Modification of interest rate model based on compound poisson process and brownian motion affected by inflation rate

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Abstract. In actuarial science, interest rate model based on compound Poisson process and Brownian motion is proposed to determine actuarial present value. We used inflation rate in interest rate model to obtain the behavioural change of interest rate model under different parameters based on inflation rate data of Bank Indonesia. The interest rate model can be used to determine actuarial present value of term life insurance for discrete life annuities. To confirm the analytical result, some numerical simulations are presented.

1. Introduction
Life insurance is a form of agreement (insurance policy) between an insurance company (guarantor) and the owner of the policy (insured) to reduce the risk caused by the risk of death, old age, and accident. Everyone who takes out insurance is required to pay actuarial present value (APV), which is the amount of money that must be paid by the insured person to the guarantor by the agreed insurance contract. Several factors must consider in calculating actuarial present value of life insurance, specifically company costs, mortality rates, and interest rates [1]. Considering the interest rates, actuarial present value has been calculated using a fixed interest rate, which does not always agree with the practice of financial market. Therefore, to determine the uncertain rate movements, stochastic perturbation methods was proposed [2].

There are two methods that can be used for perturbation methods in actuarial science. First method describes interest rate perturbation as a special stochastic process and uses stochastic integration to determine the function of interest rate accumulation. Two approaches used to model interest randomness and the first three moments of homogeneous portfolios of life insurance directly based on the Ornstein-Uhlenbeck process and a Wiener process has been studied (see reference [3,4]). Another method, include stochastic processes such as the Wiener process, Ornstein-Uhlenbeck process, and Poisson process into the interest rate model. Zhao et al. took the accumulated force of interest by Wiener process and Poisson process and studied the optimal dividen strategy in ruin theory [5]. Determining the actuarial present value of term life insurance by incorporating the Wiener process, Poisson process, and inflation rate into the interest rate model studied by Buana [6].

A new class of stochastic interest model in which the force of interest is expressed by compound Poisson process introduced by Li et al. [2] in 2017. Later Li et al. took combining stochastic continuous fluctuation from risk-free investments of insurance companies which is expressed in Brownian motion and stochastic jumps from adjustments of market interest which is explain in compound Poisson
process[7]. Based on Li’s and Buana’s works, in this paper, we present an interest rate model with Compound Poisson process and Brownian motion affected by inflation. An interest rate model with inflation rate was presented and discussed before we determine the actuarial present value of term life insurance for discrete life annuities which is illustrated through simulation study.

2. Interest Rate Model Affected by Inflation Rate
We formulate our interest rate model under compound Poisson process and Brownian motion affected by inflation rate and these following assumptions are given:

- the nominal interest rate is assumed follows the Vasicek interest rate model,
- the real interest rates is assumed to be constant are determined based on the average difference between the inflation value and interest rates of Bank Indonesia from 2009-2018,
- the market price of inflation risk and the up-jumping probability of the force of interest are assumed to be constant,
- life insurance and life annuity types are term life insurance and discrete life annuities.

The inflation rate model used in this study is based on [8] and assumptions in (a) until (c). Then, inflation rate model as follows

\[ I(t) = I_0 e^{\int_0^t r_N(s) ds} \left( e^{\gamma_0 t} - e^{-\frac{\gamma_0^2 t^2}{2}} \right) + \theta_1, \quad t \in [0, \infty) \]  

with

\[ r_N(t) = r_N(0)e^{-kt} + \theta(1 - e^{-kt}) + \sigma_N \int_0^t e^{-k(t-s)} dW_N(s) \]

where \( r_N(t) \) is Vasicek interest rate model with parameter \( \theta \) denote mean reversion of nominal interest rate, \( k \) is a speed of mean reversion nominal interest rate, \( \sigma_N \) denote volatility of nominal interest rate, and \( W_N(t) \) defined Wiener process standard. For \( \gamma_0 \) and \( c_r \) is a constant, \( \sigma_I \) expressed volatility of inflation rate, and \( B_I(t) \) denote Brownian motion with \( N(0, \sigma_I^2 t) \).

The force of interest \( \delta(t), t \geq 0 \) is defined by

\[ \delta(t) = I(t) + \sum_{i=1}^{N(t)} Y_i + \sigma B(t), \quad t \in [0, \infty) \]  

where \( I(t) \) is a inflation rate, \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda > 0 \) which denotes the jumping number of interest rate on interval \([0, t]\). \( \{Y_i\}_{i=1}^{\infty} \) are i.i.d. random variables and each \( Y_i \) denote the jumping range of the \( i \)-th jump of interest rate. Then, \( \sum_{i=1}^{N(t)} Y_i, t > 0, \) is a compound Poisson process. The stochastic process \( \{B(t), t \geq 0\} \) is a standard Brownian motion which describes fluctuations that occur continuously from interest rate movements and \( \sigma \) expressed the fluctuation intensity.

3. Expected Discount Function under Compound Poisson Process and Brownian Motion Affected by Inflation Rate
This section provides expected discount function of the stochastic interest model and illustrated through simulation study.

3.1. Expected Discount Function
The expected discount function affected by inflation rate, compound Poisson process, and Brownian motion to determine actuarial present value which is defined by

\[ v^t = E[e^{-\int_0^t \delta(s) ds}] \]  

with integration \( \delta(s) \) on the time interval \([0, t]\)

\[ \int_0^t \delta(s) ds = \int_0^t \left( I(s) + \sum_{i=1}^{N(s)} Y_i + \sigma B(s) \right) ds \]  

substitution equation (1) to equation (4), then we obtain
\[
\int_0^t \delta(s) ds = \int_0^t \left( l_0 e^{l_0 t} r_N(s) ds + (\sigma_1 \theta^t - \frac{\sigma_7^t}{2}) t + \sigma_1 W_1(t) \right) ds + \int_0^t \left( \sum_{i=1}^{N(s)} Y_i \right) ds + \int_0^t \sigma B(s) ds
\]

\[= H_0(t) + H_1(t) + H_2(t). \tag{5}\]

Furthermore, substitution equation (5) to equation (3) and there are independent mutual as follows

\[v^t = E[e^{-H_0(t)}] E[e^{-H_1(t)}] E[e^{-H_2(t)}]. \tag{6}\]

The expected discount function in equation (6) can be explained by

- \[E[e^{-H_0(t)}] = E \left[ e^{\int_0^t (l_0 e^{l_0 t} r_N(s) ds + (\sigma_1 \theta^t - \frac{\sigma_7^t}{2}) t + \sigma_1 W_1(t)) ds} \right] \]

Let \(-H_0(t) = X\), we have

\[E[e^{-H_0(t)}] = E[e^X] = M_X(u) |_{u=1} \]

\[M_X(u) \text{ is a moment generating function of } X \sim N(\mu, \sigma^2), \text{ then} \]

\[M_X(u) = E[e^{uX}] = e^{\mu u + \frac{\sigma^2 u^2}{2}} \]

where

\[\mu = -\int_0^t \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) \left( \theta + \frac{\sigma_7^2}{2 k^2} \right) + \frac{\sigma N^2 (1 - e^{-\frac{kt}{k}})^2}{4 k^3} + r_N(0) \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) + \frac{\sigma_7^t}{2} \] \[\left[ e^{(\sigma_1 \theta^t - \frac{\sigma_7^t}{2}) t} - 1 \right] \]

and \(\sigma^2 = 0\).

Therefore, expected discount function of inflation rate can be obtained

\[E[e^{-H_0(t)}] = M_X(u) |_{u=1} \]

\[= e^{\int_0^t \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) \left( \theta + \frac{\sigma_7^2}{2 k^2} \right) + \frac{\sigma N^2 (1 - e^{-\frac{kt}{k}})^2}{4 k^3} + r_N(0) \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) + \frac{\sigma_7^t}{2} \] \[\left[ e^{(\sigma_1 \theta^t - \frac{\sigma_7^t}{2}) t} - 1 \right] \]

\[= e^{\int_0^t \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) \left( \theta + \frac{\sigma_7^2}{2 k^2} \right) + \frac{\sigma N^2 (1 - e^{-\frac{kt}{k}})^2}{4 k^3} + r_N(0) \left( \frac{1 - e^{-\frac{kt}{k}}}{k} \right) + \frac{\sigma_7^t}{2} \] \[\left[ e^{(\sigma_1 \theta^t - \frac{\sigma_7^t}{2}) t} - 1 \right] \]

\[= e^{\int_0^t (\sum_{i=1}^{N(s)} Y_i) ds} \tag{7} \]

- \[E[e^{-H_1(t)}] = E \left[ e^{\int_0^t (\sum_{i=1}^{N(s)} Y_i) ds} \right] \]

The expected discount function of compound Poisson process can be obtained by changing the integral direction and use the law of total expectation (refer to subsection 3.1 in [7]), then we have

\[E[e^{-H_1(t)}] = e^{\lambda t (\beta - 1)} \tag{8} \]

where

\[\beta_t = \frac{1}{b t} \left( p (1 - e^{-bt}) + (1 - p) (e^{bt} - 1) \right) \]

- \[E[e^{-H_2(t)}] = E \left[ e^{-l_0 \sigma B(s)} ds \right] \]
The expected discount function of Brownian motion can be obtained through stochastic calculus (see [9]), and we have $H_2(t) \sim N \left(0, \frac{\sigma^2 t^3}{3}\right)$. Therefore, from subsection 3.1 in [7], expected discount function can be expressed as

$$E[e^{-H_2(t)}] = e^{\frac{\sigma^2 t^3}{6}}$$

(9)

From equation (7), (8), and (9), we have expected discount function (interest rate) as follows

$$v_t = E[e^{-H_0(t)}]E[e^{-H_1(t)}]E[e^{-H_2(t)}]$$

$$= e^{\frac{e^{-\left(\theta + \frac{\sigma^2 N^2 \beta^2}{4k} + \gamma N(0)\beta + \frac{\gamma N^2}{2}\right)} + \lambda t (\beta N - 1) t^3}{\alpha}}$$

(10)

where

$$\alpha = \sigma I_0 - \frac{\sigma^2}{2}, \quad \beta = \frac{1 - e^{-kt}}{k}$$

and

$$\beta_t = \frac{1}{b} \left( p \left(1 - e^{-bt}\right) + (1 - p) \left(e^{bt} - 1\right) \right)$$

3.2. Numerical Analysis

In this subsection, we analyze the changes of interest rate (expected discount function) with inflation rate under different parameter assumptions. In this paper, parameters required based on inflation rate data and interest rate data of Bank Indonesia from January 2009 until December 2018. The parameters as follows

| Parameters | Description | Value | Reference |
|------------|-------------|-------|-----------|
| $I_0$      | Initial of inflation rate | 0.0917 | Bank Indonesia |
| $r_N$      | Average of BI rate | 0.0631 | Bank Indonesia |
| $c_R$      | Average difference between the inflation rate and BI rate | 0.0131 | Bank Indonesia |
| $\sigma_I$ | Deviation standard of inflation rate | 0.0177 | Bank Indonesia |
| $\sigma$   | Varians of BI rate | 0.0001 | Bank Indonesia |
| $b$        | Jumping range of the $i$-th jump of the BI rate | 0.0025, 0.005, 0.0075, 0.0125 | Bank Indonesia |
| $\theta_I$ | market price of inflation risk | 0.3 | Assumed |
| $p$        | up-jumping probability of BI rate | 0.4, 0.6 | Assumed |
| $r_N(0)$   | Initial of nominal interest rate | 0.0875 | Bank Indonesia |

The parameter for Vasicek interest rate model with mean reversion of nominal interest rate ($\hat{\theta}$), speed of mean reversion nominal interest rate ($\hat{k}$), and volatility of nominal interest rate ($\hat{\sigma}_N$) can be obtained by parameter estimation of Maximum Likelihood Estimation (MLE). Then, we have $\hat{\theta} = 0.036858$, $\hat{k} = 0.102541$, and $\hat{\sigma}_N = 0.0000721549$. 

Furthermore, parameter of $\lambda$ can be obtained by considering the number of changes in inflation rate based on average of inflation rate for ten years. The average of inflation rate is 5% and the parameter of $\lambda$ per year is determined by calculating the number of inflation rate greater than or equal to 5%.

After that, the parameter of $\lambda$ for each year can be defined as:

$$\lambda(1) = 5, \lambda(2) = 7, \lambda(3) = 6, \lambda(4) = 0, \lambda(5) = 11, \lambda(6) = 8, \lambda(7) = 10, \lambda(8) = 0, \lambda(9) = 0,$$

and $\lambda(10) = 0$.

(a) $\lambda = 5$

(b) $\lambda = 11$

Figure 1. Curves of interest rate under Compound Poisson process and Brownian motion affected by inflation

To find the interest rate under Compound Poisson process and Brownian motion affected by inflation, we use the interest rate model in equation (10). When eleven times the increase in inflation ($\lambda = 11$), the interest rate for $p = 0.4$ increased than five times the increase in inflation. If $p = 0.6$, then obtained the interest rate which does not have a significant differences between five times and eleven times the increased in inflation. Then, for five times and eleven times the increased in inflation with $p = 0.4$, both show the interest rate for $b = 0.0125$ is greater than $b = 0.0075, 0.005, 0.0025$. Otherwise, the interest rate for $b = 0.0125$ is smaller than $b = 0.0075, 0.005, 0.0025$ for $p = 0.6$.

4. Actuarial Present Value of Term Life Insurance

Determining actuarial present value of term life insurance for discrete life annuities use interest rate model with inflation in equation (10). In this paper, the term life insurance is assumed 20 year term insurance for the insured age 30 with $\lambda = 5, 6, 7, 8, 10, 11$. The mortality rate is from Indonesia Life Insurance Mortality Table 1999 for male and female. Following [10], actuarial present value of term life insurance for discrete life annuities can be defined as:

$$A^1_{x:n} = E[Z] = \sum_{k=0}^{n-1} v^{k+1} k p_x q_{x+k}. \quad (11)$$

Substitution equation (10) to equation (11), then we have

$$A^1_{x:n} = \sum_{k=0}^{n-1} e^{-\frac{(k+1)\beta - \beta}{2\alpha} \left( \delta + \frac{\sigma^2}{2\alpha} \right)} \frac{e^{-\frac{\sigma^2}{2\alpha} + \sigma^2 \beta^2}}{\alpha} \left[ e^{(k+1)\frac{\sigma^2}{2\alpha}} + \lambda(k+1) \right] \frac{d_{x+k}}{t_x}. \quad (12)$$

where
\[ \alpha = \sigma_t \theta_t - c_R - \frac{\sigma_t^2}{z}, \quad \beta = \frac{1 - e^{-kt}}{k} \]

and

\[ \beta_t = \frac{1}{b_t} \left( p(1 - e^{-b_t}) + (1 - p)(e^{b_t} - 1) \right) \]

The value of actuarial present value of 20 year term insurance for the insured age 30 under interest rate with inflation and different parameter of up-jumping probability of interest rate are shown in Table 2 and Table 3.

**Table 2. APV of 20 year term insurance for the insured age 30 when \( p = 0.4 \)**

| Actuarial Present Value | \( b \) | 0.0025 | 0.005 | 0.0075 | 0.0125 |
|-------------------------|---------|--------|-------|--------|--------|
| \( \lambda = 5 \)       | Male    | 0.0003 | 0.0005| 0.0010 | 0.0055 |
|                         | Female  | 0.0002 | 0.0003| 0.0007 | 0.0035 |
| \( \lambda = 6 \)       | Male    | 0.0003 | 0.0006| 0.0015 | 0.0113 |
|                         | Female  | 0.0002 | 0.0004| 0.0010 | 0.0072 |
| \( \lambda = 7 \)       | Male    | 0.0003 | 0.0008| 0.0022 | 0.0230 |
|                         | Female  | 0.0002 | 0.0005| 0.0014 | 0.0147 |
| \( \lambda = 8 \)       | Male    | 0.0004 | 0.0010| 0.0032 | 0.0468 |
|                         | Female  | 0.0002 | 0.0007| 0.0020 | 0.0299 |
| \( \lambda = 10 \)      | Female  | 0.0005 | 0.0016| 0.0067 | 0.1942 |
|                         | Male    | 0.0003 | 0.0010| 0.0043 | 0.1242 |
| \( \lambda = 11 \)      | Male    | 0.0005 | 0.0021| 0.0098 | 0.3957 |
|                         | Female  | 0.0003 | 0.0013| 0.0063 | 0.2530 |

**Table 3. APV of 20 year term insurance for the insured age 30 when \( p = 0.6 \)**

| Actuarial Present Value | \( b \) | 0.0025 | 0.005 | 0.0075 | 0.0125 |
|-------------------------|---------|--------|-------|--------|--------|
| \( \lambda = 5 \)       | Male    | 0.9972 | 0.6849| 0.5108 | 0.3633 |
|                         | Female  | 0.6376 | 0.4379| 0.3266 | 0.2323 |
| \( \lambda = 6 \)       | Male    | 0.9098 | 0.5797| 0.4077 | 0.2709 |
|                         | Female  | 0.5817 | 0.3706| 0.2607 | 0.1732 |
| \( \lambda = 7 \)       | Male    | 0.8301 | 0.4906| 0.3254 | 0.2019 |
|                         | Female  | 0.5308 | 0.3137| 0.2081 | 0.1291 |
| \( \lambda = 8 \)       | Male    | 0.7574 | 0.4152| 0.2597 | 0.1506 |
|                         | Female  | 0.4843 | 0.2655| 0.1661 | 0.0963 |
| \( \lambda = 10 \)      | Female  | 0.4031 | 0.1902| 0.1058 | 0.0535 |
|                         | Male    | 0.5752 | 0.2517| 0.1320 | 0.0624 |
| \( \lambda = 11 \)      | Female  | 0.3678 | 0.1610| 0.0844 | 0.0399 |

It can be observed from Table 2 and Table 3 that the chance of up-jumping probability of interest rate with 0.4 and 0.6 respectively, shows a significant difference of actuarial present value. As shown in Table 2, we can obtain that the result of APV with various increase in inflation and jumping range of the \( t \)-th jump of the interest rate always increase. For instance, in Table 2, APV of 20 year term insurance for the insured age 30 pay Rp.10,000,000,- at the end of the year of death, then APV that
must be paid for \( \lambda = 5 \) and \( b = 0.0025 \) is Rp. 3,000,- for male and Rp. 2,000,- for female. When \( \lambda = 11 \)
and \( b = 0.0125 \), APV that must be paid is Rp. 3,957,000,- for male and Rp. 2,530,000 for female. From these results, it can be conclude that the value of APV for male is greater than the value of APV for female.

5. Conclusions
In this paper, we used inflation rate in the interest rate model based on Compound Poisson process and Brownian motion. The result of the model, when the increase in inflation is high with \( p = 0.4 \), then the interest rate will be increased. If \( p = 0.6 \), then obtained the interest rate which does not have a significant differences between five times and eleven times the increased in inflation. Moreover, we apply the proposed interest rate model with inflation to determine actuarial present value of term life insurance for discrete life annuities. When up-jumping probability of interest rate is high, then the value of APV will be decreased. Otherwise, up-jumping probability of interest rate is low, then the value of APV will be increased.

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