WEAK SOLUTIONS OF THE STOCHASTIC LANDAU-LIFSCHITZ-GILBERT EQUATIONS WITH NON-ZERO ANISOTROPY ENERGY

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ABSTRACT. We study a Stochastic Landau-Lifschitz Equation with non-zero anisotropy energy and multidimensional noise. The existence and some regularities of weak solution have been proved.

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1. Introduction

The ferromagnetism theory was first studied by Weiß in 1907 and then further developed by Landau and Lifshitz [32] and Gilbert [24]. According to their theory there is a characteristic of the material called the Curie’s temperature, whence below this critical temperature, large ferromagnetic bodies would break up into small uniformly magnetized regions separated by thin transition layers. The small uniformly magnetized regions are called Weiß domains and the transition layers are called Bloch walls. This fact is taken into account by imposing the following constraint:

\[ |u(t, x)|_{\mathbb{R}^3} = |u_0|_{\mathbb{R}^3}. \]

Moreover the magnetization in a domain \( D \subset \mathbb{R}^3 \) at time \( t > 0 \) given by \( u(t, x) \in \mathbb{R}^3 \) satisfies the following Landau-Lifschitz constraint:

\[ \frac{du(t, x)}{dt} = \lambda_1 u(t, x) \times \rho(t, x) - \lambda_2 u(t, x) \times (u(t, x) \times \rho(t, x)). \]

The \( \rho \) in the equation (1.2) is called the effective magnetic field and defined by

\[ \rho = -\nabla \mu \mathcal{E}, \]

where the \( \mathcal{E} \) is the so called total electro-magnetic energy which composed by anisotropy energy, exchange energy and electronic energy.

In order to describe phase transitions between different equilibrium states induced by thermal fluctuations of the effective magnetic field \( \rho \), Brzeźniak and Goldys and Jegaraj [13] introduced the Gaussian noise into the Landau-Lifschitz-Gilbert (LLG) equation to perturb \( \rho \) and then the stochastic Landau-Lifschitz-Gilbert (SLLG) equation have the following form:

\[ du(t) = [\lambda_1 u(t) \times \rho(t) - \lambda_2 u(t) \times (u(t) \times \rho(t))] dt + (u(t) \times h) \circ dW(t). \]

Their total energy is with only the exchange energy \( \int \frac{1}{2} |\nabla u|_{L^2(D)} \) taken into account, and hence their equation has the following form:

\[ \begin{cases} 
\frac{du(t)}{dt} = (\lambda_1 u(t) \times \Delta u(t) - \lambda_2 u(t) \times (u(t) \times \Delta u(t))) dt + (u(t) \times h) \circ dW(t), \\
\frac{\partial u}{\partial n}(t, x) = 0, \quad t > 0, x \in \partial D, \\
u(0, x) = u_0(x), \quad x \in D. \end{cases} \]

They concluded the existence of the weak solution of (1.5) and also proved some regularities of the solution.

There is also some research about the numerical schemes of equation (1.5), such as Bañas, Brzeźniak, and Prohl [7], Bañas, Brzeźniak, Neklyudov, and Prohl [8], Bañas, Brzeźniak, Neklyudov, and Prohl [9], and Goldys, Le, and Tran [25].

In this paper we consider the SLLG equation with the total energy \( \mathcal{E} \) defined as:

\[ \mathcal{E} = \mathcal{E}_{an} + \mathcal{E}_{ex} = \int_D \left( \phi(u(x)) + \frac{1}{2} |\nabla u(x)|^2 \right) dx, \]

where \( \mathcal{E}_{an} := \int_D \phi(u(x)) dx \) stands for the anisotropy energy and \( \mathcal{E}_{ex} := \frac{1}{2} \int_D |\nabla u(x)|^2 dx \) stands for the exchange energy.
So the SLLG equation we are going to study in this paper has the form:

\[
\begin{align*}
\frac{du(t)}{dt} &= \left\{ \lambda_1 u(t) \times [\Delta u(t) - \nabla \phi(u(t))] \\
&\quad - \lambda_2 u(t) \times (u(t) \times [\Delta u(t) - \nabla \phi(u(t))]) \right\} dt \\
&\quad + \sum_{j=1}^{N} \{u(t) \times h_j\} \circ dW_j(t)
\end{align*}
\]

(1.6)

\[
\left. \frac{\partial u}{\partial n} \right|_{\Gamma} = 0, \quad u(0) = u_0
\]

We follow the same method as used in Brzeźniak and Goldys and Jegaraj’s paper [13] to proved the existence of the weak solution of (1.6) and get some similar regularities of the weak solution (but not uniqueness).

In particular, our results give an alternative proof of the existence result from Brzeźniak, Goldys and Jegaraj’s paper [12], where large deviations principle for stochastic LLG equation on a 1-dimensional domain has been studied.

This paper is organized as follows. In Section 2 we introduce the notations and formulate the main result on the existence of the weak solution of the Equation (1.6) as well as some regularities. In Section 3 we introduce the Galerkin approximation and prove the existence of the global solutions \(\{u_n\}\) of the approximate equation of (1.6), which are in \(n\) dimensional spaces, where \(n \in \mathbb{N}\). In Section 4 we prove the global solutions of the approximate equations in finite dimensional spaces satisfy some a’priori estimates. In Section 5, we use the a’priori estimates to show the laws of the \(\{u_n\}\) are tight on a suitable space. In Section 6, we use the tightness results and the Skorohod’s Theorem to construct a new probability space and some processes \(\{u_n^\prime\}\) which have the same laws as \(\{u_n\}\). By the Skorohod’s Theorem, we also get a limit process \(u^\prime\) of \(\{u_n^\prime\}\). And we show some properties that \(u^\prime\) satisfies. In Section 7, we use two steps to show that \(u^\prime\) constructed before is a weak solution of the Equation (1.6). In Section 8, we prove some regularities of \(u^\prime\) and so finish the proof of the main Theorem which stated in Section 2.

Let us finish the introduction by remarking that all our results are formulated for \(D \subset \mathbb{R}^d, d = 3\), but they are also valid for \(d = 1\) or \(d = 2\).

Remark. This paper is from a part of the Ph.D. thesis at the University of York in UK of the second named author.

2. Notations and the formulation of the main result

Notation 2.1. For \(O = D\) or \(O = \mathbb{R}^3\), let us denote

\[
\begin{align*}
\mathbb{L}^p(O) &= L^p(O; \mathbb{R}^3), \\
L^p(O) &:= L^p(O; \mathbb{R}). \\
W^{k,p}(O) &= W^{k,p}(O; \mathbb{R}^3), \\
W^{k,p}(O) &:= W^{k,p}(O; \mathbb{R}). \\
H^k(O) &= H^k(O; \mathbb{R}^3), \\
H^k(O) &:= H^k(O; \mathbb{R}). \\
H &:= \mathbb{L}^2(D), \\
V &:= W^{1,2}(D).
\end{align*}
\]
Assumption 2.2. Let $D$ be an open and bounded domain in $\mathbb{R}^3$ with $C^2$ boundary $\Gamma := \partial D$. $n$ is the outward normal vector on $\Gamma$. $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$, $h_j \in L^\infty(D) \cap L^{1,3}(D)$, for $j = 1, \ldots, N$, $u_0 \in V$. $\phi : \mathbb{R}^3 \to \mathbb{R}^+ \cup \{0\}$ is in $C^4$ and $\phi$, $\phi'$, $\phi''$ and $\phi(3)$ are bounded. $\phi'$ is also globally Lipschitz. Moreover, we also assume that we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and this probability space satisfies the so-called usual conditions:

(i) $\mathbb{P}$ is complete on $(\Omega, \mathcal{F})$,
(ii) for each $t \geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F}, \mathbb{P})$-null sets,
(iii) the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous.

We also assume that $(W(t))_{t \geq 0} = ((W_j)_{j=1}(t))_{t \geq 0}$ is an $\mathbb{R}^N$-valued, $(\mathcal{F}_t)_{t \geq 0}$-adapted Wiener process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

The equation we are going to study in this paper is:

$$
\begin{aligned}
\begin{align*}
\frac{du(t)}{dt} &= \left[ \lambda_1 u(t) \times [\Delta u(t) - \nabla \phi(u(t))] - \lambda_2 u(t) \times [u(t) \times [\Delta u(t) - \nabla \phi(u(t))]] \right] \, dt \\
&\quad + \sum_{j=1}^{N} \left[ u(t) \times h_j \right] \circ dW_j(t)
\end{align*}
\end{aligned}
$$

(2.1)

$$
\frac{\partial u}{\partial n} |_{\Gamma} = 0
$$

Remark 2.3. Since $\phi : \mathbb{R}^3 \to \mathbb{R}$, for every $x \in \mathbb{R}^3$ the Fréchet derivative $d_x \phi = \phi'(x) : \mathbb{R}^3 \to \mathbb{R}$ is linear, and hence by the Riesz Lemma, there exists a vector $\nabla \phi(x) \in \mathbb{R}^3$ such that

$$
\langle \nabla \phi(x), y \rangle = d_x \phi(y), \quad y \in \mathbb{R}^3.
$$

Definition 2.4 (Solution of (2.1)). A weak solution of (2.1) is system consisting of a filtered probability space $(\Omega', \mathcal{F}', \mathbb{P}', \mathbb{P}')$, an $N$-dimensional $\mathbb{F}'$-Wiener process $W' = (W'_j)_{j=1}^{N}$ and an $\mathbb{F}'$-progressively measurable process

$$
\begin{aligned}
\begin{align*}
u' = (u'_j)_{j=1}^{3} : \Omega' \times [0, T] \to V \cap L^\infty(D)
\end{align*}
\end{aligned}
$$

such that for all $\psi \in C^0_{0}(D; \mathbb{R}^3)$, $t \in [0, T]$, we have, $\mathbb{P}'$-a.s.,

$$
\begin{aligned}
\langle u'(t), \psi \rangle_{H} &= \langle u_0, \psi \rangle_{H} - \lambda_1 \int_{0}^{t} \langle \nabla u'(s), \nabla \psi \times u'(s) \rangle_{L^2(D; \mathbb{R}^{3 \times 3})} \, ds \\
&\quad + \lambda_1 \int_{0}^{t} \langle u'(s) \times \nabla \phi(u'(s)), \psi \rangle_{H} \, ds \\
&\quad - \lambda_2 \int_{0}^{t} \langle \nabla u'(s), \nabla (u'(s) \times \psi)(s) \times u'(s) \rangle_{L^2(D; \mathbb{R}^{3 \times 3})} \, ds \\
&\quad + \lambda_2 \int_{0}^{t} \langle u'(s) \times (u'(s) \times \nabla \phi(u'(s)), \psi \rangle_{H} \, ds \\
&\quad + \sum_{j=1}^{N} \int_{0}^{t} \langle u'(s) \times h_j, \psi \rangle_{H} \, dW'_j(s)
\end{aligned}
$$

(2.2)
Next we will formulate the main result of this paper:

**Theorem 2.5.** Under the assumptions listed in Assumption 2.2, i.e., a system consisting of a filtered probability space \((\Omega', \mathcal{F}', P')\), and \(N\)-dimensional \(\mathbb{F}'\)-Wiener process \(W' = (W'_j)_{j=1}^N\).

(i) There exists a weak solution of (2.1).

(ii) \[\mathbb{E} \int_0^T \|u'(t) \times (\Delta u'(t) - \phi'(u'(t)))\|_H^2 \, dt < \infty.\]

(iii) For every \(t \in [0, T]\), in \(L^2(\Omega'; H)\),

\[
u'(t) = u_0 + \lambda_1 \int_0^t (u' \times [\Delta u' - \phi'(u')])(s) \, ds - \lambda_2 \int_0^t u'(s) \times (u' \times [\Delta u' - \phi'(u')])(s) \, ds + \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \circ dW'_j(s);
\]

(iv) \(|u'(t, x)|_{\mathbb{R}^3} = 1, \text{ for Lebesgue a.e. } (t, x) \in [0, T] \times D \text{ and } \mathbb{P}' - a.s..\]

(v) For every \(\alpha \in (0, \frac{1}{2})\),

\[u' \in C^\alpha([0, T]; H), \quad \mathbb{P}' - a.s..\]

**Remark 2.6.** The notation \(u' \times \Delta u'\) used in Theorem 2.5 will be defined in the Notation 6.11.

The notation \(u' \times (u' \times \Delta u')\) used in Theorem 2.5 will be defined in the Notation 6.12.

**Remark 2.7.** Our results are for the Laplace operator with Neumann boundary conditions. Without any difficult work one could prove the same result for the Laplace operator on a compact manifold without boundary. In particular, for Laplace operator with periodic boundary condition.

### 3. Galerkin approximation

Let us define \(A := -\Delta\) as the \(-\Delta\)-Laplace operator in \(D\) acting on \(\mathbb{R}^3\)-valued functions with Neumann boundary condition:

\[D(A) = \left\{ u \in H^{2,2}(D; \mathbb{R}^3) : \frac{\partial u}{\partial n} \bigg|_{\partial D} = 0 \right\} \subset H.\]

\(A\) is self-adjoint, so by ([20], p.335, Thm 1), there exists an orthonormal basis (which are eigenvectors of \(A\)) \(\{e_k\}_{k=1}^\infty\) of \(H\), such that \(e_k \in C^\infty(D)\) for all \(k = 1, 2, \ldots\). We set \(H_n = \text{span}\{e_1, e_2, \ldots, e_n\}\) and let \(\pi_n\) denote the orthogonal projection from \(H\) to \(H_n\). We also note that \(V = D(A^{\frac{1}{2}})\) and define \(A_1 := I + A\),
then \(\|u\|_V = \|A^{\frac{1}{2}}u\|_H\) for \(u \in V\).

We also have the following definition and properties relate to the operator \(A\), which will be frequently used later:

**Definition 3.1** (Fractional power spaces of \(A_1 = I + A\)). For any nonnegative real number \(\beta\) we define the Hilbert space \(X^\beta := D(A_1^{\beta})\), which is the domain of the fractional power operator \(A_1^{\beta}\). And the dual of \(X^\beta\) is denoted by \(X^{-\beta}\). See [13].

We have the following property about the relations of \(X^\gamma\) and \(H^{2\gamma}\).

**Proposition 3.2.** With \(A_1 = I + A\) as above we have, see [48, 4.3.3],

\[
X^\gamma = D(A_1^{\gamma}) = \left\{ u \in H^{2\gamma} : \frac{\partial u}{\partial n}\big|_{\partial D} = 0 \right\}, \quad 2\gamma > \frac{3}{2},
\]

\[
2\gamma < \frac{3}{2}.
\]

**Proposition 3.3.** Let \(D\) be a bounded open domain in \(\mathbb{R}^3\) with \(C^2\) boundary, \(u \in H^2(D; \mathbb{R}^3)\), \(v \in H^1(D; \mathbb{R}^3)\), and \(\frac{\partial u}{\partial n}\big|_{\partial D} = 0\) then we have

\[
\langle Au, v \rangle_H = \int_D \langle \nabla u(x), \nabla v(x) \rangle_{\mathbb{R}^{3\times 3}} \, dx.
\]

**Proposition 3.4.** If \(v \in V\) and \(u \in D(A)\), then

\[
(3.1) \quad \int_D \langle u(x) \times Au(x), Au(x) \rangle \, dx = 0.
\]

\[
(3.2) \quad \int_D (u(x) \times (u(x) \times Au(x)), Au(x)) \, dx = -\int_D |u(x) \times Au(x)|^2 \, dx.
\]

\[
(3.3) \quad \int_D \langle u(x) \times Au(x), v(x) \rangle \, dx = \sum_{i=1}^3 \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial v}{\partial x_i}(x) \times u(x) \right\rangle \, dx.
\]

\[
(3.4) \quad \int_D \langle u(x) \times (u(x) \times Au(x)), v(x) \rangle \, dx = \sum_{i=1}^3 \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (v \times u)}{\partial x_i}(x) \times u(x) \right\rangle \, dx.
\]

**Proof of (3.3) and (3.4).** The equality (3.3) follows from Brzeźniak and Goldys and Jegaraj’s paper [13]. And since \(\langle u \times (u \times Au), v \rangle = \langle u \times Au, v \times u \rangle\) and if \(u \in D(A), v \in V\), then \(v \times u \in V\), (3.4) follows from (3.3).

We consider the following equation in \(H_{n} (H_{n} \subset D(A))\) with all the assumptions in Assumption 2.2:

\[
(3.5) \quad \begin{cases}
\frac{du_{n}(t)}{dt} = -\pi_{n} \left(A_{1}u_{n}(t) \times \left[Au_{n}(t) + \pi_{n}(\nabla \phi(u_{n}(t)))\right]\right) - \pi_{2} \left(u_{n}(t) \times \left[Au_{n}(t) + \pi_{n}(\nabla \phi(u_{n}(t)))\right]\right) \, dt \\
+ \sum_{j=1}^{N} \pi_{n} \left[u_{n}(t) \times h_{j}\right] \circ dW_{j}(t), \quad t \geq 0, \\
u_{n}(0) = \pi_{n}u_{0}.
\end{cases}
\]

Let us point out that (3.5) is a suitable projection of (2.1) onto the space \(H_{n}\). In what follows, in order to simplify notation, instead of \(\nabla \phi(u)\) we will write, somehow incorrectly, \(\phi'(u)\).
Let us define the following maps:

\[ (3.6) \quad F_1^i : H_n \ni u \mapsto -\pi_n(u \times Au) \in H_n, \]
\[ (3.7) \quad F_2^i : H_n \ni u \mapsto -\pi_n(u \times (u \times Au)) \in H_n, \]
\[ (3.8) \quad F_3^i : H_n \ni u \mapsto -\pi_n[u \times \pi_n(\phi'(u))] \in H_n, \]
\[ (3.9) \quad F_4^i : H_n \ni u \mapsto -\pi_n[u \times [u \times \pi_n(\phi'(u))]] \in H_n, \]
\[ (3.10) \quad G_{jn} : H_n \ni u \mapsto \pi_n(u \times h_j) \in H_n, \quad h_j \in L^{\infty}(D) \cap W^{1,3}(D), \quad j = 1, \ldots, N. \]

Since \( A \) restrict to \( H_n \) is linear and bounded (with values in \( H_n \)) and since \( H_n \subset D(A) \subset \mathbb{L}^{\infty}(D) \), we infer that \( G_{jn} \) and \( F_1^i, F_2^i, F_3^i, F_4^i \) are well defined maps from \( H_n \) to \( H_n \).

The problem (3.5) can be written in a more compact way

\[
\begin{align*}
\left\{ \begin{array}{l}
du_n(t) = \lambda_1 [F_1^i(u_n(t)) + F_2^i(u_n(t))] + d\lambda_2 [F_3^i(u_n(t)) + F_4^i(u_n(t))]
dt + \frac{1}{2} \sum_{j=1}^N G_{jn}^j(u_n(t)) \ dt + \sum_{j=1}^N G_{jn}^j(u_n(t)) \ dW_j(t),
\end{array} \right.
\end{align*}
\]
\[ u_n(0) = \pi_n u_0. \]

**Remark 3.5.** In the Equations (2.1) and (3.5), we use the Stratonovich differential and in the Equation (3.11) we use the Itô differential, the following equality relates the two differentials: for the map \( G : H \ni u \mapsto u \times h \in H, \)

\[ (Gu) \circ dW(t) = \frac{1}{2} [G'(u)(Gu)] dt + Gu \ dW(t), \quad u \in H. \]

**Remark 3.6.** As the equality (1.3), we have

\[ -\nabla_{H_n} E(u_n) = Au_n + \pi_n \phi'(u_n), \]

so with the “\( \pi_n \)”s in the equation (3.5), our approximation keeps as much as possible the structure of the equation (2.1), and consequently we will get the a’ priori estimates.

Now we start to solve the Equation (3.11).

**Lemma 3.7.** The maps \( F_i^i, i = 1, 2, 3, 4 \) are Lipschitz on balls, that is, for every \( R > 0 \) there exists a constant \( C = C(n, R) > 0 \) such that whenever \( x, y \in H_n \) and \( \|x\|_{H} \leq R, \|y\|_{H} \leq R \), we have

\[ \|F_i^i(x) - F_i^i(y)\|_H \leq C \|x - y\|_H. \]

The map \( G_{jn} \) is linear and

\[ \|G_{jn} u\|_{H} \leq \|u\|_{H} \|h_j\|_{L^\infty}, \quad u \in H_n. \]

**Proof.** Let us notice that the maps

\[ H_n \ni u \mapsto Au \in H_n \quad \text{and} \quad H_n \ni u \mapsto \pi_n(\phi'(u)) \in H_n \]

are locally bounded and globally Lipschitz. And if the map \( \psi : H_n \longrightarrow H_n \) is locally bounded and locally Lipschitz, then the map

\[ H_n \ni u \mapsto u \times \psi(u) \in H \]
is also locally bounded and locally Lipschitz. Hence the maps $F_n^i, i = 1, 2, 3, 4$ are locally Lipschitz. The result about $G_{jn}$ is obvious. This completes the proof of Lemma 3.7. □

Since the linear operator $\pi_n : H_n \rightarrow H_n$ is self-adjoint and by the formula $(a \times b, b)_{\mathbb{R}^3} = 0$, we infer that

**Lemma 3.8.**

$$G_{jn}^* = -G_{jn}$$

Moreover for $i = 1, 2, 3, 4$ and $u \in H_n$, we have

$$\langle F_n^i(u), u \rangle_H = 0.$$  

**Corollary 3.9.** [3] The Equation (3.5) has a unique global solution $u_n : [0, T] \rightarrow H_n$.

**Proof.** By the Lemma 3.7 and Lemma 3.8, the coefficients $F_n^i, i = 1, 2, 3, 4$ and $G_{jn}$ are locally Lipschitz and one side linear growth. Hence by a result in [3], the Equation (3.5) has a unique global solution $u_n : [0, T] \rightarrow H_n$. □

Let us define functions $F_n$ and $\hat{F}_n : H_n \rightarrow H_n$ by

$$F_n = \lambda_1(F_n^1 + F_n^3) - \lambda_2(F_n^2 + F_n^4),$$  

and

$$\hat{F}_n = F_n + \frac{1}{2} \sum_{j=1}^{N} G_{jn}^2.$$  

Then the problem (3.5) (or (3.11)) can be written in the following compact way

$$\begin{equation}
(3.13) \quad du_n(t) = \hat{F}_n(u_n(t))\, dt + \sum_{j=1}^{N} G_{jn}(u_n(t))\, dW_j(t).
\end{equation}$$

4. A’priori estimates

In this section we will get some properties of the solution of Equation (3.5) especially some a’priori estimates.

**Theorem 4.1.** Assume that $n \in \mathbb{N}$. Let $u_n$ be the solution of the Equation (3.5) which was constructed earlier. Then for every $t \in [0, T]$,

$$\begin{equation}
(4.1) \quad \|u_n(t)\|_H = \|u_n(0)\|_H, \quad a.s.
\end{equation}$$  

**Proof.** Let us consider a function $\psi : H_n \ni u \mapsto \frac{1}{2}\|u\|_H^2 \in \mathbb{R}$. Since $\psi$ is a homogeneous polynomial of degree 2, $\psi$ is of $C^\infty$. Moreover we have

$$\psi'(u)(g) = \langle u, g \rangle_H, \quad \text{and} \quad \psi''(u)(g, k) = \langle k, g \rangle_H.$$
By the Itô Lemma and Lemma 3.8, we have
\[
\frac{1}{2} \, \left\| u_n(t) \right\|^2_H = \left\langle u_n(t), \hat{F}(u_n(t)) \right\rangle_H + \frac{1}{2} \, \sum_{j=1}^N \left\langle G_{jn}(u_n(t)), G_{jn}(u_n(t)) \right\rangle_H \, dt \\
+ \sum_{j=1}^N \left\langle u_n(t), G_{jn}(u_n(t)) \right\rangle_H \, dW_j(t) \\
= \frac{1}{2} \, \sum_{j=1}^N \left\langle u_n(t), G_{jn}^2(u_n(t)) \right\rangle_H + \sum_{j=1}^N \left\| G_{jn}(u_n(t)) \right\|^2_H \, dt + 0 \, dW_j(t) \\
= 0
\]
Hence for \( t \in [0, T] \),
\[
\left\| u_n(t) \right\|_H = \left\| u_n(0) \right\|_H, \quad a.s.
\]
\[ \square \]

**Lemma 4.2.** Let us define a function \( \Phi : H_n \to \mathbb{R} \) by
\[
\Phi(u) := \frac{1}{2} \int_D \| \nabla u(x) \|^2 \, dx + \int_D \phi(u(x)) \, dx, \quad u \in H_n.
\]
Then \( \Phi \in C^2(H_n) \) and for \( u, g, k \in H_n \),
\[
d_u \Phi(g) = \Phi'(u)(g) = \langle \nabla u, \nabla g \rangle_{L^2(D, \mathbb{R}^3)} + \int_D \langle \nabla \phi(u(x)), g(x) \rangle \, dx
\]
\[
= \langle Au, g \rangle_{L^2(D, \mathbb{R}^3)} + \int_D \langle \nabla \phi(u(x)), g(x) \rangle \, dx,
\]
\[
\Phi''(u)(g, k) = \langle \nabla g, \nabla k \rangle_{L^2(D, \mathbb{R}^3)} + \int_D \phi''(u(x))(g(x), k(x)) \, dx.
\]

**Proof.** Let us introduce auxiliary functions \( \Phi_0 \) and \( \Phi_1 \) by:
\[
\Phi_0(u) := \int_D \phi(u(x)) \, dx, \quad u \in H_n.
\]
\[
\Phi_1(u) := \frac{1}{2} \| \nabla u \|^2_{L^2(D, \mathbb{R}^3)}, \quad u \in H_n.
\]
It is enough to prove the results of \( \Phi_0 \) and \( \Phi_1 \).

The result about \( \Phi_0 \) is obvious and the result of \( \Phi_1 \) follows from the mean value theorem in integral form, see [11], for related result. \[ \square \]

**Proposition 4.3.** There exist constants \( a, b, a_1, b_1 > 0 \) such that for all \( n \in \mathbb{N} \),
\[
\| \nabla G_{jn} u \|^2_{L^2} \leq a \| \nabla u \|^2_{L^2} + b, \quad u \in H_n,
\]
and
\[
\| \nabla^2 G_{jn} u \|^2_{L^2} \leq a_1 \| \nabla u \|^2_{L^2} + b_1, \quad u \in H_n.
\]
Proof of (4.5). Since $A_1$ is self-adjoint and $A_1 \geq A$, we have
\[
\|\nabla G_{j_0} u\|_{L^2}^2 = (AG_{j_0}(u), G_{j_0}(u))_H \leq (A_1 G_{j_0}(u), G_{j_0}(u))_H
\]
\[
= \|A_1 \pi_n (u \times h_j)\|_H^2 = \|\pi_n A_1^T (u \times h_j)\|_H^2 \leq \|A_1^T (u \times h_j)\|_H^2
\]
\[
= \|(u \times h_j)\|_V^2 \leq N \left( \|u \times h_j\|_{L^2}^2 + \|\nabla (u \times h_j)\|_{L^2}^2 \right)
\]
\[
\leq \left[ \|h_j\|_{L^{\infty}(D)}^2 \left( \|u\|_{H^1}^2 + 2\|\nabla u\|_{L^2}^2 \right) + 2\|\nabla h_j\|_{L^{2}(\Omega)}^2 \|u\|_{L^2(\Omega)}^2 \right].
\]
Next since $L^6(D) \hookrightarrow V$ and by equality (4.1) $\|u_0(s)\|_H \leq \|u_0\|_H$, we infer that
\[
\|\nabla G_{j_0} u\|_{L^2}^2 \leq d \|\nabla u\|_{L^2}^2 + b,
\]
for some constants $a$ and $b$ which only depend on $\|h_j\|_{L^{\infty}(D)}$, $\|\nabla h_j\|_{L^{2}(\Omega)}$ and $\|u_0\|_H$, but not on $n$. \(\square\)

Proof of (4.6). The estimate (4.6) followed from double application of (4.5). \(\square\)

Remark 4.4. The previous results will be used to prove the following fundamental a’priori estimates on the sequence $\{u_n\}$ of the solution of Equation (3.5).

Theorem 4.5. Assume that $p \geq 1$, $\beta > \frac{1}{4}$. Then there exists a constant $C > 0$, such that for all $n \in \mathbb{N},$
\[
\mathbb{E} \sup_{r \in [0, t]} \left( \|\nabla u_n(r)\|_{H^1}^p + \int_D \phi(u_n(r, x)) \, dx \right)^{\frac{p}{2}} \leq C, \quad t \in [0, T],
\]
\[
\mathbb{E} \left[ \left( \int_0^T \left[ \|u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t)))]\|_{H^1}^2 \, dt \right)^{\frac{p}{2}} \right] \leq C,
\]
\[
\mathbb{E} \left[ \left( \int_0^T \|u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t)))]\|_{L^2}^2 \, dt \right)^{\frac{p}{2}} \right] \leq C,
\]
\[
\mathbb{E} \int_0^T \|\pi_n (u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]))\|_{X^{1/2}}^2 \, dt \leq C.
\]

Proof of (4.7) and (4.8). Let us define a function $\Phi$ same as in the Equation (4.2). Then by the It\'o Lemma,
\[
\Phi(u_n(t)) - \Phi(u_n(0))
\]
\[
= \int_0^t \left( \Phi'(u_n(s)) \tilde{F}_n(u_n(s)) + \frac{1}{2} \sum_{j=1}^N \Phi''(u_n(s)) G_{j\theta}(u_n(s)) \right) \, ds
\]
\[
+ \sum_{j=1}^N \int_0^t \Phi'(u_n(s)) G_{j\theta}(u_n(s)) \, dW_j(s), \quad t \in [0, T].
\]
Then we consider each term on the RHS of the Equation (4.11), and we can prove that

\[ \Phi'(u) \hat{F}_n(u) = -\lambda_2 \| u \times (\Delta u - \pi_n(\phi'(u))) \|^2_H \]

(4.12)

\[ -\frac{1}{2} \sum_{j=1}^N \langle \Delta u - \pi_n(\phi'(u)), \pi_n(u \times h_j) \times h_j \rangle_H \]

and

(4.13)

\[ \Phi'(u)[G_{jn}(u)] = -(\Delta u, u \times h_j) + \langle \phi'(u), \pi_n(u \times h_j) \rangle, \]

and

(4.14)

\[ \Phi''(u)[G_{jn}(u)^2] \]

\[ = \| \nabla \pi_n(u \times h_j) \|^2_{L^2} + \int_D \phi''(u(x)) \| \pi_n(u \times h_j)(x), \pi_n(u \times h_j)(x) \|^2 \, dx. \]

Therefore by Equations (4.2), (4.12), (4.13) and (4.14), the Equation (4.11) transforms to:

(4.15) \[
\frac{1}{2} \| \nabla u_n(t) \|^2_{L^2} + \frac{1}{2} \int_D \phi(u_n(t, x)) \, dx + \lambda_2 \int_0^t \| u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s))) \|^2_H \, ds
\]

\[ = \frac{1}{2} \| \nabla u_n(0) \|^2_{L^2} + \frac{1}{2} \int_D \phi(u_n(0, x)) \, dx - \frac{1}{2} \sum_{j=1}^N \int_0^t \langle \Delta u_n(s), \pi_n(u_n(s) \times h_j) \times h_j \rangle_H \, ds
\]

\[ + \frac{1}{2} \sum_{j=1}^N \int_0^t \langle \phi'(u_n(s)), \pi_n(u_n(s) \times h_j) \times h_j \rangle_H \, ds + \frac{1}{2} \sum_{j=1}^N \int_0^t \| \nabla \pi_n(u_n(s) \times h_j) \|^2_{H} \, ds
\]

\[ + \frac{1}{2} \sum_{j=1}^N \int_0^t \int_D \phi''(u_n(s), x)) \| \pi_n(u_n(s) \times h_j)(x), \pi_n(u_n(s) \times h_j)(x) \| \, dx \, ds
\]

\[ - \sum_{j=1}^N \int_0^t \langle \Delta u_n(s), u_n(s) \times h_j \rangle_H \, dW_j(s) + \sum_{j=1}^N \int_0^t \langle \phi'(u_n(s), \pi_n(u_n(s) \times h_j)) \rangle_H \, dW_j(s). \]

Next we will get estimates for some terms on the right hand side of Equation (4.15).

For the first term on the right hand side of Equation (4.15), we have

(4.16)

\[ \| \nabla u_n(0) \|^2_{L^2} \leq \| \pi_n u_0 \|^2_{L^2} \leq \| \pi_n u_0 \|^2_{L^2} = \| \pi_n A_1 \pi_n u_0 \|^2_H = \| \pi_n A_1 \pi_n u_0 \|^2_H = \| A_1 \pi_n u_0 \|^2_H = \| A_1 \pi_n u_0 \|^2_H = \| u_0 \|^2_{L^2}. \]

By our assumption, \( \phi \) is bounded, so there is a constant \( C_\phi > 0 \), such that for the second term on the right hand side of Equation (4.15), we have

(4.17)

\[ \left| \int_D \phi(u_n(0, x)) \, dx \right| \leq C_\phi m(D). \]
For the third term on the right hand side of Equation (4.15), by (4.6) and Cauchy-Schwartz inequality, we have
\[
\langle \Delta u_n(s), \pi_n(u_n(s) \times h_j) \rangle \leq \frac{1}{2} \| \nabla u_n(s) \|_{L^2}^2 + a_1 \| \nabla u_n(s) \|_{L^2}^2 + b_1 \leq \sqrt{a_1} \| \nabla u_n(s) \|_{L^2}^2 + \frac{b_1}{2 \sqrt{a_1}}.
\]
(4.18)

For the fourth term on the right hand side of Equation (4.15), by the equality (4.1) and Cauchy-Schwartz inequality, we have
\[
\langle \phi'(u_n(s)), \pi_n(u_n(s) \times h_j) \rangle \leq C_{\phi'} \| m(D) |u_0| \| h_j \|_{L^\infty}^2.
\]
(4.19)

For the fifth term on the right hand side of Equation (4.15), by (4.5), we have
\[
\| \nabla \pi_n(u_n(s) \times h_j) \|_{L^2}^2 = \| \nabla G_{j\phi}(u_n(s)) \|_{L^2}^2 \leq a_1 \| u_n(s) \|_{L^2}^2 + b_1.
\]
(4.20)

For the sixth term on the right hand side of Equation (4.15), we have
\[
\int_D \left[ \phi''(u_n(s, x)) \right] \left[ \pi_n(u_n(s) \times h_j)(x), \pi_n(u_n(s) \times h_j)(x) \right] \, dx
\]
\[
\leq C_{\phi''} \int_D \| \pi_n(u_n(s) \times h_j)(x) \|_{L^2}^2 \, dx
\leq C_{\phi''} \| h_j \|_{L^\infty}^2 \| u_0 \|_{H^2}^2.
\]
(4.21)

Then by the equalities (4.15)-(4.21), there exists a constant $C_2 > 0$ such that for all $n \in \mathbb{N}$, $t \in [0, T]$ and $\mathbb{P}$-almost surely:
\[
\| \nabla u_n(t) \|_{L^2}^2 + \int_0^t \phi(u_n(t, x)) \, dx + 2 \lambda_2 \int_0^t \| u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s))) \|_{L^2}^2 \, ds
\]
\[
\leq C_2 \int_0^t \| \nabla u_n(s) \|_{L^2}^2 \, ds + C_2 + 2 \sum_{j=1}^N \int_0^t \left\langle \nabla u_n(s), \nabla G_{j\phi}(u_n(s)) \right\rangle_{L^2} \, dW_j(s)
\]
\[
+ \sum_{j=1}^N \int_0^t \left\langle \phi'(u_n(s)), G_{j\phi}(u_n(s)) \right\rangle_H \, dW_j(s).
\]
(4.22)

Hence for $p \geq 1$,
\[
\mathbb{E} \sup_{r \in [0,t]} \left\{ \| \nabla u_n(r) \|_{L^2}^2 + \int_0^r \phi(u_n(r, x)) \, dx + 2 \lambda_2 \int_0^r \| u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s))) \|_{H}^2 \, ds \right\}^p
\leq 4^{p-1} C_2^p \left( \int_0^t \| \nabla u_n(s) \|_{L^2}^2 \, ds \right)^p
\]
\[
+ 4^{p-1} 2 \mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^N \int_0^r \left\langle \nabla u_n(s), \nabla G_{j\phi}(u_n(s)) \right\rangle_{L^2} \, dW_s \right|^p
\]
\[
+ 4^{p-1} \mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^N \int_0^r \left\langle \phi'(u_n(s)), G_{j\phi}(u_n(s)) \right\rangle_H \, dW_s \right|^p \leq 4^{p-1} C_2^p.
\]
By the Burkholder-Davis-Gundy inequality, there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$,

$$
\mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^{N} \int_0^r \left\langle \nabla u_n(s), \nabla G_{jk}(u_n(s)) \right\rangle_{L^2} \, dW_s \right|^p \leq K \mathbb{E} \left[ \sum_{j=1}^{N} \int_0^r \left\langle \nabla u_n(s), \nabla G_{jk}(u_n(s)) \right\rangle_{L^2}^2 \, ds \right]^\frac{p}{2},
$$

$$
\mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^{N} \int_0^r \left\langle \phi'(u_n(s)), G_{jk}(u_n(s)) \right\rangle_{H} \, dW_s \right|^p \leq K \mathbb{E} \left[ \sum_{j=1}^{N} \int_0^r \left\langle \phi'(u_n(s)), G_{jk}(u_n(s)) \right\rangle_{H}^2 \, ds \right]^\frac{p}{2}.
$$

By the inequality (4.5) we get, for any $\varepsilon > 0$,

$$
\mathbb{E} \left[ \sum_{j=1}^{N} \int_0^r \left\langle \nabla u_n(s), \nabla G_{jk}(u_n(s)) \right\rangle_{L^2}^2 \, ds \right]^\frac{p}{2} \leq \mathbb{E} \left[ \sup_{r \in [0,t]} \left\| \nabla u_n(r) \right\|_{L^2}^p \left( \sum_{j=1}^{N} \int_0^r \left\| \nabla G_{jk}(u_n(s)) \right\|_{L^2}^2 \, ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq \mathbb{E} \left[ \varepsilon \sup_{r \in [0,t]} \left\| \nabla u_n(r) \right\|_{L^2}^2 + \frac{4}{\varepsilon} \left( \sum_{j=1}^{N} \int_0^r \left\| \nabla G_{jk}(u_n(s)) \right\|_{L^2}^2 \, ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq \varepsilon \mathbb{E} \left( \sup_{r \in [0,t]} \left\| \nabla u_n(r) \right\|_{L^2}^2 \right)^{\frac{p}{2}} + \frac{4}{\varepsilon} (2t)^{p-1} a^n N^p \mathbb{E} \left( \int_0^r \left\| \nabla u_n(s) \right\|_{L^2}^2 \, ds \right) + \frac{4}{\varepsilon} \frac{2^p - 1}{(bt)^p} N^p.
$$

And

$$
\mathbb{E} \left[ \sum_{j=1}^{N} \int_0^r \left\langle \phi'(u_n(s)), G_{jk}(u_n(s)) \right\rangle_{L^2}^2 \, ds \right]^\frac{p}{2} \leq \mathbb{E} \left[ \sup_{r \in [0,t]} \left\| \phi'(u_n(r)) \right\|_{L^2}^p \left( \sum_{j=1}^{N} \int_0^r \left\| \nabla G_{jk}(u_n(s)) \right\|_{L^2}^2 \, ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq \varepsilon \left[ C_{\phi}(m(D)) \right]^{2p} + \frac{4}{\varepsilon} (2t)^{p-1} a^n N^p \mathbb{E} \left( \int_0^r \left\| \nabla u_n(s) \right\|_{L^2}^2 \, ds \right) + \frac{4}{\varepsilon} \frac{2^p - 1}{(bt)^p} N^p.
$$

Hence we infer that for $t \in [0,T]$,

$$
\mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^{N} \int_0^r \left\langle \nabla u_n(s), \nabla G_{jk}(u_n(s)) \right\rangle_{L^2} \, dW_s \right|^p \leq K \varepsilon \mathbb{E} \left( \sup_{r \in [0,t]} \left\| \nabla u_n(r) \right\|_{L^2}^p \right)^p + \frac{4K}{\varepsilon} (2t)^{p-1} a^n N^p \mathbb{E} \left( \int_0^r \left\| \nabla u_n(s) \right\|_{L^2}^2 \, ds \right) + \frac{4K}{\varepsilon} \frac{2^p - 1}{(bt)^p} N^p.
$$

and similarly for $t \in [0,T]$,

$$
\mathbb{E} \sup_{r \in [0,t]} \left| \sum_{j=1}^{N} \int_0^r \left\langle \phi'(u_n(s)), G_{jk}(u_n(s)) \right\rangle_{L^2} \, dW_s \right|^p \leq K \varepsilon \left[ C_{\phi}(\mu(D)) \right]^{2p} + \frac{4K}{\varepsilon} (2t)^{p-1} a^n N^p \mathbb{E} \left( \int_0^r \left\| \nabla u_n(s) \right\|_{L^2}^2 \, ds \right) + \frac{4K}{\varepsilon} \frac{2^p - 1}{(bt)^p} N^p.
Hence for every $t \in [0, T]$,

\[
\mathbb{E} \sup_{r \in [0, t]} \left\{ \|\nabla u_n(r)\|_{L^2}^2 + \int_D \phi(u_n(r, x)) \, dx + 2\lambda_2 \int_0^t \|u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s)))\|_H^2 \, ds \right\}^p
\]

\[
\leq 4^{p-1} C_2^p \mathbb{E} \left( \int_0^t \|\nabla u_n(s)\|_{L^2}^2 \, ds \right) + K \mathbb{E} \left( \sup_{r \in [0, t]} \|\nabla u_n(r)\|_{L^2}^2 \right) + K \mathbb{E} \left[ C_\phi \mu(D) \right]^{2p}
\]

\[
+ \frac{8K}{\varepsilon} \left( 2t \right)^{p-1} a^p N^{p} \mathbb{E} \left( \int_0^t \|\nabla u_n(s)\|_{L^2}^2 \, ds \right) + \frac{8K}{\varepsilon} \left( 2t \right)^{p-1} \left( bt \right)^{p} N^{p}
\]

Set $\varepsilon = \frac{1}{2K}$ in the above inequality, we have:

\[
\mathbb{E} \sup_{r \in [0, t]} \left\{ \|\nabla u_n(r)\|_{L^2}^2 + \int_D \phi(u_n(r, x)) \, dx 
+ 2\lambda_2 \int_0^t \|u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s)))\|_H^2 \, ds \right\}^p
\]

\[
\leq \left[ 2 \cdot 4^{p-1} C_2^p t^{p-1} + 32K^2 \left( 2t \right)^{p-1} a^p N^{p} \right] \mathbb{E} \left( \int_0^t \|\nabla u_n(s)\|_{L^2}^2 \, ds \right)
\]

\[
+ \left[ C_\phi \mu(D) \right]^{2p} + 32K^2 \left( 2t \right)^{p-1} \left( bt \right)^{p} N^{p}
\]

\[
= C_3 \mathbb{E} \left( \int_0^t \|\nabla u_n(s)\|_{L^2}^2 \, ds \right) + C_4.
\]

where the constants $C_3$ and $C_4$ are defined by:

\[
C_3 = 2 \cdot 4^{p-1} C_2^p t^{p-1} + 32K^2 \left( 2t \right)^{p-1} a^p N^{p},
\]

\[
C_4 = \left[ C_\phi \mu(D) \right]^{2p} + 32K^2 \left( 2t \right)^{p-1} \left( bt \right)^{p} N^{p},
\]

note that they do not depend on $n$.

And since $\int_0^t \phi(u_n(r, x)) \, dx$ and $\lambda_2 \int_0^t \|u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s)))\|_H^2 \, ds$ are non-negative, so by the inequality (4.25), we have

\[
\mathbb{E} \sup_{r \in [0, t]} \left\{ \|\nabla u_n(r)\|_{L^2}^2 + \int_D \phi(u_n(r, x)) \, dx 
+ 2\lambda_2 \int_0^t \|u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s)))\|_H^2 \, ds \right\}^p
\]

\[
\leq C_3 \int_0^t \mathbb{E} \sup_{r \in [0, s]} \left\{ \|\nabla u_n(r)\|_{L^2}^2 + \int_D \phi(u_n(r, x)) \, dx 
+ 2\lambda_2 \int_0^t \|u_n(\tau) \times (\Delta u_n(\tau) - \pi_n \phi'(u_n(\tau)))\|_H^2 \, d\tau \right\}^p \, ds + C_4.
\]
Let us define a function $\psi$ by:

$$\psi(s) = \mathbb{E} \sup_{r \in [0, s]} \left\{ \left\| \nabla u_n(r) \right\|^2_{L^2} + \int_D \phi(u_n(r, x)) \, dx + 2\lambda_2 \int_0^s \left\| u_n(\tau) \times (\Delta u_n(\tau) - \pi_n \phi'(u_n(\tau))) \right\|^2_H \, d\tau \right\}^p, \quad s \in [0, T].$$

Then by the inequality (4.26), we deduce that:

$$\psi(t) \leq C_3 \int_0^t \psi(s) \, ds + C_4.$$

Observe that $\psi$ is a bounded Borel function. The boundedness is because

$$\|\nabla u_n(r)\|_{L^2} \leq \|u_n(r)\|_V \leq C_n \|u_n(r)\|_H \leq C_n \|u_0\|_H, \quad r \in [0, T],$$

and

$$\left\| u_n(s) \times (\Delta u_n(s) - \pi_n \phi'(u_n(s))) \right\|_{H^n} \leq \|u_n(s)\|_{L^\infty(D)} (\|\Delta u_n(s)\|_H + \|\pi_n \phi'(u_n(s))\|_H) \leq C_n \|u_0\|_H \left( C_n \|u_0\|_H + C_{\phi, \mu}(D) \right) \leq C_n \|u_0\|_H \left( C_n \|u_0\|_H + 2C_{\phi, \mu}(D) \right).$$

where $C_n$ is from the norm equivalence in the $n$-dimensional space. Therefore

$$\|\psi(s)\| \leq \left( C_n^2 \|u_0\|_{H^n}^2 + C_{\phi, \mu}(D) + 2\lambda_2 TC_n^2 \|u_0\|^2_H \left( C_n \|u_0\|_H + 2C_{\phi, \mu}(D) \right)^2 \right)^{\frac{p}{2}}.$$

Therefore by the Gronwall inequality, we have

$$\psi(t) \leq C_3 e^{C_3 t}, \quad t \in [0, T].$$

Since $C_3$ and $C_4$ are independent of $n$, we have proved that for $T \in (0, \infty)$,

$$\mathbb{E} \sup_{r \in [0, t]} \left\{ \left\| \nabla u_n(r) \right\|^2_{L^2} + \int_D \phi(u_n(r, x)) \, dx + 2\lambda_2 \int_0^t \left\| u_n(\tau) \times (\Delta u_n(\tau) - \pi_n \phi'(u_n(\tau))) \right\|^2_H \, d\tau \right\}^p \leq C_3 e^{C_3 T} = C_T, \quad t \in [0, T]$$

where $C_T$ is independent of $n$. Therefore we infer that

$$\mathbb{E} \sup_{r \in [0, t]} \left\{ \left\| \nabla u_n(r) \right\|^2_{L^2} + \int_D \phi(u_n(r, x)) \, dx \right\}^p \leq C_T,$$

and

$$\mathbb{E} \left( \int_0^T \left\| u_n(\tau) \times (\Delta u_n(\tau) - \pi_n \phi'(u_n(\tau))) \right\|^2_H \, d\tau \right)^p \leq C_T.$$

This completes the proof of the inequalities (4.7) and (4.8). \hfill \Box

**Proof of (4.9).** By the H"older inequality and the Sobolev imbedding $\mathbb{H}^1 \hookrightarrow L^6$, we have that for some constant $c > 0$

$$\left\| u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]) \right\|_{L^\frac{5}{2}} \leq \left\| u_n(t) \right\|_{L^6} \left\| u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))] \right\|_H \leq c \left\| u_n(t) \right\|_V \left\| u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))] \right\|_H.$$
Then by (4.1), (4.7) and (4.8), there exists some constant $c_1 > 0$, such that
\[
\mathbb{E} \left[ \left( \int_0^T \| u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]) \|_{L^2}^p \ dt \right)^{\frac{2}{p}} \right] \\
\leq c_1 \mathbb{E} \left[ \sup_{r \in [0,T]} \| u_n(r) \|_V^p \left( \int_0^T \| u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]\|_{H}^p \ dt \right)^{\frac{2}{p}} \right] \\
\leq c_1 \left( \mathbb{E} \left[ \sup_{r \in [0,T]} \| u_n(r) \|_V^p \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]\|_{H}^2 \ dt \right)^{2p} \right] \right)^{\frac{1}{2p}} \leq C,
\]

Note that $C$ is independent of $n$. This completes the proof of (4.9). \hfill \Box

**Proof of (4.10).** By Sobolev imbedding theorems, if $\beta > \frac{1}{4}$, $X^\beta \hookrightarrow \mathbb{H}^{2\beta}(D)$ continuously. And if $\beta > \frac{1}{4}$, $\mathbb{H}^{2\beta}(D)$ is continuously imbedded in $L^2(D)$. Therefore $L^2(D)$ is continuously imbedded in $X^{-\beta}$. And since for $\xi \in H,$
\[
\| \pi_n \xi \|_{X^{-\beta}} = \sup_{\| \varphi \|_{H} \leq 1} \| X^{-\beta}(\pi_n \xi, \varphi) \|_{H} = \sup_{\| \varphi \|_{H} \leq 1} |\langle \pi_n \xi, \varphi \rangle|_H \\
= \sup_{\| \varphi \|_{H} \leq 1} |\langle \xi, \pi_n \varphi \rangle|_H \leq \sup_{\| \pi_n \varphi \|_{H} \leq 1} \| X^{-\beta}(\xi, \pi_n \varphi) \|_{H} = \| \xi \|_{X^{-\beta}}.
\]

Therefore we infer that there exists some constant $c > 0$ such that
\[
\mathbb{E} \int_0^T \| \pi_n(u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))])) \|_{X^{-\beta}} \ dt \\
\leq c \mathbb{E} \int_0^T \| u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \pi_n \phi'(u_n(t))]) \|_{L^2}^2 \ dt.
\]
Then (4.10) follows from (4.9). \hfill \Box

**Proposition 4.6.** Let $u_n$, for $n \in \mathbb{N}$, be the solution of the equation (3.5) and assume that $\alpha \in (0, \frac{1}{2}), \beta > \frac{1}{4}, p \geq 2$. Then the following estimates holds:
\[
\sup_{n \in \mathbb{N}} \mathbb{E}(\| u_n \|_{W^{\alpha,p}(0,T;X^{-\beta})}^2) < \infty.
\]

We need the following Lemma to prove (4.27).

**Lemma 4.7** ([22], Lem 2.1). Assume that $E$ is a separable Hilbert space, $p \in [2, \infty)$ and $a \in (0, \frac{1}{2})$. Then there exists a constant $C$ depending on $T$ and $a$, such that for any progressively measurable process $\xi = (\xi_j)^{j=1}_{j=t}$,
\[
\mathbb{E} \| I(\xi) \|_{W^{\alpha,p}(0,T;E)}^p \leq C \mathbb{E} \int_0^T \left( \sum_{j=1}^{\infty} |\xi_j(r)|_E^p \right)^{\frac{p}{2}} \ dt,
\]
where $I(\xi_j)$ is defined by
\[
I(\xi) := \sum_{j=1}^{\infty} \int_0^T \xi_j(s) \ dW_j(s), \quad t \geq 0.
\]

In particular, $\mathbb{P}$-a.s. the trajectories of the process $I(\xi_j)$ belong to $W^{\alpha,2}(0,T;E)$. 

Proof of (4.27). Let us fix $\alpha \in (0, \frac{1}{2})$, $\beta > \frac{1}{4}$, $p \geq 2$. By the equation (3.11), we get

$$u_n(t) = u_{0,n} + \lambda_1 \int_0^t \left[ F_1^n(u_n(s)) + F_3^n(u_n(s)) \right] ds - \lambda_2 \int_0^t \left[ F_2^n(u_n(s)) + F_4^n(u_n(s)) \right] ds$$

$$+ \frac{1}{2} \sum_{j=1}^N \int_0^t G_{jn}^2(u_n(s)) ds + \sum_{j=1}^N \int_0^t G_{jn}(u_n(s)) dW(s)$$

$$=: u_{0,n} + \sum_{i=1}^4 u_i^n(t), \quad t \in [0, T].$$

By Theorem 4.5, we have the following results:
There exists $C > 0$ such that for all $n \in \mathbb{N}$,

1. $$\mathbb{E} \left[ \|u_{1,n}^2\|_{\mathcal{W}^{1,2}(0,T;H)}^2 \right] \leq C.$$

2. $$\mathbb{E} \left[ \|u_{2,n}^2\|_{\mathcal{W}^{1,2}(0,T,X-\beta)}^2 \right] \leq C.$$

3. $$\|u_{3,n}^2\|_{\mathcal{W}^{1,2}(0,T;H)}^2 \leq C, \quad \mathbb{P} - \text{a.s.}$$

Moreover, by the equality (4.1),

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \|u_n(t)\|_H^p \right] = \mathbb{E} \left[ \|u_n(0)\|_H^p \right] \leq C.$$  

By the inequality (3.12) and Lemma 4.7, we have:

$$\mathbb{E} \left[ \|u_{n}^4\|_{\mathcal{W}^{0,p}(0,T,X-\beta)}^p \right] \leq C.$$

Therefore since $H^1(0,T;X^{-\beta}) \hookrightarrow W^{\alpha, p}(0,T;X^{-\beta})$ continuously, we get

$$\sup_{n \in \mathbb{N}} \mathbb{E}(\|u_n^2\|_{\mathcal{W}^{0,p}(0,T,X^{-\beta})}^2) < \infty.$$ 

This completes the proof of the inequality (4.27). \hfill \Box

5. Tightness results

In this subsection we will use the a’priori estimates (4.1)-(4.10) to show that the laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ are tight on a suitable path space.

**Lemma 5.1.** For any $p \geq 2$, $q \in [2, 6)$ and $\beta > \frac{1}{4}$ the set of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ on the Banach space

$$L^p(0, T; \mathbb{L}^q(D)) \cap C(0, T; X^{-\beta})$$

is tight.
Proof. Let us choose and fix $p \geq 2$, $q \in [2, 6)$ and $\beta > \frac{1}{4}$. Since $q < 6$ we can choose $\gamma \in \left(\frac{1}{4}, \frac{3}{2p} + \frac{1}{2}\right)$ such that $\mathbb{H}^{2\gamma}(D) \hookrightarrow L^q(D)$ continuously and choose $\gamma' \in (\frac{1}{4}, b)$, $\alpha \in (\frac{1}{p}, 1)$. Since the embedding $V = D(A^\frac{1}{2}) \hookrightarrow X' = D(A')$ is compact, the embedding

$$L^p(0, T; V) \cap W^{\alpha, p}(0, T; X^{-\beta'}) \hookrightarrow L^p(0, T; X')$$

is compact. We note that for any positive real number $r$ and random variables $\xi$ and $\eta$, since

$$\left\{ \omega : \xi(\omega) > \frac{r}{2} \right\} \cup \left\{ \omega : \eta(\omega) > \frac{r}{2} \right\} \supset \left\{ \omega : \xi(\omega) + \eta(\omega) > r \right\},$$

we have

\[
\mathbb{P}(\|u_n\|_{L^p(0,T;H^1) \cap W^{\alpha, p}(0,T;X^{-\beta'})} > r) \\
= \mathbb{P}(\|u_n\|_{L^p(0,T;H^1)} + \|u_n\|_{W^{\alpha, p}(0,T;X^{-\beta'})} > r) \\
\leq \mathbb{P}(\|u_n\|_{L^p(0,T;H^1)} > \frac{r}{2}) + \mathbb{P}(\|u_n\|_{W^{\alpha, p}(0,T;X^{-\beta'})} > \frac{r}{2}) \\
\leq \ldots
\]

then by the Chebyshev inequality,

$$\ldots \leq \frac{4}{r^2} \mathbb{E}(\|u_n\|_{L^p(0,T;V)}^2 + \|u_n\|_{W^{\alpha, p}(0,T;X^{-\beta'})}^2).$$

By the estimates in (4.27), (4.1) and (4.7), the expected value on the right hand side of the last inequality is uniformly bounded in $n$. Let $X_T := L^p(0, T; V) \cap W^{\alpha, p}(0, T; X^{-\beta'})$. There is a constant $C$, such that

$$\mathbb{P}(\|u_n\|_{X_T} > r) \leq \frac{C}{r^2}, \quad \forall r, n.$$ 

Since

$$\mathbb{E}(\|u_n\|_{X_T}) = \int_0^{\infty} \mathbb{P}(\|M_n\| > r) \, dr,$$

we can infer that

$$\mathbb{E}(\|u_n\|_{X_T}) \leq 1 + \int_1^{\infty} \frac{C}{r^2} \, dr = 1 + C < \infty, \quad \forall n \in \mathbb{N}.$$ 

Therefore the family of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $L^p(0, T; X')$. By Proposition 3.2, $X' = \mathbb{H}^{2\gamma}(D)$. Therefore $X' \hookrightarrow L^q(D)$ continuously. Hence $L^p(0, T; X') \hookrightarrow L^p(0, T; L^q(D))$ continuously. Therefore $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is also tight on $L^p(0, T; L^q(D))$. Since $\beta' < \beta$, $W^{\alpha, p}(0, T; X^{-\beta'}) \hookrightarrow C(0, T; X^{-\beta})$ compactly. Therefore by the estimates in (4.27), we can conclude that $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $C(0, T; X^{-\beta})$. Therefore $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $L^p(0, T; L^q) \cap C([0, T]; X^{-\beta})$. Hence the proof of Lemma 5.1 is completed.

From now on we will always assume $\beta > \frac{1}{4}$.
6. Construction of new Probability Space and Processes

In this section we will use Skorohod’s theorem to obtain another probability space and an almost surely convergent sequence defined on this space whose limit is a weak martingale solution of the equation (2.1).

By Lemma 5.1 and Prokhorov’s Theorem, we have the following property.

**Proposition 6.1.** Let us assume that \( W \) is a \( N \)-dimensional Wiener process and \( p \in [2, \infty), q \in [2, 6) \) and \( \beta > \frac{1}{4} \). Then there is a subsequence of \( \{u_n\} \) which we will denote it in the same way as the full sequence, such that the laws \( \mathcal{L}(u_n, W) \) converge weakly to a certain probability measure \( \mu \) on \( L^p(0, T; \mathbb{L}^\beta(D)) \cap C([0, T]; X^{-\beta}) \times C(0, T; \mathbb{R}^N) \).

Now by the Skorohod’s theorem we have:

**Proposition 6.2.** There exists a probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \) and there exists a sequence \( (u'_n, W'_n) \) of \( (\mathcal{L}(u_n, W)) \cap C([0, T]; X^{-\beta}) \times C(0, T; \mathbb{R}^N) \)-valued random variables defined on \( (\Omega', \mathcal{F}', \mathbb{P}') \) such that

(a) \( \mathcal{L}(u_n, W) = \mathcal{L}(u'_n, W'_n), \quad \forall n \in \mathbb{N} \)

(b) There exists a random variable

\[
(u', W') : (\Omega', \mathcal{F}', \mathbb{P}') \longrightarrow [\mathcal{L}(u_n, W)] \cap C([0, T]; X^{-\beta}) \times C(0, T; \mathbb{R}^N)
\]

such that

(i) \( \mathcal{L}(u', W') = \mu \),

(ii) \( u'_n \rightarrow u' \) in \( \mathcal{L}(u_n, W) \cap C([0, T]; X^{-\beta}) \) almost surely,

(iii) \( W'_n \rightarrow W' \) in \( C([0, T]; \mathbb{R}^N) \) almost surely.

**Notation 6.3.** We will use \( \mathbb{F}' \) to denote the filtration generated by \( u' \) and \( W' \) in the probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \).

From now on we will prove that \( u' \) is the weak solution of the equation (2.1). And we begin with showing that \( \{u'_n\} \) satisfies the same a’priori estimates as the original sequence \( \{u_n\} \). By the Kuratowski Theorem, we have

**Proposition 6.4.** The Borel subsets of \( C(0, T; H_n) \) are Borel subsets of \( \mathcal{L}(u_n, W) \cap C([0, T]; X^{-\beta}) \times C(0, T; X^{-\beta}) \).

So we have the following Corollary.

**Corollary 6.5.** \( u'_n \) takes values in \( H_n \) and the laws on \( C([0, T]; H_n) \) of \( u_n \) and \( u'_n \) are equal.

By the Corollary 6.5, we have
Lemma 6.6. The \{u'_n\} defined in Proposition 6.2 satisfies the following estimates:

\begin{equation}
\sup_{t \in [0,T]} \|u'_n(t)\|_H \leq \|u_0\|_H, \quad \mathbb{P}' - \text{a.s.,}
\end{equation}

\begin{equation}
\sup_{n \in \mathbb{N}} \mathbb{E}' \left[ \sup_{t \in [0,T]} \|u'_n(t)\|_{L^2}^{2r} \right] < \infty, \quad \forall r \geq 1,
\end{equation}

\begin{equation}
\sup_{n \in \mathbb{N}} \mathbb{E}' \left[ \left\| \int_0^T \left[ \Delta u'_n(t) - \pi_n \phi'(u'_n(t)) \right] \, dt \right\|^2 \mathbb{E}ight] < \infty, \quad \forall r \geq 1,
\end{equation}

\begin{equation}
\sup_{n \in \mathbb{N}} \mathbb{E}' \left[ \left\| \int_0^T \left[ \Delta u'_n(t) - \pi_n \phi'(u'_n(t)) \right] \, dt \right\|^2 \mathbb{E}ight] < \infty,
\end{equation}

\begin{equation}
\sup_{n \in \mathbb{N}} \mathbb{E}' \left[ \left\| \int_0^T \left[ \Delta u'_n(t) - \pi_n \phi'(u'_n(t)) \right] \, dt \right\|^2 \mathbb{E}ight] < \infty.
\end{equation}

Now we will study some inequalities satisfied by the limiting process \(u'\).

Proposition 6.7. Let \(u'\) be the process which is defined in Proposition 6.2. Then we have

\begin{equation}
\text{ess sup}_{t \in [0,T]} \|u'(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \mathbb{P}' - \text{a.s.}
\end{equation}

\begin{equation}
\sup_{t \in [0,T]} \|u'(t)\|_{X^{-\beta}} \leq c\|u_0\|_{L^2}, \quad \mathbb{P}' - \text{a.s.}
\end{equation}

Proof of (6.6). Since \(u'_n\) converges to \(u'\) in \(L^4(0, T; \mathbb{L}^4) \cap C(0, T; X^{-\beta})\) \(\mathbb{P}'\) almost surely,

\[
\lim_{n \to \infty} \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt = 0, \quad \mathbb{P}' - \text{a.s.}
\]

Since \(L^4(D) \hookrightarrow L^2(D)\), we infer that

\[
\lim_{n \to \infty} \int_0^T \|u'_n(t) - u'(t)\|_{L^2}^2 \, dt = 0.
\]

Hence \(u'_n\) converges to \(u'\) in \(L^2(0, T; L^2)\) \(\mathbb{P}'\) almost surely. Therefore by (6.1),

\[
\text{ess sup}_{t \in [0,T]} \|u'(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \mathbb{P}' - \text{a.s.}
\]

\[\square\]

Proof of (6.7). Since \(L^2(D) \hookrightarrow X^{-\beta}\), there exists some constant \(c > 0\), such that

\[
\|u'_n(t)\|_{X^{-\beta}} \leq c\|u'_n(t)\|_{L^2} \text{ for all } n \in \mathbb{N}.
\]

By (6.1), we have

\[
\sup_{t \in [0,T]} \|u'_n(t)\|_{X^{-\beta}} \leq c \sup_{t \in [0,T]} \|u'_n(t)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \mathbb{P}' - \text{a.s.}
\]

And by Proposition 6.2 (ii) \(u'_n\) converges to \(u'\) in \(C([0, T]; X^{-\beta})\), we infer that

\[
\sup_{t \in [0,T]} \|u'(t)\|_{X^{-\beta}} \leq c\|u_0\|_{L^2}, \quad \mathbb{P}' - \text{a.s.}
\]

\[\square\]
We continue investigating properties of the process $u'$. The next result and it’s proof are related to the estimate (6.2).

**Proposition 6.8.** Let $u'$ be the process which was defined in Proposition 6.2. Then we have

\[(6.8) \quad \mathbb{E}'[\text{ess sup}_{n\in[0,T]} \|u'(t)\|_V^2] < \infty, \quad r \geq 2.\]

**Proof.** Since $L^2(\Omega'; L^\infty(0, T; V))$ is isomorphic to $[L^2(\Omega'; L^1(0, T; X^{-\frac{1}{2}}))]^*$, by the Banach-Alaoglu Theorem we infer that the sequence $\{u'_n\}$ contains a subsequence, denoted in the same way as the full sequence, and there exists an element $v \in L^2(\Omega'; L^\infty(0, T; V))$ such that $u'_n \to v$ weakly$^*$ in $L^2(\Omega'; L^\infty(0, T; V))$. In particular, we have

\[\langle u'_n, \varphi \rangle \to \langle v, \varphi \rangle, \quad \varphi \in L^2(\Omega'; L^1(0, T; X^{-\frac{1}{2}})).\]

This means that

\[\int_{\Omega'} \int_0^T \langle u'_n(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\mathbb{P}'(\omega) \to \int_{\Omega'} \int_0^T \langle v(t, \omega), \varphi(t, \omega) \rangle \, dt \, d\mathbb{P}'(\omega).\]

On the other hand, if we fix $\varphi \in L^4(\Omega'; L^3(0, T; \mathbb{L}^4))$, by the inequality (6.2) we have

\[\sup_n \int_{\Omega'} \left( \int_0^T \|u'_n(t, \omega)\|_{L^4(0, T; \mathbb{L}^4)} \|\varphi\|_{L^3(0, T; \mathbb{L}^4)} \, dt \right)^2 \, d\mathbb{P}'(\omega) \leq \sup_n \int_{\Omega'} \left( \int_0^T \|u'_n||\varphi||_{L^\infty(0, T; \mathbb{L}^4)} \right)^2 \, d\mathbb{P}'(\omega) \leq \sup_n \|u'_n\|_{L^1(\Omega'; L^\infty(0, T; \mathbb{L}^4))} \|\varphi\|_{L^4(\Omega'; L^1(0, T; \mathbb{L}^4))} \leq \infty.

So the sequence $\int_0^T L^4(u'_n(t, \omega), \varphi(t))_{\mathbb{L}^4} \, dt$ is uniformly integrable on $\Omega'$. Moreover, by the $\mathbb{P}'$ almost surely convergence of $u'_n$ to $u'$ in $L^4(0, T; \mathbb{L}^4)$, we get $\mathbb{P}'$-a.s.

\[\left| \int_0^T L^4(u'_n(t, \omega), \varphi(t))_{\mathbb{L}^4} \, dt - \int_0^T L^4(u'(t, \omega), \varphi(t))_{\mathbb{L}^4} \, dt \right| \leq \int_0^T \|u'_n(t) - u'(t, \omega)\|_{L^4} \|\varphi\|_{L^4(0, T; \mathbb{L}^4)} \, dt \leq \int_0^T \|u'_n(t) - u'(t)\|_{L^4} \|\varphi\|_{L^4(0, T; \mathbb{L}^4)} \, dt \to 0.

Therefore we infer that $\int_0^T L^4(u'_n(t, \omega), \varphi(t))_{\mathbb{L}^4} \, dt$ converges to $\int_0^T L^4(u'(t, \omega), \varphi(t))_{\mathbb{L}^4} \, dt$ $\mathbb{P}'$-almost surely. Thus,

\[\int_{\Omega'} \int_0^T L^4(v(t, \omega), \varphi(t, \omega))_{\mathbb{L}^4} \, dt \, d\mathbb{P}'(\omega) \to \int_{\Omega'} \int_0^T L^4(u'(t, \omega), \varphi(t, \omega))_{\mathbb{L}^4} \, dt \, d\mathbb{P}'(\omega).

Hence we deduce that

\[\int_{\Omega'} \int_0^T L^4(v(t, \omega), \varphi(t, \omega))_{\mathbb{L}^4} \, dt \, d\mathbb{P}'(\omega) = \int_{\Omega'} \int_0^T L^4(u'(t, \omega), \varphi(t, \omega))_{\mathbb{L}^4} \, dt \, d\mathbb{P}'(\omega).\]
By the arbitrariness of \( \varphi \) and density of \( L^4(\Omega'; L^\frac{4}{2}(0, T; \mathbb{L}^\frac{4}{2})) \) in \( L^\frac{2}{r}(\Omega'; L^1(0, T; X^{-\frac{1}{2}})) \), we infer that \( u' = v \) and since \( v \) satisfies (6.8) we infer that \( u' \) also satisfies (6.8). In this way the proof of (6.8) is complete.

Now we will strengthen part (ii) of Proposition 6.2 about the convergence of \( u'_n \) to \( u' \).

**Proposition 6.9.**

\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt = 0.
\]

**Proof.** Since \( u'_n \to u' \) in \( L^4(0, T; \mathbb{L}^4) \cap C(0, T; X^{-\beta}) \) \( \mathbb{P}' \)-almost surely, \( u'_n \to u' \) in \( L^4(0, T; \mathbb{L}^4) \) \( \mathbb{P}' \)-almost surely, i.e.

\[
\lim_{n \to \infty} \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt = 0, \quad \mathbb{P}' - a.s.,
\]

and by (6.2) and (6.8),

\[
\sup_n \mathbb{E}' \left( \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt \right)^2 \leq 2^7 \sup_n \left( \|u'_n\|^8_{L^4(0, T; L^4(D))} + \|u'\|^8_{L^4(0, T; L^4(D))} \right) < \infty,
\]

Hence we infer that

\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt = 0.
\]

This completes the proof. \( \square \)

By the estimate (6.2), \( \{u'_n\}_{n=1}^\infty \) is bounded in \( L^2(\Omega'; L^2(0, T; \mathbb{H}^1)) \). And since \( u'_n \to u' \) in \( L^2(\Omega'; L^2(0, T; \mathbb{H}^1)) \), we have:

\[
\frac{\partial u'_n}{\partial x_i} \to \frac{\partial u'}{\partial x_i} \text{ weakly in } L^2(\Omega'; L^2(0, T; \mathbb{H}^1)), \quad i = 1, 2, 3.
\]

**Lemma 6.10.** There exists a unique \( \Lambda \in L^2(\Omega'; L^2(0, T; H)) \) such that for \( v \in L^2(\Omega'; L^2(0, T; \mathbb{H}^1(D))) \),

\[
\mathbb{E}' \int_0^T \langle \Lambda(t), v(t) \rangle_H \, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'(t), u'(t) \times D_i v(t) \rangle_H \, dt.
\]

**Proof:** We will omit “(t)” in this proof. Let us denote \( \Lambda_n := u'_n \times Au'_n \). By the estimate (6.3), there exists a constant \( C \) such that

\[
\|\Lambda_n\|_{L^2(\Omega'; L^2(0, T; H))} \leq C, \quad n \in \mathbb{N}.
\]

Hence by the Banach-Alaoglu Theorem, there exists \( \Lambda \in L^2(\Omega'; L^2(0, T; H)) \) such that \( \Lambda_n \to \Lambda \) weakly in \( L^2(\Omega'; L^2(0, T; H)) \).

Let us fix \( v \in L^2(\Omega'; L^2(0, T; \mathbb{H}^1(D))) \). Since \( u'_n(t) \in D(A) \) for almost every \( t \in [0, T] \) and \( \mathbb{P}' \)-almost surely, by the Proposition 3.4 and estimate (6.3) again, we have

\[
\mathbb{E}' \int_0^T \langle \Lambda_n, v \rangle_H \, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'_n, u'_n \times D_i v \rangle_H \, dt.
\]
Moreover, by the results: (6.10), (6.2) and (6.9), we have for \( i = 1, 2, 3 \),

\[
\left| \mathbb{E}' \int_0^T \langle D_i u', u' \times D_j v \rangle_H \, dt - \mathbb{E}' \int_0^T \langle D_i u'_n, u'_n \times D_j v \rangle_H \, dt \right|
\]

\[
\leq \mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_j v \rangle_H \, dt + \left| \mathbb{E}' \int_0^T \langle D_i u'_n, (u' - u'_n) \times D_j v \rangle_H \, dt \right|
\]

\[
\leq \mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_j v \rangle_H \, dt + \left( \mathbb{E}' \int_0^T \|D_i u'_n\|^2_H \, dt \right)^{\frac{1}{2}}
\]

\[
\times \left( \mathbb{E}' \int_0^T \|u' - u'_n\|^4_{L^4(D)} \, dt \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \|D_j v\|^4_{L^4(D)} \, dt \right)^{\frac{1}{2}} \to 0.
\]

Therefore we infer that

\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle \Lambda_n, v \rangle_H \, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u', u' \times D_j v \rangle \, dt.
\]

Since on the other hand we have proved \( \Lambda_n \to \Lambda \) weakly in \( L^2(\Omega'; L^2(0, T; H)) \) the equality (6.11) follows.

It remains to prove the uniqueness of \( \Lambda \), but this follows from the fact that \( L^2(\Omega'; L^2(0, T; \mathbb{W}^{1,4}(D))) \) is dense in \( L^2(\Omega'; L^2(0, T; H)) \) and (6.11). This complete the proof of Lemma 6.10. \( \Box \)

**Notation 6.11.** The process \( \Lambda \) introduced in Lemma 6.10 will be denoted by \( u' \times \Delta u' \). Note that \( u' \times \Delta u' \) is an element of \( L^2(\Omega'; L^2(0, T; H)) \) such that for all test functions \( v \in L^2(\Omega'; L^2(0, T; \mathbb{W}^{1,4}(D))) \) the following identity holds

\[
\mathbb{E}' \int_0^T \langle (u' \times \Delta u')(t), v(t) \rangle_H \, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'(t), u'(t) \times D_j v(t) \rangle_H \, dt.
\]

**Notation 6.12.** Since by the estimate (6.8), \( u' \in L^2(\Omega', L^\infty(0, T; V)) \) and by Notation 6.11, \( \Lambda \in L^2(\Omega'; L^2(0, T; H)) \), the process \( u' \times \Lambda \in L^2(\Omega'; L^2(0, T; L^2(D))) \). And \( u' \times \Lambda \) will be denoted by \( u' \times (u' \times \Delta u') \).

Next we will show that the limits of the following three sequences

\[
[u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)]],
\]

\[
[u'_n \times (u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)]],
\]

\[
[\pi_n(u'_n \times (u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)]))],
\]

exist and are equal respectively to

\[
u' \times [\Delta u' - \phi'(u')],
\]

\[
u' \times (u' \times [\Delta u' - \phi'(u')]],
\]

\[
u' \times (u' \times [\Delta u' - \phi'(u')]).
\]
By (6.3)-(6.5), the first sequence is bounded in $L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2))$ for $r \geq 1$, the second sequence is bounded in $L^2(\Omega'; L^2(0, T; \mathbb{L}^2))$ and the third sequence is bounded in $L^2(\Omega'; L^2(0, T; X^{-\beta})).$ And since $L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2)), L^2(\Omega'; L^2(0, T; \mathbb{L}^2))$ and $L^2(\Omega'; L^2(0, T; X^{-\beta}))$ are all reflexive Banach spaces, by the Banach-Alaoglu theorem, there exist subsequences weakly convergent. So we can assume that there exist
\[ Y \in L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2)), \]
\[ Z \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)), \]
\[ Z_1 \in L^2(\Omega'; L^2(0, T; X^{-\beta})), \]
such that
\[ u_n' \times [\Delta u_n' - \pi_n \phi'(u_n')] \rightharpoonup Y \text{ weakly in } L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2)), \]
\[ u_n' \times (u_n' \times [\Delta u_n' - \pi_n \phi'(u_n')]) \rightharpoonup Z \text{ weakly in } L^2(\Omega'; L^2(0, T; \mathbb{L}^2)), \]
\[ \pi_n (u_n' \times (u_n' \times [\Delta u_n' - \pi_n \phi'(u_n')])) \rightharpoonup Z_1 \text{ weakly in } L^2(\Omega'; L^2(0, T; X^{-\beta})). \]

**Remark.** Similar argument has been done in [13] for terms not involving $\phi'$. Our main contribution here is to show the validity of such an argument for term containing $\phi'$ (and to be more precise). This works because earlier we have been able to prove generalized estimates as in [13] as in Lemma 6.6.

**Proposition 6.13.** If $Z$ and $Z_1$ defined as above, then $Z = Z_1 \in L^2(\Omega'; L^2(0, T; X^{-\beta})).$

**Proof.** Notice that $(L^2_2)^\ast = L^3$, and by Proposition 3.2, $X^\beta = H^{2\beta}. \text{ By } X^\beta \subset L^3 \text{ for } \beta > \frac{1}{4}, \text{ we deduce that } L^3 \subset X^{-\beta}. \text{ So }$
\[ L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \subset L^2(\Omega'; L^2(0, T; X^{-\beta})). \]
Therefore $Z \in L^2(\Omega'; L^2(0, T; X^{-\beta}))$ and also $Z_1 \in L^2(\Omega'; L^2(0, T; X^{-\beta})).$

Since $X^\beta = D(A^\beta_1)$ and $A_1$ is self-adjoint, we can define
\[ X^\beta_n = \left\{ \pi_n \phi \sum_{j=1}^n x_j e_j : \sum_{j=1}^\infty \phi_j^2 \left< x_j \right> < \infty \right\}, \]
Then $X^\beta = \bigcup_{n=1}^{\infty} X_n^\beta$ and $L^2(\Omega'; L^2(0, T; X^\beta)) = \bigcup_{n=1}^{\infty} L^2(\Omega'; L^2(0, T; X_n^\beta))$. We have for $\psi_n \in L^2(\Omega'; L^2(0, T; X_n^\beta))$,

$$L^2(\Omega'; L^2(0, T; X^\beta))(\pi_n(u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))])), \psi_n)_{L^2(\Omega'; L^2(0, T; X^\beta))}$$

$$= \mathbb{E}' \int_0^T X^\beta (\pi_n(u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))])), \psi_n(t))_{X^\beta} dt$$

$$= \mathbb{E}' \int_0^T H(\pi_n(u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))])), \psi_n(t))_{H} dt$$

$$= \mathbb{E}' \int_0^T H(u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))]), \psi_n(t))_{H} dt$$

$$= \mathbb{E}' \int_0^T X^\beta (u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))]), \psi_n(t))_{X^\beta} dt$$

$$= L^2(\Omega'; L^2(0, T; X^\beta))(u'_n(t) \times (u'_n(t) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))]), \psi_n)_{L^2(\Omega'; L^2(0, T; X^\beta))}$$

Hence

$$L^2(\Omega'; L^2(0, T; X^\beta))(Z, \psi)_{L^2(\Omega'; L^2(0, T; X^\beta))} = L^2(\Omega'; L^2(0, T; X^\beta))(Z, \psi)_{L^2(\Omega'; L^2(0, T; X^\beta))}$$

$$= \lim_{n \to \infty} L^2(\Omega'; L^2(0, T; X^\beta))(Z, \psi)_{L^2(\Omega'; L^2(0, T; X^\beta))}$$

$$= \lim_{n \to \infty} L^2(\Omega'; L^2(0, T; X^\beta))(Z, \psi)_{L^2(\Omega'; L^2(0, T; X^\beta))}$$

Therefore $Z = Z_1 \in L^2(\Omega'; L^2(0, T; X^\beta))$ and this concludes the proof of Proposition 6.13.

**Lemma 6.14.** For any measurable process $\psi \in L^4(\Omega'; L^4(0, T; \mathbb{W}^1, 1, A))$, we have the equality

$$\lim_{n \to \infty} \mathbb{E} \int_0^T \langle u'_n(s) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))], \psi(s) \rangle_{L^2} ds$$

$$= \mathbb{E}' \int_0^T \langle Y(s), \psi(s) \rangle_{L^2} ds$$

$$= \mathbb{E}' \int_0^T \sum_{i=1}^3 \langle \frac{\partial u'(s)}{\partial x_i}, u'(s) \times \frac{\partial \psi(s)}{\partial x_i} \rangle_{L^2} ds + \mathbb{E} \int_0^T \langle u'(t) \times \phi'(u'(t)), \psi \rangle_{H} dt.$$ 

**Proof.** Let us fix $\psi \in L^4(\Omega'; L^4(0, T; \mathbb{W}^1, 1, A))$. Firstly, we will prove that

$$\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times \Delta u'_n(t), \psi(t) \rangle_{L^2} dt = \mathbb{E}' \int_0^T \sum_{i=1}^3 \left( \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right)_{L^2} dt.$$ 

For each $n \in \mathbb{N}$ we have

$$\langle u'_n(t) \times \Delta u'_n(t), \psi \rangle_{L^2} = \sum_{i=1}^3 \left( \frac{\partial u'_n(t)}{\partial x_i}, u'_n(t) \times \frac{\partial \psi(t)}{\partial x_i} \right)_{L^2}$$

(6.15)
for almost every \( t \in [0, T] \) and \( \mathbb{P}' \) almost surely. By Corollary 6.5, \( \mathbb{P}(u'_n \in C(0, T; H^s)) = 1 \). For each \( i \in \{1, 2, 3\} \) we may write

\[
(6.16) \quad \left\langle \frac{\partial u'_n}{\partial x_i}, u'_n \times \frac{\partial \psi}{\partial x_i} \right\rangle_{L^2} = \left\langle \frac{\partial u'_n}{\partial x_i} - \frac{\partial u'}{\partial x_i} + \frac{u'}{\partial x_i}, u'_n \times \frac{\partial \psi}{\partial x_i} \right\rangle_{L^2}
\]

Since \( L^4 \hookrightarrow L^2 \) and \( \mathcal{W}^{1,4} \hookrightarrow L^2 \), so there are constants \( C_1 \) and \( C_2 < \infty \) such that

\[
\left\langle \frac{\partial u'_n(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{L^2} \leq \left\| \frac{\partial u'_n(t)}{\partial x_i} \right\|_{L^2} \left\| \frac{\partial \psi(t)}{\partial x_i} \right\|_{L^2} \leq \left\| u'_n(t) \right\|_{H^1} C_1 \left\| u'_n(t) - u'(t) \right\|_{L^2} \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}}.
\]

Hence

\[
\mathbb{E}' \int_0^T \left\langle \frac{\partial u'_n(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{L^2} dt \leq C_1 C_2 \mathbb{E}' \int_0^T \left\| u'_n(t) \right\|_{H^1} \left\| u'_n(t) - u'(t) \right\|_{L^2} \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}} dt.
\]

Moreover by the H"older's inequality,

\[
\mathbb{E}' \int_0^T \left\| u'_n(t) \right\|_{H^1} \left\| u'_n(t) - u'(t) \right\|_{L^2} \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}} dt \\
\leq \left( \mathbb{E}' \int_0^T \left\| u'_n(t) \right\|_{H^1}^2 dt \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \left\| u'_n(t) - u'(t) \right\|_{L^2}^4 dt \right)^{\frac{1}{4}} \left( \mathbb{E}' \int_0^T \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}}^4 dt \right)^{\frac{1}{4}} \\
\leq T^{\frac{1}{2}} \left( \mathbb{E}' \sup_{t \in [0, T]} \left\| u'_n(t) \right\|_{H^1}^2 \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \left\| u'_n(t) - u'(t) \right\|_{L^2}^4 dt \right)^{\frac{1}{4}} \left( \mathbb{E}' \int_0^T \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}}^4 dt \right)^{\frac{1}{4}}.
\]

By (6.2), (6.9) and since \( \psi \in L^4(\Omega' ; L^4(0, T; \mathcal{W}^{1,4})) \), we have

\[
\lim_{n \to \infty} \left( \mathbb{E}' \sup_{t \in [0, T]} \left\| u'_n(t) \right\|_{H^1} \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \left\| u'_n(t) - u'(t) \right\|_{L^2}^4 dt \right)^{\frac{1}{4}} \left( \mathbb{E}' \int_0^T \left\| \psi(t) \right\|_{\mathcal{W}^{1,4}}^4 dt \right)^{\frac{1}{4}} = 0
\]

Hence

\[
(6.17) \quad \lim_{n \to \infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u'_n(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, \frac{\partial \psi(t)}{\partial x_i} \right\rangle_{L^2} dt = 0
\]

Both \( u' \) and \( \frac{\partial \phi}{\partial x_i} \) are in \( L^2(\Omega' ; L^2(0, T; L^2)) \), so \( u' \times \frac{\partial \phi}{\partial x_i} \in L^2(\Omega' ; L^2(0, T; L^2)) \). Hence by (6.10), we have

\[
(6.18) \quad \lim_{n \to \infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u'_n(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \phi(t)}{\partial x_i} \right\rangle_{L^2} dt = 0.
\]

Therefore by (6.16), (6.17), (6.18),

\[
(6.19) \quad \lim_{n \to \infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u'_n(t)}{\partial x_i}, u'_n(t) \times \frac{\partial \phi(t)}{\partial x_i} \right\rangle_{L^2} dt = \mathbb{E}' \int_0^T \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \phi(t)}{\partial x_i} \right\rangle_{L^2} dt
\]
Then by (6.15), we have

\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times \Delta u'_n(t), \psi(t) \rangle_{L^2} \, dt = \mathbb{E}' \int_0^T \sum_{i=1}^3 \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \phi'(t)}{\partial x_i} \right\rangle_{L^2} \, dt
\]

Secondly, we will show that

\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times \pi_n \phi'(u'_n(t)), \psi \rangle_H \, dt = \mathbb{E}' \int_0^T \langle u'(t) \times \phi'(u'(t)), \psi \rangle_H \, dt.
\]

Since

\[
\left| \langle u'_n(t) \times \pi_n \phi'(u'_n(t)), \psi \rangle_H - \langle u'(t) \times \phi'(u'(t)), \psi \rangle_H \right|
\]

\[
\leq \left( \|u'_n(t) - u'(t)\|_H \|\phi'(u'_n(t))\|_H + \|\phi'(u'(t))\|_H \|\pi_n \phi'(u'_n(t)) - \phi'(u'(t))\|_H \right)
\]

we have

\[
\mathbb{E}' \int_0^T \left| \langle u'_n(t) \times \pi_n \phi'(u'_n(t)), \psi \rangle_H - \langle u'(t) \times \phi'(u'(t)), \psi \rangle_H \right| \, dt
\]

\[
\leq \mathbb{E}' \int_0^T (\|\phi\|_H \|u'_n(t) - u'(t)\|_H+\|\phi\|_H \|u'(t)\|_H+\|\pi_n \phi'(u'_n(t)) - \phi'(u'(t))\|_H) \, dt
\]

\[
\leq \left( \mathbb{E}' \int_0^T \|\phi\|_{L^4}^2 \, dt \right)^{1/2} \left( \mathbb{E}' \int_0^T \|u'_n(t) - u'(t)\|_{L^4}^4 \, dt \right)^{1/2} \left( \mathbb{E}' \int_0^T \|\phi'(u'_n(t))\|_{L^2}^2 \, dt \right)^{1/2}
\]

\[
+ \left( \mathbb{E}' \int_0^T \|\phi\|_{L^4}^2 \, dt \right)^{1/2} \left( \mathbb{E}' \int_0^T \|u'(t)\|_{L^4}^4 \, dt \right)^{1/2} \left( \mathbb{E}' \int_0^T \|\pi_n \phi'(u'_n(t)) - \phi'(u'(t))\|_{H}^2 \, dt \right)^{1/2} \to 0.
\]

We need to prove why

\[
\mathbb{E}' \int_0^T \|\pi_n \phi'(u'_n(t)) - \phi'(u'(t))\|_H^2 \, dt \to 0
\]

This is because

\[
\left( \mathbb{E}' \int_0^T \|\pi_n \phi'(u'_n(t)) - \phi'(u'(t))\|_H^2 \, dt \right)^{1/2}
\]

\[
\leq \left( \mathbb{E}' \int_0^T \|\pi_n \phi'(u'_n(t)) - \pi_n \phi'(u'(t))\|_H^2 \, dt \right)^{1/2} + \left( \mathbb{E}' \int_0^T \|\pi_n \phi'(u'(t)) - \phi'(u'(t))\|_H^2 \, dt \right)^{1/2} \leq \cdots
\]

Since \( \phi' \) is global Lipschitz, there exists a constant \( C \) such that

\[
\cdots \leq C \left( \mathbb{E}' \int_0^T \|u'_n(t) - u'(t)\|_H^2 \, dt \right)^{1/2} + \left( \mathbb{E}' \int_0^T \|\pi_n \phi'(u'(t)) - \phi'(u'(t))\|_H^2 \, dt \right)^{1/2}.
\]

By (6.9), the first term on the right hand side of above inequality converges to 0. And since \( \|\pi_n \phi'(u'(t)) - \phi'(u'(t))\|_H^2 \to 0 \) for almost every \( (t, \omega) \in [0, T] \times \Omega \), and since \( \phi' \) is bounded, \( \|\pi_n \phi'(u'(t)) - \phi'(u'(t))\|_H^2 \) is uniformly integrable, hence the
second term of right hand side also converges to 0 as \( n \to \infty \). Therefore we have proved (6.21).

Hence we have
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times \pi_n \phi'(u'_n(t)), \psi \rangle_H dt = \mathbb{E}' \int_0^T \langle u'(t) \times \phi'(u'(t)), \psi \rangle_H dt.
\]
Therefore by the equalities (6.20) and (6.22), we have
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times [\Delta u'_n(t) - \pi_n \phi'(u'_n(t))], \psi(t) \rangle_{L^2} dt
= \mathbb{E}' \int_0^T \langle Y(t), \psi(t) \rangle_{L^2} dt, \quad \psi \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)).
\]
Hence by (6.23) and (6.24),
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle u'_n(t) \times [\Delta u'_n(t) - \pi_n \phi'(u'_n(t))], \psi(s) \rangle_H dt
= \mathbb{E}' \int_0^T \langle Y(t), \psi(t) \rangle_H dt
= \mathbb{E}' \int_0^T \sum_{i=1}^3 \left( \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i} \right)_H dt + \mathbb{E}' \int_0^T \langle u'(t) \times \phi'(u'(t)), \psi(t) \rangle_H dt.
\]
This completes the proof of Lemma 6.14. \( \square \)

**Lemma 6.15.** For any process \( \psi \in L^4(\Omega'; L^4(0, T; \mathbb{L}^4)) \) we have
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \int_{L^2} \frac{1}{3} \langle u'_n(s) \times (u'_n(s) \times [\Delta u'_n - \pi_n \phi'(u'_n(t))]), \psi(s) \rangle_{L^3} ds
= \mathbb{E}' \int_0^T \int_{L^2} \langle Z(s), \psi(s) \rangle_{L^3} ds
(6.25)
= \mathbb{E}' \int_0^T \int_{L^2} \langle u'(s) \times Y(s), \psi(s) \rangle_{L^3} ds
(6.26)
\]

**Proof:** Let us take \( \psi \in L^4(\Omega'; L^4(0, T; \mathbb{L}^4)) \). For \( n \in \mathbb{N} \), put \( Y_n := u'_n \times [\Delta u'_n + \phi'(u'_n)] \). \( L^4(\Omega'; L^4(0, T; \mathbb{L}^4)) \subset L^2(\Omega'; L^2(0, T; \mathbb{L}^3)) \). Hence (6.13) implies that (6.25) holds. So it remains to prove equality (6.26). Since by the Hölder’s inequality
\[
\|\psi \times u'\|^2_{L^2} = \int_D |\psi(x) \times u'(x)|^2 \, dx \leq \int_D |\psi(x)|^2 |u'(x)|^2 \, dx \leq \|\psi\|^2_{L^4} \|u'\|^2_{L^4} \leq \|\psi\|^4_{L^4} + \|u'\|^4_{L^4}.
\]
And since by (6.9), \( u' \in L^4(\Omega'; L^4(0, T; L^4)) \), we infer that
\[
\mathbb{E}' \int_0^T ||\psi \times u'||_{L^2}^2 \, dt \leq \mathbb{E}' \int_0^T ||\psi||_{L^4}^4 \, dt + \mathbb{E}' \int_0^T ||u'||_{L^4}^4 \, dt < \infty.
\]
This proves that \( \psi \times u' \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \) and similarly \( \psi \times u'_n \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \).

Thus since by (6.12), \( Y_n \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \), we infer that
\[
L^2 \langle u'_n \times Y_n, \psi \rangle_{L^2} = \int_D \langle u'_n(x) \times Y_n(x), \psi(x) \rangle \, dx
\]
(6.27)
\[
= \int_D \langle Y_n(x), \psi(x) \times u'_n(x) \rangle \, dx = \langle Y_n, \psi \times u'_n \rangle_{L^2}.
\]

Similarly, since by (6.12), \( Y \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \), we have
\[
L^2 \langle u' \times Y, \psi \rangle_{L^2} = \int_D \langle u'(x) \times Y(x), \psi(x) \rangle \, dx
\]
(6.28)
\[
= \int_D \langle Y(x), \psi(x) \times u'(x) \rangle \, dx = \langle Y, \psi \times u' \rangle_{L^2}.
\]

Thus by (6.27) and (6.28), we get
\[
L^2 \langle u'_n \times Y_n, \psi \rangle_{L^2} - L^2 \langle u' \times Y, \psi \rangle_{L^2} = \langle Y_n, \psi \times u'_n \rangle_{L^2} - \langle Y, \psi \times u' \rangle_{L^2}
\]
\[
= \langle Y_n - Y, \psi \times u' \rangle_{L^2} + \langle Y_n, \psi \times (u'_n - u') \rangle_{L^2}.
\]

In order to prove (6.26), we are aiming to prove that the expectation of the left hand side of the above equality goes to 0 as \( n \to \infty \). By (6.12), since \( \psi \times u' \in L^2(\Omega'; L^2(0, T; \mathbb{L}^2)) \),
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T \langle Y_n(s) - Y(s), \psi(s) \times u'(s) \rangle_{L^2} \, ds = 0.
\]

By the Cauchy-Schwartz inequality and the equation (6.9), we have
\[
\mathbb{E}' \left( \langle Y_n, \psi \times (u'_n - u') \rangle_{L^2}^2 \right) \leq \mathbb{E}' \left( \left| \int \left| Y_n \right|^2 \psi \times (u'_n - u') \right|^2 \right) \leq \mathbb{E}' \left( \left| \int \left| Y_n \right|^2 \psi \times (u'_n - u') \right|^2 \right) = \mathbb{E}' \left( \left| \int \left| Y_n \right|^2 \psi \times (u'_n - u') \right|^2 \right)
\]
\[
\leq \mathbb{E}' \left( \int_0^T \left| \int \psi(s) \right|_{L^2} \, ds \right) \leq \mathbb{E}' \left( \int_0^T \left| \int \psi(s) \right|_{L^2}^4 \, ds \right) \leq \mathbb{E}' \left( \int_0^T \left| \int u'_n(s) - u'(s) \right|_{L^2}^4 \, ds \right) \to 0.
\]

Therefore, we infer that
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^T L^2 \langle u'_n(s) \times (u'_n(s) \times \Delta u'_n(s)), \psi(s) \rangle_{L^2} \, ds = \mathbb{E}' \int_0^T L^2 \langle u'(s) \times Y(s), \psi(s) \rangle_{L^2} \, ds.
\]

This completes the proof of the Lemma 6.15.

The next result will be used to show that the process \( u' \) satisfies the condition \( \|u'(t, x)\|_{\mathbb{L}^2} = 1 \) for all \( t \in [0, T] \), \( x \in D \) and \( \mathbb{P}' \)-almost surely.
Lemma 6.16. For any bounded measurable function \( \psi : D \rightarrow \mathbb{R} \) we have

\[
\langle Y(s, \omega), \psi u'(s, \omega) \rangle_H = 0,
\]

for almost every \((s, \omega) \in [0, T] \times \Omega'\).

Proof. Let \( B \subset [0, T] \times \Omega' \) be a measurable set.

\[
\begin{align*}
\left| \mathbb{E}' \int_0^T 1_B(s) \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))], \psi u'_n(s) \rangle_H \, ds - \mathbb{E}' \int_0^T 1_B(s) \langle Y(s), \psi u'(s) \rangle_H \, ds \right| \\
= \left| \mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'_n(s) \rangle_H \, ds + \langle Y(s), \psi u'_n(s) - \psi u'(s) \rangle_H \, ds \right| \\
\leq \left| \mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'_n(s) \rangle_H \, ds \right| + \mathbb{E}' \int_0^T \langle Y(s), \psi u'_n(s) - \psi u'(s) \rangle_H \, ds.
\end{align*}
\]

\(\psi u'(s) \in H\), by (6.12) and (6.3) and (ii) of Proposition 6.2, we have

\[
\left| \mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'_n(s) \rangle_H \, ds \right| \\
\leq \left| \mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'(s) \rangle_H \, ds \right| \\
+ \left| \mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi [u'(s) - u'_n(s)] \rangle_H \, ds \right| \rightarrow 0.
\]

And since \( \psi \) is bounded and \( \mathbb{L}^4 \hookrightarrow \mathbb{L}^2 \), by (6.9), we have

\[
\left| \mathbb{E} \int_0^T \langle Y(s), \psi u'_n(s) - \psi u'(s) \rangle_H \, ds \right| \leq \mathbb{E}' \int_0^T \left| \psi u'_n(s) - \psi u'(s) \right|_H^2 \, ds \\
\leq \left( \mathbb{E}' \int_0^T |Y(s)|_H^2 \, ds \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \left| \psi u'_n(s) - \psi u'(s) \right|_H^2 \, ds \right)^{\frac{1}{2}} \\
\leq \left( \mathbb{E}' \int_0^T |Y(s)|_H^2 \, ds \right)^{\frac{1}{2}} C_1 \left( \mathbb{E}' \int_0^T \left| u'_n(s) - u'(s) \right|_{\mathbb{L}^4}^4 \, ds \right)^{\frac{1}{4}} \left( \mathbb{E}' \int_0^T 1^4 \, ds \right)^{\frac{3}{4}} \\
\leq C \left( \mathbb{E}' \int_0^T |Y(s)|_H^2 \, ds \right)^{\frac{1}{2}} \left( \mathbb{E}' \int_0^T \left| u'_n(s) - u'(s) \right|_{\mathbb{L}^4}^4 \, ds \right)^{\frac{1}{4}} \rightarrow 0.
\]

Therefore

\[
0 = \lim_{n \to \infty} \mathbb{E}' \int_0^T 1_B(s) \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))], \psi u'_n(s) \rangle_H \, ds \\
= \mathbb{E}' \int_0^T 1_B(s) \langle Y(s), \psi u'(s) \rangle_H \, ds.
\]

This concludes the proof of Lemma 6.16. \( \square \)
7. Conclusion of the proof of the existence of a weak solution

Our aim in this section is to prove that the process $u'$ from Proposition 6.2 is a weak solution of the equation (2.1) according to the definition 2.4. We define a sequence of $H$-valued process $(M_n(t))_{t \in [0,T]}$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$M_n(t) := u_n(t) - u_n(0) - \lambda_1 \int_0^t \pi_n(u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))]) \, ds$$

(7.1) $$\quad + \lambda_2 \int_0^t \pi_n(u_n(s) \times (u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))])) \, ds$$

$$- \frac{1}{2} \sum_{j=1}^N \int_0^t \pi_n[(\pi_n(u_n(s) \times h_j)) \times h_j] \, ds.$$ 

Since $u_n$ is the solution of the Equation (3.5), we have

$$u_n(t) = u_n(0) + \lambda_1 \int_0^t \pi_n(u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))]) \, ds$$

$$- \lambda_2 \int_0^t \pi_n(u_n(s) \times (u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))])) \, ds$$

$$+ \frac{1}{2} \sum_{j=1}^N \int_0^t \pi_n(\pi_n(u_n(s) \times h_j) \times h_j) \, ds + \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j) \, dW_j(s).$$

Hence we have

$$M_n(t) = \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j) \, dW_j(s), \quad t \in [0,T].$$

It will be 2 steps to prove $u'$ is a weak solution of the Equation (2.1):

Step 1 : we are going to find some $M'(t)$ defined similar as in (7.1), but with $u'$ instead of $u_n$.

Step 2 : We will show the similar result as in (7.2) but with $u'$ instead of $u_n$ and $W'_j$ instead of $W_j$.

7.1. **Step 1.** We define a sequence of $H$-valued process $(M'_n(t))_{t \in [0,T]}$ on the new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ by a formula similar as (7.1).

$$M'_n(t) := u'_n(t) - u'_n(0) - \lambda_1 \int_0^t \pi_n(u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))]) \, ds$$

(7.3) $$\quad + \lambda_2 \int_0^t \pi_n(u'_n(s) \times (u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))])) \, ds$$

$$- \frac{1}{2} \sum_{j=1}^N \int_0^t \pi_n[(\pi_n(u'_n(s) \times h_j)) \times h_j] \, ds.$$
It will be natural to ask if \( \{M_n^t\} \) has limit and if yes, what is the limit. The next result answers this question.

**Lemma 7.1.** For each \( t \in [0, T] \) the sequence of random variables \( M_n^t(t) \) converges weakly as \( n \to \infty \) in \( L^2(\Omega'; X^{-\beta}) \) to the limit

\[
M'(t) := u'(t) - u_0 - \lambda_1 \int_0^t (u'(s) \times [\Delta u'(s) - \phi'(u'(s))]) \, ds \\
+ \lambda_2 \int_0^t (u'(s) \times (u'(s) \times [\Delta u'(s) - \phi'(u'(s))]) \, ds \\
- \frac{1}{2} \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \times h \, ds.
\]

**Proof:** The dual space of \( L^2(\Omega'; X^{-\beta}) \) is \( L^2(\Omega'; X^{\beta}) \). Let \( t \in (0, T] \) and \( U \in L^2(\Omega'; X^{\beta}) \). We have

\[
\mathbb{E}'[X^{-\beta}\langle M_n^t(t), U \rangle_{X^{\beta}}] = E^t[X^{-\beta}\langle M_n^t(t), U \rangle_{X^{\beta}}] \\
= E^t[X^{-\beta}\langle u_n(t), U \rangle_{X^{\beta}} - X^{-\beta}\langle u_0(t), U \rangle_{X^{\beta}} \\
- \lambda_1 \int_0^t \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))] , \pi_n U \rangle_{L^2} \, ds \\
+ \lambda_2 \int_0^t X^{-\beta} \langle (u_n'(s) \times (u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))]) , U \rangle_{X^{\beta}} \, ds \\
- \frac{1}{2} \sum_{j=1}^N \int_0^t \langle \pi_n (u_n'(s) \times h_j) \times h_j , \pi_n U \rangle_{L^2} \, ds.
\]

We know that \( u_n' \to u' \) in \( C(0, T; X^{-\beta}) \) \( \mathbb{P}' \)-a.s., so

\[
\sup_{t \in (0, T]} \| u_n(t) - u(t) \|_{X^{-\beta}} \to 0, \quad \mathbb{P}' \text{-a.s.}
\]

so \( u_n'(t) \to u'(t) \) in \( X^{-\beta} \) \( \mathbb{P}' \)-almost surely for any \( t \in (0, T] \). And \( X^{-\beta}\langle \cdot, U \rangle_{X^{\beta}} \) is a continuous function on \( X^{-\beta} \), hence

\[
\lim_{n \to \infty} X^{-\beta}\langle u_n'(t), U \rangle_{X^{\beta}} = X^{-\beta}\langle u'(t), U \rangle_{X^{\beta}}, \quad \mathbb{P}' \text{-a.s.}
\]

By (6.1), \( \sup_{t \in [0, T]} \| u_n'(t) \|_H \leq \| u_0 \|_H \), since \( H \hookrightarrow X^{-\beta} \) continuously, we can find a constant \( C \) such that

\[
\sup_n \mathbb{E}' \left[ \| X^{-\beta}\langle u_n'(t), U \rangle_{X^{\beta}} \|_2^2 \right] \leq \sup_n \mathbb{E}' [\| U \|_{X^{\beta}}^2 \mathbb{E}' [\| u_n'(t) \|_{X^{-\beta}}^2] \\
\leq C \mathbb{E}' [\| U \|_{X^{\beta}}^2] \sup_n \mathbb{E}' [\| u_n'(t) \|_H^2] \leq C \mathbb{E}' [\| U \|_{X^{\beta}}^2] \mathbb{E}' [\| u_0 \|_H^2] < \infty.
\]

Hence the sequence \( X^{-\beta}\langle u_n'(t), U \rangle_{X^{\beta}} \) is uniformly integrable. So the almost surely convergence and uniform integrability implies that

\[
\lim_{n \to \infty} \mathbb{E}'[X^{-\beta}\langle u_n'(t), U \rangle_{X^{\beta}}] = \mathbb{E}'[X^{-\beta}\langle u'(t), U \rangle_{X^{\beta}}].
\]
By (6.12),
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^t \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))], \pi_n U \rangle_{L^2} \, ds = \mathbb{E}' \int_0^t \langle Y(s), U \rangle_{L^2}.
\]
By (6.14)
\[
\lim_{n \to \infty} \mathbb{E}' \int_0^t \int_{X^\#} \langle \pi_n (u_n'(s) \times (u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))]), U \rangle_{X^\#} \, ds = \mathbb{E}' \int_0^t \langle Z(s), U \rangle_{X^\#} \, ds.
\]
By the Hölder’s inequality,
\[
\|u_n'(t) - u'(t)\|^2_{L^2} \leq m(D)^{\frac{1}{2}} \|u_n'(t) - u'(t)\|^2_{L^1}.
\]
Hence by (6.9),
\[
\mathbb{E}' \int_0^t \langle \pi_n \langle (u_n'(s) - u'(s)) \times h \rangle \times h, \pi_n U \rangle_{L^2} \, ds
\leq \|U\|_{L^2(\Omega; L^2(0,T; L^2))} \left( \mathbb{E}' \int_0^t \|\pi_n \langle (u_n'(s) - u'(s)) \times h \rangle \times h \|_{L^2}^2 \, ds \right)^{\frac{1}{2}}
\leq \|h\|_{L^2(\Omega; L^2(0,T; L^2))} \mathbb{E}' \int_0^t \|u_n'(s) - u'(s)\|_{L^2}^2 \, ds^{\frac{1}{2}}
\leq \|h\|_{L^2(\Omega; L^2(0,T; L^2))} \left( \mathbb{E}' \int_0^t \|u_n'(s) - u'(s)\|_{L^2}^4 \, ds \right)^{\frac{1}{4}}
\to 0.
\]
The last “≤” is from the Jensen’s inequality.
Hence
\[
\lim_{n \to \infty} L^2(\Omega; X^\#) \langle M_n'(t), U \rangle_{L^2(\Omega; X^\#)}
\]
\[
= \mathbb{E}' \left[ X^\# \langle u'(t), U \rangle_{X^\#} - X^\# \langle u_0, U \rangle_{X^\#} - \lambda_1 \int_0^t \langle Y(s), U \rangle_{L^2} \, ds \right.
\]
\[
+ \lambda_2 \int_0^t \langle Z(s), U \rangle_{X^\#} \, ds - \frac{1}{2} \sum_{j=1}^N \int_0^t \langle (u'(s) \times h_j) \times h_j, U \rangle_{L^2} \, ds \right].
\]
Since by Lemma 6.14 and Lemma 6.15, we have \( Y = u' \times \Delta u' \) and \( Z = u' \times (u' \times \Delta u') \). Therefore for any \( U \in L^2(\Omega; X^\#) \),
\[
\lim_{n \to \infty} L^2(\Omega; X^\#) \langle M_n'(t), U \rangle_{L^2(\Omega; X^\#)} = L^2(\Omega; X^\#) \langle M'(t), U \rangle_{L^2(\Omega; X^\#)}.
\]
This concludes the proof of Lemma 7.1. \( \Box \)

Before we can continue to prove \( u' \) is the weak solution of equation (2.1), we need to show that the \( W' \) and \( W_n' \) in Proposition 6.2 are Brownian Motions. And that will be done in Lemma ?? and Lemma 7.2, which can be proved by considering the characteristic functions. And we will only show the proof of Lemma 7.2.
Lemma 7.2. The process \((W'(t))_{t \in [0,T]}\) is a real-valued Brownian Motion on \((\Omega', F', \mathbb{P}')\) and if \(0 \leq s < t \leq T\) then the increment \(W'(t) - W'(s)\) is independent of the \(\sigma\)-algebra generated by \(u'(r)\) and \(W'(r)\) for \(r \in [0, s]\).

Proof: We consider the characteristic functions of \(W'\). Let \(k \in \mathbb{N}\) and \(0 = s_0 < s_1 < \cdots < s_k \leq T\). For \((t_1, \ldots, t_k) \in \mathbb{R}^k\), we have for each \(n \in \mathbb{N}\):

\[
\mathbb{E}' \left[ e^{i \sum_{j=1}^{k} t_j (W_n'(s_j) - W_n'(s_{j-1}))} \right] = e^{-\frac{1}{2} \sum_{j=1}^{k} t_j^2 (s_j - s_{j-1})}.
\]

Notice that \(\left| e^{i \sum_{j=1}^{k} t_j (W_n'(s_j) - W_n'(s_{j-1}))} \right| \leq 1\), by the Lebesgue’s dominated convergence theorem.

\[
\mathbb{E}' \left[ e^{i \sum_{j=1}^{k} t_j (W_n'(s_j) - W_n'(s_{j-1}))} \right] = \lim_{n \to \infty} \mathbb{E}' \left[ e^{i \sum_{j=1}^{k} t_j (W_n'(s_j) - W_n'(s_{j-1}))} \right] = e^{-\frac{1}{2} \sum_{j=1}^{k} t_j^2 (s_j - s_{j-1})}.
\]

Hence \(W'(t)\) has the same distribution with \(W_n'(t)\) for \(t \in [0, T]\). Since random variables are independent if and only if the characteristic function of the sum of them equals to the multiplication of their characteristic functions, and here we have

\[
\mathbb{E}' \left[ e^{i \sum_{j=1}^{k} t_j (W(s_j) - W(s_{j-1}))} \right] = \prod_{j=1}^{k} \mathbb{E}' \left[ e^{i t_j (W(s_j) - W(s_{j-1}))} \right].
\]

Hence \(W'\) has independent increments. And

\[ W'(0) = \lim_{n \to \infty} W_n'(0) = 0, \quad \mathbb{P}' - a.s., \]

so \((W'(t))_{t \in [0,T]}\) is a real-valued Brownian motion on \((\Omega', F', \mathbb{P}')\).

The law of \((u_n, W)\) is same as \((u_n', W_n')\) and if \(t > s \geq r\), \(W(t) - W(s)\) is independent of \(u_n(r)\), so as the same method as before we can see \(W_n'(t) - W_n'(s)\) is independent of \(u_n'(r)\) for all \(n\). By Proposition 6.2, \(\lim_{n \to \infty} |u_n'(r)||\gamma_V| = |u'(r)||\gamma_V|\) and \(\lim_{n \to \infty} (W_n'(t) - W_n'(s)) = W'(t) - W'(s)\), hence by the Lebesgue’s dominated convergence theorem we have

\[
\mathbb{E}' \left( e^{i \alpha (|u'(r)||\gamma_V| + W'(t) - W'(s))} \right) = \lim_{n \to \infty} \mathbb{E}' \left( e^{i \alpha (|u_n'(r)||\gamma_V| + W_n'(t) - W_n'(s))} \right)
\]

= \lim_{n \to \infty} \mathbb{E}' \left( e^{i \alpha (|u'(r)||\gamma_V|)} \right) \mathbb{E}' \left( e^{i \alpha (W_n'(t) - W_n'(s))} \right) = \mathbb{E}' \left( e^{i \alpha (|u'(r)||\gamma_V|)} \right) \mathbb{E}' \left( e^{i \alpha (W(t) - W(s))} \right).
\]

So \(W'(t) - W'(s)\) is independent of \(u'(r)\). Hence this completes the proof of Lemma 7.2. \(\Box\)

Remark 7.3. We will denote \(\mathbb{F}'\) the filtration generated by \((u', W')\) and \(\mathbb{F}_n'\) the filtration generated by \((u_n', W_n')\). Then by Lemma 7.2, \(u'\) is progressively measurable with respect to \(\mathbb{F}'\) and by Lemma ??, \(u_n'\) is progressively measurable with respect to \(\mathbb{F}_n'\).

7.2. Step 2. Let us summarize what we have achieved so far. We have got our process \(M'\) and have shown \(W'\) is a Wiener process. Next we will show a similar result as in equation (7.2) to prove \(u'\) is a weak solution of the Equation (2.1).

But before that we still need some preparation. The following estimate will be used to prove Lemma 7.5.
Proposition 7.4. For every \( h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3} \), there exists \( c = c(h, \beta) > 0 \) such that for every \( u \in \mathbb{L}^2 \), we have

\[
(7.4) \quad \|u \times h\|_{X^{-\beta}} \leq c\|u\|_{X^{-\beta}} < \infty.
\]

Proof. Let \( z \in \mathbb{H}^1 \), \( h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3} \). Then

\[
|\|z \times h\|_{L^2}^2 = |\|\nabla(z \times h)\|_{L^2}^2 + |\|z \times h\|_{L^2}^2 |
\leq 2(|\|\nabla(z \times h)\|_{L^2}^2 + |\|z \times h\|_{L^2}^2 |
\leq 2(|\|h\|_{L^\infty}^2 |\|\nabla z\|_{L^2}^2 + |\|\nabla h\|_{L^2}^2 |\|z\|_{L^\infty}^2) + |\|h\|_{L^\infty}^2 |\|z\|_{L^\infty}^2|
\leq 2(|\|h\|_{L^\infty}^2 + c^2 |\|\nabla h\|_{L^2}^2 |\|z\|_{L^\infty}^2),
\]

so the map

\[
\mathbb{H}^1 \ni z \mapsto z \times h \in \mathbb{H}^1
\]
is linear and bounded. And so for \( u \in \mathbb{L}^2 \), \( z \in \mathbb{X}^\beta \),

\[
|\|X^{-\beta}(u \times h, z)\|_{X^\beta} | = |\|X^{-\beta}(u, z \times h)\|_{X^\beta} | \leq \sqrt{2(|\|h\|_{L^\infty}^2 + c^2 |\|\nabla h\|_{L^2}^2 |\|z\|_{L^\infty}^2)} |\|z\|_{X^\beta} | |\|u\|_{X^{-\beta}} < \infty,
\]

let \( c_h = \sqrt{2(|\|h\|_{L^\infty}^2 + c^2 |\|\nabla h\|_{L^2}^2)} \), we have get

\[
\|u \times h\|_{X^{-\beta}} \leq c_h |\|u\|_{X^{-\beta}} < \infty, \quad u \in \mathbb{L}^2.
\]

This completes the proof of Proposition 7.4. \( \square \)

Lemma 7.5. For each \( m \in \mathbb{N} \), we define the partition \( \{s_i^m := \frac{i}{m}, \; i = 0, \ldots, m\} \) of \( [0, T] \). Then for any \( \varepsilon > 0 \), there exists \( m_0(\varepsilon) \in \mathbb{N} \) such that for all \( m \geq m_0(\varepsilon) \), we have:

(i)

\[
\lim_{n \to \infty} \left( E' \left| \sum_{j=1}^n \int_0^t \left( \pi_n(u_j' (s) \times h_j) - \pi_n(u_j' (s_j^m) \times h_j) 1_{(s_j^m, s_{j+1}^m]}(s) \right) dW_j(s) \right|_{X^{-\beta}}^2 \right)^\frac{1}{2} < \frac{\varepsilon}{2}.
\]

(ii)

\[
\lim_{n \to \infty} \left| \sum_{i=0}^{m-1} \sum_{j=1}^N \pi_n(u_j' (s_j^m) \times h_j) (W_j' (t \wedge s_{j+1}^m) - W_j' (t \wedge s_j^m)) 
- \sum_{i=0}^{m-1} \sum_{j=1}^N \pi_n(u_j' (s_j^m) \times h_j) (W_j' (t \wedge s_{j+1}^m) - W_j' (t \wedge s_j^m)) \right|_{X^{-\beta}}^2 = 0;
\]

(iii)

\[
\lim_{n \to \infty} \left( E' \left| \sum_{j=1}^n \int_0^t \left( \pi_n(u_j' (s) \times h_j) - \pi_n(u_j' (s_j^m) \times h_j) 1_{(s_j^m, s_{j+1}^m]}(s) \right) dW_j(s) \right|_{X^{-\beta}}^2 \right)^\frac{1}{2} < \frac{\varepsilon}{2};
\]

(iv)

\[
\lim_{n \to \infty} \left| \sum_{j=1}^N \int_0^t (\pi_n(u_j' (s) \times h_j) - (u_j' (s) \times h_j)) dW_j'(s) \right|_{X^{-\beta}}^2 = 0.
\]
Now we are ready to state the Theorem which means that \( u' \) is the weak solution of the equation (2.1).

**Theorem 7.6.** For each \( t \in [0, T] \) we have \( M'(t) = \sum_{j=1}^{N} \int_{0}^{t} (u'(s) \times h_j) \, dW'_j(s) \).

**Proof.** Step 1: We will show that

\[
M'_n(t) = \sum_{j=1}^{N} \int_{0}^{t} \pi_n(u'_n(s) \times h_j) \, dW'_{jn}(s)
\]

\( \mathbb{P}' \) almost surely for each \( t \in [0, T] \) and \( n \in \mathbb{N} \).

Let us fix that \( t \in [0, T] \) and \( n \in \mathbb{N} \). Let us also fix \( m \in \mathbb{N} \) and define the partition \( \{s_i^m := \frac{j}{m}, i = 0, \ldots, m\} \) of \([0, T]\). Let us recall that \((u'_n, W'_n)\) and \((u_n, W)\) have the same laws on the separable Banach space \( C([0, T]; H_n) \times C([0, T]; \mathbb{R}^N) \). Since the map

\[
\Psi : \ C([0, T]; H_n) \times C([0, T]; \mathbb{R}^N) \rightarrow H_n
\]

\((u_n, W) \mapsto M_n(t) - \sum_{j=0}^{m-1} \sum_{j=1}^{N} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_i^m) - W_j(t \wedge s_i^m))
\]

is continuous and so measurable. By involving the Kuratowski Theorem we infer that the \( H \)-valued random variables:

\[
M_n(t) - \sum_{j=0}^{m-1} \sum_{j=1}^{N} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_i^m) - W_j(t \wedge s_i^m))
\]

and

\[
M'_n(t) - \sum_{j=0}^{m-1} \sum_{j=1}^{N} \pi_n(u'_n(s_i^m) \times h_j)(W'_jn(t \wedge s_i^m) - W'_jn(t \wedge s_i^m))
\]

have the same laws. Let us denote \( \tilde{u}_n := \sum_{i=0}^{m-1} u_n(s_i^m)1_{[s_i^m, s_i^m)} \). By the Itô isometry, we have

\[
\left\| \sum_{i=0}^{m-1} \pi_n(u_n(s_i^m) \times h_j)(W_j(t \wedge s_i^m) - W_j(t \wedge s_i^m)) - \int_{0}^{t} \pi_n(u_n(s) \times h_j) \, dW_j(s) \right\|_{L^2(\Omega; H)}^2
\]

\[
= \mathbb{E} \left\| \int_{0}^{t} [\pi_n(\tilde{u}_n \times h_j) - \pi_n(u_n(s) \times h_j)] \, dW_j(s) \right\|^2_H \leq \|h_j\|^2_{L^\infty(\mathbb{D})} \mathbb{E} \int_{0}^{t} \|\tilde{u}_n(s) - u_n(s)\|^2_H \, ds.
\]

Since \( u_n \in C([0, T]; H_n) \) \( \mathbb{P} \)-almost surely, we have

\[
\lim_{n \to \infty} \int_{0}^{t} \|\tilde{u}_n(s) - u_n(s)\|^2_H \, ds = 0, \quad \mathbb{P} \text{-a.s.}
\]

Moreover by the equality (4.1), we infer that

\[
\sup_n \mathbb{E} \left\| \int_{0}^{t} [\tilde{u}_n(s) - u_n(s)]^2_H \, ds \right\|^2 \leq \sup_n \mathbb{E} \left\| \int_{0}^{t} [2(\tilde{u}_n(s))^2_H + 2(u_n(s))^2_H] \, ds \right\|^2 \\
\leq \mathbb{E} [4(u_0)^2_{H}]^2 T^2 = 16(u_0)^4_{H} T^2 < \infty.
\]
So \( \int_0^t ||\tilde{u}_n(s) - u_n(s)||_H^2 \, ds \) is uniformly integrable. Therefore, we have

\[
\lim_{m \to \infty} \left\| \sum_{i=0}^{m-1} \pi_n(u_n(s_i) \times h_j)(W_j(t \land s_{i+1}) - W_j(t \land s_i)) - \int_0^t \pi_n(u_n(s) \times h_j) \, dW_j(s) \right\|_{L^2(\Omega; H)}^2 = 0.
\]

Similarly, because \( u'_n \) satisfies the same conditions as \( u_n \), we also get

\[
\lim_{m \to \infty} \left\| \sum_{i=0}^{m-1} \pi_n(u'_n(s_i) \times h_j)(W'_j(t \land s_{i+1}) - W'_j(t \land s_i)) - \int_0^t \pi_n(u'_n(s) \times h_j) \, dW'_j(s) \right\|_{L^2(\Omega; H)}^2 = 0.
\]

Hence, since the \( L^2 \) convergence implies the weak convergence, we infer that the random variables \( M_n(t) - \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j) \, dW_j(s) \) and \( M'_n(t) - \sum_{j=1}^N \int_0^t \pi_n(u'_n(s) \times h_j) \, dW'_j(s) \) have same laws. But \( M_n(t) - \sum_{j=1}^N \int_0^t \pi_n(u_n(s) \times h_j) \, dW_j(s) = 0 \) \( \mathbb{P} \)-almost surely, so (7.5) follows.

Step 2: From Lemma 7.5 and the Step 1, we infer that \( M'_n(t) \) converges in \( L^2(\Omega'; X^{-\beta}) \) to \( \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \, dW'_j(s) \) as \( n \to \infty \).

This completes the proof of Theorem 7.6. \( \square \)

Summarizing, it follows from Theorem 7.6 that the process \( u' \) satisfies the following equation in \( L^2(\Omega'; X^{-\beta}) \) for \( t \in [0, T] \):

\[
u'(t) = u_0 + \lambda_1 \int_0^t (u' \times [\Delta u' - \phi'(u')]) \, ds \]
\[-\lambda_2 \int_0^t (u' \times [\Delta u' - \phi'(u')]) \, ds \]
\[+ \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \, dW'_j(s).
\]

Hence by Definition 2.4, \( u' \) is a weak solution of Equation (2.1).

8. Regularities of the weak solution

Now we will start to show some regularity of \( u' \).

**Theorem 8.1.** The process \( u' \) from Proposition 6.2 satisfies:

\[
|u'(t, x)|_{L^3} = 1, \quad \text{for Lebesgue a.e. } (t, x) \in [0, T] \times D \quad \text{and } \mathbb{P}' - a.s.
\]

To prove Theorem 8.1, we need the following Lemma:

**Lemma 8.2.** ([38], Th. 1.2) Let \( (\Omega, (\mathcal{F}_t), \mathbb{P}) \) be a filtered probability space and let \( V \) and \( H \) be two separable Hilbert spaces, such that \( V \leftrightarrow H \) continuously and densely. We identify \( H \) with its dual space and have a Gelfand triple: \( V \leftrightarrow H \equiv H' \leftrightarrow V' \). We assume that

\[
u \in M^2(0, T; V), \quad u_0 \in H, \quad v \in M^2(0, T; V'), \quad z_j \in M^2(0, T; H),
\]
for every \( t \in [0, T] \).

\[
    u(t) = u_0 + \int_0^t v(s) \, ds + \sum_{j=1}^N \int_0^t z_j(s) \, dW_j(s), \quad \mathbb{P} - a.s.
\]

Let \( \psi \) be a twice differentiable functional on \( H \), which satisfies:

(i) \( \psi, \psi' \) and \( \psi'' \) are locally bounded.

(ii) \( \psi \) and \( \psi' \) are continuous on \( H \).

(iii) Let \( \mathcal{L}^1(H) \) be the Banach space of all the trace class operators on \( H \). Then

\( \forall Q \in \mathcal{L}^1(H), \ Tr(Q \circ \psi'') \) is a continuous functional on \( H \).

(iv) If \( u \in V, \psi'(u) \in V; u \mapsto \psi'(u) \) is continuous from \( V \) (with the strong topology) into \( V \) endowed with the weak topology.

(v) \( \exists k \) such that \( \|\psi''(u)\|_V \leq k(1 + \|u\|_V), \forall u \in V \).

Then for every \( t \in [0, T] \),

\[
    \psi(u(t)) = \psi(u_0) + \int_0^t \langle v(s), \psi'(u(s)) \rangle_V \, ds + \sum_{j=1}^N \int_0^t \langle \psi'(u(s)), z_j(s) \rangle_H \, dW_j(s)
\]

\[
    + \frac{1}{2} \sum_{j=1}^N \int_0^t \langle \psi''(u(s)) z_j(s), z_j(s) \rangle_H \, ds, \quad \mathbb{P} - a.s.
\]

Proof of Theorem 8.1. Let \( \xi \in C_0^\infty(D, \mathbb{R}) \). Then we consider a function

\( \psi : H \ni u \mapsto \langle u, \xi u \rangle_H \in \mathbb{R} \).

It’s easy to see that \( \psi \) is of \( C^2 \) class and \( \psi'(u) = 2\xi u, \psi''(u)(v) = 2\xi v, \forall u, v \in H \).

Next we will check the assumptions of Lemma 8.2. By previous work (see details below), \( \psi' \) satisfies:

\[
    \mathbb{E}' \int_0^T \|u'(t)\|^2_V \, dt < \infty, \quad \text{by (6.8)},
\]

\[
    \mathbb{E}' \int_0^T \|u'(t) \times [\Delta u' - \phi'(u')]\|_{X^\beta}^2 \, dt < \infty, \quad \text{by (6.12)},
\]

\[
    \mathbb{E}' \int_0^T \|u'(t) \times (u' \times [\Delta u' - \phi'(u')]\)(t)\|_{X^\beta}^2 \, dt < \infty, \quad \text{by (6.14)},
\]

\[
    \mathbb{E}' \int_0^T \|(u'(s) \times h_j) \times h_j\|_{X^\beta}^2 \, dt < \infty, \quad \text{by (6.6)},
\]

\[
    \mathbb{E}' \int_0^T \|u'(s) \times h_j\|_H^2 \, dt < \infty, \quad \text{by (6.6)}.
\]

And \( \psi \) satisfies:

(i) \( \psi, \psi', \psi'' \) are locally bounded.

(ii) Since \( \psi', \psi'' \) exist, \( \psi, \psi' \) are continuous on \( H \).
(iii) \( \forall Q \in \mathcal{L}^1(H) \),

\[
Tr[Q \circ \psi''(a)] = \sum_{j=1}^{\infty} \langle Q \circ \psi''(a)e_j, e_j \rangle_H = 2 \sum_{j=1}^{\infty} \langle Q(e_j), e_j \rangle_H,
\]

which is a constant in \( \mathbb{R} \), so the map \( H \ni a \mapsto Tr[Q \circ \psi''(a)] \in H \) is a continuous functional on \( H \).

(iv) If \( u \in V \), \( \psi'(u) \in V \); \( u \mapsto \psi'(u) \) is continuous from \( V \) (with the strong topology) into \( V \) endowed with the weak topology.

This is because: For any \( v^* \in X^{\beta'} \), we have

\[
\chi_{\beta'}(\psi'(u + v) - \psi'(u), v^*)_{X^{\beta'}} = \chi_{\beta'}(2\phi v, v^*)_{X^{\beta'}} \\
\leq 2\|\xi\|C(D,\mathbb{E},X^{\beta'})\|v\|_{X^{\beta'}} \leq 2\|\xi\|_{C(D)} \|v\|_{X^{\beta'}} ,
\]

hence \( \psi' \) is weakly continuous. Let us denote \( \tau \) as the strong topology of \( V \) and \( \tau_w \) the weak topology of \( V \). Take \( B \in \tau_w \), by the weak continuity \( (\psi')^{-1}(B) \in \tau_w \), but \( \tau_w \subset \tau \). Hence \( (\psi')^{-1}(B) \in \tau \), which implies (iv).

(v) \( \exists k = 2\|\xi\|_{C(D)} \) such that

\[
||\psi'(u)||_V = 2\|\xi u||_V \leq k(1 + ||u||_V), \quad \forall u \in V.
\]

Hence by Lemma 8.2, we have that for \( t \in [0, T] \), \( \mathbb{P}' \) almost surely,

\[
\langle u'(t), \xi u'(t) \rangle_H - \langle u_0, \xi u_0 \rangle_H \\
= \sum_{j=1}^{N} \int_0^{t} \chi_{\beta'}(\lambda_1 (u' \times [\Delta u' - \phi'(u')]) (s) - \lambda_2 u'(s) \times (u' \times [\Delta u' \\
+ \phi'(u')])(s) + \frac{1}{2}(u'(s) \times h_j) \times h_j, 2\xi u'(s)\rangle_{X^{\beta'}} ds \\
+ \sum_{j=1}^{N} \int_0^{t} \langle 2\xi u'(s), u'(s) \times h_j \rangle_H d\mathbb{W}'(s) + \sum_{j=1}^{N} \int_0^{t} \langle \xi u'(s) \times h_j, u'(s) \times h_j \rangle_H ds.
\]

By Lemma 6.16,

\[
\chi_{\beta'}(\lambda_1 (u' \times [\Delta u' - \phi'(u')]) (s), 2\xi u'(s))_{X^{\beta'}} = 0.
\]

And since

\[
\chi_{\beta'}(\lambda_2 u'(s) \times (u' \times [\Delta u' - \phi'(u')]) (s), 2\xi u'(s))_{X^{\beta'}} = 0,
\]

\[
\chi_{\beta'}((u'(s) \times h_j) \times h_j, \xi u'(s))_{X^{\beta'}} = -\chi_{\beta'}(u'(s) \times h_j, \xi u'(s) \times h_j)_{X^{\beta'}},
\]

\[
\langle 2\xi u'(s), u'(s) \times h_j \rangle_H = 0,
\]

we have

\[
\langle u'(t), \xi u'(t) \rangle_H - \langle u_0, \xi u_0 \rangle_H = 0, \quad \mathbb{P}' \text{ - a.s.}
\]

Since \( \phi \) is arbitrary and \( |u_0(x)| = 1 \) for almost every \( x \in D \), we infer that \( |u'(t, x)| = 1 \) for almost every \( x \in D \) as well. This completes the proof of Theorem 8.1. \( \square \)

By Theorem 8.1, we can show that:
Theorem 8.3. The process \( u' \) from Proposition 6.2 satisfies: for every \( t \in [0, T] \),

\[
(8.2) \quad u'(t) = u_0 + \lambda_1 \int_0^t (u' \times [\Delta u' - \phi'(u')]) (s) \, ds
- \lambda_2 \int_0^t u'(s) \times (u' \times [\Delta u' - \phi'(u')]) (s) \, ds
+ \sum_{j=1}^N \int_0^t (u'(s) \times h_j) \circ dW_j(s)
\]

in \( L^2(\Omega'; H) \).

Proof. By (6.12) and Lemma 6.14,

\[
(8.3) \quad \mathbb{E}' \left( \int_0^T \left\| (u' \times [\Delta u' - \phi'(u')]) (t) \right\|_H^2 \, dt \right)^r < \infty, \quad r \geq 1.
\]

And then by (8.1), we see that

\[
(8.4) \quad \|u'(t, \omega) \times ((u' \times [\Delta u' - \phi'(u')]) (t, \omega)) \|_H \leq \| (u' \times [\Delta u' - \phi'(u')]) (t, \omega) \|_H
\]

for almost every \((t, \omega) \in [0, T] \times \Omega' \). And so

\[
\mathbb{E}' \int_0^T \left\| u'(t) \times (u' \times [\Delta u' - \phi'(u')]) (t) \right\|_H^2 \, dt < \infty.
\]

Therefore all the terms in the equation (8.2) are in the space \( L^2(\Omega'; H) \). This completes the proof of Theorem 8.3.

\[\square\]

Theorem 8.4. The process \( u' \) defined in Proposition 6.2 satisfies: for every \( \alpha \in (0, \frac{1}{2}) \),

\[
(8.5) \quad u' \in C^\alpha([0, T]; H), \quad \mathbb{P}' \text{ a.s.}.
\]

We need the following Lemma to prove Theorem 8.4.

Lemma 8.5 (Kolmogorov test). [18] Let \( \{u(t)\}_{t \in [0, T]} \) be a stochastic process with values in a separable Banach space \( X \), such that for some \( C > 0, \varepsilon > 0, \delta > 1 \) and all \( t, s \in [0, T] \),

\[
\mathbb{E}\|u(t) - u(s)\|_X^{1+\varepsilon} \leq C|t - s|^{1+\varepsilon}.
\]

Then there exists a version of \( u \) with \( \mathbb{P} \) almost surely trajectories being Hölder continuous functions with an arbitrary exponent smaller than \( \frac{\varepsilon}{\delta} \).
Proof of Theorem 8.4. By (7.6), we have

\[
\begin{align*}
    u'(t) - u'(s) &= \lambda_1 \int_s^t (u' \times [\Delta u' - \phi'(u')]) (\tau) \, d\tau - \lambda_2 \int_s^t u'(\tau) \times (u' \times [\Delta u' - \phi'(u')]) (\tau) \, d\tau \\
    &\quad + \sum_{j=1}^N \int_s^t (u'(\tau) \times h_j) \circ dW_j(\tau) \\
    &= \lambda_1 \int_s^t (u' \times [\Delta u' - \phi'(u')]) (\tau) \, d\tau - \lambda_2 \int_s^t u'(\tau) \times (u' \times [\Delta u' - \phi'(u')]) (\tau) \, d\tau \\
    &\quad + \frac{1}{N} \sum_{j=1}^N \int_s^t (u'(\tau) \times h_j) \times h_j \, d\tau + \sum_{j=1}^N \int_s^t u'(\tau) \times h_j \, dW_j(\tau), \quad 0 \leq s < t \leq T.
\end{align*}
\]

Hence by Jensen’s inequality, for \( q > 1 \),

\[
\mathbb{E}' \left[ \left\| u'(t) - u'(s) \right\|^2_H \right] 
\leq \mathbb{E}' \left\{ \lambda_1 \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|_H \, d\tau + \lambda_2 \int_s^t \left\| u'(\tau) \times (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|_H \, d\tau \\
+ \frac{1}{N} \sum_{j=1}^N \int_s^t \left\| u'(\tau) \times h_j \right\|_H \, d\tau + \sum_{j=1}^N \left\| \int_s^t u'(\tau) \times h_j \, dW_j(\tau) \right\|^2_H \right\} 
\leq 4^q \mathbb{E}' \left\{ \lambda_1^2 q \left( \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|_H \, d\tau \right)^2 \\
+ \lambda_2^2 q \left( \int_s^t \left\| u'(\tau) \times (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|_H \, d\tau \right)^2 \right\} \\
+ \frac{1}{4^q} \left( \sum_{j=1}^N \int_s^t \left\| u'(\tau) \times h_j \right\|_H \, d\tau \right)^2 + \sum_{j=1}^N \left\| \int_s^t u'(\tau) \times h_j \, dW_j(\tau) \right\|^2_H \right\}.
\]

By (8.3), there exists \( C^1 > 0 \), such that

\[
\mathbb{E}' \left( \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|^2_H \, d\tau \right)^{\frac{q}{2}} \leq (t-s)^q \mathbb{E}' \left( \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|^2_H \, d\tau \right)^q 
\leq C^1 (t-s)^q.
\]

By (8.4)

\[
\mathbb{E}' \left( \int_s^t \left\| u'(\tau) \times (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|^2_H \, d\tau \right)^{\frac{q}{2}} \leq \mathbb{E}' \left( \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|^2_H \, d\tau \right)^q 
\leq (t-s)^q \mathbb{E}' \left( \int_s^t \left\| (u' \times [\Delta u' - \phi'(u')]) (\tau) \right\|^2_H \, d\tau \right)^q \leq C^1 (t-s)^q.
\]
And by (6.6),
\[
\mathbb{E}' \left( \sum_{j=1}^{N} \int_{s}^{t} \left\| u'(\tau) \times h_j \times h_j \right\|_{H}^{2q} d\tau \right)^{2q} \leq \|u_0\|_{H}^{2q} T \sum_{j=1}^{N} \|h_j\|_{L_{loc}^{2q}(t-s)^q}.
\]

By the Burkholder-Davis-Gundy Inequality,
\[
\mathbb{E}' \left( \sum_{j=1}^{N} \int_{s}^{t} \left\| u'(\tau) \times h_j \times dW'_j(\tau) \right\|_{H}^{2q} \right)^{q} \leq K_q \mathbb{E}' \left( \sum_{j=1}^{N} \int_{s}^{t} \left\| u'(\tau) \times h_j \right\|_{H}^{2} d\tau \right)^{q} \leq K_q \|u_0\|_{H}^{2q} \sum_{j=1}^{N} \|h_j\|_{L_{loc}^{2q}(t-s)^q}.
\]

Therefore, let
\[
C = 2C_1^{q} + \|u_0\|_{H}^{2q} T \sum_{j=1}^{N} \|h_j\|_{L_{loc}^{2q}} + K_q \|u_0\|_{H}^{2q} \sum_{j=1}^{N} \|h_j\|_{L_{loc}^{2q}},
\]

we have
\[
\mathbb{E}' \left[ \left\| u'(t) - u'(s) \right\|_{H}^{2q} \right] \leq C(t-s)^q, \quad q \geq 1.
\]

Then by Lemma 8.5,
\[
u \in C^{\alpha}([0,T];H), \quad \alpha \in \left(0, \frac{1}{2}\right).
\]

This completes the proof of Theorem 8.4. \qed
APPENDIX A. SOME EXPLANATION

This Appendix aims to clarify the meaning of the process $\Lambda$ from Notation 6.11 and Lemma 6.10. And the explanation present here goes back to Visintin [49].

**Definition A.1.** Assume that $D \subset \mathbb{R}^d$, $d \leq 3$. Suppose that $M \in H^1(D)$. We say that $M \times \Delta M$ exists in the $L^2(D)$ sense (and write $M \times \Delta M \in L^2(D)$) iff there exists $B \in L^2(D)$ such that for every $u \in W^{1,3}(D)$,

\begin{equation}
\langle B, u \rangle_{L^2} = \sum_{i=1}^{3} \langle D_i M, M \times D_i u \rangle_{L^2},
\end{equation}

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2}$.  

**Remark.** Since $H^1(D) \subset L^6(D)$ and $D_i u \in L^3(D)$, the integral on the RHS above is convergent.

**Remark.** If $M \in D(A)$, then $B = M \times \Delta M$ can be defined pointwise as an element of $L^2(D)$. Moreover by Proposition 3.4, (A.1) holds, so $M \times \Delta M$ in the sense of Definition A.1. The next result shows that this can happen also for less regular $M$.

**Proposition A.2.** Suppose that $M_n \in H^1(D)$ so that $\Lambda_n := M_n \times \Delta M_n \in L^2(D)$ and $|\Lambda_n|_{L^2} \leq C$.

Suppose that

$|M_n|_{H^1} \leq C$.

Suppose that

$M_n \rightarrow M$ weakly in $H^1(D)$.  

Then $M \times \Delta M \in L^2(D)$.

**Proof.** By the assumptions there exists a subsequence $(n_j)$ and $\Lambda \in L^2(D)$ such that for any $q < 6$ (in particular $q = 4$)

$\Lambda_{n_j} \rightarrow \Lambda$ weakly in $L^2(D)$

$M_{n_j} \rightarrow M$ strongly in $L^6(D)$

$\nabla M_{n_j} \rightarrow \nabla M$ weakly in $L^2(D)$

We will prove that $M \times \Delta M = \Lambda \in L^2$. Let us fix $u \in W^{1,4}(D)$.

First we will show that

\begin{equation}
\langle \Lambda_n, u \rangle = \sum_{i=1}^{3} \langle D_i M_n, M \times D_i u \rangle,
\end{equation}

Since

$\langle \Lambda_n, u \rangle = \sum_{i=1}^{3} \langle D_i M_n, M_n \times D_i u \rangle$,

we have...
\[-\langle \Lambda_n, u \rangle + \sum_{i=1}^{3} \langle D_i M, M \times D_i u \rangle \]

\[= - \sum_{i=1}^{3} \langle D_i M_n, M_n \times D_i u \rangle + \sum_{i=1}^{3} \langle D_i M, M \times D_i u \rangle \]

\[= \sum_{i=1}^{3} \langle D_i M - D_i M_n, M \times D_i u \rangle + \sum_{i=1}^{3} \langle D_i M_n, M \times D_i - M_n \times D_i u \rangle \]

\[= I_n + II_n \]

Since \( M \times D_i u \in L^2 \) and \( D_i M - D_i M_n \to 0 \) weakly in \( L^2 \) we infer that \( I_n \to 0 \). Moreover, by the Hölder inequality we have

\[|II_n| \leq \sum_{i=1}^{3} |D_i M_n|_{L^2} |M - M_n|_{L^4} |D_i u|_{L^4} \to 0.\]

Thus, \( \langle \Lambda_n, u \rangle \to \sum_{i=1}^{3} \langle D_i M, M \times D_i u \rangle \). On the other hand, \( \langle \Lambda_n, u \rangle \to \langle \Lambda, u \rangle \), what concludes the proof of equality (A.2) for \( u \in W^{1,4}(D) \).

Since both sides of equality (A.2) are continuous with respect to \( W^{1,3}(D) \) norm of \( u \) and the space \( W^{1,4}(D) \) is dense in \( W^{1,3}(D) \), the result follows.

\[\square\]
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