Abstract. We consider the error analysis of Lagrange interpolation on triangles and tetrahedrons. For Lagrange interpolation of order one, Babuška and Aziz showed that squeezing a right isosceles triangle perpendicularly does not deteriorate the optimal approximation order. We extend their technique and result to higher-order Lagrange interpolation on both triangles and tetrahedrons. To this end, we make use of difference quotients of functions with two or three variables. Then, the error estimates on squeezed triangles and tetrahedrons are proved by a method that is a straightforward extension of the original given by Babuška-Aziz.

Keywords: Lagrange interpolation, Babuška-Aziz’s technique, difference quotients

MSC 2010: 65D05, 65N30

1. Introduction

Lagrange interpolation on triangles and tetrahedrons and the associated error estimates are important subjects in numerical analysis. In particular, they are crucial in the error analysis of finite element methods. Let \( d = 2 \) or 3. Throughout this paper, \( K \subset \mathbb{R}^d \) denotes a triangle or tetrahedron with vertices \( x_i, i = 1, \cdots, d+1 \). We always suppose that triangles and tetrahedrons are closed sets in this paper. Let \( \lambda_i \) be its barycentric coordinates with respect to \( x_i \). By definition, we have \( 0 \leq \lambda_i \leq 1 \), \( \sum_{i=1}^{d+1} \lambda_i = 1 \). Let \( \mathbb{N}_0 \) be the set of nonnegative integers, and \( \gamma = (a_1, \cdots, a_{d+1}) \in \mathbb{N}_0^{d+1} \) be a multi-index. Let \( k \) be a positive integer. If \( |\gamma| := \sum_{i=1}^{d+1} a_i = k \), then

\[
|\gamma| := \sum_{i=1}^{d+1} a_i = k, \quad \text{then}
\]
\( \gamma/k := (a_1/k, \cdots, a_{d+1}/k) \) can be regarded as a barycentric coordinate in \( K \). The set \( \Sigma^k(K) \) of points on \( K \) is defined by

\[
\Sigma^k(K) := \left\{ \frac{\gamma}{k} \in K \mid |\gamma| = k, \gamma \in \mathbb{N}_0^{d+1} \right\}.
\]

Let \( 1 < p \leq \infty \). From Sobolev’s imbedding theorem and Morry’s inequality, we have the continuous imbeddings

\[
W^{2,p}(K) \subset C^{1,1-d/p}(K), \quad p > d,
\]
\[
W^{2,d}(K) \subset W^{1,q}(K) \subset C^{0,1-d/q}(K), \quad \forall q > d,
\]
\[
W^{2,p}(K) \subset W^{1,dp/(d-p)}(K) \subset C^{0,2-d/p}(K), \quad \frac{d}{2} < p < d.
\]

If \( d = 3 \), we also have the continuous imbeddings

\[
W^{3,3/2}(K) \subset W^{2,3}(K) \subset W^{1,q}(K) \subset C^{0,1-3/q}(K), \quad \forall q > 3,
\]
\[
W^{3,p}(K) \subset W^{2,3p/(3-p)}(K) \subset W^{1,3p/(3-2p)}(K) \subset C^{0,3-3/p}(K), \quad 1 < p < \frac{3}{2}.
\]

Although Morry’s inequality may not be applied, the continuous imbedding \( W^{d,1}(K) \subset C^0(K) \) \((d = 2, 3)\) still holds. For the imbedding theorem, see [1], [7], and [16]. In the sequel we always suppose that \( p \) is taken so that the imbedding \( W^{k+1,p}(K) \subset C^0(K) \) holds, that is,

\[
1 \leq p \leq \infty, \quad \text{if } d = 2, \; k + 1 \geq 2 \text{ or } d = 3, \; k + 1 \geq 3, \quad \text{and}
\]
\[
\frac{3}{2} < p \leq \infty, \quad \text{if } d = 3, \; k + 1 = 2.
\]

Note that our discussion includes the case \( d = k + 1, \; p = 1 \) that is sometimes ignored in literature.

We define the subset \( T^k_p(K) \subset W^{k+1,p}(K) \) by

\[
T^k_p(K) := \left\{ v \in W^{k+1,p}(K) \mid v(x) = 0, \; \forall x \in \Sigma^k(K) \right\}.
\]

Let \( \mathcal{P}_k \) be the set of polynomials with two or three variables for which the degree is at most \( k \). For a continuous function \( v \in C^0(K) \), the \( k \)th-order Lagrange interpolation \( I^k_K v \in \mathcal{P}_k \) is defined by

\[
v(x) = (I^k_K v)(x), \quad \forall x \in \Sigma^k(K).
\]

From this definition, it is clear that

\[
v - I^k_K v \in T^k_p(K), \quad \forall v \in W^{k+1,p}(K).
\]
For an integer $m$ such that $0 \leq m \leq k$, $B_{p}^{m,k}(K)$ is defined by

$$B_{p}^{m,k}(K) := \sup_{v \in T_{p}^{n}(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$ 

Note that we have

$$B_{p}^{m,k}(K) = \inf \left\{ C; |v - \mathcal{I}_{K}^{k}v|_{m,p,K} \leq C|v|_{k+1,p,K}, \ \forall v \in W^{k+1,p}(K) \right\},$$

that is, $B_{p}^{m,k}(K)$ is the best constant $C$ for the error estimation

$$|v - \mathcal{I}_{K}^{k}v|_{m,p,K} \leq C|v|_{k+1,p,K}, \ \forall v \in W^{k+1,p}(K).$$

To establish the mathematical foundation of the finite element methods, we must show that $B_{p}^{m,p}(K)$ is bounded. Many textbooks on finite element methods, such as those by Ciarlet [8], Brenner-Scott [6], and Ern-Guermond [10], present the following theorem. Let $h_{K}$ be the diameter of $K$ and $\rho_{K}$ be the radius of the inscribed ball of $K$.

**Shape-regularity.** Let $\sigma > 2$ be a constant. If $h_{K}/\rho_{K} \leq \sigma$ and $h_{K} \leq 1$, then there exists a constant $C = C(\sigma)$ that is independent of $h_{K}$ such that

$$\|v - \mathcal{I}_{K}^{1}v\|_{1,2,K} \leq Ch_{K}|v|_{2,2,K}, \ \forall v \in H^{2}(K).$$

The maximum of the ratio $h_{K}/\rho_{K}$ in a triangulation is called the chunkiness parameter [6]. The shape regularity, however, is not necessarily needed to obtain an error estimate for triangles and tetrahedrons. For triangles, the following estimations are well-known [4], [3], [11].

**The maximum angle condition.** Let $\theta_{1}$, $(\pi/3 \leq \theta_{1} < \pi)$ be a constant. If any angle $\theta$ of $K$ satisfies $\theta \leq \theta_{1}$ and $h_{K} \leq 1$, then there exists a constant $C = C(\theta_{1})$ that is independent of $h_{K}$ such that

$$\|v - \mathcal{I}_{K}^{1}v\|_{1,2,K} \leq Ch_{K}|v|_{2,2,K}, \ \forall v \in H^{2}(K).$$

Later, Krřízek [14] introduced the semiregularity condition for triangles, which is equivalent to the maximum angle condition. Let $R_{K}$ be the circumradius of $K$.

**The semiregularity condition.** Let $p > 1$ and $\sigma > 0$ be a constant. If $R_{K}/h_{K} \leq \sigma$ and $h_{K} \leq 1$, then there exists a constant $C = C(\sigma)$ that is independent of $h_{K}$ such that

$$\|v - \mathcal{I}_{K}^{1}v\|_{1,p,K} \leq Ch_{K}|v|_{2,p,K}, \ \forall v \in W^{2,p}(K).$$
For tetrahedrons, the following estimation is well-known [15], [9].

**Krížek’s maximum angle condition.** Let $\theta_2$, $(\pi/3 \leq \theta_2 < \pi)$ be a constant. Let $\gamma_k$ be the maximum angle of faces of a tetrahedron $K$ and $\varphi_K$ be the maximum angle between faces of $K$. If $\gamma_k \leq \theta_2$, $\varphi_K \leq \theta_2$, and $h_K \leq 1$, then there exists a constant $C = C(\theta_2)$ that is independent of $h_K$ such that

$$
\|v - I_K^1 v\|_{1,p,K} \leq C h_K |v|_{2,p,K}, \quad \forall v \in W^{2,p}(K), \quad 2 < p \leq \infty.
$$

Jamet [11] presented a general result which covers both triangles and tetrahedrons. Let $E_d := \{e_s\}_{s=1}^d \subset \mathbb{R}^d$ be a set of unit vectors which are linearly independent. Let $\xi \in \mathbb{R}^d$ be an unit vector and $\theta_s$, $0 \leq \theta_s \leq \frac{\pi}{2}$ be the angle between $\xi$ and the line which is defined by $e_s$. Define

$$
\theta(E_d) := \max_{\xi \in \mathbb{R}^d} \min_{e_s \in E_d} \{\theta_s\}.
$$

Let $K \subset \mathbb{R}^d$ be $d$-simplex. Let $N := (d+1)/2$ and $E_N$ be the set of $N$ unit vectors that are parallel to the edges of $K$. Define $\theta_K := \min_{E_d \subset E_N} \{\theta(E_d)\}$. Note that if $d = 2$ and $K$ is an obtuse triangle, then $2\theta_K$ is the maximum angle of $K$.

**Theorem 1.1** (Jamet). Let $1 \leq p \leq \infty$. Let $m \geq 0$, $k \geq 1$ be integers such that $k + 1 - m > 2/p$ ($1 < p \leq \infty$) or $k - m \geq 1$ ($p = 1$) if $d = 2$, or $k + 1 - m > 3/p$ if $d = 3$. Then, the following estimate holds:

$$
|v - I_K^1 v|_{m,p,K} \leq C h_K^{k+1-m} \frac{h_K}{(\cos \theta_K)^m} |v|_{k+1,p,K}, \quad \forall v \in W^{k+1,p}(K),
$$

where $C$ depends only on $k, p$.

**Remark:** Note that in [11, Théorème 3.1] the case $d = 2$ and $p = 1$ is not mentioned explicitly but clearly holds for triangles.

For further results of error estimations on “skinny elements”, readers are referred to the monograph by Apel [2].

The common idea among the above mentioned estimations is that (i) show an error estimate for a particular type of elements, then (ii) extend it for general elements by affine transformation. To prove the maximum angle condition for triangles, for example, Babuška and Aziz showed the following theorem [4, Lemma 2.2, Lemma 2.4]. Let $\hat{K}$ be the right triangle with vertices $(0,0)^\top$, $(1,0)^\top$, and $(0,1)^\top$, and $K_\alpha$ be the right triangle with vertices $(0,0)^\top$, $(1,0)^\top$, and $(0,\alpha)^\top$ ($0 < \alpha \leq 1$). That is, $K_\alpha$ is obtained by squeezing $\hat{K}$. 

Theorem 1.2 (Babuška-Aziz). There exists a constant independent of $\alpha$ ($0 < \alpha \leq 1$) such that $B_2^{m,1}(K_\alpha) \leq C$, $m = 0, 1$. As an immediate consequence, we obtain the error estimation of Lagrange interpolation $I_1^k$ on a right triangle $K$; for $m = 0, 1$,

$$|v - I_1^k v|_{m,2,K} \leq Ch_2^{-m}|v|_{2,2,K}.$$

Theorem 1.2 claims that squeezing a right isosceles triangle perpendicularly does not deteriorate the optimal approximation order of $I_1^k$. Babuška and Aziz then claim that the estimate (1.1) for general triangular elements is obtained by affine transformations. Kobayashi and Tsuchiya [12] extended Theorem 1.2 for any $p$ ($1 \leq p \leq \infty$).

Now, let $\tilde{K}$ denote also the reference tetrahedron with vertices $(0,0,0)^\top$, $(1,0,0)^\top$, $(0,1,0)^\top$, and $(0,0,1)^\top$. Let $K_{\alpha\beta}$ be the “right” tetrahedron with vertices $(0,0,0)^\top$, $(1,0,0)^\top$, $(0,\alpha,0)^\top$, and $(0,0,\beta)^\top$ ($0 < \alpha, \beta \leq 1$).

The aim of this paper is to extend Theorem 1.2 and establish the following theorem.

**Theorem 1.3.** If $d = 2$, there exists a constant $C_{k,m,p}$ such that, for $m = 0, \cdots, k$,

\begin{equation}
B_p^{m,k}(K_\alpha) := \sup_{v \in T_p^k(K_\alpha)} \frac{|v|_{m,p,K_\alpha}}{|v|_{k+1,p,K_\alpha}} \leq C_{k,m,p}, \quad k \geq 1, \ 1 \leq p \leq \infty.
\end{equation}

If $d = 3$, there exists a constant $C_{k,m,p}$ such that, for $m = 0, \cdots, k$,

\begin{equation}
B_p^{m,k}(K_{\alpha\beta}) := \sup_{v \in T_p^k(K_{\alpha\beta})} \frac{|v|_{m,p,K_{\alpha\beta}}}{|v|_{k+1,p,K_{\alpha\beta}}} \leq C_{k,m,p},
\end{equation}

\begin{align*}
\begin{cases}
  k - m = 0, & 2 < p \leq \infty, \\
  k = 1, m = 0, & \frac{3}{2} < p \leq \infty, \\
  k \geq 2, k - m \geq 1, & 1 \leq p \leq \infty.
\end{cases}
\end{align*}

Using Theorem 1.3 and affine transformations, we can derive an error estimation on general triangles. See Section 4.

The above mentioned estimations (1.1), (1.2), (1.3), (1.4) cover Theorem 1.3 partially. We also mention that Shenk [18] showed (1.5) for $p = 2$, $k \geq 1$, $m = 0, 1$, and (1.6) for $p = 2$, $k \geq 2$, $m = 0, 1$.

Because of the restrictions for $m$, $k$, and $p$ in the above mentioned estimations, it seems that (1.5) with $k = m \geq 2, 1 \leq p \leq 2$, and (1.6) with $k = m \geq 2, 1 \leq p \leq 3$ have not yet been proved. To prove Theorem 1.3, we introduce the difference quotients of functions with two or three variables in Section 2. Then, Theorem 1.3 is proved in Section 3 by a method that is a straightforward extension of Babuška-Aziz’s original argument. The notations of functional spaces used in this paper are exactly the same as those in [13].
2. DIFFERENCE QUOTIENTS FOR MULTI-VARIABLE FUNCTIONS

In this section, we define the difference quotients for two- and three-variable functions. Our treatment is based on the theory of difference quotients for one-variable functions given in standard textbooks such as [3] and [19]. All statements in this section can be readily proved.

For a positive integer $k$, the set $\widehat{\Sigma}^k \subset \widehat{K}$ is defined by

$$\widehat{\Sigma}^k := \left\{ x_\gamma := \frac{\gamma}{k} \in \widehat{K} \mid \gamma \in \mathbb{N}_0^d, \ 0 \leq |\gamma| \leq k \right\},$$

where $\gamma/k = (a_1/k, \cdots, a_d/k)$ is understood as the coordinate of a point in $\widehat{\Sigma}^k$.

For $x_\gamma \in \widehat{\Sigma}^k$ and a multi-index $\delta \in \mathbb{N}_0^d$ with $|\gamma| \leq k - |\delta|$, we define the correspondence $\Delta^\delta$ between nodes by

$$\Delta^\delta x_\gamma := x_\gamma + \delta \in \widehat{\Sigma}^k.$$

For two multi-indexes $\eta = (m_1, \cdots, m_d), \delta = (n_1, \cdots, n_d)$, $\eta \leq \delta$ means that $m_i \leq n_i$ ($i = 1, \cdots, d$). Also, $\delta - \eta$ and $\delta!$ are defined by $\delta - \eta := \sum_{i=1}^d m_i n_i$ and $\delta! := n_1! \cdots n_d!$, respectively. Using $\Delta^\delta$, we define the difference quotients on $\widehat{\Sigma}^k$ for $f \in C^0(\widehat{K})$ by

$$f^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] := k^{|\delta|} \sum_{\eta \leq \delta} \frac{(-1)^{|\delta|}}{|\eta|! (\delta - \eta)!} f(\Delta^\eta x_\gamma).$$

Let $0 := (0, \cdots, 0) \in \mathbb{N}_0^d$. For simplicity, we denote $f^{[\delta]}[x_0, \Delta^\delta x_0]$ by $f^{[\delta]}[\Delta^\delta x_0]$.

The following are examples of $f^{[\delta]}[\Delta^\delta x_0]$: if $d = 2$,

\[
\begin{align*}
    f^2[\Delta^{(2,0)} x_{(0,0)}] &= \frac{k^2}{2} (f(x_{(2,0)}) - 2f(x_{(1,0)}) + f(x_{(0,0)})), \\
    f^2[\Delta^{(1,1)} x_{(0,0)}] &= k^2 (f(x_{(1,1)}) - f(x_{(1,0)}) - f(x_{(0,1)}) + f(x_{(0,0)})), \\
    f^2[\Delta^{(2,1)} x_{(0,0)}] &= \frac{k^3}{2} (f(x_{(2,1)}) - 2f(x_{(1,1)}) + f(x_{(0,1)}) - f(x_{(2,0)}) \\
                              &\quad + 2f(x_{(1,0)}) - f(x_{(0,0)})),
\end{align*}
\]

and if $d = 3$,

\[
\begin{align*}
    f^3[\Delta^{(2,1,1)} x_{(0,0,0)}] &= \frac{k^4}{2} (f(x_{(2,1,1)}) - 2f(x_{(1,1,1)}) + f(x_{(0,1,1)}) \\
                              &\quad - f(x_{(2,0,1)}) + 2f(x_{(1,0,1)}) - f(x_{(0,0,1)}) \\
                              &\quad - f(x_{(2,1,0)}) + 2f(x_{(1,1,0)}) - f(x_{(0,1,0)}) \\
                              &\quad + f(x_{(2,0,0)}) - 2f(x_{(1,0,0)}) + f(x_{(0,0,0)})).
\end{align*}
\]
Let \( \eta \in \mathbb{N}_0^d \) be such that \( |\eta| = 1 \). The difference quotients clearly satisfy the following recursive relations:

\[
f^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] = \frac{k}{\delta \cdot \eta} \left( f^{[\delta]-1}[x_{\gamma+\eta}, \Delta^{\delta-\eta}x_{\gamma+\eta}] - f^{[\delta]-1}[x_\gamma, \Delta^{\delta-\eta}x_\gamma] \right).
\]

If \( f \in C^k(\hat{K}) \), the difference quotient \( f^{[\delta]}[x_\gamma, \Delta^\delta x_\gamma] \) is written as an integral of \( f \). Setting \( d = 2 \) and \( \delta = (0, s) \), for example, we have

\[
f^1[x_{(l,q)}, \Delta^{(0,1)} x_{(l,q)}] = k(f(x_{l,q+1}) - f(x_{l,q})) = \int_0^1 \partial_{x_2} f \left( \frac{1}{k} \cdot \frac{q}{k} + \frac{w_1}{k} \right) \, dw_1,
\]

\[
f^s[x_{(l,p)}, \Delta^{(0,s)} x_{(l,q)}] = \int_0^1 \int_0^{w_1} \cdots \int_0^{w_{s-1}} \partial^{(0,s)} f \left( \frac{1}{k} \cdot \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s) \right) \, dw_s \cdots dw_1 \, dw_1.
\]

To provide a concise expression for the above integral, we introduce the \( s \)-simplex

\[
S_s := \{(t_1, t_2, \ldots, t_s) \in \mathbb{R}^s \mid t_i \geq 0, \ 0 \leq t_1 + \cdots + t_s \leq 1 \},
\]

and the integral of \( g \in L^1(S_s) \) on \( S_s \) is defined by

\[
\int_{S_s} g(w_1, \ldots, w_k) \, dW_s := \int_0^1 \int_0^{w_1} \cdots \int_0^{w_{s-1}} g(w_1, \ldots, w_s) \, dw_s \cdots dw_1 \, dw_1.
\]

Let us temporarily set \( d = 2 \). Then, \( f^s[x_{(l,q)}, \Delta^{(0,s)} x_{(l,q)}] \) becomes

\[
f^s[x_{(l,q)}, \Delta^{(0,s)} x_{(l,q)}] = \int_{S_s} \partial^{(0,s)} f \left( \frac{1}{k} \cdot \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s) \right) \, dW_s.
\]

For a general multi-index \((t, s)\), we have

\[
f^{t+s}[x_{(l,q)}, \Delta^{(t,s)} x_{(l,q)}] = \int_{S_t} \int_{S_s} \partial^{(t,s)} f \left( \frac{1}{k} + \frac{1}{k}(z_1 + \cdots + z_t), \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s) \right) \, dZ_t \, dW_s.
\]

Let \( \Box^\delta \gamma \) be the rectangle defined by \( x_\gamma \) and \( \Delta^\delta x_\gamma \) as the diagonal points. If \( \delta = (t, 0) \) or \((0, s)\), \( \Box^\delta \gamma \) degenerates to a segment. For \( v \in L^1(\hat{K}) \) and \( \Box^\delta \gamma \) with \( \gamma = (l, q) \), we denote the integral as

\[
\int_{\Box^t \gamma} v := \int_{S_t} \int_{S_s} v \left( \frac{1}{k} + \frac{1}{k}(z_1 + \cdots + z_t), \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s) \right) \, dZ_t \, dW_s.
\]
If $\Box^\delta_\gamma$ degenerates to a segment, the integral is understood as an integral on the segment. By this notation, the difference quotient $f^{t+s}[x_\gamma, \Delta^{(t,s)}x_\gamma]$ is written as
\[
f^{t+s}[x_\gamma, \Delta^{(t,s)}x_\gamma] = \int_{\Box^{(t,s)}_\gamma} \partial^{(t,s)} f.
\]
Therefore, if $u \in T^k_p(\hat{K})$, then we have
\[
0 = u^{t+s}[x_\gamma, \Delta^{(t,s)}x_\gamma] = \int_{\Box^{(t,s)}_\gamma} \partial^{(t,s)} u, \quad \forall \Box^{(t,s)}_\gamma \subset \hat{K}.
\]
For the case $d = 3$, the integral $\int_{\Box^3_\gamma} v$ is defined in exactly the same manner.

3. Proof of Theorem 1.3

Let $S \subset \hat{K}$ be a segment. In the proof of Theorem 1.3, the continuity of the trace operator $t$ defined as $t : W^{1,p}(\hat{K}) \ni v \mapsto v|_S \in L^1(S)$ is crucial. If $d = 2$, the continuity of $t$ is standard and is mentioned in many textbooks such as [7]. For the case $d = 3$, the situation becomes a bit more complicated. If the continuous inclusion $W^{k+1,p}(\hat{K}) \subset C^0(\hat{K})$ holds, the continuity of $t$ is obvious. Even if this is not the case, we still have the following lemma. For the proof, see [1, Theorem 4.12], [9, Lemma 2.2], and [17, Theorem 2.1].

**Lemma 3.1.** Let $d = 3$ and $S \subset \hat{K}$ be an arbitrary segment. Then, the following trace operators are well-defined and continuous:

\[
t : W^{1,p}(\hat{K}) \to L^p(S), \quad 2 < p < \infty, \quad t : W^{2,p}(\hat{K}) \to L^p(S), \quad 1 \leq p < \infty.
\]

For a multi-index $\delta$, $|\delta| \geq 1$, $p$ is taken so that
\[
(3.1) \quad \begin{cases} 2 < p \leq \infty, & \text{if } k + 1 - |\delta| = 1, \ d = 3, \\ 1 \leq p \leq \infty, & \text{if } k + 1 - |\delta| \geq 2, \ d = 3 \text{ or } d = 2. \end{cases}
\]

The set $\Xi^{\delta,k}_p \subset W^{k+1-|\delta|,p}(\hat{K})$ is then defined by
\[
\Xi^{\delta,k}_p := \left\{ v \in W^{k+1-|\delta|,p}(\hat{K}) \mid \int_{\Box^\delta_{ij,p}} v = 0, \ \forall \Box^\delta_{ij,p} \subset \hat{K} \right\}.
\]

By Lemma 3.1 and (3.1), $\Xi^{\delta,k}_p$ is well-defined. Note that $u \in T^k_p(\hat{K})$ implies $\partial^\delta u \in \Xi^{\delta,k}_p$ by definition and (2.1).
Lemma 3.2. We have $\Xi_p^{δ,k} \cap P_{k−|δ|} = \{0\}$. That is, if $q \in P_{k−|δ|}$ belongs to $\Xi_p^{δ,k}$, then $q = 0$.

Proof. We notice that $\dim P_{k−|δ|} = \# \{ \Box_{l_p} \subset \hat{K} \}$. For example, if $k = 4$, $d = 2$, and $|δ| = 2$, then $\dim P_2 = 6$. This corresponds to the fact that, in $\hat{K}$, there are six squares with size $1/4$ for $δ = (1,1)$ and there are six horizontal segments of length $1/2$ for $δ = (2,0)$. All their vertices (corners and end-points) belong to $\Sigma^4(\hat{K})$ (see Figure 1). The situation is the same for $d = 3$. Now, suppose that $v \in P_{k−|δ|}$ satisfies $\int_\Box v_{l_p} \neq 0$ for all $\Box_{l_p} \subset \hat{K}$. This condition is linearly independent and determines $q = 0$ uniquely.

The constant $A_p^{δ,k}$ is defined by

$$A_p^{δ,k} := \sup_{v \in \Xi_p^{δ,k}} v_{0,p,\hat{K}} / |v|_{k+1−|δ|,p,\hat{K}}.$$  

The following lemma is an extension of [4, Lemma 2.1].

Lemma 3.3. Let $p$ be given by (3.1). Then, we have $A_p^{δ,k} < \infty$.

Proof. The proof is by contradiction. Suppose that $A_p^{δ,k} = \infty$. Then there exists a sequence $\{w_n\}_{n=1}^\infty \subset \Xi_p^{δ,k}$ such that $w_n_{[0,p,\hat{K}} = 1$ and $\lim_{n \to \infty} |w_n|_{k+1−|δ|,p,\hat{K}} = 0$. By [5] Theorem 3.1.1, there exists $\{q_n\} \subset P_{k−|δ|}$ such that

$$\|w_n + q_n\|_{k+1−|δ|,p,\hat{K}} \leq \inf_{q \in P_{k−|δ|}} \|w_n + q\|_{k+1−|δ|,p,\hat{K}} + \frac{1}{n} \leq C|w_n|_{k+1−|δ|,p,\hat{K}} + \frac{1}{n}$$

and $\lim_{n \to \infty} \|w_n + q_n\|_{k+1−|δ|,p,\hat{K}} = 0$. Because $\{w_n\} \subset W^{k+1−|δ|,p}(\hat{K})$ is bounded, $\{q_n\} \subset P_{k−|δ|}$ is bounded as well. Hence, there exists a subsequence $\{q_{n_k}\}$ such that
Because \( q_n \) converges to \( \bar{q} \in T_{k-\delta} \) and \( \lim_{n_i \to \infty} \| w_{n_i} + \bar{q} \|_{k+1-\delta, p, \hat{K}} = 0 \). By definition and the continuity of the trace operator, we have \( \int_{T_p} w_{n_i} = 0 \) and

\[
0 = \lim_{n_i \to \infty} \int_{T_p} (w_{n_i} + \bar{q}) = \int_{T_p} \bar{q}, \quad \forall \delta \subset \hat{K}.
\]

Therefore, it follows from Lemma 3.1 that \( \bar{q} = 0 \). This implies that

\[
0 = \lim_{n_i \to \infty} \| w_{n_i} \|_{k+1-\delta, p, \hat{K}} \geq \lim_{n_i \to \infty} |w_{n_i}|_{0, p, \hat{K}} = 1,
\]

which is a contradiction. □

**The proof of Theorem 7.3.** The proof is an direct extension of the proof given in [4, Lemma 2.2]. Let \( d = 2 \) initially. Define the linear transformation \( F_\alpha : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
(x^*, y^*)^T = (x, \alpha y)^T, \quad (x, y)^T \in \mathbb{R}^2, \quad 0 < \alpha \leq 1,
\]

which squeezes the reference element \( \hat{K} \) perpendicularly to \( K_\alpha := F_\alpha(\hat{K}) \). Take an arbitrary \( v \in W^{k+1, p}(K_\alpha) \) and define \( u \in W^{k+1, p}(\hat{K}) \) by \( u(x, \alpha y) := v(x, y) \). To make the formula concise, we introduce the following notation. For a multi-index \( \gamma = (a, b) \in \mathbb{N}_0^2 \) and a real \( t \neq 0, (\alpha)^{\gamma t} := \alpha^t \). Let \( 1 \leq p < \infty \) and \( 1 \leq m \leq k \). Because \( u \in T^{k+1}_p(\hat{K}) \) and \( \partial^\delta u \in \Xi^{k+1, p}_p \), we may apply Lemma 3.3 and obtain

\[
\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} |\partial^\gamma u|^p_{0, p, \hat{K}}
\]

\[
\leq \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \left( \sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta! (\alpha)^p} |\partial^\eta (\partial^\gamma u)|^p_{0, p, \hat{K}} \right)
\]

which we can rewrite as

\[
\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} |\partial^\gamma u|^p_{0, p, \hat{K}}
\]

\[
\leq \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \left( \sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta! (\alpha)^p} |\partial^\eta (\partial^\gamma u)|^p_{0, p, \hat{K}} \right)
\]

\[
\leq \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} |\partial^\gamma u|^p_{k+1-m, p, \hat{K}}
\]

\[
\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} |\partial^\gamma u|^p_{k+1-m, p, \hat{K}}
\]
Therefore, (1.5) is proved. Hence, we obtain (1.5) for this case. If

\[ C = \max_{k,m,p} \left( A_{p}^{\gamma,k} \right) \]

where \( C_{k,m,p} := \max_{|\gamma|=m} A_{p}^{\gamma,k} \). Here, we use the equality

\[ \frac{(k+1)!}{\delta!} = \sum_{\gamma+\eta=\delta, |\gamma|=m, |\eta|=k+1-m} \frac{m!}{\gamma! \eta!} \]

Hence, we obtain (1.5) for this case. If \( m = 0 \), we have

\[ (3.3) \quad \frac{|v|_{0,p,K_{\alpha}}^{p}}{|v|_{k+1,p,K_{\alpha}}^{p}} = \sum_{|\delta|=k+1} \frac{(k+1)!}{\delta!} \frac{|u|_{0,p,K_{\hat{K}}}^{p}}{|\partial^{\delta} u|_{0,p,K_{\hat{K}}}^{p}} \leq \sum_{|\delta|=k+1} \frac{(k+1)!}{\delta!} \frac{|u|_{0,p,K_{\hat{K}}}^{p}}{|\partial^{\delta} u|_{0,1,p,K_{\hat{K}}}^{p}} = \frac{|u|_{0,p,K_{\hat{K}}}^{p}}{|u|_{k+1,p,K_{\hat{K}}}^{p}} \leq B_{0,p}^{0,p}(K_{\hat{K}})^{p} < \infty. \]

Setting \( p = \infty \) and \( 1 \leq m \leq k \), we have

\[ (3.4) \quad \frac{|v|_{m,\infty,K_{\alpha}}^{p}}{|v|_{k+1,\infty,K_{\alpha}}^{p}} = \frac{\max_{|\gamma|=m} \left\{ (\alpha)^{-\gamma} |\partial^{\gamma} u|_{0,\infty,K_{\hat{K}}} \right\}}{\max_{|\delta|=k+1} \left\{ (\alpha)^{-\delta} |\partial^{\delta} u|_{0,\infty,K_{\hat{K}}} \right\}} \]

where \( C_{k,m,\infty} := \max_{|\gamma|=m} A_{\infty}^{\gamma,k} \). Now, the case with \( p = \infty \) and \( m = 0 \) is obvious. Therefore, (1.5) is proved.
Next, let $d = 3$ and repeat the above proof. Define the linear transformation $F_{\alpha\beta} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$(x^*, y^*, z^*)^T = (x, \alpha y, \beta z)^T, \quad (x, y, z)^T \in \mathbb{R}^3, \quad 0 < \alpha, \beta \leq 1,$$

which squeezes the reference tetrahedron $\hat{K}$ perpendicularly to $K_{\alpha\beta} := F_{\alpha\beta}(\hat{K})$. Take an arbitrary $v \in W^{k+1,p}(K_{\alpha\beta})$ and define $u \in W^{k+1,p}(\hat{K})$ by $u(x, y, z) := v(x, \alpha y, \beta z)$. Let $p$ be given by (3.1) with $m = |\delta|$. To make formula concise, we introduce the following notation. For a multi-index $\gamma = (a, b, c) \in \mathbb{N}_0^3$ and a real $t \neq 0$, $(\alpha, \beta)^{\gamma t} := \alpha^a \beta^b \gamma^c$. Because $u \in T_p^k(\hat{K})$ and $\partial^\gamma u \in \Xi_{p}^{k,\gamma}$, we may apply Lemma 3.3 as above. Thus, we may repeat (3.2), (3.3), and (3.4) by replacing $K_{\alpha}$ with $K_{\alpha\beta}$, $(\alpha)^{\gamma p}$ with $(\alpha, \beta)^{\gamma p}$, etc. Thus, (1.6) is proved.  

4. Concluding Remarks

Theorem 1.3 deals with only right triangles and “right” tetrahedrons. Based on Theorem 1.3, a new error estimation of Lagrange interpolation on triangles is obtained in [13]. It should be emphasized that no geometric condition on triangles is imposed in Theorem 4.1.

**Theorem 4.1** (Kobayashi-Tsuchiya [13]). Let $K$ be an arbitrary triangle. Let $1 \leq p \leq \infty$, and $k, m$ be integers such that $k \geq 1$ and $0 \leq m \leq k$. Then, for the $k$th-order Lagrange interpolation $I^k_K$ on $K$, the following estimation holds:

$$|v - I^k_K v|_{m,p,K} \leq C \left( \frac{R_K}{h_K} \right)^m h_K^{k+1-m} |v|_{k+1,p,K} = CR_R h_K^{k+1-2m} |v|_{k+1,p,K}$$

for any $v \in W^{k+1,p}(K)$, where the constant $C$ depends only on $k, p$ and is independent of the geometry of $K$.

Any tetrahedron can be obtained from a “skinny right” tetrahedron by an affine transformation. To obtain an error estimate, we need to estimate the ratio of the maximum and minimum singular values of the Jacobian matrix of the affine transformation. If we obtain an expression of the ratio in terms of geometric quantities of the tetrahedron, a new error estimation would be obtained. The authors hope that they will report further development of error estimations on tetrahedrons in near future.

**References**

[1] R.A. Adams, J.J.F. Fournier: Sobolev Spaces, 2nd edition, Pure and Applied Mathematics 140, Elsevier/Academic Press, New York, 2003. Zbl 1098.46001, MR2424078

[2] T. Apel: Anisotropic Finite Element: Local estimates and applications, Advances in Numerical Mathematics. B.G. Teubner, Stuttgart, 1999. Zbl 0934.65121, MR1716824.
[3] K.E. Atkinson: An Introduction to Numerical Analysis, 2nd edition, John Wiley & Sons, New York, 1989. Zbl 0718.65001, MR1007135

[4] I. Babuška, A.K. Aziz: On the angle condition in the finite element method, SIAM J. Numer. Anal. 13 (1976), 214–226. Zbl 0324.65046, MR0455462

[5] R.E. Barnhill, J.A. Gregory: Sard kernel theorems on triangular domains with application to finite element error bounds. Numer. Math., 25 (1976), 215-229. Zbl 0304.65076, MR0458000

[6] S.C. Brenner, L.R. Scott: The Mathematical Theory of Finite Element Methods. 3rd edition. Texts in Applied Mathematics 15, Springer, New York, 2008. Zbl 1135.65042, MR2373954

[7] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York, 2011. Zbl 1220.46002, MR1930132

[8] P.G. Clément: The Finite Element Methods for Elliptic Problems. Classics in Applied Mathematics 40, SIAM, Philadelphia, 2002, Reprint of the 1978 original (North Holland, Amsterdam). Zbl 0999.65129, MR1930132

[9] R.G. Durán: Error estimates for 3-d narrow finite elements. Math. Comp., 68 (1999), 187–199. Zbl 0910.65078, MR1489970

[10] A. Ern, J.-L. Guermond: Theory and Practice of Finite Elements. Applied Mathematical Sciences 159, Springer, New York, 2004. Zbl 1059.65103, MR2050138

[11] P. Jamet: Estimations d’erreur pour des éléments finis droits presque dégénérés. R.A.I.R.O. Anal. Numer., 10 (1976), 43–61. (In French.) Zbl 0346.65052, MR0455282

[12] K. Kobayashi, T. Tsujiya: A Babuška-Aziz type proof of the circumradius condition, Japan J. Indust. Appl. Math., 31 (2014), 193-210. Zbl 1295.65011, MR3167084

[13] K. Kobayashi, T. Tsujiya: A priori error estimates for Lagrange interpolation on triangles. Appl. Math., Praha 60 (2015), 485–499. Zbl 06486922, MR3396477

[14] M. Kráček: On semiregular families of triangulations and linear interpolation. Appl. Math., Praha 36 (1991), 223–232. Zbl 0755.41003, MR1109126

[15] M. Kráček: On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal., 29 (1992), 513–520. Zbl 0755.41003, MR1145427.

[16] A. Kufner, O. John, S. Fučík: Function Spaces. Noordhoff International Publishing, Leyden, 1977. Zbl 0364.46022, MR0482102

[17] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural’ceva: Linear and Quasi-linear Equations of Parabolic Type. Translations of Mathematical Monographs 23, AMS, Providence, 1968. Zbl 0174.15403, MR0241822

[18] N.A. Shenk: Uniform error estimates for certain narrow Lagrange finite elements. Math. Comp., 63 (1994), 105–119. Zbl 0807.65003, MR1226816

[19] T. Yamamoto: Introduction to Numerical Analysis, 2nd edition. (In Japanese.) Saiensu-sha, 2003.

Kenta Kobayashi, Graduate School of Commerce and Management, Hitotsubashi University, Kunitachi, 186-8601, Japan, e-mail: kenta.k@r.hit-u.ac.jp.

Takuya Tsuchiya, Graduate School of Science and Engineering, Ehime University, Matsuyama, 790-8577, Japan, e-mail: tsuchiya@math.sci.ehime-u.ac.jp.