Liouville type theorem for stationary Navier–Stokes equations

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Abstract
It is shown that any smooth solution to the stationary Navier–Stokes system in \( \mathbb{R}^3 \) with the velocity, belonging to \( L_6 \) and \( BMO^{-1} \), must be zero.

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1. Introduction

Let us consider smooth solutions to the stationary Navier–Stokes system
\[
\mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} = \nabla p, \quad \text{div} \mathbf{u} = 0
\]
(1.1)
in \( \mathbb{R}^3 \) with the additional condition at infinity: \( \mathbf{u}(x) \to 0 \) as \( |x| \to \infty \). Then the question is whether or not \( \mathbf{u} \) must be identically zero. The point addressed in this short note is under which additional assumptions the answer to the above question is positive.

In the monograph [3], it has been shown that the condition
\[
\mathbf{u} \in L_2(\mathbb{R}^3)
\]
(1.2)
implies \( \mathbf{u} = 0 \).

A plausible conjecture is that the sufficient condition for the positive answer could be as follows:
\[
\int_{\mathbb{R}^3} |
\Delta \mathbf{u}|^2 \, dx < \infty.
\]
(1.3)

At the moment of writing this note, a proof of whether or not (1.3) is the sufficient condition for \( \mathbf{u} \) to be identically zero is not known yet. We are going to show, however, that under an additional condition it is true. To describe that condition, we need the following definition.
We say that a divergence free vector valued field $u$ belongs to the space $BMO^{-1}(\mathbb{R}^3)$ if there exists skew symmetric tensor $d$ from $BMO(\mathbb{R}^3)$ such that

$$u = \text{div } d = (d_{ij}).$$

It is known, see for instance [7], that if $d \in BMO(\mathbb{R}^3)$, then

$$\Gamma(x) := \sup_{x_0 \in \mathbb{R}^3, 0 < r} \left( \frac{1}{|B(r)|} \int_{B(x_0, r)} |d - [d]_{x_0,r}|^2 \, dx \right)^{1/2} < \infty$$

for each $1 \leq s < \infty$. Here, we denote by $B(x_0, r)$ the ball of radius $r$ centred at point $x_0$ and $[d]_{x_0,r}$ is the mean value of $d$ over the ball $B(x_0, r)$.

Our aim is to prove the following result.

**Theorem 1.1.** The following statements are true:

(A) any smooth divergence free vector-valued field $u \in BMO^{-1}(\mathbb{R}^3)$, satisfying condition (1.3) and the system (1.1), is identically equal to zero;

(B) any smooth divergence free vector-valued field $u \in BMO^{-1}(\mathbb{R}^3)$, satisfying the condition

$$u \in L_6(\mathbb{R}^3)$$

and the system (1.1), is identically equal to zero.

By the known inequality

$$\|v\|_{L_6} \leq c\|\nabla v\|_{L_2},$$

being valid for any $v \in C_0^\infty(\mathbb{R}^3)$, statement (A) follows from statement (B).

Before proving theorem 1.1, let us show that the assumptions in (B) do not follow immediately from the condition (1.2) of [3]. Indeed, we can let

$$w = \sin \left( (|x|^2 + 1)^{1/2} - 1 \right)(1, 1, 1)$$

and

$$v = \text{rot } w.$$

Direct calculations show that $v \in BMO^{-1}(\mathbb{R}^3) \cap L_6(\mathbb{R}^3)$ but $v \notin L_2(\mathbb{R}^3)$.

Finally, we would like to mention that there are many interesting papers devoted to the above or related questions, see, for example, [2, 5, 6] and [1].

### 2. Proof of main result

#### 2.1. Caccioppoli type inequality

This is the main technical part of the proof. We take an arbitrary ball $B(x_0, R) \in \mathbb{R}^3$ and a non-negative cut off function $\varphi \in C_0^\infty(B(x_0, R))$ with the following properties $\varphi(x) = 1$ in $B(x_0, \varrho)$, $\varphi(x) = 0$ out of $B(x_0, r)$, and $|\nabla \varphi(x)| \leq c(r - \varrho)$ for any $R/2 \leq \varrho < r \leq R$.

We let

$$\pi = u - u_0, \quad \bar{d} = d - [d]_{x_0, R},$$

where $u_0$ is an arbitrary constant.
We also know that, for any \(2 < s < \infty\), there exists a constant \(c_0 = c_0(s) > 0\) and a function \(w \in W^2_0(\Omega_{\infty, r})\), vanishing on \(\partial \Omega_{\infty, r}\), such that \(\text{div} \ w = \nabla \varphi \cdot \pi\) and

\[
\int_{\Omega_{\infty, r}} |\nabla w|^2 \, dx \leq c_0 \int_{\Omega_{\infty, r}} |\nabla \varphi \cdot \pi|^2 \, dx \leq \frac{c_0}{(r - \theta)^2} \int_{\Omega_{\infty, r}} |\pi|^2 \, dx.
\]

(2.1)

Now, we can test the Navier–Stokes equations \((1.1)\) with the function \(\varphi \pi - w\), integrate by parts in \(\Omega_{\infty, r}\), and find the following identity:

\[
\int_{\Omega_{\infty, r}} \varphi |\nabla u|^2 \, dx = -\int_{\Omega_{\infty, r}} \nabla u : (\nabla \varphi \otimes \pi) \, dx + \int_{\Omega_{\infty, r}} \nabla w : \nabla u \, dx
\]

\[
- \int_{\Omega_{\infty, r}} (u \cdot \nabla u) \cdot \varphi \pi \, dx + \int_{\Omega_{\infty, r}} (u \cdot \nabla u) \cdot w \, dx = I_1 + I_2 + I_3 + I_4.
\]

\(I_1\) and \(I_2\) can be estimated easily. As a result, we find

\[
|I_1| + |I_2| \leq \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\pi|^2 \, dx \right)^{\frac{1}{2}} R^{\frac{1}{s} - \frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\pi|^2 \, dx \right)^{\frac{1}{2}}.
\]

To estimate \(I_3\) and \(I_4\), we are going to use the skew symmetry of the matrix \(d\). We have

\[
|I_3| = \left| \int_{\Omega_{\infty, r}} \tilde{d}_{jm,n} \tilde{u}_{j,n} \tilde{u}_{i} \varphi \, dx \right| = \left| \int_{\Omega_{\infty, r}} \tilde{d}_{jm,n} \tilde{u}_{j,n} \varphi \tilde{u}_{i} \, dx \right|
\]

\[
\leq \frac{1}{r - \theta} \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\tilde{d}|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\pi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\tilde{u}|^{2s - 2} \, dx \right)^{\frac{1}{2}}
\]

\[
\leq c \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\pi|^2 \, dx \right)^{\frac{1}{2}} \Gamma(2s/(s - 2)).
\]

It remains to evaluate \(I_4:\)

\[
|I_4| = \left| \int_{\Omega_{\infty, r}} \tilde{d}_{jm,n} \tilde{u}_{j,n} \tilde{w} \, dx \right| = \left| \int_{\Omega_{\infty, r}} \tilde{d}_{jm,n} \tilde{u}_{j,n} \tilde{w} \, dx \right|
\]

\[
\leq \frac{1}{r - \theta} \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\tilde{d}|^2 |\nabla w|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{r - \theta} \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\tilde{d}|^2 \nabla w^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\tilde{u}|^{2s - 2} \, dx \right)^{\frac{s - 2}{2s}}
\]

\[
\leq c(s) \left( \int_{\Omega_{\infty, r}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{\infty, r}} |\pi|^2 \, dx \right)^{\frac{1}{2}} \Gamma(2s/(s - 2)).
\]

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So, we have the following inequality
\[ \int_{B(x_0, \rho)} |\nabla u|^2 \, dx \leq c(s) \frac{R^{3-\frac{2}{p-2}}}{r - \rho} \left( \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \]

We can apply Young’s inequality and conclude
\[ \int_{B(x_0, \rho)} |\nabla u|^2 \, dx \leq \frac{1}{4} \int_{B(x_0, r)} |\nabla u|^2 \, dx + c(s) \frac{1}{(r - \rho)^2} R^{3-\frac{2}{p-2}} \left( \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \]

The later inequality is valid for any \( R/2 \leq \rho < r \leq R \). It is known, see for instance [4] (this is just a matter of suitable iterations), that such an inequality implies the following Caccioppoli type inequality
\[ \int_{B(x_0, R/2)} |\nabla u|^2 \, dx \leq c(s) R^{3-\frac{2}{p-2}} \left( \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}, \tag{2.2} \]
which holds for any \( B(x_0, R) \) in \( \mathbb{R}^3 \).

2.2. (A) implies (B)

Indeed, if we let \( s = 6 \) and \( u_0 = 0 \), then (2.2) takes the form
\[ \int_{B(x_0, R/2)} |\nabla u|^2 \, dx \leq c \left( \int_{B(x_0, R)} |u|^6 \, dx \right)^{\frac{1}{2}} \leq c \left( \int_{\mathbb{R}^3} |u|^6 \, dx \right)^{\frac{1}{2}}. \]
Passing \( R \to \infty \), we conclude that (1.3) holds.

2.3. Proof of (A)

Now, we let \( s = 3 \) and \( u_0 = |u|_{x_0, R} \) and use the Gagliardo–Nirenberg type inequality
\[ \left( \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \leq c \left( \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}, \]
with a universal positive constant \( c \). Now, (2.2) can reduced to the following reverse Hölder inequality
\[ \frac{1}{|B(R/2)|} \int_{B(x_0, R/2)} |\nabla u|^2 \, dx \leq c \left( \frac{1}{|B(R)|} \int_{B(x_0, R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \]
with a constant \( c \) that is independent of \( x_0 \) and \( R \). We let \( h := |\nabla u|^2 \in L^2(\mathbb{R}^3) \) and let \( M(h) \) be the maximal function of \( h \), i.e.
\[ M_h(x_0) = \sup_{R > 0} \int_{B(x_0, R)} h(x) \, dx. \]
Then from the above inequality, it follows that
\[ M_h^4(x_0) \leq c M_h^4(x_0). \]
It is known, see [7], that the right-hand side of the latter inequality is integrable in $\mathbb{R}^3$ and the corresponding integral is bounded from above by the quantity

$$\int_{\mathbb{R}} h^4 dx = \int_{\mathbb{R}} |\nabla u|^2 dx$$

times a universal constant. So, this means that $M_{\frac{h}{M}} \in L_4(\mathbb{R}^3)$. Since $h^4 \in L_4(\mathbb{R}^3)$, it is possible only if $h$ is identically equal to zero, see [7]. So, $\nabla u$ is identically equal to zero.

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