The fractional porous medium equation on manifolds with conical singularities II

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Abstract
This is the second of a series of two papers that studies the fractional porous medium equation,
\[ \partial_t u + (-\Delta)^\sigma(|u|^{m-1}u) = 0 \]
with \( m > 0 \) and \( \sigma \in (0, 1] \), posed on a Riemannian manifold with isolated conical singularities. The first aim of the article is to derive some useful properties for the Mellin–Sobolev spaces including the Rellich–Kondrachov theorem and Sobolev–Poincaré, Nash and Super Poincaré type inequalities. The second part of the article is devoted to the study the Markovian extensions of the conical Laplacian operator and its fractional powers. Then based on the obtained results, we establish existence and uniqueness of a global strong solution for \( L_\infty \)–initial data and all \( m > 0 \). We further investigate a number of properties of the solutions, including comparison principle, \( L_\rho \)–contraction and conservation of mass. Our approach is quite general and thus is applicable to a variety of similar problems on manifolds with more general singularities.

KEYWORDS
Markovian semigroups, Mellin–Sobolev spaces, nonlinear nonlocal diffusion, porous medium equation, Riemannian manifolds with singularities

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INTRODUCTION

The study of nonlocal diffusion equations has become an active research field, both from the point of view of mathematical theory and in relation with its wide applications to real-world problems in material sciences, finance, biology, and so forth (cf. [1, 19, 23, 43]). The fractional porous medium equation, arguably, is one of the most extensively studied nonlinear nonlocal diffusion equations. There is a vast amount of work on fractional porous medium equations in Euclidean spaces (cf. [2, 6, 15–17, 36, 48, 49, 52]), just to name a few.

In this paper, we consider the fractional porous medium equation:

\[
\begin{align*}
\partial_t u + (-\Delta_g)^\sigma(|u|^{m-1}u) &= 0 & \text{on } & M \times (0, \infty); \\
u(0) &= u_0 & \text{on } & M
\end{align*}
\]

(1.1)

for \( m > 0 \) and \( \sigma \in (0, 1] \), on an \((n + 1)\)–dimensional Riemannian manifolds \((M, g)\) with isolated conical singularities. Here, \( \Delta_g \) is the Laplace–Beltrami operator associated with the conical metric \( g \), called the conical Laplacian. A conical
manifold, in the simplest case, is a product $\mathcal{M} = (0, \varepsilon) \times N$ with a smooth and compact manifold $(N, g_N)$ near a conical singularity, equipped with the degenerate metric $g(x, y) = dx^2 + x^2 g_N(y)$ for $(x, y) \in (0, \varepsilon) \times N$.

The objective of our work is threefold. First, the theory of Mellin–Sobolev spaces is widely used in the study of differential equations on conical manifolds. Briefly speaking, Mellin–Sobolev spaces are weighted Sobolev spaces with weights defined in terms of $x$. In this article, we establish several interesting properties like Rellich–Kondrachov theorem, Sobolev–Poincaré inequality, Nash inequality, and Super Poincaré inequalities for Mellin–Sobolev spaces with various weights. Some of these properties are already present for some limited cases in previous works. However, they have never been established in this generality. In particular, the derived properties can be used to investigate the asymptotic behavior of the solution to (1.1) on conical manifolds. To keep the paper in a reasonable length, we will explore this topic in a subsequent paper.

Second, we conduct a careful study of an important class of closed extensions, the Markovian extension (cf. Section 3.2), of the conical Laplacian operator $\Delta_g$ and the Markovian property of the fractional conical Laplacian. A major difficulty in the study of the conical Laplacian operator is that it does not have a canonical choice for a closed extension, as noticed by J. Brüning and R. Seeley [21] (see also [25–27]). Indeed, the domains of the minimal and the maximal closed extensions of $\Delta_g$ differ by a non-trivial finite dimensional space. Functions in this finite dimensional space admit certain asymptotic behaviors as $x \to 0^+$, as we will see in Section 3.1. Interested readers may find more details in [30, 45, 53, 54, 60].

Among all possible closed extensions of a differential operator, the Markovian extension plays an important role in the study of partial differential equations. For example, once we find a Markovian extension of a differential operator in $L_2(M)$, then the extension or restriction of the operator onto $L_p(M)$ for any $p \in (1, \infty)$ generates a positive contraction analytic semigroup. Following a transference method argument, see for instance [64], we can show that the operator has maximal $L_p$-regularity property. It is well-known that combining with appropriate techniques, maximal $L_p$-regularity property can be used to study the well-posedness, regularity, and stability of differential equations or systems. The monograph [51] by Prüss and Simonett presents an in-depth discussion of maximal $L_p$-regularity theory and its applications. Despite its importance, very little is known about the Markovian extension of differential operators on conical manifolds. Indeed, to the best of our knowledge, even for the conical Laplacian, many properties like the uniqueness of Markovian extension are only known for the case $\dim(M) = 2$. In this article, we carefully study the existence and uniqueness of Markovian extension of the conical Laplacian (cf. Propositions 3.4 and 3.9), in arbitrary dimensions. Denote by $\tilde{\Delta}_F$ the unique Markovian extension. Then the dissipativity of the extensions or restriction of $-\tilde{\Delta}_F$ to $L_p(M)$ follows from a classic argument by Davies [33]. Based on these results, we are able to construct the fractional conical Laplacian operators and show that the fractional operator is again Markovian.

The third and most important objective of this article is to find a suitable approach to (1.1), which is capable of dealing with both possibly degenerate initial data and the presence of conical singularities. In Euclidean space $\mathbb{R}^N$, to the best of our knowledge, both the Cauchy and Dirichlet problems of (1.1) were studied via one of the following approaches or their analogues.

1. In [48, 49], the authors constructed the Fractional Laplacian via the Caffarelli–Silvestre extension [22], that is, consider the problem:

$$\nabla \cdot (y^{1-2\sigma} \nabla w) = 0 \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^+; \quad w(x, 0) = u(x), \quad x \in \mathbb{R}^N.$$ 

For some constant $C_\sigma$, it holds that

$$(-\Delta)^\sigma u(x) = -C_\sigma \lim_{y \to 0} y^{1-2\sigma} \frac{\partial w}{\partial y}(x, y).$$

Then (1.1) was transformed into a quasi-stationary problem with a dynamical boundary condition on $\mathbb{R}^N \times \mathbb{R}^+$, and solutions were obtained by means of Crandall–Liggett’s theorem. See also [6, 52].

2. In [15–17], Bonforte et al. used the Spectral Fractional Laplacian (SFL) in a bounded domain $\Omega$ defined by

$$(-\Delta)^\sigma u(x) := \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left( e^{t\Delta} u(x) - u(x) \right) \frac{dt}{t^{1+\sigma}} = \sum_{j=1}^\infty \lambda_j^\sigma \hat{u}_j \phi_j(x),$$

where $\lambda_j$ and $\phi_j$ are eigenvalues and eigenfunctions of the Laplacian in $\Omega$. The authors constructed the fractional conical Laplacian via the Caffarelli–Silvestre extension [22].
where \((\phi_k, \lambda_k)_{k=1}^{\infty}\) denote an orthonormal basis of \(L_2(\Omega)\) consisting of eigenfunctions of the Dirichlet Laplacian \(-\Delta\) in \(\Omega\) and their corresponding eigenvalues. \(\hat{u}_k\) are the Fourier coefficients of \(u\). A solution was obtained via monotone operators techniques in Hilbert spaces or approximation by mild solutions. This approach relies on either good estimates of the Green functions or certain growth condition of \(\lambda_k\). Unfortunately, both are unknown on general conical manifolds.

3. Another way to construct the Fractional Laplacian is by using the integral representation in terms of hypersingular kernels:

\[
(-\Delta)^\sigma u(x) = C_{N,\sigma} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x+y|^{N+2\sigma}} dy.
\]

It is understood that \(u\) is extended by zero outside \(\Omega\). This is called Restricted Fractional Laplacian (RFL). Approaches based on RFL are similar to those in (2).

The presence of conical singularities in the underlying space \(M\) changes Equation (1.1) in a fundamental way. Indeed, it is quite obvious that (RFL) is not applicable to non-Euclidean spaces in general. For the reason stated above, (SFL) also seems to be restrictive in the case of a conical manifold. In the Caffarelli–Silvestre extension approach, a crucial component is the trace and extension theorems between a weighted space defined on \(\mathbb{R}^N \times \mathbb{R}_+\) and \(\dot{H}^\sigma(\mathbb{R}^N)\). However, if the underlying space \(M\) is a conical manifold, \(M \times \mathbb{R}_+\) becomes a corner manifold and such trace and extension theorems do not seem to be true in this case.

In [56], we initiated the study of (1.1) on conical manifolds, where we proved that operators of the form \(w(-\Delta_g)^\sigma\) enjoys maximal \(L_p\)-regularity property for some proper strictly positive continuous function \(w\). Then combining with a fixed point theorem argument, we established the existence and uniqueness of classical solution for strictly positive Hölder continuous data. To deal with general bounded initial data, in this article, we take a completely different approach to (1.1). Our approach only relies on the existence of a Markovian extension of \(\Delta_g\). Based on the Markovian properties of \((-\Delta_g)^\sigma\) proved in Section 4, we are able to establish the following for (1.1).

**Theorem 1.1.** Suppose that \((M, g)\) is an \((n + 1)\)-dimensional conical manifold. Then for any \(u_0 \in L_\infty(M), (1.1)\) with \(\sigma \in (0, 1)\) and \(m > 0\) has a unique global strong solution \(u\). Moreover, the solution \(u\) satisfies the following properties.

1. Continuous dependence on the initial data: The solution depends continuously on \(u_0\) in the norm \(C([0, T), L_1(M))\).
2. Comparison principle: If \(\hat{u}\) and \(\check{u}\) are the unique strong solutions to (8.1) with initial data \(u_0\), \(\check{u}_0\), respectively, then \(u_0 \leq \check{u}_0 \) a.e. implies \(u \leq \check{u}\) a.e.
3. \(L_p\)-contraction: For all \(0 \leq t_1 \leq t_2\) and \(1 \leq p \leq \infty\)

\[
\|u(t_2)\|_p \leq \|u(t_1)\|_p.
\]

4. Conservation of mass: For any \(t \geq 0\), it holds that

\[
\int_M u(t) \, d\mu_g = \int_M u_0 \, d\mu_g.
\]

We would like to refer the reader to Definitions 5.4 and 7.1 for the definitions of solutions used in the paper. Analogous results also hold when \(\sigma = 1\), that is, the usual porous medium equation, see Section 8.

It is worth mentioning that the method in this article can be applied to similar problems on manifolds with more general singularities, including those with cuspidal, edge, and corner singularities. We will discuss this idea in a future work. The method in this paper also seems to give an alternative approach to the Cauchy and Dirichlet problem of the fractional porous medium equation in \(\mathbb{R}^N\).

The paper is organized as follows:

In Section 2, we give the precise definitions of conical manifolds and the Mellin–Sobolev spaces. Then we prove a Rellich–Kondrachov type theorem for Mellin–Sobolev spaces.

In Section 3.1, we give a brief review of the closed extensions of the conical Laplacian operator \(\Delta_g\). In Section 3.2, we study the Markovian property of the Friedrichs extension of \(\Delta_g\) and fix the closed extensions of \(\Delta_g\) that we used in the analysis of (1.1). These closed extensions are denoted by \(\Delta_{F,p}\) for \(1 \leq p < \infty\). In Section 3.3, we prove that \(\Delta_g\) indeed
admits only one Markovian extension. Section 3.4 is where we establish the Sobolev–Poincaré, Nash, and Super Poincaré inequalities for Mellin–Sobolev spaces.

In Section 4.1, we construct the fractional powers of $\omega - \Delta_{F, p}$ for all $\omega \geq 0$; and then in Section 4.2, we derive the crucial properties of these operators. These results form the theoretic basis for the study of (1.1).

In Section 5, we employ the results from Sections 2–4 and prove that (1.1) admits a unique global weak solution depending continuously on the initial data. In Section 6, we further investigate various properties of weak solutions ((2)–(4) of Theorem 1.1). In Section 7, we prove that (1.1) indeed has a unique global strong solution.

In Section 8, we include a discussion of the porous medium equation, that is, $\sigma = 1$ in (1.1), and show that results parallel to those in Theorem 1.1 can be established.

**Notations**

For any two Banach spaces $X, Y$, $X \trianglerighteq Y$ means that they are equal in the sense of equivalent norms. The notations

$$ X \hookrightarrow Y, \quad X \hookrightarrow Y, \quad X \hookrightarrow Y $$

mean that $X$ is continuously embedded, densely embedded and compactly embedded into $Y$, respectively. $\mathcal{L}(X, Y)$ denotes the set of all bounded linear maps from $X$ to $Y$, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. Moreover, $\mathcal{L}_{is}(X, Y)$ stands for the subset of $\mathcal{L}(X, Y)$ consisting of all bounded linear isomorphisms from $X$ to $Y$. Given a sequence $(u_k)_k := (u_1, u_2, ...)$ in $X$, $u_k \rightharpoonup u$ in $X$ means that $u_k$ converge weakly to some $u \in X$. Given a densely-defined operator $A$ in $X$, $D(A)$ stands for the domain of $A$.

In addition, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

## 2 | PRELIMINARIES

### 2.1 | Conical manifolds

We can construct an $(n + 1)$-dimensional conical manifold $M$ from a $C^\infty$-compact manifold $(\bar{M}, \bar{g})$ with possibly disconnected smooth boundary. Let $(B, h) = (\partial \bar{M}, \bar{g}_{\partial \bar{M}})$. We may assume that $B = \bigcup_{i=1}^{k_B} B_i$ for some $k_B \in \mathbb{N}$, where $B_i$ are closed, smooth and connected. We equip $M = \bar{M} \setminus B$ with a smooth metric $g = \{g_{ij}\}_{j,i \in \{0, 1, ..., n\}}$ such that in local coordinates

$$(y_0, y_1, ..., y_n) = (x, y_1, ..., y_n) = (x, y)$$

of a closed collar neighborhood $(0, 1] \times B$ of the boundary $(B, h)

$$g(x, y) = dx^2 + x^2 h(y), \quad (x, y) \in (0, 1] \times B.$$  

Here, $x$ is a boundary defining function of $\bar{M}$ and $h = \{h_{ij}\}_{j, i \in \{1, ..., n\}}$. Outside $(0, 1] \times B$, $g$ is equivalent to $\bar{g}$ in the sense that there is a constant $c \geq 1$ such that

$$\frac{1}{c}g(\xi, \xi)(p) \leq g(\xi, \xi) \leq cg(\xi, \xi)(p), \quad \xi \in T_p M, \ p \in M.$$  

**Remark 2.1.**

1. In this article, we will only consider the non-trivial case $n \geq 1$.

2. Under the compactness assumption of $(\bar{M}, \bar{g})$, we have $L_q(M) \hookrightarrow L_p(M)$ for $q > p$, since $\text{Vol}(M) < \infty$. We pose the compactness assumption mainly for the sake of simplicity. In a subsequent paper [57], we will show with slight modification the method in this article can be applied to non-compact case as well.

### 2.2 | Mellin–Sobolev spaces

Here, we will describe a scale of weighted Sobolev spaces $\mathcal{H}^{\omega, p}_{L_p} (M)$. These weighted Sobolev spaces are widely used in the analysis of conical manifolds. See [45, 53, 54].
Let $I := (0,1]$. We pick a cut-off function $\psi$ on $I$, which means $\psi \in C^\infty([0,1])$ with $\psi \equiv 1$ near 0 and $\psi \equiv 0$ near 1.

For $k \in \mathbb{N}_0$, $H^{k,v}_p(M)$ is the space of all functions $u \in H^k_{p,loc}(M)$ such that near the conical singularities, or more precisely, in $I \times B$

$$x^{\frac{n+1}{2} - r}(x\partial_x)\partial_y^j(\psi u) \in L_p \left( I \times B, \frac{dx}{x} \frac{dy}{y} \right), \quad j + |\alpha| \leq k, \alpha \in \mathbb{N}_0^n,$$

where $(x, y) \in I \times B$. Here, $\partial_y^j$ can be considered as the derivatives in local coordinates of $B$, and we will use this slight abuse of notation throughout this paper.

To understand the motivation of this somewhat unusual definition, let us consider the flat cone $M = I \times S^n$ in $\mathbb{R}^{n+1}$, where $S^n$ is the $n$-sphere. Taking polar coordinates in $M$, then $H^{0,0}_2(M)$ coincides with the usual $L_2(B_1)$ space ($B_1$ is the closed unit ball in $\mathbb{R}^{n+1}$).

For arbitrary $\gamma \in \mathbb{R}$, we define $S_\gamma : C^\infty_c(I \times B) \to C^\infty_c(\mathbb{R}_+ \times B) : u(x, y) \mapsto e^{(\gamma - \frac{n+1}{2})t}u(e^{-t}, y)$.

Then for any $s \geq 0$ and $\gamma \in \mathbb{R}$, $H^{s,\gamma}_p(M)$ is the space of all distributions on $M$ such that

$$\|u\|_{H^{s,\gamma}_p(M)} = \|S_\gamma(\psi u)\|_{H^s_2(\mathbb{R}_+ \times B)} + \|(1 - \psi)u\|_{H^s_2(M)} < \infty.$$

It is understood that $\psi$ is extended to be zero outside $I \times B$.

**Lemma 2.2.** For all $s \geq 0$, $S_\gamma \in \mathcal{L}(H^{s,\gamma}_p(I \times B), H^{s}_q(\mathbb{R}_+ \times B))$.

**Proof.** This easily follows from the definition of Mellin–Sobolev spaces. \[Q.E.D.\]

It is immediate from the definition of Mellin–Sobolev spaces that

$$H^{s_1,\gamma_1}_p(M) \hookrightarrow H^{s_0,\gamma_0}_q(M).$$

(2.1)

if (1) $s_1 - \frac{n+1}{p} = s_0 - \frac{n+1}{q}$, $s_1 \geq s_0$ and $\gamma_1 \geq \gamma_0$ or (2) $s_1 - \frac{n+1}{p} > s_0 - \frac{n+1}{q}$, $s_1 \geq s_0$ and $\gamma_1 > \gamma_0$.

We will prove a Rellich–Kondrachov type theorem on conical manifolds. See also [37] and [58, Remark 2.1(b)].

**Proposition 2.3.** Assume that $s_1 - \frac{n+1}{p} > s_0 - \frac{n+1}{q}$, $s_1 > s_0$ and $\gamma_1 > \gamma_0$. Then

$$H^{s_1,\gamma_1}_p(M) \overset{c}{\hookrightarrow} H^{s_0,\gamma_0}_q(M).$$

**Proof.** Take $\gamma_1 > \gamma > \gamma_0$ and assume that $(u_k)_k \subset H^{s_1,\gamma_1}_p(M)$ with

$$\|u_k\|_{H^{s_1,\gamma_1}_p} \leq M.$$

Since the Rellich–Kondrachov theorem holds for compact manifolds with smooth boundary, by using a partition of unity, we may assume that $(u_k)_k$ are supported in $I \times B$.

**Lemma 2.2** immediately implies that $\|S_{\gamma_1}(u_k)\|_{s_1, p} \leq M$. Put $B_i = [0, i] \times B$. It is understood that $B_0 = \emptyset$. $B_i$ are compact manifolds with smooth boundaries. Then for each $i$, there exists a subsequence, still denoted by $(u_k)_k$, such that

$$S_{\gamma_1}(u_k)|_{B_i} \to u_i \quad \text{in} \quad H^{s_0}_q(B_i).$$

Moreover, for $j > i$, we have $u_j|_{B_i} = u_i$. Hence, we obtain a function $v$ on $\mathbb{R}_+ \times B$ such that $v|_{B_i} = u_i$. Pick $s_1 \leq k \in \mathbb{N}$. By the standard point-wise multiplication theorem, we can find a constant $M_1$ independent of $i$ such that

$$\|w_1w_2\|_{H^{s_0}_q(B_i \setminus B_{i-1})} \leq M_1\|w_1\|_{BC(B_i \setminus B_{i-1})}\|w_2\|_{H^{s_0}_q(B_i \setminus B_{i-1})}$$
for all \( w_1 \in BC^k(B_1 \setminus B_{i-1}) \) and \( w_2 \in H_q^{\infty}(B_1 \setminus B_{i-1}) \), where the space \( BC^k(B_1 \setminus B_{i-1}) \) consists of all functions \( w \in C^k(B_1 \setminus B_{i-1}) \) such that

\[
\|w\|_{BC^k(B_1 \setminus B_{i-1})} := \sum_{j+l \leq k} \|\partial_j^l \nabla_h w\|_{\infty} < \infty.
\]

Here, \( \nabla_h \) is the covariant derivative with respect to \( h \). Let \( M_2 := \sum_{j=0}^{k} (\gamma_1 - \gamma_0)^j \). One can then compute that

\[
\|e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(\mathbb{R}_+ \times B)} = \sum_{i=1}^{\infty} \|e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(B_i \setminus B_{i-1})}
\]

\[
\leq M_1 M_2 \sum_{i=1}^{\infty} e^{(\gamma_0 - \gamma_1)^{(i-1)}} \|\nabla_h^l w\|_{\infty} \|e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(B_i \setminus B_{i-1})}
\]

\[
\leq MM_1 M_2 \sum_{i=1}^{\infty} e^{(\gamma_0 - \gamma_1)^{(i-1)}} = M'.
\]

Hence, \( e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot) \in H_q^{\infty}(\mathbb{R}_+ \times B) \). For any \( \varepsilon > 0 \), take \( i \) large enough such that

\[
M_1 M_2 e^{(\gamma_0 - \gamma_1)^t} \varepsilon / 2 < \varepsilon.
\]

Then letting \( B_i^r = (\mathbb{R}_+ \times B) \setminus B_i \), we have

\[
\|S_{\gamma_0}(u_k) - e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(\mathbb{R}_+ \times B)}
\]

\[
\leq \|S_{\gamma_0}(u_k) - e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(\mathbb{B})} + \|S_{\gamma_0}(u_k) - e^{(\gamma_0 - \gamma_1)^t} u(t, \cdot)\|_{H_q^{\infty}(\mathbb{B})}
\]

\[
\leq \|e^{(\gamma_0 - \gamma_1)^t} (S_{\gamma_1}(u_k) - v_i)\|_{H_q^{\infty}(\mathbb{B})} + M_1 \sum_{j=i+1}^{\infty} \| e^{(\gamma_0 - \gamma_1)^t} \|_{BC^k(\mathbb{B}) \setminus \mathbb{B}_{j-1}}\|S_{\gamma_1}(u_k) - e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot)\|_{H_q^{\infty}(\mathbb{B}) \setminus \mathbb{B}_{j-1}}
\]

\[
\leq \|e^{(\gamma_0 - \gamma_1)^t} (S_{\gamma_1}(u_k) - v_i)\|_{H_q^{\infty}(\mathbb{B})} + \varepsilon / 2 \leq \varepsilon
\]

for \( k \) large enough. This implies that

\[
S_{\gamma_0}(u_k) \rightarrow e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot) \quad \text{in} \quad H_q^{\infty}(\mathbb{R}_+ \times B).
\]

Now, it follows from Lemma 2.2 that

\[
\|u_k - S_{\gamma_0}^{-1}(e^{(\gamma_0 - \gamma_1)^t} v(t, \cdot))\|_{H_q^{\infty}} = 0 \quad \text{as} \quad k \rightarrow \infty.
\]

This establishes the compact embedding assertion. \( \square \)

For \( 0 < \theta < 1 \) and \( 1 \leq q \leq \infty \), we denote by \((\cdot, \cdot)_{\theta,q}\) the real interpolation method, cf. [7, Example I.2.4.1].

**Definition 2.4.** Recall that \( \mathbb{B} = \bigcup_{i=1}^{k_B} B_i \). Denote by \( C_{\psi} \) the space of all smooth functions \( c \) that vanish on \( M \setminus (I \times \mathbb{B}) \) and on each component \( I \times B_i, i \in \{1, \ldots, k_B\} \), they are of the form \( c_i \psi_i \), where \( c_i \in \mathbb{C} \), that is, \( C_{\psi} \) consists of smooth functions that are locally constant close to the boundary. Endow \( C_{\psi} \) with the norm \( \| c \|_{C_{\psi}} \) given by \( c \mapsto \| c \|_{C_{\psi}} := (\sum_{i=1}^{k_B} |c_i|^2)^{1/2} \).
Lemma 2.5. Suppose that \(0 \leq s_0 < s_1\) and \(\gamma_0, \gamma_1 \in \mathbb{R}\). For any \(\varepsilon > 0\) and \(0 < \theta < 1\),
\[
H^{s_0+\varepsilon, \gamma+\varepsilon}_p (M) \hookrightarrow (H^{s_0, \gamma}_p (M), H^{s_1, \gamma}_p (M))_{\varepsilon, \theta} \hookrightarrow H^{s-\varepsilon, \gamma-\varepsilon}_p (M),
\]
where \(s = (1 - \theta)s_0 + \theta s_1\) and \(\gamma = (1 - \theta)\gamma_0 + \theta \gamma_1\), and for any \(s \geq 0\)
\[
H^{s+2\varepsilon, \gamma+2\varepsilon}_p (M) + \mathbb{C}\psi \hookrightarrow (H^{s, \gamma}_p (M), H^{s+2, \gamma+2}_p (M) + \mathbb{C}\psi)_{\varepsilon, \theta} \hookrightarrow H^{s+2\varepsilon-\varepsilon, \gamma+2\varepsilon-\varepsilon}_p (M) + \mathbb{C}\psi.
\]

Proof. The first embeddings follow from [29, Lemma 5.4] and [54, Lemma 3.6]. The second embeddings were proved in [54, Lemma 5.2].

3 | THE LAPLACE–BELTRAMI OPERATOR ON CONICAL MANIFOLDS

3.1 | Closed extensions of \(\Delta_g\)

The Laplace–Beltrami operator \(\Delta_g\) induced by the conical metric \(g\) is a second order differential operator, which near the conical singularities, that is, inside \((0, 1] \times \mathbb{B}\), can be written as
\[
\Delta_g = \frac{1}{x^2}[(x \partial_x)^2 + (n - 1)(x \partial_x) + \Delta_h],
\]
where \(\Delta_h\) is the Laplace–Beltrami operator on \(\mathbb{B}\) with respect to \(h\).

The conormal symbol of \(\Delta_g\) is defined by
\[
\sigma_M(\Delta_g)(z) := z^2 - (n - 1)z + \Delta_h, \quad z \in \mathbb{C}.
\]
In particular, \(\sigma_M(\Delta_g) \in \mathcal{A}(C, \mathcal{L}(H^{s+2}_p(\mathbb{B}), H^s_p(\mathbb{B})))\), where \(\mathcal{A}(C, E)\) stands for the space of analytic \(E\)-valued functions on \(C\) for any Banach space \(E\).

If we consider \(\Delta_g\) as an unbounded operator in \(H^{s, \gamma}_p(M)\) with domain \(C^\infty_c(M)\), denote its closure by \(\Delta_{\min} = \Delta^{\gamma}_{\min}\), and its maximal closed extension by \(\Delta_{\max} = \Delta^{\gamma}_{\max}\), where
\[
D(\Delta_{\max}) = \{u \in H^{s, \gamma}_p(M) : \Delta_g u \in L^2(M)\}.
\]

We have
\[
D(\Delta_{\min}) = D(\Delta_{\max}) \cap \bigcap_{\varepsilon > 0} H^{s+2, \gamma+2-\varepsilon}_p (M)
\]
\[
= \{u \in \bigcap_{\varepsilon > 0} H^{s+2, \gamma+2-\varepsilon}_p (M) : \Delta_g u \in H^{s, \gamma}_p(M)\}. \tag{3.1}
\]
In particular, \(D(\Delta_{\min}) = H^{s+2, \gamma+2}_p(M)\) iff \(\sigma_M(\Delta_g)(z)\) is invertible on the line
\[
\left\{ z \in \mathbb{C} : \text{Re} z = \frac{n - 3}{2} - \gamma \right\}.
\]

The reader may refer to [60, Proposition 5.1] for the details of this result; note that the structure of the minimal domain in general stratified space is a highly non-trivial issue, cf. [4, 42].
We denote by \( 0 = \lambda_0 > \lambda_1 > \ldots \) the distinct eigenvalues of \( \Delta_{\mathbb{B}} \) and by \( E_0, E_1, \ldots \) the corresponding eigenspaces. Then the non-bijectivity points of \( \sigma_M(\Delta_g) \) are exactly

\[
q_j^\pm = \frac{n-1}{2} \pm \sqrt{\left( \frac{n-1}{2} \right)^2 - \lambda_j}, \quad j \in \mathbb{N}_0.
\]  

(3.2)

When \( q_j^\pm \neq \frac{n+1}{2} - 2 - \gamma \) for all \( j \in \mathbb{N}_0 \), the minimum domain of \( \Delta_g \) in \( H^2_0(M) \) is

\[
D(\Delta_{\text{min}}) = H^2_0(M).
\]

(3.3)

For \( q_j^\pm \) with \( j \neq 0 \), we define the spaces

\[
\mathcal{E}_{q_j^\pm} = \psi x^{-q_j^\pm} \otimes E_j = \{ \psi(x)x^{-q_j^\pm} e_j(y) : e_j \in E_j \}.
\]

(3.4)

When \( j = 0 \), put

\[
\mathcal{E}_{q_0^\pm} = \begin{cases} 
\psi x^{-q_0^\pm} \otimes E_0 & n > 1; \\
\psi \otimes E_0 + \psi \log x \otimes E_0 & n = 1.
\end{cases}
\]

(3.5)

We will also introduce the set \( I_\gamma \) defined by

\[
I_\gamma := \left( \frac{n+1}{2} - \gamma - 2, \frac{n+1}{2} - \gamma \right).
\]

(3.6)

As a conclusion from [39, Theorem 3.6] and [60, Proposition 5.1], we have

\[
D(\Delta_{\text{max}}) = D(\Delta_{\text{min}}) \oplus \bigoplus_{q_j^\pm \in I_\gamma} \mathcal{E}_{q_j^\pm}.
\]

(3.7)

Let \( \gamma_p = \frac{n+1}{2} - \frac{n+1}{p} \) for \( 1 \leq p \leq \infty \). Note that

\[
H^{\gamma_p}_0(M) = L_p(M) \quad \text{on } (M,g).
\]

(3.8)

For this, we will denote the norm \( \| \cdot \|_{H^{\gamma_p}_0} \) simply by \( \| \cdot \|_p \) throughout. The above results are obtained from the theory of Schulze’s cone calculus, see [61, 62]. For an analysis of stratified spaces in the abstract setting of Dirichlet spaces, we also refer to [3].

### 3.2 Markovian property of \( \Delta_g \)

We write \( \mathcal{T}M \) for the \( C^\infty(M) \)-module of all smooth sections of \( TM \), and denote by

\[
| \cdot |_g : \mathcal{T}M \to C^\infty(M), \quad a \mapsto \sqrt{(a|a)_g}
\]

the (vector bundle) norm induced by the Riemannian metric \( g = (\cdot|\cdot)_g \). In addition, we define \( \langle \cdot, \cdot \rangle_T : \mathcal{T}M \times \mathcal{T}M \to \mathbb{R} \) by

\[
\langle u, v \rangle_T := \int_M (u|v)_g \, d\mu_g,
\]

where \( d\mu_g \) is the volume element induced by \( g \), and by \( \langle \cdot, \cdot \rangle \) the inner product of \( L^2(M) \).
Let us define a quadratic form associated with $-\Delta g$ by
\[ a(u, v) = \langle \nabla u, \nabla v \rangle_{\mathcal{D}}, \]
on $\mathcal{D} := C_c^\infty(M)$. Here, $\nabla u$ is the gradient vector. We have
\[ a(u, v) = \langle -\Delta g u, v \rangle, \quad u, v \in \mathcal{D}. \]
Moreover, $-\Delta g$ is symmetric on $\mathcal{D}$. By [32, Theorem 4.14] and [33, Theorem 1.2.8], $a$ is closable and its closure, still denoted by $a$, with domain
\[ D(a) = \text{the completion of } \mathcal{D} \text{ with respect to } \| \cdot \|_{D(a) \text{ in } L_1, \text{loc}(M)}, \]
where $\|u\|_{D(a)} := \|u\|_2 + \|\nabla u\|_2$, is associated with a self-adjoint operator, that is, the Friedrichs extension of $-\Delta g$. We denote the Friedrichs extension of $\Delta g$ by $\Delta_F$. Interested readers may refer to the Sections 3.3 and 4.2 for more details of $D(a)$ and $D(\Delta_F)$.

Since $-\Delta_F$ is non-negative and self-adjoint, from the spectral theory, $\Delta_F$ generates a self-adjoint contraction $C_0$-semigroup $\{e^{t\Delta_F}\}_{t \geq 0}$ on $L_2(M)$.

In the sequel, we will establish the Markovian property of the semigroup $\{e^{t\Delta_F}\}_{t \geq 0}$. To this end, let us first introduce several concepts on Dirichlet forms and Markovian semigroups.

**Definition 3.1.** A symmetric quadratic form $\mathcal{E}$ defined in $L_2(M)$ with domain $D(\mathcal{E})$ is called Markovian if for each $\varepsilon > 0$ there exists $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that $-\varepsilon \leq \phi_\varepsilon(t) \leq 1 + \varepsilon$ for all $t \in \mathbb{R}$ and $\phi_\varepsilon(t) = t$ for $t \in [0, 1]$ and
\[ 0 \leq \phi_\varepsilon(t) - \phi_\varepsilon(s) \leq t - s \quad \text{whenever } t > s, \]
and
\[ u \in D(\mathcal{E}) \implies \phi_\varepsilon(u) \in D(\mathcal{E}), \quad \mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leq \mathcal{E}(u, u). \]
A closed symmetric Markovian quadratic form is called a Dirichlet form.

Let $X_{\mathbb{R}}$ be a real Banach lattice with an order $\leq$. See [8, Chapter C-I]. The complexification of $X_{\mathbb{R}}$ is a complex Banach lattice defined by
\[ X := X_{\mathbb{R}} \oplus iX_{\mathbb{R}}. \quad (3.9) \]
The positive cone of $X_{\mathbb{R}}$ is defined by
\[ X^+_{\mathbb{R}} := \{ x \in X_{\mathbb{R}} : 0 \leq x \}. \]

**Definition 3.2.** Let $\delta \in \mathbb{R}$, and $X$ be a complex Banach lattice defined as in (3.9). A semigroup $T(t)$ is called real if
\[ T(t)X_{\mathbb{R}} \subset X_{\mathbb{R}}, \quad t \geq 0. \]
We say that $T(t)$ is positive if
\[ T(t)X^+_{\mathbb{R}} \subset X^+_{\mathbb{R}}, \quad t \geq 0. \]

**Definition 3.3.** A strongly continuous semigroup $T(t)$ on $L_2(M)$ is called Markovian if it is both positive and $L_\infty$-contraction, that is,
\[ \|T(t)u\|_\infty \leq \|u\|_\infty, \quad t \geq 0, \quad u \in L_\infty(M) \cap L_2(M). \]
Proposition 3.4. The semigroup $\{e^{t\Delta_g}\}_{t \geq 0}$ is Markovian.

Proof. By [38, Example 1.2.1], $a$ with domain $D(a)$ is Markovian and thus is a Dirichlet form. It follows from [38, Theorem 1.4.1] that

$$0 \leq u \leq 1 \implies 0 \leq e^{t\Delta_g}u \leq 1. \quad (3.10)$$

Note that although [38, Theorem 1.4.1] is proved for the quadratic form $\mathcal{E}(u, v) = \langle (-\Delta_g)^{1/2}u, (-\Delta_g)^{1/2}v \rangle$, one can follow that proof and check that it also works for $a$.

It only remains to show that $\{e^{t\Delta_g}\}_{t \geq 0}$ is positivity-preserving. For any $0 \leq u \in L_2(M)$, take a sequence $(u_n)$ in $L_\infty(M)$ converging to $u$ in $L_2(M)$. Without loss of generality, we can take $u_n \geq 0$. By the above discussion, $e^{t\Delta_g}u_n \geq 0$. If we have $\|e^{t\Delta_g}(u_n - u)\| \geq C > 0$, then $\|e^{t\Delta_g}(u_n - u)\| \geq C > 0$. A contradiction. Therefore, $\{e^{t\Delta_g}\}_{t \geq 0}$ is Markovian. □

Remark 3.5. Although [38, Example 1.2.1, Theorem 1.4.1] are stated for real-valued function spaces, we can overcome this restriction by the following procedure. First consider $\Delta_g : C_0^\infty(M; \mathbb{R}) \to L_2(M; \mathbb{R})$ and prove that its associated quadratic form $a_{\mathbb{R}}$ is Dirichlet. Then denoting the corresponding self-adjoint extension related to $a_{\mathbb{R}}$ by $\Delta_{F, \mathbb{R}}$, it is an easy job to check that $\Delta_{F, \mathbb{R}} = \Delta_{\mathbb{R}} | D(\Delta_{F, \mathbb{R}})$ and $e^{t\Delta_{F, \mathbb{R}}} = e^{t\Delta_{\mathbb{R}}} | _{L_2(M; \mathbb{R})}$, where $D(\Delta_{F, \mathbb{R}})$ is the restriction of $D(\Delta_{\mathbb{R}})$ on real-valued functions. Now, it follows from the above proof that (3.10) still holds true.

Proposition 3.6. $\Delta_g$ generates a contraction $C_0$-semigroup on $L_p(M)$ for $1 \leq p < \infty$, and this semigroup is analytic when $1 < p < \infty$. Moreover, for any $\omega > 0$, $0 \in \rho(\Delta_g - \omega)$ and

$$\|e^{t(\Delta_g - \omega)}\|_{L(L_p(M))} \leq e^{-\omega t}. \quad (3.11)$$

Proof. The generation of semigroup part follows from a standard argument, cf. [33, Section 1.4]. The assertion that $0 \in \rho(\Delta_g - \omega)$ is a direct consequence of the Hille–Yosida theorem. (3.11) follows from the fact that $\{e^{t\Delta_g}\}_{t \geq 0}$ is a contraction. □

Throughout the rest of this paper, we will denote by $\Delta_{F,p}$ the infinitesimal generator of the semigroup obtained via the extension of $\{e^{t\Delta_g}\}_{t \geq 0}$ onto $L_p(M)$ in Proposition 3.6 and its domain by $D(\Delta_{F,p})$. In particular, $\Delta_{F,2} = \Delta_F$. Note that $D(\Delta_{F,p})$ is always a subspace of $D(\Delta_{\mathbb{R}})$, cf. (3.7).

### 3.3 Uniqueness of Markovian extension

In this subsection, we will show that $\Delta_F$ is indeed the unique Markovian extension of the unbounded operator $\Delta_g : C_0^\infty(M) \to L_2(M)$.

By [67, Theorem 5.37], the domain of $\Delta_F$ can be expressed as

$$D(\Delta_F) = D(a) \cap D(\Delta_{\mathbb{R}}),$$

where $D(a)$ is the domain of the quadratic form $a$ defined in Section 3.2.

A closed extension $A$ of the unbounded operator $\Delta_g : C_0^\infty(M) \to L_2(M)$ is called a Markovian extension of $\Delta_g$ if $A$ generates a Markovian semigroup. From [38, Theorem 1.3.1], we learn that there is a one-one correspondence between
Markovian semigroups and Dirichlet forms. Therefore, each Markovian extension $A$ of $\Delta_g$ is associated to a Dirichlet form $\mathcal{E}_A$ defined on $L_2(M)$.

Denote by $\mathcal{M}_\Delta$ the set of all Markovian extensions of $\Delta_g$. We can define a partial order in $\mathcal{M}_\Delta$ by

$$A \preceq B \text{ if } D(\mathcal{E}_A) \subseteq D(\mathcal{E}_B) \text{ and } \mathcal{E}_A(u, u) \geq \mathcal{E}_B(u, u) \text{ for all } u \in D(\mathcal{E}_A)$$

for two Markovian extensions $A, B$ of $\Delta_g$. Here, $\mathcal{E}_A$ and $\mathcal{E}_B$ are the associated Dirichlet forms to $A, B$, respectively, and $D(\mathcal{E}_A)$ and $D(\mathcal{E}_B)$ are their domains, respectively. From [38, Lemma 3.3.1], we know that $\Delta_\kappa$ is the minimal Markovian extension with respect to the order $\preceq$.

Recall that $\|v\|_{D(\mathfrak{a})} = \|v\|_2 + \|\nabla v\|_g^2$.

**Lemma 3.7.** Define

$$H^1(M) = \{ u \in L_2(M) : \|\nabla u\|_g < \infty \}.$$

Then $H^1(M)$ is the completion of $H^1(M) \cap C^\infty(M)$ with respect to $\| \cdot \|_{D(\mathfrak{a})}$ and thus is a Banach space.

**Proof.** The proof is exactly the same to that of [38, Lemma 3.3.3]. □

**Lemma 3.8.** For any $\gamma \neq \frac{n-1}{2}$,

$$\|u\|_{X^{\gamma}} := \|u\|_{H^{0,\gamma}_p} + \|\nabla u\|_{H^{0,\gamma}_p}$$

defines an equivalent norm on $H^{1,\gamma+1}_p(M)$. In particular, we have $D(\mathfrak{a}) = H^{2,1}_2(M)$ for $n \neq 1$.

**Proof.** It suffices to consider this problem inside $I \times B$ and thus, we only consider $u \in H^{1,\gamma+1}_p(M)$ vanishing outside $I \times B$. For this, in the rest of this proof, we are free to assume all functions vanish outside $I \times B$, where $I = (0, 1]$. Thus, we can omit the multiplier $\psi$ in the definition of $\| \cdot \|_{X^{\gamma}}$.

By the weighted Hardy inequality cf. [41], we have for $p > 1$ and $\alpha \neq 1$

$$\int_0^\infty |f|^p x^{-\alpha} \, dx \leq \left( \frac{p}{|\alpha - 1|} \right)^p \int_0^\infty |f'(x)|^p x^{p-\alpha} \, dx. \quad (3.12)$$

We will show that an analogue of (3.12) holds true for any $f \in C^\infty_c(M)$. Indeed, we have

$$\int_{I \times B} x^{-\alpha} |f|^p \, d\mu_g = \int_B \int_{(0, \infty)} x^{-\alpha} |f|^p \, dx \, dy$$

$$\leq \left( \frac{p}{|\alpha - 1 - n|} \right)^p \int_B \int_{(0, \infty)} x^{p+n-\alpha} |\partial_x f(x, y)|^p \, dx \, dy$$

$$= \left( \frac{p}{|\alpha - 1 - n|} \right)^p \int_{I \times B} x^{p-\alpha} |\partial_x f(x, y)|^p \, d\mu_g.$$

We infer that

$$\int_{I \times B} |f|^p x^{-\alpha} \, d\mu_g \leq \left( \frac{p}{|\alpha - 1 - n|} \right)^p \int_{I \times B} |\nabla f|^p x^{p-\alpha} \, d\mu_g \quad (3.13)$$

for all $f \in C^\infty_c(M)$ and $\alpha \neq n + 1$. Here, we have abused the notation a little and omitted the pull-back of $f$ into the local coordinates of $B$. We will use this convention in the sequel.
Recall that $f \in H_p^{1,\gamma+1}(M)$ iff $f \in H_{p,\text{loc}}^1(M)$ such that
\[ \| f \|^p_{H_p^{1,\gamma+1}} = \sum_{k=0}^{\infty} \int_{I \times \mathcal{B}} x^p (k+\frac{n-1}{2}-\gamma-p+1) \| \nabla^k f \|^p_g \, d\mu_g < \infty. \]

Given any $f \in C_c^\infty(M)$, it is clear that
\[ \| f \|_{X_p^\gamma} \leq \| f \|_{H_p^{1,\gamma+1}}. \]
Choosing $a = p(\gamma + \frac{n+1}{p} - \frac{n-1}{2})$ in (3.13), we can infer that
\[ \| f \|_{H_p^{1,\gamma+1}} \leq C(n) \| f \|_{X_p^\gamma} \]
for $\gamma \neq \frac{n-1}{2}$. It follows from Lemma 2.2 that $u \in H_p^{1,\gamma+1}(M)$ iff $S_{\gamma+1} u \in H_p^1(\mathbb{R}^+ \times \mathcal{B})$. On the infinite cylinder $\mathbb{R}^+ \times \mathcal{B}$, functions in $H_p^1(\mathbb{R}^+ \times \mathcal{B})$ can be approximated by $C_c^\infty$-functions. Therefore, the closure of $C_c^\infty(M)$ with respect to $\| \cdot \|_{H_p^{1,\gamma+1}}$ is exactly $H_p^{1,\gamma+1}(M)$. This proves the asserted statement.

We introduce a new closed symmetric quadratic form
\[ E_+(u, v) = \langle \nabla u, \nabla v \rangle_f \]
with domain $H^1(M)$, and denote the associated Markovian extension by $A_+$. One can follow the proof of [38, Theorem 3.3.1] and show that $A_+$ is the maximal element of $\mathcal{M}_\Delta$.

**Proposition 3.9.** $H^1(M) = D(\mathfrak{a})$ and consequently, $\Delta_p^+ \mathfrak{a}$ is the unique Markovian extension of $\Delta_g$.

**Proof.** As in the proof of Lemma 3.8, we assume all functions vanish outside $I \times \mathcal{B}$.

**Case 1: $n = 1$**
This result is proved in [18, Proposition 3.10] for $B = S^1$; and it is known that any one-dimensional connected closed manifold is homeomorphic to $S^1$.

**Case 2: $n > 1$**
It is not hard to show that $H^1(M)$ is the completion of $H^1(M) \cap L_\infty(M)$. Indeed, given any $k \in \mathbb{N}$ and $u \in H^1(M)$, we can find $\lambda_k > 0$ such that
\[ \| u \|_{\| u \| > \lambda_k} \|_{D(\mathfrak{a})} < 1/k. \]
Define
\[ u_k(p) = \begin{cases} \lambda_k & \text{when } u(p) > \lambda_k \\ u(p) & \text{when } |u(p)| \leq \lambda_k \\ -\lambda_k & \text{when } u(p) < -\lambda_k. \end{cases} \]
Then $u_k \to u$ in $H^1(M)$.

Next, we will show that $H^1(M) \cap L_\infty(M) \subseteq H_2^{1,1}(M)$. First, any $u \in H^1(M)$ satisfies
\[ \int_{I \times \mathcal{B}} |\nabla u|^2 \, d\mu_g < \infty. \]
On the other hand, if we assume further \( u \in L^\infty(M) \), it holds that
\[
\int_{I \times B} x^{-2} |u|^2 \, d\mu_g = \int_{I} \int_{B} x^{n-2} |u|^2 \, dx \, dy \leq \|u\|_{L^\infty} \text{vol}(B).
\]
Combining with Lemma 3.8, this shows that \( D(\mathfrak{a}) \) is dense in \( H^1(M) \) and thus \( D(\mathfrak{a}) = H^1(M) \).

### 3.4 Functional inequalities in Mellin–Sobolev spaces

**Proposition 3.10.** For any \( \gamma \neq \frac{n-1}{2} \), there exists a constant \( C > 0 \) such that
\[
\|u - (u)_{\theta, \gamma}\|_{H^\theta_p} \leq C \|\nabla u\|_{H^{\theta, \gamma}_p}, \quad u \in H^{1, \theta+1}_p(M),
\]
where \((u)_{\theta, \gamma} = \frac{\int_{M} \tilde{x}^{\theta-\gamma} u \, d\mu_g}{\int_{M} \tilde{x}^{\theta-\gamma} \, d\mu_g}\) with \( \theta > \frac{n+1}{2} - \frac{n+p}{p} \) and
\[
\tilde{x}(p) = \begin{cases} x, & p = (x, y) \in I \times B \\ 1, & \text{elsewhere} \end{cases}
\]
In particular, we have
\[
\|u - \bar{u}\|_2 \leq C \|\nabla u\|_2, \quad u \in H^{1, 1}_2(M),
\]
where \( \bar{u} \) is the mean of \( u \) on \( M \).

**Proof.** Assume, to the contrary, that the theorem fails. Then for any \( k \in \mathbb{N} \), there exists \( u_k \in H^{1, \theta+1}_p(M) \) such that
\[
\|u_k - (u_k)_{\theta, \gamma}\|_{H^\theta_p} > k \|\nabla u_k\|_{H^{\theta, \gamma}_p}.
\]
Let
\[
v_k := \frac{u_k - (u_k)_{\theta, \gamma}}{\|u_k - (u_k)_{\theta, \gamma}\|_{H^\theta_p}}.
\]
Then it holds that \((v_k)_{\theta, \gamma} = 0\), \(\|v_k\|_{H^\theta_p} = 1\) and
\[
\|\nabla v_k\|_{H^{\theta, \gamma}_p} < 1/k.
\]
Together with Lemma 3.8, this implies that \(\|v_k\|_{X_p^\gamma} \), and thus \(\|v_k\|_{H^{1, \theta+1}_p} \), are uniformly bounded. It follows from Proposition 2.3 that we can find some \( v \in H^{1, \theta+1}_p(M) \) such that \( v_k \to v \) in \( H^{\theta, \gamma}_p(M) \) and \( v_k \to v \) in \( H^{1, \theta+1}_p(M) \). The condition \( \theta > \frac{n+1}{2} - \frac{n+p}{p} \) guarantees that
\[
H^{\theta, \gamma}_p(M) \hookrightarrow H^{0, \frac{n+1-\theta}{2}}_1(M)
\]
due to the fact \(-\frac{n+1}{2} - \theta < 0\) and (2.1). Note that
\[
\left[ u \mapsto \int_M \tilde{x}^{\theta-\gamma} |u| \, d\mu_g \right]
\]
is an equivalent norm on $H^{0,-n+\frac{1}{2}}(\mathcal{M})$. We thus infer that $(v)_{0,-\gamma} = 0$. Moreover, since $\|\nabla u\|_{H^0_p} = 0$, we conclude that $v \equiv 0$. A contradiction.

Given $p \in [1, n+1)$, let $p^* = \frac{pn+p}{n+1-p}$.

**Proposition 3.11.** Assume that $\gamma \neq \frac{n-1}{2}$ and $\Theta > \frac{n+1}{2} - \frac{n+p}{p}$. Then the following inequalities hold.

(i) (Sobolev–Poincaré inequality): When $p \in [1, n+1)$,

$$\|u - (u)_{\Theta,\gamma}\|_{H^{\Theta}_{p^*}} \leq C\|\nabla u\|_{H^\gamma_p}, \quad u \in H^{1,\gamma+1}_p(\mathcal{M}).$$

(ii) (Nash inequality): When $n \geq 2$,

$$\|u - (u)_{\Theta,\gamma}\|_{H^{1}_{2}} \leq C_1 \|\nabla u\|_{H^{\gamma}_{2}}\|u - (u)_{\Theta,\gamma}\|_{H^{\gamma}_{1}}, \quad u \in H^{1,\gamma+1}_p(\mathcal{M}).$$

(iii) (Super Poincaré inequality): When $n \geq 2$

$$\|u - (u)_{\Theta,\gamma}\|_{H^{1}_{2}} \leq r\|u - (u)_{\Theta,\gamma}\|_{H^{1}_{2}} + \beta(r)\|\nabla u\|_{H^{\gamma}_{2}}, \quad u \in H^{1,\gamma+1}_p(\mathcal{M}),$$

where $\beta : (0, \infty) \to (0, \infty)$ is a decreasing function.

**Proof.** (3.14) is a direct consequence of (2.1), Lemma 3.8, and Proposition 3.10. By the Hölder inequality, we have

$$\|u - (u)_{\Theta,\gamma}\|_{H^{\Theta}_{p^*}} \leq \|u - (u)_{\Theta,\gamma}\|_{H^{\gamma}_{2}}\|u - (u)_{\Theta,\gamma}\|_{H^{\gamma}_{1}},$$

where $\Theta = \frac{n+1}{n+3}$ and $2^* = \frac{2n+2}{n-1}$. Taking $p = 2$ in (3.14) gives (3.15). Then (3.16) follows from the Young’s inequality and (3.15).

## 4 | Fractional Powers of the Laplace–Beltrami Operator

### 4.1 | Construction of the fractional Laplace–Beltrami operators

Proposition 3.6 implies that the operator $\Delta_{\mathcal{F},p} - \omega$ has a bounded inverse in $L_p(\mathcal{M})$ when $\omega > 0$. In this case, we can define $(\omega - \Delta_{\mathcal{F},p})^\sigma$ by

$$\begin{align*}
(\omega - \Delta_{\mathcal{F},p})^\sigma : = & \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty s^{\sigma-1}(\omega - \Delta_{\mathcal{F},p})(s+\omega - \Delta_{\mathcal{F},p})^{-1} \, ds \\
= & \frac{1}{\Gamma(1-\sigma)} \int_0^\infty s^{-\sigma}(\omega - \Delta_{\mathcal{F},p})e^{(\Delta_{\mathcal{F},p}-\omega)} \, ds
\end{align*}$$

(4.1)

in $D(-\Delta_{\mathcal{F},p})$, cf. [65, Formula (2.33)]. (4.2) is a direct consequence of (4.1) and [47, Formula (2.1.10)], that is,

$$(s+\omega - \Delta_{\mathcal{F},p})^{-1} = \int_0^\infty e^{-st} e^{(\Delta_{\mathcal{F},p}-\omega)} \, dt.$$
In particular, by [65, Formula (2.34)] the following resolvent formula holds.

\[
(\lambda + (\omega - \Delta_{F,p})^\sigma)^{-1} = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty \frac{s^\sigma(s + \omega - \Delta_{F,p})^{-1}}{(s^\sigma + \lambda e^{i\pi\sigma})(s^\sigma + \lambda e^{-i\pi\sigma})} \, ds, \quad \lambda > 0.
\] (4.3)

When \( \omega = 0 \), we define the pseudo-resolvent \( I(\lambda) \) for \( \lambda \in (0, \infty) \) by

\[
I(\lambda) = \frac{\sin(\pi\sigma)}{\pi} \int_0^\infty \frac{s^\sigma(s - \Delta_{F,p})^{-1}}{(s^\sigma + \lambda e^{i\pi\sigma})(s^\sigma + \lambda e^{-i\pi\sigma})} \, ds.
\]

Then \( (-\Delta_{F,p})^\sigma \) is defined as the unique closed densely defined operator such that

\[
(\lambda + (-\Delta_{F,p})^\sigma)^{-1} = I(\lambda), \quad \lambda \in (0, \infty).
\]

Moreover, (4.3) still holds for \( \omega = 0 \), see [65, Formulas (2.40) and (2.44)].

The domain of \( (\omega - \Delta_{F,p})^\sigma \) is independent of \( \omega \geq 0 \), cf. [65, Lemma 2.3.5]. For this, we will simply abbreviate it to \( D((-\Delta_{F,p})^\sigma) \). Further, [65, Lemma 2.3.5] states that

\[
\| (\omega - \Delta_{F,p})^\sigma u - (-\Delta_{F,p})^\sigma u \|_p \leq c\omega \sigma \| u \|_p, \quad u \in D((-\Delta_{F,p})^\sigma).
\] (4.4)

Next, we will show that Formula (4.1) actually holds for \( \omega = 0 \). Indeed first note that Proposition 3.6 and [47, Formula (2.1.10)] imply

\[
\| (\lambda + \omega - \Delta_{F,p})^{-1} u \|_p = \left\| \int_0^\infty e^{-\lambda t} e^{i(\Delta_{F,p} - \omega)} e^{-\lambda t} u \, dt \right\|_p
\]

\[
\leq \int_0^\infty e^{-\lambda t} \| u \|_p \, dt = \frac{1}{\lambda} \| u \|_p,
\]

which gives

\[
\| \lambda (\lambda + \omega - \Delta_{F,p})^{-1} \|_{L_p(M)} \leq 1, \quad \lambda > 0, \ \omega \geq 0.
\] (4.5)

This implies that the operator

\[
Bu := -\frac{\sin(\pi\sigma)}{\pi} \int_0^\infty s^\sigma \Delta_{F,p} (s - \Delta_{F,p})^{-1} u \, ds
\]

is well-defined for all \( u \in D(-\Delta_{F,p}) \) because we can write \( Bu \) as

\[
Bu = -\frac{\sin(\pi\sigma)}{\pi} \int_1^\infty s^\sigma \Delta_{F,p} (s - \Delta_{F,p})^{-1} u \, ds + \frac{\sin(\pi\sigma)}{\pi} \int_0^1 s^\sigma u \, ds - \frac{\sin(\pi\sigma)}{\pi} \int_0^1 s^\sigma (s - \Delta_{F,p})^{-1} u \, ds
\]

and all three terms converge absolutely in \( L_p(M) \). (4.1) yields

\[
\| (\omega - \Delta_{F,p})^\sigma u - Bu \|_p
\]

\[
\leq \frac{\sin(\pi\sigma)}{\pi} \omega \left\| \int_0^\infty s^\sigma (s + \omega - \Delta_{F,p})^{-1} u \, ds \right\|_p
\]

\[
+ \frac{\sin(\pi\sigma)}{\pi} \left\| \int_0^\infty s^\sigma \Delta_{F,p} [(s - \Delta_{F,p})^{-1} - (s + \omega - \Delta_{F,p})^{-1}] u \, ds \right\|_p.
\]
The first term on the right hand side can be estimated as follows.

\[
\omega \left\| \int_0^\infty s^{-1} (s + \omega - \Delta_{F,p})^{-1} u \, ds \right\|_p
\]

\[
\leq \int_0^\infty s^{-1} \omega \| (s + \omega - \Delta_{F,p})^{-1} u \|_p \, ds + \int_0^\omega s^{-1} (s + \omega) \| (s + \omega - \Delta_{F,p})^{-1} u \|_p \, ds
\]

\[
+ \int_0^\omega s^2 \| (s + \omega - \Delta_{F,p})^{-1} u \|_p \, ds
\]

\[
\leq c \omega^2 \| u \|_p.
\]

Similarly,

\[
\left\| \int_0^\infty s^{-1} \Delta_{F,p} \left[ (s - \Delta_{F,p})^{-1} - (s + \omega - \Delta_{F,p})^{-1} \right] u \, ds \right\|_p
\]

\[
\leq \int_0^\infty s \omega \| (s - \Delta_{F,p})^{-1} \|_p \| (s + \omega - \Delta_{F,p})^{-1} u \|_p \, ds + \int_0^\infty s^{-1} \omega \| (s + \omega - \Delta_{F,p})^{-1} u \|_p \, ds
\]

\[
\leq c \omega^2 \| u \|_p.
\]

To sum up, we have proved

\[
(\omega - \Delta_{F,p})^\sigma u \to Bu \quad \text{in } L_p(M)
\]

as \( \omega \to 0^+ \). In view of (4.4), we thus show that \( B = (-\Delta_{F,p})^\sigma \) in \( D(-\Delta_{F,p}) \). This coincides with Balakrishnan’s formula for the fractional power of a dissipative operator, cf. [9].

### 4.2 Properties of the fractional Laplace–Beltrami operator

**Lemma 4.1.** For all \( \lambda > 0 \), \( \omega \geq 0 \), \( \sigma \in (0, 1) \), and \( u \in L_1(M) \), we have

\[
\sup \{ \text{id} + \lambda (\omega - \Delta_{F,1})^\sigma \}^{-1} u \leq \max \{0, \sup u\}.
\]

**Proof.** As in the proof of (4.5), it follows from Proposition 3.4 and [47, Formula (2.1.10)] that for all \( u \in L_\infty(M) \)

\[
\left\| (\lambda - \Delta_{F,1})^{-1} u \right\|_\infty = \left\| \int_0^\infty e^{-\lambda t} e^{i\Delta_{F,1}} u \, dt \right\|_\infty
\]

\[
\leq \int_0^\infty e^{-\lambda t} \| u \|_\infty \, dt = \frac{1}{\lambda} \| u \|_\infty,
\]

because \( L_\infty(M) \hookrightarrow L_1(M) \) and \( e^{i\Delta_{F,1}} \) coincides with \( e^{i\Delta_F} \) on \( L_\infty(M) \). For any \( \lambda > 0 \) and \( u \in L_\infty(M) \), (4.3) yields

\[
\| [\lambda + (\omega - \Delta_{F,1})^\sigma]^{-1} u \|_\infty
\]

\[
\leq \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty \frac{s^{\sigma-1}}{(s^\sigma + \lambda e^{i\pi \sigma})(s^\sigma + \lambda e^{-i\pi \sigma})} \| s (s + \omega - \Delta_{F,1})^{-1} u \|_\infty \, ds
\]

\[
\leq \| u \|_\infty \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty \frac{s^{\sigma-1}}{(s^\sigma + \lambda e^{i\pi \sigma})(s^\sigma + \lambda e^{-i\pi \sigma})} \, ds = \frac{1}{\lambda} \| u \|_\infty.
\]
Therefore, for all \( u \in L_1(M) \), we have
\[
\| [\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma]^{-1} u \|_{\infty} \leq \| u \|_{\infty}.
\]

Next, we will use Proposition 3.4 to establish the positivity of \( \{e^{t\Delta_{F,1}}\}_{t \geq 0} \) on \( L_1(M) \). Pick \( u \in L_1(M, \mathbb{R}^+) \) and a sequence \( (u_k) \) in \( L_2(M) \) converging to \( u \) in \( L_1(M) \). Without loss of generality, we may assume \( u_k \in L_2(M, \mathbb{R}^+) \). If
\[
\| (e^{t\Delta_{F,1}} u) - 1 \| \geq C > 0,
\]
then by the positivity of \( \{e^{t\Delta_{F,1}}\}_{t \geq 0} \) on \( L_2(M) \) and Proposition 3.6, we have
\[
\| e^{t\Delta_{F,1}} (u - u_k) \|_1 \geq \| (e^{t\Delta_{F,1}} u)^- \|_1 \geq C > 0.
\]

A contradiction. This implies that \( \{e^{t\Delta_{F,1}}\}_{t \geq 0} \) is positive on \( L_1(M) \). For any \( u \in L_1(M, \mathbb{R}^+) \), we have
\[
(\lambda - \Delta_{F,1})^{-1} u = \int_0^\infty e^{-\lambda t} e^{t\Delta_{F,1}} u \, dt \geq 0, \quad \lambda > 0.
\]

This gives the positivity of \( (\text{id} - \lambda \Delta_{F,1})^{-1} \) on \( L_1(M) \), that is, for all \( \lambda > 0 \)
\[
(\text{id} - \lambda \Delta_{F,1})^{-1} (L_1(M, \mathbb{R}^+)) \subset L_1(M, \mathbb{R}^+).
\]

The positivity of \( (\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma)^{-1} \) on \( L_1(M) \) follows from (4.3) and (4.6).

Given any \( u \in L_1(M, \mathbb{R}) \), we decompose it into \( u = u^+ - u^- \). It holds that
\[
\sup [\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma]^{-1} u^+
\leq \| [\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma]^{-1} u^+ \|_{\infty} \leq \| u^+ \|_{\infty} = \sup u^+.
\]

On the other hand, since \( -u^- \leq 0 \), it follows from the positivity of \( (\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma)^{-1} \) on \( L_1(M) \) that
\[
\sup [\text{id} + \lambda (\omega - \Delta_{F,1})^\sigma]^{-1} (-u^-) \leq 0.
\]

Combining these two estimates, the asserted claim then follows.

**Proposition 4.2.** For \( 1 \leq p < \infty \) and \( \omega \geq 0 \), \( - (\omega - \Delta_{F,p}^\sigma) \) generates a contraction \( C_0 \)-semigroup on \( L_p(M) \). In particular, when \( 1 < p < \infty \), this semigroup is analytic; when \( p = 2 \), this semigroup is Markovian. Furthermore, \( 0 \in \rho((\omega - \Delta_{F,p}^\sigma)) \) whenever \( \omega > 0 \).

**Proof.** From Proposition 3.6, [33, Theorems 1.4.1 and 1.4.2] and the standard semigroup theory, for all \( 1 \leq p < \infty \) and \( \omega > 0 \), we can learn the following facts:

- \( \omega - \Delta_{F,p}^\sigma \) is densely defined and closed.
- \( \mathbb{R}^+ \setminus \{0\} \subset p((\Delta_{F,p}^\sigma) - \omega) \), and \( \| \lambda (\lambda + \omega - \Delta_{F,p}^\sigma)^{-1} \|_{L_p(M)} \leq 1 \) for any \( \lambda > 0 \). Particularly, \( 0 \in p(\Delta_{F,p}^\sigma - \omega) \) when \( \omega > 0 \).
- We can find some \( \theta_p \in (0, \pi] \) and \( M > 0 \) such that \( \Sigma_{\theta_p} \subset p(\Delta_{F,p}^\sigma - \omega) \) and
\[
\| \lambda (\lambda + \omega - \Delta_{F,p}^\sigma)^{-1} \|_p \leq M, \quad \lambda \in \Sigma_{\theta_p}.
\]

In particular, \( \theta_p > \pi/2 \) when \( 1 < p < \infty \).
The resolvent estimate $\|\lambda (\lambda + \omega - \Delta_{F,p})^{-1}\|_p \leq 1$ for $\lambda > 0$ is proved in (4.5). In view of [65, Formula (2.26), Definition 2.3.1, Propositions 2.3.1, Theorem 2.3.1], we infer that for any $\sigma \in (0,1)$ the following properties are satisfied by $(\omega - \Delta_{F,p})^{\sigma}$.

- $(\omega - \Delta_{F,p})^{\sigma}$ is densely defined and closed.
- $0 \in \rho((\omega - \Delta_{F,p})^{\sigma})$ whenever $\omega > 0$.
- $\Sigma(1-\sigma)\pi + \sigma\theta_p \subset \rho(-(\omega - \Delta_{F,p})^{\sigma})$, and for any $\lambda > 0$

$$\|\lambda (\lambda + (\omega - \Delta_{F,p})^{\sigma})^{-1}\|_p \leq 1.$$  

- For every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that for all $\lambda \in \Sigma(1-\sigma)\pi + \sigma\theta_p - \varepsilon$

$$\|\lambda (\lambda + (\omega - \Delta_{F,p})^{\sigma})^{-1}\|_p \leq M_\varepsilon.$$  

Now, it follows from the Lumer–Philips theorem and [35, Proposition 4.4] that $-(\omega - \Delta_{F,p})^{\sigma}$ generates a contraction $C_0$-semigroup on $L_p(M)$ and this semigroup is analytic when $1 < p < \infty$.

In Lemma 4.1, we have shown that $[(id + \lambda(\omega - \Delta_{F,1})^{\sigma}]^{-1}$ is an $L_\infty$-contraction and positive. The Markovian property of the semigroup $\{e^{-t(\omega - \Delta_{F,1})^{\sigma}}\}_{t \geq 0}$ now follows by [50, Theorem 1.8.3].

The invertibility of $(\omega - \Delta_{F,1})^{\sigma}$ for $\omega > 0$ immediately implies the following lemma.

**Corollary 4.3.** For every $\omega > 0$, there exists some $C > 0$ such that

$$C\|u\|_1 \leq \|(\omega - \Delta_{F,1})^{\sigma}u\|_1, \quad u \in D((-\Delta_{F,1})^{\sigma}).$$  

Replacing $(\omega - \Delta_{F,1})^{\sigma}$ by $\lambda(\omega - \Delta_{F,1})^{\sigma}$ for $\lambda > 0$, the new operator still satisfies Lemma 4.1 and Corollary 4.3.

Let $\Phi(x) = |x|^{-m-1}x$ and $\beta = \Phi^{-1}$. Then they are maximal monotone graphs in $\mathbb{R}^2$ containing $(0,0)$. By Proposition 4.2, Lemma 4.1, and Corollary 4.3, we can apply [20, Theorem 1] and prove the following proposition.

**Proposition 4.4.** For any $f \in L_1(M)$ and all $\lambda > 0$, there exists a unique solution $u \in D((-\Delta_{F,1})^{\sigma})$ to

$$\lambda(\omega - \Delta_{F})^{\sigma}u + \beta(u) = f.$$  

Moreover, for any $f_1, f_2 \in L_1(M)$, the corresponding solutions $u_1, u_2$ satisfy

$$\|\beta(u_1) - \beta(u_2)\|_1 \leq \|f_1 - f_2\|_1.$$  

In next two propositions, we will give some characterizations of the domain of $(\omega - \Delta_{F,p})^{\sigma}$.

**Proposition 4.5.** It holds that $D(\Delta_{F}) \hookrightarrow H^{2,1+\delta}_{\kappa}(M) + C_{\psi}$ for some $\delta > 0$ sufficiently small. The sum with $C_{\psi}$ is present only when $n = 1$, in which case it is a direct sum. As a result, for any $\varepsilon > 0$,

$$D((\omega - \Delta_{F})^{\sigma}) \hookrightarrow H^{2,1+\delta}_{\kappa}(M) + C_{\psi}.$$  

In particular, for any $1 \leq p \leq 2$

$$D((\omega - \Delta_{F})^{\sigma}) \hookrightarrow L_p(M).$$  

(4.7)
Proof. According to [60, Corollary 5.4],

\[
D(\Delta_F) = \begin{cases} 
D(\Delta_{F,\min}) \oplus \bigoplus_{q_j^+ \in I_0} \mathcal{E}_{q_j^+} & n = 1 \\
D(\Delta_{F,\min}) \oplus \bigoplus_{q_j^+ \in I_0} \mathcal{E}_{q_j^+} \oplus \mathbb{C} \phi, \quad n > 1. 
\end{cases}
\] (4.9)

Here, $I_0$ is defined in (3.6), and $\Delta_{F,\min} := \Delta_{0,0,\min}$ is the minimal closed extension of $\Delta_0$ on $L_2(M)$. (3.1) implies

\[
H_2^{2,2}(M) \hookrightarrow D(\Delta_{F,\min}) \hookrightarrow \bigcap_{\varepsilon > 0} H_2^{2,2-\varepsilon}(M).
\]

When $\frac{n-3}{2} < q_j^+ < \frac{n-1}{2}$, it is an easy job to verify that $\mathcal{E}_{q_j^+} \hookrightarrow H_2^{1,1+\delta}(M)$ for some $\delta > 0$ sufficiently small. (4.7) then follows from (4.9).

By [47, Proposition 2.2.15] and [65, Lemma 2.3.5], it holds that

\[
D((-\Delta_F)^{\alpha}) \hookrightarrow (H_2^{0,0}(M), D(\Delta_F))_{\alpha,2}, \quad 0 < \alpha < \sigma.
\]

Together with Lemma 2.5, this establishes $D((-\Delta_F)^{\alpha}) \hookrightarrow L_p(M)$ when $1 \leq p \leq 2$. To see (4.8) is compact, when $n > 1$, it is a direct consequence of Proposition 2.3. In the case $n = 1$, in view of (4.7) and noting that $\mathbb{C} \phi \hookrightarrow H_p^{1/\sigma+p,s}(M)$ for $s > 0$ sufficiently small, we conclude that compactness of the embedding (4.8) from Proposition 2.3.

Proposition 4.6. For any $1 \leq p < \infty$, it holds that $D((-\Delta_F)^{\alpha}) \hookrightarrow H_p^{1,1+\delta/\sigma}(M)$ for some $\delta > 0$ sufficiently small. As a result, for any $\varepsilon > 0$,

\[
D((-\Delta_F)^{\alpha}) \hookrightarrow H_p^{2,2-\varepsilon, \delta \sigma + \sigma/\sigma - \varepsilon}(M).
\]

In particular, for $1 \leq q \leq p$

\[
D((-\Delta_F)^{\alpha}) \hookrightarrow L_q(M).
\]

Proof. From (3.7), one can conclude that for sufficiently small $\delta > 0$

\[
D(\Delta_F) \hookrightarrow D(\Delta_{\max}) \hookrightarrow H_p^{2,1+\delta/\sigma}(M).
\]

Then the assertion can be proved in an analogous way as in Proposition 4.5. \qed

5 | WEAK SOLUTIONS TO THE FRACTIONAL POROUS MEDIUM EQUATION

In this section, based on the work in Sections 2–4, we will apply the nonlinear semigroup theory and establish the existence of a unique global weak solution to (1.1) with $\sigma \in (0,1)$ in the case $m \in (0,1) \cup (1,\infty)$. Here, $m$ is from (1.1). Note that when $m > 1$, $\Phi(u) \in D((-\Delta_F)^{\alpha}) \hookrightarrow L_1(M)$ shows that $u \in L_m(M) \hookrightarrow L_1(M)$. Then in Section 7, we will show this solution is indeed strong. At the end of Section 7, the case $m = 1$ will be handled separately.

**Definition 5.1.** [11, Chapter II.3]

(i) A nonlinear operator $A : D(A) \subseteq X \rightarrow X$ defined in a Banach space $X$ is called accretive if for all $\lambda > 0$

\[
\|(\text{id} + \lambda A)x_1 - (\text{id} + \lambda A)x_2\|_X \geq \|x_1 - x_2\|_X, \quad x_1, x_2 \in D(A).
\] (5.1)
(ii) A nonlinear operator \( A \) defined in a Banach space \( X \) is called \( m \)-accretive if \( A \) is accretive and it satisfies the range condition

\[
Rng(id + \lambda A) = X, \quad \lambda > 0.
\]

Let \( X = L_1(M) \) and \( A(u) := (\omega - \Delta_{F,1})^2 \Phi(u) \) with domain

\[
D(A) = \{ u \in L_1(M) : \Phi(u) \in D((-\Delta_{F,1})^2) \}.
\]

Note that when \( m \geq 1 \), \( u \in D(A) \) iff \( \Phi(u) \in D((-\Delta_{F,1})^2) \).

Proposition 4.4 implies that for any \( \lambda > 0 \), \((id + \lambda A)^{-1}\) is a bijection between \( D(A) \) and \( L_1(M) \) when \( m \geq 1 \), and (5.1) is fulfilled. In the case \( m < 1 \), given any \( u \in D(A) \), it holds that

\[
f := \lambda A(u) + u \in L_1(M).
\]

(5.2)

Now, assuming \( f \in L_1(M) \), let \( u \) be the unique solution of (5.2) obtained in Proposition 4.4. We have \( u = f - \lambda A(u) \in L_1(M) \) and \( \Phi(u) \in D((-\Delta_{F,1})^2) \), and thus \( u \in D(A) \). Therefore, \( A \) is \( m \)-accrete.

**Lemma 5.2.** \( D(A) \) is dense in \( L_1(M) \).

**Proof.** Note that \( \beta \in BC^{1/m}(\mathbb{R}) \) when \( m \geq 1 \) and \( \beta \in C^1(\mathbb{R}) \) when \( 0 < m < 1 \).

Case 1: \( m \geq 1 \). For any \( w \in L_m(M) \), there exists a sequence \( (u_k)_k \subset D((-\Delta_{F,1})^2) \) converging to \( \Phi(w) \) in \( L_1(M) \). It follows that

\[
\| \beta(u_n) - w \|_1 \leq C \| (u_n - \Phi(w))^{1/m} \|_1 \leq C \| u_n - \Phi(w) \|_1^{1/m}.
\]

This proves that the closure of \( D(A) \) contains \( L_m(M) \). Since \( L_m(M) \) is dense in \( L_1(M) \), we infer that \( D(A) \) is dense in \( L_1(M) \).

Case 2: \( 0 < m < 1 \). Given arbitrary \( w \in L_\infty(M) \), there exists a sequence \( (u_k)_k \subset C^\infty(M) \subset D((-\Delta_{F,1})^2) \) converging to \( \Phi(w) \) in \( L_1(M) \) satisfying that \( \| \beta(u_k) \|_\infty \leq 2 \| w \|_\infty \). So the Lipschitz continuity of \( \beta \) implies

\[
\| \beta(u_k) - w \|_1 \leq C(w) \| u_k - \Phi(w) \|_1 \to 0
\]
as \( k \to \infty \). As above, this implies the density of \( D(A) \) in \( L_1(M) \).

In the sequel, we will always use \( \Delta_g \) to denote the closed extensions \( \Delta_{F,p} \) whenever the choice of \( p \) is clear from context. With the above preparations, we can apply the Crandall–Liggett generation theorem [31, Theorem I] to prove the existence of a \( L_1 \)-global mild solution to the following \( \omega \)-fractional porous medium equation:

\[
\begin{cases}
\partial_t u + (\omega - \Delta_g)^2(|u|^{m-1}u) = 0 & \text{on } M \times (0, \infty); \\
u(0) = u_0 & \text{on } M.
\end{cases}
\]

(5.3)

Mild solutions are defined as the limit of a sequence of approximation solutions by implicit time discretization. More precisely, since a mild solution does not depend on a specific choice of discretization of a time interval \([0, T]\), we may equally divide \([0, T]\) into \( n \) subintervals. Let \( t_k = kT/n \) for \( k = 0, 1, \ldots, n \). Then the discretized problem to (5.3) is

\[
\begin{cases}
\frac{T}{n}(\omega - \Delta_g)^2\Phi(u_{n,k;\omega}) = u_{n,k-1;\omega} - u_{n,k;\omega};& \\
u_{n,0;\omega} = u_0.
\end{cases}
\]

(5.4)

We define the piecewise solution as
\( u_{\omega,0}(0) = u_0, \quad u_{\omega,t}(t) = u_{n,k,\omega} \quad \text{for} \ t \in (t_{k-1}, t_k] \)

Then the mild solution \( u_\omega \) is defined as the uniform limit of \( u_{\omega,t} \), that is, for any \( \varepsilon > 0 \),

\[
\| u_\omega(t) - u_{\omega,t}(t) \|_1 < \varepsilon, \quad t \in [0, T] \quad (5.5)
\]

for sufficiently large \( n \). [31, Theorem I] then implies that the following proposition holds true.

**Proposition 5.3.** Let \((M, g)\) be an \((n + 1)\)-dimensional compact conical manifold. Assume that \( m > 0 \). For every \( u_0 \in L_1(M) \), (5.3) has a unique global mild solution

\( u_\omega \in C([0, \infty), L_1(M)) \).

In the following, we will show that (1.1) has a global weak solution as long as \( u_0 \in L_\infty(M) \quad \text{if} \quad d \circ \rho \quad \text{on} \quad a\lambda \rightarrow L_1(M) \).

**Definition 5.4.** For \( \omega \in [0, \infty) \), we say that \( u \) is an \( L_1 \)-weak solution to (5.3) on \([0, T)\) if

- \( u \in L_\infty((0, T), L^{m+1}(M)) \) and,
- \( \Phi(u) \in L_2((0, T), D((-\Delta_F)^{\sigma/2})) \cap L_\infty, loc((0, T), D((-\Delta_F)^{\sigma/2})). \)

Moreover, for every \( \phi \in C^1_0([0, T), C_\infty^c(M)) \) it holds that

\[
\int_0^T \int_M (\omega - \Delta g)^{\sigma/2} \Phi(u)(\omega - \Delta g)^{\sigma/2} \phi \, d\mu_g \, dt = \int_0^T \int_M u_\phi \phi \, d\mu_g \, dt + \int_M u_0 \phi(0) \, d\mu_g. \quad (5.6)
\]

If in addition, \( u \) satisfies

- \( u \in C([0, T), L_1(M)) \), we call \( u \) a weak solution to (5.3).

By the self-adjointness of \((\omega - \Delta_F)^{\sigma}\), we have

\[
\langle (\omega - \Delta g)^{\sigma} u, v \rangle = \langle (\omega - \Delta g)^{\sigma/2} u, (\omega - \Delta g)^{\sigma/2} v \rangle \quad (5.7)
\]

for all \( u \in D((-\Delta_F)^{\sigma}), v \in D((-\Delta_g)^{\sigma/2}) \). In (5.7), \( \omega = 0 \) is admissible.

Note that \((\omega - \Delta_{F,1})^{\sigma}\) is an extension of \((\omega - \Delta_g)^{\sigma}\) as \( \Delta_{F,1} \) extends \( \Delta_g \). From Proposition 4.2, we infer that \( u \in D((-\Delta_F)^{\sigma}) \) and \((\omega - \Delta_g)^{\sigma} u \in L_2(M) \) implies

\( u \in D((-\Delta_F)^{\sigma}). \)

From Proposition 4.4, we already know that \( \Phi(u_{n,k,\omega}) \in D((-\Delta_{F,1})^{\sigma}) \). In addition, [20, Proposition 4] implies that

\[
\| u_{n,k,\omega} \|_\infty \leq \| u_0 \|_\infty. \quad (5.8)
\]

In view of (5.4), (5.8) reveals that \( (\omega - \Delta_g)^{\sigma} \Phi(u_{n,k,\omega}) \in L_\infty(M) \).

Multiplying the first line of (5.4) by \( \Phi(u_{n,k,\omega}) \) and integrating over \( M \), we obtain in virtue of (5.7) that

\[
\frac{T}{n} \int_M |(\omega - \Delta g)^{\sigma/2} \Phi(u_{n,k,\omega})|^2 \, d\mu_g = \int_M u_{n,k-1,\omega} \Phi(u_{n,k,\omega}) \, d\mu_g - \int_M u_{n,k,\omega} \Phi(u_{n,k,\omega}) \, d\mu_g \\
\leq \left( \int_M |u_{n,k-1,\omega}|^{m+1} \, d\mu_g \right)^{\frac{1}{m+1}} \left( \int_M |\Phi(u_{n,k,\omega})|^{\frac{m+1}{m}} \, d\mu_g \right)^{\frac{m}{m+1}} - \int_M |u_{n,k,\omega}|^{m+1} \, d\mu_g \quad (5.9)
\]
\[ \begin{align*}
&= \frac{1}{m+1} \int_{M} |u_{n,k-1;\omega}|^{m+1} d\mu_{g} + \frac{m}{m+1} \int_{M} |u_{n,k;\omega}|^{m+1} d\mu_{g} - \int_{M} |u_{n,k;\omega}|^{m+1} d\mu_{g} \\
&\leq \frac{1}{m+1} \left( \int_{M} |u_{n,k-1;\omega}|^{m+1} d\mu_{g} - \int_{M} |u_{n,k;\omega}|^{m+1} d\mu_{g} \right). \\
\end{align*} \tag{5.10} \]

We have used the Hölder inequality in (5.9) and the Young’s inequality in (5.10). Then adding from \( k = 1 \) to \( k = n \) yields

\[ \int_{0}^{T} \int_{M} |(\omega - \Delta_{g})^{\sigma/2}\Phi(u_{n,\omega})|^{2} d\mu_{g} dt \leq \frac{1}{m+1} \int_{M} |u_{0}|^{m+1} d\mu_{g}. \tag{5.11} \]

Note that \((\omega - \Delta_{g})^{\sigma/2}\) is invertible. Therefore, \(\Phi(u_{n,\omega})\) is uniformly bounded for all \( n \) in \( L_{2}((0, T), D((-\Delta_{g})^{\sigma/2}))\).

On the other hand, (5.9) or [20, Proposition 4] also implies that

\[ \int_{M} |u_{n,k;\omega}|^{m+1} d\mu_{g} \leq \int_{M} |u_{n,k-1;\omega}|^{m+1} d\mu_{g} \leq \int_{M} |u_{0}|^{m+1} d\mu_{g}. \tag{5.12} \]

This yields the uniform boundedness of \( u_{n,k;\omega}(t) \) in \( L_{m+1}(M) \) for all \( t \in [0, T) \), which implies that

\[ u_{\omega} \in L_{\infty}((0, T), L_{m+1}(M)). \tag{5.13} \]

By the definition of mild solutions, for all \( t \in [0, T) \), (up to a subsequence) \( u_{n;\omega}(t) \) converge to \( u_{\omega}(t) \) pointwise a.e. Hence, we can conclude from (5.11) that

\[ (\omega - \Delta_{g})^{\sigma/2}\Phi(u_{n;\omega}) \rightharpoonup (\omega - \Delta_{g})^{\sigma/2}\Phi(u_{\omega}) \quad \text{in} \quad L_{2}((0, T), L_{2}(M)). \]

and

\[ \|((\omega - \Delta_{g})^{\sigma/2}\Phi(u_{\omega}))\|_{L_{2}((0, T), L_{2}(M))} \leq \frac{1}{m+1} \int_{M} |u_{0}|^{m+1} d\mu_{g}. \tag{5.14} \]

Now, multiplying the first line of (5.4) by \( \phi \in C_{0}^{1}([0, T), C_{c}^{\infty}(M)) \) and integrating over \( M \), we infer that

\[ \langle (\omega - \Delta_{g})^{\sigma/2}\Phi(u_{n,k;\omega}), (\omega - \Delta_{g})^{\sigma/2}\phi \rangle = \frac{n}{T} \int_{M} (u_{n,k-1;\omega} - u_{n,k;\omega})\phi \, d\mu_{g}. \]

Then integrate over \([t_{k-1}, t_{k})\) and sum over \( k = 1, 2, \ldots, n \). The right hand side equals

\[ \frac{n}{T} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_{k}} \int_{M} (u_{n,k-1;\omega} - u_{n,k;\omega})\phi \, d\mu_{g} dt = \int_{0}^{T} \int_{M} u_{n,\omega}(t) \frac{\phi(t + T/n) - \phi(t)}{T/n} \, d\mu_{g} dt + \frac{n}{T} \int_{0}^{1} \int_{M} u_{0}\phi(t) \, d\mu_{g} dt \]

\[ - \frac{n}{T} \int_{t_{n-1}}^{t_{n}} \int_{M} u_{n;\omega}(T)\phi(t) \, d\mu_{g} dt. \]

As \((\omega - \Delta_{g})^{\sigma/2}\phi \in L^{2}((0, T), L_{2}(M))\), letting \( n \to \infty \) yields

\[ \int_{0}^{T} \int_{M} u_{\omega}\phi_t \, d\mu_{g} dt + \int_{M} u_{0}\phi(0) \, d\mu_{g} = \int_{0}^{T} \int_{M} (\omega - \Delta_{g})^{\sigma/2}\Phi(u_{\omega})(\omega - \Delta_{g})^{\sigma/2}\phi \, d\mu_{g} dt \]

\[ = \int_{0}^{T} \int_{M} \Phi(u_{\omega})(\omega - \Delta_{g})^{\sigma/2}\phi \, d\mu_{g} dt. \tag{5.15} \]

The last equality will be used to obtain a weak solution to (1.1).
For every $v \in L^2(M)$, it follows from Proposition 3.6 and the formula above [47, Formula 2.2.24] that

$$\| (\omega - \Delta g)^{-\sigma} v \|_2 \leq C \int_0^\infty t^{n-1} \| e^{t(\Delta g - \omega)} v \|_2 \, dt$$

$$\leq C \int_0^\infty t^{n-1} e^{-\omega t} \, dt \| v \|_2 \leq C \omega^{-\sigma} \| v \|_2. \quad (5.16)$$

This result, together with (5.14) and (4.4), yields

$$\| (-\Delta g)^{\sigma/2} \Phi(u_\omega) \|_{L^2((0,T),L^2(M))}$$

$$\leq \| (\omega - \Delta g)^{\sigma/2} \Phi(u_\omega) \|_{L^2((0,T),L^2(M))} + C \omega^{\sigma/2} \| \Phi(u_\omega) \|_{L^2((0,T),L^2(M))}$$

$$\leq C \| (\omega - \Delta g)^{\sigma/2} \Phi(u_\omega) \|_{L^2((0,T),L^2(M))} < \infty. \quad (5.17)$$

(5.14) and (5.17) imply that there exists a sequence $(\omega_k)_k$ with $\omega_k \to 0^+$ such that

$$(-\Delta g)^{\sigma/2} \Phi(u_{\omega_k}) \rightharpoonup w \text{ in } L^2((0,T),L^2(M)) \quad (5.18)$$

for some $w$ in $L^2((0,T),L^2(M))$; and it follows from (5.12) and (5.13) that for a.a. $t \in (0,T)$

$$u_{\omega_k}(t) \rightharpoonup u(t) \text{ in } L^{m+1}(M) \quad (5.19)$$

for some $u \in L^\infty((0,T),L^{m+1}(M))$.

Let us now refine the estimate for $\| (-\Delta g)^{\sigma/2} \Phi(u_\omega) \|_2$. Suppose that $u_\omega, \hat{u}_\omega$ are mild solutions to (5.3) with respect to the initial data $u_0, \hat{u}_0$. It follows from [20, Theorem 1] that

$$\| u_{n;\omega}(t) - \hat{u}_{n;\omega}(t) \|_1 \leq \| u_0 - \hat{u}_0 \|_1. \quad (5.20)$$

Assume, to the contrary, that

$$\| u_0 - \hat{u}_0 \|_1 < \| u_\omega(t) - \hat{u}_\omega(t) \|_1$$

for some $t > 0$. Then

$$\| u_0 - \hat{u}_0 \|_1 < \| u_\omega(t) - \hat{u}_\omega(t) \|_1$$

$$= \int_M [u_\omega(t) - \hat{u}_\omega(t)] \text{sign}(u_\omega(t) - \hat{u}_\omega(t)) \, d\mu_g$$

$$= \lim_{n \to \infty} \int_M [u_{n;\omega}(t) - \hat{u}_{n;\omega}(t)] \text{sign}(u_{n;\omega}(t) - \hat{u}_{n;\omega}(t)) \, d\mu_g$$

$$\leq \lim_{n \to \infty} \| u_{n;\omega}(t) - \hat{u}_{n;\omega}(t) \|_1.$$ A contradiction. Therefore,

$$\| u_\omega(t) - \hat{u}_\omega(t) \|_1 \leq \| u_0 - \hat{u}_0 \|_1. \quad (5.20)$$

When $m \neq 1$, following the proof of [12, Theorem 1], see also [49, Proposition 8.1], and using (5.20), it holds that as $h \to 0$

$$\frac{1}{h} \int_M |u_\omega(t + h) - u_\omega(t)| \, d\mu_g \leq \frac{2}{|m-1|t} \| u_0 \|_1 + o(1). \quad (5.21)$$
Fix \( \tau \in (0, T) \). For every \( t \in (\tau, T) \), choose \( h > 0 \) so small that \( t + h < T \). For a time discretization of \([0, T)\) with \( n \) subintervals \([t_{k-1}, t_k)\), without loss of generality, we may always assume that \( t = t_i \) and \( t + h = t_j \) for some \( i, j \in \{1, 2, \ldots, n - 1\} \). Otherwise, we can always change the end points of the closest subintervals, as the solution \( u_\omega \) does not depend on the choice of time discretization. Using (5.9), we have

\[
(t_k - t_{k-1}) \int_M |(\omega - \Delta g)^{\sigma/2} \Phi(u_{n,k;\omega})|^2 \, d\mu_g \\
\leq \frac{1}{m+1} \int_M (|u_{n,k-1;\omega}|^{m+1} - |u_{n,k;\omega}|^{m+1}) \, d\mu_g.
\]

Then we sum over all the subintervals \([t_{k-1}, t_k)\) contained in \([t, t + h)\).

\[
\int_{t}^{t + h} \int_M |(\omega - \Delta g)^{\sigma/2} \Phi(u_{\omega}(s))|^2 \, d\mu_g \, ds \\
\leq \frac{1}{m+1} \int_M (|u_{n;\omega}(t)|^{m+1} - |u_{n;\omega}(t + h)|^{m+1}) \, d\mu_g \\
\leq \frac{1}{m+1} \int_M |u_{n;\omega}(t)|^m u_{n;\omega}(t) - |u_{n;\omega}(t + h)|^m u_{n;\omega}(t + h) | \, d\mu_g \\
\leq C(u_0) \frac{1}{m+1} \int_M |u_{n;\omega}(t) - u_{n;\omega}(t + h)| \, d\mu_g
\]

for some \( C = C(u_0) > 0 \). Here, (5.22) is due to (5.8) and the fact that

\[ [x \rightarrow |x|^m x] \in C^{1-}(\mathbb{R}) \quad \text{for} \quad m > 0. \]

Dividing both sides by \( h \) and letting \( n \to \infty \) yields that

\[
\frac{1}{h} \int_{t}^{t + h} \int_M |(\omega - \Delta g)^{\sigma/2} \Phi(u_{\omega})(s)|^2 \, d\mu_g \, ds \\
\leq \frac{C(u_0)}{m+1} \frac{1}{h} \int_M |u_{\omega}(t + h) - u_{\omega}(t)| \, d\mu_g
\]

\[
\leq C(u_0) \frac{2}{(m+1)|m-1|} \|u_0\|_1 + o(1).
\]

Here, (5.23) follows directly from (5.5) and (5.22), and (5.24) is derived from (5.21).

The above inequality holds for all \( h > 0 \) small. Therefore, for a.a. \( t \in (\tau, T) \), \( |(\omega - \Delta g)^{\sigma/2} \Phi(u_{\omega})(t)|_2 \) is uniformly bounded with bound independent of \( \omega \), that is,

\[ \|(\omega - \Delta g)^{\sigma/2} \Phi(u_{\omega})(t)|_2 \leq M, \quad t \in (\tau, T). \]

In the sequel, \( M > 0 \) always denotes a constant independent of \( \omega \). By an analogous argument to (5.17), we can infer that

\[ \|(-\Delta g)^{\sigma/2} \Phi(u_{\omega})(t)|_2 \leq M, \quad t \in (\tau, T). \]

Note that (5.5) and (5.8) imply that

\[ \|u_{\omega}(t)\|_\infty \leq \|u_0\|_\infty. \]
One can conclude that
\[ \| \Phi(u_\omega) \|_{L_\infty((\tau,T),L_2(M))} < M, \]
and thus
\[ \| \Phi(u_\omega) \|_{L_\infty((\tau,T),D((-\Delta)^\sigma/2))} < M. \]
Hence, \( u_\omega \) is indeed a weak solution to (5.3) for any \( \omega > 0 \).

Now, we can apply Proposition 4.6 to show that for every \( t \in (\tau,T) \), there exists a subsequence of \((\omega_k)_k\), still denoted by \((\omega_k)_k\), such that
\[ \Phi(u_{\omega_k})(t) \to v(t) \quad \text{in} \quad L_2(M), \quad (5.27) \]
and as \( D((-\Delta)^\sigma/2) \) is a Hilbert space,
\[ \Phi(u_{\omega_k})(t) \rightharpoonup v(t) \quad \text{in} \quad D((-\Delta)^\sigma/2). \quad (5.28) \]
We can thus infer that \( \Phi(u_{\omega_k})(t) \to v(t) \) pointwise a.e. (up to a subsequence). In view of (5.18) and (5.19), we conclude that \( v(t) = \Phi(u)(t) \) a.e.; and since \( \tau \) is arbitrary, in (5.18), we have \( w = (-\Delta)^\sigma/2 \Phi(u) \).

Letting \( \omega_k \to 0^+ \) in (5.15), then (5.7), (5.19) and the dominated convergence theorem imply that
\[
\int_0^T \int_M u \frac{\partial}{\partial t} \phi \, d\mu_g \, dt + \int_M u_0 \phi (0) \, d\mu_g = \lim_{k \to \infty} \int_0^T \int_M u_{\omega_k} \frac{\partial}{\partial t} \phi \, d\mu_g \, dt + \int_M u_0 \phi (0) \, d\mu_g \\
= \lim_{k \to \infty} \int_0^T \int_M (\omega_k - \Delta_g)^{\sigma/2} \Phi(u_{\omega_k})(\omega_k - \Delta_g)^{\sigma/2} \phi \, d\mu_g \, dt \\
= \lim_{k \to \infty} \int_0^T \int_M \Phi(u_{\omega_k})(\omega_k - \Delta_g)^{\sigma} \phi \, d\mu_g \, dt \\
= \lim_{k \to \infty} \int_0^T \int_M \Phi(u_{\omega_k})(-\Delta_g)^{\sigma} \phi \, d\mu_g \, dt + \lim_{k \to \infty} \int_0^T \int_M \Phi(u_{\omega_k})(\omega_k - \Delta_g)^{\sigma} \phi \, d\mu_g \, dt. \quad (5.29)
\]
Since for any \( 1 < p < \infty \),
\[ \| (\omega_k - \Delta_g)^{\sigma} \phi - (-\Delta_g)^{\sigma} \phi \|_p \leq C \alpha_k^p \| \phi \|_p, \]
in view of (5.12), we conclude that (5.30) tends to 0 as \( k \to \infty \). (5.29) can be estimated as follows.
\[
\lim_{k \to \infty} \int_0^T \int_M \Phi(u_{\omega_k})(\omega_k - \Delta_g)^{\sigma} \phi \, d\mu_g \, dt = \lim_{k \to \infty} \int_0^T \int_M (-\Delta_g)^{\sigma/2} \Phi(u_{\omega_k})(-\Delta_g)^{\sigma/2} \phi \, d\mu_g \, dt \\
= \int_0^T \int_M (-\Delta_g)^{\sigma/2} \Phi(u)(-\Delta_g)^{\sigma/2} \phi \, d\mu_g \, dt.
\]
In the last step, we have used (5.18). This gives rise to

\[ \int_0^T \int_M u \partial_t \phi \, d\mu_g \, dt + \int_0^T u_0 \phi(0) \, d\mu_g = \int_0^T \int_M (-\Delta_g)^{\sigma/2} \Phi(u)(-\Delta_g)^{\sigma/2} \phi \, d\mu_g \, dt. \]

Therefore, we obtain an \( L_1 \)-weak solution to (1.1).

**Lemma 5.5.** There is at most one \( L_1 \)-weak solution to (1.1) with \( u_0 \in L_\infty(M) \).

**Proof.** The proof is basically the same as the porous medium equation, c.f. [66, Theorem 5.3]. First observe that the weak formulation (5.6) with \( \omega = 0 \) still holds true for functions \( \phi \in W^{1}_2((0,T), D((-\Delta_g)^{\sigma/2})) \) with \( \phi(T) = 0 \). Indeed, take a sequence

\[ C^1([0,T), C_c^\infty(M)) \ni \phi_n \to \phi \quad \text{in} \quad W^{1}_2([0,T), D((-\Delta_g)^{\sigma/2})). \]

Then

\[ \int_0^T \int_M (-\Delta_g)^{\sigma/2} \Phi(u)(-\Delta_g)^{\sigma/2} \phi_n \, d\mu_g \, dt = \int_0^T \int_M u \partial_t \phi_n \, d\mu_g \, dt + \int_0^T u_0 \phi_n(0) \, d\mu_g. \]

Letting \( n \to \infty \) on both sides yields the desired result.

Assume, to the contrary, there are two \( L_1 \)-weak solutions \( u, \bar{u} \). Define

\[ \phi(t) = \int_t^T (\Phi(u) - \Phi(\bar{u})) \, ds, \quad 0 \leq t \leq T. \]

Then \( \phi \in W^{1}_2((0,T), D((-\Delta_g)^{\sigma/2})) \) with \( \phi(T) = 0 \) is a valid test function. Multiplying \( \phi \) to the first line of (1.1) and integrating over \([0,T) \times M\) yields

\[ \int_0^T \int_M (-\Delta_g)^{\sigma/2} (\Phi(u) - \Phi(\bar{u}))(t) \left[ \int_t^T (-\Delta_g)^{\sigma/2} (\Phi(u) - \Phi(\bar{u}))(s) \, ds \right] \, d\mu_g \, dt 
+ \int_0^T \int_M (u - \bar{u})(t)(\Phi(u) - \Phi(\bar{u}))(t) \, d\mu_g \, dt = 0. \]

The first term on the left equals

\[ \frac{1}{2} \int_M | \int_0^T (-\Delta_g)^{\sigma/2} (\Phi(u) - \Phi(\bar{u}))(t) \, dt |^2 \, d\mu_g. \]

Therefore,

\[ \frac{1}{2} \int_M | \int_0^T (-\Delta_g)^{\sigma/2} (\Phi(u) - \Phi(\bar{u}))(t) \, dt |^2 \, d\mu_g 
+ \int_0^T \int_M (u - \bar{u})(t)(\Phi(u) - \Phi(\bar{u}))(t) \, d\mu_g \, dt = 0. \]

Note that \( \Phi \) is monotone. So both terms are non-negative. This implies that \( u = \bar{u} \) a.e.  \( \square \)

One can further argue that the solution \( u \in C([0,T), L_1(M)) \). To this end, we will first prove the following two lemmas.
Lemma 5.6. For any $\sigma \in (0, 1]$, assume that $u$ is the unique $L_1$-weak solution to (1.1). Then for any $\phi \in C_0^1([0, T), D((-(\Delta g)^{\sigma/2}))$,

$$\lim_{t \to 0^+} \int_M u(t) \phi(t) \, d\mu_g = \int_M u_0 \phi(0) \, d\mu_g.$$  

Proof. The proof of this lemma is partially based on Theorem 6.1(I) and (II) in Section 6. Their proofs work for $L_1$-weak solutions and thus apply here. Given any $\varepsilon \in (0, T)$ and $\phi \in C_0^1([0, T), D((-(\Delta g)^{\sigma/2}))$, we consider the problem:

$$\begin{cases}
\partial_t v + (-(\Delta g)^{\sigma/2} \Phi(v)) = 0 & \text{on } M \times (\varepsilon, T);

v(\varepsilon) = u(\varepsilon) & \text{on } M.
\end{cases} \quad (5.31)$$

By Theorem 6.1(I) and the same process we used for (1.1), there exists a unique $L_1$-weak solution to (5.31) such that

$$\int_\varepsilon^T \int_M (-(\Delta g)^{\sigma/2} \Phi(v))((-(\Delta g)^{\sigma/2} \Phi) \, d\mu_g \, dt = \int_\varepsilon^T \int_M v \partial_t \phi \, d\mu_g \, dt + \int_M u(\varepsilon) \phi(\varepsilon) \, d\mu_g. \quad (5.32)$$

It follows from Lemma 5.5 that $v(t) = u(t)$ a.e. for $t \in [\varepsilon, T]$. Because

$$\left| \int_0^\varepsilon \int_M (-(\Delta g)^{\sigma/2} \Phi(u))((-(\Delta g)^{\sigma/2} \Phi) \, d\mu_g \, dt \right| \leq C \int_0^\varepsilon \|(-(\Delta g)^{\sigma/2} \Phi(u))_2 \, dt \to 0$$

and

$$\left| \int_0^\varepsilon \int_M u \partial_t \phi \, d\mu_g \, dt \right| \leq C \int_0^\varepsilon \|u(t)\|_2 \, dt \to 0$$

as $\varepsilon \to 0$ due to Theorem 6.1(II), the asserted results follows by letting $\varepsilon \to 0$ in (5.32).  

Next, we will show the $L_1$-contraction property of $L_1$-weak solutions to (1.1).

Lemma 5.7. Suppose that $u, \hat{u}$ are $L_1$-weak solutions to (1.1) with respect to the initial data $u_0, \hat{u}_0$. Then for any $t \geq 0$

$$\|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1.$$  

Proof. This assertion follows by an analogous argument to (5.20).  

Now, we are ready to prove the continuity of $u$. Take a sequence

$$D((-\Delta g)^{\sigma/2}) \cap L_\infty(M) \ni u_{0,n} \to u_0 \quad \text{in } L_1(M).$$

We denote by $u_{0,n}$ the unique $L_1$-weak solution to (1.1) with initial data $u_{0,n}$.

In particular, Lemma 5.6 shows that

$$\lim_{t \to 0^+} \int_M u_n(t) u_{0,n} \, d\mu_g = \int_M u_{0,n}^2 \, d\mu_g.$$  

Now, we look at

$$\langle u_n(t) - u_{0,n}, u_n(t) - u_{0,n} \rangle = \|u_n(t)\|_2^2 + \|u_{0,n}\|_2^2 - 2\langle u_n(t), u_{0,n} \rangle.$$
Taking \( \limsup \) on both sides yields
\[
\limsup_{t \to 0^+} \langle u_n(t) - u_{0,n}, u_n(t) - u_{0,n} \rangle = \limsup_{t \to 0^+} \|u_n(t)\|_2^2 - \|u_{0,n}\|_2^2 \leq 0.
\]
The last inequality is due to Theorem 6.1(II). Note that the proof for Theorem 6.1(II) is completely independent of the convergence of the solutions at \( t = 0 \). This implies that
\[
\lim_{t \to 0^+} \langle u_n(t) - u_{0,n}, u_n(t) - u_{0,n} \rangle = 0,
\]
which shows that \( u_n \) is right continuous at \( t = 0 \) in the \( L_2(\mathcal{M}) \)-norm and thus in the \( L_1(\mathcal{M}) \)-norm. Now, we have
\[
\lim_{t \to 0^+} \|u(t) - u_0\|_1 \leq \lim_{n \to \infty} \lim_{t \to 0^+} \left( \|u(t) - u_n(t)\|_1 + \|u_n(t) - u_{0,n}\|_1 + \|u_{0,n} - u_0\|_1 \right)
\leq 2 \lim_{n \to \infty} \|u_{0,n} - u_0\|_1 + \lim_{n \to \infty} \lim_{t \to 0^+} \|u_n(t) - u_{0,n}\|_1 = 0
\tag{5.33}
\]
The inequality in (5.33) follows from Lemma 5.7. This shows the right continuity of \( u \) at \( t = 0 \) in the \( L_1(\mathcal{M}) \)-norm. To show the interior temporal regularity, first note that for every \( \varepsilon > 0 \), there exists a \( \tau > 0 \) such that \( 0 < h < \tau \) implies
\[
\|u(h) - u(0)\|_1 \leq \varepsilon.
\]
We apply Lemmas 5.5 and 5.7 again and obtain for every \( t > 0 \) and \( 0 < h < \tau \)
\[
\|u(t + h) - u(t)\|_1 \leq \|u(h) - u(0)\|_1 \leq \varepsilon.
\]
This shows that \( u \in C([0, T), L_1(\mathcal{M})) \), and thus \( u \) is indeed a weak solution to (1.1).

**Theorem 5.8.** Let \((\mathcal{M}, g)\) be an \((n + 1)\)-dimensional compact conical manifold. Assume that \( m > 0 \) with \( m \neq 1 \). Then for any \( T > 0 \) and every \( u_0 \in L_\infty(\mathcal{M}) \), (1.1) has a unique weak solution on the interval \([0, T)\). Moreover, the solution depends continuously on \( u_0 \) in the norm \( C([0, T), L_1(\mathcal{M})) \).

**Proof.** The continuous dependence of the weak solution on \( u_0 \) is a direct consequence of Lemma 5.7. \( \square \)

**Remark 5.9.**
(i) Well-posedness for \( L_2 \)-initial data is related to the smoothing effect, which can be established via some nonlocal logarithmic Sobolev inequality based on (3.14). The idea follows from a generalization of the argument in [57], which will be discussed in a future work.
(ii) The method used in this section can be applied to the (fractional) porous medium equation on general complete Riemannian manifolds without any difficulty to establish the uniqueness and existence of global weak solution. See [57, Theorem 2.3].

**6 SOME OTHER PROPERTIES OF THE FRACTIONAL POROUS MEDIUM EQUATION**

**Theorem 6.1.** Under the same assumptions as in Theorem 5.8, the weak solutions to (1.1) satisfy the following properties.

(I) Comparison principle: If \( u, \hat{u} \) are the unique weak solutions to (1.1) with initial data \( u_0, \hat{u}_0 \), respectively, then \( u_0 \leq \hat{u}_0 \) a.e. implies \( u \leq \hat{u} \) a.e.

(II) \( L_p \)-contraction: For all \( 0 \leq t_1 \leq t_2 \) and \( 1 \leq p \leq \infty \)
\[
\|u(t_2)\|_p \leq \|u(t_1)\|_p.
\]
(III) Conservation of mass: For all \( t \geq 0 \), it holds that
\[
\int_M u(t) \, d\mu_g = \int_M u_0 \, d\mu_g.
\]

Proof.

(I) Suppose \( u_{n,k;\omega} \) and \( \hat{u}_{n,k;\omega} \) are the solutions obtained in the kth step of the discretized problem (5.4) with respect to the initial data \( u_0, \hat{u}_0 \), respectively. Then [20, Proposition 5] implies that
\[
\int_M [u_{n,k+1;\omega} - \hat{u}_{n,k+1;\omega}]^+ \, d\mu_g \leq \int_M [u_{n,k;\omega} - \hat{u}_{n,k;\omega}]^+ \, d\mu_g \leq \int_M [u_0 - \hat{u}_0]^+ \, d\mu_g. \tag{6.1}
\]
We immediately conclude from the above inequality that \( u_0 \leq \hat{u}_0 \) a.e. implies \( u_{n,\omega} \leq \hat{u}_{n,\omega} \) a.e.

Suppose that, for some \( t \), \( m(\{u_\omega(t) > \hat{u}_\omega(t)\}) > 0 \), where \( m \) is the measure on \((M,g)\). Pick \( f \in L^\infty(M) \) such that \( f = 1 \) on \( \{u_\omega(t) > \hat{u}_\omega(t)\} \), and \( f = 0 \) elsewhere. Then
\[
0 < \int_M [u_\omega(t) - \hat{u}_\omega(t)]^+ \, d\mu_g = \int_M (u_\omega(t) - \hat{u}_\omega(t)) f \, d\mu_g = \lim_{n \to \infty} \int_M (u_{n,\omega}(t) - \hat{u}_{n,\omega}(t)) f \, d\mu_g \leq \lim_{n \to \infty} \int_M [u_{n,\omega}(t) - \hat{u}_{n,\omega}(t)]^+ \, d\mu_g \leq \int_M [u_0 - \hat{u}_0]^+ \, d\mu_g = 0,
\]

as \( u_{n,\omega}(t) - \hat{u}_{n,\omega}(t) \to u_\omega(t) - \hat{u}_\omega(t) \) in \( L_1(M) \). A contradiction. Applying the same argument to \( u(t) \) and \( \hat{u}(t) \) yields the desired result.

(II) By a similar argument to part I, we have
\[
\int_M [u(t_2) - \hat{u}(t_2)]^+ \, d\mu_g \leq \int_M [u(t_1) - \hat{u}(t_1)]^+ \, d\mu_g, \quad t_2 > t_1.
\]
This readily implies that
\[
\|u(t_2)\|_1 \leq \|u(t_1)\|_1 \quad \text{and} \quad \|u(t_2)\|_\infty \leq \|u(t_1)\|_\infty.
\]

Hence, for the remaining case \( 1 < p < \infty \), it suffices to consider \( t_1 = 0 \), for otherwise we can replace \( u_0 \) by \( u(t_1) \) in (1.1). By [20, Proposition 4], the solution of the discretized problem (5.4) satisfies
\[
\|u_{n,k+1;\omega}\|_p \leq \|u_{n,k,\omega}\|_p \leq \|u_0\|_p, \quad 1 \leq p \leq \infty.
\]
Taking limit in the above inequality yields (II) when \( 1 < p < \infty \).

(III) Since constant functions belong to \( D(\Delta_F^\omega) \) and thus are in \( D((-\Delta_F)^{\gamma/2}) \), we can take \( \phi \equiv 1 \) as a test function in (5.6) with \( \omega = 0 \), that is,
\[
\int_M [u(t_2) - u(t_1)] \phi \, d\mu_g = -\int_{t_1}^{t_2} \int_M (-\Delta_F)^{\gamma/2} \Phi(u)(-\Delta_F)^{\gamma/2} \phi \, d\mu_g \, dt = 0.
\]
This proves the conservation of mass. \( \square \)
Remark 6.2. The conservation of mass holds for all $m > 0$, and this is an essential difference from the extinction phenomenon observed for the (fractional) porous medium equation on $\mathbb{R}^N$ and in bounded domains with vanishing Dirichlet boundary condition, cf. [13] for the finite time extinction of solutions for the porous medium equation and [49] for the fractional porous medium equation.

Remark 6.3. The proof of conservation of mass actually shows that if $(M, g)$ is an anti-cone then the mass is actually conserved on both components of $M$. This means the nonlocal diffusion of (1.1) cannot transport any mass from one component to another through the conical ends.

7 | STRONG SOLUTIONS

Definition 7.1. We call a weak solution $u$ to (1.1) on $[0, T)$ a strong solution if in addition

$$\partial_t u \in L_{\infty, \text{loc}}((0, T), L_1(M))$$

and

$$(-\Delta_g)^{\sigma} \Phi(u) \in L_{\infty, \text{loc}}((0, T), L_1(M)).$$

We will use a modification of the method in [48, 49] to prove that solutions to (1.1) are indeed strong solutions. We first consider the case $m \neq 1$. By [12, Theorems 1 and 2], we have

(i) $\partial_t u$ is a Radon measure;
(ii) $\limsup_{h \to 0^+} \frac{\|u(t+h) - u(t)\|_1}{h} \leq \frac{2}{|m-1|} \|u_0\|_1$.

For any $f \in L_{1, \text{loc}}(0, T)$, define the Steklov average of $f$ by

$$f^h(t) := \frac{1}{h} \int_t^{t+h} f(s) \, ds,$$

and

$$\delta^h f(t) := \partial_t f^h(t) = \frac{f(t+h) - f(t)}{h} \text{ a.e.}$$

We can write the weak formulation (5.6) with $\omega = 0$ as

$$\int_0^T \int_M (\delta^h u) \Phi \, d\mu_g \, dt + \int_0^T \int_M (-\Delta_g)^{\sigma/2} (\Phi(u))^h (-\Delta_g)^{\sigma/2} \Phi \, d\mu_g \, dt = 0. \quad (7.1)$$

For any $[\tau, S] \subset (0, T)$, we choose $\xi \in C^1_0((0, T), [0, 1])$ such that $\xi \equiv 1$ on $[\tau, S]$ and vanishes outside $[\tau', S']$ for some $[\tau', S'] \subset (0, T)$ with $[\tau, S] \subset (\tau', S')$. Let us take $\phi = \xi \delta^h (\Phi(u))$. Then (7.1) yields

$$\int_0^T \int_M \xi (\delta^h u) \delta^h (\Phi(u)) \, d\mu_g \, dt$$

$$+ \int_0^T \int_M \xi (-\Delta_g)^{\sigma/2} (\Phi(u))^h (-\Delta_g)^{\sigma/2} \partial_t (\Phi(u))^h \, d\mu_g \, dt = 0. \quad (7.2)$$
Since \((\delta^h u)(\delta^h \Phi(u)) \geq c(\delta^h(|u|^{(m-1)/2}u))^2\), cf. [48, Section 5.3], the first term on the left hand side of (7.2) satisfies that
\[
\int_0^T \int_M \zeta (\delta^h u)(\delta^h \Phi(u)) \, d\mu_g \, dt \geq c \int_0^T \int_M \xi (\delta^h(|u|^{(m-1)/2}u))^2 \, d\mu_g \, dt.
\]

The second term on the left hand side of (7.2) can be estimated as follows.
\[
\left| \int_0^T \int_M \xi (-\Delta_g)^{\sigma/2}E(\Phi(u))h(-\Delta_g)^{\sigma/2}e\eta(\Phi(u))^h \, d\mu_g \, dt \right| \\
\leq \frac{1}{2} \int_{t'}^{S'} \int_M \xi'^2 |(-\Delta_g)^{\sigma/2}E(\Phi(u))|^2 \, d\mu_g \, dt \\
\leq C \int_{t'}^{S'} \int_M |(-\Delta_g)^{\sigma/2}E(\Phi(u))|^2 \, d\mu_g \, dt. \tag{7.3}
\]

Recall a weak solution \(\Phi(u) \in L_{\infty,loc}((0,T),D((-\Delta_g)^{\sigma/2}))\). By Minkowski’s inequality,
\[
\left( \int_M [(-\Delta_g)^{\sigma/2}E(\Phi(u))h]^2 \, d\mu_g \right)^{1/2} = \frac{1}{h} \left( \int_M \left( \int_t^{t+h} (-\Delta_g)^{\sigma/2}E(\Phi(u))(s) \, ds \right)^2 \, d\mu_g \right)^{1/2} \\
\leq \frac{1}{h} \int_t^{t+h} \left( \int_M [(-\Delta_g)^{\sigma/2}E(\Phi(u))(s)]^2 \, d\mu_g \right)^{1/2} \, ds \\
\leq \sup_{t \in [t',S']} \left( \int_M [(-\Delta_g)^{\sigma/2}E(\Phi(u))(t)]^2 \, d\mu_g \right)^{1/2}.
\]

Thus (7.3) is uniformly bounded in \(h\). This implies that
\[
\|\delta^h(|u|^{(m-1)/2}u)\|_{L^2([t,S],L^2(M))}
\]
is uniformly bounded in \(h\), and thus
\[
\hat{\delta}_t(|u|^{(m-1)/2}u) \in L^2_{2,loc}((0,T),L^2(M)).
\]

Since (i) implies \(u \in BV((\tau,T),L^1(M))\) for any \(\tau > 0\), it then follows from [14, Theorem 1.1] that
\[
\hat{\delta}_t u \in L^\infty_{2,loc}((0,T),L^1(M)) \quad \text{with} \quad \|\hat{\delta}_t u(t)\|_1 \leq \frac{2}{|m-1|t} \|u_0\|_1.
\]

The last inequality is derived from (ii). Since \((-\Delta_g)^{\sigma}E(u) = -\hat{\delta}_t u\) in the distributional sense, the above conclusion reveals that
\[
(-\Delta_g)^{\sigma}E(u) \in L^\infty_{2,loc}((0,T),L^1(M)).
\]

**Theorem 7.2.** Let \((M, g)\) be an \((n+1)\)-dimensional compact conical manifold. Assume that \(m > 0\) with \(m \neq 1\) and \(u_0 \in L^\infty(M)\). Then for any \(T > 0\), (1.1) with \(\sigma \in (0, 1)\) has a unique strong solution on \([0,T)\).

For the remaining case \(m = 1\), by [50, Theorem 4.1.4] and Proposition 4.2, we obtain the following theorem.
Theorem 7.3. Let \((M, g)\) be an \((n + 1)\)-dimensional compact conical manifold. Assume that \(m = 1\) and \(1 < p < \infty\). Then for any \(T > 0\) and every \(u_0 \in L_p(M)\), (1.1) with \(\sigma \in (0, 1)\) has a unique solution \(u\) on \([0, T)\) such that

\[
u \in C^1([0, T), L_p(M)) \cap C([0, T), D((-\Delta_{F, p})^\sigma))
\]

When \(u_0 \in L_\infty(M)\), this solution is strong, depends continuously on \(u_0\) in the norm \(C([0, T), L_1(M))\) and satisfies (I)-(III) of Theorem 6.1.

Proof. The unique solution to (1.1) when \(m = 1\) is given by

\[
u(t) = e^{-t(-\Delta)^\sigma} u_0.
\]

Then the uniqueness, regularity and continuous dependence of \(u\) on \(u_0\) follows immediately. (I) and (II) of Theorem 6.1 is a direct consequence of Proposition 4.2. (III) of Theorem 6.1 is obtained by multiplying both sides of (1.1) by \(\phi \equiv 1\) and integrating over \([t_1, t_2] \times M\).

\[\square\]

8 | THE POROUS MEDIUM EQUATION ON CONICAL MANIFOLDS

When \(\sigma = 1\), (1.1) reduces to the usual porous medium equation

\[
\begin{cases}
\partial_t u - \Delta_g(|u|^{m-1}u) = 0 & \text{on } M \times (0, \infty); \\
u(0) = u_0 & \text{on } M.
\end{cases}
\]

In this last section, we will study the well-posedness and various properties of (8.1) for \(m > 0\). These results generalize the study of the porous medium equation on manifolds with singularities in \([54, 55, 63]\).

Since the proofs for the results in this section are analogous to the fractional porous medium equation, only necessary modifications of the proofs will be presented.

Theorem 8.1. Let \((M, g)\) be an \((n + 1)\)-dimensional compact conical manifold. Assume that \(m > 0\) with \(m \neq 1\). Then for any \(T > 0\) and every \(u_0 \in L_\infty(M)\), (8.1) has a unique strong solution on \([0, T)\) in the sense that

(i) \(u \in L_\infty((0, T), L_{m+1}(M))\), and
(ii) \(\Phi(u) \in L_2((0, T), H^1(M)) \cap L_\infty,loc((0, T), H^1(M))\), and
(iii) \(u \in C([0, T), L_1(M))\), and

for every \(\phi \in C^1_0((0, T), C^\infty(M))\), it holds that

\[
\int_0^T \langle \nabla \Phi(u), \nabla \phi \rangle_T dt = \int_0^T \int_M u \partial_t \phi d\mu_g dt + \int_M u_0 \phi(0) d\mu_g.
\]

In addition,

\[
\partial_t u \in L_\infty,loc((0, T), L_1(M))
\]

and

\[
\Delta_g \Phi(u) \in L_\infty,loc((0, T), L_1(M)).
\]

Moreover, the solutions satisfy the following properties.

1. Continuous dependence on the initial data: The solution depends continuously on \(u_0\) in the norm \(C([0, T), L_1(M))\).
2. Comparison principle: If \( u, \hat{u} \) are the unique strong solutions to (8.1) with initial data \( u_0, \hat{u}_0 \), respectively, then \( u_0 \leq \hat{u}_0 \) a.e. implies \( u \leq \hat{u} \) a.e.

3. \( L_p \)-contraction: For all \( 0 \leq t_1 \leq t_2 \) and \( 1 \leq p \leq \infty \)

\[
\| u(t_2) \|_p \leq \| u(t_1) \|_p.
\]

4. Conservation of mass: For all \( t \geq 0 \), it holds that

\[
\int_M u(t) \, d\mu_g = \int_M u_0 \, d\mu_g.
\]

**Proof.** Note that by the divergence theorem

\[
\langle -\Delta_g u, v \rangle = \langle \nabla u, \nabla v \rangle_T
\]

for all \( u \in D(\Delta_g), v \in W^1_2(M) \). So following the same approach for the fractional porous medium equation in Section 5, one can show, with \( (-\Delta_g)^{\sigma/2} \) being replaced by \( \nabla \), that (8.1) has a unique solution satisfying (i)-(iii) and (8.2).

(8.3) can be proved along the same line of argument in Section 7. Once we have (8.3), note that (8.2) implies that

\[
\int_T^S \int_M \partial_t u \phi \, d\mu_g \, dt + \int_T^S \langle \nabla \Phi(u), \nabla \phi \rangle_T \, dt = 0
\]

for any \( \tau, S \in [0, T) \) Thus \( -\partial_t u(t) \) is indeed the weak divergence of \( \nabla u(t) \) for a.a. \( t \). We can conclude that (8.4) holds.

Properties (1)-(3) can be established via the same argument we used for (1.1).

The proof of (4) is the same as that of Theorem 6.1(III). \( \square \)

When \( m = 1 \), we can apply [50, Theorem 4.1.4] and Proposition 3.6 to attain an analogous result to Theorem 7.3.

**Theorem 8.2.** Let \( (M, g) \) be an \((n + 1)\)-dimensional compact conical manifold. Assume that \( m = 1 \) and \( 1 < p < \infty \). Then for any \( T > 0 \) and every \( u_0 \in L_p(M) \), (8.1) has a unique solution \( u \) on \([0, T)\) such that

\[
u \in C^1([0, T), L_p(M)) \cap C([0, T), D(\Delta_{F,p})).\]

In particular, when \( u_0 \in L_\infty(M) \), \( u \) is a strong solution and satisfies (1)-(4) of Theorem 8.1.

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