Development of invariant plasticity theory approach to construction of non-isothermal plastic flow models

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Abstract. Constitutive equations of invariant plasticity theory model with two structural parameters are presented. The method of determination of model applicability for non-isothermal cyclic loading simulation of constructional materials is described. Results of model testing on non-isothermal and non-proportional loading programs for nickel base alloy IN738LC and steel 45 specimens are presented in compare with experimental results from literature.

1. Introduction

Critical parts of modern gas turbine engines frequently work in conditions of elevated and variable temperatures under cyclic loading. Improvement of such constructions develops by the way of weight reduction with simultaneous growth of work temperature, whereas life-time requirements constantly grow. Therefore the problem of development and refinement of non-elasticity models which capable to describe structure materials behavior in real operating conditions does not lose it currency. The most suitable for practical calculations elastoplastic models class is the class of plastic flow models with combine hardening, for example [1, 2]. Invariant plasticity theory [3] can be used as a theoretical basis for construction and development of plastic flow models.

2. Non-isothermal invariant plasticity theory constitutive equations

Let us consider homogeneous and initial isotropic materials with elastic volumetric change, which properties have isotropic changes under thermal fields exposure. Elastic \(d\varepsilon_{ij}^e\), plastic \(d\varepsilon_{ij}^p\) and creep \(d\varepsilon_{ij}^c\) deformations increments are considered as independent from each other, and the increment of full deformations \(d\varepsilon_{ij}\) is equal to their sum

\[ d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p + d\varepsilon_{ij}^c, \] (1)

Elastic deformation increment is defined by the generalized Hook’s law.

Let us consider existence of plastic deformation hypersurface \(g = 0\) and loading hypersurface \(f = 0\) for plastic deformation definition,

\[ g = g(\sigma, T, \varepsilon^p, \varepsilon^e, \varepsilon^c, \chi_1, \ldots, \chi_n), \quad f = g(\sigma, T; \varepsilon^p, \varepsilon^e, \varepsilon^c, \chi_1, \ldots, \chi_n), \] (2)

\(\gamma\)
\[
\begin{align*}
\frac{d\chi}{\beta} &= H^{(p)}_\chi \frac{d\varepsilon^p}{\beta} + G_\beta \left( \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} \right)^{1/2} + M_\beta \left( \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} \right)^{1/3}, \quad \beta = 1, r, \\
\frac{d\xi^{\beta}}{\xi^{\beta}} &= A_\rho \left( \chi_1, ..., \chi_r, \sigma^{\beta}, \varepsilon^{\beta} \right) \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} d\chi, 
\end{align*}
\]

where \( \sigma^{\beta} \) – stress tensor components; \( T \) – temperature; \( \chi_\beta \) – scalar structure parameters, presenting of plastic deformation history; \( \xi^{\beta} \) – tensorial structure parameters, presented the resulted anisotropy of plastic deformation; \( H^{(p)}_\chi = H^{(p)}_\chi \left( \sigma^{\beta}, \varepsilon^{\beta} \right) \); \( G_\beta \) and \( M_\beta \) – scalar functions of invariants stress and plastic strain tensors; \( \nu \) – tensor \( \xi^{\nu} \) rank; \( \frac{d\varepsilon^p}{\beta} = \left( \frac{d\varepsilon^p}{\beta} \frac{d\varepsilon^p}{\beta} \right)^{1/2} \) – the plastic deformation intensity increment; \( A_\rho \left( \chi_1, ..., \chi_r, \sigma^{\beta}, \varepsilon^{\beta} \right) \) – material function.

With the account of invariance of plastic deformation hypersurface relative to the coordinate system, \( g \) is the function of scalars \( T \) and \( \chi_\beta \) and complete invariant system of tensors, used in (2). Let us consider invariant system

\[
\begin{align*}
J_1 &= \sigma^{\beta} \delta^{\beta}_1, \quad E_1 = e^{\beta}_1 \delta^{\beta}_1 = 0, \quad D_1 = s^{\beta}_1 e^{\beta}_1, \quad H^{(t)}_{\xi^{t}} = \xi^{t}_1 \delta^{t}_1 \xi^{t}_1, \\
J_2 &= s^{\beta}_1 s^{\beta}_2, \quad E_2 = e^{\beta}_2 e^{\beta}_2, \quad D_2 = s^{\beta}_2 s^{\beta}_2 \varepsilon^{\beta}_2, \quad R^{\gamma}_v = \xi^{\gamma}_1 e^{\gamma}_1, ..., e^{\gamma}_n, \\
J_3 &= s^{\beta}_1 s^{\beta}_2 s^{\beta}_3, \quad E_3 = e^{\beta}_2 s^{\beta}_2 e^{\beta}_3, \quad D_3 = s^{\beta}_3 e^{\beta}_2 s^{\beta}_3 e^{\beta}_3, \quad \Pi^{\gamma}_v = \xi^{\gamma}_1 s^{\gamma}_1 s^{\gamma}_m, 
\end{align*}
\]

where \( s^{\beta}_i = \sigma^{\beta}_i - \sigma_{ii} / 3 \) – stress deviator tensor components, which is not complete, assuming that it sufficiently presents properties of a material. Then

\[
g(J_1, J_2, J_3, E_1, E_2, D_1, D_2, D_3, T, \chi_1, ..., \chi_r, H^{11}_{ii}, ..., H^{11}_{ii}, R^{1}_1, ..., R^{1}_n, ..., \Pi^{1}_1, ..., \Pi^{1}_n) = 0. \tag{5}
\]

Increment of plastic deformation will exist if the condition \( f = 0 \wedge df > 0 \) is true. The point representing the loading process always locates on plastic deformation hypersurface. We received following equations for plastic deformation definition with using (2)-(5) and plastic flow law, associated with loading hypersurface [3]:

\[
\begin{align*}
\frac{d\varepsilon^{\beta}}{\beta} &= C^{\beta}_{ijkl} d\sigma_{kl} + \Psi^{\beta} dT, \\
C^{\beta}_{ijkl} &= -\frac{1}{\varphi} \sigma^{\beta}_i \sigma^{\beta}_l, \quad \Psi^{\beta} = -\frac{1}{\varphi} \sigma^{\beta}_i \sigma^{\beta}_l, \\
\sigma^{\beta}_i &= \frac{\partial f}{\partial \delta^{\beta}_i},
\end{align*}
\]

where \( \varphi \) – hardening function.

3. **Model with one scalar and one tensorial structure parameters**

Let us consider the variant of invariant plasticity model. The following hypothesis will be used: the influence of elastoplastic deformation history can be described by two independent from temperature \( T \) structure parameters, scalar \( \chi \) and tensorial \( \xi^{\beta} \),

\[
\begin{align*}
\frac{d\chi}{\chi} &= \left( 2/3 \frac{d\varepsilon^{\beta}}{\beta} \right)^{1/2}, \quad \frac{d\xi^{\beta}}{\xi^{\beta}} = K(J_2, \chi) \frac{d\varepsilon^{\beta}}{\beta},
\end{align*}
\]

where \( \varepsilon^{\beta}_i \) – plastic deformation tensor components; \( K(J_2, \chi) \) – material function.

According to invariant plasticity theory the plastic deformation and loading hypersurfaces are

\[
g = g(\sigma^{\beta}_i, T, \varepsilon^{\beta}_i, \xi^{\beta}_i, \chi), \quad f = g(\sigma^{\beta}_i, T; \varepsilon^{\beta}_i, \xi^{\beta}_i, \chi). \tag{8}
\]

For this case we can consider complete invariant system of tensors, used in (8):
\[ J_1 = \sigma_j \delta_j, \quad E_1 = e_{ij}^p \delta_{ij} = 0, \quad H_1 = \xi_j \delta_j = 0, \quad D_1 = s_j e_{ij}^p, \quad P_1 = s_j e_{ij}^p, \]
\[ J_2 = s_i \delta_{ij}, \quad E_2 = e_{ij}^p \delta_{ij} = 0, \quad H_2 = \xi_j \xi_j = 0, \quad D_2 = s_i s_k e_{ik}^p, \quad P_2 = e_{ij}^p \xi_{ij}, \]
\[ J_3 = s_i s_k s_{ik}, \quad E_3 = e_{ij}^p e_{ij}^p, \quad H_3 = \xi_j \xi_j \xi_j = 0, \quad D_3 = s_i s_k e_{ik}^p, \quad P_3 = s_i s_k \xi_{ij}. \]

With using invariant plasticity approach we can receive equations for plastic deformation definition, which are similar to equations (6). So, for model with one scalar and one tensorial parameters the complete invariant system (9) choice does not change the system of constitutive equations.

Further for approach testing let us consider the model with plastic deformation hypersurface
\[ g = J_2 - 2a_1 D_1 - 2a_2 P_2 + 2a_2 a_2 P_2 + a_i^2 E_2 + a_i^2 H_2 - \sigma_i^2 = 0, \]
where \( a_1(\chi, T), \ a_2(\chi, T), \ \sigma_i(\chi, T) \) - material functions.

According to hypothesis, which have been accepted before constructing structure of invariant plasticity theory constitutive equations, base experiment for definition of material functions should be carry out with sufficiently great deformation rate to exclude effects of creep.

Considered constitutive equations framework do not take into account multiaxiality degree in special way, so as base experiment it is possible to choose cylindrical specimen testing on cyclic symmetric strain-controlled isothermal loading for some set of fixed temperatures.

For model (10) in uniaxial isothermal case taking into account that tensors \( s_i, \ e_{ij}^p \) and \( \xi_{ij} \) are deviators let us write down condition of representing deformation process point finding on a plastic deformation hypersurface:
\[ dg = \frac{3}{2} (s_{i1} - a_i e_{i1}^p - a_i e_{i2}^p) \left[ ds_{i1} - (a_i + Ka_i) de_{i1}^p \right] + g \psi d\chi = 0. \]

Inside one loading cycle it is possible to accept \( g \psi = 0 \) perceiving the second component in (11) characterizes changing of material properties from one cycle to another. Then inside one loading cycle of base experiment there is a valid proposition
\[ ds_{i1} / de_{i1}^p = a_i(\chi, T') + K(J_2, \chi) a_i(\chi, T') \approx E_{\text{un}}(J_2, \chi, T'), \]
where \( T' \) - fixed temperature of process. Equation (12) allows to verify hypothesis about temperature independency of material function \( K(J_2, \chi) \). If hypothesis is true for considering material, then variables \( E_{\text{un}} = E_{\text{un}}(J_2, \chi, T_1') \) and \( E_{\text{un}} = E_{\text{un}}(J_2, \chi, T_2') \), gained as a results of isothermal experiments for temperatures \( T_1' \) and \( T_2' \), will be in linear dependence at any fixed \( \chi \) for all effective range of \( J_2 \) changes. According to described method, usability of model (10) is shown for a number of nickel base alloys and structural steels.

4. Simulation results
A number of simulation tests for 45 steel to estimate a developed model capacity to describe non-proportional loading effects in comparison with experimental results from literature were carried out. There are three major factors of non-proportionality of a path: curvature change, twist angles and complex unloading. The set of tests included double-link (with twist angles), multi-link (with twist angles and complex unloading) and curvilinear (one, two or three factors of non-proportionality) deformation paths.

The dependences of approach angle \( \theta \) of strain and stress vectors on the second link length \( \Delta l \) for double-link deformation paths with different twist angles \( \theta_i \) are presented in figure 1, solid line – calculation, points – experiment [1], \( \mathcal{E} \) and \( \mathcal{E}_j \) are components of II’ushin deviatoric strain vector [4].
Figure 1. The results of model testing under double-link deformation paths.

The results of simulation of loading process under planar curvilinear deformation path, consisting of circular arcs of 1% radius and link unloading segment 2% length closing a path are shown in figure 2. The specific of this path consists in absence of curvature change and twist angles on curvilinear path. So there is only one factor of non-proportionality, complex unloading, on curvilinear part of this path. Deformation path and scalar properties $S(l)$ are presented in figure 2(a) and deformation curves in axial and torsional coordinates – in figure 2(b), solid line – calculation, points – experiment [5], $S_1$ and $S_3$ are components of Il’ushin deviatoric stress vector [4], $l$ is the length of loading path.

Figure 2. The results of model testing under curvilinear deformation path.
Calculation results are in conformity with results of experiment from the beginning of loading to point A. This part of the loading path corresponds to non-proportional loading in both directions. On the part $AB$ of the loading path, which corresponds to loading in axial direction and unloading in torsional direction (complex unloading), understating of scalar properties is observed. Finally, on the part $BC$ (unloading in torsional direction) rates of scalar properties change match to the experimental.

As an example of simulation of loading process on thermomechanical trajectory, the graph of stresses changes depending on cycle number is shown in figure 3. The graph is received as a result of simulation of cylindrical specimen testing under thermomechanical loading program, characteristic for points on blade leading edge under typical loading operating cycle of first-stage blade [6]. Solid line – calculation, points – experiment [6], material – nickel based alloy IN738LC.

![Graph of stresses changes](image)

**Figure 3.** The result of simulation of thermomechanical loading process for cylindrical specimen.

5. **Conclusion**

Results of model capacity research shown, that developed model is capable to describe loading processes on non-proportional path with twist angles and curvilinear change, but there are some difficulties with description of complex unloading. These difficulties can be associated both with absence of additional hardening account, which can be carried out in considering framework, and with inability of non-linear partial unloading description, which inherent in all plastic flow models. Thus, results of the model testing allow to draw a conclusion about application possibility of developed model for engineering calculations in simulation of stress-strain state of structure parts, working in conditions of non-isothermal and non-proportional loading.

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