A FRACTIONAL GENERALIZATION OF THE POISSON PROCESSES
AND SOME OF ITS PROPERTIES

Nicy Sebastian
Indian Statistical Institute, Chennai Centre, Taramani, Chennai - 600113, India
nicy@isichennai.res.in

Rudolf Gorenflo
Department of Mathematics and Informatics, Free University of Berlin, Arnimallee 3,
D-14195 Berlin, Germany

Abstract

We have provided a fractional generalization of the Poisson renewal processes by
replacing the first time derivative in the relaxation equation of the survival probabil-
ity by a fractional derivative of order \( \alpha \) \((0 < \alpha \leq 1)\). A generalized Laplacian model
associated with the Mittag-Leffler distribution is examined. We also discuss some prop-
erties of this new model and its relevance to time series. Distribution of gliding sums,
regression behaviors and sample path properties are studied. Finally we introduce the
\( q \)-Mittag-Leffler process associated with the \( q \)-Mittag-Leffler distribution.

Keywords: Poisson process; Renewal theory; Fractional derivative; Mittag-Leffler dis-
tribution; Laplacian model; Autoregressive process; Sample path properties.

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1 Introduction

It is our intention to provide via fractional calculus a generalization of the pure and compound
Poisson processes, which are known to play a fundamental role in renewal theory. If the
waiting time is exponentially distributed we have a Poisson process, which is Markovian.
However, other waiting time distributions are also relevant in applications, in particular such
ones with a fat tail caused by a power law decay of its density. In this context we analyze
a non-Markovian renewal process with a waiting time distribution described by the Mittag-
Leffler function. This distribution, containing the exponential as particular case, is known
to play a fundamental role in the infinite thinning procedure of a generic renewal process
governed by a power-asymptotic waiting time.

The concept of renewal process has been developed as a stochastic model for describing the
class of counting processes for which the times between successive events are independently
and identically distributed non-negative random variables, obeying a given probability law.
These times are referred to as waiting times or inter-arrival times.

For a renewal process having waiting times \( T_1, T_2, \ldots \), let

\[
t_0 = 0, \quad t_k = \sum_{j=1}^{k} T_j, \quad k \geq 1,
\]

where \( t_1 = T_1 \) is the time of the first renewal, \( t_2 = T_1 + T_2 \) is the time of the second renewal
and in general \( t_k \) denotes the \( k^{th} \) renewal. The process is specified if we know the probability
law for the waiting times. A relevant quantity is the counting function \( N(t) \) defined as

\[
N(t) = \max \{ k | t_k \leq t, k = 0, 1, 2, \ldots \},
\]
that represents the effective number of events before or at instant \( t \). Also

\[
F_k(t) = P(t_k = T_1 + T_2 + \cdots + T_k \leq t), \quad f_k(t) = \frac{d}{dt} F_k(t), \quad k \geq 1,
\]

where \( F_k(t) \) represents the probability that the sum of the first \( k \) waiting times is less or equal \( t \) and \( f_k(t) \) its density. We assume the waiting times \( T_j = t_j - t_{j-1} \) to be mutually independent, all having the same probability density \( f(t) \). We introduce the function \( R(t) = P(T > t) = \int_t^\infty f(t')dt' \), the survival probability. This name comes from theory of maintenance and means the probability that the relevant piece of equipment lives at least until instant \( t \). We have

\[
f(t) = -\frac{d}{dt} R(t).
\]

In the classical Poisson process the survival probability obeys the relaxation equation

\[
\frac{d}{dt} R(t) = -\lambda R(t), \quad t \geq 0; \quad R(0+) = 1, \quad \lambda > 0
\]

(1)

with a positive constant \( \lambda \). We get

\[
R(t) = \exp(-\lambda t), \quad t \geq 0,
\]

(2)

A “fractional” generalization of the Poisson renewal process is simply obtained by generalizing the differential equation (1) replacing there the first derivative with the integro-differential operator \( t D_+^\alpha \) that is interpreted as the fractional derivative of order \( \alpha \) in Caputo’s sense ([23], [7]) and which in the case \( 0 < \alpha \leq 1 \) is defined as follows:

\[
t D_+^\alpha f(t) = \begin{cases} f'(t) & \text{if } \alpha = 1 \\ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'((t-s)^{\alpha}) d\tau}{(t-s)^\alpha} & \text{if } 0 < \alpha < 1. \end{cases}
\]

We write, taking for simplicity \( \lambda = 1 \),

\[
t D_+^\alpha R(t) = -R(t), \quad t > 0, 0 < \alpha \leq 1; \quad R(0+) = 1.
\]

(2)

The solution of (2) is known to be,

\[
R(t) = E_\alpha(-t^\alpha), \quad t \geq 0, 0 < \alpha \leq 1,
\]

\[
E_\alpha(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-z \tau} \tau^{\alpha-1} d\tau
\]

(3)
where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function (see [3]) which is given as the case $\beta = 1$ of the two-index Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, z \in \mathbb{C}.$$  

In contrast to the Poissonian case $\alpha = 1$, in the case $0 < \alpha < 1$ for large $t$ the function $R(t)$ no longer decays exponentially but algebraically. As a consequence of the power-law asymptotics the process turns to be no longer Markovian but of long-memory type. However, we recognize that for $0 < \alpha < 1$ the function $R(t)$, keeps the completely monotonic character of the Poissonian case. Complete monotonicity of a functions $g(t)$ means

$$(−1)^n \frac{d^n}{dt^n}g(t) ≥ 0, \quad n = 0, 1, 2, \ldots, \quad t ≥ 0,$$

or equivalently, its representability as real Laplace transform of nonnegative generalized function (or measure). By using the Laplace transform technique we generalize the Poisson distribution to the fractional Poisson distribution,

$$P(N(t) = k) = \frac{t^{k\alpha}}{k!} E^{(k)}_\alpha(−t^\alpha), \quad k = 0, 1, 2, \ldots.$$ 

The corresponding fractional Erlang pdf (of order $k \geq 1$) is

$$f_k(t) = \alpha \frac{t^{k\alpha−1}}{(k−1)!} E^{(k)}_\alpha(−t^\alpha),$$

and the fractional Erlang distribution function turns out to be

$$\int_0^t f_k(t')dt' = 1 − \sum_{n=0}^{k−1} \frac{(t)^{n\alpha}}{n!} E^{(n)}_\alpha(−t^\alpha) = \sum_{n=k}^{\infty} \frac{(t)^{n\alpha}}{n!} E^{(n)}_\alpha(−t^\alpha).$$

When solving certain problems in processes of decay and oscillation, diffusion and wave propagation the solution can often be obtained in terms of exponential or logarithmic functions when the orders of differentiation are integer numbers. In the case of non-integer orders Mittag-Leffler functions or more general special functions are required, see [8], [19]. Pillai [22] proved that $F_\alpha(x) = 1 − E_\alpha(−x^\alpha)$, $0 < \alpha ≤ 1$, $x > 0$ and $F_\alpha(x) = 0$ for $x ≤ 0$ are distribution functions, having the Laplace transform $(1 + s^\alpha)^{-1}, s > 0$. He called $F_\alpha(x)$, for $0 < \alpha ≤ 1$, a Mittag-Leffler distribution and showed that $1 − F_\alpha(x)$ is completely monotone for $x > 0$. For $x \in \mathbb{R}$ and $\alpha = 1$, the Mittag-Leffler function with argument $−x^\alpha$ reduces to a standard exponential decay exp(−$x$); when $0 < \alpha < 1$, the Mittag-Leffler function is approximated for small values of $x$ by a stretched exponential decay (Weibull function) $exp(−x^\alpha / \Gamma(\alpha + 1))$ and for large values of $x$ by a power law $bx^{−\alpha}$, where $b = \Gamma(\alpha) \sin(\alpha \pi) / \pi$; see Figure[1]. We obtain the density function $f_\alpha(x)$ as follows

$$f_\alpha(x) = −\frac{d}{dx} E_\alpha(−x^\alpha) = x^{\alpha−1} E_{\alpha,\alpha}(−x^\alpha), \quad 0 < \alpha ≤ 1, \quad x > 0,$$

and $f_\alpha(x) = 0$ for $x < 0$. For $0 < \alpha < 1$ the asymptotic relations can be taken from [3] as

$$E_\alpha(−x^\alpha) \sim \frac{\sin(\alpha \pi) \Gamma(\alpha)}{\pi x^\alpha}, \quad f_\alpha(x) \sim \frac{\sin(\alpha \pi) \Gamma(\alpha + 1)}{\pi x^{\alpha+1}}$$

for $x \to \infty$. 

The second asymptotic relation also comes out by formal differentiation of the first. For \(0 < x \to 0\) we have \(F_\alpha(x) = (x^\alpha / \Gamma(\alpha + 1)) + \text{smaller order terms}\), \(f_\alpha(x) \sim x^{\alpha - 1} / \Gamma(\alpha)\).

Already in the sixties of the past century [6] discovered our Mittag-Leffler waiting time density, \(f_\alpha(x)\) by finding the Laplace transform of the waiting time density of a properly scaled rarefaction (thinning) limit of a renewal process with power law waiting time. But they did not identify this transform as belonging to \(f_\alpha(x)\). In 1985 Balakrishnan found the same Laplace transform also without identifying its inverse as relevant for the time fractional diffusion process. In 1995 Hilfer and Anton were the first authors who introduced explicitly the Mittag-Leffler density \(f_\alpha(x)\) into the theory of continuous time random walk. They showed that it is required for obtaining as evolution equation the fractional variant of the Kolmogorov-Feller equation. By completely different reasoning [15] also discussed the relevance of \(f_\alpha(x)\) in theory of continuous time random walk. However all these early authors did not consider the renewal process with waiting time density \(f_\alpha(x)\) as a subject of study in its own right but only as useful for general analysis of certain stochastic processes. The detailed investigation of the renewal process with \(f_\alpha(x)\) as waiting time density and its analytic and probabilistic properties started (as far as we know) in 2000 with the paper by [24]. Then more and more researchers, often independent of each other, investigated this renewal process, its properties and its applications to other processes. Let us here cite only a few relevant papers: [12], [16], [2], [8], [21], [9].

2 A generalized Laplacian model associated with Mittag-Leffler distribution

2.1 Type-2 generalized Laplacian model

In input-output modeling, the basic idea is to model \(u = x_1 - x_2\) by imposing assumptions on the behaviors or types of \(x_1\) and \(x_2\) and assumptions about whether \(x_1\) and \(x_2\) are independently varying or not, where \(x_1\) and \(x_2\) respectively denote the input and output variables. In a study on modeling growth-decay mechanism, [17] introduced a generalized Laplacian
density of which the Laplace density is a special case. This concept is connected to bilinear forms, quadratic forms and the concept of chi-squaredness of quadratic forms, which is the basis for making inference in analysis of variance, analysis of covariance, regression and general model building areas (see [17], [18]).

So here we introduce a contrasting growth-decay mechanism by assuming that stress and strength are independently distributed Mittag-Leffler random variables. Consider the random variable, \( u = x_1 - x_2 \) which will lead to another class of generalized Laplacian model, say type-2 generalized Laplacian. The characteristic function of a type-2 generalized Laplacian model, denoted by \( \phi_u(t) \), can be obtained as follows. Here \( t \) does not denote the time-variable but the argument of the characteristic function. For \( t \geq 0 \) we have

\[
\phi_u(t) = \phi_{x_1}(t)\phi_{x_2}(-t) = \frac{1}{[1 + ((i)^\alpha + (-i)^\alpha)t^\alpha + t^{2\alpha}]^{-1}} = \frac{1}{[1 + (e^{i\frac{\pi\alpha}{2}} + e^{-i\frac{\pi\alpha}{2}})t^\alpha + t^{2\alpha}]^{-1}} = \left(1 + 2\cos\left(\frac{\pi\alpha}{2}\right)t^\alpha + t^{2\alpha}\right)^{-1}, \quad 0 < \alpha \leq 1. \tag{3}
\]

Obviously \( \phi_u(t) \) is an even function so that we finally get

\[
\phi_u(t) = \left(1 + 2\cos\left(\frac{\pi\alpha}{2}\right)|t|^\alpha + |t|^{2\alpha}\right)^{-1}, \quad t \in \mathbb{R}, \quad 0 < \alpha \leq 1. \tag{4}
\]

The importance of this model is that we can easily obtain the fractional order residual effect. When \( \alpha = 1 \), the characteristic function reduces to \((1 + t^2)^{-1}\), which is the characteristic function of a Laplace random variable whose density is \((1/2)\exp(-|x|)\). If there are several independent input variables \( x_1, \ldots, x_n \) such as the situation in reaction or production problems, and if there are several independent output variables \( x_{m+1}, \ldots, x_{m+n} \) and if they are all independently distributed Mittag-Leffler type variables with different parameters, then the residual \( v = x_1 + \cdots + x_m - x_{m+1} - \cdots - x_{m+n} \) has characteristic function

\[
\phi_v(t) = \frac{1}{(1 + (i)^\alpha_1)} \cdots \frac{1}{(1 + (i)^\alpha_m)} \frac{1}{(1 + (i)^\alpha_{m+1})} \cdots \frac{1}{(1 + (i)^\alpha_{m+n})}, \quad t \geq 0.
\]

The difference \( u = x_1 - x_2 \) or \( v = x_1 + \cdots + x_m - x_{m+1} - \cdots - x_{m+n} \) can be used to describe the behaviour of a stress-strength model. In the context of reliability the stress-strength model describes the life of a component which has a random strength \( x_1 \) and is subjected to random stress \( x_2 \). The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever \( x_1 > x_2 \). Thus \( R = \Pr(u > 0) = \Pr(x_1 > x_2) \) is a measure of component reliability.

### 2.2 Properties of type-2 generalized Laplacian distribution

The type-2 generalized Laplacian density function, denoted by \( h(u) \), can be obtained via the inverse Fourier transform of \( \phi_u(t) \). Hence

\[
h(u) = 1 \int_{-\infty}^{\infty} \frac{e^{-itu}}{(1 + 2\cos\left(\frac{\pi\alpha}{2}\right)|t|^\alpha + |t|^{2\alpha})} dt, \quad 0 < \alpha < 1, \quad u > 0.
\]
**Proposition 2.1** For any $0 < \alpha < 1$, the density function of a type-2 generalized Laplacian random variable $u$ has the representation

$$h(u) = \frac{1}{\pi} \int_0^\infty \frac{\cos tu}{(1 + 2 \cos \left( \frac{\pi \alpha}{2} \right) |t|^\alpha + |t|^{2\alpha})} dt, u > 0.$$ 

### 2.3 Asymptotic behavior

The anti-auto-convolution of a function vanishing for $x < 0$. Assume $f(x) \equiv 0$ for $x < 0$. Set $g(x) := f(-x)$. Then for $h = f * g$ we get

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy = \int_{-\infty}^{\infty} f(y)f(y - x)dy$$

$$= \int_{y=x}^{\infty} f(y)f(y - x)dy = \int_{z=0}^{\infty} f(x + z)f(z)dz,$$

and it can be shown that $(f * g)(-x) = (f * g)(x)$. With $f = f_\alpha$ we obtain (without inverting a Fourier transform) the integral representation

$$h(x) = \int_{x=0}^{\infty} f_\alpha(x + z)f_\alpha(z)dz = \int_{x}^{\infty} f_\alpha(y)f_\alpha(y - x)dy,$$

from which we can draw asymptotic relations. Because of symmetry we have $h(x) = h(-x)$ and we need only to consider $x \geq 0$. For $x \to \infty$ we have with $c_\alpha = \{\Gamma(\alpha + 1) \sin(\alpha\pi)/\pi\}$ asymptotically $h(x) \sim \int_{y=x}^{\infty} c_\alpha y^{-(\alpha + 1)} f_\alpha(y - x)dy$ and using $f_\alpha(x) = -d/dx(E_\alpha(-x^\alpha))$ we get by product integration

$$h(x) \sim c_\alpha y^{-(\alpha + 1)}E_\alpha(-(y - x)^\alpha)\Big|_{y=x}^{y=\infty} + \int_{x}^{\infty} c_\alpha(\alpha + 1)y^{-(\alpha + 2)}E_\alpha(-(y - x)^\alpha)dy$$

$$= c_\alpha x^{-(\alpha + 1)} + b_\alpha(x).$$

Because $0 < E_\alpha(-(y - x)^\alpha) \leq E_\alpha(0) = 1$ we conclude on $b_\alpha(x) \leq c_\alpha x^{-(\alpha + 1)}$ so that finally

$$h(x) = O(x^{-(\alpha + 1)}) \text{ for } x \to \infty.$$ 

**Remark 2.1** By Tauberian theory of asymptotics for Fourier transforms (see \[7\], \[8\]) we find for the tail $T(x) = \int_x^\infty h(z)dz$ the asymptotics $T(x) \sim \{\Gamma(\alpha) \sin(\alpha\pi)/\pi\} x^{-\alpha}$ from which by formal differentiation we would get $h(x) = c_\alpha x^{-(\alpha + 1)}$. However, asymptotic relations generally can be integrated, but differentiation needs additional smoothness requirements. Also we can show that $b_\alpha(x)$ tends to zero faster than $x^{-(\alpha + 1)}$ because the estimate $0 < E_\alpha(-(y - x)^\alpha) \leq E_\alpha(0) = 1$ is very rough and the Mittag-Leffler expression tends to zero. Anyway, we now know that $h(x) = O(x^{-(\alpha + 1)})$ for $x \to \infty$.

### 2.4 Moments of type-2 Laplacian distribution

The moment $M_n = \int_{-\infty}^{\infty} x^n h(x)dx$ are given via the values at 0 of the $n^{th}$ derivative of the Fourier transform

$$\left(1 + 2 \cos \left( \frac{\pi \alpha}{2} \right) |t|^\alpha + |t|^{2\alpha} \right)^{-1} = 1 - 2 \cos \left( \frac{\pi \alpha}{2} \right) |t|^\alpha + \text{smaller order terms.}$$
Because $\alpha < 1$ this Fourier transform is not differentiable at $t = 0$. Hence, the moment $M_n$ does not exist for $n \geq 1$. Clearly, the median exists and because of symmetry is at 0. Contrastingly, in the limiting case $\alpha = 1$ all moments exist. Then we have $h(x) = (1/2)\exp(-|x|)$ and $M_\beta = \Gamma(\beta + 1)$ for all real $\beta = 0$.

3 Time series model associated with the type-2 generalized Laplacian model

3.1 First order autoregressive model associated with the type-2 generalized Laplacian model

Gaver and Lewis derived the exponential solution of the first order autoregressive (abbreviated as AR(1)) equation $x_n = \rho x_{n-1} + \epsilon_n, n = 0, \pm 1, \pm 2, \cdots$, where $\{\epsilon_n\}$ is a sequence of independently and identically distributed random variables when $0 \leq \rho < 1$, see [5].

Definition 3.1 A characteristic function $\phi$ is self decomposable (belongs to class $\mathcal{L}$) if, for every $\rho, 0 < \rho < 1$, there exists a characteristic function $\phi_\rho$ such that $\phi(t) = \phi(\rho t)\phi_\rho(t), \forall t \in \mathbb{R}$.

Theorem 3.1 The type-2 generalized Laplacian distribution belongs to class $\mathcal{L}$.

Proof. The proof is obvious from (7)-(9).

In [5] it is proved that only class $\mathcal{L}$ distributions can be marginal distributions of a first order autoregressive process. Hence from Theorem 3.1 it follows that the type-2 generalized Laplacian distribution can be the marginal distribution of an AR(1) process.

The type-2 generalized Laplacian first order autoregressive process is constituted by $\{u_n; n \geq 1\}$ where the $u_n$ with some $0 < \rho \leq 1$ satisfy the equation

$$u_n = \rho u_{n-1} + \epsilon_n,$$

and $\{\epsilon_n\}$ is sequence of independently and identically distributed random variables such that $u_n$ is stationary Markovian with type-2 generalized Laplacian distribution. In terms of characteristic function, (5) can be given as

$$\phi_{u_n}(t) = \phi_{\epsilon_n}(t)\phi_{u_{n-1}}(\rho t).$$

Assuming stationarity we have,

$$\phi_{\epsilon_n}(t) = \frac{\phi_u(t)}{\phi_u(\rho t)}$$

$$= \frac{(1 + (-it)^{\alpha})(1 + (i\rho t)^{\alpha})}{(1 + (-it)^{\alpha})(1 + (-i\rho t)^{\alpha})}$$

$$= \left[\rho^{\alpha} + (1 - \rho^{\alpha})\frac{1}{(1 + (-it)^{\alpha})}\right] \left[\rho^{\alpha} + (1 - \rho^{\alpha})\frac{1}{(1 + (i\rho t)^{\alpha})}\right].$$

The distribution of innovation sequence can be obtained as

$$\epsilon_n \overset{d}{=} ML_1 - ML_2,$$
where
\[
ML_1 = \begin{cases} 
0, & \text{with probability } \rho^\alpha \\
ML_{11}, & \text{with probability } (1 - \rho^\alpha)
\end{cases}
\]
\[
ML_2 = \begin{cases} 
0, & \text{with probability } \rho^\alpha \\
ML_{21}, & \text{with probability } (1 - \rho^\alpha)
\end{cases}
\]
and \(ML_{11}\) and \(ML_{21}\) are independently distributed Mittag-Leffler random variables.

**Remark 3.1** If \(u_0 \overset{d}{=} ML_1 - ML_2\), then the process is strictly stationary.

**Proof.** For the process to be strictly stationary, it suffices to verify that \(u_n \overset{d}{=} ML_1 - ML_2\) for every \(n\). This can be proved using an inductive argument. Suppose \(u_{n-1} \overset{d}{=} ML_1 - ML_2\), then from (3), (6) and (5),
\[
\phi_{u_n}(t) = \left(1 + 2 \cos\left(\frac{\pi \alpha}{2}\right) |t|^{\alpha} + |t|^{2\alpha}\right)^{-1}, 0 < \alpha \leq 1.
\]
Hence the process is strictly stationary and Markovian, provided \(u_0\) is distributed as type-2 generalized Laplacian.

**Remark 3.2** If \(u_0\) is distributed arbitrarily and \(0 < \rho < 1\), then the process is also asymptotically Markovian with type-2 generalized Laplacian distribution, provided \(\epsilon\) is as in (9).

**Proof.**
\[
u_n = \rho^n u_0 + \sum_{k=0}^{n-1} \rho^k \epsilon_{n-k}.
\]
In terms of characteristic function it can be rewritten as,
\[
\phi_{u_n}(t) = \phi_{u_0}(\rho^n t) \prod_{k=0}^{n-1} \phi_{\epsilon}(\rho^k t).
\]
Thus the left hand side tends to \(\left(1 + 2 \cos\left(\frac{\pi \alpha}{2}\right) |t|^{\alpha} + |t|^{2\alpha}\right)^{-1}, 0 < \alpha \leq 1\), as \(n\) tends to \(\infty\). Hence it follows that, even if \(u_0\) is arbitrarily distributed, the process is asymptotically stationary Markovian with type-2 generalized Laplacian marginals. Thus the following theorem holds.

**Theorem 3.2** The AR(1) process \(u_n = \rho u_{n-1} + \epsilon_n, \rho \in (0, 1)\) is strictly stationary with type-2 generalized Laplacian marginal distributions, if and only if \(\{\epsilon_n\}\) are independently and identically distributed as defined in (3) provided \(u_0\) follows a type-2 generalized Laplacian and is independent of \(\epsilon_1\).

### 3.2 Distribution of sums and joint distribution of \((u_n, u_{n+1})\)

When a stationary sequence \(u_n\) is used, the distribution of the gliding sums \(s_r = u_n + u_{n+1} + \cdots + u_{n+r-1}\) is important. We have
\[
u_{n+j} = \rho^j u_n + \rho^{j-1} \epsilon_{n+1} + \rho^{j-2} \epsilon_{n+2} + \cdots + \epsilon_{n+j}.
\]
Hence
\[
s_r = \sum_{j=0}^{r-1} [\rho^j u_n + \rho^{j-1} \epsilon_{n+1} + \rho^{j-2} \epsilon_{n+2} + \cdots + \epsilon_{n+j}]
\]
\[
= u_n \left(1 - \frac{\rho^r}{1 - \rho}\right) + \sum_{j=0}^{r-1} \epsilon_{n+j} \left(1 - \frac{\rho^{r-j}}{1 - \rho}\right),
\]
The characteristic function of \( s_r \) is

\[
\phi_{s_r}(t) = \phi_{u_n} \left( t \frac{1 - \rho^r}{1 - \rho} \right) \prod_{j=1}^{r-1} \phi_{u} \left( t \frac{1 - \rho^{r-j}}{1 - \rho} \right);
\]

\[
\phi_{s_r}(t) = K \left[ \rho^\alpha + (1 - \rho^\alpha) \frac{1}{1 + \left( -it \frac{1 - \rho^r}{1 - \rho} \right)^\alpha} \right] \left[ \rho^\alpha + (1 - \rho^\alpha) \frac{1}{1 + \left( it \frac{1 - \rho^r}{1 - \rho} \right)^\alpha} \right],
\]

where

\[
K = \left[ 1 + \left( -it \frac{1 - \rho^r}{1 - \rho} \right)^\alpha \right]^{-1} \left[ 1 + \left( it \frac{1 - \rho^r}{1 - \rho} \right)^\alpha \right]^{-1}.
\]

The density function of \( s_r \) can be obtained by inverting the characteristic function as the characteristic function uniquely determines the distribution of a random variable. Now the joint distribution of \( (u_n, u_{n+1}) \) can be given in terms of characteristic function as

\[
\phi_{u_n,u_{n+1}}(t_1, t_2) = E \left[ e^{it_1 u_n + it_2 u_{n+1}} \right] = \phi_{u_n} \left( t_1 + \rho t_2 \right) \phi_{\epsilon_{n+1}}(t_2) = \phi_{u}(t_1 + \rho t_2) \phi_{\epsilon_{n+1}}(t_2) = I \left[ \rho^\alpha + (1 - \rho^\alpha) \frac{1}{1 + (-it_2)^\alpha} \right] \left[ \rho^\alpha + (1 - \rho^\alpha) \frac{1}{1 + (it_2)^\alpha} \right],
\]

where

\[
I = \left( 1 + 2 \cos \left( \frac{\pi \alpha}{2} \right) (t_1 + \rho t_2)^\alpha + (t_1 + \rho t_2)^{2\alpha} \right)^{-1}, t_1, t_2 \geq 0.
\]

The above characteristic function is not symmetric in \( t_1 \) and \( t_2 \) and hence the process is not time reversible.

### 3.3 Regression behaviour of type-2 generalized Laplacian process

Now we shall consider the regression behaviour of the type-2 generalized Laplacian model. Study of the regression of the model is in effect for forecasting of the process. Regression in the forward direction explains the forecasting of future values \( u_n \) while the prediction of past values of \( u_n \) can be done through regression in the backward direction. As stated in [13] the practical implication of regression will be in the statistical analysis of direction dependent data, since the type-2 generalized Laplacian process is not time reversible.

#### 3.3.1 Regression in forward direction.

The regression in the forward direction is linear since, \( E[u_n|u_{n-1} = u] = \rho u, 0 < \rho < 1 \). Furthermore, the conditional variance is constant.

#### 3.3.2 Regression in backward direction.

In the backward direction, the conditional distribution of \( u_n \) given \( u_{n+1} = u \) has non-linear regression and non-constant conditional variance. Following the steps described in [13], the joint characteristic function of \( u_n \) and \( u_{n+1} \) can be derived as,

\[
\phi_{u_n,u_{n+1}}(t_1, t_2) = \frac{\phi_u(t_1 + \rho t_2) \phi_u(t_2)}{\phi_u(\rho t_2)}.
\]
Differentiating this with respect to $t_1$ and setting $t_1 = 0, t_2 = t$;

$$iE[e^{i\alpha u_{n+1}}E[u_n|u_{n+1}]] = \frac{\phi_u'(\rho t)\phi_u(t)}{\phi_u(\rho t)} = \phi_u'(\rho t)\phi_u(t)$$

(11)

where $\phi_u(t)$ is as defined in (7), (11) reduces to,

$$iE[e^{i\alpha u_{n+1}}E[u_n|u_{n+1}]] = Q[R + S],$$

(12)

where

$$Q = \frac{1}{[1 + (-it)^{\alpha}][1 + (it)^{\alpha}][1 + (-i\rho t)^{\alpha}][1 + (i\rho t)^{\alpha}]}$$

$$R = -i\alpha(\rho t)^{\alpha-1}$$

$$S = \frac{i\alpha(-i\rho t)^{\alpha-1}}{[1 + (i\rho t)^{\alpha}][1 + (-i\rho t)^{\alpha}]}.$$

From (12), we can obtain the expression for $E[u_n|u_{n+1}]$ as in [13]. Also proceeding with the bivariate characteristic function defined in (10), the conditional expectation $E[u_n|u_{n+1}]$ can be obtained by following [5].

### 3.4 Simulation studies

#### 3.4.1 Algorithm for $ML_\alpha$ generator

The following algorithm can be used to generate $ML_\alpha$ random variables, for more details see [11].

1. Generate random variate $z$ from standard exponential
2. Generate uniform $[0,1]$ variate $u$, independent of $z$
3. Set $\alpha$
4. Set $w \leftarrow \sin(\pi \alpha) \cot(\pi \alpha u) - \cot(\pi \alpha)$
5. Set $y \leftarrow zw^{\frac{1}{\alpha}}$
6. Return $y$.

We generated type-2 generalized Laplacian random variables for fixed $\alpha = 0.9$ and the histogram for those generated values are given below.
3.4.2 Sample path properties

Here we use the generated type-2 generalized Laplacian distribution for different values of the parameters. Its sample path is observed in the following figures. In Figure 2, we fixe $\rho = 0.3$ and the $\alpha$ values are 0.3 and 1 respectively. For $\rho = 0.6$, we choose the $\alpha$ values as 0.6 and 0.9 respectively, the plot is given in Figure 3. It is evident from the figures that the process exhibits both positive and negative values with upward as well as downward trend. These figures point out the rich variety of contexts where the newly developed time series models can be applied. It is clear that the model gives rise to a wide variety of sample paths so that it can be used to model data from various contexts such as communication engineering, growth-decay mechanism, crop prices etc.

![Figure 2](image1.png)

Figure 2: Sample paths of type 2 generalized Laplacian process for $\rho = 0.3$ and $\alpha = 0.3, 1$.

4 $q$-Mittag-Leffler distribution

Recently various authors have introduced several $q$-type distributions such as $q$-exponential, $q$-Weibull, $q$-logistic and various pathway models in the context of information theory, statistical mechanics, reliability modeling etc. The $q$-exponential distribution can be viewed as

![Figure 3](image2.png)

Figure 3: Sample paths of type 2 generalized Laplacian process for $\rho = 0.6$ and $\alpha = 0.6, 0.9$. 
a stretched model for exponential distribution so that the exponential form can be reached as $q \to 1$. The $q$-exponential distribution is characterized by the density function

$$f(y) = c[1 + (q - 1)\lambda y]^{-\frac{1}{q-1}}; \lambda > 0, y > 0$$

where $c$ is the normalizing constant. In 2010 Mathai considered the Mittag-Leffler density, associated with a Mittag-Leffler function as follows [20]:

$$f(y) = \sum_{k=0}^{\infty} \frac{(-1)^k(\eta)_k}{k!\Gamma(\alpha \eta + \alpha k)} y^{\alpha \eta - 1 + \alpha k} \left(\frac{a}{1 + \frac{\lambda y}{\eta}}\right)^{\alpha \eta + \alpha k}; \eta > 0, a > 0, 0 < \alpha \leq 1, 0 \leq y < \infty$$

The distribution with Laplace transform (14) will be called $q$-Mittag-Leffler distribution and is denoted by $ML(\alpha, \eta, q^{-1})$. If $q \to 1^+$ in (14), then

$$L_g(t) = \lim_{q \to 1^+} L_f(t) = \lim_{q \to 1^+} [1 + a(q - 1)t^\alpha]^{-\frac{\eta}{\alpha}} = e^{-\eta t^\alpha} = L_{f_1}(t).$$

which is the Laplace transform of a constant multiple of a positive Lévy variable with parameter $\alpha, 0 < \alpha \leq 1, t \geq 0$. Thus here $q$ creates a pathway of going from the general Mittag-Leffler density $f$ to a positive Lévy density $f_1$ with parameter $\alpha$, the multiplying constant being $(\eta a)^{1/\alpha}$.

5 $q$-Mittag-Leffler process

The $q$-Mittag-Leffler first order autoregressive process is constituted by $\{y_n; n \geq 1\}$ where $y_n$ satisfies the equation

$$y_n = \rho y_{n-1} + \epsilon_n, \ 0 < \rho \leq 1,$$

where $\{\epsilon_n\}$ is sequence of independently and identically distributed random variables such that $y_n$ is stationary Markovian with $q$-Mittag-Leffler distribution. We consider the $AR(1)$ structure given by (6). In terms of Laplace transforms, this can be rewritten as

$$\psi_{y_n}(t) = \psi_{\epsilon_n}(t)\psi_{y_{n-1}}(\rho t).$$
Assuming stationarity we have,

$$
\psi_{\epsilon_n}(t) = \frac{\psi_y(t)}{\psi_y(\rho t)} = \frac{[1 + (q - 1)t^\alpha]^{-\eta q^{-1}}}{[1 + (q - 1)\rho^\alpha t^\alpha]^{-\eta}}
= \left[\frac{1 + (q - 1)\rho^\alpha t^\alpha}{1 + (q - 1)t^\alpha}\right]^{\eta q^{-1}}, t > 0.
\tag{16}
$$

The infinitely divisible $ML(\alpha, \eta, q - 1)$ variable is of the class $\mathcal{L}$, and therefore that (16) is the Laplace-Stieltjes transform of a distribution function follows from the class $\mathcal{L}$ theorem of [4], since the determining canonical measure $M$ of the $ML(\alpha, \eta, q - 1)$ variable is $q - 1$ (when $\eta = 1$) times the determining canonical measure of the $ML(\alpha, \eta, q - 1)$ variable. Thus we can in principle generate an autoregressive process with gamma marginals (13) by utilizing the $\{\epsilon_n\}$ process characterized by (16). Here are three simple special cases. When $q = 2$ and $\eta = 1$ in (16) we have

$$
\psi_{\epsilon_1}(t) = \left[\rho^\alpha + (1 - \rho^\alpha) \left(\frac{1}{1 + t^\alpha}\right)\right], t > 0.
\tag{17}
$$

Thus the random variable,

$$
\epsilon_1 = \begin{cases} 
0, & \text{with probability } \rho^\alpha \\
M, & \text{with probability } (1 - \rho^\alpha).
\end{cases}
$$

Hence, $\epsilon_1$ is a convolution of an atom of mass $\rho^\alpha$ at zero and $1 - \rho^\alpha$ at $M$ where $M$ is distributed as $ML(\alpha)$. When $q = 3$ and $\eta = 1$ in (16) we have

$$
\psi_{\epsilon_2}(t) = \left[\rho^{2\alpha} + 2\rho^\alpha(1 - \rho^\alpha) \left(\frac{1}{1 + 2t^\alpha}\right) + (1 - \rho^\alpha)^2 \left(\frac{1}{1 + 2t^\alpha}\right)^2\right], t > 0,
\tag{18}
$$

and $\epsilon_2 = \begin{cases} 
0, & \text{with probability } \rho^{2\alpha} \\
ML(\alpha, 3, 2), & \text{with probability } 2\rho^\alpha(1 - \rho^\alpha) \\
ML(\alpha, 4, 2), & \text{with probability } (1 - \rho^\alpha)^2.
\end{cases}$

When $q = 3/2$ and $\eta = 1/4$ in (16) we have

$$
\psi_{\epsilon_3}(t) = \left[\rho^{2\alpha} + 2\rho^\alpha(1 - \rho^\alpha) \left(\frac{1}{1 + \frac{\alpha}{2}}\right) + (1 - \rho^\alpha)^2 \left(\frac{1}{1 + \frac{\alpha}{2}}\right)^2\right], t > 0,
\tag{19}
$$

and $\epsilon_3 = \begin{cases} 
0, & \text{with probability } \rho^{2\alpha} \\
ML(\alpha, 1/2, 1/2), & \text{with probability } 2\rho^\alpha(1 - \rho^\alpha) \\
ML(\alpha, 1, 1/2), & \text{with probability } (1 - \rho^\alpha)^2.
\end{cases}$

In general the $q$-Mittag-Leffler process can give a generalization of the model given in [5]. Hence the essentials of fractional calculus according to different approaches that can be useful for our applications in the theory of probability and stochastic processes are established.
6 Applications

During the last 15 years a lot of engineers and scientists have shown very much interest in the Mittag-Leffler function and Mittag-Leffler type functions due to their vast potential of applications in several fields such as fluid flow, rheology, electric networks, probability, and statistical distribution theory. The Mittag-Leffler function arises naturally in the solution of fractional order integral or differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, anomalous diffusion transport and in the study of complex systems. In recent years the fractional generalization of the classical Poisson process has gained increasing interest. In it the waiting time between events is the Mittag-Leffler distribution function $F_\alpha$ in place of the exponential distribution. Of all the papers devoted to this special renewal process we content ourselves to cite only [15], [16] and [9]. Mittag-Leffler distributions can be used as waiting-time distributions as well as first-passage time distributions for certain renewal processes with geometric exponential as waiting-time distribution. They can also be used in reliability modeling as an alternative for exponential lifetime distribution. The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts, see [19]. Mittag-Leffler functions are also used for computation of the change of the chemical composition in stars like the Sun. Recent investigations have proved that they are useful in modelling the flux of solar neutrinos in cosmological studies, which can be expressed in terms of special functions like $G$ and $H$-functions, see [19], [26]. The Mittag-Leffler distribution finds applications in a wide range of contexts such as stress-strength analysis, growth-decay mechanisms like formation of sand dunes in nature, input-output situations in economics, industrial productions, production of melatonin in human body etc.

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