Duality–invariant Quantum Field Theories of Charges and Monopoles

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Abstract

We present a manifestly Lorentz– and $SO(2)$–Duality–invariant local Quantum Field Theory of electric charges, Dirac magnetic monopoles and dyons. The manifest invariances are achieved by means of the PST–mechanism. The dynamics for classical point particles is described by an action functional living on a circle, if the Dirac–Schwinger quantization condition for electric and magnetic charges holds. The inconsistent classical Field Theory depends on an arbitrary, but fixed, external vector field, a generalization of the Dirac–string. Nevertheless, the Quantum Field Theory, obtained from this classical action via a functional integral approach, turns out to be independent of the particular vector field chosen, and thus consistent, if the Dirac–Schwinger quantization condition holds. We provide explicit expressions for the generating functionals of observables, proving that they are Dirac–string independent. Since Lorentz–invariance is manifest at each step, the quantum theory admits also a manifestly diffeomorphism invariant coupling to external gravity. Relations with previous formulations, and with $SO(2)$–non invariant theories are clarified.

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1 Introduction

The construction of relativistic quantum field theories for electric charges and Dirac magnetic monopoles encountered in the past two main difficulties: the implementation of Lorentz–invariance and the realization of the global $SO(2)$–duality group as a manifest symmetry of the theory. While the first symmetry is a consistency requirement for the theory itself, the second appears as a supplementary property which is inherent to the classical generalized Maxwell equations, including magnetic charges. The problem of Lorentz–invariance is usually entangled with the dependence of the quantum theory on the unphysical Dirac–string. More precisely, in this paper we call “Dirac–string” of a particle the two–dimensional hypersurface swept out by the string during its time evolution; its boundary is the space–time particle trajectory.

The fundamental starting point for any theory of charges and monopoles are the classical generalized Maxwell equations:

$$\partial^\mu F_{\mu\nu} = j^e_\nu, \quad (1.1)$$

$$\partial^\mu * F_{\mu\nu} = j^g_\nu, \quad (1.2)$$

where $j^e$ and $j^g$ are the electric and magnetic currents, and $*F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual field strength. These equations imply current conservation but need to be accompanied by the equations of motion for the charged matter. For $N$ classical point–particles, dyons, with electric and magnetic charges $(e_r, g_r)$ and masses $m_r$, $r = 1, \cdots, N$, they involve the generalized Lorentz–force and read

$$m_r \frac{du^\mu_r}{d\tau_r} = (e_r F^{\mu\nu}(y_r) + g_r * F^{\mu\nu}(y_r)) u_{r\nu}, \quad (1.3)$$

where $y^\mu_r$ are the particles trajectories, $u^\mu_r = \frac{dy^\mu_r}{d\tau_r}$, and $\tau_r$ is the proper time.

For a classical field theory $(1.3)$ has to be replaced by the Dirac or Klein–Gordon equations, and this requires also the introduction of vector potentials.

In spite of the manifest Lorentz–invariance of $(1.1)–(1.3)$, the construction of a classical invariant action, from which they can be deduced, involves already some problem: one has to renounce either to locality [1, 2] or to manifest Lorentz–invariance [3]. These problematic aspects which one encounters at the very beginning – specially the missing manifest Lorentz–invariance – prevented for long time also the construction of a consistent quantum theory. That for the classical point–particle theory these features can eventually be saved [4], relies heavily on the point–particle nature itself of the charged matter.

At the quantum level the implementation of Lorentz–invariance has been achieved first by Brandt, Neri and Zwanziger [4], using a functional integral approach based on a classical local and manifestly $SO(2)$–invariant action which breaks, however, Lorentz–invariance, depending on a fixed constant four–vector $n^\mu$. Despite of the explicit breaking of the Lorentz group in this approach, the classical action for point particles turns out to
be Lorentz–invariant if it is defined on a circle, and if the Dirac–Schwinger quantization condition holds. On the other hand, the classical field theory is always inconsistent in that it breaks the Lorentz group, while, strikingly enough, the quantum field theory based on this inconsistent classical field theory action turns again out to be Lorentz–invariant, if the Dirac–Schwinger quantization condition holds. The reason for this is essentially that the quantum theory can be traced back, through a multi–path Feynman expansion of the functional integral, to classical point–like trajectories (a realization of wave/particle quantum duality), and that the classical point particle theory is consistent and admits an invariant exponentiated action.

In short, there exists a consistent quantum field theory of charges and monopoles, if the Dirac–Schwinger quantization condition holds. There exist, actually, two inequivalent classes of consistent theories: the theories of the first class are \(SO(2)\)–invariant, manifestly or not, while the theories of the second class are only invariant under the discrete duality group \(Z_4\), despite of the \(SO(2)\)–invariance of the underlying classical equations of motion, which are for both classes (1.1)–(1.3). With \(Z_4\) we mean here the discrete electromagnetic duality group generated by \((e_r, g_r) \rightarrow (g_r, -e_r)\). The difference between the two types of theories lies in the mixed quantum interactions between charges and monopoles.

For the \(SO(2)\)– and \(Z_4\)–theories we have respectively the two distinct Dirac–Schwinger quantization conditions

\[
\frac{1}{2} (e_r g_s - e_s g_r) = 2\pi n_{rs}, \tag{1.4}
\]

\[
e_r g_s = 2\pi n_{rs}, \tag{1.5}
\]

for each \(r\) and \(s\), where the \(n_{rs}\) are integer. The first is invariant under \(SO(2)\) and the second only under \(Z_4\). None of the two conditions implies the other. Only if the stronger condition

\[
\frac{1}{2} e_r g_s = 2\pi n_{rs} \tag{1.6}
\]

is satisfied, then the two theories coincide, as we will see in detail in the text.

Which kind of theory describes the monopoles depends on the model one considers. For example, for the four–dimensional effective heterotic string action [3] (1.4) is appropriate, while for the original Dirac monopole [6] one has to use Dirac’s original condition (1.5).

In this paper we concentrate mainly on the \(SO(2)\)–theory (elaborating only those details of the \(Z_4\)–theory through which it differs from the \(SO(2)\)–theory), formulated according to the Pasti–Sorokin–Tonin (PST) method [7]. In addition to an \(SO(2)\)–doublet of vector potentials, this method requires a single non propagating scalar auxiliary field, in the following called \(a\). The method appears particularly suitable for a formulation of a quantum field theory of dyons, in that it allows a clear distinction between the problems of Lorentz–invariance and of Dirac–string dependence. The principal advantage of the PST–action is, in fact, its manifest \(SO(2)\)– and Lorentz–invariance in the case of free Maxwell equations, i.e. (1.1) and (1.2) with \(j^e = 0 = j^g\). The free PST–action admits
also a natural coupling to non vanishing currents, if they are kinematically conserved \[8\], as in the case of classical point particles. In this situation the action \( S \) maintains its manifest \( SO(2) \)- and Lorentz-invariance, for arbitrary values of the charges, but depends on generalized Dirac–strings. Nevertheless, if (1.4) holds the functional \( \exp(iS) \) is string–independent and gives rise to the equations of motion (1.1)–(1.3).

On the other hand, for the construction of a quantum field theory through a functional integral approach, one needs as starting point a classical field theory action which includes also the coupling to matter fields, charged spinors or scalars. Such an action, however, is always inconsistent. On a more fundamental level the reason for this is that there is no consistent set of equations of motion which describe the coupling of gauge fields to dyonic matter fields, irrespective of the existence of an action which gives rise to them. The corresponding classical currents are not point–like, but spread out continuously and the local electric and magnetic fluxes are non–integer. As a consequence no Dirac–Schwinger condition can save the theory.

The necessarily resulting inconsistency of the field theory action can be traded in several ways, e.g. Zwanziger \[3\] and Brandt et al \[4\] sacrificed Lorentz–invariance keeping locality, while the authors of \[3\] keep Lorentz–invariance but renounce to the description of the matter field dynamics through a local action.

In this paper, relying on to the PST–mechanism, we present a classical field theory action which keeps formally Lorentz–invariance and locality, but depends on an arbitrary, nowhere light–like, external fixed vector field \( U^\mu(x) \) (see (5.7)). The interpretation of this vector field is rather simple: its unique integral curve through a given space–time point determines the Dirac–string attached to that point. This will allow us to define Dirac–strings not only for point–like particles but also for continuous current distributions. All correlators of observables in the corresponding quantum field theory will be shown to be independent of the choice of the vector field \( U \), if (1.4) holds, guaranteeing in this way the consistency of the quantum theory. (A completely analogous statement can be made also for the \( Z_4 \)–theory.) This can be regarded as the main result of the paper.

At the quantum level the particular choice \( U^\mu(x) = n^\mu \) relates our formulation to the one in \[3, 4\].

The PST–method appears rather powerful in combining manifest duality and Lorentz invariance at the classical level, and a variant of it allows also to write manifestly Lorentz–invariant classical actions for self–dual tensor fields (chiral bosons) in \( 4k + 2 \) dimensions \[10\]. It turns also out to be compatible with all relevant symmetries, in particular with diffeomorphisms as a direct consequence of its manifest Lorentz–invariance. The reliability of the method at the quantum level has been examplified in \[11\], where the gravitational anomalies for chiral bosons have been derived performing a perturbative one–loop analysis of the PST–action. The results of the present paper underline, once more, that the method is perfectly well suited also at the quantum level, and that it allows a direct control of Lorentz–invariance in quantized theories.
For the reasons just explained, in the PST–approach the coupling of the classical field theory of dyons to external gravity, represented by a classical metric $g_{\mu\nu}(x)$, can be achieved simply through the minimal prescription, at least in a topologically trivial manifold. In the presence of gravity the classical action depends again on a given vector field $U^\mu(x)$. In the corresponding quantum theory, where $g_{\mu\nu}$ and $U^\mu$ remain classical fields, under a change of $U \to U'$ the correlators of the observables change, as in the flat case, by a phase factor, the “Dirac–anomaly”; but, since this phase factor is a topological term it is metric independent, and since for the flat theory it equals unity under (1.4), it equals unity also for the theory in a curved background.

The analysis of the quantum field theory in the present paper is made at a non–perturbative (although formal) level, using a functional integral approach, but ignores ultraviolet divergences; some regulator is always implicitly assumed. For a discussion of the renormalizability of the quantum theory of dyons, in different approaches, in particular of the stability under renormalization of (1.4) and (1.5), we refer the reader to [12, 13].

In the next section we give a self–contained account of the classical PST–approach for the $SO(2)$–theory, using the concise formalism of differential forms, which we will keep throughout this paper. Section three is devoted to the classical point particle theory and establishes the Dirac–Schwinger condition as a necessary and sufficient condition for a consistent classical action principle. In section four, which is a kind of intermezzo, we give a short account of the $Z_4$–theory; the formulation of this theory does not necessitate the PST–techniques and requires only the introduction of a single vector potential. In section five we return to the $SO(2)$–theory and present its inconsistent classical field theory version, while in section six we prove the consistency of its quantum field theory version, paying particular attention to the correlation functions of observables. Section seven is devoted to a discussion of duality symmetries, $\vartheta$–angles, and the relation between the PST–approach and previous approaches to the theory of interacting dyons. Section eight contains some concluding remarks. Some basic identities for differential forms and a technical proof are relegated to the appendix.

2 The PST–approach

In this section we give a self–contained account of the classical PST–method, applied to Maxwell’s equations. First we introduce our basic notations concerning differential forms and present an equivalent form for (1.1) and (1.2), suitable for the PST approach.

The components of a $p$–form $\Phi_p$ are defined by

$$\Phi_p = \frac{1}{p!} dx^{\mu_1} \cdots dx^{\mu_p} \Phi_{\mu_1 \cdots \mu_p}.$$

The Hodge dual of $\Phi_p$, a $(4-p)$–form, has components

$$(\ast \Phi_p)_{\mu_1 \cdots \mu_{4-p}} = \frac{1}{p!} \varepsilon_{\mu_1 \cdots \mu_{4-p} \nu_1 \cdots \nu_p} \Phi^{\nu_1 \cdots \nu_p},$$
and the differential $d$ acts from the right, $d^2 = 0$. On $p$–forms the Hodge duality operator squares to $*^2 = (-1)^{p+1}$. Forms in the image of $d$ are called exact and forms in the kernel of $d$ are called closed. Since we work in $R^4$ all closed forms are also exact. The product between forms will always be an exterior (wedge) product and the wedge symbol $\wedge$ will be omitted.

In the language of forms Maxwell’s equations are written in terms of the field strength two–form $F$ and of the three–forms $(J^e, J^g)$, which are the Hodge duals of the currents $(j^e, j^g)$, as

$$d * F = J^e \equiv J^1$$
$$dF = - J^g \equiv J^2. \quad (2.1)$$

These equations imply current conservation, which reads now

$$dJ^1 = 0 = dJ^2,$$

and these, in turn, allow the introduction of a doublet of two–forms $C^I$, the “strings”, satisfying

$$J^I = dC^I, \quad I = 1, 2. \quad (2.2)$$

In the sequel with the capital indices $I, J, K, \ldots$ we will indicate $SO(2)$–doublets. The two–forms $C^I$ are defined only modulo exact 2–forms, i.e. we have invariance under

$$C^I \rightarrow C^I + dH^I. \quad (2.3)$$

For the time being we make an arbitrary but fixed choice of $C^I$ and we anticipate only that this ambiguity will correspond precisely to a Dirac–string change, as it will become clear in the next section.

Introducing a doublet of one–form gauge potentials, $A^I = dx^\mu A^I_\mu$, Maxwell’s equations for conserved external currents can be put in the equivalent manifestly $SO(2)$– and $SO(1,3)$–invariant form

$$F^I = dA^I + C^I \quad (2.4)$$
$$F^I = *\varepsilon^{IJ}F^J. \quad (2.5)$$

$\varepsilon^{IJ}$ is the $SO(2)$–invariant antisymmetric tensor, with $\varepsilon^{12} = +1$, and in the following we will suppress the $SO(2)$–indices, e.g. $(2.3)$ will be written simply as $F = *\varepsilon F$. In particular, in bilinears a contraction of those indices is always understood. The equivalence of $(2.4)$, $(2.5)$ with $(2.1)$ is established via $(2.2)$ and through the identification $F = F^2$. The eq. $(2.4)$ has to be viewed as a definition of $F^I$ while the pseudo self–duality relation $(2.5)$ produces really the dynamics.

The invariance of the curvature doublet $F$ under “string changes”, requires that $(2.3)$ has to be accompanied by

$$A^I \rightarrow A^I - H^I. \quad (2.6)$$
The PST–approach allows now to write a manifestly invariant action \[7, 8\] for the pseudo self–duality equation of motion (2.5). One introduces a single auxiliary scalar field \(a(x)\) and the corresponding one–form

\[
v = \frac{da}{\sqrt{-\partial_a \partial^a}} \equiv dx^\mu v_\mu,
\]

whose components satisfy \(v^2 = v^\mu v_\mu = -1\). It is also convenient to introduce the associated vector–field, which we indicate with the same symbol, \(v = v^\mu \partial_\mu\), and to indicate with \(i_v\) its interior product with a \(p\)–form.

The PST–action can then be written as the integral of a four–form,

\[
S_0[A, C, a] = \frac{1}{2} \int \mathcal{P}(v) \mathcal{F} + dA \varepsilon C. \tag{2.7}
\]

\(\mathcal{P}(v)\) is a symmetric operator which acts in the space of two–forms and on the \(SO(2)\)–indices as

\[
\mathcal{P}^{IJ}(v) = vi_v * \delta^{IJ} + \left(vi_v - \frac{1}{2}\right) \varepsilon^{IJ}.
\]

The operators \(vi_v\) and \(*vi_v*\) map \(p\)–forms in \(p\)–forms and project a form respectively on its components along \(v\) and orthogonal to \(v\); their basic properties are given in the appendix.

In the compact notation used in (2.7) the usual Maxwell action with a single \(F_{\mu\nu}\),

\[-\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu},\]

reads \(\frac{1}{2} \int F \ast F\).

The second term in (2.7) can also be written as

\[
\frac{1}{2} \int dA \varepsilon C = -\frac{1}{2} \int A \varepsilon J = \frac{1}{2} \int d^4 x A_\mu \varepsilon J^\mu. \tag{2.8}
\]

It is a kind of Wess–Zumino term and represents the standard “electrical” couplings of both currents. Notice, however, that the couplings carry a factor of one–half w.r.t. ordinary Q.E.D. This discrepancy is balanced by the fact that both currents carry also a “magnetic” interaction, appearing in the first term in the form \(dA + C\). In this \(SO(2)\)–symmetric formulation there is, actually, no way to distinguish between magnetic and electric currents, both are electrically and magnetically coupled.

What determines, ultimately, the form of the PST–action are its particular local symmetries. Under generic variations of the fields \(a\) and \(A\) one has, in fact

\[
\delta S_0 = \int \frac{1}{2} d \left( \frac{1}{\sqrt{-\partial a}^2} v f \varepsilon f \right) \delta a - d(vf) \varepsilon \delta A, \tag{2.9}
\]

where we defined the doublet of one–forms

\[
f = i_v(F - \ast \varepsilon F).
\]
From (2.9) one sees that the action is invariant under the following transformations:

\[ \delta A = d\Lambda \]  
\[ \delta A = \Phi da \]  
\[ \delta A = -\frac{\varphi}{\sqrt{-\left(\partial a\right)^2}} f, \quad \delta a = \varphi. \]  

(2.10)  
(2.11)  
(2.12)

The transformation parameters are the doublets of scalars \( \Lambda \) and \( \Phi \), and the single scalar \( \varphi \). The transformations in (2.10) are just ordinary \( U(1) \) gauge transformations for the vector potentials \( A \), and (2.12) states that the field \( a \) is a non-propagating auxiliary field which can be shifted to any value. The transformations (2.11) allow to reduce the second order equation of motion for the gauge fields to the first order pseudo self-duality relation (2.5). To see this we write the equations of motion for \( A \) and \( a \), which can be read directly from (2.9),

\[ d(vf) = 0 \]  
\[ d \left( \frac{v}{\sqrt{-\left(\partial a\right)^2}} f \varepsilon f \right) = 0. \]  

(2.13)  
(2.14)

The equation of motion for \( a \) (2.14) is a consequence of the \( A \)-equation (2.13) and the general solution of the latter is \( vf = da d\Phi \), for some scalar doublet \( \Phi \); through a transformation (2.11), with \( \Phi = \Phi' \), one can set this doublet to zero and obtains \( f = 0 \leftrightarrow F = \varepsilon F \), which is the desired result.

In the sequel an important role will be plaid by the “Dirac-anomaly” of the PST-action. It is defined as the (finite) variation of the action under string changes (2.3),(2.6). The first term in (2.7) is invariant by construction while the Wess-Zumino term contributes with

\[ A_D[H] \equiv \delta S_0 = \frac{1}{2} \int C \varepsilon dH = \frac{1}{2} \int J \varepsilon H. \]  

(2.15)

Despite of this anomaly the equations of motion for the gauge fields are invariant under string changes, as we saw, but this anomaly will be no longer harmless when we couple the PST-action to dynamical matter, as we will see in the next section.

### 2.1 The electromagnetic field strength

An issue left open by the PST-approach is the correct identification of the electromagnetic field-strength tensor. If the equations of motion (2.13) are satisfied and the symmetries in (2.11) have been fixed as above, then the field strength is clearly say \( F = F^2 \), and \( F^1 = \varepsilon F \). But these tensors can not be identified with the electromagnetic field strength off-shell, since they are not invariant under (2.11) and (2.12), even if (2.13) holds. The correct off-shell electromagnetic tensor is represented by the two-form doublet

\[ K^I \equiv F^I - vf^I, \]  

(2.16)
which is uniquely characterized by the following symmetry properties:

1) $\delta K = 0$ under (2.10) and (2.11) and under string changes.
2) $K$ is invariant under (2.12) on–shell (in fact, $\delta K = \frac{\sqrt{-\eta}}{(\partial \eta)^2} (1 + *\varepsilon) i_v d(vf)$).
3) $K = *\varepsilon K$.
4) $K$ reduces on–shell to $F$, if the symmetry (2.11) is fixed as above.

In particular, the $K$’s are invariant under all symmetries on–shell and this is just what is required for an observable. For example, the quantum correlators of the field strength tensors in a path integral representation are obtained by insertion of the fields $K$. The pseudo self–duality property 3) ensures then the “uniqueness” of these correlators as they involve only a single field strength, say $K^1$, and its dual; if one had used the $F'$s instead of the $K'$s then the correlators for $F^1$ would differ from the ones of $*F^2$.

Notice also that the equations of motion for $A$ (2.13) can be written alternatively as

$$dK = dC.$$ 

2.2 The effective action

We conclude this section with the derivation of the effective current–current action associated to the PST–action, which is obtained upon performing the functional integration over the gauge fields $A$ and over the auxiliary field $a$ in (2.7). These integrations require an appropriate gauge fixing of the local symmetries (2.10)–(2.12). Since $a$ is a “pure gauge” field, the integration over $a$ becomes trivial since the insertion of a $\delta$–function $\delta(a(x) - a(0)(x))$ just works. Notice that, due to the appearance of $\partial_\mu a$ in the denominators of the classical action, the choice $a(0) = \text{const.}$ is not allowed. Minimal choices are $a(0) = x^\mu n_\mu$, for some constant vector $n$, which lead to $v(0)_\mu = \frac{n_\mu}{\sqrt{-n^2}}$. We make here a generic choice for $a(0)$ which leads to a nowhere singular $v(0)$.

To integrate over $A$ one has to fix the symmetries (2.10) and (2.11). To avoid explicit gauge fixings, which make the derivation rather cumbersome for a generic $a(0)$, we proceed in the following alternative way. We rewrite (2.7) as

$$S_0[A, C, a] = \Gamma_0[C] + \frac{1}{2} \int (G + D) \Omega(v)(G + D), \quad (2.17)$$

where $G$ and $D$ are single two–forms defined by

$$G = dA^1 - *dA^2 \quad (2.18)$$

$$D = C^1 - *C^2. \quad (2.19)$$

$\Omega(v)$ is an operator, mapping two–forms in two–forms, given by

$$\Omega(v) = \frac{d * d}{\boxtimes} - v * v,$$

where $\boxtimes = \partial_\mu \partial^\mu$ is the D’Alambertian. We recall the standard Hodge decomposition on $p$–forms:

$$\boxtimes = d \delta + \delta d,$$
where $\delta$ is the co–differential, 

$$\delta = *d * .$$

$\Gamma_0$ is defined by

$$\Gamma_0[C] = -\frac{1}{2} \int \left( dC * dC - dC \frac{\varepsilon}{\Box} \delta C \right), \tag{2.20}$$

where contraction of the $SO(2)$–indices of the $C^I$ is understood. The effective action is defined by

$$e^{i\Gamma[C]} = \frac{\int \{DA\} g_f \{Da\} \delta(a - a^{(0)}) e^{iS_0[A,C,a]} \int \{DA\} g_f \{Da\} \delta(a - a^{(0)}) e^{iS_0[A,0,a]} }{\int \{DA\} g_f \{Da\} \delta(a - a^{(0)}) e^{iS_0[G+D] \Omega(v^{(0)}) \Omega(v^{(0))} G} . \tag{2.21}$$

Since there exist linear gauge fixings for the symmetries (2.10) and (2.11), as shown below, in the numerator one can now perform the shift of integration variables

$$A^1 \rightarrow A^1 - \frac{\delta}{\Box} D \tag{2.23}$$

$$A^2 \rightarrow A^2 + * d \frac{\Box}{\Box} D, \tag{2.24}$$

proving that the numerator is $C$–independent and equals the denominator, so that $\Gamma[C] = \Gamma_0[C]$.

For gauge–fixings of the kind $a^{(0)} = x^\mu n_\mu$ the functional integral over $A$ can be performed also with more conventional techniques, since in this case the kinetic term for the gauge fields in the PST–action becomes local. We have now $v_\mu = n_\mu / \sqrt{-n^2} =$ const., and the symmetries (2.10) and (2.11) can be fixed imposing

$$i_v A = 0 = d * A. \tag{2.25}$$

These gauge fixings are suitable also for a generic $a^{(0)}$. The kinetic term for the gauge fields in (2.7) reduces with these gauge fixings to

$$\frac{1}{2} \int A * \mathcal{K}(v) A = -\frac{1}{2} \int A * \begin{pmatrix} \Box + \partial_v^2 & T \partial_v \\ -T \partial_v & \Box + \partial_v^2 \end{pmatrix} A, \tag{2.26}$$

where $T = * v d$ is an operator which sends one–forms into one–forms and $\partial_v = v^\mu \partial_\mu$. The functional integral over $A$ in (2.7) is Gaussian and one needs only to compute the propagator–matrix for the gauge fields. Using the algebraic identity

$$T^3 = -\left( \Box + \partial_v^2 \right) T,$$

which holds for constant $v$, this matrix can be evaluated as

$$\mathcal{K}^{-1}(v) = -\frac{1}{\Box (\Box + \partial_v^2)} \cdot \begin{pmatrix} \Box - \frac{T^2 \partial_v^2}{T \partial_v} & -T \partial_v \\ T \partial_v & \Box - \frac{T^2 \partial_v^2}{T \partial_v} \end{pmatrix} , \tag{2.27}$$
The gaussian integral results then in
\[ \Gamma[C] = \frac{1}{2} \int \left( d \left( \mathcal{P}(v) + \frac{1}{2} \varepsilon \right) C \right) K^{-1}(v) * d \left( \mathcal{P}(v) + \frac{1}{2} \varepsilon \right) C + C \mathcal{P}(v) C. \]

It is a mere exercise to show that the \( v \)-dependence drops out from this expression and that the latter coincides with (2.20). The independence of the effective action of the particular choice one makes for \( a^{(0)} \) is clearly a consequence of the symmetry (2.12) of the PST–action.

The first term of the effective action corresponds to the usual diagonal current–current interaction (remember that \( J = dC \)), while the second term, corresponding to a mixed electric–magnetic interaction, involves not only \( J \) but exhibits also an explicit dependence on the strings \( C \). It is this mixed interaction which is responsible for the Dirac–anomaly associated to \( \Gamma_0[C] \). The effective action carries, indeed, the same Dirac–anomaly as the PST–action itself: under \( C \to C + dH \) one has
\[ \Gamma_0 \to \Gamma_0 + A_D[H]. \]

This simple result will play a fundamental role in establishing the consistency of the quantum field theory.

3 Point–like particles

Classical point–like dyons are characterized by their mass \( m_r \) and by their electric and magnetic charges, \( e_r' = (e_r, -g_r) \), and sweep out a space–time trajectory \( \gamma_r \), parametrized by arbitrary parameters \( s_r, \{ y^\mu(s_r), -\infty \leq s_r \leq \infty \} \). The extension of the PST–action to dynamical classical particles requires the addition of the standard term
\[ -\sum_{r=1}^{N} m_r \int_{\gamma_r} d\tau_r, \]
but needs also an appropriate definition of the two–forms \( C \) in terms of the particles trajectories. For this purpose we make now a short digression on Poincarè–duality and de Rham–currents.

3.1 Poincarè–Duality and Dirac–strings

Until now we did not specify the regularity properties of the differential \( p \)-forms introduced, and we assumed implicitly that their components are smooth functions. For our purposes it is, however, more convenient to consider them as “(de Rham) \( p \)-currents”, i.e. linear functionals on the space of smooth \((4 - p)\)-forms with compact support, which are continuous in the sense of distributions [14]. In other words, \( p \)-currents are \( p \)-forms with distribution–valued components; we will call them still “forms”.

In this space we can extend Poincarè duality to a map PD which associates to every \( p \)-dimensional hypersurface \( \Sigma_p \) a \((4 - p)\)-form \( \Phi_{\Sigma_p} \) according to
\[
\text{PD : } \Sigma_p \to \Phi_{\Sigma_p}, \quad \int_{\Sigma_p} \Psi_p = \int_{R^4} \Psi_p \Phi_{\Sigma_p}, \quad (3.1)
\]
for any $p$–form $\Psi_p$. This map respects in particular the exact–forms $\leftrightarrow$ boundary–hypersurfaces correspondence, i.e.

\[
\text{PD} : \Sigma_p \rightarrow \Phi_{\Sigma_p} \quad \text{implying} \quad (3.2)
\]

\[
\text{PD} : \partial \Sigma_p \rightarrow d\Phi_{\Sigma_p}, \quad (3.3)
\]

where $\partial$ indicates the boundary operator $[1]$.

A basic consequence of PD–duality is that the integral of $\Phi_{\Sigma_p}$ over a generic $(4 - p)$–dimensional surface $S_{4-p}$ is an integer, counting the intersections with sign of $\Sigma_p$ with $S_{4-p}$. This implies, in particular, that the integral over $R^4$ of a product of two such forms is also an integer,

\[
\int_{R^4} \Phi_{\Sigma_p} \Phi_{\Sigma_{4-p}} = N, \quad (3.4)
\]

which counts the number of intersections with sign of $\Sigma_p$ with $\Sigma_{4-p}$.

Linear combinations of such $p$–forms with integer coefficients are called integer forms (they are PD–dual to integer $(4 - p)$–chains [14]). The property (3.4) holds also for generic integer forms: the integral of a product of two integer forms is an integer, whenever the integral is well defined.

In this language the closed three–forms $J$ are a weighted sum of closed integer forms, the weights being given by the charges

\[
J^I = \sum_r e^I_r J_r, \quad (3.5)
\]

where the closed integer three–forms $J_r$ are the Poincarè duals of the boundaryless curves $\gamma_r$,

\[
J_r = \frac{1}{3!} dx^\sigma dx^\nu dx^\mu \epsilon_{\mu \nu \sigma} \int_{\gamma_r} \frac{dy^\sigma}{ds_r} \delta^4(x - y_r) ds_r. \quad (3.6)
\]

In this case the Wess–Zumino term of the PST–action can be written also in the standard form

\[
-\frac{1}{2} \int A \epsilon J = -\frac{1}{2} \sum_r e_r \epsilon \int_{\gamma_r} A. \quad (3.7)
\]

Once the currents have been determined as in (3.5), for the strings $C$ subject to $dC = J$ we choose a class of solutions given by

\[
C^I = \sum_r e^I_r C_r, \quad (3.8)
\]

where the $C_r$ are integer two–forms, being Poincarè duals of two–dimensional hypersurfaces, extending to infinity, whose boundaries are $\gamma_r$. This implies in particular that

\[
J_r = dC_r.
\]

For a mathematically precise formulation, involving chains, see [14].
If the surfaces are parametrized for example by \( Y^\mu_r(s, u) \) with \( Y^\mu_r(s, 0) = y^\mu_r(s) \), then one has the explicit expressions

\[
C_r = \frac{1}{2} dx^\nu dx^\mu \varepsilon_{\mu\rho\sigma} \int_0^\infty du \int_{-\infty}^{+\infty} ds \frac{dY^\rho_r}{du} \frac{dY^\sigma_r}{ds} \delta^4(x - Y_r).
\] (3.9)

The way of formalizing the Dirac–string problem presented above unifies, actually, a variety of treatments made in the literature, which differ through the particular classes of chosen surfaces. In [9, 12], where only closed curves are considered, the surfaces have been chosen as bounded ones; in [3, 4] they have been chosen to be in every point parallel to a given constant four–vector \( n^\mu: Y^\mu(s, u) = y^\mu(s) + n^\mu u \). Schwinger’s method [1, 2] amounts to picking a single space–time curve \( \xi(u) \) which extends from the origin to infinity, chosen once for ever, and to attaching this same curve to every point of the particle trajectory: \( Y^\mu(s, u) = y^\mu(s) + \xi^\mu(u) \). Schwinger considered, indeed, also double–sheeted but half–weighted surfaces. These are obtained from the single–sheeted surfaces in (3.9) with the replacement

\[
\int_0^{+\infty} du \rightarrow \frac{1}{2} \left( \int_0^{+\infty} du - \int_{-\infty}^0 du \right),
\]

and upon defining \( \xi(-u) = -\xi(u) \). In his formulation single–sheeted surfaces gave rise to the \( Z_4 \)–theory and double–sheeted ones to the \( SO(2) \)–theory. Since our theory is manifestly \( SO(2) \)–invariant one could equally well use one or the other type of surfaces; for definiteness from now on we use the type (3.9).

Among all possible surfaces of this type there are special classes which will become relevant at the field theory level later on. Each of these classes is characterized by a nowhere light–like vector field \( U^\mu(x) \)

\[
U^\mu(x)U_\mu(x) \neq 0.
\]

This condition is necessary to ensure the validity of the associated decomposition of the identity on \( p \)–forms

\[
1 = \frac{1}{U^2} \left( (-)^{p+1}UiU + *UiU * \right), \quad U \equiv dx^\mu U_\mu,
\] (3.10)

relation which allows to decompose a \( p \)–form uniquely in a \( p \)–form parallel to \( U \) and one orthogonal to \( U \).

The strings associated to \( U \) are determined requiring that they are made out of the integral curves of \( U \) which end on the particles trajectories. The related \( C \)–fields are uniquely determined through the equations

\[
dC = J
\]

\[
i_U C = 0,
\] (3.11)

and through the boundary condition

\[
C \rightarrow 0 \quad \text{for} \quad x_U \rightarrow -\infty,
\] (3.12)
meaning that we require that \( C(x) \) goes to zero if \( x \) goes to \(-\infty\) along the integral curves of the vector field \( U\).

To prove the uniqueness, and existence, of the solution of this system we observe that there exists always a diffeomorphism which maps the vector field \( U^\mu(x) \) to a constant vector \( N^\mu \). Since the system (3.11), (3.12) is diffeomorphism invariant and metric independent, in the new coordinate system it becomes

\[
dC = J \\
i_N C = 0 \\
C \to 0 \quad \text{for} \quad x_N \to -\infty,
\]

where \( x_N^\mu = uN^\mu \) is the coordinate along \( N \) and the limit above means \( u \to -\infty \). This system has now the solution

\[
C = \frac{1}{\partial_N} i_N J,
\]

as can be seen observing that

\[
i_N d + di_N = \partial_N.
\]

To satisfy the boundary condition in (3.13) the inverse operator \( \frac{1}{\partial_N} \) has to be defined by the Kernel

\[
G(x) = \Theta(x_N)\delta^3(\vec{x}_N), \quad \partial_N G(x) = \delta^4(x),
\]

where \( \vec{x}_N \) are the three coordinates orthogonal to \( x_N \) and \( \Theta \) is the step–function. The uniqueness of the solution can be inferred considering the associated homogeneous equations, which lead, thanks to (3.15), to \( \partial_N C = 0 \) and this, due to the boundary condition, implies \( C = 0 \). The solution of the original system can be obtained from (3.14) upon performing the inverse diffeomorphism.

This proof of the uniqueness of the solution for \( C \) applies equally well to integer and smooth \( J \).

If the currents are given as in (3.5) and (3.6), one can also exhibit an explicit expression for the unique solution. For each curve \( \gamma_r \) one determines the integral curves of \( U \) ending on \( \gamma_r \),

\[
\frac{d}{du} Y_r^\mu(s, u) = U^\mu(Y_r(s, u)), \quad 0 \leq u \leq \infty \\
Y_r^\mu(s, 0) = y_r^\mu(s),
\]

and one uses the associated surfaces in the general formula (3.9), to obtain

\[
C_r = \frac{1}{2} \int_0^\infty dx d\sigma \varepsilon_{\mu\rho\sigma} U^\rho(x) \int_{-\infty}^{\infty} ds \frac{dY_r^\sigma}{ds} \delta^4(x - Y_r).
\]

The total \( C \)'s are given by a weighted sum according to (3.8); one recognizes in particular that \( i_U C = 0 \). It is worthwhile to notice that the representations for \( J \) and \( C \) given above
apply equally well when the curves are closed; closed curves will indeed play a central role in the quantum field theory.

Once the $C$–fields are weighted integer two–forms, what we called previously string changes amounts now just to a change of the surface whose boundary is a given curve:

$$C_r \to C_r + dH_r,$$

where $H_r$ is the integer one–form associated, via PD, to the three–dimensional hyper–surface which is bounded by the old and the new surfaces. For the two–forms $C$ the transformations (3.19) lead to

$$C^I \to C^I + dH^I, \quad \text{with}$$

$$H^I = \sum_r e^I_r H_r.$$

### 3.2 Action principle for classical dyons

Now we can present an action principle for classical dyons. The action which describes the interaction of dyons with the electromagnetic field would be given in the PST–approach by

$$S[A, C_r, a] = S_0[A, C, a] - \sum_{r=1}^{N} m_r \int_{\gamma_r} d\tau_r,$$

where the $C_r$–fields are defined as above in terms of arbitrary surfaces; thus the dynamical variables of this action are formally the strings $C_r$ and not the curves $\gamma_r$. But the string dependence of $S$ is measured by the Dirac–anomaly. Under (3.20), together with $A_I \to A^I - H^I$, one has

$$S \to S + A_D[H],$$

where now

$$A_D[H] = \frac{1}{2} \int J_\varepsilon H = \frac{1}{2} \sum_{r,s} (e_r \varepsilon e_s) \int J_r H_s.$$

The last integral is integer, because the integrand is a product of integer forms, and $\frac{1}{2} (e_r \varepsilon e_s) = \frac{1}{2} (e_1^r e_2^s - e_2^r e_1^s)$ is an integer multiple of $2\pi$ due to the Dirac–Schwinger condition (1.4). In conclusion, for the point particle theory, the Dirac–anomaly is an integer multiple of $2\pi$. This suggests to consider as action functional the exponentiated action $e^{iS}$, which is string–independent $\ddagger$.

For regular curves the “dimension” of the set of these intersection points is $-1$.

---

$\ddagger$The classical theory could be based also on the exponential of a scaled action, $e^{i\lambda S}$, which would imply a quantization condition where $\frac{1}{2} (e_r \varepsilon e_s)$ is an integer multiple of $2\pi$ modulo an overall constant; it is quantum mechanics which eventually forces $\lambda = \hbar = 1$.

$\ddagger$For regular curves the “dimension” of the set of these intersection points is $-1$. 

---
the boundary of this three–volume. These are precisely the configurations which were
forbidden by Dirac’s veto [13], and which were counted by Schwinger [2] with weight
1/2 (instead of 0 or 1), leading him to postulate momentarily the stronger quantization
condition \( \frac{1}{2} (e_r \varepsilon e_s) = 4\pi n \), a request which he eventually abandoned.

To cope also with these configurations we modify the action (3.21) as follows. For
each configuration \( \{\gamma_r\} \), in addition to the arbitrary strings \( C_r \), we introduce a set of
“regular” strings \( \tilde{C}_r \), \( d\tilde{C}_r = J_r \), which never intersect a curve \( \gamma_s \) (this is always possible
if the curves themselves do not intersect) and define \( \tilde{C}^I = \sum_r e_r^I \tilde{C}_r \). Then we modify the
action according to
\[
\tilde{S}[A, C_r, \tilde{C}_r, a] = S[A, C_r, a] + \frac{1}{2} \int C \varepsilon \tilde{C} 
= \frac{1}{2} \int F \mathcal{P}(v) F + F \varepsilon \tilde{C} - \sum_r m_r \int_{\gamma_r} d\tau_r, \tag{3.24}
\]
and treat \( C_r \) and \( \tilde{C}_r \) as dynamical variables. The added term is \( \frac{1}{2} \sum_{r,s} (e_r \varepsilon e_s) \int C_r \tilde{C}_s \) and, since the integrand is a product of integer forms, upon exponentiation it contributes only
if the \( C_r \) are in exceptional configurations, i.e. if they intersect a curve \( \gamma_s \) which is, indeed,
the boundary of \( \tilde{C}_s \). Notice in particular that, since the added term is \( A \)–independent,
\( \tilde{S} \) possesses still the PST–symmetries (2.10)–(2.12) and leads to the same \( A \)–equations of
motion.

Since \( C \) appears in \( \tilde{S} \) only in the combination \( F = dA + C \), the modified action is now
strictly invariant under a \( C \)–string change, even if (1.4) does not hold. All the string–
dependence has now been shifted to the regular strings \( \tilde{C} \). Under \( \tilde{C} \to \tilde{C} + d\tilde{H} \) one has
\[
\tilde{S} \to \tilde{S} + A_D[\tilde{H}],
\]
and \( A_D[\tilde{H}] \) is always an integer multiple of \( 2\pi \).

The action principle can be based on the functional \( \exp(i\tilde{S}) \) in a canonical manner.
One introduces a real parameter \( \alpha \) and deformed strings \( C_\alpha, \tilde{C}_\alpha \) such that \( C_\alpha \to C \),
\( \tilde{C}_\alpha \to \tilde{C} \) for \( \alpha \to 0 \), and one evaluates the corresponding action \( \tilde{S}_\alpha \). These deformations
include in particular deformations of the curves. The equations of motion for the particles
(Lorentz–force law) are then obtained as
\[
\left. \left( \frac{d}{d\alpha} e^{i\tilde{S}_\alpha} \right) \right|_{\alpha=0} = 0. \tag{3.25}
\]
As \( \exp(i\tilde{S}) \) is invariant under string changes and under the PST–symmetries so are the
equations of motion; this implies already that the Lorentz–force can depend on \( C, A \) and
\( a \) only through the tensors \( K \), defined in the preceding section, and on \( u_r^\mu = \frac{dx_r^\mu}{d\tau_r} \). Apart
from a term proportional to the \( A \)–equations,
\[
dK^I = J^I, \tag{3.26}
\]
an explicit evaluation of (3.25) leads indeed to

\[ m_r \frac{du^\nu}{d\tau_r} = (e_r \in K^{\mu\nu})(y_r) u_{r\nu}. \]  

(3.27)

Remembering that one has also

\[ K^I = \ast \varepsilon^{IJ} K^J, \]  

(3.28)

and that \( K \) is an on–shell completely invariant tensor, the system (3.26)–(3.28) becomes a closed system for all observables in that it determines uniquely \( K^I \) and \( y_r \), once the initial conditions have been fixed. From this point of view \( K \) is truly identified with the electromagnetic field strength and, a posteriori, there is no need to introduce vector potentials or strings. Alternatively, one can use the gauge–fixing of the PST–symmetry (2.11) presented in the preceding section, which leads to \( K^I = F^I \) and \( F^1 = \ast F^2 \), and the system given above reduces to the system (1.1)–(1.3), with \( F = dA + C \).

This concludes the presentation of the extension of the PST–method to a system of classical dynamical dyons. Summarizing we can say that the functional \( \exp(\tilde{S}) \), which is manifestly \( SO(1,3) \) and \( SO(2) \)–invariant, gives rise to the correct equations of motion (1.1)–(1.3) if the Dirac–Schwinger quantization condition (1.4) holds.

The next issue would be the classical field theory, but before addressing this point we present now the essential features of the \( Z_4 \)–theory.

4 The \( Z_4 \)–theory

Let us first of all stress that this theory is equivalent to the \( SO(2) \)–theory for what concerns the classical equations of motion, which are again (1.1)–(1.3). It differs from the latter only through the classical action, on which these equations are based, and through the quantization condition, now (1.5), which ensures the consistency of the corresponding action principle. Another important difference lies in the fact that the classical PST–action is manifestly \( SO(2) \)–invariant, whether (1.4) holds or not, while for the \( Z_4 \)–theory \( Z_4 \) becomes a symmetry group of the action only at the quantum level, and requires (1.7).

4.1 The classical and effective actions

Since the \( Z_4 \) theory is not \( SO(2) \)-invariant we abandon in this section the doublet notation and introduce a single vector potential \( A = A^2 \) and the single curvature two–form \( F = F^2 = dA + C^2 \). The two–forms \( C^1 \) and \( C^2 \) maintain the meaning of the preceding section. \( J^1 = dC^1 \) represents now the electric current and \( J^2 = dC^2 \) the magnetic one. The action, analogous to \( S_0 \), is

\[ I_0[A, J^1, C^2] = \int \frac{1}{2} F \ast F - dA C^1 = \int \frac{1}{2} F \ast F + A J^1, \]  

(4.1)
where the electric current carries now only an electric coupling, with weight 1, and the magnetic current only a magnetic one. The equation of motion for $A$ and the Bianchi identity for $F$ reproduce the generalized Maxwell equations (2.1).

The action (4.1) is invariant under ordinary $U(1)$ gauge transformations, $A \rightarrow A + d\Lambda$, while under string changes

$$C^1 \rightarrow C^1 + dH^1$$
$$C^2 \rightarrow C^2 + dH^2$$
$$A \rightarrow A - H^2,$$  \hspace{1cm} (4.2)

it varies as

$$A_D[H^2] \equiv \delta I_0 = \int J^1 H^2.$$  \hspace{1cm} (4.3)

Notice that under $C^1 \rightarrow C^1 + dH^1$, $A \rightarrow A$ the action is invariant.

The quantum field theory associated to (4.1) can be constructed according to the recipe reported in section six. The resulting theory, based on a functional integral, can be considered as a formal continuum limit of the compact Q.E.D. with scalar matter fields, formulated on a lattice. In fact, due to the invariance (4.2) we can view $A \text{mod} \ H^2$ as a continuum analogue of the compact–$U(1)$ gauge field on the lattice.

The current–current effective action associated to $I_0$ is obtained upon performing the functional integration over the vector potential $A$

$$e^{iI_0^Z_4[C]} = \frac{\int \{DA\}_g e^{iI_0[A,J^1,C^2]} }{\int \{DA\}_g e^{iI_0[A,0,0]}},$$  \hspace{1cm} (4.4)

where one needs only to fix the ordinary $U(1)$ gauge invariance. The integral is Gaussian and gives

$$\Gamma_0^Z_4[C] = \int -\frac{1}{2} \left( dC^1 \square dC^1 + dC^2 \square dC^2 \right) + dC^1 \square \delta C^2,$$

which depends on $C^1$ only through $J^1$, as does $I_0$, and carries the same Dirac–anomaly (4.3) as the classical action.

The action principle for classical point–like particles can be formulated according to the strategy adopted for the $SO(2)$–theory. One introduces general strings $C_r$ and regular strings $\tilde{C}_r$ and constructs the total $C$’s and $\tilde{C}$’s as in the preceding section. The appropriate action for dynamical dyons is then

$$\tilde{I}[A,\tilde{C}^1,C^2] = \int \frac{1}{2} F \star F - FC^1 - \sum_r m_r \int_{r\tau} d\tau,$$  \hspace{1cm} (4.5)

and transforms under string changes, see (4.3), as

$$\delta \tilde{I} = \int J^1 H^2 = \sum_{r,s} e_r^1 e_s^2 \int J_r \tilde{H}_s.$$  \hspace{1cm} (4.6)

The last integral is integer and the Dirac–anomaly becomes an integer multiple of $2\pi$, if Dirac’s original quantization condition (1.3) holds. The added term, $-\int C^2 \tilde{C}^1$, contributes in the exponential again only for exceptional configurations, thanks to (1.3). The action principle is based on $exp(i\tilde{I})$ and the Lorentz–force equation is again (1.3).
4.2 Comparison with the $SO(2)$–theory

To compare the two theories, which at the classical level are equivalent, we compare the effective current–current interactions, which trigger the quantum dynamics, and assume, for consistency, that (1.4) and (1.5) are satisfied.

The diagonal interactions are identical, but the difference in the mixed interactions leads to

$$\Delta \Gamma_0 \equiv \Gamma^{Z_4}_0 - \Gamma_0 = \frac{1}{2} \int C^1 C^2.$$  \hspace{1cm} (4.7)

For arbitrary $C$–fields $\Delta \Gamma_0$ is non vanishing. If the strings are weighted integer forms, as in the point particle theory, we have

$$\Delta \Gamma_0 = \frac{1}{2} \sum_{r,s} e_r e_s \int C_r C_s = \frac{1}{2} \sum_{r,s} e_r e_s N_{rs},$$

where $N_{rs}$ is a symmetric matrix of integers. What matters in the quantum field theory is eventually the exponential of the effective action, therefore a necessary and sufficient condition for the identification of the two theories is

$$e^{i \Delta \Gamma_0} = 1.$$  

This is equivalent to

$$\frac{1}{2} \left( e_r^1 e_s^2 + e_r^1 e_s^2 \right) = 2 \pi n_{rs} \text{ for } r \neq s, \quad \frac{1}{2} e_r^1 e_r^2 = 2 \pi n_{rr}. \hspace{1cm} (4.8)$$

Taking (1.4) and (1.5) into account, these conditions are satisfied only if eq. (1.6), i.e.

$$\frac{1}{2} e_r g_s = 2 \pi n_{rs},$$

holds for each $r$ and $s$.

In conclusion we can say that the consistency conditions for the $SO(2)$ and $Z_4$–theories are respectively (1.4) and (1.5), while (1.6) are the supplementary conditions for the identification of $Z_4$ as a duality subgroup of $SO(2)$.

4.3 $Z_4$–symmetry

We address now the realization of the discrete duality group $Z_4$. Its generator acts on the $C$–fields as

$$C^1 \rightarrow C^2,$$

$$C^2 \rightarrow -C^1.$$

\hspace{1cm} (4.9) \hspace{1cm} (4.10)

\footnotesize{\textsuperscript{6}For the self–interaction of the $r$–th particle a regularization is understood, which keeps $N_{rr}$ integer.}

\footnotesize{\textsuperscript{7}While (1.6) implies (1.4) and (1.5), there are solutions to this last system of equations which do not satisfy (1.6).}
The action $I_0$ is not invariant while the transformation of the effective action can be read from (1.7). Since $\Gamma_0$ is manifestly invariant, we deduce

$$\Gamma_0^{Z_4} \to \Gamma_0^{Z_4} - \int C^1 C^2. \quad (4.11)$$

For integer currents, due to (1.5), we have that $\exp(i \Gamma_0^{Z_4})$ is indeed $Z_4$–invariant.

To display the quantum implementation of this symmetry on the action $I_0$ one has to combine the $Z_4$ transformation with an $S$–duality transformation of the vector potential in the functional integral (for a discussion of $S$–duality in the functional integral formalism, see e.g. [30]). For this purpose one introduces an auxiliary two–form $H$ and a one–form $A^1$, the new vector potential, and considers the action

$$I_H = \int \frac{1}{2} \left( H + C^2 \right) \star \left( H + C^2 \right) - H \left( dA^1 + C^1 \right).$$

Functional integration over $A^1$ leads to $dH = 0 \Rightarrow H = dA^2$ and one recovers $I_0$. On the other hand, integrating over $H$

$$\int \{DH\} e^{iH} \equiv e^{i\bar{I}_0},$$

one obtains

$$\bar{I}_0 = \int \frac{1}{2} \left( dA^1 + C^1 \right) \star \left( dA^1 + C^1 \right) + dA^1 C^2 + C^1 C^2.$$

Performing now the $Z_4$–transformation on the $C$’s, accompanied by $A^1 \to A^2$, one obtains

$$\bar{I}_0 \to I_0 - \int C^1 C^2, \quad (4.12)$$

in agreement with the transformation of the effective action, (4.11).

On the other hand it may be interesting to notice that the transformation law (4.11) can also be deduced directly for the classical action $I_0$, but using a non–local transformation for the vector potential. Under

$$C^1 \to C^2$$

$$C^2 \to -C^1$$

$$A \to A + d \left( (1 + *) C^2 - (1 - *) C^1 \right)$$

one has $I_0 \to I_0 - \int C^1 C^2$.

At this point an interesting question is whether the action $I_0$ can be amended for a symmetry enhancement from $Z_4$ to $SO(2)$ to occur. This modified action would thus represent a non–manifestly invariant alternative for the PST–action $S_0$. The suggestion for such a modification comes from (4.7). It suffices indeed to put

$$\bar{I}_0 = I_0 - \frac{1}{2} C^1 C^2,$$
which has as effective action the $SO(2)$–invariant one $\Gamma_0$. But $\bar{I}_0$ possesses now also a non–local $SO(2)$–invariance, the corresponding infinitesimal transformations, with parameter $\varphi$, being given by

$$\begin{align*}
\delta C^1 &= \varphi C^2 \\
\delta C^2 &= -\varphi C^1 \\
\delta A &= -\varphi \star d (C^2 + *C^1).
\end{align*}$$

(4.16)

Under these transformations $\bar{I}_0$ is invariant irrespectively of the validity of any quantization condition on the charges, as is $\Gamma_0$. This action possesses the same Dirac anomaly as the PST–action and upon adding the same regularizing term, $\frac{1}{2} \int C \varepsilon \tilde{C}$, produces also the correct Lorentz–force–law, if the $SO(2)$–invariant quantization condition (1.4) holds. Due to its manifest invariance, however, the PST–action is preferable in many respects.

The action $I_0$ represents a generalization to arbitrary strings of the action used by Schwinger [2] to deduce the quantization conditions (1.4) and (1.5) considering respectively double–sheeted and single–sheeted strings. For single–sheeted strings, which were always implicitly assumed by us, $I_0$ is not $SO(2)$–invariant, as we saw above. Keeping single–sheeted strings it becomes $SO(2)$–invariant upon adding the term $-\frac{1}{2} \int C^1C^2$.

On the other hand, if one uses the particular double–sheeted strings introduced by Schwinger [2], this added term can be seen to vanish identically.

4.4 The $\vartheta$–angle

The last issue of this section concerns the introduction of $\vartheta$–angles in the $Z_4$–theory. In the present context this is a question about more general consistent couplings between the currents and the electromagnetic field, then those contained in $I_0$. Taking the quantization condition (1.3) into account, at present the theory depends only on a single coupling constant (apart from the masses). In fact, the general solution of (1.3) is

$$\begin{align*}
\epsilon_r &= \epsilon_0 n_r \\
g_r &= \frac{2\pi}{\epsilon_0} m_r,
\end{align*}$$

(4.17)

with $n_r, m_r$ integers, where $\epsilon_0$ is the fundamental electric charge and represents the unique coupling constant.

For additional couplings we have to require $U(1)$ gauge invariance and string independence. For dimensional reasons the unique new term which satisfies these requirements is $\int FF$. Therefore, a $\vartheta$–term can be introduced consistently in the theory according to

$$I_\vartheta = I_0 + \frac{\epsilon_0^2 \vartheta}{8\pi^2} \int FF.$$  

(4.18)

Notice that, contrary to what happens for $\vartheta$–terms in Yang–Mills theories, in the present case it is not a topological term and hence not invariant under $\vartheta \rightarrow \vartheta + 2\pi$; we will come back to this feature in sections seven and eight.
To analyze the effect of this additional coupling we compute the new Maxwell equations

\[ d \star F = J^1 - \frac{e_0^2 \vartheta}{4\pi^2} J^2 \]  
\[ dF = J^2, \]  

which show that the electric current has been shifted by an amount proportional to the magnetic current, which is indeed the expected effect of a \( \vartheta \)-term, while the magnetic current has not been touched. The new Lorentz–force law is again of the form (4.13) but the individual charges are now given by

\[ E_r = e_r + \frac{e_0^2 \vartheta}{4\pi^2} g_r \]
\[ G_r = g_r, \]  

reflecting the shift of the total currents in the Maxwell equations.

The Dirac–anomaly, instead, does not change since the \( \vartheta \)-term is string–independent.

The effective action associated to \( I_{\vartheta} \) reflects just the shifts noted above and is given by

\[ \Gamma^Z_{\vartheta} = \int -\frac{1}{2} \left( \left( dC^1 - \frac{e_0^2 \vartheta}{4\pi^2} dC^2 \right) \star \left( dC^1 - \frac{e_0^2 \vartheta}{4\pi^2} dC^2 \right) + dC^2 \star dC^2 + dC^1 \frac{1}{\vartheta} \delta C^2. \]  

Notice, however, that the mixed term is unaffected by the \( \vartheta \)-angle as is the Dirac anomaly. Consequently the consistency condition remains (1.3), with solution (4.17), and the total individual charges become

\[ E_r = e_0 \left( n_r + \frac{\vartheta}{2\pi} m_r \right) \]
\[ G_r = \frac{2\pi}{e_0} m_r. \]  

These charges satisfy, in turn, the relation \( E_r G_s - E_s G_r = 2\pi n_{rs} \), but one should keep in mind that the fundamental equations are (1.3) and (4.21). There are, in fact, solutions of the former relation, depending on one more parameter, which are not taken into account by (4.23).

One might wonder whether \( \vartheta \)-terms can be introduced also in the \( SO(2) \)-theory. The answer, which is encoded in the different quantization condition (1.4), is that these couplings are already present because that quantization condition allows for solutions with non vanishing \( \vartheta \)-angles (see section seven).

5 Classical \( SO(2) \)-field theory

The starting point for a quantum field theory in a functional integral approach is a classical field theory action. Since for a theory of charges and Dirac–monopoles there is no natural string–independent choice for such an action, we begin by searching for a convenient set
of equations of motion, over which we have a more direct control. Our guiding principles will be \( U(1) \) gauge invariance, current conservation, and manifest \( SO(2) \)-invariance. For simplicity we consider here dyonic matter fields which are only minimally coupled to gauge fields, without additional mutual interactions.

## 5.1 A set of equations of motion

The field content is a doublet of vector potentials \( A^I \) and a finite number of bosonic or fermionic complex matter fields \( \Phi^r \), which are characterized by their electric and magnetic charges \( e_r^I \) and by their masses \( m_r \). The \( U(1) \)-covariant derivative on the \( r \)-th matter field is given by

\[
D^\mu_r (A) \Phi^r = \left( \partial^\mu + i e^I r \varepsilon^{IJ} A^J_j \right) \Phi^r.
\]

The equations of motion for the matter fields should already imply current conservation by themselves; this constrains them to be the standard ones,

\[
\begin{align*}
\left(D^\mu D_\mu + m_r^2\right) \varphi^r &= 0 \quad (5.2) \\
(i \gamma^\mu D_\mu - m_r) \psi^r &= 0, \quad (5.3)
\end{align*}
\]

for bosons and fermions respectively. The individual number–currents are defined as usual by

\[
j_\mu^r (\Phi) = \begin{cases} 
\bar{\varphi}^r D_\mu \varphi^r + c.c. & \text{for bosons} \\
\bar{\psi}^r \gamma^\mu \psi^r & \text{for fermions},
\end{cases}
\]

and the total electric and magnetic currents are represented by the doublet of one–forms

\[
J^I (\Phi) = \sum_r e_r^I d x^\mu j^\mu_r (\Phi).
\]

With \( J^I (\Phi) \) we indicate the smooth three–forms which are Hodge dual to \( j^I (\Phi) \); in the bosonic case they depend, actually, also on \( A \). Current conservation is again expressed by

\[
d J(\Phi) = 0,
\]

and we impose further the generalized Maxwell equations

\[
d F = J(\Phi), \quad F = * \varepsilon F.
\]

What is missing at this point is the link between the fields \( F \) and \( A \), which can not be independent variables. To construct this link we follow the strategy adopted for the point particle theory. Current conservation allows to introduce a doublet of \( C \)-forms satisfying

\[ J(\Phi) = d C \] and Maxwell’s equations lead then to \( F = d A + C \). But the problem arising now is that the \( C \)-fields are not uniquely determined, they are determined only modulo exact forms, corresponding to “string changes”. Moreover, under \( C \to C + d H, A \to A - H \)

\[^8\text{Actually, the gauge group is } U(1) \times U(1).\]
the matter equations are not invariant meaning that, contrary to what happens in the point particle theory, in the classical field theory the equations of motion are necessarily “string–dependent”. This difference between the two theories is due to the fact that in the point particle theory the introduction of vector potentials is optional while in the field theory it is unavoidable.

To make a specific string choice we have to impose one more condition on $C$. For this purpose we introduce a fixed external vector field $U^\mu(x)$, with properties specified in section three, and impose $i_U C = 0$. The set of equations of motion which govern the classical field theory (taking e.g. the matter fields to be bosons) are given by

\[
\left( D^\mu D_\mu + m_r^2 \right) \varphi_r = 0 \\
F = dA + C \\
F = \ast \varepsilon F \\
dC = J(\varphi) \\
i_U C = 0,
\]

with the boundary condition

\[
C \to 0 \quad \text{for} \quad x_U \to -\infty. \tag{5.5}
\]

As we showed in section three the last two equations, together with the boundary condition, determine $C$ uniquely as a function of $J$, and the remaining equations become then a closed system for the vector potentials and the matter fields.

This set of equations is $SO(2)$–invariant and formally Lorentz–invariant, but depends on an external vector field. In this sense it is inconsistent. We observed already the classical field theory is necessarily inconsistent, but the way the inconsistency shows up depends clearly on the formulation of the theory. In the set (5.4), once the auxiliary fields $C$ have been eliminated, what remains is a system of non local equations for the remaining physical fields, due to the presence of $U$. On the other hand, in [3] Zwanziger used a local action, amended by suitable boundary conditions, where only the physical fields show up; but, due to the presence of a constant vector $N^\mu$, Lorentz–invariance is explicitly broken. If the boundary conditions are imposed, Zwanziger’s equations of motion become non–local and coincide with the system (5.4), for $U^\mu(x) = N^\mu$. In the next subsection we will write an action which gives rise to (5.4); but even for $U^\mu(x) = N^\mu$ this action will not coincide with the one given in [3]. The relation between Zwanziger’s framework and ours will be explained in section seven.

One can exhibit a formal solution to the last two equations in (5.4) as

\[
C = \ast \frac{1}{U^2} U i_U \left( \frac{1}{U^2} U i_U + \frac{1}{\square} \delta d \right)^{-1} \frac{1}{\square} d \ast J, \tag{5.6}
\]

where the inverse operator in the brackets sends two–forms into two–forms and has to be defined according to the boundary condition (5.5). For the particular case $U^\mu(x) = N^\mu$ (5.6) can be seen to reduce to (3.14).
5.2 Field theory action

We present now an action which gives rise to the set of equations of motion given above, and which will be the basis for the quantum field theory. Its basic building block will be the PST-action, but the two–forms \( C \) are now treated as independent variables. The last two equations in (5.4) can be entailed by a convenient set of lagrange multiplier fields. The consistency of this method requires to show that these new fields do not propagate and become algebraic functions of the physical fields. This will be the main consistency check for the action we propose.

We introduce as Lagrange multipliers a doublet of two–forms \( \Sigma^I \) and a doublet of one–forms \( \Lambda^I \), which are viewed as \( U(1) \) vector potentials. Denoting collectively the fields as \( \phi = (A, C, a, \Sigma, \Lambda, \varphi_r) \), the proposed action, for bosons, is given by

\[
S_U[\phi] = S_0 + \int \left( \Lambda \varepsilon dC - \frac{1}{2} \sum_I U_i U \varepsilon C \right) - \sum_r \int d^4x \bar{\varphi}_r \left( D^\mu(\Lambda) D_\mu(\Lambda) + m_r^2 \right) \varphi_r. \tag{5.7}
\]

Here \( S_0[A, C, a] \) is the PST-action and the covariant derivatives on the scalars are defined as in (5.1), but with the vector potentials \( \Lambda \) replacing \( A \). It is understood that \( U^\mu(x) \) is regarded as a fixed external field and that the boundary conditions (5.5) are imposed.

The formula (5.7) for \( S_U \) can be regarded, in a sense, as the key equation of the paper.

The action (5.7) possesses, in addition to the PST–symmetries for \( a \) and \( A \), the following invariances

\[
\begin{align*}
\varphi'_r &= e^{-i e_r \varepsilon \lambda} \varphi_r \\
\Lambda' &= \Lambda + d\lambda, \tag{5.8}
\end{align*}
\]

\[
\begin{align*}
C' &= C + i U dH \\
\Sigma' &= \Sigma - dH \quad \text{for} \quad i U dH = 0, \\
A' &= A - H \\
\Sigma' &= \Sigma + U_i U \Delta, \tag{5.9}
\end{align*}
\]

where \( \Delta \) is a two–form doublet. The transformations (5.8) are ordinary \( U(1) \)–invariances for the vector fields \( \Lambda \); the transformations in (5.9) appear as a kind of constrained string changes and are quasi–local, in the sense that the parameter doublet \( H \) is constrained by \( i U dH = 0 \). The last transformations imply that the fields \( \Sigma \) are defined modulo forms parallel to \( U \), as is obvious from the manner they appear in the action.

The equations of motion for \( a \) and \( A \) amount, as in the pure PST–case, to

\[
dK = dC, \tag{5.11}
\]

where \( K^I \) are the off–shell electromagnetic tensors defined previously. Variation of \( S_U \) with respect to \( \Sigma \) and \( \Lambda \) gives respectively

\[
i_U C = 0, \quad dC = J(\varphi, \Lambda), \tag{5.12}
\]

\[
24
\]
where \( J(\varphi, \Lambda) \) are the matter current three–forms defined above but with \( A \) replaced by \( \Lambda \). The equations of motion for \( \varphi \) are the Klein–Gordon equations, again with \( A \rightarrow \Lambda \). The equation of motion for \( C \) becomes

\[
K - C - d\Lambda = -\frac{1}{2} (C + *U iU * \Sigma), \tag{5.13}
\]

and it should determine the auxiliary fields \( \Sigma \) and \( \Lambda \). To show that this, actually, happens we make the natural gauge choice for (5.10) \( i_U \Sigma = 0 \), which is equivalent to \( \Sigma = -*U i_U * \Sigma \). Then, defining \( G = \frac{1}{2}(C - \Sigma) \) (5.11), (5.12) and (5.13) imply that

\[
dG = 0 = i_U G.
\]

But since under (5.9) \( G \) transforms as \( G \rightarrow G + dH \), with \( i_U dH = 0 \), these equations mean precisely that \( G \) is “pure gauge” under this symmetry and one can set \( G = 0 \). The left hand side of (5.13) must then also vanish, and if one fixes the PST–symmetries in the standard way such as to give \( K = F = dA + C \), one determines finally the Lagrange multipliers as

\[
\Sigma = C \tag{5.14}
\]

\[
\Lambda = A, \tag{5.15}
\]

apart from a \( U(1) \) transformation (5.8).

Since now \( \Lambda = A \), the equations of motion for all the other fields amount indeed to the set (5.4), q.e.d. In particular, the physical propagating fields are made out only of one photon and of the matter fields. But, since the identification (5.15) occurs only on–shell, for a correct identification of the observables e.g. Wilson loops, the electromagnetic field tensor etc. some caution is required in picking up one vector potential or the other. As we shall see in the next section symmetry reasons will always lead one to a unique choice.

### 6 Quantum field theory

Our formulation of a quantum field theory for dyons interacting with an electromagnetic field will be based on the classical action \( S_U \), with the boundary condition (5.4), in the framework of a functional integral approach. Since the correlation functions of the gauge–invariant operators encode all physical information, it suffices to construct only these correlation functions. The classical action depends on \( U \) and, as a consequence, also these correlation functions depend a priori on this external field. The main goal of this section is to show that, if (1.4) holds, the \( U \)–dependence disappears in the correlation functions of physical fields. Manifest \( SO(2) \)– and Lorentz–invariance then follows automatically.

To prove \( U \)–independence we use a standard technique: path integral representations in terms of classical particle trajectories [16] of determinants and Green functions. For definiteness we consider bosonic matter fields and comment eventually about the fermionic case which presents, however, no additional difficulties. For more details on path integral representations for particles with arbitrary spin we refer the reader to [17].
6.1 Path integral representations

We begin by writing the representation for the determinant associated to the scalar field \( \varphi_r \) in \((6.1)\):

\[
\int \{D\varphi_r D\bar{\varphi}_r\} e^{-i \int \varphi_r (D^2_r + m_r^2) \varphi_r} = \det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) = \exp \left( -tr \ln \left( -i \left( D^2_r + m_r^2 \right) \right) \right),
\]

where the covariant derivative on \( \varphi_r \) is \( D_{\mu} = \left( \partial_{\mu} + ie_{\mu}^I \varepsilon^{IJ} \Lambda^J_{\mu} \right) \). Apart from a normalization, the basic path representation is given formally by (see e.g. [23] for a rigorous definition):

\[
\text{tr} \ln \left( -i \left( D^2_r + m_r^2 \right) \right) = - \int_0^\infty \frac{dT}{T} \int \{Dy_r\} e^{-i \int_0^T \left( \frac{1}{2} \left( \frac{dy_r}{ds} \right)^2 + m_r^2 + e_r \varepsilon \Lambda^r_{\mu} \frac{dy^\mu_r}{ds} \) ds,}
\]

where the integration is over all closed path, \( y_r(T) = y_r(0) \). We rewrite \((6.1)\) in a more concise form as

\[
\text{tr} \ln \left( -i \left( D^2_r + m_r^2 \right) \right) = - \int \{D\gamma_r\} e^{-i \int \Lambda^r \varepsilon \gamma_r},
\]

where the integrand in the exponent is written as a four–form and the classical current three–form \( \gamma_r \) is defined as the PD–dual of the closed curve \( \gamma_r \), parametrized by \( y_r(s) \). We absorbed in the path integral measure \( \{D\gamma_r\} \) the exponential prefactors in \((6.2)\) and included formally also the integral over \( T \). Expanding eventually the exponential in \((6.1)\) we arrive to the identity

\[
\det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \prod_{k=1}^{l} \int \{D\gamma_r(k)\} \right) e^{-i \int \Lambda^r \varepsilon \sum_{k=1}^{l} \gamma_r(k)},
\]

where the integer three–form \( J_r \) represents now a sum of \( l \) classical point particle currents, each one being of the form \((3.0)\), and the measure \( \{D\Gamma_r\} \) includes now also a sum over the number of loops.

The functional integration over all \( N \) matter fields leads then to

\[
\prod_{r=1}^{N} \det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) = \left( \prod_{r=1}^{N} \{D\Gamma_r\} \right) e^{-i \int \Lambda^r \varepsilon \sum_{r=1}^{N} e_r J_r} \equiv \int \{D\Gamma\} e^{-i \int \Lambda^r \varepsilon J^r},
\]

where the three–forms \( J^r \) correspond now to a weighted sum of closed integer forms, as in \((3.4)\) and \((3.5)\).

An analogous representation can be given for the matter two–point function in an external gauge field \( \Lambda \):

\[
\langle T \bar{\varphi}_r(x) \varphi_r(y) \rangle\Lambda = \int \{D\varphi_r D\bar{\varphi}_r\} e^{-i \int \varphi_r (D^2_r + m_r^2) \varphi_r} \bar{\varphi}_r(x) \varphi_r(y)
\]
\[
= \det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) \cdot \left( -i \left( D^2_r + m_r^2 \right) \right)^{-1}(x, y)
\]
\[
= \det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) \int_0^\infty dT \int \{Dy_r\} e^{-i \int_0^T \left( \frac{1}{2} \left( \frac{dy_r}{ds} \right)^2 + m_r^2 + e_r \varepsilon \Lambda^r_{\mu} \frac{dy^\mu_r}{ds} \) ds
\]
\[
\equiv \det^{-1} \left( -i \left( D^2_r + m_r^2 \right) \right) \cdot \int \{D\gamma_{r(x,y)}\} e^{-i \int \Lambda^r \varepsilon J_{r(x,y)}}. \]
Here the path integration is over open curves \( y_r(s) \) with endpoints \( y_r(0) = x, y_r(T) = y \), and the integer three–forms \( J_r^{(x,y)} \) are PD–dual to those curves.

### 6.2 Correlators of observables and \( U \)–independence

We begin by establishing the \( U \)–independence of the partition function. This is the simplest case and it encodes already the two basic ingredients which allow to establish it for all observable correlation functions. The first ingredient is the technical advice of path integral representations for the matter fields and the second is represented by the vanishing of the Dirac–anomaly for classical point particles, a result which we have already established.

#### 6.2.1 Partition function

The partition function is represented by

\[
Z_U = \int \{D\phi\} e^{iS_U[\phi]},
\]

where the boundary condition (5.5) is imposed on the functional integration over \( C \), and appropriate gauge fixings for the local symmetries are understood.

The integration over \( A \) and \( a \) has already been performed explicitly in section two and gives rise to what we called the effective current–current action \( \Gamma_0[C] \). The integration over \( \Sigma \) gives the \( \delta \)–function \( \delta(i_U C) \) while for the integration over the matter fields we can use the path–integral representations given above. Putting everything together one obtains

\[
Z_U = \int \{D\Gamma\} \{DC\} \{DA\} \delta(i_U C) \ e^{i\Gamma_0[C]+i\int \Lambda \varepsilon (dC-J)}
\]

\[
= \int \{D\Gamma\} \{DC\} \ \delta \ (i_U C) \ \delta (dC - J) \ e^{i\Gamma_0[C]}.
\]

The \( \delta \)–functions restrict the variable \( C \) to \( dC = J, i_U C = 0 \) (remember that \( J \) satisfies \( dJ = 0 \) identically) which, taken together with the boundary condition (5.3), admit a unique solution for \( C \); since \( J \) is now a classical current this solution is given by (3.17) and (3.18) and we call it \( C^I(J,U) \). Therefore

\[
Z_U = \int \{D\Gamma\} \ e^{i\Gamma_0[C^I(J,U)]}, \quad (6.7)
\]

which is our final expression.

An arbitrary change in the external field from \( U \rightarrow U' \) corresponds now, for fixed \( J \), just to a string–change \( C(J,U') = C(J,U) + dH_{U,U'} \), where the \( H_{U,U'} \) are weighted integer one–forms which correspond to the three–volumes bounded by the surfaces associated respectively to \( U \) and \( U' \) according to (3.17). But under a string–change \( \Gamma_0 \) changes by the Dirac–anomaly

\[
Z_{U'} = \int \{D\Gamma\} \ e^{i[\Gamma_0[C^I(J,U)]+\frac{1}{2}\int J \varepsilon H_{U,U'}}},
\]

which is the final expression for the partition function with a changed field configuration. In this way \( Z_{U'} \) is a polynomial in \( U' \) which is consistent with all \( U \)–independence results already obtained.
and if (1.4) holds $\exp \left( \frac{1}{2} \int J \in H_{U,U'} \right) = 1$. Thus $Z_U = Z_{U'}$.

As long as one considers observable correlation functions, i.e. correlation functions of invariant operators the strategy presented here will always lead to the desired result. Below we work out the most significant cases.

6.2.2 Current correlation functions

In this subsection we consider the correlation functions of the individual matter currents $(r = 1, \ldots, N)$,

$$j_{r\mu} = i \overline{\varphi}_r D_\mu (\Lambda) \varphi_r + c.c.$$  

(6.8)

In writing this expression we solved already the ambiguity regarding the choice between $A$ and $\Lambda$. Here one has to choose $\Lambda$ because the corresponding expressions with $A$ would violate the PST–symmetries.

The current correlators are conveniently encoded by a generating functional which can be constructed in a standard way. One introduces $N$ external currents $L_r$, one–forms, and performs in the Klein–Gordon term in $S_U$ the replacement

$$e_r \in \Lambda \rightarrow e_r \in \Lambda + L_r,$$  

(6.9)

without affecting the term $\int \Lambda \in dC$ and the other terms of the action. We call the action obtained in this way $S_U[\phi, L]$. The terms in $S_U[\phi, L]$ which are linear in $L_r$ multiply simply the currents (6.8), while the terms which are quadratic in $L_r$ – contact terms – are needed to ensure the global invariances $[U(1)]^N$, i.e. the individual current conservations.

The correlation functions are now obtained, via differentiation w.r.t. the $L_r$, from the generating functional

$$e^{iW[L]} = \int \{D\phi\} e^{iS_U[\phi, L]}.$$

This functional integral can be analyzed in the same manner as the partition function. The Klein–Gordon determinants can be obtained from (6.5) with the substitution (6.9) and their product can be written as

$$\int \{D\Gamma\} e^{-i \int (\Lambda \in J - \sum_r L_r J_r)}. $$

The integration over the other fields can be handled as above and one obtains

$$e^{iW[L]} = \int \{D\Gamma\} \{DC\} \{DL\} \delta(i_U C) \delta(\Gamma_0 [C] + \int \Lambda \in (dC - J) + \sum_r \int L_r J_r) \delta(\Gamma_0 [C] + \sum_r \int L_r J_r)$$

$$= \int \{D\Gamma\} \{DC\} \delta(\Gamma_0 [C] + \int (dC - J) \delta(\Gamma_0 [C] + \sum_r \int L_r J_r)$$

$$= \int \{D\Gamma\} e^{i(\Gamma_0 [C] + J_U) + \sum_r \int L_r J_r}.$$
6.2.3 Electromagnetic field correlation functions

We noticed already that the off-shell electromagnetic tensor is represented by the two-form doublet $K^I$ which is given as a function of $A$, $C$ and $a$ in (2.16). It satisfies in particular the pseudo self-duality relation

$$K = * \varepsilon K,$$

(6.10)

which is necessary to avoid a doubling of the electromagnetic degrees of freedom. The alternative for $K$, in terms of $\Lambda$, would in this case be given by $d\Lambda + C$ because on-shell we have $K = dA + C$ and $A = \Lambda$. But the expression $d\Lambda + C$ violates the quasi-local invariances (5.9) and, moreover, it does not satisfy (6.10).

Therefore, if we introduce as external currents a doublet of two-forms, $L^I$, the generating functional for the electromagnetic field correlators could be based on the action

$$S_U + \int K \varepsilon L,$$

(6.11)

where the $\varepsilon$-tensor has been introduced only for notational convenience. The resulting generating functional would then automatically satisfy the pseudo self-duality invariance

$$W[- * \varepsilon L] = W[L],$$

(6.12)

i.e. generate correlation functions only for a single field strength tensor.

The expression (6.11) represents already the correct starting point, apart from contact terms. As in the case of current correlators these terms, quadratic in $L$, have to be fixed in order to respect the symmetries of the underlying theory. The problem related with (6.11) is that the $K$’s are invariant under the PST-symmetry (2.12) only on-shell. Therefore, the direct use of (6.11) would introduce in the $K$-correlators a spurious dependence on the unphysical auxiliary field $a$.

The problem can be solved by basing the generating functional on the following action

$$S_U[\phi, L] = S_U + \int \left( K \varepsilon L + \frac{1}{2} L \mathcal{P}(v) L - \frac{1}{4} L * L \right)$$

$$\equiv S_L + \int \left( \Lambda \varepsilon dC - \frac{1}{2} \Sigma U_i U_j \varepsilon C \right) - \sum_r \int d^4 x \bar{\varphi}_r \left( D^\mu(\Lambda) D_\mu(\Lambda) + m_r^2 \right) \varphi_r,$$

where $S_L$ represents a deformation of the PST-action

$$S_L = S_0[A, C + L, a] + \frac{1}{2} \int \left( C \varepsilon L - \frac{1}{2} L * L \right).$$

The quadratic terms we have added depend only on $L$ and $a$, therefore they influence only the PST-symmetry (2.12), which sends $a \to a + \varphi$. From the form of $S_L$ one sees that it is invariant under the transformations (2.12) if one replaces in the $A$-transformations $C$ with $C + L$. This ensures again the decoupling of the auxiliary field $a$. To achieve
This decoupling only the $a$–dependent term $\frac{1}{2} \int L \mathcal{P}(v) L$ would be required, but this term alone is not invariant under pseudo–duality, i.e. under
\[ L \rightarrow - \ast \varepsilon L. \]

It is only the combination $\frac{1}{2} \int \left( L \mathcal{P}(v) L - \frac{1}{2} L \ast L \right)$ which is invariant.

The generating functional for the field strength correlators, defined as
\[ e^{i W[L]} = \int \{D \phi\} e^{i S_U[\phi, L]} \]
with $S_U[\phi, L]$ specified above, entails, therefore, all the required symmetry properties, in particular (6.12). It remains to show that it is $U$–independent. To this order we perform first the functional integration over the fields $A$ and $a$, which appear only in $S_L$,
\[ e^{i \Gamma_L[C]} \equiv \int \{DA\} \{Da\} e^{i S_L}. \]

The result for $\Gamma_L[C]$ can be easily read from the analogous functional integration over the PST–action, which gave the effective action $\Gamma_0[C]$,
\[ \Gamma_L[C] = \Gamma_0[C + L] + \frac{1}{2} \int \left( C \varepsilon L - \frac{1}{2} L \ast L \right). \]

What matters eventually is how this expression transforms under a generic string change $C \rightarrow C + dH$. By direct inspection one finds that it transforms precisely as $\Gamma_0[C]$,
\[ \Gamma_L[C] \rightarrow \Gamma_L[C] + A_D[H], \quad (6.13) \]
where $A_D[H]$ is the usual Dirac–anomaly. This is a consequence of the fact that also $S_L$ carries the same Dirac–anomaly as $S_0$.

The integration over the remaining fields is carried out as in the case of the partition function and the result for the generating functional is obtained from the expression for $Z_U$ through the replacement $\Gamma_0 \rightarrow \Gamma_L$,
\[ e^{i W[L]} = \int \{D \Gamma\} e^{i \Gamma[L,C(J,U)]}. \]

$U$–independence then follows from (6.13).

6.2.4 Wilson loops

Wilson loops are associated to a set of $N$ closed space–time curves $\beta_r$, one for each dyon, and represent the interaction between the vector potential and the particles circulating in the loops. The correlator of the corresponding observables is defined as
\[ W[\beta] = \left\langle T \prod_{r=1}^{N} e^{-ie_r \varepsilon \oint_{\beta_r} A} \right\rangle = \left\langle T e^{-i \int A \varepsilon J(\beta)} \right\rangle. \quad (6.14) \]
Here we defined the closed three–form doublet \( J^I = \sum_r e^I_J(r) \), where the three–forms \( J_r(\beta) \) are the PD–duals of the loops \( \beta_r \).

In the definition of the correlators we have to use again \( \Lambda \) instead of \( A \) since Wilson loops constructed with the latter would violate the PST–symmetries. On the other hand, since the loops \( \beta_r \) are closed, the classical \( U(1) \)–invariances of \( \Lambda \) would allow also for arbitrary real weights of the \( N \) individual terms in the exponentials in (6.14). But the corresponding quantum correlators would be string–dependent, that is \( U \)–dependent, as one expects also on the basis of Dirac’s original argument for charge quantization. Thus these weights have to take on integer values.

The functional integral representation for (6.14) is

\[
\mathcal{W}[\beta] = \int \{ \mathcal{D}\phi \} e^{iS_U} e^{-i \int \Lambda \varepsilon J(\beta)}.
\]

Once one represents the matter determinants as in (6.5), one sees that the presence of the Wilson loops amounts merely to a replacement of the closed currents \( J \) with the closed currents \( J + J(\beta) \). Thus

\[
\mathcal{W}[\beta] = \int \{ \mathcal{D}\Gamma \} e^{i\Gamma_0[C(J + J(\beta), U)]}.
\]

But since \( J + J(\beta) \) is also a weighted integer current the Dirac–anomaly is again a multiple of \( 2\pi \) and the correlators are \( U \)–independent.

### 6.2.5 Mandelstam–string observables

We consider here correlation functions corresponding to a particular type of neutral composite fields, formed by a certain number of particles and antiparticles, with support on a bounded string. Due to the global \( [U(1)]^N \) invariances only correlation functions in which each particle appears together with its own antiparticle are non–vanishing.

Here we work out only the one–point Green function for a “meson” field, involving say the \( r \)-th species, generalizations to multiple states being straightforward. The Green function we consider is specified by the positions \( x \) and \( y \) of the two particles and by an open curve \( \beta_{x,y} \), the “Mandelstam–string” \[18\], which connects the two points. The corresponding correlator is given by

\[
G_r[\beta_{x,y}] = \left\langle T \bar{\varphi}_r(x) e^{i e_r \varepsilon \int_{\beta_{x,y}} \Lambda \varphi_r(y)} \right\rangle = \left\langle T \psi_r(x) e^{i \int \Lambda \varepsilon e_r J_r(\beta_{x,y}) \bar{\varphi}_r(y)} \right\rangle = \int \{ \mathcal{D}\phi \} e^{i(S_U + \int \Lambda \varepsilon e_r J_r(\beta_{x,y}))} \bar{\varphi}_r(x) \varphi_r(y).
\]

The insertion of the Mandelstam line–integral along \( \beta_{x,y} \), with vector potentials \( \Lambda \), is needed to realize the local electromagnetic invariances (5.8).

The evaluation of the correlator can be based on the expansion (6.6). The determinant in that formula combines with the determinants of the other \( N - 1 \) charged fields and
leads again to (6.5). One obtains

\[
G_r[\beta_{x,y}] = \int \{D\gamma_{r}^{(x,y)}\} \int \{D\Gamma\} \{DC\} \{DA\} \delta(i_U C) e^{i \Gamma_0[C]} + i \int \Lambda \varepsilon \left( dC - \left( J + e_r \left( J_r^{(x,y)} - J_r(\beta_{x,y}) \right) \right) \right)
\]

(6.15)

Since \( J_r^{(x,y)} \) and \( J_r(\beta_{x,y}) \) are PD–dual to the open curves \( \gamma_{r}^{(x,y)} \) and \( \beta_{x,y} \) respectively, the total currents \( J + e_r \left( J_r^{(x,y)} - J_r(\beta_{x,y}) \right) \) are again closed and weighted integer forms. This is sufficient to guarantee \( U \)–independence once more.

Generalizations to \( n \)–point Green functions are straightforward since the second row in formula (6.6) admits immediate extensions to arbitrary polynomials in \( \{ \varphi_r \} \), in terms of multiple contractions.

### 6.2.6 Charged fields

We wish here to outline the construction of Green functions for charged fields, following the strategy developed in [21] (for all technical details we refer the reader to this reference). The fields we consider are required to be invariant under local \( U(1) \) gauge transformations which vanish at infinity, but to transform non–trivially, i.e. to be charged, with respect to the global \( U(1) \) transformations. In agreement with Strocchi’s theorem [24], these requirements imply that the fields are non–local, more precisely that their support is non–compact and extends to infinity.

There are two natural candidates for such fields. The first is obtained multiplying (“dressing”) a charged matter field \( \varphi_r(x) \) by a Mandelstam–string phase factor

\[
exp \left( i \int_{\gamma_x} e_r \varepsilon \Lambda \right).
\]

\( \gamma_x \) is here a curve starting from \( x \) and reaching infinity along some fixed space–direction at fixed time, where it meets a “compensating charge”.

The second candidate field is obtained, according to Dirac’s original proposal [19], dressing \( \varphi_r(x) \) by a Coulomb phase factor

\[
exp \left( ie_r \varepsilon \int d^4 y \Lambda_\mu(y) E_{r}^\mu(y) \right).
\]

\( E_{r} \) is a classical Coulomb electric field, generated by a unit charge located at \( x \), with support in a fixed–time three–space and satisfying \( \partial_\mu E_{r}^\mu(y) = \delta^4(x - y) \) [\( \Box \)]. More precisely

\[
E_{r}^0(y) = 0, \quad E_{r}^\mu(y) = \frac{1}{4\pi} \frac{\delta(y^0)}{|\vec{y} - \vec{x}|^3} \]  \( \Box \)

(6.16)

9In lattice abelian gauge theories without dynamical monopoles, it has been shown rigorously [28] that charged states and charged quantum fields can be reconstructed out of euclidean Green functions, derived as expectation values of globally neutral products of charged fields defined according to Dirac’s proposal.
From a physical point of view Dirac’s recipe describes a charged particle dressed with its Coulomb photon cloud. The latter appears energetically more favorable then the Mandelstam–string proposal in Q.E.D.–like theories, as we know that in such theories the electromagnetic field associated classically to a charged particle is indeed Coulomb–like.

However, neither of these candidates appears to be suitable in a theory where electric and magnetic dynamical charges coexist. The problem arising with Mandelstam–strings is related with infrared divergences (they are unbounded) and it can be understood as follows. Consider e.g. the ansatz two–point function for a charged particle created at \(x\) and annihilated at \(y\)

\[
\langle \overline{\varphi}_r(x) e^{i e_r \int_{\gamma_x} \Lambda} \varphi_r(y) e^{-i e_r \int_{\gamma_y} \Lambda} \rangle.
\]

It involves two strings of infinite length, one starting from \(x\) and the other from \(y\). Loosely speaking the strings would have infinite positive self–energies and infinite negative interaction–energy as a consequence of the interaction mediated by the photons. More concretely, the effective action \(\Gamma_0[C(J, U)]\) diverges if the currents \(J\) correspond to unbounded curves, like the above Mandelstam–strings. The self–energies could be in principle removed by a suitable multiplicative point–independent renormalization. But the interaction energy can not be removed as it depends on the distance between \(x\) and \(y\), and thus the ansatz Green function would be infrared divergent. (More precisely, in the euclidean formulation renormalizing away the interaction energy would spoil Osterwalder–Schrader positivity of the correlation function, a property which is crucial for the reconstruction of the quantum field, see e.g. [23].)

On the other hand, Dirac’s construction can not be used in a theory of dyons because the field \(E_x\) would give rise to a smooth non integer current \(J\) and the Dirac–anomaly would no longer be an integer multiple of \(2\pi\). This would then imply an unphysical dependence of the Green functions on Dirac–strings.

The recipe presented in [21], which overcomes these difficulties, consists in replacing a single Mandelstam–string \(\exp(i e_r \int_{\gamma_x} \Lambda)\) by a sum over Mandelstam–strings weighted by a suitable measure \(d\mu\{\gamma_{x}\}\). This measure is supported on strings \(\gamma_{x}\) which fluctuate so strongly that with probability one the corresponding interaction–energy is finite, and it entails the additional feature that on average it reproduces just the Coulomb field \(E_x\). Therefore, although on small scales the “photon cloud” associated to the charged field

\[\varphi_r(x) \int d\mu\{\gamma_{x}\} e^{i e_r \int_{\gamma_x} \Lambda}\]

respects the Dirac quantization condition, at large distances on average it scales to the classical Coulomb field \(E_x\).

The Green functions for charged fields are eventually given by

\[\left\langle T \prod_{i} \varphi_{r_i}(x_i) \int d\mu\{\gamma_{x_i}\} e^{i e_{r_i} \int_{\gamma_{x_i}} \Lambda} \right\rangle,
\]

where \(\varphi\) denotes \(\varphi\) or \(\overline{\varphi}\) and total charge neutrality is understood.
The strategy developed above can now be used in the same fashion to prove Dirac–string independence of these correlation functions too.

6.3 Comments

The procedure adopted above to prove the string–independence of the invariant correlation functions works equally well for fermions. The corresponding classical action is obtained from (5.7) upon replacing the Klein–Gordon action with the Dirac action, and the relevant path–integral representations are available also for fermions. For example, the fermionic determinants admit expansions of precisely the same type as (6.3), what changes is only the functional measure over the paths \[4\]. Appropriate expansions for the fermionic propagators in an external gauge field are also available [17].

Our formulation for the quantum field theory of dyons, based on the action (5.7), furnishes string–independent correlators for all gauge–invariant operators. It may be worthwhile to notice that correlators of operators which are not invariant under local gauge transformations, instead, turn out to vanish. Consider, for example, the scalar two–point function \[\langle \bar{\phi}_r(x)\phi_r(y) \rangle\]. This is obtained from (6.15) upon dropping the current \(J_r(\beta_{x,y})\). The integration over \(\Lambda\) leads then to the \(\delta\)–function \(\delta(dC - (J + e_r J_r(\beta_{x,y})))\) which has no solution since the current is now no longer closed. This would then give a vanishing correlator, in agreement with Elitzur’s [25] theorem\(^{10}\). The point is that in the case of currents which are not closed, occurring in the presence of non invariant operators, it is necessary to fix the local \(U(1)\)–symmetries (5.8). But this leads then to smooth non integer currents and the correlators become \(U\)–dependent.

A word is in order for what concerns the configurations which we called in the classical point particle theory “exceptional”. These configurations appear indeed also in the quantum field theory; taking, for example, the expansion of the partition function in (6.7), they occur when some curve \(\gamma_r\) intersects the string \(C_s(J_s, U)\) associated to another curve \(\gamma_s\). In this case \(\Gamma_0[C(J, U)]\) would be ill–defined. However, every regularization which preserves the integrality of the currents gives the same result for \(\exp(i\Gamma_0[C(J, U)])\). More concretely, the problem can be solved by performing in the expansion (6.7) a string change from \(C_s(J_s, U)\) to an arbitrary regular string \(\tilde{C}_s(J_s)\) (see section three). Since \(\exp(i\Gamma_0[C(J, U)])\) is string–independent, the regularized expression \(\exp(i\Gamma_0[\tilde{C}(J)])\) equals the original one for regular configurations and is, moreover, independent of the regularization. This regularization procedure fails only if the curves themselves intersect each other; but, in the models considered here, these configurations form a set of measure zero in the path integral.

A classical field theory action of the kind (5.7) can be given also for the \(Z_4\) theory. In this case one has only one vector potential \(A^2\), and only one lagrange multiplier field, \(\Sigma^1 = C^1\), is needed to impose \(dC^2 = J^2\). The fields are now \(\phi = (A^2, \Lambda^1, C^I, \varphi_r)\) and the

\(^{10}\)The same happens formally in ordinary Q.E.D. if one does not fix the \(U(1)\)–symmetry.
counterpart of (5.7) is
\[ S_U^{Z_4}[\phi] = \int \frac{1}{2} F^2 \ast F^2 + \Lambda^1 dC^2 - C^1 U_i U^2 C^2 - \sum_r \int d^4x \bar{\varphi}_r \left( D^\mu D_\mu + m_r^2 \right) \varphi_r, \]
where \( F^2 = dA^2 + C^2 \), and the covariant derivative is here \( D_\mu = \partial_\mu + i \left( e_r^1 A^2_\mu - e_r^2 \Lambda^1_\mu \right) \). For the \( C^I \) we assume the usual boundary conditions (5.5). The symmetries of this action are analogous to the ones given in (5.8)–(5.10) for the \( SO(2) \)–theory and, together with the equations of motion, they determine the auxiliary fields as follows
\[ d\Lambda^1 + C^1 = *F^2 \]
\[ dC^I = J^I(A^2, \Lambda^1, \varphi_r), \quad i_U C = 0. \] (6.18) (6.19)

Taking the matter field equations into account and setting \( \Lambda^1 \equiv A^1 \), one recovers precisely the system of equations of motion of the \( SO(2) \)–theory, (5.4).

The corresponding quantum field theory can be analyzed in the same way as the \( SO(2) \)–theory and it turns out to be consistent if (1.5) holds.

It is worthwhile to notice a puzzle which arises in comparing the inequivalent quantum field theories based on \( S_U \) and \( S_U^{Z_4} \) respectively. As we noted in section 4.2, the difference arises in the effective actions
\[ \Delta \Gamma_0 = \Gamma_0^{Z_4} - \Gamma_0 = \frac{1}{2} \int C^1 C^2. \] (6.20)

Since in both quantum field theories the strings are chosen to satisfy \( i_U C = 0 \) and one has the identity decomposition \( 1 = \frac{1}{U^2} (i_U U - U i_U) \), one would conclude
\[ C^1 C^2 = \frac{1}{U^2} C^1 (i_U U - U i_U) C^2 = 0. \]

The point is that in deriving (6.20) we performed integrations by parts and ignored boundary terms at infinity; non vanishing boundary terms arise, in general, if the strings extend to infinity, as does \( C[J,U] \). Strings extending to infinity are unavoidable if the curves themselves extend to infinity, as in the classical point particle theory. But in the partition function or in the correlation functions of local observables in the quantum field theory the curves are closed and one can also choose compact surfaces as strings. On the other hand, since \( exp(i \Gamma_0) \) is string–independent, one can switch from \( C[J,U] \) to arbitrary compact strings \( C_{\text{com}}[J] \). Since now curves \( \text{and} \) strings are compact no boundary terms arise and one gets indeed (5.24). But then \( C_{\text{com}}^1 C_{\text{com}}^2 \) is different from zero and one concludes that the two quantum field theories coincide only if the charges satisfy the stronger relations (1.6), as anticipated in section 4.2.

The extension of our formulation to include a coupling with an external classical gravitational field, represented by a metric \( g_{\mu \nu}(x) \), does not meet additional difficulties. Since the classical field theory action is formally \( SO(1,3) \)–invariant, the metric can be introduced in a standard fashion, via the minimal coupling, and the resulting action is
formally diffeomorphism invariant; the specification “formally” is referred to the presence of the external field \( U \). The quantum correlators are defined again by the functional integral and one has only to worry about the dependence of the correlators on \( U \). With respect to this issue the first key observation is that the path–integral representations for determinants and correlators, of the type (6.3) and (6.6), hold true. Since the exponentials involving \( \Lambda \) are written as integrals of products of \textit{forms}, they do not acquire any metric dependence; it is only the path–integral measure which involves now also the metric. This leads then to expansions which are of the same form as (6.7), but clearly \( \Gamma_0 \) depends now also on \( g_{\mu\nu} \). The second key observation is that, nevertheless, the Dirac–anomaly is metric–independent. This is due to the fact that the concept of PD–duality is metric independent and that the Dirac–anomaly is written as the integral of a product of \textit{forms}. Its exponential equals, therefore, again unity and the correlators are \( U \)–independent.

7 Duality symmetries and equivalences

The aim of this section is to establish which theories of a fixed number \( N \) of species of interacting dyons are physically equivalent to each other. Different actions can be connected by field redefinitions, upon elimination of auxiliary fields or through an \( S \)–duality transformation of a \( p \)–form potential (or by a combination of these operations). These operations suffice to establish the equivalences between the actions considered in this section. Theories based on classical actions which can be connected in one of these ways are physically equivalent also at the quantum level.

7.1 S–duality for the vector potentials \( A^I \)

We saw in section 4.3 that, by applying an \( S \)–duality transformation of the vector potential \( A^2 \) in the \( \mathbb{Z}_4 \)–theory, one can obtain a theory in which the electric and magnetic charges are interchanged. The electric charges are now magnetically coupled and the magnetic charges electrically. This means that \( \mathbb{Z}_4 \)–theories with charges respectively \( (e_r, g_r) \) and \( (g_r, -e_r) \), satisfying (1.5), are equivalent.

In the \( SO(2) \)–theory both types of charges are electrically and magnetically coupled. This indicates that under an \( S \)–duality of the vector potentials \( A^I \), the PST–action should turn into itself. This is indeed what happens. In fact, consider instead of the PST–action (2.7) the action

\[
S_H = \frac{1}{2} \int (H + C) \mathcal{P}(v) (H + C) + H \varepsilon (d\Lambda + C),
\]

where \( H^I \) is a doublet of two–forms and \( \Lambda^I \) a doublet of one–forms, dual to \( A^I \). Functional integration over \( \Lambda \) leads to \( dH = 0 \Rightarrow H = dA \) and one is back to the PST–action. On the other hand, one can now perform the integration over \( H \)

\[
\int \{DH\} e^{iS_H} \equiv e^{iS_0[\Lambda,C,a]}.
\]
The integral is Gaussian, and to perform it one has only to know the inverse of the \( a \)-dependent operator \( P(v) \) on two–form doublets

\[
P^{-1}(v) = -\frac{1}{4} P(v).
\]

The result is indeed

\[
\tilde{S}_0[\Lambda, C, a] = S_0[\Lambda, C, a],
\]

i.e. the PST–action for the dual vector potentials \( \Lambda \). This generalizes the result of [20], for the case of non vanishing currents.

Therefore, an \( S \)–duality transformation on \( A \), contrary to what happens in the \( \mathbb{Z}_4 \)–theory, sends the PST–action into itself, and does not lead to a new formulation. Moreover, this statement does not require any quantization condition on the charges. Clearly, the same result holds also for the total classical field theory action \( S_U \).

### 7.2 S–duality for the scalar \( a \)

A new formulation can be obtained if one performs an \( S \)–duality on the scalar auxiliary field \( a \). This transformation has been performed in [20], and we report here only the result. Due to the complicated dependence on \( a \) of the PST–action, it seems hard to implement this duality transformation at the level of the functional integral; one has to proceed through the equations of motion. This will lead in the dual action to the appearance of a *quasi*–local symmetry, needed to obtain the duality relation \( F = *\epsilon F \). Apart from this difference, the dual action shares the same good properties with the PST–action, i.e. manifest \( SO(1,3) \)– and \( SO(2) \)–invariance.

The field dual to \( a \) is a two–form potential \( b = \frac{1}{2} dx^\mu dx^\nu b_{\nu\mu} \), the relation being given by

\[
da = * db.
\]

To construct the dual PST–action one defines the one–forms \( h = * db \) and \( u = \frac{h}{\sqrt{-h^2}} \), such that \( u^2 = -1 \); \( u \) is the counterpart of the one–form \( v \). The dual action is \( (F = dA+C) \)

\[
\tilde{S}_0[A, C, b] = \frac{1}{2} \int F \ Q(u) \ F + dA \epsilon C, \quad (7.1)
\]

where \( Q(u) \) is a symmetric operator which acts in the space of two–forms and on the \( SO(2) \)–indices as

\[
Q^{IJ}(u) = *u i_u \delta^{IJ} - \left(u i_u - \frac{1}{2}\right) \epsilon^{IJ}.
\]

Apart from the formal analogies between (2.7) and (7.1) one should note, however, that the latter is not obtained from the former with the simple substitution \( v \rightarrow u \). One has rather

\[
Q(u) + P(u) = *.
\]
The symmetries of this action are
\[
\begin{align*}
\delta A &= d\Lambda \quad (7.2) \\
\delta b &= dc \quad (7.3) \\
\delta A &= \chi, \quad \text{with} \quad i_u d\chi = 0 \quad (7.4) \\
\delta A &= i_V i_u u (F - \ast \varepsilon F), \quad \delta b = i_V db. \quad (7.5)
\end{align*}
\]
c is a one–form, \(\Lambda\) a doublet of scalars, \(V\) a vector field, and \(\chi\) a constrained doublet of one–forms which entail now quasi–local transformations, as anticipated above. The symmetries (7.3) and (7.5) allow to fix \(b\) completely, which becomes again a non–propagating auxiliary field. The symmetries (7.4) allow to obtain the pseudo self–duality relation \(F = \ast \varepsilon F\). This implies that the dual action describes the same classical dynamics as the PST–action.

Even if (7.1) has not been obtained from the PST–action via functional integration, on can construct a quantum field theory, and a point particle theory, based on the dual action, in precisely the same way as we did for the PST–action: the resulting quantum field theories are, indeed, identical. The principal common features are that the two classical actions carry the same Dirac–anomaly, and entail the same effective action \(\Gamma_0\), as shown in the appendix\(^\text{11}\).

The off–shell electromagnetic fields, the counterparts to (2.16), are now given by
\[
\tilde{K} = F + \ast \varepsilon u i_u (F - \ast \varepsilon F),
\]
and have the same properties as the two–forms \(K\), but now referred to the symmetries (7.2) – (7.3).

Under an \(S\)–duality of the vector potentials \(A\), also the dual PST–action goes into itself, in particular one has \(Q^{-1}(u) = -\frac{1}{4} Q(u)\).

The advantages of the original PST–action w.r.t. the dual action are represented essentially by its simpler symmetry structure, in particular by the absence of quasi–local symmetries. The most significant difference between the two actions relies in the formulae (2.11) and (7.4). The symmetry (2.11) eliminates one transverse degree of freedom from each \(A^I\), such that each \(A^I\) carries one transverse physical photon degree of freedom. On the other hand, the quasi–local symmetry (7.4) for the dual action does not eliminate degrees of freedom; it allows, rather, to choose boundary conditions for the equations of motion, which reduce the second–order equations for the vector potentials, to the pseudo self–duality relation \(F = \ast \varepsilon F\).

As shown below, the dual action plays a significant role in establishing the relation between formulations with a single vector potential and formulations with a doublet of vector potentials.

\[^{11}\text{In the calculation of} \Gamma_0 \text{from the dual action, which can be carried out using an identity similar to (2.17), the symmetries (7.4), being quasi–local, do not need a gauge–fixing.}\]
7.3 Equivalence between the Zwanziger– and PST–actions

The classical field theory action introduced by Zwanziger \[3\] depends only on a doublet of vector potential one–forms \(\Lambda^I\), and on the matter fields \(\varphi_r\). It is local and manifestly \(SO(2)\)–invariant, but breaks explicitly Lorentz–invariance, due to the appearance of a constant vector \(n_\mu\), \(n^2 = -1\). It reads

\[
S_{ZW}[\Lambda, \varphi] = \frac{1}{2} \int d\Lambda \mathcal{Q}(n) d\Lambda - \sum_r \int d^4x \bar{\varphi}_r \left( D^\mu(\Lambda) D_\mu(\Lambda) + m_r^2 \right) \varphi_r, \tag{7.6}
\]

where the contribution of the matter fields is written as in (5.7). The operator \(\mathcal{Q}(n)\) is the one appearing in the dual PST–action, but evaluated at \(u^\mu = n^\mu\).

The equivalence between this action and the classical original PST field theory action \(S_U\) \[5.7\] can be understood as follows. Since in the PST–approach the fields \(a(x)\) and \(U(x)\) can be chosen arbitrarily, the first because it is an auxiliary field and the second because the quantum theory is \(U\)–independent, we choose

\[
a(x) = n_\mu x^\mu \Rightarrow v^\mu = n^\mu.
\]

The action \(S_U\) becomes then a functional of \(A, \Lambda, C, \Sigma, \) and \(\varphi_r\), which we call \(S_n\). Since the fields \(\Lambda\) and \(\varphi_r\) have the same symmetry properties, i.e. \(U(1)\)–invariances, in \(S_{ZW}\) and \(S_n\), it is natural to try a functional integration over \(C, A\) and \(\Sigma\) of \(S_n\):

\[
e^{i S_U[A, C, \Sigma, \varphi]} \equiv \int \{DC\} \{DA\} \{D\Sigma\} e^{i S_n[A, C, \Sigma, \varphi]}.
\]

It is understood that the integrations have to be carried out performing appropriate gauge fixings.

Since in the action \(S_U\) we have now to set \(U = v = n\), due to the presence of the term \(-\frac{1}{2} \int \Sigma n_i n^i C\) we can eliminate from the PST–action all terms proportional to \(i_n C\) (shifting \(\Sigma\)). This leads to

\[
S_n = \int \frac{1}{2} dA \mathcal{P}(n) dA + \frac{1}{2} C \ast C + \left( * \varepsilon dA - d\Lambda - \frac{1}{2} \Sigma n_i n^i \right) \varepsilon C \tag{7.7}
\]

\[- \sum_r \int d^4x \bar{\varphi}_r \left( D^\mu(\Lambda) D_\mu(\Lambda) + m_r^2 \right) \varphi_r. \tag{7.8}
\]

We perform first the gaussian integration over \(C\), which leads to an action which is quadratic in \(\Sigma\). The integration over \(\Sigma\) is again gaussian, and to perform it one has only to take the gauge–fixing \(i_n \Sigma = 0\) into account. After these integrations one obtains as intermediate result

\[
\int \{DC\} \{D\Sigma\} e^{i S_n} = e^{i \left( S_{ZW}[\Lambda, \varphi] + \frac{1}{2} \int (dA - d\Lambda) \varepsilon n_i n^i (dA - d\Lambda) \right)}.
\]

Since there exist linear gauge–fixing conditions for the symmetries \((2.10)\) and \((2.11)\), one can perform the shift \(A \rightarrow A + \Lambda\), and the integral over \(A\) gives a constant. Therefore, the result is

\[
S[\Lambda, \varphi] = S_{ZW}[\Lambda, \varphi],
\]

39
which shows the equivalence between the Zwanziger action and the original PST–action. Notice, in particular, that the operator $\mathcal{P}(n)$ of the original PST–action has turned in the operator $Q(n)$.

Similarly one can ask if there exists an analogous relation between the Zwanziger action and the dual PST field theory action $\tilde{S}_U$. The latter is obtained from $S_U$ in (5.7) upon substituting the PST–action $S_0[A, C, a]$ with the dual PST–action $\tilde{S}_0[A, C, b]$. To see that this is the case one has to choose in $\tilde{S}_U$

$$b^{(0)}_{\mu\nu}(x) = \varepsilon_{\mu\nu\rho\sigma} n^\rho x^\sigma \Rightarrow u^\mu = n^\mu \quad (7.9)$$

$$U^\mu(x) = n^\mu,$$

obtaining an action $\tilde{S}_n[A, \Lambda, C, \Sigma, \varphi]$. Modulo a suitable redefinition of $\Sigma$, one has

$$\tilde{S}_n = \int \frac{1}{2} dA \mathcal{Q}(n) dA + \left(dA - d\Lambda - \frac{1}{2} \Sigma n\Sigma\right) \varepsilon C \quad (7.10)$$

$$- \sum_r \int d^4 x \tilde{\varphi}_r \left(D^\mu(\Lambda)D_\mu(\Lambda) + m_r^2\right) \varphi_r. \quad (7.11)$$

As above we integrate over $A, C$ and $\Sigma$

$$e^{i\tilde{S}[\Lambda, \varphi]} \equiv \int \{\mathcal{D}C\} \{\mathcal{D}A\} \{\mathcal{D}\Sigma\} e^{i\tilde{S}_n[A, \Lambda, C, \Sigma, \varphi]}.$$

The integration over $C$ leads to the $\delta$–function $\delta(dA - d\Lambda - \frac{1}{2} i_n n\Sigma)$, which implies

$$dA = d\Lambda + \frac{1}{2} i_n n\Sigma, \quad d(i_n n\Sigma) = 0.$$

Substituting these relations above and taking into account that $\mathcal{Q}(n)i_n n\Sigma = \frac{1}{2} \varepsilon i_n n\Sigma$, the integration over $A$ and $\Sigma$ becomes trivial. The result is

$$\tilde{S}[\Lambda, \varphi] = S_{ZW}[\Lambda, \varphi],$$

i.e. again the Zwanziger action.

The connections presented above and in the preceding subsection ensure that the quantum field theories based on $S_U$, $\tilde{S}_U$ and $S_{ZW}$ are equivalent. The PST approach introduces additional non propagating auxiliary fields, but avoids some unpleasant features of the Zwanziger approach. Due to the presence of the vector $n$ in the last one, the consistency check of the quantum theory requires the proof of Lorentz–invariance of the correlators of observables. Therefore, the vector $n$ acquires only a posteriori the meaning of the direction of the Dirac–string. The entanglement of these two issues – Lorentz–invariance and Dirac–string dependence – in the Zwanziger theory, leads, already at the level of point particles, to a rather complicated realization of Lorentz–invariance. For point particles Zwanziger’s action becomes

$$S_{ZW}^{pp} = \frac{1}{2} \int d\Lambda \mathcal{Q}(n) d\Lambda - \sum_r c_r \varepsilon \int_{\gamma_r} \Lambda - \sum_{r=1}^N m_r \int_{\gamma_r} d\tau_r. \quad (7.12)$$
It can be obtained from the corresponding dual PST–action for dynamical dyons, setting \( u = n \) and choosing \( C \)'s satisfying \( i_n C = 0 \). A Lorentz transformation in (7.12) amounts then to make the replacement \( n^\mu \to n'^\mu = \Lambda^\mu_{\nu} n^\nu \). From the equivalence with the dual PST–action it is clear that, to recover Lorentz–invariance, one has to modify the standard Lorentz–transformation law of \( A \) as follows. First one has to add a transformation of the type (7.5), which re-rotates the vector \( n \) (the transformation of \( b \) allows, indeed, to reach an arbitrary \( n \)). Second, since Zwanziger’s action corresponds to a choice for the \( C \)'s restricted to \( i_n C = 0 \), a change of \( n \) has to be compensated by a string shift, i.e. by \( \delta A = -H_{n,n'} \); therefore, under these modified Lorentz–transformations Zwanziger’s action changes by an integer multiple of \( 2\pi \), if (1.4) holds.

These rather complicated arrangements are avoided in the classical PST–approach, due to its manifest \( SO(1,3) \)–invariance. Moreover, once the classical (exponentiated) PST–action is checked to be Dirac–string independent, the quantum string–independence follows almost immediately.

The relation between the PST–approach and [9] is less clear, since the authors of that paper renounce to construct a classical field theory action.

### 7.4 Equivalence of formulations with one and two vector potentials

In the subsection 4.1 we gave an action for the \( Z_4 \)–theory, in terms of a single vector potential, see the expression for \( I_0[A^2, J^1, C^2] \) in eq. (4.1). There we observed also that the action \( I_0 - \frac{1}{2} \int C^1 C^2 \) could be used, instead of the PST–action, for a formulation of the \( SO(2) \)–theory in terms of a single vector potential.

Vice versa, the action \( S_0 + \frac{1}{2} \int C^1 C^2 \) (or \( \tilde{S}_0 + \frac{1}{2} \int C^1 C^2 \)) could be used for a formulation of the \( Z_4 \)–theory in terms of a doublet of vector potentials.

The question, which arises naturally, is if the formulations with one and two vector potentials are directly related: the functional integration over one of the two vector potentials could give the corresponding formulation in terms of a single vector potential. Actually, we have two formulations in terms of vector potential doublets, based on \( S_0 \) and \( \tilde{S}_0 \) respectively, and it turns out that, in this respect, the two formulations behave indeed differently. The reason has been anticipated above. In the PST–action \( S_0 \) each vector potential carries one of the two transverse photon degrees of freedom. Therefore, integrating out one of the two vector potentials leads to a non–local action for the remaining vector potential, even in the absence of currents, as can be checked also directly. On the other hand, in the dual PST–approach each vector potential carries two transverse degrees of freedom, which become identified at the level of the equations of motion.

Therefore, integrating the dual PST–action, augmented by the term \( \frac{1}{2} \int C^1 C^2 \), over \( A^1 \) and \( b \), one can expect to obtain the action \( I_0[A^2, J^1, C^2] \) for the \( Z_4 \)–theory (while the
analogous calculation with the PST–action can surely not lead to \( I_0 \). We define \( \hat{I}_0 \) by

\[
e^{i\hat{I}_0[A^2,C^1,C^2]} = \int \{DA^1\} \{Db\} e^{i\left(\hat{S}_0[A,C,b] + \frac{1}{2} \int C^1C^2\right)}.
\]

The integration over \( b \) needs the insertion of a gauge–fixing \( \delta \)–function \( \delta(b(x) - b^{(0)}(x)) \), and the integration over \( A^1 \) requires only to fix its \( U(1) \)–invariance. The gaussian integral over \( A^1 \), for a fixed \( b^{(0)} \), can be conveniently evaluated through an \( S \)–duality transformation and turns out to be independent of \( b^{(0)} \). For the details of this calculation we refer the reader to the appendix.

The result is, indeed,

\[ \hat{I}_0 = I_0. \]  

(7.13)

The disappearance of \( b^{(0)} \), and the result itself, are somewhat surprising since there is no symmetry which, a priori, protects \( \hat{I}_0 \) from a \( b^{(0)} \)–dependence or from non–local terms. Clearly, the result is in agreement with the fact that a further functional integration over \( A^2 \) leads to the effective action \( \Gamma_0^{Z_4} \); but this would have happened also if one had started with the PST–action.

### 7.5 \( \vartheta \)–angles and \( SO(2) \)–symmetry

We anticipated that in the \( SO(2) \)–theory \( \vartheta \)–angles are already present, while in the \( Z_4 \)–theory they are introduced adding to the action a term proportional to \( \int \bar{F}F \). In the case of the \( SO(2) \)–theory such terms are forbidden by the PST–symmetries.

To see how \( \vartheta \)–angles arise, it suffices to search for the general solution of the \( SO(2) \)–invariant Dirac–Zwanziger quantization condition for the charge vectors \((e_r, g_r)\),

\[ e_r g_s - e_s g_r = 4\pi n_{rs}. \]  

(7.14)

Since this relation is invariant under \( Sl(2,R) \) – a three parameter group – and not only under \( SO(2) \), one expects that the general solution is characterized by three continuous parameters. To see that this is indeed the case we observe that, with an \( SO(2) \) rotation, one can eliminate the magnetic charge of say the first charge vector, \( g_1 = 0 \). This leads to

\[ g_r = \frac{4\pi}{e_1} m_r, \]  

(7.15)

with \( m_r \) integer for all magnetic charges. Substituting this back in (7.14) one gets

\[ e_r = \frac{n_{rs}}{m_s} e_1 + \frac{e_s}{m_s} m_r \]

for each \( r \) and \( s \). Choosing a particular \( s \) such that \( L \equiv m_s \neq 0 \), and defining \( e_0 = e_1/L \), one can write the general solution as

\[ e_r = e_0 \left( n_r + \frac{\vartheta}{2\pi} m_r \right) \]

\[ g_r = \frac{4\pi}{e_0} \frac{m_r}{L}, \]  

(7.16)
where \( m_r, n_r \) and \( L \) are integers restricted to
\[
n_r m_s - n_s m_r = L \cdot K_{rs}; \quad (7.17)
\]
this means that the left hand side has to be an integer multiple of \( L \). Apart from an \( SO(2) \)–rotation, the relations (7.16) and (7.17) specify the general solution of (7.14). They represent the counterpart to the relations (4.23) in the \( Z_4 \)–theory, modified by a \( \vartheta \)–term.

The integer \( L \) can, in general, not be eliminated by rescaling \( e_0 \) or \( \vartheta \). (For example, the solution corresponding to \( n_r = m_r = M \) for any \( r \) and \( L \) arbitrary, can not be traced back to the case \( L = 1 \).) Nevertheless, the physically relevant solutions correspond to \( L = 1 \). In fact, it is sufficient to assume that there exists a particle, say the electron, with the minimal electric charge \( e_0 \) and with vanishing magnetic charge: \( n_s = 1, m_s = 0 \). Equation (7.17) implies then that all \( m_r \) are integer multiples of \( L \), and this amounts to set in (7.16) \( L = 1 \), upon rescaling \( m_r \) and \( \vartheta \). Henceforth we will restrict to this case.

Taking the initial \( SO(2) \)–rotation into account, these relevant solutions can then be expressed as
\[
\begin{pmatrix} e_r \\ g_r \\ n_r \\ m_r \end{pmatrix} = T(\alpha) \begin{pmatrix} e_0 \\ 0 \\ \frac{\vartheta}{2\pi} e_0 \\ \frac{e_0}{e_0} \end{pmatrix} \equiv \Omega(\alpha, e_0, \vartheta) \begin{pmatrix} n_r \\ m_r \end{pmatrix}. \quad (7.18)
\]
Here \( T(\alpha) \) is an \( SO(2) \)–matrix, and
\[
\Omega(\alpha, e_0, \vartheta) = T(\alpha) \begin{pmatrix} e_0 \\ 0 \\ \frac{\vartheta}{2\pi} e_0 \\ \frac{1}{e_0} \end{pmatrix}
\]
parametrizes a generic element of \( Sl(2, R) \), as anticipated above.

Two theories whose charge vectors differ only by the matrix \( T(\alpha) \) are equivalent, because this difference can be eliminated by an \( SO(2) \)–redefinition of the vector potential doublets (a consequence of the \( SO(2) \)–duality invariance of the theory). This implies also that the angle \( \alpha \) is a fictitious coupling constant, and that the matrix \( T(\alpha) \) can be set equal to unity. Thus the geometric meaning of (7.18) is that the space of coupling constants is represented by the coset \( \frac{Sl(2, R)}{SO(2)} \), parametrized by \( \vartheta \) and \( e_0 \).

Apart from \( \vartheta \) and \( e_0 \), the charge vectors are characterized also by the vectors of integers \((n_r, m_r)\), which until now have been kept fixed. If we consider also \( Sl(2, Z) \)–transformations of these integer vectors, we can find more equivalence relations between different theories. It is sufficient to concentrate on the two generators \( T \) and \( S \) of \( Sl(2, Z) \). They act as
\[
T: \quad e'_0 = e_0 \\
\vartheta' = \vartheta + 2\pi \\
n'_r = n_r - m_r \\
m'_r = m_r
\]
43
\[ S : \quad e'_0 = \frac{4\pi}{e_0} \sqrt{1 + \left( \frac{\vartheta e_0^2}{8\pi^2} \right)^2} \]
\[ \vartheta' = -\frac{\left( \frac{e'_0}{4\pi} \right)^2}{1 + \left( \frac{e_0^2}{8\pi^2} \right)^2} \vartheta \]
\[ n'_r = m_r \]
\[ m'_r = -n_r. \]

Under a \( \mathcal{T} \)-transformation we have simply \((e_r, g_r) = (e'_r, g'_r)\) and the corresponding theories are trivially equivalent. Under \( S \) we have instead

\[ \begin{pmatrix} e'_r \\ g'_r \end{pmatrix} = T(\alpha) \begin{pmatrix} e_r \\ g_r \end{pmatrix}, \]

where \( T(\alpha) \) is an \( SO(2) \)-matrix, with rotation angle given by

\[ \tan \alpha = -\frac{8\pi^2}{\vartheta e_0^2}. \]

Due to \( SO(2) \)-invariance the corresponding theories are then again equivalent.

Notice, however, that the theory is not invariant under the shift \( \vartheta \to \vartheta + 2\pi \) alone, because this shift amounts to \( n_r \to n_r + m_r \), and this changes the electric charges of the charge vectors.

8 Concluding remarks

In this concluding section we mention briefly some other possible applications and natural extensions of the methods developed in this paper.

The monopoles considered in this paper are Dirac–monopoles – exhibiting a singularity at the monopole location – and are typical for abelian gauge theories. A priori, the formalism developed in the paper does not apply to the non–singular ‘t Hooft–Polyakov monopoles [27]. They appear in the semi–classical analysis of non–abelian gauge theories whose gauge group is spontaneously broken to an abelian subgroup, containing at least one \( U(1) \)-factor; a prototype is given by the Georgi–Glashow model. A way to exhibit explicitly ‘t Hooft–Polyakov monopoles in correlation functions of observables of the full quantum theory has been suggested by ’t Hooft [28], and, following ideas similar to those outlined in sect 6.2.6, a recipe to construct monopole correlation functions has been proposed in [21].

Although in the semi–classical analysis these monopoles are non–singular, it is well known [29] that they are mapped to Dirac–like monopoles in suitable unitary gauges and, therefore, in these gauges, the formalism developed in the present paper applies also to them.
By adding to the action of e.g. the Georgi–Glashow model the topological $\vartheta$–term

$$\frac{e^2 \vartheta}{16\pi^2} \int \text{Tr} (\mathcal{F}\mathcal{F}),$$

(8.1)

where $\mathcal{F}$ is the non–abelian field strength two–form, the 't Hooft–Polyakov monopoles are turned into dyons. The integral appearing in (8.1) is a topological invariant, the second Chern class, and it takes values in the integers. Therefore, the quantum field theory with $\vartheta$–term is invariant under the shift $\vartheta \rightarrow \vartheta + 2\pi$, in contrast with the remarks made at the end of the previous section for the abelian models considered in this paper. The different behaviour is related to the appearance of an infinite number of species of dyons in the Georgi–Glashow model, as discussed in [28].

In abelian models invariance under $2\pi$–shifts of $\vartheta$ can be achieved in the limiting case, where the integral $e^2_0/8\pi^2 \int FF$ is forced to be an integer, as in the Cardy–Rabinovici model [31].

One of the principal advantages of the formalism presented in this paper is the implementation of Lorentz–invariance as a manifest symmetry, at each step, while the quantum consistency of the theory relies on the Dirac–string independence. From the latter point of view we saw that the classical point–particle theory contains already the relevant informations, like the quantization conditions for the charges, the distinction between $SO(2)$–theories and $Z_4$–theories, the effect of $\vartheta$-angles, and the current–current effective actions.

At this “semi–classical” level the formalism allows immediate extensions from point–particles (0–branes) to arbitrary $p$–branes, coupled to $(p+1)$–form gauge potentials, and to arbitrary dimensions. It can be applied not only to $p$–brane dyons, living in $D = 2p+4$ dimensions, but also to a system of dual branes, for example strings and five–branes in ten dimensions. Results in this direction will be reported elsewhere [32].

9 Appendix

9.1 Useful identities

We collect here the basic identities, involving differential $p$–forms, the $*$–operator and the interior product $i_U$ with a vector field $U^\mu$, used in the paper. The one–form $U$ is given by $U = dx^\mu U_\mu$.

$$d (\Phi_p \Phi_q) = \Phi_p d\Phi_q + (-)^q (d\Phi_p) \Phi_q$$
$$i_U (\Phi_p \Phi_q) = \Phi_p i_U \Phi_q + (-)^q (i_U \Phi_p) \Phi_q$$
$$d^2 = 0 = (i_U)^2$$
$$i_U = (-)^p * U *$$
$$U = (-)^{p+1} * i_U *$$
$$* i_U = U *$$
\[ i_U* = -*U \]
\[ \delta = *d* \]
\[ *^2 = (-)^{p+1} \]
\[ 1 = \frac{1}{U^2}((-)^{p+1}Ui_U + *Ui_U*) \]
\[ * = \frac{(-)^{p+1}}{U^2}(*Ui_U + Ui_U*) \]
\[ \Box = d\delta + \delta d. \]

For constant vector fields \( n \) we have also
\[ di_n + i_n d = \partial_n = n^\mu \partial_\mu. \]

### 9.2 Proof of (7.13)

We begin by writing for \( \tilde{S}_0[A, C, b] \) an expression analogous to (2.17), keeping the notation used for that formula.

\[ \tilde{S}_0[A, C, b] = \Gamma_0[C] + \frac{1}{2} \int (G + D) \tilde{\Omega}(u)(G + D). \]  
(9.1)

The unique difference w.r.t. (2.17) lies in the operator \( \tilde{\Omega}(u) \) which is here
\[ \tilde{\Omega}(u) = \frac{d*d}{\Box} - i_u* i_u. \]

We remember that \( G = dA^1 - *dA^2 \) and \( D = C^1 - *C^2 \). The above identity shows, by the way, that the effective action associated to \( \tilde{S}_0 \) is \( \Gamma_0[C] \).

Since we want now to integrate only over \( A^1 \) (and \( b \)) it is convenient to define
\[ \tilde{A}^2 = A^2 - *dD \]
(9.2)
\[ \tilde{A}^1 = A^1 + \delta D, \]

such that \( G + D = d\tilde{A}^1 - *d\tilde{A}^2 \). In terms of \( \tilde{A}^I \) one can then write
\[ \tilde{S}_0[A, C, b] = \tilde{S}_0[\tilde{A}, 0, b] + \Gamma_0[C], \]

where \( \tilde{S}_0[0, b] \) is the free dual PST–action, i.e. with vanishing currents, as can be seen by setting in (9.1) \( C = 0 \). This means that
\[ \tilde{S}_0[A, C, b] = \frac{1}{2} \int d\tilde{A} \mathcal{Q}(u) d\tilde{A} + \Gamma_0[C]. \]

This reduces the problem to the integration over \( A^1 \) and \( b \) in the free dual PST–action,
\[ e^{iS[A^2]} = \int \{DA^I\}{Db} e^{\frac{i}{2} \int dA \mathcal{Q}(u) dA} \delta(b - b^{(0)}). \]  
(9.3)
To perform this integration it is convenient to perform an $S$–duality of $A^1$,

$$e^{iS[A^2]} = e^{\frac{i}{2} \int dA^2_u u^i dA^2} \int \{ \mathcal{D}H \} \{ \mathcal{D}A^1 \} \{ \mathcal{D}b \} \delta (b - b^{(0)}) e^{i \int \frac{1}{2} dH u^i u^j (dA^2_u u^i - d\Lambda) H}.$$  

Integration over $\Lambda$ gives, indeed, $dH = 0 \Rightarrow H = dA^1$. The quasi–local symmetry of $A^2$, $\delta A^2 = \chi^2, i_u d\chi^2 = 0$ has now to be accompanied by

$$\delta \Lambda = \chi^2. \quad (9.4)$$

We perform now the shift of integration variable $H \rightarrow H + u^i u^j (dA^2_u - d\Lambda)$ which gives

$$e^{iS[A^2]} = e^{\frac{i}{2} \int dA^2_u u^i dA^2} \int \{ \mathcal{D}H \} \{ \mathcal{D}A^1 \} \{ \mathcal{D}b \} \delta (b - b^{(0)}) e^{i \int \frac{1}{2} dH u^i u^j (dA^2_u - d\Lambda) + u^i u^j H d\Lambda}. \quad (9.5)$$

Since the two orthogonal components of $H$, $u^i u^j H$ and $u^i u^j H$, are now decoupled, the integration over these components factorizes and gives

$$e^{iS[A^2]} = e^{\frac{i}{2} \int dA^2_u u^i dA^2} \int \{ \mathcal{D}A^1 \} \{ \mathcal{D}b \} \delta (b - b^{(0)}) \delta (i_u d\Lambda) e^{i \int \frac{1}{2} d\Lambda u^i u^j (dA^2_u - d\Lambda) u^i u^j d\Lambda}. \quad (9.6)$$

But the $\delta$–function for $\Lambda$ means now that $d\Lambda$ is a pure gauge under (9.4) and amounts, therefore, to $\delta (d\Lambda)$. This implies that the integral over $\Lambda$ gives a constant, as does the final integration over $b$. Thus, the result is $b^{(0)}$–independent and reads simply

$$S[A^2] = \frac{1}{2} \int dA^2_u u^i dA^2.$$

The functional integration over $A^1$ and $b$ of $exp \left( i \left( \tilde{S}_0[A, C, b] + \frac{1}{2} \int C^1 C^2 \right) \right)$ gives therefore

$$\hat{I}_0 = \frac{1}{2} \int d\tilde{A}^2_u u^i u^j (dA^2_u + \Gamma_0[C]) + \frac{1}{2} \int C^1 C^2.$$

Upon substituting (9.2) and the explicit expression of $\Gamma_0[C]$, one obtains then easily

$$\hat{I}_0 = \int \frac{1}{2} (dA^2_u + C^2) * (dA^2_u + C^2) - dA^2 C^1 = I_0,$$

q.e.d.

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