Conformal Generators and Doubly Special Relativity Theories

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Abstract

In this paper, the relation between the modified Lorenz boosts, proposed in the doubly relativity theories and a linear combination of Conformal Group generators in $R^{1,d-1}$ is investigated. The introduction of a new generator is proposed in order to deform the Conformal Group to achieve the connection conjectured. The new generator is obtained through a formal dimensional reduction from a free massless particle living in a $R^{2,d}$ space. Due this treatment it is possible to say that even DSR theories modify light cone structure in $R^{1,d-1}$, it could remains, in some cases, untouched in $R^{2,d}$.

1 Introduction

There are some evidences that the special relativity must be modified. It has been claimed that Lorentz symmetry breaking could be observable in the future in high energy cosmic ray spectra [1], and corrections in the dispersion relation $E^2 - p^2 = m^2$ have been proposed [2]. On the other hand, quantum gravity models suggest that it could be desirable to review the Lorentz invariance relations. Indeed there are special features in these theories: the Planck longitude $l_p = \sqrt{\hbar G/c^3}$, its associated time scale $t_p = l_p/c$ and the Planck energy $E_p = \hbar/t_p$. They are thresholds beyond the physics should change dramatically. However, absolute values of longitude, time or energy are not in agreement with the Lorentz transformations, because they are not the same for different observers in different frames. Of course, whatever the deformation of the Lorentz boost introduced be, it must fit smoothly with the usual experience at energies far from Planck scales.

Several solutions for these problems have been proposed, in particular doubly special relativity (DSR) theories [9],[3],[4]. They are generalizations of the special relativity with two observer independent scales. Starting with the Fock-Lorentz transformations in the position space [7], [8], a modification of Lorentz boosts in the momentum space was proposed by Magueijo and Smolin [9], with a very similar approach. The treatment of the problem in the momentum space has been preferred because of the introduction of deformed dispersion relations, however when this choice is done, recovering the position space dynamics can be a highly problematic task due to the loss of linearity. Recently Kimberly, Magueijo and

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Medeiros, [10], have shown methods for obtaining position space of non linear relativity models from the usual momentum space formulation, using a free field theory.

Nowadays, DSR theories are of increased interest because they can be useful as a new tool in gravity theories, in Cosmology as an alternative to inflation, [5], [6], and in other fields like propagation of light [11], that is related, for instance, to cosmic microwave background radiation.

In this paper it is demonstrated that it is possible, formally, to obtain Fock-Lorentz and Magueijo-Smolin deformations, trough a reduction process and using the conformal group generators as the generators of the deformed Lorentz algebra.

More specifically, it is conjectured that the deformations of the Lorentz algebra in the Fock-Lorentz formulation, can be treated as a transformation made by a linear combination of conformal group generators and the momentum case, can be understood as the same process, but the inclusion of a new generator is needed. This new generator can be constructed from the same theory, but some new aspects of the dimensional reduction process proposed for the formers must be added. Finally the nonlinear induced algebra is shown. So, a relationship between space and momentum treatments is worth investigating as becoming from a bigger, underlaying theory that could bear the whole features of DSR theories.

The paper is organized as follows. In section 2, the conformal symmetry group of a $d$ space, massless relativistic particle is shortly reviewed. In section 3, isometries of a $d+2$ space plus a dimensional reduction process is recalled as underlying in the origin of those symmetries. In section 3, the position space DSR transformation are presented as combination of the conformal generators. Then, the momentum space DSR transformations are obtained using an extra generator. In section 4, the origin of this new generator is conjectured and the algebra including this new generator is shown. Conclusions and comments are presented in section 5.

## 2 Conformal Symmetry of Massless Particle

The infinitesimal transformations

$$\delta x^\mu = \omega^\mu_{\nu}x^\nu + \alpha^\mu + \beta x^\mu + 2(x\gamma)x^\mu - x^2\gamma^\mu \quad (2.1)$$

generate the conformal symmetry, $ds^2 \rightarrow ds'^2 = e^{2\sigma}ds^2$, on the $d$-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ with metric

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2.$$

Here the parameters $\omega^\mu_{\nu}, \alpha^\mu, \beta$ and $\gamma^\mu$ correspond to the Lorentz rotations, space-time translations, scale and special conformal transformations. Due to a nonlinear (quadratic) in $x^\mu$ nature of the two last terms in (2.1), the finite version of the special conformal transformations,

$$x'\mu = \frac{x^\mu - \alpha^\mu x^2}{1 - 2\alpha x + \alpha^2 x^2}, \quad (2.2)$$
is not defined globally, and to be well defined requires a compactification of $\mathbb{R}^{1,d-1}$ by including the points at infinity.

On the classical phase space with canonical Poisson bracket relations $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$, $\{x_\mu, x_\nu\} = \{p_\mu, p_\nu\} = 0$, the transformations (2.1) are generated by
\[
M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad P_\mu = p_\mu, \quad D = x_\mu p_\mu, \quad K_\mu = 2x_\mu(xp) - x^2 p_\mu. \tag{2.3}
\]
The generators (2.3) form the conformal algebra
\[
\{M_{\mu\nu}, M_{\sigma\lambda}\} = \eta_{\mu\sigma}M_{\nu\lambda} - \eta_{\nu\sigma}M_{\mu\lambda} + \eta_{\mu\lambda}M_{\sigma\nu} - \eta_{\nu\lambda}M_{\mu\sigma},
\]
\[
\{M_{\mu\nu}, P_\lambda\} = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu, \quad \{M_{\mu\nu}, K_\lambda\} = \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu,
\]
\[
\{D, P_\mu\} = P_\mu, \quad \{D, K_\mu\} = -K_\mu,
\]
\[
\{K_\mu, P_\nu\} = 2(\eta_{\mu\nu}D + M_{\mu\nu}), \quad \{D, M_{\mu\nu}\} = \{P_\mu, P_\nu\} = \{K_\mu, K_\nu\} = 0. \tag{2.4}
\]
The algebra (2.4) is isomorphic to the algebra $so(2, d)$, and by defining
\[
J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(P_\mu + K_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{d(d+1)} = D, \tag{2.5}
\]
can be put in the standard form
\[
\{J_{AB}, J_{LN}\} = \eta_{AL}J_{BN} - \eta_{BL}J_{AN} + \eta_{AN}J_{LB} - \eta_{BN}J_{LA} \tag{2.6}
\]
with $A, B = 0, 1, d, d+1$, and
\[
\eta_{AB} = diag(-1, +1, \ldots, +1, -1). \tag{2.7}
\]
The phase space constraint $\varphi_m \equiv p^2 + m^2 = 0$ describing the free relativistic particle of mass $m$ in $\mathbb{R}^{1,d-1}$ is invariant under the Poincaré transformations, $\{M_{\mu\nu}, \varphi_m\} = \{P_\mu, \varphi_m\} = 0$. Unlike the $P_\mu$ and $M_{\mu\nu}$, the generators of the scale and special conformal transformations commute weakly with $\varphi_m$ only in the massless case $m = 0$: $\{D, \varphi_0\} = 2\varphi_0 = 0$, $\{K_\mu, \varphi_0\} = 4x_\mu\varphi_0 = 0$.

### 3 Dimensional Reduction

It was demonstrated in [12], that a dimensional reduction process from a massless particle living in $(d + 2)$ dimensional space $\mathbb{R}^{2,d}$, with coordinates $H^A$ and metric (2.7), and introducing canonical momenta $\Pi_A$, ($\{H_A, \Pi_B\} = \eta_{AB}$) can transform the $so(2, d)$ generators
\[
\mathfrak{J}_{AB} = H_A \Pi_B - H_B \Pi_A, \tag{3.1}
\]
into the conformal generators in $(d)$-dimensional space $\mathbb{R}^{1,d-1}$. In order to achieve this, three constrains are added to the theory:
\[
\phi_0 \equiv \Pi_A \Pi^A = 0, \quad \phi_1 \equiv H^A H_A = 0, \quad \phi_2 \equiv H^A \Pi_A = 0. \tag{3.2}
\]
The process is implemented through suitable canonical transformations and then a dimensional reduction is performed onto the surface defined by $\phi_1$ and $\phi_2$. So in terms of new $\mathbb{R}^{2,d}$ canonical variables $\tilde{H}_{\mu}$, $\tilde{\Pi}_\mu$ with $\{\tilde{H}_{\mu}, \tilde{\Pi}_{\mu}\} = 1$ as defined in [12], the $so(2,d)$ generators are the following:

\[ \mathcal{J}_{\mu\nu} = \tilde{H}_{\mu} \tilde{\Pi}_{\nu} - \tilde{H}_{\nu} \tilde{\Pi}_{\mu}, \quad \mathcal{J}_{\mu+} = \tilde{\Pi}_{\mu}, \quad \mathcal{J}_{d(d+1)} = \tilde{H}_{\mu} \tilde{\Pi}_{\mu} + 2 \frac{\tilde{\Pi}_{+}}{\tilde{H}^{-}} \phi_1 - \phi_2, \tag{3.4} \]

\[ \mathcal{J}_{\mu-} = 2(\tilde{H}_{\mu} \tilde{\Pi}_{\mu}) \tilde{H}_{\mu} - (\tilde{\Pi}_{\mu} \tilde{H}_{\mu}) \tilde{\Pi}_{\mu} + \frac{1}{\tilde{H}^{-2}} (\tilde{\Pi}_{\mu} + 4 \tilde{\Pi}_{+} \tilde{H}_{-} \tilde{H}_{\mu}) \phi_1 - 2 \tilde{H}_{\mu} \phi_2. \]

To execute this process it is supposed that $\tilde{H}^{-} \neq 0$. The constraints now, look like:

\[ \tilde{\Pi}_{\mu} \tilde{\Pi}^{\mu} = 0, \quad \tilde{H}^{+} = 0, \quad \tilde{\Pi}_{-} = 0. \tag{3.5} \]

The variables $\tilde{\Pi}_{\mu}$ and $\tilde{H}_{\mu}$ are observable (gauge invariant) variables with respect to the constraints (3.5) and (3.6), whereas $\tilde{H}^{-} = H^{-}$ and $\tilde{\Pi}_{+} = \Pi_{+}$ are not. They can be removed by introducing the constraints

\[ \phi_3 \equiv H^{-} + 1 = 0, \quad \phi_4 \equiv \Pi_{+} = 0 \tag{3.7} \]

as gauge fixings of (3.6) and the reduction to the physical surface achieves the conformal generators in $\mathbb{R}_{1,d-1}$. Now, with the identification $x^\mu = \tilde{H}^{\mu}$ and $p_\mu = \tilde{\Pi}_{\mu}$ and (3.5), the original massless relativistic particle is retrieved.

4 DSR transformations

4.1 Fock-Lorentz transformations

The Fock-Lorentz transformation [7], [8], are introduced as general linear-fractional transformations between spatial and temporal coordinates and the result is led by symmetry arguments to the final form:

\[ t' = \frac{\gamma(u)(t - ur/c^2)}{1 - (\gamma(u) - 1)ct/R + \gamma(u)ur/Rc}, \tag{4.1} \]

\[ r^\parallel = \frac{\gamma(u)(r^\parallel - ut)}{1 - (\gamma(u) - 1)ct/R + \gamma(u)ur/Rc}, \tag{4.2} \]

\[ r^\perp = \frac{r^\perp}{1 - (\gamma(u) - 1)ct/R + \gamma(u)ur/Rc}. \tag{4.3} \]

where $r^\parallel$ is the component of the position vector parallel to the boost $u$ and $r^\perp$ is the perpendicular one. Where $\gamma$ is the Lorentz factor $\sqrt{1 - u^2/c^2}$ and $R$ is a constant with the dimension of length.
These transformations can be seen as Lorentz transformations for the quantities:

\[ \tilde{t} = \frac{t}{1 + ct/R}, \quad \tilde{r} = \frac{r}{1 + ct/R} \]  (4.5)

They can be treated as results of a infinitesimal transformations:

\[ \delta x^\mu = \alpha^\nu \{ x^\mu, 2x_\nu(x^p) \} \]  (4.6)

where \( x^0 = ct \).

In order to obtain this generator, it can be represented as the projection of

\[ (\mathbf{J}_\mu - \mathbf{J}_\mu + \mathbf{J}_{d(d+1)}) \],  (4.7)

in \( R^{d+2} \) space, which is a nonlinear combination of the isometries there. However \( 2x_\nu(x^p) \) is not a symmetry of the Klein-Gordon equation because \( \{2x_\nu(x^p), \varphi_0\} = 4p_\mu(x^p) + 4x_\mu \varphi_0 \neq 0 \). So, it is not possible to include it into the generator set of the massless relativistic particle symmetries.

An alternative, an linear solution is to see (4.5) as (2.2) reduced to the surface \( x^2 = 0 \) and with the choice \( \alpha = (-1/2R, 0, 0, 0) \) and identifying \( x^0 \) with \( ct \). In this sense, the Fock Lorentz transformations can be seen as a linear combination of conformal generators

\[ M_{0\mu} + K_{\mu}, \]

that is a projection of

\[ (\mathbf{J}_{\mu\nu} + \mathbf{J}_{\mu-}) \]

which is linear too, for a particle living on the cone \( x_1^2 + x_2^2 + x_3^2 = x_0^2 \).

### 4.2 Momentum space transformations

On the other hand, the Lorentz boost proposed in [9]

\[ K^i = L_{0i} + l_p p^j p_\mu \frac{\partial}{\partial p_\mu} \]  (4.8)

where \( l_p \) is the Planck length, can be exponentiated as

\[ K^i = U^{-1}(p_0)L_{0i}U(p_0) \]  (4.9)

where \( U(p_0) = exp(l_p p_0 p_\mu \frac{\partial}{\partial p_\mu}) \), and the action of \( U(p_0) \) over \( p_\mu \) is

\[ U(p_0)p_\mu = \frac{p_\mu}{1 - l_p p_0} \]  (4.10)

The generator added to the Lorentz boost in (4.8) is the operational representation of \( p^i(x^p) \) and it can be seen as part of some \( \tilde{K}_\mu = 2p_\mu(x^p) - p_\mu^2x_\mu \) generator that, with the constrain \( \varphi_0 \) in mind, produces the following transformation:

\[ p'^\mu = -p^\mu - \frac{\alpha^\mu \varphi_0}{1 - 2\alpha p + \alpha^2 \varphi_0}, \]  (4.11)
to achieve the identification with (4.10), the transformation parameter must be \( \alpha^\mu = (-l_p/2, 0, 0, 0) \).

It is possible then, to prescribe how to obtain the position and momenta transformation using conformal generators with some \( \tilde{K} \) added. Where does this new generator come from? Some clues will be seen in the next section.

### 5 Extended Non-linear Conformal Algebra

It is possible, through a rather tricky process, to introduce the new generator \( \tilde{K}_\mu \). To do this, we can perform a new canonical transformation in (3.5), \( \tilde{H}^A \rightarrow -\tilde{\Pi}^A \) and \( \tilde{\Pi}^A \rightarrow \tilde{H}^A \) and now to identify \( x^\mu = \tilde{H}_\mu \) and \( p_\mu = \tilde{\Pi}_\mu \). Doing this, \( \mathcal{J}_{\mu-} \) is projected as \( \tilde{K} \), and \( \mathcal{J}_{\mu+} \) is projected as \( x^\mu \), but the particle has now the constrain \( x^2 = 0 \).

If we want to use this generator on the \( \varphi_0 = p^2 = 0 \) surface and we wish to maintain this kind of description as underlying in the origin of the generator, it is necessary to set \( xp \neq 0 \), otherwise \( D, K \) and \( \tilde{K} \) become zero and the theory becomes trivial. This condition means that \( x^2 = 0 \) can be seen as a gauge fixing for \( p^2 = 0 \), because \( \{x^2, p^2\} = 4xp \neq 0 \).

\( \tilde{K} \) is a symmetry of the Klein-Gordon equation too, in fact: \( \{\tilde{K}, \varphi_0\} = 2\varphi_0 P_\mu = 0 \). So the relation of \( \tilde{K} \) with the other generators is.

After the inclusion of \( \tilde{K} \), aside the relations (2.4) we have:

\[
\begin{align*}
\{M_{\mu\nu}, \tilde{K}_\lambda\} &= \eta_{\mu\lambda} \tilde{K}_\nu - \eta_{\nu\lambda} \tilde{K}_\mu, \\
\{D, \tilde{K}_\mu\} &= \tilde{K}_\mu, \\
\{\tilde{K}_\mu, P_\nu\} &= 2P_\mu P_\nu - \varphi_0 \eta_{\mu\nu}, \\
\{\tilde{K}_\mu, \tilde{K}_\nu\} &= 0, \\
\{\tilde{K}_\mu, K_\nu\} &= 4(M_{\mu\nu} - \eta_{\mu\nu})D - 2P_\mu K_\nu - \varphi_0 (q^2 \eta_{\mu\nu} - 2q_\mu q_\nu) \tag{5.1}
\end{align*}
\]

These relations generate a non-linear algebra on the constrain \( \varphi_0 \) surface

### 6 Discussion and outlook

To conclude, let us summarize the obtained results and discuss shortly some problems that deserve a further attention.

Transformations of DSR theories can be partially achieved by a suitable combination of conformal generators. To accomplish this in the momentum space, an extra generator must be added to the conformal group. This one can be obtained, formally, from the same \( R^{d+2} \) massless particle theory, as the others, but a rather tricky process must be performed. Anyway, this generator is a symmetry of the Klein-Gordon equation and when it is included, a non-linear algebra arise just on mass shell.

An important feature of this treatment is the evidence that even the Lorentz invariance is broken in \( R^d \) by DSR theories, when we are dealing with some special cases as a massless relativistic particle living in the cone \( x^2 = 0 \), it could remains untouched in \( R^{d+2} \) because the deformed Lorentz boost in the former are linear combinations of the images of the isometries.
of the latter. Even although we can accept this idea, there are not clues about the conditions for the parameters, in order to include $l_p$, this must be a future task. On the other hand, it will be important to investigate the geometry of the systems here detailed and to analyze the case when totally arbitrary parameters are included.

Of course, if a discrete space-time is claimed, the notion of a point particle must be abandoned, so all classic treatments must be replaced by a suitable theory. In order to do this, the ideas proposed here must be translated to the field theories ambit.

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