Research Article

C*-Algebras Generated by a System of Unilateral Weighted Shifts and Their Application

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We study the structure C*-algebras generated by a system of unilateral weighted shifts. Finally the obtained results are applied to a class of integral equations.

1. Introduction

The structure of C*-algebras generated by isometry is determined in [1, 2]. The structure is the same with the structure of C*-algebras generated by unilateral weighted shift operators, that is, the structure of C*-algebras generated by multiplication operators with the independent variable in the Hardy space on the unit disc. The analogue of the unit disc on \( \mathbb{C}^N \) is the polydisc

\[
\Delta^N = \left\{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : |z_i| < 1, 1 \leq i \leq N \right\}
\] (1.1)

or the unit ball

\[
B^N = \left\{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 < 1 \right\}.
\] (1.2)

The structures of C*-algebras generated by multiplication operators with the independent variable in the Hardy space on the unit ball and polydisc are different. To understand this difference we study the structure of C*-algebras generated by system of unilateral weighted shifts.
Let $I$ be a multiindex $(i_1, \ldots, i_N)$ of integers and $I \pm \varepsilon_j$ denotes $(i_1, \ldots, i_j \pm 1, \ldots, i_N)$ for the multi-index $I$. Here $\varepsilon_j$ is another multi-index $(\delta_1, \ldots, \delta_N)$, where $\delta_i$ is the Kronecker symbol.

Let $\{e_I\}_{I \geq 0}$ be an orthonormal basis of a separable complex Hilbert space $H$ and let $\{\omega_{I,j} : I \geq 0, 1 \leq j \leq N\}$ be a bounded net of complex numbers. Denote by $A_j$ the bounded linear operators whose effect on the elements of basis $\{e_I\}_{I \geq 0}$ of $H$ is given as $A_j e_I = \omega_{I,j} e_{I+\varepsilon_j}$, $1 \leq j \leq N$. A family of $N$ operators, denoted by $A = (A_1, A_2, \ldots, A_N)$, is called a system of unilateral weighted shifts, and the numbers $\{\omega_{I,j} : I \geq 0, 1 \leq j \leq N\}$ are called the weights of the system. It is known from [3, page 209, Corollary 2] that for shifts with nonzero weights $\{\omega_{I,j}\}$, without loss of generality we may always assume the weights are a set of positive real numbers, that is, the system $A$ positive. It is possible to show that if there exists a solution for the multivariable moment problem for the net $\{\beta^2_I\}_{I \geq 0}$ where $\beta_{I+\varepsilon_j} = \omega_{I,j} \beta_I$, $\beta_0 = 1$, that is, if there exists the probability measure $\nu_A$ on $[0,1]^N = [0,1] \times \cdots \times [0,1]$ ($N$ times) such that

$$\beta^2_I = \int_{[0,1]^N} r_1^{2i_1} r_2^{2i_2} \cdots r_N^{2i_N} d\nu(r_1, r_2, \ldots, r_N),$$

then the system $A$ is unitarily equivalent to the system of multiplication operators by the independent variables $z_j$, $1 \leq j \leq N$, on the space

$$H^2(\Delta^N, \mu_A) = \left\{ f = \sum_{I \geq 0} f_I z^I : \sum_{I \geq 0} |f_I|^2 \beta^2_I < \infty \right\},$$

where $\mu_A = \nu_A d\theta_1 d\theta_2 \cdots d\theta_N$, $0 \leq \theta_i \leq 2\pi$.

The reader can find for more details of such operators in the article by Jewell and Lubin [3] and Ergezen and Sadik [4]. Furthermore the papers of Curto and Yoon [5] and Curto and Yan [6] are closely related to our study.

2. C*-Algebras Generated by a System of Unilateral Weighted Shifts

Let $\Omega$ denote the family of the systems $A$ which satisfy the functional model defined above. Moreover, let $\Omega_1$ be a subset of $\Omega$ defined by

$$\Omega_1 = \left\{ A \in \Omega : \nu_A(U(1,1,\ldots,1)) > 0 \text{ for arbitrary neighborhood } U(1,1,\ldots,1) \text{ of the point } (1,1,\ldots,1) \in [0,1]^N \right\}.$$  

**Theorem 2.1** (see [4, page 25, Theorem 2]). Let $A \in \Omega$. A necessary and sufficient condition for the operator algebra generated by the system $A$ to be isometrically isomorphic to the polydisc algebra is that $A$ belongs to $\Omega_1$.

This theorem will be helpful in studying the structure of C*-algebra $C^*(A)$ generated by $A \in \Omega_1$.

Let $P$ denote the orthogonal projection of $L_2(\Delta^N, \mu_A)$ onto $H^2(\Delta^N, \mu_A)$ and let $\varphi$ lie in $C(\text{supp } \mu_A)$. Then the Toeplitz operator $T_\varphi f = P(\varphi f)$ for $f$ in $H^2(\Delta^N, \mu_A)$. 


Without loss of generality we may take \( N = 2 \). The following theorems for \( N = 1 \) were given by Sadikov [7].

**Theorem 2.2.** Let \( A \in \Omega \). If the algebra generated by the system \( A \) is polydisc algebra then the commutator ideal \( \mathcal{J} \) of \( C^*(A) \) contains properly the ideal of compact operators \( \mathcal{K} \) and the quotient space \( \mathcal{J}/\mathcal{K} \) is isometrically isomorphic to \( C(T \times \{0, 1\}) \oplus \mathcal{K} \) and \( C^*(A)/\mathcal{J} = C(T \times T) \), where \( T \) is unit circle and \( \{0, 1\} \) is the two point space.

**Corollary 2.3.** Let \( T_\psi \in C^*(A) \). Then necessary and sufficient condition for \( T_\psi \) to be Fredholm is that \( \psi(z) \) is nonvanishing for \( z \in T^2 \) and \( \psi|_{T^2} \) is homotopic to constant.

Theorem 2.2 and Corollary 2.3 are proved by using the methods of Douglas and Howe in their study [8] and Curto and Muhly in [9].

Let

\[
S_1 = \{ (r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 \leq 1 \},
\]

\[
\tilde{S}_1 = \{ (r_1, r_2) \in [0, 1] \times [0, 1] : r_1^2 + r_2^2 = 1 \},
\]

and let \( \Omega_2 \) be subset of \( \Omega \) defined by

\[
\Omega_2 = \{ A \in \Omega : \nu_A(U(a)) > 0 \text{ for arbitrary neighborhood } U(a) \text{ of the arbitrary point } a \in \tilde{S}_1 \}. \tag{2.3}
\]

**Theorem 2.4** (see [10, p. 1932, Theorem 2]). Let \( A \in \Omega \). A necessary and sufficient condition for the operator algebra generated by the system \( A \) to be isometrically isometric to the ball algebra is that \( A \) belongs to \( \Omega_2 \).

**Theorem 2.5.** Let \( A \in \Omega \). If the algebra generated by the system \( A \) is ball algebra then \( C^*(A) \) contains \( \mathcal{K} \) ideal of compact operators and \( C^*(A) = \{ T_\psi + K : \psi \in C(\text{supp } \mu_A), K \in \mathcal{K} \} \). The quotient \( C^*(A)/\mathcal{K} \) is naturally identified with \( C(S_3) \) by a map \( \sigma(T_\psi + K) = \psi|_{S_3} \).

It is enough to show compactness of the operators \( A_i^*A_i - A_iA_i^* \), \( i = 1, 2 \) for proving that commutant of the algebra \( C^*(A) \) is \( \mathcal{K} \). For this, we just study the case \( i = 2 \). Then the case \( i = 1 \) is similarly showed, as well. With the basic computations, we obtain \( (A_2^*A_2 - A_2A_2^*)e_{(m,n)} = (\omega_{(m,n),2} - \omega_{(m,n+1),2}^2)e_{(m,n)} \). If we show \( \omega_{(m,n),2}^2 - \omega_{(m,n-1),2}^2 \) converges the zero when \( |I| \to \infty \) then the proof is completed. For this, we need the following lemma.

**Lemma 2.6.** Let \( \nu_A \) be a measure determined by the system \( A \in \Omega_2 \) and \( I = (m, n) \), then the expression

\[
\frac{\int_{S_1} r_1^{2m} r_2^{2n+4} d\nu(r_1, r_2)}{\int_{S_1} r_1^{2m} r_2^{2n} d\nu(r_1, r_2)} - \frac{\int_{S_1} r_1^{2m} r_2^{2n} d\nu(r_1, r_2)}{\int_{S_1} r_1^{2m} r_2^{2n-2} d\nu(r_1, r_2)} \tag{*}
\]

converges to zero when \( |I| \to \infty \).
In view of the following process, the lemma is proved. Without loss of generality, we can take $n = 1$. Hence the second fractional of the expression (*) becomes $I_1/I_2 := \int_{S_1} r_1^{2m} r_2^2 d\nu(r_1, r_2) / \int_{S_1} r_1^{2m} d\nu(r_1, r_2)$. It is enough to show $I_1/I_2$ goes to zero when $m \to \infty$.

Take $r_{20} = \sqrt{\varepsilon/2}$ and $r_{10} = \sqrt{1 - (\varepsilon/2)^2} = \sqrt{1 - r_{20}^2}$ for given $\varepsilon > 0$. Consider $S_{10} := \{(r_1, r_2) : r_{10} < r_1 \leq 1 \cap S_1$ and $S_{11} := S_1/S_{10}$. We have $I_1 = I_{10} + I_{11}$, where $I_{10} = \int_{S_{10}} \int_{S_1} r_1^{2m} r_2^2 d\nu(r_1, r_2)$. It easily shows that $I_{10} \leq (\varepsilon/2) I_2$ and $I_{11} \leq r_{10}^{2m} \nu(S_1)$ for all $m$. Moreover, take $r_{11} = (1 + r_{10})/2$; then we have $I_{12} \geq r_{11}^{2m} \nu(S_{101})$ for all $m$, where $S_{101} = \{(r_1, r_2) : r_{11} < r_1 \leq 1 \cap S_1$. Hence, there exists $M > 0$ such that for all $m > M$ it is obtained $I_1/I_2 < \varepsilon$.

It follows from Lemma 2.6 and a well-known result in $C^*$-algebras [2, page 212, Proposition 1] that $K \subset C^*(A)$.

### 3. An Application

Throughout this section, we follow the notations and definitions in the preceding section.

Let $H$ be separable complex Hilbert space, let $B(H)$ denote the algebra of linear bounded operators on $H$ and let $I$ be identity operator. Consider an operator

$$T = B_1 + B_2 S + K,$$

(3.1)

where $B_1$ and $B_2$ are the elements of a subalgebra $\Lambda$ of $B(H)$ such that the image $\tau(\Lambda)$ is a commutative subalgebra of the algebra $B(H)/K$ under the natural quotient map $\tau$ from $B(H)$ to $B(H)/K$ and $S$ denotes an automorphism in the algebra $\tau(\Lambda)$, that is, $\tau(S)\tau(B)\tau(S^{-1}) = \tau(B')$, where $B$ and $B'$ belong to $\Lambda$. It is obvious that if $B \in \Lambda$ then $SBS^{-1} = B' + K$, where $B' \in \Lambda$ and $K \in K$.

**Theorem 3.1** (see [11]). *If the operator $\tau(B_1)\tau(B_1') - \tau(B_2)\tau(B_2')$ has an inverse in $\tau(\Lambda)$ then $T = B_1 + B_2 S + K$ is Fredholm operator.*

Using the Theorem 3.1 we take $A \in \Omega_2$ and consider the operator $T_{\psi_1} + T_{\psi_2} S + K$, where $T_{\psi_1}$ and $T_{\psi_2}$ are Toeplitz operators in $C^*(A)$ and the operators $S$ and $T$ satisfy the conditions given above.

Moreover if we take into account the orthonormal projection $P$ has the form

$$(Pf)(z_1, z_2) = \int_{\Delta^2} K(z, l) f(l_1, l_2) d\mu_A(l_1, l_2),$$

(3.2)

where $K$ is the reproducing Bergman kernel of the functional space $H^2(\Delta^2, \mu_A)$, then the equation $Tf = \varphi$ is written in the form

$$\int_{\Delta^2} K(z, l) \psi_1(l_1, l_2) f(l_1, l_2) d\mu_A(l_1, l_2) + \int_{\Delta^2} K(z, l) \psi_2(l_1, l_2) f(l_1, l_2) d\mu_A(l_1, l_2) + (Kf)(z_1, z_2) = \varphi(z_1, z_2).$$

(3.3)
Hence we have the following theorem.

**Theorem 3.2.** If the function \( q_1(z_1, z_2)q_3(z_2, z_1) - q_2(z_1, z_2)q_3(z_2, z_1) \) does not vanish in \( S_3 \), then all of Noether’s theorems is true for (3.3). In particular, if we take \( \omega_{1,1} = \sqrt{(m+1)/(2+m+n)} \), \( \omega_{1,2} = \sqrt{(n+1)/(2+m+n)} \) and \( Sf (l_1, l_2) = f (l_2, l_1) \) then (3.3) has the form

\[
\int_{S_3} \frac{q_1(l_1, l_2)f(l_1, l_2)}{(1 - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2} ds + \int_{S_3} \frac{q_2(l_1, l_2)f(l_2, l_1)}{(1 - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2} ds + (Kf)(z_1, z_2) = \varphi(z_1, z_2),
\]

where \( ds \) is the surface measure in \( S_3 \).

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