\textbf{b-Functions associated with quivers of type A}

\textit{Dedicated to the 60th birthday of Professor Tatsuo Kimura}

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\textbf{Abstract}

We study the \textit{b}\textsuperscript{-}functions of relative invariants of the prehomogeneous vector spaces associated with quivers of type \(A\). By applying the decomposition formula for \textit{b}\textsuperscript{-}functions, we determine explicitly the \textit{b}\textsuperscript{-}functions of one variable for each irreducible relative invariant. Moreover, we give a graphical algorithm to determine the \textit{b}\textsuperscript{-}functions of several variables.

\textbf{Introduction}

We start with a classical formula

\begin{equation}
\text{det}\left(\frac{\partial}{\partial v}\right)\text{det}(v)^{s+1} = (s + 1)(s + 2) \cdots (s + n) \cdot \text{det}(v)^s \quad (v \in M(n)),
\end{equation}

where \(M(n)\) is the set of square matrices of degree \(n\), and \(\text{det}(\partial/\partial v)\) is the differential operator obtained by replacing each variable \(v_{ij}\) in \(\text{det}(v)\) with \(\partial/\partial v_{ij}\). From a modern viewpoint, the identity (0.1) can be regarded as an example of \textit{b}\textsuperscript{-}functions of prehomogeneous vector spaces. In the present paper, we generalize (0.1) to the relative invariants of the prehomogeneous vector spaces associated with quivers of type \(A\). For this purpose, we use the result of [17], which asserts that under certain conditions, \textit{b}\textsuperscript{-}functions of reducible prehomogeneous vector spaces have decompositions correlated to the decomposition of representations. Moreover, in the latter half, we describe a graphical algorithm to calculate the \textit{b}\textsuperscript{-}functions of several variables.

Now what are relative invariants of prehomogeneous vector spaces associated with quivers of type \(A\)? Let \(Q\) be a quiver of type \(A_r\). In Introduction, \textit{for simplicity}, we assume that \(Q\) is of the following form:

\[
Q: \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow r.
\]
Fix a dimension vector $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{>0}^r$ and put

$$GL(\underline{n}) = GL(n_1) \times GL(n_2) \times \cdots \times GL(n_r),$$

$$\text{Rep}(Q, \underline{n}) = M(n_2, n_1) \oplus M(n_3, n_2) \oplus \cdots \oplus M(n_r, n_{r-1}).$$

The action of $GL(\underline{n})$ on $\text{Rep}(Q, \underline{n})$ is given as follows: for $g = (g_1, \ldots, g_r) \in GL(\underline{n})$ and $v = (X_{2,1}, X_{3,2}, \ldots, X_{r,r-1}) \in \text{Rep}(Q, \underline{n})$, $g \cdot v = (g_2 X_{2,1}^{-1}, g_3 X_{3,2}^{-1}, \ldots, g_r X_{r,r-1}^{-1}).$

Then $(GL(\underline{n}), \text{Rep}(Q, \underline{n}))$ is a prehomogeneous vector space, i.e., there exists an open dense $GL(\underline{n})$-orbit in $\text{Rep}(Q, \underline{n})$. Now we define the set $I_\underline{n}(Q)$ by

$$(0.2) \quad I_\underline{n}(Q) = \{(p, q) : 1 \leq p < q \leq r, n_p = n_q \text{ and } n_t > n_p \text{ for any } t = p+1, \ldots, q-1\}.$$

Then, for $(p, q) \in I_\underline{n}(Q)$,

$$f_{(p,q)}(v) = \det (X_{q,q-1}X_{q-1,q-2} \cdots X_{p+1,p})$$

is a non-zero relative invariant of $(GL(\underline{n}), \text{Rep}(Q, \underline{n}))$, and $f_{(p,q)}(v) ((p, q) \in I_\underline{n}(Q))$ are the fundamental relative invariants. These polynomials are known as “determinantal semi-invariants” (Schofield [19]). By the general theory of prehomogeneous vector spaces, there exists a polynomial $b_{(p,q)}(s) \in \mathbb{C}[s]$ satisfying

$$f_{(p,q)} \left( \frac{\partial}{\partial v} \right) f_{(p,q)}(v)^{s+1} = b_{(p,q)}(s) \cdot f_{(p,q)}(v)^s.$$

Our first result is the calculation of $b_{(p,q)}(s)$.

**Theorem 0.1** (Theorem 3.4 equioriented case).

$$b_{(p,q)}(s) = \prod_{t=p+1}^{q} \prod_{\lambda=1}^{n_p} (s + n_t - n_p + \lambda).$$

In Theorem 3.4, the $b$-functions $b_{(p,q)}(s)$ for quivers $Q$ of type $A$ with arbitrary orientation are determined. This result is a simple application of the decomposition formula found by F. Sato and the author [17].

Our second result is on the $b$-functions of several variables. See Lemma 1.5 for the definition of $b$-functions of several variables. In general, two different relative invariants may have common variables, and this causes a serious combinatorial difficulty. As is easily guessed from (0.2), even in the equioriented case, it is unbelievably tedious to enumerate all the patterns of occurrence of relative invariants which have common variables. Instead of doing this, the author proposes an algorithm to use diagrams, which are recently called “lace diagrams” (Buch-Rimány [4], Knutson-Miller-Shimozono [7]). To apply our
algorithm, we draw some diagram for each irreducible relative invariant, and then simply superpose them. From the resultant diagram, the $b$-function of several variables can be easily calculated. See Figures 5, 6, 7, and 8 in Section 5.

We remark that in [8, 20], the $b$-functions of linear free divisors related to prehomogeneous vector spaces have been studied, and among them are the $b$-functions arising from some quiver representations. There are extensive studies on geometric structures of $(GL(n), \text{Rep}(Q, n))$, and of further interest is to clarify the relation between those studies and our results on $b$-functions.

The plan of the present paper is as follows. In Section 1, we recall some basic definitions and results, and in Section 2, we recall the results on the explicit construction of relative invariants. In Section 3, we calculate the $b$-functions of one variable (Theorem 3.4), and in Section 4, our lace diagrams are defined. Our algorithm to calculate $b$-functions of several variables with diagrams is explained in Section 5, and the following sections (Sections 6 and 7) are devoted to its proof.

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Notation. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. For positive integers $m, n$, we denote by $M(m, n)$ the totality of $m \times n$ complex matrices, and by $0_{m,n}$ the $m \times n$ zero matrix. However, we write simply $M(m)$ and $0_m$ instead of $M(m, m)$ and $0_{m,m}$, respectively. Further, we denote by $E_m$ the identity matrix of size $m$.

1 Preliminaries

A quiver is an oriented graph $Q = (Q_0, Q_1)$ consisting of a set $Q_0$ of vertices and a set $Q_1$ of arrows. Each arrow $a \in Q_1$ has a tail $t(a) \in Q_0$ and a head $h(a) \in Q_0$. Throughout the present paper, except in Section 7, we consider quivers of type $A$. Let $Q$ be a quiver of type $A_r$, i.e., a chain of $r$-vertices and arrows between them:

\[ Q: \begin{array}{ccc}
1 & \leftarrow & 2 \\
\leftarrow & & \leftarrow \\
\cdots & & \leftarrow \\
r & & 
\end{array} \]

We identify the vertex and arrow sets with integral intervals, as

\[ Q_0 = \{1, \ldots, r\}, \quad Q_1 = \{1, 2, \ldots, r-1\} \]
such that \( \{ t(a), h(a) \} = \{ a, a + 1 \} \) for each \( a \in Q_1 \). We also set \( \delta(a) = h(a) - t(a) \), which is equal to \(-1\) for a leftward arrow and \(+1\) for a rightward arrow.

Fix a dimension vector \( \underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{>0} \) and let \( L_i \) be a vector space of dimension \( n_i \). The set of quiver representations with dimension vector \( \underline{n} \) forms the affine space

\[
\text{Rep}(Q, \underline{n}) = \text{Hom}(L_{t(1)}, L_{h(1)}) \oplus \text{Hom}(L_{t(2)}, L_{h(2)}) \oplus \cdots \oplus \text{Hom}(L_{t(r-1)}, L_{h(r-1)}),
\]

on which the group \( GL(\underline{n}) = GL(L_1) \times \cdots \times GL(L_r) \) has a natural action. We choose a basis of \( L_i \) and identify

\[
GL(L_i) \cong GL(n_i) \quad (i = 1, \ldots, r),
\]

\[
\text{Hom}(L_{t(a)}, L_{h(a)}) \cong M(n_{h(a)}, n_{t(a)}) \quad (a = 1, \ldots, r - 1).
\]

Then, for \( g = (g_i)_{1 \leq i \leq r} \in GL(\underline{n}) \) and \( v = (X_{h(a), t(a)})_{1 \leq a \leq r-1} \in \text{Rep}(Q, \underline{n}) \), the action is given explicitly by

\[
g \cdot v = \left( g_{h(a)}X_{h(a), t(a)}g_{t(a)}^{-1} \right)_{1 \leq a \leq r-1}.
\]

**Example 1.1.** Let us consider an equioriented quiver of type \( A_5 \).

\[
Q : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5.
\]

For \( \underline{n} = (n_1, \ldots, n_5) \in \mathbb{Z}^5_{>0} \), \( GL(\underline{n}) \) and \( \text{Rep}(Q, \underline{n}) \) are given by

\[
GL(\underline{n}) = GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4) \times GL(n_5),
\]

\[
\text{Rep}(Q, \underline{n}) = M(n_2, n_1) \oplus M(n_3, n_2) \oplus M(n_4, n_3) \oplus M(n_5, n_4),
\]

and for \( g = (g_1, \ldots, g_5) \in GL(\underline{n}) \) and \( v = (X_{2,1}, X_{3,2}, X_{4,3}, X_{5,4}) \in \text{Rep}(Q, \underline{n}) \), we have

\[
g \cdot v = (g_2X_{2,1}g_1^{-1}, g_3X_{3,2}g_2^{-1}, g_4X_{4,3}g_3^{-1}, g_5X_{5,4}g_4^{-1}).
\]

**Remark 1.2.** When \( Q \) is equioriented, \( (GL(\underline{n}), \text{Rep}(Q, \underline{n})) \) can be regarded as a prehomogeneous vector space of parabolic type arising from a special linear Lie algebra \( \mathfrak{sl}(N) \) with \( N = n_1 + \cdots + n_r \) (Rubenthaler [16], Mortajine [14]).

**Example 1.3.** Let us consider an alternating quiver of type \( A_5 \).

\[
Q : 1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5.
\]

For \( \underline{n} = (n_1, \ldots, n_5) \in \mathbb{Z}^5_{>0} \), \( GL(\underline{n}) \) and \( \text{Rep}(Q, \underline{n}) \) are given by

\[
GL(\underline{n}) = GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4) \times GL(n_5),
\]

\[
\text{Rep}(Q, \underline{n}) = M(n_2, n_1) \oplus M(n_2, n_3) \oplus M(n_4, n_3) \oplus M(n_4, n_5),
\]

and for \( g = (g_1, \ldots, g_5) \in GL(\underline{n}) \) and \( v = (X_{2,1}, X_{2,3}, X_{4,3}, X_{4,5}) \in \text{Rep}(Q, \underline{n}) \), we have

\[
g \cdot v = (g_2X_{2,1}g_1^{-1}, g_2X_{2,3}g_3^{-1}, g_4X_{4,3}g_3^{-1}, g_4X_{4,5}g_5^{-1}).
\]
Two elements $v, v' \in \text{Rep}(Q, \mathbb{R})$ belong to the same $GL(\mathbb{R})$-orbit if and only if $v$ and $v'$ are isomorphic as representations of the quiver $Q$, and the isomorphism classes of representations of $Q$ can be classified by the \textit{indecomposable decompositions}. (For basic terminology on quiver representations, we refer to [3, 6].) It is well known that for the Dynkin quivers, the isomorphism classes of the indecomposable representations correspond to the positive roots of the corresponding root systems. For the quiver of type $A_r$, there is an indecomposable representation $I_{ij}$ for each pair of integers $(i, j)$ with $1 \leq i \leq j \leq r$. The dimension vector of $I_{ij}$ assigns the dimension 1 to all vertices $k \in Q_0$ if $i \leq k \leq j$, and the dimension 0 otherwise. For each arrow $a \in Q_1$ with $i \leq a < j$, the map $I_{ij}^a : \mathbb{C} \to \mathbb{C}$ is the identity. Then, as a representation of the quiver $Q$, any $v \in \text{Rep}(Q, \mathbb{R})$ can be decomposed into the direct sum of indecomposable representations as

$$v \cong \bigoplus_{1 \leq i \leq j \leq r} m_{ij} I_{ij},$$

and the multiplicities $m_{ij}$ are uniquely determined. Hence $\text{Rep}(Q, \mathbb{R})$ decomposes into a finite number of $GL(\mathbb{R})$-orbits. In particular, $(GL(\mathbb{R}), \text{Rep}(Q, \mathbb{R}))$ is a prehomogeneous vector space.

Now we give a brief review on basic properties of prehomogeneous vector spaces. We refer to [9], [12] Chapter 2, [11] for the details.

Let $G$ be a connected algebraic group and $\rho : G \to GL(V)$ a rational representation of $G$ on a finite dimensional vector space $V$. The triplet $(G, \rho, V)$ is called a \textit{prehomogeneous vector space} if $V$ has an open dense $G$-orbit, say $O_0 = \rho(G)v_0$. Let $f$ be a non-zero rational function on $V$ and $\chi \in \text{Hom}(G, \mathbb{C}^\times)$. Then we call $f$ a \textit{relative invariant} with character $\chi$ if $f(\rho(g)v) = \chi(g)f(v)$ for all $g \in G$ and $v \in O_0$. If $f_1$ and $f_2$ are relative invariants which correspond to the same character, then $f_2$ is a constant multiple of $f_1$.

We denote by $S_1, \ldots, S_l$ the irreducible components of $V \setminus O_0$ with codimension one, and let $f_i$ ($1 \leq i \leq l$) be an irreducible polynomial defining $S_i$. Then $f_1, \ldots, f_l$ are algebraically independent relative invariants. Furthermore, every relative invariant $f$ is of the form $f = cf_1^{m_1} \cdots f_l^{m_l}$ ($c \in \mathbb{C}^\times, m_i \in \mathbb{Z}$). We call $f_1, \ldots, f_l$ the \textit{fundamental relative invariants}.

A prehomogeneous vector space $(G, \rho, V)$ is called \textit{reductive} if $G$ is a reductive algebraic group. Now we assume that $(G, \rho, V)$ is a reductive prehomogeneous vector space which has a relatively invariant polynomial $f$ with character $\chi$. Let $d = \deg f, n = \dim V$. We denote by $V^*$ be the dual space of $V$, and by $\rho^* : G \to GL(V^*)$ the contragredient representation of $\rho$. Then the dual triplet $(G, \rho^*, V^*)$ is a prehomogeneous vector space and has a relatively invariant polynomial $f^*$ of degree $d$ with character $\chi^{-1}$.

Fix a basis $\{e_1, \ldots, e_n\}$ of $V$ and let $v = (v_1, \ldots, v_n)$ be the coordinate system of $V$ with respect to this basis. We identify $V$ with $\mathbb{C}^n$. Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis of $\{e_1, \ldots, e_n\}$ and $v^* = (v_1^*, \ldots, v_n^*)$ be the coordinate system of $V^*$ with respect to the dual
basis. We also identify $V^*$ with $\mathbb{C}^n$.

**Lemma 1.4.** There exists a polynomial $b_f(s) = b_0 s^d + b_1 s^{d-1} + \cdots + b_d \in \mathbb{C}[s]$ with $b_0 \neq 0$ such that

$$f^*(\text{grad}_v)f(v)^{s+1} = b_f(s)f(v)^s,$$

where $\text{grad}_v$ is given by

$$\text{grad}_v = \left( \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial v_n} \right).$$

We call $b_f(s)$ the $b$-function of $f$. By Kashiwara [11, Theorem 6.9], every root of the $b$-function $b_f(s)$ is a negative rational number.

Finally, we recall the definition of $b$-functions of several variables (cf. M. Sato [18, Proposition 14]). Let $(G, \rho, V)$ be a reductive prehomogeneous vector space and $f_1, \ldots, f_l$ the fundamental relative invariants. Let $f_1^*, \ldots, f_l^*$ the fundamental relative invariants of the dual prehomogeneous vector space $(G, \rho^*, V^*)$ such that the characters of $f_i$ and $f_i^*$ are the inverse of each other. We put $f = (f_1, \ldots, f_l)$ and $f^* = (f_1^*, \ldots, f_l^*)$. For a multi-variable $s = (s_1, \ldots, s_l)$, we define their powers by $f^s = \prod_{i=1}^l f_i^{s_i}$ and $f^{s^*} = \prod_{i=1}^l f_i^{s_i^*}$.

**Lemma 1.5.** For any $l$-tuple $m = (m_1, \ldots, m_l) \in \mathbb{Z}_{\geq 0}^l$, there exists a non-zero polynomial $b_m(s)$ of $s_1, \ldots, s_l$ satisfying

$$f^m(\text{grad}_v)f^{s+m}(v) = b_m(s)f^v(v).$$

Here $b_m(s)$ is is independent of $v$.

We call $b_m(s)$ the $b$-function of $f = (f_1, \ldots, f_l)$. We easily see that $b_{fm}(s) = b_m(ms)$, and thus $b_{fi}(s) (i = 1, \ldots, l)$ is a specialization of $b_m(s)$.

## 2 Relative invariants

In this section, we describe a condition for $(\text{GL}(n), \text{Rep}(Q, n))$ with $Q$ being of type $A$ to have relative invariants and give their explicit construction. For the details, see Abeasis [1], Koike [13]. Note that Schofield [19] constructed relative invariants for general quivers $Q$. (see Section 7.)

Let $Q$ be a quiver of type $A_r$ with arbitrary orientation. The orientation of $Q$ is determined by the sequence

$$\{1 = \nu(0) < \nu(1) < \nu(2) < \cdots < \nu(h) < \nu(h + 1) = r\}$$

which consists of the sinks and sources of $Q$. Note that if $Q^*$ is the quiver obtained from $Q$ by reversing all the arrows, then $(\text{GL}(n), \text{Rep}(Q^*, n))$ is the dual prehomogeneous vector space $(\text{GL}(n), \text{Rep}(Q, n)^*)$. 

Now we fix a dimension vector \( \underline{n} = (n_1, \ldots, n_r) \) and consider the fundamental relative invariants of \((GL(\underline{n}), \text{Rep}(Q, \underline{n}))\). First, for a given pair \((p, q)\) with \(1 \leq p < q \leq r\), we define indices \(\alpha = \alpha(p, q)\) and \(\beta = \beta(p, q)\) by the conditions
\[
\nu(\alpha - 1) \leq p < \nu(\alpha), \quad \nu(\beta) < q \leq \nu(\beta + 1).
\]
When \(p, q\) are clear from the context, we just write \(\alpha, \beta\) instead of \(\alpha(p, q), \beta(p, q)\). Then \(I_n(Q)\) is defined to be the totality of pairs \((p, q)\) with \(1 \leq p < q \leq r\) which satisfy the following conditions (I1)∼(I4):

(I1) For \(t\) with \(p < t \leq \nu(\alpha)\), we have \(n_t > n_p\),

(I2) For \(\kappa = 0, 1, \ldots, \beta - \alpha - 1\) and \(t\) with \(\nu(\alpha + \kappa) < t \leq \nu(\alpha + \kappa + 1)\), we have
\[
n_t > n_{\nu(\alpha + \kappa)} - n_{\nu(\alpha + \kappa - 1)} + \cdots + (-1)^{\kappa} n_{\nu(\alpha)} + (-1)^{\kappa+1} n_p,
\]

(I3) For \(t\) with \(\nu(\beta) < t < q\), we have
\[
n_t > n_{\nu(\beta)} - n_{\nu(\beta - 1)} + \cdots + (-1)^{\beta-\alpha} n_{\nu(\alpha)} + (-1)^{\beta-\alpha+1} n_p,
\]

(I4) \(n_q = n_{\nu(\beta)} - n_{\nu(\beta - 1)} + \cdots + (-1)^{\beta-\alpha} n_{\nu(\alpha)} + (-1)^{\beta-\alpha+1} n_p\).

By Abeasis [1], we have the following lemma.

**Lemma 2.1.** There exists a one-to-one correspondence between \(I_n(Q)\) and the set of \(GL(\underline{n})\)-orbits in \(\text{Rep}(Q, \underline{n})\) of codimension one. In particular, there exists a one-to-one correspondence between \(I_n(Q)\) and the fundamental relative invariants of \((GL(\underline{n}), \text{Rep}(Q, \underline{n}))\).

The explicit construction of an irreducible relative invariant corresponding to \((p, q) \in I_n(Q)\) is given as follows. When there exist no sink and source between two vertices \(\mu, \nu\) (\(\mu < \nu\)) of \(Q\), either the following (a) or (b) holds:

\[
\begin{align*}
\ (a) \quad & \mu \rightarrow \mu + 1 \rightarrow \cdots \rightarrow \nu - 1 \rightarrow \nu \\
\ (b) \quad & \mu \leftarrow \mu + 1 \leftarrow \cdots \leftarrow \nu - 1 \leftarrow \nu.
\end{align*}
\]

In the case of (a), we put
\[
X_{\nu,\mu} = X_{\nu,\nu-1} X_{\nu-1,\nu-2} \cdots X_{\mu+1,\mu},
\]
and in the case of (b), we put
\[
X_{\mu,\nu} = X_{\mu,\mu+1} X_{\mu+1,\mu+2} \cdots X_{\nu-1,\nu}.
\]

Now suppose that \(p\) is a source and \(q\) is a sink.
\[
\begin{align*}
\ p \rightarrow \cdots \rightarrow \nu(\alpha) \leftarrow \nu(\alpha)+1 \leftarrow \cdots \leftarrow \nu(\alpha+1) \rightarrow \cdots \rightarrow \nu(\beta) \rightarrow \cdots \rightarrow q
\end{align*}
\]
In this case, for \( v \in \text{Rep}(Q, \underline{n}) \), we define a matrix \( Y_{(p,q)}(v) \) by

\[
Y_{(p,q)}(v) = \begin{pmatrix}
X_{\nu(a),p} & X_{\nu(a),\nu(a+1)} & O & \cdots & O & O \\
O & X_{\nu(a+2),\nu(a+1)} & X_{\nu(a+2),\nu(a+3)} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & X_{\nu(\beta-1),\nu(\beta-2)} & X_{\nu(\beta-1),\nu(\beta)} \\
O & O & O & \cdots & O & X_{\nu(\beta),\nu(\beta)}
\end{pmatrix},
\]

and put \( f_{(p,q)}(v) = \det Y_{(p,q)}(v) \). Then it is easy to see that \( f_{(p,q)}(v) \) is a relative invariant of \((GL(n), \text{Rep}(Q, \underline{n}))\). 

Next we consider the case where both \( p \) and \( q \) are sources.

Then we define a matrix \( Y_{(p,q)}(v) \) by

\[
Y_{(p,q)}(v) = \begin{pmatrix}
X_{\nu(a),p} & X_{\nu(a),\nu(a+1)} & O & \cdots & O & O \\
O & X_{\nu(a+2),\nu(a+1)} & X_{\nu(a+2),\nu(a+3)} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & X_{\nu(\beta-1),\nu(\beta)} & X_{\nu(\beta),\nu(\beta)} \\
O & O & O & \cdots & O & X_{\nu(\beta),\nu(\beta)}
\end{pmatrix},
\]

and put \( f_{(p,q)}(v) = \det Y_{(p,q)}(v) \). Then it is easy to see that \( f_{(p,q)}(v) \) is a relative invariant of \((GL(n), \text{Rep}(Q, \underline{n}))\). One can easily find similar expressions of \( Y_{(p,q)}(v) \) for the other cases, i.e., where “\( p \) is a sink and \( q \) is a source” or “both \( p \) and \( q \) are sinks”.

**Example 2.2.** In Example 1.1 assume that \( n_1 < n_2 < n_3 = n_4, n_5 = n_1 \). Then we have \( I_\lambda(Q) = \{(1, 5), (3, 4)\} \). The fundamental relative invariants are given explicitly by

\[
f_{(3,4)}(v) = \det X_{4,3}, \quad f_{(1,5)}(v) = \det X_{5,1} = \det X_{5,4}X_{4,3}X_{3,2}X_{2,1}.
\]

**Example 2.3.** In Example 1.3 assume that \( n_1 + n_3 = n_2 + n_4, n_1 < n_2 < n_3, n_5 = n_1 \). Then we have \( I_\lambda(Q) = \{(1, 4), (2, 5)\} \). The fundamental relative invariants are given explicitly by

\[
f_{(1,4)}(v) = \det \begin{pmatrix} X_{2,1} & X_{2,3} \\ O & X_{4,3} \end{pmatrix}, \quad f_{(2,5)}(v) = \det \begin{pmatrix} X_{2,3} & O \\ X_{4,3} & X_{4,5} \end{pmatrix}.
\]

**Example 2.4.** Let \( Q \) be the following quiver of type \( A_8 \):

\[
Q : 1 \rightarrow \cdot \rightarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6 \leftarrow 7 \leftarrow 8
\]
In this case, \( \nu(0) = 1, \nu(1) = 3, \nu(2) = 4, \nu(3) = 6, \nu(4) = 8 \). If the dimension vector \( \mathbf{n} \) satisfies
\[
\begin{align*}
n_t &> n_1 \quad (t = 2, 3), \\
n_t &> n_3 - n_1 \quad (t = 4), \\
n_t &> n_4 - n_3 + n_1 \quad (t = 5, 6), \\
n_t &> n_6 - n_4 + n_3 - n_1 \quad (t = 7), \\
n_8 &= n_6 - n_4 + n_3 - n_1,
\end{align*}
\]
then \((1, 8) \in I_2(Q)\) and the corresponding relative invariant is given by
\[
f_{(1,8)}(v) = \det \begin{pmatrix} X_{3,1} & X_{3,4} & O \\ O & X_{6,4} & X_{6,8} \end{pmatrix} = \det \begin{pmatrix} X_{3,2}X_{2,1} & X_{3,4} & O \\ O & X_{6,5}X_{5,4} & X_{6,7}X_{7,8} \end{pmatrix}.
\]

3 \( b \)-Functions of one variable

We identify the dual space \( \text{Rep}(Q, n)^* \) of \( \text{Rep}(Q, n) \) with \( \text{Rep}(Q, n) \) itself via a non-degenerate bilinear form
\[
(v, w) = \sum_{a=1}^{r-1} \text{tr}(Y_{h(a),t(a)}X_{h(a),t(a)})
\]
for \( v = (X_{h(a),t(a)})_a \) and \( w = (Y_{h(a),t(a)})_a \in \text{Rep}(Q, n) \). Then, by Lemma 1.4, there exists a polynomial \( b_{(p,q)}(s) \in \mathbb{C}[s] \) satisfying
\[
f_{(p,q)}(\text{grad}_v) f_{(p,q)}(v)^{s+1} = b_{(p,q)}(s) \cdot f_{(p,q)}(v)^s.
\]
By using the decomposition formula for \( b \)-functions proved in F. Sato and Sugiyama [17], we determine \( b_{(p,q)}(s) \).

\textit{Notation 3.1.} As we have seen, in the case of quivers of type \( A \), our relative invariants \( f_{(p,q)}(s) \) are uniquely determined if we specify the subquiver together with dimension vector \( \mathbf{n} \). In the following, we use an informal notation
\[
f_{(p,q)}(v) = \det \begin{pmatrix} n_p & \cdots & n_{p+1} \\ \cdots & \cdots & \cdots \\ n_q & \cdots & n_{q+1} \\ & \cdots & \cdots \\ & \cdots & \cdots \\
\end{pmatrix}
\]
to denote the relative invariant \( f_{(p,q)}(v) \). Moreover, we use the following informal notation to denote the \( b \)-function \( b_{(p,q)}(s) \) of \( f_{(p,q)}(s) \):
\[
b_{(p,q)}(s) = b \begin{pmatrix} n_p & \cdots & n_{p+1} \\ \cdots & \cdots & \cdots \\ & \cdots & \cdots \\ & \cdots & \cdots \\
\end{pmatrix}
\]

\textbf{Example 3.2.} Let us consider an equioriented quiver of type \( A_r \) as below:
\[
Q : 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow r
\]
Then, for \( \mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{>0}^r \), we have
\[
\begin{align*}
\text{GL}(\mathbf{n}) &= \text{GL}(n_1) \times \text{GL}(n_2) \times \text{GL}(n_3) \times \cdots \times \text{GL}(n_r), \\
\text{Rep}(Q, \mathbf{n}) &= M(n_2, n_1) \oplus M(n_3, n_2) \oplus \cdots \oplus M(n_r, n_{r-1}),
\end{align*}
\]
and the action is given by
\[ g \cdot v = (g_2 X_{2,1} g_1^{-1}, g_3 X_{3,2} g_2^{-1}, \ldots, g_r X_{r,r-1} g_{r-1}^{-1}) \]
for \( g = (g_1, g_2, g_3, \ldots, g_r) \in GL(n) \) and \( v = (X_{2,1}, X_{3,2}, \ldots, X_{r,r-1}) \in \text{Rep}(Q, n) \). Now we assume that \( n_1 = n_r < n_2, n_3, \ldots, n_{r-1} \). Then
\[ f(v) = \det(X_{r,1}) = \det(X_{r,r-1} \cdots X_{3,2} X_{2,1}) \]
is an irreducible relative invariant corresponding to the character \( \chi(g) = \det g_r \det g_r^{-1} \).
Note that this polynomial can be expressed as
\[ f(v) = \det \left( \begin{array}{ccc} n_1 \circ & n_2 \circ & n_3 \circ & \cdots & n_r \circ \end{array} \right) \]
if we employ Notation 3.1. We shall calculate the \( b \)-function \( b_f(s) \) of \( f \) by using the result of [17]. We put
\[ G' = GL(n_3) \times \cdots \times GL(n_r), \]
\[ E = M(n_3, n_2) \oplus \cdots \oplus M(n_r, n_{r-1}), \]
\[ F = M(n_2, n_1), \]
\[ GL(m) = GL(n_2), \quad GL(n) = GL(n_1) \]
and regard \((GL(n), \text{Rep}(Q, n))\) as a prehomogeneous vector space of the form [17 (2.2)]. Then we have \( l = 0, d = 1 \) in the notation of [17 Section 2] and thus we can apply [17] Theorem 2.6 in order to obtain the decomposition
\[ b_f(s) = b_1(s)b_2(s). \]
Moreover, by [17] Theorem 3.3, we have
\[ b_2(s) = \prod_{\lambda=1}^{n_1} (s + n_2 - n_1 + \lambda). \]
Note that \( m = n_2, n = n_1, d = 1 \) in the notation of [17] Theorem 3.3]. The last step is to calculate \( b_1(s) \). Let \( X_{2,1}^0 = t(E_{n_1} | 0_{n_1, n_2-n_1}) \in F \) and put this into \( f(v) \). Then we have
\[ f(X_{2,1}^0, X_{3,2}, \ldots, X_{r,r-1}) = \det(X_{r,r-1} \cdots X_{3,2}^t), \]
where \( X_{3,2}^t \) is a part of the following block decomposition:
\[ X_{3,2} = (X_{3,2}^t | X_{3,2}^0) \in M(n_3, n_2), \quad X_{3,2}^t \in M(n_3, n_1), \quad X_{3,2}^0 \in M(n_3, n_2 - n_1). \]
Note that \( f(X_{2,1}^0, X_{3,2}, \ldots, X_{r,r-1}) \) can be regarded as the relative invariant
\[ \det \left( \begin{array}{ccc} n_1 \circ & n_2 \circ & n_3 \circ & \cdots & n_r \circ \end{array} \right) \]
of a prehomogeneous vector space arising from an equioriented quiver of type \( A_{r-1} \). By using Notation 3.1, we can summarize the above argument into a reduction formula

\[
b_f(s) = b \left( \mathcal{O} \to \mathcal{O} \to \mathcal{O} \to \cdots \to \mathcal{O} \right) = b \left( \mathcal{O} \to \mathcal{O} \to \mathcal{O} \to \cdots \to \mathcal{O} \right) \times \prod_{\lambda=1}^{n_1} (s + n_2 - n_1 + \lambda).
\]  

(3.1)

By repeating this cut-off operation, we get to the \( b \)-function of \( \det \left( \mathcal{O} \to \mathcal{O} \right) \), which is nothing but the formula (0.1). We therefore obtain

\[
b_f(s) = \prod_{t=2}^{r} \prod_{\lambda=1}^{n_1} (s + n_t - n_1 + \lambda).
\]

The reduction formula (3.1) can be generalized as the following lemma.

**Lemma 3.3.**

\[
b \left( \mathcal{O} \to \mathcal{O} \to \cdots \to \mathcal{O} \right) = \prod_{\lambda=1}^{n_p} (s + n_{p+1} - n_p + \lambda) \times b \left( \mathcal{O} \to \cdots \to \mathcal{O} \right)
\]

(3.2)

\[
b \left( \mathcal{O} \to \cdots \to \mathcal{O} \right) = \prod_{\lambda=1}^{n_p} (s + n_{p+1} - n_p + \lambda) \times \prod_{\lambda=1}^{n_{p+2}} (s + n_{p+2} - n_p + \lambda) \times b \left( \cdots \right),
\]

(3.3)

\[
b \left( \mathcal{O} \to \cdots \to \mathcal{O} \right) = \prod_{\lambda=1}^{n_p} (s + n_{p+1} - n_p + \lambda) \times b \left( \mathcal{O} \to \cdots \to \mathcal{O} \right)
\]

Proof. We first prove (3.3) in the case of \( \delta(p+2) = 1 \), in which case our quiver is of the form

\[
\mathcal{O} \to \mathcal{O} \to \cdots
\]

and we have \( \nu(\alpha) = n_{p+1} \) and \( \nu(\alpha + 1) = n_{p+2} \). The relative invariant \( f_{(p,q)}(v) \) is given by \( f_{(p,q)}(v) = \det Y_{(p,q)}(v) \) with

\[
Y_{(p,q)}(v) = \begin{pmatrix}
X_{p+1,p} & X_{p+1,p+2} & O \\
O & X_{v(\alpha+2),p+2} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]

11
and the corresponding character is \( \chi(g) = \det g_p^{-1} \det g_{p+1}^{-1} \cdots \). We put

\[
G' = GL(n_{p+2}) \times GL(n_{p+3}) \times \cdots \times GL(n_q),
\]

\[
E = M(n_{p+1}, n_{p+2}) \oplus M(n_{p+3}, n_{p+2}) \oplus \cdots, \quad F = M(n_{p+1}, n_p),
\]

\[
GL(m) = GL(n_{p+1}), \quad GL(n) = GL(n_p),
\]

and regard \((GL(n), \text{Rep}(Q, n))\) as a triplet of the form \([17, (2.2)]\). Then we have \(l = d = 1\) in the notation of \([17\text{ Section 2}]\) and thus we can apply \([17\text{ Theorem 2.5}]\) so that we have the decomposition

\[
b_{(p,q)}(s) = b_1(s)b_2(s).
\]

Moreover, by \([17\text{ Theorem 3.3}]\), we have

\[
b_2(s) = \prod_{\lambda=1}^{n_p} (s + n_{p+1} - n_p + \lambda).
\]

Note that \(m = n_{p+1}, n = n_p, d = 1\) in the notation of \([17\text{ Theorem 3.3}]\). To calculate \(b_1(s)\), we let \(X^0_{p+1,p} = \{E_{n_p} | 0_{n_p,n_{p+1}-n_p}\} \in F\) and put this into \(f_{(p,q)}(v)\). Then we have

\[
f_{(p,q)}(X^0_{p+1,p}, X_{p+1,p+2}, X_{p+3,p+2}, \ldots) = \det Y'_{(p,q)}(v),
\]

where

\[
Y'_{(p,q)}(v) = \left(\begin{array}{cccc}
X''_{p+1,p+2} & O & \cdots \\
X_{p+1,p+2} & O & \cdots \\
& & \ddots & \ddots \\
& & & \ddots
\end{array}\right),
\]

and \(X''_{p+1,p+2}\) is a part of the following block decomposition:

\[
X_{p+1,p+2} = \left(\begin{array}{c}
X''_{p+1,p+2} \\
X'_{p+1,p+2}
\end{array}\right) \in M(n_{p+1}, n_{p+2}),
\]

\[
X'_{p+1,p+2} \in M(n_p, n_{p+2}), \quad X''_{p+1,p+2} \in M(n_{p+1} - n_p, n_{p+2}).
\]

The polynomial \(f_{(p,q)}(X^0_{p+1,p}, X_{p+1,p+2}, X_{p+3,p+2}, \ldots)\) can be regarded as the relative invariant

\[
\det \left(\begin{array}{c}
n_{p+1} - n_p \\
n_{p+2} \\
& \cdots
\end{array}\right)
\]

and hence we obtain a reduction formula

\[
b_{(p,q)}(s) = b \left(\begin{array}{c}
n_p \\
n_{p+1} \\
n_{p+2} \\
& \cdots
\end{array}\right) \times \prod_{\lambda=1}^{n_p} (s + n_{p+1} - n_p + \lambda).
\]
By repeating this cut-off operation with [17, Theorem 2.5], we have
\[ b\left(\begin{array}{c}
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right) = b\left(\begin{array}{c}
\scriptscriptstyle n_{p+2} - n_{p+1} + n_p \\
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right) \times \prod_{\lambda=1}^{n_{p+1} - n_p} (s + n_{p+2} - n_{p+1} + n_p + \lambda).
\]

Combining the above two formulas, we obtain
\[ b\left(\begin{array}{c}
\scriptscriptstyle p,q \\
\scriptscriptstyle s
\end{array} \right) = b\left(\begin{array}{c}
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right) \left(\begin{array}{c}
\scriptscriptstyle s + n_{p+2} - n_{p+1} + n_p + \lambda \\
\scriptscriptstyle s + n_{p+1} - n_p + \lambda
\end{array} \right) \times \prod_{\lambda=1}^{n_{p+1} - n_p} (s + n_{p+2} - n_{p+1} + n_p + \lambda)
\times b\left(\begin{array}{c}
\scriptscriptstyle n_{p+2} - n_{p+1} + n_p \\
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right).
\]

On the other hand, when \(\delta(p + 2) = -1\), we carry out the cut-off operations as follows:
\[ b\left(\begin{array}{c}
\scriptscriptstyle n_p \\
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right) \left(\begin{array}{c}
\scriptscriptstyle s + n_{p+1} - n_p + \lambda \\
\scriptscriptstyle s + n_{p+2} - n_{p+1} + n_p + \lambda
\end{array} \right) \times \prod_{\lambda=1}^{n_{p+1} - n_p} (s + n_{p+2} - n_{p+1} + n_p + \lambda)
\times b\left(\begin{array}{c}
\scriptscriptstyle n_{p+2} - n_{p+1} + n_p \\
\scriptscriptstyle n_{p+1} - n_p \\
\scriptscriptstyle n_{p+2} \\
\hspace{1cm} \vdots
\end{array} \right) \left(\begin{array}{c}
\scriptscriptstyle n_{p+3} \\
\scriptscriptstyle \cdots
\end{array} \right) \right).
\]

In the first equality, we use [17, Theorem 2.5] and in the second equality, we use [17, Theorem 2.6]. The two formulas (3.4) and (3.5) can be summarized into one formula (3.3) in an abbreviated form. We omit the proof of (3.2), since it is quite the same as above.

Now we give a general formula for the \(b\)-function \(b\left(\begin{array}{c}
\scriptscriptstyle p,q \\
\scriptscriptstyle s
\end{array} \right)\). For \(\kappa = 0, 1, \ldots, \beta - \alpha\), we put
\[\pi_{\nu(\alpha + \kappa)} = \sum_{\tau=0}^{\kappa} (-1)^{\tau} n_{\nu(\alpha + \kappa - \tau)} + (-1)^{\kappa+1} n_p\]
\[= n_{\nu(\alpha + \kappa)} - n_{\nu(\alpha + \kappa - 1)} + \cdots + (-1)^{\kappa} n_{\nu(\alpha)} + (-1)^{\kappa+1} n_p.
\]

Then, as an immediate consequence of (3.2) and (3.3), we obtain the following theorem.
Theorem 3.4.

\[ b_{(p,q)}(s) = \prod_{t=p+1}^{\nu(\alpha)} \prod_{\lambda=1}^{n_p} (s + n_t - n_p + \lambda) \]
\[ \times \prod_{\kappa=0}^{\beta-\alpha-1} \prod_{t=\nu(\alpha+\kappa)+1}^{\nu(\alpha+\kappa)} (s + n_t - \nu(\alpha+\kappa) + \lambda) \]
\[ \times \prod_{t=\nu(\beta)+1}^{q} \prod_{\lambda=1}^{\nu(\beta)} (s + n_t - \nu(\beta) + \lambda). \]

Remark 3.5. Recently, Wachi [26] has studied the above theorem from the viewpoint of Capelli identities. In particular, he gave another proof of the above theorem for the equioriented case by using his generalized Capelli identity ([26, Theorem 5.1]).

Example 3.6. For the relative invariants in Example 2.2 we have

\[ b_{(3,4)}(s) = (s + 1)(s + 2) \cdots (s + n_3), \]
\[ b_{(1,5)}(s) = \prod_{t=2}^{5} \prod_{\lambda=1}^{n_1} (s + n_t - n_1 + \lambda) \]
\[ = (s + 1) \cdots (s + n_1) \times (s + n_2 - n_1 + 1) \cdots (s + n_2) \]
\[ \times (s + n_3 - n_1 + 1)^2 \cdots (s + n_3)^2. \]

Example 3.7. For the relative invariants in Example 2.3 we have

\[ b_{(1,4)}(s) = (s + 1) \cdots (s + n_3) \times (s + n_2 - n_1 + 1) \cdots (s + n_2), \]
\[ b_{(2,5)}(s) = (s + 1) \cdots (s + n_4) \times (s + n_3 - n_2 + 1) \cdots (s + n_3). \]

Example 3.8. The \( b \)-function \( b_{(1,8)}(s) \) of the relative invariant \( f_{(1,8)}(v) \) in Example 2.4 is given by

\[ b_{(1,8)}(s) = \prod_{t=2}^{3} (s + n_t - n_1 + 1) \cdots (s + n_t) \]
\[ \times \prod_{t=4}^{4} (s + n_t - n_3 + n_1 + 1) \cdots (s + n_t) \]
\[ \times \prod_{t=5}^{6} (s + n_t - n_4 + n_3 - n_1 + 1) \cdots (s + n_t) \]
\[ \times (s + 1) \cdots (s + n_7). \]
4 Rank parameters and lace diagrams

To describe a combinatorial method to compute the $b$-functions of several variables, we need to introduce the notion of rank parameters and lace diagrams.

Assume that $Q$ is a quiver of type $A_r$ with arbitrary orientation. For any pair $(i, j)$ of integers with $i < j$, we denote by $Q^{(i,j)}$ the subquiver of $Q$ starting at $i$ and terminating at $j$. Here $Q^{(i,j)}$ includes the vertices $i$ and $j$, and thus $i$ is either a source or a sink of $Q^{(i,j)}$, and so is $j$. Let $A = (A_{h(a), t(a)})_{1 \leq a \leq r-1} \in \text{Rep}(Q, \underline{n})$ be a given representation of $Q$ with dimension vector $\underline{n}$; recall that $A_{h(a), t(a)}$ is a linear map from $L_{t(a)}$ to $L_{h(a)}$. For $1 \leq i < j \leq r$, we define a map

$$Y_{(i,j)}(A) : \bigoplus_{\tau} L_{\tau} \rightarrow \bigoplus_{\sigma} L_{\sigma} \quad \left( \tau \text{ runs over all the sources of } Q^{(i,j)} \right)$$

$$\sigma \text{ runs over all the sinks of } Q^{(i,j)}$$

to be the collection of linear maps $\varphi^A_{\kappa}$ defined by

$$(4.1) \quad \varphi^A_{\kappa} : L_{\mu(\kappa-1)} \oplus L_{\mu(\kappa+1)} \rightarrow L_{\mu(\kappa)} ; \quad (z, w) \mapsto (A_{\mu(\kappa), \mu(\kappa-1)} z + A_{\mu(\kappa), \mu(\kappa+1)} w),$$

where $\mu(\kappa-1), \mu(\kappa+1)$ are sources of $Q^{(i,j)}$ and $\mu(\kappa)$ is a sink of $Q^{(i,j)}$.

**Example 4.1.** Let us consider the following quiver of type $A_5$:

$$1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5.$$ 

Then we have

- $Y_{(1,4)}(A) : L_1 \oplus L_3 \rightarrow L_2 \oplus L_4 ; \quad (z_1, z_3) \mapsto (A_{2,1}z_1 + A_{2,3}z_3, A_{4,3}z_3)$
- $Y_{(2,5)}(A) : L_3 \rightarrow L_2 \oplus L_5 ; \quad z_3 \mapsto (A_{2,3}z_3, A_{5,4}A_{4,3}z_3)$
- $Y_{(1,5)}(A) : L_1 \oplus L_3 \rightarrow L_2 \oplus L_5 ; \quad (z_1, z_3) \mapsto (A_{2,1}z_1 + A_{2,3}z_3, A_{5,4}A_{4,3}z_3)$.

For $A \in \text{Rep}(Q, \underline{n})$, we define $N_{ij}^A$ by

$$N_{ij}^A := \begin{cases} \text{rank } Y_{(i,j)}(A) & (i < j) \\ \dim L_i = n_i & (i = j) \end{cases}.$$ 

We call $N_A := \{N_{ij}^A\}_{1 \leq i < j \leq r}$ the rank parameter of $A \in \text{Rep}(Q, \underline{n})$. As the following lemma shows, the rank parameter is an invariant which characterizes the $GL(\underline{n})$-orbit.

**Lemma 4.2.** For $A \in \text{Rep}(Q, \underline{n})$, we denote by $O_A$ the $GL(\underline{n})$-orbit through $A$. Then for $A, B \in \text{Rep}(Q, \underline{n})$, we have $O_A \subset O_B$ if and only if $N_{ij}^A \leq N_{ij}^B$ for $1 \leq i \leq j \leq r$. That is, the partial ordering on the rank parameters coincides with the closure ordering on $GL(\underline{n})$-orbits. In particular, $O_A = O_B$ if and only if $N_{ij}^A = N_{ij}^B$ for $1 \leq i \leq j \leq r$. 

15
Lemma 4.3. Let \((p,q) \in I_\mathfrak{a}(Q)\). If we multiply all the linear forms contained in \(s + \mathcal{F}^{(p,q)}\) together, we obtain the \(b\)-function \(b_{(p,q)}(s)\).

The rest of this section is devoted to the proof of the above lemma. To calculate the rank parameters, we need to construct the lace diagrams corresponding to the locally closed orbits; let us recall the definition of the lace diagrams (cf. Gyoja [9, Lemma 1.4]). We note that Shmelkin [21] called \(A^{(p,q)} \in \mathcal{O}^{(p,q)}\) a locally semi-simple representation of \(Q\).

For \(N^{(p,q)} = \{N_{ij}^{(p,q)}\}_{1 \leq i \leq j \leq r}\), we put

\[
\mathcal{F}^{(p,q)} := \left\{ \left\{ N_{22}^{(p,q)} - N_{12}^{(p,q)} + 1, \ldots, N_{22}^{(p,q)} (= n_2) \right\}, \ldots, \left\{ N_{33}^{(p,q)} - N_{23}^{(p,q)} + 1, \ldots, N_{33}^{(p,q)} (= n_3) \right\}, \ldots, \left\{ N_{rr}^{(p,q)} - N_{r-1,r}^{(p,q)} + 1, \ldots, N_{rr}^{(p,q)} (= n_r) \right\} \right\}.
\]

Note that \(\mathcal{F}^{(p,q)}\) is a set consisting of \(r - 1\) sets, and each set consists of consecutive natural numbers. However, if \(N_{k,k-1,k}^{(p,q)} = 0\), then we regard \(\{N_{k,k}^{(p,q)} - N_{k-1,k}^{(p,q)} + 1, \ldots, N_{k,k}^{(p,q)} (= n_k)\}\) as the empty set \(\emptyset\). Moreover, we define a “set” of linear forms \(s + \mathcal{F}^{(p,q)}\) by

\[
s + \mathcal{F}^{(p,q)} := \left\{ \left\{ s + N_{22}^{(p,q)} - N_{12}^{(p,q)} + 1, \ldots, s + N_{22}^{(p,q)} (= s + n_2) \right\}, \ldots, \left\{ s + N_{33}^{(p,q)} - N_{23}^{(p,q)} + 1, \ldots, s + N_{33}^{(p,q)} (= s + n_3) \right\}, \ldots, \left\{ s + N_{rr}^{(p,q)} - N_{r-1,r}^{(p,q)} + 1, \ldots, s + N_{rr}^{(p,q)} (= s + n_r) \right\} \right\}.
\]

Then we have the following lemma.

**Lemma 4.3.** Let \((p,q) \in I_\mathfrak{a}(Q)\). We denote by \(N^{(p,q)}\) the rank parameter which is minimal (with respect to the above-mentioned partial ordering) among the rank parameters \(N_A\) such that \(Y_{(p,q)}(A)\) is an isomorphism. The orbit \(\mathcal{O}^{(p,q)}\) corresponding to \(N^{(p,q)}\) is the closed \(GL(\mathfrak{a})\)-orbit in \(\{A \in \text{Rep}(Q, \mathfrak{a}) \mid f_{(p,q)}(A) \neq 0\}\), and thus it is unique (cf. Gyoja [9, Lemma 1.4]). We note that Shmelkin [21] called \(A^{(p,q)} \in \mathcal{O}^{(p,q)}\) a locally semi-simple representation of \(Q\).
Here $I_{ij}$ is the indecomposable representation corresponding to the interval $[i, j]$ and $m_{ij}$ is the multiplicity. See (1.1).

3. Draw arrows between dots in such a way that there exist exactly $m_{ij}$ line segments starting at $i$ and terminating at $j$. In other words, if we take a suitably chosen basis of $L_i$ and identify them with the dots of the $i$-th column, then each linear map $A_{t(h(a))} : L_{t(a)} \to L_{h(a)}$ is given according to the connections between dots. Namely, if dot $j$ of column $t(a)$ is connected to dot $k$ of column $h(a)$, then $A_{t(h(a))}$ maps the $j$-th basis element of $L_{t(a)}$ to the $k$-th basis element of $L_{h(a)}$; and if dot $j$ of column $t(a)$ is not connected to any dot in column $h(a)$, then the corresponding basis element of $L_{t(a)}$ is mapped to zero.

4. Two consecutive columns connected with a rightward arrow (resp. leftward arrow) are bottom-aligned (resp. top-aligned). This convention comes from [1, p. 467].

**Example 4.5.** Let $n = (2, 5, 6, 6, 2)$ in Example 2.2 (see also Example 3.6). Then we have

\[
\begin{align*}
b_{(3,4)}(s) &= (s + 1)(s + 2)(s + 3)(s + 4)(s + 5)(s + 6), \\
b_{(1,5)}(s) &= (s + 1)(s + 2)(s + 4)(s + 5)^3(s + 6)^2.
\end{align*}
\]

Now the lace diagrams corresponding to the locally closed orbits $O^{(3,4)}$ and $O^{(1,5)}$ are given as Figure 1. Note that if any array in the diagram is erased, then the condition $f_{(p,q)}(A) \neq 0$ is not satisfied, and conversely if some extra array is added, then the minimality condition is not satisfied. Thus we see that the rank parameters $N^{(3,4)}$ and $N^{(1,5)}$ are given by (4.3).

\[
\begin{align*}
N^{(3,4)} : & \quad 2 & 0 & 0 & 0 & 0 \\
& 5 & 0 & 0 & 0 \\
& 6 & 6 & 0 \\
& 6 & 0 \\
& 2
\end{align*}
\]

\[
\begin{align*}
N^{(1,5)} : & \quad 2 & 2 & 2 & 2 & 2 \\
& 2 & 2 & 2 & 2 \\
& 6 & 2 & 2 \\
& 6 & 2 \\
& 2
\end{align*}
\]
By (4.2), the rank parameters read $\mathcal{F}^{(3,4)}$, $s + \mathcal{F}^{(3,4)}$ and $\mathcal{F}^{(1,5)}$, $s + \mathcal{F}^{(1,5)}$ as

\[
\mathcal{F}^{(3,4)} = \{ \emptyset : \emptyset ; \{ 1, 2, 3, 4, 5, 6 \} : \emptyset \}, \\
s + \mathcal{F}^{(3,4)} = \{ \emptyset : \emptyset ; \{ s + 1, s + 2, s + 3, s + 4, s + 5, s + 6 \} : \emptyset \}, \\
\mathcal{F}^{(1,5)} = \{ \{ 4, 5 \} ; \{ 5, 6 \} ; \{ 1, 2 \} \}, \\
s + \mathcal{F}^{(1,5)} = \{ \{ s + 4, s + 5 \} ; \{ s + 5, s + 6 \} ; \{ s + 5, s + 6 \} ; \{ s + 1, s + 2 \} \}.
\]

Note that if we multiply all the linear forms contained in $s + \mathcal{F}^{(3,4)}$ (resp. $s + \mathcal{F}^{(1,5)}$), then we obtain the $b$-function $b_{(3,4)}(s)$ (resp. $b_{(1,5)}(s)$).

**Example 4.6.** Let $n = (2, 5, 7, 4, 2)$ in Example 2.3 (see also Example 3.7). Then we have

\[
b_{(1,4)}(s) = (s + 1)(s + 2)(s + 3)(s + 4)^2(s + 5)^2(s + 6)(s + 7), \\
b_{(2,5)}(s) = (s + 1)(s + 2)(s + 3)(s + 4)^2(s + 5)(s + 6)(s + 7).
\]

The lace diagrams corresponding to $\mathcal{O}^{(1,4)}$ and $\mathcal{O}^{(2,5)}$ are given as in Figure 2. Here the reader is referred to the condition 4 of Definition 4.4. Now we see that the rank parameters $\mathcal{N}^{(1,4)}$ and $\mathcal{N}^{(2,5)}$ are given by (4.4).

\[
(4.4) \quad \mathcal{N}^{(1,4)} : \quad 2 \quad 2 \quad 5 \quad 9 \quad 9 \\
\quad \quad \quad \quad 5 \quad 3 \quad 7 \quad 7 \\
\quad \quad \quad \quad 7 \quad 4 \quad 4 \\
\quad \quad \quad \quad 4 \quad 0 \\
\quad \quad \quad \quad 2 \\
\mathcal{N}^{(2,5)} : \quad 2 \quad 0 \quad 5 \quad 7 \quad 9 \\
\quad \quad \quad \quad 5 \quad 5 \quad 7 \quad 9 \\
\quad \quad \quad \quad 7 \quad 2 \quad 4 \\
\quad \quad \quad \quad 4 \quad 2 \\
\quad \quad \quad \quad 2
\]

Hence we observe that $\mathcal{F}^{(1,4)}$, $s + \mathcal{F}^{(1,4)}$ and $\mathcal{F}^{(2,5)}$, $s + \mathcal{F}^{(2,5)}$ are

\[
\mathcal{F}^{(1,4)} = \{ \{ 4, 5 \} ; \{ 5, 6, 7 \} ; \{ 1, 2, 3, 4 \} : \emptyset \}, \\
s + \mathcal{F}^{(1,4)} = \{ \{ s + 4, s + 5 \} ; \{ s + 5, s + 6, s + 7 \} ; \{ s + 1, s + 2, s + 3, s + 4 \} : \emptyset \}, \\
\mathcal{F}^{(2,5)} = \{ \emptyset ; \{ 3, 4, 5, 6, 7 \} ; \{ 3, 4 \} ; \{ 1, 2 \} \}, \\
s + \mathcal{F}^{(2,5)} = \{ \emptyset ; \{ s + 3, s + 4, s + 5, s + 6, s + 7 \} ; \{ s + 3, s + 4 \} ; \{ s + 1, s + 2 \} \}.
\]
and that if we multiply all the linear forms contained in \( s + \mathcal{F}(1,4) \) (resp. \( s + \mathcal{F}(2,5) \)), then we obtain the \( b \)-function \( b_{(1,4)}(s) \) (resp. \( b_{(2,5)}(s) \)).

Now we accurately describe how to construct the lace diagrams such as Figures 1 and 2.

**Definition 4.7 (exact lace diagrams).** Let \((p, q) \in I_2(Q)\) and \(O(p,q)\) the closed orbit in \( \{ A \in \text{Rep}(Q, \underline{n}) : f_{(p,q)}(A) \neq 0 \} \). Then we construct the lace diagram corresponding to \( O(p,q) \) according to the following convention.

1. For \( \nu = p, \ldots, q \), let \( e^{(\nu)}_1, \ldots, e^{(\nu)}_{n_{\nu}} \) be a basis of \( L_{\nu} \) and we identify these vectors with the dots in column \( \nu \) in the lace diagram.

2. First we consider the case of \( \delta(p) = 1 \), i.e., we assume that the arrow \( p \in Q_1 \) is rightward. Then, from each \( e^{(p)}_j \) for \( j = 1, \ldots, n_p \), we draw a horizontal arrow to a dot in column \( (p + 1) \). Choose a suitable order of the basis of \( L_{p+1} \) so that these arrows give a one-to-one correspondence between \( e^{(p)}_1, \ldots, e^{(p)}_{n_p} \) and \( e^{(p+1)}_1, \ldots, e^{(p+1)}_{n_p} \). Stop here if \( p + 1 = q \).

3. If \( p + 1 < q \) and \( \delta(p + 1) = 1 \), then we draw a horizontal arrow from each \( e^{(p+1)}_j \) for \( j = 1, \ldots, n_p \) to a dot in column \( (p + 2) \). If \( p + 1 < q \) and \( \delta(p + 1) = -1 \), then we draw a horizontal arrow to each \( e^{(p+1)}_j \) for \( j = n_p + 1, \ldots, n_{p+1} \) from some dot in column \( (p + 2) \). Continue to draw arrows until we reach the column \( q \).

4. In the case of \( \delta(p) = -1 \), we reverse the orientations in the above 2., 3.

We remark that in general, two different lace diagrams may correspond to the same orbit. However, under the above convention, the locally closed orbit \( O(p,q) \) uniquely determines the lace diagram. We call it the exact lace diagram corresponding to \( O(p,q) \).

**Proof of Lemma 4.3.** We compare the construction of the exact lace diagrams with the reduction formulas (3.2) and (3.3). Then we see that there exists a one-to-one correspondence between the arrows in the exact lace diagrams and the factors of \( b \)-functions. See Figures 3 and 4. Thus the lemma is proved. \( \square \)
5 \textit{b-Functions of several variables}

In this section, we describe an algorithm to calculate the multi-variate \textit{b}-function \(b_{\gamma}(2)\) of \((GL(\mathcal{N}), \text{Rep}(Q, \mathcal{N}))\), and give some examples. We take an arbitrary numbering on \(I_{\mathcal{N}}(Q)\) and let

\[I_{\mathcal{N}}(Q) = \{(p_1, q_1), (p_2, q_2), \ldots, (p_1, q_l)\}\.

In the following, we write as \(f_{(p_1,q_1)} = f_1, N^{(p_2,q_2)} = N^{(2)}, \mathcal{F}^{(p_3,q_3)} = \mathcal{F}^{(3)}, \ldots\).

1°. First, for given linear forms \(s_{i_1} + \alpha, s_{i_2} + \alpha, \ldots, s_{i_t} + \alpha\) with the same constant term, we define the \textit{superposition operation} as follows:

\[
\begin{align*}
  s_{i_1} + \alpha, s_{i_2} + \alpha, \ldots, s_{i_t} + \alpha & \mapsto s_{i_1} + s_{i_2} + \ldots + s_{i_t} + \alpha \\
\end{align*}
\]

Carry out this operation on the \((k-1)\)-th components

\[
\begin{align*}
  \left\{ s_1 + N^{(1)}_{k,k-1} - N^{(1)}_{kk} + 1, \ldots, s_1 + N^{(1)}_{kk} (= s_1 + n_k) \right\}, \\
  \left\{ s_2 + N^{(2)}_{k,k-1} - N^{(2)}_{kk} + 1, \ldots, s_2 + N^{(2)}_{kk} (= s_2 + n_k) \right\}, \\
  \cdots \\
  \left\{ s_l + N^{(l)}_{k,k-1} - N^{(l)}_{kk} + 1, \ldots, s_l + N^{(l)}_{kk} (= s_l + n_k) \right\}
\end{align*}
\]

of \(s_1 + \mathcal{F}^{(1)}, s_2 + \mathcal{F}^{(2)}, \ldots, s_l + \mathcal{F}^{(l)}\). Here we ignore the empty set \(\emptyset\).

\textbf{Example 5.1.} From

\[
\begin{align*}
  \{s_1 + 3, s_1 + 4, s_1 + 5\}, \\
  \{s_2 + 4, s_2 + 5\}, \\
  \emptyset, \\
  \{s_4 + 1, s_4 + 2, s_4 + 3, s_4 + 4, s_4 + 5\},
\end{align*}
\]
we obtain the following new linear forms

\[ s_1 + 1, s_4 + 2, s_1 + s_4 + 3, s_1 + s_2 + s_4 + 4, s_1 + s_2 + s_4 + 5. \]

2°. Carry out the operation 1° for all \( k = 2, \ldots, r \).

3°. Substitute the linear forms obtained in 2° by the rule

\[ s_{i_1} + s_{i_2} + \cdots + s_{i_t} + \alpha \mapsto [s_{i_1} + s_{i_2} + \cdots + s_{i_t} + \alpha]_{m_1 + m_2 + \cdots + m_t}, \]

and then multiply all of them together. Here the square parentheses symbol stands for

\[ [A]_m := A(A + 1)(A + 2) \cdots (A + m - 1). \]

**Theorem 5.2.** The output of the operation 3° is the \( b \)-function \( b_m(s) \) of \( f = (f_1, \ldots, f_t) \).

The proof of the theorem will be given in the following sections. In the remainder of this section, we calculate the \( b \)-functions \( b_m(s) \) for the two examples discussed in the previous sections.

**Example 5.3.** In Example 4.5 put \( f_1 := f_{(3,4)} \), \( f_2 := f_{(1,5)} \). For

\[ s_1 + \mathcal{F}^{(1)} = s_1 + \mathcal{F}^{(3,4)} = \{0 \}; \{s_1 + 1, s_1 + 2, s_1 + 3, s_1 + 4, s_1 + 5, s_1 + 6\}; 0 \}, \]

\[ s_2 + \mathcal{F}^{(2)} = s_2 + \mathcal{F}^{(1,5)} \]

\[ = \{s_2 + 4, s_2 + 5\}; \{s_2 + 5, s_2 + 6\}; \{s_2 + 1, s_2 + 2\}, \]

we carry out the operation 1°. Since the 1-st, 2-nd, 4-th components of \( \mathcal{F}^{(3,4)} \) are the empty sets, we obtain \( \{s_2 + 4, s_2 + 5\}; \{s_2 + 5, s_2 + 6\}; \{s_2 + 1, s_2 + 2\} \) from the 1-st, 2-nd, 4-th components, and at the 3-rd component, we superpose the linear forms as follows:

\[ \{s_1 + 1, s_1 + 2, s_1 + 3, s_1 + 4, s_1 + 5, s_1 + 6\} \quad \{s_2 + 5, s_2 + 6\} \quad \mapsto \quad s_1 + 1, s_1 + 2, s_1 + 3, s_1 + 4, s_1 + 5, s_1 + 6. \]

All the linear forms are aligned as

\[ s_1 + 1, s_1 + 2, s_1 + 3, s_1 + 4, s_2 + 1, s_2 + 2, s_2 + 4, (s_2 + 5)^\times 2, s_2 + 6, s_1 + s_2 + 5, s_1 + s_2 + 6 \]

and by multiplying these factors according to 3°, we obtain the \( b \)-function \( b_m(s) \). That is,

\[
\begin{align*}
\mathbf{b}_m(s) &= [s_1 + 1]_{m_1}[s_1 + 2]_{m_1}[s_1 + 3]_{m_1}[s_1 + 4]_{m_1} \\
&\times [s_2 + 1]_{m_2}[s_2 + 2]_{m_2}[s_2 + 4]_{m_2}[s_2 + 5]_{m_2}^2[s_2 + 6]_{m_2} \\
&\times [s_1 + s_2 + 5]_{m_1 + m_2}[s_1 + s_2 + 6]_{m_1 + m_2}.
\end{align*}
\]

(5.2)
The aspect of the superposition can be visualized as follows: First, as in Figure 5, we attach the linear forms in \( s \) (5.3) corresponding to \( O \) to the arrows in the lace diagram. Then we superpose these two diagrams. If two linear forms are attached to the same arrow, those two linear forms are also superposed as in Figure 6.

**Example 5.4.** In Example 4.6 we put \( f_1 := f^{(1,4)} \), \( f_2 := f^{(2,5)} \). For 

\[
s_1 + F^{(1,4)} = \{ \{ s_1 + 4, s_1 + 5 \} ; \{ s_1 + 5, s_1 + 6, s_1 + 7 \} ; \{ s_1 + 1, s_1 + 2, s_1 + 3, s_1 + 4 \} ; \emptyset \}, \\
s_2 + F^{(2,5)} = \{ \emptyset ; \{ s_2 + 3, s_2 + 4, s_2 + 5, s_2 + 6, s_2 + 7 \} ; \{ s_2 + 3, s_2 + 4 \} ; \{ s_2 + 1, s_2 + 2 \} \},
\]

we perform the operations \( 1^\circ, 2^\circ, 3^\circ \), and obtain

\[
b_{m_2}(2) = [s_1 + 1]_{m_1} [s_1 + 2]_{m_1} [s_1 + 4]_{m_1} [s_1 + 5]_{m_1} \\
\times [s_2 + 1]_{m_2} [s_2 + 2]_{m_2} [s_2 + 3]_{m_2} [s_2 + 4]_{m_2} \\
\times [s_1 + s_2 + 3]_{m_1+m_2} [s_1 + s_2 + 4]_{m_1+m_2} [s_1 + s_2 + 5]_{m_1+m_2} \\
\times [s_1 + s_2 + 6]_{m_1+m_2} [s_1 + s_2 + 7]_{m_1+m_2}. 
\]

(5.3)

Also in this case, the aspect of the superposition can be interpreted visually. First, as in Figure 7, we attach the linear forms in \( s_1 + F^{(1,4)} \) (resp. \( s_2 + F^{(2,5)} \)) to the arrows in the lace diagram corresponding to \( O^{(1,4)} \) (resp. \( O^{(2,5)} \)). Here the linear forms in each column are attached upside down according as the arrow is leftward or rightward. See Figure 4.
for the way of attaching the factors. As before, we superpose two diagrams and if two linear forms are attached to the same arrow, those two linear forms are also superposed as in Figure 8.

Figure 8: Superposition of the lace diagrams in Example 5.4

In the following sections, we investigate these two examples in more detail, while the validity of the algorithm will be proved in general.

6 Calculation of the $a$-function

In this section, we discuss the method to calculate the $a$-functions. In general, if we have the explicit form of the $a$-function of a prehomogeneous vector space, we can determine the $b$-function of the space to a certain extent, by using the structure theorems on $a$-functions and $b$-functions of several variables. We refer to M. Sato [18] for the details of the structure theorems. A convenient summary of [18] is given in Ukai [25, §§1.3].

We begin by recalling a general definition of $a$-functions. We keep the notation of Section 1. Let $(G, ρ, V)$ be a reductive prehomogeneous vector space, $f$ a relative invariant,
and $\Omega(f) = V \setminus f^{-1}(0)$. Then we define the map $\text{grad log } f : \Omega(f) \to V^*$ by

\begin{equation}
(6.1) \quad \text{grad log } f(v) = \sum_{i=1}^{n} \frac{1}{f(v)} \cdot \frac{\partial f}{\partial v_i}(v) \cdot e_i^* = \left( \frac{1}{f(v)} \frac{\partial f}{\partial v_1}(v), \ldots, \frac{1}{f(v)} \frac{\partial f}{\partial v_n}(v) \right).
\end{equation}

Let $f_1, \ldots, f_l$ be the fundamental relative invariants of $(G, \rho, V)$ and $f^*_1, \ldots, f^*_l$ the fundamental relative invariants of $(G, \rho^*, V^*)$ such that the characters of $f_i$ and $f^*_i$ are the inverse of each other. We define $f_*, f^*_1, \ldots, f^*_l$ in the same manner as in Section 1.

**Lemma 6.1.** For any $l$-tuple $m = (m_1, \ldots, m_l) \in \mathbb{Z}_{\geq 0}^l$, there exists a non-zero homogeneous polynomial $a_m(s)$ of $s_1, \ldots, s_l$ satisfying

$$f^m(v)f^{*m}(\text{grad log } f^s(v)) = a_m(s).$$

Here $a_m(s)$ is independent of $v$.

We call $a_m(s)$ the $a$-function of $f$. The calculation of $a_m(s)$ can be reduced to that of $\text{grad log } f^s(v)$, and we have

\begin{equation}
(6.2) \quad \text{grad log } f^s(v) = \sum_{i=1}^{l} s_i \cdot \text{grad log } f_i(v).
\end{equation}

Moreover, since $a_m(s)$ is independent of $v$, it is enough to calculate $\text{grad log } f_i(v_0)$ ($i = 1, \ldots, l$) for a suitable generic point $v_0$.

**Example 6.2.** Before proceeding to the general case, we consider the case where $Q$ is an equioriented quiver of type $A_r$ as below.

\[ Q : 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow r \]

We keep the notation of Example 3.2. At first, we give explicitly a generic point $v_0$ of $(GL(n), \text{Rep}(Q, n))$. For $m, n \in \mathbb{Z}_{>0}$, let

\[ E(m, n) = \begin{cases} (E_m | O_{m,n-m}) & \text{if } m < n \\ E_m & \text{if } m = n \\ t(E_n | O_{n,m-n}) & \text{if } m > n \end{cases}. \]

Further, for $h \in \mathbb{Z}$ with $0 \leq h \leq \min \{m, n\}$, let

\[ E(m, n; h) = \begin{pmatrix} E_h & O_{h,n-h} \\ O_{m-h,h} & O_{m-h,n-h} \end{pmatrix} \begin{cases} & \text{if } h < \min \{m, n\} \\ E(m, n) & \text{if } h = \min \{m, n\} \end{cases}. \]

Then a direct computation shows that

\[ A_0 = (E(n_2, n_1), E(n_3, n_2), \ldots, E(n_r, n_{r-1})) \in \text{Rep}(Q, n) \]
is a generic point of \((GL(n), \text{Rep}(Q, n))\). (see also Lemma 6.4 below.) Recall that we have assumed that \(n_1 = n_r < n_2, \ldots, n_{r-1}\), and then

\[
f(v) = \det(X_{r,1}) = \det(X_{r,r-1} \cdots X_{3,2}X_{2,1})
\]

is a relative invariant of \((GL(n), \text{Rep}(Q, n))\). We will show that

\[
\text{grad } \log f(A_0) = (E(n_2, n_1), E(n_3, n_2; n_1), \ldots, E(n_{r-1}, n_{r-2}; n_1), E(n_r, n_{r-1})).
\]

In general, we have \(E(m, n) \cdot E(n, l) = E(m, l; \min\{m, n, l\})\) and thus

\[
f(A_0) = \det(E_{n_1}) = 1.
\]

For \(1 \leq i \leq n_k\) and \(1 \leq j \leq n_{k-1}\), we denote by \(x^{(k,k-1)}_{ij}\) the \((i, j)\)-th component of \(X_{k,k-1}\). In view of the definition (6.1), it is enough to calculate

\[
\frac{\partial f}{\partial x^{(k,k-1)}_{ij}}(A_0) \quad (k = 2, \ldots, r; i = 1, \ldots, n_k; j = 1, \ldots, n_{k-1}).
\]

For \(1 \leq s, t \leq n_1\), we denote by \(y_{st}\) the \((s, t)\)-th component of \(X_{r,1} = X_{r,r-1} \cdots X_{2,1}\). Then it follows from the chain rule that

\[
\frac{\partial f}{\partial y_{st}}(A_0) = \sum_{1 \leq s, t \leq n_1} \frac{\partial f}{\partial y_{st}}(A_0) \cdot \frac{\partial y_{st}}{\partial x^{(k,k-1)}_{ij}}(A_0).
\]

For a square matrix \(R\), we denote by \(\Delta_{st}(R)\) the \((s, t)\)-cofactor of \(R\). By definition of the determinant, we have \(\partial f / \partial y_{st} = \Delta_{st}(X_{r,1})\) and thus

\[
\frac{\partial f}{\partial y_{st}}(A_0) = \Delta_{st}(X_{r,1})\big|_{v = A_0} = \Delta_{st}\left(X_{r,1}\big|_{v = A_0}\right) = \Delta_{st}(E_{n_1}) = \delta_{st}
\]

where \(\delta_{st}\) is the Kronecker-delta symbol. On the other hand,

\[
y_{st} = \sum_{\alpha = 1}^{n_k} \sum_{\beta = 1}^{n_{k-1}} z^{(r,k)}_{s\alpha} \cdot z^{(k,k-1)}_{\alpha\beta} \cdot z^{(k-1,1)}_{\beta t},
\]

where \(z^{(r,k)}_{s\alpha}\) is the \((s, \alpha)\)-component of \(X_{r,k} = X_{r,r-1} \cdots X_{k+1,k}\) and \(z^{(k-1,1)}_{\beta t}\) is the \((\beta, t)\)-component of \(X_{k-1,1} = X_{k-1,k-2} \cdots X_{2,1}\). Thus we have

\[
\frac{\partial y_{st}}{\partial x^{(k,k-1)}_{ij}} = z^{(r,k)}_{si} \cdot z^{(k-1,1)}_{jt}
\]

and hence

\[
\frac{\partial y_{st}}{\partial x^{(k,k-1)}_{ij}}(A_0) = z^{(r,k)}_{si}\big|_{v = A_0} \cdot z^{(k-1,1)}_{jt}\big|_{v = A_0}.
\]
Since $X_{rk}|_{v=A_0} = E(n_1, n_k)$ and $X_{k-1,1}|_{v=A_0} = E(n_{k-1}, n_1)$, we see that
\[
z_{si}^{(r,k)}|_{v=A_0} = \begin{cases} 
\delta_{si} & (1 \leq i \leq n_1) \\
0 & (n_1 + 1 \leq i \leq n_k)
\end{cases}, \quad z_{jt}^{(k-1,1)}|_{v=A_0} = \begin{cases} 
\delta_{jt} & (1 \leq j \leq n_1) \\
0 & (n_1 + 1 \leq i \leq n_{k-1})
\end{cases}.
\]

As a result, we have
\[
\frac{\partial f}{\partial x_{ij}^{(k,k-1)}}(A_0) = \sum_{1 \leq s, t \leq n_1} \delta_{st} \cdot \frac{\partial y_{st}}{\partial x_{ij}^{(k,k-1)}}(A_0) = \sum_{s=1}^{n_1} \frac{\partial y_{ss}}{\partial x_{ij}^{(k,k-1)}}(A_0)
= \left\{ \sum_{s=1}^{n_1} \delta_{si} \cdot \delta_{js} = \delta_{ij} \quad (1 \leq i, j \leq n_1) \\
0 \quad \text{(otherwise)} \right\}.
\]

Together with (6.4), this proves (6.3).

In the case of arbitrary orientation, we use lace diagrams to give explicit descriptions of generic points of $(\text{GL}(n), \text{Rep}(Q, n))$. For a given orientation $Q$ and a given dimension vector $n$, we consider the lace diagram obtained by simply drawing all possible horizontal lines between dots of consecutive columns. We call it the complete lace diagram for $(\text{GL}(n), \text{Rep}(Q, n))$.

**Example 6.3.** Let us consider $(\text{GL}(n), \text{Rep}(Q, n))$ discussed in Examples 4.5 and 4.6. Then the complete lace diagrams are given as in Figure 9.

![Figure 9: Complete lace diagrams](image)

Then we have the following lemma.

**Lemma 6.4** (Abasis [1, Proposition 3.1]). The element $A_0$ of $\text{Rep}(Q, n)$ represented by the complete lace diagram is a generic point of $(\text{GL}(n), \text{Rep}(Q, n))$.

The matrix representation of $A_0$ depends on the choice of the basis of $\text{Rep}(Q, n)$. However, the lemma below says that without fixing the basis, it is possible to calculate the value of grad log $f(A_0)$ from the complete lace diagram.
Lemma 6.5. Let $A_0$ be a generic point of $(GL(n), \text{Rep}(Q,n))$ given by the complete lace diagram, and $f_{(p,q)}(v)$ the irreducible relative invariant given as in Section 2. Then the value of $\text{grad log} f_{(p,q)}(A_0)$ is given by the exact lace diagram corresponding to the locally closed orbits $O^{(p,q)}$.

Remark 6.6. Do not interpret the lemma above as a result on the indecomposable decompositions of $\text{grad log} f_{(p,q)}(A_0)$. If we give a matrix representation of $A_0$, then it automatically determines the matrix representation of $\text{grad log} f_{(p,q)}(A_0)$. This enables us to translate the addition in (6.2) as the superposition operation on lace diagrams. We also note that the fact $\text{grad log} f_{(p,q)}(A_0) \in O^{(p,q)}$ follows from a previously known result of Gyoja [9, Theorem 1.18].

Proof of Lemma 6.5. Let

$$Y_{(p,q)}(A_0) : \bigoplus_{\tau} L_{\tau} \longrightarrow \bigoplus_{\sigma} L_{\sigma} \quad \left( \begin{array}{c} \tau \text{ runs over all the sources of } Q^{(p,q)} \\ \sigma \text{ runs over all the sinks of } Q^{(p,q)} \end{array} \right)$$

be the linear map defined in Section 4. Since $f_{(p,q)}(A_0) = \det Y_{(p,q)}(A_0) \neq 0$, $Y_{(p,q)}(A_0)$ is an isomorphism. Let $N := \sum_\tau \dim L_\tau = \sum_\sigma \dim L_\sigma$. Now we choose basis $\{u_1, \ldots, u_N\}$ (resp. $\{u'_1, \ldots, u'_N\}$) of $\bigoplus_\tau L_\tau$ (resp. $\bigoplus_\sigma L_\sigma$). Each $u_i$ ($i = 1, \ldots, N$) (resp. $u'_i$ ($i = 1, \ldots, N$)) can be identified with a dot at some source (resp. sink) of the complete lace diagram associated with $Q^{(p,q)}$. By choosing a suitable order of the basis, we may assume that for $i = 1, \ldots, N$, there exists a path from $u_i$ to $u'_i$ in the complete lace diagram. Via this basis, we identify $\bigoplus_\tau L_\tau$ and $\bigoplus_\sigma L_\sigma$ with $\mathbb{C}^N$. Note that this identification depends on $(p,q)$. Our choice of the basis ensures that $Y_{(p,q)}(A_0)$ is of the form

$$Y_{(p,q)}(A_0) = E_N + \sum_{(s,t) \in \mathcal{B}} E_{st},$$

where $E_N$ is the identity matrix of size $N$, $E_{st}$ is the $(s,t)$-matrix unit of size $N$, and $\mathcal{B}$ is an index set. Moreover, by looking at the shape of the complete lace diagram, we see that $\mathcal{B}$ has the following property:

$$\text{any two element } (s,t), (s',t') \in \mathcal{B} \text{ satisfy } s \neq t' \text{ and } t \neq s'. \quad (6.6)$$

Example 6.7. Let us explain the reason for (6.6) by examples. In Example 4.5, $N = 6$ for $(p,q) = (3,4)$ and $N = 2$ for $(p,q) = (1,5)$. If we choose the basis $\{u_1, \ldots, u_N\}$ and $\{u'_1, \ldots, u'_N\}$ as Figure 10 then we have

$$Y_{(3,4)}(A_0) = E_6, \quad Y_{(1,5)}(A_0) = E_2.$$
The property (6.6) yields the following three formulas.

(6.7) \[ Y_{(p,q)}(A_0) = \prod_{(s,t) \in B} (E_N + E_{st}), \]

(6.8) \[ f_{(p,q)}(A_0) = 1, \]

(6.9) \[ t\left(Y_{(p,q)}(A_0)\right)^{-1} = E_N - \sum_{(t,s) \in B} E_{st}. \]

By using these formulas, we generalize the calculation in Example 6.2 to the case of arbitrary orientation. For \( a \in Q_1 = \{1, \ldots, r - 1\} \), let \( e_i^{(h(a))} \) \((i = 1, \ldots, n_{h(a)})\) be the basis of \( L_{h(a)} \) and \( e_j^{(t(a))} \) \((j = 1, \ldots, n_{t(a)})\) the basis of \( L_{t(a)} \). We take the coordinate system with respect to these basis and denote by \( x_{ij}^{(a)} \) the \((i, j)\)-th component of \( X_{h(a),t(a)}^{(a)} \).
Moreover, for \(1 \leq s, t \leq N\), we denote by \(y_{st}\) the \((s, t)\)-th component of \(Y_{(p,q)}(v)\). Then the formula (6.5) can be generalized as follows.

\[
\frac{\partial f_{(p,q)}(A_0)}{\partial x_{ij}} = \sum_{1 \leq s, t \leq N} \frac{\partial f_{(p,q)}(A_0)}{\partial y_{st}} \cdot \frac{\partial y_{st}}{\partial x_{ij}}(A_0).
\]

For a matrix \(R\), let \(\Delta(R)\) be the cofactor matrix. Then (6.8) and (6.9) imply that

\[
\begin{align*}
\frac{\partial f_{(p,q)}(A_0)}{\partial y_{st}}(A_0) & = E_N - \sum_{(t,s) \in \mathcal{B}} E_{st}, \\
\end{align*}
\]

and thus we see that

\[
\frac{\partial f_{(p,q)}(A_0)}{\partial y_{st}}(A_0) = \begin{cases} 
1 & s = t \\
-1 & (t, s) \in \mathcal{B} \\
0 & \text{otherwise}
\end{cases}.
\]

However, if \((t, s) \in \mathcal{B}\), then \(y_{st} = 0\); the reader is referred to (6.6). Hence we may assume that \(s = t\), and it is enough to calculate \(\frac{\partial y_{st}}{\partial x_{ij}}(A_0)\). By generalizing the calculation of \(\frac{\partial y_{st}}{\partial x_{ij}}(A_0)\) in Example 6.2 we see that

\[
\frac{\partial y_{st}}{\partial x_{ij}}(A_0) = \begin{cases} 
1 & \text{In the complete lace diagram, there exists an arrow } e_j^{(t(a))} \rightarrow e_i^{(h(a))} \\
0 & \text{(otherwise)}
\end{cases}.
\]

We therefore conclude that the diagram representing the value of \(\text{grad log } f_{(p,q)}(A_0)\) consists only of the paths connecting \(u_s\) and \(u'_s\) \((s = 1, \ldots, N)\). This diagram is nothing but the exact lace diagram corresponding to \(O^{(p,q)}\). This completes the proof of the lemma.

As we have mentioned in Remark 6.6 the summation

\[
\text{grad log } f_{(p,q)}(A_0) = \sum_{i=1}^{t} s_i \cdot \text{grad log } f_i(A_0)
\]

can be transferred to the superposition of the lace diagrams, and this ensures the validity of the “superposition method” for the \(a\)-functions.

Example 6.8. Let us consider the case of Example 4.5. By Lemma 6.5 \(\text{grad log } f_{(p,q)}(A_0) = s_1 \text{ grad log } f_1(A_0) + s_2 \text{ grad log } f_2(A_0)\) is given as in Figure 12. If we fix a basis of \(\text{Rep}(Q, \mathbb{N})\), then the arrows \(-s_1, -s_2, s_1 + s_2\) correspond to \(s_1 E_{ij}, s_2 E_{i'j'}, (s_1 + s_2) E_{i''j''}\) with some
Figure 12: Diagram expressing the value of \( \text{grad} \log f(A_0) \) for Example 4.5

matrix units \( E_{i,j}, E'_{i,j'}, E''_{i,j''} \). Then, by the definition of \( f_1 \) (recall that \( f_1 \) is constructed as \( f_{(3,4)} \) in Example 2.2), we have

\[
a_{(1,0)}(s) = f_1(A_0)f_1(\text{grad} \log f(A_0)) = 1 \cdot s_1^4(s_1 + s_2)^2 = s_4^1(s_1 + s_2)^2.
\]

Similarly, we have

\[
a_{(0,1)}(s) = f_2(A_0)f_2(\text{grad} \log f(A_0)) = 1 \cdot s_2^6(s_1 + s_2)^2 = s_6^2(s_1 + s_2)^2,
\]

for \( f_2 \) is constructed as \( f_{(1,5)} \) in Example 2.2. Hence we have

\[
a_m(s) = a_{(1,0)}(s)^{m_1}a_{(0,1)}(s)^{m_2} = s_4^1s_6^2(s_1 + s_2)^2(m_1 + m_2).
\]

In general, the \( a \)-functions determine the \( b \)-functions except the “constant terms”. In this case, we have

\[
b_m(s) = \prod_{r=1}^{4}(s_1 + \alpha_{1,r})^{m_1} \times \prod_{r=1}^{6}(s_2 + \alpha_{2,r})^{m_2} \times \prod_{r=1}^{2}(s_1 + s_2 + \alpha_{3,r})^{m_1 + m_2}
\]

with some \( \alpha_{j,r} \in \mathbb{Q}_{>0} \). Here we have employed the structure theorem on \( b \)-functions (see M. Sato [18, Theorem 2]) and Kashiwara’s theorem [11, Theorem 6.9]; see also Ukai [25, Theorem 1.3.5].

Similarly, in the case of Example 4.6 we see that

\[
a_m(s) = s_4^{4m_1}s_2^{4m_2}(s_1 + s_2)^{5(m_1 + m_2)},
\]

and

\[
b_m(s) = \prod_{r=1}^{4}(s_1 + \alpha_{1,r})^{m_1} \times \prod_{r=1}^{4}(s_2 + \alpha_{2,r})^{m_2} \times \prod_{r=1}^{5}(s_1 + s_2 + \alpha_{3,r})^{m_1 + m_2}
\]

with some \( \alpha_{j,r} \in \mathbb{Q}_{>0} \). In any case, the calculation of \( b_m(s) \) is reduced to the determination of \( \alpha_{j,r} \).
7 Proof of Theorem 5.2

The determination of \( \alpha_{j,r} \) can be carried out by localization of \( b \)-functions (cf. Ukai [25, p. 57], [24]). We briefly recall the localization of \( b \)-functions, to the extent necessary for the calculation in the present paper. (the idea of [25] can be applied to a more general setting.) Let \((G, V)\) be a reductive prehomogeneous vector space and \( f \in \mathbb{C}[V] \) a relatively invariant polynomial. Take an arbitrary point \( v_1 \in V \). Let \( \mathcal{O}_1 = G \cdot v_1 \) be the \( G \)-orbit through \( v_1 \), and \( G_{v_1} \) be the isotropy subgroup of \( G \) at \( v_1 \). Then \( G_{v_1} \) acts on the tangent space \( T_{v_1}(\mathcal{O}_1) = \mathfrak{g} \cdot v_1 \). Now we assume that \( G_{v_1} \) is reductive. Then there exists a \( G_{v_1} \)-invariant subspace \( W \) of \( V \) such that \( V = T_{v_1}(\mathcal{O}_1) \oplus W \). We call \((G_{v_1}, W)\) the slice representation at \( v_1 \) (cf. Kac [10], Shmelkin [21]). It is easy to see that \((G_{v_1}^{o}, W)\) is a reductive prehomogeneous vector space, where \( G_{v_1}^{o} \) is the connected component of \( G_{v_1} \).

Moreover, the function \( f_{v_1} \) on \( W \) defined by \( f_{v_1}(w) = f(v_1 + w) \) \((w \in W)\) is a relatively invariant polynomial of \((G_{v_1}^{o}, W)\). We denote by \( b_{f,\mathcal{O}_1}(s) \) the \( b \)-function of \( f_{v_1} \); in effect, if \( v'_1 \in \mathcal{O}_1 \), then the \( b \)-function of \( f_{v'_1} \) coincides with that of \( f_{v_1} \). Then we have the following lemma.

Lemma 7.1. \( b_{f,\mathcal{O}_1}(s) \) divides \( b_f(s) \), the \( b \)-function of \( f \).

To apply Lemma 7.1 to our prehomogeneous vector spaces \((GL(n), \text{Rep}(Q, \mu))\), we recall some facts from representation theory of quivers. A useful reference for this section is a survey by Brion [3]. Now, for a moment, we assume that \( Q \) is an arbitrary quiver which has no oriented cycles. For two dimension vectors \( \mu = (n_i)_{i \in Q_0}, m = (m_i)_{i \in Q_0} \), we consider quiver representation spaces

\[
\text{Rep}(Q, \mu) = \bigoplus_{a \in Q_1} \text{Hom}(L_{t(a)}, L_{h(a)}), \quad \text{Rep}(Q, m) = \bigoplus_{a \in Q_1} \text{Hom}(L'_{t(a)}, L'_{h(a)}),
\]

where \( L_i \) (resp. \( L'_i \)) is a vector space of dimension \( n_i \) (resp. \( m_i \)). Take \( A = (A_{h(a), t(a)})_{a \in Q_1} \in \text{Rep}(Q, \mu) \) and \( B = (B_{h(a), t(a)})_{a \in Q_1} \in \text{Rep}(Q, m) \). Then we define a map

\[
(7.1) \quad d_{A,B} : \bigoplus_{i \in Q_0} \text{Hom}(L_i, L'_i) \rightarrow \bigoplus_{a \in Q_1} \text{Hom}(L_{t(a)}, L'_{h(a)})
\]

by

\[
d_{A,B}(\phi) = (\phi_{h(a)}A_{h(a), t(a)} - B_{h(a), t(a)}\phi_{t(a)})_{a \in Q_1}
\]

for \( \phi = (\phi_i)_{i \in Q_0} \in \bigoplus_{i \in Q_0} \text{Hom}(L_i, L'_i) \). It is known (see Ringel [15], or Brion [3, Corollary 1.4.2]) that \( \text{Hom}_Q(A, B) \) and \( \text{Ext}_Q(A, B) \), which are defined through the Ringel resolution, are equal to \( \text{Ker} d_{A,B} \) and \( \text{Coker} d_{A,B} \), respectively. That is,

\[
\text{Hom}_Q(A, B) = \text{Ker} d_{A,B}, \quad \text{Ext}_Q(A, B) = \text{Coker} d_{A,B}.
\]
Then we have an exact sequence

\[ 0 \rightarrow \text{Hom}_Q(A, B) \rightarrow \bigoplus_{i \in Q_0} \text{Hom}_Q(L_i, L'_i) \xrightarrow{d_{A,B}} \bigoplus_{a \in Q_1} \text{Hom}(L_{t(a)}, L'_{h(a)}) \xrightarrow{\delta} \text{Ext}_Q(A, B) \rightarrow 0. \]

Taking dimensions in the exact sequence above yields

\[ \dim \text{Hom}_Q(A, B) - \dim \text{Ext}_Q(A, B) = \sum_{i \in Q_0} n_i m_i - \sum_{a \in Q_1} n_{t(a)} m_{h(a)}. \]

The Euler form of a quiver $Q$ is the bilinear form $\langle \cdot, \cdot \rangle_Q$ on $\mathbb{R}^{Q_0}$ defined by

\[ \langle n, m \rangle_Q = \sum_{i \in Q_0} n_i m_i - \sum_{a \in Q_1} n_{t(a)} m_{h(a)} \]

for any $n = (n_i)_{i \in Q_0}$ and $m = (m_i)_{i \in Q_0}$. When $A$ (resp. $B$) is an element of $\text{Rep}(Q, n)$ (resp. $\text{Rep}(Q, m)$), we often write $\langle A, B \rangle_Q$ for $\langle n, m \rangle_Q$. We thus obtain Ringel’s formula (cf. [15])

\[ \langle A, B \rangle_Q = \dim \text{Hom}_Q(A, B) - \dim \text{Ext}_Q(A, B), \]

which will be used later. Next let $n = m$ and $A = B$. We identify $\bigoplus_{i \in Q_0} \text{Hom}(L_i, L_i)$ with the Lie algebra $\mathfrak{gl}(n) = \bigoplus_{i \in Q_0} \mathfrak{gl}(n_i)$ of $GL(n)$. Then the map $d_{A,A}$ can be identified with the differential map at the identity of the orbit map

\[ GL(n) \ni g \mapsto g \cdot A \in \text{Rep}(Q, n). \]

By definition, $\text{Hom}_Q(A, A)$ is isomorphic to the isotropy subalgebra $\mathfrak{gl}(n)_A$ of $\mathfrak{gl}(n)$ at $A$, and $\text{Ext}_Q(A, A)$ is isomorphic to the normal space in $\text{Rep}(Q, n)$ to the orbit $GL(n) \cdot A$ at $A$. More precisely, we have

\[ \text{Ext}_Q(A, A) \cong \text{Rep}(Q, n)/T_A (GL(n) \cdot A). \]

These notions in representation theory of quivers are linked to the slice representations in the following way: Let $O_1$ be the closed orbit in $\{ A \in \text{Rep}(Q, n) : f(A) \neq 0 \}$, where $f$ is a relative invariant of $(GL(n), \text{Rep}(Q, n))$. Take an element $A_1$ of $O_1$; this is a locally semi-simple representation of $Q$ in the sense of Shmelkin [21]. Then, Matsushima’s theorem implies that the isotropy subgroup $GL(n)_{A_1}$ is reductive, and thus there exists a $GL(n)_{A_1}$-invariant subspace $W_1$ satisfying

\[ \text{Rep}(Q, n) = T_{A_1}(O_1) \oplus W_1. \]

Let $\mathfrak{gl}(n)_{A_1}$ be the isotropy subalgebra of $\mathfrak{gl}(n)$ at $A_1$. The argument above implies that

\[ \mathfrak{gl}(n)_{A_1} \cong \text{Hom}_Q(A_1, A_1), \]

\[ W_1 \cong \text{Ext}_Q(A_1, A_1). \]
Shmelkin [21] showed that the structure of $(GL(n)_{A_1}, W_1)$ can be described by using the local quiver. Let $A_1 = \bigoplus_{i=1}^t m_i(1) I_i(1)$ be the indecomposable decomposition of $A_1$, where $I_i(1)$ is an indecomposable representation of $Q$ and $m_i(1)$ is the multiplicity of $I_i(1)$ in $A_1$. By Shmelkin [21, Proposition 8], we have

\begin{equation}
\dim \text{Hom}_Q(I_i(1), I_j(1)) = \delta_{ij},
\end{equation}

and by Ringel’s formula (7.3), we have

$$\delta_{ij} - (I_i(1), I_j(1)) = \dim \text{Ext}_Q(I_i(1), I_j(1)) \geq 0.$$ 

The local quiver $\Sigma$ is a quiver with vertices $a_1, \ldots, a_t$ corresponding to the summands $I_1(1), \ldots, I_t(1)$, and $\delta_{ij} - (I_i(1), I_j(1))$ arrows from $a_i$ to $a_j$. We set $\gamma = (m_1(1), \ldots, m_t(1))$. Then we have the following formula for the slice representation $(GL(n)_{A_1}, W_1)$.

**Lemma 7.2** (Shmelkin [21] formula (9)).

$$(GL(n)_{A_1}, W_1) \cong (GL(\gamma), \text{Rep}(\Sigma, \gamma))$$

$$\cong \left( \prod_{i=1}^t GL(m_i(1)), \bigoplus_{1 \leq i,j \leq t} \text{Ext}_Q(I_i(1), I_j(1)) \otimes M(m_i(1), m_j(1)) \right).$$

Next we recall Schofield’s determinantal invariants, since we need to consider the restrictions of relative invariants to slice spaces. Take $A \in \text{Rep}(Q, n)$ and $B \in \text{Rep}(Q, m)$ such that $\langle A, B \rangle_Q = 0$. Then the matrix representing the map $d_{A,B}$ in (7.1) is a square matrix, and

$$c(A, B) := \det d_{A,B}$$

is a relative invariant of $(GL(n) \times GL(m), \text{Rep}(Q, n) \oplus \text{Rep}(Q, m))$. It is easy to see that $c(A, B) \neq 0$ if and only if $\text{Hom}_Q(A, B) = 0$, which is equivalent to $\text{Ext}_Q(A, B) = 0$. For a fixed $B$, the restriction of $c$ to $\text{Rep}(Q, n) \oplus \{B\}$ gives a relative invariant $c_B$ of $(GL(n), \text{Rep}(Q, n))$. Similarly, for a fixed $A$, the restriction of $c$ to $\{A\} \oplus \text{Rep}(Q, m)$ gives a relative invariant $c^A$ of $(GL(m), \text{Rep}(Q, m))$. Moreover, we define the right perpendicular category $A^\perp$ of $A$ to be the full subcategory of representations $B$ such that $\text{Hom}_Q(A, B) = 0 = \text{Ext}_Q(A, B)$. Similarly, we define the left perpendicular category $^\perp A$ as the full subcategory of representations $C$ such that $\text{Hom}_Q(C, A) = 0 = \text{Ext}_Q(C, A)$. Then we have the following lemma due to Schofield [19].

**Lemma 7.3** ([19]). Let $Q$ be a quiver with $r$ vertices, without oriented cycles. Assume that $(GL(n), \text{Rep}(Q, n))$ is a prehomogeneous vector space and take a generic point $A_0 = \bigoplus_{i=1}^p m_i I_i$. Here $I_i$ is an indecomposable representation of $Q$ and $m_i$ is the multiplicity of $I_i$ in $A_0$. Then $A_0^\perp$ and $^\perp A_0$ are equivalent to categories of representations of $Q^\perp$ and $^\perp Q$, respectively, where $Q^\perp$ and $^\perp Q$ are quivers with $l := r - p$ vertices, without oriented cycles.
The above lemma implies that both $A_\perp^0$ and $\perp A_0$ contain exactly $l = r - p$ simple objects. Let $T_1, \ldots, T_l$ be the simple objects in $A_0^\perp$.

**Lemma 7.4** ([19]). Keep the notation as above. Then $c_{T_1}, \ldots, c_{T_l}$ are the fundamental relative invariants of $(GL(\mathfrak{m}), \text{Rep}(Q, \mathfrak{m}))$.

**Remark 7.5.** Recall that in the case of quivers of type $A$, the fundamental relative invariants are parametrized by the set $I_n(Q)$ defined in Section 2. The enumeration of $I_n(Q)$ is, of course, equivalent to the calculation of the simple objects $T_1, \ldots, T_l$. In [22], Shmelkin gives an algorithm for the calculation of those simple objects, and implemented the algorithm on a computer program TETIVA [23]. By using TETIVA, one can also calculate the indecomposable decomposition $A_0 = \bigoplus_{i=1}^l m_i I_i$ of a generic point of $A_0$, and so on.

**Lemma 7.6.** Let $Q$ be a quiver without oriented cycles, $(GL(\mathfrak{m}), \text{Rep}(Q, \mathfrak{m}))$ a triplet arising from quiver representations which is not necessarily a prehomogeneous vector space, and $f$ a relatively invariant polynomial of $(GL(\mathfrak{m}), \text{Rep}(Q, \mathfrak{m}))$. Take an element $A_1$ of $\text{Rep}(Q, \mathfrak{m})$ such that $GL(\mathfrak{m}) A_1$ is closed in $\{ A \in \text{Rep}(Q, \mathfrak{m}) : f(A) \neq 0 \}$ and put $W_1 = \text{Ext}_Q(A_1, A_1)$. Then the function $f_{A_1}$ on $W_1$ defined by

$$f_{A_1}(w) := f(A_1 + w) \quad (w \in W_1)$$

is a constant function.

**Proof.** By Derksen-Weyman [5, Theorem 1], we have $f = c_B$ for some $B \in \text{Rep}(Q, \mathfrak{m})$. Then, by Shmelkin [21, Theorem 11], $A_1$ is a direct sum of simple objects in $\perp B$. Since the perpendicular categories are closed under direct sums and extensions, all the elements in $A_1 + W_1 = A_1 + \text{Ext}_Q(A_1, A_1)$ belongs to $\perp B$. By the definition of $c_B$ and $\perp B$, we observe that $c_B$ does not vanish on $A_1 + W_1$, and this proves the lemma.

Now let us return to our case; we assume that $Q$ is a quiver of type $A_r$. We keep the notation in Sections 2-4. Let $O(p,q)$ be the closed orbit in $\{ A \in \text{Rep}(Q, \mathfrak{m}) : f_{(p,q)}(A) \neq 0 \}$, and take an element $A^{(p,q)}$ of $O^{(p,q)}$. We denote by $GL(\mathfrak{m})_{A^{(p,q)}}$ the isotropy subgroup of $GL(\mathfrak{m})$ at $A^{(p,q)}$, and by $W^{(p,q)}$ a $GL(\mathfrak{m})_{A^{(p,q)}}$-invariant subspace of $\text{Rep}(Q, \mathfrak{m})$ satisfying

$$\text{Rep}(Q, \mathfrak{m}) = T_{A^{(p,q)}}(O^{(p,q)}) \oplus W^{(p,q)}. \quad (7.5)$$

Let

$$A^{(p,q)} \cong \bigoplus_{1 \leq i \leq j \leq r} m_{ij}^{(p,q)} I_{ij}$$

be the indecomposable decomposition of $A^{(p,q)}$, and set

$$C^{(p,q)} = \left\{ [i,j] : 1 \leq i \leq j \leq r, m_{ij}^{(p,q)} \neq 0 \right\}.$$
Lemma 7.7. For \([i, j], [k, l] \in C^{(p,q)}\), we have

\[
\text{Ext}_Q(I_{ij}, I_{kl}) \cong \begin{cases} 
\mathbb{C} & \text{if there exists } a \in Q_1 \text{ such that } t(a) = j, h(a) = k \\
\mathbb{C} & \text{if there exists } a \in Q_1 \text{ such that } t(a) = i, h(a) = l \\
0 & \text{otherwise}
\end{cases} .
\]

Proof. By considering the shape of exact lace diagrams and the definition \((7.2)\), we observe that for \([i, j], [k, l] \in C^{(p,q)}\),

\[
(I_{ij}, I_{kl})_Q = \begin{cases} 
1 & \text{if } [i, j] = [k, l] \\
-1 & \text{if there exists } a \in Q_1 \text{ such that } t(a) = j, h(a) = k \\
-1 & \text{if there exists } a \in Q_1 \text{ such that } t(a) = i, h(a) = l \\
0 & \text{otherwise}
\end{cases} .
\]

Combining this with \((7.3)\) and \((7.4)\), we obtain the lemma. We also note that the lemma can be proved by using \([4\text{, Lemma 3}]\).

By Lemmas \((7.2)\) and \((7.7)\) we obtain the following lemma.

Lemma 7.8. Let \(O^{(p,q)}\) be the closed orbit in \(\{A \in \text{Rep}(Q, \mathfrak{g}); f_{(p,q)}(A) \neq 0\}\). Take \(A^{(p,q)} \in O^{(p,q)}\) and let \(A^{(p,q)} \cong \oplus_{i,j} m^{(p,q)}_{ij} I_{ij}\) be the indecomposable decomposition of \(A^{(p,q)}\). Then the isotropy subgroup \(GL(n)_{A^{(p,q)}}\) and the subspace \(W^{(p,q)}\) satisfying \((7.3)\) are given as follows:

\[
GL(n)_{A^{(p,q)}} \cong \prod_{1 \leq r \leq j \leq r} GL(m^{(p,q)}_{ij}),
\]

\[
W^{(p,q)} \cong \bigoplus_{\exists a \in Q_1 \text{ s.t. } t(a) = j, h(a) = k} M(m^{(p,q)}_{ij}, m^{(p,q)}_{kl}) \oplus \bigoplus_{\exists a \in Q_1 \text{ s.t. } t(a) = i, h(a) = l} M(m^{(p,q)}_{kl}, m^{(p,q)}_{ij}).
\]

Example 7.9. Lemma \((7.8)\) tells us that the structures of slice representations emerge from the exact lace diagrams. First let us consider the locally closed orbit \(O^{(3,4)}\) in Example \((1.5)\).

Then, in view of Lemma \((7.8)\), we draw pictures like Figure \((1.3)\) and observe that

\[
GL(n)_{A^{(3,4)}} \cong GL(2) \times GL(5) \times GL(6) \times GL(2),
\]

\[
W^{(3,4)} \cong M(5, 2) \oplus M(6, 5) \oplus M(2, 6).
\]

Here the lines (not arrows!) in the right picture denote the basis of \(W^{(3,4)}\). The action of \(GL(n)_{A^{(3,4)}}\) on \(W^{(3,4)}\) is inherited from that of \((GL(n), \text{Rep}(Q, \mathfrak{g}))\), and is given by

\[
h \cdot w = \left( h_2 w_1^{(3,4)} h_1^{-1}, h_3 w_2^{(3,4)} h_2^{-1}, h_4 w_3^{(3,4)} h_3^{-1} \right)
\]

for \(h = (h_1, h_2, h_3, h_4) \in GL(n)_{A^{(3,4)}}\) and \(w = (w_1^{(3,4)}, w_2^{(3,4)}, w_3^{(3,4)}) \in W^{(3,4)}\).
Second we consider the locally closed orbit $O_{(1,5)}$ in Example 4.5. Then, we draw pictures like Figure 14 and observe that

$$GL(n)_{A(1,5)} \cong GL(2) \times GL(3) \times GL(4) \times GL(4),$$

$$W^{(1,5)} \cong M(4, 3) \oplus M(4, 4).$$

The action of $GL(n)_{A(1,5)}$ on $W^{(1,5)}$ is given by

$$h \cdot w = \left( h_3 w_1^{(1,5)} h_2^{-1}, h_4 w_2^{(1,5)} h_3^{-1} \right)$$

for $h = (h_1, h_2, h_3, h_4) \in GL(n)_{A(1,5)}$ and $w = (w_1^{(1,5)}, w_2^{(1,5)}) \in W^{(1,5)}$.

Finally, we consider the locally closed orbit $O_{(1,4)}$ in Example 4.6. Then, we draw pictures like Figure 15 and observe that

$$GL(n)_{A(1,4)} \cong GL(2) \times GL(3) \times GL(4) \times GL(2),$$

$$W^{(1,4)} \cong M(2, 4) \oplus M(4, 2).$$

The action of $GL(n)_{A(1,4)}$ on $W^{(1,4)}$ is given by

$$h \cdot w = \left( h_1 w_1^{(1,4)} h_3^{-1}, h_3 w_2^{(1,4)} h_4^{-1} \right)$$

for $h = (h_1, h_2, h_3, h_4) \in GL(n)_{A(1,4)}$ and $w = (w_1^{(1,4)}, w_2^{(1,4)}) \in W^{(1,4)}$.  

36
Now we complete the calculation of (5.2) and (5.3), which is a prototype of the proof of Theorem 5.2.

Example 7.10 (proof of (5.2)). We keep the notation of Example 7.9. First we consider the slice representation \((GL(n), W(1,4))\). By Lemma 7.6, the restriction of \(f_1 = f_{(3,4)}\) to \(W(3,4)\) is constant. On the other hand, by the construction of \(f_2 = f_{(1,5)}\), the restriction \(f'_2\) of \(f_2\) to \(W(3,4)\) is given by

\[
f'_2(w) = \det \begin{pmatrix} w_3^{(3,4)} & w_2^{(3,4)} & w_1^{(3,4)} \end{pmatrix}
\]

for \(w = (w_1^{(3,4)}, w_2^{(3,4)}, w_3^{(3,4)}) \in W(3,4)\). If we employ Notation 3.1 it follows that

\[
f'_2(w) = \det \begin{pmatrix} 2 \rightarrow 6 \rightarrow 5 \rightarrow 2 \end{pmatrix}.
\]

By Theorem 3.4 we have

\[
b \left(\begin{pmatrix} 2 \rightarrow 6 \rightarrow 5 \rightarrow 2 \end{pmatrix}\right) = (s + 1)(s + 2)(s + 4)(s + 5)^2(s + 6).
\]

Then we apply Lemma 7.1 to \(f^m = f_1^{m_1} f_2^{m_2}\) and \(O^{(3,4)}\). Since \(b_{f^m}(s) = b_m(m_2)\), we see that \(b_m(s)\) is divisible by

\[
[s_2 + 1]_{m_2} [s_2 + 2]_{m_2} [s_2 + 4]_{m_2} [s_2 + 5]_{m_2}^2 [s_2 + 6]_{m_2}.
\]

This calculation can be interpreted graphically in the following way. Let us compare two exact lace diagrams in Figure 5 and pick up the arrows of the diagram of \(O^{(1,5)}\) (right) which do not overlap with the arrows of the diagram of \(O^{(3,4)}\) (left). Transfer those arrows to the diagram representing the slice representation associated with \(O^{(3,4)}\) (see Figure 13). Then we observe that these arrows form a smaller exact lace diagram as shown in Figure 16. The local b-function (7.6) corresponds to this smaller exact lace diagram.
Second we consider the slice representation \((GL(\mathfrak{m})_{A^{(1,5)}}, W^{(1,5)})\). The restriction of \(f_2 = f^{(1,5)}\) to \(W^{(1,5)}\) is constant, and the restriction \(f'_1\) of \(f_1 = f^{(3,4)}\) to \(W^{(1,5)}\) is given by

\[
f'_1(w) = \det \left( w^{(1,5)}_2 \right)
\]

for \(w = (w_1^{(1,5)}, w_2^{(1,5)}) \in W^{(1,5)}\). It follows that \(f'_1(w) = \det \left( \varnothing \to \varnothing \right)\). Since \(b \left( \varnothing \to \varnothing \right) = (s + 1)(s + 2)(s + 3)(s + 4)\), we observe that \(b_m(\varnothing)\) is divisible by

\[
[s_1 + 1, m_1] [s_1 + 2, m_1] [s_1 + 3, m_1] [s_1 + 4, m_1].
\]

Finally, let \(m = (1, 0)\) and \(\varnothing = (s, 0)\) in \((5.10)\). Then it follows from \(b_{(1,0)}(s, 0) = f_1(s) = (s + 1) \cdots (s + 6)\) that \(\{\alpha_{3,1}, \alpha_{3,2}\} = \{5, 6\}\). Now all of \(\alpha_{j,r}\) in \((6.10)\) have been determined and this completes the proof of \((5.2)\), i.e.,

\[
b_m(\varnothing) = [s_1 + 1, m_1] [s_1 + 2, m_1] [s_1 + 3, m_1] [s_1 + 4, m_1] \\
\times [s_2 + 1, m_2] [s_2 + 2, m_2] [s_2 + 3, m_2] [s_2 + 4, m_2] [s_2 + 5, m_2^2] [s_2 + 6, m_2] \\
\times [s_1 + s_2 + 5, m_1 + m_2] [s_1 + s_2 + 6, m_1 + m_2].
\]

**Remark 7.11.** In fact, the argument in Example \((7.10)\) is redundant; it is enough to determine one of the two local \(b\)-functions \((7.6)\) and \((7.7)\).

**Example 7.12** (proof of \((5.3)\)). We consider the slice representation \((GL(\mathfrak{m})_{A^{(1,4)}}, W^{(1,4)})\). By Lemma \((7.6)\) the restriction \(f'_1\) of \(f_1 = f^{(1,4)}\) to \(W^{(1,4)}\) is constant, and by the construction of \(f_2 = f^{(2,5)}\), the restriction \(f'_2\) of \(f_2\) to \(W^{(1,4)}\) is given by

\[
f'_2(w) = \det \left( w^{(1,4)}_1, w^{(1,4)}_2 \right)
\]

for \(w = (w_1^{(1,4)}, w_2^{(1,4)}) \in W^{(1,4)}\). We have \(f'_2(w) = \det \left( \varnothing \varnothing \right)\), and since \(b \left( \varnothing \varnothing \right) = (s + 1)(s + 2)(s + 3)(s + 4)\), we see that \(b_m(s)\) is divisible by

\[
[s_2 + 1, m_2] [s_2 + 2, m_2] [s_2 + 3, m_2] [s_2 + 4, m_2].
\]
Then it follows from
\[ b_{(0,1)}(0,s) = b_{f_2}(s) = (s+1)(s+2)(s+3)^2(s+4)^2(s+5)(s+6)(s+7) \]
that \( \{\alpha_{3,1}, \ldots, \alpha_{3,5}\} = \{3,4,5,6,7\} \) in (6.11). We thus obtain (5.2), i.e.,
\[
b_m(2) = [s_1 + 1]_{m_1} [s_1 + 2]_{m_1} [s_1 + 4]_{m_1} [s_1 + 5]_{m_1}
\times [s_2 + 1]_{m_2} [s_2 + 2]_{m_2} [s_2 + 3]_{m_2} [s_2 + 4]_{m_2}
\times [s_1 + s_2 + 3]_{m_1 + m_2} [s_1 + s_2 + 4]_{m_1 + m_2} [s_1 + s_2 + 5]_{m_1 + m_2}
\times [s_1 + s_2 + 6]_{m_1 + m_2} [s_1 + s_2 + 7]_{m_1 + m_2}.
\]

Now we are in a position to finish the proof of our main theorem.

**Proof of Theorem 5.2.** Step (I). First we consider the case of \( l = 2 \), i.e., we assume that there exist two fundamental relative invariants \( f_1 \) and \( f_2 \). As shown in Section 6 the \( b \)-function \( b_m(s) \) of \( f = (f_1, f_2) \) is, in general, of the form
\[
b_m(s) = \prod_{r=1}^{\mu_1} [s_1 + \alpha_{1,r}]_{m_1} \times \prod_{r=1}^{\mu_2} [s_2 + \alpha_{2,r}]_{m_2} \times \prod_{r=1}^{\mu_3} [s_1 + s_2 + \alpha_{3,r}]_{m_1 + m_2}
\]
with some \( \alpha_{j,r} \in \mathbb{Q}_{>0} \). Thus it remains to determine \( \alpha_{j,r} \). By using \( b_{(1,0)}(s,0) = b_{f_1}(s), b_{(0,1)}(0,s) = b_{f_2}(s) \) and Theorem 3.4 we obtain \( \{\alpha_{1,1}, \ldots, \alpha_{1,\mu_1}, \alpha_{3,1}, \ldots, \alpha_{3,\mu_3}\} \) and \( \{\alpha_{2,1}, \ldots, \alpha_{2,\mu_2}, \alpha_{3,1}, \ldots, \alpha_{3,\mu_3}\} \) as sets. We will show that the arrangement such as Figures 6 and 8 gives the correct answer to our problem. For \( i = 1, 2 \), let \( O^{(i)} \) be the closed orbit in \( \{f_i \neq 0\} \) and \( (GL(n)_{A(\nu)}, W^{(i)}) \) the slice representation associated with \( O^{(i)} \). Let \( f_2' \) be the restriction of \( f_2 \) to \( W^{(1)} \) and
\[
b_{f_2'}(s) = \prod_{r=1}^{\mu_2'} (s + \beta_r)
\]
be the \( b \)-function of \( f_2' \). Then, by applying Lemma 7.1 to \( f_m = f_1^{m_1} f_2^{m_2} \) and \( O^{(1)} \), we see that \( b_m(s) \) is divisible by
\[
\prod_{r=1}^{\mu_2'} [s_2 + \beta_r]_{m_2},
\]
and hence \( \{\beta_1, \ldots, \beta_{\mu_2'}\} \subset \{\alpha_{2,1}, \ldots, \alpha_{2,\mu_2}\} \). On the other hand, let \( D^{(i)} \) (\( i = 1, 2 \)) be the exact lace diagram representing \( O^{(i)} \). Then, by the construction of the exact lace diagrams, the arrows of \( D^{(2)} \) which do not overlap with the arrows of \( D^{(1)} \) form a smaller exact lace diagram, which is contained in the diagram representing the slice representation \( (GL(n)_{A(i)}, W^{(1)}) \). The arrows which corresponds to \( s_2 + \beta_1, \ldots, s_2 + \beta_{\mu_2'} \) also form an exact lace diagram in the diagram of \( (GL(n)_{A(i)}, W^{(1)}) \). This observation proves that \( \mu_2 = \mu_2' \) and \( \{\alpha_{2,1}, \ldots, \alpha_{2,\mu_2}\} = \{\beta_1, \ldots, \beta_{\mu_2}\} \). Since \( \{\alpha_{2,1}, \ldots, \alpha_{2,\mu_2}, \alpha_{3,1}, \ldots, \alpha_{3,\mu_3}\} \) is
calculated as a set, we obtain \( \{\alpha_{3,1}, \ldots, \alpha_{3,\mu_3}\} \), and also \( \{\alpha_{1,1}, \ldots, \alpha_{1,\mu_1}\} \). This means that our arrangement is correct for the case of \( l = 2 \).

**Step (II).** For general \( l \), we use an induction on \( l \). Let \( b_m(s) \) be the \( b \)-function of \( f = (f_1, \ldots, f_l) \). We decompose \( b_m(s) \) as

\[
(7.8) \quad b_m(s) = c_m(s) \cdot \prod_{r=1}^{\mu} [s_1 + \cdots + s_l + \alpha_r]^{m_1+\cdots+m_l},
\]

where \( c_m(s) \) does not contain the factor of the form \( s_1 + \cdots + s_l + \alpha \). Let \( \tilde{D} \) be the diagram obtained by superposing all the exact lace diagrams \( D^{(i)} \) representing the locally closed orbit \( O^{(i)} \) \( (i = 1, \ldots, l) \). Fix an index \( k \) \( (k = 1, \ldots, l) \). Then the arrows of \( \tilde{D} \) which do not overlap with the arrows of \( D^{(k)} \) corresponds to the \( b \)-function \( b_m^{(k)}(s) \) of the slice representation \((GL(n)_A, W^{(k)})\). By induction hypothesis, \( b_m^{(k)}(s) \) can be calculated by the superposition method. By repeating this for \( k = 1, \ldots, l \), we obtain \( c_m(s) \). For \( \varepsilon_i = (0, \ldots, 1, \ldots, 0) \), we have \( b_{\varepsilon_i}(s) = b_{f_i}(s) \), and by using this with Theorem 5.1 we obtain \( \{\alpha_1, \ldots, \alpha_{\mu}\} \) in (7.8). This completes the proof of Theorem 5.2.

We conclude the present paper by giving an example with \( l = 3 \).

**Example 7.13.** Let us consider the following quiver of type \( A_7 \):

\[
Q: \quad 1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \leftarrow 7.
\]

We put \( n = (1, 3, 5, 4, 4, 3, 1) \). Then \((GL(n), \text{Rep}(Q, n))\) has 3 fundamental relative invariants \( f_1 := f_{(1,5)}, f_2 := f_{(3,4)}, f_3 := f_{(2,7)} \), which are given explicitly as follows:

\[
\begin{align*}
 f_1(v) &= \det \begin{pmatrix} X_{2,1} & X_{2,3} \\ O & X_{6,5}X_{5,4}X_{4,3} \end{pmatrix}, \\
 f_2(v) &= X_{5,4}, \\
 f_3(v) &= \det \begin{pmatrix} X_{2,3} & O \\ X_{6,5}X_{5,4}X_{4,3} & X_{6,7} \end{pmatrix}
\end{align*}
\]

for \( v = (X_{2,1}, X_{2,3}, X_{4,3}, X_{5,4}, X_{6,5}, X_{6,7}) \in \text{Rep}(Q, n) \). We superpose the exact lace diagrams corresponding to the locally closed orbits, and the resultant diagram is given as Figure 17. Hence we see that the \( b \)-function \( b_m(s) \) of \( f = (f_1, f_2, f_3) \) is given by

\[
b_m(s) = [s_1 + 1]^{m_1} [s_1 + 2]^{m_1} [s_1 + 3]^{m_1} [s_2 + 1]^{m_2} [s_3 + 1]^{m_3} [s_3 + 3]^{m_3} \\
\times [s_1 + s_2 + 2]^{m_1+m_2} [s_1 + s_3 + 2]^{m_1+m_3} [s_1 + s_3 + 3]^{m_1+m_3} \\
\times [s_1 + s_3 + 4]^{m_1+m_3} [s_1 + s_3 + 5]^{m_1+m_3} \\
\times [s_1 + s_2 + s_3 + 3]^{m_1+m_2+m_3} [s_1 + s_2 + s_3 + 4]^{m_1+m_2+m_3}.
\]
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