Virtual Black Holes in a Third Quantized Formalism

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Abstract

In this paper, we will analyse virtual black holes using the third quantization formalism. As the virtual black hole model depends critically on the assumption that the quantum fluctuations dominate the geometry of spacetime at Planck scale, we will analyse the quantum fluctuations for a black hole using third quantization. We will demonstrate that these quantum fluctuations depend on the factor ordering chosen. So, we will show that only certain values of the factor ordering parameter are consistent with virtual black holes model of spacetime foam.

Keywords : Third quantization, Black hole, Uncertainty relation, Operator ordering

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1 Introduction

A universal prediction from almost all approaches to quantum gravity is that the geometry of spacetime will be dominated by quantum fluctuations near Planck scale, such that it would not be possible to analyse the geometric structures below Planck scale [1]-[2]. Furthermore, as there is strong evidence of the existence of macroscopic black holes, the virtual black holes are expected to form due to these quantum fluctuations of spacetime. This is because all approaches to quantum gravity should produce classical general relativity, in the classical limit. Thus, the formation of black holes should be allowed in all approaches to quantum gravity, and hence, quantum fluctuations at Planck scale should produce virtual black holes, in all these approaches. So, even though we will study the virtual black holes using a specific approach, it is expected that such virtual black holes will also occur using other approaches to quantum gravity.

We will analyse the virtual black holes using the Wheeler-DeWitt equation, as the quantum theory of black holes has been studied using the Wheeler-DeWitt equation [3]-[6]. The mass of the black holes changes dynamically with time, so a variable relating to time is obtained from the mass of the black hole, and this Wheeler-DeWitt equation is constructed using such a variable. Now as the black holes are valid quantum states in this theory, they can also be produced from quantum fluctuations. Such quantum fluctuations in the geometry, which change the topology of spacetime would occur near Planck scale. So, it is expected that the spacetime at Planck scale would be filled by a sea of virtual black holes [7]-[8].

To analyse the formation of virtual black holes, we would require a formalism in which the geometries would be dynamically produced, and which would allow for topology changing processes to occur. Thus, we will use the third quantized formalism to analyse virtual black holes [8]. This is because second quantized Wheeler-DeWitt equation is the functional Schrödinger equation describing the quantum state of geometries, [9]-[10], and it is not possible to analyse the dynamic creation or annihilation of geometries, or even a multi-geometry state, using this second quantized Wheeler-DeWitt equation. In fact, it is well known that it is not possible to analyse topology changing processes using the second quantized Wheeler-DeWitt equation. However, the creation and annihilation of virtual black holes is a process that changes the topology of spacetime. Hence, it cannot be properly analysed using a second quantized Wheeler-DeWitt equation. This is similar to the situation in first quantized formalism with particle creation and annihilation, as it is not possible to describe the dynamic creation or annihilation of particles, or even a free multi-particle state using a first quantized single particle Schrödinger equation. However, just as we can describe the dynamic creation and annihilation of particles in second quantized formalism, we can analyse the creation and annihilation of geometries in the third quantized formalism [11]-[18]. So, in the third quantized formalism, the Wheeler-DeWitt equation is not viewed as a quantum mechanical functional Schrödinger equation describing the quantum evolution of the wave function of geometries, but rather as a classical field equation. This equation is third quantized, and the creation and annihilation operators thus obtained make it possible to create and annihilate different geometries. Thus, the third quantized Wheeler-DeWitt equation is usually used for analysing the multiverse [19]-[24], but here we will use it for analyzing virtual black holes [8].
It may be noted that this third quantized model of virtual black holes has been used to motivate an interesting solution to the problem of time in quantum gravity [8]. According to this proposal, the entropy of the universe would increase due to the interaction of virtual black holes with all other particles. This is because the particles would fall into these virtual black holes and be radiated as different particles. This would cause information to be lost, and this lost information would increase the entropy of the universe. Thus, it would be possible to define a direction of time by identifying time with the increase of the entropy of the universe. The loss of quantum coherence through scattering off virtual black holes has also been studied [7]. In this analysis, a quantum field on the C metric, was analysed. As the C metric has the same topology as a pair of virtual black holes, it was argued that such processes can be used for understanding some important features of virtual black holes. A scalar field theory was analysed on this metric, and it was explicitly demonstrated that there is a loss of quantum coherence. This calculation was generalized to include higher spin field [25]. The transmission coefficient were calculated and used for analysing the the loss of quantum coherence of an incident field through scattering off virtual black holes. Virtual black holes in two dimensional quantum gravity have also been analysed [26]-[27]. Virtual black holes have also been studied in the context using generalized dilaton theories [28]. The phenomenological aspects of virtual black holes have also been studied [29].

The virtual black holes have also been used for explaining the end state of evaporation of real black holes [30]. According to this model, the real black holes reduce in size due to Hawking radiation. This process continues till they reach Planck size. At this point they are lost in a sea of virtual black holes. However, this model implicitly relies on the assumption that the solutions to the Wheeler-DeWitt equation for a black hole is dominated by quantum fluctuations when the black hole becomes sufficiently small. So, it is important to analyse the effects of quantum fluctuations on the solutions of the Wheeler-DeWitt equation. The fluctuations in the geometry of the universe has been studied using third quantization [31]. The third quantized formalism has also been used for analysing fluctuations for Brans-Dicke theories [32], f(R) gravity theories [33]-[34], and Kaluza-Klein theories [35]. Third quantization has also been used for analysing the string perturbative vacuum [36]. However, the fluctuations in the geometry of a black hole have never been analysed using the third quantized formalism. It is important to analyse such fluctuations in the geometry of a black hole, as the virtual black hole model of spacetime foam relies on the assumption that such fluctuations dominate the geometry at Planck scale. So, in this paper, we will analyse the effect of such fluctuations. We will observe that such fluctuations depend critically on the factor ordering chosen. Hence, only certain factor orderings are consistent with virtual black hole model of spacetime foam. It has been observed that factor ordering can have real physical consequences [37]-[39]. So, such a dependence of the physics of virtual black holes on the factor ordering chosen was something that was expected. We will demonstrate this to be the case explicitly in this paper.

In section 2, we will consider the third quantization of the black hole. In section 3, we will analyse the uncertainty relation for the black hole. In section 4, we will study the operator ordering for the Wheeler-DeWitt equation of the black hole. Conclusion will be given in section 5.
2 Third Quantization of Black Hole

In this section, we will analyse the third quantization of virtual black holes. This will be done by first analyzing the Wheeler-DeWitt equation for a real black hole, in the minisuperspace approximation. As the mass of the black hole changes with time, such a mass can be used to obtain a suitable time variable, for such a system. Then the Wheeler-DeWitt equation can be constructed using this time variable. We will third quantize this second quantized Wheeler-DeWitt equation, and this will allow the dynamic creation and annihilation of such black holes. Now as the theory will allow the creation and annihilation of real black holes, it would be expected that the creation and annihilation of virtual black holes will also occur at Planck scale, due to quantum fluctuations. Furthermore, as the time variable will be expressed in terms of the mass of the black hole, such quantum fluctuations can be expressed in terms of the mass of the black hole. So, we can start with the classical Hamiltonian for a Schwarzschild black hole, \( H = \frac{p_o^2}{2a} + \frac{a}{2} \) [3]-[6], where \( a \) roughly corresponds to the size (mass) of the black hole and \( p_o \) is the momentum conjugate to \( a \). Both \( a \) and \( p_o \) are related to \( m \) and \( p_m \) through a canonical transformation. Here, we have \( m(t) := M(t, r) \) and \( p_m(t) := \int_{-\infty}^{\infty} dr p_M(t, r) \), and so \( m \) is related to the mass of the black hole, which is denoted by \( M \). This relation holds for a spherically symmetric spacetime, and so the physics of the system is expressed by the mass of the black hole. As we are interested in the process that the mass evaporates due to the Hawking radiation, it can be used to construct a suitable time variable for the black hole. The variable \( a \) satisfies \( 0 < a \leq 2M \). We consider the course that the black hole starts from the limit \( a \to 2M \), becomes smaller owing to the Hawking radiation, and ends at the limit \( a \to 0 \). So let us define \( b \equiv 2M - a \), and we regard \( b \) as the time variable for the black hole. Now we can write the Wheeler-DeWitt equation for the Schwarzschild black hole as [3]-[6],

\[
\left[ \frac{1}{ap_o+1} \frac{d}{da} a^{p_o} \frac{d}{da} - (a - 2M) \right] \psi(a) = 0 ,
\]

which is equivalent to

\[
\left[ \frac{d^2}{da^2} + \frac{p_o}{a} \frac{d}{da} - (a^2 - 2Ma) \right] \psi(a) = 0 , \tag{2.1}
\]

where \( p_o \) is the operator ordering parameter. Now we rewrite these equations by using our time variable \( b \) as

\[
\left[ \frac{1}{(2M - b)p_o+1} \frac{d}{d(2M - b)} (2M - b)^{p_o} \frac{d}{d(2M - b)} - [(2M - b) - 2M] \right] \psi(b) = 0 ,
\]

which is equivalent to

\[
\left[ \frac{d^2}{db^2} - \frac{p_o}{2M - b} \frac{d}{db} - [(2M - b)^2 - 2M(2M - b)] \right] \psi(b) = 0 . \tag{2.2}
\]
The Lagrangian for the third quantization whose variation gives Eq. (2.2) is
\[
L_{3Q} = \frac{1}{2} \left[ (2M - b)\rho_o \left( \frac{d\psi(b)}{db} \right)^2 + (2M - b)^2 \rho_o [(2M - b)^2 - 2M(2M - b)]\psi(b)^2 \right].
\] (2.3)

Thus, by defining
\[
S_{3Q} = \int db \ L_{3Q},
\] (2.4)
we can get Eq. (2.2) from \( \delta S_{3Q} = 0 \). The momentum canonically conjugate to \( \psi(b) \) is defined as
\[
\pi(b) = \frac{\partial L_{3Q}}{\partial \left( \frac{d\psi(b)}{db} \right)} = (2M - b)\rho_o \frac{d\psi(b)}{db}.
\] (2.5)

The Hamiltonian for the third quantization can be written as
\[
H_{3Q} = \pi(b) \frac{d\psi(b)}{db} - L_{3Q},
\] (2.6)

\[
= \frac{1}{2} \left[ \frac{1}{(2M - b)\rho_o} \pi(b)^2 - (2M - b)^2 \rho_o [(2M - b)^2 - 2M(2M - b)]\psi(b)^2 \right].
\]

Now we can third quantize this theory by imposing the following equal time commutation relation,
\[
[\hat{\psi}(b), \hat{\pi}(b)] = i,
\] (2.7)
where a hat denotes that we are dealing with an operator. If we take the Schrödinger picture, we have the time-independent c-number \( \psi \) for the operator \( \hat{\psi}(b) \). Therefore, we can rewrite the operators as
\[
\hat{\psi}(b) \rightarrow \psi, \quad \hat{\pi}(b) \rightarrow -i \frac{\partial}{\partial \psi}.
\] (2.8)

Thus, we have the Schrödinger equation for virtual black holes,
\[
i \frac{\partial \Psi(b, \psi)}{\partial b} = \hat{H}_{3Q} \Psi(b, \psi),
\]
\[
\hat{H}_{3Q} = \frac{1}{2} \left[ -\frac{1}{(2M - b)\rho_o} \frac{\partial^2}{\partial \psi^2} - (2M - b)^2 \rho_o [(2M - b)^2 - 2M(2M - b)]\psi^2 \right].
\] (2.9)

Here \( \Psi(b, \psi) \) is the third quantized wave function of the black hole.
3 Uncertainty Relation

In the previous section, we described the black holes using a third quantized formalism. In this section, we will analyse the uncertainty relation for such a model. It is important to analyse the uncertainty for black holes, as we expect the quantum fluctuations to dominate near the Planck scale for the virtual black hole model to be consistent. We assume that the solution to Eq. (2.9) has the Gaussian form [32]-[34], [39]

$$\Psi(b, \psi) = C \exp \left\{ -\frac{1}{2} A(b)[\psi - \eta(b)]^2 + iB(b)[\psi - \eta(b)] \right\}, \quad (3.1)$$

where $C$ is a real constant, and $A(b) = D(b) + iI(b)$. Here $A(b), B(b), \eta(b)$ must be determined from Eq. (2.9). Note that in order for Eq. (3.1) to satisfy Eq. (2.9) $D(b)$ and $I(b)$ are real functions, but $B(b)$ and $\eta(b)$ are complex functions in general. We can define the inner product of two third quantized wave functions $\Psi_1$ and $\Psi_2$ as

$$\langle \Psi_1, \Psi_2 \rangle \equiv \int_{-\infty}^{\infty} d\psi \Psi_1^*(b, \psi)\Psi_2(b, \psi). \quad (3.2)$$

Now we define the inner product of the Gaussian form wave function as

$$\langle \Psi, \Psi \rangle = C^2 \sqrt{\frac{\pi}{D(b)}} \exp E(b), \quad (3.3)$$

where $E(b)$ is a real function, such that

$$E(b) = \frac{1}{2[A(b) + A^*(b)]} \times \{-A(b)A^*(b)[\eta(b) - \eta^*(b)]^2 - [B(b) - B^*(b)]^2 - 2i[A^*(b)B(b) + A(b)B^*(b)][\eta(b) - \eta^*(b)]\}.$$

Now we calculate the Heisenberg’s uncertainty relation, and this is done by defining the dispersion of $\psi$ as

$$(\Delta \psi)^2 \equiv \langle \psi^2 \rangle - \langle \psi \rangle^2, \quad \langle \psi^2 \rangle \equiv \frac{\langle \Psi, \psi^2 \Psi \rangle}{\langle \Psi, \Psi \rangle}. \quad (3.4)$$

From Eqs. (3.1), (3.2), (3.3) and (3.4), we obtain

$$\langle \psi^2 \rangle = \frac{1}{2D(b)} + F^2(b), \quad \langle \psi \rangle = F(b), \quad \text{and} \quad (\Delta \psi)^2 = \frac{1}{2D(b)}, \quad (3.5)$$

where $F(b)$ is a real function, such that

$$F(b) = \frac{A(b)\eta(b) + A^*(b)\eta^*(b) + i[B(b) - B^*(b)]}{A(b) + A^*(b)}.$$

We define the dispersion of $\pi$ as

$$(\Delta \pi)^2 \equiv \langle \pi^2 \rangle - \langle \pi \rangle^2, \quad \langle \pi^2 \rangle \equiv \frac{\langle \Psi, \pi^2 \Psi \rangle}{\langle \Psi, \Psi \rangle}. \quad (3.6)$$
Then we can write
\[
\langle \pi^2 \rangle = \frac{D(b)}{2} + \frac{I^2(b)}{2D(b)} + G^2(b), \quad \langle \pi \rangle = G(b), \tag{3.7}
\]
and
\[
(\Delta \pi)^2 = \frac{D(b)}{2} + \frac{I^2(b)}{2D(b)},
\]
where \(G(b)\) is a real function, such that
\[
G(b) = \frac{1}{A(b) + A^*(b)} \{ A(b)B^*(b) + A^*(b)B(b) - iA(b)A^*(b)[\eta(b) - \eta^*(b)] \}.
\]

So, we can write the Heisenberg’s uncertainty as
\[
(\Delta \psi)^2(\Delta \pi)^2 = \frac{1}{4} \left( 1 + \frac{I^2(b)}{D^2(b)} \right). \tag{3.8}
\]

If we substitute the assumption (3.1) to Eq. (2.9), we can obtain the equation for \(A(b)\) as
\[
-i \frac{dA(b)}{db} = -\frac{1}{(2M - b)p_o} A(b)^2 - (2M - b)^{p_o} [(2M - b)^2 - 2M(2M - b)] . \tag{3.9}
\]
(Note that three complex equations for \(A(b), B(b), \eta(b)\) can be obtained by comparing the order of \(\psi\) in Eq. (2.9). However, Eq. (3.9) is enough for the calculation of the Heisenberg’s uncertainty relation.) Let us define
\[
\sigma \equiv (2M - b)^{1-p_o} = a^{1-p_o}, \quad (p_o \neq 1). \tag{3.10}
\]
then we have
\[
i(1-p_o) \frac{dA(\sigma)}{d\sigma} + A(\sigma)^2 + \sigma^{\frac{2p_o+2}{1-p_o}} - 2M \sigma^{\frac{2p_o+1}{1-p_o}} u(\sigma) = 0 . \tag{3.11}
\]
Defining a function \(u(\sigma)\) by the equation,
\[
A(\sigma) = i(1-p_o) \frac{d \ln u(\sigma)}{d\sigma}, \tag{3.12}
\]
we obtain
\[
\frac{d^2 u(\sigma)}{d\sigma^2} - \frac{1}{(1-p_o)^2} \sigma^{\frac{2p_o+2}{1-p_o}} u(\sigma) + \frac{2M}{(1-p_o)^2} \sigma^{\frac{2p_o+1}{1-p_o}} u(\sigma) = 0 ,
\]
\[
\text{namely}
\frac{d^2 u(\sigma)}{d\sigma^2} - \frac{1}{(1-p_o)^2} \sigma^{\frac{2p_o+1}{1-p_o}} (a - 2M) u(\sigma) = 0 . \tag{3.13}
\]
These equations are too complicated to be solved analytically. However, since we are interested in the late time limit \((b \to 2M, a \to 0)\), we can neglect the second term of the first equation in Eqs. (3.13) in this limit. So we obtain
\[
\frac{d^2 u(\sigma)}{d\sigma^2} + \frac{2M}{(1-p_o)^2} \sigma^{\frac{2p_o+1}{1-p_o}} u(\sigma) = 0 . \tag{3.14}
\]
We can solve this equation using a Bessel function as
\[ u(\sigma) = \sigma^{\frac{1}{2}} B_{n-\frac{p}{3}} \left( \frac{2}{3} \sqrt{2M} \sigma \frac{1}{n-\frac{p}{3}} \right) , \] (3.15)
where \( B \) is a Bessel function that satisfies [40]
\[ \frac{d^2B_{n-\frac{p}{3}}(z)}{dz^2} + \frac{1}{z} \frac{dB_{n-\frac{p}{3}}(z)}{dz} + \left( 1 - \frac{(1-\frac{p}{3})^2}{z^2} \right) B_{n-\frac{p}{3}}(z) = 0 . \] (3.16)

So, the general solution to Eq. (3.14), can be written as
\[ u(\sigma) = cJ_{n-\frac{p}{3}} + cY_{n-\frac{p}{3}} \left( \frac{2}{3} \sqrt{2M} \sigma \frac{1}{n-\frac{p}{3}} \right) , \] (3.17)
where \( c_J \) and \( c_Y \) are arbitrary complex constants and \( J_{n-\frac{p}{3}} \) and \( Y_{n-\frac{p}{3}} \) are Bessel functions.

Now let us define
\[ z \equiv \frac{2}{3} \sqrt{2M} \sigma \frac{1}{n-\frac{p}{3}} = \frac{2}{3} \sqrt{2M} (2 - b)^\frac{1}{3} = \frac{2}{3} \sqrt{2M} a^\frac{1}{3} , \] (3.18)
then Eq. (3.17) becomes
\[ u(z) = \left( z \frac{2}{3} \sqrt{2M} \right)^\frac{1}{n-\frac{p}{3}} \left[ c_J J_{n-\frac{p}{3}} + c_Y Y_{n-\frac{p}{3}} \right] . \] (3.19)

We can obtain from Eqs. (3.12), (3.18), (3.19)
\[ A(z) = i(1 - \frac{p}{3}) \frac{dz}{d\sigma} \frac{d \ln u(z)}{dz} \]
\[ = i \sqrt{2M} \left( z \frac{2}{3} \sqrt{2M} \right)^\frac{1}{n-\frac{p}{3}} \left[ c_J J_{n-\frac{p}{3}} + c_Y Y_{n-\frac{p}{3}} \right] . \] (3.20)

Here we have used [40]
\[ \frac{dB_{n-\frac{p}{3}}(z)}{dz} = B_{n-\frac{p}{3}}(z) - \frac{1 - \frac{p}{3}}{3z} B_{n-\frac{p}{3}}(z) . \] (3.21)

As \( A(z) = D(z) + i I(z) \), we obtain
\[ D(z) = \frac{-i \sqrt{2M} \left( z \frac{2}{3} \sqrt{2M} \right)^\frac{1}{n-\frac{p}{3}}}{\pi z |c_J J_{n-\frac{p}{3}}(z) + c_Y Y_{n-\frac{p}{3}}(z)|^2} (c_J c_Y - c_J^* c_Y) , \] (3.22)
where we have used [40]
\[ J_{n-\frac{p}{3}}(z) Y_{n-\frac{p}{3}}(z) - J_{n-\frac{p}{3}}(z) Y_{n-\frac{p}{3}}(z) = \frac{2}{\pi z} . \] (3.23)
(Note that \( c_J c_Y^* - c_Y c_Y^* \) is a pure imaginary number.) We can obtain

\[
I(z) = \frac{\sqrt{2M}}{4\sqrt{2M}c_JJ_{1-p_o}(z) + c_YY_{1-p_o}(z)}^{2p_o+1} \\
\times \left[ 2|c_J|^2J_{-2-p_o}(z)J_{1-p_o}(z) + 2|c_Y|^2Y_{-2-p_o}(z)Y_{1-p_o}(z) \right. \\
+ (c_J c_Y^* + c_Y^* c_Y)(J_{-2-p_o}(z)Y_{1-p_o}(z) \\
\left. + J_{1-p_o}(z)Y_{-2-p_o}(z)) \right].
\]

So, assuming \( c_J c_Y^* - c_Y c_Y^* \neq 0 \) (note that this means both of \( c_J, c_Y \) are nonzero), we have

\[
\frac{I(z)^2}{D(z)^2} = -\frac{\pi^2 z^2}{4(c_J c_Y^* - c_Y c_Y^*)^2} \\
\times \left[ 2|c_J|^2J_{-2-p_o}(z)J_{1-p_o}(z) + 2|c_Y|^2Y_{-2-p_o}(z)Y_{1-p_o}(z) \right. \\
+ (c_J c_Y^* + c_Y^* c_Y)(J_{-2-p_o}(z)Y_{1-p_o}(z) \\
\left. + J_{1-p_o}(z)Y_{-2-p_o}(z)) \right]^2.
\]

If we substitute Eq. (3.25) to Eq. (3.8), we can have the Heisenberg’s uncertainty relation. We can use this uncertainty relation to discuss the behavior of the black hole at the last stage of its evaporation.

At the early time limit ( \( b \to 0, a \to 2M, \sigma \to (2M)^{1-p_o} \) ), so we change the variable from \( \sigma \) to \( \alpha \),

\[
\sigma = (2M)^{1-p_o} + \alpha.
\]

This limit means \( \alpha \to 0 \), and we should consider only \( O(\alpha^1) \). Then we obtain from Eqs. (3.13) and Eq. (3.26),

\[
\frac{d^2 u(\alpha)}{d\alpha^2} - \frac{1}{(1 - p_o)^2}(2M)^{1-p_o} + \alpha \frac{2p_o+2}{1 - p_o} u(\alpha) \\
\approx \frac{d^2 u(\alpha)}{d\alpha^2} - \frac{1}{(1 - p_o)^2}(2M)^{2p_o+2} \left[ \frac{1}{1 - p_o} \frac{2p_o + 2}{(2M)^{1-p_o}} \right. \\
\left. \frac{1}{1 - p_o} \frac{1}{1 - p_o} \frac{1}{(2M)^{1-p_o}} \right] u(\alpha) \\
= \frac{d^2 u(\alpha)}{d\alpha^2} - (2M)^{3p_o+1} \alpha u(\alpha) = 0.
\]

This last equation has the solution

\[
u(\alpha) = \alpha^{1/2} \mathcal{B}_{\frac{3}{2}} \left( \frac{2i (2M)^{3p_o+1}}{3 (1 - p_o)^2} \alpha^{3/2} \right) \quad (p_o < 1),
\]

\[9\]
and
\[
  u(\alpha) = a^{\frac{3}{2}}B_{\frac{1}{2}} \left( \frac{2}{3} \frac{(2M)^{\frac{3p_o+1}{2}}}{(p_o - 1)^{\frac{1}{2}}} \alpha^{\frac{3}{2}} \right) \quad (p_o > 1),
\]

where \( B \) is the Bessel function.

When \( p_o < 1 \), if we write
\[
  z = \frac{2i (2M)^{\frac{3p_o+1}{2}}}{3 (1 - p_o)^{\frac{1}{2}}} \alpha^{\frac{3}{2}},
\]
the general solution to the last equation of Eqs. (3.27) is
\[
  u(z) = \left( -\frac{3i}{2} \left( \frac{1}{2} \right)^{\frac{3p_o+1}{2}} \right) \left[ c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z) \right] .
\]

Therefore, we obtain from Eqs. (3.12) and (3.31)
\[
  A(z) = -\left( -\frac{3i}{2} \left( \frac{1}{2} \right)^{\frac{3p_o+1}{2}} \right)^{\frac{1}{2}} \frac{c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z)}{c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z)},
\]
where we have used the similar equation as Eq. (3.21). Since \( A(z) = D(z) + iI(z) \), after a short calculation we can obtain
\[
  \frac{I(z)^2}{D(z)^2} = -\frac{(c_j^2 p_+ + |c_y|^2 q_+ + c_j c_y r_+ + c_j^2 c_y s_-)^2}{(c_j^2 p_+ + |c_y|^2 q_+ + c_j c_y r_+ + c_j^2 c_y s_-)^2},
\]
where
\[
  p_\pm = J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z) \pm J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z)
\]
\[
  q_\pm = Y_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z) \pm Y_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z)
\]
\[
  r_\pm = J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z) \pm J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z)
\]
\[
  s_\pm = J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z) \pm J_{\frac{3}{2}}(z) Y_{\frac{3}{2}}(-z).
\]

From these equations and Eq. (3.8), we will obtain the Heisenberg uncertainty relation at the early time limit when \( p_o < 1 \) in the next section.

When \( p_o > 1 \), if we write
\[
  z = \frac{2}{3} \frac{(2M)^{\frac{3p_o+1}{2}}}{(p_o - 1)^{\frac{1}{2}}} \alpha^{\frac{3}{2}},
\]
the general solution to the last equation of Eqs. (3.27) is
\[
  u(z) = \left( \frac{3}{2} \frac{(p_o - 1)^{\frac{1}{2}}}{(2M)^{\frac{3p_o+1}{2}}} z^{\frac{1}{2}} \right) \left[ c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z) \right].
\]

Therefore, we obtain from Eqs. (3.12) and (3.35)
\[
  A(z) = -i \left( \frac{3}{2} \frac{(p_o - 1)^{\frac{1}{2}}}{(2M)^{\frac{3p_o+1}{2}}} z^{\frac{1}{2}} \right) \frac{c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z)}{c_j J_{\frac{3}{2}}(z) + c_y Y_{\frac{3}{2}}(z)},
\]

10
where we have used the similar equation as Eq. (3.21). After the similar calculation as in Eq. (3.25) we have

\[
\frac{I(z)^2}{D(z)^2} = -\frac{\pi^2 z^2}{4(c_Jc_Y^* - c_J^*c_Y)^2} \times \left[ 2|c_J|^2 J_{-\frac{3}{2}}(z)J_{\frac{1}{2}}(z) + 2|c_Y|^2 Y_{-\frac{3}{2}}(z)Y_{\frac{1}{2}}(z) ight. \\
\left. + (c_Jc_Y^* + c_J^*c_Y)(J_{-\frac{3}{2}}(z)Y_{\frac{1}{2}}(z) + J_{\frac{1}{2}}(z)Y_{-\frac{3}{2}}(z)) \right] ^2,
\]

which will be used to obtain the Heisenberg uncertainty relation at the early time limit when \(p_o > 1\).

4 Operator Ordering

In this section, we will analyse the effect of operator ordering on the quantum fluctuations for the black hole. At late times namely when \(b \to 2M, a \to 0\) i.e., \(z \to 0\) from Eq. (3.18), we should divide the cases by the value of the operator ordering parameter \(p_o\). For example, when we choose

\[ p_o = 2, \]

as we obtain at late times [40]

\[
J_{-\frac{3}{2}}(z) \sim -\frac{1}{3\Gamma\left(\frac{2}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{5}{4}}, \quad J_{\frac{1}{2}}(z) \sim \frac{1}{\Gamma\left(\frac{2}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{1}{4}}, \\
Y_{-\frac{3}{2}}(z) \sim \frac{1}{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{5}{4}}, \quad Y_{\frac{1}{2}}(z) \sim -\frac{1}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{1}{4}},
\]

we can obtain from Eq. (3.25)

\[
\frac{I(z)^2}{D(z)^2} \sim -\frac{\pi^2}{9 \left(\Gamma\left(\frac{2}{3}\right)\right)^4 (c_Jc_Y^* - c_J^*c_Y)^2} \times \left[ -2|c_J|^2 - \frac{2}{3}|c_Y|^2 + \frac{2}{\sqrt{3}}(c_Jc_Y^* + c_J^*c_Y) \right] ^2 \left(\frac{z}{2}\right)^{-\frac{5}{4}} \to \infty.
\]

This and Eq. (3.8) show that at the end stage of evaporation of the black hole i.e., in the limit \(a \to 0\), the quantum fluctuations dominate the black hole. This is what is required in the virtual black hole model. It may be noted that here \(a\) is related to the mass of the black hole. Thus, as the mass of the black hole becomes small, quantum fluctuations increase, and in the limit \(a \to 0\), the black hole gets lost in a sea of virtual black holes produced by quantum fluctuations.

However, a different choice of operator ordering can produce a different result. For example, when we choose

\[ p_o = 0, \]
as we obtain at late times \[40\]

\[
J_{-\frac{2}{3}}(z) \sim \frac{1}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{2}{3}}, \quad J_{\frac{1}{3}}(z) \sim \frac{3}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{z}{2}\right)^{\frac{1}{3}},
\]

\[
Y_{-\frac{2}{3}}(z) \sim \frac{1}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{2}{3}}, \quad Y_{\frac{1}{3}}(z) \sim -\frac{\Gamma\left(\frac{1}{3}\right)}{\pi} \left(\frac{z}{2}\right)^{-\frac{1}{3}},
\]

we can have

\[
\frac{I(z)^2}{D(z)^2} \sim -\frac{1}{(c_J c^*_Y - c_J^* c_Y)^2} \left[ \frac{2|c_Y|^2 + (c_J c^*_Y + c_J^* c_Y)}{\sqrt{3}} \right]^2 \sim O(1).
\]

\[(4.5)\]

This and Eq. (3.8) indicate that at the end stage of evaporation of the black hole, i.e., in the limit \(a \to 0\), the black hole would become classical in the sense that the quantum fluctuations become minimum. Thus, at the end stage of the evaporation of the black hole, virtual black holes cannot be produced from quantum fluctuations, as the quantum fluctuations become minimum at this stage. Thus, this choice of operator ordering is not consistent with the existence of virtual black holes. It is interesting to note that we have demonstrated that only certain factor orderings are consistent with the existence of virtual black holes.

On the other hand at the early time limit (\(a \to 2M\), which means \(a \to 0\) by Eq. (3.26), we must consider the cases \(p_o < 1\) and \(p_o > 1\) separately.

When \(p_o < 1\) the early time limit means \(z \to 0\), where \(z\) is defined in Eq. (3.30). Using the relations (4.5) and those which have the argument \(-z\), we obtain from Eq. (3.33)

\[
\frac{I(z)^2}{D(z)^2} \sim -\frac{1}{\sqrt{3}|c_Y|^2} \left[ e^{-\frac{\pi}{3}i} - e^{-\frac{2\pi}{3}i} \right] \left[ c_J c^*_Y e^{-\frac{\pi}{3}i} - c_J^* c_Y e^{-\frac{2\pi}{3}i} \right]^2 \sim O(1).
\]

\[(4.6)\]

When \(p_o > 1\) the early time limit means \(z \to 0\), where \(z\) is defined in Eq. (3.34). We use Eq. (3.37) and the relations (4.5), so we obtain the same result as in the relations (4.6), though \(z\) is defined in Eq. (3.34). Hence, both in the cases \(p_o < 1\) and \(p_o > 1\), the black hole would become classical at the early time limit, since the quantum fluctuations become minimum for this limit.

## 5 Conclusion

In this paper, we have analysed the effect of quantum fluctuations on the geometry of a black hole. We demonstrated that the quantum fluctuations for the black hole depend strongly on the factor ordering chosen. So, for a certain value of the factor ordering parameter the quantum fluctuations dominate at
the Planck scale. However, for another value of factor ordering parameter, the geometry remains classical even near Planck scale. Hence, only certain values of factor ordering parameter are consistent with the proposal of describing spacetime foam in terms of virtual black holes.

It will be interesting to analyse virtual black holes in some UV complete theory of gravity, like the Horava-Lifshitz gravity [41]-[42]. In this theory, space and time have different Lifshitz scaling, and it reduces to general relativity in the IR limit. It may be noted that the Wheeler-DeWitt equation for the Horava-Lifshitz gravity has been studied [43]-[44], and the third quantization of the Horava-Lifshitz gravity has also been performed [19]. It will be interesting to analyse the Wheeler-DeWitt equation for a black hole in Horava-Lifshitz gravity, and analyse the virtual black hole model using such a Wheeler-DeWitt equation. It may also be noted that the modification of the Wheeler-DeWitt equation from the generalized uncertainty principle has also been studied [3]. In this Wheeler-DeWitt equation, we can obtain higher derivative corrections. The third quantization of such a Wheeler-DeWitt equation for cosmology has already been analysed [45]. It would be interesting to perform a similar analysis for the Wheeler-DeWitt equation of a black hole. It has also been argued that virtual black holes can lead to a vanishing of the QCD $\theta$ parameter [30]. This occurs as virtual black holes can produce a loss of coherence between the different vacuum states, and this can lead to the vanishing of the QCD $\theta$ parameter. It would be interesting to derive this result using the third quantized formalism.

References

[1] S. Das and E. C. Vagenas, Phys. Rev. Lett. 101, 221301 (2008)
[2] W. A. Christiansen, D. J. E. Floyd, Y. J. Ng and E. S. Perlman, Phys. Rev. D 83, 084003 (2011)
[3] B. Majumder, Phys. Lett. B 701, 384 (2011)
[4] J. Makela and P. Repo, Phys. Rev. D 57 4899 (1998).
[5] O. Obregon, M. Sabido and V. I. Tkach, Gen. Relativ. Gravit. 33913 (2001)
[6] S. Jalalzadeh and B. Vakili, Int. J. Theor. Phys. 51, 263 (2012)
[7] S. W. Hawking and S. F. Ross, Phys. Rev. D 56, 6403 (1997)
[8] M. Faizal, JETP. 114, 400 (2012)
[9] B. S. DeWitt, Phys. Rev. 160, 1113 (1967)
[10] J. A. Wheeler, Ann. Phys. 2, 604 (1957)
[11] N. Caderni and M. Martellini, Int. J. Theor. Phys. 23, 233 (1984)
[12] M. McGuigan, Phys. Rev. D 38, 3031 (1988)
[13] M. Faizal, Class. Quant. Grav. 29, 215009 (2012)
[14] M. Faizal, Int. J. Geom. Meth. Mod. Phys. 11, 1450010 (2014)
[15] Y. Peleg, Class. Quant. Grav. 8, 827 (1991)
[16] M. Faizal, Phys. Lett. B 727, 536 (2013)
[17] M. Faizal, Grav. Cosmol. 20, 132 (2014)
[18] Y. Peleg, Mod. Phys. Lett. A 8, 1849 (1993)
[19] M. Faizal, Mod. Phys. Lett. A 27, 1250007 (2012)
[20] P. F. G. Diaz and S. R. Perez, Int. J. Mod. Phys. D 17, 1213 (2008)
[21] S. R. Perez, Y. Hassouni and P. F. G. Diaz, Phys. Lett. B 683, 1 (2010)
[22] S. R. Perez and P. F. G. Diaz, Phys. Rev. D 81, 083529 (2010)
[23] I. Garay and S. R. Perez, Int. J. Mod. Phys. D 23, 1450043 (2014)
[24] M. Faizal, Comm. Theor. Phys. 62, 697 (2014)
[25] T Prestidge, Phys. Rev. D 58, 124022 (1998)
[26] D. Grumiller, W. Kummer and D. V. Vassilevich, Nucl. Phys. B 580, 438 (2000)
[27] D. Grumiller, Class. Quant. Grav. 19, 997 (2002)
[28] D. Grumiller, W. Kummer, D. V. Vassilevich, Eur. Phys. J. C 30, 135 (2003)
[29] D. Grumiller, Int. J. Mod. Phys. D 13, 1973 (2004)
[30] S. W. Hawking, Phys. Rev. D 53, 3099 (1996)
[31] M. McGuigan, Phys. Rev. D 39, 2229 (1989)
[32] L. O. Pimentel and C. Mora, Phys. Lett. A 280, 191 (2001)
[33] Y. Ohkuwa and Y. Ezawa, Class. Quant. Grav. 29, 215004 (2012)
[34] Y. Ohkuwa and Y. Ezawa, Class. Quant. Grav. 30, 235015 (2013)
[35] Y. Ohkuwa, Int. J. Mod. Phys. A 13, 4091 (1998)
[36] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Class. Quant. Grav. 14, L97 (1997)
[37] N. Kontoleon and D. L. Wiltshire, Phys. Rev. D 59, 063513 (1999)
[38] D. L. Wiltshire, Gen. Rel. Grav. 32, 515 (2000)
[39] Y. Ohkuwa, M. Faizal and Y. Ezawa, Ann. Phys. 365, 54 (2016)
[40] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publishing, New York (1972)
[41] P. Horava, JHEP, 0903, 020 (2009)
[42] P. Horava, Phys. Rev. Lett. 102, 161301 (2009)
[43] B. Vakili and V. Kord, Gen. Rel. Grav. 45, 1313 (2013)

[44] M. Sakamoto, Phys. Rev. D 79, 124038 (2009)

[45] M. Faizal, Int. J. Mod. Phys. A 30, 1550036 (2015)