Numerical modeling of the Kolmogorov flow in a viscous and inviscid media

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Abstract. In this paper we consider the problem proposed by A.N. Kolmogorov to study the reasons of turbulence. We consider the action of the external periodic field along one of the coordinates on a viscous and inviscid conductive media in the two-dimensional case. We propose a numerical study of this problem based on the solution of the systems of the Euler and mainly the Navier-Stokes equations. The calculations show that under certain conditions periodic vortex structures may appear in the liquid, similar to the “parquet” mode in the Kolmogorov task, with the subsequent development of the self-similar regime.

1. Introduction

Despite the fact that three-dimensional turbulence is a variety and non-linear phenomenon, interest to two-dimensional turbulence attracts the attention of many researchers [1-9]. Kraichnan and Batchelor [2, 3] established the fact that two-dimensional turbulence is not a simplified model of three-dimensional turbulence, which has its own unique properties. The existing difference between two-dimensional and three-dimensional turbulence is as follows. In three-dimensional case, flows are generated with scales smaller than the scale at which turbulence is excited (the “pumping” scale). In this case, the energy is distributed in a direct cascade with the Kolmogorov law −5/3 in the inertial range of the spectrum. In two-dimensional case, this leads to a very large scale of “pumping” with the appearance of large coherent structures. Energy can be distributed in a reverse cascade (from small structures to large ones) with the -5/3 Kraichnan's law [2], and can also be transferred from the pumping scale to small scales (direct cascade) with the -3 law due to the dissipation of enstrophy.

The inverse cascade in two-dimensional turbulence was studied experimentally [4] and numerically [5]. The peculiarity of these studies is the emergence of intensive large-scale flow, including large eddies. In [6], a vortex dipole-stable coherent structure was obtained numerically in a square cell with periodic boundary conditions. In the field experiment [7], a stable coherent structure was also obtained.

In this work, we consider the problem proposed by A.N. Kolmogorov to study the causes of turbulence in the two-dimensional case that was experimentally investigated in [8]. It is a study of the plane flow of an incompressible conductive fluid under the action of an external force, periodic in the transverse direction. In the linear formulation, this problem was studied, where the fact of the stability loss of the main laminar flow was proved. The problem of nonlinear development of perturbations and the occurrence of secondary stationary or periodic flows with a stability loss of the laminar flow is
described in [9]. However, the issues related to the existence of stable stationary or self-oscillatory regimes of the secondary flow, as well as the possibility of transition to chaos, are still largely open.

We propose a numerical study of the flat flow problem mainly of a viscous weakly compressible and an inviscid weakly compressible fluid under the action of a periodic force.

2. Problem statement

Let us consider the problem of a flat flow of an inviscid weakly compressible fluid under the action of an external periodic force directed along the Ox axis which equals \( \rho G \sin(ky) \). Here \( G = 0.01 \text{ N/kg} \) is the Lorentz force, equal to the vector product of the strength of the current passed through the liquid, by the magnetic field strength; and \( k \) is the wave number that specifies the period of the force (in our calculation \( k = 1 \)). This \( G \) force came from the experiments with the conducting fluid which were carried out by other researchers. The motion of the viscous medium in this case is described by the Navier-Stokes equations in the following form:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0, \\
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) &= -\frac{\partial p}{\partial x} + \rho G \sin ky + \mu \Delta u, \\
\frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v \mathbf{V}) &= -\frac{\partial p}{\partial y} + \mu \Delta v,
\end{align*}
\]  

(1)

Here \( u, v \) are the components of the velocity vector, along the Ox and Oy axes, respectively; \( P \) is the pressure; and \( \rho \) is the density.

The flow is investigated in a rectangular area with periodic boundary conditions. Size of this calculation domain equals \( 8\pi \times 4\pi \) along Ox and Oy axes, respectively. Calculations were performed on a grid size of 200 x 100 and 800 x 400 cells along Ox and Oy axes, respectively.

In the calculation, the initial condition was immediately set as the sum of the main flow, taken as \( u = \sin(y) \) and superimposed small disturbances which can be found in formulas (2) and (3). It is known that for an incompressible fluid, the long-wave perturbations imposed on the main flow field are the most unstable. Therefore, such a superposition of the main flow velocity and small perturbations superimposed on the main flow are presented as the initial conditions for the velocity field. Such a statement of the initial conditions allows obtaining an analogue of the self-oscillatory regime.

Other used initial conditions can be represented as follows

\[
P(t = 0) = P_0 = 10^5 \text{ Pa},
\]

\[
\mu = 2 \text{ Pa s},
\]

\[
u(t = 0) = 0.1 \sin(y) + 0.001 \sin(x/2), \quad (2)
\]

\[
u(t = 0) = 0.1 \sin(y) + 0.001 \sin(x/2). \quad (3)
\]

Here \( \mu \) is the viscosity of fluid.
3. Numerical method

The numerical solution of the inviscid problem was performed using the Hyperbolic Solver computational package, which allows solving various problems of continuous media dynamics described by the hyperbolic system of equations.

The calculation algorithm, used for viscous medium modeling, is based on the MacCormack explicit method, which is of second order in accuracy in time and space and well-proven in solving hyperbolic equations. The Navier-Stokes equations for an incompressible fluid are solved using the method of artificial compressibility.

4. Results

We will not consider the inviscid case in detail here, since more detailed information can be found in [10]. We give only the views of arising characteristic flows in comparison with the viscous case. From fig. 1 it can be seen, that a “parquet” of not so big eddies first arises from the initial state with the subsequent development of two relatively large vortexes.

![Figure 1. Formation of the “vortex parquet” and its decay in the numerical simulation of the Kolmogorov problem. Streamlines and velocity modulus for inviscid (a), (b), (c) and viscous (d), (e), (f) cases.](image)

Turning to the consideration of the viscous case, we can pay attention to fig. 2. You can see graphs of kinetic energy and enstrophy on value \( N_{\text{print}} \) normalized on the maximum value of these parameters. \( N_{\text{print}} \) is the number of output file and actually \( N_{\text{print}}=t/4 \), so, it can be considered as the time. The enstrophy can be described as the integral of the square of the vorticity.
Figure 2. Graphs of kinetic energy and enstrophy on time normalized on the maximum value of these parameters.

In fig. 3 you can see the series of the vorticity magnitude taken in different moments of time. It can be seen that from the initial state \((N_{\text{print}} = 15)\) vortices form a parquet-like pattern \((N_{\text{print}} = 50)\), which over time combine at first into four vortices \((N_{\text{print}} = 100)\), and then into two vortices \((N_{\text{print}} = 785)\), which rotate in different directions. Each of the pictures corresponds to more or less stable pattern in the time moments between peaks of kinetic energy and enstrophy in fig. 2. So each peak separates the arising quasi-stationary flows. At the moments of peak maxima, transient regimes occur, transforming one quasi-stationary state into another.

![Vorticity magnitude](image)

Figure 3. Vorticity magnitude taken in different moments of time.

In the region of time \(N_{\text{print}} = 250\) after a long transient regime, a system consisting of two vortices rotating in opposite directions begins to form. The conditional left vortex rotates counterclockwise, and the conditional right vortex rotates clockwise. After that, the system of two
vortices passes into the self-similar state. Each of the vortices from time to time form a “ring”. Further, this “ring” is divided into two vortices, rotating in one direction, and located close to each other. As a result of their interaction, a larger vortex is formed, which later, in its turn, forms a “ring”. The process is repeated cyclically throughout the observed time up to 12800 time steps or \(N_{\text{print}} = 3200\). From the graph in Fig. 2 it may be seen that the self-similar mode is observed with the minimum value of enstrophy.

**Conclusion**

The result of this work is a direct numerical simulation of the vortex flow formation regime in a layer of weakly compressible medium based on the Euler and Navier-Stokes equations. Namely, a small perturbation of the velocity components leads to the appearance of a “vortex parquet” and further to two rotating vortices.

In addition when modeling viscous fluid, several transient modes were found to be separated from each other by maxima of kinetic energy or enstrophy. In addition, the self-similar mode is observed with the minimum value of enstrophy.

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