We briefly report on the contribution of QCD-instantons to the phenomenon of saturation in deep-inelastic scattering (DIS) at small Bjorken-\(x\). The explicitly known instanton gauge field serves as a concrete realization of an underlying non-perturbative saturation mechanism associated with strong classical fields. Two independent strategies were pursued, with consistent results: On the one hand, an approach starting from instanton-perturbation theory was considered and on the other hand, the scattering of a Wilson loop in an instanton gauge field background.

In both cases, the conspicuous, intrinsic instanton size scale \(\langle \rho \rangle \approx 0.5\) fm, as known from the lattice, turns out to determine the saturation scale.

1. Setting the stage

\(eP\)-scattering at small Bjorken-\(x\) uncovers a novel regime of QCD, where the coupling \(\alpha_s\) is small, but the parton densities are so large that conventional perturbation theory ceases to be applicable. While there is experimental evidence from HERA for a strong growth of the gluon density at small \(x_{\text{Bj}}\), a mechanism is needed that eventually leads to “saturation”, i.e. to a limitation of the maximum gluon density per unit of phase space. Much interest has recently been generated through association of the saturation phenomenon with a multiparticle quantum state of high occupation numbers, the “Colour Glass Condensate” that correspondingly, can be viewed as a strong classical colour field.

In this paper, we briefly report on the promising rôle of QCD-instantons \(^2\) \((I)\) in this context. With the help of crucial information from lattice simulations \(^3\), \(^4\), we consider a background instanton as an explicitly known, truly non-perturbative classical gauge field \((\times 1/g_s)\) in the context of saturation at small \(x_{\text{Bj}}\). A crucial aspect concerns the \(I\)-size \(\rho\). On the one hand it is just a collective coordinate to be integrated over

\(^2\)Talk presented at the Workshop on Strong and Electroweak Matter (SEWM 2002), October 2-5, 2002, Heidelberg, Germany.
in any observable, with the \( I \)-size distribution \( D(\rho) = \frac{dn_I}{d^4zd\rho} \) as universal weight. On the other hand, according to lattice data, \( D(\rho) \) turns out to be sharply peaked (Fig. 1 (left)) around \( \langle \rho \rangle \approx 0.5 \) fm. Hence instantons represent truly non-perturbative gluons that bring in naturally an intrinsic size scale \( \langle \rho \rangle \) of hadronic dimension. As we shall see, \( \langle \rho \rangle \) actually determines the saturation scale \(^{6,7} \). Presumably, it is also reflected in the conspicuous geometrization of soft QCD at high energies \(^{8,6,7} \). For related approaches associating instantons with high-energy (diffractive) scattering, see Refs. \(^{9,10,11,12} \).

We know already from \( I \)-perturbation theory that the instanton contribution tends to strongly increase towards the soft regime \(^{13,14,15} \). The mechanism for the decreasing instanton suppression with increasing energy is known since a long time \(^{16,12} \): Feeding increasing energy into the scattering process makes the picture shift from one of tunneling between vacua \( (E \approx 0) \) to that of the actual creation of the sphaleron-like, coherent multi-gluon configuration \(^{17} \) on top of the potential barrier of height \(^{13} \) \( E = M_{\text{sph}} \propto \frac{1}{\alpha_s \rho_{\text{eff}}} \). In Ref. \(^{7} \), we have already argued by means of lattice results that the QCD - “sphaleron” is playing an essential rôle in building up the “Colour Glass Condensate”.

![Figure 1](image-url)

**Figure 1.** (Left) UKQCD lattice data\(^{3,4,5} \) of the \((I + \bar{I})\)-size distribution for quenched QCD \((n_f = 0)\). Both the sharply defined \( I \)-size scale \( \langle \rho \rangle \approx 0.5 \) fm and the parameter-free agreement with \( I \)-perturbation theory\(^{4,5} \) (solid line) for \( \rho \leq 0.35 \) fm are apparent. (Right) the simplest \( I \)-induced process \(^{20} \) transcribed into the colour dipole framework.
The colour dipole picture \(^{18}\) represents an intuitive framework for investigating saturation effects at small \(x_{\text{Bj}}\). In the proton’s rest frame, the virtual photon fluctuates mainly into a \(q\bar{q}\)-dipole a long distance upstream of the target proton. The large difference of the \(\gamma^* \rightarrow \bar{q}q\)-dipole formation and \((q\bar{q})P\)-interaction times at small \(x_{\text{Bj}}\) generically give rise to the familiar factorized expression of the inclusive photon-proton cross sections,

\[
\sigma_{L,T}(x_{\text{Bj}}, Q^2) = \int_0^1 dz \int d^2r |\Psi_{L,T}(z, r)|^2 \sigma_D(r, \ldots).
\]

Here, \(|\Psi_{L,T}(z, r)|^2\) denotes the modulus squared of the (light-cone) wave function of the virtual photon, calculable in pQCD, and \(\sigma_D(r, \ldots)\) is the \(q\bar{q}\)-dipole-nucleon cross section. The variables in Eq. (1) are the transverse \((q\bar{q})\)-size \(r\) and the photon’s longitudinal momentum fraction \(z\) carried by the quark. The dipole cross section is expected to include in general the main non-perturbative contributions. For small \(r\), one finds within pQCD \(^{18,19}\) that \(\sigma_D\) vanishes with the area \(\pi r^2\) of the \(q\bar{q}\)-dipole. Besides this phenomenon of “colour transparency” for small \(r\), the dipole cross section is expected to saturate towards a constant, once the \(q\bar{q}\)-separation \(r\) exceeds a certain saturation scale \(r_s\).

2. Instantons and Saturation

2.1. Starting from Instanton Perturbation Theory

Next, let us briefly illustrate that in the presence of a background instanton, the saturation scale \(r_s\) indeed equals the intrinsic \(I\)-size scale \(\langle \rho \rangle\). For reasons of space, we restrict the discussion to the simplest \(I\)-induced process \(^{20}\), \(\gamma^* g \Rightarrow qR\bar{R}\), with one flavour and no final-state gluons (Fig. 1 (right)). More details and the realistic case with gluons and three light flavours, using the \(I\)-valley approach, may be found in Ref. \(^7\). The strategy is to transform the known results on \(I\)-induced processes in DIS into the colour dipole picture. By exploiting the lattice results on the \(I\)-size distribution (Fig. 1 (left)), we then carefully increase the \(q\bar{q}\)-dipole size \(r\) towards hadronic dimensions. Let us start by recalling the results from Ref. \(^{20}\),

\[
\sigma_{L,T}(x_{\text{Bj}}, Q^2) = \int_{x_{\text{Bj}}}^1 \frac{dx}{x} \left(\frac{x_{\text{Bj}}}{x}\right) G \left(\frac{x_{\text{Bj}}}{x}, \mu^2\right) \int dt \frac{d\tilde{\sigma}_{L,T}^g(x, t, Q^2)}{dt};
\]

\[
\frac{d\tilde{\sigma}_{L,T}^g}{dt} = \frac{\pi^7}{2} \frac{e_q^2}{Q^2} \alpha_{\text{em}} \alpha_s \left[x(1 - x)\sqrt{tu} \frac{R(\sqrt{-t}) - R(Q)}{t + Q^2} - (t \leftrightarrow u)\right]^2.
\]
and a similar expression for \( d\tilde{\sigma}^2_{T} g / dt \).

Eqs. (2, 3) involve the master integral \( R(Q) \) with dimensions of a length,

\[
R(Q) = \int_{0}^{\infty} d\rho \, D(\rho) \rho^5 (Q\rho) K_1(Q\rho).
\]  

(4)

By means of an appropriate change of variables and a subsequent 2d-Fourier transformation, Eqs. (2, 3) may indeed be cast into a colour dipole form (1), e.g. (with \( \hat{Q} = \sqrt{z(1-z)} Q \))

\[
\left| \Psi_L \right|^2 \sigma_D^{(I)} \approx \left| \Psi_{L,T}^{PQCD} (z, r) \right|^2 \frac{1}{\alpha_s} x_{Bj} G(x_{Bj}, \mu^2) \frac{\pi^8}{12} \times \left\{ \int_{0}^{\infty} d\rho D(\rho) \rho^5 \left( \frac{-d}{dr^2} \left( \frac{2r^2 K_1(Q\sqrt{r^2 + \rho^2/z})}{Q\sqrt{r^2 + \rho^2/z}} \right) - \frac{1}{z \leftrightarrow 1-z} \right) \right\}^2.
\]  

(5)

The strong peaking of \( D_{\text{lattice}}(\rho) \) around \( \rho \approx \langle \rho \rangle \), implies

\[
\left| \Psi_{L,T} \right|^2 \sigma_D^{(I)} \Rightarrow \begin{cases} 
O(1) \text{ but exponentially small; } & \text{if } r \to 0, \\
\left| \Psi_{L,T}^{PQCD} \right|^2 \frac{1}{\alpha_s} x_{Bj} G(x_{Bj}, \mu^2) \frac{\pi^8}{12} R(0)^2; & \frac{r}{\langle \rho \rangle} \gtrsim 1.
\end{cases}
\]  

(6)

Hence, the association of the intrinsic instanton scale \( \langle \rho \rangle \) with the saturation scale \( r_s \) becomes apparent from Eqs. (5, 6): \( \sigma_D^{(I)} (r, \ldots) \) rises strongly as function of \( r \) around \( r_s \approx \langle \rho \rangle \), and indeed saturates for \( r/\langle \rho \rangle > 1 \) towards a constant geometrical limit, proportional to the area \( \pi R(0)^2 = \pi \left( \int_{0}^{\infty} d\rho D_{\text{lattice}}(\rho) \rho^5 \right)^2 \), subtended by the instanton. Since \( R(0) \) would be divergent within I-perturbation theory, the information about \( D(\rho) \) from the lattice (Fig. 1 (left)) is crucial for the finiteness of the result.

### 2.2. Wilson–Loop Scattering in an Instanton Background

Complementary to the above strategy of extending the known results of I-perturbation theory by means of lattice information into the saturation regime, we have followed a second promising route. As a simple semiclassical eikonal estimate of the total \( q\bar{q} \)-dipole cross section, the scattering of a pair of infinitely long Wilson lines \( (q\bar{q} \text{-dipole}) \) on the proton in an instanton background field was evaluated along the lines of Refs. 22,10,11,

\[
\sigma_D^{(I)} (r) = \int d^2 b \, \text{Im} T(b, r) = \int d^2 b \frac{2}{N_c} \text{tr} \left\{ \langle \text{Re}\{(1 - S^{(I)}(b, r)) T_r \} \rangle \right\}_{G^{(I)}}
\]  

(7)
The functional integration over the $I$-background field in Eq. (7) reduces to integrations over the $I$-collective coordinates,

$$\langle \cdots \rangle_{G^{(I)}} : \mathcal{D}G^a_{\mu} \Rightarrow d^4 z \, d\rho \, D_{\text{lattice}}(\rho) \, dU,$$

(8)

with $z_\mu$ denoting the $I$-position 4-vector and $U$ the $I$-colour orientation.

The Wilson loop in Eq. (7), associated with the $q\bar{q}$-dipole,

$$S^{(I)}(b, r) = W^+(b) \, W(b+r),$$

(9)

$$W(x_\perp) = P \exp \left\{ i g_s \int_{-\infty}^{\infty} d\tau \, v \cdot G^{(I)}(\tau \, v + x_\perp) \right\}; \quad v_\mu = q_\mu/Q, \quad x_\perp \cdot v = 0,$$

(10)

turns out to only involve an integration over the $z_+$ light cone component of the $I$-position, while the remaining $dz \, d^2 z$ integrations only act on the factor $T_P(\ldots)$, associated with the proton structure. Again, the result can be obtained in analytic form with similar features as in our previously discussed approach based on $I$-perturbation theory. The predicted ratio

$$\frac{\sigma_D^{(I)}(r)}{\sigma_D^{(I)}(\infty)}$$

displayed in Fig. 2 (left) as function of $r/\langle \rho \rangle$, illustrates the importance of $\langle \rho \rangle$ in the approach to saturation. In Fig. 2 (right), the corresponding impact parameter profile for $r = \langle \rho \rangle$, $\infty$ is shown. The important issue of understanding the $x_{Bj}$-dependence of the saturation scale within the present $I$-approach, is currently under active investigation \(^{21}\).

\[\text{Figure 2. (Left) approach to saturation around } r \gtrsim \langle \rho \rangle \text{ and (right) central } b \text{-profile}\]
Acknowledgements

We thank Igor Cherednikov and Michael Lublinsky for interesting discussions.

References

1. E. Iancu, A. Leonidov and L. D. McLerran, Nucl. Phys. A692, 583 (2001).
2. A. Belavin et al., Phys. Lett. B59, 85 (1975).
3. D.A. Smith and M.J. Teper (UKQCD), Phys. Rev. D58, 014505 (1998).
4. A. Ringwald and F. Schrempp, Phys. Lett. B459, 249 (1999).
5. A. Ringwald and F. Schrempp, Phys. Lett. B503, 331 (2001).
6. F. Schrempp and A. Utermann, Acta Phys. Polon. B33, 3633 (2002).
7. F. Schrempp and A. Utermann, Phys. Lett. B543, 197 (2002).
8. F. Schrempp, J. Phys. G28, 915 (2002).
9. D.E. Kharzeev, Y.V. Kovchegov and E. Levin, Nucl. Phys. A690, 621 (2001).
10. E. Shuryak and I. Zahed, Phys. Rev. D62, 085014 (2000).
11. M. A. Nowak, E. V. Shuryak and I. Zahed, Phys. Rev. D64, 034008 (2001).
12. D. M. Ostrovsky et al., Phys. Rev. D66, 036004 (2002).
13. A. Ringwald and F. Schrempp, Proc. Quarks ‘94, ed. D. Yu. Grigoriev et al. (Singapore: World Scientific) p 170, [arXiv:hep-ph/9411217].
14. A. Ringwald and F. Schrempp, Phys. Lett. B438, 217 (1998).
15. A. Ringwald and F. Schrempp, Comput. Phys. Commun. 132, 267 (2000).
16. H. Aoyama and H. Goldberg, Phys. Lett. B188, 506 (1987); A. Ringwald, Nucl. Phys. B330, 1 (1990); O. Espinosa, Nucl. Phys. B343, 310 (1990).
17. F. R. Klinkhamer and N. S. Manton, Phys. Rev. D30, 2212 (1984).
18. N. Nikolaev and B.G. Zakharov, Z. Phys. C49, 607 (1990); Z. Phys. C53, 331 (1992); A.H. Mueller, Nucl. Phys. B 415, 373 (1994).
19. F. E. Low, Phys. Rev. D12, 163 (1975);
L. Frankfurt, G.A. Miller and M. Strikman, Phys. Lett. B304, 1 (1993).
20. S. Moch, A. Ringwald and F. Schrempp, Nucl. Phys. B507, 134 (1997).
21. F. Schrempp and A. Utermann, in preparation.
22. A. Krämer and H. G. Dosch, Phys. Lett. B252, 669 (1990);
O. Nachtmann, Annals Phys. 209, 436 (1991);
W. Buchmüller and A. Hebecker, Nucl. Phys. B476, 203 (1996);
A. I. Shoshi, F. D. Steffen and H. J. Pirner, Nucl. Phys. A709, 131 (2002).