STOCHASTIC HEAT EQUATION WITH GENERAL ROUGH NOISE

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Abstract. In this paper, we study a nonlinear one spatial dimensional stochastic heat equations driven by Gaussian noise:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sigma(u) \dot{W},$$

where $W$ is white in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. We remove a critical and unnatural condition $\sigma(0) = 0$ previously imposed in a recent paper by Hu, Huang, Lê, Nualart and Tindel. The idea is to work on a weighted space $Z^{\lambda,T}$ for some power decay weight $\lambda(x) = c_H (1 + |x|^{2H-1})$. We obtain the weak existence of solution. With additional decay conditions on $\sigma$ we obtain the existence of strong solution and the pathwise uniqueness of the strong solution. The reason to introduce the weight function is that the solution $u(t,x)$ may explode as $|x| \to \infty$ when the “diffusion coefficient” $\sigma(u)$ does not satisfy $\sigma(0) = 0$ regardless of the initial condition. This motivates us to study the exact asymptotics of the solution $u_{add}(t,x)$ as $t$ and $x$ go to infinity when $\sigma(u) = 1$ and when the initial condition $u_0(x) \equiv 0$. In particular, we find the exact growth of $\sup_{|x| \leq L} |u_{add}(t,x)|$. Furthermore, we find the sharp growth rate for the Hölder coefficients, namely, $\sup_{|x| \leq L} |u_{add}(t,x+h)-u_{add}(t,x)|$ and $\sup_{|x| \leq L} |u_{add}(t,x+h)-u_{add}(t,x)|$. These results are interesting and fundamental themselves.

1. Introduction and main results

In this paper, we consider the following one dimensional (in space variable) nonlinear stochastic heat equation driven by the Gaussian noise which is white in time and fractional in space:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \sigma(t,x,u(t,x))\dot{W}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}, \quad (1.1)$$

where $W(t,x)$ is a centered Gaussian process with covariance given by

$$\mathbb{E}[W(t,x)W(s,y)] = \frac{1}{2}(|x|^{2H} + |y|^{2H} - |x-y|^{2H})(s \wedge t) \quad (1.2)$$

and where $\frac{1}{4} < H < \frac{1}{2}$ and $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}$.

There has been a lot of work on stochastic heat equations driven by general Gaussian noises. We refer to [10] for a short survey and for more references. The main feature of this work is that the noise is rough (e.g. $\frac{1}{4} < H < \frac{1}{2}$) in space variable. We mention three works that are directly related to this specific Gaussian noise structure. The first two are [3] and [12], where the authors study the existence,

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uniqueness and some properties such as moment bounds of the mild solution when diffusion coefficient $\sigma$ is affine ($\sigma(u) = au + b$) in [3] and linear ($\sigma(u) = au$) in [12]. After these works researchers tried to study (1.1) for general nonlinear $\sigma$. However, the method effective for affine (and linear) equations cannot no longer work. One difficulty is that we cannot no longer bound $|\sigma(x_1) - \sigma(x_2) - \sigma(y_1) + \sigma(y_2)|$ by a multiple of $|x_1 - x_2 - y_1 + y_2|$ (which is possible only in the affine case). In [11] the authors have successfully overcame this difficulty and obtained the existence and uniqueness of the strong solution to the equation (1.1) for general nonlinear $\sigma$. However, to solve the equation (1.1) the authors in [11] have to assume that $\sigma(0) = 0$, in addition to some more natural conditions such as $\sigma(t, x, u) = \sigma(u)$ is differentiable and the initial data to be integrable with respect to space variable.

While the smoothness condition on $\sigma$ looks rather natural since the noise $\tilde{W}$ is quite rough, people feel less comfortable with the condition $\sigma(0) = 0$. In fact, this condition does not even cover the affine case studied in [3]! The main motivation of this paper is to remove the condition $\sigma(0) = 0$. The main reason to assume $\sigma(0) = 0$ in [11] is the choice of function space $\mathcal{Z}_T^p$ where the solution lives (see [11] or (4.4) in Section 4 of this paper with $\lambda(x) = 1$). As we shall see that even in the simplest case $\sigma(u) \equiv 1$, the solution (to (1.1) with additive noise) is not in $\mathcal{Z}_T^p$ (see e.g. Theorem 1.1 and Proposition 4.11). Our idea is to introduce a decay weight (as the spatial variable $x$ goes to infinity) to modify the solution space $\mathcal{Z}_T^p$ to a weighted space $\mathcal{Z}_{\lambda,T}^p$ for some suitable power decay function $\lambda(x)$. This weight function will have to be chosen appropriately (not too fast and not too slow. See Section 2 for details).

The introduction of the weight makes things much more complex. As we can see we shall need much more delicate estimates compared to those of [11] to complete our program. People may wonder if one can use just $\mathcal{Z}_T^\infty$ for our solution space. This question is natural since we work in the whole Euclidean space $\mathbb{R}$ for the space variable. A constant function is in $L^\infty(\mathbb{R})$ but not in $L^p(\mathbb{R})$ for any finite $p$. If it is still possible to use $\mathcal{Z}_T^\infty$, then many computations in [11] will be valid and the problem becomes much easier.

To see if this is possible or not we consider the solution $u_{add}(t, x)$ to the equation with additive noise, which is the solution to (1.1) with $\sigma(t, x, u) = 1$ and with initial condition $u_0(x) = 0$. To see if $u_{add}(t, x)$ is in $\mathcal{Z}_T^\infty$ or not (or to see if the introduction of decay weight $\lambda$ is necessary or not), we shall find the sharp bound of the solution $u_{add}(t, x)$ as $x$ goes to infinity. In other words, we shall find the exact explosion rate of $\sup_{|x| \leq L} |u_{add}(t, x)|$ as $L$ goes to infinity. This problem has a great value of its own. To study the supremum of a family of random variables, there are two powerful tools: one is to use the independence and another one is to use the martingale inequalities. However, $u_{add}(t, x)$ is not a martingale with respect to spatial variable $x$ (nor it is a martingale with respect to time variable $t$) and since the noise $\tilde{W}$ is not independent in spatial variable either, the application of independence may be much more involved (We refer, however, to [4 5 6 7] for some successful applications of the independence in the stochastic heat equation (1.1)). In this work, we shall use instead the idea of majorizing measure to obtain sharp growth of $\sup_{|x| \leq L} |u_{add}(t, x)|$ and $\sup_{0 \leq t \leq T, |x| \leq L} |u_{add}(t, x)|$, as $L$ and $T$ go to infinity, both in term of expectation and almost surely. More precisely, we have

**Theorem 1.1.** Let the Gaussian field $u_{add}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$. Then, we have the following statements.
(1) There are two positive constants $c_H$ and $C_H$, independent of $T$ and $L$, such that
\[
c_H \Psi(T, L) \leq E \left[ \sup_{0 \leq t \leq T} |u(t, x)| \right] \leq E \left[ \sup_{0 \leq t \leq T} |u(t, x)| \right] \leq C \Psi(T, L),
\]
where
\[
\Psi(T, L) = \begin{cases} T^{\frac{H}{2}} + T^{\frac{H}{2}} \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} & \text{if } L^2 > T, \\ T^{\frac{H}{2}} & \text{if } L^2 \leq T. \end{cases}
\]

(2) There are two strictly positive random constants $c_H$ and $C_H$, independent of $T$ and $L$, such that almost surely
\[
c_H \left( T^{\frac{H}{2}} + T^{\frac{H}{2}} \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} \right) \leq \sup_{(t, x) \in \mathcal{Y}_{\varepsilon}(T, L)} |u_{add}(t, x)| \leq C_H \left( T^{\frac{H}{2}} + T^{\frac{H}{2}} \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} \right),
\]
where $\mathcal{Y}_{\varepsilon}(T, L) = \{(t, x) \in [0, T] \times [-L, L] : L \geq T^{\frac{1}{2} + \varepsilon}\}$ for any $\varepsilon > 0$.

It is well-known that the solution to the equation (1.1), if exists, is usually Hölder continuous on any bounded domain. But it usually is not Hölder continuous on the whole space. An interesting question to ask is how the Hölder coefficient depends on the size of the domain? Since the additive solution $u_{add}(t, x)$ is a Gaussian random field we will be able to obtain the sharp dependence on the size of the domain of the Hölder coefficient. In the following we first state our result on the Hölder continuity in spatial variable over unbounded domain as follows.

**Theorem 1.2.** Let $u_{add}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$ and denote
\[
\Delta_h u_{add}(t, x) := u_{add}(t, x + h) - u_{add}(t, x).
\]
Let $\theta \in (0, H)$ be given and let $L > \sqrt{T}$. Then, there are two positive constants $c_H$ and $C_{H, \theta}$ such that for sufficiently small value of $h$ satisfying $0 < |h| \leq C(\sqrt{T} \wedge 1)$ for some constant $C$, the following inequalities hold true:
\[
c_H |h|^H \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} \leq E \left[ \sup_{-L \leq t \leq L} \Delta_h u_{add}(t, x) \right] \leq E \left[ \sup_{-L \leq t \leq L} |\Delta_h u_{add}(t, x)| \right] \leq C_{H, \theta} |h|^\theta \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)}.
\]
Moreover, there are two (strictly) positive random constants $c_H$ and $C_{H, \theta}$, independent of $L \in \mathbb{R}_+$ and $h \in [-C(\sqrt{T} \wedge 1), C(\sqrt{T} \wedge 1)]$ such that
\[
c_H |h|^H \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} \leq \sup_{-L \leq t \leq L} \Delta_h u_{add}(t, x) \leq \sup_{-L \leq t \leq L} |\Delta_h u_{add}(t, x)| \leq C_{H, \theta} |h|^\theta \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)}.
\]
Next, we study the Hölder continuity in time over the unbounded domain. We state the following.

**Theorem 1.3.** Let $u_{\text{add}}(t, x)$ be the solution to (1.1) with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$ and denote

$$\Delta_\tau u_{\text{add}}(t, x) := u_{\text{add}}(t + \tau, x) - u_{\text{add}}(t, x).$$

Then, for sufficiently small value of $\tau$ such that $0 < \tau \leq C(t \wedge 1)$ for some constant $C$, we have

$$c_H \tau^{\frac{H}{2}} \log_2 \left( \frac{L}{\sqrt{t}} \right) \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} \Delta_\tau u_{\text{add}}(t, x) \right] \leq C_{H, \theta} t^{\frac{H}{2} - \theta} \tau^\theta \log_2 \left( \frac{L}{\sqrt{t}} \right),$$

where $0 < \theta < H/2$ and the positive constants $c_H$ and $C_{H, \theta}$ are independent of $L$ and $\tau$. We also have the almost sure version of the above result: if $0 < \tau \leq C(t \wedge 1)$, then we have

$$c_H \tau^{\frac{H}{2}} \log_2 \left( \frac{L}{\sqrt{t}} \right) \leq \sup_{-L \leq x \leq L} |\Delta_\tau u_{\text{add}}(t, x)| \leq C_{H, \theta} t^{\frac{H}{2} - \theta} \tau^\theta \log_2 \left( \frac{L}{\sqrt{t}} \right),$$

where $0 < \theta < H/2$, and random positive constants $c_H$ and $C_{H, \theta}$ are independent of $L$ and $\tau$.

The above Theorem 1.1-1.3 are proved in Section 3. Now let us return to the equation (1.1). To make things precise we give here the definitions of strong and weak solutions.

**Definition 1.4.** Let $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ be a real-valued adapted stochastic process such that for all $t \in [0, T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(s, y, u(s, y)) 1_{[0, t]}(s)\}$ is integrable with respect to $W$ (see Definition 2.4), where $G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right)$ is the heat kernel on $\mathbb{R}$ associated with the Laplacian operator $\Delta$.

(i) We say that $u(t, x)$ is a strong (mild) solution to (1.1) if for all $t \in [0, T]$ and $x \in \mathbb{R}$ we have

$$u(t, x) = G_t \ast u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s, y, u(s, y)) W(dy, ds)$$

almost surely, where the stochastic integral is understood in the sense of Definition 2.4.

(ii) We say (1.1) has a weak solution if there exists a probability space with a filtration $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}_t)$, a Gaussian random field $\tilde{W}$ identical to $W$ in law, and an adapted stochastic process $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ on this probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}_t)$ such that $u(t, x)$ is a mild solution with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}_t)$ and $\tilde{W}$.

Before stating our theorem on the existence of a weak solution, we make the following assumption.
Theorem 1.5. Let \( \lambda(x) = c_H (1 + |x|^2)^{H-1} \) satisfy \( \int \lambda(x) dx = 1 \). Assume that \( \sigma(t, x, u) \) satisfies hypothesis (H1) and that the initial data \( u_0 \) belongs to \( Z^p_{\lambda,0} \) for some \( p \geq \frac{6}{\gamma H} \) (see section 4.1 for the definition of \( Z^p_{\lambda,T} \)). Then, there exists a weak solution to (1.1) with sample paths in \( C([0,T] \times \mathbb{R}) \) almost surely. In addition, for any \( \gamma < H - \frac{2}{p} \) the process \( u(\cdot, \cdot) \) is almost surely Hölder continuous on any compact sets in \( [0,T] \times \mathbb{R} \) of Hölder exponent \( \gamma/2 \) with respect to the time variable \( t \) and of Hölder exponent \( \gamma \) with respect to the spatial variable \( x \).

Although the techniques of (11) is no longer effective in our new situation we still follow some spirit there. We need some very subtle bounds on the heat kernel \( G_t(x-y) \) with respect to the weight function \( \lambda(x) \), which is of interest in its own. This is done in Section 2. After these preparations, we shall show the above theorem in Section 4.

It is always interesting to have existence and uniqueness of the strong solution. As we said earlier, due to the roughness of the noise we need to handle, as in (11), the square increment \( |\sigma(x_1) - \sigma(x_2) - \sigma(y_1) + \sigma(y_2)| \). It seems too complicated for the weighted space. In fact, we will explain that the method of proving pathwise uniqueness in (11) is not applicable in our setting (see Proposition 3.11). So to show the existence and uniqueness of strong solution we assume that the derivative of diffusion coefficient in (1.1) possessing a decay itself as \( x \to \infty \). More precisely, we make the following assumptions.

(H2) Assume that \( \sigma(t, x, u) \in C^{0,1,1}([0,T] \times \mathbb{R}^2) \) satisfies the following conditions: \( |\sigma_u'(t, x, u)| \) and \( |\sigma_{uu}''(t, x, u)| \) are uniformly bounded, i.e. there is some constant \( C > 0 \) such that

\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma_u'(t, x, u)| \leq C; \tag{1.13}
\]

\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma_{uu}''(t, x, u)| \leq C. \tag{1.14}
\]

Moreover, we assume

\[
\sup_{t \in [0,T], x \in \mathbb{R}} \lambda^{-\frac{1}{2}}(x) |\sigma_u'(t, x, u_1) - \sigma_u'(t, x, u_2)| \leq C|u_2 - u_1|. \tag{1.15}
\]

Theorem 1.6. Let \( \sigma \) satisfy the above hypothesis (H2) and assume that for some \( p > \frac{6}{\gamma H} \), \( u_0 \in Z^p_{\lambda,0} \). Then (1.1) has a unique strong solution with sample paths in \( C([0,T] \times \mathbb{R}) \) almost surely. Moreover, the process \( u(\cdot, \cdot) \) is uniformly Hölder continuous a.s. on compact sets in \( [0,T] \times \mathbb{R} \) with the same temporal and spatial Hölder exponents as in Theorem 1.5.
This theorem will be proved in Section 5. Let us point out that if \( \sigma(u) \) is affine, then it satisfies the assumption \((H2)\).

2. Auxiliary Lemmas

In this section, we shall obtain some estimates about the heat kernel \( G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} \) associated with the Laplacian \( \Delta \) combined with the decay weight \( \lambda(x) \). These estimates are the key ingredients to establish our results.

2.1. Covariance structure. We start by recalling some notations used in \([11]\).

Denote by \( D = D(\mathbb{R}) \) the space of smooth functions on \( \mathbb{R} \) with compact support, and by \( D' \) the dual of \( D \) with respect to the \( L^2(\mathbb{R}, dx) \). The Fourier transform of a function \( f \in D \) is defined as

\[
\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx,
\]
and the inverse Fourier transform is then given by

\[
\mathcal{F}^{-1}g(x) = \frac{1}{2\pi} \mathcal{F}g(-x).
\]

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \) be given and fixed. Our noise \( \dot{W} \) is a zero-mean Gaussian family \( \{W(\phi), \phi \in D(\mathbb{R}_+ \times \mathbb{R})\} \) with covariance structure given by

\[
\mathbb{E}[W(\phi)W(\psi)] = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)}|\xi|^{1-2H} d\xi ds, \tag{2.1}
\]

where \( c_{1,H} \) is given below by (2.7) and \( \mathcal{F}\phi(s,\xi) \) is the Fourier transform with respect to the spatial variable \( x \) of the function \( \phi(s,x) \). Let \( \mathcal{F}_t \) be the filtration generated by \( W \). This means

\[
\mathcal{F}_t = \sigma\{W(\phi(x)1_{[0,r]}(s)) : r \in [0, t], \phi(x) \in D(\mathbb{R})\}.
\]

Equation (2.1) defines a Hilbert scalar product on \( D(\mathbb{R}_+ \times \mathbb{R}) \). To express this product without the use of Fourier transform, we recall the Marchaud fractional derivative \( D^\beta_- \) of order \( \beta \in (0, 1) \) with respect to the space variable. For a function \( \phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \), the Marchaud fractional derivative \( D^\beta_- \) is defined as:

\[
D^\beta_- \phi(t,x) = \lim_{\epsilon \downarrow 0} D^\beta_{-\epsilon} \phi(t,x) = \lim_{\epsilon \downarrow 0} \frac{\beta}{\Gamma(1-\beta)} \int_x^\infty \frac{\phi(t,x+y) - \phi(t,x)}{y^{1+\beta}} dy. \tag{2.2}
\]

We also define the Riemann-Liouville fractional integral of order \( \beta \) of a function \( \phi \)

\[
I^\beta \phi(t,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t,y)(y-x)^{\beta-1} dy.
\]

Set

\[
\mathcal{S} = \{ \phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \text{ s.t. } \phi(t,x) = I^\frac{1}{2-H} \psi(t,x) \}. \tag{2.3}
\]

With this notation we can express the Hilbert space obtained by completing \( D(\mathbb{R}_+ \times \mathbb{R}) \) with respect to the scalar product given by (2.1) in the following proposition (see e.g. \([19]\) for a proof).
Proposition 2.1. The function space $\mathcal{F}$ is a Hilbert space equipped with the scalar product

$$\langle \phi, \psi \rangle_\mathcal{F} = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi)\mathcal{F}\psi(s,\xi)\xi^{1-2H}d\xi ds$$

(2.4)

$$= c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_2^{\frac{\alpha}{2}-H}\phi(t,x)D_2^{\frac{\alpha}{2}-H}\psi(t,x)dx dt$$

(2.5)

$$= c_{3,\beta} \int_{\mathbb{R}^2} |\phi(x+y) - \phi(x)||\psi(x+y) - \psi(x)|y^{2H-2}dxdy,$$  

(2.6)

where

$$c_{1,H} = \frac{1}{2\pi} \Gamma(2H+1) \sin(\pi H);$$

(2.7)

$$c_{2,H} = \left[ \Gamma\left(H + \frac{1}{2}\right) \right]^2 \left( \int_0^\infty \left[ (1+t)^{H-\frac{\alpha}{2}} - t^{H-\frac{\alpha}{2}} \right]^2 dt + \frac{1}{2H} \right)^{-1};$$

(2.8)

$$c_{3,\beta} = (\frac{1}{2} - \beta)\beta c_{1,\beta}.$$

(2.9)

The space $\mathcal{F}(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathcal{F}$.

The Gaussian space $\mathcal{F}$ is the same as the homogeneous Sobolev space $\mathcal{H}^\beta$ for $\beta = \frac{\alpha}{2} - H \in (0,\frac{1}{2})$ in harmonic analysis (\cite{2}). The Gaussian family $W = \{W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ can be extended to an isonormal Gaussian process $W = \{W(\phi), \phi \in \mathcal{F}\}$ indexed by the Hilbert space $\mathcal{F}$. It is easy to see that $\phi(t, x) = \chi_{\{(0, t) \times (0, x)\}}, t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, is in $\mathcal{F}$ (we set $\chi_{\{(0, t) \times (0, x)\}} = -\chi_{\{(0, t) \times (0, x)\}}$ if $x$ is negative). We denote $W(t, x) = W(\chi_{\{(0, t) \times (0, x)\}})$.

2.2. Stochastic integration. We first define stochastic integral for elementary integrands and then extend it to general ones.

Definition 2.2. An elementary process $g$ is a process of the following form

$$g(t, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j}1_{(a_i, b_i]}(t)1_{(h_j, l_j]}(x),$$

where $n$ and $m$ are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{\alpha_i}$-measurable random variables for $i = 1, \ldots, n$. The stochastic integral of such an elementary process with respect to $W$ is defined as

$$\int_{\mathbb{R}_+ \times \mathbb{R}} g(t, x)W(dx, dt) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j}W(1_{(a_i, b_i]} \otimes 1_{(h_j, l_j]})$$

(2.10)

$$= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} [W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j)].$$

Proposition 2.3. Let $\Lambda_H$ be the space of predictable processes $g$ defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathcal{F}$ and $\mathbb{E}[||g||_H^2] < \infty$. Then, the space of elementary processes defined in Definition 2.2 is dense in $\Lambda_H$.  

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Remark 2.6.\[ \text{Proof.}\]

Notice that this is to estimate the decay rate of the Fourier transform of Lemma 2.7. Denote
\[ \text{This finishes the proof.} \quad \blacksquare \]

Definition 2.4. For \( g \in \Lambda_H \), the stochastic integral \( \int_{\mathbb{R}^+ \times \mathbb{R}} g(t, x)W(dx, dt) \) is defined as the \( L^2(\Omega) \) limit of stochastic integrals of the elementary processes approximating \( g(t, x) \) in \( \Lambda_H \), and we have the following isometry equality
\[ E \left[ \left( \int_{\mathbb{R}^+ \times \mathbb{R}} g(t, x)W(dx, dt) \right)^2 \right] = E \left[ ||g||_{L^2}^2 \right]. \quad (2.11) \]

2.3. Auxiliary Lemmas. We shall find a solution to equation (1.1) in the space \( \mathcal{Z}^p_{\lambda,T} \). To deal with weight \( \lambda \) we need a few technical results concerning the interaction between the weight \( \lambda(x) \) and the Green’s function \( G_t(x - y) \).

Lemma 2.5. For any \( \lambda \in \mathbb{R} \), \( \lambda(x) = \frac{1}{(1 + |x|^2)^\lambda} \), and \( T > 0 \), we have
\[ \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \frac{1}{\lambda(x)} \int_{\mathbb{R}} G_t(x - y)\lambda(y)dy < \infty. \quad (2.12) \]

Remark 2.6. To avoid using too many notations we use the symbol \( \lambda \) for a real number and the function induced. Apparently, there will be no confusion.

Proof. Let us rewrite (2.12) as
\[ \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} G_t(y) \frac{\lambda(y + x)}{\lambda(x)}dy \leq \sup_{0 \leq t \leq T} \int_{\mathbb{R}} G_t(y) \sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)}dy. \]
We discuss the cases \( \lambda \geq 0 \) and \( \lambda < 0 \) separately. When \( \lambda \geq 0 \), we have
\[ \sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)} \leq C_\lambda \sup_{x \in \mathbb{R}} \left( \frac{1 + |x|}{1 + |x + y|} \right)^{2\lambda} \leq C_\lambda (1 + |y|)^{2\lambda}. \]
On the other hand when \( \lambda < 0 \) we have
\[ \sup_{x \in \mathbb{R}} \left( \frac{1 + |x + y|^2}{1 + |x|^2} \right)^{-\lambda} \leq C_\lambda \sup_{x \in \mathbb{R}} \left( \frac{1 + |x + y|}{1 + |x|} \right)^{-2\lambda} \leq C_\lambda (1 + |y|)^{-2\lambda}. \]
In both cases we see
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}} G_t(y) \sup_{x \in \mathbb{R}} \frac{\lambda(y + x)}{\lambda(x)}dy \leq C_\lambda \sup_{t \in [0, T]} \int_{\mathbb{R}} G_t(y)(1 + |y|)^{2|\lambda|}dy < \infty. \]
This finishes the proof. \( \square \)

Lemma 2.7. Denote \( J(x) := \int_0^\infty e^{-\eta^2} \eta^\beta \cos(\eta \eta) d\eta \), where \( \beta > -1 \). We have
\[ |J(x)| \leq C_\beta \left( 1 \wedge \frac{1}{|x|^{\beta + 1}} \right). \quad (2.13) \]

Proof. Notice that this is to estimate the decay rate of the Fourier transform of \( e^{-\eta^2} \eta^\beta \) when \( |x| \) is large. Since \( J(-x) = J(x) \) and since we are only concerned with the large \( x \) behavior we may assume \( x \geq 1 \). We split the integral \( J(x) \) into two parts:
\[ J(x) = \int_0^{s(x)} e^{-\eta^2} \eta^\beta \cos(\eta \eta) d\eta + \int_{s(x)}^\infty e^{-\eta^2} \eta^\beta \cos(\eta \eta) d\eta := J_1(x) + J_2(x), \]
where \( s(x) > 0 \) is a function to be determined shortly.

First, it is easy to see
\[ |J_1(x)| \leq \int_0^{s(x)} \eta^\beta d\eta \leq C_\beta [s(x)]^{\beta + 1}. \]
For $J_2(x)$, an integration by parts implies
\[
|J_2(x)| = \left| \int_{s(x)}^{\infty} e^{-\eta^2} \eta^\beta \cos(x\eta) \, d\eta \right|
\]
\[
= \frac{1}{x} \int_{s(x)}^{\infty} e^{-\eta^2} \eta^\beta \sin(x\eta) \, d\eta
\]
\[
\leq C_{\beta} \frac{[s(x)]^{\beta}}{x} + C_{\beta} \frac{1}{x} \int_{s(x)}^{\infty} \eta^{\beta-1} e^{-\eta^2} \sin(x\eta) \, d\eta
\]
\[
+ \frac{C_{\beta}}{x} \int_{s(x)}^{\infty} \eta^{\beta+1} e^{-\eta^2} \sin(x\eta) \, d\eta.
\]

Let $k = \lceil \beta \rceil$ denote the least integer greater than or equal to $\beta$. Continuing the above application of integration by parts another $k$ times yields
\[
|J_2(x)| \leq C_{\beta} \frac{k}{x^{k+1}} + C_{\beta} \sum_{j=0}^{k-1} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^{j+1}}.
\]

Combining the estimates of $J_1(x)$ and $J_2(x)$ we have
\[
|J(x)| \leq C_{\beta}[s(x)]^{\beta+1} + C_{\beta} \sum_{j=0}^{k} \frac{[s(x)]^{\beta-j} + [s(x)]^{\beta+j}}{x^{j+1}}.
\]
The lemmas follows with the choice of $s(x) = \frac{1}{x}$.

Let us associate two increments related to the Green function $G_t(x)$, given as follows. The first one is a first order difference:
\[
D_t(x, h) := G_t(x + h) - G_t(x).
\] (2.14)

Denote $D(x, h) = \sqrt{\pi}D_{1/4}(x, h) = e^{-(x+h)^2} - e^{-x^2}$. The second one is a second order difference:
\[
\Box_t(x, y, h) := G_t(x + y + h) - G_t(x + y) - G_t(x + h) + G_t(x).
\] (2.15)

As above, we denote $\Box_t(x, y, h) = \sqrt{\pi}\Box_{1/4}(x, y, h)$:
\[
\Box_t(x, y, h) = e^{-(x+y+h)^2} - e^{-(x+h)^2} - e^{-(x+y)^2} + e^{-x^2}.
\] (2.16)

For these two increments, we have the following estimates which are needed later.

**Lemma 2.8.** For any $\alpha, \beta \in (0, 1)$, we have
\[
\int_{\mathbb{R}^2} |D_t(x, h)|^2 |h|^{-1-2\beta} \, dh \, dx = \frac{C_{\beta}}{t^{\frac{\alpha}{2} + \beta}}.
\] (2.17)

and
\[
\int_{\mathbb{R}^3} |\Box_t(x, y, h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} \, dy \, dh \, dx = \frac{C_{\alpha, \beta}}{t^{\frac{\alpha}{2} + \alpha + \beta}}.
\] (2.18)

**Proof.** With a change of variables, it suffices to show
\[
\int_{\mathbb{R}^2} |D(x, h)|^2 |h|^{-1-2\beta} \, dh \, dx < \infty;
\] (2.19)
\[
\int_{\mathbb{R}^3} |\Box(x, y, h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} \, dy \, dh \, dx < \infty.
\]
The above two inequalities will be derived from Plancherel’s identity. The Fourier transforms with respect to the variable $x$ of $D(x, h)$ and $\Box(x, y, h)$ are, respectively,

$$\hat{D}(\xi, h) = \mathcal{F}[D(\cdot, h)](\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}} \left[e^{ih\xi} - 1\right]$$

and

$$\hat{\Box}(\xi, y, h) = \mathcal{F}[\Box(\cdot, y, h)](\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}} \left[e^{iy\xi} - 1\right] \left[e^{ih\xi} - 1\right].$$

Thus, we have

$$\int_{\mathbb{R}} |D(x, h)|^2 dx = \int_{\mathbb{R}} |\hat{D}(\xi, h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} \left[1 - \cos(h\xi)\right] d\xi$$

and

$$\int_{\mathbb{R}} |\Box(x, y, h)|^2 dx = \int_{\mathbb{R}} |\hat{\Box}(\xi, y, h)|^2 d\xi = 4\pi \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} \left[1 - \cos(h\xi)\right] \left[1 - \cos(y\xi)\right] d\xi.$$

By Fubini’s theorem

$$\int_{\mathbb{R}^2} |D(x, h)|^2 |h|^{-1 - 2\beta} dhdx = C \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} d\xi \int_{\mathbb{R}} \left[1 - \cos(h\xi)\right] |h|^{-1 - 2\beta} dh$$

$$= C \int_{\mathbb{R}} e^{-\frac{\xi^2}{4}} d\xi \int_{\mathbb{R}} \left[1 - \cos(h\xi)\right] |h|^{-1 - 2\beta} dh < \infty$$

(2.20)

since $\int_0^\infty \frac{1 - \cos(\theta)}{\theta} d\theta$ is finite for all $\theta \in (1, 3)$ which requires $\alpha, \beta \in (0, 1)$. This proves the first inequality in (2.19). Same argument shows the second inequality in (2.19) under the condition of the lemma. \qed

**Remark 2.9.** In the rest of our paper, we shall use the lemma for $\alpha = \beta = \frac{1}{2} - H \in (0, 1/4)$.

**Lemma 2.10.** For $D(x, h)$ and $D_t(x, h)$ defined in (2.14), we have

$$F(x) := \int_{\mathbb{R}} |D(x, h)|^2 |h|^{2H-2} dh \leq C_H \left(1 \wedge |x|^{2H-2}\right), \quad (2.21)$$

and when $t > 0$

$$F_t(x) := \int_{\mathbb{R}} |D_t(x, h)|^2 |h|^{2H-2} dh \leq C_H \left(t^{H-\frac{3}{2}} \wedge \frac{|x|^{2H-2}}{\sqrt{t}}\right), \quad (2.22)$$

where $0 < H < \frac{1}{2}$.

**Proof.** The assertion (2.22) is an easy consequence of (2.21) by change of variables so we only need to provide a proof for (2.21).

Recall that the Fourier transform of $D(x, h)$ (as a function of $x$) is

$$\hat{D}(\eta, h) = \mathcal{F}[D(\cdot, h)](\eta) = \sqrt{\pi} e^{-\frac{\eta^2}{4}} \left[e^{ih\eta} - 1\right].$$

By the inverse Fourier transformation $D(x, h)$ can also be written as

$$D(x, h) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{D}(\eta, h) e^{ix\eta} d\eta = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-\frac{\eta^2}{4}} \left[e^{ih\eta} - 1\right] e^{ix\eta} d\eta.$$
Therefore, we can write

\[
F(x) = C_H \pi^2 \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} \left| e^{i\eta_1} - 1 \right| \left| e^{i\eta_2} - 1 \right| |h|^{2H-2} \, d\eta e^{ix(\eta_1 - \eta_2)} \, d\eta_1 \, d\eta_2.
\]

\[
= C_H \int_{\mathbb{R}} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} H(\eta_1, \eta_2) e^{ix(\eta_1 - \eta_2)} \, d\eta_1 \, d\eta_2,
\]

where similar to (2.20), we have

\[
H(\eta_1, \eta_2) = C_H \int_{\mathbb{R}} [e^{i\eta_1} - 1][e^{i\eta_2} - 1]|h|^{2H-2} \, dh
\]

where

\[
C_H(\eta_1^2 + \eta_2^2) = \int_{\mathbb{R}} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} H(\eta_1, \eta_2) e^{ix(\eta_1 - \eta_2)} \, d\eta_1 \, d\eta_2.
\]

It is easy to see that \(\sup_{x \in \mathbb{R}} |F(x)| \leq C < \infty\). Now, we want to get the desired decay estimate when \(x\) goes to infinity. We have

\[
F(x) \leq C_H \left| \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} |\eta_2|^{1-2H} e^{ix(\eta_1 - \eta_2)} \, d\eta_1 \, d\eta_2 \right|
\]

\[
+ C_H \left| \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} |\eta_1 - \eta_2|^{1-2H} e^{ix(\eta_1 - \eta_2)} \, d\eta_1 \, d\eta_2 \right|
\]

\[
\leq C_H e^{-x^2} \left| \int_{\mathbb{R}^2} e^{-\frac{\eta_1^2 + \eta_2^2}{4}} |\eta_2|^{1-2H} e^{-ix\eta_2} \, d\eta_1 \, d\eta_2 \right|
\]

\[
+ C_H \left| \int_{\mathbb{R}} |\eta_1|^{1-2H} e^{-ix\eta} \left| \int_{\mathbb{R}} e^{-|\eta_1|^2 + |\eta_2 + \eta_1|^2} \, d\eta_2 \right| \, d\eta_1 \right|
\]

since

\[
\int_{\mathbb{R}^2} e^{-|\eta_1|^2 + |\eta_2|^2} \, d\eta_2 = C e^{-|\eta|^2}.
\]

Now the estimate (2.21) follows from Lemma 2.7 \[\Box\]

**Lemma 2.11.** Recall that \(\Box_t(x, y, h)\) and \(\Box(x, y, h)\) are defined by (2.15) and (2.16). We have

\[
F(x) := \int_{\mathbb{R}^2} |\Box(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} \, dy \, dh \leq C_H \left(1 \wedge |x|^{2H-2}\right).
\]

Moreover, for any \(t > 0\) we have

\[
F_t(x) := \int_{\mathbb{R}^2} |\Box_t(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} \, dy \, dh \leq C_H \left(t^{2H-2} \wedge \frac{|x|^{2H-2}}{t^{1-H}}\right).
\]

**Proof.** As for Lemma (2.10) we only need to prove (2.24) and last inequality can be derived from (2.24) by a change of variable.

The proof of (2.24) is similar to that of Lemma (2.10) Recall the Fourier transform of \(\Box(x, y, h)\) as a function of \(x\):

\[
\Box(\eta, y, h) = \mathcal{F}[\Box(\cdot, y, h)](\eta) = \sqrt{\pi} e^{-\frac{x^2}{2}} |e^{iy} - 1| |e^{ih} - 1|.
\]

This means

\[
\Box(x, y, h) = \sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} |e^{iy} - 1| |e^{ih} - 1| e^{ix\eta} \, d\eta.
\]
Thus, we have

\[
F(x) = \int_{\mathbb{R}^4} e^{-\frac{\xi^2 + \eta^2}{2}} [e^{i\eta \xi} - 1][e^{i\eta_2} - 1] \cdot [e^{i\eta_2} - 1] \cdot [e^{i\eta_1} - 1] \\
\quad \cdot \frac{|h|^{2H-2} |y|^{2H-2} e^{i\xi(x-\eta_2)} d\eta_1 d\eta_2}{e^{i\eta_2} - 1} \\
= 2\pi^2 \int_{\mathbb{R}^2} e^{-\frac{\xi^2 + \eta^2}{2}} H^2(\eta_1, \eta_2) e^{i\eta(x-\eta_2)} d\eta_1 d\eta_2,
\]

(2.26)

where \(H(\eta_1, \eta_2)\) is defined in (2.23) or

\[
H^2(\eta_1, \eta_2) = |\eta_1|^{2-4H} + |\eta_2|^{2-4H} + |\eta_1|^{1-2H} |\eta_2|^{1-2H} + |\eta_1 - \eta_2|^{2-4H} \\
- |\eta_1|^{1-2H} |\eta_1 - \eta_2|^{1-2H} - |\eta_2|^{1-2H} |\eta_1 - \eta_2|^{1-2H}.
\]

It is easy to see that \(\sup_{x \in \mathbb{R}} |F(x)| \leq C < \infty\). Now we want to show the desired decay rate as \(x \to \infty\). By the symmetry \(F(-x) = F(x)\), we can and will assume \(x \geq 1\). The argument in the proof of Lemma 2.10 can be used to obtain the desired bound for each of the above terms expect the terms \(|\eta_1 - \eta_2|^{2-4H}\) and \(|\eta_1|^{1-2H} |\eta_1 - \eta_2|^{1-2H}\) (and \(|\eta_2|^{1-2H} |\eta_1 - \eta_2|^{1-2H}\)), which can be handled analogously.

For term \(|\eta_1 - \eta_2|^{2-4H}\), letting \(\xi_1 = \eta_1 - \eta_2\) and \(\xi_2 = \eta_1 + \eta_2\) implies

\[
\int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{2}} |\eta_1 - \eta_2|^{2-4H} e^{i\xi_1(x-\eta_2)} d\eta_1 d\eta_2
\]

\[
= C \int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{2}} |\xi_1|^{1-2H} e^{i\xi_1(x-\eta_2)} d\xi_1 d\eta_2
\]

Then using Lemma 2.7, we see that this term is bounded by \(1 \wedge |x|^{4H-3} \lesssim 1 \wedge |x|^{2H-2}\) for \(\frac{1}{3} < H < \frac{1}{2}\).

In order to deal with the second term \(|\eta_1|^{1-2H} |\eta_1 - \eta_2|^{1-2H}\), we make the substitution \(\xi = \eta_1\) and \(\eta = \frac{1}{2}(\eta_1 - \eta_2)\) to obtain

\[
J(x) := \int_{\mathbb{R}^2} e^{-\frac{\xi_1^2 + \xi_2^2}{2}} |\eta_1|^{2-2H} |\eta_1 - \eta_2|^{1-2H} e^{i\xi_1(x-\eta_2)} d\eta_1 d\eta_2
\]

\[
= C \int_{\mathbb{R}^2} \exp \left( -\frac{(|\eta| - |\eta|)^2}{2} \right) \exp \left( -\frac{|\xi|^2}{2} \right) |\xi_1|^{1-2H} |\eta_1|^{1-2H} e^{i2x\eta} d\eta_1 d\eta_2.
\]

Denote

\[
E(\eta) := \int_{\mathbb{R}} \exp \left( -\frac{(|\eta| - |\eta|)^2}{2} \right) |\xi_1|^{1-2H} d\xi.
\]

We need to show a similar inequality as that in Lemma 2.7

\[
|J(x)| = \left| \int_{0}^{\infty} e^{-\frac{\xi^2}{2}} \eta^{1-2H} E(\eta) \cos(2x\eta) d\eta \right| \leq C_H \left( 1 \wedge |x|^{2H-2} \right).
\]

First, we observe that \(|E(\eta)| \leq C_H (1 + |\eta|^{1-2H})\) and both \(|E'(\eta)|\) and \(|E''(\eta)|\) can be bounded by a multiple of

\[
\int_{\mathbb{R}} \exp \left( -\frac{(|\eta| - |\eta|)^2}{4} \right) |\xi|^{1-2H} d\xi \leq C_H \left( 1 + |\eta|^{1-2H} \right).
\]

We only need to care the case when \(x\) is large. Let us split \(J(x)\) into two parts of which one integrates from 0 to \(s(x)\), denoted by \(J_1(x)\), and another integrates
from \( s(x) \) to infinity, denoted by \( J_2(x) \), such that \( s(x) \to 0 \) as \( x \) goes to infinity and whose precise form will be given later. For the first part

\[
|J_1(x)| \leq [s(x)]^{1-2H} \int_0^{s(x)} |E(\eta)| d\eta \leq C_H \left( |s(x)|^{2-2H} + |s(x)|^{3-4H} \right).
\]

For \( J_2(x) \), an integration by parts yields

\[
|J_2(x)| = \left| \int_{s(x)}^\infty e^{-\frac{\eta^2}{4}} \eta^{-2H} E(\eta) \cos(2x\eta) d\eta \right|
= C \left( \frac{1}{x} \right) \int_{s(x)}^\infty e^{-\frac{\eta^2}{4}} \eta^{-2H} E(\eta) d(2x\eta) d\eta
\leq C_H \left( \frac{s(x)}{x} \right)^{2-2H} \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E(\eta) d\eta
+ \frac{C_H}{x} \left( \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E(\eta) d\eta \right)^2
+ \frac{C_H}{x} \left( \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E'(\eta) d\eta \right)^2
= J_{21} + J_{22} + J_{23} + J_{24}.
\]

The first term is bounded by

\[
J_{21}(x) \leq C_H \left( \frac{1}{x} \right) [s(x)]^{1-2H}.
\]

As for \( J_{22}(x) \) an integration by parts yields

\[
J_{22}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E(\eta) d\eta \right|
\leq C \frac{E(s(x))}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{-2H} E(\eta) e^{-\frac{\eta^2}{4}} \right] \right| d\eta
\leq C_H \frac{x^2}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} [s(x)]^{-4H} + \frac{C_H}{x^2}.
\]

In the same way we can bound \( J_{23}(x) \) as follows.

\[
J_{23}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{-2H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E(\eta) d\eta \right|
\leq C \frac{E(s(x))}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{-2H} E(\eta) e^{-\frac{\eta^2}{4}} \right] \right| d\eta
\leq C_H \frac{x^2}{x^2} [s(x)]^{-2H} + \frac{C_H}{x^2} [s(x)]^{-4H} + \frac{C_H}{x^2}.
\]

The term \( J_{24}(x) \) satisfies

\[
J_{24}(x) := \frac{1}{x} \left| \int_{s(x)}^\infty \eta^{-1H} e^{-\frac{\eta^2}{4}} \sin(2x\eta) E'(\eta) d\eta \right|
\leq C \frac{E'(s(x))}{x^2} [s(x)]^{-1H} + \frac{C_H}{x^2} \int_{s(x)}^\infty \left| \frac{d}{d\eta} \left[ \eta^{-1H} E'(\eta) e^{-\frac{\eta^2}{4}} \right] \right| d\eta
\leq C_H \frac{x^2}{x^2} [s(x)]^{-1H} + \frac{C_H}{x^2} [s(x)]^{-3H} + \frac{C_H}{x^2}.
\]

Noticing that \( \frac{1}{4} < H < \frac{1}{2} \), and taking \( s(x) = \frac{1}{x} \) imply our result.
Lemma 2.12. Denote \( \lambda(x) = \frac{1}{(1 + |x|^2)^{1-n}} \) and recall \( D_t(x, h) \) defined by (2.14) and \( \Box_t(x, y, h) \) defined by (2.15). We have
\[
\begin{align*}
\int_{\mathbb{R}^2} |D_t(x, h)|^2 |h|^{2H-2} \lambda(z - x) dx dh & \leq C_T H t^{H-1} \lambda(z), \\
\int_{\mathbb{R}^3} |\Box_t(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} \lambda(z - x) dx dy dh & \leq C_T H t^{2H - \frac{3}{2}} \lambda(z).
\end{align*}
\] (2.27)

Proof. Set
\[
R(x, z) = \frac{\lambda(z - x)}{\lambda(z)} \simeq \left( \frac{1 + |z|}{1 + |x - z|} \right)^{2-2H},
\]
where and throughout the paper for two functions \( f \) and \( g \), notation \( f \simeq g \) means that there exist two positive constants \( c_H \) and \( C_H \) such that \( c_H g \leq f \leq C_H g \). By Lemma 2.7, we have
\[
\int_{\mathbb{R}^2} |D_t(x, h)|^2 |h|^{2H-2} R(x, z) dx dh = C_H t^{H-1} \int_{\mathbb{R}^2} |D_t(x, h)|^2 |h|^{2H-2} R(\sqrt{tx}, \sqrt{tz}) dx dh 
\]
\[
\leq C_H t^{H-1} \int_\mathbb{R} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{tx}, \sqrt{tz}) dx. \tag{2.28}
\]

Similarly, we have
\[
\int_{\mathbb{R}^3} |\Box_t(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} R(x, z) dx dy dh = C_H t^{2H - \frac{3}{2}} \int_{\mathbb{R}^3} |\Box_t(x, y, h)|^2 |h|^{2H-2} |y|^{2H-2} R(\sqrt{tx}, \sqrt{tz}) dx dy dh 
\]
\[
\leq C_H t^{2H - \frac{3}{2}} \int_\mathbb{R} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{tx}, \sqrt{tz}) dx. \tag{2.29}
\]

From (2.28) and (2.29) to show our lemma it is suffices to show
\[
\sup_{t \in [0, T]} \sup_{z \in \mathbb{R}} \int_\mathbb{R} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{tx}, \sqrt{tz}) dx < \infty. \tag{2.30}
\]

Notice that we assume that \( t \in [0, T] \) is bounded. If \( z \) is bounded then \( R(\sqrt{tx}, \sqrt{tz}) \) is also bounded. Then, we have
\[
\sup_{t \in [0, T]} \sup_{|z| \leq 2} \int_{\mathbb{R}} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{tx}, \sqrt{tz}) dx \leq C_{T, H} \int_\mathbb{R} 1 \wedge |x|^{2H-2} dx < \infty. \tag{2.31}
\]

This means that we only need to consider the case \( |z| \geq 2 \). Due to the symmetry \( R(-\sqrt{tx}, -\sqrt{tz}) = R(\sqrt{tx}, \sqrt{tz}) \), we can assume \( z \geq 2 \).

Next we shall divide the integral into two domains.

(i) The domain \( x \leq z/2 \) or \( x \geq 2z \). On this domain \( R(\sqrt{tx}, \sqrt{tz}) \) is bounded. Thus
\[
\sup_{t \in [0, T]} \sup_{|z| \geq 1} \int_{\mathbb{R}} \left( 1 \wedge |x|^{2H-2} \right) R(\sqrt{tx}, \sqrt{tz}) dx \leq C_{T} \int_\mathbb{R} 1 \wedge |x|^{2H-2} dx \leq \infty. \tag{2.32}
\]

(ii) The domain \( z/2 \leq x \leq 2z \). On this domain we have \( x \geq z/2 \geq (z + 1)/3 \geq 1 \) and then
\[
1 \wedge |x|^{2H-2} \leq |x|^{2H-2} \leq \frac{3^{2-2H}}{(1 + z)^{2-2H}}.
\]
Thus,
\[
\int_{(\frac{1}{2} < x < 2z)} (1 \wedge |x|^{2H-2}) R(\sqrt{t}x, \sqrt{t}z) \, dx
\]
\[
\leq C_H \left( \frac{1 + \sqrt{t}z}{1 + z} \right)^{2-2H} \int_0^2 \frac{1}{(1 + \sqrt{t}x - z)^{2-2H}} \, dx
\]
\[
\leq C_H \frac{1 + (\sqrt{t}z)^{2-2H}}{1 + z^{2-2H}} \int_0^2 \frac{1}{(1 + \sqrt{t}x - z)^{2-2H}} \, dx
\]
\[
= C_H \frac{1 + (\sqrt{t}z)^{2-2H}}{\sqrt{tz} (1 + (\sqrt{t}z)^{1-2H})} \left[ 1 - (1 + \sqrt{t}z)^{2H-1} \right]
\]
\[
\leq C_H T^{2-H} \frac{1 + (\sqrt{t}z)^{2-2H}}{\sqrt{tz} (1 + (\sqrt{t}z)^{1-2H})} \left[ 1 - (1 + \sqrt{t}z)^{2H-1} \right].
\]
Consider now the function
\[
f(u) = \frac{1 + u^{2-2H}}{u(1 + u^{1-2H})} \left[ 1 - (1 + u)^{2H-1} \right], \quad u > 0.
\]
This is a continuous function on $(0, \infty)$. When $u \to 0$ and when $u \to \infty$ we have
\[
\lim_{u \to 0^+} f(u) = 1 - 2H, \quad \lim_{u \to \infty} f(u) = 1.
\]
Thus, $f(u)$ is bounded on $(0, \infty)$ and it in turn proves
\[
\sup_{u \in [0, T]} \sup_{z \geq 1} \int_{(\frac{1}{2} < x < 2z)} (1 \wedge |x|^{2H-2}) R(\sqrt{t}x, \sqrt{t}z) \, dx < \infty. \quad (2.33)
\]
Combining $(2.31)$, $(2.32)$ together with our above symmetry argument we prove $(2.30)$ and hence complete the proof of the lemma. \(\square\)

**Remark 2.13.** From this lemma, we see why we take the above decay rate for our weight function. If we consider $\lambda(x) = (1 + |x|^2)^{-\lambda}$ with $\lambda > 1 - H$, then for $|z|$ large enough one has
\[
\int_{\mathbb{R}} (1 \wedge |x|^{2H-2}) R(x, z) \, dx \gtrsim \int_{|x-z| < 1} |x|^{2H-2} R(x, z) \, dx \gtrsim \frac{(1 + |z|)^{\lambda}}{|z|^{2-2H}},
\]
which diverges as $|z| \to \infty$. This elementary fact tell us that $\lambda$ must be in $(\frac{1}{2}, 1-H]$, and it is obvious $L^p_\lambda(\mathbb{R})$ is the largest space when $\lambda = 1 - H$.

### 3. Additive noise

When the diffusion coefficient $\sigma(t, x, u) = 1$ (or a general constant), the noise is additive and the solution to equation $(1.1)$ can be written explicitly as
\[
u(t, x) = \int_{\mathbb{R}} G_t(x - y)u_0(y) \, dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y)W(dy, ds) \quad (3.1)
\]
where $G_t(x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right)$ is the heat kernel. To focus on the stochastic part we assume $u_0 = 0$. Thus, the resulted solution is written as
\[
u_{\text{add}}(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x - y)W(dy, ds). \quad (3.2)
\]
This solution $u_{add}(t, x)$ defines a (symmetric) centered Gaussian process. We shall study how it grows as the parameters $t$ and $x$ go to infinity. It is expected that $u_{add}(t, x)$ is Hölder continuous in $t$ and $x$. More precisely, for any positive constants $\gamma < H, T, L \in (0, \infty)$, there is a constant $C_{T, L, \gamma}$, depending only on $T, L$ and $\gamma$, such that

$$
\sup_{0 \leq s, t \leq T, |x|, |y| \leq L} |u_{add}(s, x) - u_{add}(t, y)| \leq C_{T, L, \gamma} \left( |t - s|^{\gamma/2} + |x - y|^{\gamma} \right)
$$

We want to consider the Hölder continuity of $u_{add}(t, x)$ on the whole space $\mathbb{R}$. Namely, we want to know how the constant $C_{T, L, \gamma}$ grows as $T$ and $L$ go to infinity (for any fixed $\gamma$).

### 3.1. Majorizing measure theorem.

To find the sharp bound for $C_{T, L, \gamma}$ we shall utilize Talagrand’s majorizing measure theorem which we recall below.

**Theorem 3.1.** (Majorizing Measure Theorem, see e.g. [21, Theorem 2.4.2]). Let $T$ be a given set and let $\{X_t, t \in T\}$ be a centered Gaussian process indexed by $T$. Denote $d(t, s) = \langle \mathbb{E}X_t - X_s \rangle_{H^2}$, the associated natural metric on $T$. Then

$$
\mathbb{E}\left[ \sup_{t \in T} X_t \right] \leq \gamma_2(T, d) := \inf_{A \subset T} \sup_{n \geq 0} 2^{n/2} \text{diam}(A_n(t)),
$$

where the infimum is taken over all increasing sequence $A := \{A_n, n = 1, 2, \cdots\}$ of partitions of $T$ such that $\#A_n \leq 2^{2^n}$ ($\#A$ denotes the number of elements in the set $A$). $A_n(t)$ denotes the unique element of $A_n$ that contains $t$, and $\text{diam}(A_n(t))$ is the diameter (with respect to the natural distance $d$) of $A_n(t)$.

This theorem provides a powerful general principle for the study of the supremum of Gaussian process.

**Remark 3.2.** The natural metric $d(t, s)$ is actually only a pseudo-metric because $d(t, s) = 0$ does not necessarily imply $t = s$ (e.g. $X_t \equiv 1$). It is also call the canonical metric.

It is more convenient for us to use the following theorem to obtain the lower bound.

**Theorem 3.3.** (Sudakov minoration theorem, see e.g. [21, Lemma 2.4.2]). Let $\{X_t, t = 1, \cdots, L\}$ be centered Gaussian family with natural distance $d$ and assume

$$
\forall p, q \leq L, p \neq q \Rightarrow d(t_p, t_q) \geq \delta.
$$

Then, we have

$$
\mathbb{E}\left( \sup_{1 \leq i \leq L} X_i \right) \geq \frac{\delta}{C} \sqrt{\log_2(L)},
$$

where $C$ is a universal constant.

The following “concentration of measure” type theorem allows us to obtain deviation inequalities for the supremum of Gaussian family.

**Theorem 3.4.** (Borell, see e.g. [11, Theorem 2.1]). Let $\{X_t, t \in T\}$ be a centered separable Gaussian process simple paths of $\{X_t, t \in T\}$ bounded a.s. on some topological index set $T$. Then

$$
\mathbb{P}\left( \sup_{t \in T} X_t - \mathbb{E}\left[ \sup_{t \in T} X_t \right] > \lambda \right) \leq 2 \exp\left( -\frac{\lambda^2}{2\sigma_t^2} \right),
$$

where $\sigma_t$ is the variance of $X_t$. This solution $u_{add}(t, x)$ defines a (symmetric) centered Gaussian process. We shall study how it grows as the parameters $t$ and $x$ go to infinity. It is expected that $u_{add}(t, x)$ is Hölder continuous in $t$ and $x$. More precisely, for any positive constants $\gamma < H, T, L \in (0, \infty)$, there is a constant $C_{T, L, \gamma}$, depending only on $T, L$ and $\gamma$, such that

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$$

We want to consider the Hölder continuity of $u_{add}(t, x)$ on the whole space $\mathbb{R}$. Namely, we want to know how the constant $C_{T, L, \gamma}$ grows as $T$ and $L$ go to infinity (for any fixed $\gamma$).

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$$

where the infimum is taken over all increasing sequence $A := \{A_n, n = 1, 2, \cdots\}$ of partitions of $T$ such that $\#A_n \leq 2^{2^n}$ ($\#A$ denotes the number of elements in the set $A$). $A_n(t)$ denotes the unique element of $A_n$ that contains $t$, and $\text{diam}(A_n(t))$ is the diameter (with respect to the natural distance $d$) of $A_n(t)$.

This theorem provides a powerful general principle for the study of the supremum of Gaussian process.

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Then, we have

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\mathbb{P}\left( \sup_{t \in T} X_t - \mathbb{E}\left[ \sup_{t \in T} X_t \right] > \lambda \right) \leq 2 \exp\left( -\frac{\lambda^2}{2\sigma_t^2} \right),
$$

where $\sigma_t$ is the variance of $X_t$. This solution $u_{add}(t, x)$ defines a (symmetric) centered Gaussian process. We shall study how it grows as the parameters $t$ and $x$ go to infinity. It is expected that $u_{add}(t, x)$ is Hölder continuous in $t$ and $x$. More precisely, for any positive constants $\gamma < H, T, L \in (0, \infty)$, there is a constant $C_{T, L, \gamma}$, depending only on $T, L$ and $\gamma$, such that

$$
\sup_{0 \leq s, t \leq T, |x|, |y| \leq L} |u_{add}(s, x) - u_{add}(t, y)| \leq C_{T, L, \gamma} \left( |t - s|^{\gamma/2} + |x - y|^{\gamma} \right).
$$

We want to consider the Hölder continuity of $u_{add}(t, x)$ on the whole space $\mathbb{R}$. Namely, we want to know how the constant $C_{T, L, \gamma}$ grows as $T$ and $L$ go to infinity (for any fixed $\gamma$).
where \( \sigma^2_t := \sup_{t \in T} E(X_t^2) \).

We have the following observation which can be deduced immediately from [21 Lemma 2.2.1]. This simple fact tells us \( E[\sup_{t \in T} |X_t|] \simeq E[\sup_{t \in T} X_t] \). So we only need to consider \( E[\sup_{t \in T} X_t] \).

**Lemma 3.5.** If the process \( \{X_t, t \in T\} \) is symmetric, then we have
\[
E\left[ \sup_{t \in T} |X_t| \right] \leq 2E\left[ \sup_{t \in T} X_t \right] + \inf_{t_0 \in T} E\left[ |X_{t_0}| \right].
\]

(3.6)

### 3.2. Asymptotics of the Gaussian solution.

For the mild solution \( u_{add}(t, x) \) to (1.1) with additive noise (e.g. \( \sigma(t, x, u) = 1 \)), defined by (3.2), we shall first obtain the sharp upper and lower bounds for its associated natural metric:
\[
d_1((t, x), (s, y)) = \sqrt{E[u_{add}(t, x) - u_{add}(s, y)]^2},
\]
where without loss of generality we assume \( 0 \leq s < t < \infty \).

The following lemma gives a sharp bounds for this induced natural metric for the Gaussian solution \( u_{add}(t, x) \).

**Lemma 3.6.** Let \( d_1((t, x), (s, y)) \) be the natural metric defined by (3.7). Then, there are positive constants \( c_H, C_H \) such that
\[
c_H(|x - y|_H \wedge (t \wedge s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}}) \leq d_1((t, x), (s, y)) \leq C_H(|x - y|_H \wedge (t \wedge s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}})
\]
for any \( (t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{R} \).

**Remark 3.7.** The above property of the natural metric can also be written as
\[
d_1((t, x), (s, y)) \simeq d_{1,H}((t, x), (s, y)) := |x - y|_H \wedge (t \wedge s)^{\frac{H}{2}} + |t - s|^{\frac{H}{2}}.
\]
(3.9)

\( d_{1,H}((t, x), (s, y)) \) is no longer a distance but it is very convenient for us to obtain the desired results.

**Proof.** Without loss of generality, let us assume \( t > s \). Plancherel’s identity and the independence of the stochastic integrals over the time intervals \([0, s]\) and \([s, t]\) give
\[
d_1^2((t, x), (s, y)) = E|u_{add}(t, x) - u_{add}(s, y)|^2
\]
\[
= E\left[ \int_0^s \int \mathbb{R} \left| G_{t-r}(x-z) - G_{s-r}(y-z) \right| W(dz, dr) \right]^2
\]
\[
+ E\left[ \int_s^t \int \mathbb{R} \left| G_{t-r}(x-z) W(dz, dr) \right|^2 \right]
\]
\[
= \int_{\mathbb{R}_+} [1 - \exp(-2s\xi^2)] \left[ 1 + \exp(-2(t-s)\xi^2) \right]
\]
\[
- 2 \exp(-(t-s)\xi^2) \cos(|x-y|\xi) \cdot \xi^{1-2H} d\xi + 2^{H-1}\kappa_H (t-s)^H,
\]
(3.10)

where
\[
\kappa_H = H^{-1}\Gamma(1-H)
\]
is a positive constant. We start to obtain the upper bound of (3.8). The triangular inequality gives
\[
d_1((t, x), (s, y)) \leq d_1((t, x), (s, s)) + d_1((s, x), (s, y)).
\]
(3.11)
Let us deal with the two term on the right hand side of the above inequality separately. For the first term, Plancherel’s identity (3.10) implies
\[
d_1^2((s, x), (s, y)) = \kappa_H \left[ 2^{H-1}H + 2^{H-1}sH - (t+s)^H \right] + (2^{H-1} + 1)\kappa_H(t-s)^H
\]
\[
\leq C_H(t-s)^H,
\]
because $2^{H-1}H + 2^{H-1}sH - (t+s)^H \leq 0$ when $t \geq s$. Again from (3.10), the second term on the right hand side of (3.11) is given by
\[
d_1^2((s, x), (s, y)) = \int_{\mathbb{R}} \int_{\mathbb{R}} \exp[-2(s-r)^2|x|^2] \cdot |\xi|^{1-2H}|1 - \cos(\xi|x-y|)|d\xi dr.
\]
which can be controlled by $C_H|x-y|^{2H}$. On the other hand, we have
\[
d_1^2((s, x), (s, y)) = \mathbb{E}[|u_{\text{add}}(s, x) - u_{\text{add}}(s, y)|^2]
\]
\[
\leq 2(\mathbb{E}[|u_{\text{add}}(s, x)|^2] + \mathbb{E}[|u_{\text{add}}(s, y)|^2]) \leq C_Hs^H.
\]
Thus, the quantity of $d_1^2((s, x), (s, y))$ is bounded by the minimum of $C_H|x-y|^{2H}$ and $C_Hs^H$. We can summarize the above argument as
\[
d_1((t, x), (s, y)) \leq C_H(|x-y|^H \wedge s^\frac{H}{2} + (t-s)^\frac{H}{2}),
\]
which is the upper bound part of (3.8).

Now we turn to show the lower bound part of (3.8). From Plancherel’s identity it is sufficient to bound the first summand in (3.10) from below by $c_H(|x-y|^H \wedge s^\frac{H}{2})$ for some constant $c_H > 0$. We denote this first summand by $I$:
\[
I := \int_{\mathbb{R}} \left[ 1 - \exp(-2s^2) \right][1 + \exp(-2(t-s))]
\]
\[-2 \exp(-(t-s)^2) \cos(|x-y|)|\xi|^{1-2H} d\xi
\]
\[
= c|x-y|^{2H} \int_{\mathbb{R}^+} \left[ 1 - \exp\left(-\frac{2\xi^2}{|x-y|^2}\right) \right] \cdot |\xi|^{-1-2H}
\]
\[
\cdot \left[ 1 - \exp\left(-\frac{(t-s)^2}{|x-y|^2}\cos(|x-y|)\right) \right]^2 d\xi.
\]
To this end, we divide our argument into two cases:
\[
|x-y| > \sqrt{\epsilon} \quad \text{and} \quad |x-y| \leq \sqrt{\epsilon}.
\]
When $|x-y| \leq \sqrt{\epsilon}$, we can bound (3.13) below by
\[
I \geq c_H|x-y|^{2H} \sum_{n=1}^{\infty} \int_{2n\pi+\frac{\pi}{2}}^{2n\pi+\frac{3\pi}{2}} \left[ 1 - \exp(-2\xi^2) \right] \cdot |\xi|^{-1-2H} d\xi
\]
\[
\geq c_H|x-y|^{2H},
\]
since $1 - \exp(-2\xi^2/|x-y|^2) \geq 1 - \exp(-2\xi^2)$ by the assumption and $\cos(|x-y|)$ is negative on the intervals $[\epsilon,\infty)$ for some constant $c_H > 0$. We denote this summand by $I_0$.

The case $|x-y| > \sqrt{\epsilon}$ is a little bit more involved. Denote
\[
n_0 := \inf \left\{ n \in \mathbb{N}_0 : 2n\pi + \frac{\pi}{2} \geq \sqrt{\frac{-\ln(1-c^*)}{2s}}|x-y| \right\}
\]
\[
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\]
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with the choice \( c^* = 1 - \exp(-\pi^2/2) \) such that

\[
\sqrt{\frac{-\ln(1 - c^*)}{2s}} |x - y| \geq \frac{\pi}{2}.
\]

It is then easy to see that \( n_0 \) is a well defined finite positive integer. This way, we have the lower bound for (3.13):

\[
I \geq \sum_{n=n_0} |x - y|^{2H} \int_{2n\pi + \frac{\pi}{2}}^{2(n+1)\pi + \frac{\pi}{2}} \left[ 1 - \exp\left(\frac{-2s\xi^2}{|x - y|^2}\right) \right] \xi^{-1-2H} d\xi 
\geq c^* |x - y|^{2H} \sum_{n=n_0} |x - y|^{2H} \int_{2n\pi + \frac{\pi}{2}}^{\infty} \xi^{-1-2H} d\xi,
\]

where the last inequality follows from the fact that \( \xi^{-1-2H} \) is a decreasing function on \((0, \infty)\). From the definition of \( n_0 \), it follows

\[
I \geq c_H |x - y|^{2H} \left( \sqrt{\frac{-\ln(1 - c^*)}{2s}} |x - y| + 2\pi \right)^{-2H} \geq c_H s^H \tag{3.15}
\]

since \( |x - y| > \sqrt{s} \) and consequently

\[
|x - y|^{2H} \left( \sqrt{\frac{-\ln(1 - c^*)}{2s}} |x - y| + 2\pi \right)^{-2H} \\
= \left( \sqrt{\frac{-\ln(1 - c^*)}{2s}} + \frac{2\pi}{|x - y|} \right)^{-2H} \geq \left( \sqrt{\frac{-\ln(1 - c^*)}{2s}} + \frac{2\pi}{\sqrt{s}} \right)^{-2H} = c_H s^H.
\]

Thus (3.14) together with (3.15) imply the desired lower bound for (3.13), which indicates

\[
d^2((t, x), (s, y)) \geq c_H (|x - y|^{H \wedge \frac{H}{2}} + (t - s)^{\frac{H}{2}}).
\]

Combining (3.12) and (3.16), we have completed the proof of this lemma. \( \square \)

Now we are ready to prove Theorem 1.1, which gives a sharp bound for

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T, -L \leq x \leq L} |u_{\text{add}}(t, x)| \right].
\]

**Proof of the first part of Theorem 1.1.** To simplify notation we denote

\[
T = [0, T] \quad \text{and} \quad \mathbb{L} = [-L, L].
\]

Since \( u_{\text{add}}(t, x) \) is a symmetric and centred Gaussian process Lemma 3.3 says that

\[
\mathbb{E} \left[ \sup_{(t, x) \in T \times \mathbb{L}} |u_{\text{add}}(t, x)| \right] \approx \mathbb{E} \left[ \sup_{(t, x) \in T \times \mathbb{L}} u_{\text{add}}(t, x) \right]. \tag{3.17}
\]

To show (1.3) it is equivalent to show

\[
c_H \rho(T, L) \leq \mathbb{E} \left[ \sup_{t \in T, x \in \mathbb{L}} u_{\text{add}}(t, x) \right] \leq C_H \rho(T, L), \tag{3.18}
\]

where \( \rho(T, L) \) is defined by (1.4). We shall prove the upper and lower bound parts of (3.18) separately. Let us first consider the upper bound part in (3.18). We shall use the majorizing measure theorem 5.1 and our bound for the natural distance
Let us separate the proof into the cases \( L > \sqrt{T} \) and \( L \leq \sqrt{T} \). First, we assume \( L > \sqrt{T} \). We choose the admissible sequences \((A_n)\) as uniform partition of \( \mathbb{T} \times \mathbb{L} = [0, T] \times [-L, L] \) such that \( \text{card}(A_n) \leq 2^n \). More precisely, we partition \([0, T] \times [-L, L]\) as

\[
[0, T] = \bigcup_{j=0}^{2^{n-1}-1} \left[ j \cdot 2^{-2^n-1} T, (j + 1) \cdot 2^{-2^n-1} T \right),
\]

\[
[-L, L] = \bigcup_{k=-2^{n-2}}^{2^{n-2}-1} \left[ k \cdot 2^{-2^n-2} L, (k + 1) \cdot 2^{-2^n-2} L \right).
\]

Theorem 3.1 states

\[
\mathbb{E} \left[ \sup_{(t, x) \in \mathbb{T} \times \mathbb{L}} u_{\text{add}}(t, x) \right] \leq C \gamma_2(T, d) \leq C \sup_{(t, x) \in \mathbb{T} \times \mathbb{L}} \sum_{n \geq 0} 2^{n/2} \text{diam } (A_n(t, x)). \quad (3.19)
\]

Here \( A_n(t, x) \) is the element of uniform partition \( A_n \) that contains \((t, x)\), i.e.

\[
A_n(t, x) = \left[ j \cdot 2^{-2^n-1} T, (j + 1) \cdot 2^{-2^n-1} T \right) \times \left[ k \cdot 2^{-2^n-2} L, (k + 1) \cdot 2^{-2^n-2} L \right)
\]

such that \( j \cdot 2^{-2^n-1} T \leq t < (j + 1) \cdot 2^{-2^n-1} T \) and \( k \cdot 2^{-2^n-2} L \leq x < (k + 1) \cdot 2^{-2^n-2} L \).

We only need to estimate diameter of each \( A_n(t, x) \). Since \((A_n)\) is a uniform partition, the diameter of \( A_n(t, x) \) with respect to \( d_{1, H}((t, x), (s, y)) \) defined in (3.9) can be estimated as

\[
\text{diam } (A_n(t, x)) \leq C_H \left( T^{\frac{\nu}{2}} \wedge (2^{-H} 2^n L^H) \right) + C_H 2^{-H} 2^n L^{\frac{\nu}{2}}.
\]

Let \( N_0 \) be the smallest integer such that \( 2^{-2^n-2} L \leq \sqrt{T} \), i.e. \( \log_2(\log_2(L/\sqrt{T})) + 2 \leq N_0 < \log_2(\log_2(L/\sqrt{T})) + 3 \). By (3.19) we have

\[
\mathbb{E} \left[ \sup_{(t, x) \in \mathbb{T} \times \mathbb{L}} u(t, x) \right] \\
\leq C_H \sup_{(t, x) \in \mathbb{T} \times \mathbb{L}} \left[ \sum_{n=0}^{N_0} 2^{n/2} \text{diam } (A_n(t, x)) + \sum_{n=N_0+1}^{\infty} 2^{n/2} \text{diam } (A_n(t, x)) \right] \\
+ C_H \sup_{(t, x) \in \mathbb{T} \times \mathbb{L}} T^{\frac{\nu}{2}} \sum_{n=0}^{\infty} 2^n \cdot 2^{-2^n-2} \\
\leq C_H T^{\frac{\nu}{2}} \left[ \sum_{n=0}^{N_0} 2^{n/2} + \sum_{n=N_0+1}^{\infty} 2^{n/2} \left( \frac{2^{2^n-2} L^H}{2^{2^n-2}} \right) \right] + C_H T^{\frac{\nu}{2}} \\
\leq C_H T^{\frac{\nu}{2}} \left[ \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} + 1 \right] + C_H T^{\frac{\nu}{2}},
\]

where \( L > \sqrt{T} \). This concludes proof of the upper bound in (3.18) when \( L > \sqrt{T} \).
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Now we prove the upper bound part in (3.18) when \( L \leq \sqrt{T} \). The same uniform partition discussed above is still applicable. We have

\[
\mathbb{E} \left[ \sup_{(t,x) \in [0,T] \times \Lambda} |u(t,x)| \right] 
\leq C_H \left[ \sum_{n=0}^{\infty} 2^{n/2} \sup_{(t,x) \in [0,T] \times \Lambda} \text{diam}(A_n(t,x)) \right] + C_H \sup_{(t,x) \in [0,T] \times \Lambda} T^{\frac{H}{4}} \sum_{n=0}^{\infty} 2^{-H-2n-2} 
\leq C_H T^{\frac{H}{4}} \sum_{n=0}^{\infty} 2^{n/2} \cdot 2^{-H-2n-1} + C_H T^{\frac{H}{4}} \leq C_H T^{\frac{H}{4}},
\]

because

\[
\sup_{(t,x) \in [0,T] \times \Lambda} \text{diam}(A_n(t,x)) \leq C_H \left[ (2^{-2n-1} L)^{H} + (2^{-2n-1} T)^{\frac{H}{2}} \right] \leq C_H 2^{-H-2n-2} T^{\frac{H}{4}}.
\]

This completes the upper bounds part of (3.18).

We will utilize Theorem 3.3 (Sudakov minoration Theorem) to prove the lower bound in (3.18). We also divide the proof into two cases: \( L > \sqrt{T} \) and \( L \leq \sqrt{T} \).

First, we consider the case \( L > \sqrt{T} \). Select \( \delta \) in Theorem 3.3 as \( c_H T^{\frac{H}{4}} \) with certain relatively small \( c_H > 0 \). For the sequence \( \{u(T,x_i), i = 0, 1, \cdots, N\} \), where \( N = \lfloor L/\sqrt{T} \rfloor \geq 1 \) by the assumption and

\[
x_0 = 0, x_{\pm 1} = \pm \sqrt{T}, \cdots, x_{\pm N} = \pm N \sqrt{T},
\]

we have

\[
d_{1,H}((T,x_i),(T,x_j)) \geq c_H T^{\frac{H}{4}} = \delta \quad \text{if} \quad i \neq j.
\]

Sudakov’s minoration theorem implies

\[
\mathbb{E} \left[ \sup_{(t,x) \in [0,T] \times \Lambda} |u(t,x)| \right] \geq \mathbb{E} \left[ \sup_{i} u(T,x_i) \right] 
\geq c_H \delta \sqrt{\log_2(2N+1)} \geq c_H T^{\frac{H}{4}} \left[ \log_2 \left( \frac{L}{\sqrt{T}} \right) + 1 \right].
\]

The lower bound in (3.18) is established when \( L > \sqrt{T} \).

Now we prove the lower bound part in (3.18) when \( L \leq \sqrt{T} \). We choose \( \delta = c_H T^{\frac{H}{4}} \) as above and we choose \( u(T/2, 0), u(T, 0) \) as our comparison set. We have \( d_{1,H}((T/2, 0), (T, 0)) \geq c_H (T/2)^{\frac{H}{4}} \geq \delta \). Theorem 3.3 gives

\[
\mathbb{E} \left[ \sup_{(t,x) \in [0,T] \times \Lambda} |u(t,x)| \right] \geq \mathbb{E}[u(T/2, 0) \vee u(T, 0)] \geq c_H T^{\frac{H}{4}}.
\]

Thus, the proof of the lower bound part in (3.18) is completed. \( \square \)

Notice that for any fixed \( t \in \mathbb{R}^+ \)

\[
d_1((t,x),(t,y)) \asymp d_{1,H}(x,y) := t^{\frac{H}{4}} |x-y|^H,
\]

and for fixed \( x \in \mathbb{R} \)

\[
d_1((t,x),(s,x)) \asymp d_{1,H}(t,s) := |t-s|^{\frac{H}{4}}.
\]

Similar to the argument in the proof of inequality (1.3) we have the following corollary.
Corollary 3.8. Let the Gaussian field \( u_{\text{add}}(t,x) \) be defined by (3.2). There are positive universal constants \( c_H \) and \( C_H \) such that

\[
\begin{aligned}
&\left\{ c_H T^\frac{\mu}{2} \sqrt{\log_2(L)} \leq \mathbb{E} \left[ \sup_{-L \leq x \leq L} |u_{\text{add}}(t,x)| \right] \\
\leq &\mathbb{E} \left[ \sup_{-L \leq x \leq L} u_{\text{add}}(t,x) \right] \leq C_H T^\frac{\mu}{2} \sqrt{\log_2(L)} ;
\end{aligned}
\tag{3.26}
\]

Next, we shall explain that the almost sure version of Theorem 1.1, which is an extension of Theorem 1.2 of [6] and Theorem 2.3 of [7] to spatial rough noise, is a consequence of (1.3) with the aid of Borell’s inequality (Theorem 3.4).

Proof of the second part of Theorem 1.1. First, we shall prove (1.3) for \( T = n^\alpha \) for some \( \alpha \) and for all sufficiently large integer \( n \). Denote \( L := [-L,L] \), \( \mathbb{T}^\alpha = [0,n^\alpha] \). For some \( \varepsilon > 0 \) and integer \( n \) sufficiently large such that \( L \geq n^{\frac{\varepsilon}{1+2\alpha}} \). We start with the lower bound, Theorem 1.1 gives

\[
\mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \right] \geq c_H \left( \sqrt{\frac{n^\mu}{2}} + n^\frac{\mu}{2} \sqrt{\log_2 \left( \frac{L}{n^{\alpha/2}} \right)} \right)
\]

for some positive number \( c_H \). Denote

\[
\lambda_H := \lambda_H(\mathbb{T}^\alpha \times L) = \frac{1}{2} \mathbb{E} \left[ \sup_{x \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \right],
\]

and

\[
\sigma_H^2 := \sigma_H^2(\mathbb{T}^\alpha \times L) = \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} |u_{\text{add}}(t,x)|^2 \right] = C_H n^\frac{\mu}{2}
\]

Then, Borell’s inequality implies

\[
\mathbb{P} \left\{ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) < \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \right] \right\} \leq 2 \exp \left( -\frac{\lambda_H^2}{2\sigma_H^4} \right)
\leq 2 \exp \left( -c_H \left[ 1 + \log_2 \left( \frac{L}{n^{\alpha/2}} \right) \right] \right) \leq C_H \left( \frac{n^\alpha}{n^{\alpha(1+\varepsilon)}} \right) \leq C_H n^{-\alpha \varepsilon \frac{\mu}{2}},
\]

where \( c_H, C_H > 0 \) are some constants independent of \( n \). Select real number \( \alpha \) sufficiently large such that \( \alpha \varepsilon \cdot \frac{\mu}{2} > 1 \) and define the events \( F_n \)

\[
F_n := \left\{ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) < \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \right] \right\}.
\]

The bound (3.27) means \( \sum_{n=1}^{\infty} \mathbb{P}(F_n) < \infty \). An application of Borel-Cantelli’s lemma yields that \( \mathbb{P}(\limsup_n F_n) = 0 \). This means that

\[
\sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \geq \frac{1}{2} \mathbb{E} \left[ \sup_{(t,x) \in \mathbb{T}^\alpha \times L} u_{\text{add}}(t,x) \right] \geq c_H \left( T^\frac{\mu}{2} + T^\frac{\mu}{2} \sqrt{\log_2 \left( \frac{L}{\sqrt{T}} \right)} \right),
\]

with \( T = n^\alpha \) almost surely for sufficiently large values of \( n \). This proves lower bound part of (1.5) for \( T = n^\alpha \).
The proof of the upper bound in [1.5] can be done in exactly the same manner as that in the proof of the lower bound except now we replace (3.27) by

\[
P \left\{ \sup_{(t,x) \in T^\alpha \times L} u_{ad}(t,x) > \frac{3}{2} \mathbb{E} \left[ \sup_{(t,x) \in T^\alpha \times L} u_{ad}(t,x) \right] \right\} \leq 2 \exp \left( -\frac{\lambda^2_H}{2\sigma_H^2} \right)
\]

(3.29)

\[
\leq 2 \exp \left( -c_H \left[ 1 + \log_2 \left( \frac{L}{n^{\alpha/2}} \right) \right] \right) \leq C_H \left[ \frac{n^\alpha}{n^{\alpha(1+\epsilon)}} \right]^{\frac{2}{\alpha+2}} \leq C_H n^{-\alpha-\frac{2\alpha}{\alpha+2}},
\]

with some positive constant \(c_H, C_H\) independent of \(n\). Similar to (3.28) we can obtain by letting \(T = n^\alpha\)

\[
\sup_{(t,x) \in T^\alpha \times L} u_{ad}(t,x) \leq \frac{3}{2} \mathbb{E} \left[ \sup_{(t,x) \in T^\alpha \times L} u_{ad}(t,x) \right] \leq C_H \left( \frac{T^2 + T^2}{\sqrt{\log_2 \left( \frac{L}{n^{\alpha}} \right)}} \right).
\]

(3.30)

almost surely for sufficiently large \(n\).

Finally, we can conclude the proof of the \(\sup_{(t,x)} u_{ad}(t,x)\) part in [1.5] by combining (3.28), (3.30), and the property that \(\sup_{(t,x) \in T \times L} u_{ad}(t,x)\) is an increasing function of \(L\) and \(T\) almost surely. Also it is easy to see

\[
\sup_x |f(x)| \leq \sup_x |f(x)| + \sup_x |f(x)|
\]

since \(|f(x)| \leq \sup_x |f(x)| + \sup_x |f(x)|\) for any function \(f(x)\). Since \(u_{ad}(t,x)\) is symmetric, we see that \(\sup_{t,x} |u_{ad}(t,x)|\) and \(\sup_{t,x} |u_{ad}(t,x)|\) have the same law.

Then, we have

\[
\sup_{t,x} |u_{ad}(t,x)| \leq 2 \sup_{t,x} |u_{ad}(t,x)|.
\]

This completes the proof of [1.5]. \(\square\)

One can show the following asymptotic (3.31) by combining (3.26) and Borell’s inequality but we omit the details.

**Corollary 3.9.** Let \(u_{ad}(t,x)\) be defined by (3.2) and \(T\) satisfies \(T < L^2\). Then, there are two positive random constants \(c_H\) and \(C_H\) such that for any fixed \(t \in [0,T]\) we have

\[
c_H t^\frac{\mu}{2} \sqrt{\log_2(L)} \leq \sup_{-L \leq x \leq L} |u_{ad}(t,x)| \leq C_H t^\frac{\mu}{2} \sqrt{\log_2(L)},
\]

(3.31)

almost surely.

**Remark 3.10.** As in [6, 7], there exist some constants \(c, C > 0\) such that

\[
c t^\frac{\mu}{2} \leq \lim_{|x| \to \infty} \frac{u_{ad}(t,x)}{\sqrt{\log_2(|x|)}} \leq \limsup_{|x| \to \infty} \frac{u_{ad}(t,x)}{\sqrt{\log_2(|x|)}} \leq C t^\frac{\mu}{2},
\]

(3.32)

for any \(t \in \mathbb{R}_+\) almost surely.

We now turn to show Theorem 1.2.

**Proof of Theorem 1.2.** \(\Delta_h u_{ad}(t,x)\) is centered symmetric and stationary Gaussian process. As before, we only need to find appropriate bounds \(t\) for \(\Delta_h u_{ad}(t,x)\) instead of \(|\Delta_h u_{ad}(t,x)|\) inside the \(\sup_{x \in \mathbb{R}}\). Our strategy to complete this is also to
apply Talagrand’s majorizing measure theorem and Sudakov minoration theorem to the following Gaussian process

$$
\Delta_h u_{\text{add}}(t, x) := u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)
$$

$$
= \int_0^t \int_{\mathbb{R}} [G_{t-s}(x + h - z) - G_{t-s}(x - z)]W(dz, ds),
$$

(3.33)

with fixed $t > 0$ and fixed $h \neq 0$. Without loss of generality, we assume $h > 0$. The natural metric is given by

$$
d_{2, t, h}(x, y) := \left( \mathbb{E}[|\Delta_h u_{\text{add}}(t, x) - \Delta_h u_{\text{add}}(t, y)|^2] \right)^{\frac{1}{2}}.
$$

We need to obtain good upper and lower bounds of $d_{2, t, h}(x, y)$. Let us first focus on the upper bound. Similar to (3.10) Plancherel’s identity yields

$$
d_{2, t, h}(x, y) = C_H \int_{\mathbb{R}_+} [1 - \exp(-2t\xi^2)] [1 - \cos(|x - y|\xi)][1 - \cos(h\xi)] \cdot \xi^{-1-2H} d\xi.
$$

By the same argument as that in the proof of the upper bound of $d_1((s, x), (s, y))$ in Lemma 3.6 it is easy to see for any $0 \leq \theta \leq 1$

$$
d_{2, t, h}^2(x, y) \leq C_H \int_{\mathbb{R}_+} [1 - \exp(-2t\xi^2)] [1 - \cos(h\xi)] \cdot \xi^{-1-2H} d\xi
$$

$$
\leq C_H t^H \land h^{2H} \leq C_H t^{H-\theta} h^{2\theta},
$$

On the other hand, an application of the elementary inequality $1 - \cos(x) \leq C_0 x^{2\theta}$, where $\theta \in (0, H)$ is as above, and a substitution $\xi \to \xi/|x - y|$ yield

$$
d_{2, t, h}^2(x, y) \leq C_{\theta, H} h^{2\theta} |x - y|^{2H-2\theta} \int_{\mathbb{R}_+} [1 - \cos(\xi)] \xi^{2\theta-1-2H} d\xi
$$

$$
\leq C_{\theta, H} h^{2\theta} |x - y|^{2H-2\theta}.
$$

In conclusion, we have the following bound analogous to upper bound part of (3.9):

$$
d_{2, t, h}(x, y) \leq C_{H, \theta} h^\theta (|x - y|^{H-\theta} \land t^{\frac{H-\theta}{2}}),
$$

(3.34)

for any $\theta \in (0, H)$.

Now we can follow the same argument as that in the proof of Theorem 1.1 by invoking Talagrand’s majorizing measure theorem (Theorem 3.1) to prove the upper bound part of (1.6):

$$
\mathbb{E} \left[ \sup_{x \in I} \Delta_h u_{\text{add}}(t, x) \right] \leq C_{H, \theta} \|h\|^\theta t^{\frac{H-\theta}{2}} \sqrt{\log_2 \left( \frac{L}{\sqrt{t}} \right)},
$$

if $L > \sqrt{t}$. To this end, we need the inverse part of (3.34), we shall use again the Sudakov minoration theorem. Observe that we only need to consider the case when $|x - y| \geq \sqrt{t}$. We claim

$$
d_{2, t, h}^2(x, y) \geq c_H h^{2H} \text{ when } |x - y| \geq \sqrt{t} \text{ and } h \leq \sqrt{\frac{\pi^2}{8 \ln 2}} \land 1.
$$

In fact, notice that

$$
1 - \exp \left( \frac{2t\xi^2}{|x - y|^2} \right) \geq \frac{1}{2} \quad \forall \xi \geq \frac{|x - y|\pi}{4h} \text{ and } h \leq \sqrt{\frac{\pi^2}{8 \ln 2}} \land 1.
$$

The simple inequality

$$
1 - \cos(x) \geq \frac{x^2}{4} \text{ if } |x| \leq \frac{\pi}{2}.
$$
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implies

\[ 1 - \cos \left( \frac{h \xi}{|x - y|} \right) \geq \frac{h^2 \xi^2}{4|x - y|^2} \quad \text{if} \quad \xi \leq \frac{|x - y|}{2h}. \]

Therefore, a substitution \( \xi \rightarrow \xi / |x - y| \) yields

\[
\begin{align*}
d^2_{2,t,h}(x, y) &= c_H |x - y|^{2H} \int_{\mathbb{R}_+} \left[ 1 - \exp \left( - \frac{2t \xi^2}{|x - y|^2} \right) \right] \left[ 1 - \cos \left( \frac{h \xi}{|x - y|} \right) \right] \cdot [1 - \cos(\xi)] \xi^{-1-2H} d\xi \\
&\geq c_H |x - y|^{2H} \int_{\mathbb{R}_+} \left[ 1 - \cos \left( \frac{h \xi}{|x - y|} \right) \right] \cdot [1 - \cos(\xi)] \xi^{-1-2H} d\xi \\
&\geq c_H h^2 |x - y|^{2H-2} \int_{\mathbb{R}_+} \left[ 1 - \cos(\xi) \right] \xi^{-1-2H} d\xi,
\end{align*}
\]

Set

\[ k_0 = \inf \left\{ k \in \mathbb{N}_0 : \frac{(2k + 1)\pi}{2} \geq \frac{|x - y|\pi}{4h} \right\}; \]

and

\[ k_1 = \sup \left\{ k \in \mathbb{N}_0 : \frac{(2k + 3)\pi}{2} \leq \frac{|x - y|\pi}{2h} \right\}. \]

If \( h \) is sufficiently small, then

\[
\begin{align*}
&\int_{\mathbb{R}_+} \left[ 1 - \cos(\xi) \right] \xi^{-1-2H} d\xi = \sum_{k \geq 0} \int_{I_k} \int_{I_{k+1}} \left[ 1 - \cos(\xi) \right] \xi^{-1-2H} d\xi \\
&\geq k_1 \int_{I_k} \left[ 1 - \cos(\xi) \right] \xi^{-1-2H} d\xi \geq \frac{1}{2} \left( \frac{(2k_1 + 3)\pi}{2} \right)^{2-2H} \xi^{1-2H} d\xi \\
&= c_H \left[ \left( \frac{(2k_1 + 3)\pi}{2} \right)^{2-2H} \right. - \left( \frac{(2k_0 + 1)\pi}{2} \right)^{2-2H} \right] \geq c_H \left( \frac{|x - y|}{h} \right)^{2-2H},
\end{align*}
\]

due to the fact that \( \xi^{1-2H} \) is an increasing function. Thus we have for \( |x - y| \geq \sqrt{t} \)

\[ d^2_{2,t,h}(x, y) \geq c_H h^H, \quad (3.35) \]

if \( h \leq C(\sqrt{t} \wedge 1) \) for some small positive quantity \( C \). On the interval \( L = [-L, L] \) for \( L \) large enough, let us select \( x_j = jL / \sqrt{t} \) for \( j = 0, \pm 1, \cdots, \pm [L / \sqrt{t}] \). Applying Sudakov minoration theorem (Theorem 3.3) with \( \delta = c_H |h|^H \) yields

\[
E \left[ \sup_{x \in L} \Delta_h u_{add}(t, x) \right] \geq E \left[ \sup_{x_i} \Delta_h u_{add}(t, x) \right] \geq c_H |h|^H \sqrt{\log_2 \left( \frac{L}{\sqrt{t}} \right)}.
\]
The proof of (1.7) follows from exactly the same argument as that in the proof of (1.6) by Borel-Cantelli’s lemma. The only difference is that now we have

\[
\sigma^2(h) = \sup_{x \in L^0} \mathbb{E}[|\Delta_h u_{add}(t, x)|^2] \leq C_{H, \theta} t^{H-\theta}|h|^{2\theta};
\]

\[
\lambda_L := \frac{1}{2} \mathbb{E} \left[ \sup_{x \in L^0} \Delta_h u_{add}(t, x) \right];
\]

\[
\exp \left( -\frac{\lambda^2}{2\sigma^2(h)} \right) \leq C_{H, \theta} \exp \left( -\frac{h^2}{t} H - \theta \log_2 \frac{n}{\sqrt{t}} \right),
\]

with \( L^0 := [-n^\alpha, n^\alpha] \). We can then complete the proof of the theorem by choosing \( \alpha \) appropriately. We omit the details here. \qed

**Proof of Theorem 1.3.** We will use the same method as before. The natural metric associated with the time increment of the solution is

\[
d_{3, t, \tau}(x, y) = (\mathbb{E}[|\Delta_{\tau} u_{add}(t, x) - \Delta_{\tau} u_{add}(t, y)|^2])^{1/2}.
\]

Using

\[
\Delta_{\tau} u_{add}(t, x) = \int_0^{t + \tau} \int_{\mathbb{R}} G_{t+s}(x-z)W(z, ds) - \int_0^t \int_{\mathbb{R}} G_{t-s}(x-z)W(z, ds),
\]

one can derive from the isometric property of stochastic integral and Plancherel’s identity

\[
d_{3, t, \tau}^2(x, y) = 2 \int_{\mathbb{R}^+} \left[ 1 - \exp(-2(t + \tau)\xi^2) \right] \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-1-2H} d\xi
\]

\[
+ 2 \int_{\mathbb{R}^+} \left[ 1 - \exp(-2t\xi^2) \right] \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-1+2H} d\xi
\]

\[
- 4 \int_{\mathbb{R}^+} \exp(-\tau\xi^2) \left[ 1 - \exp(-2t\xi^2) \right] \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-1+2H} d\xi
\]

\[
= 2 \int_{\mathbb{R}^+} f(t, \tau, \xi) \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-1+2H} d\xi,
\]

where

\[
f(t, \tau, \xi) = \left[ 1 - \exp(-2(t + \tau)\xi^2) \right] + \left[ 1 - \exp(-2t\xi^2) \right] - 2 \exp(-\tau\xi^2) \left[ 1 - \exp(-2t\xi^2) \right]
\]

\[
= \left[ 1 - \exp(-2\tau\xi^2) \right] + \left[ 1 - \exp(-2t\xi^2) \right] + \left[ 1 + \exp(-2\tau\xi^2) \right] - 2 \exp(-\tau\xi^2).
\]

Notice that when \( x \geq 0, 1 - e^{-x} \leq C_\theta x^\theta \) and \( 1 + e^{-2x} - 2e^{-x} = (1 - e^{-x})^2 \leq C_\theta^2 x^{2\theta} \) for any \( \theta \in (0, 1) \). Then, we have

\[
f(t, \tau, \xi) \leq C_\theta (\tau\xi^2)^\theta, \quad \forall \theta \in (0, 1).
\]

Inserting this bound into (3.36) yields

\[
d_{3, t, \tau}(x, y) \leq C_\theta \tau^\theta \int_{\mathbb{R}^+} \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-1-2H+2\theta} d\xi
\]

\[
\leq C_{H, \theta} \tau^\theta |x - y|^{2H-2\theta} \quad \text{for any } 0 < \theta < H.
\]
On the other hand, a substitution \( \xi \rightarrow \xi/\sqrt{\tau} \) yields
\[
d_{3,t,\tau}^2(x,y) \leq C \int_{\mathbb{R}^+} [1 - \exp(-2\tau\xi^2)]|\xi|^{-1-2H}d\xi
\]
\[
+ \int_{\mathbb{R}^+} [1 - \exp(-2t\xi^2)][1 - \exp(-\tau\xi^2)]^2|\xi|^{-1-2H}d\xi
\]
\[
\leq C_{H,\theta}^\tau\theta + C_{H,\theta}^\tau\theta t^\theta \leq C_{H,\theta}^\tau\theta t^{H-\theta},
\]
when \( \tau \leq Ct \). Thus, we have
\[
d_{3,t,\tau}(x,y) \leq C_{H,t,\theta}^\tau\theta/2(\|x-y\|^{H-\theta} \wedge t^{H-\theta}), \tag{3.37}
\]
where \( 0 < \theta < H \), which is the bound needed for us to prove the upper bound part of (1.8).

The Sudakov minoration Theorem 3.3 will still be used to prove the lower bound. We need to obtain an appropriate lower bound of \( d_{3,t,\tau}(x,y) \) for \( |x-y| \geq \sqrt{t} \). We have
\[
d_{3,t,\tau}^2(x,y) \geq c \int_{\mathbb{R}^+} [1 - \exp(-2\tau\xi^2)]\left(1 - \cos(|x-y|\xi)\right)\xi^{-1-2H}d\xi
\]
\[
\geq c\tau|x-y|^{2H-2} \int_{\mathbb{R}^+} \left(1 - \cos(\xi)^{-1-2H}d\xi. \tag{3.38}
\]
Analogous to the obtention of (3.35) we can conclude that the integral in (3.38) is bounded below by a multiple of \( (\|x-y\|/\sqrt{\tau})^{2-2H} \). Thus, we obtain
\[
d_{3,t,\tau}(x,y) \geq c_{H,t}^\tau t^{H/2}, \tag{3.39}
\]
if \( \tau \leq C(t \wedge 1) \) for some constant \( C \). This is the bound needed to use Theorem 3.3 to show the lower bound part of (1.8).

Once again, Borell’s inequality (Theorem 3.4) can be combined with Borel-Cantelli’s lemma to show the almost sure asymptotics (1.7), and the proof Theorem 1.3 is completed.

In \([11]\) (see also next section) to show the existence and uniqueness of the solution to \((1.1)\) (for Hurst parameter \( H \in (1/4, 1/2) \)) it is extensively used the following quantity
\[
\mathcal{N}_{x-H}^2 u(t,x) = \left(\int_{\mathbb{R}} |u(t,x+h) - u(t,x)|^2 \cdot |h|^{2H-2}dh \right)^{1/2}, \tag{3.40}
\]
which plays the role of fractional derivative of \( u \). It is because of the difficulty to appropriately bound this quantity (see \([11]\) or the next section) it is assumed that \( \sigma(0) = 0 \) in \([11]\). After our work on the bound of the solution \( u_{\text{add}}(t,x) \) we want to argue that
\[
\mathbb{E}[\sup_{x \in L} \mathcal{N}_{x-H}^2 u_{\text{add}}(t,x)] \geq c_{t,H} \log_2(L) \quad \text{if } L \text{ is sufficiently large.} \tag{3.41}
\]
This fact illustrates that the argument in \([11]\) for the pathwise uniqueness (see Lemma 4.9 in \([11]\) for this argument) is not applicable in the general setting when \( \sigma(0) \neq 0 \). Here is the precise statement of our result, which is also interesting for its own sake.
Proposition 3.11. Let \( u_{\text{add}}(t, x) \) be defined by (3.2) and let \( \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t, x) \) be defined be (3.40).

(i) For any fixed \( t > 0 \) and \( L \geq \sqrt{t} \) we have

\[
\mathbb{E} \left[ \sup_{-L \leq x \leq L} \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t, x) \right] \geq c_H [1 - \exp(-2t)] \log_2(L), \tag{3.42}
\]

where \( c_H \) is a positive constant.

(ii) Moreover, we have

\[
\sup_{-L \leq x \leq L} \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t, x) \leq C_H \left( t^H + t^{H-\theta} \right) \sqrt{\log_2(L)} \quad \text{almost surely}, \tag{3.43}
\]

where \( C_H \) is a positive random constant and \( \theta > \frac{1-2H}{2} \).

Proof. First, we consider the upper bound (3.43). Applying Theorem 1.2 when \( |h| \leq 1 \) and Theorem 3.9 when \( |h| > 1 \), respectively, we obtain

\[
\sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t, x) = \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)]^2 \cdot |h|^{2H-2} dh
\]

\[
\leq \int_{\mathbb{R}} \left( \sup_{x \in \mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \right)^2 \cdot |h|^{2H-2} dh
\]

\[
\leq \int_{\{|h| \leq 1\}} \left( \sup_{x \in \mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \right)^2 \cdot |h|^{2H-2} dh
\]

\[
+ \int_{\{|h| \geq 1\}} \left( \sup_{x \in \mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \right)^2 \cdot |h|^{2H-2} dh
\]

\[
\leq C_H \rho t^{H-\theta} \log_2(L) \int_{\{|h| \leq 1\}} |h|^{2H-2+2\theta} dh
\]

\[
+ C_H t^H \log_2(L) \left[ \int_{\{|h| \geq 1\}} |h|^{2H-2} dh + \int_{\{|h| \geq 1\}} \log_2(h) |h|^{2H-2} dh \right],
\]

where we applied an elementary inequality

\[
|\log_2 (L + h)| \leq \begin{cases} 
\log_2(L) + 1 & \text{when } |h| \leq 1; \\
\log_2(L) + \log_2(h) + 1 & \text{when } |h| \geq 1.
\end{cases}
\]

Letting \( \theta > \frac{1-2H}{2} \) yields (3.43).

Now we turn to the lower bound (3.42). A simple observation and an application of Jensen’s inequality give

\[
\mathbb{E} \left[ \sup_{x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u_{\text{add}}(t, x) \right]
\]

\[
\geq c_H \mathbb{E} \left[ \left( \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \varphi(h) dh \right)^2 \right] \tag{3.44}
\]

\[
\geq c_H \left( \mathbb{E} \left[ \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \varphi(h) dh \right]^2 \right),
\]

\[
28
\]
where \( \varrho(h) = |h|^{2H-\frac{1}{2}}\mathbf{1}_{|h|\leq 1} + |h|^{2H-2}\mathbf{1}_{|h|>1} \). Denote
\[
\hat{u}_\varrho(t, x) = \int_{\mathbb{R}} [u_{\text{add}}(t, x + h) - u_{\text{add}}(t, x)] \varrho(h) dh
\]
\[
= \int_0^t \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |G_t-s(x + h - z) - G_t-s(x - z)| \varrho(h) dh \right) W(dz, ds).
\]
It is a well-defined Gaussian random field since \( \varrho(h) \) is integrable for \( \frac{1}{4} < H < \frac{1}{2} \).

Introduce the induced natural metric
\[
d_{\text{add}}(x, y) := (\mathbb{E}[u_\varrho(t, x) - u_\varrho(t, y)]^2)^{\frac{1}{2}}.
\]
We need to bound this distance for \(|x - y| \geq 1\). Applying Plancherel’s identity we can find
\[
d_{\text{add}}^2(x, y) = c_H \int_{\mathbb{R}^+} \left[ 1 - \exp(-2t\xi^2) \right] \left[ 1 - \cos(|x - y|\xi) \right] \cdot \left( \int_{\mathbb{R}^+} \left[ 1 - \cos(h\xi) \right] \varrho(h) dh \right)^2 \cdot \xi^{-1-2H} d\xi.
\]
When \( \xi \geq 1 \), we have
\[
\int_{\mathbb{R}^+} \left[ 1 - \cos(h\xi) \right] \varrho(h) dh \geq \xi^{\frac{1}{2}-2H} \int_0^\xi \left[ 1 - \cos(h) \right] \cdot h^{2H-\frac{3}{2}} dh \geq c_\xi \xi^{\frac{1}{2}-2H}.
\]
Thus, we conclude that if \(|x - y| \geq 1\), then
\[
d_{\text{add}}^2(x, y) \geq c_H \left[ 1 - \exp(-2t) \right] \int_1^\infty \left[ 1 - \cos(|x - y|\xi) \right] \cdot \xi^{-6H} d\xi
\]
\[
\geq c_H \left[ 1 - \exp(-2t) \right] \tag{3.45}
\]
by the same argument as that in proof of lower bound of \( \mathbb{E}[\sup_{x \in \mathbb{L}} \Delta_h u_{\text{add}}(t, x)] \) in Theorem 1.2. Thus, an application of the Sudakov minoration Theorem 3.3 implies the lower bound.

\section{Weak Existence and Regularity of Solutions}

\subsection{Basic settings}

This section is devoted to prove the existence of a weak solution to \((1.1)\). Let us recall some notations and facts in \([11]\). Let \((B, \| \cdot \|_B)\) be a Banach space with the norm \(\| \cdot \|_B\). Let \(\beta \in (0, 1)\) be a fixed number. For any function \(f : \mathbb{R} \rightarrow B\) denote
\[
N_\beta f(x) = \left( \int_{\mathbb{R}} \| f(x + h) - f(x) \|_B^2 |h|^{-1-2\beta} dh \right)^\frac{1}{2},
\]
if the above quantity is finite. When \(B = \mathbb{R}\), we abbreviate the notation \(N^B_\beta f\) as \(N_\beta f\). With this notation, the norm of the homogeneous Sobolev space \(\dot{H}^\beta\) defined by \((2.5)\) can be given by using \(N_\beta f : \| f \|_{\dot{H}^\beta} = \| N_\beta f \|_{L^2(\mathbb{R})}\). As in \([11]\) throughout this paper we are particularly interested in the case \(B = L^p(\Omega)\), and in this case we denote \(N_\beta^B\) by \(N_{\beta,p}\):
\[
N_{\beta,p} f(x) = \left( \int_{\mathbb{R}} \| f(x + h) - f(x) \|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^\frac{1}{2}.
\]
The following Burkholder-Davis-Gundy inequality is well-known (see e.g. \([11]\)).
Proposition 4.1. Let $W$ be the Gaussian noise defined by the covariance (2.1), and let $f \in \Lambda_H$ be a predictable random field. Then for any $p \geq 2$ we have
\[
\left\| \int_0^t \int_R f(s,y)W(dy,ds) \right\|_{L^p(\Omega)} \leq \sqrt{4pc_3\sqrt{2}-H} \left( \int_0^t \int_R \left[ N_{\frac{3}{2}-H,p}f(s,y) \right]^2 dyds \right)^{\frac{1}{2}},
\]
where $c_3K_{\sqrt{2}-H}$ is a constant depending only on $H$ and $N_{\frac{3}{2}-H,p}f(s,y)$ denotes the application of $N_{\frac{3}{2}-H,p}$ to the space variable $y$.

In the work [11], the authors have already proved the existence and uniqueness result in a solution space $Z^p_T$ (see [11] or next paragraph, formula (4.4), for the definition of $Z^p_T$) under condition $\sigma(t,x,0) = 0$. When $\sigma(t,x,0) \neq 0$ or even in the simplest case $\sigma(t,x,u) = 1$ (as we see from (3.12)) we cannot expect that the solution is still in $Z^p_T$. So, the method powerful in [11] is no longer valid to solve the equation (1.1) for general $\sigma(t,x,u)$. Our idea is to add an appropriate weight $\lambda(x)$ to the space $Z^p_T$ to obtain a weighted space $Z^p_{\lambda,T}$.

Let $\lambda(x) \geq 0$ be a Lebesgue integrable positive function with $\int_R \lambda(x)dx = 1$. Introduce a norm $\| \cdot \|_{Z^p_{\lambda,T}}$ for a random field $v(t,x)$ as follows:
\[
\|v\|_{Z^p_{\lambda,T}} := \sup_{t \in [0,T]} \|v(t,\cdot)\|_{L^p(\Omega \times R)} + \sup_{t \in [0,T]} N_{\frac{3}{2}-H,p}v(t),
\]
where $p \geq 2$, $\frac{1}{2} < H < \frac{3}{2}$.

Then $Z^p_{\lambda,T}$ is the function space consisting of all the random fields $v = v(t,x)$ such that $\|v\|_{Z^p_{\lambda,T}}$ is finite. When the function is independent of $t$, the corresponding space is denoted by $Z^p_{\lambda,0}$.

4.2. Some bounds for stochastic convolutions. To prove the existence of weak solution, we need some delicate estimates of stochastic integral with respect to the weight.

Proposition 4.2. Denote the weight function
\[
\lambda(x) = \lambda_H(x) = c_H(1 + |x|^2)^{H-1},
\]
where $c_H$ is a constant such that $\int \lambda(x)dx = 1$, and denote
\[
\Phi(t,x) = \int_0^t \int_R G_{t-s}(x-y)v(s,y)W(dy,ds).
\]
We have the following estimates. [In the following $C_{T,p,H,\gamma}$ denotes a constant, depending only on $T$, $p$, $H$, and $\gamma$.]

(i) If $p > \frac{3}{H}$, then
\[
\left\| \sup_{t \in [0,T], x \in R} \lambda^{\frac{1}{p}}(x) \Phi(t,x) \right\|_{L^p(\Omega)} \leq C_{T,p,H} \|v\|_{Z^p_{\lambda,T}}.
\]
(ii) If \( p > \frac{6}{H - 1} \), then
\[
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda^\frac{p}{H} (x) N_{2-H} \Phi(t, x) \right\|_{L^p(\Omega)} \leq C_{T,p,H} \|v\|_{\mathcal{H}^{\gamma,p}_{\lambda,x}}. \tag{4.9}
\]

(iii) If \( p > \frac{3}{H} \) and \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \), then
\[
\left\| \sup_{t,t+h \in [0,T], x \in \mathbb{R}} \lambda^\frac{p}{H} (x) \left[ \Phi(t+h, x) - \Phi(t, x) \right] \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^{\gamma} \|v\|_{\mathcal{H}^{\gamma,p}_{\lambda,x}}. \tag{4.10}
\]

(iv) If \( p > \frac{3}{H} \) and \( 0 < \gamma < H - \frac{3}{p} \), then
\[
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \frac{\Phi(t, x) - \Phi(t, y)}{\lambda^\frac{p}{H} (x) + \lambda^\frac{p}{H} (y)} \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |x - y|^\gamma \|v\|_{\mathcal{H}^{\gamma,p}_{\lambda,x}}. \tag{4.11}
\]

**Remark 4.3.** The method provided here depends on the semigroup property of heat kernel because we need to use factorization method (see (4.13)). Consequently, we can not apply this approach directly to stochastic wave equation since wave kernel does not satisfy semigroup property.

**Proof.** For any \( \alpha \in (0, 1) \) we set
\[
J_{\alpha}(r, z) := \int_0^T \int_{\mathbb{R}} (r - s)^{-\alpha} G_{r-s}(z - y)v(s, y)W(dy, ds). \tag{4.12}
\]
A stochastic version of Fubini’s theorem implies
\[
\Phi(t, x) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha-1} G_{t-r}(x - z)J_{\alpha}(r, z)dzdr. \tag{4.13}
\]
We are going to show the four different parts of the proposition separately. We divide our proof into six steps.

**Step 1.** The first two steps are to prove part (i). In this step we will obtain the desired growth estimate of \( \Phi(t, x) \) in term of \( J_{\alpha}(r, z) \). From our expression (4.13) and inequality (2.22) together with (2.12) we have
\[
\sup_{t,x} \lambda^\theta(x) |\Phi(t, x)| \lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t \left| \int_{\mathbb{R}} (t - r)^{\alpha-1} G_{t-r}(x - z)J_{\alpha}(r, z)dzdr \right|
\]
\[
\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t - r)^{\alpha-1} \left( \int_{\mathbb{R}} |G_{t-r}(x - z)|^\frac{p}{\alpha}dz \right)^\frac{\alpha}{p} \left\| J_{\alpha}(r, \cdot) \right\|_{L^\frac{p}{\alpha}(\mathbb{R})}dr
\]
\[
\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t - r)^{\alpha-1} \left( \int_{\mathbb{R}} (t - r)^{\frac{1-\alpha}{2}} G_{(t-r)/q}(x - z)dz \right)^\frac{\alpha}{p} \left\| J_{\alpha}(r, \cdot) \right\|_{L^\frac{p}{\alpha}(\mathbb{R})}dr
\]
\[
\lesssim \sup_{t,x} \lambda^\theta(x) \int_0^t (t - r)^{\alpha-1} \cdot (t - r)^{\frac{1-\alpha}{2q}} \lambda^{-\frac{p}{H}}(x) \cdot \left\| J_{\alpha}(r, \cdot) \right\|_{L^\frac{p}{\alpha}(\mathbb{R})}dr.
\]
introduce the following two notations

Step 2. We shall prove the above bound (4.16) in this step and to do this let us

Setting \( \theta = \frac{1}{p} \) and then applying the Hölder inequality we obtain

\[
\sup_{t,x} \lambda^\theta(x)|\Phi(t, x)| \lesssim \sup_{t \in [0, T]} \left[ \frac{1}{T} \int_0^T (t-r)^{\alpha - \frac{3}{2} + \frac{1}{p}} \cdot \|J_\alpha(r, \cdot)\|_{L_p^\alpha(\mathcal{R})} \, dr \right]
\]

\[
\lesssim \left[ \sup_{t \in [0, T]} \left[ \frac{1}{T} \int_0^T (t-r)^{\alpha - \frac{3}{2} + \frac{1}{p}} \, dr \right] \cdot \left[ \frac{1}{T} \int_0^T \|J_\alpha(r, \cdot)\|^p_{L_p^\alpha(\mathcal{R})} \, dr \right] \right]^{\frac{1}{p}}
\]

(4.14)

if \( q(\alpha - \frac{3}{2} + \frac{1}{p}) > -1 \), i.e. if

\[
\alpha > \frac{3}{2p}, \tag{4.15}
\]

which is possible when \( p > 3/2 \). Thus to prove part (i), we only need to show that

there exists a constant \( C \), independent of \( r \in [0, T] \), such that

\[
\mathbb{E}\|J_\alpha(r, \cdot)\|^p_{L_p^\alpha(\mathcal{R})} \leq C\|v\|^p_{Z_{\alpha, T}}. \tag{4.16}
\]

Step 2. We shall prove the above bound (4.16) in this step and to do this let us introduce the following two notations

\[
\mathcal{D}_1(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |G_{r-s}(y) - G_{r-s}(y + h)|^2 \right. \]

\[
\times \|v(s, y + z)\|^2_{L_p^\alpha(\Omega)} |h|^{2H-2} dhdyds \right)^{\frac{p}{2}},
\]

and

\[
\mathcal{D}_2(r, z) := \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} |G_{r-s}(y)|^2 \right. \]

\[
\times \|v(s, z + y + h) - v(s, z + y)\|^2_{L_p^\alpha(\Omega)} |h|^{2H-2} dhdyds \right)^{\frac{p}{2}}.
\]

From the definition (4.12) of \( J \) and by Burkholder-Davis-Gundy’s inequality (4.3) stated in Lemma 4.1, we have

\[
\mathbb{E}\|J_\alpha(r, \cdot)\|^p_{L_p^\alpha(\mathcal{R})} \lesssim \int_{\mathbb{R}} \left\{ \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \left[ \mathbb{E}|G_{r-s}(y + h - z)v(s, y + h)ight. \right.
\]

\[
- G_{r-s}(y - z)v(s, y)|^2/p h^{2H-2} dhdyds \right\}^{p/2} \lambda(z) \, dz
\]

\[
= \int_{\mathbb{R}} \left\{ \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \left[ \mathbb{E}|G_{r-s}(y + h)v(s, y + z + h)ight. \right.
\]

\[
- G_{r-s}(y)v(s, y + z)|^2/p h^{2H-2} dhdyds \right\}^{p/2} \lambda(z) \, dz
\]

\[
\lesssim \int_{\mathbb{R}} \left[ \mathcal{D}_1(r, z) + \mathcal{D}_2(r, z) \right] \lambda(z) \, dz.
\]
Step 3. In this and next steps we prove (i).

If we have 
\[ \lambda < H - 1 > -1 \]  
and 
\[ -2\alpha - \frac{1}{2} > -1, \]  
i.e. \( \alpha > \frac{H}{2} \), then (4.16) follows.

However, the condition \( \alpha < H/2 \) should be combined with (4.15). This gives \( \frac{1}{2} < \frac{\alpha}{p} < \frac{H}{2} \), which implies \( p > \frac{H}{2} \). Thus, under the condition of the proposition, the inequality (4.16) holds true. This finishes the proof of (i).

**Step 3.** In this and next steps we prove (ii). The spirit of the proof will be similar to that of the proof of (i) but is more involved. In order to obtain the desired decay
rate of $\mathcal{N}_{\frac{1}{2}-H} \Phi(t, x)$, we still use the equation (4.13) to express $\Phi(t, x)$ by $J$.

$$
\begin{align*}
\Phi(t, x + h) - \Phi(t, x) &= \sin(\pi \alpha) \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} \left[ G_{t-r}(x + h - z) - G_{t-r}(x - z) \right] J_\alpha(r, z) \, dz \, dr \\
&= \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} G_{t-r}(x - z) \left[ J_\alpha(r, z) + J_\alpha(r, z) \right] \, dz \, dr.
\end{align*}
$$

Invoking Minkowski’s inequality and then Hölder’s inequality with $\frac{1}{p} + \frac{1}{q} = 1$ we get

$$
\begin{align*}
\int_{\mathbb{R}} |\Phi(t, x + h) - \Phi(t, x)|^2 |h|^{2H-2} \, dh \\
&\leq \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} G_{t-r}(x - z) \left[ J_\alpha(r, z) + J_\alpha(r, z) \right] \, dz \, dr \right)^2 |h|^{2H-2} \, dh \\
&\leq \left( \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} G_{t-r}(x - z) \left[ \int_{\mathbb{R}} |J_\alpha(r, z) + J_\alpha(r, z)|^2 |h|^{2H-2} \, dh \right]^\frac{1}{2} \, dz \, dr \right)^2 \\
&\leq \left( \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha - 1} G_{t-r}(x - z) \left[ \int_{\mathbb{R}} |J_\alpha(r, z) + J_\alpha(r, z)|^2 |h|^{2H-2} \, dh \right]^\frac{1}{2} \, dz \, dr \right)^2 \\
&\times \left( \int_0^T \left[ \int_{\mathbb{R}} |J_\alpha(r, z + h) - J_\alpha(r, z)|^2 |h|^{2H-2} \, dh \right]^\frac{1}{2} \lambda(z) \, dz \, dr \right)^\frac{1}{2},
\end{align*}
$$

where in the above last inequality we used $G_{t-r}(x - z) = (t - r)^{\frac{\alpha - 1}{2}} G_{\frac{t-r}{2}}(x - z)$ and inequality (2.12). If we take $\theta = \frac{1}{p}$, and $q(\alpha - \frac{3}{2} + \frac{1}{p}) > -1$, i.e.

$$
\alpha > \frac{3}{2p},
$$

then

$$
\sup_{t, x} \lambda(x) \theta \left( \int_{\mathbb{R}} |\Phi(t, x + h) - \Phi(t, x)|^2 |h|^{2H-2} \, dh \right)^\frac{1}{2} \\
\leq \left( \int_0^T \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |J_\alpha(r, z + h) - J_\alpha(r, z)|^2 |h|^{2H-2} \, dh \right]^\frac{1}{2} \lambda(z) \, dz \, dr \right)^\frac{1}{2},
$$

Thus to prove part (ii) we only need to prove that there exists some constant $C$, independent of $r \in [0, T]$, such that

$$
\mathcal{I} := \mathbb{E} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |J_\alpha(r, z + h) - J_\alpha(r, z)|^2 |h|^{2H-2} \, dh \right]^\frac{1}{2} \lambda(z) \, dz \leq C \|v\|_{Z_{\alpha, \tau}^p}^p.
$$

**Step 4.** In this step we show the above inequality (4.21). By the definition (4.12) of $J$ and by an application of Minkowski’s inequality and then an application of
By inequality (2.17) with \( \beta \) the Burkholder-David-Gundy inequality we have

\[
I \lesssim \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathbb{E}[J_{\alpha}(r, z + h) - J_{\alpha}(r, z)]^p \lambda(z)dz \right] \frac{1}{\pi |h|^{2H-2}} dh \right)^{\frac{1}{p}}.
\]

We introduce two notations:

\[
I_1(r, z, h) := \mathbb{E} \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} |[G_{r-s}(z + h - y) - G_{r-s}(z - l)]v(s, y + l) - [G_{r-s}(z + h - y) - G_{r-s}(z - y)]v(s, y)|^2 |l|^2 |dldyds| \right)^{\frac{1}{2}},
\]

and

\[
I_2(r, z, h) := \mathbb{E} \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} |G_{r-s}(z + h - y) - G_{r-s}(z - y)|^2 \times |v(s, y)|^2 |l|^2 |dldyds| \right)^{\frac{1}{2}}.
\]

Then we have

\[
\mathbb{E} \left[ \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathbb{E}[J_{\alpha}(r, z + h) - J_{\alpha}(r, z)]^2 |h|^{2H-2} dh \right] \lambda(z)dz \right] \lesssim \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} I_1(r, z, h) \lambda(z)dz \right] \frac{1}{\pi |h|^{2H-2}} dh \right)^{\frac{1}{2}}
\]

\[
+ \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} I_2(r, z, h) \lambda(z)dz \right] \frac{1}{\pi |h|^{2H-2}} dh \right)^{\frac{1}{2}} =: I_1^{p/2} + I_2^{p/2}.
\]

We shall bound \( I_1 \) and \( I_2 \) one by one. For the first term, a change of variables and an application of Minkowski’s inequality yield

\[
I_1 \lesssim \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \mathbb{E} \left( \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\alpha} [G_{r-s}(y + h) - G_{r-s}(y)]^2 \times |v(s, y + z + l) - v(s, y + z)|^2 |l|^2 |dldyds| \right)^{\frac{1}{2}} \lambda(z)dz \right] \frac{1}{\pi |h|^{2H-2}} dh
\]

\[
\lesssim \int_0^r \int_{\mathbb{R}^3} (r - s)^{-2\alpha} [G_{r-s}(y + h) - G_{r-s}(y)]^2 |l|^2 |h|^{2H-2} \times \left( \int_{\mathbb{R}} \mathbb{E}[v(s, z + l) - v(s, z)]^p \lambda(z-y)dz \right) dhdlds.
\]

By inequality (2.17) with \( \beta = \frac{1}{2} - H \) we see that

\[
\int_{\mathbb{R}^2} [G_{r-s}(y) - G_{r-s}(y + h)]^2 |h|^{2H-2} dhdy \lesssim (r-s)^H,
\]

which is finite. Since \( x^{2/p}, x > 0 \) is a concave function for \( p \geq 2 \) we can apply Jensen’s inequality with respect to the probability measure \( (r-s)^{1-H} [G_{r-s}(y) -
\]

\[
\int_{\mathbb{R}^2} [G_{r-s}(y) - G_{r-s}(y + h)]^2 |h|^{2H-2} dhdy \lesssim (r-s)^H,
\]

which is finite. Since \( x^{2/p}, x > 0 \) is a concave function for \( p \geq 2 \) we can apply Jensen’s inequality with respect to the probability measure \( (r-s)^{1-H} [G_{r-s}(y) -
\]

\[
\int_{\mathbb{R}^2} [G_{r-s}(y) - G_{r-s}(y + h)]^2 |h|^{2H-2} dhdy \lesssim (r-s)^H,
\]

which is finite. Since \( x^{2/p}, x > 0 \) is a concave function for \( p \geq 2 \) we can apply Jensen’s inequality with respect to the probability measure \( (r-s)^{1-H} [G_{r-s}(y) -
\]

\[
\int_{\mathbb{R}^2} [G_{r-s}(y) - G_{r-s}(y + h)]^2 |h|^{2H-2} dhdy \lesssim (r-s)^H,
\]

which is finite. Since \( x^{2/p}, x > 0 \) is a concave function for \( p \geq 2 \) we can apply Jensen’s inequality with respect to the probability measure \( (r-s)^{1-H} [G_{r-s}(y) -
Thus, we have for $p \geq 2$:

$$I_1 \lesssim \int_0^r \int_\mathbb{R} (r-s)^{-2\alpha+H-1} \left( \int_{\mathbb{R}^3} (r-s)^{1-H} \left[ G_{r-s}(y+h) - G_{r-s}(y) \right]^2 \right) ds \lesssim \int_0^r \int_\mathbb{R} (r-s)^{-2\alpha+H-1} \|v(s,\cdot + l) - v(s,\cdot)\|^2_{L^p_\alpha(\Omega \times \mathbb{R})} \|l\|^{2H-2} ds,$$

by the first inequality in Lemma 2.12.

In order to bound $I_2(t,x,h)$, we make a change of variable and then split it to two terms. More precisely, we have

$$I_2(r,z,h) \lesssim I_{21}(r,z,h) + I_{22}(r,z,h)$$

with the notation $\Box_i(y,l,h)$ being defined by (2.15). Using Minkowski’s inequality, Lemma 2.8 and Lemma 2.11, one can check that

$$I_{21} := \int_\mathbb{R} \left( \int_\mathbb{R} I_{21}(r,z,h) \lambda(z) \, dz \right)^{\frac{2}{p}} \|l\|^{2H-2} \, dh$$
\hspace{1cm} \lesssim \int_0^r \int_{\mathbb{R}^3} (r-s)^{-2\alpha} \|\Box_{r-s}(y,l,h)\|^2 \|v(s)\|^2_{L^p_\alpha(\Omega \times \mathbb{R})} \|l\|^{2H-2} \, dh \, ds,$$

and

$$I_{22} := \int_\mathbb{R} \left( \int_\mathbb{R} I_{22}(r,z,h) \lambda(z) \, dz \right)^{\frac{2}{p}} \|l\|^{2H-2} \, dh$$
\hspace{1cm} \lesssim \int_0^r \int_{\mathbb{R}^3} (r-s)^{-2\alpha} \left( \int_{\mathbb{R}^2} \|\Box_{r-s}(y,l,h)\|^2 \|v(s)\|^2_{L^p_\alpha(\Omega \times \mathbb{R})} \|l\|^{2H-2} \, dh \right) \times \|v(s,\cdot) - v(s,\cdot + y)\|^2_{L^p_\alpha(\Omega \times \mathbb{R})} \, dy \, ds,$$

Recalling the definition of $\| \cdot \|^p_{Z^p_{\alpha,T}}$, and combining (4.22), (4.23) and (4.24), we obtain

$$\mathbb{E} \left( \int_\mathbb{R} \left[ \int_\mathbb{R} |J_\alpha(r,z+h) - J_\alpha(r,z)|^2 \|l\|^{2H-2} \, dh \right]^{\frac{2}{p}} \lambda(z) \, dz \right) \leq C_2 \|v\|^p_{Z^p_{\alpha,T}} \left( \int_0^r (r-s)^{-2\alpha+2H-\frac{3}{2}} + (r-s)^{-2\alpha+H-1} \, dr \right)^{\frac{2}{p}}.$$

Once we have $-2\alpha+2H-\frac{3}{2} > -1$ and $-2\alpha+H-1 > -1$, i.e. $\alpha < H - \frac{1}{4}$, then (4.21) follows. This condition on $\alpha$ is combined with (4.20) to become $\frac{2}{3p} < \alpha < H - \frac{1}{4}$.
Therefore, we have proved that if \( p > \frac{6}{3H - 1} \), then (4.21) holds, finishing the proof of (ii).

**Step 5.** We are going to prove part (iii). We continue to use (4.13). Without loss of generality, we can assume \( h > 0 \) and \( t \in [0, T] \) such that \( t + h \leq T \). We have

\[
\Phi(t + h, x) - \Phi(t, x) = \frac{\sin(\pi \alpha)}{\pi} \left[ \int_0^{t+h} \int_{\mathbb{R}} (t + h - r)^{\alpha-1} G_{t+h-r}(x-z)J_\alpha(r, z) dr dz - \int_0^t \int_{\mathbb{R}} (t - r)^{\alpha-1} G_{t-r}(x-z) \times J_\alpha(r, z) dr dz \right] \\
\lesssim \sum_{i=1}^{3} J_i(t, h, x),
\]

where

\[
J_1(t, h, x) := \int_0^t \int_{\mathbb{R}} [(t + h - r)^{\alpha-1} - (t - r)^{\alpha-1}] G_{t-r}(x-z) J_\alpha(r, z) dr dz,
\]

\[
J_2(t, h, x) := \int_0^t \int_{\mathbb{R}} (t + h - r)^{\alpha-1} [G_{t+h-r}(x-z) - G_{t-r}(x-z)] J_\alpha(r, z) dr dz,
\]

and

\[
J_3(t, h, x) := \int_t^{t+h} \int_{\mathbb{R}} (t + h - r)^{\alpha-1} G_{t+h-r}(x-z) J_\alpha(r, z) dr dz.
\]

As in the proof of (i) and (ii), we insert additional factors of \( \lambda^{-\frac{q}{q'}}(z) \cdot \lambda^q(z) \) and apply Hölder inequality in the expression for \( J_1 \). Then, \( J_1 \) is estimated as follows.

\[
J_1(t, h, x) \leq \lambda^{-\frac{1}{q'}}(x) \int_0^t |(t + h - r)^{\alpha-1} - (t - r)^{\alpha-1}|(t - r)^{\frac{q-2}{q}} \| J_\alpha(r, \cdot) \|_{L^p(\mathbb{R})} dr \\
\leq \lambda^{-\frac{1}{q'}}(x) \left( \int_0^t |(t + h - r)^{\alpha-1} - (t - r)^{\alpha-1}|^q(t - r)^{\frac{q-2}{q}} dr \right)^{\frac{1}{q}} \\
\times \left( \int_0^T \| J_\alpha(r, \cdot) \|_{L^p(\mathbb{R})}^p dr \right)^{\frac{1}{p}}.
\]

Fix \( \gamma \in (0, 1) \). It is easy to see

\[
|(t + h - r)^{\alpha-1} - (t - r)^{\alpha-1}| \lesssim |t - r|^{\alpha-1-\gamma} h^\gamma.
\]

Thus, we have

\[
\sup_{t,x} \lambda^\theta(x)|J_1(t, h, x)| \lesssim h^\gamma \sup_{t \in [0, T]} \left( \int_0^t (t - r)^{q(\alpha-1-\gamma) + \frac{q-2}{q}} dr \right)^{\frac{1}{q}} \\
\times \left( \int_0^T \| J_\alpha(r, \cdot) \|_{L^p(\mathbb{R})}^p dr \right)^{\frac{1}{p}}.
\]

In other word, if \( \gamma + \frac{q}{2p} < \alpha < \frac{H}{2} \) or equivalently, if \( \gamma < \frac{H}{2} - \frac{q}{2p} \), then we have

\[
\mathbb{E} \left[ \sup_{t,x} \lambda^\theta(x)|J_1(t, h, x)| \right]^p \lesssim |h|^p \| v \|_{Z^p_{\mathbb{R}, T}}^p.
\]
Let us proceed to bound $J_2(t, h, x)$. One finds easily

$$J_2(t, h, x) \leq \left( \int_0^t \int_{\mathbb{R}} (t + h - r)^{q(\alpha - 1)} |G_{t + h - r}(x - z) - G_{t - r}(x - z)|^q \lambda^{-\frac{q}{p}}(z) dz dr \right)^{\frac{1}{q}} \left( \int_0^T \| J_{t}(r, \cdot) \|_{L^p(X)}^p dr \right)^{\frac{1}{p}}. \tag{4.28}$$

To bound the above first factor we use the following inequality

$$\left| \exp \left( -\frac{x^2}{t + h} \right) - \exp \left( -\frac{x^2}{t} \right) \right| \leq C_{\gamma} h^{\gamma} t^{-\gamma} \exp \left( -\frac{x^2}{2(t + h)} \right) \quad \forall \gamma \in (0, 1).$$

Combining the above inequality with (4.26) (with $\alpha = 1/2$), we have

$$|G_{t + h - r}(x - z) - G_{t - r}(x - z)| \leq C_{\gamma} h^{\gamma} (t - r)^{-\gamma} \left[ G_{t + h - r}(x - z) + G_{t + h - r}(x - z) \right]. \tag{4.29}$$

Thus, the first factor in (4.28) is bounded by

$$\int_0^t \int_{\mathbb{R}} (t + h - r)^{q(\alpha - 1)} |G_{t + h - r}(x - z) - G_{t - r}(x - z)|^q \lambda^{-\frac{q}{p}}(z) dz dr$$

$$\leq h^{q\gamma} \int_0^t \int_{\mathbb{R}} (t - r)^{q(\alpha - 1) - \gamma} \frac{1 - \gamma}{p} G_{t - r}(x - z) \lambda^{-\frac{q}{p}}(z) dz dr$$

$$+ h^{q\gamma} \int_0^t \int_{\mathbb{R}} (t - r)^{q(\alpha - 1) - \gamma} \frac{1 - \gamma}{p} G_{t - r}(x - z) \lambda^{-\frac{q}{p}}(z) dz dr$$

$$\leq h^{q\gamma} \lambda^{-\frac{q}{p}}(x) \int_0^t (t - r)^{q(\alpha - 1) - \gamma} \frac{1 - \gamma}{p} dr,$$

where the last inequality follows from Lemma 2.5. Hence, if $\gamma + \frac{1}{2p} < \alpha < \frac{H}{2}$, namely, if $\gamma < \frac{H}{2} - \frac{3}{2p}$, then we have the following estimation:

$$E \left[ \sup_{t, x} \lambda^{\theta}(x) J_2(t, h, x) \right]^{p} \lesssim |h|^{p\gamma} \| v \|_{Z_{X, T}^p}^{p}. \tag{4.30}$$

Now we are going to bound $J_3(t, x, h)$. Applying Minkowski’s inequality and then Hölder’s inequality we have

$$J_3(t, x, h) \leq \lambda^{-\frac{q}{p}}(x) \left( \int_0^{t + h} (t + h - r)^{q(\alpha - 1) - \gamma} \frac{1 - \gamma}{p} dr \right)^{\frac{1}{q}} \left( \int_0^T \| J_{t}(r, \cdot) \|_{L^p(X)}^p dr \right)^{\frac{1}{p}}.$$

If $\frac{3}{2p} < \alpha < \frac{H}{2}$, which is possible if $\gamma = \alpha - \frac{3}{2p} < \frac{H}{2} - \frac{3}{2p}$, then

$$E \left[ \sup_{t, x} \lambda^{\theta}(x) J_3(t, h, x) \right]^{p} \lesssim |h|^{p\gamma - \frac{3}{2}} \| v \|_{Z_{X, T}^p}^{p} = |h|^{p\gamma} \| v \|_{Z_{X, T}^p}^{p}. \tag{4.31}$$

Combining (4.27), (4.30) and (4.31) we prove (4.10).
\textbf{Step 6.} We prove part (iv) of the proposition. As before, we shall again use the representation formula \[(4.13)\] and then we apply the Hölder inequality to find
\[
\Phi(t, x) - \Phi(t, y) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} [G_{t-r}(x-z) - G_{t-r}(y-z)] J_\alpha(r, z) dz dr
\]
\[
\lesssim \left( \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} \|G_{t-r}(x-z) - G_{t-r}(y-z)\|^q \lambda^{-\frac{q}{2}}(z) dz dr \right)^{\frac{1}{q}}
\]
\[
\times \left( \int_0^T \int_\mathbb{R} |J_\alpha(r, z)|^p \lambda(z) dz dr \right)^{\frac{1}{p}}.
\]
Denote
\[
\mathcal{K}(t, x, y) := \int_0^t \int_\mathbb{R} (t-r)^{\alpha-1} |G_{t-r}(x-z) - G_{t-r}(y-z)|^q \lambda^{-\frac{q}{2}}(z) dz dr.
\]
Fix $\gamma \in (0, 1)$. Using Hölder’s inequality we have
\[
\mathcal{K}(t, x, y) \lesssim \int_0^t (t-r)^{\alpha-1} \left( \int_\mathbb{R} |G_{t-r}(x-z) - G_{t-r}(y-z)|^{pq(1-\gamma)} \lambda^{-\frac{q}{2}}(z) dz \right)^{\frac{1}{q}} dr
\]
\[
\lesssim |x-y|^\gamma \cdot \int_0^t (t-r)^{\alpha-1} \left[ \left( \int_\mathbb{R} \left| G_{t-r}(x-z) + G_{t-r}(y-z) \right|^2 \lambda^{-\frac{q}{2}}(z) dz \right)^{\frac{1}{2}} \right] dr
\]
\[
\lesssim |x-y|^\gamma \cdot \left[ \lambda^{-\frac{q}{2}}(x) + \lambda^{-\frac{q}{2}}(y) \right] \cdot \int_0^t (t-r)^{\alpha-1} \left( \int_\mathbb{R} \left| G_{t-r}(x-z) + G_{t-r}(y-z) \right|^2 \lambda^{-\frac{q}{2}}(z) dz \right)^{\frac{1}{2}} dr,
\]
where the last inequality follows from Lemma \([24.5]\) and the second last inequality follows from the following easy bound:
\[
\int_\mathbb{R} |G_{t-r}(x-z) - G_{t-r}(y-z)|^\rho dz
\]
\[
\approx (t-r)^{\frac{1-\rho}{2}} \int_\mathbb{R} \exp(-|\tilde{x} - \tilde{z}|^2) - \exp(-|\tilde{y} - \tilde{z}|^2)|^\rho d\tilde{z}
\]
\[
\lesssim (t-r)^{\frac{1-\rho}{2}} |\tilde{x} - \tilde{y}|^\rho = (t-r)^{\frac{1-\rho}{2}} |x-y|^\rho \quad \forall \rho > 0,
\]
where $\tilde{x} = \frac{x}{\sqrt{t-r}}$, $\tilde{y} = \frac{y}{\sqrt{t-r}}$, and $\tilde{z} = \frac{z}{\sqrt{t-r}}$.

If $q(\alpha - \frac{3}{2} + \frac{1}{2q}) - \frac{q}{2} > -1$ and $\alpha < \frac{H}{2}$, namely, $\frac{3}{2p} + \frac{q}{2} < \alpha < \frac{H}{2}$, then with $\theta = \frac{1}{p}$ we have
\[
\mathbb{E} \sup_{t \in [0, T]} (\lambda^{-\theta} x + \lambda^{-\theta} y)^{-1} |\mathcal{K}(t, x, y)|^{\frac{1}{\theta}} \times \left( \int_0^T \int_\mathbb{R} |J_\alpha(r, z)|^p \lambda(z) dz dr \right)^{\frac{1}{p}}
\]
\[
\lesssim |x-y|^\gamma \cdot \int_\mathbb{R} \mathbb{E} |J_\alpha(r, z)|^p \lambda(z) dz dr \leq C_4 |x-y|^\gamma \|v\|^p_{L_\infty^{\alpha,T}} . \quad (4.32)
\]
So we have completed the proof of (4.11). The proof of the proposition is then completed.

4.3. Weak existence of the solution. In this subsection we show the weak existence of a solution with paths in \( C([0, T] \times \mathbb{R}) \), the space of all continuous real valued functions on \([0, T] \times \mathbb{R}, \) equipped with a metric

\[
\| C(u, v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq T, |x| \leq n} (|u(t, x) - v(t, x)| + 1).
\]

(4.33)

We state a tightness criterion of probability measures on \((C([0, T] \times \mathbb{R}), B(C([0, T] \times \mathbb{R}))\) that we are going to use (see Section 2.4 in [17] for the case where \([0, T] \times \mathbb{R}\) is replaced by \([0, \infty, \)). It is also true for our case as indicated there).

**Theorem 4.4.** A sequence \( \{P_n\}_{n=1}^{\infty} \) of probability measures on \((C([0, T] \times \mathbb{R}), B(C([0, T] \times \mathbb{R}))\) is tight if and only if

1. \( \lim_{T \to \infty} \sup_{n \geq 1} P_n\{\{\omega \in C([0, T] \times \mathbb{R}) : |\omega(0, 0)| > \lambda\} = 0, \)
2. For any \( T > 0, R > 0 \) and \( \varepsilon > 0 \)

\[
\lim \sup_{\varepsilon, n \geq 1} P_n\{\{\omega \in C([0, T] \times \mathbb{R}) : m_{T, R}(\omega, \delta) > \varepsilon\} = 0
\]

where

\[
m_{T, R}(\omega, \delta) := \max_{0 \leq t, s \leq T, 0 \leq |x|, |y| \leq R} |\omega(t, x) - \omega(s, y)|
\]

is the modulus of continuity on \([0, T] \times [-R, R]\).

We approximate the noise \( W \) with respect to the space variable by the following smoothing of the noise. That is, for \( \varepsilon > 0 \) we define

\[
\frac{\partial}{\partial x} W_\varepsilon(t, x) = \int_{\mathbb{R}} G_\varepsilon (y) W(dy, t).
\]

(4.34)

The noise \( W_\varepsilon \) induces an approximation to mild solution

\[
u_\varepsilon (t, x) = G_\varepsilon * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sigma(s, y, u_\varepsilon(s, y)) W_\varepsilon(dy, ds),
\]

(4.35)

where the stochastic integral is understood in the Itô sense. As in [11] due to the regularity in space, the existence and uniqueness of the solution \( u_\varepsilon(t, x) \) to above equation is well-known.

The lemma below asserts that the approximate solution \( u_\varepsilon(t, x) \) is uniformly bounded in the space \( \mathcal{Z}^p_{\lambda,T} \). More precisely, we have

**Lemma 4.5.** Let \( H \in (\frac{1}{2}, \frac{3}{2}) \) and let \( \lambda(x) \) be defined by (4.6). Assume \( \sigma(t, x, u) \) satisfies hypothesis (H1). Assume also that the initial value \( u_0(x) \in \mathcal{Z}^p_{\lambda,0} \). Then the approximate solution \( u_\varepsilon \) satisfies

\[
\sup_{\varepsilon > 0} \|u_\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}} := \sup_{\varepsilon > 0} \sup_{t \in [0,T]} \|u_\varepsilon(t, \cdot)\|_{L^p_{\lambda}(\Omega \times \mathbb{R})} + \sup_{\varepsilon > 0} \sup_{t \in [0,T]} N_{\lambda}^x_{\frac{T}{2}-H, \delta} u_\varepsilon(t) < \infty.
\]

(4.36)

**Proof.** For notational simplicity we can assume \( \sigma(t, x, u) = \sigma(u) \) without loss of generality because of hypothesis (H1). We shall follow the same argument as in [11] but now with special attention to the weight \( \lambda(x) \). In the following steps 1 to 2 we will use Picards iteration to show that for each \( \varepsilon, u_\varepsilon \in \mathcal{Z}^p_{\lambda,T} \) for \( p \geq 2 \). Then,
in step 3 we prove that \( u_\varepsilon \) is uniformly bounded in \( Z_{\varepsilon,T}^p \) with respect to \( \varepsilon \in (0, 1] \). To this end, we define the Picard iteration sequence as follows.

\[
    u_\varepsilon^0(t, x) = G_t * u_0(x),
\]

and recursively for \( n = 0, 1, 2, \cdots \),

\[
    u_\varepsilon^{n+1}(t, x) = G_t * u_0(x) + \int_0^t \int \sigma(u_\varepsilon^n(s, y)) W_\varepsilon(dy, ds).
\]

We shall bound \( \|u_\varepsilon^n\|_{Z_{\varepsilon,T}^p} \) uniformly in \( n \) and we will complete this in the following two steps.

**Step 1.** We bound \( \|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} \) uniformly in \( n \). From Burkholder-Davis-Gundy’s inequality and from the fact that \( \sigma(\cdot) \) is a Lipschitz function it follows

\[
    \mathbb{E}[\|u_\varepsilon^{n+1}(t, x) - u_\varepsilon^n(t, x)\|^p] = \mathbb{E} \left[ \int_0^t \int \sigma(u_\varepsilon^n(s, y)) - \sigma(u_\varepsilon^{n-1}(s, y)) |W_\varepsilon(dy, ds)|^p \right] \\
    \leq C_{\varepsilon,p} \mathbb{E} \left[ \int_0^t \left( \int \sigma(u_\varepsilon^n(s, y)) - \sigma(u_\varepsilon^{n-1}(s, y)) dy \right)^2 ds \right]^{\frac{p}{2}},
\]

where \( C_{\varepsilon,p} \) is a constant depending on \( \varepsilon \) and \( p \). Integrating with respect to the space variable with the weight \( \lambda(x) \) and invoking Hölder’s inequality, Jensen’s inequality and an application of Lemma 2.5 yield

\[
    \|u_\varepsilon^{n+1}(t, \cdot) - u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^p = \int_{\Omega \times \mathbb{R}} \mathbb{E}[\|u_\varepsilon^{n+1}(t, x) - u_\varepsilon^n(t, x)\|^p] \lambda(x) dx \\
    \leq C_{\varepsilon,p} \int_0^t \int_{\mathbb{R}} \left( \int G_{t-s}(x-y) \lambda(x) dx \right) \mathbb{E}[\|u_\varepsilon^n(s, y) - u_\varepsilon^{n-1}(s, y)\|^p] dy ds \\
    \leq C_{\varepsilon,p,T} \int_0^t \int_{\mathbb{R}} \mathbb{E}[\|u_\varepsilon^n(s, y) - u_\varepsilon^{n-1}(s, y)\|^p] \lambda(y) dy ds \\
    = C_{\varepsilon,p,T} \int_0^t \|u_\varepsilon^n(s, \cdot) - u_\varepsilon^{n-1}(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^p ds \\
    \leq C_{\varepsilon,p,T} \sup_{0 \leq s \leq T} \|u_\varepsilon^1(s, \cdot) - u_\varepsilon^0(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^p,
\]

for some constant \( C_{\varepsilon,p,T} \) depending on \( \varepsilon, p \) and \( T \). This implies that

\[
    \sup_n \sup_{t \in [0, T]} \|u_\varepsilon^n(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} < \infty \quad \text{for each } \varepsilon > 0.
\]
Step 2. Next, we estimate $N_{\frac{1}{2}-H,p}^{\ast}u_\varepsilon(t)$. Since $|\sigma(u)| \lesssim |u| + 1$, we have

$$\int_{\mathbb{R}} \mathbb{E} \left[ |u_\varepsilon^{n+1}(t,x) - u_\varepsilon^{n+1}(t,x+h)|^p \right] \lambda(x)dx$$

$$\leq C \int_{\mathbb{R}} \left| G_\varepsilon \ast u_\varepsilon(0) - G_\varepsilon \ast u_\varepsilon(t,x+h) \right|^p \lambda(x)dx$$

$$+ C_\varepsilon \int_{\mathbb{R}} \left[ \int_0^t \left| G_{t-s}(x) - G_{t-s}(x+y) \right|^2 dy \right] ds \lambda(x)dx$$

$$+ C_\varepsilon \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^t \left| G_{t-s}(x) - G_{t-s}(x+y) \right| \times |u_\varepsilon^n(s,y)|dy \right] ds \lambda(x)dx.$$

(4.37)

Introduce the notation

$$e_{\varepsilon,\lambda}^n(t,h) := \left( \int_{\mathbb{R}} \mathbb{E} \left[ |u_\varepsilon^n(t,x) - u_\varepsilon^n(t,x+h)|^p \right] \lambda(x)dx \right)^{\frac{1}{p}}.$$

Thanks to Minkowski’s inequality we can bound the last term in (4.37) by

$$\int_{\mathbb{R}} \mathbb{E} \left[ \int_0^t \left( \int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right| \times |u_\varepsilon^n(s,x+y) - u_\varepsilon^n(s,x)|dy \right) dy \right] ds \lambda(x)dx$$

$$+ \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^t \left( \int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right| \times |u_\varepsilon^n(s,x)|dy \right)^2 ds \right] \lambda(x)dx$$

$$\lesssim \sup_{s \in [0,T]} \| u_\varepsilon^n(s,\cdot) \|^p_{L^p_{\varepsilon,\lambda}(\Omega \times \mathbb{R})} \times \left( \int_0^t \int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 dyds \right)^{\frac{1}{p}}$$

$$+ \left( \int_0^t \int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 e_{\varepsilon,\lambda}^n(s,y)dyds \right)^{\frac{1}{p}}.$$

Thus by the assertion obtained in Step 1, and by Lemma 2.18 and (4.37) we end up with

$$N_{\frac{1}{2}-H,p}^{\ast}u_\varepsilon^{n+1}(t) = \int_{\mathbb{R}} e_{\varepsilon,\lambda}^{n+1}(t,h) |h|^{2H-2} dh$$

$$\leq C + C_\varepsilon \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathbb{E} \left[ |u_\varepsilon^n(t,x) - u_\varepsilon^n(t,x+h)|^p \right] \lambda(x)dx \right)^{\frac{1}{p}} |h|^{2H-2} dh$$

$$+ C_\varepsilon \sup_{s \in [0,T]} \| u_\varepsilon^n(s,\cdot) \|^p_{L^p_{\varepsilon,\lambda}(\Omega \times \mathbb{R})} \times \left( \int_0^t \int_{\mathbb{R}^2} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 |h|^{2H-2} dyds \right)^{\frac{1}{p}}$$

$$+ C_\varepsilon \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(y) - G_{t-s}(y+h)|^2 \times e_{\varepsilon,\lambda}^n(s,y)|h|^{2H-2} dyds$$

$$\leq C_\varepsilon \left( 1 + \int_0^t \int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 |h|^{2H-2} dy \right) e_{\varepsilon,\lambda}^n(s,y) dyds \right).$$

Applying Lemma 2.11 one sees

$$\int_{\mathbb{R}} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 |h|^{2H-2} dy \lesssim (t-s)^{-\frac{3}{2}} |y|^{2H-2}.$$
Putting last two inequalities together, we conclude that
\[
N_{\frac{1}{2}-H,p}^n u_\varepsilon^n(t) \leq C_\varepsilon \left( 1 + \int_0^t (t-s)^{-\frac{1}{2}} N_{\frac{1}{2}-H,p}^s u_\varepsilon^2(s) ds \right).
\]

Recursively use this inequality to obtain
\[
N_{\frac{1}{2}-H,p}^n u_\varepsilon^n(t) \leq \sum_{k=0}^{n-1} C_\varepsilon^k \int_{0<s_1<\ldots<s_k<t} (t-s_k)^{-1/2} \cdots (s_2-s_1)^{-1/2} ds_1 \cdots ds_k
\]
\[+ C_\varepsilon^n \int_{0<s_1<\ldots<s_n<t} (t-s_n)^{-1/2} \cdots (s_2-s_1)^{-1/2} N_{\frac{1}{2}-H,p}^s u_\varepsilon^n(s) ds_1 \cdots ds_n \]
\[\leq \sum_{k=0}^{n} C_\varepsilon^k \int_{0<s_1<\ldots<s_k<t} (t-s_k)^{-1/2} \cdots (s_2-s_1)^{-1/2} ds_1 \cdots ds_k \]
\[\leq \sum_{k=0}^{\infty} \frac{C_\varepsilon^k k^{2k}}{\Gamma\left(\frac{1}{2} + k\right)} < \infty.
\]

This implies
\[
\sup_n \sup_{t \in [0,T]} \|N_{\frac{1}{2}-H,p}^n u_\varepsilon^n(t, \cdot)\|_{L^p_x(\Omega \times \mathbb{R})} < \infty \quad \text{for each } \varepsilon > 0.
\]

Then \(u_\varepsilon\) is in \(L^p_{\lambda,T}\) for \(p \geq 2\) for each \(\varepsilon > 0\).

**Step 3.** In this step, we prove that the norm of \(u_\varepsilon(t, x)\) in \(L^p_{\lambda,T}\) is uniformly bounded in \(\varepsilon\). Notice that \(u_\varepsilon\) satisfies
\[
u_\varepsilon(t, x) = G_t \ast u_0(x) + \int_0^t \int_\mathbb{R} \left[ (G_{t-s}(x-y) \sigma(u_\varepsilon(s, \cdot))) \ast G_\varepsilon \right] (y) W(dy, ds).
\]

Hence, by Proposition 4.33 and \(|\sigma(u)| \lesssim |u| + 1\) we have
\[
\mathbb{E}[|u_\varepsilon(t, x)|^p] \lesssim |G_t \ast u_0(x)|^p + \mathbb{E} \left( \int_0^t \int_\mathbb{R} |F[G_{t-s}(x-y) \sigma(u_\varepsilon(s, \cdot))](\xi)|^2 e^{-c|\xi|^2} |\xi|^{-1-2H} d\xi ds \right)^{\frac{p}{2}}
\]
\[\simeq |G_t \ast u_0(x)|^p + \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left| G_{t-s}(x-y-h) \sigma(u_\varepsilon(s, y+h)) - G_{t-s}(x-y) \sigma(u_\varepsilon(s, y)) \right|^2 |h|^{2H-2} dhdys \right)^{\frac{p}{2}}
\]
\[\lesssim |G_t \ast u_0(x)|^p + D_1(t, x) + D_2(t, x) + D_3(t, x),
\]
where
\[
D_1(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} \left| G_{t-s}(y) - G_{t-s}(y+h) \right|^2 d\xi \right)^{\frac{p}{2}} 
\]
\[\times \left( 1 + \|u_\varepsilon(s, x+y)\|^2_{L^p_x(\Omega)} \right) |h|^{2H-2} dhdys \]
and
\[
D_2(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} \left| G_{t-s}(y) \right|^2 \|u_\varepsilon(s, x+y+h) - u_\varepsilon(s, x+y)\|^2_{L^p_x(\Omega)} |h|^{2H-2} dhdys \right)^{\frac{p}{2}}.
\]
This means
\[ \|u_\varepsilon(t, \cdot)\|_{L^p_\lambda(\Omega \times \mathbb{R})}^2 = \left( \int_{\mathbb{R}} \mathbb{E} \left[ |u_\varepsilon(t, x)|^p \right] \lambda(x) dx \right)^{\frac{2}{p}} \lesssim \|u_0(x)\|_{L^p_\lambda(\mathbb{R})}^2 + I_1 + I_2, \] (4.39)
where \( I_1, I_2 \) and \( I_3 \) are defined and bounded as follows.
\[ I_1 := \left( \int_{\mathbb{R}} \mathcal{D}_1(t, x) \lambda(x) dx \right)^{\frac{2}{p}} \lesssim \int_0^t (t-s)^{H-1} \left( 1 + \|u_\varepsilon(s, \cdot)\|_{L^p_\lambda(\Omega \times \mathbb{R})}^2 \right) ds. \] (4.40)
and lastly,
\[ I_2 := \left( \int_{\mathbb{R}} \mathcal{D}_3(t, x) \lambda(x) dx \right)^{\frac{2}{p}} \lesssim \int_0^t \left[ \mathcal{N}_{\frac{2}{p}-H,p}^* u_\varepsilon(s) \right] \frac{2}{\sqrt{t-s}} ds. \] (4.41)
The bounds on \( I_1, I_2 \) together with (4.39) yield
\[ \|u_\varepsilon(t, \cdot)\|_{L^p_\lambda(\Omega \times \mathbb{R})}^2 \lesssim \|u_0\|_{L^p_\lambda(\omega \times \mathbb{R})}^2 + \int_0^t (t-s)^{H-1} \|u_\varepsilon(s, \cdot)\|_{L^p_\lambda(\Omega \times \mathbb{R})} ds \]
\[ + \int_0^t (t-s)^{-1/2} \left[ \mathcal{N}_{\frac{2}{p}-H,p}^* u_\varepsilon(s) \right]^2 ds. \] (4.42)
\[ \textbf{Step 4.} \text{ Next, we obtain a bound for } \mathcal{N}_{\frac{2}{p}-H,p}^* u_\varepsilon(t) \text{ analogous to (4.42). Similar to (4.38) we have} \]
\[ \mathbb{E} \left[ |u_\varepsilon(t, x) - u_\varepsilon(t, x + h)|^p \right] \lesssim |G_t * u_0(x) - G_t * u_0(x + h)|^p \]
\[ + \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left[ [G_{t-s}(x-y-z) - G_{t-s}(x-y-z+h)] \sigma(u_\varepsilon(s, y+z)) \\
- [G_{t-s}(x-z) - G_{t-s}(x-z+h)] \sigma(u_\varepsilon(s, z)) \right] |y|^{2H-2} dxdyds \right)^{\frac{2}{p}} \]
\[ \lesssim \mathcal{I}_1(t, x, h) + \mathcal{I}_2(t, x, h) + \mathcal{I}_3(t, x, h), \]
where
\[ \mathcal{I}_1(t, x, h) := |G_t * u_0(x) - G_t * u_0(x + h)|^p, \]
\[ \mathcal{I}_2(t, x, h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left[ [G_{t-s}(x-y-z) - G_{t-s}(x-y-z+h)] \\
\times \left| \sigma(u_\varepsilon(s, y+z)) - \sigma(u_\varepsilon(s, z)) \right| |y|^{2H-2} dxdyds \right)^{\frac{2}{p}}, \]
\[ \mathcal{I}_3(t, x, h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left[ G_{t-s}(x-y-z) - G_{t-s}(x-y-z+h) - G_{t-s}(x-z) \\
+ G_{t-s}(x-z+h) \right] |y|^{2H-2} dxdyds \right)^{\frac{2}{p}}. \]
Therefore, by Minkowski’s inequality we have
\[ \left[ \mathcal{N}_{\frac{2}{p}-H,p}^* u_\varepsilon(t) \right]^2 \lesssim \sum_{j=1}^3 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{I}_j(t, x, h) \lambda(x) dx \right)^{\frac{2}{p}} |h|^{2H-2} dh. \] (4.43)
SHE with general rough noise

Our strategy is to control these three quantities by using the similar ideas as those when we are dealing with the terms in proof of Proposition 4.2 (ii). At first, from Lemma 2.3, the first one is bounded as follows:

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_1(t, x, h) \lambda(x) dx \right)^{\frac{p}{2}} |h|^{2H-2}dh
\]

\[
\lesssim \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} G_t(x - y) \lambda(x) dx \right] |u_0(y) - u_0(y + h)|^p dy \right)^{\frac{p}{2}} |h|^{2H-2}dh
\]

\[
\lesssim \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |u_0(y) - u_0(y + h)|^p \lambda(y) dy \right)^{\frac{p}{2}} |h|^{2H-2}dh
\]

(4.44)

For the second term, a change of variables and an application of Minkowski’s inequality give

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_2(t, x, h) \lambda(x) dx \right)^{\frac{p}{2}} |h|^{2H-2}dh
\]

\[
\lesssim \int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \left( \int_{\mathbb{R}^3} (t-s)^{1-H} [G_t - (z) - G_{t-s}(z + h)]^2 |h|^{2H-2}
\]

\[
\times \mathbb{E} \left[ |u_0(s, x + y) - u_0(s, x)|^p \lambda(x - z) dx dz dh \right] \frac{p}{2} |y|^{2H-2}dy ds
\]

\[
\lesssim \int_0^t (t-s)^{H-1} \left[ N_{t-s}^{H, u_0}(s) \right]^2 ds.
\]

(4.45)

In order to bound \( I_3(t, x, h) \), let us do change of variable and then split it to two terms. This means

\[
I_3(t, x, h) \lesssim I_{31}(t, x, h) + I_{32}(t, x, h)
\]

\[
:= \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(y, z, h)|^2 |\sigma(u_0(s, x))|^2 |y|^{2H-2}dy dz ds \right)^{\frac{p}{2}}
\]

\[
+ \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(y, z, h)|^2 |\sigma(u_0(s, x + z)) - \sigma(u_0(s, x))|^2 |y|^{2H-2}dy dz ds \right)^{\frac{p}{2}}
\]

with the notation \( \Box_{t-s}(y, z, h) \) being defined in (2.15). Applying Minkowski’s inequality, the condition |\( \sigma(u) \)| \( \lesssim |u| + 1 \), and Lemma 2.8 one has

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_{31}(t, x, h) \lambda(x) dx \right)^{\frac{p}{2}} |h|^{2H-2}dh
\]

\[
\lesssim \int_0^t (t-s)^{H-\frac{q}{2}} \left( 1 + \|u_0(s, \cdot)\|_{L_2^{0,s} \Omega \times \mathbb{R}} \right)^2 ds
\]

(4.46)

Again by Minkowski’s inequality, the Lipshitz condition on \( \sigma(\cdot) \), and Lemma 2.11 we obtain

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_{32}(t, x, h) \lambda(x) dx \right)^{\frac{p}{2}} |h|^{2H-2}dh \lesssim \int_0^t (t-s)^{H-1} \left[ N_{t-s}^H u_0(s) \right]^2 ds.
\]

(4.47)

Thus we get

\[
\left[ \mathcal{N}_{t-s}^{H, u_0}(t) \right]^2 \lesssim \left[ \mathcal{N}_{t-s}^{H, u_0} \right]^2 + \int_0^t \left[ (t-s)^{H-1} + (t-s)^{2H-\frac{q}{2}} \left[ \mathcal{N}_{t-s}^{H, u_0}(s) \right]^2 \right] ds.
\]

(4.48)
Let us denote
\[ \Psi_\varepsilon(t) := \| u_\varepsilon(s, \cdot) \|_{L^p(\Omega \times \mathbb{R})}^2 + \left[ N^p_{\frac{1}{2} - H, p} u_\varepsilon(s) \right]^2. \]
Combining the estimates (4.42) and (4.44)-(4.47) yields
\[ \Psi_\varepsilon(t) \lesssim \| u_0 \|_{L^p(\Omega \times \mathbb{R})}^2 + \left[ N^p_{\frac{1}{2} - H, p} u_0 \right]^2 + \int_0^t \left[ (t-s)^{H-1} + (t-s)^{2H-\frac{3}{2}} \right] \Psi_\varepsilon(s) ds. \]

From the above we can easily apply the Gronwall-type Lemma to obtain
\[ \sup_{\varepsilon > 0} \| u_\varepsilon \|_{Z^{p}_{\lambda, T}} < \infty. \]
This completes the proof of the lemma. \( \square \)

Recall that \((C([0, T] \times \mathbb{R}), d_C)\) is the metric space with the metric \(d_C\) defined by (4.43).

**Lemma 4.6.** Let \( u_\varepsilon \in Z^p_{\lambda, T} \). If \( u_\varepsilon \to u \) almost surely in \((C([0, T] \times \mathbb{R}), d_C)\) as \( \varepsilon \to 0 \), then \( u \) is also in \( Z^p_{\lambda, T} \).

**Proof.** Since \( u_\varepsilon \) converges to \( u \) in \((C([0, T] \times \mathbb{R}), d_C)\) a.s., we have \( u_\varepsilon(t, x) \to u(t, x) \) for each \((t, x) \in [0, T] \times \mathbb{R}\) almost surely. Thus
\[
\| u(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} = \left( \int_{\mathbb{R}} \mathbb{E} \left[ \lim_{\varepsilon \to 0} |u_\varepsilon(t, x)|^p \right] \lambda(x) dx \right)^{\frac{1}{p}} 
\leq \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}} \mathbb{E} [|u_\varepsilon(t, x)|^p] \lambda(x) dx \right)^{\frac{1}{p}} < \infty. \quad (4.49)
\]
Thus, we can conclude that \( \sup_{t \in [0, T]} \| u(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} \) is finite.

On the other hand, for \( \forall \varepsilon, h \) we have \( |u_\varepsilon(t, x + h) - u_\varepsilon(t, x)|^2 \to u(t, x + h) - u(t, x)|^2 \) a.s., so on the domain \( |h| \leq 1 \)
\[
\int_{|h| \leq 1} |u(t, \cdot + h) - u(t, \cdot)|^2_{L^p(\Omega \times \mathbb{R})} |h|^{2H-2} dh 
\leq \lim_{\varepsilon \to 0} \int_{|h| \leq 1} |u_\varepsilon(t, \cdot + h) - u_\varepsilon(t, \cdot)|^2_{L^p(\Omega \times \mathbb{R})} |h|^{2H-2} dh.
\]
For \( |h| \geq 1 \), we simply bound \( \| u(t, \cdot + h) - u(t, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} \) by \( 2\| u(t, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} \), which is finite. When \( H < \frac{1}{2} \), \( \int_{|h| > 1} |h|^{2H-2} < \infty \). Thus we obtain that
\[
\sup_{t \in [0, T]} N^p_{\frac{1}{2} - H, p} = \sup_{t \in [0, T]} \left( \int_{\mathbb{R}} \| u(t, \cdot + h) - u(t, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} |h|^{2H-2} dh \right)^{\frac{1}{2}} < \infty.
\]
Together with (4.49), this implies that \( u \in Z^p_{\lambda, T}. \) \( \square \)

**Lemma 4.7.** Let \( u_\varepsilon \) be the approximation mild solution defined in (4.35) and assume that \( u_0(x) \) belongs to \( Z^p_{\lambda, 0} \). Then

(i) If \( p > \frac{2}{2H-1} \), then
\[
\| \lambda^\frac{2}{p}(x) N^p_{\frac{1}{2} - H} u_\varepsilon(t, x) \|_{L^p(\Omega)} \leq C_{T, H}(\| u_\varepsilon \|_{Z^p_{\lambda, T}} + 1). \quad (4.50)
\]
(ii) If \( p > \frac{3}{p} \), then
\[
\left\| \sup_{t,t+h \in [0,T], x \in \mathbb{R}} \lambda_t^\frac{1}{p}(x) \left[ u_\varepsilon(t+h, x) - u_\varepsilon(t, x) \right] \right\|_{L^p(\Omega)} \leq C_{T,H} |h|^\gamma (\|u_\varepsilon\|_{Z_{\lambda,T}^p} + 1), \tag{4.51}
\]
for all \( 0 < \gamma < \frac{H}{2} - \frac{3}{2p} \).

(iii) If \( p > \frac{3}{p} \), then
\[
\left\| \sup_{t \in [0,T], x \in \mathbb{R}} \lambda_t^\frac{1}{p}(x) \frac{u_\varepsilon(t, x) - u_\varepsilon(t, y)}{x-y} \right\|_{L^p(\Omega)} \leq C_{T,H} |x-y|^\gamma (\|u_\varepsilon\|_{Z_{\lambda,T}^p} + 1), \tag{4.52}
\]
for all \( 0 < \gamma < H - \frac{3}{p} \).

Proof. Denote for \( \alpha \in (0,1] \)
\[
J_\alpha^\varepsilon(r, \xi) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} G_{r-s}(\xi - z) \sigma(u_\varepsilon(s, z)) G_\varepsilon(z - y) dz W(dy, ds).
\]
Then Fubini's theorem (factorization method) implies
\[
u_\varepsilon(t, x) = G_t * u_0(x) + \frac{\sin(\pi \alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x - \xi) J_\alpha^\varepsilon(r, \xi) d\xi dr.
\]
Using the same method as that in the proof of Proposition \ref{prop:regularity} (ii), (iii), (iv) and Step 3 of Lemma \ref{lem:regularity}, we can conclude the result. \qed

Proof of Theorem \ref{thm:main}. We still assume \( \sigma(t, x, u) = \sigma(u) \) to simplify the notations. From Lemma \ref{lem:compactness} and Lemma \ref{lem:tightness} (ii) and (iii) it follows that the two conditions of Theorem \ref{thm:main} are satisfied. Hence, the probability measures on the space \((C([0,T] \times \mathbb{R}), \mathcal{B}(C([0,T] \times \mathbb{R}), d_\varepsilon))\) corresponding to the processes \(\{u_\varepsilon, \varepsilon \in (0,1]\}\) are tight. Thus, there is a subsequence \(\varepsilon_n \downarrow 0\) such that \(u_n = u_{\varepsilon_n}\) convergence weakly. By Skorokhod representation theorem, there is a probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) carrying the subsequence \(\bar{u}_{n_j}\) and noise \(\bar{W}\) such that the finite dimensional distributions of \((\bar{u}_{n_j}, \bar{W})\) and \((u_n, W)\) coincide. Moreover, we have
\[
\bar{u}_{n_j}(t, x) \rightarrow \bar{u}(t, x) \text{ in } \mathcal{C}([0,T] \times \mathbb{R}), d_\varepsilon \text{ } \bar{P}\text{-almost surely (4.53)}
\]
for a certain stochastic process \(\bar{u}\) as \(j \rightarrow \infty\). By Lemma \ref{lem:equivalence} we see that \(\bar{u}\) belongs to space \(Z_{\lambda,T}^p\) equipped with the new probability \(\bar{P}\). We want to show that \(\bar{u}\) is a weak solution to (1.1).

Define the filtration \(\bar{\mathcal{F}}_t\) be the filtration generated by \(\bar{W}\), we claim that \(\bar{u}_{n_j}\) satisfies (1.1) with \(W\) replaced by \(\bar{W}\), more precisely,
\[
\bar{u}_{n_j}(t, x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - \cdot) \sigma(\bar{u}_{n_j}(s, \cdot)) * G_{\varepsilon_j}(y) \bar{W}(dy, ds). \tag{4.54}
\]
To show this it is sufficient to prove that for any \(Z \in L^2(\bar{\Omega}, \bar{P})\) one has
\[
\mathbb{E}[\bar{u}_{n_j}(t, x)Z] = \mathbb{E}\left[ G_t * u_0(x)Z \right. \left. + \int_0^t \int_{\mathbb{R}} G_{t-s}(x - \cdot) \sigma(\bar{u}_{n_j}(s, \cdot)) * G_{\varepsilon_j}(y) \bar{W}(dy, ds)Z \right]. \tag{4.55}
\]
where \( \tilde{E} \) means the expectation under \( \tilde{P} \).

For any \( \phi \in \mathcal{D}(\mathbb{R}) \), denote
\[
\tilde{W}_i(\phi) = \int_{\mathbb{R}} \phi(x) \tilde{W}(t, dx); \quad W_i(\phi) = \int_{\mathbb{R}} \phi(x) W(t, dx)
\]
It is routine to argue that the set
\[
\mathcal{S} := \left\{ f(\tilde{W}_1(\phi), \cdots, \tilde{W}_n(\phi)), \ 0 \leq t_1 \leq \cdots \leq t_n \leq T \ f \in C_0(\mathbb{R}^n) \right\}
\]
are dense in \( L^2(\Omega, \tilde{P}, \tilde{\mathcal{F}}_T) \). This means that it is sufficient to choose \( Z = f(\tilde{W}_1(\phi), \cdots, \tilde{W}_n(\phi)) \) in \( \mathcal{S} \), which is true because we have the following identities:
\[
\tilde{E}[\tilde{u}_{n_j}(t, x)f(\tilde{W}_1(\phi), \cdots, \tilde{W}_n(\phi))] = E[u_{n_j}(t, x)f(W_1(\phi), \cdots, W_n(\phi))];
\]
\[
\tilde{E} \left[ G_t * u_0(x)f(\tilde{W}_1(\phi), \cdots, \tilde{W}_n(\phi)) \right] = E[G_t * u_0(x)f(W_1(\phi), \cdots, W_n(\phi))]
\]
and
\[
\tilde{E} \left[ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(\tilde{u}_{n_j}(s, \cdot)) \cdot \tilde{W}(dy, ds)f(\tilde{W}_1(\phi), \cdots, \tilde{W}_n(\phi)) \right]
\]
\[
= \tilde{E} \left[ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-\cdot)\sigma(u_{n_j}(s, \cdot)) \cdot W(dy, ds)f(W_1(\phi), \cdots, W_n(\phi)) \right]
\]
due to the fact the finite dimensional distributions of \( (\tilde{u}_{n_j}, \tilde{W}) \) coincide with that of \( (u_{n_j}, W) \). Therefore, \( \tilde{u}_{n_j}(t, x) \) satisfies \( (1.54) \).

From \( (4.53) \) and \( (4.54) \) it follows that \( \tilde{u} \) is a mild solution to \( (1.1) \) with \( W \) replaced by \( \tilde{W} \). Therefore we have completed the existence of a weak solution to \( (1.1) \).

Moreover, for any \( \gamma \in (0, H - \frac{3}{p}) \) and for any compact set \( \mathcal{T} \subseteq [0, T] \times \mathbb{R} \), Lemma \( 4.7 \) (parts (ii) and (iii)) implies that there exists constant \( C \) such that
\[
\tilde{E} \left( \sup_{t, s, x, y \in \mathcal{T}} \left| \frac{\tilde{u}(t, x) - \tilde{u}(s, y)}{|t - s|^{\frac{3}{p}} + |x - y|^{\gamma}} \right|^P \right) \leq C \|u\|_{L^p_{\mathcal{X}, \lambda, \mathcal{T}}}^p.
\]
(4.56)
This combined with Kolmogorov lemma implies the desired Hölder continuity.

\section{Pathwise Uniqueness and Strong Existence of solutions}

In this section we prove the pathwise uniqueness and the existence of strong solution for the equation \( (1.1) \). It is well known once pathwise uniqueness is achieved, together with the existence of weak solution proved in previous section, we can conclude the existence of the unique strong solutions to \( (1.1) \) by, see for example, proof of Theorem 4.3 in \( (11) \) or the Yamada-Watanabe theorem \( (13) \). Therefore we only need to focus on the proof of pathwise uniqueness.

\textbf{Proof of Theorem 4.6.} The proof follows the strategy in the proof of Theorem 4.3 of \( (11) \) combined with Proposition 4.2 (part (ii)).

Define the following stopping times
\[
T_k := \inf \left\{ t \in [0, T]: \sup_{0 \leq s \leq t} \lambda^x_\sigma(x)N_{\frac{3}{p} - H} u(s, x) \geq k, \right. \]
\[
\left. \text{or} \sup_{0 \leq s \leq t} \lambda^x_\sigma(x)N_{\frac{3}{p} - H} v(s, x) \geq k \right\}, \quad k = 1, 2, \cdots
\]

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Proposition 4.2 part (ii) implies that $T_k \uparrow T$ almost surely as $k \to \infty$. We need to find appropriate bounds for the following two quantities:

$$I_1(t) = \sup_{x \in \mathbb{R}} \mathbb{E} \left[ 1_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right]$$

and

$$I_2(t) = \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \int_{\mathbb{R}} 1_{\{t < T_k\}} |u(t, x) - v(t, x) - u(t, x + h) + v(t, x + h)|^2 |h|^{2H-2} dh \right].$$

At first, it is easy to see

$$1_{\{t < T_k\}}(u(t, x) - v(t, x)) = 1_{\{t < T_k\}} \int_0^t G_{t-s}(x-y) 1_{\{s < T_k\}}[\sigma(s, y, u(s, y)) - \sigma(s, y, v(s, y))] W(dy, ds).$$

Recall $D_r(x, h)$ defined in (2.14) and denote $\Delta(t, x, y) = \sigma(t, x, u(t, y)) - \sigma(t, x, v(t, y))$. We can decompose

$$\mathbb{E} \left[ 1_{\{t < T_k\}} |u(t, x) - v(t, x)|^2 \right] \leq \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} 1_{\{s < T_k\}} |D_{t-s}(x-y, h)|^2 |\Delta(s, y, y)|^2 |h|^{2H-2} dhdyds \right)$$

$$+ \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} 1_{\{s < T_k\}} G_{t-s}(x-y-h)[\Delta(s, y, h, y) - \Delta(s, y+h, y)]^2 |h|^{2H-2} dhdyds \right)$$

$$+ \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} 1_{\{s < T_k\}} G_{t-s}^2(x-y)[\Delta(s, y, y+h) - \Delta(s, y, y+h)]^2 |h|^{2H-2} dhdyds \right).$$

(5.1)

The assumption (1.13) of $\sigma$ and the equality (2.17) can be used to dominate the above first term. This is,

$$\mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} 1_{\{s < T_k\}} |D_{t-s}(x-y, h)|^2 |\Delta(s, y, y)|^2 |h|^{2H-2} dhdyds \right) \leq \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} 1_{\{s < T_k\}} |D_{t-s}(x-y, h)|^2 |u(s, y) - v(s, y)|^2 |h|^{2H-2} dhdyds \right)$$

$$\leq \int_0^t (t-s)^{H-1} \sup_{y \in \mathbb{R}} \mathbb{E} \left[ 1_{\{s < T_k\}} |u(s, y) - v(s, y)|^2 \right] ds = \int_0^t (t-s)^{H-1} I_1(s) ds.$$

Using the properties (1.13) of $\sigma$, we have if $|h| > 1$

$$[\Delta(s, y + h, y) - \Delta(s, y, y)]^2 \lesssim |u(s, y) - v(s, y)|^2$$

$$= \left| \int_0^u \left[ \sigma'_\xi(s, y + h, \xi) - \sigma'_\xi(s, y, \xi) \right] d\xi \right|^2 \lesssim |u(s, y) - v(s, y)|^2,$$

and if $|h| \leq 1$ (with the help of additional properties (1.14))

$$[\Delta(s, y + h, y) - \Delta(s, y, y)]^2$$

$$= \left| \int_0^u \left[ \sigma'_\xi(s, y + h, \xi) - \sigma'_\xi(s, y, \xi) \right] d\xi \right|^2 \lesssim |h|^2 |u(s, y) - v(s, y)|^2.$$
Thus, the second term in (5.1) is bounded by
\[
\mathbb{E}\left( \int_0^t \int_\mathbb{R} \int_{|h|>1} 1_{s<T_k} G_{t-s}^2(x-y-h)|u(s,y) - v(s,y)|^2 |h|^{2H-2} dh dy ds \right) \\
+ \mathbb{E}\left( \int_0^t \int_\mathbb{R} \int_{|h|\leq 1} 1_{s<T_k} G_{t-s}^2(x-y-h)|u(s,y) - v(s,y)|^2 |h|^{2H} dh dy ds \right) \\
\lesssim \int_0^t I_1(s) \left( \int_\mathbb{R} G_{t-s}^2(x-y)dy \right) ds \lesssim \int_0^t (t-s)^{-\frac{5}{2}} I_1(s) ds.
\]

For the last term in (5.1) we have by (1.13), (1.15)
\[
|\triangle(s,y,y+h) - \triangle(s,y,y)|^2 \\
= \left| \int_0^1 [u(s,y+h) - v(s,y+h)]\sigma_z'(s,y,\theta u(s,y+h) + (1 - \theta)v(s,y+h))d\theta \\
- \int_0^1 [u(s,y) - v(s,y)]\sigma_z'(s,y,\theta u(s,y) + (1 - \theta)v(s,y))d\theta \right|^2 \\
\lesssim |u(s,y+h) - v(s,y+h) - u(s,y) + v(s,y)|^2 \\
+ \lambda^{\frac{3}{2}}(y)|u(s,y) - v(s,y)|^2 \cdot \left[ |u(s,y+h) - u(s,y)|^2 + |v(s,y+h) - v(s,y)|^2 \right].
\]

Noticing the additional uniform decay assumption (1.13), we can dominate the last term in (5.1) by
\[
kC \int_0^t (t-s)^{-\frac{5}{2}} [I_1(s) + I_2(s)] ds.
\]

Summarizing the above estimates we have
\[
I_1(t) \lesssim k \int_0^t (t-s)^{H-1} [I_1(s) + I_2(s)] ds,
\]
where the constant k depends on the stopping times \( T_k \).

The similar procedure can be applied to estimate term \( I_2(t) \) above as
\[
I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{5}{2}} [I_1(s) + I_2(s)] ds.
\]

As a consequence,
\[
I_1(t) + I_2(t) \lesssim k \int_0^t (t-s)^{2H-\frac{5}{2}} [I_1(s) + I_2(s)] ds.
\]

Now Gronwall’s lemma implies \( I_1(t) + I_2(t) = 0 \) for all \( t \in [0,T] \). In particular, we have
\[
\mathbb{E}[1_{\{t<T_k\}}|u(t,x) - v(t,x)|^2] = 0.
\]

Thus, we have \( u(t,x) = v(t,x) \) almost surely on \( \{t<T_k\} \) for all \( k \geq 1 \), and the fact \( T_k \uparrow \infty \) a.s as \( k \) tends to infinity necessarily indicate \( u(t,x) = v(t,x) \) a.s. for every \( t \in [0,T] \) and \( x \in \mathbb{R} \).

It is clear that hypothesis (H2) implies the hypothesis (H1). So the existence of a Hölder continuous modification version of the solution follows from Theorem 1.6. We have then completed the proof of Theorem 1.6. \( \square \)
REFERENCES

1. Adler, R. An introduction to continuity, extrema, and related topics for general Gaussian processes. Institute of Mathematical Statistics Lecture Notes-Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA, 1990.

2. Bahouri, H., Chemin, J., and Danchin, R. Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011. xvi+523 pp.

3. Balan, R.; Jolis, M. and Quer-Sardanyons, L. SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$. Electronic Journal of Probability. 20 (2015).

4. Chen, X. Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise. Ann. Probab. 44 (2016), no. 2, 1535-1598.

5. Chen, X.; Hu, Y.; Nualart, D. and Tindel, S. Spatial asymptotics for the parabolic Anderson model driven by a Gaussian rough noise. Electron. J. Probab. 22 (2017), Paper No. 65, 38 pp.

6. Conus, D.; Joseph, M. and Khoshnevisan, D. On the chaotic character of the stochastic heat equation, before the onset of intermittency. The Annals of Probability, 41 (2013), 2225-2260.

7. Conus, D.; Joseph, M.; Khoshnevisan, D. and Shiu, S.Y. On the chaotic character of the stochastic heat equation, II. Probability Theory and Related Fields. 156 (2014), 483-533.

8. Dirksen S. Tail bounds via generic chaining. Electronic Journal of Probability. 20(2015).

9. Gyöngy, I. Existence and uniqueness results for semilinear stochastic partial differential equations. Stochastic Processes and their Applications. 73 (1998), 271-299.

10. Hu, Y. Some recent progress on stochastic heat equations. Acta Math. Sci. 39 (2019), 874-914.

11. Hu, Y.; Huang, J.; Lé, K.; Nualart, D. and Tindel, S. Stochastic heat equation with rough dependence in space. The Annals of Probability. 45 (2017), 4561-616.

12. Hu, Y.; Huang, J.; Lé, K.; Nualart, D. and Tindel, S. Parabolic Anderson model with rough dependence in space. Computation and combinatorics in dynamics, stochastics and control, 477-498, Abel Symp., 13, Springer, 2018.

13. Hu, Y. and Lé, K. Joint Hölder continuity of parabolic Anderson model. Acta Math. Sci. 39 (2019), 764-780.

14. Hu, Y. and Lé, K. Asymptotics of the density of parabolic Anderson random fields. arXiv:1801.03386 (2018).

15. Ikeda, N. and Watanabe, S. Stochastic Differential Equations and Diffusion Processes, second edition, North-Holland Mathematical Library, vol. 24, North-Holland Publishing Co., Kodansha, Ltd., Amsterdam, Tokyo, 1989.

16. Khoshnevisan D. Analysis of stochastic partial differential equations. American Mathematical Soc.; 2014 Jun 11.

17. Karatzas I, Shreve SE. Brownian Motion and Stochastic Calculus. Springer. 1998.

18. Kurtz, T. The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. Electronic Journal of Probability. 2007;12:951-65.

19. Pipiras, V. and Taqqu, M.S. Integration questions related to fractional Brownian motion. Probability theory and related fields. 2000 Oct 1;118(2):251-91.

20. Sanz-Solé, M. and Sarrà, M. Hölder continuity for the stochastic heat equation with spatially correlated noise. In Seminar on Stochastic Analysis, Random Fields and Applications III 2002 (pp. 259-268). Birkhäuser, Basel.

21. Talagrand, M. Upper and lower bounds for stochastic processes: modern methods and classical problems. Vol. 60. Springer Science and Business Media, 2014.

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