On the space of super maps between smooth super manifolds

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Abstract

Mapping spaces of supermanifolds are usually thought as exclusively in functorial terms (i.e. trough the Grothendieck functor of points). In this work we provide a geometric description of such mapping spaces in terms of infinite-dimensional super-vector bundles.

Keywords: smooth super-manifolds, mapping space of smooth supermanifolds, jets spaces, global analysis, manifolds of smooth mapping space.

1 Introduction

Let $\textbf{SVect}_\mathbb{R}$ be the category of super-vector spaces over $\mathbb{R}$ with the standard structure of tensor category and braiding corresponding to the Koszul sign rule convention (see [10]). The linear algebra in $\textbf{SVect}_\mathbb{R}$ has two notions of linear operations: categorical hom, $\text{Hom}_{\textbf{SVect}_\mathbb{R}}(V,W)$ is an ordinary vector space for each couple of super-vector spaces $V, W \in \textbf{SVect}_\mathbb{R}$ and it consists of all $\mathbb{R}$-linear operators preserving the parity; the inner hom, $\text{Hom}(V,W)$, defined as the adjoint functor to the tensor bi-funcctor, which is a super-vector space consisting of all $\mathbb{R}$-linear operators. The setting change drastically passing to the category of finite dimensional smooth supermanifold $\textbf{SMan}$. In notation of [10] each element $M \in \textbf{SMan}$ will be represented by the corresponding locally ringed space $(M, \mathcal{O}_M)$. This category has a natural monoidal category structure, with Cartesian product of two supermanifolds $M = (M, \mathcal{O}_M), N = (N, \mathcal{O}_N)$ given by the coproduct of the corresponding locally ringed space (see appendix). Analogously to $\textbf{SVect}_\mathbb{R}$, there are two notions of mapping space in $\textbf{SMan}$: the categorical hom, i.e. morphisms of locally ringed spaces, still denoted as $\text{Hom}(M,N)$ and the inner hom, $\text{Hom}(M,N)$ (see appendix, [10]). However one treats such spaces only as functors, i.e. through the Grothendieck.

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functor of points, without spelling out the structure of infinite-dimensional super-
manifolds. Many remarkable attempts to define a category of infinite dimensional super-
manifolds have been performed: \[9, 7-8\]. We intend to suggest a different
approach to the same problem, that up to our opinion has the advantage of being
less abstract and very close to the coordinate viewpoint used in physics for “super-
fields”. Our aim is to describe both mapping space as particular infinite dimensional super-vector bundle over the infinite dimensional smooth manifold mapping space
\(C^\infty(M, N)\), given by the set of all smooth mappings from \(M\) to \(N\) according to
\[4, 5\]. In paragraph §2 the starting point is the classical result known as Batchelor
theorem, which affirms that each smooth supermanifold is non-canonically isomor-
phic to a vector bundle over a classical manifold with an odd fiber. Then we shall
identify any morphism of smooth supermanifolds with a certain bundle map and fi-
nally, by the use of an auxiliary connection, with a morphism of vector bundles. This
proof consists of many steps and some preliminary work on classical jets of a smooth
manifold \(M\) is needed. In paragraph §3 the framework change: we sketchy review the
theory of infinite dimensional smooth manifold following P. Michor \([4, 5]\). In fact
it is necessary to recall some general results in the setting of global analysis in order
to describe the infinite dimensional super vector bundle structure of \(\text{Hom}(M, N)\)
and \(\text{Hom}(M, N)\). Standing this result \(\text{SM}\text{an}\) will be identified with a full subcategory
of a bigger category containing the aforementioned infinite dimensional super-vector
bundle. Finally the link between our construction and the categorical definition of
\(\text{Hom}(M, N)\) is discussed.

2 Superfields vs. vector bundles

Notation. We denote with \(\text{Man}\) the category of finite dimensional smooth mani-
folds. Let \(M, N \in \text{Man}\) we denote the mapping space \(C^\infty(M, N)\) with \(\text{Hom}_{\text{Man}}(M, N)\)
(or even neglecting \(\text{Man}\) when it is clear from the context)

Let \(M\) be a smooth supermanifold of superdimension \((p, q)\) over a base \(M\). Ac-
cording to the fundamental result of smooth supergeometry \[1\], there exists a vector
bundle \(V\) of rank \(q\) over \(M\) such that \(M\) is diffeomorphic as a supermanifold to
\(\Pi V\), that is, to the total space of \(V\) with the reversed parity of fibres, the algebra
of functions on which is canonically identified with \(\Gamma(\Lambda^*V^*)\). In other words, each
supermanifold is diffeomorphic to a bundle over a classical manifold with an odd
fiber. However, a general morphism of supermanifolds \(f: M \rightarrow N\), which covers a
base map \(f_0: M \rightarrow N\), is not a bundle map. Later on we shall identify any mor-
phism of smooth supermanifolds with a certain bundle map and then, by the use of
an auxiliary connection, with a morphism of vector bundles. We intend to prove the
following:
Proposition 1 Let $M$ and $N$ be supermanifolds which are super diffeomorphic to $\Pi W$ and $\Pi V$ for vector bundles $W \to M$ and $V \to N$. There is a one-to-one non canonical correspondence between morphisms $f \in \text{Hom}(M,N)$, which cover a base map $f_0 : M \to N$ and sections of even degree of the following super-bundle over $M$

$$\tilde{V} := S^{(k)}_+(W^*[1]) \otimes f_0^*(TN) \oplus S^{(k)}(W^*[1]) \otimes f_0^*(V[1])$$

with $k$ the rank of $W$, $S^{(k)}(W^*[1])$ the graded symmetric algebra up to order $k$ of $W^*[1]$ (resp. augmented), and $f_0^*(TN), f_0^*(V[1])$ the pull-back bundle of $TN, V[1]$ along $f_0$.

The graded morphisms, elements of $\text{Hom}(M,N)$, are obtained by the last proposition dropping off the condition about the degree and considering the projective limit of algebras.

Proposition 2 The mapping space that to each $f_0 \in \text{Hom}(M,N)$ associates the space of section of $\tilde{V}$ as in proposition $(1)$, is a locally trivial infinite dimensional vector bundle over the smooth manifold $\text{Hom}(M,N)$.

2.1 Setup and proof

Let $M$ be a smooth manifold, $\mathcal{O}_M$ be the sheaf of local smooth functions on $M$. Let $J^k(M)$ be the bundle of $k$–jets of local smooth functions on $M$, $J^k_M$ be the sheaf of its local sections, and $j^k : \mathcal{O}_M \to J^k_M$ the morphism of $\mathbb{R}$–sheaves, which associates to any (local) smooth function its $k$–jet. Given any coordinate chart $(U, \{x^i\})$, we can identify $J^k_M(U)$ with $C^\infty(U) \otimes \mathbb{R}[\delta x^1, \ldots, \delta x^n]/m^{k+1}$, where $\delta x^i$ are generators of the polynomial algebra associated to $\{x^i\}$ and $m$ is an ideal of polynomials vanishing at 0. Then for each smooth $h(x)$ on $U$ one has

$$j^k(h)(x, \delta x) = h(x + \delta x) \mod m^{k+1}$$

$$\sim \sum_{\mu_1 + \ldots + \mu_n \leq k} \frac{(\delta x^1)^{\mu_1} \ldots (\delta x^n)^{\mu_n}}{\mu_1! \ldots \mu_n!} (\partial_{x^1})^{\mu_1} \ldots (\partial_{x^n})^{\mu_n} f(x),$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with non-negative integer components. For simplicity, further we shall use the following compact form for multi-index notations:

$$\sum_{\mu_1 + \ldots + \mu_n \leq k} \frac{(\delta x^1)^{\mu_1} \ldots (\delta x^n)^{\mu_n}}{\mu_1! \ldots \mu_n!} (\partial_{x^1})^{\mu_1} \ldots (\partial_{x^n})^{\mu_n} f(x) = \sum_{|\mu| = 0}^{k} \frac{(\delta x)^{\mu}}{\mu!} \partial_{x}^{\mu} f(x).$$

\footnote{We will moreover assume $V$ and $W \mathbb{Z}$–graded.}
Let \( \{y^i\} \) be another coordinate system, such that \( y^i = f^i(x) \), then the corresponding change of polynomial generators of \( J^k_M(U) \) over \( C^\infty(U) \) is

\[
\delta y^i = (f^i(x + \delta x) - f^i(x)) \mod m^{k+1} \sim \sum_{|\mu|=1}^{k} \frac{\delta x^\mu}{\mu!} \partial_x^\mu f^i(x).
\] (4)

Besides the canonical multiplication on local functions, given by the embedding \( i^k: f \mapsto f \otimes 1 \) for each \( f \) in \( \mathcal{O}_M \), the sheaf \( J^k_M \) admits the second structure of an \( \mathcal{O}_M \)-algebra, determined by the embedding \( j^k \).

2.2 These two structures (of ringed spaces over \( M \)) have the following clear geometrical meaning (cf. [2], also [3]). Let \( M \times M \) be the Cartesian product of two copies of \( M \) with natural projections \( \sigma \) and \( \tau \) onto the factors:

\[
\begin{array}{ccc}
M \times M & \xrightarrow{\sigma} & M \\
& \xleftarrow{\tau} & \\
M & & M
\end{array}
\] (5)

Let \( M \hookrightarrow M \times M \) be the diagonal embedding and let \( m_M \) be the sheaf of local smooth functions on \( M \times M \) vanishing on the diagonal. The coset sheaf \( \mathcal{O}_{M \times M}/m^{k+1}_M \) can be thought of as the structure sheaf of the \( k \)-th order neighborhood of the diagonal, denoted as \( M^{(k)} \). The restrictions of the projections \( \sigma \) and \( \tau \) to \( M^{(k)} \), which we denote as \( \sigma_k \) and \( \tau_k \), respectively, give us the structure of two ringed spaces mentioned above, such that \( i^k = \sigma_k^k \) and \( j^k = \tau_k^k \). Further we shall treat \( \mathcal{O}_{M \times M} \) and its coset sheaves as sheaves of \( \mathcal{O}_M \)-algebras with respect to the first module structure, unless another assumption is explicitly stated. Then \( J^k_M \) is isomorphic to \( \mathcal{O}_{M \times M}/m^{k+1}_M \).

In fact, in the definition of the \( k \)-th order neighborhood one can replace \( \mathcal{O}_{M \times M} \) with \( \mathcal{O}_{\tilde{U}} \), where \( \tilde{U} \) is any tubular neighborhood of the diagonal. More precisely, let \( r(x, x') \) be a smooth function on \( U \times U \) which represents an element \([r]\) in the quotient sheaf of algebras (here \( \{x'\} \) are the same coordinates on the copy of \( U \)). Then the image of \([r]\) in \( J^k(U) \) is

\[
r(x, x + \delta x) \mod m^{k+1}_X \sim \sum_{|\mu|=0}^{k} \frac{\delta x^\mu}{\mu!} \partial_x^\mu r(x, x')_{|x = x'}.
\] (6)

2.3 Given an arbitrary (torsion free) connection \( \nabla \) in \( TM \), we can establish a diffeomorphism between a neighborhood of the zero section in \( TM \) and some \( \tilde{U} \). Indeed,

\(\footnote{We use the fact that the normal bundle to the diagonal in \( M \times M \) is naturally isomorphic to \( TM \).}
let \( z(x, \xi, t) \) be the solution of the Cauchy problem for geodesic equation with the initial data \( (x, \xi) \in TM \), then the required map is \( (x, \xi) \mapsto (x, z(x, \xi, 1)) \). The image of the zero section in \( TM \) under the map above coincides with the diagonal.

As a simple corollary, the \( k \)-th order neighborhood of the diagonal is isomorphic to the \( k \)-th order neighborhood of the zero section in \( TM \), which we denote as \( TM_0^{(k)} \).

More precisely, let \( \{x^i, \xi^i\} \) and \( \{x^i, \delta x^i\} \) be the local coordinates of \( TM_0^{(k)} \) and \( M \), respectively, which are canonically associated to coordinates \( \{x^i\} \) of \( M \), and let \( \Gamma^i_{jl}(x) \) be the Christoffel symbols of \( \nabla \) in these coordinates. Then, taking into account the property \( z^i(x, t\xi, 1) = z^i(x, \xi, t) \), we obtain the following asymptotic expansion of \( z^i(x, \xi, 1) \):

\[
z^i(x, \xi, 1) = \sum_{m=0}^{k} \frac{1}{m!} \partial_t^m z^i(x, t\xi, 1)_{|t=0} \mod \xi^{k+1}
\]

where by \( \mod \xi^{k+1} \) we mean the quotient modulo all terms of degree higher than \( k \).

Now we apply the geodesic flow equation for \( z^i(x, \xi, t) \)

\[
\partial^2_t z^i = -\Gamma^i_{jl}(z) \partial_t z^j \partial_t z^l
\]

with the initial data

\[
z^i(x, \xi, 0) = x^i, \quad \partial_t z^i(x, \xi, 0) = \xi^i
\]

to establish the required isomorphism of \( k \)-th order neighborhoods \( (x, \xi) \mapsto (x, \delta x(x, \xi)) \):

\[
\delta x^i = (z^i(x, \xi, 1) - x^i) \mod \xi^{k+1} = \\
= \xi^i - \frac{1}{2} \Gamma^i_{jl}(x) \xi^j \xi^l + \frac{1}{6} \left( -\partial_x \Gamma^i_{jl}(x) + 2 \Gamma^i_{ps}(x) \Gamma^p_{jl}(x) \right) \xi^s \xi^j \xi^l + \ldots.
\]

The structure sheaf of \( TM_0^{(k)} \) is canonically isomorphic to \( S^{(k)}(T_M^*) \) where \( T_M^* \) is the sheaf of smooth sections of the cotangent bundle and \( S^{(k)}(T_M^*) \) is the quotient of the symmetric algebra of \( T_M^* \) generated over \( O_M \) by the ideal of elements of degree higher than \( k \). Therefore \( J_M^k \simeq S^{(k)}(T_M^*) \) as sheaves of \( O_M \)-algebras. The next proposition provides us with an explicit formula for the isomorphism between \( J_M^k \) and \( S^{(k)}(T_M^*) \) (fixed by a torsion free connection \( \nabla \) in \( TM \)) which we denote by \( \Phi^k \).

Let \( \chi^k \) be a derivation of the quotient sheaf \( S^{(k)}(T_M^*) \) determined by the symmetrized covariant derivative

\[
S^*(T_M^*) \xrightarrow{\nabla} T^* \otimes S^*(T_M^*) \rightarrow S^{*+1}(T_M^*).
\]
Here the last arrow is the symmetrization map \( 3 \).

**Proposition 3** For any local function \( h \) the following identity holds: \( \Phi^k \circ j^k(h) = \exp(\chi^k)(h) \).

**Proof.** Given that the statement, we are going to prove, is coordinately independent, it is enough to check it in arbitrary local coordinates \( \{ x^s \} \) for \( h = x^i \). In the associated coordinates \( \{ x^i, \xi^i \} \) on \( TM \) the derivation \( \chi^k \) will have the following expression:

\[
\chi = \xi^s \frac{\partial}{\partial x^s} - \Gamma^s_{jl}(x) \xi^j \xi^l \frac{\partial}{\partial \xi^s} \mod \xi^{k+1},
\]

(12)

where \( \Gamma^i_{jk} \) are the Levi-Civita coefficients of \( \nabla \). Let \( \{ t, z^s, p^s, p^s_2, p^s_3, \ldots \} \) be the local coordinates on the space of jets of parameterized curves in \( M \) associated to \( \{ x^s \} \), where \( z^s \) is a copy of \( x^s \) (used for the convenience), \( p^s_k \) corresponds to the \( k \)-th derivative of \( z^s \), and \( p^s = p^s_1 \). The 2d order geodesic equation (8) will be written as follows:

\[
p^s_2 = -\Gamma^s_{jl}(z) p^j p^l.
\]

(13)

Apparently, \( p^s_k = D^k_t(z^s) \), where \( D_t \) is the total derivative with respect to \( t \). Let us consider the infinite prolongation of (13) as a subspace in the jet space. The total derivative is tangent to this subspace. Then \( p_k \) becomes a function of \( z \) and \( p \) for all \( k \geq 2 \), which allows to express \( D_t \) in the form

\[
D_t = \partial_t + p^s \frac{\partial}{\partial z^s} - \Gamma^s_{jl}(z) p^j p^l \frac{\partial}{\partial p^s}.
\]

(14)

Using (7), (9) and (14) and taking into account (12), we obtain that

\[
\Phi^k(x^i) = \sum_{m=0}^{k} \frac{1}{m!} \left( \xi^s \frac{\partial}{\partial x^s} - \Gamma^s_{jl}(x) \xi^j \xi^l \frac{\partial}{\partial \xi^s} \right)^m (x^i) \mod \xi^{k+1} = \exp(\chi^k)(x^i).
\]

(15)

\( \square \)

**2.4** The last result can be generalized to the sheaf of \( k \)-jets of sections \( J^k(A) \) of an arbitrary sheaf of graded commutative algebras \( A \) over \( O_M \) which is freely generated by some finitely graded vector bundle \( V = \oplus_{j \in \mathbb{Z}} V^j \) over \( M \), i.e. \( A = S^*(V^*) \), where \( V \) is the sheaf of smooth sections of \( V \).

**Proposition 4** The algebras \( J^k(A) \) and \( S^{(k)}(T^*_M) \otimes_{O_M} A \) are isomorphic. The choice of an isomorphism is uniquely fixed by the choice of a torsion free connection in \( TM \) and a graded connection in \( V \).

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\( ^3 \)The symmetrized covariant derivative preserves the ideal of symmetric tensors of degree \( > k \), thus \( \chi^k \) is well-defined.
Proof. One can check that $\mathcal{J}^k(A)$ is canonically isomorphic to $\tau^*_k(A)$ such that $j^k(s) = \tau^*_k(s)$ for any local section $s$ of $\mathcal{A}$, where $\tau_k$ is the second projection of $M^{(k)}$ onto $M$. Taking into account that $\mathcal{A}$ is freely generated as a graded commutative algebra over $\mathcal{O}_M$ by $V^*$ and thus $\mathcal{J}^k(A)$ is freely generated over $\mathcal{J}_k$ by $\tau^*_k(V^*)$, it is sufficient to prove that $\sigma^*_kV^* \simeq \tau^*_kV^*$ as $\mathcal{J}^k$-modules. The last statement is a corollary of the following fact. Let $\tilde{U}$ be a tubular (geodesic) neighborhood of the diagonal given by an arbitrary torsion free connection in $TM$ and let $\sigma$ and $\tau$ be the restrictions to $\tilde{U}$ of the corresponding projections onto $M$. Let us fix a connection in $V$. Then $\sigma^*V$ and $\tau^*V$ are canonically isomorphic over $\tilde{U}$. Indeed, $\tilde{U}$ is isomorphic to $TM$ as a bundle over $M$ where the first bundle structure is determined by $\sigma$. In this setting the $\sigma$-fibers are not only contractible, but moreover the contraction is uniquely fixed. This allows to trivialize $\tau^*V$ along $\sigma$-fibers in the unique way by use of parallel transport. Finally we apply the same technique as above to pass from the tabular neighborhood to the $k$-th order neighborhood of the diagonal. □

Remark 1 This procedure is actually dual to the “quantization” of symbols of differential operators by use of connections.

Let us denote by $\Phi^k$ the isomorphism between $\mathcal{J}^k(A)$ and $S^{(k)}(T_M^*) \otimes A$ fixed by a torsion free connection $\nabla^{TM}$ in $TM$ and a vector connection $\nabla^V$ in $V$. Let $\chi^k$ be the mapping determined by the symmetrized covariant derivative $S^*(T_M^*) \otimes A \to S^{*+1}(T_M^*) \otimes A$ on the quotient $S^{(k)}(T_M^*) \otimes A$.

Proposition 5

- The composition $\Phi^k \circ j^k: A \to S^{(k)}(T_M^*) \otimes A$ coincides with the restriction of $\exp(\chi^k)$.

- Let $\Phi^k_t$ be the isomorphism determined by another couple of connection. Then there exists a nilpotent automorphism $\Psi$ of the algebra $S^{(k)}(T_M^*) \otimes S^*(A)$ over $\mathcal{O}_M$, such that $\Phi^k_t = \Psi \circ \Phi^k$.

Proof. The first part of Proposition 5 is a simple generalization of Proposition 3. Let $\Phi^k_t$, $t \in [0,1]$ be the family of isomorphisms determined by the linear interpolation between the new and old connections, such that $\Phi^k_{t=1} = \Phi^k$, $\Phi^k_{t=0} = \Phi^k$. The variation $\delta^k = \partial_t \chi^k_t$ can be identified with a section of $\mathfrak{g}^1$ where $\mathfrak{g}^r = S^{r+1}(T_M^*) \otimes T_M \oplus S^r(T_M^*) \otimes \text{End}(A)$.

Apparently, $\mathfrak{g}^+ = \oplus_{r>1} \mathfrak{g}^r$ can be viewed as the sheaf of nilpotent derivations of $S^{(k)}(T_M^*) \otimes S^*(A)$ over $\mathcal{O}_M$, thus $\delta^k$ is a nilpotent derivation. Then $\Phi^k_t \circ j^k = \exp(\chi^k_t) = \exp(\chi^k_t) \circ \exp(-\chi^k) \circ \Phi^k \circ j^k$.  

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Taking into account that \( J^k(A) \) is generated by \( j^k(A) \) over \( O_M \), we immediately obtain that
\[
\Phi^k_t = \left( \exp(\chi^k_t) \circ \exp(-\chi^k_t) \right) \circ \Phi^k,
\]
which implies that \( \partial_t \Phi^k_t = \psi_t \circ \Psi^k_t \), where
\[
\psi_t = \int_0^1 \exp(\tau \chi^k_t) \circ \partial_t \chi^k_t \circ \exp(-\tau \chi^k_t) d\tau = \int_0^1 \exp \left( \tau \text{ad}_{\chi^k_t} \right)(\delta^k) d\tau.
\]
It is clear that \( \text{ad}_{\chi^k_t}(g^r) \subset g^{r+1} \), therefore \( \psi_t \in g^+ \) and hence \( \Phi^k_t = \Psi_{t=0} = \text{Id} \).

Given that \( \psi_t \) belongs to the Lie algebra of nilpotent derivations of \( S^{(k)}(\mathcal{T}^*_M) \otimes S^*(A) \) over \( O_M \), we deduce that \( \Psi_t \) must be a nilpotent automorphism of the same algebra. Setting \( \Psi = \Psi_{t=1} \) we finalize the proof. \( \square \)

2.5 Let \( M \) and \( N \) be supermanifolds which are super diffeomorphic to \( \Pi W \) and \( \Pi V \) for vector bundles \( W \to M \) and \( V \to N \), respectively.

**Proposition 6** There exists a one-to-one correspondence between morphisms of supermanifolds \( f: M \to N \), which cover a base map \( f_0: M \to N \), and super bundle morphisms \( \bar{f}: \Pi W \to \Pi(\tau^*_k V) \) such that the following diagram becomes commutative

\[
\begin{array}{ccc}
\Pi IV & \xrightarrow{f} & N \\
\downarrow{\gamma_k} & & \downarrow{\tau_k} \\
\Pi W & \xrightarrow{\tau_k(\Pi IV)} & N^{(k)} \\
\downarrow{\sigma_k} & & \downarrow{f_0} \\
M & & N \\
\end{array}
\]

Here \( \tau^*_k(\Pi IV) \) is the pull-back of \( \Pi IV \) on \( N^{(k)} \) and \( \gamma_k \) is the natural projection \( \tau^*_k(\Pi IV) \to \Pi IV \) over \( \tau_k: N^{(k)} \to N \).

**Proof.** In one direction, we lift \( f_0 \) to a morphism \( f': M \to N \) (as zero on \( W \)-fibers). Let us consider the super morphism \( f' \times f: M \to N \times N \). By definition, this super morphism fits to the following commutative diagram:
It is clear that $N \times \Pi V$ is diffeomorphic to $\Pi(\tau^* V)$, where $\tau$ is the projection of $N \times N$ onto the second factor (as before), $\sigma$ and $\gamma$ are the projections of $N \times \Pi V$ onto $N$ and $\Pi V$, respectively. The proof of Proposition 6 follows from the next Lemma.

**Lemma 1** The image of $f' \times f$ belongs to $\Pi(\tau^*_k V)$, where $k = \text{rk} W$.

**Proof.** The proof of the Lemma is a straightforward generalization of the next local observation. Let us assume for simplicity that $V = 0$. Then $\Pi(\tau^*_k V)$ is just the $k$th order neighborhood of the diagonal $M^{(k)}$ the structure sheaf of which over an open $U \subset M$ is the quotient $O_U \times O_U$ by the ideals of functions on $U \times U$ vanishing on the diagonal up to the $k$th order. We have to show that $(f' \times f)^*(F)$ for any such $F$ is zero. Indeed, if $f$ is given by

$$x^i = f^i_0(y) + \frac{1}{2} \sum_{a,b} f^i_{ab}(y)\theta^a \theta^b + \ldots$$  \hspace{1cm} (18)

where $y$ and $\theta$ are local even and odd coordinates of $\Pi W$, respectively, then

$$(f' \times f)^*(F)(y, \theta) = F(f(y), f(y)) + \frac{1}{2} \sum_j \partial_{\theta^j} F(f(y), z)|_{z = f(y)} f^j_{ab}(y)\theta^a \theta^b + \ldots .$$  \hspace{1cm} (19)

On the other hand $(\theta)^\mu = 0$ for each multi-index of length greater than $k$. Therefore, if $F$ vanishes on the diagonal up to the $k$th order its image is identically zero. □

In the opposite direction, the statement is tautologically trivial. □

2.6 As a corollary of Proposition 6 and using Proposition 4 we conclude that morphisms of supermanifolds covering $f_0$ are in one to one correspondence with section of (1) of even degree.

Thus “superfields”, elements of $\text{Hom}(M, N)$, are in one-to-one correspondence with all sections of (1) where we consider the projective limit over $k$ - i.e. the fields become graded.

\footnote{Generally, $V$ and $W$ should be also $\mathbb{Z}$-graded.}
3 Global aspects

We sketchy review the theory of infinite-dimensional manifold structure for the mapping space $\text{Hom}_{\text{Man}}(M,N)$ following [4, 5]. In particular we review the $\mathcal{D}$-topology and its refinement $\mathcal{D}^\infty$-topology, and use it to recall that $\text{Hom}(M,N)$ is a smooth manifold modelled on spaces $\mathcal{D}(F)$ of smooth sections with compact support of vector bundles $F$ over $M$. A particular type of infinite-dimensional vector bundle over $\text{Hom}(M,N)$ is introduced. Then the proof of Proposition 2 follows. The last proposition describes the link between the categorical definition of $\text{Hom}(M,N)$ and our construction.

3.7 Topology. For each integer $n \in \mathbb{N}$, let $J^n(M,N)$ be the smooth fibre bundle of $n$-jets of maps from $M$ to $N$. For each $f \in \text{Hom}(M,N)$ let $j^n f : M \to J^n(M,N)$ being a smooth section, where $j^n f(x)$ is the $n$-jet of $f$ at $x \in M$.

Definition 1 ([4]) Let $K = (K_n)$, $n \in \mathbb{N}$ be a fixed sequence of compact subset of $M$ such that $K_0 = \emptyset$, $K_{n-1} \subset K_n^\circ$, $\forall n \ X = \bigcup_n K_n$ ($K_n^\circ$ open interior of $K_n$). Then consider sequences $m = (m_n), U = (U_n)$ for $n = 0, 1, ...$ such that $m_n$ is non-negative integer and $U_n$ is open in $J^{m_n}(M,N)$. For each such pair $(m, U)$ of sequences define a set $O(m, U) \subset \text{Hom}(M,N)$

$$O(m, U) = \{ f \in \text{Hom}(M,N) | j^{m_n} f(K_n^\circ) \subset U_n \ \forall n \in \mathbb{N} \}$$

The $\mathcal{D}$-topology on $\text{Hom}(M,N)$ is given by taking all sets $O(m, U)$ as a basis for its open sets.

The $\mathcal{D}$-topology is finer than the Whitney-$C^\infty$ topology ($W^\infty$-topology). The $W^\infty$-topology is defined by

$$W^\infty = \bigcup_{k=0}^{\infty} W^k,$$

where each $W^k$ has the following properties:

1) A basis for open sets is given by all sets of the form $W^k(U) = \{ g \in C^k(M,N), j^k g(M) \subset U \}$, where $U$ is open in $J^k(M,N)$.

2) If $d_k$ is a metric on $J^k(M,N)$ ($0 \leq k \leq \infty$) generating the topology, and if $f \in C^k(M,N)$, then the following is a neighborhood basis for $f$ in the $W^k$-topology:

$$N(f, k, \epsilon) := \{ g \in C^r(M,N) : d_k(j^k g(x), j^k f(x)) < \epsilon(x), \forall x \in M \}$$

\begin{footnote}
\text{completely metrizable space}
\end{footnote}
where $\epsilon \in C(M,]0,\infty[)$.

3) A sequence $f_n$ in $C^r(M,N)$ converges to $f \in C^r(M,N)$ in $W^k$-topology iff there exists a compact set $K \subseteq M$ such that $f_n$ equals $f$ off $K$ for all but finitely many $n$’s and $j^k f_n \to j^k f$ uniformly on $K$.

At point 2) by the use of the metric $d_k$, the concept of “$k$-uniform convergence” has been introduced. Observe that if $f_n \to f$ in the $\mathfrak{D}$-topology then $f_n \to f$ in the $W$-topology, in particular:

**Lemma 2** [4] A sequence $(f_n) \in \text{Hom}(M,N)$ converges in the $\mathfrak{D}$-topology to $f$ iff there exists a compact set $K \subset M$ such that all but a finitely many of the $f_n$’s equal $f$ off $K$ and $j^l f_n \to j^l f$ uniformly on $K$, for all $l \in \mathbb{N}$.

**Definition 2** Let $M,N$ as above, if $f,g \in \text{Hom}(M,N)$ and the set

$$\{x \in M \mid f(x) \neq g(x)\}$$

is relatively compact in $M$, we call $f$ equivalent to $g$, $f \sim g$.

This is clearly an equivalence relation. The $\mathfrak{D}^\infty$-topology on the set $\text{Hom}(M,N)$ is now the weakest among all topologies on $\text{Hom}(M,N)$ which are finer than $\mathfrak{D}$-topology and for which all equivalence classes of the above relation are open.

### 3.8 local model.

For each $f \in \text{Hom}(M,N)$ define $\Gamma_f \subset M \times N$ the graph of $f$, and $\pi_f : E(f) \to \Gamma_f$ the vector bundle over $\Gamma_f$, given by (essentially pullback of the vector bundle $TN$ along $f$)

$$E(f) = \bigcup_M T_{(x,f(x))}(\{x\} \times N) \subset T_{\Gamma_f}(M \times N)$$

(with $T_{(x,f(x))}(\{x\} \times N) = \{0\}_x \times T_{f(x)}(N)$ and $\{0\}_x$ the zero section of $TM$ over $x$). $E(f)$ is a realization of the normal bundle to $T_{\Gamma_f}(\Gamma_f)$ in $T_{\Gamma_f}(M \times N)$. Now let

$$\exp^N : V \subset T(N) \to N$$

be exponential map associated to a torsion free connection, defined on a neighborhood $V$ of the zero section in $TN$. The exponential map gives a diffeomorphism of $E(f)$ onto an open neighborhood $Z_f$ of $\Gamma_f$ in $M \times N$, given by

$$(0,v_x) \in E(f) \to (x,\exp^N_{f(x)}(v_x)) \in Z_f \subset M \times N,$$

if $v_x \in V$. It is clear that $\rho_f : Z_f \to \Gamma_f$ is a vector bundle with vertical projection$^6$, i.e. $\rho_f(x,y) = (x,f(x)) \ \forall (x,y) \in Z_f$. We will denote with $C^\infty(Z_f)$ the set of

$^6$actually it is a tubular neighborhood of $\Gamma_f$ in $M \times N$
sections of \((Z_f, \rho_f, \Gamma_f)\). The exponential map gives a fibre preserving diffeomorphism 
\[ \tau_f : E(f) \to Z_f \]
\[ \begin{array}{cccc}
E(f) & \xrightarrow{\tau_f} & Z_f \\
\downarrow{\pi_f} & & \downarrow{\rho_f} \\
\Gamma_f & \end{array} \]
\[ (20) \]

For \( f \in \text{Hom}(M, N) \) we define the set \( U_f \) given by
\[ U_f = \{ h \in \text{Hom}(M, N) | \Gamma_h \subset Z_f, \ h \sim f \} \]
In particular each \( U_f \) is open in the \( D^\infty \)-topology.

In words: \( h \) is in the open neighborhood of \( f \) iff the corresponding graph \( \Gamma_h \) is contained in \( Z_f \) and \( f \sim h \) according to definition \([2]\).

The space of sections of the vector bundle \( \pi_f : (E_f) \to \Gamma_f \), with compact support endowed with \( D^\infty \)-topology, is a locally convex topological linear space (\([4]\) pag. 57). It will be denoted as \( \mathcal{D}(E_f) \).

**3.9 smooth structure.** We declare \( U_f \) being a local chart for \( f \) with the coordinate mapping \( \phi_f : U_f \to \mathcal{D}(E_f) \)
\[ \phi_f(g) = \tau_f^{-1}(x, g(x)) \quad g \in U_f, \ (x, f(x)) \in X_f \]
and \( \psi_f : \mathcal{D}(E_f) \to U_f \) defined by
\[ \psi_f(s)(x) = \pi_N \circ \tau_f \circ s(x, f(x)), \ s \in (\mathcal{D}(E_f)), x \in M, \]
with \( \pi_N : M \times N \to N \) the canonical projection. The maps \( \psi_f \) and \( \phi_f \) are inverse to each other and the chart change for \( s \in \phi_g(U_g \cap U_f) \) is defined by
\[ \phi_f \circ \psi_g(s)(x, f(x)) = \tau_f^{-1} \circ \tau_g \circ s(x, g(x)). \]

**Theorem 1** \([4]\) Hom\((M, N)\) is a smooth manifold with smooth atlas \((U_f, \psi_f)_{f \in \text{Hom}(M, N)}\).

Another useful theorem is the following

**Theorem 2** \([6]\) The infinite dimensional smooth vector bundle \( T\text{Hom}(M, N) \) (“tangent bundle of Hom\((M, N)\)”) and the space of sections with compact support of the pullback vector bundle \( C^\infty_c(M, f^*TN) \), \( \forall f \in \text{Hom}(M, N) \), are canonically isomorphic.
3.10 locally trivial vector bundles. Let \((U_f, \psi_f)_{f \in \text{Hom}(M, N)}\) be an atlas for \(\text{Hom}(M, N)\). Smooth curves in \(U_f \subset \text{Hom}(M, N)\) are just the images under \(\psi_f\) of smooth sections of the bundle \(pr_2^*E(f) \to \mathbb{R} \times M, pr_2 : \mathbb{R} \times M \to M\), which have compact support in \(M\) locally in \(\mathbb{R}\).

Let \((V, \pi_V, M)\) and \((W, \pi_W, N)\) be two smooth vector bundles of finite rank over \(M\) and \(N\) respectively. Define \((\tilde{V}, \pi, \text{Hom}(M, N))\) as the mapping space that to each \(f : M \to N\) associates the space of sections \(C^\infty(M, V \otimes f^*W)\) restricted to the graph of \(f\) with compact support (endowed with the \(\mathcal{D}^\infty\)-topology).

**Proposition 7** \((\tilde{V}, \pi, \text{Hom}(M, N))\) is a smooth vector bundle over \(\text{Hom}(M, N)\).

**Proof.** The proof consists in the introduction of a local trivialization for \((\tilde{V}, \pi, \text{Hom}(M, N))\) adapted to the atlas \((U_f, \psi_f)_{f \in \text{Hom}(M, N)}\). Besides the torsion free connection on \(N\), we consider \(W\) endowed with a connection, \(\nabla_W\). For each \(f \in \text{Hom}(M, N)\) consider \(\eta \in C^\infty(E(f))\) a smooth section with compact support of the pullback bundle \(f^*TN\), and define the associated curve in \(\text{Hom}(M, N)\),

\[
\gamma_t := \psi_f(t \cdot \eta), \quad t \in [0, 1] \subset U_f,
\]

with \(t \cdot \eta\) a section of \(pr_2^*E(f) \to [0, 1] \times M \subset \mathbb{R} \times M\) obtained by fiberwise multiplication, i.e. \(t \cdot \eta(x) = t(\eta(x))\). The parallel transport \(P_{\exp(\gamma_t, \nabla_W, t)}\) along the curve \(\gamma_t\) induces an isomorphism between the spaces of sections \(C^\infty(M, f^*W)\) and \(C^\infty(M, g^*W)\). Taking into account that the trivialization for \(V\) in \(\tilde{V}\) is obvious, we immediately get a trivialization of \(\tilde{V}\) over \(U_f\).

**Remark 2** The previous result is just a generalization of the following result in differential geometry; given \((V, \pi, M)\) a vector bundle over \(M\) with connection \(\nabla_V\), and \(U \subset M\) an open ball in \(M\): parallel transport along rays in \(U\) gives a smooth trivialization of \(V\) over \(U\), by identifying the fibres with the parallel transport map.

3.11 Follow the proof of Proposition 2. Let \((U_f, \psi_f)_{f \in \text{Hom}(M, N)}\) be a smooth atlas for \(\text{Hom}(M, N)\) as before; for each \(g \in U_f\) consider the corresponding mapping space (typical fiber) given by the space of sections of \(\tilde{V}\) in \(\text{Hom}(M, N)\) as in Proposition 6

\[
\tilde{V}_g = C^\infty(M, S_+^{(k)}(W^*[1]) \otimes g^*(TN) \oplus S^{(k)}(W^*[1]) \otimes g^*(V[1])).
\]

Since the hypothesis of Proposition 7 for the data \((\tilde{V}_g \in U_f, (U_f, \psi_f)_{f \in \text{Hom}(M, N)})\) are verified, the result follows. \(\square\)

3.12

**Proposition 8** The categorical property

\[
\text{Hom}(\mathbb{Z} \times M, N) = \text{Hom}(\mathbb{Z}, \text{Hom}(M, N)), \quad \forall \mathbb{Z}, M, N \in \text{SMan}
\]  

is verified.
Proof. Let’s denote with $\Pi L, \Pi W, \Pi V$ three vector-bundles with bases respectively the smooth manifolds $Z, M, N$, diffeomorphic to $Z, M, N$. For the right-hand side of the previous equality let’s choose a map $F \in \text{Hom}(Z, \text{Hom}(M, N))$; such a map locally taking value in a generic open set $U_f \subset \text{Hom}(M, N)$, consists of a section of the following bundle

$$pr_2^*E(f)$$

$$Z \times M$$

with $pr_2 : Z \times M \to M$ the projection in the second element. By the use of Theorem 2 the tangent bundle to $\text{Hom}(M, N)$ is identified with the sections with compact support of the pullback tangent bundle to $N$. Applying the same arguments used in the proof of Proposition 1 we get a one to one correspondence between $\text{Hom}(Z, \text{Hom}(M, N))$ and sections of

$$\{S_+^{(q)}(pr_1^*L^*[1]) \otimes pr_2^*(E(f)) \oplus S_+^{(q)}(pr_1^*L^*[1]) \otimes pr_2^*(\tilde{V}_f[1])\}_{f \in \text{Hom}}$$

$$Z \times M$$

with $q$ the rank of $\Pi L$. For left-hand side of the equality, let’s choose a map $G \in \text{Hom}(Z \times M, N)$ which corresponds to $F$ for the categorical property $\text{Hom}(Z \times M, N) = \text{Hom}(Z, \text{Hom}(M, N))$ in $\text{Man}$. Due to Proposition 1 the sections of $\text{Hom}(Z \times M, N)$ are in one to one correspondence with the sections of

$$S_+^{(q)}(pr_1^*L[1]) \otimes S_+^{(k)}(pr_2^*W[1]) \otimes G^*(TN) \oplus S_+^{(q)}(pr_1^*L[1]) \otimes S_+^{(k)}(pr_2^*W[1]) \otimes G^*(V[1])$$

$$Z \times M.$$  

The sections of (24) and (23) coincide by construction.

Addendum. Actually what is crucial in this identification is the fact that the space of sections of $\{pr_2^*E(f)\}_{f \in \text{Hom}(M, N)} \to Z \times M$ for a given $F \in \text{Hom}(Z, \text{Hom}(M, N))$ coincides with the space of sections of $G^*(TN) \to Z \times M$, with $F, G$ related by the mentioned categorical property. □

4 Appendix

In the category $\text{SMan}$ the finite product ($\times$) is defined in the following way: given $\mathcal{M} = (M, O_M), \mathcal{N} = (N, O_N) \in \text{SMan}$, $\mathcal{M} \times \mathcal{N}$ is the graded-manifold obtained attaching to each rectangular open set $U \times V \subset M \times N$ the super-algebra
\( \mathcal{O}_{M \times N}(U \times V) := \mathcal{O}_M(U) \otimes \mathcal{O}_N(V) \), where \( \mathcal{O}_M(U) \otimes \mathcal{O}_N(V) \) is the completion of \( \mathcal{O}_{M_0}(U) \otimes \mathcal{O}_{N_0}(V) \). In particular \( \text{SMan} \) is a monoidal category, with tensor product equal to the coproduct of ringed spaces (i.e. \( \otimes := \times \)) and the identity given by the terminal object \( \mathbb{R}^{0,0} \) in \( \text{SMan} \), i.e. the locally ringed space with the singleton \{\(*\)\} as topological space and \( \mathbb{R} \) as sheaf. In every closed monoidal category the internal Hom, for any two objects \( \mathcal{N}, \mathcal{M} \), is equipped with the canonical morphism (evaluation)

\[
ev_{
, \m} : \text{Hom}(\mathcal{N}, \mathcal{M}) \otimes \mathcal{N} \to \mathcal{M},
\]

which is universal (for every object \( \mathcal{X} \) and morphism \( f : \mathcal{X} \otimes \mathcal{N} \to \mathcal{M} \), there exist a unique morphism \( h : \mathcal{X} \to \text{Hom}(\mathcal{N}, \mathcal{M}) \), such that \( f = ev_{\n, \m} \circ (h \otimes id_{\mathcal{N}}) \)). The introduction of the morphism \( ev \) can be easily understood by the use of the definition of \( \text{Hom} \) applied to

\[
\text{Hom}(\text{Hom}(\mathcal{N}, \mathcal{M}) \otimes \mathcal{N}, \mathcal{M}) = \text{Hom}(\text{Hom}(\mathcal{N}, \mathcal{M}), \text{Hom}(\mathcal{N}, \mathcal{M}))
\]

and then considering on the right hand side the identity morphism \( id_{\text{Hom}(\mathcal{N}, \mathcal{M})} \). Other useful properties of internal Hom-functor of a closed monoidal category are the isomorphism

\[
\text{Hom}(\mathcal{M} \otimes \mathcal{N}, \mathcal{G}) \simeq \text{Hom}(\mathcal{M}, \text{Hom}(\mathcal{N}, \mathcal{G}));
\]

the composition of internal homomorphism

\[
\text{Hom}(\mathcal{M}, \mathcal{N}) \otimes \text{Hom}(\mathcal{N}, \mathcal{K}) \to \text{Hom}(\mathcal{M}, \mathcal{K}),
\]

and the product of internal homomorphisms by identities

\[
\text{Hom}(\mathcal{M}, \mathcal{N}) \to \text{Hom}(\mathcal{K} \otimes \mathcal{M}, \mathcal{K} \otimes \mathcal{N}).
\]

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