Noncommutative Instantons on $R^2_{NC} \times R^2_C$

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ABSTRACT

We study $U(1)$ and $U(2)$ noncommutative instantons on $R^2_{NC} \times R^2_C$ based on the ADHM construction. It is shown that a mild singularity in the instanton solutions for both self-dual and anti-self-dual gauge fields always disappears in gauge invariant quantities and thus physically regular solutions can be constructed even though any projected states are not involved in the ADHM construction. Furthermore the instanton number is also an integer.
1 Introduction

A noncommutative space is obtained by quantizing a given space with its symplectic structure, treating it as a phase space. Also field theories can be formulated on a noncommutative space. Noncommutative field theory means that fields are defined as functions over noncommutative spaces. At the algebraic level, the fields become operators acting on a Hilbert space as a representation space of the noncommutative space. Since the noncommutative space resembles a quantized phase space, the idea of localization in ordinary field theory is lost. The notion of a point is replaced by that of a state in representation space.

Instanton solutions in noncommutative Yang-Mills theory can also be studied by Atiyah-Drinfeld-Hitchin-Manin (ADHM) equation [1] slightly modified by the noncommutativity. Recently much progress has been made in this direction [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. The remarkable fact is that the deformation of the ADHM equation depends on the self-duality of the noncommutativity [5]. Anti-self-dual instantons on self-dual noncommutative $R^4_{\text{NC}}$ are described by a deformed ADHM equation adding a Fayet-Iliopoulos term to the usual ADHM equation and the singularity of instanton moduli space is resolved [2, 3]. However, self-dual instantons on self-dual $R^4_{\text{NC}}$ are described by an undeformed ADHM equation and the singularity of instanton moduli space still remains [11, 18]. This property is closely related to the BPS property of D0-D4 system [5]. The latter system is supersymmetric and BPS whereas the former is not BPS.

In this report, we study $U(1)$ and $U(2)$ noncommutative instantons on $R^2_{\text{NC}} \times R^2_{\text{C}}$, based on the ADHM construction where $R^2_{\text{NC}}$ is the noncommutative space but $R^2_{\text{C}}$ is the commutative space. It was already shown in [13] that the completeness relation in the ADHM construction is generally satisfied for $R^2_{\text{NC}} \times R^2_{\text{C}}$ as well as $R^4_{\text{NC}}$. Unlike $R^4_{\text{NC}}$, the ADHM equation for the noncommutative space $R^2_{\text{NC}} \times R^2_{\text{C}}$ is always deformed for self-dual and anti-self-dual gauge fields since both systems are not BPS states any more. This implies that the small instanton singularity of moduli space can be resolved for this case too. Actually, even though the instanton solutions for both self-dual and anti-self-dual gauge fields contain a mild singularity, i.e. a measure zero singularity, it always disappears in gauge invariant quantities and thus physically regular solutions can be constructed even though any projected states are not involved in the ADHM construction. Furthermore
the instanton number is always an integer as it should be. Our result is different from [18] by Chu, et al. claiming that there is no nonsingular \( U(N) \) instanton on \( \mathbb{R}_{NC}^2 \times \mathbb{R}_C^2 \) due to the breakdown of the completeness relation.

The paper is organized as follows. In next section we briefly review the ADHM construction of noncommutative instantons on \( \mathbb{R}_{NC}^2 \times \mathbb{R}_C^2 \). In section 3 we explicitly calculate the self-dual and anti-self-dual field strengths for single \( U(1) \) and \( U(2) \) instantons. We show that physically non-singular solutions can be constructed and they correctly give integer instanton numbers. In section 4 we discuss the results obtained and address some issues.

### 2 ADHM Equations on \( \mathbb{R}_{NC}^2 \times \mathbb{R}_C^2 \)

Let’s briefly review the ADHM construction on \( \mathbb{R}_{NC}^2 \times \mathbb{R}_C^2 \) where \( \mathbb{R}_{NC}^2 \) is the noncommutative space but \( \mathbb{R}_C^2 \) is the commutative space. This space is represented by the algebra generated by \( x^\mu \) obeying the commutation relation:

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu},
\]

where \( \mu, \nu = 1, 2, 3, 4 \) and the matrix \( \theta^{\mu\nu} \) is of rank-two. We set here \( \theta^{12} = \frac{\zeta}{2} = -\theta^{21} \) and \( \theta^{34} = 0 \). In terms of complex coordinates

\[
z_1 = x^2 + ix^1, \quad z_2 = x^4 + ix^3,
\]

the commutation relation (2.1) reduces to

\[
[\bar{z}_1, z_1] = \zeta, \quad [\bar{z}_2, z_2] = 0,
\]

which generates an operator algebra denoted as \( \mathcal{A} \). The commutation algebra on \( \bar{z}_1, z_1 \) is that of a simple harmonic oscillator and so one may use the Hilbert space \( \mathcal{H} = \sum_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}|n> \) as a representation of this algebra, where \( \bar{z}_1, z_1 \) are represented as the annihilation and the creation operators:

\[
\sqrt{\frac{1}{\zeta}}|n> = \sqrt{n}|n - 1>, \quad \sqrt{\frac{1}{\zeta}}|n> = \sqrt{n + 1}|n + 1>.
\]

Thus the integration on \( \mathbb{R}_{NC}^2 \times \mathbb{R}_C^2 \) for an operator \( \mathcal{O}(x) \) in \( \mathcal{A} \) can be replaced by

\[
\int d^4x \mathcal{O}(x) \rightarrow \zeta \pi \sum_{n \in \mathbb{Z}_{\geq 0}} \int d^2x \langle n|\mathcal{O}(x)|n\rangle,
\]
\[ d^2x = dx^3dx^4. \]

ADHM construction describes an algebraic way for finding (anti-)self-dual configurations of the gauge field in terms of some quadratic matrix equations on four manifolds \[ \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \]. The ADHM construction can be generalized to the space \( \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \) under consideration \[ [18] \). In order to describe \( k \) instantons with gauge group \( U(N) \), one starts with the following data:

1. A pair of complex hermitian vector spaces \( V = \mathbb{C}^k \), \( W = \mathbb{C}^N \).

2. The operators \( B_1, B_2 \in \text{Hom}(V, V), \ I \in \text{Hom}(W, V) \) and \( J \in \text{Hom}(V, W) \) satisfying the equations

\[
\begin{align*}
\mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta, \tag{2.5} \\
\mu_c &= [B_1, B_2] + IJ = 0. \tag{2.6}
\end{align*}
\]

3. Define a Dirac operator \( D^\dagger : V \oplus V \oplus W \to V \oplus V \) by

\[
D^\dagger = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix} \tag{2.7}
\]

where

\[
\tau_z = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} -B_1 + z_1 \\ B_2 - z_2 \\ J \end{pmatrix} \tag{2.8}
\]

for anti-self-dual instantons and

\[
\tau_z = \begin{pmatrix} B_2 - \bar{z}_2 & B_1 + z_1 & I \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} -B_1 - z_1 \\ B_2 - \bar{z}_2 \\ J \end{pmatrix} \tag{2.9}
\]

for self-dual instantons.

The origin of the ADHM equations \( (2.5) \) and \( (2.6) \) is the so-called factorization condition:

\[
\tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z, \quad \tau_z \sigma_z = 0. \tag{2.10}
\]

Note that, unlike \( \mathbb{R}^4_{NC} \), the ADHM equation \( (2.5) \) for \( \mathbb{R}^2_{NC} \times \mathbb{R}^2_C \) is always deformed for self-dual and anti-self-dual instantons. According to the ADHM construction, one can get the gauge field (instanton solution) by the formula

\[
A_\mu = \psi^\dagger \partial_\mu \psi, \tag{2.11}
\]

where \( \psi : W \to V \oplus V \oplus W \) is \( N \) zero-modes of \( D^\dagger \), i.e.,

\[
D^\dagger \psi = 0. \tag{2.12}
\]
For given ADHM data and the zero mode condition (2.12), the following completeness relation has to be satisfied to construct (anti-)self-dual instantons from the gauge field 

$$\frac{1}{D^\dagger D}D^\dagger + \psi\psi^\dagger = 1.$$  

(2.13)

It was shown in [13] that this relation is always satisfied even for noncommutative spaces.

The space $\mathbb{R}_{NC}^2 \times \mathbb{R}_C^2$ doesn’t have any isolated singularity due to the factor $\mathbb{R}_{NC}^2$. However in this case it is a measure zero singularity, so it doesn’t cause any physical trouble although we don’t project out it. This property presents a striking contrast to $\mathbb{R}_{NC}^4 \[8, 9, 13, 18\]$ since this space is two-dimensional discrete lattice, so singularities are always separable. Actually, it will be shown that, for the space $\mathbb{R}_{NC}^2 \times \mathbb{R}_C^2$, the singularity in the instanton solutions always disappears in the gauge invariant quantities, e.g. $\text{Tr}_\mathcal{H} F^n$ where $\text{Tr}_\mathcal{H}$ is the integration over $\mathbb{R}_{NC}^2$, possibly including the group trace too. So in our ADHM construction we will not project out any state in $\mathcal{H}$ and thus the zero-modes (2.12) are normalized in usual way

$$\psi^\dagger \psi = 1.$$  

(2.14)

With the above relations, the anti-self-dual field strength $F_{ASD}$ can be calculated by the following formula

$$F_{ASD} = \psi^\dagger \left( d\tau_1^\dagger \frac{1}{\Delta_z} d\tau_z + d\sigma_z \frac{1}{\Delta_z} d\sigma_z^\dagger \right) \psi$$

$$= \psi^\dagger \begin{pmatrix} dz_1 \frac{1}{\Delta_z} d\bar{z}_1 - dz_2 \frac{1}{\Delta_z} d\bar{z}_2 & -2dz_1 \frac{1}{\Delta_z} d\bar{z}_2 & 0 \\
-2dz_2 \frac{1}{\Delta_z} d\bar{z}_1 & -dz_1 \frac{1}{\Delta_z} d\bar{z}_1 + dz_2 \frac{1}{\Delta_z} d\bar{z}_2 & 0 \\
0 & 0 & 0 \end{pmatrix} \psi, \quad (2.15)$$

where $\Delta_z = \tau_2^\dagger \tau_z = \sigma_z^\dagger \sigma_z$ has no zero-modes so it is invertible. Similarly, the self-dual field strength $F_{SD}$ can be calculated by

$$F_{SD} = \psi^\dagger \left( d\tau_1^\dagger \frac{1}{\Delta_z} d\tau_z + d\sigma_z \frac{1}{\Delta_z} d\sigma_z^\dagger \right) \psi$$

$$= \psi^\dagger \begin{pmatrix} dz_1 \frac{1}{\Delta_z} d\bar{z}_1 + dz_2 \frac{1}{\Delta_z} d\bar{z}_2 & 2dz_1 \frac{1}{\Delta_z} d\bar{z}_2 & 0 \\
-2dz_1 \frac{1}{\Delta_z} d\bar{z}_1 & -dz_1 \frac{1}{\Delta_z} d\bar{z}_1 - dz_2 \frac{1}{\Delta_z} d\bar{z}_2 & 0 \\
0 & 0 & 0 \end{pmatrix} \psi. \quad (2.16)$$

\footnote{If one insists on projecting out some states in $\mathcal{H}$, e.g. $|0\rangle$, then the whole $\mathbb{R}_C^2$ plane at $|0\rangle$ is necessarily projected out. A serious trouble here is that one cannot have a projection projecting out only offending states, e.g. $|0\rangle$ at $z_2 = 0$. This too excessive projection causes the breaking of the completeness relation (2.13) as shown in [18]. We thank the authors of [18] for this discussion.}
3 Instanton Solutions on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$

In this section we will solve the ADHM equation (2.12) for single $U(1)$ and $U(2)$ instantons and calculate both the anti-self-dual field strength (2.15) and the self-dual field strength (2.16). Also we will numerically calculate the topological charge for the solutions to show it is always an integer. It naturally turns out that, even though any state in $\mathcal{H}$ is not projected out, a mild singularity in the solution doesn’t cause any physical trouble and they can define physically regular solutions.

3.1 Anti-self-dual $U(1)$ Instanton

In this case the ADHM equation (2.12) can be solved in the exactly same way as the $\mathbb{R}^4_{NC}$ case, only keeping in mind the algebra (2.2). The solution $\psi = \psi_1 \oplus \psi_2 \oplus \xi$ in $V \oplus V \oplus W$ has the same form as the anti-self-dual instanton on self-dual $\mathbb{R}^4_{NC}$ [13]:

$$\psi_1 = \bar{z}_2 \sqrt{\frac{\zeta}{\delta \Delta}}, \quad \psi_2 = \bar{z}_1 \sqrt{\frac{\zeta}{\delta \Delta}}, \quad \xi = \sqrt{\frac{\delta}{\Delta}},$$

(3.1)

where $\delta = z_1 \bar{z}_1 + z_2 \bar{z}_2$ and $\Delta = \delta + \zeta$.

It is straightforward to calculate the anti-self-dual field strength $F_{ASD}$ for the solution (3.1) from (2.15):

$$F_{ASD} = \frac{\zeta}{\delta^2 \Delta^2} (z_1 \bar{z}_1 \Delta - z_2 \bar{z}_2 \delta)(dz_2 d\bar{z}_2 - dz_1 d\bar{z}_1) + 2 \frac{\zeta}{\Delta^2 \sqrt{\delta(\Delta + \zeta)}} z_2 \bar{z}_1 dz_2 d\bar{z}_1 + 2 \frac{\zeta}{\delta^2 \sqrt{\Delta(\delta - \zeta)}} z_1 \bar{z}_2 dz_1 d\bar{z}_2.$$  

(3.2)

One can see that the above field strength contains a (mild) singularity of the type $z_2/|z_2|$ in the second term at the state $|1\rangle$ and $z_2 = 0$ and in the third term at the state $|0\rangle$ and $z_2 = 0$. These singularities are placed at single point, the origin, of $\mathbb{R}^2_C$ only at $|0\rangle$ or $|1\rangle$. So these are measure zero singularities, which is very similar situation to the usual singular gauge $SU(2)$ instantons. There is no huge plane singularity claimed in [18].

Note that the field strength $F_{ASD}$ in noncommutative gauge theory is not a gauge invariant quantity. Rather the gauge invariant quantity is $\text{Tr}_\mathcal{H} F_{ASD}$ which is definitely singularity-free. Also the instanton density defined below is singularity-free. See the Fig. 1-a. Thus one can see that the singularity in (3.2) doesn’t cause any physical trouble and the physical quantities such as $\text{Tr}_\mathcal{H} F^n$ are well-defined although there is no projected state in $\mathcal{H}$.
Finally the topological charge can be easily calculated by using the prescription (2.4)

\[ Q = -\sum_{n=0}^{\infty} \int_{0}^{\infty} d\gamma Q_n(\gamma) \]

\[ = -\sum_{n=0}^{\infty} \int_{0}^{\infty} d\gamma \left[ \frac{(n(n+1)+\gamma)(n+\gamma)^2}{(n+\gamma)(n+\gamma+2)^4} \right] \]

\[ = -1, \]

where \( n = z_1 \bar{z}_1/\zeta \) and \( \gamma = (x_3^2 + x_4^2)/\zeta \). If one projects out the state |0⟩, one could not get −1 for \( Q \).

### 3.2 Anti-self-dual \( U(2) \) Instanton

For this case also the ADHM equation (2.12) can be solved in the exactly same way and the solution has the same form as the \( R_{NC}^4 \) case [13]:

\[ \psi = (\psi^{(1)} \psi^{(2)}) = \begin{pmatrix} \bar{z}_2 f & -z_1 g \\ \bar{z}_1 f & z_2 g \\ \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad (3.4) \]

where

\[ f = \sqrt{\frac{\rho^2 + \zeta}{\delta(\Delta + \rho^2)}}, \quad g = \sqrt{\frac{\rho^2}{\Delta(\Delta + \rho^2)}}, \quad \xi_1 = \sqrt{\frac{\delta}{\Delta + \rho^2}}, \quad \xi_2 = \sqrt{\frac{\Delta}{\Delta + \rho^2}}. \quad (3.5) \]

When the instanton size vanishes, that is \( \rho = 0 \), then \( g = 0 \), and, from (2.15), one can see that \( \psi^{(2)} \) does not contribute to the field strength. Therefore the structure of the \( U(2) \) instanton at \( \rho = 0 \) is completely determined by the minimal zero-mode \( \psi^{(1)} \) in the \( U(1) \) subgroup. This property is exactly same as the \( R_{NC}^4 \) case [8, 13].

The field strength \( F_{ASD} \) can be obtained from (2.13) with the solution (3.4):

\[ F_{ASD} = (d\bar{z}_2 \wedge dz_2 - d\bar{z}_1 \wedge dz_1) \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \]

\[ + d\bar{z}_1 \wedge dz_2 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} - dz_1 \wedge d\bar{z}_2 \begin{pmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{pmatrix}, \quad (3.6) \]

where

\[ a_{11} = \frac{\rho^2 + \zeta}{\delta(\delta + \rho^2)(\Delta + \rho^2)^2} (\bar{z}_2 \bar{z}_2 - (\Delta + \rho^2)z_1 \bar{z}_1), \]

\[ a_{12} = -\frac{1}{\delta} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{(\delta + \rho^2)(\Delta + \rho^2)}} \left( \frac{1}{\delta + \rho^2} + \frac{1}{\Delta + \rho^2} \right) z_1 \bar{z}_2, \]
\[ a_{22} = \frac{\rho^2}{\Delta(\Delta + \rho^2)^2(\Delta + \rho^2 + \zeta)} \left( (\Delta + \rho^2)(\bar{z}_1 \bar{z}_1 + \zeta) - (\Delta + \rho^2 + \zeta)\bar{z}_2 \bar{z}_2 \right), \]

\[ b_{11} = \frac{2(\rho^2 + \zeta)}{\delta + \rho^2} \sqrt{\frac{1}{(\delta - \zeta)\delta(\delta + \rho^2)(\Delta + \rho^2)}} \bar{z}_1 \bar{z}_2, \quad (3.7) \]

\[ b_{12} = -\frac{2}{\delta + \rho^2} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{\delta \Delta}} \bar{z}_1 \bar{z}_1, \]

\[ b_{21} = \frac{2}{(\Delta + \rho^2)^2} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{\delta \Delta}} \bar{z}_2 \bar{z}_2, \]

\[ b_{22} = -\frac{2\rho^2}{\Delta + \rho^2} \sqrt{\frac{1}{\delta(\delta + \rho^2)\Delta(\Delta + \rho^2)}} \bar{z}_1 \bar{z}_2. \]

It can be confirmed again to recover the ordinary \( SU(2) \) instanton solution in the \( \zeta = 0 \) limit and the \( U(1) \) solution \( (3.2) \) for the limit \( \rho = 0 \) where only \( a_{11} \) and \( b_{11} \) terms in \( (3.6) \) survive. It is a pleasant property that the solution shows smooth behaviors with respect to \( \rho \) and \( \zeta \) (except only \( \rho = \zeta = 0 \)).

One can explicitly check that the field strength \( (3.6) \) has the exactly same kind of singularity appeared in \( (3.2) \) and it appears only in \( b_{11} \) and in \( b_{11}^\dagger \) which is just \( U(1) \) part. Thus this singularity doesn’t cause any physical trouble either for the exactly same reason in section 3.1.

After a little but straightforward algebra, one can determine the instanton charge density \( Q_n(\gamma) \) and calculate the topological charge of the solution \( (3.6) \):

\[ Q = -\sum_{n=0}^{\infty} \int_0^\infty d\gamma Q_n(\gamma), \quad (3.8) \]

where

\[
Q_n(\gamma) = \frac{1}{(n + 1 + \gamma + a^2)^4} \left( \frac{(1 + a^2)^2\gamma^2}{(n + \gamma)^2} + \frac{a^4 \gamma^2}{(n + 1 + \gamma)^2} + \frac{4a^2(1 + a^2)\gamma^2}{(n + \gamma)(n + 1 + \gamma)} \right) \\
- \frac{2(1 + a^2)^2n\gamma(n + 1 + \gamma + a^2)}{(n + \gamma)^2(n + \gamma + a^2)} + \frac{4a^4n\gamma(n + 1 + \gamma + a^2)}{(n + \gamma)(n + 1 + \gamma)(n + \gamma + a^2)} \\
+ \frac{(1 + a^2)^2n^2(n + 1 + \gamma + a^2)^2}{(n + \gamma)^2(n + \gamma + a^2)^2} + \frac{4(1 + a^2)^2n\gamma(n + 1 + \gamma + a^2)^3}{(n - 1 + \gamma)(n + \gamma)(n + \gamma + a^2)^3} \\
+ \frac{4a^2(1 + a^2)(n - 1)n(n + 1 + \gamma + a^2)^3}{(n - 1 + \gamma)(n + \gamma)(n - 1 + \gamma + a^2)(n + \gamma + a^2)^2} \\
+ \frac{a^4(n + 1)^2(n + 1 + \gamma + a^2)^2}{(n + 1 + \gamma)^2(n + 2 + \gamma + a^2)^2} - \frac{2a^4(n + 1)\gamma(n + 1 + \gamma + a^2)}{(n + 1 + \gamma)^2(n + 2 + \gamma + a^2)} \\
+ \frac{2a^2(1 + a^2)n\gamma(n + 1 + \gamma + a^2)(2n + 1 + 2\gamma + 2a^2)^2}{(n + \gamma)^2(n + \gamma + a^2)^3} \quad (3.9)
\]
with \( a = \rho / \sqrt{\zeta} \). The charge density \( Q_n(\gamma) \) is smooth function with respect to \( \gamma \) for all \( n \). See the Fig. 1-b, c, d. We performed the integral first and then the summation in (3.8) numerically using Mathematica and the result is summarized below (where we indicate
the summation range for each case).

\[
\begin{array}{|c|c|c|}
\hline
a & 0.1 (0 \leq n \leq 10^2) & a = 1 (0 \leq n \leq 10^2) & a = 10 (0 \leq n \leq 10^3) \\
Q & -0.998541 & -0.999791 & -0.991667 \\
\hline
\end{array}
\]

We further checked that \( Q(100) \equiv -\sum_{n=0}^{100} \int_0^{\infty} d\gamma Q_n(\gamma) = \sum_{k=0} q_k a^{2k} \), and we obtained
\( q_0 = -0.998542, \ q_1 = 9.82 \times 10^{-5}, \ q_2 = -9.63 \times 10^{-5} \), etc.

Noting that the topological charge density \( Q_n(\gamma) \) in (3.9) is rapidly convergent series with respect to \( n \) after the \( \gamma \)-integration, the above numerical results lead us to the conclusion very confidently that the topological charge of the anti-self-dual \( U(2) \) instanton is also an integer and independent of the modulus \( \rho \).

### 3.3 Self-dual \( U(1) \) Instanton

Now we will solve the ADHM equation (2.12) with the self-dual ADHM data (2.9). The solution can be found very easily:

\[
\psi_1 = z_2 \sqrt{\frac{\zeta}{\delta \Delta}}, \quad \psi_2 = -\bar{z}_1 \sqrt{\frac{\zeta}{\delta \Delta}}, \quad \xi = \sqrt{\frac{\delta}{\Delta}}.
\]

Also one can easily calculate the self-dual field strength \( F_{SD} \) for the solution (3.11) from (2.16):

\[
F_{SD} = -\frac{\zeta}{\delta^2 \Delta^2}(z_1 \bar{z}_1 \Delta - z_2 \bar{z}_2 \delta)(dz_1 dz_\bar{1} + dz_2 dz_\bar{2})
\]

\[
- \frac{2\zeta}{\Delta^2 \sqrt{\delta(\Delta + \zeta)}} z_1 \bar{z}_2 dz_1 dz_\bar{2} + \frac{2\zeta}{\delta^2 \sqrt{\Delta(\delta - \zeta)}} z_1 z_2 \bar{z}_1 dz_\bar{1} dz_\bar{2}.
\]

It can be checked explicitly that the self-dual field strength is also well-defined for all states in \( \mathcal{H} \) and on \( \mathbb{R}^2_C \) except the mild singularities of the second and the third terms. But, for the same reason as the previous cases, this singularity is never harmful and we can well define singularity-free physical quantities such as \( \text{Tr}_{\mathcal{H}} F^n \) from \( F_{SD} \) in (3.12).

The topological charge for the solution (3.12) has the same expression as (3.3) except the sign which is now +, so we get \( Q = 1 \).
### 3.4 Self-dual $U(2)$ Instanton

The self-dual $U(2)$ instantons can be obtained by solving the ADHM equation (2.12) with the data (2.9):

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} = \begin{pmatrix} z_2 f \\ -\bar{z}_1 f \\ z_1 g \\ -\bar{z}_2 g \\ \xi_1 \\ 0 \\ 0 \\ \xi_2 \end{pmatrix}$$

(3.13)

with the notation (3.5). For the same reason as section 3.2 the structure of the self-dual instanton at $\rho = 0$ is also completely determined by the minimal zero-mode $\psi^{(1)}$ in the $U(1)$ subgroup.

The field strength $F_{SD}$ for the solution (3.13) can be easily obtained from (2.16):

$$F_{SD} = (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \begin{pmatrix} c_{11} & c_{12} \\ c_{12}^* & c_{22} \end{pmatrix}$$

$$+ dz_1 \wedge dz_2 \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} + d\bar{z}_1 \wedge d\bar{z}_2 \begin{pmatrix} d_{11} & d_{21}^* \\ d_{12} & d_{22}^* \end{pmatrix},$$

(3.14)

where

$$c_{11} = \frac{\rho^2 + \zeta}{\delta(\delta + \rho^2)(\Delta + \rho^2)^2} (\delta + \rho^2)z_2 \bar{z}_2 - (\Delta + \rho^2)z_1 \bar{z}_1,$$
$$c_{12} = \frac{1}{\delta} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{(\delta + \rho^2)(\Delta + \rho^2)}} \left( \frac{1}{\delta + \rho^2} + \frac{1}{\Delta + \rho^2} \right) z_1 \bar{z}_2,$$
$$c_{22} = \frac{\rho^2}{\Delta(\Delta + \rho^2)^2(\Delta + \rho^2 + \zeta)} (\delta + \rho^2)(z_1 \bar{z}_1 + \zeta) - (\Delta + \rho^2 + \zeta)z_2 \bar{z}_2,$$
$$d_{11} = -\frac{2(\rho^2 + \zeta)}{\Delta + \rho^2} \sqrt{\frac{1}{\delta \Delta (\Delta + \rho^2)(\Delta + \rho^2 + \zeta)}} \bar{z}_1 \bar{z}_2,$$
$$d_{12} = \frac{2}{(\Delta + \rho^2)^2} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{\delta \Delta}} \bar{z}_2 \bar{z}_2,$$
$$d_{21} = -\frac{2}{\Delta + \rho^2 + \zeta} \sqrt{\frac{\rho^2(\rho^2 + \zeta)}{\Delta(\Delta + \zeta)(\Delta + \rho^2)(\Delta + \rho^2 + 2\zeta)}} \bar{z}_1 \bar{z}_1,$$
$$d_{22} = \frac{2\rho^2}{\Delta + \rho^2 + \zeta} \sqrt{\frac{1}{\Delta(\Delta + \zeta)(\Delta + \rho^2)(\Delta + \rho^2 + \zeta)}} \bar{z}_1 \bar{z}_2.$$

(3.15)

One can check again the solution (3.14) is also well-defined on the whole space $\mathbb{R}_{NC}^2 \times \mathbb{R}_{\mathbb{C}}^2$ up to a mild singularity. The above solution also reduces to the ordinary $SU(2)$ instanton in the $\zeta = 0$ limit and the $U(1)$ solution (3.12) for the limit $\rho = 0$ where only $c_{11}$ and $d_{11}$ terms in (3.6) survive.
One can check that the topological charge for the solution (3.14) has exactly the same expression as (3.8) except the sign which is now +. So we can conclude for the same reason as section 3.2 that the topological charge of the self-dual $U(2)$ instanton is also an integer and independent of the modulus $\rho$. (Actually it should be since the changes of ADHM data in (2.8) and (2.9) are only $z_2 \leftrightarrow \bar{z}_2$ and $z_1 \leftrightarrow -z_1$. However these changes should not be important since $z_2$ and $\bar{z}_2$ are commutative coordinates, i.e. $[\bar{z}_2, z_2] = 0$.)

4 Discussion

In this letter we studied anti-self-dual and self-dual noncommutative instantons on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ based on the ADHM construction. Unlike $\mathbb{R}^4_{NC}$, the ADHM equation for the noncommutative space $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ is always deformed since this system is not a BPS state any more. Remarkably, although the instanton solutions for both self-dual and anti-self-dual gauge fields contain a mild singularity, i.e. a measure zero singularity, it always disappears in gauge invariant quantities and thus physically regular solutions can be constructed even though any projected states are not involved in the ADHM construction. Furthermore the instanton number is always an integer.

Our present result is different from [18] by Chu, et al. claiming that there is no nonsingular $U(N)$ instanton on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ due to the breakdown of the completeness relation. The authors of [18] argued that if the offending state, e.g. $|0\rangle$, is not subtracted, a huge plane singularity on the whole $\mathbb{R}^2_{NC}$-plane placed at $z_2 = 0$ is developed in the solution and this huge singularity is not allowed in the semi-classical picture, drawing the conclusion that the vacuum structure of noncommutative $U(N)$ gauge theories on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ is trivial for all $N \geq 1$. However we showed that although the solutions contain mild singularities, these singularities always disappear whenever we define the gauge invariant quantities, so they don’t induce any physical singularities. Also they appear only in the $U(1)$ part of $U(N)$ gauge theory. Thus the singularity in the $U(N)$ instanton solution on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ is a gauge artifact in the sense that it appears only gauge non-invariant quantities.

The space $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ can be realized as the spatial worldvolume of D4-brane with rank-2 B field. Obviously one can put D0-branes on this D4-brane. By SUSY analysis, this system is not supersymmetric, so FI-term should be introduced in the D4-brane world volume theory. This FI-term appears as the deformation of ADHM equation as in (2.5).
This means that the D0-brane moduli space is resolved, i.e. the D0-brane is a little bit smeared out on the D4-brane. This picture is consistent with the result in this work. There is no reason why such a huge plane singularity claimed in [18] should be developed in the D4-brane and why the D0-brane on the D4-brane is so singular.

The instanton configurations on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ can be naturally explained by the topology of gauge group suggested by Harvey [21], where the gauge transformations on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ are characterized by the maps from $\mathbb{S}^1$ to $U_{\text{cpt}}(\mathcal{H})$. Here $U_{\text{cpt}}(\mathcal{H})$ denotes the unitary operators over $\mathcal{H}$ of the form $U = 1 + K$ with $K$ a compact operator. This map is nontrivial since $\pi_1(U_{\text{cpt}}) = \mathbb{Z}$. Thus the vacuum structure of noncommutative gauge theory on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ is still parameterized by an integer winding number. This fact was already noticed by Chu, et al. in [18], but rejected for a wrong reason.

Multi-instanton solutions on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ can be constructed too. As shown in [14], after separating out the center of mass, the moduli space of two $U(1)$ instantons on $\mathbb{R}^4_{NC}$ is given by the Eguchi-Hanson metric which is non-singular even at the origin where the two $U(1)$ instantons coincide. As shown in this paper the $U(1)$ instantons on $\mathbb{R}^2_{NC} \times \mathbb{R}^2_C$ are definitely non-singular. However, in our case, any projection is not involved in the solution and the commutative space $\mathbb{R}^2_C$ is still there. Thus it will be interesting to study whether or not these differences can affect the moduli space for the two $U(1)$ instantons.

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Figure 1: Topological charge densities $y = 2Q_n(\gamma)$ defined in (3.3) and (3.9) where $x = \sqrt{\gamma}$. a: $U(1)$, b: $U(2)$, $a = 0.1$, c: $U(2)$, $a = 1$, d: $U(2)$, $a = 10$. 