AN UNDECIDABILITY RESULT FOR THE ASYMPTOTIC THEORY OF $p$-ADIC FIELDS

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Abstract. Fix a prime $p$. We prove that the set of sentences true in all but finitely many finite extensions of $\mathbb{Q}_p$ is undecidable in the language of valued fields with a cross-section. The proof goes via reduction to positive characteristic, ultimately adapting Pheidas’ proof of the undecidability of $\mathbb{F}_p((t))$ with a cross-section. This answers a variant of a question of Derakhshan-Macintyre.

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Introduction

Let $L_{\text{val}, x}$ be the language of valued fields with a cross-section (see §1.3.3). Ax-Kochen [AK65] and independently Ershov [Ers65] showed that $\mathbb{Q}_p$, equipped with the normalized cross section $s : n \mapsto p^n$, is decidable in $L_{\text{val}, x}$. Every finite extension of $\mathbb{Q}_p$ is also decidable in $L_{\text{val}, x}$, for a suitable choice of a cross-section (see §2.4 [Kar22]). Combining the results of [AK65] with the theory of pseudofinite fields, Ax showed in Theorem 17 [Ax68] that the (asymptotic) theory of $\{\mathbb{Q}_p : p \in \mathbb{P}\}$ is decidable in $L_{\text{val}, x}$ and also that the (asymptotic) theory of $\{\mathbb{Q}_p(\zeta_n) : p \nmid n\}$, namely the collection of all finite unramified extensions of $\mathbb{Q}_p$, is decidable in $L_{\text{val}, x}$. Recall that the asymptotic theory of a class $\mathcal{C}$ of $L$-structures is defined to be the set of sentences $\phi \in L$ which are true in all but finitely many $M \in \mathcal{C}$.

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In sharp contrast, we have that $\mathbb{F}_p((t))$ is undecidable in $L_{\text{val},\times}$. This was already known to Ax (unpublished) and an elementary proof was given later by Becker-Denef-Lipshitz [BDL80], which was also reworked by Cherlin in §4 [Che82]. Pheidas [Phe87] generalized the result for $k((t))$, where $k$ is an arbitrary field of characteristic $p$, and showed that already the existential $L_{\text{val},\times}$-theory is undecidable. The decidability problem for $\mathbb{F}_p((t))$ in the language of rings $L_{\text{rings}}$ is still an open problem.

An important related open question in mixed characteristic is whether $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$, i.e. the theory of all finite $p$-adic extensions, is decidable. This question was first raised (in print) by Derakhshian-Macintyre in §9 [DM22] for the language of rings $L_{\text{rings}}$. In Theorem 9.1 [DM22], they showed that the theory of adele rings of all number fields in $L_{\text{rings}}$ is decidable if and only if $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$ is decidable in $L_{\text{rings}}$, for each prime $p$. It is also natural to consider $\text{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$ in any of the standard expansions $L$ of $L_{\text{rings}}$ (see e.g., pg. 20-22 [Der20]), and ask whether it is decidable (Problem 6.2 [Der20]).

In the present paper, we resolve negatively the asymptotic version of this problem in the presence of a cross-section. While each individual finite $p$-adic extension is decidable in $L_{\text{val},\times}$, the asymptotic theory of all of them is less well-behaved:

**Theorem A.** The asymptotic $L_{\text{val},\times}$-theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable.

We note that for each $K$ with $[K : \mathbb{Q}_p] < \infty$ there are many choices of a cross-section and any such choice will lead to an undecidability result (see Convention 2.2.5). By carefully keeping track of quantifiers, we will prove in §13 that already the asymptotic $\exists\forall$-theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable in $L_{\text{val},\times}$.

The key idea of the proof will be to use highly ramified $p$-adic fields to approximate $\mathbb{F}_p((t))$ à la Krasner-Kazhdan-Deligne (see Section 2) and then adapt Pheidas’ proof of the undecidability of $\mathbb{F}_p((t))$ in $L_{\text{val},\times}$ (see [Phe87]). In more detail, the proof of Theorem A consists of the following steps:

1. Encode the asymptotic theory of totally ramified $p$-adic fields, i.e., the asymptotic theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$ (see the proof of Theorem A).
2. Observe that $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^e)$, for $K/\mathbb{Q}_p$ totally ramified of degree $e$, and thereby encode the asymptotic theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ in $L_t$ with a predicate for powers of $t$ (see Corollary 2.2.7).
3. Encode the asymptotic theory of truncated fragments of $(\mathbb{N}; 0, 1, +, \mid_p)$ (see §3.2.5), where $m \mid_n n$ if and only if $n = p^s \cdot m$ for some $s \in \mathbb{N}$.
4. Show that the latter is undecidable by encoding the Diophantine problem of $(\mathbb{N}; 0, 1, +, \mid_p)$ (see §3.3).

We note that Derakhshian-Macintyre had already suggested that the difficulty in understanding the (asymptotic) theory of $p$-adic fields must lie in unbounded
ramification (see the last paragraph of [DM22]). This is precisely the ingredient that makes the transition to positive characteristic work and is the reason why this asymptotic class exhibits a different behavior from the two asymptotic theories mentioned in the first paragraph of the introduction.

1. Preliminaries

1.1. Interpretability. Our formalism follows closely §5.3 [Hod93], where details and proofs may be found.

1.1.1. Interpretations of structures. Given a language $L$, an unnested atomic $L$-formula is one of the form $x = y$ or $x = c$ or $F(x) = y$ or $Rx$, where $x, y$ are variables, $c$ is a constant symbol, $F$ is a function symbol and $R$ is a relation symbol of the language $L$.

**Definition 1.1.2.** An $n$-dimensional interpretation of an $L$-structure $M$ in the $L'$-structure $N$ is a triple consisting of:

1. An $L'$-formula $\partial_\Gamma(x_1,\ldots,x_n)$.
2. A map $\phi \mapsto \phi_\Gamma$, that takes an unnested atomic $L$-formula $\phi(x_1,\ldots,x_m)$ and sends it to an $L'$-formula $\phi_\Gamma(\overline{y}_1,\ldots,\overline{y}_m)$, where each $\overline{y}_i$ is an $n$-tuple of variables.
3. A surjective map $f_\Gamma : \partial_\Gamma(N^n) \rightarrow M$ such that for all unnested atomic $L$-formulas $\phi(x_1,\ldots,x_m)$ and all $\overline{a}_i \in \partial_\Gamma(N^n)$, we have

$$M \models \phi(f_\Gamma(\overline{a}_1),\ldots,f_\Gamma(\overline{a}_m)) \iff N \models \phi_\Gamma(\overline{a}_1,\ldots,\overline{a}_m)$$

An interpretation of an $L$-structure $M$ in the $L'$-structure $N$ is an $n$-dimensional interpretation, for some $n \in \mathbb{N}$. In that case, we also say that $M$ is interpretable in $N$. The formulas $\partial_\Gamma$ and $\phi_\Gamma$ (for all unnested atomic $\phi$) are the defining formulas of $\Gamma$.

**Proposition 1.1.3** (Reduction Theorem 5.3.2 [Hod93]). Let $\Gamma$ be an $n$-dimensional interpretation of an $L$-structure $M$ in the $L'$-structure $N$. There exists a map $\phi \mapsto \phi_\Gamma$, extending the map of Definition 1.1.2, such that for every $L$-formula $\phi(x_1,\ldots,x_m)$ and all $\overline{a}_i \in \partial_\Gamma(N^n)$, we have that

$$M \models \phi(f_\Gamma(\overline{a}_1),\ldots,f_\Gamma(\overline{a}_m)) \iff N \models \phi_\Gamma(\overline{a}_1,\ldots,\overline{a}_m)$$

**Proof.** We describe how $\phi \mapsto \phi_\Gamma$ is built, for completeness (omitting details). By Corollary 2.6.2 [Hod93], every $L$-formula is equivalent to one in which all atomic subformulas are unnested. One can then construct $\phi \mapsto \phi_\Gamma$ by induction on the complexity of formulas. The base case is handled by Definition 1.1.2. This definition extends inductively according to the following rules:

1. $(\neg \phi)_\Gamma = \neg(\phi)_\Gamma$.
2. $(\bigwedge_{i=1}^n \phi_i)_\Gamma = \bigwedge(\phi_i)_\Gamma$. 
The resulting map satisfies the desired conditions of the Proposition.

\[\begin{align*}
(3) \quad \forall \phi_T = \forall x_1, \ldots, x_n (\partial_T(x_1, \ldots, x_n) \rightarrow \phi_T) \\
(4) \quad \exists \phi_T = \exists x_1, \ldots, x_n (\partial_T(x_1, \ldots, x_n) \land \phi_T)
\end{align*}\]

\textbf{Definition 1.1.4.} The map \(\text{Form}_L \rightarrow \text{Form}_{L'} : \phi \mapsto \phi_T\) constructed in the proof of Proposition 1.1.3 is called the \textit{reduction} map of the interpretation \(\Gamma\).

1.1.5. \textit{Complexity of interpretations.} The complexity of the defining formulas of an interpretation defines a measure of complexity of the interpretation itself:

\textbf{Definition 1.1.6} (§5.4(a) [Hod93]). An interpretation \(\Gamma\) of an \(L\)-structure \(M\) in an \(L'\)-structure \(N\) is quantifier-free if the defining formulas of \(\Gamma\) are quantifier-free. Other syntactic variants are defined analogously (e.g., existential interpretation).

\textbf{Remark 1.1.7.} Note that the reduction map of an existential interpretation sends \textit{positive existential} formulas to existential formulas but does \textit{not} necessarily send existential formulas to existential formulas.

1.1.8. \textit{Recursive interpretations.}

\textbf{Definition 1.1.9} (Remark 4, pg. 215 [Hod93]). Suppose \(L\) is a recursive language. Let \(\Gamma\) be an interpretation of an \(L\)-structure \(M\) in the \(L'\)-structure \(N\). We say that the interpretation \(\Gamma\) is \textit{recursive} if the the map \(\phi \mapsto \phi_T\) on unnested atomic formulas is recursive.

\textbf{Remark 1.1.10} (Remark 4, pg. 215 [Hod93]). If \(\Gamma\) is a recursive interpretation of an \(L\)-structure \(M\) in the \(L'\)-structure \(N\), then the reduction map of \(\Gamma\) is also recursive.

1.2. \textit{Decidability.}

1.2.1. \textbf{Definition.} Fix a \textit{countable} language \(L\) and let \(\text{Sent}_L\) be the set of well-formed \(L\)-sentences, identified with \(\mathbb{N}\) via some Gödel numbering. Let \(T\) be an \(L\)-theory, not necessarily complete. Recall that \(T\) is decidable if we have an algorithm to decide whether \(T \models \phi\), for any given \(\phi \in \text{Sent}_L\). More formally, let \(\chi_T : \text{Sent}_L \rightarrow \{0, 1\}\) be the partial characteristic function of \(T \subseteq \text{Sent}_L\). We say that \(T\) is \textit{decidable} if \(\chi_T\) is a partial recursive function. Let \(M\) be an \(L\)-structure. We say that \(M\) is decidable if \(\text{Th}(M)\) is decidable.

1.2.2. \textit{Turing reducibility of theories.} See Definition 14.3 [Pap94] for the formal definition of a Turing machine with an oracle.

\textbf{Definition 1.2.3.} A theory \(T\) is \textit{Turing reducible} to a theory \(T'\) if there is a Turing machine which decides membership in \(T\) using an \textit{oracle} for \(T'\).

\textbf{Remark 1.2.4.} In particular, if \(T\) is Turing reducible to \(T'\) and \(T'\) is decidable, then so is \(T\).
Example 1.2.5. If $\Gamma$ is a recursive interpretation of an $L$-structure $M$ in the $L'$-structure $N$, then $\text{Th}(M)$ is Turing reducible to $\text{Th}(N)$. Indeed, for any given $\phi \in \text{Sent}_L$, we have that $M \models \phi \iff N \models \phi_{\Gamma}$, where $\phi \mapsto \phi_{\Gamma}$ is the reduction map of $\Gamma$. This furnish us with an algorithm to decide whether $M \models \phi$ using an oracle for $\text{Th}(N)$.

1.3. Languages. We write $L_{\text{oag}} = \{0, +, <\}$ for the language of ordered abelian groups and $L_{\text{rings}} = \{0, 1, +, \cdot\}$ for the language of rings.

1.3.1. Valued fields language. Let $L_{\text{val}}$ be the three-sorted language of valued fields, with sorts for the field, the value group and the residue field.

- The field sort $K$ is equipped with the language of rings $L_{\text{rings}}$.
- The value group sort $\Gamma$ is equipped with $L_{\text{oag}}$, together with a constant symbol for $\infty$.
- The residue field sort $k$ is equipped the language of rings $L_{\text{rings}}$.

We also have function symbols for the valuation map $v : K \to \Gamma$ and a residue map $\text{res} : K \to k$ (where $\text{res}(x) = 0$ when $vx < 0$ by convention).

Convention 1.3.2. We shall write $x \in O$ as an abbreviation of the formula $x \in K \land vx \geq 0$.

1.3.3. Ax-Kochen/Ershov language. Historically, the Ax-Kochen/Ershov formalism also included a function symbol for a cross-section, i.e. group homomorphism $s : \Gamma \to K^\times$ satisfying $v \circ s = \text{id}_{\Gamma}$ (see [AK65] and [Koc74]). One can also extend $s$ by defining $s(\infty) = 0$. We write $L_{\text{val}, x}$ for the language $L_{\text{val}}$ enriched with such a cross-section symbol $s : \Gamma \to K$.

1.4. Ax-Kochen/Ershov Theorem. Among other results, Ax-Kochen [AK65] and independently Ershov [Ers65] obtained the following:

Fact 1.4.1 (Ax-Kochen/Ershov). The field $\mathbb{Q}_p$, equipped with the normalized cross-section $s : n \mapsto p^n$, is decidable in $L_{\text{val}, x}$.

Remark 1.4.2. (a) More generally, by Theorem 4 [Koc74], any unramified henselian field $(K, v)$, equipped with a normalized cross-section (viz., $s(1) = p$) and with perfect residue field $k$, is decidable in $L_{\text{val}, x}$.

(b) Finite extensions of $\mathbb{Q}_p$ are also decidable in $L_{\text{val}, x}$, for a suitable choice of a cross-section (see §2.4 [Kar22]).

Theorem 12 [AK66] in fact shows that $\mathbb{Q}_p$ admits quantifier elimination in $L_{\text{val}, x}$ relative to the value group. The definable sets of the latter are perfectly understood by classical quantifier elimination results for Presburger arithmetic. Nevertheless, the following is worth noting:

Remark 1.4.3. The definable sets in the Ax-Kochen/Ershov language are complicated and are generally not definable in the valued field language. For instance,
the image of the cross-section is not definable without the cross-section (see Example, pg. 609 [Mac76]). For this reason, at least for the purpose of studying definable subsets of the $p$-adics, the Ax-Kochen/Ershov formalism was superseded by Macintyre’s language (see pg. 606 [Mac76]).

Despite Remark 1.4.3, the Ax-Kochen/Ershov formalism has remained relevant. The fact that $p$-adic fields are decidable in such an expressive language is a strong result. As was mentioned in the introduction, this is in stark contrast with the fact that positive characteristic local fields are undecidable in $L_{\text{val}, x}$.

2. Local field approximation

2.1. Motivation. A powerful philosophy, often referred to as the Krasner-Kazhdan-Deligne principle (due to [Kra56], [Kaz86] and [Del84]), says that a highly ramified $p$-adic field $K$ (e.g., $K = \mathbb{Q}_p(p^{1/n})$ with $n$ large) is in many respects "close" to a positive characteristic valued field. Although this is reflected in many aspects of $K$, perhaps the most elementary one is that the residue ring $\mathcal{O}_K/(p)$ "approximates" a positive characteristic valuation ring (see Lemma 2.2.1, Remark 2.2.2). Our goal in this section will be to prove Corollary 2.2.7, which will serve as a bridge between Pheidas’ work in positive characteristic and our problem in mixed characteristic (see §3.1).

2.2. Computations modulo $p$. Let $K/\mathbb{Q}_p$ be a finite totally ramified extension of degree $n$, with value group $\Gamma$ and residue field $k$. For completeness, we record here the following computation:

**Lemma 2.2.1.** For each uniformizer $\pi$ of $\mathcal{O}_K$ we have an isomorphism $\mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n)$, which maps the image of $\pi$ in $\mathcal{O}_K/(p)$ to the image of $t$ in $\mathbb{F}_p[t]/(t^n)$.

**Proof.** Write $\mathcal{O}_K = \mathbb{Z}_p[\pi]$, where $\pi$ is a root of an Eisenstein polynomial $E(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0 \in \mathbb{Z}_p[t]$ (see Proposition 11, pg.52 [Lan94]). In particular, we have that $a_i \equiv 0 \mod p\mathbb{Z}_p$ and the reduction of $E(t)$ modulo $p\mathbb{Z}_p$ is equal to $\overline{E}(t) = t^n \in \mathbb{F}_p[t]$. We now compute

$$\mathcal{O}_K/(p) = (\mathbb{Z}_p[t]/(p, E(t))) \cong \mathbb{F}_p[t]/(\overline{E}(t)) = \mathbb{F}_p[t]/(t^n)$$

and observe that the above isomorphism sends $\pi + (p)$ to $t + (t^n)$. □

**Remark 2.2.2.** If $K, \pi$ are as above, then $\mathcal{O}_K/((\pi^n)) = \mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n) \cong \mathbb{F}_p[t]/(t^n)$ and Kazhdan says that $K$ is $n$-close to $\mathbb{F}_p[t]$ (see §0 [Kaz86]). In what follows, it will indeed be useful to think of $\mathcal{O}_K/(p)$ as being very close to $\mathbb{F}_p[t]$ for large $n$ (see §3.1.2).
2.2.3. Residue rings. For each \( n \in \mathbb{N} \), we view \( \mathbb{F}_p[t]/(t^n) \) as an \( L_t \cup P \)-structure, where \( L_t \) is the language of rings \( L_{\text{rings}} \) together with a constant symbol for \( t \) and \( P \) is a unary predicate, whose interpretation is the set \( \{0, 1, t, \ldots, t^{n-1}\} \). Note that we have tacitly replaced the equivalence class \( t^k + (t^n) \) with \( t^k \), which is harmless and common when dealing with truncated/modular arithmetic.

2.2.4. Interpreting \( \mathbb{F}_p[t]/(t^n) \). As a consequence of Lemma 2.2.11 we obtain:

**Proposition 2.2.5.** Let \( K/\mathbb{Q}_p \) be totally ramified of degree \( n \in \mathbb{N} \) and \( s : \Gamma \to K^\times \) be a cross-section. The structure \( \mathbb{F}_p[t]/(t^n) \) in \( L_t \cup P \) is \( \exists \forall \)-interpretable in \( K \) in the language \( L_{val,x} \). Moreover, the reduction map \( \phi \mapsto \phi_{\Delta_K} \) does not depend on \( K \) or the choice of \( s \).

**Proof.** Let \( \gamma \) be the minimal positive element in \( \Gamma \). It is definable by the \( \forall \)-formula in the free variable \( \gamma \in \Gamma \) written below

\[
\gamma > 0 \land \forall \delta \in \Gamma^{>0}(\gamma \leq \delta)
\]

henceforth abbreviated by \((\gamma \text{ minimal positive})\).

We now define a 1-dimensional interpretation \( \Delta_K \) of \( \mathbb{F}_p[t]/(t^n) \) in \( K \). Take \( \partial_{\Delta_K}(x) \) to be the formula \( x \in \emptyset \). The reduction map on unnested atomic formulas is described as follows:

1. If \( \phi(x) \) is the formula \( x = 0 \) (resp. \( x = 1 \) and \( x = t \)), we take \( \phi_{\Delta_K}(x) \) to be the formula \( \exists y \in \mathcal{O}(x = py) \) (resp. \( \exists y \in \mathcal{O}(x = 1 + py) \) and \( \exists \gamma \in \Gamma[(\gamma \text{ minimal positive}) \land (x = s(\gamma))] \).
2. If \( \phi(x, y) \) is the formula \( x = y \), we take \( \phi_{\Delta_K}(x, y) \) to be the formula \( \exists z \in \mathcal{O}(x = y + pz) \).
3. If \( \phi(x, y, z) \) is \( x \circ y = z \), then we take \( \phi_{\Delta_K} \) to be the formula \( \exists w \in \mathcal{O}(x \circ y = z + pw) \), where \( \circ \) is either \( \cdot \) or \( + \).
4. If \( \phi(x) \) is the formula \( x \in P \), we take \( \phi_{\Delta_K}(x) \) to be the formula \( \exists \gamma \in \Gamma^{\geq0}(x = s(\gamma)) \)

The coordinate map \( f_{\Delta_K} : \mathcal{O}_K \to \mathbb{F}_p[t]/(t^n) \) is the projection modulo \( p \), given by Lemma 2.2.1. The isomorphism \( \mathcal{O}_K/(p) \cong \mathbb{F}_p[t]/(t^n) \) identifies \( \pi \) with \( t \) and the image of \( s(\Gamma^{\geq0}) \) in \( \mathcal{O}_K/(p) \) is equal to \( \{0, 1, t, \ldots, t^{n-1}\} \), via the above identification. One readily checks that the above data defines an \( \exists \forall \)-interpretation of the \( L_t \cup P \)-structure \( \mathbb{F}_p[t]/(t^n) \) in the \( L_{val,x} \)-structure \( K \).

Finally, the reduction map \( \phi \mapsto \phi_{\Delta_K} \) does not depend on \( K \), because of the inductive construction of the reduction map (see Proposition 1.1.3) and the fact that \( \phi \mapsto \phi_{\Delta_K} \) does not depend on \( K \) when \( \phi \) is any of the unnested atomic formulas listed above. \( \square \)

**Convention 2.2.6.** For the rest of the paper, fix once and for all a choice of a cross-section \( s_K : \Gamma \to K^\times \), for each finite extension \( K \) of \( \mathbb{Q}_p \). For each such \( K \), a cross-section does indeed exist and corresponds to a choice of a uniformizer for \( \mathcal{O}_K \).
Corollary 2.2.7. The asymptotic existential $L_t \cup P$-theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is Turing reducible to the asymptotic existential-universal $L_{val, x}$-theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$.

Proof. For $K/\mathbb{Q}_p$ totally ramified of degree $n$, let $\Delta_K$ be the interpretation of the $L_t \cup P$-structure $\mathbb{F}_p[t]/(t^n)$ in the $L_{val, x}$-structure $K$, provided by Proposition 2.2.5. The reduction map of $\Delta_K$ does not depend on $K$ and will simply be denoted by $\phi \mapsto \phi_\Delta$.

Claim: For any existential $\phi \in \text{Sent}_{L_t \cup P}$, the sentence $\phi_\Delta$ is equivalent to an $\exists \forall$-sentence.

Proof. For sentences $\phi$ of the form $\exists x \psi(x)$, where $\psi(x)$ is a quantifier-free formula without negations, this follows from the fact that $\Delta_K$ is an $\exists \forall$-interpretation. We may therefore focus on formulas $\psi(x)$ of the form $f(x_1, ..., x_m, t) \neq 0$ (resp. $f(x_1, ..., x_m, t) \notin P$). Such a formula is logically equivalent to $\neg f(x_1, ..., x_m, y) = 0 \land y = t$ (resp. $\neg f(x_1, ..., x_m, y) \in P \land y = t$). Now $\neg(f(x_1, ..., x_m, y) = 0)_\Delta$ (resp. $\neg(f(x_1, ..., x_m, y) \in P)_\Delta$) is universal and $(y = t)_\Delta$ is existential-universal. The conclusion follows. \(\square\) Claim

For any $\phi \in \text{Sent}_{L_t \cup P}$ and any totally ramified extension $K/\mathbb{Q}_p$ of degree $n$, we have that $\mathbb{F}_p[t]/(t^n) \models \phi \iff K \models \phi_\Delta$. For any given $n \in \mathbb{N}$, there are finitely many totally ramified extensions $K/\mathbb{Q}_p$ of degree $n$ (Proposition 14 [Lan94]). It follows that $\mathbb{F}_p[t]/(t^n) \models \phi$ for almost all $n \in \mathbb{N} \iff K \models \phi_\Delta$ for almost all $K$ with $[K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty$

The conclusion follows from the Claim. \(\square\)

3. Truncations of $(\mathbb{N}; 0, 1, +, |_{p})$

3.1. Motivation.

3.1.1. Pheidas’ work. In Theorem 1 [Phe87], Pheidas showed that the Diophantine problem for $(\mathbb{N}; 0, 1, +, |_{p})$ is undecidable. The proof goes by defining multiplication via a positive existential formula and using Matiyasevich’s negative solution to Hilbert’s tenth problem [Mat70]. In Lemma 1(c) [Phe87], it is shown that the Diophantine problem for $(\mathbb{N}; 0, 1, +, |_{p})$ can be encoded in the existential theory of $\mathbb{F}_p((t))$ in $L_t$ with a predicate $P$ for $\{0, 1, t, t^2, \ldots\}$. It follows that the latter is also undecidable.

3.1.2. Plan of action. Our intuition that $\mathbb{F}_p[t]/(t^n)$ approximates $\mathbb{F}_p[[t]]$ when $n$ is large, suggests that we can adapt Pheidas’ strategy and show that the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is also undecidable in $L_t \cup P$. This will be established in Proposition 4.1.1 and in combination with Corollary 2.2.7 will pave the way for proving Theorem A in Section 4.3.
3.2. \( p \)-divisibility. The notion of \( p \)-divisibility was introduced by Denef in [Den79], in order to show that the Diophantine problem of a polynomial ring of positive characteristic is undecidable. Given \( n, m \in \mathbb{N} \) and \( p \in \mathbb{P} \), we write \( n \mid_p m \) if \( m = p^s n \), for some \( s \in \mathbb{N} \). Let us also write \( L_{p\text{-div}} = \{0, 1, +, |_p\} \) for the language of addition and \( p \)-divisibility.

3.2.1. Encoding \( |_p \) in \( \mathbb{F}_p[t]/(t^n) \). The following result is due to Pheidas:

**Lemma 3.2.2** (Lemma 1(a) [Phe87]). Let \( n, m \in \mathbb{N} \) with \( 0 < n \leq m \). Then \( n \mid_p m \) if and only if there exists \( a \in \mathbb{F}_p[t] \) such that \( t^{-m} - t^{-n} = a^{-p} - a^{-1} \).

**Remark 3.2.3.** If \( n \mid_p m \), then the proof of Lemma 1(a) [Phe87] provides \( a = (t^{-n+p^{-1}} + t^{-n+p^{-2}} + \cdots + t^{-n})^{-1} \). Note that we have slightly rephrased the original formulation of Lemma 1(a) [Phe87], so that the witness \( a \) has positive valuation.

We shall use a truncated version of Lemma 3.2.2 whose proof is identical, modulo some additional bookkeeping:

**Lemma 3.2.4.** Let \( n, m, N \in \mathbb{N} \) with \( 0 < n \leq m < N/3 \). Then \( n \mid_p m \) if and only if there exists \( \alpha \in \mathbb{F}_p[t]/(t^N) \) such that \( \alpha^p(t^n - t^m) = t^n t^m (1 - \alpha^{p-1}) \) and \( \alpha^{3p} \neq 0 \) in \( \mathbb{F}_p[t]/(t^N) \).

**Proof.** \( \Rightarrow \) : Let \( a = (t^{-n+p^{-1}} + t^{-n+p^{-2}} + \cdots + t^{-n})^{-1} \in \mathbb{F}_p[t] \). After clearing denominators in Lemma 3.2.2, we get that \(\alpha^p(t^n - t^m) = t^n t^m (1 - \alpha^{p-1}) \). Reducing the equation modulo \( t^N \), yields \(\alpha^p(t^n - t^m) = t^n t^m (1 - \alpha^{p-1}) \), where \( \alpha \) is the image of \( a \) in \( \mathbb{F}_p[t]/(t^N) \). Note that \( v_t a^{3p} = 3m < N \) and thus \(\alpha^{3p} \neq 0 \) in \( \mathbb{F}_p[t]/(t^N) \).

\( \Leftarrow \) : Let \( n = p^i k \) and \( m = p^j l \) with \( p \nmid k, l \). Let \( \alpha \) be as in our assumption and \( a \in \mathbb{F}_p[t] \) be a lift of \(\alpha\). Since \(\alpha^{3p} \neq 0 \), we get that \( v_t a^{3p} < N/3 \). We will have by assumption that \( t^{-m} - t^{-n} = a^{-p} - a^{-1} + t^{N-m-n} z \), for some \( z \in \mathbb{F}_p[t] \). Set \( \varepsilon = t^{N-m-n} z \alpha^{-p} \) and note that \( v_t \varepsilon > 0 \) because \( m, n, v_t a^{3p} < N/3 \) and \( v_t \varepsilon \geq 0 \). By Lemma 1 [Phe87], we may find \( a_1, a_2 \in \mathbb{F}_p[t] \) such that \( t^{-m} - t^{-k} = a_1^{-p} - a_1^{-1} \) and \( t^{-m} - t^{-i} = a_2^{-p} - a_2^{-1} \). We compute that

\[
-t^{-m} - t^{-k} = b^p - b + \varepsilon
\]

where \( b = a^{-1} + a_1^{-1} - a_2^{-1} \).

We claim that \( i = k \). Otherwise, the left hand side must have negative valuation. Since \( v_t \varepsilon > 0 \), this forces the right hand side to have \( p \)-divisible valuation. This is contrary to the fact that \( p \nmid i, k \). It follows that \( i = k \) and thus \( n \mid_p m \). □

3.2.5. Interpreting \( I_n \) in \( \mathbb{F}_p[t]/(t^n) \). Motivated by Lemma 3.2.4 we introduce for each \( n \in \mathbb{N} \) the \( L_{p\text{-div}} \cup \{\infty\} \)-structure \( I_n = (\{0, 1, \ldots, n-1, \infty\}; 0, 1, \infty, \oplus, |_p) \), where:

- For \( x, y \in I_n \), we have \( x \mid_p y \) if \( y = p^s x \) for some \( s \in \mathbb{N} \) and \( 1 \leq x, y < n/3 \).
• The operation $\oplus : I_n \times I_n \to I_n$ stands for truncated addition, i.e., given $x, y \in \{0, 1, ..., n - 1\}$ we have that $x \oplus y = x + y$ if $x + y < n$ and $\infty$ otherwise. Moreover, $\infty \oplus x = x \oplus \infty = \infty$ for all $x \in \{0, 1, ..., n - 1, \infty\}$.

**Proposition 3.2.6.** For each $n \in \mathbb{N}$, there is an $\exists$-interpretation $\Gamma_n$ of the $L_p$-structure $I_n$ in the $L_t \cup \mathcal{P}$-structure $\mathbb{F}_p[t]/(t^n)$. Moreover, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n$.

**Proof.** Take $\partial_{\Gamma_n}(x)$ to be the formula $x \in P$. The reduction map of $\Gamma_n$ on unnested atomic formulas is described as follows:

1. If $\phi$ is the formula $x = 0$ (resp. $x = 1$, $x = \infty$ and $x = y$), we take $\phi_{\Gamma_n}$ to be the formula $x = 1$ (resp. $x = t$, $x = 0$ and $x = y$).
2. If $\phi(x, y, z)$ is $x + y = z$, then we take $\phi_{\Gamma_n}(x, y, z)$ to be the formula $x \cdot y = z$.
3. If $\phi(x, y)$ is the formula $x \mid_p y$, we take $\phi_{\Gamma_n}(x, y)$ to be the formula $\exists z (z^p (x - y) = x \cdot y (1 - z^{n-1}) \land z^{3p} \neq 0)$.

The coordinate map $f_{\Gamma_n} : \{0, 1, t, ..., t^{n-1}\} \to I_n$ is equal to $v \mid_P$, i.e., the valuation map $v$ restricted on $P$. Condition (3) of Definition 3.1.2 is readily verified for unnested atomic formulas of type (1). For the formula described in (2), one has to use the isomorphism of monoids $(P, \cdot, 1) \cong (\{0, 1, ..., n - 1, \infty\}, \oplus, 0)$. Using Lemma 3.2.4, one also verifies it for the unnested atomic formula of type (3). We deduce that $\Gamma_n$ is an interpretation. It is clear from the description of the reduction map on unnested formulas that $\Gamma_n$ is existential.

Finally, the reduction map $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n \in \mathbb{N}$, because of the inductive construction of the reduction map (see Proposition 1.1.3) and the fact that $\phi \mapsto \phi_{\Gamma_n}$ does not depend on $n$ when $\phi$ is any of the unnested atomic formulas listed above. 

\[\square\]

**Corollary 3.2.7.** The (resp. asymptotic) $L_p$-theory of $\{I_n : n \in \mathbb{N}\}$ is Turing reducible to the (resp. asymptotic) $L_t \cup \mathcal{P}$-theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$.

**Proof.** Immediate from Proposition 3.2.6. \[\square\]

### 3.3. Undecidability of fragments.

Using the undecidability of the Diophantine problem over $(\mathbb{N}; 0, 1, +, \mid_p)$ (Theorem 1 [Phe87]) as a black box, we shall prove that the asymptotic existential theory of $\{I_n : n \in \mathbb{N}\}$ is also undecidable. The proof becomes more transparent by using matrix norms. Recall the maximum absolute row sum norm $\| \cdot \|_\infty$ on the set of all matrices over $\mathbb{R}$, which is defined as $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, for $A \in M_{m \times n}(\mathbb{R})$. One readily checks that $\| \cdot \|_\infty$ is both sub-additive and sub-multiplicative, meaning that $\|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$ and $\|AB\|_\infty \leq \|A\|_\infty \cdot \|B\|_\infty$, whenever the operations are defined. Note also that when $A = (a_1, ..., a_n) \top \in M_{n \times 1}(\mathbb{N})$, we have $\|A\|_\infty = \max \{a_i : i = 1, ..., n\}$. 


For $A = (a_{ij})$ and $B = (b_{ij})$ in $M_{m \times n}(\mathbb{N})$, it will be convenient to use the notation $A \mid_p B$, which means that $a_{ij} \mid_p b_{ij}$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

**Proposition 3.3.1.** The asymptotic existential $L_{p-\text{div}} \cup \{\infty\}$-theory of $\{I_n : n \in \mathbb{N}\}$ is undecidable. Moreover, let $T$ be the subtheory consisting of sentences $\phi$ of the form $\exists x \psi(x)$, where $x = (x_1, ..., x_n)$ and $\psi(x)$ is a conjunction of a quantifier-free formula without negations with a formula of the form $\bigwedge_{i=1}^n N x_i \neq \infty$, for some $N \in \mathbb{N}$. Then $T$ is undecidable.

**Proof.** We shall encode the Diophantine problem of $(\mathbb{N}; 0, 1, +, |_p)$ in $T$. Let $\Sigma$ be an arbitrary system in variables $x = (x_1, ..., x_n) \in M_{n \times 1}(\mathbb{N})$ of the form

$$(\Sigma) \quad \begin{cases} A_1 x + b_1 = A_2 x + b_2 \\ A_3 x + b_3 \mid_p A_4 x + b_4 \end{cases}$$

where $A_i \in M_{m \times n}(\mathbb{N})$ and $b_i \in M_{m \times 1}(\mathbb{N})$. Consider also the formula $\Sigma(x) \in \text{Form}_{p-\text{div}}$ associated with the system $\Sigma$.

**Claim:** We have that

$$(\mathbb{N}; 0, 1, +, |_p) \models \exists x \Sigma(x) \iff T \models \exists x(\Sigma(x) \bigwedge_{i=1}^n 3M x_i \neq \infty)$$

where $M = \max_{1 \leq i \leq 4}\{|A_i|_\infty + |b_i|_\infty\}$.

**Proof.** "$\Rightarrow$": Consider a witnessing tuple $c = (c_1, ..., c_n) \in M_{n \times 1}(\mathbb{N})$. If $c = 0$, then the conclusion is clear. Otherwise, consider $m = |c|_\infty = \max\{c_i : i = 1, ..., n\} \geq 1$ and choose $N \in \mathbb{N}$ such that $N > 3 \cdot M \cdot m$. Using the sub-additive and sub-multiplicative properties of $\| \cdot \|_\infty$, we see that

$$\|A_i c + b_i\|_\infty \leq \|A_i\|_\infty \|c\|_\infty + \|b_i\|_\infty \leq m \cdot (\|A_i\|_\infty + \|b_i\|_\infty) \leq M \cdot m < N/3$$

for $i = 1, ..., 4$. In particular, we get that both $\oplus$ and $|_p$ specialize to their ordinary counterparts in $\mathbb{N}$ and that $\Sigma(c)$ holds true in $I_N$, viewing $c$ as a tuple in $I_N^n$. Each conjunct $3M x_i \neq \infty$ also holds true for $c_i$ because $3M c_i \leq 3M \cdot m < N$.

"$\Leftarrow$": Let $c = (c_1, ..., c_n) \in I_N^n$ be a witness of the sentence $\exists x(\Sigma(x) \bigwedge_{i=1}^n 3M x_i \neq \infty)$, for some $N > 3 \cdot M$. Since $3M c_i \neq \infty$, we get that $c_i < N/3M$ for $i = 1, ..., n$. We therefore get that

$$\|A_i c + b_i\|_\infty \leq \|A_i\|_\infty \|c\|_\infty + \|b_i\|_\infty < \frac{N}{3M} \cdot (\|A_i\|_\infty + \|b_i\|_\infty) \leq \frac{N}{3M} \cdot M = N/3$$

for $i = 1, ..., 4$. In particular, we get that both $\oplus$ and $|_p$ specialize to their ordinary counterparts in $\mathbb{N}$. The corresponding tuple in $\mathbb{N}^n$ is the desired witness. \(\square\)

The conclusion follows from the fact that the Diophantine problem over $(\mathbb{N}; 0, 1, +, |_p)$ is undecidable (Theorem 1 [Phe87]). \(\square\)
4. PROOF OF THE MAIN THEOREM

4.1. Truncated polynomial rings.

**Proposition 4.1.1.** The asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is undecidable.

**Proof.** Let \( T \) be as in Proposition 3.2.6. We shall argue that the asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \) is Turing reducible to \( T \). The conclusion will then follow from Proposition 3.3.1. Let \( \Gamma \) be the \( \exists \)-interpretation of the \( L_{p-\text{div}} \cup \{ \infty \} \)-structure \( I_n \) in the \( L_t \cup P \)-structure \( \mathbb{F}_p[t]/(t^n) \), provided by Proposition 3.2.6. The reduction map of \( \Gamma_n \) does not depend on \( n \in \mathbb{N} \) and will simply be denoted by \( \phi \mapsto \phi_T \). Since \( \Gamma_n \) is existential, whenever \( \psi(x) \) is a quantifier-free \( L_{p-\text{div}} \cup \{ \infty \} \)-formula, the formula \( \psi_T(x) \) is an existential formula. Moreover, from the description of the reduction map on unnested atomic formulas (see the proof of Proposition 3.2.6), we have \( (\bigwedge_{i=1}^n N x_i \neq \infty)_T = \bigwedge_{i=1}^n (N x_i \neq \infty) \). It follows that \( T \) is Turing reducible to the asymptotic existential \( L_t \cup P \)-theory of \( \{ \mathbb{F}_p[t]/(t^n) : n \in \mathbb{N} \} \). \( \square \)

4.2. Totally ramified extensions.

**Proposition 4.2.1.** The asymptotic \( \exists \forall \)-theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) is undecidable in \( L_{\text{val},x} \).

**Proof.** This follows from Corollary 2.2.7 and Proposition 4.1.1. \( \square \)

**Remark 4.2.2.** (a) The same proof applies verbatim to any infinite collection of totally ramified extensions of \( \mathbb{Q}_p \), e.g., \( \{ \mathbb{Q}_p(p^n) : n \in \mathbb{N} \} \) or \( \{ \mathbb{Q}_p(p^{1/n}) : n \in \mathbb{N} \} \).

(b) We do not know if the asymptotic existential theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) is decidable or not in \( L_{\text{val},x} \).

4.3. Finite extensions.

**Theorem A.** The asymptotic \( \exists \forall \)-theory of \( \{ K : [K : \mathbb{Q}_p] < \infty \} \) is undecidable in \( L_{\text{val},x} \).

**Proof.** Let \( T \) be the theory in question and \( T_{\text{tot}} \) be the asymptotic theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \) in \( L_{\text{val}} \) with a cross-section. We shall encode \( T_{\text{tot}} \) in \( T \). Given an \( \exists \forall \)-sentence \( \phi \), we see that

\[ T_{\text{tot}} \models \phi \iff T \models (k = \mathbb{F}_p) \rightarrow \phi \]

The formal counterpart of \( (k = \mathbb{F}_p) \rightarrow \phi \) is logically equivalent to an \( \exists \forall \)-sentence. It follows that the asymptotic existential-universal \( L_{\text{val},x} \)-theory of \( \{ K : [K : \mathbb{Q}_p] < \infty \} \) is Turing reducible to the asymptotic existential-universal \( L_{\text{val},x} \)-theory of \( \{ K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty \} \). The conclusion follows from Proposition 4.2.1. \( \square \)
5. Final remarks

5.1. $L_{\text{val}}$ vs $L_{\text{rings}}$. In view of [CDLM13] and by possibly increasing the complexity, one may replace $L_{\text{val},x}$ with the 1-sorted language of rings $L_{\text{rings}}$ together with a unary predicate $P$ for the image of the cross-section in $K$. We need to use the following:

**Fact 5.1.1** (Theorem 2 [CDLM13]). There is an $\exists\forall$-formula $\phi(x)$ in $L_r$ such that $\emptyset_K = \phi(K)$ for any henselian valued field $K$ with finite or pseudo-finite residue field.

**Remark 5.1.2.** In fact, Theorem 2 [CDLM13] is about the existence of an existential formula in $L_{\text{rings}} \cup P_{2^{AS}}$, where $P_{2^{AS}}(x) = \exists y(x = y^2 + y)$. However, any existential sentence in $L_{\text{rings}} \cup P_{2^{AS}}$ is equivalent to an $\exists\forall$-sentence in $L_{\text{rings}}$.

By Fact 5.1.1 the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in $L_{\text{val}}$ with a cross-section can be encoded in the asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ in $L_{\text{rings}} \cup P$. By Theorem A, the latter is undecidable. Our use of the cross-section/predicate formalism is essential and we do not know whether the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in $L_{\text{rings}}$.

5.2. Residue rings and $\mathbb{F}_p((t))$. If the (asymptotic) theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is decidable in $L_{\text{rings}}$, then by Corollary 2.2.7 this would also yield a positive answer to the following question:

**Question 5.2.1.** Is the (asymptotic) theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ decidable in $L_{\text{rings}}$?

**Observation 5.2.2.** The asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is decidable in $L_{\text{rings}}$.

**Proof.** Let $\mathbb{F}_p[t^{1/\infty}]$ be the direct limit of the injective system $\lim_{n \to \infty} \mathbb{F}_p[t^{1/n}]$, where $\phi_{nm} : \mathbb{F}_p[t^{1/n}] \hookrightarrow \mathbb{F}_p[t^{1/m}]$ is the natural inclusion map for $n \mid m$. We write $\mathbb{F}_p[t^{1/n}]/(t)$ and $\mathbb{F}_p[t^{1/\infty}]/(t)$ for the quotients modulo $t$. Note that $\phi_{nm}(t \cdot \mathbb{F}_p[t^{1/m}]) \subseteq t \cdot \mathbb{F}_p[t^{1/m}]$ and therefore the induced maps $\bar{\phi}_{nm} : \mathbb{F}_p[t^{1/n}] \hookrightarrow \mathbb{F}_p[t^{1/m}]$ and $\bar{\phi}_n : \mathbb{F}_p[t^{1/n}]/(t) \hookrightarrow \mathbb{F}_p[t^{1/\infty}]/(t)$ are also injective.

**Claim 1:** The asymptotic existential $L_{\text{rings}}$-theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is equal to $\text{Th}_3(\mathbb{F}_p[t^{1/\infty}]/(t))$.

**Proof.** For each $n \in \mathbb{N}$, we have a ring isomorphism

$$\mathbb{F}_p[t^{1/n}]/(t) \cong \mathbb{F}_p[t, X]/(X^n - t, t) \cong \mathbb{F}_p[X]/(X^n) \cong \mathbb{F}_p[t]/(t^n)$$

Now if $\phi \in L_{\text{rings}}$ is existential, then $\mathbb{F}_p[t^{1/\infty}]/(t) \models \phi$ if and only if $\mathbb{F}_p[t^{1/n}]/(t) \models \phi$ for all sufficiently large $n$. The conclusion follows.

Finally, we prove:

**Claim 2:** $\text{Th}_3(\mathbb{F}_p[t^{1/\infty}]/(t))$ is decidable in $L_{\text{rings}}$. 

Proof. A straightforward adaptation of Proposition 6.2.1 [Kar20] shows that

$$\mathbb{F}_p[t^{1/\infty}]/(t) \models \exists x \bigwedge_{1 \leq i,j \leq n} (f_i(x) = 0 \land g_j(x) \neq 0) \iff$$

$$\mathbb{F}_p[[t]](t^{1/\infty}) \models \exists x \bigwedge_{1 \leq i,j \leq n} (v(f_i(x)) > v(g_j(x)))$$

where $f_i(x), g_j(x) \in \mathbb{F}_p[x]$ are any multi-variable polynomials in $x = (x_1, ..., x_m)$ for $i, j = 1, ..., n$. Finally, the henselian valued field $\mathbb{F}_p((t))(t^{1/\infty})$ is existentially decidable in $L_{val}$ by Corollary 7.5 [AF16]. □

Claim 2 □

Remark 5.2.3. (a) Observation 5.2.2 should be contrasted with Proposition 4.1.1, which shows that the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is undecidable in $L_t \cup P$.

(b) The proof of Observation 5.2.2 does not go through for the language $L_t$, as the ring isomorphisms in the proof of Claim 1 do not respect $t$. We do not know if the asymptotic existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ is decidable in $L_t$.

(c) On the other hand, the asymptotic positive existential theory of $\{\mathbb{F}_p[t]/(t^n) : n \in \mathbb{N}\}$ in $L_t$ is equal to $\text{Th}_{\exists \forall} \mathbb{F}_p[t]$ and is decidable by an effective version of Greenberg’s approximation theorem (see Theorem 3.1 and Theorem 6.1 [BDLvdD79]).

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References

[AF16] Sylvy Anscombe and Arno Fehm. The existential theory of equicharacteristic henselian valued fields. Algebra & Number Theory Volume 10, Number 3 (2016), 665-683., 2016.

[AK65] James Ax and Simon Kochen. Diophantine problems over local fields II. Amer. J. Math. 87, 1965.

[AK66] James Ax and Simon Kochen. Diophantine problems over local fields III. Annals of Mathematics, Second Series, Vol. 83, No. 3, pp. 437-456, 1966.

[Ax68] James Ax. The elementary theory of finite fields. Annals of Mathematics , Second Series, Vol. 88, No. 2 , pp. 239-271, 1968.

[BDL80] J. Becker, J. Denef, and L. Lipshitz. Further remarks on the elementary theory of formal power series rings. Model Theory of Algebra and Arithmetic, Pacholski et al. eds., LNM 834, Springer-Verlag NY, pp 1-9, 1980.

[BDLvdD79] J. Becker, J. Denef, L. Lipshitz, and L. van den Dries. Ultraproducts and approximation in local rings I. Inventiones math. 51,189-203, 1979.
[CDLM13] Raf Cluckers, Jamshid Derakhshan, Eva Leenknegt, and Angus Macintyre. Uniformly defining valuation rings in henselian valued fields with finite or pseudo-finite residue fields. *Ann. Pure Appl. Logic* 164, 12, 1236-1246, 2013.

[Che82] Gregory Cherlin. Undecidability of rational function fields in nonzero characteristic. *Logic Colloq., no. 82, North-Holland, Amsterdam.,* 1982.

[Del84] Pierre Deligne. Les corps locaux de caractéristique p, limites de corps locaux de caractéristique 0. *Représentations des groupes réductifs sur un corps local, - Hermann Paris*, 1984.

[Den79] Jan Denef. The diophantine problem for polynomial rings of positive characteristic. *Logic Colloq., no. 78, North-Holland, Amsterdam*, 1979.

[Der20] Jamshid Derakhshan. Model theory of Adeles and number theory. arXiv: 2007.09237 [math.LO], 2020.

[DM22] Jamshid Derakhshan and Angus Macintyre. Model theory of adeles I. *Ann. Pure Appl. Logic*, 173(3):Paper No. 103074, 43, 2022.

[Ers65] Ju.L. Ershov. On elementary theories of local fields. *Algebra i Logika 4, No. 2, 5-30, 1965.*

[Hod93] Wilfrid Hodges. *Model theory*. Cambridge University Press, 1993.

[Kar20] Konstantinos Kartas. Decidability via the tilting correspondence. arXiv:2001.04424 [math.LO], (2020).

[Kar22] Konstantinos Kartas. Contributions to the model theory of henselian fields. PhD thesis, University of Oxford, 2022.

[Kaz86] D. Kazhdan. Representations of groups over close local fields. *Journal d’ Analyse Mathématique* volume 47, pages 175-179, 1986.

[Koc74] Simon Kochen. The model theory of local fields. *ISILC Logic Conference (Proc. Internat. Summer Inst. and Logic Colloq., Kiel),* pp. 384-425. Lecture Notes in Math., Vol. 499, Springer, Berlin., 1974.

[Kra56] Marc Krasner. Approximation des corps valués complets de caractéristique p par ceux de caractéristique 0. *Colloque d’Algebre Superleure Bruxelles,* 1956.

[Lan94] Serge Lang. *Algebraic Number Theory*. Springer, Graduate Texts in Mathematics book series (GTM, volume 110), 1994.

[Mac76] Angus J. Macintyre. On definable subsets of p-adic fields. *Journal of Symbolic Logic*, 41:605–10, 1976.

[Mat70] Yuri Matiyasevich. Enumerable sets are Diophantine. *Dokl. Akad. Nauk SSSR* 191, 279- 282, 1970.

[Pap94] Christos H. Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, Inc., 1994.

[Phe87] Thanases Pheidas. An undecidability result for power series rings of positive characteristic. II. *Proceedings of the American Mathematical Society, Vol. 100*, pp. 526-30, 1987.

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