Entropy generation at the multi-fluid MHD solar wind termination shock

H.-J. Fahr and M. Siewert

Argelander Institut für Astronomie der Universität Bonn, Abteilung f. Astrophysik und Extraterrestrische Forschung, Auf dem Huegel 71, 53121 Bonn (Germany)

Preprint online version: August 27, 2014

ABSTRACT

In a series of earlier papers, we have developed expressions for ion and electron velocity distribution functions and their velocity moments at the passage over the solar wind termination shock. As we have shown there, with introduction of appropriate particle invariants and the use of Liouville’s theorem one can get explicit solutions for the resulting total downstream pressure adding up from partial pressure contributions of solar wind protons, solar wind electrons and pick-up protons. These expressions deliver in a first step the main contributions to the total plasma pressure in the downstream plasma flow and consistently determine the shock compression ratio. Here now we start out from these individual fluid pressures downstream of the shock and thereafter evaluate for the first time the shock-induced entropy production of the different fluids, when they are passing over the shock to the downstream side. As is shown here, the resulting ion entropy production substantially deviates from earlier calculations using a pseudo-polytropic reaction of the ions to the shock compression, with polytropies selected to describe fluid-specific reactions at the shock passage similar to those seen by the VOYAGERs. From these latter models ion entropy jumps are derived that depend on the pick-up ion abundance, while our calculations, to the opposite, deliver an abundance-independent ion entropy production which only depends on the shock compression ratio and the tilt angle between the upstream magnetic field and the shock surface normal. We also do show here that only when including the strongly heated electrons into the entropy balance does one then arrives at the total entropy production that just fulfills the thermodynamically permitted limit.

Key words. Shock waves – Plasmas – Solar wind – Sun: heliosphere

1. Introduction

The plasma physics of shocks in the literature of the past was essentially reduced to the consideration of flux conservation requirements well known as Rankine-Hugoniot relations (see e.g. Serrin 1959; Landau and Lifshitz 1977; Gombosi 1998). In these relations the internal microphysics of the shock transition usually is not explicitly formulated, instead it is attempted to describe the main shock features with the help of conservation equations requiring the conservation of the mass-, the momentum-, and the energy- fluxes at the plasma passage from upstream to downstream side of the shock. Even though this naturally is the main physical request, this procedure nevertheless has the normal drawback that these conservation relations do not allow for a unique solution, since the fluid-like conservation requests do not establish a closed system of equations, thus not allowing for one unique solution. In order to arrive at specific, discrete solutions one rather has to assume something in addition to the fluid-like conservation requirements. Often this is done assuming a polytropic relation between the pressure and density, prescribing for example the rate of entropy generation at the shock passage.

Things become even much more complicated, if anisotropic plasma pressures and magnetic field stresses have to be taken into account. Then the system of conservation equations is substantially enlarged (see Hudson 1970; Baumjohann and Treumann 1996; Gombosi 1998; Erkaev et al. 2000; Diver 2001) and can only be solved by adding additional informations, such as e.g. two adiabatic equations requiring the conservation of two CGL- invariants (Chew et al. 1956), as suggested by e.g. Neubauer (1970). Furthermore, Siewert and Fahr (2008); Siewert and Fahr (2009); Fahr and Siewert (2010) have studied thereafter the action of shock-generated unstable anisotropic distribution functions that drive magneto-acoustic and Alfvénic turbulences. This was identified as a specific microphysical relaxation process which effectively operates downstream of the shock especially working in terms of efficient entropy generation.

Furthermore even on the fluid-level the system complicates substantially, if more than only one plasma fluid have to be considered. If instead of a monofluid, for instance a multifluid plasma has to be consistently described at its shock passage, then a number of additional complications have to be faced in order to arrive at appropriate solutions (see Zank et al. 1993; Le Roux and Fichtner 1997, le Roux and Fichtner 1999, with Chalov and Fahr 1994, 1995, 1996 giving descriptions for two- and three-fluid plasmas passing over the solar wind termination shock. The three fluids treated by them as being subject to the bulk motion of the solar wind are normal solar wind protons (SW’s: eV-energetic), pick-up protons (PUI’s: KeV-energetic) and anomalous cosmic ray protons (ACRs: MeV-energetic). A consistent solution of the shock passage of this three-fluid plasma is only possible, if some additional prescriptions are made about how these fluids thermodynamically interact with eachother when undergoing a shock. 

In Chalov and Fahr (1996), it is formulated that, according to the pickup proton pressure $P_{pui}$, a specific percentage $\eta$ of these ions is Fermi-1 accelerated at the shock and injected to the MeV-energetic ACR fluid regime with an average energy of $E_{inj}$, where both $\eta$ and $E_{inj}$ are unknown parameters which need to be
fixed by numbers. This then, however, has the interesting consequence that PUI’s do not react adiabatically at the shock, but rather in a quasi-isothermal mode with an effective polytropic index $\gamma_{\text{pui}}$ given by

$$\gamma_{\text{pui}} = \gamma_{\text{adia}} - \delta(\gamma_{\text{adia}} - 1)$$  \hspace{1cm} (1)$$

where $\gamma_{\text{adia}} = 5/3$ is the adiabatic index and $\delta$ is given by

$$\delta = \frac{\eta E_{\text{inj}}}{(1 - \frac{1}{\gamma}) P_{\text{pui}} \Delta}$$  \hspace{1cm} (2)$$

with $s$ being the compression ratio of the shock.

Thus one can see that the entropy production occurring at the shock passage in this case is specifically different for the different fluids and is regulated by specific assumptions for values of $\eta$ and $E_{\text{inj}}$ (see Fig. 8 of Chalov and Fahr 1996).

In the following part of the paper we shall now study in detail the entropy generation in the different fluids when they are passing over the shock, including electrons as an independent separate fluid. Hereby we try to avoid the above mentioned ad-hoc assumptions introducing instead kinetic informations on the single particle behaviour at the shock passage. For that purpose we first derive the expressions for the downstream pressures of these separate fluids after taking first a short look into general aspects of the entropy generation at the shock under Boltzmann kinetic auspices.

### 2. The Boltzmann entropic view

In order to study the entropy of multicomponent systems, we first need to understand the entropy of a single fluid that is described by an arbitrary physical velocity distribution function $f(v)$. Therefore, we need to establish the basic equations required for this first.

According to e.g. Landau and Lifshitz (1977), for a state of a system close to a thermodynamic equilibrium, assumed to prevail at some distance upstream and downstream of the shock, the kinetic ensemble entropy per particle can be given by the expression (see also Weizel 1958; Cercignani 1988; Brev and Santos 1992; Treumann 2001)

$$S(r) = \frac{S(r)}{k_B} = -\int f(r,v) \cdot \ln(f(r,v)) \cdot d^3v, \hspace{1cm} (3)$$

This equation is valid only in the rest frame of the system, i.e. a reference frame where the plasma does not possess a bulk flow speed ($U = 0$), as defined by

$$U \equiv v \equiv \int d^3v v \cdot f(v). \hspace{1cm} (4)$$

For a system in motion, the definition of the entropy has to be modified to account for the so-called bulk particle flow speed $U$.

$$S(r,U) = -\int f(r,v-U) \cdot \ln(f(r,v-U)) \cdot d^3v, \hspace{1cm} (5)$$

since otherwise, and for an arbitrary distribution function $f$, the definition of the entropy would not be unique. This configuration is commonly found in systems where the plasma flow is accelerated or decelerated, and where a “natural” reference frame for the system can not be easily found. One classical example for such a system is an MHD shock wave, where, depending on the problems under investigation, one could select the rest frame of the upstream plasma, the downstream plasma or the shock front itself.

In the following parts of the study, we are mainly interested in entropy jumps at MHD shock waves, where the distribution function $f(v)$ is not readily available. Therefore, we need to use a different approach to the entropy jump. We begin by writing down the differential expression

$$\Delta S = \frac{\partial S}{\partial U} \Delta U + \frac{\partial S}{\partial T} \Delta T + \frac{\partial S}{\partial \eta} \Delta \eta. \hspace{1cm} (6)$$

This expression can be simplified by noting that the kinetic expression for the Boltzmann entropy is only applicable for quasi-LTE conditions, i.e. sufficiently relaxed plasma states. Connected with the shock influence through shock-associated electric and magnetic fields, the plasma properties may have temporarily attained non-relaxed intermediate features like asymmetric, anisotropic distributions (which is the case when using anisotropic MHD jump conditions, see e.g. Erkaev et al. 2000; Fahr and Siewert 2006; Siewert and Fahr 2007) or jet-like velocity structures (e.g. due to a possible over-shooting of electrons or heavy ions, as discussed by Fahr et al. 2012). The standard definition of the Boltzmann entropy can thus evidently not be applied as long as these intermediate, perturbed, non-relaxed conditions have not yet reached a quasi-LTE with the help of instabilities driving isotropisation and relaxation processes. Nevertheless, Eq. (6) may provide a valuable first-order estimate of the final permitted entropy jump.

Following the line of Boltzmann’s understanding, in the above expression the distribution function should be applied as a normalized one, i.e. as a velocity-space probability distribution, so that no explicit dependence of $\Delta S$ on the density jump $\Delta \rho$ appears in the expression for $S$. However, there will be an implicit dependence on $n$ due to the temperature being directly related to the density, allowing to rewrite Eq. (6) in the form

$$\Delta S = \frac{\partial S}{\partial U} \Delta U + \frac{\partial S}{\partial T} \cdot \frac{\partial T}{\partial n} \Delta \eta. \hspace{1cm} (7)$$

This result clearly shows the “kinetic” (first term on the right) and the “thermal” (second term on the right) contributions to the total entropy jump, while at the same time replacing the temperature jump $\Delta T$ with the density jump $\Delta \eta$, allowing us to express the entropy jump as a function of parameters appearing directly in the jump conditions.

#### 2.1. The entropy jump for a Maxwell-Boltzmann distribution at the termination shock

In the following, we make explicit use of two relaxed model distribution functions to describe the plasma on the upstream and downstream sides of the solar wind termination shock (TS) in the shock frame. When introducing shock parameters, the subscripts 1 and 2 are used to denote quantities on the upstream and downstream sides of the shock.

First, we assume that the thermal upstream plasma is well described by a shifted Maxwell-Boltzmann distribution function, i.e.

$$f(v) = \frac{C}{T_{\text{inj}}^{3/2}} \exp\left(-\frac{m(U-v)^2}{2k_B T_{\text{inj}}}\right), \hspace{1cm} (8)$$

with a normalisation factor $C = n(m/2\pi k_B)^{3/2}$. For an MHD shock, this is just the standard kinetic description that is implicitly assumed to persist on the upstream side, and - after some relaxation time - also on the downstream side of the shock.
To simplify the following calculations, we introduce the short-hand notation
\[ \Psi(U, T) = \frac{m(U - v)^2}{2k_BT}, \]  
which allows us to write down the entropy in the more compact form
\[ S = \frac{C}{T^{3/2}} \int \exp(-\Psi) \cdot \left[ \ln \frac{C}{T^{3/2}} - \Psi \right] d^3v \]
\[ = -\frac{C}{T^{3/2}} + \frac{C}{T^{3/2}} \int \exp(-\Psi) \Psi d^3v. \]  
After evaluating the integrals, one obtains
\[ \bar{S}(r) = \ln \left( \frac{T^{3/2}}{C} \right) + \frac{1}{2} \Gamma(5/2). \]  
Consider that thermodynamics is only interested in entropy jumps, but not in absolute values, the constant second term in this sum can be interpreted as a normalisation constant (see e.g. Collier 1995).

This expression in principle allows us to derive the entropy jump \( \Delta S = S_2 - S_1 \) between both sides of the shock, assuming that we have solved the MHD jump conditions. However, it is also possible to derive a more explicit expression for the entropy jump that does not depend on explicit values for the shock parameters. Using Eq. 10 it is possible to evaluate the partial derivatives in Eq. 7. After some elementary operations, we obtain the relations
\[ \frac{\partial}{\partial T} \Psi(U, T) = -\frac{1}{T} \Psi(U, T) \]  
and
\[ \frac{\partial}{\partial U} \Psi(U, T) = 2 \sqrt{\frac{m}{2k_BT}} (U - \nu \cos \theta), \]  
where \( \theta = \angle(U, v) \). Using these relations, we obtain
\[ \frac{\partial}{\partial T} \bar{S} = \frac{3}{2T} + \int \frac{\partial}{\partial T} \frac{C}{T^{3/2}} \exp(-\Psi(U, T)) \Psi(U, T) d^3v. \]  
Evaluating the partial derivatives and collecting terms, we further obtain
\[ \frac{\partial}{\partial T} \bar{S} = \frac{3}{2T} \int \frac{C}{T^{3/2}} \exp(-\Psi(U, T)) \Psi(U, T) d^3v \]
\[ -\frac{5}{2T} \int \frac{C}{T^{3/2}} \exp(-\Psi(U, T)) \Psi(U, T) d^3v \]
\[ + \frac{1}{T} \int \frac{C}{T^{3/2}} \exp(-\Psi(U, T)) \Psi^2(U, T) d^3v. \]  
After some elementary substitutions, the integrals evaluate to
\[ \frac{\partial}{\partial T} \bar{S} = \frac{1}{T} \left( \frac{3}{7} - \frac{5}{4} \Gamma \left( \frac{5}{2} \right) \right) = \frac{3}{2T}. \]  
Using a similar approach, we obtain for the other partial derivatives
\[ \frac{\partial}{\partial U} \bar{S}(r) = \left( -2 \sqrt{\frac{m}{2k_BT}} \right) \int \frac{C}{T^{3/2}} (U - \nu \cos \theta) \exp(-\Psi(U, T)) \Psi(U, T) d^3v \]
\[ + 2 \sqrt{\frac{m}{2k_BT}} \int \frac{C}{T^{3/2}} (U - \nu \cos \theta) \exp(-\Psi(U, T)) d^3v. \]  
After introducing spherical coordinates and substituting \( w = U - \nu \cos \theta \), this expression evaluates into
\[ \frac{\partial}{\partial U} \bar{S} = -\sqrt{\frac{m}{2k_BT}} \left( \Gamma(3) - \Gamma(2) \right) = -\sqrt{\frac{m}{2k_BT}}. \]  
Therefore, we obtain an entropy jump for the general Maxwell-Boltzmann distribution given by
\[ \Delta S = -\sqrt{\frac{m}{2k_BT}} \Delta U + \frac{3}{2T} \frac{\partial T}{\partial n} \Delta n. \]  
The only unknown parameter entering this expression is the thermodynamic relation between temperature and density, which reflects the entire microphysics inside the shock transition layer that is inaccessible to MHD.

Depending on the behaviour of \( \frac{\partial T}{\partial n} \), the entropy jump may be negative, suggesting that the shock transition is physically impossible. However, one also has to consider that the partial derivative essentially represents an average of the entire microphysics of the shock transition, to which the single fluid approximation just may be a too strong simplification, and the presence of plasma-wave interactions or heavy ions in principle even allow a decrease in the entropy in a single part of the entire system (see also Eq. 12).

In the limit of a single component shock, one can request the entropy change to be positive (\( \Delta S > 0 \)), and obtains
\[ \frac{\partial T}{\partial n} > \frac{2}{3} \sqrt{\frac{m}{2k_BT}} \Delta U \frac{\Delta S}{\Delta n}. \]  
which can be simplified further by introducing the MHD compression ratio \( s = \frac{U_2}{U_1} = \frac{n_2}{n_1} \), resulting in
\[ \frac{\partial T}{\partial n} > \frac{2}{3} \sqrt{\frac{m}{2k_BT}} U_1 \frac{s^{-1} - 1}{s - 1} \]
\[ = \frac{2}{3} \sqrt{\frac{m}{2k_BT}} U_2 \frac{1 - s}{s - 1}. \]  
This equation provides an easy approach to the question whether a physical, entropy-increasing shock is possible without having to introduce additional nontrivial thermodynamic degrees of freedom to the shock transition.

2.2. The entropy of a \( \kappa \) function

The shock passage very likely provokes nonthermal equilibrium conditions, and therefore, a distribution function \( f(v) \) that differs from the classical thermodynamic Maxwellian (Eq. 3) is required. Following theoretical arguments studied by e.g. Treumann (1999), Treumann et al. (2004), or Livadiotis and McComas (2012) we adopt a nonthermal \( \kappa \) function to describe the quasi-stable nonthermal downstream equilibrium state that likely develops on the near downstream side of the shock. Therefore, we need to understand the entropy stored in a \( \kappa \)-function, which allows us to compare it to the entropy gain using a conventional Maxwellian distribution (Eq. 19).

Taking the standard definition of an isotropic \( \kappa \) distribution,
\[ f_{\kappa}(v) = \frac{n}{(\pi \sqrt{\Theta^2})^{3/2}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa/2)} \left[ 1 + \frac{v^2}{\Theta^2} \right]^{-(\kappa+1)/2}. \]  

H.-J. Fahr and M. Siewert: Entropy generation at the multi-fluid MHD solar wind termination shock
with

$$\Theta^2 = \frac{2kT_c}{m},$$  

(23)

we are able to calculate the $\kappa$-entropy using the same Boltzmann formalism as used above for the Maxwell-Boltzmann distribution, as long as one assumes that the $\kappa$-function does reflect an equilibrium state between classical elastic scattering and energy diffusivity (see e.g. [Collied 1995]). Under these conditions, the entropy of a $\kappa$ distribution function is given by

$$S^\kappa = \ln\left(\frac{(\Theta^2\pi\kappa)^{3/2}\Gamma(\kappa-1/2)}{\Gamma(\kappa+1)}\right)$$

$$\cdot \exp\left[(\kappa + 1)(f(\kappa + 1) - f(\kappa - 1/2))\right],$$  

(24)

where the digamma function $\Gamma$ is defined by:

$$\Gamma(z) = \frac{d}{dz} \ln(\Gamma(z)).$$  

(25)

Using elementary properties of the gamma function (see e.g. Abramowitz & Stegun), the somewhat unhandy expression can be simplified by first eliminating the $f$ symbols, leading to

$$S^\kappa = \ln\left(\frac{(\Theta^2\pi\kappa)^{3/2}\Gamma(\kappa-1/2)}{\Gamma(\kappa+1)}\right)$$

$$\cdot \exp\left[(\kappa + 1)\frac{\Gamma(\kappa)}{\Gamma(\kappa+1)} - \frac{(k-3/2)\Gamma(\kappa-3/2)}{\Gamma(\kappa-1/2)}\right].$$  

(26)

After some further evaluations, the $\gamma$ functions in the exponent cancel out, leaving only

$$S^\kappa/k_B = \ln\left(\frac{(\Theta^2\pi)^{3/2}\Gamma(\kappa-1/2)}{\Gamma(\kappa+1)}\right).$$  

(27)

One specifically interesting property of this expression is the limit for $\kappa \to \infty$, which should reproduce the classical Maxwellian limit (Eq. (11)). Using Stirlings formula (see e.g. Abramowitz & Stegun), one easily sees that

$$\Gamma(\kappa-1/2) \rightarrow k^{-3/2},$$  

(28)

and the entropy becomes

$$S^\kappa \rightarrow \ln(\Theta^2\pi)^{3/2},$$  

(29)

which (with the exception of a different normalisation, see [Collied 1995]) is just the entropy for a Maxwellian distribution function.

2.3. The classical MHD shock entropy gain and the polytropic index $\gamma$

For completeness, we also briefly mention the entropy gain found in most MHD textbooks, as is derived from classical shock relations, and without paying much attention to the kinetic origin of the entropy (see e.g. [Serrin 1959, Landau and Lifshitz 1977]). This jump in entropy per unit mass at the shock, judged in the frame of the bulk plasma flow, is given by

$$\Delta S_{MHD} = \ln\left[\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma_{adi}}\right],$$  

(30)

where $\gamma_{adi} = 5/3$ is the so-called adiabatic index. For this specific choice of $\gamma_{adi}$, it immediately follows that a gas, reacting strictly adiabatically at the shock compression, will not increase its thermal entropy in the bulk frame at all, which strongly suggests that the shock transition can not be purely adiabatic.

However, the adiabatic index $\gamma_{adi}$ is just a special case of the more general polytropic relation, which for the system studied here can be represented in the form

$$T = T_0 \cdot (n/n_0)^{\gamma-2},$$  

(31)

where $\gamma$ is the more general polytropic index, for which $\gamma \approx \gamma_{adi}$ is just one special value. In addition to the obvious impact on the MHD entropy jump (Eq. (30)), this equation also allows to quantify the previously unknown parameter in Eq. (19) i.e. the partial derivative $\frac{dT}{dn}$, which for the general polytropic relation becomes

$$\frac{dT}{dn} = (\gamma - 1)(\frac{n}{n_0})^{\gamma-2}T_0 \cdot \frac{n_0}{n_0}.$$  

(32)

Observational data studies usually treat the polytropic index $\gamma$ as a free fit parameter, while theoretical studies often apply this relation as an “ad-hoc” boundary condition; in principle, for arbitrary shocked systems, the $T$-$n$ relation could be of a form that differs from this model approach. However, at an MHD shock-wave, this approach can be justified by the fact that the microphysics of the shock can not be modeled by a fluid theory, and therefore, a polytropic index can be interpreted as an averaged description of the microphysical plasma-wave interactions in the shock transition layer. Applying Eq. (32) to Eq. (21) one easily sees that the monofluid polytropic approach to the MHD shock always increases the entropy in the system for $\gamma > 1$, which covers pretty much all polytropic indices found in the literature (where, usually, $\gamma = 1\ldots2$).

In this study, we present one possible way of connecting an effective polytropic index with a multifluid MHD shock wave, thus introducing theoretical concepts to the ad-hoc boundary condition, and compare the result with the best fitting effective polytropic index found by [Wu et al. 2009] at the solar wind termination shock.

3. The multi-fluid plasma at the termination shock

3.1. Pressures in a multifluid system

As suggested by Eq. (30), the upstream and downstream pressures of the shocked plasma provide an important quantity for the entropy problem. For an MHD approach, the upstream and downstream pressures directly enter Eq. (30), while for the kinetic approach (Eq. (7), we need to convert between pressures (that appear in the jump conditions) and temperatures (that appear in the kinetic distribution function). Due to the fact that the immediate downstream side of the shock may represent a thermal nonequilibrium, this conversion can be difficult; in this paper, we assume that the downstream side is defined as the region where a new (possibly nonthermal) equilibrium has been reached, so that we can ignore more details of the shock transition. However, we do not ignore the multifluid characteristics of the shock, for which we adopt an overshooting description, i.e. a description where particles possessing different electric charges and masses will react differently to the global electric ramp of the shock potential. While thermal and PUI protons do not require this additional detail, it becomes important when we study the entropy gain of the electron component in Sect. 4.3.

In this study, we consider MHD shocks in a one-dimensional approach with the shock normal $n$ assumed to be parallel to the upstream bulk flow velocity $U_1$ and an upstream magnetic field vector tilted by an angle $a$ with respect to $n$. We consider three
different fluids, namely solar wind protons, solar wind electrons and pick-up ions. For an explicit description of the downstream pressure $P_{2,p}$ of the thermal SW proton component, we adopt the relations given by Fahr et al. (2012) and Fahr and Siewert (2013),

$$P_{2,p} = \frac{1}{3} s \cdot (2A(s, \alpha) + B(s, \alpha)) \cdot P_{1,p},$$

(33)

where the indices 1, 2 denote upstream and downstream quantities, respectively, $s = U_1 / U_2$ denotes the shock compression ratio, and the functions $A(s, \alpha)$ and $B(s, \alpha)$ are given by

$$A(\alpha) = \sqrt{\cos^2 \alpha + s^2 \sin^2 \alpha}$$

(34)

and

$$B(\alpha) = s^2 / A^2(\alpha),$$

(35)

where the angle $\alpha$ defines the inclination between the shock surface normal $n$ and the upstream magnetic field $B_1$ (i.e. for a perpendicular shock this means $\alpha = \pi / 2$).

In our multifluid description, the downstream PUI pressure $P_{2,pui}$ is derived in the same way as the downstream proton pressure $P_{2,p}$, i.e. the main difference between both pressures is a factor $\Gamma_1 = P_{pui} / P_{1,p}$, and therefore is simply given by the following analogous formula

$$P_{2,pui} = \frac{1}{3} s [2A(\alpha) + s^2 A^2(\alpha)] \cdot \Gamma_1 P_{1,p},$$

(36)

This approach turns out to be justified because a pick-up proton cannot be physically differentiated from a solar wind proton as a different ion species, due to the following reasons. The main complication when dealing with an initial distribution of PUIs at a shock is the reflection of a certain fraction of the energetic ions from the electric shock potential. Considering the total velocity $u$ of an individual ion in the shock frame, i.e. $u = v + U$, it becomes obvious that some ions may possess a velocity vector that does not enable them to cross the shock potential at the first attempt. Instead, these ions are reflected into the shock precursor region where they induce local two-stream instabilities. Following this, they do gain energy and momentum from interactions with just these instabilities, until they finally get transported across the shock. In many MHD shock simulations, a different (i.e. purely numerical) approach to this situation has been studied, but the answer concerning the resulting final downstream plasma mixture has not yet been conclusively given (see Schoelei [1993], Liewer et al. [1993], Kucharek and Schoelei [1995], Kucharek et al. [2006], Zank et al. [2010], Matsukiyo and Schoelei [2011], Wu et al. [2010]). Results obtained in these simulations strongly depend on the shock compression ratio and especially on the upstream PUI velocity distribution used by the authors at the start of their simulations, e.g. cooled or heated shell distributions. Nevertheless, these simulations clearly demonstrate that a coupling between PUIs and thermal ions is introduced at the shock in a natural way.

To get a reliable answer to this problem for our purposes here we look back at the work by Chalov and Fahr (1996), who studied the kinetic transport of a statistical sample of PUIs at their passage over the electric and magnetic shock structure, starting from a realistic upstream PUI distribution functions taking into account the cooling of PUIs before they enter the shock according to most up-to-date theories (see Chalov and Fahr 1995). Following the method by Decker (1988) involving a de Hoffmann-Teller frame and the conservation of the magnetic moment, they found that Chalov and Fahr (1996, Figr. 6 and 7), for realistic upstream PUI distributions, the fraction of reflected (second order PUIs) over directly transmitted (first order PUIs) is less than $10^{-2}$ and is especially low for perpendicular shocks ($\alpha = \pi / 2$). Taking this result as solid, it is thus possible to assume for the rest of this paper that PUIs behave practically the same way as SW ions, and that they are all transmitted through the shock, with just a negligibly small number of reflected PUIs not taken into account in our present consideration. We study the entropy gain in this approximation of the multicomponent shock in Sect. 4.1.

3.2. The joint downstream ion distribution

In addition to this straightforward approach, we will also apply a different description of the downstream plasma, motivated by the coupling between thermal and nonthermal ions that is naturally introduced at the shock. Since both downstream ion populations (i.e. thermal protons and PUIs) are located at overlapping regions in phasespace, it is possible to describe them as one joint downstream pick-up ion and solar wind proton distribution. As demonstrated by Fahr and Siewert (2013), the main features of this combined ion distribution can be represented surprising well by a joint Kappa distribution,

$$f_2(v) = \frac{n_2}{(\pi \kappa_\theta^2)^{3/2}} \left( \frac{1}{2} \right)^{3/2} \left( 1 + \frac{\nu^2}{\kappa_\theta^2} \right)^{-3/2},$$

(37)

with a Gaussian core velocity spread $\Theta_0$ and a net Kappa index $\kappa_2$ as characteristic parameters. We interpret the downstream solar wind proton population as constituting the so-called Gaussian core $\Theta_0$ of the Kappa distribution (following Collier [1995], Heerikhuisen et al. [2008], Livadiotis and McComas [2009] and thus fix the needed Kappa function parameters as done by Fahr and Siewert (2013). The downstream thermal width $\Theta_0$ of the Gaussian core then turns out as

$$\Theta_0^2 = \frac{s}{3} \left( 2A(\alpha) + B(\alpha) \right) \frac{2}{n_s m} P_{1,p}.$$

(38)

The joint Kappa index $\kappa_2$ follows from the requirement of the pressure identity $P_2 = P_{2,p} + P_{2,pui}$, resulting in

$$\kappa_2 = \frac{3 [1 + \zeta K]}{2 \zeta [K - 1]},$$

(39)

with the parameter $K$ given by

$$K = \frac{k_{1,p} - 3/2 k_{1,pui}^2 \Theta_0^2_{pui}}{k_{1,p} - 3/2 k_{1,pui}^2 \Theta_0^2_{pui}}.$$

(40)

where the upstream PUI abundance is given by $\zeta = n_{1,pui} / n_{1,p}$. In this description, we assume separate Kappa distribution functions on the upstream side of the TS, one for the thermal ions (using the parameters $\Theta_{1,p}$ and $k_{1,p}$), and another separate function for the PUIs (using the parameters $\Theta_{1,pui}$ and $k_{1,pui}$). This allows for a greater flexibility in upstream configurations, with $K \to \infty$ for a pure Maxwellian distribution (i.e. the thermal component), and $K \to 3/2$ for a pure $\nu^{-2}$ power law (i.e. the PUI component). In the following, we will adopt the same parameters as Fahr and Siewert (2013), resulting in the parameter $K = 119$.

We study the entropy gain at the multicomponent shock in this approach in greater detail in Sect. 4.4.
3.3. Downstream pressures in magneto-adiabatic or pseudo-polytropic approaches

Before calculating actual entropy gains, we study the alternate approach by Wu et al. (2009), where the relation between downstream and upstream PUI pressures is obtained as a pseudo-adiabatic reaction of the PUIs to the shock compression given by

\[ P_{2,pui} = (\rho_{pui,2}/\rho_{pui,1})^{\gamma_p} P_{1,pui} = (s)^{\gamma_p} P_{1,pui} , \tag{41} \]

where instead of a joint \(k\) index or a kinetically derived enhancement factor, a PUI-specific polytropic index \(\gamma_p \geq \gamma_{adia} = 5/3\) is used to describe an additional heating of PUIs at the shock passage, which in the view of the authors is a need to make the simulation results better fit the Voyager-2 shock data (see Richardson et al. 2008). We can now compare their approach with our kinetic model by deriving the adequate polytropic index \(\gamma_p\) that would give equivalent PUI pressure transformations as derived from our model. This requirement leads to the following expression:

\[ \Pi(\alpha, s) = \frac{1}{3} s [2A(\alpha) + \frac{s^2}{A^2(\alpha)}] = (\rho_{pui,2}/\rho_{pui,1})^{\gamma_p} = s^{\gamma_p} , \tag{42} \]

or, for the resulting polytropic index \(\gamma_p\),

\[ \gamma_p = \frac{\ln(s(\frac{2A(s, \alpha)}{\frac{s^2}{A^2(s, \alpha)}})))}{\ln s} \tag{43} \]

For a quasiperpendicular shock as encountered by the Voyager-2 spacecraft, we need to adopt \(\alpha = 90^\circ\), where

\[ A(s, \alpha) \rightarrow s , \tag{44} \]

and Eq. (43) reduces to

\[ \gamma_p = \frac{\ln(s(\frac{2}{s} + \frac{1}{s}))}{\ln s} \tag{45} \]

For a compression ratio of \(s = 3\), we simply obtain

\[ \gamma_p(90^\circ) = \frac{\ln(7)}{\ln(3)} = 1.77 . \tag{46} \]

Evaluating the same equation for a more parallel shock, e.g. with \(\alpha = 20^\circ\), we instead obtain \(\gamma_p(20^\circ) = 1.82\). Global results for all angles and various compression ratios are presented in Fig. 1, which demonstrates that, for most magnetic field orientations, our kinetic model for the shock transition results in an over-adiabatic behaviour. As demonstrated by Figure 1, this overadiabatic behaviour with \(\gamma_p > \gamma_{adia} = 5/3\), dominates the parameter region for angles \(\alpha < 30^\circ\) and \(\alpha > 40^\circ\). However, for angles between \(30^\circ < \alpha < 40^\circ\), the effective polytropic indices are unexpectedly close to the classical adiabatic value of \(\gamma_p \approx \gamma_{adia}\). The reason for this is not directly evident from the calculations presented here, but can be understood with the help of the results published earlier by Fahr and Siewert (2010). This earlier paper demonstrates that the range of tilt angles between \(30^\circ\) and \(40^\circ\) is characterized by the absence of downstream ion temperature anisotropies (i.e. \(T_L \approx T_B\)), which can be easily seen on Fig. 3 in the mentioned earlier study. This means that, on this narrow interval, the two degrees of freedom parallel and perpendicular to \(B\) are equally heated, which is the same behaviour as the one found in the case of an unmagnetized gas. In addition, the changes of the temperature from upwind to downwind behave just as in the adiabatic shock compression, i.e. \(P_2/P_1 = (n_2/n_1)^{\gamma_{adia}}\).

These values display the same behaviour found by Wu et al. (2009), who found a general over-adiabatic behaviour when trying to best-fit the Voyager-2 shock observations. They obviously needed a preferential heating of the PUI-fluid compared to the SW fluid, and in their case obtain it by a fluid-specifically increased polytropic index.

While this may suggest that the two theoretical approaches can both deliver similar results, it must, however, be recognized that the selected \(\gamma_{p}\) value invokes an unexplained ad hoc process for PUIs. This follows from the fact that this approach treats the PUI protons and their thermodynamic reaction to a shock compression in a substantially different way from that of the normal solar wind protons. The justification for this approach may be that some of the PUI protons are reflected by the shock and later get transmitted after experiencing some energy gain. On the other hand, seen from physical grounds and argued on the basis of results presented by Chatov and Fahr (1996), who found that PUI reflection is fairly unlikely, protons should react like protons, disregarded whether they are of the PUI or of the solar wind type. In our approach, this results in PUIs being heated more efficient than solar wind protons due to the simple fact that PUIs are already hotter upstream of the shock. This means that protons of both fluids in fact do react completely alike, and the resulting pressure conversion simply is derived under conservation of kinetic particle invariants. This difference in the shock reaction also leads to different entropy production rates, as we shall demonstrate in the remaining sections. Especially the entropy generated in the pseudo-polytropic multi-ion shock turns out to be very much different from corresponding results that we obtain when using our “magneto-adiabatic” approach. We present explicit values for the entropy generated in a pseudo-polytropic multi-ion shock in Sect. 4.2.

4. The entropy production at the multifluid shock

4.1. The entropy jump in the “magneto-adiabatic” approach

Using our “magneto-adiabatic” formulae for the downstream ion pressures (Eq. (13)), we are now able to calculate the following
non-vanishing entropy jump in the MHD limit (Eq. 30):
\[ \Delta S = \ln(\frac{s}{3(1 + \zeta K)}(2A + B)(1 + \zeta K)(1 + \zeta K_s^{\gamma_w})) \]  
(47)

where \( \zeta = n_{pui}/n_p \) is the PUI abundance, and \( K \) is a parameter reflecting the upstream thermal and nonthermal proton configuration (see Sect. 3.2 for a more detailed explanation). Interestingly enough, this equation simplifies to an expression independent on the PUI abundance (and the upstream parameter \( K \)), simply given by:
\[ \Delta S = \ln(\frac{s}{3}(2A + B)s^{-\gamma_w}) \]  
(48)

This expresses the expected fact that the entropy production in our case does not depend on the abundance \( \zeta \) of upstream pick-up ions, which trivially follows from the concept that PUI protons at the TS should behave exactly like solar wind protons, i.e. depending only on the compression ratio \( s \) and the magnetic tilt angle \( \alpha \). A graphical representation of the entropy gain for this pressure model is given in Fig. 3.

We can also compare this result with the effective polytropic indices \( \gamma_p \) presented on Fig. 1. Adopting a description using polytropic indices, we obtain
\[ \Delta S = \ln(s^{\gamma_p-\gamma_w}) \]  
(49)

This relation easily proves that, for \( \gamma_p < 8/3 \), the normalized entropy increase is of the order of \( \ln s \), i.e. between 0 and 1.38, which agrees with the numerical results given in Fig. 2.

4.2. The entropy jump for non-adiabatic PUIs

In the description by Wu et al. (2009), however, the entropy jump explicitly depends on the pick-up ion abundance \( \zeta \) as we will demonstrate now. Following these authors, solar wind protons and pick-up ions do react to the shock compression in different polytropic forms, the first characterized by a polytropic index \( \gamma_w \), the latter by a larger pseudo-polytropic index \( \gamma_{pui} \). Thus, when looking for the related proton entropy jump of the joint ion population, one finds the following result that is valid for these multi-polytropic conditions:
\[ \Delta S^p = \ln(\frac{P_{pui}}{P_{p1}}^{s^{\gamma_{p1}} + P_{pui,1}^{s^{\gamma_{pui}}}(s^{-\gamma_w})}) \]  
(50)

Now, introducing the same representation of the upstream pressures \( P_{pui,1} \) and \( P_{p1} \), as used in our approach, and again introducing the PUI abundance \( \zeta \), we obtain an entropy jump given by
\[ \Delta S^p = \ln(\frac{s^{\gamma_p + \zeta K s^{\gamma_{pui}}}}{1 + \zeta K}) \]  
(51)

We now assume that solar wind protons are reacting adiabatically at the shock, so we can select \( \gamma_p = \gamma_{adia} \), and obtain an entropy jump of
\[ \Delta S^p = k_B \ln(\frac{1 + \zeta K s^{\gamma_p-\gamma_{adia}}}{1 + \zeta K}), \]  
(52)

making it evident that their expression inherently depends on the PUI abundance \( \zeta \).

A graphical representation of the entropy gain in this model is presented in Fig. 3 where one easily sees that \( \Delta S^p \) is about one order of magnitude smaller than the magneto-adiabatic entropy gains presented in the previous section. This can be understood easily, as the entropy gain in the non-adiabatic PUI description is exclusively related to the nonadiabatic PUI behaviour, which only make a fraction of the entire entropy of the system. Thermal ions, behaving adiabatically, do not increase their entropy at all in this representation.

The selection of this assumption was based on a best fit approach to the Voyager data, which does not allow to assess the behaviour of the thermal solar wind plasma component. Therefore, any theoretical modeling must take great care when making model assumptions concerning the behaviour of the thermal protons. However, our results suggest that, the thermal protons do most likely not behave adiabatically at the shock (unless average tilt angles of \( \alpha \approx 40^\circ \) are assumed, see Fig. 1), as this behaviour obviously results in a strong suppression of entropy production.

4.3. The entropy jump for a multifluid system including electrons

Finally, we also want to include the downstream electron pressure that we have derived in Fahr and Siewert (2013), joining it with the ion pressure and derive a more consistent description.

Fig. 2: The normalized ion entropy gain as a function of the magnetic field tilt angle \( \alpha \) and the compression ratio \( s \) for the magneto-adiabatic approach.

Fig. 3: The normalized ion entropy gain as a function of the PUI abundance \( \zeta \) and the PUI polytropic index \( \gamma \) for a shock where thermal protons behave adiabatically, and the PUIs behave non-adiabatically. The compression ratio is \( s = 3 \).

\( \Delta S = \ln(\frac{s}{3(1 + \zeta K)}(2A + B)(1 + \zeta K)(1 + \zeta K_s^{\gamma_w})) \)  
(47)

\( \Delta S = \ln(\frac{s}{3}(2A + B)s^{-\gamma_w}) \)  
(48)

\( \Delta S = \ln(s^{\gamma_p-\gamma_w}) \)  
(49)

\( \Delta S^p = \ln(\frac{P_{pui}}{P_{p1}}^{s^{\gamma_{p1}} + P_{pui,1}^{s^{\gamma_{pui}}}(s^{-\gamma_w})}) \)  
(50)

\( \Delta S^p = \ln(\frac{s^{\gamma_p + \zeta K s^{\gamma_{pui}}}}{1 + \zeta K}) \)  
(51)

\( \Delta S^p = k_B \ln(\frac{1 + \zeta K s^{\gamma_p-\gamma_{adia}}}{1 + \zeta K}), \)  
(52)
of the total particle entropy increase. In this earlier description, we found that the electron downstream pressure is substantially increased as a reaction of the negatively charged electrons to the electric shock potential. Therefore, one can expect that the electron entropy will be increased as well, and we can start with Eq. 27 and add the electron pressure as found in the earlier study, and obtain

\[
\Delta S^e = \ln \left[ \frac{P_{2,\text{sat}}}{P_1} \right] \\
\left( \frac{P_{1,\text{sat}}}{P_{1,\text{sat}}} \right) \right] \\
\Delta S^e = \ln \left[ \frac{P_{2,\text{sat}}}{P_1} \right],
\]

Now, reminding that \(P_{1,\text{sat}} \approx P_{1,\text{sat}}\), we can adopt \(P_{1,\text{sat}} + P_{1,\text{sat}} + P_{1,\text{sat}} = P_{1,\text{sat}} + \frac{2}{3} + \frac{1}{3} \) and obtain to the expression

\[
\Delta S^e = \ln \left[ \frac{P_{2,\text{sat}}}{P_1} \right] + \frac{2}{3} \left( 2A(\alpha) + B(\alpha) \right) (1 + \zeta K) \\
+ \frac{1}{s} \left( \frac{\rho_{2,\text{sat}}}{\rho_1} \right)^{\alpha} \left[ \sin^2 \alpha A(\alpha) + \cos^2 \alpha B(\alpha) \right],
\]

where the final identity follows from Fahr and Siewert (2013).

Here, \(U_1/c_{1,\text{sat}}\) is the ratio of the upstream plasma flow speed and the thermal proton sound speed (i.e. a sonic Mach number).

As one can see in this expression for the total entropy jump related to the particles, the dependence on the PUI abundance \(\zeta\) now reappears, since not all particles, i.e. electrons and ions, react alike when passing over the shock. However, as it turns out, the term with the resulting \(\zeta\)-dependence is negligible compared to the electron term; in fact, assuming that \(\zeta K \gg 1\), the dependence on these parameters drops out completely. Different from ions, electrons experience a strong overshooting at the shock, resulting in a strong heating by thermalisation of this kinetic overshoot energy. On the other hand, the most important point here is that the magnitude of the entropy jump now has increased significantly due to the large downstream electron pressure as shown in Fig. 4 where we demonstrate that electrons in fact provide the strongest contribution to the entropy increase, and that the Mach number \(M = U_1/c_{1,\text{sat}}\) provides the strongest influence on the overall entropy jump. In addition to this, we have also studied the dependence on the MHD compression ratio \(s\), where one easily sees that the influence of this parameter is also strong (Fig. 5).

Unfortunately, there is neither data available to check on this point, nor is there any running mission dedicated to TS electrons, so it is impossible to verify this result observationally in the forseeable future. Nevertheless, our results suggest that the electron component possesses a strong dependence on various parameters of the shock that can be difficult to observe otherwise, so any future mission to the solar wind TS would greatly benefit from a dedicated electron instrument.

### 4.4. The entropy jump using a downstream \(\kappa\) function for a combined proton component

Finally, we study the impact of using a downstream \(\kappa\) function to represent the joint thermal and PUI proton populations. In Sect. 3.2 we introduced joint downstream proton distribution function describing both a thermal core and nonthermal tail of PUI protons. Since the parameters of the \(\kappa\)-function were set up in a way that the pressure is identical to the magneto-adiabatic entropy (see Sect. 3.1), we can not expect new results from this side. Instead, we now present some selected, more general aspects of entropy jumps with \(\kappa\) functions at the shock.

First, we want to remind the reader that \(\kappa\)-functions originate in a model function for data fits (Vasyliunas 1968), and even after all of the progress that has been made with understanding \(\kappa\)-functions, one still commonly finds modeling approaches where this function and its parameters are not strongly supported by theoretical arguments. Therefore, we now close our study of the multicomponent TS with a brief overview of entropy jumps in \(\kappa\)-functions.

Taking Eq. 27 and the discussion following it, we can find the following relation between the Maxwellian and the \(\kappa\)-entropy:

\[
\tilde{S}^e = S^0 + \ln \left( \frac{\kappa - 1/2}{\Gamma(\kappa + 1)} \right).
\]
This equation easily proves an important point. As long as the upstream and downstream \(\kappa\)-values remain the same, the contribution to the entropy jump cancels out, and the entropy increase simplifies to the classical Maxwellian expression \(\Delta S^0\). Only when there are different \(\kappa\)-indices on both sides of the termination shock, one obtains an additional contribution to the power indices:

\[
\Delta S^* = \Delta S^0 + \ln \left[ \frac{\kappa_2}{\kappa_1} \left( \frac{\Gamma_2 - 1}{\Gamma_1 - 1} \right) \frac{\Gamma_1 + 1}{\Gamma_2 + 1} \right] = \Delta S^0 + \ln \left[ \frac{\kappa_2}{\kappa_1} \frac{\Gamma_1 + 1}{\Gamma_2 + 1} \right]
\]

This relation holds true for arbitrary physical values of \(\kappa\) and clearly demonstrates that the additional contribution to \(\Delta S\) depends only on the \(\kappa\)-parameters on both sides of the shock. In addition, Eq. (56) is only applicable to systems where the number of individual shock components does not change between both sides, i.e. it is not applicable to the joint downstream model adopted in Sect. 3.2.

### 4.5. An eye-guide estimate of the thermodynamically permitted total entropy jump

The total entropy jump resulting from the full conversion of the free kinetic energy of the upstream flow can, however, compared to the above considerations, be estimated much easier along the following procedure: The normalized maximum entropy jump per particle, \(\Delta S_{\text{max}}\), namely is given by the following thermodynamic expression

\[
\Delta S_{\text{max}} = \frac{Q_{1,2}}{k_b T_2} = \frac{\frac{1}{2}m(U_1^2 - U_2^2) \ln \left( \frac{\kappa_2}{\kappa_1} \right)}{k_b T_2 \ln \left( \frac{\Gamma_2}{\Gamma_1} \right)} = \frac{mU_1^2(1 - \frac{1}{\Gamma})}{2sk_b T_2},
\]

where \(T_2\) denotes the effective downstream temperature of the plasma mixture that absorbs the converted kinetic energy. Taking Voyager-2 shock crossing data, i.e. \(T_2 = 2 \cdot 10^7\text{K}; s = 2.5; U_1 = 4 \cdot 10^7\text{cm/s}\), one then would find

\[
\Delta S_{\text{max}} = 16.
\]

Comparing this with the values \(\Delta S^0\) and \(\Delta S_{\text{psu}}\) which we have displayed in our Figures 2, 3, 4 and 5, one can see that one could easily allow an increase of the effective downstream temperature by a factor of 80 to 100. Thus one could allow for instance to increase the downstream pressure by about this factor without violating any fundamental thermodynamic law. For, if, instance the downstream thermal ensemble is characterized by a temperature of the order of \(T_2 = 2 \cdot 10^7\text{K}\) (instead as for normal solar wind protons of \(2 \cdot 10^5\text{K}\) still everything would be in complete thermodynamic order without violating fundamental principles. Hence what we may learn from this view: The strongly heated downstream electrons, which in our approach (Fahr and Siewert 2013) do attain temperatures of the order of \(T_{e,2} \geq 10^7\text{K}\), are in excellent agreement with thermodynamically allowed values, and they also support the maximum entropy principle much better than a pure proton plasma.

If, instead of \(T_2 = 2 \cdot 10^5\text{K}, one would assume the effective thermal downstream plasma mixture, due to the strongly heated electrons, to be of the order of \(T_2 = 2 \cdot 10^7\text{K}\) (instead of just the electron temperature \(T_{e,2}\)), it then would bring the maximum entropy jump just down to the achieved level \(\Delta S_{\text{max}} = 0.16\) (see Figs. 2 and 3).

### 5. Conclusions

In this article we have investigated how much particle-specific entropy is produced at the passage of the multifluid solar plasma over the MHD termination shock. Hereby we have started from a consistent solution of the multifluid MHD Rankine-Hugoniot shock relations to first find the consistent value of the resulting compression ratio \(s\). With the additional help of kinetic informations on the behaviour of the particle velocity components at the shock passage and with the use of the Liouville theorem we then obtain expressions for the downstream distribution functions and the pressures of the different fluids like solar wind protons, pick-up protons and electrons. Using then standard thermodynamic expressions we can calculate fluid-specific entropy productions from the individual fluid pressures. As we can show then the calculated entropy productions per particle, both of solar wind protons and pick-up protons, amount to much lower values than allowed by thermodynamically maximal values of \(\Delta S_{\text{max}}/k_b = 16\). Only when including the electron fluid as the downstream fluid with by far the highest temperature, we can then calculate for the first time a reasonably high value for the joint entropy production of the shocked multifluid plasma. Not only do the shocked electrons represent the most important part of the whole entropy production, they also differ from all earlier representations are shown to be that plasma fluid with the highest thermal pressure. This fact allows many interesting new conclusions concerning the dynamics of the downstream heliosheath plasma flow.

Even though the main outcome of this article here can be seen in the fact that the decisive part of the entropy production at the plasma passage over the shock is represented by the strongly heated downstream electrons, we also want to emphasize the related earlier result from Fahr and Siewert (2013) who found that these latter, nearly massless particles do also represent the dominant contribution to the total downstream plasma pressure (and thus, the entropy, as seen from Eq. (30)). This eminent feature has some important consequences for the form, how the downstream plasma flow organizes itself, as we shall demonstrate below.

As is well known from the equation of motion of the multifluid plasma mixture, one can construct a typical streamline constant for the downstream plasma flow, called the Bernoulli constant \(C_B\), with the property \((U \cdot \nabla)C_B = 0\). In case of the multifluid plasma which we have considered in this article we find \(C_B\) as given by (see Landau and Lifshitz 1977)

\[
C_B = (1/2)\rho U^2 + \sum_i P_i
\]

where \(\rho = \sum \rho_i\) denotes the total mass density of the plasma and \(P_i\) denote the different downstream pressures of SW protons, PUI protons and electrons. Taking now into account that amongst the downstream pressures the electron pressure \(P_{e,2}\) is by far dominant, one can write for the stagnation streamline, i.e. approaching from downstream of the termination shock the stagnation point of the heliosheath flow at the heliopause, the following relation

\[
C_{B,s} = (1/2)\rho U^2 + P_e = P_{e,s}.
\]

where \(P_{e,s}\) is the electron pressure at the stagnation point. This relation can easily be rearranged to

\[
1 + \frac{2P_e}{\rho U^2} = \frac{2P_{e,s}}{\rho U^2}.
\]
Introducing now the effective sound velocity $c_s$ and the effective Mach number in the heliosheath flow by

\[ c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{P_e}{\rho} \right) = \frac{P_e}{\rho} = \frac{1}{\rho} \frac{\partial P_e}{\partial \rho} = \frac{\partial T_e}{\partial e} = \frac{1}{\rho} \frac{\partial (\rho U^2)}{\partial \rho} \quad (62) \]

and

\[ M_t = \frac{U}{c_s} \quad (63) \]

then brings the above relation into the following form

\[ 1 + \frac{2}{M_t^2} = \frac{2P_e}{\rho U^2} \quad (64) \]

Finally, reminding that $c_s = \sqrt{2kT_e/m_e} = 6 \times 10^7$ cm/s then shows that $M_t^2 \approx 10^{-2}$ and that hence in the above relation the first term on the left side, i.e. "1", can be neglected, then leads to the simple requirement

\[ P_e = P_{e,s} \quad (65) \]

and with a polytropic relation of gas and density like $C_p = P/\rho^\gamma$, then simply states that $\rho/\rho_1 = 1^{-7} \approx 1$, i.e. that the density along the stagnation streamline, to be generalized to other streamlines, is constant, and that the plasma consequently behaves incompressible, which then for instance allows a flow potential $\Phi$ to be used as Fahr and Fichtner (1999) have done to describe the heliosheath streamlines through $\rho U = -\text{grad}\Phi$.

Acknowledgements. M. Siewert is grateful to the Deutsche Forschungsgemeinschaft for financial support granted to him in the frame of the project Si-1550/2-2.

References

Baumjohann, W. and Treumann, R. A.: Basic space plasma physics, London: Imperial College Press, 1996.

Brej, J. J. and Santos, A.: Nonequilibrium entropy of a gas, Phys. Rev. A, 45, 8566–8572, 1992.

Cercignani, C.: The Boltzmann Equation and Its Applications, New York: Springer-Verlag, 1988.

Chalov, S. V. and Fahr, H.-J.: A two-fluid model of the solar wind termination shock modified by shock generated cosmic rays, A&A, 288, 973, 1994.

Chalov, S. V. and Fahr, H.-J.: Entropy generation at the multi-fluid solar wind termination shock, J. Geophys. Res., 98, 15211–15221, 1993.

Diver, D. A.: A Plasma Formulary for Physics, Technology and Astrophysics, Imperial College Press, 1996.

Erkaev, N. V., Vogl, D. F., and Biernat, H. K.: Solution for jump conditions at fast shocks in an anisotropic plasma, J. Plasma Physics, 64, 561–578, 2000.

Fahr, H.-J. and Fichtner, H.: Physical reasons and consequences of a three-dimensionally structured heliosphere, Science Rev., 58, 193–258, 1991.

Fahr, H.-J. and Siewert, M.: Kinetic study of the ion passage over the solar wind termination shock, A&A, 458, 13–20, 2006.

Fahr, H.-J. and Siewert, M.: Ion passage over the solar wind termination shock under conservation of particle invariants under view of Voyager-2 observations, A&A, 6, 31–38, 2010.

Fahr, H.-J. and Siewert, M.: The multi-fluid pressures downstream of the solar wind termination shock, A&A, 528, A41, 2013.

Gombosi, T. I.: Physics of the Space Environment, New York: Cambridge University Press, 1998.

Heerikhuisen, J., Pogorelov, N. V., Florinski, V., Zank, G. P., and le Roux, J. A.: Effects of a $e$-Distribution in the Heliosheath on the Global Heliosphere and ENA Flux at 1 AU, ApJ, 682, 679–689, 2008.

Hudson, P. D.: Discontinuities in an anisotropic plasma and their identification in the solar wind, Planet. Space Sci., 18, 1611–1622, 1970.

Kucharek, H. and Scholer, M.: Injection and acceleration of interstellar pickup ions at the heliospheric termination shock, J. Geophys. Res., 100, 1745–1754, 1995.

Kucharek, H., M¨obius, E., and Scholer, M.: Kinetic simulations and recent observations of ion injection and acceleration at the termination shock and in the heliosphere, in: Physics of the Inner Heliosphere, edited by Heerikhuisen, J., Florinski, V., Zank, G. P., and Pogorelov, N. V., Vol. 858 of American Institute of Physics Conference Series, pp. 196–201, 2006.

Landau, L. D. and Lifshitz, E. M.: Lehrbuch der theoretischen Physik, Berlin: Akademiker-Verlag, 1977.

Le Roux, J. A. and Fichtner, H.: The Influence of Pickup, Anomalous, and Galactic Cosmic-Ray Protons on the Structure of the Heliospheric Shock: A Self-consistent Approach, ApJ, 477, L115, 1997.

le Roux, J. A. and Fichtner, H.: Global merged interaction regions, the heliospheric termination shock, and time-dependent cosmic ray modulation, J. Geophys. Res., 104, 4709–4730, 1999.

Livadiotis, G. and McComas, D. J.: Evolution of the solar wind termination shock, J. Geophysical Research (Space Physics), 114, A11105, 2009.

Livadiotis, G. and McComas, D. J.: Non-equilibrium Thermodynamic Processes: Space Plasmas and the Inner Heliosphere, ApJ, 749, 11, 2012.

Matsukiyo, S. and Scholer, M.: Microstructure of the heliospheric termination shock: Full particle hydrodynamic simulations, Journal of Geophysical Research (Space Physics), 116, A08106, 2011.

Neubauer, F. M.: Jump relations for shocks in an anisotropic magnetized plasma, Zeitschrift fur Physik, 237, 205–223, 1970.

Richardson, J. D., Kaspar, J. C., Wang, C., Belcher, J. W., and Lazarus, A. J.: Cool heliosphere plasma and deceleration of the upstream solar wind at the termination shock, Nature, 454, 63–66, 2008.

Scholer, M.: Upstream waves, shocklets, short large-amplitude magnetic structures and the cyclic behavior of oblique quasi-parallel collisionless shocks, J. Geophys. Res., 98, 47–57, 1993.

Serrin, J.: Mathematical principles of classical fluid mechanics, vol. VIII of Handbuch der Physik, Berlin: Springer Verlag, 1959.

Siewert, M. and Fahr, H.-J.: Analytic distribution functions for an ion plasma crossing an MHD shock, A&A, 463, 799–805, 2007.

Siewert, M. and Fahr, H.-J.; A Boltzmann-kinetic description of an MHD shock with arbitrary field inclination, A&A, 485, 327–336, 2008.

Siewert, M. and Fahr, H.-J.: Modified jump conditions for anisotropic temperature plasmas at parallel shocks, A&A, 501, 407–410, 2009.

Treu, M. A.: Kinetic theoretical foundation of Lorentzian statistical mechanics, Phys. Script., 59, 19–26, 1999.

Treu, M. A.: Statistical Mechanics of Stable States Far from Equilibrium: Thermodynamics of Turbulent Plasmas, ApSS, 277, 81–95, 2001.

Treumann, R. A. and Scholer, M.: Stationary plasma states far from equilibrium, Phys. Plasmas, 11, 8566–8572, 2004.

Vasyliunas, V. M.: A survey of low-energy electrons in the evening sector of the magnetosphere with OGO 1 and OGO 3, J. Geophys. Res., 73, 2839–2884, 1968.

Wenzel, W.: Das Boltzmannsche H-Theorem, pp. 1516–1525, Springer Verlag, 1958.

Wu, P., Winske, D., Gary, S. P., Schwadron, N. A., and Lee, M. A.: Energy dissipation and ion heating at the heliospheric termination shock, Journal of Geophysical Research (Space Physics), 114, A08103, 2009.

Wu, P., Liu, K., Winske, D., Gary, S. P., Schwadron, N. A., and Funsten, H. O.: Hybrid simulations of the termination shock: Suprathermal ion velocity distributions in the heliosheath, Journal of Geophysical Research (Space Physics), 115, A11105, 2010.

Zank, G. P., Webb, G. M., and Donohue, D. J.: Particle injection and the structure of energetic-particle modified shocks, ApJ, 406, 97, 1993.

Zank, G. P., Heerikhuisen, J., Pogorelov, N. V., Burrows, R., and McComas, D.: Microstructure of the Heliospheric Termination Shock: Implications for Energetic Neutral Atom Observations, ApJ, 708, 1092–1106, 2010.