Existence and nonlinear stability of stationary states for the semi-relativistic Schrödinger-Poisson system

Walid Abou Salem
Department of Mathematics and Statistics, University of Saskatchewan,
Saskatoon, SK, S7N 5E6, Canada
e-mail: walid.abousalem@usask.ca

Thomas Chen
Department of Mathematics, University of Texas at Austin,
Austin, TX, 78712, USA
e-mail: tc@math.utexas.edu

Vitali Vougalter
University of Cape Town, Department of Mathematics and Applied Mathematics,
Private Bag, Rondebosch 7701, South Africa
e-mail: Vitali.Vougalter@uct.ac.za

Abstract. We study the stationary states of the semi-relativistic Schrödinger-Poisson system in the repulsive (plasma physics) Coulomb case. In particular, we establish the existence and the nonlinear stability of a wide class of stationary states by means of the energy-Casimir method. We generalize the global well-posedness result of [1] for the semi-relativistic Schrödinger-Poisson system to spaces with higher regularity.

Keywords: relativistic kinetic energy; Schrödinger-Poisson system; Hartree-von Neumann equation; stationary solutions; nonlinear stability; global existence and uniqueness

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1 Introduction

In this work, we prove the existence and the nonlinear stability for a certain class of stationary solutions of the semi-relativistic Schrödinger-Poisson system in a finite volume domain with Dirichlet boundary conditions. Such a system describes the mean-field dynamics of semi-relativistic quantum particles (for instance, in the case of heated plasma), with the particles moving with extremely high velocities. Let us consider semi-relativistic quantum particles confined in a domain \( \Omega \subset \mathbb{R}^3 \) which is an open set with a \( C^2 \) boundary and \( |\Omega| < \infty \). These particles are interacting by means of the electrostatic field they collectively generate. In the mean-field limit, the density matrix describing the mixed state of the system solves the Hartree-von Neumann equation

\[
\begin{cases}
   i\partial_t \rho(t) = [H_V, \rho(t)], & x \in \Omega, \ t \geq 0 \\
   -\Delta V = n(t, x), & n(t, x) = \rho(t, x, x), \ \rho(0) = \rho_0
\end{cases},
\]  

with Dirichlet boundary conditions, \( \rho(t, x, y) = 0 \) if \( x \) or \( y \) \( \in \partial \Omega \), for \( t \geq 0 \). The single particle Hamiltonian is given by

\[
H_V := T_m + V(t, x).
\]  

The relativistic kinetic energy operator \( T_m := \sqrt{-\Delta + m^2} - m \) is defined by means of the spectral calculus. In system (1.1) and further below, \( \Delta \) stands for the Dirichlet Laplacian on \( L^2(\Omega) \), and \( m > 0 \) is the single particle mass. We refer to \[3\] and \[4\] for a derivation of the analogous system of equations in the non-relativistic case. Due to the fact that \( \rho(t) \) is a nonnegative, self-adjoint and trace-class operator acting on \( L^2(\Omega) \), we can expand its kernel, for every \( t \in \mathbb{R}_+ \), with respect to an orthonormal basis of \( L^2(\Omega) \). We denote this kernel at the initial time \( t = 0 \) by \( \rho_0 \),

\[
\rho_0(x, y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_k(x)\overline{\psi_k(y)}.
\]  

Here \( \{\psi_k\}_{k \in \mathbb{N}} \) stands for an orthonormal basis of \( L^2(\Omega) \), such that \( \psi_k|_{\partial \Omega} = 0 \) for all \( k \in \mathbb{N} \), and the coefficients are given by

\[
\lambda := \{\lambda_k\}_{k \in \mathbb{N}} \in l^1, \quad \lambda_k \geq 0, \quad \sum_{k \in \mathbb{N}} \lambda_k = 1.
\]  

In \[1\], we showed that there exists a one-parameter family of complete orthonormal bases of \( L^2(\Omega) \), \( \{\psi_k(t)\}_{k \in \mathbb{N}} \), with \( \psi_k(t)|_{\partial \Omega} = 0 \) for all \( k \in \mathbb{N} \), and for \( t \in \mathbb{R}_+ \), such that the kernel of the density matrix \( \rho(t) \), which satisfies system (1.1), can be expressed as

\[
\rho(t, x, y) = \sum_{k \in \mathbb{N}} \lambda_k \psi_k(t, x)\overline{\psi_k(t, y)}.
\]  

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As a consequence of the particular commutator structure of (1.1) (where \( \rho(t) \) and \(-iH_V \) satisfy the conditions of a Lax pair), the corresponding flow of \( \rho(t) \) leaves its spectrum invariant. Accordingly, the coefficients \( \lambda \) are independent of \( t \). This isospectrality is crucial for the stability analysis for stationary states based on the Casimir energy method employed in this paper; see also [5, 7, 8, 12, 13].

By substituting the expression (1.5) in system (1.1), one can verify that the one-parameter family of orthonormal vectors \( \{ \psi_k(t) \}_{k \in \mathbb{N}} \) satisfies the semi-relativistic Schrödinger-Poisson system equivalent to (1.1) and given by

\[
i \frac{\partial \psi_k}{\partial t} = T_m \psi_k + V \psi_k, \quad k \in \mathbb{N},
\]

\[- \Delta V[\Psi] = n[\Psi], \quad \Psi := \{ \psi_k \}_{k=1}^{\infty},
\]

\[n[\Psi(t, x)] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2,
\]

with initial data \( \{ \psi_k(0) \}_{k=1}^{\infty} \). Here, the potential function \( V[\Psi] \) is the solution of the Poisson equation (1.7). Both \( V[\Psi] \) and \( \psi_k(t) \), for all \( k \in \mathbb{N} \), satisfy the Dirichlet boundary conditions

\[\psi_k(t, x), \quad V(t, x) = 0, \quad t \geq 0, \forall x \in \partial \Omega.
\]

The global well-posedness for system (1.6)-(1.9) was established in the recent work [1]. Analogous results were derived before in the nonrelativistic case in a finite volume domain with Dirichlet boundary conditions in [6], and in the whole space of \( \mathbb{R}^3 \) in [6] and [8].

In this paper, we are interested in the properties of stationary states which occur when \( \rho(t) = f(H_V) \) for some function \( f \). Substituting the latter in (1.1), the commutator on the right side of the first equation of system (1.1) vanishes, and the density matrix is time independent. The precise properties of the distribution function \( f \) will be discussed below. The solution of the Schrödinger-Poisson system corresponding to the stationary states is

\[\psi_k(t, x) = e^{-it\mu_k} \psi_k(x), \quad k \in \mathbb{N},
\]

such that the potential function \( V[\Psi] \) is time independent, \( \mu_k \in \mathbb{R} \) are the eigenvalues of the Hamiltonian (1.2) and \( \psi_k(x) \) are the corresponding eigenfunctions.

The organization of this article is as follows. In Section 2, we describe the class of stationary states we will study, and state our hypotheses and main results about nonlinear stability and existence of stationary states. In Section 3, we derive some preliminary results. In Section 4, we prove the nonlinear stability of the stationary states of the semi-relativistic Schrödinger-Poisson system via the energy-Casimir functional as a Lyapunov function (see the statement of Theorem 1). In Section 5 we define the dual functional and in Section 6 study its properties using the methods of convex analysis, and show that it admits a unique maximizer (see Theorem 2), which implies the existence of a stationary state for our Schrödinger-Poisson system. In an Appendix, we discuss generalizing the well-posedness result of [1] to spaces with higher regularity.
2 The Model and Statement of the Main Results

The state space for the Schrödinger-Poisson system is defined as

\[ \mathcal{L} := \{ (\Psi, \lambda) \mid \Psi = \{ \psi_k \}_{k=1}^{\infty} \subset H_{0}^{0}(\Omega) \cap H^{1}(\Omega) \text{ is a complete orthonormal system in } L^{2}(\Omega), \]

\[ \lambda = \{ \lambda_k \}_{k=1}^{\infty} \in l^{1}, \quad \lambda_k \geq 0, \quad k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} \lambda_k \int_{\Omega} |\nabla \psi_k|^2 \, dx < \infty \}, \]

see [1].

In order to precisely define the class of stationary states we will study, we need to introduce the Casimir class of functions. We say that a function \( f : \mathbb{R} \to \mathbb{R} \) is of Casimir class \( C \) if and only if it possesses the following properties:

(i) \( f \) is continuous, such that \( f(s) > 0 \) for \( s \leq s_0 \) and \( f(s) = 0 \) when \( s \geq s_0 \), with some \( s_0 \in [0, \infty) \],

(ii) \( f \) is strictly decreasing on \( ]-\infty, s_0[ \), such that \( \lim_{s \to -\infty} f(s) = \infty \),

(iii) there exist constants \( \varepsilon > 0 \) and \( C > 0 \), such that for \( s \geq 0 \) the estimate

\[ f(s) \leq C(1 + s)^{-5-\varepsilon} \]  

holds.

Note that \( s_0 \) acts as an “ultra-violet” cut-off, and we can take is as large as we wish.

Consider the quadruple \( (\Psi_0, \lambda_0, \mu_0, V_0) \) with \( (\Psi_0, \lambda_0) \in \mathcal{L}, \mu_0 = \{ \mu_{0,k} \}_{k=1}^{\infty} \) real valued, and the potential function \( V_0 \in H_{0}^{1}(\Omega) \cap H^{2}(\Omega) \), such that the stationary Schrödinger-Poisson system holds

\[ (T_m + V_0)\psi_{0,k} = \mu_{0,k}\psi_{0,k}, \quad k \in \mathbb{N}, \]

\[ -\Delta V_0 = n_0 = \sum_{k=1}^{\infty} \lambda_{0,k} |\psi_{0,k}|^2, \]

with

\[ \lambda_{0,k} = f(\mu_{0,k}), \quad k \in \mathbb{N}, \]

where \( f \in C \). Then, the corresponding density matrix \( \rho_0 = f(T_m + V_0) \) solves the stationary state Hartree-von Neumann equation

\[ [H_{V_0}, \rho_0] = 0. \]

Remark. In the nonrelativistic case, the Casimir class was defined similarly in [12] with the exception that the rate of decay of the distribution function \( f \) was assumed to be
smaller. A good example of \( f \in \mathcal{C} \) is the function decaying exponentially as \( s \to \infty \) with the cut-off level \( s_0 = \infty \). This is the Boltzmann distribution \( f(s) := e^{-\beta s}, \beta > 0 \).

In order to prove the nonlinear stability of the stationary states, we will use the energy-Casimir method. This method was used in \([5]\) for fluid problems, and in \([7, 13]\) for studying stationary states of kinetic equations, in particular, Vlasov-Poisson systems. Here, we extend the energy-Casimir functional used in \([12]\) to the semi-relativistic case. For \( f \in \mathcal{C} \), let us define

\[
F(s) := \int_s^\infty f(\sigma)d\sigma, \quad s \in \mathbb{R}.
\]
(2.5)

Note that the function defined via (2.5) is decreasing, continuously differentiable, nonnegative and is strictly convex on its support. Moreover, for \( s \geq 0 \)

\[
F(s) \leq C(1 + s)^{-4-\varepsilon}.
\]
(2.6)

Its Legendre (Fenchel) transform is given by

\[
F^*(s) := \sup_{\lambda \in \mathbb{R}} (\lambda s - F(\lambda)), \quad s \leq 0.
\]
(2.7)

We define the energy-Casimir functional for a given \( f \) as

\[
\mathcal{H}_C(\Psi, \Lambda) := \sum_{k=1}^\infty F^*(-\lambda_k) + \mathcal{H}(\Psi, \Lambda), \quad (\Psi, \Lambda) \in \mathcal{L}.
\]
(2.8)

In particular, \( \mathcal{H}_C \) is conserved along solutions of the Schrödinger-Poisson system, as a consequence of isospectrality of the flow of \( \rho(t) \), which is equivalent to the \( t \)-independence of \( \lambda_k \). Our main results in this paper address the existence and stability of stationary states that are given by (2.2)-(2.4), for \( f \in \mathcal{C} \). Stability is controlled by the following main theorem.

**Theorem 1.** Let \((\Psi_0, \Lambda_0, \mu_0, V_0)\) be a stationary state of the semi-relativistic Schrödinger-Poisson system, where

\[
\lambda_{0,k} = f(\mu_{0,k}), \quad k \in \mathbb{N}
\]

with some \( f \in \mathcal{C} \) and \((\Psi_0, \Lambda_0) \in \mathcal{L} \). Let \((\Psi(t), \Lambda)\) be a solution of the Schrödinger-Poisson system, such that initial datum \((\Psi(0), \Lambda) \in \mathcal{L} \). Then, for all \( t \geq 0 \), the estimate

\[
\frac{1}{2}||n[\Psi(t), \Lambda] - n_0||^2_{\dot{H}^{-1}(\Omega)} \leq \mathcal{H}_C(\Psi(0), \Lambda) - \mathcal{H}_C(\Psi_0, \Lambda_0)
\]

holds, such that the stationary state is nonlinearly stable. Here, \( \dot{H}^{-1}(\Omega) \) is the dual of \( \dot{H}^1(\Omega) \) with norm \( ||u||_{\dot{H}^{-1}(\Omega)} = (u, (-\Delta)^{-1}u)^{1/2}_{L^2(\Omega)} \).

To prove the existence of stationary states, we introduce the dual of the energy-Casimir functional. To this end, we let, for \( \Lambda > 0 \) fixed,

\[
\mathcal{G}(\Psi, \Lambda, V, \sigma) := \sum_{k=1}^\infty [F^*(-\lambda_k) + \lambda_k \int_{\Omega} [\frac{1}{2}|T_m \psi_k|^2 + V|\psi_k|^2]dx] - \frac{1}{2} \int_{\Omega} |\nabla V|^2dx + \sigma \left[ \sum_{k=1}^\infty \lambda_k - \Lambda \right].
\]
Here, $\sigma \in \mathbb{R}$ is a Lagrange multiplier.

The dual functional to $\mathcal{H}_C$ is given by

$$\Phi(V, \sigma) := \inf_{\Psi, \lambda} \mathcal{G}(\Psi, \lambda; V, \sigma).$$

(2.9)

The infimum in the formula above is taken over all $\lambda \in l^1_+$ and all complete orthonormal sequences $\Psi$ from $L^2(\Omega)$. Let us consider only non-negative potential functions and define

$$H^1_{0,+}(\Omega) := \{ V \in H^1_0(\Omega) | V \geq 0 \}.$$

The following is our main result about the existence of stationary states.

**Theorem 2.** Let $f \in C$ and $\Lambda > 0$ be fixed. The functional $\Phi$

$$(V, \sigma) \in H^1_{0,+}(\Omega) \times \mathbb{R} \to -\frac{1}{2} \int_{\Omega} |\nabla V|^2 dx - \text{Tr}[F(T_m + V + \sigma)] - \sigma \Lambda$$

is continuous, strictly concave, bounded from above and $-\Phi(V, \sigma)$ is coercive. There exists a unique maximizer $(V_0, \sigma_0)$ of $\Phi(V, \sigma)$. Let $\{\psi_{0,k}\}_{k=1}^\infty$ be the orthonormal sequence of eigenfunctions of the Hamiltonian $T_m + V_0$ corresponding to the eigenvalues $\{\mu_{0,k}\}_{k=1}^\infty$ and $\lambda_{0,k} := f(\mu_{0,k} + \sigma_0)$. Then $(\Psi_0, \Delta_0, \mu_0, V_0)$ is a stationary state of the semi-relativistic Schrödinger-Poisson system, where $\sum_{k=1}^\infty \lambda_{0,k} = \Lambda$ and $(\Psi_0, \Delta_0) \in \mathcal{L}$.

We will prove Theorem 1 in Section 4, and Theorem 2 in Section 5.

### 3 Preliminaries

We have the following elementary lemma.

**Lemma 3.** For $(\Psi, \Delta) \in \mathcal{L}$ we have

$$n_{\psi,\lambda} := \sum_{k \in \mathbb{N}} \lambda_k |\psi_k|^2 \in L^2(\Omega).$$

Let $V_{\psi,\lambda}$ stand for the Coulomb potential induced by $n_{\psi,\lambda}$, such that

$$-\Delta V_{\psi,\lambda}(x) = n_{\psi,\lambda}(x), \quad x \in \Omega; \quad V_{\psi,\lambda}(x) = 0, \quad x \in \partial \Omega.$$

Then $V_{\psi,\lambda} \in H^1_0(\Omega) \cap H^2(\Omega)$.

**Proof.** We easily express the norm as

$$\|n_{\psi,\lambda}\|_{L^2(\Omega)}^2 = \sum_{k,s=1}^\infty \lambda_k \lambda_s (|\psi_k|^2, |\psi_s|^2)_{L^2(\Omega)}.$$
Here and further below, the inner product of two functions \( f(x), g(x) \in L^2(\Omega) \) is denoted as \((f, g)_{L^2(\Omega)} := \int_\Omega f(x)g(x)dx\). Application of the Schwarz inequality to the right side of the identity above yields the upper bound

\[
\left( \sum_{k=1}^{\infty} \lambda_k \sqrt{\int_\Omega |\psi_k|^4 dx} \right)^2,
\]

which can be estimated from above by applying the Schwarz inequality as well. Thus, we obtain

\[
|\Omega|^{\frac{1}{3}} \left( \sum_{k=1}^{\infty} \lambda_k \left( \int_\Omega |\psi_k|^6 dx \right)^{\frac{1}{3}} \right)^2.
\]

We next make use of the Sobolev inequality

\[
\int_\Omega |\nabla f|^2 dx \geq c_s \left( \int_\Omega |f|^6 dx \right)^{\frac{1}{3}},
\]

in which the constant \( c_s \) is given on p.186 of \([11]\). Noting that a function compactly supported in the set \( \Omega \) can be extended by zero to the whole space of \( \mathbb{R}^3 \), we arrive at the upper bound

\[
\frac{|\Omega|^{\frac{1}{3}}}{c_s^2} \left( \sum_{k=1}^{\infty} \lambda_k \int_\Omega |\nabla \psi_k|^2 dx \right)^2 < \infty
\]

by means of the definition of the state space \( \mathcal{L} \) given above, such that \( n_{\psi,\lambda} \in L^2(\Omega) \). Note that the particle density \( n_{\psi,\lambda} \) vanishes on the boundary of the set \( \Omega \) by means of formula \([1.8]\) and boundary conditions \([1.9]\). Therefore, \( \Delta V_{\psi,\lambda} \in L^2(\Omega) \). Let \( \{\mu_k^0\}_{k \in \mathbb{N}} \) denote the eigenvalues of the Dirichlet Laplacian on \( L^2(\Omega) \) and \( \mu_1^0 \) is the lowest one of them. Note that \( \mu_k^0 > 0, \ k \in \mathbb{N} \).

Since

\[
V_{\psi,\lambda} = (-\Delta)^{-1} n_{\psi,\lambda},
\]

we have that \( \|V_{\psi,\lambda}\|_{L^2(\Omega)} \leq \frac{1}{\mu_1^0} \|n_{\psi,\lambda}\|_{L^2(\Omega)} < \infty \). Furthermore, \( V_{\psi,\lambda} \) vanishes on the boundary of the set \( \Omega \) via \([1.9]\).

According to Theorem 1 of \([11]\), for every initial state \((\Psi(0), \Delta) \in \mathcal{L}\), there exists a unique strong solution of system \([1.6]\)-\([1.9]\), where \((\Psi(t), \Delta) \in \mathcal{L}\) for all \( t \geq 0 \). Let us define the energy of a state \((\Psi, \Delta) \in \mathcal{L}\) as

\[
\mathcal{H}(\Psi, \Delta) := \sum_{k=1}^{\infty} \lambda_k \int_\Omega |T_m \psi_{\lambda_k}|^2 dx + \frac{1}{2} \int_\Omega n_{\psi,\lambda} V_{\psi,\lambda} dx = \]

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\[\sum_{k=1}^{\infty} \lambda_k \int_{\Omega} |T_{m}^{\frac{1}{2}} \psi_k|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla V_{\psi,\lambda}|^2 dx,\]

which is a conserved quantity along solutions of the Schrödinger-Poisson system (see Lemma 7 of \[1\]).

Analogously to \[1\] we assume that \(\lambda_k > 0\) via density arguments. To prove the nonlinear stability for a specified stationary state, We have the following auxiliary statements.

**Lemma 4.** Let \(f \in C\).

a) For every \(\beta > 1\) there exists \(C = C(\beta) \in \mathbb{R}\), such that for \(s \leq 0\) we have

\[F(s) \geq -\beta s + C\]

b) Let \(V \in H_0^1(\Omega)\) and \(V(x) \geq 0\) for \(x \in \Omega\). Then both operators \(f(T_m + V)\) and \(F(T_m + V)\) are trace class.

**Proof.** The part a) of the lemma comes from the fact that function \(F(s)\) is smooth with the slope varying from \(-\infty\) to 0, and convex; therefore, its graph is located above a tangent line to it.

For the Dirichlet eigenvalues of \(-\Delta\) on \(L^2(\Omega), \Omega \subset \mathbb{R}^3\), we will make use of the semiclassical lower bound

\[\mu_k^0 \geq C k^{\frac{3}{2}}, \quad k \in \mathbb{N}\]

with a constant dependent on \(|\Omega| < \infty\) (see e.g. \[10\]). Since the potential function \(V(x)\) is nonnegative in \(\Omega\) by assumption, we easily estimate from below the eigenvalues \(\mu_k\) of the Hamiltonian \(T_m + V\) for \(k \in \mathbb{N}\) as

\[\mu_k \geq \sqrt{\mu_k^0 + m^2} - m \geq \left( C k^{\frac{3}{2}} - m \right)_{+}\] (3.2)

with the right side of the inequality above positive for \(k\) large enough. For the sharp semiclassical bounds on the moments of Dirichlet eigenvalues to fractional powers see \[14\]. We express

\[\text{Tr}F(T_m + V) = \sum_{k=1}^{\infty} F(\mu_k) < \infty,\]

since \(F(s)\) is decreasing, satisfies estimate \(\text{(2.6)}\), and the series with a general term \((1 + (Ck^{\frac{3}{2}} - m)_{+})^{-4-\varepsilon}\) converges. Similarly,

\[\text{Tr}f(T_m + V) = \sum_{k=1}^{\infty} f(\mu_k) < \infty,\]

due to the fact that \(f(s)\) decreases, obeys bound \(\text{(2.1)}\) and the series with a general term \((1 + Ck^{\frac{3}{2}} - m)^{-5-\varepsilon}\) is convergent. This completes the proof of the part b) of the lemma. \(\blacksquare\)
Lemma 5. Let $\psi \in H^2_0(\Omega) \cap H^1(\Omega)$ with $\|\psi\|_{L^2(\Omega)} = 1$, the potential function $V \in H^1_0(\Omega)$ and $V(x) \geq 0$ for $x \in \Omega$. Then,

$$F(\langle \psi, (T_m + V)\psi \rangle) \leq \langle \psi, F(T_m + V)\psi \rangle$$

(3.3)

holds with equality if $\psi$ is an eigenstate of the Hamiltonian $T_m + V$.

Proof. By means of the Spectral Theorem we have

$$T_m + V = \sum_{k=1}^{\infty} \mu_k P_k,$$

where the operators $\{P_k\}_{k=1}^{\infty}$ are the orthogonal projections onto the bound states corresponding to the eigenvalues $\{\mu_k\}_{k=1}^{\infty}$. Hence

$$F(\langle \psi, (T_m + V)\psi \rangle) = F\left( \sum_{k=1}^{\infty} \mu_k \|P_k \psi\|_{L^2(\Omega)}^2 \right).$$

The right side of (3.3) can be easily written as

$$\sum_{k=1}^{\infty} F(\mu_k) \|P_k \psi\|_{L^2(\Omega)}^2.$$

Estimate (3.3) follows from Jensen’s inequality. When $\psi$ is an eigenstate of the operator $T_m + V$ corresponding to an eigenvalue $\mu_k$, for some $k \in \mathbb{N}$, both sides of (3.3) are equal to $F(\mu_k)$. Note that the converse of this statement does not hold in general. Indeed, let us consider as $\psi$ a linear combination of more than one eigenstate of the Hamiltonian with corresponding eigenvalues $\mu_k$ located outside the support of $F(s)$. Then both sides of (3.3) will be equal to zero.

The lemma below shows that a stationary solutions belong to the state space for the Schrödinger-Poisson system.

Lemma 6. Let the quadruple $(\Psi_0, \lambda_0, \mu_0, V_0)$ satisfy equations (2.2), (2.3) and (2.4), where $\Psi_0$ is a complete orthonormal system in $L^2(\Omega)$ and the distribution $f \in \mathcal{C}$. Then,

$$\sum_{k=1}^{\infty} \lambda_{0,k} \int_{\Omega} |\nabla \psi_{0,k}|^2 dx < \infty$$

holds, such that $(\Psi_0, \lambda_0) \in \mathcal{L}$.

Proof. We express the following quantity using (2.2) and (2.4) as

$$\sum_{k=1}^{\infty} \lambda_{0,k} \|T^{\frac{1}{2}}_m \psi_{0,k}\|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla V_0|^2 dx =$$
\[ \sum_{k=1}^{\infty} \lambda_{0,k} (T_m + V_0) \psi_{0,k}, \psi_{0,k})_{L^2(\Omega)} = \sum_{k=1}^{\infty} f(\mu_{0,k}) \mu_{0,k}. \quad (3.4) \]

The potential function \( V_0(x) \geq 0 \) in \( \Omega \) since it is superharmonic by means of (2.3), and vanishes on the boundary of \( \Omega \). Thus, \( \mu_{0,k} > 0, \ k \in \mathbb{N} \) and via (2.1) the right side of (3.4) can be bounded above by

\[ \sum_{k=1}^{\infty} C(1 + \mu_{0,k})^{-\epsilon} \mu_{0,k} < \infty, \]

which follows from the eigenvalue estimate (3.2). We have also obtained

\[ \nabla V_0 \in L^2(\Omega), \ T_m^\frac{1}{2} \psi_{0,k} \in L^2(\Omega), \ k \in \mathbb{N}. \quad (3.5) \]

In fact, from equation (2.2) with a nonnegative potential, we easily conclude that

\[ \| T_m^\frac{1}{2} \psi_{0,k} \|_{L^2(\Omega)}^2 \leq \mu_{0,k}, \ k \in \mathbb{N}. \quad (3.6) \]

Note that the standard requirement \( V_0 \in L^1(\Omega) \) (see e.g. p.234, 245 of [11]) is satisfied here as well. Throughout the article the operator \( |p| := \sqrt{-\Delta} \), which is the massless relativistic kinetic energy operator defined via the spectral calculus. Clearly, in the sense of quadratic forms, we have

\[ T_m \geq |p| - m, \quad (3.7) \]

such that (3.5) yields

\[ \| |p|^\frac{1}{2} \psi_{0,k} \|_{L^2(\Omega)}^2 \leq ((T_m + m) \psi_{0,k}, \psi_{0,k})_{L^2(\Omega)} = m + \| T_m^\frac{1}{2} \psi_{0,k} \|_{L^2(\Omega)}^2 < \infty. \]

Thus, \( \psi_{0,k} \in H^\frac{1}{2}_0(\Omega), \ k \in \mathbb{N} \). Moreover, let us make use of the relativistic Sobolev inequality (see e.g. p.183 of [11]) for a function compactly supported in \( \Omega \)

\[ (f, |p|f)_{L^2(\Omega)} \geq c_\delta \| f \|_{L^3(\Omega)}^2, \quad (3.8) \]

which gives us \( \psi_{0,k} \in L^3(\Omega), \ k \in \mathbb{N} \). Hence by means of Hölder’s inequality, we arrive at

\[ \int_{\Omega} |V_0|^2 |\psi_{0,k}|^2 \, dx \leq \left( \int_{\Omega} |V_0|^6 \, dx \right)^\frac{1}{3} \left( \int_{\Omega} |\psi_{0,k}|^3 \, dx \right)^\frac{2}{3} < \infty. \]

Indeed, \( V_0(x) \in L^6(\Omega) \) due to (3.5) along with (3.1). Therefore, \( V_0 \psi_{0,k} \in L^2(\Omega), \ k \in \mathbb{N} \). From equation (2.2) we easily deduce that \( T_m \psi_{0,k} \in L^2(\Omega), \ k \in \mathbb{N} \) as well. Then the identity

\[ \| T_m \psi_{0,k} \|_{L^2(\Omega)}^2 = \int_{\Omega} |\nabla \psi_{0,k}|^2 \, dx - 2m \| T_m^\frac{1}{2} \psi_{0,k} \|_{L^2(\Omega)}^2 \quad (3.9) \]
via (3.5) yields $\nabla \psi_{0,k} \in L^2(\Omega)$ and $\psi_{0,k} \in H^1(\Omega)$, $k \in \mathbb{N}$. By means of (2.4), we have $\lambda_{0,k} \geq 0$, $k \in \mathbb{N}$. Convergence of the series on the right side of (3.4) implies

$$\sum_{k=1}^{\infty} \lambda_{0,k} = \sum_{k=1}^{\infty} f(\mu_{0,k}) < \infty,$$

(3.10)

such that $\lambda_0 = \{\lambda_{0,k}\}_{k=1}^{\infty} \in l^1$. Let us make use of identity (3.9) along with (3.6), such that

$$\sum_{k=1}^{\infty} \lambda_{0,k} \int_{\Omega} |\nabla \psi_{0,k}|^2 \, dx = \sum_{k=1}^{\infty} \lambda_{0,k} \{2m \| T_m \psi_{0,k} \|^2_{L^2(\Omega)} + \| T_m \psi_{0,k} \|^2_{L^2(\Omega)} \} \leq$$

$$\leq 2m \sum_{k=1}^{\infty} f(\mu_{0,k}) \mu_{0,k} + \sum_{k=1}^{\infty} \lambda_{0,k} \| T_m \psi_{0,k} \|^2_{L^2(\Omega)}.$$

(3.11)

The first term in the right side of (3.11) is finite as it was shown above. The second expression on the right side of the inequality above can be written via (2.2) as

$$\sum_{k=1}^{\infty} \lambda_{0,k} \| \mu_{0,k} \psi_{0,k} - V_0 \psi_{0,k} \|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \lambda_{0,k} \{\mu_{0,k}^2 + \| V_0 \psi_{0,k} \|^2_{L^2(\Omega)} - 2\mu_{0,k} \int_{\Omega} V_0 |\psi_{0,k}|^2 \, dx \}. \quad (3.12)$$

Our goal is to prove that (3.12) is convergent. Indeed, (2.1) implies

$$\sum_{k=1}^{\infty} \lambda_{0,k} \mu_{0,k} \leq C \sum_{k=1}^{\infty} (1 + \mu_{0,k})^{-5-\epsilon} \mu_{0,k}^2 < \infty,$$

due to the eigenvalue bound (3.2). We estimate the second term on the right side of (3.12) using Hölder’s inequality, such that

$$\sum_{k=1}^{\infty} \lambda_{0,k} \| V_0 \psi_{0,k} \|^2_{L^2(\Omega)} \leq \left( \int_{\Omega} |V_0|^6 \, dx \right)^{\frac{1}{3}} \sum_{k=1}^{\infty} \lambda_{0,k} \left( \int_{\Omega} |\psi_{0,k}|^3 \, dx \right)^{\frac{2}{3}},$$

where $V_0(x) \in L^6(\Omega)$ as discussed above. Let us make use of inequalities (3.8) and (3.7), such that

$$\sum_{k=1}^{\infty} \lambda_{0,k} \| \psi_{0,k} \|^2_{L^2(\Omega)} \leq \frac{1}{c_s} \sum_{k=1}^{\infty} \lambda_{0,k} (|p| \psi_{0,k}, \psi_{0,k})_{L^2(\Omega)} \leq \frac{1}{c_s} \sum_{k=1}^{\infty} \lambda_{0,k} \{m + \| T_m \psi_{0,k} \|^2_{L^2(\Omega)} \} < \infty$$

due to estimates (3.10) and (3.4). The last term in the right side of (3.12) can be bounded above in the absolute value by applying the Schwarz inequality to it twice, such that

$$\sum_{k=1}^{\infty} \lambda_{0,k} \mu_{0,k} \| V_0 \psi_{0,k} \|_{L^2(\Omega)} \leq \sqrt{\sum_{k=1}^{\infty} \lambda_{0,k} \mu_{0,k}^2} \sqrt{\sum_{s=1}^{\infty} \lambda_{0,s} \| V_0 \psi_{0,s} \|^2_{L^2(\Omega)}} < \infty$$

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as it was shown above.

**Remark.** In the stationary situation, our semi-relativistic Schrödinger-Poisson problem can be easily written as

\[-\Delta V_0 = f(T_m + V_0)(x, x), \quad x \in \Omega,
\]

\[V_0(x) = 0, \quad x \in \partial \Omega.\]

Let us turn our attention to defining the corresponding Casimir functional for a fixed \(f \in \mathcal{C}\). The following elementary lemma gives the alternative representation for the Legendre transform of our integrated distribution. Note that \(f \in \mathcal{C}\) considered on the \((-\infty, s_0]\) semi-axis has an inverse \(f^{-1}\).

**Lemma 7.** For the function \(F(s)\) defined in (2.7) and \(s \leq 0\) we have

\[F^*(s) = \int_{-s}^{0} f^{-1}(\sigma)d\sigma.\]  

(3.13)

**Proof.** Let us define

\[g(\lambda) := \lambda s - F(\lambda), \quad \lambda \in \mathbb{R}, \quad s \leq 0,\]

such that via (2.5) we have \(g'(\lambda) = s + f(\lambda)\). Hence, the maximal value in the right side of (2.7) is attained at \(\lambda^* := f^{-1}(-s)\) and is equal to

\[\varphi(s) := g(\lambda^*) = f^{-1}(-s)s - \int_{f^{-1}(-s)}^{\infty} f(\sigma)d\sigma\]

with \(\varphi(0) = 0\). Although \(f\) is continuous and not necessarily differentiable, we can approximate it by a differentiable function. Let \(f_\epsilon(s) = \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} f(t)dt\), and

\[\varphi_\epsilon(s) = f_\epsilon^{-1}(-s)s - \int_{f_\epsilon^{-1}(-s)}^{\infty} f_\epsilon(\sigma)d\sigma.\]

A direct computation using the formula for the derivative of the inverse yields \(\varphi'_\epsilon(s) = f_\epsilon^{-1}(-s)\). Integrating and taking the \(\epsilon \to 0\) limit yields \(\varphi(s) = \int_{-s}^{0} f^{-1}(\sigma)d\sigma.\)  

In the next section, we prove the nonlinear stability of stationary states, using the energy-Casimir functional defined above.
4 Stability of stationary states

In this section, we prove Theorem 1, which yields lower bound in terms of the electrostatic field. The auxiliary statement below is crucial for establishing this nonlinear stability result.

**Lemma 8.** Let $V \in H^1_0(\Omega)$ and $V \geq 0$. (i) Then, for $(\Psi, \lambda) \in \mathcal{L}$, the lower bound

$$\sum_{k=1}^{\infty} \left\{ F^*(-\lambda_k) + \lambda_k \int_{\Omega} \left[ |T_m^{1/2} \psi_k|^2 + V |\psi_k|^2 \right] dx \right\} \geq -\text{Tr}[F(T_m + V)]. \quad (4.1)$$

(ii) Equality is attained for $(\Psi, \lambda) = (\Psi_V, \lambda_V)$, where $\psi_{V,k} \in H^1_0(\Omega) \cap H^1(\Omega)$, $k \in \mathbb{N}$ stands for the orthonormal sequence of eigenfunctions of the Hamiltonian $T_m + V$ with corresponding eigenvalues $\mu_{V,k}$ and $\lambda_{V,k} = f(\mu_{V,k})$, $k \in \mathbb{N}$.

**Proof.** According to definition (2.7), we have

$$F^*(s) \geq \mu s - F(\mu), \quad \mu \in \mathbb{R}, \quad s \leq 0,$$

which easily implies

$$F^*(-\lambda_k) + \lambda_k \mu_k \geq -F(\mu_k), \quad k \in \mathbb{N}. \quad (4.2)$$

Now let

$$\mu_k := \int_{\Omega} \left\{ |T_m^{1/2} \psi_k|^2 + V |\psi_k|^2 \right\} dx = \langle \psi_k, (T_m + V) \psi_k \rangle, \quad k \in \mathbb{N},$$

which proves part (i). To prove part (ii), we note that after summation, we arrive at

$$\sum_{k=1}^{\infty} \left\{ F^*(-\lambda_k) + \lambda_k \int_{\Omega} \left[ |T_m^{1/2} \psi_k|^2 + V |\psi_k|^2 \right] dx \right\} \geq -\sum_{k=1}^{\infty} F(\langle \psi_k, (T_m + V) \psi_k \rangle).$$

Lemma 5 along with the definition of trace yields the lower bound for the right side of the inequality above as

$$-\sum_{k=1}^{\infty} \langle \psi_k, F(T_m + V) \psi_k \rangle = -\text{Tr}(F(T_m + V)).$$

Suppose that $(\Psi, \lambda) = (\Psi_V, \lambda_V)$, where $\psi_{V,k}$ are eigenfunctions of the Hamiltonian $T_m + V$ and $\mu_k$, which are defined above are the corresponding eigenvalues $\mu_{V,k}$, $k \in \mathbb{N}$. Therefore, on the right side of the lower bound (4.1) we have

$$-\text{Tr}(F(T_m + V)) = -\sum_{k=1}^{\infty} F(\mu_{V,k}).$$
Next, we use the identity $\lambda_{V,k} = f(\mu_{V,k}) = -F'(\mu_{V,k})$. Then, via Lemma 7, $F^*(-\lambda_{V,k}) = f^{-1}(\lambda_{V,k}) = \mu_{V,k}$, $k \in \mathbb{N}$. Using the argument of Lemma 7, we arrive at

$$F^*(-\lambda_{V,k}) = \sup_{\lambda \in \mathbb{R}} (-\lambda \lambda_{V,k} - F(\lambda)) = -f^{-1}(\lambda_{V,k}) \lambda_{V,k} - F(f^{-1}(\lambda_{V,k})) = -\lambda_{V,k} \mu_{V,k} - F(\mu_{V,k}).$$

Therefore, the left side of (4.1) will be equal to $-\sum_{k=1}^{\infty} F(\mu_{V,k})$ as well.

Armed with the technical lemma above, we may now prove our first main statement.

**Proof of Theorem 2.** Let $(\Psi, \lambda) \in \mathcal{L}$ and the potential function $V = V_{\psi,\lambda}$ is induced by this state. Then we will use the following identity for the energy of the electrostatic field

$$\frac{1}{2} \|n - n_0\|_{H^{-1}(\Omega)}^2 = \frac{1}{2} \|\nabla V - \nabla V_0\|_{L^2(\Omega)}^2 = \frac{1}{2} \int_{\Omega} |\nabla V|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx + \int_{\Omega} V_0 \Delta V dx.$$

By the definition of the energy-Casimir functional, this can be written as

$$\mathcal{H}_C(\Psi, \lambda) = \left\{ \sum_{k=1}^{\infty} \left( F^*(-\lambda_k) + \lambda_k \int_{\Omega} |T_m^\frac{1}{2} \psi_k|^2 dx \right) - \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx - \int_{\Omega} V_0 \Delta V dx \right\},$$

which is equal to

$$\mathcal{H}_C(\Psi, \lambda) = \left\{ \sum_{k=1}^{\infty} \left[ F^*(-\lambda_k) + \lambda_k \int_{\Omega} (|T_m^\frac{1}{2} \psi_k|^2 + V_0 |\psi_k|^2) dx \right] - \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx \right\}.$$

Applying first Lemma 8(i), and subsequently Lemma 8(ii), we obtain that the expression above is bounded from above by

$$\mathcal{H}_C(\Psi, \lambda) = \left\{ - \text{Tr}[F(T_m + V_0)] - \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx \right\} =$$

$$= \mathcal{H}_C(\Psi, \lambda) - \left\{ \sum_{k=1}^{\infty} \left[ F^*(-\lambda_{0,k}) + \lambda_{0,k} \int_{\Omega} (|T_m^\frac{1}{2} \psi_{0,k}|^2 + V_0 |\psi_{0,k}|^2) dx \right] - \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx \right\} =$$

$$= \mathcal{H}_C(\Psi, \lambda) - \left\{ \sum_{k=1}^{\infty} \left[ F^*(-\lambda_{0,k}) + \lambda_{0,k} \int_{\Omega} |T_m^\frac{1}{2} \psi_{0,k}|^2 dx \right] + \frac{1}{2} \int_{\Omega} |\nabla V_0|^2 dx \right\} =$$

$$= \mathcal{H}_C(\Psi, \lambda) - \mathcal{H}_C(\Psi_0, \lambda_0).$$

Since the Casimir functional is constant along the solutions of the Schrödinger-Poisson system, which is globally well-posed (see [1]), for an initial condition $(\Psi(0), \lambda) \in \mathcal{L}$, we can use $\mathcal{H}_C(\Psi(0), \lambda)$ in the estimate above instead of $\mathcal{H}_C(\Psi(t), \lambda)$.

After establishing the nonlinear stability of the stationary states of the semi-relativistic Schrödinger-Poisson system, our main goal is show the existence of such states satisfying the assumptions of the stability theorem.
5 Dual functionals

For every distribution function \( f \in \mathcal{C} \) we will derive a corresponding stationary state as the unique maximizer of a functional defined below. Let us use the energy-Casimir functional from the stability result to obtain such a dual functional. Our tool below will be the saddle point principle. Recall that, for \( \Lambda > 0 \) fixed

\[
\mathcal{G}(\Psi, \Lambda, V, \sigma) := \sum_{k=1}^{\infty} \left( F^*(\lambda_k) + \lambda_k \int_{\Omega} \| T_m^\frac{\Lambda}{|\psi_k|^2} V |\psi_k|^2 \| dx \right) - \frac{1}{2} \int_{\Omega} |\nabla V|^2 dx + \sigma \left[ \sum_{k=1}^{\infty} \lambda_k - \Lambda \right].
\]

Here as before \( \Psi = \{\psi_k\}_{k=1}^{\infty} \subset H^\frac{1}{2}_0(\Omega) \cap H^1(\Omega) \) is a complete orthonormal system in \( L^2(\Omega) \) and \( \Lambda \in l^1_+ = \{ (\lambda_k) \in l^1 \mid \lambda_k \geq 0, \ k \in \mathbb{N} \} \). Now the function \( V \in H^1_0(\Omega) \) is allowed to vary independently of \( \Psi \) and \( \lambda \). The parameter \( \sigma \in \mathbb{R} \) here plays the role of Lagrange multipliers. The statement below demonstrates how the functional defined above is related to our energy-Casimir functional.

**Lemma 9.** For arbitrary \( \Psi, \Lambda, \sigma \),

\[
\sup_V \mathcal{G}(\Psi, \Lambda, V, \sigma) = \mathcal{H}_C(\Psi, \Lambda) + \sigma \left[ \sum_{k=1}^{\infty} \lambda_k - \Lambda \right]. \tag{5.1}
\]

The supremum is attained at \( V = V_{\psi,\lambda} \).

**Proof.** We express the functional defined above as

\[
\mathcal{G}(\Psi, \Lambda, V, \sigma) = \sum_{k=1}^{\infty} \left( F^*(\lambda_k) + \lambda_k \int_{\Omega} \| T_m^\frac{\Lambda}{|\psi_k|^2} V |\psi_k|^2 \| dx \right) + \frac{1}{2} \lambda_k \int_{\Omega} |\psi_k|^2 V_{\psi,\lambda} dx + \sum_{k=1}^{\infty} \lambda_k \int_{\Omega} V |\psi_k|^2 dx -
\]

\[
- \frac{1}{2} \int_{\Omega} |\nabla V_{\psi,\lambda}|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla V|^2 dx + \sigma \left[ \sum_{k=1}^{\infty} \lambda_k - \Lambda \right].
\]

Using the definition of the energy-Casimir functional (2.8) we arrive at

\[
\mathcal{H}_C(\Psi, \Lambda) = \int_{\Omega} V \Delta V_{\psi,\lambda} dx - \frac{1}{2} \int_{\Omega} |\nabla V_{\psi,\lambda}|^2 dx - \frac{1}{2} \int_{\Omega} |\nabla V|^2 dx + \sigma \left[ \sum_{k=1}^{\infty} \lambda_k - \Lambda \right].
\]

The expression above can be written as

\[
\mathcal{H}_C(\Psi, \Lambda) = - \frac{1}{2} \| \nabla V_{\psi,\lambda} - \nabla V \|_{L^2(\Omega)}^2 + \sigma \left[ \sum_{k=1}^{\infty} \lambda_k - \Lambda \right],
\]

which completes the proof of the lemma.
In the next Section, we will show that the functional $\Phi(V, \sigma)$ defined in (2.9) has a unique maximizer, which is a stationary state of our Schrödinger-Poisson system. Let us first prove the following auxiliary statement, which is the generalization of Lemma 8 above.

**Lemma 10.** Let $V \in H^1_0(\Omega)$ and $V \geq 0$. Then for $(\Psi, \Delta) \in \mathcal{L}$ and $\sigma \in \mathbb{R}$, the lower bound

$$
\sum_{k=1}^{\infty} \left[ F^*(-\lambda_k) + \lambda_k \left( \int_{\Omega} \left| T_m^{\frac{3}{2}} \psi_k \right|^2 + V |\psi_k|^2 \right) dx + \sigma \right] \geq -\text{Tr}[F(T_m + V + \sigma)]
$$

(5.2)
is valid. Equality in it is attained when $(\Psi, \Delta) = (\Psi_V, \Delta_V)$, where $\psi_{V,k} \in H^1_0(\Omega) \cap H^1(\Omega)$, $k \in \mathbb{N}$ is the orthonormal sequence of eigenfunctions of the operator $T_m + V$ corresponding to eigenvalues $\mu_{V,k}$. Moreover, $\lambda_{V,k} = f(\mu_{V,k} + \sigma)$, $k \in \mathbb{N}$.

**Proof.** Let us use inequality (4.2) with

$$
\mu_k := \int_{\Omega} \left( |T_m^{\frac{3}{2}} \psi_k|^2 + V |\psi_k|^2 \right) dx + \sigma = \langle \psi_k, (T_m + V + \sigma) \psi_k \rangle, \quad k \in \mathbb{N}.
$$

Therefore,

$$
F^*(-\lambda_k) + \lambda_k \left( \int_{\Omega} \left| T_m^{\frac{3}{2}} \psi_k \right|^2 + V |\psi_k|^2 \right) dx + \sigma \geq -F(\langle \psi_k, (T_m + V + \sigma) \psi_k \rangle), \quad k \in \mathbb{N}. \quad (5.3)
$$

Clearly,

$$
T_m + V + \sigma = \int_0^{\infty} (\lambda + \sigma) dE_{\lambda},
$$

where $E_{\lambda}$ is the spectral family associated with the Hamiltonian $T_m + V$, such that $d\nu_k(\lambda) := \langle \psi_k, dE_{\lambda} \psi_k \rangle$ is a probability measure for $k \in \mathbb{N}$. By means of Jensen’s inequality

$$
F(\langle \psi_k, (T_m + V + \sigma) \psi_k \rangle) = F\left( \int_0^{\infty} (\lambda + \sigma) d\nu_k(\lambda) \right) \leq \int_0^{\infty} F(\lambda + \sigma) d\nu_k(\lambda) =
$$

$$
= \langle \psi_k, F(T_m + V + \sigma) \psi_k \rangle.
$$

This upper bound along with (5.3) and summation over $k \in \mathbb{N}$ give us the desired inequality (5.2).

Then consider $\{\psi_{V,k}\}_{k=1}^{\infty} \subset H^1_0(\Omega) \cap H^1(\Omega)$ forming a complete orthonormal system in $L^2(\Omega)$, such that $(T_m + V)\psi_{V,k} = \mu_{V,k}\psi_{V,k}$ and $\lambda_{V,k} = f(\mu_{V,k} + \sigma)$, $k \in \mathbb{N}$. In this case the right side of (5.2) is equal to

$$
- \sum_{k=1}^{\infty} \langle F(T_m + V + \sigma) \psi_{V,k}, \psi_{V,k} \rangle = - \sum_{k=1}^{\infty} F(\mu_{V,k} + \sigma).
$$

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We have for \( k \in \mathbb{N} \)
\[
F^*(-\lambda_{V,k}) = \sup_{\lambda \in \mathbb{R}} (-\lambda \lambda_{V,k} - F(\lambda)) = -f^{-1}(\lambda_{V,k}) \lambda_{V,k} - F(f^{-1}(\lambda_{V,k})),
\]
since it is attained at the maximal point \( \lambda^* := f^{-1}(\lambda_{V,k}) \). The equality \( \lambda_{V,k} = f(\mu_{V,k} + \sigma) \) yields \( f^{-1}(\lambda_{V,k}) = \mu_{V,k} + \sigma \), such that
\[
F^*(-\lambda_{V,k}) = -(\mu_{V,k} + \sigma) \lambda_{V,k} - F(\mu_{V,k} + \sigma).
\]
A direct computation implies that the left side of (5.2) equals to \(-\sum_{k=1}^{\infty} F(\mu_{V,k} + \sigma)\).

Armed with the auxiliary lemma above we manage to derive the expression for the dual functional for our problem.

**Lemma 11.** The infimum in definition (2.9) is attained at \( \Psi = \{\psi_{V,k}\}_{k=1}^{\infty} \), an orthonormal sequence of eigenfunctions of the Hamiltonian \( T_m + V, V \geq 0 \) corresponding to eigenvalues \( \mu_{V,k} \) with \( \lambda_{V,k} = f(\mu_{V,k} + \sigma) \) for \( k \in \mathbb{N} \). Furthermore, the dual functional is given by
\[
\Phi(V, \sigma) = -\frac{1}{2} \int_{\Omega} |\nabla V|^2 dx - \text{Tr}[F(T_m + V + \sigma)] - \sigma \Lambda.
\]

**Proof.** Let us show that the operator \( F(T_m + V + \sigma) \) is trace class. Clearly,
\[
\text{Tr}[F(T_m + V + \sigma)] = \sum_{k=1}^{\infty} F(\mu_{V,k} + \sigma).
\]

Since the potential function \( V \geq 0 \) by assumption, we use inequalities (3.2) and (2.6) and arrive at the series with the general term \((1 + Ck^{\frac{1}{3}} - m + \sigma)^{-1-\varepsilon}\). This series is convergent. We conclude the proof of the lemma by referring to the result of Lemma 10 above.

\[\blacksquare\]

### 6 Existence of stationary states

In this section we prove, for each distribution function \( f \in \mathcal{C} \) and each value of \( \Lambda > 0 \), the existence of a unique maximizer of the functional \( \Phi \), which will be a stationary state of the semi-relativistic Schrödinger-Poisson system.

**Proof of Theorem 2.** Let us first show that the bound
\[
\text{Tr}[F(T_m + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2)) \leq \alpha \text{Tr}[F(T_m + V_1 + \sigma_1)] + (1 - \alpha) \text{Tr}[F(T_m + V_2 + \sigma_2)]
\]
holds for any \( \alpha \in (0,1) \) and \( (V_j, \sigma_j) \in H_{0,+}^1(\Omega) \times \mathbb{R} \), \( j = 1,2 \). Let \( \phi \in H_{0}^{\frac{3}{2}}(\Omega) \cap H^1(\Omega) \) and \( \|\phi\|_{L^2(\Omega)} = 1 \). We make use of the spectral decompositions
\[
T_m + V_1 = \int_{0}^{\infty} \gamma dP_\gamma, \quad T_m + V_2 = \int_{0}^{\infty} \beta dQ_\beta,
\]
where \( P_\gamma \) and \( Q_\beta \) are the spectral families associated with the operators \( T_m + V_1 \) and \( T_m + V_2 \) respectively. This enables us to introduce the probability measures

\[
d\nu(\gamma) := (\phi, dP_\gamma \phi)_{L^2(\Omega)}, \quad d\mu(\beta) := (\phi, dQ_\beta \phi)_{L^2(\Omega)}
\]

and write

\[
F((\phi, [T_m + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2)] \phi)_{L^2(\Omega)}) = \\
= F(\alpha \int_0^\infty (\gamma + \sigma_1) d\nu(\gamma) + (1 - \alpha) \int_0^\infty (\beta + \sigma_2) d\mu(\beta)).
\]

Since \( F \) is strictly convex on its support, we obtain the upper bound for the expression above using Jensen’s inequality as

\[
\alpha \int_0^\infty F(\gamma + \sigma_1) d\nu(\gamma) + (1 - \alpha) \int_0^\infty F(\beta + \sigma_2) d\mu(\beta).
\]

By means of definition \( 6.2 \) we arrive at

\[
\alpha(\phi, F(T_m + V_1 + \sigma_1) \phi)_{L^2(\Omega)} + (1 - \alpha)(\phi, F(T_m + V_2 + \sigma_2) \phi)_{L^2(\Omega)}.
\]

Let \( \{\psi_k\}_{k=1}^\infty \) be the set of eigenfunctions of the operator \( T_m + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2) \) forming a complete orthonormal system in \( L^2(\Omega) \). Then via the argument above we obtain

\[
\sum_{k=1}^\infty F((\psi_k, [T_m + \alpha(V_1 + \sigma_1) + (1 - \alpha)(V_2 + \sigma_2)] \psi_k)_{L^2(\Omega)}) \leq \\
\leq \alpha \sum_{k=1}^\infty (\psi_k, F(T_m + V_1 + \sigma_1) \psi_k)_{L^2(\Omega)} + (1 - \alpha) \sum_{k=1}^\infty (\psi_k, F(T_m + V_2 + \sigma_2) \psi_k)_{L^2(\Omega)}
\]

and arrive at inequality \( 6.1 \). Suppose equality holds. From the fact that the function \( F \) is strictly convex on its support we deduce that the operators \( T_m + V_1 + \sigma_1 \) and \( T_m + V_2 + \sigma_2 \) with potential functions \( V_1 \) and \( V_2 \) vanishing on the boundary of \( \Omega \), have the same set of eigenvalues and the corresponding eigenfunctions are \( \{\psi_k\}_{k=1}^\infty \). Therefore, \( V_1(x) = V_2(x) \) in \( \Omega \) and \( \sigma_1 = \sigma_2 \), and \( \text{Tr}[F(T_m + V + \sigma)] \) is strictly convex. Since \( -\frac{1}{2} \int_\Omega |\nabla V|^2 dx \) and \( -\sigma \Lambda \) are concave, we obtain that our functional given by \( 5.4 \) is strictly concave.

Then we turn our attention to the proof of its boundedness from above and coercivity. Obviously, by means of the Poincaré inequality

\[
\frac{1}{2} \int_\Omega |\nabla V|^2 dx \geq \frac{C_1}{2} \|V\|_{H^1_0(\Omega)}^2
\]

with a constant \( C_1 > 0 \). Let \( \mu_V \) be the lowest eigenvalue of the Hamiltonian \( T_m + V \). Clearly, we have the estimate with a trial function \( \tilde{\phi} \) as

\[
\mu_V \leq \int_\Omega \{||p|^{\frac{2}{\gamma}} \tilde{\phi}|^2 + V|\tilde{\phi}|^2\} dx, \quad ||\tilde{\phi}||_{L^2(\Omega)} = 1.
\]
Let us fix $\tilde{\phi}$ as the ground state of the negative Dirichlet Laplacian on $L^2(\Omega)$. Then
\[
\int_{\Omega} ||p|^{\frac{2}{3}} \tilde{\phi}|^2 dx = \sqrt{C_p},
\]
where $C_p$ is the constant in the Poincaré inequality. We introduce
\[
C_2 := \sqrt{\int_{\Omega} |\tilde{\phi}|^4 dx} > 0,
\]
which is finite. Indeed, $\tilde{\phi} \in L^6(\Omega)$ via the Sobolev inequality (3.1). Hence via the Schwarz inequality we arrive at
\[
\int_{\Omega} V|\tilde{\phi}|^2 dx \leq C_2 \|V\|_{H^1_0(\Omega)},
\]
such that
\[
\mu_V \leq \sqrt{C_p} + C_2 \|V\|_{H^1_0(\Omega)}.
\]
This yields the upper bound
\[
\Phi(V, \sigma) \leq -\frac{C_1}{2} \|V\|^2_{H^1_0(\Omega)} - F(\sqrt{C_p} + C_2 \|V\|_{H^1_0(\Omega)} + \sigma) - \sigma \Lambda. \tag{6.3}
\]
Let us use the convexity property, such that
\[
F(x) \geq -\beta x + C_3,
\]
where $\beta > \Lambda > 0$ is large enough. This implies the inequality
\[
\Phi(V, \sigma) \leq -\frac{C_1}{2} \|V\|^2_{H^1_0(\Omega)} + (\beta - \Lambda)\sigma + \beta C_2 \|V\|_{H^1_0(\Omega)} + \beta \sqrt{C_p} - C_3.
\]
A direct computation yields the estimate
\[
\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|^2_{H^1_0(\Omega)} + C_5 + (\beta - \Lambda)\sigma + \beta \sqrt{C_p} - C_3.
\]
Let us choose $\beta = 2\Lambda$ and define the nonnegative constant $k := \max\{C_5 + \beta \sqrt{C_p} - C_3, 0\}$. Hence
\[
\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|^2_{H^1_0(\Omega)} + \Lambda \sigma + k. \tag{6.4}
\]
Combining estimates (6.3) and (6.4), we easily arrive at
\[
\Phi(V, \sigma) \leq -\frac{C_1}{4} \|V\|^2_{H^1_0(\Omega)} - \Lambda |\sigma| + k,
\]
which shows that our functional $\Phi(V, \sigma)$ is bounded above and $-\Phi(V, \sigma)$ is coercive. Therefore, $\Phi(V, \sigma)$ has a unique maximizer $(V_0, \sigma_0)$. Let the hamiltonian $T_m + V_0$ have the sequence of eigenvalues $\{\mu_{0,k}\}_{k=1}^{\infty}$ and corresponding eigenfunctions $\{\psi_{0,k}\}_{k=1}^{\infty}$, such that

$$(T_m + V_0)\psi_{0,k} = \mu_{0,k}\psi_{0,k}, \quad k \in \mathbb{N}$$

and let $\lambda_{0,k} := f(\mu_{0,k} + \sigma_0)$. We have

$$\Phi(V_0, \sigma) = -\frac{1}{2} \int |\nabla V_0|^2 dx - \sum_{k=1}^{\infty} \int_{\mu_{0,k} + \sigma}^{\infty} f(\xi) d\xi - \sigma \Lambda,$$

such that $\sigma = \sigma_0$ is its critical point. Therefore,

$$0 = \frac{d\Phi}{d\sigma}(V_0, \sigma)|_{\sigma=\sigma_0} = -\Lambda + \sum_{k=1}^{\infty} f(\mu_{0,k} + \sigma_0) = \sum_{k=1}^{\infty} \lambda_{0,k} - \Lambda,$$

such that $\sum_{k=1}^{\infty} \lambda_{0,k} = \Lambda$. The first variation of $\Phi(V, \sigma_0)$ at $V = V_0$ vanishes as well. Thus, an easy computation gives us

$$-\Delta V_0(x) = \sum_{k=1}^{\infty} \lambda_{0,k} |\psi_{0,k}(x)|^2.$$

By direct substitution, the functions $\psi_k(x, t) = e^{-i\mu_{0,k}t}\psi_{0,k}(x), \quad k \in \mathbb{N}$ satisfy the Schrödinger equation

$$i \frac{\partial \psi_k}{\partial t} = [T_m + V_0]\psi_k, \quad x \in \Omega, \quad t \geq 0.$$

The density matrix

$$\rho_0(t, x, y) = \sum_{k=1}^{\infty} \lambda_{0,k} \psi_k(x, t) \overline{\psi_k(y, t)} = \sum_{k=1}^{\infty} \lambda_{0,k} \psi_{0,k}(x) \overline{\psi_{0,k}(y)},$$

such that $\frac{\partial \rho_0}{\partial t} = 0$ and the particle concentration $n_0(t, x) = \rho_0(t, x, x)$.

Therefore, $(\Psi_0, \Delta_0, \mu_0, V_0)$ is a stationary state of our semi-relativistic Schrödinger-Poisson system. Finally, we are in position to show that $(\Psi_0, \Delta_0) \in \mathcal{L}$, which can be done analogously to the proof of Lemma 6 above.

We have the following result relating the functional $\Phi$ and $\mathcal{H}_C$.

**Proposition 12.** Let the assumptions of Theorem 2 hold, such that $(\Psi_0, \Delta_0, \mu_0, V_0)$ is the corresponding stationary state of the semi-relativistic Schrödinger-Poisson system. Then

$$\Phi(V_0, \sigma_0) = \mathcal{H}_C(\Psi_0, \Delta_0).$$
Proof. Note that\
\[ \Phi(V_0, \sigma_0) = -\frac{1}{2} \int_\Omega |\nabla V_0|^2 dx - \text{Tr}[F(T_m + V_0 + \sigma_0)] - \sigma_0 \Lambda \]
and
\[ \mathcal{H}_C(\Psi_0, \lambda_0) = \sum_{k=1}^\infty F^*(-\lambda_{0,k}) + \sum_{k=1}^\infty \lambda_{0,k} \int_\Omega \frac{1}{2} |T_m^\dagger \psi_{0,k}|^2 dx + \frac{1}{2} \int_\Omega |\nabla V_0|^2 dx. \]

By means of Lemma 10
\[ \sum_{k=1}^\infty \left[ F^*(-\lambda_{0,k}) + \lambda_{0,k} \left( \int_\Omega \left| T_m^\dagger \psi_{0,k} \right|^2 + V_0 |\psi_{0,k}|^2 \right) dx + \sigma_0 \right] = -\text{Tr}[F(T_m + V_0 + \sigma_0)], \]

which implies the statement of the proposition.

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Appendix: Higher regularity

In this Appendix, we extend the global well-posedness result of [1] to spaces with higher regularity. For \( s \in \mathbb{N} \), let
\[ \mathcal{L}^s := \{ (\Psi, \lambda) | \Psi = \{ \psi_k \}_{k=1}^\infty \subset H^s_0(\Omega) \cap H^s(\Omega) \} \text{ is a complete orthonormal system in } L^2(\Omega), \]
\[ \Delta = \{ \lambda_k \}_{k=1}^\infty \subset \ell^1, \quad \lambda_k \geq 0, \ k \in \mathbb{N}, \quad \sum_{k=1}^\infty \lambda_k \int_\Omega |(-\Delta)^{s/2} \psi_k|^2 dx < \infty \}. \]

Note that in the state spaces defined above the Dirichlet Laplacian was used and no additional boundary conditions were required. We introduce inner product \( (\cdot, \cdot)_{\mathcal{H}^s_\Delta(\Omega)} \) which induces the generalized inhomogenous Sobolev norm
\[ \| \Phi \|_{\mathcal{H}^s_\Delta(\Omega)} := \left( \sum_{k=1}^\infty \lambda_k \| \phi_k \|_{H^s(\Omega)}^2 \right)^{\frac{1}{2}}, \]

and define the corresponding Hilbert space
\[ \mathcal{H}^s_\Delta(\Omega) := \{ \Phi = \{ \phi_k \}_{k=1}^\infty | \phi_k \in H^s_0(\Omega) \cap H^s(\Omega), \ \forall \ k \in \mathbb{N}, \ \| \Phi \|_{\mathcal{H}^s_\Delta(\Omega)} < \infty \}. \]
We also introduce the generalized homogenous Sobolev norm
\[ \| \Phi \|_{\dot{H}_s^\lambda(\Omega)} := \left( \sum_{k=1}^{\infty} \lambda_k \| (-\Delta)^{s/2} \phi_k \|^2_{L^2(\Omega)} \right)^{1/2}. \]
Clearly, \( \| \Phi \|_{\dot{H}_s^\lambda(\Omega)} \lesssim \| \Phi \|_{\dot{H}_s^\lambda(\Omega)} \). Furthermore, it follows from the Poincaré inequality that \( \| \Phi \|_{\dot{H}_s^\lambda(\Omega)} \lesssim \| \dot{\Phi} \|_{\dot{H}_s^\lambda(\Omega)} \). This implies the following result.

**Lemma A.1.** For \( \Gamma \in H_s^\lambda(\Omega) \) the norms \( \| \Gamma \|_{H_s^\lambda(\Omega)} \) and \( \| \Gamma \|_{\dot{H}_s^\lambda(\Omega)} \) are equivalent.

We know from [1] that, for \( \Psi, \Phi \in H_1^\lambda(\Omega) \),
\[ \| F[\Psi] \|_{H_1^\lambda(\Omega)} \lesssim \| \Psi \|_{\dot{H}_1^\lambda(\Omega)} \| \Psi \|_{H_1^\lambda(\Omega)} \]
and that
\[ \| F[\Psi] - F[\Phi] \|_{H_1^\lambda(\Omega)} \lesssim (\| \Psi \|_{\dot{H}_1^\lambda(\Omega)}^2 + \| \Phi \|_{\dot{H}_1^\lambda(\Omega)}^2) \| \Psi - \Phi \|_{H_1^\lambda(\Omega)}. \]
We have the following inequalities in spaces of higher regularity.

**Lemma A.2.** Let \( \Psi, \Phi \in H_s^\lambda(\Omega), \ s \geq 2 \). Then
\[ \| F[\Psi] \|_{H_s^\lambda(\Omega)} \lesssim \| \Psi \|_{\dot{H}_s^\lambda-1(\Omega)} \| \Psi \|_{H_s^\lambda(\Omega)} \]
and that
\[ \| F[\Psi] - F[\Phi] \|_{H_s^\lambda(\Omega)} \lesssim (\| \Psi \|_{\dot{H}_s^\lambda(\Omega)}^2 + \| \Phi \|_{\dot{H}_s^\lambda(\Omega)}^2) \| \Psi - \Phi \|_{H_s^\lambda(\Omega)}. \]

We start by proving the first inequality. The second inequality follows using a similar analysis.

\[
\| F[\Psi] \|_{H_s^\lambda(\Omega)}^2 \lesssim \| F[\Psi] \|_{\dot{H}_s^\lambda(\Omega)}^2 = \sum_{k,l \geq 0} \lambda_k \lambda_l \langle V[\Psi] \psi_k, (-\Delta)^s V[\Psi] \psi_l \rangle \\
= (-1)^s \sum_{0 \leq |\alpha| \leq s} \sum_{k,l \geq 0} \lambda_k \lambda_l \langle V[\Psi] \psi_k, \partial^{2\alpha} V[\Psi] \psi^{2s-2\alpha} \psi_l \rangle \\
\lesssim \left( \sum_{0 \leq |\alpha| \leq s} \sum_{k \geq 0} \lambda_k \| \partial^{\alpha} V[\Psi] \partial^{s-\alpha} \psi_k \|_{L^2(\Omega)} \right)^2 \\
\lesssim \left( \sum_{0 \leq |\alpha| \leq s-1} \sum_{k \geq 0} \lambda_k \| \partial^{\alpha} V[\Psi] \|_{L^6(\Omega)} \| \partial^{s-\alpha} \psi_k \|_{L^3(\Omega)} + \sum_{k \geq 0} \lambda_k \| V[\Psi] \|_{L^\infty(\Omega)} \| \partial^s \psi_k \|_{L^2(\Omega)} \right)^2 \\
\lesssim \left( \sum_{0 \leq |\alpha| \leq s-1} |\partial^{\alpha} V[\Psi] \|_{L^6(\Omega)} \| \Psi \|_{H_s^\lambda \frac{1}{2} (\Omega)} + \| V[\Psi] \|_{L^\infty(\Omega)} \| \Psi \|_{H_s^\lambda(\Omega)} \right)^2
\]
where we have used the generalized Leibnitz rule on the third line, Hölder’s inequality on the fourth and fifth lines, and the Sobolev inequality

$$\|f\|_{L^\infty_{\bar{\Omega}}} \lesssim \|f\|_{H^p(\Omega)}$$
onumber

on the sixth line. It follows from the Sobolev inequality that

$$\|V[\Psi]\|_{L^\infty(\Omega)} \lesssim \|p|^{-1/2}n[\Psi]\|_{L^2(\Omega)}^2.$$ nonumber

Furthermore

$$\|p|^{-1/2}n[\Psi]\|_{L^2(\Omega)}^2 = (n[\Psi], |p|^{-1}n[\Psi])_{L^2(\Omega)} \leq \|n[\Psi]\|_{L^{3/2}(\Omega)} \|p|^{-1}n[\Psi]\|_{L^3(\Omega)}$$
onumber

$$\lesssim \|\hat{n}\|_{\bar{\Delta}^{1/2}(\Omega)}^2 \|p|^{-1/2}n[\Psi]\|_{L^2(\Omega)},$$

where we have used Hölder’s inequality in the first line, and the Sobolev inequality on the second line. This yields $\|p|^{-1/2}n[\Psi]\|_{L^2(\Omega)} \lesssim \|\hat{n}\|_{\bar{\Delta}^{1/2}(\Omega)}^2$, and hence

$$\|V[\Psi]\|_{L^\infty(\Omega)} \lesssim \|\hat{n}\|_{\bar{\Delta}^{1/2}(\Omega)}^2.$$ nonumber

We now estimate $\|\partial^\alpha V(\psi)\|_{L^5(\Omega)}$, $0 \leq |\alpha| \leq s - 1$.

$$\|\partial^\alpha V(\psi)\|_{L^5(\Omega)}^2 \lesssim \|n[\Psi]\|_{H^{s-1}(\Omega)}$$

$$\lesssim \sum_{0 \leq |\beta| \leq |\alpha| - 1} \sum_{k \geq 0} \lambda_k \lambda_l \|\partial^{\alpha-\beta-1} \psi_k \partial^\beta \psi_l\|_{L^2(\Omega)}^2$$

$$\lesssim \sum_{0 \leq |\beta| \leq |\alpha| - 1} \sum_{k \geq 0} \lambda_k \lambda_l \|\partial^{\alpha-\beta-1} \psi_k\|_{L^6(\Omega)}^2 \|\partial^\beta \psi_l\|_{L^\infty(\Omega)}^2$$

$$\leq \sum_{0 \leq |\beta| \leq |\alpha| - 1} \|\hat{n}\|_{\bar{\Delta}^{s-\beta}(\Omega)}^2 \|\hat{n}\|_{\bar{\Delta}^{\alpha+\beta}(\Omega)}^2.$$ nonumber

Combining the above inequalities yields

$$\|F[\Psi]\|_{H^{\bar{\Delta}^s}(\Omega)}^2 \lesssim \left( \sum_{0 \leq |\alpha| \leq s-1} \sum_{0 \leq |\beta| \leq |\alpha| - 1} \|\hat{n}\|_{H^{\bar{\Delta}^{s-\beta}(\Omega)}} \|\hat{n}\|_{H^{\bar{\Delta}^{s-\beta}(\Omega)}} \|\hat{n}\|_{H^{\bar{\Delta}^{s-\beta}(\Omega)}} \|\hat{n}\|_{H^{\bar{\Delta}^{s-\beta}(\Omega)}} \right)^2$$

$$\lesssim \|\hat{n}\|_{H^{s-1}(\Omega)}^4 \|\hat{n}\|_{H^{s}(\Omega)}^2,$$ nonumber

and hence

$$\|F[\Psi]\|_{H^{\bar{\Delta}^s}(\Omega)} \lesssim \|\hat{n}\|_{H^{s-1}(\Omega)} \|\hat{n}\|_{H^{s}(\Omega)}.$$ nonumber

To prove the second inequality, note that

$$\|F[\Psi] - F[\Phi]\|_{H^{\bar{\Delta}^s}(\Omega)} = \|V[\Psi](\Psi - \Phi) + (V[\Psi] - V[\Phi])\Phi\|_{H^{\bar{\Delta}^s}(\Omega)}$$

$$\leq \|V[\Psi](\Psi - \Phi)\|_{H^{\bar{\Delta}^s}(\Omega)} + \|(V[\Psi] - V[\Phi])\Phi\|_{H^{\bar{\Delta}^s}(\Omega)}.$$
An analysis similar to the proof of the first inequality yields the local Lipschitz continuity and completes the proof of the lemma.

Lemma A.2 together with the fact that the operator $T_m$ generates the group $e^{-iT_m t}$, $t \in \mathbb{R}$, of unitary operators implies local well-posedness in $L^s$. Furthermore, we know from \[1\] that, for $\Psi(x,0) \in \mathcal{H}^1_\Lambda(\Omega)$, $\|\Psi(x,t)\|_{\mathcal{H}^1_\Lambda(\Omega)}$ is bounded for all times.

**Theorem A.3.** For every initial state $(\Psi(x,0), \Lambda) \in L^s$, $s \geq 2$, there is a unique mild solution $\Psi(x,t), t \in [0, \infty), \text{ of } (1.6)-(1.8) \text{ with } (\Psi(x,t), \Lambda) \in L^s$, which is also a unique strong global solution in $L^s_\Lambda(\Omega)$.

The proof follows from the blow-up alternative and the first inequality in Lemma A.2. The mild solution of the Schrödinger-Poisson system (1.6)-(1.8), given by

$$
\Psi(t) = e^{-iT_m t}\Psi(0) + e^{-iT_m t}\int_0^t e^{iT_m t'}F[\Psi(t')]dt',
$$

which implies

$$
\|\Psi(t)\|_{\mathcal{H}^1_\Lambda(\Omega)} \leq \|\Psi(0)\|_{\mathcal{H}^1_\Lambda(\Omega)} + \int_0^t \|F[\Psi(t')]\|_{\mathcal{H}^1_\Lambda(\Omega)}dt'.
$$

If $\|\Psi(t)\|_{\mathcal{H}^1_\Lambda(\Omega)} \leq 1$, it follows from Lemma A.2 that

$$
\|\Psi(t)\|_{\mathcal{H}^1_\Lambda(\Omega)} \leq C_1 + C_2 \int_0^T \|\Psi(t')\|_{\mathcal{H}^1_\Lambda(\Omega)}dt'.
$$

By Gronwall’s lemma,

$$
\|\Psi(t)\|_{\mathcal{H}^1_\Lambda(\Omega)} \leq C_1 e^{C_2t}, \quad t \in [0, T] \subset [0, T).
$$

Since $\|\Psi(t)\|_{\mathcal{H}^1_\Lambda(\Omega)}$ is bounded for all times, it follows by induction on $s$ that $\|\Psi(t)\|_{\mathcal{H}^s_\Lambda(\Omega)}$ is bounded for all times.

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