Approximate perturbed direct homotopy reduction method: infinite series reductions to two perturbed mKdV equations

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Abstract: An approximate perturbed direct homotopy reduction method is proposed and applied to two perturbed modified Korteweg-de Vries (mKdV) equations with fourth order dispersion and second order dissipation. The similarity reduction equations are derived to arbitrary orders. The method is valid not only for single soliton solution but also for the Painlevé II waves and periodic waves expressed by Jacobi elliptic functions for both fourth order dispersion and second order dissipation. The method is valid also for strong perturbations.

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It is very difficult to study nonlinear phenomena lies in the fact that there are various nonlinear systems which are usually nonintegrable. For some types of idea cases so-called integrable models one may use some types of powerful methods (such as the symmetry reduction method [1], the Darboux transformation [2], the nonlinearization [3] or symmetry constraint method [4] etc) to find some kinds of exact solutions thanks to there usually exist infinitely many symmetries. However, for real nonintegrable physical systems, there are only a little of symmetries or even there is no symmetry at all. In many cases, the nonintegrable sector of a physical system may company with some small parameters. In these cases, one may use the perturbation theory to treat the problems via different approaches. Among these approaches, the approximate symmetry reduction method may be one of the best ways [5, 6]. To find symmetry reductions, one may use the classical, nonclassical approaches [7,1] and/or the Clarkson-Kruskal’s (CK’s) direct

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method [7, 8]. The CK’s direct method is simplest one and can be used to find many group invariant solutions without using group theory. Furthermore, in more general cases, the perturbations may not be weak at all. For strong perturbations, some other types of approaches, such as the homotopy analysis method (HAM) [9] and the Linear [10] and nonlinear [11] nonsensitive homotopy approaches etc., have to be used.

In this letter, we try to combine the CK’s direct symmetry reduction method and HAM to an approximate homotopy direct reduction approach (AHDRA).

The celebrated modified Korteweg-de Vries (mKdV) equation appears in many branches of nonlinear science. As one form of approximation, the singularly perturbed form,

\[ u_t + 6au^2 u_x + u_{xxx} = \epsilon(ux^2 + ux^4), \quad (1) \]

where \( a = 1 \) or \( a = -1 \), the subscripts \( x^n \) mean the differentiations with respect to \( x \) in \( n \) times, has arisen in a number of physical fields, such as models of shallow water on tilted planes [12]. Soliton perturbation property of the mKdV equation was analyzed in [13–15].

In this letter, we consider two special forms of the above equation

\[ u_t + 6au^2 u_x + u_{xxx} = \epsilon u_{x^3 \pm 1}, \quad (2) \]

with fourth order dispersion (the up sign case) and second order dissipation (the lower sign case) in terms of APDRA which is a combination of perturbation theory, direct method and HAM [7–9].

It should be emphasize that in real physical case, the perturbation terms, say, the parameter \( \epsilon \) in (2), may not be small. When the perturbations are not weak, the HAM may be successfully applied by introducing a homotopy \( H(u, q) = 0 \) of the original model \( A(u) = 0 \). When the homotopy parameter, \( q = 0 \), the homotopy model \( H_0(u) = H(u, 0) = 0 \) should be solved via known approach. Usually, \( H_0(u) \) is selected as a linear system. In this paper, we select \( H_0(u) \) as an integrable nonlinear system. Concretely, for the perturbed mKdV system (2), we introduce the following linear homotopy model (linear for the homotopy parameter),

\[ (1 - q)(u_t + 6au^2 u_x + u_{xxx}) - q(u_t + 6au^2 u_x + u_{xxx} - \epsilon u_{x^3 \pm 1}) = 0. \quad (3) \]

It is clear that when \( q = 0 \), (3) is the well known integrable mKdV equation which can be solved via many methods. When \( q = 1 \), it is just the original model (2). Now we can solve (3) via perturbation approaches by taking \( q \) as a perturbation parameter no matter \( \epsilon \) is small or not.
For Eq. (3), according to perturbation theory, the solution can be expressed as a series of \( q \)

\[
u = \sum_{j=0}^{\infty} q^j u_j,
\]

with \( u_j \) being functions of \( x \) and \( t \). Substituting Eq. (4) into Eq. (2) and vanishing the coefficients of all different powers of \( q \), we get

\[
O(\epsilon^0): \quad u_{0t} + 6au_0^2u_{0x} + u_{0x^3} = 0,
\]

\[
O(\epsilon^1): \quad u_{1t} + 6a(u_0^2u_{1x} + 2u_0u_1u_{0x}) + u_{1x^3} - \epsilon u_{0x^3\pm 1} = 0,
\]

\[
O(\epsilon^2): \quad u_{2t} + 6a(u_0^2u_{2x} + u_1^2u_{0x} + 2u_0u_2u_{0x} + 2u_0u_1u_{2x}) + u_{2x^3} - \epsilon u_{1x^3\pm 1} = 0,
\]

\[
\ldots \ldots \ldots .
\]

\[
O(\epsilon^j): \quad u_{jt} + 6a \sum_{k=0}^{j} \sum_{l=0}^{k} u_l u_{k-l} u_{j-k,x} + u_{jx^3} - \epsilon u_{j-1,x^3\pm 1} = 0,
\]

\[
\ldots \ldots \ldots .
\]

The similarity solutions for the above equation are of the form

\[
u_j = U_j(x, t, P_j(z(x, t))), \quad (j = 0, 1, \ldots ),
\]

where \( U_j, P_j \) and \( z \) are functions with respect to the indicated variables and \( P_j(z) \) satisfy a system of ordinary differential equations, which can be obtained by substituting Eq. (6) into Eq. (5). After the substitution, it is easily seen that the coefficients for \( P_{j,zzz} \) and \( P_{j,zz} P_{j,z} \) are \( U_j P_j z_x^3 \) and \( 3aU_j P_j z_x^3 \) respectively. We reserve uppercase Greek letters for undetermined functions of \( z \) from now on. Because the functions \( P_j(z) \) dependent only on the variable \( z \), then the ratios of the coefficients are only functions of \( z \), namely,

\[
3aU_j P_j z_x^3 = U_j P_j z_x^3 \Gamma_j(z), \quad (j = 0, 1, \ldots ),
\]

with the solution \( U_j = F_j(x, t) + G_j(x, t)e^{\frac{1}{\epsilon} \Gamma_j(z)} \) \( \rightarrow F_j(x, t) + G_j(x, t)P_j'(z) \). Hence, it is sufficient to seek similarity reductions of Eq. (5) in the special form

\[
u_j = \alpha_j(x, t) + \beta_j(x, t) P_j(z(x, t)), \quad (j = 0, 1, \ldots ),
\]

instead of the general form Eq. (6).

**Remark:** Three freedoms in the determination of \( \alpha_j(x, t), \beta_j(x, t) \) and \( z(x, t) \) can be notified:

(i) If \( \alpha_j(x, t) \) has the form \( \alpha_j(x, t) = \alpha_j'(x, t) + \beta_j \Omega(z) \), then one can take \( \Omega(z) = 0; \)
(ii) If $\beta_j(x, t)$ has the form $\beta_j(x, t) = \beta_j'(x, t)\Omega(z)$, then one can take $\Omega(z) = \text{constant};$

(iii) If $z(x, t)$ is determined by $\Omega(z) = z_0(x, t)$, where $\Omega(z)$ is any invertible function, then one can take $\Omega(z) = z$.

Substituting Eq. (5) into Eq. (5a), we can see that the coefficients for $P_{0zzz}$, $P_{0z}P_0^2$, $P_{0z}P_0$ and $P_{0zz}$ are $\beta_0z_x^3$, $6\alpha\alpha_0\beta_0z_x^2$, $12\alpha\alpha_0\beta_0^2z_x$ and $3\beta_0z_x^2 + 3\beta_0z_xz_{xx}$, respectively. We require that

\[\begin{align*}
6\alpha\beta_0z_x &= \beta_0z_x^3\Psi(0), \quad 12\alpha\alpha_0\beta_0^2z_x = \beta_0z_x^3\Phi(0), \\
3\beta_0z_x^2 + 3\beta_0z_xz_{xx} &= \beta_0z_x^2\Omega(z),
\end{align*}\]

and thus, applying Remark (i), (ii) and (iii), we have

\[\alpha_0(x, t) = 0, \quad \beta_0(x, t) = z_x, \quad z(x, t) = \theta(t)x + \sigma(t).\]

Eq. (5a) is then simplified to

\[\theta^4P_{0zzz} + 6\theta^4P_0^2P_{0z} + (\theta_tz - \theta_t\sigma + \theta_t\sigma)P_{0z} + \theta_tP_0 = 0.\]

From the coefficients of $P_{0zzz}$, $zP_{0z}$ and $P_{0z}$ of the above equation, it is easily seen that

\[\theta_t = A\theta^{4}, \quad -\theta_t\sigma + \theta_t\sigma = B\theta^{4},\]

with $A$ and $B$ being arbitrary constants.

When $A \neq 0$, Eq. (12) has the solution

\[\theta = -(3A(t - t_0))^{-\frac{1}{4}}, \quad \sigma = -\frac{B}{A} + s_0(t - t_0)^{-\frac{1}{4}},\]

where $s_0$ and $t_0$ are arbitrary constants.

On substitution of Eq. (5) into the general form Eq. (5d), the coefficients of $P_{j=3}$, $P_{j-1,3\pm1}$ and $P_{0z}P_0$ are $\beta_jz_x^3 + \epsilon\beta_{j-1}z_x^4$ and $12\alpha\alpha_j\beta_0^2z_x$, respectively, which lead to

\[\beta_jz_x^4 = \pm\epsilon\beta_jz_x^3\Psi_j(z), \quad 12\alpha\alpha_j\beta_0^2z_x = \beta_jz_x^3\Phi_j(z), \quad (j = 1, 2, \cdots).\]

By $\beta_0 = z_x$, Remark (i) and (ii), we obtain

\[\alpha_j = 0, \quad \beta_j = z_x^{1+j}, \quad (j = 0, 1, \cdots).\]

Eq. (4), (8), (10), (13) and (15) determine the perturbation series solution to Eq. (2)

\[u = \sum_{j=0}^{\infty}(-1)^{j+1}(3A(t - t_0))^{-\frac{1}{4}(1+j)}e^jP_j(z),\]
with the similarity variable \( z = -x(3A(t-t_0))^{-\frac{1}{3}} + s_0(t-t_0)^{-\frac{1}{3}} - \frac{B}{t} \) and the similarity reduction equations are

\[
P_{j,z} = -(j+1)AP_j - (Az + B)P_{j,z} - 6a \sum_{k=0}^{j} \sum_{l=0}^{k} P_{l}P_{k-l} + \epsilon P_{j-1,z^3\pm 1}, \quad (j = 0, 1, \cdots), \quad (17)
\]

with \( P_{-1} = 0 \). When \( j = 0 \), Eq. (17) degenerates to Painlevé II type equation.

When \( A = 0 \), Eq. (12) has the solution

\[
\theta = t_0, \quad \sigma = Bt_0^3t + s_0, \quad (18)
\]

where \( s_0 \) and \( t_0 \) are arbitrary constants. From Eq. (10), Eq. (18) implies an equivalent travelling wave form \( z = x + ct \), so that we obtain the perturbation series travelling wave solution to Eq. (2)

\[
u = \sum_{j=0}^{\infty} \epsilon^j P_j(z), \quad z = x + ct, \quad (19)
\]

where all \( P_j(z) \) satisfy

\[
cP_j + 2a \sum_{k=0}^{j} \sum_{l=0}^{k} P_{j-k}P_{k-l} + \epsilon P_{j-1,z^2 \pm 1} + a_j = 0, \quad (j = 0, 1, \cdots), \quad (20)
\]

with \( P_{-1} = 0 \).

Taking \( j = 0 \), it is obvious that Eq. (20) becomes the zeroth order equation, the well known mKdV equation which has the general solution

\[
\int_{P_0}^{P_{\infty}} \frac{dp}{\sqrt{c_0 - ap^4 - cp^2 - 2a_0p}} = \pm (z - z_0) \quad (21)
\]

with arbitrary constants \( c, a_0, c_0 \) and \( z_0 \). It is also interesting that for the series travelling wave solution, it is not difficult to find the solution of \( P_j \) can be expressed by \( P_0 \), result reads

\[
P_j = \sqrt{c_0 - 2a_0P_0 - cP_0^2 - aP_0^4} \left[ c_{1j} + c_{2j} \int_{P_0}^{P_{\infty}} \frac{dp}{\sqrt{(c_0 - ap^4 - cp^2 - 2a_0p)^3}} - \int_{P_0}^{P'_{\infty}} \int_{P'}^{P'} \frac{F_j(p)}{(c_0 - ap^4 - cp^2 - 2a_0p)^3} dp dp' \right], \quad (22)
\]

where \( c_{1j} \) and \( c_{2j} \) are arbitrary constants while

\[
F_j(P_0) \equiv 2a \sum_{k=1}^{j-1} \sum_{l=0}^{k} P_{j-k}P_{k-l} + 2aP_0 \sum_{l=1}^{j-1} P_{j-l} + \epsilon P_{j-1,z^3 \pm 1} + a_j. \quad (23)
\]
The general solution (21) can be rewritten as some types of Jacobi elliptic functions [16]. For some special selections of the constants, it can be written as some types of soliton solutions or periodic wave solutions, for instance, if we select \( a = -1, \ c = 2k^2, \ a_0 = 0, \ c_0 = k^4 \) for the up sign (the dissipative case), then we have the hyperbolic tangent shape kink soliton solution

\[
P_0 = k\tanh(kx + 2k^3t) \equiv kT. \tag{24}
\]

**Remark:** The convergence of infinite series solution Eq. (16) is superior to the fourth order dispersion case (the up sign case), because the general terms of Eq. (16) become infinitesimal for sufficiently large time \( t \),

\[
|3A(t-t_0)| \gg 1.
\]

For the infinite series solution (16) with the lower sign (the dissipative case), the series will be convergent for not very large time, i.e. for

\[
|3A(t-t_0)| \ll 1.
\]

More specifically, for the dark solitary wave solution (26), we can easily find the closed forms for the higher order homotopy perturbation solutions. Here is a explicit form of the eighth order approximate solution (the end point condition \( q = 1 \) has been used)

\[
u = kT + \frac{\epsilon}{6} + \frac{S^2T - 2T}{48k} \epsilon^2 - \frac{2T + BS^2(BT + 1)}{2034k^3} \epsilon^4 - \frac{12T + 3B^3S^4 - 2B(B^2 - 3)S^2}{331776k^5} \epsilon^6 \\
+ \frac{3B^3S^4(BT - 1) - BS^2(TB^3 + 15 - 3BT - 2B^2) - 30T}{15925248k^7} \epsilon^8 + \cdots, \tag{25}
\]

where

\[ S \equiv \sqrt{1 - T^2}, \quad B \equiv \ln \frac{1 - T}{1 + T}. \]

It should be emphasized that homotopy approximation convergence quite well not only for weak perturbation (small \( \epsilon \)) but also for strong perturbations. Fig. 1 shows the schematic plots of the first five approximants with respect to the orders 1, 2, 4, 6 and 8 respectively from upper to lower of the right side of the figure while the parameters are fixed as

\[
\epsilon = 2.9, \ k = 1. \tag{26}
\]

From the figure, we find that the lines of the sixth order and the eighth order are almost stuck together though the “perturbed parameter” \( \epsilon = 2.9 \) which is not a small one!
Figure 1: The plots of the perturbed kink solitary wave solution for the orders 1, 2, 4, 6 and 8 respectively from upper to lower of the right side of the figure while the parameters are fixed as $\epsilon = 2.9$, $k = 1$.

Similar to the HAM, the APDRA is applicable to other perturbed nonlinear partial differential equations with and without small parameters and it is thought-provoking to explore a general principle for the perturbed nonlinear partial differential equations holding similar results. Different from the HAM, we take the zeroth order as an nonlinear integrable system instead of a linear one, which largely modified the convergence rate. Here we take the direct method as a tool to find the approximate symmetries and symmetry reductions. The similar results can also be obtained via approximate classical and nonclassical symmetry reduction approaches which may be used to the KdV-Burgers equation [6], the perturbed nonlinear Schrödinger systems [?] and the perturbed Boussinesq system [18].

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References

[1] Olver P J Application of Lie Groups to Differential Equation Graduate Texts Math, Vol. 107 New York: Springer; 1993
[2] Gu C H, Hu H S and Zhou Z X *Darboux transformation in soliton theory and its geometric applications* Shanghai Science and Technical Publishers 1999

[3] Cao C W Sci China A 33 (1990) 528

[4] Cheng Y and Li Y S Phys Lett A 175 (1991) 22

Konopelchenko B G, Sidorenko V and Strampp W Phys Lett A 175 (1991) 17

[5] Fushchich W I and Shtelen W M, J. Phys. A: Math. Gen. 22 (1989) L887

[6] Jiao X Y, Yao R X, Zhang S L and Lou S Y 2008 arXiv: 0801.0856v1

[7] Clarkson P A and Kruskal M D, J. Math. Phys. 30 (1989) 2201

[8] S. Y. Lou, Phys. Lett. A 151 (1990) 133

[9] S. J. Liao, Appl. Math. Comput. 147 (2004) 499.

[10] Z. R. Wu, Y. Gao and S. Y. Lou, Pacific J. Appl. Math. (2009) in press.

[11] Y. Gao and S. Y. Lou, *Nonsensitive nonlinear homotopy approach in physics with strong perturbations*, arXiv:0812.3480 2008.

[12] Jones C K R T, *Geometric singular perturbation theory*, Lecture Notes Math, vol. 1609, Springer-Verlag; 1994. pp. 45-118

[13] Kalyakin L A, Russ. J. Math. Phys. 5 (1997) 447

[14] Yan J R, Pan L X and Zhou G H, Commun. Theor. Phys. 34 (2000) 463

[15] Lou S Y, Chin. Phys. Lett. 16 (1999) 659

[16] Lou S Y and Tang X Y, *Methods of Nonlinear Mathematical Physics*, Science Press, (in Chinese) 2006 pp32-35.

[17] Jia M and Lou S Y, Chin. Phys. B (2009) in press.

[18] Jiao X Y, Yao R X and Lou SY, J. Math. Phys. 49 (2008) 093505.