).

The bound is asymptotically optimal in \( q \). Namely, if we fix \( \deg(X) \), then as \( q \to \infty \), the bound becomes \( d = 1 \). Thus, linear Noether normalizations exist if \( q \gg \deg(X) \). For a fixed \( q \), we expect the bound could be significantly improved. This is interesting even in the case \( \dim X = 0 \), where it is related to Kakeya type problems over finite fields [EE16, EOT10].

Theorem C provides the first explicit bound for Noetherian normalization over a finite field.

(One could potentially derive an explicit bound from Nagata’s argument in [Nag62, Chapter I.14], though the inductive nature of that construction would at best yield a bound that is multiply exponential in the largest degree of a defining equation of \( X \).)

After computing the probabilities over finite fields, we combine these analyses and characterize the distribution of parameters on projective \( B \)-schemes where \( B = \mathbb{Z} \) or \( F_q[t] \). We use standard notions of density for a subset of a free \( B \)-module; see Definition 7.1.

Theorem D. If \( X \subseteq \mathbb{P}^r_B \) is a closed subscheme whose general fiber over \( B \) has dimension \( n \), then
\[
\lim_{d \to \infty} \text{Density}\left\{ (f_0, f_1, \ldots, f_k) \text{ of degree } d \text{ that restrict to parameters on } X_p \text{ for all } p \right\} = \begin{cases} 1 & \text{if } k < n \\ 0 & \text{if } k = n \text{ and all } d. \end{cases}
\]

The density over \( B \) thus equals the product over all the fibers of the asymptotic probabilities over \( F_q \). In the case \( B = \mathbb{Z} \), our proof relies on Ekedahl’s infinite Chinese Remainder Theorem [Eke91, Theorem 1.2] combined with Proposition 5.1, which illustrates uniform convergence in \( p \) for the asymptotic probabilities in Theorem B. In the case \( B = F_q[t] \), we use Poonen’s analogue of Ekedahl’s result [Poo03, Theorem 3.1].

When \( k = n \), an analogue of Theorem D for smoothness is given by Poonen’s [Poo04, Theorem 5.13]. Moreover, it is believed that there are no smooth hypersurfaces of degree \( > 2 \) over \( \mathbb{Z} \). By contrast, the density zero subset from Theorem D turns out to always be nonempty. This leads to a new proof of a recent result about uniform Noether normalizations.

Corollary 1.3. Let \( B = \mathbb{Z} \) or \( F_q[t] \). Let \( X \subseteq \mathbb{P}^r_B \) be a closed subscheme. If each fiber of \( X \) over \( B \) has dimension \( n \), then for some \( d \), there exist homogeneous polynomials \( f_0, f_1, \ldots, f_n \in B[x_0, x_1, \ldots, x_r] \) of degree \( d \) inducing a finite morphism \( \pi : X \to \mathbb{P}^n_B \).

Theorem D shows that the collections defining a finite map \( \pi \) have density zero, even as \( d \to \infty \). Thus the existence of \( \pi \) is subtle and perhaps unexpected. Corollary 1.3 is a special case of a recent result of Chinburg-Moret-Bailly-Pappas-Taylor [CMBPT12, Theorem 1.2] and of Gabber-Liu-Lorenzini [GLL15, Theorem 8.1]. Our proof of the corollary involves two steps. We first use Theorem D to choose a submaximal collection \( f_0, f_1, \ldots, f_{n-1} \) of parameters on \( X \). This yields a scheme \( X' := X \cap \mathbb{V}(f_0, f_1, \ldots, f_{n-1}) \) with 0-dimensional fibers over \( \text{Spec}(B) \), and we then use that \( \text{Pic}(X') \) is torsion to find the final section \( f_n \).
Corollary 1.3 does not require the full strength of Theorem D, as we only need that the set of good choices for $f_0, f_1, \ldots, f_{n-1}$ is nonempty, and the proofs in [CMBPT12, GLL15] do use simpler techniques when selecting $f_0, f_1, \ldots, f_{n-1}$. For the final section $f_n$, they use essentially the same technique as we do. Since we rely on the infinite Chinese Remainder Theorems of Ekedahl and Poonen, Corollary 1.3 only recovers the results in [CMBPT12, GLL15] for $\mathbb{Z}$ and $\mathbb{F}_q[t]$, but not for more general number fields or function fields.

Corollary 1.3 can fail when $B$ is any of $\mathbb{Q}[t]$ or $\mathbb{Z}[t]$ or $\mathbb{F}_q[s,t]$, as in those cases, the Picard group of a finite cover of Spec $B$ can fail to be torsion. See §8 for explicit examples and counterexamples and see [CMBPT12, GLL15] for generalizations and applications.

It would be interesting to produce an effective version of Corollary 1.3 similar to Theorem C. One would need to bound the height of $f_0, f_1, \ldots, f_{n-1}$ when applying Theorem D, and combining this with effective bounds on the size of the class group of a number field.

There are a few earlier results related to Noether normalization over the integers. For instance [Moh79] shows that Noether normalizations of semigroup rings always exist over $\mathbb{Z}$; and [Nag62, Theorem 14.4] implies that given a family over any base, one can find a Noether normalization over an open subset of the base. Relative Noether normalizations play a key role in [Ach15, §5]. There is also the incorrect claim in [ZS75, p. 124] that Noether normalizations exist over any infinite base ring (see [AK07]). Brennan and Epstein [BE11] analyze the distribution of systems of parameters from a different perspective, introducing the notion of a generic matroid to relate various different systems of parameters.

This paper is organized as follows. §2 gathers background results and §3 involves a key lower bound on Hilbert functions. §4 contains our geometric analysis of parameters including a proof of Theorem A. §5 and §6 contain our probabilistic analysis of parameters over finite fields: §5 proves Theorem B while §6 gives the more detailed description via an analogue of the zeta function enumerating $(n-k)$-dimensional subvarieties. §7 contains our analysis over $\mathbb{Z}$ including proofs of Theorems C and 1.3 and related corollaries. §8 contains examples.

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2. Background

In this section, we gather some algebraic and geometric facts that we will cite throughout.

Lemma 2.1. Let $k$ be a field and let $R$ be a $(k+1)$-dimensional graded $k$-algebra where $R_0 = k$. If $f_0, f_1, \ldots, f_k$ are homogeneous elements of degree $d$ and $R/(f_0, f_1, \ldots, f_k)$ has finite length, then the extension $k[z_0, z_1, \ldots, z_k] \to R$ given by $z_i \mapsto f_i$ is a finite extension.

Proof. See [BH93, Theorem 1.5.17].
Definition 2.2. Let $X \subseteq \mathbb{P}^r$ be a projective scheme with minimal irreducible components $V_1, \ldots, V_s$ (considered with the reduced scheme structure). We define $\deg(X) := \sum_{i=1}^{s} \deg(V_i)$. For a subscheme $X' \subseteq \mathbb{A}^r$ with projective closure $\overline{X'} \subseteq \mathbb{P}^r$ we define $\deg(X') := \deg(\overline{X'})$.

This provides a notion of degree which ignores nonreduced structure but takes into account components of lower dimension. Similar definitions have appeared in the literature: for instance, in the language of [BM93, §3], we would have $\deg(X) = \sum_{j=0}^{\dim X} \text{geom-deg}_j(X)$. This definition is useful when bounding the number of points of a scheme over a finite field.

Lemma 2.3. Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a closed subscheme, where $\mathbb{F}_q$ is a finite field. Then
$$\#X(\mathbb{F}_q) \leq \deg(X)q^{\dim X}.$$  

Proof. This is an immediate consequence of the Schwarz-Zippel lemma, which implies that for an irreducible algebraic variety $V_i \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ we have $\#V_i(\mathbb{F}_q) \leq \deg(V_i)q^{\dim V_i}$. \hfill $\square$

Lemma 2.4. Let $k$ be any field and let $X \subseteq \mathbb{A}^r_k$. Let $f_0, f_1, \ldots, f_t$ by polynomials in $k[x_1, \ldots, x_r]$. If $X' = X \cap V(f_0, f_1, \ldots, f_t)$, then $\deg(X') \leq \deg(X) \cdot \prod_{i=0}^{t} \deg(f_i)$.

Proof. This follows from the refined version of Bezout’s Theorem [Ful98, Example 12.3.1]. \hfill $\square$

3. A uniform lower bound on Hilbert functions

For a subscheme of $\mathbb{P}^r$, the Hilbert function in degree $d$ is controlled by the Hilbert polynomial, at least if $d$ is very large related to some invariants of the subscheme. We analyze the Hilbert function of a subscheme at the opposite extreme, where the degree of the subscheme is much larger than $d$. The following lemma, which applies to subschemes of arbitrarily high degree, provides uniform lower bounds that are crucial to bounding the error in our sieves.

Lemma 3.1. Let $k$ be an arbitrary field and fix some $e \geq 0$. Let $V \subseteq \mathbb{P}^r_k$ be any closed, $m$-dimensional subscheme of degree $> e$ with homogeneous coordinate ring $R$.

(1) We have $\dim R_d \geq h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$ for all $d$.

(2) For any $0 < \epsilon < 1$, there exists a constant $C$ depending only on $e$, $m$ and $\epsilon$ such that $\dim R_d > (e + \epsilon) \cdot h^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$ for all $d \geq C e^{m+1}$.

Proof. We can assume that we are working over an infinite field, since this will not change the values of the Hilbert function of $R$. For part (1), we simply take a linear Noether normalization $k[t_0, t_1, \ldots, t_m] \subseteq R$ of the ring $R$ [Eis95, Theorem 13.3]. This yields $k[t_0, t_1, \ldots, t_m]d \subseteq R_d$, giving the statement about Hilbert functions.

We prove part (2) of the theorem by induction on $m$. Let $S = k[x_0, x_1, \ldots, x_r]$ and let $I_V \subseteq S$ be the saturated, homogeneous ideal defining $V$. Thus $R = S/I_V$. If $m = 0$, then we have $\dim R_d \geq \min\{d+1, \deg V\} \geq \min\{d+1, e+1\}$ which is at least $e + \epsilon$ for all $d \geq e$. This proves the case $m = 0$, where the constant $C$ can be chosen to be 1.
Now assume the claim holds for all closed subschemes of dimension less than \( m \). Let \( V \subset \mathbb{P}^r \) be a closed subscheme with \( \dim V = m \geq 1 \). Fix \( 0 < \epsilon < 1 \). Since we are working over an infinite field, [Eis95, Lemma 13.2(e)] allows us to choose a linear form \( \ell \) that is a nonzero divisor on \( R \). Hence we have a short exact sequence:

\[
0 \longrightarrow R(-1) \longrightarrow R \longrightarrow R/\ell \longrightarrow 0.
\]

Letting \( W = V \cap V(\ell) \) we know that \( \dim W = m - 1 \) and \( \deg W = \deg V \). Moreover, if \( I_V \) is the saturated ideal defining \( V \) and if \( I_W \) is the saturated ideal defining \( W \), then we have that \( I_W \) is the saturation of \( I_V + (\ell) \) at \( \mathfrak{m} \). In particular, since \( I_W \) contains \( I_V + (\ell) \), we have

\[
\dim(S/(I_V + (\ell)))_i \geq \dim(S/I_W)_i.
\]

By induction, for \( \epsilon' \) with \( 0 < \epsilon < \epsilon' < 1 \), there exists \( C' \) depending on \( \epsilon', \epsilon \) and \( m - 1 \) where

\[
\dim(S/I_W)_i \geq (\epsilon + \epsilon') \left( \frac{m - 1 + i}{m - 1} \right)
\]

for all \( i \geq C' e^m \). Iteratively applying the exact sequence (1) for \( d \geq C' e^m \) we obtain:

\[
\dim R_d \geq \dim R_{C' e^m} + \sum_{i=C' e^m-1}^{d} \dim(S/I_V + \ell)_i
\]

\[
\geq \dim R_{C' e^m} + \sum_{i=C' e^m-1}^{d} \dim(S/I_W)_i
\]

\[
\geq \sum_{i=C' e^m}^{d} (\epsilon + \epsilon') \left( \frac{m - 1 + i}{m - 1} \right).
\]

The identity \( \sum_{i=a}^{b} \binom{i+k}{k} = \binom{b+k+1}{k+1} - \binom{a+k}{k+1} \) implies that \( \sum_{i=C' e^m}^{d} (\epsilon + \epsilon') \left( \frac{m - 1 + i}{m - 1} \right) \) can be rewritten as \( (\epsilon + \epsilon') \left( \binom{m+d}{m} - \binom{m+1+C' e^m}{m} \right) \). There exists a constant \( C \) depending on \( \epsilon, \epsilon \), and \( m \) so that \( (\epsilon' - \epsilon) \binom{m+d}{m} \geq (\epsilon + \epsilon') \binom{m+1+C' e^m}{m} \) for all \( d \geq C e^{m+1} \). Thus, for all \( d \geq C e^{m+1} \) we have

\[
\dim R_d \geq (\epsilon + \epsilon') \left( \frac{m+d}{m} \right) - (\epsilon' - \epsilon) \left( \frac{m+d}{d} \right) = (\epsilon + \epsilon) \left( \frac{m+d}{m} \right).
\]

\[\Box\]

**Remark 3.2.** Asymptotically in \( e \), the bound of \( C e^2 \) is the best possible for curves. For instance, let \( C \subset \mathbb{P}^r \) be a curve of degree \( (e+1) \) lying inside some plane \( \mathbb{P}^2 \subset \mathbb{P}^r \). Let \( R \) be the homogeneous coordinate ring of \( C \). If \( d \geq e \) then the Hilbert function is given by

\[
\dim R_d = (e+1)d - \frac{e^2-e}{2}.
\]

Thus, if we want \( \dim R_d \geq (e+\epsilon)(d+1) \), we will need to let \( d \geq \frac{e^2+3e+2\epsilon}{2(1-\epsilon)} \approx \frac{1}{2} e^2 \). It would be interesting to know if the bound \( C e^{m+1} \) is the best possible for higher dimensional varieties.

4. Geometric Analysis

In this section we analyze the geometric picture for the distribution of parameters on \( X \). The basic idea behind the proof of Theorem A is that \( f_0, f_1, \ldots, f_k \) fail to be parameters on \( X \) if and only if they vanish along some \((n-k)\)-dimensional subvariety of \( X \). Since the Hilbert
We begin by constructing the schemes \( D_{k,d}(X) \). Fix \( X \subseteq \mathbb{P}^r_k \) a closed subscheme of dimension \( n \) over a field \( k \). Given \( k < n \) and \( d > 0 \), let \( \mathcal{A}_{k,d} \) be the affine space \( H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))^{\oplus k+1} \) and \( k[c_0,1, \ldots, c_k, (r_d)^] \) be the corresponding polynomial ring. We enumerate the monomials in \( H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \) as \( m_1, \ldots, m_{(r_d)} \), and then define the universal polynomial

\[
F_i := \sum_{j=1}^{N} c_{i,j} m_j \in k[c_0,1, \ldots, c_k, (r_d)] \otimes_k k[x_0, \ldots, x_r].
\]

Given a closed point \( c \in \mathcal{A}_{k,d} \) we can specialize \( F_0, \ldots, F_k \) and obtain polynomials \( f_0, \ldots, f_k \in k(c)[x_0, \ldots, x_r] \), where \( k(c) \) is the residue field of \( c \). We will thus identify each element of \( \mathcal{A}_{k,d}(k) \) with a collection of polynomials \( f = (f_0, f_1, \ldots, f_k) \in k[x_0, \ldots, x_r] \).

Now define \( \Sigma_{k,d}(X) \subseteq X \times \mathcal{A}_{k,d} \) via the equations \( F_0, \ldots, F_k \). Consider the second projection \( p_2 : \Sigma^{(k,d,x)} \rightarrow \mathcal{A}_{k,d} \). Given a point \( f = (f_0, \ldots, f_k) \in \mathcal{A}_{k,d} \), the fiber \( p_2^{-1}(f) \subseteq X \) can be identified with the points lying in \( X \cap \mathcal{V}(f_0, \ldots, f_k) \). For generic choices of \( f \) (after passing to an infinite field if necessary) the polynomials \( (f_0, \ldots, f_k) \) will have codimension \( k+1 \), and thus the fiber \( p_2^{-1}(f) \) will have dimension \( n - k - 1 \).

There is a closed sublocus in \( D_{k,d}(X) \subseteq \mathcal{A}_{k,d} \) where the dimension of the fiber is at least \( n-k \), and we give \( D_{k,d}(X) \) the reduced scheme structure. It follows that \( D_{k,d}(X) \) parametrizes collections \( f = (f_0, \ldots, f_k) \) of degree \( d \) polynomials which fail to be parameters on \( X \).

**Remark 4.1.** If we fix \( X_Z \subseteq \mathbb{P}^r_Z \), then we can follow the same construction to obtain a scheme \( D_{k,d}(X_Z) \subseteq \mathcal{A}_{k,d} \). Writing \( X_k \) as the pullback \( X \times_{\text{Spec}Z} \text{Spec}k \), we observe that the equations defining \( \Sigma_{k,d}(X_k) \) are obtained by pulling back the equations defining \( \Sigma_{k,d}(X_Z) \). It follows that \( D_{k,d}(X_Z) \times_{\text{Spec}Z} \text{Spec}(k) \) has the same set-theoretic support as \( D_{k,d}(X_k) \).

**Definition 4.2.** We let \( D_{k,d}^{bad}(X) \) be the locus of points in \( D_{k,d}(X) \) where \( f_0, \ldots, f_{k-1} \) already fail to be parameters on \( X \) and let \( D_{k,d}^{good}(X) := D_{k,d}(X) \setminus D_{k,d}^{bad}(X) \). We set \( D_{0,d}^{bad}(X) = \emptyset \).

**Remark 4.3.** We have a splitting:

\[
\mathcal{A}_{k,d} \rightarrow \mathcal{A}_{k-1,d} \times \mathcal{A}_{0,d} \\
(f_0, \ldots, f_k) \mapsto ((f_0, \ldots, f_{k-1}), f_k).
\]

Letting \( \pi : D_{k,d}(X) \rightarrow \mathcal{A}_{k-1,d} \) be the induced projection, we will speak about the degree of the image of \( \pi \) (considered in \( \mathcal{A}_{k-1,d} \)) and the degree of a fiber of \( \pi \) (considered in \( \mathcal{A}_{0,d} \)).

**Proof of Theorem A.** First consider the case \( k = n \). There is a natural rational map from \( \mathcal{A}_{n,d} \) to the Grassmanian \( \text{Gr}(n+1, S_d) \) given by sending the polynomials \( (f_0, \ldots, f_n) \) to the linear space that they span. Inside of the Grassmanian, the locus of choices of \( (f_0, \ldots, f_n) \) that all vanish on a point of \( X \) is a divisor in the Grassmanian defined by the Chow form; see [GKZ08, 3.2.B]. The preimage of this hypersurface in \( \mathcal{A}_{n,d} \) is a hypersurface contained in \( D_{n,d}(X) \), and thus \( D_{n,d}(X) \) has codimension 1.
For $k < n$, we will induct on $k$. Let $k = 0$. A polynomial $f_0$ will fail to be a parameter on $X$ if and only if $\dim V = \dim(X \cap \mathbb{V}(f_0))$. This happens if and only if $f_0$ is a zero divisor on a top-dimensional component of $X$. Let $V$ be the reduced subscheme of some top-dimensional irreducible component of $X$ and let $\mathcal{I}_V$ be the defining ideal sheaf of $V$. Then the set of zero divisors of degree $d$ on $V$ will form a linear subspace in $\mathcal{A}_{0,d}$ corresponding to the elements of the vector subspace $H^0(\mathcal{I}_V(d))$. The codimension of $H^0(\mathcal{I}_V(d)) \subseteq S_d$ is precisely given by the Hilbert function of the homogeneous coordinate ring of $V$ in degree $d$. By applying Lemma 3.1, we conclude that for all $d$ this linear space has codimension at least $\binom{n+d}{d}$. Since $\mathcal{D}_{0,d}(X)$ is the union of these linear spaces over all top-dimensional components of $X$, this proves that $\text{codim} \mathcal{D}_{0,d}(X) \geq \binom{n+d}{d}$.

Take the induction hypothesis that we have proven the statement for $\mathcal{D}_{j,d}(X')$ for all $X' \subseteq \mathbb{P}^r$ and all $j \leq k-1$. We separate $\mathcal{D}_{k,d}(X) = \mathcal{D}_{k,d}(X) \cup \mathcal{D}_{k,d}(X)$ and will show that each locus has sufficiently large codimension. We begin with $\mathcal{D}_{k,d}(X)$. By definition, the projection $\pi$ from Remark 4.3 maps $\mathcal{D}_{k,d}(X)$ onto $\mathcal{D}_{k-1,d}(X)$. We thus have:

$$\text{codim}(\mathcal{D}_{k,d}(X), \mathcal{A}_{k,d}) \geq \text{codim}(\mathcal{D}_{k-1,d}(X), \mathcal{A}_{k-1,d}) \geq \binom{n-k+1+d}{n-k+1} \geq \binom{n-k+d}{n-k},$$

where the middle inequality follows by induction.

Now consider an arbitrary point $f = (f_0, \ldots, f_k)$ in $\mathcal{D}_{k,d}(X)$. By definition, $f_0, \ldots, f_{k-1}$ must be parameters on $X$, and thus $\pi(f) \in \mathcal{A}_{k-1,d} \subseteq \mathcal{D}_{k-1,d}(X)$. Using the splitting of Remark 4.3, the fiber of $\mathcal{D}_{k,d}(X)$ over $f$ can be identified with $\mathcal{D}_{0,d}(X')$ where $X' : = X \cap \mathbb{V}(f_0, \ldots, f_{k-1})$. Since $(f_0, \ldots, f_{k-1}) \notin \mathcal{D}_{k-1,d}(X)$, we have that $\dim X' = n - k$. The inductive hypothesis thus guarantees that $\text{codim} \mathcal{D}_{0,d}(X') \geq \binom{\dim X' + d}{d} = \binom{n-k+d}{d}$. □

### 5. Probabilistic Analysis I: Proof of Theorem B

Throughout this section, we let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be a projective scheme of dimension $n$ over a finite field $\mathbb{F}_q$. Recall that $S_d = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$. We define

$$\text{Par}_{d,k} = \left\{ (f_0, f_1, \ldots, f_k) \text{ that are parameters on } X \right\} \subseteq S_d^{k+1}.$$

In Theorem B, we compute the following limit (which a priori might not exist):

$$\lim_{d \to \infty} \text{Prob} \left( (f_0, f_1, \ldots, f_k) \text{ of degree } d \text{ are parameters on } X \right) = \lim_{d \to \infty} \frac{\#\text{Par}_{d,k}}{\#S_d^{k+1}}.$$

As in the geometric case, there is a bifurcation between when $k = n$ and $k < n$. The case $k = n$ largely parallels Poonen’s work [Poo04]: we will sieve over closed points of $X$ and show that the asymptotic probability that $(f_0, f_1, \ldots, f_n)$ are parameters on $X$ equals the product of local probabilities at the closed points of $X$, and the resulting formula will correspond with the value of a zeta function. This approach was already worked out by Bucur and Kedlaya in [BK12]. They assume that $X$ is smooth, but their proof does not need that assumption.

When $k < n$ we need to significantly alter Poonen’s sieve. In this section, we focus on proving that the asymptotic probability converges to 1 as $d \to \infty$. For this, we will use
a coarse error bound based on the geometric picture developed in §4. In §6, we provide a deeper analysis of the limit probability based on the detailed geometry of \( X \).

**Proposition 5.1.** If \( k < n \) then

\[
\text{Prob} \left( (f_0, f_1, \ldots, f_k) \text{ of degree } d \text{ are parameters on } X \right) \geq 1 - \hat{\deg}(X)(1 + d + d^2 + \cdots + d^k)q^{-(n-k+d)}.
\]

**Proof.** We induct on \( k \) and largely follow the structure of the proof of Theorem A. First, let \( k = 0 \). A polynomial \( f_0 \) will fail to be a parameter on \( X \) if and only if it is a zero divisor on a top-dimensional component \( V \) of \( X \). There are at most \( \hat{\deg}(X) \) many such components. As argued in the proof of Theorem A, the set of zero divisors on \( V \) corresponds to the elements of \( \mathbb{H}^0(\mathbb{P}^r, \mathcal{I}_V(d)) \) which has codimension at least \( \left( \frac{n+d}{d} \right) \) in \( S_d \). It follows that

\[
\text{Prob} \left( f_0 \text{ of degree } d \text{ is not a parameter on } X \right) \leq \hat{\deg}(X)q^{-\left( \frac{n+d}{d} \right)}.
\]

Now consider the induction step. We will separately compute the probability that \( f = (f_0, f_1, \ldots, f_k) \) lies in \( D_{k,d}^{\text{bad}}(X) \) and the probability that \( f \) lies in \( D_{k,d}^{\text{good}}(X) \). By definition, the projection \( \pi \) maps \( D_{k,d}^{\text{bad}}(X) \) onto \( D_{k-1,d}(X) \), and by induction

\[
\text{Prob}(\pi(f) \in D_{k-1,d}(X)(\mathbb{F}_q)) \leq \hat{\deg}(X) \left( 1 + d + d^2 + \cdots + d^{k-1} \right) q^{-\left( \frac{n-k+1+d}{n-k+1} \right)}
\]

\[
\leq \hat{\deg}(X) \left( 1 + d + d^2 + \cdots + d^{k-1} \right) q^{-\left( \frac{n-k+d}{n-k} \right)}.
\]

We now assume \( f \notin D_{k,d}^{\text{bad}}(X) \). We thus have that \( f_0, \ldots, f_{k-1} \) are parameters on \( X \). As in the proof of Theorem A, the fiber \( \pi^{-1}(f) \) can be identified with \( D_{0,d}(X') \) where \( X' := X \cap V(f_0, f_1, \ldots, f_{k-1}) \). By construction \( \dim X' = n - k \) and by Lemma 2.4, \( \hat{\deg}(X') \leq \hat{\deg}(X) \cdot d^k \). Our inductive hypothesis thus implies that

\[
\text{Prob} \left( (f_0, \ldots, f_k) \in D_{k,d}(X)(\mathbb{F}_q) \text{ given that } (f_0, \ldots, f_{k-1}) \notin D_{k-1,d}(X)(\mathbb{F}_q) \right) \leq \hat{\deg}(X')q^{-\left( \frac{n-k+d}{n-k} \right)} \leq \hat{\deg}(X) \cdot d^k q^{-\left( \frac{n-k+d}{n-k} \right)}.
\]

Combining the estimates for \( D_{k,d}^{\text{bad}}(X) \) and \( D_{k,d}^{\text{good}}(X) \) yields the proposition. \( \square \)

**Proof of Theorem B.**

If \( k < n \), then we apply Proposition 5.1 to obtain

\[
\lim_{d \to \infty} \text{Prob} \left( (f_0, \ldots, f_k) \text{ of degree } d \text{ are parameters on } X \right) \geq \lim_{d \to \infty} 1 - \hat{\deg}(X)(d^0 + d^1 + \cdots + d^k)q^{-(n-k+d)} = 1.
\]

Now let \( k = n \). For completeness, we summarize the proof of [BK12, Theorem 1.2]. We fix \( e \), which will go to \( \infty \), and separate the argument into low, medium, and high degree cases.

**Low degree argument.** For a zero dimensional subscheme \( Y \), we have that \( S_d \) surjects on \( \mathbb{H}^0(Y, \mathcal{O}_Y(d)) \) when \( d \geq \deg Y - 1 \) [Poo04, Lemma 2.1]. So if \( d > \deg P - 1 \), the probability that \( f_0, f_1, \ldots, f_n \) all vanish at a closed point \( P \in X \) is \( 1 - q^{-(n+1)\deg P} \). If \( Y \subseteq X \) is the union of all points of degree \( \leq e \), and if \( d \geq \deg Y - 1 \), then the surjection onto \( \mathbb{H}^0(Y, \mathcal{O}_Y(d)) \)
implies that the probabilities at the points \( P \in Y \) behave independently. This yields:

\[
\text{Prob} \left( \begin{array}{c} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ are} \\ \text{parameters on } X \text{ at all points} \\ P \in X \text{ where } \deg(P) \leq e \end{array} \right) = \prod_{P \in X} \left( 1 - q^{-(n+1)\deg P} \right).
\]

**Medium degree argument.** Our argument is nearly identical to [Poo04, Lemma 2.4], and covers all points whose degree lies in the range \([e, \frac{d}{n+1}]\). For any such point \( P \in X \), \( S_d \) surjects onto \( H^0(P, \mathcal{O}_P(d)) \) and thus the probability that \((f_0, f_1, \ldots, f_n)\) all vanish at \( P \) is \( q^{-\ell(n+1)} \).

By [LW54], \( \#X(\mathbb{F}_q^\ell) \leq Kq^{\ell n} \) for some constant \( K \) independent of \( \ell \). We have

\[
\text{Prob} \left( \begin{array}{c} f_0, f_1, \ldots, f_n \text{ of degree } d \text{ all} \\ \text{vanish at some } P \in X \\ \text{where } e < \deg(P) \leq \left[ \frac{d}{n+1} \right] \end{array} \right) \leq \sum_{\ell = e}^{\left[ \frac{d}{n+1} \right]} \#X(\mathbb{F}_q^{\ell}) q^{-\ell(n+1)} \leq \sum_{\ell = e}^{\infty} Kq^{\ell n} q^{-(n+1)\ell} = \frac{Kq^{-e}}{1 - q^{-1}}.
\]

This tends to 0 as \( e \to \infty \), and therefore does not contribute to the asymptotic limit.

**High degree argument.** By the case when \( k = n - 1 \), we may assume that \( f_0, f_1, \ldots, f_{n-1} \) form a system of parameters with probability \( 1 - o(1) \). So we let \( V \) be one of the irreducible components of this intersection (over \( \mathbb{F}_q \)) and we let \( R \) be its homogeneous coordinate ring. If \( \deg V \leq \frac{d}{n+1} \), then it can be ignored as we considered such points in the low and medium degree cases. Hence, we can assume \( \deg V > \frac{d}{n+1} \). Since \( \dim R_\ell \geq \min\{\ell + 1, \deg R\} \) for all \( \ell \), the probability that \( f_n \) vanishes along \( V \) is at most \( q^{-\left(\frac{d}{n+1}\right) - 1} \). Hence the probability of vanishing on some high degree point is bounded by \( O(d^n q^{-\left(\frac{d}{n+1}\right) - 1}) \) which is \( o(1) \) as \( d \to \infty \).

Combining the various parts as \( e \to \infty \), we see that the low degree argument converges to \( \zeta_X(n + 1)^{-1} \) and the contributions from the medium and high degree points go to 0.

**Remark 5.2.** It might be interesting to consider variants of Theorem B that allow imposing conditions along closed subschemes, similar to Poonen’s Bertini with Taylor Coefficients [Poo04, Theorem 1.2]. For instance, [Ked05, Theorem 1] might be provable by such an approach, though this would be more complicated than the original proof.

Proposition 5.1 also yields an effective bound on the degree of a full system of parameters over a finite field. Sharper bounds can obtained if one allows the \( f_i \) to have different degrees.

**Corollary 5.3.**

1. If \( d_1 \) satisfies \( d_1^{n-1} q^{-d_1-1} < (n \cdot \deg(X))^{-1} \), then there exist \( g_0, g_1, \ldots, g_{n-1} \) of degree \( d_1 \) that are parameters on \( X \).

2. Let \( X' \) be 0-dimensional. If \( \max\{d_2 + 1, q\} \geq \deg(X') \) then there exists a degree \( d_2 \) parameter on \( X' \).

**Proof.** Applying Proposition 5.1 in the case \( k = n - 1 \) yields (1). For (2), let \( f \) be a random degree \( d \) polynomial and let \( P \in X' \) be a closed point. Since the dimension of the image of \( S_d \) in \( H^0(P, \mathcal{O}_P(d)) \) is at least \( \min\{d+1, \deg P\} \), the probability that \( f \) vanishes at \( P \) is at worst \( q^{-\min\{d+1, \deg P\}} \), which is at least \( q^{-1} \). It follows that the probability that a degree \( d \) function
vanishes on some point of $X'$ is at worst $\sum_{P \in X'} q^{-1} \leq \deg(X') q^{-1}$. Thus if $q > \deg(X')$, this happens with probability strictly less than 1. On the other hand, if $d + 1 \geq \deg(X')$ then polynomials of degree $d$ surject onto $H^d(X', \mathcal{O}_{X'}(d))$ and hence we can find a parameter on $X'$ by choosing a polynomial that restricts to a unit on $X'$. \hfill \square

**Proof of Theorem C.** If $\dim X = 0$, then we can directly apply Corollary 5.3(2) to find a parameter of degree $d = \max\{d, d^n\}$. So we assume $n := \dim X > 0$. Since $d > \log_q \deg(X) + \log_q n + n \log_q d$ it follows that $(n \cdot \deg(X))^{-1} > q^{-d} d^n > q^{-d-1} d^{n-1}$. Applying Corollary 5.3(1), we find $g_0, g_1, \ldots, g_{n-1}$ in degree $d$ that are parameters on $X$. Let $X' = X \cap V(g_0, g_1, \ldots, g_{n-1})$. Since $\max\{d, \frac{d}{d^n}\} \geq \deg(X)$ it follows that $\max\{d^{n+1}, q\} \geq d^n \deg(X) \geq \deg(X')$, and Corollary 5.3(2) yields a parameter $g_n$ of degree $d^{n+1}$ on $X'$.

Thus $g_0, g_1, \ldots, g_{n-1}, g_n$ are parameters of degree $d^{n+1}$ on $X$.

\hfill \square

6. **Probabilistic Analysis II: The Error Term**

In this section, we let $k < n$ and we analyze the error terms in Theorem B more precisely. In particular, we will show that the probabilities are controlled by the probability of vanishing along an $(n - k)$-dimensional subvariety, with varieties of lowest degree contributing the most. Theorem 6.2 is the estimate obtained by tracking subvarieties of degree $\leq e$.

**Notation 6.1.** *For a subscheme $Z \subseteq X$, we write $|Z|$ for the number of irreducible components of $Z$, and we write $\dim Z \equiv k$ if $Z$ is equidimensional of dimension $k$.*

**Theorem 6.2.** Let $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ be a projective scheme of dimension $n$. Fix $e$ and let $k < n$. The probability that random polynomials $f_0, f_1, \ldots, f_k$ of degree $d$ are parameters on $X$ is

$$\text{Prob}\left(f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are parameters on } X\right) = 1 - \sum_{\substack{Z \subseteq X \text{ reduced} \\ \dim Z \equiv n-k \\ \deg Z \leq e}} (-1)^{|Z|-1} q^{-(k+1) h^0(Z, \mathcal{O}_Z(d))} + o\left(q^{-e(k+1)((n-k)+d)}\right).$$

The terms of the above sum have the form $q^{-(k+1) h^0(Z, \mathcal{O}_Z(d))}$, where $Z \subseteq X$ is an $(n - k)$-dimensional subvariety. Hence, the exponents are controlled by the Hilbert polynomial of a $(n - k)$-dimensional variety, and will grow like $d^{n-k}$, converging to 0 rapidly as $d \to \infty$.

Our proof of Theorem 6.2 adapts Poonen’s sieve in a couple of key ways. The first big difference is that instead of sieving over closed points, we will sieve over $(n - k)$-dimensional subvarieties of $X$; this is because polynomials $(f_0, \ldots, f_k)$ will fail to be parameters on $X$ only if they vanish along some $(n - k)$-dimensional subvariety.

The second difference is that the resulting probability formula will not be a product of local factors. This is because the values of a function can never be totally independent along two higher dimensional varieties with a nontrivial intersection. For instance, Lemma 6.3 shows that the probability that a degree $d$ polynomial vanishes along a line is $q^{-(d+1)}$, but the probability of vanishing along two lines that intersect in a point is $q^{-(2d+1)} > (q^{-(d+1)})^2$.

The following result characterizes the individual probabilities arising in our sieve.
Lemma 6.3. If \( Z \subseteq \mathbb{P}_q^r \) is a reduced, projective scheme over a finite field \( \mathbb{F}_q \) with homogeneous coordinate ring \( R \) then

\[
\text{Prob}\left( (f_0, \ldots, f_k) \text{ of degree } d \text{ vanish along } Z \right) = \left( \frac{1}{\#R_d} \right)^{k+1}.
\]

If \( d \) is at least the Castelnuovo-Mumford regularity of the ideal sheaf of \( Z \), then

\[
\text{Prob}\left( (f_0, \ldots, f_k) \text{ of degree } d \text{ vanish along } Z \right) = q^{-(k+1)h^0(Z, \mathcal{O}_Z(d))}.
\]

Proof. Let \( I \subseteq S \) be the homogeneous ideal defining \( Z \), so that \( R = S/I \). An element \( h \in S_d \) vanishes along \( Z \) if and only if it restricts to 0 in \( R_d \) i.e. if and only if it lies in \( I_d \). Since we have an exact sequence of \( \mathbb{F}_q \)-vector spaces:

\[
0 \rightarrow I_d \rightarrow S_d \rightarrow R_d \rightarrow 0
\]

we obtain

\[
\text{Prob}(h \text{ vanishes on } Z) = \frac{\#I_d}{\#S_d} = \frac{1}{\#R_d}.
\]

For \( k + 1 \) elements of \( S_d \), the probabilities of vanishing along \( Z \) are independent and this yields the first statement of the lemma.

We write \( \tilde{I} \) for the ideal sheaf of \( Z \). If \( d \) is at least the regularity \( \tilde{I} \), then \( H^1(\mathbb{P}_q^r, \tilde{I}(d)) = 0 \). Hence there is a natural isomorphism between \( R_d \) and \( H^0(Z, \mathcal{O}_Z(d)) \). Thus, we have

\[
1/\#R_d = q^{-h^0(Z, \mathcal{O}_Z(d))},
\]

yielding the second statement. \( \square \)

Proof of Theorem 6.2. Throughout the proof, we set \( \epsilon_{e,k} \) to be the error term for a given \( e \) and \( k \), namely \( \epsilon_{e,k} := q^{-(k+1)(n-k+d)\choose n-k} \). We also set:

\[
\text{Par}_{d,k} := \left\{ \begin{array}{l}
(f_0, f_1, \ldots, f_k) \text{ are parameters on } X \\
\end{array} \right\}
\]

\[
\text{Low}_{d,k,e} := \left\{ \begin{array}{l}
(f_0, f_1, \ldots, f_k) \text{ vanish along a variety } Z \\
\text{ where dim } Z = (n-k) \text{ and } \deg(Z) \leq e \\
\end{array} \right\}
\]

\[
\text{Med}_{d,k,e} := \left\{ \begin{array}{l}
(f_0, f_1, \ldots, f_k) \notin \text{Low}_{d,k,e} \text{ which vanish along a variety } Z \\
\text{ where dim } Z = (n-k) \text{ and } e < \deg(Z) \leq e(k+1) \\
\end{array} \right\}
\]

\[
\text{High}_{d,k,e} := \left\{ \begin{array}{l}
(f_0, f_1, \ldots, f_k) \notin \text{Low}_{d,k,e} \cup \text{Med}_{d,k,e} \text{ which vanish along a variety } Z \\
\text{ where dim } Z = (n-k) \text{ and } e(k+1) < \deg(Z) \\
\end{array} \right\}.
\]

Note that if \( (f_0, f_1, \ldots, f_k) \) vanish along a variety of dimension \( > n-k \) then they will also vanish along a high degree variety, and hence we do not need to count this case separately. For \( \mathbf{f} = (f_0, f_1, \ldots, f_k) \in S_{d+1}^k \), we thus have

\[
\text{Prob}(\mathbf{f} \in \text{Par}_{d,k}) = 1 - \text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e} \cup \text{Med}_{d,k,e} \cup \text{High}_{d,k,e})
\]

\[
= 1 - \text{Prob}(\mathbf{f} \in \text{Low}_{d,k,e}) - \text{Prob}(\mathbf{f} \in \text{Med}_{d,k,e}) - \text{Prob}(\mathbf{f} \in \text{High}_{d,k,e}).
\]

12
It thus suffices to show that
\[
\text{Prob}(f \in \text{Low}_{d,k,e}) = \sum_{Z \subseteq X \text{ reduced}} (-1)^{|Z|-1} q^{-(k+1)h^0(Z,\mathcal{O}_Z(d))} + o(\epsilon_{e,k})
\]
and that \(\text{Prob(\text{Med}_{d,k,e})}\) and \(\text{Prob(\text{High}_{d,k,e})}\) are each in \(o(\epsilon_{e,k})\).

We proceed by induction on \(k\). When \(k = 0\) the condition that \(f_0\) is a parameter on \(X\) is equivalent to \(f_0\) not vanishing along a top-dimensional component of \(X\). Thus, combining Lemma 6.3 with an inclusion/exclusion argument implies the exact result:
\[
\text{Prob}(f_0 \in \text{Par}_{d,0}) = 1 - \sum_{Z \subseteq X \text{ reduced}} (-1)^{|Z|-1} q^{h^0(Z,\mathcal{O}_Z(d))} \cdot (n+d^n + o(d^n)) + o(d^n).
\]

By basic properties of the Hilbert polynomial, as \(d \to \infty\) we have
\[
h^0(Z,\mathcal{O}_Z(d)) = \deg(Z) \cdot n! \cdot d^n + o(d^n) = \deg(Z) \left( \frac{n+d}{d} \right) + o(d^n).
\]

Hence for the fixed degree bound \(e\), we obtain:
\[
\text{Prob(\text{Par}_{d,0})} = 1 - \sum_{Z \subseteq X \text{ reduced}} (-1)^{|Z|-1} q^{h^0(Z,\mathcal{O}_Z(d))} - \sum_{Z \subseteq X \text{ reduced}} (-1)^{|Z|-1} q^{h^0(Z,\mathcal{O}_Z(d))} + o(\epsilon_{e,0}).
\]

We now consider the induction step. Let \(f = (f_0,\ldots,f_k)\) drawn randomly from \(S_{d,k+1}^{k+1}\). Here we separate into low, medium, and high degree cases.

**Low degree argument.** Let \(V_{k,e}\) denote the set of integral projective varieties \(V \subseteq X\) of dimension \(n-k\) and degree \(\leq e\). We have \(f \in \text{Low}_{d,k,e}\) if and only if \(f\) vanishes on some \(V \in V_{k,e}\). Since \(V_{k,e}\) is a finite set, we may use an inclusion-exclusion argument to get
\[
\text{Prob}(f \in \text{Low}_{d,k,e}) = \sum_{Z \subseteq X \text{ a union of } V \in V_{k,e}} (-1)^{|Z|-1} \text{Prob}(f_0,\ldots,f_k \text{ of degree } d \text{ vanish along } Z).
\]

If \(\deg Z > e\) then Lemma 6.3 implies that those terms can be absorbed into the error term \(o(\epsilon_{e,k})\). Moreover, assuming that \(Z\) is a union of \(V \in V_{k,e}\) satisfying \(\deg(Z) \leq e\) is equivalent to assuming \(Z\) is reduced and equidimensional of dimension \(n-k\). We thus have:
\[
= \sum_{Z \subseteq X \text{ reduced}} (-1)^{|Z|-1} \text{Prob}(f_0,\ldots,f_k \text{ of degree } d \text{ vanish along } Z) + o(\epsilon_{e,k}).
\]
Medium degree argument. We know that $\text{Prob}(f \in \text{Med}_{d,k,e})$ is bounded by the sum of the probabilities that $f$ vanishes along some irreducible variety $V$ in $V_{e(k+1),k} \setminus V_{e,k}$.

$$\text{Prob}(f \in \text{Med}_{d,k,e}) \leq \sum_{Z \in V_{e(k+1),k} \setminus V_{e,k}} \text{Prob} \left( (f_0, f_1, \ldots, f_k) \text{ of degree } d \right. \left. \text{ vanish along } Z \right).$$

Lemma 6.3 implies that each summand on the right-hand side lies in $o(\epsilon_{e,k})$. This sum is finite and thus $\text{Prob}(f \in \text{Med}_{d,k,e})$ is in $o(\epsilon_k)$.

High degree argument. Proposition 5.1 implies that $f_0, f_1, \ldots, f_{k-1}$ are parameters on $X$ with probability $1 - o \left( q^{-(n-k+1+d)} \right) \geq 1 - o(\epsilon_{e,k})$ for any $e$. Hence we may restrict our attention to the case where $f_0, \ldots, f_{k-1}$ are parameters on $X$.

Let $V_1, V_2, \ldots, V_s$ be the irreducible components of $X' := X \cap \mathcal{V}(f_0, f_1, \ldots, f_{k-1})$ that have dimension $n - k$. We have that $f_0, f_1, \ldots, f_k$ fail to be parameters on $X$ if and only if $f_k$ vanishes on some $V_i$. We can assume that $f_k$ does not vanish on any $V_i$ where $\text{deg} V_i \leq e(k + 1)$ as we have already accounted for this possibility in the low and medium degree cases. After possibly relabelling the components, we let $V_1, V_2, \ldots, V_t$ be the components of degree $> e(k + 1)$ and $X'' = V_1 \cup V_2 \cup \cdots \cup V_t$. Using Lemma 2.4, we compute $\text{deg}(X'') \leq \tilde{\text{deg}}(X') = \tilde{\text{deg}}(X) \cdot d^{k}$. It follows that $X''$ has at most $\frac{\text{deg}(X)d^k}{e(k+1)}$ irreducible components.

Since the value of $d$ is not necessarily larger than the Castelnuovo-Mumford regularity of $V_i$, we cannot use a Hilbert polynomial computation to bound the probability that $f_k$ vanishes along $V_i$. Instead, we use the lower bound for Hilbert functions obtained in Lemma 3.1. Let $\epsilon = 1/2$, though any choice of $\epsilon$ would work. We write $R(V_i)$ for homogeneous coordinate ring of $V_i$. For any $1 \leq i \leq t$, Lemmas 3.1 and 6.3 yield

$$\text{Prob} \left( f_k \text{ of degree } d \right. \left. \text{ vanishes along } V_i \right) = q^{-\text{dim} R(V_i)d} \leq q^{-(e(k+1)+\epsilon)(n-k+d)}$$

whenever $d \geq C e^{k+1}$. Combining this with our bound on the number of irreducible components of $X''$ gives $\text{Prob} \left( f \in \text{High}_{d,k,e} \right) \leq \frac{\text{deg} X}{e(k+1)}d^{k}q^{-(e(k+1)+\epsilon)(n-k+d)}$ which is in $o(\epsilon_{e,k})$. □

Proof of Corollary 1.2. Let $N$ denote the number of $(n-k)$-planes spaces $L \subseteq \mathbb{P}^{r}_{F_q}$ such that $L \subseteq X$. Choosing $e = 1$ in Theorem 6.2, we compute that

$$\text{Prob} \left( f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are parameters on } X \right) = 1 - Nq^{-(k+1)(n-k+d)} + o \left( q^{-(k+1)(n-k+d)} \right).$$

It follows that

$$\text{Prob} \left( f_0, f_1, \ldots, f_k \text{ of degree } d \text{ are not parameters on } X \right) = Nq^{-(k+1)(n-k+d)} + o \left( q^{-(k+1)(n-k+d)} \right).$$

Dividing both sides by $q^{-(k+1)(n-k+d)}$ and taking the limit as $d \to \infty$ yields the corollary. □

7. Proofs over $\mathbb{Z}$ and $\mathbb{F}_q[t]$

In this section we prove Theorem D and Corollary 1.3.
Definition 7.1. Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and fix a finitely generated, free $B$-module $B^*$ and a subset $S \subseteq B^*$. Given $a \in B^*$ we write $a = (a_1, a_2, \ldots, a_k)$. The density of $S \subseteq B$ is

$$
\text{Density}(S) := \begin{cases} 
\lim_{N \to \infty} \frac{\# \{ a \in S \mid \text{max}\{ |a_i| \} \leq N \}}{\# \{ a \in \mathbb{Z}^k \mid \text{max}\{ |a_i| \} \leq N \}} & \text{if } B = \mathbb{Z} \\
\lim_{N \to \infty} \frac{\# \{ a \in S \mid \text{max}\{ \text{deg } a_i \} \leq N \}}{\# \{ a \in \mathbb{F}_q[t]^k \mid \text{max}\{ \text{deg } a_i \} \leq N \}} & \text{if } B = \mathbb{F}_q[t] 
\end{cases}
$$

Proof of Theorem D. For clarity, we will prove the result over $\mathbb{Z}$ in detail and at the end, mention the necessary adaptions for $\mathbb{F}_q[t]$.

We first let $k < n$. Given degree $d$ polynomials $(f_0, f_1, \ldots, f_k)$ with integer coefficients and a prime $p$, let $(\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k)$ be the reduction of these polynomials mod $p$. Then $(\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k)$ will be parameters on $X_p$ if and only if the point $\overline{p} = (\overline{f}_0, \overline{f}_1, \ldots, \overline{f}_k)$ lies $\mathcal{D}_{d,k}(X_p)$. As noted in Remark 4.1, this is equivalent to asking that $\overline{p}$ is an $\mathbb{F}_p$-point of $\mathcal{D}_{d,k}(\mathbb{Z}_p)$. Thus, we may apply [Eke91, Theorem 1.2] to $\mathcal{D}_{d,k}(X_p) \subseteq \mathcal{A}_{d,k}$ (using $M = 1$) to conclude that

$$
\text{Density} \left\{ (f_0, \ldots, f_k) \text{ of degree } d \text{ that restrict to parameters on } X_p \text{ for all } p \right\} = \prod_p \text{Prob} \left( \text{(f_0, \ldots, f_k) of degree } d \text{ restrict to parameters on } X_p \right).
$$

Applying Proposition 5.1 to estimate the individual factors; we have:

$$
\text{Density} \left\{ (f_0, \ldots, f_k) \text{ of degree } d \text{ that restrict to parameters on } X_p \text{ for all } p \right\} = \lim_{d \to \infty} \prod_p \text{Prob} \left( \text{(f_0, \ldots, f_k) of degree } d \text{ restrict to parameters on } X_p \right) \\
\geq \lim_{d \to \infty} \prod_p \left( 1 - \overline{\text{deg}}(X_p)(1 + d + \cdots + d^k)p^{-\left(n-k+d \atop n-k \right)} \right).
$$

Lemma 7.2 shows that there is an integer $D$ where $D \geq \overline{\text{deg}}(X_p)$ for all $p$. Moreover, $1 + d + \cdots + d^k \leq kd^k$ for all $d$, and hence:

$$
\geq \lim_{d \to \infty} \prod_p \left( 1 - Dk d^k p^{-\left(n-k+d \atop n-k \right)} \right).
$$

For $d \gg 0$ we can make $Dk d^k p^{-\left(n-k+d \atop n-k \right)} \leq p^{-d/2}$ for all $p$ simultaneously. Using $\zeta(n)$ for the Riemann-Zeta function, we get:

$$
\geq \lim_{d \to \infty} \prod_p \left( 1 - p^{-d/2} \right) \geq \lim_{d \to \infty} \zeta(d/2)^{-1} = 1.
$$

We now consider the case $k = n$. This follows by a “low degree argument” exactly analogous to [Poo04, Theorem 5.13]. Fix a large integer $N$ and let $Y$ be the union of all closed points $P \in X$ whose residue field $\kappa(P)$ has cardinality at most $N$. Since $Y$ is a finite union of closed, we see that for $d \gg 0$, there is a surjection:

$$
H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d)) \cong \bigoplus_{P \in X : \# \kappa(P) \leq N} H^0(P, \mathcal{O}_P(d)) \longrightarrow 0.
$$
It follows that we have a product formula:

\[
\text{Density}\left\{ (f_0, f_1, \ldots, f_n) \text{ of degree } d \text{ do not vanish on a point } P \text{ with } \#\kappa(P) \leq N \right\} = \prod_{P \in X, \#\kappa(P) \leq N} \left( 1 - \frac{1}{\#\kappa(P)^{n+1}} \right)
\]

This is certainly an upper bound on the density of \((f_0, f_1, \ldots, f_n)\) that are parameters on \(X_p\) for all \(p\). As \(N \to \infty\) the righthand side approaches \(\zeta_X(n+1)^{-1}\). However, since the dimension of \(X\) is \(n+1\), this zeta function has a pole at \(s = n+1\) [Ser65, Theorems 1 and 3(a)]. Hence this asymptotic density equals 0. This completes the proof over \(\mathbb{Z}\).

Over \(\mathbb{F}_q[t]\), the key adaptation is to use [Poo03, Theorem 3.1] in place of Ekedahl’s result. Poonen’s result is stated for a pair of polynomials, but it applies equally well to \(n\)-tuples of polynomials such as the \(n\)-tuples defining \(\mathcal{D}_{k, d}(X)\). In particular, one immediately reduces to proving an analogue of [Poo03, Lemma 5.1], for \(n\)-tuples of polynomials which are irreducible over \(\mathbb{F}_q(t)\) and which have \(\gcd\) equal to 1; but the \(n = 2\) version of the lemma then implies the \(n \geq 2\) versions of the lemma. \(^1\) The rest of our argument over \(\mathbb{Z}\) works over \(\mathbb{F}_q[t]\). \(\square\)

**Lemma 7.2.** Let \(X \subseteq \mathbb{P}_B^r\) be any closed subscheme. There is an integer \(D\) where \(D \geq \deg(X_s)\) for all \(s \in \text{Spec } B\).

**Proof.** First we take a flattening stratification for \(X\) over \(B\) [GD67, Corollaire 6.9.3]. Within each strata, the maximal degree of a minimal generator is semicontinuous, and we can thus find a degree \(e\) where \(X_s\) is generated in degree \(e\) for all \(s \in \text{Spec } B\). By [BM93, Prop. 3.5], we then obtain that \(\deg(X) \leq \sum_{j=0}^n e^r - j\). In particular defining \(D := re^r\) will suffice. \(\square\)

To prove Corollary 1.3, we use Theorem D to find a submaximal collection \((f_0, f_1, \ldots, f_{n-1})\) which restrict to parameters on \(X_s\) for all \(s \in \text{Spec } B\). This cuts \(X\) down to a scheme \(X' = X \cap \mathbb{V}(f_0, f_1, \ldots, f_{n-1})\) with 0-dimensional fibers over each point \(s\). When \(B = \mathbb{Z}\), such a scheme is essentially a union of orders in number fields, and we find the last element \(f_n\) by applying classical arithmetic results about the Picard groups of rings of integers of number fields. When \(B = \mathbb{F}_q[t]\), we use similar facts about Picard groups of affine curves over \(\mathbb{F}_q\).

An example illustrates this approach. Let \(X = \mathbb{P}_2^1 = \text{Proj}(\mathbb{Z}[x, y])\). A polynomial of degree \(d\) will be a parameter on \(X\) as long as the \(d+1\) coefficients are relatively prime. Thus as \(d \to \infty\), the density of these choices will go to 1. However, once we have fixed one such parameter, say \(5x - 3y\), it is much harder to find an element that will restrict to a parameter on \(\mathbb{Z}[x, y]/(5x - 3y)\) modulo \(p\) for all \(p\). In fact, the only possible choices are the elements which restrict to units on \(\text{Proj}(\mathbb{Z}[x, y]/(5x - 3y))\). Among the linear forms, these are

\[\pm(7x - 4y) + c(5x - 3y)\]

for any \(c \in \mathbb{Z}\).

Hence, these elements arise with density zero, and yet they form a nonempty subset.

Lemmas 7.3 and 7.4 below are well-known to experts, but we sketch the proofs for clarity.

**Lemma 7.3.** If \(X' \subseteq \mathbb{P}_Z^r\) is closed and finite over \(\text{Spec } (\mathbb{Z})\), then \(\text{Pic}(X')\) is finite.

**Proof.** We first reduce to the case where \(X'\) is reduced. Let \(\mathcal{N} \subseteq \mathcal{O}_{X'}\) be the nilradical ideal. If \(X\) is nonreduced then there is some integer \(m > 1\) for which \(\mathcal{N}^m = 0\). Let \(X''\) be the closed

\(^1\) We thank Bjorn Poonen for pointing out this reduction.
subschemes defined by $N^{m-1}$. We have a short exact sequence $0 \rightarrow N^{m-1} \rightarrow \mathcal{O}_{X'}^* \rightarrow \mathcal{O}_{X''}^* \rightarrow 1$ where the first map sends $f \mapsto 1 + f$. Since $H^1(X', N^{m-1}) = H^2(X', N^{m-1}) = 0$, taking cohomology yields an isomorphism $\text{Pic}(X') \cong \text{Pic}(X'')$. Iterating this argument, we may assume $X'$ is reduced.

We now have $X' = \text{Spec}(B)$ where $B$ is a finite, reduced $\mathbb{Z}$-algebra. If $Q$ is a minimal prime of $B$, then $B/Q$ is either zero dimensional or an order in a number field, and hence has a finite Picard group [Neu99, Theorem 12.12]. If $Q'$ is the intersection of all of the other minimal primes $B$. Then we again have an exact sequence in cohomology

$$
\ldots \rightarrow (B/(Q + Q'))^* \rightarrow \text{Pic}(X') \rightarrow \text{Pic}(B/Q) \oplus \text{Pic}(B/Q') \rightarrow \ldots
$$

Since $(B/(Q + Q'))^*$ is a finite set, and since $B/Q$ and $B/Q'$ have fewer minimal primes than $B$, we may use induction to conclude that $\text{Pic}(X')$ is finite. \qed

**Lemma 7.4.** If $C$ is an affine curve over $\mathbb{F}_q$, then $\text{Pic}(C)$ is finite.

**Proof.** If $C$ fails to be integral, then an argument entirely analogous to the proof of Lemma 7.3 reduces us to the case $C$ is integral. We next assume that $C$ is nonsingular and integral, and that $\overline{C}$ is the corresponding nonsingular projective curve. Since $C$ is affine we have $\text{Pic}(C) = \text{Pic}^0(C) \subseteq \text{Pic}^0(\overline{C}) \subseteq \text{Jac}(\overline{C})(\mathbb{F}_q)$, which is a finite set. If $C$ is singular, then the finiteness of $\text{Pic}(C)$ follows from the nonsingular case by a minor adaptation of the proof of [Neu99, Proposition 12.9]. \qed

**Proof of Corollary 1.3.** By Theorem D, for $d \gg 0$ we can find polynomials $f_0, f_1, \ldots, f_{n-1}$ of degree $d$ that restrict to parameters on $X_s$ for all $s \in \text{Spec}(B)$. Let $X' := \mathbb{V}(f_0, f_1, \ldots, f_{n-1}) \cap X$, which is finite over $B$ by construction. Let $A$ be the finite $B$-algebra where Spec $A = X'$. Lemma 7.3 or 7.4 implies that $H^0(X', \mathcal{O}_{X'}(e)) = A$ for some $e$. We can thus find a polynomial $f_n$ of degree $e$ mapping onto a unit in the $B$-algebra $A$. It follows that $\mathbb{V}(f_n) \cap X' = \emptyset$. Replace $f_i$ by $f_i^e$ for $i = 0, \ldots, n-1$ and replace $f_n$ by $f_n^d$. Then we have $f_0, f_1, \ldots, f_n$ of degree $d' := de$ and restricting to parameters on $X_s$ for all $s \in \text{Spec}(B)$ simultaneously.

We thus obtain a proper morphism $\pi : X \rightarrow \mathbb{P}^n_B$ where $X_s \rightarrow \mathbb{P}^n_{\kappa(s)}$ is finite for all $s$. Since $\pi$ is quasi-finite and proper, it is finite by [GD66, Théorème 8.11.1]. \qed

The following generalizes Corollary 1.3 to other graded rings.

**Corollary 7.5.** Let $B = \mathbb{Z}$ or $\mathbb{F}_q[t]$ and let $R$ be a graded, finite type $B$-algebra where \( \dim R \otimes_{\mathbb{Z}} \mathbb{F}_p = n + 1 \) for all $p$. Then there exist $f_0, f_1, \ldots, f_n$ of degree $d$ for some $d$ such that $B[f_0, f_1, \ldots, f_n] \subseteq R$ is a finite extension.

**Proof.** After replacing $R$ by a high degree Veronese subring $R'$, we may assume that $R'$ is generated in degree one and contains no $R'_+$-torsion submodule, where $R'_+ \subseteq R'$ is the homogeneous ideal of strictly positive degree elements. Let $r+1$ be the number of generators of $R'_+$. Then there is a surjection $\phi : B[x_0, \ldots, x_r] \rightarrow R'$ inducing an embedding of $X := \text{Proj}(R') \subseteq \mathbb{P}^r_B$. Since $R'$ contains no $R'_+$-torsion submodule, the kernel of $\phi$ will be saturated with respect to $(x_0, x_1, \ldots, x_r)$ and hence $R'$ will equal the homogeneous coordinate ring of $X$. Choosing $f_0, f_1, \ldots, f_n$ as in Corollary 1.3, it follows that $B[f_0, f_1, \ldots, f_n] \subseteq R'$ is a finite extension, and thus so is $B[f_0, f_1, \ldots, f_n] \subseteq R$. \qed
8. Examples

**Example 8.1.** By Corollary 1.2, it is more difficult to randomly find parameters on surfaces that contain lots of lines. Consider $\mathbb{V}(xyz) \subset \mathbb{P}^3$ which contains substantially more lines than $\mathbb{V}(x^2 + y^2 + z^2) \subset \mathbb{P}^3$. Using Macaulay2 [M2] to select 1,000,000 random pairs $(f_0, f_1)$ of polynomials of degree two, the proportion that failed to be systems of parameters were:

| Field $\mathbb{F}_q$ | $\mathbb{V}(xyz)$ | $\mathbb{V}(x^2 + y^2 + z^2)$ |
|---------------------|-----------------|-----------------|
| $\mathbb{F}_2$     | .2638           | .1179           |
| $\mathbb{F}_3$     | .0552           | .0059           |
| $\mathbb{F}_5$     | .0063           | .0004           |

**Example 8.2.** Let $X \subseteq \mathbb{P}^3_{\mathbb{F}_q}$ be a smooth cubic surface. Over the algebraic closure $X$ has 27 lines, but it has between 0 and 27 lines defined over $\mathbb{F}_q$. For example, working over $\mathbb{F}_4$, the Fermat cubic surface $X'$ defined by $x^3 + y^3 + z^3 + aw^3$ has 27 lines, while the cubic surface $X$ defined by $x^3 + y^3 + z^3 + aw$ where $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ has no lines defined over $\mathbb{F}_4$ [DLR15]. It will thus be more difficult to find parameters on $X$ than on $X'$. Using Macaulay2 [M2] to select 100,000 random pairs $(f_0, f_1)$ of polynomials of degree two, 0.62% failed to be parameters on $X$ whereas no choices whatsoever failed to be parameters on $X'$. This is in line with the predictions from Corollary 1.2; for instance, in the case of $X$, we have $27 \cdot 4^{-2.3} \approx 0.66\%$.

**Example 8.3.** Let $X = [1 : 4] \cup [3 : 5] \cup [4 : 5] = \mathbb{V}((4x - y)(5x - 3y)(5x - 4y)) \subseteq \mathbb{P}^2_k$ and let $R$ be the homogeneous coordinate ring of $X$. The fibers are 0-dimensional so finding a Noether normalization $X \to \mathbb{P}^2_k$ is equivalent to finding a single polynomial $f_0$ that restricts to a unit on all of the points simultaneously. We can find such an $f_0$ of degree $d$ if and only if the induced map of free $\mathbb{Z}$-modules $\mathbb{Z}[x, y]_d \to R_d$ is surjective. A computation in Macaulay2 [M2] shows that this happens if and only if $d$ is divisible by 60.

**Example 8.4.** Let $R = \mathbb{Z}[x]/(3x^2 - 5x) \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{3}]$. This is a flat, finite type $\mathbb{Z}$-algebra where every fiber has dimension 0, yet it is not a finite extension of $\mathbb{Z}$. However, if we take the projective closure of $\text{Spec}(R)$ in $\mathbb{P}^2_{\mathbb{Z}}$, then we get $\text{Proj}(\overline{R})$ where $\overline{R} = \mathbb{Z}[x, y]/(3x^2 - 5xy)$. If we then choose $f_0 := 4x - 7y$, we see that $\mathbb{Z}[f_0] \subseteq \overline{R}$ is a finite extension of graded rings.

**Example 8.5.** Let $k$ be a field and let $X = [1 : 1+t] \cup [1-t : 1] = \mathbb{V}((y-(1+t)x)(x-(1-t)y)) \subseteq \mathbb{P}^1_{k[t]}$. Let $R$ be the homogeneous coordinate ring of $X$. In degree $d$, we have the map $\phi_d : k[t][x, y]_d \cong k[t]^{d+1} \to R_d \cong k[t]^2$. Choosing the standard basis $x^d$, $x^{d-1}y$, $\ldots$, $y^d$ for the source of $\phi_d$, and the two points of $X$ for the target, we can represent $\phi_d$ by the matrix

$$\begin{pmatrix}
1 & 1 + t & (1 + t)^2 & \ldots & (1 + t)^d \\
(1 - t)^d & (1 - t)^{d-1} & (1 - t)^{d-2} & \ldots & 1
\end{pmatrix}.$$

It follows that $\text{im} \phi_d = \text{im} \begin{pmatrix} t^2 & 1 + dt \\ 0 & 1 \end{pmatrix} = \text{im} \begin{pmatrix} t^2 & 1 + dt \\ 0 & 1 \end{pmatrix}$. The image of $\phi_d$ thus contains a unit if and only if the characteristic of $k$ is $p$ and $p|d$. In particular, if $k = \mathbb{Q}$, then we cannot find a polynomial $f_0$ inducing a finite map $X \to \mathbb{P}^0_{\mathbb{Q}[t]}$.

**Example 8.6.** Let $k$ be any field, let $B = k[s, t]$, and let $X = [s : 1] \cup [1 : t] = \mathbb{V}((x - sy)(y - tx)) \subseteq \mathbb{P}^2_B$. We claim that for any $d > 0$, there does not exist a polynomial that restricts to a parameter on $X_b$ for each point $b \in B$. Assume for contradiction that we had such an
\( f = \sum_{i=0}^{d} c_i s^i t^{d-i} \) with \( c_i \in B \). After scaling, we obtain
\[
f([s : 1]) = c_0 s^d + c_1 s^{d-1} + \cdots + c_d = 1 \quad \text{and} \quad f([1 : t]) = c_0 + c_1 t + \cdots + c_d t^d = \lambda
\]
where \( \lambda \in B^* = k^* \). Substituting for \( c_d \) we obtain
\[
f([1 : t]) = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} + (1 - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s)) t^d = \lambda,
\]
which implies that
\[
\lambda - t^d = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} - (c_0 s^d + c_1 s^{d-1} + \cdots + c_{d-1} s) t^d
\]
\[
= (c_0 - c_0 s^d t^d) + (c_1 t - c_1 s^{d-1} t^d) + \cdots + (c_{d-1} t^{d-1} - c_{d-1} st^d) = (1 - st) h(s, t)
\]
where \( h(s, t) \in k[s, t] \). This implies that \( \lambda - t^d \) is divisible by \( 1 - st \), which is a contradiction.

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