An application of Pappus’ Involution Theorem in euclidean and non-euclidean geometry.

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Abstract

Pappus’ Involution Theorem is a powerful tool for proving theorems about non-euclidean triangles and generalized triangles in Cayley-Klein models. Its power is illustrated by proving with it some theorems about euclidean and non-euclidean polygons of different types. A $n$-dimensional euclidean version of these theorems is stated too.

Keywords: Pappus’ Involution Theorem, Cayley-Klein models, euclidean geometry, non-euclidean geometry, quadrilaterals, right-angled hexagons, right-angled pentagons, non-euclidean trigonometry, non-euclidean triangles

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1 Introduction.

In spite of being non-conformal, the use of the projective models of Cayley [2] and Klein [5] in the study of non-euclidean planar geometries has some advantages. Any projective theorem involving a conic could have multiple interpretations as theorems in elliptic or hyperbolic plane. Following [7], in [12] those different non-euclidean theorems emanating directly from a single projective one are called shadows of the original projective theorem. In the limit case when the conic degenerates into a single line, a non-euclidean theorem usually has a “limit” theorem which holds in euclidean plane. This property is illustrated in Section [4], and it has been exhaustively applied in [12], where the non-euclidean shadows of some classical projective planar theorems are explored: the whole non-euclidean trigonometry is deduced from Menelaus’ Theorem, and Pascal’s and Desargues’ Theorems are used.
to construct some classical and non-classical triangle centers, together with a non-euclidean version of the Euler line and the nine-point circle of a triangle\footnote{This non-euclidean version of the Euler line is the line denoted \textit{orthaxis} in [13, 14]. This is the reason why it is given the name \textit{Euler-Wildberger line} in [12].}. In [12], it is shown also the unique projective theorem hidden behind all the cosine rules of elliptic and hyperbolic triangles and \textit{generalizes triangles} in the sense of [1].

Many geometric problems can be easily proven using involutions. In particular, in many proofs and constructions of [12], a particular projective theorem has arisen as an extremely powerful tool: Pappus’ Involution Theorem (Theorem 2 below). Here we will exhibit its power by using it for giving a simple proof of a little theorem about certain euclidean and non-euclidean quadrilaterals. We say that a quadrilateral in euclidean, hyperbolic or elliptic plane is \textit{diametral}\footnote{In euclidean plane it is a cyclic quadrilateral with a diametral diagonal.} if it has two right angles located at opposite vertices.

\textbf{Theorem 1} Let $R$ be a diametral quadrilateral in euclidean, hyperbolic or elliptic plane, with vertices $A, B, C, D$ and right angles at $B$ and $D$. Let $A_*, C_*$ be the orthogonal projections of the points $A, C$ into the diagonal line $BD$, respectively. A midpoint of the segment $BD$ is also a midpoint of the segment $A_*C_*$ (see Figure 1).

Note that in the statement of this theorem we have written “a midpoint” instead of “the midpoint”. An euclidean or hyperbolic segment is uniquely determined by its endpoints, and it has a unique midpoint (\textit{the} midpoint). In
the elliptic case this concept is more subtle. Although we will not enter into this discussion, depending on how we define “segment” and/or “midpoint” an elliptic segment has one or two midpoints.

We give a projective proof of Theorem 1 in Section 3. The non-euclidean part of this proof relies essentially in Pappus’ Involution Theorem, and it can be reused for proving some other theorems (shadows) about non-euclidean polygons. This will be done in Section 4. Before all of that, in Section 2 we introduce the basic projective tools to be used in the subsequent sections.

Finally, in Section 5 we propose a \( n \)-dimensional version of Theorem 1 which until now is valid only in euclidean space.

In all figures right angles are denoted with the symbol \( \Box \).

## 2 Cayley-Klein models

We will assume that the reader is familiar with the basic concepts of real and complex planar projective geometry: the projective plane and its fundamental subsets (points, lines, pencils of lines, conics), and their projectivities. Nevertheless, we will review some concepts and results needed for a better understanding of Sections 3 and 4. For the rigorous definitions and proofs we refer to [3, 11] or [7], for example. We assume also that the reader has some elementary background in non-euclidean planar geometry (see [4, 8, 10], for example).

Although we will work with real elements, we consider the real projective plane \( \mathbb{R}P^2 \) standardly embedded in the complex projective plane \( \mathbb{C}P^2 \).

If \( A, B \) are two different points in the projective plane we denote by \( AB \) the line joining them. If \( a, b \) are two different lines, or a line and a conic, in the projective plane, we denote by \( a \cdot b \) their intersection set.

The main theorem of projective geometry that we will use is:

**Theorem 2 (Pappus’ Involution Theorem)** The three pairs of opposite sides of a complete quadrangle meet any line (not through a vertex) in three pairs of an involution.

See [3] p. 49 for a proof. This is a partial version of Desargues’ Involution Theorem (see [3] p. 81). Using this theorem, a given complete quadrangle in the projective plane determines a quadrangular involution on every line not through a vertex.

Let review briefly how the projective models of euclidean, hyperbolic and elliptic planes are constructed. We just want to show how the basic geometric concepts needed later (perpendicular lines, midpoint of a segment) are interpreted in projective terms, avoiding a full construction of these models.
2.1 The euclidean plane

For constructing the euclidean plane in projective terms [6], we fix a line \( \ell_\infty \) in the projective plane (the \textit{line at infinity}), and an elliptic involution on it, i.e., a projective involution \( \rho_\infty \) on \( \ell_\infty \) without real fixed points (the \textit{absolute involution}). Two points \( A_\infty, B_\infty \) on \( \ell_\infty \) are \textit{conjugate} if they are related by the absolute involution: \( \rho_\infty(A_\infty) = B_\infty \). The euclidean plane \( \mathbb{E}^2 \) is composed by the points of \( \mathbb{RP}^2 \) not lying in \( \ell_\infty \).

For a given line \( r \) different from \( \ell_\infty \), the intersection point \( r \cdot \ell_\infty \) is the \textit{point at infinity} of \( r \). Two lines \( r, s \) are \textit{parallel} if their points at infinity coincide, and they are \textit{perpendicular} if they points at infinity are conjugate.

Given two different points \( A, B \) on \( \mathbb{E}^2 \), the \textit{midpoint} of the segment \( \overline{AB} \) joining them is the harmonic conjugate with respect to \( A, B \) of the point at infinity of \( AB \).

2.2 The hyperbolic and elliptic planes

For constructing the non-euclidean planar models, we fix a non-degenerate conic \( \Phi_\infty \) (the \textit{absolute conic}) such that the polar of each real point with respect to \( \Phi_\infty \) is a real line. An equivalent formulation of this property is to require, working with homogeneous coordinates, that \( \Phi_\infty \) can be expressed by an equation with real coefficients. Such a conic can be of two kinds: a \textit{real conic}, if it has real points; or an \textit{imaginary conic} if it has no real points (see [11, vol. II, p. 186]).
When $\Phi_\infty$ is a real conic, the interior points of $\Phi_\infty$ compose the hyperbolic plane, and when $\Phi_\infty$ is an imaginary conic the whole $\mathbb{RP}^2$ composes the elliptic plane.

We will use the common term the non-euclidean plane $\mathbb{P}^2$ either for the hyperbolic plane (when $\Phi_\infty$ is a real conic) or for the elliptic plane (when $\Phi_\infty$ is an imaginary conic). Geodesics in these models are given by the intersection with $\mathbb{P}^2$ of real projective lines. In the hyperbolic case, we will talk always about points or lines in a purely projective sense, even if the referred elements are exterior to $\Phi_\infty$.

The polarity $\rho$ with respect to $\Phi_\infty$ is a key tool in these models, where it plays a similar role as $\rho_\infty$ does in the euclidean case: two lines not tangent to $\Phi_\infty$ are perpendicular if they are conjugate with respect to $\Phi_\infty$, that is, if each one contains the pole of the other one with respect to $\Phi_\infty$. For a point $P$ and a line $p$, we denote by $\rho(P)$ and $\rho(p)$ the polar line of $P$ and the pole of $p$ with respect to $\Phi_\infty$, respectively. The polarity $\rho$ induces a natural involution on any line $p$ not tangent to $\Phi_\infty$: the conjugacy involution, which sends each point $P \in p$ to the intersection $p \cdot \rho(P)$ of $p$ with the polar of $P$.

The double points of the conjugacy involution on $p$ are the two points on $p \cdot \Phi_\infty$. If the points $A,B$ not lying in $\Phi$ are conjugate with respect to $\Phi_\infty$ in the line $p$ that contains them, the polar of $A$ is the line perpendicular to $p$ through $B$ and vice versa.

Let $A,B$ be two points on $\mathbb{RP}^2$ not lying in $\Phi_\infty$ and such that the line $p$ joining them is not tangent to $\Phi_\infty$. Let $P = \rho(p)$ be the pole of $p$ with respect to $\Phi_\infty$, and let $a,b$ be the lines joining $A,B$ with $P$, respectively. Each of the lines $a,b$ has two (perhaps imaginary) different intersection points with $\Phi_\infty$. Let $A_1,A_2$ and $B_1,B_2$ be the intersection points of $a$ and $b$ with $\Phi_\infty$ respectively. The points

$$E_1 = A_1B_1 \cdot A_2B_2 \quad \text{and} \quad E_2 = A_1B_2 \cdot A_2B_1$$

lie at the line $p$ and they are the midpoints of the segment $\overline{AB}$. Note that this definition is projective, and so it can be interpreted in multiple ways (see [4]). For example:

- If $\Phi_\infty$ is imaginary, the points $E_1,E_2$ are the two points of $p$ which are equidistant from $A$ and $B$ in the elliptic plane $\mathbb{P}^2$.

- If $\Phi_\infty$ is a real conic and $A,B$ are interior to $\Phi_\infty$, exactly one of the two points $E_1,E_2$, say $E_1$, is interior to $\Phi_\infty$, and it is the midpoint of the hyperbolic segment $\overline{AB}$. In this case, the other point $E_2$ is the pole of the orthogonal bisector of the segment $\overline{AB}$. 

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• If $\Phi_\infty$ is a real conic and the line $p$ is exterior to $\Phi_\infty$ its pole $P$ is interior to $\Phi_\infty$. The lines $PE_1, PE_2$ are the two bisectors of the angle between the lines $PA, PB$.

An easy characterization of midpoints is:

**Lemma 3** Let $p$ be a line not tangent to $\Phi$, let $p \cdot \Phi_\infty = \{U, V\}$, and take two points $A, B \in p$ different from $U, V$. If $C, D$ are two points of $AB$ verifying the cross-ratio identities

\[(ABCD) = (UVCD) = -1,\]

then $C, D$ are the midpoints of $AB$.

3 Proof of Theorem 1

Although it is not difficult to find synthetic euclidean, elliptic or hyperbolic proofs of Theorem 1, we will limit ourselves to the use of projective techniques.

In both (euclidean and non-euclidean) cases, in the degenerate case where $A_*$ equals the point $B$ ($D$), it can be seen that $C_*$ equals the point $D$ ($B$) and vice versa, and the statement is true. Thus, we can assume that $A_*, C_*$ are different from $B, D$.

**Euclidean case.** Let $A_0, B_0, D_0$ be the points at infinity of the lines $BD, AB, AD$ respectively, and let $A_1, B_1, D_1$ be their conjugate points in $\ell_\infty$.
The perpendicular lines to $BD$ through $A, C$ are $AA_1, CA_1$ respectively, and so we have

$$A_* = AA_1 \cdot BD \quad \text{and} \quad C_* = CA_1 \cdot BD.$$ 

Let $M$ be the midpoint of the segment $BD$, and let $H$ be the point $BD_0 \cdot DB_0$.

If we consider the quadrangle $\mathcal{Q} = \{B, C, D, H\}$, the quadrangular involution $\tau$ that $\mathcal{Q}$ induces on $\ell_\infty$ sends $B_0, D_0$ into $B_1, D_1$ respectively, and vice versa. This implies that $\tau$ coincides with $\rho_\infty$ and, in particular, that $CH$ passes through $A_1$ ($H$ is the orthocenter of the triangle $\triangle BCD$).

By considering the quadrangle $\{A, B_0, D_0, H\}$, it turns out that $AH$ passes through the harmonic conjugate of $A_0$ with respect to $B, D$, that is, that $A, H$ are collinear with $M$. In particular, this implies that if $A_*$ and $C_*$ coincide, they coincide also with $M$.

By applying Pappus’ Theorem to the hexagon $BB_0C_1A_1A_0D_0$, we have that the point $F = B_0C_* \cdot D_0A_*$ is collinear with $A, H$. In the same way, using the hexagon $BB_0A_*A_1C_*D_0$ it is proved that $G = B_0A_* \cdot D_0C_*$ is collinear with $A, H$. Taking the quadrangle $\{F, G, B_0, D_0\}$, the point $M$ is also the harmonic conjugate of $A_0$ with respect to $A_*, C_*$.

The **non-euclidean case.** We consider the points $A, B, C, D$ such that the lines $AB, AD$ are conjugate to $BC, DC$ respectively with respect to $\Phi_\infty$ (Figure 3). This means that the poles $B', D'$ of the lines $AB, AD$ belong to $BC, DC$ respectively.

Let $a$ be the line $BD$, and let $A'$ be the pole of $a$ with respect to $\Phi_\infty$. The lines perpendicular to $a$ through $A$ and $C$ are $AA'$ and $CA'$, respectively, and so it is $A_* = a \cdot AA'$ and $C_* = a \cdot CA'$.

Let $B_0, D_0$ be the intersection points with $a$ of the lines $A'B', D'A'$, respectively. The triangle $A'B'D'$ is the polar triangle of $ABD$, and so $B_0, D_0$ are respectively the conjugate points of $B, D$ with respect to $\Phi_\infty$ in the line $a$. The point $N = a \cdot B'D'$ is the pole of the line $AA'$, and so it is the conjugate point of $A_*$ in $a$ with respect to $\Phi_\infty$.

Let $M_1, M_2$ be the midpoints of the segment $BD$, and consider the quadrangle $\mathcal{Q}$ with vertices $C, A', B', D'$. We will make use of three involutions in $a$:

- the conjugacy involution $\rho_a$ induced in $a$ by the polarity with respect to $\Phi_\infty$;
- the quadrangular involution $\tau_\mathcal{Q}$ induced in $a$ by $\mathcal{Q}$; and
- the harmonic involution $\sigma_a$ in $a$ with respect to $M_1, M_2$. 

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The quadrangular involution $\tau_\mathcal{D}$ sends the points $B, D, C_*$ into the points $D_0, B_0, N$ and vice versa. This implies that the composition $\rho_a \tau_\mathcal{D}$ sends $B, D, B_0, D_0$ into $D, B, D_0, B_0$ respectively. Thus, $\sigma_a$ and $\rho_a \tau_\mathcal{D}$ agree over at least three different points and so they coincide. As $\rho_a \tau_\mathcal{D}(C_*) = \rho_a(N) = A_*$, the points $A_*$ and $C_*$ are harmonic conjugate with respect to $M_1$ and $M_2$, and so by Lemma 3 the points $M_1$ and $M_2$ are the midpoints of $A_*C_*$.

\[\blacksquare\]

**Problem 4** Find a synthetic proof of Theorem 1 using the axioms of absolute geometry.
4 Generalizations

As we have seen, the proof of Theorem 1 in the non-euclidean case is projective, and it does not depend on the type of conic $\Phi_\infty$ that we have considered. Thus, the same proof is valid for the hyperbolic and elliptic cases (see Figure 1). In the same way, when $\Phi_\infty$ is a real conic, the same proof does not depend on the relative position of the vertices $A, B, C, D$ with respect to $\Phi_\infty$. Indeed, what we have proved in the non-euclidean part of the proof is the following projective theorem:

**Theorem 5** Let $\mathcal{Q} = \{A, B, C, D\}$ be a complete quadrangle in the projective plane in general position with respect to $\Phi_\infty$ (vertices and diagonal points not in $\Phi$, sides and diagonal lines not tangent to $\Phi_\infty$) such that the lines $AB, AD$ are conjugate to $BC, DC$ respectively with respect to $\Phi_\infty$. Let $A'$ be the pole of $BD$, and let $A_*, C_*$ be the intersection points of the lines $AA', CA'$ respectively with $BD$. The midpoints of $A_*C_*$ are also the midpoints of $BD$.

Although we were talking about quadrilaterals, Theorem 5 can be applied to other hyperbolic figures that appear from the same projective configuration.
4-right pentagon  If in the quadrangle $ABCD$ we assume that the vertex $C$ lies outside the absolute conic while the rest of vertices remain inside $\Phi_\infty$, the polar of $C$ appears into the figure as the common perpendicular to the lines $CB$ and $CD$. The hyperbolic polygon that appears is a 4-right pentagon: a hyperbolic pentagon with four right angles (at least) at the vertices different from $A$ (Figure 4). In this case, Theorem 5 implies:

Theorem 6  In the 4-right pentagon $ABC_1C_2D$, perhaps with non-right angle at $A$, let $A_*$ be the orthogonal projection of $A$ into $BD$, and let $C_*$ be the intersection of $BD$ with the common perpendicular of $BD$ and $C_1C_2$. The midpoint of $BD$ is also the midpoint of $A_*C_*$. 

Right-angled hexagon  If in the previous figure we push also the vertex $A$ out of $\Phi_\infty$, while $B, D$ remain interior to $\Phi_\infty$, the polar of $A$ become part of the figure as the common perpendicular to the lines $AB$ and $AD$. The figure that appears is a right-angled hexagon: an hexagon in the hyperbolic plane with six right angles as that depicted in Figure 5. With the notation of this figure, the traslation of Theorem 5 for this configuration is:
Theorem 7 Let $A_1A_2BC_1C_2D$ be a right-angled hexagon. Let $A_*$ be the intersection point with $BD$ of the common perpendicular to $BD$ and $A_1A_2$, and let $C_*$ be the intersection point with $BD$ of the common perpendicular to $BD$ and $C_1C_2$. The midpoint of $BD$ is also the midpoint of $A_*C_*$. 

4 right-pentagon II If, after pushing $C$ out of $\Phi_\infty$ for obtaining the 4-right pentagon of Figure 4 we push also $B,D$ out of $\Phi_\infty$ but still being the line $BD$ secant to $\Phi_\infty$ ($A$ remains interior to $\Phi_\infty$), the polars of $B$ and $D$ appear in the figure, drawing with the lines $AB, AD$ and $BD$ another 4-right pentagon $AB_1B_2D_2D_1$ as that of Figure 6. Because the pole of $AB$ is collinear with $B$ and $C$, the polars of $B$ and $C$ intersect at the point $B_1$ lying in $AB$, which is also the conjugate point of $C_1$ in the polar of $C$ with respect to $\Phi_\infty$. In the same way, the polars of $C$ and $D$ intersect at the point $D_1$ lying in $AD$ which is the conjugate of $C_2$ with respect to $\Phi_\infty$. On the other hand, the polars of $B, D$ intersect $BD$ at $B_2, D_2$ respectively, which are the conjugate points of $B, D$ respectively in $BD$ with respect to $\Phi_\infty$. The pentagon $AB_1B_2D_2D_1$ is a 4-right pentagon with right angles at all its vertices with the unique possible exception of $A$. By Lemma 3 it can
be deduced that the midpoints of $BD$ are also the midpoints of $B_2D_2$, and thus we have, with the notation of Figure 6:

**Theorem 8** Let $AB_1B_2D_2D_1$ be a 4-right pentagon with right angles at $B_1, B_2, D_1, D_2$. Let $A_1$ be the orthogonal projection of $A$ into $B_2D_2$, and let $C_*$ be the intersection with $B_2D_2$ of the common perpendicular to $B_1D_1$ and $B_2D_2$. The midpoint of $B_2D_2$ is also the midpoint of $A_*C_*$. 

**Right-angled hexagon II** If in the previous figure we push also $A$ out of the absolute conic, we obtain a theorem similar to Theorem 8 for right-angled hexagons. With the notation of Figure 7:

**Theorem 9** Let $A_1A_2B_1B_2D_2D_1$ be a right-angled hexagon. Let the common perpendiculars to $B_2D_2$ and $A_1A_2, B_1D_1$ be denoted by $a_*, c_*$ respectively, and let $A_*, C_*$ be the intersection points of $a_*, c_*$ with $B_2D_2$ respectively. The midpoint of $A_*C_*$ is midpoint of $B_2D_2$.

**Quadrangle II** If in the configuration “4 right-pentagon II” (Figure 6) we move $B, D$ until the line $BD$ is exterior to $\Phi_\infty$, the point $A'$ becomes...
interior to $\Phi_\infty$ and $AB_1A'D_1$ is a hyperbolic diametral quadrangle with right angles at the opposite vertices $B_1, D_1$. After a reinterpretation of the points $A_*, C_*, M_1, M_2$ for this figure, we obtain the following theorem (see Figure 8):

**Theorem 10** Let $AB_1A'D_1$ be a hyperbolic diametral quadrangle $R$ with right angles at the opposite vertices $B_1, D_1$. Consider the lines $a_* = AA'$ and the line $c_*$ perpendicular to $B_1D_1$. The bisectors of the angle $B_1A'D_1$ are also the bisectors of the angle $a_*c_*$. 

As we can expect, and as it happened with Theorem 1, this theorem is also true in the euclidean and elliptic cases.

**Problem 11** Find a synthetic proof of Theorem 10 using the axioms of absolute geometry.

5 **A higher-dimensional generalization**

A higher-dimensional generalization of Theorem 1 is:

**Theorem 12** Let $\Delta$ be a simplex in euclidean $n$-dimensional space with vertices $A_0, A_1, \ldots, A_n$. Consider the opposite face $\Delta_0$ to $A_0$ in $\Delta$, and take the hyperplane $\pi_0$ containing $\Delta_0$. Let $\pi_1, \pi_2, \ldots, \pi_n$ be the hyperplanes orthogonal to $A_0A_1, A_0A_2, \ldots, A_0A_n$ through $A_1, A_2, \ldots, A_n$ respectively, and let $C$ be the intersection point of $\pi_1, \pi_2, \ldots, \pi_n$. If $A_*, C_*$ are the orthogonal projections of $A, C$ into $\pi_0$, the midpoint of $A_*C_*$ is the circumcenter of $\Delta_0$.

We have illustrate the three-dimensional version of this theorem in Figure 9. Its proof (there are plenty of them) is left to the reader; it is just an exercise on euclidean geometry. Our interest in Theorem 12 relies on the fact that it is not valid in the hyperbolic and elliptic cases.

**Question 13** Do there exist a non-euclidean version of Theorem 12?

According to the previous paragraph, the answer to this problem is obviously “no”. Nevertheless, there are many geometric constructions that are equivalent in euclidean geometry but that are not in the non-euclidean case. For example, in [12] it is shown how we can take alternative definitions for the circumcenter and barycenter of a triangle, different to the standard ones but equivalent to them in euclidean geometry, in such a way that the Euler line does exist in the hyperbolic and elliptic planes. In Question 13, we wonder if there exists a different formulation of Theorem 12 which is valid
also in the non-euclidean cases. It must be noted that in euclidean $n$-space the set of points $A_0, A_1, \ldots, A_n, C$ is \textit{diametrally cyclic}, in the sense that all these points lie in an $(n-1)$-dimensional sphere in $\mathbb{R}^n$ in which $A_0$ and $C$ are antipodal points, while this is not the case in the non-euclidean context.

In the same way as Theorem $\ref{Thm:12}$ is a $n$-dimensional generalization of Theorem $\ref{Thm:1}$ we have tried to find a $n$-dimensional generalization of Theorem $\ref{Thm:10}$ without success. So the last problem that we propose is.

\textbf{Question 14} Do there exist a $n$-dimensional (euclidean or non-euclidean) version of Theorem $\ref{Thm:10}$?

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