1-Jet Riemann–Finsler Geometry for the Three-Dimensional Time

Gheorghe Atanasiu and Mircea Neagu

February 2010; Revised September 2010
(correction upon the canonical nonlinear connection)

Abstract

The aim of this paper is to develop on the 1-jet space \( J^1(\mathbb{R}, M^3) \) the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions and d-curvatures) of the rheonomic Berwald-Moór metric of order three

\[
\hat{F}(t, y) = \sqrt{h^{11}(t)} \cdot \sqrt{y_1^1 y_2^2 y_3^3}.
\]

Some natural geometrical field theories (gravitational and electromagnetic) produced by the above rheonomic Berwald-Moór metric are also exposed.

Mathematics Subject Classification (2000): 53C60, 53C80, 83C22.

Key words and phrases: rheonomic Berwald-Moór metric of order three, canonical nonlinear connection, Cartan canonical connection, d-torsions and d-curvatures, geometrical Einstein equations.

1 Introduction

It is a well known fact that, in order to create the Relativity Theory, Einstein was forced to use the Riemannian geometry instead of the classical Euclidean geometry, the first one representing the natural mathematical model for the local isotropic space-time. But, there are recent studies of physicists which suggest a non-isotropic perspective of the space-time. For example, in Pavlov’s opinion [14], the concept of inertial body mass emphasizes the necessity of study of local non-isotropic spaces. Obviously, for the study of non-isotropic physical phenomena, the Finsler geometry is very useful as mathematical framework.

The studies of Russian scholars (Asanov [1], Garas’ko [4] and Pavlov [5], [13], [14]) emphasize the importance of the Finsler geometry which is characterized by the total equality of all non-isotropic directions. For such a reason, Asanov, Pavlov and their co-workers underline the important role played in the theory of space-time structure and gravitation, as well as in unified gauge field theories, by the Berwald-Moór metric (whose certain Finsler geometrical properties are
studied by Matsumoto and Shimada in the works [6], [7], [15]

\[ F : TM \to \mathbb{R}, \quad F(y) = (y_1^1 y_2^1 ... y_n^1)^{1/n}. \]

Because any of such directions can be related to the proper time of an inertial reference frame, Pavlov considers that it is appropriate as such spaces to be generically called "multi-dimensional time" [14]. In the framework of the 3- and 4-dimensional linear space with Berwald-Moór metric (i.e. the three- and four-dimensional time), Pavlov and his co-workers [5], [13], [14] offer some new physical approaches and geometrical interpretations such as:

1. physical events = points in the multi-dimensional time;
2. straight lines = shortest curves;
3. intervals = distances between the points along of a straight line;
4. light pyramids ⇔ light cones in a pseudo-Euclidian space;
5. simultaneous surfaces = the surfaces of simultaneous physical events.

According to Olver’s opinion [12], we appreciate that the 1-jet fibre bundle is a basic object in the study of classical and quantum field theories. For such geometrical and physical reasons, this paper is devoted to the development on the 1-jet space \( J^1(\mathbb{R}, M^3) \) of the Finsler-like geometry (together with a theoretical-geometric gravitational and electromagnetic field theory) of the rheonomic Berwald-Moór metric

\[ \hat{F} : J^1(\mathbb{R}, M^3) \to \mathbb{R}, \quad \hat{F}(t, y) = \sqrt{h_{11}(t)} \cdot \sqrt[3]{y_1^1 y_2^1 y_3^1}, \]

where \( h_{11}(t) \) is a Riemannian metric on \( \mathbb{R} \) and \((t, x^1, x^2, x^3, y_1^1, y_2^1, y_3^1)\) are the coordinates of the 1-jet space \( J^1(\mathbb{R}, M^3) \).

The geometry (in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by an arbitrary jet rheonomic Lagrangian function \( L : J^1(\mathbb{R}, M^n) \to \mathbb{R} \) is now completely done in the second author’s paper [11]. We point out that the geometrical ideas from [11] are similar, but however distinct ones, with those exposed by Miron and Anastasiei in the classical Lagrangian geometry [8]. In fact, the geometrical ideas from [11] (which we called the jet geometrical theory of the relativistic rheonomic Lagrange spaces) were initially stated by Asanov in [2] and developed further by the second author of this paper in the book [9].

In the sequel, we apply the general geometrical results from [11] to the particular rheonomic Berwald-Moór metric \( \hat{F} \), in order to obtain what we called the 1-jet Riemann-Finsler geometry of the three-dimensional time.

## 2 Preliminary notations and formulas

Let \((\mathbb{R}, h_{11}(t))\) be a Riemannian manifold, where \( \mathbb{R} \) is the set of real numbers. The Christoffel symbol of the Riemannian metric \( h_{11}(t) \) is

\[ \chi_{11}^{11} = \frac{h_{11} \frac{dh_{11}}{dt}}{2}, \quad h_{11} = \frac{1}{h_{11}} > 0. \]
Let also $M^3$ be a manifold of dimension three, whose local coordinates are $(x^1, x^2, x^3)$. Let us consider the 1-jet space $J^1(\mathbb{R}, M^3)$, whose local coordinates are

$$(t, x^1, x^2, x^3, y^1_1, y^2_1, y^3_1).$$

These transform by the rules (the Einstein convention of summation is used throughout this work):

$$\tilde{t} = \tilde{t}(t), \quad \tilde{x}^p = \tilde{x}^p(x^q), \quad \tilde{y}^p_1 = \frac{\partial \tilde{x}^p}{\partial x^q} \frac{dt}{dt}, \quad p, q = 1, 3, \quad (2.1)$$

where $\frac{dt}{dt} \neq 0$ and rank $(\frac{\partial \tilde{x}^p}{\partial x^q}) = 3$. We consider that the manifold $M^3$ is endowed with a tensor of kind $(0, 3)$, which is totally symmetric in the indices $p, q$ and $r$. Suppose that the $d$-tensor $G_{ij1} = 6G_{ijp}y^p_1$, is non-degenerate, that is there exists the $d$-tensor $G_{ij1}G_{jkl} = 6^k$. If we denote $G_{i11} = G_{iqr}y^p_1y^q_1$, we can consider the third-root Finsler-like function $\tilde{G}_{i11}$ (it is 1-positive homogenous in the variable $y$):

$$F(t, x, y) = \sqrt[3]{G_{iqr}y^p_1y^q_1} \cdot \sqrt{h^{i1}(t)} = \sqrt[3]{G_{i11}(x, y)} \cdot \sqrt{h^{i1}(t)}, \quad (2.2)$$

where the Finsler function $F$ has as domain of definition all values $(t, x, y)$ which verify the condition $G_{i11}(x, y) \neq 0$. If we denote $G_{i11} = 3G_{i1r}y^p_1$, then the 3-positive homogeneity of the "$y$-function" $G_{i11}$ (this is in fact a $d$-tensor on the 1-jet space $J^1(\mathbb{R}, M^3)$) leads to the equalities:

$$G_{i11} = \frac{\partial G_{i11}}{\partial y^1_1}, \quad G_{i11}y^1_1 = 3G_{i11}, \quad G_{ij1}y^1_1 = 2G_{i11},$$

$$G_{ij1} = \frac{\partial G_{i11}}{\partial y^1_1} = \frac{\partial^2 G_{i11}}{\partial y^1_1 \partial y^1_1}, \quad G_{ij1}y^1_1 = 6G_{i11}, \quad \frac{\partial G_{ij1}}{\partial y^1_1} = 6G_{ij1}.$$  

The fundamental metrical $d$-tensor produced by $F$ is given by the formula

$$g_{ij}(t, x, y) = \frac{h_{i11}(t)}{2} \frac{\partial^2 F^2}{\partial y^1_1 \partial y^1_1}.$$  

By direct computations, the fundamental metrical $d$-tensor takes the form

$$g_{ij}(x, y) = \frac{G_{i11}^{1/3}}{3} \left[ G_{ij1} - \frac{1}{3G_{i11}}G_{i11}G_{j11} \right]. \quad (2.3)$$

Moreover, taking into account that the $d$-tensor $G_{ij1}$ is non-degenerate, we deduce that the matrix $g = (g_{ij})$ admits the inverse $g^{-1} = (g^{jk})$. The entries of the inverse matrix $g^{-1}$ are

$$g^{jk} = 3G_{i11}^{1/3} \left[ G^{jk1} - \frac{G^j_1G^k_1}{3(G_{i11} - G_{i11})} \right], \quad (2.4)$$

where $G^j_1 = G^{jp1}G_{p11}$ and $3G_{i11} = G^{pq1}G_{p11}G_{q11}$.
3 The rheonomic Berwald-Moór metric

Beginning with this Section we will focus only on the rheonomic Berwald-
Moór metric of order three, which is the Finsler-like metric (2.2) for the particular case
\[ G_{pq} = \begin{cases} 
\frac{1}{3!} & \{p, q, r\} \text{- distinct indices} \\
0, & \text{otherwise.} 
\end{cases} \]
Consequently, the rheonomic Berwald-Moór metric of order three is given by
\[ \hat{F}(t, y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y_1^1 y_1^2 y_1^3}. \] (3.1)
Moreover, using preceding notations and formulas, we obtain the following relations:
\[ G_{111} = y_1^1 y_1^2 y_1^3, \quad G_{i11} = \frac{G_{111}}{y_1^1}, \]
\[ G_{ij1} = (1 - \delta_{ij}) \frac{G_{111}}{y_1^j y_1^k} \text{ (no sum by } i \text{ or } j), \]
where \( \delta_{ij} \) is the Kronecker symbol. Because we have
\[ \det (G_{ij1})_{i,j=1,3} = 2G_{111} \neq 0, \]
we find
\[ G^{jk1} = \frac{(1 - 2\delta^{jk})}{2G_{111}} y_1^j y_1^k \text{ (no sum by } j \text{ or } k). \]
It follows that we have \( G_{111} = (1/2)G_{111} \) and \( G^j_1 = (1/2)y_1^i. \)
Replacing now the preceding computed entities into the formulas (2.3) and (2.4), we get
\[ y_{ij} = \frac{(2 - 3\delta_{ij})}{9} \frac{G^{2/3}_{111}}{y_1^j y_1^k} \text{ (no sum by } i \text{ or } j) \] (3.2)
and
\[ g^{jk} = (2 - 3\delta^{jk})G^{-2/3}_{111} y_1^j y_1^k \text{ (no sum by } j \text{ or } k). \] (3.3)
Using a general formula from the paper [11], we find the following geometrical result:

**Theorem 3.1** For the rheonomic Berwald-Moór metric (3.1), the energy action functional
\[ \tilde{E}(t, x(t)) = \int_{a}^{b} \frac{1}{3!} \sqrt{\left( y_1^1 y_1^2 y_1^3 \right)^2} \cdot h^{11} \sqrt{h_{11}} dt \]
produces on the 1-jet space \( J^1(\mathbb{R}, M^3) \) the canonical nonlinear connection
\[ \Gamma = \begin{pmatrix} M^{(i)}_{(1)1} = -\kappa_{11}^1 y_1^i, & N^{(i)}_{(1)j} = 0 \end{pmatrix}. \] (3.4)
Because the canonical nonlinear connection (3.4) has the spatial components equal to zero, it follows that our subsequent geometrical theory becomes trivial, in a way. For such a reason, in order to avoid the triviality of our theory and in order to have a certain kind of symmetry, we will use on the 1-jet space $J^1(\mathbb{R}, M)$, by an "a priori" definition, the following nonlinear connection:

$$\hat{\Gamma} = \begin{pmatrix} M^{(i)}_{(1)j} = -\varkappa^{11}y^i_1, & N^{(i)}_{(1)j} = -\frac{\varkappa^{11}}{2}\delta^i_j \end{pmatrix}. \quad (3.5)$$

4 Cartan canonical connection. $d$-Torsions and $d$-curvatures

The importance of the nonlinear connection (3.5) is coming from the possibility of construction of the dual adapted bases of d-vector fields

$$\begin{cases} \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \varkappa^{11}y^i_1 \frac{\partial}{\partial y^i_1}, & \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \frac{\varkappa^{11}}{2} \frac{\partial}{\partial y^i_1}, & \frac{\partial}{\partial y^i_1} \end{cases} \subset \mathcal{X}(E) \quad (4.1)$$

and d-covector fields

$$\begin{cases} dt, & dx^i, & \delta y^i_1 = dy^i_1 - \varkappa^{11}y^i_1 dt - \frac{\varkappa^{11}}{2} dx^i \end{cases} \subset \mathcal{X}^*(E), \quad (4.2)$$

where $E = J^1(\mathbb{R}, M)$. Note that, under a change of coordinates (2.1), the elements of the adapted bases (4.1) and (4.2) transform as classical tensors. Consequently, all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M)$ (such as Cartan canonical connection, torsion, curvature etc.) will be described in local adapted components.

Using a general result from [11], by direct computations, we can give the following important geometrical result:

**Theorem 4.1** The Cartan canonical $\hat{\Gamma}$-linear connection, produced by the rheonomic Berwald-Moór metric (3.1), has the following adapted local components:

$$C\hat{\Gamma} = \begin{pmatrix} x^i_{11}, & C^k_{j1} = 0, & L^i_{jk} = \frac{x^i_{11}}{2} C^{i(1)}_{j(k)} \end{pmatrix},$$

where, if we use the notation

$$A^i_{jk} = \frac{3\delta^i_j + 3\delta^i_k + 3\delta_{jk} - 9\delta^i_j\delta_{jk} - 2}{9} \quad (no \ sum \ by \ i, j \ or \ k),$$

then

$$C^{i(1)}_{j(k)} = A^i_{jk} \cdot \frac{y^i_1}{y^j_1 y^k_1} \quad (no \ sum \ by \ i, j \ or \ k).$$
Proof. Via the Berwald-Moór derivative operators (4.1) and (4.2), we use the general formulas which give the adapted components of the Cartan canonical connection, namely [11]

\[ C_{jk}^i = \frac{g^{km}}{2} \frac{\delta g_{mj}}{\delta t}, \quad L^i_j = \frac{g^{im}}{2} \left( \frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right), \]

\[ C_{j(k)}^{i(1)} = \frac{g^{im}}{2} \left( \frac{\partial g_{jm}}{\partial y^k_1} + \frac{\partial g_{km}}{\partial y^j_1} - \frac{\partial g_{jk}}{\partial y^m_1} \right) = \frac{g^{im}}{2} \frac{\partial g_{jk}}{\partial y^m_1}. \]

Remark 4.2 The below properties of the d-tensor \( C_{j(k)}^{i(1)} \) are true (sum by \( m \)):

\[ C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(m)} y^m_1 = 0, \quad C_{j(m)}^{m(1)} = 0. \]  \hspace{1cm} (4.3)

For similar properties, please see also the papers [3], [7], [10] or [15].

Remark 4.3 The coefficients \( A_{ij}^l \) have the following values:

\[ A_{ij}^l = \begin{cases} 
-\frac{2}{9}, & i \neq j \neq l \neq i \\
\frac{1}{9}, & i = j \neq l \text{ or } i = l \neq j \text{ or } j = l \neq i \\
-\frac{2}{9}, & i = j = l.
\end{cases} \] \hspace{1cm} (4.4)

Theorem 4.4 The Cartan canonical connection \( \tilde{\Gamma} \) of the rheonomic Berwald-Moór metric (3.1) has three effective adapted local torsion d-tensors:

\[ P_{(1)i(j)}^{(k)(1)} = -\frac{x_{11}^1}{2} C_{i(j)}^{1(k)}, \quad P_{i(j)}^{(1)(k)} = C_{i(j)}^{1(k)}, \]

\[ R_{(1)1j}^{(k)} = \frac{1}{2} \left( \frac{d x_{11}^1}{dt} - x_{11}^1 x_{11}^1 \right) \delta_j^k. \]

Proof. A general \( h \)-normal \( \Gamma \)-linear connection on the 1-jet space \( J^1(\mathbb{R}, M^3) \) is characterized by eight effective d-tensors of torsion (for more details, please see [11]). For our Cartan canonical connection \( \tilde{\Gamma} \) these reduce to the following three (the other five cancel):

\[ P_{(1)i(j)}^{(k)(1)} = \frac{\partial N_{(1)i(j)}^{(k)}}{\partial y_1^i} - L_{ji}^k, \quad R_{(1)1j}^{(k)} = \frac{\delta M_{(1)1j}^{(k)}}{\delta x^j} - \frac{\delta N_{(1)1j}^{(k)}}{\delta t}, \quad P_{i(j)}^{(k)(1)} = C_{i(j)}^{k(1)}. \]
Theorem 4.5  The Cartan canonical connection $\hat{\nabla}$ of the rheonomic Berwald-Moór metric $\mathcal{M}$ has three effective adapted local curvature $d$-tensors:

$$R^l_{ijk} = \frac{\kappa_1^l\kappa_1^j}{4} S^{l(1)(1)}_{i(j)(k)}, \quad P^l_{ij(k)} = \frac{\kappa_1^l}{2} S^{l(1)(1)}_{i(j)(k)}.$$ 

where

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^l_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^l_1} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)}.$$ 

Proof. A general $h$-normal $\Gamma$-linear connection on the 1-jet space $J^1(\mathbb{R}, M^3)$ is characterized by five effective $d$-tensors of curvature (for more details, please see [11]). For our Cartan canonical connection $\hat{\nabla}$ these reduce to the following three (the other two cancel):

$$R^l_{ijk} = \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^l_{ij} L^m_{mk} - L^m_{ik} L^l_{mj},$$

$$P^l_{ij(k)} = \frac{\partial L^l_{ij}}{\partial y^l_1} - \frac{\partial L^l_{ik}}{\partial y^l_1} + C^{l(1)}_{i(j)} P^{(m)}_{(1)(k)} + C^{l(1)}_{i(m)} P^{(j)}_{(1)(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^l_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^l_1} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)},$$

where

$$C^{l(1)}_{i(k)j} = \frac{\delta C^{l(1)}_{i(k)}}{\delta x^j} + C^{m(1)}_{i(k)} L^m_{mj} - C^{l(1)}_{m(k)} L^m_{ij} - C^{l(1)}_{m(k)} L^m_{ij}.$$
6. \( S^{(1)(1)}_{i(j)(l)} = \frac{1}{9 (y_1^i)^2} \) \((i \neq l \text{ and no sum by } i \text{ or } l)\);

7. \( S^{(1)(1)}_{i(l)(l)} = -\frac{1}{9 (y_1^l)^2} \) \((i \neq l \text{ and no sum by } i \text{ or } l)\);

8. \( S^{(1)(1)}_{l(j)(k)} = 0 \) \((k \neq l \text{ and no sum by } l)\);

9. \( S^{(1)(1)}_{l(j)(l)} = 0 \) \((j \neq l \text{ and no sum by } l)\).

**Proof.** For \( j \neq k \), the expression of the curvature tensor \( S^{(1)(1)}_{i(j)(k)} \) takes the form (no sum by \( i \), \( j \), \( k \) or \( l \), but with sum by \( m \))

\[
S^{(1)(1)}_{i(j)(k)} = \left[ \frac{A^l_{ij} \delta y_i^l}{y_1^i y_1^j} - \frac{A^l_{ik} \delta y_i^l}{y_1^i y_1^k} \right] + \left[ \frac{A^l_{ik} \delta y_j^l}{(y_1^l)^2 y_1^k} - \frac{A^l_{ij} \delta y_k^l}{(y_1^l)^2 y_1^j} \right] +
\]

\[
+ \left[ A^m_{ik} A^l_{mk} - A^m_{ik} A^l_{mj} \right] \frac{y_1^l}{y_1^i y_1^j y_1^k},
\]

where the coefficients \( A^l_{ij} \) are given by the relations (4.4).

\[
5 \text{ Geometrical field theories on the 1-jet three-dimensional time}
\]

5.1 Geometrical gravitational theory

From a physical point of view, on the 1-jet three-dimensional time, the rheonomic Berwald-Moëur metric (3.1) produces the adapted metrical d-tensor

\[
G = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h^{11} g_{ij}^l \delta y_1^i \otimes \delta y_1^j,
\]

(5.1)

where \( g_{ij} \) is given by (3.2). This may be regarded as a "non-isotropic gravitational potential" [8]. In such a physical context, the nonlinear connection \( \Gamma \) (used in the construction of the distinguished 1-forms \( \delta y_1^i \)) prescribes, probably, a kind of "interaction" between \((t)\)-, \((x)\)- and \((y)\)-fields.

We postulate that the non-isotropic gravitational potential \( G \) is governed by the geometrical Einstein equations

\[
\text{Ric} \left( C \Gamma \right) - \frac{\text{Sc} \left( C \Gamma \right)}{2} G = \mathcal{K} \mathcal{T},
\]

(5.2)

where \( \text{Ric} \left( C \Gamma \right) \) is the Ricci d-tensor associated to the Cartan canonical connection \( C \Gamma \) (in Riemannian sense and described in adapted bases), \( \text{Sc} \left( C \Gamma \right) \) is the scalar curvature, \( \mathcal{K} \) is the Einstein constant and \( \mathcal{T} \) is the intrinsic stress-energy d-tensor of matter.
In this way, working with the adapted basis of vector fields (4.1), we can find the local geometrical Einstein equations for the rheonomic Berwald-Moór metric (3.1). Firstly, by direct computations, we find:

**Proposition 5.1** The Ricci $d$-tensor of the Cartan canonical connection $C_\Gamma$ of the rheonomic Berwald-Moór metric (3.1) has the following effective adapted local Ricci $d$-tensors:

$$R_{ij} = R_{ijm} = \frac{\varepsilon_{11}^1 \varepsilon_{11}^1}{4} S^{(1)(1)}_{(i)(j)}, \quad P_{(i)(j)} = P_{(i)(j)} = P_{ijm} = \frac{\varepsilon_{11}^1}{2} S^{(1)(1)}_{(i)(j)},$$

$$S^{(1)(1)}_{(i)(j)} = S^{m(1)(1)}_{(i)(m)} = 3\delta_{ij} - \frac{1}{9} \cdot \frac{1}{y_1^1 y_1^1} \text{ (no sum by } i \text{ or } j).$$

**Remark 5.2** The local Ricci $d$-tensor $S^{(1)(1)}_{(i)(j)}$ has the following expression:

$$S^{(1)(1)}_{(i)(j)} = \begin{cases} -\frac{1}{9} \cdot \frac{1}{y_1^1 y_1^1}, & i \neq j \\ \frac{2}{9} \cdot \frac{1}{y_1^1}, & i = j. \end{cases}$$

**Remark 5.3** Using the last equality of (5.3) and the relation (3.3), we deduce that the following equality is true (sum by $r$):

$$S_{i11}^{m11} \overset{\text{def}}{=} g^{m}{}^{r} S^{(1)(1)}_{(r)(i)} = G_{11}^{-2/3} \cdot \frac{1}{3} \cdot \frac{y_1^m}{y_1^1} \text{ (no sum by } i \text{ or } m).$$

Moreover, by a direct calculation, we obtain the equalities

$$\sum_{m,r=1}^{3} S_{r}^{m11} e^{r(1)}_{i(m)} = 0, \quad \sum_{m=1}^{3} \frac{\partial S_{i11}^{m11}}{\partial y_1^m} = \frac{2}{3} \cdot \frac{1}{y_1^1} \cdot G_{111}^{-2/3}.$$  

**Proposition 5.4** The scalar curvature of the Cartan canonical connection $C_\Gamma$ of the rheonomic Berwald-Moór metric (3.1) is given by

$$S_{c} \left( C_\Gamma \right) = -\frac{4h_{11} + \varepsilon_{11}^1 \varepsilon_{11}^1}{2} \cdot G_{111}^{-2/3}.$$  

**Proof.** The general formula for the scalar curvature of a Cartan connection is (for more details, please see [11])

$$S_{c} \left( C_\Gamma \right) = g^{pq} R_{pq} + h_{111} g^{pq} S^{(1)(1)}_{(p)(q)}.$$  

Describing the global geometrical Einstein equations (5.2) in the adapted basis of vector fields (4.1), we find the following important geometrical and physical result (for more details, please see [11]):

---

9
Theorem 5.5  The adapted local geometrical Einstein equations that govern the non-isotropic gravitational potential (5.1), produced by the rheonomic Berwald-Moór metric (3.1), are given by:

\[
\begin{cases}
\xi_{11} \cdot G_{111}^{-2/3} \cdot h_{11} = \mathcal{T}_{11} \\
\frac{x_{11} x_{11}}{4 \mathcal{K}} S_{(i)(j)}^{(1)(1)} + \xi_{11} \cdot G_{111}^{-2/3} \cdot g_{ij} = \mathcal{T}_{ij} \\
\frac{1}{\mathcal{K}} S_{(i)(j)} + \xi_{11} \cdot G_{111}^{-2/3} \cdot h_{11} \cdot g_{ij} = \mathcal{T}_{(i)(j)}^{(1)(1)} 
\end{cases}
\] (5.6)

\[
\begin{cases}
0 = \mathcal{T}_{1i}, \\
0 = \mathcal{T}_{i1}, \\
0 = \mathcal{T}_{(i)}^{(1)} \\
0 = \mathcal{T}_{1(i)}^{(1)}, \\
\frac{x_{11}}{2 \mathcal{K}} S_{(i)(j)} = \mathcal{T}_{i(j)}^{(1)}, \\
\frac{x_{11}}{2 \mathcal{K}} S_{(i)(j)} = \mathcal{T}_{(i)(j)},
\end{cases}
\] (5.7)

where

\[
\xi_{11} = \frac{4 h_{11} + x_{11} x_{11}}{4 \mathcal{K}}.
\] (5.8)

Remark 5.6  The adapted local geometrical Einstein equations (5.6) and (5.7) impose as the stress-energy d-tensor of matter \( \mathcal{T} \) to be symmetrical. In other words, the stress-energy d-tensor of matter \( \mathcal{T} \) must verify the local symmetry conditions

\[
\mathcal{T}_{AB} = \mathcal{T}_{BA}, \quad \forall \, A, B \in \{ 1, i, (1) \}.
\]

By direct computations, the adapted local geometrical Einstein equations (5.6) and (5.7) imply the following identities of the stress-energy d-tensor (summation by \( r \)):

\[
\mathcal{T}_1 \overset{\text{def}}{=} h_{11} \mathcal{T}_{11} = \xi_{11} \cdot G_{111}^{-2/3}, \quad \mathcal{T}_m \overset{\text{def}}{=} g^{mr} \mathcal{T}_{r1} = 0,
\]

\[
\mathcal{T}_{(1)(1)} \overset{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)}^{(1)} = 0, \quad \mathcal{T}_1 \overset{\text{def}}{=} h_{11} \mathcal{T}_{1i} = 0,
\]

\[
\mathcal{T}_{(i)} \overset{\text{def}}{=} g^{mr} \mathcal{T}_{ri} = \frac{x_{11} x_{11}}{4 \mathcal{K}} S_{m11} + \xi_{11} \cdot G_{111}^{-2/3} \cdot \delta^m,
\]

\[
\mathcal{T}_{(1)(i)} \overset{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)(i)} = \frac{x_{11} x_{11}}{2 \mathcal{K}} S_{m11} + \xi_{11} \cdot G_{111}^{-2/3} \cdot \delta^m,
\]

\[
\mathcal{T}_{(1)(i)} \overset{\text{def}}{=} h_{11} g^{mr} \mathcal{T}_{(r)(i)} = \frac{x_{11} x_{11}}{4 \mathcal{K}} S_{m11},
\]

where the distinguished tensor \( S_{m11} \) is given by (5.4) and \( \xi_{11} \) is given by (5.8).
Theorem 5.7 The stress-energy d-tensor of matter $T$ must verify the following geometrical conservation laws (summation by $m$):

\[
\begin{align*}
T^{(1)}_{i_1} + T^{m(i)}_{i_1(m)} + T^{(m)(1)}_{i_1(m)}(T^{(m)(1)}_{i_1(m)}) &= \frac{(h^{11})^2}{16K} \frac{dh_{11}}{dt} + \left[ 2 \frac{d^2h_{11}}{dt^2} - \frac{3}{h_{11}} \left( \frac{dh_{11}}{dt} \right)^2 \right] \cdot G_{11}^{2/3} \\
T^{(1)}_{i_1} + T^{m(i)}_{i_1(m)} + T^{(m)(1)}_{i_1(m)} &= 0 \\
T^{(1)(i)}_{i_1} + T^{m(i)}_{i_1(m)} + T^{(m)(1)(i)}_{i_1(m)} &= 0,
\end{align*}
\]

where (summation by $m$ and $r$)

\[
\begin{align*}
T^{(1)}_{1/1} &= \frac{\delta T^{(1)}_{1/1}}{\delta t} + T^{1}_{1/1} \frac{\Delta x}{\Delta t} - T^{1}_{1/1} \frac{\Delta y}{\Delta t} = \frac{\delta T^{1}_{1/1}}{\delta t}, \\
T^{m}_{1/m} &= \frac{\delta T^{m}_{1/m}}{\delta x^m} + T^{m}_{1} \frac{L_{rm}}{\delta x^m} = \frac{\delta T^{m}_{1/m}}{\delta x^m}, \\
T^{(m)(1)}_{1(m)} &= \frac{\partial T^{(m)}_{1(m)}}{\partial y^m_1} + T^{(r)(m)}_{1(m)} C^{(m)}_{1(m)} = \frac{\partial T^{(m)}_{1(m)}}{\partial y^m_1}, \\
T^{(1)_{i_1}} &= \frac{\delta T^{(1)_{i_1}}}{\delta t} + T^{(1)_{i_1}} \frac{\Delta x}{\Delta t} - T^{(1)_{i_1}} \frac{\Delta y}{\Delta t} = \frac{\delta T^{(1)_{i_1}}}{\delta t} + T^{1}_{1/1} \frac{\Delta x}{\Delta t} \\
T^{m}_{1/m} &= \frac{\delta T^{m}_{1/m}}{\delta x^m} + T^{m}_{1} \frac{L_{rm}}{\delta x^m} - T^{m}_{1} \frac{L_{rm}}{\delta x^m} = \frac{\Delta x}{\Delta t} \frac{\partial T^{m}_{1/m}}{\partial y^m_1}, \\
T^{(m)(1)}_{1(m)} &= \frac{\partial T^{(m)(1)}_{1(m)}}{\partial y^m_1} + T^{(r)(m)}_{1(m)} C^{(m)}_{1(m)} - T^{(m)(1)(m)}_{1(m)} C^{(1)(m)}_{1(m)} = \frac{\partial T^{(m)(1)}_{1(m)}}{\partial y^m_1} \\
T^{(1)(i)} &= \frac{\delta T^{(1)(i)}}{\delta t} + 2 T^{(1)(i)} \frac{\Delta x}{\Delta t} \\
T^{m(i)}_{1/m} &= \frac{\delta T^{m(i)}_{1/m}}{\delta x^m} + T^{(r)(i)}_{1(m)} L_{rm} - T^{m(i)}_{1(m)} L_{rm} = \frac{\Delta x}{\Delta t} \frac{\partial T^{m(i)}_{1/m}}{\partial y^m_1}, \\
T^{(m)(1)(i)}_{1(m)} &= \frac{\partial T^{(m)(1)(i)}_{1(m)}}{\partial y^m_1} + T^{(r)(1)(i)}_{1(m)} C^{(m)(1)}_{1(m)} - T^{(m)(1)(i)}_{1(m)} C^{(1)(i)}_{1(m)} = \frac{\partial T^{(m)(1)(i)}_{1(m)}}{\partial y^m_1}.
\end{align*}
\]

Proof. The conservation laws are provided by direct computations, using the relations \[4.3\] and \[5.3\]. \(\blacksquare\)

11
5.2 Geometrical electromagnetic theory

In the paper [11], using only a given Lagrangian function $L(t,x,y)$ on the 1-jet space $J^1(\mathbb{R}, M^n)$, a geometrical theory for electromagnetism was also created. In the background of our geometrical electromagnetism from [11], we work with an electromagnetic distinguished 2-form (the latin letters run from 1 to $n$)

$$\mathcal{F} = F^{(1)}_{(ij)} dy^i \wedge dx^j,$$

where

$$F^{(1)}_{(ij)} = \frac{h^{11}}{2} \left[ g_{jm}N^{(m)}_{(1)i} - g_{im}N^{(m)}_{(1)j} + (g_{ir}L^r_{jm} - g_{jr}L^r_{im}) y^m_i \right],$$

which is characterized by the following geometrical Maxwell equations [11]:

$$F^{(1)}_{(ij)/1} = \frac{1}{2} A_{(i,j)} \left\{ \begin{align*}
D^{(1)}_{(i)1j} & - D^{(1)}_{(i)m} G^m_{j1} + d^{(1)(1)}_{(i)m} R^{(m)}_{(1)kj} - \\
- [C^{p1}_{j(m)} R^{(m)}_{(1)i1k} - G^p_{ri}] h^{11} g_{ip} y^m_i \end{align*} \right\},$$

$$\sum_{\{i,j,k\}} F^{(1)}_{(ij)k} = \frac{1}{4} \sum_{\{i,j,k\}} \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k} \left[ \frac{\delta N^{(m)}_{(1)i}}{\delta x^k} - \frac{\delta N^{(m)}_{(1)j}}{\delta x^k} \right] y^p_i,$$

$$\sum_{\{i,j,k\}} F^{(1)}_{(ij)1(k)} = 0,$$

where $A_{(i,j)}$ means an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and we have

$$D^{(1)}_{(i)1} = \frac{h^{11}}{2} \frac{\delta g_{im}}{\delta t} y^m_i, \quad D^{(1)}_{(i)j} = h^{11} g_{ip} \left[ -N^{(p)}_{(1)ij} + F^p_{jm} y^m_i \right],$$

$$d^{(1)(1)}_{(i)j} = h^{11} \left[ g_{ij} + g_{ip} C^{p1}_{m(j)} y^m_i \right],$$

$$D^{(1)}_{(i)ij} = \frac{\delta D^{(1)}_{(i)1}}{\delta x^j} - D^{(1)}_{(m)1} L_{ij}^m, \quad G_{i1}^k = \frac{\delta G^k_{il}}{\delta x^j} + G^m_{i1} L^k_{mj} - G^k_{mj} L^m_{ij},$$

$$F^{(1)}_{(ij)1} = \frac{\delta F^{(1)}_{(ij)1}}{\delta t} + F^{(1)}_{(ij)1} x^1 - F^{(1)}_{(ij)m} G^m_{i1} - F^{(1)}_{(i)m} G^m_{1j},$$

$$F^{(1)}_{(ij)j} = \frac{\delta F^{(1)}_{(ij)j}}{\delta x^k} - F^{(1)}_{(ij)m} L^m_{ik} - F^{(1)}_{(i)m} L^m_{jk},$$

$$F^{(1)}_{(ij)k} = \frac{\delta F^{(1)}_{(ij)k}}{\delta y^l} - F^{(1)}_{(ij)m} C^m_{i(k)} - F^{(1)}_{(i)m} C^m_{j(k)}.$$

For $n = 3$, the rheonomic Berwald-Moór metric [3,1] and the nonlinear connection [3,4], we find the electromagnetic 2-form

$$\mathcal{F} := \hat{\mathcal{F}} = 0.$$
In conclusion, our Berwald-Moór geometrical electromagnetic theory on the 1-jet three-dimensional time is trivial. In our opinion, this fact suggests that the geometrical structure of the 1-jet three-dimensional time contains rather gravitational connotations than electromagnetic ones. This is because, in our geometrical approach, the Berwald-Moór electromagnetism of order three does not exist.

References

[1] G.S. Asanov, Finslerian Extension of General Relativity, Reidel, Dordrecht, 1984.

[2] G.S. Asanov, Jet extension of Finslerian gauge approach, Fortschritte der Physik 38, No. 8 (1990), 571-610.

[3] Gh. Atanasiu, M. Neagu, On Cartan spaces with the m-th root metric $K(x,p) = \sqrt{a^{i_{1}i_{2}...i_{m}}(x)p_{i_{1}}p_{i_{2}}...p_{i_{m}}}$, Hypercomplex Numbers in Geometry and Physics (2010), in press; http://arXiv.org/math.DG/0802.2887v4 (2008).

[4] G.I. Garas’ko, Fundamentals of Finsler Geometry (for Physicists), Moscow, 2010, Preprint.

[5] G.I. Garas’ko, D.G. Pavlov, The notions of distance and velocity modulus in the linear Finsler spaces, ”Space-Time Structure. Algebra and Geometry” (D. G. Pavlov, Gh. Atanasiu, V. Balan Eds.), pp. 104-117; Russian Hypercomplex Society, Lilia Print, Moscow, 2007.

[6] M. Matsumoto, On Finsler spaces with curvature tensors of some special forms, Tensor N. S. 22 (1971), 201-204.

[7] M. Matsumoto, H. Shimada, On Finsler spaces with 1-form metric. II. Berwald-Moór’s metric $L = (y^{1}y^{2}...y^{n})^{1/n}$, Tensor N. S. 32 (1978), 275-278.

[8] R. Miron, M. Anastasiei, The Geometry of Lagrange Spaces: Theory and Applications, Kluwer Academic Publishers, 1994.

[9] M. Neagu, Riemann-Lagrange Geometry on 1-Jet Spaces, Matrix Rom, Bucharest, 2005.

[10] M. Neagu, A relativistic approach on 1-jet spaces of the rheonomic Berwald-Moór Metric, http://arXiv.org/math.DG/1002.0241v1 (2010).

[11] M. Neagu, The geometry of relativistic rheonomic Lagrange spaces, Proceedings of Workshop on Diff. Geom., Global Analysis and Lie Algebras (Grigoris Tsagas Ed.), Geometry Balkan Press, Bucharest, No. 5 (2001), 142-168.
[12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.

[13] D.G. Pavlov, *Chronometry of three-dimensional time*, Hypercomplex Numbers in Geometry and Physics **1** (1), Vol. **1** (2004), 19-30.

[14] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics **1** (1), Vol. **1** (2004), 31-39.

[15] H. Shimada, *On Finsler spaces with the metric* \( L(x, y) = \sqrt[\langle a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m} \rangle} \), Tensor N. S. **33** (1979), 365-372.

Gheorghe ATANASIU and Mircea NEAGU
University Transilvania of Braşov, Faculty of Mathematics and Informatics
Department of Algebra, Geometry and Differential Equations
B-dul Iuliu Maniu, No. 50, BV 500091, Braşov, Romania.
E-mails: gh.atanasiu@yahoo.com, mircea.neagu@unitbv.ro
Website: http://cs.unitbv.ro/website/membrii/aged/index.html