On Properties of Compact 4th order Finite-Difference Schemes for the Variable Coefficient Wave Equation

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Abstract

We consider an initial-boundary value problem for the \(n\)-dimensional wave equation with the variable sound speed, \(n \geq 1\). We construct three-level implicit in time compact in space (three-point in each space direction) 4th order finite-difference schemes on the uniform rectangular meshes including their one-parameter (for \(n = 2\)) and three-parameter (for \(n = 3\)) families. They are closely connected to some methods and schemes constructed recently by several authors. In a unified manner, we prove the conditional stability of schemes in the strong and weak energy norms together with the 4th order error estimate under natural conditions on the time step. We also give an example of extending a compact scheme for non-uniform in space and time rectangular meshes. We suggest simple effective iterative methods based on FFT to implement the schemes whose convergence rate, under the stability condition, is fast and independent on both the meshes and variable sound speed. A new effective initial guess to start iterations is given too. We also present promising results of numerical experiments.

Keywords: wave equation, variable speed of sound, compact higher-order scheme, stability, iterative methods

AMS Subject Classification: 65M06; 65M12; 65M15; 65N22.

1 Introduction

Vast literature is devoted to compact higher-order finite-difference schemes for PDEs including elliptic, parabolic, 2nd order hyperbolic and the time-dependent Schrödinger equation, etc. This is due to the fact that the formulas and implementation of compact schemes are not so complicated as compared to the most standard 2nd order schemes but the error of compact schemes is usually several orders of magnitude less than for the 2nd order ones on the same mesh leading to significantly less computational work to ensure given accuracy. In recent years, the case of initial-boundary value problems for the multidimensional wave equation with the variable sound speed \(c(x)\) has attracted much attention, see, in particular, \[3,5,7,10\], where much more relevant references can be found. Among them, in papers \[3\] for 2D case and \[10\] for 3D case, some three-term recurrent in time compact higher-order methods on the square spatial mesh have been constructed. In the case \(c(x) \equiv \text{const}\), the spectral analysis of the methods has been given. The methods are conditionally stable but implicit in time. Therefore, to implement the methods, a direct method (for \(n = 2\)) and iterative methods of the conjugate gradient and multigrid types (for \(n = 2, 3\)) have been considered and verified.

In the case \(n = 2\), another two-level vector in time 4th order compact method has been constructed in \[5\]. Here “the vector method” means that approximations for the solution \(u\) and its weighted time derivative \(\frac{1}{c^2(x)} \partial_t u\) are constructed jointly. This method is unconditional.

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stable, but it exploits rather cumbersome approximations to the Laplace operator in the wave equation involving multiple application of a mesh Laplace operator that includes the inverse operators to the Numerov averages in each spatial coordinate. Note that two-level vector methods were studied previously, in particular, in [2,15].

In a quite recent paper [14], implicit three-level in time compact in space (three-point in each space direction) finite-difference schemes on uniform rectangular meshes have been constructed by other techniques for the initial-boundary value problem (IBVP) with the nonhomogeneous Dirichlet boundary condition for the $n$-dimensional wave equation with constant coefficients, $n \geq 1$. The conditional stability together with 4th order error estimates have been rigorously proved for the schemes. An enlarging of the schemes to the case of non-uniform in space and time rectangular meshes has been also given.

In this paper, we accomplish a generalization of compact schemes from [14] to the case $c(x) \not\equiv$ const. Moreover, we present one-parameter (for $n = 2$) and three-parameter (for $n = 3$) families of compact schemes. We also show that the methods from [3,10] can be rewritten as three-level compact schemes for the wave equation, and they are included into these families of compact approximations of the wave equation in the case of square meshes up to our simpler approximation of the free term in the equation. But notice that we use another (also implicit) approximation of the second initial condition $\partial_t u|_{t=0} = u_1$ similar and closely connected to the approximation of the wave equation itself (going back, in particular, to [15]). We also apply an operator technique that greatly simplifies and shortens deriving, writing, generalization and analysis of the schemes.

In fact, we first consider three-level in time finite-difference schemes with a weight $\sigma$ and the variable coefficient in an abstract form and prove a theorem on stability of these schemes in the strong (standard) and weak energy norms with respect to the initial data and free term in several norms. The stability is unconditional for $\sigma \geq \frac{1}{4}$ and conditional for $\sigma < \frac{1}{4}$. In the latter case, practical stability conditions on the time step of the mesh are often derived by applying the spectral method in the case of the $c(x) \equiv$ const and then taking the maximal value of $c(x)$ as this constant. The theorem justifies that such an approach is correct, in particular, for constructed compact schemes where $\sigma = \frac{1}{12}$. As a corollary of the main theorem, we rigorously prove the 4th order error estimate in the strong energy norm for constructed compact schemes. Notice that the spectral analysis for $c(x) \not\equiv$ const is impossible, and we are based on the energy method; moreover, namely stability theorems of the mentioned type give a possibility to prove rigorous error estimates.

Next we consider the method from [3] mentioned above. Excluding the auxiliary unknown function approximating $\frac{1}{c(x)^2} \partial_t u$, we reduce it to the three-level method with the weight $\sigma = \frac{1}{4}$. We also generalize it to any $n \geq 1$ and easily check that the above general stability theorem ensures its unconditional stability.

We also present an example of extending a compact scheme to the case of non-uniform in space and time rectangular meshes. Note that compact schemes on non-uniform meshes for other equations were considered, in particular, in [6,8,11,12].

In the case of the uniform rectangular mesh, we construct new simple efficient iterative methods to implement the schemes at each time level using FFT at each iteration. They are fast convergent under the stability condition, and the convergence rate is independent both on the meshes and $c(x)$ (in particular, on the amplitude of its values). We also suggest how to select an initial guess, which is close to the sought solution at each time level. This choice is based on a simplified scheme of the same type and also assuming application of FFT.

The paper is organized as follows. In Section [2] we consider three-level in time finite-difference schemes with a weight $\sigma$ and the variable coefficient in an abstract form. We adapt
to such schemes theorem on stability in the strong and weak energy norms; the stability is conditional for \( \sigma < \frac{1}{4} \). The energy conservation law for these schemes is written as well. We also transform methods from [3,10] to the form of considered schemes. In Section 3 we generalize schemes from [13] to the case of the variable sound speed. Moreover, we present one-parameter (for \( n = 2 \)) and three-parameter (for \( n = 3 \)) families of compact schemes and compare the methods from [3,10] with the constructed schemes. For the schemes, we give conditional stability theorem and its corollary on the 4th order error estimates. We also explain how to extend one of the compact schemes suitable for any \( n \geq 1 \) to the case of non-uniform in space and time rectangular meshes. The last Section 4 is devoted to the fast iterative methods to implement compact schemes. We also suggest the new effective initial guess to start the iterations. Finally we present results of numerical experiments on testing constructed schemes and the iterative method in 2D case including the wave propagation in a three-layer 2D medium initiated by the Ricker wavelet.

2 Symmetric three-level method for second order hyperbolic equations with variable coefficients and its stability theorem

Let \( H_h \) be a Euclidean space of mesh functions endowed with an inner product \( \langle \cdot, \cdot \rangle_h \) and the corresponding norm \( \| \cdot \|_h \), where \( h \) is the parameter related to a spatial discretization. Let \( B_h \) and \( A_h \) be linear operators in \( H_h \) having the properties \( B_h = B_h^* > 0 \) and \( A_h = A_h^* > 0 \). For any operator \( C_h = C_h^* > 0 \) in \( H_h \), one can define the norm \( \|w\|_{C_h} = (C_hw, w)_h^{1/2} \) in \( H_h \) generated by it.

We introduce the uniform mesh \( \omega_h = \{t_m = mh_t\}_{m=0}^M \) on a segment \([0, T]\), with the step \( h_t = T/M > 0 \) and \( M \geq 2 \). Let \( \omega_h = \{t_m\}_{m=0}^{M-1} \) be the internal part of \( \omega_h \). We introduce the mesh averages and difference operators

\[
\begin{align*}
\hat{y}^m &= \frac{y + y_{m+1}}{2}, & \hat{y} = \frac{\hat{y}^m + \hat{y}^{m+1}}{2}, & \delta y = \frac{\hat{y} - y_{m+1}}{h_t}, & \delta y = \frac{\hat{y} - y_{m}}{h_t}, & \Lambda y = \delta^2 y = \frac{\hat{y} - 2y + \hat{y}}{h_t^2}
\end{align*}
\]

with \( y^m = y(t_m) \), \( \hat{y}^m = \hat{y}^{m+1} \), and \( \hat{y}^m = \hat{y}^{m+1} \), as well as the operator of summation with the variable upper limit

\[
I^m_h y = h_t \sum_{l=1}^{m} y^l \quad \text{for} \quad 1 \leq m \leq M, \quad I^0_h y = 0.
\]

Let a multiplier \( \rho \) be given such that \( \rho w \in H_h \) for any \( w \in H_h \) and \( 0 < \rho = \text{const} < \rho \). We consider the following symmetric three-level in \( t \) method with a weight (parameter) \( \sigma \):

\[
\begin{align*}
B_h(\rho \Lambda_t v) + \sigma h_t^2 A_h \Lambda_t v + A_h v &= f \quad \text{in} \quad H_h \quad \text{on} \quad \omega_h, & (2.1) \\
B_h(\rho \delta_t v^0) + \sigma h_t^2 A_h \delta_t v^0 + \frac{1}{2} h_t A_h v^0 &= u_1 + \frac{1}{2} h_t f^0 \quad \text{in} \quad H_h, & (2.2)
\end{align*}
\]

where \( v : \omega_h \to H_h \) is the sought function and the functions \( v^0, u_1 \in H_h \) and \( f : \{t_m\}_{m=0}^{M-1} \to H_h \) are given; we omit their dependence on \( h \) for brevity. Also \( \sigma \) can depend on \( h := (h, h_t) \). Note that the form of equation (2.2) for \( v^1 \) goes back to [15] and is essential for several purposes. It can be rewritten in the form close to (2.1):

\[
\frac{B_h(\rho \delta_t v^0) + \sigma h_t^2 A_h \delta_t v^0 - u_1}{0.5 h_t} + A_h v^0 = f^0;
\]

note that clearly \( \delta_t v^0 - (\partial_t u)_{t=0} \approx (\partial_t v)_{t=0} \) for any function \( u \in C^2[0, T] \).
Recall that linear algebraic systems in $H_h$ of the form

$$B_h(\rho v^m) + \sigma h_t^2 A_h w^m = b^m$$  \hspace{1cm} (2.3)

has to be solved at time levels $t_m$ to find the solution $v^{m+1}$ for all $0 \leq m \leq M - 1$. One of the possible ways is to find directly $w^0 = \delta_t v^0$ from (2.2) and set $v^1 = v^0 + h_t w^0$, then find $w = \Lambda_t v$ from (2.1) and set $\tilde{v} = 2v - \tilde{v} + h^2_t w$. We can define the “diagonal” operator $D_\rho w := \rho w$ in $H_h$, then $B_h D_\rho + \sigma h_t^2 A_h$ is the operator in the problem (2.3).

In [3,10], for $\sigma \neq 0$, a special trick is applied. The auxiliary function $b$ is introduced by the recurrent relation

$$\hat{b} = (2 - \frac{1}{\sigma})b - \frac{\rho}{\sigma h_t^2}v - \frac{1}{\sigma} \tilde{f} \text{ on } \omega_h,$$  \hspace{1cm} (2.4)

and it is suggested to solve the equation

$$-A_h \hat{v} - \frac{1}{\sigma h_t^2} B_h(\rho \hat{v}) = B_h \hat{b} \text{ on } \omega_h,$$  \hspace{1cm} (2.5)

to find $\hat{v}$ (here the notation is slightly changed, and we do not dwell on the motivation). Rewriting relation (2.4) as

$$h_t^2 \Lambda_t b = -\frac{1}{\sigma}b - \frac{\rho}{\sigma h_t^2}v - \frac{1}{\sigma} \tilde{f}$$

and applying $-\sigma B_h$ to it, we get

$$-\sigma h_t^2 \Lambda_t B_h b = B_h b + \frac{1}{\sigma h_t^2} B_h(\rho v) + B_h \tilde{f}.$$  \hspace{1cm} (2.6)

Applying equation (2.5) from right to left (we use it also for $t_0 = 0$ together with $-A_h v^0 - \frac{1}{\sigma h_t^2} B_h(\rho v^0) = B_h b^0$ for the definitions of $b^1$ and $b^0$), we obtain

$$B_h(\rho \Lambda_t v) + \sigma h_t^2 A_h \Lambda_t v = -A_h v - \frac{1}{\sigma h_t^2} B_h(\rho v) + \frac{1}{\sigma h_t^2} B_h(\rho v) + B_h \tilde{f} = -A_h v + B_h \tilde{f}$$
on $\omega_h$. This is nothing more than equation (2.1) with $f = B_h \tilde{f}$.

We also assume that $A_h$ and $B_h$ are related by the following inequality

$$\|w\|_{A_h} \leq \alpha_h \|w\|_{B_h} \text{ for all } w \in H_h \iff A_h \leq \alpha_h^2 B_h.$$  \hspace{1cm} (2.7)

Clearly the minimal value of $\alpha_h^2$ is the maximal eigenvalue of the generalized eigenvalue problem

$$A_h e = \lambda B_h e, \hspace{0.5cm} e \in H_h, \hspace{0.5cm} e \neq 0.$$  \hspace{1cm} (2.8)

For method (2.1)-(2.2), we present a theorem on uniform in time stability (conditional for $\sigma < \frac{1}{4}$) in the mesh strong (standard) and weak energy norms with respect to the initial data $v^0$ and $u_1$ and the free term $f$. Let $\|y\|_{L^2_t(H_h)} := \frac{1}{2} h_t \|y^0\|_h + I_{h_t}^{M-1} \|y\|_h$.

**Theorem 2.1.** Let the operators $A_h$ and $B_h$ commute, i.e. $A_h B_h = B_h A_h$. Let either $\sigma \geq \frac{1}{4}$ and $\varepsilon_0 = 1$, or

$$\sigma < \frac{1}{4}, \hspace{0.5cm} (\frac{1}{4} - \sigma) h_t^2 \alpha_h^2 \leq (1 - \varepsilon_0^2) \rho \text{ for some } 0 < \varepsilon_0 < 1.$$  \hspace{1cm} (2.9)

For the solution to method (2.1)-(2.2), the following bounds hold:
(1) in the strong energy norm
\[
\max_{1 \leq m \leq M} \left[ \| \sqrt{\rho} \delta_t v^m \|_h^2 + (\sigma - \frac{1}{4} \hat{h}_t^2) \| \delta_t v^m \|^2_{B^{-1}_h A_h} + \| s_t v^m \|^2_{B^{-1}_h A_h} \right]^{1/2}
\leq \left( \| v^0 \|^2_{B^{-1}_h A_h} + \varepsilon_0^{-2} \| \frac{1}{\sqrt{\rho}} B^{-1}_h u_1 \|^2_h \right)^{1/2} + 2 \varepsilon_0^{-1} \| \frac{1}{\sqrt{\rho}} B^{-1}_h f \|_{L^1_h(H_h)};
\tag{2.9}
\]
the f-term can be replaced with \(2t_{h^{-1}} \| A_h^{-1/2} B^{-1/2}_h \delta_t f \|_h + 3 \max_{0 \leq m \leq M - 1} \| A_h^{-1/2} B^{-1/2}_h f^m \|_h\).

(2) in the weak energy norm
\[
\max_{0 \leq m \leq M} \max \left\{ \left[ \| \sqrt{\rho} v^m \|_h^2 + (\sigma - \frac{1}{4} \hat{h}_t^2) \| v^m \|^2_{B^{-1}_h A_h} \right]^{1/2}, \| I_h^m \delta_t v \|_{B^{-1}_h A_h} \right\}
\leq \left[ \| \sqrt{\rho} v^0 \|_h^2 + (\sigma - \frac{1}{4} \hat{h}_t^2) \| v^0 \|^2_{B^{-1}_h A_h} \right]^{1/2} + 2 \| A_h^{-1/2} B^{-1}_h u_1 \|_h + 2 \| A_h^{-1/2} B^{-1}_h f \|_{L^1_h(H_h)}.
\tag{2.10}
\]
For \( f = \delta_t g \), one can replace the f-term with \( \frac{2 \varepsilon_0 I_h^m}{\sqrt{\rho}} B^{-1}_h (g - s_t g^0) \) in the role of \( f \).

Proof. Applying \( B^{-1}_h \) to equations (2.1)-(2.2), we get
\[
(\rho I + \sigma h_t^2 B^{-1}_h A_h) A_h v + A_h v = B^{-1}_h f \quad \text{in} \quad H_h \quad \text{on} \quad \omega_h,
\tag{2.11}
\]
\[
(\rho I + \sigma h_t^2 B^{-1}_h A_h) \delta_t v^0 + \frac{1}{2} h_t B^{-1}_h A_h v^0 = B^{-1}_h u_1 + \frac{1}{2} h_t B^{-1}_h f^0 \quad \text{in} \quad H_h.
\tag{2.12}
\]
We have \( D^{\dagger}_\rho = D_\rho > 0 \) and \( (B^{-1}_h A_h)^{-1} = B^{-1}_h A_h > 0 \). Moreover, equation in (2.7) can be rewritten as \( B^{-1}_h A_h \epsilon = \lambda e \) and therefore inequality (2.6) is equivalent to
\[
\| w \|_{B^{-1}_h A_h} \leq \alpha_h \| w \|_h \quad \forall w \in H_h.
\tag{2.13}
\]
Consequently under the imposed conditions on \( \sigma \) and \( h_t \) we also have
\[
\varepsilon_0^2 \| \sqrt{\rho} v \|_h^2 \leq \| \sqrt{\rho} v \|_h^2 + (\sigma - \frac{1}{4} \hat{h}_t^2) \| v \|^2_{B^{-1}_h A_h} \quad \forall w \in H_h.
\tag{2.14}
\]
Now one can apply [14, Theorem 1] (see also [16, Theorem 1]) concerning scheme (2.1)-(2.2) with \( \rho = 1 \) to method (2.11)-(2.12) with \( D_\rho, B^{-1}_h A_h, B^{-1}_h f \) and \( B^{-1}_h u_1 \) in the role of \( B_h, A_h, f \) and \( u_1 \), respectively, and derive the stated bounds.

We also take into account that \( (B^{-1}_h A_h)^{-1/2} B^{-1}_h = A_h^{-1/2} B^{-1}_h \) concerning the second form of the f-term in (2.9).

Stability conditions like (2.8) are often derived by applying the spectral method in the case of the \( \rho(x) \equiv \text{const} \) and then taking in the result \( \rho \) as this constant. Theorem 2.1 justifies that such an approach is correct in our case.

We indicate that the discrete energy conservation law not only implies bound (2.9) for method (2.11)-(2.12) but itself has the independent interest
\[
\| \sqrt{\rho} \delta_t v \|_h^2 + (\sigma - \frac{1}{4} \hat{h}_t^2) \| \delta_t v \|^2_{B^{-1}_h A_h} + \| s_t v^m \|^2_{B^{-1}_h A_h}
= \left( \begin{array}{c}
B^{-1}_h A_h v^0, s_t v^0 \\
B^{-1}_h u_1, \delta_t v^0 \\
\frac{1}{2} h_t \left( B^{-1}_h f^0, \delta_t v^0 \right)_h + 2 I_{h_i}^m \left( B^{-1}_h f, \delta_t v \right)_h
\end{array} \right)
\tag{2.15}
\]
on \( \omega_h \setminus \{0\} \), see proof of Theorem 1 in [16]. Note that such a natural form is obtained, in particular, due to equation (2.2) for \( v^1 \).

Bound (2.10) in the weak energy norm is less standard than (2.9) but namely it contains simple \( H_0 \)-norm of \( v^0 \) most relevant when studying stability with respect to the round-off errors; also bounds in both norms are essential when proving delicate error estimates [15] in dependence with the data smoothness.
3 Construction and properties of compact finite-difference schemes of the 4th order of approximation

3.1. We consider the following IBVP with the nonhomogeneous Dirichlet boundary condition for the wave equation in a generalized form

\[ \rho(x) \partial^2_t u(x,t) - a_i^2 \partial^2_x u(x,t) = f(x,t) \quad \text{in} \quad Q_T = \Omega \times (0,T); \]

\[ u|_{\Gamma_T} = g(x,t); \quad u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \quad x \in \Omega. \]

We assume that \( 0 < \rho \leq \rho(x), \quad a_i > 0, \ldots, a_n > 0 \) are constants, \( x = (x_1, \ldots, x_n), \quad \Omega = (0,X_1) \times \ldots \times (0,X_n), \quad n \geq 1 \), \( \partial \Omega \) is the boundary of \( \Omega \) and \( \Gamma_T = \partial \Omega \times (0,T) \) is the lateral surface of \( Q_T \). Hereafter the summation from 1 to \( n \) over the repeated indices \( i \) and \( j \) (and only over them) is assumed. Note that \( c(x) = \frac{1}{\sqrt{\rho(x)}} \) is the variable speed of sound in the case \( a_i = 1, \quad 1 \leq i \leq n \).

Define the uniform rectangular mesh

\[ \bar{\omega}_h = \{ x_k = (k_1h_1, \ldots, k_nh_n); \ 0 \leq k_1 \leq N_1, \ldots, 0 \leq k_n \leq N_n \} \]

in \( \bar{\Omega} \) with the steps \( h_1 = \frac{X_1}{N_1}, \ldots, h_n = \frac{X_n}{N_n}, h = (h_1, \ldots, h_n) \) and \( k = (k_1, \ldots, k_n) \). Let

\[ \omega_h = \{ x_k; \ 1 \leq k_1 \leq N_1 - 1, \ldots, 1 \leq k_n \leq N_n - 1 \}, \quad \partial \omega_h = \partial \omega_h \times \{ t_m \}_{m=1}^M \text{ on } \Gamma_T. \]

We introduce the well-known difference operators

\[ (\Lambda_l w)_k = \frac{1}{h_l} (w_{k+e_l} - 2w_k + w_{k-e_l}), \quad l = 1, \ldots, n, \]

on \( \omega_h \), where \( w_k = w(x_k) \) and \( e_1, \ldots, e_n \) is the standard coordinate basis in \( \mathbb{R}^n \).

Let below \( H \) be the space of functions defined on \( \bar{\omega}_h \) and equal 0 on \( \partial \omega_h \), endowed with the inner product \( (v, w)_h = h_1 \ldots h_n \sum_{x_k \in \omega_h} v_k w_k \).

We define the Numerov-type operators and approximation of \( f \)

\[ s_N := I + \frac{1}{12} h^2_i \Lambda_i, \quad s_{N,j} := I + (1 - \delta^{(ij)}) \frac{1}{12} h^2_i \Lambda_i, \quad f_N := f + \frac{1}{12} h^2_i \Lambda_i f + \frac{1}{12} h^2_i \Lambda_i f, \]

where \( I \) is the identity operator and \( \delta^{(ij)} \) is the Kronecker symbol. Note that \( s_{N,j} = I \) for \( n = 1 \). We also set

\[ u_{1N} := s_N (\rho u_1) + \frac{1}{12} h^2_i a_i^2 \Lambda_i u_1, \]

\[ f_0^N := f^{(0)}_{dh} + \frac{1}{12} h^2_i \Lambda_i f^0, \quad \text{with some} \quad f^{(0)}_{dh} = f^0_d + O(h^3), \]

on \( \omega_h \), where \( f^0_d := f_0 + \frac{1}{3} h_i (\partial f)_0 + \frac{1}{12} h^2_i (\partial^2 f)_0 \) and \( y_0 := y|_{t=0} \), see \cite{14}.

The following lemma only slightly generalizes the similar results from \cite[Lemmas 1-2]{14}. Its proof in fact simply repeat the given there second proofs of these results (based on the averaging in space and time of equation (3.1)). Remark 3.1 below is also taken from \cite{14}.

**Lemma 3.1.** Let the coefficient \( \rho \) and solution \( u \) to the IBVP (3.1)-(3.2) be sufficiently smooth in \( Q_T \). Then the following formulas hold

\[ s_N (\rho \Lambda_i u) - \frac{1}{12} h^2_i a_i^2 \Lambda_i u - a_i^2 s_{N,j} \Lambda_j u - f_N = O(|h|^4) \quad \text{on} \quad \omega_h, \]

\[ s_N (\rho \delta_i u)^0 - \frac{k^2}{12} a_i^2 \Lambda_i (\delta_i u)^0 - \frac{h_i^2}{12} a_i^2 s_{N,j} \Lambda_j u_0 - u_{1N} - \frac{h_i^2}{12} f_N = O(|h|^4) \quad \text{on} \quad \omega_h. \]
Remark 3.1. Let $0 < h_t \leq \bar{h}_t \leq T$. If $f$ is sufficiently smooth in $t$ in $Q_{\bar{h}_t}$ (or $\Omega \times [-\bar{h}_t, \bar{h}_t]$), then $f_{\bar{h}_t}^{(0)} = f_d^{(0)} + \mathcal{O}(h_t^2)$ (see (3.4)) for the following three- and two-level approximations

$$f_{\bar{h}_t}^{(0)} = \frac{7}{12} f^0 + \frac{1}{4} f_1 - \frac{1}{12} f^2; \quad f_{\bar{h}_t}^{(0)} = \frac{1}{3} f^0 + \frac{2}{3} f_1^{1/2} \quad \text{with} \quad f_1^{1/2} := f|_{t=h_t/2}. \quad (or \quad f_{\bar{h}_t}^{(0)} = f^0 + \frac{1}{2} h \tilde{\delta} f^0 + \frac{1}{12} h^2 \Lambda_t f^0 = -\frac{1}{12} f^{-1} + \frac{5}{6} f^0 + \frac{1}{4} f^1 \quad \text{with} \quad f^{-1} := f|_{t=-h_t}).$$

The construction of compact schemes for the IBVP (3.1)-(3.2) also follows [14]. Formulas (3.5)-(3.6) mean that the scheme of the form

$$s_N(\rho \Lambda_t v) - \frac{1}{12} h^2 \Lambda_n A_n v = f_N \quad \text{on} \quad \omega_h, \quad (3.7)$$

$$v|_{\partial \omega_h} = g, \quad s_N(\rho \delta v) - \frac{1}{12} h^2 \Lambda_i \delta_i v = f_N \quad \text{on} \quad \omega_h, \quad (3.8)$$

has the approximation error of the order $\mathcal{O}(|h|^4)$. For $n = 1$, it takes the simple form

$$s_N(\rho \Lambda_t v) - \frac{1}{12} h^2 A_n \Lambda_n v = f_N \quad \text{on} \quad \omega_h, \quad (3.9)$$

$$v|_{\partial \omega_h} = g, \quad s_N(\rho \delta v) - \frac{1}{12} h^2 A_i \delta_i v = f_N \quad \text{on} \quad \omega_h, \quad (3.10)$$

But for $n \geq 2$ scheme (3.7)-(3.8) is no more of type (2.1)-(2.2). Therefore we replace it with the following one

$$s_N(\rho \Lambda_t v) + \frac{1}{12} h^2 A_n \Lambda_n v + A_N v = f_N \quad \text{on} \quad \omega_h, \quad (3.11)$$

$$v|_{\partial \omega_h} = g, \quad s_N(\rho \delta v) + \frac{1}{12} h^2 A_i \delta_i v + \frac{1}{2} h A_N v = u_1 + \frac{1}{2} h f_N \quad \text{on} \quad \omega_h, \quad (3.12)$$

with $A_N := -a_i^2 s_N i \Lambda_i$, that corresponds to the case $B_h = s_N, A_h = A_N$ and $\sigma = \frac{1}{12}[14]$. Since $A_N + a_i^2 \Lambda_i = a_i^2(I - s_N i) \Lambda_i$, the approximation error of this scheme is also of the order $\mathcal{O}(|h|^4)$.

For $n = 2$, one can easily generalize this scheme by the extension

$$s_N = I + \frac{1}{12} h^2 A_1 + I \frac{1}{12} h^2 A_2 \Rightarrow s_N \beta := s_N + \beta^2 \frac{1}{12} h^2 A_1 \Lambda_2, \quad (3.13)$$

with the parameter $\beta$, keeping its approximation order. Note that $\Lambda_1 \Lambda_2 > 0$ in $H_\beta$.

But the last scheme fails for $n \geq 3$ similarly to [14]. Recall that the point is that the minimal eigenvalue of $s_N$ as the operator in $H_\beta$ is such that $\lambda_{\min}(s_N) > 1 - \frac{n}{3}$ and $\lambda_{\min}(s_N) = 1 - \frac{n}{3} + O(\delta^{(iv)} \frac{1}{N})$ that is suitable only for $n = 1, 2$, since $s_N$ becomes almost singular for $n = 3$ and even $\lambda_{\min}(s_N) < 0$ (i.e., a crucial property $s_N > 0$ is not valid any more) for $n \geq 4$, for small $|h|$. Thus for $n = 3$ it is of sense to replace the last scheme with the other one

$$\tilde{s}_N(\rho \Lambda_t v) + \frac{1}{12} h^2 A_n \Lambda_n v + A_N v = f_N \quad \text{on} \quad \omega_h, \quad (3.14)$$

$$v|_{\partial \omega_h} = g, \quad \tilde{s}_N(\rho \delta v) + \frac{1}{12} h^2 A_i \delta_i v + \frac{1}{2} h A_N v = u_1 + \frac{1}{2} h f_N \quad \text{on} \quad \omega_h. \quad (3.15)$$

Moreover, for any $n \geq 1$, we can use the following unified scheme

$$\tilde{s}_N(\rho \Lambda_t v) + \frac{1}{12} h^2 \Lambda_n A_n v + A_N v = f_N \quad \text{on} \quad \omega_h, \quad (3.16)$$

$$v|_{\partial \omega_h} = g, \quad \tilde{s}_N(\rho \delta v) + \frac{1}{12} h^2 \Lambda_i \delta_i v + \frac{1}{2} h A_N v = u_1 + \frac{1}{2} h f_N \quad \text{on} \quad \omega_h. \quad (3.17)$$

(that goes back to [4] in the case of the time-dependent Schrödinger equation, see also [14]). In the both schemes, we use the operators

$$\tilde{s}_N := \prod_{k=1}^n s_{kN}, \quad \tilde{s}_{N_i} := \prod_{1 \leq k \leq n, k \neq i} s_{kN}, \quad s_{kN} := I + \frac{1}{12} h^2 \Lambda_k, \quad \tilde{A}_N := -a_i^2 s_{N_i} i, \quad (3.18)$$

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where \( \tilde{s}_N \) is the splitting version of \( s_N \) and \( \tilde{s}_{Nl} \) is the \((n - 1)\)-dimensional case of \( \tilde{s}_N \), with \( \tilde{s}_{Nl} = I \) for \( n = 1 \). All of them are symmetric positive definite in \( H_r \).

Clearly \( A_N = A_N \) for \( n = 1, 2 \), and for \( n = 1 \) the last scheme coincides with (3.9)-(3.10), whereas

\[
\tilde{A}_N = A_N + \tilde{A}_N^{(3)}, \quad \tilde{A}_N^{(3)} := -\frac{1}{12r}(a_1^2 h_i^2 h_3^2 + a_2^2 h_1^2 h_3^2 + a_3^2 h_1^2 h_2^2)\Lambda_1\Lambda_2\Lambda_3 \quad \text{for} \quad n = 3,
\]

where \( \tilde{A}_N^{(3)} > 0 \) in \( H_r \).

We have \((\frac{2}{3})^n I < \tilde{s}_N < I \) in \( H_r \). Moreover, the following formula connects \( \tilde{s}_N \) and \( s_N \)

\[
\tilde{s}_N = s_N + \sum_{k=2}^{n} \tilde{s}_N^{(k)}, \quad \tilde{s}_N^{(k)} := \left(\frac{1}{12}\right)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} h_{i_1}^2 \ldots h_{i_k}^2 \Lambda_{i_1} \ldots \Lambda_{i_k}.
\]

(3.19)

Notice that \((-1)^k \tilde{s}_N^{(k)} > 0 \) in \( H_r, 2 \leq k \leq n \).

Due to formulas \( \tilde{A}_N - A_N = - a_i^2 (\tilde{s}_{Nl} - s_{Nl}) \Lambda_i \) and (3.19), the approximation errors of schemes (3.14)-(3.15) and (3.16)-(3.17) have the same order \( O(|h|^4) \) as the preceding scheme (3.11)-(3.12).

For \( n = 3 \), one can easily generalize scheme (3.14)-(3.15) by the extensions

\[
\tilde{s}_N = s_{N1}s_{N2}s_{N3} \mapsto s_{N\beta} := s_N + \beta \tilde{s}_N^{(2)} + \gamma \tilde{s}_N^{(3)}, \quad A_N \mapsto A_{N\theta} := A_N + \theta \tilde{A}_N^{(3)},
\]

(3.20)

with the three parameters \( \beta, \gamma \) and \( \theta \), keeping its approximation order. Here we have explicitly

\[
\tilde{s}_N^{(2)} = \frac{1}{12r}(h_{i_1}^2 h_2^2 \Lambda_i \Lambda_2 + h_{i_1}^2 h_3^2 \Lambda_1 \Lambda_3 + h_{i_2}^2 h_3^2 \Lambda_1 \Lambda_2 \Lambda_3), \quad \tilde{s}_N^{(3)} = \frac{1}{12r} h_{i_1}^2 h_{i_2}^2 h_{i_3}^2 \Lambda_1 \Lambda_2 \Lambda_3 \quad \text{for} \quad n = 3
\]

as well as

\[
s_{N\beta} = (1 - \beta)s_N + \beta \tilde{s}_N \quad \text{for} \quad n = 2; \quad s_{N\beta\beta} = (1 - \beta)s_N + \beta \tilde{s}_N, \quad A_{N\theta} := (1 - \theta)A_N + \theta \tilde{A}_N \quad \text{for} \quad n = 3.
\]

The following expansions in \( \Lambda_k \) for the operators at the upper level in (3.11) for \( n = 2 \) and (3.14) for \( n = 3 \) hold, respectively

\[
s_N(\rho w) + \frac{1}{12} h_i^2 A_N w = \rho w + \frac{1}{12} [h_i^2 \Lambda_i (\rho w) - h_i^2 a_i^2 \Lambda_i w] - \frac{1}{12} \left(\frac{1}{h_i^2}\right)^2 h_i^2 (a_i^2 h_i^2 + a_1^2 h_1^2) \Lambda_1 \Lambda_2 w,
\]

\[
\bar{s}_N(\rho w) + \frac{1}{12} h_i^2 \bar{A}_N w = \rho w + \frac{1}{12} [h_i^2 \Lambda_i (\rho w) - h_i^2 a_i^2 \Lambda_i w] + \tilde{s}_N^{(2)}(\rho w) - \frac{1}{12} \bar{h}_i^2 \bar{A}_N^{(2)} + \tilde{s}_N^{(3)}(\rho w), \quad \bar{A}_N^{(2)} := \frac{1}{12} \sum_{1 \leq i < j \leq 3} (a_i^2 h_i^2 + a_j^2 h_j^2) \Lambda_i \Lambda_j.
\]

In the particular case of \( a_i \) and \( h_i \) independent on \( i \), the formulas are simplified, and the operators on the left in them differ only up to factors from those appearing in the related formulas (21)-(22) in [3] and (11) in [10]. Moreover, turning to formulas (2.4)-(2.5) one can show that in this case equations (3.11) for \( n = 2 \) and (3.16) for \( n = 3 \) are equivalent to respective methods from (3.10) up to our simpler approximations of \( f \). But note that we prefer to supplement them by similar equations for \( v^l \) respectively (3.12) and (3.17).

Notice that, in the same particular case, also the family of methods with the operators

\[
s_{N\beta\gamma}(\rho w) + \frac{1}{12} h_i^2 A_{N\theta}, \quad \text{with} \quad \beta = 2, \quad \gamma = \theta = 12(1 - \rho), \quad -\frac{1}{2} < \rho < 3 \quad (3.21)
\]

at the upper level was also studied in [10]. These methods are related to equation (3.14) with the extended operators (3.20) in the same way (actually for any \( \beta, \gamma \) and \( \theta \)).
Theorem 3.1. Let $g = 0$ in (3.2). Consider schemes (3.11)-(3.13) for $n = 2$, (3.14)-(3.15) and (3.20) for $n = 3$ and (3.16)-(3.17) for $n \geq 1$ (for $n = 1$, this covers also scheme (3.9)-(3.10)) and set respectively $(B_h, A_h) = (s_N \beta, A_N)$, $(B_h, A_h) = (s_N \beta, A_N)$, and $(s_N, A_N)$. Let the parameters $\beta, \gamma$ and $\theta$ be chosen such that $B_h > 0$ and $A_h > 0$ in $H_h$ and inequality (2.6) hold.

Let also $0 < \varepsilon_0 < 1$. Then under the condition
\begin{equation}
\frac{1}{6} h^2 \sigma_h^2 \leq (1 - \varepsilon_0^2) \rho \quad (3.22)
\end{equation}
for the first and second schemes or the explicit condition
\begin{equation}
h^2 \frac{\sigma_h^2}{h^2} \leq (1 - \varepsilon_0^2) \rho \quad (3.23)
\end{equation}
for the third scheme, the solutions to all these schemes satisfy the following two bounds:
\begin{equation}
\max_{1 \leq m \leq M} \left( \frac{\varepsilon^2_0 \| \sqrt{p} v_m \|_{B_h}^2 + \| s_t v_m \|_{B_h^{-1}}^2}{h_{A_h}} \right)^{1/2} 
\leq \left( \left\| v^0_{B_h^{-1}} \right\|_h + \varepsilon_0^2 \left\| \frac{1}{\sqrt{p}} B_h^{-1} u_{1N} \right\|_h \right)^{1/2} + 2 \varepsilon_0 \left\| \frac{1}{\sqrt{p}} B_h^{-1} f_N \right\|_{L_h^1(H_h)} \quad (3.24)
\end{equation}
where the $f_N$-term can be taken also as $2 I_{h_1}^0 \left\| A_h^{1/2} B_h^{-1/2} \delta_t f_N \right\|_h + 3 \max_{0 \leq m \leq M-1} \left\| A_h^{1/2} B_h^{-1/2} f_{m} \right\|_h$.

\begin{equation}
\max_{0 \leq m \leq M} \left\| \left\{ \frac{\varepsilon^2_0 \| \sqrt{p} v^m \|_h^2 + \| t_m s_t v \|_{B_h^{-1}} \right\}_{B_h} \right\}_h \n
\leq \left\| \sqrt{p} v^0 \right\|_h + 2 \left\| A_h^{1/2} B_h^{-1} u_{1N} \right\| + 2 \left\| A_h^{1/2} f_{N} \right\|_{L_h^1(H_h)}
\end{equation}

for $f_N = \delta_t g$, one can replace the $f_N$-term with $\frac{2}{\varepsilon_0} I_{h_1}^0 \left\| \frac{1}{\sqrt{p}} B_h^{-1} (g - s_t g^0) \right\|_h$.

Importantly, the both bounds hold for any free terms $u_{1N} \in H_h$ and $f_N$: $\{ t_m \}_{m=0}^{M-1} \rightarrow H_h$ (not only for those defined above).

Remark 3.2. For $(B_h, A_h) = (s_N \beta, A_N)$ with $\beta \geq 0$ for $n = 2$, $(B_h, A_h) = (s_N \beta, A_N)$ with $\beta \geq 1$, $\gamma \leq 1$ and $0 \leq \theta \leq 1$ for $n = 3$ and $(B_h, A_h) = (s_N, A_N)$ for $n \geq 1$, clearly conditions $B_h > 0$ and $A_h > 0$ in $H_h$ hold, as well as condition (2.6) has recently been studied in [14]. Lemma 31 (for $\beta = 0$ and $\theta = 0.1$ that is enough here). Consequently condition (3.22) is valid under the assumption $\frac{1}{3} h^2 \sigma_h^2 \leq (1 - \varepsilon_0^2) \rho$ in the first case or (3.23) in the second and third cases. We use here that, under the assumptions made, $s_N \beta \geq s_N$ for $n = 2$ as well as $s_N \beta \geq s_N$ and $A_N \geq A_N$ for $n = 3$, in $H_h$. This is only an example, and we do not intend here to study condition (2.6) for general $\beta$, $\gamma$, and $\theta$.

Notice also that condition (3.22) for family of schemes (3.21) was studied in [10]; thus Theorem 3.1 is applicable to it as well.

Remark 3.3. Normally $\nu_0 I \leq B_h \leq \nu I$ with some $\nu \geq \nu_0 > 0$ both independent of $h$. Then one can simplify the bounds in the statement replacing the operator $B_h^{-1}$ with the constant $\nu^{-1}$ on the left and/or replacing $B_h^{-1/2}$ with $\nu^{-k/2}$ on the right, $k = 1, 3$.

Proof. The theorem follows directly from the general stability Theorem 2.1 in the particular case $\sigma = \frac{1}{12}$ specifying assumption (2.8) and inequality (2.13).
Corollary 3.1. Let the coefficient $\rho$ and solution $u$ to the IBVP (3.1)-(3.2) be sufficiently smooth in $Q_T$. Then for $v^0 = u_0$ on $\omega_h$ and under the hypotheses of Theorem 3.1 and Remark 3.3 excluding $g = 0$, for all the schemes listed in it, the following 4th order error bound in the strong energy norm holds

$$\max_{1 \leq m \leq M} \left[ \varepsilon_0^2 \| \sqrt{\rho} \delta_t (u - v)^m \|^2_h + \| s_t (u - v)^m \|_{A_h}^2 \right]^{1/2} = O(|h|^4).$$

The proof is standard according to the general idea that stability in a norm and approximation of some order imply convergence in the same norm and of the same order (for example, see [9]) and follows from the stability bound (3.24) applied to the error $r := u - v$ (herewith $r|_{\partial\omega_h} = 0$ and $r^0 = 0$). The approximation errors play the role of stability in a norm and approximation of some order imply convergence in the same norm and of the same order (for example, see [15, Section 8]).

Normally $h_t = O(|h|)$ according to conditions (3.22) and (3.23), then $O(|h|^4) = O(|h|^4)$.

Clearly under the hypotheses of Theorem 3.1, for example, for scheme (3.11)-(3.13) for $n = 2$, the general energy conservation law (2.15) takes the form

$$\| \sqrt{\rho} \delta_t v \|_h^2 = \frac{1}{6} h_t^2 \| \delta_t v \|_{N \beta A_N}^2 + \| s_t v^m \|_{N \beta A_N}^2$$

$$= (s_{N \beta}^{-1} A_N v^0, s_t v^0)_h + (s_{N \beta}^{-1} u_1, \delta_t v^0)_h + \frac{1}{2} h_t (s_{N \beta}^{-1} f^0, \delta_t v^0)_h + 2 I^{-1}_h (s_{N \beta}^{-1} f, \delta_t v)_h$$

(3.25)
on $\omega_h \setminus \{0\}$.

Now we discuss the method from [5] formulas (14), (26)] constructed for $n = 2$. For $g = 0$, in our notation it can be rewritten as a system of two operator equations in $H_h$:

$$\delta_t v = \left[ c^2 I - \frac{1}{12} h_t^2 c^2 L(c^2 I) \right] \delta_t w,$$

$$\delta_t w = \left[ L - \frac{1}{12} h_t^2 c^2 A \right] \delta_t v + \tilde{f}.$$ (3.26)
on $\omega_h \setminus \{0\}$, where $w$ approximates $\frac{1}{c^2} \partial_t u$ and originally $L := s_{1N}^{-1} A_1 + s_{2N}^{-1} A_2 = -s_N^{-1} A_N$. There is no the free term in (3.27) in [5], so that we have inserted it ourselves for completeness. It is well-known that such methods are closely related to more standard three-level methods like (2.1)-(2.2) with $\sigma = \frac{1}{4}$, for example, see [15, Section 8].

To see that, let us exclude $w$ from this system. Applying $\delta_t$ to (3.26) and dividing it by $c^2$ as well as applying $s_t$ to (3.27), we find

$$\rho \Lambda t v = \left[ I - \frac{1}{12} h_t^2 c^2 L(c^2 I) \right] \delta_t s_t w,$$

$$s_t \delta_t w = L \left[ I - \frac{1}{12} h_t^2 c^2 A \right] s_t \delta_t v + s_t \tilde{f}. (3.28)$$

Since

$$\delta_t s_t w = s_t \delta_t w = \frac{\delta - w}{2h_t}, \quad s_t \delta_t v = v^{(1/4)} \equiv \frac{1}{4} (\delta + 2v + \tilde{v}),$$

inserting $s_t \delta_t w$ from the second equation into the first one, we obtain the following individual equation for $v$

$$\rho \Lambda t v + A_h v^{(1/4)} = f_h \quad \text{on} \quad \omega_h,$$ (3.29)
Notice that $A_h^* = A_h > 0$ in $H_h$ since $(-L)^* = -L > 0$ and
\[(A_h y, y)_h = (-Lz, z)_h, \quad \text{with } z := (I - \frac{1}{12} h_t^2 \delta c^2 I) y, \quad \forall y \in H_h.
\]
Moreover, we use the formula $\bar{s}_I w = \bar{w} + \frac{1}{2} h_t \bar{\delta}_t w$ in (3.26) and divide it by $c^2$. We also use the same formula for $v$ in (3.27) and apply the operator $\frac{1}{2} h_t [I - \frac{1}{12} h_t^2 L(c^2 I)]$ to it:
\[
\rho \delta_t v = [I - \frac{1}{12} h_t^2 L(c^2 I)](\bar{w} + \frac{1}{2} h_t \bar{\delta}_t w)
= [I - \frac{1}{12} h_t^2 L(c^2 I)] \bar{w} + \frac{1}{2} h_t [I - \frac{1}{12} h_t^2 L(c^2 I)] \{[L - \frac{1}{12} h_t^2 L(c^2 L)](\bar{v} + \frac{1}{2} h_t \bar{\delta}_t v) + \tilde{f}\).
\]
Considering the first time level $t_1 = h_t$, we find
\[
(\rho I + \frac{1}{4} h_t^2 A_h) \delta_t v^0 + \frac{1}{2} A_h v^0 = u_{1h} + \frac{1}{2} h_t f_h^0,
\]
where we have set
\[
u_{1h} := [I - \frac{1}{12} h_t^2 L(c^2 I)] w^0, \quad f_h^0 = [I - \frac{1}{12} h_t^2 L(c^2 I)] f_0.
\]
Since $v^{(1/4)} = v + \frac{1}{4} h_t^2 A_h v$, equations (3.28)–(3.31) form the particular case of method (2.1).

Moreover, Theorem 2.1 ensures unconditional stability of this generalized method. But notice that the operator $A_h$ is much more complicated than the corresponding operators in the schemes constructed above, and the latter ones do not contain any non-explicit (inverse) operators.

3.3. We briefly dwell on the case of non-uniform meshes in $x$ and $t$ when the schemes can be constructed quite similarly to [14]. Define the general non-uniform meshes $\omega_{h_i}$ in $t$ and $\omega_{hk}$ in $x_k$ with the nodes
\[
0 = t_0 < t_1 < \ldots < t_M = T, \quad 0 = x_{k0} < x_{k1} < \ldots < x_{kN_k} = X_k
\]
and the steps $h_{kl} = x_{kl} - x_{k(l-1)}$, $1 \leq k \leq n$, and $h_{tm} = t_m - t_{m-1}$. Let $\omega_{hk} = \{x_{kl}\}_{l=1}^{N_k-1}$. We set
\[
h_{t,+} = h_{t(m+1)}, \quad h_{t,max} = \max_{1 \leq m \leq M}, \quad h_{t,+} = \frac{1}{2} (h_t + h_{t,+}), \quad h_{k+1} = h_{k+1}, \quad h_{s,+} = \frac{1}{2} (h_k + h_{k+1}), h_{tm}
\]
and $h_{k,max} = \max_{1 \leq l \leq N_k} h_{kl}$. Let now $\bar{\omega}_h = \bar{\omega}_{h1} \times \ldots \times \bar{\omega}_{hn}$, $\omega_h = \omega_{h1} \times \ldots \times \omega_{hn}$ and $\partial \omega_h = \partial_h \omega_h$.

Define the difference operators in $t$ and $x_k$
\[
\delta_t y = \frac{1}{h_t} (\bar{y} - y), \quad \bar{\delta}_t y = \frac{1}{h_t} (y - \bar{y}), \quad \Lambda_t y = \frac{1}{h_t} (\delta_t y - \bar{\delta}_t y),
\]
\[
\Lambda_k w_l = \frac{1}{h_{k,g}} \left[ \frac{1}{h_{k+1}} (w_{l+1} - w_l) - \frac{1}{h_{kl}} (w_l - w_{l-1}) \right],
\]
where $w_l = w(x_{kl})$. They generalize operators defined above so their notation is the same.

The operator $s_{k,N}$ is generalized as follows
\[
s_{k,N} w_l = \frac{1}{12} (\alpha_k w_{l-1} + 10 \gamma_{kl} w_l + \beta_{kl} w_{l+1}) \text{ on } \omega_{hk},
\]
\[
\alpha_k = 2 - \frac{h_{k+1}}{h_k h_{k+1}}, \quad \beta_k = 2 - \frac{h_{k+1}}{h_k h_{k+1}}, \quad \gamma_k = 1 + \frac{(h_{k+1} - h_k)^2}{5 h_k h_{k+1}}, \quad \alpha_k + 10 \gamma_k + \beta_k = 12.
\]
Its several derivations and other forms can be found in [6,8,11,14]. The similar operator in $t$ we write in another equivalent form

$$s_{tN} = I + \frac{1}{12}(h_{t+1} \beta_t \delta_t - h_t \alpha_t \delta_t), \quad \text{with} \quad \alpha_t = 2 - \frac{h_{t+1}^2}{h_t h_{t+1}}, \quad \beta_t = 2 - \frac{h_t^2}{h_{t+1} h_t}.$$  

We confine ourselves by scheme (3.10)-(3.17) only (for brevity). Its generalized form, quite similarly to [14], looks as follows

$$\frac{1}{h_t^n}\{\bar{s}_N(\rho \delta_t v) + \frac{h_t h_{t+1}}{12} \beta_t \bar{A}_N \delta_t v - \left[\bar{s}_N(\rho \delta_t v) + \frac{h_t h_{t+1}}{12} \alpha_t \bar{A}_N \delta_t v\right]\} + \bar{A}_N v = \bar{s}_N s_{tN} f,$$  

$$v|_{\partial \omega_h} = g, \quad \bar{s}_N(\rho \delta_t v)^0 + \frac{h_t}{12} \bar{A}_N (\delta_t v)^0 + \frac{h_t^2}{2} \bar{A}_N v_0 = \bar{s}_N(\rho u_1) - \frac{h_t^2}{12} \bar{A}_N u_1 + \frac{h_t^2}{2} f_N$$

with $f_N^0 = \bar{s}_N f_0 + \frac{h_t^2}{3} (\delta_t f)^0$. Here the operators $\bar{s}_N$ and $\bar{A}_N$ are defined as in (3.18) but with the generalized terms $s_{kN}$ and $\Lambda_k$. The equations are valid respectively on $\omega_h$ and $\omega_h$ and have the approximation errors of the order $O(h_{\text{max}}^3)$, where $h_{\text{max}} = \max\{h_{1\text{max}}, \ldots, h_{n\text{max}}, h_t\text{max}\}$. This is checked by repeating the arguments from [14].

For the uniform mesh in $t$, the left-hand side of (3.32) takes the previous simpler form whereas the term $\bar{s}_N s_{tN} f$ can be simplified keeping the same order of the approximation error:

$$\bar{s}_N(\rho \Lambda_t v) + \frac{1}{12} h_t^2 \bar{A}_N \Lambda_t v + \bar{A}_N v = (\bar{s}_N + \frac{1}{12} h_t^2 \Lambda_t) f.$$  

Equation (3.33) involves only the zero and first time levels thus it is simplified only in notation.

One can check also that the approximation errors still has the 4th order $O(h_{\text{max}}^4)$ for smoothly varying non-uniform meshes, cp. [12,14], provided that, for example, $f_N^0 = \bar{s}_N f^0 - f^0 + f^0_{\partial \omega_h}$.

Other above constructed schemes can be generalized to the case of non-uniform meshes similarly. Here we do not touch the stability issue in the case of the non-uniform mesh (even only in space). But this is more cumbersome (like in [12]) since the operator $s_{kN}$ is not self-adjoint as well as $s_{kN}$ and $\Lambda_k$ do not commute any more, and, moreover, this leads to stronger conditions on $h_t$, especially in the case when the corresponding eigenvalue problem (2.7) has complex eigenvalues, see [13]. On the other hand, for smoothly varying non-uniform meshes, results of some 1D numerical experiments are positive, see [12,14].

## 4 Iterative methods and numerical experiments

### 4.1. We go back to equation (2.3), or omitting the superscript $m$ and taking $\sigma = \frac{1}{12}$, to the equation

$$B_h(\rho w) + \frac{1}{12} h_t^2 A_h w = b \quad \text{in} \quad H_h,$$  

(4.1)

for any of the pairs of operators $(B_h, A_h)$ considered in the previous section. Thus we assume that the non-homogeneous boundary condition $v|_{\partial \omega_h} = g$ is reduced to the homogeneous one $v|_{\partial \omega_h} = 0$ by respective change in $f_{\omega_N}$ and $u_{1\text{h}}$ at the mesh nodes of $\omega_h$ closest to $\partial \omega_h$.

We first consider the simple iterative method with a preconditioner

$$B_h\left(\frac{w^{(l+1)} - w^{(l)}}{\theta}\right) + B_h(\rho w^{(l)}) + \frac{1}{12} h_t^2 A_h w^{(l)} = b, \quad l = 0, 1, \ldots,$$  

(4.2)

that can be effectively applied, with the parameter $\theta > 0$. Its equivalent form is

$$w^{(l+1)} = (1 - \theta) w^{(l)} - \frac{\theta}{\rho} B_h^{-1} \left(\frac{1}{12} h_t^2 A_h w^{(l)} - b\right), \quad l = 0, 1, \ldots$$

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For schemes from Subsection 3.1, taking $B_h^{-1}$ can be effectively implemented by FFT.

Let condition (3.22) be valid. Then the following spectral equivalence inequalities hold

$$D_\rho \leq A_h := D_\rho + \frac{1}{12} h_t^2 B_h^{-1} A_h \leq \bar{\lambda}(\varepsilon_0^2) D_\rho \quad \text{in} \quad H_h, \quad \text{with} \quad \bar{\lambda}(\varepsilon_0^2) := 1 + \frac{1}{2}(1 - \varepsilon_0^2).$$

(4.3)

Thus the optimal value of $\theta$ is

$$\theta_{\text{opt}} = \frac{2}{1 + \lambda} = \frac{1}{1 + \frac{1}{4}(1 - \varepsilon_0^2)}$$

that ensures the convergence rate of geometric progression (in the norms $\|\sqrt{\rho} \cdot h\|$ and $\|\cdot\|_{A_h}$) with the common ratio

$$q_0 = q_0(\varepsilon_0^2) := \frac{1 - \varepsilon_0^2}{5 - \varepsilon_0^2} \leq 0.2 \quad \text{on} \quad [0, 1)$$

independent of both the meshes and $\rho$, in particular, the ratio $\hat{\rho} = \rho/\bar{\rho}$. For example, in the typical case $\varepsilon_0^2 = \frac{1}{2}$, we have $q_0(\frac{1}{2}) = \frac{1}{10} \approx 0.1111$. For the $\sqrt{2}$ times stronger condition on $h_t$ with $\varepsilon_0^2 = \frac{3}{4}$, we have already $q_0(\frac{3}{4}) \approx 0.05882$.

Recall also the preconditioned method of steepest descent with at least the same convergence rate (in the norm $\|\cdot\|_{A_h}$), where the parameter $\theta = \theta_l$ is not fixed and related to $\varepsilon_0^2$ but is computed by the formula

$$\theta_l = \frac{(D_\rho y_l, y_l)_h}{(A_h y_l, y_l)_h} = \frac{\|\sqrt{\rho} y_l\|_{h}^2}{\|\sqrt{\rho} y_l\|_{h}^2 + \frac{1}{12} h_t^2 (B_h^{-1} A_h y_l, y_l)_h}, \quad y_l := w_l + \frac{1}{\rho} B_h^{-1} (\frac{1}{12} h_t^2 A_h w_l - b).$$

For the corresponding preconditioned conjugate gradient method which formulas and standard theory are well-known, for example, see [1] Chapter 1, and thus omitted here, we have the smaller common ratio

$$q_1 = q_1(\varepsilon_0^2) := \frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} = \frac{1 - \varepsilon_0^2}{5 - \varepsilon_0^2 + 4\sqrt{1 + \frac{1}{2}(1 - \varepsilon_0^2)}} \leq \frac{1}{5 + 4\sqrt{1.5}} \approx 0.1010 \quad \text{on} \quad [0, 1).$$

We have, in particular, $q_1(\frac{1}{2}) \approx 0.05573$ and $q_1(\frac{3}{4}) \approx 0.02944$.

It is easy to see that $0.5 < \frac{q_0}{q_1} \leq \frac{5}{5 + 4\sqrt{1.5}} \approx 0.5051$ on $[0, 1)$, as well as $q_0, q_1$ and $\frac{q_0}{q_1}$ decrease on $[0, 1)$. Moreover, $q_0(\varepsilon_0^2) \rightarrow 0$ as $\varepsilon_0 \rightarrow 1$, $l = 0, 1$, i.e., the common ratios become arbitrarily small as condition (3.22) on $h_t$ turns more and more stronger.

Clearly these methods can be generalized for equation (4.1) with any $\sigma \neq 0$ in the role of $\frac{1}{12}$.

Concerning the initial guess for (4.2), one can base simply on the formula $v^{m+1,0} = v^m$, for $0 \leq m \leq M - 1$, or $v^{m+1,0} = 2v^m - v^{m-1}$, for $1 \leq m \leq M - 1$. But it seems much better to use closely related equations (2.1) and (2.2) for $\sigma = 0$ in the form:

$$(\Lambda_t v)^{m,0} := -\frac{1}{\rho} B_h^{-1} (A_h v^m - f^m) \quad \text{in} \quad H_h, \quad 1 \leq m \leq M - 1,$$

$$(\delta_t v^0)^{(0)} := -\frac{1}{\rho} B_h^{-1} (\frac{1}{2} h_t A_h v^0 - u_1 - \frac{1}{2} h_t f^0) \quad \text{in} \quad H_h.$$

Taking $B_h^{-1}$ can be again effectively implemented by FFT. Note that a discussion on the choice of the initial guess can be found in [3].
Let us describe results of the numerical experiments. To be definite, we take \( n = 2 \) and use mainly scheme (3.11)-(3.12) that below we call scheme \( S_0 \). In order to compare the results with those presented in literature, we solve two test problems from \([5]\) and also take one more problem for the rectangular mesh. Our numerical tests have been performed on the computer with Intel® Xeon® processor E5-2670, and the algorithm is implemented using C++ language.

We rewrite the IBVP (3.1)-(3.2) for \( n = 2 \) and \( g = 0 \) as

\[
\partial_t^2 u - c^2(x,y)(\partial_x^2 u + \partial_y^2 u) = \varphi(x,y,t), \quad (x,y) \in [0, X] \times [0, Y], \; 0 < t \leq T,
\]

\[
u|_{\Gamma_T} = 0, \quad u(x,y,0) = u_0(x), \quad \partial_t u(x,y,0) = u_1(x).
\]

**Example 1.** First we take \( X = Y = T = 2, \; c^2(x,y) = 1 + \left( \frac{\pi x}{8} \right)^2 + \left( \frac{\pi y}{8} \right)^2 \). The data \( u_0(x), \; u_1 = 0 \) and \( \varphi(x,y,t) \) are chosen so that the solution is the simple standing wave 

\[
u(x,y,t) = \sin(\pi x) \sin(\pi y) \cos(\pi t)
\]

as in \([5]\).

Table 1 contains the errors \( e_{L^2}(N) \) and \( e_{L^\infty}(N) \) in the mesh \( L_2 \) and \( L_{\infty} \) norms at \( t = T \) together with the corresponding experimental convergence rates:

\[
p_{L^q}(N) = \log \frac{e_{L^q}(N)}{e_{L^q}(N/2)} / \log 2, \quad q = 2, \infty.
\]

Table 1: Example 1: errors \( e_{L^2}(N) \), convergence rates \( p_{L^q}(N) \), numbers of iterations \( N_{\text{iter}} \) and CPU times for a sequence of meshes

| \( N \) | \( h_x = h_y \) | \( e_{L^2}(N) \) | \( p_{L^2}(N) \) | \( e_{L^\infty}(N) \) | \( p_{L^\infty}(N) \) | \( N_{\text{iter}} \) | \( \text{CPU time} \) |
|-----|-------------|----------|-----------|-------------|-----------|-------|-----------|
| 8   | 1/4         | 3.3660e-3 | —         | 3.5483e-3  | —         | 6     | 0.001 s   |
| 16  | 1/8         | 2.0104e-4 | 4.065     | 2.2719e-4  | 3.965     | 6     | 0.012 s   |
| 32  | 1/16        | 1.2128e-5 | 4.051     | 1.4623e-5  | 3.958     | 5     | 0.085 s   |
| 64  | 1/32        | 7.4564e-7 | 4.023     | 9.1493e-7  | 3.998     | 5     | 0.608 s   |

Clearly scheme \( S_0 \) demonstrates the 4th order accuracy in both norms. The obtained \( L^2 \) errors are about 5 times more accurate than those in \([5\]; Table 12\). Also it can be seen that it \( N_{\text{iter}} \) is small and the CPU time is approximately proportional to the size of the discrete problem.

Next we investigate in more details the convergence of the proposed iterative algorithm (4.2) with the initial guess defined by (4.4). The given problem is solved for different values of \( c^2_0 \) and the number \( M \) defining the time step \( h_t = \frac{T}{M} \). Table 2 contains the values of \( N_{\text{iter}} \) for \( h_x = \frac{1}{16} \).

| \( N \) | \( h_x = h_y \) | \( e_{L^2}(N) \) | \( p_{L^2}(N) \) | \( e_{L^\infty}(N) \) | \( p_{L^\infty}(N) \) | \( N_{\text{iter}} \) | \( \text{CPU time} \) |
|-----|-------------|----------|-----------|-------------|-----------|-------|-----------|
| 8   | 1/4         | 3.3660e-3 | —         | 3.5483e-3  | —         | 6     | 0.001 s   |
| 16  | 1/8         | 2.0104e-4 | 4.065     | 2.2719e-4  | 3.965     | 6     | 0.012 s   |
| 32  | 1/16        | 1.2128e-5 | 4.051     | 1.4623e-5  | 3.958     | 5     | 0.085 s   |
| 64  | 1/32        | 7.4564e-7 | 4.023     | 9.1493e-7  | 3.998     | 5     | 0.608 s   |

For comparison, in brackets we also present its values when a simple guess \( w^{(0)} = w \) is used. We observe that the convergence of the iterative algorithm (4.2) with the initial guess defined by (4.4) is very fast requiring no more than 5 iterations to reach the high tolerance error \( 10^{-10} \), and its rate is only slightly sensitive to the selection of the iterative parameter \( \theta \). The role of this initial guess is essential since it reduces \( N_{\text{iter}} \) at least twice. Still this dependence can become more pronounced for not so smooth solutions when errors in high modes become more important.
Table 2: Example 1: \( N_{\text{iter}} \) for different \( M \) and parameters \( \theta \) in (4.2).

| \( M \) | \( \theta = \frac{8}{9} (\varepsilon_0^2 = \frac{1}{2}) \) | \( \theta = \frac{16}{17} (\varepsilon_0^2 = \frac{3}{4}) \) | \( \theta = \frac{32}{33} (\varepsilon_0^2 = \frac{7}{8}) \) |
|---|---|---|---|
| 256 | 5 (10) | 5 (9) | 5 (9) |
| 512 | 5 (10) | 4 (9) | 4 (8) |
| 1024 | 4 (10) | 4 (9) | 3 (8) |
| 2048 | 4 (9) | 3 (8) | 3 (8) |

Example 2. Next we take \( X = Y = T = 1, c^2(x, y) = (1+x^2+4y^2)^{-1} \). The data \( u_0, u_1 \) and \( \varphi \) are chosen so that the solution is the simple standing wave \( u(x, y, t) = \sin(\pi x) \sin(4\pi y) \exp(t) \).

In this example, the wave propagation in \( x \) and \( y \) directions is different, thus the mesh steps \( h_x = \frac{1}{N} \neq h_y = \frac{1}{4N} \) are taken.

Table 3 contains the errors \( e_{L^2}(N) \) and \( e_{L^\infty}(N) \) at \( t = 1 \) together with the corresponding experimental convergence rates for scheme \( S_0 \). Clearly the scheme is robust for \( h_x \neq h_y \) as well.

Table 3: Example 2: errors \( e_{L^2}(N) \) and convergence rates \( p_{L^2}(N) \) of the solution to scheme \( S_0 \), i.e., (3.11)-(3.12), for a sequence of meshes

| \( N \) | \( h_x \) | \( h_y \) | \( h_t \) | \( e_{L^2}(N) \) | \( p_{L^2}(N) \) | \( e_{L^\infty}(N) \) | \( p_{L^\infty}(N) \) |
|---|---|---|---|---|---|---|---|
| 4 | 1/4 | 1/16 | 1/32 | 3.3710e-3 | — | 3.6410e-3 | — |
| 8 | 1/8 | 1/32 | 1/64 | 1.9822e-4 | 4.088 | 2.3470e-4 | 3.955 |
| 16 | 1/16 | 1/64 | 1/128 | 1.1960e-5 | 4.051 | 1.5246e-5 | 3.982 |
| 32 | 1/32 | 1/128 | 1/256 | 7.2937e-7 | 4.035 | 9.2547e-7 | 4.004 |

For comparison, we solved the same problem by using the modified 4th order scheme (3.14)-(3.15) (which is applicable for any \( n \)). Table 4 contains the same type results for this scheme. The results for both schemes are very close thus for the remaining tests we apply only the former one. Nevertheless we note carefully that all the errors are (very) slightly larger for the latter scheme; this is since it exploits the more complex (more dissipative in space) operator \( \bar{s}_N = s_N + \frac{h_x^2 + h_y^2}{12} \Lambda_x \Lambda_y \) rather than \( s_N \) in the former scheme.

Table 4: Example 2: errors \( e_{L^2}(N) \) and convergence rates \( p_{L^2}(N) \) of the solution to scheme (3.14)-(3.15) for a sequence of meshes

| \( N \) | \( h_x \) | \( h_y \) | \( h_t \) | \( e_{L^2}(N) \) | \( p_{L^2}(N) \) | \( e_{L^\infty}(N) \) | \( p_{L^\infty}(N) \) |
|---|---|---|---|---|---|---|---|
| 4 | 1/4 | 1/16 | 1/32 | 3.4940e-3 | — | 3.7327e-3 | — |
| 8 | 1/8 | 1/32 | 1/64 | 2.0533e-4 | 4.088 | 2.4078e-4 | 3.956 |
| 16 | 1/16 | 1/64 | 1/128 | 1.2386e-5 | 4.051 | 1.3524e-5 | 3.981 |
| 32 | 1/32 | 1/128 | 1/256 | 7.5548e-7 | 4.035 | 9.5043e-7 | 4.004 |

Example 3. Finally, the wave propagation is studied in the three-layer medium with the sound speeds \( s_1, s_2 \) and \( s_3 = s_1 \) (unless otherwise stated) respectively in its left, middle and
right layers of the same thickness. Here we take \( X = Y = 3000 \) m. The source is defined as the Ricker wavelet known in geophysics and given by

\[
\varphi(x, y, t) = \delta(x - x_0, y - y_0) \sin(50t)e^{-200t^2},
\]

where \( \delta(x - x_0, y - y_0) \) is the Dirac distribution located at the center of domain \((x_0, y_0) = (1500 \text{ m}, 1500 \text{ m})\). Also \( u_0 = u_1 = 0 \). It was shown in [5] that the wave dynamics is complicated. The computational challenges arise due to discontinuous coefficient \( c^2 \) and the very non-smooth distributional source function \( \varphi \).

We take \( h_x = h_y = h = \frac{X}{N} \) with even \( N \) and approximate \( \delta(x - x_0, y - y_0) \) as the mesh delta-function that equals \( h^{-2} \) at the node \((x_0, y_0)\) and 0 at other nodes (in accordance with [14] where an approximation for non-smooth \( \varphi \) has been considered).

Let first \( s_1 = 1500 \) and \( s_2 = 1000 \) m/s as in [5]. Figure 1(a) shows 1D profiles of waves at \( y = 1.5 \) m for various times in the three-layer medium. At \( t = 0.25 \), the wave moves still inside the middle layer only. At \( t = 0.75 \), the wave fronts have already passed the interfaces of layers, have decreased their amplitude and move through the left and right layers towards the boundary; simultaneously the reflected waves of much smaller amplitude move back inside the middle layer. At \( t = 1.05 \), both reflected waves collide and acquire larger amplitude; then they continue their movement as shown at \( t = 1.15 \).

For comparison, Figure 1(b) shows 1D profiles of waves at \( y = 1.5 \) m in the homogeneous medium for \( s_1 = s_2 = 1000 \) m/s. Now only the refraction wave exists and moves towards the boundary with a constant velocity; the graphs on the both figures are the same at \( t = 0.25 \).

![Figure 1](image1.png)

**Figure 1:** Dynamics of the waves at different times for: (a) the three-layer medium; (b) the homogeneous medium for \( s_1 = s_2 = 1000 \) m/s

Next, in Figure 2 we present the dynamics of the waves at \( y = 1.5 \) m in the case of three different sound speeds \( s_1 = 1500 \), \( s_2 = 1000 \) and \( s_3 = 3000 \). At \( t = 0.25 \), the graph is the same once again. At \( t = 0.6 \) and \( t = 0.7 \), the wave fronts have already passed the interfaces of layers. In contrast to Figure 1 the amplitudes and speeds of the right refracted and reflected waves are higher than of the left ones.

In addition, we investigate experimentally the robustness of our iterative algorithm with respect to jumps in the sound speed and the convergence order of scheme \( S_0 \). Such an analysis was not done in [5].
Figure 2: Dynamics of the waves at different times for the three-layer medium with $s_1 = 1500$, $s_2 = 1000$ and $s_3 = 3000$ m/s

Table 5 contains the values of $N_{iter}$ for different speeds $s_1$ together with $s_2 = 1000$ m/s. In computations, the space steps are $h = 15$ and $7.5$ m; the time steps $h_t$ are respectively selected from the stability requirement. The presented results confirm that the iterative algorithm (4.2) with the initial guess defined by (4.4) is both robust and fast.

Table 5: Example 3: $N_{iter}$ for different speeds $s_1$ in the left and right layers

| $s_1$ | $T$ | $h$ | $h_t$ | $N_{iter}$ | $h$ | $h_t$ | $N_{iter}$ |
|-------|-----|-----|-------|------------|-----|-------|------------|
| 1000  | 1.0 | 15  | 0.005 | 9          | 7.5 | 0.0025 | 9          |
| 1500  | 0.8 | 15  | 0.004 | 9          | 7.5 | 0.002  | 9          |
| 3000  | 0.6 | 15  | 0.002 | 9          | 7.5 | 0.001  | 9          |
| 6000  | 0.6 | 15  | 0.0012| 9          | 7.5 | 0.0006 | 9          |

Table 6 contains the errors $\bar{e}_{L^2}(N)$ and $e_{L^\infty}(N)$ in the mesh scaled $L_2$ and $L_\infty$ norms at $t = 0.8$, for $h = \frac{X}{N}$, with $N = 100, 200, 400$, and $h_t = \frac{0.8}{N}$. The approximations to these errors are computed as

$$\bar{e}_{L^2}(N) = \frac{1}{X} ||v_h - v_{h/2}||_{L^2}, \quad e_{L^\infty}(N) = ||v_h - v_{h/2}||_{L^\infty},$$

where $X$ equals the square root of the domain area, and $v_h$ is the solution to the scheme $S_0$ for $h = \frac{X}{N}$. The computations are accomplished for the homogeneous case $s_1 = s_2 = 1000$ m/s and three-layer one with $s_1 = 1500$ and $s_2 = 1000$ m/s. We see that since the exact solution is a non-smooth function, the convergence rates are essentially reduced, and they are visibly higher in a simpler case of the constant sound speed. The results in $L^2$ norm are much better than in $L^\infty$ one. Both last details are natural.

For comparison, we also investigate the accuracy of the standard explicit 2nd order scheme $\Lambda_t z - c^2(\Lambda_x + \Lambda_y)z = \varphi$ for the same tests as described in Table 6. Table 7 contains the errors $\bar{e}_{L^2}(N)$ and $e_{L^\infty}(N)$ in the mesh scaled $L_2$ and $L_\infty$ norms at $t = 0.8$, for $h = \frac{X}{N}$, $N = 100, 200, 400$, and $h_t = \frac{0.8}{N}$. Here the errors are computed as

$$\bar{e}_{L^2}(N) = \frac{1}{X} ||z_h - v_{h0}||_{L^2}, \quad e_{L^\infty}(N) = ||z_h - v_{h0}||_{L^\infty},$$

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Table 6: Example 3: errors $\bar{e}_{L^2}(N)$ and $e_{L^\infty}(N)$ and convergence rates $p_{L^q}(N)$ of for a sequence of meshes and two speeds $s_1 = 1000$ and 1500 in the left and right layers

| $s_1$ | $N$ | $h$ | $h_t$ | $\bar{e}_{L^2}(N)$ | $p_{L^2}(N)$ | $e_{L^\infty}(N)$ | $p_{L^\infty}(N)$ |
|-------|-----|-----|-------|---------------------|--------------|-------------------|-------------------|
| 1000  | 100 | 30  | 0.008 | 1.78919e-3          | —            | 0.012093         | —                |
| 1000  | 200 | 15  | 0.004 | 4.04097e-4          | 2.146        | 0.004069         | 1.571            |
| 1000  | 400 | 7.5 | 0.002 | 9.88333e-5          | 2.032        | 0.001387         | 1.553            |
| 1500  | 100 | 30  | 0.008 | 2.01559e-3          | —            | 0.012093         | —                |
| 1500  | 200 | 15  | 0.004 | 6.18800e-4          | 1.704        | 0.005448         | 1.150            |
| 1500  | 400 | 7.5 | 0.002 | 2.11363e-4          | 1.550        | 0.002736         | 0.994            |

where $v_{h_0}$ is the solution of scheme $S_0$ for $h_0 = \frac{X}{800}$ and $h_t = 0.001$ and $z_h$ is the solution of the explicit 2nd order scheme. Clearly, for the 2nd order scheme, the errors are larger and the convergence rates are worse than for scheme $S_0$ thus the latter scheme is better in the non-smooth case as well.

Table 7: Example 3: errors $\bar{e}_{L^2}(N)$ and $e_{L^\infty}(N)$ and convergence rates $p_{L^q}(N)$ for the standard explicit 2nd order scheme for a sequence of meshes and $s_1 = 1000$

| $s_1$ | $N$ | $h$ | $h_t$ | $\bar{e}_{L^2}(N)$ | $p_{L^2}(N)$ | $e_{L^\infty}(N)$ | $p_{L^\infty}(N)$ |
|-------|-----|-----|-------|---------------------|--------------|-------------------|-------------------|
| 1000  | 200 | 15  | 0.004 | 2.57470e-3          | —            | 0.015435         | —                |
| 1000  | 400 | 7.5 | 0.002 | 9.75537e-4          | 1.400        | 0.008072         | 0.935            |
| 1000  | 800 | 3.75| 0.001 | 3.18427e-4          | 1.615        | 0.004047         | 0.996            |

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