Asymptotics for Toeplitz determinants on a circular arc

I. V. Krasovsky

Technische Universität Berlin, Germany

and

Brunel University West London, United Kingdom

1 Introduction

In this work we find an asymptotic formula for a Toeplitz determinant with the symbol supported on an arc of the unit circle in the case when the symbol has Fisher-Hartwig singularities.

Let \( f(\theta) \) be an integrable function on the unit circle. It was proved by Szegő [1] that if \( f(\theta) \) is positive and sufficiently smooth, namely, its derivative satisfies a Lipschitz condition, (these requirements have been later relaxed) then the Toeplitz determinant

\[
D_n(f) = \det \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-k)\theta} f(\theta) d\theta \right), \quad j, k = 0, 1, \ldots, n,
\]

has the asymptotic expansion

\[
D_{n-1}(f) \sim H(f)^n E(f), \quad n \to \infty,
\]

where

\[
H(f) = \exp \left( \ln f \right)_0, \quad E(f) = \exp \sum_{k=1}^{\infty} k(\ln f)_k (\ln f)_{-k},
\]

\[
(\ln f)_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \ln f(\theta) d\theta, \quad k = 0, 1, \ldots
\]

This formula is not valid for \( f(\theta) \) with zeros or singularities. The asymptotics in the case when \( f(\theta) \) has zeros or root-type singularities were conjectured by Lenard [2], Fisher and Hartwig [3] and proved by Widom [4]. Namely, let

\[
f(\theta) = \psi(\theta) \prod_{r=1}^{R} (2 - 2\cos(\theta - \theta_r))^{\alpha_r} = \psi(\theta) \prod_{r=1}^{R} \left| 2 \sin \frac{\theta - \theta_r}{2} \right|^{2\alpha_r},
\]

where \( \psi(\theta) \) satisfies the conditions of Szegő’s theorem and all \( \alpha_r > -1/2 \) (this is the most general case of real \( \alpha_r \)'s since it is a condition for existence of Fourier coefficients). Then

\[
D_{n-1}(f) \sim H(\psi)^n \sum_{s=1}^{R} \alpha_s^2 E(\psi) \prod_{r=1}^{R} \left( \frac{H(\psi)}{\psi(\theta_r)} \right)^{\alpha_r} \frac{G(\alpha_r + 1)^2}{G(2\alpha_r + 1)} \prod_{r \neq s} \left| 2 \sin \frac{\theta_r - \theta_s}{2} \right|^{-\alpha_r \alpha_s},
\]
where \( G(z) \) is Barnes’ G-function. This was further generalized for the case of \( f(\theta) \) with jumps by Basor [5], Böttcher and Silbermann [8], and other authors. An incomplete list includes the works [5]–[10]. Note that the positivity condition for \( f(\theta) \) can be removed: the asymptotics [11] are actually proved in [4] for a complex-valued \( f(\theta) \) with \( \psi(\theta) \neq 0 \) and the change of the argument over the closed circle \( \Delta_{-\pi \leq \theta \leq \pi} \arg \psi(\theta) = 0 \).

It is interesting how the asymptotics look like in case when \( f(\theta) \) is zero on an arc. This question was addressed by Widom in [11]. Suppose that \( f(\varphi) \) is supported on an arc \( \alpha \leq \varphi \leq 2\pi - \alpha \), where it is positive, smooth enough, and symmetric: \( f(\varphi) = f(2\pi - \varphi) \). Then it is proved in [11] that

\[
D_{n-1}(f) \sim \gamma \alpha^2 H(F)^n \left( n \sin \frac{\alpha}{2} \right)^{-1/4} \frac{G(3/2)^2}{\sqrt{\pi}} E(F),
\]

with \( \gamma = \cos(\alpha/2) \) and

\[
F(\theta) = f(2 \arccos(\gamma \cos \frac{\theta}{2})),
\]

the function \( f(\varphi) \) “extended” to the whole circle. Note that as is known

\[
\frac{G(3/2)^2}{\sqrt{\pi}} = 2^{1/12} e^{3\zeta'(1)}
\]

where \( \zeta'(x) \) is the derivative of Riemann’s zeta function. Widom’s proof was based on a theorem of Hirschman for a Toeplitz-like determinant, where in the matrix elements the Fourier coefficients of \( f(\theta) \) on the circle, i.e. the coefficients w.r.t. the system \( z^k \), are replaced by the coefficients of a function on the interval \([-1, 1]\) w.r.t. the Legendre polynomials.

Just like [11] this formula breaks down if \( f(\varphi) \) has zeros or singularities on the arc. Resolution of this question is presented here. Let

\[
f(\varphi) = \psi(\varphi) \prod_{r=1}^{R} (2 - 2 \cos(\varphi - \varphi_r))^{\alpha_r}, \quad \alpha_r > -1/2, \quad \varphi_1 = \alpha, \quad \varphi_R = 2\pi - \alpha,
\]

be supported on an arc \( \alpha \leq \varphi \leq 2\pi - \alpha \), \( f(\varphi) = f(2\pi - \varphi) \) (in particular, \( \varphi_r = 2\pi - \varphi_{R+1-r} \)), and let \( \psi(\varphi) \) satisfy the conditions of Szegö’s theorem. Then

\[
D_{n-1}(f) \sim \gamma^{(\alpha + \sum^R \alpha_r)^2} H(\Psi)^n \left( \frac{n}{\gamma} \right)^{2\alpha_1^2 + \sum^R \alpha_r^2} \left( n \sin \frac{\alpha}{2} \right)^{-1/4} 2^{4\alpha_1 + 2\alpha_2} \Gamma(1 + 2\alpha_1) \frac{G(3/2 + 2\alpha_1)^2}{\sqrt{\pi} G(2 + 4\alpha_1)} \times
\]

\[
E(\Psi)(\sin \alpha)^{2\alpha_2^2} \prod_{r=1}^{R} \frac{H(\Psi)}{\psi(\varphi_r)} \prod_{r=1}^{R} \prod_{r \neq r} G(\alpha_r + 1)^2 \left( \frac{\sin^2(\varphi_r/2)}{1 - \gamma^{-2} \cos^2(\varphi_r/2)} \right)^{\alpha_r^2/2} \prod_{r \neq s} 2 \sin \frac{\varphi_r - \varphi_s}{2} \left| -\alpha_r \alpha_s \right|,
\]

where

\[
\Psi(\theta) = \psi(2 \arccos(\gamma \cos \frac{\theta}{2})).
\]
In particular, when zeros and singularities are absent, we reproduce the asymptotics (5).

For technical reasons, we need to replace \( \psi(\varphi) \) by an analytic function. Smoothness of \( \psi(\varphi) \) implies existence of a trigonometric polynomial \( \psi_0(\varphi) \) such that \( \mu(\varphi) = \psi(\varphi) - \psi_0(\varphi) = O(1/C) \) uniformly in \( \varphi \) on the arc for some constant \( C(n) \) which can be chosen arbitrary large. Let, e.g., \( C(n) = n^n \). Then a simple analysis of the r.h.s. of (7) (using smoothness and positivity of \( \psi(\varphi) \)) and the l.h.s. of (7) (using Hadamard’s inequality) shows that it is sufficient to prove the theorem when \( \psi \) is replaced by \( \psi_0 \). To simplify notation, we just assume in what follows that \( \psi(\varphi) \) is analytic. (Note that we have to bear in mind that \( \psi(\varphi) \) now depends on \( n \) in a particular way, as it now stands for an analytic approximation to the original \( \psi \).)

To obtain (7), we use (4) to evaluate \( D_{n-1}(\hat{F}) \), where
\[
\hat{F}(\theta) = \frac{F(\theta) \sin(\theta/2)}{\sqrt{1 - \gamma^2 \cos^2(\theta/2)}}, \quad F(\theta) = f(2 \arccos(\gamma \cos \theta/2)),
\]
is a symbol on the whole circle, and the following relation
\[
\frac{D_{n-1}(f)}{D_{n-1}(F)} = \gamma^{n^2+n} \frac{P_n(\gamma^{-1})}{P_n(1)} \sim \gamma^{n^2} \frac{\sqrt{\bar{f}(\alpha)(1 + \sin(\alpha/2))} n^{2\alpha_1} \Gamma(1 + 2\alpha_1)}{\Delta(F) \sqrt{\pi} \sin(\alpha/2) n^{2\alpha_1}},
\]
where
\[
\bar{f}(\varphi) = \frac{(2 \gamma)^{4\alpha_1} f(\varphi)}{\prod_{r=1}^{R} (2 - 2 \cos(\varphi - \varphi_r))^{\alpha_r}},
\]
\[
\ln \Delta(F) = \frac{\sin(\alpha/2)}{4\pi} \int_0^{2\pi} \ln F(\theta) \frac{d\theta}{1 - \gamma^2 \cos^2(\theta/2)},
\]
and \( P_k(x) = x^k + \cdots, \ k = 0, 1, \ldots \) are the monic polynomials orthogonal on \([-1, 1]\) w.r.t. the weight function
\[
w(x) = \frac{f(2 \arccos \gamma x)}{\sqrt{1 - \gamma^2 x^2}} ;
\]
\[
\int_{-1}^1 P_n(x) P_m(x) w(x) dx = h_n \delta_{mn}, \quad h_n > 0, \ m, n = 0, 1, \ldots
\]
In our case of a symmetric weight \( (w(x) = w(-x)) \), these polynomials are simply related (see [12, 13, 14, 15]) to the polynomials \( \Phi_k(z, \beta) = z^k + \cdots, \ k = 0, 1, \ldots \) orthogonal w.r.t. \( g(\varphi) = f(2 \arccos(\gamma \cos \frac{\varphi}{2}/\cos \frac{\beta}{2})) \) on an arc:
\[
\frac{1}{2\pi} \int_{\beta}^{2\pi-\beta} \Phi_n(z) \Phi_m(z) g(\varphi) d\varphi = \tilde{h}_n \delta_{mn}; \quad z = e^{i\varphi}, \ \tilde{h}_n > 0, \ m, n = 0, 1, \ldots
\]
The first equation in (9) follows from the connection between 1) Toeplitz matrices, 2) orthogonal polynomials on an arc of the unit circle, and 3) those on a segment of the real
axis. To obtain the second equation, we need to analyze polynomials \( P_k(z) \) asymptotically. This is partly achieved with the help of an approach based on a matrix Riemann-Hilbert problem, a new method which turned out to be very effective for analysis of asymptotic problems \([16,17,18]\). Most Riemann-Hilbert analysis we need is contained in \([19,20]\) where the asymptotics were found for polynomials orthogonal on \([-1,1]\) with a weight \((1-x)^a(1+x)^bh(x), h(x) \) is positive and real analytic. We get the main term in the asymptotics of \( P_n(1) \) by generalizing a small part of \([19,20]\) to the weight \((12)\). The main term for \( P_n(\gamma^{-1}) \) can be obtained much easier, from the well-known Szegő asymptotics \((12), \text{Thm.12.1.2}\) which apply here because \(\gamma^{-1}\) lies in the exterior of the orthogonality interval \([-1,1]\). As will be clear from the proof, equations \((9)\) also hold when \(f(\theta)\) has jumps in addition to zeros and root singularities (see \([3]\)). Therefore the result of Basor \([5]\) can also be extended to symmetric symbols on an arc in the same way as \((4)\).

2 The ratio \( D_{n-1}(f)/D_{n-1}(\hat{F}) \).

In this section we obtain equations \((9)\). Due to the above mentioned connection between polynomials orthogonal on an interval of the real axis and on an arc of the unit circle one can write the following relations (Lemma 3.1. of \([15]\)):

\[
D_{n-1}(f) = 2^{n(n-1)} \frac{\gamma^{n^2+n}}{\pi^n} P_n(1/\gamma) \prod_{j=0}^{n-1} h_j, \quad f(\varphi) = w(\gamma^{-1} \cos \frac{\varphi}{2}) \sin \frac{\varphi}{2};
\]

\[
D_{n-1}(\hat{F}) = 2^{n(n-1)} \frac{1}{\pi^n} P_n(1) \prod_{j=0}^{n-1} h_j, \quad \hat{F}(\theta) = w(\cos \frac{\theta}{2}) \sin \frac{\theta}{2}.
\]

In the first of them the corresponding arc is \(\alpha \leq \varphi \leq 2\pi - \alpha\), and in the second one, the arc coincides with the whole circle \(0 \leq \theta < 2\pi\). Setting \(\gamma \cos(\theta/2) = \cos(\varphi/2)\) and comparing the expressions for \(f\) and \(\hat{F}\) we get \((8)\). The ratio of \(D_{n-1}\)'s yields the first equation in \((9)\). Note that we can also rewrite it as follows (using, e.g., formula (14) of \([15]\)):

\[
\frac{D_{n-1}(f)}{D_{n-1}(\hat{F})} = \frac{2\Phi_n(1,\alpha) \gamma^{n^2+n}}{e^{-im\alpha/2}\Phi_n(e^{i\alpha},\alpha) + \text{c.c.}}.
\]

Thus, to obtain the second (asymptotic) equation in \((9)\), we can either consider the asymptotics of polynomials on an arc or on an interval. We choose the second option since in that case we can use the results of Kuijlaars, McLaughlin, Van Assche, and Vanlessen \([19]\) instead of reformulating the Riemann-Hilbert solution for polynomials on an arc. As indicated in the introduction, we need these methods to obtain the main term in the asymptotics of \(P_n(1)\). In what follows, we mostly dwell on the differences of our case from \([19]\) and refer the reader to that work for other details.

Consider the \(2 \times 2\) matrix

\[
Y(z) = \begin{pmatrix}
P_n(z) & \frac{1}{2\pi i} \int_{-1}^{1} P_n(x) w(x) \frac{dx}{x-z} \\
-2\pi i k_{n-1} P_{n-1}(z) & -k_{n-1} \int_{-1}^{1} P_{n-1}(x) w(x) \frac{dx}{x-z}
\end{pmatrix},
\]
where $\kappa_n = h_n^{-1/2}$ is the leading coefficient of the orthonormal polynomials. $Y(z)$ is the unique solution of the following Riemann-Hilbert problem (the proof is almost the same as in [19], the difference being additional singularities inside $(-1, 1)$):

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus [-1, 1]$.

(b) For $x \in (-1, 1) \setminus \cup_r x_r$, $x_r = \gamma^{-1} \cos(\varphi_r/2)$, $Y$ has continuous boundary values $Y_{+}(x)$ as $z$ approaches $x$ from above, and $Y_{-}(x)$, from below. They are related by the jump condition

$$Y_{+}(x) = Y_{-}(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (-1, 1) \setminus \cup_r x_r. \tag{19}$$

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left( I + O \left( \frac{1}{z} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty. \tag{20}$$

(d) Near the points $1 = x_1$, $-1 = x_R$, $x_r$

$$Y(z) = \begin{cases} 
O \left( \frac{1}{1} \frac{|z - x_r|^{2\alpha_r}}{|z - x_r|^{2\alpha_r}} \right), & \text{if } \alpha_r < 0, \\
O \left( \frac{1}{1} \frac{\log |z - x_r|}{\log |z - x_r|} \right), & \text{if } \alpha_r = 0, \\
O \left( \frac{1}{1} \frac{1}{1} \right), & \text{if } \alpha_r > 0,
\end{cases} \tag{21}$$

as $z \to x_r$, $z \in \mathbb{C} \setminus [-1, 1]$.

Here the points $x_r$ correspond to $\varphi_r$ under the mapping $x = \gamma^{-1} \cos(\varphi/2)$. Moreover, recalling (6), (12), we have

$$w(x) = \frac{\psi(2 \arccos \gamma x)}{\sqrt{1 - \gamma^2 x^2}} \prod_{1}^{R} (2\gamma)^{2\alpha_r} |x_r - x|^{2\alpha_r}. \tag{22}$$

The proof of (21) is similar to that in [19].

Now we find the asymptotics of the matrix element $Y_{11}(z)$ in the neighbourhood of $z = 1$. To do this, we apply a series of transformations to the Riemann-Hilbert problem for $Y(z)$, as is usual in this method. These transformations $Y \mapsto T \mapsto S$ are the same as in [19] but the contour where the jump for $S(z)$ occurs is different. It consists of $R - 1$ lenses which touch at zeros and singularities $x_r$ of the weight $w_0(x)$ (see the figure). In [19] there was only one lens with the endpoints $-1$ and $1$. We now consider the neighbourhoods of the points $x_r$ and the region outside of them separately. In the outside region, a parametrix $N(z)$ is given by the same expression as in [19] where we substitute our weight $w_0(x)$. It coincides with the main asymptotic term for $Y(z)$ obtained from Szegö’s Theorem 12.1.2 in [12]. In
the neighbourhood of each point $x_r$, one could solve the local Riemann-Hilbert problem such that it fits $N(z)$ on the neighbourhood’s boundary. Of those solutions, we need only the one in the neighbourhood of $x_1 = 1$. Indeed, it is easy to show again like in Theorems 7.8, 7.9 of [21] that corrections from neighbourhoods of the other points $x_r$ contribute at most to the next-leading term in the asymptotics near $x = 1$. Therefore the leading term near $x = 1$ coincides exactly with that for the problem considered in [19, 20]. It is given by the formula (2.23) and the one after (3.7) of [20]. Taking there the limit $x \to 1$, we obtain the asymptotics of $Y_{11}(1)$ and finally, recalling (18), get the relation:

$$P_n(1) \sim D_\infty \sqrt{\frac{2}{1 + \sin(\alpha/2)}} \sqrt{\frac{\pi n}{\nu(1)}} \frac{n^{2\alpha_1}}{\Gamma(1 + 2\alpha_1)},$$

(23)

where

$$\ln D_\infty = \frac{1}{2\pi} \int_{-1}^{1} \ln f(2 \arccos \gamma x) \sqrt{1 - x^2} \, dx = \frac{1}{4\pi} \int_{0}^{2\pi} \ln F(\theta) \, d\theta$$

(24)

and $v(x) = w(x)/(1 - x^2)^{2\alpha_1}$.

One can verify that there is no problem in the fact that $\psi(\varphi)$ (and hence $w(x)$) depends on $n$ (in a special way as it approximates some smooth function).

We now turn to calculation of the asymptotics for $P_n(\gamma^{-1})$. Here it suffices to use an old result of Szegő. The leading term for asymptotics of orthonormal polynomials $p_n(x)$ for $x$ outside $[-1, 1]$ and a wide class of weights on $[-1, 1]$ (which includes our weight (22)) is given by Theorem 12.1.2 in [12]. Namely,

$$p_n(x) \sim \frac{z^n}{\sqrt{2\pi D(z^{-1})}}, \quad x = \frac{1}{2}(z + z^{-1}),$$

(25)

$$\ln D(z) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(\cos t) \frac{|\sin t|^{1 + \gamma z^{-\gamma}}}{1 - \gamma z^{-\gamma}} \, dt.$$

(26)

For $x = 1/\gamma = 1/\cos(\alpha/2)$ we have $z = (1 + \sin(\alpha/2))/\gamma \equiv z_0$. Substituting (12) into (26), separating the integral into 2 parts, one over $[0, \pi]$, the other over $[-\pi, 0]$, and changing the variables $\theta \mapsto -\theta$ in the second one, we get

$$\ln D(z_0^{-1}) = \frac{1}{4\pi} \int_{0}^{\pi} \ln \left(\frac{F(2\theta)}{\sqrt{1 - \gamma^2 \cos^2 \theta}}\right) \frac{2\sin(\alpha/2)}{1 - \gamma \cos \theta} \, d\theta = \ln \Delta(F) + I,$$

where

$$\ln \Delta(F) = \frac{\sin(\alpha/2)}{2\pi} \int_{0}^{\pi} \frac{\ln F(2\theta)}{1 - \gamma \cos \theta} \, d\theta = \frac{\sin(\alpha/2)}{4\pi} \int_{0}^{2\pi} \frac{\ln F(\theta)}{1 - \gamma^2 \cos^2(\theta/2)} \, d\theta.$$
(we used the symmetry \( F(\theta) = F(2\pi - \theta) \) to get the last equation) and

\[
I = \frac{\sin(\alpha/2)}{4\pi} \int_0^{2\pi} \ln \left| \frac{\sin \theta}{1 - \gamma^2 \cos^2 \theta} \right| \frac{d\theta}{1 - \gamma \cos \theta}.
\] (27)

The last integral can be evaluated explicitly. We have for \(|x| > 1, 0 < a < 1\)

\[
- \frac{\sqrt{x^2 - 1}}{4\pi} \int_0^{2\pi} \ln(1 - a \cos^2 \theta) \frac{x}{x - \cos \theta} d\theta = \ln \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1} + x \sqrt{1 - a}}.
\] (28)

This integral is calculated by differentiating w.r.t. \( a \) and then computing residues in the complex plane of \( e^{i\theta} \). Taking here \( a \to 1 \) and setting \( x = 1/\gamma \) gives part of the integral (27) with \( \ln |\sin \theta| \). The part with \( \ln(1 - \gamma^2 \cos^2 \theta) \) is obtained by setting \( a = \gamma^2, x = 1/\gamma \). Thus we get \( I = -\ln \sqrt{2} \), and therefore

\[
\ln D(z_0^{-1}) = \frac{\Delta(F)}{\sqrt{2}}.
\] (29)

The main term in the asymptotics of the leading coefficient \( \kappa_n \) of \( p_n(x) = \kappa_n x^n + \cdots \) is given by Theorem 12.7.1 of [12]. A calculation like the one just carried out yields

\[
\kappa_n \sim \frac{2^n}{D_\infty} \sqrt{\frac{1 + \sin(\alpha/2)}{2\pi}},
\] (30)

where \( D_\infty \) is given by (24). Putting (25), (29), and (30) together, we finally obtain

\[
P_n(\gamma^{-1}) = \frac{p_n(\gamma^{-1})}{\kappa_n} \sim \left( \frac{1 + \sin(\alpha/2)}{2\gamma} \right)^n \sqrt{\frac{2}{1 + \sin(\alpha/2)}} D_\infty.
\] (31)

Equations (29) and (31) give the asymptotics (9).

### 3 Asymptotics for Toeplitz determinants

We are now ready to obtain the asymptotics (7). An important role will be played by the map between the arc and the circle given by \( \cos(\varphi/2) = \gamma \cos(\theta/2) \). Using the symmetry of the symbol \( f(\varphi) = f(2\pi - \varphi) \), and hence \( \varphi_r = 2\pi - \varphi_{R+1-r} \), we can write (3) as \( (\theta_0 = \pi) \)

\[
F(\theta) = f(\varphi) = \psi(\varphi) \prod_{r=1}^{[R/2]} \left| 4 \sin \frac{\varphi - \varphi_r}{2} \sin \frac{\varphi + \varphi_r}{2} \right|^{2\alpha_r} \left| 2 \cos \frac{\varphi}{2} \right|^{2\alpha_0} =
\]

\[
\psi(\varphi) \prod_{r=1}^{[R/2]} \left( 4 \cos^2 \frac{\varphi}{2} - \cos^2 \frac{\varphi_r}{2} \right)^{2\alpha_r} \left| 2 \cos \frac{\varphi}{2} \right|^{2\alpha_0} =
\]

\[
\gamma^2 \sum_r \alpha_r \Psi(\theta) \prod_{r=1}^{R} \left| 2 \sin \frac{\theta - \varphi_r}{2} \right|^{2\alpha_r}.
\] (32)
Consider the function $\hat{F}$. Vanishing of $f(\varphi)$ on the arc $-\alpha \leq \varphi \leq \alpha$ is “replaced” in $\theta$ variable by a singularity of $\hat{F}$ at $\theta = 0$ given by the factor $2\sin(\theta/2)$. If $f(\varphi)$ also has singularities at the endpoints of the arc, the corresponding factor becomes $(2\sin(\theta/2))^{1+4\alpha_1}$. We see that Widom’s formula (4) holds for $\hat{F}$. The role of $\psi$ is now played by

$$\frac{\gamma^2 \sum_r \alpha_r \Psi(\theta)}{2\sqrt{1 - \gamma^2 \cos^2(\theta/2)}}.$$  

$\alpha_1$ in the formula is replaced by $1/2 + 2\alpha_1$, $R$, by $R - 1$ (as the ends of the arc merge in $\theta$ variable). Let us now simplify this expression for $D_{n-1}(\hat{F})$. We have

$$\left(\ln 2 \sqrt{1 - \gamma^2 \cos^2\frac{\theta}{2}}\right) = \begin{cases} -\frac{1}{2|k|} \left(1 - \sin\left(\frac{\alpha}{2}\right)\right)^{2|k|}, & k \neq 0 \\ \ln(1 + \sin(\alpha/2)), & k = 0 \end{cases}.$$ (33)

These integrals are calculated as (28). In particular, if we set $\alpha = 0$ for $k = 0$, we get

$$\left(\ln 2 \bigg|\sin\frac{\theta - \theta_r}{2}\bigg|\right)_0 = 0.$$

The first obvious application of these relations is the expression

$$H(\hat{F}) = H\left(\frac{\gamma^2 \sum_r \alpha_r \Psi(\theta)}{2\sqrt{1 - \gamma^2 \cos^2(\theta/2)}}\right) = \frac{\gamma^2 \sum_r \alpha_r H(\Psi)}{1 + \sin(\alpha/2)}.$$ (34)

According to (11), we now have to calculate $E(\Psi(\theta)(1 - \gamma^2 \cos^2(\theta/2))^{-1/2})$. (Note that $E(C\nu(\theta)) = E(\nu(\theta))$, where $C$ is any nonzero constant.) Using (33) and performing summation of geometric series, we obtain:

$$\sum_{k=1}^{\infty} k \left(\ln\{\Psi(\theta)(1 - \gamma^2 \cos^2\frac{\theta}{2})^{-1/2}\}\right)_k^2 = \sum_{k=1}^{\infty} k (\ln \Psi(\theta))^2_k + \frac{1}{2\pi} \int_0^{2\pi} \ln \Psi(\theta) \frac{\gamma^{-2}(1 - \sin \frac{\alpha}{2})^2 e^{-i\theta}}{1 - \gamma^{-2}(1 - \sin \frac{\alpha}{2})^2 e^{-i\theta}} d\theta - \frac{1}{4} \ln \left\{1 - \gamma^{-4} \left(1 - \sin \frac{\alpha}{2}\right)^4\right\}.$$ (35)

The imaginary part of the integral here vanishes while the real part can be written as

$$-\frac{1}{2}(\ln \Psi(\theta))_0 + \ln \Delta(\Psi),$$

where $\Delta(\cdot)$ is defined in (11).

Exponentiating (35) and multiplying the result with $(2\sin(\alpha/2)/(1 + \sin(\alpha/2)))^{1/2}$, we finish calculation of $E$:

$$\sqrt{\frac{2\sin(\alpha/2)}{1 + \sin(\alpha/2)}} E\left(\frac{\Psi(\theta)}{\sqrt{1 - \gamma^2 \cos^2(\theta/2)}}\right) = E(\Psi) \frac{\Delta(\Psi)}{\sqrt{H(\Psi)}} \sin^{1/4} \frac{\alpha}{2}.$$ (36)
Separating the contribution of the point $\theta_1 = 0$, we can now rewrite (11) as follows:

$$
D_{n-1}(F) \sim \left( \gamma^2 \sum \alpha_r H(\Psi) \right)^n \frac{1}{1 + \sin(\alpha/2)} \frac{n^{1/4 + 2\alpha_1 + 2\alpha_1^2 + \sum \alpha_r^2}}{\Gamma(1 + 2\alpha)} E(\Psi(\theta)) \frac{\Delta(\Psi)}{\sqrt{\Psi(0)}} \frac{\Delta(3/2 + 2\alpha_1)}{2 G(2 + 4\alpha_1)} \times 
$$

$$
\prod_{r\neq 1}^{R} \frac{H(\Psi)}{\Psi(\theta)} \frac{2 \sin(\varphi_r/2)}{1 + \sin(\alpha/2)} \right)^{\alpha_r} \left( \prod_{r \neq R}^{R-1} \frac{G(2\alpha_r + 1)}{G(2\alpha_r + 1)} \frac{2 \sin(\theta_r/2)}{2 \sin(\theta_r - \theta_s/2)} \right)^{\alpha_r} \right)
$$

From (10), (32) we obtain

$$
\tilde{f}(\varphi) = 2^{4\alpha_1} \gamma^2 \sum \alpha_r \Psi(\theta) \prod_{r=2}^{R-1} \frac{2 \sin(\theta_r - \theta_s/2)}{2 \sin(\theta_r/2)}
$$

The ratio (9) for this $\tilde{f}(\varphi)$ can be written as

$$
\frac{\Delta(1 + 2\alpha_1)}{n^{2\alpha_1}}
$$

where

$$
\ln K = \frac{\sin(\alpha/2)}{4\pi^2} \sum_{r=1}^{R} 2\alpha_r \int_0^{2\pi} \ln \left| \frac{2 \sin((\theta_r/2))}{1 - \gamma^2 \cos^2(\theta_r/2)} \right| d\theta.
$$

Using the symmetry of $\theta_r$’s we regroup the addends in the above expression and write

$$
\ln \left| \frac{4 \sin(\theta_r - \theta_s/2)}{2} \right| = \ln \left| 1 - 2b \cos \theta \right| - \ln b, \quad 1/b = 2 \cos \theta_r.
$$

Now the integrals can be computed as before, and we get

$$
K = \prod_{r=1}^{R} \left( \frac{2 \sin(\varphi_r/2)}{1 + \sin(\alpha/2)} \right)^{\alpha_r}
$$

Putting this together with (38) and (37), we obtain

$$
D_{n-1}(f) \sim \gamma^2 2^{n-2n} \sum \alpha_r H(\Psi) \frac{n^{2\alpha_1^2 + \sum \alpha_r^2}}{\Gamma(n \alpha)} \left( \frac{\sin(\alpha/2)}{2} \right)^{-1/4} \frac{G(3/2 + 2\alpha_1)}{\sqrt{\gamma^2 G(2 + 4\alpha_1)}} E(\Psi) \times 
$$

$$
2^{4\alpha_1} \Gamma(1 + 2\alpha_1) \prod_{r=1}^{R} \left( \frac{H(\Psi)}{\Psi(\varphi_r)} \right)^{\alpha_r} \prod_{r=1}^{R-1} \frac{G(\alpha_r + 1)}{G(2\alpha_r + 1)} \left( \frac{2 \sin(\theta_r/2)}{2 \sin(\theta_r - \theta_s/2)} \right)^{\alpha_r}
$$

Finally, a simple but cumbersome calculation which uses the symmetry of $\theta_r$’s (cf. (32)) gives

$$
\prod \left| \frac{2 \sin(\theta_r - \theta_s/2)}{2} \right|^{\alpha_r}
$$

$$
\gamma^{\sum_{r=2}^{R-1} \alpha_r^2} \prod_{r \neq s \neq 1, R} \left| \frac{2 \sin(\varphi_r - \varphi_s/2)}{2} \right|^{\alpha_r^2} \prod_{r=1}^{R/2} \left( \frac{\sin^2(\varphi_r/2)}{1 - \gamma^{-2} \cos^2(\varphi_r/2)} \right)^{\alpha_r^2}
$$

(42)
Substituting this into the previous formula completes the derivation of (7).

References

[1] U. Grenander and G. Szegő: Toeplitz forms and their applications. Berkeley, 1958

[2] A. Lenard: Some remarks on large Toeplitz determinants. Pacific J. Math. 42, 137–145 (1972)

[3] M. E. Fisher and R. E. Hartwig: Toeplitz determinants: some applications, theorems, and conjectures. Advan. Chem. Phys. 15, 333–353 (1968)

[4] H. Widom: Toeplitz determinants with singular generating functions. Amer. J. Math. 95, 333–383 (1973)

[5] E. Basor: Asymptotic formulas for Toeplitz determinants. Trans. Amer. Math. Soc. 239, 33–65 (1978)

[6] E. Basor: A localization theorem for Toeplitz determinants. Indiana Univ. Math. J. 28, 975–983 (1979)

[7] E. L. Basor, K. E. Morrison: The extended Fisher-Hartwig conjecture for symbols with multiple jump discontinuities. in “Toeplitz operators and related topics” (Santa Cruz, CA, 1992), 16–28.

[8] A. Böttcher and B. Silbermann: Toeplitz matrices and determinants with Fisher-Hartwig symbols. J. Func. Anal. 63, 178–214 (1985)

[9] A. Böttcher and B. Silbermann: Toeplitz operators and determinants generated by symbols with one Fisher-Hartwig singularity. Math. Nachr. 127, 95–124 (1986)

[10] T. Ehrhardt: A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities. Oper. Theory: Adv. Appl. 124, 217–241 (2001)

[11] H. Widom: The strong Szegő limit theorem for circular arcs. Indiana Univ.Math.J. 21, 277–283 (1971)

[12] G. Szegő: Orthogonal polynomials. AMS Colloquium Publ. 23. New York: AMS 1959

[13] P. Delsarte, Y. Genin: Tridiagonal approach to the algebraic environment of Toeplitz matrices. SIAM J.Matrix Anal. Appl. 12, 220–238, 432–448 (1991)

[14] A. Zhedanov: On some classes of polynomials orthogonal on arcs of the unit circle connected with symmetric orthogonal polynomials on an interval. J.Approx.Th. 94, 73–106 (1998)
[15] I. V. Krasovsky: Some computable Wiener-Hopf determinants and polynomials orthogonal on an arc of the unit circle. math-FA/0310172

[16] A. S. Fokas, A. R. Its, A. V. Kitaev: An isomonodromy approach to the theory of two-dimensional quantum gravity. Uspehi Mat. Nauk 45, 135 (1990)

[17] P. Deift and X. Zhou: A steepest descent method for oscillatory Riemann-Hilbert problem. Ann. Math. 137, 295–368 (1993)

[18] P. Deift: Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Math. 1998

[19] A. B. J. Kuijlaars, K. T-R McLaughlin, W. Van Assche, M. Vanlessen: The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on [−1, 1]. math.CA/0111252 To appear in Ann. Math.

[20] A. B. J. Kuijlaars, M. Vanlessen: Universality for eigenvalue correlations from the modified Jacobi unitary ensemble. math-ph/0204006.

[21] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, X. Zhou: Strong asymptotics for orthogonal polynomials with respect to exponential weights. Commun. Pure Appl.Math. 52, 1491–1552 (1999)