Set-based state estimation for discrete-time constrained nonlinear systems: an approach based on constrained zonotopes and DC programming

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Abstract

This paper proposes a new state estimator for discrete-time nonlinear dynamical systems with unknown-but-bounded uncertainties and state linear inequality and nonlinear equality constraints. Our algorithm is based on constrained zonotopes (CZs) and on a DC programming approach (DC stands for difference of convex functions). Recently, mean value extension and first-order Taylor extension have been adapted from zonotopes to propagate CZs over nonlinear mappings. Although the resulting algorithms (called CZMV and CZFO) reach better precision than the original zonotopic versions, they carry the sensitivity to the wrapping and dependency effects inherited from interval arithmetic. These interval errors can be mitigated with DC programming since the approximation error bounds are obtained solving optimization problems. A direct benefit of this technique is the elimination of the dependency effect. Our set-membership filter (called CZDC) offers an alternative solution to CZMV and CZFO. In order to demonstrate the effectiveness of the proposed approach, CZDC is experimented over two numerical examples.

Keywords: Constrained zonotopes, Set-membership filtering, Discrete-time nonlinear systems, DC programming, Nonlinear invariants, linear inequality constraints

1. Introduction

Set-based techniques have been investigated in the literature to solve problems involving parameter estimation [1, 2], state estimation [3, 4], fault diagnosis [5, 6], control design [7, 8], among others. In these cases, sets are used to represent unknown-but-bounded uncertainties. The success of set-based techniques has been exemplified by means of their use in many applications such as fault detection and isolation for industrial applications [6], fault diagnosis for wind turbines [9], dynamic robot localization and mapping [10], active localization of static features for mobile robots using range-only sensors [11], vehicle state estimation [12], and robot-assisted dressing [13].

Set-based filtering can be split in two branches: interval observers and set-membership observers [14]. We here focus on the second one, whose main difference is the presence of intersection among sets to combine forecast and measurement sets. Recently, constrained zonotopes (CZs) have motivated new advances in set membership since they can in principle represent any convex polytope efficiently. The class of CZs extends zonotopes (centrally symmetric convex polytopes) by introducing linear equality constraints. A direct gain of this extension is to propagate asymmetric polytopes, keeping the computational advantages of zonotopes. Also, from the introduction of the generalized intersection among CZs, which can be computed exactly, the loss of precision with respect to the zonotopic intersection, which demands in general an approximation, can be in principle eliminated.

The original paper on CZs [15] has considered state estimation and fault diagnosis for state-space linear uncertain systems. The authors have shown that their state estimator reaches better precision and detection ratio over other guaranteed estimators [16, 17] at the cost of a slight increase of processing time. Motivated by these benefits, the algorithm of [15] has been extended to cover more general cases as in [18]. In particular, recent contributions for nonlinear systems have been achieved with CZs in [3, 19]. [19] has considered linear output equations, while [3] has extended [19] to nonlinear measurement models and added a step to enforce algebraic equations on set-based estimates. The proposed algorithms have been developed on existing methods of zonotopes, namely: mean value extension [17] and first-order Taylor extension [20].

All aforementioned nonlinear methods rely on interval arithmetic to compute interval enclosures related to the approximation remainder. Therefore, the algorithms proposed in [3, 19] still bring up the sensitivity to the so-called wrapping and dependency effects. These effects summarize all conservatism reached by set-valued operations, with the dependency one being caused...
by the multioccurrence of variables, while the remaining conservatism can be caused by linear mapping and generalized intersection due to the wrapping effect [4]. An alternative tool to mitigate these interval issues is the DC programming approach, where nonconvex mathematical programming problems are approximated using convex analysis tools [21].

Aiming at set-membership filtering, [22] has proposed approximate solutions for DC programming problems where lower and upper bounds are provided to enclose the global solutions. To achieve that, the authors replace the exact minimization and maximization problems by approximate versions whose solutions can be obtained through the evaluation at the vertices of a convex polytope. According to this proposal, [22] has modified the zonotopic filter from [17] (based on mean value extension) to incorporate DC programming, and thereby, mitigated the conservatism generated by zonotopes over intersection, and new set-membership filter called CZDC. Unlike [22], our algorithm allows for working on asymmetric polytopes, mitigating the conservatism generated by zonotopes over intersection, and allows to account for state linear inequality and nonlinear equality constraints. Such state constraints are present in many real applications such as compartmental systems (nonnegativity, conservation laws) [23], unit-quaternion representation (holonomic constraints) [3], and water distribution networks (physical constraints, static relations) [24].

Motivated by the benefits of DC programming over interval arithmetic and of CZs over zonotopes, we here propose a new set-membership filter called CZDC. Unlike [22], our algorithm allows for working on asymmetric polytopes, mitigating the conservatism generated by zonotopes over intersection, and allows to account for state linear inequality and nonlinear equality constraints. Such state constraints are present in many real applications such as compartmental systems (nonnegativity, conservation laws) [23], unit-quaternion representation (holonomic constraints) [3], and water distribution networks (physical constraints, static relations) [24].

This paper is organized as follows. Section 2 formulates the state-estimation problem involving nonlinear state-space models with state constraints. Section 3 introduces some preliminary results. Section 4 presents the CZDC algorithm in detail. Afterwards, CZDC is executed and compared to the algorithms of [3] over two numerical examples in Section 5. Section 6 presents the concluding remarks.

Notation

The set of natural numbers is denoted as \( \mathbb{N} \). The set of positive integer numbers is denoted as \( \mathbb{Z}_+ \). The set of real numbers is denoted as \( \mathbb{R} \). An \((n \times 1)\)-dimensional vector and an \((n \times m)\)-dimensional matrix are, respectively, denoted as \( b \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{nxm} \). An \((n \times m)\)-dimensional zero matrix and an \((n \times n)\)-dimensional identity matrix are, respectively, denoted as \( 0_{nxm} \) and \( I_n \). The transpose of a matrix and the diagonal matrix obtained from a vector are, respectively, denoted as \((\cdot)\)\(^T\) and \(\text{diag}(\cdot)\). The \(i\)th row of a matrix is denoted as \((\cdot)_i\).

2. Problem Statement

Consider the discrete-time nonlinear dynamical system

\[
\begin{align*}
\dot{x}_k &= f(x_{k-1}, u_{k-1}, w_{k-1}),
\end{align*}
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m \) are the known process dynamics and measurement equations, respectively, \( u_{k-1} \in \mathbb{R}^p \) is the known deterministic input vector, \( y_k \in \mathbb{R}^m \) is the measured output vector, and \( x_k \in \mathbb{R}^n \) is the state vector to be estimated. We assume that \( x_k \) satisfies the following nonlinear equality and linear inequality constraints:

\[
\begin{align*}
g(x_k) &= 0_{mx1},
D_k x_k &\leq d_k,
\end{align*}
\]

where \( g : \mathbb{R}^n \to \mathbb{R}^m \), \( D_k \in \mathbb{R}^{n \times n} \), and \( d_k \in \mathbb{R}^m \). Regarding (4), we make the following assumption to enable the direct use of convex polytopes.

Assumption 1. The inequality constraints given by (4), if present, define a compact feasibility set \( X_k^F \subset \mathbb{R}^n \).

The process noise \( w_{k-1} \in \mathbb{R}^q \), the measurement noise \( v_k \in \mathbb{R}^r \), and the initial state \( x_0 \in \mathbb{R}^n \) are bounded by convex polytopes \( W_{k-1}, V_k \), and \( X_0 \). Our set-membership filter aims at estimating the state vector \( x_k \) through convex polytopes \( X_k \) over \( k \in \mathbb{Z}_+ \). To achieve this goal at each \( k \), given \( u_{k-1} \), we define five steps as follows:

1. Forecast: \( X_{k|k-1} \supseteq \{ f(x_{k-1}, u_{k-1}, w_{k-1}) : x_{k-1} \in X_{k-1}, w_{k-1} \in W_{k-1} \} \);
2. Data assimilation: \( \tilde{X}_k \supseteq \{ x_k \in X_{k|k-1} : h(x_k, v_k) = y_k, v_k \in V_k \} \);
3. Admissibility: \( \hat{X}_k = \tilde{X}_k \cap X_k^F \);
4. Consistency: \( \hat{X}_k \supseteq \{ x_k \in \hat{X}_k : g(x_k) = 0_{mx1} \} \);
5. Reduction: \( \hat{X}_k \supset X_k \), with \( X_k \) being a set with lower complexity than \( \hat{X}_k \).

Steps 1, 2, and 4 will be supported by a DC programming approach to obtain tight solutions. In steps 3 and 4, the state nonlinear equality and linear inequality constraints given by (3)-(4) are enforced on the estimator. Step 5 corresponds to a complexity reduction for convex polytopes, which is necessary to control the demand of computational resources; see Section 3 for further details.

3. Preliminaries

3.1. Constrained Zonotopes

A constrained zonotope \( X \subset \mathbb{R}^n \) is a convex polytope represented by the generator matrix \( G^x \in \mathbb{R}^{nxn} \), the center \( c^x \in \mathbb{R}^n \), and the linear equality constraints given by matrix \( A^x \in \mathbb{R}^{nxn} \) and vector \( b^x \in \mathbb{R}^n \). The terms \( n_g \) and \( n_b \) refer to the number of generators and constraints, respectively. Let \( B^{n_x} \triangleq [-1, 1]^{n_x} \) be the unitary box of dimension \( n_x \) and let \( \mathcal{B}(A^w, b^w) \triangleq \{ \xi \in \mathbb{R}^{n_w} : A^w \xi = b^w \} \) be the constrained unitary box. Then, a CZ is defined as [15]

\[
X \triangleq \{ G^x \xi + c^x : \xi \in \mathcal{B}(A^x, b^x) \}.
\]
Let \( m \in \mathbb{R}^r, L \in \mathbb{R}^{r \times n}, Y = \{G^x, c^x, A^x, b^x\} \subset \mathbb{R}^n, W = \{G^w, c^w, A^w, b^w\} \subset \mathbb{R}^p, \) and \( M \in \mathbb{R}^{p \times n} \). The affine transformation, Minkowski sum, generalized intersection, and Cartesian product of CZs are explicitly computed as, respectively,

\[
LX \oplus m = \{LG^x, (LC^x + m), A^x, b^x\},
\]

\[
X \mathbin{\circ} Y = \left\{ \left( \begin{array}{ccc} G^x & G^w \\ c^x & c^w \end{array} \right), \left( \begin{array}{ccc} 0 & 0_{n \times n} \\ A^x & A^w \end{array} \right), \left[ \begin{array}{cc} 0_{n \times n} & 0_{n \times n} \\ b^x & b^w \end{array} \right] \right\},
\]

\[
X \cap_M W = \left\{ \left( \begin{array}{ccc} G^x & 0_{n \times n} \\ c^x & 0_{n \times n} \\ A^x & 0_{n \times n} \end{array} \right), \left( \begin{array}{ccc} 0_{n \times n} & 0_{n \times n} \\ -G^w & 0_{n \times n} \end{array} \right), \left[ \begin{array}{cc} 0_{n \times n} & b^x \\ 0_{n \times n} & b^w \end{array} \right] \right\},
\]

\[
X \times_M W = \left\{ \left( \begin{array}{ccc} G^x & 0_{n \times n} \\ c^x & 0_{n \times n} \end{array} \right), \left( \begin{array}{ccc} 0_{n \times n} & 0_{n \times n} \\ A^x & A^w \end{array} \right), \left[ \begin{array}{cc} 0_{n \times n} & b^x \\ 0_{n \times n} & b^w \end{array} \right] \right\}.
\]

For the set operations (7)-(9), the number of constraints \( n_h \) and generators \( n_f \) for CZs increases. Recursively, this dimension growth demands a way to reduce \( n_h \) and \( n_f \) to desired values \( \varphi_e \) and \( \varphi_x \), thus keeping complexity limited at the price of conservativeness (outer approximation). Here, we employ the algorithm proposed in [15], which can be summarized in three steps: (i) rescaling; (ii) preconditioning; (iii) elimination of constraints and (partial) generators; and (iv) final elimination of generators. The first three steps lead to reduction from \( n_h \) to \( \varphi_e \), and from \( n_f \) to \( (n_f - \varphi_e) \), while the latter finishes the reduction from \( (n_f - \varphi_e) \) to \( \varphi_x \).

The following result is used to obtain the so-called interval hull of a CZ \( X \subseteq \mathbb{R}^n \), defined as

\[
\mathbb{X} = \left\{ \left( \begin{array}{ccc} c_1^x & \cdots & c_n^x \end{array} \right) \in \mathbb{R}^n \mid c_i^x \leq \varphi_e, \ i = 1, \ldots, n \right\}.
\]

As any box expressed in interval arithmetic, the interval hull of a CZ can be equivalently expressed in affine arithmetic doing \( \bigcirc \mathbb{X} = \{ \text{diag}(\text{rad}(\mathbb{X})), \text{mid}(\mathbb{X}) \} \), where \( \text{rad}(\mathbb{X}) \triangleq \frac{1}{2} (\mathbb{X}^U - \mathbb{X}^L) \) and \( \text{mid}(\mathbb{X}) \triangleq \frac{1}{2} (\mathbb{X}^L + \mathbb{X}^U) \). For interval matrices \( [M] = \{ M \in \mathbb{R}^{m \times n} : M^L \leq M \leq M^U \} \), we have \( \text{rad}([M]) \triangleq \frac{1}{2} (\text{rad}(M)^U - \text{rad}(M)^L) \) and \( \text{mid}([M]) \triangleq \frac{1}{2} (\text{mid}(M)^U + \text{mid}(M)^L) \), with \( M^L \) and \( M^U \) being known matrices with different values \( M^L_{ij} \) and \( M^U_{ij} \), respectively, for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

### 3.2. DC Programming

As shown in [3], (6)-(8) can be directly employed in state estimation when \( f, h, \) and \( g \) given by (1), (2), and (3) are linear. Conversely, the nonlinear case requires some approximation of such functions to enable the use of (6)-(8). Contributions to this topic have been proposed in [3, Lemmas 1 and 2] using CZs.

In this paper, a DC programming approach is used to compute linearization enclosures. This approach is convenient to reduce conservatism in comparison with interval methods based on Lagrange remainder as those proposed in [20, 25], which concentrate the linearization error in the quadratic term of a truncated Taylor series. For an in-depth reading about DC programming, the reader is referred to [21, 26]. Next, we define a DC function.

**Definition 1** ([22]). Consider a polytope \( \mathcal{P} \subset \mathbb{R}^n \) and a function \( \varphi : \mathbb{R}^n \to \mathbb{R}^m \). If \( \varphi \) can be rewritten as the difference between two convex functions \( \varphi^a \) and \( \varphi^b \) in \( \mathcal{P} \), then \( \varphi \) is called DC on \( \mathcal{P} \).

The determination of the convex functions \( \varphi^a \) and \( \varphi^b \) may not be a trivial task. Some procedures to guide the choice of such functions are presented in [22]. Next, we define the general form of a DC programming problem. After, two results are presented to enclose the global solutions of DC programming in intervals.

**Definition 2** ([22]). Consider that the function \( \varphi : \mathbb{R}^n \to \mathbb{R}^m \) is DC on \( \mathcal{P} \subset \mathbb{R}^n \), with \( \varphi^a \) and \( \varphi^b \) being its DC components such that \( \varphi(z) = \varphi^a(z) - \varphi^b(z) \). Then, for each component \( i = 1, \ldots, m \), the ith DC programming problems are formulated as

\[
\min_{z \in \mathcal{P}} \varphi_i(z), \quad \max_{z \in \mathcal{P}} \varphi_i(z).
\]
specific algorithms as those presented in [27]. Polytopes are here represented by CZs. Therefore, the interval hull $\square P$ is obtained with Proposition 1. In order to obtain a tight parallelotope $T$ to polytope $P$, we next propose a new result, in which a candidate parallelotope $C \supseteq P$ is tightened via linear programs. The set $C$ is here computed as in Assumption 2.

**Assumption 2.** Given a CZ $P$, a parallelotope $C \supseteq P$ is obtained by reducing all constraints and generators of $P$ with Method 4 of [28].

**Proposition 3.** Let $C = [G^c, c^c] \in \mathbb{R}^n$ be a parallelotope containing the CZ $P = [G^b, c^b, A^b, b^b] \subset \mathbb{R}^n$. By solving the linear programs

$$
\begin{align*}
\xi_i^c &= \min \xi_i^c : G^c_i \xi^c + c^c_i = G^b_i \xi^b + c^b_i, \xi^c, \xi^b \in \mathbb{B}^n, \xi^b \in \mathbb{B}(A^b, b^b), \\
\xi_i^b &= \max \xi_i^b : G^c_i \xi^c + c^c_i = G^b_i \xi^b + c^b_i, \xi^c, \xi^b \in \mathbb{B}^n, \xi^c \in \mathbb{B}(A^c, b^c),
\end{align*}
$$

for $i = 1, \ldots, n$, we obtain the optimal parallelotope $T = \{G^c \text{diag}(\{\xi_i^c, \xi_i^b\})], c^c + G^b \text{mid}(\{\xi_i^c, \xi_i^b\}) \} \supseteq P$. 

**Proof.** Given the candidate parallelotope $C \supseteq P$, a new parallelotope $T$ is investigated to minimally contain the CZ $P$. By fixing both the generator matrix $G^c$ and the center $c^c$ of $C$, the slack variables to be manipulated are $\xi_i^c \in \mathbb{B}^n$. Aiming at tightening the facets of $C$ onto the CZ $P$, $2n$ linear programs are formulated enforcing the $i$th linear equality constraint $G^c_i \xi^c + c^c_i = G^b_i \xi^b + c^b_i$, where $\xi^b \in \mathbb{B}(A^b, b^b)$, such that the minimization and maximization of $\xi_i^b$ yield the smallest possible box $[\xi_i^c, \xi_i^b] \subset \mathbb{B}^n$. Then, $T = \{G^c \text{diag}(\{\xi_i^c, \xi_i^b\}], c^c + G^b \text{mid}(\{\xi_i^c, \xi_i^b\}) \} \supseteq P$ is obtained by considering the parallelotope as an extension of $P$, which can be rewritten as a zonotope using a rescaling (see $T$ in the statement of Proposition 3).

The choice (ii) of the DC components is also important since the optimization problems (13)-(14) are other sources of conservativeness. Note that different DC functions provide different bounds [21]. Having in mind this aspect, we present in Proposition 4 a procedure to compute the DC decomposition of a function $\varphi$. Specifically, we combine quadratic functions of [22] with the choice of eigenvalue of [29]. This choice of eigenvalue is not unique, see for instance [30], but it should be made carefully since large values imply conservative results in Proposition 2.

**Proposition 4 (Adapted from [22, 29]).** Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a function of class $C^2$ in $P$, and $\square P \subset \mathbb{R}^n$ be the interval hull of $P$. Consider functions $\varphi_i^c(z) = \varphi(z) + \varphi_i^c(z)$ and $\varphi_i^b(z) = \frac{\lambda_i}{2} z^\top z$, for $i = 1, \ldots, m$, where

$$
\lambda_i = \max \{0, -\lambda_i\},
$$

with $\lambda_i \in \mathbb{R}$ being computed as in [29, Equation (12)], which is a lower bound for the smallest eigenvalue of interval Hessian matrix $[H_i] = \begin{pmatrix} \varphi_i^c(z) & \varphi_i^b(z) \end{pmatrix} \varphi_i(\square P)$. Then, $\varphi = \varphi^a - \varphi^b$ is a DC function on $\square P$.

**Remark 1.** Each function $\varphi_i^b$ could be defined with $\frac{1}{2} z^\top Q_i z$, where $Q_i$ is a diagonal matrix whose elements could be obtained by generalizing Proposition 4 via semidefinite programming. Moreover, instead of convexifying $\varphi$ (obtaining $\varphi^a$), we could convexify $-\varphi$ and place this result in $\varphi^b$.

### 4. The Novel State Estimator

Next, we present the novel set-membership filter based on CZs and DC programming, called CZDC. This algorithm solves the problem formulated in Section 2 in five steps. The general idea is to firstly linearize models. Then, operations (6)-(8) are performed on the linearized models. Finally, DC programming is employed to bound the linearization error.

In order to obtain linearization error enclosures $\mathcal{R}$, we present Lemma 1. For practical reasons, the input CZ $\mathcal{Z}$ may be outer approximated by either a box $\square \mathcal{Z}$ (Proposition 1) or a parallelotope $T$ (Proposition 3), yielding the desired polytope $P$, before solving the problems (17)-(18).

**Lemma 1** (Adapted from [22]). Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a function in the CZ $\mathcal{Z}$ whose first-order expansion is given by

$$
\hat{\varphi}(z) = \varphi(z) + F(z - \tilde{z}),
$$

where $F = \nabla \varphi(\tilde{z})$, $\tilde{z} \in \mathcal{Z}$ is any punctual estimate, and $\mathcal{P} \supseteq \mathcal{Z}$ is a convex polytope (either a box or a parallelotope for computational reasons). Let $e(z) \triangleq \varphi(z) - \hat{\varphi}(z)$ be the linearization error. Let also $\varphi^a$ and $\varphi^b$ be convex functions such that $\varphi = \varphi^a - \varphi^b$ is DC on $\mathcal{P}$. Finally, let $(-\hat{\varphi}^b - \hat{\varphi}^b)$ be a convex majorant of $\varphi^b$, and let $(\hat{\varphi}^a - \hat{\varphi}^b - \hat{\varphi})$ be a concave minorant of $\varphi$. Then, according to Proposition 2, a linearization enclosure $\mathcal{R} = [e^a, e^b] \ni e$ is given by

$$
\begin{align*}
e_i^a &= \min_{x \in \text{vert}(P)} (\varphi_i^a(z) - \varphi_i^b(z) - \tilde{\varphi}_i(z)), \\
e_i^b &= \max_{x \in \text{vert}(P)} (\varphi_i^a(z) - \varphi_i^b(z) - \tilde{\varphi}_i(z)),
\end{align*}
$$

for $i = 1, \ldots, m$. Once (17)-(18) are solved, the intervals are expressed in affine arithmetic as the zonotope $\mathcal{R} = [G^c, c^c]$ with

$$
G^c = \text{diag}(\{e^a + e^b\}),
$$

$$
c^c = \text{mid}(\{e^a, e^b\}).
$$

**Proof.** This proof is similar to [22, Proof of Lemma 1], with the difference being that $\mathcal{Z}$ is a CZ (instead of zonotope), and $\varphi$ is any function in $\mathcal{P}$.

In the following, the results to execute a loop of CZDC are presented. We emphasize that Theorem 1 extends [22, Theorem 1] by introducing computations with CZs and deterministic input vector.

**Theorem 1** (Forecast Step). Consider the CZs $X_{k-1} \subset \mathbb{R}^n$ and $W_{k-1} \subset \mathbb{R}^n$, and the deterministic input $u_{k-1} \subset \mathbb{R}^n$. Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be rewritten as $f^f : \mathbb{R}^{n \times (n+1)} \to \mathbb{R}^n$ using the augmented vector $z_{k-1} = [x_{k-1}^T, u_{k-1}^T, w_{k-1}^T]^T$. Let also $f^a = f^{fa} - f^{fb}$ be DC on the polytope $P_{k-1} \supseteq \mathcal{Z}_{k-1} = \ldots$
Theorem 2 (Data-Approximation Step). Consider the CZs $X_{ik-1} \subset \mathbb{R}^n$ and $\mathcal{V}_k \subset \mathbb{R}^m$. Let $h : \mathbb{R}^n x \mathbb{R}^m \rightarrow \mathbb{R}^m (2)$ be rewritten as $\phi : \mathbb{R}^{n+mr} \rightarrow \mathbb{R}^m$ using the augmented vector $z_k = [\begin{bmatrix} x_k^T & v_k^T \end{bmatrix}]$. Let also $\phi = \phi_h - \phi_h^b$ be DC on the polytope $P_k \supseteq X_{ik-1} \times \mathcal{V}_k$. Finally, let $R_k \subset \mathbb{R}^m$ be the set returned by Lemma 1 to compensate the linearization error of $\phi$ for a given vector step $t \in \mathcal{P}_k$. Then, the exact image $\phi(t) (Z_{ik-1})$ is outer approximated by the CZ

$$X_{ik} = \{ \phi^*(\bar{x}) - F^* \bar{x} - F^* \bar{u} \} \cap \mathcal{P}_{k-1} + \mathcal{R}_{k-1},$$

with $F^* = \nabla_x^\phi(\bar{x})$ and $F^w = \nabla_x^\phi(\bar{x})$ being Jacobians matrices evaluated at $\bar{x} = [\begin{bmatrix} \bar{x}^T & \bar{w}^T \end{bmatrix}]^T$.

Proof. This proof is similar to [22, Proof of Theorem 1], with the difference being the propagation of CZs instead of zonotopes.

Remark 2. If functions $f$ and $h$ are affine in the noise terms $w_{ik-1}$ and $v_k$, respectively, then these terms are canceled during the computation of $R$ in Theorems 1 and 2. It means that, instead of $2^{(n+mr)}$ and $2^{(n+r)}$ vertices, we need to process $2^n$ vertices only.

4.1. Complexity Analysis

The worst-case computational complexity $O(\cdot)$ for each step of CZDC (Algorithm 1) is shown in Table 1. Such complexities were derived using basic operations among CZs [19]. Regarding the forecast, data assimilation, and consistency steps, the complexity order to obtain the linearization point $z$ is not included since it depends on the employed methodology. As in [19], we also assume that the evaluation of nonlinear functions has complexity $O(1)$. In the second column of Table 1, the cubic term between parenthesis refers to either the computation of parallelotope via linear programs or to the Hausdorff distance minimization (order reduction). The term $2^n$ is related to either the computation of vertices or the DC programming problems (17)-(18). In turn, the third column of Table 1 presents the number of constraints and generators for the state $X_k$ across the different steps.

Table 1 makes the following assumptions: $X_{ik-1} = \{ g_{ik-1}, c_{ik-1}, A_{ik-1,1}, b_{ik-1} \} \subset \mathbb{R}^n$, $W_{ik-1} = \{ g_{ik-1}, c_{ik-1,1}, A_{ik-1,1}, b_{ik-1} \} \subset \mathbb{R}^n$, $\mathcal{V}_k = \{ g_{ik}, c_{ik}, b_{ik} \} \subset \mathbb{R}^r$, and $X_k^F = \{ g_{ik}, c_{ik}, b_{ik} \} \subset \mathbb{R}^n$, $G_{ik-1} \in \mathbb{R}^{n \times n}$, $G_{ik-1}^w \in \mathbb{R}^{n \times n}$, $G_{ik} \in \mathbb{R}^{n \times n}$, $G_{ik}^w \in \mathbb{R}^{n \times n}$, $c_{ik-1,1} \in \mathbb{R}^r$, $c_{ik-1} \in \mathbb{R}^r$, $c_{ik} \in \mathbb{R}^r$, $c_{ik}^w \in \mathbb{R}^r$, $A_{ik-1,1} \in \mathbb{R}^{n \times n}$, $A_{ik-1}^w \in \mathbb{R}^{n \times n}$, $A_{ik}^w \in \mathbb{R}^{n \times n}$, $b_{ik-1}^w \in \mathbb{R}^{n}$, $b_{ik-1} \in \mathbb{R}^r$, $b_{ik} \in \mathbb{R}^r$, and $b_{ik}^w \in \mathbb{R}^r$. These sets are evaluated over the functions $f : \mathbb{R}^n x \mathbb{R}^p \rightarrow \mathbb{R}^n$, $h : \mathbb{R}^n x \mathbb{R}^m \rightarrow \mathbb{R}^m$, and $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$, considering the vectors $u_{ik-1} \in \mathbb{R}^p$ and $y_{ik} \in \mathbb{R}^m$. At the end of an iteration of CZDC, the desired CZ $X_k$ is returned with $\phi_k$ constraints and $\phi_k$ generators.

Remark 3. According to Table 1, the output CZs obtained by Theorems 1-2 and Corollary 1 have smaller number of constraints and generators than those pointed out by Remarks 5, 7, and 10 from [3]. Exceptionally, the number of constraints for $X_{ik-1}$ coincides with the value indicated in [3, Remark 5] for the CZMV algorithm.
Windows 10 Pro, and AMD FX-6300 CPU 3.50 GHz. All im-
val matrix; CZMV is run with metric \([3, m] \) to minimize the diameter of an interval vector. Two performance indexes are
a box or a parallelotope, whose procedure is O(1); CZFO is run with metric \([3, C3]\) to minimize the diameter of an inter-
val matrix; CZMV is run with metric \([3, C2]\) to minimize the diameter of an interval vector. Two performance indexes are
computed, namely: (i) the mean processing time \(T_{CPU}\), given by
by \(T_{CPU} \triangleq \frac{1}{m_s} \sum_{k=1}^{m_s} \sum_{j=1}^{k_i} \tau_{x_{i,j}}\), where \(j_{i,j} \in N\) is the number of
time steps, \(m_s \in N\) is the number of Monte Carlo simulations, and \(\tau_{x_{i,j}}\) is the time to execute the \(k_i\)th iteration of a given algo-

5. Numerical Results
In this section, CZDC is experimented over two case studies. For comparison purposes, we also implement the state-of-the-
art algorithms proposed in \([3]\), called CZFO (based on Taylor expansion) and CZMV (based on mean value extension). To
yield punctual estimates \(z\), and thereby, to approximate the non-
linear models, we make the following choices: CZDC uses the
center of the polytope \(P\) associated to Lemma 1, where \(P\) is a
box or a parallelopetope, whose procedure is O(1); CZFO is run with metric \([3, C3]\) to minimize the diameter of an inter-
val matrix; CZMV is run with metric \([3, C2]\) to minimize the diameter of an interval vector. Two performance indexes are
computed, namely: (i) the mean processing time \(T_{CPU}\), given by
by \(T_{CPU} \triangleq \frac{1}{m_s} \sum_{k=1}^{m_s} \sum_{j=1}^{k_i} \tau_{x_{i,j}}\), where \(j_{i,j} \in N\) is the number of
time steps, \(m_s \in N\) is the number of Monte Carlo simulations, and \(\tau_{x_{i,j}}\) is the time to execute the \(k_i\)th iteration of a given algo-

Table 1: Complexity order of the forecast, data assimilation, admissibility, consistency, and reduction steps from CZDC using Proposition 3.

| Step          | \(O(\cdot)\) | Definition                                                                 |
|---------------|--------------|----------------------------------------------------------------------------|
| Forecast      | \(\bar{n}_0 (\bar{n}_0 + \bar{n}_s) + \bar{n} (\bar{n} + \bar{n}_s) (\bar{n} + \bar{n}_s + \bar{n}_s) + \bar{n}^2 2^8\) | \(\bar{n} = n + q, \bar{n}_0 = n_0 + n_0^s, \bar{n}_s = n_s + n_s^s\) |
| Data Assimilation | \(\bar{n}_0 (\bar{n}_0 + \bar{n}_s) + \bar{n} (\bar{n} + \bar{n}_s) (\bar{n} + \bar{n}_s + \bar{n}_s) + (\bar{n}^2 + m^2) 2^8 + m n_0 + m n_0^s + m^2\) | \(\bar{n} = n + r, \bar{n}_0 = n_0 + n_0^s + n_0^s, \bar{n}_s = n_s + n_s^s + n + n_s^s\) |
| Admissibility  | \(n^2 \bar{n}_s + m n_0^2\) | \(\bar{n}_s = n_0^s + n + n_s^s + m\) |
| Consistency    | \(n_0 (\bar{n} + \bar{n}_s) + \bar{n} (\bar{n} + \bar{n}_s + \bar{n}_s) + (\bar{n}^2 + m^2) 2^8 + m n_0 + m n_0^s + m^2\) | \(\bar{n}_s = n_0 + n_0^s + n + n_s^s + m\) |
| Reduction      | \(k_s (\bar{n} + \bar{n}_s) + n (\bar{n} + \bar{n}_s + \bar{n}_s) + (\bar{n}^2 + m^2) 2^8 + m n_0 + m n_0^s + m^2\) | \(\bar{n}_s = n_0 + n_0^s + n + n_s^s + m + m_s^s + m_c\) |

5.1. Two-State Nonlinear Process
Consider the nonlinear uncertain system \([22]\)
\[
x_k = \begin{bmatrix} -0.7 x_{1,k-1} + 0.1 x_{2,k-1}^2 + 0.1 x_{1,k-1} x_{2,k-1} + 0.1 \exp (x_{1,k-1}) \\ x_{1,k-1} + x_{2,k-1} - 0.1 x_{1,k-1}^2 + 0.2 x_{1,k-1} x_{2,k-1} \end{bmatrix}
\]
with \(x_{2,k} \in \mathcal{V} = [0.112, 0.312]\) and \(x_{2,k} \in \mathcal{V} = [0.2, 0.8]\). To simulate this system, we set \(x_0 = [1, 1]^{T} \in X_0 = [3 \times 12, 0.241]\),
\(k_f = 40\), and \(m_t = 100\). This example aims at illustrating
that CZDC is a promising option to substitute the use of CZFO and
CZMV whenever the wrapping and dependency effects imply divergence of estimates, and that CZDC reaches a better
precision than the zonotopic filter based on DC programming
(ZDC) proposed in \([22]\). To reduce order of CZs, we set \(\varphi_0 = 3\) and \(\varphi_s = 8\). This latter value is also used to reduce order of
zonotopes in ZDC with Method 4 of \([28]\). To improve both
the computational efficiency and the precision of the minimum-
volume zonotopes computed in ZDC, we employ \([4, VM3]\) and
\([1, Definition 8]\). Motivated by \([22]\), we propose the DC function
\(x_{k+1} = f^a - f^b + w_{k-1}\) such that
\[
f^a = \begin{bmatrix} 0.1 x_{1,k-1}^2 + 0.1 x_{1,k-1} x_{2,k-1} + 0.1 x_{2,k-1}^2 + 0.1 \exp (x_{1,k-1}) \\ 0.1 x_{2,k-1}^2 + x_{1,k-1} + x_{2,k-1} \end{bmatrix},
\]
\[
f^b = \begin{bmatrix} 0.1 x_{1,k-1}^2 + 0.7 x_{2,k-1} \\ 0.1 x_{1,k-1} + 0.1 x_{2,k-1} - 0.2 x_{1,k-1} x_{2,k-1} \end{bmatrix}.
\]
Since DC functions were directly defined, Proposition 4 was not
employed, and thereby, the polytope \(P_{k+1}\) associated to Theorem
1 is a parallelopetope (given by Proposition 3) that contains the CZ
\(X_{k-1} = \mathcal{Z}_{k-1}\). In Figure 1(a), we point out that both CZFO and CZMV
diverge due to the direct usage of interval arithmetic. Although
this interval extension was used to experiment CZFO and CZMV
in \([3]\), it is not enough to reach convergence in this case study.
Differently, both ZDC and CZDC achieve convergent solu-
tions because DC programming involves evaluation of element-
ary functions rather than inclusion functions. In Figure 1(b)
and (c), one-dimensional intervals are sketched to illustrate that
those algorithms provide guaranteed solutions. As shown in
Table 2, CZDC provides a significantly better precision than
ZDC at the cost of a larger \(T_{CPU}\).

5.2. Attitude Estimation
Now, we show the application of CZDC to a more challenging
and technological example, containing multiplicative process
where the considered system concerns the attitude estimation of a flying robot. By employing quaternion representation, the attitude is expressed as $x_k \in \mathbb{R}^4$ such that $\|x_k\|^2 = 1$. These states evolve at discrete time according to [3, 23]

$$x_k = \left(\cos(p(\tilde{u}_{k-1})) I_4 + \frac{T_s}{2} \sin(p(\tilde{u}_{k-1})) \Omega(\tilde{u}_{k-1})\right)x_{k-1}, \quad (26)$$

where $T_s = 0.2$ s is the sampling time, $p(\tilde{u}_k) = \frac{T_s}{2} ||\tilde{u}_k||_2$, $\Omega(\tilde{u}_k) = \begin{bmatrix} 0 & -\tilde{u}_3, k & \tilde{u}_2, k \\ \tilde{u}_3, k & 0 & -\tilde{u}_1, k \\ \tilde{u}_2, k & \tilde{u}_1, k & 0 \end{bmatrix}$, and $\tilde{u}_k = \begin{bmatrix} 0.3 \sin((2\pi/12) k T_s) \\ 0.3 \sin((2\pi/12) k T_s - 6) \\ 0.3 \sin((2\pi/12) k T_s - 12) \end{bmatrix}$ is the physical input that drives the actual system. For state-estimation purposes, we assume that $\tilde{u}_k$ is acquired by gyroscopes. Then, $\tilde{u}_k$ is corrupted by an additive noise $w_k \in \mathcal{W} = \{3 \times 10^{-3} I_3, 0_{3 \times 1}\}$, whose result is the known signal $u_k = \tilde{u}_k + w_k$. The measurement is given by

$$y_k = \begin{bmatrix} C(x_k) f^{[1]} \\ C(x_k) f^{[2]} \end{bmatrix} + v_k, \quad (27)$$

where $f^{[1]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $f^{[2]} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $v_k \in \mathcal{V} = \{0.15 I_3, 0_{3 \times 1}\}$.

To simulate the system, we consider the uncorrupted signal $\tilde{u}_k$, initial state $x_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T \in X_0 = \{0.18 I_4, 0.1, 0.9, 0.1, 0.1^T\}$, realizations of uniform noise defined in $\mathcal{V}$ for $v_k$, $k_f = 200$, and $m_t = 5$. To estimate states, we consider the corrupted signal $u_k$, fixed values $\varphi_x = 10$ and $\varphi_u = 30$, the invariant $g(x_k) = x_k^T x_k - 1$, and the feasible set $X^0 = \{I_4, 0_{4 \times 1}\}$. Since $\tilde{u}_k$ is truly unknown, the algorithms replace $u_k$ by $(u_k - w_k)$. Due to the nonlinearity of $x_k = f(x_{k-1}, u_{k-1}, w_{k-1})$ in (26), Proposition 4 is employed to yield DC functions $f = f^a - f^b$ over each time step. In this case, the polytope $\mathcal{P}_{k-1}$ related to Theorem 1 is a box (given by Proposition 1) that contains the CZ $Z_{k-1} = X_{k-1} \times U_{k-1} \times W_{k-1}$. By exploiting the quadratic nature of both $y_k = h(x_k) + v_k$ in (27) and $g(x_k) = 0$, we propose the DC functions $y_k = h^a - h^b + v_k$ and $g = g^a - g^b$ such that

$$h^a = \begin{bmatrix} x_{1,k}^2 + x_{2,k}^2 \\ x_{1,k} x_{2,k} - x_{3,k} x_{4,k} \\ x_{1,k} x_{3,k} + x_{2,k} x_{4,k} \\ x_{1,k}^2 + x_{2,k}^2 - x_{3,k}^2 - x_{4,k}^2 \end{bmatrix}, \quad h^b = \begin{bmatrix} x_{1,k}^2 + x_{3,k}^2 \\ x_{1,k} x_{3,k} + x_{2,k} x_{4,k} \\ x_{1,k} x_{4,k} - x_{2,k} x_{3,k} \\ x_{1,k}^2 + x_{3,k}^2 \end{bmatrix}, \quad g^a = g, \quad g^b = 0.$$

In this case, the polytopes $\mathcal{P}_k$ and $\mathcal{P}_k$, in Theorem 2 and Corollary 1, are paralleloptopes (given by Proposition 3) that contain the CZs $Z_{k-1}$ and $Z_k = X_k$, respectively.

Figure 2 depicts a separate simulation with the CZDC, CZFO and CZMV algorithms. Boxes were sketched rather than CZs for computational simplicity. According to the figure, CZDC generates CZs with the smallest associated interval hulls. Moreover, a faster reduction of uncertainty is expected with CZDC during the initialization effect. Table 3 corroborates the improvement of precision caused by CZDC in comparison with

| $T_{CPU}$ | ZDC | CZDC | CZFO | CZMV |
|----------|-----|------|------|------|
| 7.90 ms  | 12.6 ms (59.5%) | 3.62 ms | 1.75 ms (51.7%) |

Table 2: Results of $T_{CPU}$ and $A^2$ for the first example (Subsection 5.1).
both CZMV and CZFO. Since CZFO is, in general, more costly than CZMV [3, Table 1], it demands a larger $T_{\text{CPU}}$ as shown in Table 3. Differently, CZDC can enhance the precision of CZMV using much less computational resource, and this advantage is related to both tight linearization remainders (Lemma 1) and low-dimension CZs (Remark 3). However, the quantity of operations involved with CZDC may be larger than the CZMV one, justifying the difference of $T_{\text{CPU}}$.

In order to verify if the precision of CZDC would be enlarged with respect to Table 3 (reduction of $A_{\boxtimes}$), we also tested if convexifying each row of $f$ or $-f$, for each time step, would be better (Remark 1), selecting the strategy with the smallest lower bound of eigenvalue. However, the tests pointed out that convexifying $f$ always yielded the best solutions.

During the experiment execution, CZMV and CZFO diverged for some simulations, whose results were discarded and not included in the computation of $A_{\boxtimes}$. The increase of $\varphi_e$ and $\varphi_g$ can, in principle, improve the results. However, the generator reduction can imply conservatism for some directions due to the wrapping effect. Each simulation has different noise realizations, which affect $u_k$ and $y_k$, and thereby, the intersections.

### 6. Conclusions

This paper proposed a new set-membership filter for discrete-time nonlinear uncertain systems with state constraints, called CZDC. A DC programming approach was used to provide a new nonlinear approximation for CZs. Thus, CZDC established an alternative estimation basis with respect to the state-of-the-art algorithms, called CZMV and CZFO [3]. We showed that the performance of these two algorithms can be significantly deteriorated due to the wrapping and dependency effects, with CZDC being a good option to mitigate divergence and conservatism issues. Over two numerical examples, we discussed advantages of CZDC over CZMV and CZFO. These three algorithms can readily enforce linear inequality constraints on the state vector by using CZs. However, the nonlinear case requires investigation and will be intended in the future.

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Figure 2: 3D projection with respect to $x_{1,k} = 0$ of the interval hulls of CZs, computed by CZDC (blue color), CZFO (red color), and CZMV (magenta color), for the second case study (Subsection 5.2). The true states are sketched in black solid line.

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