Maximum mass of a barotropic spherical star

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Abstract
The ratio of total mass $m_\ast$ to the surface radius $r_\ast$ of a spherical perfect fluid ball has an upper bound, $Gm_\ast/(c^2r_\ast) \leq B$. Buchdahl (1959 Phys. Rev. 116 1027) obtained the value $B_{\text{Buch}} = 4/9$ under the assumptions that the object has a nonincreasing mass density in the outward direction and a barotropic equation of state. Barraco and Hamity (2002 Phys. Rev. D 65 124028) decreased Buchdahl’s bound to a lower value, $B_{\text{BHa}} = 3/8 (<4/9)$, by adding the dominant energy condition to Buchdahl’s assumptions. In this paper, we further decrease Barraco–Hamity’s bound to $B_{\text{new}} \approx 0.3636403 (<3/8)$ by adding the subluminal (slower than light) condition of sound speed. In our analysis we numerically solve the Tolman–Oppenheimer–Volkoff equations, and the mass-to-radius ratio is maximized by variation of mass, radius and pressure inside the fluid ball as functions of mass density.

Keywords: mass-to-radius ratio, compact object, unstable circular orbit of photons, variational method

1. Introduction and summary of result
The subject of this paper is a question regarding a self-gravitating compact object: how much weight or how small a radius can a compact object possess? A quantitative answer is given by the mass-to-radius ratio $Gm_\ast/(c^2r_\ast)$, where $m_\ast$ and $r_\ast$ are respectively the total mass and surface radius of the compact object. Assuming static spherical symmetry for simplicity, which implies the outside region of the compact object is of Schwarzschild geometry, this...
ratio is bounded above, $Gm_*/(c^2 r_*) \leq B$. The upper bound needs to satisfy $B < 1/2$ ($\equiv 2Gm_*/c^2 < r_*$), in order to avoid gravitational collapse.

An interesting issue regarding this bound is whether $B$ is less than $1/3$ or not. If an inequality, $B \geq 1/3$ ($\equiv 3Gm_*/c^2 \geq r_*$), holds for the static spherical case, unstable circular orbits of photons can appear in the outside Schwarzschild geometry. If such a super-compact object, which possesses unstable circular orbits of photons but no black hole horizon, neither emits nor reflects any radiation, then we cannot distinguish it from black holes by observing the so-called black hole shadows. Here, the black hole shadow is a dark region which is expected to appear in a fine image of optical/radio observation of black holes (see [1, 2] and references therein). Consider a case where an optical source is extended behind a black hole and does not enter inside the unstable circular orbits of photons. The observer cannot detect photons which passed through the unstable circular orbits inward, because those photons are absorbed by a black hole eventually. Hence, in the image of the optical source, the dark region should appear, on which those photons would be detected if the black hole did not exist. This dark region is the black hole shadow, and the boundary of the shadow is determined by the photons propagating on null geodesics winding many times around unstable circular orbits of photons. This means that a super-compact object possessing unstable circular orbits but no black hole horizon can provide us with the same optical image of the shadow as black holes. Since the resolution of the image by radio observation is now approaching the visible angular size of the largest black hole candidate [3, 4], an investigation of a possible super-compact object seems to be an important issue for the near future observational study of black holes. Therefore, an interesting and important issue is whether $B < 1/3$ holds or not.

Some exotic models of super-compact objects have been proposed such as gravastars, boson stars and so on. The gravastar as a super-compact object has already been examined [5], whereas the others remain to be examined. Those exotic models may be interesting. However, in this paper, we focus on a rather usual model.

Assuming that (i) the compact object is a static spherical ball of perfect fluid, (ii) its mass density is nonincreasing with respect to radial coordinates, and (iii) its equation of state is barotropic, Buchdahl [6] obtained $Gm_*/(c^2 r_*) \leq B_{\text{Buch}} = 4/9$ [6, 7]. Furthermore, by adding the dominant energy condition to Buchdahl’s assumptions, Barraco and Hamity [8] decreased Buchdahl’s bound to a lower value, $B_{\text{BHa}} = 3/8$ ($< 4/9$) [8]. However, this upper bound is greater than $1/3$ ($< B_{\text{BHa}}$).

Other works on the bound $B$ have been performed. For example, an effect of a cosmological constant has been examined [9], a case with infinite surface radius has been analyzed [10], and a tangential pressure has been considered [11]. However, Barraco–Hamity’s bound, $B_{\text{BHa}} = 3/8$, has not been lowered so far.

As explained in detail in section 2, we add the subluminal (slower than light) condition of sound speed to the assumptions of Buchdahl and Barraco–Hamity. We discuss, in section 2, how the subluminal-sound-speed condition restricts the form of the equation of state, and also that this condition is more restrictive than the dominant energy condition. Then, our upper bound is lower than the Barraco–Hamity bound,

$$\frac{Gm_*}{c^2 r_*} \leq B_{\text{new}} \approx 0.3636403 \quad (B_{\text{BHa}} = \frac{3}{8})$$

The reason why $B_{\text{new}}$ is an approximate value is that we have performed numerical integration of Tolman–Oppenheimer–Volkoff equations. Although the upper bound is lowered, it still remains greater than $1/3$ ($< B_{\text{new}}$). Therefore, at present, we cannot deny the possibility that the same shadow image can be obtained from black holes and some super-compact object.
made of fluid matter. In order to sharpen the upper bound of \( Gm_\Sigma/(c^2r_\Sigma) \), some other condition should be assigned to the form of the equation of state. (We will report the case of the polytropic equation of state in the other paper.)

Section 2 is devoted to the details of our analysis. Section 3 contains the conclusion and discussions.

2. Variational analysis of mass-to-radius ratio

2.1. Definitions and assumptions

As explained in section 1, we are interested in a static and spherically symmetric perfect fluid ball. A line element of spacetime is

\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -e^{2\Phi(r)} c^2 dt^2 + \frac{dr^2}{1 - 2Gm(r)/(c^2r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where \((t, r, \theta, \varphi)\) are spherical polar coordinates, \(\Phi(r)\) gives a lapse function, and \(m(r)\) is a mass of perfect fluid contained in the spherical region of radius \(r\). The stress-energy-momentum tensor of perfect fluid is \(T_{\mu\nu} = \sigma(r) c^2 u_\mu u_\nu + p(r) (g_{\mu\nu} + u_\mu u_\nu)\), where \(u = e^{-\Phi} \partial_t\) is a four-velocity of static perfect fluid, and \(\sigma(r)\) and \(p(r)\) are, respectively, the mass density and pressure of the perfect fluid.

By the regularity of spacetime at the centre, a condition \(m(0) = 0\) holds. This implies that the mass density at centre \(\sigma_c = \sigma(0)\) is finite. We normalize all quantities by \(\sigma_c\),

\[
R := \frac{\sqrt{Gc}}{c} \, r, \quad \Sigma(R) := \frac{\sigma(r)}{\sigma_c}, \quad M(R) := \frac{\sqrt{Gc}}{c} \, m(r), \quad P(R) := \frac{p(r)}{\sigma_c c^2}.
\]

These are dimensionless. The lapse function, \(\Phi(r) = \Phi(R)\), does not need normalization because \(\Phi\) is originally dimensionless by definition (2).

Following Buchdahl [6, 7], we adopt two assumptions. One of them is a barotropic equation of state,

\[
P = P(\Sigma).
\]

Hereafter, we regard the mass density \(\Sigma\) as an independent variable, and the others are functions of it,

\[
R = R(\Sigma), \quad M = M(\Sigma), \quad P = P(\Sigma), \quad \Phi = \Phi(\Sigma).
\]

Another assumption is a nonincreasing mass density in the outward direction,

\[
\frac{dR(\Sigma)}{d\Sigma} \leq 0.
\]

This implies the correspondence between variables \(R\) and \(\Sigma\) is one to one.

The surface of a fluid ball is defined by vanishing pressure, where the hydrostatic equilibrium holds between the fluid ball and the outside vacuum region. Then, the mass density at surface \(\Sigma_s\) is determined by

\[
P(\Sigma_s) = 0,
\]

and the total mass \(M_s\) and surface radius \(R_s\) of the fluid ball are respectively given by

\[
M_s = M(\Sigma_s), \quad R_s = R(\Sigma_s).
\]
At the centre of the fluid ball, the mass density is unity $\Sigma_0 = 1$, and conditions $R(1) = 0$ and $M(1) = 0$, should hold. Note that the mass density takes values in an interval,

$$\Sigma_0 \leq \Sigma \leq 1,$$

where the surface mass density satisfies $0 \leq \Sigma_s < 1$.

The outside region of the fluid ball, $R > R_s$, is the Schwarzschild geometry of mass $M_s$. The inside region, $R < R_s$, is determined by the Einstein equation and conservation law $T^{0\nu}_{\nu} = 0$, which are reduced to Tolman–Oppenheimer–Volkoff (TOV) equations,

$$\frac{dM(\Sigma)}{d\Sigma} = 4\pi R(\Sigma)^2 \frac{dR(\Sigma)}{d\Sigma},$$

$$\frac{dP(\Sigma)}{d\Sigma} = A(M, R, P; \Sigma) \frac{dR(\Sigma)}{d\Sigma},$$

$$\frac{d\Phi(\Sigma)}{d\Sigma} = - \frac{1}{\Sigma + P(\Sigma)} \frac{dP(\Sigma)}{d\Sigma},$$

where

$$A(M, R, P; \Sigma) := - \frac{[\Sigma + P(\Sigma)] \left[ M(\Sigma) + 4\pi R(\Sigma)^3 P(\Sigma) \right]}{R(\Sigma) \left[ R(\Sigma) - 2M(\Sigma) \right]}.$$  

Given a concrete functional form of the equation of state, two functions $R(\Sigma)$ and $M(\Sigma)$ are obtained by solving (10a) and (10b). Substituting those solutions into (10c), $\Phi(\Sigma)$ is obtained.

In addition to Buchdahl’s assumptions (4) and (6), we assume the subluminal condition of sound speed,

$$\frac{dP(\Sigma)}{d\Sigma} \leq 1.$$

Given the above formulation, the remaining freedom is the functional form of $P(\Sigma)$ under the condition (12). Since the solution of the TOV equations depend on the concrete form of $P(\Sigma)$, the value of total mass $M_s$ and surface radius $R_s$ of the fluid ball vary with the concrete form of $P(\Sigma)$. In the following sections, by the variation of the functional form of $P(\Sigma)$ under condition (12), a possible upper bound $\mathcal{B}$ of the ratio $M_s/R_s (= Gm_s/c^2r_s)$ will be calculated,

$$\frac{M_s}{R_s} \leq \mathcal{B}.$$  

Using such a variational method, our aim is to obtain a value of $\mathcal{B}$ lower than Barraco–Hamity’s bound, $B_{BHa} = 3/8$, which is lower than the famous Buchdahl’s bound, $B_{Buch} = 4/9$.

Here, note that Barraco and Hamity [8] considered not only Buchdahl’s assumptions (4) and (6) but also the dominant energy condition which gives the following inequality for the perfect fluid ball,

$$P(\Sigma) \leq \Sigma.$$  

We expect that some physically reasonable (nonexotic) condition, which is more restrictive than the dominant energy condition, can decrease Barraco-Hamity’s bound further to a lower value. Concerning this expectation, let us point out that the subluminal-sound-speed condition (12) and definition of surface (7) predict the inequality (14) as well. The subluminal-sound-speed condition restricts not only the value of pressure such as inequality (14) but also the value of its differential, such as inequality (12). This implies that the subluminal-sound-speed
condition (12) is more restrictive than the dominant energy condition. Therefore, we expect that the subluminal-sound-speed condition is sufficient for us in search of an upper bound lower than Barraco-Hamity’s bound.

2.2. Variational method: strategy

The variational method which we are going to use is known as optimal control theory. It has already been applied to an estimation of the maximum mass of neutron stars by Rhoades and Ruffini [12]. (See also [13, 14] for more details about the application of the optimal control theory to neutron star mass.) We apply the optimal control theory to the search for upper bound \( B \) of inequality (13). However, we do not require readers to have knowledge of optimal control theory. All ideas of the variational analysis are explained below.

We introduce an auxiliary variable \( U(\Sigma) \) by

\[
\sin^2 U(\Sigma) := \frac{dP(\Sigma)}{d\Sigma}.
\] (15)

Then, the subluminal-sound-speed condition (12) is automatically satisfied, \( \sin^2 U \leq 1 \). In order to make use of the variational method, it is useful to express \( M_s/R_s \) by an integral form \(^4\),

\[
\frac{M_s}{R_s} = \int_0^{R_s} dR \frac{d(M/R)}{dR} = \int_{\Sigma_a}^{1} d\Sigma \left( \frac{d(M/R)}{d\Sigma} \right) \equiv \int_{\Sigma_a}^{1} d\Sigma L,
\] (16a)

where the integrand \( L \) is arranged into the following form using relation (15) and TOV equations (10a) and (10b),

\[
L(M, R, P, U; \Sigma) = \frac{M - 4\pi R^3 \Sigma}{A(M, R, P; \Sigma) R^2} \sin^2 U.
\] (16b)

Our problem is to maximize the functional (16a) under three constraints; the relation (15) and TOV equations (10a) and (10b). In the following discussion, (10c) is not considered because \( \Phi \) does not appear in the integrand \( L \).

Because there are three constraints, we use Lagrange’s multiplier method. Define a functional,

\[
I := \int_{\Sigma_a}^{1} d\Sigma \left[ L + Y_M(\Sigma) C_M + Y_R(\Sigma) C_R + Y_P(\Sigma) C_P \right],
\] (17a)

where functions \( Y_M(\Sigma), Y_R(\Sigma) \) and \( Y_P(\Sigma) \) are the Lagrange multiplier, and \( C_M, C_R \) and \( C_P \) are defined according to the constraints (10a), (10b) and (15),

\[
C_M(M, R, P; \Sigma) = \frac{4\pi R^3 \Sigma^2}{A(M, R, P; \Sigma)} \sin^2 U(\Sigma) - \frac{dM(\Sigma)}{d\Sigma},
\] (17b)

\[
C_R(M, R, P; \Sigma) = \frac{1}{A(M, R, P; \Sigma)} \sin^2 U(\Sigma) - \frac{dR(\Sigma)}{d\Sigma},
\] (17c)

\[
C_P(M, R, P; \Sigma) = \sin^2 U(\Sigma) - \frac{dP(\Sigma)}{d\Sigma}.
\] (17d)

\(^4\) The signature of (16a) is not \(-\int d\Sigma d(M/R)/dR \) but \( +\int d\Sigma d(M/R)/dR \), because of an inequality \( d(M/R)/dR \geq 0 \) which is found by \( M/R \sim R^2 \).
The extremal value of $M_e/R_a$ is given by the condition $\delta I = 0$ under variations,

\[
M(\Sigma) \rightarrow M(\Sigma) + \delta M(\Sigma) \\
R(\Sigma) \rightarrow R(\Sigma) + \delta R(\Sigma) \\
P(\Sigma) \rightarrow P(\Sigma) + \delta P(\Sigma) \\
U(\Sigma) \rightarrow U(\Sigma) + \delta U(\Sigma).
\]  

(18)

Given the functional expression $I$ of $M_e/R_a$, the strategy of our variational analysis consists of the following steps:

**step 1.** Divide the interval $(\Sigma)$ of $\Sigma$ into many infinitesimal intervals. Then solve the equation $\delta I = 0$ in order to maximize the functional $I$ at each infinitesimal interval. 

**step 2.** Integrate the maximized $I$ of every infinitesimal interval of $\Sigma$. Then we will obtain a formal expression for the upper bound $\mathcal{B} \geq M_e/R_a$ for the global interval $(\Sigma)$. 

**step 3.** Calculate numerically the formal expression of $\mathcal{B}$ obtained in step 2.

### 2.3. Variational method: step 1

We divide the interval $(\Sigma)$ into many infinitesimal intervals. Look at one infinitesimal interval of $\Sigma$,

\[
\Sigma_d \leq \Sigma \leq \Sigma_u,
\]  

(19)

where $\Sigma_d \leq \Sigma_u < \Sigma_a \leq 1$ and $\Sigma_a - \Sigma_d \ll 1$. Then, in step 1 of our variational analysis, we maximize $M_d/R_a$ by solving $\delta I = 0$ under the variations (18) in the interval (19). Note that, since we are considering a single interval (19) in step 1, we fix variables $(M, R, P, U)$ in the remaining intervals, $\Sigma_a \leq \Sigma < \Sigma_d$ and $\Sigma_u < \Sigma \leq 1$. This indicates boundary conditions

\[
\delta X(\Sigma_d) = 0, \quad \delta X(\Sigma_u) = 0 \quad (X = M, R, P, U).
\]  

(20)

The Euler–Lagrange equations of $\delta I = 0$ are the following. Variation of $U$ gives

\[
\left[ \frac{M - 4\pi R^3\Sigma}{AR^2} + \frac{4\pi R^2\Sigma}{A} + \frac{1}{A} \right] \sin U \cos U = 0,
\]  

(21)



variations of $X$ ($= M, R, P$) give

\[
\frac{\partial Y_X(\Sigma)}{\partial \Sigma} = \left[ \frac{\partial}{\partial X} \left( \frac{M - 4\pi R^3\Sigma}{(AR^2)} \right) \right] + \frac{Y_M}{A} \frac{\partial}{\partial X} \left( \frac{4\pi R^2\Sigma}{A} \right) + \frac{Y_R}{A} \frac{\partial (1/A)}{\partial X} \right] \sin^2 U,
\]  

(22)

and variations of $Y_X$ give $C_k = 0$ which are the constraints; the TOV equations (10a), (10b), and relation (15). Equation (21) gives

\[
U = 0, \quad \frac{\pi}{2}
\]  

(23)

or an algebraic equation given by vanishing the inside of the square bracket of (21). However, the latter case (vanishing square bracket) is impossible under the boundary condition (20) as shown in appendix A. Therefore, $U$ should be constant, as given in (23), which denotes, for extremal case of $M_e/R_a$.

\[5\] In the case of Buchdahl’s analysis [6], the maximum value $M_e/R_a = \mathcal{B}_{\text{Buch}} = 4/9$ is realized for the fluid ball of constant mass density. The constant mass density implies that the sound speed is infinity. On the other hand, we are considering the subluminal-sound-speed condition. Therefore, the value $1$ of sound speed for the extremal case of $M_e/R_a$ in our analysis is consistent with Buchdahl’s analysis.
This gives a constant pressure or linear equation of state, \( P(\Sigma) = \Sigma - \Sigma_a \). It has already been revealed by Nilsson and Uggla [15] that, for the linear equation of state \( P(\Sigma) = \Sigma - \Sigma_a \), the total mass \( M_\epsilon \) and surface radius \( R_\epsilon \) are both finite for \( \Sigma_a \neq 0 \), but both infinite for \( \Sigma_a = 0 \). However, the behaviour of \( M_\epsilon/R_\epsilon \) has not been investigated so far.

Due to the extremal condition (24) and boundary condition (20), a parallelogram appears in the \( \Sigma-P \) plane as shown in figure 1. The equation of state given by the edge \( (a)+(b) \) or \( (c)+(d) \) corresponds to a maximum or minimum value of \( M_\epsilon/R_\epsilon \) under the variations (18) in the infinitesimal interval (19). In order to judge which edge corresponds to maximum or minimum, we calculate their difference,

\[
\Delta \left( \frac{M_\epsilon}{R_\epsilon} \right) := \int_{(c)+(d)} d\Sigma L - \int_{(a)+(b)} d\Sigma L, \tag{25a}
\]

where the first term corresponds to an integral with the equation of state given by the edge \( (c)+(d) \), and the second term is an integral with the edge \( (a)+(b) \). Here, note that TOV equations (10a) and (10b) result in \( M(\Sigma) \) and \( R(\Sigma) \) being constant for \( U = 0 \) (\( dP/d\Sigma = 0 \)). Therefore, the edges \( (b) \) and \( (c) \) make no contribution to \( M_\epsilon/R_\epsilon \). Then, the difference \( \Delta (M_\epsilon/R_\epsilon) \) is calculated to be

\[
\Delta \left( \frac{M_\epsilon}{R_\epsilon} \right) = \int_{(d)} d\Sigma L - \int_{(a)} d\Sigma L
\]

\[
= L(M, R, P, U = 1; \Sigma_a + \Delta \Sigma) \delta \Sigma - L(M, R, P, U = 1; \Sigma_a) \delta \Sigma
\]

\[
= \frac{\partial L(M, R, P, U = 1; \Sigma_a)}{\partial \Sigma_a} \delta \Sigma \Delta \Sigma
\]

\[
= \frac{R_d - 2M_\epsilon}{R_d(\Sigma_a + P_\epsilon)} \delta \Sigma \Delta \Sigma > 0, \tag{25b}
\]

where \( \delta \Sigma \) and \( \Delta \Sigma \) correspond to two edges of a parallelogram, as shown in figure 1, and \( M_\epsilon, R_d \) and \( P_\epsilon \) are mass, radius and pressure at \( \Sigma_a \). This inequality (25b) denotes that the equation of state with edge \( (c)+(d) \) (edge \( (a)+(b) \)) corresponds to the maximum (minimum) of \( M_\epsilon/R_\epsilon \) under variations in the infinitesimal interval (19) and boundary condition (20).
2.4. Variational method: step 2

Next, we proceed to an analysis in the global interval \( \Sigma \) of \( \Sigma_* \). Consider a case where the values of pressure at the centre, \( P_c \), and mass density at the surface, \( \Sigma_* \), are fixed. In this case, we can repeat the deformation of the equation of state using an infinitesimally small parallelogram. Finally, the functional form of the equation of state in interval \( \Sigma_* \) reaches one of the following two options:

\[
P(\Sigma) = \begin{cases} 
\Sigma - \Sigma_* & \text{in } \Sigma_* \leq \Sigma \leq \Sigma_* + P_c \\
P_c & \text{in } \Sigma_* + P_c < \Sigma \leq 1 
\end{cases} \quad (26a)
\]

\[
P(\Sigma) = \begin{cases} 
0 & \text{in } \Sigma_* \leq \Sigma < 1 - P_c \\
\Sigma - (1 - P_c) & \text{in } 1 - P_c \leq \Sigma \leq 1 
\end{cases} \quad (26b)
\]

These forms are shown in figure 2. No contribution to \( M_\text{r}/R_* \) arises from parts, \( P = \text{constant} \), as explained before calculating \((25b)\). The inequality \((25b)\), for an infinitesimally small parallelogram, indicates that the equation of state \((26b)\) gives the maximum value of \( M_\text{r}/R_* \) for given \( P_c \) and \( \Sigma_* \), whereas \((26a)\) gives the minimum for given \( P_c \) and \( \Sigma_* \).

We should emphasize that the above discussion is applicable only in the case of fixed \( P_c \) and \( \Sigma_* \). It has not been examined so far whether or not \( M_\text{r}/R_* \) for the equation of state \((26b)\) or \((26a)\) is maximum (or minimum) even when the values of \( P_c \) and \( \Sigma_* \) vary. There is a possibility that a value of \( M_\text{r}/R_* \) for equations of state \((26a)\) or \((26b)\) for certain values of \( P_c \) and \( \Sigma_* \) is neither maximum nor minimum under variation of \( P_c \) and \( \Sigma_* \).

In order to find a true maximum value of \( M_\text{r}/R_* \) under variation of \( P_c \) and \( \Sigma_* \), we regard \( M_\text{r}/R_* \) for the equation of state \((26b)\) as a function of \( P_c \):

\[
f(P_c) = \frac{M_\text{r}}{R_*} \quad \text{for equation of state (26b).} \quad (27)
\]

Here, note that the equation of state \((26b)\) in the interval, \( 1 - P_c \leq \Sigma \leq 1 \), depends only on \( P_c \). The maximum value of \( f(P_c) \), which is denoted by \( B_* \), is the desired upper bound of the mass-to-radius ratio, \( M_\text{r}/R_* \leq B_* \). A numerical plot of \( f(P_c) \) is going to be shown in section 2.5.

In order to find a true minimum value of \( M_\text{r}/R_* \) under variation of \( P_c \) and \( \Sigma_* \), we regard \( M_\text{r}/R_* \) for the equation of state \((26a)\) as a function of \( P_c \) and \( \Sigma_* \). The minimum value of this function is the lower bound of \( M_\text{r}/R_* \). Here, note that this lower bound can be read from \( f(P_c) \) by the following discussion. For the equation of state \((26a)\) for given \( P_c \) and \( \Sigma_* \), the
quantity $\Sigma_e + P_c$ can be regarded as a central mass density. Then, following the normalization (3), we transform variables by $P = P/(\Sigma_e + P_c)$, $M = \sqrt[3]{\Sigma_e + P_c} M$ and $R = \sqrt[3]{\Sigma_e + P_c} R$. This transformation changes the form of the equation of state (26a) in the interval, $\Sigma_e \leq \Sigma \leq \Sigma_e + P_c$, to the following form,

$$P\left(\frac{\Sigma}{\Sigma_e}\right) = \Sigma - \left(1 - \tilde{P}\right) \quad \text{for} \quad 1 - \tilde{P} \leq \Sigma \leq 1,$$

(28)

where $\tilde{P} = P_c/(\Sigma_e + P_c)$. Furthermore, the form of the TOV equations for transformed variables are the same for (10a) and (10b). Hence the value of $M/R_*$ for equation of state (28) is equal to the value of $M_*/R_*$ for equation of state (26a). On the other hand, the form of equation of state (28) is the same as equation of state (26b) in the interval $1 - P_c \leq \Sigma \leq 1$. Therefore, the value of $M_*/R_*$ for equation of state (26a) for given $P_c$ and $\Sigma_e$ is equal to that for equation of state (26b) when replacing $P_c$ by $\tilde{P}_c$. This implies that the value of $M_*/R_*$ for equation of state (26a) can be read from $f(P_c)$, and the minimum value of $f(P_c)$ is the lower bound of $M_*/R_*$.

2.5. Variational method: step 3

The function $f(P_c)$, defined in (27), can be calculated by following steps:

i. Solve numerically TOV equations (10a) and (10b) with the equation of state (26b) for a given value of $P_c$. A technical remark is summarized in appendix B.

ii. Calculate $M_*/R_*$ from numerical solutions of step (i).

iii. The function $f(P_c)$ is calculated by repeating steps (i) and (ii) for different values of $P_c$ in the interval $0 < P_c \leq 1$.

We should also remember that $R(\Sigma) = \text{constant}$ for $P = \text{constant}$ due to the TOV equations.
Our numerical result is shown in figure 3. A smooth graph of \( f(P_c) \) is obtained. From the plot of differentials of \( f(P_c) \) in figure 3, we find that the maximum of \( f(P_c) \) is given at

\[
P_c = P_{\text{bound}} \simeq 0.8386058.
\]

(29)

Then, as discussed in section 2.4, the upper bound \( B_{\text{new}} \geq M_b/R_b \) is given by the maximum value, \( B_{\text{new}} \simeq f(P_{\text{bound}}) \), which is evaluated numerically to be \( B_{\text{new}} \simeq 0.3636403 \). This upper bound gives our conclusion (1). On the other hand, the lower bound read from figure 3 is zero given at \( P_c = 0 \). No finite lower bound is found in our analysis.

3. Conclusion and discussions

By adding the subluminal-sound-speed condition (12) to Buchdahl’s assumptions, we obtained figure 3 and the bound of the mass-to-radius ratio of a barotropic fluid ball,

\[
0 < \frac{M_b}{R_b} \leq B_{\text{new}} \simeq 0.3636403.
\]

(30)

This upper bound is lower than Barraco–Hamity’s bound \( B_{\text{BH}} = 3/8 \). However, our upper bound is greater than 1/3 \((< B_{\text{new}})\), and there remains a possibility of the existence of a super-compact object possessing unstable circular orbits of photons but no black hole horizon.

As read from figure 3, the upper bound \( B_{\text{new}} \) corresponds to the central pressure, \( P_c = P_{\text{bound}} \) of (29). This value of central pressure may be understood as a result of the trade-off between pressure’s two effects: a contribution to mass density (attractive force) and an effect pushing fluid outward (repulsive force).

From the above, the maximum value of \( M_b/R_b \) is \( B_{\text{new}} \) of (30) which is obtained by solving the TOV equations with the central pressure \( P_c = P_{\text{bound}} \) of (29) and linear equation of state (26b). Furthermore, by our numerical integration of the TOV equations, the mass and radius of the fluid ball in the case of the maximum mass-to-radius ratio are

\[
M_b(P_{\text{bound}}) = M_{\text{bound}} \simeq 0.2014578
\]

\[
R_b(P_{\text{bound}}) = R_{\text{bound}} \simeq 0.5540028,
\]

(31)

which gives our bound \( B_{\text{new}} = M_{\text{bound}}/R_{\text{bound}} \). From these values, together with the normalization (3), we can estimate typical unnormalized values of physical quantities for the case of maximum mass-to-radius ratio. By denoting the unnormalized mass in units of solar mass, \( m_{\text{bound}} = \alpha M_\odot \), where \( \alpha \) is a dimensionless factor, then the unnormalized central mass density \( \sigma_{\text{bound}} \), central pressure \( P_{\text{bound}} \) and radius \( r_{\text{bound}} \) are calculated as

\[
\sigma_{\text{bound}} = \frac{c^6 M_{\text{bound}}^2}{G m_{\text{bound}}^2} \simeq \frac{2.5}{\alpha^2} \times 10^{16} \text{ g cm}^{-3}
\]

\[
P_{\text{bound}} = \sigma_{\text{bound}} c^2 P_{\text{bound}} \simeq \frac{1.3}{\alpha^2} \times 10^{30} \text{ GeV cm}^{-3}
\]

\[
r_{\text{bound}} = \frac{G m_{\text{bound}}}{c^2 B_{\text{new}}} \simeq 4.1 \alpha \text{ km}.
\]

(32)

Note that the case of the maximum mass-to-radius ratio corresponds to the linear equation of state (26b) giving the luminal (equal-to-light) sound speed inside the fluid ball. If the equation of state, \( P(\Sigma) \), will be restricted to some functional form other than the linear
form, then the upper bound \( B \) of \( M_p/R_s \) will take a value different from \( B_{\text{new}} \) of (30). We will report the case of the polytropic equation of state in the other paper.\(^7\)

Finally we make a mathematical comment on a limiting behaviour found from figure 3, \[
\lim_{P_2 \to 1} f \left( P_2 \right) = \frac{1}{4}. \tag{33}
\]

At the limit \( P_2 \to 1 \), the equation of state (26b) asymptotes to a simple form \( P(\Sigma) = \Sigma \) \((0 \leq \Sigma \leq 1)\). Further, functional forms of \( M(\Sigma) \) and \( R(\Sigma) \) converge to an exact solution of the TOV equations with equation of state \( P(\Sigma) = \Sigma \);
\[
M(\Sigma) = \frac{1}{4} R(\Sigma) = \frac{1}{16\sqrt{\pi} \Sigma}. \tag{34}
\]

This results in a finite limit (33). However, the total mass and surface radius of the fluid ball diverge, \( M_0 \to \infty \) and \( R_0 \to \infty \) as \( P_2 \to 1 \), because the surface mass density is zero, \( \Sigma_0 \to 0 \) as \( P_2 \to 1 \). This result is consistent with Nilsson–Uggla’s numerical result [15] that, for the linear equation of state, \( P = \Sigma - \Sigma_0 \) \((\Sigma_0 = 1 - P_2)\), the mass \( M_0 \) and radius \( R_0 \) are both finite for \( \Sigma_0 = 0 \) but diverge as \( \Sigma_0 \to 0 \).

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Appendix A. Impossibility of vanishing square bracket of (21)

In this appendix we show that, as a solution of (21), the vanishing square bracket of (21) is impossible under the boundary condition (20).

Assume that a solution of (21) is given by vanishing the square bracket, we find
\[
Y_P = -\frac{1}{A(M, R, P; \Sigma)} \left[ Y_R + \frac{M}{R^2} + 4\pi R^2 \Sigma \left( Y_M - \frac{1}{R} \right) \right]. \tag{A.1}
\]

On the other hand, we obtain from (22),
\[
\frac{dY_M}{d\Sigma} = -\frac{R + 8\pi R^3 P}{(R - 2M)(M + 4\pi R^3 P)} Y_P \sin^2 U - \frac{\sin^2 U}{A(M, R, P; \Sigma) R^2} \tag{A.2}
\]
\[
\frac{dY_R}{d\Sigma} = \left( \frac{1}{R} + \frac{1}{(R - 2M)} - \frac{12\pi R^2 P}{M + 4\pi R^3 P} \right) Y_P \sin^2 U
+ \left[ \frac{2M}{R^2} + 4\pi \Sigma (1 - 2Y_M) \right] \frac{\sin^2 U}{A(M, R, P; \Sigma)} \tag{A.3}
\]

\(^7\) We will show in the other paper that, when the equation of state is restricted to being the polytropic one \((P(\Sigma) \propto \Sigma^{p+1/\lambda})\) under the subluminal-sound-speed condition, the maximum value of \( M_0/R_s \) becomes a value lower than \( B_{\text{new}} \) of (30), \( M_0/R_s \lesssim 0.281 \). In deriving this polytrope result, we will not use the variational method, but use the other numerical search method.
\[
\frac{dY_\Sigma}{d\Sigma} = -\frac{M\Sigma + 4\pi R^2 (P\Sigma + \Sigma + P)}{(\Sigma + P) (M + 4\pi R^2 P)} Y_P \sin^2 U.
\] (A.4)

Equation (A.4) with boundary condition (20) results in \( Y_\Sigma = 0 \). Then (A.1) gives
\[
Y_R + \frac{M}{R^2} + 4\pi R^2 \Sigma \left( Y_M - \frac{1}{R} \right) = 0.
\] (A.5)

Differentiating this equation by \( \Sigma \) and substituting (A.2), (A.3) and TOV equations (10a) and (10b), we obtain
\[
Y_M(\Sigma) = \frac{1}{R(\Sigma)}, \quad Y_R(\Sigma) = -\frac{M(\Sigma)}{R(\Sigma)^2},
\] (A.6)

where \( Y_R \) is obtained by substituting \( Y_M \) in (A.5). These solutions cannot satisfy the boundary condition (20). Hence, the solution of (21) is given by (23).

**Appendix B. On numerical treatment of TOV equations**

The right hand sides of TOV equations (10a) and (10b) are of an indeterminate form at the centre because of the conditions \( M \to 0 \) and \( R \to 0 \) as \( \Sigma \to 1 \). Therefore, in solving the TOV equations numerically in step (i) of section 2.5, we have made use of perturbative solutions near the centre.

In order to consider a perturbation near the centre, we regard the radius \( R \) as an independent variable, and the mass density as a function of radius, \( \Sigma(R) \). TOV equations (10a) and (10b) are rearranged into the following forms,
\[
\frac{dM(R)}{dR} = 4\pi R^2 \Sigma(R), \quad \frac{dP(R)}{dR} = -\left[ \frac{\Sigma(R) + P(R)}{R} \right] \left[ \frac{M(R) + 4\pi R^3 P(R)}{R - 2M(R)} \right].
\] (B.1)

For a sufficiently small radius \( R \ll 1 \), we introduce perturbations,
\[
M(R) = M_{(1)}R + \Sigma R^2 + \Sigma R^3 + \cdots,
\]
\[
P(R) = P_{(1)}R + \Sigma R^2 + \Sigma R^3 + \cdots,
\]
\[
\Sigma(R) = 1 + \Sigma_{(1)}R + \Sigma_{(2)}R^2 + \Sigma_{(3)}R^3 + \cdots,
\] (B.2)

where conditions \( M(R = 0) = 0 \), \( \Sigma(R = 0) = 1 \) and \( P(R = 0) = P_{(1)} \) are included. Substituting (B.2) into (B.1), we obtain \( M_{(1)} = 0 \), \( M_{(2)} = 0 \), \( P_{(1)} = 0 \) and the remaining parts,
\[
M(R) = \frac{4}{3}\pi R^3 + \pi \Sigma_{(1)}R^4 + \cdots,
\]
\[
P(R) = P_{(1)} - \frac{2}{3}\pi (1 + 3P_{(1)}) (1 + P_{(1)})R^2 - \frac{\pi}{9} (7 + 15P_{(1)}) \Sigma_{(1)}R^3 + \cdots,
\]
\[
\Sigma(R) = 1 + \Sigma_{(1)}R + \Sigma_{(2)}R^2 + \Sigma_{(3)}R^3 + \cdots,
\] (B.3)

where the central pressure \( P_{(1)} \) and coefficients \( \Sigma_{(n)} (n = 1, 2, 3, \cdots) \) are determined by the concrete form of the equation of state.

Comparing \( P(R) \) in (B.3) with equations of state (26b), we obtain \( \Sigma_{(1)} = 0 \) and \( \Sigma_{(2)} = P_{(2)} \). Hence, denoting a small radius by \( R_\delta \ll 1 \), the mass density \( \Sigma_\delta \) and mass \( M_\delta \) at \( R = R_\delta \) are approximately given by
If the mass density near centre $\Sigma_6$ is given, the others are determined by

$$R_6 = \sqrt{\frac{3(1 - \Sigma_6)}{2\pi(1 + 3P_6)}(1 + P_6)} , \quad M_6 = \frac{4}{3} \pi R_6^3.$$  \hspace{1cm} \text{(B.5)}$$

In step (i) of section 2.5, we have solved TOV equations (10a) and (10b) in an interval, $\Sigma_8 \leq \Sigma \leq \Sigma_6$, with initial condition (B.5). Also, the convergence of numerical solutions has been checked by varying $\Sigma_8$. Our results in figure 3 are found using $\Sigma_8 = 1 - 10^{-6}$.

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