The Normal Conformal Cartan Connection and the Bach Tensor

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Abstract

The goal of this paper is to express the Bach tensor of a four–dimensional conformal geometry of an arbitrary signature by the Cartan normal conformal (CNC) connection. We show that the Bach tensor can be identified with the Yang–Mills current of the connection. It follows from that result that a conformal geometry whose CNC connection is reducible in an appropriate way has a degenerate Bach tensor. As an example we study the case of a CNC connection which admits a twisting covariantly constant twistor field. This class of conformal geometries of this property is known as given by the Fefferman metric tensors. We use our result to calculate the Bach tensor of an arbitrary Fefferman metric and show it is proportional to the tensorial square of the four–fold eigenvector of the Weyl tensor. Finally, we solve the Yang–Mills equations imposed on the CNC connection for all the homogeneous Fefferman metrics. The only solution is the Nurowski-Plebański metric.

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1 Introduction

In the general relativity, conformally invariant features of a metric are usually described by the Weyl and the Bach tensors, and in 3 dimensions (where the Weyl tensor identically vanishes) by the Cotton–York tensor. Systematic approach to the conformal geometry theory on the other hand leads to the definition of so called Cartan normal conformal connection \cite{2}. In that framework the Weyl tensor is a part of the curvature of the connection, and in 3 dimensions the curvature consists of the Cotton–York tensor. An explicit relation between the Bach tensor and the connection, however, seems not to be known in the literature. A good suggestion follows from Merkulov’s result reported in \cite{11}. The paper contains a theorem saying that the Yang–Mills equations imposed on the twistor connection amount to the condition that the Bach tensor of the corresponding metric tensor is identically zero. For the proof and calculation, however, Merkulov referred to his unpublished work.

In this paper we are not going to consider the twistor connection per se, hence we only recall that Penrose uses the Cartan normal conformal connection to define the covariant derivative of so called local twistor fields \cite{1,16} in 4–real dimensional spacetime of the Lorenzian signature. The resulting connection is called the twistor connection.

The first goal of the current work is expressing the Bach tensor – vanishing or not – by the Cartan normal conformal connection in an explicit way. We introduce here the Yang–Mills current of the Cartan normal conformal connection of a 4–dimensional conformal geometry of arbitrary signature, calculate the current and show it can be identified with the Bach tensor. In particular, it follows from our result that the Bach tensor of a conformal geometry whose Cartan normal conformal connection is reducible is degenerate for appropriate types of the reducibility. The best known example of the reducibility \cite{12} is the class of the conformally Einstein geometries; the Bach tensor is identically zero in this case. Conformal geometry which admits covariantly constant local twistor field is another example. The following features can be used as an equivalent definition of such geometries \cite{1,4}: 1) the existence of a null Killing vector field, say \( k \), and 2) the Petrov type N of the Weyl tensor. All the 4–dimensional Lorenzian geometries of this property are known \cite{4}. When the four–fold eigenvector \( k \) of the Weyl tensor is twisting, the metrics set the class of the Fefferman metrics \cite{10,6}. Each Fefferman metric tensor (or rather its conformal class) is assigned to a
3-dimensional Cauchy–Riemann geometry. In this case the $so(2, 4)$ Cartan normal conformal connection reduces to the $su(1, 2)$ Cartan–Chern connection of the corresponding Cauchy–Riemann geometry [6]. The Yang–Mills current of the Cartan–Chern connection with respect to the Fefferman metric was derived in [7]. It is given by a single real function density. The existence of a homogeneous Cartan–Chern solution of the Yang–Mills equations was mentioned in [7]. In Section 4 we correct two misprints which occurred in that work: a mistake in Yang–Mills equation and in the Bianchi type of the solution.

The Fefferman metrics are also known not to be conformally Einstein except the conformally flat metric [9]. Comparing those facts explains why the first (and only) known explicit example of the non conformally Einstein and Bach flat metric could be recently found by Nurowski and Plebański [5] in the Fefferman class.

Using the Yang–Mills current of the Cartan–Chern connection we derive below the Bach tensor of a general Fefferman metric and show it is proportional to $k \otimes k$. Finally, we solve completely the Yang–Mills equations imposed on the Cartan–Chern connection in the homogeneous case. We find out that the only solution is the one corresponding to the Nurowski–Plebański metric.

The problem of a conformally invariant condition for a metric tensor to be conformally Einstein was addressed by several authors, including Kozameh, Newman and Tod [13]. A sufficient set of conditions they formulated consists of the Bach flatness plus a certain extra condition on the Weyl tensor. However, their result concerns only the case of appropriately generic Weyl tensor, whereas the Weyl tensor of the Fefferman metric is degenerate in that sense.

Our work was motivated by the paper of Nurowski and Plebański [5] (see also the CQG Highlights of 2000/2001). We hope our result may be applied for example in the null surface formulation of gravity of Newman and collaborators [14] which involves the conformal geometry, the Bach tensor and recently also the Cartan normal conformal connection [15].

Both Cartan normal conformal connection and twistor connection found recently yet another application to General Relativity as they were used to formulate the conformal representation of the Einstein equations, very useful when dealing with characteristic initial value problems [19, 20].

Our paper should also encourage to classify and study all the possible
reductions of the Cartan normal conformal connections.

2 The Cartan normal conformal connection

We recall in this section the definition of the Cartan normal conformal connections [2, 6]. The connections correspond to conformal geometries. We consider here the 4-dimensional case of an arbitrarily fixed signature \((p,q)\).

A short, working definition formulated in the first subsection is just an appropriate assignment of a certain \(so(p+1,q+1)\) valued 1-form to every co-frame cotangent to \(M\). Whenever two co-frames are related by a conformal transformation, the corresponding 1-forms are gauge equivalent. This formulation is analogous to the definition of the twistor connection [1, 16]. The full, geometric definition of [6] is recalled in the second subsection.

We will use below a fixed real 4 by 4 symmetric matrix \(\eta = (\eta_{\mu\nu})_{\mu,\nu=1,\ldots,4}\) of the signature \(p\) minuses and \(q\) pluses, and the following 6 by 6 matrix \(Q = (Q_{IJ})_{IJ=0,\ldots,5}\)

\[
Q = \begin{pmatrix}
0 & 0 & -1 \\
0 & \eta & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

The group \(SO(q+1,p+1)\) will be identified with the group \(SO(Q)\) of 6 by 6 real matrices \(A = (A^\mu_\nu)_{\mu=1,\ldots,4}\) such that

\[
A^\alpha_\mu A^\beta_\nu Q_{\alpha\beta} = Q_{\mu\nu}.
\] (1)

2.1 The short definition

Let \((\theta^1,\ldots,\theta^4)\) be a sequence of differential 1-forms which sets a basis of \(T_m^*M\) at every point \(m\) of some open subset \(U \subset M\). We call it a co-frame and assign to \((\theta^1,\ldots,\theta^4)\) a metric tensor \(g\),

\[
g = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu.
\] (2)

The frame of tangent vectors dual to \((\theta^1,\ldots,\theta^4)\) will be denoted by \((X_1,\ldots,X_4)\).
The Cartan normal conformal (CNC) connection 1-form in the natural gauge is the following matrix of 1-forms assigned to the co-frame \((\theta^1, \ldots, \theta^4)\)

\[
\omega_{(C)} = \begin{pmatrix}
0 & \eta_{\nu\alpha} \theta^\alpha & 0 \\
P^\mu & \Gamma^\mu_{\nu} & \theta^\mu \\
0 & \eta_{\nu\alpha} P^\alpha & 0
\end{pmatrix}
\]  

(3)

where each component labeled by \(\mu (\nu)\) stands for a column (row) given by all the values \(\mu = 1, \ldots, 4 (\nu = 1, \ldots, 4)\), \(\Gamma^\mu_{\nu}\) is the Riemann connection 1-form, i.e.

\[
d\theta^\mu + \Gamma^\mu_{\nu} \wedge \theta^\nu = 0, \quad \eta_{\mu\alpha} \Gamma^{\alpha}_{\nu} = -\eta_{\nu\alpha} \Gamma^{\alpha}_{\mu},
\]  

(4)

and \(P^\mu\) is given by the Ricci tensor \(R_{\mu\nu} \theta^\mu \otimes \theta^\nu\) of the metric \(g\), and by its Ricci scalar \(R = R_{\mu\mu}\),

\[
P^\mu = \left( \frac{1}{12} R_{\mu\nu} - \frac{1}{2} R_{\mu\nu} \right) \theta^\nu
\]  

(5)

The advantage of this definition is that \(\omega'_{(C)}\) assigned to a co-frame \((\theta'^1, \ldots, \theta'^4)\) given by a point \(m \in M\) depending conformal transformation \(\theta'^\mu = \Lambda^\mu_{\nu} \theta^\nu\),

(6)

is gauge equivalent to \(\omega_{(C)}\), namely

\[
\omega'_{(C)} = h^{-1} \omega_{(C)} h + h^{-1} dh
\]  

(7)

where in the pure rescaling case (i.e. when \(\Lambda^\mu_{\nu} = c \delta^\mu_{\nu}\)) the matrix \(h\) is given by the function \(c\) in the following way,

\[
h = \begin{pmatrix}
c^{-1} & 0 & 0 \\
c^{-2} c_{,\sigma} \eta^{\mu\sigma} & \delta^\mu_{\nu} & 0 \\
\frac{1}{2} \eta^{\alpha\beta} c_{,\alpha} c_{,\beta} & c^{-1} c_{,\nu} & c
\end{pmatrix}
\]  

(8)

and in the (pseudo)rotation case (i.e. \(g' = g\)), the matrix \(h\) is

\[
h = \begin{pmatrix}
1 & 0 & 0 \\
0 & \Lambda^\mu_{\nu} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(9)
The curvature of the connection \( \omega_{(C)} \) is

\[
\Omega_{(C)} = d\omega_{(C)} + \omega_{(C)} \wedge \omega_{(C)} = \begin{pmatrix}
0 & 0 & 0 \\
DP^\mu & C^\mu_\nu & 0 \\
0 & D\eta_\alpha P^\alpha & 0
\end{pmatrix}
\] (10)

where

\[
DP^\mu = dP^\mu + \Gamma^\mu_\nu \wedge P^\nu
\] (11)

and the tensor \( X_\mu \otimes \theta^\nu \otimes C^\mu_\nu \) is the Weyl tensor of the metric \( g \) (2). Owing to the transformation law (7) the curvature \( \Omega'_N \) of the connection \( \omega'_N \) is

\[
\Omega_{(C)} = h^{-1} \Omega_{(C)} h.
\] (12)

The formula (3) for \( \omega_{(C)} \) shows the relation between the CNC connection and the Penrose twistor connection [1, 16]. The normal gauge will be applied in the next section in the derivation of the Yang–Mills current of the CNC connection.

Finally, Cartan normal conformal connection 1-form in a general gauge is given by imposing on \( \omega_{(C)} \) in a natural gauge, any transformation of the following form,

\[
\omega = h^{-1} \omega_{(C)} h + h^{-1} dh
\] (13)

where \( h \) is any matrix–valued function defined (locally) on \( M \) which takes values in the subgroup \( H \subset SO(Q) \) defined by the matrices of the form (22) (see the next subsection).

### 2.2 The geometric definition

The definitions presented in this section are based on [6].

**The conformal bundle.** Given a conformal geometry \([g]\) on a 4-manifold \( M \) we denote by \([g](m) \subset T^*_M M \otimes T^*_m M\) the set of the conformally equivalent, non–degenerate symmetric tensors corresponding to the conformal geometry \([g]\) at \( m \). Consider the principal fiber bundle \( p : C \to M \) where

\[
C = \bigcup_{m \in M} [g](m) \subset T^* M \otimes T^* M
\] (14)
and the structure group is $\mathbb{R}^+$ acting by the rescaling
\[ C \ni g(m) \mapsto \tilde{R}_c g(m) = c^2 \cdot g(m) \in C. \]

There is a natural horizontal metric tensor $\tilde{g}$ on $C$ and a vertical vector field $\zeta$ defined by
\[
\tilde{g}(X, Y) = g(p_* X, p_* Y), \\
\zeta(f) = \frac{d}{dc} \bigg|_{c=1} \tilde{R}_c^* f, 
\]
where $X, Y \in T_{g(m)} C$ and $f$ is a function defined on $C$, all arbitrary. A co-frame $(\tilde{e}^1, \ldots, \tilde{e}^4, \tilde{\phi})(g(m))$ at a point $g(m) \in C$ will be called admissible if it satisfies each of the following two conditions:

1. $\tilde{g} = \eta_{\mu\nu} \tilde{e}^\mu \otimes \tilde{e}^\nu$
2. $\tilde{\phi}(\zeta) = -1$.

The admissible co-frames set up a conformal bundle $\pi : P(C) \to M$ defined as follows. $P(C)$ is the space of all the admissible co-frames cotangent to $C$. The projection is
\[
\pi : (\tilde{e}^1, \ldots, \tilde{e}^4, \tilde{\phi})(g(m)) \mapsto m. 
\]
In $P(C)$ one considers the following three groups of maps
\[
(\tilde{e}^1, \ldots, \tilde{e}^4, \tilde{\phi})(g(m)) \mapsto (\tilde{e}'^1, \ldots, \tilde{e}'^4, \tilde{\phi}')(g'(m)) : 
\]
\[
(g', \tilde{e}^\mu, \tilde{\phi}') = (g, \tilde{e}^\mu, \tilde{\phi} + \tilde{e}^\mu b_\mu) \quad (18) \\
(g', \tilde{e}^\mu, \tilde{\phi}') = (g, \tilde{e}^\nu (\Lambda^{-1})^{\nu}_{\mu}, \tilde{\phi}) \quad (19) \\
(g', \tilde{e}^\mu, \tilde{\phi}') = (c^{-2} \cdot g, c^{-1} \cdot R^*_{\nu} \tilde{e}^\mu, R^*_{\nu} \tilde{\phi}) 
\]
where the co-vector $(b_\mu)_{\mu=1,\ldots,4}$, the matrix $(\Lambda^\mu_{\nu})_{\nu=1,\ldots,4}$ such that $\Lambda^T \eta \Lambda = \eta$ and the number $c$ are arbitrary. The maps can be considered as given by a right action in $P(C)$ of elements of $SO(p, q)$ of the following form,
\[
\left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\frac{1}{2} b_\sigma b^\sigma & \delta_{\nu}^{\mu} & 0 & 0 \\
b_{\mu} & 0 & 0 & 1 \end{array} \right), \\
\left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
c & 0 & 0 & \frac{1}{c} \end{array} \right) 
\]
respectively. In this way the subgroup $H \subset SO(Q)$ generated by all the matrices (22) acts on $P(C)$. This completes the construction of the $H$-principal fiber bundle structure of $\pi : P(C) \to M$.

Using the following map $\lambda : P(C) \to C$,

$$\lambda : (\tilde{e}^1, ..., \tilde{e}^4, \tilde{\phi})(g(m)) \mapsto g(m).$$

we lift to $P(C)$ the co-frames defined on $C$,

$$\hat{e}^\mu = \lambda^* \tilde{e}^\mu, \quad \hat{\phi} = \lambda^* \tilde{\phi}.$$  

The connection. We denote here by $\hat{R}$ the right action of the group $H$ in $P(C)$. Due to $\hat{R}$ every left invariant vector field $\xi$ tangent to the $H$ defines naturally a vector field $\hat{\xi}$ tangent to $P(C)$. Since $H$ is the subgroup of the group of the matrices $SO(Q)$, we identify the Lie algebra $h$ of the left invariant vector fields on $H$ with the corresponding algebra of matrices. Since $\dim M + \dim H = \dim SO(Q)$, one can consider $SO(Q)$-Cartan connections on the principal fiber bundle $(P(C), M, H, \pi)$. (A detailed introduction to the Cartan connections can be found in [2]). An $SO(Q)$-Cartan connection defined on $P(C)$ is an $so(Q)$ valued 1-form $\hat{\omega}$ defined on $P(C)$ such that the following three conditions are satisfied:

1. $\hat{\omega}(X) = 0 \iff X = 0$ (the non-degeneracy)

2. $\hat{\omega}(\hat{\xi}) = \xi$ for every $\xi \in h$

3. $\hat{R}_h^* \hat{\omega} = h^{-1} \hat{\omega} h$ for every $h \in H$.

One of the connections, say $\hat{\omega}(C)$, can be uniquely distinguished by the following two natural conditions. The first one concerns the first row of $\hat{\omega}(C)$, namely we require that

$$\hat{\omega}(C) = \begin{pmatrix} * & \hat{e}^\mu \eta_{\mu\nu} & * \\
* & * & * \\
* & * & * \end{pmatrix}$$

8
(* denotes those components which are arbitrary modulo the symmetries of the matrices of the Lie algebra $so(Q)$). The second condition is imposed on the curvature of $\hat{\omega}(C)$, namely one requires that

$$\hat{\Omega}(C) := d\hat{\omega}_{CNC} + \hat{\omega}_{CNC} \wedge \hat{\omega}_{CNC} = \begin{pmatrix} 0 & 0 & 0 \\ \ast & \frac{1}{2} K^\mu_{\nu\rho\sigma} \hat{e}_\rho \wedge \hat{e}_\sigma & 0 \\ 0 & \ast & 0 \end{pmatrix} \quad (25)$$

and

$$K^\mu_{\nu\mu\sigma} = 0. \quad (26)$$

A section $\sigma : \mathcal{U} \to P(C)$, where $\mathcal{U} \subset M$, defines the pullback of $\hat{\omega}(C)$ onto $\mathcal{U}$,

$$\sigma^*(\hat{\omega}(C)) = \begin{pmatrix} -\psi & \theta^\alpha \eta_{\alpha\nu} & 0 \\ V^\mu & G^\mu_{\nu\sigma} & \theta^\nu \\ 0 & V^\alpha \eta_{\alpha\nu} & \psi \end{pmatrix}.$$

The pullback is in general different than $\omega(C)$ defined in the previous subsection. However, for a 1-form $b$ defined on $\mathcal{U}$, we consider at each $m \in M$ the corresponding $h_{b(m)} \in H$ given by the first matrix in (22) and a new section

$$\sigma' = \hat{R}_{h_b}^* \sigma \quad (27)$$

The corresponding diagonal element $\psi'$ of $\sigma'^* \hat{\omega}$ is

$$\psi' = \psi + b.$$

Since $b$ is arbitrary, we can fix it such that

$$\psi' = 0. \quad (28)$$

With respect to this natural section $\sigma' =: \sigma_N$

$$\sigma_N^* \hat{\omega}(C) = \omega(C), \quad (29)$$

that is $\sigma_N \ast \hat{\omega}$ is exactly the CNC connection 1-form in the natural gauge defined in (3).
3 The Yang–Mills current of $\hat{\omega}(C)$

Given a conformal geometry $[g]$ on $M$, there is a uniquely defined Hodge $*$ acting in the space of the differential 2-forms defined on $M$ \footnote{This is true for dimension $d = 4$. Otherwise the conformal geometry is not enough to determine the action of $*$.}. We use it below to define the Yang–Mills current of the CNC connection as the source given by imposing the Yang–Mills equations. As in the previous section, we formulate the definition in two versions: directly on $M$ and, respectively, on the bundle $P(C)$. We use the first one to calculate the Yang–Mills current. Next, we find the formula for the Yang–Mills current defined on $P(C)$.

3.1 The calculation in the terms of the working definition

Given a co-frame $(\theta^1, ..., \theta^4)$ and the corresponding Cartan normal conformal connection 1-form $\omega(C)$ in the natural gauge, the Yang–Mills current of $\omega(C)$ is

$$J(C) := D*\Omega(C) = d*\Omega(C) + \omega(C) \wedge *\Omega(C) - *\Omega(C) \wedge \omega(C). \quad (30)$$

The conformal transformations (6,7) of the co-frame and the connection, respectively, are accompanied by the suitable gauge transformation of $J(C)$, namely

$$J(C) = h^{-1}J(C)h,$$

where $h$ is the same as in (7).

Substituting $\Omega(C)$ and $\omega(C)$ in (30) for those given by (10), and (3), respectively, we obtain:

$$J(C) = \left( J(C)_{KL}^{K=0,...,5} \right)_{L=0,...,5} =$$

$$\begin{pmatrix}
\theta_\sigma \wedge *DP^\sigma & ; & \theta_\sigma \wedge *C_\nu^\sigma & ; & 0 \\
D*DP^\mu - *C^\mu_\sigma \wedge P^\sigma & ; & D*C^\mu_\nu - *DP^\mu \wedge \theta_\nu + & ; & -*C^\mu_\sigma \wedge \theta^\sigma + \theta_\mu \wedge *DP_\nu \\
0 & ; & D*DP_\nu + P_\sigma \wedge *C^\sigma_\nu & ; & -*DP_\sigma \wedge \theta^\sigma
\end{pmatrix} \quad (31)$$
The Greek indeces are lowered and raised by \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \), respectively. In particular
\[
\theta_\nu = \eta_{\nu\alpha} \theta^\alpha
\]
above. The differential three–form components of the matrix (31) will be decomposed in the basis
\[
\star \theta_\mu := \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma,
\]
which satisfies the following identity, true for every differential 3-form \( W = W^\mu \star \theta_\mu \),
\[
\theta^\nu \wedge W = W^\nu \text{vol},
\]  
\[
\text{vol} = \frac{1}{4!} \epsilon_{\alpha\beta\gamma\delta} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta := \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4.
\]

Begin with the computation of the components \( J^{(C)}_0^\mu \), \( \mu = 1, ..., 4 \) of the Yang–Mills current \( J^{(C)} \) (that is the middle 4 terms of the zeroth column of \( J^{(C)} \)). They are given by
\[
\theta^\alpha \wedge D\star DP_\nu = \theta^\alpha \wedge D\star (D_\gamma P_{\nu\beta} \theta^\gamma \wedge \theta^\beta) = \frac{1}{2} \theta^\alpha \wedge D(D_\rho P_{\nu\sigma} \\
\eta^{\rho\sigma}_{\gamma\beta} \theta^\gamma \wedge \theta^\beta) = \frac{1}{2} \theta^\alpha \wedge (D_\delta D_\rho P_{\nu\sigma} \epsilon^{\rho\sigma}_{\gamma\beta} \theta^\delta \wedge \theta^\gamma \wedge \theta^\beta) = \\
= -\frac{1}{2} D_\delta D_\rho P_{\nu\sigma} \eta^{\rho\sigma}_{\gamma\beta} \eta^{\alpha\delta\gamma\beta} \text{vol} = D_\delta D_\rho P_{\nu\sigma} (\eta^{\rho\alpha} \eta^{\sigma\delta} - \\
- \eta^{\rho\delta} \eta^{\sigma\alpha}) \text{vol} = (D^\sigma D^\alpha P_{\nu\sigma} - D^\sigma D_\sigma P_\nu^\alpha) \text{vol}
\]
(34)

By a similar calculation we get
\[
\theta^\alpha \wedge P_\sigma \wedge \star C^\sigma_\nu = -(P^{\sigma\beta} C_{\nu\sigma\beta}) \text{vol}.
\]
(35)

Therefore,
\[
J^{(C)}_0^\mu = D\star DP^\mu + P_\sigma \wedge \star C^{\sigma\nu} = B^{\mu\alpha} \star \theta_\alpha
\]
(36)
where \( B^{\mu\alpha} \) is the Bach tensor that is,
\[
B^{\mu\alpha} = 2D_\sigma D^{[\alpha} P^{\sigma]\mu} - 2P^{\beta\sigma} C^{\mu}_{\sigma \beta} \alpha.
\]
(37)
By the symmetry of $J_{(C)}$, 

$$J_{(C)}^0_{\nu} = B_{\nu}^{\alpha} \star \theta_{\alpha}. \quad (38)$$

Surprisingly, it turns out that all the other elements of $J_{(C)}$ identically vanish. Indeed, we find out, that the left-upper corner component of $J_{(C)}$ is

$$J_{(C)}^0_0 = \frac{1}{2} D_{\nu} G^{\nu\alpha} \star \theta_{\alpha} \quad (39)$$

where

$$G^{\nu\alpha} = R^{\nu\alpha} - \frac{1}{2} R_{\eta^{\nu\alpha}} \quad (40)$$

is the Einstein tensor, hence $D_{\nu} G^{\nu\alpha} = 0$.

To evaluate the middle block $J_{(C)}^\mu_{\nu}$, $\mu, \nu = 1, ..., 4$, of the Yang–Mills current $J_{(C)}$ we apply the tensor identity given by the corresponding middle block of the conformal Bianchi identity

$$D \Omega_{(C)} = d \Omega_{(C)} + \omega_{(C)} \wedge \Omega_{(C)} - \Omega_{(C)} \wedge \omega_{(C)} = 0, \quad (41)$$

and the known symmetry of the Weyl tensor

$$\star C^\mu_{\nu} = \frac{1}{2} \eta^{\mu}_{\nu\gamma} \delta \gamma_{\delta} C^\nu_{\delta}. \quad (42)$$

The result is

$$J_{(C)}^\mu_{\nu} = \left( -\frac{1}{2} \eta^{\alpha\mu} D_{\delta} G^{\delta_{\nu}} + \frac{1}{2} \delta^{\alpha}_{\nu} D_{\delta} G^{\delta_{\mu}} \right) \star \theta_{\alpha} = 0.$$

In conclusion, we proved that the Yang–Mills current of the Cartan normal conformal connection $\omega_{(C)}$ in the natural gauge corresponding to a coframe $(\theta^1, ..., \theta^n)$ has the following form,

$$J_{(C)} = \begin{pmatrix} 0 & 0 & 0 \\ B_{\mu}^{\alpha} \star \theta_{\alpha} & 0 & 0 \\ 0 & B_{\nu}^{\alpha} \star \theta_{\alpha} & 0 \end{pmatrix} \quad (43)$$

where $B_{\mu}^{\alpha} \theta^\mu \otimes \theta^\alpha$ is the Bach tensor of the metric tensor $g = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu$.

**Remark** A straightforward consequence of (43) is the equivalence of the Bach equation for conformal metric

$$B_{\mu\nu} = 0$$

and the Yang–Mills equation imposed on the Cartan normal conformal connection [11].

$$D \star \Omega_{(C)} = 0$$
3.2 In terms of the geometric definition

We now turn to the geometric approach to the CNC connection of Sec. 2.2. The Hodge star can be naturally lifted to \( \hat{\star} \) acting on every horizontal\(^2\) 2-form \( \hat{\Omega}' \) defined on \( P(C) \). At \( p \in P(C) \)

\[
\hat{\star} \hat{\Omega}' := \pi^* \sigma^* \hat{\Omega}',
\]

where \( \sigma \) is any section of \( P(C) \) containing \( p \) in its image. Notice, that the curvature \( \hat{\Omega} \) of a Cartan connection is a horizontal 2-form. We define the Yang–Mills current \( \hat{J}(C) \) of \( \hat{\omega} \) in the following way,

\[
\hat{J}(C) := D \hat{\star} \hat{\Omega}(C) = d \hat{\star} \hat{\Omega}(C) + \hat{\omega}(C) \wedge \hat{\star} \hat{\Omega}(C) = \hat{\star} \hat{\Omega}(C) \wedge \hat{\omega}(C).
\]

The current \( J(C) \) introduced in the previous subsection is related to \( \hat{J}(C) \) by the natural sections of \( P(C) \) defined in Sec. 2.2. Let \( \sigma_N \) be a natural section of \( P(C) \) corresponding to the co-frame \((\theta^1, ..., \theta^n)\) used in the definition of \( J(C) \) above. Then

\[
J(C) = \sigma_N^* \hat{J}(C).
\]

Therefore our result of the previous subsection seems to refer to the natural gauge only. However, for a general (local) section \( \sigma : U \rightarrow P(C) \) of the bundle \( P(C) \) there is a natural section \( \sigma_N : U \rightarrow P(C) \) such that

\[
\sigma^* \hat{e}^\mu = \sigma_N^* \hat{e}^\mu =: \theta^\mu.
\]

Then, the corresponding pullbacks of \( J(C) \) are related by a gauge transformation

\[
\sigma^* J(C) = h^{-1} \sigma_N^* J(C) h
\]

where \( h : U \rightarrow H \) takes values in the subgroup of the structure group \( H \) given by the first family of matrices in (22). The point is, that the formula (43) is invariant with respect to those gauge transformations (see Appendix). This observation allows us to strengthen the result: for every (local) section \( \sigma \) of \( P(C) \), the pullback of the Yang–Mills current of \( \hat{\omega} \) is

\[
\sigma^* J(C) = \begin{pmatrix}
0 & 0 & 0 \\
B^\alpha \hat{\star} \theta_\alpha & 0 & 0 \\
0 & B_\nu \hat{\star} \theta_\alpha & 0
\end{pmatrix}.
\]

\(^2\) That is, differential 2-forms such that contracted with every vector tangent to a fiber of the bundle give zero.
Since $\hat{J}_{(C)}$ is a horizontal 3-form defined on $P(C)$, it follows that $\hat{J}_{(C)}$ is

$$
\hat{J}_{(C)} = \begin{pmatrix}
0 & 0 & 0 \\
B^{\mu\alpha} \hat{\theta}_\alpha & 0 & 0 \\
0 & B^{\alpha\nu} \hat{\theta}_\alpha & 0
\end{pmatrix},
$$

(50)

where

$$
\hat{\theta}_\mu := \frac{1}{3!} \eta_{\mu\alpha\beta\gamma} \hat{e}^\alpha \wedge \hat{e}^\beta \wedge \hat{e}^\gamma,
$$

(51)

and $B_{\mu\nu}$ at $p \in P(C)$ is such that for every section $\sigma$ such that $\sigma(m) = \pi(p)$, the tensor $B_{\mu\nu} \sigma^*(\hat{e}^\mu \otimes \hat{e}^\nu)$ is the Bach tensor at $m \in M$ of the metric tensor $g = \eta_{\mu\nu} \sigma^*(\hat{e}^\mu \otimes \hat{e}^\nu)$.

4 Reducible non–conformally Einstein geometries

Suppose that $[g]$ is a conformal geometry of the signature $(p, q)$ and $g \subset so(p+1, q+1)$ is a subalgebra. We say that the CNC connection of is reducible to $g$ in a neighborhood of a point $m \in M$ if there is a choice of gauge (13) such that the CNC connection 1-form $\omega_{(C)}$ takes values in $T^*_m M \otimes g$ at every point of some neighborhood of the point $m$. If the CNC connection of a given conformal geometry $[g]$ is reducible to a single subalgebra $g \subset so(p+1, q+1)$ at a neighborhood of every point $m \in M$, then we say that the connection is reducible to $g$, or just that the conformal geometry $[g]$ is reducible to $g$. In this case also the Yang–Mills current takes values in the subalgebra. Due to the result of the previous section, one may expect that the Bach tensor of a reducible conformal geometry may be degenerate in some way, depending on the Lie subalgebra $g$. An example is the family of the conformally Einstein geometries. Obviously, the Bach tensor is zero at every point $m \in M$ in this case. Via the correspondence between the twistorial [1] and the normal conformal Cartan [16] connections the reducibility of the first one implies the reducibility of the second one. Therefore, another reducible example is a conformal geometry admitting a covariantly constant local twistor. All the conformal metric tensors of this property and of the Lorenzian signature have been found in [4]. In the generic (twisting local twistor field) case they were shown to be the Fefferman geometries known in the theory of the Cauchy–Riemann (CR) structures (3–dimensional in our case).
The Yang–Mills current corresponding to such CR structures has been derived in [7]. Below we use this result to characterize the degeneracy of the Bach tensor of the Fefferman metric. Indeed, it has only one independent real component. On the other hand, the only Fefferman geometry which is conformally Einstein is the flat geometry [9]. Putting together those properties makes the class of the Fefferman metric tensors a probable source of possible Bach flat metric tensors which are not conformally Einstein. Therefore, we consider in this section the (source free) Yang–Mills equations imposed on the CNC connection of the Fefferman metric tensor. We derive all the homogeneous solutions.

4.1 The Fefferman conformal geometries

Note that given the CNC connection of an initially unknown conformal geometry it is straightforward to recover the geometry itself: four entries of the connection matrix constitute a null tetrad.

On a 4-manifold $M$, we introduce now a certain 1-form $\omega_{(CC)}$ taking values in $su(1, 2)$, next embed $R : su(1, 2) \rightarrow so(4, 2)$ (the embedding $R$ is defined below) and notice that the resulting 1-form $R(\omega_{(CC)})$ is the CNC connection 1-form of the metric tensor we can read out from $R(\omega_{(CC)})$. The space-time $M$ and the 1-form $\omega_{(CC)}$ are constructed from a 3–Cauchy–Riemann (CR) geometry $(N, [(\theta^1, \theta^3)])$; that is, from a real 3-manifold $N$ and an equivalence class of pairs of one–forms $[(\mu, \lambda)]$ such that

1. $\lambda$ is real, $\mu$ is complex
2. $\mu \wedge p \wedge \lambda \neq 0$ everywhere
3. Two pairs $(\mu, \lambda)$ and $(\mu', \lambda')$ are by definition equivalent whenever

$$\lambda' = f \lambda \quad \mu' = g \mu + h \lambda$$

where $f$ is an arbitrary real function while $g$ and $h$ are arbitrary complex-valued functions.

We assume throughout this work the following non-degeneracy condition,

$$\lambda \wedge d\lambda \neq 0$$
at every point of $N$, and given a CR geometry $(N, [(\mu, \lambda)])$ we choose a representing pair $(\mu, \lambda)$ such that the following normalization condition holds,

$$\lambda \wedge d\lambda = i\mu \wedge \bar{\mu} \wedge \lambda.$$  \hfill (53)

We follow now the definition of the Fefferman metric formulated in [6]. Given a CR geometry $(N, [(\mu, \lambda)])$, and a representative $(\mu, \lambda)$ the corresponding 4-space-time manifold $M$ is defined as follows

$$M := \{(x, e^{ir}\mu) | x \in N, \mu = \mu_{|x}, r \in [0, 2\pi]\}$$  \hfill (54)

The projection $\pi : M \to N$ is used to pull the forms $\mu$ and $\lambda$ back into each point $(x, e^{ir}\mu) \in M$,

$$\omega_0 := \pi^* \lambda, \quad \omega_1 := e^{ir}\pi^* \mu.$$  \hfill (55)

The $su(1, 2)$–valued connection 1-form $\omega_{(CC)}$ we introduce on $M$ is the Cartan–Chern (CC) connection defined by the following set of conditions: \footnote{This is again a “working” definition, compatible with the working definition of the Cartan normal conformal connection. For the full definition see [18, 17].}

$$\omega_{(CC)} = \begin{pmatrix} \psi + i\theta^4 & \omega_1 & -\omega_0 \\ i\bar{\omega}_3 & -2i\theta^4 & i\bar{\omega}_1 \\ \omega_4 & -\omega_3 & -\psi + i\theta^4 \end{pmatrix}$$  \hfill (56)

where the real 1-forms $\psi, \theta^4, \omega_3$ and the complex valued 1-form $\omega_3$ are such that the CC curvature takes the following form:

$$\Omega_{(CC)} = d\omega_{(CC)} + \omega_{(CC)} \wedge \omega_{(CC)} = \begin{pmatrix} 0 & 0 & 0 \\ i\bar{R} \omega_1 \wedge \omega_0 & 0 & 0 \\ \omega_0 \wedge (S\omega_1 + \bar{S}\omega_2) & -R \bar{\omega}_1 \wedge \omega_0 \end{pmatrix}$$  \hfill (57)

where $R$ and $S$ are any complex valued functions.

In fact, given the pair $(\mu, \lambda)$, the connection $\omega_{(CC)}$ always exists, and it is determined up to the gauge transformation $\psi \mapsto \psi + t\omega_0$, where $t \in \mathbb{R}$ is arbitrary. An outline of the derivation and the result is given in the Appendix.
Clearly, the CC connection 1-form takes values in the Lie algebra of matrices isomorphic with $su(1,2)$. If we choose another pair $(\mu', \lambda') \in [(\mu, \lambda)]$, the connection form transforms according to a law $\omega'_{CC} = a^{-1} \omega_{CC} a + a^{-1} da$, where $a : M \to H$ is a suitable.

The embedding $\mathcal{R}$ is the natural embedding of the algebra of the $n$ by $n$ complex valued matrices into the algebra of the the $2n$ by $2n$ real matrices: $\mathcal{R}(A)$ is given by replacing each entry $A_{I,J}$ by a 2 by 2 block

$$
\begin{pmatrix}
\text{Re}A_{I,J} & \text{Im}A_{I,J} \\
-\text{Im}A_{I,J} & \text{Re}A_{I,J}
\end{pmatrix}.
$$

Indeed, the resulting 1-form $\mathcal{R}(\omega_{CC})$ takes the values in the Lie algebra $so(Q)$ defined in Section 2, where in this case

$$
\eta = 
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

Note that if $\mathcal{R}(\omega_{CC})$ is the CNC connection of some metric tensor, then the corresponding co-frame $(\theta^1, \theta^2, \theta^3, \theta^4)$ and the metric tensor $g = g_F$ can be immediately identified as

$$
\mathcal{R}(\omega_{CC}) = 
\begin{pmatrix}
\psi & \theta^4 & \text{Re}\omega_1 & \text{Im}\omega_1 & -\omega_0 & 0 \\
-\theta^4 & \psi & -\text{Im}\omega_1 & \text{Re}\omega_1 & 0 & -\omega_0 \\
\text{Re}\omega_3 & \text{Im}\omega_3 & 0 & -2\theta^4 & \text{Im}\omega_1 & \text{Re}\omega_1 \\
-\text{Im}\omega_3 & \text{Re}\omega_3 & 2\theta^4 & 0 & -\text{Re}\omega_1 & \text{Im}\omega_1 \\
\omega_4 & 0 & -\text{Im}\omega_3 & -\text{Re}\omega_3 & -\psi & \theta^4 \\
0 & \omega_4 & \text{Re}\omega_3 & -\text{Im}\omega_3 & -\theta^4 & -\psi
\end{pmatrix}
$$

where $\theta^4$ and $\eta$ coincide with the ones already defined in this section. As before the dual basis of vectors at each point will be denoted by $(X_1, \ldots, X_4)$. The linear independence of the co-frame follows from (57) and (4.1). The resulting metric $g_F$ is the Fefferman metric tensor [10, 6].

The signature of the metric is $(-, +++, +, +)$. The metric $g_F$ is determined by the pair of 1-forms $(\mu, \lambda)$ whereas it is independent of the remaining ambiguity in the CC connection 1-form $\omega$. The null vector field $X_4 = k$ is a
Killing vector field for \( g_F \). It generates the flow \( R_t: (x, e^{ir} \mu) \to (x, e^{i(r+t)} \mu) \). It can be checked by inspection that the definition of the CC connection 1-form \( \omega_{(CC)} \) implies that \( \mathcal{R}(\omega_{(CC)}) \) is the CNC connection of the Fefferman metric \( g_F \),

\[
\mathcal{R}(\omega_{(CC)}) = \omega_{(C)} \tag{61}
\]

The Fefferman metric \( g'_F \) assigned to another pair \((\mu', \lambda' = e^h \lambda) \in [(\mu, \lambda)] \) (remember about the normalization (53)) and defined on \( M' \) is related to \( g_F \) by the map \( \varphi : (x, e^{ir} \mu) \mapsto (x, e^{ir} \mu') \) [10, 6]

\[
\varphi^* (g'_F) = e^h g_F. \tag{62}
\]

Another useful technical remark is that if we represent a given CR geometry by a pair \((\mu, \lambda)\) such that

\[
d\lambda = i \mu \wedge \bar{\mu}, \tag{63}
\]

and use \( \psi = 0 \) (this combination of conditions is always possible) then, the resulting CNC connection 1-form \( \mathcal{R}(\omega_{(CC)}) \) comes out in the natural gauge (3).

The CNC curvature form is

\[
\Omega_{(C)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\text{Re}(R\omega_2) & \text{Im}(R\omega_2) & 0 & 0 & 0 & 0 \\
-\text{Im}(R\omega_2) & \text{Re}(R\omega_2) & 0 & 0 & 0 & 0 \\
-\text{Re}(S\omega_1) & 0 & \text{Re}(R\omega_2) & \text{Im}(R\omega_2) & 0 & 0 \\
0 & -\text{Re}(S\omega_1) & -\text{Im}(R\omega_2), \omega_0 & \text{Re}(R\omega_2) & 0 & 0 \\
\end{pmatrix} \wedge \omega_0
\]

Note that the whole middle block vanishes iff \( R \) vanishes, that is, Fefferman metric is conformally flat iff \( R = 0 \).

4.2 The Bach tensor of the Fefferman metrics

Combining the results of the previous and the current sections, one can see that the Bach tensor of the the Fefferman metric is given by the Yang–Mills current \( J_{(CC)} \) of the CC connection coupled with the corresponding Fefferman
metric,

\[ J_{(CC)} = D \ast \Omega_{(CC)} = d \ast \Omega_{(CC)} + \omega_{(CC)} \wedge \ast \Omega_{(CC)} - \ast \Omega_{(CC)} \wedge \omega_{(CC)} \]

\[ J_{(CC)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix} \]

\[ J_{(C)} = \mathcal{R}(J_{(CC)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \kappa & 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 & 0 & 0 \end{pmatrix} \]

where

\[ \kappa = 2 (\omega_0 \wedge d \text{Im}(S \omega_1)) - 2 \text{Re}(R \bar{\omega}_1 \wedge \bar{\omega}_3) \wedge \omega_0 = \]

\[ = -2(\theta^1 \wedge d \text{Im}(S \theta^2)) - 2 \text{Re}(R \theta^3 \wedge \bar{\omega}_3) \wedge \theta^1 \quad (64) \]

As we know from Section 1, the Bach tensor may be calculated from the Yang–Mills current via

\[ B^{\mu \alpha} \text{ vol} = \theta^\alpha \wedge J_{(C) \mu} \quad (65) \]

The only non–vanishing entry of the current is \( J_{(C) \mu} \), therefore

\[ B^{1 \alpha} = B^{2 \alpha} = B^{3 \alpha} = 0 \quad B^{44} \neq 0 \text{ in general} \quad (66) \]

Since \( B^{\mu \nu} \) is a symmetric tensor, this result means that the only non–vanishing component of the Bach tensor is \( B^{44} \). In other words

\[ B^{\mu \nu} X_\mu \otimes X_\nu = T \cdot X_4 \otimes X_4 = 0 \quad (67) \]

where \( X_4 \) is the null Killing field of the Fefferman metric and \( T \) is a function which can be explicitly calculated using (64). Therefore, as it was anticipated in Introduction, the Bach tensor is degenerate.

**Remarks.**

1. \( J_{(C)} \) does not contain \( \theta^4 \) and therefore the contraction \( i(X_4) J_{(C)} \) vanishes.
2. The Yang–Mills equations imposed on the normal Cartan connection of a Fefferman metric proved to be equivalent to one single real equation

\[ \kappa = 0 \] (68)

This fact was first pointed out in [7] but there was a misprint in the explicit formula for (what we currently call) \( \kappa \). This is the reason why we state the correct formula here in detail.

Since \( X_4 \) is a Killing vector field of the metric tensor, the Yang–Mills equations for the CC connection amount to a differential equation on the forms \( \mu \) and \( \lambda \) defined on \( N \). In order to study it, in the next subsection we take the section \( \xi: N \ni \mathcal{O} \mapsto M \) defined by \( r = 0 \) in (54), and consider the pullbacks of the CC connection and curvature forms to the CR manifold \( N \). These pullbacks will be referred to as the CC connection and curvature forms on \( N \) respectively.

4.3 Homogeneous Fefferman geometries

We now turn to a specific class of highly symmetric CR geometries. We will assume the CR geometry \( N \) admits a 3 dimensional group \( G_3 \) of symmetries. Such geometries are called homogeneous. In this class we will solve the Yang–Mills equations for the CC connection.

We will use the following notation for \( \lambda \) and \( \mu \) defined on \( N \):

\[
\begin{align*}
e^1 &= -\mu = -\xi^*(\theta^2 + i\theta^3) \\
e^2 &= -\bar{\mu} = -\xi^*(\theta^2 - i\theta^3) \\
e^3 &= \lambda = -\xi^*\theta^1
\end{align*}
\]

(69)

(70)

and for the pullback of CC connection matrix elements

\[
\begin{align*}
\omega_{III} &= -\xi^*\omega_3 \\
\omega_{IV} &= \xi^*\omega_4 \\
\omega_{II} &= \xi^*(-\psi - \frac{3i}{2}\theta^4)
\end{align*}
\]

(71)

(72)

This choice is just a matter of convenience and was made to match the notation of [8].

The CC connection form on \( N \) satisfies the same condition (57) with CC curvature form on \( N \) on the right hand side. In the Appendix we present
an outline of derivation of CC connection one–form on $N$ of a given CR geometry.

The forms $e^1$, $e^2$ and $e^3$ on a homogeneous CR manifold may be chosen in such a way that their exterior derivatives were decomposable with constant coefficients in the basis of wedge products of $e^i$ [3]. The forms $e^1$, $e^2$ and $e^3$ are then the left–invariant one–forms on the homogeneous space of $G_3$.

$$
de^3 &= A e^3 \land e^1 + \bar{A} e^3 \land e^2 + iB e^1 \land e^2 \\
de^1 &= C e^3 \land e^1 + D e^3 \land e^2 + E e^1 \land e^2
$$

$A, C, D, E \in \mathbb{C}$ $\quad B \in \mathbb{R}$

This condition doesn’t fix the forms completely: a residual gauge freedom remains like in (52) with $f, g$ and $h$ constant numbers. We may reduce this freedom by a further requirement that the forms satisfy

$$
de^3 \land e^3 &= i e^1 \land e^2 \land e^3 \quad (73) \\
de^1 \land e^1 &= 0 \quad (74)
$$
or

$$
de^3 &= A e^3 \land e^1 + \bar{A} e^3 \land e^2 + i e^1 \land e^2 \\
de^1 &= C e^3 \land e^1 + E e^1 \land e^2 \quad (75)
$$

This can always be done provided that the CR structure is non–degenerate. Indeed, if

$$
de^3 \land e^3 &= i P e^1 \land e^2 \land e^3 \quad (76) \\
de^1 \land e^1 &= Q e^1 \land e^2 \land e^3 \quad (77)
$$

then the forms

$$
e^1_{New} = e^1 + h e^3 \\
e^3_{New} = \frac{1}{P} e^3
$$

satisfy the gauge conditions iff $h$ is such that

$$0 = de^1 \land e^1 + h (de^1 \land e^3 + de^3 \land e^1) + h^2 de^3 \land e^3 \quad (80)$$
(80) is a complex quadratic equation and therefore always has a solution. Moreover in the generic case it has two solutions. This gives rise to an additional discrete gauge transformation of changing of the root of (80) and causes some complications.

Calculating the CC connection and its curvature is quite easy due to the fact that $A$, $C$ and $E$ are constant. By plugging (75) into (112)–(116) (Appendix) we get the connection form elements

\[ \omega_{II2} = -E \]  
\[ \omega_{II1} = -\bar{E} \]  
\[ \omega_{III2} = 0 \]  
\[ \omega_{III1} = 0 \]  
\[ \omega_{II1} = -A + \bar{E} \]  
\[ \omega_{II2} = -\bar{A} + E \]  
\[ \omega_{II3} = \frac{i\beta}{4} \]

where we introduced for convenience

\[ \beta = 3 \text{Im} C - \text{Re} EA + 2|E|^2 \]

\[ \omega_{III1} = -C + \frac{i\beta}{4} \]  
\[ \omega_{III2} = -\bar{C} - \frac{i\beta}{4} \]  
\[ \omega_{III3} = \bar{A} \left( \frac{i\bar{C}}{3} - \frac{2iC}{3} - \frac{\beta}{6} \right) \]  
\[ \omega_{IV1} = \frac{C}{3} (A + \bar{E}) + \frac{i\beta A}{12} \]  
\[ \omega_{IV2} = \frac{\bar{C}}{3} (\bar{A} + E) - \frac{i\beta \bar{A}}{12} \]  
\[ \omega_{IV3} = \frac{i\beta C}{2} - C^2 + \frac{\beta^2}{16} + (\bar{E} + A) \left( \frac{i\bar{C} \bar{A}}{3} - \frac{2i\bar{C} E}{3} - \frac{\beta \bar{A}}{6} \right) \]

Analogous expressions for the curvature

\[ R = \bar{A} \left( \frac{i\bar{C}}{3} - \frac{2iC}{3} - \frac{\beta}{6} \right) (-2\bar{A} + E) \]
\[
S = C \left( \frac{C}{3} (A + E) + \frac{iA\beta}{12} \right) + 2A \left( \frac{i\beta C}{6} - C^2 + \frac{\beta^2}{16} + \right.
+ (\bar{E} + A) \left( \frac{i\bar{C}\bar{A}}{2} - \frac{2i\bar{C}E - \beta A}{6} \right) + \\
+ iA \left( -C + \frac{i\beta}{4} \right) \cdot \left( -\frac{iC}{3} + \frac{2i\bar{C}}{3} - \frac{\beta}{6} \right)
\]
(95)

By applying this we find out that (68) is equivalent to

\[
0 = S \left( \frac{1}{2} E + \bar{A} \right)
\]
(96)

We conclude that the CC connection satisfies YM iff \( E = -2\bar{A} \) or \( S = 0 \).

Before we proceed we will investigate the second condition in details.

In the case of \( S = 0 \) the Bianchi identity

\[
d\Omega_{(CC)} + \omega_{(CC)} \wedge \Omega_{(CC)} - \Omega_{(CC)} \wedge \omega_{(CC)} = 0
\]

implies that

\[
\bar{R} \left( 2E + \bar{A} \right) = 0
\]

Once again we get two (perhaps overlapping) cases: \( R = 0 \) i \( E = -1/2 \bar{A} \).

The first one clearly corresponds vanishing \( \Omega_{(CC)} \). As we concluded in the previous subsection this means that the Weyl tensor of the Fefferman metric is equal to 0 and hence the metric is conformally flat.

We may summarize the result as follows: (68) reduces to one algebraic equation involving \( A, C \) and \( E \). Homogeneous CR geometries satisfying (68) fall into three categories:

1. \( S = 0 \) i \( R = 0 \) (the flat case)
2. \( S = 0 \) i \( E = -1/2\bar{A} \)
3. \( E = -2\bar{A} \)

Only the latter two may actually contain non–trivial (non–conformally flat) solutions.

The final step involves using a complete classification of homogeneous, three–dimensional CR geometries as presented in [3]. The classification is
based on Bianchi classification of the group $G_3$. Before we apply it we must impose our gauge conditions to the forms presented in [3] and calculate $A$, $C$ and $E$. We will just state the results here:

| Type | $A$  | $C$  | $E$  |
|------|------|------|------|
| II [A] | 0    | 0    | 0    |
| IV [F] | $-i/2$ | 0    | $i/2$ |
| $VI_h$ including $VI_0$ and III [E,B] | $ib/2$ | 0    | $i/2$ |
| $VII_h$ including $VII_0$ | $b + i$ | $2(b + i)$ | $b + i$ |
| IX [D,L] and VIII [C,K] | $ik/2$ | $i(k^2 \pm 1)/2$ | $ik/2$ |

Some of types involve a one–parameter family of CR geometries. It’s worth mentioning that the CR structures of type $VI$ corresponding to $b$ and $b^{-1}$ are isomorphic, as well as $b$ and $-b$ in type $VII$ and $k$ and $-k$ in type $IX$. This is due to the previously mentioned discrete gauge symmetry.

Simple inspection of the table convinces us that

- type II is the flat case
- $E = -2\bar{A}$ are satisfied only by types types II and $VI_h$ with $b = \frac{1}{2}$
- $E = -1/2\bar{A}$ are satisfied only by types II and $VI_h$ with $b = 2$

Therefore type $VI_h$ with $b = \frac{1}{2}$ and type $VI_h$ with $b = 2$ are the only solutions of the YM equation which may prove non–flat. We now check if this is really the case. The general formula for the curvature coefficients of CC connection associated with type $VI_h$ homogeneous CR geometry:

$$R = -\frac{ib}{2} \left( -\frac{b + 2}{24} \right) \left( ib + \frac{ib}{2} \right) = -\frac{b(b + 2)(b + 1/2)}{48}$$

$$S = \frac{\beta^2}{16} + (\bar{E} + A) \left( -\frac{\beta A}{6} \right) + iA \cdot \frac{i\beta}{4} \left( -\frac{\beta^3}{6} \right) =$$

$$= \frac{\beta A}{3} \left( \frac{\beta}{2} - \bar{E}A - |A|^2 \right) = -\frac{b(b + 2)}{24} \left( -\frac{b^2}{4} + \frac{3}{8}b + \frac{1}{4} \right) =$$

$$= \frac{b(b + 2)(b - 2)(b + 1/2)}{96}$$
$S$ doesn’t vanish for either $b = \frac{1}{2}$ or $b = 2$. Both coefficients of the curvature vanish for $b = -\frac{1}{2}, -2$ and 0. However, as we have shown in the appendix, the CR structures $b$ and $b^{-1}$ are in fact isomorphic. This is due to the discrete gauge symmetry mentioned in section the Appendix.

Our consideration can now be summarized: in the type VI$_h$ family of CR structures the YM equation is satisfied for

- $b = 0$ – flat
- $b = -2$ and $b = -\frac{1}{2}$ (isomorphic) – flat
- $b = 2 i b = \frac{1}{2}$ (isomorphic) – the only non-flat solutions

We did not exclude the existence of trivial solutions among other types of homogeneous CR geometries.

Using the coordinates on $N$ introduced in [3] for each type of homogeneous CR geometry we can write the Fefferman metrics corresponding to non-trivial solutions

i. $b = \frac{1}{2}$:

$$g = dx^2 + dy^2 + \frac{4}{3} \left( y^{3/2} du - dx \right) \left( y dr + \frac{5}{36} y^{3/2} du + \frac{25}{36} dx \right)$$ (97)

ii. $b = 2$:

$$g = dx^2 + dy^2 + \frac{2}{3} \left( y^3 du - dx \right) \left( y dr + \frac{1}{9} y^3 du + \frac{11}{9} dx \right)$$ (98)

(97) and (98) are conformally equivalent as they are constructed from isomorphic CR geometries. Indeed, it’s straightforward to verify that the coordinate change

$$dx = -\frac{1}{2} d\tilde{u}$$
$$dy = -\frac{1}{2} y^{-3/2} d\tilde{y}$$
$$du = -\frac{1}{2} d\tilde{x}$$

transforms (98) into (97) up to a conformal factor.
Metric (98) as a Bach and non-conformally Einsteinian metric appeared for the first in [5], it was derived however in a different way.

The results of this chapter were verified using the www version of symbolic calculations program GRTensor.

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A Notation and identities

A.1 Bach tensor and Bianchi identities

Apart from the standard Weyl and Ricci tensors the following Riemann geometry objects have been used

\[ P_{\mu\nu} = \frac{1}{12} R g_{\mu\nu} - \frac{1}{2} R_{\mu\nu} \]

and the Bach tensor

\[ B_{\mu\nu} = 2 \nabla^\sigma [\nu P_\sigma]_\mu - P_{\mu\sigma} C^\sigma_{\mu\nu} \] (99)

A.2 Volume and the antisymmetric symbol

We define the volume form in an orthonormal frame by

\[ \text{vol} = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = \frac{1}{24} \epsilon_{\mu\nu\alpha\beta} \theta^\mu \wedge \theta^\nu \wedge \theta^\alpha \wedge \theta^\beta \]

where the antisymmetric symbol \( \epsilon \) is in our convention

\[ \epsilon_{0123} = 1 \quad \epsilon^{0123} = -1 \]
With this setup we have the following identities

\[
\begin{align*}
\theta^\mu \wedge \theta^\nu \wedge \theta^\alpha \wedge \theta^\beta &= -\varepsilon^\mu\nu\alpha\beta \text{ vol} \quad (100) \\
\frac{1}{2} \varepsilon^\mu\nu\rho\sigma \varepsilon^\alpha\beta_{\rho\sigma} &= -\eta^\mu\alpha \eta^\nu\beta + \eta^\mu\beta \eta^\nu\alpha \quad (101) \\
\frac{1}{6} \varepsilon^\mu\nu\rho\sigma \varepsilon^{\alpha\nu\rho\sigma} &= -\eta^\mu\alpha \quad (102)
\end{align*}
\]

A.3 The adjoint action of \(SO(2, 4)\) on \(so(2, 4)\)

Taking into account the definition of the form \(Q\) every \(A \in so(2, 4)\) has the form of

\[
A = \begin{pmatrix}
C & U_\nu & 0 \\
V^\mu & L_\nu^\mu & U_\mu \\
0 & V_\nu & -C
\end{pmatrix}
\]

The adjoint action of 3 subgroups of \(SO(1, 3)\) may easily be computed.

- **dilations**
  \[
  U^\mu \mapsto c \cdot U^\mu \\
  V_\nu \mapsto c^{-1} \cdot V_\nu
  \]
  other elements do not change

- **Lorentz transformations**
  \[
  U^\mu \mapsto \Lambda^\mu_\nu U^\nu \\
  V_\nu \mapsto (\Lambda^{-1})^\mu_\nu V_\mu \\
  L_\nu^\mu \mapsto \Lambda^\mu_\rho L_\sigma^\rho (\Lambda^{-1})^\sigma_\nu
  \]
  other elements do not change

- **Möbius transformations**
  \[
  C \mapsto C - U^\alpha \xi_\alpha \\
  L_\nu^\mu \mapsto L_\nu^\mu + \xi^\mu U_\nu - \xi_\nu U^\mu \\
  V_\nu \mapsto V_\nu + (L_\nu^\mu + \delta^\mu_\nu (C - \xi^\alpha U_\alpha)) \xi_\mu
  \]
  other elements do not change

The curvature form \(\Omega_{(C)}\) is a type \(ad\) form. The transformation law of the Cartan normal connection form is more complicated, but involves adjoint action on \(so(2, 4)\) too. Therefore the equations above may be used to derive the transformation laws for geometric objects of conformal geometry: \(P_{\mu\nu}\), \(B_{\mu\nu}\) and \(C^\mu_{\nu\rho\sigma}\).
A.4 Admissible transformations of $e^1$ and $e^3$ for homogeneous CR geometries with the gauge conditions imposed

With the conditions (73) and (74) imposed there remains one–parameter gauge freedom in the choice of forms $\{e^1, e^3\}$

$$e^1 \mapsto g \cdot e^1$$
$$e^3 \mapsto |g|^2 e^3$$

which results in changing the parameters $A$, $C$ and $E$:

$$A \mapsto \frac{A}{g} \quad C \mapsto \frac{C}{|g|^2} \quad E \mapsto \frac{E}{\bar{g}}$$

We also have an unexpected transformation of changing the root of the quadratic equation (80):

$$e^1 \mapsto e^1 + i(E - \bar{A}) e^3$$
$$e^3 \mapsto e^3$$

with the resulting change of structure constants

$$A \mapsto E \quad C \mapsto C - 2\text{Im}EA \quad E \mapsto \bar{A}$$

The transformation is clearly an involution. This fact lies behind the isomorphisms of homogeneous structures of various types mentioned in the last section.

A.5 Derivation of the elements of the CC connection form on $N$ from the exterior derivatives of $e^1$ and $e^3$

$\omega_{II}$, $\omega_{III}$, $\omega_{IV}$ and the curvature can be expressed in the terms of $e^1$ and $e^2$. The method of calculation is taken from [8].

The condition (57) yields six equations

$$de^1 = \omega_{II} \wedge e^1 + \omega_{III} \wedge e^3$$

(106)
\[
\begin{align*}
\text{de}^2 &= \bar{\omega}_{II} \wedge e^2 + \bar{\omega}_{III} \wedge e^3 \\
\text{de}^3 &= ie^1 \wedge e^2 + (\omega_{II} + \bar{\omega}_{II}) \wedge e^3 \\
\text{d}\omega_{II} &= 2ie^1 \wedge \bar{\omega}_{III} + ie^2 \wedge \omega_{III} + \omega_{IV} \wedge e^3 \\
\text{d}\omega_{III} &= \omega_{IV} \wedge e^1 + \omega_{III} \wedge \bar{\omega}_{II} + Re^2 \wedge e^3 \\
\text{d}\omega_{IV} &= \omega_{IV} \wedge (\omega_{II} + \bar{\omega}_{II}) - i\bar{\omega}_{III} \wedge \omega_{III} - Se^1 \wedge e^3 + \bar{S} e^2 \wedge e^3
\end{align*}
\]

Note that (106) is the complex conjugate of (107).

We will first consider the equations (106)–(108). By taking their relevant components we get

\[
\begin{align*}
\omega_{II2} &= -(de^1)_{12} \\
\bar{\omega}_{II1} &= (de^2)_{12} \\
\omega_{III2} &= (de^1)_{23} \\
\bar{\omega}_{III1} &= (de^2)_{123}
\end{align*}
\]

and two relations

\[
\begin{align*}
(de^3)_{13} &= \omega_{II1} + \bar{\omega}_{II1} \\
(de^3)_{23} &= \omega_{II2} + \bar{\omega}_{II2}
\end{align*}
\]

which we may combine with the previous results and find out that

\[
\begin{align*}
\omega_{II1} &= (de^3)_{13} - (de^2)_{12} \\
\bar{\omega}_{II2} &= (de^3)_{23} + (de^1)_{12}
\end{align*}
\]

Finally we note the following relations

\[
\begin{align*}
(de^1)_{13} &= -\omega_{II3} + \omega_{III1} \\
(de^2)_{23} &= -\bar{\omega}_{II3} + \bar{\omega}_{III2}
\end{align*}
\]

which we will use further on.

We calculate both sides of (109) keeping in mind that \(\omega_{IV}\) and \(e^3\) are real

\[
2i \text{ Im } d\omega_{II} \equiv d\omega_{II} - d\bar{\omega}_{II} = 3i \left( e^1 \wedge \bar{\omega}_{III} + e^2 \wedge \omega_{III} \right)
\]

or

\[
d \text{ Im } \omega_{II} = \frac{3}{2} \left( e^1 \wedge \bar{\omega}_{III} + e^2 \wedge \omega_{III} \right)
\]
The left hand side

\[ \text{l.h.s.} = d \text{ Im } (\omega_{II1} e^1 + \omega_{II2} e^2) + \text{ Im } (d\omega_{II3} \wedge e^3 + \omega_{II3} de^3) \]

The first term inside the second bracket does not contain \(e^1 \wedge e^2\). Therefore

\[ (\text{l.h.s.})_{12} = d (\text{ Im } (\omega_{II1} e^1 + \omega_{II2} e^2)_{12} + i \omega_{II3} \] (r.h.s.)_{12} = \frac{3}{2} (\bar{\omega}_{III2} - \omega_{III1}) \]

The latter equation after substituting (113) and (114) takes the form of

\[ (\text{r.h.s.})_{12} = \frac{2}{3} \left( (de^2)_{23} - (de^1)_{13} \right) - 3i \text{ Im } \omega_{II3} \]

Combining both sides yields an equation for the imaginary part of \(\omega_{II3}\) expressed in the terms of quantities we already know

\[ i \text{ Im } \omega_{II3} = \frac{3}{8} \left( (de^2)_{23} - (de^1)_{13} \right) - \frac{1}{4} \left( d \text{ Im } (\omega_{II1} e^1 + \omega_{II2} e^2) \right)_{12} \]

The real part of \(\omega_{II3}\) can be assumed to be equal to naught. Hence we have got the whole form \(\omega_{II}\) expressed in the terms of external derivatives of \(e^1\) and \(e^3\). This allows us to calculate the whole \(\omega_{II}\): components 1 and 2 from (113) and (114), while the third component from (115)

\[ \omega_{III3} = \frac{2}{3} (d \text{ Im } \omega_{II})_{23} \]

We now take the 13 component of (109)

\[ \omega_{IV1} = (d\omega_{II})_{13} - 2i\bar{\omega}_{III3} \]

\(\omega_{IV}\) is real, therefore

\[ \omega_{IV2} = \overline{\omega_{IV1}} \]

The component 13 of (110) yields

\[ \omega_{IV3} = (\omega_{III} \wedge \bar{\omega}_{II})_{13} - (d\omega_{III})_{13} \]

Finally we can use (110) and (111) to calculate the curvature coefficients

\[ R = (d\omega_{II})_{23} - (\omega_{III} \wedge \bar{\omega}_{II})_{23} \]
\[ S = (-d\omega_{IV})_{13} + (\omega_{IV} \wedge (\omega_{II} + \bar{\omega}_{II}))_{13} \] (116)
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