Central factorials under the Kontorovich–Lebedev transform of polynomials

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In this paper, we show that slight modifications of the Kontorovich–Lebedev (KL) transform lead to an automorphism of the vector space of polynomials. This circumstance along with the Mellin transformation property of the modified Bessel functions perform the passage of monomials to central factorial polynomials. A special attention is driven to the polynomial sequences whose KL transform is the canonical sequence, which will be fully characterized. Finally, new identities between the central factorials and the Euler polynomials are found.

Keywords: central factorials; Kontorovich–Lebedev transform; modified Bessel function; Fourier transform; Laguerre polynomials; Hermite polynomials; Euler polynomials; Bernoulli numbers; Euler numbers; Genocchi numbers; Stirling numbers; combinatorial identities

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1. Introduction and preliminary results

Throughout the text, \( \mathbb{N} \) will denote the set of all positive integers, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), whereas \( \mathbb{R} \) and \( \mathbb{C} \) the field of the real and complex numbers, respectively. The notation \( \mathbb{R}^+ \) corresponds to the set of all positive real numbers. The present investigation is primarily targeted at the analysis of sequences of polynomials whose degrees equal its order, which will be shortly called as PS. Whenever the leading coefficient of each of its polynomials equals 1, the PS is said to be an MPS (monic polynomial sequence). A PS or an MPS forms a basis of the vector space of polynomials with coefficients in \( \mathbb{C} \), here denoted as \( \mathcal{P} \). The convention \( \prod_{\sigma=0}^{-1} := 1 \) is assumed. Further notations are introduced as needed.

We show that slight modifications on the Kontorovich–Lebedev (KL) transform, introduced in [15], permit to transform the canonical polynomial sequence \( \{x^n\}_{n \geq 0} \) into the so-called central factorials of even or odd order [28]:

\[
\left( x - \frac{n}{2} + \frac{1}{2} \right)_n = \begin{cases} 
-1^k (1 - x)_k (1 + x)_k, & \text{if } n = 2k, \\
(-1)^k \left( \frac{1}{2} - x \right)_k \left( \frac{1}{2} + x \right)_k, & \text{if } n = 2k + 1,
\end{cases}
\]

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where the \((x)_n\) represents the Pochhammer symbol: \((x)_n \coloneqq \prod_{\sigma=0}^{n-1} (x + \sigma)\) when \(n \geq 1\) and \((x)_0 = 1\). Indeed, the set \((x - n/2 + \frac{1}{2})_{n \geq 0}\) is an Appell sequence with respect to the central difference operator \(\delta\), defined by \((\delta f)(x) = f(x + \frac{1}{2}) - f(x - \frac{1}{2})\), for any \(f \in \mathcal{P}\) \([5,28]\), since \(\delta(x - n/2 - \frac{1}{2})_{n+1} = (n + 1)(x - n/2 + \frac{1}{2})_n\).

Precisely, we define the two following modifications of the KL transform, which figure out to be our main tools \([16,30,35,36]\):

\[
\begin{align*}
\text{KL}_\nu[f](\tau) &= \frac{2 \sinh(\pi \sqrt{\tau})}{\pi \sqrt{\tau}} \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})f(x) \, dx, \\
\text{KL}_c[f](\tau) &= \frac{2 \cosh(\pi \sqrt{\tau})}{\pi} \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})f(x) \frac{dx}{\sqrt{x}},
\end{align*}
\]

where \(K_\nu(z)\) represents the modified Bessel function (also known as the MacDonald function) \([11, \text{Vol. II}; 17]\). The reciprocal inversion formulas are, respectively,

\[
\begin{align*}
xf(x) &= \frac{2}{\pi} \lim_{\lambda \to \pi^-} \int_0^\infty \sqrt{\tau} \cosh(\lambda \sqrt{\tau})K_{2i\sqrt{\tau}}(2\sqrt{x})\text{KL}_\nu[f](\tau) \, d\tau, \\
\sqrt{x}f(x) &= \frac{2}{\pi} \lim_{\lambda \to \pi^-} \int_0^\infty \sinh(\lambda \sqrt{\tau})K_{2i\sqrt{\tau}}(2\sqrt{x})\text{KL}_c[f](\tau) \, d\tau.
\end{align*}
\]

The formulas (1)–(4) are valid for any continuous function \(f \in L_1(\mathbb{R}_+, K_0(2\mu \sqrt{x}) \, dx), 0 < \mu < 1\), in a neighbourhood of each \(x \in \mathbb{R}_+\) where \(f(x)\) has bounded variation \([41, \text{Theorem 6.3}]\). Properties of KL transforms in \(L_p\)-spaces can be found in \([38,40]\).

The kernel of such transformation is the modified Bessel function (also called the MacDonald function) \(K_{2i\sqrt{\tau}}(2\sqrt{x})\) of purely imaginary index, which is real-valued and can be defined by the integral of Fourier type

\[
K_{2i\sqrt{\tau}}(2\sqrt{x}) = \int_0^\infty e^{-2\sqrt{\tau}\cosh(u)} \cos(2\sqrt{\tau} u) \, du, \quad x \in \mathbb{R}_+, \quad \tau \in \mathbb{R}_+.
\]

Moreover, it is an eigenfunction of the operator

\[
A = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - x = x \frac{d}{dx} \sqrt{x} \frac{d}{dx} - x
\]

insofar as

\[
AK_{2i\sqrt{\tau}}(2\sqrt{x}) = -\tau K_{2i\sqrt{\tau}}(2\sqrt{x}).
\]

Besides, \(K_\nu(2\sqrt{x})\) reveals the asymptotic behaviour with respect to \(x\) \([11, \text{Vol. II}; 36]\)

\[
\begin{align*}
K_\nu(2\sqrt{x}) &= \frac{\sqrt{\pi}}{2x^{1/4}}e^{-2\sqrt{x}} \left[ 1 + O \left( \frac{1}{\sqrt{x}} \right) \right], \quad x \to +\infty, \\
K_\nu(2\sqrt{x}) &= O(x^{-3/2}), \quad K_0(2\sqrt{x}) = O(\log x), \quad x \to 0.
\end{align*}
\]
From the comparison between (1) and (2), readily comes out the identity

\[
\text{KL}_s[f](\tau) = \frac{\tanh(\pi \sqrt{\tau})}{\sqrt{\tau}} \text{KL}_c[\sqrt{\tau}f(\tau)](\tau).
\]

The KL transform of the canonical sequence \(\{x^n\}_{n \geq 0}\) is also an MPS whose elements are the central factorials. Indeed, while evaluating the Mellin transform of the function \(K_{\nu}(2\sqrt{x})\) at positive integer values (i.e. the moments of this function), the output is a product of an elementary function by a polynomial whose degree is exactly the order of the moment increased by one unity. To be more specific, for positive real values of \(\tau\), we recall relation (2.16.2.2) in [27] to obtain

\[
\text{KL}_s[x^n](\tau) = \frac{2 \sinh(\pi \sqrt{\tau})}{\pi \sqrt{\tau}} \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})x^n \, dx
\]

\[
= \prod_{\sigma=1}^n (\sigma^2 + \tau) = (1 - i\sqrt{\tau})_n(1 + i\sqrt{\tau})_n, \quad n \in \mathbb{N}_0, \quad (10)
\]

\[
\text{KL}_c[x^n](\tau) = \frac{2 \cosh(\pi \sqrt{\tau})}{\pi} \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})x^{n-1/2} \, dx = \prod_{\sigma=0}^{n-1} \left( \left( \frac{1}{2} + \sigma \right)^2 + \tau \right)
\]

\[
= \left( \frac{1}{2} - i\sqrt{\tau} \right)_n \left( \frac{1}{2} + i\sqrt{\tau} \right)_n, \quad (11)
\]

for \(n \in \mathbb{N}_0\), while as \(\tau \to 0\), the moments become much more simpler

\[
\text{KL}_c[x^n](0) = (n!)^2, \quad n \in \mathbb{N}_0. \quad (12)
\]

As a matter of fact, from (10) and (11), we observe that the outcome of the KL\(_s\) and KL\(_c\) transforms of the canonical MPS \(\{x^n\}_{n \geq 0}\) is two other bases of \(\mathcal{P}\) formed by the so-called central factorial polynomials (or shortly, the central factorials) of even order \(\{(1 - i\sqrt{\tau})_n(1 + i\sqrt{\tau})_n\}_{n \geq 0}\) and those of odd order \(\{(\frac{1}{2} - i\sqrt{\tau})_n(\frac{1}{2} + i\sqrt{\tau})_n\}_{n \geq 0}\). Therefore, as explained in Section 2, the two transforms KL\(_s\) and KL\(_c\) actually behave as an isomorphic operator representing the passage between the monomials and central factorials. These latter were treated in the book by Riordan [28, pp. 212–217, 233–236] while analysing the central difference operator \(\delta\). The connection coefficients between \(\{x^n\}_{n \geq 0}\) and each one of the bases \(\{(1 - i\sqrt{\tau})_n(1 + i\sqrt{\tau})_n\}_{n \geq 0}\) and \(\{(\frac{1}{2} - i\sqrt{\tau})_n(\frac{1}{2} + i\sqrt{\tau})_n\}_{n \geq 0}\) correspond to the central factorial numbers of even and odd orders, respectively. They have appeared in other contexts from the approximation theory [3,4] to spectral theory of differential operators [12,19].

Entailed in this framework, we bring to light two MPSs, \(\{P_n\}_{n \geq 0}\) and \(\{\tilde{P}_n\}_{n \geq 0}\), having the canonical sequence as the corresponding KL\(_s\) and KL\(_c\) transforms. The characterization of these MPS is the main goal of the present work and it will be unravelled throughout Section 3, where the central factorial numbers are an asset. But the analytical properties of the KL\(_{s,c}\) transform interlaced with algebraic properties of the polynomial sequences are indeed the fabric of almost all the developments hereby made. Thus, after obtaining generating functions for \(\{P_n\}_{n \geq 0}\) and \(\{\tilde{P}_n\}_{n \geq 0}\) in Section 3.1, we focus, in Section 3.3, on the integral and algebraic relations between the aforementioned polynomial sequences and the Euler polynomials. Precisely, such connection is indeed the key ingredient not only to recognize the connection coefficients between \(\{P_n\}_{n \geq 0}\) and \(\{\tilde{P}_n\}_{n \geq 0}\), but also to grasp the recurrence relation fulfilled by each of these sequences.

In Section 3.5, we analyse the behaviour of the corresponding dual sequences. Generally speaking, the dual sequence \(\{v_n\}_{n \geq 0}\) of a given MPS \(\{Q_n\}_{n \geq 0}\) belongs to the dual space \(\mathcal{P}'\) of \(\mathcal{P}\) and its elements are uniquely defined by \(\langle v_n, Q_k \rangle := \delta_{n,k}, \quad n, k \geq 0\), where \(\delta_{n,k}\) represents the Kronecker delta function. Its first element, \(u_0\), earns the special name of canonical form of the MPS. Here, by
Whenever there is a form $v \in \mathcal{P}$ over $f \in \mathcal{P}$, but a special notation is given to the action over the elements of the canonical sequence $\{x^n\}_{n \geq 0}$ – the moments of $u \in \mathcal{P}$: $(u)_n := \langle u, x^n \rangle$, $n \geq 0$. Whenever there is a form $v \in \mathcal{P}$ such that $(v, Q_n Q_m) = k_n \delta_{n,m}$ with $k_n \neq 0$ for all $n, m \in \mathbb{N}_0$ [23,24], the PS $\{Q_n\}_{n \geq 0}$ is then said to be orthogonal with respect to $v$ and we can assume the system (of orthogonal polynomials) to be monic and the original form $v$ is proportional to $v_0$. This unique monic orthogonal polynomial sequence (MOPS) $\{Q_n(x)\}_{n \geq 0}$ with respect to the regular form $v_0$ can be characterized by the popular second-order recurrence relation

$$
Q_0(x) = 1, \quad Q_1(x) = x - \beta_0, \\
Q_{n+2}(x) = (x - \beta_{n+1})Q_{n+1}(x) - \gamma_{n+1}Q_n(x), \quad n \in \mathbb{N}_0,
$$

(13)

where $\beta_n = \langle v_0, x Q_n^2 \rangle / \langle v_0, Q_n^2 \rangle$ and $\gamma_{n+1} = \langle v_0, Q_{n+1}^2 \rangle / \langle v_0, Q_n^2 \rangle$ for all $n \in \mathbb{N}_0$.

Although the attained recursive relations for $\{P_n\}_{n \geq 0}$ and $\{\tilde{P}_n\}_{n \geq 0}$ reject their (regular) orthogonality (i.e. with respect to an $L_2$-inner product) of the two MPSSs, because they do not fulfil a second-order recursive relation of the type (13), their corresponding canonical forms $u_0$ and $\tilde{u}_0$ are positive definite, respectively, associated with the weight functions $K_0(2\sqrt{x})$ and $(1/\sqrt{x})K_0(2\sqrt{x})$. Therefore, the existence of two MOPSSs $\{Q_n\}_{n \geq 0}$ and $\{\tilde{Q}_n\}_{n \geq 0}$ with respect to $u_0$ and $\tilde{u}_0$, respectively, is ensured. The problem of characterizing the first one posed by Prudnikov [32] is still open.

As a consequence, new identities involving the Genocchi numbers, the central factorial, the Euler numbers and polynomials will be singled out, bringing an humble contribution to those already known [1,5,8,18,28,31,42], not disregarding [29]. Finally, we outline some problems to be treated in a forthcoming work.

2. The KL transform of a polynomial sequence

The importance of the central factorials in this work justifies a preliminary revision of their foremost properties, before splitting the analysis into the even and odd order cases. Thus, the set of central factorial polynomials $\{(x-n/2+\frac{1}{2})_n\}_{n \geq 0}$ is an MPS (ergo, form a basis of $\mathcal{P}$) bridged to the canonical MPS $\{x^n\}_{n \geq 0}$ via the central factorial numbers of first and second kind $\{(t(n,v), T(n,v))\}_{0 \leq v \leq n}$ [28, pp. 212–217]. The central factorial numbers of first kind $t(n,v)$ fulfil the triangular relation

$$
t(n,v) = t(n-2, v-2) - \frac{1}{4}(n-2)^2 t(n-2, v), \quad 0 \leq v \leq n, \\
t(n,0) = t(0, n) = \delta_{n,0}, \quad n \geq 0,
$$

(14)

whereas those of second kind $T(n,v)$ satisfy

$$
T(n,v) = T(n-2, v-2) - \frac{1}{4}v^2 T(n-2, v), \quad 0 \leq v \leq n, \\
T(0,n) = T(0,n) = \delta_{n,0}, \quad T(n,n) = 1, \quad n \geq 0,
$$

(15)

but whenever $v \geq n + 1$ or $(-1)^n + (-1)^v = 0$, necessarily, $t(n,v) = T(n,v) = 0$, impelling to split the analysis into the cases of even or odd order. Following up the idea, we adopt the notation

$$
t_E(n,v) := t(2n,2v), \quad T_E(n,v) := T(2n,2v), \quad t_O(n,v) := t(2n + 1,2v + 1), \\
T_O(n,v) := T(2n + 1,2v + 1), \quad 0 \leq v \leq n.
$$
Thus, according to the aforementioned properties, we have
\[
\left\{ \frac{(1-i\sqrt{\tau})^n(1+i\sqrt{\tau})^n}{2-i\sqrt{\tau}} \right\} = \sum_{i=0}^{n} k_{i}\left\{ \frac{i^{n+v} + v^{n}}{n!} \right\} \tau^v
\]
and, reciprocally,
\[
\tau^n = \sum_{i=0}^{n} k_{i}\left\{ \frac{i^{n+v} + v^{n}}{n!} \right\} \tau^v, \quad n \in \mathbb{N}_0.
\]

Closed-form expressions for the central factorial numbers of second kind (even or odd) can be read from [1, p. 824; 28, pp. 214, 216] (we also refer to [29, A008955, A008956])
\[
T(n, v) = \frac{1}{v!} \sum_{\mu=0}^{v} \binom{v}{\mu} (-1)^\mu \left( \frac{v}{2} - \mu \right)^n; \quad T(n, v)
\]
\[
= \sum_{\mu=0}^{\nu-v} \binom{n}{\mu} S(n-\mu, v) \left( \frac{v}{2} \right)^\mu, \quad 0 \leq v \leq n, n, v \in \mathbb{N}_0.
\]

Apparently, the even-order case appears more frequently than the odd-order one. For instance, in [12,19] while dealing with spectral properties of even-order differential operators having the classical orthogonal polynomials as eigenfunctions, the central factorial numbers appear as particular cases of the so-called \textit{z-modified Stirling numbers} [19] or Jacobi–Stirling numbers [12]. Notwithstanding they are just a particular case of the \textit{z-modified Stirling numbers}, the central factorial numbers of even-order interfere in the expression of the expansion of the \textit{z-modified Stirling numbers} in terms of powers of \textit{z} [13]. They have indeed received combinatorial interpretations from different aspects [2,13,14,26].

**Proposition 2.1** The KL\(_t\) and KL\(_c\) transforms are both an automorphism of the vector space of polynomials defined on \(\mathbb{R}_+\).

**Proof** Since \(\mathcal{P} := \{p(x) \in L_1(\mathbb{R}, K_0(2\mu \sqrt{|x|}) \, dx) : 0 < \mu < 1\} \) is a vector space of polynomials defined on \(\mathbb{R}_+\), then, on account of (11), the linearity and the injectivity of the KL\(_t\) transforms ensure the result. 

**Remark 2.2** Both of the modified KL transforms under analysis can be extended to any continuous function \(f \in L_1(\mathbb{R}, K_0(2\mu \sqrt{|x|}) \, dx)\), \(0 < \mu < 1\), in a neighbourhood of each \(x \in \mathbb{R}\) where \(f(x)\) has bounded variation in the following manner [41]:
\[
\text{KL}_t(f(x))(\tau) = \frac{\sinh(\pi \sqrt{|\tau|})}{\pi \sqrt{|\tau|}} \int_{-\infty}^{+\infty} K_{2i\sqrt{\tau}}(2\sqrt{|x|})f(|x|) \, dx,
\]
\[
\text{KL}_c(f(x))(\tau) = \frac{\cosh(\pi \sqrt{|\tau|})}{\pi} \int_{-\infty}^{+\infty} K_{2i\sqrt{\tau}}(2\sqrt{|x|})f(|x|) \frac{dx}{\sqrt{|x|}}.
\]
Consequently, the latter result can be extended to the vector space of polynomials \(\mathcal{P}\) without any further restrictions over the domain.
Lemma 2.3 For any MPS \( \{B_n\}_{n \geq 0} \) and any two integers \( k, m \in \mathbb{N}_0 \), we have

\[
\text{KL}_s \left[ \frac{1}{x} A^m x^{k+1} B_n(x) \right](\tau) = (-1)^m \tau^m \text{KL}_s \left[ x^k B_n \right](\tau), \quad n \in \mathbb{N}_0, \tag{18}
\]

\[
\text{KL}_c \left[ \frac{1}{\sqrt{x}} A^m x^{k+1/2} B_n(x) \right](\tau) = (-1)^m \tau^m \text{KL}_c \left[ x^k B_n \right](\tau), \quad n \in \mathbb{N}_0, \tag{19}
\]

and

\[
A^m x^{k+1} B_n(x) = \frac{2(-1)^m}{\pi} \lim_{\lambda \to \pi} \int_0^\infty \cosh(\lambda \sqrt{\tau}) K_{2i \sqrt{\tau}}(2 \sqrt{x}) \tau^{m+1/2} \text{KL}_s \left[ x^k B_n \right](\tau) \, d\tau, \tag{20}
\]

\[
A^m x^{k+1/2} B_n(x) = \frac{2(-1)^m}{\pi} \lim_{\lambda \to \pi} \int_0^\infty \sinh(\lambda \sqrt{\tau}) K_{2i \sqrt{\tau}}(2 \sqrt{x}) \tau^m \text{KL}_c \left[ x^k B_n \right](\tau) \, d\tau, \tag{21}
\]

where \( A \) represents the operator in (6) acting over the variable \( x \).

Proof Following a similar procedure of the one taken in [37], we come out with

\[
\int_0^\infty \psi(x) (A \phi(x)) \frac{dx}{x} = \int_0^\infty (A \psi(x)) \varphi(x) \frac{dx}{x}, \tag{22}
\]

whenever \( \phi, \psi \in C^2_0(\mathbb{R}_+) \) vanishing at \( \infty \) and near the origin together with their derivatives, in order to eliminate the outer terms. Thus, due to (7), we can successively write

\[
\int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) (A^m x^{k+1} B_n(x)) \frac{dx}{x} = (-1)^m \tau^m \int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) x^k B_n(x) \, dx,
\]

which provides (18). Likewise, considering that

\[
\int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) \left( \frac{1}{\sqrt{x}} (A^m x^{k+1/2} B_n(x)) \right) \frac{dx}{\sqrt{x}} = (-1)^m \tau^m \int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) x^k B_n(x) \, dx,
\]

we obtain (19). Finally, on the grounds of (1)–(3), (20) is a mere consequence of (18), just like (21) is a consequence of (19) within the framework of (2)–(4).

3. The MPS whose KL transform is the canonical MPS

The relations (10)–(12) have the straightforward, yet important, consequence that the KL transform of any MPS is again another MPS. This legitimates the question of seeking two MPSs \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \) whose KL\(_s\) and KL\(_c\) transforms correspond, respectively, to the canonical sequence \( \{\tau^n\}_{n \geq 0} \).

Proposition 3.1 The two polynomial sequences \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \) such that

\[
\text{KL}_s[P_n(x)](\tau) = \frac{2}{\pi \sqrt{\tau}} \sinh(\pi \sqrt{\tau}) \int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) P_n(x) \, dx = \tau^n, \quad n \in \mathbb{N}_0, \tag{23}
\]

and

\[
\text{KL}_c[P_n(x)](\tau) = \frac{2}{\pi \sqrt{\tau}} \cosh(\pi \sqrt{\tau}) \int_0^\infty K_{2i \sqrt{\tau}}(2 \sqrt{x}) P_n(x) \, dx = \tau^n, \quad n \in \mathbb{N}_0, \tag{24}
\]
The fact that KL's Lemma 2.3 ensures that whereas \( \tilde{c}_n \) are also accomplished, since (15) for even and odd values of \( n, k \) for \( n, k \in \mathbb{N}_0 \), under the convention \( c_{n,-1} = c_{nk} = 0 = \tilde{c}_{n,-1} = \tilde{c}_{nk} \) whenever \( k > n \). But the remaining values for \( n \in \mathbb{N} \) are also accomplished, since (15) for even and odd values of
\( n \) permits to successively write

\[
P_{n+1}(x) = \sum_{k=0}^{n} (-1)^{n+k} T_E(n+1, k+1) \left( x - \frac{d}{dx} x - \frac{d}{dx} x \right) x^k
\]

\[
= \left( x - \frac{d}{dx} x - \frac{d}{dx} x \right) P_n(x) = \frac{-1}{x} A x P_n(x), \quad n \in \mathbb{N}_0.
\]

\[
\tilde{P}_{n+1}(x) = - \sum_{k=0}^{n} (-1)^{n+k} T_O(n, k) \left( x - \sqrt{x} \frac{d}{dx} x - \sqrt{x} \right) x^k = \frac{-1}{\sqrt{x}} A x \tilde{P}_n(x), \quad n \in \mathbb{N}_0.
\]

By a finite induction process, it is straightforward to prove that this latter implies (25). Finally, by taking into account the linearity of the KL transform, (10) and (11) along with (17), we conclude that (24) is just a consequence of (29) and (30).

Meanwhile, the inverse relations of (29) and (30), that is, the expression of \( \{x^n\}_{n \geq 0} \) by means of \( \{P_n\}_{n \geq 0} \) or \( \{\tilde{P}_n\}_{n \geq 0} \), can be achieved directly from the properties of the central factorial numbers \( (t_E(n, n), T_E(n, n)) \) or \( (t_O(n, n), T_O(n, n)) \), respectively. Thus, it follows:

\[
x^n = \sum_{k=0}^{n} (-1)^{n+k} t_E(n+1, k+1) P_k(x), \quad n \in \mathbb{N}_0, \quad (31)
\]

\[
x^n = \sum_{k=0}^{n} (-1)^{n+k} t_O(n, k) \tilde{P}_k(x), \quad n \in \mathbb{N}_0. \quad (32)
\]

In the sequel of [39], the authors studied in [21] the polynomial sequence \( \{(\pm)^n e^{\pm x} A^\alpha e^{-\pm x} \}_{n \geq 0} \), where \( A - x^2 = -A - x \). The connection coefficients between this latter and the canonical sequence were essentially the set of decentralized central factorials of parameter \( \alpha \), where the choice of \( \alpha = 0 \) would give the central factorial numbers of even order. However, the hard calculus involved, especially from the analytical point of view, impelled a characterization farther less extensive as the one here taken for \( \{P_n\}_{n \geq 0} \) or \( \{\tilde{P}_n\}_{n \geq 0} \). The characterization of all the polynomial sequences generated by integral composite powers of first-order differential operators with polynomial coefficients acting on suitable analytical functions was taken in [20].

### 3.1. The generating function

The elements of \( \{P_n\}_{n \geq 0} \) or \( \{\tilde{P}_n\}_{n \geq 0} \) allow integral representations triggered by the inverse of the corresponding KL transform. While the KL\(_d\) transform of \( \{P_n\}_{n \geq 0} \) (23) has the inverse

\[
x P_n(x) = \frac{2}{\pi} \lim_{\lambda \to -\pi} \int_{0}^{\infty} \tau^{n+1/2} \cosh(\lambda \sqrt{\tau}) K_{2i\sqrt{\tau}}(2i \sqrt{\tau}) d\tau, \quad n \in \mathbb{N}_0, \quad (33)
\]

the reciprocal formula of KL\(_d\) transform of \( \{\tilde{P}_n\}_{n \geq 0} \) (24) is

\[
\sqrt{x} \tilde{P}_n(x) = \frac{2}{\pi} \lim_{\lambda \to -\pi} \int_{0}^{\infty} \tau^n \sinh(\lambda \sqrt{\tau}) K_{2i\sqrt{\tau}}(2i \sqrt{\tau}) d\tau, \quad n \in \mathbb{N}_0. \quad (34)
\]

These integral representations (33) and (34) are actually a key ingredient either to show that \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \) are the coefficients of the Taylor expansion of a certain elementary function, or to obtain their relation to the widely known Euler polynomials. This latter is of major importance.
for the achievement of the structural-recursive relations of \(\{P_n\}_{n \geq 0}\) and \(\{\tilde{P}_n\}_{n \geq 0}\), as well as for the determination of their interim connection coefficients.

**Lemma 3.2** The two polynomial sequences \(\{P_n\}_{n \geq 0}\) and \(\{\tilde{P}_n\}_{n \geq 0}\) defined in (25) and (26) can also be represented by

\[
xP_n(x) = \lim_{\lambda \to \pi} \frac{\partial^{2n+2}}{\partial \lambda^{2n+2}} e^{-2\sqrt{\gamma} \cos(\lambda/2)} \quad \text{and} \quad \sqrt{x}\tilde{P}_n(x) = \lim_{\lambda \to \pi} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} e^{-2\sqrt{\gamma} \cos(\lambda/2)}, \quad n \in \mathbb{N}_0.
\]

**Proof** The relations (33) and (34) can be restyled, respectively, to

\[
xP_n(x) = \lim_{\lambda \to \pi} \frac{1}{2^{2n+1}} \int_0^\infty \tau^{2n+2} \cosh(\lambda \tau) K_{ir}(2\sqrt{\tau}) \, d\tau, \quad n \in \mathbb{N}_0,
\]

\[
\sqrt{x}\tilde{P}_n(x) = \lim_{\lambda \to \pi} \frac{1}{2^{2n}} \int_0^\infty \tau^{2n+1} \sinh(\lambda \tau) K_{ir}(2\sqrt{\tau}) \, d\tau, \quad n \in \mathbb{N}_0.
\]

Considering that

\[
\cosh(\lambda \tau) \left(\frac{\tau}{2}\right)^{2n+2} = 2^{-(2n+2)} \frac{\partial^{2n+2}}{\partial \lambda^{2n+2}} \cosh(\lambda \tau) \quad \text{and} \quad \sinh(\lambda \tau) \left(\frac{\tau}{2}\right)^{2n+1} = 2^{-(2n+1)} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} \cosh(\lambda \tau),
\]

it is reasonable to rewrite (36) and (37) as follows:

\[
xP_n(x) = 2^{-(2n+2)} \lim_{\lambda \to \pi} \frac{1}{2^{2n+1}} \int_0^\infty \tau^{2n+2} \frac{1}{\pi} K_{ir}(2\sqrt{\tau}) \cosh(\lambda \tau) \, d\tau,
\]

\[
\sqrt{x}\tilde{P}_n(x) = 2^{-(2n+1)} \lim_{\lambda \to \pi} \frac{1}{2^{2n}} \int_0^\infty \tau^{2n+1} \frac{1}{\pi} K_{ir}(2\sqrt{\tau}) \cosh(\lambda \tau) \, d\tau,
\]

motivated by the absolute and uniform convergence by \(\lambda \in [0, \pi/2 - \epsilon]\), for a small positive \(\epsilon\). Meanwhile, from (5) and the inversion formula of the cosine Fourier transform, we deduce [36]

\[
\frac{2}{\pi} \int_0^\infty \cosh(\lambda \tau) K_{ir}(2\sqrt{\tau}) \, d\tau = e^{-2\sqrt{\gamma} \cos(\lambda/2)}, \quad x > 0,
\]

which completes the proof. \(\blacksquare\)

**Remark 3.3** By taking \(n = 0\) in (18) and on account of (10), (16), and (33), we deduce

\[
\frac{(-1)^m}{\sqrt{x}} A^m x^{k+1} = \sum_{\nu = 0}^{k} (-1)^{k+\nu} t_E(k+1, \nu+1) P_{m+\nu}(x), \quad m, k \in \mathbb{N}_0.
\]

Similarly, upon the choice of \(n = 0\) in (19) and considering (11), (16), and (34), we conclude

\[
\frac{(-1)^m}{\sqrt{x}} A^m x^{k+1/2} = \sum_{\nu = 0}^{k} (-1)^{k+\nu} t_O(k, \nu) \tilde{P}_{m+\nu}(x), \quad m, k \in \mathbb{N}_0.
\]
Making the change of variable \( u = \lambda - \pi \) under the limit sign in (36), we consider them as coefficients of the MacLaurin expansions, which provides a generating function for each of the MPSs \( \{P_n(x)\}_{n \geq 0} \) and \( \{\hat{P}_n(x)\}_{n \geq 0} \):

\[
\frac{\partial^2}{\partial u^2} \frac{e^{2\sqrt{x} \sin(\alpha/2)}}{x} = \sum_{n \geq 0} P_n(x) \frac{u^{2n}}{(2n)!}, \tag{39}
\]

\[
\frac{e^{2\sqrt{x} \sin(\alpha/2)}}{\sqrt{x}} = \sum_{n \geq 0} \hat{P}_n(x) \frac{u^{2n+1}}{(2n+1)!}. \tag{40}
\]

The expressions for these generating functions could as well be attained from the developments made in [28, Chapter 6, p. 214] and after a few steps of computations.

### 3.2. Integral relation with Bernoulli and Euler numbers

The *Genocchi numbers* are the coefficients in the Taylor series expansion of the function \( 2t/(e^t + 1) \) and they are connected to the Bernoulli numbers \( B_n \) through \( G_n = 2(1 - 2^n)B_n, \ n \in \mathbb{N}_0 \) [9, (24.15.2)]. The known integral expression [11, Vol. I; 9] of the Genocchi numbers \( G_{2n+2} \)

\[
G_{2n+2} = (-1)^{n+1}(2n + 2) \int_0^\infty \frac{\tau^n}{\sinh(\pi \sqrt{\tau})} \, d\tau, \quad n \in \mathbb{N}_0, \tag{41}
\]

\[
E_{2n} = (-1)^n 2^{2n} \int_0^\infty \frac{\tau^{n-1/2}}{\cosh(\pi \sqrt{\tau})} \, d\tau, \quad n \in \mathbb{N}_0, \tag{42}
\]

endows another integral meaning for the elements of the MPSs \( \{P_n\}_{n \geq 0} \) and \( \{\hat{P}_n\}_{n \geq 0} \).

**Corollary 3.4** The MPS \( \{P_n\}_{n \geq 0} \) is connected with the Genocchi numbers of even order via

\[
\frac{(-1)^{n+1}}{2n + 2} G_{2n+2} = \int_0^\infty e^{-2\sqrt{x}} P_n(x) \, dx, \quad n \in \mathbb{N}_0, \tag{43}
\]

whereas the MPS \( \{\hat{P}_n\}_{n \geq 0} \) is connected with the Euler numbers of even order via

\[
(-1)^n 2^{-2n} E_{2n} = \int_0^\infty e^{-2\sqrt{x}} \hat{P}_n(x) \frac{dx}{\sqrt{x}}, \quad n \in \mathbb{N}_0. \tag{44}
\]

**Proof** The equality (23) can be rewritten as

\[
\frac{\tau^n}{\sinh(\pi \sqrt{\tau})} = \frac{2}{\pi \sqrt{\tau}} \int_0^\infty K_{2\sqrt{\tau}}(2\sqrt{x})P_n(x) \, dx, \quad n \in \mathbb{N}_0,
\]

and, in turn, (24) can be restyled into

\[
\frac{\tau^{n-1/2}}{\cosh(\pi \sqrt{\tau})} = \frac{2}{\pi \sqrt{\tau}} \int_0^\infty K_{2\sqrt{\tau}}(2\sqrt{x})\hat{P}_n(x) \frac{dx}{\sqrt{x}}, \quad n \in \mathbb{N}_0.
\]

We integrate both sides of each of the latter equations over \( \mathbb{R}_+ \) by \( \tau \) and interchange the order of integration in the left-hand side of the equalities according to Fubini’s theorem on the grounds of the inequality [36]

\[
|K_{2\sqrt{\tau}}(2\sqrt{x})| \leq e^{-\delta \sqrt{\tau}}K_0(2\sqrt{x} \cos \delta), \quad x > 0, \ \tau > 0, \ \delta \in \left(0, \frac{\pi}{2}\right). \tag{45}
\]

By virtue of the relation \( \int_0^\infty K_{2\sqrt{\tau}}(2\sqrt{x}) \, (d\tau / \sqrt{\tau}) = (\pi/2)e^{-2\sqrt{x}} \) together with (41) and (42), we, respectively, achieve the identities (43) and (44). □
The relation (43) also provides another identity between the Genocchi and central factorial numbers of even order, insofar as the input of (29) in (43) yields

$$\Theta_{2n+2} = (2n+2) \sum_{\nu=0}^{n} (-1)^{\nu+1} T_E(n+1, \nu+1) 2^{-2\nu-1}(2\nu+1)! , \quad n \in \mathbb{N}_0.$$  

Likewise, recalling (30), the integral relation (44) provides

$$2^{-2n} E_{2n} = \sum_{\nu=0}^{n} (-1)^{\nu+1} T_O(n, \nu) 2^{-2\nu}(2\nu)! , \quad n \in \mathbb{N}_0.$$  

### 3.3. Structural relations

In order to figure out the structural algebraic relation for the MPS \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \), we will proceed with the computation of the KL\( s \) transform of \( xP_n(x) \) along with the KL\( c \) transform of \( x\tilde{P}_n(x) \). At a first glance, by making use of the explicit expression (29) for the polynomials \( P_n \) and the linearity of the KL transform, we straightforwardly obtain

$$\text{KL}_s[xP_n(x)](\tau) = \sum_{\nu=0}^{n} (-1)^{\nu+v} T_E(n+1, \nu+1) \prod_{\sigma=1}^{\nu+1} (\sigma^2 + \tau). \quad (46)$$

In a similar manner, we deduce, in view of (11) and (30), the identity

$$\text{KL}_c[x\tilde{P}_n(x)](\tau) = \sum_{\nu=0}^{n} (-1)^{\nu+v} T_O(n, \nu) \prod_{\sigma=0}^{\nu} \left( \left( \frac{1}{2} + \sigma \right)^2 + \tau \right) , \quad n \in \mathbb{N}_0. \quad (47)$$

Despite the simple appearance of the just obtained expressions, it seems considerably complicate to deduce from them a recursive-structural expression for the two MPSs \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \). We will indeed succeed in achieving this goal after some manipulations with the KL transforms which will result in identities with the Euler polynomials. Yet, this section comprises the determination of the connection coefficients between \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \).

On the other hand, a suitable change on the order of integration sustained by (49) permits to deduce a connection between the latter expression and the Euler polynomials, \( E_n(x) \), commonly defined by Abramowitz and Stegun [1, (23.1.1)] and Digital Library of Mathematical Functions [9] via \( 2e^{ix}/(e^x + 1) = \sum_{n \geq 0} E_n(x) (a^n / n!) \). They are a key ingredient to further our ends, specially their explicit expression in terms of the monomials [9, (24.4.14)]

$$E_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} \binom{n+1}{\nu} \Theta_{n+1-\nu} x^\nu , \quad n \in \mathbb{N}_0, \quad (48)$$

where \( \Theta_n \) represent the aforementioned Genocchi numbers (see Section 3.2).

**Theorem 3.5** Let \( f(x) \), \( x \in \mathbb{R}_+ \), be a continuous function of bounded variation such that \( f(x) \in L_1(\mathbb{R}_+, K_0(2\alpha \sqrt{x}) \, dx) \), \( 0 < \alpha < 1 \) and

$$\phi_\lambda(x) = \frac{2}{\pi} \int_0^\infty \sqrt{\mu} \cosh(\lambda, \sqrt{\mu}) K_{2i\sqrt{\mu}}(2\sqrt{x}) \text{KL}_s[f](\mu) \, d\mu , \quad x > 0.$$
\[
\psi_{\lambda}(x) = \frac{2\sqrt{x}}{\pi} \int_{0}^{\infty} \sinh(\lambda \sqrt{\mu}) K_{2i\sqrt{\mu}}(2\sqrt{x}) KL_{c}[f](\mu) \, d\mu, \quad x > 0.
\]

If the functions \( \Phi_{\tau}(\lambda) = KL_{c}[\phi_{\lambda}](\tau) \) and \( \Psi_{\tau}(\lambda) = KL_{c}[\psi_{\lambda}](\tau) \), \( \lambda \in [0, \pi] \), are continuous at the point \( \lambda = \pi \) for each \( \tau \in \mathbb{R}_+ \), then

\[
\lim_{\lambda \to \pi^{-}} KL_{c}[\phi_{\lambda}](\tau) = KL_{c}[\phi f_{\lambda}](\tau)
\]

\[
eq \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\sqrt{\mu} \sinh(\pi \sqrt{\mu}) \cosh(\pi \sqrt{\mu})(\tau - \mu)}{\cosh(2\pi \sqrt{\mu}) - \cosh(2\pi \sqrt{\mu})} KL_{c}[f](\mu) \, d\mu, \quad (49)
\]

\[
\lim_{\lambda \to \pi^{-}} KL_{c}[\psi_{\lambda}](\tau) = KL_{c}[\psi f_{\lambda}](\tau) = 2 \int_{0}^{\infty} \frac{\cosh(\pi \sqrt{\mu}) \sinh(\pi \sqrt{\mu})(\tau - \mu)}{\cosh(2\pi \sqrt{\mu}) - \cosh(2\pi \sqrt{\mu})} KL_{c}[f](\mu) \, d\mu.
\]

**Proof** \( \) Within the framework (1) the KL_{c} transform of \( \phi_{\lambda} \) is given by

\[
KL_{c}[\phi_{\lambda}](\tau) = \frac{2 \sinh(\pi \sqrt{\tau})}{\pi \sqrt{\tau}} \int_{0}^{\infty} K_{2i\sqrt{\tau}}(2\sqrt{x}) \phi_{\lambda}(x) \, dx.
\]

Likewise, recalling (2), the KL_{c} transform of \( \psi_{\lambda} \) becomes

\[
KL_{c}[\psi_{\lambda}](\tau) = \frac{2 \cosh(\pi \sqrt{\tau})}{\pi \sqrt{\tau}} \int_{0}^{\infty} K_{2i\sqrt{\tau}}(2\sqrt{x}) \psi_{\lambda}(x) \frac{dx}{\sqrt{x}}.
\]

Now, we interchange the order of integration on the right-hand side of the latter equality according to Fubini’s theorem and on account of the inequality (45), the condition \( f \in L_{1}(\mathbb{R}_+, K_{0}(2\alpha \sqrt{x}) \, dx), \ 0 < \alpha < 1 \), and the following estimate

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \sqrt{\mu} \cosh(\lambda \sqrt{\mu}) K_{2i\sqrt{\mu}}(2\sqrt{x}) K_{2i\sqrt{\mu}}(2\sqrt{y}) f(y) \, dy \, d\mu \, dx
\]

\[
\leq e^{-\delta \sqrt{\tau}} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \sqrt{\mu} \cosh(\lambda \sqrt{\mu}) e^{-(\delta_{1} + \delta_{2}) \sqrt{\mu}} K_{0}(2\sqrt{x} \cos \delta_{1}) K_{0}(2\sqrt{y} \cos \delta_{2}) \, dy \, d\mu \, dx < \infty,
\]

where \( \delta_{1} \in (0, \pi/2) \) for \( i = 1, 2 \), and \( \delta_{1} + \delta_{2} > \lambda \). As a result,

\[
KL_{c}[\phi_{\lambda}(x)](\tau) = \frac{4 \sinh(\pi \sqrt{\tau})}{\pi^{2} \sqrt{\tau}} \int_{0}^{\infty} \sqrt{\mu} \cosh(\lambda \sqrt{\mu}) KL_{c}[f(x)](\mu) K(\tau, \mu) \, d\mu,
\]

whereas

\[
KL_{c}[\psi_{\lambda}(x)](\tau) = \frac{4 \cosh(\pi \sqrt{\tau})}{\pi^{2} \sqrt{\tau}} \int_{0}^{\infty} \sinh(\lambda \sqrt{\mu}) KL_{c}[f(x)](\mu) K(\tau, \mu) \, d\mu
\]

where the kernel \( K(\tau, \mu) = \int_{0}^{\infty} K_{2i\sqrt{\tau}}(2\sqrt{x}) K_{2i\sqrt{\mu}}(2\sqrt{x}) \, dx \) is indeed an elementary function, which can be calculated via relation (2.16.33.2) in [27, Vol. 2]:

\[
K(\tau, \mu) = \int_{0}^{\infty} K_{2i\sqrt{\tau}}(2\sqrt{x}) K_{2i\sqrt{\mu}}(2\sqrt{x}) \, dx = \frac{\pi^{2}}{2} \frac{\tau - \mu}{\cosh(2\pi \sqrt{\tau}) - \cosh(2\pi \sqrt{\mu})}.
\]

Furthermore, the passage to the limit \( \lambda \to \pi^{-} \) under the integral sign on the right-hand side of either (51) or (52) can be justified similarly due to the absolutely and uniform convergence via the Weierstrass test. \( \blacksquare \)
The latter result is of the utmost importance to deduce a simple expression for the KL_{(s,e)} transforms of \( xP_n(x) \) and \( x\tilde{P}_n(x) \), which, in turn, provide the structural-recursive relation for the corresponding sequences.

**Theorem 3.6** The KL\(_s\) transform of \( xP_n(x) \) is given by

\[
\text{KL}_s[xP_n(x)](\tau) = \frac{(-1)^n}{i\sqrt{\tau}} \left\{ \tau E_{2n+2}(i\sqrt{\tau}) + E_{2n+4}(i\sqrt{\tau}) \right\}
\]

\[
= \sum_{k=0}^{n+1} \left( \frac{2n+2}{2k} \right) \frac{(n+k+2)\mathfrak{G}_{2n-2k+4}}{(2k+1)(n-k+2)} (-1)^{n+k} \tau^k, \quad (53)
\]

while the KL\(_c\) transform of \( x\tilde{P}_n(x) \) can be expressed as follows:

\[
\text{KL}_c[x\tilde{P}_n(x)](\tau) = (-1)^n \left\{ \tau E_{2n+1}(i\sqrt{\tau}) + E_{2n+3}(i\sqrt{\tau}) \right\}
\]

\[
= \sum_{k=0}^{n+1} \left( \frac{2n+2}{2k} \right) \frac{(n+k+1)\mathfrak{G}_{2n-2k+4}}{2(n+1)(n-k+2)} (-1)^{n+k} \tau^k, \quad (54)
\]

where \( E_n(\cdot) \) represents the Euler polynomials and \( \mathfrak{G}_n \) the Genocchi numbers. Consequently, MPS \( \{P_n\}_{n \geq 0} \) fulfills the structural relations

\[
P_{n+2} = (x - (4n + 8)^2)P_{n+1} - \sum_{k=0}^{n} \left( \frac{2n+4}{2k} \right) \frac{(-1)^{n+k+1}(n+k+4)\mathfrak{G}_{2n-2k+6}}{(2k+1)(n-k+4)} P_k, \quad (55)
\]

\[
\tilde{P}_{n+2} = \left( x - \left( n + \frac{3}{2} \right)^2 \right) \tilde{P}_{n+1} - \sum_{k=0}^{n} \left( \frac{2n+4}{2k} \right) \frac{(-1)^{n+k+1}(n+k+2)\mathfrak{G}_{2n-2k+6}}{2(n+2)(n-k+3)} \tilde{P}_k, \quad (56)
\]

for \( n \in \mathbb{N}_0 \), with \( P_0(x) = 1 \) and \( P_1(x) = x - 1 \).

**Proof** For \( 0 < \epsilon < 1 \), the Euler polynomials admit the following integral representations [9, (24.7.9); 11, p. 43, Vol. I]:

\[
E_{2n+2}(i\sqrt{\tau} + \epsilon) = 2i(-1)^{n+1} \int_{0}^{\infty} \frac{\sinh(\pi(\sqrt{\tau} - i\epsilon)) \cosh(\pi\sqrt{\mu})}{\cosh(2\pi\sqrt{\mu}) - \cosh(2\pi(\sqrt{\tau} - i\epsilon))} \mu^{n+1/2} d\mu, \quad n \in \mathbb{N}_0, \quad (57)
\]

\[
E_{2n+1}(i\sqrt{\tau} + \epsilon) = 2(-1)^{n+1} \int_{0}^{\infty} \frac{\cosh(\pi(\sqrt{\tau} - i\epsilon)) \sinh(\pi\sqrt{\mu})}{\cosh(2\pi\sqrt{\mu}) - \cosh(2\pi(\sqrt{\tau} - i\epsilon))} \mu^n d\mu, \quad n \in \mathbb{N}_0. \quad (58)
\]

In the light of Theorem 3.5, the replacement \( f(x) = P_n(x) \) on relation (49) and on account of (23) provides

\[
\text{KL}_s[xP_n(x)](\tau) = \frac{2 \sinh(\pi\sqrt{\tau})}{\sqrt{\tau}} \int_{0}^{\infty} \mu^{n+1/2} \frac{(\tau - \mu) \cosh(\pi\sqrt{\mu})}{\cosh(2\pi\sqrt{\tau}) - \cosh(2\pi\sqrt{\mu})} d\mu, \quad n \in \mathbb{N}_0,
\]

while the substitution of \( f(x) = \tilde{P}_n(x) \) on (50) and on account of (24) provides

\[
\text{KL}_c[x\tilde{P}_n(x)](\tau) = 2 \cosh(\pi\sqrt{\tau}) \int_{0}^{\infty} \mu^{n} \frac{(\tau - \mu) \sinh(\pi\sqrt{\mu})}{\cosh(2\pi\sqrt{\tau}) - \cosh(2\pi\sqrt{\mu})} d\mu, \quad n \in \mathbb{N}_0.
\]
The accomplishment of the proof depends on the vital argument that is now our target:

\[ \text{KL}_c[xP_n(x)](\tau) = \frac{1}{\sqrt{\tau}} \lim_{\epsilon \to 0^+} \Phi_n(\epsilon, \tau) \quad \text{along with} \quad \text{KL}_c[x\bar{P}_n(x)](\tau) = \lim_{\epsilon \to 0^+} \Psi_n(\epsilon, \tau), \quad n \in \mathbb{N}_0, \]

where

\[
\Phi_n(\epsilon, \tau) = 2 \int_0^\infty \mu^{n+1/2} \left( \frac{(\sqrt{\tau} - i \epsilon)^2 - \mu}{\cosh(2\pi(\sqrt{\tau} - i \epsilon))} \right) \frac{\sinh(\pi(\sqrt{\tau} - i \epsilon))}{\cosh(\tau - i \epsilon)} \, d\mu 
\]

\[
= \int_0^\infty \mu^n \left\{ \frac{-\left(\sqrt{\mu} + \sqrt{\tau} - i \epsilon\right)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} + \sqrt{\tau} - i \epsilon))} 
+ \frac{(\sqrt{\mu} + \sqrt{\tau} + i \epsilon)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} - \sqrt{\tau} + i \epsilon))} \right\} \, d\mu,
\]

for all \( n \in \mathbb{N}_0 \), and

\[
\Psi_n(\epsilon, \tau) = 2 \int_0^\infty \mu^n \left( \frac{(\sqrt{\tau} - i \epsilon)^2 - \mu}{\cosh(2\pi(\sqrt{\tau} - i \epsilon))} \right) \frac{\sinh(\pi(\sqrt{\tau} - i \epsilon))}{\cosh(\tau - i \epsilon)} \, d\mu 
\]

\[
= \int_0^\infty \mu^n \left\{ \frac{\left(\sqrt{\mu} + \sqrt{\tau} - i \epsilon\right)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} + \sqrt{\tau} - i \epsilon))} 
+ \frac{(\sqrt{\mu} + \sqrt{\tau} + i \epsilon)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} - \sqrt{\tau} + i \epsilon))} \right\} \, d\mu,
\]

for all \( n \in \mathbb{N}_0 \). Indeed, for each \( \tau > 0 \) and for sufficiently small \( \delta > \epsilon \geq 0 \), the integral

\[
\int_0^\infty \mu^{n+1/2} \frac{(\sqrt{\mu} + \sqrt{\tau} + i \epsilon)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} - \sqrt{\tau} + i \epsilon))} \, d\mu 
\]

\[
= \left\{ \int_{|\sqrt{\mu} - \sqrt{\tau}| < \delta} \, d\mu + \int_{|\sqrt{\mu} - \sqrt{\tau}| \geq \delta} \, d\mu \right\} \times \left( \frac{(\sqrt{\mu} + \sqrt{\tau} + i \epsilon)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} - \sqrt{\tau} + i \epsilon))} \right), \quad n \in \mathbb{N}_0,
\]

converges uniformly by \( \epsilon \in [0, \delta) \), insofar as the two integrals

\[
\int_{|y| \geq \delta} \left| \frac{(y + \sqrt{\tau})^{n+1}y^2}{\sinh(\pi y)} \right| \, dy < \infty,
\]

\[
\int_{|y| \leq \delta} \left| \frac{(y + i \epsilon)^{n+1/2}(\sqrt{\mu} + \sqrt{\tau} + i \epsilon)\left(\sqrt{\mu} - \sqrt{\tau} + i \epsilon\right)}{\sinh(\pi(\sqrt{\mu} - \sqrt{\tau} + i \epsilon))} \right| \, d\mu \leq C \int_{|y| \leq \delta} \left| \frac{y + i \epsilon}{\sinh(\pi(y + i \epsilon))} \right| \, dy < C \delta < \infty
\]

converge uniformly by \( 0 \leq \epsilon < \delta \). Thus, the integral \( \Phi_n(\epsilon, \tau) \) converges uniformly by \( \epsilon \), according to the Weierstrass test, which legitimates the passage of the limit as \( \epsilon \to 0^+ \) under the integral.
signs, so that we have motivated the validity of (59) since the integral $\Psi_n(\epsilon, \tau)$ can be treated in a similar way. On the other hand, in the light of (57) and (58), $\Phi_n(\epsilon, \tau)$ and $\Psi_n(\epsilon, \tau)$ can be written in terms of even-order Euler polynomials precisely:

\[
\Phi_n(\epsilon, \tau) = (-1)^n i((\sqrt{\tau} - i\epsilon)^2 E_{2n+2}(i\sqrt{\tau} + \epsilon) + E_{2n+4}(i\sqrt{\tau} + \epsilon)),
\]

\[
\Psi_n(\epsilon, \tau) = (-1)^n ((\sqrt{\tau} - i\epsilon)^2 E_{2n+1}(i\sqrt{\tau} + \epsilon) + E_{2n+3}(i\sqrt{\tau} + \epsilon)),
\]

which, together with (59), ensure the first equalities on (53) and (54). Meanwhile, the expansion for the Euler polynomials (48) enables

\[
- \frac{\tau E_{2n+2}(i\sqrt{\tau}) + E_{2n+4}(i\sqrt{\tau})}{i\sqrt{\tau}} = \sum_{k=0}^{n+1} \left( \frac{2n + 2}{2k} \right) \frac{(n + k + 2) \Phi_{2n-2k+4}}{(2k + 1)(n - k + 2)} (-\tau)^k, \quad (60)
\]

\[
\tau E_{2n+1}(i\sqrt{\tau}) + E_{2n+3}(i\sqrt{\tau}) = \sum_{k=0}^{n+1} \left( \frac{2n + 2}{2k} \right) \frac{(k + n + 1) \Phi_{2n-2k+4}}{2(n + 1)(n - k + 2)} (-1)^k \tau^k, \quad (61)
\]

whence, the equalities between the middle and last members on both (53) and (54) hold.

Moreover, bearing in mind the injectivity and linearity of the KL transform (and in particular KL$_s$ and KL$_c$) and on account of (23), the identities (60) and (61), respectively, give rise to

\[
xP_n(x) = \sum_{k=0}^{n+1} \left( \frac{2n + 2}{2k} \right) \frac{(n + k + 2) \Phi_{2n-2k+4}}{(2k + 1)(n - k + 2)} (-1)^{n+k} P_k(x), \]

\[
x\tilde{P}_n(x) = \sum_{k=0}^{n+1} \left( \frac{2n + 2}{2k} \right) \frac{(n + k + 1) \Phi_{2n-2k+4}}{2(n + 1)(n - k + 2)} (-1)^{n+k} \tilde{P}_k(x),
\]

providing (56), because $\Phi_2 = -1 = -\Phi_4$.

Remark 3.7 The comparison of (46) with (53) and also of (47) with (54) brings an identity between the central factorial coefficients of even order and the Euler polynomials which, as far as we are concerned, is new in the theory:

\[
\sum_{\nu=0}^{n} (-1)^{\nu} T_E(n + 1, \nu + 1) \prod_{\sigma=1}^{\nu+1} \left( \tau + \sigma^2 \right) = \frac{\tau E_{2n+2}(i\sqrt{\tau}) + E_{2n+4}(i\sqrt{\tau})}{i\sqrt{\tau}}, \quad (62)
\]

\[
\sum_{\nu=0}^{n} (-1)^{\nu} T_O(n, \nu) \prod_{\sigma=0}^{\nu} \left( \left( \frac{1}{2} + \sigma \right)^2 + \tau \right) = \tau E_{2n+1}(i\sqrt{\tau}) + E_{2n+3}(i\sqrt{\tau}). \quad (63)
\]

In particular, when $\tau \rightarrow 0+$ in (62), we recover the relation already pointed in [8,31, pp. 74–75].

On the other hand, yet the same framework, in the light of (11), from (68) and (69) we also deduce

\[
\sum_{\nu=0}^{n} (-1)^{\nu} T_E(n + 1, \nu + 1) \prod_{\sigma=0}^{\nu} \left( \tau + \left( \sigma + \frac{1}{2} \right)^2 \right) = -E_{2n+2} \left( \frac{1}{2} + i\sqrt{\tau} \right), \quad n \in \mathbb{N}_0, \quad (64)
\]

\[
\sum_{\nu=0}^{n} (-1)^{\nu} T_O(n, \nu) \prod_{\sigma=1}^{\nu} \left( \tau + \sigma^2 \right) = \frac{1}{i\sqrt{\tau}} E_{2n+1} \left( \frac{1}{2} + i\sqrt{\tau} \right), \quad n \in \mathbb{N}_0. \quad (65)
\]
Remark 3.8 Noteworthy to notice is the fact that the relation (55) shows that \( \{P_n\}_{n \geq 0} \) or \( \{\tilde{P}_n(x)\}_{n \geq 0} \) is not orthogonal (with respect to a \( L_2 \)-inner product), insofar as they do not fulfill a second-order recursive relation. Or, even more generally, neither the MPS \( \{P_n\}_{n \geq 0} \) nor \( \{\tilde{P}_n(x)\}_{n \geq 0} \) can be \( d \)-orthogonal, because it does not fulfill a recursive relation whose order is not independent of the order of the elements. For more details regarding the \( d \)-orthogonality, we refer to [22,34].

Notwithstanding these negative results, we cannot discard the possibility of the existence of an inner product with respect to which it would be possible to give some orthogonal sense to each of the MPSs \( \{P_n\}_{n \geq 0} \) or \( \{\tilde{P}_n(x)\}_{n \geq 0} \). One possibility would be the Sobolev orthogonality, but we defer the discussion for a further work, leaving this as an open problem.

3.4. Connection coefficients between \( \{P_n(x)\}_{n \geq 0} \) and \( \{\tilde{P}_n(x)\}_{n \geq 0} \)

The procedure taken in this section to obtain the connection coefficients between \( \{P_n(x)\}_{n \geq 0} \) and \( \{\tilde{P}_n(x)\}_{n \geq 0} \) is much similar to the one taken for the obtention of the structural-recursive relation for each of the sequences. The idea is to obtain integral relations between the two KL modified transforms of a certain function \( f \) satisfying the required conditions.

Motivated by similar arguments as the ones evoked in the proof of Theorem 3.5 and under the same assumptions over the function \( f(x) \), then, bearing in mind the identity (see (2.16.33.2) in [27, Vol. 2])

\[
\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\pi}}(2\sqrt{x}) \frac{dx}{\sqrt{x}} = \frac{\pi^2}{2} \frac{1}{\cosh(2\pi \sqrt{\mu}) + \cosh(2\pi \sqrt{\tau})},
\]

the application of the KL\(_c\) transform, defined in (2), over (3) leads to

\[
\text{KL}_c[xf(x)](\tau) = 2 \cosh(\pi \sqrt{\tau}) \int_0^\infty \frac{\sqrt{\mu} \cosh(\pi \sqrt{\mu})}{\cosh(2\pi \sqrt{\mu}) + \cosh(2\pi \sqrt{\tau})} \text{KL}_c[f](\mu) \, d\mu,
\]

whereas the action of the KL\(_s\) transform, defined in (1), over (4) divided by \( \sqrt{x} \) results in

\[
\text{KL}_s[f](\tau) = \frac{2 \sinh(\pi \sqrt{\tau})}{\sqrt{\tau}} \int_0^\infty \frac{\sinh(\pi \sqrt{\mu})}{\cosh(2\pi \sqrt{\mu}) + \cosh(2\pi \sqrt{\tau})} \text{KL}_c[f](\mu) \, d\mu.
\]

Both of them are valid as long as \( \phi_\lambda(x) \) and \((1/x)\psi_\lambda(x)\), with \( \lambda \in [0, \pi] \), are continuous functions at the point \( \lambda = \pi \) for each \( \tau \in \mathbb{R}_+ \).

On the other hand, mimicking the framework of Theorem 3.6, then upon the choice of \( f(x) = P_n(x) \) on the relation (66) and, afterwards, by invoking (57), we come out with the relation

\[
(-1)^{n+1}E_{2n+2} \left( i\sqrt{\tau} + \frac{1}{2} \right) = 2 \int_0^\infty \frac{\cosh(\pi \sqrt{\tau}) \cosh(\pi \sqrt{\mu})}{\cosh(2\pi \sqrt{\tau}) + \cosh(2\pi \sqrt{\mu})} \mu^{n+1/2} \, d\mu
\]

\[
= \text{KL}_c[xP_n(x)](\tau).
\]

Likewise, relation (67) upon the replacement of \( f(x) = \tilde{P}_n(x) \) together with (58) gives rise to

\[
2i(-1)^{n+1}E_{2n+1} \left( i\sqrt{\tau} + \frac{1}{2} \right) = \int_0^\infty \frac{\sinh(\pi \sqrt{\tau}) \sinh(\pi \sqrt{\mu}) \mu^{n+1/2}}{\cosh(2\pi \sqrt{\tau}) + \cosh(2\pi \sqrt{\mu})} \, d\mu = \text{KL}_s[\tilde{P}_n](\tau).
\]

Since \( |(\partial^{2n+2}/\partial \lambda^{2n+2})e^{-2\sqrt{\tau} \cos(\lambda/2)}| \leq C_nx^n \) for \( x > 0 \) and \( \lambda \in [0, \pi/2] \), where \( C_n \) depends exclusively on \( n \in \mathbb{N}_0 \), the fact that \( |K_{y+1}(2\sqrt{x})| \leq K_{y+1}(2\sqrt{x}) \) and \( \int_0^\infty K_{y+1}(2\sqrt{x})x^n \)
dx < +∞ give grounds for the passage of the limit under the outer integral in

\[ E_{2n}\left(-\frac{y}{2}\right) = \frac{2}{\pi} (-1)^{n+1} \sin \left(\frac{\pi y}{2}\right) \lim_{\lambda \to \pi^-} \frac{\partial^{2n+2}}{\partial \lambda^{2n+2}} \int_0^\infty e^{-2\sqrt{x}\cos(\lambda/2)} K_{\lambda+1}(2\sqrt{x}) \frac{dx}{\sqrt{x}}. \]

Appealing to (35) and as long as \(-2 < \Re y < 0\), we derive

\[ E_{2n}\left(-\frac{y}{2}\right) = \frac{2}{\pi} (-1)^{n+1} \sin \left(\frac{\pi y}{2}\right) \int_0^\infty \sqrt{x}P_{\lambda-1}(x) K_{\lambda+1}(2\sqrt{x}) \, dx, \quad n \in \mathbb{N}, \]

which yields (68) when \(y = -1 + 2i\sqrt{\tau}\).

Sustained by similar arguments, one can prove as well that

\[ E_{2n+1}\left(-\frac{y}{2}\right) = \frac{2}{\pi} (-1)^{n+1} \cos \left(\frac{\pi y}{2}\right) \lim_{\lambda \to \pi^-} \frac{\partial^{2n+1}}{\partial \lambda^{2n+1}} \int_0^\infty e^{-2\sqrt{x}\cos(\lambda/2)} K_{\lambda+1}(2\sqrt{x}) \frac{dx}{\sqrt{x}}, \]

which, after (35), may be restyled into

\[ E_{2n+1}\left(-\frac{y}{2}\right) = \frac{2}{\pi} (-1)^{n+1} \cos \left(\frac{\pi y}{2}\right) \int_0^\infty \tilde{P}_n(x)K_{\lambda+1}(2\sqrt{x}) \, dx, \quad n \in N_0, \]

which leads to (69) upon the replacement \(y = -1 + 2i\sqrt{\tau}\).

Actually, (70) and (71) are both valid in the vertical strip \(-4 < \Re y < 2\) because of the uniform and absolute convergence of the integral. In the meantime, since the Euler polynomials admit the following expansion [9, (24.2.10)]

\[ E_{2n+2}\left(i\sqrt{\tau} + \frac{1}{2}\right) = \sum_{k=0}^{n+1} (-1)^k \left(\frac{2n+2}{2k}\right) E_{2n+2-2k} \frac{\tilde{K}_k}{2^{2n-2k+2}} \tau^k, \]

\[ \frac{1}{i\sqrt{\tau}} E_{2n+1}\left(i\sqrt{\tau} + \frac{1}{2}\right) = \sum_{k=0}^{n} (-1)^k \left(\frac{2n+1}{2k+1}\right) E_{2n-2k} \frac{\tilde{K}_k}{2^{2n-2k}} \tau^k, \quad n \in N_0, \]

it readily follows from (68) and (69), respectively, the following

\[ xP_n(x) = \sum_{k=0}^{n+1} (-1)^{n+k+1} \left(\frac{2n+2}{2k}\right) E_{2n+2-2k} \frac{\tilde{K}_k}{2^{2n-2k+2}} \tilde{P}_k(x), \quad n \in N_0, \]

\[ \tilde{P}_n(x) = \sum_{k=0}^{n} (-1)^{n+k} \left(\frac{2n+1}{2k+1}\right) E_{2n-2k} \frac{\tilde{K}_k}{2^{2n-2k}} P_k(x), \quad n \in N_0. \]

### 3.5. The dual sequences of the MPSs whose KLₜ transforms are the canonical sequences

Despite none of the polynomial sequences \{P_n(x)\}_{n \geq 0} and \{\tilde{P}_n(x)\}_{n \geq 0} can be (regularly) orthogonal (with respect to an \(L_2\)-inner product), we show that the first element of the corresponding dual sequence is a regular form, which amounts to be same as the guarantee of the existence of an orthogonal polynomial sequence with respect to it. For a more clear understanding, we recall a few concepts of the utmost importance for this goal. Any element \(u\) of \(\mathcal{P}'\) can be written in a series of any dual sequence \(\{v_n\}_{n \geq 0}\) of an MPS \(\{P_n\}_{n \geq 0}\) [23,24]:

\[ u = \sum_{n \geq 0} (u, P_n) \, u_n. \]  

Differential equations or other kind of linear relations realized by the elements of the dual sequence can be deduced by transposition of those relations fulfilled by the elements of the corresponding
MPS, insofar as a linear operator \( T : \mathcal{P} \rightarrow \mathcal{P} \) has a transpose \( T' : \mathcal{P}' \rightarrow \mathcal{P}' \) defined by
\[
\langle T(u), f \rangle = \langle u, T(f) \rangle, \quad u \in \mathcal{P}', \ f \in \mathcal{P}.
\] (74)

For example, for any form \( u \) and any polynomial \( g \), let \( Du = u' \) and \( gu \) be the forms defined as usual by \( \langle u', f \rangle := -\langle u, f' \rangle \), \( \langle gu, f \rangle := \langle u, gf \rangle \), where \( D \) is the differential operator \([23, 24]\). Thus, \( D \) on forms is minus the transpose of the differential operator \( D \) on polynomials.

The properties of the MPSs \( \{P_n\}_{n \geq 0} \) and \( \{\bar{P}_n\}_{n \geq 0} \) trigger those of the corresponding dual sequence.

**Lemma 3.9** The dual sequences \( \{u_n\}_{n \geq 0} \) and \( \{\bar{u}_n\}_{n \geq 0} \) of the two MPSs \( \{P_n\}_{n \geq 0} \) and \( \{\bar{P}_n\}_{n \geq 0} \), respectively, fulfil
\[
(x^2u_0)'' - 3(xu_0)' + (1 - x)u_0 = 0, \quad (75)
\]
\[
(x^2u_{n+1})'' - 3(xu_{n+1})' + (1 - x)u_{n+1} = -u_n, \quad n \geq 0, \quad (76)
\]
and
\[
(x^2\bar{u}_0)'' - 2(x\bar{u}_0)' + (\frac{1}{4} - x)\bar{u}_0 = 0, \quad (77)
\]
\[
(x^2\bar{u}_{n+1})'' - 2(x\bar{u}_{n+1})' + (\frac{1}{4} - x)\bar{u}_{n+1} = -\bar{u}_n, \quad n \geq 0. \quad (78)
\]
Moreover, \( (u_0)_n = (n!)^2 \) and \( (\bar{u}_0)_n = \prod_{\sigma=0}^{n-1}(\frac{1}{2} + \sigma)^2 = 2^{-n}(2n - 1)!! \), \( n \in \mathbb{N}_0 \).

**Proof** The action of \( u_0 \) over (27) is given by
\[
\langle u_0, x^2P_n'(x) + 3xp_n(x) + (1 - x)P_n(x) \rangle = 0, \quad n \in \mathbb{N}_0,
\]
which, by transposition, on account of (74), is equivalent to
\[
\langle (x^2u_0)'' - 3(xu_0)' + (1 - x)u_0, P_n \rangle = 0, \quad n \in \mathbb{N}_0,
\]
providing (75). Likewise, the action of \( u_{k+1} \) over (27) yields
\[
\langle u_{k+1}, x^2P_n'(x) + 3xp_n(x) + (1 - x)P_n(x) \rangle = -\delta_{n,k}, \quad n, k \in \mathbb{N}_0,
\]
and, due to (74), we may write this latter as
\[
\langle (x^2u_{k+1})'' - 3(xu_{k+1})' + (1 - x)u_{k+1}, P_n \rangle = -\delta_{n,k}, \quad n, k \in \mathbb{N}_0.
\]
By virtue of (73), the relation (76) is then a consequence of this latter equality.

The action of both sides of (75) over the sequence \( \{x^n\}_{n \geq 0} \) permits to obtain the relation for the moments of \( u_0 \), since we have
\[
(u_0)_{n+1} = (n + 1)^2(u_0)_n, \quad n \in \mathbb{N}_0,
\]
and thereby \( (u_0)_n = (n!)^2(u_0)_0 = (n!)^2 \) for \( n \in \mathbb{N}_0 \).

Replicating this procedure for the MPS \( \{\bar{P}_n\}_{n \geq 0} \) based on the starting point (28), we conclude that the dual sequence \( \{\bar{u}_n\}_{n \geq 0} \) necessarily fulfils Equations (77)–(78). The moments of \( \bar{u}_0 \) can be directly computed from (77).

**Remark 3.10** Any MPS \( \{B_n\}_{n \geq 0} \) such that \( B_0 = 1 \) and \( B_{n+1}(0) = 0 \) has the Dirac delta \( \delta \) as canonical form, as it is the case of \( \{B_n(x) := \sum_{k=0}^{n} T_E(n, k)x^k\}_{n \geq 0} \).
Despite the non-(regular)orthogonality of \( \{P_n\}_{n \geq 0} \) with respect to the form \( u_0 \), we cannot exclude the existence of an orthogonal polynomial sequence, say \( \{V_n\}_{n \geq 0} \). Thus, we pose the problem of investigating whether \( u_0 \) or \( \tilde{u}_0 \) is regular or not. We bring an affirmative answer.

**Proposition 3.11** Both of the canonical forms \( u_0 \) and \( \tilde{u}_0 \) are positive definite, admitting the following integral representations:

\[
\langle u_0, f \rangle = 2 \int_0^\infty K_0(2\sqrt{x})f(x) \, dx \quad \forall f \in \mathcal{P},
\]

and

\[
\langle \tilde{u}_0, f \rangle = 2 \int_0^\infty \frac{K_0(2\sqrt{x})f(x) \, dx}{\sqrt{x}} \quad \forall f \in \mathcal{P}.
\]

**Proof** We seek functions \( U(x) \) and \( \tilde{U}(x) \) such that

\[
\langle u_0, f \rangle = \int_C U(x)f(x) \, dx \quad \text{and} \quad \langle \tilde{u}_0, f \rangle = \int_{\tilde{C}} \tilde{U}(x)f(x) \, dx,
\]

respectively, hold in a certain domain \( C \) and \( \tilde{C} \). Since \( \langle u_0, 1 \rangle = \langle \tilde{u}_0, 1 \rangle = 1 \neq 0 \), we must have

\[
\int_C U(x) \, dx = \int_{\tilde{C}} \tilde{U}(x) \, dx = 1 \neq 0. \quad (79)
\]

By virtue of (75), we have, for any \( f \in \mathcal{P} \),

\[
0 = \langle ((\phi(x)u_0)' + \psi(x)u_0)' + \chi(x)u_0, f(x) \rangle = \langle u_0, \phi(x)f''(x) + \psi(x)f'(x) + \chi(x)f(x) \rangle
\]

\[
= \int_C \left( (\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x)f(x) \right) \, dx
\]

\[
- (\phi(x)U(x)f'(x) - (\phi(x)U(x))'f(x) - \psi(x)U(x)f(x)) \bigg|_C
\]

with \( \phi(x) = x^2, \psi(x) = -3x, \chi(x) = 1 - x \). Consequently, \( U(x) \) is a function simultaneously fulfilling

\[
\int_C ((\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x)f(x)) \, dx = 0 \quad \forall f \in \mathcal{P}, \quad (80)
\]

\[
(\phi(x)U(x)f'(x) - (\phi(x)U(x))'f(x) - \psi(x)U(x)f(x)) \bigg|_C = 0 \quad \forall f \in \mathcal{P}. \quad (81)
\]

The first equation implies \( (\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) = \lambda g(x) \), where \( \lambda \) is a complex number and \( g(x) \neq 0 \) is a function representing the null form, that is, a function such that \( \int_C g(x)f(x) \, dx = 0, \forall f \in \mathcal{P} \). We begin by choosing \( \lambda = 0 \) and search a regular solution of the differential equation \( (x^2U(x))'' - (3xU(x))' + (1 - x)U(x) = 0 \), whose general solution is \( y(x) = c_1I_0(2\sqrt{x}) + c_2K_0(2\sqrt{x}), x \geq 0 \), for some arbitrary constants \( c_1, c_2 \), and \( y(x) = 0 \) when \( x < 0 \) [11]. Insofar as \( U(x) \) must be a rapidly decreasing sequence (i.e. such that \( \lim_{x \to \pm\infty} f(x)U(x) = 0 \) for any polynomial \( f \)) simultaneously realizing the condition (79), we readily conclude that \( U(x) = 2K_0(2\sqrt{x}) \), considering its asymptotic behaviour (8) and (9) together with the expression for its moments (12). Moreover, for every polynomial \( p \) that is not identically zero and is non-negative for all real \( x \), we have \( \langle u_0, p \rangle = 2 \int_0^\infty p(x)K_0(2\sqrt{x}) \, dx > 0 \) and therefore \( u_0 \) is a positive-definite form, which implies the existence of a corresponding MOPS (i.e. \( u_0 \) is a regular form).
Mimicking the arguments on the latter procedure to find out the function $U(x)$, under the nuance of the distinct differential equation (77) realized by $\tilde{u}_0$, the function $\tilde{U}(x)$ must be a solution of the differential equation $(x^2\tilde{U}(x))'' - (2x\tilde{U}(x))' + \left(\frac{1}{4} - x\right)\tilde{U}(x) = 0$ under the constraints $\lim_{x \to +\infty} f(x)\tilde{U}(x) = 0$ for any polynomial $f$. As a result, $\tilde{U}(x) = (2/\sqrt{x})K_0(2\sqrt{x})$ and the fact that $\langle u_0, g \rangle = 2\int_0^\infty g(x)K_0(2\sqrt{x})(dx/\sqrt{x}) > 0$ for any polynomial $g$ non-negative for all real $x$ and not identically zero ensure $\tilde{u}_0$ to be a positive-definite form (ergo regular).

Naturally, the affirmative answer to the regularity of $u_0$ and $\tilde{u}_0$ raises the problem of the identification of two polynomial sequences each of them, respectively, orthogonal to $u_0$ and $\tilde{u}_0$. Actually, the problem of characterizing the MOPS with respect to $u_0$ was already posed by Prudnikov [32] and is still an open problem, which can be traced back to his seminal work of 1966 [10]. Therein we may read the origins of the problem: the operational calculus associated with the differential operator $d/dt$ gives rise to the Laplace transform having the exponential function as a kernel, which we are going to represent in terms of the Mellin–Barnes integral [11, Vol. I]

$$e^{-x} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s)x^{-s} ds, \quad x, a > 0,$$

while the operator $(d/dt)t(d/dt)$ leads to the Meijer transform [41] involving the modified Bessel function (also known as MacDonald’s function) $2K_0(2\sqrt{x})$ as a positive kernel given by the formula [11, Vol. II]

$$2K_0(2\sqrt{x}) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma^2(s)x^{-s} ds, \quad x, a > 0.$$

Bearing in mind that $e^{-x}$ is the weight function of the very classical Laguerre polynomials (of parameter 0) [7], Prudnikov posed the problem of finding a new sequence of orthogonal polynomials related to the weight $2K_0(2\sqrt{x})$. The problem at issue encompasses the characterization of an MOPS, say $\{V_n\}_{n \geq 0}$, such that

$$\int_0^\infty 2K_0(2\sqrt{x})V_m(x)V_n(x) dx = N_n\delta_{n,m} \quad \text{with } N_n > 0, \quad n, m \in \mathbb{N}_0,$$

through the determination of a differential equation fulfilled by $\{V_n\}_{n \geq 0}$ or a relation for the recursive coefficients. The problem was later on broadened to other ultra-exponential weights [32] and, regarding the intrinsic difficulties, in the sequel, other approaches were considered, largely focused on the investigation of polynomial sequences being either multiple orthogonal [33] or $d$-orthogonal [6] with respect to these ultra-exponential weights.

However, we do not unravel the characterization of this Prudnikov MOPS; in the present work, we have enlightened the investigation of this MPS $\{P_n\}_{n \geq 0}$ whose KL$_d$ transform is the canonical sequence and whose canonical form is shared by the Prudnikov MOPS $\{V_n\}_{n \geq 0}$. An analogous circumstance occurs between the Bernoulli and the Legendre polynomials [25].

4. Final considerations

While looking at the KL$_d$-transformed polynomial sequence of classical polynomials of Laguerre or Hermite, we come to the conclusion that they are necessarily $d$-orthogonal sequences. The existence of such examples gives grounds to determine that all the MOPSs whose image by the
action of the KLs transform are d-MOPSs. The analysis beneath the answer involves a considerable amount of computations and because of this we defer the study for a forthcoming work. Nevertheless, we announce that if an MOPS is such that the corresponding KLs transform is a d-orthogonal sequence, then it is necessarily a semi-classical polynomial sequence and d must be an even number greater than 2. Thus, it is clear that the image of Prudnikov’s MOPS cannot be a d-MOPS. Indeed, we are still working towards the characterization of these polynomials and we believe that properties of the considered sequences \( \{P_n\}_{n \geq 0} \) and \( \{\tilde{P}_n\}_{n \geq 0} \) can play an important role. On the other hand, the Continuous Dual Hahn and Wilson orthogonal polynomials [9, Section 18.21] are images of two corresponding non-orthogonal MPSs whose explicit expressions can be directly obtained. The structural and differential relations are left to be worked on.

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