FDM analysis on pulsatile two-phase MHD rheology of blood in tapered stenotic blood vessels

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Abstract. The pulsatile two-phase flow of blood in a tapered narrow artery with overlapping stenosis in the presence of magnetic field is analysed, modelling the suspension of all the erythrocytes in the inner phase region as Herschel-Bulkley fluid and the cell depleted plasma in the outer-phase region as Newtonian fluid. Explicit finite difference method is employed to obtain the numerical solution to various flow measurements from the modelled initial boundary value problem. It is found that when the magnetic field parameter increases, the velocity increases and wall shear stress and longitudinal impedance to flow increase. The estimates of the increase in the longitudinal impedance to flow for the two-phase H-B fluid model is higher than that of the single-phase H-B fluid model.

1. Introduction
Atherosclerotic plaque, medically termed as stenosis, is a major cardiovascular disease which claims millions of human life across the globe every year and is developed in the lumen of the artery due to the deposit of saturated fats [1, 2]. Once stenosis develops in arteries, it narrows down its passage area and then alters the flow pattern and increases the flow resistance markedly [3, 4]. Several studies on hemodynamics propounded that the rheological properties of blood in constricted arteries play vital role in the diagnosis and treatment of many cardiovascular diseases [5, 6] and this it is important to analyse the blood flow characteristics in constricted arteries.

Several natural phenomena are influenced by the presence of the surrounding magnetic field [7, 8]. Since blood is an electrically conducting fluid, the rheological properties of blood are considerably affected by the external magnetic field [9]. External magnetic field is applied in medical field not only in the treatment of internal wounds, abnormal swelling, clearing blockages, correcting the abnormal flow [10, 11]. Hence, the study on the influence of external magnetic field in the blood flow is useful.

Newtonian fluid is used to model blood when it flows in larger diameter arteries at high shear rate (>10 /s), whereas non-Newtonian fluid models are used to represent blood when it flows through narrow diameter arteries at low shear rates [10, 12]. Chakravarty et al. [13] reported that it is more realistic to model blood flow in narrow diameter arteries at low rate as two-phase fluid model with the suspension of all the erythrocytes in the inner phase region as non-Newtonian fluid and the cell-free plasma in the outer phase region as Newtonian fluid. Since, Herschel-Bulkley fluid is a non-
Newtonian fluid model with yield stress which has several advantages over the other similar fluid models to represent the blood with higher suspension of erythrocytes [14, 15].

In view of the arguments given so far, it is proposed to analyse the two-phase flow of blood in tapered narrow arteries with overlapping stenosis and in the presence of external magnetic field, treating the suspension of all the erythrocytes in the inner-phase region as Herschel-Bulkley (H-B) fluid model and the cell depleted plasma in the outer-phase region as Newtonian fluid.

2. Mathematical formulation

Let us consider a uni-directional, laminar, pulsatile, axially symmetric and fully developed flow of blood (treated as incompressible) in the axial direction through a tapered circular artery with an overlapping mild stenoses in the presence of external transverse magnetic field. Blood is treated as a two-phase fluid model, modelling the suspension of all the erythrocytes in the inner-phase region as Herschel-Bulkley (H-B) fluid and the cell depleted plasma in the outer-phase region as Newtonian fluid. The geometry of segment of the artery with mild overlapping stenoses is depicted in Figure 1. Cylindrical polar coordinate system \((r, \psi, z)\) has been utilized to investigate the flow field.

\[
\begin{cases}
(R, R_j)(z, t) = \left[ \left( \frac{m z + r_0}{R} \right)^2 - \frac{1}{\alpha^2} \right] a_i(t) & \text{if } d \leq z \leq d + \left( \frac{3L_n}{2} \right), \\
(1, \alpha) & \text{otherwise}
\end{cases}
\]

where

\[
g(z) = \left\{ 1 - \left( \frac{94}{3L_n} \right) (z - d) + \left( \frac{32}{L_n^2} \right) (z - d)^2 - \left( \frac{32}{3L_n^3} \right) (z - d)^3 \right\};
\]

\[
a_i(t) = 1 - b \left( \cos \theta - 1 \right) e^{-bt}.
\]

where \(R(z, t), R_j(z, t)\) are the radius of the tapered arterial segment with overlapping stenosis in the outer-phase region and inner-phase region respectively; \(r_0\) is arterial radius in the normal flow region; \(\psi\) and \(m = \tan \psi\) are the angle of tapering of arterial segment and slope of the tapered blood vessel respectively; \(d\) and \(3L_n/2\) denote the stenosis' location and length; \(L_i\) is the length of the arterial segment; \(\delta_p \cos \psi\), and \(\delta_c \cos \psi\) are the stenosis' critical depths in the outer-phase and inner-phase

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**Figure 1.** Geometry of blood flow in a tapered narrow arterial segment with overlapping stenosis.
regions respectively; \( \delta_\omega = \alpha \delta_p, \quad a_\omega(t) \) is the time variant parameter; \( b \) is a time constant; \( \omega(=2\pi f_p) \) is the angular frequency with \( f_p \) as the pulse frequency. It is well known that in low Reynolds number flow (slow flow) in a narrow artery with mild stenosis, the radial velocity is negligibly small and can be neglected. The dimensionless form of the governing momentum equation coupled with the corresponding constitutive equation (of H-B fluid in the inner phase region and of Newtonian fluid in the outer-phase region) for the two-phase blood flow simplify respectively to [11, 15]

\[
\frac{\partial u_H}{\partial t} = -\frac{1}{Re} \left[ \frac{\partial P}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left( \frac{\partial u_H}{\partial r} \right)^2 \right\} \right] + H^2 u_H, \quad (4)
\]

\[
\frac{\partial u_N}{\partial t} = -\frac{1}{Re} \left[ \frac{\partial P}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left( \frac{\partial u_N}{\partial r} \right)^2 \right\} \right] + H^2 u_N, \quad (5)
\]

where \( u_H \) and \( u_N \) are the velocity of the fluid in the inner-phase region (H-B fluid) and outer-phase region (Newtonian fluid) of the artery respectively; \( \theta \) is the yield stress of blood; \( P \) is the fluid pressure; \( H \) is the Hartmann number (magnetic field parameter) and \( Re \) is the Reynolds number. Since the flow is assumed as pulsatile, it is appropriate to assume the pressure gradient as a periodic function as given below:

\[
\frac{\partial P}{\partial z} = 1 + \epsilon \cos \omega t, \quad (6)
\]

where \( \epsilon = \frac{A}{A_0} \), \( A_0 \) and \( A \) are the steady and pulsatile parts of amplitude of flow; \( \omega \) is the angular frequency of the flow. The dimensionless form of the appropriate initial and boundary conditions pertaining to this two-phase fluid flow are

\[
\frac{\partial u_H}{\partial r}(0,z,t) = 0; \quad u_N(R, z, t) = 0; \quad u_e(R_i, z, t) = u_e(R_o, z, t); \quad \tau_H(R, z, t) = \tau_H(R_i, z, t) \quad (7, 8, 9, 10)
\]

\[
u_e(r,z,0) = \nu_e(r,z,0) = \begin{cases} \left( \frac{A + A}{H^2} \right) \left( \frac{I_0(H)}{I_0(H)} \right) & \text{if } H \neq 0 \\ \left( \frac{A + A}{4} \right) (1 - r^2) & \text{if } H \neq 0 \end{cases}
\quad (11)
\]

Where \( I_0(x) \) is the modified Bessel function of order zero.

3. Finite Difference Method of Solution

The non-uniform cross-sectional geometry of artery is transformed into uniform cross-sectional geometry by applying radial coordinate transformation \( \xi = r/R(z,t) \) in the momentum equations (4) to (5) and the boundary and initial conditions (7) to (11). The transformed momentum equations and the boundary and initial conditions are not given here and one can refer Sankar et al. [15] to know the details of obtaining the transformed equations.

Finite difference method is an efficient computational method for solving the system of nonlinear partial differential equations (2) to (4). Central difference formula is used to express the first and second order spatial derivatives and forward difference formula is applied to express the time derivative which are given below:

\[
\frac{\partial u_e}{\partial t} = \frac{(u_e(x))_{i+1} - (u_e(x))_{i-1}}{2\Delta z}, \quad \frac{\partial^2 u_e}{\partial z^2} = \frac{(u_e(x))_{i+1} - 2(u_e(x))_{i} + (u_e(x))_{i-1}}{(\Delta z)^2}, \quad (12, 13)
\]
\[
\frac{\partial u_m}{\partial t} = \frac{(u_{m})_{j}^{k+1} - (u_{m})_{i,j}^{k}}{\Delta t} = \left(\frac{u_{m}}{\rho}\right)_{j}^{k+1},
\]

where \( m = C \) if \( \beta < x < \alpha \) and \( m = N \) if \( \alpha < x < 1 \), \( u_m(x, z, t) \) is discretized into \( u_m(\xi_j, z, t) \) and is denoted as \( (u_m)^{k+1} \), we define \( \xi_j = \beta + (j - 1) \Delta \xi \) for \( j = 1, 2, ..., Q_1, Q_1 + 1 \) such that \( \xi_{Q_1 + 1} = \alpha \) and \( \xi_j = \alpha + [j - (Q_1 + 1)] \Delta x \) for \( j = Q_1 + 1, Q_1 + 2, ..., Q + 1 \) such that \( \xi_{Q + 1} = 1 \), \( z_i = (i - 1) \Delta z \), \( i = 1, 2, ..., M + 1 \); \( t_k = (k - 1) \Delta t \), \( k = 1, 2, ..., \) for the entire arterial segment under consideration with \( \Delta x \) and \( \Delta \xi \) are the respective increments in the axial and radial directions and \( \Delta t \) is the time increment. Applying the finite difference formulae (12) – (14) into the momentum equations (4) - (5) and the boundary conditions (7) – (11), we get the simplified form of finite difference scheme for the velocity profile and boundary conditions as given below respectively:

\[
(u_H)_{i,j}^{k+1} = (u_H)_{i,j}^{k} - \frac{\Delta t}{Re} \left[ \left( \frac{\partial p}{\partial \xi} \right)_{i,j}^{k+1} + R \left( \frac{1}{n} \right) \left( \frac{u_H}{\xi_{j}} \right)_{i,j}^{k+1} \right] + \frac{\theta}{R^2} (u_H)_{i,j}^{k},
\]

\[
(u_N)_{i,j}^{k+1} = (u_N)_{i,j}^{k} + \frac{1}{\Delta \xi} \left[ \left( \frac{\partial p}{\partial \xi} \right)_{i,j}^{k+1} - \frac{1}{Re} \left( \frac{1}{n} \right) \left( \frac{u_N}{\xi_{j}} \right)_{i,j}^{k} + \frac{1}{R^2} \left( \frac{u_N}{\xi_{j}} \right)_{i,j}^{k+1} \right],
\]

\[
(u_N)_{i,Q_1+1}^{k+1} = (u_N)_{i,Q_1+1}^{k}; \quad (u_H)_{i,Q_1+1}^{k+1} = (u_H)_{i,Q_1+1}^{k},
\]

\[
(u_N)_{i,Q_1+1}^{k} = \frac{\frac{\Delta x}{\sqrt{R}}}{\left( \frac{1}{\Delta \xi} \right)} \left( \frac{u_N}{\xi_{j}} \right)_{i,Q_1+1}^{k+1} + \theta,
\]

\[
(u_N)_{i,Q_1+1}^{k+1} = \frac{A_1 + A_2}{H^2} \left( \frac{1}{R \xi_{j}} \right) \left( \frac{u_N}{\xi_{j}} \right)_{i,Q_1+1}^{k+1} \left( 1 - \frac{\Delta \xi}{\Delta z} \right) \quad \text{if} \; H \neq 0,
\]

\[
(u_N)_{i,Q_1+1}^{k+1} = \frac{A_1 + A_2}{4} \left( 1 - \frac{\Delta \xi}{\Delta z} \right) \quad \text{if} \; H = 0.
\]

After computing the numerical solution to the velocity distribution, one can make use of it to obtain the numerical solution to the flow rate, wall shear stress and longitudinal impedance from the following numerical schemes respectively:

\[
Q' = 2\pi R \left[ \int_{u} \frac{u_{m} (u_{m})_{j} d\xi}, + \frac{1}{\rho} \left( \frac{u_{m}}{\xi_{j}} \right)_{i,j} d\xi \right], \quad \tau_w = (\tau_w)_{i,Q_1+1}^{k+1} - \frac{1}{R} \left[ (u_N)_{i,Q_1+1}^{k+1} \right], \quad \Lambda' = \frac{\partial p}{\partial \xi} \left( \frac{u_N}{\xi_{j}} \right)_{i,Q_1+1}^{k+1},
\]

\[
(20, 21, 22)
\]

4. Results and Discussion

For the numerical simulation of the rheological quantities and also to validate the present results with the others’ published results, the range of the parameters used by Sankar and et al. [15] are utilized in this study. To ensure the fifth or higher order convergence of the numerical solution obtained by explicit finite difference method, the step size in the time direction is chosen as \( \Delta t = 0.0001 \) and while the step size along the axial and radial directions are taken as \( \Delta x = 0.05 \) and \( \Delta \xi = 0.025 \) respectively.
4.1 Velocity profile

The velocity profile for different fluid modes and for different values of Hartmann number (magnetic field parameter) is depicted in Figure 2. It is noted that velocity of the fluid decreases considerably with the increase of the Hartmann number \(H\). It is also observed that the velocity of two-fluid H-B model is marginally higher than single-fluid H-B model. It is of important to note that the velocity profile of single-fluid H-B model is in good agreement with the corresponding plot of Sankar et al. [15] in their Figure 2.

![Figure 2. Velocity distribution for different fluid models and for different values of Hartmann number with \(Re = 300, t = 60^\circ, b = 0.95, \delta_\theta = \theta = 0.1, e = 0.5, \omega = 1, z = 2.3\) and \(\psi = -0.05^\circ\).](image)

4.2 Wall shear stress

The variation of wall shear stress with axial distance for different values of \(b\), \(H\) and \(\psi\) with \(t = 45^\circ\), \(Re = 300\), \(n = 0.95\), \(\omega = 0.2\) and \(\delta_\theta = \theta = 0.1\) is shown in Figure 4. It is observed that the wall shear stress increases in the axial direction from \(z = 2\) to \(z = 2.3\) and also \(z = 2.75\) to \(z = 3.2\) and it decreases in the axial direction from \(z = 2.3\) to \(z = 2.75\) and also from \(z = 3.2\) to \(z = 3.5\). It is also seen that the wall shear stress increases considerably with the increase of the Hartmann number \(H\) and the wall shear stress decreases marginally with the increase of the angle of tapering of the tube \(\psi\) and stenosis shape parameter \(b\) when the Hartmann number \(H\) is held constant.

![Figure 3. Variation of wall shear stress with axial distance for different values of \(b\), \(H\) and \(\psi\) with \(t = 45^\circ\), \(Re = 300\), \(n = 0.95\), \(\omega = 0.2\) and \(\delta_\theta = \theta = 0.1\).](image)
4.3 Longitudinal impedance to flow

The estimates of the increase in longitudinal impedance to flow due to the presence of magnetic field and stenosis depth for single-phase H-B fluid model and two-phase H-B fluid model models with $e = 0.2$, $n = 0.95$, $\omega = 0.5$, $\theta = 0.1$ and $Re = 300$ are computed in Table 1. It is found that the estimate of the increase in the longitudinal impedance to flow increases considerably with the increase of the magnetic field parameter $H$ and it increases marginally with the increase of the stenosis depth. $\delta$. It is also noted that the estimates of the increase the impedance to flow of two-phase H-B fluid model is considerably lower than that of the single-phase H-B fluid model.

Table 1. Estimates of the increase in longitudinal impedance to flow due to the presence of magnetic field and stenosis for single-fluid H- B and two-fluid H-B model models with $e = 0.2$, $n = 0.95$, $\omega = 0.02$, $\theta = \delta = 0.1$ and $Re = 300$.

| $\delta$ | H | 0.05 | 0.1 | 0.15 | 0.2 |
|----------|----|------|-----|------|-----|
| Two-phase H-B fluid | $H = 1$ | 1 | 1.1532 | 1.1534 | 1.1547 | 1.1558 |
| | $H = 3$ | 2 | 1.5344 | 1.5686 | 1.5922 | 1.6465 |
| | $H = 5$ | 3 | 1.9243 | 1.9417 | 1.9728 | 1.9996 |
| Single-phase H-B model | $H = 1$ | 1 | 1.1552 | 1.1629 | 1.1692 | 1.1735 |
| | $H = 3$ | 2 | 1.6572 | 1.6827 | 1.7128 | 1.7529 |
| | $H = 5$ | 3 | 1.9821 | 2.0615 | 2.2732 | 2.4596 |

5. Conclusion

This mathematical investigation brings out several interesting results. The main outcomes of this research are listed (1) when the Hartmann number $H$ increases, the velocity of blood decreases considerably, while wall shear stress and longitudinal impedance to flow increased considerably. (2) The wall shear stress decreases with the increase of the angle of tapering $\psi$ of the artery and stenosis shape parameter $b$. (2) The estimates of the increase the impedance to flow of two-phase H-B fluid model is considerably lower than that of the single-phase H-B fluid model.

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