I. INTRODUCTION

The main purpose of this Letter is to establish a clear way to implement the equilibrium statistical mechanics based on the nonadditive entropy different from the usual Boltzmann-Gibbs statistical one. For the first time this concept was formulated by Tsallis in [1]. It is very well known that the conventional equilibrium statistical mechanics based on the Boltzmann-Gibbs entropy meets all the requirements of the equilibrium thermodynamics in the thermodynamic limit [2]. This is a necessary condition for self-consistent definition of any equilibrium statistical mechanics. In order to provide the connection of the statistical mechanics with the thermodynamics, the statistical entropy is usually used. From the mechanical and thermodynamical laws it allows one to determine a unique phase distribution function, or a statistical operator, which depends on two different sets of variables: the first set specifies the dynamic state of the microscopic system and the second one sets up the thermodynamic state of the macroscopic system. According to the Liouville and von Neumann equations, the equilibrium distribution function is a constant of motion which is expressed only through the first additive integrals of motion of the system. The ensemble averages in the statistical mechanics correspond to the concrete functions of state from the thermodynamics and depend only on the macroscopic variables of state. The statistical entropy as a function of the variables of state in the thermodynamic limit must satisfy all properties of the thermodynamic entropy: concavity, extensivity and so on. Note that the thermodynamic potentials of the system are functions fixing the norm of the phase distributions. It is known that the equilibrium thermodynamics is the theory defined in the thermodynamic limit. Therefore, the concept of the thermodynamic limit plays a crucial role in comparing the equilibrium statistical mechanics with thermodynamics. In this case, for thermodynamic systems the boundary effects must be neglected and only the short-range interaction forces can be taken into account. In the thermodynamic limit the ensemble averages with the corresponding distribution function should provide performance of the zero, first, second, and third laws of thermodynamics, and the principle of additivity that divide all variables into extensive and intensive ones. Moreover, the fundamental equation of thermodynamics, the Gibbs-Duhem relation, and the Euler theorem should be implemented.

In ref. [1], the author tried to construct the equilibrium statistical mechanics based on the nonextensive statistical entropy. The problems arisen in the proof of the thermodynamical laws resulted in the occurrence of the divergent variants of the Tsallis thermostatistics [2, 3, 4, 5]. In these investigations not only the ensemble averages and the norm equation for the distribution function but also the Lagrange function were drastically redefined. Moreover, it was found that any of these variants do not satisfy the zeroth law of thermodynamics if the parameter $q$ must be a universal constant. See ref. [6] for subsequent discussions of the critique of q-entropy for thermal statistics. To solve this problem, in Abe et al. [7] in the framework of the canonical ensemble the physical temperature and pressure satisfying the zeroth law of thermodynamics were introduced. However, as it was shown in [8, 9, 10, 11], such transformations of the variables including the entropy lead to the transition from the Tsallis thermostatistics to the extensive statistical mechanics of Gibbs or Rényi one. So such consideration does not pertain to the perception of the zeroth law of thermodynamics. In [12], a stronger assumption was evolved to make use of the nonextensive microscopic Hamiltonian of special kind like the nonextensive entropy form. In this case, the Hamiltonian depends on the temperature of the system that entails the changes of the thermodynamic relations and leads to the loss of self-consistency of the statistical mechanics. Closely connected with the problem of the zeroth law of thermodynamics is another one: the principle of additivity in the thermodynamic limit. In Abe [13], it was attempted to define the thermodynamic limit on a particular example of the perfect gas in the canonical ensemble. However, this limit was carried out incorrectly. The correct definition of the thermodynamic limit for the Tsallis thermostatistics in a particular case was given.
in Botet et al. [10, 14] and for the general case was developed in [15]. Note that an important criterion of correctness of the thermodynamic limit is the equivalence of all ensembles. Derivation of the fundamental equation of thermodynamics for the Tsallis thermostatics on the base of the canonical distribution function was performed in [9, 11]. In [15], it was proved that the microcanonical ensemble of the Tsallis statistical mechanics satisfies all requirements of the equilibrium thermodynamics if the entropic index $1/(q - 1)$ is the extensive variable of state of the system. In the present Letter, we will show that similar results are carried out also for the canonical ensemble.

The Letter is organized as follows. In the second section, the canonical ensemble and the derivation of the thermodynamic relations are given. In the third section, the performance of the thermodynamic principles in the thermodynamic limit on the example of the perfect gas is proved.

II. CANONICAL ENSEMBLE

Let us consider the equilibrium statistical ensemble of the classical dynamical systems of $N$ particles at the constant temperature $T$, the volume $V$, and the thermodynamic coordinate $z$ in a thermal contact with a heat bath. The system interacts weakly with its surroundings and only the energy can be transferred in and out of it. In order to determine the equilibrium distribution function, we consider the Tsallis equilibrium statistical entropy which is a function of the parameter $q$ and a functional of the probing phase distribution function $\rho(x, p)$:

$$S = -k \int \frac{\rho - \rho^q}{1 - q} \, d\Gamma,$$

where $d\Gamma = dx dp$ is an infinitesimal element of phase space, $k$ is the Boltzmann constant and $q \in \mathbb{R}$ is the real parameter taking values $0 < q < \infty$. The phase distribution function is normalized to unity:

$$\int \rho \, d\Gamma = 1.$$  (2)

In the classical statistical mechanics the expectation value of the Hamiltonian can be written as

$$\langle H \rangle = \int \rho H \, d\Gamma.$$  (3)

The phase distribution function depends on the first additive constants of motion of the system. Nevertheless, the mechanical laws are not sufficient to determine it unambiguously. For this reason additional postulates of the equilibrium thermodynamics are required. To express the equilibrium phase distribution function from the macroscopic variables of state, we consider the thermodynamic method explored in [15]. In the state of thermal equilibrium the macroscopic system is characterized by the fundamental equation of thermodynamics

$$TdS_{th} = dE + pdV + Xdz - \mu dN,$$  (4)

where $S_{th}(T,V,z,N)$ is the thermodynamic entropy, $z$ and $V$ are the "thermodynamic coordinates"; $X$ and $p$ are the associated "forces"; $\mu$ is the chemical potential and $E$ is the thermodynamic energy of the system. In the canonical ensemble the fundamental equation of thermodynamics at the fixed values of $T,V,z,N$ can be rewritten as

$$\left(TdS_{th} - dE\right)_{T,V,z,N} = 0.$$  (5)

Then, to express the phase distribution function $\rho(x, p)$ through the variables of state ($T,V,z,N$), let us replace in Eq. [15] the equilibrium thermodynamic entropy $S_{th}$ and energy $E$ of the macroscopic system with the statistical ones [11] and [13]. Then, one finds

$$T \frac{\delta S}{\delta q} dq + \int d\Gamma \left\{ \left[ T \frac{\delta S}{\delta \rho} - \frac{\delta \langle H \rangle}{\delta \rho} \right] dq - \frac{\delta \langle H \rangle}{\delta H} dH \right\} = 0,$$  (6)

where the symbol $d$ before the functions $H$, $\Phi$ and $q$ is the total differential in variables ($T,V,z,N$). The statistical parameters $q$ and $\rho$ must be expressed through the variables of state of the system to provide the unambiguous conformity between statistical and thermodynamic entropies. Since $dH = 0$, $dq = 0$ and $d\rho = 0$, we obtain

$$T \frac{\delta S}{\delta \rho} - \frac{\delta \langle H \rangle}{\delta \rho} = \alpha.$$  (7)
where $\alpha$ is a certain constant. For the microcanonical ensemble it was stated in ref. [15] that

$$\frac{1}{q-1} = z,$$

where the parameter $z$ takes the values $-\infty < z < -1$ for $0 < q < 1$ and $0 < z < \infty$ for $1 < q < \infty$ and in the limiting case for $q = 1$, we have $z = \pm\infty$. Substituting Eq. (1) into (7) and using again equation (1) to eliminate the parameter $\alpha$ we arrive at the following expression for the equilibrium phase distribution function:

$$\rho = \left[ 1 + \frac{1}{z + 1} \frac{\Lambda - H}{kT} \right]^{-z},$$

(9)

where $\Lambda = E - \frac{z+1}{z} TS$ and it is determined from the normalization condition (2)

$$\int \left[ 1 + \frac{1}{z + 1} \frac{\Lambda - H}{kT} \right]^{-z} d\Gamma = 1.$$  

(10)

Thus, $\Lambda$ is a function of the variables of state, $\Lambda = \Lambda(T, V, z, N)$. For the Jaynes principle derivation of the phase distribution function (9), see Appendix A. The expectation value $\langle A \rangle$ of the dynamical variable $A(x, p)$ can be defined as follows (cf. (3)):

$$\langle A \rangle = \int A \left[ 1 + \frac{1}{z + 1} \frac{\Lambda - H}{kT} \right]^{-z} d\Gamma.$$  

(11)

Using Eqs. (2), (3) and (9), we can write the entropy (1) as

$$S = \frac{z}{z + 1} \langle H \rangle - \frac{\Lambda}{T},$$

(12)

In the canonical ensemble it is convenient to introduce the free energy as a thermodynamic potential:

$$F = E - TS = \frac{\langle H \rangle + z\Lambda}{z + 1}, \quad E = \langle H \rangle.$$  

(13)

This is the Legendre transform of the energy with respect to the entropy of the system with $\partial E/\partial S = T$.

At this point we have obtained a rigorous derivation of classical statistical mechanics. The quantum statistical mechanics is constructed in analogy with the classical one. In this respect, all classical notions are replaced by the quantum mechanical ones. A dynamic state of the quantum system is defined by a vector of state $|\Psi(t)\rangle$, which is an element of the abstract Hilbert space $E_H$. The dynamic variables are represented by linear hermitian operators $A$ acting on the elements of the Hilbert space. In particular, the Hamiltonian $H$ is the linear hermitian operator acting on the vectors of state $|\Psi(t)\rangle$. In the quantum statistical mechanics the mixed states are considered. A macrostate thus appears as a set of possible microstates, which are set up by state vectors $|\Psi_r(t)\rangle$, $r = 1, 2, \ldots$, each with its own probability $w_r$ for its occurrence, which are eigenvalues of the statistical operator, $\rho(t)|\Psi_r(t)\rangle = w_r|\Psi_r(t)\rangle$. The integral over phase space of the classical functions is replaced by the trace of the corresponding quantum operators.

Let us show the connection between the statistical mechanics and equilibrium thermodynamics. In this respect, we derive the thermodynamic relations for the canonical ensemble from the general point of view. Let us consider now the classical case. Applying the total differential operator with respect to the ensemble variables $(T, V, z, N)$ on the entropy (11) and the norm equation (2), and using Eq. (9) one finds

$$TdS = Xdz + \int dqHd\Gamma,$$

(14)

where

$$X = kT \int \rho[1 - \rho^{1/z}(1 - \ln \rho^{1/z})]d\Gamma.$$  

(15)

The repeated application of this differential operator to Eq. (6) and substitution of the results into Eq. (14) leads to the formula

$$TdS = d\langle H \rangle - \int dqHd\Gamma + Xdz.$$  

(16)

By virtue of the parametrical dependence of the Hamilton function $H$ on the variables $V$ and $N$ we have the fundamental equation of thermodynamics

$$TdS = d\langle H \rangle + pdV + Xdz - \mu dN,$$

(17)
where
\[ p = \int \varrho \left( \frac{\partial H}{\partial V} \right)_{T,z,N} d\Gamma = \left\langle -\frac{\partial H}{\partial V} \right\rangle, \]
\[ \mu = \int \varrho \left( \frac{\partial H}{\partial N} \right)_{T,V,z} d\Gamma = \left\langle \frac{\partial H}{\partial N} \right\rangle. \]

Here, the property of the Hamilton function \( (\partial H/\partial T)_{V,z,N} = (\partial H/\partial z)_{T,V,N} = 0 \) is used. So we have proved that the statistical entropy \( \frac{\partial H}{\partial T} \) with the phase distribution function \( \varrho \) completely satisfies the fundamental equation of thermodynamics which was a starting point of our derivations concerning the distribution function. In fact, the differential form \( \partial H/\partial V \) is sufficient to prove the connection of the thermodynamics and statistical mechanics.

The thermodynamic potential of the canonical ensemble \( (T,V,z,N) \) is the Helmholtz free energy. Hence, the differential of \( F \) which follows from Eq. \( (17) \) can be written as
\[ dF = -SdT - pdV - Xdz + \mu dN, \]
and the thermodynamic relations are
\[ S = -\left( \frac{\partial F}{\partial T} \right)_{V,z,N}, \quad p = -\left( \frac{\partial F}{\partial V} \right)_{T,z,N}, \]
\[ X = -\left( \frac{\partial F}{\partial z} \right)_{T,V,N}, \quad \mu = \left( \frac{\partial F}{\partial N} \right)_{T,V,z}. \]

The free energy as a thermodynamic potential can easily be calculated in the framework of the canonical ensemble and it is used in order to obtain the functions of the state.

The fundamental equation of thermodynamics provides the first and second principles of thermodynamics
\[ \delta Q = TdS, \quad \delta Q = d\langle H \rangle + pdV + Xdz - \mu dN, \]
where \( \delta Q \) is a heat transfer by the system to the environment during a quasistatic transition of the system from one equilibrium state to a nearby one.

Let us now find the important quantity, the heat capacity, which is defined from the general rule, \( \delta Q = CdT \). According to the first and second laws of thermodynamics, in the canonical ensemble the heat capacity at the fixed values of \( T,V,z,N \) can be written as
\[ C_{V z N}(T,V,z,N) = \left( \frac{\delta Q}{dT} \right)_{V,z,N} = T \left( \frac{\partial S}{\partial T} \right)_{V,z,N} = \left( \frac{\partial (H)}{\partial T} \right)_{V,z,N} = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_{V,z,N}. \]

III. THE THERMODYNAMIC LIMIT. THE PERFECT GAS

The statistical mechanics must satisfy the requirements of the equilibrium thermodynamics in the thermodynamic limit when the number of particles of the system considered is very large, and the relative magnitude of surface effects becomes negligible. For thermodynamic systems, which are homogeneous at a macroscopic scale, the principle of additivity is valid: the various quantities of interest can be classified into either extensive or intensive ones under a division of the system into macroscopic parts. The extensive quantity, for instance, the entropy considered as a function of the extensive variables \( V, z \) and \( N \) is homogeneous of degree 1:
\[ S(T,\lambda V,\lambda z,\lambda N) = \lambda S(T, V, z, N), \]
where \( \lambda \) is a constant. However, the intensive quantities, like the pressure \( p \), the chemical potential \( \mu \), and \( X \), are the homogeneous functions of degree zero:

\[
\mu(T, \lambda V, \lambda z, \lambda N) = \mu(T, V, z, N).
\]  

(26)

Note that the temperature \( T \) is an intensive variable of state remaining invariant under such a subdivision of the system. This invariance property of the temperature guarantees the fulfillment of the zeroth law of thermodynamics. Here the thermodynamic limit denotes the limiting statistical procedure \( N \to \infty, v = V/N = \text{const}, \tilde{z} = z/N = \text{const} \) with keeping the main asymptotic on \( N \). It is meant to make an expansion of the functions of the state in powers of the small parameter \( 1/N \) \( (N \gg 1) \) with large finite values of the variables \( V, z \). Then the extensive variables \( A \) can be written (\( \alpha > 0 \)) as

\[
A(T, V, z, N)|_{N\to\infty, v, \tilde{z} = \text{const}} = N[a(T, v, \tilde{z}) + O(N^{-\alpha})] \equiv N a(T, v, \tilde{z}),
\]

(27)

whereas the intensive variables \( \phi \) take the following form:

\[
\phi(T, V, z, N)|_{N\to\infty, v, \tilde{z} = \text{const}} = \phi(T, v, \tilde{z}) + O(N^{-\alpha}) \equiv \phi(T, v, \tilde{z}),
\]

(28)

where \( v \) is the specific volume, \( \tilde{z} \) is the specific, and \( a = A/N \) is the specific \( A \). It is important to note that the limiting statistical procedures \( (V \to \infty, z/V = \text{const}, N/V = \text{const}) \) and \( (z \to \pm \infty, V/z = \text{const}, N/z = \text{const}) \) are equivalent with the limit, \( N \to \infty \), given above. Note that in Abe [13], the thermodynamic limit for the Tsallis statistics is not correct because the limits \( N \to \infty \) and \( |z| \to \infty \) are not coordinated among themselves. Note that after applying the thermodynamic limit to the functions of state the Boltzmann-Gibbs limit, \( \tilde{z} \to \pm \infty \), is provided by expansion of these functions in powers of the small parameter \( 1/\tilde{z} \) holding only the zero term of the power expansion.

In the canonical ensemble, how to prove from the general point of view the principle of additivity (see Eqs. (26) and (27)) and the zeroth law is not obvious. Therefore, we will illustrate explicitly the implementation of these principles on the foregoing example of the nonrelativistic ideal gas. In the framework of the nonrelativistic ideal gas of \( N \) identical particles in the canonical ensemble the functions of state can be explicitly expressed from the variables of state. Hence, the thermodynamic properties of the Tsallis statistics can thoroughly be investigated. In order to evaluate the expectation values of the dynamical variables, we use the method based on the integral representation of the Euler gamma function \[11, 16\]. Let us investigate the thermodynamic properties of the nonrelativistic perfect gas of \( N \) identical particles in the thermodynamic limit \( (N \to \infty, v = \text{const}, \tilde{z} = \text{const}) \). The exact relations for it can be found in Appendix B. In the thermodynamic limit one easily confirms that the canonical partition function of the ideal gas for Gibbs statistics is simply [17]

\[
Z_G^{1/N}(T, V, N) = (gve)^{3/2} \left( \frac{m k T}{2 \pi \hbar^2} \right)^{3/2} \equiv \bar{Z}_G(T, v),
\]

(29)

where \( \bar{Z}_G \) is the one-particle partition function, \( m \) is the particle mass and \( g \) is the spin degeneracy factor. Then in the thermodynamic limit and in the limit of Boltzmann-Gibbs statistics, \( \tilde{z} \to \pm \infty \), Eqs. (31) and (32) can be written as

\[
B(T, V, z, N) = \left( \bar{Z}_G e^{3/2} \right)^{-\frac{1}{1+z}} \left[ 1 + \frac{3}{2z} \right] = B(T, v, \tilde{z}),
\]

(30)

\[
B|_{\tilde{z} \to \pm \infty} = 1 - \frac{1}{2} \ln \bar{Z}_G + O((1/\tilde{z})^2),
\]

(31)

where \( \Lambda/N = kT \tilde{z}(B-1) \) and \( \tilde{z} < -3/2 \) or \( \tilde{z} > 0 \) with respect to the conditions of the integration method used. Note that the function of state \( B \) is intensive and \( \Lambda \) is extensive. The energy \[15\] \( \varepsilon \) takes the following form:

\[
\frac{E(T, V, z, N)}{N} = \frac{3}{2} kT \frac{B}{1 + \frac{3}{2z}} = \varepsilon(T, v, \tilde{z}),
\]

(32)

where \( \varepsilon \) is the specific energy depending only on intensive variables and it is intensive. Consequently, the energy \( E \) is extensive (cf. Eqs. (24) and (25)). Note that in the Boltzmann-Gibbs limit the energy (32) is reduced to \( E|_{\tilde{z} \to \pm \infty} = (3/2)kTN \). The free energy \[13\] \( f \) is found to be

\[
\frac{F(T, V, z, N)}{N} = kT \tilde{z}(B - 1) = f(T, v, \tilde{z}),
\]

(33)

where \( f \) is the specific free energy which is intensive and \( F = \Lambda \). In the Boltzmann-Gibbs limit the free energy \[13\] takes the usual form \( F|_{\tilde{z} \to \pm \infty} = -kT \ln Z_G \). Then in the thermodynamic limit and in the limit of
Boltzmann-Gibbs statistics the entropy \( S(T, V, z, N) \) can be written as

\[
\frac{S(T, V, z, N)}{N} = k\tilde{z} \left[ 1 - \frac{B}{1 + \frac{B}{2\tilde{z}}} \right] = s(T, v, \tilde{z}),
\]

(34)

where \( s \) is the specific entropy and \( S_G \) is the entropy of Gibbs statistics. The function \( s \), however, is an intensive function of \( (T, v, \tilde{z}) \). Thus, the Tsallis entropy \( S_B \) in the thermodynamic limit is extensive. It is important to note that the Tsallis entropy \( S_B \) for the finite values of \( N \) and \( z \) does not satisfy the homogeneous condition \( \lambda = 1/N \), for instance, in the case of \( \lambda = 1/N \), and it is nonextensive. However, the Gibbs entropy, \( S_G \), is also nonextensive for the nonrelativistic perfect gas of \( N \) identical particles in the canonical ensemble, because the canonical partition function \( Z_G \) is extensive. See Appendix B. In the thermodynamic limit the Tsallis entropy \( S_B \) of the perfect gas in contrast with its exact value is extensive, because Eq. (29) has been implemented.

In the Boltzmann-Gibbs limit the heat capacity takes its usual form

\[
C_v(T, v, \tilde{z}) = \frac{3}{2} kN + k \ln Z_G = S_G,
\]

(35)

and in the thermodynamic limit it is reduced to \( X|_{\tilde{z} \to \pm \infty} = 0 \). The chemical potential \( \mu \) is now

\[
\mu(T, V, z, N) = kT \left[ \frac{B}{1 + \frac{B}{2\tilde{z}}} \left( 1 - \ln \frac{B}{1 + \frac{B}{2\tilde{z}}} \right) \right] = \mu(T, v, \tilde{z}),
\]

(38)

where \( \mu \) is an intensive one and in the Gibbs limit we find that \( \mu|_{\tilde{z} \to \pm \infty} = kT(1 - \ln \tilde{Z}_G) \). Then, Eqs. (32), (34), and (36)-(38) in the thermodynamic limit yield the Euler theorem

\[
TS = E + pV + Xz - \mu N.
\]

(39)

This equation enables us to interpret \( X \) as the "force" observable of the system conjugate to the "position" variable \( z \). Moreover, Eq. (39) allows us to write

\[
F = E - TS = pV - Xz + \mu N.
\]

(40)

Its differential leads to the Gibbs-Duhem relation:

\[
SdT = Vdp + zdX - Nd\mu,
\]

(41)

which shows that the variables \( T, p, X \) and \( \mu \) are not independent. The heat capacity of the perfect gas of \( N \) identical particles is deduced from Eq. (24). Using Eq. (32) we obtain

\[
C_{vz}(T, v, \tilde{z}) = \frac{3}{2} kN \frac{B}{(1 + \frac{B}{2\tilde{z}})^2}.
\]

(42)

In the Boltzman-Gibbs limit the heat capacity takes its usual form \( C_{vz}|_{\tilde{z} \to \pm \infty} = (3/2)kN \).

To better understand the thermodynamic properties of the perfect gas, it is necessary to study the equilibrium distribution of the momenta \( p_i \) of the gas particles. The \( N \)-particle distribution function is defined in Appendix B. In the thermodynamic limit the single-particle distribution function \( f(p) \) takes the form

\[
f(p) = \left( \frac{1}{2\pi m_{\text{eff}} kT} \right)^{3/2} e^{-\frac{p^2}{2m_{\text{eff}} kT}},
\]

(43)
where $m_{\text{eff}}$ is the effective particle mass written as

\[ m_{\text{eff}} = m \frac{B}{1 + \frac{z}{x}} = \left\{ m \left[ g v \left( \frac{kT e^{5/3}}{2\pi h^2} \right)^{3/2} \right]^{-1/z} \right\}^{1/(1+\frac{1}{z})}. \]  

(44)

In the Boltzmann-Gibbs limit, $m_{\text{eff}} \mid z \to \pm \infty = m$, and the single-particle distribution function is reduced to the Maxwell-Boltzmann distribution $f(p) \mid z \to \pm \infty = (2\pi m kT)^{-3/2} \exp(-p^2/2m kT)$. It is clearly seen that the effective mass $m_{\text{eff}} \geq m$ for $z < -3/2$ and $m_{\text{eff}} \leq m$ for $z > 0$. At $z \to -3/2$ we have $m_{\text{eff}} \to \infty$. Let us investigate the single-particle averages. The particle mean kinetic energy with the single-particle distribution function $f$ can be written as

\[ \langle p^2 \rangle = m kT \frac{m_{\text{eff}}}{m} = \varepsilon. \]  

(45)

So the average kinetic energy is equivalent with the specific energy per particle $\varepsilon$. The average momentum of the particle and the highest probability momentum of the distribution $f(p)$ can be written as

\[ \langle p \rangle = \sqrt{\frac{8m_{\text{eff}} kT}{\pi}}, \quad p_{\text{hp}} = \sqrt{2m_{\text{eff}} kT}, \]  

(46)

where $p = |\vec{p}|$. Their ratio is $\langle p \rangle / p_{\text{hp}} = 2/\sqrt{\pi} \approx 1.13$ as in the Gibbs statistics for which the average momentum and the highest probability momentum are $\langle p \rangle_G = (8m kT/\pi)^{1/2}$ and $p_{\text{hp}} = (2mkT)^{1/2}$. The variance and the relative statistical fluctuations for the distribution $f$ are

\[ \langle (\Delta p)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = kT m_{\text{eff}} \left( 3 - \frac{8}{\pi} \right), \]  

(47)

\[ \delta_p = \frac{\sqrt{\langle (\Delta p)^2 \rangle}}{\langle p \rangle} = \sqrt{\frac{3\pi}{8} - 1} \]  

(48)

or $\delta_p \approx 0.424$. So from the investigation of the distribution function $f$ and its averages we arrive at the conclusion that in the Tsallis statistics the mean kinetic energy of the particles and the momentum are larger than their values in the Gibbs statistics, $\varepsilon \geq \varepsilon_G$ and $\langle p \rangle \geq \langle p \rangle_G$, for $z < -3/2$ and smaller, $\varepsilon \leq \varepsilon_G$ and $\langle p \rangle \leq \langle p \rangle_G$, for $z > 0$. Now we can give a physical interpretation for the variable of state $\tilde{z}$ in the framework of the Tsallis thermostatistics if we consider the system of noninteracting particles as the system of the interacting quasiparticles with the effective mass $m_{\text{eff}}$. Then the total energy of the quasiparticle, $\varepsilon$, is equal to the sum of the mean kinetic energy and the effective interaction energy $\Delta \varepsilon$ which can be written as

\[ \langle p^2 \rangle = \frac{3}{2} kT, \quad \Delta \varepsilon = \frac{3}{2} kT \left( \frac{m_{\text{eff}}}{m} - 1 \right). \]  

(49)

So the effective interaction energy is positive $\Delta \varepsilon > 0$, and the forces are repulsive for $\tilde{z} < -3/2$ and they are attractive $\Delta \varepsilon < 0$ for $\tilde{z} > 0$. Note that from Eq. (49) follows the physical interpretation for the temperature of the system $T$ as the average kinetic energy of the quasiparticles with the effective mass $m_{\text{eff}}$.

Fig. 1 shows the dependence of the single-particle distribution function (left panel) on the momentum $p$ for the perfect gas of free quarks in the nonrelativistic approximation. The calculations are carried out for the system of quarks with two flavor and three color degrees of freedom, and the constituent mass $m = 300$ MeV at temperature $T = 100$ MeV and the specific volume $v = 0.25/p_0$, where $p_0 = 0.168 fm^{-3}$. Note that the quarks at short distances are treated as almost free pointlike noninteracting particles because of the property of asymptotic freedom. It is remarkable that the single-particle distribution function at small values of the variable $|\tilde{z}|$ considerably differs from the limiting Maxwell-Boltzmann distribution. Such a behavior has really been caused by the sharp changes of the effective mass of quasiparticles in the dependence on $\tilde{z}$. This dependence can be seen even better in the right panel of Fig. 1 which shows the ratio of the effective mass to the constituent one vs. the variable $\tilde{z}$ for two values of the temperature $T$. Figure 1 clearly shows that the variable $\tilde{z}$ is the order parameter and the system is physically unstable in the region $-3/2 < \tilde{z} < 0$. For the microcanonical ensemble similar results were obtained in [14].

Our exploration of the perfect gas has brought the following points to the foreground: for the Tsallis statistics in the thermodynamic limit the principle of additivity and the zeroth law of thermodynamics are valid. The entropy [24] is a homogeneous function of the first order, it is an extensive variable satisfying the relation [25] with $\lambda = 1/N$ provided that the temperature $T$ must be the intensive variable of state. It should, however, be noted that in the thermodynamic limit the equivalence of the canonical and microcanonical ensembles is
FIG. 1: The dependence of the single-particle distribution function on the momentum $p$ (left) for the classical perfect gas of quarks with two flavor and three color degrees of freedom in the non-relativistic approximation for the different values of $\tilde{z} = -2, -3, 3$ and 2 (the curves 1, 2, 3 and 4, respectively) at the temperature $T = 100$ MeV and the specific volume $v = 0.25/\rho_0$. The ratio of the effective quark mass to the constituent one as a function of the specific $\tilde{z}$ (right) for the values of $T = 100$ MeV (dashed line) and $T = 200$ MeV (solid one). The dotted lines correspond to the conventional Boltzmann-Gibbs thermostatistics.

implemented. Giving constants of motion exactly or only as averages leads to the same results. For instance, if we express the temperature $T$ through the variables $(E, V, z, N)$ from Eq. (32) as

$$kT = \frac{2}{3} \varepsilon w^{1/\varepsilon}, \quad w = g v \left( \frac{m \varepsilon^{5/3}}{3 \pi h^2} \right)^{3/2}$$

and substitute it into Eqs. (34) and (36)-(38), then we obtain the results of the microcanonical ensemble derived in ref. [15]. It is important to note that in terms of the $z$ variable and our thermodynamic limit the results for the perfect gas in the canonical ensemble of Abe et al. [13, 18] are the same as here.

IV. CONCLUSIONS

In this Letter, the canonical ensemble of the nonextensive thermostatistics introduced by C. Tsallis has been reconsidered. It is shown that the equilibrium statistical mechanics based on the nonadditive statistical entropy completely satisfies all requirements of the equilibrium thermodynamics in the thermodynamic limit. The unique non-Gibbs phase distribution function corresponding to the Tsallis entropy is obtained from the constraints imposed by the equilibrium thermodynamics laws. The microscopic foundation of the equilibrium statistical mechanics proceeds on the Gibbs idea of the statistical ensembles for the quantum and classical mechanics. The phase distribution function and the statistical operator depend only on the first additive constants of motion of the system. Here they were derived within a formalism based on the fundamental equation of thermodynamics and statistical definition of the functions of state. It allows us to avoid introduction of the controversial Lagrange multiplies. Nevertheless, it is shown that the distribution function derived from the Jaynes principle exactly coincides with ours if the Lagrange parameters are expressed through a set of independent variables of state of the system. The unambiguous connection of the statistical mechanics with thermodynamics is established. The equilibrium distribution function satisfies the fundamental equation of thermodynamics, the first and the second principles preserving the Legendre transformation. All thermodynamic relations relative to the Helmholtz free energy for the thermodynamic system in the thermostat are carried out. The heat capacity of the system was derived from the first and the second laws of thermodynamics. Note that in the fundamental equation of thermodynamics the new term related to the work of the conjugate force $X$ at changes of the variable of state $z$ appeared. In the limit $z \rightarrow \pm \infty$ the conventional Gibbs statistics is recovered.

It is well known that the statistical mechanics should satisfy all requirements of the equilibrium thermodynamics only in the thermodynamic limit. Based on a particular example of the ideal gas we obviously proved the fulfillment of the principle of additivity and the zero law of thermodynamics for the Tsallis statistics in the thermodynamic limit. It was shown that all functions of state of the system are the homogeneous functions
of the first degree, extensive, or the homogeneous functions of the zero degree, intensive. In particular, the
temperature is an intensive variable and thus provides implementation of the zero law of thermodynamics. It
should be marked that for the finite values of the number of particles \(N\) and the parameter \(z\) of the system
both the Tsallis entropy and the Gibbs entropy are nonextensive functions of state, while in the thermodynamic
limit they become extensive variables. The homogeneous properties of the functions of state allow us to find
the Euler theorem and the Gibbs-Duhem relation. After applying the thermodynamic limit the expressions of
the Gibbs statistics are obtained by the limiting procedure \(\tilde{z} \to \pm \infty\). The one-particle distribution function
in the thermodynamic limit leads to the Maxwell-Boltzmann distribution function with the effective mass of
particles \(m_{eff}\). This distribution allows us to find the physical interpretation for the variable of state \(\tilde{z}\) as the
order parameter of the interacting system of quasiparticles with mass \(m_{eff}\) and the physical interpretation for
the temperature \(T\) as the average kinetic energy of quasiparticles. The numerical example for the ideal gas
of quarks with two flavor and three color degrees of freedom in the nonrelativistic approximation shows that
the dense system of quarks in the dependence of values of the parameter \(\tilde{z}\) can pass from the strong coupled
state to the repulsive state of quarks. In the framework of the ideal gas of identical particles the equivalence of
the canonical and microcanonical ensembles in the thermodynamic limit is proved. This property is the key to
identifying the self-consistency of the statistical mechanics and thermodynamics.

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APPENDIX A: PHASE DISTRIBUTION FUNCTION

Here, to derive the distribution function in the canonical ensemble we use the Jaynes principle \[19\]. In this
respect, the Lagrange function can be written as

\[
\Phi[g'] = \frac{S[g']}{k} - \alpha \left( \int g' \ d\Gamma - 1 \right) - \beta \left( \int g' H \ d\Gamma - \langle H \rangle \right),
\]

(A1)

where \(g'\) is the probing distribution function. After maximizing the Lagrange function \(\Phi\), \(\delta \Phi_{\mid g'=\hat{g}} = 0\), and
using Eq. \(\ref{1}\) to eliminate the parameter \(\alpha\), we arrive at the following expression for the equilibrium phase
distribution function:

\[
\hat{g} = \left[ 1 + (q - 1) \frac{\beta}{q} (\Lambda - H) \right]^{\frac{1}{q - 1}},
\]

(A2)

where \(\Lambda = \langle H \rangle - qS/k\beta\). Differentiating the function \(\Lambda\) and Eq. \(\ref{1}\) with respect to
\(\beta\), and using the distribution

\[
\frac{\partial S}{\partial \beta} = k\beta \frac{\partial \langle H \rangle}{\partial \beta}.
\]

(A3)

The parameter \(\beta\) can be related to the temperature

\[
\frac{1}{T} \equiv \frac{\partial S}{\partial E} = \frac{\partial S/\partial \beta}{\partial \langle H \rangle/\partial \beta} = k\beta, \quad \beta = \frac{1}{kT}.
\]

(A4)

Then the distribution function \(\ref{A2}\) takes the form

\[
\hat{g} = \left[ 1 + (q - 1) \frac{\Lambda - H}{kTq} \right]^{\frac{1}{q - 1}} = \left[ 1 + \frac{1}{z + 1} \frac{\Lambda - H}{kT} \right]^z,
\]

(A5)

where \(\Lambda\) is determined from Eq. \(\ref{2}\)

\[
\int \left[ 1 + (q - 1) \frac{\Lambda - H}{kTq} \right]^{\frac{1}{q - 1}} d\Gamma = 1.
\]

(A6)

Note that Eqs. \(\ref{A5}\) and \(\ref{A6}\) are identical with \(\ref{9}\) and \(\ref{10}\). So the form of the distribution function in terms
of the variables of state is independent of the method of derivation.

Let us show that the distribution function expressed through the variables of state \((T, V, z, N)\) in \(\ref{1}\) is
equivalent to Eq. \(\ref{A5}\). So in \(\ref{1}\) the Lagrange function was written as

\[
\Phi[g'] = \frac{S[g']}{k} + \alpha \int g' \ d\Gamma - \alpha \beta(q - 1) \int g' H \ d\Gamma.
\]

(A7)
After maximizing (A7) we obtain
\[ \rho = \frac{1}{Z} \left[ 1 - \beta(q - 1)H \right]^{\frac{1}{q-1}}, \quad (A8) \]
\[ Z = \int \left[ 1 - \beta(q - 1)H \right]^{\frac{1}{q-1}} d\Gamma. \quad (A9) \]

To express the Lagrange parameter \( \beta \) through the variables of state \( T, V, z, N \), we use the method described in [11]. Finally, for the Lagrange parameter \( \beta \) we get
\[ \beta = \frac{Z^{q-1}}{kTq}. \quad (A10) \]

Substituting Eq. (A10) into Eqs. (A8) and (A9) and introducing the new function \( \Lambda \) in the following form:
\[ Z^{1-q} \equiv 1 + (q - 1) \frac{\Lambda}{kTq}, \quad (A11) \]
we obtain the phase distribution function (A5) with the normalization condition (A6). So the form of the Lagrange function does not disturb the distribution function in terms of the variables of state.

**APPENDIX B: THE FINITE PERFECT GAS**

Following the arguments given in Ref. [11] we easily derive the norm function \( \Lambda \) from Eq. (10) in the case of \( z < -1 \):
\[ 1 + \frac{1}{z + 1} \frac{\Lambda}{kT} = \left[ Z_G \frac{\Gamma(-z - \frac{3}{2}N)}{(-z - 1)^{\frac{3}{2}N}\Gamma(-z)} \right]^{-\frac{1}{z + 1}} \equiv B(T, V, z, N), \quad z < -1, \quad (B1) \]
where \( Z_G = \left( \frac{(gV)^N}{N!} \right) \left( \frac{mkT}{2\pi\hbar^2} \right)^{3N/2} \) is the partition function of the conventional ideal gas of the Boltzmann-Gibbs statistics [17, 20] and \(-z - \frac{3}{2}N > 0\). In the case of \( z > 0 \), we obtain
\[ 1 + \frac{1}{z + 1} \frac{\Lambda}{kT} = \left[ Z_G \frac{(z + 1)^{\frac{3}{2}N}\Gamma(z + 1)}{(z + 1 + \frac{3}{2}N)^\Gamma(z)} \right]^{-\frac{1}{z + 1}} \equiv B(T, V, z, N), \quad z > 0, \quad (B2) \]
where the new function \( B \) is introduced for convenience. In order to determine the energy of the system, we insert the Hamilton function \( A(x, p) = H(x, p) \) into Eq. (11) and after performing integration, we obtain
\[ \langle H \rangle = \frac{3}{2}kTN \frac{B}{1 + \frac{3}{2}N}, \quad (B3) \]
where the function \( B \) is determined from Eqs. (B1) and (B2). The norm function \( \Lambda \) and the energy \( \langle H \rangle \) allow us to calculate the thermodynamic potential of the canonical ensemble, the free energy \( F \). Substituting Eqs. (B1), (B2) and (B3) into Eq. (13), we find
\[ F = -kTz \left[ 1 - \frac{(1 + \frac{3}{2}N) B}{1 + \frac{3}{2}N} \right]. \quad (B4) \]
Then, the entropy of the system can be easily obtained from Eqs. (12) or (13):
\[ S = kT \left[ 1 - \frac{B}{1 + \frac{3}{2}N} \right]. \quad (B5) \]

Let us calculate the pressure \( p \), the chemical potential \( \mu \), and the variable \( X \). Taking into account Eqs. (21) and (B4), one finds
\[ p = \frac{NkT}{V} \frac{B}{1 + \frac{3}{2}N} = \frac{E}{3V}. \quad (B6) \]

Differentiating Eq. (B4) with respect to \( z \) in conformity with Eq. (22) we obtain
\[ X = kT \left( 1 - \frac{B}{1 + \frac{3}{2}N} \right) \left[ 1 - \frac{(1 + \frac{3}{2}N) B}{1 + \frac{3}{2}N} - \ln B + \psi \left( 1 + \frac{3}{2}N \right) - \psi(a) \right], \quad (B7) \]
where $\psi(y)$ is the psi-function which depends on arguments $a = -z, \gamma = -1$ for $z < -1$ and $a = z + 1, \gamma = 1$ for $z > 0$. Taking the derivative of (B4) with respect to $N$ in conformity with Eq. (22), we get

$$
\mu = \frac{3}{2} kT \frac{B}{1 + \frac{z + 1}{2} N} \left[ - \ln(\gamma(z + 1)B) + \psi \left( a + \gamma \frac{3}{2} N \right) \right] - \ln \left[ \frac{V \left( \frac{mkT}{2\pi h^2} \right)^{3/2}} {\gamma (\gamma (z + 1) 2 mkT B)^{z + \frac{3}{2}(N-1)}} \right].
$$ (B8)

The $N$-particle distribution function of the classical ideal gas in the canonical ensemble for the Tsallis statistics can be written as

$$
f(\vec{p}_1, \ldots, \vec{p}_N) = \frac{(gV)^N}{N! h^{3N}} \left[ 1 + \frac{1}{z + 1} \frac{\Lambda - \sum_{i=1}^{N} \frac{p_i^2}{2m}}{kT} \right]^z,
$$ (B9)

which is normalized to unity

$$
\int d^3p_1 \cdots d^3p_N f(\vec{p}_1, \ldots, \vec{p}_N) = 1.
$$ (B10)

Then the reduced single-particle distribution function can be easily obtained by directly performing the integral over the momenta $(\vec{p}_2, \ldots, \vec{p}_N)$:

$$
f(\vec{p}) = \left[ \frac{\Gamma(a + \gamma \frac{3}{2} N)}{\Gamma(a + \gamma \frac{3}{2}(N - 1))} \right]^{\gamma} \left( z + 1 \right)^{\frac{\gamma}{2} 2 mkTB} \left[ 1 - \frac{1}{z + 1} \frac{p^2}{2mkTB} \right]^{z + \frac{3}{2}(N-1)}.
$$ (B11)

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