VARIANCE ESTIMATES AND ALMOST EUCLIDEAN STRUCTURE

GRIGORIS PAOURIS AND PETROS VALETTAS

Abstract. We introduce and initiate the study of new parameters associated with any norm and any log-concave measure on $\mathbb{R}^n$, which provide sharp distributional inequalities. In the Gaussian context this investigation sheds light to the importance of the statistical measures of dispersion of the norm in connection with the local structure of the ambient space. As a byproduct of our study, we provide a short proof of Dvoretzky’s theorem which not only supports the aforementioned significance but also complements the classical probabilistic formulation.

1. Introduction

The main focus of this note is to establish new distributional inequalities for convex functions with respect to log-concave measures. The new ingredient in these inequalities is that the controlling parameter is the variance rather than the Lipschitz constant or some moment of the length of the gradient of the function. The study of these inequalities has been motivated by the need to quantify efficiently the almost Euclidean subspaces of high-dimensional normed spaces. In particular, we would like to understand the dependence on $\varepsilon$ in the almost isometric version of Dvoretzky’s theorem [Dvo61]. In fact in this note, we are interested in the randomized Dvoretzky’s theorem (as has been established in the seminal work of Milman [Mil71]) which states that for any $\varepsilon \in (0, 1)$ there exists a constant $c(\varepsilon) > 0$ with the following property: for any normed space $X = (\mathbb{R}^n, \| \cdot \|)$ the random (with respect to the Haar measure on the Grassmannian) $k$-dimensional subspace $F$ satisfies

\[(1 - \varepsilon)M\|z\|_2 \leq \|z\| \leq (1 + \varepsilon)M\|z\|_2\]

for all $z \in F$ as long as $k \leq c(\varepsilon)k(X)$, where $M$ is the average of the norm on the unit Euclidean sphere with respect to the uniform probability measure (see §2 for the related definitions). The parameter $k(X)$ is referred to as the critical dimension (or the Dvoretzky number) of $X$ given by $k(X) = n(M(X)/b(X))^2$, where $b(X) = \max\{\|\theta\| : \|\theta\|_2 = 1\}$. We will write $k(X, \varepsilon)$ for the maximal dimension for which (1.1) holds with probability at least $1/2$. Thus, Milman’s formulation yields $k(X, \varepsilon) \geq c(\varepsilon)k(X)$, (For historical remarks, required background material and further extensions the reader may consult [MS86, Pis89, TJ89] and [AAGM15]). Concerning the dependence on $\varepsilon$, let us mention that Milman’s proof yields $c(\varepsilon) \simeq \varepsilon^2/\log \frac{1}{\varepsilon}$. This was subsequently improved by Gordon in [Gor85] showing that one can always have $c(\varepsilon) \simeq \varepsilon^2$ and an alternative approach was presented by Schechtman in [Sch89]. Hence, one has $k(X, \varepsilon) \geq \varepsilon^2k(X)$. The latter asymptotic formula is optimal up to universal constants as the example of the $\ell_1^n$ shows. However, there are (several) examples of spaces which show that the function $c(\varepsilon)$ can be significantly improved.

The investigation of the randomized Dvoretzky’s theorem on the almost isometric level for special cases of normed spaces was initiated by the works of Schechtman [Sch07] and Tikhomirov [Tik14] who determined asymptotically the dimension $k(\ell_\infty^n, \varepsilon)$. Their estimate $k(\ell_\infty^n, \varepsilon) \simeq \varepsilon^{2} \log \frac{1}{\varepsilon} n^{2}$ is better than $\varepsilon^{2} \log \frac{1}{\varepsilon}$ as was predicted by the optimal form of Milman’s formula due Gordon’s [Gor85]. This result has been extended by Zinn and the authors in [PVZ15] for all $\ell_p^n$ spaces. Once again the dimension $k(\ell_p^n, \varepsilon)$ is much larger than the estimate derived by the classical proof when $2 < p \leq \infty$. The proof is based on the fact that the $\ell_p$ norm is much more concentrated on Gauss’ space than the usual “concentration of measure” suggests. In that case we will say that the norm is “overconcentrated”. 

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This phenomenon is not only observed in the $\ell_p^n$ spaces, but is apparent in many other cases. In [PV15] we have shown that for all $2 < p < \infty$ and for any $n$-dimensional subspace $X$ of $L_p$, in Lewis’ position [Lew78], either $X$ almost spherical sections of proportional dimension (that is $k(X) \simeq n$) or $X$ is overconcentrated. Furthermore, among all $n$-dimensional subspaces $X$ of $L_p$, $2 < p < \infty$ the worst $k(X, \varepsilon)$ occurs for the $\ell_p^n$’s. Recently Tikhomirov [Tik17] proved that for every 1-unconditional $n$-dimensional normed space $X$, in $\ell$-position (see e.g. [BJ98] for the definition of the $\ell$-position), either $k(X)$ is polynomial with respect to $n$ or $X$ is overconcentrated. His approach shows that the worst $k(X, \varepsilon)$ in this class of spaces is attained for the $\ell_p^n$.

In some of the above cases, the new observed phenomenon is due to superconcentration (as defined by Chatterjee in [Cha14]) and can be quantified by employing Talagrand’s $L_1 - L_2$ bound for gaussian measure (see e.g. [Tal94] and [CEL12]). That was the crucial tool in the investigation in [PVZ15] as well as in [Tik17]. However, the superconcentration is not the only reason that causes the norm of a space to be more concentrated than expected. All the aforementioned cases share the common feature that $\text{Var}(Z) \ll \text{Lip}(\| \cdot \|)^2$ where $Z$ is an $n$-dimensional standard gaussian vector (recall that the classical gaussian concentration yields the bound $\text{Var}(Z) \leq \text{Lip}(\| \cdot \|)^2$).

Our aim in this note is to put on display the importance of the (normalized) variance in the study of the almost Euclidean structure in high-dimensional normed spaces and to initiate a systematic investigation of the concentration properties of convex functions with respect to this measure of dispersion. This investigation has two main directions. Firstly, we show that this new parameter can be used to provide a short proof of Dvoretzky’s theorem which can be viewed as the probabilistic and quantitative version of the topological proof due to Figiel [Fig76] and Szankowski’s analytic proof from [Sza74]. Further study of this parameter is also considered and is compared with the classical Dvoretzky number.

Secondly, we work on distributional inequalities in the context of log-concave measures. We show that (one-sided) deviation estimates, where the variance is involved, can be proved for any pair of a convex function and a log-concave probability measure on Euclidean space, by extending a machinery developed in [PV16].

The rest of the paper is organized as follows: In Section 2 we fix the notation and we recall standard background material. In Section 3 we present a simple proof that every infinite dimensional Banach space contains $\ell_2^n$’s uniformly. The local version of our approach uses the parameter of the normalized variance and yields optimal dependence on the size of the distortion. In Section 4 we study further the aforementioned parameter and we compare it with the classical Dvoretzky number. Our approach uses concentration estimates for the mean width of random projections of any convex body in $\mathbb{R}^n$. We postpone the somewhat more systematic study of this topic to §6. In Section 5 we study, in the general context of log-concave measures small deviation and small ball estimates for norms whose tightness is quantified in terms of the parameter of the normalized variance. Finally, in Section 6 we study independently distributional inequalities for the mean width of random projections of any convex body in $\mathbb{R}^n$ in terms of the Haar measure on the Grassmannian.

2. Notation and background material

We work in $\mathbb{R}^n$ equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. The Euclidean norm is given by $\|x\|_2 = \sqrt{\langle x, x \rangle}$, $x \in \mathbb{R}^n$. For the $(n - 1)$-dimensional Euclidean sphere we write $S^{n-1} = \{ x \in \mathbb{R}^n : \langle x, x \rangle = 1 \}$. Let also $\sigma$ be the uniform probability measure on $S^{n-1}$.

The orthogonal group on $\mathbb{R}^n$ is denoted by $O(n)$ and consists of all matrices which preserve the angles, i.e. $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. Thus, $O(n) = \{ U \in \mathbb{R}^{n \times n} : UU^* = I \}$.

The action of $O(n)$ on itself generates the Haar probability measure which we will denote by $\mu_n$.

Let $SO^{\pm}(n)$ be the collection of all matrices which are volume preserving,

$$SO^{\pm}(n) = \{ U \in O(n) : \det U = \pm 1 \}.$$  

Similarly, we may define the Haar probability measure on the special orthogonal group $SO(n)$. This is nothing more than the restriction of $\mu_n$ in $SO(n)$ (see e.g. [Mec14] for background details), thus we will still denote it by $\mu_n$. Note that although $SO^-(n)$ is not a group itself we will refer to the restricted measure, with some abuse of terminology, as the Haar measure on $SO^-(n)$ since it is still invariant (within $SO^-(n)$) under...
the action of $SO^+(n)$. One may check that if $U$ is Haar distributed in $SO^+(n)$ and $M$ is a fixed matrix in $SO^-(n)$, then $UM$ is Haar distributed in $SO^-(n)$. The Hilbert-Schmidt norm of a matrix $T$ is denoted by $\|T\|_{HS}$ and the operator norm of $T$ with $\|T\|_\infty$.

The Grassmann space $G_{n,k}$ is the set of all $k$-dimensional subspaces of $\mathbb{R}^n$. We consider the Haar measure $\nu_{n,k}$ on $G_{n,k}$ which is invariant under the action of the orthogonal group $O(n)$. For an arbitrary norm on $\mathbb{R}^n$ we write $\|\cdot\|$ and for the normed spaces $(\mathbb{R}^n,\|\cdot\|)$ we will use the letters $X,Y$.

The random variables will be denoted by $\zeta,\eta,\xi,...$. For random vectors on $\mathbb{R}^n$, distributed according to some law $\mu$, we write $Z,W$ or $W=(w_1,\ldots,w_n)$. For any random variable $\xi$ on some probability space $(\Omega,\mathcal{A},\mathbb{P})$ with $\xi \in L_2(\Omega)$ and $\mathbb{E}\xi \neq 0$ we define the parameter $\beta(\xi)$ as follows:

\begin{equation}
\beta(\xi) := \frac{\text{Var}(\xi)}{(\mathbb{E}\xi)^2},
\end{equation}

where $\mathbb{E}$ stands for the expectation and $\text{Var}$ for the variance. For any normed space $X = (\mathbb{R}^n,\|\cdot\|)$ and for any Borel probability measure $\mu$ on $\mathbb{R}^n$ we define the parameter:

\begin{equation}
\beta_\mu(X) := \beta(\|Z\|),
\end{equation}

where $Z$ is a random vector distributed according to $\mu$.

Recall the well known fact that if $\xi$ is a random variable in some probability space $(\Omega,\mathcal{A},\mathbb{P})$ with $\mathbb{E}\xi^2 < \infty$ and $m = \text{med}(\xi)$ is a median of $\xi$, then

$$|\mathbb{E}\xi - m| \leq \mathbb{E}|\xi - m| \leq \sqrt{\text{Var}(\xi)}.$$ 

If $\mathbb{E}\xi \neq 0$ then we readily get:

$$\left| \frac{m}{\mathbb{E}\xi} - 1 \right| \leq \frac{1}{\sqrt{\beta(\xi)}}.$$ 

This says that the parameter $\beta$ quantifies how close are the measures of central tendency, median and expectation.

Let us note that for probability space $(\mathbb{R}^n,\mu)$, where $\mu$ is a log-concave probability measure and the variable $\xi$ is $\|Z\|$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $Z$ is distributed according to $\mu$, Borell’s lemma \cite{Bor75} (see also \S 5) implies $m \simeq \mathbb{E}\|Z\|$. Thus, if we are only interested in the order of magnitude of the parameter $\beta$, the expectation can be sufficiently replaced by the median. For general facts concerned with gaussian measures we refer the reader to \cite{Bog98}.

A convex body $K$ in $\mathbb{R}^n$ is a compact, convex set with non-empty interior. The convex bodies will be denoted with $A,K,L,\ldots$. A convex body $K$ is said to be symmetric when $x \in K$ if and only if $-x \in K$. For any symmetric convex body $K$ we write $\|\cdot\|_K$ for the norm (gauge function) induced by $K$. Therefore, very frequently we identify notationally the normed space, whose norm is generated by a convex body, with the convex body itself. The volume (i.e. the Lebesgue measure) of a convex body $K$ in $\mathbb{R}^n$ is denoted by $|K|$. We define the circumsphere of $K$, as the number $R(K) := \max_{x \in B_2^n} \|x\|_K$. That is the smallest $R > 0$ for which $K \subseteq RB_2^n$. For any symmetric convex body $K$ in $\mathbb{R}^n$ we define the following averages:

$$J_q(K) := \left( \frac{1}{|K|} \int_K \|x\|_K^q \,dx \right)^{1/q}, \quad -n < q \neq 0.$$ 

The next lemma is essentially from \cite{Kla04}. The idea of the proof goes back to Rudelson (see \cite{Kla07}):

**Lemma 2.1.** Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Then, for all $q > 0$ we have:

$$J_q(K) \geq a_{n,q}^{-1} R(K), \quad a_{n,q}^{-1} := \frac{q}{2} B(q,n+1).$$

**Remarks 2.2.** (i) Let us note that by using standard asymptotic estimates for the Beta function we have:

\begin{equation}
a_{n,q} \leq \exp \left( \frac{cn}{q} \log \left( \frac{eq}{n} \right) \right)
\end{equation}

for all $q \geq n$.

(ii) The symmetry is not essential for the proof.
Next, for any symmetric convex body $K$ in $\mathbb{R}^n$ we define:

$$\text{vrad}(K) := \left( \frac{|K|}{|B^n_2|} \right)^{1/n}, \quad M_p^q(K) := \int_{S^{n-1}} \|\theta\|^q_K \, d\sigma(\theta), \quad q \neq 0.$$ 

The proof of the next lemma can be found in [Kla04].

**Lemma 2.3.** Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Then, for any $p > -n$ we have:

$$\text{vrad}(K)^n J_p^n(K) = \frac{n}{n + p} M_{-(n+p)}^{-}(K).$$

We denote by $b(K) = \max_{\theta \in S^{n-1}} \|\theta\|_K$. Note that $r(K) = 1/b(K)$ is the maximal radius of the centered inscribed ball in $K$. The polar body $K^\circ$ of a convex body $K$ is defined as $K^\circ = \{ y : \langle x, y \rangle \leq 1 \ \forall x \in K \}$. For any symmetric convex body $K$ the support function of $K$ at $\theta \in S^{n-1}$ is the half width of the body in direction $\theta$, i.e. $h_K(\theta) = \max\{ \langle x, \theta \rangle : x \in K \}$. Note that $h_K$ is the dual norm of $\| \cdot \|_K$. One may check that $b(K) = R(K^\circ)$ and $R(K) R(K^\circ) = 1$. The Banach-Mazur distance between two isomorphic normed spaces $X, Y$ is denoted by $d(X,Y)$ with

$$d(X,Y) := \inf \{ ||T|| \cdot ||T^{-1}|| : T : X \rightarrow Y, \text{ linear isomorphism} \}.$$ 

The reader may consult the monographs [MS86, Pis89, TJ89, AAGM15] for a detailed background information on the local theory of normed spaces.

Throughout the text we make use of $C, c, C_1, c_1, \ldots$ for positive absolute constants whose values may change from line to line. We also introduce the notation $Q_1 \lesssim Q_2$ for any two quantities $Q_1$ and $Q_2$ (which may depend on dimension or some geometric parameter of the normed space or the convex body) if there exists absolute constant $C > 0$ such that $Q_1 \leq CQ_2$. We write $Q_1 \simeq Q_2$ is $Q_1 \lesssim Q_2$ and $Q_2 \lesssim Q_1$. We write the above signs with a subscript $\lesssim_p, \simeq_p$, if the involved constant depends on the parameter $p$.

### 3. A Probabilistic Proof of Dvoretzky’s Theorem Without Concentration: The Variance Method

In this section we provide a probabilistic proof that $\ell_2^n$’s embed uniformly into any infinite dimensional Banach space $X$ without utilizing the concentration of measure. What is crucial in our approach is the use of the new critical parameter $\beta_{\gamma_n}(X)$ defined in (2.2). We investigate this parameter further in §3.

In order to prove the result we will use some standard lemmas such as the Dvoretzky-Rogers lemma and a net argument on the sphere which are rather standard in all probabilistic proofs. However, even if our approach is more elementary leads to better dependence on $\varepsilon$ in several interesting cases (see §3.3). Our proposed local form of Dvoretzky’s theorem reads as follows.

**Theorem 3.1.** For every $\varepsilon \in (0,1)$ and for any finite-dimensional normed space $X$, there exists $k$-dimensional subspace $F$ of $X$ with $k \geq c \log \frac{1}{\varepsilon} / \log \frac{1}{\delta}$, such that $d(F, \ell_2^k) < 1 + \varepsilon$, where $\beta = \beta(X)$.

Later this formulation will be used to compare the new parameter $\beta(X)$ with the Dvoretzky number $k(X)$ of the normed space $X = (\mathbb{R}^n, \| \cdot \|)$. For the proof we will need the following standard lemma (whose proof is included for the sake of completeness).

**Lemma 3.2.** Let $X$ be a $k$-dimensional normed space.

(i) For any $\delta \in (0,1)$ there exists a $\delta$-net $N$ on $S_X$ with $\text{card}(N) \leq (1 + \frac{3}{\delta})^k$.

(ii) Let $Y$ be a normed space and let $T : X \rightarrow Y$ be a linear mapping with the property:

$$1 - \varepsilon \leq ||Tz|| \leq 1 + \varepsilon$$

for all $z$ in a $\delta$-net $N$ of $S_X$ where $0 < \delta, \varepsilon < 1$. Then, we have:

$$\frac{1 - \varepsilon - 2\delta}{1 - \delta} \leq ||T\theta|| \leq \frac{1 + \varepsilon}{1 - \delta},$$

for all $\theta \in S_X$. 

[4]
Proof. The proof of the first assertion can be found in [MS86]. Let us proceed with (ii). Fix \( \theta \in S_X \). Then, there exists \( z \in \mathcal{N} \) with \( \| z - \theta \| < \delta \). Thus, we may write:

\[
\langle T \theta \rangle \leq \delta \| T \| + \| T z \| \leq \delta \| T \| + (1 + \varepsilon).
\]

Since \( \theta \) was arbitrary, it follows that \( \| T \| \leq \delta \| T \| + (1 + \varepsilon) \), or equivalently

\[
\| T \| \leq (1 - \delta)^{-1} (1 + \varepsilon).
\]

Plug this back in \((3.1)\) we obtain the right-hand side estimate. For the left hand side we argue as follows:

\[
\| T \theta \| \geq \| T z \| - \| T (\theta - z) \| \geq (1 - \varepsilon) - \delta \| T \| \geq (1 - \varepsilon) - \frac{\delta (1 + \varepsilon)}{1 - \delta},
\]

by the estimate \((3.2)\).

\(\square\)

Now we are ready to prove the aforementioned form of Dvoretzky’s theorem:

Proof of Theorem 3.7. Let \( \{g_{ij}\}_{i,j=1}^{n,k} \) be i.i.d. standard Gaussian random variables in some probability space \((\Omega, \mathcal{A}, \mathbb{P})\). We consider the random Gaussian operator: \( G : \ell^k_2 \to X \) with

\[
G_\omega z = \sum_{i=1}^n \sum_{j=1}^k g_{ij}(\omega) z_j e_i, \quad z = (z_1, \ldots, z_k) \in \ell^k_2, \quad \omega \in \Omega.
\]

Note that for fixed \( \theta \in S^{k-1} \) if we apply Chebyshev’s inequality we obtain:

\[
\mathbb{P} \left( \| G\theta \| - \mathbb{E}\| Z \| > \varepsilon \mathbb{E}\| Z \| \right) = \mathbb{P} \left( \| Z \| - \mathbb{E}\| Z \| > \varepsilon \mathbb{E}\| Z \| \right) \leq \beta(X)/\varepsilon^2,
\]

for \( \varepsilon > 0 \) and \( Z \) standard Gaussian random vector. Now fix \( \varepsilon \in (0,1/3) \) and employ Lemma 3.2 with \( \delta = \varepsilon/2 \), to get:

\[
\mathbb{P} \left( \exists z \in \mathcal{N} : \| Gz \| - \mathbb{E}\| Z \| > \varepsilon \mathbb{E}\| Z \| \right) \leq \left( \frac{6}{\varepsilon^2} \right)^k \frac{\beta}{\varepsilon^2} \leq (6\varepsilon)^{3k} \beta.
\]

Therefore, as long as \((6\varepsilon)^{3k} \beta < 1/2 \) or \( k \leq c \log \frac{1}{\varepsilon} / \log \frac{1}{\varepsilon} \), there exists \( \omega \in \Omega \) with the property:

\[
\| G_\omega z \| - \mathbb{E}\| Z \| \leq \varepsilon \mathbb{E}\| Z \|,
\]

for all \( z \in \mathcal{N} \). Fix the operator \( G = G_\omega \) and let \( T : \ell^k_2 \to X \) with \( T := (\mathbb{E}\| Z \|)^{-1} G \). Then, by Lemma 3.2 and the choice of \( \delta \) we conclude that:

\[
1 - 2\varepsilon < \frac{1 - \varepsilon - 2\delta}{1 - \delta} \leq \| T \theta \| \leq \frac{1 + \varepsilon}{1 - \delta} < 1 + 2\varepsilon,
\]

for all \( \theta \in S^{n-1} \). The result follows; for \( F := T(\ell^k_2) \) we get \( d(F, \ell^k_2) < 1 + 11\varepsilon \).

3.1. Quantitative form of Theorem 3.1. Using John’s position [John48] and the classical Dvoretzky-Rogers Lemma from [DR50] we can show the following estimate:

Proposition 3.3. Let \( X = (\mathbb{R}^n, \| \cdot \|) \) be a finite-dimensional normed space and assume that the Euclidean ball \( B^n_2 \) is the ellipsoid of maximal volume inside \( B_X \). Then, we have:

\[
\beta(X) \leq \frac{C}{\log n}.
\]

Therefore, taking into account Proposition 3.3 we readily get the finite representability of \( \ell_2 \) in any infinite dimensional Banach space \( X \):

Corollary 3.4. For every \( \varepsilon \in (0,1) \) and for any \( n \)-dimensional normed space \( X \) there exists \( k \geq c \log \log n / \log \frac{1}{\varepsilon} \) and \( k \)-dimensional subspace \( F \) of \( X \) with \( d(F, \ell^k_2) < 1 + \varepsilon \).

Proof of Proposition 3.3 (Sketch). In order to prove the above estimate we need the next lemma from [DR50]:

Lemma 3.5. Let \( X = (\mathbb{R}^n, \| \cdot \|) \) be \( n \)-dimensional normed space and let \( B^n_2 \) be the ellipsoid of maximal volume inside \( B_X \). Then, there exist orthonormal vectors \( v_1, \ldots, v_m \) with \( m \geq n \) such that \( \| v_j \| > 1/2 \) for \( j = 1, 2, \ldots, m \).
A proof of this result can be found in [MS86]. Using Lemma [5.3] and an averaging argument over signs (see [MS86] for details) we arrive at the following estimate:

Claim 1. If $B^n_2$ is the ellipsoid of maximal volume inside $B_1$ we have:

$$\mathbb{E}\|Z\| \geq c\sqrt{\log n}, \quad Z \sim N(0, I_n).$$

One more ingredient is needed:

Claim 2. If $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $b = \max_{\theta \in S^{n-1}} \|\theta\|$, then

$$\text{Var}\|Z\| \leq b^2.$$  

For the proof of Claim 2 we may employ the (Gaussian) Poincaré inequality:

$$(3.4) \quad \text{Var}[f(Z)] \leq \mathbb{E}\|\nabla f(Z)\|^2.$$  

The fact that $\langle \nabla\|x\|, v \rangle \leq b$ for all $x \in \mathbb{R}^n$ and $v \in S^{n-1}$ proves the claim.

The assertion follows if we combine the previous two claims. □

3.2. Probabilistic estimate. The argument in Theorem [5.1] in fact implies that for $k \leq c \log(1/\beta)/\log(1/\epsilon)$ the random $k$-dimensional subspace $F$ of $X$ is $(1 + \epsilon)$-Euclidean with probability $> 1 - e^{-3k}$. Let us first recall the definition:

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space. A subspace $F$ of $X$ is said to be $(1 + \epsilon)$-spherical if

$$\max_{z \in S_F} \|z\|/\min_{z \in S_F} \|z\| < 1 + \epsilon.$$  

In order to verify the aforementioned probabilistic statement, note that the proof of Theorem [3.1] yields the following estimate:

$$P\left(\|G\theta\| - \mathbb{E}\|Z\| \leq \epsilon \mathbb{E}\|Z\|, \forall \theta \in S^{k-1}\right) \geq 1 - e^{-3k},$$

provided that $k \leq \frac{1}{4} \log(1/\beta)/\log(6e/\epsilon)$. Note that:

$$\left\{\|G\theta\| - \mathbb{E}\|Z\| \leq \epsilon \mathbb{E}\|Z\|, \forall \theta \in S^{k-1}\right\} \subseteq \left\{\max_{\theta} \|G\theta\|/\min_{\theta} \|G\theta\| < \frac{1 + \epsilon}{1 - \epsilon}\right\} \equiv A_1$$

and similarly

$$\forall \theta \in S^{k-1}; \|G\theta\| - \mathbb{E}\|Z\| \leq \epsilon \mathbb{E}\|Z\| \equiv A_2.$$  

Furthermore, the event $A_1 \cap A_2$ satisfies:

$$A_1 \cap A_2 \subseteq \left\{G(\mathbb{R}^k) \text{ is } \left(\frac{1 + \epsilon}{1 - \epsilon}\right)^2 \text{ - spherical}\right\}.$$  

Finally, recall that $P(\omega : G_\omega(\mathbb{R}^k) \in \mathcal{B}) = \nu_{n,k}(F \in G_{n,k} : F \in \mathcal{B})$ for any Borel set $\mathcal{B}$ in $G_{n,k}$ and that

$$P\left(\{\|G\theta\| - \mathbb{E}\|Z\| \leq \epsilon \mathbb{E}\|Z\|, \forall \theta \in S^{k-1}\}\right) > 1 - Ce^{-c\epsilon^2n}.$$  

Thus, we get:

$$\nu_{n,k}\left(\left\{F \text{ is } \left(\frac{1 + \epsilon}{1 - \epsilon}\right)^2 \text{ - spherical}\right\}\right) \geq P(A_1 \cap A_2) \geq 1 - e^{-3k} - Ce^{-c\epsilon^2n} \geq 1 - C'e^{-c'k},$$

since $k \leq \epsilon^2 n$ as long as $\epsilon \gg n^{-1/2}$ (see §4).
3.3. Concentration vs Chebyshev’s inequality. The function $c(\varepsilon) \sim |\log \varepsilon|$ that appears in this simple argument of Theorem 3.1 is a nice feature. In particular, if it is combined with the fact that there exist $n$-dimensional normed spaces $X$ with critical dimension $k(X) \approx \log n$ whereas $\beta(X) \lesssim n^{-\alpha}$ for some absolute constant $\alpha > 0$ (see Proposition 3.3), then Theorem 3.1 yields the existence of almost Euclidean subspaces of the same dimension as Milman’s formulation [Mil71] provides, but with better dependence on $\varepsilon$. Moreover, for those spaces we conclude that the $k$-dimensional random subspace is $(1 + \varepsilon)$-Euclidean with probability $> 1 - e^{-ck}$ as long as $k \leq c \log n / \log \varepsilon$.

In the light of the above comments, the random version of Dvoretzky’s theorem can be complemented in the following way:

**Corollary 3.6.** Let $X$ be an $n$-dimensional normed space. Then, for any $\varepsilon \in (0, 1)$ there exists $k \geq c \max \left\{ \varepsilon^2 k(X), (\log \frac{1}{\varepsilon})^{-1} \log \frac{1}{\varepsilon(X)} \right\}$ such that the random $k$-dimensional subspace $F$ satisfies:

$$\frac{1 - \varepsilon}{M} B_F \subseteq B_X \cap F \subseteq \frac{1 + \varepsilon}{M} B_F,$$

with probability greater than $1 - e^{-ck}$.

3.4. Two examples on the dependence on $k(X, \varepsilon)$. The classical results of Milman [Mil71], Gordon [Gor85] and Schechtman [Sch89] predict that the dependence on $\varepsilon$ should be $\varepsilon^2$. Here we show that this is always the case after a small perturbation.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be the mapping:

$$f(t, x) = |t| + \|x\|_\infty, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

Then, we have the following properties:

i. $f$ is 2-equivalent norm to $\| \cdot \|_{\ell^\infty_{n+1}}$.
ii. $\text{Var}[f(Z)] \approx 1$.
iii. Moreover, we have: $k(f) \approx k(\ell^\infty_n) \approx \log n$.
iv. $\mathbb{E}[f(Z)] \approx \sqrt{\log n}$.

We also have the following:

**Proposition 3.7.** For any $r \geq 1$ we have:

$$\mathbb{E}[|f(g_1, Z_1) - f(g_2, Z_2)|^r]^{1/r} \approx \sqrt{r},$$

where $Z_1, Z_2$ are independent, standard Gaussian random vectors on $\mathbb{R}^n$.

**Proof.** Using the triangle inequality we may write:

$$\mathbb{E}[|f(g_1, Z_1) - f(g_2, Z_2)|^r]^{1/r} \geq (\mathbb{E}[|g_1| - |g_2|]^r)^{1/r} - (\mathbb{E}[||Z_1||_\infty - ||Z_2||_\infty]^r)^{1/r} \geq c_2 \sqrt{r} - C_2 \frac{r}{\sqrt{\log n}} \geq c_2' \sqrt{r}$$

for all $1 \leq r \leq c_3 \log n$. The assertion follows. \(\square\)

**Corollary 3.8.** Let $f$ be as above. Then, we have:

$$\mathbb{P}\left( |f(Z) - \mathbb{E} f(Z)| > \varepsilon \mathbb{E} f(Z) \right) \geq c_1 e^{-C_1 \varepsilon^2 \log n},$$

for all $\varepsilon > 0$, where $Z \sim N(0, I_{n+1})$.

**Proof.** It follows from Proposition 3.7 and the Paley-Zygmund inequality. \(\square\)

Arguing as in [PVZ15, Section 5] and using Corollary 3.8 we may conclude the following:

**Theorem 3.9.** For every $n$ there exists an $n$-dimensional, 1-unconditional normed space $X$ which is 2-isomorphic to $\ell^\infty_n$ and has the following property: If $k(X, \varepsilon)$ is the maximal dimension $k$ for which the $k$-dimensional random subspace of $X$ is $(1 + \varepsilon)$-Euclidean with probability greater than $1 - e^{-k}$, then $k(X, \varepsilon) \approx \varepsilon^2 \log n$. 7
At this point let us note that recently Tikhomirov [Tik17] showed that there exists 1-unconditional normed space \( X \) whose ball is in John’s position and the critical dimension \( k(X, \varepsilon) \) in the randomized Dvoretzky is of the order \( \varepsilon^2 \log n \).

In the classical paper of Figiel, Lindenstrauss and Milman [FLM77], it is proven that for spaces with cotype \( 2 \leq q < \infty \) (see e.g. [MS86] for the related definition) the critical dimension in the randomized Dvoretzky, in John’s position, is at least of the order \( \varepsilon^2 n^{2/q} \). The next example is concerned with the dependence on \( \varepsilon \) for spaces having the cotype property. It shows that there exists an 1-unconditional \( n \)-dimensional normed space \( X \) with cotype \( q \), \( 2 \leq q < \infty \) which has Dvoretzky number \( \approx n^{2/q} \), though the critical dimension \( k(X, \varepsilon) \) in the randomized Dvoretzky is of the exact order \( \approx n^{2/4} \).

Let \( 2 < q < \infty \). As before, we consider the map \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) with
\[
(f(t, x) = |t| + \|x\|_q, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.
\]

We have the following properties:

i. \( \|t, z\|_q \leq f(t, z) \leq 2\|t, z\|_q \) for all \( t \in \mathbb{R} \) and \( z \in \mathbb{R}^n \). Therefore, the \( q \)-cotype constant \( \mathcal{C}_q(X) \) of \( X \) satisfies:
\[
\mathcal{C}_q(X) \simeq \mathcal{C}_q(f^{n+1}) \simeq \sqrt{q},
\]
e.g. see [AK10].

ii. If \( X = (\mathbb{R}^{n+1}, f(\cdot)) \), then we have: \( k(X) \simeq \sqrt{q}n^{2/q} \).

iii. If \( Z \) is a standard Gaussian vector on \( \mathbb{R}^{n+1} \), we have: \( \mathbb{E}[f(Z)] \simeq \mathbb{E}\|Z\|_q \simeq \sqrt{q}n^{1/q} \).

Recall the following fact proved in [PVZ15]:

**Lemma 3.10.** Let \( 2 < q < \infty \). Then, for all large enough \( n \) we have:
\[
\left( \mathbb{E}\|Z_1\|_q - \|Z_2\|_q \right)^{1/r} \leq c_1(q)\mathbb{E}\|Z_1\|_q \max \left\{ \sqrt{\frac{r}{n}}, \frac{q^{r/2}}{n} \right\},
\]
for all \( r \geq 1 \), where \( c_1(q) > 0 \) is a constant depending only on \( q \) and \( Z_1, Z_2 \) are independent standard Gaussian random vectors on \( \mathbb{R}^n \).

Using that we obtain the following:

**Proposition 3.11.** Let \( f \) be as above. Then, we have:
\[
\left( \mathbb{E}|f(g_1, Z_1) - f(g_2, Z_2)|^r \right)^{1/r} \geq c_2(q)\sqrt{\frac{r}{k(X)}}\mathbb{E}|f(g_1, Z_1)|,
\]
for all \( r \geq 1 \), where \( g_1, g_2 \) are i.i.d. standard normals and \( Z_1, Z_2 \) are independent standard Gaussian vectors on \( \mathbb{R}^n \).

The proof is similar to that of Proposition 3.7, thus it is omitted. Finally, we get:

**Corollary 3.12.** Let \( f \) as above. Then, we have:
\[
\mathbb{P}\left( |f(Z) - \mathbb{E}f(Z)| > \varepsilon \mathbb{E}f(Z) \right) \geq ce^{-c_3(q)\varepsilon^2 n^{2/q}},
\]
for all \( \varepsilon > 0 \), where \( Z \sim N(0, I_{n+1}) \).

The next Theorem is the analogue of Theorem 3.9 in the case of spaces with cotype:

**Theorem 3.13.** Let \( 2 < q < \infty \) there exists a constant \( C(q) \gg 1 \) with the following property: For any \( n \geq C(q) \) there exists an \( n \)-dimensional, 1-unconditional normed space \( X \) which satisfies:

i. \( X \) has cotype \( q \) with \( \mathcal{C}_q(X) \simeq \sqrt{q} \) and,

ii. If for any \( \varepsilon \in (0, 1) \) we denote by \( k(X, \varepsilon) \) the largest dimension \( k \) for which the random subspace of \( X \) is \((1 + \varepsilon)\)-Euclidean with probability greater than \( 1 - e^{-ck} \), then \( k(X, \varepsilon) \simeq \varepsilon^2 n^{2/q} \).

A more detailed study of the instability of the concentration will appear in [PV17].
4. ON THE PARAMETER $\beta$

In this Section we study the parameter $\beta$ and we show connections with the one-sided inclusion in the randomized Dvoretzky theorem. Let us recall the definition of $\beta$ for a normed space $X = (\mathbb{R}^n, \| \cdot \|)$ equipped with some log-concave probability measure $\mu$:

$$\beta_\mu(X) = \frac{\text{Var}_\mu \|Z\|^p}{(\mathbb{E}_\mu \|Z\|^p)^2}.$$ 

If the prescribed measure is the Gaussian we will often omit the subscript. The first result shows that the extremal space for the parameter $\beta_n(X)$ is the Euclidean:

**Lemma 4.1.** Let $X$ be $n$-dimensional normed space. Then, one has:

$$\beta_n(X) \geq \beta_n(L_2^n) \simeq 1/n.$$ 

**Proof.** Using polar coordinates we see that:

$$\mathbb{E} \|Z\|^p = \mathbb{E} \|Z\|^p \int_{S^{n-1}} \|\theta\|^p d\sigma(\theta),$$

for all $p > 0$. Thus, we may write:

$$1 + \beta(X) = \frac{\mathbb{E} \|Z\|^2}{(\mathbb{E} \|Z\|^2)^2} = \frac{\mathbb{E} \|Z\|^2}{(\mathbb{E} \|Z\|^2)^2} \left( \int \|\theta\|^2 d\sigma(\theta) \right)^2 = (1 + \beta(L_2^n))(1 + \beta_\sigma(X)).$$

Since $\beta_\sigma \geq 0$, the assertion readily follows. $\square$

The following estimates for the parameter $\beta$ of the classical spaces can be found in [PVZ15]:

**Proposition 4.2.** There exist absolute constants $0 < c < 1 < C$ with the following property: For all $n \geq 1$ sufficiently large, one has

$$\beta_n(L_p^n) \simeq \begin{cases} \frac{2^p}{(\log n)^2}, & 1 \leq p < c \log n \\ \frac{C_p}{n}, & C \log n \leq p \leq \infty. \end{cases}$$

Another class of spaces with this property consists of those which “sit” inside $L_p$ and $p$ might be moderately growing along with the dimension of the underlying space. Namely, if we take into account the estimate for the variance for the subspaces of $L_p$, proved in [PVZ15], it follows that for all $1 \leq p < \infty$ and for all subspaces $X$ of $L_p$ with dim $X = n$, there exists a linear image $\tilde{X}$ of $X$ such that:

$$\beta(\tilde{X}) \leq \frac{C_p}{n} \quad \text{or} \quad \min_{T \in GL(n)} \beta(TX) \leq \frac{C_p}{n}. \tag{4.1}$$

General upper and lower bounds for the global parameter $\beta$ are given in the next:

**Proposition 4.3.** Let $X$ be finite-dimensional normed space. Then, we have:

$$\log \frac{1}{\beta(X)} \lesssim k(X) \lesssim \frac{1}{\beta(X)}.$$ 

Note that the upper bound follows from Claim 2 in §3.1. and the definition of $k(X)$. For the lower bound we give a proof which lies on results of independent interest.

For any norm $\| \cdot \|$ (or any symmetric convex body $K$) on $\mathbb{R}^n$ we define the following collections of subspaces: for $1 \leq k \leq n - 1$ and $0 < \varepsilon, \delta < 1$ let

$$\mathcal{S}_{k,\varepsilon} := \{ F \in G_{n,k} : F \text{ is } (1 + \varepsilon) - \text{ spherical subspace of } X = (\mathbb{R}^n, \| \cdot \|) \}$$

and

$$\mathcal{F}_{k,\delta} := \{ E \in G_{n,k} : (1 - \delta)M \leq \| \phi \| \leq (1 + \delta)M, \forall \phi \in S_E \}.$$ 

Next Proposition shows that these two descriptions are essentially equivalent:

**Proposition 4.4.** Let $X = (\mathbb{R}^n, \| \cdot \|)$, let $0 < \varepsilon < 1$ and let $1 \leq m \leq n - 1$. Then, we have:

i. $\mathcal{F}_{m,\varepsilon/2} \subset \mathcal{S}_{m,2\varepsilon}$. 

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ii. \( \nu_{n,m}(S_{m,\varepsilon/3}) \leq C \exp(-c\varepsilon^2 mk(X)) + \nu_{n,m}(F_{m,\varepsilon}) \).

For the proof we shall use a concentration result for the map \( F \mapsto M_F \) which is defined as follows: Fix \( A \) a symmetric convex body on \( \mathbb{R}^n \). Then \( F \mapsto M_F (A) = M(A \cap F) \) for any \( F \in G_{n,k}, 1 \leq k \leq n-1 \). Let us mention that a large deviation estimate for this mapping was proved by Klartag and Vershynin in [KV07, Lemma 3.2]. Their result reads as follows:

**Lemma 4.5.** Let \( A \) be a symmetric convex body on \( \mathbb{R}^n \) and let \( 1 \leq k \leq n-1 \). Then, we have:

\[
\nu_{n,k} \left( \{ F \in G_{n,k} : M_F(A) > \varepsilon M(A) \} \right) \leq C e^{-c k},
\]

for all \( t > 1 \).

Note that this estimate, when \( k = 1 \), is reduced to the large deviation estimate of the norm \( \theta \mapsto \| \theta \|_A \). But in that case the above estimate loses by a term \( k(A) \) on the exponent. In fact, one can recover this missing factor and moreover, prove a concentration inequality for this mapping:

**Theorem 4.6.** Let \( A \) be a symmetric convex body on \( \mathbb{R}^n \) and let \( 1 \leq s \leq n-1 \). Then, we have:

\[
\nu_{n,s} \left( \{ F \in G_{n,s} : |M_F(A) - M(A)| > \varepsilon M(A) \} \right) \leq C e^{-c s k(X)},
\]

for all \( \varepsilon > 0 \) where \( C, c > 0 \) are absolute constants.

We believe that distributional inequalities for functionals on the Grassmannian are interesting on their own right, thus we discuss this topic separately. In fact, we provide two proofs of the latter probabilistic estimate in §6. Taking this for granted we may proceed with the proof of Proposition 4.4.

**Proof of Proposition 4.4.** We write \( k = k(X) \) for short. The first inclusion is trivial. For the second part, note the following inclusion:

\[
S_{m,\varepsilon} \subset A_{m,\varepsilon} := \{ E \in G_{n,m} : (1 + \varepsilon)^{-1} M_E \leq \| z \| \leq (1 + \varepsilon) M_E, \forall z \in S_E \}.
\]

If we define

\[
C_{m,\delta} := \{ E \in G_{n,m} : |M_E - M| > \delta M \}, \quad 0 < \delta < 1,
\]

then we have:

\[
\nu_{n,m}(S_{m,\varepsilon/3}) \leq \nu_{n,m}(A_{m,\varepsilon/3}) \leq \nu_{n,m}(C_{m,\varepsilon/3}) + \nu_{n,m}(A_{m,\varepsilon/3} \setminus C_{m,\varepsilon/3}) \leq C e^{-c \varepsilon^2 m k} + \nu_{n,m}(F_{m,\varepsilon}),
\]

where we have used Theorem 4.6 and the fact that \( A_{m,\varepsilon/3} \setminus C_{m,\varepsilon/3} \subset F_{m,\varepsilon} \).

**Proof of Proposition 4.3.** (Lower bound). We set \( \beta = \beta(X) \). Note that Theorem 3.1 implies that the set

\[
S_{s,1/6} := \{ F \in G_{n,s} : F is 7/6 - spherical subspace of X \},
\]

has \( \nu_{n,s}(S_{s,1/6}) \geq 1 - e^{-s} \) for \( s \simeq |\log \beta| \). Furthermore, Proposition 4.4 implies that:

\[
1 - e^{-c_1 s} \leq \nu_{n,s} \left( \left\{ E \in G_{n,s} : \frac{M}{2} \leq \| z \| \leq 2M, \forall z \in S_E \right\} \right).
\]

Now we may employ a result of Milman and Schechtman from [MS97] (see also [HW16] for a recent development) to conclude.

**Note.** Let us mention that the bounds we derive in Proposition 4.3 are sharp (up to constants): For \( X = \ell_1^n \) note that we have \( k(\ell_1^n) \simeq 1/\beta(\ell_1^n) \). The example of \( \ell_1^n \) with \( p = c_0 \log n \) for some sufficiently small absolute constant \( c_0 > 0 \) (see Proposition 4.2) guarantees the existence of \( n \)-dimensional normed spaces \( X \) with critical dimension \( k(X) \simeq \log n \) and \( \beta(X) \lesssim n^{-\alpha} \).
4.1. Dvoretzky’s theorem revisited. In [PV16] we prove refined random versions of the classical dimension reduction lemma of Johnson and Lindenstrauss [LL84] and of Dvoretzky’s theorem due to V. Milman [Mil71], in terms of the parameter $\beta$ and the critical dimension $k$. More precisely, given a finite dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ we may define the parameter $\operatorname{ov}(X)$ associated with $X$ as follows:

$$\operatorname{ov}(X)^2 := \beta(X)k(X) = \frac{\operatorname{Var}\|Z\|_b}{b(X)^2},$$

where $Z \sim N(0, I_n)$. The applications in [PV16] exhibit improved one-sided behavior which takes into account the parameter $\operatorname{ov}(X)$. It turns out that the parameter $\operatorname{ov}(X)$ measures the sharpness of the concentration of the norm $\|\cdot\|_X$ in terms of the Lipschitz constant; in particular, it measures how “over-concentrated” is the norm (see [PV17]).

Below we give a more detailed formulation of the random version of Dvoretzky’s theorem in terms of this parameter. We also provide one-sided, almost isometric inclusion in large dimensions proportional to $1/\beta$.

**Theorem 4.7 (Randomized Dvoretzky).** For any symmetric convex body $A$ we set $k_* := k_*(A)$, $\beta_* := \beta_*(A)$ and $[\operatorname{ov}_s(A)]^2 = \beta_*(A)k_*(A)$. Then, for every $\varepsilon \in (0, 1)$ we have:

i. There exists a set $A$ in $G_{n,k}$ with $\kappa \simeq \varepsilon^2k_*$ and $\nu_{n,k}(A) \geq 1 - e^{-c\varepsilon^2k_*}$ such that

$$P_E(A) \subseteq (1 + \varepsilon)w(A)B_E,$$

for all $E \in A$.

ii. There exists a set $B$ in $G_{n,\ell}$ with $\ell \simeq \frac{\varepsilon^2}{\beta_* \log(1/\varepsilon)}$ and $\nu_{n,\ell}(B) \geq 1 - e^{-c\varepsilon^2/\beta_*}$ such that

$$P_F(A) \subseteq (1 - \varepsilon)w(A)B_F,$$

for all $F \in B$.

iii. There exists a set $C$ in $G_{n,m}$ with $m \simeq \frac{\varepsilon k_*}{\log(1/\varepsilon)}$ and $\nu_{n,m}(C) \geq 1 - e^{-c\varepsilon^2k_*}$ such that

$$w(A) \left(1 - \frac{\varepsilon \operatorname{ov}_s(A)}{\log(1/\varepsilon)} \log \left(\frac{e}{\varepsilon \operatorname{ov}_s(A)}\right)\right) B_G \subseteq P_G(A) \subseteq \left(1 + \frac{\varepsilon}{\sqrt{\log(1/\varepsilon)}}\right) w(A)B_G,$$

for all $G \in C$,

where $c > 0$ is an absolute constant.

We shall need some auxiliary results. The first one follows from Klartag-Vershynin estimate (Lemma 4.5) in the dual setting.

**Lemma 4.8.** Let $A$ be a symmetric convex body on $\mathbb{R}^n$ and let $1 \leq k \leq n - 1$. Then, we have:

$$\left(\int_{G_{n,k}} (w(P_F A))^k \, d\nu_{n,k}(F)\right)^{1/k} \leq Cw(A),$$

where $C > 0$ is an absolute constant.

The next Lemma is essentially from [KV07]. However, the next formulation, takes into account the magnitude of the constants involved.

**Lemma 4.9 (dimension lift).** Let $A$ be a symmetric convex body on $\mathbb{R}^n$ and let $1 \leq s \leq n - 1$. Then, for any $p \geq s$ we have:

$$\left(\int_{G_{n,s}} [r(P_E A)]^{-p/2} \, d\nu_{n,s}(E)\right)^{2/p} \leq \frac{e\varepsilon s/pa_{s,p}}{w(A) (w_{-2p}(A))} \left(\frac{w(A)}{w_{-2p}(A)}\right)^2,$$

where $c > 0$ is an absolute constant and $a_{s,p}$ is defined in Lemma 2.7.

**Proof.** Recall that Lemma 2.7 yields:

$$J_q(K) = \left(\frac{1}{|K|} \int_K \|x\|_2^q \, dx\right)^{1/q} \geq a_{n,q}^{-1} R(K), \quad a_{n,q}^{-q} = \frac{q}{2} \beta(q, n + 1).$$
Also Lemma 2.3 implies:

\[ vrad(K)^n J_n^p(K) = M_{-(n+q)}(K). \]

Recall that \( 1/r(K) = R(K^o) \), hence if we apply the above for \( K = \gamma^o \) we get:

\[
\int_{G_{n,s}} \frac{1}{r(\gamma^o)^p} \, dnu_{n,s}(E) \leq a_{s,p}^{p/2} J_{p/2}^{p/2}((\gamma^o)\nu) \, dnu_{n,s}(E) \\
= \frac{p a_{s,p}^{p/2}}{s + p} \int_{G_{n,s}} w_{-(s+p)}^{-(s+p)/2}(\gamma^o) \cdot |vrad((\gamma^o)\nu)|^{-s/2} \, dnu_{n,s}(E) \\
\leq a_{s,p}^{p/2} \left( \int_{G_{n,s}} w_{-(s+p)}^-(\gamma^o) \, dnu_{n,s}(E) \right)^{1/2} \cdot \left( \int_{G_{n,s}} |vrad((\gamma^o)\nu)|^{-s} \, dnu_{n,s}(E) \right)^{1/2} \\
\leq a_{s,p}^{p/2} w_{-(s+p)}^{-(s+p)/2}(\gamma^o) \left( \int_{G_{n,s}} |w(\gamma^o)|^s \, dnu_{n,s}(E) \right)^{1/2} \\
\leq a_{s,p}^{p/2} w_{-(s+p)}^{-(s+p)/2}(A) w(A)^{s/2} e^c s,
\]

where we have also used inequality \( w(K) vrad(K^o) \geq 1 \) (following by Hölder’s inequality), and Lemma 4.8. The result follows.

**Proposition 4.10.** Let \( A \) be a symmetric convex body on \( \mathbb{R}^n \). Then, we have:

\[
\frac{w(A)}{w_{-q}(A)} \leq \frac{I_1(\gamma_n, A^o)}{I_q(\gamma_n, A^o)} \leq \exp \left( C \max \{ \sqrt{\beta_*}, q \beta_* \} \right),
\]

for all \( 0 < q < c/\beta_* \), where \( \beta_* = \beta_* (A) := \beta(A^o) \).

For a proof of the estimate on the Gaussian averages the reader may consult [PV10]. In order to pass to the sphere, recall the well-known formula (which follows by integration in polar coordinates):

\[ I_r(\gamma_n, A^o) = I_r(\gamma_n, B_n^2) w_r(A), \]

for all \( -n < r \neq 0 \). Thus, Hölder’s inequality yields the asserted bound.

Now we turn to proving Theorem 4.7.

**Proof of Theorem 4.7.** Fix \( 1 \leq s \leq n - 1 \). Using Lemma 4.9 and restricting ourselves to the range \( 1 \leq s \leq p \leq c_1/\beta_* \) we obtain:

\[
\left( \int_{G_{n,s}} |r(\gamma^o)^{-p} dnu_{n,s}(E) \right)^{1/p} \leq \frac{1}{w(A)} \exp \left( \frac{c s}{p} \log \left( \frac{c p}{s} \right) + c \sqrt{\beta_*} \log \frac{1}{\varepsilon} \right),
\]

where we have also used Proposition 4.10.

(ii) Given \( \varepsilon \in (0, 1) \) with \( \varepsilon \geq \sqrt{\beta_*} \) (otherwise there is nothing to prove) we choose \( p \simeq \frac{c}{\beta_*} \) and \( s \simeq c p/ \log(1/\varepsilon) \). Then, we obtain:

\[
\left( \int_{G_{n,\ell}} |r(\gamma^o)^{-p} dnu_{n,\ell}(E) \right)^{1/p} \leq \frac{1}{w(A)} \exp (C \varepsilon),
\]

with \( \ell = s \simeq \varepsilon^2/ (\beta_* \log \frac{1}{\varepsilon}) \). The asserted set \( \mathcal{B} \) is defined as:

\[ \mathcal{B} := \{ F \in G_{n,\ell} : r(F \gamma) \geq w(A) e^{-2C \varepsilon} \}. \]

Note that by Markov’s inequality we have:

\[ \nu_{n,\ell}(\mathcal{B}^c) \leq e^{-c_1 c p} \leq e^{-c_2 \varepsilon^2/\beta_*} \]

(iii) Set \( \tau_* = \|w_{n,s}(A)\|^2 \). Given \( \varepsilon \in (0, 1) \) (with \( \varepsilon \geq 1/\sqrt{k_s} \)) we choose \( p \simeq \varepsilon \sqrt{k_s}/\beta_* \) and \( s \simeq p \varepsilon \sqrt{k_s} \log(1/\varepsilon) \) and we argue as before to get:

\[ \mathcal{C} := \{ G \in G_{n,m} : r(P_G(A)) \geq w(A) e^{-2C \varepsilon \sqrt{k_s} \log(1/\varepsilon) / \log \frac{1}{\varepsilon}} \}, \]

where \( m = s \simeq \varepsilon^2 k_s / \log \frac{1}{\varepsilon} \) and \( \nu_{n,m}(\mathcal{C}^c) \leq e^{-c_1 p \varepsilon \sqrt{k_s}} \log(1/\varepsilon) \leq e^{-c_2 \varepsilon^2 k_s} \). The proof is complete. \( \square \)
The aim of this paragraph is to prove tight small deviation and small ball probability estimates and reveal the critical parameters that govern such estimates. We will not restrict in Gaussian measure rather we will extend our work in the more general setting of log-concave probability measures. Our starting point is the following one-sided small deviation inequality proved in [PV16]:

**Theorem 5.1.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a convex map with \( f \in L_1(\gamma_n) \). Then,

\[
\mathbb{P} ( f(Z) < m - t ) \leq \frac{1}{2} \exp(-ct^2/\|f-m\|_{L_1}^2),
\]

for all \( t > 0 \), where \( m \) is a median of \( f(Z) \) and \( Z \sim N(0, I_n) \).

The latter can be viewed as a strengthening of the one-sided classical Gaussian concentration, inside the class of convex functions, since one can replace the Lipschitz constant by the a priori smaller quantity of the standard deviation. The main ingredients of the proof can be summarized in the following two observations:

- When \( f \) is a convex function, the map \( t \mapsto \Phi^{-1} \circ \gamma_n(f \leq t) \) is concave, (this follows by Ehrhard’s inequality [Ehr83]) and,

- the derivative of the above function at the median \( m \) of \( f \) with respect to \( \gamma_n \) is relatively large in terms of the standard deviation of \( f \).

Here we extend the above approach in the context of log-concave probability measures. Let \( \mu \) be a log-concave, Borel probability measure on \( \mathbb{R}^n \). Let \( Z \) be a random vector on \( \mathbb{R}^n \) distributed according to \( \mu \), i.e. \( \mathbb{P}(Z \in B) = \mu(B) \) for any Borel set \( B \subseteq \mathbb{R}^n \).

For any convex function \( \psi : \mathbb{R}^n \to \mathbb{R} \) we define the random variable \( \xi := \psi(Z) \). We write \( F_\xi(t) := \mu(z \in \mathbb{R}^n : \psi(z) \leq t) = \mathbb{P}(\psi(Z) \leq t) \) for the cumulative distribution function of \( \xi \). The first main step is to replace function \( \Phi \) by a suitably chosen function that yields the concavity. The next standard Lemma (for a proof e.g. see [Bol99]) shows that in the context of log-concave measures this function can be at least the exponential.

**Lemma 5.2.** Let \( \mu \) be a log-concave probability measure on \( \mathbb{R}^n \) and let \( \psi \) be a convex map on \( \mathbb{R}^n \). If \( F(t) = \mathbb{P}(\psi(Z) \leq t) \), then we have the following:

a. For \( t, s \in \mathbb{R} \) and \( 0 < \lambda < 1 \) we have:

\[
F((1 - \lambda)t + \lambda s) \geq [F(t)]^{1 - \lambda} [F(s)]^\lambda,
\]

that is \( F \) is log-concave.

b. If \( \psi \) is a semi-norm \(^1\), then we also have:

\[
1 - F((1 - \lambda)t - \lambda s) \geq (1 - F(t))^{1 - \lambda} F(s)^\lambda
\]

for all \( t, s > 0 \) and \( 0 < \lambda < \frac{1}{t+s} \).

**Sketch of proof.** For b. we may check (using the subadditivity of the seminorm) that

\[
(1 - \lambda)\{\psi > t\} + \lambda\{\psi \leq s\} \subset \{\psi > (1 - \lambda)t - \lambda s\},
\]

and we use the log-concavity of \( \mu \).

**Remark 5.3.** Note that (b) easily implies Borell’s lemma from [Bor75]: Fix \( s > 0 \) (say \( s = 1 \)). For any \( t > s \) we choose \( \lambda \in (0, \frac{t-s}{t+s}) \) such that \((1 - \lambda)t - \lambda s = s\), i.e. \( \lambda = \frac{t-s}{t+s} \) and hence,

\[
1 - F(t) \leq F(s) \left( \frac{1 - F(s)}{F(s)} \right)^{\frac{s}{t}}, \quad 0 < s < t.
\]

The next lemma shows that \( f(m) \) can be estimated in terms of the standard deviation of the function, thus fulfills the second main observation in the general context of log-concave measures.

---

\(^1\) A seminorm \( \psi : V \to [0, \infty) \) on a vector space \( V \) is a function which is positively homogeneous, i.e. \( \psi(\lambda v) = |\lambda|\psi(v) \) for all scalars \( \lambda \) and \( v \in V \) and sub-additive, that is \( \psi(u + v) \leq \psi(u) + \psi(v) \) for all \( u, v \in V \).
Lemma 5.4. Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ and let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex map with $\psi \in \mathcal{L}_1(\mu)$. If $F(t) = \mathbb{P}(\psi(Z) \leq t)$, $t \in \mathbb{R}$ and $f = F'$, then

$$f(m) \geq \frac{1}{32\|\psi-m\|_{\mathcal{L}_1(\mu)}},$$

where $m$ is a median of $\psi$ with respect to $\mu$.

Proof. Since $F$ is log-concave (Lemma 5.2) we have:

$$2f(m) = (\log F)'(m) \geq \frac{\log F(m + \delta) - \log F(m)}{\delta} = \frac{1}{\delta} \log [1 + 2\mathbb{P}(m < \psi(Z) \leq m + \delta)] \geq \frac{1}{\delta} \mathbb{P}(m < \psi(Z) \leq m + \delta) = \frac{1}{\delta} \left[ \frac{1}{2} - \mathbb{P}(\psi(Z) > m + \delta) \right].$$

The choice $\delta = 4\|\psi-m\|_{\mathcal{L}_1}$ and Markov’s inequality yield the result. \hfill \Box

Note 5.5. In the above argument we may replace the $L_1$ norm by the difference $M_q - m$ for any $q$-quantile of $\psi(Z)$ with $q > 1/2$. That is, if $\mathbb{P}(\psi(Z) \leq M_q) = q > 1/2$ then $f(m) \geq \frac{q - 1/2}{M_q - m}$. Note that if $\psi \in \mathcal{L}_{p,\infty}(\mu)$ then $M_q - m \leq (1-q)^{-1/p}\|\psi-M\|_{\mathcal{L}_{p,\infty}}$.

Theorem 5.6. Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$. Let $\psi$ be a convex function on $\mathbb{R}^n$ and let $m$ be a median for $\psi$ with respect to $\mu$. Then,

$$\mu \left\{ x : \psi(x) < m - t \int |\psi - m| \, d\mu \right\} \leq \frac{1}{2} \exp(-t/16),$$

for all $t > 0$.

Proof. Let $F(t) = P(\psi(Z) \leq t)$, $t \in \mathbb{R}$ and $f = F'$. Then, by Lemma 5.2 we have:

$$\log F(m - t) - \log F(m) \leq -t \frac{f(m)}{F(m)} = -2tf(m).$$

It follows that:

$$F(m - t) \leq F(m) \exp(-2tf(m)) = \frac{1}{2} \exp(-2tf(m)),$$

for all $t > 0$. Using Lemma 5.4 we get:

$$F(t - m) \leq \frac{1}{2} \exp\left(-t/(16\mathbb{E}_\mu|\psi(Z) - m|)\right),$$

as required. \hfill \Box

For any symmetric convex body $K$ in $\mathbb{R}^n$ and for any log-concave probability measure $\mu$ on $\mathbb{R}^n$ we write $F_K$ for the cumulative distribution function of $\|Z\|_K$, where $Z$ is distributed according to $\mu$, i.e. $F_K(t) = \mu(x : \|x\|_K \leq t)$, $t > 0$. If $f_K$ is the density of this random variable, we define:

$$\vartheta(\mu, K) \equiv \vartheta(\mu, K) = m f_K(m),$$

where $m$ is a median for $\|Z\|_K$. With this notation Lemma 5.4 shows that

$$\vartheta(\mu, K) \geq \frac{1}{\sqrt{\vartheta(\mu, K)^2}}.$$

Below, we also show that $\vartheta$ raises naturally in small deviation and small ball estimates for log-concave probability measures on $\mathbb{R}^n$. For this end, let us recall the $B$-property. A pair $(\mu, K)$ of a log-concave probability measure on $\mathbb{R}^n$ and a symmetric convex body $K$ in $\mathbb{R}^n$ is said to have the $B$-property if the map $t \mapsto \mu(e^t K)$ is log-concave. In [CEFMO4], Cordero-Erausquin, Fradelizi and Maurey proved that $(\gamma_n, K)$
has the $B$-property for any symmetric convex body $K$ in $\mathbb{R}^n$. They also proved that any pair $(\mu, K)$ of an 1-unconditional log-concave measure and 1-unconditional convex body on $\mathbb{R}^n$ also has the $B$-property. In view of Theorem 5.7, we have the following analogue of [PV16, Theorem 3.1] for log-concave measures:

**Theorem 5.7.** Let $K$ be a symmetric convex body on $\mathbb{R}^n$ and let $\mu$ be a log-concave probability measure. If $m$ is the median of $x \mapsto \|x\|_K$ with respect to $\mu$, then we have:

a. For every $\varepsilon \in (0, 1)$,
\[ \mu(\{x : \|x\|_K \leq (1 - \varepsilon)m\}) \leq \frac{1}{2} \exp(-2\varepsilon\theta\mu(K)) \leq \frac{1}{2} \exp\left(-c\varepsilon/\sqrt{\beta\mu(K)}\right). \]

b. Furthermore, if the pair $(\mu, K)$ possesses the $B$-property and $\mu(K) \leq 1/2$,
\[ \mu(\varepsilon K) \leq \varepsilon^2 f_{K}(m) \mu(K) \leq \varepsilon^{\frac{1}{1-m}} f_{K}(m) \mu(K), \]
for all $\varepsilon \in (0, 1)$. In particular,
\[ \mu(\{x : \|x\|_K \leq \varepsilon m\}) \leq \frac{1}{2} \varepsilon^2 \theta\mu(K) \leq \frac{1}{2} \varepsilon^{c}/\sqrt{\beta\mu(K)}, \]
for every $\varepsilon \in (0, 1)$.

**Proof.** The first assertion follows by (5.2) and (5.4). For the second estimate recall the fact that the $B$-property implies that for any symmetric convex body $A$ the map $t \mapsto [\mu(tA)/\mu(A)]^{1/\log(1/t)}$, $t \in (0, 1)$ is non-decreasing (e.g. see [LO05]). Therefore, if we define
\[ d := \sup_{0 < \varepsilon < 1} \frac{\log[\mu(\varepsilon K)/\mu(K)]}{\log(1/\varepsilon)}, \]
the monotonicity shows that:
\[ d = \lim_{\varepsilon \to 1^{-}} \frac{\log[\mu(\varepsilon K)/\mu(K)]}{\log(1/\varepsilon)} = \frac{d}{d\varepsilon} \big|_{\varepsilon = 1} \log F_{K}(\varepsilon) = -(\log F_{K})'(1) \]
On the other hand, the map $t \mapsto (\log F_{K})'(t)$ is non-increasing (Lemma 5.2) and since $\mu(K) \leq 1/2$ we get that $m \geq 1$, which implies
\[ (\log F_{K})'(1) \geq (\log F_{K})'(m) = 2f_{K}(m) \]
Combining the above we infer that:
\[ \frac{\log[\mu(\varepsilon K)/\mu(K)]}{\log(1/\varepsilon)} \leq -2f_{K}(m) \implies \mu(\varepsilon K) \leq \varepsilon^2 f_{K}(m) \mu(K) \]
for every $\varepsilon \in (0, 1)$ as required. \hfill \Box

**Remark 5.8.** Recently, in [ENT16], Eskenazis, Nayar and Tkocz proved that the pair $(\nu_{1}^{\alpha}, K)$ has the $B$-property for all symmetric convex bodies $K$ on $\mathbb{R}^n$. In view of Theorem 5.7 and their result one has:

\[ \nu_{1}^{\alpha}(\varepsilon K) \leq \varepsilon^{c/\mathbb{E}[\|Z\|_K - m]} \nu_{1}^{\alpha}(K), \quad 0 < \varepsilon < 1, \]
for all symmetric convex bodies $K$ on $\mathbb{R}^n$ with $\nu_{1}^{\alpha}(K) \leq 1/2$. A different estimate was proved in [ENT16, Corollary 13] in terms of the inradius $r(K)$ of the body $K$, which is in the spirit of [LO05]. Let us mention that for the class of 1-unconditional convex bodies a better estimate than (5.5) is known for the exponential distribution; see [PV16, Proposition 3.2]. Furthermore, for the Gaussian distribution $\gamma_n$ and all symmetric convex bodies $K$ in $\mathbb{R}^n$ with $\gamma_n(K) \leq 1/2$, one has:
\[ \gamma_n(tK) \leq (2t)^{\gamma_n(\mathbb{E}[\|Z\|_K - m])^2} \gamma_n(K), \]
for every $t \in [0, 1]$, where $m$ is the median of $\|Z\|_K$ with $Z \sim N(0, I_n)$ (see [PV16]). One should compare the latter with [LO05, Theorem 2].
In order to illustrate the difference, between estimate (5.5) and the one proved in [ENT16, Corollary 13], let us consider the following example. Fix $1 \leq p \leq \infty$ and let $W$ be an $n$-dimensional exponential random vector, i.e. $W \sim \nu_1^n$. Then, we have
\[
\text{Var}\|W\|_p \simeq_p \begin{cases} n^{\frac{p}{p-1}}, & 1 \leq p < \infty \\ \frac{1}{1}, & p = \infty \end{cases},
\]
for all sufficiently large $n$ (for a proof of this fact the reader may consult [PVZ15, Theorem 3.2 & §3.2]). Furthermore,
\[
\mathbb{E}\|W\|_p \simeq_p \begin{cases} \log n, & p \leq \log n \\ \frac{1}{n}, & p \geq \log n \end{cases}.
\]
For a proof of the latter fact the reader is referred to [SZ90]. Thus, one has:
\[
\beta_{\nu_1^n}(B^n_p) \simeq_p \begin{cases} n^{-1}, & 1 \leq p < \infty \\ (\log n)^{-2}, & p = \infty \end{cases},
\]
for fixed $p$ and for all sufficiently large $n$. It follows that if $m = m_{p,n}$ is the median of $\|W\|_p$ with $2 < p \leq \infty$, then the small ball estimate (5.5) (applied for $K = mB^n_p$) yields
\[
\mathbb{P}(\|W\|_p \leq \varepsilon m) \leq \frac{1}{2} e^{-\varepsilon \sqrt{k_{\nu_1^n}(B^n_p)}},
\]
while the corresponding small ball in terms of the inradius $r(B^n_p)$ yields
\[
\mathbb{P}(\|W\|_p \leq \varepsilon m) \leq \frac{1}{2} e^{-\varepsilon \sqrt{k_{\nu_1^n}(B^n_p)}},
\]
where
\[
k_{\nu_1^n}(B^n_p) := (r(B^n_p) \cdot \mathbb{E}\|W\|_p)^2 \simeq_p \begin{cases} n, & 1 \leq p \leq 2 \\ \frac{n^2}{p^2}, & 2 \leq p \leq \log n \\ (\log n)^2, & p \geq \log n \end{cases}.
\]
Actually in view of [PV16, Proposition 3.2] one can have an even better estimate than (5.6) since the $\ell_p$ norms are 1-unconditional.

For further comparison purposes, recall that if $\mu$ satisfies a Poincaré (or spectral gap) inequality with constant $\lambda_1 = \lambda_1(\mu) > 0$, i.e.
\[
\lambda_1 \text{Var}_\mu(f) \leq \int \|\nabla f\|^2_2 \, d\mu,
\]
for all smooth $f$, then
\[
\sqrt{\lambda_1} \|f - m\|_{L_1(\mu)} \leq (\mathbb{E}_\mu \|\nabla f\|^2_2)^{1/2} \leq \|f\|_{Lip},
\]
for every Lipschitz map $f$, where $m$ is a median of $f$. In the case that $f$ is the norm $\|\cdot\|_K$ induced by the symmetric convex body $K$, note that $\|f\|_{Lip} = 1/r(K)$, which indicates that the $L_1$ deviation is in general smaller than the inradius of the body.

**Examples 5.9.** 1. (The parameter $\vartheta_{\gamma_n}(B^n_\infty)$). Let $m$ be the median for $x \mapsto \|x\|_\infty$ with respect to $\gamma_n$. Then, we have:
\[
\vartheta(\gamma_n, B^n_\infty) \simeq \log n.
\]
Indeed; note that for any $s > 0$ we have:
\[
F_{B^n_\infty}(s) = \gamma_n(sB^n_\infty) = (\gamma_1([-s,s]))^n = (2\Phi(s) - 1)^n.
\]
Hence,
\[
f_{B^n_\infty}(s) = 2n\Phi(s)(2\Phi(s) - 1)^{n-1}.
\]
In particular, if $m$ is the median of $x \mapsto \|x\|_\infty$ with respect to $\gamma_n$, then
\[
f_{B^n_\infty}(m) = n\frac{\phi(m)}{2\Phi(m) - 1} = n2^{1/n}\phi(m).
\]
On the other hand we have:
\[ 2^{-1/n} = \int_{-m}^{m} \phi(t) \, dt = 1 - 2 \int_{m}^{\infty} \phi(t) \, dt \Rightarrow 2 \int_{m}^{\infty} \phi = 1 - 2^{-1/n} \simeq 1/n \]
and from standard estimates on the error function we have:
\[ \int_{m}^{\infty} \phi(t) \, dt \simeq \frac{1}{m} \phi(m), \quad m \to \infty. \]
Hence, we obtain \( f_{B^\infty_2}(m) \simeq m \) or \( \vartheta(\gamma_n, B^\infty_2) \simeq m^2 \simeq \log n \).

2. (The parameter \( \vartheta(\gamma_n, B^2_2) \)). Let \( m \) be a median for \( x \mapsto \|x\|_2 \) with respect to Gaussian. Then, we have:
\[ \vartheta = \vartheta(\gamma_n, B^2_2) \simeq \sqrt{n}. \]
Indeed; it is known that \( \beta(B^2_2) \simeq 1/n \) which implies that \( \vartheta \gtrsim \sqrt{n} \). For the upper estimate we argue as follows:
\[ e^{-c_1 \vartheta^2} \geq \gamma_n \left( \frac{m}{2} B^2_2 \right) \geq (2\pi)^{-n/2} \left| \frac{m}{2} B^2_2 \right| e^{-m^2/8} \geq e^{-c_2 n}, \]
where we have used the fact that \( m \simeq \sqrt{n} \).

In fact the upper estimate holds for any centrally symmetric convex body in \( \mathbb{R}^n \):

**Corollary 5.10.** Let \( K \) be a symmetric convex body on \( \mathbb{R}^n \). Then, one has:
\[ \vartheta(\gamma_n, K) \lesssim \sqrt{n}. \]

**Proof.** We shall need [KV07] Lemma 2.1 which states:
\[ \frac{1}{2} \sigma(S^{n-1} \cap \frac{1}{2} L) \leq \gamma_n(\sqrt{n} L), \]
for any centrally symmetric convex body \( L \). Using the dual Sudakov inequality (see e.g. [LT11]) we get:
\[ \frac{|\sqrt{n} B^2_2|}{|\sqrt{n} B^2_2 \cap \frac{m}{4} K|} \leq N \left( \sqrt{n} B^2_2, \frac{m}{4} K \right) \leq \exp \left( c_1(\mathbb{E}\|Z\|_K)^2(4\sqrt{n}/m)^2 \right) \leq e^{c_2 n}, \]
where \( m \) is a median for \( \|Z\|_K \) and \( Z \sim N(0, I_n) \). On the other hand we have:
\[ \frac{|\sqrt{n} B^2_2 \cap \frac{m}{4} K|}{|\sqrt{n} B^2_2|} = \sigma(S^{n-1} \cap \frac{m}{4\sqrt{n}} K) \leq 2\gamma_n(\frac{m}{2} K) \leq \exp(-c_3 \vartheta^2). \]
Combining all the above we get the result. \( \square \)

5.1. A remark on isoperimetry. Let \( \nu_1 \) be the 1-dimensional exponential measure with density \( d\nu_1(x) = \frac{1}{2} e^{-|x|} \, dx \). Let \( F_{\nu_1} \) be its cumulative distribution function, i.e.
\[ F_{\nu_1}(x) = \nu_1((-\infty, x]) = \frac{1}{2} \int_{-\infty}^{x} e^{-|t|} \, dt, \quad x \in \mathbb{R}. \]
Let \( X = (\mathbb{R}^n, \| \cdot \|) \) be a normed space and let \( \mu \) be a log-concave measure on \( \mathbb{R}^n \). Following [Bob99] we shall denote the induced measure by \( \mu_X \). That is the push forward of the measure \( \mu \) under the map \( x \mapsto \|x\| \), i.e.
\[ \mu_X(A) = \mu(x \in \mathbb{R}^n : \|x\| \in A), \quad A \subseteq [0, \infty), \text{ Borel.} \]

Let \( T : (0, \infty) \to \mathbb{R} \) with \( T = F^{-1}_{\nu_1} \circ F_K \) be the map which transports \( \mu_X \) to \( \nu_1 \), where \( K = \{ x : \|x\| \leq 1 \} \).

Bobkov in [Bob99] proved the following:

**Proposition 5.11.** Let \( \mu \) be a log-concave probability measure on \( X = (\mathbb{R}^n, \| \cdot \|) \). Then, one has:
\[ \text{Is}(\mu_X) = \inf_{s > 0} (F^{-1}_{\nu_1} \circ F_K)'(s) \gtrsim \frac{1}{\mathbb{E}_\mu \|Z\|}. \]
It is known that $\text{Is}(\mu_X) \lesssim 1/\sqrt{\text{Var}_\mu \|Z\|}$ (see e.g. [Bob99] and the references therein). In [Bob99], Bobkov shows that for 1-dimensional log-concave measures the reverse estimate also holds [Bob99, §4]. Moreover, he asks if a reverse estimate should be expected to be true for the measure $\mu_X$. Note that even though the measure $\mu_X$ is not necessarily log-concave, it enjoys many properties, see e.g. Proposition 5.11. Here we observe that Lemma 5.4 partially answers the aforementioned question:

**Corollary 5.12.** Let $\mu$ be a log-concave probability measure on $\mathbb{R}^n$ and let $K$ be a symmetric convex body on $\mathbb{R}^n$. Then, we have:

$$\inf_{0 < s \leq m} (F_{n_1}^{-1} \circ F_K)'(s) = (F_{n_1}^{-1} \circ F_K)'(m) \gtrsim \frac{1}{\text{Var}_\mu \|Z\|}.$$

where $m$ is the median of the function $x \mapsto \|x\|_K$ with respect to $\mu$.

Moreover, a lower estimate for the full range of $s$ shouldn’t be expected as the next argument shows: If $\inf_{s > 0} T(s) := 1/L$, then $T : (0, \infty) \to \mathbb{R}$ is homeomorphism and $1/L$-expansive, i.e. $|Tu - Tv| \geq |u - v|/L$ for all $u, v > 0$. Thus, the map $T^{-1} : \mathbb{R} \to (0, \infty)$ is $L$-Lipschitz and transports the measure $\nu_1$ to $\mu_X$, i.e. for any Borel set $A \subseteq \mathbb{R}$ we have:

$$\nu_1(A) = \mu_X(T^{-1}(A)) = \mu \left( x : \|x\|_K \in T^{-1}(A) \right).$$

**Claim.** For any $t > 0$ one has:

$$T^{-1}([-t,t]) \subseteq [m - tL, m + tL].$$

In particular, for $A = (-t, t)$, $t > 0$ in (5.7) we have:

$$1 - P(|\|Z\|_K - m| > tL) = \mu \left( x : \|x\|_K \in [m - tL, m + tL] \right) \geq \mu \left( x : \|x\|_K \in T^{-1}([-t,t]) \right) = \nu_1([-t,t]).$$

It follows that $P(|\|Z\|_K - m| > tL) \leq e^{-t}$ for all $t > 0$, or equivalently

$$P \left( |\|Z\|_K - m| \geq \varepsilon m \right) \leq \exp \left( -e\varepsilon / \sqrt{\beta_\mu(K)} \right),$$

for any $\varepsilon > 0$. Hence, if $L \simeq \sqrt{\text{Var}_\mu \|Z\|}$, we would obtain:

$$P \left( |\|Z\|_K - m| \geq \varepsilon m \right) \leq \exp \left( -e\varepsilon / \sqrt{\beta_\mu(K)} \right),$$

for all $\varepsilon > 0$. Note that for any fixed $2 < p < \infty$ one has:

$$e \exp \left( -C_p \min \{ \varepsilon^2 n, (\varepsilon m)^{2/p} \} \right) \leq P \left( |\|Z\|_p - M_{p,n}| \geq \varepsilon M_{p,n} \right),$$

for all $0 < \varepsilon < 1$, where $Z \sim N(0, I_n)$ and $M_{p,n}$ is a median for $\|Z\|_p$. Thus, for $p = 5$ say, we get a contradiction for all $\varepsilon \in (n^{-1/6}, 1)$.

### 5.2. Another small deviation inequality

In this subsection we prove a small deviation inequality similar to (5.1) with the variance replaced by the $\mathbb{E}||\nabla f||^2$. The inequality is known to specialists and the method of proof goes at least back to [Sam03] and [BG99]. The inequality is weaker than (5.1) but holds for a larger class of measures: in fact, all measures which satisfy a quadratic cost inequality à la Talagrand [Tal96] (see also [AS17] for a recent development on the related subject). First we recall the necessary definitions. For any two Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ the Wasserstein distance $W_2(\mu, \nu)$ is defined as

$$W_2^2(\mu, \nu) = \inf_{\pi} \iint \|x - y\|^2 d\pi(x, y),$$

where the infimum is taken over all couplings (or matchings) $\pi$ of $\mu$ and $\nu$, that is $\pi$ has marginals $\mu$ and $\nu$ respectively. The Kullback-Leibler divergence (or relative entropy) of $\nu$ with respect to $\mu$ is defined by

$$D(\nu||\mu) = \text{Ent}_\mu \left( \frac{d\nu}{d\mu} \right) = \int \log \frac{d\nu}{d\mu} d\mu,$$

if $\nu$ is absolutely continuous with respect to $\mu$ with Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ and $\infty$ otherwise.

---

2 Indeed; if $|z| \leq t$ then, since $T(m) = 0$, we have: $|T^{-1}(z) - m| = |T^{-1}(z) - T^{-1}(0)| \leq L|z| \leq tL$. 

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A Borel probability measure \( \mu \) on \( \mathbb{R}^n \) it is said to satisfy the \textit{quadratic transportation cost inequality} with constant \( A > 0 \) if
\[
W_2(\mu, \nu) \leq \sqrt{AD(\nu||\mu)},
\]
for any Borel probability measure \( \nu \) with \( \nu \ll \mu \). It is known that measures with this property can be characterized in terms of infimum convolution inequalities with cost function \( w(z) = \|z\|_2^2/(2A) \), \( z \in \mathbb{R}^n \) (e.g. see [Led01, Corollary 6.4.]). Furthermore, Otto and Villani in [OV00] showed that measures which satisfy a log-Sobolev inequality then also satisfy a quadratic transportation cost inequality. The main inequality of this subsection, which is in the same spirit as Theorem 5.6, reads as follows.

\[\textbf{Theorem 5.13.} \text{ Let } \mu \text{ be any Borel probability measure on } \mathbb{R}^n, \text{ which satisfies a quadratic transportation cost inequality } (5.8). \text{ Then, for any smooth, convex map } f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ we have:}
\]
\[
\log \mathbb{E}_\mu e^{-f} \leq -\mathbb{E}_\mu (f) + \frac{A}{4} \mathbb{E}_\mu \|\nabla f\|_2^2.
\]
\[\text{In particular,}
\]
\[
\mu \left( \left\{ x : f(x) - \mathbb{E}_\mu (f) \leq -t \left( \mathbb{E}_\mu \|\nabla f\|_2^{1/2} \right)^2 \right\} \right) \leq e^{-t^2/A},
\]
\[\text{for all } t > 0.
\]
\[\text{Proof.} \text{ Since } f \text{ is convex and smooth, for any } x, y \in \mathbb{R}^n \text{ we may write:}
\]
\[
f(x) - f(y) \leq \langle \nabla f(x), x - y \rangle \leq \|\nabla f(x)\|_2 \|x - y\|_2.
\]

Fix any probability measure \( \nu \) with \( \nu \ll \mu \) and let \( \pi \) be any coupling of \( \mu \) and \( \nu \). Thus, integration with respect to \( \pi \) yields:
\[
\mathbb{E}_\pi f - \mathbb{E}_\nu f \leq \int \|\nabla f(x)\|_2 \cdot \|x - y\|_2 d\pi(x, y) \leq \left( \mathbb{E}_\mu \|\nabla f\|_2^{1/2} \right)^2 \left( \int \|x - y\|_2^2 d\pi(x, y) \right)^{1/2}.
\]

Since the left-hand side is fixed for any coupling \( \pi \) of \( \mu \) and \( \nu \) we infer:
\[
\mathbb{E}_\mu f - \mathbb{E}_\nu f \leq \sqrt{\mathbb{E}_\mu \|\nabla f\|_2^2 D(\nu||\mu)},
\]
where we have used the assumption on \( \mu \). Now we employ Gibb’s variational formula (for a proof see [BLM13, Corollary 4.14]): For any \( \mu \)-measurable map \( f \) one has
\[
\log \mathbb{E}_\mu e^f = \sup_{\nu \ll \mu} \{ \mathbb{E}_\nu f - D(\nu||\mu) \}.
\]

Applying the latter for \(-f\) and taking into account (5.9) we obtain:
\[
\log \mathbb{E}_\mu e^{-f} \leq -\mathbb{E}_\mu f + \sup_{\nu \ll \mu} \left\{ \sqrt{\mathbb{E}_\mu \|\nabla f\|_2^2 D(\nu||\mu)} - D(\nu||\mu) \right\} \leq -\mathbb{E}_\mu f + \frac{A}{4} \mathbb{E}_\mu \|\nabla f\|_2^2.
\]

The result follows. \( \square \)

\section{Probabilistic estimates on the Grassmannian}

In this Section we prove Theorem 5.6. We present two approaches to derive this probabilistic estimate. The first one uses Gaussian tools and is based on the new small deviation inequality (5.1), therefore yields better tails estimate in the one-sided small deviation regime, but restricts the range of \( t \). The second approach overcomes this obstacle by working directly on the Grassmann space, but the tail estimate we obtain relies on the Lipschitz constant since we employ the Gromov-Milman theorem for \( SO(n) \). A small ball probability estimate for the mapping \( F \mapsto w(P_F A) \) is also provided.
6.1. From Gauss’ space to Grassmannian. First we provide a Gaussian proof of Theorem 4.6. Let us recall the following:

Lemma 6.1. Let $A$ be symmetric convex body on $\mathbb{R}^n$. For any matrix $T = (t_{ij})_{i,j=1}^{k,n}$, $1 \leq k \leq n$ we define the map $T \mapsto w(TA)$. Then, we have the Lipschitz condition:

$$|w(TA) - w(SA)| \leq \frac{R(A)}{\sqrt{k}} \|T - S\|_{HS},$$

for all $T = (t_{ij}), S = (s_{ij}) \in \mathbb{R}^{k \times n}$. Therefore, we have:

$$P(w(GA) \geq E[w(GA)] + t) \leq \exp(-ct^2k/R(A)^2),$$

and

$$P(w(GA) \leq E[w(GA)] - t) \leq \exp\left(-ct^2 \max\left\{\frac{1}{\Var[h_A(Z)]}, \frac{k}{R(A)^2}\right\}\right),$$

for all $t > 0$, where $G = (g_{ij})$ is $k \times n$ matrix with i.i.d. standard Gaussian entries and $Z \sim N(0,I_n)$.

Proof. For a proof of (6.1) the reader may consult with [PV16]. Then, estimate (6.2) immediately follows from the concentration on Gauss’ space. For proving (6.3) we need the next fact:

Claim. If $G = (g_{ij})_{i,j=1}^{k,n}$ is a Gaussian matrix then,

$$\Var[w(GA)] \leq \min\left\{\frac{R(A)^2}{k}, \Var[h_A(Z)]\right\},$$

where $Z \sim N(0,I_n)$.

Proof of Claim. The bound in terms of the circumradius follows from the Lipschitz condition and the Poincaré inequality (6.4). For bounding in terms of the variance we use the Cauchy-Schwarz inequality.

Finally, the estimate (6.3) follows from the small deviation inequality (5.1) applied for the convex function $T \mapsto w(TA)$.

One more ingredient is the polar decomposition of any matrix $T \in \mathbb{R}^{k \times n}$. If $T \in \mathbb{R}^{k \times n}$ we may write: $T = SQ$ where $S = (TT^*)^{1/2}$ and $Q$ is the orthogonal projection onto $F = \text{Im}T^*$.

The next Lemma follows if we take into account the above decomposition and the ideal property of the $\ell$-norm (see e.g. [TJS9]).

Lemma 6.2. If $T \in \mathbb{R}^{k \times n}$ and $A$ is a symmetric convex body on $\mathbb{R}^n$, then we have:

$$\lambda_k((TT^*)^{1/2})w(P_F A) \leq w(TA) \leq \lambda_1((TT^*)^{1/2})w(P_F A),$$

where $\lambda_j((TT^*)^{1/2}) \equiv s_j(T)$ is the $j$-th eigenvalue of $(TT^*)^{1/2}$ (or the $j$-th singular value of $T$) and $F = \text{Im}T^*$.

Proof. First note that if $S : \ell_2^k \to \ell_2^k$ is a linear map which satisfies $0 < a \leq \|S\theta\|_2 \leq b$ for all $\theta \in S^{k-1}$, then

$$a\E[h_A(Y)] \leq \E[h_A(SY)] \leq b\E[h_A(Y)]$$

where $Y \sim N(0,I_k)$. This follows by the ideal property of the $\ell$-norm, i.e. for any operator $u : \ell_2^k \to \ell_2^k$ and $v : \ell_2^k \to X$ we have $\ell(vu) \leq \ell(v)\|u\|_{op}$. Now in our setting, set $S = (TT^*)^{1/2}$. If $Y \sim N(0,I_k)$ and $c_k := \E[Y]_2$, we may write:

$$w(TA) = c_k^{-1}\E[h_{TA}(Y)] = c_k^{-1}\E[h_{P_FA}(S^*Y)] \leq c_k^{-1}\|S\|_{2 \to 2}\E[h_{P_FA}(Y)],$$

where we have used the right-hand side of (6.3). We work similarly for the lower estimate.

In the Gaussian random setting the variables $\lambda_j((GG^*)^{1/2})$ and $w(P_F A)$ with $F = \text{Im}G^*$ are independent each other. Namely, we have the following:

Lemma 6.3. Let $A$ be a symmetric convex body on $\mathbb{R}^n$ and let $G = (g_{ij})_{i,j=1}^{k,n}$ where $g_{ij}$ are i.i.d. standard normals. Then, $F = \text{Im}G^*$ is uniformly distributed over $G_{n,k}$ and the random variables $\lambda_j((GG^*)^{1/2})$ and $w(P_F A)$ are independent.
For a proof of this fact the reader is referred to [PPZ14 Proposition 4.1]. Now we are ready to prove the following:

**Theorem 6.4.** Let $A$ be a symmetric convex body on $\mathbb{R}^n$. Fix $1 \leq k \leq n - 1$. Then, we have:

$$\nu_{n,k}(F \in G_{n,k} : w(P_F A) > (1 + t)w(A)) \leq C \exp(-ct^2 kk_*(A)),$$

for all $t > c_1 \sqrt{k/n}$. Furthermore,

$$\nu_{n,k}(F \in G_{n,k} : w(P_F A) \leq (1 - t)w(A)) \leq C \exp\left(-ct^2 \max\left\{kk_*(A), \frac{1}{\beta_*(A)}\right\}\right),$$

for all $c_2 \sqrt{k/n} < t < 1$, provided that $k \leq n$.

**Proof.** Note that for any Gaussian matrix $G = (g_{ij})_{i,j=1}^{k,n}$ we have:

$$E[w(GA)] = E[h_A(Z)] = E\|Z\|_2 \cdot w(A),$$

where $Z \sim N(0, I_n)$. Fix $t > 0$. From Lemma 6.2 and Lemma 6.3 we may write:

$$\nu_{n,k}(w(P_F A) > (1 + t)w(A)) \leq \frac{\lambda_k((GG^*)^{1/2})^{1/2}}{(1 - \delta)E\|Z\|_2^2} \leq \frac{\lambda_1((GG^*)^{1/2})^{1/2}}{(1 + \delta)E\|Z\|_2^2},$$

for $0 < \delta < \frac{t}{2(1 + t)}$. Recall the following well known:

**Fact.** Let $\delta \in (0, 1)$. Then the random Gaussian matrix $G = (g_{ij})_{i,j=1}^{k,n}$ with $k \leq c\delta^2 n$ satisfies:

$$(1 - \delta)E\|Z\|_2^2 < \lambda_k((GG^*)^{1/2}) \leq \lambda_1((GG^*)^{1/2}) < (1 + \delta)E\|Z\|_2^2$$

with probability greater than $1 - e^{-c\delta^2 n}$, where $Z \sim N(0, I_n)$.

Now Lemma 6.4 yields:

$$(1 - e^{-c\delta^2 n})\nu_{n,k}(w(P_F A) > (1 + t)w(A)) \leq C \exp(-ct^2 kk_*(A)).$$

The choice $\delta \simeq \sqrt{k/n}$ yields the upper estimate. We work similarly for the lower estimate. \hfill \Box

6.1.1. A small ball estimate. Next, we prove the following:

**Theorem 6.5** (small ball for the mean width of projections). Let $A$ be a symmetric convex body on $\mathbb{R}^n$ and let $1 \leq k \leq n - 1$. Then:

$$\nu_{n,k}\left(\{F \in G_{n,k} : w(P_F A) \leq c\varepsilon w(A)\}\right) \leq (C\varepsilon)^{c\max\left\{kk_*(A), \frac{1}{\beta_*(A)}\right\}},$$

for all $\varepsilon \in (0, 1/2)$.

**Proof.** First note that the function $T \mapsto w(TA)$ is indeed a norm on $\mathbb{R}^{k \times n}$. If $C_A = \{T \in \mathbb{R}^{k \times n} : w(TA) \leq 1\}$ is its unit ball, then estimate (6.4) shows that $\beta(C_A) \leq \min\left\{\frac{1}{kk_*(A)}, \beta_*(A)\right\}$. It follows by [PV16 Theorem 3.1] that:

$$\mathbb{P}(w(GA) \leq c\varepsilon E[w(GA)]) \leq \frac{1}{2}e^{c/\beta(C_A)},$$

for all $\varepsilon \in (0, 1/2)$. Now we use Lemma 6.2 and Lemma 6.3 to get:

$$c_1\nu_{n,k}(w(P_F A) \leq c\varepsilon w(A)) \leq \nu_{n,k}(w(P_F A) \leq c\varepsilon w(A)) \mathbb{P}\left(\lambda_1((GG^*)^{1/2}) \leq C_1E\|Y\|_2\right) \leq \mathbb{P}(w(GA) \leq c\varepsilon E[w(GA)]) .$$

The result readily follows. \hfill \Box
6.2. **Breaking the barrier.** Note that the argument of the preceding proof does not allow to consider very small $t > 0$. In order to retrieve the behavior for small enough $t$ we have to work beyond the Gaussian setting. First we state Lipschitz estimates on the Grassmannian for the map $F \mapsto w(P_F A)$ with respect to the normalized metrics:

$$\sigma_\infty(E, F) = \|P_E - P_F\|_{op}, \quad \sigma_2(E, F) = \frac{\|P_E - P_F\|_{HS}}{\sqrt{k}}.$$ 

**Lemma 6.6.** Let $A$ be a (symmetric) convex body on $\mathbb{R}^n$ and let $1 \leq k \leq n - 1$. Then, for any $E, F \in G_{n,k}$ we have:

$$|w(P_E A) - w(P_F A)| \leq c \sqrt{n/k} w(A) \sigma_\infty(E, F).$$

Furthermore, we have:

$$|w(P_E A) - w(P_F A)| \leq c' R(A) \sigma_2(E, F),$$

where $c, c' > 0$ are absolute constants. Equivalently, in the dual setting, we may write:

$$|M(A \cap E) - M(A \cap F)| \leq \min \left\{ \sqrt{n/k} M(A) \sigma_\infty(E, F), b(A) \sigma_2(E, F) \right\}.$$

**Proof.** Using the formula

$$c_k w(P_F A) = c_k \int_{S_F} h_A(\phi) \, d\sigma_F(\phi) = \int_F h_A(x) \, d\gamma_F(x) = \int_{\mathbb{R}^n} h_A(P_F x) \, d\gamma_n(x),$$

where $c_k = \mathbb{E}\|Y\|_2 = \sqrt{2\Gamma(k + 1)/\Gamma(k)} = \sqrt{k}$ with $Y \sim N(0, I_k)$, we may write:

$$c_k |w(P_E A) - w(P_F A)| \leq \int_{\mathbb{R}^n} |h_A(P_E x) - h_A(P_F x)| \, d\gamma_n(x) \leq \int_{\mathbb{R}^n} h_A(P_E x - P_F x) \, d\gamma_n(x).$$

Now we proceed as follows. In order to prove the first estimate recall the ideal property of the $\ell$-norm (e.g. see [TJS9]). Applying this for $u = P_E - P_F$, $v = id$ and $X = (\mathbb{R}^n, \|\cdot\|_{A})$ we obtain:

$$\int_{\mathbb{R}^n} h_A((P_E - P_F) x) \, d\gamma_n(x) \leq \int_{\mathbb{R}^n} h_A(x) \, d\gamma_n(x) \cdot \|P_E - P_F\|_{op}.$$

Therefore we get:

$$|w(P_E A) - w(P_F A)| \leq \frac{c_n}{c_k} w(A) \cdot \|P_E - P_F\|_{op}.$$ 

For the second assertion we proceed as follows:

$$\int_{\mathbb{R}^n} h_A(P_E x - P_F x) \, d\gamma_n(x) \leq R(A) \int_{\mathbb{R}^n} \|(P_E - P_F) x\|_2 \, d\gamma_n(x) \leq R(A) \|P_E - P_F\|_{HS},$$

by the Cauchy-Schwarz inequality and the isotropicity of $\gamma_n$. 

Let us recall the concentration on $SO(n)$. The next result is due to Gromov and V. Milman from [GMS3].

**Theorem 6.7.** Let $f : SO(n) \to \mathbb{R}$ be $L$-Lipschitz map with respect to the Hilbert-Schmidt norm, i.e. $|f(U) - f(V)| \leq L\|U - V\|_{HS}$ for any $U, V \in SO(n)$. Then we have:

$$\mu_n \{ U \in SO(n) : |f(U) - \mathbb{E}f| \geq t \} \leq C_1 \exp(-c_1 nt^2/L^2),$$

for all $t > 0$.

First note that the distribution of the map $F \mapsto w(P_F A)$ can be carried out over the orthogonal group $O(n)$. This follows directly from the Haar-invariance of $\nu_{n,k}$ under $O(n)$. Indeed,

$$\nu_{n,k}(\{ F \in G_{n,k} : w(P_F A) \in B \}) = \nu_n(\{U \in O(n) : w(P_{UE} A) \in B\})$$

for some (any) fixed $E \in G_{n,k}$. That is $F \mapsto w(P_F A)$ can be equivalently viewed as function of $U \in O(n)$. So, in order to apply a concentration result for $F \mapsto w(P_F A)$ on $G_{n,k}$ it suffices to suitably apply the
Gromov-Milman theorem for $U \mapsto w(P_{U,E} A)$ for some fixed $E \in G_{n,k}$. Toward this end we shall need the Lipschitz constant of $U \mapsto \psi(U) = w(P_{U,E} A)$ with respect to the Hilbert-Schmidt norm. Then Lemma 6.6 implies:

$$|\psi(U_1) - \psi(U_2)| \lesssim \frac{R(A)}{\sqrt{k}} \|P_{U_1,E} - P_{U_2,E}\|_{HS}.$$

On the other hand we have $\|P_{U_1,E} - P_{U_2,E}\|_{HS} \leq 2\|U_1 - U_2\|_{HS}$, which proves the Lipschitz condition for $\psi$.

Now we explain how we derive the concentration estimate for $\psi$ on the full orthogonal group by following an argument from [Mec14]. Let $U_1$ be Haar distributed on $SO(n)$ and let $M_e$ be the permutation matrix which corresponds to the permutation $\pi$ such that $\pi(j) = j$ for $j \leq n-2$, $\pi(n-1) = n$ and $\pi(n) = n-1$. Then, $U_2 := U_1M_e$ is the same as $U_1$ (as matrices) but the last two rows which are switched. Therefore, $U_2$ is Haar distributed on $SO^-(n)$. Define the orthogonal map $U$ being $U_1$ with probability 1/2 and $U_2$ with probability 1/2. It follows that $U$ is Haar distributed over $O(n)$. Moreover, note that since $E$ can be considered as $E = [e_i : 1 \leq i \leq k]$ we have $M_{\pi}(E) = E$ as long as $k \leq n-2$. In that case:

$$\mathbb{E}_U \psi(U) = \mathbb{E}_{U_1} \psi(U_1) = \mathbb{E}_{U_2} \psi(U_2) = \int_{G_{n,k}} w(P_{F,A}) d\nu_{n,k}(F) = w(A).$$

Conditioning on whether $U = U_1$ or $U = U_2$ and taking into account Theorem 6.8 we get:

$$\mu_n \left( \left\{ U \in O(n) \mid \left| \psi(U) - \int_{O(n)} \psi \, d\mu_n \right| > t \right\} \right) \leq c_1 \exp(-c_2 n t^2/k R(A)^2),$$

for all $t > 0$. The preceding discussion leads us to the next:

**Theorem 6.8** (concentration for the mean width of projections). *Let $A$ be a symmetric convex body on $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. Then, one has the following concentration inequality:

$$\nu_{n,k} \left( \left\{ F \in G_{n,k} : w(P_{F,A}) - w(A) \geq tw(A) \right\} \right) \leq c_1 \exp \left( -c_2 t^2 k \pi_0(A) \right),$$

for all $t > 0$, where $c_1, c_2 > 0$ are absolute constants.*

**Proof.** The case $1 \leq k \leq n-2$ follows from the previous argument. We treat the case $k = n-1$ separately. Note that any $F \in G_{n,n-1}$ can be identified with $e_1$ for $e \in S^{n-1}$. Thus, we may write:

$$\int_{G_{n,n-1}} w(P_{F,A}) d\nu_{n,n-1}(F) = \int_{S^{n-1}} w(P_{e,A}) d\sigma(e) = w(A),$$

and as in (6.8) we have:

$$w(P_{e,A}) = \frac{c_n}{c_{n-1}} \int_{S^{n-1}} h_A(e) d\sigma(e) = \frac{c_n}{c_{n-1}} \int_{S^{n-1}} h_A(\theta - (\theta, e)e) d\sigma(\theta).$$

Since $c_n/c_{n-1} \approx 1$, it suffices to work with the map $f : S^{n-1} \to \mathbb{R}$ defined by $f(e) := \int_{S^{n-1}} h_A(\theta - (\theta, e)e) d\sigma(\theta)$. We have the following:

**Claim.** The mapping $f : S^{n-1} \to \mathbb{R}$ is Lipschitz with:

$$\text{Lip}(f) \leq R(A) \sqrt{\frac{2}{n}}.$$

---

3 Set $U = U_1^* U_2$. Using the (right) invariance under the orthogonal group and the contractive property of the Hilbert-Schmidt norm, we may write:

$$\|P_{U_1,E} - P_{U_2,E}\|_{HS} = \|P_{E} - U P_{E} U^*\|_{HS} \leq \|P_{E}(I - U^*)\|_{HS} + \|(I - U) P_{E} U^*\|_{HS} \leq \|I - U^*\|_{HS} + \|(I - U) P_{E}\|_{HS} \leq 2\|I - U\|_{HS} = 2\|U_1 - U_2\|_{HS}.$$

4 For any permutation $\pi$ of the set $\{1, 2, \ldots, n\}$ the permutation matrix $M_{\pi}$ acts as: $M_{\pi}(e_j) = e_{\pi(j)}$ for $j = 1, \ldots, n$. 


Proof of Claim. For any $u,v \in S^{n-1}$, we have:

$$|f(u) - f(v)| \leq \int_{S^{n-1}} h_A(\langle u, \theta \rangle u - \langle v, \theta \rangle v) \, d\sigma(\theta)$$

$$\leq R(A) \left( \int_{S^{n-1}} \|\langle u, \theta \rangle u - \langle v, \theta \rangle v\|_2^2 \, d\sigma(\theta) \right)^{1/2}.$$  

Note that:

$$\int_{S^{n-1}} \|\langle u, \theta \rangle u - \langle v, \theta \rangle v\|_2^2 \, d\sigma(\theta) = 2 \int_{S^{n-1}} \theta_1^2 \, d\sigma(\theta) - 2 \langle u, v \rangle \int_{S^{n-1}} \langle u, \theta \rangle \langle v, \theta \rangle \, d\sigma(\theta)$$

$$= \frac{2}{n} (1 - \langle u, v \rangle^2) \leq \frac{2}{n} \|u - v\|_2^2.$$

The result now follows from the concentration on the sphere $S^{n-1}$. 

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