Influence of vibration excitation on the zone of laminar–turbulent transition on a plate

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Abstract. Influence of thermal nonequilibrium on the laminar–turbulent transition is studied on the basis of the \(e^N\)-method for two widespread flow regimes of a supersonic boundary layer at Mach number \(M = 4.5\). The set of actual frequencies of spatial disturbances was determined from the neutral stability curves for temporal disturbances. The families of \(N\)-factor curves were calculated for selected frequencies. The transition Reynolds number was determined for a given transition factor \(N_T = 8\) from the envelopes of the families of \(N\)-factor curves. Calculations show that for the level of the vibrational excitation below the dissociation limit and for \(N_T = 8\) the transition zone shifts downstream by 12–14 % compared to a perfect gas.

1. Introduction
It was shown in papers [1, 2] in the framework of the linear stability theory that the vibration excitation increases the critical Reynolds number for the boundary layer on a plate within 10–12 %. At the same time, the effect of vibrational nonequilibrium on position of zone of the laminar–turbulent transition is of independent interest. The corresponding estimate can be obtained by the \(e^N\)-method, based on the notion of an exponential increasing spatial perturbations to a certain level of amplitude \(A_T\), calculated on the basis of the linear theory in accordance with relation \(A_T = A_0 \exp(N_T)\) [3, 4]. Here \(A_0\) is the amplitude of the initial perturbation, \(A_T\) is the amplitude at point of the laminar–turbulent transition, \(N_T\) is so-called the \(N\)-factor of transition, the value of which is taken depending on the specific problem.

In the present paper we calculate the position of zone of the laminar–turbulent transition for two types of boundary layer flows with a Mach number \(M = 4.5\). As the most dangerous perturbation, we took a two-dimensional perturbation of mode II, which coincides with the direction of the carrier flow. The value of the degree of increase in disturbances was chosen \(N_T = 8\). In contrast to works [1, 2], the stationary flow in the boundary layer was described by the locally self-similar solutions, the use of which was justified in paper [5].

2. Statement of problem and basic equations
In the framework of the linear stability theory, the development of two-dimensional disturbances is considered for a plane semi-infinite plate flow of a vibrational excited gas. The origin of the Cartesian coordinate system \((x, y)\) coincides with a leading edge of the plate, the \(x\) coordinate is oriented along the plate in the direction of the carried flow, the \(y\) coordinate is directed along
The normal to the plate, respectively. The flow is described by a system of equations of the two-temperature gas dynamics [6].

The current distance \( x = L \) along the plate and parameters of the unperturbed flow outside the boundary layer, marked by the index "\( \infty \)" were chosen for nondimensionalization — the velocity, \( U_\infty \), the density, \( \rho_\infty \), and the temperature, \( T_\infty \), the coefficients of the shear and bulk viscosities, \( \eta_\infty \), and \( \eta_\infty \), correspondingly, the thermal conductivity coefficient due to the energy transfer in translational and rotational degrees of freedom, \( \lambda_\infty = \lambda_{t,\infty} + \lambda_{r,\infty} \), the coefficient of thermal conductivity describing the diffusion transfer of the energy of vibrational quanta, \( \lambda_{v,\infty} \). For nondimensionalization of the pressure and time, the combined values of \( \rho_\infty U_\infty^2 \) and \( \eta_\infty / (\rho_\infty U_\infty^2) \), respectively were used. For the temperature dependence of the shear viscosity the Sutherland formula is used. The coefficients of thermal conductivity are analogous to [6] expressed in terms of the shear viscosity and heat capacities by means of semi-empirical Eucken relations. It is assumed that heat capacities are constant. It is also supposed the translational and rotational degrees of freedom of the molecules are in quasi-equilibrium and are determined by the equilibrium relations.

The system of equations of the problem under consideration was linearized on a stationary solution of the boundary layer equations in a locally parallel approximation. Two-dimensional disturbances of the type of traveling plane waves

\[
q(x, y, t) = q_0(y) \exp[i(\alpha x - \omega t)], \quad q_0(y) = (u, \alpha v, \rho, \theta, \theta_v, p)
\]

were investigated. Here \( \alpha = \alpha_r + i\alpha_i \) is the complex wavenumber along the periodic variable \( x \), \( c = c_r + ic_i \) is the complex phase velocity, \( i \) is the imaginary unit. The system of equations for the perturbation amplitudes \( q_0 \) has the form

\[
D\rho + \alpha \rho_s \sigma + \alpha v \frac{d\rho_s}{dy} = 0, \quad (1)
\]

\[
\frac{\mu_s}{Re} \Delta u - \rho_s D u - \alpha \rho_s v \frac{dU_s}{dy} - i\alpha \varepsilon + \frac{\mu_s \nu_s}{Re} \frac{d}{dy} \left( \frac{dv}{dy} + i\alpha v \right) + \theta \frac{d}{dy} \left( \frac{dU_s}{dy} \right) + \frac{\mu_s \nu_s}{Re} \frac{dU_s}{dy} \frac{d\theta}{dy} = 0, \quad (2)
\]

\[
\frac{\alpha \mu_s}{Re} \Delta v - \alpha \rho_s D v - \frac{dz}{dy} + \frac{\alpha \mu_s \nu_s}{Re} \frac{d}{dy} \left( \frac{dv}{dy} + i\alpha u \right) + i\alpha \mu_s \nu_s \frac{dU_s}{dy} \frac{d\theta}{dy} = 0, \quad (3)
\]

\[
\gamma_k_s \frac{d}{dy} \frac{dT_s}{dy} = \rho_s D \theta - \alpha \rho_s \frac{dT_s}{dy} v - \alpha (\gamma - 1) \sigma + \frac{2\gamma (\gamma - 1) \mu_s M^2}{Re} \frac{dU_s}{dy} \frac{du}{dy} \left( \frac{dv}{dy} + i\alpha^2 v \right) + \left[ \frac{\gamma}{Re Pr} \frac{d}{dy} \left( \frac{dT_s}{dy} \right) + \frac{\gamma (\gamma - 1) M^2 \mu_s k_{T,s}}{Re} \left( \frac{dU_s}{dy} \right)^2 \right] \frac{d\theta}{dy} - \frac{\gamma_v \sigma (T_s - T_{vs})}{\tau} - \frac{\gamma_v \rho_s (\theta - \theta_v)}{\tau} + \frac{2\gamma k_{T,s} T_s}{Re Pr} \frac{d\theta}{dy} = 0, \quad (4)
\]

\[
\frac{20 \gamma_v \mu_2 \nu_{k,v,s}}{33 Re Pr} \frac{d}{dy} \left( k_{T,v,s} \frac{dT_{vs}}{dy} \right) \frac{d\theta}{dy} + \frac{20 \gamma_v \gamma k_{T,v,s}}{33 Re Pr} \left( \frac{d\theta}{dy} \right)^2 = 0, \quad (5)
\]

\[
\gamma M^2 p = \rho_s \theta + T_s \rho. \quad (6)
\]
Here we used the following notations

\[ D = i(\alpha U_s - \omega), \quad \sigma = i u + \frac{dv}{dy}, \quad \varepsilon = p - \frac{\alpha \mu_s}{Re} \left( \mu_1 + \frac{1}{3} \right) \sigma, \quad \Delta = \frac{d^2}{dy^2} - \alpha^2, \]

\[ \mu_{T,s} = \frac{d\mu_i}{dT}|_{T=T_s}; \quad k_{T,s} = \frac{dk}{dT}|_{T=T_s}; \quad \kappa_{i,T,s} = \frac{dk_i}{dT}|_{T=T_s}, \]

and the subscript \( s \) denotes the values of the hydrodynamic variables related to the stationary flow.

In equations (1)–(6) \( \mu_1 = \frac{\eta_{\infty}}{\eta_{\infty}} \) is the ratio of the bulk viscosity to the shear viscosity, \( \gamma = \frac{c_{p,\infty}}{c_{V,\infty}} \) is the ratio of specific heats, \( c_{V,\infty} = c_{V,\infty} + c_{v,\infty} \) and \( c_{p,\infty} = c_{V,\infty} + R \) are the specific heats at constant volume and pressure correspondingly, which are presented as the sum of specific heats induced by translational and rotational motion of molecules, \( \gamma_v = \frac{c_{V,\infty}}{c_{V,\infty} + c_{V,\infty}} \) is the parameter characterizing the degree of nonequilibrium of the vibrational mode, \( c_{V,\infty} \) is the specific heat at constant volume corresponding to the relaxing vibrational mode, \( \tau \) is the characteristic time of relaxation of the vibrational mode, \( R \) is the gas constant, \( Re = \rho_{\infty} U_{\infty} L / \eta_{\infty} \) and \( M = U_{\infty} / \sqrt{\gamma RT_{\infty}} \) are the Reynolds and Mach numbers, respectively, \( Pr = \frac{\eta_{\infty} c_{p,\infty}}{\lambda_{\infty}} \) is the Prandtl number.

It was assumed that, at \( y = 0 \) and at the conditional upper boundary of the boundary layer \( y = \delta \), the amplitudes of all perturbations vanish. Two spectral problems were considered. For \( \alpha = \alpha_T + i\omega \), the eigenvalues for the disturbances developing in time were calculated, and for \( \alpha = \alpha_T + i\omega \) and \( \omega = \omega_S \), the spectra of spatial disturbances were obtained.

As profiles of the stationary flow in system (1)–(6), we used the locally self-similar solutions of the boundary layer [5]. In the calculations for a perfect gas similar to the paper [1] we use the Dorodnitsyn-Howart self-similar solutions.

For a flow outside the boundary layer, a set of conditions was considered that corresponded to two aerodynamic situations — a supersonic flight in an unperturbed atmosphere (flow mode 1) and a flow in a supersonic wind tunnel (flow mode 2). For surface temperatures, adiabatical or isothermal conditions were set. In all cases, it was shown that locally self-similar profiles with an increase in the coordinate \( x \) converge to some limit ones that approximate the corresponding profiles of the completely developed boundary layer, calculated by the finite-difference method in the full formulation. Moreover, the local relative discrepancy along coordinate \( y \) between the profiles obtained by the two methods did not exceed five percentages.

The Reynolds numbers \( Re_{IT} = \sqrt{Re_{ST}} \), which determine beginning the zone of laminar–turbulent transition, were calculated according to the following scheme. At the first stage, for a given flow regime, it is necessary to determine the frequency range of perturbations evolving in space. We used the approach justified in paper [7]. For two-dimensional spatial and temporal perturbations, provided that attenuation of the latter is small \( (|\omega_i|^2 < 1) \), it was shown that up to \( O (|\omega_i|^2) \) we have the equalities \( \omega_S = \omega_r \) and \( \alpha_r = \alpha_T \). That is, the frequency ranges of spatial and temporal perturbations coincide for the same wavelengths.

For a given flow mode, a neutral stability curve is calculated for the most unstable mode II of temporary disturbances in coordinates \( (Re_{\delta, \omega} \). The necessary frequency interval is determined by the lower branch of the curve, on which with a uniform step \( \Delta \omega \) some set of points \( \omega_{ri} \) (where \( i = 1, 2, \ldots, m \)) is selected.

For each of the selected frequencies, the N-factor curve \( N_\omega (Re_{\delta(\omega)}) \) is calculated. For this purpose on line \( \omega_{ri} = \text{const} \) inside the instability region distinguished by the neutral curve, a set of points with abscissas \( Re_{ij} \) (where \( j = 1, 2, \ldots, k \)) are taken. At each point, the spectral problem for the most unstable spatial mode II is solved. As a result, we obtain a set of local increments (decrements) of the growth of \( \alpha_{ij} \). Based on them, using the quadrature formula,
the integral
\[
N_\omega(\text{Re}_\delta(x)) = -2 \int_{\text{Re}_\delta(x_0)}^{\text{Re}_\delta(x)} \alpha_i^* d\text{Re}_\delta(x), \quad \alpha_i^* = \alpha_i \sqrt{\frac{x'\nu}{U}}
\] (7)

is calculated. Integral (7) is considered as a function of the upper limit \(\text{Re}_\delta(x)\). The lower limit \(\text{Re}_\delta(x_0)\) corresponds to the abscissa of the point on the lower branch of the neutral curve.

The \(N\)-factor curves calculated in this way for frequencies \(\omega_{r1} < \omega_{r2} < \ldots < \omega_{rk}\) are arranged on plane \(\left(\text{Re}_\delta(x), N_\omega\right)\). After that, the envelope of the family of curves is constructed. The Reynolds number that determines the beginning the zone of laminar–turbulent transition is found as abscissa of the envelope for a given value of the transition factor \(N_T\).

3. Calculation results

The position of the transition zone was calculated for two characteristic flow modes in a supersonic boundary layer. The following boundary conditions were set on the hydrodynamical parameters of the stationary flow.

The flow mode 1 is “flight in an undisturbed atmosphere” — an impenetrable and thermally insulated (adiabatic) wall: the vibrational temperature on the wall was assumed to be equal to the temperature of a “plate thermometer”; static (translational) and vibrational temperatures are equal at the conditional upper boundary of the boundary layer:

\[
U_s(0) = 0, \quad T_s'(0) = 0, \quad T_{vs}(0) = 1 + \frac{(\gamma - 1)}{2} \text{Pr} M^2, \quad U_s(\delta) = 1, \quad T_s(\delta) = T_{vs}(\delta) = 1.
\]

The flow mode 2 is “experiment in a supersonic wind tunnel” — an impenetrable wall cooled to the temperature of the external flow (isothermal wall), the vibrational temperature on the wall was assumed to be equal to the static temperature and the vibrational temperature was equal to the stagnation temperature (static temperature in prechamber of the wind tunnel) at the conditional upper boundary of the boundary layer:

\[
U_s(0) = 0, \quad T_s(0) = T_{vs}(0) = 1, \quad U_s(\delta) = 1, \quad T_s(\delta) = 1, \quad T_{vs}(\delta) = 1 + \frac{(\gamma - 1)}{2} M^2.
\]

The calculations were carried out with the following parameter values: \(\gamma = 1.4, \gamma_v = 0.667, \text{Pr} = 0.75, M = 4.5\) and \(\delta = 8\).

The locally self-similar profiles were calculated for the longitudinal coordinate \(x = 15\). All spectral problems for system (1)–(6) were solved numerically in the environment of the Matlab package based on the collocation method tested in previous papers [1, 6].

Here for calculations the \(N\)-factors are required the neutral stability curves of the time disturbances for much larger limits (\(\text{Re}_\delta, \omega\)) than was necessary to find of values \(\text{Re}_{cr}\) in paper [1]. The critical frequencies \(\omega_{cr}\) were taken as the upper limit of the frequency ranges for all cases considered. The inviscid asymptotic limit of mode I was taken as the lower limit of the frequency of mode II [1].

The value of the \(N\)-factor \(N_T = 8\) was chosen to assess the beginning the transition zone. The value \(N_T = 8\) is used for flows with a low level of incoming disturbances [3]. In this regard, to construct envelopes for each mode, a large amount of calculations of the \(N\)-factors was performed. The lower limits \(\text{Re}_\delta(x_0)\) in integrals (7) were taken on the lower branches of the neutral curves, starting from \(\text{Re}_{cr}\) with a step of \(\Delta \omega = 10^{-8}\). The values of the increments (decrements) of spatial perturbations \(\alpha_i\) were calculated on the lines \(\omega = \text{const}\) with the step \(\Delta \text{Re}_\delta = 10\). This choice of step allowed us to use the simple quadrature trapezoid
Figure 1. Curves of the $N$-factors and position of the laminar-turbulent transition at the Mach number $M = 4.5$ for the flow modes 1 (a) and 2 (b). The solid lines are a perfect gas ($\gamma_v = 0$). The dashed lines are the vibrational excited gas ($\gamma_v = 0.667$). The dash-dotted line is $N_T = 8$. $A$, $B$ and $A'$, $B'$ are the transition points for the flow modes 1 and 2 at $\gamma_v = 0$ and $\gamma_v = 0.667$, respectively.

| $\gamma_v$ | Flow mode 1 | Flow mode 2 |
|-----------|-------------|-------------|
| $\gamma_v$ | $Re_x T \times 10^{-7}$ | $Re_x T \times 10^{-7}$ |
| 0         | 2.382       | 2.053       |
| 0.667     | 2.727       | 14.484      |
| 14.369    |             |             |

Table 1. Parameters of the transition points at $N_T = 8$.

The table 1 contains the numerical values of the criteria $Re_x T$ for all considered of the flow modes. Also in the table relative shifts of the Reynolds numbers of the laminar–turbulent transition caused by the vibrational excitation of a gas are presented. The shifts were calculated using the formula

$$\varepsilon_T = \left| 1 - \frac{Re_x T(\gamma_v = 0.667)}{Re_x T(\gamma_v = 0)} \right| \times 100\%.$$
It can be seen that in all cases the onset of the laminar–turbulent transition in the vibrational excited gas is shifted downstream by about 14 % compared to a perfect gas. This completely correlates with the results of papers [1, 2], where it was shown that vibrational excitation in approximately the same proportion increases the critical Reynolds number calculated in the framework of linear theory.

4. Conclusion

Using the $e^N$-method, comparative calculations of the position of beginning zone the laminar–turbulent transition in the vibrational excited and a perfect gases have been carried out. Two characteristic flow modes of a supersonic boundary layer at $M = 4.5$ were considered, which are of practical interest. The value of the N-factor determining the laminar–turbulent transition was chosen $N_T = 8$. This value of the N-factor is recommended for flows with a low level of external disturbances. The frequency spectrum of the spatial disturbances was determined from the neutral curves of the temporary disturbances at the same wavelengths. Envelopes were constructed from the calculated curves of the N-factors. At a given $N_T = 8$, using the obtained envelopes, the transition Reynolds numbers $Re_{xT}$ were determined. As the calculations showed, in all cases the vibration excitation shifts zone of the laminar–turbulent transition downstream by about 14 % compared with a perfect gas. This corresponds to previously obtained results on the nature of influence of the vibration excitation on the critical Reynolds numbers.

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