The Sample Fréchet Mean (or Median) Graph of Sparse Graphs is Sparse

Daniel Ferguson
and
François G. Meyer*

Applied Mathematics
University of Colorado at Boulder, Boulder CO 80305
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Abstract

To characterize the “average” of a sample of graphs, one can compute the sample Fréchet mean (or median) graph, which provides an interpretable summary of the graph sample. In this paper, we address the following foundational question: does the mean or median graph inherit the structural properties of the graphs in the sample? An important graph property is the edge density. Because sparse graphs provide prototypical models for real networks, one would like to guarantee that the edge density be preserved when computing the sample mean (or median). In this paper, we prove that the edge density is an hereditary property, which can be transmitted from a graph sample to its sample Fréchet mean (or median), irrespective of the method used to estimate the mean or the median.

Specifically, we prove the following result: the number of edges of the Fréchet mean (or median) graph of a set of graphs is bounded by the maximal number of edges amongst all the graphs in the sample. We prove the result for the graph Hamming distance, and the spectral adjacency pseudometric, using very different arguments.

Keywords: graph metrics; Hamming distance; Karcher mean; sparse graphs; statistical network analysis.

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1 Introduction

The availability of large datasets composed of graphs creates an unprecedented need to invent novel tools in statistical learning for “graph-valued random variables” (e.g., Durante et al., 2017; Kolaczyk et al., 2020; Mukherjee et al., 2017; Olhede and Wolfe, 2018; Relión et al., 2019; Salter-Townshend and McCormick, 2017, and references therein). To characterize the “average” of a sample \( \{G^{(1)}, \ldots, G^{(N)}\} \) of graphs, one can compute the sample Fréchet mean, (Fréchet, 1947; Schweizer et al., 1960), solution to

\[
\hat{\mathbb{E}} [G] = \arg\min_{G \in \mathcal{G}} \frac{1}{N} \sum_{k=1}^{N} d^2(G^{(k)}, G),
\]

where \( \mathcal{G} \) is the set of all unweighted simple graphs on \( n \) nodes, and \( d \) is a distance defined on \( \mathcal{G} \). This notion of centrality is well adapted to metric spaces (since graph sets are not Euclidean spaces (e.g., Chowdhury and Mémoli, 2018; Jain, 2016a,b; Jain and Obermayer, 2012; Kolaczyk et al., 2020, and references therein). Replacing the squared distance in (1) with the distance, yields the notion of sample Fréchet median, solution to

\[
\overline{G} = \arg\min_{G \in \mathcal{G}} \frac{1}{N} \sum_{k=1}^{N} d(G^{(k)}, G).
\]

Several algorithms have been proposed to compute the sample Fréchet mean and median when the distance \( d \) is the edit distance (e.g., Bardaji et al., 2010; Ferrer et al., 2009; Jiang et al., 2001, and references therein), or the Euclidean distance (e.g., Jain and Obermayer, 2008, 2009, and references therein).

In this work, we assume that the graphs \( \{G^{(1)}, \ldots, G^{(N)}\} \) are defined on the same vertex set. As a result, we need not solve the graph isomorphism problem, and we avoid the combinatorial complexity associated with the more general problem of computing the graph mean (or median) of unlabeled graphs (e.g., Jain, 2016a,b; Kolaczyk et al., 2020, and references therein).

In this paper, we address the following foundational question: does \( \hat{\mathbb{E}} [G] \) or \( \overline{G} \) inherit the structural properties of the graphs in the sample \( \{G^{(1)}, \ldots, G^{(N)}\} \)? This question is significant, since one would expect that the sample Fréchet mean (or median) be a graph that captures the topological structure of the sample. For instance, the connectivity at
all scales (from the local scale – the degree of a vertex – to the global scale, such as community structures) of the sample mean (or median) should be similar to that of the graphs \( \{G^{(1)}, \ldots, G^{(N)}\} \).

The vital role played by the Fréchet mean as a location parameter, is exemplified in the work of Lunagómez et al. (2020), who have created novel families of random graphs by generating random perturbations around a given Fréchet mean. In practice, the Fréchet mean itself is computed from a training set of graphs that display specific topological features of interest. To take full advantage of the training set, one needs to insure that the sample Fréchet mean inherits from the training set the desired topological structure, which the algorithm of Lunagómez et al. (2020) can then reproduce.

1.1 Our main contributions

In this work, we establish that sparsity is an hereditary property, which can be transmitted from a graph sample to its sample Fréchet mean (or median), irrespective of the method used to estimate the mean, or the median. Because sparse graphs provide prototypical models for real networks, one would like to guarantee that this structural property is preserved when computing the sample mean (or median). In a similar vein, Han et al. (2016) construct a sparse median graph, which provides a more interpretable summary, from a set of graphs that are not necessarily sparse.

Our work answers the question raised by Ginestet et al. (2017): “does the average of two sparse networks/matrices need to be sparse?” Specifically, we prove the following result: the number of edges of the Fréchet mean, or median, graph of a set of graphs is bounded by the maximal number of edges amongst all the graphs in the sample. We prove this result for the graph Hamming distance, and the spectral adjacency pseudometric, using very different arguments.
2 Preliminary and Notations

In the following, \( G = (V, E) \) is a simple graph with \( n = |V| \) vertices. The graph adjacency matrix is denoted by \( A \), and the corresponding ordered eigenvalues of \( A \) are

\[
\lambda_1(A) \geq \ldots \geq \lambda_n(A).
\]

We denote by \( \lambda(A) \) the vector of eigenvalues,

\[
\lambda(A) = \left[ \lambda_1(A) \ldots \lambda_n(A) \right],
\]

and we denote by \( m(A) \) the number of edges of \( G \).

2.1 Distances between graphs

In this work we consider two metrics: the Hamming distance, and the spectral adjacency pseudometric. We briefly recall the definitions of these.

**Definition 1** Let \( G, G' \in G \) be two unweighted graphs with known vertex correspondence and with adjacency matrix \( A \) and \( A' \) respectively. We define the Hamming distance between \( G \) and \( G' \) as

\[
d_H(G, G') = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} - a'_{ij}| = \|A - A'\|_1.
\]

We recall the following lower bound on the Hamming distance.

**Lemma 1** Let \( A \) and \( B \) be the adjacency matrices of two unweighted graphs with number of edges \( m(A) \) and \( m(B) \) respectively. Then

\[
2|m(A) - m(B)| \leq d_H(A, B).
\]

**Proof of Lemma 1** We observe that

\[
d_H(A, B) = \sum_{i,j=1}^{n} a_{ij} + b_{ij} - 2a_{ij}b_{ij} = 2m(A) + 2m(B) - 2 \sum_{i,j=1}^{n} a_{ij}b_{ij}.
\]

Also,

\[
\sum_{i,j=1}^{n} a_{ij}b_{ij} \leq \sum_{i,j=1}^{n} b_{ij} = 2m(B),
\]
and therefore
\[ d_H(A, B) = \sum_{i,j=1}^{n} a_{ij} + b_{ij} - 2a_{ij}b_{ij} \geq 2(m(A) - m(B)). \] (9)

By symmetry, we obtain the other case, \( d_H(A, B) \geq 2(m(B) - m(A)) \), which completes the proof of the lemma.

While the Hamming distance can reliably monitor large scale changes in graph volumes, it often misses significant changes in connectivity. The Hamming distance is also very sensitive to random fine scale fluctuations that are not related to significant changes in the graph topology.

In contrast, a metric based on the eigenvalues of the adjacency matrix can quantify configurational changes that occur on a graph at different scales: from the local scale formed by the local neighbors of each vertex, to the largest scale that quantifies the connections between clusters, or communities (e.g., Donnat and Holmes, 2018; Wills and Meyer, 2020, and references therein) for recent surveys on graph distances.

**Definition 2** Let \( G, G' \in \mathcal{G} \) with adjacency matrix \( A \) and \( A' \) respectively. We define the adjacency spectral pseudometric as the \( \ell_2 \) norm between the vectors of eigenvalues of \( A \) and \( A' \) respectively,
\[ d_A(G, G') = ||\lambda(A) - \lambda(A')||_2. \] (10)

The pseudometric \( d_A \) satisfies the symmetry and triangle inequality axioms, but not the identity axiom. Instead, \( d_A \) satisfies the reflexivity axiom, \( d_A(G, G) = 0, \ \forall G \in \mathcal{G} \).

To detect large to mid-scale structural changes (e.g., alteration of the topology and connectivity, etc.) one might consider including only the highest-\( k \) eigenvalues of \( A \). This flexibility allows the user to target the particular scale at which she wishes to study the graph, and is a significant advantage of the adjacency spectral pseudometric (Wills and Meyer, 2020). Finally, we note that the adjacency spectral pseudometric does not require node correspondence.
3 Sample Fréchet Mean and Median Graph

In this section we consider a set of \(N\) unweighted graphs, \(\{G^{(k)}\}_{1 \leq k \leq N}\), with known vertex correspondence. We denote by \(A^{(k)}\) the adjacency matrix of graph \(G^{(k)}\). We equip the set \(\mathcal{G}\) of all unweighted simple graphs on \(n\) nodes with a pseudometric, or a metric, \(d\). We define several general notions of centrality for the sample \(\{G^{(k)}\}\). The graph \(G\) associated with each notion of centrality minimizes the following dispersion,

\[
F_q(G) = \sum_{k=1}^{N} d^q(G, G^{(k)}), \quad \text{where} \quad q = 1, \text{ or } 2.
\]

When \(q = 1\), a solution is called a sample median, whereas when \(q = 2\), a minimizer is a sample mean. The function \(F_q(G)\) in (11) is called the Fréchet function (Fréchet, 1947). For simplicity of notation, we keep the dependency on the distance \(d\) implicit in (11). The choice of \(d\) should be obvious from the context.

**Remark 1** With a slight abuse of notation, we sometimes write \(F_q(A)\) to denote \(F_q(G)\) when \(A\) is the adjacency matrix of graph \(G\).

3.1 When the metric is the Hamming distance

**Definition 3** The sample Fréchet mean, \(\bar{E}[G]\), with adjacency matrix, \(\bar{E}[A]\), is the minimizer of \(F_2(A)\), defined in (11), where the distance is the Hamming distance,

\[
\bar{E}[A] = \arg\min_{A \in S} \sum_{k=1}^{N} d_H^2(A, A^{(k)}),
\]

where the set \(S\) contains the \(n \times n\) adjacency matrices of unweighted graphs,

\[
S = \{ A \in \{0, 1\}^{n \times n}; \text{where } a_{ij} = a_{ji}, \text{ and } a_{i,i} = 0; 1 \leq i < j \leq n \}.
\]

The sample Fréchet median, \(\bar{G}\), with adjacency matrix, \(\bar{A}\), is the minimizer of \(F_1(A)\), defined in (11), where the distance is the Hamming distance,

\[
\bar{A} = \arg\min_{A \in S} \sum_{k=1}^{N} d_H(A, A^{(k)}).
\]
Remark 2 An important remark is in order. Because the focus of this work is not the computation of the sample Fréchet mean (or median), but rather a theoretical analysis of the properties that the sample Fréchet mean (or median) inherits from the sample, we can assume that the graphs \( \{G^{(1)}, \ldots, G^{(N)}\} \) are defined on the same vertex set. Consequently, we need not solve the graph isomorphism problem, and we avoid the combinatorial complexity associated with the more general problem of computing the median graph (e.g., Jain and Obermayer, 2008; Jain, 2015, 2016a,b; Kolaczyk et al., 2020, and references therein) for a comprehensive treatment of the computation of the sample Fréchet mean or median graph).

Remark 3 We note that solutions to the minimization problems (12) and (14) always exist, but may not be unique. In this work, all the results hold for any graph in the set formed by the solutions to (12) and (14). To simplify the exposition, and without any loss of generality, we therefore assume that the sample Fréchet mean (or median) contains a single element.

Remark 4 Several alternatives have been proposed to the minimization problems (12) and (14): the authors in (Ferrer et al., 2010) suggest to embed the graphs in Euclidean space, wherein they can trivially find the mean of the set. In the case of weighted graphs, the Laplacian matrix of the graph uniquely characterizes the graph. The authors in (Ginestet et al., 2017) define the mean of a set of weighted graphs using the sample Fréchet mean (computed on the manifold defined by the cone of symmetric positive semi-definite matrices) of the respective Laplacian matrices.

3.2 When the metric is the spectral pseudometric

The technical difficulty in defining the sample Fréchet mean and median according to the adjacency spectral pseudometric stems from the fact that the sample Fréchet function, defined in (11), takes the form

\[
F_q(A) = \sum_{k=1}^{N} ||\lambda(\hat{E}[A]) - \lambda(A^{(k)})||^q,
\]  

(15)
where \( q = 1 \) for the median and \( q = 2 \) for the mean. \( F_q(A) \) is defined in the spectral domain, but the domain over which the optimization takes place is the matrix domain. This leads to the definition of the set, \( \Lambda \), of real spectra that are realizable by adjacency matrices of unweighted graphs (elements of \( \mathcal{S} \)) (Johnson et al., 2018),

\[
\Lambda = \left\{ \lambda(A) = \left[ \lambda_1(A) \ldots \lambda(A) \right] ; \text{where} A \in \mathcal{S} \right\},
\]

and \( \mathcal{S} \) is defined by (13). We now define the sample Fréchet mean graph and median graph for the spectral pseudometric. Let \( \{G^{(k)}\} \) be a sample of \( N \) graphs from \( \mathcal{G} \). Let \( A^{(k)} \) be the adjacency matrix of graph \( G^{(k)} \), for \( 1 \leq k \leq N \), and let \( \lambda(A^{(k)}) \) be the spectrum of \( A^{(k)} \).

**Definition 4** We define the sample Fréchet mean graph, \( \hat{\mathbb{E}}[G] \), with adjacency matrix, \( \hat{\mathbb{E}}[A] \), and eigenvalues, \( \lambda_1(\hat{\mathbb{E}}[A]) \geq \cdots \geq \lambda_n(\hat{\mathbb{E}}[A]) \) as the minimizer of

\[
\lambda(\hat{\mathbb{E}}[A]) = \arg\min_{\lambda \in \Lambda} \sum_{k=1}^{N} ||\lambda - \lambda(A^{(k)})||^2. \tag{17}
\]

We define the sample Fréchet median graph, \( \overline{G} \), with adjacency matrix, \( \overline{A} \), and eigenvalues, \( \lambda_1(\overline{A}) \geq \cdots \geq \lambda_n(\overline{A}) \) as the minimizer of

\[
\lambda(\overline{A}) = \arg\min_{\lambda \in \Lambda} \sum_{k=1}^{N} ||\lambda - \lambda(A^{(k)})||. \tag{18}
\]

4 The Number of Edges in \( \hat{\mathbb{E}}[A] \) and \( \overline{A} \)

We now turn our attention to the main problem. We consider the following question: if the graphs \( G^{(1)}, \ldots, G^{(N)} \) all have a similar edge density, can one determine the edge density of the sample Fréchet mean graph, or median graph? and does the number of edges depend on the choice of the metric? We answer both questions in the following theorem.

**Theorem 1** Let \( \{G^{(k)}\}_{1 \leq k \leq N} \) be a sample of \( N \) graphs with known vertex correspondence, sampled from \( \mathcal{G} \). Let \( \hat{\mathbb{E}}[G] \) be the sample Fréchet mean graph, and \( \overline{G} \) be the sample Fréchet median graph, computed according to either the Hamming distance, or the adjacency spectral pseudometric. Then the number of edges of the sample Fréchet mean (or median) graph
is bounded by $m_{\infty}$, the maximal number of edges amongst all the graphs in the sample $\{G^{(k)}\}_{1 \leq k \leq N},$

$$\max \left\{ m(\widehat{E}[A]), m(A) \right\} < 9 m_{\infty}. \quad (19)$$

**Proof of Theorem 1** The proof is a direct consequence of the lemmata 6 and 10.

The following corollary provides a quantitative version of the title of the work.

**Corollary 1** Let $\{G^{(k)}\}_{1 \leq k \leq N}$ be a sample of $N$ graphs with known vertex correspondence, sampled from $G$. We further assume that the graphs are sparse, so that the number of edges of each $G^{(k)}$ satisfies

$$m(A^{(k)}) = o\left(n^2\right), \quad \text{but} \quad m(A^{(k)}) = \omega(n), \quad 1 \leq k \leq N. \quad (20)$$

Let $\widehat{E}[G]$ be the sample Fréchet mean graph, and $\text{G}$ be the sample Fréchet median graph, computed according to either the Hamming distance, or the adjacency spectral pseudometric. Then the sample Fréchet mean (or median) is sparse, as defined by (20).

**Proof of Corollary 1** The corollary is a direct consequence of the previous theorem.

Our analysis begins with the sample Fréchet mean and median, computed using the Hamming distance. In section 4.2, we extend these results to the sample Fréchet mean and median computed with the adjacency spectral pseudometric.

### 4.1 When the metric is the Hamming Distance

The Hamming distance, by nature, promotes sparsity (Donnat and Holmes, 2018; Wills and Meyer, 2020), and we therefore expect that the volume of the sample Fréchet mean computed with this distance be similar to the volumes of the graphs that constitute the sample.

Unlike the sample Fréchet mean graph, the sample Fréchet median graph can be characterized analytically. We recall the following result, which will prove useful in our analysis of the sample median and the sample mean.
Lemma 2. The adjacency matrix $\mathbf{A}$ of the sample median graph is given by the majority rule
\[
\forall i, j \in \{1, \ldots, n\}, \quad a_{ij} = \begin{cases} 
0 & \text{if } \sum_{k=1}^{N} a_{ij}^{(k)} < N/2, \\
1 & \text{otherwise}.
\end{cases}
\]  
(21)

Proof of Lemma 2. The result is classic, and we omit the proof, which can be found for instance in (Devroye et al., 2013); see also the literature on consensus string, (e.g., Bulteau and Schmid, 2020; Chen et al., 2019; Li et al., 2002, and references therein).

Finally, we denote by $\overline{m}$ the number of edges of the median graph,
\[
\overline{m} \stackrel{\text{def}}{=} m(\mathbf{A}).
\]  
(22)

In the following lemma, we derive an upper bound on the number of edges in the sample Fréchet median graph as a function of the sample mean number of edges.

Lemma 3. Let $\mathbf{A}$ be the adjacency matrix of the median graph. Then the number of edges of $\mathbf{A}$ is bounded by
\[
\overline{m} \leq 2 \hat{E}[m(\mathbf{A})],
\]  
(23)

where $\hat{E}[m(\mathbf{A})]$ is the sample mean number of edges,
\[
\hat{E}[m(\mathbf{A})] \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k=1}^{N} m(\mathbf{A}^{(k)}).
\]  
(24)

Proof of Lemma 3. Let $\mathcal{I} = \{(i, j), a_{ij} = 1\}$ be the set of edges (counted twice) of the median graph. We have $|\mathcal{I}| = 2m(\mathbf{A}) = 2\overline{m}$. Now,
\[
2 \sum_{k=1}^{N} m(\mathbf{A}^{(k)}) = \sum_{n=1}^{N} \sum_{i,j=1}^{n} a_{ij}^{(k)} = \sum_{i,j=1}^{n} \sum_{k=1}^{N} a_{ij}^{(k)} = \sum_{i,j \in \mathcal{I}} \sum_{k=1}^{N} a_{ij}^{(k)} + \sum_{i,j \in \mathcal{I}^c} \sum_{k=1}^{N} a_{ij}^{(k)}.
\]  
(25)

Neglecting the edges $(i, j)$ not in $\mathcal{I}$, we have
\[
2N\hat{E}[m(\mathbf{A})] = 2 \sum_{k=1}^{N} m(\mathbf{A}^{(k)}) \geq \sum_{i,j \in \mathcal{I}} \sum_{k=1}^{N} a_{ij}^{(k)} > \sum_{i,j \in \mathcal{I}} \frac{N}{2} = |\mathcal{I}| \frac{N}{2} = m(\mathbf{A})N = N\overline{m}.
\]  
(26)
Next, we derive a lower bound on the Fréchet function evaluated at the sample Fréchet mean, \( F_2(\hat{E} [ A ]) \).

**Lemma 4** Let \( \hat{E} [ A ] \) be the adjacency matrix of the sample Fréchet mean computed using the Hamming Distance, with \( m(\hat{E} [ A ]) \) edges. Let \( \hat{E}[m( A )] \) be the sample mean number of edges, defined in (24). Then

\[
4N \left[ m(\hat{E} [ A ]) - \hat{E}[m( A )] \right]^2 < \sum_{k=1}^{N} d_H^2(\hat{E} [ A ], A^{(k)}).
\] (27)

**Proof of Lemma 4** Because of lemma 1, we have

\[
2 | m(A^{(k)}) - m(\hat{E} [ A ]) | \leq d_H(\hat{E} [ A ], A^{(k)}),
\] (28)

and thus

\[
4 | m(A^{(k)}) - m(\hat{E} [ A ]) |^2 \leq d_H^2(\hat{E} [ A ], A^{(k)}).\] (29)

Now, the function

\[
x \mapsto (m(\hat{E} [ A ]) - x)^2
\] (30)

is strictly convex so,

\[
\left| \hat{E}[m( A )] - m(\hat{E} [ A ]) \right|^2 = \left| \frac{1}{N} \sum_{n=1}^{N} m(A^{(k)}) - m(\hat{E} [ A ]) \right|^2 < \frac{1}{N} \sum_{n=1}^{N} \left| m(A^{(k)}) - m(\hat{E} [ A ]) \right|^2,
\] (31)

and combining with (29), we get the advertised result,

\[
4N \left| \hat{E}[m( A )] - m(\hat{E} [ A ]) \right|^2 < 4 \sum_{n=1}^{N} \left| m(A^{(k)}) - m(\hat{E} [ A ]) \right|^2 \leq \sum_{k=1}^{N} d_H^2(\hat{E} [ A ], A^{(k)}).\] (32)

□

Finally, we compute an upper bound on the Fréchet function, \( F_2(G) \), evaluated at the median graph, \( G \).

**Lemma 5** Let \( m_\infty = \max_{k=1}^{N} m(A^{(k)}) \) be the maximal number of edges amongst all the graphs in the sample \( \{ G^{(1)}, \ldots, G^{(N)} \} \), and let \( m_0 = \min_{k=1}^{N} m(A^{(k)}) \) to be the minimum
number of edges among the graphs $\{G^{(1)}, \ldots, G^{(N)}\}$. Then the Fréchet function evaluated at the median graph, $F_2(G)$, is bounded by

$$\sum_{k=1}^{N} d_H^2(\overline{A}, A^{(k)}) \leq 4N \left\{ 3\overline{m}^2 + 2\overline{m} \left( \mathbb{E}[m(A)] - m_0 \right) + m_\infty^2 \right\}. \quad (33)$$

**Proof of Lemma 5** We have

$$d_H(\overline{A}, A^{(k)}) = \sum_{i,j=1}^{n} a_{ij}^{(k)} + a_{ij}^{(k)} - 2\overline{a}_{ij}a_{ij}^{(k)} = 2\overline{m} + 2m(A^{(k)}) - 2 \sum_{i,j=1}^{n} \overline{a}_{ij}a_{ij}^{(k)} = 2\overline{m} + m(A^{(k)}) - \sum_{(i,j) \in I} a_{ij}^{(k)} \quad (34)$$

where we recall that $I = \{(i, j), \overline{a}_{ij} = 1\}$ is the set of edges (counted twice) of the median graph. Taking the square of the Hamming distance given by (34), and summing over all the graphs, yields

$$\frac{1}{4} \sum_{k=1}^{N} d_H^2(\overline{A}, A^{(k)}) = \sum_{k=1}^{N} \left\{ [\overline{m} + m(A^{(k)})]^2 - 2(\overline{m} + m(A^{(k)})) \sum_{(i,j) \in I} a_{ij}^{(k)} + \left[ \sum_{(i,j) \in I} a_{ij}^{(k)} \right]^2 \right\}$$

$$= \sum_{k=1}^{N} [\overline{m} + m(A^{(k)})]^2 - 2\overline{m} \sum_{k=1}^{N} \sum_{(i,j) \in I} a_{ij}^{(k)} - 2 \sum_{k=1}^{N} m(A^{(k)}) \sum_{(i,j) \in I} a_{ij}^{(k)} + \sum_{k=1}^{N} \left[ \sum_{(i,j) \in I} a_{ij}^{(k)} \right]^2$$

$$= \sum_{k=1}^{N} \overline{m}^2 + 2\overline{m} \sum_{k=1}^{N} m(A^{(k)}) + \sum_{k=1}^{N} m^2(A^{(k)})$$

$$- 2\overline{m} \sum_{(i,j) \in I} a_{ij}^{(k)} - 2 \sum_{k=1}^{N} m(A^{(k)}) \sum_{(i,j) \in I} a_{ij}^{(k)} + \sum_{k=1}^{N} \left[ \sum_{(i,j) \in I} a_{ij}^{(k)} \right]^2$$

$$\leq N\overline{m}^2 + 2N\overline{m} \mathbb{E}[m(A)] + Nm_\infty^2$$

$$- 2\overline{m} \sum_{(i,j) \in I} a_{ij}^{(k)} - 2 \sum_{k=1}^{N} m(A^{(k)}) \sum_{(i,j) \in I} a_{ij}^{(k)} + \sum_{k=1}^{N} \left[ \sum_{(i,j) \in I} a_{ij}^{(k)} \right]^2 \quad (35)$$

Now, because of the characterization of the median graph (21), and of the corresponding set of edge indices $I$, we have the following lower bound

$$\sum_{(i,j) \in I} \sum_{k=1}^{N} a_{ij}^{(k)} > \sum_{(i,j) \in I} \frac{N}{2} = |I|\frac{N}{2} = \overline{m}N, \quad (36)$$
which leads to an upper bound on the first cross product term in (35)

\[-2\overline{m} \sum_{(i,j) \in I} \sum_{k=1}^{N} a_{ij}^{(k)} \leq -2\overline{m}^2 N. \tag{37}\]

Using \(m_0\), we get the following lower bound on the second cross-product term in (35),

\[\sum_{k=1}^{N} m(A^{(k)}) \sum_{(i,j) \in I} a_{ij}^{(k)} \geq m_0 \sum_{k=1}^{N} \sum_{(i,j) \in I} a_{ij}^{(k)} = m_0 \sum_{k=1}^{N} \sum_{(i,j) \in I} a_{ij}^{(k)} > m_0 \sum_{k=1}^{N} N \frac{1}{2} = m_0 |I| \frac{N}{2} \tag{38}\]

which leads to an upper bound on the second cross product term in (35)

\[-2 \sum_{k=1}^{N} m(A^{(k)}) \sum_{(i,j) \in I} a_{ij}^{(k)} \leq -2m_0 |I| \frac{N}{2} = -2m_0 \overline{m} N. \tag{39}\]

Finally, we can derive a coarse upper bound on the last term in (35). We have

\[0 \leq \sum_{(i,j) \in I} a_{ij}^{(k)} \leq |I| = 2\overline{m}, \tag{40}\]

and therefore

\[\sum_{k=1}^{N} \left[ \sum_{(i,j) \in I} a_{ij}^{(k)} \right]^2 \leq 4N\overline{m}^2. \tag{41}\]

We can now replace the terms in (35) with the corresponding upper bounds, (37), (39) and (41), and we get

\[\frac{1}{4} \sum_{k=1}^{N} d_H^2(\overline{A}, A^{(k)}) \leq N\overline{m}^2 + 2N\overline{m} \bar{E}[m(A)] + Nm_\infty^2 - 2N\overline{m}^2 - 2Nm_0\overline{m} + 4N\overline{m}^2, \tag{42}\]

which yields the advertised upper bound

\[\sum_{k=1}^{N} d_H^2(\overline{A}, A^{(k)}) \leq 4N \left\{ 3\overline{m}^2 + 2\overline{m} \left( \bar{E}[m(A)] - m_0 \right) + m_\infty^2 \right\}. \tag{43}\]

\[\square\]

We are now in position to bound the number of edges of the sample Fréchet mean (or median) graph.
Lemma 6 Let \( \{G^{(k)}\}_{1 \leq k \leq N} \) be a sample of \( N \) graphs with known vertex correspondence, sampled from \( G \). Let \( \widehat{E}[G] \) be the sample Fréchet mean graph, and \( \overline{A} \) be the sample Fréchet median graph, according to the Hamming distance. Then the number of edges of the sample Fréchet mean (or median) is bounded by the maximal number of edges amongst all the graphs in the sample \( \{G^{(k)}\}_{1 \leq k \leq N} \). We have,

\[
m(\widehat{E}[A]) < 6m_{\infty}, \quad \text{and} \quad m = m(\overline{A}) \leq 2m_{\infty}.
\] (44)

**Proof of Lemma 6** The bound on the number of edges in the median graph is a straightforward consequence of lemma 3. Indeed, (24) yields

\[
\overline{m} \leq \frac{2}{N} \sum_{k=1}^{N} m(A^{(k)}) \leq 2m_{\infty}.
\] (45)

We now move to the sample Fréchet mean. By definition of the sample Fréchet mean, we have

\[
\sum_{k=1}^{N} d_{H}^{2}(\widehat{E}[A], A^{(k)}) \leq \sum_{k=1}^{N} d_{H}^{2}(\overline{A}, A^{(k)}).
\] (46)

Using (27) as a lower bound and (33) as an upper bound in the equation above, we get

\[
4N \left[ m(\widehat{E}[A]) - E[m(A)] \right]^{2} \leq \sum_{k=1}^{N} d_{H}^{2}(\widehat{E}[A], A^{(k)}) \leq \sum_{k=1}^{N} d_{H}^{2}(\overline{A}, A^{(k)}) \leq 4N \left\{ 3m_{\infty}^{2} + 2m_{\infty} \left( E[m(A)] - m_{0} \right) + m_{\infty}^{2} \right\},
\]

or

\[
\left[ m(\widehat{E}[A]) - E[m(A)] \right]^{2} \leq \left\{ 3m_{\infty}^{2} + 2m_{\infty} \left( E[m(A)] - m_{0} \right) + m_{\infty}^{2} \right\}.
\] (47)

Finally, we can use the bound on \( \overline{m} \), given by (23), everywhere in the equation above,

\[
m(\widehat{E}[A]) < E[m(A)] + \left[ 12 E[m(A)]^{2} + 4 E[m(A)](E[m(A)] - m_{0}) + m_{\infty}^{2} \right]^{1/2}
\]

\[
< E[m(A)] + \left[ 12m_{\infty}^{2} + 4 m_{\infty}(m_{\infty} - m_{0}) + m_{\infty}^{2} \right]^{1/2}
\]

\[
< E[m(A)] + m_{\infty} \left[ 13 + 4 \left( 1 - \frac{m_{0}}{m_{\infty}} \right) \right]^{1/2}
\]

\[
< E[m(A)] + 5m_{\infty}
\]

\[
< 6m_{\infty}.
\]

\( \square \)
4.2 When the metric is the Spectral Pseudometric

In the following, we compute the sample Fréchet mean graph and median graph using the adjacency spectral pseudometric. We are able to derive a bound on the number of edges of the mean and median graphs, which is similar to the bound obtained with the Hamming distance, albeit using completely different arguments.

The line of attack relies on the following classic result that expresses the number of edges as a function of the $\ell^2$ norm of the spectrum of the adjacency matrix.

**Lemma 7** Let $G \in \mathcal{G}$ with adjacency matrix $A$. Let $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ be the eigenvalues of $A$. Then

$$2m(A) = \sum_{i=1}^{n} \lambda_i^2(A) = \|\lambda(A)\|^2.$$  \hspace{1cm} (48)

We recall the proof for completeness.

**Proof of Lemma 7** We consider the matrix $A^2$; its entry $ii$ on the diagonal is

$$[A^2]_{ii} = \sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{j=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} a_{ij} = d(i),$$  \hspace{1cm} (49)

where $d(i)$ is the degree of node $i$. Finally,

$$\text{Tr}(A) = \sum_{i=1}^{n} [A^2]_{ii} = \sum_{i=1}^{n} \lambda_i^2(A) = \sum_{i=1}^{n} d(i) = 2m(A).$$  \hspace{1cm} (50)

Much of the remaining work involves computing an upper bound on the norm of the vector of eigenvalues of the adjacency matrix of the sample Fréchet mean, $\lambda(\hat{E}[A])$.

We take a short detour to build some intuition about the geometric position of the spectrum of the Fréchet mean with respect to the spectra of the sample, $\lambda(A^{(1)}), \ldots, \lambda(A^{(N)})$.

4.2.1 Warm-up: The Sample Mean Spectrum

Let $\{G^{(k)}\}$ for $1 \leq k \leq N$ be a sample of $N$ graphs from $\mathcal{G}$. Let $A^{(k)}$ be the adjacency matrix of graph $G^{(k)}$, and let $\lambda(A^{(k)})$ be the spectrum of $A^{(k)}$. 

15
Lemma 8  Let
\[ \widehat{E} [\lambda(A)] = \frac{1}{N} \sum_{k=1}^{N} \lambda(A^{(k)}). \] (51)
be the sample mean spectrum. Then \( \exists k_0 \in \{1, \ldots, N\} \) such that
\[ \| \widehat{E} [\lambda(A)] \| \leq \| \lambda(A^{(k_0)}) \|. \] (52)

Proof of Lemma 8  A proof by contradiction is elementary.

Using the characterizations of the sample Fréchet mean and median, given by (17) and (18), we can extend the above lemma to the sample Fréchet mean and median, and derive the following result.

Lemma 9  Let \( \lambda(\widehat{E} [A]) \) be the spectrum of the sample Fréchet mean graph, and \( \lambda(A) \) be the spectrum of the sample Fréchet median graph. Let \( m_\infty = \max_{k=1}^{N} m(A^{(k)}) \) be the maximal number of edges amongst all the graphs in the sample \( G^{(1)}, \ldots, G^{(N)} \).

Then
\[ \max \left\{ \| \lambda(\widehat{E} [A]) \|, \| \lambda(A) \| \right\} \leq 3 \sqrt{2m_\infty}. \] (53)

Proof of Lemma 9  We first prove the bound on the spectrum of the Fréchet mean. Because of the previous lemma,
\[ \exists k_0 \in \{1, \ldots, N\}, \| \widehat{E} [\lambda(A)] \| \leq \| \lambda(A^{(k_0)}) \|. \] (54)

Now, because of (48), we have
\[ \| \lambda(A^{(k_0)}) \| = \sqrt{2m(A^{(k_0)})} \leq \sqrt{2m_\infty} \] (55)
The vector \( \lambda(A^{(k_0)}) \) is in \( \Lambda \), since it is the spectrum of \( A^{(k_0)} \). We now express that \( \lambda(\widehat{E} [A]) \) is a minimizer of the Fréchet function \( F_2(A) \),
\[ \sum_{k=1}^{N} \| \lambda(\widehat{E} [A]) - \lambda(A^{(k)}) \|^2 \leq \sum_{k=1}^{N} \| \lambda(A^{(k_0)}) - \lambda(A^{(k)}) \|^2. \] (56)
Expanding the norms squared on both sides yields
\[ N \| \lambda(\widehat{E} [A]) \|^2 - 2N \langle \lambda(\widehat{E} [A]), \widehat{E} [\lambda(A)] \rangle \leq N \| \lambda(A^{(k_0)}) \|^2 - 2N \langle \lambda(A^{(k_0)}), \widehat{E} [\lambda(A)] \rangle. \] (57)
Dividing by $N$, and adding $\| \mathbb{E} [\lambda(A)] \|^2$ on both sides we get

$$\| \lambda(\mathbb{E} [A]) \| \leq \| \lambda(A(k_0)) \| + \| \mathbb{E} [\lambda(A)] \|, \quad (58)$$

and therefore

$$\left| \| \lambda(\mathbb{E} [A]) \| - \| \mathbb{E} [\lambda(A)] \| \right| \leq \| \lambda(A(k_0)) \| + 2 \| \mathbb{E} [\lambda(A)] \|. \quad (59)$$

and thus

$$\| \lambda(\mathbb{E} [A]) \| \leq \| \lambda(A(k_0)) \| + 2 \| \mathbb{E} [\lambda(A)] \|. \quad (60)$$

Finally, using (54) and (55) in the equation above, we obtain

$$\| \lambda(\mathbb{E} [A]) \| \leq 3 \| \lambda(A(k_0)) \| \leq 3 \sqrt{2m_\infty}, \quad (61)$$

which completes the proof of the bound on the spectrum of the Fréchet mean. We now consider the spectrum of the Fréchet median.

The function $\Phi$,

$$\Phi : \mathbb{R}^n \rightarrow [0, \infty) \quad (62)$$

$$x \mapsto \Phi(x) = \| \lambda(A) - x \| \quad (63)$$

is strictly convex, and therefore

$$\Phi \left( \frac{1}{N} \sum_{k=1}^{N} \lambda(A(k)) \right) \leq \frac{1}{N} \sum_{k=1}^{N} \Phi \left( \lambda(A(k)) \right). \quad (64)$$

Now the right-hand side of (64) is the Fréchet function evaluated at its minimum $\lambda(\mathbb{A})$, $F_1(\lambda(\mathbb{A}))$, which is smaller than $F_1(\lambda(A(k_0)))$, and therefore (64) becomes

$$\| \lambda(\mathbb{A}) - \mathbb{E} [\lambda(A)] \| \leq \frac{1}{N} \sum_{k=1}^{N} \| \lambda(A) - \lambda(A(k)) \| \leq \frac{1}{N} \sum_{k=1}^{N} \| \lambda(A(k_0)) - \lambda(A(k)) \|. \quad (65)$$

Now, because of (48), we have

$$\frac{1}{N} \sum_{k=1}^{N} \| \lambda(A(k)) \| = \frac{1}{N} \sum_{k=1}^{N} \sqrt{2 \, m(\lambda(A(k)))} \leq \sqrt{2m_\infty}, \quad (66)$$

17
and therefore
\[ \| \lambda(\overline{A}) - \hat{E}[\lambda(A)] \| \leq \frac{1}{N} \sum_{k=1}^{N} \| \lambda(A^{(k)}) - \lambda(A) \| \leq \| \lambda(A^{(k)}) \| + \sqrt{2m_{\infty}} \leq 2\sqrt{2m_{\infty}}, \] (67)
where we have used (55) in the last inequality. Combining (54) and (55), we conclude that
\[ \| \lambda(\overline{A}) \| \leq 2\sqrt{2m_{\infty}} + \| \hat{E}[\lambda(A)] \| \leq 3\sqrt{2m_{\infty}}, \] (68)
which completes the proof of the bound on the spectrum of the Fréchet median, and completes the proof of the lemma. □

We are now ready to bound the sparsity of the sample Fréchet mean (or median) graph.

**Lemma 10** Let \( \{G^{(k)}\}_{1 \leq k \leq N} \) be a sample of unweighted graphs, from \( \mathcal{G} \). We consider the sample Fréchet mean, \( \hat{E}[A] \), and the sample Fréchet median, \( \overline{A} \), computed according to the spectral adjacency pseudometric. Then the number of edges of \( \hat{E}[A] \), or \( \overline{A} \), is bounded by the maximal number of edges amongst all the graphs in the sample \( \{G^{(k)}\}_{1 \leq k \leq N} \),
\[ \max \left\{ m(\hat{E}[A]), m(\overline{A}) \right\} \leq 9m_{\infty}. \] (69)

**Proof of Lemma 10** We first analyse the case of the sample Fréchet mean; the sample Fréchet median is handled in the same way. From lemma 9, we have
\[ \| \lambda(\hat{E}[A]) \|^2 \leq 18m_{\infty}. \] (70)

Now, from (48) we have
\[ m(\hat{E}[A]) = \frac{1}{2} \| \lambda(\hat{E}[A]) \|^2, \] (71)
and therefore
\[ m(\hat{E}[A]) \leq 9m_{\infty}, \] (72)
which completes the proof of the lemma. □

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