Path Integral Bosonization of Massive GNO Fermions

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ABSTRACT

We show the quantum equivalence between certain symmetric space sine-Gordon models and the massive free fermions. In the massless limit, these fermions reduce to the free fermions introduced by Goddard, Nahm and Olive (GNO) in association with symmetric spaces \( K/G \). A path integral formulation is given in terms of the Wess-Zumino-Witten action where the field variable \( g \) takes value in the orthogonal, unitary, and symplectic representations of the group \( G \) in the basis of the symmetric space. We show that, for example, such a path integral bosonization is possible when the symmetric spaces \( K/G \) are \( SU(N) \times SU(N)/SU(N); N \leq 3 \), \( Sp(2)/U(2) \) or \( SO(8)/U(4) \). We also address the relation between massive GNO fermions and the nonabelian solitons, and explain the restriction imposed on the fermion mass matrix due to the integrability of the bosonic model.

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1. Introduction

The quantum equivalence between the sine-Gordon and the massive Thirring models, which is known as abelian bosonization, is one of the noble features of two-dimensional quantum field theories [1]. In the massless case, it was subsequently generalized by Witten [2] to the nonabelian bosonization, where the group $SO(N)$ (or $U(N)$) Wess-Zumino-Witten model at level one is shown to be quantum equivalent to the free Majorana (or Dirac) fermion model. The fermion mass bilinears are identified with bosonic operators in the massless model; i.e., Witten’s nonabelian bosonization considers only the zero charge sector of the fermionic model where massive excitations are treated perturbatively. This should be contrasted to the abelian case where the massive fermion operator (charge nonzero sector) is identified with a nonperturbative bosonic soliton operator [3].

The nonabelian bosonization for the nonzero charge sector requires a generalization of the Mandelstam’s soliton operator. Recently, a systematic nonabelian generalization of the sine-Gordon model has been made in association with symmetric spaces [15] and nonabelian soliton solutions are found which extend the sine-Gordon soliton with extra internal degrees of freedom [20]. This raises a possibility of massive nonabelian bosonization for the nonzero charge sector in terms of nonabelian soliton operators. On the other hand, Goddard, Nahm and Olive (GNO) [4] have shown that the free, massless fermions having Sugawara’s energy-momentum tensor are in one to one correspondence with compact symmetric spaces. In the context of conformal embedding, this correspondence led to the quantum equivalence of the GNO’s free fermions to the Wess-Zumino-Witten (WZW) models. However, for the quantum equivalence, it was pointed out that having the same Sugawara’s energy-momentum tensor (or Virasoro symmetry) alone was not sufficient, but it also requires the same spectra of primary fields and the same operator product algebras [3]. In general, the equivalent bosonic model is not described by a (diagonal) WZW model on a simply connected group manifold, and so far, a path integral formulation of such bosonic model has been lacking. Only the trivial conformal embeddings, $\hat{so}(N)_{k=1} \subset \hat{so}(N)_{k=1}$ and $\hat{su}(N)_{k=1} \subset \hat{su}(N)_{k=1}$, admit a path integral bosonization of free Majorana and Dirac fermion theories in terms of diagonal WZW actions which is precisely the Witten’s bosonization [2]. This makes particularly difficult the association of GNO fermions with bosonic solitons.

The purpose of this letter is to show that, with certain restrictions, symmetric space sine-Gordon models indeed bosonize the massive nonabelian fermions. In the massless
limit, these fermions become GNO fermions. In particular, we present a path integral bosonization of massive GNO fermions in terms of a WZW action, where the field variable \( g \) takes value in the orthogonal, unitary, and symplectic representations of the group \( G \) in the basis of the tangent space generators of the symmetric space \( K/G \). We construct representations explicitly, and specify the kernels of each representation for various symmetric spaces. We show that certain GNO fermions admit a path integral bosonization when the partition function belongs to the A or D-series of modular invariants \([6]\). For example, if we consider the type II symmetric space; \( K/G = SU(N) \times SU(N)/SU(N) \), it corresponds to the bosonization of \( SO(3) \) Majorana fermions for \( N = 2 \) and of \( SO(8) \) Majorana fermions for \( N = 3 \). Similarly, we show that the type I symmetric spaces, \( Sp(2)/U(2) \) and \( SO(8)/U(4) \), also admit a path integral bosonization. Identifying the bosonic model with integrable symmetric space sine-Gordon models, we explain the relation between the massive GNO fermions and the nonabelian solitons, and explain the restriction imposed on the fermion mass matrix due to the integrability of the bosonic model.

2. Orthogonal representations

Consider the symmetric space \( K/G \) for Lie groups \( G \subset K \) whose associated Lie algebras \( g \subset k \) satisfy the Lie algebra commutation relations;

\[
[g, g] \subset g, \quad [g, p] \subset p, \quad [p, p] \subset g.
\] (1)

Here, \( p \) is the vector space complement of \( g \) in \( k \), i.e.,

\[
k = g \oplus p.
\] (2)

We denote orthogonal basis of \( g \) and \( p \) by \( T_i \) and \( P_\alpha \) and assume that

\[
[T_i , T_j] = i f_{ijk} T_k , \quad [T_i , P_\alpha] = i P_\beta M^i_{\beta \alpha}.
\] (3)

We scale \( T_i \) and \( P_\alpha \) such that

\[
\text{Tr}(T_i T_j) = y \delta_{ij}, \quad \text{Tr}(P_\alpha P_\beta) = y \delta_{\alpha \beta}.
\] (4)

Then, GNO’s theorem states that free fermions possess Sugawara’s energy-momentum tensor if free fermions transform under \( G \) as the generators \( P_\alpha \) of the symmetric space \( K/G \) do. These free fermions will be called as GNO fermions. The Sugawara’s energy-momentum tensor of GNO fermions also realize the conformal embedding \( \hat{g} \subset \hat{f} \) of
affine Lie algebras, that is, \( \hat{g} \) and \( \hat{f} \) resulting in the same Virasoro algebra. Here, \( \hat{f} \) is the Kac-Moody algebra associated with the embedding group \( F = SO(\dim p) \) (or \( U(\dim p/2), Sp(\dim p/4) \)) as will be explained later. In fact, it was shown that all possible subalgebras \( \hat{g} \) of affine algebras \( \hat{f} = so(n), su(n) \) or \( sp(n) \), satisfying the conformal embedding condition, can be found directly from the known classification of symmetric spaces [7].

In order to make the conformal embedding explicit, we represent the embedding \( G \subset F \) using the orthogonal representation of \( g \in G \) in the basis of the symmetric space generators \( P_\alpha \) as follows;

\[
f_{\alpha\beta} \equiv \frac{1}{y} \text{Tr} g^{-1} P_\alpha g P_\beta. \tag{5}
\]

For the case where \( g^{-1} = g^\dagger \) and \( P_\alpha^\dagger = -P_\alpha \), it is easy to see that matrices \( f_{\alpha\beta} \) are real and becomes an element of a subgroup of \( F = SO(\dim p) \). Thus, it provides an orthogonal matrix representation of the group \( G \) embedded in \( F \) which, in general, is reducible. For the Hermitian symmetric spaces (\( A_{III}, B_{DI}, CI, D_{III}, E_{III}, EV_{II} \) type), the embedding can be restricted to a unitary representation as follows; a Hermitian symmetric space, equipped with a complex structure, has a commuting \( U(1) \) factor \( e^{\theta T} \) in \( G \), i.e., \( G = e^{\theta T} \times G' \) and \( [T, G] = 0 \). \( T \) is an element of the Cartan subalgebra of \( k \) whose adjoint action introduces a \( \mathbb{Z}_2 \)-grading over \( p \). Thus, under the adjoint action of \( T \), generators of \( p \) can be regrouped into a set of pairs \((P_\alpha^{(1)}, P_\alpha^{(2)})\); \( \alpha = 1, \cdots, \dim p/2 \), such that

\[
[T, P_\alpha^{(1)}] = P_\alpha^{(2)}, \quad [T, P_\alpha^{(2)}] = -P_\alpha^{(1)}. \tag{6}
\]

In this new basis, the \( U(1) \) factor \( e^{\theta T} \) can be represented by a block diagonal matrix where each block is given by the \( 2 \times 2 \) matrix, \[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\].

Since \( G \) commutes with the \( U(1) \) factor, a group element of \( G \) takes a form of the \( \frac{1}{2} \dim p \times \frac{1}{2} \dim p \) matrix whose entries are given by \( 2 \times 2 \) matrices of the form \[
\begin{pmatrix}
x & -y \\
y & x
\end{pmatrix}
\]. Or, equivalently, by a complex number \( x + iy \). Therefore, in the case of Hermitian symmetric spaces, the orthogonal representation reduces to the unitary representation with \( F = U(\dim p/2) \), similar to the complex (Dirac) representation of fermions instead of the real (Majorana) representation. As an example, consider the case \( K/G = SU(3)/(SU(2) \times U(1)) \). Among eight Gell-Mann matrices \( \lambda_i \), the element \( T \) is given by \( \lambda_8/\sqrt{3}i \) and \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) form an \( su(2) \) subalgebra. Also, we have \( P_1^{(1)} = \lambda_4, \quad P_1^{(2)} = \lambda_5, \quad P_2^{(1)} = \lambda_6, \quad P_2^{(2)} = \lambda_7 \) so that \( g = \{\lambda_1, \lambda_2, \lambda_3, T\} \) and \( p = \{\lambda_4, \lambda_5, \lambda_6, \lambda_7\} \). In this way, \( G = SU(2) \times U(1) \) becomes
conformally embedded into the unitary group \( F = U(2) \).

Similarly, for the symmetric spaces (AIII, BDI), the orthogonal representation reduces to the simplectic one, with \( F = Sp(\dim p/4) \). In this case, group \( G \) contains \( SU(2) \) or \( SO(3) \) factors which commutes with \( G \). We may again choose a basis of the symmetric space so that generators \( P_\alpha \) can be regrouped into a set of quadruples \((P^{(1)}_\alpha, P^{(i)}_\alpha, P^{(j)}_\alpha, P^{(k)}_\alpha); \alpha = 1, \ldots, \dim p/4\), labeled by quaternion numbers. They satisfy the commutation relations:

\[
[T_A, P^{(B)}_\alpha] = \epsilon_{ABC} \epsilon^{(c)}\alpha \quad \text{for} \quad A = i, j, k, \quad \text{and} \quad B = 1, i, j, k
\]

(7)

where \( T_i, T_j \) and \( T_k \) are the generators of \( su(2) \) or \( so(3) \). The symbol \( \epsilon \) stands for

\[
\epsilon_{ABC} = \delta_{AB,C}
\]

(8)

where \( AB \) in \( \delta_{AB,C} \) denotes the quaternion product and the dummy index \( C \) runs over \( \pm1, \pm i, \pm j, \pm k \). For instance, \( \epsilon_{11} = \epsilon_{12} = \epsilon_{13} = 1 \), \( \epsilon_{21} = \epsilon_{22} = \epsilon_{23} = 1 \), \( \epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 1 \) etc. In this case, the \( SU(2) \) or \( SO(3) \) factor can be represented by a block diagonal matrix where each block is equal to the \( 4 \times 4 \) matrix \( \sum_{i=0}^{3} \rho_i L_i \) with real \( \rho_i \) satisfying \( \sum \rho_i^2 = 1 \). The matrices \( L_i \) are

\[
L_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}
\]

(9)

where \( \sigma_i \) are Pauli matrices. Since \( G \) commutes with the \( SU(2) \) or \( SO(3) \) factor, \( G \) takes a form of the \((\dim p/4) \times (\dim p/4)\) matrix whose entries are \( 4 \times 4 \) matrices commuting with \( L_i \). Thus, it takes a form, \( \sum_{i=0}^{3} a_i M_i \), for real \( a_i \) where

\[
M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}
\]

(10)

Or, equivalently, it may be identified with a quaternion number \( a_0 + a_1 i + a_2 j + a_3 k \) so that the orthogonal representation reduces to the simplectic matrix representation. In the GNO’s fermionic theory, this case corresponds to taking the complex representation of quark multiplets pseudoreal, i.e. it is equivalent to its complex conjugate but not real.

As an example, we consider the \( SU(N + 2)/(SU(2) \times U(N)) \) case. Then, we have

\[
T_i = \begin{pmatrix} 0 & -i & \cdots & 0 \\ -i & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad T_j = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad T_k = \begin{pmatrix} -i & 0 & \cdots & 0 \\ 0 & i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}
\]

(11)
and

\[
P^j_{\alpha-2} = \begin{pmatrix} 
\cdots & \alpha & \cdots 
\vdots & -i & \vdots 
\vdots & 0 & \vdots 
\vdots & \alpha & \vdots 
\vdots & \cdots & \vdots 
\end{pmatrix}, \quad P^i_{\alpha-2} = \begin{pmatrix} 
\cdots & \alpha & \cdots 
\vdots & 0 & \vdots 
\vdots & -i & \vdots 
\vdots & \alpha & \vdots 
\vdots & \cdots & \vdots 
\end{pmatrix}; \quad 3 \leq \alpha \leq N + 2
\]

These \( T_A \) and \( P^B_\alpha \) satisfy the commutation relation Eq. (7) and the representation becomes a simplectic one with \( F = Sp(N) \).

3. Path integral bosonization

Having specified the orthogonal representation of the group \( G \) as in Eq. (5), we now consider the level one WZW model given by the path integral,

\[
\int [df] \exp[S_{WZW}(f)],
\]

where the field \( f \) is given by Eq. (3) and \( S_{WZW} \) is the usual WZW action. In general, the orthogonal representation is reducible. Notable exceptions are when \( K/G = SO(N+1)/SO(N) = S^N \) or \( K/G = SU(N+1)/U(N) = CP^N \). In such cases, the orthogonal representation becomes the defining fundamental representations of \( SO(N) \) and \( U(N) \) respectively. Also, the model in Eq. (13) expresses the trivial conformal embeddings, \( \hat{\mathfrak{g}} = so(N)_{k=1} \subset \hat{\mathfrak{f}} = so(N)_{k=1} \) and \( su(N)_{k=1} \subset su(N)_{k=1} \), which are equivalent respectively to the Majorana and Dirac fermionic models [2]. However, for other cases of symmetric spaces, the embedding becomes nontrivial. The currents of the orthogonal representation, e.g. the chiral Kac-Moody current, are related to those of the defining representation in the following way:

\[
(f^{-1}\partial f)_{\alpha\beta} = \frac{1}{y} \text{Tr}(gP_{\alpha}g^{-1}P_{\gamma})\text{Tr}([g^{-1}P_{\gamma}g, g^{-1}\partial g]P_{\beta})
\]

\[
= \frac{1}{y} \text{Tr}(P_{\alpha}[g^{-1}\partial g, P_{\beta}])
\]

\[
= iM^i_{\alpha\beta}(g^{-1}\partial g)_i.
\]
The structure constants \( M_{i}^{\alpha \beta} \) are as given in Eq. (3) and possess the property,

\[
\sum_{\alpha \beta} M_{i}^{\alpha \beta} M_{j}^{\alpha \beta} = \kappa \delta_{ij},
\]

where \( \kappa = x \psi^2 \), \( \psi \) = highest root of \( g \) and \( x \) is the Dynkin index of the representation with generators \( M^{i} \). Thus, the bosonization of the fermionic current \( J^{i} = i \frac{\psi^\alpha}{2} M_{i}^{\alpha \beta} \psi^\beta \) is that

\[
J^{i} = M_{i}^{\alpha \beta} ( f^{-1} \partial f )^{\alpha \beta} = i \kappa ( g^{-1} \partial g )^{i},
\]

and a straightforward calculation shows that \( J^{i} \) satisfies the Kac-Moody algebra of group \( G \) with the level \( \kappa / 2 \). Since Eq. (14) holds also for the case where \( f^{-1} \partial f \) and \( g^{-1} \partial g \) are replaces by any infinitesimal variations \( f^{-1} \delta f \) and \( g^{-1} \delta g \), the WZW action reduces to

\[
S_{WZW}(f) = \frac{\kappa}{2} S_{WZW}(g).
\]

However, it should be emphasized that the identity, Eq. (17), implies the equivalence of the model in Eq. (13) with the level \( \kappa \), group \( G \) WZW model only at the classical level. At the quantum level, since the orthogonal, as well as the unitary and the symplectic, representation of group \( G \) is not necessarily faithful, the model in Eq. (13) is not in general equivalent to the level \( \kappa \), diagonal WZW model based on the simply connected group \( G \). For example, the orthogonal represenation of a type II symmetric space, \( G \times G/G \), is the same as the adjoint representation of \( G \), and the kernel of the adjoint representation is the center \( Z_{G} \) of the group \( G \). Thus, the model in Eq. (13) corresponds to the group \( G \) WZW model modded out by the center \( Z_{G} \). More specifically, if \( G = SU(2) \), the model in Eq. (13) possesses the level two Kac-Moody algebra \( \hat{su}(2)_{k=2} \) and the orthogonal representation corresponds to \( SO(3) \approx SU(2)/Z_{2} \). Even though, Kac-Moody algebras corresponding to \( SU(2) \) and \( SO(3) \) are equivalent, the corresponding WZW partition functions are different [8]. Thus, the model in Eq. (13) truely becomes equivalent to the level one \( SO(3) \) WZW model, or through the Witten’s bosonization, to the \( SO(3) \) massless Majorana fermions. This provides a field theoretic realization of the conformal embedding, \( \hat{su}(2)_{k=2} \subset \hat{so}(3)_{k=1} \). Previously, such Majorana fermions have also been used in representing the Kac-Moody algebra, \( \hat{su}(2)_{k=2} \). Similarly, the model Eq. (13) for the type II space with \( G = SU(3) \) realizes the conformal embedding \( \hat{su}(3)_{k=3} \subset \hat{so}(8)_{k=1} \). That is, the partition of the model in Eq. (13) in this case is that of the level three \( SU(3) \) WZW model modded by \( Z_{3} \), which is \( Z(A_{6}/Z_{3}) \) in the \( D \) series of modular invariants [3] (see e.g. Ref. [11] for the notation). On the other
hand, this is also the partition function of the level one \(SO(8)\) WZW model thus proving the equivalence of the model in Eq. (13) with the \(SO(8)\) Majorana fermions. However, for \(G = SU(N), N \geq 4\), the relevant conformal embeddings belong to the \(E\)-series. For example, if \(N = 4\), the relevant conformal embedding, \(\hat{su}(4)_{k=4} \subset (\hat{B}_7)_{k=1}\), belongs to the exceptional case \(E_8\). We do not know whether the model in Eq. (13) still realizes the conformal embedding for these exceptional cases. This remains as an open problem.

As for the type I symmetric spaces, kernels of orthogonal representations can be computed through an explicit parametrization of each symmetric spaces. The results are the following; for symmetric spaces \(\text{AI} (F/G = SU(N)/SO(N))\), \(\text{CI} (Sp(N)/U(N))\) and \(\text{DIII} (SO(2N)/U(N))\), kernels are \(Z_2\) for even \(N\). For odd \(N\), they become trivial. For symmetric spaces \(\text{AII} (SU(2N)/Sp(N))\) and \(\text{CII} (Sp(M + N)/(Sp(M) \times Sp(N)))\), kernels are \(Z_2\) while the kernel of \(\text{AIII} (SU(M + N)/(SU(M) \times SU(N) \times U(1)))\) case is \(Z_{(M,N)}\) where \((M,N)\) denotes the greatest common divisor of \(M\) and \(N\). For \(\text{BDI} (SO(M + N)/(SO(M) \times SO(N)))\), it becomes \(Z_2\) for \(M\) and \(N\) all even and otherwise it becomes trivial. We have not computed kernels for the exceptional cases leaving it for a future work. With kernels specified, we could check the quantum equivalence of the model in Eq. (13) with free fermion models. For instance, the \(N = 2\) case of the \(\text{CI}\) symmetric space corresponds to the conformal embedding \(\hat{su}(2)_{k=4} \subset \hat{su}(3)_{k=1}\). The partition function is \(Z(A_6/Z_2)\) in the \(D\)-series which shows the equivalence of the model in Eq. (13) with \(U(3)\) Dirac fermions. However, in the \(N = 3\) and \(N = 4\) cases, the relevant embeddings are \(\hat{su}(3)_{k=5} \subset \hat{su}(6)_{k=1}\) and \(\hat{su}(4)_{k=6} \subset \hat{su}(10)_{k=1}\) respectively, and partition functions belong to \(E_8\) and \(E_{10}\). Another example is the \(N = 4\) case of \(\text{DIII}\) symmetric space which has the conformal embedding \(\hat{su}(4)_{k=2} \subset \hat{su}(6)_{k=1}\) with the partition function \(Z(A_6/Z_2) = D_6\) thereby revealing the quantum equivalence of the model in Eq. (13) with \(U(6)\) Dirac fermions. The \(N \geq 5\) cases of the \(\text{DIII}\) symmetric space are again associated with the \(E\)-series so that the quantum equivalence with the fermion model is not clear.

4. Fermion mass and nonabelian soliton

Earlier attempts to extend the massive abelian bosonization to the nonabelian case resorted to \(N\) scalar fields [11, 12]. However, in this approach, the global nonabelian symmetry of fermions become obscure and the bosonic off-diagonal currents become nonlocal. In the Witten’s bosonization, such difficulties were removed by introducing a
nonlinear sigma field $g$ in terms of which bosonization is done in a local way with manifest
global symmetries. On the other hand, the existence of solitons in the bosonized model,
which generalizes the sine-Gordon soliton, has not been well understood. In the spirit of
particle vs. soliton duality in the abelian bosonization, the bosonic solitons associated with
nonabelian fermions can be of interest. In the following, we show how such nonabelian
solitons can arise in the model which bosonizes massive GNO fermions.

In the previous section, we have shown that the model in Eq. (13) for specific symmet-
ric spaces become equivalent to Majorana or Dirac fermions according to the orthogonal
or unitary representations of embedding. We note that the bosonic bilinear in
$g$
can be
identified with a fermion bilinear in our case;

$$\frac{1}{y} \text{Tr} g^{-1} P_\alpha g P_\beta \equiv f_{\alpha\beta}(z, \bar{z}) = M \psi^\alpha(z) \bar{\psi}^\beta(\bar{z})$$

(18)

where $M$ is the mass parameter dependent on the regularization scheme. Thus, the
bosonic model in Eq. (13) with a bosonized fermion mass term is given by

$$S = S_{ZW}(f) + \int M_{\alpha\beta} f^{\alpha\beta}$$

$$= \frac{\kappa}{2} S_{ZW}(g) + \int M_{\alpha\beta} \text{Tr} \left( \frac{1}{y} g^{-1} P^\alpha g P^\beta \right).$$

(19)

In general, this model is not integrable. However, when the mass matrix $M_{\alpha\beta}$ takes
a specific form, e.g. if only one component is nonvanishing, e.g. $M_{AB} \neq 0$, then the
classical equation of motion of Eq. (13) takes a zero curvature form,

$$[\partial + g^{-1} \partial g + m \lambda P^B, \bar{\partial} + \frac{1}{\lambda} g^{-1} P^A g] = 0,$$

(20)

where $\lambda$ is a spectral parameter and $m$ is a mass parameter. This demonstrates the
integrability of the model, at least classically [13]. With a particular choice of $P^A$ and
$P^B$, also with additional constraints which removes the massless degrees of freedom, this
model has been called as a symmetric space sine-Gordon model [14]-[16] and its various
properties including nonabelian solitons have been investigated [17]-[20].

In order to understand the existence of solitons, we assume that $M_{11} \neq 0$ and all other
components of $M_{\alpha\beta}$ are zero. The potential term $V = M_{11} f^{11} = M_{11} \text{Tr}(g^{-1} P^1 g P^1)/y$ is
in general quadratic in $g$. However, in the case where $K/G = SO(N + 1)/SO(N) = S^N$
or $K/G = SU(N + 1)/U(N) = CP^N$, the potential $V$ becomes $V = \text{const.}(g_{11} + g_{11}^{-1})$.
Since $g$ is unitary, it satisfies

$$|g_{11}|^2 + |g_{12}|^2 + \cdots + |g_{1N}|^2 = 1.$$

(21)
Thus, we could write $g_{11}$ by $g_{11} = \cos \phi e^{i\theta}$ for some scalar fields $\phi$ and $\theta$ so that the potential $V = \text{const.} \cos \phi \cos \theta$. This shows that the bosonized fermion mass term induces infinitely many degenerate vacua, and essentially soliton solutions interpolating different vacua. For concreteness, we write the Lagrangian of the massive bosonic model in Eq. (19) for the $CP^2$ case in terms of three scalar fields $\phi, \theta_1$ and $\theta_2$,

$$L = \frac{1}{2\pi} \left( \partial \phi \bar{\partial} \phi + \frac{1}{4} \partial \theta_1 \bar{\partial} \theta_1 + \cot^2 \phi \partial \theta_2 \bar{\partial} \theta_2 + 2m \cos \phi \cos(\theta_1 - \theta_2) \right).$$  \hspace{1cm} (22)

A consistent reduction can be made by taking $\theta_1 = \theta_2$ and then the model reduces the complex sine-Gordon model, which is known to possess both topological and non-topological solitons [21, 22]. Of course, a further consistent reduction which takes $\theta_1 = \theta_2 = 0$ brings the Lagrangian in Eq. (22) to that of the sine-Gordon model. One can readily check that in this case the coupling constant of the sine-Gordon model is fixed in such a way that the corresponding massive fermion does not have Thirring interaction term $[1]$. This shows that the fermion operator $\psi_1$ can be expressed in terms of the Mandelstam’s soliton operator. The existence of non-topological solitons in the complex sine-Gordon model and also other localized solutions with nonabelian nature suggest that the particle v.s. soliton duality could be further carried out in a nonabelian context. This will be considered elsewhere.

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