REGULARITY OF KÄHLER-RICCI FLOW

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Abstract. In this short note we announce a regularity theorem for Kähler-Ricci flow on a compact Fano manifold (Kähler manifold with positive first Chern class) and its application to the limiting behavior of Kähler-Ricci flow on Fano 3-manifolds. Moreover, we also present a partial $C^0$ estimate to the Kähler-Ricci flow under the regularity assumption, which extends previous works on Kähler-Einstein metrics and shrinking Kähler-Ricci solitons (cf. [17], [11], [20], [15]). The detailed proof will appear in [23].

1. Introduction

Let $M$ be a Fano $n$-manifold and $g_0$ be any Kähler metric with Kähler class $2\pi c_1(M)$. Consider the normalized Kähler-Ricci flow:

\[
\frac{\partial g}{\partial t} = g - \text{Ric}(g), \quad g(0) = g_0.
\]

It was proved in [1] that (1.1) has a global solution $g(t)$ for $t \geq 0$. The main problem is to understand the limit of $g(t)$ as $t$ tends to $\infty$.

By Perelman’s non-collapsing result [12], there is a uniform constant $\kappa = \kappa(g(0)) > 0$ satisfying:

\[
\text{vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n}, \quad \forall t \geq 0, r \leq 1.
\]

Since the volume stays the same along the Kähler-Ricci flow, this implies that for any sequence $t_i \to \infty$, by taking a subsequence if necessary, $(M, g(t_i))$ converge to a limiting length space $(M_{\infty}, d)$ in the Gromov-Hausdorff topology:

\[
(M, g(t_i)) \xrightarrow{d_{GH}} (M_{\infty}, d).
\]

The question is the regularity of $M_{\infty}$. A desirable picture is given in the following folklore conjecture.

Conjecture 1.1 ([18], also see [9]). $(M, g(t))$ converges (at least along a subsequence) to a shrinking Kähler-Ricci soliton with mild singularities.

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It is often referred as the Hamilton-Tian conjecture (see [18]).
Here, ”mild singularities” may be understood in two ways: (i) A singular set of codimension at least 4, and (ii) a singular set of a normal variety. By extending the partial $C^0$-estimate conjecture [19] to Kähler-Ricci flow, one can show that these two approaches are actually equivalent (see Section 3 or [23]).

As pointed out by the second named author, this conjecture implies the Yau-Tian-Donaldson conjecture, in the case of Fano manifolds. The conjecture states that if a Fano manifold $M$ admits a Kähler-Einstein metrics if it is K-stable. Recently, solutions were provided for this conjecture in the case of Fano manifolds ([20], also see [6, 7, 8]).

2. Regularity of Kähler-Ricci flow

Let $M$ be a Fano n-manifold and $g(t)$ a normalized Kähler-Ricci flow in the Kähler class $2\pi c_1(M)$. Let $(M_\infty, d)$ be a sequence limit of the Kähler-Ricci flow as phrased in (1.3). The main regularity result is the following theorem:

**Theorem 2.1.** Suppose that for some uniform $p > n$ and $\Lambda < \infty$,

$$
\int_M |\text{Ric}(g(t))|^p dv_{g(t)} \leq \Lambda.
$$

Then the limit $M_\infty$ is smooth outside a closed subset $S$ of (real) codimension $\geq 4$ and $d$ is induced by a smooth Kähler-Ricci soliton $g_\infty$ on $M_\infty \setminus S$. Moreover, $g(t_i)$ converge to $g_\infty$ in the $C^\infty$-topology outside $S$

The proof of the theorem relies on Perelman’s Pseudolocality theorem [12] of Ricci flow and a regularity theory for manifolds with $L^p$ bounded Ricci curvature ($p$ bigger than half dimension) and uniformly local volume noncollapsing condition as in (1.2). This is a generalization of the regularity theory of Cheeger-Colding [2, 3, 4] and Cheeger-Colding-Tian [5]. The proof can be carried out following the lines given in these papers under the framework established by Petersen-Wei [13, 14] on the geometry of manifolds with integral bounded Ricci curvature. Note that due to an example of Yang [25], the uniformly volume noncollapsing condition (1.2) can not be replaced by a lower bound of total volume or local volume of metric balls of a definite size.

We shall show in [23] that there is a uniform $L^1$ bound on the Ricci curvature along the Kähler-Ricci flow on any Fano manifold. Therefore, by the above regularity result, we have

**Corollary 2.2.** Conjecture [1] i.e., the Hamilton-Tian conjecture, holds for dimension $n \leq 3$.

In the case of Del-Pezzo surfaces, Conjecture [1] follows from [24] and [10].

\[2\]The convergence with these properties is usually referred as the convergence in the Cheeger-Gromov topology, see [17] for instance.
3. **Partial $C^0$ estimate of Kähler-Ricci flow**

The partial $C^0$ estimate of Kähler-Einstein manifolds plays the key role in Tian’s program to resolve the Yau-Tian-Donaldson conjecture, see [17], [18], [19], [11] and [20] for examples. An extension of the partial $C^0$ estimate to shrinking Kähler-Ricci solitons was given in [15]. These works are based on the compactness of Cheeger-Colding-Tian [5] and its generalizations to solitons by [22]. We shall generalize these to the Kähler-Ricci flow on Fano manifolds in [23] under the regularity assumption of the limit $M_\infty$.

Let $u(t)$ denote the Ricci potentials of the Kähler-Ricci flow $g(t)$ which satisfy

\[
(3.1) \quad \text{Ric}(g(t)) + \partial\bar{\partial}u(t) = g(t).
\]

The Hermitian metrics $\tilde{g}(t) = e^{-\frac{1}{2}u(t)}g(t)$ have $\omega(t)$, the Kähler forms of $g(t)$, as their Chern curvature forms. Let $H(t)$ be the induced metric on $K^{-l}_M$, the $l$-th power of the anti-canonical bundle ($l \geq 1$), and $D$ be the Chern connection of $H(t)$ on $K^{-l}_M$.

Let $\nabla$ and $\bar{\nabla}$ denote the $(1,0)$ and $(0,1)$ part of the Levi-Civita connection respectively. Then, at any time $t$, we have the Bochner type formula for $\sigma \in H^0(M, K^{-l}_M)$

\[
(3.2) \quad \Delta |\nabla \sigma|^2 = |\nabla \nabla \sigma|^2 + |\bar{\nabla} \nabla \sigma|^2 - ((n+2)l-1)|\nabla \sigma|^2 - \langle \partial\bar{\partial}u(\nabla \sigma, \cdot), \nabla \sigma \rangle
\]

and the Weitzenböck type formulas for $\xi \in C^\infty(M, T^{1,0}M \otimes K^{-l}_M)$

\[
(3.3) \quad \Delta_\partial \xi = \bar{\nabla}^* \nabla \xi + (l+1)\xi - \partial\bar{\partial}u(\xi, \cdot),
\]

\[
(3.4) \quad \Delta_\bar{\partial} \xi = \nabla^* \nabla \xi - (n-1)l\xi,
\]

where $\Delta_\partial$ is the Hodge Laplacian of $\partial$. By [20] there is a uniform bound of Sobolev constant under the Kähler-Ricci flow. Then apply the Moser iteration one can prove the gradient estimate to $\sigma \in H^0(M, K^{-l}_M)$ and $L^2$ estimate to solutions $\partial\bar{\partial} = \xi \in C^\infty(M, T^{1,0}M \otimes K^{-l}_M)$; see Lemmas 4.1 and 5.4 of [20]. The gradient estimate implies $\dim H^0(M, K^{-l}_M) \leq N_l$ uniformly at any time $t$. We remark that Perelman’s $C^1$ estimate to $u(t)$ [16] will be used in the iteration arguments to cancel the bad terms containing $\partial\bar{\partial}u(t)$.

Now, let $\{s_{t,i,l}\}_{i=1}^{N_{t,l}}$ be an orthonormal basis of $H^0(M, K^{-l}_M)$ with respect to the $L^2$ norm defined by $H(t)$ and Riemannian volume form, and put

\[
(3.5) \quad \rho_{t,i}(x) = \sum_{i=1}^{N_{t,l}} |s_{t,i,l}|_H^2(x), \quad \forall x \in M.
\]

By using arguments similar to those in [11] or [20], we can prove

**Theorem 3.1 (Partial $C^0$ estimate).** If $(M, g(t_i)) \xrightarrow{dGH} (M_\infty, g_\infty)$ as phrased in Theorem 2.1, then the partial $C^0$ estimate

\[
(3.6) \quad \inf_{t_i} \inf_{x \in M} \rho_{t_i,i}(x) > 0
\]
holds for a sequence of $l \to \infty$.

A direct corollary of this is to refine the regularity in Theorem 2.1.

**Theorem 3.2.** Suppose $(M, g(t)) \xrightarrow{dGH} (M_\infty, g_\infty)$ as phrased in Theorem 2.1. Then $M_\infty$ is a normal projective variety and $S$ is a subvariety of complex codimension at least 2.

Finally, let us indicate how to deduce the Yau-Tian-Donaldson conjecture from the Hamilton-Tian conjecture. Suppose $M$ is K-stable as defined in [15]. Then, under the Kähler-Ricci flow $g(t)$, we get a shrinking Kähler-Ricci soliton. This, together with the uniqueness theorem on shrinking solitons, we can conclude that the Lie algebra of holomorphic vector fields on $M_\infty$ is reductive. Then the K-stability implies the vanishing of Futaki invariant of $M_\infty$, consequently, the limit $(M_\infty, g_\infty)$ is Kähler-Einstein. If $M_\infty$ is not biholomorphic to $M$, then the eigenspaces of the first eigenvalues of $-\Delta_g(t) + g^{ij}(t)\partial_i u(t)\partial_j$ will converge to a subspace of potential functions on $M_\infty$ whose complex gradients are nontrivial holomorphic vector fields, cf. [27]. These vector fields induce the required degeneration of $M$ to $M_\infty$, with vanishing Futaki invariants. This gives a contradiction to the K-stability of $M$. So we have

**Theorem 3.3.** Suppose $M$ is K-stable. If $(M, g(t)) \xrightarrow{dGH} (M_\infty, g_\infty)$ as phrased in Theorem 2.1, then $M$ coincides with $M_\infty$ and admits a Kähler-Einstein $g_\infty$.

In view of the regularity of low dimensional Kähler-Ricci flow in §2 we have

**Corollary 3.4.** The Yau-Tian-Donaldson conjecture holds for dimension $n \leq 3$.

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