An Achilles’ Heel of Term-Resolution

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Abstract. Term-resolution provides an elegant mechanism to prove that a quantified Boolean formula (QBF) is true. It is a dual to Q-resolution (also referred to as clause-resolution) and is practically highly important as it enables certifying answers of DPLL-based QBF solvers. While term-resolution and Q-resolution are very similar, they’re not completely symmetric. In particular, Q-resolution operates on clauses and term-resolution operates on models of the matrix. This paper investigates what impact this asymmetry has. We’ll see that there is a large class of formulas (formulas with “big models”) whose term-resolution proofs are exponential. As a possible remedy, the paper suggests to prove true QBFs by refuting their negation (negate-refute), rather than proving them by term-resolution. The paper shows that from the theoretical perspective this is indeed a favorable approach. In particular, negation-refutation can p-simulates term-resolution and there is an exponential separation between the two calculi. These observations further our understanding of proof systems for QBFs and provide a strong theoretical underpinning for the effort towards non-CNF QBF solvers.

1 Introduction

Arguably, the interest of computer scientists in proof complexity begins with the seminal work of Cook and Reckhow who showed a relation between proof complexity and the question NP vs. co-NP [7]. This interest was further fueled by the practical success of programs for automated reasoning, such as SMT solvers or SAT solvers. Machine-verifiable proofs serve as certificates for such solvers. It is important that a solver can produce a certificate of its answer as the solver itself can contain bugs [20,3]. Moreover, proofs have turned out to be important artifacts for further computations, like invariant inference for example [14]. This paper follows this line of research, i.e. proof complexity and solver complication certification, with the focus on quantified Boolean formula (QBF). In particular, it focuses on QBFs whose propositional part is in conjunctive normal form (QCNF). QCNF is complete and widely popular input for QBF solvers due to its susceptibility to simple representation inside the solver.

A number of QCNF solvers take inspiration in the approach that turned out to be so successful for SAT; and that is conflict driven clause learning (CDCL) [17,16]. Since propositional resolution is the underlying proof principle used in SAT, an analogous proof system was developed for QCNF. In particular, Q-resolution [5] for false formulas, and term-resolution [9] for true formulas. It has been shown that CDCL-based QBF solvers [23] can be certified by these two proof systems [9]. Recently, several proof complexity analyses of Q-resolution were published. A separation result for Q-resolution and a sequent calculus by Krajíček and Pudlák [13] is shown by Egly [8]:
Van Gelder shows that enabling resolution on universal-variables in Q-resolution proofs gives an exponential advantage to Q-resolution \([21]\); Janota and Silva show some p-simulation results for fragments of Q-resolution and solving QBF by expanding universal variables \([11]\).

This paper brings the focus to term-resolution. While term-resolution is an elegant system because it provides a dual to Q-resolution, the two types of resolution are not perfectly symmetric. This is because Q-resolution can operate on the given clauses but term-resolution operates on the satisfying assignments of those clauses. This paper shows that this difference exposes an Achilles’ heel of term-resolution.

The first result of this paper is that it shows that term-resolution proofs are large for QCNFs whose propositional part have models with a large number of universal literals. More precisely, if each model has at least \(k\) universal literals, any term-resolution proof has at least \(2^k\) nodes. Subsequently, the paper investigates an alternative route to term-resolution and that is refuting the negation of the formula. The paper shows that any term-resolution proof can be translated to a negation-refutation in polynomial time. On a particular formula \(\Psi\) we show an exponential separation between negation-refutation and term-resolution, i.e. all term-resolution proofs of \(\Psi\) are exponential but there is a Q-resolution proof of \(\neg \Psi\) is polynomial.

These results have direct practical implications for QBF solving because term-resolution enables certifying DPLL-based QBF solvers. Consequently, a formula whose term-resolution proofs are exponential, will require exponential time to solve. These theoretical results further substantiate an observation already made in the QBF community and that is that QBF with propositional part in CNF are particularly harmful for solving solving \([2,22]\).

2 Preliminaries

A literal is a Boolean variable or its negation. For a literal \(l\), we write \(\bar{l}\) to denote the literal complementary to \(l\), i.e. \(\bar{x} = \neg x, \overline{\neg x} = x\). A clause is a disjunction of finitely many literals. A formula in conjunctive normal form (CNF) is a conjunction finitely many clauses. As common, whenever convenient, a clause is treated as a set of literals and a CNF formula as a set of sets of literals.

For a literal \(l = x\) or \(l = \bar{x}\), we write \(\text{var}(l)\) for \(x\). Analogously, for a clause \(C\), \(\text{var}(C)\) denotes \(\{\text{var}(l) \mid l \in C\}\) and for a CNF \(\psi\), \(\text{var}(C)\) denotes \(\{l \mid l \in \text{var}(C), C \in \psi\}\).

For a set of variables \(X\) an assignment is a function from \(X\) to the constants 0 and 1. We say that the assignment is complete for \(X\) if the function is total.

Analogously to a clause, a term is a conjunction of finitely many literals. Again, whenever convenient, a term is treated as a set of literals. We say that a term \(T\) is a model of a CNF \(\phi\) if and only if for each clause \(C \in \phi\) there is a literal \(l\) both in \(T\) and \(C\), i.e. \(C \cap T \neq \emptyset\).

There is an obvious relation between terms and assignments. A term uniquely determines a set of assignments that satisfy the term. If an assignment satisfies a model of \(\phi\), then it is a satisfying assignment. Note that some definitions require a model to be a complete assignment to the variables of \(\phi\). The aforementioned correspondence shows that there’s no substantial difference between the definitions.
Quantified Boolean Formulas (QBFs) [4] are an extension of propositional logic with quantifiers with the standard semantics that $\forall x. \Psi$ is satisfied by the same truth assignments as $\Psi[x/0] \land \Psi[x/1]$ and $\exists x. \Psi$ as $\Psi[x/0] \lor \Psi[x/1]$. Unless specified otherwise, QBFs are in closed prenex form with a CNF matrix, i.e. $Q_1X_1 \ldots Q_kX_k \phi$, where $X_k$ are pairwise disjoint sets of variables; $Q_i \in \{\exists, \forall\}$ and $Q_i \neq Q_{i+1}$. The formula $\phi$ is in CNF and is defined only on variables $X_1 \cup \ldots \cup X_k$. The propositional part $\phi$ is called the matrix and the rest the prefix. We write QCNF to talk about formulas in this form. If a variable $x$ is in the set $X_i$, we say that $x$ is at level $i$ and write $lv(x) = i$; we write $lv(l)$ for $lv(var(l))$. A closed QBF is false (resp. true), iff it is semantically equivalent to the constant 0 (resp. 1).

If a variable is universally quantified, we say that the variable is universal. For a literal $l$ and a universal variable $x$ such that $var(l) = x$, we say that $l$ is universal. The notions of existential variable and term are defined analogously.

### 2.1 Q-resolution

Q-resolution [5] is an extension of propositional resolution for showing that a QCNF is false. For a clause $C$, a universal literal $l \in C$ is blocked by an existential literal $k \in C$ iff $lv(l) < lv(k)$. \textit{\forall-reduction} is the operation of removing from a clause $C$ all universal literals that are not blocked by some literal. For two \forall-reduced clauses $x \lor C_1$ and $\bar{x} \lor C_2$, where $x$ is an existential variable, a \textit{Q-resolvent} [5] is obtained in two steps. (1) Compute $C_u = C_1 \cup C_2 \setminus \{x, \bar{x}\}$. If $C_u$ contains complementary literals, the Q-resolvent is undefined. (2) \forall-reduce $C_u$. For a QCNF $P \cdot \phi$, a \textit{Q-resolution proof} of a clause $C$ is a finite sequence of clauses $C_1, \ldots, C_n$ where $C_n = C$ and any $C_i$ in the sequence is part of the given matrix $\phi$ or it is a Q-resolvent for some pair of the preceding clauses. A Q-resolution proof is called a \textit{refutation} iff $C$ is the empty clause, denoted $\bot$.

In this paper Q-resolution proofs treated as connected directed acyclic graphs so that the each clause in the proof corresponds to some node labeled with that clause. We assume that the input clauses are already \forall-reduced. Q-resolution steps are depicted as on the right. Note that the \forall-reduction step is depicted separately.

\[
\begin{array}{c}
C_1 \lor x & C_2 \lor \bar{x} \\
\downarrow & \downarrow \\
C_u & C_u \\
\downarrow & \downarrow \\
C & C
\end{array}
\]

### 2.2 Term-Resolution

Term-resolution is analogous to Q-resolution with the difference that it operates on terms and its purpose is to prove that a QCNF is true [9]. Since the calculus operates on QBF’s with CNF matrices, it needs a mechanism to generate terms to operate on. This is done by a rule that enables using models of the given matrix in the proof.

Term-resolution, resolves on universal literals and reduces existential ones. For a term $T$ an existential literal $l$ is blocked, iff there is a universal $k \in T$ such that $lv(l) < lev(k)$. \exists-reduction removes from a term $T$ all existential literals that are not blocked by some universal literal. For two \exists-reduced terms $x \land T_1$ and $\bar{x} \land T_2$, a \textit{term-resolvent} is defined as the \exists-reduction of the term $T_1 \land T_2$, if $T_1$ and $T_2$ do not contain complementary literals; it is undefined otherwise.
For a QCNF $P, \phi$ a term-resolution proof of the term $T_m$ a is a finite sequence $T_1, \ldots, T_m$ of terms such that each term $T_i$ is a model of $\phi$ or it was obtained from the previous terms by $\exists$-reduction or term-resolution. Such proof proves $P, \phi$ iff $T_m$ is the empty term, denoted as $\top$. Those terms of the proof that are models of $\phi$ are said to be generated by a model generation rule. (Terms are sometimes referred to as “cubes”, especially in the context of DPLL QBF solvers that apply cube learning.)

2.3 Proof complexity

A proof system $P$ is relation $P(\Phi, \pi)$ computable in polynomial time such that a formula $\Phi$ is true iff there exists a proof $\pi$ for which $P(\Phi, \pi)$. A proof system $P_1$ p-simulates a proof system $P_2$ iff any proof in $P_2$ of a formula $\Phi$ can be translated into a proof in $P_1$ of $\Phi$ in polynomial time (cf. [7,19]).

As is common, we will count the sizes of Q-resolution and term-resolution as the number of resolution steps and number of $\forall/\exists$-reductions where each reduced literal is counted separately.

3 The Achilles’ Heel

This section describes a large class of formulas that have exponential term-resolution proofs. Recall that a leaf of a term-resolution proof must be generated by the model-generation rule. We exploit this by forcing the proof to generate many leafs.

First we make a simple observation that for any assignment to universal variables, there must be a leaf-term in a term-resolution proof that “agrees” with that assignment. We say that a term $T$ agrees with an assignment $\tau$ iff there is no literal $l$ such that $\bar{l} \in T$ and $\tau(l) = 1$.

**Lemma 1.** For any assignment $\tau$ to universal variables and a term-resolution proof $\pi$ there is a leaf-term $T$ of $\pi$ that agrees with $\tau$.

**Proof.** We construct a path from the root to some leaf such that each node on that path agrees with $\tau$. The root of $\pi$ agrees with $\tau$ because it does not contain any literals. If a term $T$ agrees with $\tau$ and $T$ is obtained from $T'$ by existential-reduction, then $T'$ also agrees with $\tau$ since $\tau$ assigns only to universal variables. If $T$ agrees with $\tau$ and is obtained from $T_1$ and $T_2$ by term-resolution on some variable $y$, it has to be that $y$ is in one of the $T_1$, $T_2$ and $\bar{y}$ in the other. Hence, at least one of the terms agrees with $\tau$. $\square$

**Theorem 1.** If all models of $\phi$ contain at least $k$ universal literals, then any term-resolution proof of $\Phi$ has at least $2^k$ leafs. (Recall that a formula has a term-resolution proof if and only if it is true.)

**Proof.** Let $V_u$ be the set of universal variables of $\Phi$. Since each leaf-term of any term-resolution proof has at least $k$ universal literals, it can agree with at most $2^{|V_u|-k}$ different complete assignments to the universal variables. Lemma 1 gives that for any of the $2^{|V_u|}$ total assignments to $V_u$ there must be a corresponding leaf-term. Averaging gives that $\pi$ has at least $\frac{2^{\mid V_u \mid}}{2^{\mid V_u \mid - k}} = 2^k$ leafs. $\square$
Theorem 1 gives us a powerful method of constructing formulas with large term-resolution proofs. It is sufficient to construct a true QCNF whose models have many universal literals. Let us construct one such formula. For a given parameter \( N \in \mathbb{N}^+ \) construct the following formula with \( 2N \) variables and \( 2N \) clauses.

\[
\forall x_1, \exists y_1, \ldots, \forall x_N, \exists y_N. \bigwedge_{i=1}^{N} (\bar{x}_i \lor y_i) \land (x_i \lor \bar{y}_i) \tag{1}
\]

**Proposition 1.** Any term-resolution proof of (1) is exponential in \( N \).

*Proof.* Formula (1) is true as each of the existential variables \( y_i \) can be set to the same value as the variable \( x_i \) and thus satisfying the matrix.

Let \( \psi \) denote the matrix of (1). Each pair of clauses \( \bar{x}_i \lor y_i \) and \( \bar{y}_i \lor x_i \) must be satisfied by any model \( \tau \) of \( \psi \), which can be only done in two ways: the model will contain the literals \( \{ y_i, x_i \} \) or the model contained literals \( \{ \bar{y}_i, \bar{x}_i \} \). Hence \( \tau \) contains a literal for each \( x_i \) and for each \( y_i \). Theorem 1 gives that at least \( 2^N \) models are needed in the leafs of any term-resolution proof. \( \square \)

4 A Possible Remedy—Negation

This section suggests a possible remedy to the weakness exposed in the previous section. Instead of proving a formula true by term-resolution, we propose to refute its negation by Q-resolution (an analogous approach to the one of propositional resolution).

To construct a negation of a formula, we follow the standard equalities \( \neg \forall x. \Psi = \exists x. \neg \Psi \) and \( \neg \exists x. \Psi = \forall x. \neg \Psi \). In order to bring the matrix back to conjunctive normal form, we add additional (Tseitin) variables [18]. We use the optimization by Plaisted-Greenbaum, which enables encoding variables’ semantics only in one direction [15]. In particular, for each clause we introduce a fresh variable that is forced to true when that clause becomes true. Using these variables, we construct a clause that requires that at least one of the clauses is false.

It would be correct to insert these fresh variables at the end of the prefix (existentially quantified) but we will see that it is useful to insert them further towards the outer levels, if possible.

**Definition 1.** The negation of a formula \( P.\phi \) is denoted as \( \text{Neg}(P.\phi) \) and constructed as follows. For each clause \( C \) introduce a fresh variable \( n_C \). Construct the prefix of \( \text{Neg}(P.\phi) \) from \( P \) inverting all the quantifiers in \( P \) and inserting each of the variables the variable \( n_C \) after the variable with maximal level in \( C \). Construct a matrix of \( \text{Neg}(P.\phi) \) as the following clauses.

\[
\{ \{ \bar{l} \lor n_C \mid l \in C \} \mid C \in \phi \} \cup \{ \bigvee_{C \in \phi} \bar{n}_C \}
\]

**Example 1.** The \( \text{Neg}(\forall x \exists y \exists z. (\bar{x} \lor y) \land (x \lor z)) \) is equal to \( \exists x \forall y \exists c_1 \forall z \exists c_2. (x \lor c_1) \land (y \lor c_1) \land (\bar{x} \lor c_2) \land (\bar{z} \lor c_2) \land (\bar{c_1} \lor \bar{c_2}) \).

Clearly, \( \text{Neg}(\Psi) \) is false if and only if \( \Psi \) is true. We say that a QCNF \( \Psi \) is negation-refuted by a Q-resolution proof \( \pi \) iff \( \pi \) is a refutation of \( \text{Neg}(\Psi) \).
4.1 Negation-Refutation P-simulates Term-Resolution

The first question we should ask is whether for any term-resolution proof there is a polynomial-size negation-refutation proof. We show this is indeed the case.

**Theorem 2.** Negation-refutation p-simulates term-resolution.

**Proof (sketch).** Let \( \pi \) be a term-resolution of a QCNF \( P.\phi \). Construct a Q-resolution of \( \text{Neg}(P.\phi) \) as follows. Let \( M \) be a leaf of \( \pi \). From the rules of term-resolution for each \( C \in \phi \) there is a literal \( l \) s.t. \( l \in C \) and \( l \in M \). From the definition of Neg, the QCNF \( \text{Neg}(P.\phi) \) contains the clause \( \bar{n}_C \lor \bar{l} \) for such literal.

Starting with the clause \( \bigvee_{C \in \phi} \bar{n}_C \), resolve each literal \( \bar{n}_C \) with the clause \( n_C \lor \bar{l} \), for each \( l \) s.t. \( l \in C \) and \( l \in M \). This results in the clause \( \bigvee_{l \in M} \bar{l} \). Note that this clause does not contain contradictory literals because \( M \) must not contain contradictory literals.

Repeating this process for each leaf of \( \pi \) produces clauses that are negations of those leaves. Perform Q-resolutions steps and \( \forall \)-reductions as are done term-resolutions steps and \( \exists \)-reductions in \( \pi \). This produces a proof where each node to a negation of the corresponding node in \( \pi \). Since \( \pi \) has the empty term in the root, the produced tree has the empty clause in the root. Resolutions needed to produce each of the leaf clauses requires at most \( \min(|\pi|, |\text{var}(\Phi)|) \) steps thus the resulting Q-resolution is at most of size \((|\phi| + |\pi|)^2 \).

\[ \square \]

4.2 Separation Between Term-Resolution and Negation-Refutation

The previous section shows that negation-refutation is at least as powerful as term-resolution. To show that the negation-refutation proof system is in fact stronger, we recall formula (1), whose term-resolution proofs are exponential, and show it has a negation-refutation proof of linear size.

**Proposition 2.** Formula (1) has a linear negation-refutation proof.

**Proof (sketch).** Negation of (1) introduces variables \( c_1, \ldots, c_{2N} \) representing the respective clauses. In particular, the following clauses are constructed \( x_i \lor c_{2i-1}, y_i \lor c_{2i}, \bar{y}_i \lor c_{2i-1}, \bar{x}_i \lor c_{2i}, y_i \lor c_{2i} \) for \( i \in 1..N \) and the clause \( \bar{c}_1 \lor \cdots \lor \bar{c}_{2N} \). With the prefix \( \exists x_1 \forall y_1 \exists x_2 \cdots \exists x_1 \forall y_1 \exists c_{2N-1} c_{2N} \).
We show how to resolve away the literals \( \bar{c}_{2N-1} \) and \( \bar{c}_{2N} \); the rest of the \( c_i \) literals is resolved in the same fashion. For conciseness we define \( D_{N-2} \) as \( \bar{c}_1 \lor \cdots \lor \bar{c}_{2N-3} \lor \bar{c}_{2N-2} \). Figure 1 shows how \( \bar{c}_{2N-1} \) and \( \bar{c}_{2N} \) are replaced by \( y_N \) and \( x_N \) at which point \( y_N \) is universally reduced. Analogously, the literals are replaced with \( \bar{x}_N \) and \( \bar{y}_N \), which enables resolving \( x_N \) away.

Using this construction, each of the literals \( \bar{c}_{2i-1}, \bar{c}_{2i} \) are resolved away in 7 resolution/reduction steps thus resulting in a resolution proof with \( 7N \) resolution/reduction steps in total.

\[ \square \]

### 4.3 Variable Definitions

We observe that formula (1) is an example of a formula where an existential variable \( y \) is defined, i.e. the value of the variable is determined by values of some variables with a lower level (in the case of formula (1) the value of \( y \) is determined by the value of \( x_1 \)). So the natural question to ask is whether any definition can be proven true by negation-refutation. We show that this is indeed the case but we will need QU-resolution—an extension of Q-resolution that enables resolving on universal variables [21].

We will demonstrate how negations of definitions can be refuted on the following representative example. Consider the prefix \( \exists x_1 \forall x_2 \exists x_3 o_1 o_2 o_3 \) and a matrix capturing the equalities \( o_1 = \text{NAND}(x_1, x_2), o_2 = \text{NAND}(x_2, x_3) \), and \( o_3 = \text{NAND}(o_1, o_2) \). These correspond to the following clauses (Tseitin variables that will be used for negating the clauses are indicated in parentheses).

\[
\begin{align*}
(c_1) \overline{x}_1 \lor \overline{x}_2 \lor \overline{o}_1 \\
(c_2) x_2 \lor o_1 \\
(c_3) x_2 \lor o_1 \\
(c_4) \overline{x}_2 \lor \overline{x}_3 \lor \overline{o}_2 \\
(c_5) x_2 \lor o_2 \\
(c_6) x_2 \lor o_2 \\
(c_7) \overline{o}_1 \lor \overline{o}_2 \lor \overline{o}_3 \\
(c_8) o_1 \lor o_3 \\
(c_9) o_1 \lor o_3
\end{align*}
\]

After negating this formula, we obtain the following prefix.

\[
\forall x_1 \exists x_2 \forall x_3 \forall o_1 \exists c_1 c_2 c_3 \forall o_2 \exists c_4 c_5 c_6 \forall o_3 \exists c_7 c_8 c_9
\]

We omit the negated formula’s matrix for succinctness. The Q-resolution proof proceeds in a similar fashion as the one for (1). Starting with the clause \( \bar{c}_1 \lor \cdots \lor \bar{c}_9 \), the \( \bar{c}_i \) literals are resolved away, starting with the innermost ones.

Figure 2 shows a fragment of the proof, which resolves away the literals \( c_7, \ldots, c_9 \) (certain resolution steps are collapsed). Using the clauses determining the value of \( o_3 \), the proof generates the clauses \( \bar{c}_1 \lor \cdots \lor \overline{o}_1 \lor \overline{o}_2 \) and \( \bar{c}_1 \lor \cdots \lor o_2 \). Resolving these two clauses removes the variable \( o_2 \). Note that \( o_2 \) is universal, which is why we need QU-resolution. In order to resolve away \( o_1 \), the clause \( \bar{c}_3 \lor \cdots \lor o_1 \) is generated analogously. Leaving us with the clause \( \bar{c}_1 \lor \cdots \lor \bar{c}_6 \). Note that it was possible to \( \forall \)-reduce \( o_3 \) throughout the process because it is blocked only by the variables \( c_7, \ldots, c_9 \). In contrast, the variables \( o_1 \) and \( o_2 \) could not be \( \forall \)-reduced because they are blocked by the literals \( \bar{c}_3, \ldots, \bar{c}_6 \). The literals \( \bar{c}_4, \ldots, \bar{c}_6 \) and subsequently \( \bar{c}_1, \ldots, \bar{c}_3 \) our resolved in the same fashion.

An analogous proof can be carried out for any acyclic circuit of NAND gates. One picks a topological order of the gates and resolves them away as in the example above.
5 Summary, Conclusions, and Future Work

This paper investigates the strength of term-resolution: a well-established calculus for true quantified Boolean formulas. This paper exposes a significant vulnerability in the term-resolution calculus, which stems from the fact that the number of leafs of a term-resolution proof is not bound by the size of the formula in question. Instead, the model-generation rule enables generating new leafs of the proof from models of the matrix. The paper demonstrates that this lets us force the proof to generate exponentially many leafs by constructing QBF matrices with “many” universal literals.

This theoretical observation provides a further underpinning of the well-known observation that solving quantified Boolean formula with a CNF matrix can be sometimes particularly harmful [2,22]. Indeed, we demonstrate that even a very simple formula where each clause has only two literals leads to exponential term-resolution proofs.

At the practical level, in response to this issue, Zhang proposes to reason on a formula and on its negation at the same time [22]. This idea was realized with different flavors in various solvers [10,12,1]. The second part of this paper takes a similar avenue at the theoretical level. We compare the term-resolution calculus with the negation-refutation calculus, a calculus which refutes the formula’s negation in order to show the formula true. The paper demonstrates that this proof system indeed has favorable theoretical properties, in particular it p-simulates term-resolution and there is an exponential separation between the two calculi.

This result is related to the well-known fact that enabling adding new variables in propositional resolution yields a more powerful proof system (extended resolution) [6]. Negation-refutation introduces new variables too. However, in extended resolution, the prover must come up with the variables’ definitions. In negation-refutation, the definitions are determined by the clauses of the formula.

The last part of the paper touches upon some limitations of the negation-refutation calculus. If a variable’s value is defined as a function of some other variables, through a Boolean circuit, we ask if it’s possible to prove that it is always possible to come up with the right value for the variable being defined, i.e. complete the circuit. This is something that we would hope to be proven easily. We show that it is indeed possible to prove such definitions true lineary using negation-refutation but we show so with the use of QU-resolution—extension of Q-resolution that enables resolving on universal variables. This result is important from a theoretical perspective but raises further questions because existing QBF solvers use Q-resolution. It is the subject of future work to look for linear proofs for such formulas using only Q-resolution.
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