Limit theorems for excursion sets of subordinated Gaussian random fields with long memory

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\textbf{ABSTRACT}
This paper considers the asymptotic behaviour of volumes of excursion sets of subordinated Gaussian random fields with (possibly) infinite variance. Actually, we consider integral functionals of such fields and obtain their limiting distribution using the Hermite expansion of the integrand. We consider the general non-stationary Gaussian random fields, including stationary and anisotropic special cases. The limiting random variables in our limit theorems have the form of multiple Wiener-Itô integrals. We illustrate most results with corresponding examples.

1. Introduction

For a real-valued measurable random field \( \{X(t), t \in \mathbb{R}^d\} \), the volume of excursion set \( A_u(X,W) = \{t \in W : X(t) \geq u\} \) in observation window \( W_n \subset \mathbb{R}^d \) is given by

\[ \nu_d(A_u(X,W)) = \int_W \mathbb{1}\{X(t) \geq u\} \nu_d(dt). \]

Here and further in this paper, \( \nu_d(\cdot) \) denotes the Lebesgue measure in \( \mathbb{R}^d \). Volumes and other geometric characteristics of excursions of random fields are widely used for data analysis purposes in physics and cosmology (see e.g. \cite{28}), medicine \cite{2, 43}, materials science \cite{36, 44}.

The volumes of excursion sets \( \{A_u(X,W_n), n \geq 1\} \) in observation windows \( W_n, n \geq 1 \) form a sequence of random variables. We expect the existence of the limit in distribution

\[ \lim_{n \to \infty} \frac{\nu_d(A_u(X,W_n)) - a_n}{b_n}. \]
for some number sequences $a_n$, $b_n > 0$, $n \in \mathbb{N}$, as observation windows $W_n$ grow in
van Hove sense, i.e., $\nu_d(W_n) \rightarrow \infty$, $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \nu_d^{-1}(W_n)\nu_d(\partial W_n \oplus B_r(o)) = 0$, $r > 0$.

During past decades, a significant contribution was made to find the limiting distribution in (1.1) for isotropic and/or stationary random fields, see the books [19] [24]. The result of Bulinski et. al. [9] states that limiting distribution in (1.1) is Gaussian if $X$ is a Gaussian centered stationary random field with continuous covariance function $C(t) = \mathbb{E}[X(0)X(t)]$ such that $|C(t)| = O(||t||_{2}^{-\alpha})$ for some $\alpha > d$ as $||t||_{2} \rightarrow \infty$. In case $0 < \alpha < d$, the field $X$ is long-range dependent and such result can not be used.

**Definition 1.1.** A square-integrable stationary centered random field $\{X(t), t \in \mathbb{R}^d\}$ with covariance function $C(t) = \mathbb{E}[X(0)X(t)], t \in \mathbb{R}^d$ is called long-range dependent (or with long memory) if

$$
\int_{\mathbb{R}^d} |C(t)| \, dt = +\infty
$$

and weakly dependent if this integral is finite.

Real data bring evidence of long-memory property in many fields of modern science. For example, the long-memory properties of the final energy demand in Portugal are detected in [5]. An overview of the state of the art in the theoretical findings for long range dependent stochastic processes can be found, for instance, in [6] and [38].

The random fields used in cosmology (potential, temperature, velocity, density of matter, etc.) are mostly Gaussian or derived from Gaussian random fields as their local transformation (Rayleigh, Maxwell, Lognormal and Rectangular processes), see e.g. [10] [11]. Frequently used transforms are $f(x) = x + \beta x^3, x \in \mathbb{R}, \beta > 0$ (cubic model) and $f(x) = x + \alpha(x^2 - 1), x \in \mathbb{R}, \alpha > 0$ (quadratic model), cf. [39].

For example, in [47] authors consider a model in which the gravitational potential $\Phi$ is a linear combination of a Gaussian random field $\phi$ and the square of the same random field, $\Phi = \phi + \alpha \phi^2 - \mathbb{E}\phi^2$, where $\alpha > 0$. Lognormal random field models with $X(t) = \exp(Y(t))$, where $Y$ is a Gaussian random field, are of interest in radar and image processing, see e.g. [14]. For further physical literature on Gaussian subordinated fields we refer to [3] [4].

The theory of random fields with long memory is not so developed as for stochastic processes. The first studies on this topic can be found in [19] [21] [24] [25]). They prove limit theorems for functionals of the form

$$
Z_n = \int_{W_n} G(X(s)) \, ds, \text{ as } n \rightarrow +\infty,
$$

where $W_n$ are some growing sets and $X$ is an isotropic stationary Gaussian random field with covariance function $C(s, t)$, that depends only on the distance $||s - t||_{2}$. This means that probability law of random field $X$ is invariant with respect to rigid motions.

Due to results in [19], if the Hermite rank of function $G$ is greater than 2 then the limiting distribution in (1.3) is non-Gaussian. Much earlier, the non-Gaussian limit was found in [35] due to non-summable correlations and non-linearity of function $G$. This paper led to further developments in 70s and 80s (see e.g. [12] [22]).

Consider the Hilbert space $L^2(\mathbb{R}, \varphi)$, with weight $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, x \in \mathbb{R}$. Hermite
polynomials \( \{H_k\}_{k \geq 0} \), given by
\[
H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, k \geq 0,
\]
form a complete orthogonal system in \( L^2(\mathbb{R}, \varphi) \) (see e.g. [1] Chapter 22, [13] Chapter 2, [17]), that is,
\[
\langle H_k, H_l \rangle := \int_{\mathbb{R}} H_k(x)H_l(x)\varphi(x)dx = \delta_{kl}k!, k \geq 0.
\]
The first few polynomials are
\[
H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \ldots x \in \mathbb{R}.
\]

**Definition 1.2.** For a function \( G \in L^2(\mathbb{R}, \varphi) \) its Hermite rank is
\[
\text{rank } G = \min\{k \in \mathbb{N} | \langle G, H_k \rangle \neq 0 \}.
\]

In the last few years, the active research of anisotropic linear random fields with long range dependence started with papers [22, 23] and [20]. The papers [30, 33, 40, 41] introduced the notions of scaling transition and distributional long-range dependence for stationary linear random fields on \( \mathbb{Z}^d \).

In this paper we extend the above lines of research to non-stationary random fields. They arise naturally either as weighted/transformed stationary fields or as a result of filtering. Particularly, non-stationary filtered random fields are used in astronomy, when spatial structure is studied by using both the time and wavelength dimensions and the method of Doppler tomography, see [46]. For example, a filtered random field when spatial structure is studied by using both the time and wavelength dimensions.

We start the paper with the central limit case in (1.1) for very general non-stationary random fields. More precisely, for a subordinated Gaussian random field \( \{X(t) = f(Y(t)), t \in \mathbb{R}^d\} \), where \( f : \mathbb{R} \to \mathbb{R} \) is a transformation function and \( Y \) is a centered Gaussian random field with \( \rho(t, s) = \text{corr}(Y(t), Y(s)), t, s, \in \mathbb{R}^d \), we obtain the convergence to \( N(0, 1) \) in (1.1) if rank \( \{f(\cdot) \geq u\} = 1 \) and
\[
\lim_{n \to \infty} \frac{\int_{W_n} \int_{W_n} \rho^2(t, s)\nu_d(dt)\nu_d(ds)}{\int_{W_n} \int_{W_n} \rho(t, s)\nu_d(dt)\nu_d(ds)} = 0.
\]

Since for the most applications rank \( \{f(\cdot) \geq u\} = 1 \), we simplify condition (1) for non-isotropic stationary covariance functions. We show that it is true only for long-range dependent random fields. We also pay special attention to the spatio-temporal
We also prove limit theorems for the cases rank $\{ f(\cdot) \geq u \} \geq 2$. To do so, we extend the problem to the limiting behaviour of general integral functionals of Gaussian random fields. The main technique here is the spectral theory and spectral representation of (non-)stationary random fields. The conditions ensuring our limit theorems are formulated via the asymptotic behaviour of spectral densities.

If the Hermite rank of transformation function $f$ is greater than 2, the limiting random variables in (1.1) have the form of multiple Wiener-Itô integrals (first introduced by Itô in [18]) defined as

$$I_m(f) = \int_{\mathbb{R}^d} f(x_1, \ldots, x_m) B(dx_1) \cdots B(dx_m),$$

where $\int'$ is an integral excluding diagonals $x_i = x_j$, $i \neq j$, $i, j = 1, \ldots, m$, $f \in L^2(\mathbb{R}^d)$ and $B$ is a real-valued Gaussian random measure on $\mathcal{B}(\mathbb{R}^d)$ satisfying $\mathbb{E}B(A) = 0$ and $\mathbb{E}B(A_1)B(A_2) = \nu_d(A_1 \cap A_2)$ for any bounded Borel sets $A, A_1, A_2$.

We use later the relationship between Hermite polynomials and complex-valued multiple stochastic integrals defined as follows. Let $M_1$ and $M_2$ be two independent real valued Gaussian measures on $\mathcal{B}(\mathbb{R}^d)$. Define a complex-valued Gaussian random measure $M$ by $M(A) = \frac{1}{\sqrt{2}}(M_1(A) + iM_2(A))$. In particular, for a set $A$, we have $\mathbb{E}M(A) = 0$, $\mathbb{E}|M(A)|^2 = \nu_d(A)$. Take any symmetric function $g : \mathbb{R}^d \to \mathbb{C}$, $g(x) = g(-x)$, $x \in \mathbb{R}^d$, which is invariant under permutation of its indices. Similarly to real valued case, one can define

$$I_m(g) = \int_{\mathbb{R}^d} g(y_1, \ldots, y_m) M(dy_1) \cdots M(dy_m),$$

where the integration disregards hyperplanes $|y_i| = |y_j|$, $y_i \neq y_j$.

The paper is organized as follows. In Section 2, we consider the central limit theorem for excursion sets of subordinated Gaussian random fields with long memory. Moreover, we apply it further in Section 4 to the case fractional Brownian motion, fractional Gaussian noise, and random fields with random volatility.

2. Central limit theorems

2.1. Non-stationary random fields

In this section, we prove the central limit theorem for excursion sets of subordinated Gaussian random fields with long memory. Moreover, we apply it further in Section 4 to the case fractional Brownian motion, fractional Gaussian noise, and random fields with random volatility.

We formulate the following theorem for non-stationary random fields with non-constant variance. This is motivated by seasonal models, in particular econometric...
time series, e.g \([10, 34]\).

**Theorem 2.1.** Let \(\{Y(t), t \in \mathbb{R}^d\}\) be a real valued measurable centered Gaussian random field with correlation function \(\text{Corr}(Y(t), Y(s)) = \rho(t, s), t, s \in \mathbb{R}^d\). Let \(\{X(t) = f(Y(t)), t \in \mathbb{R}^d\}\) be the corresponding subordinated field, where \(f : \mathbb{R} \to \mathbb{R}\) is a Borel-measurable function. Let \(\{W_n\}_{n \in \mathbb{N}}\) be a van Hove sequence of observation windows. For \(u \in \mathbb{R}\), let there exist a subset \(U \subset W_n, n \geq 1\) such that \(\nu_d(U) > 0\) and

\[
a_1(t) = \langle H_1, 1\{f(\sigma \cdot) \geq u\}\rangle_{\varphi} \neq 0, \quad t \in U,
\]

where \(\sigma^2(t) = \mathbb{E}Y(t)^2, t \in \mathbb{R}^d\). If

\[
\lim_{n \to \infty} \frac{\int_{W_n} \int_{W_n} \rho^2(t, s) \nu_d(dt) \nu_d(ds)}{\int_{W_n} \int_{W_n} a_1(t) a_1(s) \rho(t, s) \nu_d(dt) \nu_d(ds)} = 0,
\]

then

\[
\left( \int_{W_n} \int_{W_n} a_1(t) a_1(s) \rho(t, s) \nu_d(dt) \nu_d(ds) \right)^{1/2} \int_{W_n} 1\{\rho^2(t, s) \nu_d(dt) \nu_d(ds)} \to N(0, 1)
\]

as \(n \to \infty\).

**Proof.** Consider the function \(F_u(x, t) := 1\{f(\sigma t) \geq u\}, x \in \mathbb{R}\). It is clear that \(F_u(\cdot, t) \in L^2(\mathbb{R}, \varphi)\). So, the function \(F_u\) can be represented as

\[
F_u(x, t) = \sum_{k=0}^{\infty} a_k(t) \frac{H_k(x)}{\sqrt{k!}}, \quad x \in \mathbb{R}, \quad \text{where} \quad a_k(t) = \frac{\langle F_u(\cdot, t), H_k \varphi \rangle}{\sqrt{k!}}, \quad k \in \mathbb{N}_0.
\]

In particular, \(a_0(t) = \int_{\mathbb{R}} 1\{f(\sigma t) \geq u\} \varphi(x) dx = \mathbb{P}(X(t) \geq u)\). Due to the property \(\varphi^{(k)}(x) = (-1)^k H_k(x) \varphi(x), k \geq 0\), we have

\[
a_k(t) = \int_{\mathbb{R}} 1\{f(\sigma t) \geq u\} \frac{(-1)^k}{\sqrt{k!}} \varphi^{(k)}(x) dx, k \in \mathbb{N},
\]

so that

\[
a_1(t) = -\int_{\mathbb{R}} 1\{f(\sigma t) \geq u\} \varphi'(x) dx
\]

\[
= \int_{\mathbb{R}} 1\{f(\sigma t) \geq u\} x \varphi(x) dx = \frac{1}{\sigma(t)} \mathbb{E}[Y(t) 1\{f(Y(t)) \geq u\}].
\]

Let \(\tilde{Y}(t) = Y(t)/\sigma(t), t \in \mathbb{R}^d\). Then we have the expansion

\[
\int_{W_n} 1\{X(t) \geq u\} \nu_d(dt) = \int_{W_n} F_u(\tilde{Y}(t), t) \nu_d(dt) = \int_{W_n} \sum_{k=0}^{\infty} a_k(t) \frac{H_k(\tilde{Y}(t))}{\sqrt{k!}} \nu_d(dt)
\]

\[
= \sum_{n=0}^{\infty} \int_{W_n} a_k(t) \frac{H_k(\tilde{Y}(t))}{\sqrt{k!}} \nu_d(dt) = \int_{W_n} a_0(t) H_0(\tilde{Y}(t)) \nu_d(dt)
\]
Moreover, we get
\[ E(\text{Lemma } 10.2) \]
Since Hermite polynomials form an orthonormal system in \( L^2(\mathbb{R}, \varphi) \), we have
\[ \text{(2.6)} \]
From expansion (2.6) we have
\[ Y \]
\[ Z \]
\[ \nu_d(dt) \]
Denote
\[ Y_n := \int_{W_n} F_u(\tilde{Y}(t), t) \nu_d(dt) - \int_{W_n} \mathbb{P}(X(t) \geq u) \nu_d(dt), \]
\[ Z_n := \int_{W_n} a_1(t) \tilde{Y}(t) \nu_d(dt), \quad A_n := \sum_{k=2}^\infty \int_{W_n} a_k(t) \frac{H_k(\tilde{Y}(t))}{\sqrt{k!}} \nu_d(dt). \]
From expansion (2.6) we have \( Y_n = Z_n + A_n \). Random variables \( \{Z_n\}_{n \in \mathbb{N}} \) are Gaussian. So, \( (\text{Var} Z_n)^{-1/2}(Z_n - \mathbb{E} Z_n) \sim N(0, 1) \). Moreover, we prove that \( \frac{\text{Var} A_n}{\text{Var} Z_n} \to 0, n \to \infty \).
Denote
\[ \sigma^2_{n,k} := \text{Var} \left( \int_{W_n} a_k(t) H_k(\tilde{Y}(t)) \nu_d(dt) \right). \]  
(2.7)
Since Hermite polynomials form an orthonormal system in \( L^2(\mathbb{R}, \varphi) \), we have (cf. [37, Lemma 10.2])
\[ \mathbb{E}[H_k(\tilde{Y}(t))H_m(\tilde{Y}(s))] = \delta_{km}k!\rho^k(t,s), \quad t, s \in \mathbb{R}^d. \]  
(2.8)
Moreover, we get \( \mathbb{E} H_k(\tilde{Y}(t)) = \int_{\mathbb{R}} H_k(x)H_0(x)\varphi(x)dx = \delta_{k0} = 0, k \in \mathbb{N} \). Hence, from (2.7) we have
\[ \sigma^2_{n,k} = \mathbb{E} \left( \int_{W_n} a_k(t) H_k(\tilde{Y}(t)) \nu_d(dt) \right)^2 = \int_{W_n} \int_{W_n} a_k(t)a_k(s) \mathbb{E}[H_k(\tilde{Y}(t))H_k(\tilde{Y}(s))] \nu_d(dt) \nu_d(ds) = k! \int_{W_n} \int_{W_n} a_k(t)a_k(s) \rho^k(t,s) \nu_d(dt) \nu_d(ds), \quad k \in \mathbb{N}. \]  
(2.9)
So, \( \text{Var} Z_n = \sigma^2_{n,1} = \int_{W_n} \int_{W_n} a_1(t)a_1(s) \rho(t,s) \nu_d(dt) \nu_d(ds) \). It follows from (2.8) that
\[ \text{Var} A_n = \text{Var} \left( \sum_{k=2}^\infty \int_{W_n} a_k(t) \frac{H_k(\tilde{Y}(t))}{\sqrt{k!}} \nu_d(dt) \right) = \sum_{k,m=2}^\infty \text{Cov} \left( \int_{W_n} a_k(t) \frac{H_k(\tilde{Y}(t))}{\sqrt{k!}} \nu_d(dt), \int_{W_n} a_m(t) \frac{H_m(\tilde{Y}(t))}{\sqrt{m!}} \nu_d(dt) \right) = \sum_{k,m=2}^\infty \int_{W_n} \int_{W_n} a_k(t) a_m(t) \frac{Cov(H_k(\tilde{Y}(t))H_m(\tilde{Y}(s))) \nu_d(dt) \nu_d(ds) \sqrt{k!} \sqrt{m!}}{}} \]
$$\begin{align*}
&= \sum_{k,m=2}^{\infty} \int_{W_n} \int_{W_n} \frac{a_k(t) a_m(t)}{\sqrt{k!} \sqrt{m!}} \delta_{km} \rho^k(t,s) \nu_d(dt) \nu_d(ds) \\
&= \sum_{k=2}^{\infty} \int_{W_n} \int_{W_n} a_k(t) a_k(s) \rho^k(t,s) \nu_d(dt) \nu_d(ds).
\end{align*}$$

Since $|\rho(t,s)| \leq 1$ and $\sum_{k=0}^{\infty} a_k^2(t) \leq 1$, we have

$$\text{Var} A_n \leq \sum_{k=2}^{\infty} \int_{W_n} \int_{W_n} |a_k(t)| |a_k(s)| |\rho(t,s)|^k \nu_d(dt) \nu_d(ds)$$

$$\leq \int_{W_n} \int_{W_n} \left( \sum_{k=2}^{\infty} |a_k(t) a_k(s)| \right) \rho^2(t,s) \nu_d(dt) \nu_d(ds)$$

$$\leq \int_{W_n} \int_{W_n} \left( \sum_{k=2}^{\infty} a_k^2(t) \sum_{k=2}^{\infty} a_k^2(s) \right)^{1/2} \rho^2(t,s) \nu_d(dt) \nu_d(ds)$$

$$\leq \int_{W_n} \int_{W_n} \rho^2(t,s) \nu_d(dt) \nu_d(ds). \quad (2.10)$$

Thus, from condition (2.2) we get

$$\frac{\text{Var} A_n}{\text{Var} Z_n} = \frac{\text{Var} A_n}{\sigma_{n,1}^2} \leq \frac{\int_{W_n} \int_{W_n} \rho^2(t,s) \nu_d(dt) \nu_d(ds)}{\int_{W_n} \int_{W_n} a_1(t) a_1(s) \rho(t,s) \nu_d(dt) \nu_d(ds)} \to 0, \; n \to \infty.$$ 

It means that $\frac{A_n}{Z_n}$ converges to 0 in mean square sense, and hence it converges to 0 in distribution.

Thus, we obtain that the limiting distributions of $\frac{Y_n}{\sigma_{n,1}}$ and $\frac{Z_n}{\sigma_{n,1}}$ coincide. Combining this fact with $E Z_n = 0$ and $\frac{Z_n}{\sigma_{n,1}} \sim N(0,1)$, $n \in \mathbb{N}$, we obtain the statement of the Theorem.

A Gaussian random field is positive associated (PA) or negative associated (NA) if its covariance function is non-negative or non-positive, respectively, cf. [8].

In some cases, condition (2.2) can be formulated in terms of correlation function only.

**Corollary 2.2.** Let function $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) < u$ for all $x < 0$, $\lim_{x \to +\infty} f(x) > u$, and PA random field $Y$ satisfy conditions of Theorem 2.1 with $\inf_{t \in W_n, n \geq 1} EY^2(t) = \sigma_0^2 > 0$. Then condition (2.2) is satisfied if

$$\lim_{n \to \infty} \frac{\int_{W_n} \int_{W_n} \rho^2(t,s) \nu_d(dt) \nu_d(ds)}{\int_{W_n} \int_{W_n} \rho(t,s) \nu_d(dt) \nu_d(ds)} = 0. \quad (2.11)$$

**Proof.** Under proposed assumptions, coefficient $a_1$ can be bounded from below

$$a_1(t) = \int_{\mathbb{R}} 1\{f(\sigma(t) x) \geq u\} x \varphi(x) dx = \int_{0}^{\infty} 1\{f(z) \geq u\} \frac{x}{\sigma(t)} \varphi \left( \frac{x}{\sigma(t)} \right) dz.$$

Since $\lim_{x \to +\infty} f(x) > u$, there exists $u^* > 0$ such that $f(x) \geq u$ for $t > u^*$. Therefore,
\[ a_1(t) \geq \int_u^\infty \frac{z}{\sigma(t)} \varphi \left( \frac{z}{\sigma(t)} \right) dz = \varphi \left( \frac{u}{\sigma(t)} \right) \geq \varphi \left( \frac{u}{\sigma_0} \right). \] Thus,

\[
\frac{\int_{W_n} \int_{W_n} \rho^2(t, s) \nu_d(dt) \nu_d(ds)}{\int_{W_n} \int_{W_n} a_1(t) a_1(s) \rho(t, s) \nu_d(dt) \nu_d(ds)} \leq \frac{1}{\varphi^2 \left( \frac{u}{\sigma_0} \right)} \frac{\int_{W_n} \int_{W_n} \rho^2(t, s) \nu_d(dt) \nu_d(ds)}{\int_{W_n} \int_{W_n} \rho(t, s) \nu_d(dt) \nu_d(ds)} \to 0,
\]
as \( n \to \infty. \)

In case of monotonic function \( f \) we have the following corollary.

**Corollary 2.3.** Under the assumptions of Theorem 2.1 let \( f \) be a non-decreasing function and \( \{f^{-1}(x) = \inf\{y \in \mathbb{R}, f(y) \geq x\}\} \) be its generalized inverse function. Then

\[
\frac{\int_{W_n} \mathbbm{1}\{X(t) \geq u\} \nu_d(dt) - \int_{W_n} \Psi\left(f^{-1}(u)/\sigma(t)\right) \nu_d(dt)}{\sqrt{\int_{W_n} \varphi \left( \frac{f^{-1}(u)}{\sigma(t)} \right) \varphi \left( f^{-1}(u)/\sigma(s) \right) \rho(t, s) \nu_d(dt) \nu_d(ds)}} \overset{d}{\to} N(0, 1), \quad n \to \infty, \tag{2.12}
\]

where \( \Psi(u) = \int_u^{+\infty} \varphi(x) dx. \)

If \( f(x) = x, x \in \mathbb{R}, \sigma(t) = 1, t \in \mathbb{R}^d \) and (2.11) holds true, then

\[
\frac{\int_{W_n} \mathbbm{1}\{X(t) \geq u\} \nu_d(dt) - \nu_d(W_n) \Psi(u)}{\varphi(u) \sqrt{\int_{W_n} \rho(t, s) \nu_d(dt) \nu_d(ds)}} \overset{d}{\to} N(0, 1), \quad n \to \infty. \tag{2.13}
\]

Further in the paper, we consider normalized random fields with \( \sigma^2(t) = 1 \) and give the examples of non-stationary covariance functions \( \rho \) satisfying conditions (2.11).

### 2.2. Stationary random fields

In this section, we consider further applications of Theorem 2.1 and assume that the random field \( Y \) is stationary. Hence, its covariance function is invariant with respect to linear translations.

**Corollary 2.4.** Let \( Y \) be a centered stationary Gaussian random field with covariance function \( C(t) = \mathbb{E}[Y(t)Y(0)], t \in \mathbb{R}^d, \) and \( \mathbb{E}Y^2(0) = 1. \) If \( \langle H_1, \mathbbm{1}\{f(\cdot) \geq u\}\rangle \varphi \neq 0 \) and

\[
\frac{\int_{\mathbb{R}^d} C^2(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt)}{\int_{\mathbb{R}^d} C(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt)} \to 0, \quad n \to \infty, \tag{2.14}
\]
then for \( \{X(t) = f(Y(t)), t \in \mathbb{R}^d\} \) it holds

\[
\frac{\int_{W_n} \mathbbm{1}\{X(t) \geq u\} dt - \nu_d(W_n) \mathbb{P}(X(0) \geq u)}{\langle H_1, \mathbbm{1}\{f(\cdot) \geq u\}\rangle \varphi \sqrt{\int_{\mathbb{R}^d} C(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt)}} \overset{d}{\to} N(0, 1) \tag{2.15}
\]
as \( n \to \infty. \)
Proof. Consider integrals in (2.2)

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C^2(t-s)\nu_d(dt)\nu_d(ds) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(t-s)\nu_d(dt)\nu_d(ds) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C^2(u)\nu_d(du)\nu_d(du)
\]

Thus, conditions (2.2) and (2.14) are equivalent and the statement of the corollary follows from Theorem 2.1.

Remark 2.5. Assume that \( Y \) is \( \text{PA(NA)} \), then condition (2.14) can hold only if \( Y \) has a long memory. Indeed, if \( 0 < \int_{\mathbb{R}^d} |C(t)| dt < \infty \), and \( |C(t)| \leq 1, t \in \mathbb{R}^d \), then \( 0 < \int_{\mathbb{R}^d} C^2(t)dt < \infty \). Using \( \lim_{n \to \infty} \frac{\nu_d(W_n \cap (W_n - t))}{\nu_d(W_n)} = 1, t \in \mathbb{R}^d \), it follows that

\[
\Delta_n := \frac{\int_{\mathbb{R}^d} C^2(t)\nu_d(W_n \cap (W_n - t))\nu_d(dt)}{\int_{\mathbb{R}^d} C(t)\nu_d(W_n \cap (W_n - t))\nu_d(dt)} \to \frac{\int_{\mathbb{R}^d} C^2(t)\nu_d(dt)}{\int_{\mathbb{R}^d} C(t)\nu_d(dt)} \in (0, +\infty).
\]

Measurable function \( f : \mathbb{R} \to \mathbb{R} \) can be chosen arbitrarily, which means that \( \mathbb{E}X(0)^p < +\infty \) for some \( p > 0 \) need not be true. The normalization in limit (2.15) is not of CLT-type \( n^{-d/2} \) since it involves the square root of the integral of the weighted non-integrable function \( C \).

In the multidimensional case \( d > 1 \), the observation windows \( W_n \) can extend differently in different directions. In order to parametrize the growth of \( W_n \), we make some auxiliary notation. For some \( r_{n,l} > 0, 1 \leq l \leq d, n \in \mathbb{N} \) introduce “normalized” windows

\[
V_n := \left\{ \left( \frac{x_1}{r_{n,1}}, \ldots, \frac{x_d}{r_{n,d}} \right) : (x_1, \ldots, x_d) \in W_n \right\}
\]

such that \( \sup_{n \geq 1} \nu_d(V_n) < \infty \). For instance, if \( r_{n,l} = \sup\{|x_l|, (x_1, \ldots, x_d) \in W_n\} \), then \( V_n \subseteq [-1, 1]^d \). Moreover, we assume that there exists a “limit” \( V \) of \( V_n \), i.e., \( V_n \subseteq V \in \mathcal{B}^{(\mathbb{R}^d)} \), and \( \nu_d(W_n) \sim \nu_d(V) \prod_{i=1}^d r_{n,i}, \nu_d(V \setminus V_n) \to 0, \text{ as } n \to \infty \).

Corollary 2.6. If \( Y \) is a \( \text{PA(NA)} \) stationary random field with \( C(t) \to 0, \|t\| \to \infty \), then condition (2.14) holds if for some \( \delta \in (0, 1) \)

\[
\prod_{i=1}^d r_{n,i}^{-1+\delta} \int_{|t_i| \leq r_{n,i}} C(t)\nu_d(dt) \to +\infty, \quad n \to \infty.
\]

Proof. For \( \delta \in (0, 1) \) put \( V_n^\delta = \prod_{i=1}^d [-r_{n,i}^{-\delta}, r_{n,i}^{-\delta}] \). Changing variables \( t_i = r_{n,i}s_i \) in (2.14) we get

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C^2(t)\nu_d(W_n \cap (W_n - t))\nu_d(dt) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C(t)\nu_d(W_n \cap (W_n - t))\nu_d(dt)
\]
Using the symmetry of $C$

Take $\delta$ as $\delta \rightarrow 0$.

Example 2.7. For simplicity, let $d = 1$, $W_n = [-n, n]$, and $C(t) \sim |t|^{-\eta}$, $\eta \in (0, 1)$ as $t \rightarrow +\infty$. From the proof of Corollary 2.6 we get with $\delta = \frac{1-\eta}{2}$, $\beta = 1$, and $r_n = n$ that

$$\Delta_n \leq \frac{2^{1+(1+n)/2}(n/2)^{(1+\eta)/2}}{\int_0^{n/2} C(v)dv} + \sup_{v \geq n^{(1-\eta)/2}} C(v)$$

$$\sim 2^{(3-\eta)/2}(n/2)^{(\eta-1)/2}(1-\eta) + \sup_{v \geq n^{(3-\eta)/2}} C(v) \rightarrow 0, n \rightarrow \infty.$$
\[ B(p, q). \text{To summarize, the limit } (2.10) \text{ holds:} \]

\[
\frac{\int_{\mathbb{R}^n} \mathbf{1}\{X(t) > u\} \, dt - 2n\mathbb{P}(X(0) > u)}{\langle H_1, \mathbf{1}\{f(\cdot) \geq u\} \rangle_{\mathcal{F}} 2^{3/2-\eta/2} \sqrt{B(2, 1 - \eta)n^{1-\eta/2}}} \xrightarrow{d} N(0, 1), \quad n \to +\infty.
\]

Since \( a_1 = \langle H_1, \mathbf{1}\{f(\cdot) \geq u\} \rangle_{\mathcal{F}} \neq 0 \) the function \( f \) can not be even. Additionally, we require \( \mathbb{E} f^{1+\theta}(Y(0)) < +\infty \) for some \( \theta \in (0, 1) \). As an example, we consider

\[ f(x) = \text{sgn}(x) \left( e^{x^2/\beta^2} - 1 \right), \quad x \in \mathbb{R} \]

for some \( \beta > \sqrt{2(1 + \theta)} \). It follows that \( \mathbb{E} X^2(0) = \mathbb{E} \left( e^{Y^2(0)/\beta^2} - 1 \right)^2 = +\infty, \)

\( \mathbb{E} X^{1+\theta}(0) < \infty \). It can be calculated that in this case

\[ a_1 = \frac{1}{\sqrt{2\pi(1 + u)^{\beta^2/2}}} \]

for \( u > 0 \).

In the next Lemma we check condition (2.17) by using the asymptotic of correlation function \( C \) at \( \infty \).

**Lemma 2.8.** Let conditions of Corollaries 2.4, 2.6 hold true. Let \( W_n = \prod_{i=1}^{d} [a_{n,i}, b_{n,i}] \)

with \( r_{n,i} = (b_{n,i} - a_{n,i})/2 \) and there exist functions \( \lambda, q : \mathbb{R}^d \to \mathbb{R} \) and such that \( q \in L_1([-1, 1]^d) \), \( \kappa = \int_{[-1, 1]^d} q(v) \prod_{i=1}^{d} (1 - |v_i|) \nu_d(dv) > 0 \), and

\[ \frac{C(2r_{n,1}v_1, \ldots, 2r_{n,d}v_d)}{q(v_1, \ldots, v_d)} \sim \lambda(r_{n,1}, \ldots, r_{n,d}), \quad n \to \infty \]

uniformly on any rectangle \([a, 1]^d, a \in (0, 1)\). If there exists \( \delta \in (0, 1) \) such that

\[ \lambda(r_{n,1}, \ldots, r_{n,d}) \prod_{i=1}^{d} r_{n,i}^{\delta} \to \infty, n \to \infty, \quad (2.20) \]

then for \( \{X(t) = f(Y(t)), t \in \mathbb{R}^d\} \) it holds

\[
\frac{\int_{W_n} \mathbf{1}\{X(t) \geq u\} \, dt - \nu_d(W_n)\mathbb{P}(X(0) \geq u)}{\langle H_1, \mathbf{1}\{f(\cdot) \geq u\} \rangle_{\mathcal{F}} \nu_d(W_n) \sqrt{\kappa \lambda(r_{n,1}, \ldots, r_{n,d})}} \xrightarrow{d} N(0, 1), \quad (2.21)
\]

as \( n \to \infty \).

**Proof.** First, we compute the asymptotic variance in (2.15). From the proof of Corollary 2.6 we get that

\[
\int_{\mathbb{R}^d} C(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt) = \prod_{i=1}^{d} r_{n,i}^2 \int_{\mathbb{R}^d} C(r_{n,1}s_1, \ldots, r_{n,d}s_d) \nu_d([-1, 1]^d \cap ([0, 1]^d - s)) \nu_d(ds)
\]
Consider the second integral

\[ V_4 \lambda \]  

Example 2.9. Therefore, from condition (2.20) it follows that

\[ d \gamma r \]

In the case \( \gamma \in (0, 1) \), we put \( q(x, y, z) = |x|^{-2\alpha} e^{-c|z|}, (x, y, z) \in \mathbb{R}^3 \) and \( \lambda(r_{n,1}, r_{n,2}, r_{n,3}) = n^{-2\alpha}, n \geq 1 \). Indeed, due to Lemma 2.8

\[ C(2r_{n,1}x, 2r_{n,2}y, 2r_{n,3}z) / q(x, y, z) = \frac{(1 + x^2 n^2 + y^2 n^{2\gamma})^{-\alpha} e^{-c|z|}}{|x|^{-2\alpha} e^{-c|z|}} \sim n^{-2\alpha}, \ n \to \infty. \]
Then \( q \in L_1([-1,1]^3) \) and

\[
\int_{W_n} \mathbb{1}\{X(t) \geq u\} \, dt - cn^{1+\gamma} \mathbb{P}(X(0) \geq u) \frac{d}{\sqrt{\kappa}} N(0,1), \quad n \to \infty. \tag{2.23}
\]

Let \( \gamma = 1 \), then \( q(x,y,z) = (x^2 + y^2)^{-\alpha}e^{-c|z|}, (x,y,z) \in \mathbb{R}^3 \) and \( \lambda(r_{n,1}, r_{n,2}, r_{n,3}) = n^{-2\alpha}, n \geq 1. \) Indeed, by Lemma 2.8,

\[
C(2r_{n,1}x, 2r_{n,2}y, 2r_{n,3}z) = \frac{(1 + x^2n^2 + y^2n^2)^{-\alpha}e^{-c|z|}}{(x^2 + y^2)^{-\alpha}e^{-c|z|}} \sim n^{-2\alpha}, n \to \infty.
\]

Then \( q \in L_1([-1,1]^3) \) and (2.23) holds true.

If \( \gamma > 1 \), then \( q(x,y,z) = |y|^{-2\alpha}e^{-c|z|}, (x,y,z) \in \mathbb{R}^3 \) and \( \lambda(r_{n,1}, r_{n,2}, r_{n,3}) = n^{-2\gamma}, n \geq 1. \) Then \( q \in L_1([-1,1]^3) \) and

\[
\int_{W_n} \mathbb{1}\{X(t) \geq u\} \, dt - cn^{1+\gamma} \mathbb{P}(X(0) \geq u) \frac{d}{\sqrt{\kappa}} N(0,1), \quad n \to \infty.
\]

### 2.3. Spatio-temporal random fields

In this section, we apply Theorem 2.1 to random fields which can posses different properties with respect to the space and time coordinates. First, we consider a separable covariance function with a stationary time-component.

**Theorem 2.10.** Let \( \{Y(x,t), x \in \mathbb{R}^d, t \in \mathbb{R}\} \) be a centered PA (NA) Gaussian random field with covariance function \( \text{Cov}(Y(x,t), Y(y,s)) = C(x,y)\mathcal{C}(|t-s|), x,y \in \mathbb{R}^d, s,t \in \mathbb{R} \) and \( \mathbb{E}Y^2(x,t) = 1. \) Assume that \( \mathcal{C} \) is non-negative and \( \mathcal{C}(r) \to 0, r \to +\infty. \) Let \( W_n = U_n \times (a_n, b_n) \) be a sequence of Borel sets such that \( r_n := b_n - a_n \to +\infty, n \to \infty, \nu_d(U_n) \geq c > 0 \) and

\[
\tilde{\kappa} = \lim_{n \to \infty} \int_{U_n} \int_{U_n} C(x,y)\nu_d(dx)\nu_d(dy) \in (0, +\infty).
\]

If for some \( \delta \in (0,1) \)

\[
\frac{1}{r^\delta} \int_0^r \mathcal{C}(v)dv \to \infty, \quad r \to \infty, \tag{2.24}
\]

then

\[
\int_{a_n}^{b_n} \int_{U_n} \mathbb{1}\{X(x,t) \geq u\} \nu_d(dx) \, dt - \nu_d(U_n)(b_n - a_n)\mathbb{P}(X(0) \geq u) \frac{d}{\sqrt{2\tilde{\kappa}}} \int_0^{r_n} \mathcal{C}(r_n - s)ds \tag{2.25}
\]

tends to \( N(0,1) \) in distribution as \( n \to \infty. \) Here \( 0 \in \mathbb{R}^{d+1}. \)

**Proof.** We check the conditions of Theorem 2.1. First, it is obvious that \( |C(x,y)| \leq 1 \)
and $a_1(t) = \langle H_1, \mathbb{1}\{f(t) \geq u\}\rangle_\phi$ in (2.1). Rewrite the limit in (2.2) as

$$
\lim_{n \to \infty} \left| \frac{\int_U \int_U C^2(x,y) \nu_d(x) \nu_d(y) \int_{a_n}^{b_n} C^2(|t-s|) dt ds}{\int_U \int_U C(x,y) \nu_d(x) \nu_d(y) \int_{a_n}^{b_n} C(|t-s|) dt ds} \right| = \lim_{n \to \infty} \left| \frac{\int_U \int_U C^2(x,y) \nu_d(x) \nu_d(y) \int_{a_n}^{b_n} \tilde{C}^2(|r_n t-(1-|t|)|) dt}{\int_U \int_U C(x,y) \nu_d(x) \nu_d(y) \int_{a_n}^{b_n} C(|r_n t|) (1-|t|) dt} \right|
$$

$$
\leq \lim_{n \to \infty} \frac{\int_0^{r_n} \tilde{C}^2(s)(r_n-s) ds}{\int_0^{r_n} \tilde{C}(s)(r_n-s) ds}.
$$

(2.26)

Take $\delta \in (0,1)$ such that (2.24) holds true. Then

$$
\int_0^{r_n} \tilde{C}^2(v)(r_n-v) dv = \int_0^{r_n} \tilde{C}^2(v)(r_n-v) dv + \int_{r_n}^{2r_n} \tilde{C}^2(v)(r_n-v) dv
$$

$$
\leq \int_0^{r_n} (r_n-v) dv + \left( \sup_{v \geq r_n} \tilde{C}(v) \right) \int_{r_n}^{2r_n} \tilde{C}(v)(r_n-v) dv.
$$

So, the limit in (2.26) is bounded from above by

$$
\frac{r_n^{1+\delta}}{\int_0^{r_n} \tilde{C}(v)(r_n-v) dv} + \frac{\int_{r_n}^{2r_n} \tilde{C}(v)(r_n-v) dv}{\int_0^{r_n} \tilde{C}(v)(r_n-v) dv} \sup_{v \geq r_n} \tilde{C}(v) \leq \frac{2^{1+\delta}(r_n/2)^\delta}{\int_0^{r_n/2} \tilde{C}(v) dv} + \sup_{v \geq r_n} \tilde{C}(v).
$$

(2.27)

From (2.24) we get that $r_n \left( \int_0^{r_n} \tilde{C}(v) dv \right)^{-1} \to 0, r_n \to +\infty$. Moreover, we have $\sup_{v \geq r_n} \tilde{C}(v) \to 0$. Hence, the limit in (2.27) is equal to $0$ and condition (2.2) of Theorem 2.1 is satisfied. So, the asymptotic normality of (2.25) follows from Theorem 2.1.

Applying the approach of the above proof, we get the following evident corollary for non-separable space-time covariance functions.

Corollary 2.11. Let $W_n = U_n \times (a_n, b_n)$ be a sequence of Borel sets such that $r_n := b_n - a_n \to +\infty, n \to \infty$, and $\nu_d(U_n) \geq c > 0$. Let $\{Y(x,t), x \in \mathbb{R}^d, t \in \mathbb{R}\}$ be a centered $PA(NA)$ Gaussian random field with covariance function satisfying

$$
d_1 \tilde{C}(|t-s|) \leq \operatorname{Cov}(Y(x,t), Y(y,s)) \leq d_2 \tilde{C}(|t-s|), x, y \in \mathbb{R}^d, s, t \mathbb{R},
$$

where $d_1, d_2 > 0$ and $EY^2(x,t) = 1$. Assume that $\tilde{C}$ is non-negative and $\tilde{C}(r) \to 0, r \to +\infty$. If for some $\delta \in (0,1)$

$$
\frac{1}{r^{\delta}} \int_0^{r} \tilde{C}(v) dv \to \infty, \quad r \to \infty,
$$

(2.28)

then the sequence (2.25) is asymptotically standard normal as $n \to \infty$.

Example 2.12. Consider a stationary Gaussian random field $\{Y(x,t), x \in \mathbb{R}^d, t > 0\}$
on observation windows $W_n = [0,1]^d \times [0,n], n \geq 1$ with the covariance function

$$C_G(x,t) = \frac{1}{|t|^{2\alpha} + 1} \exp\left(- \frac{|x|^2}{(|t|^{2\alpha} + 1)^\gamma}\right), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R},$$

where parameters $\alpha \in (0,1]$ and $\gamma \in (0,1]$ govern the smoothness of the purely temporal and purely spatial covariance, see [15].

Since $Y$ is stationary, we can apply Lemma 2.8 with $q(x,t) = |t|^{-2\alpha}$ and $\lambda(k_1, \ldots, k_{d+1}) = k_{d+1}^{-2\alpha}$. Indeed, $\frac{|t|^{2\alpha}}{|nt|^{2\alpha} + 1} \exp\left(- \frac{|x|^2}{(|nt|^{2\alpha} + 1)^\gamma}\right) \sim \frac{1}{n^{2\alpha}}, \ n \to \infty$. Therefore, $\kappa = \int_{[-1,1]^d} \prod_{i=1}^d (1 - |v_i|) \nu_d(dv) \int_{-1}^1 |v|^{-2\alpha} (1 - |v|) dv = \frac{1}{(1-2\alpha)(1-\alpha)^2}$ for $\alpha \in (0,1/2)$ and (2.21) becomes

$$\int_{W_n} \phi\{X(x,t) \geq u\} \nu_d(dx) dt - n \mathbb{P}(X(0) \geq u) \langle H_1, \phi\{f(\cdot) \geq u\}\rangle \bar{\varphi} n^{-\alpha} \sqrt{\kappa} \to N(0,1), \ n \to \infty.$$

To illustrate Corollary 2.11 we consider the following example of a non-separable covariance function.

**Example 2.13.** Let $\tilde{Y}(x,t), x,t \in \mathbb{R}$ be a stationary Gaussian random field with covariance function $C_G$ as above. We make a quadratic transformation of the spatial coordinates to get a non-stationary random field $X = \{f(\tilde{Y}(x^T A x,t)), x \in \mathbb{R}^d, t > 0\}$, where $A$ is a symmetric real $d \times d$-matrix. Then

$$E[\tilde{Y}(x^T A x,t)Y(y^T A y,s)] = \frac{1}{|t-s|^{2\alpha} + 1} \exp\left(- \frac{|x^T A x - y^T A y|^{2\gamma}}{|t-s|^{2\alpha} + 1}\right)$$

and we can apply Corollary 2.11 due to

$$\frac{e^{-(2d)^\gamma \|A\|_1^2\gamma}}{|t-s|^{2\alpha} + 1} \leq E[\tilde{Y}(x^T A x,t)Y(y^T A y,s)] \leq \frac{1}{|t-s|^{2\alpha} + 1}.$$

for all $x,y \in [0,1]^d, t,s > 0$, where $\|A\|_1 = \max_{1 \leq j \leq d} \sum_{i=1}^d |a_{ij}|$ is the matrix norm induced by the sum norm $\|\cdot\|_1$ of $\mathbb{R}^d$.

### 3. Non-Gaussian Limits

Theorem 2.1 has been proved for the subordinated Gaussian random fields with rank $\mathbb{1}\{f(\cdot) \geq u\} = 1$. Now we consider the limiting behaviour of general integral functionals of Gaussian random fields. In this case, the limiting distribution in the corresponding limit theorem is a multiple Wiener-Itô integral.

#### 3.1. Non-stationary random fields

Due to Karhunen’s theorem, a Gaussian random field $Y = \{Y(t), t \in \mathbb{R}^d\}$ has the spectral representation

$$Y(t) = \int_{\mathbb{R}^d} h(x,t)M(dx), \ t \in \mathbb{R}^d,$$

(3.1)
where $M$ is a symmetric complex-valued Gaussian random measure with the Lebesgue control measure and $h(\cdot, t) \in L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}^d$.

**Remark 3.1.** If a function $h : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}$ is symmetric, $h(-x, \cdot) = \overline{h(x, \cdot)}$, $x \in \mathbb{R}^d$, then the random field $Y$ in (3.1) is real-valued due to the symmetry of $M$. In general, $M$ can have some other control measure $m(\cdot)$. If $m(\cdot)$ is absolutely continuous with density $\mu : \mathbb{R}^d \to \mathbb{R}_+$, then we can rewrite $Y(t) = \int_{\mathbb{R}^d} h_s(x,t)M_s(dx)$, where $h_s(x,\cdot) = h(x,\cdot)\sqrt{\mu(x)}$, $x \in \mathbb{R}^d$ and $M_s(A) = (m(A))^{-1/2}M(A)$, $A \in \mathcal{B}(\mathbb{R}^d)$ is a Gaussian random measure with the Lebesgue control measure.

Moreover, we assume that the system $\{h(\cdot, t), t \in \mathbb{R}^d\}$ is complete in $L^2(\mathbb{R}^d)$. Here, $h(x, t)$ can be considered as a linear filter applied to a Gaussian random field measure $M$. Consequently, the covariance function $\rho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ of $Y$ has a representation

$$\rho(t, s) = \int_{\mathbb{R}^d} h(x, t)\overline{h(x, s)}\nu_d(dx), \quad t, s \in \mathbb{R}^d,$$

which is real-valued if $h(\cdot, t)$ is symmetric for all $t \in \mathbb{R}^d$.

For $x_j = (x_{j,1}, \ldots, x_{j,d}) \in \mathbb{R}^d$, $j = 1, \ldots, m$, introduce

$$I_n(x_1, \ldots, x_m) := \sigma_{n,m}^{-1}\prod_{l=1}^d r_n^{-m/2} \int_{W_n} \prod_{j=1}^m h \left( \left( \frac{x_{j,1}}{r_{n,1}}, \ldots, \frac{x_{j,d}}{r_{n,d}} \right), t \right) \nu_d(dt), \tag{3.2}$$

where $\sigma_{n,m}^2 = m! \int_{W_n} \int_{W_n} \rho^m(t,s)\nu_d(dt)\nu_d(ds)$. Before stating the main results of this section we study the limit of $I_n$ as $n \to \infty$. We illustrate it with the help of the following example on a filtered Gaussian random field.

**Example 3.2.** Consider the special case $d = 1, m = 2$, $W_n = [0, n]$, and the filter

$$h(x, t) = \frac{1}{\sqrt{2\pi(1-2\alpha)}} e^{ig(tx)-|x|^2/2} e^{ig(tx)-|x|^2/2}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

with $\alpha \in (\frac{1}{4}, \frac{1}{2})$. Let $g \in C^2(\mathbb{R})$ be increasing, odd, $g'(x) \geq c > 0$, $x \in \mathbb{R}$ and $\{g(x), x \geq 0\}$ be convex. Then the corresponding filtered Gaussian random process $Y$ is given by

$$Y(t) = \frac{1}{\sqrt{2\pi(1-2\alpha)}} \int_{\mathbb{R}} e^{ig(tx)-|x|^2/2} M(dx), \quad t \geq 0.$$

The covariance function of $Y$ then equals

$$\rho(t, s) = \int_{\mathbb{R}} h(x, t)\overline{h(x, s)}dx = \frac{1}{2\pi(1-2\alpha)} \int_{\mathbb{R}} e^{i(g(tx)-g(sx))} e^{-|x|} |x|^{2\alpha} dx.$$

Therefore, $\sigma_{n,1}^2$ equals

$$\sigma_{n,1}^2 = \int_0^n \int_{0}^{n} \rho(t, s)dtds$$

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Thus, convergent theorem. Indeed, together with obvious inequality $-n\leq 1$ we have

$$R \leq y_0 \int \int \frac{e^{-|y/n|}}{|y|^{2\alpha}} dy \sim C_{\alpha,1}n^{2\alpha+1}, \text{ as } n \to \infty. \quad (3.3)$$

Similarly, $\sigma^2_{n,2}$ equals

$$\sigma^2_{n,2} = 2 \int \int \rho^2(t, s)dt ds = \frac{n^{4\alpha}}{2\Gamma^2(1-2\alpha)} \int \int \int e^{i\gamma(y_1) + i\gamma(y_2)} dt dy \frac{e^{-|y/n|}}{|y|^{2\alpha}} dy \sim C_{\alpha,2}n^{4\alpha}, \text{ as } n \to \infty. \quad (3.4)$$

Show that $C_{\alpha,1}$ and $C_{\alpha,2}$ are positive and finite due to the Lebesgue's dominated convergent theorem. Indeed,

$$2\Gamma(1-2\alpha)C_{\alpha,1} = \lim_{n \to \infty} \int \int \int e^{i\gamma(y)} dt dy \frac{e^{-|y/n|}}{|y|^{2\alpha}} dy \leq \int \int \int e^{i\gamma(y)} dt dy \frac{1}{|y|^{2\alpha}} dy. \quad (3.5)$$

By integration by parts, we get

$$\left| \int_0^1 e^{i\gamma(y)} dy \right| \leq \frac{1}{iy} \left( e^{i\gamma(y)} - e^{i\gamma(0)} \right) + \frac{1}{iy} \int_0^1 \frac{y\gamma''(y)}{(\gamma'(y))^2} e^{i\gamma'(y)} dy \leq \frac{1}{y} \left( \frac{1}{\gamma'(y)} + \frac{1}{\gamma'(0)} \right) + \frac{1}{y} \left| \frac{y\gamma''(y)}{(\gamma'(y))^2} e^{i\gamma'(y)} \right| \leq \frac{4}{c_1} \frac{1}{y}. \quad (3.6)$$

Together with obvious inequality $\left| \int_0^1 e^{i\gamma(y)} dy \right| \leq 1$, we bound the right-hand side of (3.5) by

$$\int_\mathbb{R} \left( \left( \frac{1}{|y|} \right)^2 \frac{1}{y} \right) \frac{1}{|y|^{2\alpha}} dy < \infty \text{ if } 2\alpha < 1. \text{ Similarly,}$$

$$\left| \int_0^1 e^{i\gamma(y_1) + i\gamma(y_2)} dy \right| \leq 2 \left( \frac{1}{|y_1\gamma'(y_1) + y_2\gamma'(y_2)|} + \frac{1}{\gamma'(y_1) + \gamma'(y_2)} \right) \leq 1.$$ 

Since $\gamma'' \geq 0$ we have

$$|y_1\gamma'(y_1) + y_2\gamma'(y_2)| \geq \left| y_1\gamma'(y_1) - y_2\gamma'(y_2) \right| \geq \left| y_1 - y_2 \right| |g'(y_2)\nabla g'(y_1)| \geq c_2 \left| y_1 - y_2 \right|.$$ 

Thus,

$$2\Gamma^2(1-2\alpha)C_{\alpha,2} \leq \int \int \int e^{i\gamma(y_1) + i\gamma(y_2)} dt dy \frac{1}{|y_1y_2|^{2\alpha}} dy \leq \frac{4}{c_2^2} \int \int \left( \left( \frac{1}{|y_1 - y_2|^2} \right) \frac{1}{1} + \left( \frac{1}{y_1 + y_2} \right) \frac{1}{1} \right) \frac{1}{|y_1y_2|^{2\alpha}} dy < \infty.$$ 

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if $2\alpha \in (1/2, 1)$.

Then sequence $I_n$ from (3.2) has the following form

$$I_n(x_1, x_2) = \frac{1}{\sigma_n,2n} \int_0^n e^{ig(x_1t/n)+ig(x_2t/n)} e^{-\frac{|x_1+x_2|}{2n}} n^{2\alpha} dt$$

$$\rightarrow \frac{1}{\sqrt{C_{\alpha,2}}} \int_0^1 e^{ig(x_1s)+ig(x_2s)} \frac{ds}{|x_1x_2|^\alpha} =: I(x_1, x_2)$$

in $L^2(\mathbb{R}^2)$ as $n \to \infty$. Moreover, $\frac{\sigma^2_{x\in R}}{\sigma_{x\in R}^2} = O(n^{2\alpha-1})$ as $n \to \infty$.

The possible examples of function $g$ are sinh and $\{x + x^{<\beta>}, x \in \mathbb{R}\}$ for $\beta \geq 1$, where $x^{<\beta>} = |x|^\beta \text{sign}(x)$.

**Example 3.3.** Let us consider the filter in the from Example 3.2 with $g(x) = x^{<\beta>}$, $m \geq 1$, $h(x, t) = \frac{1}{\sqrt{2\Gamma(1-\alpha)}} e^{(\text{sign}(x))<\beta>-|x|/2}$, $x \in \mathbb{R}$, and $\beta \geq 1$, $\alpha \in \left(\frac{m-1}{2m}, \frac{1}{2}\right)$. Here, it does not hold $g'(x) \geq c > 0$ for all $x \in \mathbb{R}$. By a substitution of variables and Jordan’s lemma, it holds

$$\left| \int_0^1 e^{i(yt)^{<\beta>}} dt \right| \leq \frac{1}{\beta|y|} \left| \int_0^{|y|} e^{is^\beta} ds \right| \leq C_\beta |y|^{\wedge 1},$$

(3.6)

where $C_\beta$ is a positive finite constant. Then, similarly to Example 3.2 for $m \geq 1$ we have $\sigma_{n,m}^2 \sim C_{\alpha,\beta,m}n^{(2\alpha-1)m+2}$, $\sigma_{n,m+1}^2 \sim C_{\alpha,\beta,m+1}n^{(2\alpha-1)m+2+(2\alpha-1)}$ as $n \to \infty$ and

$$I_n(x_1, \ldots, x_m) = \frac{1}{\sigma_{n,m}n^{m/2}} \int_0^n \exp \left( ix_1^{<\beta>} + \cdots + x_m^{<\beta>} \right) e^{-\frac{|x_1^m|}{2n}} n^{m\alpha} dt$$

$$\rightarrow \frac{1}{\sqrt{C_{\alpha,\beta,m}}} \int_0^1 e^{i(x_1^{<\beta>} + \cdots + x_m^{<\beta>})s^\beta} \frac{ds}{|x_1 \cdots x_m|^\alpha} =: I(x_1, \ldots, x_m)$$

in $L^2(\mathbb{R}^m)$ as $n \to \infty$. Indeed, $I \in L^2(\mathbb{R}^m)$ because

$$\int_{\mathbb{R}^m} \left| \int_0^1 e^{i(x_1^{<\beta>} + \cdots + x_m^{<\beta>})s^\beta} ds \right|^2 |x_1 \cdots x_m|^{-2\alpha} dx_1 \cdots dx_m$$

(3.6)

$\leq C_\beta^2 \int_{\mathbb{R}^m} \frac{1}{|x_1 + \cdots + x_m|^2} \wedge 1 \frac{1}{|x_1 \cdots x_m|^2} \alpha dx_1 \cdots dx_m < \infty$

for $2\alpha \in (1 - \frac{1}{m}, 1)$.

**Theorem 3.4.** Let $\{Y(t), t \in \mathbb{R}^d\}$ be a real valued centered Gaussian random field with $E[Y(t)]^2 = 1$ and covariance function $\rho(t, s) = \text{Cov}(Y(t), Y(s))$, $t, s \in \mathbb{R}^d$, which has spectral representation

$$\rho(t, s) = \int_{\mathbb{R}^d} h(x,t)h(x,s)\nu_d(dx), \quad t, s \in \mathbb{R}^d,$$

(3.7)

where $h(\cdot, t) \in L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}^d$. Let $(W_n)_{n \in \mathbb{N}}$ be a van Hove sequence. Then for
Assume that there exist sequences \( r_{n,l}, n \geq 1, 1 \leq l \leq d \) such that \( I_n \to I \) in \( L^2(\mathbb{R}^d) \) as \( n \to \infty \). Then

\[
\frac{1}{\sigma_{n,m}} \int_{W_n} H_m(Y(t)) \nu_d(dt) \overset{d}{\to} \int_{\mathbb{R}^d} I(x_1, \ldots, x_m) M(dx_1) \ldots M(dx_m).
\] (3.9)

**Proof.** Note that \( \|h(\cdot, t)\|_2 = 1, t \in \mathbb{R}^d \). For Hermite polynomials we have the following formula

\[
H_m(Y(t)) = \int_{\mathbb{R}^d} \prod_{j=1}^m h(x_j, t) M(dx_1) \ldots M(dx_m), t \in \mathbb{R}^d,
\] (3.10)

see, for example, [29, Proposition 1.1.4]. Integrate both sides over \( W_n \) with respect to \( \nu_d(dt) \) and apply Fubini theorem for multiple Wiener-Itô integrals (see [31, Theorem 2.1]). Indeed,

\[
\int_{W_n} \int_{\mathbb{R}^d} \prod_{j=1}^m |h(x_j, t)|^2 \nu_d(x_1) \ldots \nu_d(x_m) \nu_d(dt) = \int_{W_n} \nu_d(dt) < \infty, n \in \mathbb{N}.
\]

So,

\[
\frac{1}{\sigma_{n,m}} \int_{W_n} H_m(Y(t)) \nu_d(dt) = \frac{1}{\sigma_{n,m}} \int_{\mathbb{R}^d} \int_{W_n} \prod_{j=1}^m h(x_j, t) \nu_d(dt) M(dx_1) \ldots M(dx_m).
\]

By the scaling property of Gaussian random measures, one has

\[
M \left( d \left( \frac{y_{k,1}}{r_{n,1}}, \ldots, \frac{y_{k,d}}{r_{n,d}} \right) \right) \overset{d}{=} \frac{1}{\sqrt{r_{n,1}} \cdots \sqrt{r_{n,d}}} M \left( d \left( y_{k,1}, \ldots, y_{k,d} \right) \right), 1 \leq k \leq m.
\]

After the change of variables \( y_{k,l} = r_{n,l}x_{k,l}, 1 \leq k \leq m, 1 \leq l \leq d \), the last integral equals

\[
\frac{1}{\sigma_{n,m} \prod_{l=1}^d r_{n,l}^{m/2}} \int_{\mathbb{R}^d} \int_{W_n} \prod_{j=1}^m h \left( \left( \frac{y_{j,1}}{r_{n,1}}, \ldots, \frac{y_{j,d}}{r_{n,d}} \right), t \right) \nu_d(dt) M(dy_1) \ldots M(dy_m).
\]

Thus, we obtain

\[
\frac{1}{\sigma_{n,m}} \int_{W_n} H_m(Y(t)) \nu_d(dt) \overset{d}{=} \int_{\mathbb{R}^d} I_n(y_1, \ldots, y_m) M(dy_1) \ldots M(dy_m).
\] (3.11)
Applying the representation (3.7) we rewrite \( \sigma_{n,m}^2 \) as

\[
\sigma_{n,m}^2 = m! \int_{W_n} \int_{W_n} \rho^m(t,s)\nu_d(dt)\nu_d(ds) \\
= m! \int_{W_n} \int_{W_n} \prod_{j=1}^{m} \left( \int_{\mathbb{R}^d} h(x_j,t)h(x_j,s)\nu_d(dx_j) \right) \nu_d(dt)\nu_d(ds) \\
= m! \int_{\mathbb{R}^m} \left( \int_{W_n} \int_{W_n} \prod_{j=1}^{m} h(x_j,t)h(x_j,s)\nu_d(dt)\nu_d(ds) \right) \nu_d(dx_1)\ldots\nu_d(dx_m) \\
= m! \int_{\mathbb{R}^m} \left( \int_{W_n} \prod_{j=1}^{m} h(x_j,t)\nu_d(dt) \right)^2 \nu_d(dx_1)\ldots\nu_d(dx_m). \tag{3.12}
\]

Under the conditions of Theorem 3.4 we have the following corollary.

**Corollary 3.5.** Let \( Y = \{Y(t), t \in \mathbb{R}^d\} \) be a real valued centered measurable \( \mathcal{P} \mathcal{A} \) Gaussian random field with unit variance and covariance function \( \{\rho(s,t), s, t \in \mathbb{R}^d\} \). Let \( F \in L^2(\mathbb{R}, \varphi) \) be a Borel function on \( \mathbb{R}^d \) with \( m := \text{rank } F \geq 2 \) and \( b_m = \langle F, H_m \rangle_{\varphi}/\sqrt{m} \), \( Z \sim N(0,1) \). If \( \sigma_{n,m+1}/\sigma_{n,m} \to 0, n \to \infty \), then

\[
\frac{\sqrt{m!}}{\sigma_{n,m}b_m} \left( \int_{W_n} F(Y(t))\nu_d(dt) - \nu_d(W_n)\mathbb{E}(F(Y(0))) \right) \xrightarrow{d} \int_{\mathbb{R}^m} I(x_1,\ldots,x_m)M(dx_1)\ldots M(dx_m), \quad n \to \infty, \tag{3.13}
\]

where function \( I \) is defined in (3.2).

**Proof.** We can repeat the lines of the proof of Theorem 2.1 to get that

\[
\int_{W_n} F(Y(t))\nu_d(dt) = \nu_d(W_n)\mathbb{E}F(Y(0)) \\
+ b_m \int_{W_n} \frac{H_m(Y(t))}{\sqrt{m!}}\nu_d(dt) + \sum_{k=m+1}^{\infty} \frac{b_k}{\sqrt{k!}} \int_{W_n} H_k(Y(t))\nu_d(dt). 
\]

Since \( 0 \leq \rho(t,s) \leq 1 \), we have by relation (2.8) that

\[
\text{Var} \left( \sum_{k=m+1}^{\infty} \frac{b_k}{\sqrt{k!}} \int_{W_n} H_k(Y(t))\nu_d(dt) \right) = \sum_{k=m+1}^{\infty} \frac{b_k^2}{k!} \int_{W_n} \rho^k(t,s)\nu_d(dt)\nu_d(ds) \\
\leq \sum_{k=m+1}^{\infty} b_k^2 \int_{W_n} \int_{W_n} \rho^{m+1}(t,s)\nu_d(dt)\nu_d(ds) = \sigma_{n,m+1}^2 \sum_{k=m+1}^{\infty} b_k^2.
\]

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Therefore,\[ \lim_{n \to \infty} \frac{1}{b_n \sigma_{n,m}} \left( \int_{W_n} F(Y(t))\nu_d(dt) - \nu_d(W_n)\mathbb{E}(F(Y(0))) \right) \]
\[ = \lim_{n \to \infty} \frac{1}{\sigma_{n,m}} \int_{W_n} \frac{H_m(Y(t))}{\sqrt{m!}} \nu_d(dt). \quad (3.14) \]

It follows from Theorem 3.4 that limit (3.14) is equal to (3.13) in distribution. \[ \square \]

Evidently, we can apply the above corollary to the volumes of excursion sets by setting \( F(x) = F_u(x) := \mathbb{1}\{f(x) \geq u\} \) for \( u \in \mathbb{R} \).

**Example 3.6.** If \( f \) is symmetric, i.e., \( f(x) = f_0(|x|) \) for some \( f_0 : \mathbb{R}_+ \to \mathbb{R}_+ \), then
\[ \langle F_u, H_1 \rangle_\phi = \int_{\mathbb{R}} \mathbb{1}\{f_0(|x|) \geq u\} \phi(x) dx = 0 \]
and
\[ \langle F_u, H_2 \rangle_\phi = \int_{\mathbb{R}} \mathbb{1}\{f_0(|x|) \geq u\} (x^2 - 1) \phi(x) dx = 2 \int_{\mathbb{R}_+} \mathbb{1}\{f_0(x) \geq u\} (x^2 - 1) \phi(x) dx. \]

The last term is positive if \( f_0 \) is non-decreasing and \( f_0^{(-1)}(u) \geq 1 \). In such case \( m = 2 \).

Let \( f(x) = f_1(|x|)\mathbb{1}\{|x| \leq 1\} + f_2(|x|)\mathbb{1}\{|x| > 1\} \), \( x \in \mathbb{R} \), where \( f_1 : [0, 1] \to \mathbb{R}_+ \) and \( f_2 : [1, +\infty) \to \mathbb{R}_+ \) are non-decreasing and non-increasing functions, respectively. Then \( \mathbb{E}(F_u, H_1)_\phi = 0 \), \( \langle F_u, H_2 \rangle_\phi = 0 \), and
\[ \langle F_u, H_3 \rangle_\phi = -2 \int_{f_1^{-1}(u)} f_2^{-1}(u) (1 - x^2) \phi(x) dx + 2 \int_{f_2^{-1}(u)}^{+\infty} (x^2 - 1) \phi(x) dx. \quad (3.15) \]

Therefore, if \( f_1^{-1}(u) \) and \( f_2^{-1}(u) \) are such that the right hand side of (3.15) equals to zero, then \( m = 4 \).

### 3.2. Stationary random fields

Now, consider the case of stationary random fields. Let \( \{Y(t), t \in \mathbb{R}^d\} \) be a real-valued measurable stationary centered Gaussian random field with \( \mathbb{E}[Y(t)]^2 = 1 \) and covariance function \( C(t) = \text{Cov}(Y(t), Y(0)) \), \( t \in \mathbb{R}^d \), which is continuous at zero. It follows from Bochner’s theorem that function \( C \) has spectral representation
\[ C(t) = \int_{\mathbb{R}^d} e^{i(x,t)} G(dx), \quad t \in \mathbb{R}^d, \quad (3.16) \]
where \( G \) is its spectral measure. Moreover, we assume that there exists a spectral density \( g \in L^2(\mathbb{R}^d) \) of \( G \).

We take observation windows \( W_n \) as in (2.16) with "the limit" \( V \) of \( V_n \), i.e., \( V_n \subseteq V \in \mathcal{B}(\mathbb{R}^d) \) and \( \nu_d(V \setminus V_n) \to 0 \), \( n \to \infty \). Then for any \( x_k \in \mathbb{R}^d, 1 \leq k \leq m \) it holds
\[ \left| \int_{V_n} e^{i(x_1 + \cdots + x_m,t)} \nu_d(dt) - \int_{V} e^{i(x_1 + \cdots + x_m,t)} \nu_d(dt) \right| \leq \nu_d(V \setminus V_n) \to 0, \quad n \to \infty, \]
uniformly in \((x_1, \ldots, x_m)\). Moreover, the functions

\[ K_V(x) := \int_V e^{i(x, t)} \nu_d(dt), \quad x \in \mathbb{R}^d \]  

(3.17)

are square integrable on \(\mathbb{R}^d\) for any \(V \in \mathcal{B}(\mathbb{R}^d)\) with \(\nu_d(V) < \infty\) as a Fourier transform of indicator function \(\mathbf{1}(t \in V)\).

Let the behaviour of spectral density \(g\) at 0 be similar to function \(\lambda : \mathbb{R}^d \to \mathbb{R}_+\). For \(n, m \in \mathbb{N}\) and \(z = (z_1, \ldots, z_d) \in \mathbb{R}^d\) denote

\[ Q_n(z) := \frac{1}{\lambda(r_{n,1}, \ldots, r_{n,d})} \sqrt{g \left( \frac{z_1}{r_{n,1}}, \ldots, \frac{z_d}{r_{n,d}} \right)}, \quad z \in \mathbb{R}^d, \]  

(3.18)

and

\[ d_n := \frac{\lambda^m(r_{n,1}, \ldots, r_{n,d})}{\sigma_{n,m}} \prod_{l=1}^d r_{n,l}^{1-m/2}. \]

**Theorem 3.7.** Let the field \(\{Y(t), t \in \mathbb{R}^d\}\) be as above. Assume that there exist a point-wise limit \(Q := \lim_{n \to \infty} Q_n\) and functions \(L, K : \mathbb{R}^d \to \mathbb{R}\) such that for any \(n \in \mathbb{N}\)

\[ Q_n(x) \leq L(x), \quad x \in \mathbb{R}^d, \quad \text{and} \quad K_V(x), K_{V_n}(x) \leq K(x), \quad x \in \mathbb{R}^d, \]

and \(K(x_1 + \ldots + x_m) \prod_{j=1}^m L(x_j) \in L^2(\mathbb{R}^{md})\).

Then

\[ \sigma_{n,m}^2 \sim c_{m,V} m! \frac{\lambda^2m(r_{n,1}, \ldots, r_{n,d})}{\prod_{l=1}^d r_{n,l}^{m-2}}, \quad n \to \infty, \]  

(3.20)

where

\[ c_{m,V} = \int_{\mathbb{R}^{dm}} \prod_{j=1}^m Q^2(y_j) |K_V(y_1 + \cdots + y_m)|^2 \nu_d(dy_1) \cdots \nu_d(dy_m), \]  

(3.21)

and

\[ \frac{\int_{W_n} H_m(Y(t)) \nu_d(dt)}{\sigma_{n,m}} \overset{d}{\to} \frac{1}{\sqrt{m!c_{m,V}}} \int_{\mathbb{R}^{dm}} \prod_{j=1}^m Q(y_j) K_V(y_1 + \cdots + y_m) M(dy_1) \cdots M(dy_m). \]  

(3.22)

**Proof.** We apply Theorem 3.4 with \(h(x, t) = e^{i(x, t)} \sqrt{g(x)}\), \(x, t \in \mathbb{R}^d\). Functions \(I_n\), defined in (3.12), have the following form

\[ I_n(x_1, \ldots, x_m) \]

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The first summand in (3.24) tends to 0 as \( n \) \( \rightarrow \infty \) by the Lebesgue theorem on dominated convergence. The second term in (3.24) is bounded by

\[
\left( \int_{\mathbb{R}^d} \prod_{j=1}^m Q_n(x_j) K_V(x_1 + \cdots + x_m) \right)^2
\]

\[
- \prod_{j=1}^m Q(x_j) K_V(x_1 + \cdots + x_m) \nu_d(dx_1) \cdots \nu_d(dx_m)
\]

\[
\leq \left( \int_{\mathbb{R}^d} \left( \prod_{j=1}^m Q_n(x_j) - \prod_{j=1}^m Q(x_j) \right)^2 |K_V(x_1 + \cdots + x_m)|^2 \nu_d(dx_1) \cdots \nu_d(dx_m) \right)^{1/2}
\]

\[
+ \left( \int_{\mathbb{R}^d} \prod_{j=1}^m Q_n^2(x_j) |K_{V \setminus V_n}(x_1 + \cdots + x_m)|^2 \nu_d(dx_1) \cdots \nu_d(dx_m) \right)^{1/2}.
\]

The first summand in (3.24) tends to 0 as \( n \rightarrow \infty \) by the Lebesgue theorem on dominated convergence. The second term in (3.24) is bounded by

\[
\left( \int_{\mathbb{R}^d} \prod_{j=1}^m L^2(x_j) |K_{V \setminus V_n}(x_1 + \cdots + x_m)|^2 \nu_d(dx_1) \cdots \nu_d(dx_m) \right)^{1/2},
\]

which tends to 0 as \( n \rightarrow \infty \) by the Lebesgue theorem as well, since \(|K_{V \setminus V_n}(z)| \leq \nu_d(V \setminus V_n) \rightarrow 0, n \rightarrow \infty\).

In the case of stationary random fields, (3.8) has the form

\[
\sigma_{n,m}^2 = m! \int_{\mathbb{R}^d} \prod_{j=1}^m g(x_j) \left| \int_{W_n} e^{i(x_1 + \cdots + x_m,t)} \nu_d(dt) \right|^2 \nu_d(dx_1) \cdots \nu_d(dx_m).
\]

In the last integral, we make the change of variables \( u_l = t_l/r_{n,l}, y_{k,l} = x_{k,l}r_{n,l}, 1 \leq l \leq d, 1 \leq k \leq m \). Then \( \sigma_{n,m}^2 \) rewrites as

\[
\sigma_{n,m}^2 = \frac{m!}{\prod_{l=1}^d r_{n,l}^{m-2}} \int_{\mathbb{R}^d} \prod_{j=1}^m g \left( \frac{y_{j,1}}{r_{n,1}}, \ldots, \frac{y_{j,d}}{r_{n,d}} \right)
\]

(3.25)
Proof. The statement follows from Corollary 3.5 and Theorem 3.7. Indeed,

\[
\frac{\sigma_{n,m+1}^2}{\sigma_{n,m}^2} \sim \frac{c_{m+1,V,K}}{c_{m,V}} \frac{\lambda^{2m+2}(r_{n,1},\ldots,r_{n,d})}{\lambda^{2m}(r_{n,1},\ldots,r_{n,d})} \prod_{l=1}^{d} r_{n,l}^{m-2} \prod_{l=1}^{d} r_{n,l}^{m+1-2}(m+1)
= (m+1) \frac{c_{m+1,V}}{c_{m,V}} \frac{\lambda^2(r_{n,1},\ldots,r_{n,d})}{\prod_{l=1}^{d} r_{n,l}} \to 0, \ n \to \infty.
\]

as \( n \to \infty \). Therefore, \( d_n \to (m!c_{m,V})^{-1/2} \) as \( n \to \infty \) in (3.23). Hence, the conditions of Theorem 3.4 are satisfied, and the statement (3.22) is proved. \( \square \)

Remark 3.8. We can also rewrite \( c_{m,V} \) in terms of convolutions \( Q^k \) using the change of variables \( z_k = y_k + \cdots + y_m, 1 \leq k \leq m \).

\[
\int_{\mathbb{R}^m} \prod_{j=1}^{m} Q(y_j) |K_V(y_1 + \cdots + y_m)|^2 \nu_d(dy_1) \cdots \nu_d(dy_m)
= \int_{\mathbb{R}^{(m-1)}} |K_V(z_1)|^2 \prod_{j=1}^{m-2} Q(z_j - z_{j+1}) \int_{\mathbb{R}^d} Q(z_{m-1} - z_m) Q(z_m) \nu_d(dz_m) \cdots \nu_d(dz_1)
= \int_{\mathbb{R}^{(m-1)}} |K_V(z_1)|^2 \prod_{j=1}^{m-2} Q(z_j - z_{j+1}) Q^2(z_{m-1}) \nu_d(dz_{m-1}) \cdots \nu_d(dz_1) = \ldots
= \int_{\mathbb{R}^d} |K_V(z)|^2 Q^m(z) \nu_d(dz).
\]

Corollary 3.9. Under the assumptions of Theorem 3.7, let \( \{X(t) = f(Y(t)), t \in \mathbb{R}^d\} \) be the corresponding subordinated Gaussian random field, where \( f \) is a Borel function on \( \mathbb{R}^d \). Denote \( m := \text{rank} G_u \geq 2 \), where \( G_u(x) := 1\{f(x) \geq u\} \) and \( b_m = \langle G_u, H_m \rangle_{\phi} / \sqrt{m!} \) for \( u \in \mathbb{R} \). If

\[
\frac{\lambda^2(r_{n,1},\ldots,r_{n,d})}{\prod_{l=1}^{d} r_{n,l}} \to 0, \ n \to \infty,
\]

then

\[
\int_{\mathbb{R}^d} \frac{1\{X(t) \geq u\} \nu_d(dt) - \nu_d(W_n) P(X(0) \geq u)}{b_m \lambda^m(r_{n,1},\ldots,r_{n,d}) \prod_{l=1}^{d} r_{n,l}^{1-m/2}} \prod_{l=1}^{d} M_{n,l} \to \mathcal{L} \mathcal{N}(0, \Sigma)
\]

\[
\frac{d}{n \to \infty} \int_{\mathbb{R}^m} \prod_{j=1}^{m} Q(y_j) K_V(y_1 + \cdots + y_m) M(dy_1) \cdots M(dy_m).
\]

Proof. The statement follows from Corollary 3.5 and Theorem 3.7. Indeed,
Note that if $V = [0, 1]^d$, then

$$K_V(x) = \int_{[0,1]^d} e^{ix \cdot t} \nu_d(dt) = \prod_{l=1}^d \frac{e^{ix_l} - 1}{ix_l}, \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

and if $V = B_1(0)$ is a unit ball in $\mathbb{R}^d$ then

$$K_V(x) = \int_{\|x\| \leq 1} e^{ix \cdot t} \nu_d(dt) = (2\pi)^{d/2} J_{d/2}(\|x\|) \|x\|^{-d/2}, \ x \in \mathbb{R}^d,$$

where $J_\alpha$ is the Bessel function of the first kind of order $\alpha > -1/2$, cf. [26, Example 2].

Now consider several examples of spectral densities. The following isotropic case was considered in [19, 26].

**Theorem 3.10.** Under the conditions of Corollary 3.9, let the spectral density of $Y$ be equal to

$$g_Y(z_1, \ldots, z_d) = \frac{L_Y(\|z\|)}{\|z\|^{d-\alpha}}, \ z = (z_1, \ldots, z_d) \in \mathbb{R}^d,$$

(3.28)

where $\alpha \in (0, d/m)$ and $L_Y : \mathbb{R} \to \mathbb{R}_+$ is slowly varying at 0. Assume that $r_{n,l} = r_n, 1 \leq l \leq d$.

- If $V = [0, 1]^d$ then the limiting random variable in (3.27) is

$$\int_{\mathbb{R}^d} \prod_{l=1}^d e^{i(y_{1,l} + \cdots + y_{m,l})} \frac{1}{i(y_{1,l} + \cdots + y_{m,l})} \ |y_1|^{\frac{d-\alpha}{2}} \cdots |y_m|^{\frac{d-\alpha}{2}} M(dy_1) \cdots M(dy_m).$$

- If $V = B_1(0)$ then the limiting random variable in (3.27) is

$$(2\pi)^{d/2} \int_{\mathbb{R}^d} |y_1 + \cdots + y_m|^{d/2} M(dy_1) \cdots M(dy_m) \|y_1\|^{\frac{d-\alpha}{2}} \cdots \|y_m\|^{\frac{d-\alpha}{2}}.$$

**Proof.** The statement follows from Corollary 3.9 for functions $\lambda$ and $Q$ given by

$$\lambda(r) = \|r\|^{\frac{d-\alpha}{2}} L_Y^{1/2} \left( \frac{1}{\|r\|} \right), \ r \in \mathbb{R}^d, \ Q(x) = \|x\|^{(\alpha-d)/2}, \ x \in \mathbb{R}^d.$$

Integrability condition (3.19) holds by [26, Lemma 3].

Let us now consider the anisotropic case, where spectral densities, and consequently covariance functions, are coordinate-wise products of univariate spectral densities.

**Theorem 3.11.** Let conditions of Corollary 3.9 be satisfied with spectral density

$$g_A(z_1, \ldots, z_d) = \prod_{l=1}^d \frac{L_l(z_l)}{|z_l|^{1-\gamma_l}}, \ z = (z_1, \ldots, z_d) \in \mathbb{R}^d,$$

(3.29)

where $\gamma_l \in (0, 1/m), 1 \leq l \leq d$ and $L_l, 1 \leq l \leq d$ are slowly varying functions at 0.
• If $V = [0,1]^d$ then the limiting random variable in (3.27) is

$$\int_{\mathbb{R}^m}^\prime \prod_{l=1}^d \frac{1}{|y_{1,l} \cdots y_{m,l}|^{(1-\gamma_l)/2}} e^{i(y_1, \ldots, y_m, l)} M(dy_1) \cdots M(dy_m).$$

(3.30)

• If $V = B_1(0)$ then the limiting random variable in (3.27) is

$$(2\pi)^{d/2} \int_{\mathbb{R}^m}^\prime \prod_{l=1}^d \frac{1}{|y_{1,l} \cdots y_{m,l}|^{(1-\gamma_l)/2}} \frac{J_{d/2}(\|y_1 + \cdots + y_m\|)}{\|y_1 + \cdots + y_m\|^{d/2}} M(dy_1) \cdots M(dy_m).$$

Proof. Similarly to the proof of Theorem 3.10, functions $\lambda$ and $Q$ from Corollary 3.9 have the form

$$\lambda(r) = \prod_{l=1}^d r_l^{(1-\gamma_l)/2} L_l^{1/2} \left( \frac{1}{r_l} \right), \quad r = (r_1, \ldots, r_d) \in \mathbb{R}^d,$$

$$Q(x) = \prod_{l=1}^d |x_l|^{(\gamma_l-1)/2}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Following the lines of the proof of [26, Lemma 3], we can show that

$$\int_{\mathbb{R}^m} |K(y_1 + \cdots + y_m)|^2 \prod_{l=1}^d \frac{\nu_l(dy_1) \cdots \nu_l(dy_m)}{|y_{1,l}|^{1-\gamma_l} \cdots |y_{m,l}|^{1-\gamma_l}} < \infty$$

(3.31)

for $\sum_{j=1}^m \tau_{j,l} < 1, 1 \leq l \leq d$.

Thus, condition (3.19) of Theorem 3.7 is satisfied, and the required statements are true.

Random variables (3.30) have the distribution of the marginals of Hermite sheets of order $m$. In the case $m = 2$, those are called Rosenblatt sheets, cf. e.g. [32, 45].

Example 3.12. For $m = 2$, we apply the results of Theorem 3.7 to the spectral density

$$g(x, y) = \frac{\tilde{g}(x, y)}{(x^2 + c|y|^{2\gamma_1})^{H_1/2}}, \quad (x, y) \in \mathbb{R}^2,$$

considered in [33], where $H_1, H_2 > 0$, $H_1 H_2 < H_1 + H_2$, $c > 0$ and $g$ is a bounded positive function with $\tilde{g}(0, 0) = 1$. Put $W_n = [0, n] \times [0, n^\gamma]$, where $\gamma > 0$. Then $V_n = V = [0,1]^2$ in (2.16) and $K(x, y) = -\frac{e^{\gamma x} - 1}{x} e^{\gamma y - 1}, (x, y) \in \mathbb{R}^2$ in (3.17). Similarly to [33], the asymptotic behaviour of (3.22) depends on the value of $\gamma$. We consider several cases yielding the limit in (3.22) (up to a constant factor).
Let $\gamma < H_1/H_2$, then we have in (3.18) that $\lambda(n, n^{\gamma}) = n^{H_2\gamma/2}$ and

$$Q_n^2(x, y) = \frac{1}{n^{H_2\gamma}} \left( \frac{x^2}{n^2} + c \frac{y^{2H_2}}{n^{2H_2\gamma}} \right)^{-\frac{H_2}{2}} \tilde{g} \left( \frac{x}{n^{\gamma}}, \frac{y}{n^{\gamma}} \right) \xrightarrow{n\to\infty} c^{-\frac{H_2}{2}} |y|^{-H_2} = Q^2(x, y)$$

point-wise. Thus, the limit in (3.22) reads

$$\int_{\mathbb{R}^4} |y_1 y_2|^{-\frac{H_2}{2}} e^{i(y_1+y_2)} - 1 e^{i(x_1+x_2)} - 1 \frac{M(dx)M(dy)}{y_1 + y_2}.$$ 

Let $\gamma = H_1/H_2$, then $\lambda(n, n^{\gamma}) = n^{\frac{H_1}{2} H_2}$ and

$$Q_n^2(x, y) = \frac{1}{n^{H_1}} \left( \frac{x^2}{n^2} + c \frac{y^{2H_1}}{n^{2H_1\gamma}} \right)^{-\frac{H_1}{2}} \tilde{g} \left( \frac{x}{n^{\gamma}}, \frac{y}{n^{\gamma}} \right) \xrightarrow{n\to\infty} \left( x^2 + y^{2H_1} \right)^{-\frac{H_1}{2}}$$

as $n \to \infty$ point-wise. Here, the limit in (3.22) equals (up to a constant)

$$\int_{\mathbb{R}^4} \prod_{l=1}^2 \left( x_l^2 + c y_l^{2H_1} \right)^{-\frac{H_1}{2}} e^{i(y_1+y_2)} - 1 e^{i(x_1+x_2)} - 1 \frac{M(dx)M(dy)}{y_1 + y_2}.$$ 

If $\gamma > H_1/H_2$, then we have $\lambda(n, n^{\gamma}) = n^{\frac{H_1}{2} H_2}$ in (3.18) and

$$Q_n^2(x, y) = \frac{1}{n^{H_1}} \left( \frac{x^2}{n^2} + c \frac{y^{2H_1}}{n^{2H_1\gamma}} \right)^{-\frac{H_1}{2}} \tilde{g} \left( \frac{x}{n^{\gamma}}, \frac{y}{n^{\gamma}} \right) \xrightarrow{n\to\infty} \frac{|x|^{-H_1}}{M(dx)M(dy)}.$$ 

as $n \to \infty$ point-wise. This yields the limit in (3.22):

$$\int_{\mathbb{R}^4} |x_1 x_2|^{-\frac{H_1}{2}} e^{i(y_1+y_2)} - 1 e^{i(x_1+x_2)} - 1 \frac{M(dx)M(dy)}{(x_1 + x_2)}.$$ 

4. Examples

In this section, we provide further applications of our results to some well known classes of stationary and non-stationary random fields.

4.1. Ergodic theorem for random volatility models

Let $\{Y(t), t \in \mathbb{R}^d\}$ be a measurable centered stationary Gaussian random field with covariance function $C : \mathbb{R}^d \to \mathbb{R}_+$, $C(0) = 1$. Let $\xi$ be a non-negative random variable, independent of $Y$. Introduce a random volatility field by $X(t) = \xi Y(t), t \in \mathbb{R}^d$. Particularly, if $\xi$ is a non-negative $\alpha$-stable random variable, then $E X^2(t) = +\infty$. Denote by $\Psi(\cdot)$ the tail probability function of $N(0, 1)$.

Let us consider the asymptotic behaviour of volumes of excursion sets for random field $X$. 

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Theorem 4.1. Let \( X = \{X(t), t \in \mathbb{R}^d\} \) be a random volatility field as above, such that

\[
\frac{1}{\nu_d^2(W_n)} \int_{\mathbb{R}^d} |C(t)| \nu_d(W_n \cap (W_n - t)) \nu_d(dt) \xrightarrow[n \to \infty]{} 0
\] (4.1)

for a sequence of observation windows \( \{W_n\}_{n=1}^\infty \) growing in \textit{van Hove} sense. Then

\[
\frac{1}{\nu_d(W_n)} \int_{W_n} \mathbb{1}(X(t) > u) \nu_d(dt) \xrightarrow[n \to \infty]{} \frac{d}{\Psi \left( \frac{u}{\xi} \right)}.
\]

Proof. As in Theorem 2.1, we write the Hermite expansion of function \( F(x, \sigma) = \mathbb{1}\{x \sigma \geq u\} \) with Fourier coefficients \( a_k(\sigma) = \frac{1}{\sqrt{k!}} (F(\cdot, \sigma) H_k) \varphi \). Then

\[
\int_{W_n} F(Y(t), \xi) \nu_d(dt) = \sum_{k=0}^\infty \int_{W_n} a_k(\xi) \frac{H_k(Y(t))}{\sqrt{k!}} \nu_d(dt) \quad \text{a.s.}
\]

The summands with \( k \geq 1 \) are centered and uncorrelated due to independence of \( \xi \) and \( Y \). The variance of each summand equals

\[
\mathbb{E} \left[ a_k(\xi) \int_{W_n} \frac{H_k(Y(t))}{\sqrt{k!}} \nu_d(dt) \right]^2 = \mathbb{E} a_k^2(\xi) \int_{W_n} \int_{W_n} \frac{\mathbb{E}[H_k(Y(t))H_k(Y(s))] \nu_d(dt) \nu_d(ds)}{k!}
\]

\[
= \mathbb{E} a_k^2(\xi) \int_{W_n} C^k(t-s) \nu_d(ds)
\]

\[
= \mathbb{E} a_k^2(\xi) \int_{W_n} C^k(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt).
\]

Introduce \( \sigma_n^2 := \int_{\mathbb{R}^d} |C(t)| \nu_d(W_n \cap (W_n - t)) \nu_d(dt) \). As before,

\[
\mathbb{E} \left[ \frac{1}{\nu_d(W_n)} \int_{W_n} F(Y(t), \xi) \nu_d(dt) - a_0(\xi) \right]^2 \leq \frac{\sigma_n^2}{\nu_d^2(W_n)} \sum_{k=1}^\infty \mathbb{E} a_k^2(\xi) \leq \frac{\sigma_n^2}{\nu_d^2(W_n)} \mathbb{E} \Psi \left( \frac{u}{\xi} \right)
\]

by Parseval identity, where \( a_0(\xi) = \int_{\mathbb{R}} \mathbb{1}\{x \xi \geq u\} \varphi(x) dx = \Psi \left( \frac{u}{\xi} \right) \). So, if \( \frac{\sigma_n^2}{\nu_d^2(W_n)} \to 0 \), as \( n \to \infty \), then

\[
\frac{1}{\nu_d(W_n)} \int_{W_n} \mathbb{1}\{X(t) \geq u\} \nu_d(dt) \xrightarrow[\,\,\,n \to \infty\,\,\,]{} \Psi \left( \frac{u}{\xi} \right), \quad n \to \infty.
\]

\[\square\]

Remark 4.2. Condition (4.1) holds, in particular, for short memory random fields \( Y \), i.e., if \( \int_{\mathbb{R}^d} |C(t)| \nu_d(dt) < \infty \), since

\[
\frac{1}{\nu_d^2(W_n)} \int_{\mathbb{R}^d} |C(t)| \nu_d(W_n \cap (W_n - t)) \nu_d(dt) \leq \frac{1}{\nu_d(W_n)} \int_{\mathbb{R}^d} |C(t)| \nu_d(dt) \xrightarrow[n \to \infty]{} 0.
\]

Remark 4.3. Condition (4.1) holds also for fields \( Y \) such that \( C(r_n y) \nrightarrow 0 \) uniformly in \( y \in [-1,1]^d \) if \( W_n = r_n V, V \subseteq [-1,1]^d, r_n \to +\infty \) as \( n \to \infty \), and \( C \) is
Let \( W = \{ \text{a fractional Gaussian noise}\} \) locally integrable on \( \mathbb{R}^d \). Indeed,

\[
\frac{1}{\nu_d^2(W_n)} \int_{\mathbb{R}^d} |C(t)| \nu_d(W_n \cap (W_n - t)) \nu_d(dt) = \frac{1}{\nu_d^2(V)} \int_{[r_n, r_n]^d} |C(t)| \nu_d \left( V \cap \left( V - \frac{t}{r_n} \right) \right) \nu_d \left( \frac{d}{r_n} t \right) = |t = yr_n| = \frac{1}{\nu_d^2(V)} \int_{[-1, 1]^d} |C(yr_n)| \nu_d(V \cap (V - y)) \nu_d(y) \xrightarrow{n \to \infty} 0.
\]

4.2. Fractional Gaussian processes and fields

Definition 4.4. A centered Gaussian random field \( \{G^H(t), t \in \mathbb{R}^d\} \) is called a fractional Gaussian noise with \( H = (H_1, \ldots, H_d) \) if \( \mathbb{E}[G^H(t)G^H(s)] = C(t - s), t, s \in \mathbb{R}^d \) with

\[ C(u) = \frac{1}{2^d} \prod_{i=1}^d \left( |u_i - 1|^{2H_i} + |u_i + 1|^{2H_i} - 2|u_i|^{2H_i} \right), \quad u = (u_1, \ldots, u_d) \in \mathbb{R}^d. \tag{4.2} \]

Evidently, the fractional Gaussian noise is a stationary random field. Moreover, a fractional Gaussian noise \( \{G^H(t), t \in \mathbb{R}^d\} \) has a long memory if and only if \( \max_{1 \leq i \leq d} H_i > \frac{1}{2} \). This follows directly from Definition (4.2) and the fact that function \( |x - 1|^\alpha + |x + 1|^\alpha - 2|x|^\alpha, x \in \mathbb{R} \) is non-integrable for \( \alpha > 1 \).

Now we prove a central limit theorem for the volumes of excursion sets of the fractional Gaussian noise.

Proposition 4.5. Let \( \{G^H(t), t \in \mathbb{R}^d\} \) be a fractional Gaussian noise with index \( H = (H_1, \ldots, H_d) \in (0, 1)^d \) and there exists \( m \in \{1, \ldots, d\} \) such that \( H_m \in (1/2, 1) \). Let \( W_n = (a_n, b_n) \times U_n \) be a sequence of Borel subsets such that \( r_n := b_n - a_n \to +\infty, n \to \infty \) and \( 0 < c_1 \leq \nu_{d-1}(U_n) \leq c_2 < \infty \). Then

\[
\frac{\int_{W_n} 1\{G^H(t) \geq u\} dt - \nu_d(W_n) \varphi(u)}{\varphi(u) \sqrt{\int_{\mathbb{R}^d} C(t) \nu_d(W_n \cap (W_n - t)) dt}} \xrightarrow{d} N(0, 1), n \to \infty, \tag{4.3}
\]

where \( C(\cdot) \) is the covariance function \( (12) \).

Proof. For \( \alpha \in (0, 1) \), consider the function

\[ \rho_\alpha(s) := |s + 1|^{2\alpha} + |s - 1|^{2\alpha} - 2|s|^{2\alpha}, \tag{4.4} \]

prove that \( \rho_\alpha(s) > 0, s \geq 0 \) if \( \alpha > 1/2 \).

Consider the case \( s \in (0, 1) \). Then we have \( \rho_\alpha(s) = (1 - s)^{2\alpha} + (s + 1)^{2\alpha} - 2s^{2\alpha} \). Using binomial series representation \( (1 + x)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} x^k, |x| \leq 1 \), where \( \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \), we get

\[
\rho_\alpha(s) = 1 + \sum_{k=1}^\infty \binom{2\alpha}{k} (-1)^k s^k + 1 + \sum_{k=1}^\infty \binom{2\alpha}{k} s^k - 2s^{2\alpha} = 2(1 - s^{2\alpha}) + \sum_{k=1}^\infty \binom{2\alpha}{2k} s^{2k}.
\]

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It is easy to check that \( \frac{2\alpha}{2k} \geq 0, k \in \mathbb{N} \) if \( \alpha \in (1/2, 1) \). So, \( \rho_\alpha(s) \geq 2(1 - s^{2\alpha}) > 0 \), \( s \in (0, 1) \).

Consider the case \( s \geq 1 \). Then we have \( \rho_\alpha(s) = s^{2\alpha} \left( (1 - \frac{1}{s})^{2\alpha} + (1 + \frac{1}{s})^{2\alpha} - 2 \right) \).

Use the binomial series representation:

\[
\rho_\alpha(s) = s^{2\alpha} \left( 1 + \sum_{k=1}^{\infty} \binom{2\alpha}{k} (-1)^k \frac{1}{s^k} + 1 + \sum_{k=1}^{\infty} \binom{2\alpha}{k} \frac{1}{s^k} - 2 \right)
\]

\[
= 2s^{2\alpha} \sum_{k=1}^{\infty} \binom{2\alpha}{2k} \frac{1}{s^{2k}} > 0.
\] (4.5)

Then we estimate \( \int_0^r \rho_\alpha(s) \, ds, \quad r > 1 \):

\[
\int_0^r \rho_\alpha(s) \, ds \geq \int_1^r \rho_\alpha(s) \, ds = 2 \int_1^r s^{2\alpha} \sum_{k=1}^{\infty} \binom{2\alpha}{2k} \frac{1}{s^{2k}} \, ds
\]

\[
\geq 2 \int_1^r s^{2\alpha} \frac{2\alpha}{2} \frac{1}{s^2} \, ds = 2\alpha(2\alpha - 1) \int_1^r s^{2\alpha - 2} \, ds
\]

\[
= 2\alpha(r^{2\alpha - 1} - 1).
\]

Similarly to the proof of Corollary 2.10, we put \( \delta = (1 - H_m)/2 \) and use (2.27) to get

\[
\left| \frac{\int_{U_n} \int_{U_n} \prod_{1 \leq i \leq d, i \neq m} \rho_{H_i}^2(t_i - s_i) dt ds \int_{a_n}^{b_n} \rho_{H_m}^2(t_m - s_m) dt_m ds_m}{\int_{U_n} \int_{U_n} \prod_{1 \leq i \leq d, i \neq m} \rho_{H_i}(t_i - s_i) dt ds \int_{a_n}^{b_n} \rho_{H_m}(t_m - s_m) dt_m ds_m} \right|
\]

\[
\leq \sup_{n \geq 1} \int_{U_n} \int_{U_n} \prod_{1 \leq i \leq d, i \neq m} \rho_{H_i}^2(t_i - s_i) dt ds \int_{a_n}^{b_n} \rho_{H_m}^2(v)(r_m - v) dv
\]

\[
\leq \inf_{n \geq 1} \int_{U_n} \int_{U_n} \prod_{1 \leq i \leq d, i \neq m} \rho_{H_i}(t_i - s_i) dt ds \int_{a_n}^{b_n} \rho_{H_m}(v)(r_m - v) dv
\]

\[
\leq \text{const} \left( \frac{2^{1+\delta}(r_n/2)^{\delta}}{\rho_{H_m}(v) dv} + \sup_{v \geq r_n^\delta} \rho_{H_m}(v) \right) \rightarrow 0, \quad n \rightarrow \infty.
\]

Application of Corollary 2.4 finishes the proof.

**Proposition 4.6.** Let \( \{G^H(t), t \in \mathbb{R}^d\} \) be a fractional Gaussian noise with index \( H = (H_1, \ldots, H_d) \in (0, 1)^d \). Let \( W_n = \prod_{i=1}^d [r_{n,i}] \) be such that \( r_{n,i} \rightarrow +\infty, n \rightarrow \infty \) for all \( 1 \leq i \leq d \) and

\[
\prod_{i=1}^d r_{n,i}^{\delta_i + 1 - 2H_i} \rightarrow 0, \quad n \rightarrow \infty,
\] (4.6)

where \( \delta_i = \frac{2H_i - 1}{2H_i} 1 \{2H_i > 1\}, \quad 1 \leq i \leq d \). Then

\[
\frac{\int_{W_n} 1 \{G^H(t) \geq u\} dt - \prod_{i=1}^d r_{n,i}^{H_i} \Psi(u)}{\varphi(u) \prod_{i=1}^d r_{i,n}^{H_i}} \frac{d}{n} \rightarrow N(0, 1).
\] (4.7)
**Proof.** Apply Corollary 2.4 to $X = G^H$ and $f(x) = x$. We have

$$
\int_{\mathbb{R}^d} C(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt) = \int_{\mathbb{R}^d} C(t) \prod_{i=1}^{d} \max(r_{n,i} - |t|, 0) \nu_d(dt)
$$

$$
= \prod_{i=1}^{d} r_{n,i} \int_{-r_{n,i}}^{r_{n,i}} (|t_i + 1|^{2H_i} + |t_i - 1|^{2H_i} - 2|t_i|^{2H_i}) \left(1 - \frac{t_i}{r_{n,i}}\right) dt_i
$$

$$
= \prod_{i=1}^{d} r_{n,i} \left(\int_{-r_{n,i}}^{r_{n,i}} \rho_{H_i}(t_i) dt_i - \frac{1}{r_{n,i}} \int_{-r_{n,i}}^{r_{n,i}} \rho_{H_i}(t_i)|t_i| dt_i\right),
$$

where $\rho_{ \alpha}$ is defined in (4.4).

By direct calculation, we get for $r > 1$

$$
\int_{-r}^{r} \rho_{\alpha}(v) dv = 2 \left(0 + (v + 1)^{2\alpha} - 2v^2\alpha dv + \int_{0}^{1} (1 - v)^{2\alpha} dv + \int_{1}^{r} (v - 1)^{2\alpha} dv \right) \nonumber
$$

$$
= \frac{2}{2\alpha + 1} \left((r + 1)^{2\alpha + 1} + (r - 1)^{2\alpha + 1} - 2r^{2\alpha + 1}\right), \quad (4.8)
$$

and

$$
\int_{-r}^{r} \rho_{\alpha}(v)|v| dv = 2 \int_{0}^{r} v(v + 1)^{2\alpha} - 2v^{2\alpha + 1} dv + 2 \int_{0}^{1} v(1 - v)^{2\alpha} dv 
$$

$$
+ 2 \int_{1}^{r} v(v - 1)^{2\alpha} dv = \frac{(r + 1)^{2\alpha + 1}(2\alpha + 1)r - 1 + (r - 1)^{2\alpha + 1}(2\alpha + 1)r + 1}{\alpha(2\alpha + 1)}
$$

$$
+ \frac{2 - 2(2\alpha + 1)^{2\alpha + 2}}{\alpha(2\alpha + 1)} \quad (4.9)
$$

Therefore, combining (4.8), (4.9) and series representation (4.5), we get

$$
\int_{-r}^{r} \rho_{\alpha}(v) \left(1 - \frac{|v|}{r}\right) dv = \frac{1}{r(2\alpha + 1)(\alpha + 1)} \left((r + 1)^{2\alpha + 2} + (r - 1)^{2\alpha + 2} - 2r^{2\alpha + 2} - 2\right)
$$

$$
= \frac{2r^{2\alpha + 1}}{(2\alpha + 1)(\alpha + 1)} \left(\frac{(2\alpha + 1)(2\alpha + 2)}{2r^2} + \sum_{k=2}^{\infty} \frac{(2\alpha + 2)}{2k} \frac{1}{r^{2k}}\right)
$$

$$
- \frac{2}{(2\alpha + 1)(\alpha + 1)r} \sim 2r^{2\alpha - 1}. \quad (4.10)
$$

Thus, we obtain

$$
\int_{W_n} C(t) \nu_d(W_n \cap (W_n - t)) \nu_d(dt) \sim 2^d \prod_{i=1}^{d} r_{n,i}^{2H_i}.
$$

We check condition (2.14). Note that $\int_{\mathbb{R}} \rho^2_{H_i}(v) dv < \infty$ and $\int_{\mathbb{R}} \rho^2_{H_i}(v) dv < \infty$ if $2H_i < 1$, therefore

$$
\frac{\int_{-r_{n,i}}^{r_{n,i}} \rho_{H_i}^2(v)(r_{n,i} - v) dv}{\int_{-r_{n,i}}^{r_{n,i}} \rho_{H_i}(v)(r_{n,i} - v) dv} \sim r_{n,i}^{1-2H_i}, \quad n \to \infty.
$$
If $2H_i > 1$, then $\rho_{H_i}(v) > 0, v > 0$ and $\rho_{H_i}$ is non-increasing, $v > 1$. We can use the same arguments as in Proposition 4.5 and get

\[
\frac{\int_{r_{n,i}}^{r_{n,i}+\delta} \rho_{H_i}(v)(r_{n,i} - |v|)dv}{\int_{r_{n,i}}^{r_{n,i}+\delta} \rho_{H_i}(v)(r_{n,i} - |v|)dv} \leq \frac{2^{1+\delta}(r_{n,i}/2)^\delta}{\int_{0}^{r_{n,i}+\delta} \rho_{H_i}(v)dv} + \sup_{v \geq r_{n,i}} \rho_{H_i}(v) \sim \text{const}\left(\frac{r_i^{\delta_i+1-2H_i} + r_i^{2\delta_i(1-H_i)}}{r_i^{2\gamma_i}}\right), \quad n \to \infty.
\]

Choosing $\delta_i = \frac{2H_i-1}{2-2H_i} 1\{2H_i > 1\}, 1 \leq i \leq d$, we get that

\[
\lim_{n \to \infty} \frac{\int_{W_n} C^2(t-s)dt}{\int_{W_n} C(t-s)dt} \leq \text{const} \prod_{i=1}^{d} r_i^{\delta_i+1-2H_i},
\]

which ends the proof of the Proposition.

**Remark 4.7.** Let $r_{n,i} = r_{n,i}^\gamma_i$, with $\gamma_i = (3-2H_i)(3-2H_i-1\{2H_i > 1\})^{-1}, 1 \leq i \leq d$, and $r_n \to +\infty, n \to \infty$. Then condition (4.6) is fulfilled if $\sum_{i=1}^{d} H_i < d/2$ and (4.7) rewrites

\[
\frac{\int_{W_n} 1\{G^H(t) \geq u\}dt - \sum_{i=1}^{d} r_{n,i}^{\gamma_i}\Psi(u)}{\varphi(u)\sum_{i=1}^{d} r_{n,i}^{\gamma_i}} \xrightarrow{n \to \infty} N(0,1),
\]

(4.11)

Let us consider the case $d = 2, r_n = n$ and $H_1 < 1/2, H_2 > 1/2$, such that $H_1 + H_2 > 1$. Then we get from Remark 4.7 that $\gamma_1 = 1, \gamma_2 = \frac{3-2H_2}{2-2H_2}$. It is interesting to compare our result (4.11), which now reads as

\[
\frac{\int_{[0,n] \times [0,n^\gamma_2]} 1\{G^{H_1,H_2}(t_1, t_2) \geq u\}dt_1dt_2 - n^{1+\gamma_2}\Psi(u)}{\varphi(u)n^{H_1+H_2\gamma_2}} = \xrightarrow{n \to \infty} N(0,1),
\]

with the results of paper [33]. Note that the spectral density $f$ of $G^H$ is proportional to $|x_1|^{-2H_1}|x_2|^{-2H_2}$ as $x_1, x_2 \to 0$ (see e.g. [27]). Then application of [33 Proposition 3.2.] to partial sums of $G^{H_1,H_2}$ gives

\[
\kappa(H_1)\kappa(H_2) \sum_{1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2} G^{H_1,H_2}(k_1, k_2) \xrightarrow{n \to \infty} N(0,1),
\]

where $\kappa_1, \kappa_2$ are normalizing constants.

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