Robust Kantorovich’s theorem on Newton’s method
under majorant condition in Riemannian Manifolds

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Abstract

A robust affine invariant version of Kantorovich’s theorem on Newton’s method, for finding
a zero of a differentiable vector field defined on a complete Riemannian manifold, is presented in
this paper. In the analysis presented, the classical Lipschitz condition is relaxed by using a gen-
eral majorant function, which allow to establish existence and local uniqueness of the solution
as well as unifying previously results pertaining Newton’s method. The most important in our
analysis is the robustness, namely, is given a prescribed ball, around the point satisfying Kan-
torovich’s assumptions, ensuring convergence of the method for any starting point in this ball.
Moreover, bounds for $Q$-quadratic convergence of the method which depend on the majorant
function is obtained.

Keywords: Newton’s method, robust Kantorovich’s theorem, majorant function, vector field,
Riemannian manifold

1 Introduction

Extension of concepts and techniques as well as methods of Mathematical Programming from the
Euclidean space to Riemannian setting it is natural and has been done frequently before; see,
e.g., [1, 2, 13, 27, 38, 44]. The motivation of this extensions, which in general is nontrivial,
is either of purely theoretical nature or aims at obtaining efficient algorithms; see, e.g., [1, 2, 12,
21, 32, 27, 28, 38, 44]. Indeed, many optimization problems are naturally posed on Riemannian
manifolds, which has a specific underlying geometric and algebraic structure that could be exploited
to greatly reduce the cost of obtaining the solutions. For instance, in order to take advantage of
the Riemannian geometric structure, it is suitable to treat some constrained optimization problems
as one of finding the zeros of a gradient vector field on a Riemannian manifolds rather than use the
method of Lagrange multipliers or projection idea for solving the problem; see [1, 2, 27, 38, 44]. In
this case, constrained optimization problems can be seen as unconstrained one from the Riemannian
geometry viewpoint. Besides, the Riemannian geometry allows to induce new research directions
so as to produce competitive algorithms; see [12, 21, 32, 38]. In this paper, instead of considering
the problem of finding the zero of the gradient field on a Riemannian manifolds, let us consider the
more general problem of finding a zeros of a vector field defined on a Riemannian manifold.

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On the other hand, the Newton’s method and its variant are powerful tools for finding a zero of nonlinear function in real or complex Banach space. Besides its practical applications, Newton’s method is also a powerful theoretical tool having a wide range of applications in pure mathematics; see [8, 19, 22, 29, 30, 43]. Therefore, a couple of papers have dealt with the issue of generalization of Newton’s method and its variant from Euclidean to Riemannian setting in order to go further in the study of the convergence properties of this method. Early works dealing with the generalization of Newton’s methods to Riemannian setting include [10, 12, 18, 33, 36, 39]. Actually, the generalization of Newton’s method to Riemannian setting has been done with several different purposes, including the purpose of finding a zeros of a gradient vector field or, more generally, with the purpose of finding a zero of a differentiable vector field; see [1, 2, 5, 6, 7, 9, 10, 12, 14, 15, 24, 25, 26, 27, 28, 31, 35, 38, 40, 41, 42, 46] and the references therein.

Properties of convergence of Newton’s method have been extensively studied on several papers due to the important role that it plays in the development of numerical methods for finding a zero of a differentiable vector field defined on a complete Riemannian manifolds. In 2002 Ferreira and Svaiter in [15] extended the Kantorovich’s theorem on the Newton’s method to Riemannian setting using a new technique which simplifies the analysis and proof of this theorem. It is worth mention that, in a similar spirit, an extensions of the famous Smale’s theory; see [37], to analytic vector fields on analytic Riemannian manifolds were done in 2003 by Dedieu et al. in [9]. The basic idea of [15] was to combine a formulation of Kantorovich’s theorem by means of quadratic majorant functions, see [45] for more general majorant functions, with the definitions of good regions for the Newton’s method. In these regions, the majorant function bounds the vector fields which the zero is to be found, and the behavior of the Newton’s iteration in these regions is estimated using iterations associated to the majorant function. Moreover, as a whole, the union of all these regions is invariant under Newton’s iteration. Afterward, this technique was successfully employed for proving generalized versions of Kantorovich’s theorem in Riemannian setting. Inspired by previous work of Zabrejko and Nguyen in [45] on Kantorovich’s majorant method, a radial parametrization of a Lipschitz-type and L-average Lipschitz affine invariant majorante conditions were introduced in Riemannian setting by Alvarez et al. in [3] and Li and Wang in [25], respectively, in order to establish existence and local uniqueness of the solution as well as unifying previously convergence criterion of Newton’s method.

In the present paper, we will use the technique introduced in [15], see also [17], to present a robust affine invariant version of the Kantorovich’s theorem on the Newton’s method finding a zeros of a differentiable vector field defined on a complete Riemannian manifold. In our analysis, the classical Lipschitz condition is relaxed using a general majorant function. The analysis presented provides a clear relationship between the majorant function and the vector field under consideration. However, the most important in our analysis is the robustness, namely, we give a prescribed ball, around the point satisfying the Kantorovich’s assumptions, ensuring convergence of the method for any starting point in this ball. Moreover, we establish bounds for $Q$-quadratic convergence of the method which depend on the majorant function. Also, as in [3] and [25], this analysis allows us establish existence and local uniqueness of the solution as well as unifying previously results pertaining Newton’s method.

The organization of the paper is as follows. In Section 1.1 some notations and one basic results used in the paper are presented. In Section 2 the main result is stated, namely, the robust affine invariant Kantorovich’s theorem for Newton’s method and in Section 2 the affine invariant version, which is used for proving the robust one is stated and proved. In Section 4 we prove the main theorem. In Section 4 three special case of the main theorem is presented. Some final remarks are made in Section 6.
1.1 Notation and auxiliary results

In this section we recall some notations, definitions and basic properties of Riemannian manifolds used throughout the paper, they can be found, for example, in [11] and [23].

Throughout the paper, \(\mathcal{M}\) is a smooth manifold and \(C^1(\mathcal{M})\) is the class of all continuously differentiable functions on \(\mathcal{M}\). The space of vector fields \(C^r(\mathcal{M})\) on \(\mathcal{M}\) is denoted by \(X^r(\mathcal{M})\), by \(T_p\mathcal{M}\) we denote the tangent space of \(\mathcal{M}\) at \(p\) and by \(TM = \bigcup_{x \in \mathcal{M}} T_x\mathcal{M}\) the tangent bundle of \(\mathcal{M}\). Let \(\mathcal{M}\) be endowed with a Riemannian metric \((\cdot, \cdot)\), with corresponding norm denoted by \(||\cdot||\), so that \(\mathcal{M}\) is now a Riemannian manifold. Let us recall that the metric can be used to define the length of a piecewise \(C^1\) curve \(\zeta : [a, b] \to \mathcal{M}\) by

\[
\ell[\zeta, a, b] := \int_a^b ||\zeta'(t)|| dt.
\]

Minimizing this length functional over the set of all such curves we obtain a distance \(d(p, q)\), which induces the original topology on \(\mathcal{M}\). The open and closed balls of radius \(r > 0\) centered at \(p\) are compact. The exponential map \(\exp\) is the geodesic defined by its position and velocity \(v\) at \(p\). The Hopf-Rinow’s theorem asserts that if this is the case then any pair of points, say \(p\) and \(q\), in \(\mathcal{M}\) can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, \((\mathcal{M}, d)\) is a complete metric space and bounded and closed subsets are compact. The exponential map at \(p\), \(\exp_p : T_p\mathcal{M} \to \mathcal{M}\), is defined by \(\exp_pv = \zeta_v(1)\), where \(\zeta_v\) is the geodesic defined by its position \(p\) and velocity \(v\) at \(p\) and \(\zeta_v(t) = \exp_ptv\) for any value of \(t\).

Let \(X \in C^1(\mathcal{M})\). The covariant derivative of \(X\) determined by the Levi-Civita connection \(\nabla\) defines at each \(p \in \mathcal{M}\) a linear map \(\nabla X(p) : T_p\mathcal{M} \to T_p\mathcal{M}\) given by

\[
\nabla X(p)v := \nabla_Y X(p),
\]

where \(Y\) is a vector field such that \(Y(p) = v\).
**Definition 1.** Let $Y_1, \ldots, Y_n$ be vector fields on $\mathcal{M}$. Then, the $n$-th covariant derivative of $X$ with respect to $Y_1, \ldots, Y_n$ is defined inductively by

$$\nabla^2_{\{Y_1, Y_2\}} X := \nabla_{Y_2} \nabla_{Y_1} X, \quad \nabla^n_{\{Y_1\}_{i=1}^n} X := \nabla_{Y_n} (\nabla_{Y_{n-1}} \cdots \nabla_{Y_1} X).$$

**Definition 2.** Let $p \in \mathcal{M}$. Then, the $n$-th covariant derivative of $X$ at $p$ is the $n$-th multilinear map $\nabla^n X(p) : T_p\mathcal{M} \times \cdots \times T_p\mathcal{M} \to T_p\mathcal{M}$ defined by

$$\nabla^n X(p)(v_1, \ldots, v_n) := \nabla^n_{\{Y_1\}_{i=1}^n} X(p),$$

where $Y_1, \ldots, Y_n$ are vector fields on $\mathcal{M}$ such that $Y_1(p) = v_1, \ldots, Y_n(p) = v_n$.

We remark that Definition 2 only depends on the $n$-tuple of vectors $(v_1, \ldots, v_n)$ since the covariant derivative is tensorial in each vector field $Y_i$.

**Definition 3.** Let $p \in \mathcal{M}$. The norm of an $n$-th multilinear map $A : T_p\mathcal{M} \times \cdots \times T_p\mathcal{M} \to T_p\mathcal{M}$ is defined by

$$\|A\| = \sup \{ \|A(v_1, \ldots, v_n)\| : v_1, \ldots, v_n \in T_p\mathcal{M}, \|v_i\| = 1, i = 1, \ldots, n \}.$$ 

In particular, the norm of the $n$-th covariant derivative of $X$ at $p$ is given by

$$\|\nabla^n X(p)\| = \sup \{ \|\nabla^n X(p)(v_1, \ldots, v_n)\| : v_1, \ldots, v_n \in T_p\mathcal{M}, \|v_i\| = 1, i = 1, \ldots, n \}.$$

Now, the Fundamental Theorem of Calculus for a vector field $X$ becomes

**Lemma 1.** Let $\Omega$ be an open subset of $\mathcal{M}$, $X$ a $C^1$ vector field defined on $\Omega$ and $\zeta : [a, b] \to \Omega$ a $C^1$ curve. Then

$$P_{\zeta,t,a} X(\zeta(t)) = X(\zeta(a)) + \int_a^t P_{\zeta,s,a} \nabla X(\zeta(s)) \zeta'(s) \, ds, \quad t \in [a, b].$$

Proof. See [15].

**Lemma 2.** Let $\Omega$ be an open subset of $\mathcal{M}$, $X$ a $C^2$ vector field defined on $\Omega$ and $\zeta : [a, b] \to \Omega$ a $C^1$ curve. Then for all $Y \in \mathcal{X}(\mathcal{M})$ we have

$$P_{\zeta,t,a} \nabla X(\zeta(t)) Y(\zeta(t)) = \nabla X(\zeta(a)) Y(\zeta(a)) + \int_a^t P_{\zeta,s,a} \nabla^2 X(\zeta(s)) (Y(\zeta(s)), \zeta'(s)) \, ds, \quad t \in [a, b].$$

Proof. See [24].

**Lemma 3** (Banach’s Lemma). Let $B$ be a linear operator and let $I_p$ be the identity operator in $T_p\mathcal{M}$. If $\|B - I_p\| < 1$ then $B$ is nonsingular and $\|B^{-1}\| \leq 1/(1 - \|B - I_p\|)$.

Proof. See, for example, [37].

We also need the following elementary convex analysis result, see [20]:

**Proposition 1.** Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}$ be convex. For any $s_0 \in \text{int}(I)$, the left derivative there exist (in $\mathbb{R}$)

$$D^{-} \varphi(s_0) := \lim_{s \to s_0^-} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s} = \sup_{s < s_0} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s}.$$

Moreover, if $s, t, r \in I$, $s < r$, and $s \leq t \leq r$ then $\varphi(t) - \varphi(s) \leq [\varphi(r) - \varphi(s)] [(t - s)/(r - s)].$
2 Robust Kantorovich’s Theorem on Newton’s Method

Our goal is to state and prove a robust affine invariant version of Kantorovich’s Theorem on Newton’s Method for finding a zero of a vector field:

\[ X(p) = 0, \]

(4)

where \( \mathcal{M} \) is a Riemannian manifold, \( \Omega \subseteq \mathcal{M} \) an open set and \( X : \Omega \to T\mathcal{M} \) a continuously differentiable vector field. The most important in our analysis is the robustness, namely, we give a prescribed ball, around the point satisfying the Kantorovich’s assumptions, ensuring convergence of the method for any starting point in this ball. Moreover, we establish bounds for \( Q \)-quadratic convergence of the method which depend on the majorant function. Also, as in \([8]\) and \([25]\), this analysis allows us establish existence and local uniqueness of the solution. For state the theorem we need some definitions. We begin with the following definition which was introduced in \([3]\).

**Definition 4.** Let \( R > 0, n \in \mathbb{N}\setminus\{0\} \), \( p_0 \in \mathcal{M} \) and \( \mathcal{G}_n(p_0, R) \) be the class of all piecewise geodesic curves \( \xi : [0, T] \to \mathcal{M} \) for some \( T > 0 \) which satisfy the following conditions:

1. \( \xi(0) = p_0 \) and the length of \( \xi \) is no greater than \( R \);
2. there exist \( c_0, c_1, \ldots, c_n \in [0, T] \) with \( c_0 = 0 \leq c_1 \leq \ldots \leq c_n = T \) such that \( \xi|_{[c_0, c_1]} \), \( \ldots \xi|_{[c_{n-2}, c_{n-1}]} \) are \( n-1 \) minimizing geodesics and \( \xi|_{[c_{n-1}, c_n]} \) is a geodesic.

**Remark 1.** Since \( \mathcal{M} \) is complete, \( \mathcal{G}_n(p_0, R) \) is nonempty. Moreover, \( \mathcal{G}_n(p_0, R) \subseteq \mathcal{G}_{n+1}(p_0, R) \) for all \( n \in \mathbb{N}\setminus\{0\} \). Note that, in Definition \( \[4\] \) \( \mathcal{G}_1(p_0, R) \) is the class of all minimizing geodesic curves \( \xi : [0, T] \to \mathcal{M} \) with \( \xi(0) = p_0 \) and the length of \( \xi \) is no greater than \( R \).

We also need the following definition which was equivalently stated in (3.7) of \([3]\), for \( \mathcal{G}_2(p_0, R) \).

**Definition 5.** Let \( \Omega \subseteq \mathcal{M} \) an open set and \( R > 0 \) a scalar constant. A continuously differentiable \( f : [0, R] \to \mathbb{R} \) is said to be a majorant function at a point \( p_0 \in \Omega \) for a continuously differentiable vector field \( X : \Omega \to T\mathcal{M} \) with respect to \( \mathcal{G}_n(p_0, R) \) if \( \nabla X(p_0) \) is nonsingular, \( B(p_0, R) \subseteq \Omega \) and

\[
\left\| \nabla X(p_0)^{-1} [P_{\xi,b,0} \nabla X(\xi(b)) P_{\xi,a,b} - P_{\xi,a,0} \nabla X(\xi(a))] \right\| \leq f'(\ell(\xi, 0, b)) - f'(\ell(\xi, 0, a)),
\]

(5)

for all \( \xi \in \mathcal{G}_n(p_0, R) \) with \( a, b \in \text{dom}(\xi) \) and \( 0 \leq a \leq b \). Moreover, \( f \) satisfies the following conditions:

**h1.** \( f(0) > 0 \) and \( f'(0) = -1 \);

**h2.** \( f' \) is convex and strictly increasing;

**h3.** \( f(t) = 0 \) for some \( t \in (0, R) \).

We also need of the following condition on the majorant condition \( f \) which will be considered to hold only when explicitly stated

**h4.** \( f(t) < 0 \) for some \( t \in (0, R) \).

**Remark 2.** Since \( f(0) > 0 \) and \( f \) is continuous then condition \( \text{h4} \) implies condition \( \text{h3} \).

The statement of our main result is:
Theorem 1. Let $\mathcal{M}$ be a Riemannian manifold, $\Omega \subseteq \mathcal{M}$ an open set and $\bar{\Omega}$ its closure, $X : \bar{\Omega} \to T_{\bar{\Omega}} \mathcal{M}$ a continuous vector field and continuously differentiable on $\Omega$, $R > 0$ a scalar constant and $f : [0, R) \to \mathbb{R}$ a continuously differentiable function. Take $p_0 \in \Omega$. Suppose that $\nabla X(p_0)$ is nonsingular and $f$ is a majorant function for $X$ at $p_0$ with respect to $\mathcal{G}_3(p_0, R)$ satisfying $h_4$ and the inequality

$$\|\nabla X(p_0)^{-1} X(p_0)\| \leq f(0).$$

Define $\Gamma := \sup \{-f(t) : t \in [0, R)\}$. Let $0 \leq \rho < \Gamma / 2$ and $g : [0, R - \rho) \to \mathbb{R}$,

$$g(t) := \frac{1}{|f'(\rho)|} [f(t + \rho) + 2\rho].$$

Then $g$ has a smallest zero $t_{*,\rho} \in (0, R - \rho)$, the sequences generated by Newton’s Method for solving the equation $X(p) = 0$ and the equation $g(t) = 0$, with starting point $q_0$, for any $q_0 \in B[p_0, \rho]$, and $t_0 = 0$, respectively,

$$q_{k+1} = \exp_{q_k} (-\nabla X(q_k)^{-1} X(q_k)), \quad t_{k+1} = t_k - \frac{g(t_k)}{g'(t_k)}, \quad k = 0, 1, \ldots$$

are well defined, $\{q_k\}$ is contained in $B(q_0, t_{*,\rho})$, $\{t_k\}$ is strictly increasing, is contained in $[0, t_{*,\rho})$ and converges to $t_{*,\rho}$. Moreover, $\{q_k\}$ and $\{t_k\}$ satisfy the inequalities

$$d(q_k, q_{k+1}) \leq t_{k+1} - t_k, \quad k = 0, 1, \ldots,$$

$$d(q_k, q_{k+1}) \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} d(q_{k-1}, q_k)^2 \leq \frac{D - g'(t_{*,\rho})}{-2g'(t_{*,\rho})} d(q_{k-1}, q_k)^2, \quad k = 1, 2, \ldots$$

and $\{q_k\}$ converges to $p_* \in B[q_0, t_{*,\rho}]$ such that $X(p_*) = 0$. Furthermore, $\{q_k\}$ and $\{t_k\}$ satisfy the inequalities

$$d(q_k, p_*) \leq t_{*,\rho} - t_k, \quad t_{*,\rho} - t_{k+1} \leq \frac{1}{2} (t_{*,\rho} - t_k), \quad k = 0, 1, \ldots,$$

the convergence of $\{q_k\}$ and $\{t_k\}$ to $p_*$ and $t_{*,\rho}$, respectively, are $Q$-quadratic as follow

$$\limsup_{k \to \infty} \frac{d(p_{k+1}, p_*)}{d(q_k, p_*)^2} \leq \frac{D - g'(t_{*,\rho})}{-2g'(t_{*,\rho})}, \quad t_{*,\rho} - t_{k+1} \leq \frac{D - g'(t_{*,\rho})}{-2g'(t_{*,\rho})} (t_{*,\rho} - t_k)^2, \quad k = 0, 1, \ldots$$

and $p_*$ is the unique singularity of $X$ in $B(p_0, \tau)$, where $\tau \geq t_*$ is defined as

$$\tau := \sup \{t \in [t_*, R) : f(t) \leq 0\}.$$

To prove the above theorem we need some previous results. First, in the next section, we prove a particular instance of this theorem, and then, in the Section 3.3 we prove Theorem $\text{[1]}$

3 Kantorovich’s Theorem on Newton’s Method

In this section we will prove an affine invariant version of Kantorovich’s Theorem on Newton’s Method, it is a particular instance of Theorem $\text{[1]}$ namely, the case $\rho = 0$. We will use this theorem for proving Theorem $\text{[1]}$. The main results of this section are the bounds, depending on the majorant function, for the $Q$-quadratic convergence of the Newton’s Method, which gives an additional contribution for improving the results of Alvarez et al. in $\text{[3]}$, Ferreira and Svaiter in $\text{[15]}$ and Li and Wang in $\text{[25]}$. 


\textbf{Theorem 2.} Let $\mathcal{M}$ be a Riemannian manifold, $\Omega \subseteq \mathcal{M}$ an open set and $\bar{\Omega}$ its closure, $X : \Omega \to T\mathcal{M}$ a continuous vector field and continuously differentiable on $\Omega$, $R > 0$ a scalar constant and $f : [0,R) \to \mathbb{R}$ a continuously differentiable function. Take $p_0 \in \Omega$. Suppose that $\nabla X(p_0)$ is nonsingular and $f$ is a majorant function for $X$ at $p_0$ with respect to $\mathcal{G}_2(p_0,R)$ satisfying the inequality
\begin{equation}
\|\nabla X(p_0)^{-1}X(p_0)\| \leq f(0).
\end{equation}
Then $f$ has a smallest zero $t_* \in (0,R)$, the sequences generated by Newton’s Method for solving the equations $X(p) = 0$ and $f(t) = 0$, with starting point $p_0$ and $t_0 = 0$, respectively,
\begin{equation}
p_{k+1} = \exp_{p_k} \left(-\nabla X(p_k)^{-1}X(p_k)\right), \quad t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)}, \quad k = 0,1,\ldots
\end{equation}
are well defined, $\{p_k\}$ is contained in $B(p_0,t_*)$, $\{t_k\}$ is strictly increasing, is contained in $[0,t_*)$ and converge to $t_*$ and satisfy the inequalities
\begin{equation}
d(p_{k+1},p_k) \leq t_{k+1} - t_k, \quad d(p_{k+1},p_k) \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} d(p_k,p_{k-1})^2,
\end{equation}
for all $k = 0,1,\ldots$, and $k = 1,2,\ldots$, respectively. Moreover, $\{p_k\}$ converge to $p_* \in B[p_0,t_*]$ such that $X(p_*) = 0$,
\begin{equation}
d(p_*,p_k) \leq t_* - t_k, \quad t_* - t_{k+1} \leq \frac{1}{2}(t_* - t_k), \quad k = 0,1,\ldots
\end{equation}
and, therefore, $\{t_k\}$ converges $Q$-linearly to $t_*$ and $\{p_k\}$ converge $R$-linearly to $p_*$. If, additionally, $f$ satisfies $h4$ then the following inequalities hold:
\begin{equation}
d(p_{k+1},p_k) \leq \frac{D-f'(t_*)}{2f'(t_*)}d(p_k,p_{k-1})^2, \quad t_{k+1} - t_k \leq \frac{D-f'(t_*)}{2f'(t_*)}(t_k - t_{k-1})^2, \quad k = 1,2,\ldots,
\end{equation}
and, as a consequence, $\{p_k\}$ and $\{t_k\}$ converge $Q$-quadratically to $p_*$ and $t_*$, respectively, as follow
\begin{equation}
\limsup_{k \to \infty} \frac{d(p_*,p_{k+1})}{d(p_*,p_k)^2} \leq \frac{D-f'(t_*)}{2f'(t_*)}, \quad t_* - t_{k+1} \leq \frac{D-f'(t_*)}{2f'(t_*)}(t_* - t_k)^2, \quad k = 0,1,\ldots,
\end{equation}
and $p_*$ is the unique singularity of $X$ in $B(p_0,\bar{\tau})$, where $\bar{\tau} \geq t_*$ is defined as
\begin{equation}
\bar{\tau} := \sup\{t \in [t_*,R) : f(t) \leq 0\}.
\end{equation}

Henceforward we assume that all assumptions in above theorem hold. In this section, we will prove all the statements in Theorem 2 regarding to the majorant function and the real sequence $\{t_k\}$ associated. The main relationships between the majorant function and the vector field will be also established.

\subsection{The majorant function}

In this subsection we will study the majorant function $f$ and prove all results regarding only the real sequence $\{t_k\}$ defined by Newton’s method applied to the majorant function $f$. Define
\begin{equation}
\bar{t} := \sup\{t \in [0,R) : f'(t) < 0\}.
\end{equation}
Proposition 2. The majorant function $f$ has a smallest root $t_* \in (0, R)$, is strictly convex and
\[ f(t) > 0, \quad f'(t) < 0, \quad t < t - f(t)/f'(t) < t_*, \quad \forall t \in [0, t^*). \] (20)
Moreover, $f'(t_*) \leq 0$ and
\[ f'(t_*) < 0 \iff \exists t \in (t_*, R); \quad f(t) < 0. \] (21)
If, additionally, $f$ satisfies condition h4 then the following statements hold:
i) $f'(t) < 0$ for any $t \in [0, \bar{t})$;
ii) $0 < t_* < \bar{t} \leq R$;
iii) $0 < \Gamma < \bar{t}$, where $\Gamma := -\lim_{t \to \bar{t}^{-}} f(t)$.
iv) If $0 \leq \rho < \Gamma/2$ then $\rho < \bar{t}/2 < \bar{t}$ and $f'(\rho) < 0$.

Proof. See Propositions 2.3 and 5.2 of [17] and Proposition 3 of [15]. \hfill \square

In view of the second inequality in (20), Newton iteration is well defined in $[0, t_*)$. Let us call it $n_f : [0, t_*) \to \mathbb{R}$,
\[ n_f(t) := t - f(t)/f'(t). \] (22)

Proposition 3. Newton iteration $n_f$ maps $[0, t^*)$ into $[0, t^*)$ and there hold:
\[ t < n_f(t), \quad t_* - n_f(t) \leq \frac{1}{2}(t_* - t), \quad \forall t \in [0, t_*). \] (23)
If $f$ also satisfies (h4), i.e., $f'(t_*) < 0$, then
\[ t_* - n_f(t) \leq \frac{D^- f'(t_*)}{-2f'(t_*)}(t_* - t)^2, \quad \forall t \in [0, t^*). \] (24)

Proof. See Proposition 4 of [16]. \hfill \square

The next two results follow from above proposition.

Corollary 1. Take any $\tau_0 \in [0, t_*)$ and define, inductively, $\tau_{k+1} = n_f(\tau_k), k = 0, 1, \ldots$. The sequence \{\tau_k\} is well defined, is strictly increasing, is contained in $[0, t_*)$ and converges $Q$-linearly to $t_*$ as follows
\[ t_* - \tau_{k+1} \leq \frac{1}{2}(t_* - \tau_k), \quad k = 0, 1, \ldots \]
In particular, the definition (14) of \{t_k\} in Theorem 2 is equivalent to the following one
\[ t_0 = 0, \quad t_{k+1} = n_f(t_k), \quad k = 0, 1, \ldots \] (25)
and there holds

Corollary 2. The sequence \{t_k\} is well defined, is strictly increasing, is contained in $[0, t_*)$ and converges $Q$-linearly to $t_*$ as follows
\[ t_* - t_{k+1} \leq \frac{1}{2}(t_* - t_k), \quad k = 0, 1, \ldots \]
If $f$ also satisfies h4, then the following inequality holds
\[ t_{k+1} - t_k \leq \frac{D^- f'(t_k)}{-2f'(t_*)}(t_k - t_{k-1})^2, \quad k = 1, 2, \ldots , \] (26)
and, as a consequence, \{t_k\} converges $Q$-quadratically to $t_*$ as follow
\[ t_* - t_{k+1} \leq \frac{D^- f'(t_*)}{-2f'(t_*)}(t_* - t_k)^2, \quad k = 0, 1, \ldots , \] (27)
3.2 Relationship between the majorant function and the vector field

In this subsection we will establish the main relationship between the majorant function and the vector field necessary to prove Theorem 2.

**Proposition 4.** Let $\xi \in \mathcal{G}_2(p_0, R)$. If $\ell(\xi, 0, s) \leq t < \tilde{t}$ then $\nabla X(\xi(s))$ is nonsingular and the following inequality holds

$$\|\nabla X(\xi(s))^{-1}P_{\xi,0,s}\nabla X(p_0)\| \leq \frac{1}{|f'(\ell(\xi, 0, s))|} \leq \frac{1}{|f'(t)|}.$$ 

*Proof.* Using Definition 5 and Lemma 3, the proof follows the same pattern of Proposition 3.4 of [16], see also Lemma 4.2. of [3].

Newton iteration at a point happens to be a zero of the linearization at such a point. Therefore, we study the linearization error of the vector field and the associated majorant function. The formal definitions of these errors are:

**Definition 6.** Let $f : [0, R) \to \mathbb{R}$ be a continuously differentiable function. The linearization error of $f$ is defined by

$$e(a, b) := f(b) - [f(t) + f'(a)(b - a)], \quad \forall a, b \in [0, R). \tag{28}$$

**Definition 7.** Let $\mathcal{M}$ be a Riemannian manifold, $\Omega \subseteq \mathcal{M}$ an open set, $X : \Omega \to T\mathcal{M}$ a continuously differentiable vector field and $a, b \in [0, R)$. The linearization error of $X$ on a geodesic $\zeta : [a, b] \to \Omega$ is defined by

$$E(\zeta(a), \zeta(b)) := X(\zeta(b)) - P_{\zeta,a,b}[X(\zeta(a)) + (b - a)\nabla X(\zeta(a))\zeta'(a)]. \tag{29}$$

In the next result we compare linearization error of the vector field with the linearization error of the majorant function associated.

**Lemma 4.** Let $\xi \in \mathcal{G}_2(p_0, R)$ be a curve passing through $p = \xi(a)$ and $q = \xi(b)$ such that $\xi_{[a,b]}$ is a geodesic and $0 \leq a \leq b$. Take $0 \leq t < x < R$. If $\ell(\xi, 0, a) \leq t$ and $\ell(\xi, a, b) \leq x - t$, then

$$\|\nabla X(p_0)^{-1}P_{\xi,b,0}E(p, q)\| \leq e(t, x)\frac{\ell(x, a, b)^2}{(x - t)^2}.$$  

As a consequence, the following inequality holds: $\|\nabla X(p_0)^{-1}P_{\xi,b,0}E(p, q)\| \leq e(t, x)$.

*Proof.* Definition 7 with $\zeta = \xi_{[a,b]}$ and properties of parallel transport in 2 imply

$$E(p, q) = P_{\xi,a,b} \left[ P_{\xi,b,a} X(q) - X(p) - (b - a)\nabla X(p)\zeta'(a) \right].$$

Hence, using Lemma 4 and that $\zeta'(s) = P_{\xi,a,s}\zeta'(a)$, the last equality becomes

$$E(p, q) = P_{\xi,a,b} \int_a^b [P_{\xi,s,a}\nabla X(\xi(s))P_{\xi,a,s} - \nabla X(p)]\zeta'(a)ds,$$

which is equivalent to

$$\nabla X(p_0)^{-1}P_{\xi,b,0}E(p, q) = \int_a^b \nabla X(p_0)^{-1} [P_{\xi,s,0}\nabla X(\xi(s))P_{\xi,a,s} - P_{\xi,a,0}\nabla X(p)]\zeta'(a)ds.$$
Since \( \xi : [a, b] \to \mathcal{M} \) is a geodesic joining \( p \) and \( q \) we have \( \| \xi'(a) \| = \ell[p, a]/(b - a) \). Thus last equality implies

\[
\| \nabla X(p_0)^{-1} P_{\xi, s, 0} E(p, q) \| \leq \int_a^b \| \nabla X(p_0)^{-1} [P_{\xi, s, 0} \nabla X(\xi(s)) P_{\xi, a, s} - P_{\xi, a, 0} \nabla X(p)] \| \frac{\ell[\xi, a, b]}{b - a} \, ds. \tag{30}
\]

Because \( a \leq s \leq b \), using the assumptions \( \ell[\xi, 0, a] < t \) and \( \ell[\xi, a, b] \leq x - t \) we have

\[
\ell[\xi, 0, s] \leq \ell[\xi, 0, a] + \ell[\xi, a, b] \leq x - t < R,
\]

and as \( \xi : [0, s] \to \mathcal{M} \) is a piecewise geodesic curves joining the points \( p_0 \) to \( \xi(s) \) through \( p \), i.e., \( \xi \in \mathcal{G}_2(p_0, R) \), we may use the majorant condition in Definition 4 with \( b = s \) and \( q = \xi(s) \) together with inequality in (30) to conclude that

\[
\| \nabla X(p_0)^{-1} P_{\xi, s, 0} E(p, q) \| \leq \int_a^b \left[ f'(\ell[\xi, 0, s]) - f'(\ell[\xi, 0, a]) \right] \frac{\ell[\xi, a, b]}{b - a} \, ds.
\]

Using convexity of \( f' \), \( \ell[\xi, 0, a] \leq t \), \( \ell[\xi, a, b] \leq x - t \), \( x < R \) and Proposition 11 we have

\[
f'(\ell[\xi, 0, s]) - f'(\ell[\xi, 0, a]) = f'(\ell[\xi, 0, a] + \ell[\xi, a, s]) - f'(\ell[\xi, 0, a]) \\
\leq f'(t + \ell[\xi, a, s]) - f'(t) \\
= f'(t + \frac{s - a}{b - a} \ell[\xi, a, b]) - f'(t) \\
\leq f'(t + \frac{s - a}{b - a} (x - t)) - f'(t) \frac{\ell[\xi, a, b]}{x - t}.
\]

Therefore, combining two last inequality we obtain that

\[
\| \nabla X(p_0)^{-1} P_{\xi, s, 0} E(p, q) \| \leq \int_a^b \left[ f'(t + \frac{s - a}{b - a} (x - t)) - f'(t) \right] \frac{\ell[\xi, a, b]^2}{(x - t)(b - a)} \, ds.
\]

After performing the integral and some algebraic manipulations the above inequality becomes

\[
\| \nabla X(p_0)^{-1} P_{\xi, s, 0} E(p, q) \| \leq \left[ f(x) - f(t) - f'(t)(x - t) \right] \frac{\ell[\xi, a, b]^2}{(x - t)^2},
\]

which, Definition 5 implies the desired inequality. \( \square \)

Proposition 4 guarantees, in particular, that \( \nabla X(p) \) is nonsingular at \( p \in B(p_0, t_*) \) and, consequently, the Newton’s iteration is well defined in \( B(p_0, t_*) \). Let us call it \( N_X : B(p_0, t_*) \to \mathcal{M} \),

\[
N_X(p) := \exp_p(-\nabla X(p)^{-1} X(p)). \tag{31}
\]

One can apply a single Newton’s iteration on any \( p \in B(p_0, t_*) \) to obtain the point \( N_X(p) \) which may not is contained to \( B(p_0, t_*) \), or even may not in the domain of \( X \). Hence, this is enough to guarantee the well-definedness of only one iteration. To ensure that Newtonian iteration may be repeated indefinitely, we need some additional definitions and results. First, we define some subsets of \( B(p_0, t_*) \) in which, as we shall prove, Newton iteration (31) is “well behaved”:

\[
K(t) := \left\{ p \in \Omega : d(p_0, p) \leq t, \quad \| \nabla X(p)^{-1} X(p) \| \leq -\frac{f(t)}{f'(t)} \right\}, \quad t \in [0, t_*]. \tag{32}
\]

\[
K := \bigcup_{t \in [0, t_*]} K(t), \tag{33}
\]
In [82], $0 \leq t < t_\ast \leq \bar{t}$, hence using Proposition 2 and Proposition 3 we conclude that $f'(t) \neq 0$ and $\nabla X(p)$ is nonsingular in $B[p_0, \bar{t}] \subset B[p_0, t_\ast]$, respectively. Therefore the above definitions are consistent. It is worth point out that the above sets appeared for the first time in [15]; see also [16].

**Lemma 5.** For each $t \in [0, t_\ast)$ and each $p \in K(t)$ there hold:

i) $\|\nabla X(p)^{-1}X(p)\| \leq -\frac{f(t)}{f'(t)}$;

ii) $d(p_0, p) + \|\nabla X(p)^{-1}X(p)\| \leq n_f(t) < t_\ast$. As a consequence, $d(p_0, N_X(p)) \leq n_f(t) < t_\ast$.

iii) $\|\nabla (N_X(p))^{-1}X(N_X(p))\| \leq -\frac{f(n_f(t))}{f'(n_f(t))} \left[ \frac{\|\nabla X(p)^{-1}X(p)\|^2}{-f(t)/f'(t)} \right]$.

**Proof.** Let $t \in [0, t_\ast)$, $p \in K(t)$. Using definition of the set $K(t)$ in [82] the item i follows.

Using Proposition 3 and definition of $K(t)$ in [82] to obtain that $d(p_0, p) \leq t$ and $n_f(t) < t_\ast$, respectively. Hence, the proof of the first part of item ii follows by combination of two last inequalities with item i and definition of $n_f$ in [22]. For proving the second part of item ii use triangular inequality to obtain $d(p_0, N_X(p)) \leq d(p_0, p) + d(p, N_X(p))$, definition in [31] and then first part.

We are going to prove item iii. Let $\xi : [0, 2] \to M$ a piecewise geodesic curve obtained by concatenation of a minimizing geodesic $\xi_{[0, 1]}$ joining $p_0$ and $p$ and the geodesic curve $\xi_{[1, 2]}$ defined by

$$\xi(t) = \exp_p \left( (1 - t)\nabla X(p)^{-1}X(p) \right).$$

(34)

Note that $\xi \in G_2(p_0, R)$. From definition of the piecewise geodesic curve $\xi$ and definitions in [31] and [34] we have

$$\ell[\xi, 0, 2] = d(p_0, p) + \|\nabla X(p)^{-1}X(p)\|.$$

Since $\xi(2) = N_X(p)$, using last equality, first inequality in item ii and Proposition 3 by taking into account that the derivative $f'$ is increasing and negative in $[0, \bar{t})$, we conclude that $\nabla X(N_X(p))$ is nonsingular and there holds

$$\|\nabla (N_X(p))^{-1}P_{\xi,0,2}\nabla X(p_0)\| \leq \frac{1}{|f'(d(p_0, p) + \|\nabla X(p)^{-1}X(p)\|)|} \leq \frac{1}{|f'(n_f(t))|}. \quad (35)$$

On the other hand, as $\ell[\xi, 1, 2] = \|\nabla X(p)^{-1}X(p)\|$, combining item i with definition of $n_f$ in [22] we obtain $\ell[\xi, 1, 2] \leq n_f(t) - t$. Since second part in item ii implies $d(p_0, N_X(p)) \leq n_f(t) < t_\ast$. Thus, we may apply Lemma 4 with $x = n_f(t)$ and $q = N_X(p)$ to conclude that

$$\|\nabla X(p_0)^{-1}P_{\xi,2,0}E(p, N_X(p))\| \leq d(t, n_f(t))\frac{\|\nabla X(p)^{-1}X(p)\|^2}{(n_f(t) - t)^2}. \quad (36)$$

We know that $N_X(p)$ belongs to the domain of $X$. Hence, Newton’s iterations in [31], linearization error in Definition 7 with $\zeta = \xi_{[1, 2]}$ and [34] yield

$$E(p, N_X(p)) = X(N_X(p)) - P_{\xi,1,2} \left[ X(p) + \nabla X(p) \left( -\nabla X(p)^{-1}X(p) \right) \right],$$

which is equivalent to $E(p, N_X(p)) = X(N_X(p))$. Thus, using this equality we obtain after simples algebraic manipulation that

$$\nabla X(N_X(p))^{-1}X(N_X(p)) = \nabla X(N_X(p))^{-1}P_{\xi,0,2}\nabla X(p_0)\nabla X(p_0)^{-1}P_{\xi,2,0}E(p, N_X(p)).$$

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Taking norm is last equality and using the inequalities \( (35) \) and \( (36) \) we easily conclude that
\[
\|\nabla X(N_X(p))^{-1}X(N_X(p))\| \leq \frac{e(t, n_f(t))}{|f'(n_f(t))|} \frac{\|\nabla X(p)^{-1}X(p)\|^2}{(n_f(t) - t)^2}.
\]
Finally, since \( n_f(t) \) belongs to the domain of \( f \), using the definitions of Newton iterations on \( (22) \) and definition of the linearization error in \( (28) \), we obtain \( f(n_f(t)) = e(t, n_f(t)) \) which combined with \( n_f(t) - t = f(t)/f'(t) \) and last inequality implies the desired result. Therefore, the proof of the lemma is concluded. \( \square \)

**Lemma 6.** For each \( t \in [0, t_\ast] \) the following inclusions hold: \( K(t) \subset B(p_\ast, t_\ast) \) and
\[
N_X(K(t)) \subset K(n_f(t)).
\]
As a consequence, \( K \subset B(p_\ast, t_\ast) \) and \( N_X(K) \subset K \).

**Proof.** The first inclusion follows trivially from the definition of \( K(t) \) in \( (32) \). Combining items i and iii of Lemma 5 we have
\[
\|\nabla X(N_X(p))^{-1}X(N_X(p))\| \leq \frac{f(n_f(t))}{|f'(n_f(t))|}.
\]
Therefore, the second inclusion of the lemma follows from combination of last inequality in item ii of Lemma 5 last inequality and definition of \( K(t) \). The first inclusion on the second sentence follows trivially from definitions \( (32) \) and \( (33) \). To verify the last inclusion, take \( p \in K \). Then \( p \in K(t) \) for some \( t \in [0, t_\ast) \). Using the first part of the lemma, we conclude that \( N_X(p) \subset K(n_f(t)) \). To end the proof, note that \( n_f(t) \in [0, t_\ast) \) and use the definition of \( K \) in \( (33) \). \( \square \)

We end this session limiting the derivative of the vector field by the derivative of the majorant function.

**Proposition 5.** If \( d(p_\ast, p) \leq t < R \) then \( \|\nabla X(p)\| \leq \|\nabla X(p_\ast)\|(2 + f'(t)) \).

**Proof.** Let \( \xi : [0, 1] \rightarrow \mathcal{M} \) is a minimizing geodesic joining \( p_\ast \) to \( p \). After some algebraic manipulations we have
\[
\|\nabla X(p_\ast)^{-1}P_{\xi,1,0}\nabla X(p)\| = \|\nabla X(p_\ast)^{-1}[P_{\xi,1,0}\nabla X(p)P_{\xi,0,1} - \nabla X(p_\ast) + \nabla X(p_\ast)]\|
\leq \|\nabla X(p_\ast)^{-1}[P_{\xi,1,0}\nabla X(p)P_{\xi,0,1} - \nabla X(p_\ast)]\| + \|I_{p_\ast}\|.
\]
Since \( \xi \) is a minimizing geodesic joining \( p_\ast \) to \( p \) we have \( \ell[\xi, 0, 1] = d(p_\ast, p) \). Thus, using that \( f \) is a majorant function at a point \( p_\ast \) for the vector field \( X \), above inequality yields
\[
\|\nabla X(p_\ast)^{-1}P_{\xi,1,0}\nabla X(p)\| \leq f'(d(p_\ast, p)) - f'(0) + 1 \leq 2 + f'(t),
\]
because \( d(p_\ast, p) \leq t \) and \( f' \) is a increasing function. Finally, using last inequality and taking into account that
\[
\|\nabla X(p)\| \leq \|\nabla X(p_\ast)\| \|\nabla X(p_\ast)^{-1}P_{\xi,1,0}\nabla X(p)\|,
\]
the desired inequality follows. \( \square \)

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3.3 Convergence

In this section we establish all the convergence results stated in Theorem 2 related to \( \{p_k\} \), the sequence generated by Newton’s Method, namely, the convergence of \( \{p_k\} \) to a zero of \( X \), the bounds in \([15, 16, 17] \) and \([18] \). For establish these results we will combine conveniently the results of the previous section. We begin with the following result:

**Proposition 6.** Let \( \{z_k\} \) be a sequence in \( \mathcal{M} \) and \( C > 0 \). If \( \{z_k\} \) converges to \( z^* \) and satisfies
\[
d(z_k, z_{k+1}) \leq Cd(z_{k-1}, z_k)^2, \quad k = 1, 2, \ldots.
\] (37)
then \( \{z_k\} \) converges \( Q \)-quadratically to \( z^* \) as follows
\[
\limsup_{k \to \infty} \frac{d(z_{k+1}, z^*)}{d(z_k, z^*)^2} \leq C.
\]

**Proof.** The proof follows the same pattern as the proof of Proposition 1.2 of \([13] \). \( \square \)

Using equality in \((14) \) and \((31) \), the sequence \( \{p_k\} \) generated by Newton’s Method satisfies
\[
p_{k+1} = N_X(p_k), \quad k = 0, 1, \ldots.
\] (38)

This equivalent definition of the Newton’s sequence \( \{p_k\} \) allow us to use the results of the previous section to establishes its properties of convergence.

**Corollary 3.** The sequence \( \{p_k\} \) is well defined, is contained in \( B(p_0, t_*) \) and satisfies the inequalities in \((15) \). Moreover, \( \{p_k\} \) converges to a point \( p_* \in B[p_0, t_*] \) satisfying \( X(p_*) = 0 \) and its convergence rate is \( R \)-linear as in \((16) \). If, additionally, \( f \) satisfies \( \textbf{h4} \) then the inequality \((17) \) holds and, consequently, \( \{p_k\} \) converges \( Q \)-quadratically to \( p_* \) as in \((18) \).

**Proof.** We are going to prove that the sequence \( \{p_k\} \) is well defined. First note that, combining \( \textbf{b2}, \textbf{13} \) and \( \textbf{h1} \) we have
\[
p_0 \in K(0) \subset K,
\] (39)
where the second inclusion follows trivially from \((33) \). Using the above inclusion, the inclusion \( N_X(K) \subset K \) in Lemma 6 and \((38) \) we conclude that \( \{p_k\} \) is well defined and rests in \( K \). From the first inclusion on second part of the Lemma 6 we have trivially that \( \{p_k\} \) is contained in \( B(p_0, t_*) \).

Now we are going to prove the inequalities in \((15) \). First we will prove, by induction that
\[
p_k \in K(t_k), \quad k = 0, 1, \ldots.
\] (40)
The above inclusion for \( k = 0 \) follows from \((33) \). Assume now that \( p_k \in K(t_k) \). Thus, using Lemma 6 \((38) \) and \((25) \), we obtain that \( p_{k+1} \in K(t_{k+1}) \), which completes the induction proof of \((40) \). Using definition of \( \{t_k\} \) in \((14) \), we have \(-f(t_k)/f'(t_k) = t_{k+1} - t_k \). Hence combining definition of \( \{p_k\} \) in \((14) \) with \((40) \) and item i of Lemma 5 we obtain
\[
d(p_k, p_{k+1}) = \|\nabla X(p_k)^{-1}X(p_k)\| \leq t_{k+1} - t_k, \quad k = 0, 1, \ldots.
\] (41)
which is first inequality in \((15) \). In order to prove the second inequality in \((15) \), first note that \( p_{k-1} \in K(t_{k-1}) \), \( p_k = N_X(p_{k-1}) \) and \( t_k = n_f(t_{k-1}) \), for all \( k = 0, 1, \ldots \). Thus, apply item iii of Lemma 5 with \( p = p_{k-1} \) and \( t = t_{k-1} \) to obtain
\[
d(p_k, p_{k+1}) \leq \frac{f(t_k)}{f'(t_k)} \left[ \frac{d(p_{k-1}, p_k)}{t_k - t_{k-1}} \right]^2,
\]
which using second inequality in \([14]\) yields the desired inequality.

To prove that \(\{p_k\}\) converges to \(p_* \in B(p_0, t_*)\) with \(X(p_*) = 0\) and \([15]\) holds, first note that as \(\{t_k\}\) converges to \(t_*\), the first inequality \([15]\) implies
\[
\sum_{k=k_0}^{\infty} d(p_{k+1}, p_k) \leq \sum_{k=k_0}^{\infty} t_{k+1} - t_k = t_* - t_{k_0} < +\infty,
\]
for any \(k_0 \in \mathbb{N}\). Hence, \(\{p_k\}\) is a Cauchy sequence in \(B(p_0, t_*)\) and, thus, converges to some \(p_* \in B(p_0, t_*)\). Therefore, first inequality \([15]\) also implies that \(d(p_*, p_k) \leq t_* - t_k\) for any \(k\). Hence, the inequality \([15]\) holds and, as \(\{t_k\}\) converges \(Q\)-linearly to \(t_*\), \(\{p_k\}\) converges \(R\)-linearly to \(p_*\).

For proving that \(X(p_*) = 0\), note that first inequality in \([15]\) implies that \(d(p_0, p_k) \leq t_k - t_0 = t_k\). Thus using Proposition \([5]\) we have
\[
\|\nabla X(p_k)\| \leq \|\nabla X(p_0)\|(2 + f'(t_k)), \quad k = 0, 1, \ldots
\]
which combining inclusion \([10]\) and second inequality in \([11]\) yields
\[
\|X(p_k)\| \leq \|\nabla X(p_k)\|\|\nabla X(p_k)^{-1} X(p_k)\| \leq \|\nabla X(p_0)\|(2 + f'(t_k))(t_{k+1} - t_k), \quad k = 0, 1, \ldots
\]
Since \(X\) is continuous on \(\bar{\Omega}\), \(\{p_k\} \subset B(p_0, t_*) \subset \Omega\), \(\{p_k\}\) converges to \(p_* \in \bar{\Omega}\), the result follows by taking limit as \(k\) goes to infinite in above inequality.

Now, we assume that \([14]\) holds. Thus, combining second inequality in \([15]\) with \([26]\), we obtain the inequality in \([17]\). To establish the inequality in \([18]\), use inequality in \([17]\) and Proposition \([6]\) with \(z_k = p_k\) and \(C = D^{-1} f'(t_*)/(-2f'(t_*))\). Therefore, the proof is concluded. \(\square\)

### 3.4 Uniqueness

In this section we prove the last statement in Theorem \([2]\) namely, the uniqueness of the singularity of the vector field in consideration. The results of this section generalize \([16\), Section 3.2] for a general majorant function, see also \([3\), Section 4.2].

**Corollary 4.** Take \(0 \leq t < t_*\) and \(q \in K(t)\). Define
\[
\tau_0 = t, \quad \tau_{k+1} = \tau_k - f(\tau_k)/f'(\tau_k), \quad k = 0, 1, \ldots
\]
The sequence \(\{q_k\}\) generated by Newton’s method with starting point \(q_0 = q\) is well defined and satisfies \(q_k \in K(\tau_k)\), for all \(k\). Furthermore, \(\{\tau_k\}\) converges to \(t_*\); \(\{q_k\}\) converges to some \(q_* \in B[p_0, t_*]\) a singular point of \(X\) and \(d(q_k, q_*) \leq t_* - \tau_k\), for all \(k\).

**Proof.** The proof is a convenient combination of Lemma \([3]\) Corollary \([1]\) and Proposition \([5]\) following the same pattern of Corollary 3.6 of \([16]\). \(\square\)

The next two lemmas are most important results we need to prove the uniqueness of solution.

The idea of its proofs are similar to the corresponding results of \([16]\), see also \([3]\). In this more general approach, some technical details related to the parallel transport and the majorant function (possibly non-quadratic) should be used.

**Lemma 7.** Take \(0 \leq t < t_*\) and \(p \in K(t)\). Define for \(\theta \in \mathbb{R}\)
\[
\zeta(\theta) = exp_p(-\theta \nabla X(p)^{-1} X(p)), \quad \tau(\theta) = t - \theta \frac{f(t)}{f'(t)}.
\]
Then for \(\theta \in [0, 1]\) we have \(t \leq \tau(\theta) < t_*\) and \(\zeta(\theta) \in K(\tau(\theta))\).
Proof. The proof follows the same pattern of [16, Lemma 3.7], see also [3, Lemma 4.4].

Lemma 8. Take $0 \leq t < t_*$ and $p \in K(t)$. Suppose that $q_* \in B[p_0, t_*]$ is a singular point of $X$ and $t + d(p, q_*) = t_*$. Then $d(p_0, p) = t$. Furthermore, $t < n_f(t) < t_*$, $N_X(p) \in K(n_f(t))$ and $n_f(t) + d(N_X(p), q_*) = t_*$. 

Proof. The proof follows the same pattern of [16, Lemma 3.8], see also [3, Lemma 4.5].

The proof of Theorem 2 follow by direct combination of Corollary 2, Corollary 3 with Lemma 10.

Corollary 5. Suppose that $\tilde{q}_* \in B[p_0, t_*]$ is a singular point of $X$. If for some $\tilde{t}, \tilde{q}$

$$0 \leq \tilde{t} < t_*, \quad \tilde{q} \in K(\tilde{t}),$$

and $\tilde{t} + d(\tilde{q}, \tilde{q}_*) = t_*$, then $d(p_0, \tilde{q}_*) = t_*$. 

Lemma 9. The sequence $\{p_k\}$ has limit $p_*$ as the unique singular point of $X$ in $B[p_0, t_*]$. 

Lemma 10. Let $q \in B(p_0, R)$ and $\xi : [0, 1] \rightarrow M$ a minimizing geodesic in $G_{1}(p_0, R)$ joinning $p_0$ to $q$. Then the following inequality holds:

$$-f(d(p_0, q)) \leq \|\nabla X(p_0)^{-1}P_{\xi, 1, 0}X(q)\|.$$ 

As a consequence, $p_*$ is the unique singularity of $X$ in $B(p_0, \tau)$, where $\tau := \sup\{t \in [t_*, R] : f(t) \leq 0\}$. 

Proof. Applying second part of Lemma 4 with $p = p_0$, $a = 0$, $b = 1$, $t = 0$ and $x = d(p_0, q)$ we have

$$e(0, d(p_0, q)) \geq \|\nabla X(p_0)^{-1}P_{\xi, 1, 0}E(p_0, q)\|.$$ 

From Definition 4 last inequality becomes

$$e(0, d(p_0, q)) \geq \|\nabla X(p_0)^{-1}P_{\xi, 1, 0}X(q) - \nabla X(p_0)^{-1}X(p_0) - \xi'(0)\|.$$ 

Using triangular inequality in the right hand side of last inequality, it is easy to see that

$$e(0, d(p_0, q)) \geq \|\xi'(0)\| - \|\nabla X(p_0)^{-1}X(p_0)\| - \|\nabla X(p_0)^{-1}P_{\xi, 1, 0}X(q)\|.$$ 

Combining Definition 8 with assumption (13) and taking into account that $\|\xi'(0)\| = d(p_0, q)$ and $f'(0) = -1$, we obtain from last inequality that

$$f(d(p_0, q)) - [f(0) + f'(0)d(p_0, q)] \geq d(p_0, q) - f(0) - \|\nabla X(p_0)^{-1}P_{\xi, 1, 0}X(q)\|,$$

with is equivalent to the inequality of the lemma. Hence the first of the lemma is proved.

For the second part, first note that in the interval $(t_*, \tau)$ the sign of $f$ is negative. Hence, first part of the lemma implies that there is no singularity of $X$ in $B(p_0, \tau)\setminus B[p_0, t_*]$. Therefore, from Lemma 4 the unique singularity of $X$ in $B(p_0, \tau)$ is $p_* \in B[p_0, t_*]$.

3.5 Proof of Theorem 2

The proof of Theorem 2 follow by direct combination of Corollary 2, Corollary 3 with Lemma 10.
4 On the proof of the main theorem

In this section Theorem 3 will be used to prove a robust semi-local affine invariant theorem for Newton’s method for finding a singularity of the vector field $X$, namely, Theorem 1. The following result will be needed.

**Proposition 7.** Let $R > 0$ and $f : [0, R) \rightarrow \mathbb{R}$ a continuously differentiable function. Suppose that $p_0 \in \Omega$, $f$ is a majorant function for $X$ at $p_0$ with respect to $G_2(p_0, R)$ and satisfies $h_4$. If $0 \leq \rho < \Gamma/2$, where $\Gamma := \sup \{-f(t) : t \in [0, R]\}$, then for any $q_0 \in B[p_0, \rho]$ the derivative $\nabla X(q_0)$ is nonsingular. Moreover, the scalar function $g : [0, R - \rho) \rightarrow \mathbb{R}$

$$g(t) = \frac{1}{|f'(\rho)|} [f(t + \rho) + 2\rho],$$

is a majorant function for $X$ at $q_0$ with respect to $G_2(q_0, R - \rho)$ and also satisfies condition $h_4$.

**Proof.** Since the domain of $f$ is $[0, R)$ and $f'(\rho) < 0$ (see Proposition 2 item iv ), we conclude that $g$ is well defined. First we will prove that function $g$ satisfies conditions $h_1$, $h_2$, $h_3$ and $h_4$. Definition of $g$ and $f'(\rho) < 0$ trivially imply $g'(0) = -1$. Since $f$ is convex and $f'(0) = -1$ we have $f(t) + t \geq f(0) > 0$, for all $0 \leq t < R$, which, by using Proposition 2 item iv and that $0 \leq \rho$, yields $g(0) = [f(\rho) + 2\rho]/|f'(\rho)| > 0$, hence $g$ satisfies $h_1$. Using that $f$ satisfies $h_2$, we easily conclude that $g$ also satisfies $h_2$. Now, as $\rho < \Gamma/2$, using Proposition 2 item iii, we have

$$\lim_{t \rightarrow t - \rho} g(t) = \frac{1}{|f'(\rho)|} (2\rho - \Gamma) < 0,$$

which implies that $g$ satisfies $h_4$ and, as $g$ is continuous and $g(0) > 0$, it also satisfies $h_3$. To complete the proof, it remains to prove that $g$ satisfies (5). First of all, for any $q_0 \in B[p_0, \rho]$, from Proposition 2 item iv, we have $d(q_0, p_0) \leq \rho < \ell$. Let $\eta : [0, 1] \rightarrow \mathcal{M}$ be the minimizing geodesic joining $p_0$ to $q_0$. Since $\eta \in G_1(p_0, R) \subset G_2(p_0, R)$ and $d(p_0, q_0) = \ell[\eta, 0, 1] \leq \rho < \ell$ we can apply Proposition 4 to obtain that $\nabla X(q_0)$ is nonsingular and

$$\|\nabla X(q_0)^{-1} P_{\eta, 0, 1} \nabla X(p_0)\| \leq \frac{1}{|f'(\rho)|}. \quad (43)$$

Because $B(p_0, R) \subseteq \Omega$, for any $q_0 \in B[p_0, \rho]$, we trivially have $B(q_0, R - \rho) \subseteq \Omega$. Let $\mu : [0, T] \rightarrow \mathcal{M}$ such that $\mu \in G_2(q_0, R - \rho)$ and $c_0, c_1, c_2 \in [0, T]$ with $c_0 = 0 \leq c_1 \leq c_2 = T$ such that $\mu_{|c_0, c_1}$ is a minimizing geodesic and $\mu_{|c_1, c_2}$ is a geodesic. Take $a, b \in [0, T]$ with $0 \leq a \leq b$. Thus

$$\mu(a), \mu(b) \in B(q_0, R - \rho), \quad \ell[\mu, 0, a] + \ell[\mu, a, b] < R - \rho, \quad d(q_0, \mu(a)) = \ell[\mu, 0, a].$$

Using definitions of the curves $\eta$ and $\mu$, properties of the parallel transport, property of the norm and simple manipulation, we conclude that

$$\|\nabla X(q_0)^{-1} [P_{\mu, b, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))]\| \leq$$

$$\|\nabla X(q_0)^{-1} P_{\eta, 0, 1} \nabla X(p_0)\| \left\| \nabla X(p_0)^{-1} P_{\eta, 1, 0} [P_{\mu, b, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))] \right\|. \quad (44)$$

Now we are going to estimate the second norm of the right hand side of above inequality. First, we define $\xi : [\tilde{0}, \tilde{T}] \rightarrow \mathcal{M}$ a piecewise geodesic curve in $G_2(p_0, R)$ as concatenation between the curves $\eta$ and $\mu$, i.e., take $\tilde{c}_0 = 0 < \tilde{c}_1 < \tilde{c}_2 < \tilde{c}_3 = \tilde{T}$ such that

$$\xi_{|[\tilde{0}, \tilde{c}_1]} = \eta_{|[0, 1]}, \quad \xi_{|[\tilde{c}_1, \tilde{c}_2]} = \mu_{|[0, c_1]}, \quad \xi_{|[\tilde{c}_2, \tilde{c}_3]} = \mu_{|[c_1, c_2]}.$$
Definition of $\xi$ in (15) and definition of curve $\mu$ imply that there exist $\hat{a}, \hat{b} \in \text{dom}(\xi)$ with $0 \leq \hat{a} \leq \hat{b}$ such that $\xi(\hat{a}) = \mu(a)$ and $\xi(\hat{b}) = \mu(b)$. Therefore, properties of parallel transport yield $P_{\xi, 1, 0} P_{\mu, 0, 0} = P_{\xi, b, 0}$. Hence,

$$\|\nabla X(p_0)^{-1} P_{\xi, 1, 0} [P_{\mu, 0, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))]| =$$

$$\left\|\nabla X(p_0)^{-1} \left[ P_{\xi, 0, \hat{b}} \nabla X(\xi, \hat{b}) P_{\hat{a}, \hat{b}, 0} - P_{\xi, \hat{a}, 0} \nabla X(\xi, \hat{a}) \right] \right\|.$$  

Since $\xi \in \mathcal{G}_3(p_0, R)$ and $f$ is a majorant function for $X$ at $p_0$ with respect to $\mathcal{G}_3(p_0, R)$, applying Definition 5 with $a = \hat{a}$ and $b = \hat{b}$, last equality becomes

$$\left\|\nabla X(p_0)^{-1} P_{\mu, 0, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))\right\| \leq f'(\ell(\xi, 0, \hat{b})) - f'(\ell(\xi, 0, \hat{a})).$$  

Combining last inequality with (18), (19) and (20) we obtain

$$\left\|\nabla X(q_0)^{-1} [P_{\mu, 0, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))\right\| \leq$$

$$\frac{1}{|f'(\rho)|} \left[f'(\ell(\xi, 0, \hat{b})) - f'(\ell(\xi, 0, \hat{a}))\right].$$  

Since $f'$ is convex, the function $s \mapsto f'(t + s) - f'(s)$ is increasing for $t \geq 0$. Hence taking into account that definitions of $\xi$ in (15) and $\mu$ imply $\ell(\xi, 0, \hat{a}) = \ell(\xi, 0, \hat{c}) + \ell(\xi, \hat{c}, \hat{a}) \leq \rho + \ell[\mu, 0, a]$ and $\ell(\xi, 0, \hat{b}) = \ell(\xi, 0, \hat{a}) + \ell(\xi, \hat{a}, \hat{b}) \leq \rho + \ell[\mu, 0, a]$ and $\ell[\mu, 0, b] = \ell[\mu, 0, a] + \ell[\mu, a, b]$, we conclude that

$$f'(\ell(\xi, 0, \hat{b})) - f'(\ell(\xi, 0, \hat{a})) \leq f'(\rho + \ell[\mu, 0, a] + \ell[\mu, a, b]) - f'(\rho + \ell[\mu, 0, a]).$$  

Since $\ell[\mu, 0, b] = \ell[\mu, 0, a] + \ell[\mu, a, b]$, combining inequality in (17) and last inequality with the definition of the function $g$ we have

$$\left\|\nabla X(q_0)^{-1} \left[ P_{\mu, 0, 0} \nabla X(\mu(b)) P_{\mu, a, b} - P_{\mu, a, 0} \nabla X(\mu(a))\right] \right\| \leq g'(\ell[\mu, 0, b]) - g'(\ell[\mu, 0, a]),$$

implying that the function $g$ satisfies (15), which complete the proof of the proposition.

**Proposition 8.** Let $q \in B(p_0, R)$ and $\xi : [0, 1] \to \mathcal{M}$ a minimizing geodesic joining $p_0$ to $q$. Then the following inequality holds:

$$\|\nabla X(p_0)^{-1} P_{\xi, 1, 0} X(q)\| \leq f(d(p_0, q)) + 2d(p_0, q).$$  

**Proof.** Applying second part of Lemma 4 with $p = p_0$, $a = 0$, $b = 1$, $t = 0$ and $x = d(p_0, q)$ we have

$$e(0, d(p_0, q)) \geq \|\nabla X(p_0)^{-1} P_{\xi, 1, 0} E(p_0, q)\|.$$  

From Definition 7 last inequality becomes

$$e(0, d(p_0, q)) \geq \|\nabla X(p_0)^{-1} P_{\xi, 1, 0} X(q) - \nabla X(p_0)^{-1} X(p_0) - \xi'(0)\|.$$  

Using triangular inequality in the right hand side of last inequality, it is easy to see that

$$e(0, d(p_0, q)) \geq \|\nabla X(p_0)^{-1} P_{\xi, 1, 0} X(q)\| - \|\nabla X(p_0)^{-1} X(p_0)\| - \|\xi'(0)\|.$$  

Combining Definition 6 with assumption (13) and taking into account that $\|\xi'(0)\| = d(p_0, q)$ and $f'(0) = -1$, we obtain from last inequality that

$$f(d(p_0, q)) - [f(0) + f'(0)d(p_0, q)] \geq \|\nabla X(p_0)^{-1} P_{\xi, 1, 0} X(q)\| - f(0) - d(p_0, q),$$

which is equivalent to the inequality of the lemma. Hence the lemma is proved. \qed
4.1 Proof of Theorem 1

Proposition 7 claims that for any $q_0 \in B[p_0, \rho]$ the derivative $\nabla X(q_0)$ is nonsingular. Moreover, the scalar function $g : [0, R - \rho) \to \mathbb{R}$,

$$g(t) = \frac{1}{|f'(\rho)|} [f(t) + 2t],$$

is a majorant function for $X$ at $q_0$ with respect to $G_2(q_0, R - \rho)$ and also satisfies condition h4. Let $\xi : [0, 1] \to M$ a minimizing geodesic joining $p_0$ to $q_0$. Since item iv of Proposition 2 implies $\ell[\xi, 0, 1] = d(p_0, q_0) \leq \rho < t$, thus Proposition 4 gives us

$$\|\nabla X(q_0)^{-1}P_{\xi, 0, 1}\nabla X(p_0)\| \leq \frac{1}{|f'(\rho)|}.$$ 

Combining property of norm with last inequality and Proposition 8 with $q = q_0$, we have

$$\|\nabla X(q_0)^{-1}X(q_0)\| \leq \|\nabla X(q_0)^{-1}P_{\xi, 0, 1}\nabla X(p_0)\|\|\nabla X(p_0)^{-1}P_{\xi, 1, 0}X(q_0)\| \leq \frac{1}{|f'(\rho)|} [f(d(q_0, p_0)) + 2d(q_0, p_0)].$$

As $f' \geq -1$, the function $t \mapsto f(t) + 2t$ is (strictly) increasing. Using this fact, above inequality, $d(p_0, q_0) \leq \rho$ and (49) we conclude that

$$\|\nabla X(q_0)^{-1}X(q_0)\| \leq g(0).$$

Therefore, last inequality allow us to apply Theorem 2 for $X$ and the majorant function $g$ at point $q_0$ for obtaining the desired result.

5 Special cases

Kantorovich’s theorem under a majorant condition in Riemannian settings was used in 31, see also 25 to prove Kantorovich’s theorem under Lipschitz condition in Riemannian manifolds 15, Smale’s theorem 37 and Nesterov-Nemirovskii’s theorem 31. Using the ideas of 3 we present, as an application of Theorem 1 a robust version of these theorems.

5.1 Under Lipschitz’s condition

**Theorem 3.** Let $M$ be a Riemannian manifold, $\Omega \subseteq M$ an open set and $\bar{\Omega}$ its closure, $X : \bar{\Omega} \rightarrow TM$ a continuous vector field and continuously differentiable on $\Omega$. Take $p_0 \in \Omega$, $L > 0$, $\beta > 0$ and $R = \sup\{r > 0 : B(p_0, r) \subset \Omega\}$. Suppose that $\nabla X(p_0)$ is nonsingular, $B(p_0, 1/L) \subset \Omega$,

$$\|\nabla X(p_0)^{-1}[P_{\xi, b, 0}\nabla X(\xi(b))P_{\xi, a, b} - P_{\xi, a, 0}\nabla X(\xi(a))]\| \leq L \ell[\xi, a, b],$$

for all $\xi$ in $G_3(p_0, R)$ and $2\beta L < 1$. Moreover, assume that

$$\|\nabla X(p_0)^{-1}X(p_0)\| \leq \beta.$$

Let $0 \leq \rho < (1 - 2\beta L)/(4L)$ and $t_{*, \rho} = \left(1 - \rho L - \sqrt{1 - 2L(\beta + 2\rho)}\right)/L$. Then the sequence generated by Newton’s Method for solving the equations $X(p) = 0$, with starting point $q_0$, for any $q_0 \in B[p_0, \rho]$,

$$q_{k+1} = \exp_{q_k}(-\nabla X(q_k)^{-1}X(q_k)), \quad k = 0, 1, \ldots.$$
is well defined, \( \{ q_k \} \) is contained in \( B(q_0, t_{*, \rho}) \) and satisfy the inequality
\[
d(q_k, q_{k+1}) \leq \frac{L}{2\sqrt{1 - 2L(\beta + 2\rho)}}d(q_{k-1}, q_k)^2, \quad k = 1, 2, \ldots
\]
Moreover, \( \{ q_k \} \) converges to \( p_* \in B[q_0, t_{*, \rho}] \) such that \( X(p_*) = 0 \) and the convergence is \( Q \)-quadratic as follows
\[
\limsup_{k \to \infty} \frac{d(q_{k+1}, p_*)}{d(q_k, p_*)^2} \leq \frac{L}{2\sqrt{1 - 2L(\beta + 2\rho)}}.
\]
Furthermore, if \( B(p_0, \tau) \subset \Omega \) then \( p_* \) is the unique singularity of \( X \) in \( B(p_0, \tau) \), where \( \tau := (1 + \sqrt{1 - 2\beta L})/L \).

**Proof.** The proof follows from Theorem \( \square \) with the quadratic polynomial \( f(t) = \frac{k}{2}t^2 - t + \beta \) as the majorant function to \( X \) with respect to \( G_3(p_0, 1/L) \) and \( \Gamma = (1 - 2\beta L)/(4L) \). \( \square \)

5.2 Under Smale’s condition

**Theorem 4.** Let \( \mathcal{M} \) be an analytic Riemannian manifold, \( \Omega \subseteq \mathcal{M} \) an open set and \( X : \Omega \to T\mathcal{M} \) an analytic vector field. Let \( p_0 \in \mathcal{M} \) be such that \( \nabla X(p_0) \) is nonsingular and set \( \beta := \|\nabla X(p_0)^{-1}X(p_0)\| \). Suppose
\[
\alpha := \beta \gamma < 3 - 2\sqrt{2}, \quad \gamma := \sup_{n \geq 1} \left\| \frac{1}{n!} \nabla X(p_0)^{-1} \nabla^n X(p_0) \right\|^{1/(n-1)} < \infty,
\]
\( B(p_0, R) \subset \Omega \), where \( R := (1 - 1/\sqrt{2})/\gamma \). Let \( 0 \leq \rho < [3 - 2\sqrt{2} - \alpha]/(2\gamma) \) and
\[
t_{*, \rho} := \left( \alpha + 1 - 2\rho \gamma - \sqrt{(\alpha + 1 - 2\rho \gamma)^2 - 8\alpha - 8\rho \gamma (1 - \alpha)} \right)/(4\gamma).
\]
Then the sequences generated by Newton’s method for solving the equations \( X(p) = 0 \) with starting at \( q_0 \), for any \( q_0 \in B[p_0, \rho] \),
\[q_{k+1} = \exp_{q_k}(-\nabla X(q_k)^{-1}X(q_k)), \quad k = 0, 1, \ldots\]
are well defined, \( \{ q_k \} \) is contained in \( B[q_0, t_{*, \rho}] \) and satisfy the inequality
\[
d(q_k, q_{k+1}) \leq \frac{\gamma}{(1 - \gamma(t_{*, \rho} + \rho))[2(1 - \gamma(t_{*, \rho} + \rho))^2 - 1]}d(q_{k-1}, q_k)^2, \quad k = 1, 2, \ldots
\]
Moreover, \( \{ q_k \} \) converges to \( p_* \in B[p_0, t_{*, 0}] \) such that \( X(p_*) = 0 \) and the convergence is \( Q \)-quadratic as follows
\[
\limsup_{k \to \infty} \frac{d(q_{k+1}, p_*)}{d(q_k, p_*)^2} \leq \frac{\gamma}{(1 - \gamma(t_{*, \rho} + \rho))[2(1 - \gamma(t_{*, \rho} + \rho))^2 - 1]}.
\]
Furthermore, \( p_* \) is the unique singularity of \( X \) in \( B(p_0, R) \subset \Omega \).

We need the following results to prove the above theorem.

**Lemma 11.** Let \( \mathcal{M} \) be an analytic Riemannian manifold, \( \Omega \subseteq \mathcal{M} \) an open set and \( X : \Omega \to T\mathcal{M} \) an analytic vector field. Suppose that \( p_0 \in \Omega \), \( \nabla X(p_0) \) is nonsingular and that \( R \leq (1 - 1/\sqrt{2})\gamma^{-1} \). Then, for all \( \zeta \in G_3(p_0, R) \) there holds
\[
\|\nabla X(p_0)^{-1}P_{\zeta, s, 0}\nabla^2 X(\zeta(s))\| \leq (2\gamma)/(1 - \gamma(\zeta, 0, s))^3.
\]
Proof. The proof follows the same pattern of Lemma 5.3 of [3].

Lemma 12. Let $\mathcal{M}$ be an analytic Riemannian manifolds, $\Omega \subseteq \mathcal{M}$ an open set and $X : \Omega \to T\mathcal{M}$ an analytic vector field. Suppose that $p_0 \in \Omega$ and $\nabla X(p_0)$ is nonsingular. If there exists an $f : [0, R) \to \mathbb{R}$ twice continuously differentiable such that

$$\|\nabla X(p_0)^{-1} P_{\zeta,s,0} \nabla^2 X(\zeta(s))\| \leq f''(\ell[\zeta, 0, s]),$$

(50)

for all $\zeta \in \mathcal{G}_3(p_0, R)$ and for all $s \in \text{dom}(\zeta)$, then $X$ and $f$ satisfy [5] with $n = 3$.

Proof. Let $\zeta$ be a curve of $\mathcal{G}_3(p_0, R)$, $a, b \in \text{dom}(\zeta)$ with $0 \leq a \leq b$. From Definition [4] there exist $c_0, c_1, c_2, c_3 \in [0, T]$ with $c_0 = 0 \leq c_1 \leq c_2 \leq c_3 = T$ such that $\xi_{[c_0, c_1]}$ and $\xi_{[c_1, c_2]}$ are minimizing geodesics and $\xi_{[c_2, c_3]}$ is a geodesic. We have six possibilities:

- $a, b \in [c_i, c_{i+1}]$ for $i = 0, 1, 2$;
- $a \in [c_i, c_{i+1}]$ and $b \in [c_{i+1}, c_{i+2}]$ for $i = 0, 1$;
- $a \in [c_0, c_1]$ and $b \in [c_2, c_3]$.

We are going to analyze the possibility $a \in [c_0, c_1]$ and $b \in [c_2, c_3]$, the others are similar. Since $a \in [c_0, c_1]$ and $\xi_{[c_0, c_1]}$ is geodesic, taking $v \in T_{\zeta(a)}\mathcal{M}$ and $Y \in \mathcal{X}(\mathcal{M})$ the vector field on $\zeta$ such that $\nabla \xi_{(s)} Y = 0$ and $Y(\zeta(a)) = v$, we may apply Lemma [2] to have

$$P_{\zeta,c_1,a} \nabla X(\zeta(c_1)) Y(\zeta(c_1)) = \nabla X(\zeta(a)) Y(\zeta(a)) + \int_a^{c_1} P_{\zeta,s,a} \nabla^2 X(\zeta(s)) \left( Y(\zeta(s)), \zeta'(s) \right) ds.$$  

(51)

Using that $Y(\zeta(a)) = v$ and $Y(\zeta(c_1)) = P_{\zeta,a,c_1} v$, we obtain, after some algebraic manipulation in last equality, that

$$\nabla X(p_0)^{-1} \left[ P_{\zeta,c_1,0} \nabla X(\zeta(c_1)) P_{\zeta,a,c_1} - P_{\zeta,a,0} \nabla X(\zeta(a)) \right] v = \int_a^{c_1} \nabla X(p_0)^{-1} P_{\zeta,s,0} \nabla^2 X(\zeta(s)) \left( Y(\zeta(s)), \zeta'(s) \right) ds.$$  

Since $\|Y(\zeta(s))\| = \|v\|$ for all $s \in [a, c_1]$ and $v$ is a arbitrary, we conclude from Definition [3] that

$$\|\nabla X(p_0)^{-1} \left[ P_{\zeta,c_1,0} \nabla X(\zeta(c_1)) P_{\zeta,a,c_1} - P_{\zeta,a,0} \nabla X(\zeta(a)) \right]\| \leq \int_a^{c_1} \|\nabla X(p_0)^{-1} P_{\zeta,s,0} \nabla^2 X(\zeta(s))\| \|\zeta'(s)\| ds.$$  

Now, as $\|\zeta'(s)\| = \ell[\zeta, a, c_1]/(c_1 - a)$ and $\ell[\zeta, 0, s] = \ell[\zeta, 0, a] + ((c_1 - s)/(c_1 - a)) \ell[\zeta, a, c_1] < R$ for all $s \in [a, c_1]$, using [50] we obtain, from the last inequality, that

$$\|\nabla X(p_0)^{-1} \left[ P_{\zeta,c_1,0} \nabla X(\zeta(c_1)) P_{\zeta,a,c_1} - P_{\zeta,a,0} \nabla X(\zeta(a)) \right]\| \leq \int_a^{c_1} f'' \left( \ell[\zeta, 0, a] + \frac{c_1 - s}{c_1 - a} \ell[\zeta, a, c_1] \right) \frac{\ell[\zeta, a, c_1]}{c_1 - a} ds.$$  

Evaluating the latter integral, it follows that

$$\|\nabla X(p_0)^{-1} \left[ P_{\zeta,c_1,0} \nabla X(\zeta(c_1)) P_{\zeta,a,c_1} - P_{\zeta,a,0} \nabla X(\zeta(a)) \right]\| \leq f' \left( \ell[\zeta, 0, c_1] \right) - f' \left( \ell[\zeta, 0, a] \right).$$

(52)
On the other hand, using that $\xi_{[c_1, c_2]}$ is geodesic, similar arguments used above show that
\[
\left\| \nabla X(p_0)^{-1} [P_{\xi,c_2,0} \nabla X(\xi(c_2))P_{\xi,c_1,0} - P_{\xi,c_1,0}\nabla X(\xi(c_1))] \right\| \leq f'(\ell[\xi, 0, c_2]) - f'(\ell[\xi, 0, c_1]).
\] (53)

We may also use that $b \in [c_2, c_3]$ and $\xi_{[c_2, c_3]}$ is geodesic to obtain the following inequality
\[
\left\| \nabla X(p_0)^{-1} [P_{\xi,b,0} \nabla X(\xi(b))P_{\xi,c_2,b} - P_{\xi,c_2,0}\nabla X(\xi(c_2))] \right\| \leq f'(\ell[\xi, 0, b]) - f'(\ell[\xi, 0, c_2]).
\] (54)

Now, taking into account that the parallel transport is an isometry, the triangular inequality yields
\[
\left\| \nabla X(p_0)^{-1} [P_{\xi,b,0} \nabla X(\xi(b))P_{\xi,a,b} - P_{\xi,a,0}\nabla X(\xi(a))] \right\| \leq f'(\ell[\xi, 0, b]) - f'(\ell[\xi, 0, a]),
\]
which is the desired result.

**Proof of Theorem 4.** Since $\alpha < 3 - 2\sqrt{2}$, combining Lemma 11 and Lemma 12 we have that the analytic function $f : [0, R] \to \mathbb{R}$ defined by $f(t) = \beta - 2t + t/(1 - \gamma t)$ is a majorant function to $X$ with respect to $G_3(p_0, R)$. Hence, the proof follows from Theorem 1 with $\Gamma = (3 - 2\sqrt{2} - \alpha)/\gamma$. $\square$

### 5.3 Under Nesterov-Nemirovskii’s condition

**Theorem 5.** Let $C \subset \mathbb{R}^n$ be a open convex set and $F : C \to \mathbb{R}$ be a strictly convex function, three times continuously differentiable. Take $x_0 \in C$ with $F''(x_0)$ nonsingular. Define the norm
\[
\| u \|_{x_0} := \sqrt{\langle u, u \rangle_{x_0}}, \quad \forall u \in \mathbb{R}^n,
\]
where $\langle u, v \rangle_{x_0} = a^{-1}(F''(x_0)u, v)$ for all $u, v \in \mathbb{R}^n$ and some $a > 0$. Suppose that $F$ is a self-concordant, i.e., satisfies
\[
|F''(x)[h, h, h]| \leq 2a^{-1/2}(F''(x)[h, h])^{3/2}, \quad \forall x \in C, \ h \in \mathbb{R}^n,
\]
$W_1(x_0) = \{ x \in \mathbb{R}^n : \| x - x_0 \|_{x_0} < 1 \} \subset C$ and there exists $\beta \geq 0$ such that
\[
\| F''(x_0)^{-1} F'(x_0) \|_{x_0} \leq \beta < 3 - 2\sqrt{2}.
\]

Let $0 \leq \rho < (3 - 2\sqrt{2} - \beta)/2$ and $t_{*, \rho} := (\alpha + 1 - 2\rho - \sqrt{(\alpha + 1 - 2\rho)^2 - 8\alpha - 8\rho(1 - \alpha)})/4$. Then the sequences generated by Newton’s method for solving the equations $F'(x) = 0$ with starting at $y_0$, for any $y_0 \in W_\rho[x_0] = \{ x \in \mathbb{R}^n : \| x - x_0 \|_{x_0} \leq \rho \}$,
\[
y_{k+1} = y_k - F''(y_k)^{-1} F'(y_k), \quad k = 0, 1, ...
\]

is well defined, $\{ y_k \} \subset W_{t_{*, \rho}}[x_0] = \{ x \in \mathbb{R}^n : \| x - x_0 \|_{x_0} \leq t_{*, \rho} \}$ and satisfy the inequality
\[
\| y_{k+1} - y_k \| \leq \frac{1}{(1 - (t_{*, \rho} + \rho))(2(1 - (t_{*, \rho} + \rho))^2 - 1)}\| y_k - y_{k-1} \|^2, \quad k = 1, 2, ...$

21
Moreover, \( \{y_k\} \) converges to \( x_* \in W_{t_0}(x_0) \) such that \( F'(x_*) = 0 \) and the convergence is \( Q \)-quadratic as follows

\[
\limsup_{k \to \infty} \frac{\|x_* - y_{k+1}\|}{\|x_* - y_k\|^2} \leq \frac{1}{(1 - (t_0 + \rho))(2(1 - (t_0 + \rho))^2 - 1)}.
\]

Proof. Since \( \alpha < 3 - 2\sqrt{2} \), combining Lemma 5.1 of \cite{3} and Lemma 12 we have that the function \( f : [0, R) \to \mathbb{R} \) defined by \( f(t) = \beta - 2t + t/(1 - t) \) is a majorant function to \( G_3(x_0, R) \). Hence, the proof follows from Theorem 1 with \( \Gamma = 3 - 2\sqrt{2} - b \). \( \square \)

6 Final remark

Let us present some computational aspects of Newton’s method in Riemannin settings for solving the equation (4). Note that the first equality in (8) is equivalent to

\[
q_{k+1} = \exp_{q_k} S_k, \quad \nabla X(q_k)^{-1} S_k = -X(q_k), \quad k = 0, 1, \ldots.
\] (55)

Since the solution of the linear systems in (55) for large systems is computationally expensive, namely, at each iteration the derivative at \( q_k \) must be computed and stored. Besides, the solution of the linear system in (55) is required. To circumvent these drawbacks, we propose the inexact Newton’s method: given an initial point \( q_0 \), the method generates a sequence \( \{q_k\} \) as follows:

\[
q_{k+1} = \exp_{q_k} S_k, \quad \nabla X(q_k)^{-1} S_k = -X(q_k) + r_k, \quad \|r_k\| \leq \theta_k \|X(q_k)\| \quad k = 0, 1, \ldots
\]

for a suitable forcing sequence \( \{\theta_k\} \), which is used to control the level of accuracy. Therefore, solutions of practical problems are obtained by computational implementations of the inexact Newton-like methods. The analysis of these methods under majorant condition will be done in the near future.

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