BERTINI TYPE THEOREMS

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ABSTRACT. Let \( X \) be a smooth irreducible projective variety of dimension at least 2 over an algebraically closed field of characteristic 0 in the projective space \( \mathbb{P}^n \). Bertini’s Theorem states that a general hyperplane \( H \) intersects \( X \) with an irreducible smooth subvariety of \( X \). However, the precise location of the smooth hyperplane section is not known. We show that for any \( q \leq n + 1 \) closed points in general position and any degree \( a > 1 \), a general hypersurface \( H \) of degree \( a \) passing through these \( q \) points intersects \( X \) with an irreducible smooth codimension 1 subvariety on \( X \). We also consider linear system of ample divisors and give precise location of smooth elements in the system. Similar result can be obtained for compact complex manifolds with holomorphic maps into projective spaces.

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1. INTRODUCTION

Bertini’s two fundamental theorems concern the irreducibility and smoothness of the general hyperplane section of a smooth projective variety and a general member of a linear system of divisors. The hyperplane version of Bertini’s theorems says that if \( X \) is a smooth irreducible projective variety of dimension at least 2 over an algebraically closed field \( \mathbb{k} \) of characteristic 0, then a general hyperplane \( H \) intersects \( X \) with a smooth irreducible subvariety of codimension 1 on \( X \). But we do not know the exact location of the smooth hyperplane sections.

Let \( F \) be an effective divisor on \( X \). We say that \( F \) is a fixed component of linear system \(|D|\) of a divisor \( D \) if \( E > F \) for all \( E \in |D| \). \( F \) is the fixed part of a linear system if every irreducible component of \( F \) is a fixed component of the system and \( F \) is maximal with respect to the order \( \geq \). Every element \( E \) in the system can be written in the form \( E = E' + F \). We say that \( E' \) is the variable part of \( E \). A point \( x \in X \) is a base point of the linear system if \( x \) is contained in the supports of variable parts of all divisors in the system. The second Bertini Theorem is (\([U]\), Theorem 4.21): If \( \kappa(D, X) \geq 2 \), then the variable part of a general member of the complete linear system \(|D|\) is irreducible and smooth away from the singular locus of \( X \) and the base locus of \(|D|\). Here

\[
\kappa(D, X) = \text{tr.deg}_\mathbb{C} \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) - 1.
\]

In \([F]\), by using the theory of intersection numbers of semipositive line bundles, Fujita sharpens the above celebrated Bertini’s theorem and presents conditions on the base locus under which the general member is also nonsingular on the base locus itself. In \([X]\), Xu applies deformation of singularities to study the
singularities of a generic element of a linear system and give detailed information on the singular type of the base element. In [Z], Zak considers that under what condition the hyperplane section of a normal variety is normal. Diaz and Harbater consider the singular locus of the general member of a linear system and obtain better dimension estimate if the base locus is scheme-theoretically smooth. They successfully apply their strong Bertini theorem to complete intersection varieties. Our results and methods are different from all these known results. This work is inspired by Hartshorne’s proof ([H] Theorem 8.18, Chapter 2) and Kleiman’s very interesting article [K1].

In this paper, we assume that the ground field \( k \) is algebraically closed and of characteristic 0.

**Definition 1.1.** Let \( S = \{P_0, P_1, \ldots, P_{q-1}\} \) be \( q \) points in \( \mathbb{P}^n \). We say that they are in general position if

1. for \( q < n + 1 \), the vectors defined by the homogeneous coordinates of these \( q \) points are linearly independent;
2. for \( q = n + 1 \), any \( n \) points are linearly independent.

Let \( L \) be the linear system of hypersurfaces of degree \( a > 1 \) passing through these \( q \) points \( P_0, P_1, \ldots, P_{q-1} \) in general position. Our main result is that a general member of \( L \) is irreducible and smooth.

**Theorem 1.2.** If \( X \) is an irreducible smooth projective variety of dimension at least 2 in \( \mathbb{P}^n \), then for any \( q \leq n + 1 \) closed points \( P_0, P_1, \ldots, P_{q-1} \) on \( X \) in a general position and any degree \( a > 1 \), a general hypersurface \( H \) of degree \( a \) passing through these \( q \) points intersects \( X \) with an irreducible smooth codimension 1 subvariety on \( X \).

In fact, if some points even all points do not lie on \( X \), Theorem 1.2 still holds.

**Theorem 1.3.** If \( X \) is an irreducible smooth projective variety of dimension at least 2 in \( \mathbb{P}^n \), \( D \) is an ample divisor on \( X \), then there is an \( n_0 > 0 \) such that for all \( m \geq n_0 \) and any \( q \leq n + 1 \) closed points \( P_0, P_1, \ldots, P_{q-1} \) on \( X \), a general member of \( |mD|_q \) is irreducible and smooth, where \( |mD|_q \) is the linear system of effective divisors in \( |mD| \) passing through these \( q \) points \( P_s, s = 0, 1, \ldots, (q - 1) \).

The paper is organized as follows. In Section 2 and 3, we will deal with hypersurface sections. In Section 4 and 5, we will consider linear system of ample and big divisors. In Section 6, some applications in compact complex manifolds will be discussed.

### 2. Hypersurface Sections Passing Through a Point

Because of the lengthy calculation, we first show the case when there is only one point to indicate the idea. The general case will be proved in Section 3.

**Theorem 2.1.** If \( X \) is an irreducible smooth projective variety of dimension at least 2, then for any closed point \( P_0 \) on \( X \) and any degree \( a > 1 \), a general hypersurface \( H \) of degree \( a \) passing through \( P_0 \) intersects \( X \) with an irreducible smooth codimension 1 subvariety on \( X \).
Proof. Let $X$ be a closed subset of $\mathbb{P}^n$, $n \geq 3$. We may assume that the homogeneous coordinate of $P_0$ is $(1,0,...,0)$ after coordinate transformation. Let $x = (x_0, x_1, ..., x_n)$ be the homogeneous coordinates of $\mathbb{P}^n$.

The idea of the proof is the following. Let $V$ be the vector space of homogeneous polynomials of degree $a$ passing through the point $P_0$. For every closed point $x$, we will construct a map $\xi_x$ from $V$ to $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}$ such that $\xi_x$ is surjective for all closed points $x \neq P_0$ and $\xi_{P_0}$ is surjective from $V$ to $\mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}^2$. Let $S_x$ be the set of smooth hypersurfaces $H$ in $V$ such that $x$ is a singular point of $H \cap X$ or $X \subset H$. Let $S$ be the set of closed points of a closed subset of projective variety $X \times V$:

$$S = \{< x, H > | x \in X, H \in S_x \}.$$ 

Let $p_2 : S \rightarrow V$ be the projection. We will show that the image $p_2(S)$ is a closed subset of $V$. So a general member of $V$ intersects $X$ with a smooth subvariety of codimension 1. By standard vanishing theorems, we will obtain the irreducibility.

For the simplicity, we will first give detail when the degree is 2. Higher degree case can be proved in Step 6 by the same method.

**Step 1.** Let $V$ be the vector space of the hypersurfaces of degree 2 passing through $P_0$, then a general member of $V$ is smooth.

Let $H$ be a hypersurface defined by a homogeneous polynomial $h$ of degree 2 passing through $P_0$, then

$$h = \sum_{j=1}^{n} a_{0j} x_0 x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$ 

Since $\frac{\partial h}{\partial x_j}(P_0) = a_{0j}$, $H$ is nonsingular at $P_0$ if at least one $a_{0j} \neq 0$.

The dimension of $V$ as a vector space is

$$\dim_k V = \frac{(n+2)(n+1)}{2} - 1 = \frac{n^2 + 3n}{2}.$$ 

By Euler’s formula, the hypersurface $H$ is singular at a point $x = (x_0, x_1, ..., x_n)$ if and only if

$$\frac{\partial h}{\partial x_0} = \frac{\partial h}{\partial x_1} = ... = \frac{\partial h}{\partial x_n} = 0.$$ 

It is a system of linear equations

$$a_{01} x_1 + a_{02} x_2 + \cdots + a_{0n} x_n = 0$$

$$a_{01} x_0 + 2a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0$$

$$\cdots$$

$$a_{0n} x_0 + a_{1n} x_1 + a_{2n} x_2 + \cdots + 2a_{nn} x_n = 0.$$ 

The above system has a solution in $\mathbb{P}^n$ if and only if the determinant of the following symmetric matrix $A$ is zero,
Let $h$ be a hypersurface of degree 2 in $V$ such that the corresponding hypersurface $H$ is contained in $V$ and $H$ is nonsingular on an open subset of $\mathbb{P}^{2+3n-1}$, i.e., a general member $H$ of $V$ is smooth. Thus among the hypersurfaces of degree 2 passing through $P_0$, a general member is smooth.

**Step 2.** There is a map $\xi_x$ from $V$ to $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2$ such that $\xi_x$ is surjective for all closed points $x \neq P_0$ and $\xi_{P_0}$ is surjective from $V$ to $\mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}^2$.

Let $x$ be a closed point of $X$ and define $S_x$ to be the set of smooth hypersurfaces $H$ (defined by $h$) of degree 2 in $V$ such that $x$ is a singular point of $H \cap X$ ($\neq X$) or $X \subset H$. Fix a hypersurface $H_0$ of degree 2 in $V$ such that $x$ is not a point of $H_0$. Let $h_0$ be the defining homogeneous polynomial of $H_0$, then $h/h_0$ gives a regular function on $\mathbb{P}^a - H_0$. When restricted to $X$, it is a regular function on $X - X \cap H_0$.

Let $\mathcal{M}_{x,X}$ be the maximal ideal of the local ring $\mathcal{O}_{x,X}$ at $x$. Define a map $\xi_x$ from the vector space $V$ to $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2$ as follows: for every element $h$ in $V$ (a homogeneous polynomial of degree 2 such that the corresponding hypersurface $H$ is smooth and passes through the fixed point $P_0$), the image $\xi_x(h)$ is the image of $h/h_0$ in the local ring $\mathcal{O}_{x,X}$ modulo $\mathcal{M}_{x,X}^2$. It is easy to see that $x$ is a point of $H \cap X$ if and only if the image $\xi_x(h)$ of the defining polynomial $h$ of $H$ is contained in $\mathcal{M}_{x,X}^2$. And $x$ is singular on $H \cap X$ if and only if the image $\xi_x(h)$ is contained in $\mathcal{M}_{x,X}$, because the local ring $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}(h)$ will not be regular. So there is the following one-to-one correspondence

$$H \in S_x \iff h \in \ker \xi_x.$$

Since $x$ is a closed point and the ground field is algebraically closed of characteristic 0, the maximal ideal $\mathcal{M}_{x,X}$ is generated by linear forms in the coordinates. Let $d$ be the dimension of $X$, then the vector space $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2$ has dimension $d+1$ over $k$.

We will show that the map $\xi_x$ is surjective from $V$ to $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2$ if $x \neq P_0$ and $\xi_{P_0}$ is surjective from $V$ to $\mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}^2$.

Let $U_1 = \{ (x_0, ..., x_n) \in \mathbb{P}^n | x_i \neq 0 \}$. Then $\{ U_0, ..., U_n \}$ is an affine open cover of $\mathbb{P}^n$. In $U_1$, we choose the local coordinates in the following

$$y_1 = \frac{x_0}{x_1}, \quad y_2 = \frac{x_2}{x_1}, \quad ..., \quad y_n = \frac{x_n}{x_1}.$$

Let $P$ be a closed point in $X \cap U_1$ and $(a_1, ..., a_n)$ be the local coordinate of $P$ in $U_1$. We choose $h_0$ to be $x_1^2$, then

$$\frac{h}{h_0} = a_{01} \frac{x_0}{x_1} + a_{11} \frac{x_1}{x_1} + a_{12} \frac{x_2}{x_1} + ... + a_{1n} \frac{x_n}{x_1} + \text{other terms}.$$
= a_{01}(y_1 - a_1) + a_{12}(y_2 - a_2) + ... + a_{1n}(y_n - a_n) + c + other terms,
where the constant
\[ c = a_{11} + a_{01}a_1 + a_{12}a_2 + ... + a_{1n}a_n. \]

From the expression of \( \frac{h}{h_0} \), we know that
\[ \{ h/h_0 | h \in V \} \to \mathcal{O}_{P, \mathbb{P}^n}/\mathcal{M}^2_{P, \mathbb{P}^n} \]
is surjective. So \( \xi_P \) is surjective ([H], page 32).

For any closed point \( P = (a_1, ..., a_n) \) in \( U_i \cap X, i = 2, ..., n \), choose \( y_j = x_j/x_i \) as local coordinates, similar calculation shows that \( \xi_P \) is surjective.

In \( U_0 \), let \( h_0 = x_0^2 \) and \( y_i = x_i/x_0 \), then
\[
\frac{h}{h_0} = \sum_{j=1}^{n} a_{0j}y_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}y_iy_j.
\]

There is no constant term in \( h/h_0 \), so the map \( \xi_{P_0} \) is not surjective to \( \mathcal{O}_{P_0, X}/\mathcal{M}^2_{P_0, X} \) but surjective to \( \mathcal{M}_{P_0, X}/\mathcal{M}^2_{P_0, X} \).

If \( P = (a_1, ..., a_n) \neq P_0 = (0, ..., 0) \) is a closed point in \( U_0 \cap X \), then write \( y_i = (y_i - a_i) + a_i \), we have
\[
\frac{h}{h_0} = \sum_{j=1}^{n} a_{0j}(y_j - a_j) + \sum_{j=1}^{n} a_{0j}a_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}[(y_i - a_i) + a_i][(y_j - a_j) + a_j]
= c + I + II,
\]
where the constant term
\[
c = \sum_{j=1}^{n} a_{0j}a_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}a_i a_j,
\]
the linear term with respect to \( y_i - a_i \) is complete
\[
I = \sum_{j=1}^{n} a_{0j}(y_j - a_j) + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}a_i(y_j - a_j) + a_j(y_i - a_i),
\]
and the degree 2 term with respect to \( y_i - a_i \)
\[
II = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}(y_i - a_i)(y_j - a_j).
\]

Since \( P = (a_1, ..., a_n) \neq (0, ..., 0) \), there is at least one \( i \), such that \( a_i \neq 0, 1 \leq i \leq n \). So the arbitrary constant \( a_{ii}a_i^2 \) is a term in \( c \). The above expressions of constant \( c \) and linear term \( I \) show that on \( U_0 \), \( \xi_x \) is surjective if the closed point \( x \neq P_0 \).

Considering the kernel of the map
\[
\xi_x : V \longrightarrow \mathcal{O}_{x, X}/\mathcal{M}^2_{x, X},
\]
if \( x \neq P_0 \), the kernel as a vector space has dimension
\[
\dim_{\mathbb{C}} \ker \xi_x = \frac{n(n+3)}{2} - d - 1.
\]

Therefore \( S_x \) is a linear system of hypersurfaces with dimension \( \frac{n(n+3)}{2} - d - 2 \) if \( x \neq P_0 \). If \( x = P_0 \), then the projective dimension of \( S_{P_0} \) is \( \frac{n(n+3)}{2} - d - 1 \).

**Step 3.** If \( V \) is considered as a projective space, then \( X \times V \) is a projective variety. Let the subset \( S \subset X \times V \) consist of all pairs \(<x, H>\) such that \( x \in X \) is a closed point and \( H \in S_x \). Then the dimension of \( S \) is less than the dimension of \( V \).

\( S \) is the set of closed points of a closed subset of \( X \times V \) and we give a reduced induced scheme structure to \( S \). The first projection \( p_1 : S \to X \) is surjective. If \( x \neq P_0 \), the fiber \( p_1^{-1}(x) \) is a projective space with dimension \( \frac{n(n+3)}{2} - d - 2 \). The special fiber \( p_1^{-1}(P_0) \) is a projective space with dimension \( \frac{n(n+3)}{2} - d - 1 \).

Let \( S = \bigcup_{i=0}^{n} S_i \) be an irreducible decomposition. Then every \( p_1(S_i) \) is closed and there is an \( i \), such that \( p_1(S_i) = X \). For every \( S_i \) with \( p_1(S_i) = X \), there is an open subset \( U_i \subset S_i \) such that for every \( x \in U_i \), the fiber \( p_1^{-1}(x) \) has constant dimension \( n_i \) ([S], Chapter 1, Section 6.3, Theorem 7). Let \( x \in \cap U_i \), since the fiber \( p_1^{-1}(x) \) is irreducible, it is contained in some \( S_i \). Suppose \( p_1^{-1}(x) \in S_i \). Let \( f_1 \) be the restriction of \( p_1 \) on \( S_1 \), i.e., \( p_1|_{S_1} = f_1 \), then \( p_1^{-1}(x) \subset f_1^{-1}(x) \) since \( p_1^{-1}(x) \) is irreducible. The opposite inclusion is obvious, so \( p_1^{-1}(x) = f_1^{-1}(x) \) for \( x \in \cap U_i \) and
\[
n_1 = \frac{n(n+3)}{2} - d - 2.
\]

Since \( f_1 \) is surjective and \( S_1 \) is one irreducible component of \( S \), for every \( x \in X \), the fiber \( f_1^{-1}(x) \) is not empty and contained in \( p_1^{-1}(x) \). But the dimension of \( f_1^{-1}(x) \) is at least \( \frac{n(n+3)}{2} - d - 2 \), so for every \( x \in X \), \( x \neq P_0 \), \( p_1^{-1}(x) = f_1^{-1}(x) \).

Hence \( S_1 \) has dimension ([S], Chapter 1, Section 6.3)
\[
\left[ \frac{n(n+3)}{2} - d - 2 \right] + d = \frac{n(n+3)}{2} - 2.
\]

If there is a component \( S_j \) in \( S \) such that \( p_1(S_j) \neq X \), then the dimension of \( S_j \) is not greater than the dimension of \( S_1 \) ([S], Chapter I, Section 6.3, Theorem 7). So if \( S \) is not irreducible, then for all components \( S_i \) in \( S \), \( S_1 \) has the maximum dimension \( \left[ \frac{n(n+3)}{2} - d - 2 \right] + d = \frac{n(n+3)}{2} - 2 \), which is the dimension of \( S \).

**Step 4.** A general member \( H \) of \( V \) intersects \( X \) with a smooth codimension 1 subvariety on \( X \).

Looking at the second projection (a proper morphism) \( p_2 : S \to V \). The dimension of the image
\[
\dim p_2(S) \leq \dim S = \frac{n(n+3)}{2} - 2.
\]

Since \( S \) is closed in \( X \times V \) and the dimension of \( V \) (as a projective space) is \( \frac{n(n+3)}{2} - 1 \), \( V - p_2(S) \) is an open subset of \( V \). This implies that a general member
Step 5. $X \cap H$ is irreducible.

From the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-H) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_H \rightarrow 0,$$

since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-H)) = 0$ ([H], Page 225, Theorem 5.1), we have a surjective map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k \rightarrow H^0(H, \mathcal{O}_H).$$

$H^0(H, \mathcal{O}_H) = k$ implies that the hypersurface $H$ is connected.

Since $X$ is closed in $\mathbb{P}^n$, $H|_X$ is ample on $X$. By Kodaira Vanishing Theorem, $H^1(X, \mathcal{O}_X(-H)) = 0$ ([KM], Page 62). Applying the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{H \cap X} \rightarrow 0,$$

we get

$$H^0(H \cap X, \mathcal{O}_{H \cap X}) = H^0(X, \mathcal{O}_X) = k.$$

Thus the intersection $H \cap X$ is connected. Therefore for a general hypersurface $H$ of degree 2, $H \cap X$ is smooth and irreducible.

We have proved that a general smooth hypersurface of degree 2 passing through $P_0$ intersects $X$ with an irreducible smooth subvariety of codimension 1.

Step 6. Degree $a > 2$ case.

Let $W$ be the vector space of hypersurfaces $H$ of degree $a > 2$ such that $P_0 \in H$. Then any element of $W$ can be written in the following form

$$g = c_0x_0^{a-2}h + c_1x_1^{a-2}h + \ldots + c_nx_n^{a-2}h + \text{other terms},$$

where

$$h = \sum_{j=1}^n a_{0j}x_0x_j + \sum_{i=1}^n \sum_{j=i}^n a_{ij}x_ix_j,$$

is the hypersurface of degree 2 in Step 1.

It is easy to see from the calculation of Step 2 that in each affine open subset $U_i$, we have

$$\left\{ \frac{h}{x^a_i} \middle| h \in V \right\} \subset \left\{ \frac{g}{x^a_0} \middle| g \in W \right\}.$$

So again the map $\xi_x$ from $W$ to $\mathcal{O}_{x,X}/\mathcal{M}^2_{x,X}$ is surjective and $\xi_{P_0}$ is surjective from $W$ to $\mathcal{M}_{P_0,X}/\mathcal{M}^2_{P_0,X}$.

For any degree $a > 2$ hypersurface, by counting the dimension correctly as above, we can similarly show that a general hypersurface passing through $P_0$ intersects $X$ with an irreducible smooth projective variety of codimension 1 on $X$. In fact, the vector space $W$ has dimension greater than $(n^2 + 3n)/2$, and dimension of $p_2(S)$ is less than the dimension of $W$ as a projective space. So the whole argument works.

Q.E.D.

Remark 2.2. We only need the intersection part $X \cap H$ is irreducible and smooth. Outside $X$, $H$ being smooth or not does not play any role.
Remark 2.3. From the proof, we see that if the point $P_0$ is a point outside $X$, the theorem still holds since the map $\xi_x$ is surjective for all $x \in X$.

Remark 2.4. Let $L$ be the vector space of hyperplanes passing through $P_0$, then its dimension is $n$. If $\dim X = d = n - 1$, then the map $\xi_x$ may not be surjective. So the above proof does not work for hyperplanes. For general hyperplane sections, Theorem 2.1 is still true but the proof is different.

The proof of the following Theorem 2.5 has been communicated to me by Igor Dolgachev.

**Theorem 2.5.** Let $X$ be an irreducible smooth projective variety of dimension at least 2 in $\mathbb{P}^n$. Let $P_0$ be a closed point on $X$. Then a general hyperplane passing through $P_0$ is irreducible and smooth.

*Proof.* If $X$ is a hyperplane, then the theorem is true. So we may assume that $X$ is not a hyperplane.

Consider the dual projective space $\mathbb{P}^{n*}$. The hyperplanes passing through $P_0$ in $\mathbb{P}^n$ is a hyperplane $H^*$ in $\mathbb{P}^{n*}$. A hyperplane $B$ intersects $X$ with a singular point $x \in B \cap X$ if $B$ is tangent to $X$ at $x$. Let $X^*$ be the dual space of $X$, i.e., the set of hyperplanes tangent to $X$ at some point. Then the dimension of $X^* \leq n - 1$ ([AG], Section 2.5, 3.1). If $X^* = H^*$, then $(X^*)^* = X = H$, which is not possible by our assumption. So $X^* \cap H^*$ has dimension at most $n - 2$ and $H^* - X^*$ is an open subset of $H^*$. Any point of $H^*$ away from $X^*$ corresponds to a hyperplane in $\mathbb{P}^n$ which cuts $X$ with a smooth subvariety of codimension 1 on $X$. By Kodaira Vanishing Theorem, the intersection subvariety is irreducible.

Q.E.D.

Remark 2.6. By the proof, we see that if the point $P_0$ is not a point on $X$, then Theorem 2.5 is still true.

Remark 2.7. By Veronese embedding and considering the dual variety, we can prove Theorem 2.1 in a much easier and geometric way. The advantage of the long proof is that the idea can be used to the case of $q$ points, $q > 1$.

3. Hypersurface Sections Passing Through $q$ Points

**Theorem 3.1.** If $X$ is an irreducible smooth projective variety of dimension at least 2 in $\mathbb{P}^n$, then for any $q \leq n + 1$ closed points $P_0, P_1, \ldots, P_{q-1}$ on $X$ in a general position and any degree $a > 1$, a general hypersurface $H$ of degree $a$ passing through these $q$ points intersects $X$ with an irreducible smooth codimension 1 subvariety on $X$.

*Proof.* By the proof of Theorem 2.1, Step 5, for any hypersurface $H$, $H \cap X$ is connected. So we only need to show that $H \cap X$ is nonsingular for a general $H$ passing through these $q$ points.

**Step 1.** There is a hyperplane $H$ in $\mathbb{P}^n$ such that $H$ does not contain any point $P_s$, $s = 0, 1, \ldots, (q - 1)$. 

Let \((x_0, x_1, ..., x_n)\) be the homogeneous coordinates of \(\mathbb{P}^n\), and \(a_0 x_0 + a_1 x_1 + ... + a_n x_n = 0\) be the hyperplane \(H\).

Let \(P_s = (b_{s0}, b_{s1}, ..., b_{sn})\). Consider \((a_0, a_1, ..., a_n)\) such that \(a_0 b_{s0} + a_1 b_{s1} + ... + a_n b_{sn} = 0\). We may look at it as a hyperplane \(H^*_s\) in the dual projective space \(\mathbb{P}^{n*}\). Choose \(c_0, ..., c_n\) in the open subset \(\mathbb{P}^{n*} \setminus \cup_{s=0}^{q-1} H^*_s\), then for every \(P_s\), \(c_0 b_{s0} + ... + c_n b_{sn} \neq 0\). So there is a hyperplane \(H\) defined by \(c_0 x_0 + ... + c_n x_n = 0\) that does not contain any point \(P_s\), \(s = 0, 1, ..., q - 1\).

**Step 2.** We may change the coordinates such that \(P_0 = (1, 0, ..., 0)\) and \(H\) is defined by \(x_0 = 0\) such that \(H\) does not contain any \(P_s\), \(s = 0, 1, ..., q - 1\).

By Step 1, we may choose hyperplane \(H\): \(c_0 x_0 + ... + c_n x_n = 0\) such that \(H\) does not contain any point \(P_s\), \(s = 0, 1, ..., q - 1\). Define the new coordinates

\[
X_0 = c_0 x_0 + ... + c_n x_n,
\]

\[
X_1 = \sum_{j=0}^{n} a_{1j} x_j,
\]

\[
X_n = \sum_{j=0}^{n} a_{nj} x_j,
\]

where the coefficients \(a_{ij}\) satisfy the following system of linear equations

\[
\sum_{j=0}^{n} a_{1j} b_{1j} = 0,
\]

\[
\sum_{j=0}^{n} a_{2j} b_{2j} = 0,
\]

\[
\sum_{j=0}^{n} a_{nj} b_{nj} = 0.
\]

The \(n\) points \(P_1, ..., P_n\) being linearly independent guarantees that the above linear transformation is well-defined. Since \(c_0 b_{00} + c_1 b_{01} + ... + c_n b_{0n} \neq 0\) the new coordinate of \(P_0\) is \((1, 0, ..., 0)\) and the plane \(H : X_0 = 0\) does not contain any \(P_s\), \(s = 0, 1, ..., q - 1\).

**Step 3.** Let \(V\) be the vector space of the hypersurfaces of degree 2 passing through these \(q \leq n + 1\) closed points \(P_0, ..., P_{q-1}\) in general position, then the map \(\xi_s\) from \(V\) to \(\mathcal{O}_{x,X}/\mathcal{M}^2_{x,X}\) defined in the proof of Theorem 2.1, Step 2, is surjective for all closed points \(x \in U_0\), \(x \neq P_s\) and \(\xi_{P_s}\) is surjective from \(V\) to \(\mathcal{M}_{P_s,X}/\mathcal{M}^2_{P_s,X}\), \(s = 0, 1, ..., q - 1\), where \(\mathcal{M}_{P_s,X}\) is the maximal ideal of \(\mathcal{O}_{x,X}\).

By Step 2, we may assume that \(P_0 = (1, 0, ..., 0)\) and the hyperplane \(H_0\) defined by \(x_0 = 0\) does not contain any point \(P_s\), \(s = 0, 1, ..., q - 1\), \(q \leq n + 1\). By Step 1, proof of Theorem 2.1, a homogeneous polynomial \(h\) of degree 2 passing through \(P_0\) is of the form

\[
h = \sum_{j=1}^{n} a_{0j} x_0 x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.
\]
In $U_0 = \mathbb{P}^n - H_0$, let $h_0 = x_0^2$ and $y_i = x_i/x_0$, then
\[
\frac{h}{h_0} = \sum_{j=1}^{n} a_{0j} y_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} y_i y_j.
\]

Let $P_s = (b_{s1}, b_{s2}, ..., b_{sn})$ in $U_0 \cong \mathbb{A}^n$. Since $h(P_s) = 0$ but $h_0(P_s) \neq 0$, we have the following $q - 1$ equations
\[
\frac{h}{h_0}(P_s) = \sum_{j=1}^{n} a_{0j} b_{sj} + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} b_{si} b_{sj} = 0,
\]
for $s = 1, 2, ..., q-1$. This is a system of $(q-1) \leq n$ linear equations with $(n^2 + 3n)/2$ variables $a_{ij}$.

The coefficient matrix $A$ of this linear system is
\[
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} & b_{11}^2 & b_{11} b_{12} & \cdots & b_{1n}^2 \\
  b_{21} & b_{22} & \cdots & b_{2n} & b_{21}^2 & b_{21} b_{22} & \cdots & b_{2n}^2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{(q-1)1} & b_{(q-1)2} & \cdots & b_{(q-1)n} & b_{(q-1)1}^2 & b_{(q-1)1} b_{(q-1)2} & \cdots & b_{(q-1)n}^2
\end{pmatrix}.
\]

The matrix $B$ of the first $n$ columns and all $(q-1)$ rows is
\[
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{(q-1)1} & b_{(q-1)2} & \cdots & b_{(q-1)n}
\end{pmatrix}.
\]

Because the $q - 1$ points $P_1, ..., P_{q-1}$ are in general position, the rank of the above matrix $B$ is $q - 1 \leq n$. So the system defined by $A$ is consistent and each $a_{0j}$ varies independently. The dimension of solutions of the system $\frac{h}{h_0}(P_s) = 0$, $s = 1, ..., q-1$, is
\[
\frac{n^2 + 3n}{2} - (q - 1)
\]
as a vector space.

For any closed point $P = (a_1, a_2, ..., a_n)$ in $U_0$, we have
\[
\frac{h}{h_0} = \sum_{j=1}^{n} a_{0j} (y_j - a_j) + \sum_{j=1}^{n} a_{0j} a_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} [(y_i - a_i) + a_i][(y_j - a_j) + a_j]
\]
\[
= c + I + II,
\]
where the constant term
\[
c = \sum_{j=1}^{n} a_{0j} a_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} a_i a_j,
\]
the linear term with respect to \( y_i - a_i \)

\[
I = \sum_{j=1}^{n} a_{0j}(y_j - a_j) + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}[a_i(y_j - a_j) + a_j(y_i - a_i)],
\]

and the degree 2 term with respect to \( y_i - a_i \)

\[
II = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}(y_i - a_i)(y_j - a_j).
\]

If \( P \neq P_0 \), then there is at least one \( i \), such that \( a_i \neq 0 \), \( i = 1, \ldots, n \). So the constant \( c \) can be any number since it has a term \( a_i^2 \), where \( a_{ii} \) varies independently and \( a_i \neq 0 \). We already see that \( a_{01}, a_{02}, \ldots, a_{0n} \) vary independently, so \( c + I \) is a complete linear form with every linear term. Thus we have a surjective map from \( V \) to \( \mathcal{O}_{P, \mathbb{P}^n}/\mathcal{M}^2_{P, \mathbb{P}^n} \). Since there is a natural surjective map from \( \mathcal{O}_{P, \mathbb{P}^n}/\mathcal{M}^2_{P, \mathbb{P}^n} \) to \( \mathcal{O}_{P, X}/\mathcal{M}^2_{P, X} \) (II, Page 32), by the above expression of \( h/h_0 \), if the closed point \( P = P_s, s = 1, 2, \ldots, q - 1 \), then \( \xi_{P_s} \) is surjective from \( V \) to \( \mathcal{O}_{P_s, \mathbb{P}^n}/\mathcal{M}^2_{P_s, \mathbb{P}^n} \) so surjective to \( \mathcal{O}_{P_s, X}/\mathcal{M}^2_{P_s, X} \).

**Step 4.** The map \( \xi_x \) from \( V \) to \( \mathcal{O}_{x, X}/\mathcal{M}^2_{x, X} \) is surjective for all closed points \( x \in H_0 \).

Since on \( H_0, x_0 = 0 \), every element in \( V \) is of the same form

\[
h = \sum_{j=1}^{n} a_{0j}x_0x_j + \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij}x_i x_j.
\]

Let \( P = (0, a_1, a_2, \ldots, a_n) \) be a closed point on \( X \cap H_0 \), then at least one \( a_i \neq 0 \). Suppose that \( a_1 \neq 0 \). Let \( h_1 = x_1^2 \), define \( y_i = x_i/x_1 \), then

\[
\frac{h}{h_1} = a_{01}y_0 + a_{02}y_0y_2 + \ldots + a_{0n}y_0y_n + a_{11} + a_{12}y_2 + \ldots + a_{1n}y_n + \sum_{i=2}^{n} \sum_{j=i}^{n} a_{ij}y_i y_j.
\]

We can rewrite it into three parts

\[
\frac{h}{h_1} = c + I + II,
\]

where the constant

\[
c = a_{11} + \sum_{i=2}^{n} a_{1i}a_i + \sum_{i=2}^{n} \sum_{j=i}^{n} a_{ij}a_i a_j,
\]

the linear term

\[
I = a_{01}y_0 + \sum_{i=2}^{n} a_{1i}(y_i - a_i) + \sum_{i=2}^{n} \sum_{j=i}^{n} a_{ij}[a_j(y_i - a_i) + a_i(y_j - a_j)],
\]

and the degree 2 term

\[
II = \sum_{i=2}^{n} a_{0i}y_0(y_i - a_i) + \sum_{i=2}^{n} \sum_{j=i}^{n} a_{ij}(y_i - a_i)(y_j - a_j).
\]
Since \( h/h_1 \) contains the complete linear form with every linear term, the map from \( V \) to \( O_{P,X}/M^2_{P,X} \) at any point \( P = (0, a_1, ..., a_n) \) on \( H_0 \) with \( a_1 \neq 0 \) is surjective. In general, if \( a_i \neq 0 \), choose \( h_i = x_i^2 \), then by the same calculation, we can show that the map \( \xi_P \) is surjective from \( V \) to \( O_{P,X}/M^2_{P,X} \).

**Step 5.** Let \( x \) be a closed point of \( X \) and define \( S_x \) to be the set of smooth hypersurfaces \( H \) (defined by \( h \)) of degree 2 in \( V \) such that \( x \) is a singular point of \( H \cap X (\neq X) \) or \( X \subset H \). If \( V \) is considered as a projective space, let the subset \( S \subset X \times V \) consist of all pairs \( \langle x, H \rangle \) such that \( x \in X \) is a closed point and \( H \in S_x \). Then the dimension of \( S \) is less than the dimension of \( V \).

As a projective space, \( V \) has dimension

\[
\frac{n^2 + 3n}{2} - (q - 1) - 1 = \frac{n^2 + 3n}{2} - q = \frac{n^2 + n}{2} - 1,
\]

where \( q \leq n + 1 \). Let \( d \) be the dimension of of \( X \). Consider the first projection \( p_1 : S \rightarrow X \), for all closed points \( x \in X \), \( x \neq P_s \), \( s = 0, 1, ..., q - 1 \), as a vector space, \( O_{x,X}/M^2_{x,X} \) has dimension \( d + 1 \). By Step 3 and 4, the dimension of the fiber \( p_1^{-1}(x) \) is the dimension of the kernel of the map \( \xi_x \), so as a projective space,

\[
dim p_1^{-1}(x) = \left( \frac{n^2 + 3n}{2} - q \right) - (d + 1) = \frac{n^2 + 3n}{2} - q - d - 1.
\]

The vector space \( M_{P,X}/M^2_{P,X} \) has dimension \( d \). The dimension of the fiber over the point \( P_s \) is

\[
dim p_1^{-1}(P_s) = \left( \frac{n^2 + 3n}{2} - q \right) - d = \frac{n^2 + 3n}{2} - q - d.
\]

The dimension of \( S \) is

\[
\frac{n^2 + 3n}{2} - q - d - 1 + d = \frac{n^2 + n}{2} - q - 1.
\]

So

\[
\dim(V) - \dim(S) = (n^2 + 3n)/2 - q - (\frac{n^2 + 3n}{2} - q - 1) = 1.
\]

Let \( p_2 : S \rightarrow V \) be the second projection, then

\[
\dim(p_2(S)) \leq \dim(S) < \dim(V).
\]

Since \( p_2(S) \) is a closed subset of \( V \), a general member of \( V \) intersects \( X \) with a smooth subvariety of codimension 1 on \( X \). By Step 5 of proof of Theorem 2.1, it is also connected so irreducible. This proves the degree 2 case.

**Step 6.** Let \( h \) be an element of \( V \) in Step 3. Then any degree \( a > 2 \) homogeneous polynomial passing through the \( q \) points \( P_0, P_1, ..., P_{q-1} \) can be written as

\[
f = c_0 x_0^{a-2} h + c_1 x_1^{a-2} h + ... + c_n x_n^{a-2} h + \text{other terms},
\]

where \( c_0, ..., c_n \) are constants. Let \( W \) be the set of homogeneous polynomials of degree \( a \) passing through these \( q \) points. Considering \( f/x_i^a \), \( i = 0, 1, ..., n \), and using the same calculation, we can show that \( \xi_x \) from \( W \) to \( O_{x,X}/M^2_{x,X} \) is surjective for all closed points \( x \neq P_2 \) and is surjective from \( W \) to \( M_{P,x,X}/M^2_{P,X} \). Carrying out
the dimension calculation as in Step 5, we see that the theorem holds for all degree \( a > 2 \).

Q.E.D.

4. Linear System of Ample Divisors

**Theorem 4.1.** If \( X \) is an irreducible smooth projective variety of dimension \( d \geq 2 \), \( D \) is an ample divisor on \( X \), then there is an \( n_0 > 0 \) such that for all \( n \geq n_0 \) and any closed point \( P_0 \) on \( X \), a general member of \( |nD|_{P_0} \) is irreducible and smooth, where \( |nD|_{P_0} \) is the linear system of effective divisors in \( |nD| \) passing through the point \( P_0 \).

**Proof.** Since \( D \) is ample, there is an \( l \) such that the basis \( \{ f_0, ..., f_m \} \) in the vector space \( H^0(X, \mathcal{O}_X(lD)) \) gives an embedding \( \phi \) from \( X \) to the projective space \( \mathbb{P}^m \) by sending a point \( x \) on \( X \) to \( (f_0(x), f_1(x), \cdots, f_m(x)) \) in \( \mathbb{P}^m \). Let \( W = \phi(X) \), \( w_0 = \phi(P_0) \). Let \( M \) be the set of homogeneous polynomials \( h \) of degree \( a > 1 \) such that every hypersurface \( H \) defined by \( h \) contains the point \( w_0 \) in \( \mathbb{P}^m \). By the proof of the Theorem 2.1, if \( w \neq w_0 \), then the natural map \( \xi_w \) from \( M \) to \( \mathcal{O}_{w,X}/\mathcal{M}_{w,X}^2 \) is surjective and \( \xi_w \) is surjective from \( M \) to \( \mathcal{M}_{w_0,W}/\mathcal{M}_{w_0,W}^2 \).

Let

\[
L = \{ f \in H^0(\mathcal{O}_X(n_0D)) | f(P_0) = 0 \},
\]

where \( n_0 = a^l \), then \( \phi^*(M) \) is a subspace of \( L \).

Since \( \phi \) is an isomorphism between \( X \) and \( W \), the map \( \xi_x \) from \( L \) to \( \mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2 \) is also surjective for every point \( x \neq P_0 \) in \( X \) and \( \xi_{P_0} \) is surjective from \( L \) to \( \mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}^2 \). The dimension of the kernel of \( \xi_x \) as a projective space is \( \dim L - d - 1 \) and the dimension of the kernel of \( \xi_{P_0} \) is \( \dim L - d \). Here \( L \) is considered as a projective space.

Because there is a one-to-one correspondence between the effective divisors in \( \mathcal{M}_{n_0D} \) passing through \( P_0 \) and the elements in \( L \) up to a nonzero constant, we may think about \( L \) as a vector space of effective divisors passing through \( P_0 \) and linearly equivalent to \( n_0D \).

Let \( x \) be a closed point of \( X \) and \( S_x \) be the set of effective divisors \( E \) in \( L \) such that \( x \) is a singular point of \( X \cap E \). Let

\[
S = \{ < x, E > | x \in X, E \in S_x \}.
\]

Then \( S \) is a closed subvariety of \( X \times L \), where \( L \) is the projective space.

A point \( x \in X \) is a singular point of \( f \in L \) if and only if the image of \( f \) in the natural map

\[
\xi_x : L \rightarrow \mathcal{O}_{x,X}/\mathcal{M}_{x,X}^2
\]

is zero, i.e., the image of \( f \) is in the kernel of \( \xi_x \).

Considering the projections \( p_1 : S \rightarrow X \) and \( p_2 : S \rightarrow L \). \( p_1 \) is surjective. Every fiber \( p_1^{-1}(x) \) is a projective space of dimension \( \dim L - d - 1 \) if \( x \neq P_0 \) and \( p_1^{-1}(P_0) \) is a projective space of dimension \( \dim L - d \), where \( \dim L \) is the dimension of \( L \).
as a projective space. By the proof of Theorem 2.1, the projective variety $S$ has dimension $\dim L - d - 1 + d = \dim L - 1$.

Looking at the second projection $p_2 : S \to L$, we have $\dim p_2(S) \leq \dim S \leq \dim L - 1$. So $p_2(S)$ is a closed subset of $L$. Thus a general member of $L$ is smooth.

Let $E$ be a general smooth element in $L$, then from

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

by Kodaira Vanishing Theorem, we have

$$H^0(E, \mathcal{O}_E) = H^0(X, \mathcal{O}_X) = k.$$ 

Thus $E$ is connected.

If $n > n_0$, considering the vector space

$$L' = \{ f \in H^0(\mathcal{O}_X(nD)) \mid f(P_0) = 0 \},$$

then $L$ is a subspace of $L'$. So the map

$$\xi_x : L' \to \mathcal{O}_{x,X}/\mathcal{M}_{x,X}$$

is surjective if $x \neq P_0$ and $\xi_{P_0}$ is surjective from $L'$ to $\mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}$. The proof still works for $L'$ by counting the dimensions in the same way.

Q.E.D.

**Theorem 4.2.** If $X$ is an irreducible smooth projective variety of dimension $d \geq 2$ in $\mathbb{P}^n$, $D$ is an ample divisor on $X$, then there is an $n_0 > 0$ such that for all $m \geq n_0$ and any $q \leq n + 1$ closed points $P_0, P_1, ..., P_{q-1}$ on $X$ in general position, a general member of $|mD|_q$ is irreducible and smooth, where $|mD|_q$ is the linear system of effective divisors in $|mD|$ passing through these $q$ points $P_s$, $s = 0, 1, ..., (q - 1)$.

**Proof.** Let $\phi$ be an embedding as in the proof of Theorem 4.1. Let $M$ be the set of homogeneous polynomials $h$ of degree $a > 1$ such that every hypersurface $H$ defined by $h$ contains the points $\phi(P_s) = w_s$, $s = 0, ..., q - 1$. By the proof of the Theorem 3.1, if $w \neq w_s$, then the natural map $\xi_w$ given in the proof from $M$ to $\mathcal{O}_{w,W}/\mathcal{M}_{w,X}$ is surjective and $\xi_{w_s}$ is surjective from $M$ to $\mathcal{M}_{w_s,W}/\mathcal{M}_{w_s,W}$.

Let

$$L = \{ f \in H^0(\mathcal{O}_X(n_0D)) \mid f(P_0) = 0 \},$$

where $n_0 = l^a$, then $\phi^*(M)$ is a subspace of $L$.

Since $\phi$ is an isomorphism between $X$ and $W$, the map $\xi_x$ from $L$ to $\mathcal{O}_{x,X}/\mathcal{M}_{x,X}$ is also surjective for every point $x \neq P_0$ in $X$ and $\xi_{P_0}$ is surjective from $L$ to $\mathcal{M}_{P_0,X}/\mathcal{M}_{P_0,X}$. Carrying out the same dimension counting, it is clear that the theorem holds for $|n_0D|_q$. The rest of the proof is the same as the proof of Theorem 4.1.

Q.E.D.
5. Linear System of Big Divisors

Let $X$ be an irreducible normal projective variety and $D$ a Cartier divisor on $X$. If for all $m > 0$, $H^0(X, \mathcal{O}_X(mD)) = 0$, then the $D$-dimension $\kappa(D, X) = -\infty$. Otherwise,

$$\kappa(D, X) = \text{tr.deg}_C \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)) - 1.$$ 

If $h^0(X, \mathcal{O}_X(mD)) > 0$ for some $m \in \mathbb{Z}$ and $X$ is normal, choose a basis $\{f_0, f_1, \cdots, f_n\}$ of the linear space $H^0(X, \mathcal{O}_X(mD))$, it defines a rational map $\Phi_{|mD|}$ from $X$ to the projective space $\mathbb{P}^n$ by sending a point $x$ on $X$ to $(f_0(x), f_1(x), \cdots, f_n(x))$ in $\mathbb{P}^n$. We define the $D$-dimension $\left(\mathbb{U}, \text{Definition 5.1}\right)$,

$$\kappa(D, X) = \max_{m \in \mathbb{Z}} \{\dim(\Phi_{|mD|}(X))\}.$$

From the definition, if $D$ is an effective divisor, we have $0 \leq \kappa(D, X) \leq d$, where $d$ is the dimension of $X$. We say that $D$ is a big divisor if $\kappa(D, X)$ is equal to the dimension of $X$.

**Theorem 5.1.** Let $X$ be an irreducible smooth projective variety of dimension $d \geq 2$ and $D$ an effective big divisor on $X$ such that $\Phi_{|nD|}$ defines a birational morphism. Then there is an open subset $U$ in $X$ such that for any point $P_0$ on $U$, a general member of $|nD|_{P_0}$ is smooth, where $|nD|_{P_0}$ is the linear system of effective divisors passing through the point $P_0$.

**Proof.** Since $\Phi_{|nD|}$ is a birational morphism, its image $W$ has dimension $d$ and there is an open subset $U$ in $X$ such that $U$ is isomorphic to $\Phi_{|nD|}(U)$.

By definition of $D$-dimension, $H^0(X, \mathcal{O}_X(nD))$ has $d$ algebraically independent nonconstant elements, where $\dim X = d$. Let $(f_0, f_1, \ldots , f_m)$ be a representation of $\Phi_{|nD|}$. Here $(f_0, f_1, \ldots , f_m)$ is a basis of $H^0(X, \mathcal{O}_X(nD))$. We may arrange the order such that $f_0, \ldots , f_{d-1}$ are algebraically independent. Then any element of $H^0(X, \mathcal{O}_X(nD))$ is of the form

$$f = \sum_{i=0}^{m} c_i f_i.$$

Let $P \neq P_0$ be a point on $U$ and $x_1, \ldots , x_d$ be the local coordinates at $P$ in an open subset of $V$, $P \in V$. The conditions that $f(P_0) = f(P) = 0$ and $f$ is singular at $P$ are determined by the system of the following linear equations.

$$f(P_0) = c_0 f_0(P_0) + c_1 f_1(P_0) + \ldots + c_m f_m(P_0) = 0$$

$$f(P) = c_0 f_0(P) + c_1 f_1(P) + \ldots + c_m f_m(P) = 0$$

$$\frac{\partial f}{\partial x_1}(P) = c_0 \frac{\partial f_0}{\partial x_1}(P) + c_1 \frac{\partial f_1}{\partial x_1}(P) + \ldots + c_m \frac{\partial f_m}{\partial x_1}(P) = 0$$

$$\frac{\partial f}{\partial x_d}(P) = c_0 \frac{\partial f_0}{\partial x_d}(P) + c_1 \frac{\partial f_1}{\partial x_d}(P) + \ldots + c_m \frac{\partial f_m}{\partial x_d}(P) = 0.$$
Consider the \((d+2)\) by \(m+1\) matrix \(A\)

\[
\begin{pmatrix}
  f_0(P_0) & f_1(P_0) & f_2(P_0) & \cdots & f_m(P_0) \\
  f_0(P) & f_1(P) & f_2(P) & \cdots & f_m(P) \\
  \frac{\partial f_0}{\partial x_1}(P) & \frac{\partial f_1}{\partial x_1}(P) & \frac{\partial f_2}{\partial x_1}(P) & \cdots & \frac{\partial f_m}{\partial x_1}(P) \\
  \frac{\partial f_0}{\partial x_2}(P) & \frac{\partial f_1}{\partial x_2}(P) & \frac{\partial f_2}{\partial x_2}(P) & \cdots & \frac{\partial f_m}{\partial x_2}(P) \\
  \frac{\partial f_0}{\partial x_3}(P) & \frac{\partial f_1}{\partial x_3}(P) & \frac{\partial f_2}{\partial x_3}(P) & \cdots & \frac{\partial f_m}{\partial x_3}(P) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \frac{\partial f_0}{\partial x_d}(P) & \frac{\partial f_1}{\partial x_d}(P) & \frac{\partial f_2}{\partial x_d}(P) & \cdots & \frac{\partial f_m}{\partial x_d}(P)
\end{pmatrix}
\]

Since \(f_0, \ldots, f_{d-1}\) are algebraically independent, the determinant of the Jacobian \(J(\frac{\partial f_i}{\partial x_j})_{0 \leq i, j \leq (d-1)}(P_0)\) is not identically zero (\([R],\) Chapter 6, Proposition 6A.4).

Because \(\Phi_{[nD]}\) is a birational morphism and an isomorphism on \(U\), the two vectors \((f_0(P_0), f_1(P_0), \ldots, f_m(P_0))\) and \((f_0(P), f_1(P), \ldots, f_m(P))\) are linearly independent since \(P_0 \in U\). Thus the rank of matrix \(A\) is at least 2 at any point \(P \neq P_0\) of \(X\) except these finite points. At \(P_0\), the Jacobian \(J(\frac{\partial f_i}{\partial x_j})_{1 \leq i, j \leq (d-1)}(P_0)\) has rank \(d\).

If the rank of \(A\) at \(P \in X\) is 2, then

\[
\frac{\partial f_i}{\partial x_j}(P) = 0, \text{ for all } i = 0, 1, \ldots, d-1; j = 1, \ldots, d.
\]

Let \(X_2\) be the set on \(U\) such that the rank of \(A\) at every point \(P\) of \(U\) is 2. Then the dimension of \(X_2\) is at most 0. And the dimension of the projective space \(C_2\) of solutions \((c_0, c_1, \ldots, c_m)\) is \(m-2\) in \(\mathbb{P}^m\). So \(\dim X_2 + \dim C_2 = m - 2\).

If the rank of \(A\) at the point \(P\) is 3, then three rows including the first two are linearly independent as vectors in \(k^m\). So other \(d-1\) rows can be written as linear combinations these two rows. There are \(d-1\) equations. Let \(X_3\) be the set of points in \(U\) such that at every point \(P\) of \(X_3\), rank of \(A\) is 3. Then the dimension of \(X_3\) is at most 1. Let \(C_3\) be the corresponding set of solutions of the system, then the dimension of \(C_3\) as a projective space is \(m - 3\). So \(\dim C_3 + \dim X_3 = m - 2\).

In general, if the rank of \(A\) is \(r\), \(2 \leq r \leq d+2\), then the rank of the Jacobian \(J(f_1, \ldots, f_d)\) is \(r-2\), which give \(d-(r-2)\) conditions. So the dimension of \(X_r\) is \(r-2\). Again we have \(\dim X_r + \dim C_r = (r-2) + (m-r) = m - 2\).

Let \(S_x\) be the set of effective divisors \(E\) in \([nD]_{P_0}\) such that \(E\) is singular at \(x\) and \(S = \{< x, E > | x \in X, E \in S_x \}\). Let \(L = \{f \in H^0(X, \mathcal{O}_X(nD)) | f(P_0) = 0\}\). As a projective space, \(L\) has dimension \(m-1\). Consider the projections \(p_1 : S \rightarrow X\) and \(p_2 : S \rightarrow L\). \(p_1\) is surjective. Take an irreducible component \(S_1\) of \(S\) such that \(p_1\) is surjective on \(S_1\).

There are finitely many affine open subsets \(\{U_i\}\) covering \(X\). Since the projective dimension of every fiber over \(U_i\) is \(m-d-2\), the dimension of \(S_1\) is at most \(m-d-2+d = m-2\). So every irreducible component of \(S\) has dimension at
most \( m - 2 \). For the projection \( p_2 : S \to L \), we have \( \dim p_2(S) \leq \dim S \leq \dim L - 1 \). But the projective dimension of \( L \) is \( m - 1 \), \( p_2(S) \) is a closed subset of \( M \). Thus a general member of \( |nD|_{\mathbb{P}^1} \) is smooth.

Q.E.D.

6. Applications

**Theorem 6.1.** Let \( M \) be a compact connected complex manifold biholomorphic to an irreducible smooth projective variety \( X \) in \( \mathbb{P}^n(\mathbb{C}) \). Then

1. the analytic inverse image of a general hyperplane passing through a point \( P_0 \) in \( \mathbb{P}^n(\mathbb{C}) \) is a connected complex manifold of codimension 1 on \( M \);
2. the analytic inverse image of a general hypersurface passing through \( q \leq n + 1 \) points \( P_0, ..., P_{q-1} \) in general position in \( \mathbb{P}^n(\mathbb{C}) \) is a connected complex manifold of codimension 1 on \( M \).

**Proof.** This is a direct consequence of Theorem 2.5 and Theorem 3.1.

Q.E.D.

**Theorem 6.2.** Let \( M \) be a compact connected complex manifold and \( f : M \to X \) is a proper holomorphic surjective map with maximal rank at every point of \( M \), where \( X \) is a smooth projective variety in \( \mathbb{P}^n(\mathbb{C}) \). Then

1. the analytic inverse image of a general hyperplane passing through a point \( P_0 \) in \( \mathbb{P}^n(\mathbb{C}) \) is a connected complex manifold of codimension 1 on \( M \);
2. the analytic inverse image of a general hypersurface passing through \( q \leq n + 1 \) points \( P_0, ..., P_{q-1} \) in general position in \( \mathbb{P}^n(\mathbb{C}) \) is a connected complex manifold of codimension 1 on \( M \).

Let \( L \) be a holomorphic line bundle on a complex manifold \( M \). If for every \( x \in M \), there is a section \( s \in H^0(M, L) \) such that \( s(x) \neq 0 \), then the basis of \( H^0(M, L) \) gives a holomorphic map \( f \) from \( M \) to \( \mathbb{P}^n(\mathbb{C}) \). If this map is an (analytic) isomorphism from \( M \) to its image, then \( L \) is very ample.

For any \( q \leq n + 1 \) points \( P_0, ..., P_{q-1} \) on \( M \), we say that they are in general position if their images under the map \( f \) given by basis of \( H^0(M, L) \) are in general position in \( \mathbb{P}^n(\mathbb{C}) \).

**Corollary 6.3.** Let \( M \) be a compact connected complex manifold and \( L \) a very ample holomorphic line bundle on \( M \). Then

1. for any point \( P_0 \) on \( M \), a general section \( s \in H^0(M, L) \) with \( s(P_0) = 0 \) gives a connected complex manifold of codimension 1 on \( M \);
2. for any \( q \leq n + 1 \) points \( P_0, ..., P_{q-1} \) in general position on \( M \), a general section \( s \in H^0(M, L^\otimes a) \) passing through these \( q \) points gives a connected complex manifold of codimension 1 on \( M \), where \( a > 1 \).

**Theorem 6.4.** Let \( Y \) be an irreducible smooth affine variety in \( k^n \) contained in an irreducible smooth projective variety \( X \) in \( \mathbb{P}^n \). Let \( P_0, P_1, ..., P_{q-1} \) be \( q \leq n + 1 \) closed points in general position in \( k^n \). Then for any degree \( a > 1 \), a general hypersurface in \( k^n \) passing through these \( q \) points are irreducible and smooth.
Proof. If all \( q \) points lie on \( Y \), it is a direct consequence of Theorem 3.1. In other cases, the proof is similar to the proof of Theorem 3.1.

Q.E.D.

The following proposition should not be new. The idea can be traced back to Bertini and Severi ([K1], Section 3 and 4). I cannot find a proof anywhere so I will write one here. It is interesting relationship between the linear system of a divisor and linear system of hypersurfaces.

**Proposition 6.5.** Let \( D \) be an effective Cartier divisor on an irreducible projective variety \( X \) in \( \mathbb{P}^N \) such that \( \kappa(D, X) \geq 1 \). Then there is a positive integer \( n_0 \) such that for all \( n \geq n_0 \), the linear system \( |nD| \), except for the divisor \( nD \), can be obtained by cutting out on \( X \) by a linear system of hypersurfaces and then removing some fixed components, which are the common components of all hypersurfaces in the system.

**Proof.** Since \( s = \kappa(D, X) \geq 1 \), the dimension of the linear system \( |nD| \) as a vector space grows like \( cn^s \), where \( c > 0 \) is a constant. Let \( \{f_0, f_1, ..., f_m\} \) be a basis of \( H^0(X, \mathcal{O}_X(nD)) \), then there are rational functions \( g_i \) on \( X \) such that \( \text{div}(f_i) = \text{div}(g_i) + nD \). Let \( g_i = h_i/h'_i \), where \( h_i \) and \( h'_i \) are homogeneous polynomials of the same degree in \( \mathbb{P}^N \). Any element \( g \) of \( L \) is a linear combination of its basis. Let \( \mathbb{C}(X) \) be the function field of \( X \), then the vector space \( H^0(X, \mathcal{O}_X(nD)) \) is isomorphic to the vector space (up to a constant)

\[
L = \{ g \in \mathbb{C}(X) | g = 0 \text{ or } \text{div}(g) + nD \geq 0 \}.
\]

Since \( |nD| \) is isomorphic to \( L \) module a constant ([U], Chapter II, Lemma 4.16), there are constants \( c_0, c_1, ..., c_n \) such that we can write any element \( E \in |nD| \) as follows

\[
E = \text{div}(\sum_{i=0}^{m} c_i g_i) + nD
\]

\[
= \text{div}(\sum_{i=0}^{m} c_i h_i) + nD
\]

\[
= \text{div}(\sum_{i=0}^{m} c_i \frac{h_0'...h'_{i-1}h_i h'_{i+1}...h'_m}{h_0'...h'_{i-1}h'_i h'_{i+1}...h'_m}) + nD
\]

\[
= \text{div}(\sum_{i=0}^{m} c_i h_0'...h'_{i-1}h_i h'_{i+1}...h'_m) - \text{div}(h_0'...h'_{i-1}h'_i h'_{i+1}...h'_m) + nD
\]

\[
= \text{div}(\sum_{i=0}^{m} c_i (\alpha_i)) - \text{div}(\beta) + nD
\]
where
\[ \alpha_i = h'_0 \ldots h'_{i-1} h_i h'_{i+1} \ldots h'_m \]
and
\[ \beta = h'_0 \ldots h'_{i-1} h'_{i} h'_{i+1} \ldots h'_m \]
are homogeneous polynomials in the projective space \( \mathbb{P}^N \). From the above formula we see that \( \beta \) defines a fixed hypersurface.

The equation proves the proposition.

Q.E.D.

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