Enhanced quantization on the circle

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Abstract

We apply the quantization scheme introduced in Klauder 2012 (arXiv:1204.2870) to a particle on a circle. We find that the quantum action functional restricted to appropriate coherent states can be expressed as the classical action plus \( \hbar \)-corrections. We succeed in proving this for a Hamiltonian endowed with a general periodic potential. This result extends the examples presented in the cited paper for a system on a nontrivial configuration space topology.

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(Some figures may appear in color only in the online journal)

1. Introduction

Conventional canonical quantization promotes a classical momentum \( p \) into a Hermitian operator \( P \) and a classical position \( q \) into a Hermitian operator \( Q \) which obey \([Q, P] = i\hbar\). Other classical quantities such as the Hamiltonian find a quantum counterpart as \( H(p, q) \longrightarrow \mathcal{H} = (P, Q) \) plus possible corrections of \( O(\hbar) \). This prescription works very well for many systems but it also has its limitations.

Enhanced quantization \cite{1, 2} offers a new interpretation of the very process of quantization that encompasses the usual canonical story and offers additional features as well. As a result, enhanced quantization has achieved important points that we cannot refrain to mention: the invariance of the theory under canonical transformations, the removal of singularities in classical solutions (hydrogen atom and quantum toy models \cite{3} and other simple cosmological models \cite{4}), the triviality of certain quantum field models gets replaced by a nontrivial behavior, a central point around the metric positivity preservation in quantum gravity kinematics, to mention a few (see the review \cite{3}). For the moment, let us review the main ingredients of the enhanced quantization program.

Provided both \( P \) and \( Q \) are self-adjoint operators (a stronger condition than Hermiticity) we can generate unitary operators acting on a fiducial state \(|\eta\rangle\) to yield

\[
|p, q\rangle = e^{-i q P} e^{i p Q} |\eta\rangle
\]

as a set of coherent states which spans the Hilbert space \( \mathcal{H} \). If the quantum action functional

\[
A_Q = \int_0^T \langle \psi(t) | [i\hbar \partial_t - \mathcal{H}] |\psi(t)\rangle \, dt
\]

is used in a variational principle, it leads to the Schrödinger equation \([i\hbar \partial_t - \mathcal{H}] |\psi(t)\rangle = 0\) as an equation of motion. However, arbitrary variations of \(|\psi(t)\rangle\) for a microscopic system are not accessible to a macroscopic observer who can only change the velocity or position of the microscopic system leading to the fact that the only state she/he could make are those represented by \(|p(t), q(t)\rangle\). The restricted quantum action functional becomes

\[
A_{Q(R)} = \int_0^T \langle p(t), q(t) | [i\hbar \partial_t - \mathcal{H}] |p(t), q(t)\rangle \, dt
\]

\[
= \int_0^T \left[ p(t) \dot{q}(t) - \mathcal{H}(p(t), q(t)) \right] \, dt
\]

which, because \( \hbar > 0 \) still, may be called an enhanced classical action functional whereas the usual classical action is given by

\[
A_C = \int_0^T \left[ p(t) \dot{q}(t) - \mathcal{H}_C(p(t), q(t)) \right] \, dt,
\]

where \( \mathcal{H}_C(p(t), q(t)) = \lim_{\hbar \to 0} \mathcal{H}(p(t), q(t)) \). In this view, then, classical theory is a subset of quantum theory, and
they both co-exist just like they do in the real world where $\hbar > 0$ [1].

Other two-dimensional continuous sheets of unit vectors may also correspond to a classical canonical system, and in [1], two other sets of coherent states were shown to have this property. In this paper, we consider the enhanced quantization of a classical particle moving on a circle of finite radius. This system leads to yet another set of coherent states which serves to unite the classical and quantum theories for such a system. Furthermore, we find that at the quantum level the nontrivial topology induces a possibly shifted momentum which can be reabsorbed by a canonical shift in momentum. These are our main results.

Quantum mechanics on a nontrivial topological space has been discussed in many contexts and still attracts attention (see [5–7] and more references therein) and coherent states on the circle have been also discussed in [8–12]. Nevertheless, none of these contributions addresses the issue of the relationship between classical and quantum actions as presented here. This paper provides another instance for which the rationale introduced in [1] yields an interesting quantum/classical connection.

2. Enhanced quantization on the circle

Consider a particle on the circle $S^1$. The position space can be parametrized by $\theta \in [-\pi, \pi]$. We consider, at the quantum level, a set of quantum operators $Q$ and $P$, associated with position and momentum, satisfying the commutation relation

$$[Q, P] = i\hbar. \tag{5}$$

The spectrum of the operator $Q$ is bounded in $[-\pi, \pi]$ and, like $\theta$, is periodic. To start the program of [1], the operators $P$ and $Q$ have to be self-adjoint such that the operators $e^{-i\frac{2\pi}{\hbar} P}$ and $e^{-i\frac{2\pi}{\hbar} Q}$ keep their ordinary and useful unitary feature.

2.1. Self-adjoint extension of $P$ on the circle

Self-adjoint extensions are well known on the space of square integrable functions on any finite interval [A, B], with vanishing boundary conditions (see for instance [13]). In a streamlined analysis, we review the properties of $P = -i\hbar \partial_\theta$ as an operator acting on $L^2([-\pi, \pi], d\theta)$.

Let us investigate the domain $D(P)$ of $P$ for $P$ being symmetric, i.e. $P^\dagger = P$ on $D(P)$. Consider the inner product for any two functions $\psi, \varphi, \psi', \varphi' \in L^2([-\pi, \pi], d\theta)$ (with as yet unspecified boundary values) given by

$$(\psi, P\varphi) = (-i\hbar) \int_{-\pi}^{\pi} \psi^*(\theta) \varphi'(\theta) \, d\theta$$

$$= (-i\hbar) \int_{-\pi}^{\pi} \psi^*(\theta) \varphi'(\theta) \, d\theta + (P\psi, \varphi) \tag{6}$$

so that for $(\psi, P\varphi) = (P\psi, \varphi)$ to hold on $D(P)$, one requires

$$\psi^*(\pi)\psi(-\pi) - \psi^*(-\pi)\psi(\pi) = 0. \tag{7}$$

This condition is satisfied if we adopt $\psi(\pm \pi) = 0$ and make no restriction on $\psi$. In this case $D(P) = \{\psi; \varphi, \psi' \in L^2([-\pi, \pi], \psi(\pi) = \varphi(-\pi) = 0\}$. However, the domain of $P^\dagger, D(P^\dagger) = \{\psi; \varphi, \psi' \in L^2([-\pi, \pi], d\theta)\} \supset D(P)$.

Defining the self-adjoint extension of $P$ is a procedure aimed at rendering $D(P^\dagger) = D(P)$ by enlarging $D(P)$ and restricting $D(P^\dagger)$ so that

$$\tilde{D}(P_\alpha) = \{\psi; \varphi, \psi' \in L^2([-\pi, \pi], d\theta); \varphi(\pi) = \varphi(-\pi) = 0\} \tag{8}$$

As noticed in [1], having defined self-adjoint extensions for $P_\alpha$ and $Q$ (which is trivial here), we can define unitary operators by exponentiating these generators.

2.2. Coherent states on the circle

Let us now pursue the quantum program associated with (5) henceforth denoting $P$ by $P_\alpha$, where $\alpha \in [0, 1]$ labels the different inequivalent representations of the momentum operator. We will use units such that $Q$ is dimensionless and so the dimension of $P$ is that of $\hbar$. We define eigenvectors $|\theta\rangle$ for the operator $Q$, satisfying $\langle \theta | \theta' \rangle = \delta_\alpha(\theta - \theta')$, where $\delta_\alpha$ should be understood as periodic on $S^1$, as well as eigenvectors $|n, \alpha\rangle$ of $P_\alpha$, obeying $\langle n, \alpha | m, \alpha \rangle = \delta_{nm, \alpha}$, such that

$$Q |\theta\rangle = \theta |\theta\rangle, \quad P_\alpha |n, \alpha\rangle = \alpha |n, \alpha\rangle.$$ 

It is well known that the spectrum of $P_\alpha$ on the circle is such that $p_{n, \alpha} = h(n + \alpha), (n, \alpha) \in \mathbb{Z} \times [0, 1)$; (more features introduced by the topology of the manifold and self-adjoint properties of $P$ can be found in [5–7]). As a realization of the functions $|\theta |n, \alpha\rangle$, we find that the normalized wave functions are given by

$$|\theta |n, \alpha\rangle = \frac{1}{\sqrt{2\pi}} e^{i\theta |n, \alpha\rangle \psi}. \tag{10}$$

The self-adjoint operators $P_\alpha$ and $Q$ yield the unitary operators $e^{-i\alpha P_\alpha}$ and $e^{-i\alpha P_\alpha}Q$, where $(q, p) \in S^1 \times \mathbb{R}$. From these operators, a set of states is defined by

$$|p, q\rangle = e^{-i\alpha P_\alpha}e^{-i\alpha Q}|\eta_\alpha\rangle. \tag{11}$$

where $|\eta_\alpha\rangle$ is called the fiducial state. We verify that the set $\{|p, q\rangle\}$ satisfies

1. a normalization condition: $\langle p, q | p, q \rangle = |\eta_\alpha \rangle |\eta_\alpha\rangle = 1$, since $|\eta_\alpha \rangle |\eta_\alpha\rangle$ is normalized;

2. a resolution of unity:

$$\int_{\mathbb{R} \times S^1} |p, q\rangle \langle p, q\rangle \frac{dp \, dq}{2\pi \hbar} = I_\hbar. \tag{12}$$

Indeed, it can be shown that, for all $\theta, \theta' \in [\pi, \pi]$,

$$\int_{\mathbb{R} \times S^1} \langle \theta | p, q \rangle \langle p, q | \theta' \rangle \frac{dp \, dq}{2\pi \hbar} = \delta_\pi(\theta - \theta') \times \int_{S^1} \langle \theta - q | \eta_\alpha \rangle \langle \eta_\alpha | \theta - q \rangle dq = \langle \theta | \theta' \rangle \langle \eta_\alpha | \eta_\alpha \rangle, \tag{13}$$
where we used the fact that \( \int_{-\pi}^{\pi} dq |q\rangle \langle q| = \int_{-\pi}^{\pi} dq |q\rangle \langle q| \) because of periodicity. Thus (12) is recovered for a normalized \(|\eta_0\rangle\).

The set of states \(|p, q\rangle\) forms an overcomplete family of normalized states. Henceforth, these states will be called coherent states.

We can now discuss the dynamics associated with such states by introducing a general quantum Hamiltonian of the form \( \mathcal{H}(P, e^{iQ}, e^{-iQ}) \). It has been argued that quantum and classical mechanics should coexist (see for instance \([14-18]\)). In the present situation and using the coherent states, we establish a link between the quantum and the classical actions.

Consider the restricted quantum action associated with \(|\psi(t)\rangle \rightarrow |p(t), q(t)\rangle\) defined above, which leads to

\[
A_{Q(R)} = \int_0^T \langle p(t), q(t) \bigg| i\hbar \partial_t - \mathcal{H} \bigg| p(t), q(t) \rangle dt. \tag{14}
\]

As explained earlier, within this action functional a macroscopic observer can vary, not the entire Hilbert space of states, but only the coherent-state subset, when studying a microscopic system. We choose a class of fiducial vectors satisfying

\[
\langle \eta_0 | Q | \eta_0 \rangle = 0 \quad \text{and} \quad \langle \eta_0 | P_a | \eta_0 \rangle = \hbar \alpha \tag{15}
\]

so that \(|\eta_0\rangle\) is in the domain of the self-adjoint operators \(Q\) and \(P_a\).

In the ordinary canonical situation, the choice of the fiducial vector \(|\eta\rangle\) as the ground state of an harmonic oscillator makes it an extremal weight vector: \(|Q + iP\rangle|0\rangle = 0\), and the latter relation yields \(|0\rangle Q|0\rangle = 0\) and \(|0\rangle P|0\rangle = 0\). A wider class of fiducial vectors \(|\eta\rangle\) for the canonical case are those which are ‘physically centered’, meaning only that \(|\eta\rangle Q|\eta\rangle = 0\) and \(|\eta\rangle P|\eta\rangle = 0\). Hence, (15) can be considered an analogue of this condition modified slightly due to the topology of the configuration space. A straightforward calculation using (15) yields

\[
\langle p(t), q(t) \bigg| i\hbar \partial_t \bigg| p(t), q(t) \rangle = (\hbar \alpha + p) \dot{q}. \tag{16}
\]

Furthermore, we have

\[
H_\alpha(p, q, t) := \langle p(t), q(t) \big| \mathcal{H}(P_a, e^{iQ}, e^{-iQ}) \big| p(t), q(t) \rangle = \langle \eta_0 | \mathcal{H}(P_a + p, e^{i(Qt + \alpha q)}, e^{-i(Qt + \alpha q)}) | \eta_0 \rangle. \tag{17}
\]

Hence, the restricted quantum action reads

\[
A_{Q(R)} = \int_0^T \left[ \frac{p \dot{q} - H_\alpha(p, q, t)}{\hbar} \right] + \hbar \alpha \dot{q} \, dt, \tag{18}
\]

where the term \(\tilde{A}_C = \int_0^T [p \dot{q} - H_\alpha(p, q, t)] \, dt\), as in the ordinary situation \([1]\), can be related to a classical action \(A_C\) up to \(\hbar\) corrections since

\[
H_\alpha(p, q) = H_C(p, q) + O(\hbar; p, q). \tag{19}
\]

In the last equation, \(H_C(p, q)\) is viewed as the usual classical Hamiltonian. Interestingly, we notice that the quantum parameter \(\alpha\) induces a surface term \(\hbar \alpha \dot{q}\) in \(A_{Q(R)}\) which makes no influence on the enhanced classical equations of motion whatsoever. Finally, we can assert that

\[
A_{Q(R)} = A_C + O(\hbar). \tag{20}
\]

Let us discuss, in more detail, three examples which can be evaluated completely.

**Example 1.** We choose the quantum Hamiltonian as

\[
\mathcal{H}(P_a, e^{iQ}, e^{-iQ}) = P_a^2 + V(e^{iQ}, e^{-iQ}),
\]

\[
V(e^{iQ}, e^{-iQ}) = a_0 + \sum_{n=1}^{m} [a_n \cos nQ + b_n \sin nQ] \quad \tag{21}
\]

with mass units chosen so that \(1/2\mu = 1\) and where \(m \in \mathbb{N}\). Next, we choose a particular fiducial vector such that

\[
\eta_\alpha(\theta) := \langle \theta | \eta_\alpha \rangle = N e^{i(\sigma/\hbar)(\cos \theta - \alpha \sin \theta)},
\]

\[
N = \left[ 2\pi e^{-2r/h} I_0 \left( \frac{2r}{h} \right) \right]^{-1/2}, \tag{22}
\]

where \(r/h > 0\). \(I_0(\cdot)\) denotes a modified Bessel function \([19, chapter 9]\), and \(N\) is a normalization factor fixed such that \(|\eta_\alpha| \langle \eta_\alpha | \eta_\alpha \rangle = 1\). Note that \(|\eta_\alpha| \langle \eta_\alpha | \eta_\alpha \rangle\) is even and periodic and that (15) is satisfied. For \(r/h \gg 1\), the following approximation is valid for \(|\theta| \leq \pi:\n
\[
|\eta_\alpha(\theta)\rangle^2 = N^2 e^{i(\cos \theta - 1)} \lesssim K N^2 e^{-2\alpha^2} \tag{23}
\]

for a sufficiently large constant \(K\); hence \(|\theta| \lesssim \sqrt{\hbar/2T}\) which is small. One can therefore consider \(\eta_\alpha(\theta)\) as cutting-off large \(\theta\)-values. Evaluating the diagonal coherent state matrix elements of \(\mathcal{H}\), and setting \(\alpha' = \hbar \alpha\), leads to

\[
\langle p(t), q(t) \big| \mathcal{H}(P_a, e^{iQ}, e^{-iQ}) \big| p(t), q(t) \rangle = \langle \eta_0 | (P_a + p)^2 + V(e^{i(Qt + \alpha q)}, e^{-i(Qt + \alpha q)}) | \eta_0 \rangle = \langle \alpha' + p)^2 - \langle \eta_0 | P_a^2 | \eta_0 \rangle + a_0 + \sum_{n=1}^{m} [a_n \cos nq + b_n \sin nq] + O(\hbar) = (p + \alpha')^2 + V(e^{i\alpha}, e^{-i\alpha}) + O(\hbar), \tag{24}
\]

where, in the last equality, \(|\eta_0| \langle P_a^2 | \eta_0 \rangle\) and \(\alpha^2\) are henceforth included in \(O(\hbar)\). Thus, we can infer that

\[
A_{Q(R)} = \int_0^T \left[ (p + \alpha') \dot{q} - \left[ (p + \alpha')^2 + V(e^{i\alpha}, e^{-i\alpha}) + O(\hbar) \right] \right] \, dt. \tag{25}
\]

Therefore, up to constants and a canonical shift in momentum \((p \rightarrow p - \alpha')\),

\[
A_{Q(R)} = \int_0^T \left[ p \dot{q} - \left[ p^2 + V(e^{i\alpha}, e^{-i\alpha}) + O(\hbar) \right] \right] \, dt = A_C + O(\hbar). \tag{26}
\]

Some comments are in order at this stage. During the evaluation of (24), the expectation value \(|\langle \eta_0 | (P_a + p)^2 | \eta_0 \rangle\) contains terms \(\alpha'\) of order \(O(\hbar)\). We have preferred to remove
these terms in a unified way by making a canonical shift of momenta at the last stage (26). This is of course without any consequence for the result. Furthermore, the quantity $\langle \eta_a | V(e^{i Q q}, e^{-i Q q})) | \eta_a \rangle$ contains, strictly speaking, at the first order of approximation, terms of order $O(\hbar/r)$. However, keeping a unique small parameter $\hbar$, we have finally $O(\hbar/r) = O(\hbar)$, for a fixed $r > 0$ (for instance $r = 100$), so that (24) has a clear meaning. A way to see this is to consider $|\eta_a(\theta)\rangle^2$, a locally integrable function on $[-\pi, \pi]$; indeed, as $J = 2r/\hbar$ approaches infinity, it acts as a distribution.

Changing notation, we write

$$\langle \eta_a | V(Q + q) | \eta_a \rangle = \int_{-\pi}^{\pi} |\eta_a(\theta)\rangle^2 V(\theta + q) d\theta = V(q) + \int_{-\pi}^{\pi} |\eta_a(\theta)\rangle^2 [V(\theta + q) - V(q)] d\theta.$$ 

(27)

In the large $J = 2r/\hbar$ limit, $|\eta_a(\theta)\rangle^2 = (\sqrt{2\pi})e^{-Jq^2/2 + O(1/J)}$ hence acts like a Dirac $\delta$ function up to some corrections (see figure 1). As an analogy, consider a sequence of locally integrable functions $f_a(\theta) = \frac{1}{\sqrt{2\pi}}e^{i\theta} \theta_{-\pi}^{\pi}$ giving $\lim_{a \rightarrow \infty} f_a(\theta) = \delta(\theta)$ (for $|\theta| \leq \pi$) in the sense of distributions [20], such that

$$\lim_{a \rightarrow \infty} \int_{-\pi}^{\pi} [V(\theta + q) - V(q)] |\eta_a(\theta)\rangle^2 d\theta = \lim_{a \rightarrow \infty} \int_{-\pi}^{\pi} [V(\theta + q) - V(q)] f_a(\theta) d\theta = 0.$$  

(28)

This generates a large class of potentials for which the procedure is valid.

**Example 2.** Let us consider now a singular type of potential such that

$$V(Q) = [1 - \cos Q]^a \quad 0 < a < 1/2$$  

(29)

for eigenvalues of $Q$ in $[-\pi, 0) \cup (0, \pi)$, using the same kinetic part $P_a^2$ as well as the fiducial vector (22). The procedure computing the restricted quantum action yields the same kinetic part $p q - p^2 + O(\hbar)$ as in (26). Using again the same recipe as previously done starting by (27), the difference $[V(\theta + q) - V(q)]$ is a well defined integrable function for $0 < a < 1/2$. This limitation on $a$ is due to the fact that, for small values $|\theta| \ll 1$, $[1 - \cos(\theta)]^{-a} \propto |\theta|^{-2a}$ such that $\int |\theta|^{-2a} |\eta_a(\theta)\rangle^2 d\theta < \infty$ is defined only for $a < 1/2$. We can conclude that $\langle \eta_a | V(Q + q) | \eta_a \rangle = V(q) + O(\hbar/r)$.

**Example 3.** Finally, let us consider the Hamiltonian, written with a particular Hermitian order,

$$\mathcal{H}(P_a, Q) = P_a [1 - \cos Q] P_a$$  

(30)

with $b > 0$, for which we evaluate the expectation value for coherent states with fiducial vector (22) as

$$\langle p(t), q(t) | \mathcal{H}(P_a, Q) | p(t), q(t) \rangle = \langle \eta_a | (P_a + p)[1 - \cos(Q + q)]P_a + p | \eta_a \rangle = p^2 \langle \eta_a | [1 - \cos(Q + q)]P_a + O(\hbar) \rangle = p^2 \int_{-\pi}^{\pi} |\eta_a(\theta)\rangle^2 [1 - \cos(\theta + q)] d\theta + O(\hbar).$$  

(31)

The fact that $|\eta_a(\theta)\rangle^2$ is a $\delta$ function up to correction $O(\hbar/r)$ at large $r/\hbar$ allows us to get the classical Hamiltonian

$$H_C(p, q) = [1 - \cos q] p^2$$  

(32)

plus corrections of order $O(\hbar)$.

### 2.3. Uncertainty relation

In the present context, the deviation for $Q$ and $P_a$ on coherent states can be studied and yields $\Delta Q \Delta P > h/2$. Insisting on the saturation of the uncertainty relation, in general, this feature occurs if the choice of the fiducial state $|\eta\rangle$ is made such that it becomes extremal in the representation space of some given operators. We deduce that $(i/\hbar) P_a |\eta\rangle = (-J \sin \theta + i a) |\eta\rangle$ and define $A(Q) = (J \sin Q - i a)$ so that

$$\langle A + (i/\hbar) P_a |\eta\rangle = 0 \quad \text{and} \quad [A(Q), P_a] = i h J \cos Q.$$  

(33)

Note that $A(Q)$ is not self-adjoint unless $a = 0$. In this paragraph, we will adjust the analysis for complex operators. Consequently, since $\langle \eta_a | P | \eta_a \rangle = h a$, the following relations hold:

$$\langle \eta_a | A | \eta_a \rangle = -\langle i/\hbar | \eta_a | P_a | \eta_a \rangle = -i a$$  

$$\langle \eta_a | A^2 | \eta_a \rangle = (1/h^2) \langle \eta_a | P_a^2 | \eta_a \rangle.$$  

(34)

Now evaluating the deviation on some coherent states, one finds

$$\langle A(Q) \rangle^2 = \langle \eta_a | A^\dagger (Q + q) A(Q + q) | \eta_a \rangle = - \langle \eta_a | A(Q + q) | \eta_a \rangle^2.$$  

(35)
\[
J^2(\sin q)^2[1 - I_{10}(J)^2] + J[(\cos q)^2 - (\sin q)^2]I_{10}(J), \\
(\Delta P)^2 = J(h^2/4)I_{10}(J), \\
\langle \eta_0 | P^2 | \eta_0 \rangle = \hbar^2 \alpha^2 + J(h^2/4)I_{10}(J), \\
\text{where } I_{10}(J) = I_1(J)/I_0(J) \text{ stands for the ratio of two modified Bessel functions and } J = 2\pi/\hbar. \text{ Note that the expression (35) cannot be easily recast in terms (34) or (36) because of shifts in position and this even for the choice } \alpha = 0. \text{ We get after some algebra}
\]

\[
\frac{\Delta A(Q)^2}{\Delta P^2} = \frac{(h^2/4)J^2}{J(\sin q)^2[1 - I_{10}(J)^2]I_{10}(J)} + \cos 2q I_{10}(J)^2, \\
J^2(h^2/4)(\cos Q)^2 = J^2(h^2/4)(\cos q)^2 I_{10}(J)^2.
\]

Finally, using coherent states from the fiducial vector \(|\eta_0\rangle\), we have the relation

\[
\frac{\Delta A(Q)^2}{\Delta P^2} = (h^2/4)J^2 \left[J(\sin q)^2[1 - I_{10}(J)^2]I_{10}(J) + (1 - (\tan q)^2)\right](\cos Q)^2
\]

which provides an exact evaluation of the uncertainty relation with respect to the algebra (33). Not surprisingly, we do not recover an exact saturation of the uncertainty valid for all coherent states. Nevertheless, from (38), one checks that for \(q_0 = 0\) and \(-\pi\), on the coherent state subspace generated by \([p, q_0] = e^{\pm i\eta_0 P} e^{\pm i\eta_0 Q}|\eta_0\rangle\), the uncertainty relation is indeed saturated namely \(\Delta \sin Q \times \Delta P = (h/2)(\cos Q)\).

2.4. Canonical transformations

Canonical transformations are well defined in the present setting. These transformations involve a change of variables \((p, q) \to (\tilde{p}, \tilde{q})\) such that the symplectic structure at the classical level is preserved leading to \([\tilde{q}, \tilde{p}] = 1 = [q, p]\) as well as \(p d\tilde{q} = \tilde{p} dq + dG(\tilde{p}, \tilde{q})\), where \(G\) is the generator of such a transformation. Coherent states in the transformed coordinates, namely \(|\tilde{p}, \tilde{q}\rangle\), are chosen to be the same as before change of variables. Thus, it is clear that the restricted quantum action computed with respect to \(|\tilde{p}, \tilde{q}\rangle \equiv |p(\tilde{p}, \tilde{q}), q(\tilde{p}, \tilde{q})\rangle = (p, q)\) yields a classical action \(A_C\) corresponding to \(A_C\) up to a surface term given by \(G(\tilde{p}, \tilde{q})\), which has no influence on the equations of motion. No change of the operators is involved.

2.5. Coherent state induced geometry on phase space

The geometry of the coherent states [16] can be investigated as well using the Fubini–Study metric element, \(c \, d\sigma_a = \hbar^2/\pi \frac{d\langle p, q \rangle dl}{d(p, q)} d(p, q)^2\), for a constant \(c\). We have

\[
\frac{d(p, q)}{d(p, q)} = \frac{1}{\hbar^2} \left[(\ell + 2\alpha' + p^2) d\eta^2 + \ell d\eta^2\right], \\
\langle p, q d(p, q) \rangle^2 = \frac{1}{\hbar^2}(p + \alpha')^2 d\eta^2,
\]

where \(\ell := \langle \eta_\mu | Q^2 | \eta_\nu \rangle\) and \(\ell := \langle \eta_\mu | P^2 | \eta_\nu \rangle\). Finally, the metric on the subspace of coherent states can be written as

\[
d\sigma^2 = (\ell - \alpha^2) d\eta^2 + \ell d\eta^2 + d\eta^2 + d\eta^2,
\]

which we have set \(q_a = \sqrt{1/A_e}\), \(c = 1 = A_p\) and \(A_e = (\ell - \alpha^2)/c\). The metric \(d\sigma^2\) describes a flat geometry with a cylindrical topology. If desired, this metric can be imposed on the classical phase space as well.

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