LUDWIG EDWARD FRAENKEL
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Edward Fraenkel’s professional career began as an experimentalist at the Royal Aircraft Establishment, Farnborough, but his preoccupation with the theoretical and mathematical aspects of aerodynamics led him into academia, working initially in aerodynamics and classical applied mathematics, but later in the modern theory of nonlinear partial differential equations and its applications to fluid mechanics. He made outstanding contributions to the mathematical theories of viscous flow separation, steady vortex rings and surface waves on water.

FAMILY BACKGROUND

Edward Fraenkel was the youngest of five children born to the eminent classicist Eduard Fraenkel (1888–1970) and his wife Ruth (née von Velsen, 1892–1970): Gustav Julius (1919–1998), Edith Renate (1920–2015), Friedrich Andreas (1923–1930), Barbara Helene (1924–1953) and Ludwig Eduard, the subject of this memoir, who was born in Kiel, Germany, on Saturday 28 May 1927.

Edward’s father, Mortier David Eduard Fraenkel, was brought up in Berlin in a highly intellectual upper-middle-class Prussian environment, and, although his parents were married in the Old Synagogue, the family was not religious. In 1906 he entered Berlin University to study law but almost immediately switched to classical philology, and in 1909 moved to Göttingen before returning to Berlin to complete his doctorate in 1912. After a scholarship in Rome, he was appointed to the staff of the Thesaurus Linguae Latinae in Munich. In 1917 he became privatdozent of classics in Berlin, which is where he met Ruth von Velsen when both

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were attending the seminar of the linguist and classical philologist, Wilhelm Schulze (1863–1935). By 1920, aged 32, he was a professor extraordinarius in Berlin, and then full professor successively at the universities of Kiel (1923), Göttingen (1928) and Freiburg (1931).

Edward’s mother Ruth, one of three beautiful sisters, was brought up Lutheran. Her father was a mining engineer who worked in many countries, including England, India, Japan and America, and was honoured for bravery in the Franco–Prussian war of 1870/71, although scarred by the experience. He ended his career as the head of the Ministry of Mines in Berlin, with the title Excellenz, and ennobled to the German equivalent of knighthood.

Ruth, who was vivacious, independent, charming, highly intelligent and popular, had attended finishing school in England and her eldest child Gustav later recalled (Fraenkel 1998) that she was the only member of the family with a decent command of English when they were forced to leave Germany in 1934. She enjoyed skiing, which at that time was a novelty, especially for girls, and horse riding, astride at a time when ladies rode side-saddle. She read modern German poetry and studied classical philology, including Sanskrit, in Berlin (PhD 1917) where she met her future husband Eduard.

Both Eduard and Ruth liked music, and throughout their lives often listened to it together in the evenings. She had had professional, concert-level lessons on a full size Bechstein grand piano given to her by her parents, and Eduard played the violin occasionally, partly as a form of physiotherapy for an injured arm.

Eduard and Ruth were married in a civil ceremony on 2 February 1918. Having abandoned formal religion before they met, they agreed that their family’s general cultural background should be Christian.

**EARLY LIFE**

Gustav recorded (Fraenkel 1998) that his parents named their youngest son, with whom he shared a birthday, in honour of Ludwig Traube (1861–1907), a cousin of Eduard’s paternal grandfather Julius (1865–1919), who was a palaeographer at the Ludwig Maximilian University of Munich and the first person in Germany to hold a chair of Medieval Latin. Consequently, throughout his childhood and occasionally afterwards, Edward Fraenkel was known to his family as Ludwig.

How Ludwig came to be known as Edward is explained below. Henceforth he will be referred to as Edward, not to be confused with Eduard, his father.

With the rise of Hitler and the imposition of anti-Semitic laws in Nazi Germany, Eduard Fraenkel, a non-Aryan although not a member of the Jewish faith, was barred from entering the University of Freiburg in February 1933. In April the philosopher and Nazi sympathizer Martin Heidegger became rector,* and in November Eduard was dismissed, although he immediately received a pension.

With the aid of the Academic Assistance Council,† recently established by the Royal Society to aid academics forced to flee the Nazi regime, he obtained temporary college

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* Philosophical historian Hans Sluga wrote: ‘Though as Rector he prevented students from displaying an anti-Semitic poster at the entrance to the university and from holding a book burning, he kept in close contact with the Nazi student leaders and clearly signalled to them his sympathy with their activism.’

† See https://en.wikipedia.org/wiki/Council_for_At-Risk_Academics (accessed 29/7/20).
positions in Oxford, first at Corpus Christi, then at Christ Church, and left Germany early in 1934. He was then offered the Bevan Fellowship, for five years, at Trinity College, Cambridge, to follow on from his Oxford commitments. So when the family arrived in England later the same year, with two railway-wagon loads of household possessions, they took up residence in Cambridge to await the arrival of Eduard. In December, however, Eduard accepted the Corpus Christi Professorship of the Latin Language and Literature at Oxford, a post he held until his retirement in 1953. In later life Edward (45) quoted his mother Ruth as saying ‘Thanks to Hitler we fell upstairs’.

The family settled in Oxford in 1935 and Edward enrolled as a day pupil at the Dragon School (figure 1), where his peers included the mathematician Sir Peter Swinnerton-Dyer (FRS 1967), two months his junior.

Edward said he found the Dragon School very stimulating, making learning a game to be taken almost as seriously as cricket or rugby, but curiously found mathematics bewildering because as soon as he understood a little, he was promoted up two or three sets without an opportunity to assimilate the material in between. He later quipped that translating Latin into English without knowing either language, and coming first in the examination, was probably
the intellectual peak of his life (45). On the other hand his father deplored the incompetence of masters who had awarded full marks for Latin composition in which he had found mistakes.

From that very early age Edward was acutely aware that his father was possibly the most prominent and respected classical philologist of his generation, whom he would admire and eulogize throughout his long life while at the same time describing him as a dreadful father with whom he became friends only in the late 1950s and early 1960s, when his own career began to blossom (45). By contrast, his mother, whom he thought of as a polymath, helped him illicitly with his Greek homework, with Euclidean Geometry, made his school uniforms and taught him whatever morality and sense of style he ever acquired.

Around this time his parents considered the possibility of Edward sitting an examination for a scholarship to Winchester, an important English public (meaning private) school with a reputation for producing mathematicians, just as a few years earlier his sister Barbara, who was also at the Dragon School, had gone to Roedean, a public school for girls near Brighton.

**WAR INTERVENES**

On 24 August 1939, 10 days before Great Britain declared war on Germany, the family received their naturalization certificates and became British citizens. But nine months later, on 28 May 1940, which was Edward’s thirteenth birthday, the evacuation in small boats of 18 000 men from the beaches at Dunkirk led the nation to prepare for invasion. The possibility of German victory was on everyone’s minds and in Canada academic families were organizing to welcome as ‘war guests’ the children of university teachers in England. After anxious discussions, it was decided to abandon the Winchester plan for Edward and instead to send him (the only family member under 16) alone to Canada, thereby guaranteeing that at least one of them would survive. Arrangements for his evacuation were made, but he developed mumps and could not go; the vessel on which he would have travelled was torpedoed and sunk by U-boats. Once recovered, he sailed from Liverpool to Montreal.

On arrival in Canada, he was one of 30 war guests to end up in Toronto, where he was placed in the care of an Anglican clergyman Gerald Sandes Despard (1882–1962) and his wife Ida, who had three grown-up children. Between 1940 and 1948 Edward saw his family only once, during the summer vacation of 1946.

Although the Despards were extremely kind, Edward found the intellectual environment unsophisticated by comparison with home and he was profoundly homesick, although he did not say so (senior boys at the Dragon were never homesick). All this made him a difficult guest. On the other hand, the Despards were disappointed to find that their war guest who spoke Oxford English was in fact a German boy, embarrassingly called Ludwig. So they wrote to Oxford asking his parents for permission to call him Edward, and in the circumstances the Fraenkels felt they could not object. Since then Ludwig Eduard has been known as Edward, or sometimes even Ed, but his initials remained L.E.

At the University of Toronto Schools, 1940–1943, Edward found the teaching excellent, although his Dragon School preparation meant at the beginning that he was in short trousers while being taught among boys several years older. But he was very content at school and, aged 16, after excelling in Latin, Greek, French, mathematics, physics and chemistry, he was admitted by the University of Toronto to study aeronautical engineering. Although he was interested in aeroplanes, he would have preferred to study mathematics, but had been advised
that if he did so he would have been lucky to get a job as a school master. However, at Toronto he was pleased to discover that the aeronautics syllabus was highly mathematical and shared many options with the mathematics degree, including a course on tensor calculus from Leopold Infeld (1898–1968) (figure 2).

ACADEMIC CAREER

In 1947 Edward became a Bachelor of Applied Science with Honours and in 1948, with a thesis on the design of nozzles of supersonic wind tunnels that could transform subsonic flow to parallel supersonic flow with Mach number about 8, he graduated as a Master of Applied Science in Aeronautical Engineering. On a subsequent visit to Toronto he was pleased to find his design had been implemented and the apparatus was still functioning as planned. He never did a PhD (not uncommon for those whose education had been disrupted by war and National Service) and insisted on the title Mr Fraenkel until he became Professor Fraenkel.

Back in England, for his National Service he was assigned to the Royal Aircraft Establishment, Farnborough, where he spent four happy years (1948–1952) as a scientific officer. There, at a public dance in Farnborough Town Hall in 1949, he met Beryl (Jacqueline Margaret) Currie, a painter whose beauty and quick and versatile intellect he immediately admired (figure 3). He characterized this encounter as one of the great good fortunes of his life. They were married in January 1954 and had two daughters, Anna (1954) and Juliet (1959).

During his first two years at Farnborough he ran a supersonic wind tunnel suitable for testing conical air intakes for the English Electric Lightning, the only UK-designed-and-built fighter aircraft capable of Mach 2 (1, 2, 3, 4). In the second two years he was attached to the theoretical section headed by C. H. E. Warren. There he did what he later described as his
‘first serious mathematics, although not rigorous by my standards today’ on supersonic flows past slender bodies (5).

After four years of directed, yet enjoyable, research at the Royal Aircraft Establishment, Edward felt a need for scientific independence. So, despite an invitation from Austyn Mair to apply for a lectureship in aeronautical engineering in Cambridge, where Mair had recently been appointed professor, Edward accepted a research fellowship at the University of Glasgow for 1952/3. There he worked on the unsteady motion, both sub- and supersonic, of a slender body through a compressible fluid. The resulting publication (6) was awarded the prize for the outstanding paper of the year 1955 by The Aeronautical Quarterly.
In 1953 he accepted a lectureship in the Department of Aeronautical Engineering at Imperial College London. The position was offered by the head of department, the distinguished theoretical fluid mechanist H. B. Squire (FRS 1957), with the specific remit of delivering a graduate course, three lectures per week across the whole year, on compressible flows. This he did, but in retrospect thought he might have been too mathematical. However, he was gratified when the class, which included engineers who later worked on the supersonic airliner Concorde, said it was not, and showed their appreciation with a generous wedding present on his marriage to Beryl.

The Fraenkels spent 1957/8 in California where Edward was on sabbatical leave at Caltech (California Institute of Technology) as a visiting assistant professor. There he became interested in mathematical models of magnetohydrodynamics that might help explain experimental studies of surface waves on conducting liquids in the presence of magnetic fields (8, 9). On his return to Imperial he was promoted to reader, and in 1961 transferred from aeronautical engineering to become a reader in the Department of Mathematics at Imperial College. He returned to Caltech in the summer of 1968 as a senior visitor.

Edward spent the spring of 1964 at the Weizmann Institute in Israel, having been invited by its head of mathematics, Chaim L. Pekeris, with whom he shared an interest in surface waves and the stability of pipe flows. However, the many visits to tourist attractions and high-level receptions that had been arranged made Edward, who only wanted to do research, think this might have been an attempt to recruit him. In any case, not all was lost because during his stay he visited Nima Geffen at Technion, and at the Weizmann Institute he met J. B. Keller (ForMemRS 1986), which led to him spending three months at the end of 1966 at the Courant Institute for Mathematical Sciences in New York.

On his return from Israel, despite lucrative offers from the USA, he accepted an invitation from G. K. Batchelor (FRS 1957) and moved from a readership at Imperial College to a lectureship in the Department of Applied Mathematics and Theoretical Physics in Cambridge.

At Cambridge, for the benefit of students, but also to familiarize himself with functional analytic methodologies and to convince himself of their relevance (without hand-waving, or ‘cheating’ as he called it) to problems from continuum mechanics, Edward volunteered to give a sequence of advanced lecture courses to postgraduate students. Although unpublished, the manuscripts were influential and widely circulated under titles such as Some mathematical aspects of Navier–Stokes equations, 1968; Applications of functional analysis to continuum mechanics, 1969, 1970, 1971, 1972; and Fixed-point theorems and their application, 1972, 1973. The latter course included an analyst’s account of the infinite dimensional topological degree theory due to Leray and Schauder (Leray & Schauder 1934), and bifurcation theory as characterized in another seminal work (Crandall & Rabinowitz 1971). A debt to its treatment of degree theory was acknowledged in the Preface to Degree theory (Lloyd 1978).

To say the least, Edward was something of a perfectionist, and his commitment to scholarship, his students’ as well as his own, meant that he expended enormous amounts of energy and time conscientiously preparing material for courses and student supervisions. However, although he carried out his duties with the utmost diligence, Edward did not enjoy routine tutoring, or marking of homework or examinations. In fact, although he was in Cambridge for 11 years, despite the consequent significant loss of income, he resigned his fellowship at Queens’ College after only four years because tutoring undergraduates was severely impinging on his research time.
Moreover, Edward had become increasingly convinced that Cambridge pure mathematicians considered the theory of nonlinear partial differential equations as a branch of applied mathematics, while it was considered irrelevant by applied mathematicians, who sought a better understanding of physical phenomena. He was not alone: in 1972, T. B. Benjamin (FRS 1966), who shared Edward’s appreciation of rigour and the role of modern abstract methods, left Cambridge to set up an institute for what he called the alliance of practical and analytical insights in nonlinear fluid mechanics (Benjamin 1976) at the University of Essex.

At the same time the Mathematics Committee of the Science Research Council was conscious that the increasing volume of research in partial differential equations world-wide, especially in France, Russia, Scandinavia and the USA, meant that the UK was slipping far behind. So, with strong support from Michael Atiyah (later Sir Michael Atiyah; FRS 1962, PRS 1990–1995), an initiative to encourage theoretical studies of partial differential equations was established at the University of Sussex with Professor David Edmunds at the helm (Edmunds & Bushell 2018). During those years Sussex attracted an impressive list of world-leading experts who gave courses for staff and students. In 1971, and again in 1973/4, Edward was a visiting professorial research fellow at Sussex, in the period 1975–1978 he was a research professor there and in 1978 he became a permanent faculty member.

As part of the partial differential equations initiative, in 1976 David Edmunds and Edward were joint organizers of the sixth London Mathematical Society (LMS) Durham Symposium. Entitled *Partial differential equations*, it was an enormous success in attracting many of the leading specialists from across Europe and the United States: among senior delegates were M. F. Atiyah, H. Brézis, L. Gårding, L. Hörmander, J. J. Duistermatt, J. Leray, L. Nirenberg, P. H. Rabinowitz, J. B. Serrin, I. M. Singer, L. C. Tartar and N. S. Trudinger (FRS 1997), and among the young people were John Ball (later Sir John Ball; FRS 1989), Richard Melrose, Charles Amick and Charles Stuart. It was recalled with a photograph 25 years later, labelled with the names of participants, in the *Notices of the American Mathematical Society* (Bushell & Edmunds 2001; for the complete programme and participant lists, see Durham Symposia et al. 1976).

The significance for UK mathematics of the partial differential equation initiative at Sussex in the late 1960s, and of Edward’s impact on the Analysis group at Sussex when he joined them in the 1970s, was enormous. Although Edward abhorred administration, the role that he and David Edmunds played in organizing the Durham Symposium of 1976 cannot be overestimated (Durham 1976; Edmunds & Bushell 2018); Edmunds was awarded the London Mathematical Society’s Pólya Prize in 1996.

**Scientific interests and methods**

The University of Toronto, where Edward’s undergraduate thesis was on wind-tunnel design in the Department of Aeronautical Engineering, prepared him for his first job at the Royal Aircraft Establishment, Farnborough, 1948–1952. There he started as an experimentalist working on air intakes for jet engines (1, 2, 3, 4) before moving to the theoretical group, which worked on related mathematical issues. As his long career progressed, a quest for mathematical rigour in problems of physical relevance would become an increasingly important priority. Of course, with his background he understood the role played by formal mathematics in physical and engineering sciences, but regarded unspecified assumptions or understated hypotheses as a form of ‘cheating’, which he characterized with mock disdain.
as ‘hand-waving’. On the other hand, he was not interested in abstract mathematics for its own sake and agreed with Hilbert when he wrote ‘He who seeks for method without having a definite problem in mind seeks in the most part in vain’ (Reid 1996, p. 80). Curiously, although he was trained as an engineer, Edward showed no interest in mastering the numerical methods that, with high performance computing, could yield accurate approximations to solutions of nonlinear equations which could not be solved in closed form. In fact he had an idiosyncratic relationship with computing in general.

Around 1990 he obtained a Texas Instruments TI-81, which was then a cutting edge graphing calculator. Having mastered it and found it sufficient for his purposes, he did not waste important research time learning how to use another, and continued to use the TI-81 as his sole investigative computing resource for the rest of his life. When, some 15 years later, his by-then-obsolete TI-81 failed, a colleague found a second-hand replacement on the Internet and he continued laboriously to do important work on it. For example, all the investigative calculations that led him to a constructive theory of Stokes’ extreme wave (40) were done on the TI-81, yet he happily worked with computational scientists when he wanted numerical confirmation of his theories (43, 46). He did not have a home computer or a smart phone, and emails to his university address were printed by helpful staff at the University of Bath, to whom he dictated a response if one was needed.

Of his 31 papers before 1970, only three had a co-author, but from 1970 onward he had several significant collaborators: M. S. Berger* (16, 17, 20, 24); his brilliant student C. J. Amick who died tragically young (23, 27, 29, 30, 31); and, towards the end of his life, a significant collaboration with his near contemporary J. B. McLeod (FRS 1992), which led to two publications (35, 38) and to a copious collection of unpublished material that means significant additional progress had been made on the problem of wedge entry into water.

After McLeod died in 2014, Edward continued to work on the problem and all the unpublished outcome of that late collaboration is now catalogued and housed in the ‘Fraenkel Collection’, University of Bath Archives and Research Collections. Nevertheless, almost three-quarters of all his scientific publications, including a book, were his alone.

* Melvyn Stuart Berger, a student at Yale of Felix Browder (himself a student of Solomon Lefschetz and Witold Hurewicz) had been trained in the then rapidly developing theory of linear and nonlinear partial differential equations, and had become interested in the applicability of abstract methods to nonlinear problems in, for example, geometry and applied mathematics.
Matched asymptotic expansions

By the 1960s, the method of matched asymptotic expansions was being used widely, but sometimes in circumstances that were dubious mathematically and in certain cases the answer was obviously wrong. To confront these issues, which Milton Van Dyke (Van Dyke 1964, p. 221) described as not ‘mere mathematical quibbles’, Edward wrote three papers (12, 13, 14) identifying and illustrating sufficient conditions for correct matching, with examples to show the failure of the method when these conditions are not met. Another paper (16), which has been expanded by others, when combined with modern theory of partial differential equations in Sobolev spaces, leads to a rigorous account of solutions.

Around this time, the rigorous treatment of large amplitude water waves in work by Krasovskii (1960) and Ladyzhenskaya (1961/1963) on the Navier–Stokes equations had revived interest in the application of abstract methods, variational, topological and analytical (Lyusternik & Schnirelmann 1930; Leray 1934; Leray & Schauder 1934), to theoretical questions about the nonlinear equations and systems from hydro- and aerodynamics. Edward knew he had a lot to learn mathematically, and he found an ideal environment to do so during his 1966 visit to the Courant Institute in New York, then, as now, one of the world’s leading centres for nonlinear partial differential equations. There he met Melvyn Berger, who was to become an important collaborator.

In his early years of aeronautical research, Edward had employed methods familiar to applied mathematicians and engineers, while being ever conscious of the need for analytical rigour as exemplified in (12, 13, 14). Later, especially after spending the autumn of 1966 at the Courant Institute, he began to appreciate the possibilities offered by modern nonlinear functional analysis, in particular in the non-perturbative study of partial differential equations that were not amenable to closed form solution or asymptotic analysis. An immediate fruit of this perspective was the joint paper with Berger (16) that applied functional analysis in conjunction with asymptotic expansion to a boundary-value problem for a partial differential equation, followed by another (17) that extended some of its results to the purely abstract setting of Banach spaces.

Vortex rings

The problem of describing the shape of axisymmetric vortex rings (such as smoke rings) has been of great interest to mathematicians since H. Helmholtz (ForMemRS 1860) first described them in his celebrated paper (Helmholtz 1858). The mathematical theory is difficult because the vortex ring, which is the domain of the solution of a partial differential equation, is not prescribed. Instead, it is the most interesting part of the solution. Together, these considerations make the problem intrinsically nonlinear.

The only solutions known in closed form are the spherical vortex discovered by M. J. M. Hill (FRS 1894) for the case when the flux is zero, the vorticity function is a step function and the helicity vanishes—that is, there is no rotation about the axis—(Hill 1894), and a later extension by W. M. Hicks (FRS 1885) to the case of non-vanishing helicity (Hicks 1899), subsequently rediscovered by H. K. Moffatt (FRS 1986) (Moffatt 1969). By contrast, Edward (15) proved the existence of a family of ring-like solutions with vortex-cores of small cross-section. He was subsequently disappointed to discover that his main result had been obtained earlier (Maruhn 1934, 1957) using a similar argument, but he was later able to employ the method as a basis for approximations (18). However, around this time Edward was guiding
his Cambridge postgraduate student John Norbury to prove the existence of vortex rings with near-spherical cross-section (Norbury 1972).

There followed a successful grant application to the Science Research Council (SRC) to bring Berger to Cambridge for four months with the intention of working on ‘vortex rings other than in the small’. The outcome was their paper (20) giving the first proof of existence of solutions corresponding to large vortex rings. More precisely, by means of a constrained minimization argument formulated on a Sobolev space, theirs was a global theory that established the existence of steady vortex rings with prescribed kinetic energy. To accommodate, in a useful functional analytic framework, the essential physical features of the problem,

(a) the flow domain being unbounded (leading to mathematical problems with loss of compactness),

(b) allowing for both continuous and discontinuous vorticity functions,

(c) and the singularity along the axis,

a Sobolev space was carefully constructed, followed by an approximation argument and Steiner symmetrization. The solutions thus obtained were axi-symmetric with zero helicity.

Fraenkel and Berger visited one another, in Sussex (1982) and Amherst (1990), and prepared extensive notes on the existence of solutions with non-zero helicity. When, around 1993, they considered publishing a joint article, Edward was already deeply engaged in writing his book (37) and the project was never completed, although a sketch appeared (33).

Later, other authors had contributed existence theorems for vortex rings based on various formulations, but Edward was conscious that apart from his paper with Amick (31) little was known about the connection (if any) between these various approaches. It was his hope that the uniqueness results they had achieved (29, 31, 33) would mark the beginning of a unified theory.

For Edward, the publication of his paper with Berger in 1974 (20) represented a major departure, from an engineering mathematics to a mathematical physics viewpoint of models of physical phenomena. Until then he had almost exclusively used perturbation methods and asymptotics to describe solutions that were assumed to exist, and implicitly to be unique, where a small perturbation parameter represented deviation from a particularly simple form of the problem in hand. By contrast, in this paper (20) there is no small parameter, and the questions are:

(a) Does the model have any solution in-the-large?

(b) If the answer is yes, how many does it have? and

(c) what are the properties of these solutions and what distinguishes one from the other?

But this was not a sudden change in perspective. In their earlier paper (16) and its subsequent generalization (17), Berger and Fraenkel had shown, by rigorous perturbation methods, that nonlinear elliptic partial differential equations in a certain class have solutions on a domain $\Omega$ for all small values of a parameter $\varepsilon > 0$. Significantly, they went on to show that, as a function of $\varepsilon$, these solutions could be extended continuously, as members of a Sobolev space, to every $\varepsilon \in (0, \sqrt{\Lambda})$ but no further, where $\Lambda$ is the smallest Dirichlet eigenvalue of the Laplacian on $\Omega$. So their first paper (16) is an example of global
continuation of a local branch of solutions using abstract methods that are not perturbative, the first of many instances where Edward employed this approach.

Also around this time, Edward returned to the question of separation of plane viscous flows in channels, which he had begun to study while at Imperial College teaching aeronautics, but this time armed with functional analytic tools and demanding strict rigour in the analysis (19, 21). Shinbrot considered this later work, which he described as the only rigorous justification of Prandtl’s picture (Shinbrot 1977), as vastly important. Later, with Amick (23), he established the existence of steady, plane solutions of the Navier–Stokes equations for flows in more general channels. Significantly, the difficulties (more challenging in two dimensions than in three) of describing the velocity field far upstream and downstream were overcome and the results obtained were more general, but necessarily less explicit, than for more tightly specified channels.

**Sobolev embeddings, rooms and passages**

When $p \geq 1$, $m \in \mathbb{N}$ and $\Omega$ is an open subset of $\mathbb{R}^N$, the Sobolev space $W^m_p(\Omega)$ is the set of scalar-valued functions on $\Omega$ for which the integrals of the $p^{th}$ powers of the first $m$ generalized derivatives are finite or, when $p = \infty$, the first $m$ generalized derivatives are essentially bounded. Sobolev spaces enjoy some obvious embedding properties such as $W^m_p(\Omega) \subset W^n_p(\Omega)$ when $m \leq n$ and, when $\Omega$ has bounded measure, $W^m_p(\Omega) \subset W^q_q(\Omega)$ when $p \geq q$. However, their importance as a tool in the theory of partial differential equations comes from two less obvious facts: namely, that it is sometimes possible to assert that an element of $W^m_p(\Omega)$ can be differentiated $k$ times at every point of $\Omega$ in a classical sense ($k$ depending on $m$ and $p$), or that bounded sets in $W^m_p(\Omega)$ have compact closure in $W^q_q(\Omega)$, where $q$ and $n$ are determined by $p$ and $m$. The proofs of results of the latter types, which are generally referred to as Sobolev embedding theorems, depend crucially on the regularity (a generalization of the notion of smoothness) of the boundary of $\Omega$.

Edward was concerned that some authors were careless (or ‘cheated’) about the boundary regularity needed for the validity of embedding theorems they sought to invoke. So he made a major contribution to Sobolev embedding theory by selecting, from a bewildering variety in the literature, the most important notions of boundary regularity (22), including a new one suggested to him by Louis Nirenberg (see Epilogue). In doing so he was the first to observe, surprisingly because previously they had been considered independent, that boundaries with the ‘segment property’ are of ‘type $C$’, and vice versa. He analysed comprehensively the effect of boundary regularity on embedding theorems and used parameterized families of domains, which he called ‘rooms and passages’, to illustrate how embedding results hold or fail depending on the regularity of the boundary. Rooms and passages revealed that only the most basic embedding theorems hold in full generality. The differences between different types of boundary regularity were illustrated by other families of domains, which he called ‘dragon’s teeth’ and ‘the corrugated plane’.

On embedding issues for spaces of classically differentiable functions (28), he obtained optimal assumptions on the boundary of $\Omega$ for the embedding of $C^1(\overline{\Omega})$ in $C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha \leq 1$, to hold. Here $C^1(\overline{\Omega})$ is the space of scalar-valued functions $u$ on an open set $\Omega$ in $\mathbb{R}^N$ for which $u$ and its first-order partial derivatives have bounded continuous extensions to the closure $\overline{\Omega}$ of $\Omega$ and $C^{0,\alpha}(\overline{\Omega})$ comprises the bounded functions on $\Omega$ for which $|u(x) - u(y)| \leq \text{const.}|x - y|^\alpha$. 
Symmetry and Fraenkel asymmetry

Edward had a long-standing interest in calculating, or estimating, the electrostatic capacity of bodies (25, 34), and in unpublished notes (26), written in Chicago around 1981, he proposed a way of relating the capacity to the asymmetry of a body in \( \mathbb{R}^n \). In the mid 1980s he passed his notes to Hall, Hayman and Weitsman, who attributed to Edward (Hall et al. 1991) the notion of the asymmetry \( \alpha(F) \) of an open set \( F \) of finite \( n \)-dimensional volume. This he had defined as the least proportion of the volume of \( F \) that can remain uncovered by an arbitrarily placed spherical ball of the same volume as \( F \), and they proceeded to study \( \alpha(F) \), referring for motivation to other parts of Edward’s unpublished work. Thereafter, \( \alpha \) has been known to experts as the ‘Fraenkel modulus of asymmetry’, but the term ‘Fraenkel asymmetry’ was introduced by Hall and Hayman (Hall & Hayman 1993, p. 99).

Subsequently, stability results in terms of Fraenkel asymmetry have been obtained for: the isoperimetric inequality (Fusco et al. 2008); Saint-Venant’s inequality (Brasco & De Philippis 2017); Faber–Krahn inequality (Fusco et al. 2009; Brasco et al. 2017); Krahn–Szegő inequality (Brasco & Pratelli 2012); isodiametric inequality (Maggi et al. 2014); the anisotropic Sobolev inequality (Fusco et al. 2009; Brasco et al. 2017); the Plateau Problem (De Phillipis & Maggi 2014), and much more. The Fraenkel modulus of asymmetry was examined in great detail and cited 22 times in Brasco & Pratelli (2012). Edward himself published on it (41), without any mention of the eponymous adjective, of course.

By the mid 1980s Edward had become interested in the relevance to his work of the groundbreaking discovery by Gidas, Ni and Nirenberg (Gidas et al. 1979) that positive solutions of a wide class of nonlinear elliptic differential equations such as

\[
\Delta u(x) + f(u(x)) = 0 \quad \text{and} \quad u(x) > 0, \quad x \in \Omega; \quad u(x) = 0, \quad x \in \text{bdry } \Omega,
\]

where \( \Delta(u)(x) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(x), \quad x \in \Omega \text{ (open) } \subset \mathbb{R}^n, \)

must be symmetrical when the domain \( \Omega \) is symmetrical, provided only that the function \( f: [0, \infty) \to \mathbb{R} \) is locally Lipschitz continuous. For example, when \( \Omega \) is a ball they showed that all positive solutions must be radially symmetric about the centre of the ball. There are no other conditions on \( f \), and even continuity may not be required (36).

With his former student Charles Amick, he observed that every solution of the three-dimensional system satisfied by Hill’s spherical vortex can be obtained from the three dimensional trace of a function of five dimensions that satisfies an autonomous, non-singular equation and must therefore be radially symmetric by the methods of Gidas et al. (1979). From this they concluded (29) that, up to axial translation, all solutions of equations for vortex rings, when the vorticity is a step function and the flux constant is zero, coincide with Hill’s spherical vortex. In their references they announced an analogous result for Norbury’s problem (Norbury 1972), but it appeared under a different title in (31).

When he moved to Bath in 1988, Edward undertook to design and teach a postgraduate course, for the Masters programme on Nonlinear Mathematics, on symmetry and the maximum principle. This course evolved over the years and culminated in his monograph (37). There, in Appendix A, he joked...
There is a good case for believing that God devised
the Newtonian potential on the first day of Creation.

The word ‘devised’ was a compromise, in response to a referee who objected to the word
‘invented’ and suggested the word ‘discovered’ instead.

**Diffusing vortex circle and cat’s eyes**

Edward’s joint work with McLeod (38) considers the initial-value problem for the Navier–
Stokes equations when the initial data are given by a steady solution of Euler’s equations
for an incompressible ideal fluid, with vorticity concentrated tangentially and with constant
strength on a circle. First steps in a study of the subsequent ‘diffusing vortex circle’ are taken.
A similar study is made (42) for the ‘cat’s eyes’ flow in the plane, where the initial vorticity
is concentrated equally on a row of points spaced equally on a straight line.

**Stokes wave of extreme form**

In 1880 the Irish mathematician Sir George Gabriel Stokes (FRS 1851, PRS 1885–90) argued
that immediately before breaking, a planar, symmetric gravity wave of permanent form on
the surface of an ideal liquid should have a stagnation point at its crest, and consequently be
distinguished by having a sharp corner there with included angle $2\pi/3$ (Stokes 1889). Such
waves are nowadays called waves of extreme form.* By 1978 the existence of periodic waves
of extreme form on infinite depth had been established, but the Stokes conjecture, that on
either side of the stagnation point there are separate tangents to the surface, each at $\pi/6$ to the
horizontal, remained open.

However the conjecture has been proved correct and Edward made several contributions,
involving estimates of increasing sophistication, to this challenging problem. First, the Stokes
conjecture was shown to be correct (27) and (Plotnikov 1982), independently, and an exotic
asymptotic series that describes, to arbitrary order, the shape of an extreme wave close to its
crest was justified (30). Then, 20 years later, in a solo *tour de force* he obtained the existence
of an extreme wave for which the Stokes conjecture is automatically satisfied (40). Although
his subtle new proof amounted to showing, by the contraction mapping principle in a suitable
setting, that there is a Stokes extreme wave that is locally unique in a certain class, it is still
to be decided whether there is any sense in which Stokes’ extreme wave can be said to be
globally unique. The latter question is addressed in a paper with P. J. Harwin (43), which
contains three important but highly technical contributions to the theory of extreme waves:

(a) a solution of Nekrasov’s equation for the extreme wave that consists of an explicit
function, plus a rigorously estimated remainder that vanishes at the crest and trough
and has a maximum value less than 0.0026 times the maximum value of the explicit
part;

(b) a transformation of Nekrasov’s equation that has one and only one solution between
upper and lower bounding functions that are moderately far apart (local uniqueness);

* For physical reasons, Stokes referred to them as ‘waves of greatest height’. But because ‘height’ here does not refer
to the vertical distance between crest and trough, waves with the essential feature of a stagnation point at the crest
are better called waves of extreme form.
As a result of this paper, a description of the extreme wave profile in the vicinity of its crest that is more explicit than in any previous work either heuristic or rigorous was obtained by Edward (44) following elaborate investigative calculations on the TI-81 mentioned earlier.

Wedge entry into water

As a direct result of a consultancy at the Armaments Research Development Establishment (ARDE), Fort Halstead, in the summer of 1956, Edward produced a technical report (7) that suggested a new theory for the vertical entry at constant speed of a narrow cone into water. Although aesthetically more satisfying, the report recognized that the outcome was no more accurate than existing, more formal theories.

Forty years on, he would contribute to the rigorous theory of the analogous problem in two dimensions, the vertical entry of a wedge into water. This classical problem, formulated by Wagner (1932) and the subject of many papers since, concerns the motion of water, which initially is at rest in a half space with horizontal free surface, when an infinite symmetric wedge penetrates it, moving vertically downwards with constant speed. Wagner noted that if gravity, viscosity and surface tension are neglected, a similarity transformation eliminates time and yields a steady problem in which the given parameter is the wedge angle $2\pi \alpha$, and a significant feature of the solution is the contact angle $\pi \beta$ (figure 4).

Concerning the contact angle $\pi \beta$, the main question is whether the supremum $\bar{\beta}$ of $\beta$, over the set of solutions having $0 < 2\pi \alpha < \pi$, is $1/4$ or smaller. A definitive answer, that $\bar{\beta} < 1/4$, was obtained (39) with a proof that suggests $1/4 - \bar{\beta}$ is not small relative to the range of $\beta$. This paper introduced an integral equation of boundary-layer type that allows numerical
calculation of the limiting solution as $\alpha \to 0$, and of $\beta_0$, the value of $\beta$ corresponding to $\alpha = 0$. In particular it identifies a boundary layer effect that explains why $\pi \beta$ does not tend to $\pi/2$ as $\alpha \to 0$, as might otherwise have been expected. Numerical calculation with this equation indicates that $\beta_0 = \bar{\beta}$ and that $\beta_0 = 0.100 \pm 0.002$.

Fraenkel and McLeod (35) contains a detailed statement (Theorem 2.1) of their main result on the existence of solutions of the wedge entry problem and an account of the existence of an explicit solution for the limiting case of wedge angle $\pi$, which, they said, is surprisingly simple in greatly transformed variables. For proofs of these results, they cited a joint paper ‘in preparation’, which had not appeared at the time of their deaths. Hence, although the main result was cited again (39, Theorem 3.6), and despite the copious notes in the archives and research collections of the University of Bath Library, the undoubtedly important contribution of Fraenkel and McLeod to the existence theory for Wagner’s problem and related issues is not to be found in the published literature.

**Epilogue**

Edward used to say that there were only two categories of Fellow of the Royal Society: those who were elected for the wrong reason and those who should not have been elected at all. He put Sir Isaac Newton in the first category and, of course, relegated himself to the lower ranks of the second. Although false, this self-assessment was genuinely held (45) because, from his earliest years, knowledge of his father’s distinguished reputation had exposed him to exceptional scholarship that he felt he could not match, and later he felt overshadowed by eminent colleagues in his own field, such as Sir James Lighthill (FRS 1953), whose intellect he hugely admired—but whose taste in problems he did not (Ball 2017)! At times he might have pleaded guilty to the charge of being unduly critical of those who tried to make sense of things in the real world that were too difficult to treat with complete precision, typically describing research on inexact models as ‘squalid’. He worked extraordinarily hard, and in his own words achieved more than others who were just as slow (a self-evaluation which should be taken with a pinch of salt.)

He was an entertaining and quick-witted raconteur who had a wonderful way with words, both spoken and written. His letters, in tiny handwriting (figure 5), were valued for their charm, precision and good sense, while friends remember his highly articulated voice telling, for the entertainment for colleagues at lunchtime, stories about sloppy editors or railing against himself having lived too long and deploring the quality of individuals whose obituaries now appeared in the newspapers, and in general bemoaning the decline of civilisation. But he was very committed to the encouragement of young people, through the UK Mathematics Trust, the mathematical olympiad movement and the Scottish Mathematical Council, and made significant personal donations to support individuals.

Edward was a very principled, straightforward person who did not expect to give offence when expressing an opinion, but no one could expect to get away with shoddy thinking when he was around. His book reviews were thorough and honest: ‘the book is marred by the casual nature of many definitions (hidden in the text, and sometimes violated soon after their appearance); by a host of omitted symbols and small inaccuracies; by lapses into dreadful grammar (other passages show that the authors can write beautifully when they choose to do so) . . . ’ His referee’s reports for learned journals were easily recognizable for their
FIGURE 5. Example of Edward Fraenkel’s tiny handwriting.

FIGURE 6. Edward and Beryl Fraenkel with daughter Anna and grandson Daniel skiing, Tignes, France, ca 2000. (Online version in colour.)
combination of astute assessment, wit and helpful advice: ‘I have not checked the references, nor, it seems, have the authors.’ He had a certain disdain for scientists who attracted publicity, especially self-publicists.

He was meticulous in his dealings with others, in particular by generously citing contributions, whether something said in a casual conversation or a more significant contribution to his own work. He valued honesty highly, and was disappointed at a personal level if he judged it lacking in others. Edward was widely respected in the international mathematical community and had many loyal friends and admirers. He kept a notebook, to which he would refer about personal matters, such as names and ages of children, spouses,
promotions, pastimes, etc., to avoid embarrassing encounters if he hadn’t seen a colleague for a while.

His lectures and seminars were organized with precision and delivered with inimitable verve and style, and beautifully presented with clear writing using chalk on a blackboard. When he gave seminars he sometimes offered cash prizes, which he placed on the desk in front of him, for solutions to questions he raised (bigger prizes for students than for professors). He held that a good teacher had to be a bit stupid in order to understand the difficulties students were facing.

But first and foremost he was the mathematician who showed no interest when approached to consider a vacant vice-chancellorship in a UK university. In 1982, aged 55, he retired early from Sussex in order to have unfettered time for research, which he continued to do for another 35 years, first at Sussex and then, from 1988, at the University of Bath. Even then he remained willing to teach advanced courses.

He would also commit huge amounts of precious time to help others with their research. A striking example was the 55-page calculation (32) carried out following a question posed to him by the distinguished mechanist D. D. Joseph, Regents’ Professor in the Department of Aerospace Engineering and Mechanics, during a six-month visit to the University of Minnesota in 1986. Even lunchtimes with him were transformed by discussions of challenging questions, informed by illustrations written on paper napkins in his tiny, but impeccably
legible, handwriting with an excruciatingly fine-tipped pen or pencil.* And no one was spared: in his celebrated paper on rooms and passages (22) he thanked Louis Nirenberg (Abel Prize 2015), who was visiting Sussex when he was working on it, ‘for being kind enough to play with good grace the role of Wedding-Guest to my Ancient Mariner on the subject of irregular boundaries’.

After a cycling accident in his late seventies, Edward decided to retire completely from teaching, but subsequently accepted an invitation from University College London to give an advanced course on partial differential equations when the lecturer responsible resigned. He travelled from Bath to London and stayed overnight in order to deliver the lectures.

As a young man, Edward smoked Gauloises, a fashionable brand of strong French cigarettes with an immediately recognizable exotic aroma, but switched to Henri Winterman cigarillos, which were considered safer when the dangers of smoking began to emerge, before giving up smoking completely. In spite of this, all his life he kept extremely fit, with long cycle rides up and down steep hills (but not without serious accidents) into his seventies; grass skiing and, as a member of the Ski Club of Great Britain, he skied enthusiastically in the Alps into his eighties (figures 6 and 7). He remained in good health and mathematically active until shortly before his death.

In conversation with Sir John Ball (FRS 1989) (Ball 2017) after his ninetieth birthday (figure 8), Edward said: ‘If God put me in the world for any purpose at all, if there is a God, then it is to do Mathematics. I don’t have to urge myself. After all, mankind hasn’t devised anything else that is so beautiful and still has strict rules of right or wrong.’

**Honours awarded to Edward Fraenkel**

1989 Senior Whitehead Prize of the London Mathematical Society
2004 Membership of the European Academy of Sciences

**Doctoral students of Edward Fraenkel**

- Henry Portnoy, PhD, London, 1957
- Nicholas Sapsford Clarke, PhD, London, 1966
- Peter Mortimer Eagles, PhD, London, 1968
- Jonathan Paul Kingsland Tillet, PhD, Cambridge, 1969
- Andrew Frank Sheer, PhD, Cambridge, 1969
- John Norbury, PhD, Cambridge, 1971
- Grant Keady, PhD, Cambridge, 1973
- Johan Marthinus de Villiers, PhD, Cambridge, 1974
- Charles James Amick, PhD, Cambridge, 1977
- Adebimpe Adebiyi, DPhil, Sussex, 1978

*Edward said he learnt this use of napkins and H6 pencils from the distinguished complex analyst Walter Hayman (FRS 1956) when they were colleagues in Mathematics together at Imperial College after he had moved from Aeronautical Engineering.*
Figure 9. Edward and Beryl Fraenkel during a conference visit, Paris, France 1980/81. (Online version in colour.)

POSTDOCTORAL ASSISTANTS OF EDWARD FRAENKEL

• Dr Petra J. Harwin
• Dr Mark D. Preston

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We are very grateful to Mrs Beryl Fraenkel (figure 9) for her support, especially for her insight into family matters before the Second World War and for her gift to the University of Bath of the extensive papers left by Edward Fraenkel. This memoir would not have been possible without the enthusiastic commitment of archivists Lizzie Richmond and Adrian Nardone, at the University of Bath, to the huge task of cataloguing and curating those papers. Thanks also to Professor Michiel van den Berg, University of Bristol, for mathematical advice, especially on the matter of Fraenkel asymmetry, to Professor David Edmunds for advice about Edward’s time at Sussex and to Daniel Fraenkel for providing the photographs, including the frontispiece photograph.

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