Hyperspherical Harmonics, Separation of Variables and the Bethe Ansatz

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Abstract

The relation between solutions to Helmholtz’s equation on the sphere $S^{n-1}$ and the $[sl(2)]^n$ Gaudin spin chain is clarified. The joint eigenfunctions of the Laplacian and a complete set of commuting second order operators suggested by the $R$–matrix approach to integrable systems, based on the loop algebra $\tilde{sl}(2)_R$, are found in terms of homogeneous polynomials in the ambient space. The relation of this method of determining a basis of harmonic functions on $S^{n-1}$ to the Bethe ansatz approach to integrable systems is explained.

Introduction

In some recent papers on separation of variables in the Hamilton-Jacobi and Schrödinger equations [1–3], it has been asserted that free motion, both classical and quantum, on the sphere $S^{n-1}$ and on the negative constant curvature hyperboloid $H^{n-1}$ is equivalent to the $[sl(2)]^n$ Gaudin spin chain [4,5]. There is, in fact, a close relationship between these systems, which may best be seen through the method of moment map embeddings in loop algebras [6–9], but they actually differ significantly at the quantum level. The purpose of this work is, first of all, to clarify what this relationship is and, secondly, to show how an approach that is related to the functional Bethe ansatz [10] for spin chains leads very simply to a basis of harmonic functions on $S^{n-1}$ generalizing that provided, for the case $n = 3$, by the Lamé polynomials [11,12].

Briefly, the difference between the Laplacian on $S^{n-1}$ and the $[sl(2)]^n$ spin chain system is that, while the former does belong to a commuting family which formally is of the same type as the Gaudin systems, the permissible values of the $sl(2)$ Casimir invariants are different for the two cases. Hence, whereas the classical phase spaces may be simply related, the quantum...
Hilbert space, joint eigenstates and spectrum are quite different. Moreover, the mapping which determines this correspondence is not 1–1, but rather involves a quotient by the group \( \mathbb{Z}_2^n \) of reflections in the coordinate planes. As a consequence, the Laplacian on \( S^{n-1} \), together with the associated commuting family of second order operators provided by the loop algebra framework, is diagonalized not on a unique highest weight module of \([\mathfrak{sl}(2)]^n\), as in the Gaudin systems, but rather on the sum of \( 2^n \) invariant subspaces, each characterized by distinct \( \mathbb{Z}_2^n \) transformation properties and containing its own highest weight vector. Essentially, the \([\mathfrak{sl}(2)]^n\) spin chains are defined on tensor products of irreducible discrete series representations of \( \mathfrak{sl}(2) \), while the Laplacian on \( S^{n-1} \) involves products of the direct sum of the two lowest metaplectic representations. Thus there are, roughly speaking, \( 2^n \) times as many harmonic functions on \( S^{n-1} \) as joint eigenstates in an irreducible Gaudin system.

In the following, these joint eigenfunctions will be constructed through a separation of variables technique originally developed in the loop algebra approach to classical integrable systems [13,14]. In the quantum setting this leads directly to the functional Bethe ansatz eigenstates. The classical case is already amply treated in [7–9,13,14], so we proceed directly to the quantum case here. The canonical quantization approach leading to the construction used below is detailed in [15]. Similar formulations may be found in [1,2,10,16].

1. Quantum Moment Map Construction and Commuting Invariants

1a. \( \tilde{\mathfrak{l}}(2)_R \) Loop Algebra Representation and Commuting Invariants.

Let \((x_1, \ldots, x_n)\) be the Cartesian coordinates in \( \mathbb{R}^n \) and \((\mu_1, \ldots, \mu_n)\) a set of \( n \) real numbers. Define the operators

\[
e_i := \frac{1}{2} \left( \frac{\partial^2}{\partial x_i^2} - \frac{\mu^2_i}{x_i^2} \right) \quad (1.1a)
\]
\[
f_i := \frac{1}{2} x_i^2 \quad (1.1b)
\]
\[
h_i := \frac{1}{2} \left( x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right), \quad i = 1, \ldots, n, \quad (1.1c)
\]

which satisfy the commutation relations

\[
[h_i, f_i] = f_i, \quad [h_i, e_i] = -e_i, \quad [e_i, f_i] = 2h_i, \quad (1.2)
\]

and hence determine \( n \) representations of \( \mathfrak{sl}(2) \). The constants \( \{\mu_1, \ldots, \mu_n\} \) are related to the values of the \( \mathfrak{sl}(2) \) Casimir invariants by

\[
h_i^2 - \frac{1}{2} (e_i f_i + f_i e_i) = \frac{1}{4} \left( \mu^2_i - \frac{3}{4} \right). \quad (1.3)
\]
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We shall only be concerned here with quadratic combinations of these operators that may consistently be restricted to the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). Pick a further set \( \{ \alpha_1, \ldots \alpha_n \} \) of real constants, all distinct, and ordered with increasing values. Define now the following operator–valued rational functions of the “loop” parameter \( \lambda \)

\[
e(\lambda) := \sum_{i=1}^{n} \frac{e_i}{\lambda - \alpha_i} \quad (1.4a)
\]

\[
f(\lambda) := \sum_{i=1}^{n} \frac{f_i}{\lambda - \alpha_i} \quad (1.4b)
\]

\[
h(\lambda) := a + \sum_{i=1}^{n} \frac{h_i}{\lambda - \alpha_i}, \quad (1.4c)
\]

where \( a \) is a real constant. These satisfy the well-known commutation relations for the loop algebra \( \tilde{sl}(2)_R \) defined with respect to a rational classical \( R \)–matrix structure \([6,17]\):

\[
[h(\lambda), e(\mu)] = \frac{e(\lambda) - e(\mu)}{\lambda - \mu} \quad (1.5a)
\]

\[
[h(\lambda), f(\mu)] = -\frac{f(\lambda) - f(\mu)}{\lambda - \mu} \quad (1.5b)
\]

\[
[e(\lambda), f(\mu)] = -2\frac{h(\lambda) - h(\mu)}{\lambda - \mu}, \quad (1.5c)
\]

It follows that the coefficients of the following operator–valued rational function

\[
\Delta(\lambda) := h^2(\lambda) - \frac{1}{2} (e(\lambda)f(\lambda) + f(\lambda)e(\lambda)) \quad (1.6)
\]

commute amongst themselves:

\[
[\Delta(\lambda), \Delta(\mu)] = 0, \quad \forall \lambda, \mu. \quad (1.7)
\]

Expanding \( \Delta(\lambda) \) in partial fractions gives

\[
\Delta(\lambda) = a^2 + \sum_{i=1}^{n} \frac{H_i}{\lambda - \alpha_i} + K(\lambda) + L^2(\lambda) + L'(\lambda), \quad (1.8)
\]

where

\[
K(\lambda) := \frac{1}{4} \sum_{i=1}^{n} \frac{\mu_i^2}{(\lambda - \alpha_i)^2} \quad (1.9a)
\]

\[
L(\lambda) := \frac{1}{4} \sum_{i=1}^{n} \frac{1}{\lambda - \alpha_i} \quad (1.9b)
\]
and \( \{H_i\}_{i=1,...,n} \) are a set of commuting second order differential operators defined by

\[
H_i := \frac{1}{4} \sum_{j=1 \atop j \neq i}^{n} \frac{-L_{ij}^2 + \mu_i^2 \frac{x_i^2}{x_j^2} + \mu_j^2 \frac{x_j^2}{x_i^2}}{\alpha_i - \alpha_j} + a \left( x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right),
\]

(1.10)

where

\[
L_{ij} := x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}
\]

(1.11)
is the generator of rotations in the \((ij)\) plane. For \(a \neq 0\), the \(H_i\)'s are linearly independent

and their sum is

\[
\sum_{i=1}^{n} H_i = a(D + \frac{n}{2}),
\]

(1.12)

where

\[
D := \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}
\]

(1.13)
is the Euler homogeneity operator. For \(a = 0\), this sum vanishes, so only \(n-1\) of the
operators are independent. But since they still all commute with the Euler operator \(D\), this
has been adjoined as the \(n\)th independent commuting invariant in this case.

Contained within this framework, for particular values of the constants \(\{a, \mu_1, \ldots, \mu_n\}\), are
both the Laplacian on the sphere \(S^{n-1}\) and the Gaudin \([sl(2)]^n\) spin chains. It is important
\note{to note}, however, that they correspond to different values of the Casimir invariants for the
various \(sl(2)\) representations. To obtain the Laplacian on \(S^{n-1}\), we set \(a\) and all the \(\mu_i\)'s
\note{equal to zero}. Since the \(H_i\)’s are defined in terms of the rotation generators \(L_{ij}\), they may be
\note{restricted consistently to \(S^{n-1}\)}. Forming the sum

\[
H_0 := 4 \sum_{i=1}^{n} \alpha_i H_i = -\frac{1}{2} \sum_{i,j=1}^{n} L_{ij}^2
\]

(1.14)

and restricting to the unit sphere \(S^{n-1}\) gives the Laplacian

\[
H_0|_{S^{n-1}} = -\Delta_{S^{n-1}}.
\]

(1.15)

More generally, if the \(\mu_i\)’s are nonvanishing, we have

\[
H := 4 \sum_{i=1}^{n} \alpha_i H_i = -\frac{1}{2} \sum_{i,j=1}^{n} L_{ij}^2 + \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} \frac{\mu_j^2}{x_j^2} - \sum_{i=1}^{n} \mu_i^2,
\]

(1.16)

and restricting to \(S^{n-1}\) gives

\[
H|_{S^{n-1}} = -\Delta_{S^{n-1}} + \sum_{i=1}^{n} \frac{\mu_i^2}{x_i^2} - \sum_{i=1}^{n} \mu_i^2.
\]

(1.17)
This corresponds to a degenerate case of the quantum Rosochatius system \[15,18\], in which the harmonic oscillator potential is absent. (It may also be obtained as a reduction of the free system on a sphere of dimension \(2n - 1\) under the action of the maximal torus in the isometry group \(SO(2n)\) (cf. \[19\]). The other operators \(H_i\) in the commuting family may similarly be restricted to define commuting operators on \(S^{n-1}\).

1b. Relation to the \([\mathfrak{sl}(2)]^n\) Gaudin Chain.

To obtain the \([\mathfrak{sl}(2)]^n\) Gaudin spin chain, we must make a change of representation. Instead of considering square–integrable functions on \(\mathbb{R}^n\) or \(S^{n-1}\), we must reinterpret the operators \(\{e_i, f_i, h_i\}\) entering in the definitions (1.4a)–(1.4c), (1.6) as acting upon a highest weight module of \([\mathfrak{sl}(2)]^n\) of the type

\[
H = \otimes_{i=1}^n H_{l_i},
\]

where the individual factors \(H_{l_i}\) in the tensor product are generated by application of the ladder operator \(f_i\) to a unique highest weight vector \(|0\rangle_{l_i}\), the kernel of the operator \(e_i\). (We make no notational distinction between an operator \(O\) acting on \(H_{l_i}\) and its extension \(I \otimes I \otimes \cdots \otimes O \otimes \cdots \otimes I\) acting on \(H\).)

Note that the space of square-integrable smooth functions on \(\mathbb{R}^n\), with the representation (1.1a)–(1.1c), will not do for this purpose, since the kernel of the operator (1.1a) is two dimensional. However, ignoring for the present the question of normalizability, we may still consider the space obtained by application of polynomials in the operators \(f_i\) to any specifically chosen vector \(|0\rangle_{l_i}\) within the kernel of each \(e_i\). This kernel is of the form

\[
K_i = \{a_i x_1^{\beta_i} + b_i x_1^{-(\beta_i - 1)}\},
\]

where

\[
\beta_i(\beta_i - 1) = \mu_i^2.
\]

For \(\mu_i \neq 0\), we may choose the unique positive root of this quadratic equation and identify, up to normalization

\[
|0\rangle_{l_i} \sim x^{\beta_i}.
\]

If the domain of definition is \(\mathbb{R}^n\), to obtain single–valued functions, the \(\beta_i\)'s must be chosen to have integer values. Then \(H_{l_i}\) consists of functions of the form \(p(x_1^2)x_1^{\beta_i}\), where \(p\) is a polynomial in \(x_1^2\), and \(\mathcal{H}\) consists of functions of the form \(\mathcal{P}(x_1^2, \ldots x_n^2) \prod_{i=1}^n x_i^{\beta_i}\), where \(\mathcal{P}\) is a polynomial in its \(n\) arguments \(\{x_1^2, \ldots x_n^2\}\). In the case where the \(\mu_i\)'s vanish, however, which is the one relevant to the Laplacian on \(S^{n-1}\), each of the \(\beta_i\)'s may have the two possible values

\[
\beta_i = 0 \quad \text{or} \quad \beta_i = 1,
\]

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and we obtain distinct highest weight representations for each choice of the $\beta_i$'s.

To obtain the $\mathfrak{sl}(2)$ analogue of the highest weight representations used in [10], we must conjugate the operators $\{e_i, f_i, h_i\}$ by the factor $x_i^{\beta_i}$ and re-express the result in terms of the new variable

$$z_i := \frac{1}{2} x_i^2. \quad (1.23)$$

Thus, we define

$$\hat{e}_i := x_i^{-\beta_i} e_i x_i^{\beta_i} = z_i \frac{\partial^2}{\partial z_i^2} + 2l_i \frac{\partial}{\partial z_i} \quad (1.24a)$$

$$\hat{f}_i := x_i^{-\beta_i} f_i x_i^{\beta_i} = z_i \quad (1.24b)$$

$$\hat{h}_i := x_i^{-\beta_i} h_i x_i^{\beta_i} = z_i \frac{\partial}{\partial z_i} + l_i, \quad (1.24c)$$

where

$$l_i := \beta_i^2 + \frac{1}{4}. \quad (1.25)$$

This is the $\mathfrak{sl}(2)$ analogue of the representation used in [10] to describe the $\mathfrak{su}(2)$ Gaudin spin chain in functional terms (cf. [20,21]). The highest weight state $|0\rangle$ in this representation is just a constant, and a basis for the corresponding space $\hat{H}$ generated by application of the $\hat{f}_i$'s is given by the homogeneous polynomials in the variables $\{z_1, \ldots, z_n\}$.

Within such representations, the $\beta_i$'s need not have integer values. For any set of positive values for the $l_i$'s, a scalar product may be defined by:

$$\langle z_1^{j_1} \ldots z_n^{j_n}, z_1^{k_1} \ldots z_n^{k_n} \rangle := \prod_{i=1}^{n} \delta_{j_i,k_i} C_{l_i,j_i}, \quad j_i, k_i \in \mathbb{N}, \quad (1.26)$$

where

$$C_{l_i,j_i} := (j_i)! \prod_{j=1}^{j_i} (2l_i + j - 1) \quad \text{for} \quad j_i \geq 1, \quad C_{l_i,0} := 1. \quad (1.27)$$

With respect to this scalar product, the operators $\hat{h}_i$ are hermitian, while $\hat{e}_i$ and $\hat{f}_i$ are mutually hermitian conjugate, so we are really dealing with unitary representations of the real form $\mathfrak{su}(1,1)$. It follows that, for the case of integer $\beta_i$'s, the operators $\hat{H}_i$ obtained by conjugating the $H_i$'s by the factor $\prod_{i=1}^{n} x_i^{\beta_i}$ are all hermitian. Within such representations, these may be viewed as the Hamiltonians of the $[\mathfrak{sl}(2)]^n$ Gaudin spin chain [4,10]. However, it is only integer or half-integer $l_i$'s that define discrete series representations; integer $\beta_i$'s give quarter-integer $l_i$'s, corresponding instead metaplectic representations [22].

Moreover, for the case when all the $\mu_i$'s vanish, we must replace the factors $\mathcal{H}_{l_i}$ in (1.18), by the direct sum $\mathcal{H}_{\frac{1}{4}} \oplus \mathcal{H}_{\frac{3}{4}}$ of the two lowest metaplectic representations. Therefore, we do
not have a unique highest weight representation. Expressed in terms of the original Cartesian coordinates \( \{x_1, \ldots, x_n\} \), to each invariant subspace in the \( z_i \) representation consisting of homogeneous polynomials in \( \{z_1, \ldots, z_n\} \) of degree \( p \), there are associated \( 2^n \) different possible invariant subspaces, consisting of polynomials in \( \{x_1, \ldots, x_n\} \) of degree \( 2p + \sum_{i=1}^{n} \beta_i \), one for each choice of the binary sequence \( \{\beta_1, \ldots, \beta_n\} \). These are not only invariant under the representation of \([\mathfrak{sl}(2)]^n\) defined by (1.1a)–(1.1c), but also under the group \( \mathbb{Z}_2^n \) of reflections in the coordinate planes, and each has its own highest weight vector.

2. Determination of Joint Eigenfunctions; Separation of Variables

2a. Separating Coordinates and Irreducible Subspaces.

We now introduce a coordinate system that is specially adapted to the simultaneous diagonalization of the commuting operators \( H_i \) introduced above (cf. [13,14]), the spheroconical coordinates \( \{r, \lambda_1, \ldots, \lambda_{n-1}\} \), defined by

\[
\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \alpha_i} = \frac{r^2 Q(\lambda)}{a(\lambda)},
\]

where

\[
Q(\lambda) := \prod_{\mu=1}^{n-1} (\lambda - \lambda_\mu),
\]

\[
a(\lambda) := \prod_{i=1}^{n} (\lambda - \alpha_i),
\]

with the \( \lambda_\mu \)'s chosen in the range:

\[
\alpha_1 < \lambda_1 < \alpha_2 < \cdots < \lambda_{n-1} < \alpha_n.
\]

The functions \( x_i^2 \) are given in terms of the coordinates \( \{r, \lambda_1, \ldots, \lambda_{n-1}\} \) by

\[
x_i^2 = \frac{2r^2 \prod_{\mu=1}^{n-1} (\alpha_i - \lambda_\mu)}{a'(\alpha_i)},
\]

and hence are linear combinations of the elementary symmetric invariants

\[
\sigma_k := \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 0, \ldots, n - 1.
\]

Define the operator–valued polynomial \( \hat{P}(\lambda) \) by

\[
\frac{\hat{P}(\lambda)}{a(\lambda)} := \sum_{i=1}^{n} \frac{H_i}{\lambda - \alpha_i}.
\]
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Since the coefficients of $\hat{P}(\lambda)$ are just linear combinations of the $H_i$'s, simultaneous diagonalization of the latter is equivalent to diagonalization of $\hat{P}(\lambda)$ on functions that do not depend on the parameter $\lambda$. Noting that the operator $h(\lambda)$, evaluated at $\lambda = \lambda_\mu$ is just

$$h(\lambda)|_{\lambda=\lambda_\mu} = \frac{\partial}{\partial \lambda_\mu} + L(\lambda_\mu) + a,$$

and that $f(\lambda_\mu)$ vanishes at $\lambda = \lambda_\mu$, it follows from (1.6), (1.8) that

$$\hat{P}(\lambda)|_{\lambda=\lambda_\mu} = a(\lambda_\mu) \left( \frac{\partial^2}{\partial \lambda_\mu^2} + 2(a + L(\lambda_\mu)) \frac{\partial}{\partial \lambda_\mu} + 2aL(\lambda_\mu) - K(\lambda_\mu) \right).$$

Using also the fact that the $\lambda^{n-1}$ coefficient of $\hat{P}(\lambda)$ is given by (1.12), Lagrange interpolation shows that $\hat{P}(\lambda)$ may be expressed in terms of the $\{r, \lambda_1, \ldots, \lambda_{n-1}\}$ coordinates as

$$\hat{P}(\lambda) = \sum_{\mu=1}^{n-1} \frac{Q(\lambda)a(\lambda_\mu)}{Q'(\lambda_\mu)(\lambda - \lambda_\mu)} \left[ \frac{\partial^2}{\partial \lambda_\mu^2} + 2(a + L(\lambda_\mu)) \frac{\partial}{\partial \lambda_\mu} + 2aL(\lambda_\mu) - K(\lambda_\mu) \right]$$

$$+ aQ(\lambda) \left( r \frac{\partial}{\partial r} + \frac{n}{2} \right).$$

Now assume that the $\mu_i$'s are given by (1.20), with the $\beta_i$'s all nonnegative integers. Let $\mathcal{H}^p(\beta_1, \ldots, \beta_n)$ denote the space of homogeneous polynomials of the form

$$\mathcal{P}(x_1^2, \ldots, x_n^2) \prod_{i=1}^{n} x_i^{\beta_i},$$

where $\mathcal{P}(x_1^2, \ldots, x_n^2)$ is a homogeneous polynomial of degree $p$ in its arguments. In the following, we must distinguish between the case when all the $\mu_i$'s vanish, for which there are $2^n$ different such spaces for each $p$, and the case when they do not. For the former, the operators $H_i$ are well-defined for all values of the $x_i$'s. Restricting to the unit sphere $S^{n-1}$, the resulting operators are still well defined, because the Euler operator

$$D = r \frac{\partial}{\partial r}$$

entering in (2.10) takes a fixed value on each $\mathcal{H}^p(\beta_1, \ldots, \beta_n)$. For the case of nonvanishing $\mu_i$'s, the operators $H_i$ are singular on the hyperplanes $x_i = 0$ whenever $\mu_i \neq 0$. However, restricting to the space $\mathcal{H}^p(\beta_1, \ldots, \beta_n)$, the $H_i$'s become regularized through the conjugation by $x_i^{\beta_i}$ that was used in defining the operators $\{\hat{e}_i, \hat{f}_i, \hat{h}_i\}$ in (1.24a)–(1.24c). We see that the Hamiltonian operators (1.16) and (1.17) contain potentials that give rise to an infinite repulsion away from these coordinate hyperplanes, and hence the eigenfunctions must vanish to appropriate order at these planes.
2b. Determination of Eigenfunctions and the Functional Bethe Ansatz.

We seek joint eigenfunctions $\Psi$, satisfying

$$H_i \Psi = E_i \Psi$$

or, equivalently

$$\hat{P}(\lambda) \Psi = E(\lambda) \psi,$$

where the polynomial $E(\lambda)$ is defined by

$$\frac{E(\lambda)}{a(\lambda)} = \sum_{i=1}^{n} \frac{E_i}{\lambda - \alpha_i}.$$ (2.15)

Since the Euler operator $D$ and the generators of the group $\mathbb{Z}_2^n$ of reflections in the coordinate planes all commute with the $H_i$'s, these may be simultaneously diagonalized. This, together with the regularization requirement in the case when some $\mu_i$'s are nonzero, implies that each joint eigenfunction $\Psi$ belongs to one of the spaces $\mathcal{H}^p(\beta_1, \ldots, \beta_n)$, and the Euler operator in (2.10) may be replaced by its eigenvalue $2p + \sum_{i=1}^{n} \beta_i$. The resulting operators may consistently be restricted to the unit sphere $S^{n-1}$. Expressed in terms of the $\{r, \lambda_1, \ldots, \lambda_{n-1}\}$ coordinates, $\Psi$ is therefore of the form

$$\Psi(r, \lambda_1, \ldots, \lambda_{n-1}) = r^{(2p + \sum_{i=1}^{n} \beta_i)} \psi(\lambda_1, \ldots, \lambda_{n-1}) \prod_{i=1}^{n} \prod_{\mu=1}^{n-1} (\lambda_{\mu} - \alpha_i)^{\beta_i},$$ (2.16)

where $\psi(\lambda_1, \ldots, \lambda_{n-1})$ is a homogeneous polynomial of degree $p$ in the symmetric invariants $\{\sigma_0, \ldots, \sigma_{n-1}\}$, and hence, of degree $\leq p(n-1)$ in the $\lambda_i$'s. (For odd values of $\beta_i$, the apparent sign ambiguity in the factors $(\lambda_{\mu} - \alpha_i)^{\beta_i}$ is resolved by identifying the sign of the product over $\mu$ with that of $x_{\mu}^{\beta_i}$.) The eigenvalue equation (2.14) then reduces to

$$\sum_{\mu=1}^{n-1} \frac{a(\lambda_{\mu})}{Q'(\lambda_{\mu})(\lambda - \lambda_{\mu})} \left[ \frac{\partial^2 \psi}{\partial \lambda_{\mu}^2} + 2(a + \Lambda(\lambda_{\mu})) \frac{\partial \psi}{\partial \lambda_{\mu}} + \frac{1}{4} \sum_{j,i=1}^{n} 2\beta_i \beta_j + \beta_i + \beta_j \frac{2}{(\alpha_i - \alpha_j)(\lambda_{\mu} - \alpha_i)} \psi + 2a\Lambda(\lambda_{\mu}) \psi \right]$$

$$+ a \left( 2p + \sum_{i=1}^{n} \beta_i + \frac{n}{2} \right) \psi = \frac{E(\lambda)}{Q(\lambda)} \psi,$$ (2.17)

where

$$\Lambda(\lambda) := \frac{1}{2} \sum_{i=1}^{n} \frac{\beta_i}{\lambda - \alpha_i},$$ (2.18)

Equating residues at each $\lambda = \lambda_{\mu}$, this is equivalent to

$$\frac{\partial^2 \psi}{\partial \lambda_{\mu}^2} + 2(a + \Lambda(\lambda_{\mu})) \frac{\partial \psi}{\partial \lambda_{\mu}} + 2a\Lambda(\lambda_{\mu}) \psi + \frac{1}{4} \sum_{j,i=1}^{n} \frac{2\beta_i \beta_j + \beta_i + \beta_j}{(\alpha_i - \alpha_j)(\lambda_{\mu} - \alpha_i)} \psi = \frac{E(\lambda_{\mu})}{a(\lambda_{\mu})} \psi.$$ (2.19)
By uniqueness (up to normalization) of polynomial solutions of equations of the type (2.19), $\psi$ must have the factorized form

$$
\psi(\lambda_1, \ldots, \lambda_{n-1}) = \prod_{\mu=1}^{n-1} q(\lambda_\mu),
$$

(2.20)

where $q(\lambda)$ is a polynomial of degree $\leq p$ satisfying the equation

$$
a(\lambda)q''(\lambda) + 2b(\lambda)q'(\lambda) + c(\lambda)q(\lambda) = 0,
$$

(2.21)

with polynomial coefficients $b(\lambda), c(\lambda)$ defined by

$$
b(\lambda) := a(\lambda)(a + \Lambda(\lambda))
$$

(2.22a)

$$
c(\lambda) := a(\lambda) \left( 2a\Lambda(\lambda) + \frac{1}{4} \sum_{j, j' \neq i}^{n} \frac{2\beta_i \beta_{j'} + \beta_i + \beta_{j'}}{\alpha_i - \alpha_{j'}}(\lambda - \alpha_i) \right) - E(\lambda).
$$

(2.22b)

For the case $a = 0$, the number of distinct polynomials of degree $p$ satisfying an equation of the type (2.21) is known, by the Heine-Stieltjes theorem [23], to be $\binom{n + p - 2}{p}$. This may also be seen as follows. Note that the operators $H_i$ are all hermitian with respect to the scalar product on $H^p(\beta_1, \ldots, \beta_n)$ determined from (1.26) through the isomorphism between the space of homogeneous polynomials $P(z_1, \ldots, z_n)$ of degree $p$ and $H^p(\beta_1, \ldots, \beta_n)$ given by multiplication by the factor $\prod_{i=1}^{n} x_i^{\beta_i}$. Their joint eigenfunctions therefore provide a basis for this space, which is of dimension $\binom{n + p - 1}{p}$. Each such joint eigenfunction corresponds to a unique polynomial solution of the equation (2.21) for some set of eigenvalues $\{E_1, \ldots, E_n\}$. These include not only polynomials of degree $p$, but also those of all lower degree. The corresponding eigenfunctions are obtained from an eigenfunction in some lower degree space $H^{p'}(\beta_1, \ldots, \beta_n), \ p' < p$ by multiplication by $r^{2(p-p')}$. On the sphere, therefore, the number of eigenfunctions in $H^p(\beta_1, \ldots, \beta_n)|_{S^{n-1}}$ which do not coincide with one in some lower degree space $H^{p'}(\beta_1, \ldots, \beta_n)|_{S^{n-1}}$ is

$$
\binom{n + p - 1}{p} - \binom{n + p - 2}{p' - 1} = \binom{n + p - 2}{p'}.
$$

(2.23)

For $a \neq 0$, multiplication of the eigenfunctions in $H^{p'}(\beta_1, \ldots, \beta_n)$ by $r^{2(p-p')}$ does not produce eigenfunctions in $H^p(\beta_1, \ldots, \beta_n)|_{S^{n-1}}$, due to the last term in (2.9). Therefore the eigenfunctions in $H^p(\beta_1, \ldots, \beta_n)|_{S^{n-1}}$ must include those of lower degree, giving a total of $\binom{n + p - 1}{p'}$. 

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Hyperspherical Harmonics and Bethe Ansatz

To actually compute the eigenfunctions, we express \( q(\lambda) \) in factorized form

\[
q(\lambda) = \prod_{b=1}^{p} (\lambda - v_b),
\]

and substitute in (2.21). Dividing by \( a(\lambda) \), the resulting rational function vanishes if the singular parts at each of the poles \( \{ \lambda = \alpha_i \}_{i=1,...,n} \) and \( \{ \lambda = v_a \}_{a=1,...,p} \) vanish. This gives the secular equations determining the nodes \( \{ v_a \}_{a=1,...,p} \) as

\[
2 \sum_{c=1 \atop c \neq b}^{p} \frac{1}{v_b - v_c} + \sum_{i=1}^{n} \frac{\beta_i + \frac{1}{2}}{v_b - \alpha_i} + 2a = 0, \quad b = 1, \ldots, p,
\]

and the eigenvalues \( \{ E_i \}_{i=1,...,n} \) as

\[
E_i = \left( \beta_i + \frac{1}{2} \right) \sum_{b=1}^{p} \frac{1}{\alpha_i - v_b} + \frac{1}{4} \sum_{j=1 \atop j \neq i}^{n} \frac{2\beta_i \beta_j + \beta_i + \beta_j}{\alpha_i - \alpha_j} + a \left( \beta_i + \frac{1}{2} \right).
\]

These are the same as the Bethe-Gaudin equations occurring in the Bethe ansatz solutions to the \( \mathfrak{sl}(2) \) spin chain [4,5,10]. However, for the case of the Laplacian \( \Delta \), they differ in interpretation, since the quantities \( \{ l_i = \frac{\beta_i}{2} + \frac{1}{4} \}_{i=1,...,n} \) do not take unique integer values, but rather all possible \( 2^n \) combinations of the values \( \frac{1}{4} \) or \( \frac{3}{4} \). The resulting eigenvectors, within normalization, are precisely of the Bethe ansatz form:

\[
\Psi = \prod_{a=1}^{p} f(v_a) |0; \beta_1, \ldots, \beta_n \rangle,
\]

where

\[
|0; \beta_1, \ldots, \beta_n \rangle := \prod_{i=1}^{n} x_i^{\beta_i}
\]

denotes the highest weight vector in the \( \mathbb{Z}_2 \)-invariant subspace \( \mathcal{H}^p(\beta_1, \ldots, \beta_n) \).

Summing over the eigenvalues of the \( H_i \)'s to obtain that of \( H \) in (1.16) gives

\[
4 \sum_{i=1}^{n} \alpha_i E_i = (2p + \sum_{i=1}^{n} \beta_i)(2p + \sum_{i=1}^{n} \beta_i + n - 2) + \sum_{i=1}^{n} (\beta_i - \beta_i^2) + 4a \left( 2 \sum_{b=1}^{p} v_b + \sum_{i=1}^{n} \alpha_i \left( \beta_i + \frac{1}{2} \right) \right).
\]

For the case when \( a = 0 \) and each \( \beta_i \) is either 0 or 1, this reduces to the usual formula [24] for the eigenvalues \( E_{n,l} \) of the Laplacian on \( S^{n-1} \)

\[
E_{n,l} = -l(l + n - 2),
\]
expressed in terms of the total degree of homogeneity

\[ l := 2p + \sum_{i=1}^{n} \beta_i. \]  

To count the total number of eigenfunctions in this case with a given degree of homogeneity \( l \), let \( \epsilon = 0 \) or \( 1 \), for \( l \) even or odd, respectively, and

\[ m := \min \left( \left\lfloor \frac{n - \epsilon}{2} \right\rfloor, \left\lfloor \frac{l}{2} \right\rfloor \right). \]  

Summing over the number of eigenfunctions for each combination of values of \( p \) and \( \beta_i \)'s giving the same homogeneity \( l \), the total number is

\[ a_{n,l} = \sum_{k=0}^{m} \left( \frac{n + \left\lfloor \frac{l}{2} \right\rfloor - 2 - k}{n - 2} \right) \left( \frac{n}{2k + \epsilon} \right) = \frac{(n + 2l - 2)(n + l - 3)!}{l!(n - 2)!}, \]  

which is the number of homogeneous polynomials of degree \( l \) which may not be obtained by multiplying those of degree \( l - 2 \) by \( r^2 \). This agrees with the usual count for the number of harmonic functions on \( S^{n-1} \) obtained, e.g., through the use of Gel’fand–Tseitlin bases [24].

3. Conclusions and Discussion

We have seen that a basis of harmonic functions on \( S^{n-1} \) may be obtained in terms of homogeneous polynomials that are simultaneous eigenfunctions of the complete set of commuting second order operators \( \{ H_1, \ldots, H_n \} \) provided by the \( \tilde{sl}(2)_R \) loop algebra framework, as well as of the generators of the group \( \mathbf{Z}_2^n \) of reflections in the coordinate planes. This gives a natural generalization of the harmonic functions on \( S^2 \) provided by the Lamé polynomials [11,12]. More generally, the same method determines the joint eigenfunctions of the systems associated to the operator (1.16) for \( \mu_i = \beta_i(\beta_i - 1) \) with \( \beta_i \)'s any nonnegative integers. These eigenfunctions provide bases for the spaces \( \mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_{l_i} \) formed from metaplectic representations of \( sl(2) \) with highest weights \( l_i = \frac{\beta_i}{2} + \frac{1}{4} \). Since these are just the functional Bethe ansatz eigenstates for the \( [sl(2)]^n \) Gaudin spin chain within these representations, their completeness is equivalent to the completeness of the Bethe ansatz for this case.

It may be worthwhile noting that the decomposition of the relevant Hilbert space \( \mathcal{H} \) into the \( 2^n \) irreducible subspaces formed from tensor products of metaplectic representations of the type \( [(\frac{1}{4}) \oplus (\frac{3}{4})]^n \) suggests the presence of a supersymmetric structure involving representations of \( osp(2,1) \) (cf. [25]). Gaudin spin chains based on such superalgebras have been considered recently [26] within the Bethe ansatz approach. Extending the functional formulation to such superalgebras may serve to further clarify the rôle of the \( \mathbf{Z}_2^n \) invariance of the Laplacian and
the associated commuting operators on $S^{n-1}$, as well as to prove the completeness of the Bethe ansatz eigenstates in the supersymmetric case.

The method of solution used here may also be extended to other commuting operators on $S^{n-1}$ arising from the $\tilde{\mathfrak{sl}}(2)_{R}$ loop algebra approach, including, e.g., harmonic oscillator interactions [15,18,27] and modifications of the $-\frac{\mu^2}{x^2}$ interaction terms in (1.17) due to degeneracies in the space of parameters $\{\alpha_1, \ldots, \alpha_n\}$ [3,28]. However, since the homogeneity operator $D$ does not necessarily commute with the operators in question, the joint eigenfunctions may no longer simply be homogeneous polynomials; in fact, they need not be polynomials at all. These questions and further extensions of the loop algebra separation of variables method will be addressed in future work.

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