SHARP BOUNDS FOR GENERALIZED ELLIPTIC INTEGRALS
OF THE FIRST KIND

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Abstract. In this paper, we prove that the double inequality

\[ 1 + \alpha r^2 < \frac{K_a(r)}{\sin(\pi a) \log(e^{R(a)/2}/r')} < 1 + \beta r^2 \]

holds for all \( a \in (0, 1/2] \) and \( r \in (0, 1) \) if and only if \( \alpha \leq \pi/[R(a) \sin(\pi a)] - 1 \)
and \( \beta \geq a(1-a) \), where \( r' = \sqrt{1 - r^2} \), \( K_a(r) \) is the generalized elliptic integral
of the first kind and \( R(x) \) is the Ramanujan constant function. Besides, as the key tool, the series expression for the Ramanujan constant function \( R(x) \) is given.

1. Introduction

For \( r \in (0, 1) \), Legendre’s complete elliptic integrals [1] of the first kind and
the second kind are given by

\[ K = K(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2(t)}} \]
\[ E = E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2(t)} dt, \]

respectively. They are the particular cases of Gaussian hypergeometric function

\[ F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n (-1 < x < 1), \]

where \((a)_n = \Gamma(a + n)/\Gamma(a)\) and \( \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \) \((x > 0)\) is the gamma
function. Indeed, we have

\[ K(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad E(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \]

It is well known that the complete elliptic integrals and Gaussian hypergeometric
function have important applications in quasiconformal mappings, number theory,
and other fields of the mathematical and mathematical physics. For instance, the
Gaussian arithmetic-geometric mean \( AGM \) and the modulus of the plane Grötzsch
ring can be expressed in terms of the complete elliptic integrals of the first kind,
and the complete elliptic integrals of the second kind gives the formula of the
perimeter of an ellipse. Moreover, Ramanujan modular equation and continued

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fraction in number theory are both related to the Gaussian hypergeometric function
\( F(a, b; c; x) \). For these, and the properties for the complete elliptic integrals and
Gaussian hypergeometric function see \([2, 3, 5, 6, 9, 11, 16, 19-21]\).

For \( r \in (0, 1) \) and \( a \in (0, 1) \), the generalized elliptic integrals (see \([4, 10]\)) are
defefined by
\[
K_a = K_a(r) = \pi \frac{\Gamma(a, 1-a; r^2)}{2}
\]  
(1.1)
and
\[
E_a = E_a(r) = \pi \frac{\Gamma(a-1, 1-a; r^2)}{2}.
\]  
(1.2)
Clearly \( K_a(0) = E_a(0) = \pi/2, K_a(1^-) = \infty \) and \( E_a(1) = [\sin(\pi a)]/[2(1-a)] \). In the
particular case \( a = 1/2 \), the generalized elliptic integrals reduce to the complete
elliptic integrals. By symmetry of (1.1), we assume that \( a \in (0, 1/2] \) in the sequence.

The generalized elliptic integrals satisfy the following derivative formulas:
\[
\frac{d}{dr} K_a = \frac{2(1-a)}{r^2} (E_a - r^2 K_a), \quad \frac{d}{dr} E_a = -\frac{2(1-a)}{r} (K_a - E_a),
\]  
(1.3)
\[
\frac{d}{dr} (K_a - E_a) = \frac{2(1-a)}{r^2} E_a, \quad \frac{d}{dr} (E_a - r^2 K_a) = 2a r K_a.
\]  
(1.4)
Here and in what follows we set \( r' = \sqrt{1-r^2} \) for \( r \in (0, 1) \).

In 2000, Anderson, Qiu, Vamanamurthy and Vuorinen \([4]\) reintroduced the gen-
eralized elliptic integrals in geometry function theory, found that the generalized
elliptic integrals of the first kind \( K_a \) arises from the Schwarz-Christoffel trans-
formation of the upper half-plane onto a parallelogram, and established several
monotonicity theorems for the generalized elliptic integrals \( K_a \) and \( E_a \).

Recently, the generalized elliptic integrals have attracted the attention of many
mathematicians. In particular, many remarkable properties and inequalities for the
generalized elliptic integrals can be found in the literature \([8, 12-14, 22]\).

Very recently, Takeuchi \([18]\) discussed the generalized trigonometric function and
found a new form of the generalized complete elliptic integrals.

In \([16]\), Qiu and Vamanamurthy proved that the inequality
\[
\frac{K(r)}{\log(4/r')} < 1 + \frac{1}{4} r'^2
\]  
(1.5)
holds for all \( r \in (0, 1) \).

Alzer \([3]\) proved that the inequality
\[
1 + \left( \frac{\pi}{4 \log 2} - 1 \right) r'^2 < \frac{K(r)}{\log(4/r')}
\]  
(1.6)
holds for all \( r \in (0, 1) \). Moreover, Alzer also proved that the constant factors \( 1/4 \) in
(1.5) and \( \pi/(4 \log 2) - 1 \) in (1.6) are best possible.

The main purpose of this paper is to generalize inequalities (1.5) and (1.6) to
\( K_a \). Our main result is the following Theorem 1.1.

**Theorem 1.1.** Let \( R(x) \) be the Ramanujan constant function defined by (2.1).
Then the double inequality
\[
1 + \alpha r'^2 < \frac{K_a(r)}{\sin(\pi a) \log(e^{R(a)}/2/r')} < 1 + \beta r'^2
\]  
(1.7)
holds for all \( a \in (0, 1/2] \) and \( r \in (0, 1) \) if and only if \( \alpha \leq \alpha_0 = \pi/[R(a) \sin(\pi a)] - 1 \)
and \( \beta \geq \beta_0 = a(1-a) \).
2. Some properties for Ramanujan constant function $R(x)$

For $x \in (0, 1/2]$, the Ramanujan constant function $R(x)$ ([17]) is given by

$$R(x) = -2\gamma - \Psi(x) - \Psi(1 - x), \quad R(1/2) = 4 \log 2,$$

where $\gamma = \lim_{n \to \infty} (\sum_{k=1}^{n} 1/k - \log n) = 0.577215 \cdots$ is the Euler-Mascheroni constant, and

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

It is well known that $\Psi^{(n)}(x) \ (n \geq 0)$ has the series expansion as follows:

$$\Psi^{(n)}(x) = \begin{cases} 
-\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)}, & n = 0 \\
(-1)^{n+1}n! \sum_{k=0}^{\infty} \frac{1}{x+k}^{n+1}, & n \geq 1.
\end{cases}$$

The purpose of this section is to present the series expansion formula for $R(x) (x \in (0, 1/2])$ (Theorem 2.2) and establish two important inequalities involving $R(x)$ (Corollaries 2.4 and 2.5), which will be used in the proof of our main result.

**Lemma 2.1.** The function $\xi(x) = 1/[x(1-x)] - R(x)$ is strictly increasing from $(0, 1/2]$ onto $(1, 4 - 4\log 2]$.

**Proof.** Differentiating $\xi$ yields

$$\xi'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} + \Psi'(x) - \Psi'(1-x)$$

$$= -\frac{1}{x^2} + \frac{1}{(1-x)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+x)^2} - \sum_{k=0}^{\infty} \frac{1}{(k+1-x)^2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} - \sum_{k=1}^{\infty} \frac{1}{(k+1-x)^2} > 0$$

for $x \in (0, 1/2]$. Moreover, $\xi(1/2) = 4 - R(1/2) = 4 - 4\log 2$ and

$$\lim_{x \to 0^{-}} \xi(x) = \lim_{x \to 0^{-}} \left( \frac{1}{x} + \frac{1}{1-x} + \Psi(x) + \Psi(1-x) + 2\gamma \right)$$

$$= \lim_{x \to 0^{-}} \left( \frac{1}{x} + \frac{1}{1-x} - \gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)} - \gamma - \frac{1}{1-x} + \sum_{k=1}^{\infty} \frac{1}{k(k+1-x)} \right)$$

$$= \lim_{x \to 0^{-}} \left( \sum_{k=1}^{\infty} \frac{x}{k(k+x)} + \sum_{k=1}^{\infty} \frac{1}{k(k+1-x)} \right) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

**Theorem 2.2.** The Ramanujan constant function $R(x)$ has the following series expansion:

$$R(x) = \frac{1}{x} + 2\zeta(3)x^2 + 2\zeta(5)x^4 + \cdots + 2\zeta(2k+1)x^{2k} + \cdots$$
\[ f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 0} x (1 + \frac{1}{1-x} - \sum_{k=1}^{\infty} \frac{x}{k(k+x)} - \sum_{k=1}^{\infty} \frac{1-x}{k(k+1-x)}) = 1, \quad (2.4) \]

\[ f'(x) = R(x) + xR'(x) = -\Psi(x) - \Psi(1-x) - 2\gamma + x[-\Psi'(x) + \Psi'(1-x)] \]

\[ = \frac{1}{x} + \frac{1}{1-x} - \sum_{k=1}^{\infty} \frac{x}{k(k+x)} - \sum_{k=1}^{\infty} \frac{1-x}{k(k+1-x)} + x \left[-\sum_{k=0}^{\infty} \frac{1}{(k+x)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+1-x)^2}\right] \]

\[ = \frac{1}{1-x} - \sum_{k=1}^{\infty} \frac{x^2}{k(k+x)^2} - \sum_{k=1}^{\infty} \frac{1-x}{k(k+1-x)} + \sum_{k=0}^{\infty} \frac{x}{(k+1-x)^2}. \]

\[ \lim_{x \to 0} f'(x) = 0. \quad (2.5) \]

For \( n \geq 2 \), it follows from (2.2) that

\[ R^{(n)}(x) = -\Psi^{(n)}(x) - (-1)^n \Psi^{(n)}(1-x) \]

\[ = - n! (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} - (-1)^n n! (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{(1-x+k)^{n+1}} \]

\[ = n! \left[ \sum_{k=0}^{\infty} \frac{1}{(1-x+k)^{n+1}} + (-1)^n \frac{1}{(x+k)^{n+1}} \right]. \]

Therefore, we get

\[ f^{(n)}(x) = n R^{(n-1)}(x) + x R^{(n)}(x) \]

\[ = n! \sum_{k=0}^{\infty} \left[ \frac{x}{(1-x+k)^{n+1}} + (-1)^{n-1} \frac{k}{(x+k)^n} \right] \]

Furthermore, if \( n \) is even, then

\[ f^{(n)}(x) = n! \sum_{k=0}^{\infty} \left[ \frac{1+k}{(1-x+k)^{n+1}} - \frac{k}{(x+k)^{n+1}} \right] \]

\[ = n! \sum_{k=0}^{\infty} \left[ \frac{1+k}{(1-x+k)^{n+1}} - \frac{1+k}{(x+1+k)^{n+1}} \right], \]

\[ \lim_{x \to 0} f^{(n)}(x) = 0. \quad (2.6) \]
If \( n \) is odd, then
\[
\begin{align*}
 f^{(n)}(x) &= n! \sum_{k=0}^{\infty} \left[ \frac{1 + k}{(1 - x + k)^{n+1}} + \frac{k}{(x + k)^{n+1}} \right] \\
 &= n! \sum_{k=0}^{\infty} \left[ \frac{1 + k}{(1 - x + k)^{n+1}} + \frac{1 + k}{(x + 1 + k)^{n+1}} \right],
\end{align*}
\]
\[
\lim_{x \to 0} f^{(n)}(x) = 2n! \sum_{k=0}^{\infty} \frac{1}{(k+1)^n} = 2n! \zeta(n). \tag{2.7}
\]
Equations (2.4)-(2.7) implies that \( f(x) \) has the following Taylor series expansion
\[
f(x) = 1 + 2 \zeta(3)x^3 + 2 \zeta(5)x^5 + \ldots = 1 + \sum_{k=1}^{\infty} 2\zeta(2k + 1)x^{2k+1}, \quad x \in (0, 1).
\]
Therefore, (2.3) follows. \( \square \)

**Theorem 2.3.** The function
\[
\eta(x) = \frac{\pi/ \sin(\pi x) - R(x)}{x(1 - x)}
\]
is strictly decreasing from \((0, 1/2] \) onto \([4\pi - 16 \log 2, \pi^2/6)\).

**Proof.** Clearly (2.1) gives \( \eta(1/2) = 4\pi - 16 \log 2 \). Simple computations lead to
\[
x^2(1 - x)^2 \eta'(x) = -\left[ \frac{\pi^2 \cos(\pi x)}{\sin^2(\pi x)} + R'(x) \right] x(1 - x) - \left[ \frac{\pi}{\sin(\pi x)} - R(x) \right] (1 - 2x),
\]
\[
\eta' \left( \frac{1}{2} \right) = 0. \tag{2.8}
\]
Making use of the well-known formulas (see [1, p.75, 4.3.68] and [7, p.16 and p.56])
\[
\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} - 2}{(2k)!} B_{2k} x^{2k-1}, \quad |x| < \pi,
\]
\[
B_{2k} = 2(-1)^{k+1} \frac{\zeta(2k)(2k)!}{(2\pi)^{2k}},
\]
where \( B_k \) is the Bernoulli number, we can rewrite \( \eta(x) \) as
\[
\eta(x) = \frac{\pi x/ \sin(\pi x) - xR(x)}{x^2(1 - x)} = \sum_{k=1}^{\infty} \frac{2^{2k} - 2}{2^{2k-1}} \zeta(2k)x^{2k} - \sum_{k=1}^{\infty} 2\zeta(2k + 1)x^{2k+1}
\]
\[
= \sum_{k=0}^{\infty} \frac{2^{2k+2} - 2}{2^{2k+1}} \zeta(2k + 2)x^{2k} - \sum_{k=0}^{\infty} 2\zeta(2k + 3)x^{2k+1}
\]
\[
= \frac{1}{(1 - x)}. \]
Therefore, \( \eta(0^+) = \zeta(2) = \pi^2/6 \) and
\[
(1 - x)^2 \eta'(x)
\]
\[
= \left[ \sum_{k=1}^{\infty} \frac{2^{2k+2} - 2}{2^{2k+1}} (2k) \zeta(2k+2)x^{2k-1} - \sum_{k=0}^{\infty} 2(2k + 1) \zeta(2k + 3)x^{2k} \right] (1 - x)
\]
\[\begin{align*}
&+ \sum_{k=0}^{\infty} \frac{2^{2k+2} - 2}{2^{2k+1}} \zeta(2k + 2)x^{2k} - \sum_{k=0}^{\infty} 2\zeta(2k + 3)x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)\zeta(2k + 4)x^{2k+1} - \sum_{k=0}^{\infty} 2(2k + 1)\zeta(2k + 3)x^{2k} \\
&\quad - \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)\zeta(2k + 4)x^{2k+2} + \sum_{k=0}^{\infty} 2(2k + 1)\zeta(2k + 3)x^{2k+1} \\
&\quad + \sum_{k=0}^{\infty} \frac{2^{2k+2} - 2}{2^{2k+1}} \zeta(2k + 2)x^{2k} - \sum_{k=0}^{\infty} 2\zeta(2k + 3)x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)\zeta(2k + 4)x^{2k+1} - \sum_{k=0}^{\infty} 2(2k + 1)\zeta(2k + 3)x^{2k} \\
&\quad - \sum_{k=0}^{\infty} 2\zeta(2k + 3)x^{2k+1} - \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)\zeta(2k + 4)x^{2k+2} \\
&= g(x).
\end{align*}\]

Differentiating \(g(x)\) yields

\[\begin{align*}
g'(x) &= \sum_{k=1}^{\infty} \frac{2^{2k+2} - 2}{2^{2k+1}} (2k)\zeta(2k + 2)x^{2k-1} + \sum_{k=0}^{\infty} 2(2k + 1)^2 \zeta(2k + 3)x^{2k} \\
&\quad + \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 1)(2k + 2)\zeta(2k + 4)x^{2k} \\
&\quad - \sum_{k=1}^{\infty} 2(2k)(2k + 1)\zeta(2k + 3)x^{2k-1} - \sum_{k=0}^{\infty} 2(2k + 1)\zeta(2k + 3)x^{2k} \\
&\quad - \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)^2 \zeta(2k + 4)x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)\zeta(2k + 4)x^{2k+1} - \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 2)^2 \zeta(2k + 4)x^{2k+1} \\
&\quad + \sum_{k=0}^{\infty} 2(2k + 1)^2 \zeta(2k + 3)x^{2k} - \sum_{k=0}^{\infty} 2(2k + 1)\zeta(2k + 3)x^{2k} \\
&\quad + \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 1)(2k + 2)\zeta(2k + 4)x^{2k} \\
&\quad - \sum_{k=0}^{\infty} 2(2k + 2)(2k + 3)\zeta(2k + 5)x^{2k+1} \\
&= - \sum_{k=0}^{\infty} \frac{2^{2k+4} - 2}{2^{2k+3}} (2k + 1)(2k + 2)\zeta(2k + 4)x^{2k+1}
\end{align*}\]
Corollary 2.4. The inequality

\[ \frac{\pi}{R(x) \sin(\pi x)} - 1 > \frac{\sin(\pi x) - \pi x(1 - x)}{\sin(\pi x) [R(x) - 1]} \]

(2.13)

holds for all \( x \in (0, 1/2) \).
Proof. We clearly see that inequality (2.13) is equivalent to
\[
\frac{\pi[1 + x(1 - x)]R(x) - \pi \sin(\pi x)R(x)^2}{x(1 - x)} > 0
\]
or
\[
R(x) \sin(\pi x) \frac{\pi/\sin(\pi x) - R(x)}{x(1 - x)} - \frac{1 - x(1 - x)R(x)}{x(1 - x)} > 0.
\]

It follows from Lemma 2.1 and Theorem 2.3 together with the fact that \(x \to R(x) \sin(\pi x)\) is strictly decreasing from \((0, 1/2]\) onto \([4 \log 2, \pi]\) (see [15, Theorem 2]) that
\[
R(x) \sin(\pi x) \frac{\pi/\sin(\pi x) - R(x)}{x(1 - x)} - \frac{1 - x(1 - x)R(x)}{x(1 - x)} > 4 \log 2(4 \pi - 16 \log 2) - \pi(4 - 4 \log 2) = 20 \log 2 - 64 \log 2 = 0
\]
for \(x \in (0, 1/2]\). □

Corollary 2.5. The inequality
\[
x(1 - x) > \frac{\pi - R(x) \sin(\pi x)}{R(x) \sin(\pi x)}
\]
holds for all \(x \in (0, 1/2]\).

Proof. It follows from Theorem 2.3 that
\[
R(x) \sin(\pi x) \frac{\pi/\sin(\pi x) - R(x)}{x(1 - x)} > \log 16 - \sin(\pi x) \left[\frac{\pi/\sin(\pi x) - R(x)}{x(1 - x)}\right] > \log 16 - \sin(\pi x) \frac{\pi^2}{6} > \log 16 - \frac{\pi^2}{6} = 1.127 \cdots > 0
\]
for \(x \in (0, 1/2]\). Therefore, inequality (2.14) follows easily from (2.15). □

3. Proof of Theorem 1.1

Lemma 3.1. (see [5, Theorem 1.25]) For \(-\infty < a < b < \infty\), let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\), and be differentiable on \((a, b)\), let \(g'(x) \neq 0\) on \((a, b)\). If \(f'(x)/g'(x)\) is increasing (decreasing) on \((a, b)\), then so are
\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]
If \(f'(x)/g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 3.2. The inequality
\[
\sin(\pi x) > \frac{\pi x(1 - x)}{2} [2 + x(1 - x)]
\]
holds for all \(x \in (0, 1/2]\).
Lemma 3.3. For \( a \in (0, 1/2) \), defined the function \( F \) on \( (0, 1) \) by

\[
F(r) = r^4 \sin(\pi a) - 2a(1-a)r^2 r^2 K_a + 2(1-a)(r^2 - r^2)(E_a - r^2 K_a). \tag{3.2}
\]

Then \( F(r) \) is strictly increasing from \( (0, 1) \) onto \( (\sin(\pi a) - \pi a(1-a) - (\pi/2)2a(1-a)^2, a(1-a)\sin(\pi a)) \).

Proof. Let \( F_1(r) = \sin(\pi a) - 2a(1-a)r^2 K_a + 2(1-a)(r^2 - r^2)(E_a - r^2 K_a) \), \( F_2(r) = r^2 \). Then \( F(r) = F_1(r)/F_2(r) \), \( F_1(1) = F_2(1) = 0 \).

\[
\begin{align*}
F_1'(r) &= -2a(1-a) \left[ -\frac{2}{r^3} K_a + \frac{2}{r} \left( \frac{2(1-a)}{r^2} - 2(1-a) \right) \right] \left( \frac{E_a - r^2 K_a}{r^2} \right) \\
&\quad + 2(1-a) \left( \frac{2}{r^2} K_a + 2(1-a) \left( \frac{r^2}{r^2} - 1 \right) \frac{2ar^2 K_a - 2(E_a - r^2 K_a)}{r^3} \right) \\
&= 4a(1-a) \frac{K_a}{r^3} - 4a(1-a)^2 \frac{E_a - r^2 K_a}{r^3} - 4(1-a) \frac{E_a - r^2 K_a}{r^3} \\
&\quad + 4(1-a)(r^2 - r^2) \frac{ar^2 K_a - (E_a - r^2 K_a)}{r^3}.
\end{align*}
\]

From (3.7) and (3.8) we clearly see that there exists \( x_0 \in (0, 1/2) \) such that \( h(0^+) = 0 \) for \( x \in (0, x_0) \) and \( h(0^+) > 0 \) for \( x \in (x_0, 1/2] \). Thus \( h''(x) \) is strictly decreasing on \( (0, x_0] \) and strictly increasing on \( [x_0, 1/2] \).

Equation (3.6) and the piecewise monotonicity of \( h''(x) \) implies that \( h''(x) < 0 \) for \( x \in (0, 1/2) \). Hence \( h''(x) \) is strictly decreasing on \( (0, 1/2) \). Then (3.5) leads to the conclusion that there exists \( x_1 \in (0, 1/2) \) such that \( h''(x) \) is strictly increasing on \( (0, x_1] \) and strictly decreasing on \( [x_1, 1/2) \).

It follows from (3.3) and (3.4) together with the piecewise monotonicity of \( h''(x) \) that \( h(x) \) is strictly increasing on \( (0, 1/2) \) and \( h(x) > h(0^+) = 0 \) for \( x \in (0, 1/2) \).

Therefore, inequality (3.1) follows easily from (3.2) and \( h(x) > 0 \) for \( x \in (0, 1/2) \). \( \square \)
where

\[
F_3(r) = \left[2 - 2r^2 + a(1-a)r^2\right] \frac{E_a - r^2K_a}{r^2} - 2ar^2K_a.
\]

Making use of series expansion, we get

\[
\frac{2}{\pi} F_3(r) = \left\{2 + [a(1-a) - 2]r^2\right\} \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n} - 2a(1-a)^2 \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n+2}
\]

\[
= a \left\{2 \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n} + [a(1-a) - 2] \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n+2}
\right. \]

\[
- 2 \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n} + \frac{2}{(n+2)!} \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n+1)!} r^{2n+2}
\right\}
\]

\[
= a \left[2 + a(1-a) \right] \sum_{n=0}^{\infty} \frac{(a)_{n+1}(1-a)_{n+1}}{(n+3)!} r^{2n+4}. \tag{3.10}
\]

From (3.9) and (3.10) one has

\[
\frac{F_1'(r)}{F_2'(r)} = \pi a(1-a) \left[2 + a(1-a) \right] \sum_{n=0}^{\infty} \frac{(a)_{n+1}(1-a)_{n+1}}{(n+3)!} r^{2n}. \tag{3.11}
\]

Therefore, the monotonicity of \(F(r)\) follows from Lemma 3.1 and (3.11). Moreover, by l'Hôpital’s rule we get

\[
\lim_{r \to 1^-} F(r) = a(1-a) \sin(\pi a),
\]

\[
\lim_{r \to 0^+} F(r) = \sin(\pi a) + \lim_{r \to 0^+} \frac{2(1-a)(-r^2)(E_a - r^2K_a)}{r^4} + \lim_{r \to 0^+} \frac{2(1-a)r^2(E_a - r^2K_a) - 2a(1-a)r^2 r^2K_a}{r^4}
\]

\[
= \sin(\pi a) - 2(1-a) + \lim_{r \to 0^+} \frac{2(1-a)(E_a - r^2K_a) - ar^2K_a}{r^4}
\]

\[
= \sin(\pi a) - \pi a(1-a) - (\pi/2)a^2(1-a)^2. \quad \square
\]
Lemma 3.4. Let \( a \in (0, 1/2) \), \( \lambda_1 = [\sin(\pi a) - a(1 - a)\pi]/\{\sin(\pi a)[R(a) - 1]\} \), \( \lambda_2 = 1 - [\pi(1 - a)]/\sin(\pi a) - [\pi a^2(1 - a)^2]/[2\sin(\pi a)] \), \( \beta = a(1 - a) \), and the function \( G_\lambda(\cdot) \) be defined on \((0, 1)\) by

\[
G_\lambda(r) = \frac{1}{2} - \log \left( \frac{e^{R(a)/2}}{r'} \right) + \frac{r^2 \sin(\pi a) - 2(1-a)(E_a - r^2 K_a)}{2\lambda r^2 r' \sin(\pi a)}.
\]

Then the following statements are true:

1. \( G'_\lambda(r) > 0 \) for all \( r \in (0, 1) \) if \( 0 < \lambda \leq \lambda_2 \);
2. \( G'_\lambda(r) < 0 \) for all \( r \in (0, 1) \) if \( \lambda \geq \beta \);
3. There exists \( r_0 \in (0, 1) \) such that \( G'_\lambda(r) < 0 \) for \( r \in (0, r_0) \) and \( G'_\lambda(r) > 0 \) for \( r \in (r_0, 1) \) if \( \lambda_2 < \lambda < \beta \).

Moreover,

\[
\begin{align*}
G_\lambda(0^+) &> 0, & 0 < \lambda < \lambda_1, \\
G_\lambda(0^+) &= 0, & \lambda = \lambda_1, \\
G_\lambda(0^+) &< 0, & \lambda > \lambda_1
\end{align*}
\]

and

\[
\begin{align*}
G_\lambda(1^-) &= +\infty, & 0 < \lambda < \beta, \\
G_\lambda(1^-) &= 1 - \frac{1}{2a(1-a)} < 0, & \lambda = \beta, \\
G_\lambda(1^-) &= -\infty, & \lambda > \beta.
\end{align*}
\]

Proof. It is apparent from Lemma 3.2 that \( \lambda_2 > 0 \) for all \( a \in (0, 1/2) \). Therefore, parts (1)-(3) follows from Lemma 3.3 and the fact that

\[
G'_\lambda(r) = -\frac{r}{r'^2} + \frac{1}{2\lambda \sin(\pi a)} \times \\
\left[ 2r \sin(\pi a) - 4a(1 - a)rK_a \right] r^2 r'^2 - [r^2 \sin(\pi a) - 2(1-a)(E_a - r^2 K_a)](2rr'^2 - 2r^3)
\]

\[
\begin{align*}
&= \frac{r}{\lambda r'^2} \left[ -\lambda + \frac{1}{\sin(\pi a)} \frac{r^4 \sin(\pi a) - 2a(1-a)r^2 r'^2 K_a + 2(1-a)(r^2 - r'^2)(E_a - r^2 K_a)}{r'^2 r^4} \right] \\
&= \frac{r}{\lambda r'^2} \left[ -\lambda + \frac{1}{\sin(\pi a)} F(r) \right],
\end{align*}
\]

where \( F \) is defined as in Lemma 3.3.

Next, we calculate the limit values of \( G_\lambda(r) \) at 0 and 1. Simple computations lead to

\[
G_\lambda(0^+) = \frac{1}{2} - \frac{R(a)}{2} + \frac{\sin(\pi a) - 2(1-a)[(\pi a)/2]}{2\lambda \sin(\pi a)}
\]

\[
= \frac{\sin(\pi a)[R(a) - 1]}{2\lambda \sin(\pi a)} \left( -\lambda + \frac{\sin(\pi a) - \pi a(1-a)}{\sin(\pi a) [R(a) - 1]} \right)
\]

\[
= \frac{\sin(\pi a)[R(a) - 1]}{2\lambda \sin(\pi a)} ( -\lambda + \lambda_1 ),
\]

which implies (3.12).

It follows from

\[
\lim_{r \to 1^-} \frac{K_a(r)}{\sin(\pi a)} - \log \left( \frac{e^{R(a)/2}}{r'} \right) = 0
\]

in [17, p. 635, (2.26)] that

\[
G_\lambda(1^-) = \frac{1}{2} + \lim_{r \to 1^-} \left[ \frac{K_a}{\sin(\pi a)} - \log(e^{R(a)/2}/r') \right] + \lim_{r \to 1^-} \left[ \frac{r^2 \sin(\pi a) - \sin(\pi a)}{2 \sin(\pi a) \lambda r^2 r'^2} \right]
\]
Proof of Theorem 1.1

where

\[ G_{12} \]

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\[ G(1) \]

Case 1

We divide the proof of (3.13) into two cases.

Case I

We divide the proof of inequality (1.7) into two cases.

Case 2

\[ \lambda(1 - \lambda) = 1 - \frac{2}{r^2} \]

is defined as in Lemma 3.4.

\[ \lambda \]

\[ (1 - \lambda) = 1 - \frac{2}{r^2} \]

Then from (3.14) we have

\[ G_{1}(1) = \frac{1}{2} - \frac{1}{2\lambda} + \lim_{r \to 1^+} \left[ -1 + \frac{\sin(\pi a) - 2(1 - a)(\mathcal{E}_a - r^2 K_a) - \frac{r^2 K_a}{2(1 - a)} \frac{K_a(r)}{\sin(\pi a)} \right] \]

\[ G_{1}(1) = \frac{1}{2} - \frac{1}{2\lambda} + \lim_{r \to 1^+} \left[ -1 + \frac{a(1 - a)}{\lambda} \frac{K_a(r)}{\sin(\pi a)} \right] \]

\[ = \begin{cases} +\infty, & 0 < \lambda < \beta, \\ -\infty, & \lambda > \beta. \end{cases} \]

Case 2 \( \lambda = \beta \). Then equation (3.14) leads to

\[ G_{1}(1) = \frac{1}{2} - \frac{1}{2a(1 - a)} + \lim_{r \to 1^+} \frac{\sin(\pi a) - 2(1 - a)(\mathcal{E}_a - r^2 K_a) - 2a(1 - a)r^2 r^2 K_a}{2a(1 - a)r^2 r^2 \sin(\pi a)} \]

\[ = \frac{1}{2} - \frac{1}{2a(1 - a)} + \frac{1}{a \sin(\pi a)} \lim_{r \to 1^+} \frac{\sin(\pi a) - (\mathcal{E}_a - r^2 K_a) - 2ar r^2 K_a}{r^2} \]

\[ = 1 - \frac{1}{2a(1 - a)}. \]

\[ \square \]

Proof of Theorem 1.1. Let

\[ H_{\lambda}(r) = \sin(\pi a)(1 + \lambda r^2) \log\left(\frac{e^{R(a)/2}}{r}\right) - K_a(r), \quad \lambda \in \mathbb{R}^+. \]

Then simple computations lead to

\[ H_{\lambda}(0^+) = \left[ \sin(\pi a)(1 + \lambda) - \pi \right]/2, \]

\[ H_{\lambda}(1^-) = 0, \]

\[ H'_{\lambda}(r) = \sin(\pi a)(-2\lambda r) \log\left(\frac{e^{R(a)/2}}{r}\right) + \sin(\pi a)(1 + \lambda r^2) \frac{r}{r^2} \]

\[ - 2(1 - a) \frac{\mathcal{E}_a - r^2 K_a}{r^2} \]

\[ = 2\lambda \sin(\pi a) G_{\lambda}(r), \]

where \( G_{\lambda} \) is defined as in Lemma 3.4.

We divide the proof of inequality (1.7) into two cases.

Case 1 \( \lambda = \alpha_0 = \pi/|R(a)\sin(\pi a)| - 1 \). Then equation (3.16) reduces to

\[ H_{\alpha_0}(0^+) = 0. \]
From Corollaries 2.4 and 2.5 we know that $\beta > \alpha_0 > \lambda_1$, then (3.12) and (3.13) lead to the conclusion that $G_{\alpha_0}(0^+) < 0$ and $G_{\alpha_0}(1^-) = +\infty$. Moreover, wether $\alpha_0 \in (0, \lambda_2]$ or $\alpha_0 \in (\lambda_2, \beta)$, it follows from part (1) or (3) in Lemma 3.4 that there exists $r_0^* \in (0, 1)$ such that $G_{\alpha_0}(r) < 0$ for $r \in (0, r_0^*)$ and $G_{\alpha_0}(r) > 0$ for $r \in (r_0^*, 1)$. Hence, from (3.18) we clearly see that $H_{\alpha_0}(r)$ is strictly decreasing on $(0, r_0^*)$ and strictly increasing on $(r_0^*, 1)$.

Equations (3.17) and (3.19) together with the piecewise monotonicity of $H_{\alpha_0}(r)$ lead to the conclusion that $H_{\alpha_0}(r) < 0$ for all $r \in (0, 1)$. Therefore, the first inequality in (1.7) for $\alpha = \alpha_0$ follows easily from (3.15).

**Case II** $\lambda = \beta_0 = a(1 - a)$. Then from (3.12), (3.13), Lemma 3.4(2) and the fact that $\beta_0 > \lambda_1$ for all $a \in (0, 1/2]$ we know that $G_{\beta_0}(r)$ is strictly decreasing on $(0, 1)$, $G_{\beta_0}(0^+) < 0$ and $G_{\beta_0}(1^-) < 0$. Thus $G_{\beta_0}(r) < 0$ for $r \in (0, 1)$. It follows from (3.17) and (3.18) that $H_{\beta_0}(r)$ is strictly decreasing on $(0, 1)$ and $H_{\beta_0}(r) > H_{\beta_0}(1^-) = 0$ for $r \in (0, 1)$. Therefore, the second inequality in (1.7) for $\beta = \beta_0$ follows from (3.15).

Finally, we prove that $\alpha = \alpha_0$ and $\beta = \beta_0$ are the best possible parameters such that inequality (1.7) holds for all $\alpha \in (0, 1/2]$ and $r \in (0, 1)$. In fact, if $\lambda > \alpha_0$, then from (3.16) we know that $H_{\lambda}(0^+) > 0$. Hence there exists $r_1 \in (0, 1)$ such that $H_{\lambda}(r) > 0$ for $r \in (0, r_1)$. That is, $1 + \lambda r^2 > K_\alpha(r)/[\sin(\pi a) \log(e^{R(a)/2}/r')]$ for $r \in (0, r_1)$.

On the other hand, if $0 < \lambda < \beta_0$, then (3.13) and (3.18) imply that there exists $r_1^* \in (0, 1)$ such that $H_{\lambda}'(r) > 0$ for $r \in (r_1^*, 1)$. Thus $H_{\lambda}(r)$ is strictly increasing on $(r_1^*, 1)$ and $H_{\lambda}(r) < H_{\lambda}(1^-) = 0$. That is, $1 + \lambda r^2 < K_\alpha(r)/[\sin(\pi a) \log(e^{R(a)/2}/r')]$ for $r \in (r_1^*, 1)$.

\[\square\]

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