PERMUTATION STATISTICS OF PRODUCTS OF RANDOM PERMUTATIONS

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Abstract. Given a permutation statistic \( s : S_n \to \mathbb{R} \), define the mean statistic \( \overline{s} \) as the statistic which computes the mean of \( s \) over conjugacy classes. We describe a way to calculate the expected value of \( s \) on a product of \( t \) independently chosen elements from the uniform distribution on a union of conjugacy classes \( \Gamma \subseteq S_n \). In order to apply the formula, one needs to express the class function \( \overline{s} \) as a linear combination of irreducible \( S_n \)-characters. We provide such expressions for several commonly studied permutation statistics, including the excedance number, inversion number, descent number, major index and \( k \)-cycle number. In particular, this leads to formulae for the expected values of said statistics.

1. Introduction

Consider the symmetric group \( S_n \) of permutations of \([n] = \{1, \ldots, n\}\). For \( \Gamma \subseteq S_n \) one may study the behaviour of various permutation statistics \( s : S_n \to \mathbb{R} \) on products \( \gamma_1 \cdots \gamma_t \in S_n \) of random \( \gamma_i \in \Gamma \).

Definition 1.1. Choose a subset \( \Gamma \subseteq S_n \), a function \( s : S_n \to \mathbb{R} \) and a nonnegative integer \( t \). We denote by \( \mathbb{E}_\Gamma(s, t) \) the expected value of \( s \) on a product of \( t \) elements independently chosen from the uniform distribution on \( \Gamma \).

A product of \( t \) random elements of \( \Gamma \) corresponds to a \( t \)-step random walk on the Cayley graph of \( S_n \) induced by \( \Gamma \). Random walks on Cayley graphs form a classical and well studied subject in probability theory; a good general reference is [12]. In the present paper, we specifically address the problem of computing \( \mathbb{E}_\Gamma(s, t) \). Recent work in this vein includes the following. The case of \( \Gamma \) being the set of adjacent transpositions and \( s \) counting inversions was studied by Eriksson et al. [4], Eriksen [5] and Bousquet-Mélou [2]. Turning, instead, to \( \Gamma \) comprised of all transpositions, \( s \) being the absolute length function (essentially counting disjoint cycles) was studied in [6], whereas Jonsson [9] considered the fixed point counting function \( s \) and Sjöstrand [13] found the solution when \( s \) counts inversions.

We shall describe a method to attack the general problem. Although it could potentially be of use for more general \( \Gamma \) (as indicated by the hypotheses of Theorem 3.1 below), we shall apply it to situations where \( \Gamma \) is a union of conjugacy classes. The technique makes use of representation theory of \( S_n \). Similar ideas have been frequent in the study of random walks on Cayley graphs since the seminal paper by Diaconis and Shahshahani [3]. In particular, our method is heavily inspired by that described in [6]. We are concerned with more general \( \Gamma \), but the principal novelty here is to dispose of the apparent requisite of [6] that \( s \) be a class function. As we shall see, removing this restriction significantly improves the versatility of the method.

Here is a brief sketch of the content of the paper. In the next section, we review some facts about symmetric group characters. In Section 3 we then describe how they connect with our expected statistics problem. The main results of that section, Theorem 3.1 and Theorem 3.2, provide a recipe for computing \( \mathbb{E}_\Gamma(s, t) \) whenever \( \Gamma \)
is a union of conjugacy classes. In order to express $E_\Gamma(s,t)$ explicitly for a given statistic $s$, the remaining task is to decompose the mean statistic $\mathcal{E}$ as a linear combination of irreducible $S_n$-characters. This turns out to be a fairly straightforward task for many standard permutation statistics. We provide explicit decompositions for the mean statistics corresponding to the $k$-cycle number, excedance number, weak excedance number, inversion number, major index and descent number statistics in Sections 4, 5 and 6. In particular, $E_\Gamma(s,t)$ is determined for $s$ being any of these statistics (and conjugation invariant $\Gamma$). We conclude with a few explicit examples in Section 7.

2. Symmetric group characters

In this section, we review elements of the representation theory of $S_n$. From this vast and classical subject only a few bits and pieces that we need in the sequel are extracted in order to agree on notation. We refer to [11] for a thorough background and much more information.

Let $P_n$ denote the set of integer partitions $\lambda \vdash n$. The irreducible representations of $S_n$ are in bijection with $P_n$ in a standard way. We use the notation $\rho_\lambda$ for the representation indexed by $\lambda \vdash n$ and denote the corresponding character by $\chi_\lambda$. These irreducible characters form a basis for the $C$-vector space $Cl_n = \{ f : P_n \to C \}$ of class functions. Moreover, this basis is orthonormal with respect to the standard Hermitian inner product on $Cl_n$

$$\langle f, g \rangle = \frac{1}{n!} \sum_{\lambda \vdash n} |C_\lambda| f(\lambda) g(\lambda)^*,$$

where $C_\lambda$ denotes the conjugacy class of permutations $\pi$ with type($\pi$) = $\lambda$ as cycle type.

Abusing notation, we at times consider class functions as defined on $S_n$ rather than on $P_n$. In other words, for $\pi \in S_n$ and $f \in Cl_n$, $f(\pi)$ should be interpreted as $f(\text{type}(\pi))$. We trust the context to prevent confusion.

It is convenient to encode partitions as weakly decreasing sequences of positive integers, sometimes employing exponent notation to signal repeated parts. For example, $(7,3^4,1)$ denotes the partition of 20 which consists of one part of size 7, four parts of size 3 and one part of size 1. In this notation, a hook shape is a partition of the form $(a,1^b)$ for integers $a \geq 1$ and $b \geq 0$.

The trivial $S_n$-character is indexed by $(n)$. Thus, $\chi^{(n)}(\mu) = 1$ for all $\mu \vdash n$. The next three lemmata collect a few more values of certain irreducible characters that we shall need in the sequel. All statements are readily verified using e.g. the Murnaghan-Nakayama rule.

Define $f^\lambda = \chi^\lambda((1^n))$. This is the dimension of the irreducible representation $\rho^\lambda$.

**Lemma 2.1.** Hook shape characters satisfy

$$f^{(n-k,1^k)} = \binom{n-1}{k}$$

and

$$\chi^{(n-k,1^k)}((n)) = (-1)^k,$$

whereas $\chi^\lambda((n)) = 0$ if $\lambda$ is not a hook shape.

**Lemma 2.2.** Let $\lambda \vdash n$. If $\lambda$ has $p$ parts of size 1 and $q$ parts of size 2, then

$$\chi^{(n-1,1)}(\lambda) = p - 1$$
and
\[ \chi^{(n-2,1^2)}(\lambda) = \binom{p-1}{2} - q. \]

A frequently occurring quantity is the content of \( \lambda \vdash n \). It is defined by
\[ c_\lambda = \binom{n}{2} \frac{\chi^\lambda((2,1^{n-2}))}{f^\lambda}. \]

Lemma 2.3. The content of a hook shape is
\[ c_{(n-k,1^k)} = \frac{n(n-2k-1)}{2}. \]

3. Mean statistics and irreducible characters

Let \( s : \mathfrak{S}_n \to \mathbb{R} \) be any real-valued function on the symmetric group. In all our subsequent applications, \( s \) will be a permutation statistic associating a nonnegative integer with each permutation in \( \mathfrak{S}_n \).

Choose \( \Gamma \subseteq \mathfrak{S}_n \). We now describe a general procedure for computing the expected value of \( s \) on a product of random elements of \( \Gamma \) taken independently from the uniform distribution.

The mean statistic \( \overline{s} \) is the class function which computes the mean of \( s \) over conjugacy classes. That is, \( \overline{s} : P_n \to \mathbb{R} \) is defined by
\[ \overline{s}(\lambda) = \frac{1}{|C_\lambda|} \sum_{\pi \in C_\lambda} s(\pi). \]

Hence, thinking of class functions as being defined on \( \mathfrak{S}_n \), \( \overline{s} = s \) if and only if \( s \) is a class function. Considered in that particular setting, the remainder of this section resembles the procedure described in \[6\].

Let \( \pi \in \mathfrak{S}_n \). We need to keep track of the ways to express \( \pi \) as a product of elements from \( \Gamma \). To this end, define a permutation statistic \( n_t \) by
\[ n_t(\pi) = \#\{(\gamma_1, \ldots, \gamma_t) \in \Gamma^t \mid \gamma_1 \cdots \gamma_t = \pi\}. \]

Observe that all \( n_t \) are class functions if and only if \( \Gamma \) is a union of conjugacy classes.

Theorem 3.1. If at least one of the statistics \( n_t \) and \( s \) is a class function, then
\[ \mathbb{E}_\Gamma(s, t) = \frac{n!}{|\Gamma|^t} \overline{s}(\text{over } \text{classes of } \Gamma). \]

Proof. By definition,
\[ \mathbb{E}_\Gamma(s, t) = \frac{1}{|\Gamma|^t} \sum_{\pi \in \mathfrak{S}_n} n_t(\pi) s(\pi). \]

Assume now that \( s \) is a class function. Rewriting the right hand side by first summing over the conjugacy classes of \( \mathfrak{S}_n \), we obtain
\[ \mathbb{E}_\Gamma(s, t) = \frac{1}{|\Gamma|^t} \sum_{\lambda \vdash n} \overline{s}(\lambda) \sum_{\pi \in C_\lambda} n_t(\pi) \]
\[ = \frac{1}{|\Gamma|^t} \sum_{\lambda \vdash n} \overline{s}(\lambda) |C_\lambda| \overline{n_t}(\lambda) \]
\[ = \frac{n!}{|\Gamma|^t} \overline{s}(\text{over } \text{classes of } \Gamma), \]

as desired. If, instead, \( n_t \) is a class function, the proof is completely analogous. \( \square \)

1The context of \[6\] was that of \( s(\pi) \) being the absolute length of \( \pi \) with \( \Gamma \) the set of transpositions. The technique, however, could clearly have been applied to any class function \( s \).
Under the hypotheses of the preceding theorem, we are left with the task of evaluating the inner product of two mean statistics. This is easy if we are somehow able to express them in the orthonormal basis comprised of the irreducible $S_n$-characters. In other words, we want to find the coefficients $a_\lambda$ and $b^{(t)}_\lambda$ defined by

$$\bar{s} = \sum_{\lambda \vdash n} a_\lambda \chi^\lambda$$

and

$$\text{bar}_t = \sum_{\lambda \vdash n} b^{(t)}_\lambda \chi^\lambda,$$

respectively.

Although Theorem 3.1 applies if $n_t$ or $s$ is a class function, all our subsequent applications come from the former setting. The next theorem is the reason; it shows how to compute the $b^{(t)}_\lambda$ if $\Gamma$ consists of conjugacy classes. Variations of the formula (and its proof) are numerous in the literature. With a bit of labour, it can be extracted e.g. from [3] or [7]. When $\Gamma$ is a conjugacy class, the statement follows immediately from [10, Theorem A.1.9] which is attributed to Frobenius. We provide a self-contained proof for convenience.

**Theorem 3.2.** Suppose $\Gamma$ is a disjoint union of conjugacy classes $\Gamma_1, \ldots, \Gamma_k \subseteq S_n$. Let $\mu_i$ denote the cycle type of the permutations in $\Gamma_i$. Then,

$$b^{(t)}_\lambda = \frac{1}{n!} \left( \sum_{i=1}^k |\Gamma_i| \chi^\lambda(\mu_i) \right)^t.$$

**Proof.** Let $\delta_{\cdot,\cdot}$ denote the Kronecker delta. By the Schur orthogonality relations,

$$\frac{1}{n!} \sum_{\lambda \vdash n} f^\lambda \chi^\lambda(\mu) = \delta_{(1^n) \mu} = \text{bar}_0(\mu).$$

This proves the $t = 0$ case of the asserted statement.

Suppose $f$ is any class function and define a linear map $P_{\Gamma}$ on $\text{Cl}_n$ by declaring

$$P_{\Gamma}(f)(\pi) = \sum_{\gamma \in \Gamma} f(\pi^{-1}\gamma),$$

for $\pi \in S_n$. By definition, $\text{bar}_t = P^t_{\Gamma}(\text{bar}_0)$. Thus, it suffices to show that $\chi^\lambda$ is an eigenfunction of $P_{\Gamma}$ with eigenvalue $\frac{1}{n!} \sum_{i=1}^k |\Gamma_i| \chi^\lambda(\mu_i)$. To this end, we may by linearity assume without loss of generality that $\Gamma$ is a single conjugacy class. Denote its cycle type simply by $\mu$.

Define $M_\lambda = \sum_{\gamma \in \Gamma} \rho^\lambda(\gamma)$. Since $\Gamma$ is a conjugacy class, $M_\lambda$ and $\rho^\lambda(\pi)$ commute for all $\pi \in S_n$. By Schur’s Lemma, $M_\lambda = h(\lambda)I$, where $I$ is the identity map and $h(\lambda) \in \mathbb{C}$. Now observe that

$$P_{\Gamma}(\chi^\lambda)(\pi) = \text{trace}(\rho^\lambda(\pi)M_\lambda) = h(\lambda)\chi^\lambda(\pi)$$

for $\pi \in S_n$. Plugging in $\pi = id$ yields $|\Gamma|\chi^\lambda(\mu) = h(\lambda)f^\lambda$, proving the claim. \qed

For $\Gamma$ satisfying the hypothesis of Theorem 3.2, we conclude that the remaining challenge is to decompose the mean statistic $\bar{s}$ as a linear combination of irreducible $S_n$-characters. The upcoming three sections are essentially devoted to such computations.
4. Cycle numbers

For \( \pi \in S_n \) and a positive integer \( k \), let \( \text{cyc}_k(\pi) \) denote the number of elements that are contained in \( k \)-cycles in the disjoint cycle decomposition of \( \pi \). Thus, \( \pi \) contains \( \text{cyc}_k(\pi)/k \) \( k \)-cycles. Clearly, \( \text{cyc}_k \) is a class function so that \( \text{cyc}_k = \text{cyc}_k \).

The following result, providing a decomposition of this statistic, is due to Alon and Kozma [1]. We take this opportunity to state a shorter, independent proof.

**Theorem 4.1** (Theorem 3 in [1]). Let \( k \in [n] \). Regarded as a class function on \( S_n \), \( \text{cyc}_k \) decomposes as

\[
\text{cyc}_k = \chi^{(n)} + \sum_{i=1}^{\min(k,n-k)} (-1)^{k-i} \chi^{(n-k,1)}
\]

\[
+ \sum_{j=n-k+1}^{k-1} (-1)^{k-j} \chi^{(j,n-k+1)}
\]

**Proof.** If \( \mu \vdash n-k \) is obtained from \( \lambda \vdash n \) by removing a part of size \( k \), we of course have \( \text{cyc}_k(\lambda) = \text{cyc}_k(\mu) + k \). Thus, the symmetric function image of \( \text{cyc}_k \) under the characteristic map is

\[
\sum_{\lambda \vdash n \text{ has } k \text{-parts}} \frac{p_{\lambda} \text{cyc}_k(\lambda)}{z_{\lambda}} = \sum_{\mu \vdash n-k} \frac{p_{\mu} p_k(\text{cyc}_k(\mu) + k)}{z_{\mu} k(\text{cyc}_k(\mu) + 1)} = p_k \sum_{\mu \vdash n-k} \frac{p_{\mu}}{z_{\mu}} = p_k s_{n-k}.
\]

Using [14, 7.72], we may write

\[
p_k s_{n-k} = \sum (-1)^{\text{ht}(\lambda/(n-k))} s_{\lambda},
\]

where the sum is over all partitions \( \lambda \vdash n \) such that \( \lambda/(n-k) \) is a border strip, and \( \text{ht}(\lambda/(n-k)) \) is one less than the number of rows in the strip. Applying the inverse of the characteristic map, this is precisely the desired result. \( \square \)

Theorem 4.1 refines results from [6] and [9]. The former work essentially revolved around the total number of cycles, i.e. the statistic \( \sum_k \text{cyc}_k/k \), whereas the fixed point number \( \text{cyc}_1 \) was studied in the latter.

5. Excedances

Recall that an excedance of \( \pi \in S_n \) is an index \( i \in [n] \) such that \( \pi(i) > i \). Similarly, \( i \) is a weak excedance if \( \pi(i) \geq i \). Let \( \text{exc}(\pi) \) and \( \text{wexc}(\pi) \) denote the number of excedances and weak excedances, respectively, of \( \pi \). Clearly, neither \( \text{exc} \) nor \( \text{wexc} \) is a class function.

**Theorem 5.1.** The mean statistics \( \text{exc} \) and \( \text{wexc} \) decompose as

\[
\text{exc} = \frac{n-1}{2} \chi^{(n)} - \frac{1}{2} \chi^{(n-1,1)}
\]

and

\[
\text{wexc} = \frac{n+1}{2} \chi^{(n)} + \frac{1}{2} \chi^{(n-1,1)},
\]

respectively.

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2The proof employs standard terminology from the theory of symmetric functions. We refrain from reproducing the definitions since they are not used elsewhere in the paper. Everything can be found e.g. in [14, Chapter 7].
6. Inversions, descents and the major index

This section treats the mean statistics associated with three commonly occurring permutation statistics. First, we recall their definitions.

Let $\pi \in S_n$. A descent of $\pi$ is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. The number of descents of $\pi$ is denoted by $\text{des}(\pi)$, whereas the major index $\text{maj}(\pi)$ is the sum of all descents of $\pi$.

An index pair $1 \leq i < j \leq n$ forms an inversion of $\pi$ if $\pi(i) > \pi(j)$. Let $\text{inv}(\pi)$ be the number of inversions of $\pi$.

In order to study these statistics all at once, it is convenient to define the quantity

$$I_{\lambda}(i,j) = \# \{ \pi \in C_\lambda \mid \pi(i) > \pi(j) \}$$

for $\lambda \vdash n$ and $1 \leq i < j \leq n$.

**Lemma 6.1.** Suppose $\lambda \vdash n$. Let $p$ and $q$ denote the number of 1-parts and the number of 2-parts, respectively, in $\lambda$. Then,

$$I_{\lambda}(i,j) = \frac{\#C_\lambda}{2} \left( 1 + \frac{2q - p(p - 1)}{n(n - 1)} + \frac{2(j - i - 1)(n - p)(1 - p) - 2q}{n(n - 1)(n - 2)} \right).$$

**Proof.** Fix $\lambda \vdash n$ and indices $1 \leq i < j \leq n$. Consider the following subsets of $C_\lambda$: \[ T_1 = \{ \pi \in C_\lambda \mid \pi(i) = i \text{ and } \pi(j) = j \}, \]

\[ T_2 = \{ \pi \in C_\lambda \mid \pi(i) = j \text{ and } \pi(j) = i \}, \]

\[ T_3 = \{ \pi \in C_\lambda \mid \pi(i) = i \text{ and } \pi(j) = k \text{ for some } i < k < j \}, \]

\[ T_4 = \{ \pi \in C_\lambda \mid \pi(j) = j \text{ and } \pi(i) = k \text{ for some } i < k < j \}, \]

\[ T_5 = \{ \pi \in C_\lambda \mid \pi(i) = j \text{ and } \pi(j) = k \text{ for some } i < k < j \}, \]

\[ T_6 = \{ \pi \in C_\lambda \mid \pi(j) = i \text{ and } \pi(i) = k \text{ for some } i < k < j \}, \]

\[ T_7 = C_\lambda \setminus (T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6). \]

Thus, $C_\lambda$ is the disjoint union $C_\lambda = T_1 \cup \cdots \cup T_7$.

Let $f_{i,j} : C_\lambda \to C_\lambda$ be the involution $\pi \mapsto (i \ j)\pi(i \ j)$, where $(i \ j)$ denotes the transposition which interchanges $i$ and $j$. Then, $f_{i,j}$ restricts to an involution $T_7 \to T_7$. This restriction has no fixed points since $T_1 \cup T_2$ is the fixed point set of $f_{i,j}$. Moreover, for a permutation $\pi \in T_7$, $(i \ j)$ is an inversion if and only if it is not an inversion of $f_{i,j}(\pi)$.

Observing that $(i \ j)$ is an inversion for all $\pi \in T_2 \cup T_5 \cup T_6$, whereas it is a non-inversion for all $\pi \in T_1 \cup T_3 \cup T_4$, we thus obtain

$$I_{\lambda}(i,j) = \#T_2 + \#T_5 + \#T_6 + \frac{\#T_7}{2} \quad \text{for } \lambda \vdash n.$$
Computing
\[ \#T_1 = \frac{p(p - 1)\#C_\lambda}{n(n - 1)}, \]
\[ \#T_2 = \frac{2q\#C_\lambda}{n(n - 1)}, \]
\[ \#T_3 = \#T_4 = \frac{p(n - p)(j - i - 1)\#C_\lambda}{n(n - 1)(n - 2)}, \]
\[ \#T_5 = \#T_6 = \frac{(n - p - 2q)(j - i - 1)\#C_\lambda}{n(n - 1)(n - 2)} \]
yields the desired result. □

Next, we exploit the fact that several familiar permutation statistics are obtained by taking appropriate sums of \( I_{\lambda(i, j)} \).

**Theorem 6.2.** The mean statistics associated with des, maj and inv can be written as the following linear combinations of irreducible characters:
\[ \overline{\text{des}} = \frac{n - 1}{2} \chi(n) - \frac{1}{n} \chi(n-1) - \frac{1}{n} \chi(n-2,1), \]
\[ \overline{\text{maj}} = \frac{n - 1}{4} \chi(n) - \frac{1}{2} \chi(n-1) - \frac{1}{2} \chi(n-2,1), \]
\[ \overline{\text{inv}} = \frac{n(n - 1)}{6} \chi(n) - \frac{n + 1}{6} \chi(n-1) - \frac{1}{6} \chi(n-2,1). \]

**Proof.** Let \( \lambda \vdash n \). Applying Lemma 6.1, we obtain the identities
\[ \overline{\text{des}}(\lambda) = \sum_{i=1}^{n-1} \frac{I_{\lambda(i, i+1)}}{\#C_\lambda} = \frac{n - 1}{2} + q - \frac{1}{n} \binom{p}{2}, \]
\[ \overline{\text{maj}}(\lambda) = \sum_{i=1}^{n-1} \frac{I_{\lambda(i, i+1)}}{\#C_\lambda} = \frac{n(n - 1)}{4} + q - \frac{1}{2} \binom{p}{2}, \]
\[ \overline{\text{inv}}(\lambda) = \sum_{1 \leq i < j \leq n} \frac{I_{\lambda(i, j)}}{\#C_\lambda} = \frac{n(n - 1)}{4} - \frac{p(p - 1)}{12} - \frac{n(n - 1)}{6} + q \]
where \( p \) and \( q \) are as in Lemma 6.1. That these equations are equivalent to the asserted ones is readily shown using Lemma 2.2. □

7. Some examples

If \( \Gamma \) is a union of conjugacy classes, we may combine Theorem 3.1 with Theorem 3.2 in order to explicitly compute \( E_\Gamma(s, t) \) for any of the permutation statistics \( s \) which were studied in the previous sections. We conclude the paper with a few sample computations of this kind.

**Example 7.1.** Suppose \( \Gamma = T \subseteq S_n \) is the set of transpositions. The contents \( c_\lambda \) considered in Lemma 6.3 make an appearance as Theorem 5.1 specializes to
\[ \eta_{\lambda}^{(t)} = \frac{1}{n!} \left( \frac{n!}{f_\lambda} \chi((2,1^{n-2})) \right)^t f_\lambda = \frac{c_\lambda^t}{n!}. \]
Combining this with the decompositions found in Theorem 5.1 and Theorem 6.2, we may invoke Theorem 3.1 and show that a product of \( t \) random transpositions

\[ E_\Gamma(s, t) = \frac{1}{n!} \left( \frac{n!}{f_\lambda} \chi((2,1^{n-2})) \right)^t f_\lambda = \frac{c_\lambda^t}{n!}. \]
has the following expected values of the (weak) excedance number, descent number, major index and inversion number, respectively:

$$
\mathbb{E}_T(\text{exc}, t) = \frac{n-1}{2} \left( 1 - \left( 1 - \frac{2}{n-1} \right)^t \right),
$$

$$
\mathbb{E}_T(\text{wexc}, t) = \frac{n+1}{2} \left( 1 + \frac{n-1}{n+1} \left( 1 - \frac{2}{n-1} \right)^t \right),
$$

$$
\mathbb{E}_T(\text{des}, t) = \frac{n-1}{2} \left( 1 - \frac{2}{n} \left( 1 - \frac{2}{n-1} \right)^t \right) - \frac{n-2}{n} \left( 1 - \frac{4}{n-1} \right)^t,
$$

$$
\mathbb{E}_T(\text{maj}, t) = \frac{n(n-1)}{4} \left( 1 - \frac{2}{n} \left( 1 - \frac{2}{n-1} \right)^t \right) - \frac{2(n+1)}{3n} \left( 1 - \frac{2}{n-1} \right)^t - \frac{n-2}{n} \left( 1 - \frac{4}{n-1} \right)^t.
$$

The formula for $\mathbb{E}_T(\text{inv}, t)$ obtained in Example 7.1 was previously found by Sjöstrand [13, Theorem 5.1].

We have refrained from stating the explicit formula for $\mathbb{E}_T(\text{cyc}_k, t)$ which follows from Theorem 3.1. With $k = 1$, it recovers results on fixed points from [9], whereas the sum over all $k$ leads to expected cycle numbers that were computed in [8]. Also, note that $\mathbb{E}_T(\text{cyc}_n, t)/n$ is nothing but the probability that a product of $t$ random transpositions forms an $n$-cycle. From that probability, one easily derives the number of factorisations of an $n$-cycle into $t$ transpositions. Working out the details, one recovers the formula counting such factorisations which appears in Jackson [8].

**Example 7.2.** Let $\Gamma = C_{(n)} \subseteq S_n$ be the set of $n$-cycles. Assuming now that $t > 0$, Theorem 7.2 and Lemma 2.7 show that $b^{(t)}_{\lambda} = 0$ unless we have a hook shape $\lambda = (n-j, 1^j)$. Moreover,

$$
b^{(t)}_{(n-j, 1^j)} = \frac{1}{n^t} \binom{(n-1)!((-1)^j)}{(n-1)^j} \binom{n-1}{j}.
$$

Suppose $k < n$ and consider the statistic $\text{cyc}_k$ which counts elements that belong to $k$-cycles. According to Theorem 4.1, exactly two terms (namely $\chi^{(n)}$ and $(-1)^{k-1} \chi^{(n-k, 1^k)}$) which correspond to hook shapes appear in the expansion of $\text{cyc}_k$. Applying Theorem 7.2, we deduce that the expected number of $k$-cycles in a product of $t > 0$ random $n$-cycles in $S_n$ is

$$
\frac{1}{k} \mathbb{E}_{C_{(n)}}(\text{cyc}_k, t) = \frac{1}{k} + \frac{(-1)^{k(t+1)-1}}{k^{n-k}}.
$$

Let us, finally, consider an example where $\Gamma$ does not consist of a single conjugacy class.

**Example 7.3.** Suppose $\Gamma$ is any union of conjugacy classes in which every permutation has exactly one fixed point. In this case, Theorem 7.2 in conjunction with Lemma 2.7 shows that $b^{(t)}_{(n-1, 1)} = 0$ for all $t > 0$. Thus, by Theorem 7.2 and Theorem 5.4, the expected number of excedances of a product of $t > 0$ elements from $\Gamma$ is

$$
\mathbb{E}_T(\text{exc}, t) = \frac{n-1}{2}.
$$
whereas for weak exceedances one obtains

$$\mathbb{E}_t(\text{weex}, t) = \frac{n + 1}{2},$$

independently of $t$.

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