Linear Matrix Inequality Design of Observer-Based Controllers for port-Hamiltonian Systems

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Abstract—The design of an observer-based state feedback controller for port-Hamiltonian (pH) systems is addressed using linear matrix inequalities (LMIs). The controller is composed of the observer and the state feedback. By passivity, the asymptotic stability of the closed-loop system is guaranteed even if the controller is implemented on complex physical systems such as the ones defined by infinite-dimensional or nonlinear models. An infinite-dimensional Timoshenko beam model and a micro-electromechanical system are used to illustrate the achievable performances using such an approach under simulations.

Index Terms—Distributed port-Hamiltonian systems, State feedback, Luenberguer observer, Linear Matrix Inequalities.

I. INTRODUCTION

The port-Hamiltonian (pH) framework has been introduced in [1] and has shown to be well suited for the modelling and control of multi physical systems [2–3]. It has been widely studied for finite-dimensional systems in [2,4,5,6] and it has been generalized to infinite-dimensional systems in [7,8]. The main idea of the pH approach is to describe physical systems in terms of the energy and its exchanges between each internal component and the environment.

Stabilization of pH systems using interconnection and damping assignment (IDA) has been proposed in [4,5] and extensively developed for linear system in [6], where a linear matrix inequality (LMI) approach has been employed to obtain a solution of the IDA control problem. This LMI problem allows designing a static feedback matrix to have desired closed-loop performances. It can be seen as an alternative to traditional approaches as pole-placement, $\mathcal{L}_Q$-control or $H_\infty$-control. This result is also implemented for the dual problem, i.e., for the observer design in [9]. Further works on observer design for linear and nonlinear pH systems have been reported in [10,11,12] where the properties of the system are used to ensure the observer convergence. Nevertheless, no results are reported regarding observer based control design.

An observer-based controller designed for pH systems is proposed in [9], where the observer-based state feedback is designed from the linearization of the system and used to stabilize the non linear system by means of a feedforward term. However, the passivity of the system is not preserved in closed-loop since the observer-based controller is not passive, thus the closed-loop stability is not guaranteed using the passivity properties of pH systems. In [13], an observer-based state feedback design is proposed such that the controller is on the pH form and in [14], the same authors proposed a similar controller for infinite-dimensional port-Hamiltonian system with distributed actuation. Nevertheless, the closed-loop performances can only be modified through damping injection. Recently in [15,16], this result has been improved allowing to modify the whole structure of the plant in closed-loop and then, having more degrees of freedom in terms of control design.

In this work an observer-based state feedback design based on LMIs is proposed for linear pH system developing the LMIs presented in [6] for IDA control design. The feedback consists of a Luenberger observer and a negative feedback on the observed states. The novelty and main contribution of this paper is to recast the feedback and the Luenberger observer as a pH control system interconnected with the system to be controlled in a power preserving manner. This reinterpretation of the observer-based controller allows to use the passivity properties of the system to guarantee the closed-loop stability. A second contribution of this work is to explicitly give the conditions such that the observer based control system is strictly positive real, output strictly passive and zero state detectable. This result allows to use the proposed controller to asymptotically stabilize a large class of boundary controlled infinite dimensional pH systems [8,17] and non-linear pH system [2] when using a linear approximation of these systems to design the controller.

The paper is organized as follows. Section [II] presents the main result of the paper, namely the pH observer based control system and its design parameters in terms of a set of LMIs. Section [III] presents two examples. An infinite dimensional Timoshenko beam model on a one dimensional spatial domain and a non-linear microelectromechanical system (MEMS), which are used to show the design procedure and the achieved closed-loop performances by means of numerical simulations. Finally, Section [IV] gives some final remarks and discussions on possible future work related to this topic.

II. OBSERVER-BASED STATE FEEDBACK DESIGN

Consider the following linear pH system

$$
\begin{align*}
\dot{x}(t) &= (J-R)Qx(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= B^TQx(t)
\end{align*}
$$

(1)
where \( x(t) \in \mathbb{R}^n \) is defined for all \( t \geq 0 \), \( x_0 \in \mathbb{R}^n \) is the unknown initial condition, \( u(t) \in \mathbb{R}^m \) is the input and \( y(t) \in \mathbb{R}^m \) is the power conjugated output of \( u(t) \), which in this work is considered to be measurable. \( J = -J^T \), \( R = R^T \geq 0 \) and \( Q = Q^T > 0 \) all known real matrices of size \( n \times n \) and \( B \in \mathbb{R}^{n \times m} \). For simplicity, we refer to system (1) as the system \((A, B, C)\), with \( A = (J - R)Q \) and \( C = B^TQ \), and we assume that it is controllable and observable.

Define the following Luenberger observer
\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0
\]
for the pH system (1), where \( \hat{x} \in \mathbb{R}^n \) is the estimation of \( x \), \( \hat{x}_0 \) is a known initial condition and \( L \in \mathbb{R}^{n \times m} \) is a matrix to design.

In this work, we use the results from [6] to design the matrix \( L \) such that (2) converge asymptotically to (1). Then, we design the state feedback matrix \( K \) such that the observer based control law
\[
u(t) = r(t) - K\hat{x}(t), \quad r(t) \in \mathbb{R}^m, \quad K \in \mathbb{R}^{m \times n}
\]
leads to a closed-loop system (2)-(3) on a pH form with inputs \( r(t) \) and \( y(t) \). The importance of guaranteeing this closed-loop property is that it is instrumental to assure asymptotic stability of the closed-loop system [18].

A. Observer design by LMIs

Define the error of the state as \( \hat{\dot{x}}(t) = x(t) - \hat{x}(t) \). The error dynamics is obtained from (1) and (2):
\[
\dot{\hat{x}}(t) = (A - LC)\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0 = x_0 - \hat{x}_0,
\]
where \( \hat{x}_0 \) is an unknown initial condition.

We recall the following proposition from [6], which is instrumental for the design of the matrix \( L \) such that \( A - LC \) is Hurwitz.

**Proposition 1** [6]: Denote by \( B_\perp \) a full rank \((n - m) \times n\) matrix that annihilates \( B \), i.e. \( B_\perp B = 0 \). Let us also denote \( E_\perp = B_\perp A \). There exist matrices \( J_\perp = -J^T_\perp \), \( R_\perp = R^T_\perp \geq 0 \), \( Q_d = Q^T_d > 0 \) and \( F \) such that \((J_\perp - R_\perp)Q_d = A + BF \) if and only if there exists a solution \( X = X^T \in \mathbb{R}^{n \times n} \) to the LMIs:
\[
X > 0,
\]
\[
-[E_\perp XB^T_\perp + B_\perp XE^T_\perp] \geq 0.
\]
Given such an \( X \), compute \( S_d \) as follows:
\[
S_d = \left( B_\perp B^T \right)^{-1} \left( E_\perp X - B^T_\perp XE^T_\perp (B_\perp B^T_\perp)^{-1} B_\perp \right),
\]
then the following matrices
\[
J_d = \frac{1}{2} (S_d - S^T_d), \quad R_d = \frac{1}{2} (S_d + S^T_d), \quad Q_d = X^{-1}, \quad F = (B_\perp B^T)^{-1} B^T \left( S_d X^{-1} - A \right)
\]
satisfy \((J_d - R_d)Q_d = A + BF \), \( J_d = -J_\perp^T \), \( R_d = R_\perp^T \geq 0 \), \( Q_d = Q^T_d > 0 \) and \((J_d - R_d)Q_d = A + BF \).

**Remark 1**: Proposition 1 is related to the stabilizability of (1). In fact, the LMI (3) has a solution if and only if the pair \((A, B)\) is stabilizable [Proposition 9 in [6]].

**Remark 2**: The dual problem consists in following Proposition 1 but replacing \( A \) by \( A^T \), \( B \) by \( C^T \) and \( F \) by \( -L^T \).

**Remark 3**: Similar to Remark 1 the pair \((A, C)\) is detectable if and only if the LMI (3) has a solution with \( E_\perp = B_\perp A^T \) and \( B_\perp \in \mathbb{R}^{(n-m) \times n} \) a left annihilator of \( C^T \), i.e. \( B_\perp C^T = 0 \).

The performances obtained using Proposition 1 are in terms of \( Q_d \) (energy matrix) and \( R_d \) (dissipation matrix). As it is mentioned in [6], the LMI (3) can be slightly modified in order to keep the energy matrix in a desired interval and to have sufficient but not excessive damping. This is formalized in the following proposition.

**Proposition 2**: Under the same statements of Proposition 1 if the following LMIs:
\[
q \Lambda_2^{-1} - X < 0,
\]
\[
-\Lambda_1^{-1} + X < 0,
\]
\[
\Xi_1 + E_\perp X B_\perp^T + B_\perp X E_\perp^T \leq 0,
\]
\[
-\Xi_2 - E_\perp X B_\perp^T - B_\perp X E_\perp^T \leq 0,
\]
have a solution \( X = X^T \) for some symmetric matrices \( \Lambda_1 \), \( \Lambda_2 \in \mathbb{R}^{n \times n} \), \( \Xi_1 \in \mathbb{R}^{(n-m) \times (n-m)} \), such that \( 0 < \Lambda_1 < \Lambda_2 \) and \( 0 \leq \Xi_1 < \Xi_2 \), then \( \Lambda_1 < Q_d < \Lambda_2 \). Moreover, choosing
\[
S_d = \left( B_\perp B^T \right)^{-1} \left( -B^T X E_\perp^T (B_\perp B^T_\perp)^{-1} B_\perp - \gamma B^T \right),
\]
for some scalar \( \gamma > 0 \), and the matrices \( J_d, R_d \) and \( F \) as in (7), then \( A + BF = (J_d - R_d)Q_d \) with \( R_d > 0 \).

**Proof**: The proof of Proposition 1 is a direct application of Proposition 7 and Remark 8 in [6]. See also Proposition 1 in [9].

**Remark 4**: Matrices \( \Lambda_1 \) and \( \Lambda_2 \) allow to fix the lowest and highest eigenvalues of \( Q_d \) respectively. Matrices \( \Xi_1 \) and \( \Xi_2 \) bound the damp term, while the scalar \( \gamma \) implies \( R_d > 0 \) and then, the asymptotic behavior is ensured.

In the following section, we consider the Luenberger observer [2] already designed by Proposition 2 using the dual problem, i.e. replacing \( A \) by \( A^T \), \( B \) by \( C^T \), and \( L = -F^T \), and then we design the matrix \( K \) in the control law (3) such that the system is equivalent to control by interconnection.

B. PH observer-based controller

Consider the Luenberger observer [2] combined with the state feedback [3]. The aim is to formulate this observer-based state feedback as the power preserving interconnection
\[
u(t) = r(t) - y_e(t), \quad u_e(t) = y(t)
\]
of (1) with a pH dynamic control system, defined as
\[
\dot{\hat{x}}(t) = (J_e - R_e)Q_e \hat{x}(t) + B_e u_e(t) + Br(t),
\]
\[
y_e(t) = B^T_e Q_e \hat{x}(t),
\]
as depicted in Fig. [7]. This is possible if the control gain is defined as \( K = B^T_e Q_e \), \( B_e = L \) and the following matching equation
\[
A - LC - BK = (J_e - R_e)Q_e
\]
(12)
is satisfied for some $n \times n$ matrices $J_c = -J_c^T$, $R_c = R_c^T \geq 0$, $Q_c = Q_c^T > 0$ and $(A, B, C)$ defined in (1).

The following proposition, which presents a set of LMIs whose solution allows to define $K$, $J_c$, $R_c$ and $Q_c$ such that the observer-based controller (11) is a pH system, is the main contribution of this work.

**Proposition 3:** Given $(A, B, C)$ (1), the power preserving interconnection (10) and a matrix $L$ such that $A_L := A - LC$ is Hurwitz. Then (11) is a pH system if the LMIs

$$
2\Gamma_1 - BL^T + LB^T - A_L X - X A_L^T \leq 0, \\
-2\Gamma_2 + BL^T + LB^T - A_L X - X A_L^T \leq 0, \\
-\Delta_2^{-1} + X \leq 0,
$$

have a solution $X = X^T$, for some $n \times n$ symmetric matrices $\Gamma_1, \Gamma_2, \Delta_1$ and $\Delta_2$ such that $0 \leq \Gamma_1 < \Gamma_2$ and $0 < \Delta_1 < \Delta_2$. $S_c = A_L X - BL^T$, we have $J_c = \frac{1}{2}(S_c - S_c^T)$, $R_c = -\frac{1}{2}(S_c + S_c^T)$, $Q_c = X^{-1}$, $B_c = L$ and $K = B_c^T Q_c$. From $\Gamma_1 > 0$, (11) is strictly positive real (SPR), output strictly passive (OSP) and zero state detectable (ZSD) with respect to the input/output pair $u_c/y_c$.

**Proof.** The proof of Proposition 3 and Corollary 1 are shown here. $X$ being the solution of the LMI (13), from $S_c = A_L X - BL^T$, $J_c = \frac{1}{2}(S_c - S_c^T)$, $R_c = -\frac{1}{2}(S_c + S_c^T)$, $Q_c = X^{-1}$, $B_c = L$ and $K = B_c^T Q_c$, one can verify that $J_c = -J_c^T$, $R_c = R_c^T$ and $Q_c = Q_c^T$. To conclude that (11) is a pH system it has to be verified that $R_c \geq 0$ and $Q_c > 0$. From (13),

$$
2\Gamma_1 \leq BL^T + LB^T - A_L X - X A_L^T \leq 2\Gamma_2, \\
\Delta_2^{-1} \leq X \leq \Delta_1^{-1}.
$$

Replacing $X$, $A_L X - BL^T$ by their expression with respect to $S_c$ and $Q_c$, and inverting the second inequality we obtain

$$
2\Gamma_1 \leq -(S_c + S_c^T) \leq 2\Gamma_2, \\
\Delta_1 \leq Q_c \leq \Delta_2.
$$

Using $R_c = -(S_c + S_c^T)$ we conclude that $Q_c > 0$ and $R_c \geq 0$ since $\Delta_1 > 0$ and $\Gamma_1 \geq 0$. Implication (i) of Corollary 1 is directly obtained from (3) and the assumption that $A_L$ is Hurwitz. Implication (ii) is replacing $R_c = -(S_c + S_c^T)$ in (14). The SPR property of implication (iii) is verified applying the Kalman-Yakubovich-Popov Lemma [19]. To this end, the existence of real matrices $P = P^T > 0$, $S$ and a scalar $\varepsilon > 0$ such that $PA_c + A_c^T P = -S^T S - \varepsilon P$ and $C_c = B_c^T P$ is proved by choosing $P = Q_c$, which implies $S^T S = 2Q_c R_c Q_c - \varepsilon Q_c$, and since $\Gamma_1 > 0$ implies $R_c > 0$, we can always find a small enough $\varepsilon$ such that $2Q_c R_c Q_c - \varepsilon Q_c$ is positive definite. Then, the matrix $2Q_c R_c Q_c - \varepsilon Q_c$ can always be decomposed as $S^T S$ using for instance Cholesky factorization. The OSP property follows noting that $\Gamma_1 > 0$ implies $R_c > 0$, and taking the time derivative of the Hamiltonian of the controller $H_c = \frac{1}{2} x_c^T Q_c x_c$. It is not difficult to show that

$$
\dot{H}_c = -x_c^T Q_c R_c Q_c x_c + y_c^T u_c \\
= -x_c^T Q_c (R_c - \varepsilon B_c B_c^T) Q_c x_c + y_c^T u_c - \varepsilon \|y_c\|^2
$$

where we have added $\pm c y_c^T y_c$, with $c > 0$, to the first line of (15) and used $y_c = B_c^T Q_c x_c$ and $Q_c = Q_c^T$ in the second line of (15). Hence it is always possible to find a small enough $\varepsilon$ such that (11) is dissipative with respect to the supply rate $y_c^T u_c - \varepsilon \|y_c\|^2$, implying that (11) is OSP. The ZSD property is inferred from (15) setting $u_c = y_c = B_c^T Q_c x_c = 0$ and noting that since $R_c > 0$, the states of (11) converge exponentially to zero.

**Remark 5:** Matrix $L$ of Proposition 3 can be designed with Proposition 2 or with any other control design technique such as, for instance, Linear Quadratic Regulator (LQR) or pole-placement approaches.

**Remark 6:** A simple choice for designing matrices $\Gamma_1, \Gamma_2, \Delta_1$ and $\Delta_2$ is to use identity matrices modulated by a constant. Proposition 3 permits to assure that the observer-based controller can be formulated as a pH system. This is instrumental to guarantee the asymptotic stability of the closed-loop system in some particular cases of interest. Indeed, if (11) is the finite-dimensional approximation of a boundary controlled pH system (BC-PHS) defined on a 1-dimensional spatial domain as in [8, Theorem 4.4], or the linear approximation of a finite dimensional non-linear system (see the appendix for the precise definition of the class of considered systems), then the controller (11) from Proposition 3 asymptotically stabilizes the non-approximated systems under some very general conditions. This is formalized in the following proposition.

**Proposition 4:** Let (11) be the finite-dimensional and linear approximation of

(i) a linear boundary controlled pH system (BC-PHS) defined on a 1-dimensional spatial domain, or

(ii) an output strictly passive (OSP) and zero-state detectable (ZTD) finite dimensional non-linear system as defined by (22),

then, (11) designed using Proposition 3 asymptotically stabilizes (i), respectively (ii), if $\Gamma_1 > 0$.

**Proof.** By Corollary 1 (11) is SPR, OSP and ZSD if $\Gamma_1 > 0$. Hence the proof of (i) follows by direct application of Theorem 5.10 in [20], concerning the power preserving interconnection of a BC-PHS defined on a 1-dimensional spatial domain.
the energy balance is given by $\dot{H}(t) = u(t)^T y(t)$. The reader is referred to [21] for more details on the model, to [8] for the well-posedness of this class of systems and to [20] for stability analysis. The parameters of the model are shown in Table I.

### TABLE I

| PLANT PARAMETERS. |
|-------------------|
| **Value** | **Measurement unit** |
| --- | --- |
| $T$ | 1 Pa |
| $\rho$ | 1 kg.m$^{-1}$ |
| $EI$ | 1 Pa.m$^4$ |
| $I_\rho$ | 1 Kg.m$^2$ |
| $a$ | 0 m |
| $b$ | 1 m |

To design the passive observer-based controller using Proposition [3] the infinite-dimensional model is first approximated by a finite-dimensional system using the finite difference discretization scheme on staggered grids proposed in [22]. This is a structure preserving spatial approximation method which preserves the pH structure of the system. The matrices of the finite-dimensional approximation on the form (1) are

$$
J = \begin{bmatrix}
0 & D & 0 & -F \\
-D^T & 0 & 0 & 0 \\
0 & 0 & 0 & D \\
F^T & 0 & -D^T & 0
\end{bmatrix},
R = 0,
$$

$$
Q = \begin{bmatrix}
hQ_1 & 0 & 0 & 0 \\
0 & hQ_2 & 0 & 0 \\
0 & 0 & hQ_3 & 0 \\
0 & 0 & 0 & hQ_4
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
b_{11} & b_{12} & 0 & 0 \\
b_{12} & b_{23} & 0 & 0 \\
0 & 0 & b_{32} & 0 \\
0 & 0 & b_{43} & b_{44}
\end{bmatrix},
$$

where

$$
D = \frac{1}{h^2} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1
\end{bmatrix},
$$

$$
F = \frac{1}{2h} \begin{bmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 1
\end{bmatrix},
$$

$Q_i, i \in \{1, \cdots, 4\}$ are diagonal matrices containing the evaluation of $T(\zeta), \rho(\zeta)^{-1}, EI(\zeta)$ and $I_\rho(\zeta)^{-1}$ respectively, at the specific discretization points and

$$
b_{11} = \frac{1}{h} \begin{bmatrix}
-1 \\
0 \\
\vdots \\
0
\end{bmatrix},
 b_{12} = \frac{1}{2} \begin{bmatrix}
-1 \\
0 \\
0 \\
0
\end{bmatrix},
 b_{32} = b_{11},
$$

$$
b_{23} = \frac{1}{h} \begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix},
 b_{43} = \frac{1}{2} \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
 b_{44} = b_{23}.
$$

The state variables of the approximated model are $x(t) = (x_1^d, x_2^d, x_3^d, x_4^d)^\top$, where $x_i^d(t) \in \mathbb{R}^{n_d}, i \in \{1, \cdots, 4\}$ and the $i$–th component of $x_1^d, x_2^d, x_3^d$ and $x_4^d$ corresponds to the approximation of $z_1((i-0.5)h, t), z_2((i-0.5)h, t)$ and $z_4((i-0.5)h, t)$ respectively, with $h = 2/(2n_d+1)$, $b - a$ being the length of the beam and $n_d$ the number of element. In this example, we choose $n_d = 5$ and hence the complete state is composed of 20 elements. The reader is referred to [22] for further details about this discretization method. The observer design is done following Propositions [1] and Remark [2]. The design parameters for the observer are shown in Table II and
and 2, respectively. Since for both controllers \( \Gamma \) and \( \Delta \), the eigenvalues of the matrix \( A_L = A - LC \) are shown in Figure 2. For the state feedback design, we follow Proposition 3 with the design matrices given in Table III varying only the matrix \( \Delta_1 \). The eigenvalues of both closed-loop matrices are shown in Figure 2, where \( K_1 \) and \( K_2 \) refer to design 1 and 2, respectively. Since for both controllers \( \Gamma_1 > 0 \), the closed-loop between the low order observer-based controller and the infinite-dimensional system is asymptotically stable by Proposition 4. For the simulation we use a time interval \( t = [0, 10 \text{s}] \) with step time \( \delta_t = 0.1 \text{ ms} \) and mid point temporal discretization [22]. The simulation is done taking 100 elements per state variable for the infinite-dimensional system (in order to approach the infinite dimensional system over a large set of frequencies), 400 in total, while for the observer we only take 5 elements per state variable, i.e. 20 in total. The initialization is such that the beam is in equilibrium position with a force of \( N \) applied at the end tip, which gives the following initial conditions for the plant: \( z_1(\zeta, 0) = 0.01 \), \( z_2(\zeta, 0) = 0 \), \( z_3(\zeta, 0) = -0.01(\zeta - 1) \) and \( z_4(\zeta, 0) = 0 \). The observer is initialized with null initial conditions, i.e. \( \hat{x}(0) = 0 \). The deformation of the beam is reconstructed from the state variables \( z(\zeta, t) \) and \( \hat{z}(\zeta, t) \), taking into account that the beam is clamped at the left side. Figure 3 shows the end tip responses in open-loop and closed-loop for design 1 and 2. The settling time is improved when increasing \( \Delta_1 \).

| Matrix | Value |
|--------|-------|
| \( \Lambda_1 \) | 0.1120 |
| \( \Lambda_2 \) | 50000I20 |
| \( \Xi_1 \) | 1118 |
| \( \Xi_2 \) | 10000I18 |
| \( \gamma \) | 10 |

**TABLE II**

**Observer design parameters**

| Matrix | Design 1 | Design 2 |
|--------|----------|----------|
| \( \Gamma_1 \) | \( 1 \times 10^{-15}I_{20} \) | \( 1 \times 10^{-15}I_{20} \) |
| \( \Gamma_2 \) | \( 1 \times 10^{15}I_{20} \) | \( 1 \times 10^{15}I_{20} \) |
| \( \Delta_1 \) | \( 0.1 \times 10^{-1}I_{20} \) | \( 0.18 \times 10^{-1}I_{20} \) |
| \( \Delta_2 \) | \( 1 \times 10^{15}I_{20} \) | \( 1 \times 10^{15}I_{20} \) |

**TABLE III**

**Controller design parameters**

![Image](561x633)

**Fig. 2.** \( \lambda(A) \): Eigenvalues of \( A \), \( \lambda(A_L) \): Eigenvalues of \( A - LC \), \( \lambda(A_{K_i}) \): Eigenvalues of \( A - BK_i \), with \( i = \{1, 2\} \).
port-Hamiltonian representation in \[24\]

\[
\begin{pmatrix}
    \dot{q} \\
    \dot{p} \\
    \dot{Q}
\end{pmatrix} = \begin{pmatrix}
    0 & 1 & 0 \\
    -1 & -b & 0 \\
    0 & 0 & -\frac{1}{\tau}
\end{pmatrix} \begin{pmatrix}
    \frac{\partial H}{\partial q} \\
    \frac{\partial H}{\partial p} \\
    \frac{\partial H}{\partial Q}
\end{pmatrix} + \begin{pmatrix}
    0 \\
    0 \\
    \frac{1}{\tau}
\end{pmatrix} u
\]

\[y = \frac{1}{\tau} \frac{\partial H}{\partial q} = \frac{p^2}{2m} + \frac{1}{2} k_1 q^2 + \frac{1}{2} k_2 q^4 + \frac{Q^2}{2C(q)}
\]

\[C(q) = \frac{\varepsilon A_s}{q_{\text{max}} - q}\]

where \(q(t), p(t)\) and \(Q(t)\) are respectively, the position, the momentum, and the charge in the capacitor, \(k_1\) and \(k_2\) are the spring coefficients, \(m\) is the mass of the moving part, \(C(q)\) is the non-linear capacitance which depends on the gap of the MEMS, \(b > 0\) and \(r > 0\) are the damping and resistance constant parameters, respectively, \(\varepsilon\) is the dielectric constant, \(A_s\) is the surface of the MEMS and \(q_{\text{max}}\) is such that \(q < q_{\text{max}}\). The input of the system \(u(t)\) is the input voltage and \(y(t)\) is the supplied current. The balance equation of the Hamiltonian is

\[
\dot{H}(t) = -b \left(\frac{p(t)}{m}\right)^2 - r y(t)^2 + y(t) u(t)
\]

which implies that the system is OSP. Under realistic operation conditions we can assume that the state space of the system is such that \(Q(t) > 0\) for all \(t > 0\), allowing to conclude that the system is ZSD. The parameters of the plant are shown in Table [IV] The linearization of \[16\] around an equilibrium operation point is given by

\[
A = \begin{pmatrix}
    0 & \frac{1}{m} Q^* & 0 \\
    -3k_2(q^*)^2 - k_1 & -\frac{b}{m} & \frac{Q^*}{A_s \varepsilon r} \\
    0 & \frac{Q^*}{A_s \varepsilon r} & 0 - q^* q_{\text{max}}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    0 \\
    0 \\
    \frac{1}{\tau}
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
    \frac{Q^*}{A_s \varepsilon r} & 0 & -q^* q_{\text{max}}
\end{pmatrix}
\]

In the current example the studied equilibrium is given in Table [V] Following the design procedure of section [II] the linearized model \[17\] is used for the synthesis of an observer-based controller. For the observer design Proposition \[2\] is used with the parameters given in Table [VI]. The eigenvalues of the matrix \(A_L = A - LC\) are shown in Figure [9]. Two state feedbacks are designed using Proposition \[3\] with the parameters given in Table [VII]. Note that the first and second controller only differ by \(A_1\). Since \[16\] is OSP and for both controllers \(\Gamma_1 > 0\), the closed-loop system is asymptotically stable by Proposition \[4\]. The feedback matrices are for each controller denoted by \(R_1\) and \(K_2\), respectively, and the closed-loop eigenvalues are shown in Figure [6]. For the simulation, time \(t = [0, 0.01] s\) is used with a step time \(\delta t = 1 \mu s\). The initial conditions are set equal to \(q(0) = q^*, p(0) = p^*, Q(0) = 0.9Q^*\) for the non linear system, while for the observer all initial conditions are set exactly at the equilibrium.

**TABLE IV**  
**PLANT PARAMETERS.**

| Value | Measurement unit |
|-------|------------------|
| \(k_1\) | 0.46 Nm\(^{-1}\) |
| \(k_2\) | 0.46 Nm\(^{-1}\) |
| \(m\) | 2.4 \times 10^{-8} kg |
| \(\varepsilon\) | 8.854 \times 10^{-12} Fm\(^{-1}\) |
| \(A_s\) | 4 \times 10^{-4} m\(^2\) |
| \(q_{\text{max}}\) | 10^{-5} m |
| \(b\) | 10^{-7} Ns |
| \(r\) | 0.5 \times 10^6 Ω |

**TABLE V**  
**LINEARIZATION POINT.**

| Value | Measurement unit |
|-------|------------------|
| \(q^*\) | 0.5 \times 10^{-6} m |
| \(p^*\) | 0 kg m s\(^{-1}\) |
| \(Q^*\) | 4.0363 \times 10^{-11} C |
| \(u^*\) | 0.1083 V |
| \(y^*\) | 2.1654 \times 10^{-8} A |

**TABLE VI**  
**OBSERVER DESIGN PARAMETERS.**

| Matrix | Value |
|--------|-------|
| \(\Lambda_1\) | 1 \times 10^{-2} \times \text{diag}(1, 200, 1) |
| \(\Lambda_2\) | 1 \times 10^{-9} I_3 |
| \(\Xi_1\) | 1 \times 10^{-1} I_2 |
| \(\Xi_2\) | 1 \times 10^{-1} I_2 |
| \(\gamma\) | 30 \times 10^4 |

**TABLE VII**  
**CONTROLLER DESIGN PARAMETERS.**

| Matrix | Design 1 | Design 2 |
|--------|----------|----------|
| \(\Gamma_1\) | 1 \times 10^{-11} I_3 | 1 \times 10^{-11} I_3 |
| \(\Gamma_2\) | 1 \times 10^{15} I_3 | 1 \times 10^{15} I_3 |
| \(\Delta_1\) | 0.5 \times 10^{-1} I_3 | 1.5999 \times 10^{-1} I_3 |
| \(\Delta_2\) | 1 \times 10^{15} I_3 | 1 \times 10^{15} I_3 |
point. Figure 7 shows the open-loop response and the closed-loop responses when applying the two different controllers on the non-linear system. Figure 8 shows the closed-loop response for the second controller together with the observed variables. In both cases, the mechanical oscillations have been reduced by increasing the electrical ones. We observe that changing the lower bound of $Q_e$, i.e., $\Delta_1$, better performances for the mechanical part of the micro robot are obtained.

**IV. CONCLUSION**

An observer-based state feedback controller design based on LMIs is proposed for linear pH systems. The feedback consists on a Luenberger observer and a negative feedback on the observed states. The novelty and main contribution of this paper is to cast the feedback and the Luenberger observer as a pH control system interconnected in a power preserving manner with the system to be controlled. This reinterpretation of the observer based controller allows to use the passivity properties of the system to guarantee the closed-loop stability. The second contribution of this work is to explicitly give the conditions such that the observer-based control system is strictly positive real, output strictly passive, and zero state detectable. This result allows to use the proposed controller to asymptotically stabilize a large class of linear boundary controlled infinite dimensional pH systems and non-linear pH systems when using a linear approximation of these systems to design the controller. An infinite dimensional Timoshenko beam model and a finite dimensional non-linear model of a microelectromechanical actuator are used to illustrate the effectiveness of the proposed approach.

**APPENDIX**

**Boundary controlled PHS on 1D domain**

In this subsection the definition of boundary controlled port-Hamiltonian (BC-PHS) system is given. The reader is refered to [8, 17, 20] for further details and definitions. A BC-PHS is a dynamical system governed by the following partial differential equation

\[
\frac{\partial z}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(H(\zeta)z(\zeta,t)) + P_0 H(\zeta)z(\zeta,t),
\]

\[
z(\zeta,0) = z_0(\zeta),
\]

\[
W_B \left( f_\alpha(t), e_\alpha(t) \right) = u(t),
\]

\[
y(t) = W_C \left( f_\alpha(t), e_\alpha(t) \right).
\]

where the initial condition is given by (19), the boundary input by (20) and the boundary output by (21). Here $z(\zeta,t) \in \mathbb{R}^n$ is the state variable with initial condition $z_0(\zeta)$. $\zeta \in [a,b]$ is the 1D domain and $t \geq 0$ is the time. $P_1 = P_1^T \in \mathbb{R}^{n \times n}$ is a non-singular matrix. $P_0 = -P_0^T \in \mathbb{R}^{n \times n}$, $H(\zeta)$ is a bounded and continuously differentiable matrix-valued function satisfying for all $\zeta \in [a,b]$, $H(\zeta) = H^T(\zeta)$ and $mI < H(\zeta) < M I$ with $0 < m < M$ both scalars independent on $\zeta$. The Hamiltonian energy function of (18) is given by

\[
H(t) = \frac{1}{2} \int_a^b z(\zeta,t)^T H(\zeta) z(\zeta,t) d\zeta.
\]
\( \begin{pmatrix} f_{\partial}(t) \\ e_{\partial}(t) \end{pmatrix} \) are the boundary port variables defined as
\[
\begin{pmatrix} f_{\partial}(t) \\ e_{\partial}(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} H(b)z(b, t) \\ H(a)z(a, t) \end{pmatrix}.
\]

\( W_B, W_C \in \mathbb{R}^{n \times 2m} \) are two matrices such that if \( W_B \Sigma W_B^T = W_C \Sigma W_C^T = 0 \) and \( W_C \Sigma W_C^T = I \), with \( \Sigma = \begin{pmatrix} 0 & I \end{pmatrix} \), then \( \dot{H}(t) = u(t)^T y(t) \).

**ZSD and OSP non-linear control system**

In this subsection the definition of zero-state detectable (ZSD) and output strictly passive (OSP) non-linear systems is given. The reader is refereed to [2] for further details and definitions. Consider a non-linear controlled system

\[
\dot{x} = f(x, u), \quad y = h(x, u)
\]

with \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \) and \( f(\cdot) \) and \( h(\cdot) \) sufficiently smooth differentiable mappings, then [22] is

- OSP if there exists \( \epsilon > 0 \) such that it is dissipative with respect to the supply rate \( s(u, y) = u^T \epsilon - \epsilon \| y \|^2 \);
- ZSD if \( u(t) = 0, y(t) = 0, \forall t \geq 0 \), implies \( \lim_{t \to \infty} x(t) = 0 \).

A (non-linear) PHS is a dissipative system with storage function \( H(x) \) [2].

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