EXACT SOLUTIONS OF THE EQUATION FOR COMPOSITE SCALAR PARTICLES IN QUANTIZED ELECTROMAGNETIC WAVES

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Abstract

The equation is considered for a composite scalar particle with polarizabilities in an external quantized electromagnetic plane wave. This equation is reduced to a system of equations for infinite number of interacting oscillators. After diagonalization, we come to equations for free oscillators. As a result, exact solutions of the equation for a particle are found in a plane quantized electromagnetic wave of the arbitrary polarization. As a particular case, the solution for monochromatic electromagnetic waves is considered. The relativistic coherent states of a particle are constructed in the case of the Poisson distribution of photon numbers. In the limit when the average photon number $<n>$ and the volume $V$ of the quantization trend to infinity (but the photon density $<n>/V$ remains constant), the wavefunction converts to the solution corresponding to the external classical electromagnetic wave.

1 Introduction

In this work we find and investigate exact solutions of the equation for composite scalar particles (for example pions, kaons or charged composite Higgs bosons $H^\pm$) which possess electromagnetic polarizabilities in the external quantized plane electromagnetic wave of arbitrary polarization. It should be noted, however, that the notion of an external electromagnetic field in the relativistic theory is limited. In [1-3] some exact solutions were considered for a scalar particle in the field of the static uniform electric, magnetic fields, the monochromatic classical and quantized electromagnetic wave. It is important to study the general case of an interaction of charged composite

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scalar particles with strong non-monochromatic laser waves with an arbitrary polarization. For point-like scalar and spinor particles such solutions were derived in [4-6]. In our case we can investigate the effects which are connected with the interaction of external electromagnetic waves with charges distributed inside of a composite particles. Considering the quantized external electromagnetic waves allow us to take into account the interaction of a particle and photons with arbitrary frequencies and polarizations. Here we describe a particle and electromagnetic field quantum-mechanically. This is the most general and important case which can be compared with the semi-classical approaches. We interpret the complex, which consists of a particle and some number of photons as a quasi-particle. This is an analog of the state of an electron in the field of the crystal lattice. The solutions obtained can be used in different calculations of probabilities of composite scalar particle scattering processes in the field of quantized electromagnetic waves. We notice that some processes, however, can not be considered in the framework of one-particle theory.

Section 2 contains the discussion of the conserved quantum numbers for an arbitrary external quantized electromagnetic wave. The solution of the equation, momentum and mass for a particle in the field of a classical plane electromagnetic wave are considered in section 3. In Sec. 4 we find wavefunction of a composite particle in a quantized plane electromagnetic wave with arbitrary polarization. The momentum and mass of a particle in a plane quantized electromagnetic wave is studied in Sec. 5. Section 6 contains the solution of the equation for the particular case of monochromatic plane waves. In Sec. 7 the coherent states of a particle are investigated when the average number of photons $\langle n \rangle$ and quantization volume $V$ go to infinity. In this limit the wavefunction transforms to the solution of the equation for a particle in the external classical electromagnetic wave. Sec. 8 contains a conclusion.

We use units in which $\hbar = c = 1$. 
2 An arbitrary external quantized electromagnetic wave

The Lagrangian of a composite scalar particle interacting with electromagnetic field is given by [1]

\[ \mathcal{L} = -(D^+_{\mu} \varphi^*)(D^\mu \varphi) (\delta_{\mu\nu} - K_{\mu\nu}) - m^2_{\text{eff}} \varphi^* \varphi, \]  

\[ (1) \]

where \( m \) is the rest mass and \( m^2_{\text{eff}} = m^2 \left(1 - \beta F^2_{\mu\nu}/(2m) \right) \) is the squared effective mass of a particle, \( D_{\mu} = \partial_{\mu} - ieA_{\mu}, \) \( D^+_{\mu} = \partial_{\mu} + ieA_{\mu}, \) \( \partial_{\mu} = \partial/\partial x_{\mu}, \) \( x_{\mu} = (x, ix_0), \) \( x_0 \equiv t \) is the time, \( e \) is a charge, \( A_{\mu} \) is the vector potential of the external electromagnetic field, \( K_{\mu\nu} = (\alpha + \beta) F_{\mu\alpha} F_{\nu\alpha}/m, \) \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the strength tensor, and \( \alpha, \beta \) are electric and magnetic polarizabilities of a particle, respectively. From the Lagrangian (1) we find the equation for the wavefunction of a composite scalar particle

\[ D^2_{\mu} \varphi - D_{\mu} [K_{\mu\nu} (D^\nu \varphi)] - m^2_{\text{eff}} \varphi = 0. \]  

\[ (2) \]

Equation (2) is a generalization of a Klein-Gordon equation on the case of a composite scalar particle. This equation was obtained [7] using the instanton vacuum model (that is QCD-motivated) and the phenomenological approach [8].

Using the general expression for the density of the electric current

\[ j_{\mu} = -i \left( \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi)} \varphi - \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \varphi^*)} \varphi^* \right), \]

we find from Eq. (1)

\[ j_{\mu} = i(\varphi D^+_{\nu} \varphi^* - \varphi^* D^\nu \varphi) (\delta_{\mu\nu} - K_{\mu\nu}). \]

\[ (3) \]

The last term in Eq. (3) contributes to the usual current density of a point-like scalar particle. This contribution is due to the charge distribution inside of a composite scalar particle. The vector potential of the quantized electromagnetic field can be chosen in the form [9]

\[ A_{\mu} = \sum_{k,s} \frac{e_{s\mu}}{\sqrt{2k_0}} \left[ e_{ks} e^{i(kx)} + e_{ks}^+ e^{-i(kx)} \right], \]

\[ (4) \]

where \( e_{s\mu} = \delta_{s\mu} \) are two unit polarization vectors \( (s = 1, 2) \), \( k_{\mu} = (k, ik_0) \) is a wavevector with the properties \( (e_1 k) = (e_2 k) = k^2 = k^2 - k_0^2 = 0 \)
and \( k_0 \) is the photon energy, \( V = L^3 \) is the normalizing volume (\( L \) is the normalizing length). The creation \( c^+_{ks} \) and annihilation \( c^-_{ks} \) operators satisfy the commutation relations \([c^-_{ks}, c^+_{k's}] = \delta_{kk'}\delta_{ss'}\). The scalar product of four-vectors is defined as \((kx) = kx - k_0x_0\). We consider in (4) an arbitrary number of modes \( N \) of electromagnetic waves.

As noted in [10], the energy-momentum tensor of a hole system is the sum of tensors of energy-momentum of electromagnetic fields and particles, and the last are considered as non-interacting particles. Thus, we introduce the operator of the total four-momentum of the particle-photon system as follows

\[
\hat{Q}_\mu = -i\partial_\mu + \sum_{k,s} k_\mu \left( N_{ks} + \frac{1}{2} \right), \quad N_{ks} = c^+_{ks}c^-_{ks},
\]

where \( N_{ks} \) is the operator of the photon number corresponding to the four-momentum \( k_\mu \) and polarization vector \( e_{sp} \). As the operator \( \hat{Q}_\mu \) commutes with the covariant derivative \( D_\mu \), i.e. \([\hat{Q}_\mu, D_\nu] = 0\), an eigenvalue of the operator \( \hat{Q}_\mu \) is the integral of motion. As a result, the wavefunction of a particle in the external quantized electromagnetic field, \( \varphi \), obeys as Eq. (2) and the following equation

\[
\hat{Q}_\mu \varphi = q_\mu \varphi,
\]

where \( q_\mu \) is an eigenvalue of the operator \( \hat{Q}_\mu \). We will look for the solution of Eqs. (2), (6) which corresponds to the definite four-momentum \( q_\mu \) of a particle-photon system. It is convenient to choose the replacement as follows

\[
\varphi = U_1 \Phi, \quad U_1 = \exp \left\{ -i \sum_{k,s} (k_\mu x_\mu) \left( N_{ks} + \frac{1}{2} \right) \right\}.
\]

The operator \( U_1 \) is the unitary operator so that \( U_1^+ U_1 = 1 \). From Eqs. (5)-(7) we find

\[
-i \partial_\mu \Phi = q_\mu \Phi.
\]

The solution to Eq. (8) is given by

\[
\Phi = \exp \{i(q_\mu x_\mu)\} \chi,
\]

where the function \( \chi \) does not depend on the coordinates \( x_\mu \) and depends only on photon variables. As a result the general solution to Eqs. (2) and (6) can be written as

\[
\varphi = U \chi, \quad U = \exp \{i(q_\mu x_\mu)\} U_1.
\]
The function $\chi$ obeys the equation
\begin{equation}
U^+ D_\mu^2 U \chi - U^+ D_\mu \left[ K_{\mu\nu} (D_\nu U \chi) \right] - m^2 \chi = 0, \tag{11}
\end{equation}
where $U^+$ is the Hermitian conjugated operator:
\begin{equation}
U^+ = \exp \left\{ -i \left[ (q_\mu x_\mu) - \sum_{k,s} (k_\mu x_\mu) \left( N_{ks} + \frac{1}{2} \right) \right] \right\}. \tag{12}
\end{equation}

We took into consideration that for electromagnetic waves with the vector potential (4) $F_{\mu\nu}^2 = 0$ (and therefore $m_{\text{eff}} = m$).

Using the operator identity
\begin{equation}
U^+ c_{ks}^\pm U = \exp \{ \pm (k_\mu x_\mu) \} c_{ks}^\pm, \tag{13}
\end{equation}
we find
\begin{equation}
U^+ D_\mu U = i \left\{ q_\mu - \sum_{k,s} k_\mu \left( N_{ks} + \frac{1}{2} \right) - e \sum_{k,s} \frac{e_{s\mu}}{\sqrt{2k_0 V}} \left[ c_{ks}^- + c_{ks}^+ \right] \right\}, \tag{14}
\end{equation}
\begin{equation}
U^+ F_{\mu\nu} U = i \sum_{k,s} \frac{k_\mu e_{s\nu} - k_\nu e_{s\mu}}{\sqrt{2k_0 V}} \left[ c_{ks}^- - c_{ks}^+ \right]. \tag{15}
\end{equation}

It is convenient to use the coordinate representation for the operators $c_{ks}^-$, $c_{ks}^+$ [9]:
\begin{equation}
c_{ks}^- = \frac{1}{\sqrt{2}} \left( \xi_{ks} + \frac{\partial}{\partial \xi_{ks}} \right), \quad c_{ks}^+ = \frac{1}{\sqrt{2}} \left( \xi_{ks} - \frac{\partial}{\partial \xi_{ks}} \right). \tag{16}
\end{equation}

In this representation the operator of the photon number becomes
\begin{equation}
N_{ks} = \frac{1}{2} \left( \xi_{ks}^2 - \frac{\partial^2}{\partial \xi_{ks}^2} - 1 \right). \tag{17}
\end{equation}

Then the wavefunction $\chi$, which obeys Eq. (11), depends on photon variables $\xi_{ks}$. 

5
3 A classical plane electromagnetic wave

Equation (2) has exact solutions for a particle moving in the classical plane electromagnetic wave [1]. The vector potential of a plane electromagnetic wave depends only on \( \vartheta = k_\mu x_\mu \), i.e. \( A_\mu = A_\mu(\vartheta) \), so that the Lorentz condition \( \partial_\mu A_\mu = 0 \) is valid. For such a potential Eq. (2) takes the form

\[
\left( \partial_\mu^2 - 2i e A_\mu \partial_\mu - e^2 A_\mu^2 - W k_\mu k_\nu (A'_\alpha)^2 \partial_\mu \partial_\nu - m^2 \right) \varphi(x) = 0,
\]

(18)

where \( W = (\alpha + \beta)/m \), \( A'_\alpha = \partial A_\alpha / \partial \vartheta \). The solution to Eq. (18) can be represented as

\[
\varphi(x) = \exp(ipx) \chi(\vartheta),
\]

(19)

where \( px = p_\mu x_\mu \), \( p_\mu \) is the momentum of a free particle so that \( p^2 = -m^2 \). Using Eq. (19) we arrive from Eq. (18) at

\[
2i(pk)\chi'(\vartheta) + \left[ 2e(Ap) - e^2 A^2 + W(kp)^2 (A')^2 \right] \chi(x) = 0.
\]

(20)

After integration of Eq. (20), the wavefunction (19) takes the form

\[
\varphi(x) = C \exp \left\{ i(px) + i \int kx 0 \left[ \frac{e(Ap)}{(kp)} - \frac{e^2 A^2}{2(kp)} + \frac{1}{2} W(kp)(A')^2 \right] d\vartheta \right\}.
\]

(21)

The normalization constant \( C \) can be chosen as \( C = 1/\sqrt{2} \). From Eq. (3) taking into account the solution (21) we find the current density of a composite scalar particle

\[
j_\mu = p_\mu - e A_\mu + k_\mu \left[ \frac{e(Ap)}{(kp)} - \frac{e^2 A^2}{2(kp)} - \frac{1}{2} W(kp)(A')^2 \right].
\]

(22)

The last term in Eq. (22) came from the charge distribution inside of a composite scalar particle. For point-like particle \( W = 0 \), and this term is absent.

Using solution (21), it is easy to find also the density of kinetic momentum of a particle \( P_\mu \) in the state \( \varphi \). Taking into consideration the operator of kinetic momentum [11] \( \hat{P} = -i \partial_\mu - e A_\mu \), we find

\[
P_\mu \equiv \varphi^* (-i \partial_\mu - e A_\mu) \varphi - \varphi (i \partial_\mu + e A_\mu) \varphi^* = p_\mu - e A_\mu + k_\mu \left[ \frac{e(Ap)}{(kp)} - \frac{e^2 A^2}{2(kp)} + \frac{1}{2} W(kp)(A')^2 \right].
\]

(23)
If $A_\mu(\vartheta)$ is a periodic function, the mean value of $A_\mu(\vartheta)$ equals zero, $\overline{A_\mu(\vartheta)} = 0$. Then averaging Eq. (23) on time, we arrive at the mean value of the kinetic momentum density of a composite particle

$$P_\mu = p_\mu - k_\mu \left[ \frac{e^2 A^2}{2(kp)} - \frac{1}{2} W(kp)(A')^2 \right].$$

(24)

From Eq. (24) we find

$$P_\mu^2 = -m_*^2, \quad m_*^2 = m^2 + e^2 A^2 - W(kp)^2(A')^2,$$

(25)

where $m_*$ is the effective mass of a composite particle in the field of a classical electromagnetic wave. When we neglect the polarizabilities of a particle, i.e. $W = 0$, we come to the known expression [11]. The total momentum of the particle-photon system (see Eq. (5)) can be represented as follows

$$q_\mu = P_\mu + R_\mu,$$

(26)

where $R_\mu$ is the four-momentum of photons (the electromagnetic field) interacting with a particle. In the framework of one-particle theory both variables $P_\mu, R_\mu$ are the integrals of motion (see also [6]).

4 Plane quantized electromagnetic waves

For our purposes, we will use the approximation of a plane wave when the momenta of photons are parallel to the direction $\kappa_\mu = (\kappa, i\kappa_0)$ so that $\kappa^2 = \kappa_0^2$, $\kappa_0 = 2\pi/L$ and $k_\mu = n\kappa_\mu$, where $n$ is an integer positive number $n = 1, 2, ..., N$. All frequencies of photons are taken into consideration here. In the case of the monochromatic wave only one term survives in the sum of Eq. (4). For a plane electromagnetic wave we have instead of Eq. (4) the following vector potential

$$A_\mu = \sum_{n=1}^{N} \sum_{s=1}^{2} \frac{e_s n}{\sqrt{2n\kappa_0}} \sqrt{2n\kappa_0} V \left[ c_{ks} e^{in(kx)} + c_{ks}^* e^{-in(kx)} \right].$$

(27)

Taking into account Eqs. (13)-(17) and (27), equation (11) is transformed to

$$\left\{ \sum_{n,s} n \left( \frac{\partial^2}{\partial \xi_{ns}^2} - \xi_{ns}^2 \right) + \frac{1}{(q\kappa)} \left( \sum_{n,s} \frac{b_{ns}}{\sqrt{n}} \xi_{ns} \right)^2 - 2 \sum_{n,s} \frac{d_{ns}}{\sqrt{n}} \xi_{ns} \right\} = \cdots$$
\[- \frac{W(q\kappa)}{e^2} \left( \sum_{n,s} b_{s\mu} \sqrt{n} \frac{\partial}{\partial \xi_{ns}} \right)^2 + \frac{q^2 + m^2}{(q\kappa)} \right\} \chi(\xi) = 0, \tag{28}\]

where we use the notations

\[b_{s\mu} = b e_{s\mu}, \quad b = \frac{e}{\sqrt{\kappa_0 V}}, \quad \alpha_s = \left( \frac{q b_s}{q\kappa} \right), \quad W = \frac{\alpha + \beta}{m}, \tag{29}\]

and the scalar products \((q\kappa) \equiv q_{\mu} \kappa_{\mu}, q^2 = q_{\mu}^2 \) and so on. Taking into consideration the equality \(b_{s\mu} b_{s'\mu} = b^2 \delta_{ss'}\) and introducing

\[M = \frac{q^2 + m^2}{(q\kappa)}, \quad \delta = \frac{b^2}{(q\kappa)}, \quad \gamma = \frac{W(q\kappa)^2}{e^2} \delta, \tag{30}\]

Eq. (28) is converted into

\[\sum_{s=1}^{2} \left\{ \sum_{n=1}^{N} n \left( \frac{\partial^2}{\partial \xi_{ns}^2} - \xi_{ns}^2 \right) + \delta \left( \sum_{n=1}^{N} \xi_{ns} \sqrt{n} \right)^2 - 2\alpha_s \sum_{n=1}^{N} \xi_{ns} \sqrt{n} \right. \]

\[- \left. \gamma \left( \sum_{n=1}^{N} \sqrt{n} \frac{\partial}{\partial \xi_{ns}} \right)^2 + M \right\} \chi(\xi) = 0. \tag{31}\]

Variables with different polarization index \(s = 1, 2\) are separated in equation (31), and, therefore, we can look for a solution to Eq. (31) in the form \(\chi(\xi) = f_1(\xi) f_2(\xi)\), where the function \(f_s(\xi)\) obeys the equation

\[\left\{ \sum_{n=1}^{N} n \left( \frac{\partial^2}{\partial \xi_{ns}^2} - \xi_{ns}^2 - 2\alpha_s \xi_{ns} \sqrt{n} \xi_{ns} \right) \right. \]

\[\left. + \delta \left( \sum_{n=1}^{N} \xi_{ns} \sqrt{n} \right)^2 \right\} f_s(\xi) = 0 \tag{32}\]

with the condition \(M = \varepsilon_1 + \varepsilon_2\). Equation (32) represents the system of interacting oscillators. To avoid the terms in Eq. (32) which is linear in \(\xi_{ns}\) and to have the unit coefficient at the second derivatives we made the linear transformation

\[z_{ns} = \frac{\xi_{ns}}{\sqrt{n}} + a_{ns} \tag{33}\]

with the constraint

\[n^2 a_{ns} - \alpha_s - \delta \sum_{m=1}^{N} a_{ns} = 0. \tag{34}\]
It is easy to find the following solution to Eq. (34)

$$a_{ns} = \frac{\alpha_s}{n^2 \left( 1 - \delta \sigma_N \right)}$$

(35)

where

$$\sigma_N = 1 + \frac{1}{2^2} + \ldots + \frac{1}{N^2}.$$ 

In the case of infinite number of modes of quantized electromagnetic waves ($N \rightarrow \infty$) $\sigma_N \rightarrow \sigma_\infty = \pi^2/6$. In new variables (33) Eq. (32) takes the form

$$\left\{ \sum_{n=1}^{N} \left( \frac{\partial^2}{\partial z_{n_s}^2} - n^2 z_{n_s}^2 \right) + \delta \left( \sum_{n} z_{ns} \right)^2 \right. 

- \gamma \left( \sum_{n=1}^{N} \frac{\partial}{\partial z_{ns}} \right)^2 + \left. \frac{\alpha_s^2 \sigma_N}{1 - \delta \sigma_N} + \varepsilon_s \right\} f_s(z) = 0. \quad (36)$$

It is convenient to rewrite equation (36) as follows:

$$\left\{ \sum_{n,m} \left[ (\delta_{nm} - \gamma) \frac{\partial^2}{\partial z_{ns} \partial z_{ms}} - \left( n^2 \delta_{nm} - \delta \right) z_{ns} z_{ms} \right] 

+ \frac{\alpha_s^2 \sigma_N}{1 - \delta \sigma_N} + \varepsilon_s \right\} f_s(z) = 0. \quad (37)$$

Eq. (37) still describes the system of interacting oscillators. To diagonalize Eq. (37) let us introduce the matrices $A$, $B$, $C$ and $D$ in N-dimensional space:

$$A = I - \gamma C, \quad B = D - \delta C,$$

\begin{align*}
C = & \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
& \ldots & \ldots \\
1 & 1 & 1
\end{pmatrix}, & D = & \begin{pmatrix}
1 & 0 & 0 \\
0 & 2^2 & 0 \\
& \ldots & \ldots \\
0 & 0 & N^2
\end{pmatrix} (38)
\end{align*}

with the matrix elements $A_{nm} = \delta_{nm} - \gamma C_{nm}$, $B_{nm} = D_{nm} - \delta C_{nm}$, $D_{nm} = n^2 \delta_{nm}$; $I$ is the unit matrix. Using matrices (38) Eq. (37) is converted into equation

$$\left\{ \left( \frac{\partial}{\partial z} \right)^T A \frac{\partial}{\partial z} - (z)^T B z + \frac{\alpha_s^2 \sigma_N}{1 - \delta \sigma_N} + \varepsilon_s \right\} f_s(z) = 0, \quad (39)$$
where $\partial/\partial z$, $z$ are the columns, and $(\partial/z)^T$, $(z)^T$ are the rows of variables $\partial/\partial z_{ns}$, $z_{ns}$ ($n = 1, 2, ..., N$). Two quadratic forms for derivatives and coordinates in Eq. (39) can be diagonalized by the linear transformation

$$ z = Py, \quad A' = P^{-1} A \left( P^{-1} \right)^T, \quad B' = P^T B P, \quad (40) $$

where $P^{-1} P = 1$ and $P^T$ is the transposed matrix. Transformations (40) with diagonal matrices $A'$ and $B'$ guarantee that Eq. (39) becomes diagonal and variables $y_k$ will be separated. Transformations (40) mean the transition to normal variables $y_{ns}$ which describe non-interacting oscillators. With the help of this variables we construct creation $\bar{c}^+_k$ and annihilation $\bar{c}^-_k$ operators of non-interacting quasi-photons

$$ \bar{c}^-_k = \frac{1}{\sqrt{2}} \left( y_{ks} - \frac{\partial}{\partial y_{ks}} \right), \quad \bar{c}^+_k = \frac{1}{\sqrt{2}} \left( y_{ks} + \frac{\partial}{\partial y_{ks}} \right). $$

Then Eqs. (40) are equivalent to Bogolubov’s transformations (see also [6]).

It follows from Eq. (40) that the matrix $A^{-1}$ transforms under the transformation $z = Py$ like the matrix $B$:

$$ \left( A^{-1} \right)^T = P^T A^{-1} P. \quad (41) $$

Thus, to find a matrix $P$ we should solve the characteristic equation [12]

$$ \det \left( B - \lambda^2 A^{-1} \right) = 0. \quad (42) $$

Solutions to Eq. (42) $\lambda^2_n$ ($n = 1, 2, ..., N$) are eigenvalues, so that $BP_n = \lambda^2_n A^{-1} P_n$ or $ABP_n = \lambda^2_n P_n$, where $P_n$ ($N$-dimensional columns) are eigenvectors. At $P$-transformations matrices $B$ and $A^{-1}$ (and $A$) can be diagonalized simultaneously so that (see [12])

$$ \left( \frac{\partial}{\partial z} \right)^T A \left( \frac{\partial}{\partial z} \right) = \left( \frac{\partial}{\partial y} \right)^T A \left( \frac{\partial}{\partial y} \right) \equiv \sum_{n=1}^N \frac{\partial^2}{\partial y_{ns}^2}, \quad (z)^T B z = \sum_{n=1}^N \lambda^2_n y_{ns}^2. \quad (43) $$

From definitions (38) we obtain the following relationships

$$ C^2 = NC, \quad A^2 = I + \gamma (\gamma N - 2) C, \quad A^{-1} = I + \frac{\gamma}{1 - \gamma N} C, \quad (44) $$

"
where \( N \) is the number of modes. Taking into account equalities (44) we find from Eq. (42) the characteristic equation as follows

\[
\prod_{m=1}^{N} \left( m^2 - \lambda^2 \right) \left[ 1 - \left( \delta + \frac{\gamma \lambda^2}{1 - \gamma N} \right) \sum_{n=1}^{N} \frac{1}{n^2 - \lambda^2} \right] = 0.
\] (45)

It should be noted that by considering infinite number of modes, \( N \to \infty \), and as a result \( (\delta + \gamma \lambda^2/(1 - \gamma N)) \to \delta \). So, in this case electromagnetic polarizabilities \( \alpha, \beta \) does not enter Eq. (45) and we come to the equation for point-like particle (see [4]). As a result, the interaction of composite scalar particles and the electromagnetic waves with the infinite number of modes is similar to the interaction of scalar point-like particles. To extract the effect of composite structure of scalar particles we have to consider few number of modes or monochromatic photons when the ratio \( \gamma \lambda^2/(1 - \gamma N) \) is not negligible. The formula (45) has the singularity at \( \gamma N = 1 \), and is valid when \( \gamma N \neq 1 \). Indeed, at \( \gamma N = 1 \) we have from Eq. (44) \( A^2 = A \). It means that eigenvalues of the matrix \( A \) are 1 and 0 (because \( A (A - 1) = 0 \)) and, therefore, the matrix \( A \) is a projection matrix which does not have the inverse matrix. We skip this special case. Below we analyze two cases: (i) \( \gamma N \ll 1 \), (ii) \( \gamma N \gg 1 \).

At \( e = 0 (\delta = 0) \) and \( \gamma = 0 \) we come to non-interacting oscillators, and the solution to Eq. (45) is \( \lambda_n^2 = n^2 \) \( (n = 1, 2, ..., N) \). So eigenvalues \( \lambda_n \) are, as usual, integer numbers. In our case \( \delta \) and \( \gamma \) are small values (at \( V \to \infty, \delta \to 0 \)), and, thus, we have for eigenvalues \( \lambda_n \) some small corrections to integer numbers \( n \). Then Eq. (45) is reduced into

\[
\left( \delta + \frac{\gamma \lambda^2}{1 - \gamma N} \right) \sum_{n=1}^{N} \frac{1}{n^2 - \lambda^2} = 1.
\] (46)

Eq. (46) defines the energies of quasi-photons.

Let us consider the following cases:

I. \( \gamma N \ll 1 \). Approximate solutions to Eq. (46) can be looked for as the expansion \( \lambda_n^2 - n^2 \) in the small parameters \( \delta, \gamma \). We leave only terms in order of \( \delta, \delta^2, \gamma, \gamma \delta \) and neglect the higher powers \( \delta^3, \gamma^2 \gamma \delta^2 \) and so on. As a result, we put

\[
\lambda_n^2 = n^2 - a_n \delta - b_n \delta^2 - c_n \gamma - d_n \gamma \delta.
\] (47)

After replacing expression (47) into Eq. (46), with the accepted approximation, we obtain the coefficients \( a_n, b_n, c_n, d_n \) as follows:

\[
a_n = 1, \quad b_n = \rho_n, \quad d_n = 2n^2 \rho_n - 1, \quad c_n = n^2,
\]
\[ \rho_n = \sum_{m=1, m \neq n}^{N} \frac{1}{m^2 - n^2} \quad (n = 1, 2, \ldots, N). \quad (48) \]

The sum in Eq. (48) includes all terms besides the one with \( m = n \).

Formulas (47), (48) define an approximate, in the small parameters \( \delta, \gamma \) eigenvalues \( \lambda_n^2 \) which are the solutions of Eqs. (42), (45), (46). The parameter \( \gamma \) characterizing the internal structure of a scalar particle (electromagnetic polarizabilities) occurring eigenvalues \( \lambda_n^2 \).

Now we consider the particular cases of external quantized electromagnetic fields:

1) Two monochromatic electromagnetic waves, \( N = 2 \).

In this case \( n = 1, 2 \), and from Eq. (48) we find eigenvalues

\[ \lambda_1^2 = 1 - \delta - \frac{1}{3} \delta^2 - \gamma + \frac{1}{3} \gamma \delta, \]
\[ \lambda_2^2 = 4 - \delta + \frac{1}{3} \delta^2 - 4 \gamma + \frac{11}{3} \gamma \delta. \quad (49) \]

2) Three monochromatic electromagnetic waves, \( N = 3 \).

The eigenvalues \( \lambda_n^2 \) \( (n = 1, 2, 3) \), found from Eq. (48), are given by

\[ \lambda_1^2 = 1 - \delta - \frac{11}{24} \delta^2 - \gamma + \frac{1}{12} \gamma \delta, \]
\[ \lambda_2^2 = 4 - \delta + \frac{2}{15} \delta^2 - 4 \gamma + \frac{31}{15} \gamma \delta, \]
\[ \lambda_3^2 = 9 - \delta + \frac{13}{40} \delta^2 - 9 \gamma + \frac{137}{20} \gamma \delta. \quad (50) \]

II. \( \gamma N \gg 1 \). In this case using the known sum [13]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 - \lambda^2} = \frac{1}{2\lambda} \left( \frac{1}{\lambda} - \pi \cot \pi \lambda \right), \quad (51) \]

Eq. (45) reduces to (see [4])

\[ (2\lambda^2 - \delta) \sin \pi \lambda + \delta \pi \lambda \cos \pi \lambda = 0. \quad (52) \]

With the accuracy in order of \( \delta^2 \) the solution to Eq. (52) is given by

\[ \lambda_n = n - \frac{\delta}{2n} - \frac{\delta^2}{2n^3}. \quad (53) \]
In this case, there is not an effect of composite structure of a scalar particle and we arrive at the electromagnetic interaction of a scalar point-like particle. Therefore, this case is not of our interest.

Now we find the matrix $P$ which diagonalizes simultaneously matrices $A^{-1}$ and $B$ in accordance with formulas (40), (41). To find the eigenvectors $P_n$ of the matrix $AB$ ($ABP_n = \lambda^2_n P_n$), let us define the adjoint matrix $P$ which is the solution of the matrix equation

$$
\left( B - \lambda^2 A^{-1} \right) P = \det \left( B - \lambda^2 A^{-1} \right).
$$

Matrix elements of $P$ are minors of the det $(B - \lambda^2 A^{-1})$ [12]. If $\lambda^2$ are roots of the characteristic equation (42) ($\lambda = \lambda_n$) then columns $(P_n, n = 1, 2, ..., N)$ of the matrix $P$ are eigenvectors, so that $(B - \lambda^2_n A^{-1}) P_n = 0$. The calculation of minors mentioned gives matrix elements of the matrix $P$ as follows:

$$
P_{nm} = \frac{v_m}{n^2 - m^2},
$$

where coefficients $v_m$ can be found by the constraints

$$
P^T A^{-1} P = 1, \quad P^T B P = \| \lambda_n \| \delta_{nm},
$$

$$
\left( \begin{array}{ccc}
\lambda_1^2 & 0 & 0 \\
0 & \lambda_2^2 & 0 \\
0 & 0 & \lambda_n^2 \\
\end{array} \right),
$$

As the matrix $A^{-1}$ is positive-defined, the conditions (56) can be satisfied [12]. The first equality $P^T A^{-1} P = 1$ leads to

$$
v_n v_k \left( \Sigma_{nk} + \frac{\gamma}{1 - \gamma N} \Sigma_n \Sigma_k \right) = \delta_{nk},
$$

$$
\Sigma_{nk} = \sum_{m=1}^{N} \frac{1}{(m^2 - \lambda_n^2)(m^2 - \lambda_k^2)}, \quad \Sigma_n = \sum_{m=1}^{N} \frac{1}{m^2 - \lambda_n^2}.
$$

It easy to verify that Eq. (57) at $n \neq k$ satisfied because in accordance with Eq. (46) $\Sigma_n = (1 - \gamma N) / [\delta (1 - \gamma N) + \gamma \lambda_n^2]$. At $n = k$ Eq. (57) allows us to find $v_n$ as follows

$$
v_m = \sqrt{\frac{1 - \gamma N}{(1 - \gamma N) \Sigma_{mm} + \gamma \Sigma_m^2}}.
$$
Practically the second term in Eq. (57) is smaller than the first term due to the small factor $\gamma$. Therefore, at the approximate calculations, taking into consideration small parameters $\beta$ and $\gamma$, we can neglect the term $\gamma \sum_{m}^{2}$ in Eq. (59).

The second equation in (56) is also satisfied taking into account Eqs. (55), (59). So, the transformations (40), with the matrix $P$ (55) and coefficients (59), diagonalize the matrices $A$ and $B$ simultaneously according to Eq. (56).

After the transformations (40) with the matrix (55), and taking into consideration Eq. (43), equation (39) becomes

$$\left\{ \sum_{n=1}^{N} \left( \frac{\partial^{2}}{\partial y_{ns}^{2}} - \lambda_{n}^{2} y_{ns}^{2} \right) + \frac{\alpha_{s}^{2} \sigma_{N}}{1 - \delta \sigma_{N}} + \varepsilon_{s} \right\} f_{s}(y) = 0. \tag{60}$$

The variables $y_{ns}$ in Eq. (60) are separated and the finite normalized solution at $y_{ks} \to \infty$ is given by

$$f_{s}(y) = \prod_{k=1}^{N} f_{ks}(y), \quad f_{ks}(y) = \sqrt{\frac{\sqrt{\lambda_{k}}}{\sqrt{\pi} 2^{n_{k}} n_{k}!}} H_{n_{k}} \left( \sqrt{\lambda_{k}} y_{ks} \right) \exp \left( -\frac{\lambda_{k} y_{ks}^{2}}{2} \right), \tag{61}$$

where $H_{n_{k}} \left( \sqrt{\lambda_{k}} y_{ks} \right)$ are the Hermite polynomials [12]. The eigenvalues $\varepsilon_{s}$ obey the equality

$$\varepsilon_{s} = \sum_{k=1}^{N} \varepsilon_{ks} - \frac{\alpha_{s}^{2} \sigma_{N}}{1 - \delta \sigma_{N}}, \quad \varepsilon_{ks} = \left( 2 n_{k}^{(s)} + 1 \right) \lambda_{k}, \quad n_{k}^{(s)} = 1, 2, .... \tag{62}$$

Eq. (62) gives the energy of quasi-photons. The quantum number $n_{k}^{(s)}$ is an integer and means the number of quasi-photons with the polarization $s$ corresponding to the $k$-th mode.

5 Momentum and mass of a particle in plane quantized electromagnetic wave

Using the relation $M = \varepsilon_{1} + \varepsilon_{2}$ and Eq. (30), we find the momentum squared of the particle-photon system

$$q^{2} = (q \kappa) (\varepsilon_{1} + \varepsilon_{2}) - m^{2}. \tag{63}$$
This is the dispersion relation for the momentum \( q_\mu \) of a particle-photon system. Let us introduce the momentum of a particle as follows:

\[
p_{\mu} = q_\mu - \frac{1}{2} \kappa_\mu (\varepsilon_1 + \varepsilon_2).
\]  

(64)

Using equation (63) and equality \( \kappa^2 = 0 \) it is easy to see that \( p_\mu \) obeys the relation for the momentum of a free particle

\[
p^2 = -m^2.
\]  

(65)

From Eqs. (62), (64) we come to the expression for the momentum \( q_\mu \) of a particle-photon system

\[
q_\mu = p_\mu + \kappa_\mu \left[ \sum_{k=1}^{N} \lambda_k \left( n_k^{(1)} + n_k^{(2)} + 1 \right) - \frac{(\alpha_1^2 + \alpha_2^2) \sigma_N}{2 (1 - \delta \sigma_N)} \right].
\]  

(66)

According to Eqs. (47), (48) eigenvalues \( \lambda_k^2 \) are given by

\[
\lambda_k^2 = k^2 - \delta - \rho_k \delta^2 - k^2 \gamma - \left( 2 k^2 \rho_k - 1 \right) \gamma \delta.
\]  

(67)

From Eq. (67) we derive approximate (at small \( \delta, \gamma \) up to the orders \( O(\delta^2), O(\gamma \delta) \)) values \( \lambda_k \):

\[
\lambda_k = k - \epsilon_k,
\]

\[
\epsilon_k = \epsilon_k = \frac{1}{2k} \delta + \frac{1 + 4k^2 \rho_k}{8k^3} \delta^2 + \frac{k}{2} \gamma + \frac{4k^2 \rho_k - 1}{4k} \gamma \delta.
\]  

(68)

Let us introduce the quasi-momentum of a scalar particle (see Eq. (26)) \( P_\mu \) which is a difference between a momentum of a particle-photon system (total momentum) \( q_\mu \), a momentum of photons in the quantized electromagnetic wave

\[
P_\mu = q_\mu - \kappa_\mu \sum_{k=1}^{N} k \left( n_k^{(1)} + n_k^{(2)} + 1 \right),
\]  

(69)

where \( k_\mu = k \kappa_\mu \) is the momentum of a photon. Then from Eqs. (29), (30), (66), (68), (69) we arrive at

\[
P^2 = -m^2 - b^2 \sum_{k=1}^{N} \frac{1}{k} \left( n_k^{(1)} + n_k^{(2)} + 1 \right) - \gamma (P \kappa) \sum_{k=1}^{N} k \left( n_k^{(1)} + n_k^{(2)} + 1 \right)
\]

\[
- b^4 \sum_{k=1}^{N} \left[ \frac{4k^2 \rho_k + 1}{4 (P \kappa) k^3} + \frac{W (P \kappa) (4k^2 \rho_k - 1)}{2 e^2 k} \right] \left( n_k^{(1)} + n_k^{(2)} + 1 \right)
\]

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\[- \frac{b^2 \left[ (P e_1)^2 + (P e_2)^2 \right] \sigma_N}{(P \kappa) - b^2 \sigma_N}, \quad (70)\]

where \( W = (\alpha + \beta)/m \). We took into account that according to Eq. (69) \((q \kappa) = (P \kappa)\). Equation (70) represents the dispersion relation for a quasi-momentum \( P_\mu \) of a scalar composite particle. The second and third terms in the right side of Eq. (70) contribute to the mass of a composite particle. So, the effective mass of a scalar particle is given by

\[
m^*_s = m^2 + \frac{e^2}{\kappa_0 V} \sum_{k=1}^{N} \frac{1}{k} \left( n_k^{(1)} + n_k^{(2)} + 1 \right) + \frac{W (P \kappa)^2}{\kappa_0 V} \sum_{k=1}^{N} k \left( n_k^{(1)} + n_k^{(2)} + 1 \right). \quad (71)\]

For the vacuum state \( n_k^{(1)} = n_k^{(2)} = 0 \), and at \( N \to \infty \), we have the divergence sums in Eq. (71) which approach to zero when the volume of the quantization \( V \to \infty \). In the classical case the average number of photons \( \langle n^{(s)} \rangle \to \infty \) and the volume \( V \to \infty \) but the ratio \( \langle n \rangle / V \), where \( \langle n \rangle = \langle n^{(s)} \rangle \) remains the constant. Within this limitation, creation and annihilation operators are replaced by the c-numbers in accordance with relations

\[
c^-_{ks} = \left( n_k^{(s)} \right)^{1/2} \exp(i \theta_{ks}), \quad c^+_{ks} = \left( n_k^{(s)} \right)^{1/2} \exp(-i \theta_{ks}), \quad (72)\]

where \( \theta_{ks} \) is the phase of the classical electromagnetic wave. Then Eq. (27) represents the Fourier transformation of the classical electromagnetic plane wave. It is easy to verify that Eq. (71) agrees with Eq. (25) at classical limit. Two last terms in equation (70) at \( V \to \infty \) and \( \langle n_k \rangle / V = \text{const} \) trend to zero and do not contribute to the mass in the case of classical electromagnetic waves. These terms do not vanish only for quantized waves at the constant volume \( V \). Thus, for the classical case, the quasi-momentum of a scalar composite particle obeys the relation \( P^2 + m^*_s = 0 \). The term in Eq. (70) containing the parameter \( \gamma \) (electromagnetic polarizabilities) contributes to the mass of a composite scalar particle. At classical limit this additional term does not vanish.
6 Monochromatic quantized electromagnetic wave

Let us consider the particular case when the main contribution to the sum of electromagnetic potential (27) comes from the definite number \( n \), such that the four-momentum of all photons is \( k_\mu = n\kappa_\mu \). Then Eq. (27) becomes

\[
A_\mu = \sum_{s=1}^{2} \frac{e_{s\mu}}{\sqrt{2k_0 V}} \left[ c_s^- e^{i(kx)} + c_s^+ e^{-i(kx)} \right].
\]  

(73)

and Eq. (32) is transformed to

\[
\left\{ \left[ 1 - W(qk)a^2 \right] \frac{\partial^2}{\partial \xi_s^2} - \left[ 1 - \frac{e^2a^2}{(qk)} \right] \xi_s^2 \right. \\
\left. - 2e\frac{(qa_s)}{(qk)} \xi_s + \varepsilon_s \right\} f_s(\xi) = 0,
\]

(74)

where \( \chi(\xi) = f_1(\xi)f_2(\xi) \), and for convenience we use new notations \( a_{s\mu} = e_{s\mu}/\sqrt{k_0 V} \), \( a_{s\mu}^2 = a^2 \), and eigenvalues \( \varepsilon_s \) obey the following equation:

\[
\frac{q^2 + m^2}{(qk)} = \varepsilon_1 + \varepsilon_2.
\]

(75)

After introducing the variables

\[
\zeta_s = \frac{\tau(\xi_s + \sigma_s)}{c^{1/4}}, \quad \tau^4 = 1 - \frac{e^2a^2}{(qk)}, \quad \sigma_s = e\frac{(qa_s)}{\tau^4(qk)}, \quad c = 1 - W(qk)a^2,
\]

(76)

Eq. (74) is rewritten as

\[
\left( \frac{\partial^2}{\partial \zeta_s^2} - \zeta_s^2 + \nu_s \right) f_s(\zeta) = 0,
\]

(77)

where

\[
\nu_s = \frac{\tau^4\sigma_s^2 + \varepsilon_s}{\tau^2\sqrt{c}}.
\]

(78)

The normalized and finite at \( \zeta_s \to \infty \) solution to Eq. (77) is given by

\[
f_s(\zeta) = \frac{1}{\pi^{1/4}2^{n/2}(n!)^{1/2}} H_n(\zeta) \exp \left( -\frac{\zeta_s^2}{2} \right)
\]

(79)
with the condition \( \nu_s = 2n_s + 1 \), and \( n_s = 1, 2, \ldots \), where \( n_s \) means the number of photons with polarization \( s \). This equality leads with the help of Eqs. (75), (78) to the total squared momentum of a particle-photon system

\[
q^2 = -m^2 + 2\tau^2(qk) \left[ \sqrt{c(n_1 + n_2 + 1)} - \frac{\tau^2(\sigma_1^2 + \sigma_2^2)}{2} \right].
\] (80)

From Eq. (80) we find the four-vector of a total momentum

\[
q_\mu = p_\mu + \tau^2k_\mu \left[ \sqrt{1 - W(pk)a^2(n_1 + n_2 + 1)} - \frac{\tau^2(\sigma_1^2 + \sigma_2^2)}{2} \right],
\] (81)

where \( p_\mu^2 = -m^2 \), \( (pk) = (qk) \) and \( p_\mu \) is a momentum of a free scalar particle. Taking into consideration the smallness of the parameter \( W = (\alpha + \beta)/m \), we can write \( \sqrt{1 - W(pk)a^2} \simeq 1 - W(pk)a^2/2 \). As a result, Eq. (81) takes the form

\[
q_\mu = q_\mu^{pl} - \tau^2k_\mu \frac{W(pk)a^2}{2}(n_1 + n_2 + 1),
\]

\[
q_\mu^{pl} = p_\mu + \tau^2k_\mu \left[ (n_1 + n_2 + 1) - \frac{\tau^2(\sigma_1^2 + \sigma_2^2)}{2} \right].
\] (82)

The value \( q_\mu^{pl} \) is the total momentum of a system of point-like scalar particle interacting with \( n_s \) \((s = 1, 2)\) photons. It follows from Eq. (82) that electromagnetic polarizabilities contribute to the total momentum.

With the help of Eqs. (7), (10), (17) we write out the final solution to Eq. (2) for composite scalar particles in the external quantized electromagnetic wave:

\[
\varphi(x, \xi) = \left( \pi^{2n_1 + n_2}n_1!n_2! \right)^{-1/2} \exp \left\{ i \left[ (qk) + \frac{1}{2}(kx) \sum_{s=1}^{2} \left( \frac{\partial^2}{\partial \xi_s^2} - \xi_s^2 \right) \right] \right\}
\]

\[ \times H_{n_1}(\zeta_1) H_{n_2}(\zeta_2) \exp \left[ -\frac{1}{2} \left( \zeta_1^2 + \zeta_2^2 \right) \right]. \] (83)

7 Coherent states

For simplicity, let us consider the case of a linear polarization of an electromagnetic wave. Then the polarization index \( s = 1 \) and solution (83) and total momentum of a system (82) become

\[
\varphi_n(x, \xi) = \left( \pi^{1/2}2^n n! \right)^{-1/2} \exp \left\{ i \left[ (qk) + \frac{1}{2}(kx) \left( \frac{\partial^2}{\partial \xi^2} - \xi^2 \right) \right] \right\}
\]
\[ H_n(\zeta) \exp\left(-\frac{1}{2}\zeta^2\right), \quad (84) \]

where we omitted the polarization index. The variable \( n \) in Eqs. (84), (85) is the number of linearly polarized photons.

The wavefunction (84) describes a charged scalar composite particle with the arbitrary phase of the wave interacting with \( n \) external photons. Because in wavefunction (84) we have two different arguments \( \xi \) and \( \zeta \) let us make the expansion [15]

\[ H_n(\zeta) \exp\left(-\frac{1}{2}\zeta^2\right) = \sum_{m=1}^{\infty} \beta_{nm} H_m(\xi) \exp\left(-\frac{1}{2}\xi^2\right), \quad (86) \]

where the coefficients \( \beta_{nm} \) can be calculated from the relationship

\[ \beta_{nm} = \frac{1}{\sqrt{\pi}2^m m!} \int_{-\infty}^{\infty} d\xi H_n(\xi) H_m(\xi) \exp\left(-\frac{\xi^2 + \bar{\zeta}^2}{2}\right), \quad (87) \]

where \( \bar{\zeta} = \tau(\xi + \sigma)/c^{1/4} \). Using the property of the oscillator wavefunction [16]

\[ \left(\xi^2 - \frac{\partial^2}{\partial\xi^2}\right) H_n(\xi) \exp\left(-\frac{1}{2}\xi^2\right) = (2n + 1) H_n(\xi) \exp\left(-\frac{1}{2}\xi^2\right), \quad (88) \]

we derive taking into account (86) the wavefunction (84):

\[ \varphi_n(x, \xi) = N_0 \sum_{m=1}^{\infty} \exp\left\{ i \left[ (qk) - \frac{1}{2} (kx) (2m + 1) \right] \right\} \beta_{nm} H_m(\xi) \exp\left(-\frac{1}{2}\xi^2\right), \quad (89) \]

where \( N_0 = (\pi^{1/2}2^m n!)^{-1/2} \). The result of interacting a scalar particle with a quantized electromagnetic field is that the solution (89) represents the superposition of the oscillator wavefunctions with all quantum numbers \( m \).

For noninteracting particle, the charge \( e = 0 \), electromagnetic polarizabilities \( \alpha = \beta = 0 \) \( (W = 0) \), and parameters take the values \( \tau = 1, \sigma = 0, c = 1 \). As a result \( \zeta = \xi, \beta_{nm} = \delta_{nm} \) and wavefunction (89) has only one term with the definite \( n \). To calculate the sum in equation (89), we use the relation [16]

\[ \sum_{m=0}^{\infty} \frac{z^m}{2^m m!} H_m(x) H_m(y) = \frac{1}{\sqrt{1 - z^2}} \exp\left[\frac{2xyz - (x^2 + y^2)z^2}{1 - z^2}\right], \quad (90) \]
Replacing expression (87) into Eq. (89), and using Eq. (90), we arrive at

\[ \varphi_n(x, \xi) = N_0 \sqrt{\frac{z}{\pi (1 - z^2)}} \exp \left[ i(qk) - \frac{\xi^2 (z^2 + 1)}{2(1 - z^2)} \right] \]

\[ \times \int_{-\infty}^{\infty} d\xi H_s(\xi) \exp \left[ -\frac{\xi^2}{2} - \frac{\xi^2 (z^2 + 1) - 4\xi^2}{2(1 - z^2)} \right], \quad (91) \]

where \( z = \exp \left[ -i (kx) \right], \xi = \tau(\xi + \sigma)/c^{1/4} \). With the help of the generating function \[16\]

\[ \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x) = \exp \left( -t^2 + 2tx \right), \quad (92) \]

we can evaluate the integral in Eq. (91), and we find (see also \[15\])

\[ \varphi_n(x, \xi) = N_0 \sqrt{\frac{2z}{\pi (1 - z^2)}} \exp \left[ \frac{\tau}{c(1 + z^2)} \right] H_n \left( \frac{\tau \sigma (1 + z^2) + 2z\xi}{c(1 + z^2)^2 - \tau^4 (1 - z^2)^2 1/2} \right) \]

\[ \times \exp \left\{ i(qk) - \frac{\xi^2 \left[ (z^2 - 1)c^{1/2} - \tau^2 (z^2 + 1) \right] - 4\xi^2 \sigma z - \tau^2 \sigma^2 (z^2 + 1)}{2 \left[ (1 + z^2)c^{1/2} + \tau^2 (1 - z^2) \right]} \right\} \quad (93) \]

Wavefunction (93) corresponds to the charged scalar composite particle interacting with \( n \) external photons possessing four-momentum \( k_\mu \).

As Eq. (2) is a linear equation, the combination of solutions (93) with different filling numbers \( n \) and phases is also the solution of Eq. (2). The distribution of phases and quantum numbers \( n \) can be arbitrary. The very important case is the coherent wave when the state reduces the uncertainty relations for coordinates and momentum to a minima. The coherent state is described by the Poisson distribution of the filling numbers \( n \).

The wave packet for the Poisson distribution of the photon numbers \( n \) is given by the sum \[17\]

\[ \varphi_n(x, \xi) = \exp \left( -\frac{1}{2} \langle n \rangle \right) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \langle n \rangle^{n/2} \varphi_n(x, \xi), \quad (94) \]

where \( \langle n \rangle \) is the average photon number, and we put the phase to be zero \[15\].

Replacing Eq. (93) into Eq. (94), taking into consideration Eq. (85) and using the properties of Hermite polynomials, we arrive at the wavefunction

20
for coherent state with the average photon number $\langle n \rangle$ to be present in the volume $V$:

$$
\varphi_{(n)}(x, \xi) = \frac{\pi^{-1/4} c^{1/4} \sqrt{2} z}{\sqrt{(1 + z^2) c + \tau^2 (1 - z^2)}} \exp \left\{ -\frac{\langle n \rangle}{2} \right\}
+ i \left[ (px) + \frac{\tau^2}{2} \left( \sqrt{c} - \tau^2 \sigma^2 \right) (kx) \right] + 2^{-1} \left[ \sqrt{c} (1 + z^2) + \tau^2 (1 - z^2) \right]^{-1}
\times \left\{ \xi^2 \left[ \sqrt{c} (z^2 - 1) - \tau^2 (z^2 + 1) \right] - 4 \xi \tau^2 \sigma z - \tau^2 \sigma^2 (z^2 + 1) \right\}
- \frac{\langle n \rangle}{z^2 \tau^2 \sqrt{c}} \left[ \sqrt{c} (1 + z^2) - \tau^2 (1 - z^2) \right] + \frac{2 \sqrt{2}}{z^2 \tau^2 \sqrt{c}} \sqrt{\langle n \rangle} \tau c^{1/4} \left[ \sigma (1 + z^2) + 2 z \xi \right] \right\}.
$$

(95)

The variables $\tau$, $a$, and $\sigma$ contain the volume of the quantization of electromagnetic waves $V$. At the particular case when electromagnetic polarizabilities $\alpha = \beta = 0$ ($W = 0$, $c = 1$), we arrive at the case of pointlike scalar particles (see [15]).

Consider the classical case when the number of photons $n$ and the volume $V$ go to infinity but the photon density $\langle n \rangle/V$ remains constant. The coherent states describe the wave packet which is localized at the point $\xi_0 = \sqrt{2 \langle n \rangle}$ [17, 18], where wavefunction (95) does not vanish. Taking into account that $a_\mu = e_\mu / \sqrt{k_0 V}$ is a small parameter and expanding the variables $\tau^2 \simeq 1 - e^2 a^2 / 2(pk)$, $\sqrt{c} \simeq 1 - W(pk) a^2 / 2$, we arrive at the limit of the expression (95) ($\langle n \rangle/V = \text{const}$):

$$
\varphi(x) = \lim_{\langle n \rangle \to \infty} \varphi_{(n)}(x, \xi) = \pi^{-1/4} \exp \left\{ -\frac{(\xi - \sqrt{2 \langle n \rangle})^2}{2} \right\}
+ i \left[ (px) + \frac{e (p a_1)}{(pk)} \sin (kx) - \frac{e^2 a_1^2}{4 (pk)} \left( (kx) + \frac{1}{2} \sin 2(kx) \right) \right]
- \frac{W a_1^2 (pk)}{4} \left[ (kx) - \frac{1}{2} \sin 2(kx) \right],
$$

(96)

where $a_{1\mu} = a_\mu \sqrt{2 \langle n \rangle}$. Wavefunction (96) is the product of the oscillator eigenfunction for the ground state [18]

$$
\varphi_{\text{osc}} = \pi^{-1/4} \exp \left\{ -\frac{(\xi - \sqrt{2 \langle n \rangle})^2}{2} \right\},
$$

(97)
and the solution to Eq. (2) in the field of the classical linearly polarized electromagnetic wave [1]:

\[
\varphi_{cl}(x) = \exp \left[ ipx + \frac{e(pa_1)}{pk} \sin(kx) \right. \\
- \left. \frac{e^2 a_1^2}{4(pk)} \left( kx + \frac{1}{2} \sin(2kx) \right) - \frac{W a_1^2(pk)}{4} \left( kx - \frac{1}{2} \sin(2kx) \right) \right],
\]

(98)

where the average energy density of the coherent electromagnetic wave is \( k_0^2 a_1^2 / 2 = k_0 \langle n \rangle / V \).

8 Conclusion

The exact solutions of the equation for a composite scalar particle (pion or kaon) in the field of a plane quantized electromagnetic wave have been obtained. For the case of the monochromatic wave, we have studied the coherent states with the Poisson distribution of the photon number. When the filling number approaches to infinity with the constant density of photons, the wavefunction is converted into the solution for a composite particle in the classical electromagnetic wave. The solutions obtained can be used for the investigation of the behavior of pions in strong electromagnetic fields where nonperturbative effects are essential. The case of quantized electromagnetic fields is important when the photon number is not large enough and there is an interaction of a particle with separate photons. The found wavefunction of a composite scalar particle can be applied for solving complex problems when the semiclassical approach does not work.

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