Vacuum in quantum liquids and in general relativity

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Abstract

Quantum liquids, in which an effective Lorentzian metric and thus some kind of gravity gradually arise in the low-energy corner, are the objects where the problems related to the quantum vacuum can be investigated in detail. In particular, they provide the possible solution of the cosmological constant problem: why the vacuum energy is by 120 orders of magnitude smaller than the estimation from the relativistic quantum field theory. The almost complete cancellation of the cosmological constant does not require any fine tuning and comes from the fundamental “trans-Planckian” physics of quantum liquids. The remaining vacuum energy is generated by the perturbation of quantum vacuum caused by matter (quasiparticles), curvature, and other possible sources, such as smooth component – the quintessence. This provides the possible solution of another cosmological constant problem: why the present cosmological constant is on the order of
the present matter density of the Universe. We discuss here some properties of the quantum vacuum in quantum liquids: the vacuum energy under different conditions; excitations above the vacuum state and the effective acoustic metric for them provided by the motion of the vacuum; Casimir effect, etc.

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# 1 Introduction.

Quantum liquids, such as $^3$He and $^4$He, represent the systems of strongly interacting and strongly correlated atoms, $^3$He and $^4$He atoms correspondingly. Even in its ground state, such liquid is a rather complicated object, whose many body physics requires extensive numerical simulations. However, when
the energy scale is reduced below about 1 K, we cannot resolve anymore the
motion of isolated atoms in the liquid. The smaller the energy the better
is the liquid described in terms of the collective modes and the dilute gas
of the particle-like excitations – quasiparticles. This is the Landau picture
of the low-energy degrees of freedom in quantum Bose and Fermi liquids.
The dynamics of collective modes and quasiparticles is described in terms
of what we call now ‘the effective theory’. In superfluid $^4$He this effective
theory, which incorporates the collective motion of the ground state – the
quantum vacuum – and the dynamics of quasiparticles in the background of
the moving vacuum, is known as the two-fluid hydrodynamics [1].

Such an effective theory does not depend on details of microscopic (atomic)
structure of the quantum liquid. The type of the effective theory is deter-
mined by the symmetry and topology of the ground state, and the role of the
microscopic physics is only to choose between different universality classes
on the basis of the minimum energy consideration. Once the universality
class is determined, the low-energy properties of the condensed matter sys-
tem are completely described by the effective theory, and the information on
the microscopic (trans-Planckian) physics is lost [2].

In some condensed matter the universality class produces the effec-
tive theory, which reminds very closely the relativistic quantum field theory. For
example, the collective fermionic and bosonic modes in superfluid $^3$He-A
reproduce chiral fermions, gauge fields and even in many respects the gravi-
tational field [3].

This allows us to use the quantum liquids for the investigation of the
properties related to the quantum vacuum in relativistic quantum field theo-
ries, including the theory of gravitation. The main advantage of the quantum
liquids is that in principle we know their vacuum structure at any relevant
scale, including the interatomic distance, which plays the part of one of the
Planck length scales in the hierarchy of scales. Thus the quantum liquids
can provide possible routes from our present low-energy corner of the effec-
tive theory to the “microscopic” physics at Planckian and trans-Planckian
energies.

One of the possible routes is related to the conserved number of atoms
$N$ in the quantum liquid. The quantum vacuum of the quantum liquids
is constructed from the discrete elements, the bare atoms. The interaction
and zero point motion of these atoms compete and provide an equilibrium
ground state of the ensemble of atoms, which can exist even in the absence of
external pressure. The relevant energy and the pressure in this equilibrium ground state are exactly zero in the absence of interaction with environment. Translating this to the language of general relativity, one obtains that the cosmological constant in the effective theory of gravity in the quantum liquid is exactly zero without any fine tuning. The equilibrium quantum vacuum is not gravitating.

This route shows a possible solution of the cosmological constant problem: why the estimation the vacuum energy using the relativistic quantum field theory gives the value, which is by 120 orders of magnitude higher than its upper experimental limit. In quantum liquids there is a similar discrepancy between the exact zero result for the vacuum energy and the naive estimation within the effective theory. We shall also discuss here how different perturbations of the vacuum in quantum liquids lead to small nonzero energy of quantum vacuum. Translating this to the language of general relativity, one obtains that in each epoch the vacuum energy density must be either of order of the matter density of the Universe, or of its curvature, or of the energy density of the smooth component – the quintessence.

Here we mostly discuss the Bose ensemble of atoms: a weakly interacting Bose gas, which experiences the phenomenon of Bose condensation, and a real Bose liquid – superfluid \(^4\)He. The consideration of the Bose gas allows us to use the microscopic theory to derive the ground state energy of the quantum system of interacting atoms and the excitations above the vacuum state – quasiparticles. We also discuss the main differences between the bare atoms, which comprise the vacuum state, and the quasiparticles, which serve as elementary particles in the effective quantum field theory.

Another consequence of the discrete number of the elements comprising the vacuum state, which we consider, is related to the Casimir effects. The discreteness of the vacuum – the finite-\(N\) effect – leads to the mesoscopic Casimir forces, which cannot be derived within the effective theory. For these purposes we consider the Fermi ensembles of atoms: Fermi gas and Fermi liquid.

2 Einstein gravity and cosmological constant problem
2.1 Einstein action

The Einstein’s geometrical theory of gravity consists of two main elements \[4\].

(1) Gravity is related to a curvature of space-time in which particles move along the geodesic curves in the absence of non-gravitational forces. The geometry of the space-time is described by the metric \( g_{\mu\nu} \) which is the dynamical field of gravity. The action for matter in the presence of gravitational field \( S_M \), which simultaneously describes the coupling between gravity and all other fields (the matter fields), is obtained from the special relativity action for the matter fields by replacing everywhere the flat Minkowski metric by the dynamical metric \( g_{\mu\nu} \) and the partial derivative by \( g \)-covariant derivative. This follows from the principle that the equations of motion do not depend on the choice of the coordinate system (the so called general covariance). This also means that the motion in the non-inertial frame can be described in the same manner as motion in some gravitational field – this is the equivalence principle. Another consequence of the equivalence principle is that the the space-time geometry is the same for all the particles: the gravity is universal.

(2) The dynamics of the gravitational field is determined by adding the action functional \( S_G \) for \( g_{\mu\nu} \), which describes propagation and self-interaction of the gravitational field:

\[
S = S_G + S_M .
\]  

The general covariance requires that \( S_G \) is the functional of the curvature. In the original Einstein theory only the first order curvature term is retained:

\[
S_G = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R ,
\]  

where \( G \) is gravitational Newton constant; and \( R \) is the Ricci scalar curvature. The Einstein action is thus:

\[
S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_M
\]  

Variation of this action over the metric field \( g_{\mu\nu} \) gives the Einstein equations:

\[
\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \sqrt{-g} \left[ -\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + T^M_{\mu\nu} \right] = 0 ,
\]
where $T_{\mu\nu}^M$ is the energy-momentum of the matter fields. Bianchi identities lead to the “covariant” conservation law for matter

$$T_{\nu;\mu}^\mu = 0 \ , \ \text{or} \ \partial_\mu \left( T_{\nu}^\mu \sqrt{-g} \right) = \frac{1}{2} \sqrt{-g} T^{\alpha\beta M} \partial_\nu g_{\alpha\beta} .$$

But actually this “covariant” conservation takes place in virtue of the field equation for “matter” irrespective of the dynamics of the gravitational field.

### 2.2 Vacuum energy and cosmological term

In particle physics, field quantization allows a zero point energy, the constant energy when all fields are in their ground states. In the absence of gravity, only the difference between zero points can be measured, for example in the Casimir effect, while the absolute value in unmeasurable. However, Einstein’s equations react to $T_{\mu\nu}^M$ and thus to the value of vacuum energy itself.

If the vacuum energy is taken seriously, the energy-momentum tensor of the vacuum must be added to the Einstein equations. The corresponding action is given by the so-called cosmological term, which was introduced by Einstein in 1917

$$S_\Lambda = -\rho_\Lambda \int d^4x \sqrt{-g} , \ \ T_{\mu\nu}^\Lambda = \frac{2}{\sqrt{-g}} \frac{\delta S_\Lambda}{\delta g_{\mu\nu}} = \rho_\Lambda g_{\mu\nu} .$$

The energy-momentum tensor of the vacuum shows that the quantity $\rho_\Lambda \sqrt{-g}$ is the vacuum energy density, and the equation of state of the vacuum is

$$P_\Lambda = -\rho_\Lambda ,$$

where $P_\Lambda \sqrt{-g}$ is the partial pressure of the vacuum. The Einstein’s equations are modified:

$$\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = T_{\mu\nu}^\Lambda + T_{\mu\nu}^M .$$

### 2.3 Cosmological constant problem

The most severe problem in the marriage of gravity and quantum theory is why is the vacuum not gravitating. The vacuum energy density can be easily estimated: the positive contribution comes from the zero-point energy of the bosonic fields and the negative – from the occupied negative energy
levels in the Dirac sea. Since the largest contribution comes from the high momenta, where the energy spectrum of particles is massless, $E = cp$, the energy density of the vacuum is

$$\rho_\Lambda \sqrt{-g} = \frac{1}{V} \left( \nu_{\text{bosons}} \sum_p \frac{1}{2} cp - \nu_{\text{fermions}} \sum_p cp \right)$$

(9)

where $V$ is the volume of the system; $\nu_{\text{bosons}}$ is the number of bosonic species and $\nu_{\text{fermions}}$ is the number of fermionic species. The vacuum energy is divergent and the natural cut-off is provided by the gravity itself. The cut-off Planck energy is determined by the Newton’s constant:

$$E_{\text{Planck}} = \left( \frac{\hbar c^5}{G} \right)^{1/2},$$

(10)

It is on the order of $10^{19}$ GeV. If there is no symmetry between the fermions and bosons (supersymmetry) the Planck energy scale cut-off provides the following estimation for the vacuum energy density:

$$\rho_\Lambda \sqrt{-g} \sim \pm \frac{1}{c^3} E_{\text{Planck}}^4 = \pm \sqrt{-g} E_{\text{Planck}}^4,$$

(11)

with the sign of the vacuum energy being determined by the fermionic and bosonic content of the quantum field theory. Here we considered the flat space with Minkowski metric $g_{\mu\nu} = \text{diag}(-1, c^{-2}, c^{-2}, c^{-2})$.

The “cosmological constant problem” is a huge disparity, between the naively expected value in Eq. (11) and the range of actual values: the experimental observations show that $\rho_\Lambda$ is less than or on the order of $10^{-120} E_{\text{Planck}}^4$. In case of supersymmetry, the cut-off is somewhat less, being determined by the scale at which supersymmetry is violated, but the huge disparity persists.

This disparity demonstrates that the vacuum energy in Eq. (2) is not gravitating. This is in apparent contradiction with the general principle of equivalence, according to which the inertial and gravitating masses must coincide. This indicate that the theoretical criteria for setting the absolute zero point of energy are unclear and probably require the physics beyond general relativity. To clarify this issue we can consider such quantum systems where the elements of the gravitation are at least partially reproduced, but where the structure of the quantum vacuum is known. Quantum liquids are the right systems.
2.4 Sakharov induced gravity

Why is the Planck energy in Eq. (10) the natural cutoff in quantum field theory? This is based on the important observation made by Sakharov that the second element of the Einstein’s theory can follow from the first one due to the quantum fluctuations of the relativistic matter field [8]. He showed that vacuum fluctuations of the matter field induce the curvature term in action for $g_{\mu\nu}$. One can even argue that the whole Einstein action is induced by vacuum polarization, and thus the gravity is not the fundamental force but is determined by the properties of the quantum vacuum.

The magnitude of the induced Newton’s constant is determined by the value of the cut-off: $G^{-1} \sim \hbar E_{\text{cutoff}}^2/c^5$. Thus in this Sakharov’s gravity induced by quantum fluctuations the causal connection between the gravity and the cut-off is reversed: the physical high-energy cut-off determines the gravitational constant. The $E_{\text{cutoff}}^2$ dependence of the inverse gravitational constant explains why the gravity is so small compared to the other forces, whose running coupling “constants” have only mild logarithmic dependence on $E_{\text{cutoff}}$.

The same cut-off must be applied for the estimation of the cosmological constant, which thus must be of order of $E_{\text{cutoff}}^4$. But this is in severe contradiction with experiment. This shows that, though the effective theory is appropriate for the calculation of the Einstein curvature term, it is not applicable for the calculation of the vacuum energy: the trans-Planckian physics must be evoked for that.

The Sakharov theory does not explain the first element of the Einstein’s theory: it does not show how the metric field $g_{\mu\nu}$ appears. This can be given only by the fundamental theory of quantum vacuum, such as string theory where the gravity appears as a low-energy mode. The quantum liquid examples also show that the metric field can naturally and in some cases even emergently appear as the low-energy collective mode of the quantum vacuum.

2.5 Effective gravity in quantum liquids

The first element of the Einstein theory of gravity (that the motion of quasiparticles is governed by the effective curved space-time) arises in many condensed matter systems in the low-energy limit. An example is the motion of
acoustic phonons in distorted crystal lattice, or in the background flow field of superfluid condensate. This motion is described by the effective acoustic metric \[ g_{\mu\nu} \]. For this “relativistic matter field” (acoustic phonons with dispersion relation \( E = cp \), where \( c \) is a speed of sound, simulate relativistic particles) the analog of the equivalence principle is fulfilled. As a result the covariant conservation law in Eq. (5) does hold for the acoustic mode, if \( g_{\mu\nu} \) is replaced by the acoustic metric.

The second element of the Einstein’s gravity is not easily reproduced in condensed matter. In general, the dynamics of acoustic metric \( g_{\mu\nu} \) does not obey the equivalence principle inspite of the Sakharov mechanism of the induced gravity. In the existing quantum liquids the Einstein action induced by the quantum fluctuations of the “relativistic matter field” is much smaller than the non-covariant action induced by “non-relativistic” high-energy component of the quantum vacuum, which is overwhelming in these liquids. Of course, one can find some very special cases where the Einstein action for the effective metric is dominating, but this is not a rule.

Nevertheless, inspite of the incomplete analogy with the Einstein theory, the effective gravity in quantum liquids can be useful for investigation of the cosmological constant problem.

3 Microscopic ‘Theory of Everything’ in quantum liquids

3.1 Microscopic and effective theories

There are two ways to study quantum liquids:

(i) The fully microscopic treatment. It can be realized completely (a) by numerical simulations of the many body problem; (b) analytically for some special models; (3) perturbatively for some special ranges of the material parameters, for example, in the limit of weak interaction between the particles.

(ii) Phenomenological approach in terms of effective theories. The hierarchy of the effective theories correspond to the low-frequency long-wave-length dynamics of quantum liquids in different ranges of frequency. Examples of effective theories: Landau theory of Fermi liquid; Landau-Khalatnikov two-fluid hydrodynamics of superfluid \(^4\text{He}\); theory of elasticity in solids; Landau-Lifshitz theory of ferro- and antiferromagnetism; London theory of
superconductivity; Leggett theory of spin dynamics in superfluid phases of $^3$He; effective quantum electrodynamics arising in superfluid $^3$He-A; etc. The latter example indicates, that the existing Standard Model of electroweak, and strong interactions, and the Einstein gravity too, are the phenomenological effective theories of high-energy physics, which describe its low-energy edge, while the microscopic theory of the quantum vacuum is absent.

3.2 Theory of Everything for quantum liquid

The microscopic “Theory of Everything” for quantum liquids – “a set of equations capable of describing all phenomena that have been observed” [2] in these quantum systems – is extremely simple. On the “fundamental” level appropriate for quantum liquids and solids, i.e. for all practical purposes, the $^4$He or $^3$He atoms of these quantum systems can be considered as structureless: the $^4$He atoms are the structureless bosons and the $^3$He atoms are the structureless fermions with spin $1/2$. The Theory of Everything for a collection of a macroscopic number $N$ of interacting $^4$He or $^3$He atoms is contained in the non-relativistic many-body Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^{N} \sum_{j=i+1}^{N} U(r_i - r_j), \quad (12)$$

acting on the many-body wave function $\Psi(r_1, r_2, ..., r_i, ..., r_j, ...)$). Here $m$ is the bare mass of the atom; $U(r_i - r_j)$ is the pair interaction of the bare atoms $i$ and $j$. When written in the second quantized form it becomes the Hamiltonian of the quantum field theory

$$\mathcal{H} - \mu \mathcal{N} = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) \left[ -\frac{\nabla^2}{2m} - \mu \right] \psi(\mathbf{x}) + \frac{1}{2} \int d\mathbf{x} d\mathbf{y} U(\mathbf{x} - \mathbf{y}) \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{y}) \psi(\mathbf{y}) \psi(\mathbf{x}) \quad (13)$$

In $^4$He, the bosonic quantum field $\psi(\mathbf{x})$ is the annihilation operator of the $^4$He atoms. In $^3$He, $\psi(\mathbf{x})$ is the fermionic field and the spin indices must be added. Here $\mathcal{N} = \int d\mathbf{x} \, \psi^\dagger(\mathbf{x}) \psi(\mathbf{x})$ is the operator of particle number (number of atoms); $\mu$ is the chemical potential – the Lagrange multiplier which is introduced to take into account the conservation of the number of atoms.
3.3 Importance of discrete particle number in microscopic theory

This is the main difference from the relativistic quantum field theory, where the number of any particles is not restricted: particles and antiparticles can be created from the quantum vacuum. As for the number of particles in the quantum vacuum itself, this quantity is simply not determined today. At the moment we do not know the structure of the quantum vacuum and its particle content. Moreover, it is not clear whether it is possible to describe the vacuum in terms of some discrete elements (bare particles) whose number is conserved. On the contrary, in quantum liquids the analog of the quantum vacuum – the ground state of the quantum liquid – has the known number of atoms. If \( N \) is big, this difference between the two quantum field theories disappears. Nevertheless, the mere fact, that there is a conservation law for the number of particles comprising the vacuum, leads to the definite conclusion on the value of the relevant vacuum energy. Also, as we shall see below in Sec. [4], the discreteness of the quantum vacuum can be revealed in the mesoscopic Casimir effect.

3.4 Enhancement of symmetry in the low energy corner. Appearance of effective theory.

The Hamiltonian (3) has very restricted number of symmetries: It is invariant under translations and \( SO(3) \) rotations in 3D space; there is a global \( U(1) \) group originating from the conservation of the number of atoms: \( \mathcal{H} \) is invariant under gauge rotation \( \psi(x) \rightarrow e^{i\alpha} \psi(x) \) with constant \( \alpha \); in \(^3\text{He} \) in addition, if the relatively weak spin-orbit coupling is neglected, \( \mathcal{H} \) is also invariant under separate rotations of spins, \( SO(3)_s \). At low temperature the phase transition to the superfluid or to the quantum crystal state occurs where some of these symmetries are broken spontaneously. For example, in the \(^3\text{He}-\text{A} \) state all of these symmetries, except for the translational symmetry, are broken.

However, when the temperature and energy decrease further the symmetry becomes gradually enhanced in agreement with the anti-grand-unification scenario [13, 14]. At low energy the quantum liquid or solid is well described in terms of a dilute system of quasiparticles. These are bosons (phonons) in \(^4\text{He} \) and fermions and bosons in \(^3\text{He} \), which move in the background of
the effective gauge and/or gravity fields simulated by the dynamics of the collective modes. In particular, phonons propagating in the inhomogeneous liquid are described by the effective Lagrangian

\[ L_{\text{effective}} = \sqrt{-g} g^{\mu \nu} \partial_\mu \alpha \partial_\nu \alpha , \]  

(14)

where \( g^{\mu \nu} \) is the effective acoustic metric provided by inhomogeneity and flow of the liquid [9, 10, 12].

These quasiparticles serve as the elementary particles of the low-energy effective quantum field theory. They represent the analogue of matter. The type of the effective quantum field theory – the theory of interacting fermionic and bosonic quantum fields – depends on the universality class of the fermionic condensed matter (see review [3]). The superfluid \( ^3\text{He-A} \), for example, belongs to the same universality class as the Standard Model. The effective quantum field theory describing the low energy phenomena in \( ^3\text{He-A} \), contains chiral “relativistic” fermions. The collective bosonic modes interact with these “elementary particles” as gauge fields and gravity. All these fields emergently arise together with the Lorentz and gauge invariances and with elements of the general covariance from the fermionic Theory of Everything in Eq.(13).

The emergent phenomena do not depend much on the details of the Theory of Everything [4], in our case on the details of the pair potential \( U(x-y) \). Of course, the latter determines the universality class in which the system enters at low energy. But once the universality class is established, the physics remains robust to deformations of the pair potential. The details of \( U(x-y) \) influence only the “fundamental” parameters of the effective theory (“speed of light”, “Planck” energy cut-off, etc.) but not the general structure of the theory. Within the effective theory the “fundamental” parameters are considered as phenomenological.

4 Weakly interacting Bose gas

The quantum liquids are strongly correlated and strongly interacting systems. That is why, though it is possible to derive the effective theory from first principles in Eq.(13), if one has enough computer time and memory, this is a rather difficult task. It is instructive, however, to consider the microscopic theory for some special model potentials \( U(x-y) \). This allow us to solve the
problem completely or perturbatively. In case of the Bose-liquids the proper model is the Bogoliubov weakly interacting Bose gas, which is in the same universality class as a real superfluid $^4$He. Such model is very useful, since it simultaneously covers the low-energy edge of the effective theory, and the high-energy “transPlanckian” physics.

4.1 Model Hamiltonian

Here we follow mostly the book by Khalatnikov [1]. In the model of weakly interacting Bose gas the pair potential in Eq.(13) is weak. As a result the most of the particles at $T = 0$ are in the Bose condensate, i.e. in the state with the momentum $p = 0$. The Bose condensate is characterized by the nonzero vacuum expectation value (vev) of the particle annihilation operator at $p = 0$:

$$
\langle a_{p=0} \rangle = \sqrt{N_0} e^{i\Phi}, \quad \langle a_{p=0}^\dagger \rangle = \sqrt{N_0} e^{-i\Phi}.
$$

(15)

Here $N_0$ is the particle number in the Bose condensate, and $\Phi$ is the phase of the condensate. The vacuum is degenerate over global $U(1)$ rotation of the phase. Further we consider particular vacuum state with $\Phi = 0$.

If there is no interaction between the particles (an ideal Bose gas), all the particles at $T = 0$ are in the Bose condensate, $N_0 = N$. Small interaction induces a small fraction of particles which are not in condensate, these particle have small momenta $p$. As a result only zero Fourier component of the pair potential is relevant, and Eq.(13) has the form:

$$
\mathcal{H} - \mu \mathcal{N} = -\mu N_0 + \frac{N_0^2 U}{2V} + \sum_{p \neq 0} \left( \frac{p^2}{2m} - \mu \right) a_p^\dagger a_p + \frac{N_0 U}{2V} \sum_{p \neq 0} \left( 2a_p^\dagger a_p + 2a_{-p}^\dagger a_{-p} + a_p a_{-p} + a_p^\dagger a_{-p}^\dagger \right)
$$

(16)

Here $N_0 = a_0^\dagger a_0 = a_0 a_0^\dagger = a_0^\dagger a_0^\dagger = a_0 a_0$ is the particle number in the Bose-condensate (we neglected quantum fluctuations of the operator $a_0$ and consider $a_0$ as $c$-number); $U$ is the matrix element of pair interaction for zero momenta $p$ of particles. Minimization of the main part of the energy in Eq.(13) over $N_0$ gives $UN_0/V = \mu$ and one obtains:

$$
\mathcal{H} - \mu \mathcal{N} = -\frac{\mu^2}{2U} V + \sum_{p \neq 0} \mathcal{H}_p
$$

(18)
\[ H_p = \frac{1}{2} \left( \frac{p^2}{2m} + \mu \right) (a^\dagger_p a_p + a^\dagger_{-p} a_{-p}) + \frac{\mu}{2} (a_p a_{-p} + a^\dagger_p a^\dagger_{-p}) \]  (19)

4.2 Pseudorotation – Bogoliubov transformation

At each \( p \) the Hamiltonian can be diagonalized using the following consideration. The operators

\[ \mathcal{L}_3 = \frac{1}{2} (a^\dagger_p a_p + a^\dagger_{-p} a_{-p} + 1) , \quad \mathcal{L}_1 + i \mathcal{L}_2 = a^\dagger_p a^\dagger_{-p} , \quad \mathcal{L}_1 - i \mathcal{L}_2 = a_p a_{-p} \]  (20)

form the group of pseudorotations, \( SU(1,1) \) (the group which conserves the form \( x_1^2 + x_2^2 - x_3^2 \)), with the commutation relations:

\[ [\mathcal{L}_3, \mathcal{L}_1] = i \mathcal{L}_2 , \quad [\mathcal{L}_2, \mathcal{L}_3] = i \mathcal{L}_1 , \quad [\mathcal{L}_1, \mathcal{L}_2] = -i \mathcal{L}_3 , \]  (21)

In terms of the pseudomomentum the Hamiltonian in Eq.(19) has the form

\[ H_p = \left( \frac{p^2}{2m} + \mu \right) \mathcal{L}_3 + \mu \mathcal{L}_1 - \frac{1}{2} \left( \frac{p^2}{2m} + \mu \right) \]  (22)

In case of the nonzero phase \( \Phi \) of the Bose condensate one has

\[ H_p = \left( \frac{p^2}{2m} + \mu \right) \mathcal{L}_3 + \mu \cos(2\Phi) \mathcal{L}_1 + \mu \sin(2\Phi) \mathcal{L}_2 - \frac{1}{2} \left( \frac{p^2}{2m} + \mu \right) \]  (23)

The diagonalization of this Hamiltonian by is achieved first by rotation by angle \( 2\Phi \) around axis \( z \), and then by the Lorentz transformation – pseudorotation around axis \( y \):

\[ \mathcal{L}_3 = \tilde{\mathcal{L}}_3 \chi + \tilde{\mathcal{L}}_1 \chi , \quad \mathcal{L}_1 = \tilde{\mathcal{L}}_1 \chi + \tilde{\mathcal{L}}_3 \chi_y , \quad \text{th} \chi = \frac{\mu}{\frac{p^2}{2m} + \mu} . \]  (24)

This corresponds to Bogoliubov transformation and gives the following diagonal Hamiltonian:

\[ H_p = -\frac{1}{2} \left( \frac{p^2}{2m} + \mu \right) + \tilde{\mathcal{L}}_3 \sqrt{\left( \frac{p^2}{2m} + \mu \right)^2 - \mu^2} = \]  (25)

\[ = \frac{1}{2} E(p) \left( a^\dagger_p \tilde{a}_{p} + a^\dagger_{-p} \tilde{a}_{-p} \right) + \frac{1}{2} \left( E(p) - \left( \frac{p^2}{2m} + \mu \right) \right) , \]  (26)
where $\tilde{a}_p$ is the operator of annihilation of quasiparticles, whose energy spectrum $E(p)$ is

$$E(p) = \sqrt{\left(\frac{\mu^2}{2m} + \mu\right)^2 - \mu^2} = \sqrt{p^2 c^2 + \frac{p^4}{4m^2}} = c^2 \frac{\mu}{m}.$$  \hfill (27)

### 4.3 Vacuum and quasiparticles

The total Hamiltonian now represents the ground state – the vacuum – and the system of quasiparticles

$$\mathcal{H} - \mu \mathcal{N} = \langle \mathcal{H} - \mu \mathcal{N} \rangle_{\text{vac}} + \sum_p E(p) \tilde{a}_p^\dagger \tilde{a}_p$$ \hfill (28)

The lower the energy the more dilute is the system of quasiparticles and thus the weaker is the interaction between them. This description in terms of the vacuum state and dilute system of quasiparticle is generic for the condensed matter systems and is valid even if the interaction of the initial bare particles is strong. The phenomenological effective theory in terms of vacuum state and quasiparticles was developed by Landau both for Bose and Fermi liquids. Quasiparticles (not the bare particles) play the role of elementary particles in such effective quantum field theories.

In a weakly interacting Bose-gas in Eq.(27), the spectrum of quasiparticles at low energy (i.e. at $p \ll mc$) is linear, $E = cp$. The linear slope coincides with the speed of sound, which can be obtained from the leading term in energy: $N(\mu) = -d(E - \mu \mathcal{N})/d\mu = \mu V/U$, $c^2 = N(d\mu/dN)/m = \mu/m$. These quasiparticles are phonons – quanta of sound waves. The same quasiparticle spectrum occurs in the real superfluid liquid $^4$He, where the interaction between the bare particle is strong. This shows that the qualitative low-energy properties of the system do not depend on the microscopic (trans-Planckian) physics. The latter determines only the speed of sound $c$. One can say, that weakly and strongly interacting Bose systems belong to the same universality class, and thus have the same low-energy properties. One cannot distinguish between the two systems if the observer measures only the low-energy effects, since they are described by the same effective theory.

### 4.4 Particles and quasiparticles

It is necessary to distinguish between the bare particles and quasiparticles.
Particles are the elementary objects of the system on a microscopic “trans-Planckian” level, these are the atoms of the underlying liquid ($^3$He or $^4$He atoms). The many-body system of the interacting atoms form the quantum vacuum – the ground state. The nondissipative collective motion of the superfluid vacuum with zero entropy is determined by the conservation laws experienced by the atoms and by their quantum coherence in the superfluid state.

Quasiparticles are the particle-like excitations above this vacuum state, they serve as elementary particles in the effective theory. The bosonic excitations in superfluid $^4$He and fermionic and bosonic excitations in superfluid $^3$He represent the matter in our analogy. In superfluids they form the viscous normal component responsible for the thermal and kinetic low-energy properties of superfluids. Fermionic quasiparticles in $^3$He-A are chiral fermions, which are the counterpart of the leptons and quarks in the Standard Model [3].

4.5 Galilean transformation for particles and quasiparticles

The quantum liquids considered here are essentially nonrelativistic: under the laboratory conditions their velocity is much less than the speed of light. That is why they obey with great precision the Galilean transformation law. Under the Galilean transformation to the coordinate system moving with the velocity $\mathbf{u}$ the superfluid velocity – the velocity of the quantum vacuum – transforms as $\mathbf{v}_s \rightarrow \mathbf{v}_s + \mathbf{u}$.

As for the transformational properties of bare particles (atoms) and quasiparticles, it appears that they are essentially different. Let us start with bare particles. If $\mathbf{p}$ and $E(\mathbf{p})$ are the momentum and energy of the bare particle (atom with mass $m$) measured in the system moving with velocity $\mathbf{u}$, then from the Galilean invariance it follows that its momentum and energy measured by the observer at rest are correspondingly

$$\tilde{\mathbf{p}} = \mathbf{p} + m \mathbf{u}, \quad \tilde{E}(\mathbf{p}) = E(\mathbf{p} + m \mathbf{u}) = E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{u} + \frac{1}{2} m \mathbf{u}^2.$$  \hfill (29)

This transformation law contains the mass $m$ of the bare atom.

However, when the quasiparticles are concerned, one can expect that such characteristic of the microscopic world as the bare mass $m$ cannot enter the
transformation law for quasiparticles. This is because quasiparticles in effective low-energy theory have no information on the transPlanckian world of the bare atoms comprising the vacuum state. All the information on the quantum vacuum, which the low-energy quasiparticle has, is encoded in the effective metric $g_{\mu\nu}$. Since the mass $m$ must drop out from the transformation law for quasiparticles, we expect that the momentum of quasiparticle is invariant under the Galilean transformation: $\mathbf{p} \to \mathbf{p}$, while the quasiparticle energy is simply Doppler shifted: $E(\mathbf{p}) \to E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{u}$. Such a transformation law allows us to write the energy of quasiparticle in the moving superfluid vacuum. If $\mathbf{p}$ and $E(\mathbf{p})$ are the quasiparticle momentum and energy measured in the coordinate system where the superfluid vacuum is at rest (i.e. $\mathbf{v}_s = 0$, we call this frame the superfluid comoving frame), then its momentum and energy in the laboratory frame are

$$\tilde{\mathbf{p}} = \mathbf{p}, \quad \tilde{E}(\mathbf{p}) = E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}_s.$$  (30)

The difference in the transformation properties of bare particles and quasiparticles comes from their different status. While the momentum and energy of bare particles are determined in “empty” space-time, the momentum and energy of quasiparticles are counted from that of the quantum vacuum. This difference can be easily visualized if one considers the spectrum of quasiparticles in the weakly interacting Bose gas in Eq.(27) in the limit of large momentum $p \gg mc$, when the energy spectrum of quasiparticles approaches that of particles, $E \to p^2/2m$. In this limit the difference between particles and quasiparticles disappears, and at first glance one may expect that quasiparticle should obey the same transformation property under Galilean transformation as a bare isolated particle. To add more confusion let us consider an ideal Bose gas of noninteracting bare particles, where quasiparticles have exactly the same spectrum as particles. Why the transformation properties are so different for them?

The ground state of the ideal Bose gas has zero energy and zero momentum in the reference frame where the Bose condensate is at rest (the superfluid comoving reference frame). In the laboratory frame the condensate momentum and energy are correspondingly

$$\langle \mathcal{P} \rangle_{\text{vac}} = Nm\mathbf{v}_s,$$  (31)

$$\langle \mathcal{H} \rangle_{\text{vac}} = N\frac{m\mathbf{v}_s^2}{2}.$$  (32)
The state with one quasiparticle is the state in which \( N - 1 \) particles have zero momenta, \( p = 0 \), while one particle has nonzero momentum \( p \neq 0 \). In the comoving reference frame the momentum and energy of such state with one quasiparticle are correspondingly \( \langle P \rangle_{\text{vac}+1qp} = p \) and \( \langle H \rangle_{\text{vac}+1qp} = E(p) = p^2/2m \). In the laboratory frame the momentum and energy of the system are obtained by Galilean transformation

\[
\langle P \rangle_{\text{vac}+1qp} = (N - 1)mv_s + (p + mv_s) = \langle P \rangle_{\text{vac}} + p , \quad (33)
\]

\[
\langle H \rangle_{\text{vac}+1qp} = (N - 1)\frac{mv_s^2}{2} + \frac{(p + mv_s)^2}{2m} = \langle H \rangle_{\text{vac}} + E(p) + p \cdot v_s . \quad (34)
\]

Since the energy and the momentum of quasiparticles are counted from that of the quantum vacuum, the transformation properties of quasiparticles are different from the Galilean transformation law. The part of the Galilean transformation, which contains the mass of the atom, is absorbed by the Bose-condensate which represents the quantum vacuum.

### 4.6 Effective metric from Galilean transformation

The right hand sides of Eqs.\((33)\) and \((34)\) show that the energy spectrum of quasiparticle in the moving superfluid vacuum is given by Eq.\((30)\). Such spectrum can be written in terms of the effective acoustic metric:

\[
(\tilde{E} - p \cdot v_s)^2 = c^2p^2 , \quad \text{or} \quad g^{\mu\nu}p_{\mu}p_{\nu} = 0 . \quad (35)
\]

where the metric has the form:

\[
g^{00} = -1 , \quad g^{0i} = -v_s^i , \quad g^{ij} = c^2\delta^{ij} - v_s^i v_s^j , \quad (36)
\]

\[
g_{00} = -\left(1 - \frac{v_s^2}{c^2}\right) , \quad g_{0i} = -\frac{v_s^i}{c^2} , \quad g_{ij} = \frac{1}{c^2}\delta_{ij} , \quad (37)
\]

\[
\sqrt{-g} = c^{-3} . \quad (38)
\]

The Eq.\((35)\) does not determine the conformal factor. The derivation of the acoustic metric with the correct conformal factor can be found in Refs.\([10, 11, 12]\).
4.7 Broken Galilean invariance

The modified transformation law for quasiparticles is the consequence of the fact that the mere presence of the gas or liquid with nonzero number $N$ of atoms breaks the Galilean invariance. While for the total system, quantum vacuum + quasiparticles, the Galilean invariance is a true symmetry, it is not applicable to the subsystem of quasiparticles if it is considered independently on the quantum vacuum. This is the general feature of the broken symmetry: the vacuum breaks the Galilean invariance. This means that in the Bose gas and in the superfluid $^4$He, two symmetries are broken: the global $U(1)$ symmetry and the Galilean invariance.

4.8 Momentum vs pseudomomentum

On the other hand, due to the presence of quantum vacuum, there are two different types of translational invariance at $T = 0$ (see detailed discussion in Ref. [12]): (i) Invariance under the translation of the quantum vacuum with respect to the empty space; (ii) Invariance under translation of quasiparticle with respect to the quantum vacuum.

The operation (i) leaves the action invariant provided that the empty space is homogeneous. The conserved quantity, which comes from the translational invariance with respect to the empty space is the momentum. The operation (ii) is the symmetry operation if the quantum vacuum is homogeneous. This symmetry gives rise to the pseudomomentum. Accordingly the bare particles in empty space are characterized by the momentum, while quasiparticles – excitations of the quantum vacuum – are characterized by pseudomomentum. That is why the different transformation properties for momentum of particles in Eq. (29) and quasiparticles in Eq. (30).

The Galilean invariance is the symmetry of the underlying microscopic physics of atoms in empty space. It is broken and fails to work for quasiparticles. Instead, it produces the transformation law in Eq. (30), in which the microscopic quantity – the mass $m$ of bare particles – drops out. This is an example of how the memory on the microscopic physics is erased in the low-energy corner. Furthermore, when the low-energy corner is approached and the effective field theory emerges, these modified transformations gradually become the part of the more general coordinate transformations appropriate for the Einstein theory of gravity.
4.9 Vacuum energy of weakly interacting Bose gas

The vacuum energy of the Bose gas as a function of the chemical potential $\mu$ is

$$\langle H - \mu N \rangle_{\text{vac}} = -\frac{\mu^2}{2U} V + \frac{1}{2} \sum_P \left( E(P) - \frac{P^2}{2m} - mc^2 + \frac{m^3 c^4}{p^2} \right)$$

(39)

The last term in round brackets is added to take into account the perturbative correction to the matrix element $U$. If the total number of particles is fixed, the corresponding vacuum energy is the function of $N$:

$$\langle H \rangle_{\text{vac}} = E_{\text{vac}}(N) = \frac{1}{2} Nmc^2 +$$

(40)

$$\frac{1}{2} \sum_P \left( E(P) - \frac{P^2}{2m} - mc^2 + \frac{m^3 c^4}{p^2} \right)$$

(41)

Inspection of the vacuum energy shows that it does contain the zero point energy of the phonon field, $\frac{1}{2} \sum_P E(P)$. This divergent term is balanced by three counterterms in Eq.(41). They come from the microscopic physics (they explicitly contain the microscopic parameter – the mass $m$ of atom). This regularization, which naturally arises in the microscopic physics, is absolutely unclear within the effective theory. After the regularization, the contribution of the zero point energy of the phonon field in Eq.(41) becomes

$$\frac{1}{2} \sum_{P_{\text{reg}}} E(P) = \frac{1}{2} \sum_P E(P) - \frac{1}{2} \sum_P \left( \frac{P^2}{2m} + mc^2 - \frac{m^3 c^4}{p^2} \right) = \frac{8}{15\pi^2} Nmc^3 \frac{m^3 c^3}{n} ,$$

(42)

where $n = N/V$ is particle density in the vacuum. Thus the total vacuum energy

$$E_{\text{vac}}(N) \equiv \epsilon(n) V =$$

$$\frac{1}{2} Vmc^2 \left( n + \frac{16}{15\pi^2} \frac{m^3 c^3}{h^3} \right) =$$

$$V \left( \frac{1}{2} Un^2 + \frac{8}{15\pi^2 h^3} m^{3/2} U^{5/2} n^{5/2} \right)$$

(43)

(44)

(45)

In the weakly interacting Bose gas the contribution of the phonon zero point motion (the second terms in Eqs.(44) and (45)) is much smaller than the
leading contribution to the vacuum energy, which comes from interaction (the first terms in Eqs. (44) and (45)). The small parameter, which regulates the perturbation theory in the above procedure is \(mca/h \ll 1\) (where \(a\) is the interatomic distance: \(a \sim n^{-1/3}\)), or \(mU/h^2a \ll 1\). Small speed of sound reflects the smallness of the pair interaction \(U\).

### 4.10 Planck energy scales

The microscopic physics also shows that there are two energy parameters, which play the role of the Planck energy scale:

\[
E_{\text{Planck} 1} = mc^2, \quad E_{\text{Planck} 2} = \frac{hc}{a}.
\]

The Planck mass, which corresponds to the first Planck scale \(E_{\text{Planck} 1}\), is the mass of Bose particles \(m\), that comprise the vacuum. The second Planck scale \(E_{\text{Planck} 2}\) reflects the discreteness of the vacuum: the microscopic parameter, which enters this scale, is the mean distance between the particles in the vacuum. The second energy scale corresponds to the Debye temperature in solids. In a given case of weakly interacting particles one has \(E_{\text{Planck} 1} \ll E_{\text{Planck} 2}\), i.e. the distance between the particles in the vacuum is so small, that the quantum effects are stronger than interaction. This is the limit of strong correlations and weak interactions.

Below the first Planck scale \(E \ll E_{\text{Planck} 1} = mc^2\), the energy spectrum of quasiparticles is linear, which corresponds to the relativistic field theory arising in the low-energy corner. At this Planck scale the “Lorentz” symmetry is violated. The first Planck scale \(E_{\text{Planck} 1} = mc^2\) also determines the convergence of the sum in Eq.(41). In terms of this scale the Eq.(41) can be written as

\[
V \frac{8}{15\pi^2} \sqrt{-g} E_{\text{Planck} 1}^4.
\]

where \(g = -1/e^6\) is the determinant of acoustic metric in Eq.(38). This contribution to the vacuum energy has the same structure as the cosmological term in Eq.(31). However, the leading term in the vacuum energy, Eq.(40), is higher and is determined by both Planck scales:

\[
\frac{1}{2} V \sqrt{-g} E_{\text{Planck} 2}^3 E_{\text{Planck} 1}.
\]
4.11 Vacuum pressure and cosmological constant

The relevant vacuum energy of the grand ensemble of particles is the thermodynamic potential at fixed chemical potential: $\langle H - \mu N \rangle_{\text{vac}}$. It is related to the pressure of the liquid as (see the prove of this thermodynamic equation below, Eq.(55))

$$P = -\frac{1}{V} \langle H - \mu N \rangle_{\text{vac}}. \quad (49)$$

Such relation between pressure and energy is similar to that in Eq.(7) for the equation of state of the relativistic quantum vacuum, which is described by the cosmological constant.

This vacuum energy for the weakly interacting Bose gas is given by

$$\langle H - \mu N \rangle_{\text{vac}} = \frac{1}{2} V \sqrt{-g} \left( -E_{\text{Planck}}^2 E_{\text{Planck}}^1 + \frac{16}{15\pi^2} E_{\text{Planck}}^4 \right). \quad (50)$$

Two terms in Eq.(50) represent two contributions to the vacuum pressure in the weakly interacting Bose gas. The zero point energy of the phonon field, the second term in Eq.(50), which coincides with Eq.(42), does lead to the negative vacuum pressure as is expected from the effective theory. However, the magnitude of this negative pressure is smaller than the positive pressure coming from the microscopic “trans-Planckian” degrees of freedom (the first term in Eq.(50) which is provided by the repulsive interaction of atoms). Thus the weakly interacting Bose-gas can exist only under positive external pressure.

5 Quantum liquid

5.1 Real liquid $^4$He

In the real liquid $^4$He the interaction between the particles (atoms) is not small. It is strongly correlated and strongly interacting system, where the two Planck scales are of the same order, $mc^2 \sim \hbar c/a$. This means that the interaction energy and the energy of zero-point motion of atoms are of the same order. This is not the coincidince but reflects the stability og the liquid state. Each of the two energies depend on the particle density $n$. One can find the value of $n$ at which the two contributions to the vacuum pressure compensate each other. This means that the system can be in equilibrium.
even at zero external pressure, \( P = 0 \), i.e. the quantum liquid can exist as a completely autonomous isolated system without any interaction with environment. This is what we must expect from the quantum vacuum in cosmology, since there are no external environment for the vacuum.

In case of the collection of big but finite number \( N \) of \(^4\text{He} \) atoms at \( T = 0 \), they do not fly away as it happens for gases, but are held together to form a droplet of liquid with a finite mean particle density \( n \). This density \( n \) is fixed by the attractive interatomic interaction and repulsive zero point oscillations of atoms, only a part of this zero point motion being described in terms of the zero point energy of phonon mode.

The only macroscopic quantity which characterizes the homogeneous stationary liquid at \( T = 0 \) is the mean particle density \( n \). The vacuum energy density is the function of \( n \)

\[
\epsilon(n) = \frac{1}{V} \langle \mathcal{H} \rangle_{\text{vac}}, \tag{51}
\]

and this function determines the equation of state of the liquid. The relevant vacuum energy density – the density of the thermodynamic potential of grand ensemble

\[
\tilde{\epsilon}(n) = \epsilon(n) - \mu n = \frac{1}{V} \langle \mathcal{H} - \mu N \rangle_{\text{vac}}. \tag{52}
\]

Since the particle number \( N = nV \) is conserved, \( \tilde{\epsilon}(n) \) is the right quantity which must be minimized to obtain the equilibrium state of the liquid at \( T = 0 \) (the equilibrium vacuum). The chemical potential \( \mu \) plays the role of the Lagrange multiplier responsible for the conservation of bare atoms. Thus an equilibrium number of particles \( n_0(\mu) \) is obtained from equation:

\[
\frac{d\tilde{\epsilon}}{dn} = 0, \quad \text{or} \quad \frac{d\epsilon}{dn} = \mu. \tag{53}
\]

Here we discuss only spatially homogeneous ground state, i.e. with spatially homogeneous \( n \), since we know that the ground state of helium at \( T = 0 \) is homogeneous: it is uniform liquid, not the crystal.

From the definition of the pressure,

\[
P = -\frac{d(V\epsilon(N/V))}{dV} = -\epsilon(n) + n\frac{d\epsilon}{dn}, \tag{54}
\]
and from Eq. (53) for the density $n$ in equilibrium vacuum one obtains that in equilibrium the vacuum energy density $\tilde{\epsilon}$ and the vacuum pressure $P$ are related by

$$\tilde{\epsilon}_{\text{vac eq}} = -P_{\text{vac}}.$$  \hspace{1cm} (55)

The thermodynamic relation between the energy and pressure in the ground state of the quantum liquid $P = -\tilde{\epsilon}$, is the same as obtained for vacuum energy and pressure from the Einstein cosmological term. This is because the cosmological term also does not contain derivatives.

Close to the equilibrium state one can expand the vacuum energy in terms of deviations of particle density from its equilibrium value. Since the linear term disappears due to the stability of the superfluid vacuum, one has

$$\tilde{\epsilon}(n) \equiv \epsilon(n) - \mu n = -P_{\text{vac}} + \frac{1}{2n_0}\left(\frac{mc^2}{n_0}\right)^2(n - n_0(\mu))^2.$$ \hspace{1cm} (56)

5.2 Gas-like vs liquid-like vacuum

It is important that the vacuum of real $^4$He is not a gas-like but liquid-like, i.e. it can be in equilibrium at $T = 0$ without interaction with the environment. Such property of the collection of atoms at $T = 0$ is determined by the sign of the chemical potential, if it is counted from the energy of an isolated $^4$He atom. $\mu$ is positive in a weakly interacting Bose gas, but is negative in a real $^4$He where $\mu \sim -7 \text{ K}$ [15].

Due to the negative $\mu$ the isolated atoms are collected together forming the liquid droplet which is self sustained without any interaction with the outside world. If the droplet is big enough, so that the surface tension can be neglected compared to the volume effects, the pressure in the liquid is absent, $P_{\text{vac}} = 0$, and thus the vacuum energy density $\tilde{\epsilon}$ is zero in equilibrium:

$$\tilde{\epsilon}_{\text{vacuum of self--sustaining system}} \equiv 0.$$ \hspace{1cm} (57)

This condition cannot be fulfilled for gas-like states for which $\mu$ is positive and thus they cannot exist without an external pressure.

5.3 Model liquid state

It is instructive to discuss some model energy density $\epsilon(n)$ describing a stable isolated liquid at $T = 0$. Such a model must satisfy the following condition:
(i) $\epsilon(n)$ must be attractive (negative) at small $n$ and repulsive (positive) at large $n$ to provide equilibrium density of liquid at intermediate $n$; (ii) The chemical potential must be negative to prevent evaporation; (iii) The liquid must be locally stable, i.e. the eigen frequencies of collective modes must be real.

All these conditions can be satisfied if we modify the Eq. (43) in the following way. Let us change the sign of the first term describing interaction and leave the second term coming from vacuum fluctuations intact assuming that it is valid even at high density of particles. Due to the attractive interaction at low density the Bose gas collapses forming the liquid state. Of course, this is rather artificial construction, but it qualitatively describes the liquid state. So we come to the following model

$$\epsilon(n) = -\frac{1}{2}\alpha n^2 + \frac{2}{5}\beta n^{5/2},$$

though, in addition to $\alpha$ and $\beta$, one can use also the exponents of $n$ as the fitting parameter. An equilibrium particle density in terms of chemical potential is obtained from the minimization of the relevant vacuum energy $\tilde{\epsilon} = \epsilon - \mu n$ over $n$:

$$\frac{d\tilde{\epsilon}}{dn} = \mu \rightarrow -\alpha n_0 + \beta n_0^{3/2} = \mu$$

The equation of state of such a liquid is

$$P(n_0) = -(\epsilon(n_0) - \mu n_0) = -\frac{1}{2}\alpha n_0^2 + \frac{3}{5}\beta n_0^{5/2}$$

This equation of state allows the existence of the isolated liquid droplet, for which an external pressure is zero, $P = 0$. The equilibrium density, chemical potential and speed of sound in the isolated liquid are

$$n_0(P = 0) = \left(\frac{5\alpha}{6\beta}\right)^2,$$

$$\mu(P = 0) = -\frac{1}{6}n_0\alpha,$$

$$mc^2 = \left(\frac{dP}{dn_0}\right)_{P=0} = \left(n_0\frac{d^2\epsilon}{dn^2}\right)_{P=0} = \frac{7}{8}n_0\alpha = 5.25 |\mu|.$$
This liquid state is stable: the chemical potential $\mu$ is negative preventing evaporation, while $c^2$ is positive, i.e. the compressibility is negative, which indicates the local stability of the liquid.

The Eq.(60) shows that the quantum zero point energy produces a positive contribution to the vacuum pressure, instead of the negative pressure expected from the effective theory and from Eq.(50) for the weakly interacting Bose gas. Let us now recall that in this model we changed the sign of the interaction term, compared to that in the weakly interacting Bose gas. As a result both terms in Eq.(50) have changed sign.

The equilibrium state of the liquid is obtained due to the competition of two effects: attractive interaction of bare atoms (corresponding to the negative vacuum pressure in Eq.(60)) and their zero point motion which leads to repulsion (corresponding to the positive vacuum pressure in Eq.(60)). These effects are balanced in equilibrium, that is why the two “Planck” scales in Eq.(46) become of the same order of magnitude.

### 5.4 Quantum liquid from Theory of Everything

The parameters of liquid $^4$He at $P = 0$ have been calculated in exact microscopic theory, where the many-body wave function of $^4$He atoms has been constructed using the “Theory of Everything” in Eq.(13) with realistic pair potential [15]. For $P = 0$ one has

$$n_0 \sim 2 \cdot 10^{22} \text{ cm}^{-3}, \quad \mu = \frac{\epsilon(n_0)}{n_0} \sim -7 \text{ K}, \quad c \sim 2.5 \cdot 10^4 \text{ cm/sec},$$

$$mc^2 \sim 30 \text{ K}, \quad \hbar c n_0^{1/3} \sim 7 \text{ K},$$

These derived parameters are in a good agreement with their experimental values.

### 6 Vacuum energy and cosmological constant
6.1 Nullification of “cosmological constant” in quantum liquid

If there is no interaction with environment, the external pressure $P$ is zero, and thus in equilibrium the vacuum energy density $\epsilon - \mu n = -P$ in Eqs.(49) and (55) is also zero. The energy density $\tilde{\epsilon}$ is the quantity which is relevant for the effective theory: just this energy density enters the effective action for the soft variables, including the effective gravity field, which must be minimized to obtain the stationary states of the vacuum and matter fields. Thus $\tilde{\epsilon}$ is the proper counterpart of the vacuum energy density, which is responsible for the cosmological term in the Einstein gravity.

Nullification of both the vacuum energy density and the pressure in the quantum liquid means that $P_{\Lambda} = -\rho_{\Lambda} = 0$, i.e. the effective cosmological constant in the liquid is identically zero. Such nullification of the cosmological constant occurs without any fine-tuning or supersymmetry. Note that the supersymmetry – the symmetry between the fermions and bosons – is simply impossible in $^4$He, since there are no fermionic fields in the Bose liquid. The same nullification occurs in Fermi liquids, in superfluid phases of $^3$He, since these are also the quantum liquids with the negative chemical potential $\mu$. Some elements of supersymmetry can be found in the effective theory of superfluid $^3$He [10, 11], but this is certainly not enough to produce the nullification.

Applying this to the quantum vacuum, the mere assumption that the “cosmological liquid” – the vacuum of the quantum field theory – belongs to the class of states, which can exist in equilibrium without external forces, leads to the nullification of the vacuum energy in equilibrium at $T = 0$.

Whether this scenario of nullification of cosmological constant can be applied to the cosmological fluid (the physical vacuum) is a question under discussion (see discussion in Ref. [17], where the inflaton field is considered as the analog of the variable $n$ in quantum liquid).

6.2 Role of zero point energy of bosonic and fermionic fields

The advantage of the quantum liquid is that we know both the effective theory and the fundamental Theory of Everything in Eq. (13). That is why we can compare the two approaches. The microscopic wave function used
for microscopic calculations contains, in principle, all the information on the system, including the quantum fluctuations of the low-energy phonon degrees of freedom, which are considered in the effective theory in Eq. (67). That is why the separate treatment of the contribution to the vacuum energy of the low-energy degrees of freedom described by effective theory has no sense: this leads at best to the double counting.

The effective theory in quantum Bose liquid contains phonons as elementary bosonic quasiparticles and no fermions. That is why the analogue of Eq. (9) for the vacuum energy produced by the zero point motion of “elementary particles” is

$$\rho_\Lambda = \frac{1}{2V} \sum_{\text{phonons}} c p \sim \frac{1}{\epsilon^3} E_{\text{Planck}}^4 = \sqrt{-g} E_{\text{Planck}}^4 .$$

(67)

Here $g$ is the determinant of the acoustic metric in Eq. (38). The “Planck” energy cut-off can be chosen either as the Debye temperature $E_{\text{Debye}} = h c / a = \hbar c n_0^{1/3}$ in Eq. (35) with $a$ being the interatomic distance, which plays the role of the Planck length; or as $m c^2$ which has the same order of magnitude.

The disadvantages of such a naive calculation of the vacuum energy within the effective field theory are: (i) The result depends on the cut-off procedure; (ii) The result depends on the choice of the zero from which the energy is counted: a shift of the zero level leads to a shift in the vacuum energy.

In the microscopic theory these disadvantages are cured: (i) The cut-off is not required; (ii) The relevant energy density, $\tilde{\epsilon} = \epsilon - \mu n$, does not depend on the choice of zero level: the shift of the energy $\int d^3r \epsilon$ is exactly compensated by the shift of the chemical potential $\mu$.

At $P = 0$ the microscopic results for both vacuum energies characterizing the quantum liquid are: $\tilde{\epsilon}(n_0) = 0$, $\epsilon(n_0) = \mu n_0 < 0$. Both energies are in severe disagreement with the naive estimation in Eq. (67) obtained within the effective theory: $\rho_\Lambda$ in Eq. (67) is nonzero in contradiction with $\tilde{\epsilon}(n_0) = 0$; comparing it with $\epsilon(n_0)$ one finds that $\rho_\Lambda$ is about of the same order of magnitude, but it has an opposite sign.

This is an important lesson from the condensed matter. It shows that the use of the zero point fluctuations of bosonic or fermionic modes in Eq. (9) in the cis-Planckian effective theory is absolutely irrelevant for the calculations of the vacuum energy density. Whatever are the low-energy modes, fermionic or bosonic, for equilibrium vacuum they are exactly cancelled by the transn-
Planckian degrees of freedom, which are not accessible within the effective theory.

6.3 Why is equilibrium vacuum not gravitating?

We discussed the condensed matter view to the problem, why the vacuum energy is so small, and found that the answer comes from the “fundamental trans-Planckian physics”. In the effective theory of the low energy degrees of freedom the vacuum energy density of a quantum liquid is of order $E_{\text{Planck}}^4$ with the corresponding “Planck” energy appropriate for this effective theory. However, from the exact “Theory of Everything” of the quantum liquid, i.e. from the microscopic physics, it follows that the “trans-Planckian” degrees of freedom exactly cancel the relevant vacuum energy without fine tuning. The vacuum energy density is exactly zero, if the following conditions are fulfilled: (i) there are no external forces acting on the liquid; (ii) there are no quasiparticles (matter) in the liquid; (iii) no curvature or inhomogeneity in the liquid; and (iv) no boundaries which give rise to the Casimir effect. Each of these factors perturbs the vacuum state and induces a nonzero value of the vacuum energy density of order of the energy density of the perturbation, as we shall discuss below.

Let us, however, mention, that the actual problem for cosmology is not why the vacuum energy is zero (or very small when it is perturbed), but why the vacuum is not (or almost not) gravitating. These two problems are not necessarily related since in the effective theory the equivalence principle is not the fundamental physical law, and thus does not necessarily hold when applied to the vacuum energy. That is why, one cannot exclude the situation, when the vacuum energy is huge, but it is not gravitating. The condensed matter provides an example of such situation too. The weakly interacting Bose gas discussed above is just the proper object. This gas-like substance can exists only at positive external pressure, and thus it has the negative energy density. The translation to the relativistic language gives a huge vacuum energy is on the order of the Planck energy scale (see Eq.(50)). Nevertheless, the effective theory remains the same as for the quantum liquid, and thus even in this situation the equilibrium vacuum, which exists under an external pressure, is not gravitating, and only small deviations from equilibrium state are gravitating. Just this situation was discussed in Ref. [17].

In condensed matter the effective gravity appears as an emergent phe-
nomenon in the low energy corner. The gravitational field is not fundamen-
tal, but is one of the low energy collective modes of the quantum vacuum. 
This dynamical mode provides the effective metric (the acoustic metric in $^4$He and weakly interacting Bose gas) for the low-energy quasiparticles which 
serve as an analogue of matter. This gravity does not exist on the mi-
croscopic (trans-Planckian) level and appears only in the low energy limit 
together with the “relativistic” quasiparticles and the acoustics itself. The 
bare atoms, which live in the “trans-Planckian” world and form the vacuum 
state there, do not experience the “gravitational” attraction experienced by 
the low-energy quasiparticles, since the effective gravity simply does not exist 
at the microscopnic scale (we neglect here the real gravitational attraction of 
the atoms, which is extremely small in quantum liquids). That is why the 
vacuum energy cannot serve as a source of the effective gravity field: the 
pure completely equilibrium homogeneous vacuum is not gravitating.

On the other hand, the long-wave-length perturbations of the vacuum are 
within the sphere of influence of the low-energy effective theory; such pertur-
bations can be the source of the effective gravitational field. Deviations of 
the vacuum from its equilibrium state, induced by different sources discussed 
below, are gravitating.

6.4 Why is the vacuum energy unaffected by the phase 
transition?

It is commonly believed that the vacuum of the Universe underwent one or 
several broken symmetry phase transitions. Since each of the transitions is 
ampanied by a substantial change in the vacuum energy, it is not clear 
why the vacuum energy is (almost) zero after the last phase transition. In 
other words, why has the true vacuum the zero energy, while the energies of 
al other false vacua are enormously big?

What happens in quantum liquids? According to the conventional wis-
dom, the phase transition, say, to the broken symmetry vacuum state, is 
ampanied by the change of the vacuum energy, which must decrease in 
a phase transition. This is what usually follows from the Ginzburg-Landau 
description of phase transitions. However, let us compare the energy densi-
ties of the false and the true vacuum states. Let us assume that the phase 
transition is of the first order, and the false vacuum is separated from the
true vacuum by a large energy barrier, so that it can exist as a (meta)stable state. Since the false vacuum is stable, the Eq. (57) can also be applied to the false vacuum, and one obtains the paradoxical result: in the absence of external forces the energy density of the false vacuum must be the same as the energy density of the true vacuum, i.e. the relevant energy density $\tilde{\varepsilon}$ must be zero for both vacua. Thus the first order phase transition occurs without the change in the vacuum energy.

To add more confusion, note that the Eq. (57) can be applied even to the unstable vacuum which corresponds to a saddle point of the energy functional, if such a vacuum state can live long enough. Thus the vacuum energy density does not change in the second order phase transition either.

There is no paradox, however: after the phase transition to a new state has occurred, the chemical potential $\mu$ will be automatically adjusted to preserve the zero external pressure and thus the zero energy $\tilde{\varepsilon}$ of the vacuum. Thus the relevant vacuum energy is zero before and after transition, which means that the $T = 0$ phase transitions do not disturb the zero value of the cosmological constant. Thus the scenario of the nullification of the vacuum energy suggested by the quantum liquids survives even if the phase transition occurs in the vacuum. The first order phase transition between superfluid phases $^3\text{He}$-A and $^3\text{He}$-B at $T = 0$ and $P = 0$ gives the proper example [3].

6.5 Why is the cosmological constant nonzero?

We now come to another problem in cosmology: Why is the vacuum energy density presently of the same order of magnitude as the energy density of matter $\rho_M$, as is indicated by recent astronomical observations [7]. While the relation between $\rho_M$ and $\rho_\Lambda$ seems to depend on the details of trans-Planckian physics, the order of magnitude estimation can be readily obtained. In equilibrium and without matter the vacuum energy is zero. However, the perturbations of the vacuum caused by matter and/or by the inhomogeneity of the metric tensor lead to disbalance. As a result the deviations of the vacuum energy from zero must be on the of order of the perturbations.

Let us consider how this happens in quantum liquids for different types of perturbations, i.e. how the vacuum energy, which is zero at $T = 0$ and in complete equilibrium in the absence of external forces, is influenced by different factors, which lead to small but nonzero value of the cosmological constant.
6.6 Vacuum energy from finite temperature

A typical example derived from quantum liquids is the vacuum energy produced by temperature. Let us consider for example the superfluid $^4$He in equilibrium at finite $T$ without external forces. If $T \ll -\mu$ one can neglect the exponentially small evaporation and consider the liquid as in equilibrium. The quasiparticles – phonons – play the role of the hot relativistic matter, and their equation of state is $P_M = (1/3)\rho_M = (\pi^2/30\hbar^3c^3)T^4$, with $c$ being the speed of sound $[\mathbb{I}]$. In equilibrium the pressure caused by thermal quasiparticles must be compensated by the negative vacuum pressure, $P_\Lambda = -P_M$, to support the zero value of the external pressure, $P = P_\Lambda + P_M = 0$. In this case one has the following nonzero values of the vacuum pressure and vacuum energy density:

$$\rho_\Lambda = -P_\Lambda = P_M = \frac{1}{3}\rho_M = \sqrt{-g}\frac{\pi^2}{30\hbar^3c^3}T^4,$$

(68)

where $g = -c^{-6}$ is again the determinant of acoustic metric. In this example the vacuum energy density $\rho_\Lambda$ is positive and always on the order of the energy density of matter. This indicates that the cosmological constant is not actually a constant but is adjusted to the energy density of matter and/or to the other perturbations of the vacuum discussed below.

6.7 Vacuum energy from Casimir effect

Another example of the induced nonzero vacuum energy density is provided by the boundaries of the system. Let us consider a finite droplet of $^4$He with radius $R$. If this droplet is freely suspended then at $T = 0$ the vacuum pressure $P_\Lambda$ must compensate the pressure caused by the surface tension due to the curvature of the surface. For a spherical droplet one obtains the negative vacuum energy density:

$$\rho_\Lambda = -P_\Lambda = -\frac{2\sigma}{R} \sim -\frac{E_{\text{Debye}}^3}{\hbar^2c^2R} \equiv -\sqrt{-g}E_{\text{Planck}}^3\frac{\hbar c}{R},$$

(69)

where $\sigma$ is the surface tension. This is an analogue of the Casimir effect, in which the boundaries of the system produce a nonzero vacuum pressure. The strong cubic dependence of the vacuum pressure on the “Planck” energy $E_{\text{Planck}} \equiv E_{\text{Debye}}$ reflects the trans-Planckian origin of the surface tension.
\[ \sigma \sim \frac{E_{\text{Debye}}}{a^2} \sim \frac{\hbar c}{a^3} \]: it is the energy (per unit area) related to the distortion of atoms in the surface layer of the size of the interatomic distance \( a \).

Such term of order \( E_{\text{Planck}}^3/R \) in the Casimir energy has been considered in Ref. [18]. In Ref. [19] such vacuum energy, with \( R \) being the size of the horizon, has been connected to the energy of the Higgs condensate in the electroweak phase transition.

This form of the Casimir energy – the surface energy \( 4\pi R^2 \sigma \) normalized to the volume of the droplet – can also serve as an analogue of the quintessence in cosmology [20]. Its equation of state is

\[ P_\sigma = -(\frac{2}{3})\rho_\sigma \]

The equilibrium condition within the droplet can be written as \( P = P_\Lambda + P_\sigma = 0 \). In this case the quintessence is related to the wall – the boundary of the droplet. In cosmology the quintessence with the same equation of state, \( \langle P_\sigma \rangle = -(2/3) \langle \rho_\sigma \rangle \), is represented by a wall wrapped around the Universe or by a tangled network of cosmic domain walls [21]. The surface tension of the cosmic walls can be much smaller than the Planck scale.

### 6.8 Vacuum energy induced by texture

The nonzero vacuum energy density, with a weaker dependence on \( E_{\text{Planck}} \), is induced by the inhomogeneity of the vacuum. Let us discuss the vacuum energy density induced by texture in a quantum liquid. We consider here the twist soliton in \( ^3\text{He}-\text{A} \), since such texture is related to the Riemann curvature in general relativity [3]. Within the soliton the field of the \( ^3\text{He}-\text{A} \) order parameter – the unit vector \( \hat{l} \) – has a form \( \hat{l}(z) = \hat{x} \cos \phi(z) + \hat{y} \sin \phi(z) \).

The energy of the system in the presence of the soliton consists of the vacuum energy \( \rho_\lambda(\phi) \) and the gradient energy:

\[ \rho = \rho_\lambda(\phi) + \rho_{\text{grad}} \], \[ \rho_\lambda(\phi) = \rho_\lambda(\phi = 0) + \frac{K}{\xi_D^2} \sin^2 \phi \], \[ \rho_{\text{grad}} = K(\partial_z \phi)^2 \].

where \( \xi_D \) is the so-called dipole length [22]. Here we denoted the energy \( \tilde{\epsilon} \) by \( \rho \) to make the connection with general relativity, and omitted \( \sqrt{-g} \) assuming that \( c = 1 \).
The solitonic solution of the sine-Gordon equation, \(\tan(\phi/2) = e^{z/\xi_D}\), gives the following spatial dependence of vacuum and gradient energies:

\[
\rho_\Lambda(z) - \rho_\Lambda(\phi = 0) = \rho_{\text{grad}}(z) = \frac{K}{\xi_D^2 \cosh^2(z/\xi_D)}.
\]  

Let us consider for simplicity the 1+1 case. Then the equilibrium state of the whole quantum liquid with the texture can be discussed in terms of partial pressure of the vacuum, \(P_\Lambda = -\rho_\Lambda\), and that of the inhomogeneity, \(P_{\text{grad}} = \rho_{\text{grad}}\). The latter equation of state describes the so called stiff matter in cosmology. In equilibrium the external pressure is zero and thus the positive pressure of the texture (stiff matter) must be compensated by the negative pressure of the vacuum:

\[
P = P_\Lambda(z) + P_{\text{grad}}(z) = 0.
\]

This equilibrium condition produces another relation between the vacuum and the gradient energy densities

\[
\rho_\Lambda(z) = -P_\Lambda(z) = P_{\text{grad}}(z) = \rho_{\text{grad}}(z).
\]

Comparing this Eq. (74) with Eq. (72) one finds that in equilibrium

\[
\rho_\Lambda(\phi = 0) = 0,
\]

i.e., as before, the main vacuum energy density – the energy density of the bulk liquid far from the soliton – is exactly zero if the isolated liquid is in equilibrium. Within the soliton the vacuum is perturbed, and the vacuum energy is induced being on the order of the energy of the perturbation. In this case \(\rho_\Lambda(z)\) is equal to the gradient energy density of the texture.

The induced vacuum energy density in Eq. (72) is inversely proportional to the square of the size of the region where the field is concentrated:

\[
\rho_\Lambda(R) \sim \sqrt{-g} E_{\text{Planck}}^2 \left(\frac{hc}{R}\right)^2.
\]

In case of the soliton soliton \(R \sim \xi_D\). Similar behavior for the vacuum energy density in the interior region of the Schwarzschild black hole, with \(R\) being the Schwarzschild radius, was discussed in Ref. [23].
In cosmology, the vacuum energy density obeying the Eq. (76) with \( R \) proportional to the Robertson-Walker scale factor has been suggested in Ref. [24], and with \( R \) being the size of the horizon, \( R = R_H \), in Ref. [19]. Following the reasoning of Ref. [19], one can state that the vacuum energy density related to the phase transition is determined by Eq. (76) with \( R = R_H(t) \) at the cosmological time \( t \) when this transition (or crossover) occurred. Applying this to, say, the cosmological electroweak transition, where the energy density of the Higgs condensate is of order of \( T_{ew}^4 \), one obtains the relation \( T_{ew}^2 = E_{\text{Planck}} hc/R_H(t = t_{ew}) \). It also follows that the entropy within the horizon volume at any given cosmological temperature \( T \) is \( S_H \sim E_{\text{Planck}}^3/T^3 \) for the radiation-dominated Universe.

### 6.9 Vacuum energy due to Riemann curvature

The vacuum energy \( \sim R^{-2} \), with \( R \) proportional to the Robertson-Walker scale factor, comes also from the Riemann curvature in general relativity. It appears that the gradient energy of a twisted \( \hat{l} \)-texture is equivalent to the Einstein curvature term in the action for the effective gravitational field in \(^3\text{He-A} \): 

\[
- \frac{1}{16\pi G} \int d^3r \sqrt{-g} \mathcal{R} \equiv K \int d^3r ((\hat{l} \cdot (\nabla \times \hat{l}))^2.
\]

(77)

Here \( \mathcal{R} \) is the Riemann curvature calculated using the effective metric experienced by fermionic quasiparticles in \(^3\text{He-A} \):

\[
ds^2 = -dt^2 + c_\perp^{-2} (\hat{l} \times dr)^2 + c_\parallel^{-2} (\hat{l} \cdot dr)^2 .
\]

(78)

The order parameter vector \( \hat{l} \) plays the role of the Kasner axis; \( c_\parallel \) and \( c_\perp \) correspond to the speed of “light” propagating along the direction of \( \hat{l} \) and in transverse direction; \( c_\parallel \gg c_\perp \).

The analogy between the textural (gradient) energy in \(^3\text{He-A} \) and the curvature in general relativity allows us to interprete the result of the previous section, Eq. (74), in terms of the vacuum energy induced by the curvature of the space. It appears that in cosmology this effect can be described within the general relativity. We must consider the stationary cosmological model, since the time dependent vacuum energy is certainly beyond the Einstein theory. The stationary Universe was obtained by Einstein in his work where he first
introduced the cosmological term $\mathcal{F}$. It is the closed Universe with positive curvature and with matter, where the effect of the curvature is compensated by the cosmological term, which is adjusted in such a way, that the Universe remains static. This is just the correct and probably unique example, of how the vacuum energy is induced by curvature and matter within the general relativity.

Let us recall this solution. In the static state of the Universe two equilibrium conditions must be fulfilled:

$$\rho = \rho_M + \rho_\Lambda + \rho_\mathcal{R} = 0, \quad P = P_M + P_\Lambda + P_\mathcal{R} = 0.$$  \tag{79}$$

The first equation in (79) reflects the gravitational equilibrium, which requires that the total mass density must be zero: $\rho = \rho_M + \rho_\Lambda + \rho_\mathcal{R} = 0$ (actually the "gravineutrality" corresponds to the combination of two equations in (79), $\rho + 3P = 0$, since $\rho + 3P$ serves as a source of the gravitational field in the Newtonian limit). This gravineutrality is analogous to the electroneutrality in condensed matter. The second equation in (79) is equivalent to the requirement that for the "isolated" Universe the external pressure must be zero: $P = P_M + P_\Lambda + P_\mathcal{R} = 0$. In addition to matter density $\rho_M$ and vacuum energy density $\rho_\Lambda$, the energy density $\rho_\mathcal{R}$ stored in the spatial curvature is added:

$$\rho_\mathcal{R} = -\frac{\mathcal{R}}{16\pi G} = -\frac{3k}{8\pi GR^2}, \quad P_\mathcal{R} = -\frac{1}{3}\rho_\mathcal{R}. \tag{80}$$

Here $R$ is the cosmic scale factor in the Friedmann–Robertson–Walker metric

$$ds^2 = -dt^2 + R^2\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{81}$$

the parameter $k = (-1, 0, +1)$ for an open, flat, or closed Universe respectively; and we again removed the factor $\sqrt{-g}$ from the definition of the energy densities.

For the cold Universe with $P_M = 0$, the Eqs. (79) give

$$\rho_\Lambda = \frac{1}{2}\rho_M = -\frac{1}{3}\rho_\mathcal{R} = \frac{k}{8\pi GR^2}, \tag{82}$$

and for the hot Universe with the equation of state $P_M = (1/3)\rho_M$,

$$\rho_\Lambda = \rho_M = -\frac{1}{2}\rho_\mathcal{R} = \frac{3k}{16\pi GR^2}. \tag{83}$$
Since the energy of matter is positive, the static Universe is possible only for positive curvature, $k = +1$, i.e. for the closed Universe.

This is the unique solution, which describes an equilibrium static state of the Universe, where the vacuum energy is induced by matter and curvature. And this solution is obtained within the effective theory of general relativity without invoking the trans-Planckian physics and thus does not depend on details of the trans-Planckian physics.

6.10 Necessity of Planck physics for time-dependent cosmology

The condensed matter analog of gravity provides a natural explanation, why the cosmological constant is zero with a great accuracy, when compared with the result based on naive estimation of the vacuum energy within the effective theory. It also shows how the small effective cosmological constant of the relative order of $10^{-120}$ naturally arises as the response to different perturbations. We considered the time-independent perturbations, where the minimum energy consideration and equilibrium condition provided the solution of the problem.

For the time-dependent situation, such as an expansion of the Universe, the calculation of the vacuum response is not as simple even in quantum liquids. One must solve self-consistently the coupled dynamical equations for the motion of the vacuum and matter fields. In case of general relativity this requires the equation of motion for the vacuum energy $\rho_\Lambda$, but this is certainly beyond the effective theory since the time dependence of $\rho_\Lambda$ violates Bianchi identities. Probably some extension of general relativity towards the scalar-tensor theory of gravity such as discussed in Ref. [25] will be more relevant for that.

On the other hand the connection to the Planck physics can help to solve the other cosmological problems. For example there is the flatness problem: To arrive at the Universe we see today, the Universe must have begun extremely flat, which means that parameter $k$ in the Robertson-Walker metric must be zero. In quantum liquids the general Robertson-Walker metric in Eq.(81) describes the spatially homogeneous space-time as viewed by the low-energy quasiparticles within the effective theory. However, for the external or high-energy observer the quantum liquid is not homogeneous if $k \neq 0$. The
same probably happens in gravity: If general relativity is the effective theory, the invariance under the coordinate transformations exists only at low energy. For the “Planck” observer the Robertson-Walker metric in Eq. (81) is viewed as space dependent if $k \neq 0$. That is why the condition, that the Universe must be spatially homogeneous not only on the level of the effective theory but also on the fundamental level, requires that $k = 0$. Thus, if general relativity is the effective theory, the truly homogeneous Universe must be flat.

7 Effects of discrete number $N$ of particles in the vacuum

7.1 Casimir effect in quantum liquids

Till now we used the conservation law for the particle number $N$, the number of bare atoms in the quantum vacuum, to derive the nullification of the vacuum energy in the grand ensemble of particles. Now we consider another possible consequence of the discrete nature of the quantum vacuum in quantum liquids. This is related to the Casimir effect.

The attractive force between two parallel metallic plates in vacuum induced by the quantum fluctuations of the electromagnetic field has been predicted by Casimir in 1948 [26]. The calculation of the vacuum pressure is based on the regularization schemes, which allows to separate the effect of the low-energy modes of the vacuum from the huge diverging contribution of the high-energy degrees of the freedom. There are different regularization schemes: Riemann’s zeta-function regularization; introduction of the exponential cutoff; dimensional regularization, etc. People are happy when different regularization schemes give the same results. But this is not always so (see e.g. [27, 18, 28], and in particular the divergencies occurring for spherical geometry in odd spatial dimension are not cancelled [29, 30]). This raises some criticism against the regularization methods [31] or even some doubts concerning the existence and the magnitude of the Casimir effect.

The same type of the Casimir effect arises in condensed matter, due to thermal (see review paper [32]) or and quantum fluctuations. When considering the analog of the Casimir effect in condensed matter, the following correspondence must be taken into account, as we discussed above. The
ground state of quantum liquid corresponds to the vacuum of quantum field theory. The low-energy bosonic and fermionic excitations above the vacuum – quasiparticles – correspond to elementary particles forming the matter. The low energy modes with linear spectrum \( E = cp \) can be described by the relativistic-type effective theory. The analog of the Planck energy scale \( E_{\text{Planck}} \) is determined either by the mass \( m \) of the atom of the liquid, \( E_{\text{Planck}} \equiv mc^2 \), or by the Debye energy, \( E_{\text{Planck}} \equiv \hbar c/a \) (see Eq.(46)).

The traditional Casimir effects deals with the low energy massless modes. The typical massless modes in quantum liquid are sound waves. The acoustic field is described by the effective theory in Eq.(14) and corresponds to the massless scalar field. The walls provide the boundary conditions for the sound wave modes, usually these are the Neumann boundary conditions. Because of the quantum hydrodynamic fluctuations there must be the Casimir force between two parallel plates immersed in the quantum liquid. Within the effective theory the Casimir force is given by the same equation as the Casimir force acting between the conducting walls due to quantum electromagnetic fluctuations. The only modifications are: (i) the speed of light must be substituted by the spin of sound \( c \); (ii) the factor \( 1/2 \) must be added, since we have the scalar field of the longitudinal sound wave instead of two polarizations of light. If \( d \) is the distance between the plates and \( A \) is their area, then the \( d \)-dependent contribution to the ground state energy of the quantum liquid at \( T = 0 \) which follows from the effective theory must be

\[
E_C = -\frac{\hbar c \pi^2 A}{1440d^3}
\]

(84)

Such microscopic quantities of the quantum liquid as the mass of the atom \( m \) and interatomic space \( a \) do not enter explicitly the Eq.(84): the traditional Casimir force is completely determined by the “fundamental” parameter \( c \) of the effective scalar field theory.

### 7.2 Finite-size vs finite-\( N \) effect

However, we shall show that the Eq.(84) is not always true. We shall give here an example, where the effective theory is not able to predict the Casimir force, since the microscopic high-energy degrees of freedom become important. In other words the “transPlanckian physics” shows up and the “Planck” energy
scale explicitly enters the result. In this situation the Planck scale is physical and cannot be removed by any regularization.

The Eq.(84) gives a finite-size contribution to the energy of quantum liquid. It is inversely proportional to the linear dimension of the system, $E_C \propto 1/R$ for the sphere of radius $R$. However, for us it is important that it is not only the finite-size effect, but also the finite-$N$ effect, $E_C \propto N^{-1/3}$, where $N$ is the number of atoms in the liquid in the slab. As distinct from $R$ the quantity $N$ is a discrete quantity. Since the main contribution to the vacuum energy is $\propto R^3 \propto N$, the relative correction of order $N^{-4/3}$ means that the Casimir force is the mesoscopic effect. We shall show that in quantum liquids, the essentially larger mesoscopic effects, of the relative order $N^{-1}$, can be more pronounced. This is a finite-$N$ effect, which reflects the discreteness of the vacuum and cannot be described by the effective theory dealing with the continuous medium, even if the theory includes the real boundary conditions with the frequency dependence of dielectric permeability.

We shall start with the simplest quantum vacuum – the ideal one-dimensional Fermi gas – where the mesoscopic Casimir forces can be calculated exactly without invoking any regularization procedure.

### 7.3 Vacuum energy from microscopic theory

We consider the system of $N$ bare particles, each of them being one-dimensional massless fermions, whose continuous energy spectrum is $E(p) = cp$, with $c$ playing the role of speed of light. We assume that these fermions are either ‘spinless’ (this means means that they all have the same direction of spin and thus the spin degrees of freedom can be neglected) or the 1+1 Dirac fermions. If the fermions are not interacting the microscopic theory is extremely simple: in vacuum state fermions simply occupy all the energy levels below the chemical potential $\mu$. In the continuous limit, the total number of particles $N$ and the total energy of the system in the one-dimensional “cavity” of size $d$ are expressed in terms of the Fermi momentum $p_F = \mu/c$ in the following way

$$N = nd \quad n = \int_{-p_F}^{p_F} \frac{dp}{2\pi\hbar} = \frac{p_F}{\pi\hbar}, \quad (85)$$

$$E = \epsilon(n)d \quad \epsilon(n) = \int_{-p_F}^{p_F} \frac{dp}{2\pi\hbar} cp = \frac{cp_F^2}{2\pi\hbar} = \frac{\pi}{2} hcn^2. \quad (86)$$
Here $\epsilon(n)$ is the vacuum energy density as a function of the particle density. The relation between the particle density and chemical potential $\mu = \pi \hbar c n = p_F c$ also follows from minimization of the relevant vacuum energy: $d(\epsilon(n) - \mu n)/dn = 0$. In the vacuum state the relevant vacuum energy density and the pressure of the Fermi gas are

$$\tilde{\epsilon} = \epsilon(n) - \mu n = -\frac{\pi}{2} \hbar c n^2, \quad P = -\tilde{\epsilon} = \frac{\pi}{2} \hbar c n^2.$$  \hspace{1cm} (87)

Fermi gas can exist only at positive external pressure provided by the walls.

7.4 Vacuum energy in effective theory

As distinct from the microscopic theory, which deals with bare particles, the effective theory deals with the quasiparticles – fermions living at the level of the chemical potential $\mu = c p_F$. There are 4 different quasiparticles:

(i) quasiparticles and quasiholes living in the vicinity of the Fermi point at $p = +p_F$ have spectrum $E_{qp}(p_+) = |E(p) - \mu| = c|p_+|$, where $p = p_z - p_F$;

(ii) quasiparticles and quasiholes living in the vicinity of the other Fermi point at $p = -p_F$ have the spectrum $E_{qp}(p_-) = |E(p) - \mu| = c|p_-|$, where $p_- = p + p_F$.

In the effective theory the energy of the system is the energy of the Dirac vacuum

$$E = -\sum_{p_+} c|p_+| - \sum_{p_-} c|p_-|. \hspace{1cm} (88)$$

This energy is divergent and requires the cut-off. With the proper cut-off provided by the Fermi-momentum, $p_{\text{Planck}} \sim p_F$, the negative vacuum energy density $\epsilon(n)$ in Eq.(87) can be reproduced. This is a rather rare situation when the effective theory gives the correct sign of the vacuum energy.

7.5 Vacuum energy as a function of discrete $N$

Now let us discuss the Casimir effect – the change of the vacuum pressure caused by the finite size effects in the vacuum. We must take into account the discreteness of the spectrum of bare particles or quasiparticles (depending on which theory we use, microscopic or effective) in the slab. Let us start with the microscopic description in terms of bare particles (atoms). We can
use two different boundary conditions for particles, which give two kinds of discrete spectrum:

\[ E_k = k \frac{hc\pi}{d}, \quad (89) \]
\[ E_k = \left( k + \frac{1}{2} \right) \frac{hc\pi}{d}. \quad (90) \]

Eq. (89) corresponds to the spinless fermions with Dirichlet boundary conditions at the walls, while Eq. (90) describes the energy levels of the 1+1 Dirac fermions with no particle current through the wall; the latter case with the generalization to the \( d + 1 \) fermions has been discussed in [33].

The vacuum is again represented by the ground state of the collection of the \( N \) noninteracting particles. We know the structure of the completely and thus the vacuum energy in the slab is well defined: it is the energy of \( N \) fermions in 1D box of size \( d \)

\[ E(N, d) = \sum_{k=1}^{N} E_k = \frac{hc\pi}{2d} N(N + 1), \quad \text{for} \quad E_k = k \frac{hc\pi}{d}, \quad (91) \]
\[ E(N, d) = \sum_{k=0}^{N-1} E_k = \frac{hc\pi}{2d} N^2, \quad \text{for} \quad E_k = \left( k + \frac{1}{2} \right) \frac{hc\pi}{d}. \quad (92) \]

### 7.6 Leakage of vacuum through the wall.

To calculate the Casimir force acting on the wall, we must introduce the vacuum on both sides of the wall. Thus let us consider three walls: at \( z = 0 \), \( z = d_1 < d \) and \( z = d \). Then we have two slabs with sizes \( d_1 \) and \( d_2 = d - d_1 \), and we can find the force acting on the wall separating the two slabs, i.e. on the wall at \( z = d_1 \). We assume the same boundary conditions at all the walls. But we must allow the exchange the particles between the slabs, otherwise the main force acting on the wall between the slabs will be determined simply by the difference in bulk pressure in the two slabs. This can be done due to, say, a very small holes (tunnel junctions) in the wall, which do not violate the boundary conditions and thus do not disturb the particle energy levels, but still allow the particle exchange between the two vacua.

This situation can be compared with the traditional Casimir effect. The force between the conducting plates arises because the electromagnetic fluctuations of the vacuum in the slab are modified due to boundary conditions.
imposed on electric and magnetic fields. In reality these boundary conditions are applicable only in the low-frequency limit, while the wall is transparent for the high-frequency electromagnetic modes, as well as for the other degrees of freedom of real vacuum (fermionic and bosonic), that can easily penetrate through the conducting wall. In the traditional approach it is assumed that those degrees of freedom, which produce the divergent terms in the vacuum energy, must be cancelled by the proper regularization scheme. That is why, though the dispersion of dielectric permeability does weaken the real Casimir force, nevertheless in the limit of large distances, \( d_1 \gg c/\omega_0 \), where \( \omega_0 \) is the characteristic frequency at which the dispersion becomes important, the Casimir force does not depend on how easily the high-energy vacuum leaks through the conducting wall.

We consider here just the opposite limit, when (almost) all the bare particles are totally reflected. This corresponds to the case when the penetration of the high-energy modes of the vacuum through the conducting wall is highly suppressed, and thus one must certainly have the traditional Casimir force. Nevertheless, we shall show that due to the mesoscopic finite-\( N \) effects the contribution of the diverging terms to the Casimir effect becomes dominating. They produce highly oscillating vacuum pressure in quantum liquids. The amplitude of the mesoscopic fluctuations of the vacuum pressure in this limit exceeds by factor \( p_{\text{Planck}} d/\hbar \) the value of the conventional Casimir pressure. For their description the continuous effective low-energy theories are not applicable.

### 7.7 Mesoscopic Casimir force in 1d Fermi gas

The total vacuum energy in two slabs for spinless and Dirac fermions is correspondingly

\[
E(N, d_1, d_2) = \frac{\hbar c \pi}{2} \left( \frac{N_1(N_1 + 1)}{d_1} + \frac{N_2(N_2 + 1)}{d_2} \right),
\]

\[
E(N, d_1, d_2) = \frac{\hbar c \pi}{2} \left( \frac{N_1^2}{d_1} + \frac{N_2^2}{d_2} \right),
\]

where \( N_1 \) and \( N_2 \) are the particle numbers in each of the two slabs:

\[
N_1 + N_2 = N, \quad d_1 + d_2 = d
\]
Since particles can transfer between the slabs, the global vacuum state in this geometry is obtained by minimization over the discrete particle number \( N_1 \) at fixed total number \( N \) of particles. If the mesoscopic \( 1/N \) corrections are ignored, one obtains \( N_1 \approx (d_1/d)N \) and \( N_2 \approx (d_2/d)N \); the two vacua have the same pressure, and thus there is no force acting on the wall between the two vacua.

However, \( N_1 \) and \( N_2 \) are integer valued, and this leads to mesoscopic fluctuations of the Casimir force. The global vacuum with given values of \( N_1 \) and \( N_2 \) is realized only within a certain range of parameter \( d_1 \). If \( d_1 \) increases, it reaches some threshold value above which the energy of the vacuum with the particle numbers \( N_1 + 1 \) and \( N_2 - 1 \) has lower energy and it becomes the global vacuum. The same happens if \( d_1 \) decreases and reaches some threshold value below which the vacuum with the particle numbers \( N_1 - 1 \) and \( N_2 + 1 \) becomes the global vacuum. The force acting on the wall in the state \((N_1, N_2)\) is obtained by variation of \( E(N_1, N_2, d_1, d - d_1) \) over \( d_1 \) at fixed \( N_1 \) and \( N_2 \):

\[
F(N_1, N_2, d_1, d_2) = -\frac{dE(N_1, N_2, d_1, d_2)}{dd_1} + \frac{dE(N_1, N_2, d_1, d_2)}{dd_2}. \tag{96}
\]

When \( d_1 \) increases reaches the threshold, where \( E(N_1, N_2, d_1, d_2) = E(N_1 + 1, N_2 - 1, d_1, d_2) \), one particle must cross the wall from the right to the left. At this critical value the force acting on the wall changes abruptly (we do not discuss here an interesting physics arising just at the critical values of \( a_1 \), where the degeneracy occurs between the states \((N_1, N_2)\) and \((N_1 + 1, N_2 - 1)\); at these positions of the wall (or membrane) the particle numbers \( N_1 \) and \( N_2 \) are undetermined and are actually fractional due to the quantum tunneling between the slabs \cite{34}). Using for example the spectrum in Eq.(94) one obtains for the jump of the Casimir force:

\[
F(N_1 \pm 1, N_2 \mp 1) - F(N_1, N_2) = \hbar c \pi \left( \frac{\pm 2N_1 + 1}{2d_1^2} + \frac{\pm 2N_2 - 1}{2d_2^2} \right) \approx \pm \frac{\hbar c \pi N}{d_1 d_2}. \tag{97}
\]

The same result for the amplitude of the mesoscopic fluctuations is obtained if one uses the spectrum in Eq.(93).

In the limit \( d_1 \ll d \) the amplitude of the mesoscopic Casimir force

\[
|\Delta F_{\text{meso}}| = \frac{\hbar c \pi n}{d_1} = \frac{\hbar c \pi n^2}{N_1} \equiv \frac{E_{\text{Planck}}}{d_1}. \tag{98}
\]
It is by factor $1/N_1 = (\pi \hbar / d_1 p_F)^3 \equiv (\pi \hbar / d_1 p_{Planck})^3$ smaller than the vacuum energy density in Eq.(98). On the other hand it is by the factor $p_F d_1 \equiv p_{Planck} d_1$ larger than the traditional Casimir pressure, which in one-dimensional case is $P_C \sim \hbar c / d_1^2$. The divergent term which linearly depends on the Planck momentum cutoff $p_{Planck}$ as in Eq.(98) has been revealed in many different calculations (see e.g. [30]), and attempts have made to invent the regularization scheme which would cancel the divergent contribution.

### 7.8 Mesoscopic Casimir pressure in quantum liquids

The equation (98) for the amplitude of the mesoscopic fluctuations of the vacuum pressure can be immediately generalized for the $d$-dimensional space: if $V_1$ is the volume of the internal region separated by almost impenetrable walls from the outside vacuum, then the amplitude of the mesoscopic vacuum pressure must be of order

$$|P_{meso}| \sim \frac{E_{Planck}}{V_1}.$$  \hspace{1cm} (99)

The mesoscopic random pressure comes from the discrete nature of the underlying quantum liquid, which represents the quantum vacuum. The integer value of the number of atoms in the liquid leads to the mesoscopic fluctuations of the pressure: when the volume $V_1$ of the vessel changes continuously, the equilibrium number $N_1$ of particles changes in step-wise manner. This results in abrupt changes of pressure at some critical values of the volume:

$$P_{meso} \sim P(N_1 \pm 1) - P(N_1) = \pm \frac{dP}{dN_1} = \pm \frac{mc^2}{V_1} \equiv \pm \frac{E_{Planck}}{V_1},$$  \hspace{1cm} (100)

where again $c$ is the speed of sound, which plays the role of the speed of light. The mesoscopic pressure is determined by microscopic “transPlanckian” physics, and thus such microscopic quantity as the mass $m$ of the atom, the “Planck mass”, enters this force.

For the spherical shell of radius $R$ immersed in the quantum liquid the mesoscopic pressure is

$$P_{meso} \sim \pm \frac{mc^2}{R^3} \equiv \pm \sqrt{-\frac{g}{E_{Planck}}} \left(\frac{\hbar c}{R} \right)^3.$$  \hspace{1cm} (101)
7.9 Mesoscopic vacuum pressure vs conventional Casimir effect.

Let us compare the mesoscopic vacuum pressure in Eq. (101) with the traditional Casimir pressure obtained within the effective theory for the same spherical shell geometry. In the effective theory (such as electromagnetic theory in case of the original Casimir effect, and the low-frequency quantum hydrodynamics in quantum liquids) the Casimir pressure comes from the bosonic and fermionic low-energy modes of the system (electromagnetic modes in the original Casimir effect or quanta of sound waves in quantum liquids). In superfluids, in addition to phonons the other low-energy sound-like collective are possible, such as spin waves. These collective modes with linear ("relativistic") spectrum in quantum liquids play the role of the relativistic massless scalar field. They obey typically the Neumann boundary conditions, corresponding to the (almost) vanishing mass or spin current through the wall (almost, because the vacua inside and outside the shell must be connected).

If we believe in the traditional regularization schemes which cancel out the divergent terms, the effective theory gives the Casimir pressure for the spherical shell is

\[ P_C = -\frac{dE_C}{dV} = \frac{K}{8\pi} \sqrt{-g} \left( \frac{\hbar c}{R} \right)^4, \tag{102} \]

where \( K = -0.4439 \) for the Neumann boundary conditions; \( K = 0.005639 \) for the Dirichlet boundary conditions [30]; and \( c \) is the speed of sound or of spin waves. The traditional Casimir pressure is completely determined by the effective low-energy theory, it does not depend on the microscopic structure of the liquid: only the "speed of light" \( c \) enters this force. The same pressure will be obtained in case of the pair correlated fermionic superfluids, if the fermionic quasiparticles are gapped and their contribution to the Casimir pressure is exponentially small compared to the contribution of the collective massless bosonic modes.

However, at least in our case, the result obtained within the effective theory is not correct: the real Casimir pressure is given by Eq. (101): (i) It essentially depends on the Planck cut-off parameter, i.e. it cannot be determined by the effective theory; (ii) it is much bigger, by factor \( p_{\text{Planck}} R/\hbar \), than the traditional Casimir pressure in Eq. (102); and (iii) it is highly oscil-
lating. The regularization of these oscillations by, say, averaging over many measurements; by noise; or due to quantum or thermal fluctuations of the shell; etc., depend on the concrete physical conditions of the experiment.

This shows that in some cases the Casimir vacuum pressure is not within the responsibility of the effective theory, and the microscopic (trans-Planckian) physics must be evoked. If two systems have the same low-energy behavior and are described by the same effective theory, this does not mean that they necessarily experience the same Casimir effect. The result depends on many factors, such as the discrete nature of the quantum vacuum, and the ability of the vacuum to penetrate through the boundaries. It is not excluded that even the traditional Casimir effect which comes from the vacuum fluctuations of the electromagnetic field is renormalized by the high-energy degrees of freedom.

Of course, the extreme limit, which we consider, is not applicable to the original (electromagnetic) Casimir effect, since the situation in the electromagnetic Casimir effect is just opposite. The overwhelming part of the fermionic and bosonic vacuum easily penetrates the conducting wall, and thus the mesoscopic fluctuations are small. But do they negligibly small? In any case this example shows that the cut-off problem is not the mathematical, but the physical one, and the Planck physics dictates the proper regularization scheme or the proper choice of the cut-off parameters.

8 Conclusion.

We discussed the problems related to the properties of quantum vacuum in general relativity using the known properties of the quantum vacuum in quantum liquids, where some elements of the Einstein gravity arise in the low-energy corner. We found that in both systems there are similar problems, which arise if the effective theory is exploited. In both systems the naive estimation of the vacuum energy density within the effective theory gives \( \rho_\Lambda \sim E_{\text{Planck}}^4 \) with the corresponding “Planck” energy appropriate for each of the two systems. However, as distinct from the general relativity, in quantum liquids the fundamental physics, “The Theory of Everything”, is known, and it shows that the “trans-Planckian” degrees of freedom exactly cancel this divergent contribution to the vacuum energy. The relevant vacuum energy is zero without fine tuning, if the vacuum is stable (or metastable), isolated
and homogeneous.

Quantum liquids also demonstrate how the small vacuum energy is generated, if the vacuum is disturbed. In particular, thermal quasiparticles – which represent the matter in general relativity – induce the vacuum energy of the order of the energy of the matter. This example shows the possible answer to the question, why the present cosmological constant is of the order of the present matter density in our Universe. It follows that in each epoch the vacuum energy density must be of order of either the matter density of the Universe, or of its curvature, or of the energy density of the smooth component – the quintessence. However, the complete understanding of the dynamics of the vacuum energy in the time-dependent regime of the expanding Universe cannot be achieved within the general relativity and requires the extension of this effective theory.

In principle, one can construct the artificial quantum liquid, in which all the elements of the general relativity are reproduced in the low energy corner. The effective metric $g^{\mu\nu}$ acting on “relativistic” quasiparticles arises as one of the low-energy collective variables of the quantum vacuum, while the Sakharov mechanism leads to the Einstein curvature and cosmological terms in the action for this dynamical variable. In this liquid the low energy phenomena will obey the Einstein equations (8), with probably one exception: the dynamics of the cosmological “constant” will be included. It would be extremely interesting to realize this programme, and thus to find out the possible extension of general relativity, which takes into account the properties of the quantum vacuum.

The most important property of the quantum vacuum in quantum liquids is that this vacuum consists of discrete elements – bare atoms. The interaction and zero point oscillations of these elements lead to the formation of the equilibrium vacuum, and in this equilibrium vacuum state the cosmological constant is identically zero. Thus the discreteness of the quantum vacuum can be the possible source of the (almost complete) nullification of the cosmological constant in our present Universe. If so, one can try to exploit the other possible consequences of the discrete nature of the quantum vacuum, such as the mesoscopic Casimir effect discussed in Sec. 7.

Analogy with the quantum vacuum in quantum liquids allows us to discuss the other problems related to the quantum vacuum in general relativity: the flatness problem; the problem of a big entropy in the present Universe; the horizon problem, etc.
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