Precision Speckle Pattern Reconstruction for High-contrast Imaging

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Abstract

In high-contrast imaging, a large instrumental, technological, and algorithmic effort is made to reduce residual speckle noise and improve the detection capabilities. In this work, we explore the potential of using a precise physical description of speckle images, in conjunction with the optimal detection statistic to perform high-contrast imaging. Our method uses short-exposure speckle images, reconstructing the point-spread function (PSF) of each image with phase retrieval algorithms. Using the reconstructed PSFs, we calculate the optimal detection statistic for all images. We analyze the arising bias due to the use of a reconstructed PSF and correct for it completely up to its accumulation over $10^4$ images. We measure in simulations the method’s sensitivity loss due to overfitting in the reconstruction process and get to an estimated $5\sigma$ detection limit of $5 \times 10^{-7}$ flux ratio at angular separations of $0^\prime\!1-0^\prime\!5$ for a $1h$ observation of Sirius A with a 2 m telescope.

1. Introduction

High-contrast imaging (HCI) is a family of techniques to image two objects that are greatly different in their brightness—for example, detecting exoplanets by directly seeing them as additional sources near their host star—is an important method for current and future characterization of exoplanets atmospheres (Currie et al. 2011; Konopacky et al. 2013) as it measures light from the exoplanet itself.

HCI requires separating the faint exoplanet from its bright nearby host star, which is highly dependent on the telescope’s angular resolution. For ground-based observatories, the atmosphere presents an additional constraint on the telescope’s angular resolution. The turbulent flow of air in the atmosphere creates an index of refraction variations, which create rapidly changing phase aberrations and amplitude variations; the latter are neglected in this work. The phase aberrations degrade the telescope’s angular resolution, creating speckle images for short-exposure imaging and a broad seeing disk image for long exposure.

A way to overcome atmospheric seeing is the use of adaptive optics (AO) systems and coronagraph. AO systems sense and correct, in real-time, the atmospheric phase aberrations, and coronagraph block the light from the host star, by that suppressing the diffraction features that extend to the limit of the exoplanet. Such systems bring the angular resolution from the seeing limit close to the telescope’s diffraction scale and are used by the leading HCI instruments on state-of-the-art telescopes (VLT-SPHERE; Beuzit et al. 2019; SCExAO; Lozi et al. 2018; Gemini-GPI; Nielsen et al. 2019; MagAO-X; Males et al. 2018).

The biggest challenge for those AO-based HCI instruments are Non-Common Path Aberrations (NCPA), which are aberrations in the optical train that are not sensed by the WaveFront Sensor (WFS), and, therefore, not corrected (Mawet et al. 2012). Those aberrations create quasi-static speckles, slowly varying speckles that are confused as point sources.

A wide range of efforts are made to tackle this challenge, better wave front sensing (e.g., Baudoz et al. 2005; Skaf et al. 2021, 2022), deformable mirror technological improvements (e.g., Madec 2012), nulling coronagraphy (e.g., Ruane et al. 2018), and postprocessing methods (e.g., Racine et al. 1999; Marois et al. 2006; Lafrenière et al. 2007).

An additional type of method uses millisecond images rather than the widely used long exposure, such as Frazin & Rodack (2021) and Rodack et al. (2021), that use AO-corrected millisecond images with WFS telemetry to estimate the planet image and the NCPA in conjunction. Alternatively, Primot et al. (1990) reconstruct the speckle image point-spread function (PSF) based on WFS telemetry and use it for deconvolution. In Mugnier et al. (2001), they simultaneously reconstruct the phase aberrations and the object intensity map for speckle images, but focus on sharpening images rather than measuring contrasts.

In this work, we limit the analysis to detection of a secondary source that allows for a more quantitative discussion, and higher-order treatment of the bias correction problem similar to the one in Frazin & Rodack (2021) and Rodack et al. (2021). It is important to emphasize that detection of a secondary source is not deconvolution, and does not suffer from division by small, noisy, Fourier components (more discussion can be found in Zackay et al. 2016).

The problem of phase retrieval, which is often tackled when performing PSF reconstruction, asks to find the phase (atmospheric aberrations) from its Fourier modulus (image). The problem is known to be hard and was the target for many studies and algorithms aimed to solve it (for example, Fienup 1999, 2013; Shechtman et al. 2015). In this work we used a combination of popular algorithms and a direct measurement approach, with further discussion in 4.1.

Analyzing the idealized case of HCI with perfectly known—yet uncorrected—PSF, we obtain that even a modestly sized telescope in the seeing limited case can reach a fantastic contrast, comparable to the best performance reported by state-
of-the-art facilities, as can be seen in the dashed lines in Figure 1.

Motivated by this computation, in this work, we propose a method to perform HCI using short-exposure speckle images. In our proposed method, which is illustrated in Figure 2, we reconstruct the phase aberrations to fit the speckle image (Section 4.1), using a WFS only as an initial guess to a sequence of optimization algorithms.

Using the reconstructed phase aberrations, we calculate the optimal detection statistic (Section 2.3). We present a novel method to correct for the arising bias due to the use of estimated phase aberrations using a Parametric Bootstrap method (Section 4.2). Analysis of the expected performance is presented in Sections 2.3 and 4.3, and the results presented in Figure 1.

2. High-contrast Imaging through the Atmosphere

2.1. Atmospheric Seeing

The PSF is a characteristic of an optical system and describes how a point source would look when imaged using it. Atmospheric turbulence creates variations in the index of refraction, which, when the light passes through, create phase aberrations. The resulting PSF of the system can be calculated for monochromatic illumination as the Fourier transform of the aperture function \( B \) and the phase aberration \( \phi \)

\[
P = |\mathcal{F}[B e^{i\phi}]|^2.
\]

In the case of perfect imaging from a circular aperture, the PSF will take the shape of an Airy disk with an angular scale of the
diffraction limit \( \lambda/D \). The phase aberrations caused by the atmosphere degrade the PSF significantly and are a topic of many theoretical and observational studies. As a model for the atmospheric phase aberrations, we will use the classical Kolmogorov phase aberration structure function \( D_\alpha \) with a typical scale \( r_0 \) called the Fried parameter Roddier (1981)

\[
D_\alpha(\Delta x) := \langle (\phi(r) - \phi(r + \Delta x))^2 \rangle = 6.88 \left( \frac{\Delta x}{r_0} \right)^{5/3} \tag{2}
\]

The Fried parameter can also be understood as the size of the telescope that transitions from being diffraction limited to seeing limited.

To model imaging in a finite wavelength band, we integrate the single wavelength PSFs incoherently and include the two leading effects that change the PSF as a function of the wavelength:

1. The diffraction scale increases \( \lambda_0/D \rightarrow \lambda/D \).
2. The aberration in phase units decreases \( \phi \rightarrow \phi\lambda_0/\lambda \).

\[
P = \int_{\lambda_0}^{\lambda_1} E_\lambda \left| S_{\lambda/\lambda_0} \mathcal{F} \left[ B e^{i\phi} \right] \right|^2 d\lambda,
\]

where \( E_\lambda \) is a term representing the spectrum of the source, and \( S_{\lambda/\lambda_0} \) an operator that stretches the coordinates as the diffraction scale increases.

2.2. Statistical Model

To push the limits of direct imaging we have to stay as close as possible to optimality, therefore we will start by mathematically formulating the high-contrast-imaging detection problem rigorously.

Directly imaging exoplanets can be formulated as a hypothesis testing question in which we want to distinguish between the null hypothesis of the images made of a single star with some flux \( f \)

\[
H_0: \ T = f \delta(0), \tag{4}
\]
and the alternative hypothesis of an additional source with flux fraction $\epsilon$ at relative position $\mathbf{q}$

$$H_1: \, T = f \left( (1 - \epsilon) \delta_{\mathbf{0}} + \epsilon \delta_{\mathbf{q}} \right).$$

(5)

The mapping between the true sky $T$ and the measured images is by the PSF, with additive noise we will assume has a shot-noise component related to the source, and detector read noise, taking the Gaussian approximation of the Poisson distribution (justified as we have a high number of photons per pixel)

$$I \sim \mathcal{N}(P \otimes T, P \otimes T + b^2),$$

(6)

where $\mathcal{N}$ is the normal distribution, $b$ is the noise from sky background and read noise, and $\otimes$ is the convolution operator.

2.3. Detection Statistic

To distinguish between the two hypotheses we will use the following statistic (with further discussion in Appendix A)

$$S(\mathbf{q}) = \left( \widetilde{P}^2 \otimes \frac{I - fP}{fP + b^2} \right)_{\mathbf{q}} - \left( \frac{P}{fP + b^2} \right)_{\mathbf{0}} \hat{\phi} = \arg\max_{\phi} \{ P(f|H_0, P_\phi)P(\phi) \}$$

(7)

where $\widetilde{P}(x) = P(-x)$ is the reverse of $P$, and $\hat{\phi}$ is the maximum-a-posteriori (MAP) estimator of the atmospheric phase aberrations $\phi$. We can read this statistic as a filter matching the image’s deviation from a point source, and subtracting the location of the primary source as the unknown flux would create a flux deficit at its location. We can calculate the mean of the statistic under $H_1$

$$E[S(\mathbf{q})|H_1(\epsilon, \mathbf{q})] = \epsilon f^2 \left( \frac{\widetilde{P}^2 \otimes \frac{1}{fP + b^2}}{fP + b^2} \right)_{\mathbf{0}} \equiv V_\epsilon(\mathbf{q})$$

(8)

and the it is variance associated with the image’s variance is

$$V_\epsilon(\mathbf{q}) = f^2 \left( \frac{\widetilde{P}^2 \otimes \frac{1}{fP + b^2}}{fP + b^2} \right)_{\mathbf{0}} \equiv V_\epsilon(\mathbf{q})$$

(9)

We can estimate the minimal contrast $\epsilon$, which achieves $5 \sigma$ detection using this statistic for the ideal case in which we know $\phi$ exactly

$$\min \epsilon: \, 5 \leq \frac{E[S(\mathbf{q})|H_1(\epsilon, \mathbf{q})]}{\sqrt{V_\epsilon(\mathbf{q})}} = \epsilon \sqrt{V_\epsilon(\mathbf{q})}.$$

(10)

by simulating atmospheric phase screens from the power spectrum calculated by Noll (1976), the resulting contrast thresholds as a function of separation for different bandwidths are shown in the colored dashed lines in Figure 1.

| Parameter                                    | Value                  |
|----------------------------------------------|------------------------|
| Pixel grid                                   | $256 \times 256$       |
| Aperture diameter (physical)                 | $2 \text{ m}$          |
| Aperture diameter (pixels)                   | $852 \pm \text{ px}$   |
| Nyquist oversampling ratio                   | 1.5                    |
| Fried parameter                              | 15 cm                  |
| Flux (per image, finite bandwidth)           | $4 \times 10^7 \text{ @ 500–700 nm}$ |
| Exposure time (per image)                    | 4 ms                   |
| Background/read noise                        | 1 photon per pixel     |

3. Simulations

To test our method we used end-to-end numerical simulations of the proposed measurement instruments.

We generate atmospheric aberrations by sampling the Fourier modes with amplitudes corresponding to the Kolmogorov spectrum as derived by Noll (1976), and select a fraction of the phase screen, as done by McGlamery (1976).\footnote{We simulate a screen three times bigger than the aperture, while McGlamery (1976) suggest a factor of four. This does not change the spectrum much, and we use a corresponding prior.} We take the frozen flow approximation for the temporal evolution, using the same big phase aberration screen and moving the aperture between different random locations shifting the aperture by 1 pixel in the $x$ and $y$ directions between each two consecutive images (corresponds to $v \sim 8 \text{ m s}^{-1}$). Scintillating is neglected in this work. Images are calculated with the assumed true sky image and PSF as calculated in Equation (3), and then sampled according to Poisson with additional Gaussian noise. The simulation of the wavefront sensor neglects chromatic behavior, as only the centroid position will be used and is synchronized with the imaging.

Reference values for simulation parameters are listed in Table 1.

4. Methods

In order to use this statistic, we first need to show a method to calculate $\hat{\phi}$, and even though we know the statistic’s distribution given the correct $\phi$, we need to examine carefully its distribution when using $\hat{\phi}$.

4.1. Recovering the Atmospheric Phase Aberrations

In order to calculate the statistic from Equation (7) we have to find $\arg\max \{ P(f|H_0, P_\phi)P(\phi) \}$, the MAP estimator for the atmospheric phase aberrations.

The general problem of phase retrieval, recovering a phase from its Fourier modulus in the presence of noise is known to be hard and a topic of many studies (for example, Fienup 1999, 2013; Shechtman et al. 2015). Therefore we employ a simple yet powerful sequence of algorithms. Starting from direct measurement of the phase using a Shack–Hartmann WaveFront Sensor (SHWFS; Platt & Shack 2001), then improving that estimator using the Gerchberg–Saxton (Gerchberg & Saxton 1972) algorithm, and finally using Gradient Descent to ensure we get the MAP.

This procedure converges to the correct MAP for most instances in the case of enough photons ($\sim 10^9$ per image) and a reasonable atmosphere ($r_0 = 15 \text{ cm@}\!D = 2 \text{ m}$), but only for

Table 1

Reference Parameters for Simulations
imaging in a single wavelength. Further discussion and elaborated results are presented in Appendix B.

4.2. Detection in the Presence of Learned PSF - $H_0$

When applying the statistic from Equation (7) on estimated PSF we have to deal with the effect of its inevitable errors. From simulations, we learn, as expected, that our phase estimator can be modeled as distributing normally with some bias $\mu$ and covariance $\Sigma$ around the correct atmospheric phase aberration

$$\hat{\phi} \sim N(\phi + \mu, \Sigma).$$

From Equations (3) to (7) we can see that our statistic is a nonlinear function of the phase estimator. Therefore, our estimator for the statistic, calculated using our estimator of the phase, is slightly biased

$$\Delta S_{\phi} \equiv E[S_{\phi} - S_{\phi}] = 0,$$

as can be seen in Figure 3(a). But as it is a bias shared by all images, when we accumulate the statistic over many images it accumulates too and put an upper limit on the number of images we can use once it becomes significant.

As the notation suggests, this bias depends on $\phi$, and for simplicity, we will assume it varies slowly relative to our phase estimation error and therefore can be expanded in a series

$$\Delta S_{\phi}[q] = \Delta S_\phi[q] + \frac{d\Delta S_\phi[q]}{d\phi}\Delta \phi + \frac{1}{2}\Delta \phi^2 \frac{d^2\Delta S_\phi[q]}{d\phi^2}\Delta \phi,$$

and its expectation value

$$E_{\hat{\phi}}[\Delta S_{\phi}[q]] = \Delta S_\phi[q] + \frac{d\Delta S_\phi[q]}{d\phi}\mu + \frac{1}{2}\mu^2 \frac{d^2\Delta S_\phi[q]}{d\phi^2}\mu + \frac{1}{2}\mu^2 \frac{d^2\Delta S_\phi[q]}{d\phi^2}\Sigma.$$  

We extend the method of parametric bootstrap (Dekking 2005) to estimate this bias. By sampling images $I_i' \sim N(P(\hat{\phi}) \otimes T_{H_0}, P(\hat{\phi}) \otimes T_{H_0} + b^2)$ and solving their corresponding phases based on the recovered phase estimator, we can sample the distribution in Equation (11) and calculate the following linear combination that has the expectation value as the bias we want to correct (detailed calculation in Appendix C)

$$E_{\hat{\phi}}[\Delta S_{\phi}[q]] = \Delta S_\phi[q] + \frac{d\Delta S_\phi[q]}{d\phi}\mu + \frac{1}{2}\mu^2 \frac{d^2\Delta S_\phi[q]}{d\phi^2}\mu + \frac{1}{2}\mu^2 \frac{d^2\Delta S_\phi[q]}{d\phi^2}\Sigma,$$

and calculate the following linear combination that has the expectation value as the bias we want to correct (detailed calculation in Appendix C)

$$E_{\hat{\phi}}[\Delta S_{\phi}[q] + \Delta S_{\phi}[\hat{\phi} + \phi_{1}], \Delta S_{\phi}[\hat{\phi} + \phi_{2}], \Delta S_{\phi}[\hat{\phi} + \phi_{3}]] = \Delta S_\phi,$$

with its performance demonstrated in Figure 3(b). In practice, we calculate this term a few times to get better convergence to the mean and have a handle on its variance, which is empirically smaller than the inherent variance of the statistic, calculated in Equation (9), and can be reduced as $1/\sqrt{N}$ with more simulations.

It is important to note that this bias correction scheme requires a good knowledge of the optical system and atmosphere statistics. Using an incorrect prior would result in worse bias correction.

4.3. Signal Loss Due to Overfitting - $H_1$

After we have made sure that we are recovering null detections for images sampled from $H_0$, we need to test the expected performance of the method for signals from $H_1$.

An inevitable tension is present in our phase-aberration estimation process. The posterior probability of the phase aberration, $\phi$, is dominated by the deviation of its corresponding PSF, $P(\phi)$, from the image. In the presence of an additional source, this will lead to overfitting of the secondary as part of the primary. Detecting using an overfitted $\hat{\phi}$ subtracts part of the secondary and therefore loses some of the signal available with perfect knowledge of $\phi$. To quantify this effect, we perform injection-recovery simulations for different separations and different brightnesses.
In general, we expect the recovered signal-to-noise ratio \( \text{S/N} \) to be some function of the injected \( \text{S/N} \) and the separation from the star at which the signal is injected. In the weak signal limit, at which this method is designed to work, the \( \text{S/N} \) per image is small \( \text{S/N}_{\text{inj}} \ll 1 \), and we expect the recovered \( \text{S/N} \) to be proportional to the injected \( \text{S/N} \). In Figure 4(a) we can see the recovery ratio plateau as we go to lower \( \text{S/N} \), and we assume this plateau can be extrapolated to a much smaller injected \( \text{S/N} \). We also expect the dependence on the location to be some smooth function that takes into account the assumed phase-aberrations spectrum and the degeneracy between the phase aberrations and the additional source, as can be seen in Figure 4(b).

\[
\text{SNR}_{\text{rec}} = \alpha(|q|)\text{SNR}_{\text{inj}} \quad (17)
\]

We use the values of \( \alpha(|q|) \) from Figure 4 to convert the optimal dotted lines in Figure 1 to the more realistic solid lines. In the future, we will investigate using several wavelength bands in tandem to mitigate this loss, using the a-chromatic position of the secondary and the chromatic effect of the phase aberrations.

5. Conclusion and Outlook

We presented a method to reconstruct the atmospheric phase aberrations using phase retrieval algorithms, and use its corresponding PSFs to calculate the optimal statistic to detect the presence of a secondary faint source in a sequence of short-exposure speckle images. We presented a method to correct the arising bias in our statistic, showing its complete correction for stacks of up to \( 10^4 \) images, and a pathway to increase it by expanding Equation (13) beyond second order, which is crucial for the use of the detection method.

The leading idea for our method is to optimize the PSF of the image through parameters of a physically constrained model, use the optimal statistic, and correct for its arising bias or other artifacts. The analysis presented in this work treats the case of imaging with no AO, but the method is not limited to such observations. A probable avenue for improvement is to use AO to some extent, which will reduce the photon noise of the host star, and concentrate the light of the planet to a diffraction-limited spot, thus enabling even better detection of the secondary source.

We also note that since the estimation of the PSF and the detection are performed on the same observation, our method does not suffer from the limitations of NCPAs,\(^3\) if they are not bigger than the WFS error and prevent the phase reconstruction from converging.

In this work, we analyzed a simplistic model for an imaging system and frozen, independent phase screens, and one of the brightest stars in the sky. Currently, the limiting part in our method is the atmospheric phase recovery, which requires a lot of information, and, therefore, a lot of photons. Some changes that might allow for use on fainter stars are:

1. Using a redder band and a site with better seeing would decrease the information in the atmospheric phase aberrations that is needed to recover.
2. Generalizing the method for simultaneous imaging in a few bands will allow for the collection of more photons and create “natural” phase diversity.
3. Utilizing the correlation between phase screens in the frozen wind model would allow for longer effective integration.

We therefore expect that with these improvements, high-contrast imaging of nearby stars will be possible with this method.

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\(^3\) Though we might need to track the slow changes in the optical system as an amplitude bias.
**Appendix A**  
**Approximated Statistic**

As shown by Neyman & Pearson (1933), when testing hypotheses the likelihood-ratio test achieves maximal detection power for a given false-positive probability, therefore we will employ it for our detection problem

\[
\Lambda(I) = \frac{P(l|H_1)}{P(l|H_0)},
\]

which we can marginalize over the atmospheric aberrations

\[
\Lambda(I) = \frac{\int P(l|H_1, \phi) d\phi}{\int P(l|H_0, \phi) d\phi}.
\]

Calculating explicitly both marginalized likelihoods is impractical, so we employ the Gaussian approximation. At the limit of well-measured phase aberrations, in which we are interested, the posterior distribution of the phase aberrations is well approximated as a Gaussian.

Under the Gaussian approximation, the integrals reduce to the value at maximum times the square root of its covariance determinant. We further approximate the MAP estimate under \( H_1 \) as the MAP estimate under \( H_0 \). This approximation is justified as in a single image we cannot favor \( H_1 \), so, specifically, we cannot detect a difference in the MAP.

The covariance also does not contribute, as it presents a logarithmic correction, while being nearly identical for the null and alternative hypotheses.

\[
\Lambda(I) \approx \frac{P(l|H_1, \hat{\phi})}{P(l|H_0, \hat{\phi})};
\]

\[
\hat{\phi} = \text{argmax} \{ P(l|H_0, \phi) P(\phi) \}. \quad \text{(A3)}
\]

For the case of a companion detection with a known PSF, as we assume here, the likelihood-ratio test can be simplified to the expression in Equation (7), as calculated by Nir et al. (2019).

**Appendix B**  
**Phase Recovery Procedure**

**B.1. Shack–Hartmann Wave-front Sensor**

The SHWFS is a device that images small areas of the aperture separately. For sufficiently small areas (of the order of \( r_0 \)) the atmospheric phase aberrations can be estimated as simply linear. When imaging a wave front with a constant slope, the image is an off-centered point, with the tip-tilt of the point related linearly to the slope of the wave front.

In our simulation, we used an array of 26 × 26 sub-apertures, which, for our reference seeing conditions and telescope size, resulted in each subaperture being of the size \( \sim r_0/2 \times r_0/2 \). To recover the phase aberrations from the WFS, we used a maximum-a-posteriori estimator in the Gaussian regime and a measured design matrix, as done by Clare (2004).

The phase aberrations recovered by the WFS have significant errors—with the PSF calculated using them having order-unity deviations from the input PSF, as can be seen in Figure 5—therefore cannot be used to calculate the statistic. To improve the phase estimator we use further optimization techniques.

**B.2. Gerchberg–Saxton Algorithm**

The Gerchberg–Saxton algorithm is an algorithm that was designed and proved to solve the phase retrieval problem under the \( l_2 \) norm by Gerchberg & Saxton (1972). We started the algorithm with the initial guess of the pupil function and the phase measured by the WFS. From that, the algorithm transforms the guessed field to the image plane, constraining the amplitude by the measured image, then returning to the pupil plane constraining the pupil function again, and repeating until convergence or some maximal number of iterations.

To quantify the convergence of the algorithm, we compare an observable, likelihood-based criteria

\[
\sum \frac{(I - fP)^2}{fP + b^2} \lesssim N_{\text{pixel}}, \quad \text{(B4)}
\]

and an unobservable, phase-error criteria. We simulated images in the single wavelength case for different seeing and flux conditions we examine the performance of the algorithm in the observable criteria and its agreement with the unobservable one.

In Table 2 we show the fraction of phase aberrations that are successful in converging to a solution that passes the likelihood threshold. We see that the algorithm converges more for smaller phase aberrations or flux. However, in Table 3 we see the phase error of runs that converged successfully in the likelihood sense is high for the small flux case, which we can understand as not constraining enough. To ensure the gradient-based method will work, we want the phase error to be smaller.
Table 2

| r0 [cm] | N_{photons} | 10^5 | 3 \times 10^5 | 10^6 | 3 \times 10^6 |
|---------|-------------|------|--------------|------|--------------|
| 15      | 100         | 90   | 67           | 65   |              |
| 18      | 100         | 98   | 93           | 89   |              |
| 21      | 100         | 100  | 98           | 96   |              |
| 24      | 100         | 100  | 100          | 100  |              |

Note. The success rate was determined according to the likelihood threshold for 10^5 attempts.

Table 3

| r0 [cm] | N_{photons} | 10^3 | 3 \times 10^3 | 10^6 | 3 \times 10^6 |
|---------|-------------|------|--------------|------|--------------|
| 15      | 0.79        | 0.45 | 0.16         | 0.05 |              |
| 18      | 0.81        | 0.43 | 0.18         | 0.06 |              |
| 21      | 0.71        | 0.41 | 0.19         | 0.07 |              |
| 24      | 0.80        | 0.40 | 0.19         | 0.08 |              |

Note. The phase root-mean-square was averaged for runs that passed the likelihood threshold.

than a radian so the problem will be close to quadratic or some low-order polynomial.

In future works we will expand this optimization step to allow convergence for finite bandwidth imaging.

B.3. Gradient Descent

To go from the \( l_2 \) solution to the MAP estimator we optimize a function proportional to the log posterior

\[
\mathcal{L}(\phi) := \sum_{x,y} \frac{(l - fp_o)^2}{p_o + b^2} + \phi^T \mathbf{C}^{-1} \phi,
\]

using gradient descent with the method to rapidly calculate the gradient as done by Fienup (1999). This method takes advantage of the Fourier transform in Equation (3) to calculate it with only 2 FFT operations

\[
\frac{\partial \mathcal{L}}{\partial \phi_{u1}} = [2\phi^T \mathbf{C}^{-1}]_{u1}
\]

\[
+ \int_0^{\lambda_h} \text{Re} \left\{ -i B \frac{\lambda_o}{\lambda} e^{i \omega \phi} \mathcal{F}^{-1} \left[ 2f \left( S^{-1}_{\lambda o} \frac{\partial \mathcal{L}}{\partial \phi} \mathcal{F} \left[ B e^{i \omega \phi} \right] \right) \right] \right\} d\omega.
\]

Appendix C

Second-order Bias Estimate

To prove Equation (16), we start from \( \hat{\phi}_1 \) and \( \hat{\phi}_2 \), which are estimates for \( \phi \) and i.i.d, according to \( \mathcal{N}(\phi + \mu, \Sigma) \). Examining each term separately, first an unbiased phase estimate

\[
2\hat{\phi} - \hat{\phi}_1 = \hat{\phi} + (\hat{\phi} - \hat{\phi}_1) \sim \mathcal{N}(\phi, 2\Sigma)
\]

\[
E_{\hat{\phi}, \hat{\phi}_1} \left[ \Delta \mathcal{S}(2\hat{\phi} - \hat{\phi}_1) \right] = \Delta \mathcal{S}(\phi) + 3\mu \frac{\partial \Delta \mathcal{S}}{\partial \phi} \bigg|_{\phi} + \frac{9}{2} \mu^2 \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} + \frac{3}{2} \left( \Sigma^{-1} \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} \right) \left( \Sigma^{-1} \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} \right),
\]

and two biased phase estimates with different variance

\[
2\hat{\phi}_1 - \phi = 2(\hat{\phi}_1 - \phi) + \phi \sim \mathcal{N}(\phi + 3\mu, 3\Sigma)
\]

\[
E_{\hat{\phi}_1, \hat{\phi}_2} \left[ \Delta \mathcal{S}(2\hat{\phi}_1 - \hat{\phi}_2) \right] = \Delta \mathcal{S}(\phi) + 3\mu \frac{\partial \Delta \mathcal{S}}{\partial \phi} \bigg|_{\phi} + \frac{9}{2} \mu^2 \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} + \frac{3}{2} \left( \Sigma^{-1} \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} \right) \left( \Sigma^{-1} \frac{\partial^2 \Delta \mathcal{S}}{\partial \phi^2} \bigg|_{\phi} \right),
\]

Inserting those relations to the LHS of Equation (16) easily leads to the RHS.

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