STABILITY AND BIFURCATION ANALYSIS IN A DELAY-INDUCED PREDATOR-PREY MODEL WITH MICHAELIS-MENTEN TYPE PREDATOR HARVESTING

MING LIU, DONGPO Hu AND FANWEI MENG*

School of Mathematical Sciences, Qufu Normal University
Qufu 273165, Shandong, China

Abstract. The present paper considers a delay-induced predator-prey model with Michaelis-Menten type predator harvesting. The existence of the non-trivial positive equilibria is discussed, and some sufficient conditions for locally asymptotically stability of one of the positive equilibria are developed. Meanwhile, the existence of Hopf bifurcation is discussed by choosing time delays as the bifurcation parameters. Furthermore, the direction of Hopf bifurcation and the stability of the bifurcated periodic solutions are determined by the normal form theory and the center manifold theorem for functional differential equations. Finally, some numerical simulations are carried out to support the analytical results.

1. Introduction. Any population in an ecosystem does not exist in isolation, and there must be certain relationships between different populations, such as competition [3, 12], symbiosis [11], predation [2] and so on. Predation relationship plays a very important role in ecosystems and has attracted significantly and widely attention from lots of biologists, mathematicians and ecologists [9, 13, 20]. Predator-prey models are basic differential or difference equation models for describing the predation relationship between two or more species, and are among the most interested topics in mathematical biology and ecology. The researchers want to reveal the possible complex dynamics between predators and their prey by studying the dynamical behavior of predator-prey models. There are many factors which affect dynamical properties of biological and mathematical models, such as harvesting and time delays.

On the one hand, from the view of human needs, the exploitation of biological resources and the harvesting of populations are usually human purpose of achieving the economic interest in fishery and wildlife management. The harvesting is a very realistic and meaningful topic in nature and it also has crucial effect on the dynamics of model and ecological balance [29]. Therefore, it is important to study the harvesting in many predator-prey models [41]. Effects of harvesting in various types of prey-predator models have been considered by many researchers [7, 15, 33, 34, 39, 47]. It has shown that harvesting has a strong influence on the dynamical behavior of a predator-prey model [1, 19, 21, 37, 42]. Such as appearance

2010 Mathematics Subject Classification. Primary: 92D25; Secondary: 34K18, 34K20, 34K60.
Key words and phrases. Predator-prey model, Michaelis-Menten type harvesting, time delays, stability, Hopf bifurcation.

*Corresponding author: Fanwei Meng.
of numerous kinds of bifurcations, including saddle-node bifurcation, Hopf bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimensions 2 and 3. Zhu and Lan [47] studied a Leslie-Gower predator-prey system with constant harvesting in prey. They demonstrated that the predator free equilibria are saddles, saddle-nodes or unstable nodes relying on the choices of the parameters involved while the interior positive equilibria in the first quadrant are saddles, stable or unstable nodes, saddle-nodes, centers, foci or cusps. Sen et al. [34] have paid great attention to study the global dynamics of a predator-prey model when predator is provided with additional food as well as harvested at a constant rate. Xiao et al. [42] also have focused on the effect of constant-yield harvesting in predator-prey model. In [21], the existence of Bogdanov-Takens bifurcations of codimensions 2 and 3 for predator-prey model with harvesting was proved analytically.

As is known to all, because of the complexity of the actual ecosystem, constant-yield harvesting and constant-effort harvesting [31] can not completely describe all the real situations, and the nonlinear harvesting can offset some deficiencies correspondingly, nonlinear harvesting is more realistic from biological and economical points of view. In [8], a harvesting term $h = qEx/(cE + lx)$ which is named Michaelis-Menten type functional form of catch rate is put forward for the first time, where $q$ is the catch ability coefficient, $E$ is the external effort devoted to harvesting, $c$ and $l$ are constants. In [19], the authors considered saddle-node bifurcation, transcritical bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation of a predator-prey model with Michaelis-Menten type harvesting in predator, of the following form

$$\begin{align*}
\dot{x} &= r_1 x \left(1 - \frac{x}{K}\right) - ax y, \\
\dot{y} &= r_2 y \left(1 - \frac{y}{bx}\right) - \frac{qEy}{cE + ly},
\end{align*}$$

(1)

where $x(t)$ and $y(t)$ represent prey and predator densities at time $t$, respectively. All the parameters in the model are positive. $a$ is the maximum value attained by the per capita reduction rate of the prey. $b$ is the role of a prey-dependent carrying capacity for predators. $r_1$ and $r_2$ are the growth rates of the prey and predator. $K$ is the carrying capacity of the prey in the ecosystem.

On the other hand, time delays often occur in nature. In population ecology, especially in predator-prey models, time delay effects are ubiquitous, such as the time it takes for predators to hunt prey, the time it takes predators and prey to mature, and the time it takes predators to digest prey and convert it into energy. A lot of delayed models are formulated to investigate ecological systems, more and more scholars have studied delayed predator-prey models and affirmed that delays can have very complex impacts on the dynamics of a model [5, 14, 30, 32, 40, 44, 45]. Actually, any model of species dynamics without time delays is an approximation at best [22] and time delays may significantly influence the overall properties of dynamical systems [4, 24, 25, 35, 38]. The time delays will cause the stable configuration of the equilibrium points of model to change greatly. Generally, the loss of stability, various oscillations and periodic solutions can be caused by time delays.

In model (1), there exist the delays in maturation time of predator and gestation time of prey, and these types of time delays should be considered. Motivated by these, this present work will propose a delay-induced predator-prey model with
Michaelis-Menten type predator harvesting as follows

\[
\begin{align*}
x &= r_1 x \left(1 - \frac{x(t - \tau_1)}{K}\right) - a xy, \\
y &= r_2 y \left(1 - \frac{y}{b x}\right) - \frac{q E y(t - \tau_2)}{c E + l y(t - \tau_2)},
\end{align*}
\]  

(2)

where \(\tau_1\) represents the delay in gestation time of prey and \(\tau_2\) is the delay in maturation time of predator.

This paper is organized as follows. In Section 2, some sufficient conditions for the existence of the nontrivial positive equilibria are obtained. In Section 3, by analyzing the characteristic equation of the linearized model of (2) at one of the positive equilibria, the sufficient conditions ensuring the local stability of the positive equilibrium and the existence of Hopf bifurcation are obtained. Some explicit formulas determining the direction and stability of periodic solutions bifurcating from Hopf bifurcations are demonstrated by applying the normal form method and center manifold theory in Section 4. To verify our theoretical predictions, some numerical simulations are also included in Section 5.

2. The equilibria of model (2). In order to simplify model (2), the following transformations as [19] are taken

\[
\begin{align*}
\bar{t} &= r_1 t, \quad \bar{x} = \frac{x}{K}, \quad \bar{y} = \frac{a y}{r_1},
\end{align*}
\]

dropping the bars, then the model (2) can be rewritten as

\[
\begin{align*}
\dot{x} &= x(1 - x(t - \tau_1)) - x y, \\
\dot{y} &= y \left(\delta - \beta y\right) - \frac{\alpha y(t - \tau_2)}{\gamma + y(t - \tau_2)},
\end{align*}
\]  

(3)

where \(\delta = \frac{r_2}{r_1}, \beta = \frac{r_2}{ab K}, \alpha = \frac{q E}{ir_1}, \) and \(\gamma = \frac{a E}{ir_1}\) are positive constants. We can regard \(\alpha\) as the maximum harvesting rate of the predator species. When \(\gamma\) equals the number of predator species, harvested biomass reaches one-half of the maximum harvesting rate in model (3). This model characterizes the behavior of a commercial harvesting company when both the revenue and cost of harvesting determined the company’s harvesting strategy.

About the number of equilibria of model (3), we have the following theorem [19].

**Theorem 2.1.** The equilibria of model (3) are as follows: model (3) always has a boundary equilibrium given by \(E_0(1, 0)\) for all positive parameters. And for the positive equilibrium point(s), we have:

(a) If \(\alpha > \alpha_1\), then model (3) has no positive equilibria.

(b) If \(\alpha = \alpha_1\) and \(\gamma < \frac{\delta}{\beta}\), then model (3) has a unique positive equilibrium

\[
E_1(x_1, y_1), \text{ where } x_1 = \frac{\sqrt{\beta(\beta+\delta)(\gamma+1)}}{\beta+\delta}, \quad y_1 = \frac{\beta+\delta-\sqrt{\beta(\beta+\delta)(\gamma+1)}}{\beta+\delta}.
\]

(c) If \(\gamma\delta < \alpha < \alpha_1\) and \(\gamma < \frac{\delta}{\beta}\), then model (3) has two distinct positive equilibria

\[
E_2(x_2, y_2), E_3(x_3, y_3), \text{ where } x_{2,3} = \frac{\gamma+2\beta+\gamma+\delta-\alpha+\sqrt{\Delta}}{2(\beta+\delta)}, \quad y_{2,3} = 1 - x_{2,3}.
\]

(d) If \(\alpha = \gamma\delta\) and \(\gamma < \frac{\delta}{\beta}\), then \(E_3\) coincides with \(E_0\) and model (3) has a unique positive equilibrium \(E_2(x_2, y_2)\), where \(x_2\) and \(y_2\) can be simplified as

\[
x_2 = \frac{\gamma+\beta}{\beta+\delta}, \quad y_2 = \frac{\delta-\beta}{\delta+\beta}.
\]

(e) If \(0 < \alpha < \gamma\delta\), then model (3) has a unique positive equilibrium \(E_2(x_2, y_2)\), where \(x_2\) and \(y_2\) are the same as in case (c).
where \( \alpha_1 = \beta \gamma + 2 \beta + \gamma \delta + \delta - 2\sqrt{\beta (\beta + \delta) (\gamma + 1)} \), \( \alpha_2 = \beta \gamma + 2 \beta + \gamma \delta + \delta + 2\sqrt{\beta (\beta + \delta)(\gamma + 1)} \), \( \Delta = \alpha^2 - 2(\beta \gamma + 2 \beta + \gamma \delta + \delta) \alpha + (\delta + \gamma \delta + \beta \gamma)^2 \).

3. The stability of equilibrium \( E_2 \) and existence of Hopf bifurcation. We will discuss the stability of the equilibrium \( E_2(x_2, y_2) \) of model (3). The equilibrium is translated to the origin by letting \( \bar{x} = x - x_2 \), \( \bar{y} = y - y_2 \), then

\[
\begin{align*}
\dot{x} &= -\left(\bar{x} + x_2\right)(\bar{x}(t - \tau_1) + \bar{y}), \\
\dot{y} &= \left(\bar{y} + y_2\right)\left(\bar{y} - \frac{\beta(y + y_2)}{x + x_2}\right) - \frac{\alpha(\bar{y}(t - \tau_2) + y_2)}{\gamma y(t - \tau_2) + y_2}.
\end{align*}
\]

Let \( f(\bar{x}, \bar{y}) = (\bar{y} + y_2)\left(\delta - \frac{\beta(y + y_2)}{x + x_2}\right) h(\bar{y}(t - \tau_2)) = -\frac{\alpha(\bar{y}(t - \tau_2) + y_2)}{\gamma y(t - \tau_2) + y_2} \). Taylor expanding \( f(\bar{x}, \bar{y}) \) at \( \bar{x} = 0, \bar{y} = 0, h(\bar{y}(t - \tau_2)) \) at \( \bar{y}(t - \tau_2) = 0 \) and in order to simplify the discussion rewriting \( \bar{x} \) and \( \bar{y} \) as \( x, y \), respectively, we can obtain

\[
\begin{align*}
\dot{x} &= -\left(x + x_2\right)(x(t - \tau_1) + y), \\
\dot{y} &= \beta \left(\frac{y_2}{x_2}\right)^2 x + \left(\delta - \frac{2\beta y_2}{x_2}\right) y - \frac{\alpha \gamma}{\gamma y_2} t y(t - \tau_2) + \beta(y^2 + 2y_2 y) \frac{1}{(x_2)^2} x \\
& \quad - \beta(y + y_2)^2 \sum_{n=3}^{\infty} (-1)^{n-1} \frac{1}{(x_2)^n} x^{n-1} + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) y^n(t - \tau_2),
\end{align*}
\]

then the origin is an equilibrium of model (5), where \( h^{(i)}(0) \) denotes the \( n \)th derivative of \( h(\bar{y}(t - \tau_1)) \) evaluated at the equilibrium \( \bar{y} = 0 \), and model (5) linearized about this equilibrium is given by

\[
\begin{align*}
\dot{x} &= -x_2 x(t - \tau_1) - x_2 y, \\
\dot{y} &= \beta \left(\frac{y_2}{x_2}\right)^2 x + \left(\delta - \frac{2\beta y_2}{x_2}\right) y - \frac{\alpha \gamma}{\gamma y_2} t y(t - \tau_2).
\end{align*}
\]

Let \( M \) be the Jacobian matrix of model (6) and \( \lambda \) be an eigenvalue of \( M \), then

\[
M = \begin{pmatrix}
-x_2 e^{-\lambda \tau_1} & \beta \left(\frac{y_2}{x_2}\right)^2 \\
\delta - \frac{2\beta y_2}{x_2} & -x_2 - \frac{\alpha \gamma}{\gamma y_2} e^{-\lambda \tau_2}
\end{pmatrix},
\]

hence the characteristic equation of \( M \) is given by

\[
\lambda^2 + h_1 \lambda + h_2 \lambda e^{-\lambda \tau_1} + h_3 \lambda e^{-\lambda \tau_2} + h_4 e^{-\lambda \tau_1} + h_5 e^{-\lambda (\tau_1 + \tau_2)} + h_6 = 0,
\]

where

\[
\begin{align*}
h_1 &= -\delta + \frac{2\beta y_2}{x_2} h_2 = x_2, h_3 = -\frac{\alpha \gamma}{\gamma y_2}, \\
h_4 &= 2\beta y_2 - x_2 \delta, h_5 = -\frac{\alpha \gamma}{\gamma y_2}, h_6 = \frac{\beta y_2^2}{x_2^2}.
\end{align*}
\]

We know that the stability of zero equilibrium \( (0, 0) \) of model (5) is relying on the real parts of roots of (8). If all the roots of (8) have negative real parts, then the equilibrium \( (0, 0) \) is asymptotic stable, or else, the equilibrium \( (0, 0) \) is unstable. For this purpose, we will study the distribution of roots of (8) to confirm the stability of the equilibrium.

In the following, we will consider the distribution of roots of (8) while \( \tau_1 \) and \( \tau_2 \) are different given values. Five cases are considered as follows [18].

**Case 1.** \( \tau_1 = \tau_2 = 0 \). Then the characteristic equation (8) becomes

\[
\lambda^2 + (h_1 + h_2 + h_3) \lambda + h_4 + h_5 + h_6 = 0.
\]

Employing the Routh-Hurwitz stability criterion, we get the following necessary and sufficient conditions of local stability of the zero equilibrium of model (5) at once.
Lemma 3.1. If constants $h_i$ ($i = 1, 2, \ldots, 6$) defined by (9) satisfy
\[
(H_1): h_1 + h_2 + h_3 > 0 \quad \text{and} \quad h_4 + h_5 + h_6 > 0,
\]
then all the roots of (10) have negative real parts, and hence the equilibrium (0, 0) of model (5) is locally asymptotically stable. That is, the equilibrium $E_2(x_2, y_2)$ of model (3) is locally asymptotically stable.

Case 2. $\tau_1 = 0, \tau_2 > 0$. Then the characteristic equation (8) becomes
\[
\lambda^2 + (h_1 + h_2)\lambda + h_3 e^{-\lambda \tau_2} + h_5 e^{-\lambda \tau_2} + h_4 + h_6 = 0.
\]
To obtain the critical values of stability change, supposing $\lambda = i\varpi(\varpi > 0)$ is a root of (12), we can have that $\varpi$ satisfies the following equation
\[
(i\varpi)^2 + (h_1 + h_2)(i\varpi) + h_3(i\varpi)e^{-(i\varpi)\tau_2} + h_5e^{-(i\varpi)\tau_2} + h_4 + h_6 = 0,
\]
i.e.,
\[
(i\varpi)^2 + (h_1 + h_2)(i\varpi) + h_3(i\varpi)(\cos(\varpi \tau_2) - i \sin(\varpi \tau_2)) + h_5(\cos(\varpi \tau_2) - i \sin(\varpi \tau_2)) + h_4 + h_6 = 0.
\]
We separate the real and imaginary parts of (13), and have the following equations
\[
\begin{cases}
\varpi(h_1 + h_2) + \varpi h_3 \cos(\varpi \tau_2) - h_5 \sin(\varpi \tau_2) = 0, \\
-\varpi^2 + h_4 + h_6 + h_3 \varpi \sin(\varpi \tau_2) + h_5 \cos(\varpi \tau_2) = 0,
\end{cases}
\]
then
\[
\begin{align*}
\sin(\varpi \tau_2) &= \frac{\varpi(h_2 h_3 - h_2 h_5 - h_3 h_6)}{h_3^2 \varpi^2 + h_5^2}, \\
\cos(\varpi \tau_2) &= -\frac{h_1 h_3 \varpi^2 + h_4 h_5 + h_5 h_6}{h_3^2 \varpi^2 + h_5^2}.
\end{align*}
\]
By using $\sin^2(\varpi \tau_2) + \cos^2(\varpi \tau_2) = 1$, we can obtain
\[
\varpi^4 + \epsilon_1 \varpi^2 + \epsilon_2 = 0,
\]
where $\epsilon_1 = h_1^2 + h_2^2 - h_3^2 - 2h_6, \epsilon_2 = (h_4 + h_5)^2 - h_5^2$. Denote $z = \varpi^2$, then (15) can be denoted simply as the following equation
\[
z^2 + \epsilon_1 z + \epsilon_2 = 0.
\]
For (16) we have the following lemma.

Lemma 3.2. For the polynomial equation (16), we have the following results:
\begin{itemize}
\item[(i)] If $\epsilon_2 \geq 0$ and $\epsilon_1 \geq 0$, equation (16) has no positive roots;
\item[(ii)] If $\epsilon_2 < 0$, or $\epsilon_2 > 0, \epsilon_1 < 0, \epsilon_1^2 - 4\epsilon_2 \geq 0$, equation (16) has at least one positive root.
\end{itemize}
Suppose that (16) has positive roots, without loss of generality, we assume that it has two positive roots, denoted by $z_1$ and $z_2$, respectively. Hence, equation (15) has two positive roots $\varpi_1 = \sqrt{z_1}, \varpi_2 = \sqrt{z_2}.$ According to (14), we have
\[
\cos(\varpi_i \tau_2) = -\frac{h_2 h_3 \varpi_i^2 + h_4 h_5 + h_5 h_6}{h_3^2 \varpi_i^2 + h_5^2}, \quad (i = 1, 2).
\]
Define
\[
\tau_{2i}^{(j)} = \frac{1}{\varpi_i} \left\{ \arccos \left[ -\frac{h_2 h_3 \varpi_i^2 + h_4 h_5 + h_5 h_6}{h_3^2 \varpi_i^2 + h_5^2} \right] + 2j\pi \right\}, \quad (i = 1, 2; j = 0, 1, 2, \ldots),
\]
where $\varpi_i = \sqrt{z_i}$.
then when \( \tau_2 = \tau_{2i}^{(j)} (i = 1, 2; j = 0, 1, 2, \ldots) \), equation (12) has a pair of purely imaginary roots \( \pm \iota \omega_i \). Thus, a possible Hopf bifurcation occurs when \( \tau_2 = \tau_{2i}^{(j)} (i = 1, 2; j = 0, 1, 2, \ldots) \), defining

\[
\tau_{20} = \min \left\{ \tau_{2i}^{(j)} \right\}, \omega_{20} = \omega_{i0}, (i = 1, 2; j = 0, 1, 2, \ldots).
\]

Let \( \lambda(\tau_2) = \alpha(\tau_2) + i \omega(\tau_2) \) be a root of (12) satisfied \( \alpha(\tau_{20}) = 0, \omega(\tau_{20}) = \omega_{20} \), near \( \tau_2 = \tau_{20} \). For this pairs of conjugate complex roots, we have the following result.

**Lemma 3.3.** If constants \( h_i(i = 1, 2, \ldots, 6) \) defined by (9) and the following assumption

\[
(H_2): [h'(z)]_{\tau_2 = \tau_{2i}^{(j)}} > 0, (i = 1, 2; j = 0, 1, 2, \ldots)
\]

hold, then \( \frac{d(\text{Re}\lambda)}{d\tau_2} \big|_{\tau_2 = \tau_{2i}^{(j)}} > 0 (i = 1, 2; j = 0, 1, 2, \ldots) \), where \( h(z) = z^2 + \epsilon_1 z + \epsilon_2 \) and \( h'(z) = 2z + \epsilon_1 \).

**Proof.** Differentiating (12) with respect to \( \tau_2 \) and by direct calculation, we derive that

\[
\left( \frac{d\lambda}{d\tau_2} \right)^{-1} = \frac{(2\lambda + h_1 + h_2)e^{-\lambda \tau_2} + h_3}{h_3 \lambda^2 + h_5 \lambda} - \frac{\tau_2}{\lambda},
\]

which leads to

\[
\text{Re} \left( \frac{d\lambda}{d\tau_2} \right)^{-1} = \text{Re} \left\{ \frac{(2\lambda + h_1 + h_2)e^{-\lambda \tau_2} + h_3}{h_3 \lambda^2 + h_5 \lambda} - \frac{\tau_2}{\lambda} \right\}
\]

\[
= \text{Re} \left\{ \frac{(2\lambda + h_1 + h_2)e^{-\lambda \tau_2} + h_3}{h_3 \lambda^2 + h_5 \lambda} \right\}
\]

\[
= \text{Re} \left\{ \frac{(2i \omega + h_1 + h_2)(\cos(\omega \tau_2) + i \sin(\omega \tau_2)) + h_3}{h_3(i \omega)^2 + h_5 i \omega} \right\}
\]

\[
= \frac{\omega^2(h_2^2 + \omega^2)(h_1^2 + h_2^2 - h_3^2 + 2\omega^2 - 2h_6)}{\omega^4 + h_5^2 \omega^2 - 2h_6^2}
\]

\[
= \omega^2(h_2^2 + \omega^2)(h_1^2 + h_2^2 - h_3^2 + 2\omega^2 - 2h_6)
\]

Since \( \text{sgn} \left\{ \frac{d(\text{Re}\lambda)}{d\tau_2} \right\} \big|_{\tau_2 = \tau_{2i}^{(j)}} = \text{sgn} \left\{ \text{Re} \left( \frac{d\lambda}{d\tau_2} \right)^{-1} \right\} \big|_{\tau_2 = \tau_{2i}^{(j)}} = \text{sgn} \{h'(z)\} \), if the condition \( (H_2) \) holds, then \( \text{sgn} \left\{ \frac{d(\text{Re}\lambda)}{d\tau_2} \right\} \big|_{\tau_2 = \tau_{2i}^{(j)}} > 0 (i = 1, 2; j = 0, 1, 2, \ldots) \). \( \square \)

From the above discussion, we can obtain the following theorem.

**Theorem 3.4.** If constants \( h_i(i = 1, 2, \ldots, 6) \) defined by (9) satisfy the conditions \((H_1)\) and \((H_2)\), then the following results hold:

(i) If \( \epsilon_2 \geq 0 \) and \( \epsilon_1 \geq 0 \), all the roots of (12) have negative real parts when \( \tau_1 = 0 \), \( \tau_2 > 0 \), the equilibrium \( E_2(x_2, y_2) \) of model (3) is locally asymptotically stable.

(ii) If \( \epsilon_2 < 0 \), or \( \epsilon_2 > 0, \epsilon_1 < 0, \epsilon_1^2 - 4\epsilon_2 \geq 0 \), equation (16) has at least one positive root, and all the roots of (12) have negative real parts when \( \tau_1 = 0 \), \( 0 < \tau_2 < \tau_{20} \), the equilibrium \( E_2(x_2, y_2) \) of model (3) is locally asymptotically stable for all \( \tau_1 = 0, 0 < \tau_2 < \tau_{20}; \) when \( \tau_1 = 0, \tau_2 > \tau_{20} \), the equilibrium \( E_2(x_2, y_2) \) of model (3) is unstable.
Then when $\tau = \tau_2$, the equilibrium $E_2(x_2, y_2)$ becomes

$$\mu^2 + \mu^4 + (h_1 + h_3)\lambda + h_2\lambda e^{-\lambda\tau_2} + (h_4 + h_5)e^{-\lambda\tau_2} + h_6 = 0.$$  \hfill (23)

Supposing $\lambda = i\omega(\omega > 0)$ is a root of (23), we hence obtain that $\omega$ satisfies the following equation

$$\left\{ \begin{array}{l}
(i\omega)^2 + (h_1 + h_3)(i\omega) + h_2(i\omega)(\cos(\omega\tau_1) - i\sin(\omega\tau_1)) \\
+ (h_4 + h_5)(\cos(\omega\tau_1) - i\sin(\omega\tau_1)) + h_6 = 0.
\end{array} \right.$$  \hfill (24)

By separating the real and imaginary parts of (24) yields the following equations

\begin{align*}
\sin(\omega\tau_1) &= \frac{\omega(h_2\omega^2 - h_1h_4 - h_1h_5 - h_2h_6 + h_4h_3 + h_3h_5)}{h_2^2\omega^2 + (h_4 + h_5)^2}, \\
\cos(\omega\tau_1) &= \frac{(h_1h_2 + h_4)\omega^2 - h_4h_6 - h_5h_6}{h_2^2\omega^2 + (h_4 + h_5)^2}.
\end{align*}  \hfill (25)

Using $\sin^2(\omega\tau_1) + \cos^2(\omega\tau_1) = 1$, we have

$$\omega^4 + \varepsilon_1 \omega^2 + \varepsilon_2 = 0,$$  \hfill (26)

where $\varepsilon_1 = (h_1 - h_3)^2 - 2h_6 - h_2^2$, $\varepsilon_2 = -(h_4 + h_5)^2 + h_6^2$. Denoting $v = \omega^2$, then (26) can be denoted simply as the following equation

$$v^2 + \varepsilon_1 v + \varepsilon_2 = 0.$$  \hfill (27)

**Lemma 3.5.** For the polynomial equation (27), we have the following results:

(i) If $\varepsilon_2 \geq 0$ and $\varepsilon_1 \geq 0$, equation (27) has no positive roots;

(ii) If $\varepsilon_2 < 0$, or $\varepsilon_2 > 0$, $\varepsilon_1 < 0$, $\varepsilon_1^2 - 4\varepsilon_2 \geq 0$, equation (27) has at least one positive root.

Suppose that (27) has positive roots, without loss of generality, we assume that it has two positive roots, denoted by $\omega_1, \omega_2$, respectively. Hence, equation (26) has two positive roots $\omega_1 = \sqrt{\varepsilon_1}$, $\omega_2 = \sqrt{-\varepsilon_2}$.

According to (25), define

$$\tau_{3i}^{(j)} = \frac{1}{\omega_i} \left\{ \arccos \left[ \frac{(h_1h_2 + h_4)\omega_i^2 - h_4h_6 - h_5h_6}{h_2^2\omega_i^4 + (h_4 + h_5)^2} \right] + 2j\pi \right\},$$  \hfill (28)

then when $\tau_1 = \tau_{3i}^{(j)}(i = 1, 2; j = 0, 1, 2, ...)$, equation (23) has a pair of purely imaginary roots $\pm i\omega_i$. Thus, a possible Hopf bifurcation occurs when $\tau_1 = \tau_{3i}^{(j)}(i = 1, 2; j = 0, 1, 2, ...)$, defining

$$\tau_{30} = \min \left\{ \tau_{3i}^{(j)} \right\}, \omega_{30} = \omega_{i0}, (i = 1, 2; j = 0, 1, 2, ...).$$  \hfill (29)

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\omega(\tau_1)$ be a root of (23) satisfied $\alpha(\tau_{30}) = 0, \omega(\tau_{30}) = \omega_{30}$ near $\tau_1 = \tau_{30}$. For this pairs of conjugate complex roots, we have the following result.
Lemma 3.6. If constants $h_i (i = 1, 2, ..., 6)$ defined by (9) and the following assumption

$$(H_3) : |g'(v)|_{	au_1=r_{3i}^{(i)}} > 0, (i = 1, 2; j = 0, 1, 2,...)$$

hold, then $$\left[ \frac{d(\text{Re} \lambda)}{d\tau} \right]_{\tau_1=r_{3i}^{(i)}} > 0 (i = 1, 2; j = 0, 1, 2,...),$$

where $g(v) = v^2 + \varepsilon_1 v + \varepsilon_2$ and $g'(v) = 2v + \varepsilon_1$.

Then we can obtain the following results.

Theorem 3.7. If constants $h_i (i = 1, 2, ..., 6)$ defined by (9) satisfy the conditions $(H_1)$ and $(H_3)$, then the following results hold:

(i) If $\varepsilon_2 \geq 0$ and $\varepsilon_1 \geq 0$, all the roots of (23) have negative real parts when $\tau_1 > 0$, $\tau_2 = 0$, thus the equilibrium $E_2(x_2, y_2)$ of model (3) is locally asymptotically stable.

(ii) If $\varepsilon_2 < 0$ or $\varepsilon_2 > 0$, $\varepsilon_1 < 0$, $\varepsilon_1^2 - 4\varepsilon_2 \geq 0$, (27) has at least one positive root, and all the roots of (23) have negative real parts when $\tau_2 = 0$, $0 < \tau_1 < \tau_{3_0}$, the equilibrium $E_2(x_2, y_2)$ of model (3) is locally asymptotically stable for all $\tau_2 = 0$, $0 < \tau_1 < \tau_{3_0}$; when $\tau_2 = 0$, $\tau_1 > \tau_{3_0}$, the equilibrium $E_2(x_2, y_2)$ of model (3) is unstable.

(iii) If the condition of (ii) is satisfied, model (3) exhibits a Hopf bifurcation at $E_2(x_2, y_2)$ when $\tau_2 = 0$, $\tau_1$ crosses through each critical value $r_{3i}^{(i)} (i = 1, 2; j = 0, 1, 2,...)$. That is, a family of nonconstant periodic solutions can bifurcate from the equilibrium $E_2(x_2, y_2)$.

Case 4. $\tau_1 = \tau_2 = \tau \neq 0$. Then the characteristic equation (8) becomes

$$\lambda^2 + h_1 \lambda + (h_2 + h_3) \lambda e^{-\lambda \tau} + h_4 e^{-\lambda \tau} + h_5 e^{-2\lambda \tau} + h_6 = 0.$$  \hspace{1cm} (31)

Clearly, $i\varpi$ ($\varpi > 0$) is a root of (31) if and only if $\varpi$ satisfies the following equation

$$(\cos(\varpi \tau) + i\sin(\varpi \tau))(i\varpi)^2 + h_1(i\varpi) + h_6) + (h_2 + h_3)(i\varpi) + h_4$$

$$+ h_5(\cos(\varpi \tau) - i\sin(\varpi \tau)) = 0,$$  \hspace{1cm} (32)

by separating the real and imaginary parts of (32) and using $\sin^2(\varpi \tau) + \cos^2(\varpi \tau) = 1$, we have

$$\varpi^8 + m_1 \varpi^6 + m_2 \varpi^4 + m_3 \varpi^2 + m_4 = 0,$$  \hspace{1cm} (33)

where

$$m_1 = 2h_1^2 - h_2^2 - 2h_2h_3 - h_3^2 - 4h_6;$$

$$m_2 = h_1^4 - h_1^2h_3^2 - 4h_2^2h_6 - 2h_1h_2h_4 - 2h_1h_3h_4 + 2h_2^2h_5 + 2h_2^2h_6$$

$$+ 4h_2h_3h_4 + 4h_2^2h_5h_6 + 2h_2^2h_5 + 2h_3^2h_6 - 2h_2^2 + 6h_6^2;$$

$$m_3 = - h_1^2h_4^2 - 2h_1^2h_5^2 - 2h_2h_4h_5^2 - 2h_1h_2h_4h_5 + 2h_1h_2h_4h_6 + 4h_1h_3h_4h_5$$

$$- h_2^2h_5h_6 - h_2^2h_5h_6 + h_3^2h_6 - 2h_2h_3h_4h_5 + 2h_2h_3h_4h_6 - 2h_2h_3h_5h_6 - 2h_2h_3h_6^2$$

$$- h_3^2h_5^2 - 2h_2h_6^2 - 2h_3^2h_6^2 + 4h_3^2h_6 - 4h_6^3;$$

$$m_4 = - h_1^2h_5^2 + 2h_2h_4h_5h_6 - h_4^2h_6^2 + h_3^2 - 2h_2^2h_5^2 + h_4^2.$$  \hspace{1cm} (34)

We assume

$$(H_{4.1}) : \text{Equation (33) has at least one positive real root.}$$

Since (33) can be treated as a fourth order equation by the transformation $u = \varpi^2$, equation (33) has at most four real roots $u$ which implies that there exist at
most four positive roots \( \varpi_i (i = 1, 2, 3, 4) \). Without loss of generality, we assume that it has four positive roots, denoted by \( \varpi_i (i = 1, 2, 3, 4) \).

If we define
\[
\tau_{4i}^{(j)} = \frac{1}{\varpi_i} \left\{ \arccos \left[ \frac{-(h_1 h_2 + h_1 h_3 - h_4) \varpi_i^2 - h_4 h_5 + h_4 h_6}{\varpi_i^4 + (h_7^2 - 2h_6) \varpi_i^2 + h_6^2 - h_7^2} \right] + 2j\pi \right\}, \quad (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots),
\]
when \( \tau_1 = \tau_2 = \tau_{4i}^{(j)} (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots) \), equation (31) has a pair of pure imaginary roots \( \pm i\varpi_i \), defining
\[
\tau_{40} = \min \left\{ \tau_{4i}^{(j)} \right\}, \quad \varpi_{40} = \varpi_{40}, (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots). \tag{36}
\]

We assume
\[(H_{4i}) : U_R V_R + U_I V_I > 0,\]
then \[\left[ \frac{d(\Re \lambda)}{d\tau} \right]_{\tau = \tau_{4i}^{(j)}} > 0 (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots), \]
where \( U_R, U_I, V_R, V_I \) represent the real part of \( U \), the imaginary part of \( U \), the real part of \( V \) and the imaginary part of \( V \), respectively. If the condition \((H_{4i}) : U_R V_R + U_I V_I > 0 \) holds, then \[\left[ \frac{d(\Re \lambda)}{d\tau} \right]_{\tau = \tau_{4i}^{(j)}} > 0 (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots). \tag{38}\]

**Theorem 3.8.** If model (3) satisfies the conditions \((H_1), (H_{4i})\) and \((H_{4i})\) when \( \tau_1 = \tau_2 = \tau \), then

(i) If \( 0 < \tau < \tau_{40} \), the equilibrium \( E_2(x_2, y_2) \) of model (3) is locally asymptotically stable; if \( \tau > \tau_{40} \), the equilibrium \( E_2(x_2, y_2) \) of model (3) is unstable.

(ii) Model (3) exhibits a Hopf bifurcation at \( E_2(x_2, y_2) \) when \( \tau \) crosses through each critical value \( \tau_{4i}^{(j)} (i = 1, 2, 3, 4; j = 0, 1, 2, \ldots) \). That is, a family of nonconstant periodic solutions can bifurcate from the equilibrium \( E_2(x_2, y_2) \).

**Case 5.** \( \tau_1 > 0, \tau_2 > 0 \) and \( \tau_1 \neq \tau_2 \). We suppose characteristic equation (8) with \( \tau_2 \) in its stable interval, i.e., \( \tau_2 \in [0, \tau_{20}] \), and regard \( \tau_1 \) as a parameter. Let \( i \varpi \) \( (\varpi > 0) \) be a root of (8), then we have
\[
(\varpi)^2 + h_1 (i \varpi) + h_2 (i \varpi) (\cos(\varpi \tau_1) - i \sin(\varpi \tau_1)) + h_3 (i \varpi) (\cos(\varpi \tau_2) - i \sin(\varpi \tau_2)) + h_4 (\cos(\varpi \tau_1) - i \sin(\varpi \tau_1)) + h_5 (\cos(\varpi \tau_1 + \varpi \tau_2) - i \sin(\varpi \tau_1 + \varpi \tau_2)) + h_6 = 0, \tag{39}
\]
separating the real and imaginary parts of the above equation yields the following equations

\[
\begin{aligned}
&\begin{cases}
-h_5 \cos(\varpi \tau_2) - h_4 \sin(\varpi \tau_1) + (-h_5 \sin(\varpi \tau_2) - h_2 \varpi) \cos(\varpi \tau_1) \\
+h_3 \varpi \cos(\varpi \tau_2) + h_1 \varpi = 0,
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
&\begin{cases}
(h_5 \sin(\varpi \tau_2) + h_2 \varpi) \sin(\varpi \tau_1) + (-h_5 \cos(\varpi \tau_2) - h_4) \cos(\varpi \tau_1) \\
+h_3 \varpi \sin(\varpi \tau_2) - \varpi^2 + h_6 = 0.
\end{cases}
\end{aligned}
\]

(40)

Adding up the squares of the corresponding sides of (40) yields the following algebra equation with respect to \( \varpi \)

\[
M_1 \varpi^6 + M_2 \varpi^5 + M_3 \varpi^4 + M_4 \varpi^3 + M_5 \varpi^2 + M_6 \varpi + M_7 = 0,
\]

(41)

where

\[
M_1 = h_2^2;
\]

\[
M_2 = -4h_3^2 h_3 \sin(\varpi \tau_2);
\]

\[
M_3 = h_4^2 \left(4h_3^2 \cos^2(\varpi \tau_2) + 4h_1 h_3 \cos(\varpi \tau_2) + 2h_1^2 - h_2^2 + 6h_3^2 - 2h_6\right);
\]

\[
M_4 = -4h_3^2 h_3 \sin(\varpi \tau_2) \left(2h_1 h_3 \cos(\varpi \tau_2) + h_1^2 - h_2^2 + h_3^2 - \frac{3}{2} h_6\right);
\]

\[
M_5 = h_4^2 \left[4h_3^2 (h_1^2 + h_2^2 + h_6) \cos^2(\varpi \tau_2) + 4h_1 h_3 (h_1^2 - h_2^2 + h_3^2 - h_6) \cos(\varpi \tau_2)
\right.
\]

\[
\left. + h_4^2 - (2h_1^2 + 2h_6)h_3^2 + h_1^2 - (2h_1^2 + 2h_6)h_3^2 + h_6^2\right];
\]

\[
M_6 = 4h_3^2 h_3 \sin(\varpi \tau_2) \left[\left(h_2 + \frac{1}{2} h_6\right) (2h_1 h_3 \cos(\varpi \tau_2) + h_3^2) - \frac{1}{2} h_6^2\right];
\]

\[
M_7 = -h_2^2 \left(2h_1 h_3 \cos(\varpi \tau_2) + h_4^2 + h_3^2\right) \left(2h_1 h_3 \cos(\varpi \tau_2) + h_4^2 h_2^2 + h_4^2 h_3^2 - h_6^2\right).
\]

(42)

We assume

\[(H_{5i}) : \text{Equation (41) has at least one positive real root.}\]

Without loss of generality, we assume that it has six positive roots, denoted by \( \varpi_i (i = 1, 2, ..., 6) \). If we define

\[
\tau_{5i}^{(j)} = \frac{1}{\varpi_i} \arccos \left[- \frac{(h_3 h_4 - h_1 h_5) \varpi_i \sin(\varpi_i \tau_2) + h_5 h_6 \cos(\varpi_i \tau_2) + h_4 h_6}{h_2^2 \varpi_i^2 - 2h_2 h_5 \varpi_i \sin(\varpi_i \tau_2) + 2h_4 h_5 \cos(\varpi_i \tau_2) + h_4^2 + h_5^2}\right]
\]

\[
+ 2j\pi \right), (i = 1, 2, ..., 6; j = 0, 1, 2, ...),
\]

(43)

when \( \tau_1 = \tau_{5i}^{(j)} (i = 1, 2, ..., 6, j = 0, 1, 2, ...) \), equation (8) has a pair of pure imaginary roots \( \pm i \varpi_i \), defining

\[
\tau_{50} = \min \left\{ \tau_{5i}^{(j)} \right\}, \varpi_{50} = \varpi_{i0}, (i = 1, 2, ..., 6; j = 0, 1, 2, ...).
\]

(44)

We assume

\[(H_{5z}) : \frac{d(\text{Re} \lambda)}{d\tau_1} |_{\tau_1 = \tau_{5i}^{(j)}} > 0, \quad (i = 1, 2, ..., 6; j = 0, 1, 2, ...),
\]

then we have the following theorem.

**Theorem 3.9.** If model (3) satisfies the conditions \((H_1), (H_2), (H_{5i})\) and \((H_{5z})\), when \( \tau_2 \in [0, \tau_{20}) \), then
(i) If \(0 < \tau_1 < \tau_{50}, \tau_2 \in [0, \tau_{20})\), the equilibrium \(E_2(x_2, y_2)\) of model (3) is locally asymptotically stable, if \(\tau_1 > \tau_{50}, \tau_2 \in [0, \tau_{20})\), the equilibrium \(E_2(x_2, y_2)\) of model (3) is unstable.

(ii) Model (3) exhibits a Hopf bifurcation at \(E_2(x_2, y_2)\) when \(\tau_2 \in [0, \tau_{20})\) and \(\tau_1\) crosses through each critical value \(\tau^{(j)}_{50}\) \((i = 1, 2, \ldots, 6; j = 0, 1, 2, \ldots)\), that is, a family of nonconstant periodic solutions can bifurcate from the equilibrium \(E_2(x_2, y_2)\).

4. Properties of Hopf bifurcation. In the fifth case of the previous section, we have obtained the sufficient conditions that the Hopf bifurcation occurs at \(\tau_1 = \tau_{50}\), when \(\tau_2 \in [0, \tau_{20})\). In this section, we shall discuss properties of the Hopf bifurcation obtained by Theorem 3.9 and the stability of bifurcated periodic solutions occurring through the Hopf bifurcation by using the normal form theory and the center manifold reduction for retarded functional differential equations (RFDES) due to Hassard, Kazarinoff and Wan [17].

Without loss of generality, we assume \(\tau^*_2 < \tau_{50}(\tau^*_2 \in [0, \tau_{20})\)), \(\tau_1 = \tau_{50} + \mu\), then \(\mu = 0\) is the Hopf bifurcation value of model (3) at the positive equilibrium \(E_2(x_2, y_2)\). Since model (3) is equivalent to model (5), in the following discussion we will mainly study model (5). Letting \(u_1(t) = x(\tau_1 t), u_2(t) = y(\tau_1 t)\) to normalize the delays so that model (5) can be rewritten as a model of RFDEs in phase space \(\mathbb{C} = \mathbb{C}([-1, 0], \mathbb{R}^2)\) of the following form

\[
\begin{align*}
\dot{u}_1(t) &= -(u_1(t) + \tau_1 u_2(t - 1) + u_2), \\
\dot{u}_2(t) &= \beta \left( \frac{u_2}{x_2} \right)^2 u_1 + \left( \delta - \frac{2y_2 u_2}{x_2} \right) u_2 - \frac{\alpha}{(\gamma + y_2) x_2} u_2 \left( t - \tau^*_2 \right) + \beta \left( u_2^2 + 2y_2 u_2 \right) \frac{1}{(x_2)^2} u_1 - \beta (u_2 + y_2)^2 \sum_{n=3}^{\infty} (-1)^{n-1} \frac{1}{(x_2)^n} u_1^{n-1} \\
& \quad + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) \left( u_2 \left( t - \frac{\tau^*_2}{\tau_{50}} \right) \right)^n.
\end{align*}
\]

We assume that \(L_{\mu} : \mathbb{C} \to \mathbb{R}^2\) is a one-parameter family of continuous linear operators and the operator \(\mathcal{F} : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2\) contains the nonlinear terms, beginning with at least quadratic terms, model (45) can be written in a more compact form:

\[
\dot{u}(t) = L_{\mu}(u_t) + \mathcal{F}(\mu, u_t),
\]

where \(u(t) = (u_1(t), u_2(t))^T \in \mathbb{R}^2, u_t(\theta) = u(t + \theta) = (u_1(t + \theta), u_2(t + \theta))^T \in \mathbb{C}\) and \(L_{\mu}, \mathcal{F}\) are given respectively by

\[
L_{\mu} \phi = (\tau_{50} + \mu) A_1 \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_{50} + \mu) A_2 \begin{pmatrix} \phi_1 \left( -\frac{\tau^*_2}{\tau_{50}} \right) \\ \phi_2 \left( -\frac{\tau^*_2}{\tau_{50}} \right) \end{pmatrix} + (\tau_{50} + \mu) A_3 \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix},
\]

and

\[
\mathcal{F}(\mu, \phi) = (\tau_{50} + \mu) \left( \frac{-\phi_1(0) \phi_1(-1) - \phi_1(0) \phi_2(0)}{\tau_{50}} \right) + \beta \sum_{n=3}^{\infty} \frac{1}{(x_2)^n} \phi_1^{n-1}(0) + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) \left( \phi_2^n \left( t - \frac{\tau^*_2}{\tau_{50}} \right) \right)
\]
where \( \phi = (\phi_1(\theta), \phi_2(\theta))^T \in \mathbb{C} \), and

\[
A_1 = \begin{pmatrix} 0 & -x_2 \\ \beta \left( \frac{y_2}{x_2} \right)^2 & \delta - 2 \beta \frac{y_2}{x_2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -\alpha \gamma \left( \frac{y_2}{x_2} \right) & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -x_2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{49}
\]

By the Riesz representation theorem, there exists a \( 2 \times 2 \) matrix function \( \eta(\theta, \mu) \), \(-1 \leq \theta \leq 0\), whose elements are of bounded variation such that

\[
\mathcal{L}_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta) \quad \text{for} \quad \phi \in \mathbb{C}(\mathbb{R}^2).
\]

In fact, we can choose

\[
\eta(\theta, \mu) = \begin{cases} (\tau_{50} + \mu)(A_1 + A_2 + A_3), & \theta = 0, \\ (\tau_{50} + \mu)(A_2 + A_3), & \theta = \left[-\frac{\tau^2_{50}}{\tau_{50}}, 0\right), \\ (\tau_{50} + \mu)A_3, & \theta = \left[-1, -\frac{\tau^2_{50}}{\tau_{50}}\right), \\ 0, & \theta = -1. \end{cases} \tag{50}
\]

For \( \phi \in \mathbb{C}^1([-1, 0], \mathbb{R}^2) \), define

\[
\mathcal{N}(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\xi, \mu) \phi(\xi) = \mathcal{L}_\mu \phi, & \theta = 0 \end{cases} \tag{51}
\]

and

\[
\mathcal{R}(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ \mathcal{F}(\mu, \phi), & \theta = 0, \end{cases} \tag{52}
\]

since \( du_i/d\theta = du_i/dt \), then model (46) becomes

\[
\dot{u}_t = \mathcal{N}(\mu)u_t + \mathcal{R}(\mu)u_t. \tag{53}
\]

For \( \varphi \in \mathbb{C}^1([0, 1], (\mathbb{R}^2)^* \), define

\[
\mathcal{N}^*(\mu)\varphi(s) = \begin{cases} -\frac{d\varphi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, \mu)\varphi(-t), & s = 0 \end{cases} \tag{54}
\]

and a bilinear inner product

\[
\langle \varphi(s), \phi(\theta) \rangle = \varphi(0) \cdot \phi(0) - \int_{\theta = -1}^{0} \int_{\xi = 0}^{\theta} \varphi^T(\xi - \theta) \eta(\theta) \phi(\xi) d\xi, \tag{55}
\]

where \( \eta(\theta) = \eta(\theta, 0) \). Here, for \( a \) and \( b \) in \( \mathbb{C}^n \), \( a \cdot b \) means \( \sum_{i=1}^{n} a_i b_i \), where the \( a_i \) and \( b_i \) are the components of the vectors \( a \) and \( b \), respectively. Then \( \mathcal{N}(0) \) and \( \mathcal{N}^*(0) \) are adjoint operators. Furthermore, \( \langle \varphi, \mathcal{N} \phi \rangle = \langle \mathcal{N}^* \varphi, \phi \rangle \).

Noting that the above scaling transformation, the corresponding characteristic exponents and the associated frequencies are transformed into \( \tau_1 \lambda \) and \( \tau_1 \varpi \). Hence, when \( \mu = 0 \), \( \pm \varpi \tau_{50} \) are the eigenvalues of \( \mathcal{N}(0) \). Therefore, they are also eigenvalues of \( \mathcal{N}^*(0) \). Let \( q(\theta) \) be the eigenvector for \( \mathcal{N}(0) \) corresponding to \( \varpi \tau_{50} \) and \( q^*(s) \) be the eigenvector for \( \mathcal{N}^*(0) \) corresponding to \( -\varpi \tau_{50} \). That is

\[
\mathcal{N}(0)q(\theta) = i\varpi \tau_{50} q(\theta), \tag{56}
\]

\[
\mathcal{N}^*(0)q^*(s) = -i\varpi \tau_{50} q^*(s). \tag{57}
\]
From (51), we can immediately rewrite (56) as follows
\[
\begin{cases}
\frac{dq(\theta)}{d\theta} = i\omega \tau_{50} q(\theta), & \theta \in [-1, 0), \\
\mathcal{L}_0 q(0) = i\omega \tau_{50} q(0), & \theta = 0.
\end{cases}
\] (58)

Using (58), we have
\[q(\theta) = Ke^{i\omega \tau_{50} \theta}, \quad \theta \in [-1, 0],\] (60)
where \( K = (k_1, k_2)^T \) is a constant vector, and from (59), the constant vector \( K \) must satisfy
\[\left( A_1 + A_2 e^{-i\omega \tau_{50}^*} + A_3 e^{-i\omega \tau_{50}} - i\omega I \right) K = 0,\]
i.e.,
\[\begin{pmatrix}
-x_2 e^{-i\omega \tau_{50}} - i\omega \\
\beta \left( \frac{y_2}{x_2} \right)^2 - 2\frac{\beta y_2}{x_2} - \frac{-x_2}{(\gamma + y_2)^2} e^{-i\omega \tau_{50}^*} - i\omega
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2
\end{pmatrix}
= 0,
\]
where \( I \) denotes the \( 2 \times 2 \) identity matrix, the above algebraic equation has an infinite number of solutions. Without loss of generality, setting \( k_1 = 1 \), we have
\[k_2 = \frac{i\omega + x_2 e^{-i\omega \tau_{50}}}{-x_2}.
\]

Similarly, from (54), we rewrite (57) as follows
\[
\begin{cases}
\frac{dq^*(s)}{ds} = i\omega \tau_{50} q^*(s), & s \in (0, 1], \\
\int_{-1}^0 \eta^T(t, 0) \varphi(-t) = \tau_{50} A_1^T \varphi(0) + \tau_{50} A_2^T \varphi \left( \frac{\tau_{50}^*}{\tau_{50}} \right)
\end{cases}
\] (61)
where \( \mathcal{P} \) and \( K^* = (k_1^*, k_2^*)^T \) are a constant and constant vector, respectively. From (62), the constant vector \( K^* \) satisfies
\[\left( A_1^T + A_2^T e^{i\omega \tau_{50}^*} + A_3^T e^{i\omega \tau_{50}} + i\omega I \right) K^* = 0,\]
i.e.,
\[\begin{pmatrix}
-x_2 e^{-i\omega \tau_{50}} + i\omega \\
-x_2 \\
\delta - 2\frac{\beta y_2}{x_2} - \frac{\beta \left( \frac{y_2}{x_2} \right)^2}{(\gamma + y_2)^2} e^{-i\omega \tau_{50}^*} + i\omega
\end{pmatrix}
\begin{pmatrix}
k_1^* \\
k_2^*
\end{pmatrix}
= 0,
\]
where \( I \) denotes the \( 2 \times 2 \) identity matrix. Setting \( k_1^* = 1 \), we have
\[k_2^* = \frac{-i\omega + x_2 e^{i\omega \tau_{50}}}{\beta \left( \frac{y_2}{x_2} \right)^2}.
\]

Next, we will find an appropriate constant \( \mathcal{P} \) to normalize \( q(\theta) \) and \( q^*(s) \) by the condition \( \langle q^*(s), q(\theta) \rangle = 1 \). Since
\[\langle q^*(s), q(\theta) \rangle = \mathcal{P} (k_1, k_2^*) (k_1, k_2)^T - \int_{\theta = -1}^0 \int_{\xi = 0}^\theta \mathcal{P} (k_1, k_2^*) e^{-i\omega \tau_{50} (\xi - \theta)} d\eta(\theta) (k_1, k_2)^T e^{i\omega \tau_{50} \xi} d\xi
\]
and then it follows from (65) that

\[ \langle q^*(s), q(\theta) \rangle = 1 \quad \text{and} \quad \langle q^*(s), \bar{q}(\theta) \rangle = 0, \]

thus, we can choose \( \mathcal{P} \) as

\[
\mathcal{P} = \left( \begin{array}{c}
\kappa_1 \kappa_2^* \\
\kappa_2 \kappa_1^*
\end{array} \right) - \frac{\alpha \gamma}{(\gamma + y_2)^2} \left( \begin{array}{c}
k_2^* \kappa_2^* e^{-i\omega \tau_2} \\
k_2 \kappa_1^* e^{i\omega \tau_2}
\end{array} \right)
\]

for assuring \( \langle q^*(s), q(\theta) \rangle = 1 \) and \( \langle q^*(s), \bar{q}(\theta) \rangle = 0 \), we, thus, can choose \( \mathcal{P} \) as

\[
\mathcal{P} = \frac{1}{\kappa_1 \kappa_2^* + \kappa_2 \kappa_1^* - \frac{\alpha \gamma}{(\gamma + y_2)^2} \left( \begin{array}{c}
k_2^* \kappa_2^* e^{-i\omega \tau_2} \\
k_2 \kappa_1^* e^{i\omega \tau_2}
\end{array} \right)}
\]

We first compute the coordinates to describe the center manifold \( C_0 \) at \( \mu = 0 \) [17]. Let \( X_\delta \) be the solution of (53) when \( \mu = 0 \). Define

\[
z(t) = \langle q^*, X_\delta \rangle, \ V(t, \theta) = X_\delta(\theta) - z(t) q(\theta) - \bar{z}(t) \bar{q}(\theta) = X_\delta(\theta) - 2\text{Re}\{z(t)q(\theta)\},
\]

On the center manifold \( C_0 \), we have \( V(t, \theta) = V(z(t), \bar{z}(t), \theta) \), where

\[
V(z(t), \bar{z}(t), \theta) = V_{20}(\theta) \frac{z^2(t)}{2} + V_{11}(\theta) \bar{z}(t) z(t) + V_{02}(\theta) \frac{\bar{z}^2(t)}{2} + V_{30}(\theta) \frac{z^3(t)}{6} + \cdots,
\]

\[
z(t) \text{ and } \bar{z}(t) \text{ are local coordinates for center manifold } C_0 \text{ in the directions of } q^* \text{ and } \bar{q}^*.
\]

Note that \( V \) is real if \( X_\delta \) is, we shall deal with real solutions only. For the solution \( X_\delta \in C_0 \) of (53), since \( \mu = 0 \), we have

\[
\dot{z}(t) = \langle q^*, X_\delta \rangle = \langle q^*, \mathcal{N}(0) X_\delta + \mathcal{R}(0) X_\delta \rangle = \langle q^*, \mathcal{N}(0) X_\delta \rangle + \langle q^*, \mathcal{R}(0) X_\delta \rangle
\]

\[
= \langle \mathcal{N}^*(0) q^*, X_\delta \rangle + \mathcal{R}^*(0) \cdot \mathcal{F}(0, X_\delta)
\]

\[
= i\omega \tau_0 z(t) + \bar{q}^*(0) \cdot \mathcal{F}(0, \mathcal{V}(z(t), \bar{z}(t), 0) + 2\text{Re}\{z(t)q(0)\})
\]

\[
= i\omega \tau_0 z(t) + \bar{q}^*(0) \cdot \mathcal{F}(0, \mathcal{V}(z(t), \bar{z}(t), 0) + 2\text{Re}\{z(t)q(0)\}).
\]

We rewrite this equation as

\[
\dot{z}(t) = i\omega \tau_0 z(t) + h(z(t), \bar{z}(t)),
\]

where \( h(z(t), \bar{z}(t)) = \bar{q}^*(0) \cdot \mathcal{F}(0, \mathcal{V}(z(t), \bar{z}(t), 0)) \), and expand \( h(z(t), \bar{z}(t)) \) in powers of \( z(t) \) and \( \bar{z}(t) \), that is

\[
h(z(t), \bar{z}(t)) = h_{20} \frac{z^2(t)}{2} + h_{11} z(t) \bar{z}(t) + h_{02} \frac{\bar{z}^2(t)}{2} + h_{21} \frac{z^2(t)}{2} \bar{z}(t) + \cdots,
\]

then it follows from (65) that

\[
\mathcal{X}_\delta(\theta) = \mathcal{V}(t, \theta) + 2\text{Re}\{z(t)q(\theta)\} = V_{20}(\theta) \frac{z^2(t)}{2} + V_{11}(\theta) z(t) \bar{z}(t)
\]

\[
+ V_{02}(\theta) \frac{\bar{z}^2(t)}{2} + (k_1, k_2)^T e^{i\omega \tau_0 \theta} z(t) + (\bar{k}_1, \bar{k}_2)^T e^{-i\omega \tau_0 \theta} \bar{z}(t) + \cdots.
\]
It follows together with (48) that

\[ h(z(t), \dot{z}(t)) = q^*(0) \mathcal{F}_0(z(t), \dot{z}(t)) = q^*(0) \mathcal{F}_0(0, \tau_0) \]

\[ = q^*(0) \tau_{50} \left\{ \begin{array}{l}
- (\nu^{(1)}(0) + z k_1 + \tau K_1) \left( \nu^{(1)}(-1) + z k_1 e^{-i \omega \tau_{50}} + \tau K_1 e^{i \omega \tau_{50}} \right) \\
- (\nu^{(1)}(0) + z k_1 + \tau K_1) \left( \nu^{(2)}(0) + z k_2 + \tau K_2 \right) \\
- \frac{\beta}{x_2} \left( \nu^{(2)}(0) + z k_2 + \tau K_2 \right)^2 - \beta \left( \nu^{(2)}(0) + z k_2 + \tau K_2 \right) + y_2^2 \\
\times \left( \sum_{n=1}^{\infty} (\nu^{(1)}(0) + z k_1 + \tau K_1) (\nu^{(2)}(0) + z k_2 + \tau K_2)^2 + 2y_2 \\
\times (\nu^{(2)}(0) + z k_2 + \tau K_2) + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) \left( \nu^{(2)} \left( - \frac{\tau_i^2}{\tau_{50}} \right) \right) \\
\left( + z k_2 e^{-i \omega \tau_i^2} + \tau K_2 e^{i \omega \tau_i^2} \right)^n \right) \left( - \frac{\nu^{(1)}(0) + z k_1 + \tau K_1}{2} + \nu^{(1)}(0) z \bar{z} + \nu^{(1)}(0) \frac{z^2}{2} + z k_1 + \tau K_1 + \ldots \right) \times \\
\left( - \frac{\nu^{(1)}(0) + z k_1 + \tau K_1}{2} + \nu^{(1)}(0) z \bar{z} + \nu^{(1)}(0) \frac{z^2}{2} + z k_1 + \tau K_1 + \ldots \right) \\
- \left( \nu^{(1)}(0) + z k_1 + \tau K_1 \right) \left( \nu^{(1)}(0) + z k_1 + \tau K_1 \right) \left( \nu^{(2)}(0) + z k_2 + \tau K_2 \right) + y_2^2 \\
\times \left( \sum_{n=1}^{\infty} (\nu^{(1)}(0) + z k_1 + \tau K_1) (\nu^{(2)}(0) + z k_2 + \tau K_2)^2 + 2y_2 \\
\times (\nu^{(2)}(0) + z k_2 + \tau K_2) + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) \left( \nu^{(2)} \left( - \frac{\tau_i^2}{\tau_{50}} \right) \right) \\
\left( + z k_2 e^{-i \omega \tau_i^2} + \tau K_2 e^{i \omega \tau_i^2} \right)^n \right) \left( - \frac{\nu^{(1)}(0) + z k_1 + \tau K_1}{2} + \nu^{(1)}(0) z \bar{z} + \nu^{(1)}(0) \frac{z^2}{2} + z k_1 + \tau K_1 + \ldots \right) \times \\
\left( - \frac{\nu^{(1)}(0) + z k_1 + \tau K_1}{2} + \nu^{(1)}(0) z \bar{z} + \nu^{(1)}(0) \frac{z^2}{2} + z k_1 + \tau K_1 + \ldots \right) \\
- \left( \nu^{(1)}(0) + z k_1 + \tau K_1 \right) \left( \nu^{(1)}(0) + z k_1 + \tau K_1 \right) \left( \nu^{(2)}(0) + z k_2 + \tau K_2 \right) + y_2^2 \\
\times \left( \sum_{n=1}^{\infty} (\nu^{(1)}(0) + z k_1 + \tau K_1) (\nu^{(2)}(0) + z k_2 + \tau K_2)^2 + 2y_2 \\
\times (\nu^{(2)}(0) + z k_2 + \tau K_2) + \sum_{n=2}^{\infty} \frac{1}{n!} h^{(i)}(0) \left( \nu^{(2)} \left( - \frac{\tau_i^2}{\tau_{50}} \right) \right) \\
\left( + z k_2 e^{-i \omega \tau_i^2} + \tau K_2 e^{i \omega \tau_i^2} \right)^n \right) \right) \right\}. \]
Comparing the coefficients with (68), we obtain
\[ h_{20} = \mathcal{P}_{\tau_0} \left[ \left( \frac{2\alpha \gamma}{(\gamma + y_2)^2} - \frac{2\beta}{x_2} \right) k_2^2 \kappa^* - \left( \frac{2(y_2)^2}{(x_2)^3} \kappa^* \bar{\kappa}_2 + e^{-i\pi\tau_0} \bar{\kappa}_1 \right) k_1^2 \right] + \left( \frac{-2k_1^* + 4y_2^2 \beta}{(x_2)^2} \kappa^* \right) k_1 k_2 ; \]
\[ h_{11} = \mathcal{P}_{\tau_0} \left[ \left( \frac{-2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) k_2 \kappa^* \kappa^*_1 - \frac{2\beta(y_2)^2}{(x_2)^3} k_1 \kappa^*_1 \kappa^*_2 + \frac{2\beta y_2}{(x_2)^2} k_1 \kappa^*_2 \kappa^*_1 e^{-i\pi\tau_0} \right] ; \]
\[ h_{02} = \mathcal{P}_{\tau_0} \left[ \left( \frac{-2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) \kappa^* \kappa^*_1 - \left( \frac{2\beta(y_2)^2}{(x_2)^3} \kappa^* \kappa^*_1 \right) k_1 \right] + \left( \frac{-2k_1^* + 4\beta y_2 \kappa^*}{(x_2)^2} \kappa^*_1 \kappa^*_2 \right) k_1 k_2 ; \]
\[ h_{21} = \mathcal{P}_{\tau_0} \left[ \left( -2V_{11}(0) + 2V_{11}^{(2)}(0) \right) k_1 \kappa^*_1 + \left( -\frac{2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) \kappa^*_1 \kappa^*_2 + \frac{2\beta}{(x_2)^3} \kappa^*_1 \kappa^*_1 \kappa^*_2 \right] ; \]
\[ h_{12} = \mathcal{P}_{\tau_0} \left[ \left( -2V_{01}(0)e^{-i\pi\tau_0} + 2V_{11}(0) \right) \kappa^*_1 + \left( -\frac{2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) \kappa^*_1 \kappa^*_2 + \frac{2\beta}{(x_2)^3} \kappa^*_1 \kappa^*_1 \kappa^*_2 \right] ; \]
\[ h_{03} = \mathcal{P}_{\tau_0} \left[ \left( -2V_{01}(0)e^{-i\pi\tau_0} + 2V_{11}(0) \right) \kappa^*_1 + \left( -\frac{2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) \kappa^*_1 \kappa^*_2 + \frac{2\beta}{(x_2)^3} \kappa^*_1 \kappa^*_1 \kappa^*_2 \right] ; \]
\[ h_{30} = \mathcal{P}_{\tau_0} \left[ \left( -2V_{01}(0)e^{-i\pi\tau_0} + 2V_{11}(0) \right) \kappa^*_1 + \left( -\frac{2\beta}{x_2} + \frac{2\alpha \gamma}{(\gamma + y_2)^2} \right) \kappa^*_1 \kappa^*_2 + \frac{2\beta}{(x_2)^3} \kappa^*_1 \kappa^*_1 \kappa^*_2 \right] . \]

Since there are \( V_{20}(\theta) \) and \( V_{11}(\theta) \) in \( h_{21} \), we still need to compute them. From (53), (65) and (67), we have
\[ \hat{\mathcal{N}} = \hat{X}_1 - \hat{z}q - \hat{\bar{z}}q \]
\[ = \left\{ \begin{array}{ll}
\mathcal{N} \mathcal{V} - 2\text{Re}\{\bar{\mathcal{F}}(0) \cdot \mathcal{F}_0(z, \bar{z})q(\theta)\}, & \theta \in [-1, 0) \\
\mathcal{N} \mathcal{V} - 2\text{Re}\{\mathcal{F}(0) \cdot \mathcal{F}_0(z, \bar{z})q(0)\} + \bar{\mathcal{F}}_0(z, \bar{z}), & \theta = 0 \\
\end{array} \right. \quad (70) \]
\[ \triangleq \mathcal{N} \mathcal{V} + \mathcal{G}(z, \bar{z}, \theta), \]
where
\[ \mathcal{G}(z, \bar{z}, \theta) = \mathcal{G}_{20} \left( \frac{z^2}{2} + \mathcal{G}_{11}(\theta)z\bar{z} + \mathcal{G}_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots \right). \quad (71) \]

On the other hand, on \( C_0 \) near to the origin, we have \( \hat{\mathcal{V}} = V_z \hat{z} + V_{\bar{z}} \hat{\bar{z}} \). Using (66) and (67) to replace \( V_z \) and \( \hat{z} \) and their conjugates by their power series expansions, comparing the coefficients with the right hand side of (70), we obtain
\[ (2i\pi\tau_0 I - \mathcal{N}) V_{20}(\theta) = \mathcal{G}_{20}(\theta), \quad -\mathcal{N} V_{11}(\theta) = \mathcal{G}_{11}(\theta), \quad (72) \]
where \( I \) denotes the \( 2 \times 2 \) identity matrix.
From (70), we know that for $\theta \in [-1,0)$,
\[
G(z, \bar{z}, \theta) = -\bar{\eta}(0) \cdot \mathcal{F}(z, \bar{z}) q(\theta) - q^*(0) \cdot \mathcal{F}(z, \bar{z}) \bar{q}(\theta)
\]
\[
= -h(z, \bar{z}) q(\theta) - \bar{h}(z, \bar{z}) \bar{q}(\theta).
\]
(73)
Substituting (68) into (73) gives
\[
G(z, \bar{z}, \theta) = \left[-h_{20} q(\theta) - \bar{h}_{02} \bar{q}(\theta)\right] \frac{z^2}{2} + \left[-h_{11} q(\theta) - \bar{h}_{11} \bar{q}(\theta)\right] z \bar{z}
\]
\[
+ \left[-h_{02} q(\theta) - \bar{h}_{02} \bar{q}(\theta)\right] \frac{z^2}{2} + \cdots.
\]
(74)
Comparing the coefficients in (74) with those in (71) gives that
\[
\mathcal{G}_{20}(\theta) = -h_{20} q(\theta) - \bar{h}_{02} \bar{q}(\theta), \quad \mathcal{G}_{11}(\theta) = -h_{11} q(\theta) - \bar{h}_{11} \bar{q}(\theta).
\]
(75)
From (72), (75) and the definition of $\mathcal{N}$, we have
\[
\dot{V}_{20}(\theta) = 2i\varpi \tau_{50} \mathcal{V}_{20}(\theta) + h_{20} q(\theta) + \bar{h}_{02} \bar{q}(\theta).
\]
(76)
Notice that $q(\theta) = (k_1, k_2)^T e^{i\varpi \tau_{50}} = q(0)e^{i\varpi \tau_{50}}$, hence, using the method of variation of constants, the solution of (76) is given by
\[
V_{20}(\theta) = \frac{ih_{20}}{\varpi \tau_{50}} q(0)e^{i\varpi \tau_{50}} + \frac{i\bar{h}_{02}}{3\varpi \tau_{50}} \bar{q}(0)e^{-i\varpi \tau_{50}} + \mathcal{W}_1 e^{2i\varpi \tau_{50}},
\]
(77)
where $\mathcal{W}_1 = \left(\mathcal{W}_1^{(1)}, \mathcal{W}_1^{(2)}\right)^T \in \mathbb{R}^2$ is a constant vector.
Similarly, from (72), (75) and the definition of $\mathcal{N}$, we have
\[
\dot{V}_{11}(\theta) = h_{11} q(\theta) + \bar{h}_{11} \bar{q}(\theta),
\]
(78)
and
\[
V_{11}(\theta) = -\frac{ih_{11}}{\varpi \tau_{50}} q(0)e^{i\varpi \tau_{50}} + \frac{i\bar{h}_{11}}{\varpi \tau_{50}} \bar{q}(0)e^{-i\varpi \tau_{50}} + \mathcal{W}_2,
\]
(79)
where $\mathcal{W}_2 = \left(\mathcal{W}_2^{(1)}, \mathcal{W}_2^{(2)}\right)^T \in \mathbb{R}^2$ is also a constant vector.
In what follows, we shall seek appropriate constant vectors $\mathcal{W}_1$ and $\mathcal{W}_2$ in (77) and (79), respectively. From (72) and the definition of $\mathcal{N}$, we know that when $\theta = 0$,
\[
\mathcal{N}(0) \mathcal{V}_{20}(\theta) = \int_{-1}^{0} d\eta(\theta) \dot{V}_{20}(\theta) = 2i\varpi \tau_{50} \mathcal{V}_{20}(0) - \mathcal{G}_{20}(0)
\]
(80)
and
\[
\mathcal{N}(0) \mathcal{V}_{11}(\theta) = \int_{-1}^{0} d\eta(\theta) \dot{V}_{11}(\theta) = -\mathcal{G}_{11}(0),
\]
(81)
where $\eta(\theta) = \eta(\theta, 0)$. And from (70), we can obtain when $\theta = 0$,
\[
\mathcal{G}_{20}(0) = -h_{20} q(0) - \bar{h}_{02} \bar{q}(0) + 2\tau_{50}
\]
\[
\times \begin{pmatrix}
-\frac{\beta}{x_2} k_2^2 - \frac{\beta(y_2)^2}{(x_2)^3} k_1^2 + \frac{2\beta y_2}{(x_2)^2} k_1 k_2 + \frac{\alpha y_2^3}{(y_2 + y_2)^2} e^{-2i\varpi \tau_{50}}
\end{pmatrix},
\]
(82)
\[
\mathcal{G}_{11}(0) = -h_{11} q(0) - \bar{h}_{11} \bar{q}(0) + \tau_{50}
\]
\[
\times \begin{pmatrix}
-\frac{2\beta}{x_2} k_2 \bar{k}_2 - \frac{2\beta y_2^2}{(x_2)^2} k_1 \bar{k}_1 + \frac{2\beta y_2}{(x_2)^2} (k_1 \bar{k}_1 + \bar{k}_1 k_2) + \frac{2\alpha y_2^3}{(y_2 + y_2)^2} k_2 \bar{k}_2
\end{pmatrix},
\]
(83)
therefore, when \( \mu = 0 \), we have
\[
\int_{-1}^{0} e^{2i\pi \tau_{50} \theta} d\eta(\theta) = \mathcal{N}(\mu) e^{2i\pi \tau_{50} \theta} = \mathcal{L}_{\mu} e^{2i\pi \tau_{50} \theta},
\]
therefore, we have
\[
W_{1} = 2 \left( \begin{array}{cc}
2i\omega + x_{2} e^{-2i\pi \tau_{50}} & x_{2} \\
-\beta (x_{2})^{2} & 2i\omega - \delta + 2\beta \frac{y_{2}}{x_{2}} + \frac{\alpha \gamma k_{2}^{2}}{(\gamma + y_{2})^{2}} e^{-2i\pi \tau_{50}}
\end{array} \right)
\]
\phantom{1} \times \left( \begin{array}{cc}
-\beta \frac{k_{2}^{2}}{x_{2}} + \frac{\alpha \gamma y_{2}^{2}}{(x_{2})^{2}} k_{2}^{2} + \frac{2\beta y_{2}}{(x_{2})^{2}} k_{1} k_{2} + \frac{\alpha \gamma k_{2}^{2}}{(\gamma + y_{2})^{2}} e^{-2i\pi \tau_{50}}
\end{array} \right).
\]

Similarly, substituting (77) and (79) into (81), we obtain
\[
\int_{-1}^{0} d\eta(\theta) W_{2} = - \left( \begin{array}{cc}
-\frac{x_{2}}{2x_{2} k_{2}} e^{-2i\pi \tau_{50}} & \frac{x_{2}}{2x_{2} k_{2}} \\
\beta (x_{2})^{2} & \delta - 2\beta \frac{y_{2}}{x_{2}} - \frac{\alpha \gamma}{(\gamma + y_{2})^{2}}
\end{array} \right)_{1} + \frac{2\beta y_{2}}{(x_{2})^{2}} (k_{1} k_{2} + \bar{k}_{1} k_{2}) + \frac{2\alpha \gamma}{(\gamma + y_{2})^{2}} k_{2} \bar{k}_{2},
\]
\phantom{1} \times \left( \begin{array}{cc}
-\frac{\beta \gamma y_{2}^{2}}{(x_{2})^{2}} k_{2} + 2\beta y_{2} k_{1} k_{2} + \frac{\alpha \gamma k_{2}^{2}}{(\gamma + y_{2})^{2}} e^{-2i\pi \tau_{50}}
\end{array} \right).
\]

Thus, we can compute the following quantities:
\[
C_{1}(0) = \frac{i}{2 \pi \tau_{50}} \left( h_{11} h_{20} - 2|h_{11}|^{2} - \frac{|h_{02}|^{2}}{3} \right) + \frac{h_{21}}{2},
\]
\[
\beta_{2} = 2 \text{Re}(C_{1}(0)),
\]
\[
\mu_{2} = - \frac{\text{Re}(\mathcal{C}(\tau_{50}))}{\text{Re}(\mathcal{C}(\tau_{50}))},
\]
\[
T_{2} = \frac{\text{Im}(C_{1}(0)) + \mu_{2} \text{Im}(\mathcal{C}(\tau_{50}))}{\pi},
\]
5. **Numerical simulations.** In this section, we will give some numerical simulations of model (3) to support our analytical results obtained in Sections 3 and 4. When \( \tau_1 = \tau_2 = 0 \), we consider model (3) with coefficients \( \alpha = 0.1, \beta = 2, \gamma = 1 \). Obviously, model (3) has a positive equilibrium \( E_2(0.68, 0.32) \). Then we can obtain \( h_1 + h_2 + h_3 = 1.59 > 0, h_4 + h_5 + h_6 = 0.91 > 0 \). Hence, the condition (H1) is satisfied. From **Lemma 3.1**, we know that the equilibrium point \( E_2 \) is locally asymptotically stable when \( \tau_1 = \tau_2 = 0 \), as illustrated in Fig. 1.

![Figure 1](image)

**Figure 1.** The diagram (a) shows the time series of \( x(t) \), \( y(t) \) and the diagram (b) shows the phase portrait of model (3) with \( \tau_1 = \tau_2 = 0 \). The positive equilibrium point \( E_2(0.68, 0.32) \) is locally asymptotically stable. Here the initial value is \((0.8, 0.6)\).

First, we verify the correctness of the **Theorem 3.4.** In this case, we consider model (3) with coefficients \( \alpha = 9, \beta = 0.4, \delta = 0.9, \gamma = 19 \). We can compute: \( h_1 + h_2 + h_3 = 0.90 > 0, h_4 + h_5 + h_6 = 0.43 > 0, h'(\omega_{30}) = 0.42 > 0, \tau_{30} = 2.91 \). Hence, conditions (H1) and (H2) are satisfied. Based on the **Theorem 3.4**, we know that the positive equilibrium point \( E_2 \) is locally asymptotically stable when \( \tau_1 = 0, \tau_2 = 2.8 < \tau_{30} = 2.91 \), as shown in Fig. 2. And it is unstable when \( \tau_1 = 0, \tau_2 = 2.92 > \tau_{20} = 2.91 \), and model (3) undergoes a Hopf bifurcation at the positive equilibrium \( E_2 \) when \( \tau_2 \) crosses through the critical value \( \tau_{20} \), a periodic solution bifurcated from \( E_2 \), as illustrated in Fig. 3.

Now, we are going to check the correctness of the **Theorem 3.7.** In this case, we consider model (3) with coefficients \( \alpha = 0.3, \beta = 0.4, \delta = 0.8, \gamma = 0.2 \). We can compute: \( h_1 + h_2 + h_3 = 0.57 > 0, h_4 + h_5 + h_6 = 0.18 > 0, g'(\omega_{30}) = 0.73 > 0, \tau_{30} = 2.11 \). Hence, conditions (H1) and (H3) are satisfied. Based on the **Theorem 3.7**, we know that the positive equilibrium point \( E_2 \) is locally asymptotically stable when \( \tau_2 = 0, \tau_1 = 2 < \tau_{30} = 2.11 \), as shown in Fig. 4. And it is unstable when \( \tau_2 = 0, \tau_1 = 2.1185 > \tau_{30} = 2.11 \), and model (3) undergoes a Hopf bifurcation at the positive equilibrium \( E_2 \) when \( \tau_1 \) crosses through the critical value \( \tau_{30} \), a periodic solution bifurcated from \( E_2 \), as illustrated in Fig. 5.

Next, we will examine the conditions of the **Theorem 3.8.** In this case, we consider model (3) with coefficients \( \alpha = 0.3, \beta = 0.4, \delta = 0.8, \gamma = 0.2 \). Obviously, condition (H1) is satisfied. We can compute: \( \omega_1 = 0.12, \omega_2 = 0.78 \) (two roots of \((33)\)), \( \tau_4^{(0)} = 11.27, \tau_4^{(0)} = 1.92 \) and \( \tau_{30} = 1.92 \), \( (U_R V_R + U_I V_I)_{\tau = \tau_{30}} = 0.22 > 0 \). Hence, from **Theorem 3.8**, we know that the positive equilibrium point \( E_2 \) is locally asymptotically stable when \( \tau_1 = \tau_2 = \tau \) increase from zero to the critical
Figure 2. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 0$, $\tau_2 = 2.8 < \tau_{20} = 2.91$. The positive equilibrium point $E_2(0.48, 0.52)$ is locally asymptotically stable. Here the initial value is $(0.5, 0.5)$.

Figure 3. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 0, \tau_2 = 2.92 > \tau_{20} = 2.91$. The positive equilibrium point $E_2(0.48, 0.52)$ is unstable. Here the initial value is $(0.5, 0.5)$.

value $\tau_{40} = 1.92$, say $\tau_1 = \tau_2 = \tau = 1.85$ (see Fig. 6). Then the equilibrium point will lose its stability and a Hopf bifurcation occurs once $\tau_1 = \tau_2 = \tau > \tau_{40} = 1.92$, say $\tau_1 = \tau_2 = \tau = 1.926$ (see Fig. 7).

Finally, we give some numerical simulations for the Theorem 3.9 to verify its validity. In this case, we consider model (3) with coefficients $\alpha = 9$, $\beta = 0.4$, $\delta = 0.9$, $\gamma = 19$. Hence, the conditions (H$_1$) and (H$_2$) are satisfied. When $\tau_2 = 1.8 \in [0, \tau_{20}) = [0, 2.91)$, we obtain one root of (41) is $\omega_0 = 0.05, \tau_{50} = 4.90. \left[d\lambda(\tau_1)\right]_{\tau_1 = \tau_{50}} = 0.00003 - 0.00387i$. Thus, from the Theorem 3.9, we know that the equilibrium point $E_2$ is locally asymptotically stable when $\tau_2 = 1.8 \in [0, \tau_{20}) = [0, 2.91)$, $\tau_1 = 3 < \tau_{50} = 4.90$, and unstable when $\tau_1 = 6 > \tau_{50} = 4.90$. When $\tau_1$ crosses through the critical value $\tau_{50} = 4.90$, the positive equilibrium $E_2$ loses its stability and a Hopf bifurcation occurs, i.e., a periodic solution bifurcates from the
Figure 4. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 2.0 < \tau_3 = 2.11$, $\tau_2 = 0$. The positive equilibrium point $E_2(0.53, 0.47)$ is locally asymptotically stable. Here the initial value is $(0.5, 0.5)$.

Figure 5. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 2.1185 > \tau_3 = 2.11$, $\tau_2 = 0$. The positive equilibrium point $E_2(0.53, 0.47)$ is unstable. Here the initial value is $(0.5, 0.5)$.

equilibrium $E_2$. The corresponding waveforms and the phase plots are shown in Figs. 8-9.

When $\tau_{50} = 4.90$, $\tau_2 = 1.8$, we may obtain $C_1(0) = -241.46 + 633.60i$, $\mu_2 = 7.10 \times 10^6 > 0$, $\beta_2 = -482.91 < 0$, $T_2 = 5.94 \times 10^5 > 0$. Therefore, we know that model (3) can undergo a supercritical Hopf bifurcation at the equilibrium $E_2$. $\beta_2 < 0$ demonstrated that the bifurcated periodic solution occurring through the Hopf bifurcation is orbitally asymptotically stable on the center manifold. $T_2 > 0$ means that the period of bifurcating periodic solutions are increasing.

6. Discussion. This paper mainly studies a delay-induced predator-prey model with Michaelis-Menten type predator harvesting. Firstly, the existence of the non-trivial positive equilibrium of the proposed model are given. Secondly, some sufficient conditions are proposed to ensure the local asymptotic stability of one of the
positive equilibria. The existence of Hopf bifurcation is analyzed in details if taking the delays as the bifurcation parameters. Meanwhile, according to the normal form theory and the center manifold reduction for RFDEs, some explicit formulas are given for determining the direction of the Hopf bifurcation and the stability of bifurcated periodic solutions. Some numerical simulations are carried out for illustrating these analytical results.

Compared with the model (1) without time delays, we have the following observations. First, this study can be thought as a supplement to model (1). The main purpose of our present work is to investigate the effect that delays have been placed in the model (1), as far as stability of equilibria and the nature of oscillations are considered. Generally, the population growth rate of a single species is often described as the famous logistic or Pearl-Verhulst equation. However, it is difficult to fit data
Figure 8. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 3 < \tau_{50} = 4.90$, $\tau_2 = 1.8$. The positive equilibrium point $E_2(0.48, 0.52)$ is locally asymptotically stable. Here the initial value is $(0.55, 0.6)$.

Figure 9. The diagram (a) shows the time series of $x(t)$, $y(t)$ and the diagram (b) shows the phase portrait of model (3) with $\tau_1 = 6 > \tau_{50} = 4.90$, $\tau_2 = 1.8$. The positive equilibrium point $E_2(0.48, 0.52)$ is unstable. Here the initial value is $(0.55, 0.6)$.

to a logistic curve even for a simple, controlled laboratory experiments[10]. As pointed out by F. Smith[36], the main reason is due to time delays in the growth rate response to density changes. Second, the periodic oscillations generated by Hopf bifurcation caused by the system parameter $\alpha$ were investigated in previous work of model (1). Actually, periodic oscillations are often observed in such laboratory experiments and attributed to time delayed responses[6, 10]. Hence, it is necessary to consider periodic oscillation caused by time delays. However, our current work only makes a theoretical and qualitative analysis and gives some numerical simulations to reveal the rationality of our improved model (2) and the correctness of analysis. It will be very interesting and meaningful to find some more realistic examples that show the applicability and utility of the new model and new conditions and we leave them for future consideration.
Acknowledgments. This work is supported by Natural Science Foundation of China (NSFC) under Project No. 11671227, NSF of Shandong Province under Project No. ZR2018BF018 and China Postdoctoral Science Foundation under Project No. 2019M652349.

REFERENCES

[1] E. Ávila-Vales, Á. Estrella-González and E. Rivero-Esquivel, Bifurcations of a Leslie Gower predator prey model with Holling type III functional response and Michaelis-Menten prey harvesting, arXiv:1711.08081v1.
[2] A. A. Berryman, The origins and evolution of predator-prey theory, Ecology, 73 (1992), 1530–1535.
[3] Å. Brännström and D. Sumpter, The role of competition and clustering in population dynamics, Proc. Biol. Sci, 272 (2005), 2065–2072.
[4] J. Z. Cao and H. Y. Sun, Bifurcation analysis for the Kaldor-Kalecki model with two delays, Adv. Differ. Eq., 107 (2019), 1–27.
[5] J. Z. Cao and R. Yuan, Bifurcation analysis in a modified Leslie-Gower model with Holling type II functional response and delay, Nonlinear Dynamics, 84 (2016), 1341–1352.
[6] J. Caperon, Time lag in population growth response of Isochrysis Galbana to a variable nitrate environment, Ecology, 50 (1969), 188–192.
[7] B. S. Chen and J. J. Chen, Complex dynamic behaviors of a discrete predator-prey model with stage structure and harvesting, Int. J. Biomath., 10 (2017), 1750013, 25 pp.
[8] C. W. Clark and M. Mangei, Aggregation and fishery dynamics: A theoretical study of schooling and the purse seine tuna fisheries, Fish. Bull., 77 (1979), 317–337.
[9] S. Creed, E. Dröge, J. M’soka, D. Smit, M. Becker, D. Christianson and P. Schuette, The relationship between direct predation and antipredator responses: a test with multiple predators and multiple prey, Ecology, 98 (2017), 2081–2092.
[10] J. M. Cushing, Integrodifferential Equations and Delay Models in Population Dynamics, Springer-Verlag, Berlin Heidelberg New York, 1977.
[11] V. Doudoumis, U. Alam and E. Aksoy, et al., Tsetse-Wolbachia symbiosis: Comes of age and has great potential for pest and disease control, J. Invertebr. Pathol., 112 (2013), S94–S103.
[12] M. K. A. Gavina, T. Tahara and K. Tainaka, et al., Multi-species coexistence in Lotka-Volterra competitive systems with crowding effects, Sci. Rep., 8 (2018), 1198.
[13] F. Groenewoud, J. G. Frommen, D. Josi, H. Tanaka, A. Jungwirth and M. Taborsky, Predation risk drives social complexity in cooperative breeders, Proc. Natl. Acad. Sci., 113 (2016), 4104–4109.
[14] Y. X. Guo, N. N. Ji and B. Niu, Hopf bifurcation analysis in a predator-prey model with time delay and food subsidies, Adv. Differ. Eq., 2019 (2019), Paper No. 99, 22 pp.
[15] R. P. Gupta, M. Banerjee and P. Chandra, Bifurcation analysis and control of Leslie-Gower predator-prey model with Michaelis-Menten type prey-harvest, Differ. Eq. Dyn. Syst., 20 (2012), 339–366.
[16] R. P. Gupta and P. Chandra, Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting, J. Math. Anal. Appl, 398 (2013), 278–295.
[17] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge, 1981.
[18] D. P. Hu and H. J. Cao, Stability and Hopf bifurcation analysis in Hindmarsh-Rose neuron model with multiple time delays, J. Math. Anal. Appl., 11 (2016), 1650187, 27pp.
[19] D. P. Hu and H. J. Cao, Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvest, Nonlinear Anal-RWA, 33 (2017), 58–82.
[20] S. Khajanchi, Modeling the dynamics of stage-structure predator-prey system with Monod-Haldane type response function, Appl. Math. Comput., 302 (2017), 122–143.
[21] L. Kong and C. R. Zhu, Bogdanov-Takens bifurcations of codimensions 2 and 3 in a Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting, Math. Method. Appl. Sci., 40 (2017), 6715–6731.
[22] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.
[23] P. Lenzini and J. Rebaza, Nonconstant predator harvesting on ratio-dependent predator-prey models, Appl. Math. Sci., 4 (2010), 791–803.
[24] L. Z. Li, F. W. Meng and P. J. Ju, Some new integral inequalities and their applications in studying the stability of nonlinear integro-differential equations with time delay, *J. Math. Anal. Appl.*, 377 (2011), 853–862.

[25] Y. N. Li, Y. G. Sun and F. W. Meng, New criteria for exponential stability of switched time varying systems with delays and nonlinear disturbances, *Nonlinear Anal-Hybri.*, 26 (2017), 284–291.

[26] Y. Li and M. X. Wang, Dynamics of a diffusive predator-prey model with modified Leslie-Gower term and Michaelis-Menten type prey harvesting, *Acta Appl. Math.*, 140 (2015), 147–172.

[27] B. Liu, R. C. Wu and L. P. Chen, Patterns induced by super cross-diffusion in a predator-prey system with Michaelis-Menten type harvesting, *Math. Biosci.*, 298 (2018), 71–79.

[28] Y. Liu, L. Zhao, X. Y. Huang and H. Deng, Stability and bifurcation analysis of two species amensalism model with Michaelis-Menten type harvesting and a cover for the first species, *Adv. Differ. Equ.*, 2018 (2018), Paper No. 295, 19 pp.

[29] J. F. Luo and Y. Zhao, Stability and bifurcation analysis in a predator-prey system with constant harvesting and prey group defense, *Int. J. Bifurcat. Chaos*, 27 (2017), 1750179, 26pp.

[30] Z. H. Ma and S. F. Wang, A delay-induced predator-prey model with Holling type functional response and habitat complexity, *Nonlinear Dyn.*, 93 (2018), 1519–1544.

[31] R. M. May, J. R. Beddington, C. W. Clark, S. J. Holt and R. M. Law, Management of multispecies fisheries, *Science*, 205 (1979), 267–277.

[32] M. Peng, Z. D. Zhang and X. D. Wang, Hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two delays, *Adv. Differ. Equ.*, 2017 (2017), Paper No. 387, 20 pp.

[33] S. N. Raw, P. Mishra, R. Kumar and S. Thakur, Complex behavior of prey-predator system exhibiting group defense: A mathematical modeling study, *Chaos Soliton Fract.*, 100 (2017), 74–90.

[34] M. Sen, P. D. N. Srinivasu and M. Banerjee, Global dynamics of an additional food provided predator-prey system with constant harvest in predators, *Appl. Math. Comput.*, 250 (2015), 193–211.

[35] J. Shao and F. W. Meng, Oscillation theorems for second order forced neutral nonlinear differential equations with delayed argument, *Int. J. Differ. Equ.*, 2010 (2010), article ID 181784, 1–15.

[36] F. E. Smith, Population dynamics in Daphnia Magna and a new model for population growth, *Ecology*, 44 (1963), 651–663.

[37] Q. N. Song, R. Z. Yang, C. R. Zhang and L. Y. Tang, Bifurcation analysis in a diffusive predator-prey system with Michaelis-Menten-type predator harvesting, *Adv. Differ. Equ.*, 2018 (2018), Paper No. 329, 15 pp.

[38] Y. G. Sun and F. W. Meng, Reachable set estimation for a class of nonlinear time varying systems, *Complexity*, 2017 (2017), Article ID 5876371, 6pp.

[39] J. M. Wang, H. D. Cheng, H. X. Liu and Y. H. Wang, Periodic solution and control optimization of a prey-predator model with two types of harvesting, *Adv. Differ. Equ.*, 2018 (2018), Paper No. 41, 14 pp.

[40] Z. Wang, Y. K. Xie, J. W. Lu and Y. X. Li, Stability and bifurcation of a delayed generalized fractional-order prey-predator model with interspecific competition, *Appl. Math. Comput.*, 347 (2019), 360–369.

[41] R. C. Wu, M. X. Chen, B. Liu and L. P. Chen, Hopf bifurcation and Turing instability in a predator-prey model with Michaelis-Menten functional response, *Nonlinear Dyn.*, 91 (2018), 2033–2047.

[42] D. M. Xiao, W. X. Li and M. A. Han, Dynamics in a ratio-dependent predator-prey model with predator harvesting, *J. Math. Anal. Appl.*, 324 (2006), 14–29.

[43] R. Z. Yang, C. R. Zhang and Y. Z. Zhang, A delayed diffusive predator-prey system with Michaelis-Menten type predator harvesting, *Int. J. Bifurcat. Chaos*, 28 (2018), 1850099, 14pp.

[44] R. Yuan, W. H. Jiang and Y. Wang, Saddle-node-Hopf bifurcation in a modified Leslie-Gower predator-prey model with time-delay and prey harvesting, *J. Math. Anal. Appl.*, 422 (2015), 1072–1090.

[45] S. L. Yuan, X. H. Ji and H. P. Zhu, Asymptotic behavior of a delayed stochastic logistic model with impulsive perturbations, *Math. Biosci. Eng.*, 14 (2017), 1477–1498.
[46] C. H. Zhang, X. P. Yan and G. H. Cui, Hopf bifurcations in a predator-prey system with a discrete delay and a distributed delay, *Nonlinear Anal-RWA*, 11 (2010), 4141–4153.

[47] C. R. Zhu and K. Q. Lan, Phase portraits, Hopf-bifurcations and limit cycles of Leslie-Gower predator-prey systems with harvesting rates, *Discrete Contin. Dyn. Syst. Ser. B*, 14 (2010), 289–306.

Received May 2019; revised September 2019.

E-mail address: mliu2013@126.com
E-mail address: hudongpo2006@126.com
E-mail address: fwmeng@qfnu.edu.cn