K-MODULES OF FANO 3-FOLDS CAN HAVE EMBEDDED POINTS

ANDREA PETRACCI

Abstract. We exhibit an example of obstructed K-polystable Fano 3-fold X such that the K-moduli stack of K-semistable Fano varieties and the K-moduli space of K-polystable Fano varieties have an embedded point at [X].

1. Introduction

A Fano variety is a normal projective variety X such that its anticanonical divisor \(-K_X\) is \(\mathbb{Q}\)-Cartier and ample. In some sense, a projective variety is Fano if it has ‘positive curvature’. Fano varieties occupy a prominent role in algebraic geometry from many points of view.

Moduli of Fano varieties are quite elusive, as they are highly non-separated. Recently K-stability [12,23] (i.e. the study of the existence of Kähler–Einstein metrics) has been spectacularly applied to construct ‘reasonable’ (i.e. separated and proper) moduli spaces of Fano varieties [2,7–9,15,17,18,26,27]. More precisely, it is known that, for each integer \(n \geq 1\), \(\mathbb{Q}\)-Gorenstein families of K-semistable Fano \(n\)-folds form an Artin stack \(\mathcal{M}_{n}^{\text{Kss}}\), called the K-moduli stack, which is locally of finite type over \(\mathbb{C}\); furthermore, this stack has a good moduli space which is denoted by \(\mathcal{M}_{n}^{\text{Kps}}\), is called the K-moduli space, parametrises K-polystable Fano \(n\)-folds, and is a countable disjoint union of projective schemes over \(\mathbb{C}\). We refer the reader to [25] for a survey on these topics.

Since Fano varieties of dimension 2 are unobstructed [1,14], the K-moduli stack \(\mathcal{M}_{2}^{\text{Kss}}\) is smooth and each connected component of K-moduli space \(\mathcal{M}_{2}^{\text{Kps}}\) is normal. In joint work with Kaloghiros [16] we exhibited the first examples of singular points on \(\mathcal{M}_{n}^{\text{Kss}}\), for each \(n \geq 3\). More precisely, we showed that, if \(n \geq 3\), \(\mathcal{M}_{n}^{\text{Kss}}\) and \(\mathcal{M}_{n}^{\text{Kps}}\) can be locally reducible or non-reduced (or both). These examples are constructed via toric geometry.

In [20] we have showed that the deformation space of every isolated Gorenstein toric 3-fold singularity appears, in a weak sense, as a singularity on \(\mathcal{M}_{n}^{\text{Kss}}\), for each \(n \geq 3\). In this note we analyse the construction of [20] in a particular example and we prove:

Theorem 1.1. There exists a K-polystable toric Fano 3-fold \(X\) with canonical singularities, such that the miniversal ring of \(\mathcal{M}_{3}^{\text{Kss}}\) at the point corresponding to \(X\) is

\[
\mathbb{C}[t_1, \ldots, t_8]/(t_1^2, t_2^2, t_3^2, t_4^2),
\]

and the completion of the structure sheaf of \(\mathcal{M}_{3}^{\text{Kps}}\) at the point corresponding to \(X\) is

\[
\mathbb{C}[u_1, \ldots, u_6]/(u_1^2, u_2^2, u_3^2, u_4^2, u_5^2, u_6^2).
\]

An immediate consequence is:
Corollary 1.2. The stack $M_{\Delta}^{Kss}$ and the scheme $M_{\Delta}^{Kps}$ have embedded points.

Notation and conventions. We always work over an algebraically closed field of characteristic 0, which is denoted by $\mathbb{C}$. We assume that the reader is familiar with toric geometry; every toric variety or toric singularity is assumed to be normal.

Throughout this note we use the following notation.

- $P$ the 3-dimensional lattice polytope $P$ in Figure 1
- $Q$ the polar of $P$
- $\text{conv \{\cdot\}}$ convex hull of a set
- $X$ the toric variety associated to the face fan of $P$
- $\mathcal{X}$ the canonical cover stack of $X$
- $\text{Def}^G_X$ the $\mathbb{Q}$-Gorenstein deformation functor of $X$
- $A$ the hull of $\text{Def}^G_X$
- $T$ the torus acting on $X$
- $M$ the character lattice of $T$
- $G$ the automorphism group of $X$

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2. The polytopes $P$ and $Q$

Consider the points

\[
\begin{align*}
a &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & b &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & c &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & d &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} & e &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\end{align*}
\]

in the lattice $N = \mathbb{Z}^3$. They are the vertices of a lattice pentagon which lies on the horizontal plane $\mathbb{R}^2 \times \{1\}$. Let $P$ be the convex hull of the 10 points

\[
a, \ b, \ c, \ d, \ e, \ -a, \ -b, \ -c, \ -d, \ -e
\]

in $N$ (see Figure 1). It is clear that $P$ is a centrally symmetric Fano polytope and has 10 facets:

(i) the pentagon $\text{conv \{a, b, c, d, e\}}$ which lies on the horizontal plane $\mathbb{R}^2 \times \{1\}$;
(i') the pentagon $\text{conv \{-a, -b, -c, -d, -e\}}$ which lies on the horizontal plane $\mathbb{R}^2 \times \{-1\}$;
(ii) the vertical rectangle $\text{conv \{a, e, -c, -d\}}$;
(ii') the vertical rectangle $\text{conv \{-a, -e, c, d\}}$;
(iii) the triangle $\text{conv \{e, -b, -c\}}$;
(iii') the triangle $\text{conv \{-e, b, e\}}$;
(iv) the triangle $\text{conv \{a, b, -d\}}$;
(iv') the triangle $\text{conv \{-a, -b, d\}}$;
(v) the triangle $\text{conv \{-b, d, e\}}$;
(v') the triangle $\text{conv \{b, -d, -e\}}$.

Moreover, one sees that the origin is the unique interior lattice point of $P$, so $P$ is a canonical polytope.
Consider the dual lattice $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ and the polar polytope of $P$; this is the polytope $Q$ which is the convex hull of the points
\[
\pm (1, 0, 0) \\
\pm (0, 1, 0) \\
\pm (0, 0, 1) \\
\pm (1, -1, 0) \\
\pm \left( \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right)
\]
in $M_{\mathbb{R}}$. One can check that $Q$ has exactly 9 lattice points, namely the origin and $\pm (1, 0, 0), \pm (0, 1, 0), \pm (0, 0, 1), \pm (1, -1, 0)$. The normalised volume of $Q$ is $\frac{40}{3}$. The intersection of $Q$ with the horizontal plane $\mathbb{R}^2 \times \{0\} \subset M_{\mathbb{R}}$ is the hexagon with vertices $\pm (1, 0, 0), \pm (0, 1, 0), \pm (-1, 1, 0)$.

3. The variety $X$

Let $X$ be the toric variety associated to the face fan (also called spanning fan) of $P$. Since $P$ is a canonical 3-dimensional polytope, $X$ is a Fano 3-fold with canonical singularities. The polytope $Q$ is the polytope associated to the toric boundary of $X$, which is an anticanonical divisor. The anticanonical degree $(-K_X)^3$ of $X$ coincides with the normalised volume of $Q$, so it is $\frac{40}{3}$. Since $Q$ is centrally symmetric, the origin is the barycentre of $Q$, thus $X$ is $K$-polystable by [6].

The face fan of $P$ gives an affine open cover of $X$. We denote by $U_{a,b,-d}$ the toric affine open subscheme of $X$ associated to cone over the facet of $P$ with vertices $a, b, -d$, and similarly for the other facets. Now we analyse the singularities which appear on $X$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\caption{The polytope $P$}
\end{figure}
(i) $U_{a,b,c,d,e}$ and $U_{-a,-b,-c,-d,-e}$ are isomorphic to the affine cone over the anticanonical embedding in $\mathbb{P}^7$ of the smooth del Pezzo surface of degree 7, i.e. the blowup of $\mathbb{P}^2$ in 2 points. This is an isolated Gorenstein singularity.

(ii) $U_{a,c,-e,d}$ and $U_{-a,-e,c,d}$ are isomorphic to the hypersurface
\[ V = \text{Spec } \mathbb{C}[x, y, z, w]/(x^2 y^2 - zw), \]
whose singular locus is 1-dimensional and has 2 irreducible components, which correspond to the edges with lattice length 2.

(iii+iv) $U_{e,-b,-c,-d}$ and $U_{a,b,-d}$ are isomorphic to the hypersurface
\[ cA_1 = \text{Spec } \mathbb{C}[x, y, z, w]/(xy - z^2), \]
whose singular locus is 1-dimensional and irreducible and corresponds to the edge with lattice length 2.

(iv) $U_{b,d,e}$ and $U_{b,-d,-e}$ are isomorphic to the isolated cyclic quotient singularity $\frac{1}{2}(1, 1, 2)$. This is a $\mathbb{Q}$-Gorenstein non-Gorenstein singularity which is rigid, both with respect to flat deformations and to $\mathbb{Q}$-Gorenstein deformations.

The singular locus of $X$ has 6 connected components:

(i) the 0-stratum of $U_{a,b,c,d,e}$;
(ii) the 0-stratum of $U_{-a,-b,-c,-d,-e}$;
(iii) the union of the singular loci of $U_{a,c,-e,d}$, $U_{e,-b,-c,-d}$, $U_{a,b,-d}$, which is the union of two smooth rational curves meeting transversally at the 0-stratum of $U_{a,c,-e,d}$;

(iv) the union of the singular loci of $U_{-a,-e,c,d}$, $U_{-e,b,c}$, $U_{-a,-b,d}$, which is the union of two smooth rational curves meeting transversally at the 0-stratum of $U_{-a,-e,c,d}$;

(v) the 0-stratum of $U_{-b,d,e}$;
(vi) the 0-stratum of $U_{b,-d,-e}$.

It is clear that the non-Gorenstein locus of $X$ consists of the 0-strata of $U_{-b,d,e}$ and $U_{b,-d,-e}$. Let $\varepsilon : \mathcal{X} \to X$ be the canonical cover stack of $X$. Therefore $\varepsilon$ is an isomorphism away from the 0-strata of $U_{-b,d,e}$ and $U_{b,-d,-e}$. One immediately sees that $\mathcal{X}$ is lci away from the 0-strata of $U_{a,b,c,d,e}$ and $U_{-a,-b,-c,-d,-e}$.

4. Some computations

4.1. Cotangent sheaves and groups. For each $i = 0, 1, 2$, consider the coherent sheaves $\mathcal{F}_X = \mathcal{Ext}_X^i(\Omega_X, \mathcal{O}_X)$ and $\mathcal{T}^G_{X,i} = \mathcal{Ext}_X^i(\Omega_X, \mathcal{O}_X)$ on $X$, and the $\mathbb{C}$-vector spaces $T_X^i = \text{Ext}_X^i(\Omega_X, \mathcal{O}_X)$ and $T^G_{X,i} = \text{Ext}_X^i(\Omega_X, \mathcal{O}_X)$. All of them are $M$-graded, because the torus $T = N \otimes \mathbb{Z} \times M^\vee$ acts on $X$.

The sheaves $\mathcal{F}_X^0$ and $\mathcal{T}^G_{X,0}$ coincide everywhere on $X$. For $i > 0$ the sheaves $\mathcal{F}_X^i$ and $\mathcal{T}^G_{X,i}$ coincide on the Gorenstein locus of $X$; therefore when considering restrictions to Gorenstein open subschemes we will suppress the superscript $qG$. For $i > 0$, the set-theoretical support of $\mathcal{T}^G_{X,i}$ is contained in the singular locus of $X$.

4.2. The sheaf $\mathcal{T}^G_{X,1}$. We analyse the restriction of $\mathcal{T}^G_{X,1}$ to the affine charts given by the facets of $P$.

(i) The restriction to $U_{a,b,c,d,e}$ is supported on the 0-stratum of $U_{a,b,c,d,e}$ and its global sections are $\mathbb{C}^2$ in degree $(0, 0, -1) \in M$ by [4].
Figure 2. A portion of the horizontal plane $\mathbb{R}^2 \times \{0\}$ in $M_\mathbb{R} = \mathbb{R}^3$. The green hexagon is the slice of the polytope $Q$. The centre of the hexagon is the origin. The degrees of the homogeneous components of $T^1_{U_{a,c,-e,-d}}$ are denoted by red circles.

(ii) If one considers the hypersurface $V = \text{Spec} \, \mathbb{C}[x, y, z, w]/(x^2 y^2 - z w)$, one has that $T^1_V$ is the $O_V$-module
\[
\mathbb{C}[x, y, z, w]/(x y^2, x^2 y, z, w) = \mathbb{C} \oplus \mathbb{C} z w \oplus \bigoplus_{n \geq 1} (\mathbb{C} z^n \oplus \mathbb{C} w^n).
\]
The homogeneous components of $T^1_{U_{a,e,-c,-d}}$ with respect to the $M$-grading are:
\[
T^1_{U_{a,e,-c,-d}}(m) = \begin{cases} 
\mathbb{C} & \text{if } m = (-1, 1, 0) \text{ or } m = (-2, 2, 0), \\
\mathbb{C} & \text{if } m = (-2 + n, 2, 0) \text{ and } n \geq 1,
\mathbb{C} & \text{if } m = (-2, 2 - n, 0) \text{ and } n \geq 1,
0 & \text{otherwise.}
\end{cases}
\]
See Figure 2.

(iii) If one considers the hypersurface $cA_1 = \text{Spec} \, \mathbb{C}[x, y, z, w]/(x y - z^2)$, one has that $T^1_{cA_1}$ is the $O_{cA_1}$-module
\[
\mathbb{C}[x, y, z, w]/(x, y, z) = \bigoplus_{n \geq 0} \mathbb{C} w^n.
\]
The homogeneous components of $T^1_{U_{e,-b,-c}}$ with respect to the $M$-grading are:
\[
T^1_{U_{e,-b,-c}}(m) = \begin{cases} 
\mathbb{C} & \text{if } m = (-2, 2 + n, 0) \text{ and } n \geq 0,
0 & \text{otherwise.}
\end{cases}
\]
See the left of Figure 3.

(iv) The homogeneous components of $T^1_{U_{a,b,-d}}$ with respect to the $M$-grading are:
\[
T^1_{U_{a,b,-d}}(m) = \begin{cases} 
\mathbb{C} & \text{if } m = (-2, 2 + n, 0) \text{ and } n \geq 0,
0 & \text{otherwise.}
\end{cases}
\]
See the right of Figure 3.

(v) The restriction to $U_{-b,d,e}$ is zero.
The discussion about (i'), (ii'), (iii'), (iv'), (v') is completely analogous and is omitted: it is enough to apply the reflection $-\text{id}$.

Now we want to understand the restriction of the sheaf $\mathcal{F}_X^{G,1}$ to the double intersections of the toric charts given by the facets of $P$. We will ignore the smooth ones, because there the sheaf $\mathcal{F}_X^{G,1}$ vanishes.

(ii\&iii) The intersection $U_{a,e,-c,-d} \cap U_{e,-b,-c}$ is isomorphic to the hypersurface $\text{Spec} \mathbb{C}[x, y, z, w^\pm] / (xy - z^2)$, so its $\mathbb{T}^1$-module is isomorphic to
\[
\mathbb{C}[x, y, z, w^\pm] / (x, y, z) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}w^n.
\]

The homogeneous components with respect to the $M$-grading are:
\[
\mathbb{T}^1_{U_{a,e,-c,-d} \cap U_{e,-b,-c}}(m) = \begin{cases} 
\mathbb{C} & \text{if } m = (-2 + n, 2, 0) \text{ and } n \in \mathbb{Z}, \\
0 & \text{otherwise}.
\end{cases}
\]

See the left of Figure 4.
We have proved that $\bigcap U_{a,e,-e,-d} \cap U_{a,b,-d}$ is isomorphic to $\text{Spec} \mathbb{C}[x, y, z, w^\pm] / (xy - z^2)$. The $M$-homogeneous components of the restriction of $\mathcal{F}^G_{X}$ to the intersection $U_{a,e,-e,-d} \cap U_{a,b,-d}$ are

$$\mathbb{T}_U^{1, U_{a,e,-e,-d} \cap U_{a,b,-d}}(m) = \begin{cases} \mathbb{C} & \text{if } m = (-2, 2 + n, 0) \text{ and } n \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

See the right of Figure 4.

The discussion about the intersections (ii'∩iii') and (ii'∩iv') is omitted because it can easily obtained from above by applying the reflection $-\text{id}$.

Now we have all information to construct the Čech complex of the sheaf $\mathcal{F}^G_{X}$ with respect to the affine cover given by the charts associated to the facets of $P$. Notice that, since triple intersections are smooth, this complex is concentrated in degrees 0 and 1:

$$\mathbb{C}^0 \xrightarrow{d} \mathbb{C}^1.$$

We analyse its homogeneous components with respect to the $M$-grading.

- If $m = \pm(0, 0, 1)$, then $d(m)$ is the zero map $\mathbb{C}^2 \to 0$.
- If $m = \pm(-1, 1, 0)$, then $d(m)$ is the zero map $\mathbb{C} \to 0$.
- If $m = \pm(-2, 2, 0)$, then $d(m)$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} : \mathbb{C}^3 \to \mathbb{C}^2.$$

- If $m$ is none of the degrees above, then $d(m)$ is the identity of 0, of $\mathbb{C}$ or of $\mathbb{C}^2$.

We have proved that $d$ is surjective, therefore $H^1(X, \mathcal{F}^G_{X}) = 0$. Furthermore, we have computed the homogeneous components of $H^0(X, \mathcal{F}^G_{X})$:

$$H^0(X, \mathcal{F}^G_{X})(m) = \begin{cases} \mathbb{C}^2 & \text{if } m = \pm(0, 0, 1), \\ \mathbb{C} & \text{if } m = \pm(-1, 1, 0), \\ \mathbb{C} & \text{if } m = \pm(-2, 2, 0), \\ 0 & \text{otherwise}. \end{cases}$$

In particular $\dim H^0(X, \mathcal{F}^G_{X}) = 8$.

4.3. The hull of $\text{Def}^G_X$. Let $\text{Def}^G_X$ denote the functor of $\mathbb{Q}$-Gorenstein deformations of $X$. Let $A$ denote its hull.

Since $H^1(X, \mathcal{F}^0_X) = 0$ and $H^2(X, \mathcal{F}^0_X) = 0$ by [24, Proof of Theorem 5.1] (see also [21]), the natural homomorphism $\mathbb{T}^G_{X} \to H^0(X, \mathcal{F}^G_{X})$ is an isomorphism. From the conclusion of §4.2 we deduce that $\mathbb{T}^G_{X}$ has dimension 8 and we can choose coordinates $t_1, \ldots, t_8$ with weights $(0, 0, 1), (0, 0, 1), (0, 0, -1), (0, 0, -1), (-1, 1, 0), (-1, 1, 0), (2, -2, 0), (2, -2, 0)$ in $M$, respectively. This implies that $A$ is a quotient of the power series ring $\mathbb{C}[t_1, \ldots, t_8]$ with respect to an ideal which is contained in $(t_1, \ldots, t_8)^2$.

In order to understand this ideal we need to compute the obstructions. Since $X$ is lci away from the 0-strata of $U_{a,b,c,d,e}$ and $U_{-a,-b,-c,-d,-e}$, the sheaf $\mathcal{F}^G_{X}$ is set-theoretically supported on these two points, hence

$$H^0(X, \mathcal{F}^G_{X}) = \mathbb{T}^2_{U_{a,b,c,d,e}} \oplus \mathbb{T}^2_{U_{-a,-b,-c,-d,-e}}.$$
Moreover, the vanishing of $H^2(X, \mathcal{T}_X^0)$ and of $H^1(X, \mathcal{T}_X^{G,1})$ implies that the natural homomorphism $T^q_{X,G} \to H^0(X, \mathcal{T}_X^{G,2})$ is an isomorphism.

We deduce that the product of restriction maps
\begin{equation}
\text{Def}_{X,G}^q \longrightarrow \text{Def}_{U_{a,b,c,d,e}} \times \text{Def}_{U_{-a,-b,-c,-d,-e}}
\end{equation}
induces an isomorphism on tangent spaces and on obstruction spaces. Therefore this map is smooth.

By $[5]$ $\mathbb{C}[u,v]/(u^2, uv)$ is the hull of the deformation functor of the affine cone over the anticanonical embedding in $\mathbb{P}^7$ of the smooth del Pezzo surface of degree 7. From the smoothness of (4.1) we deduce
\begin{equation}
A = \mathbb{C}[t_1, \ldots, t_8]/(t_1^2, t_1 t_2, t_2^2, t_3 t_4).
\end{equation}
This shows that the base of the miniversal $\mathbb{Q}$-Gorenstein deformation of $X$ is irreducible and non-reduced.

4.4. The action the automorphism group. The automorphism group of the polytope $P$ is generated by the two involutions
$$
\sigma = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\quad \text{and} \quad
\tau = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$
which commute in $\text{GL}(N) = \text{GL}_3(\mathbb{Z})$. Let $G$ denote the automorphism group of $X$ and let $T = N \otimes \mathbb{Z} G_m = \text{Spec} \mathbb{C}[M]$ be the torus acting on $X$. Since every facet of $Q$ has no interior lattice point, by [16] Proposition 2.8 $G$ is the semidirect product $T \times (C_2 \times C_2)$, where $C_2$ denotes the group of order 2. The algebraic group $G$ acts on $A$ as follows:
- the torus $T$ acts linearly on $A$ via the weights of the $T$-representation $T^q_{X,G,1}$, so $t_1, \ldots, t_8$ have weights $(0, 0, 1), (0, 0, 1), (0, 0, -1), (0, 0, -1), (-1, 1, 0), (-2, 2, 0), (1, -1, 0), (2, -2, 0)$, respectively;
- $\sigma$ acts by permuting the coordinates $t_1, \ldots, t_8$ as follows: $t_1 \leftrightarrow t_3$, $t_2 \leftrightarrow t_4$, $t_5 \leftrightarrow t_7$, $t_6 \leftrightarrow t_8$;
- $\tau$ permutes the coordinates $t_1, \ldots, t_8$ as follows: $t_1, t_2, t_3, t_4$ are left fixed and $t_5 \leftrightarrow t_7$, $t_6 \leftrightarrow t_8$.

We need to compute the invariant subring $A^G$. Actually there is no harm in considering polynomials instead of power series:
$$
B = \mathbb{C}[t_1, \ldots, t_8]/(t_1^2, t_1 t_2, t_2^2, t_3 t_4).
$$
Moreover, in the $G$-action the two sets of variables $\{t_1, t_2, t_3, t_4\}$ and $\{t_5, t_6, t_7, t_8\}$ are not mixed. So we can consider
$$
R = \mathbb{C}[t_1, t_2, t_3, t_4]/(t_1^2, t_1 t_2, t_2^2, t_3 t_4) \quad \text{and} \quad
S = \mathbb{C}[t_5, t_6, t_7, t_8]
$$
and study the $G$-actions on $R$ and on $S$ separately.

The invariant subring $\mathbb{C}[t_1, t_2, t_3, t_4]^G$ is generated by $x_1 = t_1 t_3$, $x_2 = t_2 t_4$, $x_3 = t_2 t_3$ and $x_4 = t_1 t_4$, which satisfy the relation $x_1 x_2 - x_3 x_4 = 0$.

```
\begin{tikzpicture}
  \node (R) at (0,0) {$R$};
  \node (A) at (0,-2) {$\mathbb{C}[t_1, t_2, t_3, t_4] \longrightarrow R$};
  \node (S) at (0,-4) {$\mathbb{C}[t_1, t_2, t_3, t_4]^T \longrightarrow R^\vee$};
  \node (B) at (0,-6) {$\mathbb{C}[x_1, x_2, x_3, x_4] \longrightarrow \mathbb{C}[t_1, t_2, t_3, t_4]$};
  \draw[->] (R) -- (A);
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (S);
\end{tikzpicture}
```
One can show that the kernel of $\mathbb{C}[x_1, x_2, x_3, x_4] \to R^T$ is generated by $x_1^2 = t_1^2 t_4^2$, $x_2^2 = t_2^2 t_4^2$, $x_3^2 = t_3^2 t_4^2$, $x_4 = t_1 t_2 t_3 t_4$, $x_1 x_2 = x_3 x_4 = t_1 t_2 t_3$, $x_1 x_3 = t_1 t_2 t_4$, $x_1 x_4 = t_1 t_3 t_4$, $x_2 x_3 = t_2 t_3$, $x_2 x_4 = t_1 t_2 t_3$, i.e. by all degree 2 monomials in $x_1, x_2, x_3, x_4$ with the exception of $x_2^2$. Therefore we have an isomorphism

$$R^T \simeq \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4).$$

The involution $\sigma$ keeps $x_1$ and $x_2$ fixed and swaps $x_3$ and $x_4$. The involution $\tau$ acts trivially on $R$, and therefore also on $R^T$. Hence $R^G = (R^T)^{\sigma}$ is generated by $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3 + x_4$, $y_4 = x_3 x_4$.

$$\mathbb{C}[x_1, x_2, x_3, x_4] \xrightarrow{\sigma} R^G$$

The kernel of $\mathbb{C}[y_1, y_2, y_3, y_4] \to R^G$ is generated by $y_1^2$, $y_1 y_2$, $y_4$, $y_1 y_3$, $y_2 y_3$. Therefore we have isomorphisms

$$(4.3) \quad R^G \simeq \frac{\mathbb{C}[y_1, y_2, y_3, y_4]}{(y_1^2, y_1 y_2, y_4, y_1 y_3, y_2 y_3)} \simeq \frac{\mathbb{C}[y_1, y_2, y_3]}{(y_1^2, y_1 y_2, y_1 y_3, y_2 y_3)}.$$  

The invariant subring $S^T$ is generated by $z_1 = t_1 t_6$, $z_2 = t_5 t_7$, $z_3 = t_2 t_8$, $z_4 = t_6 t_8$, which satisfy the relation $z_1 z_3 - z_2 z_4 = 0$. Therefore we have an isomorphism

$$S^T \simeq \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_3 - z_2 z_4).$$

The action of $\tau$ on $S$ coincides with the action of $\sigma$ on $S$, therefore we can ignore $\tau$. One immediately sees that $\sigma$ swaps $z_1$ and $z_3$ and leaves $z_2$ and $z_4$ fixed. This implies that $S^G = (S^T)^{\sigma}$ is generated by $v_1 = z_1 + z_3$, $v_2 = z_1 z_3$, $v_3 = z_2$, $v_4 = z_4$.

$$\mathbb{C}[z_1, z_2, z_3, z_4] \xrightarrow{\sigma} S^T$$

The kernel of $\mathbb{C}[v_1, v_2, v_3, v_4] \to S^G$ is generated by $v_2 - v_2^3 v_4$. Therefore we have isomorphisms

$$(4.4) \quad S^G \simeq \frac{\mathbb{C}[v_1, v_2, v_3, v_4]}{(v_2 - v_2^3 v_4)} \simeq \mathbb{C}[v_1, v_3, v_4].$$

By (4.3) and (4.4), by taking the completion of $B^G \simeq R^G \otimes_{\mathbb{C}} S^G$, and by changing the names of the variables, we get an isomorphism

$$(4.5) \quad A^G \simeq \mathbb{C}[u_1, \ldots, u_6]/(u_1^2, u_2^2, u_1 u_2, u_1 u_3, u_2 u_3).$$

4.5. Conclusion. By the Luna étale slice theorem for algebraic stacks there exists a cartesian diagram

$$\begin{array}{ccc}
\text{Spec } A^G & \xrightarrow{\varphi} & \mathcal{M}_3^\text{Kps} \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{\varphi} & \mathcal{M}_3^\text{Kss}
\end{array}$$
where the horizontal arrows are formally étale and map the closed point to the point corresponding to $X$. The isomorphisms (4.2) and (4.5) imply Theorem 1.1.

5. Mirror symmetry

Let us consider Laurent polynomials in 3 variables $x, y, z$ whose Newton polytope is $P$. If we insist that the coefficient of the monomial 1 (which corresponds to the origin) is 0 and that the coefficient of the monomial corresponding to every other lattice point of $P$ is 1, we get the Laurent polynomial

$$f = (x + y + xy)(1 + x^{-1}y^{-1})z + (x^{-1} + y^{-1} + x^{-1}y^{-1})(1 + xy)z^{-1}.$$  

This polynomial is quite peculiar, as it is the unique Laurent polynomial with Newton polytope $P$ such that the restriction of $f$ to each facet of $P$ is 0-mutable in the sense of [11].

One can consider the classical period of $f$, which is a power series $\pi_f \in \mathbb{C}[t]$ defined in terms of variations of Hodge structures of the fibration $f : \mathbb{G}_m^3 \rightarrow \mathbb{A}^1$. We refer the reader to [10, 13] for the precise definition of $\pi_f$. In this particular case the first terms of $\pi_f$ are:

$$\pi_f = 1 + 12t^2 + 636t^4 + 51600t^6 + 4942140t^8 + 517143312t^{10} + \cdots.$$  

From the results in §4.3 it is easy to prove that the general fibre of the miniversal $\mathbb{Q}$-Gorenstein deformation of $X$ is a terminal Fano 3-fold $X'$ with two $13(1, 1, 2)$ points. According to [10] (see also [22] for a survey), it is conjectured that $X'$ is mirror to $f$, in the sense that $\pi_f$ coincides with regularised quantum period of $X'$. Here the regularised quantum period $\hat{G}_{X'}$ is a certain generating function for genus 0 Gromov–Witten invariants of $X'$; we refer the reader to [19, Definition 3.2] for the precise definition.

Unfortunately we are not able to prove the equality $\hat{G}_{X'} = \pi_f$ because we do not know how to compute Gromov–Witten invariants of $X'$: indeed, we do not know a presentation of $X'$ as a complete intersection in a toric variety or in a partial flag variety. Nonetheless the reader should appreciate that this type of mirror symmetry conjectures establishes a useful correspondence between algebraic geometry (i.e. the connected component of moduli containing $X$ and $X'$, and the Gromov–Witten theory of $X'$) and combinatorics (i.e. the polynomial $f$ and its classical period $\pi_f$).

References

[1] Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten, Mirror symmetry and the classification of orbifold del Pezzo surfaces, Proc. Amer. Math. Soc. 144 (2016), no. 2, 513–527.
[2] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, Reductivity of the automorphism group of $K$-polystable Fano varieties, Invent. Math. 222 (2020), no. 3, 995–1032.
[3] Jarod Alper, Jack Hall, and David Rydh, A Luna étale slice theorem for algebraic stacks, Ann. of Math. (2) 191 (2020), no. 3, 675–738.
[4] Klaus Altmann, Computation of the vector space $T^1$ for affine toric varieties, J. Pure Appl. Algebra 95 (1994), no. 3, 239–259.
[5] ———, The versal deformation of an isolated toric Gorenstein singularity, Invent. Math. 128 (1997), no. 3, 443–479.
[6] Robert J. Berman, $K$-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics, Invent. Math. 203 (2016), no. 3, 973–1025.
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[7] Harold Blum, Daniel Halpern-Leistner, Yuchen Liu, and Chenyang Xu, On properness of $K$-moduli spaces and optimal degenerations of Fano varieties. arXiv:2011.01895

[8] Harold Blum, Yuchen Liu, and Chenyang Xu, Openness of $K$-semistability for Fano varieties. arXiv:1907.02408

[9] Harold Blum and Chenyang Xu, Uniqueness of $K$-polystable degenerations of Fano varieties, Ann. of Math. (2) 190 (2019), no. 2, 609–656.

[10] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander Kasprzyk, Mirror symmetry and Fano manifolds, European Congress of Mathematics, 2013, pp. 285–300.

[11] Alessio Corti, Matej Filip, and Andrea Petracci, Mirror Symmetry and smoothing Gorenstein toric affine 3-folds. arXiv:2006.16885

[12] S. K. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), no. 2, 289–349.

[13] Sergey Galkin and Alexandr Usnich, Mutations of potentials, preprint IPMU (2010), 10–0100.

[14] Paul Hacking and Yuri Prokhorov, Smoothable del Pezzo surfaces with quotient singularities, Compos. Math. 146 (2010), no. 1, 169–192.

[15] Chen Jiang, Boundedness of $Q$-Fano varieties with degrees and alpha-invariants bounded from below, Ann. Sci. Éc. Norm. Supér. (4) 53 (2020), no. 5, 1235–1248.

[16] Anne-Sophie Kaloghiros and Andrea Petracci, On toric geometry and $K$-stability of Fano varieties. arXiv:2009.02271

[17] Chi Li, Xiaowei Wang, and Chenyang Xu, Algebraicity of the metric tangent cones and equivariant $K$-stability. arXiv:1805.03393

[18] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, Finite generation for valuations computing stability thresholds and applications to $K$-stability. arXiv:2102.09405

[19] Alessandro Oneto and Andrea Petracci, On the quantum periods of del Pezzo surfaces with $\frac{1}{2}(1,1)$ singularities, Adv. Geom. 18 (2018), no. 3, 303–336.

[20] Andrea Petracci, On deformation spaces of toric varieties. arXiv:2105.01174

[21] ________, On deformations of toric Fano varieties. arXiv:1912.01538

[22] ________, An example of Mirror Symmetry for Fano threefolds, Birational Geometry and Moduli Spaces, 2020, pp. 173–188.

[23] Gang Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1–37.

[24] Burt Totaro, Jumping of the nef cone for Fano varieties, J. Algebraic Geom. 21 (2012), no. 2, 375–396.

[25] Chenyang Xu, $K$-stability of Fano varieties: an algebro-geometric approach. arXiv:2011.10477

[26] ________, A minimizing valuation is quasi-monomial, Ann. of Math. (2) 191 (2020), no. 3, 1003–1030.

[27] Chenyang Xu and Ziquan Zhuang, On positivity of the CM line bundle on $K$-moduli spaces, Ann. of Math. (2) 192 (2020), no. 3, 1005–1068.

Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, Berlin 14195, Germany

Email address: andrea.petracci@fu-berlin.de