On the linear independence of the special values of a Dirichlet series with periodic coefficients

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Abstract

A lower bound for the dimension of the $\mathbb{Q}$-vector space spanned by special values of a Dirichlet series with periodic coefficients is given. As a corollary, it is deduced that both special values at even integers and at odd integers contain infinitely many irrational numbers. This result is proved by T.Rivoal if the function considered is the Riemann zeta function, and this paper gives its generalization to more general Dirichlet series.

1 Introduction

The special values of the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

at even integers are transcendental, since they are rational multiples of powers of $\pi$. On the other hand, arithmetic nature of the special values $\zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ at odd integers are still not well-understood enough. In this direction, the following results are known:

- $\zeta(3)$ is irrational. (R.Apéry [1], 1978)
- $\dim_{\mathbb{Q}}(\mathbb{Q} \text{-span}\{1, \zeta(3), \zeta(5), \zeta(7), \ldots\}) = \infty$. In particular, infinitely many of the numbers $\zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ are irrational. (T.Rivoal [2], 2000)
- For each odd integer $s \geq 1$, at least one of the numbers $\zeta(s + 2), \zeta(s + 4), \ldots, \zeta(8s - 1)$ is irrational. (W.Zudilin [5], 2001)
- At least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. (W.Zudilin [7], 2001)

Similarly, let us consider the arithmetic nature of values of a Dirichlet $L$-function

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s},$$

where $\chi$ is a Dirichlet character modulo $d$. A special value of $L(s, \chi)$ is well-understood if $s$ satisfies $\chi(-1) = (-1)^s$. In this case, the inclusion

$$\frac{L(s, \chi)}{\pi^s} \in \mathbb{Q}(\epsilon^{2\pi i/d}, i)$$

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holds, and in particular, \( L(s, \chi) \) is transcendental. For example, let \( \chi_3 \) (resp. \( \chi_4 \)) be the Dirichlet character modulo 3 (resp. modulo 4) which is not trivial, then the following formulas are known:

\[
\begin{align*}
L(1, \chi_3) &= \frac{\pi}{3\sqrt{3}}, & L(3, \chi_3) &= \frac{4\pi^3}{81\sqrt{3}}, & L(5, \chi_3) &= \frac{4\pi^5}{729\sqrt{3}}, \\
L(1, \chi_4) &= \frac{\pi}{4}, & L(3, \chi_4) &= \frac{\pi^3}{32}, & L(5, \chi_4) &= \frac{5\pi^5}{1536}.
\end{align*}
\]

On the other hand, special values at positive integers \( s \) satisfying \( \chi(-1) \neq (-1)^s \) (for example, even integers \( s \) for \( \chi = \chi_3, \chi_4 \)) are not well-understood. In this direction, there are a few results, as following:

- Let denote by \( \chi_5 \) the real Dirichlet character modulo 5 which is not trivial. Then the inclusion \( 8\zeta(3) - 5\sqrt{5}L(3, \chi_5) \notin \mathbb{Q}(\sqrt{5}) \) holds. (F. Beukers [3], 1987)
- For \( L(s) = L(s, \chi_4) \), the following results are shown (T. Rivoal and W. Zudilin [5], 2002)
  - At least one of the numbers \( L(2), L(4), L(6), L(8), L(10), L(12), L(14) \) is irrational.
  - \( \dim_{\mathbb{Q}}(\mathbb{Q}\text{-span}\{1, L(2), L(4), L(6), \ldots\}) = \infty \). In particular, infinitely many of the numbers \( L(2), L(4), L(6), \ldots \) are irrational.

The aim of this paper is to generalize the results for \( \zeta(s), L(s, \chi_4) \) to general Dirichlet series with periodic coefficients. The main theorem of this paper is Theorem 1.2

**Definition 1.1** A Dirichlet series

\[
L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s} \quad (a_k \in \mathbb{C})
\]

is called Dirichlet series of period \( d \) if \( a_{k+d} = a_k \) holds for each \( k = 1, 2, \ldots \), and we denote

\[
\delta(a; L) = \dim_{\mathbb{Q}}(\mathbb{Q}\text{-span}\{a_m, L(j) \mid 1 \leq m \leq d, 2 \leq j \leq a, j \equiv a(\text{mod} 2)\}).
\]

**Theorem 1.2** Let \( L \neq 0 \) be a Dirichlet series of period \( d \), and \( C \) a positive constant satisfying \( C > d + \log 2 \). Then we have

\[
\delta(a; L) \geq \frac{\log a}{C}
\]

for sufficiently large integers \( a \). In particular, we have

- \( \dim_{\mathbb{Q}}(\mathbb{Q}\text{-span}\{L(2), L(4), L(6), \ldots\}) = \infty \).
- \( \dim_{\mathbb{Q}}(\mathbb{Q}\text{-span}\{L(3), L(5), L(7), \ldots\}) = \infty \).

To obtain a lower bound for the dimension of the \( \mathbb{Q} \)-vector space spanned by \( m \) real numbers \( \theta_1, \ldots, \theta_m \), \( \mathbb{Z} \)-linear forms \( I = \sum_{j=1}^{m} A_j \theta_j \) such that \( |I| \) is very small with respect to the absolute values of the coefficients \( |A_j| \) are used. For example, the dimension is not less than 2 if we can take \( |I| \) arbitrary small. For higher dimensional cases, a criterion was shown by Nesterenko ([1], 1985).

Therefore, it is necessary to construct \( \mathbb{Z} \)-linear forms \( I \), consist of the values of \( L(s) \) at even (or odd) integers, such that \( |I| \) is very small with respect to absolute values of its coefficients. This is
equivalent to constructing \( \mathbb{Q} \)-linear forms \( I \) such that \( |I| \) is very small with respect to the absolute values and denominators of the coefficients.

In this paper, the \( \mathbb{Q} \)-linear forms \( I \) are constructed in section 2. This construction is a direct generalization of that of \([6]\). Absolute values and denominators of the coefficients are estimated in section 3, and \( |I| \) is estimated in section 4. To estimate \( |I| \), an integral representation of \( I \) and the saddle point method are used. The main theorem is proved in section 5 by applying the criterion of Nesterenko.

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## 2 Construction of the linear forms

Let

\[
L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}
\]

be a Dirichlet series of period \( d \). Assume that \( a_1, a_2, \ldots \) are reals, and not all \( a_k \) are 0. We denote by \( \zeta_m(s) \) the function

\[
\sum_{k \geq 1, \ k \equiv m(d)} \frac{1}{k^s}.
\]

Choose positive integers \( a, b \) satisfying \( a \geq 2b \). For each positive integer \( n \), consider the rational function \( P_n(t) \) defined by the equations

\[
P_n(t) = \frac{Q_n(t)}{R_n(t)} \cdot (2n!)^{a-2b} \cdot d^{2na},
\]

\[
Q_n(t) = \prod_{dn < l \leq (d+2b)n} (t - l)(t + l), \quad R_n(t) = \left( \prod_{-n \leq l \leq n} (t - dl) \right)^a.
\]

By the assumption on \( a \) and \( b \), we have

\[
deg R_n = 2an + a \geq 4bn + 2b \geq 4bn + 2 = \deg Q_n + 2. \tag{2.1}
\]

Therefore we can decompose \( P_n(t) \) into partial fractions:

\[
P_n(t) = \sum_{j=1}^{a} \sum_{l=-n}^{n} \frac{A_{l,j}(n)}{(t - dl)} \quad (A_{l,j}(n) \in \mathbb{Q}).
\]

Moreover, (2.1) implies

\[
\sum_{l=-n}^{n} A_{l,1}(n) = 0. \tag{2.2}
\]

**Remark 2.1** Although \( P_n(t), Q_n(t), R_n(t), A_{l,j}(n) \) depend on \( n \), we omit \( n \) from the notation if no confusion is possible, and denote them simply by \( P(t), Q(t), \) etc.
Lemma 2.2 For each integer $1 \leq j \leq a$, $j \not\equiv a \pmod{2}$, we have
\[ \sum_{l=-n}^{n} A_{l,j} = 0. \] (2.3)

Proof. Since $P(t)$ satisfies the relation $P(-t) = (-1)^a P(t)$, we have
\[ \sum_{j=1}^{a} \sum_{l=-n}^{n} \frac{(-1)^a A_{l,j}}{(t-dl)^j} = P(-t) = \sum_{j=1}^{a} \sum_{l=-n}^{n} \frac{(-1)^j A_{-l,j}}{(t-dl)^j}. \]

Since the decomposition into partial fractions is unique, we have $(-1)^a A_{l,j} = (-1)^j A_{-l,j}$. Hence the equation $A_{l,j} + A_{-l,j} = 0$ holds for each $-n \leq l \leq n$ if $j \not\equiv a \pmod{2}$. Therefore, we obtain (2.3). ■

Proposition 2.3 We have
\[ I_m = \sum_{k \equiv m(d)} \sum_{2 \leq j \leq a, \ j \not\equiv a(2)} A_j \zeta_m(j) - B_m, \]
where the coefficients $A_j$, $B_m$ are rationals defined by the following equations:
\[ A_j = \sum_{l=-n}^{n} A_{l,j}, \quad B_m = \sum_{1 \leq j \leq a, \ -n \leq l \leq n} A_{l,j} \left( \frac{1}{m^j} + \cdots + \frac{1}{(d(n-l-1)+m)^j} \right). \]

Proof. For each $N \geq n$, we have
\[ \sum_{d(N+1) \geq k > dn, \ k \equiv m(d)} \frac{1}{(k-dl)^j} = \left( \frac{1}{m^j} + \frac{1}{(d+m)^j} + \cdots + \frac{1}{(dN+m)^j} \right) + O(N^{-j}) \]
\[ - \left( \frac{1}{m^j} + \frac{1}{(d+m)^j} + \cdots + \frac{1}{(d(n-l-1)+m)^j} \right). \]

By taking the sum and using (2.2) and Lemma 2.2, we obtain
\[ P(k) = \sum_{d(N+1) \geq k > dn, \ k \equiv m(d)} \sum_{2 \leq j \leq a, \ j \not\equiv a(2)} \sum_{l=-n}^{n} A_{l,j} \left( \frac{1}{m^j} + \frac{1}{(d+m)^j} + \cdots + \frac{1}{(dN+m)^j} \right) + O(N^{-1}) \]
\[ - \sum_{j=1}^{a} \sum_{l=-n}^{n} A_{l,j} \left( \frac{1}{m^j} + \frac{1}{(d+m)^j} + \cdots + \frac{1}{(d(n-l-1)+m)^j} \right). \]

By taking the limit of both sides as $N \to \infty$, we obtain the assertion. ■
Let $I = \sum_{m=1}^{d} a_m I_m$.

**Proposition 2.4** We have

$$I = \sum_{2 \leq j \leq a, j \equiv a(2)} A_j L(j) - \sum_{m=1}^{d} B_m a_m,$$

where the coefficients $A_j$ and $B_m$ are rationals defined by the following equations:

$$A_j = \sum_{l=-n}^{n} A_{l,j}, \quad B_m = \sum_{\substack{1 \leq j \leq a, \, -n \leq l \leq n \atop s \leq l \leq n}} A_{l,j} \left( \frac{1}{m^j} + \cdots + \frac{1}{(d(n-l-1)+m)^j} \right).$$

**Proof.** This follows immediately from the definition of $I$ and Proposition 2.3. 

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### 3 Estimation for the coefficients

We denote by $\Delta_j$ the differential operator

$$\frac{1}{j!} \left( \frac{d}{dt} \right)^j.$$

Let $D_{2dn} = \text{lcm}\{1, 2, \ldots, 2dn\}$ and $R_0(t) = \prod_{-n \leq l \leq n} (t - dl)$.

**Lemma 3.1** Let $q$ be a polynomial of degree $\leq 2n$. Assume that the rational function $p(t) = q(t)/R_0(t)$ satisfies

$$p_k = \left( p(t)(t - dk) \right)|_{t=dk} \in \mathbb{Z}, \quad |p_k| \leq C$$

for each $-n \leq k \leq n$ and some positive constant $C$. Then we have

$$\left( D_{2dn} \right)^j \left( \Delta_j \left( p(t)(t - dk) \right) \right) |_{t=dk} \in \mathbb{Z}, \quad \left| \left( \Delta_j \left( p(t)(t - dk) \right) \right) |_{t=dk} \right| \leq \frac{2n}{d^j} C$$

for arbitrary integer $j \geq 0$.

**Proof.** It is trivial for $j = 0$, hence we may assume $j \geq 1$. Since $\deg q \leq 2n$, we can decompose $p(t)$ into partial fractions:

$$p(t) = \sum_{l=-n}^{n} \frac{p_l}{t - dl}. $$

By computation, we have

$$\Delta_j \left( \frac{p_l(t - dk)}{(t - dl)} \right) = \Delta_j \left( \frac{-p_l(dk - dl)}{(t - dl)} \right) = \frac{(-1)^{j+1} p_l(dk - dl)}{(t - dl)^{j+1}}.$$
for each $l \neq k$, therefore

$$
(\Delta_j(p(t)(t - dk)))_{t = dk} = (-1)^{j+1} \left( \sum_{-n \leq l \leq n, l \neq k} \frac{p_l}{(dk - dl)^j} \right).
$$

Thus, we have (3.1) since the inclusion

$$
\frac{D_{2dn}}{dk - dl} \in \mathbb{Z}
$$

holds for each $l \neq k$, and we have (3.2) since we have

$$
\frac{1}{(dk - dl)^j} \leq \frac{1}{d^j}
$$

for each $l \neq k$.

\[\Box\]

Let

$$
Q_i(t) = d^{2n} \prod_{(d+2(i-1))n < l \leq (d+2i)n} (t - l) \quad (1 \leq i \leq b),
$$

$$
Q_{-i}(t) = d^{2n} \prod_{(d+2(i-1))n < l \leq (d+2i)n} (t + l) \quad (1 \leq i \leq b).
$$

**Lemma 3.2** The following pairs $(q(t), C)$ satisfy the assumption of Lemma 3.1:

(i) $q(t) = Q_i(t) \quad (1 \leq i \leq b)$, $C = \frac{1}{(n!)^2} \prod_{(2d+2(i-1))n < l \leq (2d+2i)n} l$.

(ii) $q(t) = Q_{-i}(t) \quad (1 \leq i \leq b)$, $C = \frac{1}{(n!)^2} \prod_{(2d+2(i-1))n < l \leq (2d+2i)n} l$.

(iii) $q(t) = d^{2n} \cdot (2n)!$, $C = \frac{(2n)!}{(n!)^2}$.

**Proof.** (i) Let $p_k = Q_i(t)(t - dk))_{t = dk}$ for each $-n \leq k \leq n$, then we have $p_k \in \mathbb{Z}$ since

$$
p_k = \pm \left( \frac{d(n - k) + 2in}{2n} \right) \cdot \binom{2n}{n - k}.
$$

Moreover

$$
|p_k| \leq \left( \frac{(2d + 2i)n}{2n} \right) \cdot \binom{2n}{n} = \frac{1}{(n!)^2} \prod_{(2d+2(i-1))n < l \leq (2d+2i)n} l.
$$

Thus the pair $(q(t), C)$ of (i) satisfies the assumption of Lemma 3.1 (ii) is similar. (iii) We define $p_k$ as above, then we have $p_k = \pm \binom{2n}{n - k}$. Therefore we obtain $p_k \in \mathbb{Z}$ and

$$
|p_k| \leq \binom{2n}{n} = \frac{(2n)!}{(n!)^2},
$$

as required. \[\Box\]
Fix integers $l, j$ satisfying $-n \leq l \leq n$, $2 \leq j \leq a$, and $j \equiv a \pmod{2}$. Let us estimate the absolute value and the denominator of $A_{l,j}$. Define rational functions $p_1, \ldots, p_a$ as follows:

$$p_i(t) = \frac{Q_i(t)(t-dl)}{R_0(t)} \quad (1 \leq i \leq b),$$

$$p_i(t) = \frac{Q_{i-b}(t)(t-dl)}{R_0(t)} \quad (b + 1 \leq i \leq 2b),$$

$$p_i(t) = \frac{(2n!)d^{2n}(t-dl)}{R_0(t)} \quad (2b + 1 \leq i \leq a).$$

Then we have

$$P(t)(t-dl)^a = p_1(t) \cdots p_a(t).$$

**Proposition 3.3** The followings are true:

$$\frac{(D_{2dn})^{a-j} A_{l,j}}{n} \in \mathbb{Z}. \quad (3.3)$$

$$\log \left| A_{l,j} \right| n \leq 2a \log 2 + 4(b + d) \log(b + d) - 4d \log d + o(1) \quad \text{as } n \to \infty. \quad (3.4)$$

**Proof.** By the Leibniz rule, we have

$$A_{l,j} = \left. (\Delta_{a-j} P(t)(t-dl)^a) \right|_{t=dl} = \sum (\Delta_{j_1} p_1(dl) \cdots \Delta_{j_a} p_a(dl)),$$

where the sum is taken over all pairs $(j_1, \ldots, j_a)$ satisfying $a - j = j_1 + \cdots + j_a$. By Lemma 3.1 and Lemma 3.2 we have $(D_{2dn})^{a-j} \Delta_{j_i} p_i(dl) \in \mathbb{Z}$ for each $i,j_i$; hence we obtain (3.3).

Next, let us estimate $|A_{l,j}|$. Let

$$c_i = c_{b+i} = \frac{1}{(n!)^2} \prod_{(2i+2)(i-1)n < l \leq (2i+2)n} l \quad (1 \leq i \leq b),$$

$$c_i = \frac{(2n!)n}{(n!)^2} \quad (2b + 1 \leq i \leq a).$$

Then, by Lemma 3.1 and Lemma 3.2 we have

$$|\Delta_{j_i} p_i(dl)| \leq \frac{2nc_i}{d^i}$$

for each $i,j_i$. Therefore we have

$$|\Delta_{j_1} p_1(dl) \cdots \Delta_{j_a} p_a(dl)| \leq \frac{(2n!)^a}{d^{a-j} c_1 \cdots c_a}$$

for each pair $(j_1, \ldots, j_a)$. Since the number of the pairs $(j_1, \ldots, j_a)$ considered is not greater than \( \binom{2a-j-1}{a-1} \), we have

$$|A_{l,j}| \leq \binom{2a-j-1}{a-1} \frac{(2n!)^a}{d^{a-j} c_1 \cdots c_a}.$$

Therefore we have

$$\frac{\log |A_{l,j}|}{n} \leq \frac{\log(c_1 \cdots c_a)}{n} + o(1) \quad \text{as } n \to \infty.$$
Since 
\[ c_1 \cdots c_n = \frac{(2(b + d)n)! ((2n)!)^{a-2b}}{((2dn)!)^2 (n!)^{2a}}, \]
we obtain (3.4) by the Stirling formula. ■

**Proposition 3.4** The followings properties of \( A_j, B_m \) are true:

\[ (D_{2dn})^a A_j, (D_{2dn})^a B_m \in \mathbb{Z}. \] (3.5)

\[ \log \frac{|A_j|}{n}, \log \frac{|B_m|}{n} \leq 2a \log 2 + 4(b + d) \log (b + d) - 4d \log d + o(1) \quad \text{as } n \to \infty. \] (3.6)

**Proof.** By the definition of \( A_j \), we have \(|A_j| \leq (2n + 1) \cdot \max_{-n \leq l \leq n} |A_{l,j}|. \) Therefore we have

\[ \frac{\log |A_j|}{n} \leq \max_{-n \leq l \leq n} \frac{\log |A_{l,j}|}{n} + o(1) \quad \text{as } n \to \infty. \]

By Proposition 3.3, we obtain (3.5) for \( A_j \). Similarly, we can show (3.6) for \( B_m \).

Let us prove (3.5). By the definition of \( A_j \) and Proposition 3.3, the inclusion \((D_{2dn})^{a-j} A_j \in \mathbb{Z}\) holds. In particular, we have \((D_{2dn})^a A_j \in \mathbb{Z} \). Similarly, we have the inclusion

\[ (D_{2dn})^a \cdot \frac{A_{l,j}}{(dk + m)^j} = ((D_{2dn})^{a-j} A_{l,j}) \cdot \left( \frac{D_{2dn}}{dk + m} \right)^j \in \mathbb{Z}, \]

for each \( 0 \leq k < 2n \). Therefore we obtain \((D_{2dn})^a B_m \) by the definition of \( B_m \). ■

**4 Estimation of \(|I|\)**

**4.1 Integral representation of \( I \)**

Let \( r = (d + 2b)/d \).

**Proposition 4.1** The following integral representation hold for the sum \( I_m \):

\[ I_m = -\frac{n}{2i} \int_{x-i\infty}^{x+i\infty} P(dnt) \cot \left( \frac{(dnt - m)\pi}{d} \right) dt, \] (4.1)

where \( x \) is an arbitrary real number satisfying \( 1 < x < r \).

**Proof.** The well-known formula

\[ \pi \cot \pi t = \frac{1}{t} + \sum_{k=1}^{\infty} \left( \frac{1}{t - k} + \frac{1}{t + k} \right) \]

implies that the poles of the function

\[ \frac{\pi}{d} \cot \frac{\pi(t-m)}{d} \]
are \( t = k \ (k \equiv m \pmod{d}) \), and its principal part is \( \frac{1}{k} \).

Since \( P(t) \) has a zero of order 1 at each integer \( k \ (dn < k \leq drn) \), we have

\[
I_m = \sum_{k>dn, \ k\equiv m(d)} P(k) = \sum_{k>drn, \ k\equiv m(d)} P(k).
\]

Let \( M \) be a real number satisfying \( dn < M < (d+2b)n \), and \( N \) a sufficiently large integer. Consider the integral

\[
\frac{1}{2\pi i} \int_{\mathcal{R}} P(t) \cdot \frac{\pi}{d} \cot \frac{\pi(t-m)}{d} \, dt \tag{4.2}
\]

along the contour \( \mathcal{R} \) of the rectangle in Fig. 1.

![Figure 1: The contour \( \mathcal{R} \)](image)

Since \( P(t) \) has no poles in the region \( \text{Re}(t) > 1 \), the residue theorem implies that the integral (4.2) converges to \(-I_m \) as \( N \to \infty \). Moreover, the integral over the right, lower, and upper edge of the rectangle converges to 0 as \( N \to \infty \). Indeed, the length of the path is \( O(N) \), and the absolute value of the integrand is \( O(N^{-2}) \). Thus, we obtain the equation

\[
I_m = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{-M-Ni}^{M+Ni} P(t) \cdot \frac{\pi}{d} \cot \frac{\pi(t-m)}{d} \, dt = -\frac{1}{2\pi i} \int_{-M-Ni}^{M+Ni} P(t) \cdot \frac{\pi}{d} \cot \frac{\pi(t-m)}{d} \, dt.
\]

By substituting \( dnt \) for \( t \), we obtain (4.1). \( \square \)

**Lemma 4.2** The following formula holds uniformly in the strip \( 1 < \text{Re}(t) < r \):

\[
-\frac{n}{2i} \cdot \frac{P(dnt)}{\sin dnt \pi} = (1 + o(1))\varphi(n)e^{nJ(t)}g(t) \quad \text{as} \ n \to \infty. \tag{4.3}
\]
Here, $\varphi(n)$, $f(t)$, and $g(t)$ are functions defined by the following equations:

$$
\varphi(n) = \frac{-n}{2i} \cdot (-1)^{dn} \cdot 2^{2(a-2b)n+1+a-2b} \cdot \pi^{\frac{a-2b}{2}} \cdot d^{4bn+2-a} \cdot n^{\frac{4-a-2b}{2}},
$$

$$
f(t) = \frac{(t+r)\log(t+r) - (-t+r)\log(-t+r)}{2} + (a+d)((t-1)\log(t-1) - (t+1)\log(t+1)),
$$

$$
g(t) = \frac{(t+r)^{\frac{1}{2}}(-t+r)^{\frac{1}{2}}}{(t+1)^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}.
$$

**Proof.** We can express $Q(dnt)$, $R(dnt)$ by Gamma functions:

$$
Q(dnt) = \frac{\Gamma(dnt + drn + 1)}{\Gamma(dnt + dn)} \cdot \frac{\Gamma(dnt - dn + 1)}{\Gamma(dnt - drn)} = \frac{\Gamma(dnt - drn)}{\Gamma(nt - n)} \cdot \frac{\Gamma(nt - n + 1)}{\Gamma(dnt - dn + 1)} \cdot \frac{\Gamma(dnt - drn + 1)}{\Gamma(dnt + dn + 1)} \cdot \frac{\Gamma(dnt + dn + 1)}{\Gamma(nt - n + 1)}^{a}.
$$

By the functional equations of Gamma function, we have

$$
\Gamma(dnt - drn) = \frac{\Gamma(dnt + drn + 1)}{\Gamma(dnt - dn + 1)} \cdot \frac{\Gamma(nt - n + 1)}{\Gamma(dnt - dn + 1)} = \frac{(1-dn\pi)}{\sin(dnt\pi)} \cdot \frac{(1+dn\pi)}{\sin(nt\pi)} \cdot \frac{(1-nt\pi)}{\sin(nt\pi)} \cdot \frac{(1+nt\pi)}{\sin(nt\pi)}.
$$

Therefore we have

$$
P(dnt) = \frac{(-1)^{dn}(dnt + dn)}{\pi d^{a}(nt - n)^{a}} \cdot \frac{\Gamma(dnt + drn + 1)}{\Gamma(dnt - dn + 1)} \cdot \frac{\Gamma(nt - n + 1)}{\Gamma(dnt - dn + 1)} \cdot \frac{\Gamma(dnt + dn + 1)}{\Gamma(nt - n + 1)}^{a}.$$

By applying the Stirling formula, we obtain

$$
I = \left(1 + o(1)\right) \varphi(n) \left(\sum_{-d \leq \lambda \leq d, \lambda \equiv d(2)} b_{\lambda} J_{\lambda}\right) \quad \text{as } n \to \infty.
$$

For each $\lambda \in \mathbb{R}$, we denote by $J_{\lambda}$ the integral

$$
\int_{x-i\infty}^{x+i\infty} e^{n(f(t) - i\lambda \pi)} g(t) dt,
$$

where $x$ is a real number satisfying $1 < x < r$. It is easily verified that the definition of $J_{\lambda}$ doesn’t depend on the choice of $x$.

**Proposition 4.3** We have

$$
I = \left(1 + o(1)\right) \varphi(n) \left(\sum_{-d \leq \lambda \leq d, \lambda \equiv d(2)} b_{\lambda} J_{\lambda}\right) \quad \text{as } n \to \infty.
$$
where \( b_\lambda \) is a constant defined by
\[
b_\lambda = \begin{cases} 
\sum_{m=1}^{d} (-1)^{m} a_m e^{i m \lambda \pi / d} & (\lambda \neq \pm d) \\
\frac{1}{2} \sum_{m=1}^{d} (-1)^{m} a_m e^{i m \lambda \pi / d} & (\lambda = \pm d).
\end{cases}
\]

**Proof.** By the definition of \( I \) and Proposition 4.1, we have
\[
I = -\frac{n}{2i} \int_{x-i\infty}^{x+i\infty} \frac{P(dnt)}{\sin dnt \pi} \left( \sum_{m=1}^{d} a_m (\sin dnt \pi) \cot \left( \frac{(dnt - m \pi)}{d} \right) \right) \, dt.
\]
This equation and Lemma 4.2 implies
\[
I = \left( 1 + o(1) \right) \varphi(n) \int_{x-i\infty}^{x+i\infty} e^{nf(t)} \left( \sum_{m=1}^{d} a_m (\sin dnt \pi) \cot \left( \frac{(dnt - m \pi)}{d} \right) \right) \, dt \quad \text{as} \quad n \to \infty.
\]
Therefore, it is sufficient to prove the equation
\[
\left( \sum_{m=1}^{d} a_m (\sin dnt \pi) \cot \left( \frac{(dnt - m \pi)}{d} \right) \right) = \sum_{-d \leq \lambda \leq d, \lambda \equiv \pm 2d} b_\lambda e^{-n \lambda \pi it}.
\]
Let \( \omega_m = e^{i(dnt-m)\pi/d} \). Since
\[
\sin dnt \pi = (-1)^m \sin(dnt-m)\pi = (-1)^m \frac{\omega_m^d - \omega_m^{-d}}{2i},
\]
we have
\[
\sin dnt \cdot \cot \left( \frac{(dnt - m \pi)}{d} \right) = \left( -1 \right)^m \frac{\omega_m^d - \omega_m^{-d}}{2i} \frac{(\omega_m + \omega_m^{-1})/2}{(\omega_m - \omega_m^{-1})/2i} = \frac{(-1)^m}{2} (\omega_m^d + 2\omega_m^{d-2} + 2\omega_m^{d-4} + \cdots + 2\omega_m^{4-d} + 2\omega_m^{2-d} + \omega_m^{-d}).
\]
By substituting \( \omega_m = e^{int \pi} \cdot e^{-im \pi / d} \), and taking the sum, we obtain (4.4).

**Lemma 4.4**

(i) For each \( \lambda \), we have \( b_{-\lambda} = \overline{b_\lambda} \).

(ii) \( b_{\pm d} \in \mathbb{R} \).

(iii) Not all \( b_\lambda \) are 0.

**Proof.** (i) and (ii) follows immediately from the definition of \( b_\lambda \) and our assumption \( a_m \in \mathbb{R} \). To prove (iii), it is sufficient to show that \( b_{-\lambda} = b_{-\lambda+2} = \cdots = b_{-2} = 0 \) implies \( a_1 = \cdots = a_m = 0 \). The determinant of the matrix \( (e^{im\lambda \pi / d})_{\lambda,m} \) \((-d \leq \lambda \leq d, \lambda \equiv d(2), 1 \leq m \leq d)\) is factorized as
\[
\prod_{\lambda} e^{i \lambda \pi / d} \cdot \prod_{\lambda < \lambda'} (e^{i \lambda \pi / d} - e^{i \lambda' \pi / d})
\]
(the Vandermonde determinant), hence it’s not 0 since \( e^{i \lambda \pi / d} \neq e^{i \lambda' \pi / d} \) for each \(-d \leq \lambda < \lambda' \leq d-2\).
4.2 Lemmas concerning $f'(t)$

To obtain the asymptotic behavior of the integral $J_\lambda$, we examine $f'(t)$ in detail. We assume that the functions $f(t), g(t)$ are defined in the domain in Fig. 2.

![Figure 2: The domain of $f$ and $g$](image)

Since $f'(t) = d(\log(t+r) - \log(-t+r)) + (a+d)(\log(t-1) - \log(t+1))$, we have

\[
\begin{align*}
\text{Re}(f'(t)) &= \log \left| \frac{t + r}{-t + r} \right|^a + d, \\
\text{Im}(f'(t)) &= d(\arg(t+r) - \arg(-t+r)) + (a+d)(\arg(t-1) - \arg(t+1)).
\end{align*}
\]

Here each arg is chosen so that its value is 0 for each $t \in (1, r)$.

**Lemma 4.5**

(i) For $1 < \text{Re}(t) < r$, $\text{Im}(t) < 0$, we have $\text{Im}(f'(t)) < 0$.

(ii) For $t \in (1, r)$, we have $\text{Im}(f'(t)) = 0$.

(iii) For $t \in (r, \infty)$, we have $\text{Im}(f'(t)) = d\pi$.

(iv) For $\text{Im}(t) > 0$, we have $0 < \text{Im}(f'(t)) < (a+d)\pi$.

**Proof.**

(i) We have $\arg(t+r) < 0$, $\arg(-t+r) > 0$, and $\arg(t-1) < \arg(t+1)$, hence $\text{Im}(f'(t)) < 0$.

(ii) $\text{Im}(f'(t)) = d(0 - 0) + (a + d)(0 - 0) = 0$.

(iii) $\text{Im}(f'(t)) = d(0 - (-\pi)) + (a + d)(0 - 0) = d\pi$.

(iv) Let $\alpha(t) = \pi - (\arg(t+r) - \arg(-t+r))$, and $\beta(t) = \arg(t-1) - \arg(t+1)$. Then these are angles as in Fig.

Thus, we have $0 < \beta(t) < \alpha(t) < \pi$, and we obtain

\[0 < (a + d)\beta(t) + d(\pi - \alpha(t)) = a\beta(t) + d\pi - d(\alpha(t) - \beta(t)) < a\pi + d\pi = (a + d)\pi.\]
Lemma 4.6 Let $x \in [1,r]$ and $\theta \in (0,\pi)$, then $\text{Im}(f'(t))$ increases monotonically on the segment $t = x + ue^{i\theta}$ ($0 < u \leq \sqrt{x^2-1}$).

**Proof.** Let $\alpha(t)$, $\beta(t)$ be as in Lemma 4.5. Since $\text{Im}(f'(t)) = d(\pi - \alpha(t)) + (a + d)\beta(t)$, and obviously $\alpha(t)$ decreases monotonically on the segment considered, it is sufficient to show that $\beta(t)$ increases monotonically on the segment. Assume $0 < u_1 < u_2 \leq \sqrt{x^2-1}$, and let us prove $\beta(x + u_1e^{i\theta}) \leq \beta(x + u_2e^{i\theta})$. Define points $P$, $A$, $B$, $C$, $D$, and $E$ on the complex plane by

$$P(x), \quad A(-1), \quad B(1), \quad C(x + u_1e^{i\theta}), \quad D(x + u_2e^{i\theta}), \quad E\left(x + \frac{x^2-1}{u_1}u_1e^{i\theta}\right).$$

By assumption, $P$, $C$, $D$, $E$ are collinear in this order. By the power of a point theorem, $A$, $B$, $C$, $E$ are concyclic. See Fig. 4.

![Figure 4: The points P, A, B, C, D, and E](image)

Since $D$ lies inside the circle, we have $\angle ADB > \angle ACB$, that is, $\beta(x + u_2e^{i\theta}) > \beta(x + u_1e^{i\theta})$. ■

Let $R = (a + d)/d$. Assume that the inequality $R \geq 3r$ holds.

Lemma 4.7 For each $0 \leq \theta \leq \frac{\pi}{2}$, there exists a unique $u_\theta > 0$ satisfying $\text{Re}(f'(r + u_\theta e^{i\theta})) = 0$. Moreover, we have $\text{Re}(f'(r + u e^{i\theta})) > 0$ for $0 < u < u_\theta$, and $\text{Re}(f'(r + u e^{i\theta})) < 0$ for $u > u_\theta$. 13
Proof. Let us consider the function
\[ f_\theta(u) = \frac{2}{d} \text{Re}(f'(r + ue^{i\theta})) \]
\[ = (\log |2r + ue^{i\theta}|^2 - \log u^2) - R(\log |r + 1 + ue^{i\theta}|^2 - \log |r - 1 + ue^{i\theta}|^2). \]

By computing the derivative of the function \( f_\theta(u) \), we have
\[ f'_\theta(u) = \left( \frac{2u + 4r \cos \theta}{|2r + ue^{i\theta}|^2} - \frac{2}{u} \right) - R\left( \frac{2u + 2(r + 1) \cos \theta}{|r + 1 + ue^{i\theta}|^2} - \frac{2u + 2(r - 1) \cos \theta}{|r - 1 + ue^{i\theta}|^2} \right) \]
\[ = 4\left( \frac{-r(\cos \theta u + 2r)}{|2r + ue^{i\theta}|^2 u} - R \cdot \frac{(\cos \theta u^2 + 2ru + (r^2 - 1) \cos \theta)}{|r + 1 + ue^{i\theta}|^2 r - 1 + u e^{i\theta}|^2} \right) \]
\[ = \frac{4(c_5 u^5 + c_4 u^4 + c_3 u^3 + c_2 u^2 + c_1 u + c_0)}{u|2r + ue^{i\theta}|^2 r + 1 + ue^{i\theta}|^2 r - 1 + u e^{i\theta}|^2}, \]
where the coefficients \( c_5, \ldots, c_0 \) are defined by
\[ c_5 = (R - r) \cos \theta, \quad c_4 = (R - r)(2r + 4r \cos^2 \theta), \]
\[ c_3 = R \cos \theta(13r^2 - 1) - r \cos \theta(10r^2 + 2 + 4(r^2 - 1) \cos^2 \theta), \]
\[ c_2 = R(8r^3 + 4r(r^2 - 1) \cos^2 \theta) - r(4r(r^2 + 1) + 12r(r^2 - 1) \cos^2 \theta), \]
\[ c_1 = R \cos \theta 4r^2(r^2 - 1) - r \cos \theta(r^2 - 1)(9r^2 - 1), \]
\[ c_0 = -2r^2(r^2 - 1)^2. \]

By our assumption \( R \geq 3r \) and \( 0 \leq \theta \leq \frac{\pi}{4} \), we have the following inequalities:
\[ c_5, c_3, c_1 \geq 0, \quad c_4, c_2 > 0, \quad c_0 < 0. \]

Hence the function \( c_5 u^5 + c_4 u^4 + c_3 u^3 + c_2 u^2 + c_1 u + c_0 \) takes a negative value at \( u = 0 \), and increases monotonically as \( u \) increases from \( 0 \) to \( \infty \). Therefore, there exists \( u_0 > 0 \) such that
\[ f'_\theta(u) < 0 \quad (0 < u < u_0), \quad f'_\theta(u) > 0 \quad (u_0 < u). \]

Moreover, it is easily verified that
\[ \lim_{u \to +0} f_\theta(u) = \infty, \quad \lim_{u \to +\infty} f_\theta(u) = 0. \]

Thus, \( f_\theta \) changes as in Table 1, and we obtain the assertion.

| \( u \)  | \( +0 \) | \( \cdots \) | \( u_0 \) | \( \cdots \) | \( \infty \) |
|-----|-----|-----|-----|-----|-----|
| \( f'_\theta(u) \) | \( -\infty \) | \( - \) | 0 | \( + \) | 0 |
| \( f_\theta(u) \) | \( \infty \) | \( \searrow \) | \( \nearrow \) | 0 | 0 |

Table 1: The change of \( f_\theta(u) \)
Lemma 4.8 There exists a unique \( x_0 \in (1, r) \) which satisfies \( f'(x_0) = 0 \). There exists a unique \( x_1 \in (r, \infty) \) which satisfies \( f'(x_1) = d\pi i \). Moreover, for each \( t \in (1, \infty) \setminus \{r\} \), we have

\[
\Re(f'(t)) < 0 \quad (1 < t < x_0), \quad \Re(f'(t)) > 0 \quad (x_0 < t < x_1), \quad \Re(f'(t)) < 0 \quad (x_1 < t).
\]

Proof. In the interval \((1, r)\), the function

\[
\Re(f'(t)) = \frac{2r}{r - t} \left(1 + \frac{2}{t - 1}\right)^{-dR}
\]

monotonically increases as \( x \) increases from 1 to \( r \). Since

\[
\lim_{t \to 1^+} \Re(f'(t)) = -\infty, \quad \lim_{t \to r^-} \Re(f'(t)) = \infty,
\]

there exists a unique \( x_0 \) satisfying \( \Re(f'(x_0)) = 0 \). By Lemma 4.5, \( x_0 \) satisfies \( f'(x_0) = 0 \). The unique existence of \( x_1 \) follows by applying Lemma 4.7 for \( \theta = 0 \), and by Lemma 4.5. The rest of the Lemma is immediate. \( \blacksquare \)

Lemma 4.9 Let \( x \in (1, \infty) \).

(i) If \( x \in [x_0, x_1] \) then there exists a unique \( y_x \geq 0 \) which satisfies \( \Re(f'(x + y_x i)) = 0 \). We have

\[
\Re(f'(x + y)) > 0 \quad \text{for} \quad 0 < y < y_x, \quad \text{and} \quad \Re(f'(x + y)) < 0 \quad \text{for} \quad y_x < y.
\]

(ii) If \( x \notin [x_0, x_1] \) then the inequality \( \Re(f'(x + yi)) < 0 \) holds for arbitrary \( y \geq 0 \).

Proof. Let us consider the function

\[
f_x(y) = \frac{d}{2} \Re(f'(x + yi)) = \log((x + r)^2 + y^2) - \log((x - r)^2 + y^2) - R(\log((x + 1)^2 + y^2) - \log((x - 1)^2 + y^2)).
\]

By computing the derivative of the function \( f_x(y) \), we have

\[
f'_x(y) = 2y \left(\frac{1}{(x + r)^2 + y^2} - \frac{1}{(x - r)^2 + y^2} - R \left(\frac{1}{(x + 1)^2 + y^2} - \frac{1}{(x - 1)^2 + y^2}\right)\right)
\]

\[
= 2y \left(\frac{4Rx}{((x + r)^2 + y^2)((x - r)^2 + y^2)} + \frac{8xy((R - r)y^4 + 2(R(x^2 + r^2) - r(x^2 + 1)))y^2 + R(x^2 - r^2)^2 - r(x^2 - 1)^2}{(x + r)^2 + y^2)((x - r)^2 + y^2)(x + 1)^2 + y^2(x - 1)^2 + y^2}\right).
\]

Since \((R - r)y^4 + 2(R(x^2 + r^2) - r(x^2 + 1))y^2 + R(x^2 - r^2)^2 - r(x^2 - 1)^2\) increases monotonically, there are two possibilities as follows:

(a) \( f_x(y) \) monotonically decreases.

(b) There exists some \( y_0 > 0 \), such that \( f_x(y) \) monotonically decreases in the interval \( 0 < y < y_0 \), and monotonically increases in the interval \( y_0 < y \).

If \( x \in [x_0, x_1] \) then we have \( f_x(0) \geq 0 \), and \( \lim_{y \to \infty} f_x(y) = 0 \). Hence (a) can’t occur, and (b) holds in this case. Thus we obtain the assertion of (i). If \( x \notin [x_0, x_1] \) then we have \( f_x(0) < 0 \) and \( \lim_{y \to \infty} f_x(y) = 0 \). In this cases, we have \( f_x(y) < 0 \) for arbitrary \( y \geq 0 \) whether (a) or (b) is true. \( \blacksquare \)
Proposition 4.10 For each $0 \leq \lambda \leq d$, there exists some $t \in \mathbb{C}$ which satisfies $f'(t) = \lambda \pi i$, Re$(t) > 1$, and Im$(t) > 0$.

Proof. We can regard $y_x$ in the Lemma 4.9 as a continuous function of $x$. Thus Im$(f'(x + y_x i))$ is a continuous function of $x$ defined in the interval $[x_0, x_1]$. Then the assertion follows from $f'(x_0) = 0$, $f'(x_1) = d\pi i$, and the intermediate value theorem. ■

Definition 4.11 For each $0 < \lambda < d$, we choose a complex $t$ satisfying the condition of Proposition 4.10, and denote it by $t_\lambda$. For $\lambda = 0, d$, Let $t_0 = x_0$, $t_d = x_1$.

Remark 4.12 $t_\lambda$ is in fact uniquely determined, but we do not need the fact in this paper.

Let $\rho = x_1 - r$.

Lemma 4.13 We have $\rho < \frac{r - 1}{2}$.

Proof. We have
\[
\frac{1}{d} \text{Re}\left( f'(r + \frac{r - 1}{2}) \right) < 0
\]

since
\[
\frac{1}{d} \text{Re}\left( f'(r + \frac{r - 1}{2}) \right) = \log \frac{5r - 1}{r - 1} - \log \left(1 + \frac{4}{3r - 3}\right) \leq \log \left(\frac{5r - 1}{r - 1}\right) - \log \left(1 + \frac{4R}{3r - 3}\right) \\
\leq \log \left(\frac{5r - 1}{r - 1}\right) - \log \left(1 + \frac{12r}{3r - 3}\right) = 0.
\]

Therefore the assertion follows from Lemma 4.8. ■

Lemma 4.14 On the semicircle $t = x + \sqrt{\rho^2 - (x - r)^2} i$ $(r - \rho \leq x \leq r + \rho)$ of radius $\rho$ with center $r$, Re$(f'(t))$ monotonically increases as $x$ increases.

Proof. By computing the derivative of the function Re$(f'(t))$ with respect to $x$, we have
\[
\frac{2}{d} \frac{d}{dx} \text{Re}(f'(t)) = \frac{4r}{|t + r|^2} - R\left(\frac{2(r + 1)}{|t + 1|^2} - \frac{2(r - 1)}{|t - 1|^2}\right) \\
= \frac{4r}{|t + r|^2} + \frac{4R(r^2 - 1 - \rho^2)}{|t + 1|^2|t - 1|^2}.
\]

By Lemma 4.13 we have
\[
r^2 - 1 - \rho^2 > r^2 - 1 - \frac{(r - 1)^2}{4} = \frac{(r - 1)(3r + 5)}{4} > 0.
\]

Therefore we obtain the inequality $\frac{d}{dx} \text{Re}(f'(t)) > 0$, as required. ■
Lemma 4.15 Let $\text{Re}(t) > 1$, $\text{Im}(t) \geq 0$ and $|t - r| > \rho$, then we have $\text{Re}(f'(t)) < 0$. In particular, we have $|t_\lambda - r| \leq \rho$ for each $0 \leq \lambda \leq d$.

Proof. By Lemma 4.14, we have $\text{Re}(f'(r - \rho)) < 0$. Therefore we have $r - \rho < x_0$. Let $t = x + yi$. If $x \notin [x_0, x_1]$, the conclusion comes from Lemma 4.13. Let us assume $x \in [x_0, x_1]$. Since $r - \rho < x_0$, there exists some $y' \geq 0$ satisfying $|(x + y'i) - r| = \rho$. By Lemma 4.14, we have $\text{Re}(f'(x + y'i)) \leq 0$. By the assumption $|t - r| > \rho$, we have $y > y'$. Hence $\text{Re}(f'(t)) < 0$ by Lemma 4.9.

4.3 The asymptotic behavior of $J_\lambda$

We compute the asymptotic behavior of $J_\lambda$ by the saddle point method. According to the following Lemma, we can limit our consideration to the case where $\lambda \geq 0$.

Lemma 4.16 $J_{-\lambda} = -\overline{J_\lambda}$.

Proof. By the Schwarz reflection principle, we have $f(\overline{t}) = \overline{f(t)}$ and $g(\overline{t}) = \overline{g(t)}$. Therefore, we have

$$\overline{J_\lambda} = \int_{x-i\infty}^{x+i\infty} e^{n(f(t)-i\lambda \pi t)} g(t) dt = \int_{x-i\infty}^{x+i\infty} e^{n(f(\overline{t})-i\lambda \pi \overline{t})} \overline{g(t)} dt = \int_{x-i\infty}^{x+i\infty} e^{n(f(t)+i\lambda \pi t)} g(t) dt = -J_{-\lambda}.$$

Lemma 4.17 We have $\sqrt{x^2 - 1} > \rho$ for $x > r - \rho$.

Proof. It is sufficient to show $(r - \rho)^2 - 1 > \rho^2$. By Lemma 4.13, we have

$$(r - \rho)^2 - 1 - \rho^2 = r^2 - 1 - 2r\rho > r^2 - 1 - r(r - 1) = r - 1 > 0,$$

thus, the proof is completed.

Let $t_\lambda = x_\lambda + y_\lambda i = r + u_\lambda e^{i\theta_\lambda}$, where $x_\lambda, y_\lambda, u_\lambda \in \mathbb{R}$ and $\theta_\lambda \in [0, \pi]$. For each $0 \leq \lambda \leq d$, we define the path $C_\lambda$ as follows:

(i) The case where $0 \leq \lambda < d$ and $x_\lambda < r$. Let $C_\lambda$ be a polyline connecting $x_\lambda - i\infty$, $x_\lambda + i\rho$, and $\infty + i\rho$ in this order.

(ii) The case where $0 < \lambda < d$ and $r \leq x_\lambda$. Take a sufficiently small positive $\varepsilon > 0$. Let $C_\lambda$ be a polyline connecting $r - \varepsilon - i\infty$, $r - \varepsilon$, $r + \varepsilon e^{i\theta_\lambda}$, $r + \rho e^{i\theta_\lambda}$, and $\infty + \rho e^{i\theta_\lambda}$ in this order.

(iii) The case where $\lambda = d$. Take a sufficiently small positive $\varepsilon > 0$. Let $C_\lambda$ be a polyline connecting $r - \varepsilon - i\infty$, $r - \varepsilon$, $r + i\varepsilon, r + \varepsilon$, and $r + \infty$ in this order.

Since $|t_\lambda - r| \leq \rho$, each path $C_\lambda$ pass through the point $t_\lambda$.

Lemma 4.18 On the path $C_\lambda$, $\text{Re}(f(t) - i\lambda \pi t)$ takes a unique maximal value at $t = t_\lambda$. 

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Proof. For the path $\phi: I \to \mathbb{C}; u \mapsto z_0 + u e^{i\theta} (I \subset \mathbb{R})$, we have the equation

$$\frac{d}{du} \text{Re}(f(\phi(u)) - i\lambda \pi \phi(u)) = \text{Re}(e^{i\theta}(f'(\phi(u)) - \lambda \pi i))$$

$$= \cos \theta \text{Re}(f'(\phi(u))) - \sin \theta \text{Im}(f'(\phi(u)) - \lambda \pi i).$$

(4.5)

Let us prove that the value of (4.5) changes positive to negative at $t_{\lambda}$.

(i) The case where $0 \leq \lambda < d$ and $x_{\lambda} < r$. On the segment connecting $x_{\lambda} - i\infty$ and $x_{\lambda}$, Lemma 4.5 implies that the sign of (4.5) is positive. On the segment connecting $x_{\lambda}$ and $x_{\lambda} + i\rho$, Lemma 4.6 and Lemma 4.17 implies that the value of (4.5) changes positive to negative at $t_{\lambda}$. On the segment connecting $x_{\lambda} + i\rho$ and $\infty + i\rho$, Lemma 4.15 implies that the value of (4.5) is negative.

(ii) The case where $0 < \lambda < d$ and $r \leq x_{\lambda}$. On the segment connecting $r - \varepsilon - i\infty$ and $r - \varepsilon$, Lemma 4.5 implies that the value of (4.5) is negative. On the polyline connecting $r - \varepsilon$ and $r + \rho e^{i\theta \lambda}$, Lemma 4.6, Lemma 4.7 and Lemma 4.17 implies that the value of (4.5) changes positive to negative at $t_{\lambda}$. On the segment connecting $r + \rho e^{i\theta \lambda}$ and $\infty + \rho e^{i\theta \lambda}$, Lemma 4.13 implies that the value of (4.5) is negative.

(iii) The case where $\lambda = d$. The proof is similar to that of (ii), except on the segment connecting $r + \varepsilon i$ and $r + \varepsilon$. By taking $\varepsilon$ sufficiently small, we have $\text{Re}(f'(t)) > d\pi$ on the segment connecting $r + \varepsilon i$ and $r + \varepsilon$. Then we have

$$\cos \left(-\frac{\pi}{4}\right) \text{Re}(f'(t)) - \sin \left(-\frac{\pi}{4}\right) \text{Im}(f'(t) - d\pi i) > \frac{1}{\sqrt{2}} \cdot d\pi + \frac{1}{\sqrt{2}}(0 - d\pi) = 0.$$

Thus, the proof is completed.

Lemma 4.19 For each $0 \leq \lambda \leq d$, we have

$$J_\lambda = \int_{C_\lambda} e^{n(f(t) - i\lambda \pi t)} g(t) dt.$$
Proof. It is sufficient to prove the equation
\[ \lim_{N \to \infty} \int_{x + Ni}^{N + yi} e^{n(f(t) - \lambda \pi it)} g(t) \, dt = 0 \]
for each \( 1 < x < r, y \geq 0 \), where the path of the integral is taken to be a segment.

By the Taylor expansion of log, the following equations hold for \( \text{Im}(t) \geq 0 \):
\[
\begin{align*}
\log(t + r) &= \log t + \frac{r}{t} + O(|t|^{-2}), \\
\log(-t + r) &= -\pi i + \log t - \frac{r}{t} + O(|t|^{-2}), \\
\log(t + 1) &= \log t + \frac{1}{t} + O(|t|^{-2}), \\
\log(t - 1) &= \log t - \frac{1}{t} + O(|t|^{-2}).
\end{align*}
\]
Therefore, we have
\[
\begin{align*}
f(t) - \lambda \pi it &= d \left( t + r \left( \log t + \frac{r}{t} \right) + \left( -t + r \left( -\pi i + \log t - \frac{r}{t} \right) \right) \right) \\
&\quad + dR \left( (t - 1) \left( \log t - \frac{1}{t} \right) - (t + 1) \left( \log t + \frac{1}{t} \right) \right) - \lambda \pi it + O(|t|^{-1}) \\
&\quad + 2d(R - r)(\log(t + 1) + d(t - r)\pi i - \lambda \pi it + O(|t|^{-1}),
\end{align*}
\]
\[
\text{Re}(f(t) - \lambda \pi it) = -2d(R - r)(\log|t| + 1) - (d - \lambda)\pi \text{Im}(t) + O(|t|^{-1}) \leq O(|t|^{-1}).
\]
Since, we obviously have
\[ |g(t)| = O(|t|^{-a}) \leq O(|t|^{-2}), \]
we have \( |e^{n(f(t) - \lambda \pi it)} g(t)| \leq O(N^{-2}) \) on the segment connecting \( x + Ni \) and \( N + yi \). Hence we have
\[ \left| \int_{x + Ni}^{N + yi} e^{n(f(t) - \lambda \pi it)} g(t) \, dt \right| = O(N^{-1}) \]
since the length of the path of the integral is \( O(N) \). Thus, the proof is completed.

Lemma 4.20 Let
\[ h(t) = dr(\log(t + r) + \log(-t + r)) - dR(\log(t - 1) + \log(t + 1)) \]
Then, we have \( f(t_{\lambda}) - \lambda \pi it_{\lambda} = h(t_{\lambda}) \) for each \( 0 \leq \lambda \leq d \).

Proof. Since \( f'(t_{\lambda}) = \lambda \pi i \), we have \( f(t_{\lambda}) - \lambda \pi it_{\lambda} = f(t_{\lambda}) - f'(t_{\lambda})t_{\lambda} = h(t_{\lambda}) \).
**Proposition 4.21** For each $0 \leq \lambda \leq d$, we have

$$J_\lambda = (1 + o(1)) e^{nh(t_\lambda)} g(t_\lambda) \sqrt{\frac{2\pi}{n|f''(t_\lambda)|}} \cdot e^{i \pi - \text{arg } f''(t_\lambda)}/2$$

$$= (1 + o(1)) e^{n \text{Re}(h(t_\lambda))} |g(t_\lambda)| \sqrt{\frac{2\pi}{n|f''(t_\lambda)|}} \cdot e^{i \psi_\lambda(n)}$$

as $N \to \infty$,

where

$$\psi_\lambda(n) = \frac{\pi - \text{arg } f''(t_\lambda)}{2} + \text{arg } g(t_\lambda) + n \text{ Im}(h(t_\lambda)),$$

and the arg $f''(t_\lambda)$ is chosen so that

$$-\frac{\pi}{4} \leq \frac{\pi - \text{arg } f''(t_\lambda)}{2} \leq \frac{3\pi}{4}.$$

**Proof.** By applying the saddle point method for the integral

$$J_\lambda = \int_{C_\lambda} e^{n(f(t) - \lambda \pi i t)} g(t) dt,$$

we obtain the equation

$$J_\lambda = (1 + o(1)) e^{n(f(t_\lambda) - \lambda \pi i t_\lambda)} g(t_\lambda) \sqrt{\frac{2\pi}{n|f''(t_\lambda)|}} \cdot e^{i \pi - \text{arg } f''(t_\lambda)}/2.$$

By using Lemma 4.20, we obtain the assertion. ■

**Proposition 4.22** For $0 \leq \lambda \leq d$, we have

$$\lim_{n \to \infty} \frac{\log |J_\lambda(n)|}{n} = \text{Re}(h(t_\lambda))$$

**Proof.** It is immediate from Proposition 4.21. ■

### 4.4 The asymptotic behavior of $I$

For each $\lambda$, let $\varepsilon_\lambda = -t_\lambda + r$. Assume $r \geq 2$.

**Lemma 4.23** We have the following estimates:

(i) $\rho < \frac{5r}{2eR/r}$,

(ii) $|\log \varepsilon_\lambda - \left( \log 2r + R \log \frac{r-1}{r+1} - \frac{\lambda \pi i}{d} \right) | \leq \frac{5R\rho}{r}$. 

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(iii) \[ h(t_\lambda) - (2dr \log 2r + dR(r - 1) \log(r - 1) - dR(r + 1) \log(r + 1) - r\lambda\pi i - d\varepsilon_\lambda) \leq \frac{4dR}{r} |\varepsilon_\lambda|^2. \]

**Proof.**

(i) By Lemma 4.13 and the equation \( \text{Re}(f'(r + \rho)) = 0 \), we have

\[ \rho = (\rho + 2r) \left( \frac{\rho + r - 1}{\rho + r + 1} \right)^R \leq \frac{5r - 1}{2} \cdot \left( \frac{3r - 3}{3r + 1} \right)^R < \frac{5r}{2} \cdot \left( \frac{1 - \frac{1}{r^2}}{r} \right)^{R/r} < \frac{5r}{2} \cdot e^{-R/r}. \]

(ii) Since \( f'(t_\lambda) = \lambda\pi i \), we have

\[ \log \varepsilon_\lambda = \log(2r - \varepsilon_\lambda) + R(\log(r - 1 - \varepsilon_\lambda) - \log(r + 1 - \varepsilon_\lambda)) - \frac{\lambda\pi i}{d}, \]

therefore

\[ \log \varepsilon_\lambda - \left( \log 2r + R\log \frac{r - 1}{r + 1} - \frac{\lambda\pi i}{d} \right) = \log \left( 1 - \frac{\varepsilon_\lambda}{2r} \right) + R\log \left( 1 - \frac{\varepsilon_\lambda}{r - 1} \right) - R\log \left( 1 - \frac{\varepsilon_\lambda}{r + 1} \right). \quad (4.6) \]

Moreover, by using the estimate

\[ |\log(1 + t)| < \frac{3}{2}|t| \quad \left( |t| \leq \frac{1}{2} \right) \]

(this estimate follows immediately from the Taylor expansion of \( \log(1 + t) \)), we can show that the absolute value of the right hand side of (4.6) is not greater than

\[ \frac{3|\varepsilon_\lambda|}{2} \left( \frac{1}{2r} + \frac{R}{r - 1} + \frac{R}{r + 1} \right) \leq \frac{3R\rho}{2r} \left( \frac{1}{2R} + \frac{r}{r - 1} + \frac{r}{r + 1} \right) \leq \frac{3R\rho}{2r} \left( \frac{1}{12} + \frac{8}{3} \right) \leq \frac{5R\rho}{r}. \]

Thus we obtain (ii).

(iii) By the equation \( f'(t_\lambda) = \lambda\pi i \), we have

\[ h(t_\lambda) = 2dr \log(2r - \varepsilon_\lambda) + dR(r - 1) \log(r - 1 - \varepsilon_\lambda) - dR(r + 1) \log(r + 1 - \varepsilon_\lambda) - r\lambda\pi i, \]

therefore

\[
\begin{align*}
&h(t_\lambda) - (2dr \log 2r + dR(r - 1) \log(r - 1) - dR(r + 1) \log(r + 1) - r\lambda\pi i - d\varepsilon_\lambda) \\
&= 2dr \left( \log \left( 1 - \frac{\varepsilon_\lambda}{2r} \right) - \frac{\varepsilon_\lambda}{2r} \right) + dR(r - 1) \left( \log \left( 1 - \frac{\varepsilon_\lambda}{r - 1} \right) - \frac{\varepsilon_\lambda}{r - 1} \right) \\
&\quad - dR(r + 1) \left( \log \left( 1 - \frac{\varepsilon_\lambda}{r + 1} \right) - \frac{\varepsilon_\lambda}{r + 1} \right). \quad (4.7)
\end{align*}
\]

Moreover, by using the estimate

\[ |\log(1 + t) - t| < |t|^2 \quad \left( |t| \leq \frac{1}{2} \right) \]

(this estimate also follows immediately from the Taylor expansion of \( \log(1 + t) \)), we can show that the absolute value of the right hand side of (4.7) is not greater than

\[ d|\varepsilon_\lambda|^2 \left( \frac{1}{2r} + \frac{R}{r - 1} + \frac{R}{r + 1} \right) = \frac{Rd}{r} |\varepsilon_\lambda|^2 \left( \frac{1}{2R} + \frac{r}{r - 1} + \frac{r}{r + 1} \right) \leq \frac{Rd}{r} |\varepsilon_\lambda|^2 \left( \frac{1}{12} + \frac{8}{3} \right) \leq \frac{3Rd}{r} |\varepsilon_\lambda|^2. \]

Thus, we obtain (iii).
Assume that $\rho$ satisfies the inequality

$$\rho < \min\left\{ \frac{r \pi}{10Rd}, \frac{r}{2d^2}, \frac{r}{4R}\sin\frac{\pi}{2d}, \frac{r}{38R}\left( \frac{\cos\frac{\pi}{2d} - \cos\frac{3\pi}{2d}}{} \right) \right\}. \quad (4.8)$$

Then we have

$$|\log \varepsilon_\lambda - \left( \log 2r + R \log \frac{r - 1}{r + 1} - \frac{\lambda \pi i}{d} \right) | \leq \frac{\pi}{2d}$$

by Lemma 4.23 (ii). Thus, we have

$$-\left( \lambda + \frac{1}{2} \right) \frac{\pi}{d} \leq \arg \varepsilon_\lambda \leq -\left( \lambda - \frac{1}{2} \right) \frac{\pi}{d}. \quad (4.9)$$

**Lemma 4.24** We have $\text{Re}(h(t_\lambda)) < \text{Re}(h(t_{\lambda+2}))$ for $0 \leq \lambda \leq d - 2$.

**Proof.** According to Lemma 4.23 (iii), it is sufficient to show the inequality

$$-d \text{Re}(\varepsilon_\lambda) + \frac{4dR}{r} |\varepsilon_\lambda|^2 \leq -d \text{Re}(\varepsilon_{\lambda+2}) - \frac{4dR}{r} |\varepsilon_{\lambda+2}|^2.$$

Moreover, since

$$\text{Re}(\varepsilon_\lambda) = |\varepsilon_\lambda| \cos(\arg \varepsilon_\lambda) \geq |\varepsilon_\lambda| \cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right),$$

$$\text{Re}(\varepsilon_{\lambda+2}) = |\varepsilon_\lambda| \cos(\arg \varepsilon_{\lambda+2}) \leq |\varepsilon_{\lambda+2}| \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right),$$

it is sufficient to show the inequality

$$|\varepsilon_\lambda| \left( - \cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right) + \frac{4R}{r} |\varepsilon_\lambda| \right) \leq |\varepsilon_{\lambda+2}| \left( - \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right) - \frac{4R}{r} |\varepsilon_{\lambda+2}| \right).$$

This inequality is equivalent to the inequality

$$\frac{|\varepsilon_\lambda| + |\varepsilon_{\lambda+2}|}{2} \left( \cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right) - \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right) - \frac{4R}{r} (|\varepsilon_\lambda| + |\varepsilon_{\lambda+2}|) \right)$$

$$- \frac{|\varepsilon_\lambda| - |\varepsilon_{\lambda+2}|}{2} \left( - \cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right) - \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right) + \frac{4R}{r} (|\varepsilon_\lambda| - |\varepsilon_{\lambda+2}|) \right) \geq 0.$$

Therefore, it is sufficient to show the stronger inequality

$$\cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right) - \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right) \geq \frac{4R}{r} (|\varepsilon_\lambda| + |\varepsilon_{\lambda+2}|) + \frac{|\varepsilon_\lambda| - |\varepsilon_{\lambda+2}|}{|\varepsilon_\lambda| + |\varepsilon_{\lambda+2}|} \left( \cos\left( \frac{(\lambda + \frac{1}{2}) \pi}{d} \right) + \cos\left( \frac{(\lambda + \frac{3}{2}) \pi}{d} \right) + \frac{4R}{r} (|\varepsilon_\lambda| - |\varepsilon_{\lambda+2}|) \right). \quad (4.10)$$
By (4.8), the left hand side of (4.10) is not less than $\frac{38R\rho}{r}$. Besides, we have $\frac{4R}{r} (|\varepsilon\lambda| + |\varepsilon_{\lambda+2}|) \leq 8\rho$. Moreover, we have

$$4\rho r (|\varepsilon\lambda| + |\varepsilon_{\lambda+2}| + 2 |\varepsilon\lambda|) \leq 8\rho.$$

Therefore, to prove the inequality (4.10), it is sufficient to show the inequality

$$\left| \frac{|\varepsilon\lambda| - |\varepsilon_{\lambda+2}|}{|\varepsilon\lambda| + |\varepsilon_{\lambda+2}|} \right| < \frac{10R\rho}{r}.$$

If $|\varepsilon\lambda| \leq |\varepsilon_{\lambda+2}|$, we have

$$\left| \frac{|\varepsilon\lambda| - |\varepsilon_{\lambda+2}|}{|\varepsilon\lambda| + |\varepsilon_{\lambda+2}|} \right| \leq \frac{1}{2} \left( \frac{|\varepsilon_{\lambda+2}|}{|\varepsilon\lambda|} - 1 \right).$$

Here, we have

$$\left| \log \left| \frac{\varepsilon_{\lambda+2}}{\varepsilon\lambda} \right| \right| \leq \frac{10R\rho}{r}$$

by Lemma 4.23 (ii). By using the estimate

$$|1 - e^{t}| < 2|t| \quad (|t| < 1),$$

we obtain the inequality

$$\left| 1 - \frac{\varepsilon_{\lambda+2}}{\varepsilon\lambda} \right| < \frac{20R\rho}{r},$$

as required. The case where $|\varepsilon\lambda| > |\varepsilon_{\lambda+2}|$ is similarly shown. □

**Lemma 4.25** For $1 \leq \lambda \leq d - 1$, we have $\text{Im}(h(t)) \neq 0 \pmod{\pi\mathbb{Z})}.$

**Proof.** By (4.9), we have

$$-\frac{(2d - 1)\pi}{2d} \leq \arg \varepsilon\lambda \leq -\frac{\pi}{2d}.$$

Therefore we have

$$|\varepsilon\lambda| \leq \frac{-\text{Im}(\varepsilon\lambda)}{\sin \frac{\pi}{2d}}.$$

Hence we obtain

$$\frac{4R|\varepsilon\lambda|^2}{r} \leq \frac{4R\rho|\varepsilon\lambda|}{r} \leq \frac{-4R\rho \text{Im}(\varepsilon\lambda)}{r \sin \frac{\pi}{2d}} < -\text{Im}(\varepsilon\lambda).$$

This inequality and Lemma 4.23 (iii) implies $|\text{Im}(h(t)) - (-r\lambda\pi - d\text{Im}(\varepsilon\lambda))| < -d\text{Im}(\varepsilon\lambda)$. Thus, we have

$$-r\lambda\pi < \text{Im}(h(t)) < -r\lambda\pi - 2d\text{Im}(\varepsilon\lambda). \quad (4.11)$$

Moreover, by using (4.8), we have

$$|\text{Im}(\varepsilon\lambda)| < \rho < \frac{\pi}{2d^2}. \quad (4.12)$$
By (4.8) and (4.12), we obtain the inequality
\[-r\lambda \pi - \frac{\pi}{d} < \text{Im}(h(t_\lambda)) < -r\lambda \pi.\]
Since \(r\lambda \pi \equiv 0 \pmod{\frac{\pi}{d}Z}\), we have \(\text{Im}(h(t_\lambda)) \neq 0 \pmod{\frac{\pi}{d}Z}\), in particular, \(\text{Im}(h(t_\lambda)) \neq 0 \pmod{\pi Z}\).

**Proposition 4.26** If \(b_d \neq 0\), then we have
\[
\lim_{n \to \infty} \frac{\log \left| b_d J_d(n) + b_{-d} J_{-d}(n) \right|}{n} = \text{Re}(h(t_d)).
\]

**Proof.** By Lemma 4.4 and Lemma 4.16, we have \(b_d J_d(n) + b_{-d} J_{-d}(n) = 2b_d \text{Im}(J_d(n))\). Since \(\arg f''(t_d) = \pi\), \(\arg g(t_d) = -\frac{\pi}{2}\), and \(\text{Im}(h(t_d)) = -d\pi\), we have
\[
\psi_d(n) = -\frac{\pi}{2} - d\pi \equiv \pm \frac{\pi}{2} \pmod{2\pi Z},
\]
Therefore, Proposition 4.21 implies \(|\text{Im}(J_d)| = (1 + o(1))|J_d|\). Then the assertion immediately follows from Proposition 4.22.

**Lemma 4.27** Assume that \(b_\lambda \neq 0\) for some \(1 \leq \lambda \leq d - 1\), \(\lambda \equiv d \pmod{2}\). Then there exists a sequence \(n_1 < n_2 < \cdots\) of positive integers such that:
- \(n_{k+1} - n_k\) is bounded. In particular, we have \(\lim_{k \to \infty} \frac{n_{k+1}}{n_k} = 1\).
- \(\lim_{k \to \infty} \frac{\log \left| b_\lambda J_\lambda(n_k) + b_{-\lambda} J_{-\lambda}(n_k) \right|}{n_k} = \text{Re}(h(t_\lambda))\).

**Proof.** By Lemma 4.4 and Lemma 4.16, we have \(b_\lambda J_\lambda(n) + b_{-\lambda} J_{-\lambda}(n) = 2 \text{Im}(b_\lambda J_\lambda(n))\). Moreover, we have
\[
\arg(b_\lambda J_\lambda(n)) = \arg b_\lambda + \psi_\lambda(n) + o(1), \quad \psi_\lambda(n) = \frac{\pi - \arg f''(t_\lambda)}{2} + \arg g(t_\lambda) + n \text{Im}(h(t_\lambda))
\]
By Lemma 4.26 there exists a positive integer \(w\) satisfying
\[
\frac{\pi}{3} \leq w \text{Im}(h(t)) \leq \frac{2\pi}{3} \pmod{\pi Z}.
\]
Then for arbitrary positive integer \(n\), at least one of the equations
\[
\frac{\pi}{6} \leq \arg b_\lambda + \psi_\lambda(n) \leq \frac{5\pi}{6} \pmod{\pi Z}, \quad \frac{\pi}{6} \leq \arg b_\lambda + \psi_\lambda(n + w) \leq \frac{5\pi}{6} \pmod{\pi Z}
\]
holds. Let
\[ \{n_1, n_2, n_3, \ldots \} = \left\{ n \mid \frac{\pi}{6} \leq \arg b_{\lambda} + \psi_{\lambda}(n) \leq \frac{5}{6}\pi \pmod{\pi} \right\}. \]

Then we have \( n_{k+1} - n_k \leq 1 + w \), and the first assertion of Lemma is satisfied. Since \( n_k \) satisfies the inequality
\[ \left( 1 + o(1) \right) \left| b_{\lambda J_{\lambda}}(n_k) \right| < \left| \Im \left( b_{\lambda J_{\lambda}}(n_k) \right) \right| < \left| b_{\lambda J_{\lambda}}(n_k) \right|, \]
the second assertion of Lemma follows from Proposition 4.22.

Let \( \lambda_0 = \max \{0 \leq \lambda \leq d \mid b_{\lambda} \neq 0\} \). By Lemma 4.4, \( \lambda_0 \) is well-defined.

**Proposition 4.28** There exists a sequence \( n_1 < n_2 < \cdots \) of positive integers such that:

- \( \lim_{k \to \infty} \frac{n_{k+1} - n_k}{n_k} = 1 \).
- \( \lim_{k \to \infty} \frac{\log|\sum_{-d \leq \lambda \leq d, \lambda \equiv d(2)} b_{\lambda J_{\lambda}}(n_k)|}{n_k} = \Re(h(t_{\lambda_0})). \)

**Proof.** By Proposition 4.22 and Lemma 4.24, we have
\[
\sum_{-d \leq \lambda \leq d, \lambda \equiv d(2)} b_{\lambda J_{\lambda}}(n_k) = \begin{cases} b_{\lambda_0 J_{\lambda_0}} + b_{-\lambda_0 J_{-\lambda_0}} + o(e^{n \Re(h(t_{\lambda_0}))}) & (\lambda_0 \geq 1) \\ b_0 J_0 + o(e^{n \Re(h(t_0))}) & (\lambda_0 = 0). \end{cases}
\]

If \( \lambda_0 = d \), Lemma 4.26 implies that the sequence \( n_k = k \) satisfies the conditions. If \( \lambda_0 = 0 \), Proposition 4.22 implies that the sequence \( n_k = k \) satisfies the conditions. If \( 1 \leq \lambda \leq d - 1 \), the assertion follows from Lemma 4.27.

The following proposition is the conclusion of section 4.

**Proposition 4.29** Let \( a \) and \( b \) be integers satisfying \( a \geq 2b \). Let \( r = (d + 2b)/d \), and \( R = (a + d)/d \). Assume that the inequalities \( r \geq 2 \), \( R \geq 3r \), and
\[ \frac{5r}{2e^{R/r}} < \min \left\{ \frac{r\pi}{10Rd}, \frac{\pi}{2d^2}, \frac{\pi}{5d}, \frac{5r}{38R} \left( \cos \frac{\pi}{2d} - \cos \frac{3\pi}{2d} \right) \right\} \]
hold, then there exists a sequence \( n_1 < n_2 < \cdots \) of positive integers such that:

- \( \lim_{k \to \infty} \frac{n_{k+1}}{n_k} = 1 \).
- \( \lim_{k \to \infty} \frac{\log|I(n_k)|}{n_k} = 2(a - 2b) \log 2 + 4b \log d + \Re(h(t_{\lambda_0})). \)

**Proof.** By Lemma 4.23 (i), all our assumptions on \( r, R, \rho \) are satisfied. Thus, this Proposition follows from Proposition 4.28 and Proposition 4.3.
5 Proof of Theorem 1.2

We use the criterion of Nesterenko [4]. The following theorem is the original form of the criterion proved by Nesterenko:

**Theorem 5.1 (Nesterenko’s linear independence criterion)** Let \( c_1, c_2, \tau_1, \tau_2 > 0 \). Let \( N_0 \) be a positive integer, and assume that a monotonically increasing function \( \sigma : \mathbb{Z}_{\geq N_0} \rightarrow \mathbb{R} \) satisfies the conditions

\[
\lim_{t \to \infty} \sigma(t) = \infty, \quad \limsup_{t \to \infty} \frac{\sigma(t + 1)}{\sigma(t)} = 1.
\]

Let \( \theta_1, \ldots, \theta_m \in \mathbb{R} \), and assume that there exists a \( \mathbb{Z} \)-linear form \( I_N = \sum_{j=1}^{m} A_{j,N} \theta_j \) for each positive integer \( N \geq N_0 \) such that

\[
\max_{1 \leq j \leq m} \log |A_{j,N}| \leq \sigma(N), \quad c_1 e^{-\tau_1 \sigma(N)} \leq |I_N| \leq c_2 e^{-\tau_2 \sigma(N)}.
\]

Then we have the inequality

\[
\dim_{\mathbb{Q}} \left( \mathbb{Q} \text{-span}(\theta_1, \ldots, \theta_m) \right) > \tau_1 + 1 + \tau_1 - \tau_2.
\]

In this paper, we use it in the following form:

**Theorem 5.2** Let \( \theta_1, \ldots, \theta_m \in \mathbb{R} \). Assume that there exists a sequence \( n_1 < n_2 < \cdots \) of positive integers satisfying \( \lim_{k \to \infty} \frac{n_{k+1}}{n_k} = 1 \), and \( \mathbb{Z} \)-linear form

\[
I(n_k) = \sum_{j=1}^{m} A_j(n_k) \theta_j
\]

for each \( k \), such that

\[
\lim_{k \to \infty} -\frac{\log |I(n_k)|}{n_k} = \alpha, \quad \limsup_{k \to \infty} \frac{\max_j \log |A_j(n_k)|}{n_k} \leq \beta
\]

for some \( \alpha, \beta \in \mathbb{R} \). Then we have the inequality

\[
\dim_{\mathbb{Q}} \left( \mathbb{Q} \text{-span}(\theta_0, \ldots, \theta_m) \right) \geq 1 + \frac{\alpha}{\beta}.
\]

**Proof.** Let \( \varepsilon > 0 \) be a positive constant. If we set

\[
\sigma(k) = n_k (\beta + \varepsilon), \quad \tau_1 = \frac{\alpha + \varepsilon}{\beta + \varepsilon}, \quad \tau_2 = \frac{\alpha - \varepsilon}{\beta + \varepsilon}, \quad c_1 = c_2 = 1,
\]

then the assumptions of Theorem 5.1 are satisfied for sufficiently large \( N_0 \). Therefore, we have

\[
\dim_{\mathbb{Q}} \left( \mathbb{Q} \text{-span}(\theta_0, \ldots, \theta_m) \right) > \frac{\tau_1 + 1}{1 + \tau_1 - \tau_2} = \frac{\frac{\alpha + \varepsilon}{\beta + \varepsilon} + 1}{1 + \frac{\alpha + \varepsilon}{\beta + \varepsilon} - \frac{\alpha - \varepsilon}{\beta + \varepsilon}}.
\]

By taking the limit of each side as \( \varepsilon \to +0 \), we obtain the assertion. \( \blacksquare \)
Proposition 5.3 Assume that $a, b$ satisfies the assumptions of Proposition 4.29. Then we have

$$\delta(a; L) \geq 1 + \frac{\alpha}{\beta},$$

where

$$\alpha = -(2ad + 2(a - 2b) \log 2 + 4b \log d + \Re(h(t_{\lambda_0})),$$

$$\beta = 2ad + 2a \log 2 + 4(b + d) \log(b + d) - 4d \log d.$$  

Proof. We apply Theorem 5.2 for the linear form

$$(D_{2dn})^a I(n) = \sum_{2 \leq j \leq a, \ j \equiv a(2)} (D_{2dn})^a A_j(n)L(j) - \sum_{m=1}^d (D_{2dn})^a B_m(n)a_m$$

of $L(j)$ and $a_m$. By Proposition 3.4, this is a $\Z$-linear form. Moreover, by the well-known formula

$$\log D_{2dn} = 2dn \left(1 + o(1)\right)$$

and Proposition 3.4, we obtain

$$\limsup_{n \to \infty} \frac{(D_{2dn})^a |A_j(n)|}{n} \leq 2ad + 2a \log 2 + 4(b + d) \log(b + d) - 4d \log d,$$

$$\limsup_{n \to \infty} \frac{(D_{2dn})^a |B_m(n)|}{n} \leq 2ad + 2a \log 2 + 4(b + d) \log(b + d) - 4d \log d. \quad (5.1)$$

By (5.1) and Proposition 4.29, we can apply Theorem 5.2, and we obtain the assertion. ■

Lemma 5.4 Under the assumptions of Proposition 5.3, we have

$$\alpha \geq dR((r + 1) \log(r + 1) - (r - 1) \log(r - 1)) - 2a(d + \log 2) + 4b(\log d - \log 2) - 2d \log 2r - \frac{1}{3}.$$  

Proof. It is sufficient to prove the following inequality:

$$\Re(h(t_{\lambda_0})) \leq 2dr \log(2r) + dR(r - 1) \log(r - 1) - dR(r + 1) \log(r + 1) + \frac{1}{3}.$$  

By Lemma 4.24 we have $\Re(h(t_{\lambda_0})) \leq \Re(h(t_d))$. Moreover by Lemma 4.23 we have

$$\Re(h(t_d)) \leq 2dr \log(2r) + dR(r - 1) \log(r - 1) - dR(r + 1) \log(r + 1) + d\rho + \frac{4dR \rho^2}{r}.$$  

Since

$$\frac{4dR \rho^2}{r} \leq \frac{4dR \rho}{r} \cdot \frac{r \pi}{10dR} \leq \frac{2 \pi}{5} \rho \leq 2d\rho,$$

we obtain

$$d\rho + \frac{4dR \rho^2}{r} \leq d\rho + 2d\rho \leq 3d \rho \leq 3d \cdot \frac{r \pi}{10dR} \leq \frac{r}{R} \leq \frac{1}{3}.$$  

Thus, the proof is completed. ■
Theorem 5.5 Let $L \neq 0$ be a Dirichlet series of period $d$. For positive integers $a$, $b$ satisfying $a \geq 2b$, put $r = (d + 2b)/d$, $R = (a + d)/d$. If the inequalities $r \geq 2$, $R \geq 3r$ and

$$\frac{5r}{2e^{R/r}} < \min\left\{ \frac{r\pi}{10Rd}, \frac{\pi}{2d}, \frac{r}{4R} \sin \frac{\pi}{2d}, \frac{r}{38R} \left( \cos \frac{\pi}{2d} - \cos \frac{3\pi}{2d} \right) \right\}$$

hold, then we have

$$\delta(a; L) \geq 1 + \frac{\alpha(a, b)}{\beta(a, b)},$$

where

$$\alpha(a, b) = dR((r + 1)\log(r + 1) - (r - 1)\log(r - 1))$$
$$- 2a(d + \log 2) - 4b(\log d - \log 2) - 2dr\log(2r) - \frac{1}{3}$$
$$\beta(a, b) = 2ad + 2a\log 2 + 4(b + d)\log(b + d) - 4d\log d.$$

Proof. Let

$$L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}.$$

If all $a_k$ are real, the assertion follows from Proposition 5.3 and Lemma 5.4. Let us prove the general case ($a_k \in \mathbb{C}$). Let

$$L_r(s) = \sum_{k=1}^{\infty} \frac{\Re(a_k)}{k^s}, \quad L_i(s) = \sum_{k=1}^{\infty} \frac{\Im(a_k)}{k^s},$$

then we have $\Re(L(j)) = L_r(j)$, and $\Im(L(j)) = L_i(j)$. Therefore we have

$$\delta(a; L) \geq \max\{\delta(a; L_r), \delta(a; L_i)\}.$$ 

Thus, the general case follows from the real coefficient case. ■

Theorem 5.6 Let $L \neq 0$ be a Dirichlet series of period $d$. For any $\mu > 1$, we have

$$\delta([t^\mu]; L) \geq \frac{\log t}{d + \log 2} \left( 1 + o(1) \right), \quad \text{as } t \to \infty.$$

Proof. Let $a = [t^\mu]$, $b = [t]$. The assumptions of Theorem 5.5 are satisfied if $t$ is sufficiently large. By easy computation, we have $\beta([t^\mu], [t]) = t^\mu(d + \log 2 + o(1))$ and

$$\alpha([t^\mu], [t]) = dR((r + 1)\log(r + 1) - (r - 1)\log(r - 1)) + O(t^\mu).$$

Since

$$(r + 1)\log(r + 1) - (r - 1)\log(r - 1) = 2\log(r + 1) + (r - 1)\log \frac{r + 1}{r - 1} = 2\log t + O(1),$$

we have

$$\alpha([t^\mu], [t]) \geq dR((r + 1)\log(r + 1) - (r - 1)\log(r - 1)) + \frac{dR}{d} \left( \log t - \frac{\log 3}{3} \right) + O(t^\mu).$$

Thus,
we have
\[ \alpha([t^\mu],[t]) = 2t^\mu \log t(1 + o(1)). \]
Thus, we obtain the inequality
\[ \delta([t^\mu]; L) \geq 1 + \frac{2t^\mu \log t(1 + o(1))}{t^\mu (2(d + \log 2) + o(1))} = \frac{\log t}{d + \log 2}(1 + o(1)). \]

\[\Box\]

**Proof of Theorem 1.2.**

Let us take \( \mu > 1 \) such that \( C > \mu(d + \log 2) \). By putting \( t = a^{1/\mu} \) in Theorem 5.6, we have
\[ \delta(a; L) \geq 1 + \frac{\log a}{\mu(d + \log 2)}(1 + o(1)) \geq \frac{\log a}{C} \cdot \left( \frac{C}{\mu(d + \log 2)} + o(1) \right). \]
Since
\[ \frac{C}{\mu(d + \log 2)} > 1, \]
we have
\[ \delta(a; L) \geq \frac{\log a}{C} \]
for sufficiently large \( a \), as required.

Finally, I write down the estimate obtained from Theorem 5.5 for small \( d \).

If \( d = 1 \), we have \( \delta(a; L) = \dim_{\mathbb{Q}}(\mathbb{Q} \cdot \{1, \zeta(j) \mid 2 \leq j \leq a, j \equiv a \, (\text{mod} \, 2)\}) \). For even \( a \), we have \( \delta(a; L) = \frac{a+2}{2} \) since the value \( \zeta(j) \) at even integer \( j \) is rational multiple of \( \pi^j \). For odd \( a \), we obtain the estimates as in Table 2.

We can prove \( \delta(5; L) \geq 2 \) by more precise estimation for \( a = 5, b = 1 \). In fact, better estimates are known in the case of \( d = 1 \). For example, we have \( \delta(3; \zeta) = 2 \) according to Apéry’s theorem ([II]). and the estimate \( \delta(15; \zeta) \geq 3 \) is proved in [VI].

For \( d \geq 2 \), we obtain the estimates as in Table 3 by Theorem 5.5.

| \( a \) | \( b \) | \( 1 + \frac{\alpha(a,b)}{\beta(a,b)} \) | \( \delta(a; L) \) |
|----|----|----------------|--------|
| 9  | 1  | 1.08700873     | \( \geq 2 \) |
| 173 | 11 | 2.003058        | \( \geq 3 \) |
| 2187| 67 | 3.00028164      | \( \geq 4 \) |
| 21609 | 379 | 4.00001320 | \( \geq 5 \) |
| 186491 | 2119 | 5.00000046 | \( \geq 6 \) |
| 1476727 | 11735 | 6.00000012 | \( \geq 7 \) |

| \( a \) | \( b \) | \( 1 + \frac{\alpha(a,b)}{\beta(a,b)} \) | \( \delta(a; L) \) |
|----|----|----------------|--------|
| 88 | 10 | 1.00176867     | \( \geq 2 \) |
| 89 | 10 | 1.00412440     | \( \geq 2 \) |
| 4936 | 187 | 2.00003131 | \( \geq 3 \) |
| 4937 | 187 | 2.00008696 | \( \geq 3 \) |
| 159854 | 2894 | 3.00000007 | \( \geq 4 \) |
| 159855 | 2894 | 3.000000194 | \( \geq 4 \) |

Table 2: The case of \( d = 1 \)

Table 3: The case of \( d = 2 \)
Table 4: The case of \( d = 3 \)

\[
\begin{array}{c|c|c|c}
 a & b & 1 + \frac{\alpha(a,b)}{\beta(a,b)} & \delta(a;L) \\
549 & 48 & 1.00024059 & \geq 2 \\
550 & 48 & 1.00057135 & \geq 2 \\
78235 & 2165 & 2.00000009 & \geq 3 \\
78236 & 2165 & 2.00000285 & \geq 3 \\
\end{array}
\]

Table 5: The case of \( d = 4 \)

\[
\begin{array}{c|c|c|c}
 a & b & 1 + \frac{\alpha(a,b)}{\beta(a,b)} & \delta(a;L) \\
2594 & 186 & 1.00003443 & \geq 2 \\
2595 & 186 & 1.00009445 & \geq 2 \\
990205 & 21832 & 2.00000005 & \geq 3 \\
990206 & 21832 & 2.00000023 & \geq 3 \\
\end{array}
\]

References

[1] R. Apéry, *Irrationalité de \( \zeta(2) \) et \( \zeta(3) \)*, Astérisque 61, (1979), 11–13.

[2] K. Ball, T. Rivoal. *Irrationalité d’une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. 146, (2001), 193–207.

[3] F. Beukers, *Irrationality proofs using modular forms*, Journées arithmétiques de Besançon, Astérisque 147-148, (1987), 271–283.

[4] Y.V. Nesterenko, *On the linear independence of numbers*, Mosc. Univ. Math. Bull. 40, no.1, (1985), 69–74.

[5] T. Rivoal and W. Zudilin, *Diophantine properties of numbers related to Catalan’s constant*, Mathematische Annalen 326, no. 4, (2003), 705–721.

[6] W. Zudilin, *On irrationality of values of Riemann zeta-function*, Izvestia of Russian Acad. Sci. 66, (2002), 1–55.

[7] W. Zudilin, *One of the numbers \( \zeta(5) \), \( \zeta(7) \), \( \zeta(9) \), \( \zeta(11) \) is irrational*, Russian Math. Surveys 56:4, (2001), 774–776.