Controlled Drift Estimation in the Mixed Fractional Ornstein-Uhlenbeck Process

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Abstract  
This paper is devoted to the determination of the asymptotical optimal input for the estimation of the drift parameter in a directly observed but controlled fractional Ornstein-Uhlenbeck process. Large sample asymptotical properties of the Maximum Likelihood Estimator is deduced using the Laplace transform computations.

Keywords: mixed fractional Brownian motion, fundamental martingale, Laplace Transform, optimal input

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1. Introduction

The experiment design has been given a great deal of interest over the last decades from the early statistics literature as well as in the engineering literature. In the statistical aspect, the classical approach for experiment design consists on a two-step procedure: maximize the Fisher information under energy constraint of the input and find an adaptive estimation procedure. Ovseevich et al. [17] has first consider this type problem for the diffusion equation with continuous observation. When the kernel in [17] is not with explicit formula in the fractional diffusion case, Brouste et al. [3, 4] deduce the lower bound and upper bound with the method of spectral gap and solve the same problem. Base on this method, Brouste and Cai [5] have extended the result to the partially observed fractional Ornstein-Uhlenbeck process, in this work the asymptotical normality has been demonstrated with linear filtering of Gaussian processes and Laplace Transform presented in [13, 2, 11, 12, 13]. These previous work, the common point is that: the optimal input does not depend on the unknown parameter and maximum likelihood estimator can be found directly from the likelihood equation. The one-step estimator will be used following the Newton-Raphson method and this work was introduced by Cai and LV [23].

In this paper, we will consider a similar problem but with the noise of mixed fractional Brownian motion. Let \( X = (X_t, t \geq 0) \) be a real-valued process, representing the observation, which is governed by:

\[
    dX_t = -\vartheta X_t dt + u(t)dt + d\xi_t
\]

(1)
with $X_0 = 0$. Here $\xi_t = B_t + B_t^H$ where $B_t$ is a standard Wiener process and $B_t^H$ is an independent fractional Brownian motion (fBm for short) with $H > \frac{1}{2}$, is called the mixed fractional Brownian motion first presented in [20]. $u = (u(t), t \geq 0)$ is the deterministic real-valued function. $\vartheta > 0$ is the unknown parameter.

For a fixed value of parameter $\vartheta$, let $P^\vartheta_T$ denote the probability measure, induced by $X_T$ on the function space $C[0, T]$ and let $F^X_T$ be the nature filtration of $X, F^X_T = \sigma(X_s, 0 \leq s \leq t)$. Let $\mathcal{L}(\vartheta, X_T)$ be the likelihood, i.e. the Radon-Nikodym derivative of $P^\vartheta_T$, restricted to $F^Y_T$ with respect to some reference measure on $C[0, T]$. In this setting, Fisher information stands for

$$I_T(\vartheta, u) = -E_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}(\vartheta, X_T).$$

Let us denote $U_T$ some functional space of controls, that is defined by Eqs. (10) and (9). Let us therefore note

$$\mathcal{J}_T(\vartheta) = \sup_{u \in U_T} I_T(\vartheta, u),$$

our main goal is to find estimator $\hat{\vartheta}_T$ of the parameter $\vartheta$ which is asymptotically efficient in the sense that, for any compact $K \subset \mathbb{R}^+$, $\vartheta > 0$,

$$\sup_{\vartheta \in K} \mathcal{J}_T(\vartheta) E_\vartheta (\hat{\vartheta}_T - \vartheta)^2 = 1 + o(1),$$

as $T \to \infty.$

As the optimal input does not depend on $\vartheta$ (see Proposition 2.1), a possible candidate is the Maximum Likelihood Estimator (MLE) $\hat{\vartheta}_T$, defined as the maximizer of the likelihood:

$$\hat{\vartheta}_T = \arg \max_{\vartheta > 0} \mathcal{L}(\vartheta, X_T).$$

We want to find the asymptotical normality of the MLE of $\vartheta$ and establish the large deviation principle for this estimator.

The interest to mixed fractional Brownian motion was triggered by Cheridito [20]. The recent works of Cai, Chigansky, Kleptsyna and Marushkevych ([22] [26] [24] [25]) present a great value for the purpose of this paper. The process $\xi_t$ satisfies a number of curious properties with applications in mathematical finance, see [19]. In particular, as shown in [20, 21], it is a semimartingale if and only if $H \in \{\frac{1}{2}\} \cup \left(\frac{3}{4}, 1\right]$ and the measure $\mu^\xi$ induced by $\xi$ on the space of continuous functions on $[0, T]$, is equivalent to the standard Wiener measure $\mu^B$ for $H > \frac{3}{4}$. On the other hand, $\mu^\xi$ and $\mu^{B_H}$ are equivalent if and only if $H < \frac{1}{4}$.

The paper falls into four parts. In the second part, we present some main results of this paper and the third part will contribute to the proofs of the main results. Some Lemmas will be given in Appendix.

2. Main Results

2.1. Transformation of the model

Even if the mixed fractional Brownian motion $\xi$ is a semimartingale when $H > \frac{3}{4}$, it is hard to write the likelihood function directly. We will try to transform our model with the fundamental martingale in [22] and get the explicit representation of the likelihood function. In what follows, all
random variables and processes are defined on a given stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})\) satisfying the usual conditions and processes are \((\mathcal{F}_t)\)–adapted. Moreover the natural filtration of a process is understood as the \(\mathbf{P}\)-completion of the filtration generated by this process.

From the canonical innovation representation in \([22]\), the fundamental martingale is defined as \(M_t = \mathbf{E}(B_t | \mathcal{F}^\xi_t)\). Then for \(H > 1/2\) this martingale satisfies

\[
M_t = \int_0^t g(s,t) \, d\xi_s, \quad \langle M \rangle_t = \int_0^t g(s,t) \, ds = \int_0^t g^2(s,s) \, ds
\]

where \(g(s,t)\) is the solution of the integro-differential equation

\[
g(s,t) + c_H \int_0^t g(r,t) |r-s|^{2H-2} \, dr = 1, \quad c_H = H(2H - 1)
\]

It can be shown that \(g(t,t) > 0\) for all \(t \geq 0\).

Following from \([22]\), let us introduce a process \(Z = (Z_t, 0 \leq t \leq T)\) the fundamental semi-martingale associated to \(X\), defined as

\[
Z_t = \int_0^t g(s,t) \, dX_s.
\]

Note that \(X\) can be represented as \(X_t = \int_0^t \hat{g}(s,t) \, dZ_s\) where

\[
\hat{g}(s,t) = 1 - \frac{d}{d \langle M \rangle_s} \int_0^t g(r,t) \, dr
\]

for \(0 \leq s \leq t\) and there for the nature filtration of \(X\) and \(Z\) coincide. Moreover, we have the following representations:

\[
dZ_t = -\vartheta Q_t \, d\langle M \rangle_t + v(t) \, d\langle M \rangle_t + dM_t,
\]

where

\[
Q_t = \frac{d}{d \langle M \rangle_t} \int_0^t g(s,t) X_s \, ds, \quad v(t) = \frac{d}{d \langle M \rangle_t} \int_0^t g(s,t) u(s) \, ds.
\]

Let us define the space of control for \(v(t)\):

\[
\mathcal{V}_T = \left\{ h : \frac{1}{T} \int_0^T |v(t)|^2 \, d\langle M \rangle_t \leq 1 \right\}.
\]

Remark that with \([23]\) the following relationship between control \(u\) and its transformation \(v\) holds:

\[
u(t) = \frac{d}{dt} \int_0^t \hat{g}(t,s) v(s) \, d\langle M \rangle_s
\]

we can set the admissible control as \(\mathcal{U}_T = \{ u | v \in \mathcal{V}_T \} \). Note that these set are non-empty.

From \([23]\), we know \(Q_t = \int_0^t \psi(s,t) \, dZ_s\) where

\[
\psi(s,t) = \frac{1}{2} \left( \frac{dt}{d \langle M \rangle_t} + \frac{ds}{d \langle M \rangle_s} \right)
\]
Moreover, \( Q_t = \frac{1}{2} \ell(t)^* \zeta_t \), where \( \ell(t) = \begin{pmatrix} \psi(t, t) \\ 1 \end{pmatrix} \), * standing for the transposition and \( \zeta = (\zeta_t, t \geq 0) \) is the solution of the stochastic differential equation
\[
d\zeta_t = -\frac{\vartheta}{2} A(t) \zeta_t d\langle M \rangle_t + a(t) \psi(t) d\langle M \rangle_t + b(t) dM_t, \zeta_0 = 0_{2 \times 1}, \tag{12}\]
with
\[
A(t) = \begin{pmatrix} \psi(t, t) & 1 \\ \psi^2(t, t) & \psi(t, t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 \\ \psi(t, t) \end{pmatrix}. \tag{13}\]

2.2. Likelihood function and Fisher information

The classical Girsanov theorem gives
\[
\mathcal{L}(\vartheta, Z^T) = \mathbb{E}_{\vartheta} \exp \left\{ -\int_0^T (-\vartheta Q_t + a(t)) dZ_t - \frac{1}{2} \int_0^T (-\vartheta Q_t + a(t))^2 d\langle M \rangle_t \right\}, \tag{14}\]
then the Fisher information stands for
\[
\mathcal{I}_T(\vartheta, a) = -\mathbb{E}_{\vartheta} \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}(\vartheta, Z^T) = \frac{1}{4} \mathbb{E}_{\vartheta} \int_0^T (\ell(t)^* \zeta_t)^2 d\langle M \rangle_t.
\]
Then we have the following results for the optimal input:

**Theorem 2.1.** The asymptotic optimal input in the class of controls \( U_T \) is \( u_{\text{opt}}(t) = \frac{d}{dt} \int_0^t \hat{\vartheta} g(s, t) \psi(s, s) d\langle M \rangle_s \), where \( \hat{\vartheta} \) are defined in \( 4, 10, 11 \). Moreover,
\[
\lim_{T \to +\infty} \frac{\mathcal{J}_T(\vartheta)}{T} = \mathcal{I}(\vartheta),
\]
where
\[
\mathcal{I}(\vartheta) = \frac{1}{2\vartheta} + \frac{1}{\vartheta^2}. \tag{15}\]

The \( \mathcal{J}_T(\vartheta) \) is defined in \( 3 \).

**Remark 2.2.** In fact, this result is the same as in \( 3 \). For \( a(t) = 0 \), we will find the Fisher Information in \( 25 \) that \( \mathcal{I}(\vartheta) = \frac{1}{2\vartheta} \).

**Remark 2.3.** For the case of \( H < 1/2 \), different from the pure fractional case in \( 5 \) we can not change \( H < 1/2 \) to \( H > 1/2 \) without changing any structure. A possible method similar of \( 25 \) can be used and we will leave it for further study.

2.3. Asymptotical Normality of The MLE

From the theorem 2.1 we can see that the optimal input \( u_{\text{opt}}(t) \) does not depend on the unknown parameter \( \vartheta \), we can easily obtain the estimator error of the MLE of the \( \vartheta_T \):
\[
\hat{\vartheta}_T - \vartheta = \frac{\int_0^T Q_t dM_t}{\int_0^T Q_t^2 d\langle M \rangle_t} \tag{16}
\]

Then, the MLE reaches efficiency and we deduce its large sample asymptotic properties:
Theorem 2.4. The MLE is uniformly consistent on compacts $K \subset \mathbb{R}_+^*$, i.e. for any $\nu > 0$,
\[
\lim_{T \to \infty} \sup_{\vartheta \in K} P_T^\vartheta \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \nu \right\} = 0,
\]
uniformly on compacts asymptotically normal: as $T$ tends to $+\infty$,
\[
\lim_{T \to \infty} \sup_{\vartheta \in K} \left| E_{\vartheta} f (\sqrt{T} (\hat{\vartheta}_T - \vartheta)) - E f(\eta) \right| = 0 \quad \forall f \in \mathcal{C}_b
\]
and $\xi$ is a zero mean Gaussian random variable of variance $\mathcal{I}(\vartheta)^{-1}$ (see (15) for the explicit value) which does not depend on $H$ and we have the uniform on $\vartheta \in K$ convergence of the moments: for any $p > 0$,
\[
\lim_{T \to \infty} \sup_{\vartheta \in K} \left| E_{\vartheta} \left| \sqrt{T} (\hat{\vartheta}_T - \vartheta) \right|^p - E |\eta|^p \right| = 0.
\]
Finally, the MLE is efficient in the sense of (3).

Theorem 2.5. The MLE $\hat{\vartheta}_T$ is strong consistency that is
\[
\hat{\vartheta}_T \xrightarrow{a.s.} \vartheta, \; T \to \infty.
\]

3. Proofs of Main Results

3.1. Proof of Theorem 2.1

We will compute the Fisher information with the same method in [5], that is to separate the Fisher information into two parts, on into the control, the other without, we focus on the following decomposition:
\[
\mathcal{I}_T(\vartheta,v) = \frac{1}{4} E_{\vartheta} \left\{ \int_0^T (\ell(t)^* \zeta_t - E_{\vartheta} \ell(t)^* \zeta_t + E_{\vartheta} \ell(t)^* \zeta_t)^2 \right\}
\]
\[
= \mathcal{I}_{1,T}(\vartheta,v) + \mathcal{I}_{2,T}(\vartheta,v)
\]
where
\[
\mathcal{I}_{1,T}(\vartheta,v) = \frac{1}{4} \int_0^T E_{\vartheta} (\ell(t)^* \zeta_t - E_{\vartheta} \ell(t)^* \zeta_t)^2 \langle M \rangle_t
\]
and
\[
\mathcal{I}_{2,T}(\vartheta,v) = \frac{1}{4} \int_0^T (\ell(t)^* E_{\vartheta} \zeta_t)^2 d\langle M \rangle_t.
\]
The deterministic function $(\mathcal{P}(t) = E_{\vartheta} \zeta_t, \; t \geq 0)$ satisfies the following equation:
\[
\frac{d\mathcal{P}(t)}{d\langle M \rangle_t} = -\frac{1}{2} \vartheta A(t) \mathcal{P}(t) + b(t)v(t), \mathcal{P}(0) = \mathbf{0}_{2 \times 1},
\]
at the same time the process $\overline{\mathcal{P}} = (\overline{\mathcal{P}}_t = \zeta_t - E_{\vartheta} \zeta_t, \; t \geq 0)$ satisfies the following stochastic equation:
\[
\frac{d\overline{\mathcal{P}}_t}{d\langle M \rangle_t} = -\frac{1}{2} \vartheta A(t) \overline{\mathcal{P}}_t d\langle M \rangle_t + b(t) dM_t,
\]
which is just the \( \zeta_t \) with \( v(t) = 0 \) which can be found in \([25]\).

With the technical separation of (17) and the precedent remarks, we have

\[
\mathcal{J}_T(\vartheta) = \mathcal{I}_{1,T}(\vartheta) + \mathcal{J}_{2,T}(\vartheta),
\]

where

\[
\mathcal{J}_{2,T}(\vartheta) = \sup_{v \in \mathcal{V}_T} \mathcal{I}_{2,T}(\vartheta, v).
\]

From \([25]\), we know

\[
\lim_{T \to \infty} \frac{\mathcal{I}_{1,T}(\vartheta)}{T} = \frac{1}{2\vartheta},
\]

so we just need to check that \( \lim_{T \to \infty} \frac{\mathcal{J}_{2,T}(\vartheta)}{T} = \frac{1}{\vartheta} \). From (20), we get

\[
P(t) = \varphi(t) \int_0^t \varphi^{-1}(s)b(s)v(s)d\langle M \rangle_s,
\]

where \( \varphi(t) \) is the matrix defined by

\[
\frac{d\varphi(t)}{d\langle M \rangle_t} = -\vartheta A(t)\varphi(t), \quad \varphi(0) = \text{Id}_{2 \times 2}
\]

with \( \text{Id}_{2 \times 2} \) the \( 2 \times 2 \) identity matrix. Substituting into (19), we get

\[
\mathcal{I}_{1,T}(\vartheta, v) = \int_0^T \int_0^T \mathcal{K}_T(s, \sigma) \frac{1}{\sqrt{\psi(s, s)}} \varphi(s) \frac{1}{\sqrt{\psi(\sigma, \sigma)}} v(\sigma) ds d\sigma,
\]

where the operator

\[
\mathcal{K}_T(s, \sigma) = \int_{\max(s, \sigma)}^T \mathcal{G}(t, s)\mathcal{G}(t, \sigma) dt
\]

and

\[
\mathcal{G}(t, \sigma) = \frac{1}{2} \left( \frac{1}{\sqrt{\psi(t, t)}} f(t)^* \varphi(t) \varphi^{-1}(\sigma)b(\sigma) \frac{1}{\sqrt{\psi(\sigma, \sigma)}} \right).
\]

Then

\[
\mathcal{J}_{2,T}(\vartheta) = T \sup_{\bar{v} \in L^2[0, T], \|\bar{v}\| \leq 1} \int_0^T \int_0^T \mathcal{K}_T(s, \sigma) \bar{v}(s)\bar{v}(\sigma) ds d\sigma,
\]

where \( \bar{v}(s) = \frac{v(s)}{\sqrt{T}} \) and \( \| \cdot \| \) stands for the usual norm in \( L^2[0, T] \). Thus, Lemma 4.1 completes our proof.
3.2. Proof of Theorem 2.4

Taking \( v_{\text{opt}}(t) = \sqrt{\psi(t,t)} \) into the equation (14), then the likelihood function is

\[
\mathcal{L}(\vartheta, Z^T) = \mathbf{E}_\vartheta \exp \left\{ - \int_0^T (-\vartheta Q_t + v_{\text{opt}}(t)) dZ_t - \frac{1}{2} \int_0^T (-\vartheta Q_t + v_{\text{opt}}(t))^2 d\langle M \rangle_t \right\},
\]

then the maximum likelihood estimator (MLE) will be

\[
\hat{\vartheta}_T = \frac{\int_0^T v_{\text{opt}}(t)Q_t d\langle M \rangle_t - \int_0^T Q_t dZ_t}{\int_0^T Q_t^2 d\langle M \rangle_t},
\]

and the estimation error has the form

\[
\hat{\vartheta}_T - \vartheta = - \frac{\int_0^T Q_t dM_t}{\int_0^T Q_t^2 d\langle M \rangle_t},
\]

just take attention that here the \( Q_t \) will be with the relationship with \( v_{\text{opt}}(t) \). Because \( \int_0^T Q_s dM_s, 0 \leq t \leq T \) is a martingale and \( \int_0^T Q_t^2 d\langle M \rangle_t \) is its quadratic variation. In order to prove the Theorem 2.4, we only need to check the Laplace Transform of the quadratic variation and Lemma 4.2 achieves the proof.

3.3. Proof of Theorem 2.5

With the law of large numbers, in order to obtain the strong consistency of \( \vartheta \), we only need to prove that

\[
\lim_{T \to \infty} \int_0^T Q_t^2 d\langle M \rangle_t = +\infty
\]

or there exists a positive constant \( \mu \) such that the limit of the Laplace Transform

\[
\lim_{T \to \infty} \mathbf{E} \exp \left( -\mu \int_0^T Q_t^2 d\langle M \rangle_t \right) = 0.
\]

In Lemma 4.3 if we take a big enough \( \mu > 0 \) such that the limit is negative (the \( \mu \) can be easily found), then the equation (29) is directly from this Lemma which implies the strong consistency.

4. Appendix

**Lemma 4.1.** For the kernel \( K_T(s, \sigma) \) defined in equation (26)

\[
\lim_{T \to \infty} \sup_{\tilde{\vartheta} \in L^2[0,T], ||\tilde{\vartheta}|| \leq 1} (K_T \tilde{\vartheta}, \tilde{\vartheta}) = \frac{1}{\vartheta^2}
\]

with an optimal input \( v_{\text{opt}}(t) = \sqrt{\psi(t,t)} \)
Proof. When we take \( v(t) = v_{opt}(t) = \sqrt{\psi(t,t)} \), then

\[
\frac{dP(t)}{d\langle M \rangle_t} = -\frac{1}{2} \partial A(t) P(t) + b(t) v_{opt}(t), P(0) = 0_{2 \times 1}.
\]

Because for \( H > 1/2 \), \( \frac{d\langle M \rangle_t}{dt} = g^{2}(t,t) \). From [27]

\[
\langle M \rangle_T \sim T^{2-2H} \lambda^{-1}_H, \quad T \rightarrow \infty, \quad \lambda_H = \frac{2H \Gamma(3 - 2H) \Gamma(H + 1/2)}{\Gamma(3/2 - H)}.
\]

then with the calculus of [4] we can easily obtain

\[
\lim_{T \rightarrow \infty} \frac{1}{4T} \int_0^T (\ell(t)^* P(t))^2 d\langle M \rangle_t = \frac{1}{\vartheta^2}.
\]

That is to say the lower bound at least will be \( \frac{1}{\vartheta^2} \).

Now we will try to find the upper bound. Let us introduce the Gaussian process \( (\xi_t, 0 \leq t \leq T) \)

\[
\xi_t = \left( \frac{1}{\sqrt{\psi(s,s)}} \ell(s) * \varphi(s) \otimes dW_s \right) \varphi^{-1}(t), \quad \xi_T = 0
\]

where \((W_s, s \geq 0)\) is a Wiener process and \( \otimes \) denotes the Itô backward integral (see [18]). It is worth emphasizing that

\[
K_T(s, \sigma) = \frac{1}{4} E \left( \xi_s b(s) \frac{1}{\sqrt{\psi(s,s)}} \xi_\sigma b(\sigma) \frac{1}{\sqrt{\psi(\sigma,\sigma)}} \right) = E(\mathcal{X}_s \mathcal{X}_s),
\]

where \( \mathcal{X} \) is the centered Gaussian process defined by \( \mathcal{X}_t = \frac{1}{2} \xi_t b(t) \frac{1}{\sqrt{\psi(t,t)}} \). The process \((\xi_t, 0 \leq t \leq T)\) satisfies the following dynamic

\[
-d\xi_t = \frac{\vartheta}{2} \xi_t A(t) d\langle M \rangle_t + \ell(t)^* \frac{1}{\sqrt{\psi(t,t)}} \otimes dW_t, \quad \xi_T = 0.
\]

Obviously, \( K_T(s, \sigma) \) is a compact symmetric operator for fixed \( T \), so we should estimate the spectral gap (the first eigenvalue \( \nu_1(T) \)) of the operator. The estimation of the spectral gap is based on the Laplace transform computation. Let us compute, for sufficiently small negative \( a < 0 \) the Laplace transform of \( \int_0^T \mathcal{X}_t^2 dt \):

\[
L_T(a) = E_a \exp \left( -a \int_0^T \mathcal{X}_t^2 dt \right) = E_a \exp \left( -a \int_0^T \left( \frac{1}{2} \xi_t b(t) \frac{1}{\sqrt{\psi(t,t)}} \right)^2 dt \right)
\]

On one hand, for \( a > -\frac{1}{\nu_1(T)} \), since \( \mathcal{X} \) is a centered Gaussian process with covariance operator \( K_T \), using Mercer’s theorem and Parseval’s inequality, \( L_T(a) \) can be represented as:

\[
L_T(a) = \prod_{i \geq 1} (1 + 2a \nu_i(T))^{-\frac{1}{2}}, \quad (31)
\]
where \( \nu_i(T), i \geq 1 \) is the sequence of positive eigenvalues of the covariance operator. On the other hand,

\[
L_T(a) = E_\theta \left( -\frac{a}{4} \int_0^T \xi_i b(t)b(t)^* \xi_i^* d(M)_t \right)
= \exp \left( \frac{1}{2} \int_0^T \text{trace}(\mathcal{H}(t)M(t)d(M)_t) \right)
\]

where \( \mathcal{M}(t) = \ell(t)^*\ell(t) \) and \( \mathcal{H}(t) \) is the solution of Ricatti differential equation:

\[
\frac{d\mathcal{H}(t)}{d(M)_t} = \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{A}(t)\mathcal{H}(t) + \mathcal{H}(t)\mathcal{M}(t)\mathcal{H}(t) - \frac{a}{2} b(t)b(t)^*,
\]

with \( \mathcal{A}(t) = -\frac{a}{2} A(t) \) and the initial condition \( \mathcal{H}(0) = 0_{2 \times 2} \), provided that the solution of this equation exists for any \( 0 \leq t \leq T \).

It is well known that if \( \det \Psi_1(t) > 0 \), for any \( t \in [0, T] \), then \( \mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t) \), where the pair of \( 2 \times 2 \) matrices \( (\Psi_1, \Psi_2) \) satisfies the system of linear differential equations:

\[
\begin{align*}
\frac{d\Psi_1(t)}{d(M)_t} &= -\Psi_1(t)\mathcal{A}(t) - \Psi_2(t)\mathcal{M}(t), & \Psi_1(0) = \text{Id}_{2 \times 2}, \\
\frac{d\Psi_2(t)}{d(M)_t} &= -\frac{a}{2} \Psi_1(t)b(t)b(t)^* + \Psi_2(t)\mathcal{A}(t)^*, & \Psi_2(0) = 0_{2 \times 2},
\end{align*}
\]

and

\[
L_T(a) = \exp \left( -\frac{1}{2} \int_0^T \text{trace}(\mathcal{A}(t) d(N)_t) \right) (\det \Psi_1(T))^{-\frac{a}{2}}. \tag{33}
\]

Rewriting the system \((32)\) in the following form

\[
\frac{d(\Psi_1(t), \Psi_2(t)J)}{d(M)_t} = (\Psi_1(t), \Psi_2(t)J) \cdot (\Upsilon \otimes A(t)), \tag{34}
\]

where \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \Upsilon = \begin{pmatrix} \frac{a}{2} & \frac{-a}{2} \\ \frac{-a}{2} & -1 \end{pmatrix} \)

When \(-\frac{\sqrt{2}}{2} \leq a \leq 0\), we have two real eigenvalues of the matrix \( \Upsilon \), we denote them \((x_i)_{i=1,2}\). It can be checked that there exists a constant \( C > 0 \) such that

\[
\det \Psi_1(T) = \exp \left( (x_1)T \right) \left( C + O_{T \to \infty} \left( \frac{1}{T} \right) \right)
\]

where \( x_1 = \sqrt{\frac{a^2}{4} + \frac{a}{2}} \). Therefore, due to the \((33)\), we have \( \prod_{i \geq 1} (1 + 2a\nu_i(T)) > 0 \) for any \( a > -\frac{\sqrt{2}}{2} \).

It means that

\[
\nu_i(T) \leq \frac{1}{\sqrt{2}}
\]

\(\square\)
Lemma 4.2. For \( v(t) = v_{opt}(t) \) defined in Lemma 4.1 the Laplace Transform

\[
\mathcal{L}_T(\mu) = \mathbb{E}_\phi \exp \left( -\frac{\mu}{T} \int_0^T Q_t^2 \, d\langle M \rangle_t \right) \xrightarrow{T \to \infty} \exp \left( -\mu \left( \frac{1}{2\theta} + \frac{1}{\theta^2} \right) \right)
\]

for every \( \mu > 0 \).

Proof. First, we replace \( Q_t \) with \( \zeta_t \) and rewrite the Laplace transform, that is

\[
\mathcal{L}_T(\mu) = \mathbb{E}_\phi \exp \left\{ -\frac{\mu}{T} \int_0^T \zeta_t R(t) \zeta_t^* \, d\langle M \rangle_t \right\}
\]

where \( \zeta_t \) is defined in (12) and \( R(t) = \frac{1}{2} \begin{pmatrix} \psi^2(t,t) & \psi(t,t) \\ \psi(t,t) & 1 \end{pmatrix} \). Following from (11), we have

\[
\mathcal{L}_T(\mu) = \exp \left\{ -\frac{\mu}{T} \int_0^T \left[ \text{tr}(\Gamma(t)R(t)) + Z^*(t)R(t)Z(t) \right] d\langle M \rangle_t \right\}
\]

where

\[
\frac{d\Gamma(t)}{d\langle M \rangle_t} = \frac{\theta}{2} A(t) \Gamma(t) - \frac{\theta}{2} \Gamma(t) A(t)^* + b(t)b(t)^* - \frac{2\mu}{T} \Gamma(t) R(t) \Gamma(t)
\]

and

\[
Z(t) = \mathbb{E}_\phi \zeta_t - \frac{\mu}{T} \int_0^t \varphi(t) \varphi^{-1} \Gamma(s) R(s) Z(s) d\langle M \rangle_s \tag{36}
\]

with

\[
\frac{d\varphi(t)}{d\langle M \rangle_t} = -\frac{\theta}{2} A(t) \varphi(t).
\]

From (25) we know that

\[
\lim_{T \to \infty} \exp \left( -\frac{\mu}{T} \int_0^T (\text{tr} (\Gamma(t)R(t))) d\langle M \rangle_t \right) = \exp \left( \frac{\mu}{2\theta} \right)
\]

On the other hand we know \( \mathbb{E}_\zeta = \mathcal{P}(t) \) defined in Lemma 4.1 with \( v(t) = v_{opt}(t) \), thus

\[
\lim_{T \to \infty} \exp \left( -\frac{\mu}{T} \mathbb{E}_\zeta R(t) (\mathbb{E}_\zeta)^* \right) = \lim_{T \to \infty} \exp \left( -\frac{\mu}{T} \int_0^T (\ell^*(t) \mathcal{P}(t))^2 d\langle M \rangle_t \right) = \exp \left( -\frac{\mu}{\theta^2} \right)
\]

Now, the conclusion is true provided that

\[
\lim_{T \to \infty} \left( \frac{\mu}{T} \int_0^t \varphi(t) \varphi^{-1} \Gamma(s) R(s) Z(s) d\langle M \rangle_s \right) R(t) \left( \frac{\mu}{T} \int_0^t \varphi(t) \varphi^{-1} \Gamma(s) R(s) Z(s) d\langle M \rangle_s \right)^* = 0.
\]

On one hand, from (4) and (25) when \( t \) is large enough

\[
\int_0^t |F(t,s)| \, ds = \left| \frac{\mu}{T} \int_0^t \varphi(t) \varphi^{-1} \Gamma(s) R(s) \right| = O \left( \frac{1}{T} \right), \ T \to \infty \tag{37}
\]
where
\[ F(t, s) = \left| \frac{\mu}{T} \varphi(t) \varphi^{-1}(s) \Gamma(s) R(s) \right| \]
and \(| \cdot |\) denotes \(L^1\) norm of the vector. On the other hand, If we define the operator \(S\) by
\[ S(f)(t) = \int_0^t \int_0^t |F(t, s)| f(s) ds \]
then equation (36) leads to
\[ |Z(t)| \leq |P(t)| + S(|Z|)(t) \]
or we can say \((I - S)(|Z|)(t) \leq |P(t)| \leq \text{Const.}\) From Equation (37) we have for \(t\) and \(T\) large enough
\[ |Z(t)| \leq (I - S)^{-1} \text{(Const)}(t) = \prod_{n=1}^{\infty} S^n(\text{Const})(t) \leq \text{Const.} \quad (38) \]
The \(\text{Const.}\) means some constant, but in different equation they may be different. Combining (37) and (38) we have for \(t\) large enough
\[ \int_0^t |F(t, s)||Z(s)| = O \left( \frac{1}{T} \right), \quad T \to \infty \]
which achieves the proof.

**Lemma 4.3.** For the controlled mixed fractional Ornstein-Uhlenbeck process with the drift parameter \(\vartheta\), we have the following limit:
\[ K_T(\mu) = -\mu \frac{\vartheta}{T} \log E \exp \left( -\mu \int_0^T Q_s^2 d\langle M \rangle_s \right) \to \frac{\mu}{\vartheta^2} + \frac{\vartheta}{2} - \frac{\sqrt{\vartheta^2 + 4 \mu^2}}{2}, \quad T \to \infty. \]
for all \(\mu > -\frac{\vartheta^2}{2}\).

**Proof.** This proof is directly from [26] and Lemma 4.2 or more specially, the term \(\frac{\vartheta}{2} - \frac{\sqrt{\vartheta^2 + 4 \mu^2}}{2}\) comes from [26] and \(\frac{1}{T}\) from Lemma 4.2.

**References**

[1] B. Bercu, L. Coutin and N. Savy (2011) *Sharp large deviations for the fractional Ornstein-Uhlenbeck process*, SIAM Theory of Probability and its Applications, 55, 575-610.

[2] A. Brouste and M. Kleptsyna (2010) *Asymptotic properties of MLE for partially observed fractional diffusion system*, Statistical Inference for Stochastic Processes, 13(1), 1-13.

[3] A. Brouste, M. Kleptsyna and A. Popier (2011) *Fractional diffusion with partial observations*, Communications in Statistics - Theory and Methods, 19-20(40), 3479-3491.

[4] A. Brouste, M. Kleptsyna and A. Popier (2012) *Design for estimation of drift parameter in fractional diffusion system*, Statistical Inference for Stochastic Process, 15, 133-149.
[5] A. Brouste and C. Cai (2013) Controlled drift estimation in fractional diffusion linear systems, Stochastic and Dynamics, 13(3).

[6] P. Cheridito, H. Kawaguchi and M. Maejima (2003) Fractional Ornstein-Uhlenbeck processes, Electronic Journal of Probability, 8(3), 1–14.

[7] I. Cialenco and S. Lototsky and J. Pospisil (2009) Asymptotic properties of the Maximum Likelihood Estimator for stochastic parabolic equations with additive fractional Brownian motion, Stochastics and Dynamics, 9(2), 169-185.

[8] I. Ibragimov and R. Khasminskii (1981) Statistical Estimation. Asymptotic Theory, Springer.

[9] J. Istas and G. Lang (1997) Quadratic variations and estimation of the local Hölder index of a Gaussian process, Ann. Inst. Henri Poincaré, 33(4), 407-436.

[10] C. Jost (1999) Transformation formulas for fractional brownian motion, Stochastic Process. Appl, 116(10), 1341-1357.

[11] M. Kleptsyna and A. Le Breton (2001) Optimal linear filtering of general multidimensional Gaussian process - application to Laplace transforms of quadratic functionals, Journal of Applied Mathematics and Stochastic Analysis, 14(3), 215-226.

[12] M. Kleptsyna and A. Le Breton (2002) Statistical Analysis of the Fractional Ornstein-Uhlenbeck type Process, Statistical Inference for Stochastic Processes, 5, 229–241.

[13] M. Kleptsyna and A. Le Breton (2002) Extension of the Kalman-Bucy filter to elementary linear systems with fractional Brownian noises. Statistical Inference for Stochastic Process, 5, 249-271.

[14] M. Kunita (1984) Equations and stochastic flow of diffeomorphisms, Lectures Notes on Mathematics, 1097, 143-303.

[15] R. Liptser and A. Shiryaev (2001) Statistics of Random Processes, Springer.

[16] I. Norros, E. Valkeila and J. Virtamo (1999) An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motion, Bernoulli, 5, 571-587.

[17] A. Ovseevich, R. Khasminskii and P. Chow (2000) Adaptative design for estimation of unknown parameters in linear systems, Problems of Information Transmission, 36(2), 125-153.

[18] B. Rozovskii (1990) Stochastic Evolution System, Kluwer.

[19] C. Bender, T. Sottinen and E. Vlakeila (2011) Functional processes as models in stochastic finance, Advanced Mathematical Methods for Finance, 75-103.

[20] P. Cheridito (2001) Mixed fractional Brownian motion, Bernoulli, 7(6), 913-934.

[21] P. Cheridito (2003) Representation of Gaussian measures that are equivalent to Wiener measure, Séminaire de Probabilité XXXVII, volume 1832 of Lecture Notes in Maths, 81-89.

[22] C. Cai, P. Chigansky and M. Kleptsyna (2016) Mixed gaussian processes: a filtering approach, Annals of Probability, 44(4), 3032-2075.
[23] C. Cai and W. Lv (2020) Adaptative design for estimation of parameter of second order differential equation in fractional diffusion system, Physica A, 541.

[24] P. Chigansky and M. Kleptsyna (2018) Exact asymptotic in eigenproblems for fractional Brownian motion covariance operators, Stochastic Processes and their Applications, 128(6), 2007-2059.

[25] P. Chigansky and M. Kleptsyna (2019) Statistical analysis of the mixed fractional Ornstein-Uhlenbeck process, Theory of Probability and Its Applications, 63(3), 408-425.

[26] D. Marushkevych (2016) Large deviations for drift parameter estimator of mixed fractional Ornstein-Uhlenbeck process, Modern Stochastic: Theory and Applications, 3(2), 107-117.

[27] Y. A. Kutoyants (2004) Statistical inference for ergodic diffusion processes, Springer Series in Statistics