COLORING LINKS BY SYMMETRIC GROUP OF ORDER 3

KAZUHIRO ICHIHARA AND ERI MATSUDO

Abstract. We consider the number of colors for the colorings of links by the symmetric group $S_3$ of order 3. For knots, such a coloring corresponds to a Fox 3-coloring, and thus the number of colors must be 1 or 3. However, for links, there are colorings by $S_3$ with 4 or 5 colors. In this paper, we show that if a 2-bridge link admits a coloring by $S_3$ with 5 colors, then the link also admits such a coloring with only 4 colors.

1. Introduction

One of the most well-known invariants of knots and links would be the Fox $n$-coloring for an integer $n \geq 3$, originally introduced by R. Fox in [1]. In this paper, as a generalization of the Fox 3-coloring, we consider the colorings of links by the symmetric group of order 3, which we denote by $S_3$.

Throughout the paper, we set $S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = e, \sigma \tau \sigma = \tau \sigma \tau \rangle$, where $e$ denotes the identity element of $S_3$. Note that $S_3 = \{e, \sigma, \tau, \sigma \tau \sigma, \sigma \tau, \tau \sigma\}$ as a set.

Definition 1. Let $D$ be a diagram of a link. A map $C : \text{arcs on } D \rightarrow S_3 - \{e\}$ is called an $S_3$-coloring on $D$ if it satisfies $C(x)C(y) = C(z)C(x)$ (respectively, $C(x)C(z) = C(y)C(x)$) at a positive (resp. negative) crossing on $D$, where $x$ denotes the over arc, $y$ and $z$ the under arcs at the crossing supposing $y$ is the under arc before passing through the crossing and $z$ is the other. The image $C(a)$ of an arc $a$ on $D$ by an $S_3$-coloring $C$ is said to be a color on $a$ for $C$.

![Crossing conditions for $S_3$-coloring](image)

Note that an $S_3$-coloring on a diagram $D$ of a link $L$ gives a representation of the link group $G_L$ of $L$ to $S_3$, and a representation of $G_L$ to $S_3$ gives an $S_3$-coloring on a diagram $D$ of a link $L$.

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In this paper, for an integer $n$, when an $S_3$-coloring $C$ on a link diagram uses $n$ colors, we call $C$ an $(S_3, n)$-coloring. An $(S_3, 1)$-coloring is defined as a trivial $S_3$-coloring. A link $L$ is said to be $S_3$-colorable (resp. $(S_3, n)$-colorable) if $L$ has a diagram which admits a non-trivial $S_3$-coloring (resp. an $(S_3, n)$-coloring).

For a diagram $D$ of a knot, there is a one-to-one correspondence between a non-trivial Fox 3-coloring on a diagram $D$ and an $(S_3, 3)$-coloring on $D$ (Lemma 2.2). Thus a knot $K$ is $S_3$-colorable if and only if $K$ is Fox 3-colorable. In particular, if a knot is $(S_3, n)$-colorable, then $n = 1$ or 3. See the next section for details.

On the other hand, if a link $L$ has at least 2 components, then $L$ can be $(S_3, n)$-colorable with $n \geq 4$. See Figure 2 for an example.

In the next section, we describe the local behavior of $S_3$-colorings for links in detail. It enables us to show the following.

**Proposition 1.1.** Any $(S_3, 4)$-colorable link is also $(S_3, 5)$-colorable. Precisely, if a link $L$ has a diagram which admits an $S_3$-coloring with 4 colors, then $L$ also has another diagram which admits an $S_3$-coloring with 5 colors.

One can ask if the converse does hold: Is an $(S_3, 5)$-colorable link always $(S_3, 4)$-colorable? It seems to expect too much naively, but there are some results on the Fox coloring related to this question. For example, it is known that if a knot $K$ is Fox 5-colorable, then $K$ has a diagram which admits a Fox 5-coloring with only 4 colors [6]. Also the second author [4] and independently M. Zhang, X. Jin and Q. Deng [7] proved that if a link $L$ is $Z$-colorable, then $L$ has a diagram which admits a $Z$-coloring with only 4 colors.

About the question above, in this paper, we obtain the following for 2-bridge links.

**Theorem 1.1.** Any $(S_3, 5)$-colorable 2-bridge link $L$ is $(S_3, 4)$-colorable. In fact, a 2-bridge link $L$ is $(S_3, n)$-colorable with $n = 4, 5$ if and only if $L$ has a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ satisfying $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$.

A Conway diagram of a 2-bridge link is depicted in Figure 3. See [3] for details.
In the next section, we describe the local behavior of $S_3$-colorings on links preparing lemmas. Then, in Section 3, we give a proof of Theorem 1.1.

By Theorem 1.1, all the $(S_3, 5)$-colorable 2-bridge links are $(S_3, 4)$-colorable. Some of them actually are also $(S_3, 3)$-colorable, but some others are not. In the last section, among 2-bridge links, we determine the double twist links and the torus links that are $(S_3, 4)$-colorable but not $(S_3, 3)$-colorable.

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2. Local behavior of $S_3$-colorings

In this section, we observe the local behavior of $S_3$-colorings on links, and prepare some lemmas used in the remaining sections.

Let $L$ be a link with a diagram $D$. Suppose that $D$ admits a non-trivial $S_3$-coloring $C$. At a crossing of $D$, let $x$ denote the over arc, $y$ and $z$ the under arcs at the crossing supposing $y$ is the under arc before passing through the crossing and $z$ is the other. See Figure 1. Then the possible colors of the arcs $x, y, z$ assigned by $C$ can be summarized in the following table.

| Color on $x$ | $\sigma$ | $\tau$ | $\sigma\tau\sigma$ | $\sigma\tau$ | $\tau\sigma$ |
|--------------|---------|--------|---------------------|--------------|--------------|
| $\sigma$     | $\sigma$ | $\sigma\tau\sigma$ | $\tau$     | $\tau\sigma$ | $\sigma\tau$ |
| $\tau$       | $\sigma\tau\sigma$ | $\tau$ | $\sigma$ | $\tau\sigma$ | $\sigma\tau$ |
| $\sigma\tau\sigma$ | $\tau$ | $\sigma$ | $\tau\sigma$ | $\sigma\tau$ |
| $\sigma\tau$ | $\sigma\tau\sigma$ | $\tau$ | $\sigma$ | $\tau\sigma$ | $\sigma\tau$ |
| $\tau\sigma$ | $\tau$ | $\sigma\tau\sigma$ | $\sigma$ | $\tau\sigma$ | $\sigma\tau$ |

In the table above, $\alpha/\beta$ means that the color on $y$ is $\alpha$ (resp. $\beta$) if the crossing is positive (resp. negative).

Remark 1. Based on Table 1 in a similar way to the case for the Fox coloring [5], we obtain a graph for $S_3$-colorings of links as follows. The vertices of the graph are the elements of $S_3 - \{e\}$, that is, $\{\sigma, \tau, \sigma\tau\sigma, \sigma\tau, \tau\sigma\}$. A pair of vertices $\alpha$ and $\beta$ are connected by an oriented edge with label $\gamma$ if there exists a link diagram $D$ and an $S_3$-coloring on $D$ such that, at a crossing on $D$, the over arc is colored by $\gamma$ and the under arcs are colored by $\alpha, \beta$ by that coloring. The graph so obtained is called the palette graph of $S_3$-colorings for links. Actually, based on Table 1 we can illustrate the palette graph of $S_3$-colorings as in Figure 4.
Remark 2. Also, from Table 1 we see that any link with at least 2 components admits an $S_3$-coloring with 2 colors $\{\sigma, \tau\}$. See Figure 5.

The next is a fundamental lemma we will use repeatedly, which follows from Table 1.

**Lemma 2.1.** For any $S_3$-coloring on a diagram $D$ of a link $L$, the arcs of $D$ corresponding to the same component of $L$ are colored by $C$ with the colors $\{\sigma, \tau, \sigma \tau \sigma\}$ or the colors $\{\sigma \tau, \tau \sigma\}$. This property is unchanged by performing Reidemeister moves.

**Proof.** From Table 1 if one of the under arcs at a crossing of a link diagram is colored by one of $\{\sigma, \tau, \sigma \tau \sigma\}$ or one of $\{\sigma \tau, \tau \sigma\}$ by an $S_3$-coloring, then the other is also. Thus the first statement holds. One can check the local behavior of $S_3$-colorings by Reidemeister moves to keep the set of colors on the related arcs. This implies the second statement. \( \square \)

For $(S_3,3)$-colorings on knots and links, we have the following.

**Lemma 2.2.** For a non-splitable $(S_3,3)$-colorable link $L$, the set of colors for an $(S_3,3)$-coloring on a diagram of $L$ is $\{\sigma, \tau, \sigma \tau \sigma\}$. For a knot $K$, there is a one-to-one correspondence between a Fox 3-coloring on a diagram $D$ and an $S_3$-coloring on $D$ of $K$. Thus a knot $K$ is $S_3$-colorable if and only if $K$ is Fox 3-colorable. In particular, if a knot is $(S_3,n)$-colorable, then $n = 1$ or 3.

**Proof.** Suppose that a diagram of a link admits an $(S_3,3)$-coloring $C$. From Lemma 2.1 the set of colors on each component of the link are either of $\{\sigma, \tau, \sigma \tau \sigma\}$ or $\{\sigma \tau, \tau \sigma\}$.

Suppose that $C$ uses exactly 3 colors, say $\alpha, \beta, \gamma$. If $C$ uses $\alpha \in \{\sigma, \tau, \sigma \tau \sigma\}$ and $\beta, \gamma \in \{\sigma \tau, \tau \sigma\}$, then, by Table 1 the arc colored by $\alpha$ is constantly an over arc, or
an under arc at the crossing with the over arc colored by \( \alpha \). Thus the component with an arc colored by \( \alpha \) is splittable, implying that \( L \) is splittable. Similarly, the same is shown for the case \( \alpha \in \{ \sigma, \tau, \sigma \tau \sigma \} \) and \( \beta, \gamma \in \{ \sigma, \tau, \sigma \tau \sigma \} \). Thus, if \( L \) is non-splittable, the set of 3 colors for \( C \) is \( \{ \sigma, \tau, \sigma \tau \sigma \} \).

Suppose that a diagram of a knot \( K \) admits an \( S_3 \)-coloring. From the above, the set of colors for the coloring is \( \{ \sigma, \tau, \sigma \tau \sigma \} \). Then, by replacing the colors \( \{ \sigma, \tau, \sigma \tau \sigma \} \) to the colors \( \{ 0, 1, 2 \} \), an \( S_3 \)-coloring on a knot diagram is transformed to a Fox 3-coloring on the diagram. Conversely, one can obtain an \( S_3 \)-coloring from a Fox 3-coloring on a knot diagram by setting the colors \( \{ 0, 1, 2 \} \) to the colors \( \{ \sigma, \tau, \sigma \tau \sigma \} \). See Table 1.

\[ \square \]

Remark 3. For splittable links, the lemma above does not hold. See Figure 6 for example.

![Figure 6. An \((S_3,3)\)-coloring on a splittable link with colors \( \{ \sigma, \sigma \tau, \tau \sigma \} \)](image)

Also from Table 1 together with Lemmas 2.1 and 2.2 we have the following.

**Lemma 2.3.** Let \( L \) be a non-splittable link and \( D \) a diagram of \( L \). (i) If \( D \) admits an \((S_3,4)\)-coloring, then the set of colors of the coloring contains 2 colors from \( \{ \sigma, \tau, \sigma \tau \sigma \} \) and 2 colors from \( \{ \sigma \tau, \tau \sigma \} \). (ii) If \( D \) admits an \((S_3,4)\)-coloring or an \((S_3,5)\)-coloring, then any \( S_3 \)-coloring on a diagram of \( L \) obtained by Reidemeister moves from \( D \) with the coloring has at least 4 colors.

**Proof.** (i) Suppose that \( D \) admits an \( S_3 \)-coloring with four colors \( \alpha, \beta, \gamma, \delta \). If \( \alpha, \beta, \gamma \in \{ \sigma, \tau, \sigma \tau \sigma \} \) and \( \delta \in \{ \sigma \tau, \tau \sigma \} \), then, by Table 1, the arc colored by \( \delta \) is constantly an over arc, or an under arc at the crossing with the over arc colored by \( \delta \). This means that the component is splittable, and it contradicts that \( L \) is non-splittable. Therefore any \((S_3,4)\)-coloring on \( D \) uses 2 colors from \( \{ \sigma, \tau, \sigma \tau \sigma \} \) and 2 colors from \( \{ \sigma \tau, \tau \sigma \} \).

(ii) Suppose that \( D \) admits an \((S_3,4)\)-coloring or an \((S_3,5)\)-coloring. Let \( C \) be the \( S_3 \)-coloring on a diagram of \( L \) obtained by Reidemeister moves from \( D \) with the given coloring. By (i) above and Lemma 2.1, \( L \) has at least two components, one of which is colored by \( \{ \sigma, \tau, \sigma \tau \sigma \} \), and the other is by \( \{ \sigma \tau, \tau \sigma \} \). Also, by Lemma 2.1 such sets of colors on the components are unchanged by Reidemeister moves. It follows that \( C \) has at least one color in \( \{ \sigma, \tau, \sigma \tau \sigma \} \) and one color in \( \{ \sigma \tau, \tau \sigma \} \). Moreover, since \( L \) is non-splittable, there exists at least one crossing where the pair of the colors above appear. Then, by Table 1 there has to be one more color at the crossing. Thus \( C \) uses at least 3 colors. However, by Lemma 2.2, together with above, the coloring \( C \) is not an \((S_3,3)\)-coloring. Therefore, if \( D \) admits an \((S_3,4)\)-coloring or an \((S_3,5)\)-coloring, then any \( S_3 \)-coloring on a diagram of \( L \) obtained by Reidemeister moves from \( D \) with the coloring has at least 4 colors. \[ \square \]

Now we give a proof of Proposition 1.1.
Proof of Proposition 1.1. Let $L$ be an $(S_3, 4)$-colorable link and $D$ a diagram of $L$ with an $(S_3, 4)$-coloring $C$.

If $L$ is non-splittable, then there exist 2 colors in $\{\sigma, \tau, \sigma\tau\sigma\}$ and 2 colors in $\{\sigma\tau, \tau\sigma\}$ on $D$ from Lemma 2.3 (i). Let $\alpha \in \{\sigma, \tau, \sigma\tau\sigma\}$ be the color which $C$ does not use. Consider an arc on $D$ colored by $\beta, \gamma \in \{\sigma, \tau, \sigma\tau\sigma\}$ with $\beta, \gamma \neq \alpha$. Then one can deform $D$ and $C$ to a diagram with a coloring so that $\alpha$ appears by using Reidemeister move II repeatedly, as illustrated in Figure 7.

![Figure 7. Making $\tau$ appear from $\{\sigma, \sigma\tau, \sigma\tau\}$](image)

When $L$ is splittable, we also have to consider the case such that there exists 3 colors in $\{\sigma, \tau, \sigma\tau\sigma\}$ and 1 color in $\{\sigma\tau, \tau\sigma\}$ on $D$. In this case, let $\alpha \in \{\sigma\tau, \tau\sigma\}$ be a color which $C$ does not use. On the other hand, $D$ contains an arc colored by $\beta \in \{\sigma\tau, \tau\sigma\}$ with $\beta \neq \alpha$. Then one can deform $D$ with the coloring to a diagram with a coloring such that $\alpha$ appears by using Reidemeister move II repeatedly, as illustrated in Figure 8.

![Figure 8. Making $\tau\sigma$ appear from $\{\sigma, \tau, \sigma\tau, \sigma\tau\}$](image)

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. Recall that it is known that a 2-bridge link always has a Conway diagram $C(2a_1, 2b_1, \ldots, 2b_m, 2a_{m+1})$ depicted in Figure 3. See [3] for example.

Proof of Theorem 1.1. We first show that a Conway diagram $C(2a_1, 2b_1, \ldots, 2b_m, 2a_{m+1})$ admits an $(S_3, 4)$-coloring if and only if $\Sigma_{i=1}^{m+1}|a_i| \equiv 0 \pmod{2}$ holds. Then, by Proposition 1.1, the diagram represents a 2-bridge link which is $(S_3, 5)$-colorable. In the following, under the assumption, we will construct an $(S_3, 4)$-coloring on the diagram from the left end.

We fix colors on arcs $x, y$ in Figure 3 as $\sigma$ and $\sigma\tau$ respectively, and make a coloring by setting the colors on the arcs next to the right, repeatedly, by using...
Table 1. Then, on the arcs in the twist with $2b_i$ crossings, the only two colors $\sigma\tau, \tau\sigma$ appear. Moreover, the colors on the parallel arcs before and after the twisting are the same. See Figure 9.

![Figure 9. colors in the twist with 2b_i crossings](image)

Next we see the colors in the twist with $2a_i$ crossings. Since $2a_i$ is even, pairs of colors at before and after $2a_i$ crossings are the same or another color pair. Precisely, if $a_i$ is even, the pairs of colors before and after $2a_i$ crossings are coincide. If $a_i$ is odd, the pairs of colors before and after $2a_i$ crossings are distinct, but in a fixed pattern. For example, if a pair of colors $\{\sigma, \sigma\tau\}$ appears before the twist, then the pairs of colors on the parallel arcs during the twist are $\{\sigma, \sigma\tau\}$ or $\{\tau, \tau\sigma\}$ alternately as illustrated in Figure 10. In particular, during the twists, only 4 colors can appear.

![Figure 10. colors in the twist with 2a_i crossings](image)

This implies that it is necessary that $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$ holds to create an $S_3$-coloring on whole the Conway diagram. Conversely we can obtain an $S_3$-coloring on the Conway diagram if $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$ holds in such a way. By the construction, the coloring uses only 4 colors.

We next show that if a 2-bridge link $L$ is $(S_3, n)$-colorable with $n \geq 4$, then $L$ has a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ satisfying $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$.

For a given 2-bridge link $L$, we deform a diagram $D$ of $L$ with an $(S_3, n)$-coloring ($n \geq 4$) to a Conway diagram $D_C$ with the induced $S_3$-coloring $C$. By Lemma 2.3 (i) and (ii), the coloring uses at least 2 colors from $\{\sigma, \tau, \sigma\tau\}$ and 2 colors from $\{\sigma\tau, \tau\sigma\}$. Moreover, by Lemma 2.1, the arcs contained in one component are all colored by either of $\{\sigma, \tau, \sigma\tau\}$ or $\{\sigma\tau, \tau\sigma\}$. This implies that the $S_3$-coloring $C$ on the Conway diagram looks same locally as that previously constructed.

When $C(x) \in \{\sigma, \tau, \sigma\tau\}$ and $C(y) \in \{\sigma\tau, \tau\sigma\}$ for the arcs $x, y$ of $D_C$, then, by retaking the colors if necessary, the coloring is completely the same as that previously constructed. That is, $C$ is an $(S_3, 4)$-coloring on the diagram, and $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$ must hold.

Consider the case that $C(x) \in \{\sigma\tau, \tau\sigma\}$ and $C(y) \in \{\sigma, \tau, \sigma\tau\}$. Then one can deform the diagram and the coloring so that $C(x) \in \{\sigma, \tau, \sigma\tau\}$ by Reidemeister moves (and rotations) as illustrated in Figure 11.
After such modifications, in the same way as above, the condition $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$ have to be satisfied, otherwise the colors on the twists of $2a_i$ crossings must conflict. It concludes that if a 2-bridge link $L$ is $(S_3, n)$-colorable with $n = 4, 5$, then $L$ has a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ satisfying $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$.

In particular, if a 2-bridge link $L$ is $(S_3, 5)$-colorable, then $L$ has a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ satisfying $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod{2}$, and so, the diagram admits, an $(S_3, 4)$-coloring, i.e., the link $L$ is $(S_3, 4)$-colorable. This completes the proof of Theorem 1.1. □

4. Examples

From Theorem 1.1 any $(S_3, 5)$-colorable 2-bridge link is $(S_3, 4)$-colorable. However there exists some links which is $(S_3, 4)$-colorable, but is not $(S_3, 3)$-colorable. In this section, we collect some examples of $S_3$-colorings for 2-bridge links, and determine which double twist links enjoys that property.

One of the simplest 2-bridge links would be 2-bridge torus links, that are the torus links with only two strands.

**Example 1** (The torus link $T(2, q)$). By Theorem 1.1 the torus link $T(2, q)$ is $(S_3, 4)$-colorable if and only if $q \equiv 0 \pmod{4}$. The next figure depicts a torus link with $(S_3, 4)$-coloring which is not $(S_3, 3)$-colorable.

![Figure 12. Torus link $T(2, 4)$ with an $(S_3, 4)$-coloring](image)

In fact, by using Table 1 one can see that the standard torus diagram of $T(2, q)$ (Figure 12) is $(S_3, 4)$-colorable if and only $q \equiv 0 \pmod{4}$ and $T(2, q)$ is $(S_3, 3)$-colorable if and only if $q \equiv 0 \pmod{3}$. See Figures 10 and 13.
Also, by [2], the determinant of $T(2, q)$ is $q$, and so, $T(2, q)$ is Fox 3-colorable, equivalently, is $(S_3, 3)$-colorable if and only if $q \not\equiv 0 \pmod{3}$.

For example, the torus link $T(2, 12)$ is $(S_3, n)$-colorable for $n = 3, 4, 5$.

Next, we consider double-twist links, which are the links admitting the diagrams shown in Figure 14.
Figure 15. A diagram of a double twist link $J(k,l)$

An example of the double twist link with $(S_3,4)$-coloring which is not $(S_3,3)$-colorable is depicted in Figure 16.

Figure 16. Double twist link $J(3,5)$ with an $(S_3,4)$-coloring

Actually, for double twist links, we have the following.

**Proposition 4.1.** A double twist link $J(k,l)$ depicted in Figure 15 is $(S_3,4)$-colorable if and only if $kl \equiv 3 \pmod{4}$, and is $(S_3,3)$-colorable if and only if $kl \equiv 2 \pmod{3}$.

**Proof.** To see which $J(k,l)$ is $(S_3,4)$-colorable, we need to consider Conway diagrams to apply Theorem 1.1, but here, we directly consider the diagram $D$ of $J(k,l)$ shown in Figure 15.

First we show that $D$ is $(S_3,4)$-colorable if $kl \equiv 3 \pmod{4}$. We set colors $C(x), C(y)$ of arcs $x,y$ on Figure 15 as $C(x) = \sigma, C(y) = \sigma\tau$. Then the pair of colors $(C(z), C(w))$ on arcs $(z,w)$ is fixed as $(\tau\sigma, \sigma)$ with $k \equiv 1 \pmod{4}$, or $(\sigma\tau, \tau)$ with $k \equiv 3 \pmod{4}$ to make a coloring on $D$ by Table 1. For the case of $k \equiv 1 \pmod{4}$, $l \equiv 3 \pmod{4}$ also holds, and so $D$ is $S_3$-colorable as $(C(x), C(y), C(z), C(w)) = (\sigma, \sigma\tau, \tau\sigma, \sigma)$. See Figure 17. Note that $\sigma\tau\sigma$ does not appear during the twists, that is, the coloring is an $(S_3,4)$-coloring.
In the same way, in the case of $k \equiv 3 \pmod{4}$, $D$ is shown to be $(S_3, 4)$-colorable.

Conversely, suppose that $J(k, l)$ is $(S_3, 4)$-colorable. In the same argument as the proof of Theorem 1.1, the diagram $D$ of $J(k, l)$ admits a $S_3$-coloring such that the arcs contained in one component are all colored by either of $\{\sigma, \tau, \sigma \tau \sigma\}$ or $\{\sigma \tau, \tau \sigma\}$. Then, as above, by seeing the colors on the arcs from the left end, one can check that the condition $kl \equiv 3 \pmod{4}$ is necessary.

For $(S_3, 3)$-colorability, again, by [2], the determinant of $J(k, l)$ is shown to be $1 + kl$, and so, $J(k, l)$ is Fox 3-colorable, equivalently, is $(S_3, 3)$-colorable if and only if $kl \equiv 2 \pmod{3}$. \hfill \qed

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