Smoothing And Dispersive Estimates For 1d Schrödinger Equations With Bv Coefficients And Applications

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Abstract

We prove smoothing estimates for Schrödinger equations $i\partial_t \phi + \partial_x(a(x)\partial_x \phi) = 0$ with $a(x) \in BV$, real and bounded from below. We then bootstrap these estimates to obtain optimal Strichartz and maximal function estimates, all of which turn out to be identical to the constant coefficient case. We also provide counterexamples showing $a \in BV$ to be in a sense a minimal requirement. Finally, we provide an application to sharp wellposedness for a generalized Benjamin-Ono equation.

Introduction

Let us consider

\begin{equation}
    i\partial_t u + \partial_x(a(x)\partial_x u) = 0, \quad u(x,t=0) = u_0(x).
\end{equation}

We take $a \in BV$, the space of bounded functions whose derivatives are Radon measures. Moreover, we assume $a$ to be real-valued and bounded from below: $0 < m \leq a(x) \leq M$. We are interested in proving smoothing and dispersive estimates for the function $u$. This type of equations has been recently studied by Banica [3] who considered the case where the metric $a$ is piecewise constant (with a finite number of discontinuities). In [3], Banica proved that the solutions of the Schrödinger equation associated to such a metric enjoy the same dispersion estimates (implying Strichartz) as in the case of the constant metric, and conjectured it would hold true for general $a \in BV$ as well. Unfortunately, her method of proof (which consists in writing a complete description for the evolution problem) leads to constants depending upon the number of discontinuities rather than on the norm in $BV$ of the metric and consequently does not extend to more general settings. On the other hand, Castro and Zuazua [7] show that the space $BV$ is more or less optimal: they construct metrics $a \in C^{0,\beta}$ for all $\beta \in [0,1]$ (but not in $BV$) and solutions of the corresponding Schrödinger equation for which any local dispersive estimate of the type

\[ \|u(t,x)\|_{L^1_{loc}\left(L^q_{loc,x}\right)} \leq C \|u_0\|_{H^s} \]

fail if $1/p < 1/2 - s$ (otherwise, the estimate is a trivial consequence of Sobolev embeddings). In this article, we prove the natural conjecture, namely that for BV metrics, the Schrödinger equation enjoys...
the same smoothing, Strichartz and maximal function estimates as for the constant coefficient case, globally in time. In the context of variable coefficients, this appears to be the first case where such a low regularity (including discontinuous functions) is allowed, together with a translation invariant formulation of the decay at infinity (no pointwise decay). Previous works on dispersive estimates, while applying equally to any dimension, dealt with $C^2$ compact perturbations of the Laplacian ($27$), short range perturbations with symbol-like decay at infinity ($24$), and very recently long range perturbations, still with symbol-like decay ($14$). The idea to use local smoothing to derive Strichartz, however, goes back to Staffilani-Tataru ($27$) in the context of variable coefficients, and was used earlier to obtain full dispersion in $15$ where a potential perturbation was treated. All recent works on this topic make definitive use of resolvent estimates for the elliptic operator, see e.g. $25$. Finally, it has to be noticed that Salort $26$ recently obtained dispersion (hence, Strichartz) (locally in time) for 1D Schrödinger equations with $C^2$ coefficients through a completely different approach involving commuting vector fields.

We now say a word on the relevance of non-trapping conditions. In higher dimension, it has been known since the works of Doï $12, 13$ and the first author $5$ that the non trapping assumption is necessary for the optimal smoothing effect to hold and the study of eigenfunctions on compact manifolds somewhat shows that a non trapping condition is also necessary for Strichartz estimates. In the one dimensional case, a smooth metric is always non trapping as can be easily seen by a simple change of variables. However, some trapping-related behaviours (namely the existence of waves localized at a point) appear for metric with regularity below BV (see the work by Castro Zuazua $7$ and the appendix C). In fact the assumption $a \in BV$ ensures some kind of non trappingness and this fact has been known for a while in the different context of control theory $10$. Let us picture this on the model case of piecewise constant metrics: consider a wave coming from minus infinity. Then the wave propagates freely (at a constant speed) until it reaches the first discontinuity. At this point some part of the wave is reflected whereas some part is transmitted. It is easy to see that a fixed amount of the energy (depending on the size of the jump of the velocities) is transmitted. Then the transmitted wave propagate freely until it reaches the second discontinuity, and so on and so forth... Finally, we get that a fixed part of the energy of the incoming wave is transmitted at the other end and propagates freely to plus infinity. Whereas some part of the energy can remained trapped by multiple reflections, this shows that some part is not trapped. As a consequence, our geometry is (at least weakly) non trapping. This phenomenon is clearly specific to the one dimensional case as can be easily seen (using Snell law of refraction) with simple models involving only two speeds.

The structure of our paper is as follows:

- In section 1 we prove a smoothing estimate which is the key to all subsequent results, by an elementary integration by parts argument, reminiscent of the time-space symmetry for the 1D wave equation. Transferring results from the wave to Schrödinger is sometimes called a transmutation and has been used in different contexts ($19$).

- We then obtain Strichartz and maximal function estimates in section 2 by combining our smoothing estimate with known estimates for the flat case.

- Finally, we provide an application in section 3 obtaining sharp wellposedness for a generalized Benjamin-Ono equation. Further applications to the Benjamin-Ono hierarchy of equations,
including the true Benjamin-Ono, will be addressed elsewhere ([6]). The methods developed in this paper are likely to apply to other 1D dispersive models and quasilinear equations.

- The first appendix is a short recollection of some results of Auscher-Tchamitchian [2] and Auscher-MacIntosh-Tchamitchian [1] which imply that the spectral localization with respect to the operators \( \partial_x a(x) \partial_x \) and \( \partial_x^2 \) are reasonably equivalent.

- In a second appendix we give a self-contained proof of a suitably modified version of Christ-Kiselev Lemma (see [8]).

- In a third appendix we prove that the BV regularity threshold is optimal in a different direction from [7]: there exist a metric \( a(x) \) which is in \( L_\infty \cap W^{s,1} \) for any \( 0 \leq s < 1 \), bounded from below by \( c > 0 \), and such that no smoothing effect nor (non trivial) Strichartz estimates are true (even with derivatives loss). This construction is very close in spirit to the one by Castro-Zuazua [7].

Besov spaces will be a convenient tool to state and prove many of our results; we end this introduction by recalling their definition via frequency localization ([4] for details).

**Definition 1** Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \) such that \( \hat{\phi} = 1 \) for \( |\xi| \leq 1 \) and \( \hat{\phi} = 0 \) for \( |\xi| > 2 \), \( \phi_j(x) = 2^{nj} \phi(2^j x) \), \( S_j = \phi_j \ast \cdot \), \( \Delta_j = S_{j+1} - S_j \). Let \( f \in \mathcal{S}'(\mathbb{R}^n) \). We say \( f \) belongs to \( \dot{B}^{s,q}_{p} \) if and only if

- The partial sum \( \sum_{m=-\infty}^{\infty} \Delta_j(f) \) converges to \( f \) as a tempered distribution (modulo polynomials if \( s > n/p \) and \( q > 1 \)).

- The sequence \( \varepsilon_j = 2^{js} \|\Delta_j(f)\|_{L^p_x} \) belongs to \( l^q \).

A suitable modification will be of interest, to handle the additional time variable.

**Definition 2** Let \( u(x,t) \in \mathcal{S}'(\mathbb{R}^{n+1}) \), \( \Delta_j \) be a frequency localization with respect to the \( x \) variable. We will say that \( u \in \dot{B}^{s,q}_{p}(\mathcal{L}_t^p) \) iff

\[
2^{js} \|\Delta_j u\|_{L^p_x(L^p_t)} = \varepsilon_j \in l^q,
\]

and other requirements are the same as in the previous definition.

Notice that whenever \( q = \rho \), the Besov space \( \dot{B}^{s,q}_{p}(\mathcal{L}_t^\rho) \) is nothing but the usual “Banach valued” Besov space \( \dot{B}^{s,q}_{p}(F) \) with \( F = L_t^\rho \).

Finally, through this article we will denote by \( S(t) = e^{it\partial_x^2} \) and \( S_a(t) = e^{it\partial_x a(x) \partial_x} \) the (1D) group-evolution associated to the constant and variable coefficients equations respectively.

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1 Local smoothing

For the (flat) Schrödinger equation on the real line, we have the following estimate:
\[ \| \partial_x S(t) \phi_0 \|_{L^\infty_x L^2_t} \simeq \| \phi_0 \|_{H^{1/2}}. \]

It can be proved directly using the Fourier transform (see [16]). With this in mind, one can of course write a similar estimate for the 1D wave equation, which is also a trivial consequence of the explicit representation as a sum of traveling waves; however, one can prove it as well by integration by parts on the inhomogeneous equation, exchanging \( t \) and \( x \) which play equivalent roles. This last procedure is flexible enough to allow variable coefficients and will lead to our first result. We start by stating once and for all our hypothesis on the coefficient \( a \).

**Definition 3** We call \( a \) an \( m \)-admissible coefficient when the following requirements are met:
- the function \( a \) is real-valued, belongs to \( BV \), namely
  \[ \partial_x a \in \mathcal{M} = \{ \mu \text{ s.t. } \int_R d|\mu| < +\infty \}, \]
- the function \( a \) is bounded from below almost everywhere by \( m \).

We will denote by \( M \) its maximum and \( \| a \|_{BV} \) its bounded variation (\( a(x) \leq M \leq \| a \|_{BV} \)).

After this preliminary definition, we can state the main theorem.

**Theorem 1** Let \( m > 0 \) and \( a \) be an \( m \)-admissible coefficient. There exist \( C(\| a \|_{BV}, m) > 0 \) such that
- If \( u, f \) are solutions of
  \[ (i\partial_t + \partial_x a(x)\partial_x)u = f, \]
  with zero Cauchy data then
  \[ \| \partial_x u \|_{L^\infty_x L^2_t} + \| (-\partial_t^2)^{1/4} u \|_{L^\infty_x L^2_t} \leq C \| f \|_{L^1}_t L^2_t. \]
- If
  \[ (i\partial_t + \partial_x a(x)\partial_x)u = 0, \text{ with } u|_{t=0} = u_0 \in L^2 \]
  then
  \[ \| u \|_{B^{\frac{1}{2}, 2}_t (L^2)} \leq C \| u_0 \|_{L^2}. \]
Remark 1 One may wonder why we chose to consider $\partial_x a(x)\partial_x$ as opposed to, say, $g(x)\partial_x^2$. It turns out that one may obtain one from another through an easy change of variable, and we elected to keep the divergence form as the most convenient for integration by parts. The astute reader will check that $b(x)\partial_x a(x)\partial_x$ can be dealt with as well, and the additional requirement will be for $b$ to be $m$-admissible. Remark also that our method can handle first order terms of the kind $b(x)\partial_x$ with $b \in L^1$ (see section 3).

Proof: In order to obtain (4), we will reduce ourselves to a situation akin to a wave equation and perform an integration by parts. Obtaining (5) from (4) is then a simple interpolation and $TT^*$ argument. We first reduce the study to smooth $a$.

Proposition 1 Denote by $A = \partial_x a(x)\partial_x$. Assume that the evolution semi-group $S_a(t)$ satisfies for any smooth ($C^\infty$) $m$-admissible $a$:

$$\forall u_0 \in L^2, \quad \|S_a(t)u_0\|_B \leq C\|u_0\|_{L^2}$$

with $B$ a Banach space (weakly) continuously embedded in $D'(\mathbb{R}^2)$, whose unit ball is weakly compact, and $C$ a constant depending only on $m$ and $\|\partial_x a\|_{L^1}$. Then the same result holds (with the same constant) for any $m$-admissible $a$.

Proof: Let us consider $\rho \in C_0^\infty(\mathbb{R})$ a non-negative function such that $\int \rho = 1$, and $\rho_\frac{1}{\varepsilon} = \varepsilon^{-1}\rho(x/\varepsilon)$. Denote by $a_\varepsilon = \rho_\varepsilon * a$ and $A_\varepsilon = -\partial_x a_\varepsilon(x)\partial_x$. The sequence $a_\varepsilon$ is bounded in $W^{1,1}$. Furthermore, $a_\varepsilon$ converges to $a$ for the $L^\infty$ weak $*$ topology. According to the weak compactness of the unit ball of $B$, taking a subsequence, we can assume that $S_{a_\varepsilon}(t)u_0$ converges weakly to a limit $v$ in $B$ (and consequently in $D'(\mathbb{R}^2)$). To conclude, it is enough to show that $v = S_a(t)u_0$ in $D'(\mathbb{R}^2)$. We first remark that as a (multiplication) operator on $L^2$, $a_\varepsilon$ converges strongly to $a$ (but of course not in operator norm) and consequently $\partial_x a_\varepsilon(x)\partial_x$ converges strongly to $\partial_x a(x)\partial_x$ as operators from $H^1$ to $H^{-1}$. On the other hand the bound $0 < m \leq a(x) \leq M$ and the fact that $\rho$ is non-negative imply that $a_\varepsilon$ satisfy the same bound and consequently that the family $(A_\varepsilon + i)^{-1}$ is bounded from $H^{-1}$ to $H^1$ by $1/m$. From the resolvent formula

$$(A_\varepsilon + i)^{-1} - (A + i)^{-1} = (A_\varepsilon + i)^{-1}(A - A_\varepsilon)(A + i)^{-1},$$

given $(A_\varepsilon + i)^{-1}$ is uniformly bounded from $H^{-1}$ to $H^1$, we obtain that $(A_\varepsilon + i)^{-1}$ converges strongly to $(A + i)^{-1}$ as an operator from $H^{-1}$ to $H^1$, and consequently as an operator on $L^2$. This convergence implies (see [23 Vol I, Theorem VIII.9]) that $A_\varepsilon$ converges to $A$ in the strong resolvent sense and (see [23 Vol I, Theorem VIII.21]) that for any $t \in \mathbb{R}$, $S_{a_\varepsilon}(t)u_0$ converges strongly to $S_a(t)u_0$. Finally, from the boundedness of $S_{a_\varepsilon}(t)u_0$ in $L^1_t(L^2_x)$, we deduce by dominated convergence that $S_{a_\varepsilon}(t)u_0$ converges to $S_a(t)u_0$ in $L^1_{t,loc}(L^2_x)$ and hence in $D'$. Similarly, we can handle non-homogeneous estimates. \qed

Remark 2 Alternatively, we can perform our argument for $a$ a step function with finite $BV$ norm. We will briefly sketch this at the end of this section.
We are now considering the following equation (for \( a \in C_0^\infty \)):

\[
-\sigma v + \partial_x (a(x) \partial_x v) = g.
\]

where \( v, g \) will be chosen later to be the time Fourier transform of \( u, f \).

**Proposition 2** There exist \( C(m, \| a \|_{BV}) \) such that for any \( \sigma = \tau + i\varepsilon, \varepsilon \neq 0 \) the resolvent \((-\sigma + \partial_x a(x) \partial_x)^{-1}\), which is a well defined operator from \( L^1 \subset H^{-1} \) to \( H^1 \subset L^\infty \) and from \( L^2 \) to \( H^2 \) satisfies

\[
\| (-\sigma + \partial_x a(x) \partial_x)^{-1}\|_{L^1 \rightarrow L^\infty} \leq C.
\]

It should be noticed that since this and all further estimates are scale invariant (including the constants which are dependent on scale invariant quantities of \( a \)), we could reduce the study to the case \( \tau = \pm 1 \) by changing \( a(x) \) into \( a(\sqrt{\pm \tau} - 1 x) \). We elected to keep \( \tau \) through the argument as it helps doing book keeping.

**Remark 3** The elliptic case (\( \tau > 0 \)) is more or less understood and as a corollary, the associated heat equation as well. In fact these results apply to a larger class of \( a \) than the one we consider here: \( a \in L^\infty, \Re a > 0 \). More specifically, the heat kernel (and its derivatives) associated to the operator \( A = -\partial_x (a(x) \partial_x) \) is known to be of Gaussian type, a fact which will be of help to handle derivatives. A very nice and thorough presentation of this (and a lot more !) can be found in [1]. We refer to Appendix A for a short recollection of the facts we will need later.

In the sequel we will perform integrations by parts. We can assume \( g \in L^2 \). Consequently \( v \in H^2 \) and these integrations by parts are licit (in particular, the boundary terms near \( \pm \infty \) vanish). We first multiply (6) by \( \tau \), integrate by parts and take the real part and imaginary parts. This yields

\[
|\varepsilon| \int_R |v|^2 \leq \| g \|_{L^1} \| v \|_{L^\infty}
\]

\[
|\varepsilon| \int_R a(x)|\partial_x v|^2 \leq |\varepsilon| |\tau| \int_R |v|^2 + |\varepsilon| \| g \|_{L^1} \| v \|_{L^\infty} \leq (|\varepsilon| + |\tau|) \| g \|_{L^1} \| v \|_{L^\infty}.
\]

We now proceed in the hyperbolic region \(-\tau > 0\). Multiplying (6) by \( a(x) \partial_x \overline{v} \) and integrating, we get

\[
\int_{-\infty}^{x} -\sigma a v \partial_x (\overline{v}) + \int_{-\infty}^{x} \partial_x (a \partial_x v) a \partial_x \overline{v} = \int_{-\infty}^{x} ga \partial_x \overline{v}.
\]

Integration by parts and taking the real part yields

\[
-\tau a|v|^2(x) + |a \partial_x v|^2(x) + 2 \int_{-\infty}^{x} \tau (\partial_x a)|v|^2 \leq 2|\varepsilon| \int_R |v||\partial_x v| + 2 \| g \|_{L^1} \| a \partial_x v \|_{L^\infty}.
\]
We now use (8) to estimate the right hand side in (10) and obtain
\[- \tau a|v|^2(x) + |a \partial_x v|^2(x) + 2 \int_{-\infty}^{x} \tau (\partial_x a)|v|^2 \]
\[\leq 2 \max(1, \|a\|_{L^1}^2) \|g\|_{L^1} \left( \|a \partial_x v\|_{L^\infty} + (|\varepsilon| + |\tau|)^{1/2} \|v\|_{L^\infty} \right).\]

On the other hand we are in 1D and,
\[\|v\|_{L^\infty}^2 \leq 2 \|v\|_{L^2} \|\partial_x v\|_{L^2}\]
which implies, using (8),
\[\varepsilon \|v\|_{L^\infty}^2 \leq 2m^{-\frac{1}{2}} \|g\|_{L^1} \sqrt{|\varepsilon| + |\tau|} \|v\|_{L^\infty}\]
Consequently we get
\[(|\varepsilon| + |\tau|)a|v|^2(x) + |a \partial_x v|^2(x) + 2 \int_{-\infty}^{x} \tau (\partial_x a)|v|^2 \]
\[\leq C(m, \|a\|_{BV}) \|g\|_{L^1} \left( \|a \partial_x v\|_{L^\infty} + (|\varepsilon| + |\tau|)^{1/2} \|v\|_{L^\infty} \right).\]
Setting
\[\Omega_+(x) = \sup_{y<x} (|\varepsilon| + |\tau|)a(y)|v|^2(y) + |a(y) \partial_x v|^2(y)\]
\[k(x) = a(x)^{-1} |\partial_x a|,\]
we have
\[\Omega_+(x) \leq C(m, \|a\|_{BV}) \sqrt{\Omega_+(+\infty)} \|g\|_{L^1} + 2 \int_{-\infty}^{x} k(y) \Omega_+(y) dy.\]
Given that $\Omega_+$ is positive, we obtain by Gronwall inequality
\[\int_{-\infty}^{x} k(y) \Omega_+(y) dy \leq C(m, \|a\|_{BV}) \left( \int_{-\infty}^{x} e^{\int_y^x 2k(z)dz} k(y) dy \right) \|g\|_{L^1} \sqrt{\Omega_+(+\infty)}\]
\[\leq 2C(m, \|a\|_{BV}) e^{\int_{-\infty}^{x} 2k(y)dy} \|g\|_{L^1} \sqrt{\Omega_+(+\infty)}\]
and consequently, coming back to (12)
\[\sqrt{\Omega_+(+\infty)} \leq C(m, \|a\|_{BV}) \|g\|_{L^1} (2 + 8e^{2|k(x)|_{L^1}}).\]
Now we proceed with the elliptic region $\tau > 0$, for which the above line of reasoning fails. We perform the usual elliptic regularity estimate and multiply the equation by $\overline{v}$, to obtain
\[\int_{\mathbb{R}} \tau |v|^2 + a|\partial_x v|^2 = -\text{Re} \int_{\mathbb{R}} g \overline{v}, \quad \varepsilon \int_{\mathbb{R}} |v|^2 = -\text{Im} \int_{\mathbb{R}} g \overline{v}\]
which gives

\[ (14) \quad \int_{\mathbb{R}} (|\tau| + |\varepsilon|)|v|^2 + a|\partial_x v|^2 \leq 2\|g\|_{L^1}\|v\|_{L^\infty}. \]

In order to conclude, we go back to the (beginning of) the estimate we made in the hyperbolic case, i.e. (9) and integrate by parts only the second term in the left hand side,

\[ |a\partial_x v|^2(x) \leq 2\int_{-\infty}^{\infty} |g|a|\partial_x v| + 2\int_{-\infty}^{x} |\sigma|a|v||\partial_x v| \]

and to bound the last term we use (14),

\[ (15) \quad \|a\partial_x v\|_{L^\infty}^2 \leq \|g\|_{L^1}(2\|a\partial_x v\|_{L^\infty} + 4|\tau|^{1/2}\|v\|_{L^\infty}). \]

Adding $\tau a|v|^2$ to (15) and using (14), (11), we obtain

\[
\Omega_-(x) = \sup_{y \leq x}(|\varepsilon| + |\tau|)|a|v|^2(y) + |a\partial_y v|^2(y) \\
\leq 2M|\tau||v|_{L^2}\|\partial_x v\|_{L^2} + 4(|\varepsilon| + |\tau|)^{1/2}\|g\|_{L^1}\|v\|_{L^\infty} \\
\leq \|g\|_{L^1}(2M + 4)|||\varepsilon| + |\tau||^{1/2}\|v\|_{L^\infty}
\]

which gives again

\[ (16) \quad \sup_x \Omega_-(x) \leq \frac{(\|a\|_{L^\infty} + 4)^2}{m}\|g\|_{L^1}^2. \]

This ends the proof of Proposition 2 \hfill \square

REMARK 4 Notice that for this elliptic estimate, we only used $a \in L^\infty$ and nothing else.

We now come back to the proof of Theorem 1. Consider $u$, $f$ solutions of (3). We can assume that $f$ (and consequently $u$) is supported in $t > 0$ (the contribution of negative $t$ being treated similarly). Then for any $\varepsilon > 0 u_\varepsilon = e^{-\varepsilon t}u$ is solution of

\[ (i\partial_t + i\varepsilon + \partial_x a(x)\partial_x)u_\varepsilon = f, \]

Assuming that $f$ has compact support (in time), we can consider the Fourier transforms with respect to $t$ of $f$ and $u_\varepsilon$, $g(\tau)$ and $v_\varepsilon(\tau)$ which satisfy

\[ (-\tau + i\varepsilon + \partial_x a(x)\partial_x)v_\varepsilon = g. \]

We may now apply Proposition 2 take $L^2_t$ norms, switch norms and revert back to time by Plancherel, and get

\[
\|\partial_x u_\varepsilon\|_{L^\infty_t(L^2_x)} + \|(-\partial^2_t)^{1/4}u_\varepsilon\|_{L^\infty_t(L^2_x)} = \|\partial_x v_\varepsilon\|_{L^\infty_t(L^2_x)} + \|(-\partial^2_t)^{1/4}v_\varepsilon\|_{L^\infty_t(L^2_x)} \\
\leq \|\partial_x v_\varepsilon\|_{L^2_x(L^\infty_t)} + \|(-\partial^2_t)^{1/4}v_\varepsilon\|_{L^2_x(L^\infty_t)} \\
\leq C\|g_\varepsilon\|_{L^1_t(L^2_x)} \leq C\|g_\varepsilon\|_{L^1_t(L^2_x)} = C\|f_\varepsilon\|_{L^1_t(L^2_x)},
\]
where \( C = C(m, \| \partial_x a \|_{L^1}) \) is uniform with respect to \( \varepsilon > 0 \). Letting \( \varepsilon > 0 \) tend to 0, we obtain the same estimate for \( u \), which is exactly (4) in Theorem (up to replacement of \( BV \) by \( \dot{W}^{1,1} \), which was dealt with in Proposition (1)). Finally we easily drop the compact in time assumption for \( f \) by a density argument.

We are left with proving the homogeneous estimate (5). As usual, estimates on the homogeneous problem follow from the estimate with a fractional time derivative: by a \( TT^* \) argument, and using the commutation between time derivatives and the flow, we get

\[
\| (-\partial_t^2) \frac{1}{2} u \|_{L^{p_1}(L^q_1)} \lesssim \sqrt{C} \| u_0 \|_{L^2}.
\]

Then, using the equation, \( i\partial_t u = Au \) where \( A = -\partial_x a(x) \partial_x \), we can replace \( (i\partial_t)^{1/4} \) by \( A^{1/4} \). However, we will need real derivatives later, rather than powers of \( A \). We postpone the issue of equivalence between the two and take another road: notice that we obtained (7) for solutions of (6)

\[
\| \partial_x v \|_{L^{\infty}_x} \lesssim \| g \|_{L^1_x},
\]

which immediately implies

\[(17) \quad \| v \|_{\dot{B}^{1/2}_{\infty}} \lesssim \| g \|_{\dot{B}^{1}_{1}}.
\]

Call \( R_\sigma = (\partial_x a(x) \partial_x - \sigma)^{-1} \). Its adjoint is \( R_\tau \) and according to Proposition (applied to \( \sigma = \tau - i\varepsilon \)), we get

\[(18) \quad \| v \|_{\dot{B}^{1/2}_{\infty}} \lesssim \| g \|_{\dot{B}^{-1/2}_{1}}.
\]

By real interpolation (recall \( \left( \dot{B}^{s_1,q_1}_p, \dot{B}^{s_2,q_2}_p \right)_{\theta,r} = \dot{B}^{s,r}_p \)), we obtain (with \( \theta = 1/2, r = 2 \))

\[
\| v \|_{\dot{B}^{1/2}_{\infty}} \lesssim \| g \|_{\dot{B}^{1}_{2}}.
\]

Given that the third index is 2, we can again take \( L^2_t \) norms, switch them (Minkowski) and by Plancherel (and letting \( \varepsilon \) tend to 0), we get the desired estimate:

\[
\| u \|_{\dot{B}^{1/2}_{2}(L^2_1)} \lesssim \| f \|_{\dot{B}^{1/2}_{2}(L^2_1)},
\]

denote by \( S_a(t) \) the evolution group for the homogeneous equation, we have

\[
u = \int_{s<t} S_a(t-s)f(s)ds,
\]
solution of the inhomogeneous problem, and we can as well treat the \( s > t \) case. Hence we have obtained

\[
\| \int S_a(t-s)f(s)ds \|_{\dot{B}^{1/2}_{2}(L^2_1)} \lesssim \| f \|_{\dot{B}^{-1/2}_{2}(L^2_1)}.
\]
The usual $TT^*$ argument applies and gives
\[
\|S_\alpha(t)u_0\|_{B_{2q}^{\infty}(L^2)} \lesssim \|u_0\|_{L^2}.
\]
This ends the proof of Theorem 1. \hfill \Box

We now provide an alternative argument which directly proves the resolvent estimate for $a$ a step function, bounded from below and with bounded variation. For the sake of conciseness, we take directly $\sigma = \tau \in \mathbb{R}$ and will not justify the validity of the integration by parts (and in particular the vanishing of the boundary terms at $\pm \infty$). As before, the justification consists in taking $\sigma = \tau + i\varepsilon$ and passing to the limit $\varepsilon \to 0$. We set, $m = 1$ for simplicity, and rescale to obtain $\tau = \pm 1$. Starting from (5), with $\tau = +1$ (the difficult case) and denoting
\[
a(x) = \sum_i a_i \chi_{[x_i,x_{i+1}]}(x), \quad \text{with} \quad \sum_i |a_i - a_{i-1}| < +\infty,
\]
we have, with $x \in [x_I, x_{I+1}[$
\[
\sum_{i < I} \int_{x_i}^{x_{i+1}} a_i v \partial_x \bar{v} + \int_{x_I}^{x} a_I v \partial_x \bar{v} + a_I^2 |\partial_x v|^2(x) = \int_{-\infty}^{x} ga \partial_x \bar{v}.
\]
Integrating by parts,
\[
\sum_{i < I} a_i (|v|^2(x_{i+1}) - |v|^2(x_i)) + a_I (|v|^2(x) - |v|^2(x_I)) + a_I^2 |\partial_x v|^2(x) \leq 2\|g\|_1 \sup_{y \leq x} a(y) |\partial_x v(y)|.
\]
\[
\sum_{i-1 < I} (a_{i-1} - a_i) |v|^2(x_i) + |v|^2(x) + a^2(x) |\partial_x v|^2(x) \leq 2\|g\|_1 \sup_{y \leq x} a(y) |\partial_x v(y)|.
\]
We rewrite this as
\begin{equation}
(19) \quad \sup_{y \leq x} (2|v|^2(y) + |\partial_x v|^2(y)) \leq 4\|g\|_1^2 + \sum_{i \leq I} |a_{i-1} - a_i||v|^2(x_i)
\end{equation}
at which point one may simply replace the left hand side (noting that $x < x_{I+1}$) by its weaker discrete counterpart
\[
\sup_{i \leq I+1} (2|v|^2(x_i) + |\partial_x v|^2(x_i)) \leq 4\|g\|_1^2 + \sum_{i \leq I} |a_{i-1} - a_i||v|^2(x_i)
\]
which becomes a discrete analog of Gronwall: call $\gamma_i = \sup_{j \leq i} (2|v|^2(x_j) + |\partial_x v|^2(x_j))$ and $\alpha_i = |a_{i-1} - a_i|$, we have
\[
\gamma_{I+1} \leq C + \sum_{i \leq I} \alpha_i \gamma_i.
\]
Therefore,
\[
\sum_{i \leq I} \frac{\alpha_{i+1}\gamma_{i+1}}{C + \sum_{j \leq i} \alpha_j \gamma_j} \leq \sum_{i \leq I+1} \alpha_i < +\infty,
\]
Call \( S_I = \sum_{i \leq I} \alpha_i \gamma_i \),

\[
\sum_{i \leq I} \int_{S_i}^{S_{i+1}} \frac{dx}{1 + x} \leq \sum_{i \leq I} \frac{S_{i+1} - S_i}{1 + S_i} \leq \sum_i \alpha_i,
\]

yielding

\[
S_I \lesssim \exp(\sum_i \alpha_i),
\]

which is nothing but the desired bound: recall (19) and notice we just bounded the right hand side. \( \Box \)

Notice that up to this point we avoided to use any of the machinery presented in Appendix A, thus keeping the proof self-contained. However, a rather natural question is now how one can handle (fractional) derivatives: i.e., replace \( u_0 \in L^2 \) by \( u_0 \in \dot{H}^s \). In order to deal with commutation, we will rely in a very natural way on Appendix A.

**Proposition 3** Assuming \( a \) is \( m \)-admissible, we have:

- if \( u, f \) are solutions of
  
  \[
  (i\partial_t + \partial_x a(x) \partial_x)u = f,
  \]
  
  then, for \( 0 < s < 1 \),

  \[
  \|u\|_{\dot{B}^{s,2}_\infty(L^2_t)} \lesssim \|f\|_{\dot{B}^{s-1,2}_1(L^2_t)}.
  \]

- if \( -1 < s < \frac{1}{2} \) and
  
  \[
  (i\partial_t + \partial_x a(x) \partial_x)u = 0, \text{ with } u|_{t=0} = u_0
  \]
  
  then

  \[
  \|u\|_{\dot{B}^{s+\frac{1}{2},2}_\infty(L^2_t)} \lesssim \|u_0\|_{\dot{H}^s}.
  \]

**Proof:** Recall that by interpolation between (17) and (18) we have

\[
\|v\|_{\dot{B}^{s,2}_\infty(L^2_t)} \lesssim \|g\|_{\dot{B}^{s-1,2}_1(L^2_t)},
\]

for all \( 0 < s < 1 \), which immediately gives (20). For the homogeneous problem, we simply rely on the equivalence properties stated in Appendix A.2, we apply (5) to \( \Delta_j^A u_0 \), a datum localized with respect to \( A \) (see the Appendix for a definition) and use commutation between \( \Delta_j^A \) and \( S_a(t) \) to obtain

\[
\|\Delta_j^A u\|_{\dot{B}^{s+\frac{1}{2},2}_\infty(L^2_t)} \lesssim \|\Delta_j^A u_0\|_{L^2_x}.
\]

Equivalence between \( \dot{B}^{s,2}_\infty(L^2_t) \) and \( \dot{B}^{s+\frac{1}{2},2}_{\infty,A}(L^2_t) \) yields

\[
2^{\frac{js}{2}} \|\Delta_j^A u\|_{L^\infty_x L^2_t} \lesssim \|\Delta_j^A u_0\|_{L^2_x},
\]

for which multiplying by \( 2^{js} \) and summing over \( j \) provides the desired result, after switching back from \( A \) based Besov spaces to the usual ones. Hence \( s > -1 \) from the right hand side, and \( s+1/2 < 1 \) from the left hand side. \( \Box \)
2 Strichartz and maximal function estimates

We now prove Strichartz and maximal function estimates by combining the smoothing effect from the previous section with a change of variable and corresponding estimates for the flat Schrödinger equation.

**Theorem 2** Let $a$ be an $m$-admissible coefficient. Let $u$ be a solution of (1) with $u_0 \in L^2$. Then for $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, $p \geq 4$, we have

\[ \|S_a(t)u_0\|_{L^p_t(\dot{B}^{0,q}_q)} \lesssim \|u_0\|_{L^2}. \]

When $p > 4$ ($q < +\infty$),

\[ \|S_a(t)u_0\|_{L^p_t(L^q_x)} \lesssim \|S_a(t)u_0\|_{L^p_t(\dot{B}^{0,q}_q)} \lesssim \|u_0\|_{L^2}. \]

**Remark 5** Notice that the end-point $(4, \infty)$ is missing. This can be seen as an artifact of the proof. It will be clear that in this section, we only use $a \in L^\infty \cap \dot{B}^{1,\infty}_1$ and bounded from below (together with the estimates of Theorem 1). Adding a technical hypothesis like $a \in \dot{B}^{1,2}_1$ (which does not follow from $a \in BV$) would allow to recover the end-point, at the expense of extra technicalities which we elected to keep out (see [6] for further developments).

One may state a corollary including fractional derivatives as well.

**Proposition 4** Let $u$ be a solution of (1), and $u_0 \in \dot{H}^s$, $|s| < 1$. Then for $\frac{2}{p} + \frac{1}{q} = \frac{1}{2}$, $p \geq 4$, we have

\[ \|S_a(t)u_0\|_{L^p_t(\dot{B}^{s,q}_q)} \lesssim \|u_0\|_{\dot{H}^s}. \]

Similarly, we also obtain maximal function estimates.

**Theorem 3** Let $u$ be a solution of (1), and $u_0 \in \dot{H}^s$, $-3/4 < s < 1$. Then

\[ \|S_a(t)u_0\|_{\dot{B}^{s+\frac{1}{4},q}(C^\infty_x)} \lesssim \|u_0\|_{\dot{H}^s}. \]

**Proof:** We aim at taking advantage of an appropriate new formulation for our original problem and proving Theorems 2 and 3 at once. The operator $\partial_x a \partial_x$ may be rewritten as $(\sqrt{a} \partial_x)^2 + (\partial_x \sqrt{a}) \partial_x$, and one would like to “flatter out” the higher order term through a change of variable. However, performing directly a change of variable leads to problems when dealing with the newly appeared first order term. Therefore, we have to paralinearize the equation. Let us rewrite $a$:

\[ a = \frac{m}{2} + b^2, \text{ with } \partial_x b \in L^1_x, \]

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given that \( a \) is \( m \)-admissible. Writing
\[
b^2 \partial_x u = \sum_k (S_{k-3}b)^2 \partial_x S_k u - (S_{k-4}b)^2 \partial_x S_{k-1} u
\]
\[
= \sum_k (S_{k-3}b)^2 \partial_x \Delta_k u + \sum_k \Delta_{k-3}b(S_{k-3} + S_{k-4}) b \partial_x S_{k-1} u
\]
and applying \( \Delta_j \) to the equation,
\[
i \partial_t \Delta_j u + \frac{m}{2} \partial_x^2 \Delta_j u + \Delta_j \partial_x \sum_{k \sim j} ((S_{k-3}b)^2 \partial_x \Delta_k u) + \Delta_j \partial_x \sum_{j \leq k \sim l} (\Delta_{k-3}b S_{l-3}b \partial_x S_{k-1} u) = 0.
\]
From now on, we ignore shifts in indices for the last term as they won’t play any role. Thus we get
\[
i \partial_t \Delta_j u + \left( \frac{m}{2} + (S_{j-3}b)^2 \right) \partial_x \left( \left( \frac{m}{2} + (S_{j-3}b)^2 \right) \partial_x \Delta_j u \right) = R_j,
\]
and, with \( \tilde{\Delta}_j \) an enlargement of the localization,
\[
R_j = -\Delta_j \partial_x \sum_{j \leq k} \Delta_k b S_k b \partial_x S_k u - \tilde{\Delta}_j \partial_x \sum_{k \sim j} [\Delta_j, (S_{k-3}b)^2] \partial_x \Delta_k u
\]
\[
- S_{j-3}b (\partial_x S_{j-3}b) (\partial_x \Delta_j u).
\]
Assuming the smoothing effect from Theorem 1 we can effectively estimate the reminder.

**Proposition 5** Assume that the hypothesis of Theorem 1 hold: then
\[
\sum_j R_j \in \dot{B}_{1,2}^{\frac{1}{2}}(L^2_t).
\]

**Proof:** Let us do the first term: relabeling \( k = j \) for simplicity,
\[
2^{-j} \partial_x S_j u \in \dot{L}^2 L^\infty L^2_t, \quad S_j b \in \dot{L}^\infty L^\infty_x, \quad \text{and } 2^j \Delta_j b \in \dot{L}^1 L^1_x \text{ (recall that } \dot{W}^1_1 \hookrightarrow \dot{B}_{1,\infty}^{\frac{1}{2}}). \]
Before applying the remaining \( \partial_x \), we have a summand \( P_j \) which is such that \( 2^j P_j \in \dot{L}^1 L^1_t \) and frequency localized in a ball of size \( 2^j \), hence \( \sum_j P_j \in \dot{B}_{1,2}^{\frac{1}{2}}(L^2_t) \); the result follows by derivation. The commutator term is essentially the same, thanks to the following lemma.

**Lemma 1** Let \( g(x,t) \) be such that \( \| \partial_x g \|_{L^p_1(L^q_\infty)} < +\infty \), and \( f(x,t) \in L^p t(\dot{L}^q t) \), with \( \frac{1}{p_1} + \frac{1}{p_\infty} = 1 \) and \( \frac{1}{q_\infty} + \frac{1}{q_2} = \frac{1}{2} \), then \( h(x,t) = [\Delta_j, g] f \) is in \( L^1_1(\dot{L}^2_t) \).
Proof: We first take $p_1 = 1, p_\infty = \infty$: set $h(x) = [\Delta_j, g]f$, recall $\Delta_j$ is a convolution by $2^j \phi(2^j \cdot)$, and denote $\psi(z) = z|\phi|(z)$:

$$h(x) = \int_y 2^j \phi(2^j (x - y))(g(y) - g(x))f(y)dy$$

$$= \int_{y, \theta \in [0, 1]} 2^j \phi(2^j (x - y))(x - y)g'(x + \theta(y - x))f(y)d\theta dy$$

$$|h(x)| \leq 2^{-j} \int_{y, \theta \in [0, 1]} 2^j \psi(2^j (x - y))|g'(x + \theta(y - x))| |f(y)|d\theta dy$$

and then take successively time norms and space norms,

$$\|h(x, t)\|_{L^t_t} \leq 2^{-j} \int_{y, \theta \in [0, 1]} 2^j \psi(2^j (x - y))\|g'(x + \theta(y - x, t))\|_{L^\infty_t} \|f(y, t)\|_{L_1^\theta} d\theta dy$$

$$\int_x |h(x)| |f|_{L_1^\infty(L_2^2)} \int_{y, \theta \in [0, 1], y, x} 2^j \psi(2^j (x - y))\|g'(x + \theta(y - x))\|_{L^\infty_1} dx dy d\theta$$

$$\leq 2^{-j} \|f\|_{L_1^\infty(L_2^2)} \int_{y, \theta \in [0, 1], y, x} 2^j \psi(2^j (x - y))\|g'(x + \theta(y - x))\|_{L^\infty_1} dx dy d\theta$$

$$\leq 2^{-j} \|f\|_{L_1^\infty(L_2^2)} \int_{z} 2^j \psi(2^j (x - y))\|g'(x + \theta(y - x))\|_{L_1^\theta(L_2^2)}.$$}

The case $p_1 = \infty, p_\infty = 1$ is identical, exchanging $f$ and $g'$ (in fact, this would be the usual commutator estimate). The general case then follows by bilinear complex interpolation. □

Thus, the lemma allows us to effectively proceed with the second term in $R_j$ as if the derivative on $\Delta_k u$ was in fact on an $S_{k-\beta} b$ factor, and then it becomes a term “like”

$$\partial_x \sum_{k \sim j} S_{k} b \partial_x S_{k} b \Delta_k u,$$

for which the computation done with the first term holds as well. We are left with the third term: this is nothing but a paraproduct which is easily estimated: $\partial_x S_{j-\beta} b \in L_x^1$ and $2^{-\beta} \partial_x \Delta_j u \in L_x^\infty L_\theta^2$. This completes the proof of Proposition 5. □

After the paralinearization step, we perform a change of variable. We have, denoting by $\omega = \sqrt{\frac{\beta}{2}} + (S_{j-\beta} b)^2$, and $u_j = \Delta_j u$,

$$i \partial_t u_j + \omega(x) \partial_x (\omega(x) \partial_x u_j) = R_j.$$}

Now we set $x = \phi(y)$ through $\partial_y = \omega(x) \partial_x$, in other words

$$\omega(x) = \frac{dx}{dy}, \quad y = \int_0^x \omega(\rho) d\rho = \phi^{-1}(x),$$

which is a $C^1$ diffeomorphism (uniformly with respect to $j$): $\omega$ is bounded in the range $[\frac{m}{2}, 2M]$. Denote by $v_j(y) = u_j \circ \phi(y)$ and $T_j(y) = R_j \circ \phi(y)$,

$$i \partial_t v_j + \partial_y^2 v_j = T_j(y).$$

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Given that our change of variable leaves $L^p$ spaces invariant, from Proposition 5 we have that

$$T_j \in L_y^1 L_t^2, \text{ with } \|T_j\|_{L_y^1 L_t^2} \lesssim 2^j \mu_j, \quad (\mu_j)_j \in l^2.$$  

By using Duhamel,

$$v_j = S(t)v_j(0) + \int_0^t S(t-s)T_j(y,s)ds$$

for which we can apply Christ-Kiselev Lemma; first, let us obtain Strichartz estimates: according to (26), (27) and Theorem 6, we obtain

$$\|v_j\|_{L_y^2 B^{1/2,2}_\infty} \lesssim \|v_j(0)\|_{H^{1/2}} + 2^j \mu_j.$$  

Now we would like to go back to $u_j$ from $v_j$. While frequency localizations wrt $x$ and $y$ do not commute, they “almost” commute.

**Proposition 6** Let $x = \phi(y)$ be our diffeomorphism, $|s| < 1$ and $1 \leq p, q \leq +\infty$. Then the Besov spaces $\hat{B}^{s,q}_p(x)$ and $\hat{B}^{s,q}_p(y)$ are identical, with equivalent norms.

**Proof:** For any $p \in [1, +\infty]$, the $\hat{W}^1_p$ norms are equivalent: the two Jacobians $|\partial_y \phi(y)|$ or $|\partial_x \phi^{-1}(x)|$ are bounded. Therefore, with obvious notations,

$$\|\Delta^j_\phi \Delta^\varphi \|_p \sim 2^{-j} \|\Delta^j_\phi \Delta^\varphi \|_{\hat{W}^1_p(y)} \lesssim 2^{-j} \|\Delta^\varphi \|_{\hat{W}^1_p(x)} \lesssim 2^{-j} \|\Delta^\varphi \|_p \sim 2^{-j} \|\varphi \|_p.$$  

Since $x$ and $y$ play the same part, by duality we obtain

$$\|\Delta^j_\phi \Delta^\varphi \|_p \lesssim 2^{-|k-j|} \|\varphi \|_p.$$  

This essentially allows to exchange $x$ and $y$ in Besov spaces, as long as we are using spaces involving strictly less than one derivative: say $\varphi(x) \in \hat{B}^{s,q}_p(x)$, then $\varphi(y) \in \hat{B}^{s,q}_p(y)$, as

$$\|\Delta^j_\varphi \|_p \lesssim \sum_k 2^{-|k-j|} \|\Delta^\varphi \|_p \lesssim \sum_k 2^{-|k-j|} 2^{-k} \varepsilon_k$$

$$2^j \|\Delta^j_\varphi \|_p \lesssim \sum_k 2^{-|1-s|k-j} \varepsilon_k \lesssim \mu_j,$$

where $(\mu_j)_j \in l^q$ as an $l^{1-l}$ convolution. \hfill $\square$

**Remark 6** Proposition 6 is nothing but the invariance of Besov spaces under diffeomorphism. Given that we only have a $C^1$ diffeomorphism, we are restricted to Besov spaces with $|s| < 1$ regularity.

Going back to (28), we immediately obtain by inverting the change of variable,

$$\|u_j\|_{L^4_y(B^{1/2}_\infty)} \lesssim \|u_j(0)\|_{H^{1/2}} + 2^j \mu_j,$$

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and given that \( u_j = \Delta_j u \),
\[
\|u_j\|_{L^4_t(L^\infty_x)} \lesssim \|\Delta_j u_0\|_2 + \mu_j,
\]
which, by summing over \( j \), gives the desired Strichartz estimate. All other Strichartz estimates are obtained directly in the same way or by interpolation with the conservation of mass. This ends the proof of Theorem 2.

The strategy is exactly similar for the maximal function estimate. Recall (see [18]) that for the (flat) Schrödinger equation, we have
\[
\left\|( -\partial_y^2 )^{\frac{1}{8}} S(t) u_0 \right\|_{L^4_t(L^\infty_x)} \sim \left\| \int_\mathbb{R} e^{iy\xi - it|\xi|^2} \frac{\hat{u}_0}{|\xi|^\frac{1}{4}} \, d\xi \right\|_{L^4_t(L^\infty_x)} \lesssim \|u_0\|_{L^2(\mathbb{R})},
\]
from which we may obtain (by combining (29) with smoothing and Christ-Kiselev) an inhomogeneous estimate for the flat case,
\[
\left\|( -\partial_y^2 )^{\frac{1}{8}} \int_0^t S(t-s) f(s) \, ds \right\|_{L^4_t(L^\infty_x)} \lesssim \|f\|_{L^2(L^1_t)}.
\]
Therefore applying this estimate on (27) (at the frequency-localized scale) we get
\[
\|v_j\|_{\dot{B}^{\frac{1}{2},2}_q(L^\infty_x)} \lesssim \|v_j(0)\|_{\dot{H}^{\frac{1}{2}}} + 2^\frac{4}{q} \mu_j,
\]
and then
\[
\|u_j\|_{L^2_t(L^\infty_x)} \lesssim 2^\frac{4}{q}(\|\Delta_j u_0\|_2 + \mu_j),
\]
which we can then sum up.

**Remark 7** Here we are using an equivalence between Besov spaces wrt \( x \) and Besov spaces wrt \( y \) with value in \( L^\infty_t \). The reader will easily check that the argument we used to obtain Proposition 6 applies with any Besov spaces with value in \( L^q_t \) for any \( 1 \leq q \leq +\infty \). As an alternative, one could use the definition with moduli of continuity (which is the usual way to prove invariance by diffeomorphism) to obtain the \( 0 < s < 1 \) range (and duality if one needs \( -1 < s < 0 \)).

This completes the proof of Theorem 3 for the special case \( s = \frac{1}{4} \). We are left with shifting regularity in the appropriate range: but this is again nothing but a consequence of the equivalence from Appendix A. We therefore obtain the full range in Theorem 3 as well as Proposition 4, where the restriction on \( s \) follows from book keeping. \( \square \)

## 3 Application to a generalized Benjamin-Ono equation

Benjamin-Ono reads
\[
(\partial_t + H\partial_x^2)u \pm u^p \partial_x u = 0,
\]
\[
(30)
\]
with real data \( u_0 \) at time \( t = 0 \) (thus, it stays real). Here \( H \) denotes the Hilbert transform (Fourier multiplier \( i \text{sign}(\xi) \)). Given that the solution is real-valued, we can recover it from its positive spectrum; by projecting on positive frequencies, we get a Schrödinger equation. In particular, smoothing, Strichartz, maximal function estimates are strictly the same for both linear operators.

There are several cases of interest: mainly \( p = 1 \), \( p = 2 \) and \( p = 4 \). We will restrict ourselves to \( p = 4 \). Other cases will be dealt with elsewhere [6], as they require extra developments and significantly new ideas in addition to the techniques we developed in the present paper.

The study of the IVP for (30) with low regularity data was initiated in [16, 17]. The best results to date were obtained recently in [20], where they prove (among other results for different \( p \)) (30) to be locally wellposed in \( H^{1/2} \). The authors were able to remove the (rather natural with the techniques at hand) restriction on the size of the data by adapting the renormalization procedure from [28] (where global wellposedness for the \( p = 1 \) case is obtained in \( H^1 \)). The same authors proved earlier in [21] that (30) was globally wellposed for small data in \( \dot{B}_{1/4}^{1/4} \) (and extended this result to \( \dot{H}^\frac{1}{4} \) in [20]). We refer to [20] for a very nice presentation of the Benjamin-Ono family of equations and of the context in which they arise.

We intend to remove the restriction on the size of the data all the way down to \( s = 1/4 \) (which is the scaling exponent).

**Theorem 4** Let \( u_0 \in \dot{H}^{1/4} \), then the generalized Benjamin-Ono equation (30), for \( p = 4 \) is locally wellposed, i.e. there exists a time \( T(u_0) \) such that a unique solution \( u \) exists with

\[
\| u \|_2 = \| u_0 \|_2 \text{ and } E(u) = \| u \|^2_{H^{1/2}} = \frac{1}{15} \int_{\mathbb{R}} u^6 = E(u_0)
\]

Moreover, the flow map is locally Lipschitz.

Combining this local wellposedness result, which is subcritical with respect to the “energy norm” \( \dot{H}^{1/2} \), with the conservation of mass and energy,

\[
\| u(t) \|_2 = \| u_0 \|_2 \text{ and } E(u) = \| u \|^2_{H^{1/2}} = \frac{1}{15} \int_{\mathbb{R}} u^6 = E(u_0)
\]

and Gagliardo-Nirenberg, we also obtain global wellposedness in the energy space when the energy controls the \( \dot{H}^{1/2} \) norm, which occurs in the defocusing case (minus sign in (30)) or if the \( L^2 \) norm is small enough (focusing: plus sign in (30)).

**Theorem 5** Let \( u_0 \in \dot{H}^{1/2} \), then the defocusing generalized Benjamin-Ono equation (30), for \( p = 4 \), is globally wellposed, i.e. there exists a unique solution \( u \) such that

\[
\| u \|_2 = \| u_0 \|_2 \text{ and } E(u) = \| u \|^2_{H^{1/2}} = \frac{1}{15} \int_{\mathbb{R}} u^6 = E(u_0)
\]

**Proof:** We first prove Theorem 4. For local well-posedness, the sign in (30) is irrelevant and we take + for convenience. Let us sketch our strategy: the restriction on small data is induced by the maximal function estimate (29); even on the linear part, \( \| S(t)u_0 \|_{L^2(L^\infty_t)} \) will be small only if \( \| u_0 \|_{H^{1/2}_+} \) is
small as well. Here and hereafter, $S(t)$ denote the linear operator, which we recall reduces to the Schrödinger group on positive frequencies. Now, if we consider instead the difference $S(t)u_0 - u_0$, then the associated maximal function is small provided we restrict ourselves to a small time interval $[0, T]$:

**Lemma 2** Let $u_0 \in \dot{H}^{\frac{3}{4}}$, then for any $\varepsilon > 0$, there exists $T(u_0)$ such that

$$
(31) \quad \| \sup_{|t|<T} |S(t)u_0 - u_0|\|_{L^2} < \varepsilon.
$$

**Proof:** For the linear flow,

$$
\|S(t)u_0 - u_0\|_{L^2(L^\infty_T)} \leq \sum_{|j|<N} \|\Delta_j(S(t)u_0 - u_0)\|_{L^2(L^\infty_T)} + 2\left( \sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \right)^{\frac{1}{2}}
$$

$$
\leq \sum_{|j|<N} 2^{2j} \|\int_0^t S(s)\Delta_j u_0 ds\|_{L^2(L^\infty_T)} + 2\left( \sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \right)^{\frac{1}{2}}
$$

$$
\leq T \sum_{|j|<N} 2^{2j} \|\Delta_j u_0\|_{L^2(L^\infty_T)} + 2\left( \sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \right)^{\frac{1}{2}}
$$

$$
\leq T 2^{2N} \|u_0\|_{\dot{H}^{\frac{3}{4}}} + 2\left( \sum_{|j|>N} 2^{\frac{j}{2}} \|\Delta_j u_0\|_2 \right)^{\frac{1}{2}}
$$

and by choosing first $N$ large enough and then $T$ accordingly, we get arbitrary smallness. □

Given that local in time solutions do exist ([16]), we could set up an a priori estimate and pass to the limit. However, in order to get the flow to be Lipschitz, one has essentially to estimate differences of solutions, and in turn this provides the required estimates to set up a fixed point procedure.

Firstly, we proceed with an appropriate paralinearization of the equation itself. All computations which follow are justified if we consider smooth solutions. We have, denoting $u_j = \Delta_j u$, $u_{<j} = S_{j-10} u$ and $u_{\geq j} = S_j u$

$$
\partial_t u_j + H \Delta u_j + \Delta_j (u^4 \partial_x u) = 0.
$$

Rewriting $u^4 \partial_x u = \partial_x (u^5)/5$ and using a telescopic series $u = \sum_k S_k u - S_{k-1} u$, we get by standard paraproduct-like rearrangements

$$
5 \Delta_j (u^4 \partial_x u) = \Delta_j \partial_x (u^5) = \Delta_j \left( (u_{<j})^4 \sum_{k \sim j} \partial_x u_k \right) + \partial_x \Delta_j \left( \sum_{j \leq k \sim k'} (u_{k'})^2 (u_{<k'})^3 \right)
$$

$$
+ \Delta_j \left( \sum_{k \sim j} (u_{<j})^3 u_k \partial_x u_{<j} \right) = \Delta_j \left( (u_{<j})^4 \sum_{k \sim j} \partial_x u_k \right) - R_j (u).
$$

We will now consider the original equation as a system of frequency localized equations,

$$
\partial_t u_j + H \Delta u_j + \Delta_j \left( \sum_{k \sim j} (u_{<j})^4 \partial_x u_k \right) = R_j (u).
$$
If we set \( \pi(f_1, f_2, f_3, f_4, g) = \sum_j \Delta_j \left( \sum_{k \sim j} f_1, \sim j f_2, \sim j f_3, \sim j f_4, \sim j g_k \right) \) we can rewrite our model (abusing notations for \( \pi \))

\[
\partial_t u + H \Delta u + \pi(u^{(4)}, \partial_x u) = R(u),
\]

and we intend to solve (32) by Picard iterations.

Now, let us consider \( u_L \) the solution to the linear BO equation, and the following linear equation:

\[
\partial_t v + H \Delta v + \pi(u_L^{(4)}, \partial_x v) = 0, \quad \text{and } v_{t=0} = u_0.
\]

At the frequency localized level, this is almost what we can handle, except for a commutator term.

Therefore we have

\[
\partial_t v_j + H \Delta v_j + (u_{L, \sim j})^4 \partial_x v_j = - \left( \sum_{k \sim j} [\Delta_j, (u_{L, \sim j})^4] \partial_x v_k \right),
\]

\[
\partial_t v_j + H \Delta v_j + (u_{0, \sim j})^4 \partial_x v_j = \left( (u_{0, \sim j})^4 - (u_{L, \sim j})^4 \right) \partial_x v_j - \left( \sum_{k \sim j} [\Delta_j, (u_{L, \sim j})^4] \partial_x v_k \right),
\]

for which we aim at using the estimates from Section 1.

The iteration map will therefore be

\[
\partial_t u_{n+1} + H \Delta u_{n+1} + \pi(u_L^{(4)}, \partial_x u_{n+1}) = \pi(u_L^{(4)}, \partial_x u_n) - \pi(u_0^{(4)}, \partial_x u_n) + R(u_n).
\]

Hence we need estimates for the linear equation

\[
\partial_t v + H \Delta v + \pi(u_L^{(4)}, \partial_x v) = f(x,t), \quad \text{and } v_{t=0} = u_0.
\]

Restrict time to \([0, T]\) with \( T \) to be chosen later, let \( 0^+ \) denote a small number close to 0, and define

\[
E_s = \cap_{0^+ \leq \theta \leq 1} B^{s + \frac{30}{4} - 1 - \theta^2} (\mathcal{L}_1^2) \quad \text{as well as} \quad F_s = \sum_{0^+ \leq \theta \leq 1, \text{finite}} B^{s + \frac{30}{4} - 1 - \theta^2} (\mathcal{L}_1^2)
\]

(we left out the maximal function part, \( \theta = 0 \) because we need a slightly different estimate).

**Proposition 7** Let \( v \) be a solution of equation (33), \( u_0 \in \dot{H}^s \cap \dot{H}^{\frac{1}{2}} \) with \(-3/4 < s < 1/2\) and \( f \in F_s \). Then there exists \( T(u_0) \) such that on the time interval \([-T, T]\), we have

\[
\|v\|_{E_s} \lesssim_T \|u_0\|_{\dot{H}^s} + \|f\|_{F_s}.
\]

Moreover,

\[
\|v - u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 (\mathcal{L}_1^\infty)} \lesssim_T \|S(t)u_0 - u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 (\mathcal{L}_1^\infty)} + \|f\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s} + \|u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s} + \|u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s},
\]

and

\[
\|v - u_0\|_{L^2(\mathcal{L}_1^\infty)} \lesssim_T \|S(t)u_0 - u_0\|_{L^2(\mathcal{L}_1^\infty)} + \|f\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s} + \|u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s} + \|u_0\|_{B^{s + \frac{30}{4} - 1 - \theta^2}_4 \dot{H}^s}.
\]
Proof: Let us consider the equation at the frequency localized level,
\[ \partial_t v_j + H \Delta v_j + (u_{0,j})^4 \partial_x v_j = \left( (u_{0,j})^4 - (u_{L,j})^4 \right) \partial_x v_j - \left( \sum_{k \sim j} [\Delta_j, (u_{L,j})^4] \partial_x v_k \right) + f_j, \]
and we will denote by \( R_j \) the right hand side. Notice \( R_j \) is spectrally localized. In order to connect this equation with the model worked upon in Section 1, denote by
\[ b(x) = (u_{0,j})^4 \in L^1_x, \] and consider \( i \partial_t w + \partial_x^2 w + b(x) \partial_x w = g. \)

By reversing the procedure we used in Section 2, we can reduce the operator \( \partial_x^2 + b(x) \partial_x \) to \( \partial_y a(y) \partial_y \) and apply all the estimates we already know: set
\[ \frac{dy}{dx} = A(x) = \phi'(x), \] with \( A(x) = \exp\left( \int_{-\infty}^{x} (u_{0,j})^4 (\rho) \, d\rho \right), \)
then \( y = \phi(x) \) is a diffeomorphism and \( \sqrt{a(y)} = A \circ \phi^{-1}(y) \) which insures \( a \in W^{1,1} \), and \( a \) is 1-admissible. A simple calculation shows that under this change of variables,
\[ \partial_y a(y) \partial_y \to \partial_x^2 + b(x) \partial_x \]
Note that everything is uniform wrt \( j \). Interpolation between all the various bounds which one can deduce from Proposition 3 and Theorem 3 yields estimates for \( w \circ \phi^{-1} \) which are identical to the flat case (or, to get a better sense of perspective, to linear estimates for the linear Benjamin-Ono equation, see e.g. [21]):
\[ \| w \circ \phi^{-1} \|_{E_s} \lesssim \| w_0 \circ \phi^{-1} \|_{\dot{H}^s} + \| g \circ \phi^{-1} \|_{F_s}, \]
with \(-3/4 \leq s < 1/2\). Using Proposition 6, we can revert back to the \( x \) variable and obtain the exact same estimates for \( w \):
\[ \| w \|_{E_s} \lesssim \| w_0 \|_{\dot{H}^s} + \| g \|_{F_s}. \]

Recalling that \( w = v_j = \Delta_j v \) and \( g = R_j \) is frequency localized as well, hence for any \( 0^+ \leq \theta \leq 1, \)
\[ 2^{j \left( s + \frac{\theta + 1}{2} \right)} \| v_j \|_{L^2_x \left( L^\infty_y (L^2_x (L^2_y))^{-\frac{1}{2}} \right)} \leq 2^j \| u_{0,j} \|_{2} + \sum_{k, \text{finite}} 2^{j \left( s + \frac{1-3\theta}{2} \right)} \| R_j \|_{L^2_x \left( \frac{1}{L^\infty_y (L^2_x (L^2_y))^{\frac{1}{2}} \right)} \].

All is left is to estimate \( R_j \) in order to contract the \( v_j \) term:
\[ R_j = \left( (u_{0,j})^4 - (u_{L,j})^4 \right) \partial_x v_j - \left( \sum_{k \sim j} [\Delta_j, (u_{L,j})^4] \partial_x v_k \right) + f_j. \]

From the smoothing estimate for the flat Schrödinger equation and Lemma 2, there exist \( T(u_0) \) such that
\[ \left( \sum_j \left( 2^{j \frac{\theta}{2}} \| \partial_x u_{L,j} \|_{L^\infty_y (L^2_x )} \right)^2 \right)^{\frac{1}{2}} + \| u_{L,j} - u_{0,j} \|_{L^2_x (L^\infty_y )} < \eta(u_0), \]

(34)
where \( \eta(u_0) \) can be made as small as needed by choice of a smaller \( T(u_0) \). This allows to write, picking a \( \theta_B \) close to 1 and abusing notations,

\[
2^{js} \|v_j\|_{L^\infty_x(L^2_T)} + 2^{(s-j)}\|v_j\|_{L^\infty_x(L^2_T)} \leq \frac{1}{2} 2^{(s-1)j} \|\partial_x v_j\|_{L^\infty_x(L^2_T)} + \frac{1}{2K} \sum_{|\nu-j|<K} 2^{(s-j)}\|w_\nu\|_{L^\infty_x(L^2_T)} + 2^{(s-1)j} \|f_j\|_{L^1_x(L^2_T)},
\]

where we used Lemma 1 to estimate the commutator with \( 2^{-(s-j)}\partial_x u_L, \sim j \in L^4_x(L^\infty_T) \) small enough by (34) and interpolation with \( u_L \in L^4_x(L^\infty_T) \). We have therefore obtained, after summing over \( j \),

\[
\|v\|_{E_s} \leq C(u_0)(\|u_0\|_{H^s} + \|f\|_{F_s}).
\]

We only have a local in time estimate for the linearized equation, but it depends only on the data and nothing else, through lemma 2. At our desired level of regularity, namely \( s = 3/4 \),

\[
\|v\|_{B^{3/4}_4(L^2_T)} \leq C(u_0)(\|f\|_{B^{3/4}_4(L^2_T)} + \|u_0\|_{H^{3/4}}).
\]

We also need the maximal function, or more accurately, \( v - u_0 \): but this is now very easy, simply reverting back to writing (S(t) being here the group associated to the linear BO)

\[
v = u_L + \int_0^t S(t-s)(f - \pi(u_L, \partial_x v))ds,
\]

and we therefore get (using the third case in Theorem for the special case \( s = 0 \))

\[
\|v - u_0\|_{B^{3/4}_4(L^2_T)} \lesssim \|u_L - u_0\|_{B^{3/4}_4(L^2_T)} + \|f\|_{B^{3/4}_4(L^2_T)} + \|u_0\|_{H^{3/4}}\|v\|_{B^{3/4}_4(L^2_T)}^2,
\]

and

\[
\|v - u_0\|_{L^\infty_x(L^\infty_T)} \lesssim \|u_L - u_0\|_{L^\infty_x(L^\infty_T)} + \|f\|_{B^{3/4}_4(L^2_T)} + \|u_0\|_{H^{3/4}}\|v\|_{B^{3/4}_4(L^2_T)}^2.
\]

This achieves the proof of Proposition 7. □

Everything is now ready for a contraction in a complete metric space, which will be the intersection of two balls,

\[
B_M(u_0, T) = \{ u \text{ s.t. } \|u - u_0\|_{B^{3/4}_{L^\infty}(L^\infty_T)} < \varepsilon(u_0) \},
\]

and

\[
B_S(u_0, T) = \{ u \text{ s.t. } \|u\|_{B^{3/4}_{L^\infty}(L^\infty_T)} < \varepsilon(u_0) \}.
\]

We first check that the mapping \( K \) is from \( B_M \cap B_S \) to itself, where \( K(v) = u \) with

\[
\partial_t u + H\Delta u + \pi(u^{(4)}_L, \partial_x u) = \pi(u^{(4)}_L, \partial_x v) - \pi(v^{(4)}, \partial_x v) + R(v).
\]

For this we use Proposition 7 with \( s = 3/4 \) and standard (para)product estimates. The \( B_S \) part is trivial (one doesn’t even need to take advantage of the difference on the right). The \( B_M \) part follows from the ability to factor an \( u_L - u \) while rewriting the difference of the \( \pi \) on the right.

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The next step is then to contract, i.e. estimate $K(v_1) - K(v_2)$ in terms of $v_1 - v_2$. But this is again trivial given we have a multilinear operator, it will be exactly as the $v \to u$ mapping. This ends the proof of Theorem 4.

We now briefly sketch the proof of Theorem 5. We now have a minus sign in (30) but this doesn’t change the local in time contraction. Given a datum in the (inhomogeneous) space $H^s$, with $s > 1/4$, a standard modification of the fixed point provides that the solution whenever $v \to u$ will be exactly as the proof of Theorem 4.

Change the local in time contraction. Given a datum in the (inhomogeneous) space $H^s$, with $s > 1/4$, a standard modification of the fixed point provides that the solution $u$ is $C_t(H^s)$. In order to iterate whenever $s = 1/2$, we need to check that the local time $T(u_0)$ can be repeatedly chosen in a uniform way. All is required is an appropriate modification of Lemma 2 recall we can write

$$
\sum_j \|\Delta_j(S(t)u_0 - u_0)\|_{L^2(L^\infty)} \leq T^{2N}\|u_0\|_{\dot{H}^{1/2}} + 2\left(\sum_{|j| > N} 2^{\frac{j}{2}}\|\Delta_j u_0\|_2\right),
$$

from which we get, taking advantage of $u_0 \in L^2 \cap \dot{H}^{1/2}$,

$$
\sum_j \|\Delta_j(S(t)u_0 - u_0)\|_{L^2(L^\infty)} \leq T^{2N}\left(\|u_0\|_2\|u_0\|_{\dot{H}^{1/2}}\right)^{1/2} + 2^{-N}\left(\|u_0\|_2 + \|u_0\|_{\dot{H}^{1/2}}\right).
$$

Obviously, picking $T = 2^{-\frac{N}{2}}$ gives the bound $2^{-N}\left(\|u_0\|_2 + \|u_0\|_{\dot{H}^{1/2}}\right)$, which by an appropriate choice of $N$ can be made as small as we need with respect to $\left(\|u_0\|_2 + \|u_0\|_{\dot{H}^{1/2}}\right)$. However, both the $L^2$ and $\dot{H}^{1/2}$ norms are controlled, thus the local time $T(u_0)$ is uniform and we can iterate the local existence result to a global result.

\[\square\]

A Localization with respect to $\partial_x$ versus localization with respect to $\left(-\partial_x(a(x)\partial_x)\right)^{1/2}$

A.1 The heat flow associated with $-\partial_x(a(x)\partial_x)$

We would like to define an analog of the Littlewood-Paley operator $\Delta_j$, but using $A = -\partial_x(a(x)\partial_x)$ rather than $-\partial_x^2$. In the first 2 sections, this turns out to be useful because such a localization wrt $A$ will commute with the Schrödinger flow. Through spectral calculus, we can easily define $\phi(A)$ for a smooth $\phi$, but we need various properties on $L^p$ spaces for all $1 \leq p \leq +\infty$, which requires a bit more of real analysis. Fortunately, all the results we need are more or less direct consequences of (part of) earlier work related to the Kato conjecture, and we simply give a short recollection of the main facts we need, skipping details and referring to [2,1]. We call $S_A(t)$ the heat flow, namely $S_A(t)f$ solves

$$
\partial_t g + Ag = 0, \text{ with } g(0) = f,
$$

and define $\Delta_{-j} f = 4^{-j} A S_A(4^{-j}) f$. Again, in $L^2$ all of this makes sense through spectral considerations, and were $a$ to be just 1, we would just get a localization operator based on the Mexican hat $\hat{\xi} \exp(-\xi^2)$. In [2], such a semi-group $S_A(t)$ is proved to be analytic, and moreover the square-root of $A$ can be factorized as $R\partial_x$, where $R$ is a Calderon-Zygmund operator, under rather mild hypothesis: $a \in L^\infty$, complex valued, with $\text{Re} \ a > 1$. On the other hand, in [11], the authors prove Gaussian
bounds for the kernel of the semi-group as well as its derivatives, and this provides everything which is needed here. Such bounds are obtained through the following strategy:

- Derive bounds for the operator \((1 + A)^{-1}\); given that it maps \(H^{-1}\) to \(H^1\), it follows that it maps \(L^1\) to \(L^\infty\) by Sobolev embeddings.
- Obtain bounds for \((\lambda + A)^{-1}\), \(\text{Re} \lambda > 0\), by rescaling, given the hypothesis on \(a\) are invariant.
- Obtain bounds for \(A(1 + A)^{-1}\) by algebraic manipulations, proving it maps \(L^1\) to \(L^\infty\).
- Obtain again an \(L^1 - L^\infty\) bound for \(\partial_x (1 + A)^{-1}\) by “interpolation” between the two previous bounds. This specific bound we did prove directly in Section I, namely (16).
- Use a nifty trick (see Davies (11)): remark that provided \(\omega\) is sufficiently small (wrt the lower bound of \(\text{Re} a\)), all previous estimates hold as well for \(A_\omega = \exp(\omega \cdot A) \exp(-\omega \cdot)\).

Then any of the new kernels \(K_\theta(x, y)\) are just \(K(x, y) \exp(-\omega|x-y|)\), which gives exponential decay pointwise from the \(L^1 - L^\infty\) bound.

- Use the representation of \(S_A(t)\) in term of \(R_\lambda(A) = (\lambda + A)^{-1}\) to obtain that \(S_A(t)\) maps \(L^1\) to \(L^\infty\) and that its kernel verifies Gaussian bounds, as well as its derivatives.

We can summarize with the following proposition.

**PROPOSITION 8 (11)** Let \(K_A(x, y, t)\) be the kernel of the heat flow \(S_A(t)\). There exists \(c\) depending only on the lower bound of \(\text{Re} a\) and its \(L^\infty\) norm, such that

\[
|K_A(x, y, t)| \lesssim \frac{1}{\sqrt{t}} e^{\frac{-|x-y|^2}{t}},
\]

\[
(36)
\]

\[
|\partial_y K_A(x, y, t)| + |\partial_x K_A(x, y, t)| \lesssim \frac{1}{t} e^{\frac{-|x-y|^2}{t}},
\]

\[
(37)
\]

and

\[
|AK_A(x, y, t)| \lesssim \frac{1}{t^{\frac{3}{2}}} e^{\frac{-|x-y|^2}{t}}.
\]

\[
(38)
\]

Once we have all the Gaussian bounds, it becomes very easy to prove that \(S_A(t)\) is continuous on \(L^p\) (from (36)), as well as \(\Delta_j^b\) (from (38)). We are, in effect, reduced to the usual heat equation, with appropriate Bernstein type inequalities.
A.2 Equivalence of Besov norms

We first define Besov spaces using the $A$ localization rather the usual one:

**Definition 4** Let $f$ be in $S'(\mathbb{R}^n)$, $s < 1$. We say $f$ belongs to $\dot{B}^{s,q}_{p,A}$ if and only if

- The partial sum $\sum_{m}^{m-1} \Delta_j^A(f)$ converges to $f$ as a tempered distribution (modulo constants if $s \geq 1/p, q > 1$).
- The sequence $\varepsilon_j = 2^{js} \| \Delta_j^A(f) \|_{L^p}$ belongs to $l^q$.

Alternatively, one could replace the discrete sum with a continuous one, which is somewhat more appropriate when using the heat flow. Both can be proved to be equivalent, exactly as in the usual situation.

Now, our aim is to prove these spaces to be equivalent to the ones defined by Definition 1. In order to achieve this, we would like to estimate $\Pi_{jk}$ and its adjoint. The adjoint can be dealt with by duality, so we focus on $\Pi_{jk}$: there are obviously 2 cases,

- when $j > k$, we write
  $$\Pi_{jk} = 4^{-j} S_A(4^{-j}) \partial_x a(x) \partial_x \Delta_k,$$
  which immediately yields, for any $1 \leq p \leq +\infty$,
  $$\| \Pi_{jk} f \|_p = 2^{-j} \| S_A(4^{-j}) 2^{-j} \partial_x a(x) \partial_x \Delta_k f \|_p \lesssim 2^{-j} \| a(x) \partial_x \Delta_k f \|_p \lesssim 2^{-j} \| \partial_x \Delta_k f \|_p \lesssim 2^{k-j} \| \Delta_k f \|_p,$$
  where we used the bound (37) on $S_A(1) \partial_x$.

- In the same spirit, when $k > j$,
  $$\Pi_{jk} = 4^{-j} S_A \left( \frac{4^{-j}}{2} \right) S_A \left( \frac{4^{-j}}{2} \right) \partial_x (\partial_x)^{-1} \Delta_k,$$
  and then
  $$\| \Pi_{jk} f \|_p \lesssim 2^j \| S_A \left( \frac{4^{-j}}{2} \right) 2^{-j} \partial_x (\partial_x)^{-1} \Delta_k f \|_p \lesssim 2^j \| (\partial_x)^{-1} \Delta_k f \|_p \lesssim 2^{j-k} \| \Delta_k f \|_p,$$
  where we used (again) the bound (37) on $S_A(1) \partial_x$.

Therefore,

**Proposition 9** Let $|s| < 1$, $1 \leq p, q \leq +\infty$, then $\dot{B}^{s,q}_{p}$ and $\dot{B}^{s,q}_{p,A}$ are identical, with equivalence of norms.

**Remark 8** In previous sections, we actually used Besov spaces taking values in the separable Hilbert space $L^2_t$: as a matter of fact, one can reduce to the scalar case by projecting over an Hilbert basis, hence the Hilbert-valued result holds as well.
B  Christ-Kiselev lemma for reversed norms

As observed in [22] and further exploited in [21], Christ and Kiselev Lemma works also with reversed norms. In this appendix, we prove the versions of this result we need in the previous sections. The proof is very much inspired from [8]:

**Theorem 6**  
Let $1 \leq \max(p, q) < r \leq +\infty$, $B$ a Banach space, and $T$ a bounded operator from $L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ to $L^r(\mathbb{R}^d; B)$ with norm $C$. Let $K(y, s, t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,t})$ taking values in the class of bounded operators on $B$. Define $T_R$ to be the operator with kernel $1_{s<t}K(y, s, t)$. Then $T_R$ is bounded from $L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ to $L^r(\mathbb{R}^d; B)$ with norm smaller than $C/(1 - 2^{1/r-1/\max(p,q)})$.

- If $max(p, q) < \min(\alpha, \beta)$ and $T$ is a bounded operator from the space $L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ to $L^\alpha(\mathbb{R}^d; L^\beta(\mathbb{R}))$ with norm $C$. Let $K(y, s, x, t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,x,t})$. Define $T_R$ to be the operator with kernel $1_{s<t}K(y, s, x, t)$. Then $T_R$ is bounded from $L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ to $L^\alpha(\mathbb{R}^d; L^\beta(\mathbb{R}))$ with norm smaller than $C/(1 - 2^{1/\min(\alpha,\beta)-1/\max(p,q)})$.

- If $T$ is a bounded operator from $\dot{B}^{0,2}_1(L^2_t)$ to $L^4(\mathbb{R}^d; L^\infty(\mathbb{R}))$ with norm $C$. Let $K(y, s, x, t)$ be its kernel, and $K \in L^1_{loc}(\mathbb{R}^3_{y,s,x,t})$. Define $T_R$ to be the operator with kernel $1_{s<t}K(y, s, x, t)$. Then $T_R$ is bounded from $\dot{B}^{0,2}_1(L^2_t)$ to $L^4(\mathbb{R}^d; L^\infty(\mathbb{R}))$ with norm smaller than $C/(1 - 2^{-1/4})$.

**Proof:** We study the first case in Theorem 6. For any (smooth) function $f \in L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ such that $\|f\|_{L^p(\mathbb{R}^d; L^q(\mathbb{R}))} = 1$, the function $F(t) = \|1_{s<t}f(s, y)\|_{L^p(\mathbb{R}^d; L^q(\mathbb{R}))}$ is an increasing function from $\mathbb{R}$ to $[0, 1]$, and without loss of generality we can take it to be injective (hence, invertible). We have

**Lemma 3**  
For any $f \in L^p(\mathbb{R}^d; L^q(\mathbb{R}))$, such that $\|f\|_{L^p(\mathbb{R}^d; L^q(\mathbb{R}))} = 1$,

$$\|1_{F^{-1}([a,b])} f\|_{L^p(\mathbb{R}^d; L^q(\mathbb{R}))} \leq C|b - a|^{\frac{1}{\max(p,q)}}$$

Indeed denote by $|t_a, t_b| = F^{-1}([a, b])$ and

$$G(t, x) = \left(\int_{s < t} |f(s, x)|^q ds\right)^{\frac{1}{q}}$$

1. If $p \geq q$, using that for $a, b \geq 0$ we have $(a + b)^{p/q} \geq a^{p/q} + b^{p/q}$ we obtain

$$\|1_{F^{-1}([a,b])} f\|_{L^p(\mathbb{R}^d; L^q(\mathbb{R}))} = \int_x \left(\int_{t_a \leq s \leq t_b} |f(s, x)|^q ds\right)^{\frac{1}{q}} dx$$

$$= \int_x \left(\int_{t_a \leq s \leq t_b} |f(s, x)|^q ds - \int_{s \leq t_a} |f(s, x)|^q ds\right)^{\frac{1}{q}} dx$$

$$\leq \int_x \left(\int_{s \leq t_b} |f(s, x)|^q ds\right)^{\frac{1}{q}} dx - \int_{s \leq t_a} |f(s, x)|^q ds\right)^{\frac{1}{q}} dx$$

$$\leq F(t_b) - F(t_a) = b - a$$
2. If \( p \leq q \), using that for \( x, y \geq 0 \), \((x^q - y^q) \leq \frac{q}{p}(x^p - y^p)(\max(x, y)^{q-p})\), we obtain

\[
\|1_{F^{-1}([a,b])}f\|_{L^p(\mathbb{R};L^q(\mathbb{R}))}^p = \int_x \left( \int_{s \leq t} |f(s, x)|^q ds \right)^\frac{p}{q} dx
\]
\[
\leq \frac{p}{q} \int_x (G(t_b, x)^q - G(t_a, x)^q)\frac{q}{p} dx
\]
\[
\leq \frac{p}{q} \left( \int_x G(t_b, x)^p - G(t_a, x)^p \right)\frac{p}{q} \left( \int_x (G(t_b, x)^{(q-p)/(q-p)}) dx \right) \frac{2-p}{q}
\]
\[
\leq \frac{p}{q} (b-a)^\frac{q}{p} \|f\|_{L^p(\mathbb{R};L^q(\mathbb{R}))} \leq \frac{p}{q} (b-a)^\frac{p}{q}.
\]

Consider now the dyadic decomposition of the real axis given by

\[ \mathbb{R} = -\infty, t_{n,1} \cup t_{n,1}, t_{n,2} \cup \cdots \cup t_{n,2^n-1}, +\infty = \bigcup_{j=1}^{2^n} I_j \]

such that

\[ \|f\|_{L^{p'}(\mathbb{R};B)} = 2^{-n} \]

with the convention \( t_{n,0} = -\infty \) and \( t_{n,2^n+1} = +\infty \). Remark that \( F(t_{n,j}) = j2^{-n} \) is the usual dyadic decomposition of the interval \([0, 1]\). We have

\[ 1_{s < t} = \sum_{n=1}^{+\infty} \sum_{j=1}^{2^n-1} 1_{(s, t) \in Q_{n,j}} \]

where \((s, t) \in Q_{n,j} \iff (F(s), F(t)) \in \tilde{Q}_{n,j} \) and \( \tilde{Q}_{n,j} \) is as in Figure.[1]

Remark that \( 1_{(s, t) \in Q_{n,j}} = 1_{I_{n,j}} 1_{s \in I'_{n,j}} \) for suitable dyadic intervals \( I_{n,j} \) and \( I'_{n,j} \).

We are now ready to prove the main estimate:

\[ \|TRf\|_{L^p(\mathbb{R};B)} = \| \sum_{n} \sum_{j=1}^{2^n-1} T_{n,j} f \|_{L^p(\mathbb{R};B)} \]

where the kernel of the operator \( T_{n,j} \) is equal to \( K(y, s, t) \times 1_{(s, t) \in Q_{n,j}} \). Consequently \( T_{n,j} \) is (uniformly) bounded from \( L^{p'}(\mathbb{R};B) \) to \( L^{p'}(\mathbb{R};(\mathbb{R}^q)) \) with norm smaller than \( C \).

Since \( p' \geq q' \) and for fixed \( n \), the functions \( T_{n,j} f \) have disjoint support (in the variable \( t \)) we have

\[
\|TRf\|_{L^p(\mathbb{R};B)} \leq C \left( \sum_{n=1}^{2^n-1} \|1_{s \in I_{n,j}} 1_{t \in I'_{n,j}} f\|_{L^p(\mathbb{R};L^{2^n}(\mathbb{R}))} \right)^\frac{1}{p}
\]
\[
\leq C \left( \sum_{n=1}^{2^n-1} 2^{-\frac{n}{\max(p,q)}} \right)^\frac{1}{p} = (1 - 2^{-\frac{1}{\max(p,q)}})^{-1}.
\]

We now study the second case in Theorem.[3] The proof relies on
LEMMA 4  Assume that \((f_k)_{k \in \mathbb{N}}\) have disjoint supports in \(t\). Then
\[
\left\| \sum_k f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \leq \left( \sum_k \left\| f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))}^{\min(\alpha, \beta)} \right)^{\frac{1}{\min(\alpha, \beta)}}
\]

We distinguish two cases:

\(\beta \geq \alpha\)
\[
\left\| \sum_k f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} = \left( \int_x \left( \sum_k \int_t |f_k(t, x| t \beta \right)^{\frac{\alpha}{\beta}} dt \right)^{\frac{1}{\alpha}}
\]

but since \(\alpha \leq \beta\), we have \((\sum_k a_k)^{\alpha/\beta} \leq \sum_k a_k^{\alpha/\beta}\) and we obtain
\[
\left\| \sum_k f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \leq \left( \int_x \left( \sum_k \int_t |f_k(t, x) t \beta \right)^{\frac{\alpha}{\beta}} dt \right)^{\frac{1}{\alpha}}
\]

\(\beta \leq \alpha\)
\[
\left\| \sum_k f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} = \left( \sum_k \int_t |f_k(t, x) t \beta \right)^{\frac{1}{\beta}} \leq \left( \sum_k \left\| f_k \right\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))}^{\frac{\alpha}{\beta}} \right)^{\frac{1}{\beta}}
\]

To prove the second case in Theorem B, we use the same dyadic decomposition of \(\mathbb{R}\) as before and use Lemma 4 to estimate \(\|T_R f\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))}\). This gives
\[
\|T_R f\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \leq \sum_n \left( \sum_j \|T_{n,j} f\|_{L^\alpha(\mathbb{R}_x; L^\beta(\mathbb{R}_t))} \right)^{\frac{1}{\min(\alpha, \beta)}}
\]
and we conclude as in the previous case.

Finally, to prove the last case in Theorem \[\text{[III]}\] we need to combine Lemma \[\text{[III]}\] with \(\alpha = 4, \beta = +\infty\) to deal with the \(L^4(\mathbb{R}_x; L^\infty(\mathbb{R}_t))\) norm with a choice of a suitable dyadic decomposition and prove the analog of Lemma \[\text{[III]}\] for the Besov space \(\dot{B}^{0,2}_1(\mathbb{R}_x)\). The dyadic decomposition is based on

\[
F(t) = \sum_j \left( \int_{s<t} |\Delta_j f(s)|^2 \, ds \right)^{1/2} = \sum_j \gamma_j(t)^2.
\]

**Lemma 5** For any function \(f\) such that \(\|f\|_{\dot{B}^{0,2}_1(\mathcal{L}^2_t)} = \left( \sum_j \|\Delta_j f\|_{L^2_t}^2 \right)^{1/2} = 1\), we have \(\|1_{F^{-1}(a,b)} f\|_{\dot{B}^{0,2}_1(\mathcal{L}^2_t)} \leq C(b-a)^{1/2}\).

**Proof:** Denote by \(J_j(t,x) = (\int_{s<t} |\Delta_j f(s)|^2 \, ds)^{1/2}\). Then (using \(2 \geq 1\))

\[
\|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|_{L^2_t}^2 = \int_{t_a}^{t_b} |\Delta_j f(s)|^2 \, ds = J_j(t_b, x)^2 - J_j(t_a, x)^2,
\]

\[
\leq (J_j(t_b, x) - J_j(t_a, x))(J_j(t_b, x) + J_j(t_a, x)).
\]

Then we add the \(L^1_x\) norm, to get (using Cauchy-Schwarz at the second line)

\[
\int_x \|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|_{L^2_t}^2 \, dx \lesssim \int_x (J_j(t_b, x) - J_j(t_a, x))^2 (J_j(t_b, x) + J_j(t_a, x))^2 \, dx
\]

\[
\lesssim \left( \int_x J_j(t_b, x) - J_j(t_a, x) \, dx \right)^{1/2} \left( \int_x (J_j(t_b, x) + J_j(t_a, x)) \, dx \right)^{1/2}
\]

and consequently

\[
\sum_j \left( \int_x \|\Delta_j \chi_{F^{-1}(I)}(s) f(s)\|_{L^2_t}^2 \, dx \right)^2 \lesssim \sum_j (\gamma_j(t_b) - \gamma_j(t_a))(\gamma_j(t_b) + \gamma_j(t_a))
\]

\[
\lesssim \sum_j (\gamma_j^2(t_b) - \gamma_j^2(t_a)) = F(t_b) - F(t_a) = |I|.
\]

The rest of the proof of Theorem \[\text{[III]}\] is as in the previous cases.

**C  A singular metric**

In this section we construct a metric on \(\mathbb{R}\), which is in \(W^{s,1}\) for any \(0 \leq s < 1\) (but not in \(BV\)), bounded from below and above and for which no smoothing estimate and no (non trivial) Strichartz estimates hold. In fact this construction is a simplification of an argument of Castro and Zuazua \[\text{[Z]}\] (whose proof relies in turn upon some related works in semi-classical analysis and unique continuation theories), who, in the context of wave equations, provide counter examples with \(C^{0,\alpha}\), \(0 \leq \alpha < 1\)
metrics (continuous Hölder of exponent $\alpha$ metrics). As noticed by Castro and Zuazua, these counter examples extend to our setting. Figure 2 shows the range where full Strichartz/smoothing are true or no Strichartz/smoothing holds. A most interesting range of regularity is $a \in W^{s,1}$ and in particular $H^{1/2} = W^{1/2,2}$ because these regularities are scale invariant. A natural question would be to ask whether some Strichartz/smoothing estimates might hold (possibly with derivatives loss) at these levels of regularity. Remark that neither our counter examples nor Castro-Zuazua’s lie in this range (except for $s = 0$).

PROPOSITION 10 There exist a metric $\beta(x) \in W^{s,1}$ for any $0 \leq s < 1$, bounded from below and above $0 < m \leq \beta(x) \leq M$ (so that $\beta \in W^{s,p}, s < 1/p$), a sequence of functions $\phi_k \in C^\infty_0([2^{-k-1/2}, 2^{-k+1/2}])$ and a sequence $(x_k = 2^{-k}, \lambda_k = 2^{k})$ such that

\begin{equation}
(\partial_x \beta(x) \partial_x + \lambda_k)\phi_k = O(\lambda_k^{-\infty})_{H^1}
\end{equation}

\begin{equation}
\|\phi_k\|_{L^2} = 1,
\end{equation}

COROLLARY 1 For the density constructed above, we have for any $r < (q - 2)/2q$ (recall that by the usual Sobolev embedding, $H^{(q-2)/2q} \to L^q$),

\begin{equation}
\lim_{k \to +\infty} \frac{\|e^{it(\partial_x \beta(x) \partial_x + \lambda_k)}\phi_k\|_{L^1(-\varepsilon, \varepsilon) \times L^q(\mathbb{R})}}{\|\phi_k\|_{H^r}} = +\infty
\end{equation}

We first show that Proposition 10 implies 44. According to 43, $\|\phi_k\|_{H^1} \leq C\lambda_k$ and, by interpolation,

\begin{equation}
\|\phi_k\|_{H^r} \leq C\lambda_k^r \quad (0 \leq r \leq 1).
\end{equation}

According to 43,

\[ e^{it(\partial_x \beta(x) \partial_x + \lambda_k)}\phi_k = e^{it\lambda_k^2}\phi_k + v \]
where \( \|v\|_{L_t^\infty L_x^1(\mathbb{R})} = O(\lambda_k^{-\infty}) \). Using the Sobolev embedding \( H^1 \to L^q \), we can drop the contribution of \( v \) in (44). Using H"older inequality (and the fact that \( \phi_k \) is supported in a ball of radius \( 2^{-k} \)), we obtain
\[
1 = \|\phi_k\|_{L^2} \leq C 2^{-k(q-2)/q} \|\phi_k\|_{L^q}
\]
and consequently, according to (45),
\[
\|e^{it(\partial_y \beta(y))} \phi_k\|_{L^1(-\epsilon,\epsilon);L^q} \geq \frac{c 2^k(q-2)/q}{(k2^k)^r} \to +\infty, \text{ when } k \to +\infty.
\]
We now come back to the proof of Proposition 10. The starting point is the interval instability of the Hill equation (see for example [9]):

**Lemma 6** There exist \( w, \alpha \in C^\infty \) such that
\[
w'' + \alpha w = 0 \text{ on } \mathbb{R},
\]
\( \alpha \) is 1-periodic on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), equal to \( 4\pi^2 \) in a neighborhood of 0, and
\[
|\alpha - 4\pi^2| \leq 1,
\]
\( w(x) = pe^{-|x|} \) where \( p \) is 1-periodic on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \), and \( \|w\|_{L^2} = 1 \).

Changing variables and setting
\[
y(x) = \int_0^x \alpha(s) \, ds, \quad v(y) = w(x(y)), \quad \beta(y) = \alpha(x(y))
\]
we get
\[
\frac{\partial}{\partial y} = \alpha^{-1}(x) \frac{\partial}{\partial x} \text{ and } (\partial_y \beta(y) \partial_y + 1)v = 0
\]
Denote by
\[
v^{\lambda,m}(y) = v(\lambda(y - m)), \quad \beta^{\lambda,m}(y) = \beta(\lambda(y - m))
\]
solutions of
\[
(\partial_y \beta^{\lambda,m}(y) \partial_y + \lambda^2) v^{\lambda,m} = 0 \tag{46}
\]
\[
|v^{\lambda,m}(y)| \leq Ce^{-\lambda|y-m|} \tag{47}
\]
Consider \( \Psi_1 \in C^\infty(\mathbb{R}) \) equal to 1 on \([-1/4, 1/4]\), \( \Psi_2 \in C^\infty(\mathbb{R}) \) equal to 1 on \([-1/6, 1/6]\), sequences \( m_n = 2^{-n} \), \( \lambda_n = n 2^n \).

Using (47), we see that \( v_n = v^{\lambda_n,m}(y) \Psi_2(2^n(y - m_n)) \) is solution of
\[
(\partial_y \beta^{\lambda_n,m}(y) \partial_y + \lambda_n^2) v_n = O(\lambda_n e^{-cn}) \quad \text{in } H^1.
\]

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Remark also that on the support of \( v_n \), \( \Psi_1(2^n(y - m_n)) = 1 \) and consequently we can replace in (48) \( \beta^{\lambda_n,m_n}(y) \) by \( \beta_n(y) = \beta^{\lambda_n,m_n}(y)\Psi_1(2^n(y - m_n)) \). Remark also that for \( p \neq n \), the support of \( v_n \) is disjoint from the support of \( \Psi_1(2^p(y - m_p)) \). Consequently, we can replace in (48) \( \beta^{\lambda_n,m_n}(y) \) by

\[
\beta(y) = \sum_{n \in \mathbb{N}} \beta_n(y) + 4\pi (1 - \sum_{n \in \mathbb{N}} \Psi_1(2^n(y - m_n)))
\]

(the last term being here only to ensure that \( \beta(y) \geq 2\pi \)).

To prove Proposition 10, it is now enough to show that \( \beta \) is in \( W^{s,1} \) for any \( 0 \leq s < 1 \). A direct calculation shows that,

\[
\|\beta_n\|_{W^{1,1}} \sim n, \quad \|\beta_n\|_{L^1} \sim 2^{-n} \Rightarrow \|\beta_n\|_{W^{s,1}} \leq C n^s 2^{-(1-s)n}
\]

which implies that the series defining \( \beta \) converges in \( W^{s,1} \).

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