Renormalization of Multiple $q$-Zeta Values

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Abstract. In this paper we shall define the renormalization of the multiple $q$-zeta values ($M_qZV$) which are special values of multiple $q$-zeta functions $\zeta_q(s_1, \ldots, s_d)$ when the arguments are all positive integers or all non-positive integers. This generalizes the work of Guo and Zhang [12] on the renormalization of Euler-Zagier multiple zeta values. We show that our renormalization process produces the same values if the $M_qZV$s are well-defined originally and that these renormalizations of $M_qZV$ satisfy the $q$-stuffle relations if we use shifted-renormalizations for all divergent $\zeta_q(s_1, \ldots, s_d)$ (i.e., $s_1 \leq 1$). Moreover, when $q \uparrow 1$ our renormalizations agree with those of Guo and Zhang.

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1 Introduction

The Euler-Zagier multiple zeta functions are defined as nested generalizations of the Riemann zeta function:

$$\zeta(s_1, \ldots, s_d) := \sum_{k_1 > \cdots > k_d > 0} k_1^{-s_1} \cdots k_d^{-s_d} \quad (1)$$

for complex variables $s_1, \ldots, s_d$ satisfying $\Re(s_1 + \cdots + s_j) > j$ for all $j = 1, \ldots, d$. The special values of this function at positive integers are called multiple zeta values (MZVs) and were first studied systematically by Euler [9] when $d = 2$. Nevertheless, only in the past fifteen years these values have been found to have significant arithmetic, algebraic, geometric and physics meanings and have since been under intensive investigation (see [5, 10, 13, 18, 21]).

In another direction in [22] we show by using generalized functions that multiple zeta functions can be analytically continued to $\mathbb{C}^d$ as a meromorphic function with simple poles. We will henceforth always refer to this analytic continuation when we speak of multiple zeta functions in the rest of this paper. The precise location of the simple poles form the following set (see [3]):

$$S_d = \left\{(s_1, \ldots, s_d) \in \mathbb{C}^d \left| \begin{array}{l}
s_1 = 1, \text{ or } s_1 + s_2 \in \{1\} \cup 2\mathbb{Z}_{\leq 1}, \text{ or } \\
s_1 + \cdots + s_j \in \mathbb{Z}_{\leq j} \text{ for } 3 \leq j \leq d
\end{array} \right. \right\}. \quad (2)$$

Hence MZVs at non-positive integers are not always defined. Recently, Guo and his collaborators [8, 12] have applied the Rota-Baxter algebra technique to the study of MZVs after noticing that the stuffle (stuffing+shuffle) relations reflect exactly the Rota-Baxter property. In [12] the renormalization is carried out for the MZVs and they show that when $\zeta(s_1, \ldots, s_d)$ is defined then its renormalization agrees with the value itself, provided that $s_i$'s are all positive or all non-positive. Moreover, these renormalizations satisfy the stuffle (or quasi-shuffle) relations. The importance of this result is related to the conjecture (see [15]) that to obtain all the relations among MZVs of the same weight it suffices to use all the double shuffle relations including those of the renormalization of MZVs at positive integers.

On the other hand, we can define the $q$-analog ($0 < q < 1$) of multiple zeta functions as follows (see [23]). For complex variables $s_1, \ldots, s_d$ satisfying $\Re(s_1 + \cdots + s_j) > j$ for all $j = 1, \ldots, d$, set

$$\zeta_q(s_1, \ldots, s_d) := \sum_{k_1 > \cdots > k_d > 0} q^{k_1(s_1-1)+\cdots+k_d(s_d-1)} \left/ \prod_{k_1}^{k_d} \left[k_1^{s_1} \cdots k_d^{s_d} \right] \right. \quad (3)$$
where for any real number \( r \) we write \([r] = (1 - q^r)/(1 - q)\). When \( d = 1 \) this is the same as the \( q \)-analog of the Riemann zeta function defined in [16]. By using Euler-Maclaurin summations we obtained their meromorphic continuations to \( \mathbb{C}^d \) with following singularities (which are all simple):

\[
\mathcal{S}_d = \left\{ (s_1, \ldots, s_d) \in \mathbb{C}^d \bigg| s_1 \in 1 + \frac{2\pi i}{\ln q} \mathbb{Z}, \text{ or } s_1 \in \mathbb{Z}_{\leq 0} + \frac{2\pi i}{\ln q} \mathbb{Z}_{\neq 0}, \right.
\]

\[
\left. \text{or } s_1 + \cdots + s_j \in \mathbb{Z}_{\leq j} + \frac{2\pi i}{\ln q} \mathbb{Z} \text{ for } j > 1 \right\} \supset \mathcal{S}_d.
\]

Here the last part in \( \mathcal{S}_d \) is vacuous if \( d = 1 \). One can see that these \( q \)-analogs have much more poles than their ordinary counterparts. But when \( q \) approaches 1 we indeed recover exactly the poles of the multiple zeta functions. In fact, by [23, Main Theorem] for all \( (s_1, \ldots, s_d) \in \mathbb{C}^d \setminus \mathcal{S}_d \) \( \lim_{q \uparrow 1} \zeta_q(s_1, \ldots, s_d) = \zeta(s_1, \ldots, s_d) \), which shows that our \( q \)-analogue is the correct choice.

The analytical continuation of multiple zeta functions [22] utilizes generalized functions. But Euler-Maclaurin summation can also be used instead which actually provides the main idea of special value computations contained in this paper. For future reference we define the Bernoulli polynomials \( B_k(x) \) and its periodic analogue \( \tilde{B}_k(x) \) by

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \tilde{B}_k(x) = B_k(\{x\}), \quad x \geq 1,
\]

(5)

where \( \{x\} \) is the fractional part of \( x \). We can then prove the analytic continuation of \( \zeta(s_1, \ldots, s_d) \) using these functions. See [23, Theorem 3.2] for more details.

Similarly, the analytic continuation of multiple \( q \)-zeta functions \( \zeta_q(s_1, \ldots, s_d) \) can be obtained by using Euler-Maclaurin summation formula. The major difference between ordinary MZVs and \( \text{MqZVs} \) is the appearance of the shifting operators \( \delta_j \) (1 \( \leq j \leq d \)) defined as follows:

\[
\delta_j \zeta_q(s_1, \ldots, s_d) := \zeta_q(s_1, \ldots, s_d) + (1 - q)\zeta_q(s_1, \ldots, s_j - 1, \ldots, s_d).
\]

In general we may iterate the operator and get

\[
\delta_j^n \zeta_q(s_1, \ldots, s_d) = \sum_{r=0}^{n} \binom{n}{r} (1 - q)^r \zeta_q(s_1, \ldots, s_j - r, \ldots, s_d).
\]

(6)

Using these operators we proved the analytic continuation of multiple \( q \)-zeta functions in [23].

The \( q \)-analogue of MZVs will be called multiple \( q \)-zeta values (\( \text{MqZVs} \)). In this paper we will consider the renormalization problem for \( \text{MqZVs} \) motivated by the ideas of Guo and Zhang in [12]. From physics point of view these values can be regarded as the quantization of MZVs. Furthermore these \( \text{MqZVs} \) also have number theoretical significance. For instance, it is well known that \( \zeta(0) = -1/2 \) and \( \zeta(1 - 2n) = -B_{2n}/(2n) \) for positive integers \( n \) where \( B_{2n} \) are Bernoulli numbers defined by \( x/(e^x - 1) = \sum_{n=0}^{\infty} B_n x^n/n! \). What are the right \( q \)-analogue of these numbers? It turns out that one of the ways to find the answer is to consider Riemann \( q \)-ZVs at negative integers (see [16], [18]), which shows that the odd indexed \( q \)-analogs of Bernoulli numbers are actually nonzero. Is it possible to generalize Bernoulli numbers to multi-Bernoulli numbers and their \( q \)-analogs? Maybe this problem can be solved when we carry out further studies of the renormalization of MZVs at negative integers.

The major behavioral difference between MZVs and \( \text{MqZVs} \) is the appearance of the shifting operator in the \( q \)-analogs defined by (6). As we mentioned in the above it is very fruitful to study the stuffle relations between MZVs. The \( q \)-analogue of this is a little more complicated because of the shifting operator which can still be handled by setting things up carefully. The main result of this paper is that we can define the renormalization of \( \text{MqZVs} \) when \( s_j \)’s are all positive integers or all non-positive integers such that (i) they coincide with the \( \text{MqZV} \) if it is defined originally, (ii) they satisfy a shifted version of \( q \)-stuffle relations, and (iii) they become the renormalization of MZVs defined by Guo and Zhang in [12] when \( q \uparrow 1 \).

To conclude this introduction we remark that currently there are two ways to order the variables in \( \text{MVZs} \) and our \( \zeta(s_1, \ldots, s_d) \) in this paper is denoted by \( \zeta(s_d, \ldots, s_1) \) in [23]. We change our
notation system because it is more convenient now for the readers to compare results in this paper to their classical counterparts in (12) which serves as the major motivation for us.

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2 The Rota-Baxter algebra and the $q$-stuffle product

Let $k$ be a subring of $\mathbb{C}$ which is usually taken to be $\mathbb{R}$ or $\mathbb{C}$. For any fixed $\lambda \in k$ a Rota-Baxter $k$-algebra of weight $\lambda$ (previously called a Baxter algebra) is a pair $(R, P)$ in which $R$ is a $k$-algebra and $P : R \to R$ is a $k$-linear map, such that

$$P(x)P(y) = P(xy) + P(xP(y)) + P(P(x)y), \quad \forall x, y \in R. \quad (7)$$

In this paper we are going to concentrate on the following two examples, both of which are contained in (12).

Example 2.1. Let $\varepsilon$ be a complex variable such that $\Re(\varepsilon) < 0$. Let $\mathbb{C}[[\varepsilon, \varepsilon^{-1}]]$ be the algebra of convergent Laurent series in a neighborhood of $\varepsilon = 0$ with at worst finite order pole at 0. Write $T = -\ln(-\varepsilon)$ which is transcendental over $\mathbb{C}[[\varepsilon, \varepsilon^{-1}]]$. Then we can regard $R := \mathbb{C}[[\varepsilon, \varepsilon^{-1}]][T]$ as the polynomial algebra with the variable $T$ and with coefficients in $\mathbb{C}[[\varepsilon, \varepsilon^{-1}]]$. Let $P$ be the operator on $R$ which takes the pole part. Then it’s not hard to verify that $(R, P)$ is a Rota-Baxter $\mathbb{C}$-algebra of weight $-1$.

Example 2.2. Let $\mathcal{H}$ be a connected filtered Hopf algebra over $k$ (see (12) §2.1 for the definition) and let $(R, P)$ be a commutative Rota-Baxter algebra of weight $\lambda$. Define the $k$-algebra $\mathcal{R} := \text{Hom}_k(\mathcal{H}, R)$ of linear maps from $\mathcal{H}$ to $R$ with the product compatible with the coproduct of the Hopf algebra $\mathcal{H}$. Then the operator $\mathcal{P}$ on $\text{Hom}(\mathcal{H}, R)$ defined by $\mathcal{P}(L) = P \circ L$ is a Rota-Baxter operator on $\mathcal{R}$ of weight $\lambda$.

In this section we will construct one such Hopf algebra of Example 2.2. For any subset $\mathcal{Z}$ of $\mathbb{C}$ closed under addition and shifting by $-1$ we define the commutative semigroup

$$\mathcal{N}(\mathcal{Z}) := \left\{ \left[ \begin{array}{c} s \\ r \end{array} \right] \mid n \in \mathbb{Z}_{\geq 0}, (s, r) \in \mathcal{Z} \times \mathbb{R}_{> 0} \right\} \quad (8)$$

with the binary operation given by $\left[ \begin{array}{c} s \\ r \end{array} \right] \cdot \left[ \begin{array}{c} s' \\ r' \end{array} \right] = \left[ \begin{array}{c} s + s' \\ r + r' \end{array} \right]$. We will only have two different choices for $\mathcal{Z}$ in this paper: $\mathbb{Z}$ or $\mathbb{Z}_{> 0}$. The reason to require $\mathcal{Z}$ to be closed under shifting by $-1$ is because of the effects of shifting operators on $\mathbb{Q}ZVs$. To study other renormalization at other poles in the future we need to set $\mathcal{Z} = \mathbb{Z} + (2\pi i/\ln q)\mathbb{Z}$ (see (1)).

Define the $\mathbb{C}$-bilinear pairing $\langle \cdot, \cdot \rangle$ on the $\mathbb{C}$-algebra $\mathcal{N}(\mathcal{Z})$ by

$$\langle \left[ \begin{array}{c} s \\ r \end{array} \right], \left[ \begin{array}{c} s' \\ r' \end{array} \right] \rangle := \left[ \begin{array}{c} s + s' \\ r + r' \end{array} \right] + (1 - q) \left[ \begin{array}{c} s + s' - 1 \\ r + r' \end{array} \right]. \quad (9)$$

Recall from (12) §3.1 that we can define the algebra:

$$\mathcal{H}_\mathcal{Z} := \sum_{n \geq 0} \mathcal{N}(\mathcal{Z})^n$$

where $\mathcal{N}(\mathcal{Z})^0 = \{1\}$ is the multiplicative identity and $\mathbb{C} \mathcal{N}(\mathcal{Z})^n$ is the free $\mathbb{C}$-module with basis $\mathcal{N}(\mathcal{Z})^n$. Then we may equip the $q$-stuffle product, which is the $q$-analog of the quasi-shuffle product $*$ for $\mathbb{Q}ZVs$ (see (13) Thm. 2.1 or (12) Thm. 2.2), on $\mathcal{H}_\mathcal{Z}$ as follows: for $a = (a_1, \ldots, a_m) \in \mathcal{N}(\mathcal{Z})^m$ and $b = (b_1, \ldots, b_n) \in \mathcal{N}(\mathcal{Z})^n$ we set $a' = 1$ if $m = 1$ and $a' = (a_2, \ldots, a_m)$ otherwise. Then we define $1 \ast_q a = a \ast_q 1 = a$ and recursively

$$a \ast_q b = (a_1, a' \ast_q b) + (b_1, a \ast_q b') + ((a_1, b_1), a' \ast_q b') \quad (10)$$
where \( \langle a_1, b_1 \rangle \) is given by (10). It has a connected filtered Hopf algebra structure over \( \mathbb{C} \) when we define the deconcatenation coproduct suitably. If \( \tilde{s} \star_q \tilde{s}' = \sum_{m,n \geq 0} (1 - q)^n \tilde{s}_{m,n} \) then we proved in [23, Theorem 5.1] that

\[
\zeta_q(\tilde{s}) = \sum_{m \geq 0} (1 - q)^m \zeta_q(\tilde{s}_{m,0}).
\]

\section{Regularized multiple \( q \)-zeta values}

Let’s recall the classical process of renormalization. For example, let’s consider the divergent series \( \sum_{n=1}^\infty n \), which is the series we would get if we tried to plug \( s = -1 \) into \( \zeta(s) \) by using definition (11). We may tamper this series by multiplying a controlling factor on each term: \( \sum_{n=1}^\infty ne^{\varepsilon n} \), for some \( \varepsilon \) such that \( \Re(\varepsilon) < 0 \) so that we get a convergent series. By easy manipulation (see [12, (34)])

\[
\sum_{n=1}^\infty ne^{\varepsilon n} = \frac{2}{\varepsilon^2} + \sum_{j=0}^\infty \frac{B_{2j+2}}{j+2} \varepsilon^j.
\]

We then call this the \textit{regularized zeta value} at \(-1\). To recover the finite value \( \zeta(-1) \) we only need to drop the pole part \( 2/\varepsilon^2 \) and then take \( \varepsilon = 0 \). This process is called the \textit{renormalization}. Because there are more than one variable in multiple zeta functions it turns out that we need to introduce a concept called “directional vector” (see Definition 4.4) in the regularization process in order to get well-behaved regularized values so that the normalization works as desired.

Turning to our \( MqZVs \), as in the previous section we let \( \mathcal{Z} \) be a subset of \( \mathbb{C} \) which is closed under addition and shifting by \(-1\) (which will be either \( \mathbb{Z} \) or \( \mathbb{Z}_{\leq 0} \)). For \( s \in \mathcal{Z}, \ r > 0, \ \Re(\varepsilon) < 0, \) and \( x \in \mathbb{R} \) we first define

\[
f(s, r; \varepsilon, x) := \frac{q^{x(s-1)} \exp(r \varepsilon x)}{|x|^s}.
\]

Note that the controlling factor becomes \( \varepsilon x \) when \( r = 1 \) and \( q \uparrow 1 \). For every vector \( \tilde{s} = (s_1, \cdots, s_d) \in \mathcal{Z}^d \) and \( \vec{r} = (r_1, \cdots, r_d) \in (\mathbb{R}_{>0})^d \) we now set

\[
Z_q \left( \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right]; \varepsilon, x \right) := \sum_{n_1 > \cdots > n_d > 0} \prod_{j=1}^d f(s_j, r_j; \varepsilon, n_j + x)
\]

(12)

It is clear that \( Z_q \left( \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right]; \varepsilon, x \right) \) is also given by the recursive definition for \( \tilde{s} = (s_1, \cdots, s_d) \in \mathcal{Z}^d \) and \( \vec{r} = (r_1, \cdots, r_d) \in (\mathbb{R}_{>0})^d \) in (14). Following the setup of Example 2.2 we may define the \( \mathbb{C} \)-linear map

\[
\mathcal{L} : \mathcal{H}_\mathcal{Z} \rightarrow \mathcal{R}_1 := \left\{ Z_q \left( \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right]; \varepsilon, x \right) \bigg| \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right] \in \mathcal{H} \right\};
\]

(13)

where \( Q \) is the summation operator (denoted by \( P \) in [8, 24])

\[
Q(f)(x) = \sum_{n \geq 1} f(x + n).
\]

\textbf{Definition 3.1.} For \( \tilde{s} \in \mathcal{Z}^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \) by setting \( x = 0 \) in \( Z_q \left( \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right]; \varepsilon, x \right) \) we define

\[
Z_q \left( \left[ \begin{array}{c} \tilde{s} \\ \vec{r} \end{array} \right]; \varepsilon \right) := \sum_{k_1 > \cdots > k_d > 0} \prod_{j=1}^d \frac{q^{k_j(s_j-1)} \exp(r_j k_j)}{|k_j|^s}.
\]

(15)

These values are called the \textit{regularized multiple \( q \)-zeta values} (at \( \mathcal{Z} \)).
Because of the assumption $\Re(\varepsilon) < 0$ we see that $Z_q\left(\begin{bmatrix} \vec{s} \\ \vec{r} \end{bmatrix}; \varepsilon \right)$ is well-defined for all $\vec{s}$ and $\vec{r}$. In particular we don’t need to restrict to a MZV-algebra as constructed in [5 §3.2] by Ebrahimi-Fard and Guo. Moreover, when $q \uparrow 1$ we recover the definition of regularized general MZV defined in [12].

### 3.1 Regularized $q$-Riemann zeta values

In this subsection we deal with the $q$ analogue of the Riemann zeta function first. Taking $d = 1$ in Definition 3.1 we find that $Z_q\left(\begin{bmatrix} 1 \\ \varepsilon \end{bmatrix}; r \varepsilon \right) = Z_q\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}; \varepsilon \right)$ so that it suffices to consider $Z_q(s; \varepsilon) := Z_q\left(\begin{bmatrix} s \\ \varepsilon \end{bmatrix}; \varepsilon \right)$. We will first put $\mathcal{Z} = \mathcal{Z}$. To study these regularized values we set, similar to [16],

$$F(x, s; \varepsilon) = q^{\varepsilon(s-1)} \exp(\varepsilon q^{-x}[x]) = \frac{q^{\varepsilon(s-1)} \exp(\varepsilon q^{-x} - 1)/(1-q)}{(1-q)^s[x]^s}, \quad \Re(\varepsilon) < 0.$$  

Then taking derivatives of $F$ with respect to $x$ we get

$$F'(x, s; \varepsilon) = (\ln q) q^{\varepsilon(s-1)} \left( s - 1 + \frac{q^x}{1-q} \right) \exp(\varepsilon q^{-x}[x]) - \frac{\varepsilon}{1-q} (\ln q) q^{\varepsilon(s-2)} \exp(\varepsilon q^{-x}[x]) \frac{(1-q)}{(1-q)^s}.$$  

$$F''(x, s; \varepsilon) = (\ln q)^2 q^{\varepsilon(s-1)} \left( s(1-s) - 3 \frac{q^x}{1-q} + (1-q)^2 \right) \exp(\varepsilon q^{-x}[x])$$  

$$- \frac{\varepsilon}{1-q} (\ln q)^2 q^{\varepsilon(s-2)} \frac{2s - 3 + 3q^x}{(1-q)^{s+1}} \exp(\varepsilon q^{-x}[x]) + \frac{\varepsilon}{1-q} \left[ (\ln q)^2 q^{\varepsilon(s-3)} \exp(\varepsilon q^{-x}[x]) \right] \frac{(1-q)}{(1-q)^s}.$$  

Let $t = \varepsilon(q^{-x} - 1)/(1-q)$, then $dt = -\varepsilon(\ln q)q^{-x}/(1-q) \, dx$. We get

$$\int_{-\infty}^{\infty} F(x, s; \varepsilon) \, dx = \frac{1}{\ln q} \left( \frac{\varepsilon}{1-q} \right)^{s-1} \int_{-\infty}^{\infty} e^t \frac{1}{t^s} \, dt.$$  

When $s = 0$ and $k = 1$ we have

$$\int_{-\infty}^{\infty} F(x, 0; \varepsilon) \, dx = \frac{q - 1}{\ln q} \left( \frac{-1}{\varepsilon} \right) + \sum_{l=0}^{\infty} q - 1 \cdot \frac{1}{\ln q} \cdot \frac{q^{l-1}}{-l-1} \cdot \frac{\varepsilon^l}{l!}.$$  

Simple computations yields

$$F(1, 0; \varepsilon) = q^{-1} \exp(\varepsilon/q) = \sum_{l=0}^{\infty} q^{l-1} \frac{\varepsilon^l}{l!}$$  

$$F'(1, 0; \varepsilon) = (-\ln q)q^{-1} + \sum_{l=1}^{\infty} \frac{\ln q}{1-q} q^{-1-l} (1-l+q) \frac{\varepsilon^l}{l!}$$  

$$F''(x, 0; \varepsilon) = (\ln q)^2 q^{-x} \sum_{l=2}^{\infty} \left( \left( \frac{q^x - x}{q - 1} \right)^l - \frac{3lq^{-x}}{q - 1} \left( q^{-x} [x] \right)^{l-1} + \frac{l(l-1)q^{-2x}}{(q - 1)^2} \left( q^{-x} [x] \right)^{l-2} \right) \frac{\varepsilon^l}{l!}$$  

$$+ (\ln q)^2 q^{-x} + (\ln q)^2 q^{-2x} \frac{3 + 1 - q^{-x}}{1-q} \varepsilon.$$  

Note that for all $0 < q < 1$ and $\Re(\varepsilon) < 0$, $Z_q(0; \varepsilon) = \sum_{n=1}^{\infty} F(n, 0; \varepsilon)$ converges. By Euler-Maclaurin summation formula and the analytic continuation of $\zeta_q(s)$ given by [16] (12)

$$Z_q(0; \varepsilon) = \int_{-\infty}^{\infty} F(x, 0; \varepsilon) \, dx + \frac{1}{2} F(1, 0; \varepsilon) - \frac{1}{12} F'(1, 0; \varepsilon) - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{B}_2(x) F''(x, 0; \varepsilon) \, dx$$  

$$= \frac{q - 1}{\ln q} \left( \frac{-1}{\varepsilon} \right) + \sum_{l=0}^{\infty} \zeta_q(-l) \frac{\varepsilon^l}{l!},$$  

(19)
where $\tilde{B}_k(x)$ is the periodic Bernoulli polynomial defined by (5). Further, we have (20 Ch. IX, Misc. Ex. 12) for $k \geq 2$

$$\tilde{B}_k(x) = -k! \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi inx}}{(2\pi in)^k}, \quad \tilde{B}_k^{(1)}(x) = (k+1)\tilde{B}_k(x). \quad (20)$$

To determine the regularized normalization for $Z_q(s; \varepsilon)$ for positive $s$ we begin with the case $s = 1$. First we have

$$\int_1^\infty F(x, 1; \varepsilon) \, dx = -\frac{1}{\ln q} \int_{-\frac{\varepsilon}{q}}^\infty \frac{e^{-t}}{t} \, dt.$$ 

Therefore

$$(1 - q) \int_1^\infty F(x, 1; \varepsilon) \, dx = \frac{q - 1}{\ln q} \int_{-\frac{\varepsilon}{q}}^\infty \frac{e^{-t}}{t} \, dt. \quad (21)$$

Now for any real number $a > 0$ integration by parts yields

$$\int_a^\infty \frac{e^{-t}}{t} \, dt = \left[ e^{-t} \ln t \right]_a^\infty + \int_a^\infty e^{-t} \ln t \, dt = -\gamma - \ln a - \sum_{l=1}^{\infty} \frac{(-a)^l}{l!}$$

where $\gamma \approx 0.577216$ is the Euler’s $\gamma$ constant. Here we have used the fact that (see [1] or [2])

$$\Gamma'(1) = \int_0^\infty e^{-t} \ln t \, dt = -\gamma.$$ 

Hence

$$(1 - q) \int_1^\infty F(x, 1; \varepsilon) \, dx = \frac{1 - q}{\ln q} \left[ \ln \left( \frac{-\varepsilon}{q} \right) + \gamma + \sum_{l=1}^{\infty} \frac{(-\varepsilon)^l}{l!} \right].$$

Further,

$$F'(1; \varepsilon) = (\ln q) \frac{q}{(1 - q)^2} \exp(\varepsilon/q) - \frac{\varepsilon}{1 - q} (\ln q) q^{-1} \frac{\exp(\varepsilon/q)}{1 - q},$$

$$F''(x, 1; \varepsilon) = (\ln q)^2 \frac{2 - 3(1 - q^x) + (1 - q^x)^2}{(1 - q^x)^3} \exp(\varepsilon q^{-x}[x])$$

$$- \frac{\varepsilon}{1 - q} (\ln q)^2 q^{-x} - 1 + \frac{3q^x}{(1 - q^x)^2} \exp(\varepsilon q^{-x}[x]) + \left( \frac{\varepsilon}{1 - q} \right)^2 (\ln q)^2 q^{-2x} \exp(\varepsilon q^{-x}[x]) \frac{1}{1 - q^x}.$$ 

Consequently

$$Z_q(1; \varepsilon) = (1 - q) \sum_{n=1}^{\infty} F(n, 1; \varepsilon)$$

$$= (1 - q) \left( \int_1^\infty F(x, 1; \varepsilon) \, dx + \frac{1}{2} F(1, 1; \varepsilon) - \frac{1}{12} F'(1, 1; \varepsilon) - \frac{1}{2} \int_1^\infty \tilde{B}_2(x) F''(x, 1; \varepsilon) \, dx \right)$$

$$= \frac{1 - q}{\ln q} \ln(-\varepsilon) + M(q) + \frac{1 - q}{\ln q} \gamma + O(\varepsilon), \quad (22)$$

where we have set

$$M(q) = q - \frac{1}{2} + \frac{q}{12} \frac{\ln q}{q - 1} - \frac{(1 - q)(\ln q)^2}{2} \int_1^\infty \tilde{B}_2(x) \frac{2 - 3(1 - q^x) + (1 - q^x)^2}{(1 - q^x)^3} \, dx. \quad (23)$$

As a comparison we now take a look at the behavior of $\zeta_q(s)$ near $s = 1$. It is clear that

$$F(x, s, 0) = \frac{q^x(s-1)}{(1 - q^x)^s}.$$
Thus from formula \([23 \ (12)]\) we have, near \(s = 1\)

\[
\zeta_q(s) = (1 - q)^s \left( \sum_{n=1}^{\infty} F(n, s, 0) \right)
= (1 - q)^s \left( \int_1^{\infty} F(x, s, 0) \, dx + \frac{1}{2} F(1, s, 0) - \frac{1}{2} F'(1, s, 0) + \frac{1}{2} \int_1^{\infty} \tilde{B}_2(x) F''(x, s, 0) \, dx \right)
= \frac{(q - 1)}{(s - 1) \ln q} + M(q) + \frac{1 - q}{\ln q} \gamma + O(s - 1).
\]

(24)

Taking \(q \uparrow 1\) we have near \(s = 1\)

\[
\zeta(s) = \frac{1}{s - 1} + \lim_{q \uparrow 1} M(q) - \gamma + O(s - 1).
\]

(25)

On the other hand, by applications of Euler-Maclaurin summation formula ([17 p. 531] or [20 7.21]), for all integers \(k, l \geq 1\), we get

\[
\sum_{j=1}^{k} \frac{1}{j} = \ln k + \gamma + \frac{1}{2k} - \sum_{r=1}^{l} \frac{B_{2r}}{2r} k^{-2r} + \int_{k}^{\infty} \frac{\tilde{B}_{2r+1}(x)}{x^{2r+2}} \, dx.
\]

Since \(B_2 = 1/6\), putting \(k = l = 1\), using equation \([20]\) and integration by parts once in \([28]\) we get \(\lim_{q \uparrow 1} M(q) = \gamma\) which is consistent with \([25]\).

Now we can prove the following result:

**Theorem 3.2.** The value of \(Z_q\left(\left[\begin{array}{c} s \\ r \end{array}\right]; \varepsilon\right)\) at \(s = 1 - n \ (n \in \mathbb{Z}_{>0})\) is

\[
Z_q(1 - n; r \varepsilon) = \frac{q - 1}{\ln q} \left(\frac{1}{r \varepsilon}\right)^n n! + \sum_{l=0}^{\infty} \zeta_q(1 - n - l) \frac{(r \varepsilon)^l}{l!}.
\]

(26)

The value of \(Z_q\left(\left[\begin{array}{c} s \\ r \end{array}\right]; \varepsilon\right)\) at \(s = n \in \mathbb{Z}_{>0}\) is

\[
Z_q(n; r \varepsilon) = \frac{q - 1}{\ln q} u_n(r \varepsilon) + \left( M(q) + \frac{1 - q}{\ln q} \right) (r \varepsilon)^{n-1} \frac{(n-1)!}{(n-1)!} + \sum_{l=0, l \neq n-1}^{\infty} \zeta_q(n - l) \frac{(r \varepsilon)^l}{l!}.
\]

(27)

where \(u_n+1(\varepsilon) = \varepsilon^n \sum_{m=0}^{n} (H_m - \ln(-\varepsilon))\) for \(n \geq 0\), \(H_0 = 0\) and \(H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\) for \(n \geq 1\).

**Proof.** Equation \([26]\) follows from \([19]\) by taking derivatives because \(\frac{d}{d \varepsilon} Z_q(s; \varepsilon) = Z_q(s - 1; \varepsilon)\).

When \(s = 1\) equation \([27]\) follows from \([19]\) and \([22]\) by integration and Abel’s Theorem because \(\frac{d}{d \varepsilon} Z_q(1; \varepsilon) = Z_q(0; \varepsilon)\). The rest follows immediately.

**3.2 The range of regularized MqZV**

We now turn to the general MqZVs. Although the proof of Theorem 3.3 below is similar to the proof of [12 Theorem 3.3] some new phenomena arise because of the shifting principle for MqZV.

Let

\[
\mathbb{C}\{\{\varepsilon, \varepsilon^{-1}\} \mid a_n \in \mathbb{C}, N \in \mathbb{Z}\}
\]

be the algebra of Laurent series, regarded as a subalgebra of the algebra of (the germs of) meromorphic functions in a neighborhood of \(\varepsilon = 0\) with at worst finite order poles at 0. Choose \(\ln \varepsilon\) to be analytic on \(\mathbb{C}\setminus(-\infty, 0]\). Observe that the analytic function \(\ln(-\varepsilon)\) on \(\mathbb{C}\setminus[0, \infty)\) is transcendental over \(\mathbb{C}\{\{\varepsilon, \varepsilon^{-1}\}\}\) by [12 Lemma 3.1] and hence there is a natural algebra injection (see [12 (19)])

\[
u : \mathbb{C}\{\{\varepsilon, \varepsilon^{-1}\} \mid -\ln(-\varepsilon)\} \rightarrow \mathbb{C}[T]\{\{\varepsilon, \varepsilon^{-1}\}\}
\]

(28)

sending \(-\ln(-\varepsilon)\) to \(T\). This provides an identification of \(\mathbb{C}\{\{\varepsilon, \varepsilon^{-1}\} \mid -\ln(-\varepsilon)\}\) as a subalgebra of \(\mathbb{C}[T]\{\{\varepsilon, \varepsilon^{-1}\}\}\).
Theorem 3.3. (a) For any $\mathbf{s} \in (\mathbb{Z}_{\leq 0})^d$ and $\mathbf{r} \in (\mathbb{R}_{> 0})^d$, $Z_q\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) \in \mathbb{C}\{\varepsilon, \varepsilon^{-1}\}$.

(b) For any $\mathbf{s} \in \mathbb{Z}^d$ and $\mathbf{r} \in (\mathbb{R}_{> 0})^d$, $Z_q\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) \in \mathbb{C}\{\varepsilon, \varepsilon^{-1}\}[-\ln(-\varepsilon)]$.

Proof. (a) The key is the following computation of the tail of the $Z_q(s; \varepsilon)$ for $s \in \mathbb{Z}_{\leq 0}$. First define

$$\xi_i(\varepsilon) := \sum_{k > i} \varepsilon^k \sum_{k > 0} \frac{\exp(\varepsilon[k]/q^k)}{q^k} = \sum_{k > 0} \frac{\exp(\varepsilon[k+i]/q^{k+i})}{q^{k+i}}.$$ Since $[k+i] = [k] + q^k[i]$ we get

$$\xi_i(\varepsilon) = \frac{\exp(\varepsilon[i]/q^i)}{q^i} \sum_{k > 0} \frac{\exp(\varepsilon[k]/q^{k+i})}{q^k} = \frac{\exp(\varepsilon[i]/q^i)}{q^i} Z_q(0; \varepsilon/q^i).$$

Therefore for $s \in \mathbb{Z}_{\leq 0}$ and $t = -s$,

$$\xi_i(t; \varepsilon) := \sum_{k > i} q^{k(t-i)} \varepsilon^k \sum_{k > 0} \frac{\exp(\varepsilon[k]/q^k)}{q^k} = \left(\frac{d}{d\varepsilon}\right)^t \xi_i(\varepsilon) = \sum_{j=0}^t \left(\frac{d}{d\varepsilon}\right)^j \xi_i(\varepsilon) \sum_{k > i} q^{k(t-i)} \frac{\exp(\varepsilon[i]/q^{k+i})}{q^k} \left(\frac{d}{d\varepsilon}\right)^{t-j} Z_q(0; \varepsilon/q^i).$$

$$= \frac{q-1}{\ln q} \sum_{j=0}^t \frac{t!}{j!} \left(\frac{1}{\varepsilon}\right)^{1+t-j} \frac{\varepsilon^i}{q^j} \sum_{k > i} \frac{\exp(\varepsilon[i]/q^{k+i})}{q^k} \left(\frac{d}{d\varepsilon}\right)^{t-j} Z_q(0; \varepsilon/q^i).$$

Now we first prove (a) by induction on the length $d$. The case $d = 1$ corresponds to the Riemann $q$-zeta function which has been dealt with in the last section. Suppose (a) is true for length $d-1 (d > 1)$ and let $\mathbf{s} = (s_1, \ldots, s_d)$. Let $\mathbf{s}_{i-1}$ denote $\mathbf{s}$ with its $i$-th component removed. Let $\mathbf{e}_i$ denote the $i$-th unit vector of length $d-1$ with 1 at the $i$-th component. By definition (10) and the shifting operator (12) for every $i$ such that $1 \leq i \leq d$,

$$Z_q\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) = \sum_{k_1 > \cdots > k_d > 0} \prod_{m=1}^d q^{k_m(s_m-1)/k_m} \exp(\varepsilon r_m[k_m]/q^{k_m})$$

$$= \sum_{k_1 > \cdots > k_{i-1} > k_{i+1} > \cdots > k_d > 0} \prod_{m=1}^d q^{k_m(s_m-1)/k_m} \exp(\varepsilon r_m[k_m]/q^{k_m})$$

$$\cdot \left[\xi_{k_{i+1}}(-s_i; r_i \varepsilon) - \xi_{k_{i-1}}(-s_i; r_i \varepsilon) - q^{k_{i-1}(s_i-1)/k_{i-1}} \exp(\varepsilon r_i[k_{i-1}]/q^{k_{i-1}})\right]$$

$$= -s_{i-1} Z_q\left(\left[\mathbf{s}_{i-1}/\mathbf{r}_{i-1}\right]; \varepsilon\right) + \frac{q-1}{\ln q} \sum_{j=0}^{s_i} \left(-s_i/j\right) \left(-\frac{1}{r_j \varepsilon}\right)^{1-s_i-j} \left[Z_q\left(\left[\mathbf{s}_{i-1}/\mathbf{r}_{i-1} + r_i \mathbf{e}_i\right]; \varepsilon\right) - Z_q\left(\left[\mathbf{s}_{i-1}/\mathbf{r}_{i-1} + r_i \mathbf{e}_i\right]; \varepsilon\right)\right].$$

Here if $i = 1$ then the terms with $\mathbf{e}_{i-1}$ are 0. If $i = d$ then the terms with $\mathbf{e}_i$ are 0 and the first term becomes $Z_q\left(\left[\mathbf{s}_d/\mathbf{r}_d\right]; \varepsilon\right) Z_q\left(\left[\mathbf{s}_d/\mathbf{r}_d\right]; \varepsilon\right)$. By induction assumption we see that $Z_q\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) \in \mathbb{C}\{\varepsilon, \varepsilon^{-1}\}$.

Now we prove (b) by induction on length $d$ again.

**Case 1.** $s_i \leq 0$ for some $1 \leq i \leq d$. Then the proof of (a) above carries over word for word here.

**Case 2.** Suppose $s_i > 0$ for all $1 \leq i \leq d$. We use induction on the sum $s := \sum_{i=1}^d s_i$. Clearly $s \geq d$. If $s = d$, then $s_i = 1$ for $1 \leq i \leq d$. Thus

$$Z_q'\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) = \sum_{i=1}^d r_i Z_q\left(\left[\mathbf{s}_{i-1} - \mathbf{e}_i\right]/\mathbf{r}_{i-1}; \varepsilon\right) \in \mathbb{C}\{\varepsilon, \varepsilon^{-1}\}$$ by (a). Integrating we get $Z_q\left(\left[\mathbf{s}/\mathbf{r}\right]; \varepsilon\right) \in \mathbb{C}\{\varepsilon, \varepsilon^{-1}\}[-\ln(-\varepsilon)]$. The general case follows from the fact that $\mathbb{C}\{\varepsilon, \varepsilon^{-1}\}[-\ln(-\varepsilon)]$ is closed under integration. □
Later on, for several times we are going to need the special case of (30) when \( i = 1 \) and \( i = d \). For convenience we list them as

**Corollary 3.4.** Let \( d \geq 2 \) be a positive integer. For any \( \vec{s} \in (\mathbb{Z}_{\leq 0})^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \),

\[
Z_q\left( \left[ \frac{\vec{s}}{\vec{r}} \right]; \varepsilon \right) = \frac{q-1}{\ln q} \sum_{j=0}^{\vec{s}_1} \frac{(-1)^{\vec{s}_1-j}}{j!} \left( \frac{-1}{r_1} \right)^{1-\vec{s}_1-j} Z_q\left( \left[ \frac{s_2-j, s_3, \ldots, s_d}{r_1 + r_2, r_3, \ldots, r_d} \right]; \varepsilon \right) + \sum_{j=0}^{\vec{s}_1} \frac{(-1)^{\vec{s}_1-j}}{j!} \sum_{l=-\vec{s}_1-j}^{\infty} S_{l+1} Z_q\left( \left[ \frac{s_2-j, s_3, \ldots, s_d}{r_1 + r_2, r_3, \ldots, r_d} \right]; \varepsilon \right) \frac{\zeta_q(-l)(r_1 \varepsilon)^{l+\vec{s}_1+j}}{(l+\vec{s}_1+j)!}.
\]

and

\[
Z_q\left( \left[ \frac{\vec{s}}{\vec{r}} \right]; \varepsilon \right) = Z_q\left( \left[ \frac{s_d}{r_d} \right]; \varepsilon \right) Z_q\left( \left[ \frac{s_1, \ldots, s_{d-1}}{r_1, \ldots, r_{d-1}} \right]; \varepsilon \right) - \frac{q-1}{\ln q} \sum_{j=0}^{\vec{s}_d} \frac{(-1)^{\vec{s}_d-j}}{j!} \left( \frac{-1}{r_d} \right)^{1-\vec{s}_d-j} Z_q\left( \left[ \frac{s_1, \ldots, s_{d-2}, s_{d-1}-j}{r_1, \ldots, r_{d-2}, r_{d-1} + r_d} \right]; \varepsilon \right) - \sum_{j=0}^{\vec{s}_d} \frac{(-1)^{\vec{s}_d-j}}{j!} \sum_{l=-\vec{s}_d-j}^{\infty} S_{l+1} Z_q\left( \left[ \frac{s_1, \ldots, s_{d-2}, s_{d-1}-j}{r_1, \ldots, r_{d-2}, r_{d-1} + r_d} \right]; \varepsilon \right) \frac{\zeta_q(-l)(r_d \varepsilon)^{l+s_d+j}}{(l+s_d+j)!}.
\]

The next result tells us some very useful information about the general shape of the coefficients of the Laurent series \( Z_q\left( \left[ \frac{\vec{s}}{\vec{r}} \right]; \varepsilon \right) \). This will be used crucially in the proof of the existence of the renormalizations of MqZVs at non-positive integers.

**Corollary 3.5.** Let \( \vec{s} \in (\mathbb{Z}_{\leq 0})^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \). Then the coefficient \( c_i \) of the Laurent series of \( Z_q\left( \left[ \frac{\vec{s}}{\vec{r}} \right]; \varepsilon \right) \) is an \( \mathbb{R} \)-linear combination of rational function of the form \( P(\vec{r})/Q(\vec{r}) \) in \( r_1, \ldots, r_d \), where \( P, Q \) have no common factors. Moreover, both \( P \) and \( Q \) are homogeneous polynomials in \( r_1, \ldots, r_d \). \( Q \) is the product of linear factor \( r_1 + \cdots + r_d \) (and \( r_1 \) if \( d = 2 \)) with repetition allowed, and \( \text{deg}(P/Q) := \text{deg}(P) - \text{deg}(Q) = i \).

**Proof.** When \( d = 1 \) this follow from Theorem 3.24. Suppose the claim in the corollary is true when the length of the vector \( \vec{s} \) is \( < d \) for some \( d \geq 2 \). It follows from (31) and induction assumption that the only possible factors in the denominator of \( P/Q \) are of the form \( (r_1 + \cdots + r_d)^j \) and \( r_k^l \) for some \( j, k \in \mathbb{Z}_{\geq 0} \). By (32) we see that the only possibilities are \( (r_1 + \cdots + r_d)^j \), \( (r_1 + \cdots + r_{d-1})^k \), and \( r_d^l \). Hence if \( d > 2 \) then none of \( r_1, r_1 + \cdots + r_{d-1} \) or \( r_d \) can appear as a factor in the denominator of \( c_i \). But if \( d = 2 \) then both \( r_1 + r_2 \) and \( r_1 \) could appear (and indeed they do by the following formula (33).)

The statements about the degrees are clear from (31) by induction assumption. \( \square \)

4 Renormalization of MqZV

Theorem 3.28 together with the map \( u \) of (25) shows that there is an algebra homomorphism

\[
\tilde{Z}_q : \mathcal{H}_Z \to \mathbb{C}[T][\{\varepsilon, \varepsilon^{-1}\}]
\]

\[
\left[ \frac{\vec{s}}{\vec{r}} \right] \mapsto u\left( Z_q\left( \left[ \frac{\vec{s}}{\vec{r}} \right]; \varepsilon \right) \right)
\]

which restricts to an algebra homomorphism

\[
\tilde{Z}_q : \mathcal{H}_{\mathbb{R}_{\leq 0}} \to \mathbb{C}[\{\varepsilon, \varepsilon^{-1}\}].
\]

Note that only when there is a positive \( s_i \) can \( \tilde{Z}_q \) really differ from \( Z_q \). It is well-known that the Birkhoff decomposition (see [19, Theorem II.5.1]) yields two maps \( \tilde{Z}_q^- \) and \( \tilde{Z}_q^+ \) such that \( \tilde{Z}_q = \tilde{Z}_q^- \ast \tilde{Z}_q^+ \). The properties of this decomposition implies the following result immediately.
Proposition 4.1. The map \( \tilde{\mathcal{Z}}_{q^+} : \mathcal{K}_\mathbb{Z} \to \mathbb{C}[T][\varepsilon] \) is an algebra homomorphism which restricts to an algebra homomorphism \( \tilde{\mathcal{Z}}_{q^+} : \mathcal{K}_{\mathbb{Z}_\geq 0} \to \mathbb{C}[\varepsilon] \).

To write down \( \tilde{\mathcal{Z}}_{q^+} \) explicitly we need the following definition which is slightly different from [12 Definition 3.7].

Definition 4.2. Let \( \Pi_d \) be the set of increasing sequences \( i_0 = 0 < i_1 < \cdots < i_p = d \). For \( 1 \leq j \leq p \), define the partition vectors of \( \vec{s} \in \mathbb{C}^d \) from the sequence \( (i_1, \cdots, i_p) \) to be the vectors \( \vec{s}^{(j)} := (s_{i_j - 1}, \cdots, s_{i_j}), \ 1 \leq j \leq p \).

The following explicit formula for the renormalization of regularized \( M_q \text{ZV} \) is the \( q \)-analog of [12 Theorem 3.8] which follows from [19 Theorem II.5.1], which, in turn, is built upon the idea of [7 Theorem 4].

Proposition 4.3. Let \( P \) be the operator sending a Laurent series to its pole part: \( P(\sum_{n=0}^\infty a_n \varepsilon^n) = \sum_{n=-\infty}^{-N} a_n \varepsilon^n \) and let \( \tilde{P} = -P \). Then for any \( \vec{s} \in \mathbb{Z}^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \)

\[
\tilde{\mathcal{Z}}_{q^+} \left( \left[ \vec{s} \right]_{\vec{r}} \right) = \sum_{(i_1, \cdots, i_p) \in \Pi_d} \tilde{P} \left( \tilde{\mathcal{Z}}_q \left( \left[ \vec{s}^{(p)} \right]_{\vec{r}(p)} \right) \right) \cdots \tilde{P} \left( \tilde{\mathcal{Z}}_q \left( \left[ \vec{s}^{(2)} \right]_{\vec{r}(2)} \right) \right) \tilde{P} \left( \tilde{\mathcal{Z}}_q \left( \left[ \vec{s}^{(1)} \right]_{\vec{r}(1)} \right) \right). \quad (33)
\]

We are going to use the map \( \tilde{\mathcal{Z}}_{q^+} \) to define the renormalization of \( M_q \text{ZVs} \). But before doing so we recall that at the beginning of section 3 we mention that the multiple variable cases are different from the single variable case, just like the situation where a function of two variables can have all the directional derivatives at some point but yet is not differentiable there. Such phenomenon won’t happen to functions of one variable. The renormalization process is essentially a limit process just like taking the derivatives. The behavior of multiple zeta functions at poles are not so bad in that if we take appropriate “paths” to renormalize then we can produce values compatible with both the stuffle relations and the original values if they are defined originally. All the above remarks are still valid for the \( q \)-analogues. The appropriate “paths” in our case is given by Definition 4.7 and Definition 4.8 later in this section. We first define the directional version of these as follows:

Definition 4.4. For \( \vec{s} \in \mathbb{Z}^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \), the renormalized directional \( M_q \text{ZV} \) is defined by

\[
\zeta_q \left( \left[ \vec{s} \right]_{\vec{r}} \right) = \lim_{\varepsilon \to 0} \tilde{\mathcal{Z}}_{q^+} \left( \left[ \vec{s} \right]_{\vec{r}} \right),
\]

and \( \vec{r} \) is called the directional vector.

Corollary 4.5. The renormalized directional \( M_q \text{ZVs} \) satisfy the \( q \)-stuffle relations

\[
\zeta_q \left( \left[ \vec{s} \right]_{\vec{r}} \right) \zeta_q \left( \left[ \vec{s}' \right]_{\vec{r}'} \right) = \zeta_q \left( \left[ \vec{s} \right]_{\vec{r}} * q \left[ \vec{s}' \right]_{\vec{r}'} \right) := \sum_{n \geq 0} (1 - q)^n \sum_{m \geq 0} \zeta_q \left( \left[ \vec{s}_{m,n} \right]_{\vec{r}_{m,n}} \right) \quad (34)
\]

where \( \left[ \vec{s} \right]_{\vec{r}} * q \left[ \vec{s}' \right]_{\vec{r}'} = \sum_{n \geq 0, m \geq 0} (1 - q)^n \left[ \vec{s}_{m,n} \right]_{\vec{r}_{m,n}} \).

Proof. It follows directly from Definition 4.4 by Proposition 4.1. \qed

The following proposition is the \( q \)-analogue of [12 Corollary 4.13]. This and the length three case provide us a hint at the general shapes of the renormalization of \( M_q \text{ZVs} \) at non-positive integers which will be given in Proposition 5.4.

Proposition 4.6. For \( s_1, s_2 \leq 0 \), \( r_1, r_2 > 0 \), set \( t = 1 - s_1 - s_2 \). Then

\[
\zeta_q \left( \left[ \frac{s_1, s_2}{r_1, r_2} \right] \right) = \frac{q - 1}{\ln q} \left[ \zeta_q(s_1 + s_2 - 1)_{s_1 - 1} + \sum_{j=0}^{s_1} \left( -s_1 \right)_j \sum_{l=0}^{\infty} \frac{(l + 1) (1 - q)^{l-t}}{l (l+1) \cdot l + s_1 + j} \zeta_q(-l) \right]
\]

\[
+ \sum_{j=0}^{s_1} \left( -s_1 \right)_j \sum_{i=0}^{1-s_1} \left( 1 - j - s_1 \right)_i (1 - q)^i \zeta_q(s_2 - j - i) \zeta_q(j + s_1 - 1).
\]
Proof. From equation (31) in Corollary 3.4 we have

\[
\tilde{Z}_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]\right) = \frac{q - 1}{\ln q} \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\left(\frac{-1}{r_1^i}\right)^{1-s_1-j} Z_q\left(\left[\frac{s_2 - j}{r_1 + r_2}\right]; \varepsilon\right) \right.
\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \sum_{k=0}^{\infty} \zeta_q(s_2 - j - k) \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
\left. + \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\right)
\]
By Theorem 3.2 we get

\[
\tilde{Z}_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]\right) = \left(\frac{q - 1}{\ln q} \right)^2 \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\left(\frac{-1}{r_1^i}\right)^{1-s_1-j} \left(\frac{-1}{r_1 + r_2^i}\right)^{j-s_2+1} (j - s_2)!\right)
\]
\[
+ \frac{q - 1}{\ln q} \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\right)^{1-s_1-j} \sum_{k=0}^{\infty} \zeta_q(s_2 - j - k) \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
+ \frac{q - 1}{\ln q} \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\right)^{1-s_1-j} \sum_{k=0}^{\infty} \zeta_q(s_2 - j - k) \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\right)^{1-s_1-j} \sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
\left. + \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\right)^{1-s_1-j} \sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
By Definition 4.3 and Proposition 4.3 we have

\[
\zeta_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]\right) = \lim_{\varepsilon \to 0} \left[\tilde{P}\left(\tilde{Z}_q\left(\left[\frac{s_2}{r_2}\right]; \varepsilon\right)\right) \tilde{P}\left(\tilde{Z}_q\left(\left[\frac{s_1}{r_1}\right]; \varepsilon\right)\right)\right] + \tilde{P}\left(\tilde{Z}_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]; \varepsilon\right)\right).
\]

It follows from Theorem 3.2 and the above expression for \(Z_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]; \varepsilon\right)\) that

\[
\zeta_q\left(\left[\frac{s_1, s_2}{r_1, r_2}\right]\right) = \frac{q - 1}{\ln q} \left(-\frac{r_2}{r_1}\right)^{1-s_1} \zeta_q\left(s_1 + s_2 - 1\right) + \frac{-s_1}{j!} \left(-\frac{r_1 + r_2}{r_1}\right)^{1-s_1-j} \zeta_q\left(s_1 + s_2 - 1\right)
\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
\[
+ \sum_{j=0}^{s_1} \left(-\frac{s_1}{j!}\sum_{i=0}^{\infty} \frac{l + 1}{i!} (1 - q)^i \frac{(r_1 + r_2)\varepsilon^k}{(k)!}\right)\]
A simple combinatorial formula quickly reduces this to the formula in the proposition. □

We are now ready to define the renormalization of \(M_q\)ZVs.

**Definition 4.7.** For \(\vec{s} \in (\mathbb{Z}_{\geq 0})^d \cup (\mathbb{Z}_{\leq 0})^d\) define

\[
\tilde{\zeta}_q(\vec{s}) = \lim_{\delta \to 0^+} \zeta_q\left(\left[\frac{\vec{s}}{\left|\vec{s}\right| + \delta}\right]\right),
\]
where, for \(\vec{s} = (s_1, \cdots, s_d)\) and \(\delta \in \mathbb{R}_{>0}\), we write \(\vec{s} = (|s_1|, \cdots, |s_d|)\) and \(|\vec{s}| + \delta = (|s_1| + \delta, \cdots, |s_d| + \delta)\). These values \(\tilde{\zeta}_q(\vec{s})\) are called the renormalized \(M_q\)ZV of the \(d\)-tuple \(q\)-zeta function \(\zeta_q(u_1, \cdots, u_d)\) at \(\vec{s}\).

In order to deal with the \(q\)-stuffle relations we need the shifted version of the about definition.

**Definition 4.8.** Let \(\vec{f} = (f_1, \ldots, f_l)\) be a binary vector with entries equal to either 0 or 1. Let \(\vec{s} \in (\mathbb{Z}_{\leq 0})^l\) or \(\vec{s} \in (\mathbb{Z}_{\geq 0})^l\) such that \(|\vec{s}| - \vec{f} \in \mathbb{Z}_{\geq 0}^l\) (i.e., if \(s_i = 0\) then \(f_i = 0\)). We call such a binary vector \(\vec{f}\) a shifting vector of \(\vec{s}\). We define the shifted renormalization of \(\zeta_q(\vec{s})\) by \(\vec{f}\) as the limit

\[
\tilde{\zeta}_q^\vec{f}(\vec{s}) := \lim_{\delta \to 0^+} \zeta_q\left(\left[\delta + |\vec{s}| - (\vec{s} + (\delta - 1)\vec{f})\right]\right).
\]
For example, $\zeta_q^{(1,0,0)}(-1,-2,0) = \lim_{l \to 0^+} \zeta_q\left(\left[\begin{array}{c} -1, -2, 0 \\ \delta, 2 + \delta, \delta \end{array}\right]\right)$, where $2\delta$ appears because of the stuffings at the first position which is hinted by the shifting vector $(1,0,0)$.

We will compute the limit (39) in Theorem 4.12 and show that the limits (37) exist for $\vec{s} \in (\mathbb{Z}_{>0})^d$ in Theorem 4.13 for $\vec{s} \in (\mathbb{Z}_{\leq 0})^d$ in Theorem 5.2. First we consider the case of Riemann $q$-zeta function.

**Theorem 4.9.** Let $\gamma$ be Euler’s gamma constant and define $M(q)$ by (28). Then

$$\tilde{\zeta}_q(1) = \frac{q-1}{\ln q} T + M(q) + \frac{1-q}{\ln q} \gamma$$

and, for integers $s > 1$, $\tilde{\zeta}_q(s)$ is the usual Riemann $q$-zeta value $\zeta_q(s)$ defined by the series (3). If $s = -l$ is a non-positive integer then

$$\tilde{\zeta}_q(-l) = \zeta_q(-l) = (1-q)^{-l} \left\{ \sum_{r=0}^{l} (-1)^r {l \choose r} \frac{1}{q^{s+r} - 1} + \frac{(-1)^{l+1}}{(l+1) \ln q} \right\}. \quad (38)$$

**Proof.** The expression of $\tilde{\zeta}_q(-l)$ is given by (16) (60)]. The rest follows from Theorem 4.22.

Our primary goals are to show that our definition of renormalizations of $\text{MqZVs}$ are well-defined, these values agree with the usual $\text{MqZVs}$ whenever the usual values are defined, they satisfy the $q$-stuff relations, and they become renormalizations of $\text{MZVs}$ when $q \uparrow 1$.

**Theorem 4.10.** Let $\vec{s} = (s_1, \cdots, s_d) \in \mathbb{Z}_{>0}^d$ such that $s_1 > 1$. Let $\vec{r} \in (\mathbb{Z}_{>0})^d$ be an arbitrary vector. Then $\tilde{\zeta}_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} \end{array}\right]\right) = \zeta_q(\vec{s})$ which is independent of the choice of $\vec{r}$. Here $\zeta_q(\vec{s})$ denotes the usual definition of multiple $q$-zeta values. In particular, $\tilde{\zeta}_q(\vec{s}) = \zeta_q(\vec{s})$ satisfy the $q$-stuff relation.

**Proof.** By definition $Z_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} \end{array}\right]; \varepsilon\right)$ converges uniformly for $\varepsilon \in (-\infty, 0]$ and therefore continuous as a function of $\varepsilon$. In particular it is a power series and $\lim_{\varepsilon \to 0} Z_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} \end{array}\right]; \varepsilon\right) = Z_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} \end{array}\right]; 0\right) = \zeta_q(\vec{s})$.

Now we consider the divergent case $s_1 = 1$. For $f(\varepsilon), g(\varepsilon) \in \mathbb{C}[T][\varepsilon]$, denote $f(\varepsilon) = g(\varepsilon) + O(\varepsilon)$ if $g(\varepsilon) - f(\varepsilon) \in \varepsilon \mathbb{C}[T][\varepsilon]$.

**Lemma 4.11.** For $c > 0$ set $X = \frac{q-1}{\ln q} (\ln c + T)$. Let $\vec{s} \in (\mathbb{Z}_{>0})^d$ where either $l = 0$ or $s_1 > 1$. Let $\vec{r} \in (\mathbb{R}_{>0})^l$ and $\vec{1}_d = (1,1,\cdots,1) \in \mathbb{Z}^d$. Then

$$\tilde{Z}_q\left(\left[\begin{array}{c} \vec{1}_d, \vec{s} \\ \vec{r} \end{array}\right]\right) = P_{d,\vec{r}}(X) + O(\varepsilon), \quad (39)$$

where $P_{d,\vec{r}}(X)$ is a degree $d$ polynomial in $X$ with leading coefficient $\zeta_q(\vec{s})/d!$ (by convention, if $l = 0$ we set $\zeta_q(\vec{s}) = 1$). Moreover, $P_{d,\vec{r}}(X)$ is independent of $\vec{r}$.

**Proof.** We prove the lemma by induction similar to that of [12] Lemma 4.4]. Notice that the terms produced by shifting will produce polynomials of degrees less than that of the leading term.

When $d = 0$ the lemma follows from the proof of Theorem 1.10 as $\varepsilon \mathbb{C}[\varepsilon] \subset \varepsilon \mathbb{C}[T][\varepsilon]$. We now fix $d = 1$ and prove the lemma by induction on $l$. When $l = 0$ the lemma readily follows from (27). Assume equation (39) is true when the length of the vector $\vec{s}$ is $\leq l$ for $l \geq 0$. Then by (27) and Theorem 4.10 we have

$$(X + O(\varepsilon))\tilde{Z}_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} \end{array}\right]\right) - \tilde{Z}_q\left(\left[\begin{array}{c} \vec{1}, \vec{s} \\ \vec{r} \end{array}\right]\right) = u\left(Z_q\left(\left[\begin{array}{c} \vec{1} \\ \vec{c}, \vec{r} \end{array}\right]; \varepsilon\right) \cdot Z_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{c}, \vec{r} \end{array}\right]; \varepsilon\right) - Z_q\left(\left[\begin{array}{c} \vec{1}, \vec{s} \\ \vec{c}, \vec{r} \end{array}\right]; \varepsilon\right)\right)$$

$$= \sum_{j=1}^{l-1} \tilde{Z}_q\left(\left[\begin{array}{c} s_1, \cdots, s_j, 1, s_{j+1}, \cdots, s_l \\ r_1, \cdots, r_j, c, r_{j+1}, \cdots, r_l \end{array}\right]\right) + \sum_{j=1}^{l} \tilde{Z}_q\left(\left[\begin{array}{c} \vec{s} + e_j \\ \vec{r} + ce_j \end{array}\right]\right) (1-q) \sum_{j=1}^{l} \tilde{Z}_q\left(\left[\begin{array}{c} \vec{s} \\ \vec{r} + ce_j \end{array}\right]\right) = O(\varepsilon)$$
where $e_j$ is the $j$-th unit vector of length $l$. Hence by induction assumption,

$$\tilde{Z}_q\left(\left[\frac{1, s}{c, \vec{r}}\right]\right) = \zeta_q(\vec{s})X + O(\varepsilon).$$

Assume equation (39) is true for every $l > 0$ when the length of the vector $(1, \ldots, 1)$ is $\leq d$ for $d \geq 1$. Then by (27)

$$\tilde{Z}_q\left(\left[\frac{1}{c \Gamma_d, \vec{r}}\right]\right) = (P_d(X) + O(\varepsilon))(X + O(\varepsilon)) = f_{d+1}(X) + O(\varepsilon)$$

for some polynomial $f_{d+1}(X)$ of degree $d+1$ with leading coefficient $\zeta_q(\vec{s})/d!$, independent of $\vec{r}$. On the other hand, by the $q$-stuffle relation

$$\tilde{Z}_q\left(\left[\frac{1}{c \Gamma_d, \vec{r}}\right]\right) = (d + 1)\tilde{Z}_q\left(\left[\frac{1}{c \Gamma_{d+1}, \vec{r}}\right]\right) + \sum_{j=1}^{l-1} Z_q\left(\left[\frac{1}{c \Gamma_{d+1}, \vec{r}}\right]\right) + \sum_{j=1}^l Z_q\left(\left[\frac{1}{c \Gamma_{d+1}, \vec{r} + e_j}\right]\right) + (1-q) \sum_{j=1}^l Z_q\left(\left[\frac{1}{c \Gamma_{d+1}, \vec{r} + c e_j}\right]\right).$$

By induction assumption

$$\tilde{Z}_q\left(\left[\frac{1}{c \Gamma_{d+1}, \vec{r}}\right]\right) = P_{d+1, \vec{s}}(X) + O(\varepsilon)$$

where $P_{d+1, \vec{s}}(X)$ is some polynomial of degree $d+1$ with leading coefficient $\zeta_q(\vec{s})/(d+1)!$, independent of $\vec{r}$. This completes the proof of the lemma.

Now we can show

**Theorem 4.12.** For $\vec{s} \in (\mathbb{Z}_{>0})^m$, $\vec{\zeta}_q(\vec{s}) = \zeta_q\left(\left[\frac{\vec{s}}{\vec{r}}\right]\right)$.

**Proof.** In view of Theorem 4.10 we only need to consider the case $s_1 = 1$. Assume then $\vec{s} = (\vec{1}_d, \vec{s}')$ where $\vec{s}' = (s_1, \ldots, s_l)$ and $s_1 > 1$. Observe that for any substring $\vec{s}''$ of $\vec{s}$ and substring $\vec{r}''$ of $\vec{r}$ by the above lemma $Z_q\left(\left[\frac{s''}{r''}\right]\right)$ has no pole part. This means that in equation (33) of Proposition 4.3 there is only one non-trivial term which gives the finite part

$$Z_{q_+}\left(\left[\frac{s}{r}\right]\right) = (Id - P)\left(Z_q\left(\left[\frac{s}{r}\right]\right), \varepsilon\right) = P_{d, \vec{s}}(X) + O(\varepsilon).$$

Hence

$$\zeta_q\left(\left[\frac{s}{r}\right]\right) = P_{d, \vec{s}}(X).$$

By Definition 4.4

$$\vec{\zeta}_q(\vec{s}) = \lim_{c \to 1, \vec{r} \to \vec{s}} \zeta_q\left(\left[\frac{\vec{1}_d, \vec{s}'}{c \Gamma_{d+1}, \vec{r}}\right]\right) = P_{d, \vec{s}}\left(\frac{(q-1)T}{\ln q}\right) = \zeta_q\left(\left[\frac{\vec{s}}{\vec{r}}\right]\right).$$
by Theorem 3.2. This implies $\tilde{\zeta}_q^{(1)}(1) = \tilde{\zeta}_q(1) + (1 - q) \ln 2 / \ln q$ and

$$
\tilde{\zeta}_q(1^* q) = 2\tilde{\zeta}_q(1, 1) + \tilde{\zeta}_q(2) + (1 - q)\tilde{\zeta}_q(1)
$$

$$
\tilde{\zeta}_q(1^2) = 2\tilde{\zeta}_q(1, 1) + \tilde{\zeta}_q(2) + (1 - q)\tilde{\zeta}_q^{(1)}(1).
$$

Of course, when $q \uparrow 1$ these two values are both equal to $\tilde{\zeta}(1)^2 = T^2$.

**Theorem 4.13.** Let $\tilde{s} \in (Z_{>0})^l$ with $s_1 > 1$, then for any positive integer $d$ the limit of $\tilde{\zeta}_q^{T_d}(\tilde{I}_d, \tilde{s})$ in (37) exists. Moreover all the shifted renormalization of $MqZV$s at all positive arguments satisfy the $q$-stuffle relations in the following sense. Let $\tilde{s}_1 \in (Z_{>0})^k, \tilde{s}_2 \in (Z_{>0})^l$ and assume there are $W_i$ vectors of length $k + \ell - t$ produced by the $q$-stuffle with $t$ stuffings and let $f_j$ be the binary vector representing the stuffing positions of the $j$-th such vector:

$$
\tilde{s}_1 \ast q \tilde{s}_2 = \sum_{t=0}^{\min(k, \ell)} (1 - q)^t \sum_{j=1}^{W_t} (\bar{v}_j - \bar{f}_j).
$$

Then

$$
\tilde{\zeta}_q(\tilde{s}_1)\tilde{\zeta}_q(\tilde{s}_2) = \sum_{t=0}^{\min(k, \ell)} (1 - q)^t \sum_{j=1}^{W_t} \tilde{c}_q(f_j)(\bar{v}_j - \bar{f}_j).
$$

**Proof.** By the very definition of the shifted renormalization and the $q$-stuffle relation it suffices to show the first part of the theorem, namely the existence of the limit in (37). Indeed, if the leading component in $\tilde{s}$ is greater than 1 then $\tilde{\zeta}$ is independent of the directional vector by Theorem 4.10.

In the following we consider the shifted renormalization $\tilde{\zeta}_q^{T_d}(\tilde{I}_d, \tilde{s})$ only. Let the vector $\tilde{r} = (r_1, \ldots, r_{k+i}) = (\tilde{I}_d, \tilde{s}) + \tilde{f}$ be the shifted directional vector of length $l + d$. Then

$$
Z_q\left(\left[\tilde{I}_d, \tilde{s}, \tilde{r}\right]; \varepsilon\right) = \sum_{k_1, \ldots, k_d, j=1}^{d} \prod_{k_1, \ldots, k_d, j=1}^{d} \frac{\exp(r_j \varepsilon[k_j]/q^{k_j})}{[k_j]} \prod_{k_1, \ldots, k_d, j=1}^{d} \frac{q^{n_j} \varepsilon^{(s_j - 1)} \exp(r_j \varepsilon[n_j]/q^{n_j})}{[n_j]}.
$$

We want to show this series is good (used just in this proof) in the sense that it is in $C[\varepsilon][\ln(-\varepsilon)]$ and is finite when $r_1, \ldots, r_d$ are equal to either 1 or 2. We prove this by induction on $d$. If $d = 1$ this follows from (39) in Lemma 4.11. Assume the series in (40) is good when the length of $(1, \ldots, 1)$ in front of $\tilde{s}$ is less than $d$ for some $d \geq 2$. Set as before

$$
F(x, s; \varepsilon) = \frac{q^{x(s-1)} \exp(-q^x)}{(1 - q)^x} = \frac{q^{x(s-1)} \exp(\varepsilon (q^{-x} - 1) / (1 - q))}{(1 - q)^x}.
$$

As in the proof of Theorem 3.2 we can use Euler-Maclaurin summation formula to find a closed expression for

$$
\sum_{k>i} F(k, 1; \varepsilon) = \sum_{k>i} \frac{\exp(\varepsilon[k]/q^k)}{(1 - q)[k]} = \int_1^\infty F(x + i, 1; \varepsilon) dx + \frac{1}{2} F(1 + 1, 1; \varepsilon) - \frac{1}{2} \int_1^\infty \tilde{B}_2(x) F' (x + 1, 1; \varepsilon) dx.
$$

The following integral

$$
\int_1^\infty F(x + i, 1; \varepsilon) dx = \frac{-1}{\ln q} \int_{\varepsilon^{|i|+1} / q^{i+1}}^{\infty} e^{-t} dt.
$$

can be evaluated using (21) by the substitution $\varepsilon \to \varepsilon^{|i|+1} / q^i$. Therefore we get

$$
(1 - q) \int_1^\infty F(x + i, 1; \varepsilon) dx = \frac{1 - q}{\ln q} \left[ \ln \left(\frac{-\varepsilon^{|i|+1}}{q^{i+1}}\right) + \gamma + \sum_{l=1}^{\infty} \frac{\varepsilon^{|i|+1} l!}{l! q^{i(l+1)}} \right]
$$

$$
= \frac{1 - q}{\ln q} \left[ \ln(-\varepsilon) - \ln q + \ln \left(\frac{|i|}{q^{i}}\right) + \ln \left(1 + \frac{q^i}{|i|}\right) + \gamma + \sum_{l=1}^{\infty} \frac{\varepsilon^{|i|} l!}{l! q^{i}} \left(\frac{|i|}{q^i} + [1]\right) \right].
$$

(42)
Let \( i = k_2 \) in the above two formulas (11) and (12). It is straightforward to see that the middle two terms of (11) both contribute to good series by Theorem 4.10 and the induction assumption. The last term of (11) can be handled similarly using (10) after we notice that \( B_2(x) \) is bounded by 1/6 from an easy computation from the series expansion (20) and the fact that \( \zeta(2) = \pi^2/6 \). So the main divergence term of (10) when \( \varepsilon \) is near zero comes from the first term of (11). Thus by (12) it is enough to show that all the sums below are good:

\[
\sum_{k_2 > \cdots > k_d} \ln \left( \frac{k_2}{qk_2} \right) \prod_{j=2}^d \frac{\exp(r_j \varepsilon [k_j]/q^{k_j})}{[k_j]} \sum_{k_d > n_1 > \cdots > n_i} \cdots \quad (43)
\]

\[
\sum_{k_2 > \cdots > k_d} \frac{q^{mk_2}}{[k_2]^m} \prod_{j=2}^d \frac{\exp(r_j \varepsilon [k_j]/q^{k_j})}{[k_j]} \sum_{k_d > n_1 > \cdots > n_i} \cdots \quad m \in \mathbb{Z}_{\geq 0} \quad (44)
\]

\[
\sum_{k_2 > \cdots > k_d} \frac{\varepsilon^i [k_2]^m}{q^{mk_2}} \prod_{j=2}^d \frac{\exp(r_j \varepsilon [k_j]/q^{k_j})}{[k_j]} \sum_{k_d > n_1 > \cdots > n_i} \cdots \quad 1 \leq m \leq l. \quad (45)
\]

In (44) if \( m = 0 \) then this follows from induction and if \( m > 0 \) then it follows from Theorem 4.10. For (45), we can mimic the argument for finding \( Z_q(s, \varepsilon) \) at non-positive integers by differentiating (12) and see immediately that (45) is actually in \( \mathbb{C}[\varepsilon] \) and \( r_2 \) can appear in the denominator only in the form of some pure power. So the series in (45) is good by induction.

The most difficult one is (43) in which case we again apply Euler-Maclaurin summation formula using the following modified version of \( F(x, s; \varepsilon) \):

\[
G_j(x, s; \varepsilon) = \ln^j \left( \frac{[x]}{q^x} \right) \frac{q^{x(s-1)} \exp(\varepsilon q^{-x} [x])}{(1 - q)^s [x]^s}.
\]

Then (42) changes to

\[
\int_1^\infty G_j(x + i, 1; \varepsilon) \, dx = \int_{-\varepsilon + i + 1}^\infty (\ln t)^{j+1} e^{-t} \, dt
\]

\[
= - \frac{1}{j+1} \ln^{k+1} \left( - \frac{i+1}{q^i} \right) \exp \left( \frac{i+1}{q^i} \right) + \frac{1}{j+1} \int_{-\varepsilon + i + 1}^\infty (\ln t)^{j+1} e^{-t} \, dt.
\]

This integral can be treated similarly as (42) by using polygamma functions (see [1]) which is closely related to

\[
f_k(s) := \int_0^\infty (\ln t)^k t^{-s} e^{-t} \, dt = \Gamma^{(k)}(s).
\]

In particular, to proceed by induction we need the fact that all the values of \( f_k(1) \) are finite which are in fact \( \mathbb{Q} \)-linear combinations of the \( k \) products of \( \zeta(n) \)'s and the Euler constant \( \gamma \), where we take the weight of \( \gamma \) to be 1. For example, the first few values are \( f_1(1) = -\gamma \), \( f_2(1) = \zeta(2) + \gamma^2 \), \( f_3(1) = -2\zeta(3) - 3\gamma\zeta(2) - \gamma^3 \), and \( f_4(1) = \frac{7}{2}\zeta(4) + 8\gamma\zeta(3) + 6\gamma^2\zeta(2) + \gamma^4 \). This shows that the sum in (43) lies in \( \mathbb{C}[\varepsilon] \) and the only form that \( r_j \)'s can occur in the denominator of any coefficient of the series is in a factor of some form \( r_1 + \cdots + r_i \). Note that \( r_j \)'s can also occur in logarithms in the form \( \ln(r_j) \) which implies that we can take \( r_j \)'s to be either 1 or 2. This finishes the proof of the theorem.

\[
5 \quad \text{Renormalization of } M_qZVs \text{ at nonpositive integers}
\]

The following result is straightforward:

**Theorem 5.1.** For \( \vec{s} \in (\mathbb{Z}_{<0})^d \), we have

\[
\bar{\zeta}_q(\vec{s}) = \zeta_q(\left[ \frac{\vec{s}}{-\vec{s}} \right]) = \lim_{\vec{r} \to -\vec{s}} \zeta_q(\left[ \frac{\vec{s}}{\vec{r}} \right]),
\]
Proposition 5.4. Suppose \( \vec{r} \) and tailored rational function tailored fractions as above then we call it a tailored Laurent series rational functions then we say the Lauren series is a (from N is a tailored fraction \( \frac{P}{Q} \) of \( \frac{r}{r} \) for any nonnegative integers \( v \) and \( w \) such that \( w \geq v \). For convenience, we regard all polynomials of \( \vec{r} \) as tailored fractions too. If a rational function \( P/Q \) is a product of tailored fractions as above then we call it a tailored rational function. Note that \( \text{deg}(P/Q) \geq 0 \) always holds. If every coefficient of a Lauren series \( \sum c_i \varepsilon^i \) is an \( \mathbb{R} \)-linear combination of tailored rational functions then we say the Lauren series is a tailored Laurent series.

Theorem 5.2. Let \( \vec{s} \in (\mathbb{Z}_{\leq 0})^d \) and \( \vec{f} \) be a shifting vector of \( \vec{s} \). Then the limit in \( \sum \vec{r} \) exists.

In the rest of the section we are going to prove

The major complication of the proof of Theorem 5.2 arises from the possibility that \( s_i \) can be 0 or \(-1\). To prove the theorem we need some more information of \( \tilde{Z} \).

\begin{align*}
\tilde{Z}_{\vec{r}}(\vec{s}) := \lim_{\delta \to 0^+} \zeta_q(\left[\delta + -\vec{s} + (\delta - 1)\vec{f}\right])
\end{align*}

exists.

Definition 5.3. Let \( \vec{r} \in (\mathbb{R}_{>0})^d \). Let \( d \) and \( j \) be two positive integers such that \( j \leq d \). Let \( N(\vec{r}) = r_1 + r_{i+1} + \cdots + r_d \), \( D(\vec{r}) = r_j + r_{j+1} + \cdots + r_d \) for \( 1 \leq j < i \leq d \). Then we say \( N(\vec{r})/D(\vec{r}) \) is a tailored fraction for any nonnegative integers \( v \) and \( w \) such that \( w \geq v \). For convenience, we regard all polynomials of \( \vec{r} \) as tailored fractions too. If a rational function \( P/Q \) is a product of tailored fractions as above then we call it a tailored rational function. Note that \( \text{deg}(P/Q) \geq 0 \) always holds. If every coefficient of a Lauren series \( \sum c_i \varepsilon^i \) is an \( \mathbb{R} \)-linear combination of tailored rational functions then we say the Lauren series is a tailored Laurent series.

Proposition 5.4. Suppose \( m \) and \( n \) are two integers and \( d \) is a positive integer. Let \( \vec{s} \in (\mathbb{Z}_{\leq 0})^d \) and \( \vec{r} \in (\mathbb{R}_{>0})^d \). For \((i_1, \ldots, i_p) \in \Pi_d\), let \( \vec{s}^{(j)}, 1 \leq j \leq p \) be the partition vectors of \((s_2, \ldots, s_d)\) from \((i_1, \ldots, i_p)\) in Definition 4.2 and similarly define \( \vec{s}^{(j)} \), \( 1 \leq j \leq p \). Let

\begin{align*}
\tilde{Z}_{\vec{s}}(\vec{r}) := \sum_{i \geq 0} c_i \varepsilon^i := \sum_{(i_1, \ldots, i_p) \in \Pi_d} \tilde{P}(\varepsilon^n \tilde{Z}_q(\left[\vec{s}^{(p)}\right])) \cdots \tilde{P}(\varepsilon^m \tilde{Z}_q(\left[\vec{s}^{(1)}\right]))) \cdots .
\end{align*}

Then for every \( i \) the coefficient \( c_i \) can be expressed as an \( \mathbb{R} \)-linear combination of rational functions \( P/Q \) of \( r_1, \ldots, r_d \) with \( P \) and \( Q \) having no common factors such that the following must hold:

(a) \( P \) and \( Q \) are homogeneous polynomials such that \( \text{deg}(P/Q) + n + m = i \).

(b) Either \( Q \) is a constant or every factor of \( Q \) has the form \( r_j + \cdots + r_d \) for some \( j = 1, \ldots, d \).

So \( Q \) is uniquely determined if we require \( Q \) to be a monic polynomial with respect to \( r_d \). Then either \( Q = 1 \) or \( Q \) is a product of linear factors of the form \( r_j + \cdots + r_d \) \( (j = 1, \ldots, d) \) with possible repetitions.

(c) If \( m = 0 \) then \( r_d \) divides \( Q \) only if \( n > 1 - s_d \).

(d) If \( m = 0 \) and \( n \leq 0 \) then \( P/Q \) is tailored.

Proof. (a) By Corollary 3.3 for \( 1 \leq t \leq p \) every coefficient \( a_t \), \( \tilde{Z}_q(\left[\vec{s}^{(t)}\right]) = \sum_t a_t \varepsilon^t \) is a homogeneous rational function \( P/Q \) of \( r_1, \ldots, r_d \) such that \( \text{deg}(P/Q) + i = 0 \). Now the product of such terms

\begin{align*}
(P_1/Q_1)\varepsilon^i \cdot (P_2/Q_2)\varepsilon^j = (P_3/Q_3)\varepsilon^{i+j}
\end{align*}

still satisfies \( \text{deg}(P_3/Q_3 + i + j) = 0 \). Part (a) follows easily.

(b) We proceed by induction on \( d \). If \( d = 1 \) then this follows immediately from Theorem 5.2.

Now assume part (b) is true when the length of the vector \( s \) is \( d - 1 \) for some \( d \geq 2 \). Let \( \vec{s} \in (\mathbb{Z}_{\leq 0})^d \).

We have the disjoint union

\begin{align*}
\Pi_d = \{(1, i_1 + 1, \ldots, i_p + 1) \mid (i_1, \ldots, i_p) \in \Pi_{d-1}\} \\
\quad \cup \{(1 + i_2 + 1, \ldots, i_p + 1) \mid (i_1, \ldots, i_p) \in \Pi_{d-1}\}.
\end{align*}

Note that \( i_p = d - 1 \) in the above by definition. In the rest of the proof of (b), for \((i_1, \ldots, i_p) \in \Pi_{d-1}\) let \( \vec{s}^{(j)}, 1 \leq j \leq p \) be the partition vectors of \((s_2, \ldots, s_d)\) from \((i_1 + 1, \ldots, i_p + 1)\) in Definition 4.2.
and similarly define \( r^{(j)} \), \( 1 \leq j \leq p \). Putting \( \sum r = \sum (i_1, \cdots, i_p) \in \Pi_{d-1} \) we have

\[
\hat{Z}_{q, m}^{\hat{r}, p}(\left[ \hat{r}^{(1)} \right]) = \sum \hat{P}(\varepsilon^n \hat{Z}_q(\left[ \hat{r}^{(p)} \right]) \cdots \hat{P}(\hat{Z}_q(\left[ \hat{r}^{(1)} \right]) \hat{P}(\varepsilon^n \hat{Z}_q(\left[ r_{\hat{r}_{\hat{r}_{\hat{r}_{\hat{r}1}}}} \right]) \cdots ) \\
+ \sum \hat{P}(\varepsilon^n \hat{Z}_q(\left[ \hat{r}^{(p)} \right]) \cdots \hat{P}(\hat{Z}_q(\left[ \hat{r}^{(2)} \right]) \hat{P}(\varepsilon^n \hat{Z}_q(\left[ r_{\hat{r}_{\hat{r}_{\hat{r}_{\hat{r}1}}, \hat{r}^{(1)}} \right]) \cdots )).
\]

By Theorem 3.2 we have

\[
\hat{P}(\varepsilon^n \hat{Z}_q(\left[ \hat{r}_{\hat{r}_{\hat{r}_{\hat{r}_{\hat{r}1}}} \right])) = \begin{cases} 
0 & \text{if } m > -s_1, \\
\frac{q - 1}{\ln q} A(r_1)(-s_1)! - \sum_{i=0}^{-m-1} \zeta(q(s_1 - i)) \frac{r_1^{m+k}}{i!} & \text{if } m \leq -s_1,
\end{cases}
\]

where the sum is vacuous if \( m \geq 0 \) and

\[
A(r_1) := -\left( \frac{-1}{r_1} \right)^{1-s_1}.
\]

Write \( k = i_1 + 1 \). Then by (31) we get

\[
e^n \hat{Z}_q(\left[ \hat{r}_{\hat{r}_{\hat{r}_{\hat{r}_{\hat{r}1}}} \right]) = \frac{q - 1}{\ln q} B(r_1)(-s_1)! \\
+ \sum_{j=0}^{-s_1} \left( \frac{-s_1}{j} \right) \sum_{l=s_1-j}^{\infty} \hat{S}_{k+1}^{l} Z_q(\left[ \hat{r}_{l+1}, r_{l+2}, r_{l+3}, r_{l+4} \right]; \varepsilon) \zeta(q(-l)) \frac{1}{(l + s_1 + j)!} \frac{l^{m+1}}{l^{m}} \\
\]

where

\[
B(r_1) := \sum_{j=0}^{-s_1} \left( \frac{-s_1}{j} \right) \left( \frac{-1}{r_1} \right)^{1-s_1-j} Z_q(\left[ \hat{r}_{l+1}, r_{l+2}, r_{l+3}, r_{l+4} \right]; \varepsilon) \varepsilon^j.
\]

By induction assumption, the linear factors of \( Q \) can only be \( r_1, r_{l+2}, r_{l+3}, r_{l+4} \). Let us exclude the possibility of \( r_1 \). It suffices to prove that

\[
\lim_{r_1 \to 0} \left( A(r_1) \hat{Z}_q(\left[ \hat{r}_{\hat{r}_{\hat{r}_{\hat{r}_{\hat{r}1}}} \right]) + B(r_1) \right) \text{ is finite.}
\]

Set \( f(x) = Z_q(\left[ \hat{r}_{l+1}, r_{l+2}, r_{l+3}, r_{l+4} \right]; \varepsilon) \). Put \( D = d/dx \) and \( t = -s_1 \geq 0 \). By L’Hopital’s rule

\[
\lim_{x \to 0} f(0) A(x) + B(x) = (-1)^t \lim_{x \to 0} \frac{f(0) - \sum_{j=0}^{t} \frac{1}{j!} \left( -x \right)^j D^j f(x)}{x^{t+1}} \\
= (-1)^t \lim_{x \to 0} \frac{\sum_{j=0}^{t} \frac{1}{j!} \left( -x \right)^j D^j f(x)}{(t + 1)!} < \infty
\]

after cancellations in the numerator.

**Remark 5.5.** By exactly the same argument we see that \( r_d \) can not appear in the denominator of (52) when \( d \geq 2 \).

(c) and (d). Assume \( m = 0 \). We use induction on \( d \) again. Theorem 3.2 yields the case \( d = 1 \) easily for both (c) and (d). Suppose part (d) holds when \( \hat{s} \) has length \( d - 1 \) for some \( d \geq 2 \). Now let \( \hat{s} \in (\mathbb{Z}_{\leq 0})^d \). Observe that we have another disjoint union of \( \Pi_d \) given by

\[
\Pi_d = \{(i_1, \cdots, i_p, d) \mid (i_1, \cdots, i_p) \in \Pi_{d-1}\} \\
\cup \{(i_1, \cdots, i_{p-1}, i_p + 1) \mid (i_1, \cdots, i_p) \in \Pi_{d-1}\}.
\]
By Theorem 3.2 and Remark 5.5 we see that similar to part (b), in the rest of the proof, for any positive integer \( t \) and \( (i_1, \ldots, i_p) \in \Pi_l \), let \( \mathbf{s}^{(j)}, 1 \leq j \leq p \) be the partition vectors of \( (s_1, \ldots, s_i) \) from \( (i_1, \ldots, i_p) \) in Definition 4.2. Define \( r^{(j)} \) in the same fashion. Put \( \sum r(l) = \sum_{(i_1, \ldots, i_p) \in \Pi_l} \). Then

\[
\tilde{Z}^{n,0}_{\mathbf{s}^{(j)}} = \sum_{\mathbf{r}^{(j)} \in \Pi_l} \hat{P} \left( e^n \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) \hat{P} \left( \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) \hat{P} \left( \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) \cdots \right) + \sum_{\mathbf{r}^{(j)} \in \Pi_l} \hat{P} \left( e^n \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) \hat{P} \left( \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) X \right) + e^n \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) X \right) \right)
\]

\[
= \sum_{\mathbf{r}^{(j)} \in \Pi_l} \hat{P} \left( e^n \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) \hat{P} \left( \tilde{Z}_{q} \left( \left[ \mathbf{s}^{(j)}_{\mathbf{r}^{(j)}} \right] \right) X \right) \right)
\]

\[
- \frac{q - 1}{\ln q} \sum_{j=0}^{s_d - 1} \frac{(-s_d)!}{j!} \sum_{\mathbf{r}^{(j)} \in \Pi_l} \hat{P} \left( e^n \left( -1 \right)^{s_d - j} \tilde{Z}_{q} \left( \left[ s_k, \ldots, s_d - 2, s_d - 1 - j \right] ; \varepsilon \right) X \right)
\]

\[
- \frac{s_d}{j} \sum_{l=-s_d-j}^{\infty} \sum_{\mathbf{r}^{(j)} \in \Pi_l} \hat{P} \left( e^n \tilde{Z}_{q} \left( \left[ s_k, \ldots, s_d - 2, s_d - 1 - j \right] ; \varepsilon \right) X \right)
\]

By Theorem 5.2 and Remark 5.5 we see that \( r_d \) can appear essentially in the denominator of some term in (50) only if \( n > 1 - s_d \) which proves (c). As a matter of fact, we don’t need induction assumption to prove (c) so we will use (c) freely in what follows.

Suppose now \( n \leq 0 \). By part (b) every denominator in (51) can only have linear factors of the form \( r_d \) or \( R_j - r_d \) for some \( j = 1, \ldots, d - 1 \). By part (b) and (c) we know that these terms will be cancelled out in the end so (51) doesn’t contribute to any un-tailored terms. In fact, negative powers of \( R_j - r_d \) can not really appear by the reasoning in the next paragraph.

As the exponent of \( \varepsilon \) in front of \( Z_{q} \) is now negative in (52) every term has \( r_d \) in the denominator so that all of these will be cancelled by corresponding terms in (52) (note that there is no negative powers of \( r_d \)). In fact by part (b) the only linear factors of denominators that can appear are of the form \( R_j - r_d \) and of the form either \( R_j - r_d \) or \( r_d \) in (51). So to cancel terms with negative \( r_d \)-power no factors \( R_j - r_d \) can appear in denominators of (51) and no factors \( R_j \) can appear in denominators of (52). (Actually, positive powers \( R_j \) and \( R_j - r_d \) must appear so that \( r_d = R_j - (R_j - r_d) \) is produced on the numerator to cancel negative powers of \( r_d \).) This shows that the sum in (51) and (52) is tailored.

Let’s turn to (53). If \( l + j \) is small, say \( l + j \leq -s_d - n \) then by induction assumption the Laurent series is tailored. When \( l + j \) increase gradually it seems that un-tailored terms may appear. We will show this actually can not happen. Roughly speaking, in order for some linear factor \( r_j + r_{j+1} + \cdots + r_d \) to appear in the denominator, the power of \( \varepsilon \) in front of \( Z_{q} \) has to be large by part (c). But this will produce high powers of \( r_d \) which results in tailored fractions.

It is easy to see that in (53) the highest power of \( r_{d-1} + r_d \) that can appear in the denominator is \( 1 - s_d - j \) by part (c). This takes place if \( l + s_d + j + n > 1 - s_{d-1} + j \). Similarly, we can expand (53) by iteratively using the formulas from (51) to (53) and see that the highest power of \( R_j \) that can appear in the denominator is \( 1 - s_i + j_{d-i} \) where \( j_{d-i} \) is the corresponding summation index in (52) and (53). Indeed iterating once we see that for fixed \( j \) and \( l \) the essential part of (53)
\[
\begin{align*}
&\quad r_d^{l+s_d+j} \sum_{\bar{r}(d-1)} \tilde{P}(\varepsilon^{n+l+s_d+j} Z_q\left(\left[\frac{s_k, \ldots, s_{d-2}, s_{d-1} - j}{r_k, \ldots, r_{d-2}, R_{d-1}}\right]; \varepsilon\right) X) \\
&= \sum_{\bar{r}(d-2)} r_d^{l+s_d+j} \tilde{P}(\varepsilon^{n+l+s_d+j} Z_q\left(\left[\frac{s_{d-1} - j}{R_{d-1}}\right]\right) \tilde{P}(\tilde{Z}_q\left(\left[\frac{s_{d-1}^p}{R_{d-1}}\right]\right) X)) \\
&\quad - \frac{q-1}{\ln q} \sum_{j_2=0}^{j-s_d-1} \left(\frac{j - s_d - 1}{j_2!}\right) \sum_{\bar{r}(d-2)} r_d^{l+s_d+j} \varepsilon^{n+l+s_d+j} \left(\frac{-1}{R_{d-1} \varepsilon}\right)^{1+j-s_{d-1}-j_2} Z_q\left(\left[\frac{s_k, \ldots, s_{d-3}, s_{d-2} - j_2}{r_k, \ldots, r_{d-3}, R_{d-2}}\right]; \varepsilon\right) X) \\
&\quad - \sum_{j_2=0}^{j-s_d-1} \left(\frac{j - s_d - 1}{j_2}\right) \sum_{l_2=j-s_d-1-j_2}^{\infty} \sum_{\bar{r}(d-2)} r_d^{l+s_d+j} \varepsilon^{n+l+s_d+j} (R_{d-1} \varepsilon)^{l_2+s_d-j_2} Z_q\left(\left[\frac{s_k, \ldots, s_{d-3}, s_{d-2} - j_2}{r_k, \ldots, r_{d-3}, R_{d-2}}\right]; \varepsilon\right) X) X.
\end{align*}
\]

For (55) observe that \(\tilde{P}(\tilde{Z}_q\left(\left[\frac{s_{d-1}^p}{R_{d-1}}\right]\right) X)\) can not produce legitimate un-tailored fractions by part (b). So the only possible un-tailored terms are the result of negative powers of \(R_{d-1}\) coming from \(\tilde{Z}_q\left(\left[\frac{s_{d-1} - j}{R_{d-1}}\right]\right)\). When \(n \leq 0\) the only way \(R_{d-1}\) can appears in the denominator is when \(n + l + s_d + j > 1 - s_{d-1}\). Then these terms are multiplied by \(r_d^{l+s_d+j}\) to become tailored.

Let’s consider (56) next. Put \(w := n + l + s_d + j - (1 + j - s_{d-1} - j_2)\). We treat the two cases \(w \leq 0\) and \(w > 0\) separately. If \(w \leq 0\), then the induction assumption takes care of all the terms except negative powers \(R_{d-1}\). But by breaking (56) into two parts like in (54) we see that when \(w \leq 0\) negative powers of \(R_{d-1}\) can not appear by part (c). So we can assume now \(w > 0\). As \(n \leq 0\) this means that \(l + s_d + j \geq n + l + s_d + j > 1 + j - s_{d-1} - j_2\) which implies that

\[
r_d^{l+s_d+j} \varepsilon^{n+l+s_d+j} \left(\frac{-1}{R_{d-1} \varepsilon}\right)^{1+j-s_{d-1}-j_2} = \left(\frac{r_d}{R_{d-1}}\right)^{l+s_d+j} \varepsilon^n (R_{d-1} \varepsilon)^{w-n}
\]

is tailored. Hence we see that (56) is now tailored by setting \(a = w - n > 0\) and \(t = d - 2\) in the following Claim.

**Claim.** Let \(a\) and \(t\) be two positive integers such that \(1 \leq t \leq d - 1\). Then the series

\[
\sum_{\bar{r}(t)} \tilde{P}(\varepsilon^n (R_{t+1} \varepsilon)^a Z_q\left(\left[\frac{s_k, \ldots, s_{t-1}, s_t}{r_k, \ldots, r_{t-1}, R_t}\right]; \varepsilon\right) X)
\]

is tailored.

We use induction on \(t\). Notice that this is the inner induction loop. The outer induction loop is on \(d\). When \(t = 1\) the claim follows from Theorem 3.2 easily. This already proves the proposition if \(d = 2\). We now assume \(d > 2, t > 1\), and the claim is true if the length of the vector in the claim is less than \(t\) for some \(1 < t < d\).

As before we may break \(\Pi_t\) into two parts and form the disjoint union

\[
\Pi_t = \{ (i_1, \cdots, i_p, t) \mid (i_1, \cdots, i_p) \in \Pi_{t-1}\} \\
\quad \cup \{ (i_1, \cdots, i_{p-1}, i_p + 1) \mid (i_1, \cdots, i_p) \in \Pi_{t-1}\}.
\]
Adopting the same argument we used to obtain (50) to (53) we have
\[
\sum_{\tilde{n}(l)} \hat{P}(\varepsilon^n(R_{t+1}\varepsilon)^a Z_q\left(\left[s_k, \ldots, s_{t-1}, s_t; \varepsilon\right]; X\right)
\]
\[
= \sum_{\tilde{n}(l-1)} \hat{P}(\varepsilon^n(R_{t+1}\varepsilon)^a \tilde{Z}_q\left(\left[s_t; R_{t}\right]\right) \hat{P}(\tilde{Z}_q\left(\left[s_t; r_p\right]\right) X) + \varepsilon^n(R_{t+1}\varepsilon)^a \tilde{Z}_q\left(\left[s_t; R_{t+1}\right]\right) X)
\]
(59)
\[
= \sum_{\tilde{n}(l-1)} \hat{P}(\varepsilon^n(R_{t+1}\varepsilon)^a \tilde{Z}_q\left(\left[s_t; R_{t}\right]\right) \hat{P}(\tilde{Z}_q\left(\left[s_t; r_p\right]\right) X)
\]
\[
- \sum_{j=0}^{\min} (s_t)_j \sum_{l=-s_t-j}^{\tilde{n}(l-1)} \hat{P}(\varepsilon^n(R_{t+1}\varepsilon)^a (1 - R_{t+1}\varepsilon)^j Z_q\left(\left[s_k, \ldots, s_{t-2}, s_{t-1} - j; \varepsilon\right] X\right) + \varepsilon^n(R_{t+1}\varepsilon)^a (R_{t+1}\varepsilon)^j Z_q\left(\left[s_k, \ldots, s_{t-1} - j; \varepsilon\right] X\right).
\]
(62)

By part (c) $R_t$ appears in some denominator of the first term of (59) only if $n + a > 1 - s_t$ in which case the exponent of $R_t$ is $1 - s_t$. Then multiplied by $R_{t+1}^a$ this becomes tailored as $a \geq a + n$. Un-tailed terms with $R_t$ appearing in the denominator can not be produced by the second term by Remark 5.5.

The same argument for (51) above using part (b) and part (c) shows that (60) does not contribute to any un-tailed term in the end except possible powers of $1/R_t$. However, all such terms are multiplied by high enough powers of $R_{t+1}$ resulting in tailored terms again.

Similar argument for (52) implies that in (61) we only need to consider the case when $n + a > 1 - s_t$. In this case we see that
\[
\varepsilon^n(R_{t+1}\varepsilon)^a \left(1 - R_{t+1}\varepsilon\right)^j Z_q\left(\left[s_k, \ldots, s_{t-2}, s_{t-1} - j; \varepsilon\right] X\right)
\]
is tailored and therefore (61) is tailored Laurent series by inner loop induction assumption. The same argument by manipulating the powers of $\varepsilon$ works for (62) in the same fashion and this finishes the proof of the claim.

Finally, (57) follows from the claim immediately by setting $a = l + 2s_{d} + j + j > 0$ and $t = d - 1$ since
\[
\varepsilon^{n+l+s_{d}+j} (R_{d-1}\varepsilon)^{l+s_{d}+j} = \varepsilon^{n} \left(\frac{r_d}{R_{d-1}}\right)^{l+s_{d}+j} (R_{d-1}\varepsilon)^{l+s_{d}+j}.
\]
This finishes the proof of the proposition.

Proof of Theorem 5.2 Clearly Theorem 5.2 is true when $d = 1$ by Theorem 3.2. When $d \geq 2$ by setting $m = n = 0$ in Proposition 5.3 in (d) we see that the constant term of $\tilde{Z}_q^0\left(\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\right)$ must be the product of linear tailored fractions of the form
\[
\frac{r_i + r_{i+1} + \cdots + r_d}{r_j + r_{j+1} + \cdots + r_d}
\]
for some $i$ and $j$ such that $d \geq i > j \geq 1$. Note that in the limit (57) each $r_k$ can only be either $\delta + n_k$, or $2\delta + n_k$ for some nonnegative integer $n_k$. If $n_k > 0$ for some $j \leq k \leq d$ then the limit
\[
\lim_{\delta \to 0^+} \frac{r_i + r_{i+1} + \cdots + r_d}{r_j + r_{j+1} + \cdots + r_d}
\]
is clearly finite. If $n_k = 0$ for all $k = j, j+1, \ldots, d$ then the limit is still finite because $\delta$ will be cancelled out.

This completes the proof of Theorem 5.2.
One of the most important properties of MZVs and MqZVs is that they satisfy the \((q)\)-stuffle relations. In order to show that our renormalization for MqZVs at all non-positive arguments is correct we need to show some type of \((q)\)-stuffle relation hold for them. However, we must use the shifted renormalization following the shifting principle of MqZVs.

**Theorem 5.6.** The shifted renormalizations of \(\zeta_q(\vec{s})\) satisfy the \((q)\)-stuffle relations for all \(\vec{s} \in \mathbb{Z}_{\leq 0}^k\). Explicitly, let \(\vec{s}_1 \in \mathbb{Z}_{\leq 0}^k, \vec{s}_2 \in \mathbb{Z}_{\leq 0}^\ell\) and assume there are \(W_i\) vectors of length \(k + \ell - t\) produced by the \(q\)-stuffle and let \(\vec{f}_j\) be the binary vector representing the positions of the stuffing in \(j\)-th such vector:

\[
\vec{s}_1 *_q \vec{s}_2 = \sum_{t=0}^{\min(k, \ell)} (1 - q^t) \sum_{j=1}^{W_t} (\vec{v}_j - \vec{f}_j).
\]

Then

\[
\bar{\zeta}_q(\vec{s}_1) \bar{\zeta}_q(\vec{s}_2) = \sum_{t=0}^{\min(k, \ell)} (1 - q^t) \sum_{j=1}^{W_t} \bar{\zeta}^{\vec{f}_j}_q (\vec{v}_j - \vec{f}_j).
\]

When \(q \uparrow 1\) we obtain \cite{[12], Theorem 4.11}.

**Proof.** By the definition of renormalization in Definition 4.7 and Corollary 4.5 we have

\[
\bar{\zeta}_q(\vec{s}_1) \bar{\zeta}_q(\vec{s}_2) = \lim_{\vec{r}_1 \to -\vec{s}_1, \vec{r}_2 \to -\vec{s}_2} \zeta(\vec{s}_1) \zeta(\vec{s}_2) = \lim_{\vec{r}_1 \to -\vec{s}_1, \vec{r}_2 \to -\vec{s}_2} \zeta^{\vec{r}_1}_q (\vec{s}_1) \zeta^{\vec{r}_2}_q (\vec{s}_2).
\]

From the definition of \(q\)-stuffle \cite{[10]} and the shifted renormalization \cite{[37]} the theorem follows immediately. \(\square\)

To conclude our paper we observe that a multiple \(q\)-zeta function has more singularities than its classical counterpart. We don’t know how to renormalize MqZVs at these points at present.

**References**

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, (1964).

[2] E. Artin, *The Gamma Function*, New York, Holt, Rinehart and Winston, (1964). Reprinted in *Exposition by Emil Artin: A Selection*, ed. by Michael Rosen. History of Mathematics Sources Subseries. American Mathematical Society/London Mathematical Society, 2007

[3] S. Akiyama, S. Egami, and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith., 98 (2001), 107–116.

[4] D. M. Bradley, *Multiple \(q\)-zeta values*, J. Algebra, 283 (2)(2005), 752–798, arXiv: math.QA/0402093

[5] D. J. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B, 393 (1997), 403–412.

[6] P. Cartier, *On the structure of free Baxter algebras*, Adv. in Math., 9 (1972), 253–265.

[7] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*, Comm. Math. Phys., 210 (1)(2000), 249–273, hep-th/9912092

[8] K. Ebrahimi-Fard and L. Guo, *Multiple zeta values and Rota–Baxter algebras*, math.NT/0604558.

[9] L. Euler, *Meditationes circa singulare serierum genus*, Novi. Comm. Acad. Sci. Petropolitanae, 20 (1775), 140–186.

[10] A. B. Goncharov, Yu. I. Manin, *Multiple zeta-motives and moduli spaces \(M_{0,n}\)*, Compositio Math. 140 (2004), 114.
[11] L. Guo and W. Keigher, Free Baxter algebras and shuffle products, Adv. in Math., 150 (2000), 117–149.

[12] L. Guo and B. Zhang, Renormalization of multiple zeta values, math.NT/0606076v3.

[13] M. E. Hoffman, Multiple harmonic series, Pacific J. Math., 152 (2)(1992), 275–290.

[14] M. E. Hoffman, Quasi-shuffle products, J. Algebraic Combin., 11 (2000), 49–68.

[15] K. Ihara, M. Kaneko and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math., 142 (2006), 307-338

[16] M. Kaneko, N. Kurokawa, and M. Wakayama, A variation of Euler’s approach to values of the Riemann zeta function, Kyushu J. Math. 57 (2003), 175–192.

[17] K. Knopp, Theory and Application of Infinite Series, 2nd ed., Blackie and Son Limited, London and Glasgow, 1951.

[18] T. Q. T. Le and J. Murakami, Kontsevich’s integral for the Homfly polynomial and relations between values of the multiple zeta functions, Topology Appl. 62 (1995), 193-206.

[19] D. Manchon, Hopf algebras, from basics to applications to renormalization, Comptes-rendus des Rencontres mathématicques de Glanon 2001. math.QA/0408405.

[20] E.T. Whittaker and G.N. Watson, A course of modern analysis, 4th ed., Cambridge University Press, 1996.

[21] D. Zagier, Values of zeta function and their applications, Proceedings of the First European Congress of Mathematics, 2 (1994), 497–512.

[22] J. Zhao, Analytic continuation of multiple zeta functions, Proc. of AMS, 128 (1999), 275–1283.

[23] J. Zhao, q-Multiple zeta functions and q-multiple polylogarithms, Ramanujan J., 14 (2)(2007).

[24] W. Zudlin, Algebraic relations for multiple zeta values, (Russian), Uspekhi Mat. Nauk, 58 (1)(2003), 3–32, translation in Russian Math. Survey, 58 (1)(2003), 1–29.

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