On the Classification of $LS$-Sequences

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Abstract

This paper addresses the question whether the $LS$-sequences constructed in [Car12] yield indeed a new family of low-discrepancy sequences. While it is well known that the case $S = 0$ corresponds to van der Corput sequences, we prove here that the case $S = 1$ can be traced back to symmetrized Kronecker sequences and moreover that for $S \geq 2$ none of these two types occurs anymore. In addition, our approach allows for an improved discrepancy bound for $S = 1$ and $L$ arbitrary.

1 Introduction

There are essentially three classical families of low-discrepancy sequences, namely Kronecker sequences, digital sequences and Halton sequences (compare [Lar14], see also [Nie92]). In [Car12], Carbone constructed a class of one-dimensional low-discrepancy sequences, called $LS$-sequences with $L \in \mathbb{N}$ and $S \in \mathbb{N}_0$. The case $S = 0$ corresponds to the classical one-dimensional Halton sequences, called van der Corput sequences. However, the question whether $LS$-sequences indeed yield a new family of low-discrepancy sequences for $S \geq 1$ or if it is just a different way to write down already known low-discrepancy sequences has not been answered yet. In this paper, we address this question and thereby derive improved discrepancy bounds for the case $S = 1$.

Discrepancy. Let $S = (z_n)_{n \geq 0}$ be a sequence in $[0,1)^d$. Then the discrepancy of the first $N$ points of the sequence is defined by

$$D_N(S) := \sup_{B \subset [0,1)^d} \left| \frac{A_N(B)}{N} - \lambda_d(B) \right|,$$
where the supremum is taken over all axis-parallel subintervals $B \subset [0,1)^d$ and $\mathcal{A}_N(B) := \# \{ n \mid 0 \leq n < N, z_n \in B \}$ and $\lambda_d$ denotes the $d$-dimensional Lebesgue-measure. In the following we restrict to the case $d = 1$. If $D_N(S)$ satisfies

$$D_N(S) = O(N^{-1} \log N)$$

then $S$ is called a low-discrepancy sequence. In dimension one this is indeed the best possible rate as was proved by Schmidt in [Sch72], that there exists a constant $c$ with

$$D_N(S) \geq cN^{-1} \log N.$$  

The precise value of the constant $c$ is still unknown (see e.g. [Lar14]). For a discussion of the situation in higher dimensions see e.g. [Nie92], Chapter 3.

A theorem of Weyl and Koksma’s inequality imply that a sequence of points $(z_n)_{n \geq 0}$ is uniformly distributed if and only if

$$\lim_{N \to \infty} D_N(z_n) = 0.$$ 

Thus, the only candidates for low-discrepancy sequences are uniformly distributed sequences. A specific way to construct uniformly distributed sequences goes back to the work of Kakutani [Kak76] and was later on generalized in [Vol11] in the following sense.

**Definition 1.1.** Let $\rho$ denote a non-trivial partition of $[0,1)$. Then the $\rho$-refinement of a partition $\pi$ of $[0,1)$, denoted by $\rho \pi$, is defined by subdividing all intervals of maximal length positively homothetically to $\rho$.

Successive application of a $\rho$-refinement results in a sequence which is denoted by $\{\rho^n \pi\}_{n \in \mathbb{N}}$. The special case of Kakutani’s $\alpha$-refinement is obtained by successive $\rho$-refinements where $\rho = \{[0,\alpha),[\alpha,1)\}$. If $\pi$ is the trivial partition $\pi = \{[0,1)\}$ then we obtain Kakutani’s $\alpha$-sequence. In many articles Kakutani’s $\alpha$-sequence serves as a standard example and the general results derived therein may be applied to this case (see e.g. [CV07], [DII2], [IZ15], [Vol11]). Another specific class of examples of $\rho$-refinement was introduced in [Car12].

**Definition 1.2.** Let $L \in \mathbb{N}, S \in \mathbb{N}_0$ and $\beta$ be the solution of $L\beta + S\beta^2 = 1$. An $LS$-sequence of partitions $\{\rho^0_{L,S} \pi\}_{n \in \mathbb{N}}$ is the successive $\rho$-refinement of the trivial partition $\pi = \{[0,1)\}$ where $\rho_{L,S}$ consists of $L+S$ intervals such that the first $L$ intervals have length $\beta$ and the successive $S$ intervals have length $\beta^2$. 

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The partition \( \{ \rho_{L,S}^n \} \) consists of intervals only of length \( \beta^n \) and \( \beta^{n+1} \). Its total number of intervals is denoted by \( t_n \), the number of intervals of length \( \beta^n \) by \( l_n \) and the number of intervals of length \( \beta^{n+1} \) by \( s_n \). In [Car12], Carbone derived the recurrence relations

\[
\begin{align*}
t_n &= Lt_{n-1} + St_{n-2} \\
l_n &= Ll_{n-1} + Sl_{n-2} \\
s_n &= Ls_{n-1} + ss_{n-2}
\end{align*}
\]

for \( n \geq 2 \) with initial conditions \( t_0 = 1, t_1 = L + S, l_0 = 1, l_1 = L, s_0 = 0 \) and \( s_1 = S \). Based on these relations, Carbone defined a possible ordering of the endpoints of the partition yielding the LS-sequence of points. One of the observations of this paper is that this ordering indeed yields a simple and easy-to-implement algorithm but also has a certain degree of arbitrariness.

**Definition 1.3.** Given an LS-sequence of partitions \( \{ \rho_{L,S}^n \}_{n \in \mathbb{N}} \), the corresponding **LS-sequence of points** \( (\xi^n)_{n \in \mathbb{N}} \) is defined as follows: let \( \Lambda_{L,S}^1 \) be the first \( t_1 \) left endpoints of the partition \( \rho_{L,S}^1 \) ordered by magnitude. Given \( \Lambda_{L,S}^n = \{ \xi^{(n)}_1, \ldots, \xi^{(n)}_{t_n} \} \) an ordering of \( \Lambda_{L,S}^{n+1} \) is then inductively defined as

\[
\begin{align*}
\Lambda_{L,S}^{n+1} &= \{ \xi^{(n)}_1, \ldots, \xi^{(n)}_{t_n} , \\
&\quad \psi_{1,0}^{(n+1)}(\xi^{(n)}_1), \ldots, \psi_{L,0}^{(n+1)}(\xi^{(n)}_{t_n}) , \ldots, \psi_{L,0}^{(n+1)}(\xi^{(n)}_{t_n}) , \\
&\quad \psi_{L,1}^{(n+1)}(\xi^{(n)}_1), \ldots, \psi_{L,1}^{(n+1)}(\xi^{(n)}_{t_n}) , \ldots, \psi_{L,S-1}^{(n+1)}(\xi^{(n)}_{t_n}) \} ,
\end{align*}
\]

where

\[
\psi_{i,j}^{(n)}(x) = x + i\beta^n + j\beta^{n+1}, \quad x \in \mathbb{R}.
\]

As the definition of LS-sequences might not be completely intuitive at first sight, we illustrate it by an explicit example.

**Example 1.4.** For \( L = S = 1 \) the LS-sequence coincides with the so-called **Kakutani-Fibonacci sequence** (see [CTV13]). We have

\[
\begin{align*}
\Lambda_{1,1}^1 &= \{0, \beta\} \\
\Lambda_{1,1}^2 &= \{0, \beta, \beta^2\} \\
\Lambda_{1,1}^3 &= \{0, \beta, \beta^2, \beta^3, \beta + \beta^3\} \\
\Lambda_{1,1}^4 &= \{0, \beta, \beta^2, \beta^3, \beta + \beta^3, \beta^4, \beta + \beta^4, \beta^2 + \beta^4\} \\
\end{align*}
\]

and so on.
Theorem 1.5 (Carbone, [Car12]). If $L \geq S$, then the corresponding LS-sequence has low-discrepancy.

Carbone’s proof is based on counting arguments but does not give explicit discrepancy bounds. These have been derived later by Iacò and Ziegler in [IZ15] using so-called generalized LS-sequences. A more general result implicating also the low-discrepancy of LS-sequences can be found in [AH13].

Theorem 1.6. [Iacò, Ziegler, [IZ15], Theorem 1, Section 3] If $(\xi_n)_{n \in \mathbb{N}}$ is an LS-sequence with $L \geq S$ then

$$D_N(\xi_n) \leq \frac{B \log(N)}{N |\log(\beta)|} + \frac{B + 2}{N},$$

where

$$B = (2L + S - 2) \left( \frac{R}{1 - S\beta} + 1 \right),$$

with

$$R = \max \left\{ |\tau_1|, |\tau_1 + (L + S - 2)\lambda_1| \right\},$$

$$\tau_1 = -\frac{L-2S+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}} \quad \text{and} \quad \lambda_1 = -\frac{L+\sqrt{L^2+4S}}{2\sqrt{L^2+4S}}.$$

It has been pointed out that for parameters $S = 0$ and $L = b$, the corresponding LS-sequence coincides with the classical van der Corput sequence, see e.g. [AHZ14] However, for higher values of $S$ it has been not been proved if LS-sequences indeed yield a new family of examples of low-discrepancy sequences or are just a new formulation of some of the well-known ones. We close this gap to a certain extent by showing the following main result:

Theorem 1.7. For $S = 1$, the LS-sequences is a reordering of the symmetrized Kronecker sequences $(\{n\beta\})_{n \in \mathbb{Z}}$. For $S \geq 2$ the LS-construction neither yields a (r-e-)ordering of a van der Corput sequence nor of a (symmetrized) Kronecker sequence.

Let us make the notion of symmetrized Kronecker sequences more precise: given $z \in \mathbb{R}$, let $\{z\} := z - \lfloor z \rfloor$ denote the fractional part of $z$. A (classical) Kronecker sequence is a sequence of the form $(z_n)_{n \geq 0} = (\{nz\})_{n \geq 0}$. If $z \notin \mathbb{Q}$ and $z$ has bounded partial quotients in its continued fraction expansion (see Section 2) then $(z_n)$ has low-discrepancy ([Nie92], Theorem 3.3). By a symmetrized Kronecker sequence we simply mean a sequence indexed over $\mathbb{Z}$ of the form $(\{nz\})_{n \in \mathbb{Z}}$ with ordering

$$0, \{z\}, \{-z\}, \{2z\}, \{-2z\}, \ldots.$$

\footnote{If the reader is not familiar with the Definition of van der Corput sequences, he may consult [Nie92], Section 3.1.}
Note that it is still open, whether for $S \geq 2$ an LS-sequence is a reordering of some other well-known low-discrepancy sequence such as a digital-sequence or if the LS-construction really yields a new class of examples.

Our approach does not only give a significantly shorter proof of low-discrepancy of LS-sequences for $L = 1$ but also improves the known discrepancy bounds by Iacó and Ziegler in this case.

**Corollary 1.8.** For $S = 1$ the discrepancy of the LS-sequence $(\xi_n)_{n \in \mathbb{N}}$ is bounded by

$$D_N(\xi_n) \leq \frac{3}{N} + \left( \frac{1}{\log(\alpha)} + \frac{L}{\log(L + 1)} \right) \frac{\log(N)}{N},$$

where $\alpha = (1 + \sqrt{5})/2$.

Corollary 1.8 indeed improves the discrepancy bounds for LS-sequences given in Theorem 1.6 in the specific case $S = 1$. Both results yield inequalities of the type

$$D_N(\xi_n) \leq \gamma \frac{N}{N} + \delta \frac{\log(N)}{N}.$$

For instance, if $L = S = 1$ then Corollary 1.8 implies $\gamma = 3$ and $\delta = 2.776$ while according to Theorem 1.6 the discrepancy can be bounded by $\gamma = 3.447$ and $\delta = 3.01$. The difference between the two results gets the more prominent the larger $L$ is: If $L = 10$ and $S = 1$ we get $\gamma = 3$ and $\delta = 5.51$ while Theorem 1.6 only implies $\gamma = 22.87$ and $\delta = 9.03$.

2 Proof of the main results

**Continued fractions.** Recall that every irrational number $z$ has a uniquely determined infinite continued fraction expansion

$$z = a_0 + \frac{1}{1 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}}, =: [a_0; a_1; a_2; \ldots],$$

where the $a_i$ are integers with $a_0 = \lfloor z \rfloor$ and $a_i \geq 1$ for all $i \geq 1$. The sequence of *convergents* $(r_i)_{i \in \mathbb{N}}$ of $z$ is defined by

$$r_i = [a_0; a_1; \ldots; a_i].$$

\footnote{We obtain different numerical values than in [IZ15]. We checked our result on different computer algebra systems.}
The convergents $r_i = p_i/q_i$ with $\gcd(p_i, q_i) = 1$ can also be calculated directly by the recurrence relation

\[
p_{-1} = 0, \quad p_0 = 1, \quad p_i = a_ip_{i-1} + p_{i-2}, \quad i \geq 0
\]
\[
q_{-1} = 1, \quad q_0 = 0, \quad q_i = a_iq_{i-1} + q_{i-2}, \quad i \geq 0.
\]

**Remark 2.1.** If $S = 1$, then $\beta^2 + L\beta - 1 = 0$ or equivalently

\[
\frac{1}{\beta} = L + \beta
\]

holds. Thus it follows that $a_i = L$ in the continued fraction expansion of $\beta$ for all $i = 1, 2, \ldots$.

From now on the continued fraction expansion of $\beta$ is studied and it is always tacitly assumed, that the $q_i$’s are the denominators of the convergents of $\beta$. Although the proof of the following lemma is rather obvious we write it down here explicitly because our proof of the main theorem is based on this arithmetic observation.

**Lemma 2.2.** Let $n \in \mathbb{N}_0$. If $S = 1$ then we have

(i) $\beta^{2n+1} + q_{2n} = q_{2n+1}\beta$.

(ii) $\beta^{2n} - q_{2n-1} = -q_{2n}\beta$

**Proof.** We prove both claims by induction.

(i) The identity is trivial for $n = 0$. So we come to the induction step

\[
\beta^{2n+1} + q_{2n} = \beta^2\beta^{2n-1} + q_{2n}\left(\beta^2 + L\beta\right)
\]
\[
= \beta^2(\beta^{2n-1} + q_{2n}) + Lq_{2n}\beta
\]
\[
= \beta^2(q_{2n-1}\beta - q_{2n-2} + q_{2n}) + Lq_{2n}\beta
\]
\[
= \beta^2(q_{2n-1}\beta + Lq_{2n-1}) + Lq_{2n}\beta
\]
\[
= q_{2n-1}\beta(\beta^2 + L\beta) + Lq_{2n}\beta
\]
\[
= q_{2n+1}\beta.
\]

(ii) The proof works analogously as in (i). We have $\beta^2 + 1 = -L\beta$ and

\[
\beta^{2n} - q_{2n-1} = \beta^2\beta^{2(n-1)} - q_{2n-1}\left(\beta^2 + L\beta\right)
\]
\[
= \beta^2(\beta^{2(n-1)} - q_{2n-1}) - Lq_{2n-1}\beta
\]
\[
= \beta^2(-q_{2n-2}\beta + q_{2n-3} - q_{2n-1}) - Lq_{2n-1}\beta
\]
\[
= \beta^2(-q_{2n-2}\beta - Lq_{2n-2}) - Lq_{2n-1}\beta
\]
\[
= -q_{2n-2}\beta(\beta^2 + L\beta) - Lq_{2n-1}\beta
\]
\[
= -q_{2n}\beta.
\]
Example 2.3. Consider the Kakutani-Fibonacci sequence from Example [L.4]. If we denote by \((f_n)_{n\geq 0}\) the Fibonacci sequence, i.e. the sequence inductively defined by \(f_0 = 0, f_1 = 1\) and \(f_n = f_{n-1} + f_{n-2}\) for \(n \geq 2\), we have that \(q_i = f_i\) for all \(i = 1, 2, \ldots\).

If \(S = 1\), then we can furthermore deduce from Definition [L.3] that \(t_{n+1} = t_n + Ll_n\) and that \(q_{n-1} = l_n\). Starting from \(\xi_1\) we split the \(LS\)-sequence into consecutive blocks where the first block \(B_1\) is of length 1 and the \(n\)-th block \(B_n\) for \(n \geq 2\) is of length \(Ll_n = Lq_{n-1} = t_n - t_{n-1}\). We now study the blocks \(B_n\)

\[
B_n = \psi_1^{(n)}(\xi_1), \ldots, \psi_{L,0}^{(n)}(\xi_{n-1}), \ldots, \psi_{L,0}^{(n)}(\xi_1), \ldots, \psi_{L,0}^{(n)}(\xi_{n-1})
\]

\[
= \xi_1 + \beta^{n-1}, \ldots, \xi_{n-1} + \beta^{n-1}, \ldots, \xi_1 + L\beta^{n-1}, \ldots, \xi_{n-1} + L\beta^{n-1}.
\]

Lemma 2.4. Let \(n \in \mathbb{N}\).

(i) If \(n = 2k+1\) is odd, then \(B_n\) considered as a set consists of the \(L \cdot q_{2k}\) elements \((-q_{2k-1})\beta, \ldots, -(q_{2k+1} - 1)\beta\) (respectively of the element 0 if \(n = 1\)).

(ii) If \(n = 2k\) is even, then \(B_n\) considered as a set consists of the \(L \cdot q_{2k-1}\) elements \((q_{2k-2} + 1)\beta, \ldots, \{q_{2k-2} + 2\}\beta\).

Before going into the rather technical details of the proof, let us explain its idea for the example of the Kakutani-Fibonacci sequence \((L = S = 1)\). This sequence of points is given by

\[
\begin{array}{cccccccccccc}
  0 & B_0 & \beta & B_1 & \beta^2 & B_2 & \beta^3 & B_3 & \beta^4 & B_4 & \beta^5 & \ldots
\end{array}
\]

Using \(\beta + \beta^2\) this can be easily re-written as

\[
\begin{array}{cccccccccccc}
  0 & B_0 & \beta & 1 - \beta & 2\beta - 1 & 3\beta - 1 & 2 - 3\beta & 2 - 2\beta & 3 - 4\beta & B_4 & \beta^5 & \ldots
\end{array}
\]

Proof. The two assertions are proved simultaneously by induction on \(k\). For \(n = 1, 2\) the claim is obvious from definition, since \(\xi_1 = 0\) and \(\xi_2 = \beta, \ldots, \xi_l = L\beta\). Let \(k \geq 2\) and \(n = 2k + 1\) be odd. If we denote by \(\equiv\) equivalence modulo 1 we have for \(m \in \{0, \ldots, l_n - 1\}\) by Lemma [2.2] and induction hypothesis

\[
\xi_m + j\beta^{2k+1-1} \equiv \xi_m - jq_{2k}\beta \equiv (r - jq_{2k})\beta,
\]
with $-q_{2k-1}+1 \leq r \leq -q_{2k-3}$ and $q_{2k-2} + 1 \leq r \leq q_{2k}$ and $1 \leq j \leq L$. Thus it follows that

$$-q_{2k-1} + 1 - Lq_{2k} \leq r - jq_{2k} \leq q_{2k} - q_{2k} \iff -(q_{2k+1} - 1) \leq r - jq_{2k} \leq 0.$$ 

Since the sequence is injective, the claim follows for odd $n$. So let $n = 2k + 2$ be even. Then we use again Lemma 2.2 and induction hypothesis to derive

$$\xi_m + j\beta^{2k+2-1} \equiv \xi_m + jq_{2k+1}\beta \equiv (r + jq_{2k+1})\beta,$$

with $-q_{2k-1} + 1 \leq r \leq -q_{2k-3}$ and $q_{2k-2} + 1 \leq r \leq q_{2k}$ and $1 \leq j \leq L$. This completes the induction since

$$-q_{2k-1} + 1 + q_{2k+1} \leq r + jq_{2k+1} \leq q_{2k} + Lq_{2k+1} \iff 1 \leq r + jq_{2k+1} \leq q_{2k+2}.$$

\[\Box\]

**Proof of Theorem 1.7.** If $S = 1$ the LS-sequence is indeed a reordering of the symmetrized Kronecker sequence by Lemma 2.4. So let $S \geq 2$ and $L \geq S$. Then $\beta$ is irrational and the recurrence relation

$$\beta^2 = \frac{1 - L\beta}{S}. \quad (1)$$

holds. Hence the LS-sequence cannot be a reordering of a van der Corput sequence (which consists only of rational number).

Now assume that the LS-sequence is the reordering of a (possibly symmetrized) Kronecker sequence $\{n\alpha\}$ for some $\alpha \in \mathbb{R}$. Since $\alpha$ itself has to be an element of the LS-sequence, there exists an $n \in \mathbb{N}$ such that $\alpha$ can be uniquely written in the form

$$\alpha = \sum_{k=1}^{n} \alpha_k\beta^k$$

with $\alpha_k \in \{0,\ldots,L\}$ for $k = 1,\ldots,n$ and $\alpha_n \neq 0$. By (1) we have the equality $\beta^k = x_k\beta + y_k$ with $x_k, y_k \in \mathbb{Q}$ and $s^k x_k, s^k y_k \in \mathbb{Z}$. Thus, $\alpha$ itself can be rewritten as $\alpha = x_\alpha \beta + y_\alpha$ with $x_\alpha, y_\alpha \in \mathbb{Q}$ and $s^n x_\alpha, s^n y_\alpha \in \mathbb{Z}$. However, $\beta^{n+1}$, which is an element of the LS-sequence, cannot be an element of $\{n\alpha\}$, since $\beta^{n+1} = x_{n+1}\beta + y_{n+1}$, where at least one of $x_{n+1}$ and $y_{n+1}$ has denominator $s^{n+1}$. This is a contradiction. \[\Box\]
A main advantage of the approach via symmetrized Kronecker sequence is that it yields a possibility to calculate improved discrepancy bounds, namely Corollary 1.8.

Proof of Corollary 1.8. We imitate the proofs in [Nie92], Theorem 3.3 and [KN74], Theorem 3.4 respectively and leave away here the technical details that are explained therein very nicely: The number $N$ can be represented in the form

$$N = \sum_{i=0}^{l(N)} c_i q_i,$$

where $l(N)$ is the unique non-negative integer with $q_{l(N)} \leq N < q_{l(N)+1}$ and where the $c_i$ are integers with $0 \leq c_i \leq L$. Let $LS_N$ denote the set consisting of the first $N$ numbers of the $LS$-sequence. We decompose $LS_N$ into blocks of consecutive terms, namely $c_i$ blocks of length $q_i$ for all $0 \leq i \leq l(N)$. Consider a block of length $q_i$ and denote the corresponding point set by $A_i$. If $i$ is odd, $A_i$ consists of the fractional parts $\{nz\}$ with $n = n_i, n_i + 1, \ldots, n_i + q_i - 1$ according to Lemma 2.4. As shown in the proof of [Nie92], Theorem 3.3., this point set has discrepancy

$$D_{q_i}(A_i) < \frac{1}{q_i-1} + \frac{1}{q_i}.$$

If $i$ is even, $A_i$ consists of the fractional parts $\{-nz\}$ with again $n = n_i, n_i + 1, \ldots, n_i + q_i - 1$ by Lemma 2.4. Since $z$ and $-z$ have the same continued fraction expansion up to signs, we also have

$$D_{q_i}(A_i) < \frac{1}{q_i-1} + \frac{1}{q_i}.$$

Analogous calculations as in [KN74] then yield the assertion.

Asymptotically we deduce the following behaviour, again improving the more general result of [IZ15] in the special case $S = 1$.

Corollary 2.5. If $S = 1$, then we obtain

$$\lim_{N \to \infty} \frac{ND_N(\xi_n)}{\log N} \sim \frac{L}{\log(L)}$$

as $L \to \infty$.

Finally, we would like to point out the fact that it follows immediately from our approach that the Kakutani-Fibonacci sequence is the reordering of an orbit of an ergodic interval exchange transformation. In [CIV14], it was shown that a much more complicated interval exchange transformation is necessary in order to get the original ordering given in Definition 1.3.
**Corollary 2.6.** For $L = 1$, the $LS$-sequence is always a reordering of an orbit of an ergodic interval exchange transformation.

**Proof.** The map $R_\alpha : x \mapsto x + \alpha \pmod{1}$, the rotation of the circle by $\alpha$, is ergodic for $\alpha \notin \mathbb{Q}$, see e.g. [EW11], Example 2.2. Moreover, it is an interval exchange transformation, compare e.g. [Via06].

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**References**

[AH13] Aistleitner, C., Hofer, M.: “Uniform distribution of generalized Kakutani’s sequences of partitions”, Ann. Mat. Pura Appl. 192 (4), 529–538 (2013).

[AHZ14] Aistleitner, C., Hofer, M., Ziegler, V.: “On the uniform distribution modulo 1 of multidimensional $LS$-sequences”, Ann. Mat. Pura Appl. (4) 193, no. 5, 1329–1344 (2014).

[Car12] Carbone, I.: “Discrepancy of $LS$-sequences of partitions and points”, Ann. Mat. Pura Appl. 191, 819–844 (2012).

[CIV14] Carbone, I., Iacò M., Volčič, A.: “A dynamical systems approach to the the Kakutani-Fibonacci sequence”, Ergodic Th. & Dynam. Sys., 1794–1806 (2014).

[CV07] Carbone, I., Volčič, A.: “Kakutani’s splitting procedure in higher dimension”, Rend. Ist. Mathem. Univ. Trieste, XXXIX: 1–8 (2007).

[DI12] Drmota, M., Infusino, M.: “On the discrepancy of some generalized Kakutani’s sequences of partitions”, Unif. Distrib. Theory 7 (1), 75–104 (2012).

[EW11] Einsiedler, M., Ward, T.: “Ergodic Theory”, Springer, Berlin (2011).

[IZ15] Iacò, M., Ziegler, V.: “Discrepancy of generalized $LS$-sequences”, arXiv:1503.07299 (2015).

[Kak76] Kakutani, S.: “A problem on equidistribution on the unit interval $[0, 1]$”, in: Measure Theory (Proc. Conf. Oberwolfach, 1975), Lecture Notes in Mathematics, 541, Springer, Berlin, 369–375 (1975).
[KN74] Kuipers, L., Niederreiter, H.: “Uniform distribution of sequences”, John Wiley & Sons, New York (1974).

[Lar14] Larcher, G.: “Discrepancy estimates for sequences: new results and open problems”, in: Kritzer, P., Niederreiter, H., Pillichshammer, F., Winterhof, A. (eds.) Uniform Distribution and Quasi-Monte Carlo Methods, Radon Series in Computational and Applied Mathematics, 171–189. DeGruyter, Berlin (2014).

[Nie92] Niederreiter, H.: “Random Number Generation and Quasi-Monte Carlo Methods”, Number 63 in CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia (1992).

[Sch72] Schmidt, W. M.: “Irregularities of distribution VII”, Acta Arith., 21, 45-50 (1972).

[Via06] Viana, M.: “Ergodic Theory of Interval Exchange Maps”, Rev. Mat. Complut 19(1), 7-100 (2006).

[Vol11] Volčič, A.: “A generalization of Kakutani’s splitting procedure”, Ann. Mat. Pura Appl. (4) 190(1), 45-54 (2011).

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