The Algebra of Strand Splitting. I.
A Braided Version of Thompson’s Group $V$

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1. Introduction

We construct a braided version $BV$ of Thompson’s group $V$ that surjects onto $V$. The group $V$ is the third of three well known groups $F$, $T$ and $V$ created by Thompson in the 1960s that have been heavily studied since. See [6] and Section 4 of [5] for an introduction to Thompson’s groups.

The group $V$ is a subgroup of the homeomorphism group of the Cantor set $C$. It is generated by involutions [2, Section 12] and, if the metric on $C$ is ignored, $V$ can be viewed somewhat as a “Coxeter group” of permutations of $C$. In [4] we find presentations for $BV$ and $V$ that differ only in that the presentation for $V$ has relations of the form $x^2 = 1$ that are not present in the presentation for $BV$. Thus $BV$ can be thought of as an “Artinification” of $V$.

Our motivation for creating $BV$ is a relationship between $BV$ and the Thomp- son’s groups $F$ and $V$ on the one hand, and categories with multiplication on the other. Given a category $\mathcal{C}$ with multiplication, an isomorphism expressing associativity up to equivalence, and perhaps an isomorphism expressing commutativity up to equivalence, there are groups and epimorphisms

$$i : G_1(\mathcal{C}) \to F, \quad j : G_2(\mathcal{C}) \to V, \quad k : G_3(\mathcal{C}) \to BV,$$

that can be calculated from $\mathcal{C}$ and its attached data for which $i$ is an isomorphism if and only if $\mathcal{C}$ satisfies the axioms of a monoidal category, $j$ is an isomorphism if and only if $\mathcal{C}$ satisfies the axioms of a symmetric, monoidal category, and $k$ is an isomorphism if and only if $\mathcal{C}$ satisfies the axioms of a braided tensor category. See [12] and [13] for definitions. These results will be written up elsewhere.

As an intermediate step in understanding the group $BV$, we also construct a “larger” group $\hat{BV}$ that contains (and can also be shown to be contained in) $BV$ that is somewhat more tractable. If $BV$ is regarded as a braided version of $V$, then $\hat{BV}$ is a braided version of $\hat{V}$ that contains (and can also be shown to be contained in) $V$ and that is also somewhat more tractible than $V$.

The group $\hat{V}$ acts on countably many copies of the Cantor set, and another view of $\hat{BV}$ is that it is the group $B_\infty$ of finitary braids on countably many strands that has been modified by allowing the strands to split and recombine. This explains the first part of the title of this paper. The group $BV$ is the subgroup in which all splitting, braiding and recombining is confined to the first strand. The group $BV$ is thus the “braid group with splitting on one strand.”

The results in the paper are geometric and algebraic descriptions of $\hat{BV}$ and $BV$ that reveal their algebraic structure, a derivation of a normal form for the elements of the groups, an infinite presentation for $\hat{BV}$, and sketches of arguments that the geometric and algebraic descriptions of each group are of isomorphic groups. All that we say applies with trivial modification to $V$, and we get a similar normal form for $V$. This normal form for $V$ is general knowledge but has never been recorded.

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we derive finite presentations for $\hat{BV}$ and $BV$, and also a new finite presentation for $V$ that is similar to that of $BV$.

Patrick Dehornoy [9] has independently discovered $BV$ and $\hat{BV}$. Dehornoy’s motivation is not that removed from ours—he builds the groups as braided versions of structure groups of algebraic identities—but his techniques of construction, analysis, and set of observations about the groups are different. The group $BV$ is the main focus in the current paper, and we construct $\hat{BV}$ primarily to get to $BV$. The focus in [9] is on the group $\hat{BV}$ (called $FB_\infty$ in that paper) and its strong relationship with the law of left self distributivity. See [8] for more information and consequences of the relationship between this law and the ordinary braid groups.

The groups $BV$ and $\hat{BV}$ are related to other groups in the literature. The group $BV$ injects into the “universal mapping class group” $\mathcal{B}$ of [10]. There is also a non-finitely generated braided version of $V$ constructed in [11] that is different from $BV$ and $\hat{BV}$ since $BV$ and $\hat{BV}$ have finite presentations. The group $\hat{V}$ could have been defined in a single word the proof of Proposition 4.1 of [5] (where it would have been $G_{2,\infty}$ in the notation of that paper).

1.1. Descriptions of $V$, $BV$ and $\hat{BV}$. Elements of $V$ are most easily described using the standard “deleted middle thirds” description of the Cantor set $C$. The set $C$ is a limit of a sequence of collections of closed intervals in the unit interval $[0,1]$. The first few collections are

\[
\left\{ [0,1] \right\}, \\
\left\{ [0, \frac{1}{3}], \left[\frac{2}{3}, 1\right] \right\}, \\
\left\{ [0, \frac{1}{9}], \left[\frac{2}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right] \right\}, \\
\vdots
\]

Elements of $V$ are usually coded by pairs of labeled binary trees. For example, the map $f$ above is coded by the pair

\[
\begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
3 & 1 & 2 \\
\end{pmatrix}
\]

The map $f$ is the restriction of the following affine surjections

\[
\left[0, \frac{1}{3}\right] \rightarrow \left[\frac{2}{9}, \frac{1}{3}\right], \quad \left[\frac{2}{3}, \frac{7}{9}\right] \rightarrow \left[\frac{2}{9}, 1\right], \quad \left[\frac{8}{9}, 1\right] \rightarrow \left[0, \frac{1}{3}\right]
\]

to the portions of $C$ contained in the given intervals.

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\[
\begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
3 & 1 & 2 \\
\end{pmatrix}
\]
The structure of the left tree indicates that the interval \([0, 1]\) (corresponding to the root at the top of the tree) is to be split, and the resulting right interval \([\frac{1}{2}, 1]\) is to be split again. This describes the intervals in the domain of \(f\). The right tree codes the splittings needed to describe the intervals in the range of \(f\).

To obtain an element of \(BV\), we embed \(C\) in the plane \(\mathbb{R}^2\) as a subset of the \(x\)-axis. Let \(C\) be covered by a collection \(A\) of pairwise disjoint intervals from (1) and also a similar collection \(B\) with the same number of intervals. An element of \(BV\) will take intervals in \(A\) to intervals in \(B\) exactly as described above, but the move will be accomplished by an isotopy of \(\mathbb{R}^2\) with compact support. That is, the move will be accomplished by braiding if we view the isotopy as a level preserving homeomorphism from \(\mathbb{R}^2 \times [0, 1]\) to itself and letting the braid strands be the images of the components of \(C \times [0, 1]\). A restriction that must be observed is that during the isotopy, each interval in \(A\) must have its image during the isotopy parallel to the \(x\)-axis at all times. Isotopies \(u\) and \(v\) are equivalent if there is a level preserving isotopy of \(\mathbb{R}^2 \times [0, 1]\) from \(u\) to \(v\) (adhering to the restriction that the images of intervals from \(A\) be kept parallel to the \(x\)-axis throughout) that are fixed on the Cantor set at the 0 and 1 levels. Thus \(BV\) is seen to be a subgroup of a braid group on a Cantor set of strands.

The surjection from \(BV\) to \(V\) is obtained by taking each element of \(BV\) to the homeomorphism of \(C\) obtained at the end of the isotopy.

An element of \(BV\) can also be coded by pairs of binary trees, but now the connection from the leaves of the first to the leaves of the second is given by a braid and not a bijection. This is most easily pictured by drawing the second tree upside down below the first and drawing the braid connecting the leaves between the two trees.

As an example, the following is one element of \(BV\) (out of infinitely many) that maps to the element \(f\) of \(V\) in the example above. We draw both the “trees and braid” encoding of the element as well as a picture of a braiding of intervals.

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow 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number of intervals be equal. However, this might result in a “shift” taking place for large values of $x$. It is required that this shift be done by an isotopy that, outside a compact set, consists of $(x, y) \mapsto (x + td(1 - |y|), y)$ for $|y| < 1$ and $x$ greater than some $K$ and is the identity otherwise. The integer $d$ is the total amount of shift and $t$ is the parameter of the isotopy. Isotopies between isotopies that create the equivalence classes of the braids are required to have compact support, and, as above, are required to keep the images of the defining intervals parallel to the $x$-axis throughout.

We are in a position to explain some remarks made above. The structure of the trees is what keeps track of the “order of splitting.” The restriction that the isotopies used by elements of $BV$ keep certain intervals parallel to the $x$-axis explains why $BV$ is a proper of the group $B$ of [10] since the group $B$ allows the intervals to rotate.

### 1.2. Multiplication in $BV$.

Because strands are not really split and combined (bundles of strands over the Cantor set are simply degrouped and regrouped), the following relations are clear.

\[(2) \quad \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \leftrightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \quad \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \leftrightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array}
\]

The pictures in (2) simply express the fact that certain splitting and recombining operations are inverses of each other.

We are in a position to multiply pictorially. As in the braid group, the product $uv$ of $u$ and $v$ is drawn by putting $u$ over $v$. If $u$ is the element \[
\begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array}
\]
then $u^2$ is calculated as follows.

\[
\begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array}
\]

The above example has a trivial braid between two trees. A product of three elements, two of which involve non-trivial braids is shown below.

\[
\begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array} \rightarrow \begin{array}{c} \nearrow \swarrow \\
\downarrow & \downarrow \\
\end{array}
\]
The restrictions that we impose make a difference. Imagine that the following moves are allowed in a part of a diagram.

The move on the left can be accomplished by a 180 degree rotation inside a 2-sphere that intersects the diagram in exactly two points. This move would correspond to an illegal rotation of an interval during an isotopy. The move on the right is accomplished by an isotopy that reverses the slope of the line joining the two strands but that does not interchange the two strands. This move essentially alters a tree that is part of the diagram. Note that the first move can also be realized as an application of the second move to the line in the middle of the square.

The moves in (3) have the following consequence.

However, the figure on the left represents a non-trivial element of $BV$ and of $V$.

The goal of the paper is to expose as much of the algebraic structure of $BV$ as possible. For this reason, our construction and analysis of $BV$ is more algebraic than geometric. The next section reviews some of the techniques that we will use in the remainder of the paper and contains a bit of an outline as to how the techniques will be put to use.

2. Conventions, definitions and needed constructions

2.1. Outline. Working backwards from the group $BV$, we get $BV$ as a subgroup of the better behaved group $\hat{BV}$. The group $BV$ is obtained as a group of right fractions of a cancellative monoid $\mathcal{F} \bowtie B_\infty$ and a normal form (reduced terms) for the elements of $\hat{BV}$ is obtained from the fact that pairs of elements of $\mathcal{F} \bowtie B_\infty$ have unique greatest common right factors. The monoid $\mathcal{F} \bowtie B_\infty$ is a Zappa-Szép product (a generalization of the semidirect product in which neither factor need be normal) of the monoid $\mathcal{F}$ and the group $B_\infty$. The group $B_\infty$ is the familiar braid group on infinitely many strands. The monoid $\mathcal{F}$ is a monoid of binary forests and is understood by obtaining a normal form for its elements. The normal form is obtained by a standard technique using the concept of terminating and confluent relations—a technique that is used more than once in this paper.

In the rest of this section, we first review the techniques for getting normal forms from confluent and terminating relations. Next we review groups of fractions and the use of greatest common right factors to obtain reduced terms. Lastly, we review Zappa-Szép products. Since we will need information about these products that lead to certain properties (such as the existence of greatest common factors) and also lead to presentations, we review what is needed to get such information from the products.
2.2. Distinguished representatives. Normal forms will be used to establish properties of groups with presentations. The normal forms will come from standard techniques from string rewriting which arise in turn from properties of relations.

A binary relation \( \rightarrow \) on a set \( A \) is called terminating if there is no infinite sequence \( x_0 \rightarrow x_1 \rightarrow \cdots \). Note that a reflexive relation cannot be terminating. An element \( a \in A \) is said to be irreducible in \( A \) if \( a \rightarrow x \) is false for all \( x \in A \).

We let \( \rightarrow^* \) denote the reflexive, transitive closure of \( \rightarrow \). The relation \( \rightarrow \) is locally confluent if for every \( x, y \) and \( z \) satisfying \( x \rightarrow y \) and \( x \rightarrow z \), there is a \( w \) satisfying \( y \rightarrow^* w \) and \( z \rightarrow^* w \). If we let \( \sim \) denote the equivalence relation generated by \( \rightarrow \) (the symmetric, reflexive, transitive closure of \( \rightarrow \)), then we get the following result of Newman [14].

**Proposition 2.1.** If a binary relation \( \rightarrow \) on a set \( A \) is terminating and locally confluent, then every equivalence class under \( \sim \) contains a unique element that is irreducible in \( A \). Further, \( x \rightarrow^* a \) for every \( x \) where \( a \) is the unique irreducible in \( A \) that is in the equivalence class containing \( x \).

A terminating, binary relation that is also locally confluent is called complete.

Binary relations will often come from rewriting rules. If \( \Sigma \) is a set (which we call an alphabet in this situation), then \( \Sigma^* \) denotes the set of strings (finite sequences) of elements of \( A \). The empty string (sequence of length zero) is also in \( \Sigma^* \). This is a monoid under concatenation and we will refer to it as the free monoid on \( \Sigma \). A binary relation \( \rightarrow \) on \( \Sigma^* \) can be referred to as a rewriting rule which terminology implies that another binary relation \( \theta \) on \( \Sigma^* \) is to be regarded as a consequence of \( \rightarrow \) as follows. If \( u, v, p \) and \( q \) are in \( \Sigma^* \), then we write \( puq \theta pvq \) if \( u \rightarrow v \). Confusingly, \( \rightarrow \) is often used for both the rewriting rule and its consequence. The confusion is usually not crippling.

A rewriting rule is called complete if its consequence is complete.

If \( \langle X \mid R \rangle \) is a presentation, then we regard \( R \) as a relation on words in \( X \). As a relation, \( R \) is usually thought of as either symmetric or its symmetric closure is implied. However, if the symmetric closure is not taken, then \( R \) can also be thought of as a set of rewriting rules. If it turns out that \( R \) is complete as a set of rewriting rules, then we say that the presentation is complete. The power of a complete presentation is that Proposition 2.1 gives each element of the presented object a distinguished representative.

2.3. Properties of monoids and the Ore theorem. The group of fractions construction will be important. We set out the necessary definitions and results.

A semigroup is a set with a binary, associative product. A monoid is a semigroup with a global, two-sided identity. A group is a monoid in which every element has a two-sided inverse.

A semigroup is left cancellative if \( ax = ay \) always implies \( x = y \) and is right cancellative if \( xa = ya \) always implies \( x = y \). A semigroup is cancellative if it is both right and left cancellative. A semigroup is strongly left cancellative if it is left cancellative and if \( ab = a \) implies that \( b \) is a global, two-sided identity. The reader can define strongly right cancellative.

**Lemma 2.2.** Let \( S \) be a cancellative semigroup. Then (1) \( S \) is strongly left cancellative and strongly right cancellative and (2) if \( S \) has a global, two-sided identity \( 1 \), and \( ab = 1 \), then \( ba = 1 \).
In a semigroup $S$, a right multiple of an element $x \in S$ is an element $y \in S$ for which there is an element $p \in S$ so that $y = xp$. A subset $C$ of $S$ has a common right multiple $z$ if $z$ is a right multiple of every element of $C$. Usually, $C$ has two elements and we refer to the common right multiple of $C$ as the common right multiple of the two elements. A common right multiple $z$ for $C$ is a least common right multiple if every common right multiple for $C$ is a right multiple of $z$. A semigroup has common right multiples if every pair of elements has a common right multiple. A semigroup has least common right multiples if every pair of elements with a common right multiple has a least common right multiple. Note that the last definition has been carefully worded so that a semigroup with least common right multiples need not have common right multiples.

In the previous paragraph, every appearance of the word right can be replaced by the word left to give a corresponding set of definitions. It will turn out that the important concepts for this paper will be “common right multiples” and “least common left multiples.” The first concept will lead to groups of fractions and the second concept will lead to distinguished representatives in the group of fractions.

Least common multiples are often associated with greatest common factors. An element $r$ in a semigroup $S$ is a right factor of an element $x \in S$ if there is a $p \in S$ so that $x = pr$. Two elements $x$ and $y$ in $S$ have a common right factor $r \in S$ if $r$ is a right factor of both $x$ and $y$. The common right factor $r$ is a greatest common right factor if every common right factor of $x$ and $y$ is a right factor of $r$. A semigroup $S$ has greatest common right factors if every pair of elements with a common right factor has a greatest common right factor.

A length function on a semigroup $S$ is a homomorphism to the natural numbers $\mathbb{N}$ so that the preimage of 0 is contained in the set of those $x \in S$ for which there is a $y \in S$ so that $xy = yx$ is a global, two-sided identity for $S$.

The following is a pleasant exercise for the reader.

**Lemma 2.3.** A semigroup with a length function has least common left multiples if and only if it has greatest common right factors.

In the following, a presentation $\langle X \mid R \rangle$ is thought of as a set $X$ and a relation $R$ on either the free monoid $X^*$ on $X$ if the presentation is a monoid presentation or the free group $F_X$ on $X$ if the presentation is a group presentation. As a relation, $R$ is thought of as a set of ordered pairs. The two entries an ordered pair in $R$ are regarded as equal in the object being presented.

We now discuss groups of fractions. Let $S$ be a cancellative semigroup with common right multiples. Let pairs in $S \times S$ be denoted by $\frac{x}{y}$ instead of $(x,y)$. Let $\sim$ be the equivalence relation generated by $\frac{x}{y} \sim \frac{y}{x}$ and define a product on representatives by $\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$. It follows from the existence of common right multiples that any two classes have representatives that can be multiplied. The product is well defined and we have the following.

**Proposition 2.4** (Ore). Let $S$ be a cancellative semigroup with common right multiples. Then the following hold.

(a) The multiplication above turns the equivalence classes on $S \times S$ into a group $G$.

(b) If $a$ is a fixed element of $S$, then sending $x \in S$ to $\frac{ax}{z}$ is a homomorphic embedding of $S$ into $G$. 

(c) Every element of $G$ is representable in the form $pn^{-1}$ where $p$ and $n$ are in the image of the embedding in (b).

(d) If $(X \mid Y)$ is a semigroup presentation of $S$, then $(X \mid Y)$ is a group presentation of $G$.

Items (a)–(c) of Proposition 2.4 comprise a mirror image of Theorem 1.23 of [X] where the existence of common left multiples (there called right reversible) is used instead. Item (d) is well known and straightforward.

If $U$ is a cancellative semigroup with common right multiples, and if $i : U \to G$ is an injective homomorphism into a group so that every element of $G$ is of the form $uv^{-1}$ with both $u$ and $v$ in the image of $i$, then we call $i$ an Ore embedding of $U$ into a group of right fractions of $U$ and write $\frac{u}{v}$ for $uv^{-1}$. Theorem 1.25 of [X] justifies calling $G$ the group of right fractions of $U$.

We get distinguished representatives (corresponding to reduced fractions) in the group of fractions under certain circumstances. To state the next lemma, we make a definition. Let $i : U \to G$ be an embedding of a semigroup into a group of right fractions. We say that a representative $\frac{w}{v}$ as above is in reduced terms if whenever $\frac{u}{v}$ is another such representative with $\frac{u}{v} \sim \frac{w}{v}$, then there is an $x$ in the image of $i$ with $u = vx$ and $z = vx$. It follows that if $\frac{u}{v}$ and $\frac{u'}{v'}$ are two representatives of the same element and both are in reduced terms, then there are $x$ and $y$ in the image of $i$ so that $u' = ux$, $u = u'y$, $v' = vx$ and $v = v'y$. This implies that $u = uxv$ and $u' = uyv$. We identify $U$ with $i(U)$ and use Lemma 2.2 to conclude that $xy$ and $yx$ are both global, two-sided identities for $U$. That is, both $x$ and $y$ are invertible.

To give brief terminology to this situation, we say that the representatives $\frac{w}{v}$ and $\frac{w'}{v'}$ differ by invertible elements of $U$.

**Lemma 2.5.** Let $U$ be a cancellative semigroup with common right multiples, least common left multiples and a length function. Let $i : U \to G$ be an Ore embedding of $U$ into a group of right fractions. Then every element of $G$ has a representative in reduced terms and any two representatives in reduced terms of one element of $G$ differ by invertible elements of $U$. In particular, any representative $uv^{-1}$ of an element $g \in G$ with $u$ and $v$ in $i(U)$ and the length of $u$ minimal among such representatives of $g$ is in reduced terms.

**Proof.** This can be given a short direct proof, taking the last sentence as a starting point. □

2.4. **Zappa-Szép products.** In spite of the fact that the previous discussion mentioned semigroups repeatedly, we will work entirely with monoids. Zappa-Szép products work in even greater generality than semigroups (see [X]), but we will restrict our discussion to monoids for practical reasons.

The Zappa-Szép product is a generalization of the semidirect product. It is a generalization in that no normality is required. Proofs of the statements in this section and a short history of the product can be found in [X]. We first show how the ingredients of the product arise. We start with groups to give a familiar setting and then generalize to monoids.

Let $G$ be a group with identity $1$, and with subgroups $U$ and $A$ satisfying $U \cap A = \{1\}$ and $G = UA$. Then each $g \in G$ is uniquely expressible as $g = u\alpha$ with $u \in U$ and $\alpha \in A$. With $u \in U$ and $\alpha \in A$, consider $\alpha u \in G$. There are unique elements $u' \in U$ and $\alpha' \in A$ so that $\alpha u = u'\alpha'$. This defines two functions $(\alpha, u) \mapsto \alpha u \in A$...
and \((\alpha, u) \mapsto \alpha \cdot u \in U\) that are unique in that they satisfy \(\alpha u = (\alpha \cdot u)(\alpha^u)\) for all \(u \in U\) and \(\alpha \in A\). We now move to monoids.

**Lemma 2.6.** Let \(M\) be a monoid and let \(U\) and \(A\) be submonoids of \(M\). Assume that every \(x \in M\) is uniquely expressible in the form \(x = u\alpha\) with \(u \in U\) and \(\alpha \in A\). Then there are functions \((A \times U) \to A\) written \((\alpha, u) \mapsto \alpha^u\), and \((A \times U) \to U\) written \((\alpha, u) \mapsto \alpha \cdot u\) defined by the property that \(\alpha u = (\alpha \cdot u)(\alpha^u)\).

In the setting of the above lemma, the functions \((\alpha, u) \mapsto \alpha^u\) and \((\alpha, u) \mapsto \alpha \cdot u\) defined on \(A \times U\) will be called the *mutual actions defined by the multiplication*. These actions are “internally” generated by the multiplication. We can also impose actions “externally.”

Let \(U\) and \(A\) be monoids. Assume there are functions \(A \times U \to A\) written \((\alpha, u) \mapsto \alpha \cdot u\) and \(A \times U \to U\) written \((\alpha, u) \mapsto \alpha^u\). We will call such functions mutual actions between \(A\) and \(U\).

We now define a multiplication on \(U \times A\) by
\[
(4) \quad (u, \alpha)(v, \beta) = (u(\alpha \cdot v), \alpha^v \beta).
\]
This multiplication is well defined. The following ties it to the setting of Lemma 2.6 and is essentially Lemma 3.9 of [3].

**Lemma 2.7.** Let \(M\) be a monoid and let \(U\) and \(A\) be submonoids of \(M\). Assume that every \(x \in M\) is uniquely expressible in the form \(x = u\alpha\) with \(u \in U\) and \(\alpha \in A\), and let \((\alpha, u) \mapsto \alpha^u\) and \((\alpha, u) \mapsto \alpha \cdot u\) defined on \(A \times U\) be the mutual actions defined by the multiplication. Use these mutual actions and \((4)\) to build a multiplication on \(U \times A\). Then sending \((u, \alpha)\) in \(U \times A\) to \(u\alpha\) in \(M\) is an isomorphism of monoids.

Assuming the setting, hypotheses and notation of Lemma 2.7, we say that \(M\) is the *(internal)* Zappa-Szép product of \(U\) and \(A\) and write \(M = U \bowtie A\).

In the setting of monoids, the mutual actions are not arbitrary. The next lemma gives sample properties that they satisfy. The statement that \(U\) is a submonoid of \(M\) contains the assumption that the identity for \(U\) is the identity for \(M\). The Lemma combines Lemma 3.2 and Corollary 3.3.1 from [3].

**Lemma 2.8.** Let \(M\) be a monoid and let \(U\) and \(A\) be submonoids of \(M\). Assume that every \(x \in M\) is uniquely expressible in the form \(x = u\alpha\) with \(u \in U\) and \(\alpha \in A\), and let \((\alpha, u) \mapsto \alpha^u\) and \((\alpha, u) \mapsto \alpha \cdot u\) defined on \(A \times U\) be the mutual actions defined by the multiplication. Let \(\alpha\) and \(\beta\) come from \(A\) and \(u\) and \(v\) come from \(U\). Let \(1_U\) and \(1_A\) denote the identities of \(U\) and \(A\), respectively. Then the following hold.

\[
\begin{align*}
(a) \quad (\alpha \beta) \cdot u &= \alpha \cdot (\beta \cdot u), \\
(b) \quad (\alpha \beta)^u &= \alpha^{(\beta \cdot u)} \beta^u, \\
(c) \quad \alpha \cdot (uv) &= (\alpha \cdot u)(\alpha^u \cdot v), \\
(d) \quad \alpha^{uv} &= (\alpha^u)^v, \\
(e) \quad \alpha^{1v} &= \alpha, \\
(f) \quad 1_A \cdot u &= u, \\
(g) \quad \alpha \cdot 1_u &= 1_u. \\
(h) \quad (1_A)^u &= 1_A.
\end{align*}
\]

The properties of Lemma 2.8 are exactly those needed to define an external Zappa-Szép product of monoids. The next lemma is item (xv) of Lemma 3.13 of [3].
Lemma 2.9. Let $U$ and $A$ be monoids with mutual actions $(\alpha, u) \mapsto \alpha \cdot u$ and $(\alpha, u) \mapsto \alpha^u$ defined on $A \times U$. Assume (a)-(h) of Lemma 2.8. Then the multiplication (4) makes $U \times A$ a monoid $M$. Further $\alpha \mapsto (1_U, \alpha)$ and $u \mapsto (u, 1_A)$ are homomorphic embeddings of $A$ and $U$, respectively, into $M$ so that $M$ is the internal Zappa-Szép product of the images.

We can refer to the monoid $M$ of Lemma 2.9 as the (external) Zappa-Szép product of $U$ and $A$ and again write $M = U \bowtie A$.

We need to know when Zappa-Szép products have certain properties. In order to discuss this, we need to look at mutual actions as families of functions. If a function $(\alpha, u) \mapsto \alpha^u$ is defined from $A \times U$ to $A$, then we think of this as a family of functions from $A$ to itself parametrized by elements of $U$. We say that $(\alpha, u) \mapsto \alpha^u$ forms a surjective family of functions if for every $u \in U$ and $\alpha \in A$ there is a $\beta \in A$ so that $\alpha = \beta^u$. A family is coconfluent if whenever $\alpha^u = \beta^v$, there are $\gamma$, $p$ and $q$ so that $\alpha = \gamma^p$ and $\beta = \gamma^q$. A family satisfying (d) of Lemma 2.8 is strongly coconfluent if whenever $\alpha^u = \beta^v$ and $u$ and $v$ have a common left multiple, there are $\gamma$, $p$ and $q$ so that $\alpha = \gamma^p$, $\beta = \gamma^q$ and $pu = qv$. Similar definitions can be made for $(\alpha, u) \mapsto \alpha \cdot u$ defined from $A \times U$ to $U$.

The following is Lemma 3.6 of [3].

Lemma 2.10. Let $A \times B \rightarrow A$ written $(\alpha, u) \mapsto \alpha^u$ be strongly coconfluent. Assume that $B$ is a right cancellative semigroup, assume that $\alpha^v = \beta^v$ and assume that $u$ and $v$ have a least common left multiple $l = au = bv$. Then there is a $\gamma \in A$ so that $\alpha = \gamma^a$ and $\beta = \gamma^b$.

The following comprises items (viii) and (ix) of Lemma 3.12 of [3] where the proof is left to the reader.

Lemma 2.11. Assume the notation, hypotheses and conclusion of Lemma 2.8. Then the following hold.

1. If $U$ and $A$ are both right cancellative and $(\alpha, u) \mapsto \alpha^u$ is an injective family, then $U \bowtie A$ is right cancellative.
2. If $U$ and $A$ both have common right multiples and $(\alpha, u) \mapsto \alpha \cdot u$ is a surjective family, then $U \bowtie A$ has common right multiples.

Least common left multiples are a bit more complicated. The following is Lemma 3.14 of [3].

Lemma 2.12. Assume the hypotheses, notation and conclusion of Lemma 2.8. If $U$ is cancellative with least common left multiples, if $A$ is a group, and if $(\alpha, u) \mapsto \alpha^u$ is strongly coconfluent, then $M = U \bowtie A$ has least common left multiples. Further, the least common left multiple $r(\alpha)(u, \theta) = (s, \beta)(v, \phi)$ of $(u, \theta)$ and $(v, \phi)$ in $U \bowtie A$ can be constructed so that $r(\alpha \cdot u) = s(\beta \cdot v)$ is the least common left multiple of $(\alpha \cdot u)$ and $(\beta \cdot v)$. If $M$ is cancellative (e.g., $(\alpha, u) \mapsto \alpha^u$ is an injective family), then any least common left multiple $r(\alpha)(u, \theta) = (s, \beta)(v, \phi)$ of $(u, \theta)$ and $(v, \phi)$ in $U \bowtie A$ has the property that $r(\alpha \cdot u) = s(\beta \cdot v)$ is the least common left multiple of $(\alpha \cdot u)$ and $(\beta \cdot v)$.

We copy from [3] some very specialized results about presentations of Zappa-Szép products that fit the needs of this paper.

Assume that presentations $(X \mid R)$ and $(Y \mid T)$ of monoids $U$ and $A$, respectively, are given with $X \cap Y = \emptyset$, and that functions $Y \times X \rightarrow Y^*$ written $(\alpha, u) \mapsto \alpha^u$
and $Y \times X \to X$ written $(\alpha, u) \mapsto \alpha \cdot u$ are given. The unequal treatment of the codomains ($X$ in one case and $Y^*$ in the other) is deliberate.

We extend these to functions $Y^* \times X^* \to Y^*$ and $Y^* \times X^* \to X^*$ as follows. Form the monoid presentation

$$(5) \quad \langle X \cup Y \mid Z \rangle$$

in which $Z$ is regarded as a set of rewriting rules and consists of all pairs $(\alpha u \mapsto (\alpha \cdot u)(\alpha^u))$ for $(\alpha, u) \in Y \times X$. The following easy lemma is Lemma 3.18 of [3].

**Lemma 2.13.** The presentation (5) is complete.

The irreducibles are of the form $ua$ with $u$ a word in the alphabet $X$ and $a$ a word in the alphabet $Y$. This expresses the monoid presented by (5) as a Zappa–Szép product of the free monoids $X^*$ and $Y^*$. From Lemma 3.13 we get our desired extensions and the fact that they satisfy the conclusions of that lemma.

The following combines Lemmas 3.17 and 3.19 of [3].

**Lemma 2.14.** Assume that presentations $(X \mid R)$ and $(Y \mid T)$ of monoids $U$ and $A$, respectively, are given with $X \cap Y = \emptyset$, and that functions $Y \times X \to Y^*$ written $(\alpha, u) \mapsto \alpha^u$ and $Y \times X \to X$ written $(\alpha, u) \mapsto \alpha \cdot u$ are given. Let $\sim_R$ and $\sim_T$ denote the equivalence relations on $X^*$ and $Y^*$, respectively, imposed by the relation sets $R$ and $T$, respectively.

Let the functions be extended to $Y^* \times X^*$ as above and assume that they satisfy the following. If $(u, v)$ is in $R$, then for all $\alpha \in Y$ we have $(\alpha \cdot u, \alpha \cdot v)$ or $(\alpha \cdot v, \alpha \cdot u)$ is in $R$ and $\alpha^u \sim_T \alpha^v$. If $(\alpha, \beta)$ is in $T$, then for all $u \in X$ we have $\alpha \cdot u = \beta \cdot u$ and $\alpha^u \sim_T \beta^u$. Then the extensions induce well defined functions $A \times U \to A$ and $A \times U \to U$ that satisfy the hypotheses (and thus the conclusions) of Lemma 2.13 and the restriction of the function $A \times U \to U$ to $A \times X$ has its image in $X$. Further a presentation for the structure defined on $U \bowtie A$ is

$$(6) \quad \langle X \cup Y \mid R \cup T \cup W \rangle$$

in which $W$ consists of all pairs $(\alpha u, (\alpha \cdot u)(\alpha^u))$ for $(\alpha, u) \in Y \times X$.

3. The monoid of forests

Our first algebraic structure will be a monoid whose objects are forests. For us a forest is a sequence of finite trees, only finitely many of which are non-trivial. We now give detailed definitions. None are surprising, but we give details to bring reader and author into agreement on terminology.

The complete binary tree $T$ is the set of all finite sequences with values in the set $\{0, 1\}$. The sequence of length 0, denoted $\phi$, is included. The sequences will be referred to as strings, and we will concatenate string $\alpha$ and string $\beta$ to give the string $\alpha \beta$ in which $\alpha$ comes first and $\beta$ comes last. The most important relation is prefix defined by “$\alpha$ is a prefix of $\alpha \beta$.” The transitive closure of proper prefix is ancestor and the inverse of ancestor is descendant. The children of $u$ are exactly $u0$ and $u1$.

A finite binary tree $T$ is a non-empty subset of $T$ that is closed under ancestor and for which $u0$ is in $T$ if and only if $u1$ is in $T$. Every tree in this paper except $T$ will be finite and binary, so we will stop using those words as adjectives for trees.

Every tree $T$ includes $\phi$ which is called the root of $T$. Elements of $T$ will be called nodes, and the leaves of $T$ are the nodes of $T$ with no children. A tree is non-trivial if it has more than one node.
We define trees this way so that if $T$ and $U$ are trees, then $T \cup U$ and $T \cap U$ make sense. It is elementary that both $T \cup U$ and $T \cap U$ are trees.

There is a unique total ordering of the nodes of a tree $T$ so that every triad $\{u, u0, u1\}$ in $T$ is ordered $u0 < u < u1$. We call this the left-right ordering of $T$. The restriction of this order to the leaves of $T$ is the left-right ordering of the leaves of $T$.

A forest $F$ is an infinite sequence (function with domain $\mathbb{N}$) of trees so that all but finitely many are trivial. We write $F_i$ for the $i$-th tree in $F$. The set $F$ of all forests will be endowed shortly with a binary operation.

If $v$ is a node of $F_i$ for a forest $F$, we distinguish it from nodes of other trees in $F$ by writing $i.v$. We order all the leaves of $F$ by giving the leaves of each tree in $F$ the left-right ordering and then insisting that all leaves in $F_i$ come before all the leaves in $F_j$ when $i < j$.

We number the leaves of $F$ by the unique order preserving function from the leaves to $\mathbb{N}$. The roots are numbered in the obvious way: the $i$-th root is the root of $F_i$.

If $F$ and $G$ are two forests, then we form $FG$ by identifying the $i$-th root of $G$ with the $i$-th leaf of $F$. Defining $v_jG_j$ to mean $\{v_j, u | u \in G_j\}$ where $v_j$ is the $j$-th leaf of $F$, then we can formally define the $i$-th tree of $FG$ as the union of $F_i$ with all $v_jG_j$ where $v_j$ is a leaf of $F_i$.

Below we give an example of a product $FG$ of forests $F$ and $G$. For clarity, we have numbered the leaves of $F$ and the roots of $G$ and $FG$.

We leave it to the reader to verify that this product is associative and that the trivial forest is both a left and right identity. Thus finite forests form a monoid under this operation. We extend the meaning of the symbol $F$ to include this product.

A triple of vertices $(u, u0, u1)$ is called a caret and is pictured here: $\wedge$. A non-trivial tree is a union of carets. We add the trivial tree to the discussion by describing it as the unique tree with zero carets.

Since every finite tree is a union of carets, we can describe a finite forest as a finite union of carets. From this it is clear that the monoid $F$ is generated by
the single caret forests. Let $\lambda_i$ be the unique forest with one caret whose only non-trivial tree is the $i$-th tree (which consequently has only one caret).

From the definition of the product of forests, it is clear that $F\lambda_i$ is exactly $F$ with an extra caret hung from the $i$-the leaf of $F$. From this, the following is obvious.

**Lemma 3.1.** The forests $\{\lambda_i \mid i \geq 0\}$ form a generating set for $F$.

We let the reader verify the next statement.

**Lemma 3.2.** The generators $\{\lambda_i \mid i \geq 0\}$ satisfy the relations $\lambda_q\lambda_m = \lambda_m\lambda_{q+1}$ whenever $m < q$.

To argue that the generating set and relation set of the last two lemmas form a presentation for $F$, we replace the relation $\lambda_q\lambda_m = \lambda_m\lambda_{q+1}$ by the rewriting rule

$$(7) \quad \lambda_q\lambda_m \rightarrow \lambda_m\lambda_{q+1} \quad \text{whenever } m < q.$$  

It is a pleasant exercise to show that the relation is terminating and locally confluent and thus complete. The words that are reduced with respect to $\rightarrow$ are the words $\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_k}$ for which $i_0 \leq i_1 \leq \cdots \leq i_k$. We will say that words reduced with respect to $\rightarrow$ are in normal form. If $w$ and $u$ are two different words in normal form, then by looking at the leftmost position where they differ, it is easy to argue that they correspond to two different forests. This proves the following.

**Proposition 3.3.** Each element of $F$ is represented uniquely by a word in the form $\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_k}$ for which $i_0 \leq i_1 \leq \cdots \leq i_k$. The monoid $F$ is presented by

$$(\lambda_0, \lambda_1, \ldots \mid \lambda_q\lambda_m = \lambda_m\lambda_{q+1} \text{ whenever } m < q).$$

In the lemma below, we claim that $F$ is cancellative. This implies that the equation $XA = B$ has a unique solution if it has one at all. In the case that the equation has a solution, we write it as $X = A\setminus B$. We say that forests $F$ and $G$ are disjoint, if for each $i \in \mathbb{N}$, at least one of $F_i$ or $G_i$ is trivial.

**Lemma 3.4.** The following are true.

(I) The monoid $F$ has common right multiples.

(II) The monoid $F$ is cancellative.

(III) The number of generators that compose to a given element is a well defined length function on the monoid $F$.

(IV) The monoid $F$ has only trivial units.

(V) The monoid $F$ has greatest common right factors, and thus also has least common left multiples.

(VI) The monoid $F$ has greatest common left factors and the greatest common left factor of $F$ and $G$ is $F \cap G$.

(VII) Let $F$ and $G$ be forests with a common left multiple, and let $L = PF = QG$ be the least common left multiple. The following are true.

(a) $L$ is the only least common left multiple of $F$ and $G$.

(b) If $AF = BG$ is a common left multiple of $F$ and $G$, then $P = (A \cap B)\setminus A$ and $Q = (A \cap B)\setminus B$.

(c) The forests $P$ and $Q$ are disjoint.

(d) Each tree in $L$ is equal either to a single tree from the forest $F$ hung on a trivial tree from $P$, or to a single tree from the forest $G$ hung on a trivial tree from $Q$.  

Proof. Mostly left to the reader, and the method of proof can be done to the reader’s taste. All statements are geometrically clear from the structure of forests and the nature of the multiplication, and they can also be given algebraic proofs from the relations in lemma 3.2. For example (1) can be proven by noting that $F \cup G$ is a common right multiple for $F$ and $G$, or it can be given an elegant inductive algebraic proof using the relations of Lemma 3.2. An algebraic proof for (V) can be built by defining a relation on pairs in $\mathcal{F}$ by $(x, y) \rightarrow (z, w)$ if there is a $\lambda_i$ so that $x = z\lambda_i$ and $y = w\lambda_i$ and showing that it is complete. Any common right factor of $x$ and $y$ can be built from a chain from $(x, y)$ to the unique irreducible in the class containing $(x, y)$. □

4. THE MONOID OF HEDGES

Structures derived from $\mathcal{F}$ will use functions defined on $\mathcal{F}$ that factor through a quotient of $\mathcal{F}$. It will be convenient to be familiar with that quotient.

We said in the introduction that trees will keep track of the order of splitting of a strand. If we do not keep track of the order, then the data in a tree is reduced to a “shrub.” A sequence of shrubs is a hedge and we are about to define the monoid of hedges.

There are many equivalent definitions of a hedge and they each have their own advantages and disadvantages. We are less interested in the details of hedges then we are in their relation to forests, and we will make all definitions by referring to forests.

We start with the definition that makes the product clear. Unfortunately, we will rarely refer to this definition in spite of its advantages. Let $F$ be a forest, and let $l_F : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $l_F(i) = j$ if the $i$-th leaf of $F$ is in $F_j$. The function $l_F$ is a surjection from $\mathbb{N}$ to $\mathbb{N}$, each preimage is finite and non-empty, and only finitely many preimages have more than one element. Further $l_F$ preserves $\leq$ on $\mathbb{N}$. We get a monoid from the set of all such functions under composition. It is clear that $F \mapsto l_F$ is an epimorphism. We call $l_F$ the “leaf-root” function of $F$.

For the next definition, we note that it is clear that the function $l_F$ is completely determined by knowing the size of each set $l_F^{-1}(i)$. This gives a sequence of positive integers, only finitely many of which are greater than one. This sequence is just the sequence $c_F$ for which $c_F(j)$ is the number of leaves of $F_j$. Let $\mathcal{H}$ be the set of such sequences. A formula can be worked out for the product to make $\mathcal{H}$ a monoid isomorphic the the monoid in the previous paragraph. In this definition the sequence simply gives the number of leaves of each “shrub.” The epimorphism $F \mapsto c_F$ is the “leaf count” epimorphism from $\mathcal{F}$ to $\mathcal{H}$. In spite of the less pleasant product on $\mathcal{H}$, we will refer to it more often than the others and will use the word “hedge” to refer to an element of $\mathcal{H}$.

The third definition takes more information from $l_F$ and notes that $l_F$ is determined by the sets $l_F^{-1}(i)$. This is a partition of $\mathbb{N}$ into sets each of which is finite and an interval under $\leq$ on $\mathbb{N}$. Further only finitely many sets have more than one element. Let $\mathcal{P}$ be the set of such partitions. Again, a formula can be worked out for the product to make $\mathcal{P}$ a monoid isomorphic to the previous two. This definition has $l_F^{-1}(i)$ the set of leaf numbers in $F_i$.

The proofs of the lemmas in this section are left as exercises for the reader.

In $\mathcal{H}$, we define the hedge $\nu_i$ by setting $\nu_i(i) = 2$ and all other values 1.

**Lemma 4.1.** The hedges $\{\nu_i \mid i \geq 0\}$ form a generating set for $\mathcal{H}$. 
Lemma 4.2. The generators \( \{ \nu_i \mid i \geq 0 \} \) satisfy the relations

\[ \nu_q \nu_m = \nu_m \nu_{q+1} \quad \text{when} \quad m \leq q. \]

From the relations in Lemma 4.2, we derive the rewriting rules

\[ \nu_q \nu_m \rightarrow \nu_m \nu_{q+1} \quad \text{when} \quad m \leq q. \]

Lemma 4.3. The rewriting rules \( \{ \text{S} \} \) are locally confluent and terminating and thus complete. In addition the inverse rules are also locally confluent and terminating and thus complete.

We end up with two normal forms. The irreducibles under \( \{ \text{S} \} \) are easy to identify as those words \( \nu_1 \nu_2 \cdots \nu_k \) with \( i_1 < i_2 < \cdots < i_k \). It is now easy to show that two different irreducible words represent different hedges. Thus we have a presentation.

Proposition 4.4. The monoid \( \mathcal{H} \) is presented by

\[ \langle \nu_0, \nu_1, \ldots \mid \nu_q \nu_m = \nu_m \nu_{q+1} \quad \text{whenever} \quad m \leq q \rangle. \]

Lemma 4.5. Taking a forest \( F \) to the leaf count function \( c_F : \mathbb{N} \rightarrow \mathbb{N} - \{0\} \) gives the homomorphism from \( \mathcal{P} \) onto \( \mathcal{H} \) that takes each \( \lambda_i \) to \( \nu_i \).

The irreducibles under the inverse of \( \{ \text{S} \} \) are those words \( \nu_1 \nu_2 \cdots \nu_k \) with \( i_1 \geq i_2 \geq \cdots \geq i_k \). We will call the normal form obtained from \( \{ \text{S} \} \) the ascending normal form, and the normal form obtained from the inverse of \( \{ \text{S} \} \) the descending normal form. It is more compact to write the descending normal form as \( \nu_{i_1}^n \nu_{i_2}^{n_2} \cdots \nu_{i_k}^{n_k} \) where \( i_1 > i_2 > \cdots > i_k \) and all \( n_j \) are at least one.

It is a triviality to relate the descending normal form to the structure of the hedge as a function from \( \mathbb{N} \) to \( \mathbb{N} - \{0\} \).

Lemma 4.6. Let the hedge \( H = \nu_{i_1}^{n_1} \nu_{i_2}^{n_2} \cdots \nu_{i_k}^{n_k} \) be in descending normal form. Then \( H(k) = n_j + 1 \) if \( k = i_j \) for some \( j \) and \( H(k) = 1 \) otherwise.

Lemma 4.7. The monoid \( \mathcal{H} \) is right cancellative.

Since \( \nu_q \neq \nu_{q+1} \) and \( \nu_q \nu_q = \nu_q \nu_{q+1} \), left cancellativity fails in \( \mathcal{H} \).

There is a natural isomorphism from the monoid \( \mathcal{H} \) consisting of sequences in \( \mathbb{N} - \{0\} \) to the monoid \( \mathcal{P} \) consisting of partitions of \( \mathbb{N} \). The next discussion uses the advantages of each monoid and we let \( H_P \) be the image in \( \mathcal{P} \) of \( H \in \mathcal{H} \) under the isomorphism.

We define some relations and let the reader verify some claims. If \( H \) and \( K \) are hedges, we write \( H \leq K \) if for each \( i \in \mathbb{N} \), we have \( H(i) \leq K(i) \). If \( P \) and \( Q \) are partitions in \( \mathcal{P} \), then we write \( P \leq Q \) if each set in \( P \) is contained in some set in \( Q \).

Lemma 4.8. For hedges \( H \) and \( K \) the following hold.

(a) \( H \leq K \) if and only if \( H \) is a left factor of \( K \) (equivalently, \( K \) is a right multiple of \( H \)).

(b) \( H_P \leq K_P \) if and only if \( H \) is a right factor of \( K \) (equivalently, \( K \) is a left multiple of \( H \)).

If \( H \) and \( K \) are hedges, then \( \max(H,K) \) is the hedge whose value at \( i \) is \( \max(H(i),K(i)) \) and \( \min(H,K) \) is the hedge whose value at \( i \) is \( \min(H(i),K(i)) \). If \( P \) and \( Q \) are partitions in \( \mathcal{P} \), then there are equivalence relations \( \sim_P \) and \( \sim_Q \) whose equivalence classes are \( P \) and \( Q \), respectively. We set \( P \lor Q \) to be the partition of classes given by the equivalence relation generated by \( \sim_P \) and \( \sim_Q \) (that is, by
$\sim_P \cup \sim_Q$). We set $P \wedge Q$ to be the partition of classes given by the equivalence relation $\sim_P \cap \sim_Q$. Note that $P \wedge Q$ and $P \vee Q$ must be in $\mathcal{P}$.

**Lemma 4.9.** Let $H$ and $K$ be hedges.

(a) The greatest common left factor of $H$ and $K$ is $\min(H, K)$.

(b) The least common right multiple of $H$ and $K$ is $\max(H, K)$.

(c) The greatest common right factor of $H$ and $K$ is the hedge corresponding to $H \wedge K$.

(d) The least common left multiple of $H$ and $K$ is the hedge corresponding to $H \vee K$.

**Lemma 4.10.** The homomorphism of Lemma 4.5 takes least common left multiples to least common left multiples.

5. Incorporating Permutations and Braids

We wish to create Zappa-Szép products of forests or hedges with braids or permutations. The braid group $B_n$ on $n$ strands is as discussed in [1]. Since we number things from 0, the strands in $B_n$ are numbered $0, 1, \ldots, n - 1$, and the generators of $B_n$ are $\sigma_0, \ldots, \sigma_{n-2}$. The infinite braid group $B_\infty$ is the direct limit of the $B_n$ where $B_n$ injects into $B_{n+1}$ by adding a trivial strand at position $n$. The presentation of $B_\infty$ as a group has generating set

$$\Sigma = \{\sigma_0, \sigma_1, \ldots\}$$

and relations

(9) \quad \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2,

(10) \quad \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0.

The monoid presentation of $B_\infty$ has generating set to $\Sigma \cup \overline{\Sigma}$ where

$$\overline{\Sigma} = \{\sigma_0^{-1}, \sigma_1^{-1}, \ldots\}$$

are the formal inverses of the elements of $\Sigma$ and we need add the relations

(11) \quad \sigma_m \sigma_m^{-1} = \sigma_m^{-1} \sigma_m = 1, \quad m \geq 0.

We follow the convention of [1] in drawing crossings as the following picture of $\sigma_0 \sigma_2^{-1}$ shows.

```
  0  1  2  3  4  5  6  ...
  \   \   \   \   \   
  \   \   \   \   \   
  \   \   \   \   \   
  \   \   \   \   \   
  \   \   \   \   \   
```

Also as in [1], reading a word in $\Sigma \cup \overline{\Sigma}$ from left to right corresponds to reading a braid diagram from top to bottom.

The infinite symmetric group $S_\infty$ is the direct limit of the finite symmetric groups $S_n$ and the presentation of $S_\infty$ has the generators and relations of $B_\infty$ in addition to the relations

(12) \quad \sigma_m^2 = 1, \quad m \geq 0.

Sending each $\sigma_m$ in $B_n$ or $B_\infty$ to the generator of the same name in $S_n$ or $S_\infty$ gives the standard surjections from braid groups to symmetric groups. If $\sigma$ is in $B_\infty$, then the notation $\sigma(j)$, will always refer to the image of $\sigma$ in $S_\infty$ under this surjection and will give the image of $j$ under the permutation. Context will determine whether $\sigma_m$ is a generator of $B_\infty$ or $S_\infty$. The effort it takes to keep
track of context will be worth it since we will be able to deal with both braids and permutations simultaneously by always using the generating set $\Sigma \cup \Sigma$.

We regard the $i$-th strand of a braid as an arc in 3-space with top at $(i,0,1)$ and bottom at some $(j,0,0)$. We say that this strand has top at $i$ and bottom at $j$. With the usual interpretation of $\sigma_m$ in $S_\infty$ as the transposition $m \leftrightarrow m+1$, we get that the $i$-th strand of a braid $\sigma$ has bottom at $\sigma(i)$.

5.1. **Zappa-Szép products.** To define a Zappa-Szép product $\mathcal{F} \bowtie B_\infty$, we need to put a multiplication on $\mathcal{F} \times B_\infty$ where a generic element will be a forest followed by a braid. Following the convention that turns left to right in word order into top to bottom in a picture, we think of the braid as hanging from the leaves of the forest. We build a Zappa-Szép product by telling how $\beta F$ should be replaced by $F' \beta'$ with $\beta$ and $\beta'$ from $B_\infty$ and $F$ and $F'$ from $\mathcal{F}$. The following pictures motivate the relations we will write down.

\begin{align*}
(13) & \quad \begin{array}{c}
\cdots \\
\cdots
\end{array} \rightarrow \begin{array}{c}
\cdots \\
\cdots
\end{array} \\
(14) & \quad \begin{array}{c}
\quad \\
\quad
\end{array} \rightarrow \begin{array}{c}
\quad \\
\quad
\end{array} \\
& \quad \begin{array}{c}
\quad \\
\quad
\end{array} \rightarrow \begin{array}{c}
\quad \\
\quad
\end{array}
\end{align*}

Similar pictures motivate relations needed for $\mathcal{H} \bowtie B_\infty$ and $\mathcal{H} \bowtie S_\infty$.

To define $\mathcal{F} \bowtie B_\infty$ and $\mathcal{F} \bowtie S_\infty$, we let $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$, and to define $\mathcal{H} \bowtie B_\infty$ and $\mathcal{H} \bowtie S_\infty$, we let $N = \{\nu_0, \nu_1, \ldots\}$. The products $\mathcal{F} \bowtie B_\infty$ and $\mathcal{F} \bowtie S_\infty$ will be specified by functions $(\Sigma \cup \Sigma) \times \Lambda \rightarrow \Lambda$ written $(\sigma, \lambda) \mapsto \sigma \cdot \lambda$ and $(\Sigma \cup \Sigma) \times N \rightarrow (\Sigma \cup \Sigma)$ written $(\sigma, \nu) \mapsto \sigma^\nu$. Products with $\mathcal{F}$ will be specified by similar functions with $\Lambda$ replaced by $N$ and $\lambda$ replaced by $\nu$. These functions are defined by the following where $\epsilon$ represents either $+1$ or $-1$:

\begin{align*}
(15) & \quad \sigma_q^\epsilon \cdot \lambda_m = \lambda_{\sigma_q(m)}, \\
(16) & \quad \sigma_q^\epsilon \cdot \nu_m = \nu_{\sigma_q(m)}, \\
(17) & \quad (\sigma_q^\epsilon)^{\nu_m} = (\sigma_q^\epsilon)^{\lambda_m} =
\begin{cases} 
\sigma_q^\epsilon, & m > q + 1, \\
\sigma_q^\epsilon \sigma_q^{\nu_m}, & m = q + 1, \\
\sigma_q^{\nu_m} \sigma_q^\epsilon, & m = q, \\
\sigma_q^{\nu_m + 1} \sigma_q^\epsilon, & m < q,
\end{cases}
\end{align*}

The reader can check that the relations $\sigma_q \lambda_m = (\sigma_q \cdot \lambda_m) (\sigma_q)^{\lambda_m}$ are realizations of the pictures in (13) and (14).

We also define a monoid that combines braids with deletion operators as a Zappa-Szép product. It turns out that deletions from a sequence form a monoid isomorphic to hedges. We introduce the monoid presentation

\begin{equation}
Z = \langle \delta_0, \delta_1, \ldots \mid \delta_q \delta_m = \delta_m \delta_{q+1} \text{ whenever } m \leq q \rangle,
\end{equation}

and we let $\Delta = \{\delta_0, \delta_1, \ldots\}$. We will think of $\delta_q$ as deleting the strand with top at position $q$ from a braid, starting at the top. The following picture of a spark burning a strand from the top of a braid motivates the relations we will write down.
Products $B_\infty \bowtie \mathbb{Z}$ and $S_\infty \bowtie \mathbb{Z}$ will be specified by functions $\Delta \times (\Sigma \cup \overline{\Sigma}) \to \Delta$ written \((\delta, \sigma) \mapsto \delta \cdot \sigma\) and $\Delta \times (\Sigma \cup \overline{\Sigma}) \to (\Sigma \cup \overline{\Sigma})$ written \((\delta, \sigma) \mapsto \delta^\sigma\). These functions are defined by the following where $\epsilon$ represents either $+1$ or $-1$:

\begin{equation}
(\delta_q)^\sigma_n = \delta_{\sigma_m(q)},
\end{equation}

\begin{equation}
\delta_q \cdot \sigma_m' = \begin{cases} 
\sigma_{m-1}', & q < m, \\
1, & q = m, m+1, \\
\sigma_{m}', & q > m + 1.
\end{cases}
\end{equation}

The illustration in (19) shows the truth of $\delta_1 \sigma_0 \sigma_1 \sigma_0 = \delta_0 \sigma_0 \sigma_0 = \sigma_0 \delta_0 \sigma_0 = \sigma_0 \delta_1$. The $\delta_j$ at the end is to be interpreted as ready to delete strand 1 from any braid that might be multiplied on the right of the original.

In (17) and (21), the consistent treatment of the exponent $\epsilon$ allows restriction of domain and codomain from $\Sigma \cup \overline{\Sigma}$ to $\Sigma$ when working with $S_\infty$ instead of $B_\infty$.

The formula (20) has to be interpreted carefully. It is only a statement about expressions involving generators. It is not meant to imply that it applies to words in these generators. In fact, if (20) is followed literally, then we get

\begin{equation}
((\delta_q)^\sigma_m)^\sigma_n = (\delta_{\sigma_m(q)})^\sigma_n = \delta_{\sigma_n(\sigma_m(q))}
\end{equation}

which is to be compared with the incorrect

\begin{equation}
(\delta_q)^{\sigma_m \sigma_n} = \delta_{(\sigma_m \sigma_n)(q)}.
\end{equation}

From Lemma 5.1, we know we get a consistent action of words in the $\sigma$ on the various $\delta$ if we define the left side of (23) to equal the right side of (22). The various elements $\sigma$ as show up in the subscripts in (22) are to be interpreted as permutations (and are transpositions) and the reverse of a string of transpositions is the inverse of the original string. This gives one conclusions of the following lemma. The other conclusions follow from Lemma 2.14 and from (15) and (16) without the complications relating to (20) since in (15) and (16), the various $\sigma$ act on the left.

**Lemma 5.1.** For any $\lambda_j \in \Lambda$, $\nu_j \in N$ or $\delta_j \in \Delta$ and $\tau$ in $S_\infty$ or $B_\infty$, we have

$$
\tau \cdot \lambda_j = \lambda_{\tau(j)}, \quad \tau \cdot \nu_j = \nu_{\tau(j)}, \quad (\delta_j)^\tau = \delta_{\tau^{-1}(j)}.
$$

**Proposition 5.2.** The functions defined by (15), (16), (17), (20) and (21) define Zappa products $F \bowtie S_\infty$, $F \bowtie B_\infty$, $H \bowtie S_\infty$, $H \bowtie B_\infty$, $S_\infty \bowtie \mathbb{Z}$ and $B_\infty \bowtie \mathbb{Z}$.

**Proof.** This is an orgy of checking the requirements of Lemma 2.14 which is left to the reader. The number of cases is large. Note that the roles of $X | R$ and $Y | T$ in that lemma must be reversed in dealing with the products with $\mathbb{Z}$. We point out that the flexibility of the hypothesis of Lemma 2.14 that allows either of $(\sigma \cdot u, \sigma \cdot v)$ or $(\sigma \cdot v, \sigma \cdot u)$ to be a relation for $F$ if $(u, v)$ is a relation for $F$ must be used when showing that the related pair $(\lambda_{m+1} \lambda_m, \lambda_m \lambda_{m+2})$ in $F$ is carried to the related pair $(\sigma_m \cdot (\lambda_{m+1} \lambda_m), \sigma_m \cdot (\lambda_m \lambda_{m+2})) = (\lambda_m \lambda_{m+2}, \lambda_{m+1} \lambda_m)$. \qed
We show a calculation (only the first tree of each forest is shown)

\[
\begin{pmatrix}
\text{tree 1} \\
\text{tree 2}
\end{pmatrix}^2 = \begin{pmatrix}
\text{result 1} \\
\text{result 2}
\end{pmatrix} = \begin{pmatrix}
\text{result 3} \\
\text{result 4}
\end{pmatrix}
\]

or equivalently, \( \lambda_0 \sigma_0 \lambda_0 \sigma_0 = \lambda_0 \lambda_1 \sigma_0 \sigma_1 \sigma_0 \).

The check that the relation \( \nu_m \nu_{m+1} = \nu_{m+1} \nu_m \) cooperates with \( \text{(17)} \) shows that the action of forests on braids successfully factors through the action of hedges on braids.

### 5.2. Some algebraic properties of the products.

We would like to prove that \( \mathcal{F} \rtimes S_\infty \) and \( \mathcal{F} \rtimes B_\infty \) share some of the properties that are possessed by \( \mathcal{F} \). We will make use of Lemma \( \text{2.11} \) so we start by verifying some of the properties needed by that lemma. We first need some technical lemmas.

**Lemma 5.3.** The following equalities hold concerning the actions \( B_\infty \times \mathcal{F} \to B_\infty \) used in creating \( \mathcal{F} \rtimes B_\infty \) and \( \mathcal{F} \rtimes B_\infty \to B_\infty \) used in creating \( B_\infty \rtimes \mathcal{S} \):

\[
\delta_{\sigma_q(m)} \cdot ((\sigma_q)^{\lambda_m}) = \sigma_q, \quad \text{and} \quad (\delta_{\sigma_q(m)})((\sigma_q)^{\lambda_m}) = \delta_m.
\]

**Proof.** We write out the calculation for \( m = q \).

\[
\begin{align*}
\delta_{\sigma_q(q)} \cdot (\sigma_q)^{\lambda_q} &= \delta_{q+1} \cdot (\sigma_q \sigma_{q+1}) = (\delta_{q+1} \cdot \sigma_q)((\delta_{q+1})^{\sigma_q} \cdot \sigma_{q+1}) \\
&= (1)(\delta_q \cdot \sigma_{q+1}) = \sigma_q.
\end{align*}
\]

\[
\begin{align*}
(\delta_{\sigma_q(q)})((\sigma_q)^{\lambda_q}) &= (\delta_{q+1})(\sigma_q \sigma_{q+1}) = (\delta_{\sigma_{q+1}(q+1)})^{\sigma_{q+1}} \\
&= (\delta_q)^{\sigma_{q+1}} = \delta_{\sigma_{q+1}(q)} = \delta_q = \delta_q.
\end{align*}
\]

The other cases, \( m < q, m = q + 1 \) and \( m > q + 1 \) are left to the reader. \( \square \)

**Lemma 5.4.** In the setting of Lemma \( \text{5.3} \) we have

\[
\delta_{\tau(m)} \cdot (\tau^{\lambda_m}) = \tau
\]

for any \( \tau \in B_\infty \).

**Proof.** From Lemma \( \text{5.3} \) the result holds if \( \tau \) is a single generator. Consider \( \tau = \sigma \omega \) for some generator \( \sigma \) so that \( \omega \) has shorter length than \( \tau \). Then

\[
\begin{align*}
\delta_{(\sigma \omega)(m)}((\sigma \omega)^{\lambda_m}) &= \delta_{(\sigma \omega)(m)}((\sigma \omega)^{\lambda_m} \cdot \omega^{\lambda_m}) \\
&= (\delta_{\sigma(\omega(m))} \cdot (\sigma^{\lambda_{\omega(m)}})(\delta_{\sigma(\omega(m))})(\omega^{\lambda_m})) \\
&= (\sigma)(\delta_{\omega(m)} \cdot (\omega^{\lambda_m})) \\
&= \sigma \omega = \tau
\end{align*}
\]

where the next to last line is justified by the two parts of Lemma \( \text{5.3} \) and the last line is by induction since the length of \( \omega \) is less than that of \( \tau \). \( \square \)

**Corollary 5.4.1.** In the setting of Lemma \( \text{5.3} \) for each \( u \in \mathcal{F} \), the family of functions \( B_\infty \times \mathcal{F} \to B_\infty \) given by \( (\tau, u) \mapsto \tau^u \) is a family of injections.
Proof. By Lemma 5.4, this is true if $u$ is some $\lambda m$. The claim follows since $\tau^u = \tau^u v$ and a composition of injections is an injection. \qed

We are now in a position to prove the following facts about $\mathcal{F} \triangleright B_\infty$.

**Proposition 5.5.** The monoid $\mathcal{F} \triangleright B_\infty$ is cancellative and has common right multiples. Further, declaring the length of $G\tau$ with $G \in \mathcal{F}$ and $\tau \in B_\infty$ to be the length of $G$ as given in Lemma 3.4(III) gives a length function on $\mathcal{F} \triangleright B_\infty$.

**Proof.** For cancellativity, Lemma 2.11 and the unstated version for left cancellativity says that we need that $(\tau, u) \mapsto \tau u$ and $(\tau, u) \mapsto \tau \cdot u$ are injective families. We get one from Corollary 5.4.1 and the other from the fact that $B_\infty$ is a group and that $(\tau, u) \mapsto \tau \cdot u$ is an action. For common right multiples, Lemma 2.11 requires that $(\tau, u) \mapsto \tau \cdot u$ is a surjective family. This follows from the fact that we have a group action. That the claimed length function for $\mathcal{F} \triangleright B_\infty$ is truly a length function follows from the fact that $B_\infty$ is a group whose action on $\mathcal{F}$ takes generators to generators and is thus length preserving. \qed

Identical arguments over the last few lemmas repeated for $S_\infty$ give the following.

**Proposition 5.6.** The monoid $\mathcal{F} \triangleright S_\infty$ is cancellative and has common right multiples. Further, declaring the length of $G\tau$ with $G \in \mathcal{F}$ and $\tau \in S_\infty$ to be the length of $G$ as given in Lemma 3.4(III) gives a length function on $\mathcal{F} \triangleright S_\infty$.

5.3. **Least common left multiples.** Least common left multiples are needed to get reduced terms in groups of fractions. Here they require extra work.

We need more information than given in Lemmas 5.3 and 5.4. The content of Lemma 5.4 is that going from $\tau \in B_\infty$ to $\tau^\lambda m$ splits a strand into two parallel strands, and $\delta_{\tau(m)}$ restores $\tau$ by deleting one of the parallel strands. We elaborate on that. What follows are discussions about inductive extensions of Lemmas 5.3 and 5.4 from statements about the behavior of generators to statements about the behavior of arbitrary elements.

If $\sigma$ is a braid representative, then we say strands $i$ and $i + 1$ are companions if $\sigma(i + 1) = \sigma(i)$. In this case there is a circle $J$ built from strands $i$ and $i + 1$, the straight line segment joining the tops of the strands, and the straight line segment joining the bottoms of the strands. If there is a disk that meets each plane $z = t$, $0 \leq t \leq 1$, in a single line segment of length 1 that is parallel to the $x$-axis, whose boundary is $J$ and which does not meet any strand of $\sigma$ in its interior, then we say that strands $i$ and $i + 1$ are parallel.

If a braid has a representative in which strands $i$ and $i + 1$ are parallel, then we say that strands $i$ and $i + 1$ are weakly parallel in any other representative. It is possible to characterize weakly parallel strands by defining a winding number of two strands that are companions and showing that companion strands are weakly parallel if they have winding number 0 and the circle $J$ of the previous paragraph bounds a disk whose interior is disjoint from the strands in $\sigma$.

If $\sigma$ is a braid representative, then a partition of the strands of $\sigma$ is into weak parallel classes if any two consecutive braids in a class are weakly parallel. We insist that elements of the partition be finite. Since we consider braids in $B_\infty$, we do not insist that these classes be maximal. We further insist that only finitely many classes have more than one strand.

If we label strands by their strand numbers, then a partition of the strands of a braid is identified with a partition of $\mathbb{N}$. The following is straightforward.
Lemma 5.7. If \( \sigma \) is a braid representative, and \( C \) is a partition of \( \mathbb{N} \) into weak parallel classes of \( \sigma \), then there is a representative \( \sigma' \) of the same braid in which any two consecutive strands in the same class of \( C \) are parallel.

We now drop the phrase “weakly parallel” and “weak parallel classes” and only refer to parallel strands and parallel classes, and we think of the property “parallel” as being attached to strands of of a braid not the strands of a representative.

Recall that partitions of \( \mathbb{N} \) into finite sets of consecutive numbers with all but finitely many sets of size one can be viewed as hedges. The next lemma refers to the operations on hedges as used in Lemma 4.9.

Lemma 5.8. If partitions \( C \) and \( D \) of \( \mathbb{N} \) are partitions of the strands of \( \sigma \in B_\infty \) into parallel classes, then so is the partition \( C \lor D \).

The following is geometrically “obvious” and is proven inductively, first on the number of generators in the braid, and then on the number of generators in the hedge. The start of the induction is from the pictures in (14).

Lemma 5.9. Let \( \sigma \) be a braid and let \( u \) be a hedge corresponding to partition \( Q \). Then \( Q \) is a partition of \( \sigma u \) into parallel classes.

If \( \sigma \) is a braid, and \( Q \) a partition into parallel classes of strands, then we can “collapse” each class into a single strand. This is accomplished by deleting all strands but one in each class. It is clear that it does not matter which strand is the one in each class chosen to remain. We use \( \sigma/Q \) to denote the result of this operation. The next lemma is again proven by induction, first on the number of generators of the braid and then on the number of generators of the hedge corresponding to \( Q \).

Lemma 5.10. Let \( \sigma \) and \( \tau \) be braids so that a partition \( Q \) into finite sets, only finitely many of which are not singletons, is a partition of both \( \sigma \) and \( \tau \) into parallel classes. If \( \sigma/Q = \tau/Q \), then \( \sigma = \tau \).

The next lemma is built inductively from Lemma 5.4.

Lemma 5.11. Let \( \sigma \) be a braid and let \( u \) be a hedge with corresponding partition \( Q \) of \( \mathbb{N} \). Then \( (\sigma^u)/Q = \sigma \).

Lemma 5.12. Let \( \sigma \) be a braid with a partition \( Q \) into parallel classes with each class finite and only finitely many classes not singletons. Let \( u \) be the hedge corresponding to \( Q \). Then \( (\sigma/Q)^u = \sigma \).

Proof. We have \( ((\sigma/Q)^u)/Q = \sigma/Q \) by Lemma 5.11. But \( (\sigma/Q)^u \) has \( Q \) as a partition into parallel classes by Lemma 5.9. Now we get the conclusion from Lemma 5.10. \( \square \)

Lemma 5.13. The family of functions \( B_\infty \times \mathcal{F} \to B_\infty \) written \( (\sigma, u) \mapsto \sigma^u \) used in creating \( \mathcal{F} \propto B_\infty \) is a strongly coconfluent family of injections.

Proof. The injective properties follow from Lemma 5.11. For the coconfluence, we must show that if \( \sigma^u = \tau^v \) where \( u \) and \( v \) have a common left multiple, then there is a braid \( \beta \) and \( p \) and \( q \) so that \( pu = qv \), \( \beta^p = \sigma \) and \( \beta^q = \tau \) all hold. From Lemma 2.10 we know that if this holds, then it will hold when \( w = pu = qv \) is the least common left multiple of \( u \) and \( v \), so we assume that it is. (We know that the least common left multiple of \( u \) and \( v \) must exist.) We denote the homomorphism from
$F$ to $\mathcal{H}$ by $u \mapsto \pi$, and we note that $w = \overline{uw} = \overline{wv}$ is the least common left multiple of $\pi$ and $\pi$ in $\mathcal{H}$ by Lemma 4.10.

Let $\alpha = \sigma^n = \tau^m$. Since the action of $F$ factors through $\mathcal{H}$, we note that $\alpha = \sigma^\pi = \tau^\pi$. We let $C_u$ and $C_v$ denote the partition of $\mathbb{N}$ corresponding to $\pi$ and $\pi$, respectively, and we note that both $C_u$ and $C_v$ are partitions of the strands of $\alpha$ into weak parallel classes. Thus $C_u \cup C_v$ must be a partition of the strands of $\alpha$ into weak parallel classes.

From Lemma 1.9 we know that the hedge corresponding to $C_u \cup C_v$ is the least common left multiple of $\pi$ and $\pi$, which is $\overline{w}$. Let $C = C_u \cup C_v$. From Lemma 5.12 we have $(\alpha/C)^\pi = \alpha$. Let $\beta = \alpha/C$.

Now $\sigma^\pi = \alpha = \beta^\pi = \beta^\sigma$ and by Corollary 6.4.1 we get $\sigma = \beta^\pi = \beta^\sigma$. Similarly, we get $\tau = \beta^\sigma$. This completes the proof.

**Proposition 5.14.** The monoids $F \bowtie B_\infty$ and $F \bowtie S_\infty$ have least common left multiples. Further, the least common left multiple $(p, \alpha)(u, \sigma) = (q, \beta)(v, \tau)$ of $(u, \sigma)$ and $(v, \tau)$ can be constructed so that $p(\alpha \cdot u) = q(\beta \cdot v)$ is the least common left multiple of $(\alpha \cdot u)$ and $(\beta \cdot v)$ in $F$.

**Proof.** This follows from Lemma 2.12 and from Lemma 5.13 and a corresponding lemma for $F \bowtie S_\infty$ which has an identical proof given the almost identical behavior of $S_\infty$ and $B_\infty$.

Lemma 2.14 gives presentations for $F \bowtie S_\infty$ and $F \bowtie B_\infty$ as follows:

$$F \bowtie S_\infty = \langle \Lambda \cup \Sigma \mid \lambda_q \lambda^m = \lambda_m \lambda_{q+1}, \quad m < q, \sigma^2 = 1, \quad m \geq 0, \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0, \sigma_q \lambda^m = (\sigma_q \cdot \lambda_m)(\sigma_q^\lambda)^m \rangle,$$

and

$$F \bowtie B_\infty = \langle \Lambda \cup \Sigma \cup \sum \mid \lambda_q \lambda^m = \lambda_m \lambda_{q+1}, \quad m < q, \sigma^2 = 1, \quad m \geq 0, \sigma_m \sigma^{-1}_m = 1, \quad m \geq 0, \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0, \sigma_q^\epsilon \lambda^m = (\sigma_q^\epsilon \cdot \lambda_m)(\sigma_q^\epsilon)^\lambda^m \rangle.$$

6. **Groups of fractions**

The monoids $F \bowtie B_\infty$ and $F \bowtie S_\infty$ are cancellative with common right multiples and thus have groups of right fractions. We let $\overline{BV}$ be the group of right fractions for $F \bowtie B_\infty$ and we let $\overline{V}$ be the group of right fractions for $F \bowtie S_\infty$.

6.1. **Embeddings.** The group of fractions construction and the Zappa-Szép product both involve embeddings (Proposition 2.4 and Lemma 2.4). This is reflected in the following where we use notation based on the fact that if $M$ is cancellative monoid with common right multiples, then elements of the group of right fractions of $M$ are represented by elements of $M \times M$. 

Theorem 2. Lemma 3.4(III).

Proposition 6.1. Sending $F$ to $(F,1)$ embeds $F$ into $F \rtimes S_\infty$ and $F \rtimes B_\infty$. Sending $\sigma$ to $(1,\sigma)$ embeds $S_\infty$ into $F \rtimes S_\infty$ and embeds $B_\infty$ into $F \rtimes B_\infty$. Sending $(F,\beta)$ to $((F,\beta),1)$ embeds $F \rtimes S_\infty$ in $\tilde{V}$ and embeds $F \rtimes B_\infty$ into $\tilde{BV}$.

6.2. Some presentations. It is easy to give infinite presentations of $\tilde{V}$ and $\tilde{BV}$.

From Propositions 2.4 and 3.3 and from $\{15\}$, $\{17\}$, $\{23\}$ and $\{25\}$, we get the following where $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$ and $\Sigma = \{\sigma_0, \sigma_1, \ldots\}$.

Theorem 1. The groups $\tilde{V}$ and $\tilde{BV}$ are presented as groups by

$\tilde{V} = \langle \Lambda \cup \Sigma \mid \lambda_q \lambda_m = \lambda_m \lambda_{q+1}, \quad m < q, \quad \sigma_m^2 = 1, \quad m \geq 0, \quad \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \quad \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0, \quad \sigma_q \lambda_m = \lambda_m \sigma_{q+1}, \quad m < q, \quad \sigma_m \lambda_m = \lambda_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0, \quad \sigma_m \lambda_{m+1} = \lambda_m \sigma_{m+1} \sigma_m, \quad m \geq 0, \quad \sigma_q \lambda_m = \lambda_m \sigma_q, \quad m > q + 1 \rangle$,

$\tilde{BV} = \langle \Lambda \cup \Sigma \mid \lambda_q \lambda_m = \lambda_m \lambda_{q+1}, \quad m < q, \quad \sigma_m \sigma_n = \sigma_n \sigma_m, \quad |m - n| \geq 2, \quad \sigma_m \sigma_{m+1} \sigma_m = \sigma_{m+1} \sigma_m \sigma_{m+1}, \quad m \geq 0, \quad \sigma_q^\epsilon \lambda_m = \lambda_m \sigma_{q+1}^\epsilon, \quad m < q, \quad \epsilon = \pm 1, \quad \sigma_m^\epsilon \lambda_m = \lambda_{m+1} \sigma_m^\epsilon \sigma_{m+1}^\epsilon, \quad m \geq 0, \quad \epsilon = \pm 1, \quad \sigma_m^\epsilon \lambda_{m+1} = \lambda_m \sigma_{m+1}^\epsilon \sigma_m^\epsilon, \quad m \geq 0, \quad \epsilon = \pm 1, \quad \sigma_q^\epsilon \lambda_m = \lambda_m \sigma_q^\epsilon, \quad m > q + 1, \quad \epsilon = \pm 1 \rangle$.

Some of the relations are redundant. The relations $\sigma_m \lambda_{m+1} = \lambda_m \sigma_{m+1} \sigma_m$ follow from the relations $\sigma_m \lambda_m = \lambda_{m+1} \sigma_m \sigma_{m+1}$ in $\tilde{V}$ by bringing each $\sigma_m$ and $\sigma_{m+1}$ to the other side of the equality. Similarly, the relations $\sigma_m^\epsilon \lambda_{m+1} = \lambda_m \sigma_{m+1}^\epsilon \sigma_m^\epsilon$ follow from the relations $\sigma_m^\epsilon \lambda_m = \lambda_{m+1} \sigma_m^\epsilon \sigma_m^\epsilon$ in $\tilde{BV}$. Also, the exponents $\epsilon$ can be eliminated from several of the relations in $\tilde{BV}$ because of the group setting.

6.3. Normal forms. The monoids have least common left multiples and length functions. Thus elements in $\tilde{BV}$ and $\tilde{V}$ have representatives of the fractions in reduced terms. The next lemma gives the details of the normal form that comes from the reduced terms. We refer to the length function on the monoid $F$ from Lemma 3.4(III).

Theorem 2. (I) Each element $x$ of $\tilde{V}$ or $\tilde{BV}$ is represented uniquely by a triple $(G, \alpha, H)$ with the conditions that

(a) $G$ and $H$ are in $\mathcal{F}$,
(b) $\alpha$ is in $S_\infty$ if $x \in \tilde{V}$, and is in $B_\infty$ if $x \in \tilde{BV}$,
(c) $x = (G\alpha)(H)^{-1}$,
(d) the length of $G$ is minimal among all triples satisfying (a–c).
(II) Any other representative of \( x \) is of the form \((G_o J \gamma)(H J \gamma)^{-1}\) for some \( J \) in \( \mathcal{F} \) and \( \gamma \) in the appropriate one of \( S_\infty \) or \( B_\infty \).

(III) The triple \((G, \alpha, H)\) of (I) is characterized by the fact that \( x = G_o H^{-1} \) and for no \( G', H' \) and \( \lambda_i \) in \( \mathcal{F} \) and \( \alpha' \) in \( S_\infty \) or \( B_\infty \), as appropriate, is it true that \( G_o = G'o' \lambda_i \) and \( H = H' \lambda_i \).

Proof. We can do both groups at once if we consider expressions such as \( G_\mu, H_\tau \) or \( J_\gamma \) in which \( G, H \) and \( J \) are in \( \mathcal{F} \) and \( \mu \), \( \tau \) and \( \gamma \) are either in \( S_\infty \) or \( B_\infty \) depending on whether we are discussing, respectively, \( \tilde{V} \) or \( \overline{BV} \). Take an element \( x \) in one of the groups and represent it as \( x = (G_\mu)(H_\tau)^{-1} \) so that the length of \( G_\mu \) is minimal among the representatives of \( x \). From Lemma 5.4 and Propositions 5.5 and 6.1, the length of \( G_\mu \) is the length of \( G \) as a word in the symbols \( \lambda_i \). Thus \( x = (G_\mu \tau^{-1})(H)^{-1} \) is another representative of \( x \) with the same properties, and we take the desired triple to be \((G, \alpha, H)\) with \( \alpha = \mu \tau^{-1} \). This satisfies (a–d) and we now need to consider uniqueness.

Since the length of \( G_\alpha \) is minimal, we know from Lemma 2.4 that \((G_\alpha)(H)^{-1}\) is in reduced terms. If \((G'o')(H')^{-1}\) is another representative of \( x \) coming from a triple \((G', \alpha', H')\) satisfying (a–d), then there is some \( J_\gamma \) so that \( G'o' = G_o J_\gamma = (G(\alpha \cdot J))(\alpha' \gamma) \) and \( H' = H J_\gamma \). From (15), the action of \( \alpha \) on \( J \) preserves the length of \( J \), so the minimality of the lengths of \( G \) and \( G' \) force the length of \( J \) to be zero. The only element of \( \mathcal{F} \) with length zero is the identity. The uniqueness of representation in a Zappa-Szép product forces \( \gamma \) to be the identity. This finishes (I).

Item (II) follows from the fact that \((G_o)(H)^{-1}\) is in reduced terms.

To see (III), we note that from (II), the length of \( G \) is minimal if the test in (III) is satisfied, and the length is not minimal if the test in (III) is not satisfied. \( \square \)

6.4. An isomorphism. We argue that each element \( F_\beta \) of \( \mathcal{F} \bowtie B_\infty \) gives an element in the geometric description of \( \overline{BV} \) from Section 1.1. The forest \( F \) tells how to break the intervals in \( \mathcal{J} = \{2i, 2i + 1\} | i \geq 0 \) into smaller intervals. The tree \( F_i \) gives instructions on breaking the interval \([2i, 2i + 1]\). The braid \( \beta \) tells how to reorder the intervals by an isotopy of \( \mathbb{R}^2 \). The image intervals are now resized and moved horizontally on the \( x \)-axis until each maps affinely onto one of the \([2i, 2i + 1]\) so that they are all covered. Thus \( F_\beta \) can be thought of as a braiding that takes the subdivided intervals from \( \mathcal{J} \) onto the unsubdivided intervals from \( \mathcal{J} \). The fact that the forest \( F \) is finite corresponds to the fact that in the description of Section 1.1 all but finitely many intervals of the cover of \( X \) must be intervals from \( \mathcal{J} \). It is clear that any element of the group from Section 1.1 is of the form \((F_\beta)(G_\gamma)^{-1}\) for some pair of elements \( F_\beta \) and \( G_\gamma \) from \( \mathcal{F} \bowtie B_\infty \).

The multiplication of forests corresponds to successive subdivisions of intervals and the relations from Lemma 5.2 on \( \mathcal{F} \) are seen to hold among the subdivision operations. The braid relations (I) and (II) hold for the braiding and the Zappa-Szép relations (15) and (17) hold as pictured in (13) and (14). Thus the association of elements of \( \mathcal{F} \bowtie B_\infty \) to braiding of intervals is a homomorphism. By Proposition 2.4(d), the homomorphism extends to one defined on \( \overline{BV} \). As remarked in the previous paragraph, the homomorphism is a surjection.

If \((F, \beta, G)\) from \( \overline{BV} \) is taken to the identity in the group of Section 1.1, then we have a bijection of interval collections that must be the identity. It is easy to argue that different forests give different collections of intervals, so \( F = G \). By
conjugating by $F$, we see that $(1, \beta, 1)$ is also taken to the identity. But this is just a braiding of the intervals in $\beta$ and must be the trivial braid. We have given a sketch of a proof of the following.

**Theorem 3.** The group $\hat{BV}$ as defined in this section and the group called $\hat{BV}$ as described in Section 7.1 are isomorphic.

7. **Distinguished subgroups**

In the group $BV$, a single strand (corresponding to a single Cantor set $C$) is split into a finite number $n$ of strands (corresponding to a cover of $C$ by $n$ intervals from $\Pi$) which are then braided and recombined into one strand. This section picks out the appropriate subgroup of $\hat{BV}$ and the corresponding subgroup of $\hat{V}$.

7.1. **Simple elements.** We say that a hedge $H$ is simple if $H(i) = 1$ for all $i > 0$. We say that a forest $F$ is simple if its corresponding hedge $c_F$ is simple. Thus a simple forest $F$ has at most one non-trivial tree, and this non-trivial tree must be $F_0$. The type of a simple forest $F$ is the length of $F$. Note that the type of a simple forest $F$ is also the number of carets in $F$ and is one less than the number of leaves of $F_0$. Thus the type of the simple forest $F$ is $c_F(0) - 1$. The trivial forest is the only simple forest of type 0 and $\lambda_0$ is the only simple forest of type 1.

From Lemma 7.1, we have that $H(i) \leq (HK)(i)$ for all $i$ for any hedges $H$ and $K$. From this it follows that if $HK$ is simple, then $H$ is simple. Sending a forest $F$ to the corresponding hedge $c_F$ is a homomorphism, so we get the following.

**Lemma 7.1.** If $FG$ is simple for forests $F$ and $G$, then $F$ is simple.

If $F$ is a simple forest of type $k$, then the leaves of $F_0$ are numbered from 0 through $k$. We have that $F\lambda_j$ is simple if and only if $i \leq k$. Inductively, we get the following.

**Lemma 7.2.** A forest $F = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ is simple of type $k$ if and only if $i_j < j$ for all $j$ with $1 \leq i \leq k$.

Recall that $S_k$ (respectively, $B_k$) is the subgroup of $S_\infty$ (respectively, $B_\infty$) generated by $(\sigma_0, \ldots, \sigma_{k-2})$.

If $F$ is a forest and $\beta$ is in $S_\infty$ or $B_\infty$, then $F\beta$ is simple of type $k$ if $F$ is simple of type $k$ and $\beta$ is in $S_{k+1}$ or $B_{k+1}$. Intuitively, $\beta$ permutes braids only the leaves of $F_0$.

**Lemma 7.3.** Let $F$ and $G$ be in $F$ and $\beta$ and $\gamma$ be from one of $S_\infty$ or $B_\infty$ and assume that $F\beta$ is simple of type $k$ and $G = \lambda_{i_1}\cdots\lambda_{i_n}$. Then the following are equivalent.

(a) $(F\beta)G$ is simple.
(b) $F(\beta \cdot G)$ is simple.
(c) $i_j \leq k + j - 1$ whenever $1 \leq j \leq n$.
(d) $(F\beta)\lambda_{i_1}\cdots\lambda_{i_j}$ is simple whenever $1 \leq j \leq n$.

Further, $(F\beta)(G\gamma)$ is simple if and only if $(F\beta)G$ is simple and $\gamma$ is in $S_{k+n+1}$ or $B_{k+n+1}$.

**Proof.** The definition of simple gives (a) $\Rightarrow$ (b).

We prove (b) $\Rightarrow$ (c) $\Rightarrow$ (a) by induction on $n$. 


If \( n = 1 \) and \( G = \lambda_i \), then \( \beta \cdot \lambda_i = \lambda_{\beta(i)} \). Since \( \beta \) can permute non-trivially only the leaves of \( F_0 \), we have that \( F(\beta \cdot \lambda_i) \) is simple if and only if \( i \leq k \). Further, when \( F(\beta \cdot \lambda_i) \) is simple, it is of type \( k + 1 \).

Now assume \( i \leq k \). If \( \beta = \beta' \sigma_j \) with \( \beta' \) and \( \sigma_j \) in \( S_{k+1} \) or \( B_{k+1} \), then \( \beta \lambda_i = \beta' \lambda_{\sigma_j(i)}(\sigma_j)^{\lambda_i} \). Since \( \sigma_j \) is in \( S_{k+1} \) or \( B_{k+1} \), we have \( j \leq k - 1 \) and \( (17) \) gives us that \((\sigma_j)^{\lambda_i} \) is in \( S_{k+2} \) or \( B_{k+2} \). With \( r = \sigma_j(i) \leq k \) since \( j \leq k - 1 \), we inductively get that \((\beta')^{\lambda_r} \) is in \( S_{k+2} \) or \( B_{k+2} \) and thus \((\beta)^{\lambda_i} \) is in \( S_{k+2} \) or \( B_{k+2} \).

We have proven \((b) \Rightarrow (c) \Rightarrow (a) \) in the case \( n = 1 \). The general case follows by induction and the equivalence of \((a)\) and \((b)\) when \( n = 1 \). The equivalence of \((d)\) with the other statements is immediate.

The last claim follows from the equivalence of \((a)\) and \((b)\) and the definitions. \( \square \)

**Corollary 7.3.1.** Let \( F, F' \) and \( G \) be in \( \mathcal{F} \) and let \( \beta, \beta' \) and \( \gamma \) be from one of \( S_\infty \) or \( B_\infty \). If \( F\beta \) and \( F'\beta' \) are simple of the same type, then \( (F\beta)(G\gamma) \) is simple if and only if \( (F'\beta')(G\gamma) \) is simple. Further, if one (and thus both) of \( (F\beta)(G\gamma) \) and \( (F'\beta')(G\gamma) \) is (are) simple, then they are of the same type.

**Proof.** The first claim is a direct application of Lemma 7.3 and the second follows from the definition of type. \( \square \)

**Lemma 7.4.** Let \( F \) and \( G \) be in \( \mathcal{F} \) and \( \beta \) and \( \gamma \) be from either \( S_\infty \) or \( B_\infty \). If \( F\beta \) and \( G\gamma \) are simple, then they have a simple common right multiple.

**Proof.** This is easier than a reference to Lemma 2.11 since the fact that \( S_\infty \) and \( B_\infty \) are groups implies that any common right multiple of \( F \) and \( G \) in \( \mathcal{F} \) is a common right multiple of \( F\beta \) and \( G\gamma \). We know that \( F \cup G \) is a common right multiple. It is also clear that \( F \cup G \) is simple if both \( F \) and \( G \) are simple. \( \square \)

### 7.2. Balanced, simple subgroups. The groups \( \hat{\mathcal{F}} \) and \( \text{BV} \) are groups of fractions and elements are represented by pairs of elements from a monoid. We pick out elements represented by certain pairs.

Let \((F\beta, F'\beta')\) be a pair of elements from \( \mathcal{F} \bowtie S_\infty \) or \( \mathcal{F} \bowtie B_\infty \) with \( F \) and \( F' \) from \( \mathcal{F} \) and \( \beta \) and \( \beta' \) from \( S_\infty \) or \( B_\infty \) as appropriate. We say that the pair is *simple and balanced* if both entries in the pair are simple, and if the two entries are of the same type.

Let \( V \) be the set of elements in \( \hat{\mathcal{F}} \) that have at least one representative that is simple and balanced. Let \( BV \) be the set of elements in \( \text{BV} \) that have at least one representative that is simple and balanced. The point of Corollary 7.3.1 and Lemma 7.4 is the following.

**Theorem 4.** Both \( V \) and \( BV \) are groups.

**Proof.** The inverse of a pair \((u, v)\) is \((v, u)\), so the groups of the statement are closed under inversion. If \((u, v)(w, z)\) is a product of pairs that must be calculated, then we must find a common right multiple \( vp = wq \) of \( v \) and \( w \) and get the product \((up, zq)\). We know from Lemma 7.3 that \( vp = wq \) can be made simple and we know from Corollary 7.3.1 that \( up \) and \( zq \) will be as well. Since type equals length of the forest part, since length of forests is multiplicative, and since \( u \) and \( v \) share a type, and \( w \) and \( z \) share a type, we get that the types of \( up \), \( vp = wq \) and \( zq \) are the same. Thus both groups of the statement are closed under product. \( \square \)

**Theorem 5.** The group \( BV \) from this section and the group \( BV \) as described in Section 4.1 are isomorphic.
Sketch of proof. The groups under discussion are subgroups of the group \( \hat{BV} \) realized as a group of fractions and as described in Section \textcolor{red}{13}. By Theorem \textcolor{red}{8} the two versions of \( \hat{BV} \) are isomorphic. The two versions of \( BV \) are the corresponding subgroups. \( \square \)

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