Bounds on absolutely maximally entangled states from shadow inequalities, and the quantum MacWilliams identity

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Abstract
A pure multipartite quantum state is called absolutely maximally entangled (AME), if all reductions obtained by tracing out at least half of its parties are maximally mixed. Maximal entanglement is then present across every bipartition. The existence of such states is in many cases unclear. With the help of the weight enumerator machinery known from quantum error correction and the shadow inequalities, we obtain new bounds on the existence of AME states in dimensions larger than two. To complete the treatment on the weight enumerator machinery, the quantum MacWilliams identity is derived in the Bloch representation. Finally, we consider AME states whose subsystems have different local dimensions, and present an example for a $2 \times 3 \times 3 \times 3$ system that shows maximal entanglement across every bipartition.

Keywords: multipartite entanglement, quantum error correcting codes, absolutely maximally entangled states, quantum weight enumerators

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum states of many particles show interesting non-classical features, foremost the one of entanglement. A pure state of $n$ parties is called absolutely maximally entangled (AME), if all reductions to $\left\lfloor \frac{n}{2} \right\rfloor$ parties are maximally mixed. Here, $\lfloor \cdot \rfloor$ is the floor function. Then
maximal possible entanglement is present across each bipartition. Well-known examples are the Bell and GHZ states on two and three parties respectively. AME states have been shown to be a resource for a variety of quantum information-theoretic tasks that require maximal entanglement amongst many parties, such as open-destination teleportation, entanglement swapping, and quantum secret sharing [1, 2]. They also represent building blocks for holographic quantum error-correcting codes, and are often called perfect tensors in this context [3–5]. Thus, it is a natural question to ask for what number of parties and local dimensions such states may exist [6–8].

The existence of AME states composed of two-level systems was recently solved: Qubit AME states do only exist for $n = 2, 3, 5,$ and 6 parties, all of which can be expressed as graph or stabilizer states [6, 9]. Concerning larger local dimensions however, the existence of such states is only partially resolved. AME states exist for any number of parties, if the dimension of the subsystems is chosen large enough [2]. Furthermore, different constructions for such states have been put forward, based on graph states [10, 11], classical maximum distance separable codes [2, 12], and combinatorial designs [13, 14]. However, for many cases it is still unknown whether or not AME states exist5.

In this article, we give results on the question of AME state existence when the local dimension is three or higher. Namely, we show that, additionally to the known non-existence bounds, three-level AME states of $n = 8, 12, 13, 14, 16, 17, 19, 21, 23$, four-level AME states of $n = 12, 16, 20, 24, 25, 26, 28, 29, 30, 33, 37, 39$, and five-level AME states of $n = 28, 32, 36, 40, 44, 48$ parties do not exist.

To this end, we make use of the weight enumerator machinery known from quantum error correcting codes (QECC). With it, bounds can also be obtained for one-dimensional codes, which are pure quantum states [6]. We will also make use of the so-called shadow inequalities, which constrain the admissible correlations of multipartite states, to exclude the existence of the above-mentioned AME states. Along the way, we will prove a central theorem, the quantum MacWilliams identity, originally derived by Shor and Laflamme for qubits [15] and by Rains for arbitrary finite-dimensional systems in [16]. Thus our aim is twofold: On the one hand, we provide an accessible introduction into the weight enumerator machinery in terms of the Bloch representation, in order to gain physical intuition. On the other hand, we apply this machinery to exclude the existence of certain higher-dimensional AME states by making use of the shadow inequalities.

This article is organized as follows. In the next section, we introduce the shadow inequalities, from which we eventually obtain the bounds mentioned above. In section 3, the Bloch representation of quantum states is introduced, followed by a short discussion of QECC and their relation to AME states in section 4. In section 5, we introduce the shadow enumerator, the Shor–Laflamme enumerators are explained in section 6, followed by the derivation of the quantum MacWilliams identity in section 7. The shadow enumerator in terms of the Shor–Laflamme enumerator is derived in section 8, from which one can obtain bounds on the existence of QECC and of AME states in particular, which is presented in section 9. After considering AME states in mixed dimensions in section 10, we conclude in section 11.

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5 For the current status of this question, see problem 35 in the list of Open Quantum Problems, IQOQI Vienna (November 2017), http://oqp.iqoqi.univie.ac.at/existence-of-absolutely-maximally-entangled-pure-states
2. Motivation

Originally introduced by Shor and Laflamme [15], Rains established the notion of weight enumerators in a series of landmark articles on quantum error correcting codes [16–18]. With it, he stated some of the strongest bounds on the existence of QECC known to date [17]. In particular, in his paper on polynomial invariants of quantum codes [18], Rains showed an interesting theorem, which proved to be crucial to obtain those bounds. These are the so-called shadow inequalities: For all positive semi-definite Hermitian operators $M$ and $N$ on parties $(1 \ldots n)$ and any fixed subset $T \subseteq \{1 \ldots n\}$, it holds that

$$
\sum_{S \subseteq \{1 \ldots n\}} (-1)^{|S \cup T|} \text{Tr}_S[\text{Tr}_S(M)\text{Tr}_S(N)] \geq 0.
$$

(1)

Here and in what follows, $S'$ denotes the complement of subsystem $S$ in $\{1 \ldots n\}$, and the sum is performed over all possible subsets $S$. Note that if $M = N = \rho$ is a quantum state, the generalized shadow inequalities are consistency equations involving the purities of the marginals, i.e. they relate terms of the form $\text{Tr}[\text{Tr}_S(\rho)^2]$, which in turn can be expressed in terms of linear entropies. Thus, these inequalities form an exponentially large set of monogamy relations for multipartite quantum states, applicable to any number of parties and local dimensions.

To state bounds on the existence of AME states of $n$ parties having local dimension $D$ each, one could in principle just evaluate this expression by inserting the purities of AME state reductions. However, in order to understand the connections to methods from quantum error correcting codes, let us first recall the quantum weight enumerator machinery, including the so-called shadow enumerator, which is derived from equation (1). We will then rederive the central theorem, namely the quantum MacWilliams identity. Finally, we obtain new bounds for AME states with the help of the shadow inequalities. In order to remain in a language close to physics, we will work exclusively in the Bloch representation.

3. The Bloch representation

Let us introduce the Bloch representation. Denote by $\{e_j\}$ an orthonormal basis for operators acting on $\mathbb{C}^D$, such that $\text{Tr}(e_j^* e_k) = \delta_{jk}D$. We require that $\{e_j\}$ contains the identity (e.g. $e_0 = 1$), and therefore all other basis elements are traceless (but not necessarily Hermitian). Then, a local error-basis $E$ acting on $(\mathbb{C}^D)^{\otimes n}$ can be formed by taking tensor products of elements in $\{e_j\}$. That is, each element $E_\alpha \in E$ can be written as

$$
E_\alpha = e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}.
$$

(2)

Because the single-party basis $\{e_j\}$ is orthonormal, the relation $\text{Tr}(E_\alpha^* E_\beta) = \delta_{\alpha \beta}D^n$ follows. For qubits, $E$ can be thought of to contain all tensor products which can be built from the identity and the Pauli matrices; in higher dimensions, a tensor-product basis can be formed from elements of the Heisenberg–Weyl or the generalized Gell–Mann basis [19]. Further, denote by $\text{supp}(E)$ the support of operator $E$, that is, the set of parties on which $E$ acts non-trivially. The weight of an operator is then size of its support, and we write $\text{wt}(E) = |\text{supp}(E)|$.

Having defined a local error-basis $E$ acting on $(\mathbb{C}^D)^{\otimes n}$, every operator on $n$ systems that have $D$ levels each can in the Bloch representation be decomposed as

$$
M = \frac{1}{D^n} \sum_{E \in E} \text{Tr}(E^\dagger M)E.
$$

(3)
As in the above decomposition, we will often omit the subindex $\alpha$, writing $E$ for $E_\alpha$. Also, most equations that follow contain sums over all elements $E$ in $E$, subject to constraints. In those cases we will often denote the constraints only below the summation symbol.

Given an operator $M$ expanded as in equation (3), its reduction onto a given subsystem $S^c$ tensored by the identity on the complement $S$ reads

$$\text{Tr}_S(M) \otimes \mathbb{I}_S = D^{[S]} \sum_{\text{supp}(E) \subseteq S} \text{Tr}(E^\dagger M) E.$$  

This follows from $\text{Tr}_S(E) = 0$ whenever $\text{supp}(E) \not\subseteq S^c$. Interestingly, this can also be written as a quantum channel whose Kraus operators form a unitary 1-design [20].

**Observation 1.** The partial trace over subsystem $S$ tensored by the identity on $S$ can also be written as a channel,

$$\text{Tr}_S(M) \otimes \mathbb{I}_S = D^{[S]} \sum_{\text{supp}(E) \subseteq S} E E^\dagger.$$  

The proof can be found in appendix A.

### 4. Quantum error correcting codes

Let us introduce quantum error correcting codes and their relation to absolutely maximally entangled states. A quantum error correcting code with the parameters $((n,K,d))_D$ is a $K$-dimensional subspace $Q$ of $(\mathbb{C}^D)^{\otimes n}$, such that for any orthonormal basis $\{|i_Q\rangle\}$ of $Q$ and all errors $E \in E$ with $\text{wt}(E) < d$ [6, 21],

$$\langle i_Q | E | j_Q \rangle = \delta_{ij} C(E).$$  

Note that the constant $C(E)$ only depends on the error $E$. Above, $d$ is called the distance of the code. If $C(E) = \text{Tr}(E)/D^n$, the code is called pure. By convention, codes with $K = 1$ are only considered codes if they are pure.

From these definitions it follows that a one-dimensional code (also called self-dual), described by a projector $|\psi\rangle \langle \psi|$, must fulfill $\text{Tr}(E|\psi\rangle \langle \psi|) = 0$ for all $E \neq \mathbb{I}$ of weight smaller than $d$. Thus, pure one-dimensional codes of distance $d$ are pure quantum states whose reductions onto $(d-1)$ parties are all maximally mixed. AME states, whose reductions onto $\lfloor \frac{n}{2} \rfloor$ parties are maximally mixed, are QECC having the parameters $((n,1,\lfloor \frac{n}{2} \rfloor + 1))_D$.

### 5. The shadow enumerator

Let us introduce the shadow enumerator $S_{MN}(x,y)$, and point out its usefulness. Following Rains [16], we first define

$$A'_S(M,N) = \text{Tr}_S[\text{Tr}_S(M)\text{Tr}_S(N)],$$  

$$B'_S(M,N) = \text{Tr}_S[\text{Tr}_S(N)\text{Tr}_S(M)].$$  

Naturally, $A'_S = B'_S$. With this, we define

$$S_j(M,N) = \sum_{|T|=j} \sum_{S \subseteq \{1...n\}} (-1)^{|S-T|} A'_S(M,N),$$  

6 We use the symbol $Q$ to denote both the subspace and the projector onto it.
where the sum is over all subsets $T \subseteq \{1 \ldots n\}$ of size $j$. Equation (1) states that all $S_j$ must be non-negative. Note however, that there is the term $T^c$ instead of $T$ in the exponent, compared to equation (1), but this does not matter, as equation (1) holds for any $T$.

The shadow enumerator then is the polynomial

$$S_{MN}(x,y) = \sum_{j=0}^n S_j(M,N) x^{n-j} y^j.$$  \hspace{1cm} (10)

Given a hypothetical QECC or an AME state in particular, its shadow enumerator must have non-negative coefficients. If this is not the case, one can infer that such a code or state cannot exist. However, how do we obtain this enumerator? Two paths come to mind: First, if we are interested in a one-dimensional code ($K=1$), the purities of the reductions determine all $A'_j(Q) \equiv A'_j(Q,Q)$. For AME states of local dimension $D$, the situation is particularly simple: from the Schmidt decomposition, it can be seen that all reductions to $k$ parties must have the purity

$$\text{Tr}((\rho^2)_k) = D^{-\min(kn-k)}.$$  \hspace{1cm} (11)

Second, the coefficients of the so called Shor–Laflamme enumerator $A_j(Q)$ may be known (see also below), from which the shadow enumerator can be obtained.

Generally, when dealing with codes whose existence is unknown, putative weight enumerators can often be obtained by stating the relations that follow as a linear program (see appendix F) [6, 22, 23]. If, for a set of parameters $((n, K, d))_D$, no solution can be found, a corresponding QECC cannot exist.

In the following three sections, we aim to give a concise introduction as well as intuition to this enumerator theory.

6. Shor–Laflamme enumerators

In this section, we introduce the protagonists of the enumerator machinery, the Shor–Laflamme (weight) enumerators [15, 16]. These are defined for any two given Hermitian operators $M$ and $N$ acting on $(C^D)^{\otimes n}$, and are invariants under local unitary operations. Their (unnormalized) coefficients are given by\footnote{For dimensions larger than two, this definition is different, but equivalent, to the original definition as found in [16].}

$$A_j(M,N) = \sum_{\text{wt}(E)=j} \text{Tr}(EM)\text{Tr}(E^\dagger N),$$  \hspace{1cm} (12)

$$B_j(M,N) = \sum_{\text{wt}(E)=j} \text{Tr}(EME^\dagger N).$$  \hspace{1cm} (13)

The corresponding enumerator polynomials are

$$A_{MN}(x,y) = \sum_{j=0}^n A_j(M,N)x^{n-j}y^j,$$  \hspace{1cm} (14)

$$B_{MN}(x,y) = \sum_{j=0}^n B_j(M,N)x^{n-j}y^j.$$  \hspace{1cm} (15)
While it might not be obvious from the definition, these enumerators are independent of the local error-basis $\mathcal{E}$ chosen, and are thus local unitary invariants. This follows from the fact that they can expressed as linear combinations of terms having the form of equation (7). The exact relation will be made clear in section 7.

When dealing with weight enumerators, there is the following pattern, as seen above: First define a set of coefficients (e.g. $A_j(M, N)$), from which the associated polynomial, the enumerator, is constructed (e.g. $A_j(M, N)$). If $M = N$, we will often write the first argument only, e.g. $A_j(M)$, or leave it out altogether. In table 1, we give an overview of the coefficients and enumerators used in this article.

Considerating a QECC with parameters $((n, K, d), d)$, one sets $M = N$ to be equal to the projector $\mathcal{Q}$ onto the code space. The following results concerning QECC and their Shor–Laflamme enumerators are known [16]: The coefficients $A_j = A_j(\mathcal{Q})$ and $B_j = B_j(\mathcal{Q})$ are non-negative, and

$$KB_0 = A_0 = K^2,$$

$$KB_j \geq A_j,$$

with equality in the second equation for $j < d$. In fact, these conditions are not only necessary but also sufficient for a projector $\mathcal{Q}$ to be a QECC (see appendix B.). The distance of a code can thus be obtained in the following way: if a projector $\mathcal{Q}$ fulfills the above conditions with equality for all $j < d$, then $\mathcal{Q}$ is a quantum code of distance $d^8$. For pure codes, additionally $A_j = B_j = 0$ for all $1 < j < d$. In particular, AME states have $A_j = 0$ for all $1 < j < \lfloor \frac{n}{2} \rfloor + 1$; the remaining $A_j$ can be obtained in an iterative way from equation (11) [6, 9].

In the case of $\mathcal{Q} = |\psi\rangle\langle\psi|$, the weight enumerators have a particularly simple interpretation: The coefficient $A_j$ measures the contribution to the purity of $|\psi\rangle\langle\psi|$ by terms in $|\psi\rangle\langle\psi|$ having weight $j$ only, while the dual enumerator measures the overlap of $|\psi\rangle\langle\psi|$ with itself, given an error-sphere of radius $j$. Furthermore, we have $A_j \geq B_j$ for all $j$, as a direct evaluation shows.

In the entanglement literature, $A_j(\rho)$ is also called the correlation strength, or the squared two-norm of the $j$-body correlation tensor [24, 25]. Concerning codes known as stabilizer codes, $A_j$ and $B_j$ count elements of weight $j$ in the stabilizer and in its normalizer respectively [26].

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**Table 1.** An overview on the different weight enumerator polynomials and their coefficients, which are local unitary invariants.

| Coefficient | Enumerator |
|-------------|------------|
| $A_j(M, N) = \sum_{w \in \mathcal{E}_{M,N}} \Tr(E^{j}M)\Tr(E^{j}N)$ | $A_{MN}(x, y) = \sum_{j=0}^{\infty} A_j(M, N)x^{d_j}y^{j}$ |
| $B_j(M, N) = \sum_{w \in \mathcal{E}_{M,N}} \Tr(E^{j}M)\Tr(E^{j}N)$ | $B_{MN}(x, y) = \sum_{j=0}^{\infty} B_j(M, N)x^{d_j}y^{j}$ |

Rain’s unitary enum.:  
$A_j'(M, N) = \Tr_\mathcal{E}[\Tr_\mathcal{E}(M)\Tr_\mathcal{E}(N)]$  
$B_j'(M, N) = \Tr_\mathcal{E}[\Tr_\mathcal{E}(M)\Tr_\mathcal{E}(N)]$  
$A_j''(M, N) = \sum_{i=|j|} A_i(M, N)$  
$B_j''(M, N) = \sum_{i=|j|} B_i(M, N)$  
$A_{MN}''(x, y) = \sum_{j=0}^{\infty} A_j'(M, N)x^{d_j}y^{j}$  
$B_{MN}''(x, y) = \sum_{j=0}^{\infty} B_j'(M, N)x^{d_j}y^{j}$

Shadow enumerator:  
$S_j(M, N) = \sum_{i=|j|} (-1)^{|j|} A_i(M, N)$  
$S_{MN}(x, y) = \sum_{j=0}^{\infty} S_j(M, N)x^{d_j}y^{j}$
Let us now try to give some intuition for these enumerators for general Hermitian operators $M$ and $N$. Note that the coefficients of the primary enumerator $A_j(M,N)$ form a decomposition of the inner product $\text{Tr}(MN)$. This can be seen by writing $M$ and $N$ in the Bloch representation (equation (3)),

\[\text{Tr}(MN) = D^{-2n}\text{Tr}\left(\sum_E \text{Tr}(EM)E^\dagger \sum_{E'} \text{Tr}(E^\dagger N)E'\right) \]

\[= D^{-2n}\text{Tr}\left(\sum_E \text{Tr}(EM)\text{Tr}(E^\dagger N)E\right)\]  \hspace{1cm} (18)

\[= D^{-n}\sum_{j=0}^{n} A_j(M,N).\]

On the other hand, the coefficients of the dual enumerator $B_j(M,N)$ can be seen as a decomposition of $\text{Tr}(M)\text{Tr}(N)$. To see this, recall that by definition of the partial trace,

\[\text{Tr}_S[\text{Tr}_S(M)\text{Tr}_S(N)] = \text{Tr}[\text{Tr}_S(M)\otimes I_S N].\]  \hspace{1cm} (19)

As shown in observation 1, the partial trace over parties in $S$ tensored by the identity on $S$ can also be written as a quantum channel,

\[\text{Tr}_S(M)\otimes I_S = D^{-|S|} \sum_{\text{supp}(E) \subseteq S} EME^\dagger.\]  \hspace{1cm} (20)

Thus $B_j(M,N)$ decomposes $\text{Tr}(M)\text{Tr}(N)$,

\[\text{Tr}(M)\text{Tr}(N) = \text{Tr}[\text{Tr}(M)I N] \]

\[= D^{-n}\text{Tr}\left(\sum_{E} EME^\dagger N\right)\]

\[= D^{-n}\sum_{j=0}^{n} \sum_{\text{wt}(E)=j} \text{Tr}(EME^\dagger N)\]  \hspace{1cm} (21)

\[= D^{-n}\sum_{j=0}^{n} B_j(M,N).\]

The insight gained from writing the partial trace in two different ways (see equations (4) and (5)), and the decomposition of $\text{Tr}(MN)$ and $\text{Tr}(M)\text{Tr}(N)$ in terms of the coefficients of the Shor–Laflamme enumerators (see equations (14) and (15)) will prove to be the essence of the MacWilliams identity, which we rederive in the following section.

7. The quantum MacWilliams identity

In this section, we prove the quantum MacWilliams identity. It relates the two Shor–Laflamme enumerators $A_{MN}(x,y)$ and $B_{MN}(x,y)$ (equations (14) and (15)) of arbitrary Hermitian operators $M$ and $N$.

**Theorem 2 (Quantum MacWilliams identity [16, 22]).** For any two Hermitian operators $M$ and $N$ acting on $n$ systems having $D$ levels each, the following identity holds,

\[A_{MN}(x,y) = B_{MN}\left(\frac{x + (D^2 - 1)y}{D}, \frac{x - y}{D}\right).\]  \hspace{1cm} (22)
Proof. In order to prove this identity, one has to express the trace inner product of reductions in two different ways: given the operator $M$ expanded as in equation (3), its reduction tensored by the identity reads (see equation (4))

$$\text{Tr}_{S}(M) \otimes I_{S'} = D^{|S'| - n} \sum_{\text{supp}(E) \subseteq S} \text{Tr}(EM)E^\dagger.$$  \hspace{1cm} (23)

Therefore,

$$\text{Tr}[\text{Tr}_{S}(M) \otimes I_{S'} N] = \text{Tr}\left(D^{|S'| - 2n} \sum_{\text{supp}(E) \subseteq S} \text{Tr}(EM)E^\dagger \sum_{E'} \text{Tr}(E'^\dagger N)E'\right)$$

$$= D^{|S'| - n} \sum_{\text{supp}(E) \subseteq S} \text{Tr}(EM)\text{Tr}(E^\dagger N).$$  \hspace{1cm} (24)

Summing over all subsystems $S$ of size $m$, one obtains

$$\sum_{|S| = m} \text{Tr}[\text{Tr}_{S}(M) \otimes I_{S'} N] = D^{|S'| - n} \sum_{|S| = m} \sum_{\text{supp}(E) \subseteq S} \text{Tr}(EM)\text{Tr}(E^\dagger N)$$

$$= D^{|S'| - n} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} \binom{n}{j}^{-1} A_j(M, N)$$

$$= D^{|S'| - n} \sum_{j=0}^{m} \binom{n - j}{n - m} A_j(M, N).$$  \hspace{1cm} (25)

Above, the binomial factors account for multiple occurrences of terms having weight $j$ in the sum. Note that equation (25) forms the coefficients of Rains’ unitary enumerator (see (7)), defined as [16]

$$A'_m(M, N) = \sum_{|S| = m} \mathcal{A}'_S(M, N)$$

$$= \sum_{|S| = m} \text{Tr}_S[\text{Tr}_{S'}(M)\text{Tr}_{S'}(N)].$$  \hspace{1cm} (26)

On the other hand, by expressing the partial trace as a quantum channel (see observation 1) and again summing over subsystems of size $m$, we can write

$$\sum_{|S| = m} \text{Tr}[\text{Tr}_{S}(M) \otimes I_{S'} N] = \sum_{|S| = m} \text{Tr}(D^{-|S|} \sum_{\text{supp}(E) \subseteq S} EME^\dagger N)$$

$$= D^{-m} \sum_{j=0}^{m} \binom{n}{m} \binom{m}{j} \binom{n}{j}^{-1} B_j(M, N)$$

$$= D^{-m} \sum_{j=0}^{m} \binom{n - j}{n - m} B_j(M, N).$$  \hspace{1cm} (27)

Similar to above, equation (27) forms the coefficients of the unitary enumerator (see equation (7)).
\[ B'_m(M,N) = \sum_{|S| = m} E'_S(M,N) \]
\[ = \sum_{|S| = m} \text{Tr}_S \left[ \text{Tr}_S(M) \text{Tr}_S(N) \right]. \]  
\( (28) \)

Naturally, the corresponding unitary enumerator polynomials read
\[ A'_{MN}(x,y) = \sum_{j=0}^n A'_j(M,N) x^{n-j} y^j \]  
\( (29) \)
\[ B'_{MN}(x,y) = \sum_{j=0}^n B'_j(M,N) x^{n-j} y^j. \]  
\( (30) \)

Using relations \( (25) \) and \( (27) \), one can establish with the help of generating functions that
\[ A'_{MN}(x,y) = A_{MN} \left( x + \frac{y}{D} \frac{y}{D} \right), \]  
\( (31) \)
\[ B'_{MN}(x,y) = B_{MN} \left( x + \frac{y}{D} \frac{y}{D} \right). \]  
\( (32) \)

This is somewhat tedious but straightforward (see appendix C). It remains to use that \( B'_S(M,N) = A'_S(M,N) \), from which follows that \( B'_k(M,N) = A'_{n-k}(M,N) \), and
\[ A'_{MN}(x,y) = B'_{MN}(y,x). \]  
\( (33) \)

Thus the quantum MacWilliams identity is established,
\[ A_{MN}(x,y) = A_{MN} \left( x - y, Dy \right) = B'_{MN}(Dy, x - y) \]
\[ = B_{MN} \left( \frac{x + (D^2 - 1)y}{D}, \frac{x - y}{D} \right). \]  
\( (34) \)

This ends the proof. \( \Box \)

Because the relations \( (31) \) and \( (32) \) are symmetric, one also has that
\[ B_{MN}(x,y) = A_{MN} \left( \frac{x + (D^2 - 1)y}{D}, \frac{x - y}{D} \right). \]  
\( (35) \)

Thus the quantum MacWilliams transform is involutory.

Recall that for \( M = N = |\psi\rangle\langle\psi| \), one has \( A_j(|\psi\rangle) = B_j(|\psi\rangle) \). Therefore the enumerator \( A_{|\psi\rangle}(x,y) \) must stay invariant under the transform
\[ x \mapsto x + (D^2 - 1)y, \]
\[ y \mapsto \frac{x - y}{D}. \]  
\( (36) \)

In this case, a much simpler interpretation of the MacWilliams identity can be given: It ensures that the purities of complementary reductions, averaged over all complementary reductions of fixed sizes, are equal.
As shown above, the quantum MacWilliams identity is in essence a decomposition of the trace inner product of reductions of operators $M$ and $N$ in two different ways. The motivation lies in the decomposition of $\text{Tr}(MN)$ and $\text{Tr}(M)\text{Tr}(N)$, using different ways to obtain the partial trace in the Bloch picture (see equation (4) and observation 1). Finally, note that the derivation of the identity did not require $M, N$ to be positive semi-definite. Therefore the quantum MacWilliams identity holds for all, including non-positive, pairs of Hermitian operators.

8. The shadow enumerator in terms of the Shor–Laflamme enumerator

So far, we have introduced the Shor–Laflamme and the shadow enumerators. Let us now see how to express one in terms of the other. The strategy is the following: the shadow inequalities are naturally expressed in terms of $A_S'$ (see equations (1) and (9)), which we then write as a transformation of $A_{MN}(x, y)$.

**Theorem 3 (Rains)** Given $A_{MN}(x, y)$, the shadow enumerator is given by

$$S_{MN}(x, y) = A_{MN}\left(\frac{(D - 1)x + (D + 1)y}{D}, \frac{y - x}{D}\right).$$  \hspace{1cm} (37)

**Proof** Recall from equation (9), that for Hermitian operators $M, N \geq 0$, the coefficients of the shadow enumerator are

$$S_j(M, N) = \sum_{|T|=j} \sum_{S} (-1)^{|S\cap T|} A_S'(M, N).$$  \hspace{1cm} (38)

As a first step, let us understand what combinatorial factor a given $A_S'(M, N)$ receives from the sum over the subsets $T \subseteq \{1 \ldots n\}$ of size $j$, or subsets $T^c$ of size $m = n - j$ respectively. For a fixed subsystem $S$ of size $k$, we can evaluate the partial sum

$$f(m = |T^c|, k = |S|; n) = \sum_{|P|=m} (-1)^{|S\cap T^c|}.$$  \hspace{1cm} (39)

By considering what possible subsets $T^c$ of size $m$ have a constant overlap of size $\alpha$ with $S$, yielding a sign $(-1)^\alpha$, we obtain the expression

$$f(m, k; n) = \sum_{\alpha} \binom{n-k}{m-\alpha} \binom{k}{\alpha} (-1)^\alpha := K_m(k; n),$$  \hspace{1cm} (40)

where $K_m(k; n)$ is the so-called Krawtchouk polynomial (see appendix D). Above, $\binom{k}{\alpha}$ accounts for the different combinatorial possibilities of elements $T^c$ having overlap $\alpha$ with $S$. Necessarily, $T^c$ must then have a part of size $m - \alpha$ lying outside of $S$; there are $\binom{n-k}{m-\alpha}$ ways to obtain this. This is illustrated in figure 1.

---

9 See theorem 8 in [17] and theorem 13.5.1. on p 383 in [22] for $D = 2$. Also section 5 in [16] states this result, but contains a sign error in the second argument of $A_C$. 

---
Therefore, one obtains
\[ S_j(M, N) = \sum_{k=0}^{n} K_{n-j}(k; n) A'_k(M, N). \]  
(41)

Again, one can write this relation in a more compact form in terms of the unitary enumerator (see appendix E),
\[ S_{MN}(x, y) = A'_{MN}(x + y, y - x). \]  
(42)

To obtain the shadow enumerator in terms of the Shor–Laflamme enumerator, we take advantage of equation (31). Then
\[ S_{MN}(x, y) = A'_{MN}(x + y, y - x) \]
\[ = A_{MN}\left(\frac{(D - 1)x + (D + 1)y}{D}, \frac{y - x}{D}\right). \]  
(43)

This ends the proof.

Thus, given the Shor–Laflamme enumerator, one can obtain the shadow enumerator simply by a transform. If any of its coefficients are negative, a corresponding QECC cannot exist.

Given the parameters \((n, K, d)_{D}\) of a hypothetical QECC, one can formulate a linear program to find possible enumerators which satisfy all the relations derived, namely equations (16) and (17), as well as the quantum MacWilliams identity (theorem 2) and the quantum shadow identity (theorem 3) (see appendix F) [22, 23]. If no valid weights \(A'(Q)\) can be found, a code with the proposed parameters cannot exist. This provides a method to prove the non-existence of certain hypothetical states and QECC; on the other hand, the existence of a valid enumerator however does not imply the existence of a corresponding code.

An overview on the relations between the enumerators is given in appendix G.

9. New bounds on absolutely maximally entangled states

In this last section, let us return to the question of the existence of absolutely maximally entangled (AME) states. Scott showed in [6] that a necessary requirement for an AME state of \(n\) parties having \(D\) levels each to exist, is
\[ n \leq \begin{cases} 2(D^2 - 1) & n \text{ even}, \\ 2D(D + 1) - 1 & n \text{ odd}. \end{cases} \] (44)

We explain now shortly how this bound was obtained by requiring the positivity of the Shor–Laflamme coefficient \( A_{\frac{1}{2}} \). Recall that complementary reductions of pure states share the same spectrum and therefore also the same purity. Thus if \(|\phi_{n,D}\rangle\) is a putative AME state of \( n \) parties having \( D \) levels each, then the coefficients of the unitary enumerator as defined in equation (26) are given by

\[ A'_k(|\phi_{n,D}\rangle) = \binom{n}{k} D^{-\min(k,n-k)}. \] (45)

Considering the unitary enumerator coefficient \( A'_{\frac{1}{2}} \), only the terms \( A_0 = 1 \), \( A_{\frac{1}{2}} \), and \( A_{\frac{3}{2}} \) contribute, with appropriate combinatorial prefactors. From equation (25) (or from the transform in equation (31)), one obtains

\[ A'_{\frac{1}{2}} = D^{-(\frac{1}{2}+1)} \left[ \binom{n}{\frac{1}{2}+1} A_0 + \binom{n-(\frac{1}{2}+1)}{\frac{1}{2}+1} A_{\frac{1}{2}} + A_{\frac{3}{2}} \right]. \] (46)

The term \( A_{\frac{1}{2}} \) in the above equation is fixed by the knowledge of \( A'_{\frac{3}{2}} \),

\[ A'_{\frac{3}{2}} = D^{-(\frac{3}{2}+1)} \left[ \binom{n}{\frac{3}{2}+1} A_0 + A_{\frac{3}{2}} \right]. \] (47)

Combining equations (45)–(47), solving for \( A_{\frac{1}{2}} \), and requiring its non-negativity yields the bound of equation (44). One may wonder if stronger bounds can be obtained by treating the non-negativity of \( A_j \) for \( j > \frac{1}{2} + 2 \) in a similar manner. However, this does not seem to be the case.

Let us now see what the additional constraints from the shadow enumerator yield. Having knowledge of all the unitary enumerator coefficients (equation (45)), all that is left is to evaluate equation (41) (or equation (42) respectively), which relates the shadow enumerator to the unitary enumerator. If any coefficient \( S_j(|\phi_{n,D}\rangle) \) happens to be negative, an AME state on \( n \) parties having \( D \) levels each cannot exist. We should mention that one could also evaluate the shadow inequalities (equation (1)) for a suitable choice of \( T \subseteq \{1 \ldots n\} \) directly—the shadow coefficients simply represent symmetrized forms of these inequalities. To give an example, consider a putative AME state on four qubits, whose non-existence was proven by [8]. Choosing \( T = \{1, 2, 3, 4\} \) leads to

\[ S_0(|\phi_{4,2}\rangle) = A'_0 - A'_1 + A'_2 - A'_3 + A'_4 = -\frac{1}{2} \neq 0, \] (48)

in contradiction to the requirement that all \( S_j \) be non-negative. For general AME states, the coefficient \( S_0 \) reads

\[ S_0(|\phi_{n,D}\rangle) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{-\min(k,n-k)}. \] (49)

The complete set of coefficients is given by (see equation (41))

\[ S_j(|\phi_{n,D}\rangle) = \sum_{k=0}^{n} K_{n-j}(k;n) \binom{n}{k} D^{-\min(k,n-k)}, \] (50)

where \( K_n(k;n) \) is the so-called Krawtchouk polynomial (see appendix D).
In figure 2, the parameters of hypothetical AME states are shown: In dark blue, AME states are marked which are already excluded by the bound from Scott (equation (44)); in light blue, those AME states are shown for which the negativity of the shadow enumerator coefficients $S_j(|\phi_n, D\rangle)$ (equation (50)) gives stronger bounds. The non-existence of AME states having parameters $n = 4, 9, 11$ with $D = 2$ was already known [6, 8]. The AME state with $n = 7$ and $D = 2$ (marked with a cross) is neither excluded by the Scott bound nor by the shadow enumerator, but by [9]. The symbol $\exists$ marks states which are known to exist, constructions can be found in [11–14, 27–31]. In particular, AME states always exist for $n \leq D$ if $D$ is a prime-power [12].

In figure 2, the parameters of hypothetical AME states are shown: In dark blue, AME states are marked which are already excluded by the bound from Scott (equation (44)); in light blue, those AME states are shown for which the negativity of the shadow enumerator coefficients $S_j(|\phi_n, D\rangle)$ (equation (50)) gives stronger bounds. For figure 2, all shadow coefficients of hypothetical AME states with local dimension $D \leq 9$ and $n$ not violating the Scott bound have been evaluated. For $3 \leq D \leq 5$, we found 27 instances where the shadow enumerator poses a stronger constraint than the bound from Scott.

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We conclude, that additionally to the known non-existence bounds, three-level AME states of $n = 8, 12, 13, 14, 16, 17, 19, 21, 23$, four-level AME states of $n = 12, 16, 20, 24, 25, 26, 28, 29, 30, 33, 37, 39$, and five-level AME states of $n = 28, 32, 36, 40, 44, 48$ parties do not exist.

10. Mixed-dimensional AME states

One might wonder about the existence of absolutely maximally entangled states also in systems that have mixed local dimensions: does a pure state exist, such that every bipartition
shows maximal entanglement\textsuperscript{10}. For this to be true, every subsystem whose dimension is not larger than that of its complement must be maximally mixed.

Let us give examples of four-partite systems that consist of qubits and qutrits. As already shown in [8], AME states on four qubits do not exist. On the other hand, an AME state on four qutrits does exist and is given by a stabilizer state [11]. How about other configurations? Using the shadow inequality, it can be seen that AME states in systems having the local dimensions $2 \times 2 \times 2 \times 3$ and $2 \times 2 \times 3 \times 3$ are not allowed. The last remaining case, a system with dimensions $2 \times 3 \times 3 \times 3$, allows for such a state, which we could find using an iterative semi-definite program (see below) with analytical post-processing. The state we found reads

$$|\phi_{2333}\rangle = -\alpha|0011\rangle - \beta|0012\rangle + \beta|0021\rangle + \alpha|0022\rangle$$

$$- \beta|0101\rangle + \alpha|0102\rangle + \beta|0110\rangle + \alpha|0120\rangle$$

$$- \alpha|0201\rangle + \beta|0202\rangle - \alpha|0210\rangle - \beta|0220\rangle$$

$$- \beta|1011\rangle + \alpha|1012\rangle - \alpha|1021\rangle + \beta|1022\rangle$$

$$+ \alpha|1101\rangle + \beta|1102\rangle - \alpha|1110\rangle + \beta|1120\rangle$$

$$- \beta|1201\rangle - \alpha|1202\rangle - \beta|1210\rangle + \alpha|1220\rangle.$$ 

Two possible sets of coefficients are given by

$$\alpha = \frac{1}{6} \sqrt{\frac{3}{2} + \sqrt{65}} = \frac{1}{54} \sqrt{3} \sqrt{2} + \sqrt{65}$$

$$\beta = \frac{1}{6} \sqrt{\frac{3}{2} - \sqrt{65}} = \frac{1}{54} \sqrt{3} \sqrt{2} - \sqrt{65},$$ 

which are (up to a global sign) the two solutions to the constraints

$$12(\alpha^2 + \beta^2) = 1, \quad 54 \alpha \beta = 1. \quad \text{(52)}$$

Both solutions are equivalent under local unitaries; the gate $U = \exp(i \varphi \sigma_y) \otimes 1$ with

$$\varphi = 2 \arctan\left(\frac{1}{2(3+54\alpha^2)}\right) = -2 \arctan\sqrt{\frac{5}{13}}$$

maps the first to the second solution.

We found this state with an iterative semi-definite program that works in the following way [34]:

(1) Choose a random initial state $|\psi(0)\rangle$.

(2) Solve the following semidefinite program,

maximize $\langle \psi(i) | \varrho | \psi(i) \rangle$

subject to $\rho_{AB} = \rho_{AC} = \rho_{AD} = 1/6$,

$\text{tr}[\rho] = 1$, \quad $\rho = \rho^\dagger$, \quad $\rho \geq 0$.

(Note that $\rho_{AB} = 1/6$ implies $\rho_A = 1/2$ and $\rho_B = 1/3$, etc., and latter constraints do not need to be stated separately.)

(3) Set $|\psi(i+1)\rangle$ equal to the eigenvector corresponding to the maximal eigenvalue of $\rho$.

(4) Repeat steps (2) & (3) until convergence.

Thus after each iteration, the state is projected onto the eigenvector corresponding to its largest eigenvalue. Note that the reductions onto two qutrits need to have rank 6 in a 9-dimensional space, with all non-vanishing eigenvalues being equal. While this requirement cannot be stated as a semidefinite constraint, the maximal mixedness of the complementary reductions can. In that way, the above constraints guarantee maximal entanglement across every bipartition. This

\textsuperscript{10}For mixed-dimensional states, this differs in definition to the one given by [33], which demands that every reduction of size $\left\lfloor \frac{n}{2} \right\rfloor$ be maximally mixed. The choice of definition depends on the desired feature in applications.
iterative program may also be used for related problems, e.g. for those which are presented in [35].

11. Conclusion

Using the quantum weight enumerator machinery originally derived by Shor, Laflamme and Rains, we obtained bounds on the existence of absolutely maximally entangled states, excluding 27 open cases of dimensions larger than two. For this, we used the so-called shadow inequalities, which constrain the possible correlations arising from quantum states. Additionally, we provided a proof of the quantum MacWilliams identity in the Bloch representation, clarifying its physical interpretation. We furthermore raised the question of mixed-dimensional AME states, of which it may be possible to form interesting mixed-dimensional QECC\(^{11}\).

For future work, it would be interesting to see what the generalized shadow inequalities involving higher-order invariants [18] imply for the distribution of correlations in QECC and multipartite quantum states.

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Appendix A. Proof of observation 1

Let us proof observation 1.

Observation A.1. The partial trace over subsystem \( S \) tensored by the identity on \( S \) can also be written as a channel,

\[
\text{Tr}_S(M) \otimes I_S = D^{\lfloor \frac{|S|}{2} \rfloor} \sum_{\text{supp}(E) \subseteq S} EME^E^\dagger.
\] (A.1)

Proof. Consider a bipartite system with Hilbert space \( \mathcal{H} = \mathbb{C}^D \otimes \mathbb{C}^D \) with a local orthonormal operator basis \( \{e_j\} \) on \( \mathbb{C}^D \). Define the SWAP operator as

\[
\text{SWAP} = \sum_{j=0}^{D-1} |j\rangle \langle k|.
\] (A.2)

It acts on pure states as \( \text{SWAP}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle \).

\(^{11}\) As an example, the state \( |\phi_{2333}\rangle \) can be regarded as a ((4, 1, 3))\(_{2333}\) code, where the lower indices denote the local dimensions. A partial trace over the last particle yields a ((3, 3, 2))\(_{233}\) code, which can be used to correct one error if its position is known. This follows from theorem 19 in [16].
It can also be expressed in terms of any orthonormal basis \( \{e_j\} \) as [32]

\[
\text{SWAP} = \frac{1}{D} \sum_{j=0}^{D^2-1} e_j^\dagger \otimes e_j.
\]  

(A.3)

Therefore we can express \( 1 \otimes N \) as

\[
1 \otimes N = \text{SWAP} \cdot (N \otimes 1) \cdot \text{SWAP}
\]

\[
= (\frac{1}{D} \sum_{j=0}^{D^2-1} e_j \otimes e_j^\dagger)(N \otimes 1)(\frac{1}{D} \sum_{k=0}^{D^2-1} e_k^\dagger \otimes e_k)
\]

\[
= D^{-2} \sum_{j,k=0}^{D^2-1} (e_j^\dagger Ne_k^\dagger) \otimes (e_j^\dagger e_k).
\]  

(A.4)

Tracing over the second party gives

\[
\text{Tr}(N) 1 = \frac{1}{D} \sum_{j=0}^{D^2-1} e_j^\dagger Ne_j^\dagger.
\]  

(A.5)

The claim follows from the linearity of the tensor product. This ends the proof. \(\square\)

Note that the proof is independent of the local orthonormal operator basis \( \{e_j\} \) chosen.

**Appendix B. Shor–Laflamme enumerators and the distance of QECC**

Here, we prove that for any QECC, the coefficients \( A_j = A_j(Q) \) and \( B_j = B_j(Q) \) are non-negative and fulfill

\[
KB_0 = A_0 = K^2,
\]  

(B.1)

\[
KB_j \geq A_j,
\]  

(B.2)

Furthermore, any projector \( Q \) of rank \( K \) is a QECC of distance \( d \) if and only if \( KB_j(Q) = A_j(Q) \) for all \( j < d \).

Let us start with the first claim. The equation \( KB_0 = A_0 = K^2 \) follows by direct computation. Let us show that for a QECC having the parameters \((n,K,d)\), the coefficients of the Shor–Laflamme enumerator fulfill (see equation (17)) \( A_j(Q) \leq KB_j(Q) \), where equality holds for \( j < d \). Recall that

\[
A_j(Q) = \sum_{w(E) = j} \text{Tr}(EQ)\text{Tr}(E^\dagger Q),
\]  

(B.3)

\[
B_j(Q) = \sum_{w(E) = j} \text{Tr}(EQE^\dagger Q).
\]  

(B.4)

Let us check the inequality for each term appearing in the sum, namely for those of the form...
\[ \text{Tr}(EQ)\text{Tr}(E^\dagger Q) \leq K\text{Tr}(EQE^\dagger Q), \] 

(B.5)

where \( E \) is a specific error under consideration. For later convenience, let us choose the error-basis \( \mathcal{E} \) to be Hermitian, e.g. formed by tensor products of the generalized Gell-Mann matrices [19]. Using \( Q = \sum_{i=1}^{K} |i_Q\rangle \langle i_Q| \), write

\[ \text{Tr}(EQ)\text{Tr}(E^\dagger Q) = \left( \sum_{i=1}^{K} \langle i_Q|E|i_Q\rangle \right) \left( \sum_{j=1}^{K} \langle j_Q|E^\dagger |j_Q\rangle \right), \] 

(B.6)

\[ \text{Tr}(EQE^\dagger Q) = \sum_{i,j=1}^{K} \langle i_Q|E|j_Q\rangle \langle j_Q|E^\dagger |i_Q\rangle. \]

(B.7)

For the case of \( \text{wt}(E) < d \), let us recall the definition of a QECC (equation (6)),

\[ \langle i_Q|E|j_Q\rangle = \delta_{ij}C(E), \quad \text{if } \text{wt}(E) < d. \] 

(B.8)

This leads for \( j < d \) to

\[ \text{Tr}(EQ)\text{Tr}(E^\dagger Q) = K^2C(E)C^\ast(E), \]

\[ \text{Tr}(EQE^\dagger Q) = KC(E)C^\ast(E). \] 

(B.9)

Therefore, \( A_j(Q) = KB_j(Q) \) for all \( j < d \).

If on the other hand \( \text{wt}(E) \geq d \), let us define the matrix \( \mathcal{C} \) having the entries \( C_{ij} = \langle i_Q|E|j_Q\rangle \).

Note that \( \mathcal{C} \) is a Hermitian matrix of size \( K \times K \). Then

\[ \text{Tr}(EQ)\text{Tr}(E^\dagger Q) = [\text{Tr}(\mathcal{C})]^2, \]

\[ \text{Tr}(EQE^\dagger Q) = \text{Tr}(\mathcal{C}^2). \] 

(B.10)

Consider the diagonalization of \( \mathcal{C} \). By Jensen’s inequality, its eigenvalues must fulfill

\[ \left( \sum_{i=1}^{K} \lambda_i \right)^2 \leq K \sum_{i=1}^{K} \lambda_i^2, \] 

(B.11)

from which the inequality \( A_j(Q) \leq KB_j(Q) \) follows.

Let us now show that a projector \( Q \) of rank \( K \) is a QECC of distance \( d \) if and only if \( A_j(Q) = KB_j(Q) \) for all \( j < d \). This can be seen in the following way:

\[ '\Rightarrow': \text{Use the definition of QECC, equation (6).} \]

\[ '\Leftarrow': \text{Note that in order to obtain } A_j(Q) = KB_j(Q), \text{ there must be equality in equation (B.5) for all } E \text{ with } \text{wt}(E) = j. \text{ Thus, also equality in equation (B.10) is required. However, this is only possible if all eigenvalues } \lambda_i \text{ of } \mathcal{C} \text{ are equal. Then, } \mathcal{C} \text{ is diagonal in any basis, and we can write} \]

\[ \langle i_Q|E|j_Q\rangle = \delta_{ij}C(E). \] 

(B.12)

Because above equation must hold for all errors \( E \) of weight less than \( d \), we obtain equation (6) defining a quantum error correcting code:

\[ \langle i_Q|E|j_Q\rangle = \delta_{ij}C(E), \] 

for all \( E \) with \( \text{wt}(E) < d \). This ends the proof.
Appendix C. Relating the unitary enumerators to the Shor–Laflamme enumerators

Let us relate the unitary enumerators to the Shor–Laflamme enumerators by means of a polynomial transform.

\[
A'_{\text{MN}}(x, y) = \sum_{m=0}^{n} \sum_{j=0}^{m} \binom{n-j}{n-m} A_j(M, N)D^{-m} x^{n-m} y^m
\]

\[
= \sum_{j=0}^{n} \sum_{m=0}^{n} \binom{n-j}{n-m} A_j(M, N) x^{n-m} (y/D)^m y^m
\]

\[
= \sum_{j=0}^{n} \sum_{m=0}^{n} \binom{n-j}{n-m} A_j(M, N) x^{n-m} (y/D)^m (y/D)^j
\]

\[
= \sum_{j=0}^{n} \sum_{m=0}^{n-j} \binom{n-j}{n-j-m} A_j(M, N) x^{n-j-m} (y/D)^m (y/D)^j
\]

\[
= \sum_{j=0}^{n} A_j(M, N)(x+y/D)^{n-j}(y/D)^j
\]

\[
= A_{\text{MN}}(x + \frac{y}{D}, y/D).
\]

(C.1)

In an analogous fashion (replace \(A'_{\text{m}}\) by \(B'_{\text{m}}\), and \(A_j\) by \(B_j\)), one obtains

\[
B'_{\text{MN}}(x, y) = B_{\text{MN}}(x + \frac{y}{D}, y/D).
\]

(C.2)

Appendix D. Krawtchouk polynomials

The Krawtchouk (also Kravchuk) polynomials can be seen as a generalization of the binomial coefficients. They are, for \(n, k \in \mathbb{N}_0\) and \(n - k \geq 0\), defined as

\[
K_m(k; n) = \sum_{\alpha} (-1)^{\alpha} \binom{n-k}{m-\alpha} \binom{k}{\alpha}.
\]

(D.1)

If \(m < 0\), \(K_m(k; n) = 0\). The generating function of the Krawtchouk polynomial is

\[
\sum_m K_m(k; n)z^m = (1 + z)^{n-k}(1 - z)^k.
\]

(D.2)

In this work, we need a closely related expression,

\[
\sum_m K_m(k; n)x^{n-m}y^m = (x+y)^{n-k}(x-y)^k.
\]

(D.3)

That above equation holds, can be seen in the following way.

\[12\] See p 42 in [22] or chapter 5, section 7 in [36].
\[(x + y)^{n-k}(x - y)^k = \sum_{\alpha} \binom{n-k}{\alpha} x^{n-k-\alpha} y^\alpha \sum_{\beta} \binom{k}{\beta} x^{k-\beta} y^\beta (-1)^\beta \]
\[= \sum_{\alpha} \sum_{\beta} \binom{n-k}{\alpha} \binom{k}{\beta} x^{n-(\alpha+\beta)} y^{(\alpha+\beta)} (-1)^\beta \]
\[= \sum_m \left[ \sum_{\beta} \binom{n-k}{m-\beta} \binom{k}{\beta} (-1)^\beta \right] x^{n-m} y^m \]
\[= \sum_m K_m(k; n) x^{n-m} y^m, \quad \text{(D.4)} \]

where we set \(m = \alpha + \beta\) in the third line. Of course, setting \(x = 1\) recovers equation (D.2).

We will also need the Krawtchouk-like polynomial
\[\tilde{K}_m(k; n, \gamma, \delta) = \sum\binom{n-k}{m-\alpha} \binom{k}{\delta} x^{n-(\alpha+\beta)} y^{\gamma(\alpha+\beta)} (-1)^\beta \]
which are the coefficients of
\[(\gamma x + \delta y)^{n-k}(x - y)^k = \sum_{\alpha} \binom{n-k}{\alpha} \binom{k}{\beta} x^{n-(\alpha+\beta)} y^{\gamma(\alpha+\beta)} (-1)^\beta \]
\[= \sum_{\alpha} \sum_{\beta} \binom{n-k}{\alpha} \binom{k}{\beta} x^{n-(\alpha+\beta)} y^{\gamma(\alpha+\beta)} (-1)^\beta \]
\[= \sum_m \left[ \sum_{\beta} \binom{n-k}{m-\beta} \binom{k}{\beta} (-1)^\beta \gamma^{(n-k)-(m-\alpha)} \delta^{(m-\alpha)} \right] x^{n-m} y^m \]
\[= \sum_m \tilde{K}_m(k; n, \gamma, \delta) x^{n-m} y^m, \quad \text{(D.5)} \]

where we set \(m = \alpha + \beta\) in the second last line.

**Appendix E. The shadow enumerator in terms of the unitary enumerator**

Let us now transform the shadow enumerator into the unitary enumerator:
\[S_{MN}(x, y) = \sum_{m=0}^n S_m x^{n-m} y^m \]
\[= \sum_{m=0}^n \sum_{k=0}^n K_{n-m}(k; n) A'_{k}(M, N) x^{n-m} y^m \]
\[= \sum_{k=0}^n A'_{k}(M, N) \left[ \sum_{m=0}^n K_{n-m}(k; n) x^{n-m} y^m \right] \]
\[= \sum_{k=0}^n A'_{k}(M, N) \left[ \sum_{m'=0}^n K_{m'}(k; n) x^{m'} y^{n-m'} \right] \]
\[= \sum_{k=0}^n A'_{k}(M, N) y + x)^{n-k} (y - x)^k \]
\[= A'_{MN}(x + y, y - x). \quad \text{(E.1)} \]
Above, the second last equality follows from equation (D.3).

Appendix F. Linear programming bound

For completeness, we provide the linear programming bound that was first established by [15, 17]. Given the parameters \((n, K, d)\) of a hypothetical QECC, one can formulate a linear program to find possible enumerators which satisfy all the relations derived, namely equations (16) and (17), as well as the quantum MacWilliams identity (theorem 2) and the quantum shadow identity (theorem 3). If no valid weights \(A_j(Q)\) can be found, a code with the proposed parameters cannot exist. This provides a method to prove the non-existence of certain hypothetical states and QECC; on the other hand, the existence of a valid enumerator however does not imply the existence of a corresponding code.

**Theorem F.1 (LP bound for general QECC\(^{13}\)).** If a \(((n, K, d))_D\) exists, then there is a solution to the following set of linear equations and inequalities:

\[
\begin{align*}
KB_0 &= A_0 = K^2 \\
KB_i &= A_i \geq 0 & (i < d) \\
KB_i \geq A_i \geq 0 & (i \geq d) \\
B_i &= D^{-n} \sum_{0 \leq k \leq n} \tilde{K}_i(k; n, 1, D^2 - 1)A_k \\
S_i &= D^{-n} \sum_{0 \leq k \leq n} (-1)^k \tilde{K}_i(k; n, D - 1, D + 1)A_k \\
S_i &\geq 0,
\end{align*}
\]

where Krawtchouk-like polynomial \(\tilde{K}_i(k; n, \gamma, \delta)\) is given by equation (D.5). For pure codes, the second constraint above is strengthened to the equality

\[
KB_i = A_i = 0 & \quad (0 < i < d).
\]

For qubit stabilizer codes, one additionally has that either one of the two below conditions is satisfied

\[
\sum_{i \text{ even}} A_i = \begin{cases} 
2^{n-\log_2(K)-1} & \text{ (type I codes)} \\
2^{n-\log_2(K)} & \text{ (type II codes)}
\end{cases}
\]

For self-dual codes (where \(K = 1\)), additionally \(S_{n-j} = 0\) holds for all odd \(j\).

Appendix G. An overview on the identities between the weight enumerator polynomials

In table G.1, we summarize the known relations between the weight enumerator polynomials.

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\(^{13}\) For \(D = 2\), this was stated in theorem 21 in [23], in theorems 10 and 12 in [17], and on p 383 in [22].
Table G1. An overview on the identities between the weight enumerator polynomials.

| Identity Type                        | Identity                                                                 |
|--------------------------------------|--------------------------------------------------------------------------|
| Shor–Laflamme and unitary enum.:     | $A_{MN}(x,y) = A_{MN}(x + \frac{y}{D}, \frac{x}{D})$                    |
|                                      | $B'_{MN}(x,y) = B_{MN}(x + \frac{y}{D}, \frac{x}{D})$                    |
|                                      | $A'_{MN}(x,y) = B'_{MN}(y,x)$                                            |
| MacWilliams identity.:               | $A_{MN}(x,y) = B_{MN}\left(\frac{(D-1)x+1}{D}, \frac{x-y}{D}\right)$    |
|                                      | $B_{MN}(x,y) = A_{MN}\left(\frac{(D-1)y+1}{D}, \frac{x-y}{D}\right)$    |
| Shadow identity.:                    | $S_{MN}(x,y) = A_{MN}'\left(\frac{(D-1)x+1}{D}, \frac{x-y}{D}\right)$   |
|                                      | $S_{MN}(x,y) = A_{MN}'(x+y, y-x)$                                        |

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