Nonstatic $AdS_2$ Branes and the Isometry Group of $AdS_3$ Spacetime

Shijong Ryang

Department of Physics  
Kyoto Prefectural University of Medicine  
Taishogun, Kyoto 603-8334 Japan  
ryang@koto.kpu-m.ac.jp

Abstract

For the D-branes on the $SL(2,R)$ WZW model we present a particular choice of outer automorphism for the gluing condition of currents that leads to a special $AdS_2$ brane configuration. This configuration is shown to be a static solution in the cylindrical coordinates, and a nonstatic solution in the Poincaré coordinates to the nonlinear equation of motion for the Dirac-Born-Infeld action of a D-string. The generalization of it gives a family of nonstatic $AdS_2$ brane solutions. They are demonstrated to transform to each other under the isometry group of $AdS_3$ spacetime.

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1 Introduction

Recently the D-branes on the group manifolds have attracted a great deal of attention. The D-branes of the WZW models have been studied by using the algebraic methods of the boundary conformal field theory or the geometric methods where the geometries of the associated D-branes are specified in terms of the gluing conditions for the currents \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\]. Owing to the conditions the endpoints of open strings are forced to stay on the (twined) conjugacy classes of the group. For the compact SU(2) group a D2-brane wrapped on a two-sphere subspace of the three-sphere manifold has been shown to be stabilized against shrinking by the flux of worldvolume U(1) gauge field \[8\]. The stable spherical D2-brane configuration constructed by the geometric and semiclassical analysis to the Dirac-Born-Infeld (DBI) action can be identified with a Cardy state which is a linear superposition of Ishibashi states in the boundary conformal field theory.

The D-branes on the noncompact group SL(2, R) have been investigated by the geometric approach \[12\]. Gluing the currents by the inner automorphisms leads to the regular conjugacy classes that are associated with a two-dimensional hyperbolic plane (\(H_2\)), the light cone and a de Sitter brane (\(dS_2\)) which describes a tachyonic unphysical D-string. On the other hand gluing the currents by an outer automorphism leads to the twined conjugacy class that is associated with a two-dimensional anti de Sitter brane (\(AdS_2\)) which is a physical D-string stretched between two points on the boundary of \(AdS_3\). From the algebraic approach it has been indicated that the \(dS_2\) brane corresponds to the continuous representation of \(SL(2, R)\), while the \(H_2\) and \(AdS_2\) branes correspond to the discrete representation \[13\]. Further there have been various studies of the classical motions and the spectrum of open strings attached to the \(AdS_2\) branes \[14, 15, 16\] and the constructions of the Cardy states based on the discrete \[17\] and continuous representations \[18\].

In Ref. \[12\] the static \(AdS_2\) brane configuration corresponding to a particular outer automorphism was constructed in the \(SL(2, R)\) WZW model. In order to try to generalize this special \(AdS_2\) brane on the \(SL(2, R)\) manifold we will propose the other type of outer automorphism and show how the static or nonstatic D-brane configuration is described by the twined conjugacy class associated with this type of automorphism and satisfies the equation of motion for the DBI action of the D-string accompanied with the U(1) gauge field. The generalization of these automorphisms is performed and the general time-dependent solutions are constructed. These stationary configurations are shown to be related to each other by the isometry group of the \(AdS_3\) spacetime.

2 General \(AdS_2\) branes

We consider the D-branes in the \(AdS_3\) spacetime with the radius \(L\) that is isomorphic to the group manifold of \(SL(2, R)\) and is represented by the hyperboloid
\[
- (X^0)^2 - (X^3)^2 + (X^1)^2 + (X^2)^2 = -L^2
\]
embedded in \(\mathbb{R}^{2,2}\). A general group element can be parametrized as
\[
g = \frac{1}{L} \begin{pmatrix}
X^0 + X^1 & X^2 + X^3 \\
X^2 - X^3 & X^0 - X^1
\end{pmatrix},
\]
whose determinant is equal to one through \( (\mathbb{I}) \). The metric on \( AdS_3 \), \( ds^2 = -(dX^0)^2 - (dX^3)^2 + (dX^1)^2 + (dX^2)^2 \) is expressed in the cylindrical coordinates \((\tau, \rho, \phi)\)

\[
X^0 + iX^3 = L \cosh \rho e^{i\tau}, \quad X^1 + iX^2 = L \sinh \rho e^{i\phi}
\]

as

\[
ds^2 = L^2(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2)
\]

and the Neveu-Schwarz antisymmetric tensor is given by

\[
H = dB = L^2 \sinh(2\rho) d\rho \wedge d\phi \wedge d\tau.
\]

The radial coordinate \( \rho \), the angular coordinate \( \phi \) and the global time coordinate \( \tau \) range over \( 0 \leq \rho < \infty \), \( 0 \leq \phi < 2\pi \) and \(-\infty < \tau < \infty \), where the boundary of \( AdS_3 \) is at \( \rho \to \infty \). The radius of \( AdS_3 \) is specified by the level of the \( SL(2, R) \) current algebra as \( L^2 \sim |k|\alpha' \) in the semiclassical limit. Alternatively there are the Poincaré coordinates \((t, x, u)\) defined by

\[
X^0 + X^1 = Lu, \quad X^2 \pm X^3 = L u w^\pm, \quad X^0 - X^1 = L \left( \frac{1}{u} + uw^+ w^- \right)
\]

with \( w^\pm \equiv x \pm t \). In these coordinates that range over the entire \( \mathbb{R}^3 \), the metric is expressed as

\[
ds^2 = L^2 \left( \frac{du^2}{u^2} + u^2 dw^+ dw^- \right),
\]

where the boundary of \( AdS_3 \) is specified by \( |u| \to \infty \). From the gluing condition \( J = R\bar{J} \) identifying the left and right moving \( SL(2, R) \) currents of the WZW model modulo the Lie algebra automorphism \( R \), the worldvolumes of D-branes containing a fixed element \( g \) of the group are represented by the (twined) conjugacy classes

\[
\mathcal{W}_g^\omega = \{ \omega(h)gh, \forall h \in SL(2, R) \},
\]

where \( \omega \) is a group automorphism induced near the identity from \( R \) \([3, 6, 7]\). For the inner automorphism of \( SL(2, R) \), \( \omega(h) = g_0^{-1}hg_0 \) with \( g_0 \in SL(2, R) \) the worldvolumes of D-branes are characterized by the regular conjugacy classes and expressed as \( H_2 \), the light cone and \( dS_2 \). When \( \omega \) is a nontrivial outer automorphism

\[
\omega(h) = \omega_0^{-1}h\omega_0, \quad \text{with } \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( \omega_0 \) has minus one determinant so as not to be an element of \( SL(2, R) \), the D-brane worldvolume lies on the twined conjugacy class

\[
\text{tr}(\omega_0 g) = \frac{2X^2}{L} = 2C
\]

for some constant \( C \) and describes the \( AdS_2 \) geometry, \((X^0)^2 + (X^3)^2 - (X^1)^2 = L^2(1 + C^2)\) which is embedded in \( AdS_3 \). In Ref. \([12]\) from the DBI action of a D-string the static \( AdS_2 \)
brane solution was derived by analysing the continuity equation of the energy-momentum tensor or minimizing the energy of the static state in the Poincaré coordinates.

Instead of the particular outer automorphism (9) we first consider another choice

\[ \omega_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(11)

whose determinant is minus one and manipulate directly the equation of motion for the DBI action. In this choice the outer automorphism is the operation that changes the sign of \( X^2 \) and \( X^3 \), while leaving \( X^0 \) and \( X^1 \) unchanged. The twined conjugacy class given by

\[ \text{tr}(\omega_0 g) = \frac{2X^1}{L} = 2C \]  

(12)

with some constant \( C \) yields the \( AdS_2 \) geometry expressed \((X^0)^2 + (X^3)^2 - (X^2)^2 = L^2(1 + C^2)\). The DBI action of a D-string with tension \( T_D \) is given by

\[ I = -T_D \int d^2\sigma \sqrt{L^4 \cosh^2 \rho (\sinh^2 \rho + (\partial \phi \rho)^2) - F_{\phi \tau}^2}. \]  

(13)

From (5) we take a convenient gauge for the Neveu-Schwarz field to have \( B = L^2 \sinh \rho d\phi \wedge d\tau \) so that the invariant combination \( F_{\phi \tau} \) is given by \( F_{\phi \tau} = L^2 \sinh^2 \rho - 2\pi \alpha' \hat{A}_\phi \). Since the Wilson line \( A_\phi \) is a cyclic variable, its conjugate momentum is a quantized constant of motion

\[ \frac{1}{2\pi} \Pi_\phi \equiv \frac{\partial L}{\partial A_\phi} = -\frac{2\pi \alpha' T_D F_{\phi \tau}}{L^2 \sqrt{D}} = -q \in \mathbb{Z}, \]  

(16)

where \( D = \cosh^2 \rho (\sinh^2 \rho + (\partial \phi \rho)^2) - F_{\phi \tau}^2 / L^4 \). The integer \( q \) expresses the number of fundamental strings bound to the D-string, since an electric field on the D-string is equivalent to a fundamental string. The equation of motion for \( \rho \) is given by

\[ \frac{\sinh^2 \rho + \cosh^2 \rho + (\partial \phi \rho)^2}{\sqrt{D}} = \frac{1}{\sinh \rho \cosh \rho} \partial_\phi \left( \frac{\cosh^2 \rho \partial_\phi \rho}{\sqrt{D}} \right) = \frac{2(\sinh^2 \rho + f)}{\sqrt{D}}, \]  

(17)

where \( f \equiv 2\pi \alpha' F_{\phi \tau} / L^2 \). Although we must solve simultaneously the nonlinear equations (16) and (17), here we will seek a condition for the configuration (14) to satisfy them. Through
the right hand side of the formidable-looking equation (17) is written in terms of the tension $T_F$ of the fundamental string as $2qT_F/T_D$. Moreover, from (16) $D$ is tautologically expressed as

$$D = \frac{1}{1 + \left(\frac{qT_F}{T_D}\right)^2 D_0} \quad (18)$$

with $D_0 = \cosh^2 \rho \sinh^2 \rho + \cosh^2 \rho (\partial_\phi \phi)^2$. The substitution of (14) into $D_0$ yields $D_0 = (C^4 + C^2)/\cos^4 \phi$. This simplified expression can then be used to derive

$$\frac{C^2 \sqrt{1 + \left(\frac{qT_F}{T_D}\right)^2}}{\cos^2 \phi \sqrt{D_0}} = \frac{qT_F}{T_D}, \quad (19)$$

when (14) is substituted into the nonlinear equation (17). As long as $q \geq 0$, this equation can produce a solution $C = \pm qT_F/T_D$. We have demonstrated that the $AdS_2$ static configuration (14) indeed satisfies the stationary equation for the DBI action of a D-string only when $C$ takes the special value in proportion to $q$.

For the twined conjugacy class (10) the $AdS_2$ static configuration given by

$$\sinh \rho \sin \phi = C \quad (20)$$

satisfies the stationary equation only when $C = \pm qT_F/T_D$ that is obtained from

$$\frac{C^2 \sqrt{1 + \left(\frac{qT_F}{T_D}\right)^2}}{\sin^2 \phi \sqrt{D_0}} = \frac{qT_F}{T_D}, \quad (21)$$

with $D_0 = (C^4 + C^2)/\sin^4 \phi$. The obtained $C$ in the cylindrical coordinates agrees with $C$ for the static $AdS_2$ brane solution $u = C/x$ in the Poincaré coordinates (12).

Here we turn to the Poincaré coordinates. For the twined conjugacy class (12) the $AdS_2$ brane configuration is specified by

$$(\frac{1}{u} + C)^2 + x^2 - t^2 = 1 + C^2. \quad (22)$$

From it $u$ is so expressed in terms of $t$ and $x$ as

$$u = \frac{1}{\pm \sqrt{A - C}}, \quad \text{with} \quad A \equiv t^2 - x^2 + C^2 + 1 \quad (23)$$

that the $AdS_2$ brane is nonstatic. We choose a gauge for the Neveu-Schwarz potential to be $B = L^2 u^2 dx \wedge dt$. Hence the gauge invariant two-form is given by $F = L^2 (u^2 + f) dx \wedge dt$ with $f = 2\pi \alpha' F_{xt}/L^2$. A static gauge $\sigma^0 = t, \sigma^1 = x$ is taken so that the DBI action for a D-string is expressed as

$$I = -T_D L^2 \int dt dx \sqrt{D}, \quad (24)$$

where $D = u^4 + (\partial_x u)^2 - (\partial_t u)^2 - (u^2 + f)^2$. The equation of motion for the $U(1)$ gauge field gives the Gauss constraint

$$\frac{2\pi \alpha' T_D F_{xt}}{L^2 \sqrt{D}} = q \in \mathbb{Z}, \quad (25)$$
where $q$ is the number of electric flux quanta turned on along the $AdS_2$. On the other hand the equation of motion for $u$ is provided by

$$\frac{2u^3}{\sqrt{D}} + \partial_t \left( \frac{\partial_t u}{\sqrt{D}} \right) - \partial_x \left( \frac{\partial_x u}{\sqrt{D}} \right) = \frac{2u(u^2 + f)}{\sqrt{D}} , \quad (26)$$

whose right hand side simplifies to $2uqT_F/T_D$ due to the Gauss constraint (25). From (25) $D$ is also equivalently rewritten by

$$D = \frac{1}{1 + \left(\frac{qT_F}{T_D}\right)^2}D_0, \text{ with } D_0 = u^4 + (\partial_x u)^2 - (\partial_t u)^2 . \quad (27)$$

Plugging the nonstatic configuration (23) into $D_0$ we have also a simplified expression $D_0 = (1 + C^2)/A(\pm \sqrt{A} - C)^4$. Owing to this expression the nonlinear equation (26) into which (23) is substituted, becomes to take the compact form

$$\pm \frac{2Cu\sqrt{1 + \left(\frac{qT_F}{T_D}\right)^2}}{\sqrt{1 + C^2}} = \frac{2uqT_F}{T_D} . \quad (28)$$

In view of this expression we extract $C = \pm qT_F/T_D$ whose sign corresponds to the the sign of (28). Thus we have an $AdS_2$ brane classical solution in the Poincaré coordinates

$$u = \pm \frac{1}{\sqrt{t^2 - x^2 + (\frac{qT_F}{T_D})^2 + 1 - \frac{qT_F}{T_D}}} , \quad (29)$$

which exhibits the motion of one D-string bound to $q$ fundamental strings.

Now we consider the other choice for the outer automorphism

$$w_0 = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix} , \quad (30)$$

which interpolates between $w_0 = \sigma_1$ in (9) and $w_0 = \sigma_3$ in (11) where $\sigma_1$ and $\sigma_3$ are the Pauli matrices. The twined conjugacy class characterized by

$$\text{tr}(w_0g) = \frac{2(\sin \theta X^1 + \cos \theta X^2)}{L} = 2C \quad (31)$$

specifies the D-brane worldvolume which is expressed as the hyper surface

$$(X^0)^2 + (X^3)^2 - (\sec \theta X^1 - LC\tan \theta)^2 = L^2(1 + C^2) \quad (32)$$

for $\cos \theta \neq 0$ or

$$(X^0)^2 + (X^3)^2 - (\cosec \theta X^2 - LC\cot \theta)^2 = L^2(1 + C^2) \quad (33)$$

for $\sin \theta \neq 0$. In the cylindrical coordinates the Eq. (31) reads

$$\sinh \rho \sin(\phi + \theta) = C , \quad (34)$$
which is generated by making a rotation of the $\phi$-direction by $\theta$ from the $AdS_2$ solution (20) associated with $\omega_0 = \sigma_1$. If we make the following choice for $\omega_0$

$$\omega_0 = \begin{pmatrix} \sinh \varphi & \cosh \varphi \\ \cosh \varphi & \sinh \varphi \end{pmatrix},$$

which is not connected to (11) but reduced to (3) at $\varphi = 0$, the worldvolume geometry of D-brane is described by

$$\sinh \varphi \cosh \rho \cos \tau + \cosh \varphi \sinh \rho \sin \phi = C.$$  \hspace{1cm} (36)

Since in the cylindrical coordinates $\rho$ is the involved function of $\tau$ and $\phi$, we will analyse this type of configuration in the Poincaré coordinates by making the more general parametrization.

Let us consider the following general outer automorphism expressed in terms of four real parameters as

$$\omega_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ with } \alpha \delta - \beta \gamma = -1,$$

which includes (30) and (35) as special cases. The worldvolume of D-brane lying on the twined conjugacy class $\text{tr}(w_0g) = 2C$ in the Poincaré coordinates is given by

$$u = \frac{-\delta}{\pm \sqrt{A - C}}$$

with $A \equiv C^2 - \delta(\alpha + \beta w^- + \gamma w^+ + \delta w^+ w^-)$. In order to see that this general trajectory is indeed the solution of the stationary nonlinear equation for the DBI action, we calculate $D_0$ in (27) similarly as

$$D_0 = \frac{\delta^4(C^2 + \beta \gamma - \alpha \delta)}{A(\pm \sqrt{A - C})^4},$$

where remarkable cancellations happen without using $\alpha \delta - \beta \gamma = -1$ and obtain a simple equation for $C$ from (26)

$$\pm \frac{C}{\sqrt{\beta \gamma - \alpha \delta + C^2}} = \frac{qT_F}{T_D}.$$  \hspace{1cm} (40)

For $\alpha \delta - \beta \gamma = -1$, the constant $C$ is so evaluated as $\pm qT_F/T_D$ again that we have a general class of nonstatic solutions

$$u = \pm \frac{-\delta}{\sqrt{(\frac{qT_F}{T_D})^2 - \delta(\alpha + \beta w^- + \gamma w^+ + \delta w^+ w^-) - \frac{qT_F}{T_D}}}.  \hspace{1cm} (41)$$

If we make a choice $\alpha \delta - \beta \gamma = 1$ instead, the D-brane worldvolume belongs to the regular conjugacy class. From (40) we derive $C^2 = -(qT_F/T_D)^2$ which implies that $q$ becomes imaginary and this solution represents an unphysical brane with the supercritical electric
field in the same way as the $dS_2$ brane. From the general expression (41) the stationary solutions for the choices (30) and (35) are respectively given by

\begin{align}
  u &= \pm \frac{1}{\sqrt{t^2 - (x - \cot \theta)^2 + (1 + C_0^2)\csc^2 \theta - C_0 \csc \theta}}, \text{ for } \sin \theta > 0, \\
  u &= \pm \frac{-1}{\sqrt{t^2 - (x + \coth \varphi)^2 + (1 + C_0^2)\cosech^2 \varphi - C_0 \cosech \varphi}}, \text{ for } \varphi > 0
\end{align}

with $C_0 \equiv qT_F/T_D$. In the limits $\theta \to 0$ and $\varphi \to 0$ these time-dependent solutions are confirmed to reduce to the same static solution $u = \pm C_0/x$ that is associated with $\omega_0 = \sigma_1$. If we consider the more general case that $\omega_0$ is parametrized as (37) but with $\alpha \delta - \beta \gamma = -z < 0$ for $z \neq 1$, $C$ is obtained by $C = \pm C_0 \sqrt{z}$ from (40). Hence we have the general solutions also expressed by (41) where $qT_F/T_D$ is replaced by $\sqrt{z}qT_F/T_D$. They agree with the solutions obtained by starting with the parametrization

\[ \omega_0 = \frac{1}{\sqrt{z}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \]

whose determinant is minus one. Therefore it is sufficient to restrict ourselves to the $\det \omega_0 = -1$ sector. When we combine a special outer automorphism $\omega_0 = \sigma_1$ in (41) with the general inner automorphism specified by $h$ we have $\text{tr}(\omega_0 hgh^{-1}) = 2C$ that describes the D-brane worldvolume characterized by a combined automorphism $\omega'_0 = h^{-1}\sigma_1 h$. The general parametrization of $h$

\[ h = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ with } ps - qr = 1 \]

leads to

\[ \omega'_0 = \begin{pmatrix} rs - pq & s^2 - q^2 \\ p^2 - r^2 & -(rs - pq) \end{pmatrix} \]

which provides some class of outer automorphism since its determinant is minus one and its trace is zero. The outer automorphisms such as (41) and (30) belong to the class of (45) but the other outer automorphism (35) does not belong to it.

3 $SL(2, R)_L \times SL(2, R)_R$ transformations

We elucidate the relations among the above solutions. The metric (7) in the Poincaré coordinates is rewritten in terms of $y = 1/u$ as

\[ ds^2 = \frac{L^2}{y^2}(dy^2 + dw^+ dw^-), \]

where the $AdS_3$ boundary is at $y = 0$. In Ref. (19) it was presented that the $AdS_3$ metric (46) has the following $SL(2, R)_L \times SL(2, R)_R$ isometry group. The $SL(2, R)_L$ transformation
is given by

\begin{align}
    w^+ & \rightarrow w'^+ = \frac{aw^+ + b}{cw^+ + d}, \quad w^- \rightarrow w'^- = w^- + \frac{cy^2}{cw^+ + d}, \\
    y & \rightarrow y' = \frac{y}{cw^+ + d},
\end{align}

with real $a, b, c, d$ obeying $ad - bc = 1$, while the $SL(2, R)_L$ transformation is

\begin{align}
    w^+ & \rightarrow w'^+ = w^+ + \frac{cy^2}{cw^- + d}, \quad w^- \rightarrow w'^- = w^- + \frac{a}{cw^- + d}, \\
    y & \rightarrow y' = \frac{y}{cw^- + d}.
\end{align}

Both the transformations map the $AdS_3$ boundary at $y = 0$ to itself and act on the boundary as the usual conformal transformations of 1 + 1 dimensional Minkowski spacetime. In the $SL(2, R)_L$ transformation the actions on $y$ and $w^-$ are translated into $u \rightarrow u' = u(cw^+ + d)$ and $w^- \rightarrow w'^- = w^- + c/u^2(cw^+ + d)$ for the metric (47), while the action on $w^+$ is not altered. Under the $SL(2, R)_L$ transformation the static solution of the form $u = C/x = 2C/(w^+ + w^-)$ for (10) corresponding to the choice $\omega_0 = \sigma_1$ in (9) is mapped to

\begin{align}
    u' = \frac{2C}{(a - cw'^+)w'^- - (b - dw'^+) - \frac{c}{u'^2}},
\end{align}

which reads

\begin{align}
    u' = \frac{c}{\pm \sqrt{C^2 + c(-b + aw'^- + dw'^- - cw'^+w'^-)} - C}.
\end{align}

Under the $SL(2, R)_R$ transformation with $u \rightarrow u' = u(cw^- + d)$ the static solution is similarly mapped to

\begin{align}
    u' = \frac{c}{\pm \sqrt{C^2 + c(-b + dw'^- + aw'^- - cw'^+w'^-)} - C}.
\end{align}

If we choose

\begin{align}
    \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\end{align}

whose determinant is equal to one, both the transformed solutions (50), (51) are identical to the nonstatic solution of the form (23) corresponding to the choice $\omega_0 = \sigma_3$ in (11). It is interesting to note that the $SL(2, R)_L$ transformed solution (50) is just identical to the general form of solution (38), if we choose

\begin{align}
    \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\delta & \gamma \end{pmatrix},
\end{align}

whose determinant is equal to one through the relation $\alpha \delta - \beta \gamma = -1$ in (37). In the same way the $SL(2, R)_R$ transformed solution (51) also agrees with the general form of solution (38), when the transformation parameters are chosen as

\begin{align}
    \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma & -\alpha \\ -\delta & \beta \end{pmatrix},
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\begin{align}
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\end{align}
which has unit determinant. Thus there are nontrivial $SL(2, R)_L \times SL(2, R)_R$ actions on the $\omega_0 = \sigma_1$ solution that give not only the $\omega_0 = \sigma_3$ solution but also the general class of solutions.

4 Conclusions

We have observed that in the $SL(2, R)$ WZW model the $AdS_2$ brane configuration specified by an outer automorphism $\omega_0 = \sigma_3$ is static in the cylindrical coordinates and nonstatic in the Poincaré coordinates, which is compared to the $AdS_2$ brane configuration specified by $\omega_0 = \sigma_1$ that is static in both the coordinates. By manipulating the DBI action of a D-string carrying the $U(1)$ electric field in the $AdS_3$ spacetime we have demonstrated that these static and nonstatic configurations satisfy amazingly the involved nonlinear equation of motion, only when the parameter $C$ characterizing the shape of each trajectory takes the same quantized value. These classical solutions are labelled by the integer $q$, the number of fundamental strings which provides the quantized $C$. From the general outer automorphism we have found a family of time-dependent analytic solutions to the formidable-looking DBI equation. As far as the determinant of $\omega_0$ that specifies the general outer automorphism is minus one, the parameter $C$ associated with the general class of nonstatic solutions is independent of the parametrization of $\omega_0$. This general outer automorphism is more wide class than the automorphism generated by combining a special outer automorphism specified by $\omega_0 = \sigma_1$ and the general inner automorphism. We have shown that the nontrivial static and nonstatic configurations are transformed to each other by the isometry group $SL(2, R)_L \times SL(2, R)_R$ of $AdS_3$ spacetime in the Poincaré coordinates.

Our demonstration of the nontrivial solutions is expected to give a clue to the construction of the similar time-dependent classical analytic solutions for the DBI action of the low-dimensional D-branes such as D-string and D2-brane with the $U(1)$ gauge field in the noncompact part $AdS_{p+2}$ of the near horizon geometry created by the D$p$-branes in the string theory. The $U(1)$ electric field may play an important role for the existence of such classical solutions and simplification of the nonlinear DBI equation of motion.

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