Practical Tera-scale Walsh-Hadamard Transform

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Abstract
In the mid-second decade of new millennium, the development of IT has reached unprecedented new heights. As one derivative of Moore’s law, the operating system evolves from the initial 16 bits, 32 bits, to the ultimate 64 bits. Most modern computing platforms are in transition to the 64-bit versions. For upcoming decades, IT industry will inevitably favor software and systems, which can efficiently utilize the new 64-bit hardware resources. In particular, with the advent of massive data outputs regularly, memory-efficient software and systems would be leading the future.

In this paper, we aim at studying practical Walsh-Hadamard Transform (WHT). WHT proves to be powerful in a variety of applications in image and video coding, speech processing, data compression, digital logic design, communications, just to name a few. The popularity and simplicity of WHT has stimulated research efforts and interests in sparse WHT within multidisciplinary areas including (but is not limited to) Signal Processing, Cryptography. Loosely speaking, sparse WHT refers to the case that the number of nonzero Walsh coefficients is much smaller compared with the dimension. In this work, we study efficient implementations of very large dimensional WHT. Our work is believed to shed light on noisy sparse WHT, which remains to be a big open challenge. Meanwhile, the main idea behind will help to study parallel data-intensive computing, which has a broad range of applications.

Keywords. Moore’s law, 64-bit computing, Walsh-Hadamard Transform, Noisy sparse WHT, Parallel data-intensive computing.

1 Introduction
Nowadays, we can perform the typical in-memory computing workloads for the problem size as large as $O(2^{30})$ within minutes. Suppose that the complexity of such workloads grows linear in the size. From now on, the performance of these workloads will be dominantly limited by memory latency. The latter is on the order of 100 ns currently and grows at a slow rate of about 5.5% per year (i.e., doubles every ten years). This makes that the typical in-memory processing power tends to converge to an interesting critical threshold $O(2^{32})$.

Given the exponential growth of the size of demanding workloads and the convergence of the threshold for in-memory processing power, our project aims at making practical memory-constrained system and algorithms to achieve near-optimal performance. In this work, we study efficient tera-scale Walsh-Hadamard Transform (WHT) for the first time. We also intend to make it a long-term international not-for-profit project. WHT has gained non-decreasing research popularity and found a variety of scientific and engineering applications over the past (cf. [1, 15]). In particular, the topic of sparse WHT has attracted most academia attention from various areas: signal processing [3, 7, 10], cryptography (and coding theory) [2, 8, 12]. Informally, sparse WHT
refers to the case that the number of nonzero Walsh coefficients is much smaller compared with the dimension. In signal processing, more research efforts are taken in order to efficiently obtain those nonzero Walsh coefficients and the index positions using less number of time-domain components of the signal. In cryptography (and coding theory), two long-term research themes exist: 1) researchers study super-sparse WHT of super-large dimension with emphasis on better time and memory complexities than Fast Walsh Transform (e.g., [2]). 2) researchers aim at studying practical implementation of noisy WHT with both largest possible dimension and strongest possible noise, and somehow can tolerate with solving part of those nonzero Walsh coefficients and index positions (e.g., [8,12]). Our results show that WHT of dimension $2^{40}$ can be done on the PC with 2.2GHz CPU and 16GB RAM within 3 weeks using 8TB disk space. With dimension $2^{35}$, WHT can be done with time 7 hours, 5.3 hours over the rotation disk and the flash disk respectively. This compares favorably with the MPI implementation results [11, Table 5, Page 109] of sparse WHT in cryptography (in Theme 1).

The rest of the paper is organized as follows. In Section 2, we give briefs on WHT; moreover, we give a formal definition on the signal-to-noise ratio (SNR), which is very important for the multi-disciplinary topic of sparse WHT. In Section 3, we present architecture-unaware parallel WHT. We give our results on practical large-scale external WHT for the first time in Section 4; we consider both rotation-based disks and flash disks. In Section 5, we give more results on practical tera-scale WHT. We give summary in Section 6.

2 Briefs on Walsh-Hadamard Transform

Given a real-valued function $f : GF(2)^n \rightarrow \mathbb{R}$, which is defined on an $n$-tuple binary vector of input, the Walsh-Hadamard Transform of $f$, denoted by $\hat{f}$, is another real-valued function defined as

$$\hat{f}(i) = \sum_{j \in GF(2)^n} (-1)^{<i,j>} f(j), \quad (1)$$

for all $i \in GF(2)^n$, where $<i,j>$ denotes the inner product between two $n$-tuple binary vectors $i,j$. For later convenience, we give an alternative definition below. Given an input array $x = (x_0, x_1, \ldots, x_{2^n-1})$ of $2^n$ reals in the time-domain, the Walsh-Hadamard Transform $y = \hat{x} = (y_0, y_1, \ldots, y_{2^n-1})$ of $x$ is defined by

$$y_i = \sum_{j \in GF(2)^n} (-1)^{<i,j>} x_j, \quad (2)$$

for any $n$-tuple binary vector $i$. We call $x_i$ (resp. $y_i$) the time-domain component (resp. transform-domain coefficient or simply Walsh coefficient) of the signal with dimension $2^n$. We refer the reader to [6,14] for basic properties and references on Walsh-Hadamard Transforms and [10] for newly-found interesting properties.

Assume that $x$ is corrupted by additive noise $w = (w_0, w_1, \ldots, w_{2^n-1})$, where $w_j$’s are i.i.d. with zero mean and variance $\sigma^2$. Note that we do not assume that the time-domain components of noise follow the Gaussian distribution. However, when the signal dimension $2^n$ becomes large, using orthogonality of columns of Walsh-Hadamard matrix, we can deduce that the Walsh coefficients of $w$ are approximately i.i.d. Gaussian with zero mean and variance $\sigma'^2 = 2^n \sigma^2$. Denote the noise-corrupted signal by $u = x + w$ and denote the Walsh-Hadamard transform of $u$ by $y$. Define the
**Input:** the time-domain signal $buf[\cdot]$ of dimension $2^n$

**Output:** the transform-domain signal $buf[\cdot]$

1: for $k = 0, \ldots, n - 1$ do
2: \hspace{1em} $pt \leftarrow 0, j \leftarrow 2^k$
3: \hspace{1em} for $i = 0, \ldots, 2^n - 1$ do
4: \hspace{2em} $temp_1 \leftarrow buf[pt] + buf[pt + j]$
5: \hspace{2em} $temp_2 \leftarrow buf[pt] - buf[pt + j]$
6: \hspace{2em} $buf[pt] \leftarrow temp_1$
7: \hspace{2em} $buf[pt + j] \leftarrow temp_2$
8: \hspace{1em} $pt \leftarrow pt + 1$
9: \hspace{1em} if $LSB_k(pt) = 0$ then
10: \hspace{2em} $pt \leftarrow pt + j$
11: \hspace{1em} end if
12: end for
13: end for

Figure 1: Baseline implementation of $WHT(buf, 2^n)$

signal-to-noise ratio (SNR) by

$$SNR = \frac{\|\hat{x}\|^2}{2^n \sigma^2}. \quad (3)$$

Note that Eq. (3) is consistent with the quantity [7, Eq.(5)]. The main difference is that unlike [3,7] in signal processing, our definition is not based on the assumption of Gaussian distributions of the time-domain components of the noise.

In signal processing, noisy (sparse) WHT can be very efficiently solved recently (see [3,7]) with $SNR > 0$ dB. Further, it has been identified that

$$SNR > 10 \log_{10} \left( \frac{8 \log 2}{2^n} \right) \text{ dB} \quad (4)$$

has greatest significance in cryptography (and coding theory) (see [2,8,12]). In this paper, we are motivated to *practical* Walsh-Hadamard Transform with focus on a very large signal dimension $2^n$.

## 3 Architecture-unaware Parallel WHT

In this section, we restrict ourselves to in-place WHT, i.e., the inputs and the outputs are both stored at the same place. The baseline implementation of WHT is shown in Fig. 1, where $LSB_k(\cdot)$ denotes the least significant $k$ bits of the input. Our experiences show that this serial version has fairly well performance as long as the required storage does not exceed the available RAM amount.

To speed up computations, we would like to use multi-cores to parallelize above serial version of WHT. Notably, WHT is a representative example of data-intensive computing; the latter becomes the typical work task nowadays. In general, we are guided by the two heuristic principles to make our parallel WHT based on Fig. 1. First, try to use data parallelism strategy to split the data computations into $m$ independent subtasks of equal loads for the $m$-core parallel computing. Second, try to keep the total number of necessary synchronizations among the $m$ running cores small. For $m = 4$, Fig. 2 illustrates this idea. Here, we managed to split WHT to the serial run
Input: the time-domain signal \(buf[\cdot]\) of dimension \(2^n\)
Output: the transform-domain signal \(buf[\cdot]\)

\[
\text{WORKLOAD}(buf_0, pt_0, j_0)
\]
\[
\{
buf \leftarrow buf_0, \quad pt \leftarrow pt_0, \quad j \leftarrow j_0
\]
\[
\text{for } i = 0, \ldots, 2^{n-3} - 1 \text{ do}
\]
\[
\text{temp}_1 \leftarrow buf[pt] + buf[pt + j]
\]
\[
\text{temp}_2 \leftarrow buf[pt] - buf[pt + j]
\]
\[
buf[pt] \leftarrow \text{temp}_1
\]
\[
buf[pt + j] \leftarrow \text{temp}_2
\]
\[
pt \leftarrow pt + 1
\]
\[
\text{end for}
\]
\[
\text{synchronize}
\]
\[
\text{subtask}_i: \text{run } \text{WHT}(buf + i \times 2^{n-2}, 2^{n-2}), \quad i = 0, \ldots, m - 1
\]
\[
\text{until } m \text{ subtasks are completed by } m \text{ cores}
\]
\[
\text{synchronize}
\]
\[
\text{subtask}_0: \text{run } \text{WORKLOAD}(buf, 0, 2^{n-2})
\]
\[
\text{subtask}_1: \text{run } \text{WORKLOAD}(buf, 2^{n-3}, 2^{n-2})
\]
\[
\text{subtask}_2: \text{run } \text{WORKLOAD}(buf, 2^{n-1}, 2^{n-2})
\]
\[
\text{subtask}_3: \text{run } \text{WORKLOAD}(buf, 2^{n-1} + 2^{n-3}, 2^{n-2})
\]
\[
\text{until } m \text{ subtasks are completed by } m \text{ cores}
\]
\[
\text{synchronize}
\]
\[
\text{subtask}_i: \text{run } \text{WORKLOAD}(buf, i \times 2^{n-3}, 2^{n-1}), \quad i = 0, \ldots, m - 1
\]
\[
\text{until } m \text{ subtasks are completed by } m \text{ cores}
\]

Figure 2: Example of \(\text{PARALLELWHT}(buf, 2^n, m)\) with \(m = 4\)

of three synchronizations in all. Each synchronization assigns equal workloads (which are called subtask) to \(m\) cores to be run in parallel. Specifically, the first synchronization let each core run WHT with the reduced dimension \(2^{n-2}\); it thus leaves two more rounds to be done in order to complete the original WHT with dimension \(2^n\), which can be done by twice synchronizations. For convenience, we define the common \(\text{WORKLOAD}(\cdot)\) (see Fig. 2) so that each core can run with different parameters for the last two synchronizations. Note that each subtask always executes equal amount of computations yet on different part of data, i.e., \(buf[\cdot]\). In the next section, we present large-scale external WHT when the data cannot be held all in the main memory.

4 Practical Large-scale External WHT

4.1 Performance Modeling

Denote the dimension of WHT by \(2^n\). Let \(T\) denote the runtime. We write it by

\[
T = T_{CPU} + T_{I/O}.
\] (5)
It is the sum of in-memory processing time, denoted by $T_{CPU}$, and the external I/O processing time, denoted by $T_{I/O}$. For reference implementation\(^1\) on the modern PC equipped with 2.2GHz CPU, we will use $T_{CPU} = 2.5$ seconds for $n = 26$. Note that $T_{CPU}$ should always scale regularly with the problem size (herein $2^n$), regardless of the implementation details. For example, with $n = 32$, we expect $T_{CPU} \approx 2^{32} - 2^{26} \times 2.5 \approx 160$ seconds for the same implementation choice.

Next, we discuss the factor $T_{I/O}$ in (5). Let $T_{cp}$ denote the time to copy the whole dataset to another over the same disk. Given the fixed size of the dataset, $T_{cp}$ is a stable system-dependent parameter, and it does not depend on WHT implementation details. We use the disk speed testing tool dd. To test disk writing speed (i.e., not cache writing), we use

\[
\text{dd if=/dev/zero of=bigfile bs=100k count=10k conv=fdatasync.}
\]

To test disk reading speed, we use

\[
\text{dd if=anotherbigfile of=/dev/null bs=100k count=10k.}
\]

The local disk is found to have a real reading and writing speed of 140 MB/s, 83 MB/s respectively. This means reading (resp. writing) the 32 GB file takes 229 seconds (resp. 386 seconds). And our tests of using the system command to copy a 32 GB file (which corresponds to the size of the dataset for $n = 32$), result in $T_{cp} \approx 506$ seconds, which is a lot faster than the sum of $229 + 386 = 615$ seconds (see Sect. 4.2 for more discussions).

Given $n$ and in-memory processing capability for problem size up to $2^B$, suppose that external WHT needs $q$-pass of accessing (i.e., reading and writing) the dataset on the disk. Then, we have

\[
T_{I/O} = q \times T_{cp}.
\]

We can utilize the just-in-memory processing power of $2^B$ to get the optimal $q$ with

\[
q = n - B + 1.
\]  

We give an external WHT in Fig. 3, which is directly adapted from in-memory WHT in Fig. 1. It works as follows. After running an initial WHT of the reduced dimension $2^B$ which can put the whole sub-dataset in memory, we do not read/write blocks of data at one time. Instead, we read two entries from the dataset of non-contiguous locations, which are separated by distance $j$. After the butterfly-like calculations, we then write the two updated entry-values to the previous read-locations respectively. The next read-pointer is updated. The whole procedure is iterated and the distance $j$ is increased regularly. Obviously, the movement of the disk I/O pointer (i.e., to read/write the dataset) does not take the optimal strategy with respect to the total jump distance.

For the rotation-based disks, we consider that it is an important factor to the disk I/O performance in general. We propose two methods to find the optimal strategy for the external WHT using the rotation-based disk(s). First, perform two separated blocks of reading/writing from disjoint dataset locations. Vary the block size to find out the optimal size. Note that the block size is independent of the parameter $B$. Secondly, re-arrange different orders of disk I/O operations, calculations for less number and shorter jump distance of non-contiguous reading/writing.

\(^1\)The operating system is Ubuntu 14.04.1, and the compiler version is gcc 4.8.4 with the optimization flag `-O3`.
Input: the time-domain signal $buf[\cdot]$ of dimension $2^n$, the parameter $B$
Output: the transform-domain signal $buf[\cdot]$

1: for $u = 0, \ldots, 2^{n-B} - 1$ do
2: read from dataset from index $u \cdot 2^B$ with block size $2^B$ to $buf[\cdot]$
3: run WHT($buf, 2^B$)
4: write back $buf[\cdot]$ into dataset from index $u \cdot 2^B$ with block size $2^B$
5: end for
6: for $k = B, \ldots, n - 1$ do
7: $pt \leftarrow 0, j \leftarrow 2^k$
8: for $i = 0, \ldots, 2^{n-1} - 1$ do
9: read from dataset at index $pt$ to $buf_1$
10: read from dataset at index $pt + j$ to $buf_2$
11: $temp_1 \leftarrow buf_1 + buf_2$
12: $temp_2 \leftarrow buf_1 - buf_2$
13: $buf_1 \leftarrow temp_1$
14: $buf_2 \leftarrow temp_2$
15: write back $buf_1$ into dataset at index $pt$
16: write back $buf_2$ into dataset at index $pt + j$
17: $pt \leftarrow pt + 1$
18: if $\text{LSB}_k(pt) = 0$ then
19: $pt \leftarrow pt + j$
20: end if
21: end for
22: end for

Figure 3: An external WHT directly adapted from Fig. 1

4.2 Current Results

For $n = 32$, we use $B = 30$ on our PC, and we get $q = 3$, $T_{I/O} \approx 3 \times 506 = 1518$ seconds. It makes $T \approx 160 + 1518 = 1678$ seconds (about 28 mins). We see that $T$ is dominated by the factor $T_{I/O}$ rather than $T_{CPU}$. Table 1 gives the reference runtime on the PC with 2.2GHz CPU and 16GB RAM using 4GB, 8GB RAM respectively (i.e., corresponding to $B = 29, 30$). Notably, performing truly tera-scale WHT (i.e., with the dimension $2^{40}$) can be done within 3 weeks using 8TB disk space. It is worth pointing out that the required RAM amount does not make the performance improvement in proportion.

In our experiments with $B = 29, n = 32$, the runtime 2468 seconds is obtained, in contrast to the reference time 0.61 hours in Table 1, which is obtained by $(160 + 4 \times 506) = 2184$ seconds. By detailed analysis, we found that for one pass, the real $T_{I/O}$ is always around 577 seconds, while the theoretical $T_{I/O}$ is 506 seconds. This difference accounts solely for the total runtime difference of around $4 \times (577 - 506) = 284$ seconds. Recall as we have mentioned in Sect. 4.1 that the system copy command (for a 32 GB file) takes 506 seconds, which is much faster than the sum of reading the file and then writing a file of equal size. To solve this problem, we decide to write low-level I/O tests in order to know more accurate time of copying a file. For the block size of 2 MB, 8

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\(^{2}\)In this case, multiple disks might be required.
Table 1: Reference runtime for external WHT on the PC with 2.2GHz CPU, 16GB RAM

| disk space | 32GB | 64GB | 128GB | 256GB | 512GB | 1TB  | 2TB  | 4TB  | 8TB  |
|------------|------|------|-------|-------|-------|------|------|------|------|
| runtime using (hrs.) | 0.61 | 1.49 | 3.55  | 8.23  | 18.7  | 41.9 | 92.8 | 203.6| 443.2|
| 4GB RAM (days) | 0.025 | 0.06 | 0.15  | 0.34  | 0.78  | 1.75 | 3.9  | 8.5  | 18.5 |
| runtime using (hrs.) | 0.47 | 1.2  | 3.0   | 7.1   | 16.5  | 37.4 | 83.8 | 185.6| 407.2|
| 8GB RAM (days) | 0.019 | 0.05 | 0.12  | 0.30  | 0.69  | 1.56 | 3.49 | 7.73 | 17.0 |

MB, 32 MB, 128 MB, 512 MB respectively, we copy a different file of 8 GB separately and measure the total time. In our tests, we use the direct I/O writing to eliminate cache writing effects. The results are plotted in Fig. 4 (in red). Note that when we tried with a larger block size of 2 GB as the system allows physically, the file reading operation fails each time we run. So, we attempted with the next larger block size of 1 GB. Our experiences show that a block size ranging from 128 MB to 1 GB is amenable for the rotation-based disk I/O.

Figure 4: Low-level I/O measurements

4.3 Faster Portable External WHT

It is clear that the rotation-based disks suffer severely data-intensive computing. Our results with the 7200RPM Seagate disk shows that its performance is far from satisfactory, considering specifically the fact that the SATA6.0 interface is supposed to offer a much higher throughput, that
is, 6 Gb/s corresponds to 750 MB/s.

We decided to experiment with the USB3.0 flash disk which offers more than 200 MB/s speed for both read and write by the specification. For the flash disk, we did the same low-level I/O tests to know more accurate time of copying a file as done before. The results are shown in Fig. 4 (in blue). We notice two remarkable facts. First, both the rotation disk and the flash disk exhibit very similar performance, i.e., the total file copy time decreases and converges with increasing block sizes. Interestingly, the larger block size 2 GB always fails with file reading. And we consider that it is the internal memory I/O fault rather than the external I/O fault that causes the trouble. This implies 1 GB block size the best. For highly data-intensive computing, it is suggested to choose the block size 128 MB to 512 MB to guarantee error-free I/O operations. Secondly, for each block size we have tested, the flash disk improves the performance over the rotation disk by a fairly stable constant factor of around 1/3. We have extensive experiences with rotation disks over various workstations. The best I/O performance these disks can offer under our focused data-intensive computing model seems to be no better than our results in Fig. 4. On the other hand, we find that various choices of business-class flash disks are available, which offer different rates between reading and writing. In this work, we chose a flash disk with a balanced rate, which claims an official reading and writing rate of 260 MB/s, 240 MB/s respectively.

In general, the flash disk has two different properties than rotation disks with respect to the performance: the former has much quicker access time than the latter; the former is insensitive to the jump distance between two non-contiguous access regions, which is not the case for the latter. It is therefore expected that we can further improve the performance of external WHT.

Figure 5: Optimal External WHT

![Figure 5: Optimal External WHT](image)

Finally, we give the optimal performance results of the external WHT in Fig. 5. Current USB flash technology can support high transfer rate of large capacity ranging from 128 GB, 256 GB, 512 GB. This limits the dimension of portable external WHT to smaller than $2^{36}$. Nevertheless, we
see that high-performance flash disks always beat rotation disks whenever the disk space allows. Particularly, giga-scale/tera-scale WHT has notable cryptographic significance (cf. [9]). With the dimension $2^{42}$, using 4 GB and 8 GB RAM, one can run WHT with time 2160 seconds, 1660 seconds respectively over the rotation disk; on our flash disk, this can be done with time 1628 seconds, 1261 seconds respectively. As another example, with the dimension $2^{35}$, using 8 GB RAM and 256 GB disk space, WHT can be done with time 7 hours, 5.3 hours over the rotation disk, the flash disk respectively. Currently, we are experimenting with WHT over practical large distributions. Our first target distributions are the number of prime factors of each natural number. The results will be available in the final version of the paper. In next section, we will see that flash disks can be used to compute WHT of dimension $2^{40}$ within 17 hours in some very practical scenarios.

5 More Results on Practical Tera-scale WHT

According to our results, we can immediately build very practical systems to efficiently compute WHT of dimension larger than $2^{40}$.

5.1 WHT with On-the-fly Signal Source

Suppose that each PC can have independent parallel on-line access to the signal source. This is the case when we have a digital signal source of some cryptographic function for example. We note that the dominant part of computations is the external I/O. For $n = 2^{40}$, the optimal external I/O and the in-memory processing time takes time around $(5500 \times 2^8)$ seconds, $(160 \times 2^8)$ seconds respectively (on the rotation disk), according to our analysis in Sect. 4. So, trying to increasing the in-memory processing time and maintaining the external I/O loads will not lose much in performance. And we propose to use $2^5$ PCs of same settings to compute WHT of dimension $2^{45}$ as follows. Each PC receives the same copy of the time-domain signal on-the-fly. It pre-processes the dataset of its own copy by a different linear transformation and stores an initial sub-dataset of size $2^{40}$ to the external disk(s). Then, each PC does WHT with the initial sub-dataset and reduced dimension $n = 2^{40}$. We can show that each PC now has disjoint sets of Walsh coefficients of the original source. That is, the Walsh coefficients are calculated and stored in a distributed way. Suppose that each PC uses 8GB RAM and 8TB hard disk space. We estimate the total time by $(160 \times 2^{8+5} + 5500 \times 2^8)$ seconds, which is approximately 32 days. Similarly, we remark that we can do WHT of dimension $2^{40}$ using flash disks of 512 GB and $2^5$ PCs with estimated time by $(160 \times 2^{3+5} + 6 \times 367 \times 2^4)$ seconds, i.e., within 17 hours. This compares favorably with previous external WHT based on rotation disks (see Fig. 5 in Sect. 4).

5.2 Advanced Techniques

In Sect. 3, we have presented a parallel WHT for a system without cache. It is intended to speed up in-memory WHT by multi-cores, and thus it is not helpful when the dimension becomes larger. Now, we discuss advanced techniques to extend parallel WHT to larger dimensions. We are interested in the general case that the signal source is not on-the-fly.

In High-Performance Computing (HPC), there exist two different systems to solve extremely large scale problems (cf. [5]). They are the shared-memory systems and the distributed-memory systems (see Fig. 6). The former has a single shared address space (i.e., main memory) that can be accessed by any processor. For the latter, the system memory is packaged with individual nodes of
one or more processors, and communication is required to access data from one processor’s memory by another processor via the interconnect interface.

![Diagram showing shared-memory and distributed-memory architectures]

Figure 6: Two HPC architectures: the shared-memory systems (Left) and the distributed-memory systems (Right).

We use a model of the shared-memory system architecture\(^3\) as follows. Like its HPC counterpart, it has a single shared address space (i.e., main memory) that can be accessed by any core. Suppose that the system consists of two-level caches, namely, L1 cache and L2 cache. Each core has both an L1 (data) cache and a bigger L2 cache of its own. We choose the SMP architecture with the non-shared last-level cache as our best multi-core platforms. Note that though elaborate fine-tuned techniques (such as machine-level vectorization, latency hiding) can be used to improve the parallel performance in Sect. 3, to parallelize WHT on multiple computing nodes, data transfer rate between multiple nodes becomes the main bottleneck. Based on our previous analysis, we need the node-to-node data transfer rate many times higher than the data transfer rate that the local storage can afford; otherwise, we can simply make each node run the reduced computing task independently as discussed in Sect. 5.1. Obviously, the typical network setting of 1GbE (which stands for Gigabit Ethernet) is insufficient. While the cost of switching to 10GbE network is still broadly prohibitive, we propose a custom solution to use multiple GbE network cards instead for each node. This way, we can make full use of several nodes to do parallel computing. We will report the MPI implementation details in the final version of the paper.

6 Summary

WHT is powerful in a variety of scientific and engineering applications due to its popularity and simplicity. Most recently, multidisciplinary research efforts and interests emerge that center around the topic of sparse WHT. In this work, we study efficient implementations of very large dimensional WHT. Our results show for the first time that WHT of dimension \(2^{35}\) can be done within a quarter of a day; WHT of dimension \(2^{40}\) can be done within around 400 hours; in very practical scenarios, this can be done over 32 modern PCs within 17 hours. Undoubtedly, WHT plays an important role in sparse WHT, as the former can be seen as the first solution to (noisy) sparse WHT with large dimension and strong noise. Our work is believed to shed light on noisy sparse WHT, which remains a big open challenge in academia. Further, the main idea behind also helps to study parallel data-intensive computing, which has a broad range of applications.

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\(^3\)It is often associated with SMP, which stands for Shared-Memory Processing/Programming.
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