Preferred foliation effects in quantum general relativity

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Received 17 November 2009, in final form 1 March 2010
Published 17 May 2010
Online at stacks.iop.org/CQG/27/135014

Abstract
We investigate the infrared (IR) effects of Lorentz violating terms in the gravitational sector using functional renormalization group methods similar to Reuter and collaborators (Reuter 1998 Phys. Rev. D 57 971). The model we consider consists of pure quantum gravity coupled to a preferred foliation, described effectively via a scalar field with non-standard dynamics. We find that vanishing Lorentz violation is a UV attractive fixed point of this model in the local potential approximation. Since larger truncations may lead to differing results, we study as a first example effects of additional matter fields on the RG running of the Lorentz violating term and provide a general argument why they are small.

PACS numbers: 11.10.Hi, 11.30.Cp, 11.30.Qc, 98.80.Qc

1. Introduction and motivation

One of the major unsolved problems of theoretical physics is the construction of a physically acceptable and predictive UV completion of quantum gravity. The perturbative non-renormalizability of gravity hints that its quantization requires new physics, such as a preferred foliation modeled, e.g., in Einstein–aether theory [2]. Preferred foliations gained a lot of attention recently after Hořava’s proposal [3] of a perturbatively renormalizable quantum theory of gravity. Hořava’s model is a higher derivative gravity theory that avoids troublesome unitary ghosts by retaining second derivatives in time while allowing higher derivatives in space. The preferred foliation can be encoded [4] in a Stäckelberg-type field \( \phi \) which defines an irrotational unit time-like vector field \( n_\mu = \phi_\mu / \sqrt{g^{\nu\rho} \phi_\nu \phi_\rho} \), where the preferred foliation is defined by constant \( \phi \) surfaces. The extrinsic curvature of the preferred foliation is \( K_{\mu\nu} = \mathcal{L}_n (g_{\mu\nu} + n_\mu n_\nu) \), where \( \mathcal{L}_n \) denotes the Lie derivative in the direction of \( n \). The low
energy limit of Hořava’s theory also contains in addition to the usual Einstein–Hilbert action
the Lorentz violating action given by
\[ S_{LV} = \frac{b}{16\pi G_N} \int d^4x \sqrt{|g|} K^2 + \text{higher derivative terms}, \tag{1} \]
where \( G_N \) denotes Newton’s constant and \( K = g^{\mu\nu} K_{\mu\nu} \). \( b \) is a new coupling constant that
vanishes in general relativity. A possible \( K^{\mu\nu} K_{\mu\nu} \) term is absorbed in Hořava’s definition
of the speed of light. Modification (1) is a low-energy effective field theory, describing the
effects of quantum gravity at a scale \( k \) where higher derivative terms are negligible. In order to
bring this theory in contact with the observation, we use the renormalization group equation
to evolve this action to IR scales at which experimental tests for Lorentz violation are being
performed.

A Wilsonian approach to the renormalization group equation for (1) leads to considering
nonperturbative renormalization of Einstein–aether theories, which is very much involved. A
much simpler on-shell equivalent to low energy Hořava theory has been found in [5] in terms
of the cuscuton field:
\[ S_{cusc} = \int d^4x \sqrt{|g|} \left( \sqrt{g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}} - \frac{a_1}{2} \phi^2 \right). \tag{2} \]
This model generates the \( K^2 \) term dynamically and, given sufficient boundary conditions, leads
to the dynamical emergence of a constant mean curvature foliation. The quadratic cuscuton
coupling \( a_1 \) (which has dimension mass\(^{-2}\) in our normalization) is related to the Lorentz
violating parameter \( b \) by
\[ a_1 = \frac{8\pi G_N}{b}. \tag{3} \]
The observational constraints on the gravitational Lorentz violating parameter \( b \) are much
weaker than the bounds on Lorentz violating parameters in extensions of the standard model.
A cosmological bound constraining \(-b = 0.003 \pm 0.014\) at 95% CL can be found in [6].
Note that the Lorentz violating parameter \( b \) is dimensionless, so there is no power counting
argument for its running.

The advantages of Reuter’s programme, which is found to be at the same time
nonperturbative and background independent, suggest to use this Wilsonian framework for
the investigation of Lorentz violating models. As a start into the Wilsonian investigation of
Lorentz violating models of this type, we will consider an adapted version of the cuscuton
model (2) generalized to an arbitrary even power-series potential \( V(\phi) \) and discuss the \( \beta \)-
function of the quadratic cuscuton coupling in the presence of pure quantum gravity. The main result is that in the local potential approximation the \( \beta \)-function of the Lorentz violating
parameter \( b \) has a UV attractive Gaussian fixed point \( b = 0 \), i.e. the \( \beta \)-function of the absolute
value \(|b|\) is negative in the vicinity of this fixed point. Fortunately, the corresponding IR
growth of \( b \) is bounded due to an IR decreasing \( \beta \)-function \( \beta(b) \), if we assume a sufficiently
semi-classical RG flow in the gravitational sector. We show that coupling additional matter
fields to this theory has only minor effects on the RG flow of the Lorentz violating parameter,
least for small matter couplings and Lorentz violations \( b \). Note that since the cuscuton model
differs from Hořava’s theory, both through off-shell terms and higher derivative contributions,
one cannot apply these results to Hořava’s theory itself, but one should consider them as a toy
model to gain a first insight.

The outline of this paper is as follows. In section 2 we provide the background on
Wetterich’s functional renormalization group equation [7] necessary in order to be self-
contained. Section 3 explains the basic idea of IR attractive gauge symmetries using a
toy model. In section 4 we calculate explicitly the flow of the parameter \( b \) within the cuscuton
model coupled to quantized gravity. The introduction of matter into this model is discussed in section 5. We give an outlook toward further investigations in this direction in section 6.

2. The functional renormalization group equation (FRGE)

In this section we present a brief review of Wetterich’s functional renormalization group equation (FRGE) [7] to provide the prerequisites for this paper using a Euclidean scalar field theory; detailed introductions can e.g. be found in [8–11] and reviews on the application to gravity in [12–14].

The starting point for formulating the FRGE is the partition function defined by the regularized Euclidean path integral

\[
\exp(W_k[J]) = \int D\chi e^{-S(\chi) - \Delta S_k(\chi) + J \chi},
\]

where \(\Lambda\) denotes an overall UV cutoff, \(S\) the bare action (at the cutoff) and \(\Delta S_k\) an IR regulator that suppresses momentum modes with \(p^2 < k^2\) by giving them a mass of order \(k\) while effectively vanishing for modes with \(p^2 > k^2\). A typical suppression term is given by

\[
\Delta S_k[\chi] = \frac{1}{2} \int d^4x \chi(x) R_k(-\Box) \chi(x) = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{\chi}(p) R_k(p^2) \tilde{\chi}(-p).
\]

The effective average action \(\Gamma_k\) is defined via the Legendre transformation and subsequent subtraction of the IR regulator \(\Gamma_k[\phi] = \int d^4x J(x) \phi(x) - W_k[J] - \Delta S_k[\phi]\),

\[
\Gamma_k[\phi] := \int d^4x J(x) \phi(x) - W_k[J] - \Delta S_k[\phi],
\]

where \(\phi(x) = \langle \chi(x) \rangle = \delta W_k[J]/\delta J(x)\) is the expectation value of the field \(\chi\). To calculate the scale derivative \(\partial_t = k \partial_k\) of \(\Gamma_k\) we take the scale derivative of (6), substitute \(W_k\) with (4) and re-express the rhs in terms of variations of \(\Gamma_k\), resulting in Wetterich’s FRGE:

\[
\dot{\Gamma}_k[\phi] = \frac{1}{2} \text{Tr} \left[ \dot{R}_k \cdot \left( \Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right],
\]

where a dot denotes \(\partial_t\) and \(\Gamma_k^{(2)}\) denotes the second variation of \(\Gamma_k\) w.r.t. the fields. The trace extends over all fields.

The usual effective action \(\Gamma\) is attained in the limit \(k \to 0\), since \(\Delta S_k\) vanishes in this limit. The usual bare action is attained in the limit \(k, \Lambda \to \infty\), because \(\Delta S_k\) diverges in this limit, making the saddle point approximation for (4) exact. The effective average action \(\Gamma_k\) is thus an interpolation between the usual bare action \(S\) (for \(\Lambda \to \infty\)) and the usual effective action \(\Gamma\) and effectively describes physics at the scale \(k\), because its tree-level accurately describes the effects of momentum modes above \(k\). This allows one to derive an effective action without explicitly solving a path integral by specifying an initial condition \(\Gamma_k = \Lambda\) and using (7) to evolve \(\Gamma_k\) to the physically relevant scale \(k\).

To calculate \(\beta\)-functions from (7), one expands \(\Gamma_k\) in terms of field monomials in the flow equation and equates the coefficients of the field monomials on the lhs with the corresponding coefficient on the rhs. However, the trace on the rhs of the flow equation produces in general an infinite number of field monomials, meaning that one cannot stick with an action of a particular form, but is forced to make the most general ansatz compatible with field content and symmetries. Using the most general ansatz for the effective average action however yields an insurmountable calculation, so for practical purposes one will have to make an ansatz by selecting the ‘most important’ monomials and expand the trace on the rhs only within this ansatz. The strength of the flow equation is that one can use expansions that are not necessarily perturbative, in particular an expansion in the number of derivatives of the field operator.
lowest order of the derivative expansion is the local potential ansatz, which reads for a standard scalar field
\[
\Gamma_1[\phi] = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V_1(\phi) \right).
\] (8)
Practically, one will consider only the lowest dimensional monomials, so imposing a \( \mathbb{Z}_2 \) symmetry, we may consider
\[
V_1(\phi) = m^2 \phi^2 + \frac{\lambda_4}{4!} \phi^4 + \cdots.
\] (9)
The focus of this paper is the renormalization of the cuscuton field coupled to gravity, which we will do in the local potential approximation. For this we have to give a generalization of the FRGE methods so gravity can be included. An important ingredient is background-field methods, as used in various applications to gauge and gravity theories. This is particularly important in gravity, since coarse graining in a background-independent theory needs a split of the metric \( g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \) into an arbitrary background metric \( \bar{g}_{\mu\nu} \) and fluctuations \( h_{\mu\nu} \), which are not assumed to be small. Following Reuter [1], we use the background Laplacian to define the momentum scale \( k \) of the fluctuation field. In contrast to the perturbative graviton expansion, however, one does neither assume a fixed background nor that the fluctuations are small; it is hence not necessary to make an expansion in terms of gravitons for the price that the effective action of gravity depends on both \( \bar{g}_{\mu\nu} \) and \( h_{\mu\nu} \).

The trace on the rhs of the flow equation can then be expanded background independently in terms of curvature invariants using the heat kernel expansion. The first two terms in this expansion yield the invariants occurring in the Einstein–Hilbert action. The Einstein–Hilbert truncation with harmonic background field gauge fixing reads
\[
\Gamma_{EH}[\bar{g}, h] = 2\kappa^2 \int d^4x \sqrt{\bar{g}} (-R[\bar{g}] + 2\Lambda) \\
+ \kappa^2 \int d^4x \sqrt{\bar{g}} D^\rho \left( h_{\rho\sigma} - \bar{g}_{\rho\sigma} \frac{h}{2} \right) \bar{g}^{\mu\nu} D^\sigma \left( h_{\mu\nu} - \bar{g}_{\mu\nu} \frac{h}{2} \right) + \text{ghosts},
\] (10)
where \( h = \bar{g}^{\mu\nu} h_{\mu\nu} \) and \( \kappa^{-2} = 32\pi G_N \). We did not give the explicit form of the ghost action, since it does not explicitly contribute to the running of the quadratic cuscuton coupling in our truncation.

The \( R_k \) and \( \Gamma_k^{(2)} \) terms appearing in the FRGE (7) include only the functional derivatives with respect to the fluctuation fields. The background fields set the scale \( k \) present in the regulator function, i.e. we define for \( h_{\mu\nu} \), as well as for all other dynamical fields, the regulator action with respect to \( \bar{g}_{\mu\nu} \) and the corresponding Laplacian \( \square \):
\[
\Delta S_k[\bar{g}, h] = \kappa^2 \int d^4x \sqrt{\bar{g}} h_{\mu\nu} R_k^{\mu\nu\rho\sigma}(\bar{g}, -\square) h_{\rho\sigma}.
\] (11)
The background gauge condition is satisfied for vanishing fluctuations, so we may, after the FRGE has been derived, set the averaged fluctuations to zero, i.e. \( h_{\mu\nu} = 0 \). Then the background field \( \bar{g}_{\mu\nu} \) can be interpreted as the averaged quantum metric field.

A useful trick [15] in the evaluation of the trace on the rhs of the flow equation is to obtain a formal expression \( \frac{1}{2} \text{Tr}(\hat{O} \phi) = F[\phi] \), so the flow equation reads \( \sum \lambda_i O_i[\phi] = \Gamma[\phi] \), where \( \lambda_i \) denotes a coupling and \( O_i[\phi] \) a field monomial. It is now useful to insert families of field configurations that (1) project onto the truncation on the lhs; (2) identify the terms in the truncation uniquely and (3) simplify the evaluation of \( F \), e.g., constant matter fields to project onto a local potential ansatz or flat metrics to project onto monomials that do not involve curvature invariants.
3. IR symmetries: general scenario

Let us assume that we integrate out short-distance physics above a very high scale $\Lambda$ and hence produce an effective average action $\Gamma_\Lambda[\phi_i]$ with the field content $\{\phi_i\}_{i \in \mathbb{Z}}$. Let us furthermore consider a gauge group $G$ acting on the field content $\{\phi_i\}$, so the gauge invariant field content is $\{\phi_i\}/G$. The question that we want to consider is whether this particular gauge symmetry is IR attractive or whether one needs fine-tuning to retain gauge symmetry in the IR. To answer this question we use the exact renormalization group equation and evolve $\Gamma_\Lambda/\Lambda$ down to an effective scale $k$ and observe whether or not the $\beta$-functions of the couplings between physical and gauge degrees of freedom are positive for a finite range of couplings. Since the exact renormalization group for the effective average action requires us to consider the most general action that is compatible with our field content and symmetries, we have to consider a truncation that contains all couplings that are ‘low dimensional’. Similar ideas have been investigated in [16–18].

Let us consider an ultralocal toy example consisting of a real scalar multiplet $\{\phi_i\}_{i=1}^n$ with an ultralocal gauge group acting as

$$\phi_i(x) \mapsto \lambda(x) + \phi_i(x). \quad (12)$$

Let us change to the variables

$$\varphi_i = \phi_i - \phi_{i+1} \quad \text{for} \quad 1 \leq i < n,$$

$$\Phi = \sum_{i=1}^n \phi_i, \quad (13)$$

where $\{\phi_i\}_{i=1}^{n-1}$ are physical and $\Phi$ is the pure gauge. To break this gauge invariance in a controlled manner, let us introduce an additional ‘Stückelberg field’ $\chi$ transforming as $\chi(x) \mapsto n\lambda(x) + \chi(x)$ under gauge transformations, so the most general local potential ansatz with a standard kinetic term allowed by gauge symmetry is

$$\Gamma_k = \int d^4x \left( \frac{1}{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_i}{\partial x} + \frac{1}{2} \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} - \frac{1}{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \Phi}{\partial x} + V_k(\varphi_i, \Phi - \chi) \right). \quad (14)$$

Ultralocality allows us to fix a gauge $\chi \equiv 0$, making $\Phi$ physical in this gauge such that the original gauge symmetry means that $\Phi$ decouples from the theory. Let us simplify the discussion by considering $n = 2$ and a truncation of the local potential to at most dimensionless couplings and let us impose an individual $\mathbb{Z}_2$ symmetry, such that the truncation becomes

$$\Gamma_k = \int d^4x \left( \frac{1}{2} \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_i}{\partial x} + \frac{1}{2} \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{k^2 m_k^2}{2} \varphi_i^2 + \frac{k^2 M_k^2}{2} \Phi^2 + \frac{a_k}{4!} \varphi_i^4 + \frac{b_k}{4} \varphi_i^2 \Phi^2 + \frac{c_k}{4!} \Phi^4 \right). \quad (15)$$

The $\beta$-functions can be extracted from (A.2):

$$\beta(M_k^2) = -2M_k^2 - \frac{1}{32\pi^2} \left( -\frac{b_k}{(1+m_k^2)^2} + \frac{c_k}{(1+M_k^2)^2} \right), \quad (16a)$$

$$\beta(b_k) = \frac{1}{4\pi^2} \left( \frac{a_k b_k}{1+m_k^2} + \frac{b_k c_k}{1+M_k^2} + \frac{2b_k^2}{(1+m_k^2)(1+M_k^2)^2} + \frac{2c_k^2}{(1+m_k^2)^3 (1+M_k^2)^2} \right), \quad (16b)$$

$$\beta(c_k) = \frac{3}{4\pi^2} \left( \frac{b_k^2}{(1+m_k^2)^3} + \frac{c_k^2}{(1+M_k^2)^3} \right). \quad (16c)$$
We see that the $\beta$-function of the dimensionless mass $M^2_k$ is always negative whereas the $\beta$-functions of the four-point couplings $b_k$ and $c_k$ are always positive if $a_k, b_k, c_k > 0$, meaning that the decoupling limit $M_k$ is large and $b_k \to 0$ is IR attractive in this regime. This implies that the renormalization group flow drives toward original gauge symmetry, so no fine-tuning is necessary.

The opposite scenario is more familiar: if we consider a generic gauge theory, but forget about gauge symmetry, then there are in general many low-dimensional field monomials that are forbidden in the gauge theory that may enter the theory without gauge symmetry. It is very unlikely that the unphysical degrees of freedom will decouple from the physical degrees of freedom, since this requires that all new field monomials be RG irrelevant, whereas low-dimensional field monomials have the tendency to be RG relevant.

The most robust predictions of Wilsonian renormalization come from the discussion of fixed points, which corresponds to the determination of universality classes. The IR-symmetry scenario can be seen at a glance when one considers the Gaussian fixed point at which the masses are relevant and the four point couplings are (marginally) irrelevant. The IR symmetries in the vicinity of any fixed point are given by the symmetries of the relevant, marginally relevant and completely marginal directions at the fixed point. Note that e.g. the IR symmetry may be affected by a nontrivial wavefunction renormalization. However, this is not the case if we are dealing with a Gaussian fixed point due to the following argument: if we introduce wavefunction renormalizations $Z_1, Z_2$ in equation (15), then the $\beta$-function of the symmetry breaking coupling $B = Z_1 Z_2 b$ is $\beta(B) = (\eta_1 + \eta_2) B + Z_1 Z_2 \beta(b)$, where $\eta_i = \partial_t Z_i$. We obtain that $B$ has a Gaussian fixed point if and only if $b$ has one and that $\partial B \beta(B) = \partial b \beta(b)$ at the Gaussian fixed point $b = B = 0$. Note that the stability of a non-Gaussian fixed point may be affected by $\eta_i$. The running of $Z_i$ was however minimized in test models by using the optimized cutoff \cite{19}, so to gain a first insight, one may neglect the effect of $\eta_i$, even for the discussion of generic fixed points.

4. Explicit renormalization of the cuscuton model

In this section we discuss the renormalization of an adapted version of the cuscuton model \cite{5, 6} coupled to the Einstein–Hilbert truncation of general relativity in order to investigate the quantum behavior of the $K^2$ term in the Lorentz violating action (1). We make a general $\mathbb{Z}_2$-symmetric power-series potential ansatz for the effective action

\[ \Gamma_k[\bar{g}, h, \phi] = \Gamma_{EH}[\bar{g}, h] + \int d^4x \sqrt{\bar{g}} \left( \sqrt{\bar{g}}^{\mu \nu} \phi,_{\mu} \phi,_{\nu} + \lambda^2 + \infty \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} \phi^{2n} \right) \]

including the quadratic cuscuton term (2) for $n = 1$. Note that, in order to have a background-independent FRGE, we introduced $\lambda = k^2 \tilde{\lambda}$ with an arbitrarily small $\tilde{\lambda}$ in the kinetic term. This parameter is treated as an external parameter and defines a family of quantum field theories. We show that the limit $\tilde{\lambda} \to 0$ is well defined after the FRGE has been derived, leading to the cuscuton model. The sole purpose of this parameter is to regularize the cuscuton propagator without introducing a cuscuton background, as it will become clear below. We will treat gravity within the Einstein–Hilbert truncation and focus on the renormalization of the coupling constants $\kappa^2, \Lambda$ and $a_n$.

We start by collecting all terms required to evaluate the rhs of the flow equation (7). The second variations $\Gamma_k^{(2)}$ are given in appendix B. Without loss of generality, we may insert families of fields that project onto our truncation and which allow us to distinguish the field monomials therein. For extracting the $\beta$-functions of the coupling constants $a_n$, it is therefore sufficient to insert a flat Euclidean background metric $\bar{g}_{\mu \nu}(x) = \delta_{\mu \nu}$ as well as a constant
cuscuton field $\phi(x) = \phi$. Note that this is only a convenient choice that simplifies our calculations; the $\beta$-functions constructed therewith are generally valid.

As regulators $(R_k)_{AB}$ for the individual fields $A, B \in \{\phi, h_{\mu\nu}\}$ we use the properly normalized optimized regulator [20] and define

$$
(R_k)_{\phi\phi} = \frac{1}{\lambda} (\Box + k^2) \Theta \left(1 + \frac{\Box}{k^2}\right),
$$

$$
(R_k)_{hh} = 2k^2 K_{\mu\nu\rho\sigma} (\Box + k^2) \Theta \left(1 + \frac{\Box}{k^2}\right),
$$

where all indices are raised and lowered with the background metric and where

$$
K_{\mu\nu\rho\sigma} = \frac{1}{4} \left( \bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} + \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho} - \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \right).
$$

The regularized second variations $(\dot{\Gamma}_k^{(2)} + R_k)_{AB}$ can be split into a $\phi$-dependent part $(F_k)_{AB}$ and $\phi$-independent part $(G_k)_{AB}$. The inverse is then given by the geometrical series

$$
\frac{1}{\Gamma_k^{(2)} + R_k} = \sum_{l = 0}^{\infty} (-F_k \cdot P_k)^l_{CB},
$$

where $(P_k)_{AB} := (G_k)^{-1}_{AB}$ are the regularized propagators given by

$$
(P_k)_{\phi\phi} = \frac{\lambda}{k^2 + \lambda \alpha_1},
$$

$$
(P_k)_{hh} = \frac{4K_{\mu\nu\rho\sigma}}{2\pi^2 (k^2 - 2\lambda - 2\pi^2)},
$$

for $k^2 \neq 0$ and $k^2 - 2\lambda \neq 0$ we can always choose $\lambda$ sufficiently small in order to expand (21) in $\lambda$. Thus, the propagators are the power series in $\lambda$ with non-negative powers.

It turns out that the cuscuton does not contribute to the flow of the coupling constants $\alpha_1$ in the limit $\lambda \to 0$, since all internal cuscuton propagators come with $\lambda^1$ and all field terms $(F_k)_{AB}$ come with non-negative powers of $\lambda$ as well. Even traces including $(\dot{R}_k)_{\phi\phi} \sim \lambda^{-1}$ do not contribute to the flow of the cuscuton potential, since they occur in the following combination:

$$
\text{Tr}[(\dot{R}_k)_{\phi\phi} \cdot (P_k)_{\phi\phi} \cdot (F_k)_{\phi A} \cdot \ldots \cdot (P_k)_{\phi\phi}] = O(\lambda^1).
$$

The cuscuton only contributes to the flow of the gravitational sector, i.e. $\kappa$ and $\Lambda$, via the trace

$$
\frac{1}{2} \text{Tr}[(\dot{R}_k)_{\phi\phi} \cdot (P_k)_{\phi\phi}] = O(\lambda^0).
$$

In the limit $\lambda \to 0$ this contribution to the renormalization of $\kappa$ and $\Lambda$ is equal to the contribution of a canonical massless scalar field and does not depend on the cuscuton potential. Thus, we find that in the limit $\lambda \to 0$ only gravity contributes to the $\beta$-functions of $\alpha_1$. In particular, only one term is contributing to the $\beta$-function of the quadratic term $\alpha_1$:

$$
- \frac{1}{2} \text{Tr}[(\dot{R}_k)_{hh} \cdot (P_k)_{hh} \cdot (F_k)_{hh} \cdot (P_k)_{hh}],
$$

with the field term

$$
(F_k)_{hh} = -\alpha_1 K_{\mu\nu\rho\sigma} \phi^2 + O(\phi^4).
$$

This leads to the dimensionless $\beta$-function for $\tilde{\alpha}_1 = k^2 \alpha_1$:

$$
\beta(\tilde{\alpha}_1) = \tilde{\alpha}_1 \left(2 + \frac{5}{192\pi^2} \frac{\beta(k^2) + 8k^2}{(1 - 2\lambda)^2}\right),
$$

7
where we introduced the dimensionless coupling constants \( \tilde{\lambda}_i \) defined by \( \lambda_i = k^{\text{dim}_{\lambda_i}} \tilde{\lambda}_i \). The Lorentz violation is governed by the parameter \( b = (4a_1 \kappa^2)^{-1} \), see (3). Its \( \beta \)-function is

\[
\beta(b) = -b \left( \frac{\beta(\tilde{\kappa}^2) + 2\tilde{\kappa}^2}{\tilde{\kappa}^2} + \frac{5}{192\pi^2} \frac{\beta(\tilde{\kappa}^2) + 8\tilde{\kappa}^2}{(1 - 2\tilde{\Lambda})^2} \right).
\]  

(27)

The \( \beta \)-functions of \( \kappa^2 \) and \( \Lambda \) are equal to Reuter’s \( \beta \)-functions [1] plus the additional contribution of a massless free scalar due to (23), which can e.g. be found in [13, 21].

We observe that the \( \beta \)-function of \( b \) has a Gaussian fixed point. Assuming real fixed-point values \( \kappa^*, \Lambda^* \), we see that \( \partial_b \beta(b) < 0 \), meaning that the \((b = 0, \kappa = \kappa^*, \Lambda = \Lambda^*)\) fixed point is UV attractive. Assuming a, not necessarily, Gaussian fixed point \( \kappa^*, \Lambda^* \), there appears to be the possibility for \( \beta(b) \) to vanish for arbitrary values of \( b \) due to vanishing of the bracket in (27). This bracket can however not vanish for real values of \( \kappa^*, \Lambda^* \), meaning that the only physically acceptable fixed point for \( b \) is \( b = 0 \). This means that the scenario of IR attractivity of Lorentz symmetry, as explained in section 3, is not realized in the cuscuton model. This does however not mean that this scenario may not be realized in models with a different implementation of the Lorentz violation.

If we make the simplifying assumption that the \( \beta \)-functions of \( \tilde{\kappa}^2 \) and \( \tilde{\Lambda} \) behave classically in some finite interval of the scale \( k \), meaning that they have no anomalous scaling

\[
\beta(\tilde{\kappa}^2) = -2\tilde{\kappa}^2, \quad \beta(\tilde{\Lambda}) = -2\tilde{\Lambda},
\]  

(28)

then we obtain for (27)

\[
\beta(b) = -\frac{5b e^{\delta t}}{32\pi^2 \tilde{\kappa}_0^2 (e^{2t} - 2\Lambda_0)^2},
\]  

(29)

where \( t = \ln k \) and \( \tilde{\kappa}_0, \tilde{\Lambda}_0 \) denote the ‘initial conditions’ at \( t = 0 \). The physics described by this \( \beta \)-function (29) is as follows. The absolute value of the Lorentz violating parameter grows in the IR, i.e. as \( t \to -\infty \). This increase is bounded due to an exponentially decreasing \( \beta \)-function. Thus, one can in principle force \( b \) to experimentally valid values by a suitable fine-tuning, but the phenomenologically optimal scenario of a highly Lorentz violating UV theory which flows down to a sufficiently Lorentz invariant IR theory cannot be realized without seemingly unnatural assumptions or an unexpectedly large wavefunction renormalization, which we find unlikely due to our use of the optimized cutoff [19]. The practical calculation of such a wavefunction renormalization would however require new techniques for the evaluation of the traces, since the cuscuton kinetic term cannot be generated with the derivative expansion nor with a vertex (number of fields) expansion.

5. Adding matter fields

Since the renormalization of the quadratic cuscuton term coupled to pure gravity obtained in the previous section does not lead to IR attractivity of Lorentz symmetry, we will now study whether this can be achieved in the presence of additional matter fields. The simplest standard matter model is an additional scalar field \( \chi \) with canonical kinetic and mass term, together with a coupling to the cuscuton field via the lowest order interaction,

\[
S_{\text{int}} = g_{\text{mat}} \int d^4x \sqrt{g} \chi^2 \phi^2.
\]  

(30)

Note that even if this interaction would not be present at some scale, it would be induced radiatively via the graviton interaction at a different scale. It is thus natural to consider such an operator. We find that one additional trace contributes to the flow of \( b \), namely

\[
-\frac{1}{2} \text{Tr}[(\mathcal{R})_{XX} \cdot (P_{\chi})_{XX} \cdot (F_{\chi})_{XX} \cdot (P_{\chi})_{XX}].
\]  

(31)
The terms involved in this trace are given by

\[(R_k)_{\chi \chi} = (\Box + k^2) \Theta \left( 1 + \frac{\Box}{k^2} \right), \tag{32a}\]

\[(P_k)_{\chi \chi} = \frac{1}{k^2 + m_{\chi}^2}, \tag{32b}\]

\[(F_k)_{\chi \chi} = 2 g_{\text{mat}} \phi^2, \tag{32c}\]

and the resulting $\beta$-function for $b$ is given by

\[\beta(b) = -b \left( \frac{\beta(k^2) + 2k^2}{k^2} + \frac{5}{192\pi^2 k^4} \frac{\beta(k^2) + 8k^2}{(1 - 2\Lambda)^2} - \frac{b g_{\text{mat}} k^2}{2\pi^2 (1 + m_{\chi}^2)^2} \right). \tag{33}\]

We observe that this $\beta$-function can in principle be positive for a suitable choice of $b$ and $g_{\text{mat}}$. Since we are phenomenologically interested in small Lorentz violation $|b| \ll 1$ and small matter couplings $|g_{\text{mat}}| \ll 1$ (the $K^2$ interpretation is invalid at the scale at which the dimensionless matter couplings are not small), we obtain that the matter e.g. does not change the sign of the $\beta$-function in our parameter range. On the other hand (33) leads to a condition for a non-Gaussian fixed point $b = b^*$:

\[b^* = \frac{\pi^2 (1 + m_{\chi}^2)^2}{g_{\text{mat}} k_s^2} \left( 4 + \frac{5}{12\pi^2 k_s^4} \frac{1}{(1 - 2\Lambda_s)^2} \right). \tag{34}\]

This means that if one had found a gravity-matter system with non-Gaussian fixed point, then there could be a non-Gaussian fixed point for the gravity–matter–cuscuton system. However, even if one finds such a system, one would still generically have a negative $\beta$-function in the phenomenologically required regime, where $|b| \ll 1$ as well as $|g_{\text{mat}}| \ll 1$.

Similar effects can be obtained by introducing other matter fields, if they give rise to positive contributions to the $\beta$-function of $b$. In particular, all matter effects will enter with $b^2$ and will be required to couple weakly to the cuscuton field, so the $K^2$ interpretation holds.

6. Conclusions and outlook

Motivated by the possibility of constructing UV-complete theories of quantum gravity in explicitly Lorentz violating theories with preferred foliation, we investigated the low energy effect of a preferred foliation in pure quantum gravity. The particular model that we investigated was a version of cuscuton theory that has recently been shown to be on-shell equivalent to Hořava’s theory at low energies. We used a truncation ansatz consisting of Einstein–Hilbert gravity coupled to a cuscuton field with local $\mathbb{Z}_2$-symmetric potential in Wetterich’s exact renormalization group equation and applied Reuter’s background field methods for gravity. We were particularly interested in the nonperturbative and background-independent renormalization of the quadratic cuscuton term which is related to the Lorentz violation. The nondynamical nature of the cuscuton field leads to very stable predictions for the renormalization of the cuscuton potential, which can be traced to the fact that only gravity propagates. We find in particular the following.

(i) The $\beta$-function of the quadratic cuscuton coupling is universal in the sense that it is independent of the rest of the cuscuton potential parameters.

(ii) The only fixed point of the Lorentz violating parameter is a UV-attractive Gaussian fixed point.
(iii) The inclusion of generic scalar matter has very weak effect on the UV attractivity of the Gaussian fixed point.

(iv) Although having a quadratic term, the effect of the cuscuton on the renormalization of Newton’s constant and the cosmological constant is equivalent to the effect of a massless scalar field.

The physical interpretation of these results is that there might be some Lorentz violation at very low energy scales, such as the Hubble scale, that may lead to observable cosmological effects. But using current cosmological bounds on Lorentz violation at the Hubble scale we expect no observable Lorentz violation in the high-energy regime. This result contradicts the optimal scenario in which the renormalization group drives an initially large Lorentz violation in the UV theory to an unobservable value in the IR; hence, if Lorentz violation is necessary to the UV theory one needs (within the universality class of the cuscuton model) an additional mechanism to tame this Lorentz violation in the IR or an unexpectedly large wavefunction renormalization.

To gain further insight into the effect of preferred foliations one has to investigate whether different gravity models with preferred foliations lie in the same universality class as the cuscuton field. Within the cuscuton theory itself one could gain further insight by performing calculations in truncations beyond the local potential ansatz, in particular the wavefunction renormalization. The investigation of the cuscuton wavefunction renormalization cannot be calculated with standard techniques, because it is neither accessible in a derivative expansion nor in a vertex expansion.

Acknowledgments

The authors thank Thorsten Ohl for valuable comments. AS wants to thank the Perimeter Institute for Theoretical Physics for their warm hospitality. This research was supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. AS is supported by Graduiertenkolleg 1147 ‘Theoretical Astrophysics and Particle Physics’.

Appendix A. Toy model calculation

Using the same regulator $R_k$ for all fields, the fluctuation matrix $\Gamma_k^{(2)} + R_k$ of (15) can be split into $F+G$ with

$$G = \begin{pmatrix} R_k + p^2 + k^2 m_k^2 & 0 \\ 0 & R_k + p^2 + k^2 M_k^2 \end{pmatrix}, \quad (A.1a)$$

$$F = \begin{pmatrix} \frac{\psi^2}{2} + \frac{\Phi^2}{2} & b_k \psi \Phi \\ b_k \psi \Phi & \frac{b_k}{2} \psi^2 + \frac{\Phi^2}{2} \end{pmatrix}. \quad (A.1b)$$

To project the rhs of Wetterich’s equation onto the local potential ansatz, we may insert constant fields, and extract the $\beta$-functions for the masses from the trace $-\frac{1}{2} \text{Tr}(P R P F)$ and for the four-point couplings from $\frac{1}{2} \text{Tr}(P R P F P F)$ respectively, where $P = G^{-1}$. Using the optimized cutoff $R_k = (k^2 - p^2) \Theta(1 - \frac{p^2}{k^2})$, we find the following for constant fields:
\[-\frac{1}{2} \text{Tr}(P \dot{R} P F) = -\frac{k^2}{64\pi^2} \int d^4x \left( \left( \frac{a_k}{(1+m_k^2)} + \frac{b_k}{(1+M_k^2)} \right) \varphi^2 + \left( \frac{b_k}{(1+m_k^2)} + \frac{c_k}{(1+M_k^2)} \right) \Phi^2 \right) \]

\[\frac{1}{2} \text{Tr}(P \dot{R} P F F) = \frac{1}{64\pi^2} \int d^4x \left( \frac{(a_k \varphi^2 + b_k \Phi^2)^2}{2(1+m_k^2)^3} + \frac{2b_k^2 \varphi^2 \Phi^2}{(1+m_k^2)^2(1+M_k^2)} \right) \]

\[\frac{1}{2} \text{Tr}(P \dot{R} F F F) = \frac{1}{64\pi^2} \int d^4x \left( \frac{(a_k \varphi^2 + b_k \Phi^2)^2}{2(1+m_k^2)^3} + \frac{2b_k^2 \varphi^2 \Phi^2}{(1+m_k^2)^2(1+M_k^2)} \right) \] (A.2)

### Appendix B. Second variation of the cuscuton model

We give the complete second variation of the effective action (17). It is obtained by plugging in the background field expansion \( g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu} \) and \( \phi = \bar{\phi} + \epsilon f \) into the effective action (17) and collecting the \( \epsilon^2 \) terms\(^3\). For a better readability we give independently the individual parts of the second variation.

The second variation of the kinetic cuscuton action is given by

\[
\Gamma^{(2)}_{\text{kin}} = \int d^4x \sqrt{\bar{g}} \left\{ \left( \frac{h^2}{8} - \frac{h_{\mu\nu} h_{\mu\nu}}{4} \right) \sqrt{\partial^\mu \bar{\phi} \partial_\mu \bar{\phi} + \lambda^2} - \frac{h}{4} \frac{h^\mu h^\nu h_{\mu\nu}}{\sqrt{\partial^\mu \bar{\phi} \partial_\mu \bar{\phi} + \lambda^2}} \right. \\
+ \frac{1}{2} \frac{h h^\mu h^\nu h_{\mu\nu}}{\sqrt{\partial^\mu \bar{\phi} \partial_\mu \bar{\phi} + \lambda^2}} - \frac{1}{2} \left( \frac{h^\mu h^\nu h_{\mu\nu}}{\sqrt{\partial^\mu \bar{\phi} \partial_\mu \bar{\phi} + \lambda^2}} \right) \left( \frac{1}{\sqrt{\partial^\mu \bar{\phi} \partial_\mu \bar{\phi} + \lambda^2}} \right) \right) \\
\] (B.1)

The second variation of the cuscuton potential part is

\[
\Gamma^{(2)}_{\text{pot}} = \sum_{n=1}^{\infty} \int d^4x \sqrt{\bar{g}} \bar{a}_n \left\{ \left( \frac{h^2}{8} - \frac{h_{\mu\nu} h_{\mu\nu}}{4} \right) \frac{\bar{\phi}^{2n}}{(2n)!} + \frac{h f}{2} \frac{\bar{\phi}^{2n-1}}{(2n-1)!} + \frac{f^2}{2} \frac{\bar{\phi}^{2n-2}}{(2n-2)!} \right\} . \] (B.2)

The second variation of the Einstein–Hilbert part can be found in [1] for general background metric fields. Insertion of a flat background metric into the general variation yields

\[
\Gamma^{(2)}_{\text{grav}} = -\kappa^2 \int d^4x h_{\mu\nu} K^{\mu\nu\rho\sigma} (\Box + 2\Lambda) h_{\rho\sigma}, \] (B.3)

where \( K^{\mu\nu\rho\sigma} \) was defined in (19).

The functional derivatives with respect to the individual fluctuation fields \( \{ f, h_{\mu\nu} \} \) of the sum of the above expressions enter the rhs of the FRGE (7).

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\(^3\) Note that this procedure also produces the second variation w.r.t. fields not having backgrounds, because the second variation can be obtained as the second variation with respect to the fluctuations of a functional of background field plus fluctuation evaluated at vanishing background.
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