THE FFRT PROPERTY OF TWO-DIMENSIONAL NORMAL GRADED RINGS AND ORBIFOLD CURVES

NOBUO HARA AND RYO OHKAWA

Abstract. This study examines the finite F-representation type (abbr. FFRT) property of a two-dimensional normal graded ring R in characteristic \( p > 0 \), using notions from the theory of algebraic stacks. Given a graded ring \( R \), we consider an orbifold curve \( \mathcal{C} \), which is a root stack over the smooth curve \( C = \text{Proj} R \), such that \( R \) is the section ring associated with a line bundle \( L \) on \( \mathcal{C} \). The FFRT property of \( R \) is then rephrased with respect to the Frobenius push-forwards \( F^e(L') \) on the orbifold curve \( \mathcal{C} \). As a result, we see that if the singularity of \( R \) is not log terminal, then \( R \) has FFRT only in exceptional cases where the characteristic \( p \) divides a weight of \( \mathcal{C} \).

The notion of finite F-representation type for a ring \( R \) of characteristic \( p > 0 \) was introduced in [SVdB]. Its definition requires some technical assumption that \( R \) is \( F \)-finite and either a complete local domain or an Noetherian \( \mathbb{N} \)-graded domain. For each \( e \in \mathbb{N} \) we identify the ring \( R^{1/p^e} \) of \( p^e \)-th roots of \( R \) with the \( e \)-times iterated Frobenius push-forward of the structure sheaf of \( \text{Spec} R \). We say that \( R \) has finite F-representation type (FFRT for short), if the set of isomorphism classes of indecomposable modules appearing as a direct summand of \( R^{1/p^e} \) as an \( R \)-module for some \( e \), is finite.

If \( R \) is a regular local ring or a polynomial ring, then it has FFRT, since \( R^{1/p^e} \) is a free \( R \)-module. It is shown in [SVdB] that a finite direct summand of a ring of FFRT also has FFRT. In particular, \( R \) has FFRT, if \( R \) has a tame quotient singularity such as the invariant subring of a finite group of order not divisible by \( p \) acting on a regular local ring. Also, it is known that a Frobenius sandwich singularity such as \( R = k[x,y,z]/(z^p - f(x,y)) \) has FFRT (cf. [Sh]). On the other hand, simple elliptic singularities, and more generally, cone singularities over a smooth curve of genus \( g \geq 1 \), are known not to have FFRT [SVdB].

In this paper, we will explore the FFRT property for normal surface singularities with \( k^* \)-action; in other words, two-dimensional normal graded ring \( R \) over a field \( k = k_0 \). In this case, \( C = \text{Proj} R \) is a smooth curve and there is an ample \( \mathbb{Q} \)-divisor on \( C \) such that \( R \cong R(C,D) = \bigoplus_{m \geq 0} H^0(C,O_C([mD])) \). The result for cone singularities implies that we cannot expect for the FFRT property unless \( C \cong \mathbb{P}^1 \). Thus the critical case is when \( R = R(\mathbb{P}^1,D) \) and the singularity is not log terminal. As far as the authors are aware, the FFRT property of such an \( R \) is wide open. Specifically we aim to answer the following:

**Question 0.1** (Holger Brenner, 2007). Does the ring \( R = k[x,y,z]/(x^2 + y^3 + z^7) \) have FFRT?

We note that the ring \( R \) in Brenner’s question is given by a \( \mathbb{Q} \)-divisor \( D = \frac{1}{2}(\infty) - \frac{1}{3}(0) - \frac{1}{7}(1) \) on \( \mathbb{P}^1 \). If \( p = 2, 3, 7 \), it is a Frobenius sandwich and so has FFRT [Sh]. Our main result in this paper is the following, which implies that \( R = k[x,y,z]/(x^2 + y^3 + z^7) \) does not have FFRT unless \( p = 2, 3, 7 \).

**Theorem 0.2.** Let \( R = R(\mathbb{P}^1,D) \) for an ample \( \mathbb{Q} \)-divisor \( D \) on \( \mathbb{P}^1 \). If \( R \) is not a log terminal singularity and if the characteristic \( p \) does not divide any denominator appearing in the rational coefficients of \( D \), then \( R \) does not have FFRT.

In the study of the structure of \( R^{1/p^e} \) we have a difficulty with the non-integral rational coefficient of \( D \). To overcome this difficulty we will introduce an orbifold curve (called a weighted

The first author is partially supported by Grant-in-Aid for Scientific Research 16K05092, JSPS.
projective line when $C \cong \mathbb{P}^1$), which is a sort of Deligne-Mumford stack $\mathcal{C}$ with coarse moduli map $\pi : \mathcal{C} \to C$. This is the “minimal covering” of $C = \text{Proj} \, R$ on which $D$ becomes integral. Thus it serves as a very useful tool to deal with rational coefficient divisors (cf. [MO]). The FFRT property of $R$ is then rephrased in terms of a global analogue of FFRT property for a pair $(\mathcal{C}, L)$ associated with the line bundle $L = \mathcal{O}_C(\pi^*D)$. In particular, by studying the structure of the Frobenius push-forwards $F^*_e \mathcal{O}_\mathcal{C}$ on the orbifold curve $\mathcal{C}$, we prove our results.

Let us give an overview of the proof of our main theorem a bit more in detail. The assumption that $R$ is not log terminal is equivalent to the condition $\delta_\mathcal{C} \geq 0$, where $\delta_\mathcal{C}$ is the degree of the canonical bundle on $\mathcal{C}$. When $\delta_\mathcal{C} = 0$, we have an étale covering $\varphi : E \to \mathcal{C}$ from an elliptic curve $E$, via which the Frobenius push-forward $F^*_e \mathcal{O}_\mathcal{C}$ is related to that on $E$. Since the structure of $F^*_e \mathcal{O}_E$ on an elliptic curve $E$ is well-understood [A, Od], we can deduce the result for $F^*_e \mathcal{O}_\mathcal{C}$, whose structure differs according to whether $E$ is ordinary or supersingular. When $\delta_\mathcal{C} > 0$, we prove the stability of $F^*_e \mathcal{O}_\mathcal{C}$ by the method of [KS, Su] for non-orbifold curves of genus $>1$, from which follows that $F^*_e \mathcal{O}_\mathcal{C}$ is indecomposable.

This paper is organized as follows. In Section 1 we review some fundamental facts on normal graded rings and root stacks. In Section 2 we rephrase the FFRT property of $R = R(C, D)$ in terms of a global FFRT property on the orbifold curve $\mathcal{C}$ constructed from $(C, D)$. Sections 3 and 4 are devoted to the study of weighted projective lines with $\delta_\mathcal{C} \leq 0$. In Section 3, we apply Crawley-Boevey’s result [CB] to deduce the FFRT property of $R = R(\mathbb{P}^1, D)$ when $\delta_\mathcal{C} < 0$. In Section 4, we study the case $\delta_\mathcal{C} = 0$, using a covering $\varphi$ mentioned above, to prove that $\mathcal{C}$ does not have global FFRT property. In Section 5, we slightly generalize Sun’s result [Su] on the stability of Frobenius push-forwards to orbifold curves with $\delta_\mathcal{C} > 0$. In Section 6, we synthesize the result obtained in the previous sections with the main theorem (Theorem 6.2) and discuss the exceptional cases where $p$ divides denominators of the $\mathbb{Q}$-divisor $D$.

Acknowledgement. The first author thanks Holger Brenner for many useful comments to the manuscript. The second author thanks Masao Aoki for calling his attention to the book [Ol] which was helpful in writing this paper. He also thanks Shunsuke Takagi for informing of an example in [TT].

1. Preliminaries

The definition of the FFRT property requires the ring $R$ under consideration to be complete local or graded. In this paper we focus on the graded case, as follows. Let $R = \bigoplus_{m \geq 0} R_m$ be a Noetherian normal $\mathbb{N}$-graded ring over an algebraically closed field $R_0 = k$ with $\dim R \geq 2$. We denote by $X$ the normal projective variety $X = \text{Proj} \, R$.

1.1. Pinkham-Demazure construction of a normal graded ring. By [D, P] the graded ring $R$ is described as follows: There exists an ample $\mathbb{Q}$-Cartier divisor $D$ on $X$ such that

$$R \cong R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X([mD])) t^m,$$

where $t$ is a homogeneous element of degree 1. We write the $\mathbb{Q}$-divisor $D$ as

$$D = \sum_{i=1}^{n} \frac{s_i}{r_i} D_i,$$

where $D_1, \ldots, D_n$ are distinct prime divisors on $X$, and $r_i > 0$ and $s_i$ are coprime integers.

In the notation above, let

$$Y = \text{Spec}_X \left( \bigoplus_{m \geq 0} \mathcal{O}_X([mD]) t^m \right) \quad \text{and} \quad U = \text{Spec}_X \left( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X([mD]) t^m \right).$$
Then $U$ is an open subset of $Y$ and we have the following commutative diagram.

\[
\begin{array}{ccc}
U & \xrightarrow{\cong} & Z \setminus V(R_{+}) \\
\cap & & \cap \\
\text{Ex}(\varphi) & \leftrightarrow & Y \\
\cong & \sigma & Z = \text{Spec } R \\
& \varphi & X
\end{array}
\]

(1.1)

Here $\text{Ex}(\varphi) = Y \setminus U$ is endowed with reduced closed subscheme structure. Then $\text{Ex}(\varphi)$ is a section of the structure morphism $\sigma: Y \to X$ and also the exceptional divisor of the graded blowup $\varphi: Y \cong \text{Proj} (\bigoplus_{m \geq 0} R_{m}) \to \text{Spec } R$, where $R_{m} = \bigoplus_{m' \geq m} R_{m'}$. Also, $\sigma: Y \to X$ has an $\mathbb{A}^{1}$-bundle structure apart from the divisors $D_{i}$ on $X$. On the other hand, if we denote by $F_{i}$ the reduced fiber of $\sigma$ over the prime divisor $D_{i}$, then $\sigma^{*}D_{i} = r_{i}F_{i}$; see [D].

1.2. Finite $F$-representation type. We assume that the characteristic of $k$ is $p > 0$. Then any scheme $S$ over $k$ admits the Frobenius morphism $F: S \to S$ associated with the $p$-th power ring homomorphism $O_{S} \to F_{*}O_{S}$. By our assumption, the graded ring $R$ is $F$-finite, i.e., the Frobenius on $Z = \text{Spec } R$ is a finite morphism. For each $e = 0, 1, 2, \ldots$, the $e$-times Frobenius push-forward $F^{e}_{*}R$ of the graded ring $R$ is identified with the ring $R^{1/p^{e}}$, which has a natural $\frac{1}{p^{e}}\mathbb{Z}$-grading. Hence we can consider $R^{1/p^{e}}$ as an object of the category of finitely generated $\mathbb{Q}$-graded $R$-modules. In this category, we define an equivalence $\sim$ of objects to be a graded isomorphism which admits a degree shift: Namely, for $\mathbb{Q}$-graded modules $M, N$, we define $M \sim N$ if $N \cong M(\alpha)$ via a degree-preserving isomorphism for some $\alpha \in \mathbb{Q}$. Now by the Krull-Schmidt theorem, we have a unique decomposition

\[
R^{1/p^{e}} = M^{(e)}_{1} \oplus \cdots \oplus M^{(e)}_{m_{e}}
\]

in the category of finitely generated $\mathbb{Q}$-graded $R$-modules for $e = 0, 1, 2, \ldots$, with $M^{(e)}_{i}$ indecomposable.

**Definition 1.1** ([SVdB]). We say that $R$ has finite $F$-representation type (FFRT) if the set of equivalence classes $\{M^{(e)}_{i} | e = 0, 1, 2, \ldots; i = 1, \ldots, m_{e}\}/\sim$ is finite.

For $q = p^{e}$ we want to know the decomposition of the $R$-module $R^{1/q}$. The graded ring structure of $R = R(X, D)$ allows us to decompose $R^{1/q} = \bigoplus_{t \geq 0} H^{0}(X, F^{e}_{*}\mathcal{O}_{X}([tD]))^{t!/q}$ as $R^{1/q} = \bigoplus_{t = 0}^{q-1} (R^{1/q})_{i/q \mod \mathbb{Z}}$, where

\[
(R^{1/q})_{i/q \mod \mathbb{Z}} = \bigoplus_{0 \leq t \equiv i \mod q} H^{0}(X, F^{e}_{*}\mathcal{O}_{X}([tD]))^{t!/q} \cong \bigoplus_{m \geq 0} H^{0}(X, F^{e}_{*}\mathcal{O}_{X}(([qm + i]D))
\]

is an $R$-summand of $R^{1/q}$ for $i = 0, 1, \ldots, q - 1$. If $D$ is an integral Cartier divisor, then $F^{e}_{*}\mathcal{O}_{X}(([qm + i]D)) \cong \mathcal{O}_{X}(D)^{\otimes m} \otimes F^{e}_{*}\mathcal{O}_{X}(iD)$ by the projection formula. Thus in this case, the decomposition of the $R$-module $(R^{1/q})_{i/q \mod \mathbb{Z}}$ depends on the decomposition of the vector bundle $F^{e}_{*}\mathcal{O}_{X}(iD)$ on $X$. However, this observation fails when $D$ is not integral. To overcome this difficulty, we will introduce a root stack associated with the pair $(X, D)$, which allows us to treat $D$ as if it is an integral divisor.

1.3. Root stacks. The exposition and notation in this subsection is based on generalities on stacks; see Olsson [Ol] and references therein for more details. For a scheme $T$ with a group scheme $G$ acting on $T$, we denote by $[T/G]$ the quotient stack of $T$ by $G$ as in [Ol] Example 8.1.12. For a stack $\mathcal{Y}$ and a scheme $S$, we write by $\mathcal{Y}(S)$ the groupoid of $S$-valued points of $\mathcal{Y}$ as is standard.

Here we briefly review the notion of root stacks (cf. [Ol] 10.3]). Let $X$ be a $k$-scheme and $D_{1}, \ldots, D_{n}$ Cartier divisors on $X$. We consider the associated section $\xi_{i}: \mathcal{O}_{X}(-D_{i}) \to \mathcal{O}_{X}$ and
the induced morphism $\xi: X \to [\mathbb{A}^1/k^n]$, where $\mathbb{A}^1 = \mathbb{A}^1_k$ is the affine line on which $k^*$ acts naturally.

For $r = (r_1, \ldots, r_n) \in \mathbb{Z}_{\geq 0}^n$, an $r$-th root stack $\pi: \mathcal{X} \to X$ of $(D_1, \ldots, D_n)$ is defined by the pull-back of $r$ by $\xi$

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\xi} & [\mathbb{A}^1/k^n] \\
\pi & & \downarrow r \\
X & \xrightarrow{\xi} & [\mathbb{A}^1/k^n],
\end{array}
$$

where $r: [\mathbb{A}^1/k^n] \to [\mathbb{A}^1/k^n]$ is induced by $r_i$-th power $\mathbb{A}^1 \to \mathbb{A}^1; z \mapsto z^{r_i}$ for $i = 1, \ldots, n$. We also write $\mathcal{X} = X[\sqrt[d_1]{D_1}, \ldots, \sqrt[d_n]{D_n}]$, and call $r_1, \ldots, r_n$ the weights. Let $E_i$ be the integral Cartier divisor on $X$ defined by $\xi_x x_i$, where $\xi_x$ is in the diagram (1.2), and $x_i$ is the coordinate of the $i$-th component in $[\mathbb{A}^1/k^n]$. Then we have $r_i E_i = \pi^* D_i$.

By definition, for a $k$-scheme $S$, the groupoid $\mathcal{X}(S)$ consists of data

$$
\xi = (f: S \to X, (\mathcal{L}_1, s_1), \ldots, (\mathcal{L}_m, s_n), \alpha_1, \ldots, \alpha_n),
$$

where $\mathcal{L}_i$ are line bundles on $S$, and $s_i \in \Gamma(S, \mathcal{L}_i)$, and $\alpha_i: \mathcal{L}_i^{\otimes r_i} \cong f^* \mathcal{O}_X(D_i)$ such that $\alpha_i(s_i^{\otimes r_i}) = \xi_i$.

For another object $\xi' = (f': S \to \mathbb{P}^1, (\mathcal{L}'_1, s'_1), \ldots, (\mathcal{L}'_n, s'_n), \alpha'_1, \ldots, \alpha'_n) \in \mathcal{X}(S)$, an isomorphism $\eta: \xi \cong \xi'$ in $\mathcal{X}(S)$ consists of isomorphisms of line bundles $\eta_i: \mathcal{L}_i \to \mathcal{L}'_i$ for $i = 1, \ldots, n$ such that $\eta_i(s_i) = s'_i$, and the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{L}_i^{\otimes r_i} & \xrightarrow{\alpha_i} & f^* \mathcal{O}_X(D_i) \\
\eta_i^{\otimes r_i} & \xrightarrow{\eta_i} & f'^* \mathcal{O}_X(D_i)
\end{array}
$$

Locally, we take an affine open subset $W = \text{Spec } A$ of $X$ such that $D_i|_W = \{f_i = 0\}$ for $f_i \in A$ and $i = 1, \ldots, n$. Hence it is isomorphic to $[\text{Spec } B/\mu_{r_1} \times \cdots \times \mu_{r_n}]$, where

$$
B = A[w_1, \ldots, w_n]/(w_1^{r_1} - f_1, \ldots, w_n^{r_n} - f_n),
$$

and $\mu_{r_1} \times \cdots \times \mu_{r_n}$ acts on Spec $B$ by $(w_1, \ldots, w_n) \mapsto (\eta_1 w_1, \ldots, \eta_n w_n)$ for $(\eta_1, \ldots, \eta_n) \in \mu_{r_1} \times \cdots \times \mu_{r_n}$ (cf. [01 Theorem 10.3.10]).

For later use, we summarize a few fundamental properties of root stacks $\pi: \mathcal{X} \to X$ in the following:

**Lemma 1.2.** Under the notation as above we have the following.

1. $\pi: \mathcal{X} \to X$ is an isomorphism apart from $E_i$ and $D_i$ ($i = 1, \ldots, n$).
2. $E_i$ is an integral Cartier divisor on $X$ with $\pi^* D_i = r_i E_i$ for $i = 1, \ldots, n$.
3. If $\pi': \mathcal{X}' \to X$ is a morphism such that there exists a Cartier divisor $E'_i$ on $X'$ with $r_i E'_i = (\pi')^* D_i$ for $i = 1, \ldots, n$, then there is a morphism $\varphi: \mathcal{X}' \to \mathcal{X}$ such that $\pi' = \pi \circ \varphi$ and $E'_i = \varphi^* E_i$. This is unique up to 2-isomorphisms.
4. For a Cartier divisor $\sum_{i=1}^n l_i E_i$ on $\mathcal{X}$, one has

$$
\pi_* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n l_i E_i) = \mathcal{O}_X(\sum_{i=1}^n \frac{l_i}{r_i} D_i) \quad \text{and} \quad R^i \pi_* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^n l_i E_i) = 0 \quad \text{for} \; i > 0.
$$

**Proof.** (1) and (2) follow from the definition of the root stack $\mathcal{X}$ and Cartier divisors $E_i$ on $\mathcal{X}$.

(3) follows from descriptions of $\mathcal{X}(S)$ for schemes $S$.

As for (4), by the local description as above, coherent sheaves are considered as $\prod_{i=1}^n \mu_{r_i}$-equivariant coherent sheaves on Spec $B$, where $B$ is in (1.3), and the push-forward $\pi_*$ corresponds
to taking $\prod_{i=1}^{n} \mu_{e_i}$-invariant parts. This is exact, hence $R^i\pi_*F = 0$ for any coherent sheaf $F$ on $X$ and $i > 0$.

We write $l_i = r_i k_i + m_i$ for $0 \leq m_i < r_i$, that is, $k_i = \lfloor \frac{l_i}{r_i} \rfloor$. For a line bundle $O_X(E)|_{\pi^{-1}W}$, the corresponding $B$-module is

$$Bu_1^{-l_1} \cdots u_n^{-l_n} = \bigoplus_{j_1=1}^{r_1-1} \cdots \bigoplus_{j_n=0}^{r_n-1} A \frac{u_1^{j_1} \cdots u_n^{j_n}}{f_1^{k_1} u_1^{m_1} \cdots f_n^{k_n} u_n^{m_n}}.$$ 

Hence, as desired, the push-forward $\pi_*O_X(\pi^*D)|_W$ is $O_W(\sum_{i=1}^{n} \lfloor \frac{l_i}{r_i} \rfloor D_i)$ which correspond to the degree 0 part $A^{-1}_{\Pi_{i=1}^{R_1} \cdots f_n^{R_n}}$. \qed

We also have another equivalent description of $X$ as follows. Let $L_1, \ldots, L_n$ be the total spaces of line bundles $O_X(-D_1), \ldots, O_X(-D_n)$. We consider the complement $L_i^\times$ of the zero section. We consider $(k^*)^n$-action on $U = L_1^\times \times \cdots \times L_n^\times \times \mathbb{A}^n$ by $((t_i, s_i), (x_i, x_i))$ for $(t_i, s_i) \in (k^*)^n$, and $(k^*)^n$-equivariant vector bundle $E = U \times \prod_{i=1}^{n} k t_i$ over $U$. We define an equivariant section $s: U \to E$ defined by

$$((s_1, (x_i)), (\xi(s_i) - x_i^r))$$

and a closed subscheme $V = s^{-1}(0)$ of $U$. We put $X' = [V/(k^*)^n]$, and consider the natural projection $\pi': X' \to X$.

We consider Cartier divisors $E_i = \{ x_i = 0 \}$ on $X'$. If we write $E_i = O_X(E'_i)$, then we have an isomorphism $L_i^{\otimes r_i} \cong (\pi')^*O_X(D_i)$ from the construction of $X'$. This gives an homomorphism $X' \to X$ by Lemma [2.3] (3). By the above local description of $X$, this gives an isomorphism $X' \cong X$.

By [Mo, Proposition 2.3.3], we see that the cotangent complex $L_X/k$ is the following complex of $(k^*)^n$-equivariant vector bundles on $V$:

$$0 \to E^\vee|_V \to \Omega_U|_V \to O_V^\oplus_1 \to 0,$$

where we consider $O_V^\oplus_1$ as a trivial bundle with the fibers equivalent to the cotangent space of $(k^*)^n$ at the unit. Since $E^\vee|_V = \bigoplus_{i=1}^{n} O_V t_i^{-r_i}$ and $\Omega_U = \bigoplus_{i=1}^{n} (O_U \oplus O_U t_i^{-1}) \oplus \Omega_X|_U$, we have

$$\det \Omega_X = \pi^* \det \Omega_X \otimes O_X(\sum_{i=1}^{n} (r_i - 1)E_i).$$

2. FFRT PROPERTY OF R(C, D) VIA ORBIFOLD CURVES

2.1. Orbifold curves. By an “orbifold curve,” we mean a one-dimensional smooth separated Deligne-Mumford stack $\mathcal{C}$ whose coarse moduli map $\pi: \mathcal{C} \to C$ to a smooth projective curve $C$ is generically isomorphism. As in [B 1.3.6], an orbifold curve is a root stack over $C$.

We fix the notation to be used throughout this section. Given integers $r_1, \ldots, r_n \geq 2$ and closed points $P_1, \ldots, P_n$ on a smooth projective curve $C$, let $\mathcal{C} = C[\sqrt{P_1}, \ldots, \sqrt{P_n}]$ be the root stack of weight $(r_1, \ldots, r_n)$ and let $\pi: \mathcal{C} \to C$ be the coarse moduli map. For $i = 1, \ldots, n$, we denote by $Q_i$ the stacky point over $P_i$, that is, the integral Cartier divisor on $C$ with $\pi^*P_i = r_i Q_i$. The next lemma follows from ([1.4]) and [N, Theorem 2.22].

Lemma 2.1. $\mathcal{C}$ has a dualizing sheaf $\omega_\mathcal{C} \cong \pi^*\omega_C \otimes O_\mathcal{C}(\sum_{i=1}^{n} (r_i - 1)Q_i)$, which is isomorphic to the canonical bundle $\Omega_\mathcal{C}$.

We use the notation $\omega_\mathcal{C}$ for the dualizing sheaf, $\Omega_\mathcal{C}$ for the differential or canonical sheaf, and $K_\mathcal{C}$ for the canonical divisor interchangeably according to the context. We consider the Chow ring $A(\mathcal{C})$ with $\mathbb{Q}$-coefficient, and the map $\deg: A(\mathcal{C}) \to A(\text{Spec } k) \cong \mathbb{Q}$ induced by the push-forward by the structure morphism $\mathcal{C} \to \text{Spec } k$. We put $\delta_\mathcal{C} := \deg \omega_\mathcal{C} \in \mathbb{Q}$. This is equal to $n + 2g - 2 - \sum_{i=1}^{n} \frac{1}{r_i}$ by Lemma [2.1] where $g$ is the genus of $C$. 


Now let $R = R(C, D)$ for an ample $\mathbb{Q}$-Cartier divisor $D = \sum_{i=1}^{n} (s_i/r_i) P_i$ on $C$ as in subsection 1.1. Then Lemma 1.2 (4) allows us to think of $R = R(C, D)$ as the section ring associated with an integral Cartier divisor $\pi^*D$ or equivalently a line bundle $L = \mathcal{O}_C(\pi^*D)$ on $\mathcal{C}$:

$$R = R(C, D) \cong R(\mathcal{C}, L) = \bigoplus_{m \geq 0} H^0(\mathcal{C}, L^m)t^m.$$ 

We will extend the fundamental diagram (1.1) to the stacky situation.

$$\tilde{Y} = \text{Spec}_\mathcal{C} \left( \bigoplus_{m \geq 0} L^m t^m \right) \quad \text{and} \quad \tilde{U} = \text{Spec}_\mathcal{C} \left( \bigoplus_{m \in \mathbb{Z}} L^m t^m \right).$$

Then $\tilde{Y}$ is an $\mathbb{A}^1$-bundle over $\mathcal{C}$ and we have the following extended fundamental diagram.

$$\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\bar{\psi}} & Y \\
\cap_{n} & \cap_{n} & \cap_{n} \\
\bar{\psi} & \xrightarrow{\psi} & \tilde{Y} \\
\bar{\pi} & \xrightarrow{\pi} & \mathcal{C} \\
\end{array}$$

Here $\psi: \tilde{Y} \to Y$ and $\tilde{U} \to U$ are induced by isomorphisms $\pi_*L^m \cong \mathcal{O}_C([mD])$ for $m \in \mathbb{Z}$ (cf. [Ol] Theorem 10.2.4). In the local description (1.3), we see that $(π \circ \tilde{σ})^{-1}W$ is a quotient stack $[V/\prod_{i=1}^{n} \mu_{r_i}]$ of a $\prod_{i=1}^{n} \mu_{r_i}$-equivariant line bundle $V$ over $\text{Spec} \, B$, and

$$\sigma^{-1}(W) = \text{Spec} \, Γ(\mathcal{V}, \mathcal{O}_\mathcal{V})\prod_{i=1}^{n} \mu_{r_i}$$

by Lemma 1.2 (4). Furthermore $\psi|_{(π \circ \tilde{σ})^{-1}W}$ is given by the inclusion $Γ(\mathcal{V}, \mathcal{O}_\mathcal{V})\prod_{i=1}^{n} \mu_{r_i} \to Γ(\mathcal{V}, \mathcal{O}_\mathcal{V})$. Hence $ψ$ is a coarse moduli map (cf. [Ol] Chapter 6).

**Lemma 2.2.** In the situation as above we have the following.

1. $ψ$ induces an isomorphism $\tilde{U} \cong U$ preserving the $\mathbb{Z}$-grading.
2. We have an isomorphism

$$\mathcal{C} \cong \left[ \tilde{U}/k^* \right],$$

where $k^* = \text{Spec} \, k[t, t^{-1}]$ is the multiplicative group, and $k^*$-action on $\tilde{U}$ is induced by the fiberwise multiplication on the line bundle $\tilde{Y}$ over $\mathcal{C}$.

Consequently, we have an equivalence between the category of vector bundles on $\mathcal{C}$ and the category of reflexive $\mathbb{Z}$-graded $R$-modules given by

$$\mathcal{E} \mapsto Γ_*(\mathcal{E}) = \bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{C}, \mathcal{E} \otimes L^m)t^m.$$ 

**Proof.** (1) Since $r_i$ and $s_i$ are co-prime to each other, the automorphism functors of all closed points of $\tilde{U}$ are trivial. Hence by [C] Theorem 2.2.5], $\tilde{U}$ is an algebraic space, and the coarse moduli map $ψ|_{\tilde{U}}: \tilde{U} \to U$ is an isomorphism. (2) is obvious.

It follows from (2) that there is a one-to-one correspondence between vector bundles on $\mathcal{C}$ and $\mathbb{Z}$-graded vector bundles on $\tilde{U}$ given by $\mathcal{E} \to \bigoplus_{m \in \mathbb{Z}} \mathcal{E} \otimes L^m$. On the other hand, since $\tilde{U} \cong Z \setminus V(R_+)$ by (1) and $\text{codim}(V(R_+), Z) = 2$, we have a one-to-one correspondence between $\mathbb{Z}$-graded vector bundles on $\tilde{U}$ and reflexive graded $R$-modules, from which the required correspondence follows.

**Definition 2.3.** For a line bundle $L$ on $\mathcal{C}$, we define an equivalence $\sim_L$ in $\text{Coh}(\mathcal{C})$ as follows. For $\mathcal{E}, \mathcal{F} \in \text{Coh}(\mathcal{C})$, we write $\mathcal{E} \sim_L \mathcal{F}$ if $\mathcal{E} \cong \mathcal{F} \otimes L^m$ for some $m \in \mathbb{Z}$. 
Corollary 2.4. For a vector bundle $\mathcal{E}$ on $\mathcal{C}$, $$\mathcal{E} \mapsto \Gamma_\ast(\mathcal{E}) = \bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{E}, \mathcal{E} \otimes L^m)t^m$$
gives a one-to-one correspondence between the set of equivalence classes of vector bundles on $\mathcal{C}$ with respect to $\sim_L$ and the set of equivalence classes of reflexive $\mathbb{Z}$-graded $R$-modules with respect to the equivalence $\sim$ admitting degree shift as in subsection 1.2.

2.2. FFRT property of $R(C, D)$ via orbifold curves. Let us now work over a field $k$ of characteristic $p > 0$. Then for a $k$-scheme $S$ we have the Frobenius morphism $F: S \to S$. We define a Frobenius morphism $F: \mathcal{C} \to \mathcal{C}$ on the stack $\mathcal{C}$ by the pull-back functor $F^\ast: \mathcal{C}(S) \to \mathcal{C}(S)$ by the Frobenius morphism $F: S \to S$. Since $F: S \to S$ is an affine morphism, the Frobenius push-forward $F_\ast: \text{Coh}(\mathcal{C}) \to \text{Coh}(\mathcal{C})$ is exact, where Coh $\mathcal{C}$ is the category of coherent sheaves on $\mathcal{C}$.

Recall that $R = R(C, D)$ is the section ring $R = R(\mathcal{C}, L) = \bigoplus_{l \geq 0} H^0(\mathcal{C}, L^l)t^l$ associated with the line bundle $L = \mathcal{O}_\mathcal{C}(\pi^*D)$. Then for $0 \leq i < q = p^e$, the $R$-summand $(R^{1/q})_{i \mod q}$ of $R^{1/q}$, which is described as $$(R^{1/q})_{i \mod q} = \bigoplus_{0 \leq l \leq i \mod q} H^0(\mathcal{C}, F_*^e(L^l))t^{l/q} \cong \bigoplus_{m \geq 0} H^0(\mathcal{C}, F_*^e(L^l) \otimes L^m)t^{i/q + m},$$
is equivalent to the $\mathbb{Z}$-graded $R$-module $\Gamma_\ast(F_*^e(L^l))$ with respect to $\sim$. In view of Corollary 2.4, it is important for our purpose to know the decomposition of the Frobenius push-forwards $F_*^e(L^l)$ with $0 \leq i < q - 1$ on the orbifold curve $\mathcal{C}$.

Given any line bundle $L$ on $\mathcal{C}$ and integers $e$, $i \geq 0$, we have a unique decomposition $$F_*^e(L^l) = F_1^{(e,i)} \oplus \cdots \oplus F_{m_{e,i}}^{(e,i)}$$
in $\text{Coh}(\mathcal{C})$ with $F_j^{(e,i)}$ indecomposable.

Definition 2.5. Let $L$ be a line bundle on $\mathcal{C}$. We say that the pair $(\mathcal{C}, L)$ has globally finite $F$-representation type (FFRT for short), if the set of isomorphism classes $\{F_j^{(e,i)} | e = 0, 1, 2, \ldots; i = 0, 1, \ldots, p^e - 1; j = 1, \ldots, m_{e,i}\}/\cong$ is finite. We say that $\mathcal{C}$ has FFRT if the pair $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ does.

Corollary 2.6. Let $R = R(C, D) = R(\mathcal{C}, L)$, where $L = \mathcal{O}_\mathcal{C}(\pi^*D)$ as above. Then $R$ has FFRT if and only if $(\mathcal{C}, L)$ has FFRT.

Proof. By Corollary 2.4, $R$ has FFRT if and only if the set of equivalence classes $\{F_j^{(e,i)} | e = 0, 1, 2, \ldots; i = 0, 1, \ldots, p^e - 1; j = 1, \ldots, m_{e,i}\}/\sim_L$ is finite. Hence the sufficiency follows immediately. For the necessity, it is sufficient to prove the following claim.

Claim 2.6.1. For any vector bundle $\mathcal{F}$ on $\mathcal{C}$, the set $$\left\{ m \in \mathbb{Z} \left| \begin{array}{c} \mathcal{F} \otimes L^m \text{ is isomorphic to a direct summand of} \\ F_*^e(L^l) \text{ for some } e \geq 0 \text{ and } 0 \leq i \leq p^e - 1 \end{array} \right. \right\}$$
is finite.

To prove the claim we first note that there exists an integer $m_0$ such that $H^0(\mathcal{C}, F_*^e(L^l) \otimes L^{m_0}) \cong H^0(\mathcal{C}, L^{i + p^e m_0}) = H^0(C, \mathcal{O}_C(\lfloor (i + p^e m_0)D \rfloor)) = 0$ for all $e \geq 0$ and $0 \leq i \leq p^e - 1$. This follows if we choose $m_0$ small enough, e.g., $m_0 = -1$. Hence, if $\mathcal{F} \otimes L^l$ is a direct summand of $F_*^e(L^l)$, then we must have $H^0(\mathcal{C}, \mathcal{F} \otimes L^{i + m_0}) = 0$. On the other hand, there exists an integer $l_0$ such that $H^0(\mathcal{C}, \mathcal{F} \otimes L^l) \neq 0$ for all $l \geq l_0$. Indeed, if
we choose an integer \( r > 0 \) such that \( rD \) is integral and write \( l = rs + i \) with \( 0 \leq i < r \), then we see that
\[
H^0(\mathcal{C}, F \otimes L^i) = H^0(C, \pi_*(\mathcal{F} \otimes L^{i+r})) = H^0(C, \pi_*(\mathcal{F} \otimes L^i) \otimes \mathcal{O}_C(rD)^{\otimes s})
\]
is non-zero for \( s \gg 0 \), since \( rD \) is ample. Thus we conclude that if \( l \geq l_0 - m_0 \), then \( \mathcal{F} \otimes L^i \) cannot be a direct summand of \( F^e_s(L^i) \) for any \( e \geq 0 \) and \( 1 \leq i \leq p^e - 1 \). This implies that the set in Claim 2.6.1 is bounded above. Similarly, a dual argument with the first cohomology \( H^1 \) gives a lower bound of the set. \( \square \)

Finally we rephrase the \( F \)-purity of \( R \) in terms of the \( F \)-splitting of an orbifold curve. We say that \( \mathcal{C} \) is \( F \)-split if the Frobenius ring homomorphism \( F: \mathcal{O}_\mathcal{C} \to F \cdot \mathcal{O}_\mathcal{C} \) splits as an \( \mathcal{O}_\mathcal{C} \)-module homomorphism, and that a ring \( R \) is \( F \)-pure if Spec \( R \) is \( F \)-split. Note that the Frobenius morphism on \( \mathcal{C} \) induces \( F: \omega_{\mathcal{C}} \to \omega_{\mathcal{C}} \otimes F \cdot \mathcal{O}_\mathcal{C} \cong F_*(\omega_{\mathcal{C}}^p) \), and that on the first cohomology \( F: H^1(\mathcal{C}, \omega_{\mathcal{C}}) \to H^1(\mathcal{C}, F_*(\omega_{\mathcal{C}}^p)) \cong H^1(\mathcal{C}, \omega_{\mathcal{C}}^p) \).

**Proposition 2.7.** In the notation as above, the following three conditions are equivalent.
1. \( R(C, D) \cong R(\mathcal{C}, L) \) is \( F \)-pure.
2. \( \mathcal{C} \) is \( F \)-split.
3. The induced Frobenius map \( F: H^1(\mathcal{C}, \omega_{\mathcal{C}}) \to H^1(\mathcal{C}, \omega_{\mathcal{C}}^p) \) is injective.

**Proof.** First note that \( \text{Hom}(F_* \mathcal{O}_\mathcal{C}, \omega_{\mathcal{C}}) \cong F_* \omega_{\mathcal{C}} \) by the adjunction formula [HM, Ex. III.7.2], so that \( \text{Hom}(F_* \mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong F_*(\omega_{\mathcal{C}}) \otimes \omega_{\mathcal{C}}^{-1} \cong F_*(\omega_{\mathcal{C}}^{1-p}) \). Now \( \mathcal{C} \) is \( F \)-split if and only if the dual Frobenius map \( F^v: \text{Hom}(F_* \mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \to \text{Hom}(\mathcal{O}_\mathcal{C}, F_* \mathcal{O}_\mathcal{C}) \) is surjective. Since this map is identified with \( F^v: H^0(\mathcal{C}, F_*(\omega_{\mathcal{C}}^{1-p})) \to H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}) \), which is dual to the induced Frobenius \( F: H^1(\mathcal{C}, \omega_{\mathcal{C}}) \to H^1(\mathcal{C}, F_*(\omega_{\mathcal{C}}^p)) \) by the Serre duality, the equivalence of (2) and (3) follows. On the other hand, the equivalence of (1) and (3) follows from [W], since Lemma 1.2 allows us to identify the induced Frobenius map in (3) with \( F: H^1(C, \mathcal{O}_C(K_C)) \to H^1(C, \mathcal{O}_C([p(K_C + D']))) \), where \( D' \) is the “fractional part” of \( D \) so that \( \pi^i(K_C + D') = K_C \). \( \square \)

### 3. Weighted projective lines

In this section, we consider an orbifold curve \( \mathcal{C} = C[\sqrt[n]{P_1}, \ldots, \sqrt[n]{P_n}] \) with \( C = \mathbb{P}^1 \), that is, we have a coarse moduli map \( \pi: \mathcal{C} \to \mathbb{P}^1 \). In this case, \( \mathcal{C} \) is called a weighted projective line.

#### 3.1. Homogeneous coordinate ring

Here we construct \( \mathcal{C} \) as a quotient stacks \([U/G]\) following [CL]. We take the homogeneous coordinate ring \( T = k[z_1, z_2] \) of the projective line \( \mathbb{P}^1 \) such that \( P_1 = \{ z_1 = 0 \} \), \( P_2 = \{ z_2 = 0 \} \), and \( P_i = \{ z_2 - \lambda_i z_1 = 0 \} \) for \( \lambda_i \in k \) and \( i = 1, \ldots, n \). We consider a \( T \)-algebra
\[
S = T[x_1, \ldots, x_n]/(x_1^{r_1} - z_1, x_2^{r_2} - z_2, \ldots, x_n^{r_n} - (z_2 - \lambda_3 z_1)),
\]
and take an open subset \( U = \text{Spec } S \setminus \{(x_1, x_2) = (0, 0)\} \). We define a group \( G \) acting on \( U \) by \( G = \text{Hom}_{\text{group}}(\Gamma, k^*) \) for
\[
\Gamma = \bigoplus_{i=1}^n \mathbb{Z}a_i \oplus \mathbb{Z}c/(r_i a_i - c \mid i = 1, \ldots, n).
\]
This means
\[
G = \text{Spec } k[a_1^{\pm 1}, \ldots, a_n^{\pm 1}, c^{\pm 1}]/(a_i^{r_i} - c \mid i = 1, \ldots, n).
\]
Here \( G \) acts on \( U \) diagonally
\[
(x_1, \ldots, x_n) \mapsto (a_1 x_1, \ldots, a_n x_n).
\]
In other words, the \( G \)-action is given by \( \Gamma \)-grading of \( S \) defined by \( \deg \Gamma x_i = a_i \) for \( i = 1, \ldots, n \).

This action is compatible with the natural morphism \( U \to \mathbb{P}^1 \) induced by the \( T \)-algebra structure of \( S \), and gives \( \pi: [U/G] \to \mathbb{P}^1 \). Furthermore, by \( G \)-weight spaces \( ka_i \) with \( G \)-action given by the multiplication of \( a_i \), we have line bundles \( L_i = [U \times ka_i/G] \) on the quotient
The FFRT property of graded rings and orbifold curves

9

stack $[U/G]$, and sections $s_i \in \Gamma([U/G], L_i)$ defined by $x_i$. Since we have natural isomorphisms $\alpha: L^{\oplus r_i} \cong (\pi')^*\mathcal{O}_G(P_i)$ sending $x^{\alpha}_{i1}$ to $\lambda_i z_i - z_2$, these data defines a morphism $\varphi: [U/G] \to \mathcal{C}$ such that $\pi' = \pi \circ \varphi$ by Lemma 1.2 (3). By the local description (1.3), we see that $\varphi$ is an isomorphism $[U/G] \cong \mathcal{C}$. In the following, we identify $\mathcal{C}$ with the quotient stack $[U/G]$ via this isomorphism, and we call the $\Gamma$-graded algebra $S$ a homogeneous coordinate ring of $\mathcal{C}$. We have $Q_i = \{x_i = 0\}$ on $\mathcal{C} = [U/G]$, and $\mathcal{O}_\mathcal{C}(Q_i) \cong L_i$.

By this construction, we have an identification $\text{Pic} \mathcal{C} \cong \Gamma$ sending $\mathcal{O}_\mathcal{C}(Q_i)$ to $\tilde{a}_i$. We write by deg: $\text{Pic} \mathcal{C} \to \mathbb{Q}$ the map taking degrees of Chern classes of line bundles on $\mathcal{C}$. We have $\text{deg} \tilde{a}_i = \frac{1}{r_i}$ for $i = 1, \ldots, n$, and $\text{deg} \tilde{c} = 1$, and in particular, $\delta_\mathcal{C} = \text{deg} \omega_\mathcal{C} = n - 2 - \sum_{i=1}^n \frac{1}{r_i}$ by Lemma 2.1.

3.2. Indecomposable vector bundles on a weighted projective line. We introduce a classification of indecomposable vector bundles on $\mathcal{C}$ by [CB]. We consider a lattice

$$\mathcal{L} = \mathbb{Z}\alpha_s \oplus \bigoplus_{i=1}^n \bigoplus_{j=1}^{r_i-1} \mathbb{Z}\alpha_{ij},$$

and put $\hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{Z}\delta$. Here $\alpha_s, \alpha_{ij}$ corresponds to the following graph consisting of vertices $\ast, ij$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r_i - 1$, and edges joining $\ast$ and $i1$, and $ij$ and $ij + 1$, for $i = 1, \ldots, n$, $j = 1, \ldots, r_i - 2$. The Cartan matrix is defined by $C = 2E - A$, where $E$ is the identity matrix and $A$ is the adjacency matrix of the above graph. This defines an inner product $(\cdot, \cdot): \mathcal{L} \times \mathcal{L} \to \mathbb{L}$.

We define the set $\Pi = \{\alpha_s\} \cup \{\alpha_{ij} | i = 1, \ldots, n, j = 1, \ldots, r_i - 1\}$ of simple roots as follows. An element $\alpha \in \Pi$ defines a reflection $\mathcal{L} \to \mathcal{L} \; \lambda \mapsto \lambda - (\alpha, \lambda)\alpha$. We define the Weyl group $W$ by the subgroup of $\text{Aut} \mathcal{L}$ generated by these reflections, and put $\Delta^\text{re} = W\Pi$. We define the fundamental set

$$M = \{v \in \mathcal{L}^+ | v \neq 0, (\alpha, v) \leq 0 \text{ for } \alpha \in \Pi, \text{ support of } v \text{ is connected}\}$$

where $\mathcal{L}^+ = \{v = l_s \alpha_s + \sum l_{ij} \alpha_{ij} \in \mathcal{L} \mid l_s, l_{ij} \geq 0\}$. We put $\Delta = \Delta^\text{re} \cup WM \cup W(-M)$, and $\hat{\Delta} = \{\alpha + m\delta \in \hat{\mathcal{L}} \mid \alpha \in \Delta, m \in \mathbb{Z}\} \cup \{m\delta \in \hat{\mathcal{L}} \mid m \in \mathbb{Z}, m \neq 0\}$. An element $v$ of $\hat{\Delta}$ is called a root. It is called a real root, if $v = \alpha + m\delta$ for $\alpha \in \Delta^\text{re}$, otherwise it is called imaginary root.

For a vector bundle $\mathcal{E}$ on $\mathcal{C}$, we associate a vector bundle $\mathcal{F} = \pi_* \mathcal{E}$ on $\mathbb{P}^1$ and flags

$$\{0 = \mathcal{F}_{i'r_i} \subset \cdots \subset \mathcal{F}_{ij} \subset \cdots \subset \mathcal{F}_{0} = \mathcal{F}|_{P_i}\}_{i=1}^n$$

defined by $\mathcal{F}_{ij} = \pi_* \mathcal{E}(-jQ_i)|_{P_i}$ for $i = 1, \ldots, n, j = 1, \ldots, r_i - 1$. We define a type $t(\mathcal{E})$ in $\hat{\mathcal{L}}$ of $\mathcal{E}$ by

$$t(\mathcal{E}) = (\text{rk} \mathcal{F}) \alpha_s + \sum_{i=1}^n \sum_{j=1}^{r_i-1} (\dim \mathcal{F}_{ij}) \alpha_{ij} + (\text{deg} \mathcal{F}) \delta.$$

This defines a map $t: K(\mathcal{C}) \to \hat{\mathcal{L}}$ from the Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$. We consider the subset $\hat{\mathcal{L}}^+ \subset \hat{\mathcal{L}}$ of positive linear combinations of $\alpha_s + m\delta$, $\delta$, $\alpha_{ij}$ and $-\sum_{j=1}^{r_i-1} \alpha_{ij} + \delta$ for $m \in \mathbb{Z}$ and $i = 1, \ldots, n$. We call elements in $\hat{\Delta} \cap \hat{\mathcal{L}}^+$ positive roots.

The following are due to [CB] Theorem 1.

**Theorem 3.1.** For an element $t \in \hat{\mathcal{L}}$, there exists an indecomposable sheaf on $\mathcal{C}$ with the type $t$ if and only if $t$ is a positive root. There is a unique isomorphism class of indecomposable sheaf for a real root, infinitely many for an imaginary root.

We take an ample Cartier divisor $D = \sum_{i=1}^n s_i P_i$ on $\mathbb{P}^1$, and put $L = \pi^* \mathcal{O}_{\mathbb{P}^1}(D)$. Combining Theorem 3.1 with Corollary 2.6 we have the following:

**Theorem 3.2.** For a weighted projective line $\mathcal{C}$, the set of equivalence classes of indecomposable vector bundles with respect to $\sim_L$ is finite, if and only if $\delta_\mathcal{C} = \text{deg} \omega_\mathcal{C} < 0$. In this case, the graded ring $R$ has FFRT.
Proof. By direct computations, we see that \( \deg \omega_\mathcal{C} < 0 \) if and only if the corresponding graphs are of finite type. As in [K] Chapter I, it is equivalent to \( \Delta = \Delta^w \). It is also known that in this case \( \Delta = \Delta^w \) is finite.

Hence if \( \deg \omega_\mathcal{C} < 0 \), then ranks of indecomposable vector bundles are bounded. We put

\[
\rho_{\text{max}} = \max \{ \text{rk } \mathcal{E} \mid \mathcal{E} \text{ indecomposable vector bundle on } \mathcal{C} \}.
\]

Since \( D \) is ample, we have a positive integer \( d \) such that \( \mathcal{O}_\mathcal{C}(r_1 \cdots r_n \pi^* D) \cong \pi^* \mathcal{O}_\mathbb{P}_1(d) \). Then after tensoring \( L \) suitably many times, the coefficient of \( \delta \) in the type of any indecomposable vector bundle lies between 0 and \( dr_{\text{max}} \). Hence the set of equivalence classes is finite by Theorem 4.1.

On the other hand, if \( \deg \omega_\mathcal{C} \geq 0 \), then we have infinitely many isomorphism classes of indecomposable vector bundles whose type is a fixed imaginary root. Since tensoring \( L \) changes types, these vector bundles are not equivalent to each other with respect to \( \sim_L \).

Finally from the description of grading structure of \( R^1_{\mathcal{F}} \) below Definition 1.1, we see that the last statement follows from Corollary 2.3. \( \square \)

Remark 3.3. As a special case of \( \delta_\mathcal{C} = \deg \omega_\mathcal{C} < 0 \), we have the toric case, in which the weighted projective line \( \mathcal{C} \) has at most two stacky points. In this case, for every line bundle \( L \) on \( \mathcal{C} \), the Frobenius push-forward \( F_\mathcal{F}^* L \) is decomposed into direct sum bundles (cf. [OU, Theorem 4.5]).

4. Frobenius summands on weighted projective lines with \( \delta_\mathcal{C} = 0 \)

In this section we study the structure of the Frobenius push-forward \( F_\mathcal{F}^* \mathcal{O}_\mathcal{C} \) on the weighted projective line \( \mathcal{C} \) when \( \delta_\mathcal{C} \) is equal to 0. In this case \( \mathcal{C} \) has three or four stacky points and the weight \( (r_1, \ldots, r_n) \) with \( r_1 \leq \cdots \leq r_n \) is either one of the following: \( (2, 3, 6) \), \( (2, 4, 4) \), \( (3, 3, 3) \), \( (2, 2, 2, 2) \). Also the canonical bundle \( \omega_\mathcal{C} \) is torsion of order \( m := \text{lcm}(r_1, \ldots, r_n) = r_n \). In what follows, we assume that the stacky points \( Q_1, \ldots, Q_n \) on \( \mathcal{C} \) are lying over \( \lambda_1, \ldots, \lambda_n \in \mathbb{P}^1 \), respectively.

Lemma 4.1. Let \( \mathcal{C} \) be a weighted projective line with \( \delta_\mathcal{C} = 0 \) as above, and suppose \( \text{char } k = p \) does not divide \( m = r_n \). Then there exists an elliptic curve \( E \) with \( \mu_m \cong \mathbb{Z}/m\mathbb{Z} \)-action and an \( m \)-fold covering \( f: E \to \mathbb{P}^1 \) which factors through \( \mathcal{C} \) as

\[
f = \pi \circ \varphi: E \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\pi} \mathbb{P}^1,
\]

satisfying the following conditions.

1. \( \mathcal{C} = [E/\mu_m] \) and \( \mathbb{P}^1 = E/\mu_m \) via \( \varphi \) and \( f \), respectively.
2. \( \varphi \) is unramified and \( \varphi_* \mathcal{O}_E \cong \bigoplus_{\ell=0}^{m-1} \omega^{\otimes(-\ell)}_\mathcal{C} \).
3. There exist exactly \( m/r_i \) points of \( E \) lying over the stacky point \( Q_i \) whose ramification index with respect to \( f \) is equal to \( r_i \).
4. Choose the point \( P_n \in E \) lying over \( Q_n \) as the zero element of \( E \) as a group. If \( P \in E \) is a ramification point of \( f \) lying over one of the stacky points \( Q_i \), then \( P \) is an \( m \)-torsion point with respect to the group law of \( (E, P_n) \).
5. \( \mathcal{C} \) is \( F \)-split if and only if \( E \) is ordinary (or equivalently, \( F \)-split), and in this case, \( p \equiv 1 \) (mod \( m \)).

Proof. Let \( \mathcal{A} = \bigoplus_{\ell \in \mathbb{Z}/m} \omega^{\otimes(-\ell)}_\mathcal{C} \) with an \( \mathcal{O}_\mathcal{C} \)-algebra structure defined by \( \omega^{\otimes(-m)}_\mathcal{C} \cong \mathcal{O}_\mathcal{C} \) and let \( \varphi: E = \text{Spec } \mathcal{A} \to \mathcal{C} \) be the induced morphism. We recall the description of \( \mathcal{C} = [U/G] \) in subsection 3.1. Then the \( \mathcal{O}_\mathcal{C} \)-algebra \( \mathcal{A} \) corresponds to the \( \Gamma \)-graded \( S \)-algebra \( S[\xi]/(\xi^{m} - 1) \).

Here \( \deg_\Gamma \xi = - \deg_\Gamma \omega_\mathcal{C} = -(n-2)e + \sum_{i=1}^{n} d_i \in \Gamma \).

By local computations, we see that every closed point in \( E \) has a trivial automorphism functor. Hence by [OI, Theorem 2.2.5], \( E \) is an algebraic space, and the coarse moduli map
Let $E \to \text{Spec}_{\mathbb{P}^1}(\pi_*, \mathcal{A})$ be an isomorphism. Thus $f = \pi \circ \varphi$ is identified with the structure morphism $\text{Spec}_{\mathbb{P}^1}(\pi_*, \mathcal{A}) \to \mathbb{P}^1$.

We have a $\mu_m$-action on $E = \text{Spec}_k \mathcal{A}$ by deg $\xi = 1 \in \mathbb{Z}/m\mathbb{Z} = (\mu_m)\vee$. This gives a proof of (1) and (2).

We prove (3) and (4) examining the $m$-fold covering $f: E \cong \text{Spec}_{\mathbb{P}^1}(\pi_*, \mathcal{A}) \to \mathbb{P}^1$ case by case for each weight. Then it follows that $E$ is an elliptic curve from Hurwitz’ formula.

The cases for weight $(3, 3, 3)$, $(2, 2, 2, 2)$ are easy: Indeed, $f$ is totally ramified at all the ramification points $P_1, \ldots, P_n$ in these cases, so that $mP_i = f^*(\lambda_i) = f^*(\lambda_j) = mP_j$ for $1 \leq i, j \leq n$. Hence $\mathcal{O}_E(P_i - P_n)^\otimes m \cong \mathcal{O}_E$ for all $i$, which means that $P_i$ is $m$-torsion.

As for weight $(2, 4, 4)$, we choose $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1 \in \mathbb{P}^1$ in an affine coordinate $u$ of $\mathbb{P}^1$ and a $\mathbb{Q}$-divisor $B = -\frac{1}{2}(0) + \frac{1}{4}(1) + \frac{1}{2}(-1)$ on $\mathbb{P}^1$ so that $\pi^*B \sim -K_E$. We give an $\mathcal{O}_{\mathbb{P}^1}$-algebra structure of $\pi_* \mathcal{A} \cong \bigoplus_{\ell=0}^{r} \mathcal{O}_{\mathbb{P}^1}([\ell B])z^\ell$ by the isomorphism $\mathcal{O}_{\mathbb{P}^1}(4B)z^4 \cong \mathcal{O}_{\mathbb{P}^1}$, via which $\frac{u^2}{u^2 - 1}z^4$ corresponds to 1. Then for an affine open subset $U = \text{Spec} k[u]$ of $\mathbb{P}^1$, $H^0(U, \pi_* \mathcal{A}) \cong k[u, uz, uz^2]/(u^2z^4 - u^2 + 1)$, and the 4-fold covering $f$ locally looks like

$$f: f^{-1}U \cong \text{Spec} k[u, v, w]/(u^2 - u^2 + 1, uv - v^2) \to U = \text{Spec} k[u].$$

It follows that $f$ has four ramification points $P_1, P_2, P_3, P_4$ with ramification indices 2, 2, 4, 4 whose affine coordinates with respect to $u, v, w$ are $(0, 0, -\sqrt{-1})$, $(0, 0, -\sqrt{-1})$, $(1, 0, 0)$, $(-1, 0, 0)$, respectively. Clearly $4P_3 \sim 4P_1$ and $P_2, P_4$ are 4-torsion points with respect to the group law of $(E, P)$. On the other hand, choosing $\varphi = (w - \sqrt{-1})/(u + 1) \in k(E)$, we see that $\text{div}_E(\varphi) = 4P_1 - 4P_4$, so that $4P_1 \sim 4P_4$. Similarly, $4P_2 \sim 4P_4$ and we see that $P_1, P_2$ are also 4-torsion. Thus (3) and (4) are proved for weight $(2, 4, 4)$.

The case for weight $(2, 3, 6)$ is proved similarly, but we omit detailed computations.

To prove (5) recall that the elliptic curve $E$ is ordinary if and only if the Frobenius

$$F: H^1(E, \mathcal{O}_E) \to H^1(E, F_* \mathcal{O}_E) \cong H^1(E, \mathcal{O}_E)$$

is injective (cf. Proposition 2.7). Since $\varphi_* \mathcal{O}_E \cong \bigoplus_{\ell \in \mathbb{Z}_m} \omega_E^{\otimes (-\ell)}$, this is equivalent to the injectivity of $F: H^1(\mathcal{E}, \omega_E^{\otimes (-\ell)}) \to H^1(\mathcal{E}, \omega_E^{\otimes (-\ell)})$ for all $\ell \in \mathbb{Z}_m$. Thus $\mathcal{E}$ is $F$-split if so is $E$, by Proposition 2.7

Conversely, if $\mathcal{E}$ is $F$-split, then we must have $H^1(\mathcal{E}, \omega_E^p) \neq 0$. Since $\omega_E$ is an $m$-torsion line bundle, this implies that $p \equiv 1 \pmod{m}$ and the Frobenius on $H^1(E, \mathcal{O}_E) \cong k$ is identified with $F: H^1(\mathcal{E}, \omega_E) \to H^1(\mathcal{E}, \omega_E^p)$. Therefore the $F$-splitting of $\mathcal{E}$ implies that $E$ is ordinary. \hfill $\square$

Remark 4.2. Lemma 4.1 (5) is also verified with explicit computations of the induced Frobenius map $F: H^1(\mathcal{E}, \omega_E) \to H^1(\mathcal{E}, \omega_E^p)$ and Fedder’s criterion \cite{F} applied to the defining equation of $E$.

Now we state the main result of this section.

Theorem 4.3. Let $\mathcal{E}$ be a weighted projective line with $\delta_\mathcal{E} = 0$, and assume that the characteristic $p$ does not divide any weight $r_i$. Then $\mathcal{E}$ does not have GFFRT.

Proof. Let $f: E \to \mathbb{P}^1$ be the $m$-fold covering from an elliptic curve constructed in Lemma 4.1 and let $\varphi: E \to \mathcal{E}$ be the induced morphism. We divide the proof into two cases, according to whether $E$ is ordinary or supersingular. First we recall the following:

Lemma 4.4 (\cite{A}, \cite{HSY} Lemma 4.12). Let $E$ be an elliptic curve in characteristic $p$ and let $q = p^e$ for $e \geq 0$.

1. If $E$ is ordinary, then $F^e_* \mathcal{O}_E$ splits into $q$ distinct $q$-torsion line bundles.
(2) If $E$ is supersingular, then $F^k E^\ast \mathcal{O}_E$ is isomorphic to Atiyah’s vector bundle $\mathcal{F}_q$ of rank $q$; see subsection 4.1 below.

4.1. Supersingular case. On the elliptic curve $E$ (which we do not yet assume to be supersingular), we have indecomposable vector bundles $\mathcal{F}_r$ of rank $r$ and degree 0 such that $H^0(E, \mathcal{F}_r) \cong k$ for all integer $r > 0$. This bundle is determined inductively by $\mathcal{F}_1 = \mathcal{O}_E$ and a unique non-trivial extension

$$0 \to \mathcal{O}_E \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0$$

as in [A, Theorem 5]. In what follows, we construct inductively vector bundles $\mathcal{G}_r$ on $\mathcal{C}$ of rank $r = 1, 2, \ldots$ such that

$$\text{Ext}_1^k(\mathcal{G}_r, \omega^i_\mathcal{C}) \cong \begin{cases} k & \text{if } i \equiv 1 \pmod{m} \\ 0 & \text{otherwise.} \end{cases}$$

We put $\mathcal{G}_1 = \mathcal{O}_E$. Then we can easily verify condition (4.2) for $r = 1$ by computing $\text{Ext}_1^k(\mathcal{G}_1, \omega^i_\mathcal{C}) \cong H^1(\mathcal{C}, \omega^i_\mathcal{C})$ with Lemma 4.2 (4) and Lemma 2.1.

Now let $r \geq 2$ and assume condition (4.2) for $r - 1$. Since $\text{Ext}_1^k(\mathcal{G}_{r-1}, \omega_\mathcal{C}) \cong k$, we have a vector bundle $\mathcal{G}_r$ sitting in a unique non-trivial extension

$$0 \to \mathcal{O}_\mathcal{C} \to \mathcal{G}_r \to \mathcal{G}_{r-1} \otimes \omega^{-1}_\mathcal{C} \to 0.$$ 

We apply the functor $\text{Ext}(\cdot, \omega^i_\mathcal{C})$ to this exact sequence to verify condition (4.2). For $i = 0$ we have an exact sequence

$$\text{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \xrightarrow{\delta} \text{Ext}_1^k(\mathcal{G}_{r-1}, \omega_\mathcal{C}) \to \text{Ext}_1^k(\mathcal{G}_r, \mathcal{O}_\mathcal{C}) \to \text{Ext}_1^k(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) = 0,$$ 

where the connecting homomorphism $\delta$ is an isomorphism by the non-triviality of the extension (4.3). Thus we have $\text{Ext}_1^k(\mathcal{G}_r, \mathcal{O}_\mathcal{C}) = 0$. For $i \not\equiv 0 \pmod{m}$, we have

$$0 = \text{Ext}_1^k(\mathcal{G}_{r-1}, \omega^{i+1}_\mathcal{C}) \to \text{Ext}_1^k(\mathcal{G}_r, \omega^i_\mathcal{C}) \to \text{Ext}_1^k(\mathcal{O}_\mathcal{C}, \omega^i_\mathcal{C}) \to 0$$

by induction, so that $\text{Ext}_1^k(\mathcal{G}_r, \omega^i_\mathcal{C}) \cong \text{Ext}_1^k(\mathcal{O}_\mathcal{C}, \omega^i_\mathcal{C})$. Thus condition (4.2) holds for $r$.

**Proposition 4.5.** Suppose that $m$ is not divisible by $p$. Then $\varphi^* \mathcal{G}_r \cong \mathcal{F}_r$. In particular, $\mathcal{G}_r$ is indecomposable.

**Proof.** The assertion is clear if $r = 1$. Let $r \geq 2$ and let the exact sequence (4.3) be given by a non-zero extension class $\varepsilon \in \text{Ext}_1^k(\mathcal{G}_{r-1}, \omega_\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{G}^\vee_{r-1}(K_\mathcal{C}))$. Since $\varphi^* \omega_\mathcal{C} \cong \mathcal{O}_E$ (note that $\varphi$ is étale) and $\varphi^* \mathcal{G}_{r-1} \cong \mathcal{F}_{r-1}$ by induction, the pull-back of sequence (4.3) under $\varphi$ turns out to be

$$0 \to \mathcal{O}_E \to \varphi^* \mathcal{G}_r \to \mathcal{F}_{r-1} \to 0.$$ 

This extension is given by the image $\varphi^* \varepsilon$ of $\varepsilon$ under the natural map

$$\varphi^* : \text{Ext}_1^k(\mathcal{G}_{r-1}, \omega_\mathcal{C}) \to \text{Ext}_1^k(\varphi^* \mathcal{G}_{r-1}, \varphi^* \omega_\mathcal{C}) \cong \text{Ext}_1^k(\mathcal{F}_{r-1}, \mathcal{O}_E).$$

This map is injective, since it is identified with the map

$$H^1(\mathcal{C}, \mathcal{G}^\vee_{r-1}(K_\mathcal{C})) \to H^1(\mathcal{C}, \mathcal{G}^\vee_{r-1}(K_\mathcal{C}) \otimes \varphi_* \mathcal{O}_E) \cong H^1(\mathcal{F}_{r-1}, \mathcal{O}_E)$$

induced by the splitting map $\mathcal{O}_E \to \varphi_* \mathcal{O}_E (= \bigoplus_{l=0}^{m-1} \omega^{-l}_\mathcal{C})$. Thus $\varphi^* \varepsilon \neq 0$ and it sits in $\text{Ext}_1^k(\mathcal{F}_{r-1}, \mathcal{O}_E) \cong k$. Comparing extensions (4.1) and (4.3) we see that $\mathcal{F}_r \cong \varphi^* \mathcal{G}_r$, as required. \qed

We now consider the case where $\mathcal{C}$ is not $F$-split, or equivalently, $E$ is supersingular.

**Proposition 4.6.** Under the hypothesis of Proposition 4.5, assume further that $E$ is supersingular. Then we have $F^k \mathcal{O}_\mathcal{C} \cong \mathcal{G}_q$, where $q = p^k$. 
4.2. Ordinary case. We now consider the case where the weighted projective line $\mathcal{C}$ with $\delta_{\mathcal{C}} = 0$ is $F$-split. In this case, $p \equiv 1 \pmod{m}$ and we have an $m$-fold covering $f: E \rightarrow \mathbb{P}^1$ from an ordinary elliptic curve $E$. Recall that $f$ factors as

$$f: E \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\pi} \mathbb{P}^1,$$

where $\varphi: E \rightarrow \mathcal{C}$ is unramified and $\pi: \mathcal{C} \rightarrow \mathbb{P}^1$ is the coarse moduli map. There is a ramification point $P_0 \in E$ of $f$ with ramification index $m$. We choose $P_0$ as the identity point for the group structure of $E$. Since $E$ is an ordinary elliptic curve, for any $q = p^i$ there exists exactly $q$ distinct $q$-torsion points $P_0, P_1, \ldots, P_{q-1} \in E$, among which $P_1, \ldots, P_{q-1}$ are not ramification points of $f$ by Lemma 4.1 (3). By Lemma 4.4 the $e$-th Frobenius push-forward $F^e_\varphi \mathcal{O}_E$ on $E$ splits into $q$ non-isomorphic $q$-torsion line bundles $L_i = \mathcal{O}_E(P_i - P_0)$ with $i = 0, 1, \ldots, q - 1$. Thus we have the following decomposition

$$\varphi_* F^e_\varphi \mathcal{O}_E \cong \varphi_* \mathcal{O}_E \oplus \varphi_* L_1 \oplus \cdots \oplus \varphi_* L_{q-1}$$

into rank $m$ bundles $\varphi_* L_i$. On the other hand, we have $\varphi_* \mathcal{O}_E \cong \mathcal{O}_E \oplus \omega_{\mathcal{C}}^{-1} \oplus \cdots \oplus \omega_{\mathcal{C}}^{1-m}$ by Lemma 4.4 so that

$$\varphi_* F^e_\varphi \mathcal{O}_E \cong F^e_\varphi \mathcal{O}_E \oplus F^e_\varphi (\omega_{\mathcal{C}}^{-1}) \oplus \cdots \oplus F^e_\varphi (\omega_{\mathcal{C}}^{1-m}).$$

The group $G = \mu_m$ is the Galois group of the $m$-fold Galois covering $f: E \rightarrow \mathbb{P}^1$. If we define the equivalence $\sim$ by $L_i \sim L_j$ if and only if $L_i \cong \sigma^i L_j$ for some $\sigma \in \mu_m$, then the line bundles $\mathcal{O}_E = L_0, L_1, \ldots, L_{q-1}$ are divided into $r + 1$ equivalence classes, where $r = \frac{q-1}{m}$. Re-numbering the line bundles, we may and will assume that the complete representatives are $\mathcal{O}_E = L_0, L_1, \ldots, L_r$. Under this notation we have the following:

**Proposition 4.7.** Let the notation be as above. Then $\varphi_* L_i$ is an indecomposable bundle for $1 \leq i \leq q - 1$, and $\varphi_* L_i \cong \varphi_* L_j$ if and only if $L_i \cong \sigma^i L_j$ for some $\sigma \in \mu_m$. We then have a decomposition

$$F^e_\varphi (\omega_{\mathcal{C}}^{\sigma^i}) \cong \omega_{\mathcal{C}}^{\sigma^i} \oplus \varphi_* L_1 \oplus \cdots \oplus \varphi_* L_r$$

into $r + 1$ non-isomorphic indecomposable bundles for $i \in \mathbb{Z}/m\mathbb{Z}$.

**Proof.** First note that $F^e_\varphi \omega_{\mathcal{C}}^{\sigma^i} \cong \omega_{\mathcal{C}}^{\sigma^i} \otimes F^e_\varphi \mathcal{O}_E$ and $\varphi_* L \cong \omega_{\mathcal{C}}^{\sigma^i} \otimes \varphi_* L$ for any $i \in \mathbb{Z}/m\mathbb{Z}$ and line bundle $L$ on $E$.

Let $L, M$ be $q$-torsion line bundles on $E$ with $L$ non-trivial. By the preceding argument it is enough to show that

$$\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M) \cong \begin{cases} k & \text{if } \sigma^* L \cong M \text{ for some } \sigma \in \mu_m \\ 0 & \text{otherwise}. \end{cases}$$

The proof goes along the same line as Oda’s [Od]. First, we have the Cartesian diagram

$$\begin{array}{ccc}
E \times \mu_m & \xrightarrow{\mu} & E \\
p_1 \downarrow & & \downarrow \varphi \\
E & \xrightarrow{\varphi} & \mathcal{C}
\end{array}$$

where $\mu: E \times \mu_m \rightarrow E$ is the map induced by the action of $\mu_m = \text{Spec } k[\xi]/(\xi^m - 1)$ on $E$, and $p_1$ is the projection. This follows from the fact that $[E/\mu_m] \cong \mathcal{C}$ via $\varphi$ (see Lemma 4.4 (1))
and \[\nu\] (7.21). Since \(G = \mu_m\) is finite, \(\varphi\) is affine, so that \(\varphi^* \varphi_* L \cong p_1 \mu^* L\). Hence by the adjointness of \(\varphi^*\) and \(\varphi_*\) we obtain

\[
\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M) \cong \text{Hom}_{\mathcal{O}_E}(\varphi^* \varphi_* L, M)
\]

\[
\cong \text{Hom}_{\mathcal{O}_E}(p_1 \mu^* L, M) \cong H^0(E, (p_1 \mu^* (L \otimes M^{-1}))^\vee).
\]

By the Serre duality this is dual to

\[
H^1(E, p_1 \mu^* L) \cong H^1(E, p_1 \mu^* (L \otimes p_1^* M^{-1})) \cong \text{Hom}_{\mathcal{O}_E}(p_1 \mu^* L, M) \cong H^0(E, (p_1 \mu^* (L \otimes M^{-1}))^\vee).
\]

Now let \(\lambda: G \to \hat{E} = \text{Pic}^0(E)\) be the morphism sending \(\sigma \in G\) to \(\sigma^* (L) \otimes L^{-1}\), which is injective by our assumption that \(L\) is a non-trivial \(q\)-torsion line bundle (cf. Lemma \[\text{[Mum]}\] III.13, p. 125, Theorem), where \(P\) is the normalized Poincaré line bundle on \(E \times \hat{E}\).

Thus \(\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M)\) is dual to

\[
H^1(E \times G, (1 \times \lambda)^* P \otimes_{\mathcal{O}_{E \times G}} p_1^* (L \otimes M^{-1})) \cong H^1(E \times G, (1 \times \lambda)^* (P \otimes_{\mathcal{O}_{E \times \hat{E}}} p_1^* (L \otimes M^{-1}))),
\]

where we abuse the notation \(p_1\) to denote the first projection from both \(E \times G\) and \(E \times \hat{E}\). It follows from the Leray spectral sequence \(H^i(G, R^j p_{2*}(1 \times \lambda)^* (P \otimes p_1^* (L \otimes M^{-1}))) \Rightarrow H^{i+j}(E \times G, (1 \times \lambda)^* (P \otimes p_1^* (L \otimes M^{-1})))\) that this is isomorphic to

\[
H^0(G, R^1 p_{2*}(1 \times \lambda)^* (P \otimes p_1^* (L \otimes M^{-1}))).
\]

Furthermore, we have \(R^1 p_{2*}(1 \times \lambda)^* (P \otimes p_1^* (L \otimes M^{-1})) \cong \lambda^* R^1 p_{2*}(P \otimes p_1^* (L \otimes M^{-1})).\) To see this let \(\mathcal{F} = P \otimes p_1^* (L \otimes M^{-1}).\) Since the problem is local on \(\hat{E}\), we may replace \(\lambda: G \to \hat{E}\) by \(\lambda: \text{Spec} B \to \text{Spec} A\) to show that \(H^1(E_B, \mathcal{F} \otimes_{A} B) \cong H^1(E_A, \mathcal{F}) \otimes_{A} B\), where \(E_A = E \times \text{Spec} A\) and \(E_B = E \times \text{Spec} B\). Since \(\text{dim} E = 1\) we have an open covering of \(E\) consisting of two affine open subsets \(V_1, V_2\). Then \(E_A\) is covered by \(U_i = V_i \times \text{Spec} A\) with \(i = 1, 2\) and \(H^1(E_A, \mathcal{F})\) is computed with the Čech complex \(\mathcal{E}^* = [0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to 0]\) associated with \([U_1, U_2]\) and \(\mathcal{F}\), i.e., there is an exact sequence

\[
\mathcal{E}^0 \to \mathcal{E}^1 \to H^1(E_A, \mathcal{F}) \to 0.
\]

Similarly, \(H^1(E_B, \mathcal{F} \otimes_{A} B)\) is computed with the Čech complex \(\mathcal{E}^* \otimes_{A} B\), i.e.,

\[
\mathcal{E}^0 \otimes_{A} B \to \mathcal{E}^1 \otimes_{A} B \to H^1(E_B, \mathcal{F} \otimes_{A} B) \to 0
\]

is exact. Thus the right exactness of the functor \(- \otimes_{A} B\) leads us to the conclusion.

Thus we see that \(\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M)\) is dual, as a \(k\)-vector space, to

\[
H^0(G, \lambda^* R^1 p_{2*}(P \otimes p_1^* (L \otimes M^{-1}))).
\]

Let \(b \in \hat{E}\) be the point representing the class of \(L \otimes M^{-1}\) and let \(T_b: \hat{E} \to \hat{E}\) be the translation by \(b\). Then we have \(P \otimes p_1^* (L \times M^{-1}) \cong (1 \times T_b)^* P\) again by [Mum, ibid]. Therefore \(\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M)\) is dual to

\[
H^0(G, \lambda^* R^1 p_{2*}(1 \times T_b)^* P) \cong H^0(G, (T_b \circ \lambda)^* R^1 p_{2*} P).
\]

Since \(R^1 p_{2*} P\) is supported at the origin \(0 \in \hat{E}\) with \(R^1 p_{2*}(P)_0 = k\) (Mum, Ogd, Lemma 1.1) and since \(T_b \circ \lambda\) is injective, \(\text{Hom}_{\mathcal{O}_E}(\varphi_* L, \varphi_* M)\) is one-dimensional if \(T_b \circ \lambda(G)\) contains the origin 0 of \(\hat{E}\), and otherwise it is zero. Finally it is easy to see that \(0 \in T_b \circ \lambda(G)\) if and only if \(\sigma^* L \cong M\) for some \(\sigma \in G\).

\[\square\]

Remark 4.8. Theorem 4.3 fails without the assumption that the weights are not divisible by \(p\); see Section 6.
5. Stability of Frobenius push-forwards: The case where $\delta_\mathcal{E} > 0$

In this section, we assume that $\mathcal{E} = C[\sqrt[r]{T_1}, \ldots, \sqrt[r]{T_n}]$ for a smooth curve $C$, and that $p$ does not divide any weight $r_i$. Our goal is to show the slope stability of Frobenius push-forward of line bundles on $\mathcal{E}$ with $\delta_\mathcal{E} > 0$.

As a corollary, we show that orbifold curves $\mathcal{E}$ with $\delta_\mathcal{E} > 0$ do not have GFFRT. This gives a negative answer to Brenner’s question [Sh, Question 2] in characteristic $p \neq 2, 3, 7$; see Introduction and Section 6.

5.1. First Chern class of $F_\ast \mathcal{E}$. To study slope stability of $F_\ast \mathcal{E}$, we compute the degree of $c_1(F_\ast \mathcal{E})$. To this end, recall that we have

$$\text{Hom}_\mathcal{E}(F_\ast \mathcal{E}, \omega_\mathcal{E}) \cong F_\ast \omega_\mathcal{E}$$

by [Ha, Ex III 7.2]. We also consider the Frobenius push-forward of the differential map $F_\ast(d): F_\ast \mathcal{E} \to F_\ast \omega_\mathcal{E}$, which is $\mathcal{E}$-linear. We write by $\mathcal{B}$ its image in $F_\ast \omega_\mathcal{E}$. We have a homomorphism $C^{-1}: \omega_\mathcal{E} \to F_\ast \omega_\mathcal{E}/\mathcal{B}$ from the similar arguments as in [EV, 9.14]. When $p$ does not divide any weight $r_i$, then this is an isomorphism, and the inverse $C: F_\ast \omega_\mathcal{E}/\mathcal{B} \cong \omega_\mathcal{E}$ is called the Cartier operator.

In the following, we assume that the characteristic $p$ does not divide any weight $r_i$. Then we have exact sequences

$$0 \to \mathcal{E} \xrightarrow{F_\ast} F_\ast \mathcal{E} \to \mathcal{B} \to 0$$

$$0 \to \mathcal{B} \to F_\ast \omega_\mathcal{E} \xrightarrow{C} \omega_\mathcal{E} \to 0.$$

By these facts, we have

$$\text{det } F_\ast \mathcal{E} \cong \text{det } \mathcal{B} \cong \text{det } F_\ast (\omega_\mathcal{E}) \otimes \omega_\mathcal{E}^{-1} \cong \omega_\mathcal{E}^p \otimes (\text{det } F_\ast \mathcal{E})^{-1} \otimes \omega_\mathcal{E}^{-1}.$$

Hence we have

$$c_1(F_\ast \mathcal{E}) = \frac{p-1}{2} K_\mathcal{E}$$

in $A(\mathcal{E})$.

**Proposition 5.1.** For any vector bundle $\mathcal{E}$ on $\mathcal{E}$, we have

$$c_1(F_\ast \mathcal{E}) = \frac{p-1}{2} r K_\mathcal{E} + c_1(\mathcal{E})$$

in $A(\mathcal{E})$.

**Proof.** We have a full flag of sub-bundles

$$0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

corresponding to a section of the full flag bundle of $\mathcal{E}$ over an orbifold curve $\mathcal{E}$. Hence it is enough to prove for a line bundle.

We take a line bundle $\mathcal{O}_\mathcal{E}(\sum s_i Q_i^+ - \sum t_j Q_j^-)$, where $s_i, t_j > 0$, and $Q_i^+, Q_j^-$ are closed points on $\mathcal{E}$. We show the assertion by induction on $\sum s_i + \sum t_j$. The first step for the induction follows from (5.1). For the next step, it is enough to show that for any line bundle $\mathcal{L}$ and any closed point $Q$ on $\mathcal{E}$, the assertions for $\mathcal{L}$ and $\mathcal{L}(-Q)$ are equivalent. This follows from $c_1(F_\ast \mathcal{L}) = c_1(F_\ast \mathcal{L}(-Q)) + c_1(\mathcal{O}_\mathcal{E}(Q))$, since we have the following exact sequence

$$0 \to \text{det } F_\ast \mathcal{L}(-Q) \to \text{det } F_\ast \mathcal{L} \to k(Q) \otimes \rho \to 0.$$

Here $\rho$ is a one-dimensional representation of the automorphisms group of a closed point $Q \in \mathcal{E}(\text{Spec } k)$. \qed
5.2. **Slope stability.** For a vector bundle $E$ on $C$, we define the slope $\mu(E)$ of $E$ by

$$
\mu(E) = \frac{\deg c_1(E)}{\text{rk} E}.
$$

As in the previous subsection, we assume that $p$ does not divide any weight $r_i$.

**Proposition 5.2.** We have

$$
\mu(F^*F_\ast E) = \mu(E) + \frac{p-1}{2} \delta_E.
$$

**Proof.** By Proposition 5.1, we have $\mu(F_\ast E) = p^{-1}(\mu(E) + \frac{p-1}{2} \delta_E)$. Since $\mu(F^*F) = p\mu(F)$ for any vector bundle $F$ on $C$, the assertion follows. \hfill $\square$

**Definition 5.3.** We say that a vector bundle $E$ on $C$ is *semi-stable* if for any non-trivial proper sub-bundle $E'$ of $E$, we have an inequality

$$
\mu(E') \leq \mu(E).
$$

If the inequality is always strict, we say that $E$ is *stable*.

It is equivalent to that the same inequality holds for any non-trivial subsheaf of $E$ as in [OSS, 1.2.2]. We also remark that it is different from the stability defined in [N]. But it is enough for our purpose to show the indecomposability of Frobenius push-forwards $F_\ast^i \mathcal{O}_C$ for $e > 0$. For this purpose, we follow the arguments in [KS, Su]. It is straightforward to modify their arguments to our situation. Only difference is that we must consider grading even in local situation.

For a vector bundle $E$ on $C$, there exists a connection

$$
\nabla = \text{id} \otimes d: F^*E = F^{-1}E \otimes_{F^{-1}O_C} O_C \to F^*E \otimes \Omega_C = F^{-1}E \otimes_{F^{-1}O_C} \Omega_C
$$

called *canonical connection* similarly for a variety over $k$ as in [KS, Su]. Here $d: O_C \to \Omega_C$ is the differential, which is $F^{-1}O_C$-linear. This is locally written as

$$
M \otimes_B B \to M \otimes_B B \otimes_{B/k} \Omega_{B/k} \cong M \otimes_B B \otimes_{B/k} \Omega_{B/k}; \; m \otimes f \mapsto m \otimes df,
$$

where $B$ has a grading from the local description (1.3), and $\nabla$ preserves the grading.

When $E = F_\ast W$ for a vector bundle $W$ on $C$, we introduce the canonical filtration of $F = F^*F_\ast W$ due to [KS, Su]. We put $F_0 = F, F_1 = \ker(F^*F_\ast W \to W)$, and $F_\ell = \ker(F_{\ell-1} \nabla, F \otimes \Omega_C \to (F/F_{\ell-1}) \otimes \Omega_C)$. We have the filtration:

$$
\cdots \subset F_3 \subset F_2 \subset F_1 := F \subset F_0 := F^*F_\ast W.
$$

**Definition 5.4.** We call this filtration $F_\ast$ the *canonical filtration* on $F = F^*F_\ast W$.

Since local computations in [KS, Su] holds equivariantly, we have the following lemma.

**Lemma 5.5.**

1. $F_0/F_1 \cong E, \nabla(F_{t+1}) \subset F_t \otimes \mathcal{O}_C(K_e)$ for $\ell \geq 1$.
2. $\nabla: F_{t}/F_{t+1} \cong F_{t-1}/F_t \otimes \mathcal{O}_C(K_e)$.
3. If $E$ is stable (resp. semi-stable), then $F_t/F_{t+1}$ is stable (resp. semi-stable) for any $t$.

**Proof.** (1) follows from the Definition. (2) follows from [Su, Lemma 2.1 (ii)] and the fact that this is a graded isomorphism. (3) follows from (1) and (2). \hfill $\square$

By this lemma, we have $F_p = 0$ and $F_{p-1} \neq 0$.

**Theorem 5.6.** Let $W$ be a vector bundle on an orbifold curve $C$, and suppose that $p$ does not divide any weight $r_i$. If $\delta_E > 0$ and $W$ is stable (resp. $\delta_E \geq 0$ and $W$ is semi-stable), then $F^\ast W$ is stable (resp. semi-stable).
Proof. It follows from the similar argument as in the proof of Theorem 2.2. But we give a proof for the convenience of readers.

We take a non-trivial sub-bundle \( \mathcal{E}' \subset W \), and show \( \mu(\mathcal{E}') < \mu(W) \) \((\text{resp.} \mu(\mathcal{E}') \leq \mu(W))\). By Proposition 5.2, we have

\[
\mu(F^*F_*W) = \mu(W) + \frac{p-1}{2}\delta_{\mathcal{E}}.
\]

We consider the induced filtration

\[
0 \subset \mathcal{F}^m \cap F^*\mathcal{E}' \subset \cdots \subset \mathcal{F}^1 \cap F^*\mathcal{E}' \subset F^*\mathcal{E}',
\]

where we assume \( \mathcal{F}^{m+1} \cap F^*\mathcal{E}' = 0 \) and \( \mathcal{F}^m \cap F^*\mathcal{E}' \neq 0 \) for \( m < p \). If we put \( r_\ell = \text{rk} \left( \frac{\mathcal{F}^\ell \cap F^*\mathcal{E}'}{\mathcal{F}^{\ell+1} \cap F^*\mathcal{E}'} \right) \),

then we have

\[
\mu(\mathcal{E}') = \frac{1}{\text{rk}(\mathcal{E}')} \sum_{\ell=0}^{m} \mu \left( \frac{\mathcal{F}^\ell \cap F^*\mathcal{E}'}{\mathcal{F}^{\ell+1} \cap F^*\mathcal{E}'} \right) r_\ell \leq \frac{1}{\text{rk}(\mathcal{E}')} \sum_{\ell=0}^{m} (\mu(W) + \ell\delta_{\mathcal{E}}) r_\ell,
\]

where the last inequality follows from Lemma 5.5 (2), (3).

Putting (5.2) and (5.3) together, we have

\[
\mu(F_*W) - \mu(\mathcal{E}') = \frac{1}{p}(\mu(F^*F_*W) - \mu(\mathcal{E}')) \geq \frac{\delta_{\mathcal{E}}}{\text{rk}(\mathcal{E}')}p \sum_{\ell=0}^{m} \left( \frac{p-1}{2} - \ell \right) r_\ell.
\]

If \( m \leq \frac{p-1}{2} \), then the last sum is greater than 0, and we get the desired inequality. Hence we may assume \( m > \frac{p-1}{2} \).

Then the last sum is equal to

\[
\sum_{\ell=m+1}^{p-1} \left( \ell - \frac{p-1}{2} \right) r_{p-1-\ell} + \sum_{\ell=\frac{p-1}{2}}^{m} \left( \ell - \frac{p-1}{2} \right) (r_{p-1-\ell} - r_\ell).
\]

Since the isomorphism in 5.5 (2) induces an inclusion \( \frac{\mathcal{F}^\ell \cap F^*\mathcal{E}'}{\mathcal{F}^{\ell+1} \cap F^*\mathcal{E}'} \subset \frac{\mathcal{F}^{\ell-1} \cap F^*\mathcal{E}'}{\mathcal{F}^{\ell} \cap F^*\mathcal{E}'} \), we have

\[
\frac{r_0}{r_1} \geq \cdots \geq \frac{r_0}{r_m}.
\]

Hence (5.5) is greater than, or equal to 0. This implies the semi-stability of \( F_*W \).

Finally we assume that \( \delta_{\mathcal{E}} > 0 \) and \( W \) is stable. If \( \mu(F_*W) - \mu(\mathcal{E}') = 0 \), then we have an equality in (5.3). This implies \( \frac{\mathcal{F}^\ell \cap F^*\mathcal{E}'}{\mathcal{F}^{\ell+1} \cap F^*\mathcal{E}'} = \mathcal{F}^\ell/\mathcal{F}^{\ell-1} \) for all \( \ell = 0, 1, \ldots, m \), and we have \( r_0 = r_1 = \cdots = r_m = \text{rk}(W) \). Furthermore since (5.5) must be equal to 0, we have \( m = p - 1 \).

This implies \( \text{rk}(\mathcal{E}') = \text{rk} F_*W \), and a contradiction. \( \square \)

As a direct corollary, we have the following theorem.

**Theorem 5.7.** We assume that \( \delta_{\mathcal{E}} \) is greater than 0. Then for any \( e \), the Frobenius push-forward \( F^e_*\mathcal{O}_{\mathcal{E}} \) is indecomposable.

Proof. For a contradiction, we assume that \( F^e_*\mathcal{O}_{\mathcal{E}} \) is decomposed into non-trivial vector bundles \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). Then both \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are sub-bundles of \( F^e_*\mathcal{O}_{\mathcal{E}} \). Hence we get inequalities \( \mu(\mathcal{E}_1) < \mu(F^e_*\mathcal{O}_{\mathcal{E}}) < \mu(\mathcal{E}_2) \) and \( \mu(\mathcal{E}_2) < \mu(F^e_*\mathcal{O}_{\mathcal{E}}) < \mu(\mathcal{E}_1) \), and a contradiction. \( \square \)
6. The FFRT Property of $R(\mathbb{P}^1, D)$ and Concluding Remarks

In this section we prove our results on the FFRT property of $R(C, D)$.

**Proposition 6.1.** Let $C$ be a smooth projective curve of genus $g \geq 1$ over an algebraically closed field of characteristic $p > 0$ and $D$ an ample $\mathbb{Q}$-Cartier divisor on $C$. Then the graded ring $R(C, D)$ does not have FFRT.

**Proof.** First note that $C$ does not have GFFRT. This follows from Lemma 4.5 when $g = 1$ and [**Su**] when $g > 1$. Now let $\pi: C \to C$ be the orbifold curve constructed with respect to the fractional part of $D$ and let $L = \mathcal{O}_C(\pi^*D)$. Since $\pi_*\mathcal{O}_C \cong \mathcal{O}_C$, it follows that $C$ does not have FFRT as well. Then $(C, L)$ does not have GFFRT, and the result follows from Corollary 2.6. \qed

It follows from the proposition above that $R(C, D)$ has FFRT only if $C \cong \mathbb{P}^1$. To state our main theorem let us fix the notation used through the remainder of this paper. Let $R = R(\mathbb{P}^1, D)$ be a two-dimensional normal graded ring with $R_0 = k$ an algebraically closed field of characteristic $p > 0$. Here

$$D = \sum_{i=1}^n \frac{s_i}{r_i}P_i$$

is an ample $\mathbb{Q}$-divisor on $\mathbb{P}^1$, where $P_1, \ldots, P_n$ are distinct closed points on $\mathbb{P}^1$, and $r_i > 0$ and $s_i$ are coprime integers.

Let $\mathcal{C} = \mathbb{P}^1[\sqrt[n]{P_1}, \ldots, \sqrt[n]{P_n}]$ be the weighted projective line with weight $(r_1, \ldots, r_n)$. The following are well-known.

1. $R$ has log terminal singularity if and only if $\delta_{\mathcal{C}} = \deg \omega_{\mathcal{C}} = -2 + \sum_{i=1}^n \frac{r_i-1}{r_i} < 0$.
2. $R$ has log canonical singularity if and only if $\delta_{\mathcal{C}} \leq 0$.

In the case of (1) above, it is known that $R$ has finite representation type, and so it has FFRT (see Theorem 3.2). On the other hand, we have the following

**Theorem 6.2.** In the notation as above, suppose that $\delta_{\mathcal{C}} \geq 0$ and that $p$ does not divide any $r_i$. Then $R = R(\mathbb{P}^1, D)$ does not have FFRT.

**Proof.** It follows from Theorems 4.3 and 5.7 that $\mathcal{C}$ does not have GFFRT. Then for $L = \mathcal{O}_C(\pi^*D)$, the pair $(\mathcal{C}, L)$ does not have GFFRT, and the result follows from Corollary 2.6. \qed

In Theorem 6.2 the assumption that $p$ does not divide any $r_i$ is really necessary as we will see in the following examples.

**Example 6.3.** Let $R = k[x, y, z]/(x^2 + y^3 + z^7)$. This is not a rational singularity but $\text{Proj} \ R \cong \mathbb{P}^1$ and $R \cong R(\mathbb{P}^1, D)$ for a $\mathbb{Q}$-divisor $D = \frac{1}{2}(\infty) - \frac{1}{2}(0) - \frac{1}{3}(1)$ on $\mathbb{P}^1$. By Theorem 6.2, $R$ does not have FFRT if $p \neq 2, 3, 7$. On the other hand, $R$ has FFRT if $p = 2, 3, 7$ ([**Su**]).

**Example 6.4.** Let $R = R(\mathbb{P}^1, D)$ for a $\mathbb{Q}$-divisor $D = \frac{1}{3}(\infty) + \frac{1}{3}(0) - \frac{1}{3}(1)$ on $\mathbb{P}^1$. This is a rational log canonical singularity but not log terminal. By Theorem 6.2, $R$ does not have FFRT if $p \neq 3$. On the other hand, $R$ has FFRT if $p = 3$, by the following proposition.

**Proposition 6.5.** Let $\mathcal{C} = \mathbb{P}^1[\sqrt[n]{T_1}, \sqrt[n]{T_2}, \sqrt[n]{T_3}]$ be the weighted projective line of weight $(3, 3, 3)$. Then for any line bundle $L$ on $\mathcal{C}$, $(\mathcal{C}, L)$ has FFRT if and only if $p = 3$.

**Proof.** Let $C = \text{Proj} \ k[x, y, z]/(z^3 - xy(x - y))$, on which $\mu_3 = \text{Spec} \ k[m]/(m^3 - 1)$ acts by $z \mapsto \alpha z$ for $\alpha \in \mu_3$. Here $\mu_3$ is the group of cube roots of unity in $k$ if $p \neq 3$. When $p = 3$, we regard $\mu_3$ as a group scheme and the above $\mu_3$-action also makes sense. Then $[C/\mu_3] \cong \mathcal{C}$ and we have a triple covering $f: C \to \mathbb{P}^1 = \text{Proj} \ k[x, y]$, which factors through $\mathcal{C}$.

If $p \neq 3$, then $C$ is an elliptic curve as constructed in Lemma 4.1 and $(\mathcal{C}, L)$ does not have GFFRT by Theorem 6.2.
In characteristic $p = 3$, $C$ is a singular rational curve and $f : C \to \mathbb{P}^1$ is a purely inseparable triple covering. Composing with the normalization $\mathbb{P}^1 \to C$, we see that $\mathcal{C}$ is a “Frobenius sandwich,” that is, the Frobenius morphism of $\mathbb{P}^1$ is factorized as

$$F_{\mathbb{P}^1} : \mathbb{P}^1 \xrightarrow{\varphi} \mathcal{C} \xrightarrow{\pi} \mathbb{P}^1$$

and the Frobenius $F = F_\varphi$ of $\mathcal{C}$ is factorized as $F = \varphi \circ \pi$. Then for $e \geq 1$, the $e$-th Frobenius on $\mathcal{C}$ is factorized as $F^e = \varphi \circ (F_{\mathbb{P}^1})^{e-1} \circ \pi$. Thus for a line bundle $L$ on $\mathcal{C}$ of deg $L > 0$ and $0 \leq i \leq p^e - 1$, we have

$$F^e_*(L) = \varphi_*(F_{\mathbb{P}^1})^{e-1}_* \pi_*(L) = \varphi_*(F_{\mathbb{P}^1})^{e-1}_* \mathcal{O}_{\mathbb{P}^1}(a_i),$$

where $-1 \leq a_i \leq (p^e - 1) \deg L < p^e - 1 \cdot 3 \deg L$. Hence $(F_{\mathbb{P}^1})^{e-1}_* \mathcal{O}_{\mathbb{P}^1}(a_i)$ splits into line bundles $\mathcal{O}_{\mathbb{P}^1}(-1), \ldots, \mathcal{O}_{\mathbb{P}^1}(3 \deg L - 1)$, so that $F^e_*(L)$, with $0 \leq i \leq p^e - 1$, splits into finitely many vector bundles $\varphi_* \mathcal{O}_{\mathbb{P}^1}(-1), \ldots, \varphi_* \mathcal{O}_{\mathbb{P}^1}(3 \deg L - 1)$. We therefore conclude that $(\mathcal{C}, L)$ has GFFRT. □

**Remark 6.6.** In Examples 6.3 and 6.4 the ring $R = R(\mathbb{P}^1, D)$ does not have finite representation type in any characteristic $p > 0$, since $\delta C \geq 0$. However, $R$ has FFRT in exceptional characteristics, that is, $p = 2, 3, 7$ in Example 6.3 and $p = 3$ in Example 6.4. In these exceptional cases $\mathcal{C}$ turns out to be a Frobenius sandwich, as we have seen in the proof of Proposition 6.5.

In the two dimensional case, it is known that $F$-regular rings have log terminal singularities. Hence $F$-regular implies FFRT property as we saw in Section 4. However, this statement cannot hold in the higher dimensional case. For there exists an example which is $F$-regular and does not have FFRT property [SS, ITI, Remark 3.4. (2)].

**Question 6.7.** Let $X$ be a root stack in arbitrary dimensions, and $L$ a line bundle on $X$. Is there any difference between the GFRT properties of $X$ and the pair $(X, L)$? Here the latter property is equivalent to the FFRT property of the section ring $R = R(X, L)$ (cf. Corollary 2.6).

**References**

[A] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. 7(3), 414–452 (1957)

[B] K. Behrend, *Introduction to algebraic stacks*, 1–131, in *Moduli Spaces*, edited by L. Brambila-Paz et al., London Math. Soc. Lecture Note series 411, Cambridge University Press, Cambridge (2014)

[C] B. Conrad, *Arithmetic moduli of generalized elliptic curves*. (English summary) J. Inst. Math. Jussieu 6(2), 209–278 (2007)

[CB] W. Crawley-Boevey, *Kac’s theorem for weighted projective lines*, J. Eur. Math. Soc. 12, 1331–1345 (2010)

[D] M. Demazure, *Anneaux gradués normaux*, 35–68, in *Introduction à la théorie des singularités II*, edited by L. D. Tráng, Hermann, Paris (1979)

[EV] H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, 20. Birkhäuser Verlag, Basel, (1992)

[F] R. Fedder, *F-purity and rational singularity*, Trans, Amer. Math. Soc. 278, 461–480(1983)

[GL] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, 265–297, in *Singularities, Representations of Algebras, and Vector Bundles*, edited by G.-M. Greuel and G. Trautmann, Lecture Notes in Math. 1273, Springer, Berlin, 1987.

[HSY] N. Hara, T. Sawada and T. Yasuda, *F-blowups of normal surface singularities*, Algebra Number Theory 7, 733–763(2013)

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York-Heidelberg (1977)

[K] V. G. Kac, *Root systems, representations of quivers and invariant theory*, Invariant theory (Montecatini, 1982), F. Gherardelli (ed.), Lect. Notes Math. 996, Springer, Berlin, 74–108 (1983)

[KS] Y. Kitadai and H. Sumihiro, *Canonical filtrations and stability of direct images by Frobenius morphisms*, Tohoku Math. J. 60 , 287–301(2008)

[Mo] T. Mochizuki, *Donaldson Type Invariants for Algebraic Surfaces: Transition of Moduli Stacks*, Lecture Notes in Math. 1972, Springer, Berlin (2009)

[MO] K. Matsuki and M. Ohsaka, *Kawamata-Viehweg vanishing as Kodaira vanishing for stacks*, Math. Res. Lett. 12, 207–217 (2005)

[Muk] S. Mukai, *An introduction to Invariants and Moduli*, Cambridge Studies in Advanced Mathematics 81, Cambridge University Press, Cambridge (2003)
[Mum] D. Mumford, *Abelian varieties*, Tata Institute Studies in Mathematics 5, Oxford Univ. Press, London (1970)

[N] F. Nironi, *Moduli spaces of semistable sheaves on projective Deligne-Mumford stacks*, arXiv:0811.1949.

[Od] T. Oda, *Vector bundles on an elliptic curve*, Nagoya Math. J. **43**, 41–72 (1971)

[Ol] M. Olsson, *Algebraic spaces and stacks*, American Mathematical Society, Colloquium Publications **62**, American Mathematical Society, Providence (2016)

[OU] R. Ohkawa and H. Uehara, *Frobenius morphisms and derived categories on two dimensional toric Deligne-Mumford stacks*, Adv. Math. **244**, 241–267 (2013)

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics **3**, Birkhäuser, Boston, Mass (1980)

[P] H. Pinkham, *Normal surface singularities with \(C^*\) action*, Math. Ann. **227**, 183–193 (1977)

[Sh] T. Shibuta, *One-dimensional rings of finite F-representation type*, J. Algebra **332**, 434–441 (2011)

[SVdB] K. E. Smith and M. Van den Bergh, *Simplicity of rings of differential operators in prime characteristic*, Proc. London Math. Soc. **75**(1), 32–62 (1997)

[Su] X. Sun, *Direct images of bundles under Frobenius morphism*, Invent. Math. **173**, 427–447 (2008)

[SS] A. Singh and I. Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. **33**, 1703–1733 (2004)

[TT] S. Takagi and R. Takahashi, *D-modules over rings with finite F-representation type*, Math. Res. Lett. **15**(3), 563–581 (2008)

[V] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97**(3), 613–670 (1989)

[W] K.-i. Watanabe, *F-regular and F-pure normal graded rings*, J. Pure Appl. Algebra **71**, 341–350 (1991)

Tokyo University of Agriculture and Technology, 2–24–16 Nakacho, Koganei, Tokyo 184–8588, Japan

E-mail address: nhara@cc.tuat.ac.jp

Department of Mathematics, School of Fundamental Science and Engineering, Waseda University, 3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan

E-mail address: ohkawa.ryo@aoni.waseda.jp