The Dyson equation with linear self-energy: spectral bands, edges and cusps

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We study the unique solution \( m \) of the Dyson equation

\[
-m(z)^{-1} = z1 - a + S[m(z)]
\]
on a von Neumann algebra \( \mathcal{A} \) with the constraint \( \text{Im} \, m \geq 0 \). Here, \( z \) lies in the complex upper half-plane, \( a \) is a self-adjoint element of \( \mathcal{A} \) and \( S \) is a positivity-preserving linear operator on \( \mathcal{A} \). We show that \( m \) is the Stieltjes transform of a compactly supported \( \mathcal{A} \)-valued measure on \( \mathbb{R} \). Under suitable assumptions, we establish that this measure has a uniformly \( 1/3 \)-Hölder continuous density with respect to the Lebesgue measure, which is supported on finitely many intervals, called bands. In fact, the density is analytic inside the bands with a square-root growth at the edges and internal cubic root cusps whenever the gap between two bands vanishes. The shape of these singularities is universal and no other singularity may occur. We give a precise asymptotic description of \( m \) near the singular points. These asymptotics play a key role in the companion paper on the Tracy-Widom universality for the edge eigenvalue statistics for correlated random matrices [8]. We also show that the spectral mass of the bands is topologically rigid under deformations and we conclude that these masses are quantized in some important cases.

**Keywords:** Dyson equation, positive operator-valued measure, Stieltjes transform, band rigidity.

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1. Introduction

An important task in random matrix theory is to determine the eigenvalue distribution of a random matrix as its size tends to infinity. Similarly, in free probability theory, the scalar-valued distribution of operator-valued semicircular elements is of particular interest. In both cases, the distribution can be obtained from a **Dyson equation**

\[
-m(z)^{-1} = z1 - a + S[m(z)]
\]
on some von Neumann algebra \( \mathcal{A} \) with a unit \( 1 \) and a tracial state \( \langle \cdot \rangle \). Here, \( z \in \mathbb{H} := \{ w \in \mathbb{C} : \text{Im} \, w > 0 \} \), \( a = a^* \in \mathcal{A} \) and \( S : \mathcal{A} \to \mathcal{A} \) is a positivity-preserving linear operator. There is a unique solution \( m : \mathbb{H} \to \mathcal{A} \) of (1.1) under the assumption that \( \text{Im} \, m(z) := (m(z) - \overline{m(z)})/2i \) is a strictly positive element of \( \mathcal{A} \) for all \( z \in \mathbb{H} \) [23]. For suitably chosen \( a \) and \( S \) as well as \( \mathcal{A} \), this solution characterizes the distributions in the applications mentioned above. In fact, in both cases, the distribution will be the measure \( \rho \) on \( \mathbb{R} \) whose Stieltjes transform is given by \( z \mapsto \langle m(z) \rangle \). The measure \( \rho \) is called the **self-consistent density of states** and its support is the **self-consistent spectrum**. This terminology stems from the physics literature on the Dyson equation, where \( z \) is often called **spectral parameter** and \( S \) is the **self-energy operator**. The linearity of \( S \) is a distinctive feature of our setup.

We first explain the connection between the eigenvalue density of a large random matrix and the Dyson equation. Let \( H \in \mathbb{C}^{n \times n} \) be a \( \mathbb{C}^{n \times n} \)-valued random variable, \( n \in \mathbb{N} \), such that \( H = H^* \). A central objective is the analysis of the **empirical spectral measure** \( \mu_H = n^{-1} \sum_{i=1}^{n} \delta_{\lambda_i} \), or its expectation, the **density of states**, for large \( n \), where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( H \). Clearly, \( n^{-1} \text{Tr}(H - z)^{-1} \) is the Stieltjes transform of \( \mu_H \) at \( z \in \mathbb{H} \). Therefore, the resolvent \( (H - z)^{-1} \) is commonly studied to obtain information about \( \mu_H \). In fact, for many random matrix ensembles, the resolvent \( (H - z)^{-1} \) is well-approximated for large \( n \) by the solution

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m(z) of the Dyson equation (1.1). Here, we choose \( A = \mathbb{C}^{n \times n} \) equipped with the operator norm induced by the Euclidean distance on \( \mathbb{C}^n \) and the normalized trace \( \langle \cdot \rangle = n^{-1} \text{Tr}(\cdot) \) as tracial state as well as

\[
a := \mathbb{E} H, \quad S[x] := \mathbb{E}[(H - a)x(H - a)], \quad x \in \mathbb{C}^{n \times n}.
\]

(1.2)

If \( (H - z)^{-1} \) is well-approximated by \( m(z) \) for large \( n \) then \( \mu_H \) will be well-approximated by the deterministic measure \( \mu \), whose Stieltjes transform is given by \( z \mapsto \langle m(z) \rangle \). The importance of the Dyson equation (1.1) for random matrix theory has been realized by many authors on various levels of generality [10, 12, 18, 24, 31, 40], see also the monographs [17, 29] and the more recent works [3, 4, 6, 7, 9, 14, 21, 25].

Secondly, we relate the Dyson equation to free probability theory by noticing that the Cauchy transform of a shifted operator-valued semicircular element is given by \( m \). More precisely, let \( B \) be a unital \( C^* \)-algebra, \( A \subset B \) be a \( C^* \)-subalgebra with the same unit \( 1 \) and \( E: B \to A \) is a conditional expectation (we refer to Chapter 9 in [28] for notions from free probability theory). Pick an \( a = a^* \in A \) and an operator-valued semicircular element \( s = s^* \in B \) then \( G(z) := E[(z - s - a)^{-1}] \) is the Cauchy-transform of \( s + a \). In this case, \( m(z) = -G(z) \) satisfies (1.1) with \( S[x] := E[ssx] \) for all \( x \in A \) [36]. If \( A \) is a von Neumann algebra with a tracial state, then our results yield information about the scalar-valued distribution \( \rho = \rho_{s+a} \) of \( s + a \) with respect to this state. The study of qualitative regularity properties for this distribution has a long history in free probability. For example, the question of whether \( \rho \) has atoms or not is intimately related to non-commutative identity testing (see [16, 26] and references therein) and the notions of free entropy and Fischer information (see [35, 37] and the survey [39]). We also refer to the recent preprint [27], where the distribution of rational functions in noncommutative random variables is studied with the help of linearization ideas from [20, 19] and [22]. Under certain assumptions, our results provide extremely detailed information about the regularity properties of \( \rho \), thus complementing these more general insights. In particular, we show that \( \rho_s \) is absolutely continuous with respect to the Lebesgue measure away from zero for any operator-valued semicircular element \( s \). For other applications of the Dyson equation (1.1) in free probability theory, we refer to [23, 33, 36, 38] and the recent monograph [28].

In this paper, we analyze the regularity properties of the self-consistent density of states \( \rho \) in detail. More precisely, under suitable assumptions on \( S \), we show that the boundedness of \( m \) already implies that \( \rho \) has a \( 1/3 \)-Hölder continuous density \( \rho(\tau) \) with respect to the Lebesgue measure. We provide a broad class of models for which the boundedness of \( m \) is ensured. Furthermore, the set where the density is positive, \( \{ \tau : \rho(\tau) > 0 \} \), splits into finitely many connected components, called bands. The density is real-analytic inside the bands with a square root growth behavior at the edges. If two bands touch, however, a cubic root cusp emerges. These are the only possible types of singularities. In fact, \( m(z) \) is the Stieltjes transform of a positive operator-valued measure \( v \) and we establish the properties mentioned above for \( v \) as well. We also provide a novel formula for the masses that \( \rho \) assigns to the bands. We use it to infer a certain quantization of the band masses that we call band rigidity, because it is invariant under small perturbations of the data \( a \) and \( S \) of the Dyson equation. In particular, we extend a quantization result from [19] and [32] to cover limits of Kronecker random matrices. We remark that in the context of random matrices the analogous phenomenon was coined as “exact separation of eigenvalues” in [11].

In the commutative setup, the band structure and singularity behavior of the density have been obtained in [1, 2], where a detailed analysis of the regularity of \( \rho \) was initiated. In the special noncommutative situation \( A = \mathbb{C}^{n \times n} \) and \( \langle \cdot \rangle = n^{-1} \text{Tr}(\cdot) \), it has been shown that \( \rho \) is Hölder-continuous and real-analytic wherever it is positive [4]. The main novelty of the current work is to give an effective regularity analysis for the general noncommutative case, including a precise description of all singularities, i.e., edges and cusps. One of the main applications is the proof of the eigenvalue rigidity on optimal scale and the Tracy-Widom universality of the local spectral statistics near the spectral edges for random matrices with general correlation structure [8].

The key strategy behind the current paper as well as its predecessors [1, 2, 4] is a refined stability analysis of the Dyson equation (1.1) against small perturbations. It turns out that the equation is stable in the bulk regime, i.e., where \( \rho(\text{Re } z) \) is separated away from zero, but is unstable near the points, where the density vanishes. Even the stability in the bulk requires an unconventional idea; it relies on rewriting the stability operator, i.e., the derivative of the Dyson equation with respect to \( m \), through the use of a positivity-preserving symmetric map, called the saturated self-energy operator, \( F \). We then extract information on the spectral gap of \( F \) by a Perron-Frobenius argument using the positivity of \( \text{Im } m \) [1, 2]. In the non-commutative setup this transformation was based on a novel balanced polar decomposition formula [4]. In the small density regime, in particular near the edges, the stability deteriorates due to an unstable direction, which is related to the Perron-Frobenius eigenvector of \( F \). The analysis boils down to a scalar quantity, \( \Theta \), the overlap between the solution and the unstable direction. For the commutative case in [1, 2], it is shown that \( \Theta \) approximately satisfies a cubic equation. The structural property of this cubic equation is its stability, i.e., that the coefficients of the
cubic and quadratic terms do not simultaneously vanish. This guarantees that higher order terms are negligible and the order of any singularity is either cubic root or square root.

Now we synthesize both analyses in the previous works to study the small density regime in the most general setup. The major obstacle is the noncommutativity that already substantially complicated the bulk analysis in [4] but there the saturated self-energy operator, $F$, governed all estimates. However, near the edges the unstable direction is identified via the top eigenvector of a non-symmetric operator that coincides with the symmetric $F$ only in the commutative case. Thus we need to perform a non-symmetric perturbation expansion that requires precise control on the resolvent of the non-selfadjoint stability operator in the entire complex plane. We still work with a cubic equation for $\Theta$, but the analysis of its coefficients is considerably more involved. Along all estimates, the noncommutativity is a permanent enemy; in some cases it can be treated perturbatively, but for the most critical parts new non-perturbative proofs are needed. Most critically, the stability of the cubic equation is proven with a new method.

Another novelty of the current paper, in addition to handling the non-commutativity and lack of symmetry, is that we present the cubic analysis in a conceptually clean way that will be used in future works. Our analysis strongly suggests that our cubic equation for $\Theta$ is the key to any detailed singularity analysis of Dyson-type equations and its remarkable structure is responsible for the universal behavior of the singularities in the density.

2. Main results

Let $\mathcal{A}$ be a finite von Neumann algebra with unit $\mathds{1}$ and norm $\|\cdot\|$. We recall that a von Neumann algebra $\mathcal{A}$ is called finite if there is a state $\langle \cdot : \mathcal{A} \to \mathbb{C}$ which is (i) tracial, i.e., $(xy) = (yx)$ for all $x, y \in \mathcal{A}$, (ii) faithful, i.e., $\langle x^* x \rangle = 0$ for some $x \in \mathcal{A}$ implies $x = 0$, and (iii) normal, i.e., continuous with respect to the weak* topology. In the following, $\langle \cdot : \mathcal{A} \to \mathbb{C}$ through

$$ \langle x, y \rangle := \langle x^* y \rangle $$

for $x, y \in \mathcal{A}$. The induced norm is denoted by $\|x\|_2 := \langle x, x \rangle^{1/2}$ for $x \in \mathcal{A}$. Clearly, $\|x\|_2 \leq \|x\|$ for all $x \in \mathcal{A}$. We follow the convention that small letters are elements of $\mathcal{A}$ while capital letters denote linear operators on $\mathcal{A}$. The spectrum of $x \in \mathcal{A}$ is denoted by $\text{Spec} x$, i.e., $\text{Spec} x = \mathbb{C} \setminus \{z \in \mathbb{C} : (x - z)^{-1} \in \mathcal{A}\}$.

For an operator $T : \mathcal{A} \to \mathcal{A}$, we will work with three norms. We denoted these norms by $\|T\|$, $\|T\|_2$ and $\|T\|_{2 \to \|\cdot\|}$ if $T$ is considered as an operator $(\mathcal{A}, \|\cdot\|) \to (\mathcal{A}, \|\cdot\|)$, $(\mathcal{A}, \|\cdot\|_2) \to (\mathcal{A}, \|\cdot\|_2)$ or $(\mathcal{A}, \|\cdot\|_2) \to (\mathcal{A}, \|\cdot\|)$, respectively.

We denote by $\mathcal{A}_{sa}$ the self-adjoint elements of $\mathcal{A}$, by $\mathcal{A}_+$ the cone of positive definite elements of $\mathcal{A}$, i.e.,

$$ \mathcal{A}_{sa} := \{x \in \mathcal{A} : x^* = x\}, \quad \mathcal{A}_+ := \{x \in \mathcal{A}_{sa} : x > 0\}, $$

and by $\overline{\mathcal{A}_+}$, the $\|\cdot\|$-closure of $\mathcal{A}_+$, the cone of positive semidefinite elements (or positive elements). We now introduce two classes of linear operators on $\mathcal{A}$ that preserve the cone $\overline{\mathcal{A}_+}$. Such operators are called positivity-preserving (or positive) maps. We define

$$ \Sigma := \{S : \mathcal{A} \to \mathcal{A} : S \text{ is linear, symmetric wrt. (2.1) and preserves the cone } \overline{\mathcal{A}_+}\}, $$

$$ \Sigma_{\text{flat}} := \{S \in \Sigma : \varepsilon \mathds{1} \leq \inf_{x \in \mathcal{A}_+} \frac{S[x]}{(x)} \leq \sup_{x \in \mathcal{A}_+} \frac{S[x]}{(x)} \leq \varepsilon^{-1}\mathds{1} \text{ for some } \varepsilon > 0\}. $$

Moreover, if $S : \mathcal{A} \to \mathcal{A}$ is a positivity-preserving operator, then $S$ is bounded, i.e., $\|S\|$ is finite (see e.g. Proposition 2.1 in [30]).

Let $a \in \mathcal{A}_{sa}$ be a self-adjoint element and $S \in \Sigma$. For the data pair $(a, S)$, we consider the associated Dyson equation

$$ -m(z)^{-1} = z \mathds{1} - a + S[m(z)], $$

with spectral parameter $z \in \mathbb{H} := \{w \in \mathbb{C} : \text{Im } w > 0\}$, for a function $m : \mathbb{H} \to \mathcal{A}$ such that its imaginary part is positive definite,

$$ \text{Im } m(z) = \frac{1}{2i}(m(z) - m(z)^*) \in \mathcal{A}_+. $$

There always exists a unique solution $m$ to the Dyson equation (2.3) satisfying $\text{Im } m(z) \in \mathcal{A}_+$ [23]. Moreover, this solution is holomorphic in $z$ [23]. For Dyson equations in the context of renormalization theory, $a$ is called the bare matrix and $S$ the self-energy (operator). In applications to free probability theory, $S$ is usually denoted by $\eta$ and called the covariance mapping or covariance matrix [28].
We now introduce positive operator-valued measures with values in \( \mathcal{X}_+ \). If \( v \) maps Borel sets on \( \mathbb{R} \) to elements of \( \mathcal{X}_+ \) such that \( \langle x, v(\cdot)x \rangle \) is a positive measure for all \( x \in \mathcal{A} \) then we say that \( v \) is a measure on \( \mathbb{R} \) with values in \( \mathcal{X}_+ \) or an \( \mathcal{X}_+ \)-valued measure on \( \mathbb{R} \).

First, we list a few propositions that are necessary to state our main theorem. They will be proven in Section 3, Section 4.2 and Section 4.3, respectively.

**Proposition 2.1** (Stieltjes transform representation). Let \((a, S) \in \mathcal{A}_{sa} \times \Sigma \) be a data pair and \( m \) the solution to the associated Dyson equation. Then there exists a measure \( v \) on \( \mathbb{R} \) with values in \( \mathcal{X}_+ \) such that \( v(\mathbb{R}) = 1 \) and

\[
m(z) = \int_{\mathbb{R}} \frac{v(\mathrm{d}r)}{\tau - z} \tag{2.4}
\]

for all \( z \in \mathbb{H} \). The support of \( v \) and the spectrum of \( a \) satisfy the following inclusions

\[
\text{supp } v \subset \text{Spec } a + [-2\|S\|^{1/2}, 2\|S\|^{1/2}],
\]

\[
\text{Spec } a \subset \text{supp } v + [-\|S\|^{1/2}, \|S\|^{1/2}].
\]

Furthermore, if \( a = 0 \) then, for any \( z \in \mathbb{H} \), \( m(z) \) satisfies the bound

\[
\|m(z)\|_2 \leq \frac{2}{|z|}.
\]

Our goal is to obtain regularity results for the measure \( v \). We first present some regularity results on the self-consistent density of states introduced in the following definition.

**Definition 2.2** (Density of states). Let \((a, S) \in \mathcal{A}_{sa} \times \Sigma \) be a data pair, \( m \) the solution to the associated Dyson equation, (2.3), and \( v \) the \( \mathcal{X}_+ \)-valued measure of Proposition 2.1. The positive measure \( \rho = \langle v \rangle \) on \( \mathbb{R} \) is called the self-consistent density of states or short density of states.

We have \( \text{supp } \rho = \text{supp } v \) due to the faithfulness of \( \langle \cdot \rangle \). Moreover, the Stieltjes transform of \( \rho \) is given by \( \langle m \rangle \) since, by (2.3), for any \( z \in \mathbb{H} \), we have

\[
\langle m(z) \rangle = \int_{\mathbb{R}} \frac{\rho(\mathrm{d}\tau)}{\tau - z}.
\]

**Proposition 2.3** (Regularity of density of states). Let \((a, S) \) be a data pair with \( S \in \Sigma_{\text{flat}} \) and \( \rho_{a, S} \) the corresponding density of states. Then \( \rho_{a, S} \) has a uniformly Hölder-continuous, compactly supported density with respect to the Lebesgue measure,

\[
\rho_{a, S}(\mathrm{d}\tau) = \rho_{a, S}(\tau) \mathrm{d}\tau.
\]

Furthermore, there exists a universal constant \( c > 0 \) such that the function \( \rho: \mathcal{A}_{sa} \times \Sigma_{\text{flat}} \times \mathbb{R} \to [0, \infty) \), \((a, S, \tau) \mapsto \rho_{a, S}(\tau)\) is locally Hölder-continuous with Hölder exponent \( c \) and analytic whenever it is positive, i.e., for any \((a, S, \tau) \in \mathcal{A}_{sa} \times \Sigma_{\text{flat}} \times \mathbb{R} \) such that \( \rho_{a, S}(\tau) > 0 \) the function \( \rho \) is analytic in a neighbourhood of \((a, S, \tau)\). Here, \( \mathcal{A}_{sa} \) and \( \Sigma_{\text{flat}} \) are equipped with the metrics induced by \( \| \cdot \| \) on \( \mathcal{A} \) and its operator norm on \( \mathcal{A} \to \mathcal{A} \), respectively.

The following proposition is stated under a boundedness assumption on \( m \) (see (2.7) below). In the random matrix context, in Section 9, we provide a sufficient condition for this assumption to hold purely expressed in terms of \( a \) and \( S \) for a large class of random matrix models.

**Proposition 2.4** (Regularity of \( m \)). Let \((a, S) \) be a data pair with \( S \in \Sigma_{\text{flat}} \) and \( m \) the solution to the associated Dyson equation. Suppose that for a nonempty open interval \( I \subset \mathbb{R} \) we have

\[
\limsup_{\eta \downarrow 0} \sup_{\tau \in I} \|m(\tau + i\eta)\| < \infty. 
\]

Then \( m \) has a 1/3-Hölder continuous extension (also denoted by \( m \)) to any closed interval \( I' \subset I \), i.e.,

\[
\sup_{z_1, z_2 \in I \times [0, \infty)} \frac{\|m(z_1) - m(z_2)\|}{|z_1 - z_2|^{1/3}} < \infty.
\]

Moreover, \( m \) is real-analytic in \( I \) wherever \( \rho \) is positive.

The purpose of the interval \( I \) in Proposition 2.4 (see also Theorem 2.5 below) is to demonstrate the local nature of these statements and their proofs; if \( m \) is bounded on \( I \) in the sense of (2.7) then we will prove
regularity of \( m \) and later its behaviour close to singularities on a genuine subinterval \( I' \subset I \). At first reading, the reader may ignore this subtlety and assume \( I' = I = \mathbb{R} \).

In Proposition 4.7 below, we provide a quantitative version of (2.8) under slightly weaker conditions than those of Proposition 2.4.

For the following main theorem, we remark that if \( m \) has a continuous extension to an interval \( I \subset \mathbb{R} \) then the restriction of the measure \( v \) from (2.4) to \( I \) has a density with respect to the Lebesgue measure, i.e., for each Borel set \( A \subset I \), we have
\[
v(A) = \frac{1}{\pi} \int_A \text{Im} m(\tau) \, d\tau. \tag{2.9}
\]
The existence of a continuous extension can be guaranteed by (2.7) in Proposition 2.4.

**Theorem 2.5** (Im \( m \) close to its singularities). Let \( (a, S) \) be a data pair with \( S \in \Sigma_{\text{flat}} \) and \( m \) the solution to the associated Dyson equation. Suppose \( m \) has a continuous extension to a nonempty open interval \( I \subset \mathbb{R} \). Then any \( \tau_0 \in \text{supp} \rho \cap I \) with \( \rho(\tau_0) = 0 \) belongs to exactly one of the following cases:

- **Edge:** The point \( \tau_0 \) is a left/right edge of the density of states, i.e., there is some \( \varepsilon > 0 \) such that \( v(\tau_0 \mp \varepsilon) = 0 \) for \( \omega \in [0, \varepsilon] \) and for some \( v_0 \in \mathcal{A}_+ \) we have
  \[
  \text{Im} m(\tau_0 \pm \omega) = v_0 \omega^{1/2} + O(\omega), \quad \omega \downarrow 0.
  \]
  Moreover, \( \text{supp} \rho \cap I = \text{supp} v \cap I \) is a finite union of closed intervals with nonempty interior.

  Theorem 2.5 is a simplified version of our more detailed and quantitative Theorem 7.1 below. We can treat all small local minima of \( \rho \) on \( \text{supp} \rho \cap I \) – not only those ones, where \( \rho \) vanishes – and provide precise expansions corresponding to those in Theorem 2.5 which are valid in some neighbourhood of \( \tau_0 \). Moreover, the coefficients \( v_0 \) in Theorem 2.5 are bounded from above and below in terms of the basic parameters of the model.

Finally, we present our quantization result.

**Proposition 2.6** (Band mass formula). Let \( (a, S) \in \mathcal{A}_a \times \Sigma \) be a data pair and \( m \) the solution to the associated Dyson equation, (2.3). We assume that there is a constant \( C > 0 \) such that \( S[x] \leq C \langle x \rangle \mathbf{1} \) for all \( x \in \mathcal{A}_+ \). Then we have

- (i) For each \( \tau \in \mathbb{R} \setminus \text{supp} \rho \), there is \( m(\tau) \in \mathcal{A}_a \) such that \( \lim_{\eta \downarrow 0} \| m(\tau + i \eta) - m(\tau) \| = 0 \). Moreover, \( m(\tau) \) determines the mass of \( (-\infty, \tau) \) and \( (\tau, \infty) \) with respect to \( \rho \) in the sense that
  \[
  \rho((-\infty, \tau)) = \langle \mathbf{1}_{(-\infty,0)}(m(\tau)) \rangle, \tag{2.10}
  \]
  where \( \mathbf{1}_{(-\infty,0)} \) denotes the characteristic function of the interval \( (-\infty,0) \).

- (ii) If \( \pi : \mathcal{A} \to \mathbb{C}^{n \times n} \) is a faithful representation such that \( \langle x \rangle = n^{-1} \text{Tr}(\pi(x)) \) for all \( x \in \mathcal{A} \) and \( J \subset \text{supp} \rho \) is a connected component of \( \text{supp} \rho \) then we have
  \[
  n \rho(J) \in \{1, \ldots, n\}.
  \]

In particular, \( \text{supp} \rho \) has at most \( n \) connected components.

We will prove Proposition 2.6 in Section 8 below. A result similar to part (ii) has been obtained by a different method in [19], see also [32]. In fact, we will use the band mass formula, (2.10), in Corollary 9.4 below to strengthen the quantization result in (ii) for a large class of random matrices (Kronecker matrices, see Section 9).

### 2.1. Examples

We now present some examples that show the different types of singularities described by Theorem 2.5. These examples are obtained by considering the Dyson equation, (2.3), on \( \mathbb{C}^{n \times n} \) with \( \langle \cdot \rangle = n^{-1} \text{Tr} \) for large \( n \) and choosing \( a = 0 \) as well as \( S = S_\alpha \), where
\[
S_\alpha[x] := \frac{1}{n} \text{diag}(r_\alpha \text{diag}(x))
\]

Figure 1: Structure of \( r_\alpha \in \mathbb{C}^{n \times n} \).
for any $x \in \mathbb{C}^{n \times n}$. Here, for $x \in \mathbb{C}^{n \times n}$, $\text{diag}(x)$ denotes the vector of diagonal entries, $r_\alpha \in \mathbb{C}^{n \times n}$ is the symmetric block matrix from Figure 1 with $\alpha \in (0, \infty)$. All elements in each block are the indicated constants. Moreover, we write $\text{diag}(v)$ with $v \in \mathbb{C}^n$ to denote the diagonal matrix in $\mathbb{C}^{n \times n}$ with $v$ on its diagonal. In fact, this example can also be realized on $\mathbb{C}^2$ with entrywise multiplication. Here, we choose $(\langle x_1, x_2 \rangle) = \delta x_1 + (1 - \delta)x_2$, where $\delta$ is the relative block size of the small block in the definition of $r_\alpha$. In this setup on $\mathbb{C}^2$, the Dyson equation can be written as

$$-egin{pmatrix} m_1^{-1} \\ m_2^{-1} \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \end{pmatrix} + R_\alpha \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad R_\alpha = \begin{pmatrix} \alpha \delta & 1 - \delta \\ \delta & \alpha(1 - \delta) \end{pmatrix}$$

(2.11)

for $(m_1, m_2) \in \mathbb{C}^2$. We remark that $R_\alpha$ is symmetric with respect to the scalar product (2.1) induced by $\langle \cdot, \cdot \rangle$. Figure 2 contains the graphs of some self-consistent densities of states $\rho$ obtained from (2.11) for $\delta = 0.1$ and different values of $\alpha$. As the self-consistent density of states is symmetric around zero in these cases, only the internal nonzero local minimum in Figure 2 (c). This nonzero local minimum is covered by Theorem 7.1 (d) below.

Figure 2: Examples of the self-consistent density of states $\rho$ from (2.11) for $\delta = 0.1$ and several values of $\alpha$.

2.2. Main ideas of the proofs

In this subsection, we informally summarize several key ideas in the proofs of Proposition 2.4 and Theorem 2.5.

**Hölder-continuity of $m$.** To simplify the notation, we assume in this outline that $\|m(z)\| \lesssim 1$ for all $z \in \mathbb{H}$, i.e., we assume (2.7) with $I = \mathbb{R}$. We first show that $\text{Im} m(z)$ is $1/3$-Hölder continuous and then conclude the same regularity for $m = m(z)$. To that end, we now control $\partial_\tau \text{Im} m(z)$ by differentiating the Dyson equation, (2.3), with respect to $\tau$. This yields

$$2i\partial_\tau \text{Im} m = (\text{Id} - C_m S)^{-1}[m^2].$$

Here, $\text{Id}$ denotes the identity map on $\mathcal{A}$ and $C_m : \mathcal{A} \to \mathcal{A}$ is defined by $C_m[x] := mxm$ for any $x \in \mathcal{A}$.

In order to control the norm of the stability operator $(\text{Id} - C_m S)^{-1}$, we rewrite it in a more symmetric form. We find an invertible $V$ with $\|V\|, \|V^{-1}\| \lesssim 1$, a unitary operator $U$ and a self-adjoint operator $T$ acting on $\mathcal{A}$ such that

$$\text{Id} - C_m S = V^{-1}(U - T)V.$$  

The Rotation-Inversion Lemma from [2] (see Lemma 4.4 below) is designed to control $(U - T)^{-1}$ for a unitary operator $U$ and a self-adjoint operator $T$ with $\|T\|_2 \leq 1$. Applying this lemma in our setup yields $\|(\text{Id} - C_m S)^{-1}\| \lesssim \|\text{Im} m\|^{-2}$.

Since $\|m\| \lesssim 1$, we thus obtain

$$\|\partial_\tau \text{Im} m\| \lesssim \|\text{Im} m\|^{-2}.$$  

(2.12)

This bound implies that $(\text{Im} m)^3 : \mathbb{H} \to \mathcal{A}_c$ is uniformly Lipschitz-continuous. Hence, we can extend $\text{Im} m$ to a
1/3-Hölder continuous function on $\mathbb{R} \cup \mathbb{H}$ and we obtain
\[
m(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\text{Im} m(\tau)\,d\tau}{\tau - z}.
\]
This also implies that $m$ is uniformly 1/3-Hölder continuous on $\mathbb{R} \cup \mathbb{H}$. Furthermore, $m(\tau)$ and $\text{Im} m(\tau)$ are real-analytic in $\tau$ around $\tau_0 \in \mathbb{R}$, wherever $\rho(\tau_0)$ is positive.

**Behaviour of $\text{Im} m$ where it is not analytic.** Owing to (2.12), some unstable behaviour of the Dyson equation is expected close to points $\tau_0 \in \mathbb{R}$, where $\text{Im} m(\tau_0)$ is zero or small. In order to analyze this behaviour of $\text{Im} m(\tau)$, we compute $\Delta := m(\tau + \omega) - m(\tau)$ from the Dyson equation, (2.3). Since $m$ has a continuous extension to $\mathbb{R}$, (2.3) holds true for $\tau \in \mathbb{R}$ as well. We evaluate (2.3) at $\tau = \tau_0$ and $\tau = \tau_0 + \omega$ and obtain the quadratic $\mathcal{A}$-valued equation
\[
B[\Delta] = mS[\Delta] + \omega m\Delta + \omega m^2, \quad B := \text{Id} - C_mS.
\]
(2.13)
The blow-up of the stability operator $B^{-1}$ close to $\tau_0$ requires analyzing the contributions of $\Delta$ in the unstable direction of $B^{-1}$ separately. In fact, $B$ possesses precisely one unstable direction denoted by $b$ since we will show that $\|T\|_2$ is a non-degenerate eigenvalue of $T$. We decompose $\Delta$ into $\Delta = \Theta b + r$, where $\Theta$ is the scalar contribution of $\Delta$ in the direction $b$ and $r$ lies in the spectral subspace of $B$ complementary to $b$.

We view $\tau_0$ as fixed and consider $\omega \ll 1$ as the main variable. Projecting (2.13) onto $b$ and its complement yields the scalar-valued cubic equation
\[
\psi \Theta(\omega)^3 + \sigma \Theta(\omega)^2 + \pi \omega = O(|\omega| |\Theta(\omega)| + |\Theta(\omega)|^2)
\]
(2.14)
with two parameters $\psi \geq 0$ and $\sigma \in \mathbb{R}$. In fact, the 1/3-Hölder continuity of $m$ implies $\Theta = O(|\omega|^{1/3})$ and, hence, the right-hand side of (2.14) is indeed of lower order than the terms on the left-hand side. Analyzing (2.14) instead of (2.13) is a more tractable problem since we have reduced a quadratic $\mathcal{A}$-valued equation, (2.13), to the scalar-valued cubic equation, (2.14).

The essential feature of the cubic equation (2.14) is its stability. By this, we mean that there exists a constant $c > 0$ such that
\[
\psi + \sigma^2 \geq c.
\]
This bound will follow from the structure of the Dyson equation and prevents any singularities of higher order than $\omega^{1/2}$ or $\omega^{1/3}$. Obtaining more detailed information about $\Theta$ from (2.14) requires applying Cardano’s formula with an error term. Therefore, we switch to normal coordinates, $(\omega, \Theta(\omega)) \to (\lambda, \Omega(\lambda))$, in (2.14). We will study four normal forms, one quadratic $\Omega(\lambda)^2 + \Lambda(\lambda) = 0$, and three cubics, $\Omega(\lambda)^3 + \Lambda(\lambda) = 0$ and $\Omega(\lambda)^3 \pm 3\Omega(\lambda) + 2\Lambda(\lambda) = 0$, where $\Lambda(\lambda)$ is a perturbation of the identity map $\lambda \mapsto \lambda$. The first case corresponds to the square root singularity of the isolated edge, the second is the cusp. The last two cases describe the situation of almost cusps, see later.

The correct branches in Cardano’s formula are identified with the help of four selection principles for the solution $\Omega(\lambda)$ corresponding to $\Theta$ of the cubic equation in normal form (see SP1 to SP4’ at the beginning of Section 7.1 below). These selection principles are special properties of $\Omega$ which originate from the continuity of $m$, $\text{Im} m \geq 0$ and the Stieltjes transform representation, (2.4), of $m$. Once the correct branch is chosen, we obtain the precise behaviour of $\text{Im} m$ around $\tau_0$, where $\tau_0 \in \text{supp} \rho$ satisfies $\rho(\tau_0) = 0$ or even $\rho(\tau_0) \ll 1$, from Cardano’s formula and careful estimates of $r$ in the decomposition $\Delta = \Theta b + r$ (see Theorem 7.1 below).

3. The solution of the Dyson equation

In this section, we first introduce some notations used in the proof of Proposition 2.1, then prove the proposition and finally give a few further properties of $m$.

For $x, y \in \mathcal{A}$, we introduce the bounded operator $C_{x,y} : \mathcal{A} \to \mathcal{A}$ defined through $C_{x,y}[h] := xhy$ for $h \in \mathcal{A}$. We set $C_x := C_{x,x}$. For $x, y \in \mathcal{A}$, the operator $C_{x,y}$ satisfies the simple relations
\[
C_{x,y}^* = C_{x^*, y^*}, \quad C_{x,y}^{-1} = C_{x^{-1}, y^{-1}},
\]
where $C_{x,y}$ is the adjoint with respect to the scalar product defined in (2.1). Here, the second identity holds if $x$ and $y$ are invertible in $\mathcal{A}$. In fact, $C_{x,y}$ is invertible if and only if $x$ and $y$ are invertible in $\mathcal{A}$.

In the following, we will often use the functional calculus for normal elements of $\mathcal{A}$. As we will explain now,
our setup allows for a direct way to represent $A$ as a subalgebra of the bounded operators on a Hilbert space. Therefore, one can think of the functional calculus being performed on this Hilbert space. The Hilbert space is the completion of $A$ equipped with the scalar product defined in (2.1) and denoted by $L^2$. In order to represent $A$ as subalgebra of the bounded operators $B(L^2)$ on $L^2$, we denote by $\ell_x$ for $x \in A$ the left-multiplication on $L^2$ by $x$, i.e., $\ell_x: L^2 \rightarrow L^2$, $\ell_x(y) = xy$ for $y \in L^2$. The inclusion $A \subset L^2$ and the Cauchy-Schwarz inequality yield the well-definedness of $\ell_x$ and $\ell_x \in B(L^2)$, the bounded linear operators on $L^2$. In fact,

$$A \rightarrow B(L^2), \quad x \mapsto \ell_x$$

defines a faithful representation of $A$ as a von Neumann algebra in $B(L^2)$ [34, Theorem 2.22].

We now introduce balanced polar decomposition of $m$. If $w = w(z) \in A$, $q = q(z) \in A$ and $u = u(z) \in A$ are defined through

$$w := (\text{Im } m)^{-1/2}(\text{Re } m)(\text{Im } m)^{-1/2} + i1, \quad q := |w|^{1/2}(\text{Im } m)^{1/2}, \quad u := \frac{w}{|w|} \quad (3.1)$$

via the spectral calculus of the self-adjoint operator $(\text{Im } m)^{-1/2}(\text{Re } m)(\text{Im } m)^{-1/2}$ then we have

$$m(z) = \text{Re } m(z) + i\text{Im } m(z) = q^* u q. \quad (3.2)$$

Here, $u$ is unitary and commutes with $w$. The decomposition $m = q^* u q$ was already introduced and also called balanced polar decomposition in [4] in the special setting of matrix algebras. The operators $|w|^{1/2}$, $q$ and $u$ correspond to $W$, $W\sqrt{\text{Im } M}$ and $U^*$ in the notation of [4], respectively. With the definitions in (3.1), (2.3) reads as

$$-u^* = q(z - a)q^* + F[u], \quad (3.3)$$

where we introduced the saturated self-energy operator

$$F := C_{q,q^*}SC_{q^*,q}. \quad (3.4)$$

It is positivity-preserving as well as symmetric, $F = F^*$, and corresponds to the saturated self-energy operator $F$ in [4].

Proof of Proposition 2.1. The existence of $v$ will be a consequence of the following lemma which will be proven in Appendix A below.

Lemma 3.1. Let $A$ be a von Neumann algebra with unit $1$ and a tracial, faithful, normal state $\langle \cdot \rangle: A \rightarrow \mathbb{C}$. If $h: \mathbb{H} \rightarrow A$ is a holomorphic function satisfying $\text{Im } h(z) \in A_+$ for all $z \in \mathbb{H}$ and

$$\lim_{\eta \rightarrow \infty} \text{im } h(i\eta) = -1 \quad (3.5)$$

then there exists a unique measure $v: B \rightarrow A$ on the Borel sets $B$ of $\mathbb{R}$ with values in $A_+$ such that

$$h(z) = \int_{\mathbb{R}} \frac{v(dx)}{\tau - z} \quad (3.6)$$

for all $z \in \mathbb{H}$ and $v(\mathbb{R}) = 1$.

In order to apply Lemma 3.1, we have to verify (3.5) for $h = m$. To that end, we take the imaginary part of (2.3) and use $\text{Im } m \geq 0$ as well as $S \in \Sigma$ to conclude

$$-\text{Im } m^{-1}(z) = \text{Im } z 1 + S[\text{Im } m] \geq \text{Im } z 1.$$

Hence, $\|m(z)\| \leq (\text{Im } z)^{-1}$ as for any $x \in A$ we have $\|x\| \leq 1$ if $x$ is invertible and $\text{Im } x^{-1} \geq 1$. Therefore, evaluating (2.3) at $z = i\eta$, $\eta > 0$, and multiplying the result by $m$ from the left yield

$$\text{im } mn(i\eta) = -1 + m(i\eta) a - m(i\eta) S[\text{im } (i\eta)] \rightarrow -1$$

for $\eta \rightarrow \infty$ as $S$ is bounded. Hence, Lemma 3.1 implies the existence of $v$, i.e., the Stieltjes transform representation of $m$ in (2.4).

This representation has the following well-known bounds as a direct consequence (e.g. [1, 4, 7]).
Lemma 3.2. Let \( v \) be the measure in Proposition 2.1 and \( \rho = (v) \). Then, for any \( z \in \mathbb{H} \), we have
\[
\|m(z)\| \leq \frac{1}{\text{dist}(z, \text{supp}\rho)}, \quad \text{Im} m(z) \leq \frac{\text{Im} z}{\text{dist}(z, \text{supp}\rho)^2} 1.
\]
\[
(3.7)
\]
For the proofs of \((2.5a)\) and \((2.5b)\), we refer to the proofs of Proposition 2.1 in [4] and \((3.4)\) in [7] in the matrix setup, the same argument works for our general setup as well.

We now prove \((2.6)\) and hence assume \( a = 0 \). Taking the imaginary part of the Dyson equation, \((3.3)\), yields
\[
\text{Im} u = (\text{Im} z)q^* + F[\text{Im} u] \geq \max\{(\text{Im} z)q^*, F[\text{Im} u]\}.
\]
Thus, \( \text{Im} u \geq (\text{Im} z)\|(q^*)^{-1}\|^{-1} 1 \). We remark that \( q^* \) is invertible since \( \text{Im} m(z) > 0 \) for \( z \in \mathbb{H} \). Therefore, the following Lemma 3.3 with \( h = \text{Im} u/\|\text{Im} u\|_2 \) implies \( \|F\|_2 \leq 1 \).

Lemma 3.3. Let \( T: \mathcal{A} \rightarrow \mathcal{A} \) be a positivity-preserving operator which is symmetric with respect to \((2.1)\). If there are \( h \in \mathcal{A} \) and \( \varepsilon > 0 \) such that \( h \geq \varepsilon 1 \) and \( Th \leq h \) then \( \|T\|_2 \leq 1 \).

Proof. The argument in the proof of Lemma 4.6 in [1] also yields this lemma in our current setup.

We rewrite the Dyson equation \((3.3)\) in the form
\[
q^* = -\frac{1}{z}(u^* + F[u]).
\]
We take the \( \| \cdot \|_2 \)-norm on both sides of \((3.8)\) and use that \( \|u\|_2 = 1 \) (since it is unitary) and \( \|F\|_2 \leq 1 \) to find
\[
\|q^*\|_2 \leq \frac{2}{|z|}.
\]
\[
(3.9)
\]
Then we take the \( \| \cdot \|_2 \)-norm of \( m \) and use the balanced polar decomposition \( m = q^*uq \) again,
\[
\|m\|^2_2 = \langle m^*m \rangle = \langle q^*u^*qq^*uq \rangle = \langle q^*, Cu^*, u[qq^*] \rangle \leq \|qq^*\|^2_2,
\]
where the operator \( Cu^*, u \) is unitary with respect to the scalar product \((2.1)\). With \((3.9)\) we conclude \((2.6)\).

From now on until the end of Section 4.2, we will always assume that \( S \) is flat, i.e., \( S \in \Sigma_{\text{flat}} \) (cf. \((2.2b)\)). In fact, all of our estimates will be uniform in all data pairs \((a, S)\) that satisfy
\[
c_1(x)1 \leq S[x] \leq c_2(x)1, \quad \|a\| \leq c_3
\]
\[
(3.10)
\]
for all \( x \in \mathcal{A}_+ \) with the same fixed constants \( c_1, c_2, c_3 \geq 0 \). Therefore, the constants \( c_1, c_2, c_3 \) from \((3.10)\) are called model parameters and we introduce the following convention.

Convention 3.4 (Comparison relation). Let \( x, y \in \mathcal{A}_+ \). We write \( x \preceq y \) if there is \( c > 0 \) depending only on the model parameters \( c_1, c_2, c_3 \) from \((3.10)\) such that \( cy - x \) is positive definite, i.e., \( cy - x \in \mathcal{A}_+ \). We define \( x \succeq y \) and \( x \sim y \) accordingly. We also use this notation for scalars \( x, y \). Moreover, we write \( x = y + o(\alpha) \) for \( x, y \in \mathcal{A} \) and \( \alpha > 0 \) if \( \|x - y\| \lesssim \alpha \).

We remark that we will choose a different set of model parameters later and redefine \( \sim \) accordingly (cf. Convention 4.6).

Proposition 3.5 (Properties of the solution). Let \((a, S)\) be a data pair satisfying \((3.10)\) and \( m \) be the solution to the associated Dyson equation, \((2.3)\). We have
\[
\|m(z)\|_2 \lesssim 1,
\]
\[
(3.11)
\]
\[
\|m(z)\| \lesssim \frac{1}{(\text{Im} m(z)) + \text{dist}(z, \text{supp}\rho)}.
\]
\[
(3.12)
\]
\[
\|m(z)^{-1}\| \lesssim 1 + |z|,
\]
\[
(3.13)
\]
\[
\langle \text{Im} m(z) \rangle 1 \lesssim \text{Im} m(z) \lesssim (1 + |z|^2)\|m(z)\|^2\langle \text{Im} m(z) \rangle 1
\]
\[
(3.14)
\]
uniformly for \( z \in \mathbb{H} \).

These bounds are immediate consequences of the flatness of \( S \) exactly as in the proof of Proposition 4.2 in [4] using \( \text{supp}\rho = \text{supp}v \) by the faithfulness of \( (\cdot, \cdot) \). We omit the details.

Note that \((3.13)\) implies a lower bound \( \|m(z)\| \gtrsim (1 + |z|)^{-1} \) since \( \|m\|\|m^{-1}\| \gtrsim 1 \).
4. Regularity of the solution and the density of states

In this section, we will prove Proposition 2.3 and Proposition 2.4. Their proofs are based on a bound on the stability operator \((\text{Id} - C_m S)^{-1}\) of the Dyson equation, (2.3), which will be given in Proposition 4.1 below.

4.1. Linear stability of the Dyson equation

For the formulation of the following proposition, we introduce the harmonic extension of the density of states \(\rho\) defined in Definition 2.2 to \(\mathbb{H}\). The harmonic extension at \(z \in \mathbb{H}\) is denoted by \(\rho(z)\) and given by

\[
\rho(z) := \frac{1}{\pi} \langle \text{Im} m(z) \rangle.
\]

**Proposition 4.1** (Linear Stability). There is a universal constant \(C > 0\) such that, for the solution \(m\) to (2.3) associated to any \(a \in A_a\) and \(S \in \Sigma\) satisfying (3.10), we have

\[
\|(\text{Id} - C_m(z) S)^{-1}\|_2 \lesssim 1 + \frac{1}{(\rho(z) + \text{dist}(z, \text{supp} \rho))^2}
\]

uniformly for all \(z \in \mathbb{H}\).

Before proving Proposition 4.1, we will explain how the linear stability yields the Hölder-continuity and analyticity of \(\rho\) in Proposition 2.3. Indeed, assuming that \(m\) depends differentiably on \((z, a, S)\), we can compute the directional derivative \(\nabla_{(\delta, d, D)}\) at \((z, a, S)\) of both sides in (2.3). The result of this computation is

\[
(\text{Id} - C_m S)[\nabla_{(\delta, d, D)} m] = m(\delta - d + D[m]) m.
\]

Using the bound in Proposition 4.1 and \(\rho(z) = \pi^{-1} \langle \text{Im} m(z) \rangle\), we conclude from (3.12) that

\[
|\nabla_{(\delta, d, D)} \rho| \leq \frac{1}{\rho^2 C} (|\delta| + |d|) + |D|
\]

with a possibly larger \(C\). Therefore, it is clear that the control on \((\text{Id} - C_m S)^{-1}\) will be the key input in the proof of Proposition 2.3.

In order to prove Proposition 4.1, we will use the representation

\[
\text{Id} - C_m S = C_q F C_u (C_u^* - F) C_q^{-1}
\]

where \(q, u\) and \(F\) were defined in (3.1) and (3.4), respectively. This representation has the advantage that \(C_u^*\) is unitary and \(F\) is symmetric. Hence, it is much easier to obtain some spectral properties for \(C_u^* - F\) compared to \(\text{Id} - C_m S\). Now, we will first analyze \(q\) and \(F\) in the following two lemmas and then use this knowledge to verify Proposition 4.1.

**Lemma 4.2.** If (3.10) holds true then we have

\[
\langle q(z) \rangle \lesssim (1 + |z|)^{1/2} \|m(z)\|, \quad \|q(z)^{-1}\| \lesssim (1 + |z|)\|m(z)\|^{1/2}
\]

uniformly for \(z \in \mathbb{H}\).

**Proof.** For \(q = q(z)\), we will show below that

\[
\frac{A^{1/2}}{B^{1/2}} \|m(z)^{-1}\|^{-1} \leq q^* q \leq \frac{B^{1/2}}{A^{1/2}} \|m(z)\| \|\mathbf{1}\|
\]

if \(A \leq \text{Im} m(z) \leq B\|\mathbf{1}\|\) for some \(A, B \in (0, \infty)\) and \(z \in \mathbb{H}\). Choosing \(A\) and \(B\) according to (3.14), using the \(C^*\)-property of \(\|\cdot\|, \|q^* q\| = \|q\|^2\), and (3.13), we immediately obtain Lemma 4.2.

For the proof of (4.4), we set \(q := \text{Re} m\) and \(h := \text{Im} m\). Using the monotonicity of the square root, we compute

\[
q^* q = h^{1/2} (1 + h^{-1/2} g h^{-1/2} g^{-1} h^{-1/2})^{1/2} h^{1/2}
\]

\[
\leq A^{-1/2} h^{1/2} (h^{-1/2} (h^2 + g^2) h^{-1/2})^{1/2} h^{1/2}
\]

\[
\leq \|m\| A^{-1/2} h^{1/2}.
\]
Here, we employed $h^{-1} \leq A^{-1}1$ as well as $1 \leq A^{-1}h$ in the first step and $(\text{Re } m)^2 + (\text{Im } m)^2 = (m^*m + mm^*)/2 \leq \|m\|^2$ in the second step. Thus, $h \leq B1$ yields the upper bound in (4.4). Similar estimates using $1 \geq B^{-1}h$ and $\|m^{-1}\|^{-2} \leq (m^*m + mm^*)/2$ prove the lower bound in (4.4) which completes the proof of the lemma.  

**Lemma 4.3 (Properties of $F$).** If the bounds in (3.10) are satisfied then $\|F\|_2$ is a simple eigenvalue of $F : \mathcal{A} \to \mathcal{A}$ defined in (3.4). Moreover, there is a unique eigenvector $f \in \mathcal{A}_+$ such that $F[f] = \|F\|_2f$ and $\|f\|_2 = 1$. This eigenvector satisfies

$$1 - \|F\|_2 = (\text{Im } z) \langle f, qq^* \rangle / \langle f, \text{Im } u \rangle.$$  

(4.5)

In particular, $\|F\|_2 \leq 1$. Furthermore, the following properties hold true uniformly for $z \in \mathcal{H}$ satisfying $|z| \leq 3(1 + \|a\| + \|S\|^{1/2})$ and $\|F(z)\|_2 \geq 1/2$:

1. The eigenvector $f$ has upper and lower bounds

$$\|m\|^{-4}1 \lesssim f \lesssim \|m\|^41.$$  

(4.6)

2. The operator $F$ has a spectral gap $\theta \in (0, 1]$ satisfying $\theta \gtrsim \|m\|^{-28}$ and

$$\text{Spec}(F/\|F\|_2) \subset [-1 + \theta, 1 - \theta] \cup \{1\}.$$  

(4.7)

**Proof.** The definition of $F$ in (3.4), (3.10) and Lemma 4.2 imply

$$(1 + |z|)^{-4}\|m(z)\|^{-2}(a)1 \lesssim F[a] \lesssim (1 + |z|)^2\|m(z)\|^4(a)1$$  

(4.8)

for all $a \in \mathcal{A}_+$ and all $z \in \mathcal{H}$. We will use Lemma B.1 (ii) from Appendix B. The condition (B.1) with $T = F$ is satisfied by (4.8) with constants depending on $\|m\|$ and $|z|$. Hence, Lemma B.1 (ii) implies the existence and uniqueness of the eigenvector $f$. We compute the scalar product of $f$ with the imaginary part of (3.3). Since $F$ is symmetric, this immediately yields (4.5).

We now assume that $z \in \mathbb{H}$ satisfies $|z| \leq 3(1 + \|a\| + \|S\|^{1/2})$ and $\|F(z)\|_2 \geq 1/2$. Then $|z| \lesssim 1$ and, by using this in (4.8), we thus obtain (4.6) and (4.7) from Lemma B.1 (ii) since $\|m\| \gtrsim 1$ by (3.13).

The following proof of Proposition 4.1 proceeds similarly to the one of Proposition 4.4 in [4].

**Proof of Proposition 4.1.** We will distinguish several cases. If $|z| \geq 3(1 + \kappa)$ with $\kappa := \|a\| + 2\|S\|^{1/2}$ then we conclude from (2.4) and supp $\rho \subset [-\kappa, \kappa]$ by (2.5a) that $\|m(z)\| \lesssim (|z| - \kappa)^{-1}$. Thus,

$$\|C_{m(z)}S\|_2 \leq \frac{\|S\|_2}{(|z| - \kappa)^2} \leq \frac{\|S\|}{4(1 + \kappa)^2} \leq \frac{1}{4}.$$  

Here, we used $\|S\|_2 \leq \|S\|$ since $S$ is symmetric and $\kappa \geq \|S\|^{1/2}$. This shows (4.1) for large $|z|$.

Next, we assume $|z| \leq 3(1 + \kappa)$. In this regime, we use the alternative representation of $\text{Id} - C_{m}S$ in (4.3) and the spectral properties of $F$ from Lemma 4.3. Indeed, from (4.3) and Lemma 4.2, we conclude

$$\|(\text{Id} - C_{m}S)^{-1}\|_2 \lesssim \|m\|\|((C_{u} - F)^{-1}\|_2 \lesssim \frac{1}{(\rho(z) + \text{dist}(z, \text{supp } \rho)^2)}\|((C_{u} - F)^{-1}\|_2$$  

(4.9)

as $u \in \mathcal{A}$ is unitary. Here, we used (3.12) in the last step. If $\|F(z)\|_2 \leq 1/2$ then this immediately yields (4.1) as $\|C_{u}\|_2 = 1$. We now assume $\|F(z)\|_2 \geq 1/2$. In this case, we will use the following lemma.

**Lemma 4.4 (Rotation-Inversion Lemma).** Let $U$ be a unitary operator on $L^2$ and $T$ a symmetric operator on $L^2$. We assume that there is a constant $\theta > 0$ such that

$$\text{Spec } T \subset [-\|T\|_2 + \theta, \|T\|_2 - \theta] \cup \{\|T\|_2\}$$  

with a non-degenerate eigenvalue $\|T\|_2 \leq 1$. Then there is a universal constant $C > 0$ such that

$$\|(U - T)^{-1}\|_2 \leq \frac{C}{\theta|1 - \|T\|_2(t, U[t])|},$$  

where $t \in L^2$ is the normalized, $\|t\|_2 = 1$, eigenvector of $T$ corresponding to $\|T\|_2$.  

11
The proof of this lemma is identical to the proof of Lemma 5.6 in [2], where a result of this type was first applied in the context of vector Dyson equations.

We start from the estimate (4.9), use the Rotation-Inversion Lemma, Lemma 4.4, with $U = C_u^*$ and $T = F$ as well as (4.7) and (3.12) and obtain

$$
\| (\text{Id} - C_m S)^{-1} \|_2 \lesssim \frac{(\rho(z) + \text{dist}(z, \text{supp} \rho))^{31}}{|1 - \|F\| z(f, C_u^* f)|} \lesssim \frac{(\rho(z) + \text{dist}(z, \text{supp} \rho))^{31}}{\max \{1 - \|F\| z, 1 - \langle f C_u^* f \rangle \}}.
$$

In order to complete the proof of (4.1), we now show that

$$
\max \{1 - \|F\| z, 1 - \langle f C_u^* f \rangle \} \gtrsim (\rho(z) + \text{dist}(z, \text{supp} \rho))^C
$$

for some universal constant $C > 0$. We first prove auxiliary upper and lower bounds on $\text{Im} u = (q^*)^{-1}(\text{Im} m)q^{-1}$. We have

$$
\rho(z)(\rho(z) + \text{dist}(z, \text{supp} \rho))^2 \gtrsim \text{Im} u \lesssim \frac{\text{Im} z\|m\|}{\text{dist}(z, \text{supp} \rho)^2}.
$$

(4.11)

For the lower bound, we used the lower bound in (3.14), Lemma 4.2 and (3.12). The upper bound is a direct consequence of (3.7) as well as Lemma 4.2. Since $(f, q q^*) \gtrsim \|qq^*\|^{-1}(f) \gtrsim \|m\| (f)$ by Lemma 4.2, the relation (4.5) and the upper bound in (4.11) yield

$$
1 - \|F\| z \gtrsim \text{dist}(z, \text{supp} \rho)^2.
$$

As $1 - \langle f C_u^* u f \rangle \geq 0$ and $\|f^2\| = 1$, we obtain from the lower bound in (4.11) that

$$
|1 - \langle f C_u^* u f \rangle| \geq \text{Re} |1 - \langle f C_u^* u f \rangle| = 1 - \langle f C_u^* u f \rangle + \langle f C_m u f \rangle \gtrsim \rho(z)^2(\rho(z) + \text{dist}(z, \text{supp} \rho))^4.
$$

(4.12)

This completes the proof of (4.10) and hence of Proposition 4.1.

\[\square\]

4.2. Proof of Proposition 2.3

The following proof of Proposition 2.3 is similar to the one of Proposition 2.2 in [4].

Proof of Proposition 2.3. We first show that $\rho : \mathbb{H} \to (0, \infty)$ has a uniformly Hölder-continuous extension to $\mathbb{H}$, which we will also denote by $\rho$. This extension restricted to $\mathbb{R}$ will be the density of the measure $\rho$ from Definition 2.2. Since $\text{Id} - C_m S$ is invertible for each $z \in \mathbb{H}$ by (4.1), the implicit function theorem allows us to differentiate (2.3) with respect to $z$. This yields

$$
(\text{Id} - C_m S)[\partial_z m] = m^2.
$$

(4.13)

Since $z \mapsto \langle m(z) \rangle$ is holomorphic on $\mathbb{H}$ as remarked below (2.3), we have $2\pi i \partial_z \rho(z) = 2i\partial_z \text{Im} \langle m(z) \rangle = \partial_z \langle m(z) \rangle$.

Thus, we obtain from (4.13) that

$$
|\partial_z \rho| \lesssim \|\partial_z m\|_2 \lesssim \|(\text{Id} - C_m S)^{-1}\|_2 \|m\|_2 \lesssim \rho^{-(C+2)}.
$$

(4.14)

Here, we used (4.1), $\rho(z) \lesssim \|m(z)\|_2 \lesssim 1$ by (3.11) and (3.12) in the last step. Hence, $\rho^{C+3}$ is a uniformly Lipschitz-continuous function on $\mathbb{H}$. Therefore, $\rho$ defines uniquely a uniformly $1/(C + 3)$-Hölder continuous function on $\mathbb{R}$ which is a density of the measure $\rho$ from Definition 2.2 with respect to the Lebesgue measure on $\mathbb{R}$.

Next, we show the Hölder-continuity with respect to $a$ and $S$. As before in (4.2), we compute the derivatives and use (3.12) and (4.1) to obtain

$$
|\nabla_{(a, D)} \rho(a, S)(z)| \lesssim |\nabla_{(a, D)} m| \lesssim \frac{\|d\| + \|D\|}{\rho^{C+3}}.
$$

Since the constants in (4.1) and (3.12) depend on the constants in (3.10), we conclude that $\rho$ is also a locally $1/(C + 4)$-Hölder continuous function of $a$ and $S$.

We are left with showing that $\rho$ is real-analytic in a neighbourhood of $(\tau_0, a, S)$ if $\rho_{a, S}(\tau_0) > 0$. Since $\rho(\tau_0) > 0$, we can extend $m$ to $\tau_0$ by (4.14). Moreover, $m_{\tau_0}$ is invertible as $\text{Im} m_{\tau_0} > 0$ and, thus, solves (2.3) with $z = \tau_0$. Since (2.3) depends analytically on $z = \tau$, $a$ and $S$ in a small neighbourhood of $(\tau_0, a, S)$, the solution $m$ and thus $\rho$ will depend analytically on $(\tau, a, S)$ in this neighbourhood by the implicit function theorem. This completes the proof of Proposition 2.3.

\[\square\]
4.3. Proof of Proposition 2.4

For $I \subset \mathbb{R}$ and $\eta_* > 0$, we define

$$\mathbb{H}_{I, \eta_*} := \{ z \in \mathbb{H}: \text{Re} \, z \in I, \text{ Im} \, z \in (0, \eta_*) \}$$

(4.15)

and its closure $\mathbb{H}_{I, \eta_*}$.

**Assumptions 4.5.** Let $m$ be the solution of (2.3) for $a = a^* \in \mathcal{A}$ satisfying $\| a \| \leq k_1$ with a positive constant $k_1$ and $S \in \Sigma$ satisfying $\| S \|_{2 \rightarrow \mathbb{H}} \leq k_2$ for some positive constant $k_2$. For an interval $I \subset \mathbb{R}$ and some $\eta_* \in (0, 1]$, we assume that

(i) There are positive constants $k_3, k_4$ and $k_5$ such that

$$\| m(z) \| \leq k_3,$$

$$k_4 \| \text{Im} \, m(z) \| \leq \text{Im} \, m(z) \leq k_5 \| \text{Im} \, m(z) \|,$$

(4.16)

(4.17)

uniformly for all $z \in \mathbb{H}_{I, \eta_*}$.

(ii) The operator $F := C_{q,q'} S C_{q',q}$ has a simple eigenvalue $\| F \|_2$ with eigenvector $f \in \mathcal{A}$ that satisfies (4.5) for all $z \in \mathbb{H}_{I, \eta_*}$. Moreover, (4.7) holds true and there are positive constants $k_6, k_7$ and $k_8$ such that

$$k_6 \| f \| \leq k_7 \| f \|,$$

(4.18)

uniformly for all $z \in \mathbb{H}_{I, \eta_*}$.

We remark that $S \in \Sigma_{\text{flat}}$ is not necessarily required in Assumptions 4.5. In fact, we will show in Lemma 4.8 below that $S \in \Sigma_{\text{flat}}$ and (4.16) imply all other conditions in Assumptions 4.5.

**Convention 4.6** (Model parameters, Comparison relation). For the remainder of the Section 4 as well as Section 5 and Section 6, we will only consider $k_1, \ldots, k_8$ as model parameters and understand the comparison relation $\sim$ from Convention 3.4 with respect to this set of model parameters.

We remark that all of our estimates will be uniform in $\eta_* \in (0, 1]$. Therefore, $\eta_*$ is not considered a model parameter. At the end of this section, we will directly conclude Proposition 2.4 from the following proposition.

**Proposition 4.7** (Regularity of $m$). Let Assumptions 4.5 hold true on an interval $I \subset \mathbb{R}$ for some $\eta_* \in (0, 1]$. Then, for any $\theta \in (0, 1]$, $m$ can be uniquely extended to $I_0 := \{ \tau \in I: \text{dist}(\tau, \partial I) \geq \theta \}$ such that it is uniformly $1/3$-Hölder continuous, indeed,

$$\| m(z_1) - m(z_2) \| \leq \theta^{-4/3} |z_1 - z_2|^{1/3}$$

(4.19)

for all $z_1, z_2 \in I_0 \times i[0, \infty)$. Moreover, if $\rho(\tau_0) > 0$, $\tau_0 \in I$, then $m$ is real-analytic in a neighbourhood of $\tau_0$ and

$$\| \partial_\tau m(\tau_0) \| \leq \rho(\tau_0)^{-2}.$$

(4.20)

In the following lemma, we establish a very helpful consequence of (i) in Assumptions 4.5 and prove that (ii) is satisfied if we assume the flatness of $S$ in addition to (4.16).

**Lemma 4.8.** Let $m$ be the solution to (2.3) for some data pair $(a, S) \in \mathcal{A}_{sa} \times \Sigma$. We have

(i) Let $\| a \| \lesssim 1, \| S \| \lesssim 1$ and $U \subset \mathbb{H}$ such that $\sup \{ |z|: z \in U \} \lesssim 1$. If (4.16) and (4.17) hold true uniformly for $z \in U$ then, uniformly for $z \in U$, we have

$$\| q \|, \| q^{-1} \| \sim 1, \quad \text{Im} \, u \sim (\text{Im} \, u) \| f \| \sim \rho \| f \|.$$

(4.21)

(ii) Let $I \subset [-C, C]$ for some $C \sim 1$ and (4.16) hold true uniformly for all $z \in \mathbb{H}_{I, \eta_*}$. If $S \in \Sigma_{\text{flat}}$ and $\| a \| \lesssim 1$ then $\| S \|_{2 \rightarrow \mathbb{H}} \lesssim 1$, (4.17) holds true uniformly for all $z \in \mathbb{H}_{I, \eta_*}$, and part (ii) of Assumptions 4.5 is satisfied.

(iii) If Assumptions 4.5 hold true then, uniformly for $z \in \mathbb{H}_{I, \eta_*}$, we have

$$\| (\text{Id} - C_{m(z)} S)^{-1} \|_2 + \| (\text{Id} - C_{m(z)} S)^{-1} \| \lesssim \rho(z)^{-2}.$$  

(4.22)
Proof of Lemma 4.8. For the proof of (i), we use \( ||a|| \lesssim 1, ||S|| \lesssim 1 \) and (2.3) to show \( ||m(z)^{-1}|| \lesssim 1 \) uniformly for all \( z \in U \). Thus, following the proof of Lemma 4.2 immediately yields the estimates on \( q \) and \( q^{-1} \) in (4.21) due to (4.16) and (4.17). Thus, as \( ||q||, ||q^{-1}|| \sim 1 \), we obtain the missing relations in (4.21) from (4.17) since

\[
\Im u = (q^*)^{-1}(\Im m)q^{-1} \sim \Im m \sim (\Im m) \sim (\Im u).
\]

We now show (ii). By Lemma B.2 (i), the upper bound in the definition of flatness, (3.10), implies \( ||S||_{2-\gamma} \lesssim 1 \). Hence, (4.17) follows from (3.14) since \( ||z|| \leq C + 1 \) for all \( z \in \mathbb{H}_{I, \eta_0} \). Moreover, (ii) in Assumptions 4.5 is a consequence of Lemma 4.3.

To prove (4.22), we follow the proof of Proposition 4.1 and replace the use of (3.12) as well as (4.6) and (4.7) from Lemma 4.3 by (4.16) and (4.18), respectively. This yields

\[
\| (\Id - C_m S)^{-1} \| \lesssim 1 + \| F \|_2 \langle f C^*_m[f] \rangle^{-1} \lesssim 1 - \| F \|_2 \langle f C^*_m[f] \rangle^{-1},
\]

where we used in the last step that (4.16) implies \( \rho(z) \lesssim 1 \) on \( \mathbb{H}_{I, \eta_0} \). Since \( \Im u \sim \rho \) by (4.21) and \( \| F \|_2 \lesssim 1 \) by (4.5) that holds under Assumptions 4.5, we conclude

\[
\| (\Id - C_m S)^{-1} \| \lesssim 1 - \| F \|_2 \langle f C^*_m[f] \rangle^{-1} \lesssim \rho^{-2}
\]

as in (4.12) in the proof of Proposition 4.1. This shows \( (\Id - C_m S)^{-1} \| \lesssim \rho(z)^{-2} \). Using \( ||S||_{2-\gamma} \| \lesssim 1 \) and Lemma B.2 (ii), we obtain the missing \( || \cdot || \)-bound in (4.22). This completes the proof of Lemma 4.8.

Proof of Proposition 4.7. Similarly to the proof of Proposition 2.3, we obtain

\[
\| \partial_2 \Im m(z) \| \lesssim \| \partial_2 m(z) \| \lesssim \| (\Id - C_m S)^{-1} \| \| m(z) \|^2 \lesssim \rho(z)^{-2} \sim \| \Im m(z) \|^{-2}
\]

for \( z \in \mathbb{H}_{I, \eta_0} \) from (4.16), (4.22) and (4.17). By the submultiplicativity of \( \| \cdot \|_2 \), \( (\Im m(z))^3 : \mathbb{H}_{I, \eta_0} \rightarrow (\mathcal{A}, \| \cdot \|) \) is a uniformly Lipschitz-continuous function. Hence, \( \Im m(z) \) is uniformly 1-3-Hölder continuous on \( \mathbb{H}_{I, \eta_0} \), see e.g. Theorem X.1.1 in [13] and, thus, has a uniformly 1-3-Hölder continuous extension to \( \mathbb{H}_{I, \eta_0} \). We conclude that the measure \( \nu \) restricted to \( I \) has a density with respect to the Lebesgue measure on \( I \), i.e., (2.9) holds true for all measurable \( A \subset I \). Now, (A.3) in Lemma A.1 implies the uniform 1-3-Hölder continuity of \( m \) on \( I_\theta \times (0, \infty) \). In particular, \( m \) can be uniquely extended to a uniformly 1-3-Hölder continuous function on \( I_\theta \times (0, \infty) \) such that (4.19) holds true.

To prove the analyticit of \( m \), we refer to the proof of the analyticit of \( \rho \) in Proposition 2.3. The bound (4.20) can be read off from (4.24). This completes the proof of the proposition.

Proof of Proposition 2.4. By (2.7), there are \( C_0, \eta_0 \in (0, 1] \) such that \( ||m(\tau + i\eta)|| \leq C_0 \) for all \( \tau \in I \) and \( \eta \in (0, \eta_0] \). Hence, by Lemma 4.8 (ii), the flatness of \( S \) implies Assumptions 4.5 on \( I \cap [-C, C] \) for \( C = 3(1 + ||a|| + ||S||^{1/2}) \), i.e., \( C \sim 1 \). Therefore, Proposition 4.7 yields Proposition 2.4 on \( I \cap [-C, C] \).

Owing to (3.7) and \( \sup(v) = \sup \rho \), we have \( \text{dist}(\tau, \sup v) \geq 1 \) for \( \tau \in I \) satisfying \( \tau \notin [-C + 1, C - 1] \). Hence, for these \( \tau \), the Hölder-continuity follows immediately from (A.4) in Lemma A.1. By (2.5a), we have \( \Im m(\tau) = 0 \) for \( \tau \in I \) satisfying \( \tau \notin [-C, C] \). Therefore, the statement about the analyticit is trivial outside of \( [-C, C] \). This completes the proof of Proposition 2.4.

5. Spectral properties of the stability operator for small self-consistent density of states

In this section, we study the stability operator \( B^{-1} \), where \( B = B(z) := \Id - C_m(z)S \), when \( \rho = \rho(z) \) is small and Assumptions 4.5 hold true. Note that we do not require \( S \) to be flat, i.e., to satisfy (3.10). We will view \( B \) as a perturbation of the operator \( B_0 \), which we introduce now. We define

\[
s := \text{sign} \Re u, \quad B_0 := C_{q^*q}(\Id - C_s F)C_{q^*-q}^{-1}, \quad E := (C_{q^*q} - C_m)S = C_{q^*q}(C_s - C_u)FC_{q^*-q}^{-1}
\]

with \( u \) and \( q \) defined in (3.1). Note \( B_0 = \Id - C_{q^*q}S \), i.e., in the definition of \( B, u \) in \( m = q^*uq \) is replaced by \( s \). Thus, we have \( B = B_0 + E \). Under Assumptions 4.5, (4.21) holds true which we will often use in the following. Since \( 1 - \Re u = 1 - (\Im m)^2 \leq (\Im m)^2 \lesssim \rho^2 \), we also obtain

\[
\Re u = s + O(\rho^2), \quad \Im u = O(\rho), \quad \Re m = q^*sq + O(\rho^2)
\]
and with \( C_s - C_u = \mathcal{O}(\|s - u\|) = \mathcal{O}(\rho) \) we get

\[
E = \mathcal{O}(\rho). \quad (5.3)
\]

Here, we use the notation \( R = T + \mathcal{O}(\alpha) \) for operators \( T \) and \( R \) on \( \mathcal{A} \) and \( \alpha > 0 \) if \( \|R - T\| \lesssim \alpha \). We introduce

\[
f_u := \rho^{-1} \text{Im} u. \quad (5.4)
\]

By the functional calculus for the normal operator \( u \), \( \text{Re} u \), \( s \) and \( f_u \) commute. Hence, \( C_s[f_u] = f_u \). From the imaginary part of \((3.3)\) and \((4.21)\), we conclude that

\[
(\text{Id} - F)[f_u] = \rho^{-1} \text{Im} zqq^* = \mathcal{O}(\rho^{-1} \text{Im} z). \quad (5.5)
\]

In the following, for \( z \in \mathbb{C} \) and \( \varepsilon > 0 \), we denote by \( D_\varepsilon(z) := \{w \in \mathbb{C} : |z - w| < \varepsilon\} \) the disk in \( \mathbb{C} \) of radius \( \varepsilon \) around \( z \).

**Lemma 5.1** (Spectral properties of stability operator). Let \( T \in \{\text{Id} - F, \text{Id} - C_sF, B_0, B, \text{Id} - C_{m^*,m}S\} \). If Assumptions 4.5 are satisfied on an interval \( I \subset \mathbb{R} \) for some \( \eta_* \in (0,1] \), then there are \( \rho_\ast \sim 1 \) and \( \varepsilon \sim 1 \) such that

\[
\|(T - \omega \text{Id})^{-1}\|_2 + \|(T - \omega \text{Id})^{-1}\|_2 + \|(T^* - \omega \text{Id})^{-1}\|_2 \lesssim 1
\]

uniformly for all \( z \in \mathbb{H}_{I,\eta_*} \) satisfying \( \rho(z) + \rho(z)^{-1} \text{Im} z \leq \rho_\ast \) and for all \( \omega \in \mathbb{C} \) with \( \omega \notin D_\varepsilon(0) \cup D_{1-2\varepsilon}(1) \).

Furthermore, there is a single simple (algebraic multiplicity 1) eigenvalue \( \lambda \) in the disk around 0, i.e.,

\[
\text{Spec}(T) \cap D_\varepsilon(0) = \{\lambda\} \quad \text{and} \quad \text{rank } P_T = 1, \quad \text{where} \quad P_T := -\frac{1}{2\pi i} \int_{\partial D_\varepsilon(0)} (T - \omega \text{Id})^{-1} d\omega. \quad (5.7)
\]

If Assumptions 4.5 are satisfied on \( I \) for some \( \eta_* \in (0,1] \) then we have

\[
f_u = \rho^{-1} \text{Im} u \sim 1. \quad (5.8)
\]

uniformly for \( z \in \mathbb{H}_{I,\eta_*} \) due to \((4.21)\). This fact will often be used in the following without mentioning it.

**Proof.** First, we introduce the bounded operators \( V_t : \mathcal{A} \to \mathcal{A} \) for \( t \in [0,1] \) interpolating between \( \text{Id} \) and \( C_s \) by

\[
V_t := (1 - t)\text{Id} + t C_s.
\]

We will perform the proof one by one for the choices \( T = \text{Id} - F, \text{Id} - V_t F, B_0, B, \text{Id} - C_{m^*,m}S \) in that order. The operator \( \text{Id} - F \) has a spectral gap above the single eigenvalue around 0, so for this choice the statements are easy. Then we perform two approximations. First, we interpolate between \( \text{Id} - F \) and \( \text{Id} - C_s F \) via \( \text{Id} - V_t F \).

This gives Lemma 5.1 for \( T = B_0 \). Then we use perturbation theory to get the results for \( T = B = B_0 + \mathcal{O}(\rho) \) and for \( T = \text{Id} - C_{m^*,m}S = B_0 + \mathcal{O}(\rho) \). Note that for all these choices of \( T \) the bound \( \|T - \omega \text{Id}\|_2 \lesssim 1 \) holds due to \( \|S\|_2 \lesssim 1 \) \((4.16)\) and \((4.21)\). Hence, the invertibility of \( T - \omega \text{Id} \) as an operator on \( \mathcal{A} \) and on \( L^2 \) are therefore closely related by Lemma B.2 (ii). In particular, it suffices to show \((5.7)\) and the \( \|\cdot\|_2 \)-norm bound

\[
\|(T - \omega \text{Id})^{-1}\|_2 \lesssim 1, \quad (5.9)
\]

for \( \omega \notin D_\varepsilon(0) \cup D_{1-2\varepsilon}(1) \) to establish the lemma. For \( T = \text{Id} - F \) both of these assertion are true due to Lemma 4.3. In particular, we find

\[
f = \|f_u\|_2^{-1} f_u + \mathcal{O}(\rho^{-1} \text{Im} z), \quad (5.10)
\]

where \( f \) is the single top eigenvector of \( F \), \( Ff = \|F\|_2 f \) (see Lemma 4.3). The proof of \((5.10)\) follows from \((5.5)\) and \( \|F\|_2 = 1 + \mathcal{O}(\rho^{-1} \text{Im} z) \) (cf. \((4.5)\)) by straightforward perturbation theory of the simple isolated eigenvalue \( \|F\|_2 \).

Now we consider the choice \( T = T_t = \text{Id} - V_t F \). Once \((5.9)\), and with it \((5.6)\), is established for \( T_t \), the statement about the single isolated eigenvalue \((5.7)\) follows. Indeed, assuming \((5.6)\) for \( T = T_t \), we obtain that \( T_t \) and, hence, the rank of \( P_{T_t} \) is a continuous function of \( t \) on \([0,1] \). Hence, the rank of \( P_{T_t} \) is constant along this interpolation. On the other hand, rank \( P_{T_t} = 1 \) by Lemma 4.3. Therefore, for each \( t \in [0,1] \), \( \text{Spec}(T_t) \cap D_\varepsilon(0) \) consists of precisely one simple eigenvalue. We are thus left with establishing \((5.9)\) for \( T_t \). As \( \|V_t\|_2 \leq 1 \) and \( \|F\|_2 \leq 1 \) the bound \((5.9)\) is certainly satisfied for \( |\omega| \geq 3 \). Thus, we now assume \( |\omega| \leq 3 \). In order to conclude \((5.9)\), we now show a lower bound on \( \|(1 - \omega \text{Id} - V_t F)[x]\|_2 \) for all normalized, \( \|x\|_2 = 1 \), elements \( x \in \mathcal{A} \). We
decompose $x \in L^2$ as $x = \alpha f + y$, where $y \perp f$ and $\alpha \in \mathbb{C}$. Then
\[ \|(1 - \omega) \text{Id} - V_1 F[x]\|_2^2 = |\alpha|^2 |\omega|^2 + \|(1 - \omega) \text{Id} - V_1 F[y]\|_2^2 + \mathcal{O}(\rho^{-1} \text{Im} z), \]
(5.11)
because of $\|F\|_2 = 1 + \mathcal{O}(\rho^{-1} \text{Im} z)$, $V_1[f_u] = f_u$ together with (5.10), and because the mixed terms are negligible due to
\[ \langle f, V_1 F[y] \rangle = \langle F V_1 f, y \rangle = \mathcal{O}(\|y\|_2 \rho^{-1} \text{Im} z). \]
Using the spectral gap $\vartheta \sim 1$ of $F$ from (4.7) and $y \perp f$ we infer (5.9) from (5.11) by estimating
\[ \|(1 - \omega) \text{Id} - V_1 F[y]\|_2^2 \geq \text{dist}(\omega, D_{1 - \vartheta}(1)) \|y\|_2^2 \geq (\vartheta - \varepsilon)(1 - |\alpha|^2), \]
optimizing in $\alpha$ and choosing $\delta \leq \vartheta/2$. This shows the lemma for $T = \text{Id} - V_1 F$.

Since $B_0$ is related by the similarity transform (5.1) to $\text{Id} - V_1 F = \text{Id} - C_s F$ and $\|q\|q^{-1} \|1 \| \leq 1$ (cf. (4.21)), the operator $B_0$ inherits the properties listed in the lemma from $\text{Id} - C_s F$. Finally, we can perform analytic perturbation theory for the simple isolated eigenvalue in $D_z(0)$ of $B_0$ to verify the lemma for $T = B = B_0 + E$ with $E = \mathcal{O}(\rho)$ (cf. (5.3)) and $T = \text{Id} - C_{s^*} S = B_0 + E_*$ with $E_* = \mathcal{O}(\rho)$ if $\rho_*$ is sufficiently small. Here, we introduced
\[ E_* := (C_{s^*} - C_{s^*, u}) S = C_{s^*} (C_s - C_{s^*, u}) F C_{s^*}^{-1}, \]
and used $C_s - C_{s^*, u} = \mathcal{O}(\|s - u\|) = \mathcal{O}(\rho)$ due to (5.2).

If $z \in \mathbb{H}_{I_0}$ satisfies $\rho(z) + (\rho(z))^{-1} \text{Im} z \leq \rho_*$ for $\rho_* \sim 1$ from Lemma 5.1 then we denote by $P_{s,F}$ the spectral projection corresponding to the isolated eigenvalue of $\text{Id} - C_s F$, i.e., $P_{s,F}$ equals $P_T$ in (5.7) with $T = \text{Id} - C_s F$. We also set $Q_{s,F} := \text{Id} - P_{s,F}$. Moreover, for such $z$, we define $\psi$ and $\sigma$ by
\[ \psi(z) := \langle s f_u^2, (\text{Id} + F)(\text{Id} - C_s F)^{-1} Q_{s,F} [s f_u^2] \rangle, \quad \sigma(z) := \langle s f_u^2 \rangle. \]
(5.12)

**Corollary 5.2.** Let $z \in \mathbb{H}_{I_0}$ satisfy $\rho(z) + (\rho(z))^{-1} \text{Im} z \leq \rho_*$ for $\rho_* \sim 1$ from Lemma 5.1. Let $(\beta_0, b_0, l_0)$ and $(\beta, b, l)$ be the triple of eigenvalue, right and left eigenvector for the operators $B_0$ and $B$ corresponding to the isolated eigenvalue in $D_z(0)$ from Lemma 5.1, respectively. Then with a properly chosen normalization of the eigenvectors we have
\[ b_0 = C_{s^*} [f_u] + \mathcal{O}(\rho^{-1} \text{Im} z), \quad l_0 = C_{s^*}^{-1} [f_u] + \mathcal{O}(\rho^{-1} \text{Im} z), \]
(5.13a)
\[ \beta_0 = \frac{\Im z - \pi}{\rho} + \mathcal{O}(\rho^{-2} (\text{Im} z)^2) = \mathcal{O}(\rho^{-1} \text{Im} z), \]
(5.13b)
as well as
\[ b = b_0 + 2i \rho C_{s^*} (\text{Id} - C_s F)^{-1} Q_{s,F} [s f_u^2] + \mathcal{O}(\rho^2 + \text{Im} z), \]
(5.14a)
\[ l = l_0 - 2i \rho C_{s^*} (\text{Id} - F C_s)^{-1} Q_{s,F}^* [s f_u^2] + \mathcal{O}(\rho^2 + \text{Im} z), \]
(5.14b)
\[ \beta(l, b) = \pi \rho^{-1} \text{Im} z - 2i \rho \sigma + 2 \rho^2 \left( \psi + \frac{\sigma^2}{\langle f_u^2 \rangle} \right) + \mathcal{O}(\rho^3 + \text{Im} z + \rho^{-2} (\text{Im} z)^2). \]
(5.14c)
Furthermore, let $P_0$ and $P$ be the projections corresponding to the isolated eigenvalue of $B_0$ and $B$, respectively. Then with $Q_0 := \text{Id} - P_0$ and $Q := \text{Id} - P$ we have
\[ \| B^{-1} Q \| + \| B^{-1} Q_0 \| + \| B_0^{-1} Q_0 \| \lesssim 1. \]
(5.15)
Moreover, we have
\[ \|b\| \lesssim 1, \quad \|l\| \lesssim 1. \]
(5.16)
For later use, we record some identities here. From (5.10) in the proof of Lemma 5.1 with $C_s[f_u] = f_u$, we obtain the first relation in
\[ P_{s,F} = \langle f_u, \cdot \rangle f_u + \mathcal{O}(\rho^{-1} \text{Im} z), \quad P_{s,F}^* = P_{s,F} + \mathcal{O}(\rho^{-1} \text{Im} z), \quad Q_{s,F}^* = Q_{s,F} + \mathcal{O}(\rho^{-1} \text{Im} z). \]
(5.17)
This first relation together with $f_u = f_u^*$ implies the second and third one. Moreover, the definitions of $B_0$ and $Q_0$ yield
\[ B_0^{-1} Q_0 = C_{s^*} (\text{Id} - C_s F)^{-1} Q_{s,F} C_{s^*}^{-1}. \]
(5.18)
By a direct computation starting from the definitions of \( f_u \) in (5.4) and \( u \) in (3.1), we obtain
\[
\langle f_u, qq^* \rangle = \rho^{-1}(\text{Im } m) = \pi. \tag{5.19}
\]

**Proof.** Using (5.5) and \( C_s[f_u] = f_u \), we see that
\[
B_0^*C_{q,q}^{-1}[f_u] = \rho^{-1}(\text{Im } z)I, \quad B_0C_{q,q}[f_u] = O(\rho^{-1}\text{Im } z). \tag{5.20}
\]

We set \( b_0 := P_0C_{q,q}[f_u] \) and \( l_0 := P_0^*C_{q,q}^{-1}[f_u] \) which amounts to a normalization as \( \beta_0 \) is a nondegenerate eigenvalue. The representations of \( b_0 \) and \( l_0 \) in (5.13a) follow by simple perturbation theory because \( \beta_0 \) is a nondegenerate isolated eigenvalue. The expression for \( \beta_0 \) in (5.13b) is seen by taking the scalar product with \( b_0 \) in the first identity of (5.20) as well as using (5.19).

The expansions (5.14) follow by analytic perturbation theory. Indeed, \( b = b_0 + b_1 + O(\rho^2) \) and \( l = l_0 + l_1 + O(\rho^2) \) with \( b_1 := -(B_0 - \beta_0 I)^{-1}Q_0E[l_0] \) and \( l_1 := -(B_0^* - \beta_0 I)^{-1}Q_0E^*[l_0] \) (cf. Lemma C.1 with \( E \) satisfying (5.3)). Here the invertibility of \( B_0 - \beta_0 I \) on the range of \( Q_0 \) is seen from the second part of Lemma 5.1 with \( T = B_0 \). In fact,
\[
(B_0 - \beta_0 I)^{-1}Q_0 = B_0^{-1}Q_0 + O(\beta_0). \tag{5.21}
\]

Furthermore, we use (5.13a) and obtain the first equalities below:
\[
E[l_0] = C_{q,q}^{-1}F[C_s - C_u] + O(\text{Im } z) = -2ipC_{q,q}[sf_u^2] + 2p^2C_{q,q}[sf_u^2] + O(\rho^3 + \text{Im } z), \tag{5.22a}
\]
\[
E^*[l_0] = C_{q,q}^{-1}F^*[C_s - C_u^*][f_u] + O(\text{Im } z) = 2ipC_{q,q}^{-1}F[sf_u^2] + 2p^2C_{q,q}^{-1}F[sf_u^2] + O(\rho^3 + \text{Im } z). \tag{5.22b}
\]

For the second equality in (5.22a), we used (5.5), \( \|C_s - C_u\| = O(\rho) \) and \( \|C_s - C_u\| = 2(\text{Im } u - i\text{Re } u)(\text{Im } u)\rho \) to due (5.2). For the second equality in (5.22b), we applied \( (C_s - C_u^*)[f_u] = 2ip\rho f_u^2 + 2\rho^2 f_u^2 + O(\rho^3) \).

For the proof of (5.14c), we start from (3.3), use \( E = O(\rho) \) and obtain
\[
\beta(l, b) = \beta_0(l_0, b_0) + \langle l_0, E[l_0] \rangle - \langle l_0, EB_0(B_0 - \beta_0 I)^{-2}Q_0E[l_0] \rangle + O(\rho^3). \tag{5.23}
\]

Each of the terms on the right-hand side is computed individually. For the first term, we use \( \langle l_0, b_0 \rangle = \langle f_u^2 \rangle + O(\rho^{-1}\text{Im } z) \) due to (5.13a) and thus obtain from (5.13b) that
\[
\beta_0(l_0, b_0) = \pi\rho^{-1}\text{Im } z + O(\rho^{-2}(\text{Im } z)^2). \tag{5.24}
\]

Using (5.13a) and (5.22) yields for the second term
\[
\langle l_0, E[l_0] \rangle = -2ip(sf_u^2) + 2\rho^2(f_u^2) + O(\rho^3 + \text{Im } z) = -2ip\sigma + 2\rho \left( \frac{\sigma^2}{f_u^2} + (s f_u^2, Q_s, F[s f_u^2]) \right) + O(\rho^3 + \text{Im } z),
\]
where we used \( \text{Id} = P_s F + Q_s F \) and \( (s f_u^2, P_s F[f_u^2]) = \sigma^2/(f_u^2) + O(\rho^{-1}\text{Im } z) \) by (5.17) in the last step.

For the third term, we use (5.13b) and \( E = O(\rho) \) which yields
\[
\langle l_0, EB_0(B_0 - \beta_0 I)^{-2}Q_0E[l_0] \rangle = \langle E[l_0], (B_0 - \beta_0 I)^{-2}Q_0E[l_0] \rangle + O(\beta_0||E||^2)
\]
\[
= \langle E[l_0], B_0^{-1}Q_0E[l_0] \rangle + O(\rho \text{Im } z)
\]
\[
= -4\rho^2(s f_u^2, F(\text{Id} - C_s F)^{-1}Q_s, F[s f_u^2]) + O(\rho \text{Im } z + \rho^3).
\]

Here, we used (5.21) in the second step and (5.22) as well as (5.18) in the last step. Collecting the results for the three terms in (5.23) and using \( C_s = C_s^* \) as well as \( C_s[s f_u^2] = s f_u^2 \) yield (5.14c).

The bounds in (5.15) and (5.16) follow directly from the analytic functional calculus and Lemma 5.1. \( \Box \)

**Corollary 5.3** (Improved bound on \( B^{-1} \)). Let Assumptions 4.5 hold true on an interval \( I \subset \mathbb{R} \) for some \( \eta_* \in (0, 1) \). Then, uniformly for all \( z \in \mathbb{H}_{1, \eta_*} \), we have
\[
\|B^{-1}(z)\|_2 + \|B^{-1}(z)\| \lesssim \frac{1}{\rho(z) + |\sigma(z)| + \rho(z)^{-1}\text{Im } z}. \tag{5.24}
\]

**Proof.** If \( \rho \geq \rho_* \) for some \( \rho_* \sim 1 \) then (5.24) have been shown in (4.22) as \( |\sigma| \lesssim 1 \). Therefore, we prove (5.24) for \( \rho \leq \rho_* \) and a sufficiently small \( \rho_* \sim 1 \). By \( \|S\|_{2 \rightarrow 1} \lesssim 1 \) and Lemma B.3 (ii), it suffices to show the bound
for \( \| \cdot \|_2 \). We follow the proof of (4.22) until (4.23). Hence, for the improved bound, we have to show that

\[
|1 - \|F\|_2 (f C_{\rho}^* [f])| \geq \rho |\rho + |\epsilon|| + \rho^{-4} \text{Im } z.
\]

We have \( |1 - \|F\|_2 (f C_{\rho}^* [f])| \geq \max \{1 - \|F\|_2, |1 - (f C_{\rho}^* [f])| \} \geq \rho^{-1} \text{Im } z + |1 - (f C_{\rho}^* [f])| \) by (4.5). We continue

\[
|1 - (f C_{\rho}^* [f])| = |1 - (f u \ast f u^*)| \geq (\text{Im } u \text{Im } u) + |(\text{Im } u f \text{Re } u)| \geq \rho^2 + |\rho| |\epsilon| + O(\rho^3 + |\text{Im } z|).
\]

Here, we used 1 \( \geq (\text{Re } u f \text{Re } u) \) due to \( \|f\|_2 = 1 \) (4.21) as well as \( (\text{Im } u f \text{Re } u) = \rho \|f\|_2^2 (f d_s^2) + O(\rho^3 + |\text{Im } z|) \) by (5.10) and (5.2). By possibly shrinking \( \rho_* \sim 1 \), we thus obtain (5.25). This completes the proof of (5.24).

The remainder of this section is devoted to several results about the behaviour of \( \rho(z), \sigma(z) \) and \( \psi(z) \) close to the real axis. They will be applied in the next section. We start with some new notation and then prepare these results by extending \( q, u, f_u \) and \( s \) to the real axis. For any \( \epsilon > 0 \) and \( \theta > 0 \), we introduce

\[
D_{\epsilon, \theta} := \{ \tau \in \text{supp } \rho \cap I : \rho(\tau) \in [0, \epsilon], \text{ dist}(\tau, \Omega) \geq \theta \}, \quad D_{\epsilon, \theta}^\rho := \{ \tau \in D_{\epsilon, \theta} : \rho(\tau) > 0 \}.
\]

**Lemma 5.4** (Extensions of \( q, u, f_u \) and \( s \)). Let \( I \subset \mathbb{R} \) be an interval, \( \theta \in (0, 1) \) and Assumptions 4.5 hold true on \( I \) for some \( \eta_* \in (0, 1) \). We set \( I_\theta := \{ \tau \in I : \text{ dist}(\tau, \Omega) \geq \theta \} \). Then we have

(i) The functions \( q, u \) and \( f_u \) have unique extensions to \( \mathbb{H} \cup (\text{supp } \rho \cap I_\theta) \) such that they are continuous on \( \mathbb{H} \cup \{ \tau \in I_\theta : \rho(\tau) > 0 \} \) and Hölder-continuous on \( \text{supp } \rho \cap I_\theta \). In fact, \( q, u \) and \( f_u \) are \( 1/24 \)-, \( 1/12 \)- and \( 1/24 \)-Hölder-continuous on \( \text{supp } \rho \cap I_\theta \), respectively.

(ii) There is a threshold \( \rho_* \sim 1 \) such that \( s \) has a unique extension to \( \mathbb{H} \cup D_{\rho_*} \) which is continuous on \( \mathbb{H} \cup D_{\rho_*}^\rho \) and \( 1/24 \)-Hölder-continuous on \( D_{\rho_*} \).

**Proof.** We start the proof of (i) by showing that \( q \) and \( u \) can be extended to uniformly Hölder-continuous functions on \( \text{supp } \rho \cap I_\theta \). Note \( D_{\epsilon, \theta} \subset I_\theta \) for any \( \epsilon > 0 \). If \( \rho(\tau) > \rho_* \) for some \( \rho_* \sim 1 \) and \( \tau \in I_\theta \) then this extension follows directly from the definitions in (3.1) and the Hölder-continuity of \( m \) on \( I_\theta \) due to Proposition 4.7. For the other regime, we will show below that

\[
\rho(\tau) = \rho^{-1} (\text{Im } m(\tau))
\]

is uniformly Hölder-continuous on \( D_{\rho_*} \) for some suitably chosen \( \rho_* \sim 1 \). Indeed, this suffices since the definitions of \( q \) and \( u \) in (3.1) can be rewritten as

\[
q = |h^{-1/2} g h^{-1/2} + \mathbf{1}|^{1/2} h^{1/2} = (\rho(z)^2 \mathbf{1} + b_\rho^{-1/2} g b_\rho^{-1/2} g b_\rho^{-1/2})^{1/4},
\]

\[
u = \rho(z) u + \rho^{-1/2} g b_\rho^{-1/2} g b_\rho^{-1/2} g b_\rho^{-1/2} |b_\rho^{-1/2} g b_\rho^{-1/2} g b_\rho^{-1/2}|
\]

where \( g = \text{Re } m \) and \( h = \text{Im } m \) and \( z \in \mathbb{H} \) arbitrary. Since \( b_\rho(\tau) \sim 1 \) uniformly for \( \tau \in I \) by (4.17) and \( m \), hence \( \rho \) and \( \text{Re } m \) are Hölder-continuous on \( I_\theta \), it thus suffices to show that \( b_\rho \) is uniformly Hölder-continuous on \( D_{\rho_*} \) to conclude from (5.27) that \( q \) and \( u \) are Hölder-continuous on \( D_{\rho_*} \).

In the following, for \( \tau \in D_{\rho_*} \), let \( b_\rho(\tau) \in \mathbb{A}_+ \) be the unique positive definite eigenvector of \( \text{Id} - C_{m^*, m} S \) associated to its eigenvalue in \( D_\theta(0) \) (we choose \( \rho_* \sim 1 \) small enough so that Lemma 5.1 is applicable) such that \( (b_\rho(\tau)) = \pi \). Owing to the 1/3-Hölder continuity of \( m \), \( b_\rho \) is uniformly 1/3-Hölder continuous on \( D_{\rho_*} \) for some \( \rho_* \sim 1 \) by contour integration. We now prove that \( b_\rho(\tau) = b_\rho(\tau) \) for all \( \tau \in I_\theta \) satisfying \( 0 < \rho(\tau) \leq \rho_* \). We take the imaginary part of (2.3) and obtain

\[
b_\rho(z) = \rho(z)^{-1} (\text{Im } z) m^* m + C_{m^*, m} S[b_\rho(z)]
\]

for any \( z \in \mathbb{H} \), where \( b_\rho(z) := \text{Im } m(z)/\rho(z) \) and \( m = m(z) \). We fix \( \tau \in \text{supp } \rho \cap I_\theta \) satisfying \( \rho(\tau) > 0 \). By using (4.10) and performing the limit \( \text{Im } z \to 0 \) in (5.28), we obtain \( b_\rho(\tau) = C_{m^*, m} S[b_\rho(\tau)] \)

\[
m = m(\tau).
\]

Because of the uniqueness of \( b_\rho \) and \( b_\rho(\tau) = \pi \), we conclude \( b_\rho(\tau) = b_\rho(\tau) \) for all \( \tau \in \text{supp } \rho \cap I_\theta \) satisfying \( 0 < \rho(\tau) \leq \rho_* \). Therefore, since \( b_\rho \) is uniformly 1/3-Hölder continuous on \( D_{\rho_*} \), it is the unique 1/3-Hölder continuous extension of \( b_\rho \) to \( D_{\rho_*} \). Hence, owing to (5.27), \( q \) and \( u \) have extensions to \( \text{supp } \rho \cap I_\theta \) which are \( 1/24 \)- and \( 1/12 \)-Hölder-continuous, respectively.

Since \( f_u = \rho^{-1} \text{Im } u = (q^{-1} b_\rho)^{-1} \), the Hölder-continuity of \( b_\rho \), the Hölder-continuity of \( q \) and the upper and lower bounds on \( q \) from (4.21) imply that \( f_u \) can be extended to a 1/24-Hölder continuous function on \( D_{\rho_*} \).
The function \( f_u \) is trivially 1/24-Hölder continuous on the set \( \{ \tau \in I_\theta : \rho(\tau) > \rho_* \} \). This completes the proof of (i).

We now turn to the proof of (ii) and show that \( s = \text{sign}(\text{Re} \, u) \) is a Hölder-continuous function of \( \tau \) on \( \mathbb{D}_{\rho_*, \theta} \) if \( \rho_* \sim 1 \) is chosen sufficiently small. Owing to the first relation in (5.2), we have \( |\text{Re} \, u| \geq 1/2 \) if \( \tau \in \mathbb{D}_{\rho_*, \theta} \) with some small \( \rho_* \sim 1 \). Therefore, we find a smooth function \( \varphi : \mathbb{R} \to [-1, 1] \) such that \( \varphi(t) = 1 \) for all \( t \in [1/2, \infty) \), \( \varphi(t) = -1 \) for all \( t \in (-\infty, -1/2] \) and \( s(\tau) = \text{sign}(\text{Re} \, u(\tau)) = \varphi(\text{Re} \, u(\tau)) \) for all \( \tau \in \mathbb{D}_{\rho_*, \theta} \). Since \( \varphi \) is smooth, we conclude that \( \varphi \) is an operator Lipschitz function [5, Theorem 1.6.1], i.e., \( ||\varphi(x) - \varphi(y)|| \leq C||x - y|| \) for all self-adjoint \( x, y \in \mathcal{A} \). Hence, we conclude

\[
||s(\tau_1) - s(\tau_2)|| = ||\varphi(\text{Re} \, u(\tau_1)) - \varphi(\text{Re} \, u(\tau_2))|| \lesssim ||\tau_1 - \tau_2||^{1/12};
\]

where we used that \( \varphi \) is operator Lipschitz and \( u \) is 1/12-Hölder continuous in the last step. This completes the proof of Lemma 5.4.

---

**Lemma 5.5** (Properties of \( \psi \) and \( \sigma \)). Let \( I \subset \mathbb{R} \) be an interval and \( \theta \in (0, 1] \). If \( m \) satisfies Assumptions 4.5 on \( I \) for some \( \eta_* \in (0, 1] \) then there is a threshold \( \rho_* \sim 1 \) such that, with

\[
\mathbb{H}_{\text{small}} := \{ w \in \mathbb{H}_{I, \eta_*} : \rho(w) + \rho(w)^{-1} \text{Im} \, w \leq \rho_* \},
\]

we have

(i) The functions \( \psi \) and \( \sigma \) defined in (5.12) have unique extensions to \( \mathbb{H}_{\text{small}} \cup \mathbb{D}_{\rho_*, \theta} \) such that \( \psi \) and \( \sigma \) are continuous on \( \mathbb{H}_{\text{small}} \cup \mathbb{D}_{\rho_*, \theta} \) and Hölder-continuous on \( \mathbb{D}_{\rho_*, \theta} \). In fact, \( \sigma \) is uniformly 1/24-Hölder continuous on \( \mathbb{D}_{\rho_*, \theta} \).

(ii) Uniformly for all \( z \in \mathbb{H}_{\text{small}} \cup \mathbb{D}_{\rho_*, \theta} \), we have

\[
\psi(z) + \sigma(z)^2 \sim 1.
\]  

**Proof.** For the proof of (i), we choose \( \rho_* \sim 1 \) so small that all parts of Lemma 5.4 are applicable. Thus, Lemma 5.4 and \( \sigma = (sf_u^2) \) yield (i) for \( \sigma \). Similarly, since \( q \) is now defined on \( \mathbb{H} \cup (\text{supp} \, \rho \cap I_\theta) \), we can define \( F \) via (3.4) on this set as well. Moreover, owing to the regularity properties of \( q \) from Lemma 5.4, \( F \) is continuous on \( \mathbb{H} \cup (\text{supp} \, \rho \cap I_\theta) \) and uniformly Hölder-continuous on \( \text{supp} \, \rho \cap I_\theta \). Hence, using the Hölder-continuity of \( s \) and \( f_u \), the function \( \psi \) has a unique Hölder-continuous extension to \( \mathbb{D}_{\rho_*, \theta} \). This completes the proof of (i) for \( \psi \).

We now turn to the proof of (ii). In fact, we will show (5.29) only on \( \{ w \in \mathbb{H}_{I, \eta_*} : \rho(w) + \rho(w)^{-1} \text{Im} \, w \leq \rho_* \} \), where \( \rho_* \sim 1 \) is chosen small enough such that Lemma 5.1 is applicable. By the continuity of \( \sigma \) and \( \psi \), the bound (5.29) immediately extends to \( \mathbb{D}_{\rho_*, \theta} \). Instead of (5.29), we will prove that

\[
\langle x, (\text{Id} + F)(\text{Id} - C_s F)^{-1}Q_{s,F}[x]\rangle + (f_u, x)^2 \sim \|x\|^2_2
\]  

for all \( x \in \mathcal{A} \) satisfying \( C_s[x] = x \) and \( x = x^* \). Since these conditions are satisfied by \( x = sf_u^2 \), (5.30) immediately implies (5.29). In fact, the upper bound in (5.30) follows from \( ||(\text{Id} - C_s F)^{-1}Q_{s,F}\|_2 \lesssim 1 \) by Lemma 5.1, \( ||F\|_2 \leq 1 \) and \( f_u \sim 1 \) due to (5.8).

From \( C_s[x] = x \), we conclude

\[
\langle x, (\text{Id} + F)(\text{Id} - C_s F)^{-1}Q_{s,F}[x]\rangle = \langle x, (\text{Id} + C_s F)(\text{Id} - C_s F)^{-1}Q_{s,F}[x]\rangle
= \langle x, (-\text{Id} + 2(\text{Id} - C_s F)^{-1}Q_{s,F}[x]\rangle
\]  

Using (5.17) and \( C_s[f_u] = f_u \), we see that

\[
C_s P_{s,F}[x] = P_{s,F}[x] + O(\rho^{-1} \text{Im} \, z), \quad C_s Q_{s,F}[x] = Q_{s,F}[x] + O(\rho^{-1} \text{Im} \, z)
\]  

for \( x \in \mathcal{A} \) satisfying \( C_s[x] = x \). When applied to (5.31), the expansion (5.32) and (Id − FC_s)^{-1} = C_s(Id −
to a cubic equation, (6.3) below, close to points \( \tau \) all

\[ \| \langle f, \cdot \rangle f \|_2^2 + O(\|x\|_2^2\rho^{-1}\text{Im } z) \]

\[ \| \langle f, \cdot \rangle f \|_2^2 + O(\|x\|_2^2\rho^{-1}\text{Im } z) \]

Here, in the first step, we also used the second and third relation in (5.17). In the third step, we then defined the orthogonal projections \( P_f := \langle f, \cdot \rangle f \) and \( Q_f := \text{Id} - P_f \), where \( \|f\|_2 \) (cf. Assumptions 4.5 (ii)), and inserted \( Q_f \) using

\[ P_f Q_f = O(\rho^{-1}\text{Im } z) \]

which follows from (5.10) and (5.17). We also used that \( Q_{s,F} \) commutes with \( (\text{Id} - C_s F)^{-1} \). The fourth step is a consequence of (4.7) and (4.18). In the last step, we employed \( Q_f Q_{s,F} = Q_{s,F} + O(\rho^{-1}\text{Im } z) \) by (5.34) and \( \|\text{Id} - C_s F\|_2 \leq 2 \).

By (5.17), we have \( \|P_{s,F} f\|_2^2 = (f_s, x)^2 + O(\|x\|_2^2\rho^{-1}\text{Im } z) \) if \( x = x^* \). Combining this observation with (5.33) proves (5.30) up to terms of order \( O(\|x\|_2^2\rho^{-1}\text{Im } z) \). Hence, possibly shrinking \( \rho_* \sim 1 \) and requiring \( \rho(z)^{-1}\text{Im } z \leq \rho_* \) complete the proof of the lemma.

6. The cubic equation

The following Proposition 6.1 is the main result of this section. It asserts that \( m \) is determined by the solution to a cubic equation, (6.3) below, close to points \( \tau_0 \in \text{supp } \rho \) of small density \( \rho(\tau_0) \). In Section 7, this cubic equation will allow for a classification of the small local minima of \( \tau \mapsto \rho(\tau) \).

The leading order terms of the cubic and quadratic coefficients in (6.3) are given by \( \psi(\tau_0) \) and \( \sigma(\tau_0) \), respectively. For their definition, we refer to Lemma 5.5 (i) and (5.12). We recall the definition of \( \mathbb{D}_{c, \theta} \) from (5.26).

**Proposition 6.1** (Cubic equation for shape analysis). Let \( I \subset \mathbb{R} \) be an open interval and \( \theta \in (0,1] \). If Assumptions 4.5 hold true on \( I \) for some \( \eta \in (0,1] \) then there are thresholds \( \rho_* \sim 1 \) and \( \delta_* \sim 1 \) such that, for all \( \tau_0 \in \mathbb{D}_{\rho_*, \theta} \), the following hold true:

(a) For all \( \omega \in [-\delta_*, \delta_*] \), we have

\[ m(\tau_0 + \omega) - m(\tau_0) = \Theta(\omega) b + r(\omega), \]

where \( \Theta: [-\delta_*, \delta_*] \to \mathbb{C} \) and \( r: [-\delta_*, \delta_*] \to \mathbb{A} \) are defined by

\[ \Theta(\omega) := \left\{ \frac{l}{(b,l)} , m(\tau_0 + \omega) - m(\tau_0) \right\}, \quad r(\omega) := Q[m(\tau_0 + \omega) - m(\tau_0)]. \]

Here, \( l = l(\tau_0) \), \( b = b(\tau_0) \) and \( Q = Q(\tau_0) \) are the eigenvectors and spectral projection of \( B(\tau_0) \) introduced in Corollary 5.2. We have \( b = b^* + O(\rho) \) and \( l = l^* + O(\rho) \) as well as \( b + b^* \sim 1 \) and \( l + l^* \sim 1 \) with \( \rho = \rho(\tau_0) = (\text{Im } m(\tau_0))/\pi \).

(b) The function \( \Theta \) satisfies the cubic equation

\[ \mu_3 \Theta^3(\omega) + \mu_2 \Theta^2(\omega) + \mu_1\Theta(\omega) + \omega \Xi(\omega) = 0 \]

for all \( \omega \in [-\delta_*, \delta_*] \). The complex coefficients \( \mu_3, \mu_2, \mu_1 \) and \( \Xi \) in (6.3) fulfill

\[ \mu_3 = \psi + O(\rho), \]

\[ \mu_2 = \sigma + i\rho \left( 3\psi + \frac{\sigma^2}{(F_2)^2} \right) + O(\rho^2), \]

\[ \mu_1 = 2i\rho \sigma - 2\rho^2 \left( \psi + \frac{\sigma^2}{(F_2)^2} \right) + O(\rho^3), \]

\[ \Xi(\omega) = \pi(1 + \nu(\omega)) + O(\rho), \]
where $\sigma = \sigma(t_0)$ as well as $\psi = \psi(t_0)$. For the error term $\nu(\omega)$, we have

$$|\nu(\omega)| \lesssim |\Theta(\omega)| + |\omega| \lesssim |\omega|^{1/3}.$$  \hspace{1cm} (6.5)

for all $\omega \in [-\delta, \delta]$. Uniformly for $t_0 \in \mathbb{D}_{\rho, \theta}$, we have

$$\psi + \sigma^2 \sim 1.$$  \hspace{1cm} (6.6)

(c) Moreover, $\Theta(\omega)$ and $r(\omega)$ are bounded by

$$|\Theta(\omega)| \lesssim \min \left\{ \frac{|\omega|}{\rho^2}, |\omega|^{1/3} \right\},$$  \hspace{1cm} (6.7a)

$$||r(\omega)|| \lesssim |\Theta(\omega)|^2 + |\omega|,$$  \hspace{1cm} (6.7b)

uniformly for all $\omega \in [-\delta, \delta]$.

(d1) If $\rho > 0$ then $\Theta$ and $r$ are differentiable in $\omega$ at $\omega = 0$.

(d2) If $\rho = 0$ then we have

$$\text{Im } \Theta(\omega) \geq 0, \quad |\text{Im } \nu(\omega)| \lesssim \text{Im } \Theta(\omega), \quad ||\text{Im } r(\omega)|| \lesssim (|\Theta(\omega)| + |\omega|) \text{Im } \Theta(\omega),$$

for all $\omega \in [-\delta, \delta]$ and $\text{Re } \Theta$ is non-decreasing on the connected components of $\{ \omega \in [-\delta, \delta] : \text{Im } \Theta(\omega) = 0 \}$.

(e) The function $\sigma : \mathbb{D}_{\rho, \theta} \to \mathbb{R}$ is uniformly $1/24$-Hölder continuous.

The previous proposition is the analogue of Lemma 9.1 in [1]. The cubic equation for $\Theta$, (6.3), will be obtained from an $A$-valued quadratic equation for $\Delta = m(t_0 + \omega) - m(t_0)$ and the results of Section 5. In fact, we have

$$(\text{Id} - C_m S) [\Delta] = \omega m^2 + \frac{\omega}{2} (m \Delta + \Delta m) + \frac{1}{2} (m S [\Delta] \Delta + \Delta S [\Delta] m),$$  \hspace{1cm} (6.9)

where $t_0, t_0 + \omega \in I_0 := \{ \tau \in I : \text{dist}(\tau, \partial I) \geq \theta \}$ and $m := m(t_0)$ (see the proof of Proposition 6.1 in Section 6.3 below for a derivation of (6.9)). Projecting (6.9) onto the direction $b$ and its complement, where $b$ is the unstable direction of $B$ defined in Corollary 5.2, yields the cubic equation, (6.3), for the contribution $\Theta$ of $\Delta$ parallel with $b$. In the next subsection, this derivation is presented in a more abstract and transparent setting of a general $A$-valued quadratic equation. After that, the coefficients of the cubic equation are computed in Lemma 6.3 in the setup of (6.9) before we prove Proposition 6.1 in Section 6.3.

### 6.1. General cubic equation

Let $B, T : A \to A$ be linear maps, $A : A \times A \to A$ a bilinear map and $K : A \times A \to A$ a map. For $\Delta, e \in A$, we consider the quadratic equation

$$B[\Delta] - A[\Delta, \Delta] - T[e] - K[e, \Delta] = 0.$$  \hspace{1cm} (6.10)

We view this as an equation for $\Delta$, where $e$ is a (small) error term. This quadratic equation is a generalization of the stability equation (6.9) for the Dyson equation, (2.3) (see (6.20) and (6.25) below for the concrete choices of $B, T, A$ and $K$ in the setting of (6.9)).

Suppose that $B$ has a non-degenerate isolated eigenvalue $\beta$ and a corresponding eigenvector $b$, i.e., $B[b] = \beta b$ and $D_r(\beta) \cap \text{Spec}(B) = \{ \beta \}$ for some $r > 0$. We denote the spectral projection corresponding to $\beta$ and its complementary projection by $P$ and $Q$, respectively, i.e.,

$$P = -\frac{1}{2\pi i} \oint_{D_r(\beta)} (B - \omega \text{Id})^{-1} \, d\omega = \frac{\langle \cdot, e \rangle}{\langle I, b \rangle} b, \quad Q = \text{Id} - P.$$  \hspace{1cm} (6.11)

Here, $l \in A$ is an eigenvector of $B^*$ corresponding to its eigenvalue $\overline{\beta}$, i.e., $B^*[l] = \overline{\beta} l$. In the following, we will assume that

$$\|B^{-1} Q [x]\| \lesssim \|x\|, \quad \|l, b\|^{-1} + \|b\| + \|l\| \lesssim 1, \quad \|A[x, y]\| \lesssim \|x\| \|y\|, \quad \|T[e]\| \lesssim \|e\|, \quad \|K[e, y]\| \lesssim \|e\| \|y\|$$  \hspace{1cm} (6.12)
for all $x, y \in A$ and the $e \in A$ from (6.10). The guiding idea is that the main contribution in the decomposition

$$
\Delta = \Theta b + Q[\Delta], \quad \Theta := \frac{(l, \Delta)}{(l, b)}
$$

(6.13)
is given by $\Theta$, i.e., the coefficient of $\Delta$ in the direction $b$, under the assumption that $\Delta$ is small. If $A = K = 0$ then this would be a simple linear stability analysis of the equation $B[\Delta] = \text{small}$ around an isolated eigenvalue of $B$. The presence of the quadratic terms in (6.10) requires to follow second and third order terms carefully. In the following lemma, we show that the behaviour of $\Theta$ is governed by a scalar-valued cubic equation (see (6.14) below) and that $Q[\Delta]$ is indeed dominated by $\Theta$. The implicit constants in (6.12) are the model parameters in Section 6.1.

**Lemma 6.2** (General cubic equation). Let $\beta$ be a non-degenerate isolated eigenvalue of $B$. Let $\Delta \in A$ and $e \in A$ satisfy (6.10), $\Theta$ be defined as in (6.13) and the conditions in (6.12) hold true. Then there is $\varepsilon \sim 1$ such that if $\|\Delta\| \leq \varepsilon$ then $\Theta$ satisfies the cubic equation

$$
\mu_3 \Theta^3 + \mu_2 \Theta^2 + \mu_1 \Theta + \mu_0 = \tilde{e},
$$

(6.14)

with some $\tilde{e} = O(|\Theta|^4 + |\Theta||e|| + \|e\|^2)$ and with coefficients

$$
\begin{align*}
\mu_3 &= \langle l, A[b, B^{-1}QA[b, b]] + A[B^{-1}QA[b, b], b] \rangle, \\
\mu_2 &= \langle l, A[b, b] \rangle, \\
\mu_1 &= -\beta(l, b), \\
\mu_0 &= \langle l, T[e] \rangle.
\end{align*}
$$

(6.15)

Moreover, we have

$$
Q[\Delta] = B^{-1}QT[e] + O(|\Theta|^2 + \|e\|^2).
$$

(6.16)

If we additionally assume that $\text{Im} \Delta \in \mathcal{A}_+$, $l = l^*$ and $b = b^*$ as well as

$$
B[x]^* = B[x^*], \quad A[x, y]^* = A[x^*, y^*], \quad T[e]^* = T[e], \quad K[e, y]^* = K[e, y^*]
$$

(6.17)

for all $x, y \in A$ then there are $\varepsilon \sim 1$ and $\delta \sim 1$ such that $\|\Delta\| \leq \varepsilon$ and $\|e\| \leq \delta$ also imply

$$
\begin{align*}
\|\text{Im} Q[\Delta]\| &\lesssim (|\Theta| + \|e\|)|\text{Im} \Theta|, \\
\|\text{Im} \tilde{e}\| &\lesssim (|\Theta|^3 + \|e\|)|\text{Im} \Theta|.
\end{align*}
$$

(6.18a)

(6.18b)

**Proof.** Setting $r := Q[\Delta]$, the quadratic equation (6.10) reads as

$$
\Theta b + Br = T[e] + A[\Delta, \Delta] + K[e, \Delta].
$$

(6.19)

By applying $Q$ and afterwards $B^{-1}$ to the previous relation, we conclude that

$$
r = B^{-1}QT[e] + \Theta^2 B^{-1}QA[b, b] + e_1, \quad e_1 := \Theta B^{-1}Q(A[b, r] + A[r, b]) + B^{-1}QA[r, r] + B^{-1}KQ[e, \Delta].
$$

(6.20)

We have $\|e_1\| \lesssim \|r\||\Theta| + \|r\|^2 + \|e\||\Delta\|$ and $\|r\| \lesssim \|e\| + |\Theta|^2 + \|e_1\|$. From the second bound in (6.12), we conclude $\|P\| + \|Q\| \lesssim 1$ and, thus, $\|r\| \lesssim \|\Delta\|$. By choosing $\varepsilon \sim 1$ small enough, assuming $\|\Delta\| \leq \varepsilon$ and using $\|r\| \lesssim \|\Delta\|$, we obtain

$$
\|r\| \lesssim |\Theta|^2 + \|e\|, \quad \|e_1\| \lesssim |\Theta|^3 + \|e\||\Theta| + \|e\|^2.
$$

(6.21)

This proves (6.16). Defining $e_2 := e_1 + B^{-1}QT[e]$ yields $\Delta = \Theta b + \Theta^2 B^{-1}QA[b, b] + e_2$. By plugging this into (6.19) and computing the scalar product with $(l, \cdot)$, we obtain

$$
\Theta (l, b) = \langle l, T[e] \rangle + \Theta^2 (l, A[b, b]) + \Theta^3 (l, A[b, B^{-1}QA[b, b]] + A[B^{-1}QA[b, b], b]) - \tilde{e},
$$

(6.22a)

$$
\tilde{e} := -\langle l, K[e, \Delta] \rangle - \Theta^4 A[B^{-1}QA[b, b]] + B^{-1}QA[b, b] - A[\Delta, e_2] + A[e_2, \Delta] + A[e_2, e_2].
$$

(6.22b)

Since $\|e_2\| \lesssim |\Theta|^3 + \|e\|$ and $|\Delta| \lesssim |\Theta| + \|e\|$ by (6.21) and (6.16), we conclude $\tilde{e} = O(|\Theta|^4 + |\Theta||e|| + \|e\|^2)$. Therefore, $\Theta$ satisfies (6.14) with the coefficients from (6.15).

For the rest of the proof, we additionally assume that the relations in (6.17) hold true. Taking the imaginary
part of (6.20) and arguing similarly as after (6.20) yield
\[\|\text{Im} \epsilon_1\| \lesssim (\|r\| + \|\Theta\| + \|e\|)(\text{Im} \Theta + \|\text{Im} r\|), \quad \|\text{Im} r\| \lesssim |\Theta|\text{Im} \Theta + \|\text{Im} \epsilon_1\|.\]

Hence, (6.18a) and \|\text{Im} \epsilon_1\| \lesssim (\|\Theta\| + \|e\|)|\text{Im} \Theta|\) follow for \|\Delta\| \leq \varepsilon and \|e\| \leq \delta with some sufficiently small \varepsilon \sim 1 and \delta \sim 1. From this and taking the imaginary part in (6.22b), we conclude (6.18b) as \|\text{Im} \Delta\| \lesssim \text{Im} \Theta by (6.18a) and \text{Im} \epsilon_2 = \text{Im} \epsilon_1. This completes the proof of Lemma 6.2.

\[\Box\]

6.2. Cubic equation associated to Dyson stability equation

Owing to (6.15), the coefficients \(\mu_3, \mu_2\) and \(\mu_1\) are completely determined by the bilinear map \(A\) and the operator \(B\). For analyzing the Dyson equation, (2.3), owing to (6.9), the natural choices for \(A\) and \(B\) are
\[B := \text{Id} - C_mS, \quad A[x, y] := \frac{1}{2}(mS[x]y + yS[x]m)\] (6.23)
with \(x, y \in A\). In particular, \(Q\) in (6.11) has to be understood with respect to \(B = \text{Id} - C_mS\). In the next lemma, we compute \(\mu_3, \mu_2\) and \(\mu_1\) with these choices. This computation involves the inverse of \(\text{Id} - C_sF\).

In order to directly ensure its invertibility, we will assume \(\text{Im} z > 0\). This assumption will be removed in the proof of Proposition 6.1 in Section 6.3 below.

Lemma 6.3 (Coefficients of the cubic for Dyson equation). Let \(A\) and \(B\) be defined as in (6.23). If Assumptions 4.5 hold true on an interval \(I \subseteq \mathbb{R}\) for some \(\gamma_k \in (0,1]\) then there is a threshold \(\rho_s \sim 1\) such that, for \(z \in \mathbb{H}_{I,0}\), satisfying \(\rho(z) + \rho(z)^{-1}\text{Im} z \leq \rho_s\), the coefficients of the cubic (6.14) have the expansions
\[\mu_3 = \psi + O(\rho + \rho^{-1}\text{Im} z),\] (6.24a)
\[\mu_2 = \sigma + i\rho(3\psi + \frac{\sigma^2}{fu}) + O(\rho^2 + \rho^{-1}\text{Im} z),\] (6.24b)
\[\mu_1 = -\pi \rho^{-1}\text{Im} z + 2i\rho \sigma - 2\rho^2 \left(\psi + \frac{\sigma^2}{fu}\right) + O(\rho^3 + \text{Im} z + \rho^{-2}(\text{Im} z)^2).\] (6.24c)

Moreover, we also have
\[(l, mS[b][b]) = \sigma + i\rho(3\psi + \frac{\sigma^2}{fu}) + O(\rho^2 + \rho^{-1}\text{Im} z).\] (6.25)

Proof. In this proof, we use the convention that concatenation of maps on \(A\) and evaluation of these maps in elements of \(A\) are prioritized before the multiplication in \(A\), i.e.,
\[AB[c] := (A[B[c]])c\]
if \(A\) and \(B\) are maps on \(A\) and \(b, c \in A\). We will obtain all expansions in (6.24) from (6.15) by using the special choices for \(A\) and \(B\) from (6.23). Before starting with the proof of (6.24a), we establish a few identities. Recalling \(m = q^*[uq\) from (3.2) and (3.4), we first notice the following alternative expression for \(A\)
\[A[x, y] = \frac{1}{2}C_{q^*[u}FC_{q^*[u}[x]C_{q^*[u]}^{-1}[y] + C_{q^*[u]}^{-1}[y]FC_{q^*[u]}[x]u\] (6.26)
with \(x, y \in A\). Owing to (4.21), the operators \(C_{q^*[u}\) and \(C_{q^*[u]}^{-1}\) are bounded. We choose \(\rho_s \sim 1\) small enough so that Lemma 5.1 is applicable. By using \(u = s + i\text{Im} u + O(\rho^2)\) due to (5.2) as well as (5.4), (5.5) and (5.13a) in (6.26), we obtain
\[A[b_0, b_0] = C_{q^*[u}[s]fu^2 + i\rho fu^3] + O(\rho^2 + \rho^{-1}\text{Im} z).\] (6.27)
Combining (6.27) and (5.18) implies
\[B_0^{-1}Q_0A[b_0, b_0] = C_{q^*[u}[\text{Id} - C_sF]^{-1}Q_sF[s]fu^2\] + \(O(\rho + \rho^{-1}\text{Im} z).\)

We now prove the expansion (6.24a) for \(\mu_3\) by starting from (6.15) and using \(l = l_0 + O(\rho), b = b_0 + O(\rho)\) by
In this subsection, we will prove Proposition 6.1 by using Lemma 6.2 and Lemma 6.3. Therefore, in addition to the choices of $A$ and $B$ in (6.23), we choose $\Delta = m(\tau_0 + \omega) - m(\tau_0)$, $\tau_0, \tau_0 + \omega \in I$, $e = \omega I$ and

$$T[x] = x m^2, \quad K[x, y] = \frac{1}{2}(x m y + y m x) \quad (6.28)$$

for $x, y \in A$ with $m = m(\tau_0)$ in (6.10).

Proof of Proposition 6.1. We choose $\rho_c \sim 1$ such that Lemma 5.1 and Corollary 5.2 are applicable. We fix $\tau_0 \in \mathbb{D}_{\rho_c, \theta}$ and set $m = m(\tau_0)$. The statements about $l$ and $b$ in (a) of Proposition 6.1 follow from Corollary 5.2. In particular, $|\langle l, b \rangle| \sim 1$. Thus, the conditions in (6.12) are a direct consequence of Assumptions 4.5, (4.21), Lemma 5.1 and Corollary 5.2. Furthermore, if $\rho = 0$ then we have $m = m^*$ and thus, (6.17) follows. For $\omega \in [-\delta_c, \delta_c]$, $\delta_c = \theta/2$, we set $\Delta = m(\tau_0 + \omega) - m$. Since $\Theta(\omega) b = P[\Delta], r(\omega) = Q[\Delta]$ and $P + Q = Id$, we immediately obtain (6.11). This proves (a).

Next, we derive (6.9) for $\Delta := m(\tau_0 + \omega) - m(\tau_0)$ and $m := m(\tau_0)$ with $z_0 := \tau_0 + i \eta$, $\eta \in [-\delta_c, \delta_c]$ and $\eta \in (0, \eta_c)$. We subtract (2.3) evaluated at $z = z_0$ from (2.3) evaluated at $z = z_0 + \omega$ and obtain (6.9) with the original choices of $\Delta$ and $m$ at $z_0 = \tau_0$ by the Hölder-continuity of $m$ on $\mathbb{D}_{\rho_c, \theta}$, $I' := \{ \tau \in I : \text{dist}(\tau, \partial I) \geq \theta/2 \}$, due to Proposition 4.7.

Lemma 6.2 is applicable for $|\omega| \leq \delta_c$ with some sufficiently small $\delta_c \sim 1$ since this guarantees $|\Delta| \leq \varepsilon$ owing to the Hölder-continuity of $m$. Hence, Lemma 6.2 yields a cubic equation for $\Theta$ as defined in (6.2) with $l = l(z_0), b = b(z_0)$ and $z_0 = \tau_0 + i \eta$. The coefficients of this cubic equation are given in Lemma 6.2. Owing to the uniform $1/3$-Hölder continuity of $z \to m(z)$ on $\mathbb{D}_{\rho_c, \theta}$, we conclude from the definition of $\Theta$ and $r := Q[\Delta]$ in (6.2), the boundedness of $Q$ and $B^{-1}Q$ as well as (6.16) that $|\Theta(\omega)| \lesssim |\omega|^{1/2}$, i.e., the second bound in (6.7a), and (6.7b) uniformly for $\eta \in [0, \eta_c]$.
We now compute the coefficients of the cubic in (6.3) for \( \tau_0 \in \mathbb{D}_{\rho^*, 0}^\times \), where \( \mathbb{D}_{\rho^*, 0}^\times \) was defined in (5.26). Set 
\[ z_0 := \tau_0 + i\eta. \]
Note that for \( \eta = \text{Im} z_0 > 0 \) these coefficients were already given in (6.24), so the only task is to check their limit behaviour as \( \eta \downarrow 0 \). Since \( \rho^{-1}(z_0) \eta \rightarrow 0 \) for \( \eta \downarrow 0 \) as \( \rho(\text{Re} z_0) = \rho(\tau_0) > 0 \), the expansions in (6.4a), (6.4b) and (6.4c) follow from (6.24a), (6.24b) and (6.24c), respectively, using the continuity of \( \psi \), \( \eta \), and \( f_u \) on \( \mathbb{H}_{\text{small}} \cap \mathbb{D}_{\rho^*, 0}^\times \) by Lemma 5.5 and Lemma 5.4, respectively. We now show (6.4d). With the definitions of \( \tilde{c} \) and \( \mu_0 \) from Lemma 6.2, we set \( \Xi(\cdot) := \omega^{-1}(\mu_0 - \tilde{c}) \) for arbitrary \( |\omega| \leq \delta \). Since \( I = C_{q, q}^{-1} f_u + O(\rho + \rho^{-1} \eta) \) due to (5.13a) and (5.14b), as well as \( m^2 = (\text{Re} m)^2 + O(\rho) = C_{q, q} C_{\Psi(q, q')} + O(\rho) \) due to \( \text{Im} m \sim \rho \mathbb{I} \) and (5.2), we have
\[ \omega^{-1}\mu_0 = (l^* m^2) = (f_u q q^* + O(\rho + \rho^{-1} \eta)) = \pi + O(\rho + \rho^{-1} \eta). \] (6.29)

Here, we also used \( C_{u}(f_u) = f_u \) in the second step and (5.19) in the last step. We set \( \nu(\omega) := -(\omega \pi)^{-1} \tilde{c} \). We recall \( e = \omega \mathbb{I} \). Since \( \tilde{c} = O((|\Theta(\omega)|^2 + |\Theta(\omega)| |\omega| + |\omega|^2) \) and \( |\Theta(\omega)| \leq |\omega|^{1/3} \), we obtain (6.5). This yields (6.4d) by using \( \rho^{-1}(\tau_0) \eta \rightarrow 0 \) for \( \eta \downarrow 0 \) as \( \rho(\text{Re} z_0) > 0 \) in (6.29). Since (6.29) implies (6.6), this completes the proof of (b) for \( \tau_0 \in \mathbb{D}_{\rho^*, 0}^\times \).

We now fix \( \tau_0 \in \mathbb{D}_{\rho^*, 0} \setminus \mathbb{D}_{\rho^*, 0}^\times \). In particular, \( \rho(\tau_0) = 0 \). Since \( \sigma \) and \( \psi \) are Hölder-continuous on \( \mathbb{D}_{\rho^*, 0} \) and \( f_u \sim 1 \) on \( \mathbb{D}_{\rho^*, 0} \), we obtain the expansions for \( \mu_3 \), \( \mu_2 \) and \( \mu_1 \) at \( \tau_0 \) from the corresponding ones on \( \mathbb{D}_{\rho^*, 0}^\times \). Similarly, we have \( \omega^{-1} \mu_0 = \pi \) at \( \tau_0 \). Finally, the definition of \( \tilde{c} \) in the proof of Lemma 6.2 shows that it is continuous in \( \tau \). This proves (b) in all cases and we assume \( \eta = 0 \) in the following.

If \( \rho = \rho(\tau_0) > 0 \) then (4.20) yields the missing first bound in (6.7a) completing the proof of part (c). Moreover, in this case, the definitions of \( \Theta \) and \( r \) imply their differentiability at \( \omega = 0 \) due to Proposition 4.7. This shows (d1).

We now verify (d2). Since \( \rho = 0 \), we have \( \text{Im} m(\tau_0) = 0 \) and thus \( \text{Im} \Theta(\omega) \geq 0 \) by the positive semidefiniteness of \( \text{Im} m(\tau_0 + \omega) \). Since \( \rho \) is real as \( l \) and \( T[e] \) are self-adjoint, we obtain the second bound in (6.8) directly from (6.18b) and \( (\Theta(\omega)) \leq |\omega|^{1/3} \). The third bound in (6.8) follows from (6.18a) and \( e = \omega \mathbb{I} \). Since \( \rho = 0 \) and hence \( b = C_{q, q} f_u \) by (5.14a) and \( l = C_{\Psi(q)} f_u \) by (5.14b) are positive definite elements of \( \mathcal{A} \), \( \text{Re} \Theta(\omega) + (l, m(\tau_0))/|l| b \) is the real part of the Stieltjes transform of a positive measure \( \mu \) evaluated on the real axis. The real part of a Stieltjes transform is non-decreasing on the connected components of the complement in \( \mathbb{R} \) of the support of its defining measure. Therefore, as the support of \( \mu \) is contained in \( \mathbb{R} \setminus [\omega = [-\delta, \delta] \}: \text{Im} \Theta(\omega) = 0 \} \) due to \( \text{Im} m(\tau_0) = 0 \), we conclude that \( \text{Re} \Theta(\omega) \) is non-decreasing on the connected components of \( \omega \in [-\delta, \delta] \): \( \text{Im} \Theta(\omega) = 0 \).

Lemma 5.5 (i) directly implies the Hölder-continuity in (e), which completes the proof of Proposition 6.1. \( \square \)

7. Cubic analysis

The main result of this section, Theorem 7.1 below, implies Theorem 2.5 and gives even effective error terms. Theorem 7.1 describes the behaviour of \( \text{Im} m \) close to local minima of \( \rho \) inside of \( \text{supp} \rho \). This behaviour is governed by the universal shape functions \( \Psi_{\text{edge}} : [0, \infty) \rightarrow \mathbb{R} \) and \( \Psi_{\text{min}} : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[ \Psi_{\text{edge}}(\lambda) := \frac{\sqrt{(1 + \lambda) \lambda}}{(1 + 2\lambda + 2\sqrt{(1 + \lambda) \lambda}^{2/3} + (1 + 2\lambda - 2\sqrt{(1 + \lambda) \lambda}^{2/3}) + 1}, \]
\[ \Psi_{\text{min}}(\lambda) := \frac{\sqrt{1 + \lambda^2}}{(\sqrt{1 + \lambda^2} + \lambda^{2/3} + (\sqrt{1 + \lambda^2} - \lambda)^{2/3} - 1}. \]

For the definition of the comparison relation \( \lesssim, \gtrsim \) and \( \sim \) in the following Theorem 7.1, we refer to Convention 3.4 and remark that the model parameters in Theorem 7.1 are given by \( c_1, c_2 \) and \( c_3 \) in (3.10), \( k_3 \) in (4.16) and \( \theta \) in the definition of \( I_\theta \) in (7.2) below.

Theorem 7.1 (Behaviour of \( \text{Im} m \) close to local minima of \( \rho \)). Let \( (a, S) \) be a data pair such that (3.10) is satisfied. Let \( m \) be the solution to the associated Dyson equation (2.3) and assume that (4.16) holds true on \( \mathbb{H}_{l, u} \) for some interval \( I \subset \mathbb{R} \) and some \( \tau_0 \in (0, 1) \). We write \( v := \pi^{-1} \text{Im} m \) and, for some \( \theta \in (0, 1] \), we set
\[ I_\theta := \{ \tau \in I : \text{dist}(\tau, \partial I) \geq \theta \}. \]

Then there are thresholds \( \rho_* \sim 1 \) and \( \delta_* \sim 1 \) such that if \( \tau_0 \in \text{supp} \rho \cap I_\theta \) is a local minimum of \( \rho \) and \( \rho(\tau_0) \leq \rho_* \) then
\[ v(\tau_0 + \omega) = v(\tau_0) + h \Psi(\omega) + O\left(\rho(\tau_0) |\omega|^{1/3} \mathbf{1}(|\omega| \leq \rho(\tau_0)^3) + \Psi(\omega)^2 \right) \]

(7.3)
for $\omega \in [-\delta, \delta] \cap D$ with some $h = h(\tau_0) \in A$ satisfying $h \sim 1$. Moreover, the set $D$ and the function $\Psi$ depend only on the type of $\tau_0$ in the following way:

(a) Left edge: If $\tau_0 \in (\partial \text{supp} \rho) \setminus \{\inf \text{supp} \rho\}$ is the infimum of a connected component of $\text{supp} \rho$ and the lower edge of the corresponding gap is in $I$, i.e., $\tau_1 := \sup((-\infty, \tau_0) \cap \text{supp} \rho) \in I$, then (7.3) holds true with $v(\tau_0) = 0$, $D = [0, \infty)$ and

$$\Psi(\omega) = \Delta^{1/3} \Psi_{\text{edge}}\left(\frac{\omega}{\Delta}\right)$$

where $\Delta := \tau_0 - \tau_1$. If $\tau_0 = \inf \text{supp} \rho$, or more generally $\rho(\tau) = 0$ for all $\tau \in [\tau_0 - \varepsilon, \tau_0]$ with some $\varepsilon \sim 1$, then the same conclusion holds true with $\Delta := 1$.

(b) Right edge: If $\tau_0 \in \partial \text{supp} \rho$ is the supremum of a connected component then a similar statement as in the case of a left edge holds true.

(c) Cusp: If $\tau_0 \notin \partial \text{supp} \rho$ and $\rho(\tau_0) = 0$ then (7.3) holds true with $D = \mathbb{R}$ and $\Psi(\omega) = |\omega|^{1/3}$.

(d) Internal minimum: If $\tau_0 \notin \partial \text{supp} \rho$ and $\rho(\tau_0) > 0$ then there is $\tilde{\rho} \sim \rho(\tau_0)$ such that (7.3) holds true with $D = \mathbb{R}$ and

$$\Psi(\omega) = \tilde{\rho} \Psi_{\text{min}}\left(\frac{\omega}{\tilde{\rho}^2}\right).$$

Theorem 7.1 contains the most important results of the shape analysis. Auxiliary statements about the size of the connected components of $\text{supp} \rho$ and the distance between local minima are also available; they will be stated precisely in Proposition 7.3 below. We remark that $\Psi_{\text{min}}(\omega) = \Psi_{\text{min}}(-\omega)$ for $\omega \in \mathbb{R}$ and, for $\omega > 0$, $\Delta > 0$ and $\tilde{\rho} > 0$, we have

$$\Delta^{1/3} \Psi_{\text{edge}}\left(\frac{\omega}{\Delta}\right) \sim \min\left\{\frac{\omega^{1/2}}{\Delta^{1/6}}, |\omega|^{1/3}\right\},$$

(7.4a)

$$\tilde{\rho} \Psi_{\text{min}}\left(\frac{\omega}{\tilde{\rho}^2}\right) \sim \min\left\{\frac{\omega^2}{\tilde{\rho}^2}, |\omega|^{1/3}\right\}.$$  

(7.4b)

In the following definition, we introduce the notion of a shape regular point which collects the properties of $m$ necessary for the proof of Theorem 7.1. Proposition 7.3 below explains how the statements of Theorem 7.1 are transferred to this more general setup. In fact, Lemma 4.8 (ii) and Proposition 6.1 show that, under the assumptions of Theorem 7.1, any point $\tau_0 \in \text{supp} \rho \cap I$ of sufficiently small density $\rho(\tau_0)$ is a shape regular point for $m$ in the sense of Definition 7.2 below. By explicitly spelling out the properties of $m$ really used in the proof of Theorem 7.1 we made our argument modular because a similar analysis around shape regular points will be applied in later works as well.

This modularity, however, requires to reinterpret the concept of comparison relations. In earlier sections we used the comparison relation $\sim$, \lesssim and the $O$-notation introduced in Convention 3.4 to hide irrelevant constants in various estimates that depended only on the model parameters $c_1$, $c_2$, $c_3$ from (3.10), $k_3$ from (4.16) and $\theta$ from (7.2), these are also the model parameters in the conditions of Theorem 7.1.

The formulation of Definition 7.2 also involves comparison relations instead of carrying constants; in the application these constants depend on the original model parameters. When Proposition 7.3 is proven, the corresponding constants directly depend on the constants in Definition 7.2, hence they also indirectly depend on the original model parameters when we apply it to the proof of Theorem 7.1. Since these dependences are somewhat involved and we do not want to overload the paper with different concepts of comparison relations, for simplicity, for the purpose of Theorem 7.1, the reader may think of the implicit constants in every $\sim$-relation depending only on the original model parameters $c_1$, $c_2$, $c_3$, $k_3$ and $\theta$.

**Definition 7.2** (Admissibility for shape analysis, shape regular points). Let $m$ be the solution of the Dyson equation (2.3) associated to a data pair $(a, S) \in A_{sa} \times \Sigma$.

(i) Let $\tau_0 \in \mathbb{R}$, $J \subset \mathbb{R}$ be an open interval with $0 \in J$, $\Theta : J \to \mathbb{C}$ and $r : J \to A$ be continuous functions and $b \in A$. We say that $m$ is $(J, \Theta, b, r)$-admissible for the shape analysis at $\tau_0$ if the following conditions are satisfied:

(a) The function $m : H \to A$ has a continuous extension to $\tau_0 + J$, which we also denote by $m$. The relation (6.1) and the bounds (6.7a) as well as (6.7b) hold true for all $\omega \in J$. 

26
(b) The function $\Theta$ satisfies the cubic equation (6.3) for all $\omega \in J$ with the coefficients
\[
\begin{align*}
\mu_3 &= \psi + O(\rho), \\
\mu_2 &= \sigma + 13\psi \rho + O(\rho^2 + \rho |\sigma|), \\
\mu_1 &= -2\rho^2 \psi + i\kappa_1 \rho \sigma + O(\rho^3 + \rho^2 |\sigma|), \\
\Xi(\omega) &= \kappa(1 + \nu(\omega)) + O(\rho),
\end{align*}
\]
where $\rho := (\text{Im } m(\tau_0))/\pi$ and $\psi, \kappa \geq 0$ as well as $\sigma, \kappa_1 \in \mathbb{R}$ are some parameters satisfying (6.6) and $\kappa, |\kappa_1| \sim 1$. The function $\nu: J \to \mathbb{C}$ satisfies (6.5).
(c) The element $b \in A$ in (6.1) fulfills $b = b^* + O(\rho)$ and $b + b^* \sim 1$.

(i) Let $\tau_0 \in \mathbb{R}$ and $J \subset \mathbb{R}$ be an open interval with $0 \in J$. We say that $\tau_0$ is a shape regular point for $m$ on $J$ if $m$ is $(J, \Theta, b, r)$-admissible for the shape analysis at $\tau_0$ for some continuous functions $\Theta: J \to \mathbb{C}$ and $r : J \to A$ as well as $b \in A$.

The key technical step in the proof of Theorem 7.1 is the following Proposition 7.3: its first part shows that Theorem 7.1 holds under more general weaker conditions, in fact shape admissibility is sufficient. For the proof of Theorem 7.1 we will first check shape regularity from Proposition 6.1 and then we will prove part (i) of Proposition 7.3: both steps are done in Section 7.3 below. Moreover, Proposition 7.3 contains additional information on the size of the connected components of $\text{supp } \rho$ and the distance between local minima; these are collected in part (ii). Using Proposition 6.1, they can also be transferred to the setup of Theorem 7.1 however we did not list them under Theorem 7.1 for brevity. Note that the same information were also proven in the commutative setup in Theorem 2.6 of [1] and Proposition 7.3 shows that they are also available in our general von Neumann algebra setup.

**Proposition 7.3 (Theorem 7.1 under weaker assumptions; Structure of the set of minima in $\text{supp } \rho \cap I$).** For the solution $m$ to the Dyson equation (2.3), we write $v := \pi^{-1} \text{Im } m$, $\rho = \langle v \rangle$.

(i) There are thresholds $\rho_* \sim 1$ and $\delta_\xi \sim 1$ such that if $\rho(\tau_0) \leq \rho_*$ and $\tau_0 \in \partial \text{supp } \rho$ is a local minimum of $\rho$ as well as a shape regular point for $m$ on $J$ with an open interval $J \subset \mathbb{R}$ satisfying $0 \in J$ then (7.3) holds true for all $\omega \in [-\delta_\xi, \delta_\xi] \cap J \cap D$. Here, as in Theorem 7.1, $h = h(\tau_0) \in A$ with $h \sim 1$ and $D$ as well as $\Psi$ depend only on the type of $\tau_0$ in the following way:

Suppose that $\tau_0 \in \partial \text{supp } \rho$ is the infimum of a connected component of $\text{supp } \rho$. If $\rho(\tau) = 0$ for all $\tau \in [\tau_0 - \varepsilon, \tau_0]$ with some $\varepsilon \sim 1$ (e.g. $\tau_0 = \inf \text{supp } \rho$) and $|\inf J| \gtrsim 1$, then the conclusion of case (a) in Theorem 7.1 holds true with $\Delta = 1$ and $v(\tau_0) = 0$.

If $\tau_0 \neq \inf \text{supp } \rho$ and $\tau_1 := \sup((-\infty, \tau_0) \cap \text{supp } \rho)$ is a shape regular point for $m$, $\Delta \lesssim 1$ with $\Delta := \tau_0 - \tau_1$ and $|\sigma(\tau_0) - \sigma(\tau_1)| \lesssim |\tau_0 - \tau_1|^\zeta$ for some constant $\zeta \in (0, 1/3]$ then the conclusion of case (a) in Theorem 7.1 holds true with this choice of $\Delta$ as well as $v(\tau_0) = 0$.

Similarly to (a), the statement of case (b) in Theorem 7.1 can be translated to the current setup. The cases (c) and (d) of Theorem 7.1, cusp and internal minimum, respectively, hold true without any changes.

Furthermore, suppose that $\tau_0 \in \text{supp } \rho$ is a shape regular point for $m$ and $\rho(\tau_0) = 0$, then $\tau_0$ is a cusp if $\sigma(\tau_0) = 0$ and $\tau_0$ is an edge, in particular $\tau_0 \in \partial \text{supp } \rho$, if $\sigma(\tau_0) \neq 0$.

(ii) Let $I \subset \mathbb{R}$ be an open interval with $\text{supp } \rho \cap I \neq \emptyset$ and $|I| \lesssim 1$ and let $m$ have a continuous extension to the closure $\overline{T}$ of $I$. Let $J \subset \mathbb{R}$ be an open interval with $0 \in J$ and $\text{dist}(0, \partial J) \gtrsim 1$ such that $J + (\partial \text{supp } \rho) \cap I \subset J$. We assume that all points in $(\partial \text{supp } \rho) \cap I$ are shape regular points for $m$ on $J$ and all estimates in Definition 7.2 hold true uniformly on $(\partial \text{supp } \rho) \cap I$. If $|\sigma(\tau_0) - \sigma(\tau_1)| \lesssim |\tau_0 - \tau_1|^\zeta$ for some $\zeta \in (0, 1/3]$ and uniformly for all $\tau_0, \tau_1 \in (\partial \text{supp } \rho) \cap I$ then $\text{supp } \rho \cap I$ consists of $K \sim 1$ intervals, i.e., there are $\alpha_1, \ldots, \alpha_K \in \partial \text{supp } \rho \cup \partial I$ and $\beta_1, \ldots, \beta_K \in \partial \text{supp } \rho \cup \partial I$, $\alpha_1 < \beta_1 < \alpha_{i+1}$, such that
\[
\text{supp } \rho \cap I = \bigcup_{i=1}^K [\alpha_i, \beta_i]
\]
and $\beta_i - \alpha_i \sim 1$ if $\beta_i \neq \text{sup } I$ and $\alpha_i \neq \text{inf } I$. 

27
For $\rho > 0$, we define the set $\mathbb{M}_\rho$, of small local minima $\tau$ of $\rho$ which are not edges of $\text{supp}\rho$, i.e.,

$$
\mathbb{M}_\rho := \{ \tau \in (\text{supp}\rho \setminus \partial \text{supp}\rho) \cap I : \rho(\tau) \leq \rho_* \text{, } \rho \text{ has a local minimum at } \tau \}.
$$

There is a threshold $\rho_* \sim 1$ such that if, in addition to the previous conditions in (ii), all points of $(\mathbb{M}_\rho \cup \partial \text{supp}\rho) \cap I$ are shape regular points for $m$ on $I$ and all estimates in Definition 7.2 hold true uniformly on $(\mathbb{M}_\rho \cup \partial \text{supp}\rho) \cap I$ then, for $\gamma \in \mathbb{M}_\rho$, we have $|\alpha_i - \gamma| \sim 1$ and $|\beta_i - \gamma| \sim 1$ if $\alpha_i \neq \inf I$ and $\beta_i \neq \sup I$. Moreover, for any $\gamma_1, \gamma_2 \in \mathbb{M}_\rho$, we have either

$$
|\gamma_1 - \gamma_2| \sim 1, \quad \text{or} \quad |\gamma_1 - \gamma_2| \lesssim \min\{\rho(\gamma_1), \rho(\gamma_2)\}^4.
$$

(7.5)

If $\rho(\gamma_1) = 0$ or $\rho(\gamma_2) = 0$ then, for $\gamma_1 \neq \gamma_2$ only the first case occurs.

An important step towards Theorem 7.1 and Proposition 7.3 will be to prove similar behaviours for $\Theta$ as $\text{Im } \Theta$ is the leading term in $v$. These behaviours are collected in the following theorem, Theorem 7.4. It has weaker assumptions than those of Theorem 7.1 and those required in Proposition 7.3—in particular, on the coefficient $\mu_1$ in the cubic equation (6.3). However, these assumptions will be sufficient for the purpose of Theorem 7.4.

**Theorem 7.4** (Abstract cubic equation). Let $\Theta(\omega)$ be a continuous solution to the cubic equation

$$
\mu_3 \Theta(\omega)^3 + \mu_2 \Theta(\omega)^2 + \mu_1 \Theta(\omega) + \Theta(\omega) = 0
$$

(7.6)

for $\omega \in J$, where $J \subset \mathbb{R}$ is an open interval with $0 \in J$. We assume that the coefficients satisfy

$$
\mu_3 = \psi + O(\rho),
$$

$$
\mu_2 = \sigma + 3i\psi + O(\rho^2 + \rho|\sigma|),
$$

$$
\mu_1 = -2i\rho^2 \psi + O(\rho^3 + \rho|\sigma|),
$$

$$
\Theta(\omega) = \kappa(1 + \nu(\omega)) + O(\rho)
$$

with some fixed parameters $\psi \geq 0, \sigma \geq 0, \sigma \in \mathbb{R}$ and $\kappa \sim 1$. The cubic equation is assumed to be stable in the sense that

$$
\psi + \sigma^2 \sim 1.
$$

(7.7)

Moreover, for all $\omega \in J$, we require the following bounds on $\nu$ and $\Theta$:

$$
|\nu(\omega)| \lesssim |\omega|^{1/3},
$$

(7.8a)

$$
|\Theta(\omega)| \lesssim |\omega|^{1/3}.
$$

(7.8b)

Then the following statements hold true:

(i) $(\rho > 0)$ For any $\Pi_* \sim 1$, there is a threshold $\rho_* \sim 1$ such that if $\rho \in (0, \rho_*]$ and $|\sigma| \leq \Pi_* \rho^2$ then we have

$$
\text{Im } \Theta(\omega) = \rho \Psi(\omega) \left(\frac{\omega}{\rho}\right) + O\left(\frac{\min\{|\omega|, |\omega|^{2/3}\}}{\rho^3}\right),
$$

where $\Gamma := \sqrt{2}\kappa/(2\psi)$. Note that $\Gamma \sim 1$ if $\rho_* \sim 1$ is small enough.

(ii) $(\rho = 0)$ If $\rho = 0$ and we additionally assume $\text{Im } \Theta(\omega) \geq 0$ for $\omega \in J$, $\text{Re } \Theta$ is non-decreasing on the connected components of $\{\omega \in J : \text{Im } \Theta(\omega) = 0\}$ as well as

$$
|\text{Im } \nu(\omega)| \lesssim \text{Im } \Theta(\omega)
$$

(7.10)

for all $\omega \in J$ then we have

(a) If $\sigma = 0$ then $\text{Im } \Theta(\omega)$ has a cubic cusp at $\omega = 0$, i.e.,

$$
\text{Im } \Theta(\omega) = \frac{\sqrt{3}}{2} \left(\frac{\kappa}{\psi}\right)^{1/3} |\omega|^{1/3} + O(|\omega|^{2/3}).
$$

(7.11)
(b) If $\sigma \neq 0$ then $\Im \Theta(\omega)$ has a square root edge at $\omega = 0$, i.e., there is $c_* \sim 1$ such that

$$\Im \Theta(\omega) = \begin{cases} e^{\Delta/3 \Psi_{\text{edge}}} \left( \frac{1}{\Delta} \right) + O \left( (|\nu(\omega)| + \varepsilon(\omega)) \varepsilon(\omega) \right), & \text{if } \text{sign } \omega = \text{sign } \sigma, \\ 0, & \text{if } \omega \in - \text{sign } [0, c_* |\sigma|^3], \end{cases} \tag{7.12}$$

where $\Delta \in (0, \infty), c \in (0, \infty)$ and $\varepsilon : \mathbb{R} \to [0, \infty)$ are defined by

$$\Delta := \min \left\{ \frac{4}{27 \kappa} \frac{1}{\psi^3}, 1 \right\}, \quad c := 3\sqrt{\frac{\Delta^{1/6}}{|\sigma|^{1/2}}}, \quad \varepsilon(\omega) := \min \left\{ \frac{|\omega|^{1/2}}{\Delta^{1/6}}, |\omega|^{1/3} \right\}. \tag{7.13}$$

We have $\Delta \sim |\sigma|^3$ and $c \sim 1$. Moreover, for $\text{sign } \omega = \text{sign } \sigma$, we have

$$|\Theta(\omega)| \leq \varepsilon(\omega). \tag{7.14}$$

### 7.1. Cubic equations in normal form

The core of the proof of Theorem 7.4 is to bring (7.6) into a normal form by a change of variables. We will first explain the analysis of these normal forms, especially the mechanism of choosing the right branch of the solution based upon selection principles that will be derived from the constraints on $\Theta$ given in Theorem 7.4.

Then, in Section 7.2, we show how to bring (7.6) to these normal forms.

In the following proposition, we study a special solution $\Omega(\lambda)$ to a one-parameter family of cubic equations in normal forms with constant term $\Lambda(\lambda)$ (or $2\Lambda(\lambda)$), where $\Lambda(\lambda)$ is a perturbation of the identity map $\lambda \mapsto \lambda$. Here, a-priori, the real parameter $\lambda$ is always contained in an (possibly unbounded) interval around 0. This range of definition will not be explicitly indicated in the statements but will be explicitly restricted for their conclusions. We compare the solution to this perturbed cubic equation with the solution to the cubic equation with constant term $\lambda$. Depending on the precise type of the cubic equation, the choice of the solution is based on some of the following selection principles

**SP1** $\lambda \mapsto \Omega(\lambda)$ is continuous

**SP2** $\Omega(0) = \Omega_0$ for some given $\Omega_0 \in \mathbb{C}$

**SP3** $\Im (\Omega(\lambda) - \Omega(0)) \geq 0$,

**SP4'** $|\Im \Lambda(\lambda)| \leq \gamma |\lambda| \Im \Omega(\lambda)$ for some $\gamma > 0$ and $\Re \Omega(\lambda)$ is non-decreasing on the connected components of $\left\{ \lambda : \Im \Omega(\lambda) = 0 \right\}$.

We use the notation **SP4'** to distinguish this selection principle from **SP-4** which was introduced in Lemma 9.9 of [1].

We will make use of the following standard convention for complex powers.

**Definition 7.5** (Complex powers). We define $\mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}, \zeta \mapsto \zeta^\gamma$ for $\gamma \in \mathbb{C}$ by $\zeta^\gamma := \exp(\gamma \log \zeta)$, where $\log : \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ is a continuous branch of the complex logarithm with $\log 1 = 0$.

With this convention, we record Cardano’s formula as follows:

**Proposition 7.6** (Cardano). The three roots of $\Omega^3 - 3\Omega + 2\zeta, \zeta \in \mathbb{C}$, are $\widehat{\Omega}_+(\zeta), \widehat{\Omega}_-(\zeta)$ and $\widehat{\Omega}_0(\zeta)$ which are defined by

$$\widehat{\Omega}_\pm(\zeta) := \frac{1}{2} (\Phi_+(\zeta) + \Phi_-(\zeta)) \pm \frac{i\sqrt{3}}{2} (\Phi_+(\zeta) - \Phi_-(\zeta)), \quad \widehat{\Omega}_0(\zeta) := - (\Phi_+(\zeta) + \Phi_-(\zeta)), \tag{7.15}$$

where

$$\Phi_\pm(\zeta) = \begin{cases} (\zeta \pm i\sqrt{\zeta^2 - 1})^{1/3}, & \text{if } \Re \zeta \geq 1, \\ (\zeta \pm \sqrt{1 - \zeta^2})^{1/3}, & \text{if } |\Re \zeta| < 1, \\ (-\zeta \pm i\sqrt{\zeta^2 - 1})^{1/3}, & \text{if } \Re \zeta \leq -1. \end{cases}$$

**Proposition 7.7** (Solution to the cubic in normal form). Let $\Omega(\lambda)$ satisfy **SP1** and **SP2**.

(i) (Non-zero local minimum) Let $\Omega_0 = \sqrt{3}(i + \chi_1)$ in **SP2** and $\Omega(\lambda)$ satisfy

$$\Omega(\lambda)^3 + 3\Omega(\lambda) + 2\Lambda(\lambda) = 0, \quad \Lambda(\lambda) = (1 + \chi_2 + \mu(\lambda))\lambda + \chi_3, \tag{7.16}$$
with $|\mu(\lambda)| \lesssim \alpha|\lambda|^{1/3}$, $\alpha > 0$. Then there exist $\delta \sim 1$ and $\chi_* \sim 1$ such that if $\alpha,|\chi_1|,|\chi_2|,|\chi_3| \leq \chi_*$ then

$$
\Omega(\lambda) - \Omega_0 = \hat{\Omega}(\lambda) - i\sqrt[3]{\gamma} + O((\alpha + |\chi_2| + |\chi_3|) \min\{|\lambda|,|\lambda|^{2/3}\})
$$

(7.17)

for all $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq \delta/\alpha^3$, where $\hat{\Omega}(\lambda) := \Phi_{\text{odd}}(\lambda) + i\sqrt[3]{3}\Phi_{\text{even}}(\lambda)$ and $\Phi_{\text{odd}}$ and $\Phi_{\text{even}}$ are the odd and even part of the function $\Phi: \mathbb{C} \to \mathbb{C}$, $\Phi(\zeta) := (\sqrt[3]{1 + \zeta^2} + \zeta)^{1/3}$, respectively.

Moreover, we have for $|\lambda| \leq \delta/\alpha^3$ that

$$
|\Omega(\lambda) - \Omega_0| \lesssim \min\{|\lambda|,|\lambda|^{1/3}\}.
$$

(7.18)

In the following, we assume that $\Omega(\lambda)$, in addition to $SP_1$ and $SP_2$, also satisfies $SP_3$ and $SP_4^*$.

(ii) (Simple edge) Let $\Omega_0 = 0$ in $SP_2$ and $\Omega(\lambda)$ be a solution to

$$
\Omega^2(\lambda) + \Lambda(\lambda) = 0, \quad \Lambda(\lambda) = (1 + \mu(\lambda)) \lambda.
$$

(7.19)

If $|\mu(\lambda)| \lesssim \gamma^{2/3}|\lambda|^{1/3}$ for the $\gamma > 0$ of $SP_4^*$ then there is $c_* \sim 1$ such that

$$
\Omega(\lambda) = \tilde{\Omega}(\lambda) + O(|\mu(\lambda)||\lambda|^{1/2}), \quad \tilde{\Omega}(\lambda) := \begin{cases} i\lambda^{1/2}, & \text{if } \lambda \in [0,c_*\gamma^{-2}], \\ (-\lambda)^{1/2}, & \text{if } \lambda \in [-c_*\gamma^{-2},0]. \end{cases}
$$

(7.20)

Moreover, we have $\text{Im} \Omega(\lambda) = 0$ for $\lambda \in [-c_*\gamma^{-2},0]$.

(iii) (Sharp cusp) Let $\Omega_0 = 0$ in $SP_2$, $\gamma \sim 1$ in $SP_4^*$ and $\Omega(\lambda)$ be a solution to

$$
\Omega^3(\lambda) + \Lambda(\lambda) = 0, \quad \Lambda(\lambda) = (1 + \mu(\lambda))\lambda.
$$

(7.21)

If $|\mu(\lambda)| \lesssim |\lambda|^{1/3}$ then there is $\delta \sim 1$ such that

$$
\Omega(\lambda) = \tilde{\Omega}(\lambda) + O(|\mu(\lambda)||\lambda|^{1/3}), \quad \tilde{\Omega}(\lambda) := \frac{1}{2} \begin{cases} (-1 + i\sqrt[3]{3})\lambda^{1/3}, & \text{if } \lambda \in (0,\delta], \\ (1 + i\sqrt[3]{3})|\lambda|^{1/3}, & \text{if } \lambda \in [-\delta,0]. \end{cases}
$$

(7.22)

(iv) (Two nearby edges) Let $\Omega_0 = s$ for some $s \in \{\pm 1\}$ in $SP_2$, $\gamma \sim 1$ in $SP_4^*$ and $\Omega(\lambda)$ be a solution to

$$
\Omega^3(\lambda) - 3\Omega(\lambda) + 2\Lambda(\lambda) = 0, \quad \Lambda(\lambda) = (1 + \mu(\lambda))\lambda + s.
$$

(7.23)

Then there are $\delta \sim 1$, $\vartheta \sim 1$ and $\gamma_* \sim 1$ such that if $|\mu(\lambda)| \lesssim \hat{\gamma}|\lambda|^{1/3}$ for some $\hat{\gamma} \in [0,\gamma_*]$ then

(a) We have

$$
\Omega(\lambda) = \hat{\Omega}_+(1 + |\lambda|) + O(|\mu(\lambda)| \min\{|\lambda|^{1/2},|\lambda|^{1/3}\}),
$$

(7.24)

for all $\lambda \in s(0,2\delta/\hat{\gamma}^3]$. (Recall the definition of $\hat{\Omega}_+$ from (7.15).) Moreover, for all $\lambda \in s(0,2\delta/\hat{\gamma}^3]$, we have

$$
|\Omega(\lambda) - \Omega_0| \lesssim \min\{|\lambda|^{1/2},|\lambda|^{1/3}\}.
$$

(7.25)

(b) For all $\lambda \in -s(0,2 - \vartheta \hat{\gamma}]$, we have

$$
\text{Im} \Omega(\lambda) \lesssim \hat{\gamma}^{1/2}.
$$

(7.26)

(c) We have

$$
\text{Im} \Omega(-s(2 + \vartheta \hat{\gamma})) > 0.
$$

(7.27)

The core of each part in Proposition 7.7 is choosing the correct cubic root. For the most complicated part (iv), we state this choice in the following auxiliary lemma. For its formulation, we introduce the intervals

$$
I_1 := -s[-\lambda_1,0], \quad I_2 := -s(0,\lambda_2], \quad I_3 := -s[\lambda_3,\lambda_1],
$$

(7.28)

where we used the definitions

$$
\lambda_1 := 2\frac{\delta}{\hat{\gamma}^3}, \quad \lambda_2 := 2 - \vartheta \hat{\gamma}, \quad \lambda_3 := 2 + \vartheta \hat{\gamma}.
$$

(7.29)
These definitions are modelled after (9.105) in [1]. We will choose \( \hat{\gamma} = \hat{\Delta}^{1/3} \) in the proof of Theorem 7.4 below. Then \( \lambda_1 \) corresponds to an expansion range \( \delta \) in the \( \omega \) coordinate. Note that with the above choice of \( \hat{\gamma} \), we obtain the same \( \lambda_1 \) as in (9.105) of [1]. However, \( \lambda_2 \) and \( \lambda_3 \) differ slightly from those in [1], where \( \lambda_{2,3} \) were set to be \( 2 \pm \delta' \). Nevertheless, we will see below that \( \hat{\gamma} \sim |\sigma| \) but they are not equal in general.

For given \( \delta, \rho \sim 1 \), we will always choose \( \gamma_* \sim 1 \) so small that \( \hat{\gamma} \leq \gamma_* \) implies

\[
\lambda_1 \geq 4, \quad 1 \leq \lambda_2 < 2 < \lambda_3 \leq 3.
\]

Therefore, the intervals in (7.28) are disjoint and nonempty.

**Lemma 7.8** (Choice of cubic roots in Proposition 7.7 (iv)). Under the assumptions of Proposition 7.7 (iv), there are \( \delta, \rho, \gamma_* \sim 1 \) such that if \( \hat{\gamma} \leq \gamma_* \) then we have

\[
\Omega|_{I_k} = \hat{\Omega} \circ \Lambda|_{I_k}
\]

for \( k = 1, 2, 3 \). Here, \( \hat{\Omega}_+ \) is defined as in (7.15).

**Proof.** The proof is the same as the one in Lemma 9.14 in [1] but SP-4 in [1] is replaced by SP-4' above. In that proof, SP-4 is used only in the part titled “Choice of \( a_2 \)”. We redo this part here. Recall that \( a_2 = 0, \pm \) and denote the index such that \( \Omega|_{I_k} = \hat{\Omega} \circ \Lambda|_{I_k} \) and our goal is to show \( a_2 = + \). Similarly as in [1], we assume without loss of generality \( s = -1 \). Since \( \lim_{\lambda \to 0} \Omega(\lambda) = 2 \) and \( \Omega(0) = -1 \) by SP-2, we conclude \( a_2 \neq -1 \). (In the corresponding step in [1], there was a typo: \( \hat{\Omega}_+(-1+0) = 2 \) should have been \( \hat{\Omega}_--(-1+0) = 2 \), resulting in the choice \( a_2 = + \).) This conclusion is only used in the bound (9.137) of [1] which still holds true. The rest of the proof is unaffected.

We now prove \( a_2 \neq 0 \). To that end, we take the imaginary part of the cubic equation, (7.23), and obtain

\[
3((\text{Re}\Omega)^2 - 1)\text{Im}\Omega = -2\text{Im}\mu(\lambda) + (\text{Im}\Omega)^3.
\]

(7.30)

Suppose that \( a_2 = 0 \). From the definition of \( \hat{\Omega}_0, \Lambda(\lambda) = (1 + \mu(\lambda))\lambda - 1 \) and \( |\mu(\lambda)| \lesssim \hat{\gamma}|\lambda|^{1/3} \) we obtain

\[
\text{Re}\hat{\Omega}_0(\Lambda(\lambda)) \leq -1 - c|\lambda|^{1/2} + C\hat{\gamma}^{1/2}\lambda^{2/3}, \quad |\text{Im}\hat{\Omega}_0(\Lambda(\lambda))| \lesssim \hat{\gamma}^{1/2}\lambda^{2/3},
\]

(7.31)

(compare (9.120) in [1]). Thus, from (7.30), we conclude

\[
|\lambda|^{1/2}\text{Im}\Omega \lesssim |\lambda|\text{Re}\Omega
\]

for small \( \lambda \) as \( |\text{Im}\mu(\lambda)| \lesssim |\text{Im}\Omega| \) by SP-4' and \( |\text{Im}\Lambda| = |\lambda||\text{Im}\mu| \). Hence, \( \text{Im}\Omega(\lambda) = 0 \) for small enough \( |\lambda| \). Thus, \( \text{Re}\Omega \) is non-decreasing for such \( \lambda \) by SP-4', but from \( \Omega(0) = -1 \) and the first bound in (7.31) we conclude that \( \text{Re}\Omega \) has to be decreasing if \( \Omega(\lambda) = \hat{\Omega}_0(\Lambda(\lambda)) \). This contradiction shows \( a_2 \neq 0 \), hence, \( a_2 = + \). The rest of the proof in [1] is unchanged.

\( \square \)

**Proof of Proposition 7.7.** For the proof of (i), we mainly follow the proof of Proposition 9.3 in [1] with \( \gamma_4 = \chi_1, \gamma_5 = \chi_2 \) and \( \gamma_6 = \lambda_3 \) in (9.35) and (9.37) of [1].

Following the careful selection of the correct solution of (7.16) (cf. (9.36) in [1]) by the selection principles till above (9.50) in [1] yields \( \Omega(\lambda) = \hat{\Omega}(\Lambda(\lambda)) \) and hence, in particular, \( \hat{\Omega}(\chi_3) = \hat{\Omega}_0 = \sqrt{3}(i + \chi_1) \). \( \hat{\Omega}_+ = \hat{\Omega}_0 \) in [1]. By defining

\[
\Lambda(\lambda) = (1 + \chi_2 + \mu(\lambda))\lambda
\]

and using \( |\mu(\lambda)| \lesssim \alpha|\lambda|^{1/3} \) instead of (9.54) in [1], we obtain

\[
\hat{\Omega}(\lambda) - \hat{\Omega}(0) = \hat{\Omega}_0 - \hat{\Omega}(0) + \mathcal{O}\left((|\chi_2| + |\mu(\lambda)|)\frac{|\lambda|}{1 + |\lambda|^{2/3}}\right) = \hat{\Omega}(\lambda) - \hat{\Omega}(0) + \mathcal{O}\left((\alpha + |\chi_2|)|\min(|\lambda|, |\lambda|^{2/3})\right)
\]

instead of (9.56) in [1]. Thus, (9.57) in the proof of Proposition 9.3 in [1] yields

\[
\hat{\Omega}(\chi_3 + \Lambda(\lambda)) - \hat{\Omega}(\chi_3) = \hat{\Omega}_0 - \hat{\Omega}(0) + \mathcal{O}\left((\alpha + |\chi_2|)|\min(|\lambda|, |\lambda|^{2/3})\right).
\]

Thus, we obtain (7.17) since \( \hat{\Omega}(\chi_3) = \hat{\Omega}_0 \) and \( \hat{\Omega}(0) = i\sqrt{3} \). We remark that (7.18) is exactly (9.53) in [1].

The proof of (ii) resembles the proof of Lemma 9.11 in [1] but we replace assumption SP-4 of [1] by SP-4'. Since \( \Omega(\lambda) \) solves (7.19), there is a function \( A : \mathbb{R} \to \{\pm\} \) such that \( \Omega(\lambda) = \hat{\Omega}_0(\Lambda(\lambda)) \) for all \( \lambda \in \mathbb{R} \). Here,
\[ \tilde{\Omega}_\pm : \mathbb{C} \to \mathbb{C} \text{ denote the functions} \]

\[
\tilde{\Omega}_\pm(\zeta) := \begin{cases} 
\frac{i\zeta^{1/2}}{2}, & \text{if } \text{Re} \zeta \geq 0, \\
-\frac{(\zeta)^{1/2}}{2}, & \text{if } \text{Re} \zeta < 0.
\end{cases}
\]

(Note that they were denoted by \(\tilde{\Omega}_\pm\) in (9.78) of [1].) By assumption, there is \(c_\ast \sim 1\) such that \(|\mu(\lambda)| < 1\) for all \(|\lambda| \leq c_\ast \gamma^{-2}\). Hence, by \textbf{SP1}, we find \(a_+, a_- \in \{\pm\}\) such that \(A(\lambda) = a_\pm\) for \(\lambda \in \pm[0, c_\ast \gamma^{-2}]\).

For \(\lambda \geq 0\), we have

\[
\text{Im} \tilde{\Omega}_-(\Lambda(\lambda)) = -\lambda^{1/2} + \mathcal{O}(\mu(\lambda)^{1/2}).
\]

Thus, possibly shrinking \(c_\ast \sim 1\), we obtain \(\text{Im} \tilde{\Omega}_-(\Lambda(\lambda)) < 0\) for \(\lambda \in (0, c_\ast \gamma^{-2}]\). Therefore, the choice \(a_+ = -\) would contradict \textbf{SP3} and we conclude \(a_\ast = +\).

We now prove that \(a_- = +\). Assume to the contrary that \(a_- = -\). For small enough \(c_\ast \sim 1\), we have

\[
\begin{align*}
\text{Re} \tilde{\Omega}_-(\Lambda(\lambda)) &= |\lambda|^{1/2} \text{Re} (1 + \mu(\lambda))^{1/2} \sim |\lambda|^{1/2}, \\
\text{Im} \tilde{\Omega}_-(\Lambda(\lambda)) &= |\lambda|^{1/2} \text{Im} ((1 + \mu(\lambda))^{1/2}) \lesssim |\lambda|^{1/2}
\end{align*}
\]

for \(\lambda \in [-c_\ast \gamma^{-2}, 0)\) by the definition of \(\tilde{\Omega}_-\) and \(\Lambda\). Hence, taking the imaginary part of (7.19) and using \textbf{SP4}' yield

\[
|\lambda|^{1/2} \text{Im} \Omega(\lambda) \lesssim |\gamma| |\text{Im} \Omega(\lambda)|
\]

for \(\lambda \in [-c_\ast \gamma^{-2}, 0)\). By possibly shrinking \(c_\ast \sim 1\), we obtain \(\text{Im} \Omega(\lambda) = 0\) for \(\lambda \in [-c_\ast \gamma^{-2}, 0)\). Thus, \textbf{SP4}' implies that \(\text{Re} \Omega\) is non-decreasing on \([-c_\ast \gamma^{-2}, 0)\) which contradicts \(\text{Re} \tilde{\Omega}_-(0) = 0\) and \(\text{Re} \tilde{\Omega}_-(\Lambda(\lambda)) \sim |\lambda|^{1/2} > 0\) for \(\lambda \in [-c_\ast \gamma^{-2}, 0)\) with small enough \(c_\ast \sim 1\). Hence, \(a_- = +\) which completes the selection of the main term \(\tilde{\Omega} = \Omega_-\) in (7.20). The error term in (7.20) follows by estimating \(\tilde{\Omega}(\Lambda(\lambda))\) directly.

For the proof of (iii), we select the correct root of (7.21) as in the proof of Lemma 9.12 in [1] under \textbf{SP4}' instead of \textbf{SP-4}. Since \(\Omega(\lambda)\) solves (7.21) there is a function \(A : \mathbb{R} \to \{0, \pm\}\) such that

\[
\Omega(\lambda) = \tilde{\Omega}_A(\Lambda(\lambda))
\]

for all \(\lambda \in \mathbb{R}\). Here, we introduced the functions \(\tilde{\Omega}_a : \mathbb{C} \to \mathbb{C}\), \(a \in \{0, \pm\}\), defined by

\[
\tilde{\Omega}_a : = -\begin{cases}
\frac{e^{1/3}}{2}, & \text{if } \text{Re} \zeta \geq 0, \\
\frac{(\zeta)^{1/3}}{2}, & \text{if } \text{Re} \zeta < 0,
\end{cases}
\tilde{\Omega}_\pm(\zeta) := \frac{1 \pm i \sqrt{3}}{2} \tilde{\Omega}_0(\zeta).
\]

(Note that they were denoted by \(\tilde{\Omega}_a\), \(a \in \{0, \pm\}\), in (9.87) of [1].) By \textbf{SP1}, \(A\) can only change its value at \(\lambda\) if \(\Lambda(\lambda) = 0\). By choosing \(\delta \sim 1\) small enough and using \(|\mu(\lambda)| \lesssim |\lambda|^{1/3}\), we have \(A(\lambda) = a_+\) and \(A(-\lambda) = a_-\) for some constants \(a_\pm\) and for all \(\lambda \in (0, \delta]\).

We will now use \textbf{SP3} and \textbf{SP4}' to determine the value of \(a_+\) and \(a_-\). As in (9.91) of the proof of Lemma 9.12 in [1], we have

\[
\pm (\text{sign } \lambda) \text{Im} \tilde{\Omega}_\pm(\Lambda(\lambda)) = \frac{\sqrt{3}}{2} |\lambda|^{1/3} + \mathcal{O}(\mu(\lambda)^{1/3}) \geq |\lambda|^{1/3} - C|\lambda|^{2/3}.
\]

By possibly shrinking \(\delta \sim 1\), we conclude \(\text{Im} \tilde{\Omega}_\pm(\Lambda(\lambda)) < 0\) for \(\lambda \in (0, \delta]\) and \(\text{Im} \tilde{\Omega}_\pm(\Lambda(\lambda)) < 0\) for \(\lambda \in [-\delta, 0)\). Hence, owing to \textbf{SP3}, we conclude \(a_+ \neq -\) and \(a_- \neq +\).

Next, we will prove \(a_+ \neq 0\). For \(\lambda \geq 0\), we have

\[
\text{Re} \tilde{\Omega}_0(\Lambda(\lambda)) \leq -\lambda^{1/3} + C|\lambda|^{2/3}, \quad \text{Im} \tilde{\Omega}_0(\Lambda(\lambda)) \lesssim \lambda^{2/3}.
\]

Thus, assuming \(\Omega(\lambda) = \tilde{\Omega}_0(\Lambda(\lambda))\) and estimating the imaginary part of (7.21) yield

\[
\lambda^{2/3} \text{Im} \Omega(\lambda) \lesssim (\text{Im} \Omega(\lambda)^2)^{1/3} + |\text{Im} \Lambda(\lambda)| \lesssim |\lambda| \text{Im} \Omega(\lambda).
\]

Hence, we possibly shrink \(\delta \sim 1\) and conclude \(\text{Im} \Omega(\lambda) = 0\) for \(\lambda \in [0, \delta]\). Therefore, \(\text{Re} \Omega(\lambda)\) is non-decreasing on \([0, \delta]\) by \textbf{SP4}'. Combined with \(\Omega_0 = 0\) and \(\text{Re} \tilde{\Omega}_0(\Lambda(\lambda)) \lesssim -\lambda^{1/3}\), we obtain a contradiction. Hence, this implies \(a_+ \neq 0\), i.e., \(a_+ = +\).

A similar argument excludes \(a_- = 0\) and we thus obtain \(a_- = -\). Now, (7.22) is obtained from the definition
of $\hat{\Omega} = \hat{\Omega}_+$, which completes the proof of (iii).

For the proof of (iv), we remark that all estimates follow from Lemma 7.8 in the same way as they followed in [1] from Lemma 9.14 in [1]. Indeed, (7.24) is the same as (9.129) in [1]. The bound (7.25) is shown analogously to (9.129) and (9.130) in [1]. Moreover, (7.26) is (9.137) in [1] and (7.27) is obtained as (9.109) in [1]. This completes the proof of Proposition 7.7.

\section*{7.2. Proof of Theorem 7.4}

Before we prove Theorem 7.4, we collect some properties of $\Psi_{\text{edge}}$ and $\Psi_{\text{min}}$ which will be useful in the following. We recall that $\Psi_{\text{edge}}$ and $\Psi_{\text{min}}$ were defined in (7.1).

\begin{lemma}[Properties of $\Psi_{\text{min}}$ and $\Psi_{\text{edge}}$] \label{lem:7.9}
(i) Let $\hat{\Omega}$ be defined as in Proposition 7.7 (i). Then, for any $\lambda \in \mathbb{R}$, we have
\begin{equation}
\Psi_{\text{min}}(\lambda) = \frac{1}{\sqrt{3}} \Im [\hat{\Omega}(\lambda) - \hat{\Omega}(0)].
\end{equation}

(ii) Let $\hat{\Omega}_+$ be defined as in (7.15). Then, for any $\lambda \geq 0$, we have
\begin{equation}
\Psi_{\text{edge}}(\lambda) = \frac{1}{2\sqrt{3}} \Im \hat{\Omega}_+(1 + 2\lambda).
\end{equation}

(iii) There is a function $\tilde{\Psi}: [0, \infty) \to \mathbb{R}$ with uniformly bounded derivatives and $\tilde{\Psi}(0) = 0$ such that, for any $\lambda \geq 0$, we have
\begin{equation}
\Psi_{\text{edge}}(\lambda) = \frac{\lambda^{1/2}}{3}(1 + \tilde{\Psi}(\lambda)), \quad |\tilde{\Psi}(\lambda)| \leq \min\{\lambda, \lambda^{1/3}\}.
\end{equation}

(iv) There is $\varepsilon_* \sim 1$ such that if $|\varepsilon| \leq \varepsilon_*$ then, for any $\lambda \geq 0$, we have
\begin{equation}
\Psi_{\text{edge}}((1 + \varepsilon)\lambda) = (1 + \varepsilon)^{1/2}\Psi_{\text{edge}}(\lambda) + O(\varepsilon \min\{\lambda^{1/2}, \lambda^{1/3}\}).
\end{equation}
\end{lemma}

We remark that (7.33) was present in (9.127) of [1] but the coefficient $1/(2\sqrt{3})$ was erroneously missing there. The relation in (7.35) is identical to (9.145) in [1]. Moreover, we use the proof of [1].

\begin{proof}
The parts (i), (ii) and (iii) are direct consequences of the definitions of $\Psi_{\text{min}}, \hat{\Omega}, \Psi_{\text{edge}}$ and $\hat{\Omega}_+$.

For the proof of (iv), we choose $\varepsilon_* \leq 1/2$ such that $1 + \varepsilon \sim 1$ for $|\varepsilon| \leq \varepsilon_*$. If $0 \leq \lambda \leq 1$ then (7.35) follows from (7.34). For $\lambda \gtrsim 1$, we choose $\varepsilon_* = 1/3$ and then (7.35) is a consequence of (7.33) above as well as the stability of Cardano’s solutions, (9.111) in Lemma 9.17 of [1].
\end{proof}

In the following proof of Theorem 7.4, we will choose appropriate normal coordinates $\Omega$ and $\Lambda$ in each case such that (7.6) turns into one of the cubic equations in normal form from Proposition 7.7. This procedure has been similarly performed in the proofs of Proposition 9.3, Lemma 9.11, Lemma 9.12 and Section 9.2.2 in [1]. However, owing to the weaker error bounds here, we include the proof for the sake of completeness.

\begin{proof}[of Theorem 7.4]
We start with the proof of part (i) (cf. Proposition 9.3 in [1]). Owing to (7.8b) and $|\Psi_{\text{min}}(\lambda)| \lesssim |\lambda|^{1/3}$, the statement of (7.9) is trivial for $|\omega| \gtrsim 1$ since the error term dominates. Therefore, it suffices to prove (7.9) for $|\omega| \leq \delta$ with some $\delta \sim 1$.

By possibly shrinking $\rho_\ast \sim 1$, we can assume that $|\sigma| \leq \Pi_\ast \rho^2$ is small enough such that $\psi \sim 1$ by (7.7). In the following, we will choose $\omega$-independent complex numbers $\gamma_\nu, \gamma_0, \gamma_1, \ldots, \gamma_7 \in \mathbb{C}$ such that certain relations hold. For each choice, it is easily checked that $|\gamma_k| \lesssim \rho$ for $k = \nu, 0, 1, \ldots, 7$. We divide (7.6) by $\mu_3$ and obtain
\begin{equation}
\Theta^3 + 13\rho(1 + \gamma_2)\Theta^2 - 2\rho^2(1 + \gamma_1)\Theta + (1 + \gamma_0 + (1 + \gamma_\nu)\nu(\omega))\frac{K}{\psi}\omega = 0,
\end{equation}
using $|\mu_3| \sim 1$ and $|\sigma| \leq \Pi_\ast \rho^2$. We introduce the normal coordinates
\begin{equation}
\lambda := \Gamma^\omega \rho, \quad \Omega(\lambda) := \sqrt{3}[1 + \gamma_3] \frac{1}{\rho} \Theta\left(\frac{\rho^3}{\Gamma}\lambda\right) + i + \gamma_4,
\end{equation}
using $|\mu_3| \sim 1$ and $|\sigma| \leq \Pi_\ast \rho^2$. We introduce the normal coordinates
Lemma 7.10 (Simple edge). Let the assumptions of Theorem 7.4 (ii) hold true. If \( \sigma \neq 0 \) then there is \( c_\ast \sim 1 \) such that

\[
\text{Im } \Theta(\omega) = \frac{1}{\sqrt{\kappa}} \left( \frac{\omega}{\sigma} \right)^{1/2} + O \left( \left( |\sigma| + |\sigma^{-1}| |\Theta(\omega)| \right) \left( \frac{\omega}{\sigma} \right)^{1/2} \right),
\]

if \( \text{sign } \omega = \text{sign } \sigma, \ |\omega| \leq c_\ast |\sigma|^3 \)

and

\[
|\Theta(\omega)| \leq |\omega/\sigma|^{1/2} \text{ for } |\omega| \leq c_\ast |\sigma|^3.
\]

Proof. Dividing (7.38) by \( \kappa \sigma \) yields

\[
\frac{1}{\kappa} \left( 1 + \frac{\psi}{\sigma} \Theta(\omega) \right) \frac{\Theta(\omega)^2}{\kappa} + (1 + \nu(\omega)) \frac{\omega}{\sigma} = 0.
\]

We introduce \( \lambda, \Omega(\lambda) \) and \( \mu(\lambda) \) defined by

\[
\lambda := \frac{\omega}{\sigma}, \quad \Omega(\lambda) := \frac{1}{\sqrt{\kappa}} \Theta(\sigma \lambda), \quad \mu(\lambda) := \frac{1 + \nu(\sigma \lambda)}{1 + \psi \sigma^{-1} \Theta(\sigma \lambda)} - 1.
\]

In the normal coordinates \( \lambda \) and \( \Omega(\lambda), \) (7.40) viewed as a quadratic equation, fulfills (7.19) with the above choice of \( \mu(\lambda) \). Since \( |\sigma|^{-1} |\Theta(\sigma \lambda)| \lesssim |\sigma|^{-2/3} |\lambda|^{1/3} \) by (7.8b), there is \( c_\ast \sim 1 \) such that

\[
|\mu(\lambda)| \lesssim |\sigma(\sigma \lambda)| + |\sigma^{-1}| |\Theta(\sigma \lambda)| \lesssim |\sigma|^{-2/3} |\lambda|^{1/3}, \quad |\text{Im } \mu(\lambda)| \lesssim |\sigma|^{-1} |\Theta(\sigma \lambda)|
\]

for \( |\lambda| \leq c_\ast |\sigma|^2 \) by (7.8a), (7.8b) and (7.10). Hence, we apply Proposition 7.7 (ii) with \( \gamma \sim |\sigma|^{-1} \) in SP4' and obtain (7.39) with an error term \( O(|\mu(\lambda)||\lambda|^{1/2}) \) instead, as well as \( |\Theta(\omega)| \lesssim |\sigma|^{-1/2} |\omega|^{1/2} \). Thus, the first bound in (7.41) completes the proof of (7.39).

From the second case in (7.39), we conclude the second case in (7.12). The first case in (7.12) and (7.14) are trivial if \( |\omega| \geq 1 \) due to (7.8b) and (7.4a). Hence, it suffices to prove this case for \( |\omega| \leq \delta \) with some \( \delta \sim 1 \). If \( |\sigma| \gtrsim 1 \) then the first case in (7.12) also follows from (7.39) with \( \delta := c_\ast |\sigma|^3 \). Indeed, from (7.34), we conclude

\[
\sqrt{\kappa} \frac{\omega}{\sigma}^{1/2} = c \Delta^{1/3} \Psi_{\text{edge}} \left( \frac{|\omega|}{\lambda} \right) + O(|\omega|^{3/2}),
\]
where $c$ and $\tilde{\Delta}$ are defined as in (7.13). Since $|\omega| \leq \varepsilon(\omega)$ for $|\omega| \leq \delta$ and $\varepsilon(\omega)$ defined as in (7.13) we obtain the first case in (7.12) if $|\sigma| \geq 1$. Similarly, $|\Theta(\omega)| \leq |\omega|^{1/2}$ by Lemma 7.10 yields (7.14) if $|\omega| \leq \delta$ and $|\sigma| \geq 1$. Hence, it remains to show the first case in (7.12) and (7.14) if $|\sigma| \leq \sigma_*$ for some $\sigma_* \sim 1$. In fact, we choose $\sigma_* \sim 1$ so small that $\psi \sim 1$ by (7.7) and $\Delta \leq 1$ for $|\sigma| \leq \sigma_*$. In order to apply Proposition 7.7 (iv), we introduce

$$\lambda := \frac{2}{\Delta} \omega, \quad \Omega(\lambda) := 3 \frac{\psi}{|\sigma|} \Theta\left(\frac{\Delta}{2} \lambda \right) + \text{sign} \sigma, \quad \mu(\lambda) := \nu'\left(\frac{\Delta}{2} \lambda \right)$$

(cf. (9.96) and (9.99) in [1]). The cubic (7.38) takes the form (7.23) in the normal coordinates $\lambda$ and $\Omega(\lambda)$ with the above choice of $\mu(\lambda)$ and $s = \text{sign} \sigma$ in (7.23). By (7.8a), we have $|\mu(\lambda)| \leq |\lambda|^{1/3}$. We set $\tilde{\gamma} := \Delta^{1/3}$. Therefore, Proposition 7.7 (iv) and (7.33) yield $\delta \sim 1$ and possibly smaller $\sigma_* := \min(\sigma_*, \gamma_* \sim 1)$ such that the first case in (7.12) holds true for $|\sigma| \leq \sigma_*$ and $|\omega| \leq \delta$ as $\mu(\lambda) = \nu(\omega)$ and $\Delta \sim |\sigma|^3$. Moreover, (7.25) implies (7.14) for $|\omega| \leq \delta$. This completes the proof of (ii) (b) and hence of Theorem 7.4.

7.3. Proof of Theorem 7.1 and Proposition 7.3.

In this section, we prove Theorem 7.1 and Proposition 7.3. Some parts of the following proof resemble the proofs of Theorem 2.6, Proposition 9.3 and Proposition 9.8 in [1]. However, owing to the weaker assumptions, we present it here for the sake of completeness.

Proof of Theorem 7.1 and Proposition 7.3. We will only prove the statements in Proposition 7.3. Theorem 7.1 is a direct consequence of this proposition as well as Lemma 4.8 (ii) and Proposition 6.1.

We start with the proof of part (i). Along its proof, we will shrink $\delta_* \sim 1$ such that (7.3) holds true for all $\omega \in [-\delta_*, \delta_*] \cap J \cap D$. We will transfer the expansions of $\Theta$ in Theorem 7.4 to expansions of $v$ by means of (6.1). To that end, we take the imaginary part of (6.1) and obtain

$$v(\tau_0 + \omega) = v(\tau_0) + \pi^{-1} \text{Re} b \text{Im} \Theta(\omega) + \pi^{-1} \text{Im} b \text{Re} \Theta(\omega) + \pi^{-1} \text{Im} r(\omega).$$

We first establish (7.3) at a shape regular point $\tau_0 \in \text{supp} \rho \setminus \partial \text{supp} \rho$ which is a local minimum of $\tau \mapsto \rho(\tau)$. If $\rho = \rho(\tau_0) = 0$, i.e., the case of a cusp at $\tau_0$, case (c), then $\sigma = 0$. Indeed, if $\sigma$ were not 0, then, by the second case in (7.12), $\text{Im} \Theta(\omega)$ would vanish on one side of $\tau_0$. By the third bound in (6.8), this would imply the vanishing of $\rho$ as well, contradicting to $\tau_0 \in \text{supp} \rho \setminus \partial \text{supp} \rho$. Hence, for any $\delta_* \sim 1$, (7.11) and (7.43) immediately yield (7.3) for $\omega \in [-\delta_*, \delta_*] \cap J \cap D$ with $h = \pi^{-1} b \sqrt{\frac{\delta}{2\psi}}$ using (6.7a), (6.7b) and $b = b^*$ due to $\rho = 0$.

We now assume $\rho \geq 0$ which corresponds to an internal nonzero minimum at $\tau_0$, case (d). Thus, the following lemma implies that the condition $|\sigma| \leq \Pi_* \rho^2$, $\sigma = \sigma(\tau_0)$, needed to apply Theorem 7.4 (i) is fulfilled. We will prove Lemma 7.11, at the end of this section.

Lemma 7.11 (Bound on $|\sigma|$ at nonzero local minimum). There are thresholds $\rho_* \sim 1$ and $\Pi_* \sim 1$ such that

$$|\sigma(\tau_0)| \leq \Pi_* \rho(\tau_0)^2$$

for all shape regular points $\tau_0 \in \text{supp} \rho$ which are a local minimum of $\rho$ and satisfy $0 < \rho(\tau_0) \leq \rho_*$.

Hence, (7.9), (7.43) and (6.7b) yield (7.3) with $\tilde{b} = \rho \tilde{b}^{-1/3}$ and $h = \pi^{-1} \tilde{b}^{1/3} \text{Re} b$. Here, we also used

$$|\rho| \Theta(\omega) + |\Theta(\omega)|^2 + |\omega| + \min\{\rho^{-1}|\omega|, |\omega|^{3/2}\} \lesssim \frac{|\omega|}{\rho} \text{Im} b (|\omega| \lesssim \rho^2) + \Psi(\omega)^2,$$

which is a consequence of (6.7a), (7.4b) for $|\omega| \lesssim 1$, as well as $\text{Re} b \sim 1$ and $\text{Im} b = \mathcal{O}(\rho)$. This completes the proof of (7.3) for shape regular points $\tau_0 \in \text{supp} \rho \setminus \partial \text{supp} \rho$, cases (e) and (d).

We now turn to the proof of (7.3) at an edge $\tau_0$, case (a), i.e., for a shape regular point $\tau_0 \in \partial \text{supp} \rho$. We first prove a version of (7.3) with $\Delta$ in place of $\Delta$. (7.44) below. In a second step, we then replace $\Delta$ by $\tilde{\Delta}$ to obtain (7.3).

Since $\tau_0 \in \partial \text{supp} \rho$, we have $\rho = \rho(\tau_0) = 0$. Therefore, $v(\tau_0) = 0$ since $\cdot$ is a faithful trace and $v(\tau_0)$ is positive semidefinite. As $\tau_0 \in \partial \text{supp} \rho$, we have $\sigma(\tau_0) \neq 0$. Indeed, assuming $\sigma(\tau_0) = 0$, using Theorem 7.4 (ii) (a), taking the imaginary part of (6.1) as well as applying the third bound in (6.8) and the second bound in (6.7a) yield the contradiction $\tau_0 \in \text{supp} \rho \setminus \partial \text{supp} \rho$. Recalling the definitions of $\Delta$ and $c$ from (7.13), (7.43) and (7.12) yield

$$v(\tau_0 + \omega) = \pi^{-1} c \tilde{\Psi}(\omega) + \mathcal{O}(\tilde{\Psi}(\omega)^2), \quad \tilde{\Psi}(\omega) := \Delta^{1/3} \Psi_{\text{edge}}\left(\frac{\omega}{\Delta}\right).$$
for any \( \omega \in [-\delta_0, \delta_0] \cap J \cap D \) with \( \text{sign} \omega = \text{sign} \sigma \) and some \( \delta_0 \sim 1 \). Here, we also used \( b = b^* \sim 1 \), the first bound in (6.5), (7.14) and \( \varepsilon(\omega) \sim \tilde{\Psi}(\omega) \) by (7.4b) to obtain
\[
|\Theta(\omega)|^2 + |\omega| + |(\Theta(\omega)| + |\omega| + \varepsilon(\omega)| \varepsilon(\omega) \lesssim \tilde{\Psi}(\omega)^2
\]
for any \( \omega \in [-\delta_0, \delta_0] \cap J \cap D \) with \( \text{sign} \omega = \text{sign} \sigma \) and some \( \delta_0 \sim 1 \). This means that we have shown (7.3) with \( \Psi \) replaced by \( \tilde{\Psi} \).

We now replace \( \tilde{\Delta} \) by \( \Delta \) in (7.44) to obtain (7.3). To that end, we first assume that \( |\sigma| \geq 1 \) and \( \Delta \lesssim 1 \). The second part of (7.12) implies \( |\sigma|^3 \lesssim \Delta \lesssim 1 \) and thus \( |\sigma|^3 \sim \Delta \sim 1 \). Since \( |\sigma|^3 \sim \Delta \) we conclude \( \Delta \sim \Delta \). Therefore, we obtain
\[
\Delta^{1/3} \Psi_{\text{edge}}(\frac{|\omega|}{\Delta}) = \left( \frac{\Delta}{\Delta} \right)^{1/6} \Delta^{1/3} \Psi_{\text{edge}}(\frac{|\omega|}{\Delta}) + O(\min\{|\omega|^{3/2}, |\omega|^{1/3}\}).
\]
Here, we used \( \Psi_{\text{edge}}(|\lambda|) \lesssim |\lambda|^{1/3} \) for \( |\lambda| \gtrsim 1 \) and (7.34) otherwise. Applying this relation to (7.44) yields (7.3) for \( \omega \in [-\delta_0, \delta_0] \cap J \cap D \) with \( \text{sign} \omega = \text{sign} \sigma, \delta_0 \sim 1 \) and \( h := \pi^{-1}c(\Delta/\tilde{\Delta})^{1/6} \sim 1 \) for \( |\sigma| \gtrsim 1 \) and \( \Delta \lesssim 1 \).

The next lemma shows that \( |\sigma| \gtrsim 1 \) at the edge of a gap of size \( \Delta \gtrsim 1 \). We postpone its proof until the end of this section.

**Lemma 7.12 (σ at an edge of a large gap).** Let \( \tau_0 \in \partial \text{supp} \rho \) be a shape regular point for \( m \) on \( J \). If \( |\inf J| \gtrsim 1 \) and there is \( \varepsilon \sim 1 \) such that \( \rho(\tau) = 0 \) for all \( \tau \in [\tau_0 - \varepsilon, \tau_0] \) then \( |\sigma| \sim 1 \). We also have \( |\sigma| \sim 1 \) if \( \sup J \gtrsim 1 \) and \( \rho(\tau) = 0 \) for all \( \tau \in [\tau_0, \tau_0 + \varepsilon] \) and some \( \varepsilon \sim 1 \).

Under the assumptions of the previous lemma, we set \( \Delta := 1 \) and obtain trivially \( \Delta \sim 1 \sim \Delta \). Thus, (7.44) implies (7.3) by the same argument as in the case \( \Delta \lesssim 1 \).

For \( |\sigma| \leq \sigma_* \), with some sufficiently small \( \sigma_* \sim 1 \), we will prove below with the help of the following Lemma 7.13 and (7.35) that replacing \( \tilde{\Delta} \) by \( \Delta \) in (7.44) yields an affordable error. We present the proof of Lemma 7.13 at the end of this section.

**Lemma 7.13 (Size of small gap).** Let \( \tau_0, \tau_1 \in \partial \text{supp} \rho \), \( \tau_1 < \tau_0 \), be two shape regular points for \( m \) on \( J_0 \) and \( J_1 \), respectively, where \( J_0, J_1 \subseteq \mathbb{R} \) are two open intervals with \( 0 \in J_0 \cap J_1 \). We assume \( |\inf J_0| \gtrsim 1 \) and \( \sup J_1 \gtrsim 1 \) as well as \( (\tau_1, \tau_0) \cap \text{supp} \rho = \varnothing \). We set \( \Delta(\tau_0) := \tau_0 - \tau_1 \). Then there is \( \tilde{\sigma} \sim 1 \) such that if \( |\sigma(\tau_0)| \leq \tilde{\sigma} \) and \( |\sigma(\tau_0) - \sigma(\tau_1)| \lesssim |\tau_0 - \tau_1| \) for some \( \zeta \in (0, 1/3) \) then
\[
\frac{\Delta(\tau_0)}{\Delta(\tau_0)} = 1 + O(\sigma(\tau_0)).
\]
The same statement holds true when \( \tau_0 \) is replaced by \( \tau_1 \) with \( \Delta(\tau_1) := \tau_0 - \tau_1 \).

From Lemma 7.13, we conclude that there is \( \gamma \in \mathbb{C} \) such that \( |\gamma| \lesssim 1 \) and \( \Delta = (1 + \gamma |\sigma|) \tilde{\Delta} \). By possibly shrinking \( \sigma_\ast \sim 1 \), we can assume that \( |\gamma| \leq \varepsilon \ast \) for \( |\sigma| \leq \sigma_\ast \), where \( \varepsilon \sim 1 \) is chosen as in Lemma 7.9 (iv).

Thus, (7.35) yields
\[
\Delta^{1/3} \Psi_{\text{edge}}(\frac{|\omega|}{\Delta}) = \left( \frac{\Delta}{\Delta} \right)^{1/6} \Delta^{1/6} \Psi_{\text{edge}}(\frac{|\omega|}{\Delta}) + O\left( \min\{|\omega|^{3/2}, |\omega|^{1/3}\} \right).
\]
Hence, choosing \( h := \pi^{-1}c(\Delta/\tilde{\Delta})^{1/6} \) as before and noticing \( h \sim 1 \) yields (7.3) in the missing regime. This completes the proof of part (i) of Proposition 7.3.

We now turn to the proof of part (ii) of Proposition 7.3 and assume that all points of \( (\partial \text{supp} \rho) \cap I \) are shape regular for \( m \) and all estimates in Definition 7.2 hold true uniformly on this set. As in the proof of part (i) of Proposition 7.3, we conclude \( \sigma(\tau_0) \neq 0 \) for all \( \tau_0 \in (\partial \text{supp} \rho) \cap I \). Owing to \( \text{dist}(0, \partial J) \gtrsim 1 \) and the Hölder-continuity of \( \sigma \) on \( (\partial \text{supp} \rho) \cap I \), part (i) of Proposition 7.3 is applicable to every \( \tau_0 \in (\partial \text{supp} \rho) \cap I \).

Hence, (7.4a) and \( \text{dist}(0, \partial J) \gtrsim 1 \) imply the existence of \( \delta_1, \varepsilon \sim 1 \) such that
\[
\rho(\tau_0 + \omega) \geq c_1|\omega|^{1/2}
\]
for all \( \omega \in -\text{sign} \sigma(\tau_0)[0, \delta_1] \) and \( \tau_0 \in (\partial \text{supp} \rho) \cap I \). In particular, \( \tau_0 - \text{sign} \sigma(\tau_0)[0, \delta_1] \subset \text{supp} \rho \) for all \( \tau_0 \in (\partial \text{supp} \rho) \cap I \). Since \( |I| \lesssim 1 \), this implies that \( \sup \rho \cap I \) consists of finitely many intervals \( [\alpha_i, \beta_i] \) with lengths \( \gtrsim 1 \), and, thus, their number \( K \) satisfies \( K \sim 1 \) as \( \delta_1 \sim 1 \) and \( \beta_i - \alpha_i \gtrsim \delta_1 \) if \( \beta_i \neq \sup I \) and \( \alpha_i \neq \inf I \).
Additionally, we now assume that the elements of $\mathbb{M}_\rho$ are shape regular points for $m$ on $J$ and all estimates in Definition 7.2 hold true uniformly on $\mathbb{M}_\rho$. By possibly shrinking $\rho_*$ 1, we conclude from (7.45) that $|\alpha_i - \gamma| \sim 1$ and $|\beta_i - \gamma| \sim 1$ for any $i = 1, \ldots, K$ and $\gamma \in \mathbb{M}_\rho$.

Suppose now that $\tau_0 \in \mathbb{M}_\rho$, with $\rho(\tau_0) = 0$. Then part (i) and $\text{dist}(0, \partial J) \geq 1$ yield the existence of $\delta_2, c_2 \sim 1$ such that

$$\rho(\tau_0 + \gamma) \geq c_2|\gamma|^{1/3}$$

for all $|\gamma| \leq \delta_2$. By possibly further shrinking $\rho_*$, we thus obtain $|\tau_0 - \gamma| \sim 1$ for all $\gamma \in \mathbb{M}_\rho \setminus \{\tau_0\}$. Therefore, in this case, the first case in (7.5) always holds true.

Finally, let $\gamma_1, \gamma_2 \in \mathbb{M}_\rho$, with $\rho(\gamma_1), \rho(\gamma_2) > 0$. Then applying (i) with $\tau_0 = \gamma_1$ and $\tau_0 = \gamma_2$ yields

$$\Psi_1(\omega) + \Psi_2(\omega) \leq |\omega|^{1/3}\left(\rho(\gamma_1)|\omega| \leq \rho(\gamma_1)^3 + \rho(\gamma_2)|\omega| \leq \rho(\gamma_2)^3\right) + \Psi_1(\omega)^2 + \Psi_2(\omega)^2,$$

where we defined $\omega = \gamma_2 - \gamma_1$ and

$$\Psi_1(\omega) := \tilde{\rho}_1\Psi_{\min}\left(\frac{|\omega|}{\rho_1}\right), \quad \Psi_2(\omega) := \tilde{\rho}_2\Psi_{\min}\left(\frac{|\omega|}{\rho_2}\right)$$

with $\tilde{\rho}_1 \sim \rho(\gamma_1)$ and $\tilde{\rho}_2 \sim \rho(\gamma_2)$ (cf. Corollary 9.4 in [1]). Thus, we obtain either $|\omega| \sim 1$ or $|\omega| \lesssim \min\{\rho(\gamma_1), \rho(\gamma_2)\}^4$. This completes the proof of (7.5) and hence the one of Proposition 7.3. As we have already explained, Theorem 7.1 follows immediately.

The core of the proof of Lemma 7.11 is an effective monotonicity estimate on $v$, see (7.46) below, which is the analogue of (9.20) in Lemma 9.2 of [1]. Owing to the weaker assumptions on the coefficients of the cubic equation, we need to present an upgraded proof here. In fact, the bound in (9.20) of [1] contained a typo. It should have read as

$$(\text{sign } \sigma(\tau))\partial_\tau v(\tau) \gtrsim \frac{1}{(v(\tau))(1 + |\sigma(\tau)|)}$$

for $\tau \in D_\circ$, satisfying $\Pi(\tau) \geq \Pi_*$. However, this does not affect the correctness of the argument in [1].

Proof of Lemma 7.11. In the whole proof, we will use the notation of Definition 7.2. We will show below that there are $\rho_* \sim 1$ and $\Pi_* \sim 1$ such that

$$(\text{sign } \kappa_1 \sigma(\tau))\partial_\tau v(\tau) \gtrsim \rho(\tau)^{-1}$$

(7.46)

for all $\tau \in \mathbb{R}$ which satisfy $\rho(\tau) \in (0, \rho_*]$ and $|\sigma(\tau)| \geq \Pi_* \rho(\tau)^2$ and are admissible points for the shape analysis.

Now, we first conclude the statement of the lemma from (7.46) through a proof by contradiction. If $\tau_0$ satisfies the conditions of Lemma 7.11 then $\partial_\tau \rho(\tau_0) = 0$ as $\tau_0$ is a local minimum of $\rho$. Assuming $|\sigma(\tau_0)| \geq \Pi_* \rho(\tau_0)^2$ and applying (\_) to (7.46) yield the contradiction $\partial_\tau \rho(\tau_0) > 0$.

For the proof of (7.46) we start by proving a relation for $\partial_\tau v(\tau)$. We divide (6.1) by $\omega$, use $\Theta(0) = 0$ and $r(0) = 0$ as well as take the limit $\omega \to 0$ to obtain $\partial_\omega m(\tau) = b\partial_\omega \Theta(0) + \partial_\omega r(0)$. Taking the imaginary part of the previous relation yields

$$\pi \partial_\tau v(\tau) = \text{Im}[b\partial_\omega \Theta(0)] + \text{Im} \partial_\omega r(0).$$

(7.47)

We divide (6.7b) by $\omega$, employ the first bound in (6.7a) and obtain

$$\left|\frac{\tau(\omega)}{\omega}\right| \lesssim 1 + \left|\frac{\Theta(\omega)}{\omega}\right|^2 \lesssim 1 + \left|\frac{|\omega|}{\rho^3}\right|.$$

By sending $\omega \to 0$ and using $r(0) = 0$, we conclude

$$\|\text{Im} \partial_\omega r(0)\| \lesssim 1.$$

(7.48)

We divide (6.3) by $\mu_1 \omega$, take the limit $\omega \to 0$ and use $\lim_{\omega \to 0} \Theta(\omega) = \Theta(0) = 0$ to obtain

$$\partial_\omega \Theta(0) = -\frac{\Xi(0)\mu_1}{[\mu_1]^2} = \frac{(\kappa + O(\rho))(i\kappa_1 \rho \sigma + 2\rho^2 \psi + O(\rho^3 + \rho^2|\sigma|))}{4\rho^3|\psi + O(\rho + |\sigma|)|^2 + |\rho^2|\kappa_1 \sigma + O(\rho^2 + \rho|\sigma|)|^2} \lesssim \kappa \rho^2|\psi + O(\rho + |\sigma|)|^2 + |\kappa_1 \sigma + O(\rho^2 + \rho|\sigma|)|^2.$$

(7.49)
where we employed $|\mu_1|^2 = 4\rho^4|\psi + \mathcal{O}(\rho + |\sigma|)|^2 + \rho^2|\kappa_1 \sigma + \mathcal{O}(\rho^2 + \rho|\sigma|)|^2$ as $\rho, \psi, \kappa_1, \sigma \in \mathbb{R}$. Thus, we obtain
\[ \rho |\text{Re} \partial_\nu \Theta(0)| \lesssim \frac{\rho + |\sigma|}{\rho^2|\psi + \mathcal{O}(\rho + |\sigma|)|^2 + |\kappa_1 \sigma + \mathcal{O}(\rho^2 + \rho|\sigma|)|^2}. \] (7.50)

Therefore, using $b = b^* + \mathcal{O}(\rho)$, $b + b^* \sim 1$, $\kappa \sim 1$ and $|\kappa_1| \sim 1$ yields
\[ (\text{sign } \kappa_1 \sigma) \text{Im} [b \partial_\nu \Theta(0)] \gtrsim \frac{\rho^{-1}|\sigma| + \mathcal{O}(\rho + |\sigma|) + \mathcal{O}(\rho + |\sigma|)}{|\mathcal{O}(\rho^2 + \rho|\sigma|)|^2 + \rho^2|\psi + \mathcal{O}(\rho + |\sigma|)|^2} \gtrsim \frac{|\sigma|}{|\sigma|^2 + \rho^2 \rho}. \] (7.51)

Here, in the first step, the error term $\mathcal{O}(\rho + |\sigma|)$ in the numerator originates from the second term in
\[ (\text{sign } \kappa_1 \sigma) \text{Im} [b \partial_\nu \Theta(0) - \rho |\text{Re} \partial_\nu \Theta(0)|] \]
and applying (7.50) to it. We applied (7.49) to the first term on the right-hand side of (7.51). In the last estimate, we used $\psi, |\sigma|, \rho \lesssim 1$ and $|\sigma| \gtrsim \Pi_1 \rho^2$ for some large $\Pi_1 \sim 1$ as well as small $\rho \lesssim \rho_*$ for some small $\rho_* \sim 1$. Employing $|\sigma| \gtrsim \Pi_1 \rho^2$ once more, the factor $|\sigma|/(|\sigma|^2 + \rho^2)$ on the right-hand side scales like $(1 + |\sigma|)^{-1} \gtrsim 1$.

Hence, we conclude from (7.47) and (7.48) that
\[ (\text{sign } \kappa_1 \sigma) \partial_\nu v(\tau) \gtrsim \frac{1}{\rho} + \mathcal{O}(1). \]

By choosing $\rho_* \sim 1$ sufficiently small, we obtain (7.46). This completes the proof of Lemma 7.11. \qed

**Proof of Lemma 7.12.** We prove both cases, $\rho(\tau) = 0$ for all $\tau \in [\tau_0 - \varepsilon, \tau_0]$ or for all $\tau \in [\tau_0, \tau_0 + \varepsilon]$ in parallel. We can assume that $|\sigma| \leq \tilde{\sigma}$ for any $\tilde{\sigma} \sim 1$ as the statement trivially holds true otherwise.

We choose $(\delta, \varrho, \gamma_*)$ as in Proposition 7.7 (iv), $\tilde{\Delta}$ as in (7.13), normal coordinates $(\lambda, \Omega(\lambda))$ as in (7.42) as well as $\tilde{\gamma} = \tilde{\Delta}^{1/3}$ and $s = \text{sign } \sigma$. We set $\lambda_3 := 2 + \varrho \tilde{\Delta}^{1/3}$ (cf. (7.29)) and $\omega_3 := \tilde{\Delta} \lambda_3/2$. There is $\tilde{\sigma} \sim 1$ such that $\Delta \leq \gamma_*^3$ for $|\sigma| \leq \tilde{\sigma}$ due to $\tilde{\Delta} \sim |\sigma|^3$ by (6.6) and the definition of $\tilde{\Delta}$ in (7.13). Hence, $\omega_3 \leq C |\sigma|^3$ and, by possibly shrinking $\tilde{\sigma} \sim 1$, we obtain $-\omega_3 \text{sign } \sigma \in J$ for $|\sigma| \leq \tilde{\sigma}$ due to the assumption on $J$ ([inf $J$] $\gtrsim 1$ or sup $J$ $\gtrsim 1$). From (7.27), we obtain $\text{Im} \Omega(-\omega_3 \text{sign } \sigma) > 0$. Hence, $\text{Im} \Theta(-\omega_3 \text{sign } \sigma) > 0$. From the third bound in (6.8), the second bound in (6.7a) and $\omega_3 \lesssim |\sigma|^3$, we conclude $v(-\omega_3 \text{sign } \sigma) > 0$ for $|\sigma| \leq \tilde{\sigma}$ and sufficiently small $\tilde{\sigma} \sim 1$. Thus, $\rho(-\omega_3 \text{sign } \sigma) > 0$ which implies $\omega_3 > \varepsilon$. Therefore, $|\sigma|^3 \gtrsim \omega_3 > \varepsilon \sim 1$ which completes the proof of Lemma 7.12. \qed

We finish this section by proving Lemma 7.13. It is similarly proven as Lemma 9.17 in [1]. We present the proof due to the weaker assumptions of Lemma 7.13. The main difference is the proof of (7.53) below (cf. (9.138) in [1]). In [1], $\Theta$ could be explicitly represented in terms of $m$, i.e.,
\[ \Theta(\omega) = \langle f, m(\tau_0 + \omega) - m(\tau_0) \rangle \]
(cf. (9.8) and (8.10c) in [1] with $\alpha = 0$). In our setup, $b$ and $r$ do not necessarily define an orthogonal decomposition (cf. (6.1)).

**Proof of Lemma 7.13.** Let $(\delta, \varrho, \gamma_*)$ be chosen as in Proposition 7.7 (iv). We choose $\tilde{\Delta}$ as in (7.13) and normal coordinates as in (7.42) as well as $\tilde{\gamma} = \tilde{\Delta}^{1/3}$ and $s = \text{sign } \sigma$. We assume $\tilde{\Delta} \leq \gamma_*^3$ in the following and define $\lambda_3$ as in (7.29). By using [inf $J_0$] $\gtrsim 1$ as in the proof of Lemma 7.12, we find $\tilde{\sigma} \sim 1$ such that $-\omega_3 \in J_0$ for $\omega_3 := \lambda_3 \tilde{\Delta}/2$ and $|\sigma| \leq \tilde{\sigma}$. Thus, $-\Delta = \tau_1 - \tau_0 \in J_0$. We set
\[ \lambda_0 := \inf \{ \lambda > 0: \text{Im} \Omega(\lambda) > 0 \} \]
and remark that $\lambda_0 = 2\Delta/\tilde{\Delta}$ due to the definition of $\Delta$ and the third bound in (6.8). From (7.27), we conclude $\lambda_0 \leq \lambda_3$. Thus, $\Delta \leq \tilde{\Delta}(1 + \mathcal{O}(\gamma_*)) = \tilde{\Delta}(1 + \mathcal{O}(|\sigma|))$ as $\varrho \sim 1$ and $\tilde{\gamma} \sim |\sigma|$. Therefore, it suffices to show the opposite bound,
\[ \Delta \geq \tilde{\Delta}(1 + \mathcal{O}(|\sigma|)). \] (7.52)

If $\lambda_0 \geq \lambda_2 := 2 - \varrho \tilde{\Delta}^{1/3}$ (cf. (7.29)) then we have (7.52) as $\tilde{\Delta}^{1/3} \sim |\sigma|$ and $\varrho \sim 1$. If $\lambda_0 < \lambda_2$ then we will prove below that
\[ \text{Im} \Omega(\lambda_0 + \xi) \gtrsim \xi^{1/2} \] (7.53)
for \( \xi \in [0, 1] \). From (7.26), we then conclude
\[
\frac{c_0}{2} (\lambda_2 - \lambda_0)^{1/2} \leq \text{Im} \Omega(\lambda_2) \leq C_1 |\sigma|^{1/2}
\]
as \( \tilde{\gamma} \sim |\sigma| \). Hence,
\[
\lambda_0 \geq \lambda_2 - (C_1/c_0)^2 |\sigma| \geq 2 - C|\sigma|,
\]
where we used \( \lambda_2 = 2 - \varrho \tilde{\gamma} \) and \( \varrho \sim 1 \) in the last step. This shows (7.52) also in the case \( \lambda_0 < \lambda_2 \). Therefore, the proof of the lemma will be completed once (7.53) is proven.

In order to prove (7.53), we translate it into the coordinates \( \omega \) relative to \( \tau_0 \) and \( v \). From \( \lambda_0 < \lambda_2 \), we obtain
\[
\Delta < (1 - \varrho \tilde{\Delta}^{1/3}) \tilde{\Delta} \lesssim |\sigma|^3.
\] (7.54)

Since
\[
\pi v(\tau_0 - \Delta - \tilde{\omega}) = b \text{Im} \Theta(-\Delta - \tilde{\omega}) + \text{Im} r(-\Delta - \tilde{\omega}),
\]
the bound (7.53) would follow from
\[
v(\tau_0 - \Delta - \tilde{\omega}) \gtrsim \tilde{\Delta}(\tau_0)^{-1/6} |\tilde{\omega}|^{1/2}
\] (7.55)
for sufficiently small \( \Delta \lesssim |\sigma|^3 \leq \tilde{\sigma}^3 \) and \( \tilde{\omega} \leq \tilde{\delta} \) due to the third bound in (6.8). Since \( v(\tau_1) = 0 \) and \( \tau_1 = \tau_0 - \Delta \) is a shape regular point, we conclude from (7.44) that
\[
v(\tau_1 - \tilde{\omega}) \gtrsim \tilde{\Delta}(\tau_1)^{-1/6} |\tilde{\omega}|^{1/2}
\]
for \( |\tilde{\omega}| \leq \delta \). Therefore, it suffices to show that
\[
\tilde{\Delta}(\tau_1) \lesssim \tilde{\Delta}(\tau_0)
\] (7.56)
in order to verify (7.55). Owing to \(|\sigma(\tau_1) - \sigma(\tau_0)| \lesssim \Delta^\zeta \) and (7.54), we have
\[
|\sigma(\tau_1)| \lesssim |\sigma(\tau_0)| + \Delta^\zeta \lesssim |\sigma(\tau_0)|^{1/3}.
\]

We allow for a smaller choice of \( \tilde{\sigma} \sim 1 \) and assume \( \psi(\tau_1) \sim \psi(\tau_0) \sim 1 \) by (6.6). Assuming without loss of generality \( \tilde{\Delta}(\tau_0) < 1 \) and \( \tilde{\Delta}(\tau_1) < 1 \), we obtain (7.56) by the definition of \( \tilde{\Delta} \) in (7.13). We thus get (7.56) and hence (7.55). This proves (7.53) and completes the proof of Lemma 7.13.

\section{8. Band mass formula – Proof of Proposition 2.6}

Before proving Proposition 2.6, we state an auxiliary lemma which will be proven at the end of this section.

\begin{lemma}
Let \((a, S)\) be a data pair, \( m \) the solution of the associated Dyson equation (2.3) and \( \rho \) the corresponding self-consistent density of states. We assume \( |a| \leq k_0 \) and \( S[x] \leq k_1(x) \) for all \( x \in \mathbb{A}_+ \) and for some \( k_0, k_1 > 0 \). Then we have
\begin{enumerate}[(i)]
\item If \( \tau \in \mathbb{R} \setminus \text{supp} \rho \) then there is \( m(\tau) = m(\tau)^* \in A \) such that
\[
\lim_{\eta \downarrow 0} \| m(\tau + i\eta) - m(\tau) \| = 0.
\]
Moreover, \( m(\tau) \) is invertible and satisfies the Dyson equation, (2.3), at \( z = \tau \). There is \( C > 0 \), depending only on \( k_0, k_1 \) and \( \text{dist}(\tau, \text{supp} \rho) \), such that \( \| m(\tau) \| \leq C \) and \( \| (\text{Id} - (1 - t)C m(\tau) S)^{-1} \| \leq C \) all \( t \in [0, 1] \).
\item Let \( m_t \) be the solution of (2.3) associated to the data pair
\[
(a_t, S_t) := (a - tS[m(\tau)], (1 - t)S)
\]
for \( t \in [0, 1] \) and \( \rho_t \) the corresponding self-consistent density of states. Then, for any \( t \in [0, 1] \), we have
\[
\lim_{\eta \downarrow 0} \| m_t(\tau + i\eta) - m(\tau) \| = 0.
\] (8.1)
Moreover, there is \( c > 0 \) such that \( \text{dist}(\tau, \text{supp} \rho_t) \geq c \) for all \( t \in [0, 1] \).
\end{enumerate}
\end{lemma}
Proof of Proposition 2.6. We start with the proof of (i) and notice that the existence of $m(\tau)$ has been proven in Lemma 8.1 (i). In order to verify (2.10), we consider the continuous flow of data pairs $(\alpha_i, S_i)$ from Lemma 8.1 (ii) and the corresponding solutions $m_t$ of the Dyson equation, (2.3), and prove

$$\rho_t((-\infty, \tau)) = (1_{(-\infty,0)}(m_t(\tau)))$$

(8.2)

for all $t \in [0,1]$. Note that $\text{dist}(\tau, \text{supp } \rho_t) \geq c$ for all $t \in [0,1]$ by Lemma 8.1 (ii).

In particular, by Lemma 8.1 (ii), $m_t(\tau) = m(\tau)$ is constant along the flow, and with it the right-hand side of (8.2). The identity (8.2) obviously holds for $t = 1$, because $m_1(z) = (a - Sm(\tau) - z)^{-1}$ is the resolvent of a self-adjoint element and $m(\tau)$ satisfies (2.3) at $z = \tau$ by Lemma 8.1 (i). Thus it remains to verify that the left-hand side of (8.2) stays constant along the flow as well. This will show (8.2) for $t = 0$ which is (2.10).

First we conclude from the Stieltjes transform representation (2.4) of $m_t$ that

$$\rho_t((-\infty, \tau)) = -\frac{1}{2\pi i} \oint m_t(z) \, dz,$$

(8.3)

where the contour encircles $[\text{min supp } \rho_t, \tau)$ counterclockwise, passing through the real line only at $\tau$ and to the left of $\text{min supp } \rho_t$, and we extended $m_t(z)$ analytically to a neighbourhood of the contour (set $m_t(z) := m_t(z)^*$ for $z \in \mathbb{H}$ and use Lemma D.1 (iv) close to the real axis to conclude analyticity in a neighbourhood of the contour).

We now show that the left-hand side of (8.3) does not change along the flow. Indeed, differentiating the right-hand side of (8.3) with respect to $t$ and writing $m_t = m_t(z)$ yield

$$\frac{d}{dt} \oint m_t(z) \, dz = \oint (\partial_t m_t(z)) \, dz = \oint (\langle (C_{m_t}^{-1} - S_t)^{-1}[1], S[m(\tau)] - S[m_t] \rangle) \, dz = \oint (\partial_t m_t(S[m(\tau)] - S[m_t]) \, dz = \oint \partial_t (m_t(S[m(\tau)] - \frac{1}{2} m_t S[m_t])) \, dz = 0.$$

Here, in the second step, we used $\partial_t m_t(z) = (C_{m_t}^{-1} - S_t)^{-1}[-S[m_t] - S[m(\tau)]]$ obtained by differentiating the Dyson equation, (2.3), for the data pair $(\alpha_t, S_t)$ defined in Lemma 8.1 (ii) and the definition of the scalar product, (2.1). In the third step, we employed $(C_{m_t}^{-1} - S_t)^{-1}[1] = (\partial_t m_t(z))^*$ which follows from differentiating the Dyson equation, (2.3), for the data pair $(\alpha_t, S_t)$ with respect to $z$. Finally, we used that $m_t$ is holomorphic in a neighbourhood of the contour. This completes the proof of (i) of Proposition 2.6.

For the proof of (ii), we fix a connected component $J$ of supp $\rho$. Let $\tau_1, \tau_2 \in \mathbb{R} \setminus \text{supp } \rho$ satisfy $\tau_1 < \tau_2$ and $[\tau_1, \tau_2] \cap \text{supp } \rho = \emptyset$. By (2.10), we have

$$n\rho(J) = n(\rho((-\infty, \tau_2)) - \rho((-\infty, \tau_1))) = \text{Tr}(P_2) - \text{Tr}(P_1) = \text{rank } P_2 - \text{rank } P_1,$$

where $P_i := \pi(1_{(-\infty,0)}(m(\tau_i)))$ are orthogonal projections in $\mathbb{C}^{n \times n}$ for $i = 1,2$. Hence, $n\rho(J) \in \mathbb{Z}$. Since $0 < n\rho(J) \leq n$ by definition of supp $\rho$, we conclude $n\rho(J) \in \{1, \ldots, n\}$, which immediately implies that supp $\rho$ has at most $n$ connected components. This completes the proof of Proposition 2.6. □

Proof of Lemma 8.1. In part (i), the existence of the limit $m(\tau) \in \mathcal{A}$ follows immediately from the implication (v) $\Rightarrow$ (iii) of Lemma D.1. The invertibility of $m(\tau)$ can be seen by multiplying (2.3) at $z = \tau + i\eta$ by $m(\tau + i\eta)$ and taking the limit $\eta \downarrow 0$. This also implies that $m(\tau)$ satisfies (2.3) at $z = \tau$. In order to bound $\|\text{Id} - (1-t)C_{m(\tau)}S\|$, we recall the definitions of $q, u$ and $F$ from (3.1) and (3.4), respectively, and compute

$$\text{Id} - (1-t)C_m S = C_{q,t,q} \text{Id} - (1-t)C_u F \text{C}_{q,t,q}^{-1}$$

for $m = m(z)$ with $z \in \mathbb{H}$. Hence, by (D.1), Lemma 8.3 (i) and Lemma B.2, we obtain $\|\text{Id} - (1-t)C_m S\| \leq (1 - (1-t)\|F\|)^{-1} \leq (1 - (1-t)\|F\|)^{-1} \leq C$ for all $z \in \tau + i\mathbb{N}$, where the set $N \subset (0,1]$ with an accumulation point at 0 is given in Lemma D.1 (ii). Taking the limit $\eta \downarrow 0$ under the constraint $\eta \in N$ and possibly increasing $C$ yield the desired uniform bound. This completes the proof of (i).

We start the proof of (ii) with an auxiliary result. Similarly as in the proof of (i), we see that Id $- (1-t)C_{m*,m} S$ is invertible for $m = m(z), z \in \tau + i\mathbb{N}$ with $N$ as before. Moreover, we obtain

$$\|\text{Id} - (1-t)C_{m*,m} S\| = C_{q,t,q} \|\text{Id} - (1-t)C_{u,t} F \text{C}_{q,t,q}^{-1} \| \|\text{Id} - (1-t)C_{u,t} F \text{C}_{q,t,q}^{-1} \|$$

$$= 1 + C_{q,t,q} \left( \sum_{k=1}^{\infty} (1-t)k^2 (C_{u,t} F )^k \right) \|q\|^{-1} \geq 1,$$

(8.4)
where $m = m(z)$ and $z \in \tau + i\mathbb{N}$. In the second step, we expanded the inverse as a Neumann series using $\|F(z)\|_2 \leq 1 - C^{-1}$ for $z \in \tau + i\mathbb{N}$ by Lemma D.1 (ii). In particular, the series converges with respect to $\|\cdot\|_2$. In the last inequality in (8.4), we used that $(C_{\alpha \beta}F)^k$ is a positivity-preserving operator for all $k \in \mathbb{N}$. Since (8.4) holds true uniformly for $z \in \tau + i\mathbb{N}$ and $t \in [0, 1]$, taking the limit $\eta = \text{Im} \ z \downarrow 0$ in $N$, we obtain

$$-(\text{Id} - (1 - t)C_m(\tau)S)^{-1}[m] \geq 1$$

(8.5)

for all $t \in [0, 1]$. A similar argument shows that $(\text{Id} - (1 - t)C_m(\tau)S)^{-1}$ is positivity-preserving for any $t \in [0, 1]$.

We fix $t \in [0, 1]$. We write $m = m(\tau)$ and define $\Phi_t : \mathcal{A} \times \mathbb{R} \rightarrow \mathcal{A}$ through

$$\Phi_t(\Delta, \eta) := (\text{Id} - (1 - t)C_mS)[\Delta] - \frac{i}{2}(m\Delta + m\Delta) - i\eta m^2 - \frac{1}{2}(1 - t)(\Delta S[m]m + mS[\Delta]\Delta)$$

In order to show (8.1), we apply the implicit function theorem (see e.g. Lemma D.4 below) to $\Phi_t(\Delta, \eta) = 0$. It is applicable as $\Phi_t(0, 0) = 0$ and $\partial_t \Phi_t(0, 0) = (\text{Id} - (1 - t)C_mS$ which is invertible by (i). Hence, we obtain an $\varepsilon > 0$ and a continuously differentiable function $\Delta_t : (-\varepsilon, \varepsilon) \rightarrow \mathcal{A}$ such that $\Phi_t(\Delta_t, \eta) = 0$ for all $\eta \in (-\varepsilon, \varepsilon)$ and $\Delta_t(0) = 0$. We now show that $\Delta_t(\eta) + m(\tau) = m_t(\tau + iy)$ for all sufficiently small $\eta > 0$ by appealing to the uniqueness of the solution to the Dyson equation, (2.3), with the choice $z = \tau + iy$, $a = a_t$, and $S = S_t = (1 - t)S$. In fact, $m = m(\tau)$ and $m_t = m_t(\tau + iy)$ with $\eta > 0$ satisfy the Dyson equations

$$-m^{-1} = \tau - a + S[m], \quad -m_t^{-1} = \tau + iy - a + tS[m] + (1 - t)S[m_t]$$

(8.6)

and $m_t$ is the unique solution of the second equation under the constraint $\text{Im} \ m_t > 0$ (compare the remarks around (2.3)). A straightforward computation using the first relation in (8.6) and $\Phi_t(\Delta_t(\eta), \eta) = 0$ reveals that $\Delta_t(\eta) + m(\tau)$ solves the second equation in (8.6) for $m_t$. Moreover, differentiating $\Phi_t(\Delta_t(\eta), \eta) = 0$ with respect to $\eta$ at $\eta = 0$ yields

$$\partial_t \text{Im} \ \Delta_t(\eta) = 0 = (\text{Id} - (1 - t)C_mS)^{-1}[m^2] \geq \|m^{-1}\|^{-2}(\text{Id} - (1 - t)C_mS)^{-1}[1] \geq \|m^{-1}\|^{-2}.$$}

Here, we used that $(\text{Id} - (1 - t)C_mS)^{-1}$ is compatible with the involution $^*$ and $m = m^*$ in the first step. Then we employed the invertibility of $m$, $m^2 \geq \|m^{-1}\|^{-2}$, and the positivity-preserving property of $(\text{Id} - (1 - t)C_mS)^{-1}$ in the second step and, finally, (8.5) in the last step. Hence, $\text{Im} \ (\Delta_t(\eta) + m(\tau)) = \text{Im} \ \Delta_t(\eta) > 0$ for all sufficiently small $\eta > 0$. The uniqueness of the solution to the Dyson equation for $m_t$, the second relation in (8.6), implies $\Delta_t(\eta) + m(\tau) = m_t(\tau + iy)$ for all sufficiently small $\eta > 0$ and all $t \in [0, 1]$. Therefore, the continuity of $\Delta_t$ as a function of $\eta$, $\Delta_t(\eta) \rightarrow \Delta_t(0) = 0$, yields (8.1).

We now conclude from the implication (iii) $\Rightarrow$ (v) of Lemma D.1 that $\text{dist}(\tau, \text{supp} \, \rho_t) \geq \varepsilon$ for some $\varepsilon > 0$. Lemma D.1 is applicable since $\|a_t\| \leq k_0 + k_1C$ (cf. Lemma B.2 (i) and Lemma 8.1 (ii)) and $S_t[x] \leq S[x] \leq k_1(x)1$ for all $x \in \mathcal{A}_t$. For any $t \in [0, 1]$, statement (iii) in Lemma D.1 holds true with the same $m = m(\tau)$ by (8.1) and $S$ replaced by $S_t = (1 - t)S$. By (i), $\|m\| \leq C$ and $\|\text{Id} - (1 - t)C_mS\| \leq C$ for all $t \in [0, 1]$. Hence, owing to Lemma D.1 (v), there is $\varepsilon > 0$ such that $\text{dist}(\tau, \text{supp} \, \rho_t) \geq \varepsilon$ for all $t \in [0, 1]$. The uniformity of $\varepsilon$ in $t$ is a consequence of the effective dependence of the constants in Lemma D.1 on each other (see final remark in Lemma D.1) and the uniform upper bound on $\|\text{Id} - (1 - t)C_mS\|$. This completes the proof of Lemma 8.1.

9. Dyson equation for Kronecker random matrices

In this section we present an application of the theory presented in this work to Kronecker random matrices, i.e., block correlated random matrices with variance profiles within the blocks, and their limits. In particular, in Lemma 9.1 and Lemma 9.3 below, we will provide some sufficient checkable conditions that ensure the flatness of $S$ and the boundedness of $\|m(z)\|$, the main assumptions of Proposition 2.4, Theorem 2.6 and Theorem 7.1 for the self-consistent density of states of Kronecker random matrices introduced in [7].

9.1. The Kronecker setup

We fix $K \in \mathbb{N}$ and a probability space $(\mathcal{X}, \pi)$ that we view as a possibly infinite set of indices. We consider the von Neumann algebra

$$\mathcal{A} = \mathbb{C}^{K \times K} \otimes L^\infty(\mathcal{X}),$$

(9.1)
with the tracial state

\[ \langle \kappa \otimes f \rangle = \frac{\text{Tr} \kappa}{K} \int f \, d\tau. \]

For \( K = 1 \) the algebra \( \mathcal{A} \) is commutative and this setup was previously considered in \([1, 2]\). Now let \((\alpha_\mu)_{\mu=1}^{\ell_1}, (\beta_\nu)_{\nu=1}^{\ell_2}\) be families of matrices in \( \mathbb{C}^{K \times K} \) with \( \alpha_\mu = \alpha_\mu^* \) self-adjoint and let \((s^\mu)_{\mu=1}^{\ell_1}, (T^\nu)_{\nu=1}^{\ell_2}\) be families of non-negative bounded functions in \( L^\infty(\mathbb{X}^2) \) and suppose that all \( s^\mu \) are symmetric, \( s^\mu(x,y) = s^\mu(y,x) \). Then we define the self-energy operator \( \mathcal{S} : \mathcal{A} \to \mathcal{A} \) as

\[ \mathcal{S}(\kappa \otimes f) := \sum_{\mu=1}^{\ell_1} \alpha_\mu \kappa \alpha_\mu \otimes S_\mu f + \sum_{\nu=1}^{\ell_2} (\beta_\nu \kappa \beta_\nu^* \otimes T_\nu f + \beta_\nu^* \kappa \beta_\nu \otimes T_\nu^* f), \]

where the bounded operators \( S_\mu, T_\nu, T_\nu^* : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X}) \) act as

\[ (S_\mu f)(x) = \int_\mathbb{X} s^\mu(x,y) f(y) \pi(dy), \quad (T_\nu f)(x) = \int_\mathbb{X} t^\nu(x,y) f(y) \pi(dy), \quad (T_\nu^* f)(x) = \int_\mathbb{X} t^\nu(y,x) f(y) \pi(dy). \]

Furthermore we fix a self-adjoint \( a = a^* \in \mathcal{A} \). With these data we will consider the Dyson equation, \((2.3)\).

The following lemma provides sufficient conditions that ensure flatness of \( \mathcal{S} \) and boundedness of \( \|m(z)\| \) uniformly in \( z \) up to the real line. We begin with some preparations. We use the notation \( x \mapsto v_x \) for \( x \in \mathbb{X} \) and an element \( v \in \mathbb{C}^{K \times K} \otimes L^\infty(\mathbb{X}) \), interpreting it as a function on \( \mathbb{X} \) with values in \( \mathbb{C}^{K \times K} \). We also introduce the functions \( \gamma \in L^\infty(\mathbb{X}^2) \) via

\[ \gamma(x, y) := \left( \int_\mathbb{X} \left| (s^\mu(x, \cdot) - s^\mu(y, \cdot))^2 + |t^\nu(x, \cdot) - t^\nu(y, \cdot)|^2 + |t^\nu(\cdot, x) - t^\nu(\cdot, y)|^2 \right| \, d\pi \right)^{1/2} \]

and \( \Gamma : (0, \infty)^2 \to L^\infty(\mathbb{X}), (\Lambda, \tau) \mapsto \Gamma_{\Lambda, \tau}(\cdot) \) through

\[ \Gamma_{\Lambda, \tau}(x) := \int_\mathbb{X} \left( \frac{1}{\tau} + \|a_x - a_y\| + \gamma(x, y) \Lambda \right)^{-2} \pi(dy). \]

Here, we denoted by \( \| \cdot \| \) the operator norm on \( \mathbb{C}^{K \times K} \) induced by the Euclidean norm on \( \mathbb{C}^K \). The two functions \( \gamma \) and \( \Gamma \) will be important to quantify the modulus of continuity of the data \((a, \mathcal{S})\).

**Lemma 9.1.** Let \( m \) be the solution of the Dyson equation, \((2.3)\), on the von Neumann algebra \( \mathcal{A} \) from \((9.1)\) associated to the data \((a, \mathcal{S})\) with \( \mathcal{S} \) defined as in \((9.2)\).

(i) Define \( \Gamma(\tau) := C_{K\tau}, \text{ess inf}_x \Gamma_{1, \tau}(x) \) with \( C_{K\tau} := 4 + 4K(\ell_1 + \ell_2) \max_{\mu, \nu}(\|\alpha_\mu\|^2 + \|\beta_\nu\|^2) \), where \( \Gamma_{\Lambda, \tau}(x) \) was introduced in \((9.4)\) and assume that for some \( z \in \mathbb{H} \) the \( L^2\)-upper bound \( \|m(z)\|_2 \leq \Lambda \) for some \( \Lambda \geq 1 \) is satisfied. Then we have the uniform upper bound

\[ \|m(z)\| \leq \frac{\Gamma^{-1}(\Lambda)}{\Lambda}, \]

where we interpret the right-hand side as \( \infty \) if \( \Lambda \) is not in the range of the strictly monotonously increasing function \( \Gamma \).

(ii) Suppose that the kernels of the operators \( S^\mu \) and \( T^\nu \), used to define \( \mathcal{S} \) in \((9.2)\), are bounded from below, i.e., \( \text{ess inf}_{x,y} s^\mu(x,y) > 0 \) and \( \text{ess inf}_{x,y} t^\nu(x,y) > 0 \). Suppose further that

\[ \inf_{\kappa} \frac{1}{\text{Tr} \kappa} \left( \sum_{\mu=1}^{\ell_1} \alpha_\mu \kappa \alpha_\mu + \sum_{\nu=1}^{\ell_2} (\beta_\nu \kappa \beta_\nu^* + \beta_\nu^* \kappa \beta_\nu) \right) > 0, \]

where the infimum is taken over all positive definite \( \kappa \in \mathbb{C}^{K \times K} \). Then \( \mathcal{S} \) is flat, i.e., \( \mathcal{S} \in \Sigma_{\text{flat}} \) (cf. \((2.2)\)).

(iii) Let \( \mathcal{S} \) be flat, hence, \( \Lambda := 1 + \sup_{z \in \mathbb{R}} \|m(z)\|_2 < \infty \). Then \((9.5)\) holds true with this \( \Lambda \).

(iv) If \( a = 0 \) then, for each \( \varepsilon > 0 \), \((9.5)\) holds true on \( |z| \geq \varepsilon \) with \( \Lambda := 1 + 2\varepsilon^{-1} \).

**Proof of Lemma 9.1.** We adapt the proof of Proposition 6.6 in \([1]\) to our noncommutative setting in order to
prove (i). Recall the definition of $\gamma(x, y)$ in (9.3). Estimating the norm $\|m\|_2$ from below, we find

$$\|m\|_2^2 = \frac{1}{K} \text{Tr} \int \frac{\pi(dy)}{m_y^{-1}(m_y^*)^{-1}} \geq \text{Tr} \int_X \frac{C\pi(dy)}{m_x^{-1}(m_x^*)^{-1} + \|a_x - a_y\|^2 + \gamma(x, y)^2\|m\|^2_2} \geq C\pi_\|m\|_2, (m_x^*)|, \quad (9.7)$$

for $\pi$-almost all $x \in X$, where we used

$$\frac{1}{4} m_y^{-1}(m_y^*)^{-1} \leq m_x^{-1}(m_x^*)^{-1} + (a_y - a_x)(a_y - a_x)^* + ((Sm)_x - (Sm)_y)((Sm)_x - (Sm)_y)^* \leq m_x^{-1}(m_x^*)^{-1} + \|a_x - a_y\|^2 + K(\ell_1 + \ell_2) \max(\|\alpha\|^2 + \|\beta\|^2) \gamma(x, y)^2\|m\|^2_2. \quad (9.8)$$

We conclude $\Lambda^2 \geq \Lambda^{-2}\Gamma(\|m\|_2)$ for any upper bound $\Lambda \geq 1$ on $\|m\|_2$. In particular, (9.5) follows.

We turn to the proof of (ii). We view a positive element $r \in A_+$ as a function $r : [0, 1] \to \mathbb{C}^{K \times K}$ with values in positive semidefinite matrices. Then we find

$$(Sr)_x \geq c \int_X \left(\sum_{\mu=1}^{\ell_x} \alpha_{\mu} r_{\mu} a_{\mu} + \sum_{\nu=1}^{\ell_y} (\beta_{\nu} r_{\nu} \beta_{\nu}^* + \beta_{\nu}^* r_{\nu} \beta_{\nu})\right) \pi(dy),$$

as quadratic forms on $\mathbb{C}^{K \times K}$ for almost every $x \in X$. The claim follows now immediately from (9.6). Part (iii) is a direct consequence of (i) and (ii) as well as (3.11). For the proof of part (iv), we use part (i) and (2.6) if $a = 0$.

9.2. $N \times N$-Kronecker random matrices

As an application of the general Kronecker setup introduced above, we consider the matrix Dyson equation associated to Kronecker random matrices. Let $X_\mu, Y_\nu \in \mathbb{C}^{N \times N}$ be independent centered random matrices such that $Y_\nu = (y_{ij}^\nu)$ has independent entries and $X_\mu = (x_{ij}^\mu)$ has independent entries up to the Hermitian symmetry constraint $X_\mu = X_\mu^*$. Suppose that the entries of $\sqrt{N}X_\mu, \sqrt{N}Y_\nu$ have uniformly bounded moments, $E(|x_{ij}^\mu|^p + |y_{ij}^\nu|^p) \leq N^{-p/2}C_p$ and define their variance profiles through

$$s^\mu(i, j) := N E|x_{ij}^\mu|^2, \quad t^\nu(i, j) := N E|y_{ij}^\nu|^2.$$

Then we are interested in the asymptotic spectral properties of the Hermitian Kronecker random matrix

$$H := A + \sum_{\mu=1}^{\ell_x} \alpha_{\mu} \otimes X_\mu + \sum_{\nu=1}^{\ell_y} (\beta_{\nu} \otimes Y_\nu + \beta_{\nu}^* \otimes Y_\nu^*) \in \mathbb{C}^{K \times K} \otimes \mathbb{C}^{N \times N}, \quad (9.9)$$

as $N \to \infty$. Here the expectation matrix $A$ is assumed to be bounded, $\|A\| \leq C$, and block diagonal, i.e.

$$A = \sum_{i=1}^{N} a_i \otimes E_{ii}, \quad (9.10)$$

with $E_{ii} = (\delta_{ik} \delta_{il})_{i,k=1}^N \in \mathbb{C}^{N \times N}$ and $a_i \in \mathbb{C}^{K \times K}$. In [7] it was shown that the resolvent $G(z) = (H - z)^{-1}$ of the Kronecker matrix $H$ is well approximated by the solution $M(z)$ of a Dyson equation of Kronecker type, i.e., on the von Neumann algebra $A$ in (9.1) with self-energy $S$ from (9.2) and $a = A \in A$, when we choose $X = \{1, \ldots, N\}$ and $\pi$ the uniform probability distribution. In other words, $L^\infty(X) = \mathbb{C}^N$ with entrywise multiplication.

9.3. Limits of Kronecker random matrices

Now we consider limits of Kronecker random matrices $H \in \mathbb{C}^{N \times N}$ with piecewise Hölder-continuous variance profiles as $N \to \infty$. In this situation we can make sense of the continuum limit for the solution $M(z)$ of the associated matrix Dyson equation. The natural setup here is $(X, \pi) = ([0, 1], dx)$. We fix a partition $(I_l)_{l=1}^L$ of $[0, 1]$ into intervals of positive length, i.e., $[0, 1] = \cup_l I_l$ and consider non-negative profile functions $s^\mu, t^\nu : [0, 1]^2 \to \mathbb{R}$ that are Hölder-continuous with Hölder exponent 1/2 on each rectangle $I_l \times I_k$. We also fix a function $a : [0, 1] \to \mathbb{C}^{K \times K}$ that is 1/2-Hölder continuous on each $I_l$. In this piecewise Hölder-continuous setup the Dyson equation on $A$ with data pair $(a, S)$ describes the asymptotic spectral properties of Kronecker
random matrices with fixed variance profiles $s^a$ and $t^v$, i.e., the random matrices $H$ introduced in Subsection 9.2 if their variances are given by

$$E|x_{ij}^a|^2 = \frac{1}{N}s^a\left(\frac{i}{N}, \frac{j}{N}\right), \quad E|y_{ij}^v|^2 = \frac{1}{N}t^v\left(\frac{i}{N}, \frac{j}{N}\right),$$

and the matrices $a_i$ in (9.10) by $a_i = a(B_i)$.  

**Lemma 9.2.** Suppose that $a$, $s^a$ and $t^v$ are piecewise Hölder-continuous with Hölder exponent $1/2$ as described above. The empirical spectral distribution of the Kronecker random matrix $H$ follows from (9.9), with eigenvalues $(\lambda_i)_{i=1}^{KN}$ converging weakly to the self-consistent density of states $\rho$ associated to the Dyson equation with data pair $(a, S)$ as defined in (9.2), i.e., for any $\varepsilon > 0$ and $\varphi \in C(\mathbb{R})$ we have

$$P\left(\left|\frac{1}{KN}\sum_{i=1}^{KN} \varphi(\lambda_i) - \int_{\mathbb{R}} \varphi d\rho\right| > \varepsilon\right) \rightarrow 0, \quad N \rightarrow \infty.$$

**Proof of Lemma 9.2.** It suffices to prove convergence of the Stieltjes transforms, i.e., in probability $\frac{1}{KN}\text{Tr}_{KN}G(z) \rightarrow \langle m(z) \rangle$ for every fixed $z \in \mathbb{H}$, where $G(z) = (H - z)^{-1}$ is the resolvent of the Kronecker matrix $H$ and $m(z)$ is the solution to the Dyson equation with data $(a, S)$.  

First we use the Theorem 2.7 from [7] to show that $\frac{1}{N}\text{Tr}_{KN}G(z) - \frac{1}{N}\sum_{i=1}^{N}\text{Tr}_{K}m_i(z) \rightarrow 0$ in probability, where $M_N = (m_1, \ldots, m_N) \in (C^{K \times K})^N$ denotes the solution to a Dyson equation formulated on the von Neumann algebra $C^{K \times K} \otimes \mathbb{C}^N$ with entrywise multiplication on vectors in $\mathbb{C}^N$ as explained in Subsection 9.2. We recall that in this setup the discrete kernels for $S_a$ and $T_c$ from the definition of $S$ in (9.2) are given by $N\mathbb{E}|x_{ij}^a|^2$ and $N\mathbb{E}|y_{ij}^v|^2$, respectively, and $a = \sum_{i=1}^{N}a(B_i) \otimes e_i$. To distinguish this discrete data pair from the continuum limit over $C^{K \times K} \otimes L^2(0, 1)$, we denote it by $(a_N, S_N)$. Note that in Theorem 2.7 of [7] the test functions were compactly supported in contrast to the function $\tau \mapsto 1/(\tau - z)$ that we used here. However, by Theorem 2.4 of [7] and since the self-consistent density of states is compactly supported (cf. (2.5a) and $||S|| \leq 1$) no eigenvalues can be found beyond a certain bounded interval, ensuring that non compactly supported test functions are allowed as well.

Now it remains to show that $\langle M_N \rangle \rightarrow \langle m \rangle$ as $N \rightarrow \infty$ for all $z \in \mathbb{H}$. For this purpose we embed $\mathbb{C}^N$ into $L^\infty([0,1])$ by $P_N = \sum_{i=1}^{N}1_{\{(i-1)/N,j/N\}}$. With this identification $M_N$ and $m$ satisfy Dyson equations on the same space $C^{K \times K} \otimes L^\infty([0,1])$. Evaluating these two equations at $z + in$, for a fixed $z \in \mathbb{H}$ and any $\eta \geq 0$, and subtracting them from each other yield

$$B[\Delta] = m(S_N - S)[m]\Delta + c_m(S_N - S)\Delta + mS_N[\Delta] + c_m(S_N - S)[m] - m(a_N - a)\Delta - c_m(a_N - a),$$

where $m = m(z + in)$, $M_N = M_N(z + in)$, $B = \text{Id} - c_mS$ and $\Delta = M_N - m$. Using the imaginary part of $z$ we have $\text{dist}(z + in, \text{supp} \rho) \geq \text{Im} z > 0$. By (3.22), (3.23), (3.11a) and (3.11c) of [7] we infer $||m|| + ||B^{-1}|| \leq C$ for all $\eta \geq 0$ with a constant $C$ depending on $\text{Im} z$. Note that although the proofs in [7] were performed on $C^{N \times N}$ all estimates were uniform in $N$ and all algebraic relations in these proof translate to the current setting on a finite von Neumann algebra. Using $||S_N - S||_2 \leq ||S_N - S||$ as well as $||S_N|| \leq C$ and possibly increasing $C$, we thus obtain

$$||\Delta||_2 \leq C(\Psi_N + ||\Delta||^2_2), \quad \Psi_N := ||a_N - a|| + ||S_N - S||,$$

where $\Delta = \Delta(z + in)$, for all $\eta \geq 0$. We choose $N_0$ sufficiently large such that $2\Psi_N C^2 \leq 1/4$ for all $N \geq N_0$ and define $\eta_* := \sup\{\eta \geq 0 : ||\Delta(z + in)||_2 \geq 2C\Psi_N\}$. Since $||M_N|| + ||m|| \rightarrow 0$ for $\eta \rightarrow \infty$, we conclude $\eta_* < \infty$. We now prove $\eta_* < \infty$. For a proof by contradiction, we suppose $\eta_* > \infty$. Then, by continuity, $||\Delta(\tau + i\eta_*)||_2 \leq 2C\Psi_N$. Since $2\Psi_N C^2 \leq 1/4$, we have $||\Delta(z + i\eta_*)||_2 \leq 4C\Psi_N/3 < 2C\Psi_N = ||\Delta(z + i\eta_*)||_2$. From this contradiction, we conclude $\eta_* = 0$. Therefore, for $N \geq N_0$, we have

$$|M_N(z) - m(z)| \leq ||\Delta(z)||_2 \leq 2C\Psi_N = 2C(||S_N - S|| + ||a_N - a||).$$

Since the right-hand side converges to zero as $N \rightarrow \infty$, due to the piecewise Hölder-continuity of the profile functions, and since $z$ was arbitrary, we obtain $\langle M_N \rangle \rightarrow \langle m \rangle$ as $N \rightarrow \infty$ for all $z \in \mathbb{H}$. This completes the proof of Lemma 9.2.

The boundedness of the solution to the Dyson equation in $L^2$-norm already implies uniform boundedness in the piecewise Hölder-continuous setup.

**Lemma 9.3.** Suppose that $a$, $s^a$ and $t^v$ are piecewise $1/2$-Hölder continuous and that $\sup_{z \in \mathbb{D}}||m(z)||_2 < \infty$ for some domain $\mathbb{D} \subseteq \mathbb{H}$. Then we have the uniform bound $\sup_{z \in \mathbb{D}}||m(z)||_2 < \infty$.  

44
In particular, if the random matrix $H$ is centered, i.e., $a = 0$, then $m(z)$ is uniformly bounded as long as $z$ is bounded away from zero; and if $H$ is flat in the limit, i.e., $S$ is flat, then $\sup_{x \in \mathbb{R}} \|m(z)\| < \infty$.

**Proof.** By (i) of Lemma 9.1 the proof reduces to checking that $\lim_{\tau \to \infty} \Gamma(\tau) = \infty$ for piecewise $1/2$-Hölder continuous data in the special case $(X, \pi) = ([0, 1], dx)$. But this is clear since in that case $\|a_x - a_y\|^2 + \gamma(x, y)^2 \leq C|x - y|$ implies that the integral in (9.4) is at least logarithmically divergent as $\tau \to \infty$. □

**Corollary 9.4** (Band mass quantization). Let $\rho$ be the self-consistent density of states for the Dyson equation with data pair $(a, S)$ and $\tau \in \mathbb{R} \setminus \supp \rho$. Then

$$\rho((\infty, \tau)) \in \left\{ \frac{1}{K} \sum_{l=1}^{L} k_l |I_l| : k_l = 1, \ldots, K \right\}.$$  

In particular, in the $L = 1$ case when $s^t, t^a$ and $a$ are $1/2$-Hölder continuous on all of $[0, 1]^2$ and $[0, 1]$, respectively, then $\rho(J)$ is an integer multiple of $1/K$ for every connected component $J$ of $\supp \rho$ and there are at most $K$ such components.

**Proof.** Fix $\tau \in \mathbb{R} \setminus \supp \rho$. We denote by $x \mapsto m_x(\tau)$ the self-adjoint solution $m(\tau)$ viewed as a function of $x \in [0, 1]$ with values in $C^K \times \mathbb{R}$. As is clear from the Dyson equation this function inherits the regularity of the data, i.e., it is continuous on each interval $I_l$. By the band mass formula (2.10) we have

$$\rho((\infty, \tau)) = \frac{1}{K} \sum_{l=1}^{L} \int_{I_l} \text{Tr} (m_x(\tau)) dx = \frac{1}{K} \sum_{l=1}^{L} k_l |I_l|,$$

where $k_l = \text{Tr} (m_x(\tau)) \in \{0, \ldots, K\}$ is continuous in $x \in I_l$ with discrete values and therefore does not depend on $x$. □

**Remark 9.5.** We extend the conjecture from Remark 2.9 of [2] to the Kronecker setting. We expect that in the piecewise $1/2$-Hölder continuous setting of the current section, the number of connected components of the self-consistent spectrum $\supp \rho$ is at most $K(2L - 1)$.

### 10. Perturbations of the self-energy operator

In this section, as an application of our results in Sections 4 to 7, we show that the Dyson equation is stable away from almost cusp points against small perturbations of the self-energy operator $S$. In fact, for a flat self-energy operator $S$ (cf. (3.10) for the definition of flatness), we consider a perturbation $t\tilde{S}$, where $t$ is a small positive parameter and $\tilde{S} : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator which keeps $\mathcal{A}_{\text{flat}}$ invariant. We will always assume that there are $c_1, \ldots, c_4 > 0$ such that

$$c_1 \langle x \rangle \leq S[\tau] \leq c_2 \langle x \rangle, \quad \|a\| \leq c_3, \quad \|\tilde{S}[\tau]\| \leq c_4 \langle x \rangle \tag{10.1}$$

for all $x \in \mathcal{A}_{\text{flat}}$. Combining the first and last bound in (10.1) yields that $S_t := S + t\tilde{S}$ is positivity-preserving for all $t \in [0, t_1]$, where $t_1 := c_1/c_4$. In the following, we will always assume that $t \in [0, t_1]$.

We denote by $m$ the solution of (2.3) with the data pair $(a, S) \in \mathcal{A}_{\text{flat}} \times \Sigma_{\text{flat}}$. Moreover, for $t \in [0, t_1]$, let $m_t$ be the solution to the Dyson equation associated to the data pair $(a, S_t)$, i.e.,

$$-m_t(z)^{-1} = z - a + S_t[m_t] = z - a + S[m_t(z)] + t\tilde{S}[m_t(z)]. \tag{10.2}$$

The main result of this section, Proposition 10.1 below, states that $\|m_t(z) - m(z)\|$ is small for sufficiently small $t$ and all $z$ away from possible (almost) cusp points and from points, where $m(z)$ blows up. We now introduce these concepts precisely. Since $S \in \Sigma_{\text{flat}}$, the self-consistent density of states of $m$ (cf. Definition 2.2) has a continuous density $\rho : \mathbb{R} \to [0, \infty)$ with respect to the Lebesgue measure (cf. Proposition 2.3). For $\tau \in \mathbb{R} \setminus \supp \rho$, let $\Delta(\tau)$ denote the size of the largest interval that contains $\tau$ and is contained in $\mathbb{R} \setminus \supp \rho$. For $\rho_0 > 0$, we define the set $P_{\text{cusp}} = P_{\text{cusp}}^0 \subset \mathbb{R}$ of almost cusp points through

$$P_{\text{cusp}}^0 = \left\{ \tau \in \supp \rho \setminus \partial \supp \rho : E \text{ is a local minimum of } \rho, \rho(\tau) \leq \rho_0 \right\} \cup \left\{ \tau \in \mathbb{R} \setminus \supp \rho : \Delta(\tau) \leq \rho_0 \right\}.$$

Moreover, for a given $m_0 > 0$, we define the set $P_{m} := P_{m}^0 \subset \mathbb{H}$, where $\|m(z)\|$ is larger than $m_0$, i.e.,

$$P_{m}^0 := \{ z \in \mathbb{H} : \|m(z)\| > m_0 \}.$$
For any fixed $\rho_*, m_*, \delta > 0$, we set $d_{\text{cusp}}(z) := \text{dist}(z, P_{\text{cusp}})$ and $d_m(z) := \text{dist}(z, P_m)$ and introduce
\[ D := D^{\rho_*, m*, \delta} := \{ z \in \mathbb{H} : d_m(z) \geq \delta, d_{\text{cusp}}(z) \geq \delta \}. \tag{10.3} \]

In this section, the model parameters are given by $c_1, \ldots, c_4$ from (10.1) as well as the fixed parameters $\rho_*$, $m_*$ and $\delta$ from the definitions of $P_{\text{cusp}}$, $P_m$ and $D$, respectively. Thus, the comparison relation $\sim$ (compare Convention 3.4) is understood with respect to these parameters throughout this section.

**Proposition 10.1.** If $a \in A_{\text{sa}}$ and $S, \tilde{S} : A \to A$ satisfy (10.1) then there is $t_* \sim 1$ such that $S_t = S + i\tilde{S}$ is positivity-preserving for $t \in [0, t_*)$ and
\[ \|m_t(z) - m(z)\| \lesssim t^{1/2} \]
uniformly for all $z \in \mathbb{D}$ and for all $t \in [0, t_*]$. In particular, $\|m_t(z)\| \lesssim 1$ uniformly for all $z \in \mathbb{D}$ and for all $t \in [0, t_*]$.

Proof of Proposition 10.1. We choose $\varepsilon_1 \sim 1$ such that if $\|\Delta_t(z)\| \leq \varepsilon_1$ for some $z \in \mathbb{D}$, $t \in [0, t_1]$, then there are $l, b \in A$ depending on $z$ such that $\Theta_t := (l, \Delta_t)/(l, b)$ satisfies a quadratic inequality
\[ |\xi_2 \Theta_t^2 + \xi_1 \Theta_t| \lesssim t \tag{10.4} \]
with complex (z-dependent) coefficients $\xi_1$ and $\xi_2$ satisfying
\[ |\xi_1| \sim \min \{ \sqrt{\kappa + \text{Im} z}, 1 \}, \quad |\xi_1| + |\xi_2| \sim 1 \tag{10.5} \]
and $|\Theta_t| \lesssim \|\Delta_t\|$ as well as $\|\Delta_t\| \lesssim |\Theta_t| + t$. Here, $\kappa = \kappa(z) := \text{dist}(\text{Re} z, \partial \text{supp} \rho)$ and all implicit constants are uniform for any $t \in [0, t_1]$.

**Lemma 10.2.** Let $\tilde{D}$ be defined as in (10.3). Let $a \in A_{\text{sa}}$ and $S, \tilde{S} : A \to A$ satisfy (10.1). Then there is $\varepsilon_2 \sim 1$ such that if $d(z_0) \lesssim \varepsilon_2$ for some $z_0 \in \tilde{D}$, and a function $\tilde{\Theta} : z_0 + i(0, \infty) \to \mathbb{C}$ satisfies
\[ |\xi_0 \tilde{\Theta}^2 + \xi_1 \tilde{\Theta}| \lesssim d \quad \text{for all} \quad z \in z_0 + i(0, \infty), \quad \text{and} \quad |\tilde{\Theta}| \lesssim \min \{ d/\sqrt{\kappa + \text{Im} z}, \sqrt{d} \} \tag{10.6} \]
for some $z = z_0 + i\eta$ with $\eta > 0$, then $|\tilde{\Theta}| \lesssim \min \{ d/\sqrt{\kappa + \text{Im} z}, \sqrt{d} \}$ for all $z \in z_0 + i(0, \infty)$.

Proof of Proposition 10.1. We choose $t_* \sim t_1 \sim 1$. We will prove below that there are $t_* \sim 1$ and $C \sim 1$ such that, for any fixed $t \in (0, t_*)$ and $z \in \mathbb{D}$, we have the implication
\[ \|\Delta_t(z)\| \leq \varepsilon_1 \quad \Rightarrow \quad \|\Delta_t(z)\| \leq Ct^{1/2}, \tag{10.7} \]
where $\varepsilon_1 \sim 1$ is from Lemma 10.2.

Armed with (10.7), by possibly shrinking $t_* \sim 1$, we can assume that $2Ct_*^{1/2} \leq \varepsilon_1$. We fix $\tau \in \mathbb{R}$ and $t \in (0, t_*)$ and set
\[ \eta_\tau := \sup \{ \eta > 0 : \|\Delta_t(\tau + i\eta)\| \geq 2Ct^{1/2} \}. \]
Here, we use the convention $\eta_\tau = -\infty$ if the set is empty. Note that $\|\Delta_t(\tau + i\eta)\| \leq 2\eta^{-1}$ since $m$ and $m_t$ are Stieltjes transforms. Hence, $\eta_\tau < \infty$.

We prove now that $\eta_\tau = \inf \{ \text{Im} z : z \in \mathbb{D}, \text{Re} z = \tau \}$. For a proof by contradiction, we suppose there is $z_\tau \in \mathbb{D}$ such that $\text{Re} z_\tau = \tau$ and $\text{Im} z_\tau = \eta_\tau$ (note that if $\tau + i\eta \in \mathbb{D}$ then $\tau + i\eta' \in \mathbb{D}$ for any $\eta' \geq \eta$). Since $\Delta_t$ is continuous in $z$, we have $\|\Delta_t(z_\tau)\| = 2Ct^{1/2}$. Thus, $\|\Delta_t(z_\tau)\| \leq \varepsilon_1$ by the choice of $t_*$. From (10.7), we conclude $\|\Delta_t(z_\tau)\| \leq Ct^{1/2}$, which contradicts $\|\Delta_t(z_\tau)\| = 2Ct^{1/2}$. Therefore, $\|\Delta_t(z)\| \geq 2Ct^{1/2}$ for all $z \in \mathbb{D}$.

Hence, it suffices to show (10.7) to complete the proof of Proposition 10.1. In order to prove (10.7), we use Lemma 10.3 with $\hat{\Theta} = \Theta_t$ from Lemma 10.2, $\hat{D} = D$ and $d = t$. Here, we choose $t_* \sim 1$ small enough such that $d = t \leq t_* \leq \varepsilon_2$. Since $\|\Delta_t(z + i\eta)\| \leq 2\eta^{-1}$ as remarked above, the second condition in (10.6) is always satisfied for some sufficiently large $\eta > 0$. As $\|\Delta_t(z)\| \leq \varepsilon_1$, we conclude that the first condition in (10.6) is met with $d = t$ due to (10.4). Hence, by Lemma 10.3, there is $C \sim 1$ such that $\|\Theta_t(z)\| \leq Ct^{1/2}$. Possibly increasing
We remark that a straightforward computation starting from (2.3) and (10.2) yields \( \sim \) paper on the edge universality [8]. We say that

\[
\| \Delta_t(z) \| \leq C t^{1/2} \text{ due to Lemma 10.2. This completes the proof of (10.7) and, hence, the one of Proposition 10.1.}
\]

Before we prove Lemma 10.2, we list a few auxiliary statements that hold true for \( z \in \mathbb{D} \) near the edges, i.e., if \( \kappa(z) + \text{Im } z \sim \text{dist}(z, \partial \text{supp } \rho) \) is small. They will be used in the proof of Lemma 10.2 and in the companion paper on the edge universality [8]. We say that \( \tau_0 \in \partial \text{supp } \rho \) is a regular edge if there is \( \varepsilon \sim 1 \) such that \( \rho(\tau) = 0 \) for all \( \tau \in [\tau_0 - \varepsilon, \tau_0] \) or \( \tau \in [\tau_0, \tau_0 + \varepsilon] \). By definition of \( \mathbb{D}, \mathbb{T} \cap \partial \text{supp } \rho \) consists only of regular edges.

**Lemma 10.4.** There is \( \varepsilon_* \sim 1 \) such that if \( z \in \mathbb{D} \) satisfies \( \text{dist}(z, \partial \text{supp } \rho) \leq \varepsilon_* \) then

(i) For the self-consistent density of states \( \rho \), we have

\[
\rho(z) \sim \begin{cases} \sqrt{\kappa(z) + \text{Im } z}, & \text{if } \text{Re } z \in \text{supp } \rho, \\ \text{Im } z/\sqrt{\kappa(z) + \text{Im } z}, & \text{if } \text{Re } z \notin \text{supp } \rho. \end{cases} \tag{10.8a}
\]

\[
\rho(z) + \rho(z)^{-1} \text{Im } z \sim \sqrt{\kappa(z) + \text{Im } z}. \tag{10.8b}
\]

(ii) Let \( l \) and \( b \) be defined as in Corollary 5.2. Setting \( \mu_2 := (l, mS[b]b + bS[b]m)/2 \), we have

\[
|\langle l, mS[b]b \rangle| \sim 1, \quad |\mu_2(z)| \sim 1. \tag{10.9}
\]

(iii) Let \( B := \text{Id} - C_mS \) and \( \beta \) be its eigenvalue of smallest modulus (cf. Corollary 5.2). We have

\[
\| B^{-1}(z) \| + \| B^{-1}(z) \|_2 \lesssim \langle \kappa(z) + \text{Im } z \rangle^{-1/2}, \quad |\beta(z)| \sim \sqrt{\kappa(z) + \text{Im } z}. \tag{10.10}
\]

**Proof of Lemma 10.4.** By assumption, \( z \) is \( \varepsilon_* \)-close to a regular edge. Thus, owing to \( \| m \| \leq 1 \) by definition of \( \mathbb{D} \), Theorem 7.1 immediately implies (10.8a). Moreover, (10.8b) is a direct consequence of (10.8a).

For the proof of (ii), we shrink \( \varepsilon_* \sim 1 \) as well as use (10.8a), (10.8b) and \( \text{dist}(z, \partial \text{supp } \rho) \sim \kappa + \text{Im } z \) to guarantee that Lemma 5.1 and Corollary 5.2 are applicable. Furthermore, we use Lemma 7.12 and the definition of \( \mathbb{D} \) to obtain \( |\sigma(\tau_0)| \sim 1 \), where \( \tau_0 \in \partial \text{supp } \rho \) is the point in \( \partial \text{supp } \rho \) closest to \( z \). The Hölder-continuity of \( m \), Lemma 5.1 and the definition of \( \mu_2 \) in (ii) imply that \( \langle l, mS[b]b \rangle \) and \( \mu_2 \) are Hölder-continuous. Therefore, evaluating (6.24b) and (6.25) at \( \tau_0 \) as well as using \( |\sigma(\tau_0)| \sim 1 \) yield \( |\mu_2(z)| \sim 1 \) and \( |\langle l, mS[b]b \rangle| \sim 1 \) if \( \varepsilon_* \) is sufficiently small, i.e., \( z \) is sufficiently close to \( \tau_0 \).

For the proof of (iii), we evaluate (6.24b) at \( z \) and conclude \( |\sigma(z)| \sim 1 \) from \( |\mu_2(z)| \sim 1 \) by (10.9). Therefore, (5.24) and (10.8b) yield the first bound in (10.10). Similarly, we obtain the second bound in (10.10) by using \( |\sigma(z)| \sim 1 \) and (10.8b) in (5.14c). This completes the proof of Lemma 10.4.

**Proof of Lemma 10.2.** We remark that a straightforward computation starting from (2.3) and (10.2) yields

\[
B[\Delta_t] = A[\Delta_t, \Delta_t] + K[t, \Delta_t] + T[t], \tag{10.11}
\]

where \( B := \text{Id} - C_mS \), \( A, y := (mS[x]y + yS[x]m)/2 \) are defined as in (6.23) and

\[
K[t, \Delta_t] := \frac{t}{2}(mS[\Delta_t]A + A, \Delta_tS[\Delta_t])m + \frac{t}{2}(2mS[\Delta_t]m + mS[m]A + A, \Delta_tS[m]m), \quad T[t] := tmS[m]m.
\]

In the following, we will split \( \mathbb{D} \) into two regimes and choose \( l \) and \( b \) according to the regime. In both cases, we use the definitions

\[
\Theta = \Theta_t = \langle l, \Delta_t \rangle / (l, b), \quad r = r_t := Q[\Delta_t], \quad Q := \text{Id} - \langle l, \cdot \rangle (l, b).
\]

In particular, \( \Delta_t = \Theta b + r \). We denote by \( \rho(z) \) the harmonic extension of \( \rho \), i.e., \( \rho(z) = (\text{Im } m(z))/\pi \).

**Case 1:** We first assume \( \text{dist}(z, \partial \text{supp } \rho) \gtrsim \varepsilon_* \) for some \( \varepsilon_* \sim 1 \). Owing to the definition of \( \mathbb{D} \), this condition is equivalent to \( \rho(z) \gtrsim \varepsilon_* \) or \( \text{dist}(z, \partial \text{supp } \rho) \gtrsim \varepsilon_* \). This implies that \( B \) is invertible and \( \| B^{-1} \| \lesssim 1 \) due to (4.1), \( \| S \|_{2 \rightarrow 2} \lesssim 1, \| m(z) \| \lesssim 1 \) for \( z \in \mathbb{D} \) and Lemma B.2 (ii). In this case, we choose \( l = b = 1 \) and apply \( QB^{-1} \) to (10.11) to obtain

\[
r = QB^{-1}(A[\Delta_t, \Delta_t] + K[t, \Delta_t] + T[t]) = O(\| \Theta \|^2 + \| r \| \| \Delta_t \| + t),
\]

where we used that \( \| m \| \lesssim 1 \) on \( \mathbb{D} \). Shrinking \( \varepsilon_1 \sim 1 \), using \( \| \Delta_t \| \lesssim \varepsilon_1 \) and absorbing \( \| r \| \| \Delta_t \| \) into the left-hand side yield \( \| r \| \lesssim \| \Theta \|^2 + t \). Thus, \( \| \Delta_t \| \lesssim \| \Theta \| + t \). Hence, applying \( B^{-1} \) and \( \langle \cdot \rangle \) to (10.11) and using \( \langle r \rangle = 0 \) as
well as $\| \Delta \| \lesssim |\Theta| + t$, we find $\xi_2 \in \mathbb{C}$ such that $|\xi_2| \lesssim 1$ and

$$\Theta = -\xi_2 \Theta^2 + \mathcal{O}(t|\Theta| + t) = -\xi_2 \Theta^2 + \mathcal{O}(t).$$

This proves (10.4) with $\xi_1 = 1$, which yields $|\xi_1| \sim \min\{s, 1\}$ in case 1.

Case 2: We now prove (10.4) for $z \in \mathbb{D}$ satisfying $\text{dist}(z, \partial \text{supp} \rho) \leq \varepsilon_*$ for some sufficiently small $\varepsilon_* \sim 1$. Owing to (10.8b), for sufficiently small $\varepsilon_* \sim 1$, Lemma 5.1 and Corollary 5.2 are applicable and yield $l, b \in \mathcal{A}$ which we use to define $\Theta$ and $r$ by $\Delta_t = \Theta b + r$ and $\Theta = (l, \Delta_t)/(l, b)$ as usual. In order to derive (10.4), we now apply Lemma 6.2. In fact, by Lemma 5.1 and Corollary 5.2, the first two bounds in (6.12) are fulfilled. Since $||m|| \leq 1$ and $||\Delta_t|| \leq 1$, the last three bounds in (6.12) are trivially satisfied. In fact, the last bound will not hold true for a general $y \in \mathcal{A}$ but in the proof of Lemma 6.2 it is only used with the special choice $y = \Delta_t$ and $||\Delta_t|| \leq 1$. We choose $\varepsilon_1 \leq \varepsilon$ for $\varepsilon$ from Lemma 6.2 and obtain the cubic equation (10.14) from Lemma 6.2 with $|\mu_0| \leq t$ by (6.15). We decompose the error term $\hat{e} = \mathcal{O}(||\Theta||^3 + t|\Theta| + t^2)$ from (10.14) into $\hat{e} = \hat{e}_1 \Theta^2 + \hat{e}_2$ with $\hat{e}_1, \hat{e}_2 \in \mathbb{C}$ satisfying $\hat{e}_1 = \mathcal{O}(||\Theta||^2)$ and $\hat{e}_2 = \mathcal{O}(t|\Theta| + t^2)$. With the notation of Lemma 6.2, the cubic equation (6.14) can be viewed as the quadratic equation

$$\xi_2 \Theta^2 + \xi_1 \Theta = -\mu_0 + \hat{e}_2 = \mathcal{O}(t), \quad \xi_2 := \mu_3 \Theta - \hat{e}_1 + \mu_2, \quad \xi_1 = -\beta(l, b).$$

We complete the proof by establishing $|\xi_2| \sim 1$ and $|\xi_1| \sim \sqrt{k + \text{Im} z}$. We remark that the definitions of $\mu_2$ in Lemma 6.2 and in Lemma 10.4 (ii) agree. As $|\mu_3 \Theta| \lesssim ||\Delta|| \leq \varepsilon_1$ and $|\hat{e}_1| \lesssim ||\Delta|| \leq \varepsilon_1^2$, by the second relation in (10.9), we conclude $|\xi_2| \sim |\mu_2| \sim 1$ for sufficiently small $\varepsilon_1 \sim 1$. Moreover, using the definition of $\xi_1$, the last relation in (10.10) and $||\langle l, b \rangle|| \sim 1$ by (5.15a) and (5.15b), we obtain $|\xi_1| \sim \sqrt{k + \text{Im} z} = \min\{\sqrt{k + \text{Im} z}, 1\}$ which completes the proof of Lemma 10.2.

Proof of Lemma 10.3. We remark that the approximate quadratic equation in (10.6) has been analysed before, see e.g. Chapter 9 of [15]. We sketch the proof here for the convenience of the reader. We distinguish the regimes $z \in \mathbb{D}$, where $|\xi_1(z)| \geq \varepsilon_*$ and $|\xi_1(z)| \leq \varepsilon_*$ for some $\varepsilon_* \sim 1$. The statement is trivial in the first regime as $1 \sim |\xi_1| \lesssim \sqrt{k + \text{Im} z}$ in the second regime, $|\xi_2| \sim 1$ and we use that $(k + \text{Im} z)/d$ is monotonically increasing in $\text{Im} z$. We further distinguish two cases. If $(k + \text{Im} z)/d \gg 1$ then, from the solution formula of quadratic equations, we conclude $|\hat{e}| \lesssim d/\sqrt{k + \text{Im} z} \lesssim \sqrt{d}$. The relevant branch is determined by the bound on $|\Theta|$ at $z_0 + i\eta$. In the second case, $(k + \text{Im} z)/d \lesssim 1$, we have $|\xi_1| \sim \sqrt{k + \text{Im} z}$. Thus, $|\hat{e}| \lesssim \sqrt{k + \text{Im} z} + \sqrt{d} \lesssim \sqrt{d} \lesssim d/\sqrt{k + \text{Im} z}$. Here, we used in the last step that $\sqrt{d} \lesssim d/\sqrt{k + \text{Im} z}$ as $(k + \text{Im} z)/d \lesssim 1$. This completes the proof of Lemma 10.3.

A. Stieltjes transforms of positive operator-valued measures

In this appendix, we will show some results about the Stieltjes transform of a positive operator-valued measure on $\mathcal{A}$.

We first prove Lemma 3.1 by generalizing existing proofs in the matrix algebra setup. Since we have not found the general version in the literature, we provide a proof here for the convenience of the reader. In the proof of Lemma 3.1, we will use that a von Neumann algebra is always isomorphically isomorphic as a Banach space to the dual space of a Banach space. In our setup, this Banach space and the identification are simple to introduce which we will explain now. Analogously to $L^2$ defined in Section 4, we define $L^1$ to be the completion of $\mathcal{A}$ when equipped with the norm $||x||_1 := \langle (x^* x)^{1/2} \rangle = \langle |x| \rangle$ for $x \in \mathcal{A}$. Moreover, we extend $\langle \cdot \rangle$ to $L^1$ and remark that $xy \in L^1$ for $x \in \mathcal{A}$ and $y \in L^1$. It is well-known (e.g. [34, Theorem 2.18]) that the dual space $(L^1)'$ of $L^1$ can be identified with $\mathcal{A}$ via the isometric isomorphism

$$\mathcal{A} \rightarrow (L^1)', \ x \mapsto \psi_x, \ \psi_x : L^1 \rightarrow \mathbb{C}, \ y \mapsto \langle xy \rangle. \quad (A.1)$$

We stress that the existence of this isomorphism requires the state $\langle \cdot \rangle$ to be normal.

Proof of Lemma 3.1. From (3.5), we conclude that

$$\lim_{\eta \rightarrow \infty} \text{Im} \langle x, h(\eta)x \rangle = -\langle x, x \rangle$$

for all $x \in \mathcal{A}$. Hence, $z \mapsto \langle x, h(z)x \rangle$ is the Stieltjes transform of a unique finite positive measure $v_x$ on $\mathbb{R}$ with $v_x(\mathbb{R}) = ||x^* x||_1$. 

48
For any \( x \in \mathcal{A} \), we can find \( x_1, \ldots, x_4 \in \mathcal{A}_+ \) such that \( x = x_1 - x_2 + ix_3 - ix_4 \). We define
\[
\varphi_B(x) := v_{\sqrt{\tau_1}}(B) - v_{\sqrt{\tau_2}}(B) + iv_{\sqrt{\tau_3}}(B) - iv_{\sqrt{\tau_4}}(B)
\] (A.2)
for \( B \in \mathcal{B} \). This definition is independent of the representation of \( x \). Indeed, for fixed \( x \in \mathcal{A} \), any representation \( x = x_1 - x_2 + ix_3 - ix_4 \) with \( x_1, \ldots, x_4 \in \mathcal{A}_+ \) defines a complex measure \( \varphi(x) \) through \( B \mapsto \varphi_B(x) \) on \( \mathbb{R} \) via (A.2). However, extending \( h \) to the lower half-plane by setting \( h(z) := h(\bar{z})^* \) for \( z \in \mathbb{C} \) with \( \text{Im} \, z < 0 \), the Stieltjes transform of \( \varphi(x) \) is given by
\[
\int_\mathbb{R} \frac{\varphi_B(x)}{\tau - z} = \langle \sqrt{x_1}, h(z) \sqrt{x_1} \rangle - \langle \sqrt{x_2}, h(z) \sqrt{x_2} \rangle + i\langle \sqrt{x_3}, h(z) \sqrt{x_3} \rangle - i\langle \sqrt{x_4}, h(z) \sqrt{x_4} \rangle = \langle h(z)x \rangle
\]
for all \( z \in \mathbb{C} \setminus \mathbb{R} \). This formula shows that the Stieltjes transform of \( \varphi(x) \) is independent of the decomposition \( x = x_1 - x_2 + ix_3 - ix_4 \). Hence, \( \varphi_B(x) \) is independent of this representation for all \( B \in \mathcal{B} \) since the Stieltjes transform uniquely determines even a complex measure. A similar argument also implies that, for fixed \( B \in \mathcal{B} \), \( \varphi_B \) defines a linear functional on \( \mathcal{A} \).

Since \( v_{\sqrt{\tau}}(\mathbb{R}) = \langle y \rangle \) for \( y \in \mathcal{A}_+ \), we obtain for \( x = (\text{Re} \, x)_+ - (\text{Re} \, x)_- + i(\text{Im} \, x)_+ - i(\text{Im} \, x)_- \in \mathcal{A} \)
\[
|\varphi_B(x)| \leq v_{\sqrt{\text{Re}(x)_+}}(\mathbb{R}) + v_{\sqrt{\text{Re}(x)_-}}(\mathbb{R}) + v_{\sqrt{\text{Im}(x)_+}}(\mathbb{R}) + v_{\sqrt{\text{Im}(x)_-}}(\mathbb{R})
\]
\[
\leq \langle (\text{Re} \, x)_+ + (\text{Re} \, x)_- + (\text{Im} \, x)_+ + (\text{Im} \, x)_- \rangle \leq 2 \|x\|_1,
\]
where we used that \( (\text{Re} \, x)_+ + (\text{Re} \, x)_- = [\text{Re} \, x] \) and \( (\text{Im} \, x)_+ + (\text{Im} \, x)_- = |\text{Im} \, x| \). Therefore, \( \varphi_B \) extends to a bounded linear functional on \( L^1 \) as \( \mathcal{A} \) is a dense linear subspace of \( L^1 \). Using the isomorphism in (A.1), for each \( B \in \mathcal{B} \), there exists a unique \( v(B) \in \mathcal{A} \) such that
\[
\varphi_B(x) = \langle v(B)x \rangle
\]
for all \( x \in \mathcal{A} \). For \( y \in \mathcal{A} \), we conclude \( v_B(B) = v_{\sqrt{yy^*}}(B) = \varphi_B(yy^*) = \langle y, v(B)y \rangle \geq 0 \), where we used that \( v_y = v_{\sqrt{yy^*}} \) since they have the same Stieltjes transform. Since \( \langle v(B)y \rangle \geq 0 \) for all \( y \in \mathcal{A}_+ \), we have \( v(B) \in \mathcal{A}_+ \) for all \( B \in \mathcal{B} \). Moreover, \( v_x = \langle x, v(\cdot)x \rangle \), in particular, \( \langle x, v(B)x \rangle = v_x(B) = \langle x, x \rangle \), for all \( x \in \mathcal{A} \). The polarization identity yields that \( v \) is an \( \mathcal{A}_+ \)-valued measure on \( \mathcal{B} \) satisfying (3.6) and \( v(\mathbb{R}) = 1 \). This completes the proof of Lemma A.1. \( \Box \)

**Lemma A.1** (Stieltjes transform inherits Hölder regularity). Let \( v \) be an \( \mathcal{A}_+ \)-valued measure on \( \mathbb{R} \) and \( h: \mathbb{H} \to \mathcal{A} \) be its Stieltjes transform, i.e., \( h \) satisfies (3.6) for all \( z \in \mathbb{H} \). Let \( f: I \to \mathcal{A}_+ \) be a \( \gamma \)-Hölder continuous function on an interval \( I \subset \mathbb{R} \) with \( \gamma \in (0, 1) \) and \( f \) be a density of \( v \) on \( I \) with respect to the Lebesgue measure, i.e.,
\[
\|f(\tau_1) - f(\tau_2)\| \leq C_0|\tau_1 - \tau_2|^{\gamma}, \quad v(A) = \int_A f(\tau) \, d\tau
\]
for all \( \tau_1, \tau_2 \in I \), some \( C_0 > 0 \) and for all Borel sets \( A \subset I \). Moreover, we assume that \( \|f(\tau)\| \leq C_1 \) for all \( \tau \in I \). Let \( \theta \in (0, 1] \).

Then, for \( z_1, z_2 \in \mathbb{H} \) satisfying \( \text{Re} \, z_1, \text{Re} \, z_2 \in I \) and \( \text{dist}(\text{Re} \, z_k, \partial I) \geq \theta, k = 1, 2 \), we have
\[
\|h(z_1) - h(z_2)\| \leq \left( \frac{21C_0}{\gamma(1 - \gamma)} + \frac{4\|v(\mathbb{R})\|}{\theta^{1 + \gamma}} + \frac{14C_1}{\gamma \theta^{\gamma}} \right)|z_1 - z_2|^\gamma.
\] (A.3)

Furthermore, for \( z_1, z_2 \in \mathbb{H} \) satisfying \( \text{dist}(z_k, \text{supp} \, v) \geq \theta, k = 1, 2 \), we have
\[
\|h(z_1) - h(z_2)\| \leq \frac{2\|v(\mathbb{R})\|}{\theta^2}|z_1 - z_2|^\gamma.
\] (A.4)

We omit the proof of Lemma A.1 since it is very similar to the one of Lemma A.7 in [1].

**B. Positivity-preserving, symmetric operators on \( \mathcal{A} \)**

**Lemma B.1.** Let \( T: \mathcal{A} \to \mathcal{A} \) be a positivity-preserving, symmetric operator.

(i) If \( T[a] \leq C(a)1 \) for some \( C > 0 \) and all \( a \in \mathcal{A}_+ \) then \( ||T|| \leq 2C \). Moreover, \( ||T||_2 \) is an eigenvalue of \( T \) and there is \( x \in \mathcal{A}_+ \setminus \{0\} \) such that \( T[x] = ||T||_2x \).
(ii) We assume \( \|T\|_2 = 1 \) and that there are \( c, C > 0 \) such that
\[
c(a)1 \leq T[a] \leq C(a)1
\]
for all \( a \in \mathcal{A}_+ \). Then 1 is an eigenvalue of \( T \) with a one-dimensional eigenspace. There is a unique \( x \in \mathcal{A}_+ \) satisfying \( T[x] = x \) and \( \|x\|_2 = 1 \). Moreover, \( x \) is positive definite,
\[
cC^{-1/2}1 \leq x \leq C1.
\]
Furthermore, the spectrum of \( T \) has a gap of size \( \theta := c^6/(2(c^3 + 2C^2)C^2) \), i.e.,
\[
\text{Spec}(T) \subset [-1 + \theta, 1 - \theta] \cup \{1\}.
\]

Lemma B.1 is the analogue of Lemma 4.8 in [4]. Here, we explain how to generalize it to the context of von Neumann algebras. In the proof of Lemma B.1, we will use the following lemma. We omit its proof since the first part is obtained as in (4.2) of [4] and the second part as in (5.28) of [1].

**Lemma B.2.** Let \( T: \mathcal{A} \to \mathcal{A} \) be a linear map.

(i) If \( T \) is positivity-preserving such that \( T[a] \leq C(a)1 \) for all \( a \in \mathcal{A}_+ \) and some \( C > 0 \) then \( \|T\| \leq \|T\|_{2\to\|\cdot\|} \leq 2C \).

(ii) If \( T - \omega \text{Id} \) is invertible on \( \mathcal{A} \) for some \( \omega \in \mathcal{C} \setminus \{0\} \) and \( \|(T - \omega \text{Id})^{-1}\|_2 < \infty, \|T\|_{2\to\|\cdot\|} < \infty \) then we have
\[
\|(T - \omega \text{Id})^{-1}\|_2 \leq |\omega|^{-1}(1 + \|T\|_{2\to\|\cdot\|})(\|T - \omega \text{Id}\|^{-1}_2).
\]

**Proof of Lemma B.1.** For the proof of (i), we remark that Lemma B.2 (i) implies \( \|T\|_2 \leq \|T\|_{2\to\|\cdot\|} \leq 2C \).

Without loss of generality, we assume \( \|T\|_2 = 1 \). Since \( T \) is positivity-preserving, we have \( T[b] \in \mathcal{A}_+ \) for all \( b \in \mathcal{A}_+ \). It is easy to check that, for each \( a \in \mathcal{A} \), one may find \( b \in \mathcal{A}_+ \) such that \( \|a\|_2 = \|b\|_2 \) and \( \|T[a]\|_2 \leq \|T[b]\|_2 \).

Hence, \( \|T[a]\|_2 = \|T\|_2 = 1 \) and 1 is contained in the spectrum of \( T: L^2_{\mathcal{A}_+} \to L^2_{\mathcal{A}_+} \), where \( L^2_{\mathcal{A}_+} := \mathcal{A}_+^{\mathcal{A}_+} \), due to the variational principle for the spectrum of self-adjoint operators and \( \|b, T[b]\| \leq \|b, \text{Id}\| \) for all \( b \in \mathcal{A}_+ \).

This last inequality can be checked easily by decomposing \( b = b_+ - b_- \) into positive and negative part.

Hence, due to the symmetry of \( T \), there is a sequence \( (y_n)_n \) of approximating eigenvectors in \( \mathcal{A}_+ \), i.e., \( y_n \in \mathcal{A}_+ \), \( \|y_n\|_2 = 1 \) and \( T[y_n] = y_n \) for all \( n \in \mathbb{N} \). We set \( x_n := y_n \). By using \( \|T[y_n]\|_2 = 1 \) and \( \langle b, T[b] \rangle \leq \|(b, T[b])\| \) for all \( b \in \mathcal{A}_+ \), we obtain \( \|T[x_n] - x_n\|_2^2 \leq 2\|y_n\|_2 T[y_n] - y_n\|_2^2 \) and, thus,
\[
\lim_{n \to \infty} \|T[x_n] - x_n\|_2 = 0.
\]

Since the unit ball in the Hilbert space \( L^2 \) is relatively sequentially compact in the weak topology, we can assume by possibly replacing \( (x_n)_n \) by a subsequence that there is \( x \in L^2 \) such that \( x_n \to x \) weakly in \( L^2 \). From \( T[x_n] \leq C(x_n)1 \), we conclude
\[
x_n \leq (\text{Id} - T)[x_n] + C(x_n)1.
\]

Multiplying this by \( \sqrt{x_n} \) from the left and the right and applying \( \langle \cdot, \cdot \rangle \) yields
\[
1 \leq \langle x_n, (\text{Id} - T)[x_n] + C(x_n) \rangle^2.
\]

Taking the limit \( n \to \infty \), we obtain \( x \geq C^{-1/2} \), due to (B.4). Hence, \( x \neq 0 \) and we can replace \( x \) by \( x/\|x\|_2 \) and \( x_n \) by \( x_n/\|x\|_2 \). For any \( b \in L^2 \), we have
\[
\langle b, (\text{Id} - T)[x] \rangle = \lim_{n \to \infty} \langle b, (\text{Id} - T)[x_n] \rangle = 0
\]
due to \( x_n \to x \) and (B.4). Hence, \( T[x] = x \). Since \( \|T\|_{2\to\|\cdot\|} \leq 2C \), we have \( T[b] \in \mathcal{A} \) for all \( b \in L^2 \) and thus \( x = T[x] \in \mathcal{A} \). Owing to \( x_n \to x \) and \( x_n \in \mathcal{A}_+ \), we obtain \( x \in \mathcal{A}_+ \). This completes the proof of (i).

We start the proof of (ii) by using (B.1) with \( a = x \) which immediately yields the upper bound in (B.2). As \( x \geq C^{-1/2} \), the first inequality in (B.1) then yields the lower bound in (B.2).

In order to prove the spectral gap, (B.3), we remark that \( \|T\|_{2\to\|\cdot\|} \leq 2C \) due to the upper bound in (B.1) and Lemma B.2 (i). Hence, by Lemma B.2 (ii), the spectrum of \( T \) as an operator on \( \mathcal{A} \) is contained in the union of \( \{0\} \) and the spectrum of \( T \) as an operator on \( L^2 \). Therefore, we will consider \( T \) as an operator on \( L^2 \) in the following and exclusively study its spectrum as an operator on \( L^2 \). Hence, to prove the spectral gap, it suffices
to establish a lower bound on \( \langle y, (\text{Id} + T)[y] \rangle \) for all self-adjoint \( y \in A \) satisfying \( \|y\|_2 = 1 \) and \( \langle x, y \rangle = 0 \). Fix such \( y \in A \). Since \( y \) is self-adjoint we have

\[
y = \lim_{N \to \infty} y^N, \quad y^N = \sum_{k=1}^{N} \lambda_k^N p_k^N \tag{B.5}
\]

for some \( \lambda_k^N \in \mathbb{R} \) and \( p_k^N \in A \) orthogonal projections such that \( p_k^N p_l^N = p_k^N \delta_{k,l} \). Here, the convergence \( y^N \to y \) is with respect to \( \|\cdot\| \). We can assume that \( \|y^N\|_2 = 1 \) for all \( N \) as well as \( \langle y_k^N \rangle > 0 \) for all \( k \) and \( \langle p_k^N + \ldots + p_N^N \rangle = 1 \) for all \( N \).

We will now reduce estimating \( \langle y, (\text{Id} + T)[y] \rangle \) to estimating a scalar product on \( \mathbb{C}^N \). On \( \mathbb{C}^N \), we consider the scalar product \( \langle \cdot, \cdot \rangle_N \) induced by the probability measure \( \pi(A) = \sum_{k \in A} \langle p_k^N \rangle \) on \( [N] \), i.e.,

\[
(\lambda, \mu)_N = \sum_{k=1}^{N} \lambda_k \mu_k \langle p_k^N \rangle \tag{B.5}
\]

for \( \lambda = (\lambda_k)_{k=1}^{N}, \mu = (\mu_k)_{k=1}^{N} \in \mathbb{C}^N \). The norm on \( \mathbb{C}^N \) and the operator norm on \( \mathbb{C}^{N \times N} \) induced by \( \langle \cdot, \cdot \rangle_N \) are denoted by \( \|\cdot\| \) and \( \|\cdot\| \), respectively. Moreover, \( \text{Id}_N \) is the identity map on \( \mathbb{C}^N \). With this notation, we obtain from (B.5) that

\[
\langle y, (\text{Id} + T)[y] \rangle = \lim_{N \to \infty} \sum_{k,l=1}^{N} \lambda_k^N \lambda_l^N \langle p_k^N, (\text{Id} + T)[p_l^N] \rangle = \lim_{N \to \infty} (\lambda^N, (\text{Id}_N + S^N)[\lambda^N])_N,
\]

where we introduced \( \lambda^N = (\lambda_k^N)_{k=1}^{N} \in \mathbb{C}^N \) and the \( N \times N \) symmetric matrix \( S^N \) viewed as an integral operator on \( ([N], \pi) \) with the kernel \( s_{kl}^N \) given by

\[
s_{kl}^N = \frac{(p_k^N, T[p_l^N])}{(p_k^N)^2}.
\]

Since \( \|y^N\|_2 = 1 \), we have \( \|\lambda^N\|_N = 1 \). By the flatness of \( T \), we have

\[
e \leq s_{kl}^N \leq C. \tag{B.6}
\]

In the following, we will omit the \( N \)-dependence of \( \lambda_k, s_{kl} \) and \( p_k \) from our notation. By the definition of \( \langle \cdot, \cdot \rangle_N \), we have

\[
(\lambda, s\lambda)_N = \sum_{k,l=1}^{N} \lambda_k \langle p_k \rangle s_{kl} \langle p_l \rangle \lambda_l = (\lambda^N, T[y^N]).
\]

Let \( s \in \mathbb{C}^N \) be the Perron-Frobenius eigenvector of \( S \) satisfying \( Ss = \|S\|s, \|s\|_N = 1 \). From (B.6), we conclude

\[
e \leq (e, Se)_N \leq \|S\| = (s, Ss)_N \leq \|T\|_2 = 1, \tag{B.7}
\]

where \( e = (1, \ldots, 1) \in \mathbb{C}^N \). Since \( \|s\|_N = 1 \) and \( e \leq \|S\| \), we have

\[
\max s_i = \frac{(Ss)_i}{\|S\|} \leq C c \sum_{k=1}^{N} s_k \langle p_k \rangle \leq C c \left( \sum_{k=1}^{N} \langle p_k \rangle \right)^{1/2} \left( \sum_{k=1}^{N} s_k^2 \langle p_k \rangle \right)^{1/2} = \frac{C}{c}.
\]

As \( \inf_{k,l} s_{kl} \geq c \) by (B.6), Lemma 5.7 in [1] yields

\[
\text{Spec}(S) \subset \left[ -\|S\| + \frac{c^3}{C^2}, \|S\| - \frac{c^3}{C^2} \right] \cup \{\|S\|\}.
\]

We decompose \( \lambda = (1 - \|w\|_N^2)^{1/2}s + w \) with \( w \perp s \) and obtain

\[
\|\lambda, S\lambda\|_N \leq \|S\|(1 - \|w\|^2_N) + \left( \|S\| - \frac{c^3}{C^2} \right)\|w\|^2_N \leq 1 - \frac{c^3}{C^2} \|w\|_N^2, \tag{B.8}
\]

where we used \( \|S\| \leq 1 \) in the last step. Hence, it remains to estimate \( \|w\|_N \).
Recalling $T[x] = x$, we set $\bar{x} = (\langle xp_k \rangle / \langle p_k \rangle)_{k=1}^N$ and compute

$$\langle x, y^N \rangle = \sum_k \lambda_k \langle xp_k \rangle = \langle \bar{x}, \lambda \rangle_N.$$ 

Since the left-hand side goes to $\langle x, y \rangle = 0$ for $N \to \infty$, we can assume that $|\langle \bar{x}, \lambda \rangle_N| \leq \sqrt{\varepsilon/2}$ for any fixed $\varepsilon$ and all sufficiently large $N$. As $\bar{x}_k \geq c/\sqrt{C}$ by (B.2), we obtain

$$(1 - \|w\|_N^2) \left( \sum_k s_k \langle p_k \rangle \right)^2 \leq (1 - \|w\|_N^2) \langle x, s \rangle_N^2 = \langle \bar{x}, \lambda \rangle_N - \langle \bar{x}, w \rangle_N^2 \leq 2\|\bar{x}\|_N \|w\|_N^2 + \varepsilon. \quad (B.9)$$

Now, we use $c \leq \langle s, Ss \rangle_N$ from (B.7) to get

$$c \leq \langle s, Ss \rangle_N = \sum_{k,l} s_k s_l \langle p_k \rangle \langle p_l \rangle \leq C \left( \sum_k s_k \langle p_k \rangle \right)^2.$$ 

By plugging this and $\|\bar{x}\|_N^2 \leq \|x\|^2 \sum_k \langle p_k \rangle = 1$ into (B.9), solving the resulting estimate for $\|w\|_N^2$ and choosing $\varepsilon = c^3/(2C^2)$, we obtain

$$\|w\|_N^2 \geq \frac{c^3}{2(c^3 + 2C^2)}.$$ 

Therefore, from (B.8), we conclude

$$|\langle \lambda, S\lambda \rangle_N| \leq 1 - \frac{c^6}{2(c^3 + 2C^2)C^2}$$

uniformly for all sufficiently large $N \in \mathbb{N}$. We thus obtain that

$$\langle y, (\text{Id} \pm T)[y] \rangle \geq \frac{c^6}{2(c^3 + 2C^2)C^2}$$

if $y \perp x$ and $\|y\|_2 = 1$. We conclude (B.3), which completes the proof of the lemma. \qed

### C. Non-Hermitian perturbation theory

Let $B_0 : \mathcal{A} \to \mathcal{A}$ be a bounded operator with an isolated, single eigenvalue $\beta_0$ and an associated eigenvector $b_0$, $\|b_0\|_2 = 1$, i.e.,

$$B_0[b_0] = \beta_0 b_0.$$ 

Moreover, we denote by $P_0$ and $Q_0$ the spectral projections corresponding to $\beta_0$ and $\text{Spec}(B_0) \setminus \{\beta_0\}$. Note that $P_0 + Q_0 = \text{Id}$ but they are not orthogonal projections in general. If $l_0$ is a normalized eigenvector of $B_0^*$ associated to its eigenvalue $\beta_0$, then we obtain

$$P_0 = \frac{\langle l_0, \cdot \rangle}{\langle l_0, b_0 \rangle} b_0. \quad (C.1)$$

For some bounded operator $E : \mathcal{A} \to \mathcal{A}$, we consider the perturbation

$$B = B_0 + E.$$ 

We assume $E$ to be sufficiently small such that there is an isolated, single eigenvalue $\beta$ of $B$ close to $\beta_0$ and that $\beta$ and $\beta_0$ are separated from $\text{Spec}(B) \setminus \{\beta\}$ and $\text{Spec}(B_0) \setminus \{\beta_0\}$ by an amount $\Delta > 0$. Let $P$ be the spectral projection of $B$ associated to $\beta$.

**Lemma C.1.** We define $b := P[b_0]$ and $l := P^*[l_0]$. Then $b$ and $l$ are eigenvectors of $B$ and $B^*$ corresponding to $\beta$ and $\beta_0$, respectively. Moreover, we have

$$b = b_0 + b_1 + b_2 + O(||E||^3), \quad l = l_0 + l_1 + l_2 + O(||E||^3), \quad (C.2)$$

52
where we introduced

\[ b_1 = -Q_0(B_0 - \beta_0 \text{Id})^{-1}E[b_0], \]
\[ b_2 = Q_0(B_0 - \beta_0 \text{Id})^{-1}E(B_0 - \beta_0 \text{Id})^{-1}Q_0E[b_0] - Q_0(B_0 - \beta_0 \text{Id})^{-2}EP_0E[b_0] - P_0EPQ_0(B_0 - \beta_0 \text{Id})^{-2}E[b_0], \]
\[ l_1 = -Q_0(B_0^* - \beta_0 \text{Id})^{-1}E^*[l_0], \]
\[ l_2 = Q_0^*(B_0^* - \beta_0 \text{Id})^{-1}E^*(B_0^* - \beta_0 \text{Id})^{-1}Q_0^*E^*[l_0] - Q_0^*(B_0^* - \text{Id})^{-2}E^*P_0^*E^*[l_0] - P_0^*E^*Q_0^*(B_0^* - \text{Id})^{-2}E^*[l_0]. \]

In particular, we have \( b_i, l_i = \mathcal{O}(\|E\|^4) \) for \( i = 1, 2 \). Furthermore, we obtain

\[ \beta(l, b) = \beta_0(l_0, b_0) + (l_0, E[b_0]) - (l_0, EB_0(B_0 - \beta_0 \text{Id})^{-2}Q_0E[b_0]) + \mathcal{O}(\|E\|^4). \]  
(C.3)

The implicit constants in the error terms depend only on the separation \( \Delta \).

**Proof.** In this proof, the difference \( B - \omega \) with an operator \( B \) and a scalar \( \omega \) is understood as \( B - \omega \text{Id} \). We first prove that

\[ P = P_0 + P_1 + P_2 + \mathcal{O}(\|E\|^3), \]  
(C.4)

where we defined

\[
\begin{align*}
P_1 &:= -\frac{Q_0}{B_0 - \beta_0}EP_0 - P_0E\frac{Q_0}{B_0 - \beta_0}, \\
P_2 &:= P_0E\frac{Q_0}{B_0 - \beta_0}EP_0 - \frac{Q_0}{B_0 - \beta_0}EP_0E\frac{Q_0}{B_0 - \beta_0} + \frac{Q_0}{B_0 - \beta_0}E\frac{Q_0}{B_0 - \beta_0}EP_0 - \frac{Q_0}{B_0 - \beta_0}^2EP_0EP_0 - P_0E\frac{Q_0}{(B_0 - \beta_0)^2}EP_0.
\end{align*}
\]

The analytic functional calculus yields that

\[ P = -\frac{1}{2\pi i} \int_{\Gamma} \frac{d\omega}{B - \omega} = \frac{1}{2\pi i} \int_{\Gamma} \left( -\frac{1}{B_0 - \omega} + \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} - \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} - \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} d\omega + \mathcal{O}(\|E\|^3), \]  
(C.5)

where \( \Gamma \) is a closed path that encloses only \( \beta \) and \( \beta_0 \) both with winding number +1 but no other element of the spectra of \( B \) and \( B_0 \). Integrating the first summand in the integrand of (C.5) yields \( P_0 \). In the second and third summand, we expand \( \text{Id} = P_0 + Q_0 \) in the numerators. Applying an analogue of the residue theorem yields \( P_1 \) and \( P_2 \) for the second and third summand, respectively. For example, for the second summand, we obtain

\[ P_1 = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} d\omega = -\frac{Q_0}{B_0 - \beta_0}EP_0 - P_0E\frac{Q_0}{B_0 - \beta_0}. \]

The other two combinations of \( P_0, Q_0 \) vanish. Using a similar expansion for the third term, we get (C.4).

Starting from (C.4) as well as observing \( b_i = P_i[b_0] \) and \( l_i = P_i^*[l_0] \) for \( i = 1, 2 \), the relations (C.2) are a direct consequence of the definitions \( b = P[b_0] \) and \( l = P^*[l_0] \) and (C.1).

We will show below that

\[ BP = B_0P_0 + B_1 + B_2 + \mathcal{O}(\|E\|^3), \]  
(C.6)

where we defined

\[
\begin{align*}
B_1 &:= P_0EP_0 - \beta_0\left( \frac{Q_0}{B_0 - \beta_0}EP_0 + P_0E\frac{Q_0}{B_0 - \beta_0} \right), \\
B_2 &:= \beta_0\left( P_0E\frac{Q_0}{B_0 - \beta_0}EP_0 - \frac{Q_0}{B_0 - \beta_0}EP_0E\frac{Q_0}{B_0 - \beta_0} + \frac{Q_0}{B_0 - \beta_0}E\frac{Q_0}{B_0 - \beta_0}EP_0 \right) - \frac{B_0Q_0}{(B_0 - \beta_0)^2}EP_0EP_0 - P_0E\frac{B_0Q_0}{(B_0 - \beta_0)^2}EP_0 - P_0EP_0E\frac{B_0Q_0}{(B_0 - \beta_0)^2}.
\end{align*}
\]

Now, we obtain (C.3) by applying (C.2) as well as (C.6) to \( \beta(l, b) = (l, BPb) \).

In order to prove (C.6), we use the analytic functional calculus with \( \Gamma \) as defined above to obtain

\[ BP = -\frac{1}{2\pi i} \int_{\Gamma} \omega d\omega = \frac{1}{2\pi i} \int_{\Gamma} \omega \left( -\frac{1}{B_0 - \omega} + \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} - \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} - \frac{1}{B_0 - \omega} E\frac{1}{B_0 - \omega} \right) d\omega + \mathcal{O}(\|E\|^3). \]

Proceeding similarly as in the proof of (C.4) yields (C.6) and thus completes the proof of Lemma C.1. \( \square \)
D. Characterization of supp ρ

The following lemma gives equivalent characterizations of supp ρ in terms of m. Note supp ρ = supp v due to the faithfulness of (·).

We denote the disk of radius ε > 0 centered at z ∈ C by D_ε(z) = \{ w ∈ C : |z − w| < ε \}.

**Lemma D.1** (Behaviour of m on R \ supp ρ). Let m be the solution of the Dyson equation (2.3) for a data pair (a, S) ∈ A_α x Σ with |a| ≤ k_0 and S |x| ≤ k_1(x) I for all x ∈ A_+ and some k_0, k_1 > 0. Then, for any fixed τ ∈ R, the following statements are equivalent:

(i) There is c > 0 such that
\[ \limsup_{η \downarrow 0} η \| m(τ + iη) \|^{-1} ≥ c. \]

(ii) There are C ≥ 0 and N ∈ (0, 1] with an accumulation point 0 such that
\[ \| m(z) \| ≤ C, \quad \| m(z)^{-1} \| ≤ C, \quad C^{-1} (\text{Im } m(z)) I ≤ m(z) I ≤ C (\text{Im } m(z)) I, \quad \| F(z) \|_2 ≤ 1 − C^{-1} \quad (D.1) \]
for all z ∈ τ + iN. (The definition of F was given in (3.4).)

(iii) There is m = m* ∈ A such that
\[ \lim_{η \downarrow 0} η \| m(τ + iη) − m \| = 0. \] (D.2)
Moreover, there is C > 0 such that \| m \| ≤ C and \| (\text{Id} − C_m S)^{-1} \| ≤ C.

(iv) There are ε > 0 and an analytic function f : D_ε(τ) → A such that f(z) = m(z) for all z ∈ D_ε(τ) \ intersection \ B_ε(0) and f(z) = f(\overline{z})* for all z ∈ D_ε(τ). In particular, f(z) = f(\overline{z})* for z ∈ D_ε(τ) \ intersection \ R.

In other words, m can be analytically extended to a neighbourhood of τ.

(v) There is ε > 0 such that dist(τ, supp ρ) = dist(τ, supp v) ≥ ε.

(vi) There is c > 0 such that
\[ \lim inf_{η \downarrow 0} η \| m(τ + iη) \|^{-1} ≥ c. \]

All constants in (i) – (vi) depend effectively on each other as well as possibly k_0, k_1 and an upper bound on |τ|. For example, in the implication (iii) ⇒ (v), ε in (v) can be chosen to depend only on k_1 and C in (iii).

We remark that m in (iii) above is invertible and satisfies (2.3) at z = τ.

As a direct consequence of the equivalence of (i) and (v), we spell out the following simple characterization of supp ρ.

**Corollary D.2** (Characterization of supp ρ). Under the conditions of Lemma D.1, we have
\[ \lim_{η \downarrow 0} η \| \text{Im } m(τ + iη) \|^{-1} = 0. \] (D.3)
if and only if τ ∈ supp ρ(= supp v).

**Remark D.3.** In the proof of Lemma D.1, the condition S |x| ≤ k_1(x) I for all x ∈ A_+ is only used to guarantee the following two weaker consequences: First, this condition implies \| S \|_{2→2} ≤ 2k_1. Moreover, this condition yields, by Lemma B.1 (i), that F = F(τ + iη) has an eigenvector \ f ∈ A_+ corresponding to \| F \|_2. Ff = \| F \|_2 f.

for any fixed τ ∈ R \ supp ρ and any η ∈ (0, 1]. If both of these consequences are verified, then the condition S |x| ≤ k_1(x) I may be dropped from Lemma D.1 without any changes in the proof.

**Lemma D.4** (Quantitative implicit function theorem). Let X, Y, Z be Banach spaces, U ⊆ X and V ⊆ Y open subsets with 0 ∈ U, V. Let Φ : U x V → Z be continuously Fréchet-differentiable map such that the derivative \( \overline{\partial}_1 \Phi(0, 0) \) with respect to the first variable has a bounded inverse in the origin and \( \Phi(0, 0) = 0 \). Let δ > 0 such that \( B_δ^X \subseteq U, B_δ^Y \subseteq V \) and
\[ \sup_{(x,y) \in B_δ^X x B_δ^Y} \| \text{Id}_X − (\overline{\partial}_1 \Phi(0, 0))^{-1} \overline{\partial}_1 \Phi(x, y) \| ≤ \frac{1}{2}, \] (D.4)
where \( B_δ^X \) and \( B_δ^Y \) denote the δ-ball around 0 in X and Y, respectively. We also assume that
\[ \| (\overline{\partial}_1 \Phi(0, 0))^{-1} \| ≤ C_1, \quad \sup_{(x,y) \in B_δ^X x B_δ^Y} \| \overline{\partial}_2 \Phi(x, y) \| ≤ C_2 \]
for some constants $C_1$, $C_2$, where $\partial_2$ denotes the derivative of $\Phi$ with respect to the second variable. Then there is a constant $\varepsilon > 0$, depending only on $\delta$, $C_1$ and $C_2$, and a unique function $f : B^X \to B^X$ such that $\Phi(f(y), y) = 0$ for all $y \in B^X$. Moreover, $f$ is continuously Fréchet-differentiable and if $\Phi(x, y) = 0$ for some $(x, y) \in B^X \times B^Y$ then $x = f(y)$. If $\Phi$ is analytic then $f$ will be analytic.

Proof. The proof is elementary and left to the reader. \hfill \square

For $x, y \in A$ and $\omega \in C$, we define

$$
\Phi_2(y, \omega) := (\text{Id} - C_z S)[y] - \omega x^2 - \frac{\omega}{2} (xy + yx) - \frac{1}{2} (xS[y]y + yS[y]x).
$$

(D.5)

We remark that $\Phi_m(z)(m(z) + \omega - m(z), \omega) = 0$ for all $z \in \mathbb{H}$ and $z + \omega \in \mathbb{H}$ (see (6.9)).

Proof of Lemma D.1. Lemma B.2 (i) yields $\|S\|_{2\to\|\|} \lesssim 1$ due to $S[x] \leq k_1(x) \mathbb{1}$ for all $x \in \mathcal{A}_\cdot$. Therefore, $\|a\| \lesssim 1$ and $\|S\| \leq \|S\|_{2\to\|\|} \lesssim 1$ imply that $\text{supp} v = \text{supp} \rho$ is bounded, i.e., $\sup\{|\tau| : \tau \in \text{supp} \rho\} \lesssim 1$ by (2.5a).

First, we assume that (i) holds true. We set $N := \{\eta \in (0, 1] : \eta \|\text{Im} m(\tau + i\eta)\|^{-1} \geq c/2\}$. By assumption, $N$ is nonempty and has 0 as an accumulation point. In particular, we have

$$
\|\text{Im} m(z)\| \leq \frac{2\eta}{c}, \quad \eta \mathbb{1} \lesssim \text{Im} m(z) \lesssim \eta \mathbb{1}
$$

(D.6)

for all $z \in \tau + iN$. The first bound is a direct consequence of the definition of $N$. The second bound follows from (2.4) and the bounded support of $v$. Moreover, the first bound immediately implies the third bound. By averaging the two last bounds in (D.6) and using $\|m(\tau + i\eta)\| \lesssim \eta$ for $\eta \in N$, we obtain the third and fourth estimates in (D.1). In particular, $\rho(z) \sim \|\text{Im} m(z)\|$ for $z \in \tau + iN$. Owing to (2.4), for any $z \in \mathbb{H}$ and $x, y \in L^2$, we have

$$
|\langle x, m(z)y \rangle| \leq \frac{1}{2} \int_{\mathbb{R}} \frac{|\langle x, v(dr)x \rangle + \langle y, v(dr)y \rangle|}{|\tau - z|} \leq \frac{1}{\eta} \left( \|x, \text{Im} m(z)x\| + \|y, \text{Im} m(z)y\| \right) \leq \frac{2}{c} (\|x\|^2 + \|y\|^2).
$$

Here, we used that $v$ has a bounded support and (2.4) in the second step and the first bound in (D.6) in the last step. This proves the first bound in (D.1). The second estimate in (D.1) is a consequence of (2.3) as well as $\|a\| \lesssim 1$, $\|S\| \leq \|S\|_{2\to\|\|} \lesssim 1$ and the first bound in (D.1). We recall the definitions of $q = q(z)$ and $u = u(z)$ in (3.1). Owing to Lemma 4.8 (i), the bounds in (D.1) yield

$$
\|q\| \lesssim 1, \quad \|q^{-1}\| \lesssim 1, \quad \text{Im} u \sim \langle \text{Im} u \rangle \mathbb{1} \sim \rho \mathbb{1}
$$

(D.7)

uniformly for all $z \in \tau + iN$. Thus, for all $x \in \mathcal{A}_\cdot$ and $z = \tau + i\eta$ and $\eta \in N$, $F = F(z)$ satisfies $F[x] \lesssim \langle z \rangle \mathbb{1}$ due to $S[x] \lesssim \langle z \rangle \mathbb{1}$. Hence, Lemma B.1 (i) yields the existence of an eigenvector $f \in \mathcal{A}_\cdot$, i.e., $F \mathbb{1} = \|F\|_2 \mathbb{1}$. By taking the imaginary part of (3.3) and then the scalar product with $f$ as well as using the symmetry of $F$, we get

$$
1 - \|F\|_2 = \eta \langle f, qf \rangle \sim \eta \|\text{Im} m(z)\|^{-1} \gtrsim c
$$

(D.8)

for $z = \tau + i\eta$ and $\eta \in N$ (compare (4.5)). Here, we also used $f \in \mathcal{A}_\cdot$, (D.7), $\rho(z) \sim \|\text{Im} m(z)\|$ and the definition of $N$. This completes the proof of (i) $\Rightarrow$ (ii).

Next, let (ii) be satisfied. As before, Lemma 4.8 (i) implies (D.7) for all $z \in \tau + iN$ due to the first four bounds in (D.1). Thus, inspecting the proofs of Lemma 4.8 (iii) and Proposition 4.1 and using $\|S\|_{2\to\|\|} \lesssim 1$ via Lemma B.2 (ii) yield

$$
\|\langle \text{Id} - C_m(z)S\rangle^{-1}\| \lesssim 1
$$

(D.9)

uniformly for all $z \in \tau + iN$. Thus, we can apply the implicit function theorem, Lemma D.4, to $\Psi_q(V, \omega) := \Phi_m(\tau + i\eta) (\Delta, \omega)$ ($\Phi$ has been defined in (D.5)) for each $\eta \in N$ with $\omega \in C$. Since $\Psi_q(0, 0) = 0$ for all $\eta \in N$, there are $\varepsilon > 0$ and unique analytic functions $\Delta_\eta : D_\varepsilon(0) \to B^A_\varepsilon$ by Lemma D.4 such that $\Psi_q(\Delta_\eta, \omega) = 0$ for all $\omega \in D_\varepsilon(0)$ and all $\eta \in N$. We now explain why $\varepsilon$ can be chosen uniformly for all $\eta \in N$. By (B.1) and (D.9), there are bounds on $m(z)$ and $\|\text{Id} - C_m(z)\mathbb{1}\$ which hold uniformly for $z \in \tau + iN$. Hence, it is easy to find $\delta > 0$ such that (D.4) holds true uniformly for all $\eta \in N$. These uniform bounds yield the uniformity of $\varepsilon$. Since 0 is an accumulation point of $N$, there is $\eta_0 \in N$ such that $\eta_0 < \varepsilon$. We set $z := \tau + i\eta_0$. An easy computation using (2.3) at spectral parameters $z$ and $z + \omega$ shows $\Psi_{\rho_0}(m(\omega + z) - m(\omega), 0) = 0$ for all $\omega \in C$ such that $\omega + z \in \mathbb{H}$. Owing to the continuity of $m$, we find $\varepsilon' \leq (\varepsilon, 0)$ such that $m(\omega + z) - m(\omega) \in B^A_{\varepsilon}$ for all $\omega \in D_\varepsilon(0)$. Thus, by the uniqueness of $\Delta_{\rho_0}$ (cf. Lemma D.4), $\Delta_{\rho_0}(\omega) = m(\omega + z) - m(\omega)$ for all $\omega \in D_\varepsilon(0)$.  

55
As $\Delta_0$ and $m(\cdot + z)$ are analytic, owing to the identity theorem, we obtain $\Delta_\eta(0) + m(z) = m(\omega + z)$ for all $\omega \in D_z(0)$ satisfying $\omega + z \in \mathbb{H}$. Using $\eta_0 < \varepsilon$, we set $m := \Delta_\eta(-i\eta_0) + m(z)$. For this choice of $m$, the continuity of $\Delta_\eta(\omega)$ for $\omega \rightarrow -i\eta_0$ and $\Delta_\eta(\omega) + m(z) = m(\omega + z)$ yield (D.2). It remains to show that $m$ is self-adjoint. Since (D.7) holds true under (ii) as we have shown above, we obtain

$$\eta \| m(z) \|^{-1} \sim 1 - \| F \|_2 \geq C^{-1}$$

for $z = \tau + i\eta$ and $\eta \in N$ as in (D.8). Thus, $\liminf_{\eta \downarrow 0} \| m(\tau + i\eta) \| \leq 0$. Hence, we obtain $m = 0$, i.e., $m = m^\ast$. This completes the proof of (ii) $\Rightarrow$ (iii).

If (iii) holds true then $\text{Id} - C_m S$ has a bounded linear inverse on $A$ for $m$. Hence, we can apply the implicit function theorem, Lemma D.4, to $\Phi_m(\Delta, \omega) = 0$ (see (D.5) for the definition of $\Phi$) as $\Phi_m(0, 0) = 0$ and $\partial_1 \Phi_m(0, 0) = \text{Id} - C_m S$. It is easy to see that there is $\delta > 0$ such that (D.4) is satisfied. Therefore, there are $\varepsilon > 0$ and an analytic function $\Delta: D_z(0) \rightarrow B^0_S$ such that $\Phi_m(\Delta(\omega), \omega) = 0$ for all $\omega \in D_z(0)$. In particular, $f: D_z(\tau) \rightarrow A$, $f(w) := \Delta(w - \tau) + m$ is analytic. From (D.2) and (2.3), we see that $m$ is invertible and satisfies (2.3) at $z = \tau$. Thus, a straightforward computation using (2.3) at $z = \tau$ and $\tau = \tau + i\eta$ yields $\Phi_m(m(\tau + i\eta) - m, i\eta) = 0$ for all $\eta \in (0, \varepsilon)$. Therefore, $m(\tau + i\eta) = \Delta(i\eta) + m = f(\tau + i\eta)$ for all $\eta \in (0, \eta_\ast)$ and some $\eta_\ast \in (0, \varepsilon]$ due to the uniqueness part of Lemma D.4 and (D.2). Since $m$ and $f$ are analytic on $D_z(\tau) \cap \mathbb{H}$, the identity theorem implies $m(z) = f(z)$ for all $z \in D_z(\tau) \cap \mathbb{H}$. A simple computation shows $\Phi_m(\Delta(\omega)^\ast, \omega) = \Phi_m(\Delta(\omega)^\ast, \omega)^\ast = 0$ for all $\omega \in D_z(0)$ as $m = m^\ast$. Hence, $\Delta(\omega) = \Delta(\omega)^\ast$ for all $\omega \in D_z(0)$ by the uniqueness part of Lemma D.4. Thus, $f(w) = f(\tau)^\ast$ for all $w \in D_z(\tau)$ and $f(w) = f(w)^\ast$ for all $w \in D_z(\tau) \cap \mathbb{H}$. This proves (iii) $\Rightarrow$ (iv). Clearly, (iv) implies (v) by (2.4).

If the statement in (v) holds true then dist$(\tau, \text{supp} \rho) \geq \varepsilon$. In particular, by (3.7), we have

$$\liminf_{n \downarrow 0} \| m(\tau + i\eta) \|^{-1} \geq \liminf_{n \downarrow 0} \| m(\tau + i\eta) \|^{-1} \geq \varepsilon^2$$

for all $\eta > 0$. Here, we used (3.7) in the first step. This immediately implies (vi) with $c = \varepsilon^2$. Moreover, (i) is immediate from (vi).

Inspecting the proofs of the implications above shows the additional statement about the effective dependence of the constants in (i) – (vi). In particular, the application of the implicit function theorem, Lemma D.4, in the proof of (iv) shows that $\varepsilon$ can be chosen to depend only on $k_1$ and $C$ from (iii). This completes the proof of Lemma D.4.

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