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Vanishing of the anchored isoperimetric profile in bond percolation at $p_c$

Raphaël Cerf† Barbara Dembin‡§

Abstract

We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at $p_c$ exists, it has to vanish.

1 Introduction

The most well–known open question in percolation theory is to prove that the percolation probability vanishes at $p_c$ in dimension three. In fact, the interesting quantities associated to the model are very difficult to study at the critical point or in its vicinity. We study here a very modest intermediate question. We consider the anchored isoperimetric profile of the infinite open cluster, defined for $p > p_c$, whose existence has been recently proved in [3]. We extend adequately the definition for $p = p_c$, in finite boxes. We prove a partial result which implies that, if the limit defining the anchored isoperimetric profile at $p_c$ exists, it has to vanish.

The Cheeger constant. For a graph $G$ with vertex set $V$ and edge set $E$, we define the edge boundary $\partial_G A$ of a subset $A$ of $V$ as

$$\partial_G A = \left\{ e = (x, y) \in E : x \in A, y \notin A \right\}.$$ 

We denote by $|B|$ the cardinal of the finite set $B$. The Cheeger constant of the graph $G$ is defined as

$$\varphi_G = \min \left\{ \frac{|\partial_G A|}{|A|} : A \subset V, 0 < |A| \leq \frac{|V|}{2} \right\}.$$ 

†DMA, Ecole Normale Supérieure, CNRS, PSL University, 75005 Paris.
‡LMO, Université Paris-Sud, CNRS, Université Paris–Saclay, 91405 Orsay.
§LPSM UMR 8001, Université Paris Diderot, Sorbonne Paris Cité, CNRS, F-75013 Paris.

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This constant was introduced by Cheeger in his thesis [2] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian.

**The anchored isoperimetric profile** $\varphi_n(p)$. Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $C_\infty$ [5]. We say that $H$ is a valid subgraph of $C_\infty$ if $H$ is connected and $0 \in H \subset C_\infty$. We define the anchored isoperimetric profile $\varphi_n(p)$ of $C_\infty$ as follows. We condition on the event \{$0 \in C_\infty$\} and we set

$$\varphi_n(p) = \min \left\{ \frac{|\partial C_\infty H|}{|H|} : H \text{ valid subgraph of } C_\infty, 0 < |H| \leq n^d \right\}.$$

The following theorem from [3] asserts the existence of the limit of $n\varphi_n(p)$ when $p > p_c(d)$.

**Theorem 1.1.** Let $d \geq 2$ and $p > p_c(d)$. There exists a positive real number $\varphi(p)$ such that, conditionally on \{$0 \in C_\infty$\},

$$\lim_{n \to \infty} n\varphi_n(p) = \varphi(p) \text{ almost surely.}$$

We wish to study how this limit behaves when $p$ is getting closer to $p_c$. To do so, we need to extend the definition of the anchored isoperimetric profile so that it is well defined at $p_c(d)$. We say that $H$ is a valid subgraph of $C(0)$, the open cluster of $0$, if $H$ is connected and $0 \in H \subset C(0)$. We define $\tilde{\varphi}_n(p)$ for every $p \in [0,1]$ as

$$\tilde{\varphi}_n(p) = \min \left\{ \frac{|\partial C(0) H|}{|H|} : H \text{ valid subgraph of } C(0), 0 < |H| \leq n^d \right\}.$$

In particular, if $0$ is not connected to $\partial[−n/2,n/2]^d$ by a $p$-open path, then $|C(0)| < n^d$ and taking $H = C(0)$, we see that $\tilde{\varphi}_n(p)$ is equal to $0$. Thanks to theorem 1.1, we have

$$\forall p > p_c \lim_{n \to \infty} n\tilde{\varphi}_n(p) = \theta(p)\varphi(p) + (1 - \theta(p))\delta_0,$$

where $\theta(p)$ is the probability that $0$ belongs to an infinite open cluster. The techniques of [3] to prove the existence of this limit rely on coarse-graining estimates which can be employed only in the supercritical regime. Therefore we are not able so far to extend the above convergence at the critical point $p_c$. Naturally, we expect that $n\tilde{\varphi}_n(p_c)$ converges towards $0$ as $n$ goes to infinity, unfortunately we are only able to prove a weaker statement.

**Theorem 1.2.** With probability one, we have

$$\lim \inf_{n \to \infty} n\tilde{\varphi}_n(p_c) = 0.$$
revealed in the exploration of the cluster of 0 will grow fast enough of order $n^{d-1}$. Then, we can prove that the intersection of the cluster that we have explored with the boundary of the box $[-n,n]^d$ is of order $n^{d-1}$. Using the fact that there is no percolation in a half-space, we obtain a contradiction. Before starting the precise proof, we recall some results from [3] on the meaning of the limiting value $\varphi(p)$.

**The Wulff theorem.** We denote by $\mathcal{L}^d$ the $d$-dimensional Lebesgue measure and by $\mathcal{H}^{d-1}$ denotes the $(d-1)$-Hausdorff measure in dimension $d$. Given a norm $\tau$ on $\mathbb{R}^d$ and a subset $E$ of $\mathbb{R}^d$ having a regular enough boundary, we define $\mathcal{I}_\tau(E)$, the surface tension of $E$ for the norm $\tau$, as

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(n_E(x)) \mathcal{H}^{d-1}(dx).$$

We consider the anisotropic isoperimetric problem associated with the norm $\tau$:

$$\text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1. \tag{1}$$

The famous Wulff construction provides a minimizer for this anisotropic isoperimetric problem. We define the set $\hat{W}_\tau$ as

$$\hat{W}_\tau = \bigcap_{v \in S^{d-1}} \{ x \in \mathbb{R}^d : x \cdot v \leq \tau(v) \},$$

where $\cdot$ denotes the standard scalar product and $S^{d-1}$ is the unit sphere of $\mathbb{R}^d$.

Up to translation and Lebesgue negligible sets, the set

$$\frac{1}{\mathcal{L}^d(\hat{W}_\tau)^{1/d}} \hat{W}_\tau$$

is the unique solution to the problem (1).

**Representation of $\varphi(p)$**. In [3], we build an appropriate norm $\beta_p$ for our problem that is directly related to the open edge boundary. We define the Wulff crystal $W_p$ as the dilate of $\hat{W}_{\beta_p}$ such that $\mathcal{L}^d(W_p) = 1/\theta(p)$, where $\theta(p) = P(0 \in C_\infty)$. We denote by $\mathcal{I}_p$ the surface tension associated with the norm $\beta_p$. In [3], we prove that

$$\forall p > p_c(d) \quad \varphi(p) = \mathcal{I}_p(W_p).$$

## 2 Proofs

We prove next the following lemma, which is based on two important results due to Zhang [9] and Rossignol and Théret [6]. To alleviate the notation, the critical point $p_c(d)$ is denoted simply by $p_c$.

**Lemma 2.1.** We have

$$\lim_{p \to p_c} \left( \theta(p) \delta_{\mathcal{I}_p(W_p)} + (1 - \theta(p)) \delta_0 \right) = \delta_0.$$
Proof. If \( \lim_{p \to p_c} \theta(p) = 0 \), then the result is clear. Otherwise, let us assume that

\[
\lim_{p \to p_c} \theta(p) = \delta > 0.
\]

Let \( B \) be a subset of \( \mathbb{R}^d \) having a regular boundary and such that \( \mathcal{L}^d(B) = 1/\delta \). As the map \( p \mapsto \theta(p) \) is non-decreasing and \( \mathcal{L}^d(W_p) = 1/\theta(p) \), we have

\[
\forall p > p_c \quad \mathcal{L}^d(W_p) \leq \mathcal{L}^d(B).
\]

Moreover as \( W_p \) is the dilate of the minimizer associated to the isoperimetric problem (1), we have

\[
\forall p > p_c \quad \mathcal{I}_p(W_p) \leq \mathcal{I}_p(B).
\]

In [9], Zhang proved that \( \beta_{p_c} = 0 \). In [6], Rossignol and Théret proved the continuity of the flow constant. Combining these two results, we get that

\[
\lim_{p \to p_c} \beta_p = \beta_{p_c} = 0 \quad \text{and so} \quad \lim_{p \to p_c} \mathcal{I}_p(B) = 0.
\]

Finally, we obtain

\[
\lim_{p \to p_c} \mathcal{I}_p(W_p) = 0.
\]

This yields the result.

Proof of theorem 1.2. We assume by contradiction that

\[
\mathbb{P} \left( \liminf_{n \to \infty} n \tilde{\varphi}_n(p_c) = 0 \right) < 1.
\]

Therefore there exist positive constants \( c \) and \( \delta \) such that

\[
\mathbb{P} \left( \liminf_{n \to \infty} n \tilde{\varphi}_n(p_c) > c \right) = \lim_{n \to \infty} \mathbb{P} \left( \inf_{k \geq n} k \tilde{\varphi}_k(p_c) > c \right) = \delta.
\]

Therefore, there exists a positive integer \( n_0 \) such that

\[
\mathbb{P} \left( \inf_{k \geq n_0} k \tilde{\varphi}_k(p_c) > c \right) \geq \frac{\delta}{2}.
\]

In what follows, we condition on the event

\[
\left\{ \inf_{k \geq n_0} k \tilde{\varphi}_k(p_c) > c \right\}.
\]

Note that on this event, 0 is connected to infinity by a \( p_c \)-open path. For \( H \) a subgraph of \( Z^d \), we define

\[
\partial^o H = \left\{ e \in \partial H, \; e \text{ is open} \right\}.
\]

Note that if \( H \subset C_\infty \), then \( \partial_{C_\infty} H = \partial^o H \). Moreover, if \( H \) is equal to \( C(0) \), the open cluster of 0, then \( \partial_{C(0)} H = \partial^o H = \emptyset \). We define next an exploration
process of the cluster of 0. We set $C_0 = \{0\}$, $A_0 = \emptyset$. Let us assume that $C_0, \ldots, C_l$ and $A_0, \ldots, A_l$ are already constructed. We define

$$A_{l+1} = \{ x \in \mathbb{Z}^d : \exists y \in C_l \quad \langle x, y \rangle \in \partial C_l \}$$

and

$$C_{l+1} = C_l \cup A_{l+1}.$$  

We have

$$\partial C_l \subset \{ \langle x, y \rangle \in \mathbb{E}^d : x \in A_{l+1} \}$$

so that $|\partial C_l| \leq 2d |A_{l+1}|$. Since $A_{l+1}$ and $C_l$ are disjoint, we have

$$|C_{l+1}| = |C_l| + |A_{l+1}| \geq |C_l| + \frac{|\partial C_l|}{2d}.$$  

Let us set $\alpha = 1/n^d$ so that $|C_0| = \alpha n^d$. Let $k$ be the smallest integer greater than $2d+1$. We recall that $c$ and $n_0$ were defined in (2) and (3). Let us prove by induction on $n$ that

$$\forall n \geq n_0 \quad |C_{(n-n_0)k}| \geq \alpha n^d.$$  

This is true for $n = n_0$. Let us assume that this inequality is true for some integer $n \geq n_0$. If $|C_{(n+1-n_0)k}| \geq n^d$, then we are done. Suppose that $|C_{(n+1-n_0)k}| < n^d$. In this case, for any integer $l \leq k$, we have also $|C_{(n-n_0)k+l}| < n^d$, and since $C_{(n-n_0)k+l}$ is a valid subgraph of $C(0)$ and $\hat{\varphi}_n(p_c) > c/n$, we conclude that

$$\frac{|\partial C_{(n-n_0)k+l}|}{|C_{(n-n_0)k+l}|} \geq \frac{c}{n}$$

and so $|\partial C_{(n-n_0)k+l}| \geq \alpha n^{d-1}$. Thanks to inequality (4) applied $k$ times, we have

$$|C_{(n+1-n_0)k}| \geq \alpha \left( n^d + \frac{ck}{2d} n^{d-1} \right).$$

As $k \geq 2d+1$, we get

$$|C_{(n+1-n_0)k}| \geq \alpha (n^d + 2d n^{d-1}) \geq \alpha (n+1)^d.$$  

This concludes the induction.

Let $\eta > 0$ be a constant that we will choose later. In [1], Barsky, Grimmett and Newman proved that there is no percolation in a half-space at criticality. An important consequence of the result of Grimmett and Marstrand [4] is that the critical value for bond percolation in a half-space equals to the critical parameter $p_c(d)$ of bond percolation in the whole space, i.e., we have

$$\mathbb{P}(0 \text{ is connected to infinity by a } p_c\text{-open path in } \mathbb{N} \times \mathbb{Z}^{d-1} = 0),$$

so that for $n$ large enough,

$$\mathbb{P}(\exists \gamma \text{ a } p_c\text{-open path starting from 0 in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } |\gamma| \geq n) \leq \eta.$$  

In what follows, we will consider an integer $n$ such that the above inequality holds. By construction the set $C_n$ is inside the box $[-n, n]^d$. Starting from this
cluster, we are going to resume our exploration but with the constraint that we do not explore anything outside the box $[-n,n]^d$. We set $C_0' = C_n$ and $A_0' = \emptyset$.

Let us assume $C_0', \ldots, C_l'$ and $A_0', \ldots, A_{l+1}'$ are already constructed. We define

$$A_{l+1}' = \{ x \in [-n,n]^d : \exists y \in C_{l}' \ (x,y) \in \partial C_{l}' \}$$

and

$$C_{l+1}' = C_{l}' \cup A_{l+1}' .$$

We stop the process when $A_{l+1}' = \emptyset$. As the number of vertices in the box $[-n,n]^d$ is finite, this process of exploration will eventually stop for some integer $l$. We have that $|C_l'| \leq n^d$ and $n \tilde{\phi}_k(p_\alpha) > c$ so that

$$|\partial C_l'| \geq \frac{c}{n} |C_l'| \geq \frac{c}{n} |C_n| .$$

Moreover, for $n \geq kn_0$, we have, thanks to inequality (5),

$$|C_n| \geq |C_{l+1}'| \geq |C_{l+1}' \cap -n \times [n] | \geq \alpha \left( \frac{n}{|F|} \right)^d .$$

We suppose that $n$ is large enough so that $n \geq kn_0$ and $|F| \geq n/2k$. Combining the two previous display inequalities, we conclude that

$$|\partial C_l'| \geq \frac{\alpha}{2^d k^d} n^{d-1} .$$

Therefore, for $n$ large enough, there exists one face of $[-n,n]^d$ such that there are at least $c n^{d-1}/(2^d k^d 2d)$ vertices that are connected to 0 by a $p_\alpha$-open path that remains inside the box $[-n,n]^d$ and so

$$\Pr \left( \text{there exists one face of } [-n,n]^d \text{ with at least } \left( \frac{c}{2^d k^d} \right) n^{d-1} \text{ vertices that are connected to } 0 \text{ by a } p_\alpha \text{-open path that remains inside the box } [-n,n]^d \right) \geq \frac{\delta}{2} . (6)$$

Let us denote by $X_n$ the number of vertices in the face $\{-n\} \times [-n,n]^{d-1}$ that are connected to 0 by a $p_\alpha$-open path inside the box $[-n,n]^d$. We have

$$\mathbb{E}(X_n) \leq \left| \{-n\} \times [-n,n]^{d-1} \cap \mathbb{Z}^d \right| \cdot \Pr \left( \exists \gamma \text{ a } p_\alpha \text{-open path starting from } 0 \text{ in } \mathbb{N} \times \mathbb{Z}^{d-1} \text{ such that } \left| \gamma \right| \geq n \right) \leq (2n + 1)^{d-1} \eta . (7)$$

Moreover, we have

$$\mathbb{E}(X_n) \geq \frac{\alpha}{2^d k^d} n^{d-1} \cdot \Pr \left( X_n > \frac{\alpha}{2^d k^d} n^{d-1} \right) . (8)$$

Finally, combining inequalities (7) and (8), we get

$$\Pr \left( X_n > \frac{\alpha}{2^d k^d} n^{d-1} \right) \leq \frac{2d \eta 3^{d-1} 2^d k^d}{\alpha} .$$

Therefore, we can choose $\eta$ small enough such that

$$\Pr \left( X_n > \frac{\alpha}{2^d k^d} n^{d-1} \right) \leq \frac{\delta}{10d} .$$
and so using the symmetry of the lattice
\[
\Pr\left( \text{there exists one face of } [-n, n]^d \text{ such there are at least } \frac{cn^{d-1}}{(2^d k^d 2^d)} \text{ vertices that are connected to 0 by a } p_c \text{-open path that remains inside the box } [-n, n]^d \right) \\
\leq 2d \Pr\left( X_n > \frac{c \alpha}{2d 2^d k^d} n^{d-1} \right) \leq \frac{\delta}{5}.
\]
This contradicts inequality (6) and yields the result. \hfill \Box

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