We investigate the transient dynamics of photon statistics through two-time correlation functions for optical fields. We find that the transient correlations at different \( t \) yield a smooth transition from antibunching to bunching photon statistics in the weak system-environment coupling regime. In the strong-coupling regime, the two-time correlations exhibit bunching-antibunching oscillations that persists both in the transient process and in the steady-state limit. The photon bunching-antibunching oscillations is a manifestation of strong non-Markovian dynamics, where the system remains in nonequilibrium from its environment. We also find that the antibunching to bunching transition in the weak-coupling regime and the bunching-antibunching oscillation in the strong-coupling regime are strongly influenced by the initial environment temperature.

\[ g^{(2)}(t, t + \tau) = \lim_{t \to \infty} g^{(2)}(t, t + \tau) = 1 + e^{-2\kappa \tau}. \]
for which the Born-Markov approximation is not applicable. Here we show a nontrivial transient dynamics of $g^{(2)}(t, t + \tau)$ in connection to photon bunching and antibunching when the optical field interacts with a general non-Markovian environment. By solving the exact quantum Langevin equation\cite{17,22}, we can obtain the exact time evolution of the optical field operator $a(t)$. Then, the two-time correlation function is given by

\begin{equation}
\langle a^\dagger(t_1)a(t_2)\rangle = v(t)v(t_2) + |v(t, t_2)|^2 |u(t_2)|^2 \alpha + \{v(t)|u(t)|^2 + v(t_2)|u(t_2)|^2 + 2\text{Re}[v(t, t_2)u^*(t_2)u(t_2)]\} \beta
\end{equation}

which is determined by two basic Green’s functions $u(t, 0) = \langle a(t), a^\dagger(0)\rangle$ and $v(t, t_2) = \langle a^\dagger(t_1)a(t_2)\rangle$ (plus an initial state dependent term) in nonequilibrium quantum systems\cite{23–26}. We have denoted $u(t) = u(t, 0)$, $v(t) = v(t, t)$, and $\alpha$, $\beta$ are already given after Eq. (2).

The nonequilibrium Green’s function $u(t, 0)$ satisfies the following integro-differential equation

\begin{equation}
\dot{u}(t, 0) + i\omega_0 u(t, 0) + \int_0^t d\tau g(t, \tau) u(\tau, 0) = 0,
\end{equation}

where $\omega_0$ is the frequency of the optical field. The integral kernel $g(t, \tau)$ describes the non-Markovian back-action between the system and the environment, and can be determined by the spectral density $J(\omega)$ through the relations: $g(t, \tau) = \int_0^\infty d\omega J(\omega)e^{-i\omega(t-\tau)}$. The spectral density of the environment is defined by $J(\omega) = \sum_k |V_k|^2 \delta(\omega - \omega_k)$, where $V_k$ specifies the coupling between the system and the $k$-th mode of the environment.

The exact analytic solution of the integro-differential equation\cite{23} is recently given in\cite{22} as $u(t, t_0) = \int_{t_0}^\infty d\omega D(\omega) \exp[-i\omega(t-t_0)]$ with

\begin{equation}
D(\omega) = D_L(\omega) + D_c(\omega),
\end{equation}

where $D_L(\omega) = Z\delta(\omega - \omega_b)$ is the contribution of a dissipationless localized mode, and $D_c(\omega) = J(\omega)/\{\omega - \omega_b - \Delta(\omega)\}^2 + \pi\beta^2 F(\omega)$ is the continuous part of the spectra. Here $\Delta(\omega) = \mathcal{P} \int_0^\infty d\omega' J(\omega')/\{\omega' - \omega_b\}^2$ is a principal-value integral. The localized mode frequency $\omega_b$ is determined by the pole condition $\omega_b - \omega_b - \Delta(\omega_b) = 0$, and $Z = [1 - \Sigma'(\omega_b)]^{-1}$ corresponds to the residue at the pole, which gives the amplitude of the localized mode.

Here, $\Sigma(\omega + i0^+) = \int_0^\infty d\omega' \frac{J(\omega')}{\omega - \omega'} = \Delta(\omega) + i\pi J(\omega)$ is the self-energy correction induced by the system-environment coupling. We consider an Ohmic spectral density $J(\omega) = \eta\omega\exp(-\omega/\omega_c)$, where $\eta$ is the coupling strength between the system and the environment, and $\omega_c$ is the frequency cutoff of the environment spectra. For this case, a localized mode appears when the coupling strength $\eta$ exceeds some critical value $\eta_c = \omega_b/\omega_c$. With the above specification, the exact second-order correlation function $g^{(2)}(t, t + \tau)$ can be calculated explicitly and exactly through Eq. (10), where the numerator $\langle a^\dagger(t)a(t + \tau)a(t_1)a(t_2)\rangle$ is given by Eq. (11) and the denominator is determined through $\langle a^\dagger(t)a(t)\rangle = |u(t)|^2 \alpha + v(t)$.

![FIG. 1. (Color online) Transient dynamics of $g^{(2)}(t, t + \tau)$ is shown for two different system-environment coupling strengths: (a) the weak coupling ($\eta = 0.5\eta_c$) and (b) the strong coupling ($\eta = 1.5\eta_c$). Different curves represent different values of $t$ shown by the color legends. The other parameters are taken as $\omega_c = 5\omega_0$, $k_B T = 2.0\hbar\omega_0$, and the system is considered to be in an initial photon number state $|n\rangle = |5\rangle$.](image-url)
the long delay-time limit ($\omega_0 \tau > 20$), and the results also become independent of $t$ (the steady-state solution). Physically, this is intuitive from the density matrix evolution of the optical field. For the weak coupling ($\eta < \eta_c$), the initial Fock state will always evolve to the following steady-state at thermal equilibrium
\[ \rho(t_s) = \frac{\sum_{n=0}^{\infty} [v(t_s)]^n}{1 + [v(t_s)]^{n+1}} |n\rangle\langle n|, \]
which is solely determined by the steady-state value $v(t_s) = \tilde{n}(\omega_0, T)$ as the steady-state value of $u(t_s) \to 0$. For a thermal field given by Eq. (8), the steady-state correlation $g^{(2)}_{ss}(\tau)$ decays monotonically with increasing $\tau$, showing the familiar bunching statistics with $g^{(2)}(0) \to 2$ and $g^{(2)}(\infty) \to 1$. This result is consistent with the weak coupling steady-state limit of $g^{(2)}$ given by Eq. (9).

However, for the strong system-environment coupling ($\eta > \eta_c$), the transient dynamics of $g^{(2)}(t, t + \tau)$ is significantly different (see Fig. 1b). In this case, the magnitude of $g^{(2)}(t, t + \tau)$ decays first with increasing the delay-time $\tau$. This corresponds to the bunching statistics. Then $g^{(2)}(t, t + \tau)$ starts rising with increasing the delay-time $\tau$ and oscillate in $\tau$ (see Fig. 1b). This result is very different from the result obtained in the weak-coupling case, where $g^{(2)}(t, t + \tau)$ exhibits the antibunching statistics in the short-$\tau$ regime, as discussed above. Physically, such bunching-antibunching oscillation is a manifestation of non-Markovian dynamics of the optical field characterized by the reduced or enhanced correlation (4), originating from a localized mode contribution of $u(t, t_0)$ given in Eq. (4). In fact, the two-time correlation function (4) correlates a past event with its future providing useful information about the system-environment back-action memory effect. It was shown [23] that the non-Markovian dynamics of an open quantum system is fully characterized by the two-time correlation functions $u(t, t_0)$ and $v(t, t')$, the two-time correlation function is recently used to define a measure of non-Markovianity [31].

In Fig. 2 we show how the transient dynamics of $g^{(2)}(t, t + \tau)$ depends on the initial environment temperature $T$. We consider again a weak-coupling case $\eta < \eta_c$ (see Fig. 2a) and a strong-coupling case $\eta > \eta_c$ (see Figs. 2b). The temperature dependence of $g^{(2)}(t, t + \tau)$ comes from the non-equilibrium fluctuation-dissipation theorem through the initial particle number distribution $\tilde{n}(\omega, T)$, given by Eq. (8). Different curves in Fig. 2 represent different initial environment temperatures as shown by the color legend of each plot, where $T_s$ is defined as a dimensionless temperature $T_s = k_B T/\hbar \omega_0$. When the coupling strength is weak ($\eta < \eta_c$), we observe photon antibunching for low temperatures ($T_s < 5$) in the short delay-time regime ($\omega_0 \tau < 10$). Here, a rising magnitude of $g^{(2)}(t, t + \tau)$ is observed with increasing delay-time $\tau$ (see Fig. 2a). The transient antibunching effect is gradually suppressed when the initial environment temperature is increased. For a high initial environment temperature ($T_s > 5$), the transient dynamics of $g^{(2)}(t, t + \tau)$ shows a monotonous decay of magnitude with $\tau$, manifesting a photon bunching statistics. Figure 2b essentially manifests the transition from antibunching to bunching photon statistics through the initial environment temperature dependence of the correlation $g^{(2)}(t, t + \tau)$.

For the strong coupling case ($\eta > \eta_c$), $g^{(2)}(t, t + \tau)$ shows again a short-$\tau$ oscillatory behavior due to the non-Markovian dynamics from the localized mode contribution (see Fig. 2b). This bunching-antibunching oscillation in the strong-coupling regime becomes more visible as the initial environment temperature becomes higher and higher. In this case, we also find that the correlation function $g^{(2)}(t, t + \tau)$ will saturate to various long-time steady-state values (memory effect). This is unlike the case for the weak-coupling, where $g^{(2)}(t, t + \tau)$ asymptotically approaches to unity in the steady-state limit.

Moreover, we observed an unusual nonequilibrium steady-state situation in the strong-coupling case. Usually, the steady-state limit of $g^{(2)}(t, t + \tau)$ discussed in the literature [2] is valid only for the weak-coupling regime, where the two-time correlation function is calculated in the Markov limit. In such a situation, one always obtains photon bunching statistics [1] in the steady-state limit, as discussed above, and photon antibunching never shows up. In Fig. 3 going beyond the weak coupling, we show the exact dynamics of $g^{(2)}(t, t + \tau)$ in the steady-state limit. Different solid curves in Fig. 3 demonstrate different initial temperature-dependence of the environment. The steady-state limit of $g^{(2)}(t, t + \tau)$ in the strong coupling shows the striking photon bunching-antibunching oscillations. These oscillations survive longer (long-$\tau$ regime) at low temperatures, but also persist even at a high temperature ($k_B T = 100 \hbar \omega_0$). Actually, for the
FIG. 3. (Color online) We plot \( g^{(2)}(t, t + \tau) \) in the steady-state limit \( (t \to \infty) \) for strong system-environment coupling. Different curves represent different values of temperature \( T \), shown by the color legends. The other parameters are taken as \( \omega_c = 5\omega_0 \), and the system is considered to be in an initial photon number state \( \langle n_0 \rangle = |5\rangle \).

strong-coupling case (\( \eta > \eta_c \)), the initial Fock state will evolve to a more complex steady-state determined by the steady-state values of both \( u(t_s) = Z \exp(-i\omega_0 t_s) \) and \( v(t_s) = \int_0^\infty d\omega [\bar{D}_a(\omega) + \bar{D}_c(\omega)] \tilde{n}(\omega, T) \), where \( \bar{D}_a(\omega) = J(\omega) 2/(\omega - \omega_b)^2 \):

\[
\rho(t) = \sum_{n=0}^\infty p_n(t_s)|n\rangle\langle n|,
\]

where

\[
p_n(t_s) = \frac{[v(t_s)]^n}{1 + [v(t_s)]} \Omega(t_s)^n \times \sum_{k=0}^{\min(n,n_0)} \binom{n_0}{k} \binom{n}{k} \frac{\Omega(t_s)}{[v(t_s) - 1 + \Omega(t_s)]^k},
\]

and \( \Omega(t_s) = |u(t_s)|^2/[1 + v(t_s)] \). This is a nonequilibrium state that always depends on the initial state \( \langle n_0 \rangle \). In other words, the system cannot approach to a thermal equilibrium state, due to the existence of the localized mode, as shown in [21,32,33].

In conclusion, we have shown the exact transient dynamics of photon statistics for an optical field coupled to a general non-Markovian environment. We observe a nontrivial transition from antibunching to bunching statistics in the transient regime when the field interacts weakly with the environment. For the strong system-environment coupling, we find an interesting nonequilibrium oscillatory dynamics between the photon bunching and the antibunching statistics that persists for arbitrary initial temperature of the environment. Because experimentally one can prepare the system in a Fock state [28,29], the nontrivial nonequilibrium dynamics of the photon statistics discovered in this work can be experimentally measured.

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