INTER-ORGANIZATIONAL CONTRACT CONTROL OF ADVERTISING STRATEGIES IN THE SUPPLY CHAIN

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Abstract. Advertising has a crucial impact on a product’s goodwill. To further improve a product’s goodwill and make more profit, member firms in the supply chain use various contracts to coordinate the channel. Considering the dynamic effect of advertising, this paper studies a two-level supply chain consisting of one manufacturer and one retailer. The two members focus on maximizing their profits through advertising and pricing strategies under two types of contracts: the wholesale price contract and the consignment contract. The Stackelberg differential game is introduced, and the optimal advertising effort, wholesale and retail pricing strategies in the two situations are studied. Numerical examples and sensitivity analyses are conducted to explore the models further. The results show that the retailer’s revenue proportion and the product’s goodwill according to consumers significantly affect the strategies and the contract choice of the partner firms in the supply chain. A proportion of too high or too low revenue may lead to a contract selection conflict between the two partner firms. However, when consumers care more about the product’s goodwill, this contract selection conflict can be weakened.

1. Introduction. Recently, with the rise of large retailers like Walmart, Amazon, and Costco, the dominant status in the supply chain has shifted from manufacturers to retailers (Thomas and Wilkinson [42]; Buratto et al. [9]). Retailers with more bargaining power are increasingly inclined to use the consignment contract to transfer demand uncertainty to their manufacturers or suppliers (De Giovanni et al. [17]). Under this contract, the manufacturer keeps ownership of the products and bears the risk of sluggish sales. The retailer charges a specific proportion of the retail price and shares the rest with the supplier/manufacturer for each product sold (Bolen [8]; Hackett [22]). Compared with the traditional wholesale price contract, the consignment contract enables the retailer to avoid market risks and reduce double marginalization better (Li and Hua [30]). Considering the advantages...
of the consignment contract, increasing numbers of large retailers use this approach with their suppliers/manufacturers (Zhao et al. [52]). For example, the pay-by-scan program allows Home Depot to price its products and keep a specific proportion of their retail prices. The suppliers still hold ownership of the products and decide the Home Depot inventory allocation (Adida and Ratisoontorn [1]). Moreover, retailers like Amazon and Walmart keep between 6% and 20% of the retail prices if the supplier/manufacturer’s products are sold (Wang et al. [43]). However, as there is a certain ratio between the wholesale price and the retail price, the consignment contract is also considered to be less effective than the wholesale price contract (Choi et al. [14]; Cachon [10]; De Giovanni and Roselli [18]). Overall, the effects of consignment contracts on supply chain coordination remain controversial (Lu et al. [32]; Hu et al. [25]). Therefore, comparing the effects of the traditional wholesale price contract and the consignment contract in the supply chain has become a burning issue.

Advertising is one of the most important factors affecting consumer buying behaviors, brand image, and firms’ profits (Huang et al. [26]; Zhang et al. [50]). Advertising strategies can be categorized as static or dynamic. The static advertising strategy is adopted when the firm is myopic, regardless of the cumulative influence of its decision. However, this strategy remains unchanged once implemented throughout the selling period. It is convenient but cannot adapt to a rapidly changing environment. With the development of information technology and the increase in market uncertainty, the dynamic advertising strategy is increasingly used (He et al. [23]; Jørgensen and Zaccour [27]; Zhang et al. [50]). Firms that adopt the dynamic advertising strategy tend to focus on long-term development and adjust their advertising strategies timely according to the environment (Nerlove and Arrow [37]; Feichtinger and Sethi [21]). For example, Pepsi changes its product advertisements continuously and timely based on the preferences of young people (Lu et al. [33]). Although many previous studies focus on the dynamic advertising strategy, few consider its determinants in terms of supply chain coordination. Especially in today’s market environment, with the dominant status shifting from manufacturers to retailers, consignment contract (instead of the traditional wholesale price contract) has been used widely to coordinate the supply chain. Therefore, how to use different contracts to coordinate the advertising strategies of partner firms in the supply chain has become more and more critical in contemporary practice (Bai et al. [6]; Heydari et al. [24]).

Based on the research background presented above and considering the dynamic property of advertising, we focus on the following questions: (1) Aiming at profit maximization, how do partner firms make decisions about advertising and pricing strategies under the wholesale price contract and the consignment contract? What are the differences? (2) Why do partner firms prefer the wholesale price contract or the consignment contract in a specific market environment? (3) As the system parameters change, what happens to the strategies and profits of partner firms? To answer these questions, we analyze the advertising strategies of a two-level supply chain, which consists of one manufacturer and one retailer under two types of contracts: the wholesale price contract and the consignment contract. Both the manufacturer (referred to as he) and the retailer (referred to as she) are far-sighted and fully consider dynamic advertising and pricing strategies. We consider the dynamic effect using the differential Stackelberg game model, in which the retailer is the leader, and the manufacturer is the follower. The two partner firms trade
through the wholesale price contract and the consignment contract separately. In the wholesale price contract situation, the retailer sets her retail margin first. Then, the manufacturer sets his wholesale price and decides on his advertising effort. In the consignment contract situation, the retailer's proportion of sales revenue is exogenous. She decides the retail price first. Then the manufacturer decides on his advertising effort as a response. Our study derives the optimal advertising and pricing strategies and associated profits of the retailer and the manufacturer under the wholesale price contract and the consignment contract separately. These strategies are also compared.

This study contributes to addressing research gaps in the following three aspects. (1) It examines the dynamic advertising strategy in the context of the wholesale price and consignment contracts and compares their differences. Previous research on the dynamic advertising strategy mainly focuses on one specific type of contract (usually the wholesale price contract). It neglects the impacts of different contracts on advertising in the supply chain. With the shift in dominant power from manufacturers to retailers, the consignment contract has become increasingly popular in practice. However, studying the traditional wholesale price contract theoretically cannot reflect current practice. Based on this research background, our study compares the optimal advertising strategy under the wholesale price contract and the consignment contract separately, thereby filling a gap between theory and practice in this field. (2) This study constructs an integrated model that combines a macroscopic factor (namely, consumer preference for goodwill) and a microscopic factor (namely channel coordination through contracts). Most previous studies focus on the advertising problem from the perspective of market preferences and channel coordination separately. However, in practice, these two sources of influence on advertising are concurrent. Furthermore, the complementary effects of consumer preference and contract coordination on advertising differ from their separate effect. In contrast, our study combines these macroscopic and microscopic factors. It analyzes their overall effects on advertising strategies in the supply chain under the wholesale price contract and the consignment contract. (3) This study fully considers current practice by considering a finite selling period and a relatively stable proportion of revenue sharing to examine the problem of dynamic advertising. Most previous studies analyze the dynamic advertising problem in an infinite period. However, this does not reflect practice because the selling period is generally limited. Moreover, as most partner firms cooperate in the long term, their respective revenue sharing proportions must be stable and cannot be changed frequently. Therefore, our study uses a finite selling period and a stable revenue sharing proportion, which are closer to reality, to study the dynamic advertising strategy under the wholesale price contract and the consignment contract.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature, and Section 3 develops the baseline model. In Section 4, the Stackelberg game models under the wholesale price contract and the consignment contract are constructed and compared. In Section 5, numerical simulations and sensitivity analyses are conducted. Finally, Section 6 concludes the paper and discusses directions for future research.

2. Literature review. This study is closely related to three main streams of literature: (1) contract coordination in the supply chain, (2) consignment contracts, and (3) dynamic advertising strategies. Relevant studies are reviewed below.
Coordinating the supply chain through contracts is an effective way to reduce the double marginalization effect (Cachon [11]). Common types of contracts mainly include wholesale price contracts, buy-back contracts, price discount contracts, quantity discount contracts, consignment contracts, and revenue-sharing contracts. Many scholars investigate the coordination effects of different contracts in the supply chain. For example, Cachon and Lariviere [13] analyze the advantages and disadvantages of several types of contracts, including the buy-back contract, the sales rebate contract, the price/quantity discount contract, and the revenue-sharing contract. Similar to using a two-level supply chain consisting of competing manufacturers and one retailer, Cachon and Kök [12] compare the impacts of different contracts (namely, the wholesale price contract, the quantity discount contract, and a two-part tariff) on supply chain management. Wang and Shin [44] investigate the supply chain coordination effect using the wholesale price contract, the revenue-sharing contract, and the quality-dependent wholesale price contract. They find that the quality-dependent wholesale price contract can lead to under-investment in innovation, whereas the revenue-sharing contract is beneficial for supply chain coordination. Considering the difference between the buy-back contract and the revenue-sharing contract, Zhang et al. [48] analyze the contract choice of a loss-averse supplier. They find that when the critical ratio is high, it is more profitable for the supplier to choose the revenue-sharing contract than the buy-back contract. Dai et al. [16] compare the cost-sharing contract between two partner firms with cartelization in the supply chain. They find that the cost-sharing contract increases the profits of all supply chain members, whereas cartelization leads to a Pareto improvement when the environment is specific. Taking a supply chain with competing retailers and only one manufacturer as their research object, Xue et al. [45] investigate the effect of a buy-back contract. They find that when the competition level between retailers is high, the buy-back contract can lead to a Pareto improvement for all partner firms in the supply chain. However, these studies mainly examine the coordination problem statically and neglect the influence of macroscopic factors. Our study introduces the effect of time and compares the wholesale price contract and the consignment contract with dynamic advertising strategies. Moreover, we consider the overall effects of macroscopic and microscopic factors. In this way, the problem of contract coordination is closer to reality.

The topic of supply chain management through consignment contracts has received much attention in recent years. However, the effects of consignment contracts on supply chain coordination remain unclear. For example, Wang et al. [43] introduce a cost-sharing-based consignment contract in a supply chain consisting of one supplier and one retailer. They analyze the impact of different retailer cost-sharing proportions on the overall performance of the supply chain and find that this type of contract is beneficial to improve performance when the retailer shares the lowest cost with the supplier. Li and Hua [30] investigate the efficacy of the consignment contract under different types of market demands. They show that using a consignment contract to manage the supply chain is adequate and robust. Following this idea, Adida and Ratisoontorn [1] compare the impact of two consignment contracts: the price-based consignment contract and the revenue-sharing-based consignment contract. Hu et al. [25] suggest that the revenue-sharing-based consignment contract can improve profits for the retailer and the entire supply chain. This result is consistent with the conclusions of Ben-Daya et al. [7], Lu et al. [32], and De Giovanni et al. [17]. Furthermore, Li et al. [31] identify the application condition
for using the consignment contract for supply chain coordination. They adopt the Nash bargaining model to study the impact of the consignment contract on channel coordination and find that when partner firms are risk-neutral, using this type of contract leads to better coordination in the supply chain. Avinadav et al. [3, 4] also discuss the specific conditions of consignment contracts in a supply chain consisting of an app developer and a seller. In contrast, some studies such as Dong and Xu [19], Cachon [10], and De Giovanni and Roselli [18] show that it is not enough to prove that the consignment contract can coordinate the supply chain well, which leaves an open problem. In addition, most of these studies only focus on the specific characteristics of consignment contracts and do not compare different contract types. To further explore the coordination effect of the consignment contract and analyze its advantages and disadvantages, our study compares this type of contract with the wholesale price contract. It focuses on the contract preferences of partner firms in the supply chain. In addition, by introducing the specific scenario of dynamic advertising, we explain the conflicting results regarding the coordination effect of the consignment contract.

The third stream of literature is related to the analysis of dynamic advertising. In 1962, Nerlove and Arrow first constructed a dynamic advertising model to study the impact of current advertising activities on future demand. This study brought the problem of dynamic advertising to the attention of researchers. Since then, most scholars have used the (modified) Nerlove-Arrow model to compare the different impacts of dynamic and static advertising strategies on product goodwill or market demand (Feichtinger and Sethi [20]; He et al. [23]; Jørgensen and Zaccour [27]; Zhang et al. [50]; Zhang et al. [49]). Among these studies, some consider dynamic advertising with different contracts, which are closely related to our study. For example, Zaccour [47] studies the dynamic problem of marketing and advertising in the supply chain. He finds that the two-part wholesale tariff policy cannot result in vertical integration in a dynamic market environment. Furthermore, Bai et al. [5] compare the coordination effect of the two-part tariff contract with revenue and the promotional cost-sharing contract and find that the two-part tariff contract is more robust. Although these studies consider dynamic advertising with contracts, few studies pay attention to its impact on contract selection in the supply chain. In particular, prior studies mainly focus on the traditional wholesale price contract, overlooking the consignment contract. However, this setting is not suitable when the retailer occupies the leading position in the supply chain. In addition, some studies explore dynamic advertising using a two-stage game instead of a differential game. For example, Sigué and Chintagunta [40] indicate that the costs and margins of advertising activities significantly influence advertising arrangements in the supply chain. Yue et al. [46] further find that the manufacturer’s advertising allowance, price discount, and brand investment can motivate the retailer to increase her advertising expenses in the two-stage game setting. It should be noted that in the two-stage game, a firm’s advertising strategy is not time-related and is iteratively derived. With this method, the steady-state strategy (which is more robust and has significant practical implications) cannot be investigated. However, as the market environment changes rapidly, it would be better for firms to change their advertising strategies timely in accordance with the guiding standards (namely the steady-state strategy). Therefore, our study considers the consignment contract, using the differential game method to study the problem of dynamic advertising. As a result, our setting is closer to reality and can guide firms in practice.
Figure 1. The overall mechanism.

Table 1. Notations and definitions

| Notation | Definition |
|----------|------------|
| Decision variables | |
| $A(t)$ | Manufacturer’s advertising effort for the product, $A(t) > 0$. |
| $s(t)$ | Retailer’s markup (i.e., retail margin) under the wholesale price contract, $s(t) > 0$. |
| $w(t)$ | Wholesale price of the product under the wholesale price contract, $w(t) > 0$. |
| $p(t)$ | Retail price of the product under the consignment contract, $p(t) > 0$. |
| Parameters and other variables | |
| $\alpha$ | The coefficient associated with the product’s retail price in the demand function, $\alpha > 0$. |
| $\beta$ | The coefficient associated with the product’s goodwill in the demand function, $\beta > 0$. |
| $\theta$ | The coefficient associated with the manufacturer’s advertising effort in the product goodwill function, $\theta > 0$. |
| $\varphi$ | Retailer’s proportion of sales revenue under the consignment contract, $0 < \varphi < 1$. |
| $\delta$ | The decay rate of the product’s goodwill, $\delta > 0$. |
| $\mu$ | Cost parameter associated with the manufacturer’s advertising effort, $\mu > 0$. |
| $D(t)$ | Demand for the product at time $t$, with initial demand $D_0 > 0$. |
| $G(t)$ | Goodwill of the product at time $t$, with $G_0 \geq 0$. |
| $J_R, J_M$ | Objective functions (expressed in net profit) of the retailer and the manufacturer, respectively, for $t \in [0, +\infty)$. |

It is noteworthy that Lu et al. [32] take into consideration the effect of dynamic advertising and investigate its impact on two types of contracts (wholesale price contract and consignment contract) in a two-level supply chain. They use an infinite selling period and an exogenous proportion of revenue sharing. However, in practice, the selling period is generally limited. Moreover, as most partner firms cooperate in the long term, the proportion of revenue sharing between the retailer and the manufacturer should be stable and cannot be changed frequently. Thus, unlike their models, our study fully considers current practice, taking into account a finite selling period and a relatively stable revenue sharing proportion. To the best of our knowledge, our study is among the first to analyze the inter-organizational contract control of advertising strategies by considering both a finite selling period and a stable revenue sharing proportion.
3. **Baseline model.** Our study examines a two-level supply chain consisting of one retailer and one manufacturer, in which the retailer dominates the system. The manufacturer follows the retailer. Considering the status gap and the dynamic property of advertising activities, we adopt the differential Stackelberg game to analyze this problem. In our study, the manufacturer produces products and sells them to consumers through the retailer. He is also responsible for making an effort to advertise that product, as advertising can improve the product’s goodwill. In this problem, consumers also play an important role because their purchasing preferences significantly affect final demand. Specifically, consumers prefer products with higher goodwill and tend to pay more for them. As mentioned above, product advertising can significantly improve goodwill. Thus, consumer preference is the driving force that motivates the manufacturer to put more effort into advertising. To better respond to the market and maximize profit, partner firms in the supply chain use two types of contracts to coordinate the channel: the wholesale price contract and the consignment contract. The overall mechanism of this problem is illustrated in Figure 1. The relevant notations and definitions in our study are shown in Table 1.

Advertising activities are time-varying processes. Advertising is considered a differential function related to the manufacturer’s specific effort and decay rate (Nerlove and Arrow [37]). This advertising effort can improve the product’s goodwill. The goodwill contributed by advertising is a cumulative process and develops following the Nerlove-Arrow model. Moreover, when the manufacturer does not make an effort to advertise his product, the product’s goodwill will decay because competition in the market is fierce, and consumers tend to forget (Lu et al. [32]). Therefore, the product’s goodwill can be expressed differentially as follows,

\[ \dot{G}(t) = \theta A(t) - \delta G(t), \quad G(0) = G_0. \]  

As mentioned above, consumers prefer products with high goodwill. Hence, market demand is significantly affected by the product’s goodwill. Our study supposes that market demand increases directly with the level of goodwill of the product, following Amrouche et al. [2], Karray and Martín-Herrán [29], and Karray [28]. At the same time, we follow traditional studies and assume that there is a linear negative relationship between the retail price and market demand (Singh and Vives [41]; Lus and Muriel [34]; Zhou et al. [54]). The demand function in our study mainly captures the characteristics of the direct effects of the product price and goodwill on market demand. It has some real applications in practice. For example, CLEAR is a famous French shampoo. In August 2020, a well-known film star Yangqianxi Yi (Jackson Yee), became its brand ambassador in China. Then, with the advertisement of CLEAR and Yangqianxi Yi all over the place, there is a 41.2% increment in the sales of this shampoo. Especially, on November 11, 2020 (a promotion day in China), together with the hefty discount, the one-day sales of CLEAR shampoo even reached 2,668,500 yuan (Annual Shampoo Market Overview and Trend Insight Report 2020 [38]). This shows the direct influence of advertisement and price discount on market demand\(^1\). The relevant assumption for us to use the linear demand function here is that consumers are rational and only care about

\(^1\)However, it should be noted that this kind of linear demand function is an ideal form of the actual demand function in practice. Considering that various factors are influencing the market demand, the actual demand may not be linear. Nevertheless, the positive (negative) relationship between goodwill (price) and market demand remains unchanged.
product price and goodwill. Furthermore, the macroscopic market environment remains stable. Based on the traditional studies and many real applications, we adopt a linear function of product goodwill and retail price to represent market demand. The linear demand function proposed here helps us to focus on the direct effects of product goodwill and retail price better. Furthermore, we use a deterministic market demand function and assume that market demand can be satisfied without inventory. This market demand function setting is typical in practice, especially for make-to-order products such as in the software industry. Therefore, market demand is given by

\[ D(t) = D_0 - \alpha p(t) + \beta G(t), \quad D(t) \geq 0. \]  

(2)

The manufacturer is responsible for advertising his product. The cost of his advertising effort is denoted by \( C(t) \). It increases when the manufacturer makes more effort and follows the law of diminishing returns so that \( C'(t) > 0, C''(t) > 0 \) (De Giovanni et al. [17]). Thus, we use a quadratic function to represent the cost of advertising, which is consistent with many studies such as Zhang et al. [49], Zhou et al. [53], Lu et al. [32], and Martin-Herrán and Sigue [35]. Inventory and related manufacturing costs are not considered in our study. The manufacturer’s advertising cost at time \( t \) can be expressed as

\[ C(t) = \frac{\mu}{2} A^2(t). \]  

(3)

4. Differential game equilibrium analysis. Managing product advertising takes time because it is a dynamic, long-term process. In our study, the two partner firms, i.e., the manufacturer and the retailer, in the supply chain are far-sighted. That is, they consider the long-term impact of their recent decisions. In this way, the decision-making process becomes dynamic and consecutive. Given the discussion above, it is necessary to use a differential method to study this dynamic advertising problem. In addition, as there is a status gap between the two partner firms, the Stackelberg game (in which the retailer is the leader and the manufacturer is the follower) is adopted. In the following subsections, we study and compare the optimal strategies of the manufacturer and the retailer under the wholesale price contract and the consignment contract.

4.1. Wholesale price contract. In this subsection, we study the optimal strategies of the manufacturer and the retailer under the wholesale price contract. The wholesale price contract is one of the most common contracts because it is easy to implement and suitable for traditional relationships between channel members (El Ouardighi and Kim [20]; Shi et al. [39]). In this situation, the manufacturer is responsible for deciding the wholesale price \( w(t) \) and sells his product to the retailer. Then, the retailer adds a markup \( s(t) \) (namely the retail margin) and sells the product to consumers at a retail price \( p(t) \). In this setting, the relationship between the three variables is as follows,

\[ p(t) = w(t) + s(t). \]  

(4)

The game sequence is the following. (1) First, as the leader, the retailer sets her retail margin \( s(t) \). (2) Second, after observing the retailer’s decision, the manufacturer sets his wholesale price \( w(t) \) and decides on his advertising effort \( A(t) \). The purpose of the retailer and the manufacturer is to maximize their profits. Based on the sales cycle assumption, the planning interval in our study is \( t \in (0, T) \). Static feedback solutions are adopted in the Stackelberg game. Thus, in the wholesale
price contract situation, the objective functions of the retailer and the manufacturer, respectively, are given by

\[
\max_{s(\cdot)} \int_0^T [s(D_0 - \alpha w - \alpha s + \beta G)] \, dt, \quad (5a)
\]

\[
\max_{w(\cdot), A(\cdot)} \int_0^T \left[ w(D_0 - \alpha w - \alpha s + \beta G) - \frac{\mu}{2} A^2 \right] \, dt. \quad (5b)
\]

Then, the Hamilton-Jacobi-Bellman (HJB) equations of the retailer and the manufacturer (denoted by \(H_R, H_M\), respectively) for all \(G(t) \geq 0\) are given by

\[
H_R = s(D_0 - \alpha w - \alpha s + \beta G) + \lambda_R (\theta A - \delta G), \quad (6a)
\]

\[
H_M = w(D_0 - \alpha w - \alpha s + \beta G) - \frac{\mu}{2} A^2 + \lambda_M (\theta A - \delta G). \quad (6b)
\]

We can solve the optimal strategies by differentiating the HJB equations with respect to \(s, w, A\) as follows,

\[
s^* = \frac{D_0 + \beta G}{2\alpha}, \quad (7a)
\]

\[
w^* = \frac{D_0 + \beta G}{4\alpha}, \quad (7b)
\]

\[
A^* = \frac{\theta \lambda_M}{\mu}. \quad (7c)
\]

Based on the theory of optimal control, the adjoint variables \(\lambda_R, \lambda_M\) meet the following conditions: \(\dot{\lambda}_R = -\partial H_R/\partial G, \dot{\lambda}_M = -\partial H_M/\partial G\). Thus, we have

\[
\dot{\lambda}_R = \delta \lambda_R - \beta s, \quad \lambda_R(T) = 0, \quad (8a)
\]

\[
\dot{\lambda}_M = \delta \lambda_M - \beta s, \quad \lambda_M(T) = 0, \quad (8b)
\]

where \(\lambda_R(T) = 0, \lambda_M(T) = 0\) are the transversality conditions. They show that the product’s goodwill \(G(t)\) is arbitrary at the final time \(T\).

Then, the optimal advertising effort, wholesale price, and retail markup in the wholesale price contract situation can be obtained in Proposition 1 (superscript \(W\) indicates the wholesale price contract situation).

**Proposition 1.** In the wholesale price contract situation, the manufacturer’s optimal advertising effort and wholesale price and the retailer’s retail markup are given by

\[
s^W = \frac{\delta - \frac{r_2}{\beta} k_2 e^{r_2 t}}{\beta} + \frac{\delta - \frac{r_3}{2\beta} k_3 e^{r_3 t}}{2\beta} + \frac{2\delta^2 \mu D_0}{4\alpha \delta^2 \mu - \beta^2 \theta^2}, \quad (9a)
\]

\[
w^W = \frac{\delta - \frac{r_2}{\beta} k_2 e^{r_2 t}}{2\beta} + \frac{\delta - \frac{r_3}{2\beta} k_3 e^{r_3 t}}{2\beta} + \frac{\frac{\delta^2 \mu D_0}{4\alpha \delta^2 \mu - \beta^2 \theta^2}}{4\alpha \delta^2 \mu - \beta^2 \theta^2}, \quad (9b)
\]

\[
A^W = \frac{\theta}{\mu} k_1 e^{r_1 t} + \frac{\theta}{2\mu} k_2 e^{r_2 t} + \frac{\theta}{2\mu} k_3 e^{r_3 t} + \frac{\beta \delta \theta D_0}{4\alpha \delta^2 \mu - \beta^2 \theta^2}, \quad (9c)
\]

where the constants \(k_1, k_2, k_3\) are defined in Appendix 1 and simultaneously satisfy the following equations,

\[
\begin{align*}
G(0) &= G_0, \\
\lambda_R(T) &= 0, \\
\lambda_M(T) &= 0.
\end{align*}
\]
From Proposition 1, we know that the two partner firms tend to adopt different pricing strategies, including the penetration pricing strategy and the skimming pricing strategy, based on their different situations. Specifically, the pricing strategy is closely related to the initial goodwill of the product $G_0$. Both the retailer and the manufacturer tend to adopt the skimming pricing strategy when $G_0$ is relatively high. However, they prefer the penetration pricing strategy when $G_0$ is relatively low. Consequently, both the retailer and the manufacturer must fully consider $G_0$ before determining their pricing strategies.

4.2. Consignment contract. In this subsection, we study the optimal strategies of the manufacturer and the retailer under the consignment contract. The manufacturer produces and sells his product to the retailer through the consignment contract. In this way, the wholesale price and the retail price are connected. Specifically, the retailer decides the retail price $p(t)$ and sells her product to final consumers. At the same time, the retailer has a specific percentage (denoted by $\varphi$) of sales revenue and shares $1 - \varphi$ percent of that revenue with the manufacturer as the wholesale price $w(t)$. In this setting, the wholesale price of the product is a proportion of its retail price. As previously mentioned, stable cooperation between partner firms is widespread in practice, so the retailer cannot change her revenue percentage frequently. Thus, in our study, the retailer’s revenue percentage $\varphi \in (0, 1)$ is set to be exogenous. The relationship between the three variables is as follows,

$$(1 - \varphi)p(t) = w(t). \quad (11)$$

The game sequence is the following. (1) First, as the leader, the retailer decides the retail price $p(t)$. (2) Second, after observing the retailer’s decision, the manufacturer decides on his advertising effort $A(t)$. The purpose of both the retailer and the manufacturer is to maximize their profits. Based on the sales cycle assumption, the planning interval in our study is $t \in (0, T)$. Static feedback solutions are adopted in the Stackelberg game. Thus, in the consignment contract situation, the objective functions of the retailer and the manufacturer are given by

$$\max_{p(\cdot)} \int_0^T [\varphi p(D_0 - \alpha p + \beta G)] dt, \quad (12a)$$

$$\max_{A(\cdot)} \int_0^T [(1 - \varphi)p(D_0 - \alpha p + \beta G) - \frac{\mu}{2} A^2] dt. \quad (12b)$$

Then the HJB equations of the retailer and the manufacturer (denoted by $H_R$, $H_M$, respectively) for all $G(t) \geq 0$ are given by

$$H_R = \varphi p(D_0 - \alpha p + \beta G) + \lambda_R(\theta A - \delta G), \quad (13a)$$

$$H_M = (1 - \varphi)p(D_0 - \alpha p + \beta G) - \frac{\mu}{2} A^2 + \lambda_M(\theta A - \delta G). \quad (13b)$$

We can solve the optimal strategies by differentiating the HJB equations with respect to $p, A$ as follows,

$$p^* = \frac{D_0 + \beta G}{2\alpha}, \quad (14a)$$

$$A^* = \frac{\theta \lambda_M}{\mu}. \quad (14b)$$
According to the theory of optimal control, the adjoint variables $\lambda_R, \lambda_M$ meet the following conditions: $\lambda_R = -\partial H_R / \partial G$, $\lambda_M = -\partial H_M / \partial G$. Thus, we have

$$
\dot{\lambda}_R = \delta \lambda_R - \varphi \beta p, \quad \lambda_R(T) = 0, \tag{15a}
$$

$$
\dot{\lambda}_M = \delta \lambda_M - (1 - \varphi) \beta p, \quad \lambda_M(T) = 0, \tag{15b}
$$

where $\lambda_R(T) = 0, \lambda_M(T) = 0$ are the transversality conditions. They show that the product’s goodwill $G(t)$ is arbitrary at the final time $T$.

Then, the optimal advertising effort and retail price in the consignment contract situation can be expressed in Proposition 2 (superscript $C$ indicates the consignment contract situation).

**Proposition 2.** In the consignment contract situation, the manufacturer’s optimal advertising effort and the retailer’s optimal retail price are given by

$$
p^{C*} = \frac{\delta - r_2}{\varphi \beta} k_2 e^{2t} + \frac{\delta - r_3}{\varphi \beta} k_3 e^{3t} + \frac{\delta^2 \mu D_0}{\Omega}, \tag{16a}
$$

$$
A^{C*} = \frac{(1 - \varphi) \theta}{\varphi \mu} k_2 e^{2t} + \frac{(1 - \varphi) \theta}{\varphi \mu} k_3 e^{3t} + \frac{(1 - \varphi) \beta \delta \theta D_0}{\Omega}, \tag{16b}
$$

where the constants $k_2, k_3$ are defined in Appendix 2, and they simultaneously satisfy the following equations,

$$
\begin{align*}
G(0) &= G_0, \\
\lambda_R(T) &= 0, \\
\lambda_M(T) &= 0.
\end{align*} \tag{17}
$$

4.3. Comparison between situations. In this section, we compare the optimal strategies under the wholesale price contract and the consignment contract. As the parameters $k_1, k_2, k_3$ are not analytical, the optimal strategies in the steady-state are compared. The differences between the optimal strategies in the steady-state of the wholesale price contract and the consignment contract can be expressed as follows,

$$
\begin{align*}
\delta^{w*}_{SS} - \delta^{C*}_{SS} &= \frac{\delta^2 \mu D_0}{4 \alpha \delta^2 \mu - \beta^2 \theta^2} \cdot \frac{\varphi(3 \beta^2 \theta^2 - 4 \alpha \delta^2 \mu)(4 \alpha \delta^2 \mu - 2 \beta^2 \theta^2)}{\varphi \beta^2 \theta^2 + (2 \alpha \delta^2 \mu - \beta^2 \theta^2)}, \tag{18}
\\
w^{w*}_{SS} - w^{C*}_{SS} &= \frac{\delta^2 \mu D_0}{4 \alpha \delta^2 \mu - \beta^2 \theta^2} \cdot \frac{\varphi 4 \alpha \delta^2 \mu - 2 \alpha \delta^2 \mu}{\varphi \beta^2 \theta^2 + (2 \alpha \delta^2 \mu - \beta^2 \theta^2)}, \tag{19}
\\
A^{w*}_{SS} - A^{C*}_{SS} &= \frac{\delta^2 \mu D_0}{4 \alpha \delta^2 \mu - \beta^2 \theta^2} \cdot \frac{\varphi 4 \alpha \delta^2 \mu - 2 \alpha \delta^2 \mu}{\varphi \beta^2 \theta^2 + (2 \alpha \delta^2 \mu - \beta^2 \theta^2)}. \tag{20}
\end{align*}
$$

Then, the differences in Equations (18)-(20) between these optimal strategies can be uniformly written as

$$
y = \kappa \frac{\varphi a + b}{\varphi c + d} = \frac{\kappa a}{c} + \frac{\kappa (bc - ad)}{c(c^2 + d)}, \tag{21}
$$

for the sake of simplicity. It can be observed that $\kappa > 0, c > 0$, and the signs of $a, b, d$ are not defined in Equation (18); and the sign of $d$ is not defined in Equation (19); $\kappa > 0, a > 0, b < 0, c > 0$ and the sign of $d$ is not defined in Equation (20). The signs of these differences depend on the relationship between the parameters $bc$ and $ad$. The specific discussion of this function is presented in Table 2.
Table 2. Discussion of the function.

| Parameter condition | Condition of $\varphi$ | Sign of $y$ |
|---------------------|------------------------|-------------|
| $bc > ad$           | $\varphi < -b/a$ or $\varphi > -d/c$ | $y > 0$     |
|                     | $-b/a < \varphi < -d/c$ | $y < 0$     |
|                     | $\varphi = -b/a$       | $y = 0$     |
| $bc < ad$           | $\varphi < -d/c$ or $\varphi > -b/a$ | $y > 0$     |
|                     | $-d/c < \varphi < -b/a$ | $y < 0$     |
|                     | $\varphi = -b/a$       | $y = 0$     |
| $bc = ad$           | Arbitrary               | $y = a/c > 0$ |

1 Parameter $\varphi$ needs to meet the condition $\varphi \in (0, 1)$ first to ensure the practical implications.

expressions for these solutions are omitted here due to limited space. We turn to numerical simulations to study the difference in profits in the two situations in Section 5.

5. Simulations and sensitivity analyses. This section first uses specific numerical examples to illustrate the solution procedure and the theoretical results. Then, to further explore the relationships between critical factors in this optimization problem, the sensitivity of the critical parameters in the wholesale price contract situation and the consignment contract situation is analyzed. MATLAB R2019b is adopted to conduct the simulations and sensitivity analyses in this section.

5.1. Numerical examples. In this subsection, we use specific numerical examples to illustrate the solution procedure and the theoretical results. Specifically, we collect data from Firm Y and its manufacturer in China. Firm Y is a giant online sales platform. Many third-party manufacturers sell their products on this platform. For each item sold, Firm Y charges a specific proportion of its revenue. According to the empirical data of Firm Y and its manufacturer, we know that the advertising effort sensitivity coefficient of product goodwill needs to meet the condition $\theta = 0.8$ and its decay rate follows $\delta = 0.4$. Other parameters are defined as follows: $D_0 = 10, G_0 = 0, T = 1, \alpha = 0.7, \beta = 0.5, \varphi = 0.2, \mu = 1$. These parameters and their relationships meet the requirements for parameter constraints. In this setting, both the global stability and non-negativity of the strategies are ensured.

According to Proposition 1, the optimal retail markup, wholesale price, and advertising effort in the wholesale price contract situation can be given by

$$s^W = 11.111 - 1.4524e^{0.32071t} - 2.5159e^{-0.32071t},$$  \hfill (22a)
$$w^W = 5.5556 - 0.7262e^{0.32071t} - 1.2579e^{-0.32071t},$$  \hfill (22b)
$$p^W = 16.667 - 2.1786e^{0.32071t} - 3.7738e^{-0.32071t},$$  \hfill (22c)
$$A^W = 5.5556 - 3.6637e^{0.32071t} - 0.69816e^{-0.32071t}$$  \hfill (22d)
$$G^W = 11.111 - 4.0667e^{0.32071t} - 7.0444e^{-0.32071t},$$  \hfill (22e)

and the optimal profits of the two partner firms are $J^W_R = 18.380, J^W_M = 8.926$. The total profit of the entire supply chain under the wholesale price contract is $J^W = 27.306$. 

According to Proposition 2, the optimal retail price and advertising effort in the consignment contract situation can be obtained as follows,

\[ s^C = 3.3334 - 0.74224e^{0.26186t} - 1.1625e^{-0.26186t}, \quad (23a) \]
\[ w^W = 13.334 - 2.969e^{0.26186t} - 4.6501e^{-0.26186t}, \quad (23b) \]
\[ p^W = 16.667 - 3.7112e^{0.26186t} - 5.8126e^{-0.26186t}, \quad (23c) \]
\[ A^W = 13.333 - 8.597e^{0.26186t} - 2.8103e^{-0.26186t}, \quad (23d) \]
\[ G^W = 26.667 - 10.391e^{0.26186t} - 16.275e^{-0.26186t}, \quad (23e) \]

and the optimal profits of the two partner firms are \( J^C_R = 7.483, J^C_M = 29.241 \).

The total profit of the entire supply chain under the consignment contract is \( J^C = 36.724 \).

From the solutions above, we know that the product’s retail price in the wholesale price contract situation is higher than that in the consignment contract situation. In addition, the trajectory of the retail price increases over time. Thus, in this case, the penetration pricing strategy is more suitable for the product. Moreover, the manufacturer’s advertising effort in both situations decreases over time. At the final time \( T \), his effort is 0. However, at time 0, the manufacturer’s advertising effort is higher in the consignment contract situation than in the wholesale price contract situation. This indicates that the retailer’s revenue-sharing activity can better motivate the manufacturer to put more effort into advertising the product. Furthermore, the decline in the manufacturer’s advertising effort in the consignment contract situation is more significant than in the wholesale price contract situation. This means that the incentive effect of revenue sharing is limited and decreases rapidly over time.

There is a Pareto improvement for the entire supply chain from the wholesale price contract situation to the consignment contract situation in the current parameter setting. The increase is 34.5\%. However, the changing rules of individual partner firms are not the same. According to these numerical examples, when the retailer has 20\% of the sales revenue, her profit in the consignment contract situation will be lower than in the wholesale price contract situation. As for the manufacturer, after receiving his share of the retailer’s sales revenue, his profit in the consignment contract situation is higher than in the wholesale price contract situation. Thus, it is better for the manufacturer to use the consignment contract in the current setting, but the retailer should choose the wholesale price contract.

5.2. Sensitivity analyses. To further study the relationships between critical factors in this optimization problem, the sensitivity of the critical parameters is analyzed in this subsection. As our objective is to study the optimal advertising and pricing strategies in the wholesale price contract situation and the consignment contract situation, the following parameters are essential in our analysis: retail price sensitivity coefficient \( \alpha \), goodwill sensitivity coefficient \( \beta \), advertising effort sensitivity coefficient \( \theta \), advertising decay rate \( \delta \), initial goodwill \( G_0 \), and selling period \( T \). Specifically, we analyze the sensitivity of the retailer’s proportion of sales revenue \( \varphi \) under the consignment contract. We change the value of a specific parameter once and keep all other parameters at their baseline values. Both the global stability and non-negativity of the strategies are ensured for all values. The sensitivity analysis results are reported in Table 3 and 4. The optimal strategies are analyzed at time 0 and at the final time \( T \) of the selling period. The relationships
Table 3. Sensitivity analysis: wholesale price contract.

|   | p(0) | p(T) | w(0) | w(T) | A(0) | A(T) | J_R  | J_M  |
|---|------|------|------|------|------|------|------|------|
| α | 0.8  | 9.3750 | 9.5367 | 3.1250 | 3.1789 | 1.0427 | 0.0000 | 16.0238 | 7.8110 |
|   | 0.7  | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
|   | 0.6  | 12.5003 | 12.7892 | 4.1667 | 4.2630 | 1.3959 | 0.0000 | 21.5476 | 10.4131 |
| β | 0.6  | 10.7143 | 11.0215 | 3.5714 | 3.6738 | 1.4413 | 0.0000 | 18.6181 | 8.9239 |
|   | 0.5  | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
|   | 0.4  | 10.7143 | 10.8489 | 3.5714 | 3.6163 | 0.9502 | 0.0000 | 18.1885 | 8.9277 |
| θ | 0.9  | 10.7143 | 10.9834 | 3.5714 | 3.6611 | 1.3479 | 0.0000 | 18.5230 | 8.9250 |
|   | 0.8  | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
|   | 0.7  | 10.7143 | 10.8756 | 3.5714 | 3.6252 | 1.0411 | 0.0000 | 18.2548 | 8.9273 |
| δ | 0.6  | 10.7143 | 10.8901 | 3.5714 | 3.6300 | 1.0869 | 0.0000 | 18.3113 | 8.9269 |
|   | 0.5  | 10.7143 | 10.9070 | 3.5714 | 3.6357 | 1.1385 | 0.0000 | 18.3439 | 8.9267 |
|   | 0.4  | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
| G₀| 0    | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
|   | 5    | 13.3929 | 12.7621 | 4.4643 | 4.2540 | 1.4428 | 0.0000 | 26.7325 | 12.9890 |
|   | 10   | 16.0714 | 14.5983 | 5.3571 | 4.8661 | 1.6918 | 0.0000 | 36.6622 | 17.8197 |
| T | 1    | 10.7143 | 10.9259 | 3.5714 | 3.6420 | 1.1937 | 0.0000 | 18.3797 | 8.9264 |
|   | 2    | 10.7143 | 11.3267 | 3.5714 | 3.7756 | 2.0493 | 0.0000 | 39.0824 | 17.8136 |
|   | 3    | 10.7143 | 11.7422 | 3.5714 | 3.9141 | 2.6798 | 0.0000 | 62.9809 | 26.5708 |

between the key coefficients and the optimal strategies in the wholesale price contract situation are shown in Table 3. The relationships between the key coefficients and the optimal strategies in the consignment contract situation are shown in Table 4.

Based on Table 3 and 4, we first focus on each contract situation. Table 3 and 4 show that the retail price (both \( p(0) \) and \( p(T) \)) shares a negative relationship with the consumer price coefficient (\( \alpha \)). The increase in the retail price from time 0 to final time \( T \) decreases as consumers become more sensitive to the retail price. For the wholesale price contract, for both \( w(0) \) and \( w(T) \) and the manufacturer’s advertising effort (only \( A(0) \)), their changing trends are similar to the retail price except that the manufacturer’s advertising effort (\( A(T) \)) at final time \( T \) remains 0. The profits of the retailer and the manufacturer decrease when consumers care more about the price of the product. Second, we compare the strategies of the two situations and find that when consumers care more about the product’s retail price, the retailer tends to set a higher retail price under the wholesale price contract. However, the wholesale price charged by the manufacturer is higher under the consignment contract. Correspondingly, the manufacturer’s profit and advertising effort are higher under the consignment contract than under the wholesale price contract. This suggests that by using the consignment contract, the retailer can better motivate the manufacturer.

The consumers’ sensitivity to the product goodwill (\( \beta \)) also significantly influences the strategies and the contract choice of the retailer and the manufacturer. First, we focus on each contract situation. The initial states of the retail price (\( p(0) \)) and the wholesale price (\( w(0) \)) are not correlated with the consumers’ sensitivity to goodwill. However, their final states (\( p(T) \) and \( w(T) \)) are positively correlated...
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Table 4. Sensitivity analysis: consignment contract.

|     | \( p(0) \) | \( p(T) \) | \( w(0) \) | \( w(T) \) | \( A(0) \) | \( A(T) \) | \( J_R \)   | \( J_M \)   |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( \alpha \) | 0.8       | 6.2500    | 6.4241    | 5.0000    | 5.1393    | 1.6806    | 0.0000    | 6.5087    | 25.5112   |
|     | 0.7       | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 0.6       | 8.3333    | 8.6454    | 6.6667    | 6.9164    | 2.2555    | 0.0000    | 8.7988    | 34.2495   |
| \( \beta \) | 0.6       | 7.1429    | 7.4757    | 5.7143    | 5.9805    | 2.3344    | 0.0000    | 7.6410    | 29.5481   |
|     | 0.5       | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 0.4       | 7.1429    | 7.2874    | 5.7143    | 5.8299    | 1.5255    | 0.0000    | 7.3570    | 28.9960   |
| \( \theta \) | 0.9       | 7.1429    | 7.4339    | 5.7143    | 5.9471    | 2.1798    | 0.0000    | 7.5775    | 29.4257   |
|     | 0.8       | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 0.7       | 7.1429    | 7.3164    | 5.7143    | 5.8531    | 1.6765    | 0.0000    | 7.4004    | 29.0811   |
| \( \delta \) | 0.6       | 7.1429    | 7.3321    | 5.7143    | 5.8657    | 1.7515    | 0.0000    | 7.4374    | 29.1536   |
|     | 0.5       | 7.1429    | 7.3505    | 5.7143    | 5.8804    | 1.8358    | 0.0000    | 7.4589    | 29.1955   |
|     | 0.4       | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
| \( G_0 \) | 0         | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 5         | 8.9286    | 8.6119    | 7.1429    | 6.8896    | 2.3277    | 0.0000    | 10.8800   | 42.5342   |
|     | 10        | 10.7143   | 9.8528    | 8.5714    | 7.8823    | 2.7293    | 0.0000    | 14.9184   | 58.3372   |
| \( T \) | 1         | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 2         | 7.1429    | 7.8186    | 5.7143    | 6.2549    | 3.3651    | 0.0000    | 16.5499   | 61.4808   |
|     | 3         | 7.1429    | 8.3091    | 5.7143    | 6.6473    | 4.4927    | 0.0000    | 28.0220   | 97.9091   |
| \( \varphi \) | 0.2       | 7.1429    | 7.3711    | 5.7143    | 5.8969    | 1.9260    | 0.0000    | 7.4825    | 29.2415   |
|     | 0.4       | 7.1429    | 7.3128    | 4.2857    | 4.3877    | 1.4365    | 0.0000    | 14.7899   | 21.8029   |
|     | 0.6       | 7.1429    | 7.2553    | 2.8571    | 2.9021    | 0.9523    | 0.0000    | 21.9276   | 14.4509   |

with the consumers’ sensitivity to goodwill. This means that when consumers pay more attention to the product’s goodwill, both the retailer and the manufacturer tend to increase their price gradually during the selling period. Moreover, it is better for the manufacturer to put more effort on advertising at time 0 (\( A(0) \)) when the consumers’ goodwill sensitivity is high. Second, we compare the strategies of the two situations and find that when consumers care more about the product’s goodwill, the retailer tends to set a higher retail price under the wholesale price contract. However, the wholesale price charged by the manufacturer is higher under the consignment contract. The increase in these two prices is much higher under the consignment contract than under the wholesale price contract. The profit values suggest that the contract choice of the retailer and the manufacturer are opposite. However, the conflict of contract choice between the manufacturer and the retailer can be weakened when consumers care more about the product’s goodwill.

The characteristic of the manufacturer’s advertising effort sensitivity coefficient \( (\theta) \) is similar to the consumer sensitivity coefficient to the product’s goodwill \( (\beta) \). To avoid redundancy, we only discuss the managerial implications here. From the perspective of a single contract, when the manufacturer’s advertising effort can improve product goodwill, both the retailer and the manufacturer tend to increase their price gradually during the selling period. Therefore, it is preferable for the manufacturer to put more effort into advertising at time 0 when consumers’ sensitivity to goodwill is high. When comparing the contracts, we find that the increase
in the retail price and the wholesale price under the consignment contract is higher than under the wholesale price contract. Under the wholesale price contract, it is better for the retailer to set a higher price to make more profit. However, the wholesale price charged by the manufacturer should be higher under the consignment contract.

The parameter $\delta$ represents the decay rate of the product’s goodwill. First, we focus on each contract situation. The initial states of the retail price ($p(0)$) and the wholesale price ($w(0)$), and the final state of advertising ($A(T)$) are not correlated with the decay rate of goodwill ($\delta$). However, their final states ($p(T)$ and $w(T)$) are negatively correlated with $\delta$. When consumers are forgetful, or the advertising market is highly competitive, it is better for both the retailer and the manufacturer to increase their price gradually during the selling period. Moreover, it is better for the manufacturer to put less effort into advertising at time 0 ($A(0)$) when goodwill decreases rapidly. Second, we compare the strategies of the two situations and find that when the product’s goodwill decreases rapidly, it is better for the retailer to choose the wholesale price contract because she can set a higher price and make more profit. However, the retailer should choose the consignment contract to make more wholesale profit.

$G_0$ is the initial state of the product’s goodwill. From the perspective of a single contract, $G_0$ has a positive relationship with the retail price (both $p(0)$ and $p(T)$), the wholesale price (both $w(0)$ and $w(T)$), and the manufacturer’s advertising effort (only $A(0)$). This indicates that when the product’s goodwill is high at time 0, higher prices and more advertising benefit both the retailer and the manufacturer, as these strategies increase their profits. However, when $G_0$ is high enough, both the retailer and the manufacturer tend to gradually reduce their price during the selling period. When comparing the contracts, it can be seen that when $G_0$ is high, it is better for the retailer to choose the wholesale price contract and for the manufacturer to choose the consignment contract. In other words, their choice of contract is the opposite.

The characteristic of the selling period ($T$) is similar to the decay rate of the product’s goodwill ($\delta$). To avoid redundancy, we only discuss the managerial implications here. From the perspective of a single contract, when the selling period becomes more extended, both the retailer and the manufacturer tend to increase their price gradually during the selling period. Therefore, it is better for the manufacturer to put more effort into advertising at time 0 during a more extended selling period. When comparing the contracts, we find that the increase in the retail price and the wholesale price is higher under the consignment contract than under the wholesale price contract. Under the wholesale price contract, the retailer should set a higher price to make more profit. However, the wholesale price charged by the manufacturer should be higher under the consignment contract.

Regarding the retailer’s proportion of sales revenue $\varphi$ in the consignment contract situation, it is not correlated with the retail price at time 0 and the manufacturer’s advertising effort at the final time $T$. However, $\varphi$ significantly influences other decision variables. Specifically, when the retailer has a higher proportion of sales revenue, this will lead to a lower retail price and wholesale price at the end of the selling period. This suggests that the retailer sets a higher price when her proportion of revenue is lower to maintain her profit. Moreover, the retailer’s higher revenue proportion results in less advertising effort from the manufacturer at time 0. This indicates that the manufacturer can be motivated effectively if the retailer
keeps a lower proportion of the sales revenue. In terms of profit in the consignment contract situation, the retailer’s profit is positively related to her revenue proportion. In contrast, the manufacturer’s profit is negatively related to the retailer’s revenue proportion. In addition, the increase in the retailer’s profit due to her higher revenue proportion is more significant than the decrease in the manufacturer’s profit.

5.3. **Comparison of situations with a different \( \varphi \).** In this subsection, to further examine the impact of \( \varphi \) on channel members’ behaviors in the wholesale price contract situation and the consignment contract situation, we compare the two situations by changing \( \varphi \) as follows.

Figures 2(a)-2(d) show that the optimal strategies for the retailer and the manufacturer under the consignment contract change with the retailer’s proportion of revenue (\( \varphi \)). In contrast, their optimal strategies under the wholesale price contract are not correlated with \( \varphi \). Specifically, according to Figure 2(a), the optimal retail price at the final time \( T \) under the consignment contract slowly decreases when \( \varphi \) increases. It is always lower than the optimal retail price under the wholesale price contract in the interval \( \varphi \in (0, 1) \). However, the relationship between the wholesale price and the retailer’s proportion of revenue is the opposite. As shown in Figure 2(b), the optimal wholesale price at the final time \( T \) under the consignment contract increases when \( \varphi \) increases. Notably, the point of intersection of the wholesale price in the two situations is \( X(0.50, 3.64) \). This means that when the retailer has less than 50% of her sales revenue, the wholesale price under the consignment contract is lower than under the wholesale price contract. When the retailer has no sales revenue, the wholesale price is 0. This indicates that the retailer cannot increase her revenue proportion in a discretionary manner, even if she is the supply chain leader. Because when the retailer’s revenue proportion is high, the manufacturer tends to increase his profit by increasing the wholesale price, which is not beneficial to the retailer or the entire supply chain.

Figures 2(c) and 2(d) show the changing trends in the manufacturer’s advertising effort and the product’s goodwill, respectively. Their relationships with the retailer’s proportion of revenue (\( \varphi \)) are similar because, in our theoretical model, the product’s goodwill depends entirely on the manufacturer’s advertising effort. Both the manufacturer’s optimal advertising effort and the product’s goodwill are negatively related to \( \varphi \). This means that as the retailer’s proportion of sales revenue increases, the manufacturer will put less effort into advertising the product, so the product’s goodwill will decrease. Furthermore, their intersection points with \( \varphi \) are at 50%. This indicates that when the retailer has 50% of sales revenue and shares 50% with the manufacturer, the manufacturer’s advertising effort and the product’s goodwill in both situations are the same. However, if the retailer has more than 50% of sales revenue, the manufacturer tends to reduce his level of effort, and the product’s goodwill decreases accordingly. As the increase in the retailer’s proportion of sales revenue can lead to decreases in the manufacturer’s advertising effort and the product’s goodwill, the retailer should be careful when deciding her proportion of sales revenue under the consignment contract. It is better for the retailer not to set her revenue proportion above 50%. Otherwise, the product’s goodwill will be lower under the consignment contract than under the wholesale price contract.

The profits of the retailer and the manufacturer in both situations are shown in Figures 2(e) and 2(f), respectively. As shown, the optimal profits for the retailer and the manufacturer under the consignment contract change with the retailer’s proportion of revenue (\( \varphi \)), whereas their optimal profits under the wholesale price contract
Figure 2. Comparison of situations with a different $\varphi$.

are not correlated with $\varphi$. Specifically, according to Figure 2(e), the retailer’s profit under the consignment contract increases sharply when $\varphi$ increases. The point of intersection of the retailer’s profits in the two situations is $X(0.50, 18.38)$. This indicates that when the retailer has less than 50% of sales revenue, her profit will be less in the consignment contract situation than in the wholesale price contract situation. In this case, it is better for the retailer to choose the wholesale price contract. However, suppose the retailer has more than 50% sales revenue and shares the rest with the manufacturer. In that case, her profit will be higher in the consignment
contract situation than in the wholesale price contract situation. In this case, choosing the consignment contract is better for the retailer. According to Figure 2(f), the manufacturer’s profit decreases when \( \varphi \) increases under the consignment contract and reaches 0 when \( \varphi = 1 \). The point of intersection of the manufacturer’s profits in the two situations is \( X(0.75, 8.93) \). This indicates that when the retailer has less than 75% of sales revenue, the manufacturer’s profit in the consignment contract situation will be higher than in the wholesale price contract situation. In this case, the manufacturer should choose the consignment contract. However, suppose the retailer has more than 75% sales revenue and shares the rest with the manufacturer. In that case, the manufacturer’s profit in the consignment contract situation will be lower than in the wholesale price contract situation. In this case, choosing the wholesale price contract is better for the manufacturer.

Based on the profit comparison above, we know that the retailer’s revenue proportion (\( \varphi \)) significantly affects both the retailer’s and manufacturer’s choice of contract. In general, when the retailer has less than 50% or more than 75% of sales revenue, the two partner firms choose different types of contracts. However, their choice of contract can be consistent when the retailer’s proportion of revenue is between 50% and 75%. It is better for both the retailer and the manufacturer to choose the consignment contract in this interval.

6. Conclusions. Advertising has a significant impact on a product’s goodwill. To further improve goodwill and make more profit, partner firms in the supply chain use different contracts to coordinate the channel. The wholesale price contract is one of the most commonly used contracts due to its adaptability to traditional channel relationships. However, with the recent transformation of the business model, more and more retailers (especially online sales platforms) play a dominant role in the supply chain and favor the consignment contract. Therefore, the impacts of these two types of contracts on partner firms’ advertising and pricing strategies in the supply chain require further research.

In this context, considering the dynamic effect of advertising, this study focuses on a two-level supply chain consisting of one manufacturer and one retailer under two contracts: the wholesale price contract and the consignment contract. The two partner firms play a Stackelberg game in which the retailer is the leader, and the manufacturer is the follower in a certain selling period. We use differential game theory to study this dynamic optimization problem. We investigate the optimal strategies for retail price, wholesale price, and advertising effort. We also compare these optimal strategies in the wholesale price contract situation and the consignment contract situation. Finally, we present numerical examples and sensitivity analyses to support and supplement our analytical results.

The main conclusions and suggestions are as follows. (1) The retailer’s revenue proportion significantly affects the strategies and the contract choice of the two partner firms in the supply chain. Specifically, when the retailer has less than 50% or more than 75% of sales revenue, the two partner firms choose different types of contracts. However, the contract choice of the two partner firms can be consistent when the retailer’s proportion of revenue is between 50% and 75%. It is better for both the retailer and the manufacturer to choose the consignment contract in this interval. (2) Although the retailer is the leader in the supply chain, she cannot increase her revenue proportion in a discretionary manner. A higher revenue proportion will increase the manufacturer’s wholesale price and decrease
his advertising effort. These results are detrimental to the interests of the retailer and the entire supply chain. (3) The product’s goodwill according to consumers has a significant influence on the retailer and the manufacturer’s strategies and choice of contract. For example, the retail price and the wholesale price at the final time T, and the manufacturer’s advertising effort at time 0 under the wholesale price contract and the consignment contract increase with consumers’ sensitivity to the product’s goodwill. However, the conflict of contract selection between the manufacturer and the retailer can be weakened when consumers care more about the product’s goodwill.

This study makes several contributions to research on advertising strategies in the supply chain. However, some areas should be further explored in further research. For instance, the retailer’s proportion of sales revenue is assumed to be exogenous in our study, although this revenue-sharing proportion can be endogenous and determined by the retailer. Moreover, in our study, consumers’ preferences and market demand are determined ex-ante and known by both partner firms. However, they can be unpredictable or unknowable in practice. It would be interesting to use stochastic variables to represent their properties. These extensions are left for further discussion.

Appendix.

Proof of Proposition 1. By replacing Equations (7a)-(7c) in Equations (1),(8a), (8b), the following expressions can be obtained,

\[
\dot{G} = \frac{\theta^2 \lambda_M}{\mu} - \delta G, \quad (24a)
\]
\[
\dot{\lambda}_R = \delta \lambda_R - \frac{\beta}{2\alpha} (D_0 + \beta G), \quad (24b)
\]
\[
\dot{\lambda}_M = \delta \lambda_M - \frac{\beta}{4\alpha} (D_0 + \beta G). \quad (24c)
\]

Their matrix form is

\[
\begin{bmatrix}
\dot{G} \\
\dot{\lambda}_R \\
\dot{\lambda}_M
\end{bmatrix} = \begin{bmatrix}
G \\
\lambda_R \\
\lambda_M
\end{bmatrix} + Y, \quad (25)
\]

where

\[
X = \begin{bmatrix}
-\delta & 0 & \frac{\theta^2}{\mu} \\
-\frac{\beta^2}{2\alpha} & \delta & 0 \\
-\frac{\beta^2}{4\alpha} & 0 & \delta
\end{bmatrix}, \quad Y = \begin{bmatrix}
0 \\
-\frac{\beta D_0}{2\alpha} \\
-\frac{\beta D_0}{4\alpha}
\end{bmatrix}. \quad (26)
\]

The eigenvalues of X and the matrix of X’s eigenvectors are

\[
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = \begin{bmatrix}
\delta \\
\sqrt{\alpha \mu (4\alpha \delta^2 \mu - \beta^2 \theta^2)} \\
-\sqrt{\alpha \mu (4\alpha \delta^2 \mu - \beta^2 \theta^2)}
\end{bmatrix}, \quad (27)
\]

where \(4\alpha \delta^2 \mu - \beta^2 \theta^2 \geq 0\). This condition is to ensure the mathematical non-negativity of the square root. It further ensures the existence and rationality of
solutions. This condition is adopted by many prior studies, including Martín-Herrán and Taboubi [36], Dai and Zhang [15], Zhang et al. [51] and so on.

\[
H = \begin{bmatrix}
0 & \frac{2\alpha}{\beta^2}(\delta - r_2) & \frac{2\alpha}{\beta^2}(\delta - r_3) \\
1 & 1 & 1 \\
1 & \frac{1}{\delta} & \frac{1}{\delta}
\end{bmatrix}
\]  

(28)

Hence,

\[
\begin{bmatrix}
G \\
\lambda_R \\
\lambda_M
\end{bmatrix} = H \begin{bmatrix}
e^{r_1 t} & 0 & 0 \\
0 & e^{r_2 t} & 0 \\
0 & 0 & e^{r_3 t}
\end{bmatrix} \begin{bmatrix}k_1 \\ k_2 \\ k_3\end{bmatrix} - X^{-1}Y,
\]

(29)

where

\[
X^{-1} = \begin{bmatrix}
-\frac{4\alpha\delta\mu}{4\alpha\delta^2\mu - \beta^2\theta^2} & 0 & \frac{4\alpha\delta^2}{4\alpha\delta^2\mu - \beta^2\theta^2} \\
\frac{4\alpha\delta^2\mu - \beta^2\theta^2}{2\beta^2\theta^2} & \frac{1}{5} & \frac{2\beta\delta\mu D_0}{4\alpha\delta^2\mu - \beta^2\theta^2} \\
\frac{4\alpha\delta^2\mu - \beta^2\theta^2}{2\beta^2\theta^2} & 0 & \frac{4\alpha\delta\mu D_0}{4\alpha\delta^2\mu - \beta^2\theta^2}
\end{bmatrix}.
\]

(30)

According to the following conditions

\[
\begin{cases}
G(0) = G_0, \\
\lambda_R(T) = 0, \\
\lambda_M(T) = 0,
\end{cases}
\]

(31)

the parameters \(k_1, k_2, k_3\) can be calculated. Thus, the product’s optimal goodwill \(G\) and two adjoint variables \(\lambda_R, \lambda_M\) can be obtained as follows

\[
G = \frac{2\alpha}{\beta^2}(\delta - r_2)k_2e^{r_2 t} + \frac{2\alpha}{\beta^2}(\delta - r_3)k_3e^{r_3 t} + \frac{\beta\delta^2 D_0}{4\alpha\delta^2\mu - \beta^2\theta^2},
\]

(32a)

\[
\lambda_R = k_1e^{r_1 t} + k_2e^{r_2 t} + k_3e^{r_3 t} + \frac{2\beta\delta\mu D_0}{4\alpha\delta^2\mu - \beta^2\theta^2},
\]

(32b)

\[
\lambda_M = k_1e^{r_1 t} + \frac{1}{2}k_2e^{r_2 t} + \frac{1}{2}k_3e^{r_3 t} + \frac{\beta\delta\mu D_0}{4\alpha\delta^2\mu - \beta^2\theta^2}.
\]

(32c)

Then, by replacing Equations (32a)-(32c) in Equations (7a)-(7c), we obtain Proposition 1.

**Proof of Proposition 2.** By replacing Equations (14a),(14b) in Equations (1), (15a), (15b), the following expressions can be obtained,

\[
\hat{G} = \frac{\theta^2\lambda_M}{\mu} - \delta G,
\]

(33a)

\[
\hat{\lambda}_R = \delta \lambda_R - \frac{\varphi\beta}{2\alpha}(D_0 + \beta G),
\]

(33b)

\[
\hat{\lambda}_M = \delta \lambda_M - \frac{(1 - \varphi)\beta}{2\alpha}(D_0 + \beta G).
\]

(33c)

Their matrix form is

\[
\begin{bmatrix}
\hat{G} \\
\hat{\lambda}_R \\
\hat{\lambda}_M
\end{bmatrix} = X \begin{bmatrix}G \\ \lambda_R \\ \lambda_M\end{bmatrix} + Y,
\]

(34)
where
\[
X = \begin{bmatrix}
-\delta & 0 & \frac{\phi^2}{\mu} \\
-\frac{\phi \beta^2}{2\alpha} & \delta & 0 \\
\frac{(1-\phi) \beta^2}{2\alpha} & 0 & \delta
\end{bmatrix}, \quad
Y = \begin{bmatrix}
0 \\
-\frac{\phi \beta D_0}{2\alpha} \\
-\frac{(1-\phi) \beta D_0}{2\alpha}
\end{bmatrix}.
\]
(35)

The eigenvalues of \(X\) and the matrix of \(X\)'s eigenvectors are
\[
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = \begin{bmatrix}
\delta \\
\sqrt{\frac{\Omega}{2\alpha \mu}} \\
-\sqrt{\frac{\Omega}{2\alpha \mu}}
\end{bmatrix},
\]
(36)
where \(\Omega = 2\alpha \delta^2 \mu + (\varphi - 1) \beta^2 \theta^2 \geq 0\).

\[
H = \begin{bmatrix}
0 & \frac{2x}{\beta^2} (\delta - r_2) & \frac{2x}{\beta^2} (\delta - r_3) \\
1 & 1 & 1 \\
0 & \frac{1-\varphi}{\varphi} & \frac{1-\varphi}{\varphi}
\end{bmatrix}
\]
(37)

Hence,
\[
\begin{bmatrix}
G \\
\lambda_R \\
\lambda_M
\end{bmatrix} = H \begin{bmatrix}
e^{r_1 t} & 0 & 0 \\
e^{r_2 t} & 0 & 0 \\
e^{r_3 t} & 0 & 0
\end{bmatrix} \begin{bmatrix}
k_1 \\
k_2 \\
k_3
\end{bmatrix} - X^{-1} Y,
\]
(38)

where
\[
X^{-1} = \begin{bmatrix}
-\frac{2\alpha \delta \mu}{\Omega} & 0 & \frac{2\alpha \theta^2}{\Omega} \\
-\frac{\phi \beta^2 \mu}{\Omega} & \frac{1}{\delta} & \frac{\phi \beta \theta^2}{\delta \Omega} \\
\frac{(1-\phi) \beta^2 \mu}{\Omega} & 0 & \frac{2\alpha \mu}{\Omega}
\end{bmatrix}.
\]
(39)

According to the following conditions
\[
\begin{cases}
G(0) = G_0, \\
\lambda_R(T) = 0, \\
\lambda_M(T) = 0,
\end{cases}
\]
(40)
the parameters \(k_1, k_2, k_3\) can be calculated. Thus, the product’s optimal goodwill \(G\) and two adjoint variables \(\lambda_R, \lambda_M\) can be obtained as follows,

\[
G = \frac{2\alpha}{\varphi \beta^2} (\delta - r_2) k_2 e^{r_2 t} + \frac{2\alpha}{\varphi \beta^2} (\delta - r_3) k_3 e^{r_3 t} + \frac{(1-\varphi) \beta \theta D_0}{\Omega},
\]
(41a)
\[
\lambda_R = k_1 e^{r_1 t} + k_2 e^{r_2 t} + k_3 e^{r_3 t} + \frac{\varphi \beta \delta \mu D_0}{\Omega},
\]
(41b)
\[
\lambda_M = \frac{1-\varphi}{\varphi} k_2 e^{r_2 t} + \frac{1-\varphi}{\varphi} k_3 e^{r_3 t} + \frac{(1-\varphi) \beta \delta \mu D_0}{\Omega}.
\]
(41c)

Then, by replacing Equations (41a)-(41c) in Equations (14a), (14b), we obtain Proposition 2.

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