Harmonic analysis of random number generators

OLIVER SCHNETZ

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Abstract

The spectral test of random number generators (R.R. Coveyou and R.D. McPherson, 1967) is generalized. The sequence of random numbers is analyzed explicitly, not just via their $n$-tupel distributions. We find that the mixed multiplicative generator with power of two modulus does not pass the extended test with an ideal result. Best qualities has a new generator with the recursion formula $X_{k+1} = aX_k + c \text{int}(k/2) \mod 2^d$. We discuss the choice of the parameters $a, c$ for very large moduli $2^d$ and present an implementation of the suggested generator with $d = 256$, $a = 2^{128} + 2^{64} + 2^{32} + 62181$, $c = (2^{160} + 1) \cdot 11463$.

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*Institut für theoretische Physik III, Staudtstraße 7, 91058 Erlangen, Germany, e-mail: schnetz@pest.physik.uni-erlangen.de

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1 Introduction

The spectral test was proposed by R.R. Coveyou and R.D. McPherson in 1967 [1]. The advantage of this test is to present an algebraic criterion for the quality of the generator. For the mixed multiplicative generator \( X_{k+1} = aX_k + c \mod M \)

\[
\min \{ |s| = \sqrt{s_1^2 + \ldots + s_n^2} \text{ with } s_a = s_1 + as_2 + \ldots + a^{n-1}s_n = 0 \mod M \} \quad (1)
\]

should be as large as possible [1, 2]. The criterion is that simple since the \( n \)-tupels of random numbers form an \( n \)-dimensional lattice (cf. e.g. Fig. 2). A good generator has uniformly distributed \( n \)-tupels which refers to an almost cubic lattice [3, 4, 5].

The lattice is a consequence of the (affine) linear dependence of \( X_{k+1} \) on \( X_k \). From the figures on the left (type I) we see that the relation between \( k \) and \( X_k \) is much more complicated. This is however one of the most fundamental aspects of randomness. In order to judge whether a sequence \( X_k \) takes random values one would first plot the sequence itself and then maybe \( X_{k+1} \) over \( X_k \).

Of course, the correlation between \( k \) and \( X_k \) is not independent from the distribution of pairs \((X_k, X_{k+1})\). E.g., a poor ‘random’ sequence \( X_k = ak \) lying on a line with gradient \( a \) leads to pairs \((X_k, X_{k+1} = X_k + a)\) lying on a line with gradient 1 shifted by \( a \) off the origin. This makes it reasonable to judge randomness by only looking at the \( n \)-tupel distributions. However, random number generators which have identical valuation by the spectral test may still look quite different. The generators \( X_{k+1} = 41X_k + 3 \mod 1024 \) (Fig. 3) and \( X_{k+1} = 41X_k + 1 \mod 1024 \) (Fig. 4), e.g., differ only by the additive constant which does not enter Eq. (1). The lattices of pair distributions (type II in the figures) are similar whereas the plots of \( X_k \) over \( k \) show different behavior. The spectral test not even makes a difference between a prime number and a power of two modulus (cf. Fig. 1 vs. Fig. 2).

Therefore it is desirable to include the analysis of the correlation between \( k \) and \( X_k \) into the valuation of the test. In fact it is possible to analyze the accumulation of random numbers along certain lines (which is often seen in the figures) by Fourier transformation. More generally we extend the spectral test by analyzing the correlation between \( k \) and the \( n \)-tupel \((X_k, X_{k+1}, \ldots, X_{k+n-1})\). The generators mentioned above (Figs. 3, 4) acquire different valuations. Fig. 4 is preferred since the random numbers spread more uniformly in Fig. 4I than in Fig. 3I (cf. Sec. 3.2 1).

We will find that the commonly used mixed multiplicative generator always shows correlations along certain lines if the modulus is not a prime number. We will present an improved generator which is almost free from these correlations (Fig. 7). It has the recursion formula

\[
X_{k+1} = aX_k + \text{int}(k/2) \mod 2^d \quad (2)
\]

with the parameters

\[
d = 2^k d_0, \quad a = 2^{k-1}d_0 + 2^{k-2}d_0 + \ldots + 2d_0 + \left( 3 580 621 541 \equiv 2^d \mod 2^d \right),
\]

\[
c = \left( 2^{\text{int}(2^{k+1}/3)}d_0 + 1 \right) \left( 3 370 134 727 \equiv 2^d \mod 2^d \right), \quad (3)
\]
In particular, the case \( d_0 = 16, k = 4 \) is discussed in Ex. 7.1.

This generator is supposed to be a good choice with respect to the following three criteria.

Firstly, the sequence of numbers provided by the generator should behave as close to a true random sequence as possible.

Secondly, the calculation of random numbers should be as fast as possible. The generator given in Eqs. (2), (3) is explicitly constructed to have best performance. It is important to note that this is not independent from the first criterion. It is possible to produce better random numbers the more effort one spends in calculating the numbers. Figs. 2 and 6 show how a simple doubling of the digits of the modulus improves the randomness of the generator. In general we can produce arbitrarily good random numbers with e.g. \( d_0 = 32 \) and large \( k \) in Eq. (3).

Thirdly, the properties of the random numbers should be known as detailed as possible. It is not sufficient to use a messy, opaque formula. It has often been seen that this leads to numbers which are far from being random [2]. As long as one is not familiar with the qualities of the generator one can never rely on the results gained with it. The full evaluation of the generalized spectral test is supposed to provide a profound knowledge of the generator.

We start with the development of the generalized spectral test in the next section. In Sec. 3 we apply the test to a series of commonly used and some new generators. Finally we discuss the choice of parameters in Sec. 4.

2 The generalized spectral test

2.1 Review of the spectral test

We start with a short review of the spectral test [1, 2] in which we try to stress its geometrical meaning. The idea is to plot all \( n \)-tupels of successive random numbers in an \( n \)-dimensional diagram. This is done, e.g. for \( n = 2 \) in the figures of type II.

Mathematically a figure is presented as a function \( g \) which is 1 at every dot and 0 elsewhere. If \( N_X \) is the period of the generator \( X \), that is the smallest number with \( X_{k+N_X} = X_k \ \forall k \), then

\[
g(x_1, \ldots, x_n) = \sum_{k=1}^{N_X} \delta_{x_1, X_k} \cdots \delta_{x_n, X_{k+n-1}} \equiv \sum_{k \in \mathbb{Z}_{N_X}} \delta_{x, X_k}, \quad (4)
\]

where \( \delta_{a,b} = 1 \) if \( a = b \) and \( \delta_{a,b} = 0 \) if \( a \neq b \) (for later convenience we also write the Kronecker \( \delta \) as \( \delta_{a=b} \)). Moreover we have introduced the notation

\[
X_k = (X_k, X_{k+1}, \ldots, X_{k+n-1}), \quad x = (x_1, x_2, \ldots, x_n), \quad \mathbb{Z}_{N_X} = \mathbb{Z}/N_X\mathbb{Z}. \quad (5)
\]

We want to check whether the dots accumulate along certain hyper-planes (see e.g. the lines in Fig. 3II). To this end we select a hyper-plane and project all the dots onto a
line perpendicular to it. If points accumulate along the plane many dots will lie on top of each other, otherwise the dots are spread uniformly over the line.

The hyper-plane $H$ is determined by its Hesse normal form

$$H = \{ \mathbf{x} : s_1 x_1 + s_2 x_2 + \ldots + s_n x_n \equiv \mathbf{s} \cdot \mathbf{x} = 0 \} , \quad |\mathbf{s}| \equiv \sqrt{\mathbf{s} \cdot \mathbf{s}} \equiv \sqrt{s_1^2 + \ldots + s_n^2} \neq 0 . \quad (6)$$

The line is stretched by the factor $|\mathbf{s}|$. The position of a point $\mathbf{X}_k$ on the line perpendicular to the plane is given by the number $\mathbf{s} \cdot \mathbf{X}_k$.

Next we wind the line up to a circle so that the modulus $M$ as point on the line lies on top of the 0. For a suitable choice of $\mathbf{s}$, namely $\mathbf{s} = (-1, 25)$ all points in Fig. 3II lie now on the point represented by the number 25.

The points are realized as complex phases on the unit circle. We obtain the assignment

$$\mathbf{X}_k \mapsto \exp \left( \frac{2\pi i}{M} \mathbf{s} \cdot \mathbf{X}_k \right) . \quad (7)$$

Finally we draw arrows from the center of the circle to all the dots and add them. The length of the resulting vector describes how the dots are balanced on the circle. If the dots spread uniformly the arrows cancel each other and the resulting vector is small. If, on the other hand, all dots lie on top of each other the length of the arrows sums up to $N_X$.

If we restrict ourselves to integer $s_1, \ldots, s_n$ (accumulation of random numbers always occur along hyper-planes given by integer $s_i$) the resulting vector is given by the Fourier transform of $g$,

$$\hat{g}(\mathbf{s}) = \frac{1}{\sqrt{N_X}} \sum_{\mathbf{x} \in \mathbb{Z}^n_X} g(\mathbf{x}) \exp \left( \frac{2\pi i}{M} \mathbf{s} \cdot \mathbf{x} \right) = \frac{1}{\sqrt{N_X}} \sum_{k \in \mathbb{Z}_{N_X}} \exp \left( \frac{2\pi i}{M} \mathbf{s} \cdot \mathbf{X}_k \right) , \quad (8)$$

where we have introduced the normalization factor $N_X^{-1/2}$. The information about accumulations along the hyper-plane is contained in $|\hat{g}|^2$, the phase of $\hat{g}$ is irrelevant.

We remember that for the mixed multiplicative generator the $n$-tupels form a lattice (which is displaced off the origin). So $|\hat{g}|^2(\mathbf{s})$ will assume the maximum value $N_X$ if $\mathbf{s}$ lies in the dual lattice, $\mathbf{s} \cdot \mathbf{X}_k = C + \ell M$, $C, \ell \in \mathbb{Z}$, otherwise $|\hat{g}|^2(\mathbf{s})$ is zero. Since $X_k = c(a^k - 1)/(a - 1) \mod M$ this means, if $\gcd(c, M) = 1$ and $X$ has full period, that $\forall k : (a^k - 1)s_\alpha = 0 \mod M \gcd(a - 1, M)$ from which Eq. (1) follows (cf. Sec. 3.2).

### 2.2 Generalization of the spectral test

We generalize the spectral test by caring for the sequence in which the $n$-tupels are generated. The index $k$ is added to the $n$-tupel $\mathbf{X}_k$ as zeroth component and we define $g$ as

$$g(x_0, \mathbf{x}) = \sum_{k \in \mathbb{Z}_{N_X}} \delta_{x_0, k} \delta_{\mathbf{x}, \mathbf{X}_k} = \delta_{\mathbf{x}, \mathbf{x}_{x_0}} . \quad (9)$$
The geometrical interpretation remains untouched but now we consider also the figures of type I. The Fourier transform of $g$ is given by

$$
\hat{g}(s_0, s) = \frac{1}{\sqrt{N_X}} \sum_{k \in \mathbb{Z}_{N_X}} \exp \left( \frac{2\pi i}{N_X} s_0 k + \frac{2\pi i}{M} s \cdot X_k \right).
$$

(10)

The sum over $k$ is hard to evaluate since in the exponential $k$ is combined with $X_k$. However in fact we are interested in $|\hat{g}|^2$ and find

$$
|\hat{g}|^2(s_0, s) = \frac{1}{N_X} \sum_{k, k' \in \mathbb{Z}_{N_X}} \exp \left( \frac{2\pi i}{N_X} (s_0 - s_0) (k' - k) + \frac{2\pi i}{M} s \cdot (X_{k'} - X_k) \right)
$$

$$
= \frac{1}{N_X} \sum_{\Delta k \in \mathbb{Z}_{N_X}} \exp \left( \frac{2\pi i}{N_X} s_0 \Delta k \right) \sum_{k \in \mathbb{Z}_{N_X}} \exp \left( \frac{2\pi i}{M} s \cdot (X_{k+\Delta k} - X_k) \right).
$$

(11)

The sum over $k$ has no linear $k$-dependence, only differences of random numbers occur. Like in the standard spectral test in many cases the sum over $k$ can be evaluated. The result is often simple enough to be able to evaluate the sum over $\Delta k$ also.

Note that the standard spectral test corresponds to $s_0 = 0$. We give some simple results on $|\hat{g}|^2$ in the following lemma.

**Lemma 2.1.**

$$
|\hat{g}|^2[X_{k+c_2} + c_3](s_0, s) = |\hat{g}|^2[X_k](s_0, s),
$$

(12)

$$
|\hat{g}|^2(s_0, 0) = N_X \delta_{s_0=0 \mod N_X},
$$

(13)

$$
\sum_{s_0 \in \mathbb{Z}_{N_X}} |\hat{g}|^2(s_0, s) = N_X
$$

(14)

$$
\sum_{s \in \mathbb{Z}_{M}} |\hat{g}|^2(s_0, s) = M^n \quad \text{if } X_k = X_k' \Rightarrow k = k' \mod N_X.
$$

(15)

One may also be interested in correlations between non-successive random numbers like $X_k$ and $X_{k+\tau}$. In general it is possible to study $n$-tupels $X_{k+\tau} \equiv (X_{k+\tau_1}, \ldots, X_{k+\tau_n})$. This amounts to replacing $X_k$ by $X_{k+\tau}$ and $s_\alpha$ by $s_{\alpha, \tau} \equiv s_1 \alpha_1 + \ldots + s_n \alpha_n$ in our results.

### 2.3 Valuation with the generalized spectral test

Now we have to clarify how the calculation of $|\hat{g}|^2$ leads to a valuation of the generator.

We can not expect that $|\hat{g}|^2$ vanishes identically outside the origin since in this case $g$ would be constant. Eq. (14) shows that the mean value of $|\hat{g}|^2$ is 1.

What would we expect for a sum of truly random phases? Real and imaginary part of a random arrow with length 1 have equal variance $1/2$. For large $N_X$ the sum of arrows is therefore normally distributed with density $1/(\pi N_X) \cdot \exp(-r^2/N_X)dr^2/N_X$. Thus $z = |\hat{g}|^2$ has the density $\exp(-z)$ for a true random sequence and the expected value for $|\hat{g}|^2$ is 1.

This means that values of $|\hat{g}|^2 \leq 1$ can be accepted. It is clear that for a given $(s_0, s) \neq (0, 0)$ the correlations are worse the higher $|\hat{g}|^2(s_0, s) > 1$ is. But what does the location of an $(s_0, s)$ with $|\hat{g}|^2(s_0, s) > 1$ mean for the generator?
We remember that \((s_0, s)\) may be seen as normal vector on the hyper-plane along which the accumulations occur. If e.g. \(n = 1\) and \((s_0, s_1) = (1, 1)\) the corresponding 1-plane has the equation \(x_0 + x_1 = 0\) (cf. e.g. Figs. 2I, 3I). If the \(k\)-axis and the \(X_k\)-axis are normalized to length 1 this line has length \(\sqrt{2}\). With the normal vector \((3,1)\) (cf. Fig. 4I) one obtains the equation \(3x_0 + x_1 = 0 \mod 1\) which intersects the unit cube three times and therefore has the length \(\sqrt{3^2 + 1} = \sqrt{10}\). Accumulations along this longer line are less important than along the short line. In the extreme case where the line fills the whole unit cube it is hard to achieve \(Q\) as measure for the importance of the accumulations detected. The area is given by \(|(s_0, s)| = (s_0^2 + s^2)^{1/2}\), the Euclidean length of the normal vector.

We can relate both mechanisms by defining the quality parameter

\[
Q_n (s_0, s) \equiv \frac{|(s_0, s)|}{|\tilde{g}(s_0, s)|}^2, \quad Q_n \equiv \max_{(s_0, s) \in \mathbb{Z}_{N_X} \times \mathbb{Z}_M \setminus \{(0, 0)\}} Q_n (s_0, s).
\] (16)

Good generators have \(Q_1 \approx 1\). It is hard to achieve \(Q_n \approx 1\) for \(n > 1\) (see however Sec. 3.1). More realistic is \(Q_n \approx M^{1/n-1}\) (cf. Sec. 3) which means that the distribution of \(n\)-tupels deteriorates for higher \(n\). In general small \(n\) are more important than large \(n\). Apart from the value of \(Q_n\) also the number of sites \((s_0, s)\) at which \(Q_n(s_0, s) = Q_n\) is relevant (cf. Sec. 3.2 1.).

Let us try to find an interpretation for \(Q_n\). Assume the generator produces only multiples of \(t|M\). Then \(|\tilde{g}|^2(0, s_1 = M/t, 0, \ldots, 0) = N_X\), thus \(Q_n(0, M/t, 0, \ldots, 0) = M/tN_X\), and \(N_XQ_n\) determines the number of non-trivial digits. In general \(N_XQ_n\) may be larger than \(M\) and therefore we say that \(\tilde{M}_n(0) \equiv \max(N_XQ_n, M)\) determines the number of digits we can rely on. Analogously \(\tilde{N}_n(0) \equiv \max(N_XQ_n, N_X)\) gives the quantity of random numbers for which the \(n\)-tupel distributions are reasonably random. Specifically \(\tilde{M}_n(s_0, s) = \max(N_XQ_n(s_0, s), M)\) determines the digits and \(\tilde{N}_n(s_0, s) = \max(N_XQ_n(s_0, s), N_X)\) the quantity of random numbers not affected by accumulations perpendicular to \((s_0, s)\) (cf. [2], p. 90).

Note however that these are only crude statements. If, e.g., the ‘period’ of the generator is enlarged by simply repeating it then \(N_XQ_n(s_0, s)\) remains unaffected only if \(s_0 = 0\). Moreover a high \(|\tilde{g}|^2(s_0, s)\) may be harmful even if \(|(s_0, s)|\) is large.

Note that \(Q_n\) is a relative quality parameter. Although \(Q_n\) usually does not increase with larger modulus (for \(n > 1\) is actually decreases) the quality of the generator gets better since \(N_X\) grows (cf. Fig. 2 vs. Fig. 6).

3 Generators

Here we restrict ourselves to the analysis of the most important generators. More examples are found in [2].
3.1 $X_0 = 1, \ X_{k+1} = aX_k \text{ mod } P$

Let $P$ be a prime number and $a$ a primitive element of $\mathbb{Z}_P^\times$, the multiplicative group of $\mathbb{Z}_P$ (Fig. 1).

We start the analysis of this multiplicative generator with Eq. (11). We find $s \cdot (X_{k+\Delta k} - X_k) = s_a a^k (a^{\Delta k} - 1) = s_a \hat{k}(a^{\Delta k} - 1) \text{ mod } P$ for some $0 \neq \hat{k} \in \mathbb{Z}_P$. If $k$ runs through the $P - 1$ values of $\mathbb{Z}_P^\times$, then $\hat{k}$ sweeps out the whole $\mathbb{Z}_P \setminus \{0\}$. Assume $s_a \neq 0$ (the case $s_a = 0$ is trivial), then the sum over $\hat{k}$ can be evaluated yielding $P\delta_{\Delta k=0} - 1$. Finally the sum over $\Delta k$ gives together with the normalization $|\hat{g}|^2 = P/(P - 1) - \delta_{s_0=0}$.

| $|\hat{g}|^2(s_0, s)$ | $s_0 = 0$ | $s_0 \neq 0$ |
|---------------------|----------|-------------|
| $s_a = 0$           | $P - 1$  | $1/(P - 1)$ |
| $s_a \neq 0$        | $0$      | $P/(P - 1)$ |

(17)

We find that $Q_1 = Q_1(1, 1) = \sqrt{2}(P - 1)/P$ is independent of $a$ and even greater than 1. For $n \geq 2$ only the case $(s_0, s_a) = (0, 0)$ contributes to $Q_n$ and the discussion is equal to the case with power of two modulus presented in Sec. 4. We find

$$N_X = (P - 1)P^{d-1}, \quad Q_1 = \sqrt{2}(P - 1)/P, \quad Q_{n \geq 2} \approx P^{1/n-1}.$$  

(18)

From a mathematical point of view odd prime number moduli give good random number generators. In particular $\hat{N}_1 = P - 1, \hat{M}_1 = P$ whereas for the mixed multiplicative generator with a power of two modulus $M$ we will find $\hat{N}_1 = \hat{M}_1 = \sqrt{2}M/4$. So we need $M > 4P/\sqrt{2}$ to obtain power of two generators which behave better than generators with prime number moduli. However, one has to take into account that computers calculate automatically modulo powers of two. Moreover, the power of two generator will be improved in Sec. 3.3 until we achieve $Q_1 = 1$. As a byproduct a better behavior of $Q_n$ for $n \geq 2$ is obtained, too.

Best performance allow prime numbers of the form $P = 2^k \pm 1$ [4]. In this case $a \cdot b = c_1 2^k + c_2$ leads to $a \cdot b = c_2 \equiv c_1 \text{ mod } P$. The extra effort, compared with a calculation mod $2^k$ is one addition and, which is more important, the calculation of $c_1$. In Ex. 7.1 we discuss the improved generator with $M = 2^{256}$ which can most easily be changed to $M = 2^{128}$. Alternatively one may construct a generator mod $2^{127} - 1$ which is a prime number. This generator will however be more time consuming and moreover it has worse quality $\hat{N}_1 \approx 2^{127}, \hat{N}_2 \approx 2^{63.5}, \hat{N}_3 \approx 2^{42.3}$, etc. vs. $\hat{N}_1 = 2^{129}, \hat{N}_2 \approx 2^{86.3}, \hat{N}_3 \approx 2^{65}$, etc. for the generator with power of two modulus (cf. Sec. 3.3, Sec. 4). So, power of two generators are more efficient than multiplicative generators with prime number modulus. The situation is slightly different if one considers multiply recursive generators with prime number modulus, analyzed in Sec. 3.3 and Ex. 7.2.

Multiplicative generators with prime number modulus and primitive $a$ produce every random number $\neq 0 \text{ mod } P$ exactly once in a period. We recommend to use a prime number modulus only if one needs this quality.
3.2 \( X_0 = 0, X_{k+1} = aX_k + c \mod M \)

Let \( M \) be any non-prime modulus, \( \gcd(c, M) = 1 \) and let \( a \neq 1 \) have the following properties

1. \( b \equiv \gcd(a - 1, M) \) contains every prime factor of \( M \), 2. \( 4|b \) if \( 4|M \). (19)

It was shown by Greenberger \[7\] for \( M = 2^d \) and by Hull and Dobell \[8\] for general \( M \) that this leads to the quality that every random number occurs exactly once in a period. This theorem can also be obtained by harmonic analysis in a little more general framework \[3\].

The generator is called mixed multiplicative generator with full period.

**Proposition 2.** Let \( M_1 \) be a divisor of \( M \), then

\[
X_{kM_1} = ckM_1 \cdot \begin{cases} 1 & , M_1 \text{ odd} \\ 1 + b/2 & , M_1 \text{ odd} \end{cases} \mod bM_1 .
\]

**Proof.** With \( a \equiv db + 1 \) we find

\[
X_{kM_1} = c\frac{a^{kM_1} - 1}{a - 1} = ckM_1 \left( 1 + \frac{kM_1 - 1}{2} \right) + cdb \sum_{j=3}^{M/k} \left( \frac{M_1k}{j} \right) (db)^{j-2} .
\]

Obviously \( \mathbb{N} \ni \left( \frac{M_1k}{j} \right) = \left( \frac{M_1k-1}{j-1} \right)^{M/k} \). Since \( j \) has at most \( j - 2 \) prime factors for \( j \geq 3 \) and \( b \) has by definition every prime factor of \( M_1 \) we obtain \( \mathbb{N} \ni \left( \frac{M_1k-1}{j-1} \right)^{M/k} = \left( \frac{M/k}{j} \right)(db)^{j-2}/M_1 \). Therefore the latter term drops \( \mod bM_0 \) and the former gives the result. \( \square \)

**Theorem 3.** Let \( s_{a,M/b} \equiv \gcd(s_a, M/b) \). The Fourier transform of the mixed multiplicative generator with full period is

\[
|\hat{g}|^2 (s_0, s) = bs_{a,M/b} \delta_{s_0 + cs_a = 1} (s) = bs_{a,M/b} \frac{2\pi i}{M} s_0 \Delta k_{\mod bs_{a,M/b}} ,
\]

where the Kronecker \( \delta \) gives 1 if the equation in the argument holds and 0 otherwise.

**Proof.** Since \( (c(a^k - 1)/(a - 1))_{k \in \mathbb{Z}_M} \) gives every number \( \mod M \) exactly once, in \( (a^k - 1)_{k \in \mathbb{Z}_M} \mod M \) every multiple of \( b \) occurs \( b \) times. With \( s \cdot (X_{\Delta k + k} - X_k) = s_0 a^k X_{\Delta k} \) we get from Eq. (11) after a rearrangement of the \( k \)-sum

\[
|\hat{g}|^2 (s_0, s) = \frac{1}{M} \sum_{\Delta k \in \mathbb{Z}_M} \exp \left( \frac{2\pi i}{M} s_0 \Delta k \right) \cdot b \sum_{k \in \mathbb{Z}_{M/b}} \exp \left( \frac{2\pi i}{M} (kb + 1) s_0 X_{\Delta k} \right) .
\]

The \( k \)-sum gives \( M/b \) if \( bs_aX_{\Delta k} = 0 \mod M \) and vanishes otherwise. Since \( \gcd(bs_a, M) = bs_{a,M/b} \) only that \( \Delta k \) contribute to \( |\hat{g}|^2 \) for which \( X_{\Delta k} = 0 \mod M/bs_{a,M/b} \). The generator has a full period, thus there are \( bs_{a,M/b} \) such \( \Delta k \). From Prop. 2 with \( M_1 = M/bs_{a,M/b} \) we find that these have the form \( \Delta k = kM_1 \). Moreover, since \( M/s_a bM_1 \),

\[
|\hat{g}|^2 (s_0, s) = \sum_{k \in \mathbb{Z}_{bs_a,M/b}} \exp \left( \frac{2\pi i}{M} s_0 kM_1 + s_0 c kM_1 \left( 1 + \frac{b}{2} \delta_{\Delta k \mod M_1} \right) \right)
\]

\[
= bs_{a,M/b} \delta_{s_0 + cs_a = 1} (1 + \frac{b}{2} \delta_{\Delta k \mod M_1}) \mod bs_{a,M/b} .
\]

We get the result since \( cs_a b/2 = -bs_{a,M/b}/2 \mod bs_{a,M/b} \) if \( 2|M_1 \). \( \square \)
Now the proper choice of the parameters $a$ and $c$ can be discussed.

1. Choice of $c$. We can restrict ourselves to $1 \leq c \leq b/2$ since every $c$ emerges from a $c \in (0, b/2)$ via translations ($X_k \mapsto X_{k+\Delta k} = a^{\Delta k}X_k$) or reflection ($X_k \mapsto -X_k$). We can determine $c$ by the condition that there should be no small $(s_0, s_a)$, $\gcd(M, s_a) = 1$ with $s_0 + s_ac = b/2 \cdot \delta_{2|M/b}\mod b$. If $M/b$ is odd $c \approx \sqrt{b}$ gives $Q_1(s_0 \approx \sqrt{b}, -1) \approx \sqrt{b+1}/b \approx b^{-1/2}$. In the case where $M/b$ is even and $b$ is small the choice $c=1$ is best with the result $Q_1(s_0 \approx s_1 \approx b/4) \approx \sqrt{2} \cdot (b/4)/b = \sqrt{2}/4 \approx 0.35$.

In the case of Fig. 3 with the 'wrong' choice $c=3$ one has $a = 9 \mod 16$, $b = 8$ and the smallest $(s_0, s_1)$ with non-vanishing $\hat{g}$ is $(1,1)$. Since $|\hat{g}|^2(1,1) = 8$ we obtain $Q_1(1,1) = \sqrt{2}/8 \approx 0.18$ (notice the correlations perpendicular to the (1,1)-direction in Fig. 3I). With the right choice $c=1$ (Fig. 4) it takes an $(s_0, s_1) = (1,3)$ (or $(3,1)$) to get $|\hat{g}|^2 = 8$. Therefore $Q_1(1,3) = \sqrt{10}/8 \approx 0.40$ which means that the random numbers are more uniformly distributed in Fig. 4I. The large value of $Q_1(1,3)$ is yet misleading since $Q_1 = Q_1(-M/8, M/8) = \sqrt{2}/8$. However $|\hat{g}|^2$ assumes the small value of $Q_1$ at much less sites as in the case of $c=3$ which means that the choice $c=1$ is better than $c=3$.

Notice the similar pair distributions in Fig. 3II and Fig. 4II. In general the quality dependence on $c$ can not be obtained by the standard spectral test (corresponding to $s_0=0$) since the $n$-tuple distributions are only shifted by a change of $c$.

2. Choice of $b$. In general $b$ should be as small as possible in order to prevent $|\hat{g}|^2$ from being concentrated on too few points. If $M/b$ is odd, $c \approx \sqrt{b}$ then $Q_1$ behaves like $b^{-1/2}$. If $M/b$ is even, $c = 1$ then $Q_1(s_0, s_1) = Q_1(s_0 \approx s_1 \approx b/4) \approx \sqrt{2}/4$ for small $(s_0, s_1)$ (cf. 1.). However $Q_1(-M/b, M/b) = \sqrt{2}/b$ which forbids large values of $b$.

For a power of two modulus the smallest value possible is $b = 4$ which implies $Q_1 = \sqrt{2}/4$ (Fig. 2). In particular $M = 10^d$ (Fig. 5) should be avoided since in this case $b \geq 20$.

These arguments require $s_0 \neq 0$. They are not obtained by the standard spectral test.

3. Choice of $a$. Up to now we have evaluated $|\hat{g}|^2(s_0, s_1)$ for $n = 1$ which is given by $b$ and $c$. In order to determine $a$ more precisely we have to look at the distribution of $n$-tupels for $n \geq 2$. In this case $Q_n = \min_{s}Q_n(s_0 = 0, s_a = 0) = \min_{s:s_a=0}|s|/M$. A further discussion of the choice of $a$ is postponed to Sec. 4. We will see that a reasonable $a$ gives $Q_n \approx M^{1/n-1}$.

Since $s_0 = 0$ this part of the choice of $a$ is identical with the standard spectral test.

Let us summarize the result for $M = 2^d$,

$$c = 1, \quad a = 5 \mod 8, \quad \max_{s_a} \min_{s:s_a=0}|s|, \quad \text{for } n = 2, 3, \ldots \text{ gives} \quad (22)$$

$$N_X = M, \quad Q_1 = \sqrt{2}/4, \quad Q_{n \geq 2} \approx M^{1/n-1}. \quad (23)$$

A loss of randomness for $n$-tupels is avoided if one takes $\gcd(n, M) = 1$. 

3.3 \( X_0 = 0, X_{k+1} = aX_k + c \operatorname{int}(k/2) \mod M \)

Let \( M = 2^d, 1 \neq a, a = 1 \mod 4 \) and \( c \) be odd (cf. Fig. 7).

We have seen that the quality parameter \( Q_1 \) is not greater than \( \sqrt{2}/4 \) for the mixed multiplicative generator with power of two modulus. This results in correlations along certain lines in the figures (cf. e.g. Fig. 2I). This does not mean that mixed multiplicative generators can not be used if one takes large enough moduli (cf. Fig. 6 and Sec. 4). Nevertheless it is worth to look for a generator with better performance. The generator presented in this section can be algebraically analyzed and it has \( Q_1 = 1 \). The implementation presented in Ex. 7.1 shows that it has good performance. It is possible to motivate the generator by geometrical arguments [6].

Theorem 4. Let \( s_{a,M} = \gcd(s_a, M) \).

\[
|\hat{g}|^2(s_0, s) = \delta_{s_0+s_1X_0^{\text{mm}}=0} \text{mod } s_{a,M} \cdot \begin{cases} s_{a,M} & \text{if } s_{a,M} \neq M \\ M \left(1 + \cos \left(\frac{\pi}{M} \left(s_0 + 2c \sum_{j=3}^{n} \frac{a^{j-1} - a^{2j} - a^{j-1} - s_j}{a^2 - 1} \right)\right)\right) & \text{else.} \end{cases}
\]

(24)

\( X_k^{\text{mm}} = c(a^k - 1)/(a - 1) \) is the mixed multiplicative generator related to \( X \). The period is \( N_X = 2M \).

It is useful to put the proof in a more general context. It is given in [3]. Here we only discuss the result. We obtain

\[
N_X = 2M, \quad Q_1 = 1, \quad Q_2 \approx M^{-1/3}, \quad Q_{n \geq 3} \approx M^{1/(n-1)-1}
\]

(25)
as will be shown in Sec. 4. It is advantageous to have two parameters \( a \) and \( c \) at hand to optimize the quality of higher \( n \)-tuples and not only \( a \) as in the case of the mixed multiplicative generator.

The generator does not provide full periods since \( |\hat{g}|^2(0, s_1) = s_{1,M} \neq 2M\delta_{s_1=0} \). However the deviation from an exact uniform distribution is not larger than in a finite true random sequence. If one does not use the entire period of the generator (and this is not recommended because of the \( n \)-tupel distribution) the feature of having a full period is anyway irrelevant. If, for some reasons, one insists in a full period we recommend to use a multiplicative generator with prime number modulus or the multiply recursive generator which will be analyzed next.

The generator of this section behaves in every aspect better than the widely used mixed multiplicative generator. This is also confirmed by the figures (cf. Fig. 2 and Fig. 7). The extra effort in calculating random numbers is little (cf. Ex. 7.1). If \( n \)-tuples are used one should take odd \( n \) and occasionally omit one random number.

Nevertheless the most essential step for producing good random numbers is to use large moduli (cf. Fig. 6 and Sec. 4).

3.4 \( X_0 = 1, X_{-1} = \ldots = X_{r+1} = 0, X_{k+1} = a_{r-1}X_k + \ldots + a_0X_{k-r+1} \mod P \)

Let \( P \) be a prime number and \( X_k \) have the maximum period of \( P^r - 1 \) (Fig. 8). In this case the random number generator has a full period in the sense that every \( r \)-tuple \((X_0, \ldots, X_{r-1}) \neq (0, \ldots, 0) \) occurs exactly once in a period.
The corresponding generator has maximum period if and only if $P$ is a primitive polynomial over $\mathbb{Z}_{P^r}$.

The proof is found in [4, Satz 2.1].

**Theorem 6.** Let $X_k$ be defined as above and $s_k \equiv (s \cdot X_k)_{0 \leq k < r} = (\sum_{j=1}^{n} s_j X_{k+j-1})_{0 \leq k < r}$ then $|\hat{g}|^2$ is given by the following table.

| $|\hat{g}|^2 (s_0, s)$ | $s_0 = 0$ | $s_0 \neq 0$ |
|-----------------------|----------|-------------|
| $s_a = 0$             | $P^r - 1$ | $0$         |
| $s_a \neq 0$          | $1/(P^r - 1)$ | $P^r/(P^r - 1)$ |

**Proof.** First we notice that $(s \cdot X_k)_k$ obeys the same recursion relation as $(X_k)_k$ since $s \cdot X_{k+1} = \sum_{j=1}^{n} s_j X_{k+j} = \sum_{\ell=1}^{r} a_{r-\ell} \sum_{j=1}^{n} s_j X_{k+j-\ell} = \sum_{\ell=1}^{r} a_{r-\ell} s \cdot X_{k+1-\ell}$. So, $(s \cdot X_k)_k$ is either identically zero or it has maximum period. In the latter case every number $n \in \mathbb{Z}_{P^r}$ is produced $P^r - 1$ times in a period and the zero is generated $P^r - 1 - 1$ times. Since the same holds for $(s \cdot (X_{k+\Delta k} - X_k))_k$ we get

$$\sum_{k \in \mathbb{Z}_{P^r - 1}} \exp \left( \frac{2\pi i}{P} s \cdot (X_{k+\Delta k} - X_k) \right) = P^r \delta_{s \cdot (X_{k+\Delta k} - X_k) = 0} \forall 0 \leq k < r - 1.$$ 

If $s_a = 0$ then $s \cdot X_k = 0 \forall k$, the Kronecker $\delta$ gives 1 and from Eq. (11) we obtain $|\hat{g}|^2(s_0, s_a = 0) = (P^r - 1)\delta_{s_a = 0}$. If on the other hand $s_a \neq 0$ then $(s \cdot X_k)_k$ has maximum period and the Kronecker $\delta$ vanishes unless $\Delta k = 0$. In this case Eq. (11) yields $P^r/(P^r - 1) - \delta_{s_a = 0}$.

The choice of parameters is determined by avoiding small $s$ with $s_a = 0$. For practical purposes it is more convenient to replace the condition $s_a = (s \cdot X_k)_{0 \leq k < r} = 0$ by the equivalent requirement $0 = (s \cdot X_{j-1})_{1 \leq k \leq r} \iff \sum_{j=1}^{n} s_j X_{j-k} = 0 \mod P$, $k = 1, 2, \ldots, \min(r, n)$. If $n \leq r$ the only solution is $s = 0 \mod P$. For $n > r$ one has the problem of finding the smallest lattice vector of an $n$-dimensional lattice. The unit cell of this lattice has the volume $P^r$ (cf. Sec. 4). Thus for proper parameters the quality of the generator is

$$N_X = P^r - 1, \quad Q_1 = Q_2 = \ldots = Q_r = \sqrt{2}/\left(1 - P^{-r}\right), \quad Q_{n>r} \approx P^{r/n}.$$ 

This is the first generator which has $Q_n \geq 1$ for $1 \leq n \leq r > 1$. For $2 \leq n \leq r$ the generator has higher $\bar{N}_n = P^r - 1$ but lower $\bar{M}_n = P$ than the multiplicative generator $\mod P^r$ (Sec. 5). With $\bar{N}_n \approx \bar{M}_n \approx P^{r/n}$.

In particular if one needs the full periods this generator may be recommended. For prime numbers of the form $P = 2^k \pm 1$ the generator has good performance, too. If the prime factors of $P^r - 1$ are known it is no problem to find multipliers which lead to a full period. A short discussion of the choice of parameters for large $P$ and $r$ is given in the next section and an implementation is presented in Ex. 7.2.
4 Choice of parameters

We start with a discussion of the mixed multiplicative generator (the multiplicative generator is analogous). For practical purposes we can restrict ourselves to $M = 2^d$ and $a = 5 \mod 8$. We set $c = 1$ which is equivalent to any other odd $c$ and assume $n \geq 2$ since the case $n = 1$ depends only on $b$ which is $4$ for $a = 5 \mod 8$.

From Eq. (21) we obtain, as long as $bs_{a,M/b} < M$, that $\hat{g}$ vanishes unless $s_{a,M/b}s_0 \neq 0$ and therefore (A) $Q_n(s_0,s) = \sqrt{1 + s^2/s_{a,M/b}^2/4} > 1/4$. However if $s_{a,M/b} = M/b$ we get $|\hat{g}|^2 = M$ for (B) $s_0 = s_a = 0 \mod M$. This leads to $Q_n = |s|/M$ which for some $s$ is much smaller than $1/4$.

So Eq. (B) is more important. We solve it for $s_1$ yielding $s_1 = kM - a s_2 - a^2 s_3 - \ldots - a^{n-1} s_n$ depending on the free integer constants $k$, $s_2$, $s_3$, $\ldots$, $s_n$ which give rise to an $n$-dimensional lattice (cf. [3]). The lattice is given by an $n$ by $n$ matrix $A$ according to $s = A \cdot (k, s_2, \ldots, s_n)^T$, and we read off

$$A = \begin{pmatrix} M & -a & -a^2 & \ldots & -a^{n-1} \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} M & -a \\ 1 & -a \\ & 1 & \ddots & -a \\ & & & 1 \end{pmatrix},$$

(29)

where zeros have been omitted and both matrices define the same lattice since they differ only by $SL(n, \mathbb{Z})$ lattice transformations.

We denote the length of the smallest non-vanishing lattice vector by $\nu_n$. Since the quality of the random numbers is determined by $\nu_n = MQ_n$ we search for an $a$ which large $\nu_n$. Most important are small $n$, in particular the pair correlation $n = 2$. In the best case the lattice has a cubic unit-cell and $\nu_n$ is determined by the dimension of the lattice and the volume of the unit-cell. Since the volume is given by the determinant of $A$ we get as an approximate upper bound $\nu_n \lesssim M^{1/n}$. The calculation of $\nu_n$ is a standard problem in mathematics for which efficient algorithms exist [4].

To simplify the search for reasonable multipliers it is useful to have also a lower bound for $\nu_n$. Due to the specific form of $A$ it is easy to see that $\nu_n$ has to be larger than the smallest ratio $> 1$ between two elements of then set $\{1, a, a^2 \mod M, \ldots, a^{n-1} \mod M, M\}$. If we take e.g. $a \approx M^{1/2}$ we find $\nu_2 \gtrsim M^{1/2}$ which is identical with the upper bound.

Similarly we obtain $\nu_3 \approx M^{1/3}$ if we take $a \approx M^{1/3}$ or $a \approx M^{2/3}$. However this is not compatible with $a \approx M^{1/2}$ and we only get $\nu_2 \gtrsim M^{1/3}$. On the other hand we can take $a \approx M^{1/2} + 1/2 M^{1/4}$ which differs little from $M^{1/2}$. Therefore $\nu_2 \approx M^{1/2}$ and since $a^2 \approx M + M^{3/4} + 1/2 M^{1/4} \approx M^{3/4} \mod M$ we have $\nu_3 \gtrsim M^{1/4}$. Generally, with $a \approx M^{1/2} + \frac{1}{k} M^{1/4} + \ldots + \frac{1}{k-1} M^{1/2k-1}$ (the plus signs may as well be replaced by minus signs) we get $\nu_n \gtrsim M^{1/2n-1}$ as long as $k \geq n$ and $M^{1/2n-1} \gg 1$. Note that $M^{1/2n-1}$ is only a lower bound for $\nu_n$. In the generic case $\nu_n$ will be close to $M^{1/n}$ (cf. Ex. 7.1).

Obviously $\nu_n$ increases with $M$. For all practical purposes the magnitude of $M$ is only limited by the performance of the generator. In practice one has to split $M$ into groups of digits (16 or 32 bit) that can be treated on a computer. The multiplication by
4 CHOICE OF PARAMETERS

a performs best if the pre-factors $1/j$ are omitted. This should be done even though for $a \approx M^{1/2} + M^{1/4} + \ldots + M^{1/2^{k-1}}$ the lower bounds for $\nu_n$ decrease, $\nu_n \gtrapprox M^{1/2^{n-1}}/(n-1)!$. Note that the number of digits of $M$ is much more important for randomness than the fine-tuning of $a$.

Finally, we have to add not too small a constant $a_0 = 5 \mod 8$ (16 or 32 bit) to the sum of powers of $M$. This constant can be fixed by explicit calculation of the $\nu_n$ or by looking at (A) from the beginning of this section which implies that $a_0$ should have large $|s|$ for all $16 < m = s_{a,M} |M^{1/2k-1}$. A suitable choice is e.g. $a_0 = 3 \, 580 \, 621 \, 541 = 62 \, 181 \mod 2^{16}$. With this value of $a_0$ we find $|s| \approx m^{1/n}$ for $n = 2, 3$.

We summarize the result for the parameters of the mixed multiplicative generator:

\[
M = 2^{2kd_0}, \quad c = 1, \quad a = 2^{2k-1}d_0 + 2^{2k-2}d_0 + \ldots + 2d_0 + a_0, \quad \text{with}
\]

\[a_0 = 5 \mod 8, \quad a_0 \approx 2^{d_0}, \quad \text{e.g.} \quad a_0 = 3 \, 580 \, 621 \, 541 \mod 2^{d_0}, \quad \text{leads to}
\]

\[N_X = 2^{256}, \quad Q_1 = \sqrt{2}/4, \quad Q_n \approx M^{1/n-1}.
\]

Now we turn to the improved generator of Sec. 3.3. The Fourier transform of the generator is given by Eq. (24). We set $n \geq 2$ since independently of the parameters $Q_1 = 1$. Further on, we fix an $m \mid M$ and find that $|\hat{g}|^2 = m$ if and only if (C) $s_a = km, k$ odd if $m < M$, and (D) $s_0 + \sum_{j=2}^{n} (a^{j-1} - 1)s_j = \ell m$. (We neglect here that $|\hat{g}|^2$ may even be $2M$ for $m = M$.) Eq. (C) can be solved for $s_1$ and Eq. (D) for $s_0$ depending on the integer parameters $k, \ell, s_2, \ldots, s_n$. This gives rise to an $(n+1)$-dimensional lattice (for $m < M$ we actually obtain an affine sub-lattice since $k$ has to be odd) determined by the matrix $B$ via $(s_0, s) = B \cdot (\ell, k, s_2, \ldots, s_n)^T$,

\[
B = \begin{pmatrix}
m & -c & \ldots & -c (a^{n-2} + \ldots + 1) \\
n & -a & \ldots & -a^{n-1} \\
1 & \ddots & \ddots & \\
1 & & \ddots & \\
1 & & & \\
1 & & & & \\
1 & & & & & & \\
\end{pmatrix} \approx \begin{pmatrix}
m & -c & -c & \ldots & -c \\
m & -a & \ldots & & & & \\
1 & & & & & & \\
1 & & & & & & \\
\end{pmatrix}
\]

\[
B \approx \begin{pmatrix}
m & -c & & & & & \\
m & -a & a^2 + a & \ldots & a^{n-2} + \ldots + a & & \\
1 & -a - 1 & -a^2 - a - 1 & \ldots & -a^{n-2} - \ldots - 1 & & \\
1 & & & & & & \\
1 & & & & & & \\
\end{pmatrix},
\]

where again zeros have been omitted.

$B$ describes an $(n+1)$-dimensional lattice which has a unit-cell with volume $m^2$. However this does not imply that the smallest lattice vector $\nu_n$ has length of about $m^{2/(n+1)}$. We see from (32) that there exists an $(n-1)$-dimensional sub-lattice with $s_0 = 0$ and $s_2 = -s_1 - s_3 - \ldots - s_n$ (delete the first and the third row and column in (32)). The unit-cell of
the sub-lattice has volume $m$ and $v_n \approx m^{1/(n-1)}$, which, for $n \geq 4$, is smaller than $m^{3/(n+1)}$. The smallest lattice vector for $n \geq 4$ will have the form $(0, s_1, -s_1 - s_3 - \ldots - s_n, s_3, \ldots, s_n)$ with the length $(s_1^2 + s_3^2 + \ldots + s_n^2 + (s_1 + s_3 + \ldots + s_n)^2)^{1/2}$. Since this is of about the same magnitude as $(s_1^2 + s_3^2 + \ldots + s_n^2)^{1/2}$ we may simply omit $s_2$ and reduce the problem to the $(n-1)$-dimensional case given by $(s_1, s_3, \ldots, s_n)$. Geometrically this means that the lattice corresponding to $B$ for $n \geq 4$ never has an approximately cubic unit-cell. Note moreover that the sub-lattice is independent of $c$ which means that $c$ can not be fixed by looking at the $n$-tuple distributions for $n \geq 4$.

The smallest value of $Q_n = v_n/m$ is obtained for $m = M$ which is thus the most important case. For $m = M$ we are not restricted to odd $k$. The situation is similar to the $(n-1)$-dimensional case of the mixed multiplicative generator, Eq. (29), with $a^2$ replaced by $a^2 + a^2 - 1 + \ldots + a$. This allows us to use $a = M^{1/2} + M^{1/4} + \ldots + M^{1/2k-1} + a_0$ again. Since $1 \ll a \approx M^{1/2} \ll a^2 \bmod M \approx 2M^{1/4} \ll \ldots \ll a^{n-2} \bmod M \approx (n-2)!M^{1-2-n} \ll M$ we have $a^2 + a^2 - 1 + \ldots + a \approx a^2 \bmod M$. The minus sign is irrelevant, thus we can copy the corresponding lower bounds from the mixed multiplicative generator: $v_n \gtrapprox M^{1/2-2}/(n-2)!$ for $k+1 \geq n \geq 4$. The constant $a_0$ is given by the case $m < M$ as will be discussed below.

The constant $c$ can be fixed by the case $n = 2$. We have to meet two equations (E) $s_1 + as_2 = 0 \bmod M$ and (F) $s_0 + cs_2 = 0 \bmod M$ to get $|\hat{g}|^2 = M$. Both equations are solved by e.g. $s_0 = -c, s_1 = -a \approx -M^{1/2}, s_2 = 1$ with $|(s_0, s)| \approx (c^2 + M)^{1/2}$. In order to reach the theoretical limit $v_n \approx M^{1/2}$ one needs $c \gtrapprox M^{2/3}$. So, the simplest ansatz for $c$ is $c = M^\lambda + 1$ for $\lambda \geq 2/3, M^\lambda \in \mathbb{N}$. On the other hand, if $s_1 = M^{1-\lambda} s'_1, s_2 = M^{1-\lambda} s'_2$ then $s'_1 + as'_2 = 0 \bmod M^\lambda$ has a solution with $|s'| \lesssim M^{\lambda/2}$. Since (F) is solved by $s_0 = -M^{1-\lambda} s'_2$ we find $|(s_0, s)| \approx |s_0| \lesssim M^{1-\lambda} M^{\lambda/2} = M^{1-\lambda/2}$. To allow for the maximum value $M^{2/3}$ one needs $\lambda \leq 2/3$. In general, $c$ should not have more successive zero digits than $M^{2/3}$ has.

The simplest reasonable choice is therefore $c = M^{2/3} + 1$. We can generalize this slightly to $c = (2^{d_1} + 1)c_0$, where $c_0$ is a 16 or 32 bit number and $2^{d_1} \leq M^{2/3} \leq c_0 2^{d_1}$. This choice of $c$ leads to best performance among all reasonable $c$. We will see in Ex. 7.1 that it actually gives $\nu_2 \approx M^{2/3}$ and $\nu_3 \approx M^{1/2}$. As a lower bound for $\nu_2, \nu_3$ one has only the values $M^{1/2}, M^{1/4}$ that are obtained from Eqs. (E), (C) alone.

Now we determine $c_0$ and $a_0$ by looking at $s_{a,M} = m < M$. The case $m < M$ is more important than for the mixed multiplicative generator since $Q_n$ is not limited by $1/4$. To some extent the smaller $Q_n$ for $m < M$ is compensated by the fact that for small $m$ there are more points with $s_a = \text{odd} \cdot m$. We use $a_0 = 3 \ 580 \ 621 \ 541$ as for the mixed multiplicative generator and find with $c_0 = 3 \ 370 \ 134 \ 727 = 11 \ 463 \mod 2^{16}$ that $Q_2(s_0, s) \approx m^{2/3-1}$ and $Q_3(s_0, s) \approx m^{1/2-1}$ if $s_{a,M} = m$.

We summarize the result for the generator of Sec. 3.3: $M = 2^{2^k d_0}, a = 2^{2^{k-1} d_0} + 2^{2^{k-2} d_0} + \ldots + 2^{2d_0} + a_0, c = \left(2^{\text{int}(2k+1/3)} d_0 + 1\right)c_0$, with $a_0 = 5 \mod 8, a_0 \approx 2^{d_0}, c_0 \text{ odd}, c_0 \geq 2^{2/3-d_0}$, e.g. $a_0 = 3 \ 580 \ 621 \ 541 \mod 2^{d_0}, c_0 = 3 \ 370 \ 134 \ 727 \mod 2^{d_0}$ leads to $N_X = 2^{257}, Q_1 = 1, Q_2 \approx M^{2/3-1}, Q_{n \geq 3} \approx M^{1/(n-1)}$.

Finally we give a short discussion of the multiple recursive generator of Sec. 3.4 (cf.
Ex. 7.2). P should not be taken too small to provide enough digits for the random numbers. To optimize the performance one should use a prime number of the form \( P = 2^d + 1 \), e.g. \( P = 2^{31} - 1 \). Moreover we set \( a_{r-1} = 1, a_{r-2} = \ldots = a_1 = 0 \).

The most severe problem is to find the prime factors of \( P^r - 1 \). To this end it is useful to take \( r = 2^k \) since in this case \( P^{2^k} - 1 = (P^{2^{k-1}} + 1) \cdot \ldots \cdot (P + 1) \cdot (P - 1) \) and one is basically left with the problem to determine the prime factors of \( P^{2^{k-1}} + 1 \).

Afterwards it is easy to find an \( a_0 \in \mathbb{Z}_P^* \) that makes the polynomial \( P(\lambda) = \lambda^r - \lambda^{r-1} - a_0 \) primitive over \( \mathbb{Z}_{P^r} \). Since

\[
\begin{pmatrix}
X_k \\
X_{k-1} \\
\vdots \\
X_{k-r+1}
\end{pmatrix} = X^k \cdot \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad \text{with } X \equiv \begin{pmatrix}
a_{r-1} & a_{r-2} & \ldots & a_1 & a_0 \\
1 & 0 & \ddots & \vdots & \vdots \\
& & 1 & 0
\end{pmatrix},
\]

a necessary and sufficient condition for a maximum period is \( X^{(P^r-1)} = 1 \mod P \) and \( X^{(P^r-1)/p} \neq 1 \mod P \) for all prime factors \( p \) of \( P^r - 1 \). High powers of \( X \) are easily computed. If \( N = \sum b_i 2^i, b_i \in \{0, 1\} \) then \( X^N = \prod_{i:b_i=1} X^{2^i} \) and \( X^2 = (X^{2^{i-1}})^2 \).

Now one has to check the \( n \)-tupel distributions for \( n > r \). We found (Eq. (27)) that \( |g|^2 = P^r - 1 \) if and only if \( s_0 = 0 \) and \( s_a = 0 \). The latter equation is equivalent to \( 0 = \sum_{j=k}^n s_j X_{j-k} = \ell_k P, k = 1, 2, \ldots, r, \ell_k \in \mathbb{Z} \) (cf. Sec. 3.4) and gives rise to an \( n \)-dimensional lattice determined by \( C \) via \( s = C \cdot (\ell_1, \ldots, \ell_r, s_{r+1}, \ldots, s_n)^T, C = C_1 \cdots C_r, \)

\[
C_k = \begin{pmatrix}
1 \\
\vdots \\
1 \\
P & -X_1 & \ldots & -X_{n-k} \\
1 & \ddots & \ddots & \ddots \\
1
\end{pmatrix} \quad C \sim C_0 \equiv \begin{pmatrix}
P & -a_0 \\
\vdots \\
P & -1 & \ldots & -a_0 \\
1 & \ddots & \ddots & \ddots \\
1
\end{pmatrix}
\]

(after some lattice transformations) if \( r < n \leq 2r \) and \( a_{r-1} = 1, a_{r-2} = \ldots = a_1 = 0 \). The determinant of \( C_0 \) is \( P^r \), however the symmetry of \( C_0 \) leads to \( \nu_{r+1} = \ldots = \nu_{2r} \equiv \nu \) which is given by the shortest lattice vector of the 2 by 3 matrix \( \begin{pmatrix} P & 0 & 0 \\ -a_0 & -1 & 1 \end{pmatrix}^T \). Since the second and third row are identical (up to a minus sign) the problem is analogous to the calculation of \( \nu_2 \) in the case of the mixed multiplicative generator. We obtain \( \nu \approx 2^{1/4} P^{1/2} \) with \( a_0 \approx 2^{1/4} P^{1/2} \approx 55109 \) for \( P = 2^{31} - 1 \).

We summarize the result for the generator of Sec. 3.4:

\[
P = 2^d - 1, \text{ prime }, \quad r = 2^k, \quad a_{r-1} = 1, \quad a_{r-2} = \ldots = a_1 = 0, \quad a_0 \approx 2^{1/4} P^{1/2}, \text{ with } \quad X^{(P^r-1)} = 1 \mod P \text{ and } X^{(P^r-1)/p} \neq 1 \mod P \quad \forall p | (P^r - 1), \quad p \text{ prime}, \end{equation} \)

\[
N_X = (P^r - 1), \quad Q_1 = \ldots = Q_r \approx \sqrt{2}, \quad Q_{r+1} = \ldots = Q_{2r} \approx 2^{1/4} P^{1/2-r}. \]
Notice that the effort for calculating random numbers does not increase with $r$.

Let us finally mention that the quality of the $n$-tupel $X_\ell$ of the (non-successive) random numbers $X_\ell, X_{k+\ell}, \ldots, X_{k+n\ell}$ deteriorates to $Q_n \lesssim P^{(r-d)/n-r}$ if there exist $d > r-n$ values of $j \in \{-1, \ldots, -r\}$ with $X_j = 0$ (see the remark at the end of Sec. 2.2). In particular if $X_{k-1} = \ldots = X_{k-r+1} = 0$ the pair $(X_0, X_k)$ has quality of less than $P^{1/2-r}$ because $aX_\ell = bX_{k+\ell} \forall \ell$ if $a = bX_k \mod P$. From Eq. (34) we see immediately that this happens for multiples of $k = (P^r - 1)/(P - 1)$ (notice the equidistant zeros in Fig. 8I). This makes it not desirable to use more than $(P^r - 1)/(P - 1)$ multiply recursive random numbers.

**Example 7.**

1. We set $M = 2^{256} = 2^{24 \cdot 16}$, $a = 2^{128} + 2^{64} + 2^{32} + 62^{181}$ and in case of the generator of Sec. 3.3 $c = (2^{160} + 1) \cdot 11463$. In the following table we compare the mixed multiplicative generator with the generator of Sec. 3.3. The results can easily be obtained with a computer algebra program and Eq. (24).

| $Q_n \equiv M^{a_n-1}$ | $X_{k+1} = aX_k + 1$ | Eq. (30) | $X_{k+1} = aX_k + c \cdot \text{ink}(k/2)$ | Eq. (32) |
|-------------------------|----------------------|----------|---------------------------------|----------|
| $\alpha_1$              | 0.99414              | 0.99414  | 1.00000                         | 1.00000  |
| $\alpha_2$              | 0.50000              | 0.50000  | 0.65658                         | 0.66667  |
| $\alpha_3$              | 0.33203              | 0.33333  | 0.49783                         | 0.50000  |
| $\alpha_4$              | 0.24859              | 0.25000  | 0.33436                         | 0.33333  |
| $\alpha_5$              | 0.19721              | 0.20000  | 0.24636                         | 0.25000  |
| $\alpha_6$              | 0.16335              | 0.16667  | 0.19882                         | 0.20000  |

(37)

We see a good agreement of the quality parameters with the approximate upper bounds. This means that our choice of parameters is satisfactory. Moreover the table confirms that the quality parameter of the generator of Sec. 3.3 lies above the quality of the mixed multiplicative generator.

Finally we present an implementation of the generator in Pascal. We group the digits of $X_k$ to 16 blocks of 16 digits $X[1], \ldots, X[16]$ starting from the highest digits.

```pascal
unit random1;
interface
const n=16; n0=(n+2) div 3; a0=62181; c0=11463;
var X:array[1..n] of longint;
procedure nextrandom;
implementation
var even:boolean; i:word; c:longint;
procedure nextrandom;
var j,k:word;
begin
if even then inc(c,c0); even:=not even;
for j:=1 to n do begin
```
CHOICE OF PARAMETERS

\[ X[j] := X[j] \times a_0; \]
\[ k := 2; \text{while } j + k \leq n \text{ do begin } \text{inc}(X[j], X[j+k]); k := k \text{ shl 1} \text{ end end; } \]
\[ \text{inc}(X[n-1], X[n] \text{ shr 16};); X[n] := (X[n] \text{ and } \$FFFF) + c; \]
\[ \text{for } j := n \text{ downto 2 do begin } \text{inc}(X[j-1], X[j] \text{ shr 16};); X[j] := X[j] \text{ and } \$FFFF \text{ end; } \]

The corresponding mixed multiplicative generator is obtained by omitting or changing the lines containing \( c \). On a 100MHz Pentium computer this (not optimized) program produces 19 563 random numbers per second whereas 20 938 mixed multiplicative random numbers can be produced. A loss of speed of about 6.6% seems us worth the gain of better random numbers. Note that the number \( c \) suffers an overflow every about 750 000th random number. This does not affect randomness and it is not worth the effort to correct this flaw.

2. We set \( P = 2^{31} - 1 \), \( r = 8 \) which leads to \( P^r - 1 = 2^{31} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 31 \cdot 41 \cdot 151 \cdot 331 \cdot 733 \cdot 1709 \cdot 21529 \cdot 368140581013 \cdot 70865169462727115232673724657 \). Moreover we take \( a_{r-1} = 1, a_{r-2} = \ldots = a_1 = 0, a_0 = 60 \ 045 \) yielding

\[
X_0 = 1, X_{-1} = \ldots = X_{-7} = 0, \quad X_{k+1} = X_k + 60 \ 045 X_{k-7} \mod 2^{31} - 1, \quad (38)
\]

\[
N_X = P^8 - 1 \approx 2^{248}, \quad Q_1 = \ldots = Q_8 = (P^8)^{1.00202-1}, \quad Q_9 = \ldots = Q_{16} = (P^8)^{0.06368-1}.
\]

The following program gives on a 100MHz Pentium 74 473 random numbers \((X[k])\) per second.

\[
\text{unit random2;}
\]
\[
\text{interface}
\]
\[
\text{var } X: \text{array}[0..7] \text{ of longint}; k: \text{integer};
\]
\[
\text{procedure nextrandom;}
\]
\[
\text{implementation}
\]
\[
\text{const } a_0 = 60045;
\]
\[
\text{var } i: \text{integer}; x0, x1, x2: \text{longint};
\]
\[
\text{procedure nextrandom;}
\]
\[
\text{begin}
\]
\[
x0 := X[(k+1) \text{ and 7}];
\]
\[
x2 := (x0 \text{ and } \$FFFF) \times a_0; \quad x1 := (x0 \text{ shr 16}) \times a_0 + (x2 \text{ shr 16});
\]
\[
x2 := (x2 \text{ and } \$FFFF) + (x1 \text{ shr 15}) + ((x1 \text{ and } \$7FF) \text{ shr 16});
\]
\[
\text{if } (x2 \text{ shr 31}) = 1 \text{ then } x2 := (x2 \text{ xor } 8000000) + 1;
\]
\[
\text{inc}(x2, X[k]);
\]
\[
\text{while } (x2 \text{ shr 31}) = 1 \text{ do } x2 := (x2 \text{ xor } 8000000) + 1;
\]
\[
k := (k+1) \text{ and 7};
\]
\[
\text{if } x2 = \$7FFFFFFF \text{ then } X[k] := 0 \text{ else } X[k] := x2
\]
end;
begin k:=0; X[0]:=1; for i:=1 to 7 do X[i]:=0 end.

5 Results and outlook

We have generalized the spectral test. As the new feature we analyze the sequence of random numbers (I in the figures) not only the distribution of $n$-tupels (II in the figures).

We saw that the mixed multiplicative generator did not pass the test with an ideal result. We were able to construct an improved generator which has the recursion formula

$$X_0 = 0, \quad X_{k+1} = aX_k + c \text{int}(k/2) \mod 2^d.$$  \hspace{1cm} (39)

For the choice of the parameters $a$, $c$, $d$ we made suggestions in Eq. (33). This generator (or the multiply recursive generator given in Eq. (38)) seems us to be the best choice in quality and performance. An implementation of a generator of this type with modulus $2^d = 2^{256} \approx 10^{77}$ was presented in Ex. 7.1. The calculation of random numbers is fast even though the modulus is that large. We think that for all practical purposes pseudo random numbers generated with this generator can not be distinguished from a true random sequence.

We were able to analyze this and several other generators. The choice of parameters was discussed in Sec. 4.

For practical purposes there is essentially no need for further improvements. From a purely mathematical point of view however there are lots of open questions.

Some further generators are discussed in [6]. However there is still little known about multiplicative generators with prime number modulus and a non-primitive multiplier. In this case $N|\hat{g}|^2(s_0, s_1)$ is given as zero of the polynomial

$$P_{s_0}(Y) \equiv \prod_{s_1 \in \mathbb{Z}_M} \left( Y - N|\hat{g}|^2(s_0, s_1) \right).$$  \hspace{1cm} (40)

For multiplicative generators $P_{s_0}(Y) = Y^M - MNY^{M-1} + \ldots$. Numerical calculations show that $P_{s_0}$ has integer coefficients. We were not able to prove this for $s_0 \neq 0$ nor to analytically determine the coefficients for non-trivial examples.

Further on, the Fourier analysis of generators involving polynomials may lead to interesting results. Here exist some connections to the theory of Gauß sums.

Finally we would be interested in multiply recursive generators. Those generators are given by a matrix-valued multiplier. The simplest example with a prime number modulus was presented in Sec. 3.4. In this section we saw that multiply recursive generators are also the best candidates for being even more efficient than the generator given in (39). In this connection multiply recursive generators with power of two modulus may be of special interest.
Aknowledgement

I am grateful to Manfred Hück who motivated me to this work by showing me some figures of random number generators.

Figures

Some graphs of random number generators are presented to give a visual impression of what the generator looks like. There are two possibilities to draw a two-dimensional plot: first (I), to plot the k-th random number \(X_k\) over \(k\) and second (II), to plot \(X_{k+1}\) over \(X_k\) presenting the pair correlation. The third part of the figures give the absolute of the Fourier transform of \(I\). \(|\hat{g}|^2(s_0, s_1)\) is a measure for the correlations along a line perpendicular to \((s_0, s_1)\) in \(I\) (cf. Eq. [10]). For ideal generators \(|\hat{g}|^2\) should be \(\leq 1\) and Figs. I and II should look like first rain drops on a dry road.

Fig. 1: \(X_0 = 1, X_{k+1} = 195X_k\mod 1009\)

\[
\begin{array}{c|c}
X_k & X_{k+1} \\
\hline
I & II \\
\end{array}
\]

\[
\begin{array}{cccccccc}
s_1 & 1 & 1009 & 1009 & 1009 & 1009 & 1009 \\
0 & 1008 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Fig. 2: \(X_0 = 0, X_{k+1} = 37X_k + 1\mod 1024\)

\[
\begin{array}{c|c}
X_k & X_{k+1} \\
\hline
I & II \\
\end{array}
\]

\[
\begin{array}{cccccccc}
s_1 & 0 & 0 & 0 & 0 & 16 \\
0 & 4 & 0 & 0 & 0 \\
0 & 0 & 8 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 \\
0 & 2^{10} & 0 & 0 & 0 \\
\end{array}
\]
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Fig. 3: $X_0 = 0$, $X_{k+1} = 41X_k + 3 \mod 1024$

| $s_1$   | 0 | 0 | 0 | 0 | 32 |
|---------|---|---|---|---|----|
|         | 0 | 8 | 0 | 0 | 0  |
|         | 0 | 0 | 16| 0 | 0  |
|         | 0 | 8 | 0 | 0 | 0  |
| $|\hat{g}|^2$ | 0 | 2$^{10}$ | 0 | 0 | 0  |

Fig. 4: $X_0 = 0$, $X_{k+1} = 41X_k + 1 \mod 1024$

| $s_1$   | 0 | 0 | 0 | 0 | 0  |
|---------|---|---|---|---|----|
|         | 0 | 8 | 0 | 0 | 0  |
|         | 0 | 0 | 0 | 0 | 0  |
|         | 0 | 0 | 0 | 8 | 0  |
| $|\hat{g}|^2$ | 0 | 2$^{10}$ | 0 | 0 | 0  |

Fig. 5: $X_0 = 0$, $X_{k+1} = 21X_k + 1 \mod 1000$

| $s_1$   | 0 | 1000 | 0 | 0 | 0 | 0 |
|---------|---|-------|---|---|---|---|
|         | 0 | 0    | 0 | 0 | 0 | 0 |
|         | 0 | 0    | 40| 0 | 0  |
|         | 0 | 0    | 0 | 0 | 0  |
| $|\hat{g}|^2$ | -4 | 0   | 0 | 0 | 40  |

Fig. 6: $X_0 = 0$, $X_{k+1} = (37 + 1024)X_k + 1 \mod 1024^2$

| $s_1$   | 0 | 0 | 0 | 0 | 0  |
|---------|---|---|---|---|----|
|         | 0 | 0 | 0 | 0 | 0  |
|         | 0 | 0 | 40| 0 | 0  |
|         | 0 | 0 | 0 | 0 | 0  |
| $|\hat{g}|^2$ | 0 | 2$^{10}$ | 0 | 0 | 0  |

cf. Fig. 2
with $N_X = 10^{20}$
Fig. 7: $X_0 = 0$, $X_{k+1} = 37X_k + 129 \text{int}(k/2) \mod 1024$

Fig. 8: $X_0 = 1$, $X_{-1} = 0$, $X_{k+1} = X_k + 7X_{k-1} \mod 31$

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