Constraining subleading soft gluon and graviton theorems

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Abstract

We show that the form of the recently proposed subleading soft graviton and gluon theorems in any dimension are severely constrained by elementary arguments based on Poincaré and gauge invariance as well as a self-consistency condition arising from the distributional nature of scattering amplitudes. Combined with the assumption of a local form as it would arise from a Ward identity the orbital part of the subleading operators is completely fixed by the leading universal Weinberg soft pole behavior. The polarization part of the differential subleading soft operators in turn is determined up to a single numerical factor for each hard leg at every order in the soft momentum expansion. In four dimensions, factorization of the Lorentz group allows to fix the subleading operators completely.
1 Introduction

Gluon and graviton scattering amplitudes display a universal factorization behavior when a gluon (respectively photon) \[^1\] or a graviton \[^2\] becomes soft, as was shown more than 50 years ago. This leading soft pole behavior is known as Weinberg’s soft theorem \[^2,3\]. Recently an interesting proposal was put forward by Cachazo and Strominger \[^4\] in which they conjectured the extension of this theorem for gravitons to subleading and sub-subleading orders in the soft momentum expansion. The proposal was shown to hold at tree-level using the BCFW recursion relations \[^5\]. Tree-level gluon amplitudes exhibit a very similar subleading universal behavior as pointed out in ref. \[^6\] using a proof identical to that of gravitons. In fact such a subleading gluon relation was argued to exist already in refs. \[^1,7\]. Similarly, the subleading soft graviton behavior was reported already in 1968 \[^8\], see also the more recent discussion \[^9\].

Collectively these (new) subleading soft theorems state the existence of certain universal differential operators in momenta and polarizations acting on a hard \(n\)-point amplitude, which capture the subleading or even sub-subleading terms in the soft limit of the associated \((n+1)\)-point amplitude with one leg taken soft. For the case of gravity the subleading soft theorems have been conjectured to be Ward identities of a new symmetry of the quantum gravity \(S\)-matrix \[^10,11,4\], namely the extension of the Bondi, van der Burg, Metzner and Sachs (BMS) symmetry \[^12\] to a Virasoro symmetry \[^13\] acting on a sphere at past and future infinity. This connection was first established in ref. \[^11\] for the leading soft Weinberg pole term \[^3\]. Recently a connection of the first subleading graviton theorem to the superrotation symmetry of extended BMS symmetry \[^13\] was reported \[^14\]. Interesting steps towards a better understanding of such a relation through dual holographic \[^15\] or ambitwistor \[^16\] string models also appeared recently.

Inspired by these results a series of papers appeared \[^6,17–23\]. Very interestingly the validity of the gluon and graviton subleading theorems was shown to hold at any dimension for tree-level amplitudes \[^17,22\]. This is puzzling in the context of the conjectured relation between gravity and extended BMS symmetry which is clearly special to four dimensions. Similarly, it has been claimed in \[^20\] that the subleading soft theorem for gauge theory is related to the conformal symmetry of tree-level gluon amplitudes, which again contradicts the existence of the subleading theorem in general dimensions.

An important question is whether the subleading soft theorems receive radiative corrections. Loop-level modifications of the leading soft-gluon theorem are known to arise due to infrared singularities \[^24\]. Whereas the leading Weinberg soft graviton is protected, the subleading operators were shown to be corrected in refs. \[^18,19\]. This argument, however, was challenged in the recent work \[^21\], where the authors argue for an order-of-limits problem: Taking the soft limit prior to sending the dimensional regulator to zero would not cause any corrections to the soft theorems.

In this note we hope to shed some light on the above questions from a different point of view. We will show that rather elementary arguments can take one quite far. Beyond the obvious Poincaré and gauge invariance we will assume a certain local form of the soft operators (as it would follow from a Ward identity). In conjunction with a self-consistency condition of the theorems arising from the distributional nature of scattering amplitudes, the form of
the subleading operators is strongly constrained. Our argument applies to all dimensions and determines the orbital part of the subleading operators uniquely from the form of the known leading pole functions.

While our argument does not prove the existence of a universal subleading soft gluon and graviton theorems, it states that if a such a behavior exists, it is inevitably of the form proposed recently. Therefore the only input needed from a potential new symmetry of the quantum gravity or gauge theory S-matrix is the mere existence of a Ward identity pertaining to subleading orders in the soft limit. The form of the orbital part of the theorems is then fixed – at least at tree level.

This paper is organized as follows. In section 2 we provide our general arguments and derive the central distributional constraint linking the subleading operator in the soft theorems to the leading one. In section 3 we apply the established constraints to the subleading soft operators for gluons and gravitons and show that they are capable of fixing the orbital piece while strongly constraining the polarization part. In section 4 we apply the same reasoning to the sub-subleading soft graviton operator yielding identical results. In section 5 we specialize to four dimensions and employ the spinor-helicity formalism in order to find that the same line of arguments now entirely determine the subleading soft operators. We end with a discussion in section 6.

2 General arguments

Let us briefly summarize the subleading soft theorems and our central argument. We will consider amplitudes in \( D \)-dimensional pure gauge and gravity theories denoted by \( A_n = \delta^{(D)}(P) A_n \), where \( P = \sum_{n=1}^{n} p_n \) is the total momentum. The soft momentum of leg \( n + 1 \) is taken to be \( \epsilon q^\mu \), which allows us to control the soft limit by sending \( \epsilon \) to zero. The subleading soft theorems may be stated as

\[
A_{n+1}(p_1, \ldots, p_n, \epsilon q) = S^{[l]}(p_1, \ldots, p_n, \epsilon q) A_n(p_1, \ldots, p_n) + \mathcal{O}(\epsilon^l).
\]  

where we call \( S^{[l]} \) a soft operator. The integer parameter \( l \) controls the expansion in powers of the soft momentum to which the theorem holds.

This theorem has been known to hold at leading order \( (l = 0) \) for more than 50 years. The corresponding soft factors in gauge theory \([1]\) and gravity \([2]\) read

\[
S^{[0]}(\epsilon q) = \begin{cases} 
\frac{1}{\epsilon} S_{YM}^{(0)} = \frac{1}{\epsilon} \left( \frac{E \cdot p_1}{p_1 \cdot q} - \frac{E \cdot p_n}{p_n \cdot q} \right) & \text{Yang–Mills theory (color ordered)} \\
\frac{1}{\epsilon} S_{G}^{(0)} = \frac{1}{\epsilon} \sum_{a=1}^{n} E_{\mu \nu} p_1^\mu p_a^\nu \frac{p_n \cdot q}{p_n \cdot q} & \text{Gravity}
\end{cases}
\]

where \( E_{\mu} \) and \( E_{\mu \nu} \) denote the gluon or graviton polarization of the soft leg respectively and the arguments \( \{ p_a \} \) of \( S^{[0]}(\epsilon q) \) have been suppressed for brevity. Note that we are working with color

\footnote{We only consider amplitudes where the external particles are of the same type.}
ordered gauge theory amplitude. The soft limit is singular and the pole terms are universal. The graviton pole function \( S_{G}^{(0)} \) does not receive radiative corrections [3, 26]. In refs. [4, 6, 17, 22] the theorem in eqn. (1) has been demonstrated to extend to \( l = 1 \) in \( D \)-dimensional gauge theory and even \( l = 2 \) in \( D \)-dimensional gravity at least at tree-level

\[
S^{[l]}(\epsilon q) = \begin{cases} 
\frac{1}{\epsilon} S_{YM}^{(0)} + S_{YM}^{(1)} & \text{Yang-Mills theory (} l = 1 \text{)} \\
\frac{1}{\epsilon} S_{G}^{(0)} + S_{G}^{(1)} + \epsilon S_{G}^{(2)} & \text{Gravity (} l = 2 \text{)}.
\end{cases}
\]  

(3)

The operators \( S_{YM}^{(1)}, S_{G}^{(1)} \) and \( S_{G}^{(2)} \) are differential operators in the kinematical data of the hard legs and take a local form. Here with locality we want to refer to the fact that they are sums over terms depending on a single hard leg and the soft data only, i.e.

\[
S^{[l]} = \sum_{a} S_{a}^{[l]}(E, \epsilon q; p_{a}, \partial p_{a}, E_{a}, \partial E_{a}).
\]  

(4)

This situation is just as one would expect it to arise from a Ward identity. Naturally, the form of \( S^{[l]} \) is strongly restricted by Poincaré and gauge invariance. While Poincaré invariance implies linearity in the polarization tensors, gauge invariance demands vanishing of \( S^{[l]}A_{n} \) order by order in \( \epsilon \) upon replacing the polarizations by a gauge transformation.

There is, however, a further less obvious but elementary constraint on \( S^{[l]} \) emerging from the distributional nature of amplitudes. The left hand and the right hand side of the soft theorem eqn. (1) depend on Dirac delta functions which differ in their arguments by the soft momentum \( \epsilon q \). While this is no issue at leading order (\( l = 0 \)), it becomes relevant for the subleading corrections. Therefore, in order for the subleading soft theorems to be consistent, we need to require that

\[
S^{[l]}(\epsilon q) \delta^{D}(P) = \delta^{D}(P + \epsilon q) S^{[l]}(\epsilon q),
\]  

(5)

where the soft operator \( S^{[l]}(\epsilon q) \) acting on the reduced amplitude \( A_{n} \) could differ a priori from the soft operator \( S^{[l]}(\epsilon q) \) acting on the full amplitude \( A_{n} \). Interestingly, the results reported in the literature so far indicate that the \( S^{[l]}(\epsilon q) \) and \( S^{[l]}(\epsilon q) \) are equivalent. We shall show that this has to be the case.

Distinguishing different orders of \( \epsilon \), the soft theorem eqn. (1) implies the relations

\[
\lim_{\epsilon \to 0} \left( \epsilon A_{n+1}(\epsilon) \right) = S^{(0)} A_{n},
\]  

(6a)

\[
\lim_{\epsilon \to 0} \left( A_{n+1}(\epsilon) - \frac{1}{\epsilon} S^{(0)} A_{n} \right) = S^{(1)} A_{n},
\]  

(6b)

\[
\lim_{\epsilon \to 0} \left( \frac{1}{\epsilon} A_{n+1}(\epsilon) - \frac{1}{\epsilon^2} S^{(0)} A_{n} - \frac{1}{\epsilon} S^{(1)} A_{n} \right) = S^{(2)} A_{n}.
\]  

(6c)

\footnote{See e.g. [25] for a textbook treatment.}

\footnote{See in particular [13, 21] for a discussion of different prescriptions related to this issue.}
In order to derive the implications of eqns. (3) for the soft operators \( S^{(0)} \) and \( S^{(1)} \), it is useful to
Laurent expand both the reduced amplitude \( A_{n+1} \) as well as its associated delta function:

\[
A_{n+1}(\epsilon) = \frac{1}{\epsilon} A^{(-1)}_{n+1} + A^{(0)}_{n+1} + \epsilon A^{(1)}_{n+1} + \mathcal{O}(\epsilon^2)
\]

\[
\delta^{(D)}(P + \epsilon q) = \delta^{(D)}(P) + \epsilon(q \cdot \partial) \delta^{(D)}(P) + \mathcal{O}(\epsilon),
\]

where we introduced the shorthand notation \( q \cdot \partial = q^\mu \frac{\partial}{\partial q^\mu} \). Let us now substitute these expansions into eqns. (6a) and (6b). After noting that

\[
[S^{(0)}, \delta^{(D)}(P)] = 0,
\]

due to the form of \( S^{(0)} \) in eqn. (2) being a mere function, one finds for the reduced amplitude from (6b)

\[
\lim_{\epsilon \to 0} A_{n+1}(\epsilon) = \frac{1}{\epsilon} S^{(0)} A_n + A^{(0)}_{n+1} + \mathcal{O}(\epsilon),
\]

For now, we will leave the form of the subleading contribution \( A^{(0)}_{n+1} \) undetermined. Eqn. (6a) then simply implies

\[
A^{(-1)}_{n+1} = S^{(0)} A_n.
\]

Eqn. (6b) leads to

\[
S^{(1)} A_n = \lim_{\epsilon \to 0} \left[ A_{n+1} - \frac{1}{\epsilon} S^{(0)} A_n \right],
\]

\[
= \lim_{\epsilon \to 0} \left[ \left( \delta^{(D)}(P) + \epsilon(q \cdot \partial) \delta^{(D)}(P) \right) \left( \frac{1}{\epsilon} A^{(-1)}_{n+1} + A^{(0)}_{n+1} \right) - \frac{1}{\epsilon} S^{(0)} A_n \right],
\]

where we kept only terms not vanishing as \( \epsilon \to 0 \). Now we can remove the limit on the right-hand side, whereas in the left-hand side we can commute \( S^{(1)} \) past the delta function to obtain

\[
[S^{(1)}, \delta^{(D)}(P)] A_n + \delta^{(D)}(P) S^{(1)} A_n = \delta^{(D)}(P) A^{(0)}_{n+1} + S^{(0)} \left( (q \cdot \partial) \delta^{(D)}(P) \right) A_n.
\]

At this point, several comments are in order. Most importantly, \( \delta \) and \( \delta' \) may be treated as independent distributions if one takes partial integration identities into account. Therefore we will have to match their respective coefficients in order for this equation to be satisfied. Next, \( S^{(1)} \) must be a differential operator in the momenta \( p_a \); eqn. (13) implies then that

\[
[S^{(1)}, \delta^{(D)}(P)] = S^{(0)} \left( q \cdot \partial \delta^{(D)}(P) \right) + \chi \delta^{(D)}(P),
\]

\[
A^{(0)}_{n+1} = \left( S^{(1)} - \chi \right) A_n,
\]

where \( \chi \) is an undetermined function. Repeating the analysis for eqn. (6c) (extracting the singular behavior from the reduced amplitude, expanding in \( \epsilon \) and matching coefficients of the delta function and its derivatives) leads to

\[
[S^{(2)}, \delta^{(D)}(P)] = \frac{1}{2} S^{(0)} \left( (q \cdot \partial)^2 \delta^{(D)}(P) \right) + \left( q \cdot \partial \delta^{(D)}(P) \right) S^{(1)} + \chi' \delta^{(D)}(P),
\]

\[
A^{(1)}_{n+1} = \left( S^{(2)} - \chi' \right) A_n.
\]
We see that, the above equations constrain the subleading soft terms by relating their form to the leading soft function \( S^{(0)} \). We will refer to those equations as *distributional constraints*. Note also that the difference of the soft operators \( S^{(l)}(\epsilon q) \) and \( \tilde{S}^{(l)}(\epsilon q) \) mentioned in (5) is captured by – a priori – arbitrary functions \( \chi \) and \( \chi' \).

It is clear that the distributional constraints can only constrain the part of \( S^{(l)} \) that contains the derivatives with respect to the hard momenta. We call this piece the orbital part of \( S^{(l)} \) and write

\[
S^{(l)} = S^{(l)}_{\text{orb}} + S^{(l)}_{\text{polar}} + S^{(l)}_{\text{function}}
\]

with the orbital part

\[
S^{(l)}_{\text{orb}} = \sum_a S^{(l)}_{a \mu_1 \ldots \mu_l}(E, q; p_a) \frac{\partial}{\partial p_a^\mu_1} \ldots \frac{\partial}{\partial p_a^\mu_l}
\]

and the polarization part \( S^{(l)}_{\text{polar}} \) containing derivatives w.r.t. the polarizations \( E_a \). Finally \( S^{(l)}_{\text{function}} \) is a pure function of the soft and hard momenta linear in the soft polarization \( E \). It is not constrained by the distributional constraint as it commutes with the Dirac delta function.

As we are going to show below, distributional constraints, Poincaré, gauge invariance and the assumption of locality of \( S^{(l)} \) completely determine the orbital part \( S^{(l)}_{\text{orb}} \) of the soft operators in gauge theory and gravity in any dimensions. We now give a simple argument how to constrain also the remaining polarization part.

In order to treat gluon and graviton polarizations on an equal footing let us agree upon rewriting the graviton polarization of leg \( a \) as

\[
E_{a \mu \nu} \to E_{a \mu} E_{a \nu} \quad \text{with} \quad E_a \cdot E_a = 0 = p_a \cdot E_a.
\]

In four dimensions this no restriction at all, in general dimensions it is a formal agreement which we can always undo at any stage due to the fact that an amplitude is linear in the polarizations of all its legs. This replacement unifies gauge and gravity theory in the sense that the same operators act on the polarization degrees of freedom in both theories. Using this prescription, the operator representing a gauge transformation on leg \( a \) takes the form

\[
W_a := p_a \cdot \frac{\partial}{\partial E_a}
\]

and the Lorentz generators are represented as

\[
J^\mu\nu = \sum_a p^\mu_a \frac{\partial}{\partial p_a^\nu} + E_a^\mu \frac{\partial}{\partial E_a^\nu} - \mu \leftrightarrow \nu
\]

in both theories in any dimension\(^4\). In this language the polarization part \( S^{(l)}_{\text{polar}} \) depends on the differential operators \( E_{a \mu} \frac{\partial}{\partial E_a^\nu} \) in order to preserve linearity of the amplitude in the polarization \( E_a \).

\(^4\)Strictly speaking this operator does not generate the correct infinitesimal Lorentz transformation rule for the polarizations as these do not transform as vectors, see e.g. [2, 27]. Next to the vector transformation law there is an additional piece proportional to a gauge transformation in the form of \( W_a \). As this additional piece vanishes acting on amplitudes, the form of (22) is effectively correct.
Let us now consider gauge invariance of a fixed hard leg $a$ for the soft theorem eqn. (1)

$$0 = W_a A_{n+1}(p_1, \ldots, p_n, \epsilon q) = W_a \left( S^{[I]}(\epsilon q) A_n(p_1, \ldots, p_n) \right) = \left[ W_a, S^{[I]}(\epsilon q) \right] A_n = 0 .$$  \hspace{1cm} (23)$$

The orbital part $S^{[I]}_{\text{orb}}$ does not commute with $W_a$ due to the presence of operators $\partial_\mu p_\nu$. Therefore it needs to be completed to a gauge invariant structure. Employing the commutators

$$\left[ W_a, p_\mu \frac{\partial}{\partial p_\nu} \right] = -p_\mu \frac{\partial}{\partial E_a \nu}, \quad \left[ W_a, E_\mu \frac{\partial}{\partial E_a \nu} \right] = +p_\mu \frac{\partial}{\partial E_a \nu} .$$  \hspace{1cm} (24)$$

the unique linear differential operator in $p_a$ and $E_a$ commuting with $W_a$ reads

$$\Lambda_\mu^\nu := p_\mu \frac{\partial}{\partial p_a \nu} + E_\mu \frac{\partial}{\partial E_a \nu} ,$$  \hspace{1cm} (25)$$

which we shall use as building block in constraining $S^{[I]}$ below\footnote{Note that in fact we only need the weaker condition of $[W_a, S^{[I]}(\epsilon q)] \sim W_a$ in eqn. (23) as $W_a$ annihilates the amplitudes $A_n$. This is achieved by the operator $E_a \cdot \frac{\partial}{\partial E_a}$ which obeys $[W_a, E_a \cdot \frac{\partial}{\partial E_a}] = W_a$. However, as any amplitude is an eigenstate of the operator $E_a \cdot \frac{\partial}{\partial E_a}$ with eigenvalue one, including this operator in an ansatz for $S^{[I]}$ is tantamount to writing a function. We may therefore discard it in our analysis as functions of the kinematical data cannot be constrained.} Let us now turn to the explicit analysis.

### 3 Subleading soft operators

In this section we will apply the general framework outlined in the previous section to determine the subleading soft operators in both gauge theory and gravity. As derived in sec. 2 above, the subleading contribution should be fixed upon requiring locality, the distributional constraint and gauge invariance for the soft leg. The last two requirements translate into

$$\left[ S^{(1)}, \delta^{(D)}(P) \right] = S^{(0)}(q \cdot \partial) \delta^{(D)}(P) + \chi \delta^{(D)}(P) ,$$  \hspace{1cm} (26)$$

$$\left[ S^{(1)}, q \cdot \frac{\partial}{\partial E} \right] \cdot A_n = 0 .$$  \hspace{1cm} (27)$$

**Gauge theory.** In gauge theory the leading order soft factor is given by the universal Weinberg soft gluon function\footnote{Note that in fact we only need the weaker condition of $[W_a, S^{[I]}(\epsilon q)] \sim W_a$ in eqn. (23) as $W_a$ annihilates the amplitudes $A_n$. This is achieved by the operator $E_a \cdot \frac{\partial}{\partial E_a}$ which obeys $[W_a, E_a \cdot \frac{\partial}{\partial E_a}] = W_a$. However, as any amplitude is an eigenstate of the operator $E_a \cdot \frac{\partial}{\partial E_a}$ with eigenvalue one, including this operator in an ansatz for $S^{[I]}$ is tantamount to writing a function. We may therefore discard it in our analysis as functions of the kinematical data cannot be constrained.} $S^{(0)}_{\text{YM}}$

$$S^{(0)}_{\text{YM}} = \frac{p_1 \cdot E}{p_1 \cdot q} - \frac{p_n \cdot E}{p_n \cdot q}$$  \hspace{1cm} (28)$$

with the polarization vector $E_\mu$ for the soft particle. We begin with an ansatz for $S^{(1)}_{\text{YM}}$ reflecting the reasoning in sec. 2

$$S^{(1)}_{\text{YM}} = \sum_a E_\mu \Omega^{\mu\rho}_a \left( p_\nu a \cdot \frac{\partial}{\partial p_a \rho} + E_\nu a \cdot \frac{\partial}{\partial E_a \rho} \right) .$$  \hspace{1cm} (29)$$

Note that in fact we only need the weaker condition of $[W_a, S^{[I]}(\epsilon q)] \sim W_a$ in eqn. (23) as $W_a$ annihilates the amplitudes $A_n$. This is achieved by the operator $E_a \cdot \frac{\partial}{\partial E_a}$ which obeys $[W_a, E_a \cdot \frac{\partial}{\partial E_a}] = W_a$. However, as any amplitude is an eigenstate of the operator $E_a \cdot \frac{\partial}{\partial E_a}$ with eigenvalue one, including this operator in an ansatz for $S^{[I]}$ is tantamount to writing a function. We may therefore discard it in our analysis as functions of the kinematical data cannot be constrained.
Before imposing the constraints (26) and (27) we note that dimensional analysis and soft scaling requires $\Omega^{\mu\nu\rho}$ to be of mass dimension -1 and to be scale invariant w.r.t. the soft momentum $q$ respectively. In conjunction with the assumption of locality and $E \cdot q = E_a \cdot p_a = p_a^2 = 0$ we are left with the compact ansatz

$$\Omega_a^{\mu\nu\rho} = c_1^{(a)} \frac{p_a^\mu q^\nu q^\rho}{(q \cdot p_a)^2} + c_2^{(a)} \eta^{\mu\nu} q^\rho - c_3^{(a)} \eta^{\mu\rho} q^\nu$$

where the numbers $c_i^{(a)}$ are to be determined. In order to do so we first impose gauge invariance via (27) which leads to

$$0 = \sum_a (c_1^{(a)} + c_2^{(a)} + c_3^{(a)} ) \eta^{\mu\nu} q^\rho ( p_a^\mu \partial / \partial q_a^\rho + E_a^\nu \partial / \partial E_a^\rho ) A_n .$$

There is no way for this term to conspire to yield a Lorentz charge. Hence we conclude that $c_1^{(a)} + c_2^{(a)} + c_3^{(a)} = 0$. Turning to the distributional constraint (26) one now easily establishes

$$[ S_{YM}^{(1)} , \delta^{(D)} (P) ] = \left( \sum_a c_3^{(a)} \right) E \cdot \partial \delta^{(D)} (P) - \left( \sum_a c_3^{(a)} \frac{E \cdot p_a}{q \cdot p_a} \right) q \cdot \partial \delta^{(D)} (P)$$

where we have inserted eqn. (28) for $S_{YM}^{(0)}$. Solving for the undetermined coefficients, we find

$$c_3^{(1)} = -1 , \quad c_3^{(n)} = 1 , \quad c_3^{(a)} = 0 \quad \text{for} \quad a = 2 , \ldots , n - 1 ,$$

along with the vanishing of $\chi$, which implies the identity $S^{(1)} = \tilde{S}^{(1)}$ (cf. eqn. (5)). As $c_3^{(a)} = -c_1^{(a)} - c_2^{(a)}$ the differences of the remaining coefficients $c_i^{(a)} := c_i^{(a)} - c_3^{(a)}$ remain unconstrained. In fact they only couple to the polarization degrees of freedom, the orbital part of $S^{(1)}$ is completely determined. In summary we have established that

$$S_{YM}^{(1)} = \sum_{a=1,n,signed} \frac{E_a q_a}{p_a \cdot q} \left( p_a^\mu \partial / \partial p_a^\nu + E_a^\mu \partial / \partial E_a^\nu \right) - \mu \leftrightarrow \nu$$

$$+ \sum_a (E_a \cdot p_a + E_a \cdot q) \frac{1}{p_a \cdot q} q \cdot \partial / \partial E_a ,$$

where the undetermined coefficients $\tilde{c}^{(a)}$ are related to the previous ones via $\tilde{c}^{(1)} = c_1^{(1)} + \frac{1}{2}$, $\tilde{c}^{(n)} = c_1^{(n)} - \frac{1}{2}$ and $\tilde{c}^{(a)} = c_1^{(a)}$ for $a = 2 , \ldots , n - 1$. Note that the second sum is manifestly gauge invariant with respect to the soft and hard legs. Hence, the orbital part of the subleading soft operator is entirely determined by our constraints and coincides with the explicit tree-level computations in the literature. The polarization piece is constrained up to a single numerical factor for every hard leg.

Finally, let us briefly comment on the possible functional contribution $S_{YM}^{(1)}$ function at the subleading level. Assuming locality and dimensional arguments quickly rules out any contribution here.
Gravity. The analysis of the graviton soft operator is almost a carbon copy of the gauge theory one. The leading universal soft function for gravitons reads\cite{3}

\[ S_{G}^{(0)} = \sum_{a=1}^{n} \frac{E_{\mu\nu} p_{a}^{\mu} p_{a}^{\nu}}{q \cdot p_{a}}. \]  

We again start with an ansatz for \( S_{G}^{(1)} \) of the form

\[ S_{G}^{(1)} = \sum_{a} E_{\mu\nu} \Omega_{\mu\nu}^{a} \left( \frac{p_{a}^{\rho}}{q \cdot p_{a}} \frac{\partial}{\partial p_{a\sigma}} + \frac{E_{a}^{\rho}}{q \cdot p_{a}} \frac{\partial}{\partial E_{a\sigma}} \right). \]

Dimensional analysis requires \( \Omega_{\mu\nu}^{a} \) to be of mass dimension zero and to be scale invariant w.r.t. the soft momentum \( q \). This together with the assumption of locality and the relations \( E_{\mu\nu} q^{\nu} = E_{a} \cdot p_{a} = p_{a}^{2} = 0 \) leads us to the most general ansatz

\[ \Omega_{\mu\nu}^{a} = c_{1}^{(a)} \frac{p_{a}^{\mu} p_{a}^{\nu} q^{\rho} q^{\sigma}}{(q \cdot p_{a})^{2}} + c_{2}^{(a)} \frac{\eta^{\rho\mu}(p_{a}^{\nu} q^{\sigma})}{q \cdot p_{a}} + c_{3}^{(a)} \frac{\eta^{\sigma\mu}(p_{a}^{\nu} q^{\rho})}{q \cdot p_{a}} + c_{4}^{(a)} \frac{\eta^{\rho\nu}(\eta^{\mu\sigma})}{q \cdot p_{a}}, \]

with four undetermined numerical coefficients for each hard leg \( a \). Imposing gauge invariance for the soft leg amounts to the replacement \( E_{\mu\nu} \rightarrow \Lambda_{\mu}(q_{\nu}) \) in \eqref{35}. We then obtain the condition

\[ 0 = \frac{1}{2} \sum_{a} \left[ (2c_{1}^{(a)} + c_{2}^{(a)} + c_{3}^{(a)}) q^{\rho} q^{\sigma} \frac{\Lambda \cdot p_{a}}{q \cdot p_{a}} + (c_{3}^{(a)} + c_{4}^{(a)}) q^{\rho} \Lambda^{\sigma} + (c_{2}^{(a)} + c_{4}^{(a)}) q^{\sigma} \Lambda^{\rho} \right] \left( \frac{p_{a}^{\rho}}{q \cdot p_{a}} \frac{\partial}{\partial p_{a\sigma}} + \frac{E_{a}^{\rho}}{q \cdot p_{a}} \frac{\partial}{\partial E_{a\sigma}} \right) A_{n}. \]

The first term in the above requires \( 2c_{1}^{(a)} + c_{2}^{(a)} + c_{3}^{(a)} = 0 \). For the second and third term we have to be somewhat more careful. Here we have the possibility of these two terms conspiring to build up the total Lorentz generator \( J^{\rho\sigma} \) of \eqref{22} which annihilates \( A_{n} \). We thus require

\[ c_{3}^{(a)} + c_{4}^{(a)} = -c_{2}^{(a)} - c_{4}^{(a)} = c \]

with a universal constant \( c \) identical for all hard legs. We now move on to pose our distributional constraint \eqref{26} linking \( S_{G}^{(1)} \) to \( S_{G}^{(0)} \). One finds

\[ [S_{G}^{(1)}, \delta^{(D)}(P)] = \sum_{a} \left( c_{1}^{(a)} + c_{2}^{(a)} \right) \frac{\left( p_{a}^{\mu} p_{a}^{\nu} E_{\mu\nu} \right)}{q \cdot p_{a}} q \cdot \partial \delta^{(D)}(P) + c P^{\mu} E_{\mu}^{\nu} \frac{\partial}{\partial P^{\nu}} \delta^{(D)}(P) \\ = \frac{1}{S_{G}^{(0)}} q \cdot \partial \delta^{(D)}(P) + \chi \delta^{(D)}(P). \]

One nicely sees that the first term on the r.h.s. of the first line forms the leading Weinberg soft function for the uniform choice

\[ c_{1}^{(a)} + c_{2}^{(a)} = 1. \]
The following term vanishes in the distributional sense by the tracelessness of $E_{\mu \nu}$. And finally we again learn that the function $\chi = 0$ implying again the identity of $S^{[1]} = \tilde{S}^{[1]}$ in the sense of eqn. (3). The established three equations for the four unknowns may now be solved upon expressing everything in terms of $c^{(a)}_4$

$$c^{(a)}_1 = c^{(a)}_4, \quad c^{(a)}_2 = 1 - c^{(a)}_4, \quad c^{(a)}_3 = -1 - c^{(a)}_4. \quad (42)$$

One also checks that $c^{(a)}_3 + c^{(a)}_4 = 1$ in line with the above reasoning. Inserting this into the ansatz (36) yields the final result

$$S^{(1)}_G = \sum_{a=1}^{n} \left( \frac{p_a \cdot E}{p_a \cdot q} \left( \frac{1}{\partial p_a} \frac{\partial}{\partial p_a} + \frac{E_a}{\partial E_a} - \rho \leftrightarrow \sigma \right) \right) + \sum_{a=1}^{n} \tilde{c}^{(a)} \left( \frac{(E \cdot p)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a \right) \left[ \frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a} \right], \quad (43)$$

where we have renamed $c^{(a)}_4 = \tilde{c}^{(a)}$ and written the soft polarization $E_{\mu \nu} \rightarrow E_\mu E_\nu$ for compactness of notation.

We thus see that again the orbital part is completely determined and coincides with the results established in the literature for tree-level amplitudes. The polarization-dependent parts are constrained to one numerical factor for every hard leg, just as it was the case in gauge theory.

Finally, let us also comment on the possible functional contribution $S^{(1)}_G$ function in gravity. Assuming locality and dimensional arguments again rules out any contribution here as well.

### 4 Sub-subleading soft graviton operator

The discussion for the sub-subleading soft operator for graviton amplitudes is analogous to the subleading case. The starting point is an ansatz for $S^{(2)}_G$ of the form

$$S^{(2)}_G = \sum_{a=1}^{n} E_{\mu \nu} \Omega^a_{\mu \nu \rho \sigma \gamma \lambda} \Lambda_{a, \rho \sigma} \Lambda_{a, \gamma \lambda} \quad (44)$$

where we used $\Lambda_{a, \rho \sigma} := p_{a, \rho} \frac{\partial}{\partial p_a} + E_{a, \rho} \frac{\partial}{\partial E_a}$ as in (23). Again, $\Omega_a$ must obey some constraints; specifically, it must have mass dimension zero, it must vanish linearly in the limit $q \rightarrow 0$, it must be symmetric in the exchange $\mu \leftrightarrow \nu$ and it must be symmetric in the simultaneous exchange $\rho \leftrightarrow \sigma$. 

\[\text{In fact it is in accordance with the expression for } S^{(1)}_G \text{ given in ref. [19] and differs by an overall normalization factor in the expression of ref. [4].}\]
\[\rho \leftrightarrow \gamma, \ \sigma \leftrightarrow \lambda.\]  
The most general ansatz satisfying these constraints is

\[
\Omega^{\mu\nu\rho\sigma\gamma\lambda}_{a} = \frac{c_{1}^{(a)}}{(q \cdot p_{a})^{3}} \rho_{\mu} \rho_{\nu} q^{\rho} q^{\sigma} q^{\gamma} q^{\lambda} + \frac{c_{2}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\lambda} q^{\rho} q^{\gamma} + \frac{c_{3}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\gamma} q^{\rho} q^{\gamma}
\]

\[
+ \frac{c_{4}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\sigma} q^{\rho} q^{\gamma} + \frac{c_{5}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\gamma} q^{\rho} q^{\gamma} + \frac{c_{6}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\sigma} q^{\rho} q^{\gamma} + \frac{c_{7}^{(a)}}{(q \cdot p_{a})} \eta^{\rho(\mu} \eta^{\nu)\gamma} q^{\rho} q^{\gamma}.
\]  

(45)

Furthermore \(S_{G}^{(2)}\) must obey the distributional constraint eqn. (16) and the gauge invariance constraint on the soft leg. We recall these constraints here

\[
[S_{G}^{(2)}, \delta^{(D)}(P)] = \frac{1}{2} S_{G}^{(0)} \left((q \cdot \partial)^{2} \delta^{(D)}(P)\right) + \left(q \cdot \partial \delta^{(D)}(P)\right) S_{G}^{(1)} + \chi' \delta^{(D)}(P), \tag{46}
\]

\[
S_{G}^{(2)}[E \rightarrow \Lambda q] \cdot A_{n} = 0. \tag{47}
\]

Imposing these constraints yields a total of 5 linear equations for the seven unknowns \(c_{j}^{(a)}\) residing in the ansatz (45) at each leg. A tedious but straightforward computation shows that the solution is \(\chi' = 0\) and

\[
S_{G}^{(2)} = \frac{1}{2} \sum_{a=1}^{n} \frac{1}{q \cdot p_{a}} E^{\lambda \sigma} q^{\rho} q^{\gamma} J_{a, \rho \sigma} J_{a, \gamma \lambda} +
\]

\[
+ \sum_{a=1}^{n} \left\{ \frac{c_{1}^{(a)}}{q \cdot p_{a}} \left( \frac{(p_{a} \cdot E)(q \cdot E_{a})}{q \cdot p_{a}} - E \cdot E_{a} \right) \left( \frac{q \cdot \partial}{\partial E_{a}} \right)^{2}
\]

\[
+ \frac{c_{2}^{(a)}}{q \cdot p_{a}} \left( \frac{(p_{a} \cdot E)(q \cdot E_{a})}{q \cdot p_{a}} - E \cdot E_{a} \right) \left( \frac{E \cdot p_{a}}{q \cdot p_{a}} q \cdot \frac{\partial}{\partial E_{a}} - E \cdot \frac{\partial}{\partial E_{a}} \right) \times
\]

\[
\times \left( \frac{E_{a} \cdot q}{q \cdot p_{a}} q \cdot \frac{\partial}{\partial E_{a}} + q \cdot \frac{\partial}{\partial p_{a}} \right) \right\}
\]  

(48)

where for convenience we expressed the soft graviton polarization as \(E_{\mu \nu} = E_{\mu} E_{\nu}\) and \(J_{a, \rho \sigma} := p_{a, \rho} \frac{\partial}{\partial p_{a}} + E_{a, \rho} \frac{\partial}{\partial E_{a}} - (\rho \leftrightarrow \sigma)\) as before. In the notation of the ansatz eqn. (13) we have \(c_{1}^{(a)} = -c_{4}^{(a)} + c_{6}^{(a)}\) and \(c_{2}^{(a)} = c_{7}^{(a)}\) the undetermined parameter of \(S_{G}^{(1)}\).

Again we see that the orbital part of the sub-subleading soft graviton operator \(S_{G}^{(2)}\) is entirely determined. The polarization-dependent parts on the other hand are now determined up to two numerical factors for every hard leg; as already stated, the coefficient \(c_{2}^{(a)}\) equals \(\tilde{c}^{(a)}\), where the latter are the undetermined coefficients appearing in the final form of \(S_{G}^{(1)}\). We thus have one additional free coefficient for each hard leg. It is also worth noticing that the additional, polarization-dependent terms are manifestly gauge invariant.

\footnote{There is actually one additional equation that identifies the coefficients \(c_{4}^{(a)}\) in the ansatz with the undetermined coefficients \(\frac{1}{2} \tilde{c}^{(a)}\) appearing in the final form of \(S_{G}^{(1)}\). eqn. (13)}
5 Four dimensions and spinor helicity formalism

Let us now consider the four-dimensional case, where we can use the spinor-helicity formalism and obtain additional constraints from little-group scalings. Those constraints are particularly easy to access in four dimensions, because the Lorentz group factorizes into two parts acting on holomorphic and antiholomorphic spinors respectively.

**Gauge theory.** Taking a positive-helicity soft gluon for concreteness, its polarization vector can be expressed in terms of a holomorphic reference spinor \( \mu \) as

\[
E_{a\dot{a}}^{(+)\mu} = \frac{\mu_\alpha \tilde{\lambda}_{a\dot{\alpha}}}{\langle \mu | q \rangle}.
\]

(49)

The ansatz for \( S_{\text{YM}}^{(1)} \) in spinor-helicity variables reads

\[
S_{\text{YM}}^{(1)} = \sum_{a=1}^{n} E_{a\dot{a}}^{(+)} \left[ \Omega_a^{\alpha\dot{\alpha} \beta} \frac{\partial}{\partial \lambda_a^\beta} + \tilde{\Omega}_a^{\alpha\dot{\alpha} \dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\beta}}} \right].
\]

(50)

In order to yield the correct mass dimension for \( S_{\text{YM}}^{(1)} \), the coefficients \( \Omega_a \), \( \tilde{\Omega}_a \) must be of mass dimension \(-\frac{1}{2}\). In addition \( \Omega_a \) carries helicity \( \frac{1}{2} \) and \( \tilde{\Omega}_a \) helicity \( -\frac{1}{2} \) on leg \( a \). Moreover, terms where the open index \( \dot{\alpha} \) comes from \( \tilde{\lambda}_q \) do not contribute to \( S_{\text{YM}}^{(1)} \), as those terms vanish after contracting with the polarization tensor eqn. (49). Combining these constraints, the most general ansatz reads

\[
\Omega_a^{\alpha\dot{\alpha} \beta} = \frac{c_1^{(a)}}{\langle a | q \rangle} \langle a | q \rangle \lambda_a^\alpha \lambda_a^{\dot{\alpha}} \gamma_a^\beta,
\]

(51)

\[
\tilde{\Omega}_a^{\alpha\dot{\alpha} \dot{\beta}} = \frac{\tilde{c}_1^{(a)}}{\langle a | q \rangle} \langle a | q \rangle \lambda_a^{\alpha} \lambda_a^{\dot{\alpha}} \gamma_a^{\dot{\beta}} + \frac{c_2^{(a)}}{\langle a | q \rangle} \langle a | q \rangle \lambda_q^{\alpha} \lambda_a^{\dot{\alpha}} \gamma_q^{\dot{\beta}} + \frac{\tilde{c}_2^{(a)}}{\langle a | q \rangle} \langle a | q \rangle \lambda_q^{\alpha} \lambda_a^{\dot{\alpha}} \gamma_q^{\dot{\beta}}.
\]

(52)

Gauge invariance on the soft leg implies that the operator obtained by the substitution \( E_q \rightarrow q \) in \( S_{\text{YM}}^{(1)} \) annihilates the amplitude. The resulting operator is

\[
S_{\text{YM}}^{(1)}[E_q \rightarrow q] = -\sum_{a=1}^{n} \left[ c_1^{(a)} \lambda_a^{\beta} \frac{\partial}{\partial \lambda_a^\beta} + \tilde{c}_1^{(a)} \lambda_a^{\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\beta}}} \right].
\]

(53)

In principle we can allow the above operator to be any operator annihilating the \( n \)-point tree-level gluon amplitude. Since tree-level gluon amplitudes in four dimension are invariant under conformal transformations, we could in principle allow \( c_1^{(a)} = \tilde{c}_1^{(a)} = c \) for some fixed constant \( c \) for all hard legs \( a \), so that the above operator is the dilation operator \( \mathcal{D} \) [28].

---

8See e.g. [25] for a textbook treatment.

9Strictly speaking, what appears here is \( \mathcal{D} - 2 \); however, the constant piece could be in principle restored by adding a constant term \( \sum_a E_{a\dot{a}} \lambda_a^{\alpha} \lambda_{a\dot{\alpha}} \) to \( S_{\text{YM}}^{(1)} \). We will however see that, although allowed by gauge invariance, these terms are ruled out by the distributional constraint.
The distributional constraint eqn. (14) then reads
\[
\sum_{a=1}^{n} \left[ 2c_{1} \frac{\langle \alpha a \rangle}{\langle a q \rangle} \lambda^{\alpha}_{a} \hat{\lambda}^{\dot{\alpha}}_{a} + (\bar{c}_{2}^{(a)} + \bar{c}_{3}^{(a)}) \frac{1}{\langle a q \rangle} \lambda^{\alpha}_{a} \hat{\lambda}^{\dot{\alpha}}_{a} \right] \frac{\partial}{\partial P^{\alpha \dot{\alpha}}} \delta^{4}(P) = \frac{1}{\langle n q \rangle} \left( \lambda^{\alpha}_{q} \hat{\lambda}^{\dot{\alpha}}_{q} \frac{\partial}{\partial P^{\alpha \dot{\alpha}}} \delta^{4}(P) \right) + \chi \delta^{4}(P),
\]
where we have inserted the spinor-helicity form of \( S^{(0)}_{YM} \) on the r.h.s. Since the first term in the l.h.s. cannot conspire to build a Lorentz generator, we see that upon using Schouten’s identity the solution to this equation is
\[
k = 0 , \quad \chi = 0 , \quad \bar{c}_{2}^{(a)} + \bar{c}_{3}^{(a)} = \begin{cases} 1 & \text{for } a = 1, n , \\ 0 & \text{otherwise} . \end{cases} \quad (54)
\]
This leads to the known form for the four-dimensional subleading soft factor for gluon amplitudes,
\[
S_{YM}^{(1)} = \frac{\hat{\lambda}^{\dot{\alpha}}_{q}}{\langle q 1 \rangle} \frac{\partial}{\partial \lambda^{\alpha}_{1}} - \frac{\hat{\lambda}^{\dot{\alpha}}_{q}}{\langle q n \rangle} \frac{\partial}{\partial \lambda^{\alpha}_{n}} . \quad (55)
\]
**Gravity.** For what concerns graviton amplitudes, we consider a positive-helicity soft graviton; the polarization vector is expressed in terms of two reference spinors \( \lambda_{x} \) and \( \lambda_{y} \) as
\[
E_{\alpha \dot{\alpha} \beta \dot{\beta}}^{(+)} := \frac{1}{\langle x q \rangle \langle y q \rangle} \left( \lambda_{x,\alpha} \lambda_{y,\beta} + \lambda_{y,\alpha} \lambda_{x,\beta} \right) \hat{\lambda}_{s,\dot{\alpha}} \hat{\lambda}_{s,\dot{\beta}} , \quad (56)
\]
We now consider the usual local first order ansatz for \( S_{G}^{(1)} \)
\[
S_{G}^{(1)} = \sum_{a=1}^{n} E_{\alpha \dot{\alpha} \beta \dot{\beta}}^{(+)} \left[ \Omega_{a}^{\alpha \dot{\alpha} \beta \dot{\beta}} \frac{\partial}{\partial \lambda^{\alpha}_{a}} + \tilde{\Omega}_{a}^{\alpha \dot{\alpha} \beta \dot{\beta}} \frac{\partial}{\partial \lambda^{\dot{\alpha}}_{a}} \right] . \quad (57)
\]
Again \( \Omega_{a} \) and \( \tilde{\Omega}_{a} \) must obey some constraints. The mass dimensions are \( [\Omega_{a}] = [\tilde{\Omega}_{a}] = \frac{1}{2} \), the helicity of the soft leg should be zero for both \( \Omega_{a} \) and \( \tilde{\Omega}_{a} \) and the helicities for leg \( a \) are \( \frac{1}{2} \) and \( -\frac{1}{2} \) for \( \Omega_{a} \) (\( \tilde{\Omega}_{a} \)). Moreover, the open indices \( \dot{\alpha}, \dot{\beta} \) cannot come from \( \tilde{\lambda}_{a} \) and both \( \Omega_{a} \) and \( \tilde{\Omega}_{a} \) must be symmetric in the pairs \( (\alpha, \beta) \) and \( (\dot{\alpha}, \dot{\beta}) \) These constraints imply that the possible forms of \( \Omega_{a} \) and \( \tilde{\Omega}_{a} \) are
\[
\Omega_{a}^{\alpha \dot{\alpha} \beta \dot{\beta}} = \frac{c_{1}^{(a)}}{\langle a q \rangle} \hat{\lambda}_{a}^{\dot{\alpha}} \lambda_{a}^{\alpha} \lambda_{a}^{\dot{\beta}} \lambda_{a}^{\beta} , \quad (58)
\]
\[
\tilde{\Omega}_{a}^{\alpha \dot{\alpha} \beta \dot{\beta}} = \frac{\bar{c}_{1}^{(a)}}{\langle a q \rangle} \hat{\lambda}_{a}^{\dot{\alpha}} \lambda_{a}^{\alpha} \lambda_{a}^{\dot{\beta}} \lambda_{a}^{\beta} + \frac{\bar{c}_{2}^{(a)}}{\langle a q \rangle} \hat{\lambda}_{a}^{\dot{\alpha}} \lambda_{a}^{\alpha} \lambda_{a}^{\dot{\beta}} \hat{\lambda}_{a}^{\beta} + \frac{\bar{c}_{3}^{(a)}}{\langle a q \rangle} \bar{c}^{(a)} \hat{\lambda}_{a}^{\dot{\beta}} \lambda_{a}^{\alpha} \lambda_{a}^{\dot{\alpha}} \hat{\lambda}_{a}^{\beta} . \quad (59)
\]
\(^{10}\)Notice that in \( S_{YM}^{(1)} \) only the combination \( \bar{c}_{2}^{(a)} + \bar{c}_{3}^{(a)} \) appears once \( \Omega_{a} \) is contracted with the polarization \( E^{(+)} \), therefore we can consider this sum as a single coefficient.
An infinitesimal gauge transformation amounts to the shift
\[ \lambda_x \rightarrow \lambda_x + \eta \lambda_q , \quad \lambda_y \rightarrow \lambda_y + \eta' \lambda_q , \]
for some (infinitesimal) \( \eta, \eta' \). Gauge invariance then implies that
\[ c_1^{(a)} = \bar{c}_1^{(a)} = 0 , \quad c_3^{(a)} - c_2^{(a)} = c , \quad \forall a \]
for some universal constant \( c \). \( S_G^{(1)} \) then reads\(^{11}\)
\[ S_G^{(1)} = c \sum_{a=1}^{n} \frac{[q a]}{\langle a q \rangle} \frac{1}{\langle x q \rangle \langle y q \rangle} \left( \langle a x \rangle \langle q y \rangle + \langle a y \rangle \langle q x \rangle \right) \tilde{\lambda}_q^\gamma \frac{\partial}{\partial \tilde{\lambda}_a^\gamma} \]
(62)
We can now impose the distributional constraint again, eqn. (14). This reads (after using Schouten’s identity)
\[ \left( \frac{\partial}{\partial P^{\gamma\gamma}} \delta^4(P) \right) \left\{ \sum_{a=1}^{n} 2k \frac{\langle x a \rangle \langle y a \rangle [q a]}{\langle a q \rangle} \lambda_q^\gamma \tilde{\lambda}_q^\gamma - \frac{k}{\langle x q \rangle \langle y q \rangle} \tilde{\lambda}_q^\gamma \lambda_q^\gamma + \chi \delta^4(P) \right\} \]
\[ = \left( \lambda_q^\gamma \tilde{\lambda}_q^\gamma \frac{\partial}{\partial P^{\gamma\gamma}} \delta^4(P) \right) \sum_{a=1}^{n} \frac{\langle x a \rangle \langle y a \rangle [q a]}{\langle a q \rangle} + \chi \delta^4(P) , \]
(63)
where we wrote explicitly the form of \( S_G^{(0)} \) in spinor-helicity variables. Notice that the second term in the first line is zero in the distributional sense. We may therefore conclude that the solution is
\[ \chi = 0 , \quad c = -\frac{1}{2} . \]
(64)
This fixes the form of \( S_G^{(1)} \) in eqn. (62) to be
\[ S_G^{(1)} = \frac{1}{2} \sum_{a=1}^{n} \frac{[q a]}{\langle a q \rangle} \left( \frac{\langle a x \rangle}{\langle q x \rangle} + \frac{\langle a y \rangle}{\langle q y \rangle} \right) \tilde{\lambda}_q^\gamma \frac{\partial}{\partial \tilde{\lambda}_a^\gamma} . \]
(65)
In fact, we have also checked that the sub-subleading soft factor \( S_G^{(2)} \) in four dimensions is completely fixed by gauge invariance and the distributional constraint.

6 Discussion

In this note we analyzed constraints arising for the novel subleading soft gluon and graviton theorems in general dimensions. Next to the obvious demands of Poincaré and gauge invariance we pointed out a slightly less obvious distributional constraint arising from the unbalanced arguments of the total momentum conserving Dirac delta functions on both sides of the soft theorems.

\(^{11}\)Again, only the combination \( \bar{c}_3^{(a)} - \bar{c}_2^{(a)} \) appears in \( S_G^{(1)} \) once the contraction with \( E^{(+)} \) is performed.
The distributional constraint requires the subleading soft operators to be differential operators of degree one (subleading) or two (sub-subleading) in the hard momenta and relates them to the leading Weinberg soft pole function.

In the $D$-dimensional case we started from an ansatz compatible with dimensional analysis and soft momentum scaling. We demonstrated that the entity of those constraints determines the subleading soft gluon and graviton differential operators as well as the sub-subleading soft graviton differential operator up to a single numerical constant for every leg. The undetermined constant is related to derivatives with respect to polarizations. Arbitrary functions commuting with the delta distributions could be added to these operators and are generally unconstrained. However, taking scalings and mass dimension constraints into account assuming locality, there is nothing which can be written down at tree-level.

Specializing to the four-dimensional case and employing the spinor-helicity formalism, the same line of arguments was shown to entirely fix the subleading differential operators. This can be traced back to the factorization of the Lorentz group in four dimensions. Upon fixing a unitary gauge, however, there might be similar arguments from little-group scalings in other dimensions.

The operators so determined match the forms established in the literature at tree level. Given that our arguments are very general the question arises whether they apply to loop amplitudes as well: certainly, Poincaré invariance, gauge invariance as well as the distributional constraint eqn. (5) continue to hold.

However, in the loop scenario we have to consider at least four novel circumstances, which are not reflected in our ansätze for the subleading soft operators in eqns. (30), (37) and (45). First, the loop corrections may contribute to the unconstrained functional parts of $S^{(1)}$ and $S^{(2)}$, as is in fact the case in the one-loop corrections reported in [18, 19]. Interestingly, the operator $S_G^{(2)}$ may also receive first-order differential corrections which should be related to the functional corrections to $S_G^{(1)}$ by the distributional constraint eqn. (16). Second, we construct our ansätze employing dimensional analysis to constrain the possible terms. The dimensionality of the couplings, however, allows for dimensionless quantities such as $\log(-q^2_{fa}/\mu^2)$ or $q^2_{fa}/\mu^2$, which so far have not been accounted for in our ansätze. These terms arise in the IR-divergent one-loop corrections to the soft operators reported in [18]. In fact, this also introduces contributions of the form $(\log \epsilon)$ in the soft momentum expansion. Third, the loop-corrections may not respect our central assumption of locality. Fourth, for gauge theory the leading soft factor $S_{YM}^{(0)}$ receives loop corrections, which feed into the subleading constraint equations.

After incorporating the issues pointed out in the last paragraph, the distributional constraint might be of use in the future in order to constrain possible loop corrections to soft theorems. While our work constrains the possible forms of the subleading soft gluon and graviton operators, it would be desirable to have a deeper understanding towards the origin of the soft theorems.

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