SUPERCHARACTERS, ELLIPTIC CURVES, AND THE SIXTH MOMENT OF KLOOSTERMAN SUMS

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Abstract. We connect the sixth power moment of Kloosterman sums to elliptic curves. This yields an elementary proof that $K_u$ with $p \nmid u$ are $O(p^{2/3})$.

1. Introduction

Let $e_p(x) = \exp(2\pi ix/p)$, in which $p$ is an odd prime. A Kloosterman sum is

$$K(a, b) = \sum_{x=1}^{p-1} e_p(ax + bx^{-1}),$$

in which $x^{-1}$ is the inverse of $x$ modulo $p$. Kloosterman sums are real and satisfy $K(a, b) = K(1, ab)$ if $p \nmid a$. Consequently, we write $K_u = K(1, u)$. The celebrated Weil bound asserts that $|K(u)| \leq 2\sqrt{p}$ for $p \nmid u$.

The first several power moments

$$V_n(p) = \sum_{u=1}^{p-1} K_u^n$$

of the Kloosterman sums are

$V_1(p) = 1, \quad V_2(p) = p^2 - p - 1,$

$$V_3(p) = \left(\frac{p}{3}\right) p^2 + 2p + 1, \quad V_4(p) = 2p^3 - 3p^2 - 3p - 1,$$

in which $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol modulo $p$. See [10] for simple proofs of the preceding and [1] for additional mixed-moment evaluations. An expression for $V_5(p)$ was found by Livné [20] and by Peters, Top, and van der Vlugt [22]:

$$V_5(p) = \left(\frac{p}{3}\right) 4p^3 + (a_p + 5)p^2 + 4p + 1, \quad p > 5,$$

in which $|a_p| < 2p$ depends upon $p$; see [26] p. 1234] or [28] p. 112] for details. In the early 1930s, H. Salié [23] and H. Davenport [6] proved that $V_6(p) = O(p^4)$. A precise evaluation of $V_6(p)$ was obtained in 2001 by Hulek, Spandaw, van Geemen, and van Straten [14]. They showed that

$$V_6(p) = 5p^4 - 10p^3 - (b_p + 9)p^2 - 5p - 1, \quad p > 7, \quad (1)$$

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in which \(b_p\) is an integer with \(|b_p| < 2p^{3/2}\) that is derived from the Dedekind eta function. Consequently,

\[ V_6(p) = 5p^4 + O(p^{7/2}) \]

and hence \(K_u = O(p^{2/3})\). In 2010, Evans conjectured formulas for \(V_7(p)\) and \(V_8(p)\) \[8, 9\], which were ultimately proved by Yun \[27\] by attaching Galois representations to Kloosterman sums. It should be noted that the fifth through eighth moments can be related to Hecke eigenvalues; see Section 4. Exact formulas for \(V_n(p)\) for \(n \geq 9\) appear difficult to obtain.

Our main result is a formula that relates the sixth power moment of Kloosterman sums to elliptic curves. This particular connection appears novel for \(p \geq 5\), although for \(p = 2\) and \(p = 3\) some links between Kloosterman sums and elliptic curves have been discovered \[19\].

**Theorem 2.** For \(p \geq 5\),

\[ V_6(p) = 4p^4 - 8p^3 + \left[ 4 \left( \frac{p}{3} \right) + 2 \right] p^2 - 5p - 1 + p^2 \sum_{k=2, k \neq 3}^{p-1} \left( a_p(E_k) + 1 \right)^2, \]

in which \(a_p(E_k)\) denotes the Frobenius trace of the elliptic curve

\[ E_k(\mathbb{F}_p) = \{ (x, y) \in \mathbb{F}_p^2 : y^2 = 4kx^3 + (k^2 - 6k - 3)x^2 + 4x \}. \]

The restriction \(k \neq 1, 9\) is natural since these yield the non-elliptic curves \(y^2 = 4x(x-1)^2\) and \(y^2 = 4x(3x+1)^2\), respectively. For \(p = 5, 7\), we interpret these restrictions modulo \(p\). For example, if \(p = 7\), then the terms corresponding to \(k = 1\) and \(k = 2\) are omitted in (3). For \(k \neq 1, 9\), the number of points on \(E_k(\mathbb{F}_p)\), including the point at infinity, is

\[ |E_k(\mathbb{F}_p)| = p + 1 - a_p(E_k), \]

in which the Frobenius trace \(a_p(E_k)\) satisfies Hasse’s inequality \[13\]

\[ |a_p(E_k)| \leq 2\sqrt{p}. \]

An elementary inductive proof of this was found by Y. Manin \[21\]. It is simple enough that it appeared in the American Mathematical Monthly in 2008 \[3\].

To prove Theorem 2 we employ basic supercharacter theory to realize Kloosterman sums as eigenvalues of a certain matrix whose entries encode combinatorial information about a certain group action. This completely elementary, linear-algebraic perspective provides a convenient method for keeping track of various expressions that arise throughout our computations. This approach was first undertaken to study Ramanujan sums \[11\]; see also \[12\].

As a consequence of Theorem 2 we obtain an elementary proof that

\[ |K_u| \leq 1.43p^{2/3} \tag{4} \]

whenever \(p \nmid u\). In particular, this breaks the “\(O(p^{3/4})\) barrier,” which folklore suggested cannot be passed without deep techniques or difficult point-counting arguments. To obtain (4), use Hasse’s inequality in (3) and compute

\[ V_6(p) \leq 4p^4 - 8p^3 + \left[ 4 \left( \frac{p}{3} \right) + 2 \right] p^2 - 5p - 1 + p^2(p-3)(4p + 4\sqrt{p} + 1) \]
\begin{align*}
&= 8p^4 + 4p^{7/2} - 19p^3 - 12p^{5/2} + \left[ 4 \left( \frac{p}{3} \right) - 1 \right] p^2 - 5p - 1 \\
&\leq 8p^4 + 4p^{7/2} - 19p^3 - 12p^{5/2} + 3p^2 - 5p - 1 \\
&\leq 8.5p^4
\end{align*}
for \( p \geq 5 \). Taking sixth roots and verifying the cases \( p = 2, 3 \) yields (4).

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2. Kloosterman sums as supercharacters

The theory of supercharacters was introduced in 2008 by P. Diaconis and I.M. Isaacs [7], building upon previous work of C. André [1] on the representation theory of unipotent matrix groups over finite fields. We are concerned only with the special case in which the underlying group is abelian, for which the details are much simpler. A variety of exponential sums that are relevant to the theory of numbers can be realized as supercharacters on abelian groups. The following setup is from [2].

Let \( \Gamma \) be a subgroup of \( GL_d(\mathbb{Z}/n\mathbb{Z}) \) that is closed under the transpose operation and let \( X_1, X_2, \ldots, X_N \) denote the orbits in \( G = (\mathbb{Z}/n\mathbb{Z})^d \) under the action of \( \Gamma \). The functions

\[ \sigma_i(y) = \sum_{x \in X_i} e\left( \frac{x \cdot y}{n} \right) \tag{5} \]

in which \( x \cdot y \) denotes the formal dot product of two elements of \( (\mathbb{Z}/n\mathbb{Z})^d \) and \( e(x) = \exp(2\pi ix) \), are supercharacters on \( (\mathbb{Z}/n\mathbb{Z})^d \) and the sets \( X_i \) are superclasses. One can show that supercharacters are constant on superclasses, so we may write \( \sigma_i(X_j) \) without confusion. The \( N \times N \) matrix

\[ U = \frac{1}{\sqrt{n^d}} \left[ \frac{\sigma_i(X_j) \sqrt{|X_i|}}{\sqrt{|X_j|}} \right]_{i,j=1}^{N} \tag{6} \]

is complex symmetric (i.e., \( U = U^T \)) and unitary [2, Lem. 1], [11, Sect. 2.1]. It represents, with respect to a particular orthonormal basis, the restriction of the discrete Fourier transform (DFT) to the subspace of \( L^2(G) \) that consists of functions that are constant on each \( \Gamma \)-orbit in \( G \).

The following lemma identifies the set of matrices that are diagonalized by the unitary matrix (6) as the span of a certain family of normal matrices that contain combinatorial information about the group action. The proof is completely elementary and is similar to the corresponding result from classical character theory [5, Section 33]. In fact, the matrices (8), (9), (10) below and their properties can be obtained with classical character theory in a more contrived, tedious, and long-winded manner [10, Lem. 3.1]. A more general version of this lemma, in which \( G \) need not be abelian, is [11, Thm. 4.2]. The simple version that we present below is [2, Thm. 1].

**Lemma 7.** Let \( \Gamma = \Gamma^T \) be a subgroup of \( GL_d(\mathbb{Z}/n\mathbb{Z}) \), let \( \{X_1, X_2, \ldots, X_N\} \) denote the set of \( \Gamma \)-orbits in \( G = (\mathbb{Z}/n\mathbb{Z})^d \) induced by the action of \( \Gamma \), and let \( \sigma_1, \sigma_2, \ldots, \sigma_N \) denote
the corresponding supercharacters \( \sigma \). For each fixed \( z \) in \( X_k \), let \( c_{i,j,k} \) denote the number of solutions \((x_i, y_j) \in X_i \times X_j\) to the equation \( x + y = z \).

(a) \( c_{i,j,k} \) is independent of the representative \( z \) in \( X_k \) that is chosen.

(b) The identity

\[
\sigma_i(X_{k}) \sigma_j(X_{\ell}) = \sum_{k=1}^{N} c_{i,j,k} \sigma_k(X_{\ell})
\]

holds for \( 1 \leq i, j, k, \ell \leq N \).

(c) The matrices \( T_1, T_2, \ldots, T_N \), whose entries are given by

\[
[T_{i}]_{j,k} = \frac{c_{i,j,k} \sqrt{|X_k|}}{|X_j|},
\]

each satisfy

\[
T_i U = U D_i,
\]

in which \( D_i = \text{diag} (\sigma_i(X_1), \sigma_i(X_2), \ldots, \sigma_i(X_N)) \).

In particular, the \( T_i \) are simultaneously unitarily diagonalizable.

(d) Each \( T_i \) is a normal matrix (i.e., \( T_i^{*} T_i = T_i T_i^{*} \)) and the set \( \{T_1, T_2, \ldots, T_N\} \) forms a basis for the commutative algebra of all \( N \times N \) complex matrices \( T \) such that \( U^{*} TU \) is diagonal.

Let \( p \) be an odd prime. Then the action of the diagonal matrix group

\[
\Gamma = \{ \text{diag}(u, u^{-1}) : u \in \mathbb{F}_p^\times \}
\]

on the additive group \( G = \mathbb{F}_p^2 \) induces a supercharacter theory that is related to Kloosterman sums. There are \( N = p + 2 \) superclasses (that is, \( \Gamma \)-orbits in \( G \)):

\[
\begin{align*}
X_1 &= \{(x, x^{-1}) : x \in \mathbb{F}_p^\times \}, \\
X_2 &= \{(x, 2x^{-1}) : x \in \mathbb{F}_p^\times \}, \\
&\vdots \\
X_{p-1} &= \{(x, (p - 1)x^{-1}) : x \in \mathbb{F}_p^\times \}, \\
X_p &= \{(0, u) : u \in \mathbb{F}_p^\times \}, \\
X_{p+1} &= \{(u, 0) : u \in \mathbb{F}_p^\times \}, \\
X_{p+2} &= \{(0, 0)\}.
\end{align*}
\]

If \( 1 \leq i, j \leq p - 1 \), then we select the representative \( y = (1, j) \in X_j \) and compute:

\[
\sigma_i(X_j) = \sum_{x \in X_i} e_p(x \cdot y) = \sum_{u=1}^{p-1} e_p\left((u, iu^{-1}) \cdot (1, j)\right)
\]

\[
= \sum_{u=1}^{p-1} e_p(u + iju^{-1}) = K_{ij}.
\]

A few more computations complete the supercharacter table (Table\[\text{Table}1\]).
The formula (6) provides the \((p + 2) \times (p + 2)\) real-symmetric unitary matrix

\[
U := \frac{1}{p} \begin{bmatrix}
K_1 & K_2 & \cdots & K_{p-1} & -1 & -1 & \sqrt{p-1} \\
K_2 & K_4 & \cdots & K_{2(p-1)} & -1 & -1 & \sqrt{p-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
K_{p-1} & K_{2(p-1)} & \cdots & K_{(p-1)^2} & -1 & -1 & \sqrt{p-1} \\
-1 & -1 & \cdots & -1 & p-1 & -1 & \sqrt{p-1} \\
-1 & -1 & \cdots & -1 & p-1 & -1 & \sqrt{p-1} \\
\sqrt{p-1} & \sqrt{p-1} & \cdots & \sqrt{p-1} & \sqrt{p-1} & \sqrt{p-1} & 1
\end{bmatrix}; \quad (8)
\]

this is [10] eq. 3.13. Define \(D = D_1\) and \(T = T_1\) as in Lemma [7]. These are

\[
D = \text{diag}(K_1, K_2, \ldots, K_{(p-1)^2}, -1, -1, p-1)
\]

and

\[
T = \begin{bmatrix}
t_{1,1} & t_{1,2} & \cdots & t_{1,p-1} & 0 & 0 & \sqrt{p-1} \\
t_{2,1} & t_{2,2} & \cdots & t_{2,p-1} & 1 & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
t_{p-1,1} & t_{p-1,2} & \cdots & t_{p-1,p-1} & 1 & 1 & 0 \\
0 & 1 & \cdots & 1 & 0 & 0 & 0 \\
0 & 1 & \cdots & 1 & 1 & 0 & 0 \\
\sqrt{p-1} & 0 & \cdots & 1 & 0 & 0 & 0
\end{bmatrix}, \quad (10)
\]

in which

\[
t_{ij} = 1 + \left( \frac{f_j(i)}{p} \right) \quad (11)
\]

and

\[
f_j(x) = x^2 - 2(j + 1)x + (j - 1)^2. \quad (12)
\]

The symmetry \(f_j(i) = f_j(j)\) will be important later. The matrix \(T\) is [10] eq. 3.12] and it satisfies \(T = UDU\) by Lemma [7] since \(U^* = U\). These matrices and the diagonalization above can also be derived, with more effort, using classical character theory [10].

| \((\mathbb{Z}/p\mathbb{Z})^2\) | \(X_1\) | \(X_2\) | \(\cdots\) | \(X_{p-1}\) | \(X_p\) | \(X_{p+1}\) | \(X_{p+2}\) |
|---|---|---|---|---|---|---|---|
| \(\Gamma\) | \((1,1)\) | \((1,2)\) | \(\cdots\) | \((1,p-1)\) | \((0,1)\) | \((0,0)\) |
| \# | \(p\) | \(p\) | \(p\) | \(p\) | \(p\) | \(p\) |
| \(e_1\) | \(K_1\) | \(K_2\) | \(\cdots\) | \(K_{p-1}\) | \(-1\) | \(-1\) | \(p-1\) |
| \(e_2\) | \(K_2\) | \(K_4\) | \(\cdots\) | \(K_{2(p-1)}\) | \(-1\) | \(-1\) | \(p-1\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\ddots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |
| \(e_{p-1}\) | \(K_{p-1}\) | \(K_{2(p-1)}\) | \(\cdots\) | \(K_{(p-1)^2}\) | \(-1\) | \(-1\) | \(p-1\) |
| \(e_p\) | \(-1\) | \(-1\) | \(\cdots\) | \(-1\) | \(p-1\) | \(-1\) |
| \(e_{p+1}\) | \(-1\) | \(-1\) | \(\cdots\) | \(-1\) | \(-1\) | \(p-1\) |
| \(e_{p+2}\) | 1 | 1 | \(\cdots\) | 1 | 1 | 1 |

Table 1. Supercharacter table for the action of \(\Gamma = \{\text{diag}(u, u^{-1}) : u \in \mathbb{F}_p^*\}\) on \(G = \mathbb{F}_p^2\).
Before proceeding, a brief explanation of (12) is in order. Let $1 \leq i, j \leq p - 1$. In the notation of Lemma 7, $t_{i,j}$ is the number of solutions to $x + y = z$, in which $z = (1,j) \in X_j$ is fixed, $x = (x,x^{-1}) \in X_1$ and $y = (y, iy^{-1}) \in X_i$. This yields the system

$$(x,x^{-1}) + (y,iy^{-1}) = (1,j).$$

Since $x = 1$ implies $y = 0$, we may assume that $x \neq 1$. The first equation $x + y = 1$ suggests the substitution $y = 1 - x$. The second equation then yields

$$jx^2 + (i - j - 1)x + 1 = 0,$$

the discriminant of which is $f_j(i)$. This establishes (12).

3. Proof of Theorem 2

We prove the desired identity (3) by computing $[T^4]_{1,1} = [UD^4U]_{1,1}$ in two different ways. The evaluation of $[UD^4U]_{1,1}$ is relatively simple and involves $V_{\kappa}(p)$; we save this for later. To compute $[T^4]_{1,1}$ requires more work. Some of the expressions that arise involve Frobenius traces of certain elliptic curves over $\mathbb{F}_p$.

Lemma 13. For $k = 1, 2, \ldots, p - 1$,

$$\sum_{x=0}^{p-1} \left( \frac{f_k(x)}{p} \right) = -1 \quad \text{and} \quad \sum_{x=0}^{p-1} \left( \frac{f_k(x)}{p} \right)^2 = p - 1 - \left( \frac{k}{p} \right). \tag{14}$$

Proof. Since $f_k(x + k + 1) = x^2 - 4k$ and $p \nmid 4k$, the first equation in (14) follows from [15] Ex. 8, p. 63). The number of solutions to $x^2 - 4k \equiv 0 \pmod{p}$ is $1 + (k/p)$, from which the second equation in (14) follows. □

Let

$$\epsilon_k = \sum_{x=0}^{p-1} \left( \frac{f_1(x)f_k(x)}{p} \right). \tag{15}$$

In what follows, the Legendre symbol $(p/3)$ occurs frequently. Consequently, we adopt the shorthand $\ell_p = (p/3)$. Since $p \geq 5$, it follows that $\ell_p^2 = 1$.

The quadratic formula confirms that

$$f_1(x) = x^2 - 4x \quad \text{and} \quad f_k(x) = x^2 - 2(k+1)x + (k - 1)^2$$

share a common root if and only if $k = 1$ or $k = 9$. This causes some minor complications later on when we attempt to write (15) in terms of Frobenius traces. We therefore evaluate $\epsilon_1$ and $\epsilon_9$ explicitly here.

Lemma 16. $\epsilon_1 = p - 2$ and $\epsilon_9 = -1 - \ell_p$.

Proof. Since $f_1(x) = x(x - 4)$ has two distinct roots modulo $p$,

$$\epsilon_1 = \sum_{x=0}^{p-1} \left( \frac{f_1(x)^2}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right)^2 = p - 2.$$ 

Since $f_1$ and $f_k$ share the common factor $x - 4$, (14) ensures that

$$\epsilon_9 = \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) \left( \frac{f_9(x)}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{x(x-4)}{p} \right) \left( \frac{(x-4)(x-16)}{p} \right).$$
Lemma 17. \( \sum_{k=2}^{p-1} \epsilon_k = 4 + \ell_p - p. \)

Proof. The symmetry \( f_k(x) = f_x(k), \) (14), and quadratic reciprocity imply that
\[
\sum_{k=1}^{p-1} \epsilon_k = \sum_{k=1}^{p-1} \sum_{x=0}^{p-1} \left( \frac{f_1(x)f_1(x)}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) \sum_{x=1}^{p-1} \left( \frac{f_x(k)}{p} \right)
= \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) \left[ 1 - \left( \frac{f_x(0)}{p} \right) \right] = \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) \left[ 1 - \left( \frac{x - 1}{p} \right)^2 \right]
= \left( \frac{f_1(1)}{p} \right) - 2 \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) = 2 + \left( \frac{-3}{p} \right) = 2 + \left( \frac{p}{3} \right). \]

Subtract \( \epsilon_1 = p - 2 \) and obtain the desired result. \( \square \)

The following is a special case of a formula due to K.S. Williams [25]. In this instance, it concerns the birational equivalence between a quartic and a cubic elliptic curve. We provide an independent proof of the relevant case.

Lemma 18. Let \( p \) be prime,
\[ D = B^2 - 4C, \quad d = b^2 - 4c, \quad \text{and} \quad \delta = 4C - 2bB + 4c. \]
If \( B - b \neq 0 \) and the polynomials \( x^2 + bx + c \) and \( x^2 + Bx + C \) share no common roots modulo \( p \), then
\[
\sum_{x=0}^{p-1} \left( \frac{(x^2 + bx + c)(x^2 + Bx + C)}{p} \right) = -1 + \sum_{x=0}^{p-1} \left( \frac{x(Dx^2 + \delta x + d)}{p} \right).
\]

Proof. For \( x^2 + Bx + C \neq 0 \), define the \( \mathbb{F}_p \)-valued function
\[
\theta(x) = \frac{x^2 + bx + c}{x^2 + Bx + C}.
\]
Suppose that \( y \neq 0 \), then since \( x^2 + bx + c \) and \( x^2 + Bx + C \) share no common roots modulo \( p \), the number of solutions to \( \theta(x) = y \) equals the number of solutions to
\[
(y - 1)x^2 + (By - b)x + (Cy - c) = 0. \tag{19}
\]
If \( y \neq 1 \), then the number of solutions to (19) is
\[
1 + \left( \frac{(By - b)^2 - 4(y - 1)(Cy - c)}{p} \right) = 1 + \left( \frac{Dy^2 + \delta y + d}{p} \right).
\]
If \( y = 1 \), then (19) is
\[
(B - b)x + (C - c) = 0,
\]
which has exactly
\[ 1 = \left( \frac{(B - b)^2}{p} \right) = \left( \frac{1(D(1)^2 + \delta(1) + d)}{p} \right) \]
solutions since \( B - b \neq 0 \). Then
\[
\sum_{x \in \mathbb{F}_p} \left( \frac{(x^2 + bx + c)(x^2 + Bx + C)}{p} \right) = \sum_{x \in \mathbb{F}_p, x^2 + Bx + C \neq 0} \left( \frac{\theta(x)}{p} \right) = \sum_{y \in \mathbb{F}_p \setminus \{0\}} \left( \frac{y}{p} \right) \sum_{x \in \mathbb{F}_p, \theta(x) = y} 1
\]
\[
= \sum_{x \in \mathbb{F}_p, \theta(x) = 1} 1 + \sum_{y \in \mathbb{F}_p \setminus \{0,1\}} \left( \frac{y}{p} \right) \sum_{x \in \mathbb{F}_p, \theta(x) = y} 1
\]
\[
= \left( \frac{1(D(1)^2 + \delta(1) + d)}{p} \right) + \sum_{y \in \mathbb{F}_p \setminus \{0,1\}} \left( \frac{y}{p} \right) \left( 1 + \left( \frac{Dy^2 + \delta y + d}{p} \right) \right)
\]
\[
= \sum_{y \in \mathbb{F}_p \setminus \{0,1\}} \left( \frac{y}{p} \right) + \sum_{y \in \mathbb{F}_p \setminus \{0\}} \left( \frac{y(Dy^2 + \delta y + d)}{p} \right)
\]
\[
= -1 + \sum_{y \in \mathbb{F}_p \setminus \{0\}} \left( \frac{y(Dy^2 + \delta y + d)}{p} \right). \quad \square
\]

**Lemma 20.** For \( k \neq 1,9 \),
\[
\epsilon_k = -1 - a_p(E_k), \quad (21)
\]
in which
\[
a_p(E_k) = p + 1 - |E_k(\mathbb{F}_p)|
\]
denotes the Frobenius trace of the elliptic curve
\[
E_k(\mathbb{F}_p) = \{ (x,y) \in \mathbb{F}_p^2 : y^2 = g_k(x) \},
\]
where
\[
g_k(x) = x(4kx^2 + (k^2 - 6k - 3)x + 4).
\]

**Proof.** Suppose that \( k \neq 1,9 \). First observe that the discriminant of the quadratic factor of \( g_k(x) \) is \((k - 9)(k - 1)^3\). Thus, \( g_k(x) \) has distinct roots and \( y^2 = g_k(x) \) defines an elliptic curve \( E_k \) over \( \mathbb{F}_p \). Since \( f_1(x) \) and \( f_k(x) \) share no common roots in \( \mathbb{F}_p \), we apply Lemma [18] with
\[
a = 1, \quad b = -4, \quad c = 0, \quad A = 1, \quad B = -2(k + 1), \quad C = (k - 1)^2,
\]
so that
\[
D = 16k, \quad d = 16, \quad \text{and} \quad \delta = 4(k^2 - 6k - 3).
\]
Then
\[
\epsilon_k = \sum_{x=0}^{p-1} \left( \frac{f_1(x)|f_k(x)}{p} \right) = \sum_{x=0}^{p-1} \left( \frac{(x^2 - 4x)(x^2 - 2(k + 1)x + (k - 1)^2)}{p} \right)
\]
For \(k\)

\[
-1 + \sum_{x=0}^{p-1} \left( \frac{x(4kx^2 + (k^2 - 6k - 3)x + 4)}{p} \right) = -1 + \sum_{x=0}^{p-1} \left( \frac{g_k(x)}{p} \right)
\]

\[
= -p - 2 + \left[ p + 1 + \sum_{x=0}^{p-1} \left( \frac{g_k(x)}{p} \right) \right] = -p - 2 + |E_k(F_p)|
\]

\[
= -p - 2 + (p + 1 - a_p(E_k)) = -1 - a_p(E_k).
\]

\(\square\)

Lemma 22. For \(k = 1, 2, \ldots, p+2\),

\[
[T^2]_{1,k} = \begin{cases} 
3p - 6 & \text{if } k = 1, \\
p - 4 + \epsilon_k & \text{if } k = 2, \ldots, p - 1, \\
p - 3 - \ell_p & \text{if } k = p, p + 1, \\
(1 + \ell_p)\sqrt{p - 1} & \text{if } k = p + 2.
\end{cases}
\]

Proof. Use (10) and (14) to compute

\[
[T^2]_{1,1} = \sum_{u=1}^{p-1} t_{1,u} + (p - 1) = (p - 1) + \sum_{u=1}^{p-1} \left[ 1 + \left( \frac{f_1(u)}{p} \right) \right]^2
\]

\[
= p - 2 + \sum_{x=0}^{p-1} \left[ 1 + \left( \frac{f_1(x)}{p} \right) \right]^2
\]

\[
= (p - 2) + p + 2 \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) + \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right)^2
\]

\[
= 2p - 2 + 2(-1) + (p - 2)
\]

\[
= 3p - 6.
\]

For \(k = 2, 3, \ldots, p - 1\), a similar computation and (14) yield

\[
[T^2]_{1,k} = \sum_{u=1}^{p-1} t_{1,u}t_{u,k} = \sum_{u=1}^{p-1} \left[ 1 + \left( \frac{f_1(u)}{p} \right) \right] \left[ 1 + \left( \frac{f_k(u)}{p} \right) \right]
\]

\[
= -2 + \sum_{x=0}^{p-1} \left[ 1 + \left( \frac{f_1(x)}{p} \right) \right] \left[ 1 + \left( \frac{f_k(x)}{p} \right) \right]
\]

\[
= p - 2 + \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) + \sum_{x=0}^{p-1} \left( \frac{f_k(x)}{p} \right) + \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right) \left( \frac{f_k(x)}{p} \right)
\]

\[
= p - 4 + \epsilon_k.
\]

For \(k = p, p + 1\), (14) and quadratic reciprocity provide

\[
[T^2]_{1,k} = \sum_{u=2}^{p-1} t_{1,u} = \sum_{u=2}^{p-1} \left[ 1 + \left( \frac{f_1(u)}{p} \right) \right]
\]

\[
= (p - 2) - \left( \frac{f_1(0)}{p} \right) - \left( \frac{f_1(1)}{p} \right) + \sum_{x=0}^{p-1} \left( \frac{f_1(x)}{p} \right)
\]

\[
= (p - 2) - \left( \frac{0}{p} \right) - \left( \frac{-3}{p} \right) - 1
\]
Finally,

\[ [T^2]_{1,k} = \begin{cases} 
9p^2 - 36p + 36 & \text{if } k = 1, \\
(p - 4)^2 + 2(p - 4)e_k + e_k^2 & \text{if } k = 2, 3, \ldots, p - 1, \\
p^2 - p(6 + 2\ell_p) + 10 + 6\ell_p & \text{if } k = p, p + 1, \\
2(1 + \ell_p)(p - 1) & \text{if } k = p + 2. 
\end{cases} \]

A computation yields

\[
[T^4]_{1,1} = \sum_{k=1}^{p+2} [T^2]_{1,k}^2
\]

\[
= (9p^2 - 36p + 36) + \sum_{k=2}^{p-1} \left( (p - 4)^2 + 2(p - 4)e_k + e_k^2 \right) \\
+ 2(p^2 - p(6 + 2\ell_p) + 10 + 6\ell_p) + (1 + \ell_p)^2 p - 2(1 + \ell_p) \\
= p^3 + p^2 - 2(7 + \ell_p)p + 2(11 + 5\ell_p) + 2(p - 4) \sum_{k=2}^{p-1} e_k + \sum_{k=2}^{p-1} e_k^2 \\
= p^3 + p^2 - 2(7 + \ell_p)p + 2(11 + 5\ell_p) + 2(p - 4)(4 + \ell_p - p) + \sum_{k=2}^{p-1} e_k^2 \\
= p^3 - p^2 + 2p + 2(\ell_p - 5) + \sum_{k=2}^{p-1} e_k^2 \\
= p^3 - p^2 + 2p + 2(\ell_p - 5) + (-1 - \ell_p)^2 + \sum_{k=2, k \neq 9}^{p-1} e_k^2 \\
= p^3 - p^2 + 2p + 2(\ell_p - 5) + 2(\ell_p + 1) + \sum_{k=2, k \neq 9}^{p-1} e_k^2 \\
= p^3 - p^2 + 2p + 4\ell_p - 8 + \sum_{k=2, k \neq 9}^{p-1} e_k^2.
\]
The definitions (8) and (9) imply that

\[ [UD^4U]_{1,1} = \frac{1}{p^2} \left( \sum_{u=1}^{p-1} K_u^6 + 2 + (p-1)^5 \right). \]

The equality \( T^4 = UD^4U \) reveals that

\[ \sum_{u=1}^{p-1} K_u^6 + 2 + (p-1)^5 = p^2 \left( p^3 - p^2 + 2p(p - 8 + \sum_{k=2, k \neq 9}^{p-1} \epsilon_k^2) \right). \]

Consequently,

\[ \sum_{u=1}^{p-1} K_u^6 = \left( p^5 - p^4 + 2p^3 + (4p - 8)p^2 + p^2 \sum_{k=2, k \neq 9}^{p-1} \epsilon_k^2 \right) - 2 - (p-1)^5 \]

\[ = 4p^4 - 8p^3 + (4p + 2)p^2 - 5p - 1 + p^2 \sum_{k=2, k \neq 9}^{p-1} \epsilon_k^2 \]

\[ = 4p^4 - 8p^3 + (4p + 2)p^2 - 5p - 1 + p^2 \sum_{k=2, k \neq 9}^{p-1} (a_F(E_k) + 1)^2. \]

This is the desired formula (3). \( \square \)

4. Future work

It follows from (8) and (9) that the \( n \)th Kloosterman power moment is

\[ V_n(p) = p^2[T_1]_{1,1}^{n-2} + 2(-1)^{n-1} - (p-1)^{n-1}. \]

Thus, the problem of calculating \( V_n(p) \) reduces to the evaluation of sums and products of Legendre symbols. For the sixth power moment, Lemmas 17 and 18 allowed us to provide an evaluation in terms of power moments of Frobenius traces. For higher moments, similar, but more complicated, techniques are needed. There is much work to be done in this direction.

In [16], Kaplan and Petrow provide a method to evaluate power moments of Frobenius traces of families of elliptic curves whose group of \( k \)-points contains a particular subgroup. Their evaluation is in terms of traces of Hecke operators. Given that is possible to relate the constants appearing in the evaluation of the fifth through eighth power moments of Kloosterman sums in terms of Hecke operators, one wonders if the Kaplan–Petrow method can be used to evaluate the power moment of the Frobenius traces that appear in Theorem 2 and whether or not such terms will appear in higher power moments when evaluated using (8) and (9). In particular, it follows from [20, 22], that \( V_5(p) \) can be expressed in terms of Hecke eigenvalues for a weight 3 newform on \( \Gamma_0(15) \). That \( V_5(p) \) can be expressed in terms of Hecke eigenvalues for a weight 4 newform on \( \Gamma_0(6) \) follows from [14]. Evans conjectured that \( V_7(p) \) and \( V_8(p) \) can be evaluated in terms of Hecke eigenvalues for a weight 3 newform on \( \Gamma_0(525) \) and for a weight 6 newform on \( \Gamma_0(525) \), respectively [8, 9]. Yun proved Evans’ conjectures in [27].
Mixed Kloosterman moments are also of interest and have been studied in [4, 10, 17, 18]. From (8) and (9), we have
\[
p_{-1} \sum_{u=1}^{p-1} K_u K_{a_1 u} K_{a_2 u} \cdots K_{a_n u} = p^2 \left[ T_{a_1} T_{a_2} \cdots T_{a_{n-1}} \right]_{1,a_n} + 2(-1)^n - (p-1)^n. \tag{23}
\]

The second and third mixed moments are given by
\[
p_{-1} \sum_{u=1}^{p-1} K_u K_{a_1 u} = -p,
\]
\[
p_{-1} \sum_{u=1}^{p-1} K_u K_{a_1 u} K_{a_2 u} K_{b_1 u} = \left( \frac{f_a(b)}{p} \right) p^2 + 2p;
\]
see [10, 17, 18]. In [4], Á. Chávez and the second author showed that
\[
p_{-1} \sum_{u=1}^{p-1} K_u K_{a_1 u} K_{b_1 u} K_{c_1 u} = \delta_{a_1,1} \delta_{b_1,c} p^3 - \left[ \left( \frac{bc}{p} \right) a_p + 2 \right] p^2 - 3p - 1,
\]
in which \(a_p\) is a certain Frobenius trace. In light of (23), a similar evaluation for higher mixed moments appears within reach.

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