One-dimensional thermal equation modeled on a two-dimensional heat exchanger with variable solid-fluid interface

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Abstract. In this study, we are to present that a one-dimensional equation for vertically averaged temperature, modeled on a vertically thin, two-dimensional heat exchanger with variable top solid-fluid interface, recovers the two-dimensional thermal information, i.e. steady temperature and flux distribution on the top and temperature-fixed bottom faces. The relative error of these quantities is less than 5% with the maximum gradient of the height kept approximately below 0.5, while the computational time is reduced to 0.1–5%, when compared with direct two-dimensional computations, depending on the shape of the top face. The model equation, derived by the vertical averaging of the two-dimensional thermal conduction equation, is closed by an approximation that the heat exchanger is sufficiently thin in the sense that the second derivative of temperature with respect to the horizontal coordinate depends only on the coordinate. In this model equation, the fluid equation above the exchanger is decoupled by a conventional equation for the normal heat flux on the top surface. In principle, however, the coupling of the model and the fluid equation is possible through the temperature and heat flux on the top interface, recovered by the model equation. The type of mathematical modeling can be applicable to a wide variety of bodies with extremely small dimensions in some (coordinate-transformed) directions.

Keywords: heat exchanger, variable shape, heat conduction equation, one-dimensional model, solid-fluid interface

1. Introduction
Numerical analyses on the solid-to-liquid heat transfer around heat exchangers have been intensively investigated [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Such a coupled analysis involves different time-spatial scale of heat transfer for each phase, needing time-consuming computations. In particular, it is noticeable when the heat exchangers have complicated shapes [1, 2, 3, 4], or the shape itself is optimized [5, 6, 7, 8, 9]. Therefore, it is desirable to make a simplified model equation that mimics the heat transfer phenomena.

There are several models of heat exchangers. To our best knowledge, however, they intend to model the total system constructed by a heat exchanger and neighboring fluid flow, and most of them [10, 11, 12] are not based on not dynamical equations but integrated balance equations.

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for some practical system components in some unknown constants to be determined by some numerical and experimental analyses.

Some models are based on dynamic equations and, particularly, the volume averaging technique (VAT) leads to closed model equations for some heat exchangers [13, 14]. However, the models also involve the above-mentioned unknown constants. It is preferred, therefore, that a model equation does not have such unknowns while it is applicable to many practical situations.

In this study, we are to present a one-dimensional equation for vertically averaged temperature, modeled solely on a two-dimensional heat exchanger with variable top solid-fluid interface. Once the heat exchanger is assumed to be sufficiently thin, its mathematical form is derived analytically without any unknown constants to be determined by two-dimensional analyses, even though the shape of the top face is arbitrary. The applicability of the model was assessed by the direct two-dimensional simulations.

2. One-dimensional model and numerical analyses

2.1. Physical model

The physical model and coordinate definitions of this study are shown in Fig. 1. A two-dimensional solid heat exchanger is submerged into a uniform fluid flow from left to right. The dimensionless $x$ velocity component and temperature of the outer flow are held fixed at the Reynolds number $Re$, based on the fluidic dynamic viscosity, and zero, respectively. At the left- and right-hand sides the heat exchanger is insulated and the temperature at the bottom face is kept constant at unity.

The shape of the wavy top face, solid-fluid interface, is actually optional, identified by the height function $H(x)$ of the form

$$H(x) = \overline{H}(1 + \epsilon b(x)),$$

where $\overline{H}$ denotes the representative height, and $\epsilon$ a constant minute amplitude of the top-face variation $b$.

In order to decouple the heat conduction equation in the domain of heat exchanger from the outer fluid equation, the dimensionless heat flux $q_n$ in the normal direction at the top face is introduced as the conventional form
\[ q_n (= n_x q_x^{(s)} + n_y q_y^{(s)}) = Bi \theta_s, \]  

(2)

where \( Bi \) is dimensionless heat transfer rate based on the solid properties, i.e. the Biot number, depending on \( H \) as follows

\[ Bi = Bim(1 + aH'(x)), \]

(3)

\[ Bim = (k_f/k_s) Nu_m \]

(4)

\[ Nu_m = 2C(Pr)Re^{1/2}, \]

(5)

\[ C(Pr) = 0.332Pr^{1/3}, \]

(6)

and \( \theta_s(= \theta_s(x)) \) is the temperature at the top surface. In this study, \( Re \), the Prandtl number \( Pr \), the ratio \( k_f/k_s \) of the thermal conductivity of fluid to that of solid, and a constant \( a \) are fixed at 5.0, 0.71, unity and unity, respectively. The equation (3) is based on the fact that the Nusselt number \( Nu \), another dimensionless heat transfer rate based on the fluidic properties, on the heated plane increases with the angle to the flow direction [15, 16]. The equation (5) for \( Nu_m \) is a correlation equation for the surface-mean Nusselt number on the plane heated by constant temperature in a laminar flow [17]. Note that the resultant \( Bim \) of 1.33 can be obtained from many other realistic combinations of physical properties.

It should be also noted that the heat-flux model equation (2), a Robin boundary condition, is not essential for the one-dimensional modeling of this study. In order for the coupling of the inner heat conduction equation and the outer fluid equation, i.e. the Navier-Stokes equation, the so-called thermal contact condition is subjected to the solid-fluid interface: the temperature and heat flux are continuous at the face. Such a coupling computation, however, takes too much time to validate the (one-dimensional) modeling of this study and it is not treated in this study. The application of the model to the coupling problem is discussed in Appendix A.

2.2. One-dimensional model

The thermal, temperature \( \theta \), field on the above-mentioned heat exchanger is governed by the two-dimensional heat conduction equation. Its dimensionless form is

\[ \frac{\partial \theta}{\partial t} = \Delta \theta, \]  

(7)

where \( \Delta \) is the (two-dimensional) Laplacian.

In this form, the dimensional quantities are normalized by the representative length \( L(= \text{horizontal length of the heat exchanger}) \), thermal diffusivity \( a \), and the representative temperature difference \( \Delta \Theta(= \text{fixed temperature at the bottom face minus fluid temperature}) \).

If the heat exchanger is sufficiently thin, we are certain that some kind of vertical averaging must be effective even if its top face has a complicated shape. Let us begin with, therefore, the governing equation for the vertical mean temperature, defined by

\[ \overline{\theta}(x, t) \equiv \frac{1}{H(x)} \int_0^{H(x)} \theta(x, y, t) dy. \]

(8)

It follows from the vertical averaging of equation (7) that

\[ \frac{\partial (H \overline{\theta})}{\partial t} = \frac{\partial^2 (H \overline{\theta})}{\partial x^2} + 2q_x^{(s)}H'(x) + (H'^2 - 1)q_y^{(s)} - H'' \theta_s + q_y^{(0)}, \]

(9)

where
\[ q_x^{(s)} = -\left. \frac{\partial \theta}{\partial x} \right|_{y=H(x)}, \quad q_y^{(s)} = -\left. \frac{\partial \theta}{\partial y} \right|_{y=H(x)}, \quad q_y^{(0)} = -\left. \frac{\partial \theta}{\partial y} \right|_{y=0}, \]

denoting some heat flux components at the top and bottom faces.

Considering that the boundary condition (2) is given and that the normal vector \( n \) at the top face can be expressed as
\[ n \equiv (n_x, n_y) = \frac{1}{\sqrt{1 + H^2}} (-H', 1), \]  
(10)
equation (9) is rewritten as follows
\[
\frac{\partial (H\bar{\theta})}{\partial t} = \frac{\partial^2 (H\bar{\theta})}{\partial x^2} + (H^2 + 1)q_y^{(s)} - (2\sqrt{1 + H^2}Bi + H')\theta_s + q_y^{(0)}. \]  
(11)

The equations (9) and (11) are rigorous but not closed as the equation for \( \theta \). That is the reason why we need a mathematical modeling.

In what follows, we are to derive the model equation to obtain steady solutions by an approximation that the heat exchanger is sufficiently thin in the sense that
\[ \Theta(x, y) \equiv \frac{\partial^2 \theta}{\partial x^2} = \Theta_0(x) + \Theta_1(x)y + \Theta_2(x)y^2 + \cdots \approx \Theta_0(x), \]  
(12)
that is, the second derivative of temperature with respect to \( x \), \( \Theta \), depends only on \( x \).

The approximation allows us to integrate the heat conduction equation (7) in the steady state to be
\[ \theta = A(x)y + B(x) - \frac{\Theta(x)}{2}y^2. \]  
(13)

All of the quantities, including \( \theta_s, q_x^{(s)}, q_y^{(s)} \), that we should evaluate to close equations (9) or (11), or to restore the thermal information on the top and bottom faces, can be described by six unknowns, i.e. \( A, B, \Theta, A', B' \) and \( \Theta' \).

On the other hand, we have two boundary conditions on the top (2) and bottom (\( \theta = 1 \)) faces, and two their differential forms with respect to \( x \):
\[
Bi'\theta_s = (Bi + n_x')q_x^{(s)} + (BiH' + n_y')q_y^{(s)} - n_x\theta_x^{(s)} - (n_xH' + n_y)\theta_{xx}^{(s)} - n_yH'\theta_{yy}^{(s)}, \]  
(14)
\[
\left. \frac{\partial \theta}{\partial x} \right|_{y=0} = 0, \]  
(15)

where
\[ \theta_{xx}^{(s)} = \left. \frac{\partial^2 \theta}{\partial x^2} \right|_{y=H(x)}, \quad \theta_{xy}^{(s)} = \left. \frac{\partial^2 \theta}{\partial x \partial y} \right|_{y=H(x)}, \quad \theta_{yy}^{(s)} = \left. \frac{\partial^2 \theta}{\partial y^2} \right|_{y=H(x)}, \]

and the approximation (12) in the steady state enables us to evaluate
\[ \theta_{xx}^{(s)} = -\theta_{yy}^{(s)} = \Theta. \]

Moreover, we can derive the following two relations

\[ \bar{\theta} = \frac{AH}{2} + B - \frac{\Theta H^2}{6}, \]  
\[ \frac{\partial \bar{\theta}}{\partial x} = -\frac{H'H}{H} + \frac{H'\theta}{H} + \frac{A'H}{2} - \frac{\Theta'H^2}{6}, \]  

where the latter equation (17) is obtained by substituting equation (13) into the exact relation as follows

\[ \frac{\partial \bar{\theta}}{\partial x} = -\frac{H'H}{H} + \frac{H'\theta}{H} + \frac{1}{H} \int_0^H \frac{\partial \theta}{\partial x} \, dy. \]  

It follows that the six unknowns can be expressed by \( \bar{\theta} \) and \( \partial \bar{\theta}/\partial x \) without any unknown constants, and we have

\[ q_y^{(s)} = \alpha/\beta, \]  

where

\[ \alpha = \left[ \frac{Bi'n_x - Bin'_x - Bi^2}{2} - \frac{2Bi(H'n_x + n_y)}{H} \right] (3\bar{\theta} - 1) + \frac{3(H'^2 + 1)n_x^2}{H} (\bar{\theta} - 1) - \frac{6n_x(H'n_x + n_y)}{H} \frac{\partial \bar{\theta}}{\partial x}, \]  
\[ \beta = \frac{HBi'n_x - (Bi' + n'_y)(BiH + 4n_y) + 4n_x(Bi'H' + n'_y)}{4} + \frac{3(H'^2 - 1)n_x^2 - 2H'n_xn_y - 8n_y^2 - 2BiH(H'n_x + n_y)}{2H}. \]

By use of \( q_y^{(s)} \) (19), we have the following relations

\[ \theta_s = \frac{3\bar{\theta} - 1}{2} - \frac{H}{4} q_y^{(s)}, \]  
\[ q_y^{(0)} = \frac{3(1 - \bar{\theta})}{H} - \frac{q_y^{(s)}}{2}, \]  
\[ q_x^{(s)} = \frac{Bi}{2n_x} (3\bar{\theta} - 1) - \frac{BiH + 4n_y}{4n_x} q_y^{(s)}, \]

and, consequently, the model equation (9) (or equation (11)) is closed. The equation is linear with respect to \( \bar{\theta} \) while the geometric nonlinearity with \( H \) appears. It is worth noting that the form is independent of the model (3) of the Biot number. As long as the number is independent of temperature and given explicitly as the function of \( x \), the form is identical. If it is not the case, a coupling problem occurs, described in Appendix A.

We can easily confirm that the model equation has the steady solution for the case of \( \epsilon = 0 \) as follows
\[ \theta = \frac{BiH + 2}{2(BiH + 1)}, \]

in coincidence with the exact solution of equation (7). Moreover, equation (19) leads that \( q_{(s)} \) behaves like \( 2(3\theta - 1)/H \) as \( Bi_{m} \) approaches to infinity and, therefore, \( \theta_{s} \) approaches to the fluid temperature zero for any top shapes. It is reasonable behavior when the fluid velocity \( (Re) \) approaches to infinity. We can also confirm that the model has the steady solution \( \theta \) of unity when the variable top face is adiabatic, i.e. \( Bi = 0 \). These facts imply that the model equation is physically appropriate.

Does the closed model equation make no physical sense out of the steady state? As a matter of fact, it is valid for the direct component of the mean temperature \( \theta \). However, we are to temporally regard the dimensionless time \( t \) in the model equation as a pseudo parameter to obtain (linearly stable) steady solutions of \( \theta \) after sufficiently long time-evolution.

2.3. One- and two-dimensional numerical analyses

In this study, the above-mentioned one-dimensional model equation (11) is numerically solved by the finite difference method. The relation (18), in conjunction with equation (20), yields the boundary condition at both side walls \( (x = 0, 1) \) as follows

\[ \frac{\partial \theta}{\partial x} = -\frac{H'\theta}{H} + \frac{H'\theta_{s}}{H}. \] (23)

As shown in Tables 1 and 2, six types of top-face shape was utilized to validate the model equation. The amplitude \( \epsilon \) was raised up to 0.9~6.0, dependent on the shape, until a relative error reaches around 5\% when compared with direct two-dimensional computations.

The direct simulation was performed by the finite difference method. The \( x - y \) physical space is transformed into a \( \xi - \eta \) computational space, defined by

\[ \xi \equiv x, \eta \equiv \frac{Hy}{H(x)}, \] (24)

and the physical domain of the heat exchanger is resultantly transformed into a rectangular, unity by \( H \), computational one, on which the equation (7) was discretized. In the \( \xi - \eta \) space the Laplacian is expressed by

\[ \frac{\partial^{2}}{\partial \xi^{2}} + \frac{2H'^{2} - H''H\eta}{H^{2}} \frac{\partial}{\partial \eta} - \frac{2H'\eta}{H} \frac{\partial^{2}}{\partial \xi \partial \eta} + \frac{H'^{2}\eta^{2} + \Pi'^{2}}{H^{2}} \frac{\partial^{2}}{\partial \eta^{2}}, \] (25)

and the central difference scheme was utilized to transform the differential operator into a difference one.

The horizontal \( N_{x} \) and vertical \( N_{y} \) division numbers are summarized in Table 2. The time increment \( \Delta t \) was set so that the stability condition holds for the differenced equation.

Initial state is \( \theta = \theta = 0 \), and the one- and two-dimensional equations were timely evolved to be steady states. On the one-dimensional model the temperature and heat-flux distribution on the top and bottom faces are computed by the relations (19), (20) and (21), compared with the direct two-dimensional results.
Figure 2. Isotherms in the steady state for the case that $b(x) = \cos(2\pi x)$. $\epsilon$ and $\overline{H}$ are 0.5 and 0.1, respectively. The increment is 0.01, and lowermost is 1.0.

Figure 3. Comparison of various quantities in the steady state obtained by one- and two-dimensional computations: (a) vertical mean temperature $\overline{\theta}$, (b) top-surface temperature $\theta_s$, and vertical components (c) $q_y^{(s)}$ and (d) $q_y^{(0)}$ of heat flux on the top and bottom faces. Solid line indicates the results of direct two-dimensional computation, and plus symbol indicates those of model. The shape of the top face is identical with the case of Fig. 2.

3. Results and discussions

A typical two-dimensional steady temperature distribution in a heat exchanger is shown in Fig. 2, computed by the two-dimensional direct method. We can confirm that the normal temperature gradient on the top surface increases with the gradient $H'(x)$ of the height, based on the heat-flux model (3): the temperature gradient is large at around $x=0.75$ where $H'(x)$ is maximum.
Figure 4. Relative error $\varepsilon_{\text{max}}$ computed for some shapes of top face: (a) $b = \cos(2\pi x)$; (b) $b = \exp(\sin(2\pi x))$; (c) $b = \exp(x) + \sin(2\pi x)$; (d) $b = x^3$. The plus, triangle, cross and square symbols indicate $\varepsilon_{\text{max}}$ of vertical mean temperature $\bar{\theta}$, top-surface temperature $\theta_s$, and the vertical component $q_y^{(s)}$ and $q_y^{(0)}$ of heat flux, respectively. In these cases $H$ is fixed at 0.1.

The one-dimensional model equation (11) can estimate the distribution of the vertical mean temperature for any top-face shapes when the thickness of the exchanger is small enough. For the same case of Fig. 2 four quantities are evaluated by the model and the direct simulation, shown in Fig. 3. The quantities computed by both methods agree very well. It should be stressed that the coincidence in $\theta_s$ and $q_y^{(s)}$ indicates the coincidence in heat flux on the top face. Because the normal flux $q_n$, independent of the vertical component $q_y^{(s)}$, is determined by $\theta_s$ via the relation (2).

The agreement, however, depends on the thickness or the shape of the heat exchanger. That is the reason why we must introduce a relative error $\varepsilon_{\text{max}}$ of a quantity $A(x)$, defined by

$$\varepsilon_{\text{max}} = \frac{|A_{\text{model}}(x_{\text{max}}) - A_{\text{2dim}}(x_{\text{max}})|}{A_{\text{2dim}}(x_{\text{max}})},$$

where $A_{\text{model}}$ and $A_{\text{2dim}}$ are two values of a quantity $A$, computed by the one-dimensional model and the direct two-dimensional simulation, respectively. The position $x$ that maximizes the numerator is denoted by $x_{\text{max}}$ and, therefore, $\varepsilon_{\text{max}}$ is not the maximum relative error. Such relative errors are shown in Fig. 4. For the case that $b = \cos(2\pi x)$, the relative error of $q_y^{(0)}$ first reaches to 0.05 at $\epsilon = 0.9$. In contrast, the relative error of $q_y^{(s)}$ reaches first to 5% for the case that $b = x^3$. Such critical points can be also dependent on $H$

In order to summarize these results let us introduce the following index for evaluating the extent of the relative error, defined by

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Table 1. $H'_{\text{max}}$ and its components at which one of relative errors $\varepsilon_{\text{max}}$ computed from the four quantities shown in Fig. 3 exceeds 5%.

| $b(x)$ | $b'_{\text{max}}$ | $\varepsilon$ | $\overline{H}$ | $H'_{\text{max}}$ |
|--------|------------------|----------------|----------------|------------------|
| $\cos(2\pi x)$ | $2\pi$ | 0.90 | 0.1 | 0.60 |
|         |       | 0.30 | 0.2 | 0.40 |
| $\sin(2\pi x)$ | $2\pi$ | 0.90 | 0.1 | 0.60 |
|         |       | 0.30 | 0.2 | 0.40 |
| $\exp(\sin(2\pi x))$ | 9.16 | 0.62 | 0.1 | 0.57 |
|         |       | 0.20 | 0.2 | 0.40 |
| $\exp(x) + \sin(2\pi x)$ | 9.00 | 0.70 | 0.1 | 0.60 |
|         |       | 0.20 | 0.2 | 0.40 |
| $x^3$ | 3.00 | 3.0 | 0.1 | 0.90 |
|         |       | 1.5 | 0.2 | 0.90 |
| $x$ | 1.00 | 6.0 | 0.1 | 0.60 |
|         |       | 2.5 | 0.2 | 0.50 |

\[ H'_{\text{max}} = \max_x H'(x) = \epsilon \overline{H} \max_x b'(x) = \epsilon \overline{H} b'_{\text{max}}. \] (27)

The above index comes from the fact that the model equation provides the exact solution when the height $H(x)$ is uniform, i.e. $H(x) = \overline{H}$, and, therefore, that the maximum gradient $H'_{\text{max}}$ can be the index to explain the relative errors.

Table 1 shows the values of $H'_{\text{max}}$ and its components when one of relative errors $\varepsilon_{\text{max}}$ of four quantities shown in Fig. 3 exceeds 5%. In this table $\overline{H}$ is treated as a variable. These values depend on the shape of top face, amplitude $\epsilon$, representative height $\overline{H}$. However, we can find that the relative error $\varepsilon_{\text{max}}$ is less than 5% when $H'_{\text{max}}$ is reduced to approximately 0.5. Thus, the index $H'_{\text{max}}$ provides the measure of applicability of the one-dimensional model equation (11).

Finally, the computational time to solve one- and two-dimensional equations numerically is shown in Table 2. The computational time is reduced to 0.1~5% when compared with the two-dimensional computations.

In this table $\epsilon$ and $\overline{H}$ are kept constant at 0.5 and 0.1, respectively. The larger the volume of the heat exchanger becomes, the longer time the computation takes. Because a large heat exchanger takes much time to be steady state. As long as the vertical division number $N_y$ is fixed, however, the reduction rate is largely determined by the ratio between the loads of one- and two-dimensional computations at each time step, independent of the value of $\epsilon$ and $\overline{H}$.

4. Concluding remarks

This study presents a one-dimensional equation for vertically averaged temperature, modeled on a vertically thin, two-dimensional heat exchanger with variable top solid-fluid interface. The model equation, derived by the vertical averaging of the two-dimensional thermal conduction equation, is closed by an approximation that the heat exchanger is sufficiently thin in the sense that the second derivative of temperature with respect to the horizontal coordinate depends only on the coordinate. The steady solutions computed by the model are validated by direct two-dimensional computations. The main results are as follows:

(i) The model is derived analytically without any unknown constants to be determined by some two-dimensional computations. It yields rigorous results when the thickness of the
Table 2. Computational time to solve numerically the one-dimensional model equation and the direct two-dimensional equation for various shape functions. $N_x$ and $N_y$ denote the division numbers in the $x$ and $y$ direction, respectively. The time reduction rate by use of the one-dimensional model is also included. In this table, $\varepsilon$ and $H$ are constants at 0.5 and 0.1, respectively.

| $b(x)$         | dimension | $N_x$ | $N_y$ | computational time [s] | reduction rate [%] |
|----------------|-----------|-------|-------|------------------------|--------------------|
| $\cos(2\pi x)$| 1         | 200   | –     | 9.625                  | 99.74              |
|                | 2         | 200   | 40    | 3720                   |                    |
| $\sin(2\pi x)$| 1         | 200   | –     | 9.097                  | 99.87              |
|                | 2         | 200   | 50    | 6887                   |                    |
| $\exp(\sin(2\pi x))$ | 1         | 200   | –     | 15.20                  | 99.77              |
|                | 2         | 200   | 80    | 6571                   |                    |
| $\exp(x) + \sin(2\pi x)$ | 1         | 200   | –     | 11.45                  | 98.62              |
|                | 2         | 200   | 40    | 831.7                  |                    |
| $x^3$          | 1         | 100   | –     | 7.325                  | 95.13              |
|                | 2         | 100   | 20    | 150.5                  |                    |
| $x$            | 1         | 200   | –     | 5.310                  | 99.29              |
|                | 2         | 200   | 40    | 744.3                  |                    |

heat exchanger is uniform, and has desirable asymptotic behavior when the heat transfer rate on the top surface approaches to zero and infinity.

(ii) The model recovers the two-dimensional thermal information, i.e. steady temperature and flux distribution on the top and (temperature-fixed) bottom faces. The relative error of these quantities is less than 5% with the maximum gradient of the height with respect to the horizontal coordinate kept approximately below 0.5.

(iii) In contrast, the computational time by the model can be reduced to 0.1–5% when compared with direct two-dimensional computations, depending on the shape of the top face.

In this model equation, the fluid equation, i.e. the Navier-Stokes equation, above the exchanger is decoupled by a conventional equation for normal heat flux on the top surface. In principle, however, the coupling of the model and the fluid equation is possible through the temperature and heat flux on the top interface, recovered by the model equation. The coupling problem is addressed in Appendix A.

As a result, the model equation can be treated as if it were a boundary condition, enabling us to focus solely on the computations of fluid flow in the scheme of solid-fluid coupled analyses. The type of mathematical modeling can be applicable to a wide variety of bodies with extremely small dimensions in some (coordinate-transformed) directions.

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Appendix A. Coupling of the model equation and the outer fluid equation
In general, the analysis of the heat transfer between different materials or phases needs to solve a heat-flux balance equation on the contact surface. In this case the relation (2) of heat transfer is given implicitly by the balance equation in conjunction with the outer fluid equation, and the model equation described in Sec. 2.2 should be modified. In what follows, we are to derive a one-dimensional model equation without the condition (2).
Actually, the model equation is simplified because the unknowns are confined to three quantities, i.e. $A$, $B$ and $\Theta$. All we have to do is express these variables by $\bar{\theta}$ and $\theta_s$. This can be done by use of three conditions: boundary condition, $\theta = 1$, on the bottom face, two relations between the two quantities, $\bar{\theta}$ and $\theta_s$, and the three variables. The situation is caused by the fact that the heat-flux component $q_x(s)$ in the $x$ direction is given by the fluid side as the function of $\theta_s$. Once the temperature is given, the heat-flux component can be evaluated by the negative product of the thermal conductivity of the fluid and the numerical derivative by use of neighboring fluid temperatures at a time step. Consequently, the contact (continuity) condition of $q_y(s)$ at the surface determines the value of $\theta_s$.

First of all, we should notice that equations (9) and (16) hold because the condition (2) is not used for the derivation. And the former equation is regarded as the one-dimensional model equation for the fluid-to-solid heat transfer.

Equation (13) and the boundary condition on the bottom face lead that $B = 1$. Substituting $H$ for $y$ in equation (13) yields

$$\theta_s = AH + B - \frac{\Theta H^2}{2},$$

(A.1)

and from these relations the above three variables can be expressed by $\bar{\theta}$ and $\theta_s$.

The resultant relation that

$$\theta_s = \frac{3(\bar{\theta} - 1)}{2} - \frac{q_y(s)H}{4} + 1,$$

(A.2)

or equivalently

$$q_y(s) = \frac{6(\bar{\theta} - 1) - 4(\theta_s - 1)}{H},$$

(A.3)

can be utilized to determine the surface temperature $\theta_s$ from the above-mentioned contact condition.

Given the remaining relation that

$$q_y^{(0)} = \frac{6(1 - \bar{\theta}) - 2(1 - \theta_s)}{H},$$

(A.4)

we have a closed equation system constructed by the one-dimensional model equation and the outer fluid equation.

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