A CONSTRUCTION OF GORENSTEIN PROJECTIVE $\tau$-TILTING MODULES

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Abstract. We give a construction of Gorenstein projective $\tau$-tilting modules in terms of tensor products of modules. As a consequence, we give a class of non-self-injective algebras admitting non-trivial Gorenstein projective $\tau$-tilting modules. Moreover, we show that a finite dimensional algebra $A$ over an algebraically closed field is $CM$-$\tau$-tilting finite if $T_n(A)$ is $CM$-$\tau$-tilting finite which gives a partial answer to a question on $CM$-$\tau$-tilting finite algebras posed by Xie and Zhang.

1. Introduction

In 2014, Adachi, Iyama and Reiten [AIR] introduced $\tau$-tilting theory as a generalization of tilting theory from the viewpoint of mutation. It has been showed by Adachi, Iyama and Reiten that $\tau$-tilting theory is closely related to silting theory and cluster tilting theory. In $\tau$-tilting theory, (support) $\tau$-tilting modules are the most important objects. Therefore it is interesting to study (support) $\tau$-tilting modules for given algebras. For recent development on this topics, we refer to [AIR, IZ, KK, W, Z, Zi].

On the other hand, Gorenstein projective modules form the main body of Gorenstein homological algebra; their origins can be traced back to Auslander-Bridger’s modules of $G$-dimension zero [AuB]. The definition of Gorenstein projective modules over an arbitrary ring was given by Enochs and Jenda [EJ1, EJ2]. From then on, Gorenstein projective modules have gained a lot of attention in both homological algebra and the representation theory of finite-dimensional algebras. Throughout this paper, we focus on the finitely generated Gorenstein projective modules over finite dimensional algebras over an algebraically closed field $K$. For recent development of this topics, we refer to [CSZ, HuLXZ, RZ1, RZ2, RZ3].

Recently, Xie and the second author [XZ] combined Gorenstein projective modules with $\tau$-tilting modules and built a bijection map from Gorenstein projective support $\tau$-tilting modules to Gorenstein injective support $\tau^{-1}$-tilting modules which is analog to Adachi-Iyama-Reiten’s bijection map from support $\tau$-tilting modules to support $\tau^{-1}$-tilting modules. But there is little reference to show the existence of non-trivial Gorenstein projective (support) $\tau$-tilting modules except support $\tau$-tilting modules over self-injective algebras. It is natural to ask the following question.

Question 1.1. Are there non-self-injective algebras admitting non-trivial Gorenstein projective support $\tau$-tilting modules?

In this note, we give a construction of non-trivial Gorenstein projective (support) $\tau$-tilting modules. As a consequence, we can construct a large class of non-self-injective algebras admitting non-trivial Gorenstein projective (support) $\tau$-tilting modules. Our first main result is the following:

Theorem 1.2. (Theorem 3.10) Let $A$ and $B$ be finite dimensional algebras over an algebraically closed field $K$. Let $M \in \text{mod} B$ be a Gorenstein projective (support) $\tau$-tilting module. Then $A \otimes_K M$ is a Gorenstein projective (support) $\tau$-tilting module in $\text{mod}(A \otimes_K B)$.

Keywords: Gorenstein projective module, $\tau$-rigid module, support $\tau$-tilting module.

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As a consequence of Theorem 1.2, we can give a partial answer to Question 1.1 as follows.

Corollary 1.3. (Corollary 3.12) Let $A$ be a non-semisimple self-injective algebra which is not local. Then there are non-trivial Gorenstein projective modules in $\text{mod}T_n(A)$.

Recall from [XZ] that an algebra is called Cohen-Macaulay-$\tau$-tilting finite (CM-$\tau$-tilting finite for short) if it admits finitely many isomorphism classes of indecomposable Gorenstein projective $\tau$-rigid modules. The CM-$\tau$-tilting finite algebras are the generalizations of both algebras of finite Cohen-Macaulay type (CM-finite algebras for short) [B, U, LZ1] and $\tau$-tilting finite algebras [DIJ]. As an application of Theorem 1.2 we get the following characterization of CM-$\tau$-tilting finite algebras which gives a partial answer to [XZ] Question 5.7.

Theorem 1.4. (Theorem 3.13) Let $A$ be a finite dimensional algebra over an algebraically closed field $K$ and let $n \geq 2$ be a positive integer. If $T_n(A)$ is CM-$\tau$-tilting finite, then $A$ is CM-$\tau$-tilting finite.

Now we show the organization of this paper as follows:

In Section 2, we recall some preliminaries on Gorenstein projective modules and $\tau$-tilting modules. In Section 3, we show the main results.

Throughout this paper, all algebras are finite-dimensional algebras over an algebraically closed field $K$ and all modules are finitely generated right modules. We use $\tau$ to denote the Auslander-Reiten translation functor.

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2. Preliminaries

In this section, we recall definitions and basic facts on $\tau$-tilting modules, tensor products of algebras and Gorenstein projective modules.

For an algebra $A$, denote by $\text{mod}A$ the category of finitely generated right $A$-modules. We use $\mathcal{P}(A)$ to denote the subcategory of $\text{mod}A$ consisting of projective modules. Now we recall the following definition of Gorenstein projective modules from [EJ].

Definition 2.1. Let $A$ be an algebra and $M \in \text{mod}A$. $M$ is called Gorenstein projective, if there is an exact sequence $\cdots \rightarrow P_{-1} \rightarrow P_0 \rightarrow P_1 \rightarrow \cdots$ in $\mathcal{P}(A)$, which stays exact under $\text{Hom}_A(-, A) = (-)^*$, such that $M \cong \text{Im}(P_{-1} \rightarrow P_0)$.

Denote by $\Omega$ the syzygy functor and $\text{Tr}$ the Auslander-Bridger transpose functor. The following properties of Gorenstein projective modules [AH] are essential.

Proposition 2.2. Let $A$ be an algebra. A module $M \in \text{mod}A$ is Gorenstein projective if and only if $M \cong M^{**}$ and $\text{Ext}^i_A(M, A) = \text{Ext}^i_A(M^*, A) = 0$ hold for all $i \geq 1$ if and only if $\text{Ext}^i_A(M, A) = 0$ and $\text{Ext}^i_A(\text{Tr}M, A) = 0$ hold for all $i \geq 1$.

For a module $M \in \text{mod}A$, denote by $|M|$ the number of pairwise non-isomorphic indecomposable summands of $M$. We recall the definitions of $\tau$-rigid modules and $\tau$-tilting modules from [S] and [AIR].

Definition 2.3. Let $A$ be an algebra and $M \in \text{mod}A$.

1. We call $M$ $\tau$-rigid if $\text{Hom}_A(M, \tau M) = 0$. Moreover, $M$ is called a $\tau$-tilting module if $M$ is $\tau$-rigid and $|M| = |A|$.

2. We call $M$ support $\tau$-tilting if there exists an idempotent $e$ of $A$ such that $M$ is a $\tau$-tilting $A/(e)$-module.

The following result [AIR] Proposition 2.4(c)] is essential in this paper.
Proposition 2.4. Let $M \in \text{mod}A$ and $P_1(M) \xrightarrow{f} P_0(M) \rightarrow M \rightarrow 0$ be a minimal projective presentation of $M$. Then $M$ is $\tau$-rigid if and only if $\text{Hom}_A(f, M)$ is epic.

We also need the following definitions of Gorenstein projective support $\tau$-tilting modules and Gorenstein projective $\tau$-tilting modules [XZ].

Definition 2.5. Let $A$ be an algebra and $M \in \text{mod}A$.
(1) We call $M$ Gorenstein projective $\tau$-rigid if it is both $\tau$-rigid and Gorenstein projective.
(2) We call $M$ Gorenstein projective $\tau$-tilting if it is both $\tau$-tilting and Gorenstein projective.
(3) We call $M$ Gorenstein projective support $\tau$-tilting if it is both support $\tau$-tilting and Gorenstein projective.

Let $A$ and $B$ be algebras over an algebraically closed field $K$. Denote by $A \otimes_K B$ the tensor products of algebras. For modules $M \in \text{mod}A$ and $N \in \text{mod}B$, we have $M \otimes_K N \in \text{mod}(A \otimes_K B)$. In the rest of the paper, we use $M \otimes N$ to denote $M \otimes_K N$. We need the following properties on the tensor products of algebras in [CE].

Lemma 2.6. Let $A$ and $B$ be two algebras with $M_i \in \text{mod}A$ and $N_i \in \text{mod}B$ for $i = 1, 2$. Then we have the following.
(1) $\text{Hom}_{A \otimes B}(M_1 \otimes N_1, M_2 \otimes N_2) \cong \text{Hom}_A(M_1, M_2) \otimes \text{Hom}_B(N_1, N_2)$.
(2) $\text{Ext}^m_{A \otimes B}(M_1 \otimes N_1, M_2 \otimes N_2) \cong \bigoplus_{i+j=m} \text{Ext}_A^i(M_1, M_2) \otimes \text{Ext}_B^j(N_1, N_2)$ holds for $m \geq 1$.

For a right $A$-module $M$, denote by $\text{pd}_A M$ (resp. $\text{id}_A M$) the projective (resp. injective) dimension of $M$. We also need the following on the injective (resp. projective) dimension of tensor products of modules.

Lemma 2.7. Let $A$ and $B$ be two algebras with $M \in \text{mod}A$ and $N \in \text{mod}B$. Then
(1) $\text{pd}_{A \otimes B} M \otimes N = \text{pd}_A M + \text{pd}_B N$
(2) $\text{id}_{A \otimes B} M \otimes N = \text{id}_A M + \text{id}_B N$

The following results are well-known.

Proposition 2.8. Let $A$ and $B$ be two algebras over an algebraically closed field $K$. Then
(1) $P \otimes Q$ is an indecomposable projective module in $\text{mod}(A \otimes B)$ if $P$ and $Q$ are indecomposable projective in $\text{mod}A$ and $\text{mod}B$, respectively.
(2) Every indecomposable projective module in $\text{mod}(A \otimes B)$ has the form $P \otimes Q$, where $P$ and $Q$ are indecomposable projective in $\text{mod}A$ and $\text{mod}B$, respectively.
(3) Every simple module in $\text{mod}(A \otimes B)$ has the form $S \otimes S'$, where $S$ and $S'$ are simple modules over $A$ and $B$, respectively.

3. Main results

In this section, we study the intersections among tensor products of algebras, $\tau$-rigid modules and Gorenstein projective modules. We give a method in constructing non-trivial Gorenstein projective support $\tau$-tilting modules. As a consequence, we can give a partial answer to the question posed by Xie and Zhang [XZ] Question 5.7.

The following properties on the indecomposable direct summands of tensor products of modules are essential in this paper.

Proposition 3.1. Let $A$ and $B$ be two algebras. If $M \in \text{mod}A$ and $N \in \text{mod}B$ are indecomposable modules, then $M \otimes N$ is an indecomposable module in $\text{mod}(A \otimes B)$.

Proof. By Lemma 2.3(1), there is an algebra isomorphism
$$\text{End}_{A \otimes B}(M \otimes N) \cong \text{End}_A(M) \otimes \text{End}_B(N)$$

Denote by $I = \text{End}_A(M) \otimes \text{rad}([\text{End}_B(N)]) + \text{rad}([\text{End}_A(M)]) \otimes \text{End}_B(N)$, the radical of $\text{End}_{A \otimes B}(M \otimes N)$. Then tensoring the short exact sequences
$$0 \rightarrow \text{rad}([\text{End}_A(M)]) \rightarrow \text{End}_A(M) \rightarrow \text{End}_A(M)/\text{rad}([\text{End}_A(M)]) \rightarrow 0$$
Proposition 3.2. Let $A$ and $B$ be two algebras with $M \in \mod A$ and $N \in \mod B$.

1. $|M \otimes N| = |M||N|$ holds,
2. $|A \otimes B| = |A||B|$ holds.

Proof. It is easy to see that $M \otimes N_1 \simeq M \otimes N_2$ in $\mod (A \otimes B)$ implies that $N_1 \simeq N_2$ in $\mod B$.

Now we show the following proposition on tensor products of Gorenstein projective modules which is shown in [HuLXZ, Proposition 2.6]. We give a different proof in terms of functors.

Proposition 3.3. Let $A$ and $B$ be two algebras. Let $M \in \mod A$ and $N \in \mod B$ be Gorenstein projective modules. Then $M \otimes N \in \mod (A \otimes B)$ is Gorenstein projective.

Proof. Following Ringel and Zhang [RZ1], we call a module $M \in \mod A$ semi-Gorenstein projective if $\Ext^i_A(M, A) = 0$ for all $i \geq 1$. By Proposition 2.2, we divide the proof into three steps.

1. We show that $M \otimes N \in \mod (A \otimes B)$ is semi-Gorenstein projective if $M \in \mod A$ and $N \in \mod B$ are semi-Gorenstein projective.

Since $M$ and $N$ are both semi-Gorenstein projective, then $\Ext^i_A(M, A) = \Ext^j_B(N, B) = 0$ holds for all $i \geq 1$ and $j \geq 1$. By Lemma 2.6 we get that $\Ext^n_{A \otimes B}(M \otimes N, A \otimes B) \simeq \bigoplus_{i+j=m} \Ext^i_A(M, A) \otimes \Ext^j_B(N, B) = 0$ holds for $m \geq 1$.

2. We show $(M \otimes N)^* \simeq \Hom_{A \otimes B}(M \otimes N, A \otimes B)$ is semi-Gorenstein projective if both $M^*$ and $N^*$ are semi-Gorenstein projective.

By Lemma 2.6(1), we get that $(M \otimes N)^* \simeq \Hom_{A \otimes B}(M \otimes N, A \otimes B) \simeq \Hom_A(M, A) \otimes \Hom_B(N, B) \simeq M^* \otimes N^*$. Then the assertion follows from (1).

3. We show that $M \otimes N$ is reflexive, that is, $M \otimes N \simeq (M \otimes N)^{**}$.

By (2) $(M \otimes N)^* \simeq M^* \otimes N^*$. Then one gets the assertion by using (2) once more.

It has been shown in [XZ] the quotient algebras of $CM$-finite algebras need not be $CM$-finite. However, we have the following result.

Corollary 3.4. Let $A$ be an algebra and let $T_n(A)$ be the lower triangular matrix algebra for $n \geq 2$. If $T_n(A)$ is $CM$-finite, then $A$ is $CM$-finite.

Proof. Let $M_1, M_2$ be two indecomposable Gorenstein projective modules in $\mod A$ such that $T_n(K) \otimes M_1 \simeq T_n(K) \otimes M_2 \in \mod T_n(A)$. Then one gets $M_1 \simeq M_2$ since $T_n(K)$ is projective over $K$.

In the following we focus on the tensor products of $\tau$-rigid modules. In general, the tensor products of $\tau$-rigid modules need not be $\tau$-rigid. However, we have the following proposition.

Proposition 3.5. Let $A$ and $B$ be two algebras. Let $M \in \mod B$ be a $\tau$-rigid module and $P \in \mod A$ be a projective module. Then $P \otimes M \in \mod (A \otimes B)$ is a $\tau$-rigid module.

Proof. Let $P_1 \rightarrowtail P_0 \rightarrow M \rightarrow 0$ be a minimal projective presentation of $M$. Then one gets the following minimal projective presentation of $P \otimes M$: $P \otimes P_1 \rightarrowtail P \otimes P_0 \rightarrow P \otimes M \rightarrow 0$. By Proposition 2.4, it suffices to show that $\Hom_{A \otimes B}(Id_P \otimes f, P \otimes M) : \Hom_{A \otimes B}(P \otimes P_0, P \otimes M) \rightarrow \Hom_{A \otimes B}(P \otimes P_1, P \otimes M)$ is a surjective map. By Lemma 2.6(1), the map above can be seen as: $\Hom_A(P, P) \otimes \Hom_A(P_0, M) \rightarrow \Hom_A(P, P) \otimes \Hom_B(P_1, M)$ via $g \otimes h \rightarrow gId_P \otimes hf$. Since $M$ is $\tau$-rigid, we get that $\Hom(f, M) : \Hom_B(P_0, M) \rightarrow \Hom_B(P_1, M)$ is a surjective map. For any
generator $k \otimes l \in \text{Hom}_A(P, P) \otimes \text{Hom}_B(P, M)$, we get a morphism $h$ such that $hf = l$. Therefore, $(k \otimes h)(Id_P \otimes f) = k \otimes l$ which implies the map $\text{Hom}_{A\otimes B}(Id_P \otimes f, P \otimes M)$ is surjective. Then the assertion holds. \qed

The following proposition on tensor products of algebras is essential.

**Proposition 3.6.** Let $A$ and $B$ be two algebras. Let $(a)$ be a principal ideal of $A$ and $(b)$ be a principal ideal of $B$. Then the principal ideal $(a \otimes b) = (a) \otimes (b)$.

**Proof.** We first show $(a \otimes b) \subseteq (a) \otimes (b)$. For any element $m \in (a \otimes b) \subseteq A \otimes B$, one gets that $m = \sum_{i=1}^{n} a_i \otimes b_i (c_i \otimes d_i) = \sum_{i=1}^{n} a_i ac_i \otimes b_i bd_i$. Since $a_i ac_i \otimes b_i bd_i \in (a) \otimes (b)$ and $(a) \otimes (b)$ is an ideal of $A \otimes B$, we get that $m \in (a) \otimes (b)$.

Conversely, for any $n \in (a) \otimes (b)$, one gets that $n = \sum_{i=1}^{n} a_i \otimes b_i$, where $a_i = \sum_{k=1}^{s_i} a_{ik} ac_{ik} b_i = \sum_{j=1}^{t_i} b_{ij} bd_{ij}$. Thus $n = \sum_{i=1}^{n} \sum_{j=1}^{t_i} \sum_{k=1}^{s_i} a_{ik} ac_{ik} \otimes b_{ij} bd_{ij}$. Since $a_{ik} ac_{ik} \otimes b_{ij} bd_{ij} = (a_{ik} \otimes b_{ij})(a \otimes b)$ and $(a \otimes b)$ is an ideal, then the assertion holds. \qed

Now we are in a position to show our main result on support $\tau$-tilting modules.

**Theorem 3.7.** Let $A$ and $B$ be two algebras. Let $M \in \text{mod}B$ be a support $\tau$-tilting module. Then $A \otimes M \in \text{mod}(A \otimes B)$ is a support $\tau$-tilting module.

**Proof.** We divide the proof into two parts.

1. We show that $A \otimes B/(e) \simeq (A \otimes B)/(1 \otimes e)$, where $e$ is an idempotent of $B$.

   Note that there is an exact sequence $0 \rightarrow (e) \rightarrow B \rightarrow B/(e) \rightarrow 0$. Applying the functor $A \otimes -$ to the exact sequence above, one gets the following exact sequence $0 \rightarrow A \otimes (e) \rightarrow A \otimes B \rightarrow A \otimes B/(e) \rightarrow 0$. By Proposition 3.6, one gets the assertion.

2. We show that $A \otimes M$ is a $\tau$-tilting module over $(A \otimes B)/(1 \otimes e)$. Since $M$ is a support $\tau$-tilting module, then $M$ is a $\tau$-tilting module over $B/(e)$. Then $|B/(e)| = |M|$. By Proposition 3.5, $A \otimes M$ is a $\tau$-rigid module. By Proposition 3.2, $|A \otimes M| = |A||M| = |A||B/(e)| = |(A \otimes B)/(1 \otimes e)|$ by (1). The assertion holds. \qed

Now we have the following corollary on $\tau$-tilting modules.

**Corollary 3.8.** Let $A$ and $B$ be two algebras. Let $M \in \text{mod}B$ be a $\tau$-tilting module. Then $A \otimes M \in \text{mod}(A \otimes B)$ is a $\tau$-tilting module.

**Proof.** This is an immediate result of Theorem 3.7. \qed

Recall from [DL] that an algebra $A$ is called $\tau$-tilting finite if it admits finite number of isomorphism classes of indecomposable $\tau$-rigid modules. We have the following corollary on $\tau$-tilting finite algebras.

**Corollary 3.9.** Let $A$ be an algebra and let $T_n(A)$ be the lower triangular matrix algebra. If $T_n(A)$ is $\tau$-tilting finite, then $A$ is $\tau$-tilting finite.

**Proof.** This is clear since $A$ is a quotient algebra of $T_n(A)$. \qed

Now we are in a position to state the following main result on constructing Gorenstein projective support $\tau$-tilting modules.

**Theorem 3.10.** Let $A$ and $B$ be two algebras. If $M \in \text{mod}B$ is a Gorenstein projective support $\tau$-tilting module, then $A \otimes M \in \text{mod}(A \otimes B)$ is a Gorenstein projective support $\tau$-tilting module.

**Proof.** This is an immediate result of Proposition 3.6 and Theorem 3.7. \qed

As a corollary, we get the following property on the existence of non-trivial Gorenstein projective support $\tau$-tilting modules.

**Corollary 3.11.** Let $A$ and $B$ be two algebras. If $M \in \text{mod}B$ is a non-trivial Gorenstein projective support $\tau$-tilting module, then $A \otimes M \in \text{mod}(A \otimes B)$ is a non-trivial Gorenstein projective support $\tau$-tilting module.
Theorem 3.13. Let \( A \) be a non-semisimple self-injective algebra which is not local. Then there are non-trivial Gorenstein projective support \( \tau \)-tilting modules in \( \text{mod} \mathcal{A} \).

Proof. It is well-known that for an algebra \( B \) if all \( \tau \)-rigid modules in \( \text{mod} B \) are projective, then \( B \) is local. By the assumption, one gets that there are non-projective \( \tau \)-rigid modules in \( \text{mod} \mathcal{A} \). And hence there are non-trivial \( \tau \)-tilting modules \( M \in \text{mod} \mathcal{A} \). Then the assertion holds by Corollary 3.12. \( \square \)

By using Corollary 3.12, one is able to construct a large class of non-trivial Gorenstein projective support \( \tau \)-tilting modules by taking a class of non-trivial Gorenstein projective \( \tau \)-tilting modules over a self-injective algebra.

Recall from [XZ] that an algebra \( A \) is called \( CM-\tau \)-tilting finite if it admits a finite number of isomorphism classes of indecomposable \( \tau \)-rigid modules. It is an open question that whether \( CM-\tau \)-tilting finite algebras are closed under quotients. In the following we give a partial positive answer to the question.

Theorem 3.13. Let \( A \) be an algebra and let \( T_n(A) \) be the lower triangular matrix algebra for \( n \geq 2 \). If \( T_n(A) \) is \( CM-\tau \)-tilting finite, then \( A \) is \( CM-\tau \)-tilting finite.

Proof. It is well-known that \( T_n(A) \cong T_n(K) \otimes A \). For any Gorenstein projective support \( \tau \)-tilting module \( M \in \text{mod} A \), by Theorem 3.10, one gets a Gorenstein projective support \( \tau \)-tilting module \( T_n(K) \otimes M \in \text{mod} T_n(A) \). Since \( K \) is a field, we get \( T_n(K) \otimes M \cong M \otimes N \in \text{mod} A \). Therefore, the fact that \( T_n(A) \) is \( CM-\tau \)-tilting finite implies that \( A \) is \( CM-\tau \)-tilting finite. \( \square \)

Putting \( n = 2 \), one gets the following result on the representation of Gorenstein projective \( \tau \)-tilting modules which combines the results in [PMH] Corollary 4.7 and [LZ2] Theorem 1.1.

Proposition 3.14. Let \( A \) be a Gorenstein algebra and let \( T_2(A) \) be the lower triangular matrix algebra. If \( M = (\frac{Y}{X}) \) is a Gorenstein projective (support) \( \tau \)-tilting module in \( \text{mod} T_2(A) \), then \( Y \) is a Gorenstein projective (support) \( \tau \)-tilting module.

Proof. It is shown in [LZ2] that \( M \) is Gorenstein projective if and only if both \( X \) and \( Y \) are Gorenstein projective, \( f \) is a monomorphism and \( \text{Coker} f \) is Gorenstein projective. By [PMH] Corollary 4.7, one gets that \( Y \) should be a (support) \( \tau \)-tilting module. The assertion holds. \( \square \)

For more details on support \( \tau \)-tilting modules over triangular matrix rings, we refer to [CH]. [PMH]. For more details on tensor product algebras, we refer to [L]. We end this paper with the following example to show our main results.

Example 3.15. Let \( A \) be the algebra given by the quiver \( Q : 1 \xrightarrow{a_1} 2 \) with the relations \( a_1 a_2 = a_2 a_1 = 0 \). Let \( n \geq 2 \) be an integer. Then

1. \( A \) is a self-injective algebra and \( T_n(A) \) is not a self-injective algebra.
2. \( 1 \oplus 1 \) and \( 1 \) are two non-trivial Gorenstein projective support \( \tau \)-tilting modules in \( \text{mod} \mathcal{A} \).
3. \( T_n(K) \otimes (1 \oplus 1) \) and \( T_n(K) \otimes 1 \) are non-trivial Gorenstein projective support \( \tau \)-tilting modules in \( \text{mod} T_n(A) \).
4. \( T_n(T_n(A)) \) is also a non-self-injective algebra which admits non-trivial Gorenstein projective \( \tau \)-tilting modules.
5. By using the construction above, one can get infinite number of non-self-injective algebras admitting non-trivial Gorenstein projective \( \tau \)-tilting modules.
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