On the second order finite-difference scheme for the solution of the system of one-dimensional equations of hemodynamics

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Abstract. The paper is devoted to the analysis and optimization of the second order Lax – Wendroff finite-difference scheme with artificial viscosity for the system of 1D equations of hemodynamics. The scheme is constructed for the conservative form of this system. For the elimination of numerical dispersion the artificial viscosity is used. The value of the viscosity is obtained by the approach, based on simultaneous optimization of the dispersive and dissipative characteristics.

1. Introduction
In the last years the mathematical modeling of the blood flow plays an important role in physiology and medicine [1]. With the applications of the models, the properties of blood flow in vessels with stenosis, aneurysms and prostheses may be predicted [2]. For the modeling of the blood flow on a vessel network, the one-dimensional (1D) equations of hemodynamics may be used. Every vessel of such network is represented as a graph edge and the blood dynamics is described by 1D equations, obtained by the averaging of the hydrodynamic equations on the vessel cross-section [3].

The system of 1D hemodynamics is formed by a system of two differential equations, which represents the conservation of mass and momentum, and by an algebraic equation, the so-called equation-of-state (EOS), which represent the dependency of pressure on the area of the vessel cross-section. This system is nonlinear, so in the general case it must be solved only by numerical methods. There are many papers, devoted to the numerical solution of hemodynamic equations. In [3] the Taylor – Galerkin finite-element method is used for the numerical solution. Boileau et al in [4] compare the results obtained by discontinuous Galerkin and locally conservative Galerkin finite-element methods, finite-volume method, trapezium rule and MacCormack finite-difference method. Results show a good agreement among all schemes and their abilities to capture the all features of the blood flow. In [5] the time implicit solver for computations of blood flow in arterial networks is proposed. In [6] a solver based on upwind discretization and Roe type scheme is proposed. Muller and Toro in [7] propose a fifth order well-balanced scheme for the solution of Riemann problems for hemodynamic equations. Audebert et al in [8] propose a scheme, based on kinetic equations for the distribution functions, which may be applied for both venous and arterial flows.
The presented paper is devoted to the construction and optimization of Lax–Wendroff (LW) finite-difference scheme for the system of 1D hemodynamical equations. The scheme is constructed for the conservative form of this system. For the elimination of numerical dispersion the artificial viscosity is used. The value of the viscosity is obtained by the approach, based on optimization of dispersive and dissipative characteristics, proposed in [9].

The paper has a following structure. In Section 2 the LW scheme is constructed and its stability conditions are formulated. In Section 3 the problem of optimization is stated and the results of its solution are discussed. Some concluding remarks are made in Section 4.

2. Second order LW scheme

Let the blood is considered as a Newtonian incompressible fluid. For this model the equations of hemodynamics are obtained after the averaging of the Navier–Stokes system and are written as [3]:

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \frac{Q^2}{A} \right) + \frac{A \frac{\partial p}{\partial z} + K \frac{Q}{A} = 0, \right.
\]

(1)

where \(A = \frac{A}{A} (t, z)\) is the area of vessel cross-section, \(Q = Q(t, z)\) is a flow rate from the cross-section, \(t\) is a time, \(z\) is a space variable (in cylindrical system), \(p = p(t, z)\) is a pressure, \(\rho\) is a constant density, \(\alpha\) is a Bouissinesq coefficient (\(\alpha = 1.1\) for the case of Newtonian fluid), \(K\) is a friction coefficient.

To the system (1) the EOS is added: \(p = p(A)\). In most of works the following EOS is used [3, 4]:

\[
p - p_{\text{ext}} = p_0 + \frac{\beta}{A_0} (\sqrt{A} - \sqrt{A_0}), \quad \beta = \frac{4}{3} \sqrt{\pi} E h,
\]

(2)

where \(p_{\text{ext}}\) is the external pressure, \(p_0\) and \(A_0\) are the diastolic pressure and area of cross-section, \(E\) is the elastic modulus, \(h\) is a vessel wall thickness.

On the first stage of the testing of numerical schemes for system (1) the case of inviscid fluid (corresponds to \(\alpha = 1\)) is constructed [4, 5, 8], so it it proposed that \(K = 0\). For this situation system (1) is rewritten in the so-called quasilinear form:

\[
\frac{\partial U}{\partial t} + \mathbf{H}(U) \frac{\partial U}{\partial z} = 0,
\]

(3)

where \(U = (A, Q)^T\) and matrix \(\mathbf{H}\) is represented as:

\[
\mathbf{H} = \begin{pmatrix}
0 & 1 \\
\frac{A \frac{\partial p}{\partial A} - \alpha \left( \frac{Q}{A} \right)^2}{2 \alpha A} & \frac{\alpha Q}{A}
\end{pmatrix}
\]

This matrix has a real eigenvalues and linearly independent eigenvectors, so the system (3) may be related to the hyperbolic systems of partial differential equations.

Another form of system (1) is a conservative form:

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial z} = 0,
\]

(4)

where \(F = (Q, Q^2/A + \varphi(A))^T, \varphi(A) = \frac{1}{2} \int_A^A \frac{A}{A_0} (\tau) d\tau\).

One of the widely used schemes for the solution of hyperbolic system is a LW scheme [10]. It has a second order of accuracy on time and space variables. The scheme is based on two
Figure 1. Numerical solution of the test problem by LW scheme stages. On the first stage the computations of the values in nodes with fraction indexes $t_{s+1/2}$

is performed:

$$U_{j+1/2}^{s+1} = \frac{1}{2}(U_j^s + U_{j+1}^s) - \frac{\Delta t}{2h}(F(U_{j+1}^s) - F(U_j^s)), \quad (5)$$

where $\Delta t$ is a time step and $h$ is a step on space variable. On the second stage the computations

in nodes $t_s$ are performed:

$$U_j^{s+1} = U_j^s - \Delta t \left( \frac{F(U_{j+1/2}^{s+1}) - F(U_{j-1/2}^{s+1})}{h} \right). \quad (6)$$

Let us consider the simple test problem for a system (1), presented in the form (4). Let

the following initial conditions are stated: $p(0,z) = p_0$, $Q(0,z) = Q_0$. The following boundary

conditions are stated on the left and right boundaries (on the inlet and outlet of the blood vessel):

$$p(t,0) = \begin{cases} p_0 + P, & t \leq t_1, \\ p_0, & t > t_1. \end{cases} \quad Q(t,L) = Q_0.$$

Let in (2) $p_{ext} = 0$ and $p(A)$ is linearized: $p = \gamma A + \delta$. We perform all calculations in

dimensionless variables and the following constants are used: $Q_0 = 1$, $p_0 = 1000$, $P = 4000$, $t_1 = 0.02$, $L = 20$, $\gamma = 39000$, $\delta = -38500$. The calculations are performed at time interval $[0, 0.2]$. The space grid consists of 1000 nodes and time grid is formed by $10^4$ nodes. At fig. 1

the plot of the numerical solution is presented. It must be noted, that in despite of stability

condition realization, the fictitious numerical oscillations takes place in the numerical solution.

This parasitic effect is called numerical dispersion. For the elimination of this effect, the artificial

viscosity may be used. In the right part of (6) the additional term is added:

$$U_j^{s+1} = U_j^s - \frac{\Delta t}{h} \left( F(U_{j+1/2}^{s+1}) - F(U_{j-1/2}^{s+1}) \right) + \Delta t \nu (U_{j+1}^s - 2U_j^s + U_{j-1}^s), \quad (7)$$
where \( \nu \) is an artificial viscosity coefficient. Stability condition of the LW scheme with artificial viscosity is written as:

\[
\frac{\Delta t}{h} \leq \sqrt{\frac{1 - 2\nu}{Q^2 + 4\frac{df_p}{dA}}} ,
\]

so \( \nu \in [0, 1/2] \). The following problem exists for the scheme (5)–(7) — how to choose the value of \( \nu \)? The answer to this question may be done with an approach, proposed in [9].

3. Optimization of dispersive and dissipative characteristics of LW scheme

Let us consider a system (3) in dimensionless form and perform its linearization near some stationary solution. After this operation the following linear system is obtained:

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial z} = 0 ,
\]

where the matrix \( A \) is represented as:

\[
A = \begin{pmatrix} 0 & 1 \\ \chi A_0 - Q^2 \frac{\rho}{A_0^2} & 2Q_0 \frac{\rho}{A_0} \end{pmatrix} ,
\]

where \( \chi = A_{max} \frac{\delta}{(\rho U_{max})} \), where \( A_{max} \) and \( U_{max} \) are the maximal vessel cross-section and velocity of the blood.

The solution of (8) may be represented in the form of travelling wave:

\[
U(t, z) = U_0 \exp(i(\Omega t - kz)) ,
\]

where \( \Omega \) is the frequency, \( k \) is a wave number.

After the substitution of (9) into (8), the following system for components of \( U_0 \) is obtained:

\[
C(\Omega)U_0 = 0 ,
\]

where \( C = i(\Omega E - kA) \), where \( E \) is a unity matrix. System (10) has a non-zero solution only in the case, when:

\[
\det C(\Omega) = 0 .
\]

Eq. (11) is considered as a dispersive relation for values of frequency \( \Omega_{1,2} \). The roots of (11) are written as \( \Omega_{1,2} = \zeta_{1,2} k \), where \( \zeta_{1,2} \in \mathbb{R} \). So, as it can be seen, the group velocities of the wave (9) are independent on \( \zeta \), so the dispersion of the solution of (8) can’t take place.

If we construct the LW scheme (5),(7) for the system (8), we may represent it in following form:

\[
U_j^{s+1} = BU_{j-1}^s + DU_j^s + FU_{j+1}^s ,
\]

where \( \dim B = \dim D = \dim F = 2 \times 2 \).

The solution of (12) may be represented by a discrete analogues of (9):

\[
U_j^s = U_0 \exp(i(\omega s \Delta t - k j h)) = U_0 q^s \exp(-i\xi j) ,
\]

where \( \xi = kh, q = \exp(i\omega \Delta t) \).

After the substitution of (13) into (12), the following linear system is obtained:

\[
(qE - B \exp(i\xi) - D - F \exp(-i\xi))U_0 = 0 .
\]

The non-zero solution of (14) take place only if the determinant of its matrix is equal to zero. So this condition is considered as an equation for finding of \( q \). After the obtaining of the roots
A) B)

Figure 2. The plots of the dispersive (A) and dissipative (B) characteristics of the LW scheme of this equation, the real parts of frequencies $\omega_i$ are represented as: $\text{Re}(\omega_i) = \psi_i(\gamma, \xi, \nu) = \arctan\left(\frac{\text{Im}(q_i)}{\text{Re}(q_i)}\right) / \Delta t$, $i = 1, 2$, where $\gamma = \Delta t / h$. The expressions of this real parts are nonlinearly depends on $\xi$ (so on $k$), so the dispersion of the discrete waves (13) takes place. But it must be noted, that the effect of the dispersion may be damped by the proper choice of $\nu$. For this purpose the so-called dispersive and dissipative characteristics are introduced [9]. Let us rewrite $\Omega_i$ as a functions of $\gamma$ and $\xi$: $\Omega_i = \zeta_i k = \zeta_i \xi \gamma / \Delta t$. In all operations, realized below, the case of $\Delta t = 1$ is considered.

The dispersive characteristic is introduced as:

$$I(\nu) = \max_{i=1,2} \max(\text{Re}(\omega_i) - \Omega_i),$$

and this function characterizes the maximal deviation of the real parts of $\omega_i$ on $\Omega_i$ in the domain, where the both functions are defined.

Another fictitious effect, which may occur in the numerical solution of hyperbolic equations, is a numerical dissipation — the effect, when the amplitude of the wave (13) is decreased at $s \to +\infty$. As it can be seen, the behavior of amplitude is characterized by $|q|^s$, so the following dissipative characteristic may be introduced:

$$F(\nu) = \max_{i=1,2} \max(|q_i| - 1)).$$

This function characterizes the maximal deviation of the amplitude $|q|$ on its maximal value in the domain, where this function is defined.

At fig. 2 the plots of $I$ and $F$ are presented. As it can be seen, some antagonism takes place in the behavior of these functions — they have the various extremal points. But it must be noted, that function $I(\nu)$ is decreased at the considered interval. At the same time, the fictitious oscillations in numerical solution may be damped by the proper choice of the values of $\nu$. So it is proposed to use the value of $\nu$, which corresponds to the part of the interval, where $F$ is increased — as it can be seen from fig. 2, B, it corresponds to the interval $\nu \in [0, 0.3]$

If we choose $\nu$ as 0.1 for the problem, presented in Section 2, we obtain the solution, presented in fig. 3. As it can be seen, the fictitious oscillations are damped and the correct profile of the pressure takes place at this value of $\nu$. 

![Graph A) B) with plotted functions](image-url)
4. Conclusion
In the presented paper the approach to optimization of the LW scheme for the solutions of the system of 1D hemodynamics is presented. It must be noted, that unlike more complex schemes, LW scheme has a very simple algorithm of its numerical realization, and it has a second order on both independent variables, which provides a good accuracy of the obtained results. The simplicity and good accuracy of LW scheme may be considered as an important factor for its using in computations of blood flow in the complex vessel networks.

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