Supersymmetric non-Hermitian topological interface laser

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We investigate laser emission at the interface of a topological and trivial phases with loss and gain. The system is described by a Su-Schrieffer-Heeger model with site-dependent hopping parameters. We study numerically and analytically the interface states. The ground state is described by the Jackiw-Rebbi mode with a pure imaginary energy, reflecting the non-Hermiticity of the system. It is strictly localized only at the A sites. We also find a series of analytic solutions of excited states based on SUSY quantum mechanics, where the A and B sites of the bipartite lattice form SUSY partners. We then study the system containing loss and gain with saturation. The Jackiw-Rebbi mode is extended to a nonlinear theory, where B sites are also excited. The relative phases between A and B sites are fixed, and hence it will serve as a large area coherent laser.

I. INTRODUCTION

Topological physics is one of the most exciting fields[1, 2]. The Su-Schrieffer-Heeger (SSH) model is a simplest example of topological insulators[3]. The topological phase is characterized by the emergence of zero-energy states at the edges of a sample. A zero-energy state emerges also at an interface between a topological phase and a trivial phase, which is called a topological interface state. The Jackiw-Rebbi solution[4] is an analytic solution for the topological interface state. Now, non-Hermitian topological physics is an emerging field. The Jackiw-Rebbi solution seems to be not valid because the energy of the topological interface state is nonzero in general.

Topological photonics is an ideal playground of studying topological physics[5–22]. Topological laser is one of the most successful applications of topological physics[23–33]. A strong lasing from a single coherent mode is possible due to a topological edge or interface state. In topological photonics, loss is inevitable and hence leading to non-Hermitian topological physics[34, 35]. We need to add a gain in order to obtain a laser. Especially, a topological interface laser has enabled a large area coherent lasing by using a smooth interface[36].

In this paper, in order to understand laser emission at the interface between a topological and trivial phases, we analyze a non-Hermitian SSH model first by including linear loss and gain terms. We solve numerically a set of nonlinear differential equations. We also make an analytical study of the Jackiw-Rebbi mode to describe the topological interface state, upon which we construct a series of excitation states at the interface based on supersymmetric (SUSY) quantum mechanics generalized to a non-Hermitian system. We call them SUSY Jackiw-Rebbi modes because they preserve SUSY although the original Jackiw-Rebbi mode breaks SUSY. Not only the topological interface state but also the SUSY Jackiw-Rebbi modes are shown to have pure imaginary energies. Here, SUSY partners are formed by the A and B sites of the bipartite lattice, where only A sites are excited in the original Jackiw-Rebbi mode. We confirm that the analytical solutions well coincide with numerical solutions. Next, we include a saturation term to the gain, which is a nonlinear term. Such a system well describes a large area stable laser emission from an interface of a topological system. The Jackiw-Rebbi topological mode is solely stimulated in laser emission. We extend the Jackiw-Rebbi mode to the nonlinear regime. Excitations at B sites are induced in the Jackiw-Rebbi mode by a nonlinear effect, where the wavefunction at B sites is fixed to be pure imaginary. The relative phases between the saturated wavefunctions at the A and B sites are fixed. Since the Jackiw-Rebbi mode extends over a wide region around the interface, it will give a large area coherent laser.

II. MODEL

We investigate the dynamics of a laser system governed by[23]

\[ i \frac{d\psi_n}{dt} = \sum_{m} M_{nm} \psi_m - i \gamma \left( 1 - \frac{1 - (-1)^n}{2} \right) \left| \psi_n \right|^2 / \eta \psi_n , \tag{1} \]

with a site dependent hopping matrix

\[ M_{nm} = \kappa_{A,n} (\delta_{2n,2m-1} + \delta_{2m,2n-1}) + \kappa_B (\delta_{2n,2m+1} + \delta_{2m,2n+1}) , \tag{2} \]

where \( \psi_n \) is the amplitudes at the site \( n \), where \( n = 1, 2, 3, \ldots, N \) in the system composed of \( N \) sites; \( \gamma \) represents the loss in each resonator; \( \gamma \chi \) represents the amplitude of the optical gain via stimulated emission induced only at the odd site; \( \eta \) represents the nonlinear saturation constant[23]. All these parameters are taken positive semidefinite. The lattice structure of the SSH model is bipartite, where the odd and even sites are called the A and B sites, respectively. The system turns out to be a linear model in the limit \( \eta \to \infty \). On the other hand, \( \gamma \) controls the non-Hermiticity, where the system is Hermitian for \( \gamma = 0 \).

The hopping amplitudes are explicitly given by

\[ \kappa_{A,n} = \kappa \left( 1 + \lambda \tanh \frac{n-n_W+1/2}{\xi} \right) , \quad \kappa_B = \kappa , \tag{3} \]
with $\lambda > 0$, where $n_{\text{IF}}$ is the smallest odd number larger than or equal to $N/2$. Then, $n - n_{\text{IF}} + 1/2 > 0$ for $n \geq n_{\text{IF}}$, and $n - n_{\text{IF}} + 1/2 < 0$ for $n < n_{\text{IF}}$. We call the site $n = n_{\text{IF}}$ the interface of the chain. See Fig.1(a1) and (b1) for an illustration in the case of $N = 10$ and 9.

The explicit equations for a finite chain with length $N$ follow from Eq.(1) as

$$i \frac{d\psi_{2n-1}}{dt} = \kappa_B \psi_{2n-2} + \kappa_{A,n} \psi_{2n} - i \gamma \left( 1 - \frac{\chi}{1 + |\psi_{2n-1}|^2 / \eta} \right) \psi_{2n-1}, \quad (4)$$

$$i \frac{d\psi_{2n}}{dt} = \kappa_B \psi_{2n+1} + \kappa_{A,n} \psi_{2n-1} - i \gamma \psi_{2n}. \quad (5)$$

We solve this set of equations together with the initial condition

$$\psi_n (t = 0) = \delta_{n,n_0}. \quad (6)$$

This is a quench dynamics starting from the interface site by giving an input to it initially. The initial input triggers the gain effect in Eq.(4) because $n_{\text{IF}}$ is an odd number.

III. LINEAR THEORY

We start with the linear model ($\eta \to \infty$). Then, Eq.(1) is reduced to

$$i \frac{d\psi_n}{dt} = \sum_m \tilde{M}_{nm} \psi_m, \quad (7)$$

where

$$\tilde{M}_{nm} = M_{nm} - i \gamma \left( 1 - \frac{\chi}{2} \right) \delta_{nm}. \quad (8)$$

with

$$\bar{M}_{nm} = M_{nm} - i \gamma \frac{(-1)^n}{2} \delta_{nm}. \quad (9)$$

Since $\tilde{M}_{nm}$ and $\bar{M}_{nm}$ are different only by a c-number term, they describe the identical physics. Hereafter, we use $\tilde{M}_{nm}$ for the study of dynamics and $\bar{M}_{nm}$ for the analytical study of the system.

A. Topological edge and interface states

1. SSH model

We analyze the SSH model $M_{nm}$ by taking the negligible penetration depth ($\xi \to 0$). Then, Eq.(3) amounts to

$$\kappa_{A,n} = \kappa (1 + \lambda) \quad \text{for} \quad n \geq n_{\text{IF}},$$

$$\kappa_{A,n} = \kappa (1 - \lambda) \quad \text{for} \quad n < n_{\text{IF}}. \quad (10)$$

The hopping amplitudes are constant $\kappa_{A,n} = \kappa (1 + \lambda)$ for the segments with $n \geq n_{\text{IF}}$, while they are constant $\kappa_{A,n} = \kappa (1 - \lambda)$ for the segments with $n < n_{\text{IF}}$, separately. Note that $\kappa_B = \kappa$. The hopping matrix $M_{nm}$ defines the SSH model in each segment.

The SSH model with constant hopping amplitudes $\kappa_A$ and $\kappa_B$ has a topological phase for $\kappa_A < \kappa_B$ and the trivial phase for $\kappa_A > \kappa_B$. The topological phase is characterized by the emergence of zero-energy states at the edges of a finite chain, as demonstrated numerically in Fig.1(a2) for $N = 100$. This is the standard bulk-edge correspondence. It is illustrated in Fig.1(a1) for $N = 10$. See Appendix for details.

There is an intriguing phenomenon in the SSH model with respect to the even-odd effect of the number of the sites within the chain [30, 36]. We may remove the edge site at $n = N$ from an SSH chain with even $N$ to obtain an SSH chain with odd total number $N - 1$. See an illustration in Fig.1(a1) and (b1), where two chains with $N = 10$ and 9 are shown. We demonstrate numerically that there is only one zero-mode state in the odd chain with $N = 99$ in Fig.1(b2), which is the topological interface state illustrated in Fig.1(a2). This is also a bulk-edge correspondence. Recall that the topological number is defined for the unit cell of the bulk.

In the rest of this work, we focus on the topological interface state by taking an SSH chain with odd $N$. Furthermore, we do not take the limit $\xi \to 0$ any longer.

2. Non-Hermitian SSH model

We investigate the system $\bar{M}_{nm}$ with a finite loss ($\gamma \neq 0$) and gain ($\gamma \chi \neq 0$). Diagonalizing the hopping matrix $\bar{M}_{nm}$ in Eq.(8) numerically, we obtain the energy spectrum $\tilde{E}$ as a function of $\chi$ while setting $\gamma = 0.1$. We show the results in the $(\chi, \text{Re}[\tilde{E}], \text{Im}[\tilde{E}])$ space for $\xi = 10$ in Fig.2(a). See also Fig.2(c1) and (c2) for its cross section at $\text{Im}[\tilde{E}] = 0$ and...
Analyzing the dynamics of the system, it is convenient to study modes, with respect to which we discuss based on the SUSY pure imaginary energies. We call them SUSY Jackiw-Rebbi modes with almost equal spacing and characterized by their interface mode and the bulk spectrum for $\xi = 10$. We have set $\gamma = 0.1$ and $\lambda = 0.5$. We have used the chain with $N = 99$. We clearly observe a straight line passing through the point $(0,0,0)$ in the $(\chi, \text{Re}[E], \text{Im}[E])$ space, which represents the energy of the topological interface state.

Similarly, we show the energy spectrum for $\xi = 1$ and 100 in Fig. 2(b1), (b2), (d1) and (d2). We also find a straight line passing through the point $(0,0,0)$ in the $(\chi, \text{Re}[E], \text{Im}[E])$ space.

The energy of the topological interface state is well fitted for any system parameters by the formula

$$E_{\text{IF}} = i\tilde{\gamma} \quad \text{with} \quad \tilde{\gamma} = \gamma \chi / 2.$$  

(11)

The eigenvalue (11) and the associated eigenfunction are derived as a Jackiw-Rebbi solution later in Section IV: See Eq.(29).

In addition, we observe a band-edge mode[36] between the interface mode and the bulk spectrum for $\xi = 10$. In the case of $\xi = 100$, in addition to the band-edge mode, there are many modes with almost equal spacing and characterized by their pure imaginary energies. We call them SUSY Jackiw-Rebbi modes, with respect to which we discuss based on the SUSY quantum mechanics in Section IV.

\section*{B. Dynamics}

The quench dynamics is a powerful tool to distinguish topological phase even for nonlinear systems[37–40]. Before analyzing the dynamics of the system, it is convenient to study the eigenvalues and the eigenfunctions of the hopping matrix $\tilde{M}_{nm}$ given by Eq.(8). We diagonalize it as

$$\tilde{M}\phi_p = \tilde{E}_p \phi_p,$$  

(12)

where $p$ labels the eigen index, $1 \leq p \leq N$, and $\phi_p$ is the eigenfunction. We show the eigenvalues $\tilde{E}_p$ in Fig. 3(a1), (b1) and (c1). Let the wavefunction of the topological interface state be $\phi_{\text{IF}}$. Its eigenvalue is

$$\tilde{E}_{\text{IF}} = \tilde{E}_{\text{IF}} - i\gamma \left(1 - \frac{\chi}{2}\right) \delta_{nm} = i\gamma (\chi - 1),$$  

(13)

with the use of Eq.(8) and Eq.(11).

Decoupled equations follow from Eq.(7) for the eigenfunctions,

$$i\frac{d\phi_p}{dt} = \tilde{E}_p \phi_p,$$  

(14)

whose solutions are given by

$$\phi_p(t) = \exp\left[-it\tilde{E}_p\right] \phi_p.$$  

(15)

In particular, for the topological interface state, we have

$$\phi_{\text{IF}}(t) = \exp\left[\gamma (\chi - 1) t\right] \phi_{\text{IF}},$$  

(16)

with the use of Eq.(13). It has no dynamics for $\gamma = 0$ or $\chi = 1$. On the other hand, it grows exponentially for $\chi > 1$.

The initial state (6) is expanded in terms of the eigenfunctions as

$$\psi_n(t = 0) = \delta_{n,n_{\text{IF}}} = \sum_p c_p \phi_p.$$  

(17)

We show the coefficient $|c_p|$ in Fig. 3(a2), (b2) and (c2), which is determined by

$$c_p = \sum_n \delta_{n,n_{\text{IF}}} \phi_p.$$  

(18)
which is found large for \( \gamma \) as in Fig.4(a1) but the peak is tiny for \( \gamma \). This is because the topological interface state is strictly localized at the interface for small \( \xi \), but broad for large \( \xi \).

We now investigate the quench dynamics of the system by imposing the initial condition (6).

First, we neglect the loss and gain terms by setting \( \gamma = 0 \). We numerically solve a set of differential equations (4) and (5), whose results are shown in Fig.4(a1), (a2) and (a3). The input given initially at the site \( n = n_{IF} \) spreads over the chain, but the component \( c_{IF} \) remains as it is, because \( \phi_{IE} (t) = \phi_{IF} \) in Eq.(16) for \( \gamma = 0 \). There is a peak at the interface for \( \xi = 2 \) as in Fig.4(a1) but the peak is tiny for \( \xi = 100 \) as in Fig.4(a3).

Second, we include the linear loss and gain terms \( (\gamma \chi \neq 0) \), whose results are shown in Fig.4(b1), (b2) and (b3). The topological interface state has a maximum value at the site with gain. As a result, the state exponentially evolves and becomes infinite. However, this is not physical. Indeed, there is a saturation of the gain in actual experiments, about which we discuss in Section VII.

We diagonalize the matrix \( \mathcal{M}_{n,m} \) by employing an approximation similar to the one made by Jackiw and Rebbi. The hopping amplitude (3) becomes constant as in Eq.(10) far away from the interface. Then, the hopping matrix \( \mathcal{M}_{n,m} \) can be presented in the momentum space as

\[
\mathcal{H} = \begin{pmatrix}
i\gamma & \Delta_0 + ikk' \\
\Delta_0 - ikk' & -i\gamma
\end{pmatrix},
\]

with

\[
\Delta_0 = \kappa_A - \kappa_B, \quad k' = k - \pi.
\]

We bring back this Hamiltonian to the continuous coordinate space as

\[
\mathcal{H} = \begin{pmatrix}
i\gamma & \Delta (x) - \kappa \partial_x \\
\Delta (x) + \kappa \partial_x & -i\gamma
\end{pmatrix} = \begin{pmatrix}
i\gamma & A^1 \\
A & -i\gamma
\end{pmatrix},
\]

with

\[
A \equiv \Delta (x) + \kappa \partial_x, \quad A^1 \equiv \Delta (x) - \kappa \partial_x,
\]

and

\[
\Delta (x) = \kappa \lambda \tanh \frac{x - x_{IF}}{a\xi},
\]

where we have recovered the site dependent hopping amplitude from Eq.(3).

The eigenvalues of the Hamiltonian for \( \rho \)th eigenindex (23) reads

\[
\mathcal{H} \left( \begin{pmatrix}
\psi_{\rho}^A (x) \\
\psi_{\rho}^B (x)
\end{pmatrix} \right) = E_{\rho} \left( \begin{pmatrix}
\psi_{\rho}^A (x) \\
\psi_{\rho}^B (x)
\end{pmatrix} \right),
\]

IV. JACKIW-REBBI SOLUTION IN NON-HERMITIAN MODEL

Supersymmetric quantum mechanics is a method to obtain an analytic solution originally proposed by Witten[41–44]. It has also been applied to laser systems[45–50].

We continue to study the linear model but based on the PT-symmetric non-Hermitian SSH model \( \mathcal{M}_{n,m} \) from now. The two matrices \( \mathcal{M}_{n,m} \) and \( \mathcal{M}_{n,m} \) are different only by a c-number as in Eq.(9). Hence, the eigenfunctions are identical with the eigenvalues different only by this c-number.

We show the time evolution of the amplitude \( |\psi_{nIF}| \) in Fig.4(a4), (b4) and (c4). It becomes stationary after a certain time in the absence of the loss and gain terms \( (\gamma = 0) \) as shown in Fig.4(a4). On the other hand, the amplitude exponentially becomes large once the loss and gain terms are present \( (\gamma \chi \neq 0) \), as shown in Fig.4(b4). It becomes stationary by the saturation term \( (\eta < \infty) \) as in Fig.4(c4), about which we discuss in Section VII.
with (23), where we have defined the wavefunctions with the eigenvalue \( E_p \) at the A and B sites as \( \psi^A(x) \) and \( \psi^B(x) \), respectively.

We derive the eigenfunction representing the topological interface state. Its eigenenergy \( E_{IF} \) is given by Eq.(11) in the \( M_{nm} \) basis, which reads \( E_{IF} = i\gamma \) in the \( \mathcal{H} \) basis. Hence, Eq.(26) yields

\[
\mathcal{H} \left( \frac{\psi^A_0(x)}{\psi^B_0(x)} \right) = i\gamma \left( \frac{\psi^A_0(x)}{\psi^B_0(x)} \right),
\]

with \( E_0 = E_{IF} = i\gamma \) and (23) for \( \mathcal{H} \). It is easy to obtain one solution by setting \( \psi^B_0(x) = 0 \). The equation for \( \psi^A(x) \) reads

\[
A \psi^A_0(x) = \left( \Delta(x) + \kappa \partial_x \right) \psi^A_0(x) = 0,
\]

for which the Jackiw-Rebbi solution follows,

\[
\psi^A_0(x) = c \exp \left[ -\frac{1}{\kappa} \int^x \Delta(x') \, dx' \right], \quad \psi^B_0(x) = 0,
\]

with \( c \) is a normalization constant. This is a non-Hermitian generalization of the Jackiw-Rebbi mode with a pure imaginary eigenvalue. It is the unique solution because there is no degeneracy in the topological interface state.

![Diagram](image)

**FIG. 5.** Illustration of the energy levels and the SUSY quantum mechanics. Wave functions are shown in the case of \( h^A_n = h^B_n \) for the Hermitian system.

**V. SUSY QUANTUM MECHANICS**

When an operator \( A \) is given, we may define the supercharges \( Q, Q^\dagger \) and the Hamiltonian \( \hat{H} \) by [41–44] \( \quad Q \equiv \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger \equiv \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \), \quad (31)

\[
\hat{H} = \{Q, Q^\dagger\} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}. \quad (32)
\]

The superalgebra follows.

\[
\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = [\hat{H}, Q] = [\hat{H}, Q^\dagger] = 0. \quad (33)
\]

A representation of the algebra is constructed as follows.

We define the operators

\[
H_A = A^\dagger A, \quad H_B = AA^\dagger. \quad (34)
\]

The eigenvalue equations are

\[
H_A \phi^A_p = E^A_p \phi^A_p, \quad H_B \phi^B_p = E^B_p \phi^B_p. \quad (35)
\]

Using these we obtain

\[
H_B(A \phi^A_p) = AA^\dagger A \phi^A_p = E^A_p(A \phi^A_p), \quad (36)
\]

\[
H_A(A^\dagger \phi^B_p) = A^\dagger AA^\dagger \phi^B_p = E^B_p(A^\dagger \phi^B_p), \quad (37)
\]

and hence, \( \phi^A_p \) is an eigenstate of \( H_B \) with the eigenvalue \( E^A_p \). If we assume \( E^A_0 = 0 \) and \( E^B_0 \neq 0 \), we may choose \( q = p + 1 \). Then, \( \phi^B_p(x) \propto A \phi^A_{p+1}(x) \) and \( \phi^A_p(x) \propto A^\dagger \phi^B_p(x) \) so that

\[
E^B_p = E^A_{p+1}, \quad E^A_0 = 0. \quad (38)
\]
The standard commutation relation of the annihilation and creation operators follows,

\[ [b, b^\dagger] = 1, \]  

in terms of the scaled operators \( b \) and \( b^\dagger \) defined by

\[ A \equiv \sqrt{\alpha} b, \quad A^\dagger \equiv \sqrt{\alpha} b^\dagger. \]  

Eqs.(41) and (42) are rewritten as

\[ \alpha b^\dagger b \Psi_p = (E_p + \gamma^2) \Psi_p, \quad \alpha (1 + b^\dagger b) \Psi_p = (E_p + \gamma^2) \Psi_p. \]  

These are solved as

\[ \Psi_p^A(x) = h_p^A(x|p), \quad \Psi_p^B(x) = h_p^B(x|p - 1), \]  

for \( p \geq 1 \), where \( h_p^A \) and \( h_p^B \) are c-numbers. For \( p = 0 \), we have \( E_0 = i\gamma \), and the wavefunctions are given by the non-Hermitian Jackiw-Rebbi solutions (29) and (30).

We note that the energy \( E_p \) of the \( p \)th level is pure imaginary when

\[ p < \frac{\gamma^2}{\alpha} = \frac{\gamma \chi \xi}{4\kappa^2 \lambda}. \]  

We call the mode \( |p\rangle \) the SUSY Jackiw-Rebbi mode, because we create it from the Jackiw-Rebbi mode \( |0\rangle \) by the operation of \( b^\dagger \). The SUSY Jackiw-Rebbi modes are supersymmetric, while the Jackiw-Rebbi mode breaks it.

We determine the relation between two c-numbers \( h_p^A \) and \( h_p^B \). We write down the eigenvalue equations (26) explicitly,

\[ i\gamma \Psi_p^A + A^\dagger \Psi_p^B = E_p \Psi_p^A, \quad A \Psi_p^A - i\gamma \Psi_p^B = E_p \Psi_p^B, \]  

which we rewrite with the use of (53) as

\[ i\gamma h_p^A|p\rangle + \sqrt{\alpha} h_p^B b^\dagger|p - 1\rangle = E_p h_p^A|p\rangle, \quad \sqrt{\alpha} h_p^A|p\rangle - i\gamma h_p^B|p - 1\rangle = E_p h_p^B|p - 1\rangle. \]  

It follows that

\[ i\gamma h_p^A + \sqrt{\alpha} h_p^B = E_p h_p^A, \quad \sqrt{\alpha} h_p^A - i\gamma h_p^B = E_p h_p^B. \]  

or

\[ \sqrt{\alpha} h_p^A = (E_p - i\gamma) h_p^A, \quad \sqrt{\alpha} h_p^A = (E_p + i\gamma) h_p^A, \]  

which leads to

\[ h_p^B = \left( \frac{E_p - i\gamma}{E_p + i\gamma} \right)^{1/2} h_p^A. \]  

Hence, the wavefunction \( \Psi_p^B \) is determined once the wavefunction \( \Psi_p^A \) is given.

Here we recall that there are two series of eigenfunctions corresponding to \( E_p^\pm = \pm \sqrt{-\gamma^2 + \alpha p} \) for \( p \geq 1 \) and \( E_0^+ = i\gamma \). We focus on SUSY Jackiw-Rebbi modes, where \( \gamma^2 > \alpha p \).
In the parameter region with $\bar{\gamma}^2 \gg \alpha \eta$, we would expand $E^\pm_p = \pm i\bar{\gamma} + \cdots$. Then, we have

$$
\left( \frac{h^B_p}{h^A_p} \right)^2 = \frac{E^+_p - i\bar{\gamma}}{E^+_p + i\bar{\gamma}} \ll 1 \quad \text{for} \quad E^+_p = i\bar{\gamma} + \cdots, \quad (64)
$$

and

$$
\left( \frac{h^B_p}{h^A_p} \right)^2 = \frac{E^-_p - i\bar{\gamma}}{E^-_p + i\bar{\gamma}} \gg 1 \quad \text{for} \quad E^-_p = -i\bar{\gamma} + \cdots. \quad (65)
$$

Hence,

$$
|\Psi^A_p| \gg |\Psi^B_p| \quad \text{for the series} \quad E^+_p, \quad (66)
$$

and

$$
|\Psi^A_p| \ll |\Psi^B_p| \quad \text{for the series} \quad E^-_p. \quad (67)
$$

This explains a huge difference numerically found between the amplitudes at the A and B sites in Fig.6.

We comment on the SUSY quantum mechanics. First of all, there are two series of energies $E^\pm_p = \pm \sqrt{-\gamma^2 + \alpha \eta}$, although the relevant energies are $E^\pm_{p-1} = E^\pm_p = \alpha \eta$ for both the series in SUSY quantum mechanics. However, the magnitudes of the amplitudes are very different,

$$
|\phi^A_p| \gg |\phi^B_{p-1}| \quad \text{for the series} \quad E^+_p, \quad (68)
$$

and

$$
|\phi^A_p| \ll |\phi^B_{p-1}| \quad \text{for the series} \quad E^-_p, \quad (69)
$$

which follows from (43) and (67). These two series are shown in Fig.6.

The wavefunction is given by $\langle x|p \rangle$ apart from the normalization constant, and hence it is written in terms of the Hermite polynomials precisely as in the Hermitian model, where $h^B_p$ is given by Eq.(63) while $h^A_p$ is to be determined numerically.

There is the Jackiw-Rebbi mode only for A site, whose wavefunctions are

$$
\Psi^A_0(x) = h^A_0 \exp \left[ -\frac{\lambda}{2\xi^2} x^2 \right], \quad \Psi^B_0(x) = 0. \quad (72)
$$

This is the SUSY-broken state.

Finally, we compare the analytic solutions and the numerical solutions in Fig.6. The coincidence is very well between the analytic solution and the numerical results except for a minor difference, where the mirror symmetry is slightly broken in the numerical results. It is due to the difference between the hopping parameters $\kappa_{A,n}$ and $\kappa_B$ in Eq.(3), where the band widths are different between the topological and trivial phases. This difference is taken care of in the numerical calculation but ignored in the analytical study.

VII. GAIN WITH NONLINEAR SATURATION

A. Quench dynamics

We have so far studied the linear model containing loss and gain. The amplitude increases infinitely as time passes. Actually, there must be a saturation effect in gain, which makes the amplitude finite. We include the saturation effect by keeping $\eta$ finite in Eq.(1). We show the results in Fig.4(a1), (c2) and (c3). The amplitudes remain finite due to the saturation effect. It is a topological interface laser stabilized by nonlinear and non-Hermicity effects. We also show the time evolution of the amplitude $|\Psi_{n=0}|$ in Fig.4(c4).

We show the spatial profile of the saturated amplitude $|\Psi_n|$ for various $\eta$ in Fig.7. Main excitations are localized at the A sites in the vicinity of the interface, whose wavefunction is real. However, there are also excitations at the B sites in the vicinity of the interface as in Fig.7(b), (c) and (d), whose wavefunction is pure imaginary. Hence, the relative phases between the A and B sites are fixed to be $\pm i$ and hence it will serve as a large area coherent laser.

B. Nonlinear Jackiw-Rebbi theory

We have numerically revealed the excitations at the B sites in the presence of the saturation term. We now show that they form the Jackiw-Rebbi mode generalized to the nonlinear regime. Replacing the linear gain term with the nonlinear gain term in Eq.(47), we have

$$
\left( \begin{array}{c}
\frac{i\gamma x}{1+|\Psi_n(x)|^2/\eta} A - i\gamma A^\dagger \\
-\bar{i}\gamma \end{array} \right) \left( \begin{array}{c}
\Psi_A(x) \\
\Psi_B(x)
\end{array} \right) = E \left( \begin{array}{c}
\Psi_A(x) \\
\Psi_B(x)
\end{array} \right). \quad (73)
$$
We analyze a small excitation at the B sites. Using a mean-field approximation, we obtain $\Psi_A (x)$ and $\Psi_B (x)$ as

\[
\Psi_A (x) = c \exp \left[ -\frac{\kappa \lambda}{2\xi} (1 + c_2) x^2 \right],
\]

\[
\Psi_B (x) = -ic x \frac{c_2 \kappa \lambda}{\eta} \exp \left[ -\frac{\kappa \lambda}{2\xi} (1 + c_2) x^2 \right],
\]

where $c$ is a normalization constant, and

\[
c_2 = \frac{\gamma^2 \lambda^2 \xi}{\kappa^2 \lambda} \left[ \frac{1}{1 + |\Psi_A|^2/\eta} - \frac{1}{1 + |\Psi_A (0)|^2/\eta} \right],
\]

with $\Psi_A$ the mean of $\Psi_A (x)$. See Appendix B for detailed derivation. We note that

\[
\frac{\Psi_B (x)}{\Psi_A (x)} = -ic \frac{\gamma^2 \lambda^2}{\kappa} \left[ \frac{1}{1 + |\Psi_A|^2/\eta} - \frac{1}{1 + |\Psi_A (0)|^2/\eta} \right].
\]

The relative phases between the A and B are fixed to be $\pm i$. Furthermore, this formula well explains three key properties of the wavefunctions revealed in Fig.7: (1) $|\Psi_A (x)|$ is proportional to $\sqrt{\eta}$; (2) $\Psi_B (x)/\Psi_A (x)$ is independent of $\eta$; $|\Psi_A (x)| = 0$ at $x = 0$.

VIII. CONCLUSION AND DISCUSSION

We have explored a SUSY structure in the SSH model with a topological interface as a model of topological interface laser with gain and loss. By extending a SUSY quantum mechanics to non-Hermitian systems, we have found a series of analytic solutions which extend the original Jackiw-Rebbi solution. They have pure imaginary energies and their wavefunctions are given by those of a harmonic oscillator. We also derived an analytic form of the Jackiw-Rebbi mode in non-linear regime by using a mean-field approximation.

We have applied quench dynamics to investigate a topological interface laser with gain and loss. However, it may be hard to observe the time evolution in actual optical experiments because the time scale is too short. The same physics is executed by the coupled-wave-guide arrays along the $z$ direction[51], simply by replacing time $t$ by coordinate $z$ in the equation of motion.

We have developed an analysis based on the basic equation (1). On the other hand, it is well known that the dynamics of a laser is described by the rate equations. It is actually possible to derive Eq.(1) from the rate equations in a certain limit provided the carrier population is saturated. See details for Appendix C.

Large-area single-mode lasers are realized by suppressing the appearance of higher order modes. In order to increase the output power of the laser, it is necessary to enlarge the emitting area. However, it causes the multi-mode and degrades the brightness at the same time in general. There are several proposals on the single mode laser using photonic crystals have been reported by using double-lattice photonic-crystal resonators[52], accidental Dirac-point[53, 54] and Kekulé modulation[55, 56] in the photonic lattice mostly over the past few years. Our results give a deeper understanding of a large area single mode laser from a topological interface[36].

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Appendix A: Topological property of the non-Hermitian SSH model

We consider a homogeneous system. The Hamiltonian in the momentum space corresponding to the hopping matrix (8) is

\[
\tilde{H} = \left( \begin{array}{cc} -i\gamma (1 - \chi) & \kappa_A + \kappa_B e^{-iak} \\ \kappa_A + \kappa_B e^{iak} & -i\gamma \end{array} \right),
\]

\[
= -i\gamma \left( 1 - \frac{\chi}{2} \right) I_2 + \tilde{H}_{SSH}
\]

(A1)

with $a$ the lattice constant and

\[
\tilde{H}_{SSH} = \left( \begin{array}{cc} i\gamma \sqrt{2} \kappa_A + \kappa_B e^{iak} & \kappa_A + \kappa_B e^{-iak} \\ \kappa_A + \kappa_B e^{-iak} & -i\gamma \sqrt{2} \end{array} \right).
\]

(A2)

The Hamiltonian $\tilde{H}_{SSH}$ is non-Hermitian for $\gamma \neq 0$. The relation between the eigenenergy of the Hamiltonians (A1) and (A2) is

\[
\tilde{E} = -i\gamma \left( 1 - \frac{\chi}{2} \right) + E_{SSH}.
\]

(A3)

The energy spectrum reads

\[
\tilde{E} (k) = \pm \sqrt{\kappa_A^2 + \kappa_B^2 + 2\kappa_A \kappa_B \cos ak - \gamma^2}.
\]

(E4)

Especially, we have

\[
\tilde{E} (\pi/a) = \pm \sqrt{(\kappa_A - \kappa_B)^2 - \gamma^2}.
\]

(A5)

The system is the PT preserved phase for $\gamma < |\kappa_A - \kappa_B|$, where the bulk energy is real even though the system is non-Hermitian, while the system is the PT broken phase for $\gamma > |\kappa_A - \kappa_B|$, where the bulk energy becomes pure imaginary for certain range of the momentum $k$.

We recall that the PT symmetry operation is defined by

\[
PT = \sigma_z K,
\]

(A6)

with $K$ the complex conjugation. Since we have

\[
PT \tilde{H} (k) (PT)^{-1} = \tilde{H} (k),
\]

(A7)

and hence $\tilde{H}_{SSH}$ is a PT symmetric Hamiltonian.
The topological number is defined with respect to the Hamiltonian (A2). We define the right and left eigenvectors by
\[ H |\psi^R\rangle = E |\psi^R\rangle, \quad H^\dagger |\psi^L\rangle = E |\psi^L\rangle. \tag{A8} \]

The non-Hermitian Zak phase is a topological number[57]
\[ W = \frac{i}{2\pi/a} \int_0^{2\pi/a} \langle \psi^L | \frac{\partial}{\partial k} |\psi^R\rangle dk. \tag{A9} \]

It is straightforward to show that \( W = 1 \) for \( \kappa_A < \kappa_B \) and \( W = 0 \) for \( \kappa_A > \kappa_B \) irrespective of \( \gamma \). Hence, the system is topological for \( \kappa_A < \kappa_B \) and trivial for \( \kappa_A > \kappa_B \).

**Appendix B: Nonlinear Jackiw-Rebbi solution**

We derive a set of the saturated distribution (74) and (75) from Eq.(73). First, we write Eq.(73) explicitly as
\[ i\gamma \left( \frac{\Psi_A(x)}{1 + |\Psi_A(x)|^2/\eta} - 1 \right) \Psi_A(x) + A^\dagger \Psi_B(x) = E \Psi_A(x), \tag{B1} \]
\[ A \Psi_A(x) - i\gamma \Psi_B(x) = E \Psi_B(x), \tag{B2} \]

where \( A \) and \( A^\dagger \) are given by Eq.(24) with Eq.(25). The second equation is solved as
\[ \Psi_B(x) = \frac{A \Psi_A(x)}{E + i\gamma}, \tag{B3} \]

which we insert into the first equation to derive
\[ A^\dagger A \Psi_A(x) = (E + i\gamma) \left( E - i\gamma \left( \frac{1}{1 + |\Psi_A(x)|^2/\eta} - 1 \right) \right) \Psi_A(x). \tag{B4} \]

We assume that the energy is modified from Eq.(13) as
\[ E = i\gamma (\chi - 1) + c_1, \tag{B5} \]
where \( c_1 \) is a constant to be determined. Inserting it and we have
\[ A^\dagger A \Psi_A(x) \simeq i\gamma \left( c_1 + i\gamma \left( \frac{1}{1 + |\Psi_A(x)|^2/\eta} \right) \right) \Psi_A(x). \]

where we have used an approximation \( |\Psi_A(x)|^2 \simeq |\Psi_A(0)|^2 \) because \( \Psi_A(x) \) rapidly decreases except at \( x = 0 \). We choose
\[ c_1 = i\gamma \left( \frac{1}{1 + |\Psi_A|/\eta} - 1 \right), \tag{B6} \]

where \( \Psi_A \) is the mean value of \( \Psi_A(x) \). We obtain
\[ A^\dagger A \Psi_A(x) = -\gamma^2 \chi^2 \left( \frac{1}{1 + |\Psi_A|^2/\eta} - \frac{1}{1 + |\Psi_A(0)|^2/\eta} \right) \Psi_A(x). \tag{B7} \]

On the other hand, we assume a wavefunction modified from Eq.(72) as
\[ \Psi_A(x) = c \exp \left[ -\frac{\kappa \lambda}{2\xi}(1 + c_2)x^2 \right], \tag{B8} \]

where \( c \) is a normalization constant and \( c_2 \) is a constant to be determined. Applying \( A \) and \( A^\dagger \) to \( \Psi_A(x) \), we obtain
\[ A \Psi_A(x) \simeq -c_2 \frac{\kappa \lambda}{\xi} x \Psi_A(x), \tag{B9} \]
\[ A^\dagger A \Psi_A(x) \simeq -c_2 \frac{\kappa \lambda}{\xi} \Psi_A(x). \tag{B10} \]

Comparing (B10) with Eq.(B7), we obtain
\[ c_2 = \frac{\gamma^2 \chi^2 \xi^2}{\kappa \lambda} \left[ \frac{1}{1 + |\Psi_A|^2/\eta} - \frac{1}{1 + |\Psi_A(0)|^2/\eta} \right]. \tag{B11} \]

With the use of Eqs.(B3) and (B9), \( \Psi_B(x) \) is derived as
\[ \Psi_B(x) = -ic \frac{c_2 \kappa \lambda}{\xi} \exp \left[ -\frac{\kappa \lambda}{2\xi}(1 + c_2)x^2 \right]. \tag{B12} \]

It is the saturated distribution (75) in the main text. We then have
\[ \frac{\Psi_B(x)}{\Psi_A(x)} = -ic \frac{\gamma^2 \chi^2}{\kappa} \left[ \frac{1}{1 + |\Psi_A|^2/\eta} - \frac{1}{1 + |\Psi_A(0)|^2/\eta} \right], \tag{B13} \]

which is Eq.(77) in the main text.
Appendix C: Rate equation

The rate equations read[29, 58]

\[
\frac{dE_n^A}{dt} = \frac{1}{2} \left[ -\gamma_0 + \sigma \left( N_n^A - 1 \right) \right] (1 - i\alpha_H) E_n^A + i\kappa_A^0 E_n^B + i\kappa_B^0 E_{n-1}^B, \tag{C1}
\]

\[
\frac{dE_n^B}{dt} = \frac{1}{2} \left[ -\gamma_0 + \sigma \left( N_n^B - 1 \right) \right] (1 - i\alpha_H) E_n^B + i\kappa_A^0 E_n^A + i\kappa_B^0 E_{n+1}^A, \tag{C2}
\]

\[
\frac{dN_n^A}{dt} = R_A - \frac{N_n^A}{\tau_r} - F (N_n^A - 1) |E_n^A|^2, \tag{C3}
\]

\[
\frac{dN_n^B}{dt} = R_B - \frac{N_n^B}{\tau_r} - F (N_n^B - 1) |E_n^B|^2, \tag{C4}
\]

where \(E_n^A\) and \(E_n^B\) are electric field amplitudes in sublattices \(A\) and \(B\) and \(N_n^A\) and \(N_n^B\) are carrier population densities.

We assume the carrier is saturated

\[
\frac{dN_n^A}{dt} = 0, \quad \frac{dN_n^A}{dt} = 0,
\]

or

\[
N_n^A - 1 = \frac{F |E_n^A|^2 + R_A}{F |E_n^A|^2 + 1/\tau_r} - 1 = \frac{R_A - 1/\tau_r}{F |E_n^A|^2 + 1/\tau_r}, \tag{C6}
\]

\[
N_n^B - 1 = \frac{F |E_n^B|^2 + R_B}{F |E_n^B|^2 + 1/\tau_r} - 1 = \frac{R_B - 1/\tau_r}{F |E_n^B|^2 + 1/\tau_r}. \tag{C7}
\]

By inserting them into the rate equations, we have

\[
\frac{dE_n^A}{dt} = \frac{1}{2} \left[ -\gamma_0 + \sigma \left( \frac{R_A - 1/\tau_r}{F |E_n^A|^2 + 1/\tau_r} \right) \right] (1 - i\alpha_H) E_n^A + i\kappa_A^0 E_n^B + i\kappa_B^0 E_{n-1}^B, \tag{C8}
\]

\[
\frac{dE_n^B}{dt} = \frac{1}{2} \left[ -\gamma_0 + \sigma \left( \frac{R_B - 1/\tau_r}{F |E_n^B|^2 + 1/\tau_r} \right) \right] (1 - i\alpha_H) E_n^B + i\kappa_A^0 E_n^A + i\kappa_B^0 E_{n+1}^A, \tag{C9}
\]

or

\[
\frac{dE_n^A}{dt} = \frac{i}{2} \left[ -\gamma_0 + \sigma \left( \frac{\tau_r R_A - 1}{1 + \tau_r F |E_n^A|^2} \right) \right] (1 - i\alpha_H) E_n^A - \kappa_A^0 E_n^B - \kappa_B^0 E_{n-1}^B, \tag{C10}
\]

\[
\frac{dE_n^B}{dt} = \frac{i}{2} \left[ -\gamma_0 + \sigma \left( \frac{\tau_r R_B - 1}{1 + \tau_r F |E_n^B|^2} \right) \right] (1 - i\alpha_H) E_n^B - \kappa_A^0 E_n^A - \kappa_B^0 E_{n+1}^A. \tag{C11}
\]

When \(\alpha_H\) is negligible and \(\tau_r R_B = 1\), by setting

\[
\psi_n^A = E_n^A, \quad \psi_n^B = E_n^B, \quad \kappa_A = -\kappa_A^0, \quad \kappa_B = -\kappa_B^0, \tag{C12}
\]

\[
\gamma = -\gamma_0/2, \quad \eta = \tau_r F, \quad \gamma \chi = \sigma (\tau_r R_A - 1), \tag{C13}
\]

they are reduced to Eq.(1) in the main text.
