Strongly zero product determined Banach algebras

J. Alaminos, J. Extremera, M.L.C. Godoy, A.R. Villena

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

Abstract

C*-algebras, group algebras, and the algebra A(X) of approximable operators on a Banach space X having the bounded approximation property are known to be zero product determined. In this paper we give a quantitative estimate of this property by showing that, for the Banach algebra A, there exists a constant α with the property that for every continuous bilinear functional ϕ: A × A → C there exists a continuous linear functional ξ on A such that

$$\sup_{\|a\| = \|b\| = 1} |\varphi(a, b) - \xi(ab)| \leq \alpha \sup_{\|a\| = \|b\| = 1, \ ab = 0} |\varphi(a, b)|$$

in each of the following cases: (i) A is a C*-algebra, in which case α = 8; (ii) A = L1(G) for a locally compact group G, in which case $\alpha = 60\sqrt{27 \frac{1+\sin \frac{\pi}{10}}{1-\sin 2\sin \frac{\pi}{10}}}$; (iii) A = A(X) for a Banach space X having property (A) (which is a rather strong approximation property for X), in which case $\alpha = \ldots$

 MSC: primary 47H60, 42A20, 47L10
Keywords: Zero product determined Banach algebra, Group algebra, Algebra of approximable operators

Article history:
Received 5 July 2021
Accepted 2 September 2021
Available online 6 September 2021
Submitted by P. Semrl

The authors were supported by MCIU/AEI/FEDER grant PGC2018-093794-B-I00, Junta de Andalucía grant FQM-185. The first, second and fourth authors were supported by Proyectos I+D+i del programa operativo FEDER-Andalucía Grant A-FQM-484-UGR18. The third named author was also supported by MIU PhD scholarship Grant FPU18/00419. Funding for open access charge: Universidad de Granada / CBUA.

* Corresponding author.
E-mail addresses: alaminos@ugr.es (J. Alaminos), jlizana@ugr.es (J. Extremera), mgodoy@ugr.es (M.L.C. Godoy), avillena@ugr.es (A.R. Villena).

https://doi.org/10.1016/j.laa.2021.09.002
0024-3795/© 2021 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
1. Introduction

Let $A$ be a Banach algebra. Then $\pi : A \times A \to A$ denotes the product map, we write $A^*$ for the dual of $A$, and $B^2(A, \mathbb{C})$ for the space of continuous bilinear functionals on $A$.

The Banach algebra $A$ is said to be zero product determined if every $\varphi \in B^2(A, \mathbb{C})$ with the property

$$a, b \in A, \ ab = 0 \Rightarrow \varphi(a, b) = 0$$

(1)

belongs to the space

$$B^2_\pi(A, \mathbb{C}) = \{ \xi \circ \pi : \xi \in A^* \}.$$  

This concept implicitly appeared in [1] as an additional outcome of the so-called property $\mathbb{B}$ which was introduced in that paper, and was the basis of subsequent Jordan and Lie versions (see [2–4]). For a comprehensive survey of the theory of the zero product determined Banach algebras we refer the reader to [10]. The algebra $A$ is said to have property $\mathbb{B}$ if every $\varphi \in B^2(A, \mathbb{C})$ satisfying (1) belongs to the closed subspace $B^2_\mathbb{B}(A, \mathbb{C})$ of $B^2(A, \mathbb{C})$ defined by

$$B^2_\mathbb{B}(A, \mathbb{C}) = \{ \psi \in B^2(A, \mathbb{C}) : \psi(ab, c) = \psi(a, bc) \ \forall a, b, c \in A \}.$$  

In [1] it was shown that this class of Banach algebras is wide enough to include a number of examples of interest: $C^*$-algebras, the group algebra $L^1(G)$ of any locally compact group $G$, and the algebra $A(X)$ of approximable operators on any Banach space $X$.

Throughout, we confine ourselves to Banach algebras having a bounded left approximate identity. Then $B^2_\pi(A, \mathbb{C}) = B^2_\mathbb{B}(A, \mathbb{C})$ (Proposition 2.1), and hence $A$ is a zero product determined Banach algebra if and only if $A$ has property $\mathbb{B}$. For example, this applies to $C^*$-algebras, group algebras and the algebra $A(X)$ on any Banach space $X$ having the bounded approximation property, so that all of them are zero product determined Banach algebras.

For each $\varphi \in B^2(A, \mathbb{C})$, the distance from $\varphi$ to $B^2_\pi(A, \mathbb{C})$ is

$$\text{dist} \left( \varphi, B^2_\pi(A, \mathbb{C}) \right) = \inf \left\{ \| \varphi - \psi \| : \psi \in B^2(A, \mathbb{C}) \right\},$$  

which can be easily estimated through the constant
\[ |\varphi|_b = \sup \{ |\varphi(ab, c) - \varphi(a, bc)| : a, b, c \in A, \|a\| = \|b\| = \|c\| = 1 \} \]

(Proposition 2.1 below). Our purpose is to estimate \( \text{dist} (\varphi, \mathcal{B}_\pi^2(A, \mathbb{C})) \) through the constant

\[ |\varphi|_{zp} = \sup \{ |\varphi(a, b)| : a, b \in A, \|a\| = \|b\| = 1, \ ab = 0 \}. \]

Note that \( A \) is zero product determined precisely when

\[ \varphi \in \mathcal{B}^2(A, \mathbb{C}), \ |\varphi|_{zp} = 0 \Rightarrow \varphi \in \mathcal{B}_\pi^2(A, \mathbb{C}). \] (2)

We call the Banach algebra \( A \) strongly zero product determined if condition (2) is strengthened by requiring that there is a distance estimate

\[ \text{dist}(\varphi, \mathcal{B}_\pi^2(A, \mathbb{C})) \leq \alpha |\varphi|_{zp} \ \forall \varphi \in \mathcal{B}^2(A, \mathbb{C}) \] (3)

for some constant \( \alpha \); in this case, the optimal constant \( \alpha \) for which (3) holds will be denoted by \( \alpha_A \). The inequality \( |\varphi|_{zp} \leq \text{dist}(\varphi, \mathcal{B}_\pi^2(A, \mathbb{C})) \) is always true (Proposition 2.1 below). We also note that \( A \) has property \( \mathbb{B} \) exactly in the case when

\[ \varphi \in \mathcal{B}^2(A, \mathbb{C}), \ |\varphi|_{zp} = 0 \Rightarrow |\varphi|_b = 0, \]

and the algebra \( A \) is said to have the strong property \( \mathbb{B} \) if there is an estimate

\[ |\varphi|_b \leq \beta |\varphi|_{zp} \ \forall \varphi \in \mathcal{B}^2(A, \mathbb{C}) \] (4)

for some constant \( \beta \); in this case, the optimal constant \( \beta \) for which (4) holds will be denoted by \( \beta_A \). The inequality \( |\varphi|_{zp} \leq M |\varphi|_b \) is always true for some constant \( M \) (Proposition 2.1 below). The spirit of this concept first appeared in [6], and was subsequently formulated in [14] and refined in [15]. This property has proven to be useful to study the hyperreflexivity of the spaces of continuous derivations and, more generally, continuous cocycles on \( A \) (see [7,8,13–15]).

From [5, Corollary 1.3], we obtain the following result.

**Theorem 1.1.** Let \( A \) be a \( C^* \)-algebra. Then \( A \) is strongly zero product determined, has the strong property \( \mathbb{B} \), and \( \alpha_A, \beta_A \leq 8 \).

It is shown in [15] that each group algebra has the strong property \( \mathbb{B} \) and so (by Corollary 2.2 below) it is also strongly zero product determined. In Theorem 3.3 we prove that, for each group \( G \),

\[ \alpha_{L^1(G)} \leq \beta_{L^1(G)} \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}}. \]
This gives a sharper estimate for the constant of the strong property $\mathcal{B}$ of $L^1(G)$ to the one given in [15, Theorem 3.4]. The estimates given in Theorems 1.1 and 3.3 can be used to sharpen the upper bound given in [15, Theorem 4.4] for the hyperreflexivity constant of $\mathcal{Z}^n(A, X)$, the space of continuous $n$-cocycles from $A$ into $X$, where $A$ is a $C^*$-algebra or the group algebra of a group with an open subgroup of polynomial growth and $X$ is a Banach $A$-bimodule for which the $n^{th}$ Hochschild cohomology group $\mathcal{H}^{n+1}(A, X)$ is a Banach space.

Finally, in Theorem 4.1 we prove that the algebra $A(X)$ is strongly zero product determined for each Banach space $X$ having property (A) (which is a rather strong approximation property for the space $X$). Further, we will use this result to show that the space $\mathcal{Z}^n(A(X), Y^*)$ is hyperreflexive for each Banach $A(X)$-bimodule $Y$.

There is no reason for an arbitrary zero product Banach algebra to be strongly zero product determined. However, as yet, we do not know an example of a zero product determined Banach algebra which is not strongly zero product determined.

Throughout, our reference for Banach algebras, and particularly for group algebras, is the monograph [11].

2. Elementary estimates

In the following result we gather together some estimates that relate the seminorms $\text{dist} (\cdot, B^2_\pi(A, \mathbb{C}))$, $|\cdot|_b$, and $|\cdot|_{zp}$ on $B^2_\pi(A, \mathbb{C})$ to each other.

**Proposition 2.1.** Let $A$ be a Banach algebra with a left approximate identity of bound $M$. Then $B^2_\pi(A, \mathbb{C}) = B^2_\pi(A, \mathbb{C})$ and, for each $\varphi \in B^2(A, \mathbb{C})$, the following properties hold:

(i) The distance $\text{dist} (\varphi, B^2_\pi(A, \mathbb{C}))$ is attained;
(ii) $\frac{1}{2} |\varphi|_b \leq \text{dist} (\varphi, B^2_\pi(A, \mathbb{C})) \leq M |\varphi|_b$;
(iii) $|\varphi|_{zp} \leq \text{dist} (\varphi, B^2_\pi(A, \mathbb{C}))$.

**Proof.** Let $(e_\lambda)_{\lambda \in \Lambda}$ be a left approximate identity of bound $M$.

(i) Let $(\xi_n)$ be a sequence in $A^*$ such that

$$\text{dist} (\varphi, B^2_\pi(A, \mathbb{C})) = \lim_{n \to \infty} \|\varphi - \xi_n \circ \pi\|.$$

For each $n \in \mathbb{N}$ and $a \in A$, we have

$$|\xi_n(e_\lambda a)| = |(\xi_n \circ \pi)(e_\lambda, a)| \leq M \|\xi_n \circ \pi\| \|a\| \quad \forall \lambda \in \Lambda$$

and hence, taking limit in the above inequality and using that $\lim_{\lambda \in \Lambda} e_\lambda a = a$, we see that $|\xi_n(a)| \leq M \|\xi_n \circ \pi\| \|a\|$, which shows that $\|\xi_n\| \leq M \|\xi_n \circ \pi\|$. Further, since

$$\|\xi_n \circ \pi\| \leq \|\varphi - \xi_n \circ \pi\| + \|\varphi\| \quad \forall n \in \mathbb{N},$$

$$\lim_{n \to \infty} \|\varphi - \xi_n \circ \pi\| = 0.$$
it follows that the sequence $(\|\xi_n\|)$ is bounded. By the Banach–Alaoglu theorem, the sequence $(\xi_n)$ has a weak*-accumulation point, say $\xi$, in $A^*$. Let $(\xi_\nu)_{\nu \in N}$ be a subnet of $(\xi_n)$ such that $w^*\text{-}\lim_{\nu \in N} \xi_\nu = \xi$. The task is now to show that

$$\|\varphi - \xi \circ \pi\| = \text{dist} (\varphi, B^2_\pi (A, \mathbb{C})) .$$

For each $a, b \in A$ with $\|a\| = \|b\| = 1$, we have

$$|\varphi(a, b) - \xi_\nu(ab)| \leq \|\varphi - \xi_\nu \circ \pi\| \quad \forall \nu \in N,$$

and so, taking limits on both sides of the above inequality and using that

$$\lim_{\nu \in N} \xi_\nu(ab) = \xi(ab)$$

and that $(\|\varphi - \xi_\nu \circ \pi\|)_{\nu \in N}$ is a subnet of the convergent sequence $(\|\varphi - \xi_n \circ \pi\|)$, we obtain

$$|\varphi(a, b) - \xi(ab)| \leq \text{dist} (\varphi, B^2_\pi (A, \mathbb{C})) .$$

This implies that $\|\varphi - \xi \circ \pi\| \leq \text{dist} (\varphi, B^2_\pi (A, \mathbb{C}))$, and the converse inequality $\text{dist} (\varphi, B^2_\pi (A, \mathbb{C})) \leq \|\varphi - \xi \circ \pi\|$ trivially holds.

(ii) For each $\lambda \in \Lambda$ define $\xi_\lambda \in A^*$ by

$$\xi_\lambda(a) = \varphi(e_\lambda, a) \quad \forall a \in A.$$

Then $\|\xi_\lambda\| \leq M \|\varphi\|$ for each $\lambda \in \Lambda$, so that $(\xi_\lambda)_{\lambda \in \Lambda}$ is a bounded net in $A^*$ and hence the Banach–Alaoglu theorem shows that it has a weak*-accumulation point, say $\xi$, in $A^*$. Let $(\xi_\nu)_{\nu \in N}$ be a subnet of $(\xi_\lambda)_{\lambda \in \Lambda}$ such that $w^*\text{-}\lim_{\nu \in N} \xi_\nu = \xi$. For each $a, b \in A$ with $\|a\| = \|b\| = 1$, we have

$$|\varphi(e_\nu a, b) - \varphi(e_\nu, ab)| \leq M |\varphi|_b \quad \forall \nu \in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu \in N}$ is a subnet of the convergent net $(e_\lambda a)_{\lambda \in \Lambda}$ and that $\lim_{\nu \in N} \varphi(e_\lambda, ab) = \xi(ab)$, we see that

$$|\varphi(a, b) - \xi(ab)| \leq M |\varphi|_b .$$

This gives $\|\varphi - \xi \circ \pi\| \leq M |\varphi|_b$, whence

$$\text{dist} (\varphi, B^2_\pi (A, \mathbb{C})) \leq M|\varphi|_b .$$
Set $\xi \in A^*$. For each $a, b, c \in A$ with $\|a\| = \|b\| = \|c\| = 1$, we have
$$
|\varphi(ab, c) - \varphi(a, bc)| = |\varphi(ab, c) - (\xi \circ \pi)(ab, c) + (\xi \circ \pi)(a, bc) - \varphi(a, bc)|
\leq |\varphi(ab, c) - (\xi \circ \pi)(ab, c)| + |(\xi \circ \pi)(a, bc) - \varphi(a, bc)|
\leq \|\varphi - \xi \circ \pi\| \|ab\| \|c\| + \|\varphi - \xi \circ \pi\| \|a\| \|bc\|
\leq 2\|\varphi - \xi \circ \pi\|
$$
and therefore $|\varphi|_b \leq 2\|\varphi - \xi \circ \pi\|$. Since this inequality holds for each $\xi \in A^*$, it follows that
$$
|\varphi|_b \leq 2\text{ dist } (\varphi, B^2_\pi(A, \mathbb{C})) .
$$

(iii) Let $a, b \in A$ with $\|a\| = \|b\| = 1$ and $ab = 0$. For each $\xi \in A^*$, we see that
$$
|\varphi(a, b)| = |\varphi(a, b) - (\xi \circ \pi)(a, b)| \leq \|\varphi - \xi \circ \pi\| ,
$$
and consequently $|\varphi|_{zp} \leq \|\varphi - \xi \circ \pi\|$. Since the above inequality holds for each $\xi \in A^*$, we conclude that
$$
|\varphi|_{zp} \leq \text{ dist } (\varphi, B^2_\pi(A, \mathbb{C})) .
$$

Finally, it is clear that $B^2_\pi(A, \mathbb{C}) \subset B^2_0(A, \mathbb{C})$. To prove the reverse inclusion take $\varphi \in B^2_0(A, \mathbb{C})$. Then $|\varphi|_b = 0$, hence (ii) shows that dist $(\varphi, B^2_\pi(A, \mathbb{C})) = 0$, and (i) gives $\psi \in B^2_\pi(A, \mathbb{C})$ such that $\|\varphi - \psi\| = 0$, which implies that $\varphi = \psi \in B^2_\pi(A, \mathbb{C})$. \qed

The following result is an immediate consequence of assertion (ii) in Proposition 2.1.

**Corollary 2.2.** Let $A$ be a Banach algebra with a left approximate identity of bound $M$. Then $A$ is a strongly zero product determined Banach algebra if and only if has the strong property $\mathcal{B}$, in which case
$$
\frac{1}{2} \beta_A \leq \alpha_A \leq M \beta_A .
$$

Let $X$ and $Y$ be Banach spaces, and let $n \in \mathbb{N}$. We write $B^n(X, Y)$ for the Banach space of all continuous $n$-linear maps from $X \times \cdots \times X$ to $Y$. As usual, we abbreviate $B^1(X, Y)$ to $B(X, Y)$, $B(X, X)$ to $B(X)$, and $B(X, \mathbb{C})$ to $X^*$. The identity operator on $X$ is denoted by $I_X$. Further, we write $\langle \cdot, \cdot \rangle$ for the duality between $X$ and $X^*$. For each subspace $E$ of $X$, $E^\perp$ denotes the annihilator of $E$ in $X^*$.

For a Banach algebra $A$ and a Banach space $X$, and for each $\varphi \in B^2(A, X)$, we continue to use the notations
$$
|\varphi|_b = \sup \{ |\varphi(ab, c) - \varphi(a, bc)| : a, b, c \in A, \|a\| = \|b\| = \|c\| = 1 \},
$$
$$
|\varphi|_{zp} = \sup \{ |\varphi(a, b)| : a, b \in A, \|a\| = \|b\| = 1, \ ab = 0 \} .
$$
Proposition 2.3. Let $A$ be a Banach algebra with a left approximate identity of bound $M$ and having the strong property $\mathcal{B}$. Let $X$ be a Banach space, and let $\varphi \in \mathcal{B}^2(A, X)$. Then the following properties hold:

(i) $|\varphi|_{zb} \leq \beta_A |\varphi|_{zp}$;

(ii) If $X$ is a dual Banach space, then there exists $\Phi \in \mathcal{B}(A, X)$ such that $\|\varphi - \Phi \circ \pi\| \leq M\beta_A$.

Proof. (i) For each $\xi \in X^*$, we have

$$|\xi \circ \varphi|_{zb} \leq \beta_A |\xi \circ \varphi|_{zp}. $$

It follows from the Hahn-Banach theorem that

$$|\varphi|_{zb} = \sup \{|\xi \circ \varphi|_{zb} : \xi \in X^*, \|\xi\| = 1\},$$

$$|\varphi|_{zp} = \sup \{|\xi \circ \varphi|_{zp} : \xi \in X^*, \|\xi\| = 1\}. $$

In this way we obtain (i).

(ii) Suppose that $X$ is the dual of a Banach space $X_\ast$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a left approximate identity for $A$ of bound $M$, and define a net $(\Phi_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{B}(A, X)$ by setting

$$\Phi_\lambda(a) = \varphi(e_\lambda, a) \quad \forall a \in A, \ \forall \lambda \in \Lambda.$$ 

Since each bounded subset of $\mathcal{B}(A, X)$ is relatively compact with respect to the weak* operator topology on $\mathcal{B}(A, X)$ and the net $(\Phi_\lambda)_{\lambda \in \Lambda}$ is bounded, it follows that there exist $\Phi \in \mathcal{B}(A, X)$ and a subnet $(\Phi_\nu)_{\nu \in N}$ of $(\Phi_\lambda)_{\lambda \in \Lambda}$ such that $w^{\ast}-\lim_{\nu \in N} \Phi_\nu = \Phi$. For each $a, b \in A$ with $\|a\| = \|b\| = 1$, and $x_\ast \in X_\ast$ with $\|x_\ast\| = 1$, we have

$$|\langle x_\ast, \varphi(e_\nu a, b) \rangle - \langle x_\ast, \varphi(e_\nu, ab) \rangle| \leq \|\varphi(e_\nu a, b) - \varphi(e_\nu, ab)\| \leq M\beta_A \quad \forall \nu \in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu \in N}$ is a subnet of the net $(e_\lambda a)_{\lambda \in \Lambda}$ (which converges to $a$ with respect to the norm topology) and that $\lim_{\nu \in N} \langle x_\ast, \varphi(e_\nu, ab) \rangle = \langle x_\ast, \Phi(ab) \rangle$ (by definition of $\Phi$), we see that

$$|\langle x_\ast, \varphi(a, b) - \Phi(ab) \rangle| = M\beta_A.$$ 

This gives $\|\varphi - \Phi \circ \pi\| \leq M\beta_A$. □

3. Group algebras

In this section we prove that the group algebra $L^1(G)$ of each locally compact group $G$ is a strongly zero product determined Banach algebra and we provide an estimate of the constants $\alpha_{L^1(G)}$ and $\beta_{L^1(G)}$. Our estimate of $\beta_{L^1(G)}$ improves the one given in [15].
For the basic properties of this important class of Banach algebras we refer the reader to [11, Section 3.3].

Throughout this section, \( T \) denotes the circle group, and we consider the normalized Haar measure on \( T \). We write \( A(T) \) and \( A(T^2) \) for the Fourier algebras of \( T \) and \( T^2 \), respectively. For each \( f \in A(T) \), \( F \in A(T^2) \), and \( j, k \in \mathbb{Z} \), we write \( \hat{f}(j) \) and \( \hat{F}(j, k) \) for the Fourier coefficients of \( f \) and \( F \), respectively. Let \( 1, \zeta \in A(T) \) denote the functions defined by

\[
1(z) = 1, \quad \zeta(z) = z \quad \forall z \in T.
\]

Let \( \Delta: A(T^2) \to A(T) \) be the bounded linear map defined by

\[
\Delta(F)(z) = F(z, z) \quad \forall z \in T, \forall F \in A(T^2).
\]

For \( f, g \in A(T) \), let \( f \otimes g: T^2 \to \mathbb{C} \) denote the function defined by

\[
(f \otimes g)(z, w) = f(z)g(w) \quad \forall z, w \in T,
\]

which is an element of \( A(T^2) \) with \( \|f \otimes g\| = \|f\|\|g\| \).

**Lemma 3.1.** Let \( \Phi: A(T^2) \to \mathbb{C} \) be a continuous linear functional, and let the constant \( \varepsilon \geq 0 \) be such that

\[
f, g \in A(T), \ f g = 0 \implies |\Phi(f \otimes g)| \leq \varepsilon \|f\|\|g\|.
\]

Then

\[
|\Phi(\zeta \otimes 1 - 1 \otimes \zeta)| \leq \|\Phi\|_{\ker \Delta} 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon.
\]

**Proof.** Set

\[
E = \left\{ e^{\theta i} : -\frac{1}{5}\pi \leq \theta \leq \frac{1}{5}\pi \right\},
\]

\[
W = \left\{ (z, w) \in T^2 : zw^{-1} \in E \right\},
\]

and let \( F \in A(T^2) \) be such that

\[
F(z, w) = 0 \quad \forall (z, w) \in W.
\]

Our objective is to prove that

\[
|\Phi(F)| \leq 30\sqrt{27} \|F\| \varepsilon.
\]

For this purpose, we take
\[ a = e^{\frac{1}{15} \pi i}, \]
\[ A = \{ e^{\theta i} : 0 < \theta \leq \frac{1}{15} \pi \}, \]
\[ B = \{ e^{\theta i} : \frac{2}{15} \pi < \theta \leq \frac{29}{15} \pi \}, \]
\[ U = \{ e^{\theta i} : -\frac{1}{30} \pi < \theta < \frac{1}{30} \pi \}, \]

and we define functions \( \omega, \upsilon \in A(T) \) by
\[ \omega = 30 \chi_A \ast \chi_U, \quad \upsilon = 30 \chi_B \ast \chi_U. \]

We note that
\[ \{ z \in T : \omega(z) \neq 0 \} = AU = \{ e^{\theta i} : -\frac{1}{30} \pi < \theta < \frac{1}{10} \pi \}, \]
\[ \{ z \in T : \upsilon(z) \neq 0 \} = BU = \{ e^{\theta i} : \frac{1}{10} \pi < \theta < \frac{59}{30} \pi \}, \]

and, with \( \| \cdot \|_2 \) denoting the norm of \( L^2(T) \),
\[ \| \omega \| \leq 30 \| \chi_A \|_2 \| \chi_U \|_2 = 30 \frac{1}{\sqrt{30}} \frac{1}{\sqrt{30}} = 1, \]
\[ \| \upsilon \| \leq 30 \| \chi_B \|_2 \| \chi_U \|_2 = 30 \frac{\sqrt{27}}{\sqrt{30}} \frac{1}{\sqrt{30}} = \sqrt{27}. \]

Since
\[ \bigcup_{k=0}^{29} a^k A = T, \quad \bigcup_{k=2}^{28} a^k A = B, \]

it follows that
\[ \sum_{k=0}^{29} \delta_{a^k} \ast \chi_A = \sum_{k=0}^{29} \chi_{a^k A} = 1, \quad \sum_{k=2}^{28} \delta_{a^k} \ast \chi_A = \sum_{k=2}^{28} \chi_{a^k A} = \chi_B, \]

and thus, for each \( j \in \mathbb{Z} \), we have
\[ \sum_{k=j}^{j+29} \delta_{a^k} \ast \omega = 30 \delta_{a^j} \ast \sum_{k=0}^{29} \delta_{a^k} \ast \chi_A \ast \chi_U = 30 \delta_{a^j} \ast 1 \ast \chi_U = 1, \]  
(7)
\[ \sum_{k=j+2}^{j+28} \delta_{a^k} \ast \omega = 30 \delta_{a^j} \ast \sum_{k=2}^{28} \delta_{a^k} \ast \chi_A \ast \chi_U = 30 \delta_{a^j} \ast \chi_B \ast \chi_U = \delta_{a^j} \ast \upsilon. \]  
(8)

If \( j \in \mathbb{Z} \), \( k \in \{ j - 1, j, j + 1 \} \), and \( z, w \in T \) are such that \( (\delta_{a^j} \ast \omega)(z)(\delta_{a^k} \ast \omega)(w) \neq 0 \), then
\[ zw^{-1} \in a^j AU(a^k AU)^{-1} \subset a^{j-k} \{ e^{\theta i} : -\frac{2}{15} \pi < \theta < \frac{2}{15} \pi \} \subset E, \]
whence \( \{(z, w) \in \mathbb{T}^2 : (\delta_{a^j} \ast \omega) \otimes (\delta_{a^k} \ast \omega)(z, w) \neq 0\} \subset W \) and (5) gives

\[
F(\delta_{a^j} \ast \omega) \otimes (\delta_{a^k} \ast \omega) = 0. \tag{9}
\]

Since \( AU \cap BU = \emptyset \), it follows that \( \omega \upsilon = 0 \), and therefore

\[
(\delta_{a^k} \ast \omega)(\delta_{a^k} \ast \upsilon) = 0 \quad \forall k \in \mathbb{Z}. \tag{10}
\]

From (7), (8), and (9) we deduce that

\[
F = \sum_{j=0}^{29} \sum_{k=j-1}^{j+29} (\delta_{a^j} \ast \omega) \otimes (\delta_{a^k} \ast \omega) + \sum_{j=0}^{29} \sum_{k=j+2}^{j+28} (\delta_{a^j} \ast \omega) \otimes (\delta_{a^k} \ast \omega)
\]

\[
= \sum_{j=0}^{29} \sum_{k=j+2}^{j+28} (\delta_{a^j} \ast \omega) \otimes (\delta_{a^k} \ast \omega) = \sum_{j=0}^{29} (\delta_{a^j} \ast \omega) \otimes (\delta_{a^j} \ast \upsilon).
\]

As

\[
F = \sum_{j,k=\infty}^{\infty} \hat{F}(j, k) \zeta^j \otimes \zeta^k
\]

we have

\[
F = \sum_{j,k=\infty}^{\infty} \sum_{l=0}^{29} \hat{F}(j, k)(\zeta^j(\delta_{a^l} \ast \omega)) \otimes (\zeta^k(\delta_{a^l} \ast \upsilon)),
\]

so that

\[
\Phi(F) = \sum_{j,k=\infty}^{\infty} \sum_{l=0}^{29} \hat{F}(j, k) \Phi\left((\zeta^j(\delta_{a^l} \ast \omega)) \otimes (\zeta^k(\delta_{a^l} \ast \upsilon))\right).
\]

By (10), for each \( j, k, l \in \mathbb{Z} \),

\[
(\zeta^j(\delta_{a^l} \ast \omega))(\zeta^k(\delta_{a^l} \ast \upsilon)) = 0
\]

and therefore

\[
|\Phi \left((\zeta^j(\delta_{a^l} \ast \omega)) \otimes (\zeta^k(\delta_{a^l} \ast \upsilon))\right)| \leq \varepsilon \|\zeta^j(\delta_{a^l} \ast \omega)\| \|\zeta^k(\delta_{a^l} \ast \upsilon)\|
\]

\[
= \varepsilon \|\omega\| \|\upsilon\| \leq \sqrt{27} \varepsilon.
\]

We thus get
\[
|\Phi(F)| = \sum_{j,k=-\infty}^{29} \sum_{l=0}^{29} \left| \widehat{F}(j,k) \right| \left| \Phi \left( (\zeta^j (\delta_{0,l} \ast \omega)) \otimes (\zeta^k (\delta_{0,l} \ast v)) \right) \right| \\
\leq \sum_{j,k=-\infty}^{29} \sum_{l=0}^{29} \left| \widehat{F}(j,k) \right| \sqrt{27} \varepsilon = 30 \sqrt{27} \|F\| \varepsilon ,
\]
and (6) is proved.

Let \(f \in A(\mathbb{T})\) be such that \(f(z) = 0\) for each \(z \in E\), and define the function \(F : \mathbb{T}^2 \to \mathbb{C}\) by

\[
F(z, w) = f(zw^{-1})w = \sum_{k=-\infty}^{\infty} \hat{f}(k) z^k w^{-k+1} \quad \forall z, w \in \mathbb{T}.
\]

Then \(F \in A(\mathbb{T}^2)\), \(\|F\| = \|f\|\), \(\zeta \otimes 1 - 1 \otimes \zeta - F \in \ker \Delta\), and

\[
(\zeta \otimes 1 - 1 \otimes \zeta - F)(z, w) = \left(1 - \hat{f}(1)\right) z + \left(-1 - \hat{f}(0)\right) w - \sum_{k \neq 0,1} \hat{f}(k) z^k w^{-k+1},
\]

which certainly implies that

\[
\|\zeta \otimes 1 - 1 \otimes \zeta - F\| = \left|1 - \hat{f}(1)\right| + \left|-1 - \hat{f}(0)\right| + \sum_{k \neq 0,1} \left|\hat{f}(k)\right| = \|\zeta - 1 - f\|.
\]

According to (6), we have

\[
|\Phi(\zeta \otimes 1 - 1 \otimes \zeta)| \leq |\Phi(\zeta \otimes 1 - 1 \otimes \zeta - F)| + |\Phi(F)| \\
\leq \|\Phi\|_{\ker \Delta}\|\zeta \otimes 1 - 1 \otimes \zeta - F\| + 30 \sqrt{27} \|F\| \varepsilon \\
= \|\Phi\|_{\ker \Delta}\|\zeta - 1 - f\| + 30 \sqrt{27} \|f\| \varepsilon \\
\leq \|\Phi\|_{\ker \Delta}\|\zeta - 1 - f\| + 30 \sqrt{27}(2\|\zeta - 1 - f\| + 2) \varepsilon
\]

(as \(\|f\| \leq \|\zeta - 1 - f\| + \|\zeta - 1\|\)). Further, this inequality holds for each function from the set \(\mathcal{I}\) consisting of all functions \(f \in A(\mathbb{T})\) such that \(f(z) = 0\) for each \(z \in E\). Consequently,

\[
|\Phi(\zeta \otimes 1 - 1 \otimes \zeta)| \leq \|\Phi\|_{\ker \Delta}\|\text{dist}(\zeta - 1, \mathcal{I})\| + 30 \sqrt{27} (\text{dist}(\zeta - 1, \mathcal{I}) + 2) \varepsilon.
\]

On the other hand, it is shown at the beginning of the proof of [9, Corollary 3.3] that

\[
\text{dist}(\zeta - 1, \mathcal{I}) \leq 2 \sin \frac{\pi}{10},
\]

and we thus get

\[
|\Phi(\zeta \otimes 1 - 1 \otimes \zeta)| \leq \|\Phi\|_{\ker \Delta}\|2 \sin \frac{\pi}{10}\| + 30 \sqrt{27} (2 \sin \frac{\pi}{10} + 2) \varepsilon,
\]

which completes the proof. \(\square\)
Lemma 3.2. Let $\Phi: A(\mathbb{T}^2) \to \mathbb{C}$ be a continuous linear functional, and let the constant $\varepsilon \geq 0$ be such that

$$f, g \in A(\mathbb{T}), \ f g = 0 \Rightarrow |\Phi(f \otimes g)| \leq \varepsilon \|f\| \|g\|.$$  

Then

$$|\Phi(F - 1 \otimes \Delta F)| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} \varepsilon \|F\|$$

for each $F \in A(\mathbb{T}^2)$.

Proof. Fix $j, k \in \mathbb{Z}$. We claim that

$$|\Phi(\zeta^j \otimes \zeta^k - 1 \otimes \zeta^{j+k})| \leq \|\Phi\|_{\ker \Delta} \|2\sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon.$$  

(11)

Of course, we are reduced to proving (11) for $j \neq 0$. We define $d_j: A(\mathbb{T}) \to A(\mathbb{T})$, and $D_j, L_k: A(\mathbb{T}^2) \to A(\mathbb{T}^2)$ by

$$d_j f(z) = f(z^j) \ \forall f \in A(\mathbb{T}), \forall z \in \mathbb{T}$$

and

$$D_j F(z, w) = F(z^j, w), \ L_k F(z, w) = F(z, w^k) \ \forall F \in A(\mathbb{T}^2), \forall z, w \in \mathbb{T},$$

respectively. Further, we consider the continuous linear functional $\Phi \circ L_k \circ D_j$. If $f, g \in A(\mathbb{T})$ are such that $f g = 0$, then $(d_j f)(\zeta^k d_j g) = \zeta^k d_j (f g) = 0$, and so, by hypothesis,

$$|\Phi \circ L_k \circ D_j (f \otimes g)| = |\Phi(d_j f \otimes \zeta^k d_j g)| \leq \varepsilon \|d_j f\| \|\zeta^k d_j g\| = \varepsilon \|f\| \|g\|.$$

By applying Lemma 3.1, we obtain

$$|\Phi(\zeta^j \otimes \zeta^k - 1 \otimes \zeta^{j+k})| = |\Phi \circ L_k \circ D_j (\zeta \otimes 1 - \zeta)|$$

$$\leq \|\Phi \circ L_k \circ D_j\|_{\ker \Delta} \|2\sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon.$$  

We check at once that $(L_k \circ D_j)(\ker \Delta) \subset \ker \Delta$, which gives

$$\|\Phi \circ L_k \circ D_j\|_{\ker \Delta} \leq \|\Phi\|_{\ker \Delta},$$

and therefore (11) is proved.

Take $F \in A(\mathbb{T}^2)$. Then

$$F = \sum_{j, k = -\infty}^{\infty} \hat{F}(j, k) \zeta^j \otimes \zeta^k$$
\[ \Delta F = \sum_{j,k=-\infty}^{\infty} \hat{F}(j,k)\zeta^{j+k}. \]

Consequently,

\[ \Phi(F - 1 \otimes \Delta F) = \sum_{j,k=-\infty}^{\infty} \hat{F}(j,k)\Phi(\zeta^j \otimes \zeta^k - 1 \otimes \zeta^{j+k}), \]

and (11) gives

\[ |\Phi(F - 1 \otimes \Delta F)| \leq \sum_{j,k=-\infty}^{\infty} |\hat{F}(j,k)| |\Phi(\zeta^j \otimes \zeta^k - 1 \otimes \zeta^{j+k})| \]
\[ \leq \sum_{j,k=-\infty}^{\infty} |\hat{F}(j,k)| \left[ \|\Phi|_{\ker \Delta}\| 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon \right] \]
\[ = \|F\| \left[ \|\Phi|_{\ker \Delta}\| 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon \right]. \] \[(12)\]

In particular, for each \( F \in \ker \Delta \), we have

\[ \|\Phi(F)\| \leq \|F\| \left[ \|\Phi|_{\ker \Delta}\| 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon \right]. \]

Thus

\[ \|\Phi|_{\ker \Delta}\| \leq \|\Phi|_{\ker \Delta}\| 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon, \]

so that

\[ \|\Phi|_{\ker \Delta}\| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \varepsilon. \]

Using this estimate in (12), we obtain

\[ |\Phi(F - 1 \otimes \Delta F)| \leq \|F\| \left[ 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \varepsilon 2 \sin \frac{\pi}{10} + 60\sqrt{27} (1 + \sin \frac{\pi}{10}) \varepsilon \right] \]
\[ = \|F\| 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \varepsilon \]

for each \( F \in A(T^2) \), which completes the proof. \( \square \)

**Theorem 3.3.** Let \( G \) be a locally compact group. Then the Banach algebra \( L^1(G) \) is strongly zero product determined and
\[ \alpha_{L^1(G)} \leq \beta_{L^1(G)} \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}}. \]

**Proof.** On account of Corollary 2.2, it suffices to prove that \( L^1(G) \) has the strong property \( \mathbb{B} \) with

\[ \beta_{L^1(G)} \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}}, \tag{13} \]

because \( L^1(G) \) has an approximate identity of bound 1. For this purpose set \( \varphi \in \mathcal{B}^2(L^1(G), \mathbb{C}) \).

Let \( t \in G \), and let \( \delta_t \) be the point mass measure at \( t \) on \( G \). We define a contractive homomorphism \( T: A(\mathbb{T}) \to M(G) \) by

\[ T(u) = \sum_{k=-\infty}^{\infty} \hat{u}(k)\delta_{tk} \quad \forall u \in A(\mathbb{T}). \]

Take \( f, h \in L^1(G) \) with \( \|f\| = \|h\| = 1 \), and define a continuous linear functional \( \Phi: A(\mathbb{T}^2) \to \mathbb{C} \) by

\[ \Phi(F) = \sum_{(j,k) \in \mathbb{Z}^2} \hat{F}(j,k)\varphi(f \ast \delta_{tj}, \delta_{tk} \ast h) \quad \forall F \in A(\mathbb{T}^2). \]

Further, if \( u, v \in A(\mathbb{T}) \), then

\[ \Phi(u \otimes v) = \sum_{(j,k) \in \mathbb{Z}^2} \hat{u}(j)\hat{v}(k)\varphi(f \ast \delta_{tj}, \delta_{tk} \ast h) = \varphi(f \ast T(u), T(v) \ast h); \]

in particular, if \( uv = 0 \), then \( (f \ast T(u)) \ast (T(v) \ast h) = f \ast T(uv) \ast h = 0 \), and so

\[ |\Phi(u \otimes v)| = |\varphi(f \ast T(u), T(v) \ast h)| \leq |\varphi|_{zp} \|f \ast T(u)\| \|T(v) \ast h\| \leq |\varphi|_{zp} \|u\| \|v\|. \]

By applying Lemma 3.2 with \( F = \zeta \otimes 1 \), we see that

\[ |\varphi(f \ast \delta_t, h) - \varphi(f, \delta_t \ast h)| = |\Phi(\zeta \otimes 1 - 1 \otimes \zeta)| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} |\varphi|_{zp}. \]

We now take \( g \in L^1(G) \) with \( \|g\| = 1 \). By multiplying the above inequality by \( |g(t)| \), we arrive at

\[ |\varphi(g(t) \ast \delta_t, h) - \varphi(f, g(t) \delta_t \ast h)| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} |\varphi|_{zp} |g(t)|. \tag{14} \]

Since the convolutions \( f \ast g \) and \( g \ast h \) can be expressed as
\[ f \ast g = \int_{G} g(t) f \ast \delta_{t} \, dt, \]
\[ g \ast h = \int_{G} g(t) \delta_{t} \ast h \, dt, \]

where the expressions on the right-hand side are considered as Bochner integrals of \( L^{1}(G) \)-valued functions of \( t \), it follows that

\[ \varphi(f \ast g, h) - \varphi(f, g \ast h) = \int_{G} [\varphi(g(t) f \ast \delta_{t}, h) - \varphi(f, g(t) \delta_{t} \ast h)] \, dt. \]

From (14) we now deduce that

\[ |\varphi(f \ast g, h) - \varphi(f, g \ast h)| \leq \int_{G} |\varphi(g(t) f \ast \delta_{t}, h) - \varphi(f, g(t) \delta_{t} \ast h)| \, dt \]

\[ \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} |\varphi|_{z_{p}} \int_{G} |g(t)| \, dt \]

\[ = 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} |\varphi|_{z_{p}}. \]

We thus get

\[ |\varphi|_{b} \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} |\varphi|_{z_{p}}, \]

and (13) is proved. \( \Box \)

4. Algebras of approximable operators

Let \( X \) be a Banach space. Then we write \( \mathcal{F}(X) \) for the two-sided ideal of \( B(X) \) consisting of finite-rank operators, and \( \mathcal{A}(X) \) for the closure of \( \mathcal{F}(X) \) in \( B(X) \) with respect to the operator norm. For each \( x \in X \) and \( \phi \in X^{*} \), we define \( x \otimes \phi \in \mathcal{F}(X) \) by \( (x \otimes \phi)(y) = \langle y, \phi \rangle x \) for each \( y \in X \). A finite, biorthogonal system for \( X \) is a set

\[ \{(x_{j}, \phi_{k}) : j, k = 1, \ldots, n\} \]

with \( x_{1}, \ldots, x_{n} \in X \) and \( \phi_{1}, \ldots, \phi_{n} \in X^{*} \) such that

\[ \langle x_{j}, \phi_{k} \rangle = \delta_{j,k} \quad \forall j, k \in \{1, \ldots, n\}. \]

Each such system defines an algebra homomorphism
\( \theta : \mathbb{M}_n \to \mathcal{F}(X), \quad (a_{j,k}) \mapsto \sum_{j,k=1}^n a_{j,k} x_j \otimes \phi_k, \)

where \( \mathbb{M}_n \) is the full matrix algebra of order \( n \) over \( \mathbb{C} \). The identity matrix is denoted by \( I_n \).

The Banach space \( X \) is said to have property \((A)\) if there is a directed set \( \Lambda \) such that, for each \( \lambda \in \Lambda \), there exists a finite, biorthogonal system

\[ \{ (x_{j}^{\lambda}, \phi_{k}^{\lambda}) : j, k = 1, \ldots, n_{\lambda} \} \]

for \( X \) with corresponding algebra homomorphism \( \theta_{\lambda} : \mathbb{M}_{n_{\lambda}} \to \mathcal{F}(X) \) such that:

\begin{enumerate}
  \item[(i)] \( \lim_{\lambda \in \Lambda} \theta_{\lambda}(I_{n_{\lambda}}) = I_X \) uniformly on the compact subsets of \( X \);
  \item[(ii)] \( \lim_{\lambda \in \Lambda} \theta_{\lambda}(I_{n_{\lambda}})^{\ast} = I_{X^{\ast}} \) uniformly on the compact subsets of \( X^{\ast} \);
  \item[(iii)] for each index \( \lambda \in \Lambda \), there is a finite subgroup \( G_{\lambda} \) of the group of all invertible \( n_{\lambda} \times n_{\lambda} \) matrices over \( \mathbb{C} \) whose linear span is all of \( \mathbb{M}_{n_{\lambda}} \), such that
\end{enumerate}

\[
\sup_{\lambda \in \Lambda} \sup_{t \in G_{\lambda}} \| \theta_{\lambda}(t) \| < \infty. \tag{15}
\]

Property \((A)\) forces the Banach algebra \( \mathcal{A}(X) \) to be amenable. For an exhaustive treatment of this topic (including a variety of interesting examples of spaces with property \((A)\)) we refer to [12, Section 3.3].

The notation of the above definition will be standard for the remainder of this section. Furthermore, our basic reference for this section is the monograph [12].

**Theorem 4.1.** Let \( X \) be a Banach space with property \((A)\). Then the Banach algebra \( \mathcal{A}(X) \) is strongly zero product determined. Specifically, if \( C \) denotes the supremum in (15), then

\[
\frac{1}{2} \beta_{\mathcal{A}(X)} \leq \alpha_{\mathcal{A}(X)} \leq 60 \sqrt{27 \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}}} C^2.
\]

**Proof.** For each \( \lambda \in \Lambda \) we define \( \Phi_{\lambda} : \ell^1(G_{\lambda}) \to \mathcal{F}(X) \) by

\[
\Phi_{\lambda}(f) = \sum_{t \in G_{\lambda}} f(t) \theta_{\lambda}(t) \quad \forall f \in \ell^1(G_{\lambda}).
\]

We claim that \( \Phi_{\lambda} \) is an algebra homomorphism. It is clear the \( \Phi_{\lambda} \) is a linear map and, for each \( f, g \in \ell^1(G_{\lambda}) \), we have
\[
\Phi_\lambda (f * g) = \sum_{t \in G_\lambda} (f * g)(t) \theta_\lambda (t) = \sum_{t \in G_\lambda} \sum_{s \in G_\lambda} f(s) g(s^{-1} t) \theta_\lambda (t)
\]

\[
\begin{align*}
&= \theta_\lambda \left( \sum_{t \in G_\lambda} \sum_{s \in G_\lambda} f(s) g(s^{-1} t) \right) = \theta_\lambda \left( \sum_{s \in G_\lambda} f(s) s \sum_{t \in G_\lambda} g(s^{-1} t) s^{-1} t \right) \\
&= \theta_\lambda \left( \sum_{s \in G_\lambda} f(s) s \sum_{r \in G_\lambda} g(r) r \right) = \theta_\lambda \left( \sum_{s \in G_\lambda} f(s) s \theta_\lambda \left( \sum_{r \in G_\lambda} g(r) r \right) \right) \\
&= \Phi_\lambda (f) \Phi_\lambda (g).
\end{align*}
\]

Of course, \(\Phi_\lambda\) is continuous because \(\ell^1(G_\lambda)\) is finite-dimensional, and, further, for each \(f \in \ell^1(G_\lambda)\), we have

\[
\|\Phi_\lambda (f)\| \leq \sum_{t \in G_\lambda} |f(t)| \|\theta_\lambda (t)\| \leq \sum_{t \in G_\lambda} |f(t)| C = C \|f\|_1.
\]

Hence \(\|\Phi_\lambda\| \leq C\).

Let \(\varphi \in B^2(A(X), \mathbb{C})\). Let us prove that

\[
|\varphi(S\theta_\lambda (t), \theta_\lambda (t^{-1}) T) - \varphi(S\theta_\lambda (I_{n_\lambda}), \theta_\lambda (I_{n_\lambda}) T)| \leq \beta_{\ell^1(G_\lambda)} C^2 \|S\| \|T\| |\varphi|_{z^p}
\]  

(16)

for all \(\lambda \in \Lambda, S, T \in A(X),\) and \(t \in G_\lambda\). For this purpose, take \(\lambda \in \Lambda\) and \(S, T \in A(X)\), and define \(\varphi_\lambda : \ell^1(G_\lambda) \times \ell^1(G_\lambda) \to \mathbb{C}\) by

\[
\varphi_\lambda (f, g) = \varphi(S\Phi_\lambda (f), \Phi_\lambda (g) T) \quad \forall f, g \in \ell^1(G_\lambda).
\]

Then \(\varphi_\lambda\) is continuous and, for each \(f, g \in \ell^1(G_\lambda)\) such that \(f * g = 0\), we have \((S\Phi_\lambda (f))(\Phi_\lambda (g) T) = S(\Phi_\lambda (f * g)) T = 0\) and therefore

\[
|\varphi_\lambda (f, g)| \leq |\varphi|_{z^p} \|S\Phi_\lambda (f)\| \|\Phi_\lambda (g) T\| \leq |\varphi|_{z^p} C^2 \|S\| \|T\| \|f\|_1 \|g\|_1,
\]

whence

\[
|\varphi_\lambda|_{z^p} \leq C^2 \|S\| \|T\| |\varphi|_{z^p}.
\]

For each \(t \in G_\lambda\), we have

\[
|\varphi_\lambda(\delta_t, \delta_{t^{-1}}) - \varphi_\lambda(\delta_{I_{n_\lambda}}, \delta_{I_{n_\lambda}})| = |\varphi_\lambda(\delta_{I_{n_\lambda}} \ast \delta_t, \delta_{t^{-1}}) - \varphi_\lambda(\delta_{I_{n_\lambda}} \ast \delta_{t^{-1}}, \delta_t)| \leq |\varphi_\lambda|_b \leq \beta_{\ell^1(G_\lambda)} |\varphi_\lambda|_{z^p} \leq \beta_{\ell^1(G_\lambda)} C^2 \|S\| \|T\| |\varphi|_{z^p},
\]

which gives (16).

The projective tensor product \(A(X) \hat{\otimes} A(X)\) becomes a Banach \(A(X)\)-bimodule for the products defined by
\[ R \cdot (S \otimes T) = (RS) \otimes T, \quad (S \otimes T) \cdot R = S \otimes (TR) \quad \forall R, S, T \in \mathcal{A}(X). \]

We define a continuous linear functional \( \hat{\varphi} \in (\mathcal{A}(X) \hat{\otimes} \mathcal{A}(X))^* \) through
\[
\langle S \otimes T, \hat{\varphi} \rangle = \varphi(S, T) \quad \forall S, T \in \mathcal{A}(X).
\]

For each \( \lambda \in \Lambda \), set \( P_\lambda = \theta_\lambda(I_{n_\lambda}) \) and
\[
D_\lambda = \frac{1}{|G_\lambda|} \sum_{t \in G_\lambda} \theta_\lambda(t) \otimes \theta_\lambda(t^{-1}).
\]

Then \( (P_\lambda)_{\lambda \in \Lambda} \) is a bounded approximate identity for \( \mathcal{A}(X) \) and \( (D_\lambda)_{\lambda \in \Lambda} \) is an approximate diagonal for \( \mathcal{A}(X) \) (see [12, Theorem 3.3.9]), so that \( (\|S \cdot D_\lambda - D_\lambda \cdot S\|)_{\lambda \in \Lambda} \to 0 \) for each \( S \in \mathcal{A}(X) \).

For each \( \lambda \in \Lambda \) and \( S, T \in \mathcal{A}(X) \), (16) shows that
\[
|\langle S \cdot D_\lambda \cdot T, \hat{\varphi} \rangle - \varphi(SP_\lambda, P_\lambda T)|
\leq \beta_{\ell^1(G_\lambda)}C^2 \|S\| \|T\| \|\varphi\|_{zp}
\]
and Theorem 3.3 then gives
\[
|\langle S \cdot D_\lambda \cdot T, \hat{\varphi} \rangle - \varphi(SP_\lambda, P_\lambda T)| \leq 60\sqrt{2} \left( \frac{1 + \sin \frac{n}{10}}{1 - 2 \sin \frac{n}{10}} \right) C^2 \|S\| \|T\| \|\varphi\|_{zp}. \quad (17)
\]

For each \( \lambda \in \Lambda \), define \( \xi_\lambda \in \mathcal{A}(X)^* \) by
\[
\langle T, \xi_\lambda \rangle = \langle D_\lambda \cdot T, \hat{\varphi} \rangle \quad \forall T \in \mathcal{A}(X).
\]

Note that
\[
\|\xi_\lambda\| \leq ||\hat{\varphi}|| \|D_\lambda\| \leq \|\varphi\| \ C^2 \quad \forall \lambda \in \Lambda
\]
and therefore \( (\xi_\lambda)_{\lambda \in \Lambda} \) is a bounded net in \( \mathcal{A}(X)^* \). By the Banach–Alaoglu theorem the net \( (\xi_\lambda)_{\lambda \in \Lambda} \) has a weak*-accumulation point, say \( \xi \), in \( \mathcal{A}(X)^* \). Take a subnet \( (\xi_\nu)_{\nu \in N} \) of \( (\xi_\lambda)_{\lambda \in \Lambda} \) such that \( w^*\text{-lim}_{\nu \in N} \xi_\nu = \xi \). Take \( S, T \in \mathcal{A}(X) \). For each \( \nu \in N \), we have
\[
\varphi(SP_\nu, P_\nu T) - \xi_\lambda( ST ) = \varphi(SP_\nu, P_\nu T) - \langle S \cdot D_\nu \cdot T, \hat{\varphi} \rangle + \langle (S \cdot D_\nu - D_\nu \cdot S) \cdot T, \hat{\varphi} \rangle
\]
so that (17) gives
\begin{equation}

|\varphi(SP_\nu, P_\nu T) - \langle ST, \xi \rangle| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} C^2 \|S\| \|T\| \|\varphi\|_zp + \|\varphi\| \|S \cdot D_\nu - D_\nu \cdot S\| \|T\|.
\end{equation}

Taking limits on both sides of the above inequality, and using that \((SP_\nu)_\nu \in N \to S,
(P_\nu T)_\nu \in N \to T,\) and \((\|S \cdot D_\nu - D_\nu \cdot S\|)_\nu \in N \to 0,\) we see that

\begin{equation}

|\varphi(S, T) - \langle ST, \xi \rangle| \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} C^2 \|S\| \|T\| \|\varphi\|_zp.
\end{equation}

We thus get

\begin{equation}

\text{dist}(\varphi, B_2^2(A(X), \mathbb{C})) \leq 60\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} C^2 \|\varphi\|_z p,
\end{equation}

which proves the theorem. \(\square\)

The hyperreflexivity of the space \(Z^n(A, X)\) of continuous \(n\)-cocycles from \(A\) into \(X\), where \(A\) is a \(C^*\)-algebra or a group algebra and \(X\) is a Banach \(A\)-bimodule has been already studied in [15, Theorem 4.4]. We conclude this section with a look at the hyperreflexivity of the space \(Z^n(A(X), Y^*)\). For this purpose we introduce some terminology.

Let \(A\) be a Banach algebra, and let \(X\) be a Banach \(A\)-bimodule. Set

\[ L_X = \sup \left\{ \| a \cdot x \| : x \in X, \ a \in A, \ ||x|| = ||a|| = 1 \right\} \]

and

\[ R_X = \sup \left\{ \| x \cdot a \| : x \in X, \ a \in A, \ ||x|| = ||a|| = 1 \right\}. \]

For each \(n \in \mathbb{N}\), let \(\delta^n : B^n(A, X) \to B^{n+1}(A, X)\) be the \(n\)-coboundary operator defined by

\[ (\delta^n T)(a_1, \ldots, a_{n+1}) = a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{k=1}^{n} (-1)^k T(a_1, \ldots, a_k a_{k+1}, \ldots, a_{n+1}) + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1} \]

for all \(T \in B^n(A, X)\) and \(a_1, \ldots, a_{n+1} \in A\). Further, \(\delta^0 : X \to B(A, X)\) is defined by

\[ (\delta^0 x)(a) = a \cdot x - x \cdot a \quad \forall x \in X, \ A \in A. \]

The space of continuous \(n\)-cocycles, \(Z^n(A, X)\), is defined as \(\ker \delta^n\). The space of continuous \(n\)-coboundaries, \(N^n(A, X)\), is the range of \(\delta^{n-1}\). Then \(N^n(A, X) \subset Z^n(A, X)\), and
the quotient $H^n(A, X) = Z^n(A, X)/N^n(A, X)$ is the $n^{th}$ Hochschild cohomology group. For each $T \in B^n(A, X)$, the constant

$$\text{dist}_r(T, Z^n(A, X)) := \sup_{\|a_1\| = \cdots = \|a_n\| = 1} \inf \{ \|T(a_1, \ldots, a_n) - S(a_1, \ldots, a_n)\| : S \in Z^n(A, X) \}$$

is intended to estimate the usual distance from $T$ to $Z^n(A, X)$, and, in accordance with [14,15], the space $Z^n(A, X)$ is called hyperreflexive if there exists a constant $K$ such that

$$\text{dist}(T, Z^n(A, X)) \leq K \text{dist}_r(T, Z^n(A, X)) \quad \forall T \in B^n(A, X).$$

The inequality $\text{dist}_r(T, Z^n(A, X)) \leq \text{dist}(T, Z^n(A, X))$ is always true.

**Proposition 4.2.** Let $A$ be a C-amenable Banach algebra, and let $X$ be a Banach $A$-bimodule. Then there exist projections $P, Q \in \mathcal{B}(X^*)$ onto $(X \cdot A)^\perp$ and $(A \cdot X)^\perp$, respectively, with $\|P\| \leq 1 + R_X C, \|Q\| \leq 1 + L_X C$, and such that

$$\text{dist}(T, Z^1(A, X^*)) \leq C(R_X + L_X \|P\| + \|P\|\|Q\|)\|\delta^1 T\|$$

for all $T \in \mathcal{B}(A, X^*)$. In particular, if the module $X$ is essential, then

$$\text{dist}(T, Z^1(A, X^*)) \leq R_X C\|\delta^1 T\|$$

for all $T \in \mathcal{B}(A, X^*)$.

**Proof.** The Banach algebra $A$ has a virtual diagonal $D$ with $\|D\| \leq C$. This is an element $D \in (A \hat{\otimes} A)^{**}$ such that, for each $a \in A$, we have

$$a \cdot D = D \cdot a \quad \text{and} \quad a \cdot \hat{\pi}^{**}(D) = a.$$ (18)

Here, the Banach space $A \hat{\otimes} A$ turns into a contractive Banach $A$-bimodule with respect to the operations defined through

$$(a \otimes b)c = a \otimes bc, \quad c(a \otimes b) = ca \otimes b \quad \forall a, b, c \in A,$$

and both $(A \hat{\otimes} A)^{**}$ and $A^{**}$ are considered as dual $A$-bimodules in the usual way. The map $\hat{\pi}: A \hat{\otimes} A \to A$ is the projective induced product map defined through

$$\hat{\pi}(a \otimes b) = ab \quad \forall a, b \in A.$$

For each $\varphi \in \mathcal{B}^2(A, C)$ there exists a unique element $\hat{\varphi} \in (A \hat{\otimes} A)^*$ such that

$$\hat{\varphi}(a \otimes b) = \varphi(a, b) \quad \forall a, b \in A,$$
and we use the formal notation
\[ \int_{A \times A} \varphi(u, v) dD(u, v) := \langle \hat{\varphi}, D \rangle. \]

Using this notation, the properties (18) can be written as
\[ \int_{A \times A} \varphi(au, v) dD(u, v) = \int_{A \times A} \varphi(u, va) dD(u, v) \tag{19} \]
and
\[ \int_{A \times A} \langle auv, \xi \rangle dD(u, v) = \langle a, \xi \rangle \tag{20} \]
for all \( \varphi \in B^2(A, C) \), \( a \in A \), and \( \xi \in A^* \); further, it will be helpful noting that
\[ \left| \int_{A \times A} \varphi(u, v) dD(u, v) \right| \leq \|D\| \|\hat{\varphi}\| \leq C\|\varphi\|. \tag{21} \]

We proceed to define the projections \( P \) and \( Q \). For this purpose we first define \( P_0, Q_0 \in B(X^*) \) by
\[ \langle x, P_0 \xi \rangle = \int_{A \times A} \langle x \cdot (uv), \xi \rangle dD(u, v), \]
\[ \langle x, Q_0 \xi \rangle = \int_{A \times A} \langle (uv) \cdot x, \xi \rangle dD(u, v) \]
for all \( x \in X \) and \( \xi \in X^* \), and set
\[ P = I_{X^*} - P_0, \quad Q = I_{X^*} - Q_0. \]

From (21) we obtain \( \|P_0\| \leq R_X C \) and \( \|Q_0\| \leq L_X C \), so that \( \|P\| \leq 1 + R_X C \) and \( \|Q\| \leq 1 + L_X C \).

We claim that
\[ a \cdot P_0 \xi = P_0(a \cdot \xi) = a \cdot \xi, \tag{22} \]
\[ P_0 \xi \cdot a = P_0(\xi \cdot a) \tag{23} \]
for all \( a \in A \) and \( \xi \in X^* \). Indeed, for \( a \in A \), \( \xi \in X^* \), and each \( x \in X \), (19) and (20) gives
\[ \langle x, a \cdot P_0 \xi \rangle = \langle x \cdot a, P_0 \xi \rangle = \int_{A \times A} \langle x \cdot (auv), \xi \rangle \, dD(u, v) \]

\[ = \langle x \cdot a, \xi \rangle = \langle x, a \cdot \xi \rangle, \]

\[ \langle x, P_0(a \cdot \xi) \rangle = \int_{A \times A} \langle x \cdot (uv), a \cdot \xi \rangle \, dD(u, v) \]

\[ = \int_{A \times A} \langle x \cdot (uva), \xi \rangle \, dD(u, v) \]

\[ = \int_{A \times A} \langle x \cdot (uv), \xi \rangle \, dD(u, v) = \langle x, a \cdot \xi \rangle, \]

and

\[ \langle x, P_0 \xi \cdot a \rangle = \langle a \cdot x, P_0 \xi \rangle = \int_{A \times A} \langle (a \cdot x) \cdot (uv), \xi \rangle \, dD(u, v) \]

\[ = \int_{A \times A} \langle x \cdot (uv), \xi \cdot a \rangle \, dD(u, v) = \langle x, P_0(\xi \cdot a) \rangle, \]

which proves (22) and (23). From (22) we deduce that

\[ \langle x \cdot a, P \xi \rangle = \langle x, a \cdot \xi - a \cdot P_0 \xi \rangle = 0, \]

and so \( P \xi \in (X \cdot A)^\perp \). Further, if \( \xi \in (X \cdot A)^\perp \), then

\[ \langle x, P_0 \xi \rangle = \int_{A \times A} \langle x \cdot (uv), \xi \rangle \, dD(u, v) = 0, \]

and so \( P \xi = \xi \). The operator \( P \) is a projection onto \((X \cdot A)^\perp\). From (22) we deduce immediately that

\[ P(A \cdot X^*) = \{0\}. \] (24)

The operator \( Q \) can be handled in much the same way as \( P \), and we obtain

\[ Q_0 \xi \cdot a = Q_0(\xi \cdot a) = \xi \cdot a, \]

\[ a \cdot Q_0 \xi = Q_0(a \cdot \xi) \]

for all \( a \in A \) and \( \xi \in X^* \), the operator \( Q \) is a projection onto \((A \cdot X)^\perp \), and

\[ Q(X^* \cdot A) = \{0\}. \] (25)
Set $T \in \mathcal{B}(A, X^*)$, and define $\phi \in X^*$ by

$$\langle x, \phi \rangle = \int_{A \times A} \langle x, u \cdot T(v) \rangle \, dD(u, v) \quad \forall x \in X.$$ 

For each $x \in X$ and $a \in A$ we have

$$\langle x, P_0 T(a) \rangle = \int_{A \times A} \langle x \cdot (uv), T(a) \rangle \, dD(u, v) = \int_{A \times A} \langle (uv) \cdot T(a) \rangle \, dD(u, v)$$

and

$$\langle x, (\delta^0 \phi)(a) \rangle = \langle x, a \cdot \phi - \phi \cdot a \rangle = \langle x \cdot a - a \cdot x, \phi \rangle$$

$$= \int_{A \times A} \langle x \cdot a - a \cdot x, u \cdot T(v) \rangle \, dD(u, v)$$

$$= \int_{A \times A} \langle x, (au) \cdot T(v) - u \cdot T(v) \cdot a \rangle \, dD(u, v)$$

$$= \int_{A \times A} \langle x, u \cdot T(va) - u \cdot T(v) \cdot a \rangle \, dD(u, v),$$

so that

$$\langle x, (P_0 T - \delta^0 \phi)(a) \rangle = \int_{A \times A} \langle x, u \cdot (\delta^1 T)(v, a) \rangle \, dD(u, v)$$

$$= \int_{A \times A} \langle x \cdot u, (\delta^1 T)(v, a) \rangle \, dD(u, v).$$

From the latter identity and (21) we conclude that

$$|\langle x, (P_0 T - \delta^0 \phi)(a) \rangle| \leq CR_X \|\delta^1 T\| \|a\| \|x\|,$$

whence

$$\|P_0 T - \delta^0 \phi\| \leq CR_X \|\delta^1 T\|. \quad (26)$$

Write $S = PT$. From (22) and (23) it follows that $\delta^1 S(a, b) = P\delta^1 T(a, b)$, and so

$$\|\delta^1 S\| \leq \|P\| \|\delta^1 T\|. \quad (27)$$

We now define $\psi \in X^*$ by
\[ \langle x, \psi \rangle = \int_{A \times A} \langle x, S(u) \cdot v \rangle dD(u, v) \quad \forall x \in X. \]

For each \( x \in X \) and \( a \in A \) we have

\[ \langle x, Q_0 S(a) \rangle = \int_{A \times A} \langle (uv) \cdot x, S(a) \rangle dD(u, v) = \int_{A \times A} \langle x, S(a) \cdot (uv) \rangle dD(u, v) \]

and

\[ \langle x, (\delta^0 \psi)(a) \rangle = \langle x, a \cdot \psi - \psi \cdot a \rangle = \langle x \cdot a - a \cdot x, \psi \rangle \]

\[ = \int_{A \times A} \langle x \cdot a - a \cdot x, S(u) \cdot v \rangle dD(u, v) \]

\[ = \int_{A \times A} \langle x, a \cdot S(u) \cdot v - S(u) \cdot (va) \rangle dD(u, v) \]

\[ = \int_{A \times A} \langle x, a \cdot S(u) \cdot v - S(au) \cdot v \rangle dD(u, v), \]

and hence

\[ \langle x, (Q_0 S + \delta^0 \psi)(a) \rangle = \int_{A \times A} \langle x, (\delta^1 S)(a, u) \cdot v \rangle dD(u, v) \]

\[ = \int_{A \times A} \langle v \cdot x, (\delta^1 S)(a, u) \rangle dD(u, v). \]

From the latter identity and (21) we conclude that

\[ |\langle x, (Q_0 S + \delta^0 \psi)(a) \rangle| \leq C L_X \| \delta^1 S \| \| a \| \| x \|. \]

Thus \( \| Q_0 S + \delta^0 \psi \| \leq C L_X \| \delta^1 S \| \) and (27) then gives

\[ \| Q_0 S + \delta^0 \psi \| \leq C L_X \| P \| \| \delta^1 T \|. \]

(28)

Our next goal is to estimate \( \| Q P T \| \). For each \( u, v, a \in A \), we have

\[ \delta^1 T(a, uv) = a \cdot T(uv) - T(auv) + T(a) \cdot (uv), \]

(23) and (24) gives

\[ P(\delta^1 T(a, uv)) = P(a \cdot T(uv)) - PT(auv) + PT(a) \cdot (uv), \]

\[ = 0 \]
and finally (25) yields
\[ QP(\delta^1 T(a, uv)) = -QPT(auv) + Q(P^T(a) \cdot (uv)) = -QPT(auv). \]

We thus get
\[
\langle x, QPT(a) \rangle = \int_{A \times A} \langle x, QPT(auv) \rangle dD(u, v)
= \int_{A \times A} \langle x, -Q(\delta^1 T)(a, uv) \rangle dD(u, v)
\]
and (21) implies
\[
|\langle x, QPT(a) \rangle| \leq C\|QP(\delta^1 T)\| \|x\| \|a\| \leq C\|Q\|\|P\| \|\delta^1 T\| \|x\| \|a\|. \]

Hence
\[
\|QPT\| \leq C\|Q\|\|P\| \|\delta^1 T\|. \tag{29}
\]

Finally, since
\[
T - \delta^0 \phi + \delta^0 \psi = QPT + (P_0 T - \delta^0 \phi) + (Q_0 PT + \delta^0 \psi),
\]
(26), (28), and (29) show that
\[
\|T - \delta^0 \phi + \delta^0 \psi\| \leq \|P_0 T - \delta^0 \phi\| + \|Q_0 PT + \delta^0 \psi\| + \|QPT\|
\leq CR_X \|\delta^1 T\| + CL_X \|P\| \|\delta^1 T\| + C\|Q\|\|P\| \|\delta^1 T\|.
\]

Since \(-\delta^0 \phi + \delta^0 \psi \in Z^1(A, X^*)\), it follows that
\[
\text{dist}(T, Z^1(A, X^*)) \leq CR_X \|\delta^1 T\| + CL_X \|P\| \|\delta^1 T\| + C\|Q\|\|P\| \|\delta^1 T\|
\]
as required. \(\square\)

**Corollary 4.3.** Let \(A\) be a \(C\)-amenable Banach algebra, let \(X\) be a Banach \(A\)-bimodule, and let \(n \in \mathbb{N}\). Then
\[
\text{dist}(T, Z^n(A, X^*)) \leq 2(n + L_X)(1 + R_X)C^3 \|\delta^n T\|
\]
for each \(T \in B^n(A, X^*)\).
Proof. Of course, we need only consider the case where $A$ is a non-zero Banach algebra, which implies that $C \geq 1$.

Suppose that $n = 1$, and $T \in B(A, X^*)$. By Proposition 4.2,

$$\text{dist}(T, Z^1(A, X^*)) \leq C(R_X + L_X(1 + R_X C) + (1 + L_X(1 + R_X C)))\|\delta^1 T\|$$

$$\leq 2(1 + L_X)(1 + R_X C)\|\delta^1 T\|,$$

as $C \geq 1$.

The Banach space $B^n(A, X^*)$ is a Banach $A$-bimodule with respect to the operations

$$(a \cdot T)(a_1, \ldots, a_n) = a \cdot T(a_1, \ldots, a_n)$$

and

$$(T \cdot a)(a_1, \ldots, a_n) = T(aa_1, \ldots, a_n)$$

$$+ \sum_{k=1}^{n-1} (-1)^k T(a, a_1, \ldots, a_1 a_k, a_{k+1}, \ldots, a_n)$$

$$+ (-1)^n T(a, a_1, \ldots, a_{n-1}) \cdot a_n$$

for all $T \in B^n(A, X^*)$, and $a, a_1, \ldots, a_n \in A$. Let

$$\Delta^1 : B(A, B^n(A, X^*)) \to B^2(A, B^n(A, X^*))$$

be the 1-coboundary operator. We also consider the maps

$$\tau_1^n : B^{1+n}(A, X^*) \to B(A, B^n(A, X^*)),$$

$$\tau_2^n : B^{2+n}(A, X^*) \to B^2(A, B^n(A, X^*))$$

defined by

$$(\tau_1^n T)(a)(a_1, \ldots, a_n) = T(a, a_1, \ldots, a_n),$$

$$(\tau_2^n T)(a, b)(a_1, \ldots, a_n) = T(a, b, a_1, \ldots, a_n).$$

Then:

- $\tau_1^n$ and $\tau_2^n$ are isometric isomorphisms;
- $\Delta^1 \circ \tau_1^n = \tau_2^n \circ \delta^{n+1};$
- $\tau_1^n Z^{n+1}(A, X^*) = Z^1(A, B^n(A, X^*)).$

For each $T \in B^{1+n}(A, X^*)$ we have

$$\text{dist}(T, Z^{n+1}(A, X^*)) = \text{dist}(\tau_1^n T, \tau_1^n Z^{n+1}(A, X^*))$$

$$= \text{dist}(\tau_1^n T, Z^1(A, B^n(A, X^*))).$$

(30)
Our next objective is to apply Proposition 4.2 to estimate the distance of the last term in (30). To this end, we realize that $\mathcal{B}^n(A, X^*)$ is a dual Banach $A$-bimodule by setting

$$Y = A\widehat{\otimes} \cdots \widehat{\otimes} A \widehat{\otimes} X.$$ 

Then:

- $Y$ is a Banach $A$-bimodule with respect to the operations

$$(a_1 \otimes \cdots \otimes a_n \otimes x) \cdot a = a_1 \otimes \cdots \otimes a_n \otimes (x \cdot a)$$

and

$$a \cdot (a_1 \otimes \cdots \otimes a_n \otimes x) = (aa_1) \otimes \cdots \otimes a_n \otimes x$$

$$+ \sum_{k=1}^{n-1} (-1)^k a \otimes a_1 \otimes \cdots \otimes (a_k a_{k+1}) \otimes \cdots \otimes a_n \otimes x$$

$$+ (-1)^n a \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n \cdot x)$$

for all $a, a_1, \ldots, a_n \in A$, and $x \in X$;

- we have the estimates

$$L_Y \leq n + L_X, \quad R_Y \leq R_X;$$

- the Banach $A$-bimodule $\mathcal{B}^n(A, X^*)$ is isometrically isomorphic to the Banach $A$-bimodule $Y^*$ through the duality

$$\langle a_1 \otimes \cdots \otimes a_n \otimes x, T \rangle = \langle x, T(a_1, \ldots, a_n) \rangle$$

for all $T \in \mathcal{B}^n(A, X^*)$, $a_1, \ldots, a_n \in A$, and $x \in X$.

Proposition 4.2 now leads to

$$\text{dist}(\tau_1^n T, \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))) = \text{dist}(\tau_1^n T, \mathcal{Z}^1(A, Y^*))$$

$$\leq 2(1 + L_Y)(1 + R_Y)C^3 \|\Delta^1 \tau_1^n T\|$$

$$\leq 2(1 + n + L_X)(1 + R_X)C^3 \|\Delta^1 \tau_1^n T\|$$

$$= 2(1 + n + L_X)(1 + R_X)C^3 \|\tau_2^n \delta^{n+1} T\|$$

$$= 2(1 + n + L_X)(1 + R_X)C^3 \|\delta^{n+1} T\|.$$ 

Combining (30) with the inequality above, we obtain precisely the estimate of the corollary. $\Box$
Theorem 4.4. Let $X$ be a Banach space with property (A), let $Y$ be a Banach $A(X)$-bimodule, and let $n \in \mathbb{N}$. Then the space $Z^n(A(X), Y^*)$ is hyperreflexive. Specifically, if $C$ denotes the supremum in (15), then

$$\text{dist}(T, Z^n(A(X), Y^*)) \leq (n + L_Y)(1 + R_Y)C^6 2^n (C^2 \beta_{A(X)} + (C + 1)^2)^{n+1} \text{dist}_T(T, Z^n(A(X), Y^*))$$

for each $T \in B^n(A(X), Y^*)$, where

$$\beta_{A(X)} \leq 120\sqrt{27} \frac{1 + \sin \frac{\pi}{10}}{1 - 2\sin \frac{\pi}{10}} C^2.$$

Proof. From Theorem 4.1 we see that $A(X)$ has the strong property B and the estimate for $\beta_{A(X)}$ holds.

The Banach algebra $A(X)$ has an approximate identity of bound $C$. Further, for each $T \in F(X)$ there exists $S \in F(X)$ such that $ST = TS = T$, and [14, Proposition 5.4] then shows that $A(X)$ has bounded local units.

By [12, Theorem 3.3.9], $A(X)$ is $C^2$-amenable, and Corollary 4.3 now gives

$$\text{dist}(T, Z^n(A(X), Y^*)) \leq 2(n + L_Y)(1 + R_Y)C^6 \|\delta^n T\|$$

for each $T \in B^n(A(X), Y^*)$. This estimate shows that the map

$$B^n(A(X), Y^*)/Z^n(A(X), Y^*) \to N^{n+1}(A(X), Y^*),
T + Z^n(A(X), Y^*) \mapsto \delta^n T$$

is an isomorphism, hence $N^{n+1}(A(X), Y^*)$ is closed in $B^{n+1}(A(X), Y^*)$ and this implies that the $n^{th}$ Hochschild cohomology group $H^{n+1}(A(X), Y^*)$ is a Banach space. By applying [15, Theorem 4.3] we obtain the hyperreflexivity of the space $Z^n(A(X), Y^*)$ as well as the statement about the estimate of $\text{dist}(T, Z^n(A(X), Y^*))$. □

Declaration of competing interest

There is no competing interest.

References

[1] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Maps preserving zero products, Stud. Math. 193 (2009) 131–159.
[2] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Lie product determined Banach algebras, Stud. Math. 239 (2017) 189–199.
[3] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Lie product determined Banach algebras, II, J. Math. Anal. Appl. 474 (2) (2019) 1498–1511.
[4] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Jordan product determined Banach algebras, J. Aust. Math. Soc. (2020) 1–14, https://doi.org/10.1017/S1446788719000478.
[5] J. Alaminos, J. Extremera, M.L.C. Godoy, A.R. Villena, Hyperreflexivity of the space of module homomorphisms between non-commutative $L^p$-spaces, J. Math. Anal. Appl. 498 (2) (2021) 124964.

[6] J. Alaminos, J. Extremera, A.R. Villena, Approximately zero product preserving maps, Isr. J. Math. 178 (2010) 1–28.

[7] J. Alaminos, J. Extremera, A.R. Villena, Hyperreflexivity of the derivation space of some group algebras, Math. Z. 266 (2010) 571–582.

[8] J. Alaminos, J. Extremera, A.R. Villena, Hyperreflexivity of the derivation space of some group algebras, II, Bull. Lond. Math. Soc. 44 (2012) 323–335.

[9] G.R. Allan, T.J. Ransford, Power-dominated elements in a Banach algebra, Stud. Math. 94 (1) (1989) 63–79.

[10] M. Brešar, Zero Product Determined Algebras, Frontiers in Mathematics, Birkhäuser, Basel, 2021.

[11] H.G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, New Series, vol. 24, Oxford Science Publications, the Clarendon Press, Oxford University Press, New York, 2000.

[12] V. Runde, Amenable Banach algebras. A Panorama, Springer Monographs in Mathematics, Springer-Verlag, New York, 2020.

[13] E. Samei, Reflexivity and hyperreflexivity of bounded $n$-cocycles from group algebras, Proc. Am. Math. Soc. 139 (2011) 163–176.

[14] E. Samei, J. Soltani Farsani, Hyperreflexivity of the bounded $n$-cocycle spaces of Banach algebras, Monatshefte Math. 175 (2014) 429–455.

[15] E. Samei, J. Soltani Farsani, Hyperreflexivity constants of the bounded $n$-cocycle spaces of group algebras and $C^*$-algebras, J. Aust. Math. Soc. 109 (2020) 112–130.