Modified perturbation theory for pair production and decay of fundamental unstable particles

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Abstract

We construct an asymptotic expansion in powers of the coupling constant directly of the cross-section for pair production and decay of fundamental unstable particles. The resonant and kinematic singularities arising in the expansion we treat in the sense of distributions. This mode allows us to transform formally divergent integrals into absolutely convergent ones with keeping the asymptotic property of the expansion. The appropriate procedure is elaborated up to an arbitrary order of the expansion. The peculiarity of application of the procedure in the threshold region is analysed. The scheme of the calculations within the NNLO approximation is explicitly discussed.

1 Introduction

The processes of pair production and decay of fundamental unstable particles such as the top quarks or W bosons provide a powerful tool for the determination of their masses and parameters of their interaction vertices. At the LHC and at an International linear collider (ILC) [1, 2] the mentioned determination is planned to be carried out through measurings of the absolute value of the cross-section and its dependence on the center-of-mass energy, and of the angular distributions and spin correlations. At the ILC, where the higher precision is projected, the corresponding theoretical calculations must be carried out in many cases with the two-loop accuracy in matrix elements [3, 4]. The calculation of the cross-sections thereafter must be carried out with the equivalent accuracy. However, the calculation of the cross-sections is obstructed by resonant contributions of the unstable particles since such contributions are nonintegrable in the framework of conventional perturbation theory (PT).

The common practice of a solution to this problem consists in the Dyson resummation of the imaginary parts of self-energies in the denominators of unstable-particles propagators. Simultaneously this cures the Coulomb singularities, the other problematic contributions in the cross-sections [5]–[9]. However, the application of the Dyson resummation is rather dangerous in gauge theories because of risk to break the gauge cancellations necessary for unitarity. For this reason the one-loop calculations at LEP2 were carried out in the double-pole approximation (DPA), which guaranteed the gauge cancellations [10, 11]. However in the ILC case the accuracy of DPA is no longer sufficient [12, 13]. The application of the higher orders of the “pole expansion”
is ineffective, too, because of systematic inaccuracy inherent in this approach (see discussion in [11] and in the references therein). So in the ILC case the methods are required that can provide completely systematic calculations.

Among such methods as most perspective ones long time the methods were considered that were based on the background-field and pinch-technique formalisms. Both they imply a reconstruction of the conventional PT series; through the formation of the gauge-invariant effective action or through the Dyson resummation after fulfilling the gauge cancellations (see Refs. [14, 15] and [16]–[19], respectively, and the references therein). Unfortunately some difficulties are inherent in the mentioned methods, too. In particular, in the background-field formalism a dependence on the quantum gauge parameter remains, which cannot be fixed on physical grounds. The pinch-technique method does not have this problem, but there is another difficulty which is the common one in both cases. Namely for reaching the $O(\alpha^n)$ precision without violating the gauge cancellation in the framework of these methods it is not enough to calculate corrections up to the $O(\alpha^n)$ order; it is necessary to calculate all the $O(\alpha^{n+1})$ loop corrections, too, at least their imaginary-part contributions, which is impractical (see, in particular, discussion in [20]).

Another prominent approach is the “complex-mass scheme” (CMS), first considered in [21, 22] and then worked out in [12, 13]. Actually, CMS is a renormalization scheme defined through the identification of the renormalized mass of an unstable particle with the complex pole of its full propagator. So in the CMS the renormalized mass absorbs the imaginary contributions of the conventional self-energy, becoming thus a complex quantity. Unfortunately, this triggers the complex-valued counterterms, which violates unitarity. However, at the one-loop level the unitarity-violating contributions manifest themselves effectively at the higher orders with the gauge cancellations being explicitly maintained [12, 13]. So the one-loop calculations get legitimate. Whether the same behavior takes place at the two-loop level, this is not clear. For this reason the CMS cannot be considered as a rigorous procedure [13], at least at present. So a development of alternative approaches still is an actual problem.

In this paper we consider a modified perturbation theory (MPT), first proposed in [24] and further discussed in [25]–[28]. The basic idea of this approach is a systematic expansion in powers of the coupling directly of the probability instead of the amplitude. This mode allows one to impart the sense of distributions [29] to the propagators squared of the unstable particles, and on this basis to asymptotically expand the propagators squared without the appearance of the divergences in the cross-section. (Simultaneously, the rest of the amplitude squared is to be expanded in the conventional sense.) In doing so one gets the MPT expansion of the cross-section.

It must be emphasized that the appearance of the divergences in the cross-section when it is considered as an integral of the expanded amplitude squared, in fact does not yet mean impossibility to expand the cross-section. Immediately this means only an invalidity of the expansion of the integrand (the amplitude squared), because in many cases the expansion becomes possible after the calculation of the integral. In such cases the distributions method is found to be a power tool to do that in advance, before explicit calculation of the integral. The advantage of the distributions
method is caused by its possibility to make divergent contributions convergent by basing on the extension principle. Unfortunately, this procedure is accompanied by ambiguities emerging in the integration rules. However, the ambiguities may be reduced to the sum composed of the δ-functions and of their derivatives with arbitrary coefficients, which should be added to the integrand [29, 30]. So the ambiguities can be eliminated via the fixing of the coefficients only. (Let us remember that the similar phenomenon occurs in the case of the UV renormalization at the rigid fixing of the renormalization scheme [31]). In the context of asymptotic expansion the coefficients may be fixed by means of establishing the asymptotic property of the expansion that should appear after the integration is carried out [30, 32, 33]. Furthermore, in some cases the expansion may take the form of a complete series in powers of the expansion parameter. At the expansion of the cross-section this would mean that each term of the expansion is proportional to a power of the coupling constant $\alpha$ and contains no other dependence on $\alpha$. The latter property is extremely important since it means that in the cross-section the gauge cancellations must take place automatically at each power of $\alpha$ because the exact solution is gauge-invariant.

At present it is generally known how to implement the MPT expansion in the case of pair production and decay of unstable particles, but up to the next-to-leading order (NLO) only and in the context of models without massless particles [28]. (This modelling, however, in practice has shown the existence of a smooth line-shape for the observable cross-section in the framework of MPT approach.) A further development of the method should imply a construction of the cross-section in the higher orders of the MPT expansion in realistic cases. Solving this problem, one should overcome difficulties caused by the Coulomb and kinematic singularities. Recall that the Coulomb singularities arise as universal corrections caused by the exchanges by soft massless particles (photons, gluons) between outgoing massive particles in the limit of small relative velocities. Such contributions are nonintegrable in the cross-section in the higher orders of the conventional PT. However, there is a general method of summing up the Coulomb singularities, which makes the result integrable [6]–[9]. The kinematic singularities originate on account of non-analyticity of the phase-volume factor. The non-analyticity, in turn, gives rise to nonintegrable singularities in the cross-section in the presence of the MPT-expanded propagators squared. Consequently the latter singularities is an essential problem in the MPT approach.

In this paper we find a solution to the problem of kinematic singularities in an arbitrary order of the MPT expansion. The Coulomb singularities are taken into consideration rather qualitatively, in the spirit of [6]–[9], which is sufficient for our purposes. We emphasize that we search for a principle solution to the problem of curing non-integrable contributions in the cross-section. Correspondingly, we carry out only analytical calculations necessary for reducing further calculations to the realizable ones by means of numerical methods. In doing so we mean that the conventional regular contributions to the cross-section can be independently calculated in the framework of conventional PT.

The paper is organized as follows. In Sect. 2 we formulate the problem and do its preparatory analysis. An algorithm of the calculation of the MPT expansion in the case of the hard-scattering cross-section, is worked out in Sect. 3. In Sect. 4
we calculate singular contributions to the observable cross-section and introduce a necessary modification to the method in the near-threshold regions. In Sect. 5 we discuss the peculiarities of the calculations within the NNLO approximation. Sect. 6 outlines the results. In Appendix A we derive a formula for the characteristic change of variables in the $\delta$-function with derivatives. In Appendix B we carry out analytical calculations of basic singular integrals.

## 2 Statement of the problem

We analyse pair production and subsequent decay of unstable particles, in particular in the process of $e^+e^-$ annihilation. For simplicity we consider the case when only stable particles are immediately produced at the decays of the unstable particles. (The cascade decays should not lead to new significant difficulties.) The observable cross-section of the whole process is determined via the convolution of the hard-scattering cross-section with the flux function describing contributions of nonregistered photons emitted in the initial state [10]:

$$
\sigma(s) = \int_{s_{\text{min}}}^{s} \frac{ds'}{s} \phi(s'/s) \hat{\sigma}(s').
$$

(2.1)

Here $s$ is the energy squared in the center-of-mass system, $\sigma(s)$ and $\hat{\sigma}(s)$ are the observable and the hard-scattering cross-sections, respectively. Apart from $s$ both cross-sections can depend on spin and angular variables. (We do not consider explicitly this option as the corresponding modifications are unessential for our analysis.) The ratio $z = s'/s$ characterizes a fraction of the energy expended on the production of unstable particles. For our purposes it is sufficient to take flux function $\phi(z; s)$ in the leading logarithm approximation. So we put

$$
\phi(z; s) = \beta_e(1 - z)(\beta_e - 1)\left(1 + \frac{1}{2}\beta_e(1 + z)\right), \quad \beta_e = \frac{2\alpha}{\pi}\left(\ln \frac{s}{m_e^2} - 1\right).
$$

(2.2)

The hard-scattering cross-section $\hat{\sigma}(s)$ must be determined in the PT framework, though in some cases indirectly because of divergences arising in the phase-space integral. To avoid the divergences, usually the Dyson resummation is applied in the unstable-particles propagators. Starting conceptually from an exact solution, we initially imply the application of the Dyson-resummation, too. However further we will asymptotically expand the propagators squared in the sense of distributions.

Generally, there are two types of contributions to the hard-scattering cross-section, the factorizable and non-factorizable ones. The factorizable contributions retain the structure of the process as the sequential production and decay of unstable particles. The non-factorizable contributions connect these subprocesses. Typically the non-factorizable contributions are represented by configurations with contributions of one of the unstable particles or of both of them being components of one-particle irreducible subdiagrams. Such contributions do not generate singularities in the amplitude and therefore are integrable in the cross-section. Generally, the non-factorizable
contributions can be related to the cases of inclusive single production and decay of unstable particles or direct production of final states. The corresponding calculations in the MPT framework are simpler as compared to the cases with factorizable contributions. For this reason we postpone the consideration of non-factorizable contributions to the end of the next section. The only exception we make is for soft massless-particles contributions. The matter is that they can retain the double-resonant structure even if they formally are non-factorizable \cite{34}–\cite{36}. (Actually this property is a manifestation of general theorem \cite{37, 38} which states that the soft massless-particles contributions are collected in a factor in the cross-section.) For this reason we will consider the soft massless-particles contributions in parallel with the consideration of factorizable contributions.

So, at first we consider the piece of the cross-section that possesses the double-resonant structure. The hard-scattering cross-section in this case is conveniently written as follows:

$$\hat{\sigma}(s) = \int_{s_{1\min}}^{s_{2\min}} \mathrm{d}s_1 \int_{s_{2\min}}^{\infty} \mathrm{d}s_2 \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \hat{\sigma}(s; s_1, s_2) (1 + \delta_c) \quad (2.3)$$

Here $\hat{\sigma}(s; s_1, s_2)$ is an exclusive cross-section, the $(1 + \delta_c)$ stands for contributions of soft massless particles, $s_1$ and $s_2$ are the virtualities of the unstable particles. The $s_{1\min}$ and $s_{2\min}$ are the minimums of $s_1$ and $s_2$, respectively, defined as squared sums of the masses of the decay products of the unstable particles. Hereinafter we imply that the square roots of $s_{1\min}$ and $s_{2\min}$ are less than the masses of the corresponding unstable particles. The exclusive cross-section $\hat{\sigma}(s; s_1, s_2)$, we write in the form with extracted Breit-Wigner (BW) factors:

$$\hat{\sigma}(s; s_1, s_2) = \frac{1}{s^2} \sqrt{\lambda(s, s_1, s_2)} \Phi(s; s_1, s_2) \rho_1(s_1) \rho_2(s_2) . \quad (2.4)$$

Here $\rho_i(s_i)$ are the BW factors, $\lambda(s, s_1, s_2) = [s - (\sqrt{s_1} + \sqrt{s_2})^2][s - (\sqrt{s_1} - \sqrt{s_2})^2]$ is the so-called kinematic function. The product of the $\sqrt{\lambda(s, s_1, s_2)}$ with the $\theta$-function in (2.3) constitutes the kinematic factor. Let us note at once that the kinematic factor has a singularity at $\sqrt{s} = \sqrt{s_1} + \sqrt{s_2}$. The function $\Phi(s; s_1, s_2)$ is the rest of the amplitude squared. By construction it does not include singularities associated with the production and decay of unstable particles. Correspondingly, we consider $\Phi$ determined in the framework of conventional PT. The BW factors, we define as follows:

$$\rho_i(s_i) = \frac{M_i \Gamma_i}{\pi} \times |\Delta_i(s_i)|^2 . \quad (2.5)$$

Here $M_i$ is the renormalized mass and $\Gamma_i$ is the Born width of the $i$th unstable particle, $\Delta_i$ is a scalar part of the Dyson-resummed propagator (so the spin factors are referred to $\Phi$). Further we omit index "$i$" if its presence is not necessary. The $\Delta(s)$ has the form

$$\Delta(s) = \frac{1}{s - M^2 + \Re \Sigma(s) + i \Im \Sigma(s)} , \quad (2.6)$$
where \( \text{Re}\Sigma(s) \) and \( \text{Im}\Sigma(s) \) are the real and imaginary parts of the renormalized self-energy. We assume that \( \Sigma(s) \) is determined in the on-mass-shell (OMS) scheme of the UV renormalization, or rather in some version of its generalization to the unstable-particle case [39]–[43]. Recall that the naive expansion of \( \Delta(s) \) in powers of the coupling leads to divergences in integral (2.3).

Now we proceed directly to arguing the MPT approach. Its most important feature is the assignment of the distribution sense to the factors that explicitly or potentially include singularities. In the given case such factors are the BW factors, the kinematic factor, and the factor \( 1 + \delta_c \). Function \( \Phi \) may be considered as a representative of trial functions since \( \Phi \) is smooth in the vicinity of prospective singularities (located actually on the thresholds). By the next step, it is necessary to asymptotically expand the product of the mentioned factors in the sense of distributions.

For methodological reasons at first we consider the asymptotic expansion of the BW factors by considering each of them isolated. A general structure of their expansion is as follows [24]:

\[
\rho(s) = \delta(s-M^2) + PV \mathcal{T}_N \{ \rho(s) \} + \sum_{n=0}^{N} c_n(\alpha) \delta_n(s-M^2) + O(\alpha^{N+1}).
\] (2.7)

Here the expansion is carried out in powers of \( \alpha \) up to neglected terms of order \( O(\alpha^{N+1}) \). The leading term originates by virtue of the relation \( \alpha/(x^2 + \alpha^2) \to \pi \delta(x) \) as \( \alpha \to 0 \) and the unitarity relation \( \text{Im}\Sigma(M^2)|_{\text{1-loop}} = M \Gamma|_{\text{Born}} \). The second term denotes the Taylor (\( \mathcal{T} \)) expansion in powers of \( \alpha \) up to \( \alpha^N \) with the poles in \( s-M^2 \) defined in the sense of principal value (\( PV \)). Recall that the principal value may be defined by the relation

\[
P V \frac{1}{x^n} = (\frac{-1}{n-1}) \frac{d^n}{dx^n} \ln |x|,
\] (2.8)

where the derivatives are understood in the distributions sense, i.e. at an integration they should be switched to the weight function via formal integration by parts. The sum in (2.7) corrects the contributions of the \( PV \)-poles in the points where the \( PV \)-prescription operates. The \( \delta_n(\ldots) \) denotes the \( n \)th derivative of the \( \delta \)-function taken with a typical coefficient,

\[
\delta_n(x) \equiv (\frac{-1}{n}) \frac{d^n}{dx^n} \delta(x).
\] (2.9)

The highest degree of the derivatives of the \( \delta \)-functions in (2.7) is equal to the highest degree minus 1 of the \( PV \)-poles in the Taylor expansion. The coefficients \( c_n(\alpha) \) are defined so that to provide the asymptotic property for the expansion. In this way the contributions of \( \delta \)-functions in (2.7) mean finite counterterms. As for \( c_n(\alpha) \) themselves, they are the polynomials in \( \alpha \) with the exponents ranging from \( n \) to \( N \). The coefficients of the polynomials are defined by self-energy of the unstable particle and by its derivatives calculated on the mass shell. (The contributions of soft massless particles are considered regularized at this stage.)

For foreseeable applications, seemingly, the expansion within the NNLO approxi-
information will be sufficient. With this precision formula (2.7) takes the form
\[
\rho(s) = \delta(s-M^2) + \frac{M\Gamma_0}{\pi} \left\{ PV \frac{1}{(s-M^2)^2} - PV \frac{2\alpha \Re \Sigma_1(s)}{(s-M^2)^3} \right\} + \sum_{n=0}^{2} c_n(\alpha) \delta_n(s-M^2) + O(\alpha^3).
\] (2.10)

Here $\Sigma_1$ is the one-loop self-energy. The $k$-loop self-energy $\Sigma_k$ is defined through the formula
\[
\Sigma(s) = \alpha \Sigma_1(s) + \alpha^2 \Sigma_2(s) + \alpha^3 \Sigma_3(s) + \cdots.
\] (2.11)

The polynomials $c_n(\alpha)$ within the NNLO approximation actually have been calculated in [24], though have been presented not quite explicitly. Below we represent $c_n(\alpha)$ in completely expanded form in an arbitrary version of the above mentioned generalization of the OMS scheme. All these versions are characterized by the one-loop conditions $\Re \Sigma_1(M^2) = 0$, $\Re \Sigma'_1(M^2) = 0$. So we have
\[
c_0 = -\alpha \frac{I_2}{I_1} + \alpha^2 \left( \frac{I_2^2}{I_1} - \frac{I_3}{I_1} - \frac{1}{2} I_1 I''_1 + R_2 \frac{I''_1}{I_1} - R'_2 \right),
\] (2.12)
\[
c_1 = -\alpha^2 (I_1 I'_1 + R_2), \quad c_2 = -\alpha^2 I_1^2.
\]

Here $I_k = \Im \Sigma_k(M^2)$, $R_k = \Re \Sigma_k(M^2)$, and the primes mean the derivatives at $s = M^2$. The $R_2$ and $R'_2$ are determined by the renormalization conditions, too, though differently in different versions of the generalization of the OMS scheme [42]. In particular, in the OMS or “pole” scheme [42, 43], $R_2 = -I_1 I'_1$ and $R'_2 = -I_1 I''_1/2$.

So, we have determined the asymptotic expansion of isolated BW factors. Now we consider the product of BW factors. It is clear that in the case of smooth weight the asymptotic expansion of the product has the form of the formal product of the expansions of separately taken BW factors. However, this is not the case if the weight includes singularities which intersect with the singularities of the expanded BW factors. Unfortunately, this occurs in the case of integral (2.3) because of the soft massless-particles contributions and the singularity of the kinematic factor. In order to asymptotically expand, nevertheless, the integrand in (2.3), we should independently make definition of the expansion of the product of BW factors in the vicinities of intersections of singularities.

At first we discuss the case of the singularities caused by the soft massless-particles contributions. Recall that we symbolically expressed them through the factor $1 + \delta_c$ in formula (2.3). Actually these contributions become crucial when the soft massless-particles momenta merge with the momenta of the unstable particles (owing to the emission or absorption of massless particles) and when the unstable-particles momenta turn out to be on the mass shell. The asymptotic expansion in this case can be fulfilled if the additional singularities, intersecting with the singularities of the BW factor, are cancelled in the cross-section. In reality the cancellation takes place in the case of the ordinary IR-divergent contributions, but not in the case of Coulomb singularities. Therefore a solution to the problem of soft massless-particles contributions can exist in the case only when Coulomb singularities are somehow regularized.
Let us postpone the problem of the Coulomb singularities and concentrate on the ordinary IR-divergent contributions. A solution to this problem actually is proposed in [24]. It is based on the introduction of dimension regularization for the IR singularities and then on the calculation of additional counterterms at the intersections of singularities. Apart from the \(\delta\)-functions of virtualities of the unstable particles, these additional counterterms include also the \(\delta\)-functions of the massless-particles momenta. Moreover, the coefficients at these counterterms are found to be singular in the dimension-regularization parameter. However the later singularities are to be cancelled after the integration over the massless-particles momenta.\(^1\) In this way the counterterms become identical in form with those in the case without the contributions of soft massless particles (though they appear with modified coefficients). An alternative solution is based on the introduction of the IR regularization through inserting of a technical mass for massless particles. A remarkable property of this regularization is the absence in its framework of the above mentioned intersections of singularities [26]. By virtue of this fact the extra counterterms do not arise at all in this case. So, again, after the integration over the massless-particles momenta the structure of the expansion becomes the same as in the case without the contributions of massless particles. For this reason we further do not consider explicitly the contributions of soft massless particles except contributions that lead to Coulomb singularities.

Proceeding to the Coulomb singularities, we note first of all that they arise in a more restricted kinematic region and have different nature as compared to the ordinary IR-divergent contributions. So from the very beginning they can be isolated from the ordinary soft massless-particles contributions and thereupon can be separately investigated. By this means it was found that the Coulomb singularities arise on the mass shell and simultaneously in the limit of small relative velocities of outgoing massive particles. At the level of the cross-section they may be collected in the so-called Coulomb factor. For the first time this factor was discovered in the case of stable particles and in electrodynamics with the result [44, 45] (hereinafter we omit the ordinary IR-divergent contributions)

\[
1 + \delta_c = |\psi(0)|^2 = \frac{X}{1 - \exp(-X)}, \quad X = \frac{\kappa \alpha}{\beta_0}. \tag{2.13}
\]

Here \(\psi(0)\) is the wave function at the origin of the charged massive particles moving in the center-of-mass frame (c.m.f.) with velocity \(\beta_0, \kappa\) is a positive coefficient. (In the case with color particles in QCD the Coulomb factor is the same with the modification in the coefficient \(\kappa\) only [46, 47]). In the expanded form factor (2.13) can be represented with the aid of the formula

\[
\frac{X}{1 - \exp(-X)} = 1 + \frac{X}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{X^{2n}}{(2n)!}, \quad (2.14)
\]

\(^1\)The basic part of the calculation of the additional counterterms in the case of pair production and decay of unstable particles, at the level of the one-photon contributions, has been made by the author of this article in collaboration with F.Tkachov [25].
where $B_n$ are Bernoulli numbers.

Returning to the MPT expansion, we note that the $PV$-poles and the derivatives of the $\delta$-functions in the expansions of the BW factors are ill-determined in the presence of the Coulomb singularities. Really, in accordance with the integration rules of the $PV$-poles and the derivatives of the $\delta$-functions the Coulomb factor must be taken into consideration not only on-shell, but also with the derivatives with respect to virtualities of unstable particles. However the derivatives of the Coulomb singularities cannot be determined as they appear precisely on the mass shell.

In this connection further we use an extended representation of the Coulomb factor available in the unstable-particle case. Namely we mean a particular resummation of the singular and some kind of nonsingular Coulomb contributions, which should be made at a given number of exchanges by soft massless particles. The resummation should be made for the off-shell outgoing unstable particles and, simultaneously, with the Dyson resummation in those propagators of the unstable particles that are directly involved in the generation of the Coulomb singularities. The practical extraction of the mentioned contributions is a rather subtle task. For details we refer to [6]–[9], where the extraction is made both for one- and multi-photon contributions. Below we adduce the result with the one-photon contribution. In the notation of [8], but omitting some superfluous term [9] which is inessential for the validity of the ultimate result, we have

$$
\delta_c|\text{one-photon} = \frac{\kappa\alpha}{2\beta} \left[1 - \frac{2}{\pi} \arctan\left(\frac{|\beta_M|^2 - \beta^2}{2\beta \text{Im}\beta_M}\right)\right].
$$

(2.15)

Here $\beta = s^{-1}\sqrt{\lambda(s, s_1, s_2)}$ is the velocity of outgoing unstable particles in the center-of-mass frame, $\beta_M = \sqrt{1 - 4(M^2 - i\Gamma)/s}$, $\Gamma$ is the width of unstable particles. Notice that with nonzero $\Gamma$ the r.h.s. in (2.15) is a smooth function of $\beta$. At the same time at going to the mass-shell and subsequently taking the limit $\Gamma \to 0$, the r.h.s. becomes the $\kappa\alpha/(2\beta_0)$, which is the one-photon Coulomb singularity.

We emphasize that in fact only imaginary part of the on-shell self-energy may be involved in the above resummation. Therefore the regularized Coulomb factor may be made gauge-invariant in the OMS or “pole” scheme of the UV renormalization [42, 43], because in this scheme both the renormalized mass and the imaginary part of the on-shell self-energy are gauge invariant quantities. The gauge invariance of the regularized Coulomb factor, in turn, implies the gauge invariance of the piece of the cross-section that remains after the extraction of the Coulomb factor. So provided this remaining piece is completely expanded in powers of $\alpha$, the contributions at each power of $\alpha$ must possess the property of gauge cancellations. In the next section we show the existence of the complete expansion of the above mentioned remaining piece of the cross-section.

So we have gained the regularity property of the Coulomb factor and the possibility to differentiate it with respect to the virtualities of the unstable particles. In addition, we have a regularity of the derivatives of the Coulomb factor, too. The latter property most easily may be shown by noting that the regularized Coulomb factor

\footnote{The mentioned propagators belong to one-particle irreducible subdiagrams. Therefore they by no means are involved in the BW factors that are to be MPT-expanded.}
is an even function of $\beta$. In the multi-photon case this is evident from the appropriate explicit expression [9]. In the one-photon case this becomes clear if proceeding to the representation

$$\delta_c|_{\text{one-photon}} = \frac{\kappa \alpha}{\pi \beta} \arctan \left( \frac{2 \beta \text{Im} \beta M}{|\beta M|^2 - \beta^2} \right),$$

(2.16)

which is equivalent to (2.15) at small enough $\beta$. From the even property it follows that $\delta_c$ depends effectively on $\beta^2$. (In particular this is evident from the Taylor expansion.) This implies that both the $\delta_c$ and the derivatives of $\delta_c$ with respect to $s_i$ are smooth functions of $s_i$.

The property of smoothness of the regularized Coulomb factor and also of its derivatives with respect to the virtualities allows us to refer the Coulomb factor to the weight in formula (2.3). Correspondingly, at the further calculations we consider factor $1 + \delta_c$ as absorbed by function $\Phi$. In doing so, the $1 + \delta_c$ may be taken into consideration in the expanded form with respect to the number of exchanges by soft massless particles. Unfortunately, this is not the complete expansion since some dependence on $\alpha$ is involved in the parameter of the regularization. Nevertheless with the fixed parameter $\Gamma$, the expansion is an asymptotic one with respect to $\alpha$.

Finally, we turn to the kinematic factor $\theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \sqrt{\lambda(s, s_1, s_2)}$. As noted above, its singularity located on the point set $\sqrt{s} - \sqrt{s_1} - \sqrt{s_2} = 0$ can intersect with the singularities of the expanded BW factors. Most easily this may be shown by imposing conditions on $s$. So, at $\sqrt{s} = \sqrt{s_j} + M_i$ ($i \neq j$) the point set $\sqrt{s} - \sqrt{s_1} - \sqrt{s_2} = 0$ is reduced to the isolated point $\sqrt{s_i} = M_i$, where the singularity of one of the expanded BW factors is located. At $\sqrt{s} = M_1 + M_2$ the above point set is reduced to $\sqrt{s_1} + \sqrt{s_2} = M_1 + M_2$, where the singularities of both BW factors are simultaneously located.

At first sight this property makes senseless the rules of the integration of the $PV$-poles and of the derivatives of the $\delta$-functions, and hence the very possibility of the calculation of the asymptotic expansion of integral (2.3) becomes questionable. However, the distribution method in this case again allows us to add a sense to the integrals that arise in the course of the expansion. Moreover, the ambiguities which are peculiar to this method may be again removed by means of the imposing of the asymptotic condition of the expansion. Furthermore, both these requirements are automatically satisfied by application of the analytical regularization of the kinematic factor. In the next section we proceed to a systematic discussion of this issue.

### 3 The scheme of calculation of $\hat{\sigma}(s)$

First of all we reformulate the problem in dimensionless variables. For this purpose we introduce dimensionless energy variables $x, x_1, x_2$ counted off from thresholds,

$$\sqrt{s} = 2M + \frac{M}{2} x, \quad \sqrt{s_i} = M_i + \frac{M}{2} x_i.$$

(3.1)

Here $i = 1, 2$, and $M = (M_1 + M_2)/2$. Then formula (2.3) takes the form

$$\hat{\sigma}(x) = \int \int dx_1 dx_2 \theta(x_1 + a_1)\theta(x_2 + a_2)\theta(x - x_1 - x_2)\sqrt{x - x_1 - x_2}$$

(3.2)
Here \( \tilde{\sigma}(x) = M_i M_j \tilde{\sigma}(s), \tilde{\rho}_i(x_i) = M M_i \rho_i(s_i), \) and \( a_i = 2(M_i - \sqrt{s_{\text{min}}})/M. \) At once we notice that the parameters \( a_i \) are strictly positive at \( M_i^2 > s_{\text{min}} \) and that in any case \( a_1 + a_2 < 4. \) Notice also that in view of the \( \theta \)-functions, the \( \tilde{\sigma}(x) \) is effectively proportional to \( \theta(x+a_1+a_2). \) The \( 1 + \delta_i \) in (3.2) is equivalent to within the change of variables to \( 1 + \delta_c \) in (2.3). Functions \( \tilde{\Phi} \) and \( \Phi \) are related each other by the formula:

\[
\tilde{\Phi}(x_1, x_2) = \frac{1}{M^2} ds_1 ds_2 \frac{1}{s^2} \sqrt{\lambda(s, s_1, s_2)} \Phi(s, s_1, s_2). \tag{3.3}
\]

Since the factors \( \sqrt{x-x_1-x_2} \) and \( \sqrt{\lambda(s, s_1, s_2)} \) have identical analytical properties in the integration area, the analytical properties of \( \tilde{\Phi} \) and \( \Phi \) are identical in this area, too. Let us remember that by the construction \( \Phi \) is smooth in the vicinity of physical thresholds associated with the production and decay of unstable particles. So \( \Phi(s, s_1, s_2) \) has no singularities in the points \( s = 2M, s_i = M_i, s - \sqrt{s_{\text{min}}} = 0, \) \( s - s_{\text{min}} = 0, \) \( s - s_{\text{min}} - M_j = 0, \) \( s - \sqrt{s_{\text{min}}} - s_j = 0, \) \( s - \sqrt{s_i} - M_j = 0, \) \( s - \sqrt{s_i} - s_j = 0 \) (\( i \neq j \)). Correspondingly, \( \tilde{\Phi}(x_1, x_2) \) has no singularities in the points \( x = 0, x_i = 0, x + a = 0, x + a_i = 0, x + a_i - x_j = 0, x - x_i = 0, x - x_1 - x_2 = 0. \)

The asymptotic expansion of isolated BW factors \( \tilde{\rho}_i(x_i) \) most easily may be obtained by means of change of variables in formula (2.7). So, let us rewrite the second relation in (3.1) in the form \( s_i - M_i^2 = M M_i (1 + x_i M/(4M_i)) \), and note that \( [1 + x_i M/(4M_i)] \neq 0 \) everywhere in the integration area. This implies that the singular point \( s_i - M_i^2 = 0 \) transforms into the point \( x_i = 0 \) in new variables. So the \( \delta(s_i - M_i^2) \) becomes the \( (M M_i)^{-1} \delta(x_i) \), while the derivatives of \( \delta(s_i - M_i^2) \) transforms into the sum of \( \delta(x_i) \) and its derivatives. The final formula valid within the integration area is derived in Appendix A,

\[
\delta_n(s_i - M_i^2) = \frac{1}{(M M_i)^{n+1}} \sum_{k=0}^{n} \binom{n+k}{n} \left( -\frac{M}{4M_i} \right)^k \delta_{n-k}(x_i). \tag{3.4}
\]

Recall that the \( \delta_n \) means the \( n \)-th derivative of the \( \delta \)-function with a typical coefficient, cf. (2.9). As long as the PV-regularization is a canonical one \([30]\), the change of variables in the PV-poles is made as in conventional functions, i.e. without the adding of a sum of the \( \delta \)-function and its derivatives:

\[
\text{PV} \frac{1}{(s_i - M_i^2)^n} = \frac{1}{(M M_i)^n} \left( 1 + \frac{M}{4M_i} x_i \right)^{-n} \text{PV} \frac{1}{x_i^n}. \tag{3.5}
\]

Substituting (3.4) and (3.5) into (2.7), we get the asymptotic expansion of \( \tilde{\rho}_i(x_i) \).

Unfortunately, the presence of the kinematic factor \( \theta(x-x_1-x_2) \sqrt{x-x_1-x_2} \) does not allow us to substitute the expansions for \( \tilde{\rho}_i(x_i) \) in formula (3.2), because the derivatives of the kinematic factor are non-integrable, or even completely undetermined in the point \( \{x-x_i = 0, x_j = 0\}; \ i \neq j \). Nevertheless the situation changes if to proceed to the analytically regularized kinematic factor, namely to
\[ \theta(x-x_1-x_2) \left( x-x_1-x_2 \right)^\lambda \] with non-integer \( \lambda \). With large enough \( \lambda \) this gives a regularity property for the derivatives. As a result, after the substitution of the expansions for \( \tilde{\rho}_1(x_i) \), the integral (3.2) considered as iterated becomes calculable. Further, after the integral is calculated, we have to take the limit \( \lambda \to 1/2 \) in order to match the original integral. So the problem is reduced to the taking of the limit of the already calculated integral. Notice that the above procedure means the treatment of the kinematic factor as the distribution \( (x-x_1-x_2)^{1/2} \) [30].

Fortunately, if the above mentioned limit exists then this guarantees the asymptotic property of the expansion of the whole of integral (3.2). This follows, first of all, from the fact that prior to any expansion the substitution of the distribution \( (x-x_1-x_2)^{1/2} \) for the kinematic factor does not mean a modification of the integral. This, in particular, means that the asymptotic expansion after the substitution is a completely rightful operation. Technically the expansion may be carried out through the proceeding at intermediate stages of calculations to the large enough \( \lambda \). (The infimum of \( \lambda \) is dependent on the order up to which the expansion is to be made.)

This allows us to treat the regularized kinematic factor as a smooth weight and on this basis to asymptotically expand the BW factors. The integrals that will appear after the expansion by our construction will be calculable. Further, the analyticity will allow us to vary \( \lambda \) with keeping the asymptotic property of the expansion. The existence of the limit at each term of the expansion will imply the implementation of asymptotic property of the expansion.

Now let us proceed to the calculation of the above mentioned integrals at arbitrary but large enough \( \lambda \). For this purpose, we consider any one term of the resulting expansion of the integrand in (3.2) that appears after the formal expansion of the BW factors. Let this term include \( PV x_1^{-n_1} \) or \( \delta_{n_1-1}(x_1) \) from one of the BW factors and \( PV x_2^{-n_2} \) or \( \delta_{n_2-1}(x_2) \) from another BW factor \( (n_{1,2} \geq 1) \). Recall that we consider the kinematic factor in the form \( (x-x_1-x_2)^\lambda \) having in mind that at the intermediate stage the \( \lambda \) is large enough. The \( \theta(x_1+a_1) \theta(x_2+a_2) \), we consider as a potentially singular factor, as these \( \theta \)-functions via the integration by parts can generate non-integrable singularities at \( \lambda = 1/2 \). As a pure weight we consider the function \( \tilde{\Phi}(x ; x_1, x_2) \) multiplied by the regularized Coulomb factor and by the definite regular coefficients that arise in the course of application of the formulas (2.7), (3.4) and (3.5). For brevity we denote the mentioned weight by the same symbol \( \tilde{\Phi}(x ; x_1, x_2) \).

Further, let us subtract and add to \( \tilde{\Phi} \) the \( n_1 \times n_2 \) of the first terms of its Taylor expansion in powers of \( x_1 \) and \( x_2 \),

\[
T_{n_1n_2} \tilde{\Phi}(x ; x_1, x_2) = \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} {\binom{n_1-1}{k_1}} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \tilde{\Phi}(k_1, k_2)(x ; 0, 0). \tag{3.6}
\]

Then \( \tilde{\Phi} \) takes the form of a sum of two terms, the first one is (3.6) and the second

\(^3\)Let us remark that \( (x-x_1-x_2)^\lambda \) instead of \( \sqrt{x-x_1-x_2} \) arises automatically if the phase-volume integration is carried out at the space dimension \( d = 4 + 2\varepsilon \), \( \varepsilon = \lambda - 1/2 \).
one is the difference $\Delta \tilde{\Phi} = \tilde{\Phi} - T \{ \tilde{\Phi} \}$. The difference, we represent in the form

$$\Delta \tilde{\Phi}(x; x_1, x_2) = \sum_{k_1=0}^{n_1-1} \frac{x_1^{k_1}}{k_1!} \left[ x_2^{n_2} \varphi_{n_2}^{(k_1)}(x, x_2) \right] + \sum_{k_2=0}^{n_2-1} \frac{x_2^{k_2}}{k_2!} \left[ x_1^{n_1} \varphi_{n_1}^{(k_2)}(x, x_1) \right]$$

$$+ x_1^{n_1} x_2^{n_2} \varphi(x; x_1, x_2). \quad (3.7)$$

Here the first sum represents the Taylor expansion up to $x_1^{n_1-1}$ of the remainder of the Taylor expansion of $\tilde{\Phi}(x; x_1, x_2)$ in $x_2$ up to $x_2^{n_2-1}$. The second sum has the same meaning with the replacement $x_1 \leftrightarrow x_2$. The third term has the meaning of the common remainder. Note that by the construction the functions $\varphi_{n_i}^{(k_i)}(x, x_j)$ and $\varphi(x; x_1, x_2)$ are regular in some neighborhoods of the points $x_1 = 0$, $x_2 = 0$.

It is clear that $T \{ \tilde{\Phi} \}$ leads to singular contributions in the presence of the $PV$-poles and the $\delta$-functions. However, thanks to its power-like dependence on $x_1$ and $x_2$ the integral (3.2) of these contributions can be analytically calculated. Namely, owing to the relations

$$(y - x)^\lambda \, x^k \, \delta_n(x) = (y - x)^\lambda \, \delta_{n-k}(x), \quad 0 \leq k \leq n, \quad (3.8)$$

$$(y - x)^\lambda \, x^k \, PV \frac{1}{x^n} = (y - x)^\lambda \, PV \frac{1}{x^{n-k}}, \quad 0 \leq k < n, \quad (3.9)$$

which are valid in the framework of the analytical regularization, the mentioned contributions to (3.2) are reduced to linear combinations of the basic integrals

$$A_{k_1 k_2}(x) = \int \int dx_1 \, dx_2 \, \theta(x_1 + a_1) \theta(x_2 + a_2) \, (x - x_1 - x_2)^\lambda \, \delta_{k_1-1}(x_1) \, \delta_{k_2-1}(x_2), \quad (3.10)$$

$$B_{k_1 k_2}(x) = \int \int dx_1 \, dx_2 \, \theta(x_1 + a_1) \theta(x_2 + a_2) \, (x - x_1 - x_2)^\lambda \, PV \frac{1}{x_1} \, \delta_{k_2-1}(x_2), \quad (3.11)$$

$$C_{k_1 k_2}(x) = \int \int dx_1 \, dx_2 \, \theta(x_1 + a_1) \theta(x_2 + a_2) \, (x - x_1 - x_2)^\lambda \, PV \frac{1}{x_1^k} \, PV \frac{1}{x_2^k}. \quad (3.12)$$

Here subscripts $k_i$ range over $1 \leq k_i \leq n_i$. Notice that among the $B$-type integrals that include both the $PV$-pole and the $\delta$-function, there are integrals with the opposite order of $PV$ and $\delta$. Let us denote these integrals by $\overline{B}_{k_1 k_2}^\lambda$. We do not write down them explicitly as they may be obtained immediately by substitutions $\{k_1, a_1\} \leftrightarrow \{k_2, a_2\}$ in formula (3.11).

The analytical calculation of integrals (3.10)–(3.12) is carried out in Appendix B. In the general case the outcomes have the form of a sum of regular and singular at $\lambda = 1/2$ contributions. The singular contributions are defined as the power-like distributions of the type $x_1^\lambda$ multiplied by some regular factors, see Table 1. The calculation of the observable cross-section in the presence of the singular distributions is considered in the next section. Here we remark only that with a complicated weight the integral of the distribution $x_1^\lambda$ cannot be analytically calculated, but can be calculated numerically with the aid of formula [30]

$$\int dx \, x_1^\lambda \psi(x) = \int_0^\infty dx \, x^\lambda \left\{ \psi(x) - \sum_{k=0}^{K-1} \frac{x^k}{k!} \psi^{(k)}(0) \right\}. \quad (3.13)$$
Table 1: The singular contributions to $\tilde{\sigma}(x)$ arising via the two-fold basic integrals ($r$ is a non-negative integer such that $5/2 - n_2 + r < 0$).

| $A_{n_1 n_2}^{1/2}$ | $x_+^{5/2-n_1-n_2}$ |
|----------------------|----------------------|
| $B_{n_1 n_2}^{1/2}$ | $(-x)_+^{5/2-n_1-n_2}$ | $(x + a_1)_+^{5/2-n_2+r}$ |
| $C_{n_1 n_2}^{1/2}$ | $x_+^{5/2-n_1-n_2}$ |

Here $\lambda < -1$ and $\lambda$ is non-integer, $K$ is a positive integer such that $-K-1 < \lambda < -K$, $\psi(x)$ is a test function which must be smooth enough in the vicinity of $x = 0$.

In the difference term $\Delta \tilde{\Phi}(x; x_1, x_2)$, we consider separately the contributions of the sums and of the common remainder. In the common remainder in view of the relations

$$x^n \delta_{n-1}(x) = 0, \quad x^n \ PV x^{-n} = 1,$$

the multipliers $x_1^{n_1}$ and $x_2^{n_2}$ make vanish contributions of the $\delta$-functions and cancel the $PV$-poles. In order to calculate the remaining integral, we proceed to the cone variables $x_1 + x_2 = \xi$, $x_1 - x_2 = 2\eta$ and make a shift $\xi \to x - \xi$. After the calculation of the integral $d\eta$, the relevant contribution to (3.2) takes the form of

$$\int d\xi \ \theta(x + a_1 + a_2 - \xi) \xi^\lambda \varphi(x, \xi).$$

Here $\varphi(x, \xi)$ is a regular function that arises after the calculation of the integral $d\eta$. In the case $\lambda = 1/2$ integral (3.15) is absolutely convergent and therefore can be numerically calculated.

In the sums in $\Delta \tilde{\Phi}(x; x_1, x_2)$, the multiplier $x_j^{n_j}$ makes vanish contributions of $\delta_{n_j-1}(x_j)$ and cancels $PV x_j^{-n_j}$. The contributions of $\delta_{n_j-1}(x_i)$ and $PV x_i^{-n_i}$ remain not touched upon, but the dependence on $x_i$ is simpler. Therefore at first we calculate the integral $dx_i$. In view of (3.8) and (3.9), it is reduced to the sum of the basic integrals ($1 \leq k \leq n_i$)

$$\Gamma_k^\lambda(x - x_j) = \int dx_i \ \theta(x_i + a_i) (x_i - x_j)^\lambda \delta_{k-1}(x_i),$$

$$\Theta_k^\lambda(x - x_j) = \int dx_i \ \theta(x_i + a_i) (x_i - x_j)^\lambda PV \frac{1}{x_i^k}.$$

The analytical calculation of these integrals is carried out in Appendix B, as well. Having analytical expressions for $\Gamma_k^\lambda$ and $\Theta_k^\lambda$, we can calculate the remaining integral $dx_j$. In the general case it is reduced to the sum of integrals of the kind

$$\int dx_j \ \theta(x_j + a_j) \varphi_n^{(k)}(x, x_j) P_k^\lambda(x - x_j).$$
Here $P_k^\lambda$ stands for $I_k^\lambda$ or $J_k^\lambda$ and $\varphi_{n_j}^{(k_i)}(x, x_j)$ is introduced in (3.7). By making the change of variable $x - x_j \rightarrow \xi$, we rewrite (3.18) in a more convenient form,

$$
\int d\xi \, \theta(x + a_j - \xi) \varphi_{n_j}^{(k_i)}(x, x - \xi) P_k^\lambda(\xi). \tag{3.19}
$$

At first we consider the case $P_k^\lambda = I_k^\lambda$. In view of $I_k^\lambda(\xi) \sim \xi^{1+\lambda-k}$, integral (3.19) is absolutely convergent at $\lambda > k - 2$. At $\lambda < k - 2$ and $x + a_j \neq 0$ the integral can be calculated with the aid of (3.13). The condition $x + a_j \neq 0$ is necessary for the providing of smoothness for the weight at $\xi = 0$. As $x + a_j \rightarrow 0$, integral (3.19) tends to infinity. Nevertheless, the singular contribution can be analytically calculated and, simultaneously, the regular background can be numerically calculated. Namely we represent $\varphi_{n_j}^{(k_i)}(x, x - \xi)$ in the form of the Taylor expansion with a remainder,

$$
\varphi_{n_j}^{(k_i)}(x, x - \xi) = \sum_{r=0}^{K-1} \frac{\xi^r}{r!} \varphi_{n_j}^{(k_i, r)}(x, x) + \Delta \varphi_{n_j}^{(k_i)}(x, x - \xi). \tag{3.20}
$$

Here $-K-1 < 1+\lambda-k < -K$, and $\varphi_{n_j}^{(k_i, r)}(x, x) = \partial^r / \partial \xi^r \varphi_{n_j}^{(k_i)}(x, x - \xi)|_{\xi=0}$. Owing to $\Delta \varphi_{n_j}^{(k_i)}(x, x - \xi) = O(\xi^K)$, the integral with $\Delta \varphi_{n_j}^{(k_i)}$ is absolutely convergent, and this contribution can be numerically calculated. The integral of the sum in (3.20) gives

$$
\sum_{r=0}^{K-1} \frac{(x + a_j)^{2+\lambda-k+r}}{2+\lambda-k+r} \frac{\varphi_{n_j}^{(k_i, r)}(x, x)}{r!}. \tag{3.21}
$$

Here $2+\lambda-k+r < 0$, and we note the distribution sense of the singular contributions. Up to a coefficient, formula (3.21) determines the appropriate singular contributions to $\tilde{\sigma}(x)$.

In the case $P_k^\lambda = J_k^\lambda$, we have generally two singularities in $P_k^\lambda(\xi)$, the $(-\xi)^{1+\lambda-k}$ and $\xi^{1+\lambda-k}$, cf. (B14). However, the latter singularity does not make contributions at $\lambda = 1/2$ in view of the factor $\cos(\pi \lambda)$. To calculate the contribution of the former singularity, we do the change of variable $\xi \rightarrow -\xi$ and take advantage of the relation $\theta(x + a_j + \xi) = 1 - \theta(-x - a_j - \xi)$. It is clear that the unity gives regular contribution. The $\theta(-x - a_j - \xi)$ gives a regular contribution outside of a neighborhood of $x + a_j = 0$. Inside the neighborhood, by complete analogy with the previous case, it gives a sum of the regular contribution and the power-like singular contributions $(-x - a_j)^{2+\lambda-k+r}$. The singularities arising in both cases are shown in Table 2.
Thus, we have reduced integral (3.2) to the sum of regular and singular contributions. The regular contributions are described by the absolutely convergent integrals and thereupon can be numerically calculated. The singular contributions are represented by the power-like distributions with regular weights. So their further integration can be carried out with the aid of formula (3.13). It is worth noticing that the singularities that remain at the level of hard-scattering cross-section \( \tilde{\sigma}(x) \) are located on the physical thresholds \( x = 0 \) and \( x = -a_i \).

So, we have discussed the piece of the hard-scattering cross-section that initially possessed the double-resonant structure. However, this piece includes not only the double-resonant contributions but through the difference term (3.7) also the single-resonant and the non-resonant contributions. Now we consider the piece of the cross-section which initially possesses the single-resonant structure and which is originated by the integral of the type (3.2) without one of the BW factors. The kinematic factor in this case is present as before, but one of its variables, say, \( x_j \) describes immediately the invariant mass of the appropriate particles in the final state. On the contrary, the \( x_i \) variable describes first of all the virtuality of the unstable particle and only then the invariant mass of the appropriate stable particles. In this case we do at first the MPT expansion of the BW factor which depends on \( x_i \) variable. Then the weight at each term of the resulting expansion, we represent in the form of a finite power series in \( x_i \) with a remainder giving regular contribution. The latter contribution may be reduced to an integral of the type (3.15), which is to be numerically calculated. The integral \( dx_i \) of each term of the finite power series, we reduce to the sum of the basic integrals (3.16) and (3.17). After that the integrals \( dx_j \) we reduce to the sums of the integrals of the type (3.18). So we get again the sum of the regular and singular contributions with the singular contributions represented by the power-like distributions represented in Table 2. Finally, the piece of the cross-section that initially possesses the non-resonant structure, we consider in the conventional fashion.

Now let us consider the interference contributions. In the case of the interference between the contributions with the single-resonant and the double-resonant structure, in the cross-section the non-regular factors \((x_j \pm i0)^{-k}\) appear instead of one of the BW factors \((k \text{ is a positive integer})\). This factor is reduced via Sokhotsky formula to the sum of \( PV(x_j)^{-k} \) and \( \delta_{k-1}(x_j) \). Thus we obtain contributions that are similar to those which have been already discussed in the case with the double-resonant structure. The other cases of the interference are considered by similar fashion.

### 4 The observable cross-section

Having calculated the hard-scattering cross-section \( \tilde{\sigma} \), we proceed to the calculation of the convolution integral (2.1). In what follows we concentrate on the singular contributions only. (Recall that the regular contributions can be numerically calculated after the complete definition of the weights.) In the dimensionless variables integral (2.1) takes the form

\[
\sigma(s) = \frac{M^2}{2M_1M_2 s} \int dx' \left( 4 + x' \right) \theta(x' - x_{\min}) \tilde{\phi}(x', x) \tilde{\sigma}(x'). \tag{4.1}
\]
Here $\sqrt{s} = 2M (1 + x/4)$, $\sqrt{s'} = 2M (1 + x'/4)$, $\sqrt{s_{\text{min}}} = 2M (1 + x\text{_{min}}/4)$. The $\tilde{\phi}(x', x)$ is the flux function,

$$
\tilde{\phi}(x', x) = \beta_ε (4 + x)^{2(1-\beta_ε)} (8 + x + x')^{\beta_ε - 1} (x - x')^{\beta_ε - 1} - \frac{1}{2} \beta_ε \left[ 1 + \left( \frac{4 + x'}{4 + x} \right)^2 \right] \theta(x - x') \tag{4.2}
$$

In view of $\tilde{\sigma}(x') \sim \theta(x' + a_1 + a_2)$ and $a_1 + a_2 < 4$, the integration area in (4.1) is restricted so that $x' > -4$. Consequently only the factors $(x - x')^{\beta_ε - 1}$ and $\theta(x - x')$ are unsmooth in the integration area. Owing to $\beta_ε > 0$ both they are integrable.

Further, we consider separately two cases, the first one when the value of external variable $x$ is outside of the physical thresholds $x = 0$ and $x = -a_i$, and the second one when $x$ falls precisely on one of the thresholds. In the former case the singularities of $(x - x')^{\beta_ε - 1}$ and $\theta(x - x')$ do not intersect with the singularities of $\tilde{\sigma}(x')$. So all singularities are isolated and integral (4.1) can be numerically calculated with the aid of formula (3.13). In the case when $x$ coincides with one of the thresholds, the singularities intersect. However, owing to the relations $(\pm x)^{\gamma_1}_+ (\mp x)^{\gamma_2}_+ = 0$ and $(\pm x)^{\lambda_1}_+ (\pm x)^{\lambda_2}_+ = (\pm x)^{\lambda_1 + \lambda_2}_+$ valid in the framework of the analytical regularization, we get either zero contribution or a single power-like distribution instead of the product of singular distributions. So integral (4.1) can again be numerically calculated by means of formula (3.13).

Unfortunately, in the vicinity of the thresholds the behavior of the convolution integral is greatly unsmooth by the above procedure, and therefore the above description cannot be considered satisfactory. Namely the integral (4.1) is unboundedly increasing with $s$ approaching the thresholds (although remaining finite precisely on the thresholds). For definiteness let us consider the case $x \to 0$, which means $s \to 4M^2$ in dimensional variables. In this case the dangerous singularities in $\tilde{\sigma}(x')$ are $(\pm x')^{\gamma}_+$, where $\gamma$ is half-integer, because with $x \to 0$ the position of singularity of $(x - x')^{\nu}_+$, where $\nu$ is either $\beta_ε - 1$ or 0, is approaching that of $(\pm x')^{\gamma}_+$. At $\gamma + \nu < -1$ this implies the increasing of the integral as $|x|^{1+\gamma+\nu}$. The similar behavior takes place in the vicinities of the thresholds $x = -a_i$, or $s = (\sqrt{s_{\text{min}}} + M_j)^2$ in dimensional variables.

In effect the mentioned behavior means a rapid divergence of the asymptotic series when $s$ is approaching the thresholds, and thus the expansion near thresholds becomes senseless. (Actually this behavior have already been detected in [28] where also it has been revealed that the area of bad behavior of the cross-section is rather small.) Nevertheless, the situation may be corrected by means of doing an additional expansion of the cross-section $\sigma(s)$ near thresholds, namely the expansion in powers of the distance between $s$ and the appropriate threshold. For definiteness let us consider the physically important case when $s$ is close to but a little bit lower of the threshold $s = 4M^2$. In the dimensionless variables this means $x \lesssim 0$. In this case in

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4In the strict sense, the factor $\theta(x' - x_{\text{min}})$ in (4.1) is unsmooth, too. However if $x_{\text{min}}$ is the minimal kinematically-allowed energy, this factor already is present in $\tilde{\sigma}(x')$ and therefore can be omitted in formula (4.1). In the case $x_{\text{min}} > -(a_1 + a_2)$, we suppose that $x_{\text{min}}$ does not fall on the physical thresholds.
the singularity \((-x')^\gamma_+\) is crucial. After changing the variable \(x' \rightarrow x - \xi\), the corresponding contribution to \(\sigma(s)\) takes the form

\[
\sigma(s) \sim s^{-1} \int d\xi \psi(\xi, x) \xi^\nu_+ (\xi - x)^\gamma_+.
\] (4.3)

Here \(\psi(\xi, x)\) is a regular contribution. Taking Taylor of \((\xi - x)^\gamma_+\) in powers of \(x\) (there is no necessity to expand also the weight) gives the series

\[
\sum_{r=0,1,\ldots} \frac{(-x)^r}{r!} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - r)} \xi^\nu_+ \xi^{-\gamma - r} \] (4.4)

instead of \(\xi^\nu_+ (\xi - x)^\gamma_+\). Since the weight is regular in the vicinity of \(\xi = 0\), the integral of each term of this series can be numerically calculated. Truncating the infinite series, we obtain an asymptotic estimate of \(\sigma\) at small \(x\). Notice that this procedure provides a continuity of the result at deviating \(s\) from threshold. The number of terms in the truncated series should be chosen depending on the value of the deviation of \(s\) and of the current order of the expansion in \(\alpha\). Ultimately this number may be determined by the joining of the solutions outside and inside the interval where the second expansion of \(\sigma(s)\) is made.

So, outside of the physical thresholds the asymptotic expansion of the cross-section can be numerically calculated with the aid of formula (3.13). Near thresholds one should do the two-fold expansion, one in powers of \(\alpha\) and the other in powers of the distance between \(s\) and the appropriate threshold.

## 5 Calculations up to NNLO

In this section we discuss the characteristic features of the application of the MPT within the NNLO approximation. For methodological reasons we consider the expansion in the coupling constant in a sequential manner. By the first we consider the expansion of the BW factors, which by our definition let constitute the MPT expansion in the narrow sense. From this point it is convenient to trace more carefully the field of subsequent calculations. Then we consider the expansion of the rest of contributions to the cross-section.

First of all we consider contributions that possess the double-resonant structure. In this case the LO in the narrow sense, in accordance with (2.10), is determined by the substitution of the \(\delta\)-functions (without the derivatives) for the BW factors in formula (3.2). Correspondingly, the \(\tilde{\Phi}\) must be determined by the on-shell amplitude multiplied by the branching factors, both calculated as a whole with the two-loop precision. The soft massless-particles contributions are to be taken into consideration by conventional fashion. In particular, at calculating the amplitude the Coulomb singularities are to be extracted in favor of the factor in the cross-section. However, contrary to the wide belief, this factor can be taken into consideration not in the form (2.13) only. In fact, it can be taken into consideration in the expanded form, as well,
though with certain modifications in subsequent calculations. Up to the NNLO three terms of the expansion are needed only, so we put

\[ 1 + \tilde{\delta}_c = 1 + \frac{\kappa \alpha}{2\beta_0} + \frac{1}{12} \left( \frac{\kappa \alpha}{\beta_0} \right)^2. \]  

(5.1)

The subsequent calculations can be carried out with replacing \( \beta_0 \) by \( \beta \) in the above formula, where \( \beta \sim \sqrt{x-x_1-x_2} \) is the off-shell velocity in the c.m.f. Since the process in the end is considered on-shell, this replacement should not affect the ultimate result. However at the intermediate stage of calculations it implies an effective modification of the kinematic factor. At the calculation of the hard-scattering cross-section, the second and the third term of (5.1) manifest themselves through the basic integrals \( A_0^1 \) and \( A_{11}^{-1/2} \). From (B17) we have \( A_0^1(x) = \theta(x) \) and \( A_{11}^{-1/2}(x) = x_{+}^{-1/2} \). The convolution integral of such contributions without problems can be calculated.

An alternative mode of taking into consideration Coulomb singularities has been discussed in Sect. 2. It implies a regularization of the inverse powers of \( \beta_0 \) in (5.1) by means of the special resummation. As applied to the second term in (5.1) this means a substitution

\[ \frac{\kappa \alpha}{2\beta_0} \rightarrow \text{r.h.s. of (2.15)}. \]  

(5.2)

The substitution of this kind is inevitable at the calculation in the NLO of the MPT in the narrow sense and in the higher orders, because in this case not only the on-shell contributions of the Coulomb factor must be taken into account but its derivatives, too. Fortunately, in the NLO in the narrow sense, but remaining within the NNLO in the general sense, it is sufficient to take into consideration only the one-photon contributions to the Coulomb factor, which is given by formula (2.15). Simultaneously, the weight \( \tilde{\Phi} \) must be taken into consideration off-shell and within the one-loop precision. The soft massless-particles contributions that are not involved in the Coulomb factor can be taken into consideration by means of technique discussed in [25].

In the NNLO in the narrow sense the weight \( \tilde{\Phi} \) should be taken off-shell in the Born approximation. The contributions of soft massless particles are absent in this approximation. Nevertheless, as long as the Coulomb factor makes considerable contributions in the threshold region, its contribution in the advance can be taken into account via formula (2.15).

Further, we discuss the contributions that have single-resonant structure. At the amplitude level they arise either due to the process of inclusive single production of one of the unstable particles, or due to non-factorizable corrections in the mode of pair production. To the cross-section such processes can contribute in the pure form without the interference, or in the interference with the process of pair-production. Within the NNLO in the general sense the mentioned contributions can appear in the LO and NLO of the MPT in the narrow sense, and with the weight calculated within the NLO of the conventional PT and in the Born approximation, respectively. The unconventional counting of the orders is explained by an occurrence of an additional power in \( \alpha \) when one of the unstable-particle propagators squared disappears in the cross-section; cf. (2.5) and (2.7), and see discussion in [26, 27]. For the same
reason the interference between the non-resonant process and the single- or double-
resonant process contributes to the NNLO in the general sense when the processes
are calculated in the Born approximation.

After the orders of the calculations are determined and the corresponding weights
are calculated, the further calculations are to be made in accordance with the scheme
of Sects. 3 and 4. Here we notice only that in the NLO in the general sense the
following basic integrals are required:

\[ \frac{1}{2} I_1(x), A_{\frac{1}{2} \frac{1}{2} 1 1}, B_{\frac{1}{2} \frac{1}{2} 1 1}, B_{\frac{1}{2} \frac{1}{2} 2 1}, B_{\frac{1}{2} \frac{1}{2} 1 2}. \]

In the
NNLO the additionally required basic integrals are

\[ \frac{1}{2} J_1(x), J_{\frac{1}{2} 2}(x), A_{\frac{1}{2} \frac{1}{2} 2 1}, A_{\frac{1}{2} \frac{1}{2} 1 2}, A_{\frac{3}{2} \frac{1}{2} 1 3}, A_{\frac{3}{2} \frac{1}{2} 1 3}, C_{\frac{1}{2} \frac{2}{2} 1 1}, C_{\frac{2}{2} \frac{1}{2} 1 1}, C_{\frac{2}{2} \frac{1}{2} 2 1}, C_{\frac{2}{2} \frac{1}{2} 2 1}. \]

The explicit expressions for them follow immediately
from the appropriate formulas of Appendix B. The distribution sense is strongly
required of the basic integrals \( A_{\frac{1}{2} \frac{1}{2}}, A_{\frac{1}{2} \frac{1}{2}}, \) and \( C_{\frac{1}{2} \frac{1}{2}} \).

6 Discussion and conclusion

We have constructed an asymptotic expansion in powers of the coupling constant \( \alpha \)
of the cross-section for pair production and decay of fundamental unstable particles.
The algorithm of the calculation of the coefficient functions of the expansion is elab-
orated in an arbitrary order. The outcomes are presented in the form of absolutely
convergent integrals with precisely definite structure. Certain components of the inte-
grals correspond to singular contributions in the amplitude and such components are
process-independent. The MPT is intended for handling these singular contributions
at the level of the cross-section, while the regular contributions are to be determined
by means of conventional PT. The key moment of the MPT is the interpretation of
the singular contributions in the distributions sense, so that they become integrable.
In doing so, the asymptotic property of the expansion is strictly maintained. The
exception is made for the Coulomb singularities, which are considered extracted into
a factor and represented in the regularized form in the sense of conventional func-
tions. However the regularization implies no insertion of extrinsic parameter but
special resummation of intrinsic contributions [6]–[9] So the asymptotic property of
the expansion in full measure is maintained.

In effect, the difference between the conventional PT and the MPT appears at
calculating the cross-sections and becomes apparent at taking into consideration the
resonant contributions of the unstable particles. At comparing the MPT with the
other approaches intended for the description of the production and decay of unsta-
ble particles, an essential moment distinguishing the MPT is the complete expansion
in powers of the coupling constant \( \alpha \) of the resonant contributions of the unstable
particles. In view of this property one can expect the gauge cancellations at calculating
the cross-section in the framework of the MPT approach.

The MPT method in the equal measure is applicable for the calculation of the
total cross-sections, the angular distributions, and spin correlations. This follows
immediately from the fact that the angular and spin variables are not involved in
the BW factors. The calculation of the invariant-mass distributions is workable in
the MPT, too, but in conjunction with some additional operations. Namely since
the BW factors in the MPT do not have the sense of conventional functions, an
additional integration of the BW factors with some weight is necessary. In any case the integration should be made in view of the jets and the contributions of unregistered photons in the final state. Unfortunately, the convergence property of the MPT becomes worse at the proceeding to the invariant-mass distributions. For this reason we do not suggest the use of the MPT in this case, at least in the considered above mode. For similar reason we do not suggest the use of this mode of the MPT for the description of a generic $2 \to 2$ process with a single resonance. Instead in these cases it is better to use a modification of the MPT approach discussed in [26, 27], which is based on the secondary Dyson resummation in the framework of MPT.

The question of the great importance in the case of pair production of unstable particles is the near-threshold behavior of the cross-section. Fortunately, this behavior can be calculated in the MPT framework, as well, though through the double expansion of the cross-section. Namely apart from the ordinary MPT expansion in powers of $\alpha$, one should do expansion in powers of the distance between $s$ and the appropriate threshold. The necessary number of terms that must be retained in the latter expansion depends on the value of the distance and on the current order of the expansion in $\alpha$. Ultimately it is determined by the claimed precision of the description.

The convergence properties of the MPT in the case of pair production of unstable particles have been preliminary studied in the framework of model calculations, in the above-threshold region within the NLO [28]. (The modelling has concerned the weights, but not the structure of the MPT itself.) The similar calculations within the NNLO and with taking into consideration the Coulomb singularities have been recently made, as well [48]. In both cases a satisfactory behavior of outcomes has been detected with essential improving of the description at the proceeding to the NNLO. Numerical analysis in the threshold region will be carried out elsewhere. The calculations of the completely realistic processes will follow.

In conclusion, the basic result of the given work is that the very existence is proved of an asymptotic expansion in powers of the coupling constant of the cross-section for pair production and decay of fundamental unstable particles. Furthermore, a scheme of the calculation of the expansion in an arbitrary order has been explicitly constructed and its specific ingredients have been calculated. On the whole, the MPT method turns out to be a real candidate for carrying out high-precision calculations necessary for the realization of programs at colliders of next generation.

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**Appendix A**

Here we derive formula (3.4) for the change of variables in the $\delta$-function with derivatives. More precisely, we derive a formula for the distribution $\delta_n(p(x))$ resolved
with respect to \( x \) variable with

\[
p(x) = x + bx^2, \quad b \neq 0. \tag{A1}
\]

At once we note that since the distribution \( \delta_n(p) \) is concentrated at \( p = 0 \) (i.e. gives zero contributions outside a neighborhood of \( p = 0 \)), and since at the inverse mapping \( p \rightarrow x \) the point \( p = 0 \) turns into two points \( x = 0 \) and \( x = -1/b \), the sought-for formula generally has the form

\[
\delta_n(p) = \sum_{k=0}^{n} z_k \delta_k(x) + \sum_{k=0}^{n} \zeta_k \delta_k(x + 1/b). \tag{A2}
\]

Here \( z_k \) and \( \zeta_k \) are coefficients. However, our actual interest is with the formula considered in a neighborhood of \( x = 0 \). For this reason we reduce the problem to the calculation of the coefficients \( z_k \) only.

First we note that in the case of small enough neighborhoods of \( x = 0 \) and \( p = 0 \), formula (A1) defines the one-to-one mapping \( p \leftrightarrow x \), with the inverse mapping determined by formula

\[
x = \frac{\sqrt{1 + 4bp} - 1}{2b}. \tag{A3}
\]

Further we note that by virtue of

\[
\int dx \ x^n \delta_m(x) = \delta_{nm} \tag{A4}
\]

and in view of (A2), the coefficients \( z_k \) are determined by the formula

\[
z_k = \int dx \ x^k \delta_n(p(x)), \tag{A5}
\]

where \( 0 \leq k \leq n \) and the integration is carried out over the above mentioned neighborhood of the point \( x = 0 \). Making the change of variable \( x \rightarrow p \), we get

\[
z_k = \int dp \ \frac{1}{\sqrt{1 + 4bp} - 1} \left( \frac{\sqrt{1 + 4bp} - 1}{2b} \right)^k \delta_n(p). \tag{A6}
\]

An elementary calculation based on the formula for binomial differentiation yields

\[
z_k = \frac{(-4b)^n}{(2b)^k n!} \sum_{r=0}^{k} \binom{k}{r} (-)^{k+r} \frac{1}{\pi} \sin \left( \pi \frac{r+1}{2} \right) \Gamma \left( \frac{r+1}{2} \right) \Gamma \left( 1 + n - \frac{r+1}{2} \right). \tag{A7}
\]

In this formula at odd \( r \) the sine under the sum gives zero contributions. So we can set \( r = 2m, \ m = 0, \ldots, \lfloor k/2 \rfloor \), where \( \lfloor k/2 \rfloor \) is the integer part of \( k/2 \). After further manipulations with the gamma-functions, we obtain

\[
z_k = \frac{(-)^{k+n} b^n \ k!}{(2b)^k \ n!} \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-)^m}{m!} \frac{(2n - 2m)!}{(k - 2m)!(n - m)!}. \tag{A8}
\]
The calculation of the sum yields [49]

\[ z_k = \binom{2n-k}{n} (-b)^{n-k}. \]  

(A9)

Now we substitute (A9) into formula (A2) considered without the second sum. Making the change of variables \( k \rightarrow n - k \), we obtain

\[
\delta_n(p(x)) = \sum_{k=0}^{n} \binom{n+k}{n} (-b)^k \delta_{n-k}(x). \]  

(A10)

Let us remember that the contributions located at \( x = -1/b \) are omitted in this formula.

A generalization to the case of presence of a common scale factor in formula (A1) is trivial and is practically implemented in (3.4).

**Appendix B**

In this Appendix we analytically calculate the single-subscript basic integrals \( I_{n}^{\lambda} \), \( J_{n}^{\lambda} \) and the double-subscript basic integrals \( A_{n_1n_2}^{\lambda}, B_{n_1n_2}^{\lambda}, C_{n_1n_2}^{\lambda} \), considered with positive parameters \( a_i \) and non-integer \( \lambda \). In the single-subscript integrals that depend on one of the parameters \( a_i \), we omit the lower index “\( i \)” for simplicity of the notation. In the case of the double-subscript integrals we introduce \( a = a_1 + a_2 \) and \( n = n_1 + n_2 \).

**Calculation of \( I_{n}^{\lambda} \)**

Let us discard the superfluous variable \( x_j \) in formula (3.16) and omit the index “\( i \)” in the parameter \( a_i \). Then we come to an equivalent definition,

\[
I_{n}^{\lambda}(x) = \int d\xi \, \theta(\xi + a) \, (x - \xi)^{\lambda} \delta_{n-1}(\xi). \]  

(B1)

Here \( n \) is a positive integer, \( a \) and \( \lambda \) are continuous parameters, \( a > 0 \) and \( \lambda \) is non-integer. Recall that if \( \lambda \) is large enough, the \( (x - \xi)^{\lambda} \) means \( \theta(x - \xi) \, (x - \xi)^{\lambda} \).

At first stage we consider \( \lambda \) as large enough. By making the change of variable \( \xi \rightarrow x - \xi \) and resolving the \( \theta \)-functions, we come to the equivalent formula for the integral written in finite limits:

\[
I_{n}^{\lambda}(x) = \frac{\theta(x + a)}{(n-1)!} \int_{0}^{x+a} d\xi \, \xi^{\lambda} \delta^{(n-1)}(\xi - x). \]  

(B2)

Notice that at \( x < 0 \) integral (B2) vanishes owing to the strict positiveness of the argument in the \( \delta \)-function. At \( x > 0 \) the calculation by parts yields

\[
I_{n}^{\lambda}(x) = \frac{\theta(x + a)}{(n-1)!} \, (-)^{n-1} \theta(x) \, (x^{\lambda})^{(n-1)}. \]  

(B3)
Further we can omit factor $\theta(x + a)$, because at $a > 0$ the condition $x + a > 0$ is automatically satisfied in the presence of $\theta(x)$. After calculation of the derivatives, we obtain

$$I^\lambda_n(x) = \frac{(-)^{n-1} \Gamma(1 + \lambda)}{\Gamma(n) \Gamma(2 + \lambda - n)} \theta(x) x^{1+\lambda-n}. \quad (B4)$$

It is readily seen that at non-integer $\lambda$ this expression is well defined for any $x$ except $x = 0$. At $x = 0$ the above expression, generally, is ill defined. However eventually we do not need a value of the function $I^\lambda_n(x)$ at $x = 0$, but a rule of its integration in a neighborhood of $x = 0$ with smooth weight. In the framework of analytical regularization the $\theta(x) x^{1+\lambda-n}$ means the distribution $x^{1+\lambda-n}$. So in the case of analytical regularization we have

$$I^\lambda_n(x) = \frac{(-)^{n-1} \Gamma(1 + \lambda)}{\Gamma(n) \Gamma(2 + \lambda - n)} x^{1+\lambda-n}. \quad (B5)$$

With transiting to another regularization the result may change. To make this more clear, let us consider an alternative derivation of formula (B5) by basing on the direct calculation of the integral in (B1). In this way at first we formally obtain

$$I^\lambda_n(x) = \frac{1}{(n-1)!} \frac{d^{n-1}}{d\xi^{n-1}} \left[ \theta(\xi + a) \theta(x - \xi) (x - \xi)^\lambda \right]_{\xi=0}. \quad (B6)$$

Since with $a > 0$ the differentiation of $\theta(\xi + a)$ gives zero contribution at $\xi = 0$, we omit this $\theta$-function. The differentiation of other factors gives

$$I^\lambda_n(x) = \frac{(-)^{n-1}}{(n-1)!} \theta(x) (x^\lambda)^{(n-1)} + \frac{(-)^{n-1}}{(n-1)!} \sum_{r=1}^{n-1} \binom{n-1}{r} \delta^{(r-1)}(x) (x^\lambda)^{(n-r-1)}. \quad (B7)$$

In this formula the sum by definition equals zero if $n = 1$. If $n > 1$, the contributions under the sum generally are ill-defined and so they need a regularization. Generally, the different regularizations lead to different outcomes. For instance, under the cut-off regularization the sum takes the form of $\sum_{r=1}^{n-1} C_r \delta^{(r)}(x)$, where $C_r$ are non-vanishing coefficients. Under the analytical regularization all coefficients $C_r$ vanish. Really, in the framework of the analytical regularization formula (B7) must be considered at first at $\lambda > n - 2$, when all the terms in the r.h.s. are integrable. In this case all terms under the sum become zero in view of the presence of $\delta^{(r-1)}(x)$. So only the first term in (B7) survives. Having calculated the derivatives in this term, we come to formula (B5) again.

**Calculation of $J^\lambda_n$**

Proceeding on analogy to the previous case, we rewrite definition (3.17) in the form

$$J^\lambda_n(x) = \int d\xi \theta(\xi + a) (x - \xi)^\lambda_+ \text{PV} \frac{1}{\xi^n}. \quad (B8)$$
Then, making the change of variable $\xi \to x - \xi$ and resolving the $\theta$-functions, we get

$$J_\lambda^m(x) = \theta(x + a) (-)^n \int_0^{x + a} \xi^n d\xi \xi^\lambda PV \frac{1}{(\xi - x)^n}. \tag{B9}$$

Further let us consider an auxiliary integral with $b > 0$,

$$J_\lambda^m(b, x) = \int_0^b d\xi \xi^\lambda PV \frac{1}{(\xi - x)^n}. \tag{B10}$$

At $\lambda > n - 1$ and $x \neq 0, x \neq b$ this integral can be calculated by means of reducing the improper integral to the derivatives of the absolutely convergent integral:

$$\int_0^b d\xi \xi^\lambda PV \frac{1}{(\xi - x)^n} = - \frac{1}{(n - 1)!} \frac{d^n}{dx^n} \int_0^b d\xi \xi^\lambda \ln |\xi - x|. \tag{B11}$$

The integral in the r.h.s. can be explicitly calculated. At $b > x$ and non-integer $\lambda > -1$, we have

$$\int_0^b d\xi \xi^\lambda \ln |\xi - x| = \frac{b^{1 + \lambda}}{1 + \lambda} \left[ \ln(b - x) - \frac{1}{1 + \lambda} F\left(1, -\lambda - 1; -\lambda; \frac{x}{b}\right) \right] + \frac{\Gamma(1 + \lambda) \Gamma(-\lambda) 1 + \lambda}{\Gamma(n)} \left[ (-)^{n-1} \cos(\pi \lambda) \theta(x) x^{1 + \lambda - n} + \theta(-x) (-x)^{1 + \lambda} \right]. \tag{B12}$$

Hereinafter $F$ means the hypergeometric function $2F_1 \ [50]$. Substituting (B12) into (B11) and carrying out the differentiation, we get

$$J_\lambda^m(b, x) = \frac{b^{1 + \lambda - n}}{1 + \lambda - n} F\left(n, n - \lambda - 1; n - \lambda; \frac{x}{b}\right) \tag{B13}$$

$$+ \frac{\Gamma(1 + \lambda) \Gamma(n - \lambda - 1)}{\Gamma(n)} \left[ (-)^n \cos(\pi \lambda) \theta(x) x^{1 + \lambda - n} + \theta(-x) (-x)^{1 + \lambda - n} \right].$$

Comparing (B9) with (B10) and (B13), we come to the formula:

$$J_\lambda^m(x) = (-)^n \frac{(x + a)^{1 + \lambda - n}}{1 + \lambda - n} F\left(n, n - \lambda - 1; n - \lambda; \frac{x}{x + a}\right) \tag{B14}$$

$$+ \theta(x + a) \frac{(1 + \lambda) \Gamma(n - \lambda - 1)}{\Gamma(n)} \left[ (-)^n (-x)^{1 + \lambda - n} \right].$$

We see that $J_\lambda^m(x)$ has, in general, two singular points, the $x = 0$ and $x = -a$. At $x = 0$ the singularity is controlled by the second term in (B14). To find the behavior at $x \to -a$, we take advantage of the analytic-continuation formula for the hypergeometric function with inverse argument [50]. In this way at $x < 0$ we obtain

$$J_\lambda^m(x) = \frac{(x + a)^{1 + \lambda}}{(-x)^{n (1 + \lambda)}} F\left(n, 1 + \lambda; 2 + \lambda; \frac{x + a}{x}\right). \tag{B15}$$

This formula is equivalent to (B14) at $-a < x < 0$. With $1 + \lambda > 0$ the above expression is regular at $x + a \to 0$. 

25
Calculation of $A^\lambda_{n_1 n_2}$

Let us consider integral (3.10) as the iterated one, and let us at first calculate integral in $x_2$. With the aid of (B1) and (B5) we obtain

$$A^\lambda_{n_1 n_2}(x) = \frac{(-)^{n-1} \Gamma(1 + \lambda)}{\Gamma(n_2) \Gamma(2 + \lambda - n_2)} \int dx_1 \theta(x_1 + a_1) (x - x_1)^{1+\lambda-n_2} \delta_{n_1-1}(x_1). \quad (B16)$$

The similar calculation of the remaining integral yields

$$A^\lambda_{n_1 n_2}(x) = \frac{(-)^n \Gamma(1 + \lambda)}{\Gamma(n_1) \Gamma(n_2) \Gamma(3 + \lambda - n)} x_+^{2+\lambda-n}. \quad (B17)$$

Here $n = n_1 + n_2$. The result is symmetric and does not depend on the (positive) parameters $a_i$.

Calculation of $B^\lambda_{n_1 n_2}$

As in the previous case, we consider integral (3.11) as the iterated one and at first calculate integral $dx_2$. With the aid of (B1) and (B5) we get

$$B^\lambda_{n_1 n_2}(x) = \frac{(-)^{n-1} \Gamma(1 + \lambda)}{\Gamma(n_2) \Gamma(2 + \lambda - n_2)} \int dx_1 \theta(x_1 + a_1) (x - x_1)^{1+\lambda-n_2} \text{PV} \frac{1}{x_1^{n_1}}. \quad (B18)$$

Then, with the aid of (B8) and (B14) we obtain

$$B^\lambda_{n_1 n_2}(x) = (-)^{n-1} \frac{\Gamma(1 + \lambda)}{\Gamma(n_2) \Gamma(2 + \lambda - n_2)} \frac{(x + a_1)^{2+\lambda-n}}{2 + \lambda - n}$$

$$\times F\left(n_1, n; n_2; x; \frac{x}{x_1 + a_1}\right)$$

$$- \theta(x + a_1) \frac{\Gamma(1 + \lambda) \Gamma(n - \lambda - 2)}{\Gamma(n_1) \Gamma(n_2)} \left[ \cos(\pi \lambda) x_+^{2+\lambda-n} + (-)^n (-x)^{2+\lambda-n} \right]. \quad (B19)$$

The result is independent from the order of the calculation of integrals, which may be verified by direct calculations. Notice also that (B19) is independent from the parameter $a_2$. The conjugate integral $\overline{B}^\lambda_{n_1 n_2}$ is derived by the substitution $\{n_1, n_2, a_1\} \rightarrow \{n_2, n_1, a_2\}$.

In general, there are two singular points in $B^\lambda_{n_1 n_2}(x)$, the $x = 0$ and $x + a_1 = 0$. At $x \rightarrow 0$ the singular behavior is completely controlled by the second term in (B19). The behavior at $x + a_1 \rightarrow 0$ can be found on the basis of the representation of the hypergeometric function $F$ in the form with inverse argument [50]. At $x < 0$ we obtain:

$$B^\lambda_{n_1 n_2}(x) = (-)^{n-1} \frac{\Gamma(1 + \lambda) (x + a_1)^{2+\lambda-n_2}}{\Gamma(n_2) \Gamma(3 + \lambda - n_2) (-x)^{n_2}} F\left(n_1, 2 + \lambda - n_2; 3 + \lambda - n_2; \frac{x + a_1}{x}\right). \quad (B20)$$

At $2 + \lambda - n_2 > 0$ this expression is regular as $x \rightarrow -a_1$. At $2 + \lambda - n_2 < 0$ it contains singularity $(x + a_1)^{2+\lambda-n_2}$ and also the associated set of singularities $(x + a_1)^{2+\lambda-n_2+r}$, where $r$ is a positive integer such that $2 + \lambda - n_2 + r < 0$. 

26
Calculation of $C_{n_1n_2}^\lambda$

The calculation of $C_{n_1n_2}^\lambda$ is a more complicated task. We begin with the observation that at large enough $\lambda$ the singularities in the integrand in (3.12) do not intersect. So the integral is well defined as the iterated one and theoretically can be calculated on the basis of the above results. However in view of complexity of the intermediate expressions, the analytical calculation in this way unlikely is efficient. An efficient method from the very beginning should allow for symmetrical proper-

intermediate expressions, the analytical calculation in this way unlikely is efficient.

So let us consider integral (3.12) as the double one. For the reasons that become clear later, we begin calculations with adding and subtracting a certain term proportional to $A_{n_1n_2}^\lambda$. Namely we rewrite (3.12) in the form:

\[
C_{n_1n_2}^\lambda(x) = (-)^{n_1+1} \pi^2 A_{n_1n_2}^\lambda + \int \int dx_1 \, dx_2 \, \theta(x_1 + a_1) \theta(x_2 + a_2) (x-x_1-x_2)^\lambda_+
\]

\[
\times \left\{ PV \frac{1}{x_1^{n_1}} \, PV \frac{1}{x_2^{n_2}} + (-)^n \pi^2 \delta_{n_1-1}(x_1) \delta_{n_2-1}(x_2) \right\}. \tag{B21}
\]

Note that the integration in (B21) goes over a simplex. So for providing symmetric integration, we make a shift $x_i \to x_i - a_i$ and then a transition to the cone variables $x_1 + x_2 = \xi$, $x_1 - x_2 = 2\eta$. After that, formally representing the double integral again as the iterated one, we obtain

\[
C_{n_1n_2}^\lambda(x) = (-)^{n_1+1} \pi^2 A_{n_1n_2}^\lambda + \theta(x+a) (-)^{n_2} \int_0^{x+a} dx \, (x + a - \xi)^\lambda \tag{B22}
\]

\[
\times \int_{-\xi/2}^{\xi/2} d\eta \left\{ PV \frac{1}{(\eta + \xi/2 - a_1)^{n_1}} \, PV \frac{1}{(\eta - \xi/2 + a_2)^{n_2}} \right. \\
\left. + (-)^{n_1+1} \pi^2 \delta_{n_1-1}(\eta + \xi/2 - a_1) \delta_{n_2-1}(\eta - \xi/2 + a_2) \right\}.
\]

Unfortunately, in this formula the expression in the curly brackets, generally, is ill-defined. Really, with $x > 0$ the variable $\xi$ may become equal to $a = a_1 + a_2$ with the consequence that there appear the intersecting singularities. However this behavior, actually, is an artefact of the transition to cone variables and the problem may be removed on the basis of the following relation [51] (which, in turn, is a generalization of the relation obtained in a simpler case [52]):

\[
PV \frac{1}{(z-z_1)^{n_1}} \, PV \frac{1}{(z-z_2)^{n_2}} + (-)^{n_1+1} \pi^2 \delta_{n_1-1}(z-z_1) \delta_{n_2-1}(z-z_2) \tag{B23}
\]

\[
= \sum_{r=0}^{n_1-1} \binom{n_2+r-1}{r} (-)^r PV \frac{1}{(z_1-z)^{n_2+r}} \, PV \frac{1}{(z-z_1)^{n_1-r}}
\]
This yields
\( + \sum_{r=0}^{n_2-1} \binom{n_1+r-1}{r} (-)^r PV \frac{1}{(z_2 - z_1)^{n_1+r}} \frac{1}{PV} \frac{1}{(z - z_2)^{n_2-r}}. \)

The meaning of this relation is as follows. Let us assume that some integral with a weight of the expression in the l.h.s. is to be calculated at first in \( z \) and then in \( z_1 \) or \( z_2 \). Then the calculation of the integral in this sequence should be carried out with the substitution of the expression in the r.h.s in this formula. In our case, we need a projection of the relation (B23) obtained by substitutions \( z = \eta, \quad z_1 = -\xi/2 + a_1, \quad z_2 = \xi/2 - a_2 \). After the substitutions, the l.h.s in (B23) becomes the same expression as the in the curly brackets in (B22). So the initial integral can be converted to the form of (B22) with substituting the following expression for the expression in the curly brackets:

\[
\sum_{r=0}^{n_1-1} \binom{n_2+r-1}{r} (-)^r PV \frac{1}{(a - \xi)^{n_2+r}} PV \frac{1}{(\eta + \xi/2 - a_1)^{n_1-r}} \]

\[
+ \sum_{r=0}^{n_2-1} \binom{n_1+r-1}{r} (-)^r PV \frac{1}{(\xi - a)^{n_1+r}} PV \frac{1}{(\eta - \xi/2 + a_2)^{n_2-r}}. \]

A consideration of completely solvable example [51] independently shows that the substitution of (B24) solves the problem of the ghost singularities emerging at the transition to the cone variables.

So in the integral \( d\eta \) in (B22) we gain the contribution of a sum of isolated \( PV \)-poles instead of intersecting singularities. The integral of the poles, we calculate with the aid of the formulas

\[
\int_a^b d\eta \ PV \frac{1}{\eta - \xi} = \ln |b - \xi| - \ln |a - \xi|, \quad (B25)
\]

\[
\int_a^b d\eta \ PV \frac{1}{(\eta - \xi)^n} = -\frac{1}{n-1} \left[ PV \frac{1}{(b - \xi)^{n-1}} - PV \frac{1}{(a - \xi)^{n-1}} \right]. \quad (B26)
\]

After the calculation of the integral \( d\eta \), we make the change of variables \( \xi \rightarrow x + a - \xi \) in the remaining integral \( d\xi \) and simultaneously the change \( r \rightarrow n_i - r - 1 \) in (B24). This yields \((a = a_1 + a_2)\)

\[
C_{n_1n_2}^\lambda(x) = (-)^{n_1+n_2} \pi^2 A_{n_1n_2}^\lambda(x) \quad (B27)
\]

\[
\int_0^{x+a} d\xi \ PV \frac{1}{(\xi - x)^{n_1+n_2-1}}
\]

\[
+ \theta(x+a) (-)^n \frac{\Gamma(n-1)}{\Gamma(n_1) \Gamma(n_2)} \sum_{r=1}^{n_1-1} \binom{n-r-2}{n_2-1} \frac{a_{1-r}}{r} \int_0^{x+a} d\xi \ PV \frac{1}{(\xi - x)^{n_1+n_2-1}}
\]

\[
+ \theta(x+a) (-)^n \sum_{r=1}^{n_2-1} \binom{n-r-2}{n_1-1} \frac{a_{2-r}}{r} \int_0^{x+a} d\xi \ PV \frac{1}{(\xi - x)^{n_1+n_2-1}}
\]

\[
\int_0^{x+a} d\xi \ PV \frac{1}{(\xi - x)^{n_1+n_2-1}}
\]
\[ + \theta(x + a)(-)^n \sum_{r=1}^{n_2-1} \frac{1}{r} \int_0^{x+a} d\xi \frac{\xi^\lambda \text{PV}}{(\xi - x)^{n-r-1}} \text{PV} \frac{1}{(\xi - x - a)^r}. \]

In this formula if the upper bound in any sum is less than the lower bound, then by definition the sum is zero.

Let us discuss the obtained outcome. First we note that, actually, we have calculated the integral in the third term in (B27) while calculating the basic integral \(J^\lambda_n\). The integrals in the two last terms can be easily calculated, as the singularities of the \(\text{PV}\)-poles do not coincide at \(a_i \neq 0\). So the only problem integral remains in the second term, which contains logarithm contributions. To calculate it, we subtract and add to each logarithm the \(n - 1\) of the first terms of its expansion in powers of \(\xi - x\), so that to represent the logarithm in the form

\[
\ln \left| \frac{\xi - x - a_i}{a_i} \right| = \left\{ \ln \left| \frac{\xi - x - a_i}{a_i} \right| + \sum_{r=1}^{n_2-1} \frac{1}{r} \left( \frac{\xi - x}{a_i} \right)^r \right\} - \sum_{r=1}^{n_2-1} \frac{1}{r} \left( \frac{\xi - x}{a_i} \right)^r. \tag{B28} \]

Here the contribution in the curly brackets is of order \(O((\xi - x)^{n-1})\) as \(\xi - x \to 0\). Therefore the appropriate contribution to the integral is regular in the vicinity of \(\xi = x\). The contribution of the last term in (B28) is reduced to the sum of integrals considered at the calculation of \(J^\lambda_n\).

After carrying out all above mentioned calculations, we come to the following result:

\[
C^\lambda_{n_1 n_2}(x) = (-)^{n+1} \pi^2 A^\lambda_{n_1 n_2}(x) + \frac{(-)^{n-1} \Gamma(n-1)}{\Gamma(n_1) \Gamma(n_2)} \int d\xi \theta(x + a - \xi) \xi^\lambda \chi_n(\xi, x), \tag{B29} \]

\[
\chi_n(\xi, x) = \frac{1}{(\xi - x)^{n-1}} \left\{ \ln \left| \frac{\xi - x - a_1}{a_1} \right| + \sum_{r=1}^{n_2-1} \frac{1}{r} \left( \frac{\xi - x}{a_1} \right)^r + (a_1 \to a_2) \right\}. \tag{B30} \]

Since \(\chi_n(\xi, x)\) by the construction is a regular function, integral (B29) is absolutely convergent at \(\lambda > -1\). In this case the singularity of \(C^\lambda_{n_1 n_2}(x)\) is contained only in the \(A^\lambda_{n_1 n_2}(x)\) which is present in (B29).

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