Quantum theory of two-photon vector vortex beams

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We develop the formal approach to a scheme for simultaneous generation of nonlocal quantum correlations in the spin and orbital degrees of freedom of hyperentangled photon pairs. The scheme is based on an entangled photon source pumped by a laser beam structured with a vector vortex polarization profile. This creates a two-photon quantum state exhibiting polarization dependent transverse correlations that can be accessed through coincidence measurements on the spatially separated photons. The calculated correlations are shown to present a spin-orbit profile typical of vector beams, however distributed on separate measurement regions. Our approach allows the design of the pump beam vector spatial structure and measurement strategies for potential applications of these entangled states such as in quantum communication.

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I. INTRODUCTION

Quantum technologies based on photonic devices require coherent control of different optical degrees of freedom. The interplay between polarization and transverse modes of a laser beam has been successfully exploited in a number of quantum and classical experiments. In the quantum domain we can quote different applications such as simulations of quantum algorithms, quantum random walks, environment induced entanglement, decoherence, non-Markovianity, quantum communications and quantum sensing. The structural nonseparability between polarization and transverse modes has been approached from different points of view and has been used for investigating important properties of entangled states. Quantum inspired experiments in classical optics has led to interesting applications such as the mode transfer between different degrees of freedom using the teleportation algorithm. Tripartite nonseparability has also been studied in classical optics using polarization, transverse and longitudinal modes. All these developments were considerably favored by the development of important tools for spin-orbit coupling in laser beams. These developments made possible the implementation of alignment-free quantum cryptography with vector beams.

In the quantum domain, the interplay between polarization and spatial coherence of entangled photon pairs was approached and has demonstrated quantum image control through polarization entanglement in spontaneous parametric down conversion (SPDC). SPDC is a reliable source of photon pairs entangled in different degrees of freedom. The phase matching conditions fulfilled by parametric interaction impose time-energy, space-momentum and polarization constraints that are at the origin of multiple quantum correlations characterizing entanglement between the generated photons. This multiple entanglement in different degrees of freedom is sometimes referred to as hyperentanglement.

In this work we present the theoretical description of photon pairs, simultaneously entangled in spin and orbital angular momentum, generated by the two-crystal-sandwich SPDC source pumped by a vector vortex beam. The polarization dependent spatial correlations between the down converted beams are calculated using the formal approach developed in Ref., where a setup with similar characteristics was implemented. While the individual intensity distributions of the down converted beams do not carry the pump spin-orbit properties, the quantum correlations between them exhibit the typical polarization dependent spatial distribution of a vector beam, as expected and experimentally observed in Ref.

Moreover, we also calculate the polarization dependent correlations in the orbital angular momentum (OAM) basis of the two-photon quantum state.

II. EXPERIMENTAL SCHEME FOR SPIN-ORBIT HYPERENTANGLEMENT

Let us consider a frequently used source of polarization entangled photon pairs also known as two-crystal-sandwich SPDC source. It is composed by two identical nonlinear crystals placed close together with their optical axes rotated by 90° relative to each other, as shown in Fig. 1. A laser beam at frequency ωp and wavevector kp is used to pump the crystals and generate photon pairs by spontaneous parametric down conversion (SPDC). The down converted signal (s) and idler (i) photons are generated with frequencies ωs and ωi, and wave vectors ks and ki, constrained by ωs + ωi = ωp and ks + ki = kp, that express energy and momentum conservation, respectively.

The pump beam is assumed to be prepared in a vector-vortex mode of the kind

$$\Psi(r) = \frac{\psi_1(r) \hat{e}_H + \psi_2(r) \hat{e}_V}{\sqrt{2}}, \quad (1)$$
where $\Psi(r)$ is the classical field amplitude, $\hat{e}_H$ and $\hat{e}_V$ are the horizontal and vertical polarization unit vectors, respectively, and $\psi_1(r)$ and $\psi_2(r)$ are two orthonormal functions that are solutions of the paraxial wave equation \[10\]. Those can be either Hermite-Gaussian (HG) or Laguerre-Gaussian (LG) modes, for example. This kind of internal non separability between the spin-orbit degrees of freedom can be evidenced by measuring the spatial mode after polarization filtering. For example, if the vector-vortex mode given by \[1\] passes through a polarization analyzer (a sequence of a quarter waveplate (QWP) and a half waveplate (HWP) followed by a horizontal polarizer, for example) characterized by angles $\gamma$ (QWP) and $\theta/2$ (HWP) with respect to the horizontal, the transmitted beam will exhibit a spatial function that is the linear combination

$$\psi_{sH}(r) = \cos \theta \psi_1(r) + e^{i\gamma} \sin \theta \psi_2(r). \quad \text{(2)}$$

Therefore, a variable spatial profile is manifested after polarization projection. Interestingly, this feature can be transferred to the spatial quantum correlations between signal and idler photons generated by SPDC.

Under type-I phase matching, the vertically polarized component of the pump beam generates a pair of horizontally polarized photons in the first crystal and transfers its accompanying transverse mode $\psi_1$ to the spatial quantum correlations of the down converted photons \[11\]. In the same way, the transverse mode $\psi_2$ is transferred to the spatial quantum correlations of the vertically polarized down converted photons generated in the second crystal. If the coherence length of the pump laser is larger than the length of the two-crystal source, vertically and horizontally polarized signal-idler modes will add coherently and produce a two-photon vector vortex state. A subtle and interesting effect takes place when the polarization information of the down converted photons is erased by means of a variable polarization measurement. Two polarizers are used to set the measurement bases before the photocounts are acquired by intensified charge coupled device (iCCD) cameras, which register the polarization-dependent spatial correlations. As we will show, these correlations exhibit typical features of vector beams, where polarization filtering is accompanied by a variable spatial profile. However, in our proposal this spin-orbit cross-talk is nonlocal.

### III. THEORETICAL MODEL FOR SPIN-ORBIT QUANTUM CORRELATIONS

Following the sketch of section \[11\] we now develop a theoretical approach for the spin-orbit quantum correlations between signal and idler fields. Our strategy will be first to obtain the quantum state produced by the SPDC process and then use it to evaluate the spatial distribution of signal-idler intensity correlations. Let us start by writing the positive and negative frequency parts of the electric field operator of pump, signal and idler beams as a superposition of plane waves with vertical and horizontal polarization

$$\hat{E}^+(j, r, t) = \left(\hat{E}^+_{jH}(r) \hat{e}_H + \hat{E}^+_{jV}(r) \hat{e}_V\right) e^{-i\omega_j t},$$

$$\hat{E}^-_{jH}(r, t) = \left[\hat{E}^+_{jH}(r, t)\right]^\dagger,$$

$$\hat{E}^-_{jV}(r) = i \mathcal{E}_j \int \hat{a}_{\mu}^+(\mathbf{k}_j) e^{i\mathbf{k}_j \cdot \mathbf{r}} d^3\mathbf{k}_j,$$

where $j = p, s, i$; $\hat{a}_{\mu}^+(\mathbf{k}_j)$ is the annihilation operator of photons with wave vector $\mathbf{k}_j$ and polarization $\mu = H, V$, and $\mathcal{E}_j$ is a constant resulting from the quantization process and having units of electric field.

The non linear coupling between pump, signal and idler is described by the interaction Hamiltonian

$$H_I(t) = \chi e^{-i\Delta \omega t} O_I + \chi e^{i\Delta \omega t} O_I^\dagger,$$

$$O_I = \int_V \left(\hat{E}^+_{pV} \hat{E}^+_{sH} \hat{E}^-_{iH} + \hat{E}^+_{pH} \hat{E}^-_{sV} \hat{E}^-_{iV}\right) d^3\mathbf{r}, \quad \text{(4)}$$

where $\chi$ is the nonlinear susceptibility, $V$ is the crystal volume and $\Delta \omega = \omega_p - \omega_s - \omega_i$. The first term inside the integral describes the annihilation of a $V$ polarized photon of the pump and creation of $H$ polarized signal and idler photons in the first crystal. The second term describes the annihilation of a $H$ polarized photon of the pump and creation of $V$ polarized signal and idler photons in the second crystal.
A. Two-photon spin-orbit quantum state

Up to first order in perturbation theory, the time evolution operator in the interaction picture is

$$U^{(1)}(\tau) = 1 - \frac{i}{\hbar} \int_{0}^{\tau} H_I(t) \, dt$$

(5)

$$= 1 - \frac{i\chi}{\hbar} \sin\left(\frac{\Delta\omega \tau}{2}\right) \left( e^{-\frac{i}{2}\Delta\omega \tau} O_I + e^{\frac{i}{2}\Delta\omega \tau} O_I^\dagger \right),$$

where $\tau$ is the interaction time, which is assumed to be the same for all three fields, pump, signal and idler, under phase matching conditions. As we can see, the longer the interaction time, the tighter the energy conservation condition ($\Delta\omega = 0$). At this point we can make the monochromatic approximation for the pump laser and assume the interaction time long enough to impose practically perfect energy conservation. In this case, the time evolution operator becomes

$$U^{(1)}(\tau) = 1 - \frac{i\chi\tau}{\hbar} \left( O_I + O_I^\dagger \right).$$

(6)

After passing through the crystals, the quantum state of the interacting beams is given by

$$|\Psi(\tau)\rangle = U^{(1)}(\tau) |\Psi(0)\rangle,$$

(7)

where $|\Psi(0)\rangle = |\psi_0\rangle_p |0\rangle_s |0\rangle_i$ is the input state of pump, signal and idler fields. Since signal and idler are initially in the vacuum state, no contribution to the time evolution can appear from the $O_I^\dagger$ term because its action involves the annihilation of signal and idler photons. The pump laser will be treated as a monochromatic beam, described by a multimode coherent state

$$|\psi_0\rangle_p = \prod_{k_p} |v_H(k_p)\rangle_H |v_V(k_p)\rangle_V,$$

(8)

where

$$\hat{a}_\mu(k) |v_\mu(k)\rangle_\mu = v_\mu(k) |v_\mu(k)\rangle_\mu,$$

(9)

and $v_\mu(k)$ is the coherent state amplitude associated with wave vector $k$ and polarization $\mu$. After parametric interaction, the quantum state of the pump, signal and idler modes is given by

$$|\Psi(\tau)\rangle = |\Psi(0)\rangle - \frac{i\chi\tau}{\hbar} O_I |\Psi(0)\rangle.$$

(10)

Let us work out the interaction term:

$$O_I = -i \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \int d^3k_p \int d^3k_s \int d^3k_i \, F(k_p, k_s, k_i)$$

$$\left[ \hat{a}_\mu^p(k_p) \hat{a}_\mu^s(k_s) \hat{a}_\mu^i(k_i) + \hat{a}_\mu^p(k_p) \hat{a}_\mu^s(k_s) \hat{a}_\mu^i(k_i) \right],$$

(11)

where we defined the phase matching function as

$$F(k_p, k_s, k_i) = \int \, e^{i(k_p - k_s - k_i) \cdot \mathbf{r}} \, d^3\mathbf{r}$$

$$\approx L_z \prod_{l=x,y} \frac{2 \sin\left((k_{pl} - k_{sl} - k_{il}) \cdot \mathbf{L}_l/2\right)}{k_{pl} - k_{sl} - k_{il}},$$

(12)

with $L_l$ being the crystal width along the $l$ direction. Note that we have assumed a longitudinally thin crystal satisfying $(k_{pz} - k_{sz} - k_{iz}) L_z \ll 1$. This results in

$$O_I |\Psi(0)\rangle = -i \mathcal{E}_p \mathcal{E}_s \mathcal{E}_i \int d^3k_p \int d^3k_s \int d^3k_i \, F(k_p, k_s, k_i)$$

$$\left[ v_\mu^p(k_p) |1_{k_p, H}\rangle |1_{k_s, V}\rangle |1_{k_i, V}\rangle + v_\mu^p(k_p) |1_{k_s, V}\rangle |1_{k_i, V}\rangle |1_{k_p, H}\rangle \right],$$

where $|1_{k,\mu}\rangle$ is a single-photon Fock state with wave vector $k$ and polarization $\mu$. We assume that the pump beam comes from a collimated and monochromatic laser propagating along the $z$ direction, so that we can approximate

$$v_\mu^p(k_p) \approx v_\mu^p(q_p) \delta(k_{pz} - k_0),$$

(14)

where $q_p$ is the transverse and $k_0$ is the longitudinal wave vector component. Moreover, signal and idler photons are detected at small solid angles along specific directions compatible with the phase matching condition. This geometric configuration, together with interference filters placed before the detectors, fix the selected wavelengths of signal and idler. This also restricts their detected wave vectors to a small neighborhood around their respective solid angles $\Omega_j (j = s, i)$. It will be useful to decompose the wavevectors into longitudinal ($k_j^L$) and transverse ($q_j$) parts $k_j = k_j^L + q_j$, and apply the paraxial approximation $|q_j| \ll |k_j^L|$. In this case,

$$k_j^L = \sqrt{k_j^2 - q_j^2} \approx k_j - \frac{q_j^2}{2k_j^L}.$$

(15)

In the paraxial regime and with fixed wavelengths for pump, signal and idler, the longitudinal wave vector components are fixed and the relevant plane wave modes can be labeled by the transverse wave vector $q$. The angular spectrum $v_\mu^p(q)$ represents the angular spectrum carried by the pump polarization mode $\mu$. The triple integrals in Eq. (13) are reduced to double integrals over a small domain $\Omega_j$ around the main longitudinal component, as illustrated in Fig. 2. Under these assumptions, the two-photon quantum state can be written in terms...
of the transverse wavevector distributions,
\[ O_{i} \left| \Psi(0) \right> = \left( -i \mathcal{E}_{p} \delta_{s,i} \right) \int_{\Omega_{p}} d^{2} q_{p} \int_{\Omega_{s}} d^{2} q_{s} \int_{\Omega_{i}} d^{2} q_{i} F(k_{p}, k_{s}, k_{i}) \left| \psi_{0} \right>_{p} \]
\[ \otimes \left[ v_{p}^{p}(q_{p}) \left| 1_{q_{s},H} \right> \left| 1_{q_{i},H} \right> + v_{p}^{i}(q_{p}) \left| 1_{q_{s},V} \right> \left| 1_{q_{i},V} \right> \right] . \]

In most experiments for producing spatial quantum correlations in SPDC, one deals with transversely wide \((k_{pl} - k_{sl} - k_{il})L_{l} \gg 1\) for \(l = x, y\) and longitudinally thin \((k_{pz} - k_{sz} - k_{iz})L_{z} \ll 1\) crystals. In this case, a tight phase matching condition is imposed on the transverse components of the wave vectors and a lose condition applies to the longitudinal component. This approximation can be expressed as
\[ F(k_{p}, k_{s}, k_{i}) \approx L_{z} \delta(q_{p} - q_{s} - q_{i}) . \]  

In this case, the integration over \(q_{p}\) can be readily performed and the final expression for the quantum state produced by the SPDC process is
\[ |\Psi(\tau)\rangle = |\psi_{0}\rangle_{p} \otimes (|0\rangle_{s} |0\rangle_{i} + |\Phi\rangle_{si}) , \]  

where
\[ |\Phi\rangle_{si} = \kappa \int d^{2} q_{s} \int d^{2} q_{i} v_{p}^{p}(q_{s} + q_{i}) \left| 1_{q_{s},H} \right> \left| 1_{q_{i},H} \right> \]
\[ + \kappa \int d^{2} q_{s} \int d^{2} q_{i} v_{p}^{i}(q_{s} + q_{i}) \left| 1_{q_{s},V} \right> \left| 1_{q_{i},V} \right> \]  

and \(\kappa \equiv -\chi(\hbar)\tau L_{z} \mathcal{E}_{p} \mathcal{E}_{s} \mathcal{E}_{i} .\) The expression in [19] encompasses the transverse momentum constraint between signal and idler that gives rise to spatial quantum correlations.

### B. Spatial quantum correlations

In the paraxial regime, we can adopt scalar diffraction theory to study the propagation of the interacting beams after leaving the crystals. The relevant plane wave modes are labeled by the transverse wave vector \(q\) and the pump amplitude distribution \(v_{p}^{p}(q)\) represents the pump angular spectrum carried by the polarization mode \(\mu\). At the crystals’ center \((z = 0)\), the spatial distribution of the pump beam in each polarization component is given by the following Fourier decomposition,
\[ W_{\mu}(\rho, 0) = \int v_{p}^{p}(q) e^{iq \cdot \rho} d^{2} q , \]  

and the propagated field distribution is given by the Fresnel integral
\[ W_{\mu}(\rho, z) = e^{ikz} \int v_{p}^{p}(q) e^{i(q \cdot \rho - \frac{\rho^{2}}{2}z)} d^{2} q . \]  

The multimode coherent state \(|\psi_{0}\rangle_{p}\) carries the spatial properties of the pump beam in the Fourier domain through the angular spectra \(\psi_{H}(q)\) and \(\psi_{V}(q)\), independently imprinted in each pump polarization component. This will be crucial for the polarization-dependent spatial correlations between signal and idler photons.

The longitudinal positions of signal and idler detectors are fixed and the spatial quantum correlations are measured as a function of their transverse position \(\rho_{j}\) \((j = s, i)\). We recall that a polarization analyzer is placed before each detector. If we assume that these analyzers are set at angles \(\gamma_{j}\) and \(\delta_{j}/2\), then the electric field operator in each detector is
\[ \hat{E}(\rho_{j}) = \cos \theta_{j} \hat{E}_{j,H} \rho_{j} + \sin \theta_{j} e^{i\gamma_{j}} \hat{E}_{j,V} \rho_{j} . \]

The intensity distribution in each detection arm is
\[ I(\rho_{j}) = \langle \hat{E}^{\dagger}_{j,-}(\rho_{j}) \hat{E}_{j,+}(\rho_{j}) \rangle = \| \hat{E}_{j,+}(\rho_{j}) |\Psi(\tau)\rangle\|^{2} , \]

and the intensity correlations between the two detection arms are
\[ C(\rho_{s}, \rho_{i}) = \langle \hat{E}^{\dagger}_{i,-}(\rho_{s}) \hat{E}_{i,-}(\rho_{s}) \hat{E}_{i,+}(\rho_{i}) \hat{E}_{i,+}(\rho_{i}) \rangle = \| \hat{E}_{i,+}(\rho_{s}) |\Psi(\tau)\rangle\|^{2} . \]

In the monochromatic and paraxial approximations the electric field operators can be written as
\[ \hat{E}_{j,\mu}(\rho) = i \mathcal{E}_{j} \int a_{\mu}^{\dagger}(q_{j}) e^{i(q_{j} \cdot \rho + \varphi(q_{j}))} d^{2} q_{j} , \]
\[ \varphi(q_{j}) = \sqrt{k_{j}^{2} - q_{j}^{2}} z_{j} \approx k_{j} z_{j} - \frac{q_{j}^{2} z_{j}}{2k_{j}} , \]

where \(z_{j}\) is the longitudinal distance between the crystals’ center and detector \(j\).

Note that no contribution comes from the vacuum component in \(|\Psi(\tau)\rangle\), so that we only need to care about contributions coming from \(|\Phi\rangle_{si}\). The calculation of the intensity distributions and correlations will be significantly simplified by the definition of the following vectors
\[ |\alpha_{j,\mu}\rangle = \hat{E}_{j,\mu}(\rho_{j}) |\Phi\rangle_{si} , \]
\[ |\beta_{\mu}\rangle = \hat{E}_{i,\mu}(\rho_{s}) \hat{E}_{i,\mu}(\rho_{s}) |\Phi\rangle_{si} . \]

We can easily workout these auxiliary vectors using
\[ a_{j}(q_{j}) |n_{q_{j}}\rangle = \delta_{j,\mu} \delta(q_{j} - q_{j}) \sqrt{n_{q_{j}}} |n_{q_{j}} - 1\rangle . \]

As detailed in Appendix A, the result for the individual intensities of signal and idler is
\[ I_{j} = \cos^{2} \theta_{j} \mathcal{I}_{H} + \sin^{2} \theta_{j} \mathcal{I}_{V} , \]

where \(j = s, i\) and
\[ \mathcal{I}_{H(V)} = \kappa^{2} \mathcal{E}_{s}^{2} \int d^{2} \rho \| W_{V(H)}(\rho, z) \|^{2} . \]

Note that the pump spatial properties are washed out in the individual intensities. In contrast, the coincidence
count distribution carries the spatial profile of the pump beam, distributed in the joint coordinates of signal and idler. The result derived in Appendix B is

\[
C(\rho_s, \rho_i) = K^2 \left| \cos \theta_s \cos \theta_i W_V(\rho_+, z) + \sin \theta_s \sin \theta_i e^{i(\gamma_s + \gamma_i)} W_H(\rho_+, z) \right|^2 , \tag{30}
\]

where we assume degenerate SPDC \((k_s = k_i = k_p/2)\) and equidistant signal and idler detectors \((z_s = z_i = z)\). In this case, we have

\[
\rho_+ = \frac{\rho_s + \rho_i}{2}. \tag{31}
\]

Note that Eq. \((30)\) shows simultaneous dependence on the joint coordinates of signal and idler detectors and on the joint orientations of their respective polarization analyzers.

### IV. NONLOCAL VECTOR VORTEX BEAM

We can now investigate the polarization dependent spatial correlations when the pump beam is prepared in a vector mode of the kind expressed in Eq.\((1)\). For example, let the pump beam be prepared in a superposition of the Hermite-Gaussian mode \((0, 1)\) with horizontal polarization and mode \((1, 0)\) with vertical polarization so that

\[
W_V(\rho) = \frac{\psi_{10}(\rho)}{\sqrt{2}} = \frac{x}{\sqrt{\pi} w^2} e^{-(x^2+y^2)/w^2} ,
\]

\[
W_H(\rho) = \frac{\psi_{01}(\rho)}{\sqrt{2}} = -\frac{y}{\sqrt{\pi} w^2} e^{-(x^2+y^2)/w^2}. \tag{32}
\]

The longitudinal dependence has been made implicit in the variation of the mode width \(w(z) = w_0 \sqrt{1 + (z/z_R)^2}\), where \(w_0\) is the mode waist and \(z_R = \pi w_0^2 / \lambda\) is the Rayleigh distance.

Then, the expression given in \((30)\) brings us to

\[
C(\rho_s, \rho_i) = \frac{K^2}{\pi w^4} e^{-\frac{2|x_s + x_i|^2}{w^2}} e^{-\frac{2|y_s + y_i|^2}{w^2}} \times \left| \cos \theta_s \cos \theta_i (x_s + x_i) + \sin \theta_s \sin \theta_i e^{i(\gamma_s + \gamma_i)} (y_s + y_i) \right|^2 . \tag{33}
\]

This expression shows simultaneous nonlocal behavior on position and polarization settings. The spatial distribution of the coincidence counts depends on the joint orientations of the detection polarizers and on the joint transverse positions of the detectors. The physical consequence of this double nonlocal behavior can be revealed by a simple measurement strategy. Let us set the signal polarization analyzer at \(\gamma_s = 0\) and \(\theta_s = \pi/4\), so that the polarization information of the signal photons is erased, and the position of the idler detector kept at its origin \(x_i = y_i = 0\). In this case, the resulting coincidence pattern becomes

\[
C(\rho_s, 0) = \frac{K^2}{2\pi w^4} e^{-\frac{2|x_s|^2}{w^2}} e^{-\frac{2|y_s|^2}{w^2}} \left( \cos \theta_i x_s + \sin \theta_i e^{i\gamma_i} y_s \right)^2 . \tag{34}
\]

We can see that the resulting coincidence pattern corresponds to the intensity distribution of a first order Hermite-Gaussian mode function of the signal position, transformed according to the parameters of the idler polarization analyzer. In Fig. 3 we plot this coincidence pattern as a function of the signal coordinates \(\rho_s = (x, y)\), indicating that it follows the rotation of the idler polarization analyzer. This situation is similar to the one exhibited in Eq.\((1)\), where the spatial profile after transmission of a single vector beam through a polarization depends on the transmission angle. However, here we have the orientation of the spatial pattern determined by the angle of a remote polarizer. This effect can be useful for remote alignment of quantum cryptography stations or as a gyroscope, for example.

### V. PUMPING WITH LAGUERRE-GAUSSIAN BEAMS

It is also interesting to see how orbital angular momentum affects the nonlocal correlations. For this end we assume the pump mode to be prepared in a superposition of Laguerre-Gaussian modes with zero radial order and OAM \(-l\) with horizontal polarization and \(+l\) with vertical polarization so that

\[
W_{V(H)}(\rho) = \sqrt{\frac{2^{\pm l}}{\pi w^2}} \left( \frac{x \pm iy}{w} \right)^{|l|} e^{-(x^2+y^2)/w^2}. \tag{35}
\]
their mathematical expression in polar coordinates is
\[
\psi_i^p(\rho) = R_{li}^p(\rho) e^{i\phi_i},
\]
\[
R_{li}^p(\rho) = \sqrt{\frac{(2\pi)^l}{(p+|l|)!}} \frac{\rho^{|l|}}{w_0} L_p^{|l|}(\rho) e^{-\rho^2/2},
\]
where \( p \) is the radial order, \( l \) is the topological charge, \((\rho, \phi)\) are the transverse coordinates in the polar system, \( w_0 \) is the mode waist and \( L_p^{|l|} \) are the associated Laguerre polynomials. Pump, signal and idler beams are assumed to be mode matched, so that their wavefront radii are equal along the interaction length. This imposes a common Rayleigh distance \( z_R = z_{R_s} = z_{R_i} \), which requires a different waist for each interacting beam according to \( z_R = \pi w_0^2/\lambda_j \).

The two-photon quantum state generated by the SPDC process can be cast as a superposition of different partitions of the pump OAM between signal and idler. First, let us derive the LG expansion of the correlated transverse momentum distribution of signal and idler,
\[
v(q_s + q_i) = \sum_{p_i,l_i} A_{p_i,l_i}^p \tilde{\psi}_{l_i}^{p_i}(q_s) \tilde{\psi}_{l_2}^{p_2}(q_i),
\]
where \( \{ \tilde{\psi}_l^p(q) \} \) are the LG mode functions in Fourier domain,
\[
\tilde{\psi}_l^p(q) = \frac{1}{2\pi} \int \psi_l^p(\rho) e^{-i q \cdot \rho} d^2 \rho, \\
\psi_l^p(\rho) = \frac{1}{2\pi} \int \tilde{\psi}_l^p(q) e^{i q \cdot \rho} d^2 q .
\]
Both in the Fourier and position domains, the LG modes satisfy the following orthonormality
\[
\int \psi_l^{p*}(\rho) \psi_p^l(\rho) d^2 \rho = \delta_{pp'} \delta_{ll'},
\]
\[
\int \tilde{\psi}_l^{p*}(q) \tilde{\psi}_l^p(q) d^2 q = \delta_{pp'} \delta_{ll'},
\]
and completeness
\[
\sum_{p,l} \psi_l^{p*}(\rho) \psi_l^p(\rho') = \delta(\rho - \rho'),
\]
\[
\sum_{p,l} \tilde{\psi}_l^{p*}(q) \tilde{\psi}_l^p(q') = \delta(q - q'),
\]
relations. By using them, the LG expansion coefficients in Eq. 39 are given by
\[
A_{p_i,l_i}^{p_2} = \int v(q_s + q_i) \tilde{\psi}_{l_i}^{p_i*}(q_s) \tilde{\psi}_{l_2}^{p_2*}(q_i) d^2 q_s d^2 q_i .
\]
These coefficients are more easily calculated in the position domain. This can be achieved by plugging the inverse Fourier transform of \( \tilde{\psi}_l^p(q) \) and \( v(q) \) into Eq. 43.
and using the Fourier representation of the Dirac delta function. The resulting expression is

$$A_{l_1l_2}^{p_1p_2} = \int W(\rho) \psi_{l_1}^{p_1*}(\rho) \psi_{l_2}^{p_2*}(\rho) d^2\rho. \quad (44)$$

With the expansion coefficients in hands, we can rewrite the two-photon state \cite{19} in the Fock basis of OAM modes

$$|\Phi\rangle_{s_1} = \kappa \sum_{p_1, l_1, p_2, l_2} [A_{l_1l_2}^{p_1p_2} |p_1, l_1, H\rangle_{s_1} |p_2, l_2, H\rangle_{i} + B_{l_1l_2}^{p_1p_2} |p_1, l_1, V\rangle_{s_1} |p_2, l_2, V\rangle_{i}], \quad (45)$$

where

$$|p, l, \mu\rangle = \int d^2q \tilde{\psi}_{l}^\mu(q) |l_{q\mu}\rangle \quad (\mu = H, V),$$

$$\langle p, l, \mu|p', l', \mu'\rangle = \delta_{pp'}\delta_{ll'}\delta_{\mu\mu'}, \quad (46)$$

are single-photon OAM states with polarization \(\mu\) and the coefficients \(A_{l_1l_2}^{p_1p_2}\) and \(B_{l_1l_2}^{p_1p_2}\) are given by Eq. (44) with \(W_{V}\) and \(W_{H}\) respectively.

Let us assume that the pump beam is prepared in a vector mode of the kind considered in the second example of section IV, a superposition of the LG mode \((0, -l)\) with horizontal polarization and mode \((0, +l)\) with vertical polarization

$$\Psi(\rho) = \frac{\psi_{l=1}^0(\rho) \hat{e}_H + \psi_{l=1}^0(\rho) \hat{e}_V}{\sqrt{2}}. \quad (47)$$

In this case, we have set \(W_{V} = \psi_{l=1}^0/\sqrt{2}\) and \(W_{H} = \psi_{l=1}^0/\sqrt{2}\) and the OAM expansion coefficients are

$$A_{l_1l_2}^{p_1p_2} = \sqrt{2}\pi\delta_{l_1+l_2} \int R_{l_1l_2}^{p_1p_2}(\rho) R_{l_1l_2}^{p_1p_2*}(\rho) \hat{e}_H \rho d\rho,$$

$$B_{l_1l_2}^{p_1p_2} = \sqrt{2}\pi\delta_{l_1-l_2} \int R_{l_1l_2}^{p_1p_2}(\rho) R_{l_1l_2}^{p_1p_2*}(\rho) \hat{e}_V \rho d\rho, \quad (48)$$

where the Kronecker deltas, \(\delta_{l_1+l_2}\) and \(\delta_{l_1-l_2}\), result from the angular integration. They impose the OAM conservation condition in the two-photon state \cite{45}. Moreover, the following symmetry relations hold

$$A_{m, l-m}^{p_1p_2} = B_{m, l-m}^{p_1p_2},$$

$$A_{m, l-m}^{p_1p_2} = A_{m, l-m}^{p_1p_2}. \quad (49)$$

They allow us to rewrite the two-photon state in a more convenient way that makes more evident the simultaneous OAM and polarization entanglement

$$|\Phi\rangle_{s_1} = \kappa \sum_{p_1, m, p_2, m-l} A_{m, l-m}^{p_1p_2} \times \left( |p_1, m, H\rangle |p_2, l-m, H\rangle + |p_1, -m, V\rangle |p_2, m-l, V\rangle \right). \quad (50)$$

This form of the two-photon vector beam quantum state exhibits explicitly the simultaneous OAM and polarization entanglement. It is useful for measurement schemes where OAM sorting is implemented in each detection arm, as depicted in Fig. 5. While this representation of the OAM sorter is idealized for pedagogic purposes, there are several types of architectures being developed to this end, meaning that the OAM sorting operation is already viable with increasing efficiency and resolution \cite{44}.

It is instructive to obtain the OAM decomposition of the spatial correlations. We start by writing the positive frequency component of the electric field operator in terms of annihilation operators \(a_{pl}^j\) and \(b_{pl}^j\) \((j = s, i)\) of photons in Laguerre-Gaussian modes \((p, l)\) with horizontal and vertical polarizations, respectively,

$$\hat{E}_{jH}^+ = \sum_{pl} a_{pl}^j \tilde{\psi}_{l}^j(\rho),$$

$$\hat{E}_{jV}^+ = \sum_{pl} b_{pl}^j \tilde{\psi}_{l}^j(\rho). \quad (51)$$

These annihilation operators are related to those in the transverse momentum basis through

$$a_{H}^j(q) = \frac{1}{2\pi i \tilde{E}_j} \sum_{pl} a_{pl}^j \tilde{\psi}_{l}^j(q),$$

$$a_{pl}^j = 2\pi i \tilde{E}_j \int a_{pl}^j(q) \tilde{\psi}_{l}^{j*}(q) d^2q, \quad (52)$$

and the equivalent relations for \(a_{V}^j(q)\) and \(b_{pl}^j\). The action of the OAM annihilation operators on the corresponding Fock states is given by

$$a_{pl}^j |p', l', H\rangle = b_{pl}^j |p', l', V\rangle = \delta_{pp'}\delta_{ll'} |\text{vac}\rangle, $$

$$a_{pl}^j |p', l', V\rangle = b_{pl}^j |p', l', H\rangle = 0. \quad (53)$$

In terms of the OAM eigenfunctions, the auxiliary vectors defined in Eq. \cite{20} for calculating the quantum correla-
tions become
\[ |\beta_H\rangle = \hat{E}_{iH}^+(\rho_i)\hat{E}_{iH}^+(\rho_i)|\Phi_{si}\rangle, \]
\[ = \kappa \sum_{p_1, p_2, m} A_{m l m}^{p_1 p_2} \psi_{m}^{p_1} (\rho_s) \psi_{m}^{p_2} (\rho_i) |\text{vac}\rangle, \]
\[ \beta_V\rangle = \hat{E}_{iV}^+(\rho_i)\hat{E}_{iV}^+(\rho_s)|\Phi_{si}\rangle, \]
\[ = \kappa \sum_{p_1, p_2, m} A_{m l m}^{p_1 p_2} \psi_{m}^{p_1} (\rho_s) \psi_{m}^{p_2} (\rho_i) |\text{vac}\rangle. \]

Finally, the coincidence counts can be calculated from Eq. (B1) giving
\[ C(\rho_s, \rho_i) = \kappa^2 \left| \sum_{p_1, p_2, m} A_{m l m}^{p_1 p_2} F_{m l m}^{p_1 p_2} (\rho_s, \rho_i) \right|^2, \]
where
\[ F_{m l m}^{p_1 p_2} (\rho_s, \rho_i) = \cos \theta_s \cos \theta_i \psi_{m}^{p_1} (\rho_s) \psi_{m}^{p_2} (\rho_i) + \]
\[ + e^{i\gamma_+} \sin \theta_s \sin \theta_i \psi_{m l}^{p_1} (\rho_s) \psi_{m l}^{p_2} (\rho_i), \]
and \( \gamma_+ \equiv \gamma_s + \gamma_i \). Note that Eq. (55) is the Schmidt decomposition of the coincidence pattern derived in Eq. \[ \text{[36]} \] in terms of factorized OAM eigenfunctions for signal and idler. The different components \( F_{m l m}^{p_1 p_2} (\rho_s, \rho_i) \) can be accessed by mode sorting before each detector, as indicated in Fig. 6a. First, the \( H \) and \( V \) polarizations of signal and idler are separated by polarizing beam splitters (PBS) and pass through OAM sorters. Then, the \( m \) component with \( H \) polarization is recombined with the \( -m \) component with \( V \) polarization in a second PBS at the signal arm. In the same way, the \( l - m \) component with \( H \) polarization and \( m - l \) with \( V \) polarization are recombined at the idler arm. Finally, the polarization information is erased in each arm by analyzers (PA).

Note that two correlation channels are involved. The OAM components of \( H \)-polarized photons add up to \( l \) while those of \( V \)-polarized photons add up to \( -l \). After polarization erasure, the two channels interfere and the resulting coincidence pattern is \( |F_{m l m}^{p_1 p_2} (\rho_s, \rho_i) |^2 \). They differ fundamentally from those calculated in section \[ \text{IV} \] (see Fig. 1). There, similar patterns are obtained when either \( \rho_s \) or \( \rho_i \) is scanned while the other remains fixed. Here, due to transverse mode analysis before detection, the resulting coincidence patterns will present different shapes, depending on which detector is scanned. In Fig. 6a we show different OAM correlated images \( |F_{m l m}^{00} (\rho_s, \rho_i) |^2 \) for a pump topological charge \( l = 2 \) and polarization analysis set to \( \theta_s = \theta_i = \pi/4 \), \( \gamma_s = \gamma_i = 0 \).

VI. CONCLUSION

In conclusion, we present the quantum theory of polarization dependent spacial correlations between entangled photons generated by spontaneous parametric down conversion. The scheme studied is based on a hyper-entangled photon source composed by two nonlinear crystals to produce simultaneous nonlocal correlations in the spin and orbital angular momentum of entangled photon pairs. A structured pump beam, carrying internal non separability between its polarization and transverse mode, can generate spin-orbit hyperentangled photon pairs through SPDC. The two-photon spin-orbit quan-
tum state generated by the process is derived and the corresponding quantum correlations are calculated both in position and OAM domains. We show how the spatial correlations between signal and idler fields can be shaped by different settings of remote polarizers placed before their respective detectors. As an example, we show that Hermite-Gaussian and Laguerre-Gaussian coincidence distribution over the signal coordinates can be transformed by the remote control of the idler polarization settings. This resembles the behavior of a vector beam and, in this sense, we interpret this type of two-photon spin-orbit structure as a nonlocal vector beam. Moreover, the Schmidt decomposition in terms of two-photon spin-orbit structure as a nonlocal vector beam and, in this sense, we interpret this type of polarization dependent spatial correlations that can be accessed with mode filtering techniques. This subject can be further exploited with recently reported developments based on machine learning to recognize and classify non-trivial two-photon polarization patterns \[45\].

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Appendix A: Individual Intensities

We will workout explicitly the intensity of horizontally polarized signal photons. The extension to vertical polarization and to idler photons is straightforward.

\[ I_j = ||\cos \theta_j |\alpha_{jH}| + \sin \theta_j e^{i \gamma_j} |\alpha_{jV}|||^2, \]  

(A1)

The corresponding auxiliary vector becomes

\[ |\alpha_{sH}| = i \kappa E_s \int d^2 q_i \ G_V(q_i, \rho_s) |1_{q_i, V}|, \]  

(A2)

where

\[ G_V(q_i, \rho_s) \equiv \int d^2 q_s \ \psi_V^*(q_s + q_i) e^{i |q_i, \rho_s + \varphi(q_s)|} \]  

(A3)

\[ = e^{-i (q_i \rho_s + \frac{\pi}{2} z_s)} W_V \left( \rho_s + \frac{z_s}{2 k_s} q_i, z_s \right). \]

Using the orthonormality condition for the Fock states \( \langle n_{q, \mu}|m_{q', \nu} \rangle = \delta_{nm} \delta_{\mu \nu} \delta(q - q') \), we have \( \langle \alpha_{sH}|\alpha_{sV} \rangle = 0 \) and

\[ \langle \alpha_{sH}|\alpha_{sH} \rangle = \kappa^2 E_s^2 \int d^2 q_i \left| W_V \left( \rho_s + \frac{z_s}{2 k_s} q_i, z_s \right) \right|^2 \]

\[ = \kappa^2 E_s^2 \int d^2 \rho \left| W_V (\rho, 0) \right|^2, \]  

(A4)

assuming \( W_V \) to be normalizable. Along the same lines we can easily arrive at

\[ \langle \alpha_{sV}|\alpha_{sV} \rangle = \kappa^2 E_s^2 \int d^2 q_i \left| W_H \left( \rho_s + \frac{z_s}{2 k_s} q_i, z_s \right) \right|^2 \]

\[ = \kappa^2 E_s^2 \int d^2 \rho \left| W_H (\rho, 0) \right|^2. \]  

(A5)

Moreover, the same deduction can be applied to the idler individual intensity and we finally get

\[ I_j = \cos^2 \theta_j I_H + \sin^2 \theta_j I_V, \]  

(A6)

with \( j = s, i \) and

\[ I_{H(V)} = \kappa^2 E_s^2 \int d^2 \rho \left| W_{H(V)} (\rho) \right|^2. \]  

(A7)

Note that after integration, no spatial dependence is left in the individual intensities. This means that the pump spatial properties are washed out in the individual intensity of the signal beam.

Appendix B: Coincidence Counts

The coincidence count can be obtained from the following norm,

\[ C(\rho_s, \rho_i) = \left| \left( \cos \theta_s \cos \theta_i |\beta_H \rangle + \sin \theta_s \sin \theta_i e^{i (\gamma_s + \gamma_i)} |\beta_V \rangle \right) \right|^2. \]  

(B1)
This norm can be calculated by writing the auxiliary vectors in terms of the vacuum state as follows,

\[ |\beta_\mu\rangle = -\kappa \mathcal{E}_s \mathcal{E}_i \int d^2 q_s \int d^2 q_i \, v^p_\mu (q_s + q_i) \times e^{i(q_i \cdot \rho_s + q_s \cdot \rho_i + \phi(q_s) + \phi(q_i))} |\text{vac}\rangle \]

\[ = -\kappa \mathcal{E}_s \mathcal{E}_i |J|^2 \int d^2 q_+ \int d^2 q_- e^{i(q_+ \cdot \rho_{s+} + \phi(q_+))} \times \int d^2 q_+ v^p_\mu (q_+) \times e^{i(q_- \cdot \rho_{s+} + \phi(q_-))} |\text{vac}\rangle , \quad (B2) \]

where \( \mu \neq \mu' \) and we defined

\[ q_+ = q_s + q_i \quad \text{and} \quad q_- = \frac{k_i z_s}{k_s z_i} q_s - q_i , \]

\[ \rho_+ = \frac{k_i z_s}{k_s z_i} (\rho_s + \rho_i) \quad \text{and} \quad \rho_- = \frac{k_i z_s}{k_s z_i} (\rho_s - \rho_i) , \]

\[ \phi(q_+) = k_s z_s + k_i z_i - \left( \frac{k_s z_i}{k_s z_i + k_i z_s} \right) \frac{z_s}{2k_s} q^2_+ , \]

\[ \phi(q_-) = -\left( \frac{k_i z_i}{k_s z_i + k_i z_s} \right) \frac{z_i}{2k_s} q^2_- , \]

\[ J = -\frac{k_i z_i}{k_s z_i + k_i z_s} . \quad (B3) \]

The integrals in \([B2]\) are considerably simplified by assuming \( z_s = z_i = z \) and using \( k_s + k_i = k_p \),

\[ \int d^2 q_- \, e^{i(q_- \cdot \rho_{s+} + \phi(q_-))} = e^{-i \frac{k_i k_s}{k_p} \frac{z^2}{2}} , \]

\[ \int d^2 q_+ v^p_\mu (q_+) \times e^{i(q_+ \cdot \rho_{s+} + \phi(q_+))} = W_{\mu'} (\rho_{s+}, z) \quad (B4) \]

The integration on \( q_- \) gives an irrelevant phase factor, while that on \( q_+ \) brings the pump spatial properties as a function of the joint positions of signal and idler. Thus, we arrive at the following result for the auxiliary vectors

\[ |\beta_\mu\rangle = -\kappa \mathcal{E}_s \mathcal{E}_i |J|^2 \, e^{-i \frac{k_i k_s}{k_p} \frac{z^2}{2}} W_{\mu'} (\rho_{s+}, z) |\text{vac}\rangle . \quad (B5) \]

Finally, the intensity correlations are given by

\[ C(\rho_{s+}, \rho_i) = K^2 \left| \cos \theta_s \cos \theta_i W_V (\rho_{s+}, z) + \sin \theta_s \sin \theta_i e^{i(\gamma_s + \gamma_i)} W_H (\rho_{s+}, z) \right|^2 , \quad (B6) \]

where \( K \equiv \kappa \mathcal{E}_s \mathcal{E}_i |J|^2 \).
