Some (3+1) dimensional vortex solutions of the \( CP^N \) model

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Abstract

We present a class of solutions of the \( CP^N \) model in (3+1) dimensions. We suggest that they represent vortex-like configurations. We also discuss some of their properties. We show that some configurations of vortices have a divergent energy per unit length while for the others such an energy has a minimum for a very special orientation of vortices. We also discuss the Noether charge densities of these vortices.
1 Introduction

Exact topological soliton solutions are rare especially in dimensions higher than two. Their existence depends on very special structures, and the most well known examples are those provided by self-dual or BPS solutions like instantons and monopoles. In this paper we present exact vortex like solutions in $(3 + 1)$ dimensions for the well known $CP^N$ model [1, 2]. The class of exact solutions we construct is in fact very large, and consists of arbitrary functions of two variables, namely the combinations $x^1 + i x^2$ and $x^3 + x^0$, where $x^\mu$, $\mu = 0, 1, 2, 3$ are the Cartesian coordinates of four dimensional Minkowski space-time. Among these solutions there is a special subset which corresponds to straight vortex solutions with waves traveling along them with the speed of light. Our results correspond, in fact, to a specialization to $CP^N$ of those obtained in [3, 4] for extensions of the Skyrme-Faddeev model.

One of the features we explore in detail in this paper is the structure of the energy for parallel but separated vortices. The total energy of such a solution is obviously infinite since it does not depend on one of the space coordinates. However, the energy per unit of length of the vortex is finite and is made out of two parts. The first one is purely topological and corresponds to the energy of the $(2 + 0)$ dimensional $CP^N$ lumps. This part does not depend on the separation of the vortices and is proportional to the two dimensional topological charge. The second part involves the energy coming from the waves and it does depend on the separation of the vortices. It is, in fact, proportional to a combination of the Noether charges associated with the $U(1)^N$ subgroup of $SU(N + 1)$. We make a detailed study of this part of the energy and discover that the solutions can be put in two main classes according to the behavior of the energy for large separation of the vortices. For one class the energy per unit of length goes to zero for large separations and for the other it diverges. In addition, both classes possess local minima of the energy per unit of length for finite separations of the vortices. We would like to stress that such a dependence of the energy on the separation does not lead to a force among the vortices. The total energy is infinite for all such solutions and only the energy per unit of length is finite and dependent on the separation. Our analysis shows that the densities of topological and Noether charges have practically the same location for any given solution. In addition, their topological charge densities are peaked at the location of vortices but the densities of the Noether charges are concentrated only near the location of each vortex. As far as we are aware such exact solutions for $(3 + 1)$ dimensional solutions of the $CP^N$ are novel and may be important in some applications.
We consider the $CP^N$ model in $3 + 1$ dimensional Minkowski spacetime defined by the Lagrangian density

$$\mathcal{L} = M^2 (D_\mu \mathcal{Z})^\dagger D^\mu \mathcal{Z}, \quad \mathcal{Z}^\dagger \cdot \mathcal{Z} = 1,$$

(1.1)

where $M^2$ is a constant with the dimension of mass, $\mathcal{Z} = (Z_1, \ldots, Z_{N+1}) \in C^{N+1}$ and it satisfies the constraint $\mathcal{Z}^\dagger \cdot \mathcal{Z} = 1$. The covariant derivative $D_\mu$ acts on $\mathcal{Z}$ according to

$$D_\mu \mathcal{Z} = \partial_\mu \mathcal{Z} - (\mathcal{Z}^\dagger \cdot \partial_\mu \mathcal{Z}) \mathcal{Z}.$$ 

The index $\mu$ runs over the set $\mu = \{0, 1, 2, 3\}$. The Lagrangian (1.1) is invariant under the global transformation $\mathcal{Z} \rightarrow U \mathcal{Z}$, with $U$ being a $(N + 1) \times (N + 1)$ unitary matrix. One of the advantages of the $\mathcal{Z}$ parametrization is that it makes that $U(N+1)$ symmetry explicit \[1, 2\]. However, one can also use the parametrization

$$\mathcal{Z} = \frac{(1, u_1, \ldots, u_N)}{\sqrt{1 + |u_1|^2 + \ldots + |u_N|^2}},$$

(1.2)

which solves the constraint $\mathcal{Z}^\dagger \cdot \mathcal{Z} = 1$. The $u$-field parametrization does not make the $U(N + 1)$ symmetry explicit, however they transform under the $U(N)$ subgroup as $u \rightarrow \tilde{U} u$, with $\tilde{U} \in U(N)$. In terms of $u_i$’s the Lagrangian density (1.1) takes the form

$$\mathcal{L} = \frac{4M^2}{(1 + u^\dagger \cdot u)^2} \left[(1 + u^\dagger \cdot u)\partial^\mu u^\dagger \cdot \partial_\mu u - (\partial^\mu u^\dagger \cdot u)(u^\dagger \cdot \partial_\mu u)\right]$$

(1.3)

and the Euler-Lagrange equations read

$$(1 + u^\dagger \cdot u) \partial^\mu \partial_\mu u_k - 2(u^\dagger \cdot \partial^\mu u) \partial_\mu u_k = 0$$

(1.4)

for each $k = 1, \ldots, N$. The dot product denotes $u^\dagger \cdot u = \sum_{i=1}^{N} u_i^* u_i$. The simplest $CP^1$ case is given just by one function $u$: $\mathcal{Z} = \frac{(1, u)}{\sqrt{1 + |u|^2}}$.

### 2 The solutions

In what follows we shall use the notation

$$z \equiv x^1 + i \varepsilon_1 x^2, \quad \bar{z} \equiv x^1 - i \varepsilon_1 x^2, \quad y_\pm \equiv x^3 \pm \varepsilon_2 x^0$$

(2.5)

with $\varepsilon_a = \pm 1, a = 1, 2$. 

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It is easy to check that any set of functions $u_k$ that depend on coordinates $x^\mu$ in a special way, namely
$$u_k = u_k(z, y_+)$$

is a solution of the system of equations (1.4). The Minkowski metric in the coordinates (2.5) becomes $ds^2 = -dz\,d\bar{z} - dy_+\,dy_-$. It then follows that (2.6) satisfies simultaneously $\partial^\mu \partial_\mu u_i = 0$ and $\partial^\mu u_i \partial_\mu u_j = 0$ for all $i, j = 1, \ldots, N$. Hence this class of solutions is very large.

Amongst the very many solutions of the type (2.6) we have those that describe vortices with waves traveling along them. Such solutions have already been studied in the $CP^1$ case in [3], whereas the generalization to the case $CP^N$ has been considered in [4].

The solutions presented there are also exact solutions of an extension of the Skyrme-Faddeev model for a special choice of their parameters. From the point of view of the present work what matters most is that (2.6) satisfy the equations of the $CP^N$ model in (3+1) dimensions.

In the notation of (2.5) the vortex solution studied in [4] takes the form
$$u_i(z, y_+) = z^{n_i} e^{ik_i y_+},$$

where $k_i$ are arbitrary real numbers and $n_i$ are positive or negative integer numbers. A slightly modified vortex-like solution would be given by
$$u_i(z, y_+) = (z - a_i)^{n_i} e^{ik_i y_+},$$

where $a_i$ are some constant numbers (in general complex) and, in the most general ones, we would replace the expression $(z - a_i)^{n_i}$ by a rational map.

In this paper, for simplicity and the clarity of the interpretation, we shall consider only some very special maps, leaving more complex ones for future work. Thus we shall consider solutions of the form
$$u_i(z, y_+) = (z - \delta)^{n_i} (z + \delta)^{m_i} e^{ik_i y_+},$$  

where $\delta$ is a real constant and $m_i$ are integers.

### 3 The energy density of the solutions

It is easy to check that the energy density of solutions (2.6) takes the form
$$\mathcal{H} = \frac{8M^2}{(1 + u^\dagger \cdot u)^2} \left[ \partial_z u^\dagger \cdot \Delta^2 \cdot \partial_z u + \partial_{y_+} u^\dagger \cdot \Delta^2 \cdot \partial_{y_+} u \right],$$  

(3.8)
where
\[ \Delta_{ij}^2 \equiv (1 + u^\dagger \cdot u)\delta_{ij} - u_i u_j^*. \] (3.9)

The first term in (3.8), when evaluated for the solutions of the type (2.6), is a total derivative, i.e.
\[ \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \ln \left( 1 + u^\dagger \cdot u \right) = \frac{\partial \Delta \cdot \partial z u}{(1 + u^\dagger \cdot u)^2}. \] (3.10)

When integrated over the plane \( x^1 x^2 \) the first term in (3.8) becomes proportional to the topological charge of the vortex solution as we explain below.

The second term in (3.8) is related, for some special solutions of the type (2.6), to some Noether charges of the \( CP^N \) model. To see this we note that the \( CP^N \) Lagrangian (1.1) is invariant under global \( U(N + 1) \) transformations. However, the parametrization of the fields in terms of the \( u \) fields given by (1.2), gauge fixes this symmetry to \( SU(N) \otimes U(1) \). We are interested in the subgroup \( U(1)^N \) of \( SU(N) \otimes U(1) \) under which the fields \( u \) transform as
\[ u_i \to e^{i \alpha_i} u_i \quad i = 1, 2, \ldots N. \]

The Noether currents associated with these symmetries are given by
\[ J^{(i)}_\mu = -\frac{4 i M^2}{\vartheta^4} \sum_{j=1}^N \left[ u_i^* \left( \Delta^2 \right)_{ij} \partial_\mu u_j - \partial_\mu u_j^* \left( \Delta^2 \right)_{ji} u_i \right]. \] (3.11)

If one now considers solutions of the class (2.6) of the form
\[ u_i = v_i(z) e^{i k_i y^+} \] (3.12)

with \( k_i \) being the inverse of a wavelength, then one can show that
\[ \frac{8 M^2}{(1 + u^\dagger \cdot u)^2} \partial_{y^+} u^\dagger \cdot \Delta^2 \cdot \partial_{y^+} u = \varepsilon_2 \sum_{i=1}^N k_i J_0^{(i)}, \] (3.13)

where \( \varepsilon_2 \) was introduced in (2.5).

Therefore, the energy per unit of length of our vortices solutions of the type (3.12) has the form
\[ \mathcal{E} = \int dx^2 \mathcal{H} = 8 \pi M^2 Q_{\text{Top}} + \varepsilon_2 \sum_{i=1}^N k_i Q^{(i)}_{\text{Noether}}, \] (3.14)
where $Q_{\text{Top.}}$ is the topological charge and

$$Q_{\text{Noether}}^{(i)} \equiv \int dx^1 dx^2 J_0^{(i)}.$$  

(3.15)

To discuss the topological properties of (3.14) we split it into two parts; $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ corresponding to its two terms. For a solution of the type (2.7) the first term reduces to

$$\mathcal{H}^{(1)} = \frac{8M^2}{(1 + \bar{u} \cdot u)^2} \left[ \sum_i \psi_{ii} |u_i|^2 + \sum_{i,j} (\psi_{ii} - \psi_{ij}) |u_i| |u_j|^2 \right]$$  

(3.16)

where, if we define $w_\delta \equiv z - \delta$ and $w_{-\delta} \equiv z + \delta$, the functions $\psi_{ij}$ are given by

$$\psi_{ij}(z, \bar{z}) \equiv \frac{n_i n_j}{|w_\delta|^2} + \frac{m_i m_j}{|w_{-\delta}|^2} + (n_i m_j + m_i n_j) \frac{\bar{\delta} w_{-\delta} + w_\delta \bar{w}_{-\delta}}{2 |w_\delta|^2 |w_{-\delta}|^2}.$$  

(3.17)

The second contribution $\mathcal{H}^{(2)}$ is given by

$$\mathcal{H}^{(2)} = \frac{8M^2}{(1 + \bar{u} \cdot u)^2} \left[ \sum_i k_i^2 |u_i|^2 + \sum_{i<j} (k_i - k_j)^2 |u_i| |u_j|^2 \right]$$  

(3.18)

and the total energy density is given by $\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$.

In the $CP^1$ case i.e. for

$$u(z, y_+) = (z - \delta)^m (z + \delta)^n e^{iky_+}$$  

(3.19)

the energy density is just

$$\mathcal{H}^{(1)} + \mathcal{H}^{(2)} = 8M^2 \left[ \psi(x^1, x^2) + k^2 \right] \frac{|u|^2}{(1 + |u|^2)^2}$$

with

$$\psi(x^1, x^2) \equiv \frac{n^2}{|w_\delta|^2} + \frac{m^2}{|w_{-\delta}|^2} + nm \frac{\bar{\delta} w_{-\delta} + w_\delta \bar{w}_{-\delta}}{|w_\delta|^2 |w_{-\delta}|^2}.$$  

Two examples of integrands $\frac{|u|^2 |\psi(x^1, x^2)|^2}{(1 + |u|^2)^2}$ and $\frac{|u|^2}{(1 + |u|^2)^2}$ are sketched in Figs. 1 and 2. They correspond to the cases of $n = 3$ and $m = 2$ (Fig. 1) and $m = -1$ (Fig. 2).

If we interpret the solution $u(z, y_+)$ as describing a vortex we see that $\mathcal{H}$ describes its energy density per unit length of the vortex. Then this energy per unit...
length is given by a two dimensional integral (with integration over the coordinates $x^1$ and $x^2$) and is given by

$$E = 8\pi M^2 \left[ K_{(n,m)} + k^2 I_{(n,m)} \right], \quad (3.20)$$

where

$$K_{(n,m)} \equiv \frac{1}{\pi} \int_{R^2} dx^1 dx^2 \psi(x^1, x^2) \frac{|u|^2}{(1 + |u|^2)^2} \quad (3.21)$$

and

$$I_{(n,m)} \equiv \frac{1}{\pi} \int_{R^2} dx^1 dx^2 \frac{|u|^2}{(1 + |u|^2)^2}. \quad (3.22)$$

So, according to (3.14), $K_{(n,m)}$ is related to the topological charge and $I_{(n,m)}$ to the Noether charge.

Note that in this vortex interpretation $K_{(n,m)}$ is the total energy of the solution (3.19) with $k = 0$, which is then also a solution of the (2 + 0) dimensional $CP^1$ model. This solution is purely topological and so the integrand $H^{(1)}$ is a total derivative, as is clear from (3.10), and moreover, is also the density of the topological charge. Hence, this integrand tells us where the vortices are positioned (thus for the configuration in Fig. 1 one vortex is located at $(-1.3, 0)$ and two vortices are on top of each other at $(1.3, 0)$). Similarly, the vortices in Fig. 2 are located around $(5.0, 0)$).

The integral $K_{(n,m)}$ describes the total (2+0) dimensional energy of these topological solitons (which is also proportional to their topological charge) and so is independent of the parameter $\delta$ which characterizes the “distance” between the two vortices. As is well known, and it is easy to check, $K_{(n,m)}(\delta)$ depends on $n$ and $m$ in the following way:

1. when $nm > 0$ then $K_{(n,m)}(\delta) = |n + m|$ for any $\delta$,
2. when $nm < 0$ then $K_{(n,m)}(\delta \neq 0) = \max(|n|, |m|)$, otherwise one gets $K_{(n,m)}(\delta = 0) = |n + m|$.

In Fig. 3 we present similar densities for a field configuration (also a solution of the equations of motion) of the form

$$u(z, y_+) = e^{iky_+} \frac{z(z^2 - a^2)}{z - b} \quad (3.23)$$
for \( a = (2.2, 0.0) \) and \( b = (1.0, 0.0) \). We see clearly 3 peaks in each picture, although each peak on the right of the picture corresponds to a peak with a ‘crater’ at the centre of it. In Fig. 4 we present magnified pictures of the most to the right peaks of Fig. 3. This shows this ‘crater’ very clearly. As the first term describes the topological charge density we note that the energy density of the topological charge of each vortex has a maximum at the position of each vortex and then decreases as you move away from this position. The density of the other charge (or charges for \( CP^N \) with \( N > 1 \)), which are described by the other term is also located roughly at the positions of the vortices. However, the crater like-structure, shows that at the exact positions of the vortices it is zero, then increases to a maximum in a small circle around the exact position of the vortex and finally falls to zero as we move away from the vortices.

All this can be explained by the form of the functions in (3.21) and (3.22), especially in the limit of well separated vortices. In this case the positions are given by the zeros of \( u \). As \( \psi|u|^2 \) goes to a constant when \( u \to 0 \) the positions correspond to the maxima of the topological charge density (see (3.21)). For the Noether charge densities the situation is different. The last term of the expression given in (3.22) does not have the \( \psi \) factor and hence the integrand of (3.22) vanishes at this point. The maximum is nearby and, when \( u \to \infty \), the integrand, again, goes to zero. Thus the two charges are located roughly at the same places but their charge distributions are slightly different; for the topological charge we have a simple peak while for the Noether charge a ‘crater’ like structure.

As we have already said before, taking into consideration only the first term \( K_{(n,m)} \) one can already see that the total energy of our solutions is infinite (as the integrand does not depend on \( x^3 \) we clearly have a linear divergence). This is, of course, consistent with our interpretation of the solutions are describing vortices, i.e. structures whose energy is infinite but the energy density per unit length is finite.

However, the contribution to the total energy that comes from the second term, proportional to \( k^2 \), is also infinite. Bearing in mind our vortex interpretation of the solution the behaviour of \( I_{(n,m)} \) is worth studying in more detail. First, we observe that even the integral \( I_{(n,m)} \) can be divergent; in fact, it is easy to check that \( I_{(n,m)} \) diverges when the pairs \( (n, m) \) satisfy the inequality \( |n + m| < 2 \). Hence, the vortex-like interpretation has to be reconsidered in a new light. Some vortex-like configurations appear to have infinite energy per unit length. However, looking at this problem in more detail we note something even more interesting - the integral \( I_{(n,m)}(\delta) \) can depend on \( \delta \) in a nontrivial way.

To discuss this we consider the pairs \( (n, m) \) which satisfy \( |n + m| \geq 2 \) with
both \( n \) and \( m \neq 0 \). Note that our solutions possess two qualitatively different asymptotic behaviours of \( I_{(n,m)}(\delta) \) for large \( \delta \). Namely, when \( nm > 0 \) the integral tends to zero as \( \delta \to \infty \), (see Fig. 5), and when \( nm < 0 \), in the same limit, it diverges (see Fig. 6). Moreover, in the last case the integral \( I_{(n,m)}(\delta) \) has a local minimum for some finite \( \delta \).

The \( CP^2 \) case is qualitatively very similar to the \( CP^1 \) one. As an illustration in Figs. 7 and 8 we present their plots of the energy per unit length for the \( CP^2 \) solutions.

At first sight this looks like a paradox, since one would naively expect that the dependence of the energy on the distance \( \delta \) would lead to a force between the vortices. Therefore, only the configurations corresponding to the minimum of the energy would be expected to correspond to true solutions of the equations of motion. The resolution of this ‘paradox’ is based on the observation that the total energy of all these solutions is infinite. The solutions of the equations of motion minimize the total energy but here the total energy is infinite. It does not matter that the value of the energy per unit length depends on \( \delta \) and is finite - the total energy is still infinite. Hence we are in the situation in which the vortices can have different energies per unit length and still be at rest with respect to each other. Another way of phrasing the resolution of the paradox is that even though there is a finite force between the vortices (due to the dependence of the energy per unit of length on the distance \( \delta \)) their masses are infinite and so there is no acceleration.

The dependence of the energy per unit of length on the distance \( \delta \) raises some possibilities for further investigations. Notice that \( \delta \) is a distance in the \( x^1 \times x^2 \) plane, and the dependence of the energy on \( \delta \) occurs only due to the term depending on the two other dimensions, namely \( x^3 \) and \( x^0 \). Suppose now that one considers the same model but in a four dimensional space that contains the \((2+1)\) Minkowski space as a border (for instance AdS\(_4\)). It would be interesting to verify if the same effect takes place here too. If it does, then one could interpret the solution as \( CP^N \) lumps on a \((2+1)\) Minkowski space (the border) which acquire an interaction due to the part of the Hamiltonian which lives in the bulk of the four dimensional space. That would be a novel feature which certainly could have many interesting applications.
Figure 1: The energy density contributions $|u|^2 \psi(x^1,x^2)$ (on left) and $\frac{|u|^2}{(1+|u|^2)^2}$ (on right) for the case $nm > 0$. Here $n = 3$ and $m = 2$. The separation parameter $\delta = 1.3$.

Figure 2: The energy density contributions $|u|^2 \psi(x^1,x^2)$ (on left) and $\frac{|u|^2}{(1+|u|^2)^2}$ (on right) for the case $nm < 0$. Here $n = 3$ and $m = -1$. The separation parameter $\delta = 5.0$. 
Figure 3: The expressions $|u|^2\psi(z^2, x^2)$ (on left) and $\frac{|u|^2}{(1+|u|^2)^2}$ (on right) for the solution $u(z, y) = z(z^2 - a^2) e^{iky} + z$ with $a = 2.2$ and $b = 1.0$.

Figure 4: A magnification of some region of Fig. 3 where the most to right peak is located. The function $\frac{|u|^2\psi(z^2, x^2)}{(1+|u|^2)^2}$ (on left) has a local maximum at the point (2.2, 0) whereas the function $\frac{|u|^2}{(1+|u|^2)^2}$ (on right) takes value zero which is simultaneously its local minimum.
Figure 5: The integrals (from the top) $I_{(4,1)}(\delta)$, $I_{(4,2)}(\delta)$, $I_{(4,3)}(\delta)$, $I_{(4,4)}(\delta)$.

Figure 6: The integrals (from the bottom) $I_{(6,-1)}(\delta)$, $I_{(6,-2)}(\delta)$, $I_{(6,-3)}(\delta)$, $I_{(6,-4)}(\delta)$. 
Figure 7: The integral $I(\delta) \equiv \frac{1}{8\pi M^2} \int_{R^2} dx^1 dx^2 \mathcal{H}^{(2)}$ for the $CP^2$ solution $u_1(z, y_+) = (z - \delta)^{-1}(z + \delta)^{-1}e^{-iy_+}$ and $u_2(z, y_+) = (z - \delta)^2(z + \delta)^4e^{iy_+}$.

Figure 8: The integral $I(\delta) \equiv \frac{1}{8\pi M^2} \int_{R^2} dx^1 dx^2 \mathcal{H}^{(2)}$ for the $CP^2$ solution $u_1(z, y_+) = (z - \delta)^{-1}(z + \delta)e^{-iy_+}$ and $u_2(z, y_+) = (z - \delta)^{-2}(z + \delta)^4e^{iy_+}$. 
4 Some Conclusions

In this paper we have demonstrated that the $CP^N$ model in (3+1) dimensions has many classical solutions. The ones we have discussed here correspond to field configurations described by functions of two variables ($x^1 + i\epsilon_1 x^2$ and $x^3 + \epsilon_2 x_0$). As the energy of these configurations is infinite (as the energy density is independent of $x^3$) these solutions describe vortices with the time dependence corresponding to the rotation of the vortices in the internal space (in this property they resemble a little ‘Q-lumps’[5].) We have shown that they possess many interesting properties (like interesting distributions of topological charge and of the Noether charge). One of the more unusual properties is their dependence on the distance between the lines of charge: the energy density (of energy per unit length) of one vortex is infinite but of two vortices can depend on the distance between them and possess a minimum at a specific value of this distance. This suggests that vortices which are located at non-minimal distances may be unstable and so could reduce their energy per unit length by moving towards this optimal configurations. However, their configurations are solutions for any distance as their infinite ‘inertia’ stops them from moving towards them without an external push.

We are now looking at other properties of these and other solutions.

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