Generalized de Sitter solution in multidimensional cosmology with static internal spaces

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Abstract

A multidimensional cosmological model with space-time consisting of \( n(n \geq 2) \) Einstein spaces \( M_i \) is investigated in the presence of a cosmological constant \( \Lambda \) and a homogeneous minimally coupled free scalar field. Generalized de Sitter solution was found for \( \Lambda > 0 \) and Ricci-flat external space for the case of static internal spaces with fine tuning of parameters.

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1 INTRODUCTION

The de Sitter solution (De Sitter (1917a, 1917b)) in 4-dimensional space-time is remarkable from different points of view. Firstly, it describes the unique curved space-time with maximum of symmetry (Weinberg (1972), Misner, Thorne, and Wheeler (1974), Hawking, Ellis (1973), Birrell, and Davies (1982)). Secondly, this solution simulates the inflationary stages of the evolution of universe (Guth (1981), Linde (1984)). From the other hand it is well known that modern theories of unified physical interactions take place in multidimensional spaces only. Thus, it will be important to generalize the de Sitter solution for multidimensional cosmological models (MCM). In all MCM a mechanism of dimensional reduction (compactification) of extra dimensions should be present. One of the possibilities to solve this problem consists in the proposal that all extra dimensions are static and small (a few orders of Planck’s length) from the very beginning. In spite of
smallness of extra dimensions the internal spaces have strong influence on the parameters of external space, for example, on the rate of the evolution of external space.

In this paper we shall find the generalization of the de Sitter solution in the case of MCM with static internal spaces where internal spaces are Einstein ones.

2 THE MODEL

We consider a cosmological model with $n(n > 1)$ Einstein spaces containing a massless minimally coupled free scalar field and a positive cosmological constant $\Lambda$. The metric of the model

$$g = -\exp[2\gamma(\tau)]\,d\tau \otimes d\tau + \sum_{i=1}^{n} \exp[2\beta^i(\tau)]g(i)$$

is defined on the manifold

$$M = R \times M_1 \times \cdots \times M_n,$$

where the manifold $M_i$ with the metric $g(i)$ is an Einstein space of dimension $d_i$, i.e.

$$R_{m_i n_i}[g(i)] = \lambda_i g(i)_{m_i n_i}$$

$i = 1, \ldots, n; n \geq 2$. In the particular case of the spaces of constant curvature $\lambda_i = k_i(d_i-1)$, where $k_i = \pm 1, 0$. The total dimension of the space-time $M$ is $D = 1 + \sum_{i=1}^{n} d_i$.

The action of the model is

$$S = \frac{1}{2} \int d^D x \sqrt{|g|} \left\{ R[g] - (\nabla \varphi)^2 - 2\Lambda \right\} + S_{GH},$$

where $R[g]$ is the scalar curvature of the metric $g$ and $S_{GH}$ is the standard Gibbons-Hawking boundary term (Gibbons, and Hawking (1977)). This type of MCM (without scalar field and cosmological constant) was considered first in (Ivashchuk, Melnikov, and Zhuk (1989)). Some integrable cases were investigated in (Zhuk (1992a), Zhuk (1992b), Bleyer, Ivashchuk, Melnikov, and Zhuk (1994)). Following paper (Ivashchuk, Melnikov, and Zhuk (1989)) we get that the field equations, corresponding to the action (4), for the metric (1) in the harmonic time gauge $\gamma = \sum_{i=1}^{n} d_i \beta^i$ are equivalent to the Lagrange equations, corresponding to the Lagrangian

$$L = \frac{1}{2} \left( \sum_{i,j=1}^{n} G_{ij} \dot{\beta}^i \dot{\beta}^j + \dot{\varphi}^2 \right) - V$$

with the energy constraint imposed

$$E = \frac{1}{2} \left( \sum_{i,j=1}^{n} G_{ij} \dot{\beta}^i \dot{\beta}^j + \dot{\varphi}^2 \right) + V = 0$$

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Here, the overdot denotes differentiation with respect to the harmonic time \( \tau \). The components of minisuperspace metric read \( G_{ij} = d_i \delta_{ij} - d_i d_j \) and the potential is given by

\[
V = \exp \left( 2 \sum_{i=1}^{n} d_i \beta^i \right) \left[ -\frac{1}{2} \sum_{i=1}^{n} d_i \lambda_i e^{-2\beta^i} + \Lambda \right].
\] (7)

In our paper we consider MCM with static internal spaces. Let the factor space \( M_1 \) be dynamical external space. All the other factor spaces \( M_i \) \((i = 2, \ldots, n)\) are considered as internal and static. They should be compact and the internal dimensions have the size of order of Planck’s length \( L_{PL} \sim 10^{-33} cm \). The scale factors of the internal factor spaces should be constant: \( a_i = e^{\beta_i} \equiv a_{(0)i} \) \((i = 2, \ldots, n)\). The problem of compactification with static internal spaces for the models with the topology (2) was considered in (Bleyer and Zhuk (1994)) . According to the theorems proved there the static compactification for our model takes place only in the case of fine tuning of the parameters:

\[
\frac{\lambda_i}{a_{(0)i}^2} = \frac{2 \Lambda}{D-2}, \quad i = 2, \ldots, n.
\] (8)

It follows from (8) that for \( \Lambda > 0 \) parameters \( \lambda_i \) should be positive also. If \( M_i \) \((i = 2, \ldots, n)\) are spaces of constant curvature, they should have positive constant curvature.

Let us investigate the model where our external space \( M_1 \) is Ricci-flat, i.e. \( \lambda_1 = 0 \). Then the constraint (6) has form

\[
d_1(d_1 - 1)(\dot{\beta}^1)^2 = \nu^2 + e^{2d_1 \beta^1} \left[ \frac{2(d_1 - 1)\Lambda}{D-2} \prod_{k=2}^{n} a_{(0)k}^{2d_k} \right],
\] (9)

where \( \dot{\varphi} = \nu = const \) is the first integral of the equation \( \ddot{\varphi} = 0 \). Parameter \( \nu^2 \) plays the role of energy (Zhuk (1992b)).

3 GENERALIZED DE SITTER SOLUTION

We can rewrite the equation (9) as follows

\[
(\dot{\beta}^1)^2 = \tilde{\nu}^2 + \tilde{\Lambda} e^{2d_1 \beta^1},
\] (10)

where the constants are

\[
\tilde{\nu}^2 = \frac{\nu^2}{d_1(d_1 - 1)}
\] (11)

and

\[
\tilde{\Lambda} = \frac{2\Lambda}{d_1(D-2)} \prod_{k=2}^{n} a_{(0)k}^{2d_k}.
\] (12)
It is clear from (10) that the dynamical behaviour of the scale factor $a_1 = \exp \beta^1$ depends on the sign of $\nu^2$.

a) $\nu^2 > 0$: real scalar field.

The solution of equation (10) has the form

$$a_1(\tau) = \left[ \frac{\nu^2}{\Lambda} \right]^{1/2d_1} \sinh \nu d_1 (\tau - \tau_0)^{-1/d_1}, \quad -\infty < \tau < +\infty$$

where $\tau_0$ is the constant of integration.

The synchronous time $t$ and the harmonic time $\tau$ are connected by the differential equation (Ivashchuk, Melnikov, and Zhuk (1989))

$$e^{\gamma(\tau)} d\tau = dt$$

where $\gamma(\tau) = \sum_{i=1}^{n} d_i \beta^i$. It is not difficult to get the connection

$$[\sinh \nu d_1 (\tau - \tau_0)]^{-1} = \sinh \sqrt{2\Lambda d_1 / \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - t_0)$$

With the help of this connection we obtain the expression for the scale factor $a_1$ with respect to the synchronous time

$$a_1(t) = \left[ \frac{\nu^2}{\Lambda} \right]^{1/2d_1} \left| \sinh \sqrt{2\Lambda d_1 / \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - t_0) \right|^{1/d_1}, \quad -\infty < t < +\infty$$

b) $\nu^2 < 0$: imaginary scalar field.

The solution of equation (10) in this case is

$$a_1(\tau) = \left[ |\nu^2| / \Lambda \right]^{1/d_1} \left\{ \cos |\nu^2|^{1/2} d_1 (\tau - \tau_0) \right\}^{-1/d_1}, \quad |\tau - \tau_0| \leq \pi / \left( 2d_1 |\nu^2|^{1/2} \right)$$

The connection between harmonic and synchronous times is

$$\cos |\nu^2|^{1/2} d_1 (\tau - \tau_0) = \left[ \cosh \sqrt{2\Lambda d_1 / \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - t_0) \right]^{-1}$$

In the synchronous time $t$ we have the solution

$$a_1(t) = \left[ |\nu^2| / \Lambda \right]^{1/2d_1} \left\{ \cosh \sqrt{2\Lambda d_1 / \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - t_0) \right\}^{1/d_1}, \quad -\infty < t < +\infty$$
It is clear that for $\nu^2 < 0$ the solution of the Euclidean analog of the equation (10) exists also. It means on quantum level that the transitions with changing of the topology take place in this case, for example, birth from nothing (Vilenkin (1983)).

c) $\nu^2 = 0$: absence of scalar field.

In this case the solution of the equation (10) in the harmonic time $\tau$ reads

$$a_1(\tau) = \left[ d_1^2 \Lambda \right]^{-1/2d_1} |\tau - \tau_0|^{-1/d_1}, \quad -\infty < \tau < +\infty$$

(19)

The harmonic and synchronous times are connected by formula

$$|\tau - \tau_0| = \exp \left[ \pm \sqrt{2\Lambda d_1 / \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - \tau_0) \right]$$

(20)

The solution (19) with the help of the expression (20) may be rewritten in the form

$$a_1(t) = \left[ d_1^2 \Lambda \right]^{-1/2d_1} \exp \left[ \pm \sqrt{2\Lambda / d_1 \left( \sum_{k=1}^{n} d_k - 1 \right)} (t - \tau_0) \right] \equiv$$

$$\equiv a_{(0)1} \exp \left[ \pm \sqrt{2\Lambda / d_1 \left( \sum_{k=1}^{n} d_k - 1 \right) t} \right]$$

(21)

where $t_0$ and $a_{(0)1}$ are constants of integration. It is easy to see that the solutions (15) and (18) tend asymptotically to (21) when $|t| \to \infty$.

The formula (21) (with the sign " + ") represents a multidimensional generalization of the de Sitter solution. If we put formally $n = 1, d_1 = 3$ in (21) we get the usual form of the scale factor in 4-dimensional de Sitter space-time: $a \sim \exp \left( \sqrt{\Lambda / 3t} \right)$. In the case $d_1 = 3, n > 1$ the higher dimensions have an imprint in the external space $M_1$ through the exponent: $\Lambda / 3 \to 2\Lambda / [3 (\sum_{k=2}^{n} d_k + 2)]$. We would like to stress once more, that $\Lambda$ and the parameters of the internal spaces are fine tuned according to equation (8).

To show more explicitly that (21) represent a multidimensional generalization of the de Sitter solution, let us consider $M_1$ as a $d_1$-dimensional flat space with the metric $g_{(1)} = \sum_{i=1}^{d_1} dx^i \otimes dx^i$. Then, the multidimensional metric (1) in synchronous system reads

$$g = -dt \otimes dt + a_1^2(t) \sum_{i=1}^{d_1} dx^i \otimes dx^i + \sum_{i=2}^{n} a_{(0)i}^2 g_{(i)}$$

(22)

where

$$a_1(t) = \exp Ht$$

(23)

We introduced the Hubble constant $H = \sqrt{2\Lambda / d_1 \left( \sum_{k=1}^{n} d_k - 1 \right)}$ and without loss of generality we have put $a_{(0)1} = 1$. 

5
The coordinates transformation

\[
y^0 = \frac{1}{H} \sinh Ht + \frac{H}{2} e^{Ht} |\vec{x}|^2
\]

\[
y^{d_1+1} = \frac{1}{H} \cosh Ht - \frac{H}{2} e^{Ht} |\vec{x}|^2
\]

\[
y^i = e^{Ht} x^i, \quad i = 1, \ldots, d_1
\]

(24)

gives \((D + 1)\)-dimensional metric

\[
g = -dy^0 \otimes dy^0 + \sum_{i=1}^{d_1+1} dy^i \otimes dy^i + \sum_{i=2}^{n} a^2_{(0)i} g(i)
\]

(25)

where the coordinates \(y\) satisfy the equation of the \((d_1 + 1)\)-dimensional hyperboloid

\[- (y^0)^2 + \sum_{i=1}^{d_1+1} (y^i)^2 = 1/H^2
\]

(26)

Thus, the generalized de Sitter solution is the hypersurface \(\mathcal{H}^{d_1+1} \times M_2 \times \cdots \times M_n\) in the \((D + 1)\)-dimensional space with the topology \(M^{d_1+2} \times M_2 \times \cdots \times M_n\) when \(\mathcal{H}^{d_1+1}\) is \((d_1 + 1)\)-dimensional hyperboloid, \(M^{d_1+2}\) is \((d_1 + 2)\)-dimensional Minkowski space and \(M_i (i = 2, \ldots, n)\) are freezed compact Einstein spaces.

The solution (22), (23) is the metric in representation of the stationary universe (Weinberg (1972), Misner, Thorne, and Wheeler (1974), Hawking, Ellis (1973), Birrell, and Davies (1982)) . We can choose another section of the hyperboloid if we take new coordinates \((\bar{t}, \theta^1, \ldots, \theta^{d_1})\):

\[
y^0 = \frac{1}{H} \sinh H\bar{t}
\]

\[
y^1 = \frac{1}{H} \cosh H\bar{t} \cos \theta^1
\]

\[
y^2 = \frac{1}{H} \cosh H\bar{t} \sin \theta^1 \cos \theta^2
\]

\[
y^3 = \frac{1}{H} \cosh H\bar{t} \sin \theta^1 \sin \theta^2 \cos \theta^3
\]

\[\vdots\]

\[
y^{d_1} = \frac{1}{H} \cosh H\bar{t} \sin \theta^1 \sin \theta^2 \cdots \cos \theta^{d_1}
\]

\[
y^{d_1+1} = \frac{1}{H} \cosh H\bar{t} \sin \theta^1 \sin \theta^2 \cdots \sin \theta^{d_1}
\]

(27)

In this coordinate system the metric (25) reads

\[
g = -d\bar{t} \otimes d\bar{t} + \frac{1}{H^2} \cosh^2 H\bar{t} \bar{g}(1) + \sum_{i=2}^{n} a^2_{(0)i} \bar{g}(i)
\]

(28)

where \(\bar{g}(1)\) is the metric on \(S^{d_1}\):

\[
\bar{g}_1 = d\theta^1 \otimes d\theta^1 + \sin^2 \theta^1 d\theta^2 \otimes d\theta^2 + \cdots + \sin^2 \theta^1 \cdots \sin^2 \theta^{d_1-1} d\theta^{d_1} \otimes d\theta^{d_1},
\]

(29)
Thus, for this section of the hyperboloid $H^{d_{1}+1}$ the factor space $M_{1}$ has positive constant curvature. It is well known property of the de Sitter space (Hawking, Ellis (1973), Birrell, and Davies (1982)).

4 CONCLUSION

In this paper we investigated multidimensional cosmological models with $n(n > 1)$ Einstein spaces in the presence of the cosmological constant $\Lambda$ and a homogeneous minimally coupled free scalar field as a matter source. Generalized de Sitter solution was found for positive cosmological constant and Ricci-flat external space for the case of static internal spaces with fine tuning of parameters. All internal spaces have positive curvature.

The solutions obtained here give us an interesting example of the development of Einstein’s idea concerning $\Lambda$-term. Although Einstein considered his original assumption the "biggest blunder" of his life, in multidimensional theories this idea has new important features. We saw here that the cosmological constant plays the role of a "double agent". From one side, it serves to keep the internal factor spaces $M_{i} (i = 2, \ldots, n)$ freezed through the fine tuning of parameters (see equation (8)) in the spirit of Einstein idea. From other side, the cosmological constant provides the de Sitter-like behaviour for the external space.

We would like to note that the solutions with exponential behaviour of the scale factors were found also in (Bleyer, Ivashchuk, Melnikov, and Zhuk (1994)) for the model with all Ricci-flat factor spaces and positive cosmological constant. However, the isotropization condition for all dimensions takes place in this model and there is no compactification of the internal spaces. For $\Lambda < 0$ classical as well as quantum wormhole solutions were found there. Classical wormhole solutions were obtained also in this paper if $\Lambda < 0$ and only one of the $M_{i}$ being Ricci-flat for the case of freezed internal dimensions with fine tuning parameters similar to the equation (8). In this case all internal spaces should have negative curvature.

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