1. Introduction

In [Pug67], Pugh proved a fundamental property of the periodic orbits of $C^1$ dynamical systems, called the $C^1$ closing lemma. Morally speaking, Pugh’s closing lemma states that nearly periodic points become periodic after a slight $C^1$-perturbation of the dynamical system. Precisely, it is stated as follows.
Theorem 1. [Pug67] Let $V$ be a $C^1$ vector-field on a closed manifold $X$ and let $x$ be a non-wandering point of $V$. Then there is a $C^1$ vector-field $V'$ that is $C^1$-close to $V$ such that $x$ is on a closed orbit of $V'$.

Note that a point $x$ is non-wandering if, for any open neighborhood $U$ of $x$ and every time $S$, there exists a $T \geq S$ such that $\phi^T(U) \cap U$ is non-empty, where $\phi$ is the flow of $V$. If $\phi$ preserves a volume form, then every point is non-wandering.

A key question in smooth dynamical systems (posed, for instance, by Smale [Sma98]) is whether or not Theorem 1 extends to the $C^\infty$ setting. However, in [Gut87] Gutierrez proved that any compact manifold $X$ containing an embedded punctured torus $\Sigma \subset X$ possesses a vector-field $V$ and a non-wandering point $p$ that does not become periodic under any small $C^\infty$-perturbation of $V$. This follows the work of Herman [Her79] in the Hamiltonian setting, where he proved that all sufficiently small smooth Hamiltonian perturbations of a Diophantine rotation $\phi$ of the two-torus $\mathbb{T}^2$ have no periodic orbits. These negative results suggested that, for general smooth diffeomorphisms and flows, there is no analogue of the closing lemma.

Recently, dramatic progress has been made for area preserving diffeomorphisms of surfaces and Reeb flows of contact 3-manifolds. In [Iri15], Irie used spectral invariants coming from embedded contact homology (ECH) [Hut10] to prove the $C^\infty$ closing lemma for Reeb flows on closed contact 3-manifolds. In fact, Irie’s proof implied a strong closing property.

Definition 1.1 ([Iri22]). A manifold $Y$ with contact form $\alpha$ satisfies the strong closing property if, for any non-zero smooth function $f : Y \to [0, \infty)$

there is a $t \in [0, 1]$ such that $(1 + tf)\alpha$ has a closed Reeb orbit passing through the support of $f$.

Theorem 2. [Iri15] Every closed 3-manifold with contact form $(Y, \alpha)$ has the strong closing property.

Strong versions of the closing lemma were later proven for Hamiltonian surface maps [AI16] and more generally, area preserving surface maps [CGPZ21, EH21].

ECH is a fundamentally low-dimensional theory, and so the methods in [Iri15] are not directly applicable to studying the dynamics of higher-dimensional symplectomorphisms or Reeb flows. On the other hand, ECH is part of a family of Floer theories collectively called symplectic field theory (or SFT) [EGH00], and other flavors of SFT (e.g. contact homology) generalize naturally to any dimension. In a recent work [Iri22], Irie described an abstract framework for proving strong closing properties using invariants satisfying formal properties in the spirit of SFT.

In this paper, we use contact homology to prove that the Reeb flow of any ellipsoid satisfies the strong closing property, as conjectured by Irie in [Iri22]. This is a first step towards applying the machinery of SFT to prove closing properties for more general classes of Reeb flows in higher dimensions.

![Diagram](image1.png)

Figure 1. Morally speaking, the strong closing property states that any positive perturbation of a Reeb flow supported in $U$ must produce a closed orbit $\gamma$ through $U$. 
1.1. **Spectral Gaps.** The strong closing property for Reeb flows is a consequence of an abstract criterion on contact homology. In order to explain this criterion, let us briefly review the structure of contact homology (for a detailed discussion, see §2).

The contact homology of a closed contact manifold \((Y, \xi)\) with contact form \(\alpha\) is a \(\mathbb{Z}/2\)-graded vector-space over \(\mathbb{Q}\), denoted by

\[
CH(Y, \xi) \quad \text{with a filtration} \quad CH^i(Y, \xi) \subset CH(Y, \xi) \quad \text{determined by} \ \alpha.
\]

If \(\alpha\) is non-degenerate (i.e. if the linearized Poincaré return map of every closed Reeb orbit of \(\alpha\) does not have 1 as eigenvalue) then \(CH(Y, \xi)\) can be computed as the homology of a dg-algebra freely generated by good Reeb orbits. The differential counts genus 0 holomorphic curves in \(\mathbb{R} \times Y\) with one puncture near \(+\infty \times Y\) and any number of punctures near \(-\infty \times Y\).

The contact homology algebra \(CH(Y)\) comes with the additional structure of \(U\)-maps, which can be constructed as follows. An abstract constraint \(P\) of codimension \(\text{codim}(P)\) is a graded map

\[
P : CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}[\text{codim}(P)].
\]

Here \(S^{2n-1}, \xi_{\text{std}}\) is the standard tight contact sphere. There is a filtered, graded map associated to any abstract constraint \(P\), denoted by

\[
U_P : CH(Y, \xi) \to CH(Y, \xi)[\text{codim}(P)].
\]

Intuitively, the \(U\)-map \(U_P\) counts holomorphic curves \(C\) in \(\mathbb{R} \times Y\) passing through a point \(p \in \Sigma\) in a small codimension 2 symplectic sub-manifold \(\Sigma\) of \(\mathbb{R} \times Y\), where the number of branches of \(C\) through \(p\) and order of tangency of \(C\) at \(\Sigma\) is determined by \(P\). Rigorously, \(U_P\) can be most easily constructed using the maps on contact homology induced by exact symplectic cobordisms (and this approach is related to the point constraint approach by Siegel [Sie19]).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A cartoon of a U-map curve in contact homology. In general, the tangency of the curve \(C\) at the sub-manifold \(\Sigma\) can be arbitrarily complex.}
\end{figure}

Floer homologies typically have associated spectral invariants that track the minimal filtration at which a particular homology class appears. In symplectic geometry, these invariants have become pivotal tools in the study of quantitative and dynamical questions (cf. [GH18, Hut10, Sie19, CGHS21]). There are spectral invariants associated to contact homology, denoted by

\[
s_{\sigma}(Y, \alpha) \quad \text{for each} \quad \sigma \in CH(Y, \xi).
\]

The \(U\)-map \(U_P\) decreases this spectral invariant, in the sense that

\[
s_{U_P \sigma}(Y, \alpha) \leq s_{\sigma}(Y, \alpha) \quad \text{for any} \ \sigma \ \text{and} \ P.
\]

In particular, we can formulate an invariant that measures the minimal gap between the spectral invariants of a class \(\sigma\) and \(U_P \sigma\).

**Definition 1.2.** The spectral gap of a contact homology class \(\sigma \in CH(Y, \xi)\) is given by

\[
\text{gap}_\sigma(Y, \alpha) := \inf_P \{s_{\sigma}(Y, \alpha) - s_{U_P \sigma}(Y, \alpha)\} \in [0, \infty).
\]
The contact homology spectral gap of a closed contact manifold $Y$ with contact form $\alpha$ is given by
\[
\text{gap}(Y, \alpha) := \inf_{\omega} \{ \text{gap}_\omega(Y, \alpha) \}.
\]

The criterion for the strong closing property using the spectral gap can now be stated as follows.

**Theorem 3.** Let $(Y, \xi)$ be a closed contact manifold with contact form $\alpha$, and suppose that
\[
\text{gap}(Y, \alpha) = 0.
\]

Then $(Y, \alpha)$ satisfies the strong closing property.

This criterion is formulated in abstract terms in [Iri22]. We will provide our own concrete discussion of this condition, along with properties of the spectral gap, in §2.

1.2. Spectral Gap Of Ellipsoids. Given the spectral gap framework discussed above, we are naturally lead to the following question.

**Question 4.** Let $(Y, \xi)$ be a closed contact manifold with non-trivial contact homology. Does the contact homology spectral gap vanish for any contact form $\alpha$?

Even in simple cases, computing the $U$-map involves a difficult analysis of $J$-holomorphic curves, and so Question 4 is extremely difficult. As a first step, Irie conjectured an affirmative answer to Question 4 in the following family of examples of contact manifolds.

**Example 1.3 (Ellipsoids).** An ellipsoid boundary $(\partial E, \lambda|_{\partial E})$ is a contact manifold $\partial E$ given as the boundary of an ellipsoid $E$ in $\mathbb{C}^n$ of the form
\[
E = \{ z \in \mathbb{C}^n : \langle z, Az \rangle \leq 1 \} \quad \text{where} \quad A \text{ is symmetric and positive definite.}
\]

The contact form $\lambda|_{\partial E}$ is the restriction of the standard Liouville form $\lambda$ on $\mathbb{C}^n$.
\[
\lambda = \frac{1}{2} \sum_{j=1}^{n} x_j dy_j - y_j dx_j.
\]

The Reeb vector-field $R$ is given by $R(z) = 2JAz$ where $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is multiplication by $i$.

\[\begin{array}{c}
\mathbb{C}^n \\
\includegraphics[width=0.5\textwidth]{ellipsoid.png}
\end{array}\]

**Figure 3.** Two different visualizations of the Reeb flow of an ellipsoid $E \subset \mathbb{C}^n$. On the left, a flow on the boundary of the higher-dimensional domain $E$. On the right, as the dynamics of $n$ independent harmonic oscillators (e.g. springs).

**Remark 1.4 (Harmonic Oscillator).** From the perspective of classical mechanics, the Reeb flow of an ellipsoid is the Hamiltonian dynamics of a harmonic oscillator on a fixed energy surface.

Indeed, up to a linear symplectomorphism, every ellipsoid is equivalent to one of the form
\[
E(a) = E(a_1, \ldots, a_n) := \{(z_1, \ldots, z_n) : \pi \cdot \sum_{i=1}^{n} \frac{|x_i|^2 + |y_i|^2}{a_i} \leq 1 \} \quad \text{where} \quad 0 < a_1 \leq \cdots \leq a_n.
\]
The Reeb flow on the boundary of $E(a)$ is simply the Hamiltonian flow of the Hamiltonian

$$H : \mathbb{R}^{2n} \to \mathbb{R} \quad \text{given by} \quad H(x, y) = \pi \cdot \sum_i \frac{|x_i|^2 + |y_i|^2}{a_i}.$$  

This is precisely the Hamiltonian for $n$ independent harmonic oscillators with periods $a_1, \ldots, a_n$.

Ellipsoid boundaries are some of the most well-studied contact manifolds (cf. [CGHS21, Sie19, GH18]) and have provided a useful testing ground for many conjectures. Irie’s conjecture from [Iri22] can be stated as follows.

**Conjecture 5** ([Iri22, Conjecture 5.1]). The boundary $\partial E$, $\lambda|_{\partial E}$ of an ellipsoid $E \subset \mathbb{C}^n$ has the strong closing property.

1.3. **Main Results.** The purpose of this paper is to prove Irie’s conjecture via the following spectral gap result.

**Theorem 6.** The boundary $\partial E$, $\lambda|_{\partial E}$ of an ellipsoid $E \subset \mathbb{C}^n$ has vanishing spectral gap, and thus satisfies the strong closing property.

Our approach to Theorem 6 has two parts: the periodic (or integer) case and the general case.

1.3.1. **Periodic Case.** An **integer ellipsoid** $E$ is an ellipsoid that is linearly symplectomorphic to a standard ellipsoid $E(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are all integers,

$$(a_1, \ldots, a_n) \in \mathbb{Z}^n.$$  

The Reeb flow of an ellipsoid with integer $a_i$ is periodic, i.e., the flow $q_T$ is the identity for some $T$. The period is given by the least common multiple of $a_1, \ldots, a_n$,

$$T = \text{lcm}(a_1, \ldots, a_n).$$  

The strong closing property is automatically satisfied for these flows, since every point goes through a periodic orbit of period bounded by $T$. We first prove that this property is reflected in contact homology as follows.

**Theorem 7.** Let $E = E(a) \subset \mathbb{C}^n$ be an ellipsoid with $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then, there is a contact homology class $\sigma \in CH(\partial E)$ and a $U$-map $U = U_p$ such that

$$s_{U}(\partial E, \lambda|_{\partial E}) = s_{\sigma}(\partial E, \lambda|_{\partial E}) = \text{lcm}(a_1, \ldots, a_n).$$

Theorem 7 is proven by a direct holomorphic curve calculation in contact homology. Let us briefly sketch the proof, as carried out in §4.

**Proof Sketch.** Note that $\partial E$ does not have a non-degenerate contact form. Instead, the contact form is Morse-Bott, and the set of closed Reeb orbits of a given period $T$ forms a sub-manifold $N_T \subset \partial E$ with quotient $S_T := N_T/\mathbb{R}$ by the Reeb flow.

Any Morse-Bott contact form admits an arbitrarily small non-degenerate perturbation so that every closed Reeb orbit $\gamma$ of period less than some fixed $L > 0$ corresponds to a pair

$$(T, p) \quad \text{where} \quad p \quad \text{is a critical point of a Morse function} \quad f : S_T \to \mathbb{R}.$$  

The period of $\gamma$ is also approximately $T$. Note that $S_T$ is an orbifold in general, and so morally we must work with orbifold Morse functions (in the appropriate sense). Moreover, gradient flow lines $\eta$ between critical points on a fixed Morse-Bott family $S_T$ lift to holomorphic cylinders $u_\eta$ in the symplectization of $\partial E$ between the corresponding orbits.

When $T$ is the period of the Reeb flow, i.e., the least common multiple of $a_1, \ldots, a_n$, the sub-manifold $N_T \subset \partial E(a)$ is simply the ellipsoid boundary $\partial E(a)$ itself and $S_T$ is a closed orbifold of
dimension $2n - 2$. If $p_+$ and $p_-$ are the unique maximum and minimum of a Morse function $f$ on $S_T$, then for any point $z \in S_T$, there is a unique gradient flow line

$$\eta : \mathbb{R} \to S_T \quad \text{from } p_+ \text{ to } p_- \text{ passing through } z.$$ 

This flow line lifts to a cylinder $u_\eta$ from the orbit $\gamma_+$ of $(T, p_+)$ to the orbit $\gamma_-$ of $(T, p_-)$ passing through a point whose projection to $S_T$ is $z$.

On the other hand, there is a $\mathcal{U}$-map $U_{p_0}$ that counts holomorphic curves satisfying a point constraint. Using intersection theory from [Sie11, MS19] and Wendl’s automatic transversality [Wen10], we prove that the cylinder $u_\eta$ is unique and transversely cut out. Therefore,

$$U_{p_0}(\gamma_+) \quad \text{has a non-zero } \gamma_- \text{ coefficient.}$$

The orbits $\gamma_+$ and $\gamma_-$ are both closed and non-exact in contact homology, and thus, the spectral invariant of $\sigma = [\gamma_+]$ and $U_{p_0}(\sigma)$ have the same action $T$, proving Theorem 7. □

Remark 1.5. In principle, one could use Morse-Bott formulations of contact homology (cf. [Bou02]) to compute the $\mathcal{U}$-map in Theorem 7 directly using gradient flow lines (and holomorphic cascades more generally). For the sake of completeness we provide here a direct analysis of the relevant moduli space and do not rely on the Morse-Bott formulation of contact homology.

1.3.2. General Case. The second step of our proof transfers the vanishing spectral gap of integer ellipsoids to general ellipsoids via the following approximation property.

**Proposition 8.** Let $(Y, \xi)$ be a closed contact manifold with a sequence of

- contact forms $\alpha_i$,
- homology classes $\sigma_i \in CH(Y, \xi),$
- and real numbers $\epsilon_i > 0$.

Suppose that, as $i \to \infty$, these sequences satisfy

$$\alpha_i \leq \alpha \leq (1 + \epsilon_i) \cdot \alpha_i \quad \text{and} \quad \epsilon_i \cdot s_{\sigma_i}(Y, \alpha_i) \to 0.$$ 

Then, the contact homology spectral gaps satisfy

$$\text{gap}(Y, \alpha) \leq \lim_{i \to \infty} \text{gap}_{\sigma_i}(Y, \alpha_i).$$ 

Proposition 8 can be proven in an entirely formal way from Definition 1.2. On the other hand, elementary results in Diophantine approximation can be used to prove the following result.

**Proposition 9.** Let $E$ be any ellipsoid. Then there exists a linear symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ and a sequence of rational ellipsoids $E_i$ of period $T_i$ such that

$$(1 - \epsilon_i) \cdot E_i \subset \phi(E) \subset (1 + \epsilon_i) \cdot E_i \quad \text{and} \quad \epsilon_i \cdot T_i \to 0.$$

Rational ellipsoids are rescalings of integer ellipsoids and so by Theorem 7, there exist classes

$$\sigma_i \in CH(\partial E_i, \lambda|_{\partial E_i}) \quad \text{with} \quad s_{\sigma_i}(\partial E_i, \lambda|_{\partial E_i}) = T_i.$$

Theorem 6 is therefore an immediate consequence of Propositions 8 and 9.

1.4. Generalizing Irie’s Conjecture. We expect the methods developed in this paper to apply to more general contact manifolds than ellipsoids. In particular, we now discuss several conjectures generalizing our results, which will be the subject of future work.

A contact form $\alpha$ on a contact manifold $(Y, \xi)$ is periodic if the Reeb flow $\varphi : \mathbb{R} \times Y \to Y$ satisfies

$$\varphi_T = \text{Id} \quad \text{for some time} \quad T > 0.$$ 

The closed Reeb orbits of a periodic contact form on a closed manifold $Y$ form closed orbifolds

$$S_T := N_T / \mathbb{R} \quad \text{where} \quad N_T := \{ y \in Y : \varphi_T(y) = y \}.$$
Every connected component $S$ of $S_T$ has an associated grading shift $gr_S$, given by the formula

$$gr_S := RS(S) - \frac{1}{2} \cdot \dim(S) \mod 2.$$  

Here $RS$ is the Robbin-Salamon index (see §2.1.3) of the linearized flow along any Reeb orbit in $S$. To each component $S$, we also associate a filtered, graded vector-space

$$V(S) := H_{\star + gr_S}(S; \mathbb{Q}).$$

We equip $V(S)$ with the homology grading shifted by $gr_S$ and the trivial filtration where $V_L(S)$ is 0 if $L$ is less than the period of the orbits in $S$ and $V(S)$ otherwise.

Our first conjecture provides a simple formula for contact homology in the periodic setting.

**Conjecture 10.** Let $(Y, \xi)$ be a closed contact manifold with a periodic contact form $\alpha$. Then

$$CH(Y, \xi) \cong \text{Sym} \left( \bigoplus_S V(S) \right)$$

as filtered, graded vector-spaces.

We expect that Conjecture 10 can be proven without Morse-Bott theory using Pardon’s formulation of contact homology [Par15] using an $S^1$-localization argument, analogous to the proof of the Arnold conjecture given in [Par16].

Our next conjecture generalizes Theorem 7 to general periodic contact forms.

**Conjecture 11.** Let $(Y, \xi)$ be a closed contact manifold with a periodic contact form $\alpha$ of period $T$. Then, there exists a class $\sigma \in CH(Y, \xi)$ and a $U$-map $U_P$ such that

$$s_{U_P}(Y, \alpha) = s_{\sigma}(Y, \alpha) = T.$$  

We expect that Conjecture 11 admits a similar proof to Theorem 7. Namely, one should lift a gradient flow line on the orbifold of closed orbits $S = S_T$ between the maximum and minimum of an (orbifold) Morse function on $S$ to a $J$-holomorphic cylinder counted by the $U$-map and then verify uniqueness and transversality using intersection theory and automatic transversality.

**Remark 1.6.** A key technical difficulty is our use of a flag of Reeb invariant contact sub-manifolds of $\partial E$ in Theorem 7. A possible analogue of this flag can be constructed using the orbifold Donaldon divisor construction [GMZ22] on the orbifold quotient $Y/\mathbb{R}$.

We conclude by noting that the spectral gap provides a criterion for periodicity; a similar statement for ECH spectral invariants was proved by Cristofaro-Gardiner–Mazzuchelli [CGM20]. The proof of this criterion appears in §2. It follows from a simple modification of the formal arguments used to prove Theorem 6.

**Theorem 12** (Periodicity Criterion). Let $(Y, \xi)$ be a closed contact manifold with contact form $\alpha$ where

$$\text{gap}_{\sigma}(Y, \alpha) = 0$$

for some $\sigma \in CH(Y, \xi)$.

Then the Reeb flow of $\alpha$ is periodic.

Note that Conjecture 11 states that (1.1) is also a necessary condition for periodicity. Therefore, a slightly weaker version of Conjecture 11 can be reformulated as follows.

**Conjecture 11’.** If $(Y, \xi)$ is a closed contact manifold with contact form $\alpha$, then the following are equivalent.

(a) The Reeb flow of the contact form $\alpha$ is periodic.

(b) There is a class $\sigma \in CH(Y, \xi)$ such that $\text{gap}_{\sigma}(Y, \alpha) = 0$.  

Outline. The paper is organized as follows. In Section 2 we review necessary preliminaries from contact homology. Section 3 contains the exposition of abstract constraints in contact homology and the spectral gap. It also contains the proofs of Theorem 3, Proposition 8 and Theorem 12. Section 4 contains an analysis of the moduli space counted by the U₁-map. In Section 5 we use this analysis to prove Theorem 7. We also provide a proof of Proposition 9 and thus conclude the proof of the closing property for all ellipsoids, as stated in Theorem 6.

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2. Contact Homology

In this section, we review the formalism of contact homology, which is a simple variant of symplectic field theory originally introduced by Eliashberg-Givental-Hofer [EGH00].

Remark 2.1. We will work with the transversality framework developed by Pardon [Par15], although all of the results discussed here should be independent of the specific transversality scheme.

2.1. Reeb Orbits. We start by discussing some preliminaries about Reeb dynamics. Throughout this section, we fix a contact manifold (Y,ξ) with a contact form α. We let R denote the Reeb vector-field of α and φ : R × Y → Y denote the Reeb flow.

2.1.1. Reeb Orbits. A closed or periodic Reeb orbit γ is a closed trajectory of the Reeb vector-field R, that is,

γ : R/TZ → Y satisfying \( \frac{dγ}{dt} = R ∘ γ \).

Here T is called the the period or action of γ, and for any Reeb trajectory one has

\[ T = A(γ) \quad \text{where} \quad A(γ) := \int_γ α. \]

Two Reeb orbits γ and η are equivalent if they are related by translation in t, that is, if

γ(t + t₀) = γ(t) for some t₀ ∈ R and all t ∈ R.

Any closed trajectory γ factors into a covering map φ and a simple (i.e., injective) closed orbit η.

\[ \mathbb{R}/TZ \xrightarrow{φ} \mathbb{R}/T'Z \xrightarrow{η} Y. \]

The covering multiplicity κγ of γ is the degree of the covering map φ, namely,

\[ κγ := \text{deg}(φ). \]

(2.1)

We will consider tuples of Reeb orbits, possibly with repetition, of the form

\[ Γ = (γ₁, \ldots, γₙ). \]

If Γ consists of m distinct orbits η₁, \ldots, ηₘ occurring with multiplicity \( μᵢ \) in the sequence and having covering multiplicity \( κᵢ \), respectively, then we let

\[ A(Γ) = \sum_𝑖 A(γᵢ), \quad μΓ = μ₁μ₂ \cdots μₘ, \quad \text{and} \quad κΓ = κ₁κ₂ \cdots κₘ. \]
2.1.2. Non-Degeneracy. Let \( N_T \) denote the set of fixed points of the time \( T \) Reeb flow.

\[
N_T := \{ y \in Y : \varphi_T(y) = y \} \subset Y.
\]

We say that \( N_T \) is Morse-Bott if it is a closed sub-manifold of \( Y \) with tangent bundle given by

\[
TN_T = \ker(d\varphi_T - \text{Id})|_{N_T}.
\]

A Morse-Bott family \( S \subset N_T/S^1 \) of Reeb orbits is connected component of the quotient \( N_T/S^1 \) for some period \( T \). Here \( S^1 \approx \mathbb{R}/T\mathbb{Z} \) acts on \( N_T \) by the Reeb flow. Note that any Morse-Bott family is automatically an effective orbifold. As a special case, a Reeb orbit \( \gamma \) is non-degenerate if

\[
\ker(d\varphi_T - \text{Id})|_{\gamma} = T\gamma = \text{span}(R).
\]

That is, if \( S = \gamma/S^1 \) is a 0-dimensional Morse-Bott family.

A contact form \( \alpha \) is said to be non-degenerate below action \( L \) if every closed Reeb orbit \( \gamma \) with \( A(\gamma) \leq L \) is non-degenerate. The form \( \alpha \) is called simply non-degenerate if every closed Reeb orbit is non-degenerate. Finally, \( \alpha \) is Morse-Bott if every closed Reeb orbit is in a Morse-Bott family.

2.1.3. Linearized Flow And Indices. A trivialization \( \tau : \gamma^*\xi \cong \mathbb{C}^{n-1} \) of \( \xi \) along a Reeb orbit \( \gamma \) is a trivialization of \( \gamma^*\xi \) as a symplectic vector-bundle, that is, \( \tau \) is a 1-parameter family of symplectic diffeomorphism \( \tau_{\gamma(t)} : \xi_{\gamma(t)} \to \mathbb{C}^{n-1} \). The linearized flow \( \Phi_{\tau,\gamma} \) associated to \( \gamma \) and \( \tau \) is the path of symplectic matrices

\[
\Phi_{\tau,\gamma} : [0, T] \to \text{Sp}(2n-2) \quad \text{given by} \quad \Phi_{\tau,\gamma}(t) = \tau_{\gamma(t)} \circ d\varphi_T^t|_{\xi} \circ \tau_{\gamma(0)}^{-1}.
\]

This path depends on the trivialization, but if \( \sigma \) and \( \tau \) are isotopic trivializations, then the paths \( \Phi_{\sigma,\gamma} \) and \( \Phi_{\tau,\gamma} \) are isotopic via paths \( \Phi^s : [0, T] \to \text{Sp}(2n-2) \) for \( t \in [0, 1] \) such that

\[
\text{rank}(\ker(\Phi^s(T) - \text{Id})) \quad \text{is constant in } s.
\]

The path \( \Phi_{\tau,\gamma} \) allows us to associate a Robbin-Salamon index (introduced by Robbin-Salamon in [RS93]) to any orbit with a trivialization.

**Proposition 2.2.** [[Gut14]] For each \( n \geq 1 \), there exists an integer valued Robbin-Salamon index of paths of symplectic matrices

\[
\text{RS} : \mathbb{C}^0([0, 1], \text{Sp}(2n)) \to \mathbb{Z},
\]

that is characterized by the following axioms.

(a) (Homotopy) If \( \Phi^s : [0, 1] \to \text{Sp}(2n) \) for \( s \in [0, 1] \) is a family of paths such that \( \Phi^s(0) \) and \( \Phi^s(1) \) are independent of \( s \), then

\[
\text{RS}(\Phi^0) = \text{RS}(\Phi^1).
\]

(b) (Additive) \( \text{RS} \) is additive under concatenation and direct sum, that is,

\[
\text{RS}(\Phi \ast \Psi) = \text{RS}(\Phi) + \text{RS}(\Psi) \quad \text{and} \quad \text{RS}(\Phi \oplus \Psi) = \text{RS}(\Phi) + \text{RS}(\Psi).
\]

(c) (Vanishing) If \( \Phi : [0, 1] \to \text{Sp}(2n) \) is such that \( \text{rank}(\Phi(t) - \text{Id}) \) is constant in \( t \), then \( \text{RS}(\Phi) = 0 \).

The Robbin-Salamon index generalizes the Conley-Zehnder index, in the sense that \( \text{RS}(\Phi) = \text{CZ}(\Phi) \) when \( \ker(\Phi(1) - \text{Id}) \) is 0-dimensional and \( \Phi(0) \) is the identity.

For sufficiently nice paths, the Robbin-Salamon index can be explicitly computed using crossings of the Maslov cycle. To be precise, for a given path \( \Phi \) and any \( t \in [0, 1] \), let \( \Gamma_t \) be the symmetric bilinear form on \( \ker(\Phi(t) - \text{Id}) \) given by

\[
\Gamma_t(v, w) = a_0 \left( \frac{d\Phi}{dt}(t)\Phi^{-1}(t)v, w \right).
\]
Let $\text{sign}(\Gamma_t)$ be $-1$ raised to the power of the signature of $\Gamma_t$. A crossing is a time $t \in [0, T]$ such that $\det(\Phi(t) - \text{Id}) = 0$. A crossing $t$ is non-degenerate if $\Gamma_t$ is non-degenerate. When every crossing time of $\Phi$ is non-degenerate, the Robbin-Salamon index is given by

$$\text{RS}(\Phi) = \frac{1}{2} \text{sign}(\Gamma_0) + \sum_{0 < t < T} \text{sign}(\Gamma_t) + \frac{1}{2} \text{sign}(\Gamma_T)$$

where the sum is over all crossings of $\Phi$.

The Robbin-Salamon index of a Morse-Bott family of Reeb orbits $S$ with respect to a trivialization $\tau$ is given by

$$\text{RS}_\tau(S) = \text{RS}(\Phi_{\tau, \gamma})$$

where $\Phi_{\tau, \gamma}$ is the linearized flow of any closed Reeb orbit $\gamma$ in $S$. If $\gamma$ is non-degenerate, then we define the Conley-Zehnder index of $\gamma$ with respect to $\tau$ as

$$\text{CZ}_\tau(\gamma) := \text{RS}(\Phi_{\tau, \gamma}).$$

Finally, the SFT grading $|\gamma|_\tau$ is given by

$$|\gamma|_\tau = (n - 3) + \text{CZ}_\tau(\gamma) \mod 2$$

where $\text{dim}(Y) = 2n - 1$.

2.2. Holomorphic Buildings. We next establish basic notation for $J$-holomorphic curves and, more generally, $J$-holomorphic buildings.

2.2.1. Symplectic Cobordisms And Liouville Domains. A symplectic cobordism $X : Y_+ \to Y_-$ between closed contact manifolds $(Y_+, \alpha_+)$ and $(Y_-, \alpha_-)$ is a compact symplectic manifold $(X, \omega)$ with boundary such that

$$\partial X = Y_+ \cup (-Y_-) \quad \text{and} \quad \omega|_{TY_\pm} = d\alpha_\pm.$$

The symplectic cobordism $X$ is exact if $\omega = d\lambda$ where $\lambda|_{TY_\pm} = \alpha_\pm$. A deformation of exact cobordisms is simply a smooth 1-parameter family of exact cobordisms $(X, \lambda_t) : Y_+ \to Y_-$ parametrized by $t \in [0, 1]$. Two exact cobordisms are deformation equivalent if they are (up to isomorphism) connected by a deformation.

Given symplectic cobordisms $X : Y_1 \to Y_2$ and $X' : Y_2 \to Y_3$, there is a well-defined composition given by

$$X \circ X' := X \cup_{Y_2} X'.$$

This cobordism inherits a symplectic cobordism structure (induced by a standard collar neighborhood of the boundary) and is exact if $X$ and $X'$ are exact.

Any symplectic cobordism $X$ can be completed to a non-compact, cylindrical manifold called the completion of $X$, denoted by

$$\hat{X} = (-\infty, 0] \times Y_- \cup_{Y_{+ \cup Y_+}} [0, \infty) \times Y_+.$$

As a special case, if $(Y, \alpha)$ is a contact manifold with contact form, we have a trivial exact cobordism

$$[a, b]\times Y : (Y, e^b \alpha) \to (Y, e^a \alpha) \quad \text{with Liouville form} \quad \lambda = e^r \alpha.$$

The completion of the trivial cobordism is called the symplectization and is denoted by $\hat{Y}$.

A Liouville domain $(W, \lambda)$ is an exact symplectic cobordism from $\partial W$ to the emptyset. If $X$ is an exact symplectic manifold with Liouville form $\theta$, then a symplectic embedding $\iota : W \to X$ from a Lioiville domain is called exact if

$$\iota^* \theta = \lambda.$$

More generally, $\iota$ is weakly exact if

$$[\lambda|_{\partial W} - \iota^* \theta|_{\partial W}] = 0 \in H^1(\partial W; \mathbb{R}).$$
For any weakly exact embedding $\iota : W \to X$, the Liouville form $\theta$ of $X$ is homotopic through Liouville forms in a neighborhood of $W$ so that $\iota$ is exact. After such a deformation, $X\setminus W$ is a symplectic cobordism

$$X : Y_+ \to Y_- \cup \partial W$$

and we may write $X = (X\setminus W) \circ (W \cup [0,1] \times Y_-)$ (up to deformation).

2.2.2. Homology Classes. Given a symplectic cobordism $X : Y_+ \to Y_-$ and sequences of Reeb orbits $\Gamma_\pm$ in $Y_\pm$, let

$$\Xi_\pm \subset Y_\pm$$

denote the 1-manifold in $Y_\pm$ given as the union of the underlying simple orbits of $\Gamma_\pm$. We may identify $\Xi_+ \cup \Xi_-$ as a sub-manifold of $\partial X$. We denote the following subset of the relative homology by

$$S(X; \Gamma_+, \Gamma_-) := \{ A \in H_2(X, \Xi_+ \cup \Xi_-) : \partial A = [\Gamma_+] - [\Gamma_-] \in H_1(\partial X) \}.$$

Given a homology class $A \in S(X; \Gamma_+, \Gamma_-)$ and a trivialization $\tau : \xi|_{\Gamma_+ \cup \Gamma_-} \simeq \mathbb{R}^{2n-2}$ of $\xi$ over the collections of Reeb orbits $\Gamma_+$ and $\Gamma_-$, there is a well-defined relative Chern number (Definition 5.1 in [Wen15])

$$c_1(A, \tau) \in \mathbb{Z}.$$

Moreover, given a choice of genus $g$, there is a well-defined Fredholm index given by

$$\text{ind}(A, g) := (n-3) \cdot (2 - 2g - |\Gamma_+| - |\Gamma_-|) + c_1(A, \tau) + \sum_{\gamma_+ \in \Gamma_+} CZ_\gamma(\gamma_+) - \sum_{\gamma_- \in \Gamma_-} CZ_\gamma(\gamma_-).$$

Note that if $X : Y_0 \to Y_1$ and $X' : Y_1 \to Y_2$ are symplectic cobordisms, then there is a map of homology classes in $X$ and $X'$ to homology classes in the composition $S(X; \Gamma_0, \Gamma_1) \times S(X'; \Gamma_1, \Gamma_2) \to S(X \circ X'; \Gamma_0, \Gamma_2)$ denoted by $(A, A') \mapsto A + A'$.

This map is associative. Moreover, the Chern class and Fredholm index are both additive with respect to this operation.

2.2.3. Complex Structures. A compatible almost complex structure on a contact manifold $(\gamma, \xi)$ is a bundle endomorphism $J : \xi \to \xi$ such that

$$J^2 = -\text{Id}$$

and $\alpha(-, J(-))|_\xi$ is a metric for any contact form $\alpha$.

Any such almost complex structure extends to an almost complex structure $\hat{J}$ on $\hat{Y}$ by

$$\hat{J}|_\xi = J \quad \text{and} \quad \hat{J}(\partial_r) = R \quad \text{where} \quad \hat{Y} = \mathbb{R}_r \times Y.$$

Likewise, for $J_\pm$ compatible almost complex structures on $Y_\pm$, a compatible almost complex structure $\hat{J}$ on the completion $\hat{X}$ of a symplectic cobordism $X : Y_+ \to Y_-$ is a bundle endomorphism $\hat{J} : T\hat{X} \to T\hat{X}$ such that

$$\hat{J}^2 = -\text{Id}, \quad \omega(-, \hat{J}(-)|_X \text{ is a metric on } X, \quad \hat{J}|_{(-\infty, 0] \times Y_-} = \hat{J}_-, \quad \text{and} \quad \hat{J}|_{[0, \infty) \times Y_+} = \hat{J}_+.$$

2.2.4. Holomorphic Maps. Fix a symplectic cobordism $X : Y_+ \to Y_-$ and a compatible almost complex structure $\hat{J}$ on $\hat{X}$. Consider a closed Riemann surface

$$(\Sigma, j) \quad \text{with a finite set of punctures} \quad P \subset \Sigma.$$

A $J$-holomorphic map from the punctured surface $\Sigma := \Sigma \setminus P$ to the symplectization $(\hat{X}, \hat{J})$ is a map $u : \Sigma \to \hat{X}$ satisfying $J \circ du = du \circ j.$
The $\lambda$-energy $E_\lambda(u)$ and area $A(u)$ of a $\bar{f}$-holomorphic map are defined as follows.

\[
E_\lambda(u) = \sup_{\phi_-} \int_{u^{-1}((-\infty,0) \times Y)} u^*(\phi_- ds \wedge \alpha_-) + \sup_{\phi_+} \int_{u^{-1}((0,\infty) \times Y)} u^*(\phi_+ ds \wedge \alpha_+)
\]

\[
A(u) = \int_{u^{-1}((-\infty,0) \times Y)} u^*(d\alpha_-) + \int_{u^{-1}(0,\infty) \times Y} u^*(d\alpha_+).
\]

Here the supremums are over compactly supported functions $\phi_- : (-\infty,0] \to [0,\infty)$ and $\phi_+ : [0,\infty) \to [0,\infty)$ with integral 1. The energy $E(u)$ is simply the sum

\[
E(u) = E_\lambda(u) + A(u).
\]

A holomorphic map is said to have finite energy if $E(u)$ is well-defined. Any finite energy, proper holomorphic map $u$ is asymptotic to sequences of closed Reeb orbits $\Gamma_\pm$ as $r \to \pm \infty$, $u \to \Gamma_\pm$ as $\pm \infty$. To be precise, for each puncture $p \in P$, there is a neighborhood $U$ of $p$ and a holomorphic chart $\phi : [0,\infty)_s \times (S^1)_t \cong U \setminus p \subset \Sigma$ such that

\[
u_\pm \circ u \circ \phi([0,\infty) \times S^1) \subset (0,\infty) \times Y_+ \quad \text{or} \quad u \circ \phi([-\infty,0) \times S^1) \subset (-\infty,0) \times Y_-,
\]

and

\[
\pi_R \circ u \circ \phi(s,-) \underset{s \to \pm \infty}{\longrightarrow} \pm \infty, \quad \pi_Y \circ u \circ \phi(s,-) \underset{s \to \pm \infty}{\longrightarrow} \gamma(\pm T^-) \quad \text{as a map} \quad S^1 \to Y_\pm.
\]

Here $\gamma$ is a closed Reeb orbit of $Y_\pm$, $\pi_R : \hat{X} \to \mathbb{R}$ is the projection to the $\mathbb{R}$-factor, and $\pi_Y : \hat{X} \to Y_\pm$ is the projection to the $Y_\pm$-factor where both projections are defined on the cylindrical ends of $\hat{X}$. Hence, Stokes theorem implies that

\[
A(u) = A(\Gamma_+) - A(\Gamma_-).
\]

Since the area of holomorphic maps is always non-negative, this implies that $A(\Gamma_+) \geq A(\Gamma_-)$. Finally, any holomorphic curve $u$ from $\Gamma_+$ to $\Gamma_-$ represents a class

\[
[u] \in S(X;\Gamma_+,\Gamma_-)
\]

acquired as the fundamental class of the composition $\pi \circ u$ where $\pi : \hat{X} \to X$ is the continuous map sending $X$ to itself and the ends $(-\infty,0) \times Y_-$ and $(0,\infty) \times Y_+$ to $Y_-$ and $Y_+$, respectively.

2.2.5. Asymptotic Markers and Matchings. Consider a cobordism $X : Y_+ \to Y_-$ and equip each simple Reeb orbit $\eta$ in $Y_+$ and $Y_-$ with a basepoint $b_\eta \in \eta \subset Y_\pm$.

Given a finite energy holomorphic map $u : \Sigma \to \hat{X}$ asymptotic to a Reeb orbit $\gamma$ at a puncture $p$, let $S_p$ denote the unit circle bundle at the puncture $p$

\[
S_p := (T_p \hat{\Sigma} \setminus 0)/\mathbb{R}_+.
\]

The complex structure on $\hat{\Sigma}$ induces an $S^1$-action on $S_p$ and there is a natural map of the form

\[
\pi_{u,p} : S_p \to \eta, \quad \text{where} \quad \eta \text{ is the simple orbit of } \gamma.
\]

An asymptotic marker $m_\gamma$ at a puncture $p \in P$ is a choice of element

\[
m_\gamma \in S_p, \quad \text{such that} \quad \pi_{u,p}(m_\gamma) = b_\eta.
\]

Given two holomorphic curves $u$ and $v$ asymptotic to $\gamma$ at punctures $p$ and $q$, respectively, a matching isomorphism $\mu$ between their ends is a map

\[
\mu : S_p \to S_q, \quad \text{such that} \quad \pi_{v,q} \circ \mu(\pi_{u,p}^{-1}(b_\eta)) = b_{\eta}.
\]

A holomorphic map $u$ is said to be equipped with asymptotic markers if each of its punctures is.

Note that a biholomorphism $\phi : \Sigma \to \Sigma'$ induces a map on the set of asymptotic markers and matching isomorphisms along the punctures. We denote these maps by $\phi_*$. 

2.2.6. Holomorphic Curves. Given an integer $g$, a homology class $A \in S(X; \Gamma_+, \Gamma_-)$, and tuples of Reeb orbits $\Gamma_\pm$ in $Y_\pm$, we have an associated moduli space of finite energy $J$-holomorphic curves

$$\mathcal{M}_{g,A}(X, J; \Gamma_+, \Gamma_-)$$

or

$$\mathcal{M}_{g,A}(\hat{X}, \hat{J}; \Gamma_+, \Gamma_-).$$

The points in this moduli space are $J$-holomorphic maps $u : \Sigma \to \hat{X}$ equipped with asymptotic markers that satisfy

$$g(\tilde{\Sigma}) = g, \quad [u] = A \quad \text{and} \quad u \to \Gamma_\pm \text{ as } r \to \pm \infty,$$

modulo the relation that $u$ is equivalent to $u'$ if there is a biholomorphism $\phi : \Sigma' \cong \Sigma$ such that

$$u' = u \circ \phi$$

and the induced $\phi^*$'s respect the asymptotic markers. We refer to such an equivalence class $u$ as a $J$-holomorphic curve (with asymptotic markers).

In the case where $\hat{X} = \hat{Y}$ is the symplectization of a contact manifold $Y$ and $\hat{J}$ is the symplectization of a compatible almost complex structure on $\xi$, the moduli space admits a natural $\mathbb{R}$-action given by $\mathbb{R}$-translation in $\mathbb{R} \times Y$. Then we adopt the notation

$$\mathcal{M}_{g,A}(Y, J; \Gamma_+, \Gamma_-) := \mathcal{M}_{g,A}(\mathbb{R} \times Y, \hat{J}; \Gamma_+, \Gamma_-)/\mathbb{R}.$$

**Example 2.3.** A trivial cylinder in a symplectization $(\hat{Y}, \hat{J})$ is any map $u : \mathbb{R} \times S^1 \to \hat{Y}$ of the form $u(s, t) = (s, \gamma(T \cdot t))$ where $\gamma$ is a closed Reeb orbit of period $T$.

2.2.7. Holomorphic Buildings. A $J$-holomorphic building $\bar{u}$ in a contact manifold $Y$ from $\Gamma_+$ to $\Gamma_-$ is a finite sequence of orbit tuples

$$\Gamma_1, \ldots, \Gamma_m \quad \text{with} \quad \Gamma_+ = \Gamma_1 \quad \text{and} \quad \Gamma_- = \Gamma_m,$$

and a sequence of finite energy $J$-holomorphic maps in $\hat{Y}$, called the levels of $\bar{u}$, denoted by

$$u_i : \Sigma_i \to \hat{Y} \quad \text{with} \quad u_i \to \Gamma_{i+1} \text{ at } -\infty \quad \text{and} \quad u_i \to \Gamma_i \text{ at } +\infty,$$

where each level $u_i$ is non-trivial, i.e., not a union of trivial cylinders. Moreover, the levels asymptotic to $\Gamma_i$ are equipped with asymptotic markers for all $i \in \{1, \ldots, m\}$, and the (pairs of) punctures of the levels asymptotic to orbits in $\Gamma_i$ for $i = 2, \ldots, m-1$ are equipped with matching isomorphisms.

Two buildings $\bar{u}$ and $\bar{v}$ are equivalent if, up to $\mathbb{R}$-translations, there is a biholomorphism of domains on each level respecting the holomorphic map, asymptotic markers and matching isomorphisms.

Any building $\bar{u}$ has a well-defined genus $g = g(\bar{u})$ and homology class $[\bar{u}] = A$ determined by gluing the curves along the matching punctures. In particular, the homology class is given by

$$A = \sum_i A_i \in S(Y; \Gamma_+, \Gamma_-).$$

The moduli space of equivalence classes of $J$-holomorphic buildings in $Y$ from $\Gamma_+$ to $\Gamma_-$ of genus $g$ and homology class $A$ is denoted by

$$\overline{\mathcal{M}}_{g,A}(Y, J; \Gamma_+, \Gamma_-).$$

More generally, a $J$-holomorphic building $\bar{u}$ in a symplectic cobordism $X : Y_+ \to Y_-$ from $\Gamma_+$ to $\Gamma_-$ is a sequence of Reeb orbit tuples

$$\Gamma_1^+, \ldots, \Gamma_a^+, \Gamma_1^-, \ldots, \Gamma_b^-$$

and a sequence of finite energy $J$-holomorphic maps of the form

$$u_i^\pm : \Sigma_i^\pm \to \hat{Y}_\pm, \quad \text{with} \quad u_i^\pm \to \Gamma_i^\pm \text{ at } +\infty \quad \text{and} \quad u_i^\pm \to \Gamma_{i+1}^\pm \text{ at } -\infty,$$

$$u_X : \Sigma_X \to \hat{X}, \quad \text{with} \quad u_X \to \Gamma_a^- \text{ at } +\infty \quad \text{and} \quad u_X \to \Gamma_1^- \text{ at } -\infty,$$
A simplified picture of a possible holomorphic building. Note that all of the holomorphic buildings and curves of interest in this paper will be genus 0.

Equipped with the same asymptotic markers and matching isomorphisms as in the symplectization case. Equivalence of a pair of buildings is defined as in the symplectization case, but we only quotient by the \( \mathbb{R} \)-direction in the symplectization levels. The moduli space of \( J \)-holomorphic buildings in \( X \) from \( \Gamma_+ \) to \( \Gamma_- \) of genus \( g \) and homology class \( A \) is denoted by

\[
\overline{\mathcal{M}}_{g,A}(X,J;\Gamma_+;\Gamma_-).
\]

The moduli spaces \( \overline{\mathcal{M}}_{g,A}(Y,J;\Gamma_+;\Gamma_-) \) and \( \overline{\mathcal{M}}_{g,A}(X,J;\Gamma_+;\Gamma_-) \) admit a Gromov topology described by [BEH+03, §9.1]. Moreover, both spaces are compact [BEH+03, §10.1].

2.2.8. Marked Moduli Spaces. \( J \)-holomorphic curves and buildings can be decorated to include marked points. More precisely, we can formulate a moduli space

\[
\mathcal{M}_{g,A,m}(X,J;\Gamma_+;\Gamma_-).
\]

The points in this moduli space are genus \( g \) \( J \)-holomorphic maps \( u : \Sigma \to \hat{X} \) in homology class \( A \) (with asymptotic markers) and an ordered tuple of \( m \) marked points \( s_1, \ldots, s_m \) in \( \Sigma \), modulo reparametrizations that respect the marked points. There is an evaluation map

\[
ev : \mathcal{M}_{g,A,m}(X,J;\Gamma_+;\Gamma_-) \to \hat{X}^m
\]

that takes an equivalence class \([u, s_1, \ldots, s_m]\) to the point \((u(s_1), \ldots, u(s_m)) \in \hat{X}^m\). One may also form a moduli space of buildings

\[
\overline{\mathcal{M}}_{g,A,m}(X,J;\Gamma_+;\Gamma_-)
\]

of \( J \)-holomorphic buildings where each level is a \( J \)-holomorphic curve with marked points and the total number of marked points over all the levels is \( m \). This moduli space is compact with respect to the topology in [BEH+03].

2.2.9. Parametric Moduli Spaces. Let \( P \) be a compact manifold with boundary and let \( J_P \) be a \( P \)-parameter family of compactible complex structures on \( X \), consisting of a compatible complex structure \( J_{p} \) for each \( p \in P \).

There is a parametric moduli space of \( J_P \)-holomorphic curves ranging over all \( p \), namely,

\[
\mathcal{M}_{g,A,m}(X,J_P;\Gamma_+;\Gamma_-) := \bigcup_{p \in P} \{p\} \times \mathcal{M}_{g,A,m}(X,J_p;\Gamma_+;\Gamma_-)
\]

That is, a point in this moduli space is a pair consisting of a point \( p \in P \) and a \( J_p \)-holomorphic curve (with marked points). Likewise, there is a compactified moduli space of buildings

\[
\overline{\mathcal{M}}_{g,A,m}(X,J_P;\Gamma_+;\Gamma_-) := \bigcup_{p \in P} \{p\} \times \overline{\mathcal{M}}_{g,A,m}(X,J_p;\Gamma_+;\Gamma_-)
\]
These parametric moduli spaces inherit evaluation maps \( \text{ev} \) and an additional continuous projection map

\[
\pi : \overline{\mathcal{M}_{g,A,m}}(X, J_P; \Gamma_+, \Gamma_-) \to P, \quad (p, u) \mapsto p
\]
given by projection to the \( P \)-factor. We will consider parametric moduli spaces of buildings with marked points in §4.4.

### 2.2.10. Generic Transversality.

Let \( \mathcal{M} \) be a compact manifold with boundary \( \partial \mathcal{M} \) and let \( J_P := \{ J_p \}_{p \in P} \) be a \( P \)-family of compatible almost complex structures on \( X \). We now briefly review some generic transversality results that are standard in the literature on SFT.

Recall that a point \((p, [u, s_1, \ldots, s_m])\) in the moduli space

\[
\mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)
\]
is called parametrically regular or parametrically transverse if the parametric linearized operator \( D_{u,p} \) incorporating both variations in the map \( u \) and variations in the parameter space \( P \), is surjective.

**Proposition 2.4.** (cf. [Wen15, Thm. 7.1 and Rmk. 7.4]) The set of parametrically regular points

\[
\mathcal{M}_{\text{reg}} \subset \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)
\]
is an open set and a smooth orbifold of dimension

\[
\dim(\mathcal{M}_{\text{reg}}) = v\dim(\mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-)) := \text{ind}(A, g) + \dim(P) + 2m.
\]

The local isotropy group at an orbifold point \((s, [u, s_1, \ldots, s_m]) \in \mathcal{M}_{\text{reg}}\) is given by

\[
\text{Aut}(p, u, s_1, \ldots, s_m) := \{ \phi : \Sigma \to \Sigma : j \circ d\phi = d\phi \circ j, u \circ \phi = u, u(s_i) = s_i \}.
\]

Finally, the evaluation map \( \text{ev} \) and projection map \( \pi \) are both smooth on \( \mathcal{M}_{\text{reg}} \).

As a special case, an unparametrized \( J \)-holomorphic curve \([u, s_1, \ldots, s_m]\) is simply called regular if it is parametrically regular with respect to the 0-parameter family \( J \).

Given a compact, closed submanifold \( Z \subset \hat{X}_m \), a parametrically regular \((p, [u_1, \ldots, u_m])\) is parametrically \( Z \)-regular if the evaluation map

\[
\text{ev} : \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) \to \hat{X}_m
\]
from the parametric moduli space is transverse to \( Z \) at \( u \).

**Proposition 2.5.** There is a comeager set \( J_{\text{reg}}(X, P) \) of \( P \)-families of compatible almost complex structures \( J_P \) such that the space of somewhere injective curves

\[
\mathcal{M}^i_{g,A,m}(X, J_P; \Gamma_+, \Gamma_-) := \{ u \in \mathcal{M}_{g,A,m}(X, J_P; \Gamma_+, \Gamma) : u \text{ is somewhere injective} \}
\]
consists of (parametrically) \( Z \)-regular curves.

The parametric regularity part of this result (without accounting for the evaluation map) is proven in [Wen15, Thm. 7.1-7.2, Rmk. 7.4]. The transversality of the evaluation map is proven in [Wen15, §4.6] for the case of closed curves in a closed symplectic manifold \( X \). The approach used in [Wen15, §4.6] is a standard one, using the Sard-Smale theorem, and can be adapted to the symplectization case with minimal modifications. Mainly, we need to work in the appropriate analytic setup, for example, by working with weighted Sobolev spaces instead of Sobolev spaces.
2.2.11. Buildings in Contact Homology. We primarily consider holomorphic buildings arising in contact homology. These are genus 0 buildings with a single positive end in an exact cobordism \((X, \lambda)\) or in (the symplectization of) a contact manifold \(Y\). These are curves in the moduli spaces
\[
\mathcal{M}_{0,A}(X, J; \gamma, \Gamma) \quad \text{and} \quad \mathcal{M}_{0,A}(Y, J; \gamma, \Gamma).
\]

**Lemma 2.6.** Let \(\bar{u}\) be a \(J\)-holomorphic building in one of the moduli spaces (2.3). Then, each level \(u_i\) is a disjoint union of curves
\[
v : \Sigma \to \hat{Y}_\pm \quad \text{or} \quad v : \Sigma \to \hat{X}
\]
where \(\Sigma\) is connected and of genus 0, and \(v\) has exactly one positive puncture.

**Proof.** Let \(\bar{u} = (u_1, \ldots, u_m)\) be a holomorphic building in (the symplectization of) \(Y\). The case of buildings in a cobordism is similar. Let \(\bar{u}_k\) denote the building
\[
\bar{u}_k = (u_1, \ldots, u_k) \quad \text{for} \quad 1 \leq k \leq m.
\]
We prove by induction on \(k\) that each building \(\bar{u}_k\) is connected and that each component curve \(v\) satisfies the conclusion of the lemma.

For the base case, note that every component \(v\) of every level \(u_i\) is genus 0, since otherwise the entire building would have positive genus. Moreover, since \(\hat{X}\) and \(\hat{Y}\) are both exact, every non-constant finite energy holomorphic curve \(v\) must have at least one positive puncture. Thus, the top level \(\bar{u}_1 = u_1\) is a connected genus 0 curve with one positive puncture.

For the induction case, assume that \(\bar{u}_k\) satisfies the induction hypothesis. By the above reasoning, each component \(v\) of the level \(u_{k+1}\) is genus 0 with at least one positive puncture. This positive puncture must connect to a negative puncture of \(\bar{u}_k\) so that \(\bar{u}_{k+1}\) is connected. If a component \(v\) has more than one positive puncture, then attaching \(v\) to the connected building \(\bar{u}_k\) contributes genus to \(\bar{u}_{k+1}\). Therefore, \(v\) has exactly one puncture. \(\square\)

**Remark 2.7.** In [Par15], a slightly different compactification of \(M = M_{g,A}(X, J; \Gamma_+, \Gamma_-)\) is used (and similarly for \(M_{g,A}(Y, J; \Gamma_+, \Gamma_-)\)). In Pardon’s compactification, if \(u_i : \Sigma_i \to \hat{Y}_\pm\) is a symplectization level of a building \(\bar{u}\) and \(\Sigma_i\) breaks into disconnected components \(\Sigma_{i,1}, \ldots, \Sigma_{i,k}\), then each component \(u_i|_{\Sigma_{i,j}}\) is separately regarded as a holomorphic curve modulo translation and any trivial cylinder components are eliminated.

The differences between the BEHWZ compactification of [BEH+03] and the Pardon compactification [Par15] will not be important for this paper. In particular, we will treat them as equivalent in Constructions 2.8 and 2.9 below.

However, we do note that any BEHWZ building corresponds to a unique Pardon building by only remembering the constituent maps of the building on each connected component of the levels (and eliminating the trivial components of every level). Conversely, any Pardon building can be lifted to a BEHWZ building by adding trivial levels and grouping the connected components of the building appropriately. In particular
\[
\mathcal{M} = \overline{\mathcal{M}} \quad \text{in the BEHWZ topology} \quad \iff \quad \mathcal{M} = \overline{\mathcal{M}} \quad \text{in the Pardon topology}.
\]

2.3. Basic Formalism. We can now discuss the basic construction of the contact dg-algebra of a contact manifold and the cobordism map of an exact cobordism. This construction was introduced in [EGH00]. Here we discuss the specific foundational setup of Pardon [Par15].

**Construction 2.8.** The contact dga of a closed contact manifold \(Y\) with a non-degenerate contact form \(\alpha\), denoted by
\[
(A(Y, \alpha), \partial_{J, \theta}) \quad \text{or more simply} \quad A(Y)
\]
is the filtered dg-algebra formulated as follows. Associate a generator $x_\gamma$ to each good Reeb orbit $\gamma$ (see [Par15, Definition 2.49] for a definition). Each generator $x_\gamma$ is given a standard SFT grading and action filtration
\begin{equation}
|x_\gamma| = -\text{CZ}(\gamma) - n + 3 \mod 2 \quad \text{and} \quad \mathcal{A}(x_\gamma) = \int_\gamma \alpha,
\end{equation}
respectively.

The algebra $A(Y, \alpha)$ is the graded-symmetric algebra freely generated by these generators,
\begin{equation}
A(Y, \alpha) := \text{Sym}_* [x_\gamma : \gamma \text{ is a good orbit}].
\end{equation}

The SFT-grading $| \cdot |$ and the action filtration $\mathcal{A}$ on $x_\Gamma = x_{\gamma_1} \ldots x_{\gamma_n}$ are given by
\[|x_\Gamma| := \sum |\gamma_i| \quad \text{and} \quad \mathcal{A}(x_\Gamma) := \sum \mathcal{A}(\gamma_i).
\]
We let $A^L(Y, \alpha) \subset A(Y, \alpha)$ denote the graded subspace given by
\[A^L(Y, \alpha) := \mathbb{Q} \langle x_\Gamma : \mathcal{A}(x_\Gamma) \leq L \rangle.
\]
The differential on $A(Y, \alpha)$ is the unique derivation such that, for any good orbit $\gamma$, we have
\[\partial_{f, \theta}(x_\gamma) := \sum_{A, \Gamma} \frac{\#_{\theta \mathcal{M}_{0, A}(Y ; \gamma, \Gamma)} \mu \cdot \kappa_{\Gamma}}{\mu_{\Gamma} \cdot \kappa_{\Gamma}} \cdot x_\Gamma.
\]
Here, $\#_{\theta \mathcal{M}_{0, A}(Y ; \gamma, \Gamma)}$ is a (virtual) point count of index 1 holomorphic buildings in the symplectization of $Y$ with one positive puncture at $\gamma$ and negative punctures at $\Gamma$ (see [Par15]). This count depends on a choice of the VFC data $\theta$. The sum is over all ordered lists $\Gamma$ of good orbits and all homology classes $A \in S(Y; \gamma, \Gamma)$ such that $\text{ind}(A, 0) = 1$.

**Construction 2.9.** The cobordism map of an exact cobordism $X: (Y_+, \alpha_+) \rightarrow (Y_-, \alpha_-)$, denoted
\[\Phi_{X, \lambda, f, \theta} : A(Y_+, \alpha_+) \rightarrow A(Y_-, \alpha_-),
\]
and more simply $\Phi_X$, is the unique filtered dg-algebra map such that, for any good closed orbit $\gamma$ of $Y_+$, we have
\[\Phi_{X, \lambda, f, \theta}(x_\gamma) := \sum_{A, \Gamma} \frac{\#_{\theta \mathcal{M}_{0, A}(X ; \gamma, \Gamma)} \mu \cdot \kappa_{\Gamma}}{\mu_{\Gamma} \cdot \kappa_{\Gamma}} \cdot x_\Gamma.
\]
Here $\#_{\theta \mathcal{M}_{0, A}(X ; \gamma, \Gamma)}$ is a (virtual) point count of index 0 holomorphic buildings in the completion of $X$ with 1 positive puncture at $\gamma$ and negative punctures at $\Gamma$ (see [Par15]), and the sum is over all ordered lists $\Gamma$ of good orbits in $Y_-$ and all homology classes $A \in S(X; \gamma, \Gamma)$ such that $\text{ind}(A, 0) = 0$.

**Remark 2.10.** In both Constructions 2.8 and 2.9, the virtual point count is equal to an actual (oriented) point count when the relevant moduli space $\mathcal{M}$ is compact (i.e. $\overline{\mathcal{M}} = \mathcal{M}$) and each holomorphic curve is transversely cut out (i.e. the relevant linearized $\partial$ operator is surjective). See [Par15, Thm. 1.1(iv)].

The main results of Pardon’s construction [Par15] can now be summarized as follows.

**Theorem 2.11.** [Par15] Let $(Y, \alpha)$ and $X: (Y_+, \alpha_+) \rightarrow (Y_-, \alpha_-)$ be as in Construction 2.8-2.9. Then the following hold.

(a) The map $\partial_{f, \theta} : A(Y) \rightarrow A(Y)$ is a filtered differential. That is,
\[\partial_{f, \theta}^2 = 0 \quad \text{and} \quad \mathcal{A}(\partial_{f, \theta}(x)) \leq \mathcal{A}(x).
\]
(b) The map $\Phi_{X, \lambda, f, \theta} : A(Y_+) \rightarrow A(Y_-)$ is a filtered chain map. That is,
\[\Phi_{X, \lambda, f, \theta} \circ \partial_{f, \theta} = \partial_{f, \theta} \circ \Phi_{X, \lambda, f, \theta} \quad \text{and} \quad \mathcal{A}(\Phi_{X, \lambda, f, \theta}(x)) \leq \mathcal{A}(x).
\]
Furthermore, $\Phi_{X, \lambda, f, \theta}$ is independent of $f$ and $\theta$ up to filtered chain homotopy.
(c) The composition of cobordism maps is filtered homotopic to the cobordism map of the composition,

\[ \Phi_{X;X'} \cong \Phi_X \circ \Phi_{X'}. \]

(d) If \((X, \lambda)\) is the trivial cobordism \(X = [a, b] \times Y, \lambda = e^t \alpha, J\) is a translation invariant almost complex structure induced by a compatible complex structure on \(\xi\), and \(\theta\) is any VFC data, then

\[ \Phi_{X, \lambda, J, \theta} : H(A(Y), \partial J, \theta) \to H(A(Y), \partial J, \theta) \]

is the identity map on the level of unfiltered graded dg-algebras.

By Theorem 2.11, we can now define contact homology as an invariant of contact manifolds.

**Definition 2.12.** The contact homology \(CH(Y)\) of a closed contact manifold \((Y, \xi)\) is given by

\[ CH(Y) := H(A(Y), \partial J, \theta), \quad \text{for any choice of } \alpha, J, \theta. \]

The map \(\Phi_X : CH(Y_+) \to CH(Y_-)\) induced by an exact cobordism \(X : Y_+ \to Y_-\) is similarly defined with respect to any choice of \(J, \theta\). Any choice of contact form \(\alpha\) induces a filtration of \(CH(Y)\) by sub-spaces

\[ CH^L(Y) \quad \text{or} \quad CH^L(Y, \alpha) \quad \text{for any} \quad L \in [0, \infty) \]

Here \(CH^L(Y, \alpha)\) is the image of the map

\[ H(A^L(Y)) \to H(A(Y)) = CH(Y) \]

if the contact form \(\alpha\) is non-degenerate. In general, the \(CH^L(Y)\) is defined as the colimit

\[ CH^L(Y, \alpha) = \colim_{\beta} CH^L(Y, \beta) \]

where the colimit is taken over all non-degenerate contact forms \(\beta = f \alpha\) with \(f > 1\) pointwise. The cobordism maps \(\Phi_X\) are filtered with respect to this filtration.

**Remark 2.13.** More generally, given a contact form \(\alpha\) on \(Y\) that is non-degenerate below action \(T\) and any \(L < T\), we can still define the graded algebra

\[ A^L(Y) := \mathbb{Q} \langle x^L : A(\Gamma) \leq L \rangle. \]

We may equip this algebra with a differential \(\partial J, \theta\) given a choice of compatible complex structure and VFC data. Likewise, if \(X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)\) are non-degenerate below action \(T\), then we have a cobordism dg-algebra map

\[ \Phi_X : A^L(Y_+) \to A^L(Y_-) \]

for any \(L \leq T\), well-defined up to filtered chain homotopy. In particular, \(CH^L(Y, \alpha)\) is the image of the map

\[ H(A^L(Y, \alpha)) \to CH(Y) \]

Moreover, for any contact forms \(\alpha_+\) and \(\alpha_-\) that are non-degenerate up to action \(T\), the following diagram commutes.

\[
\begin{array}{ccc}
H(A^L(Y_+)) & \xrightarrow{\Phi_X} & H(A^L(Y_-)) \\
\downarrow & & \downarrow \\
CH^L(Y_+) & \xrightarrow{\Phi_X} & CH^L(Y_-)
\end{array}
\]

(2.6)
2.4. The Tight Sphere. We now calculate the contact homology algebra of the standard sphere, which is a key example in later constructions.

Consider \( C^n \) equipped with the standard Liouville form and associated Liouville vector-field

\[
\lambda_{\text{std}} = \frac{1}{2} \sum_i x_i dy_i - y_i dx_i, \quad \text{and} \quad Z_{\text{std}} = \frac{1}{2} \sum_i x_i \partial_{x_i} + y_i \partial_{y_i}.
\]

**Definition 2.14.** A star-shaped domain \( W \subset C^n \) is an embedded Liouville sub-domain of \( C^n \). Equivalently, \( W \) is a codimension zero submanifold with smooth the boundary that is transverse to the Liouville vector field \( Z, Z \cap \partial W \).

Every pair of star-shaped domains \( W \) and \( W' \) are equivalent through a canonical deformation, i.e., by deforming the boundary along the radial direction. In particular, the contact boundaries are all contactomorphic to the standard tight sphere.

**Definition 2.15.** The standard tight sphere \( (S^{2n-1}, \xi_{\text{std}}) \) is the unit sphere \( S^{2n-1} \subset C^n \) equipped with the contact structure \( \xi_{\text{std}} = \ker(\lambda_{\text{std}}|_{S^{2n-1}}) \).

If \( W \subset W' \) is an inclusion of star-shaped domains, then the exact symplectic cobordism

\[
X : (\partial W', \lambda|_{\partial W'}) \to (\partial W, \lambda|_{\partial W}) \quad \text{given by} \quad X := W \setminus W
\]

is isomorphic (as an exact cobordism) to a cylindrical cobordism \( X : (S^{2n-1}, \alpha) \to (S^{2n-1}, \beta) \) for a pair of contact forms \( \alpha \) and \( \beta \) on \( (S^{2n-1}, \xi) \).

**Example 2.16.** Choose a sequence of rationally independent, positive real numbers

\[0 < a_1 \leq a_2 \leq \cdots \leq a_n.\]

The standard ellipsoid \( E = E(a_1, \ldots, a_n) \subset C^n \) is the star-shaped domain given by

\[E(a_1, \ldots, a_n) = \{ z \in C^n : \sum_{i} \frac{\pi}{a_i} \cdot |z_i|^2 \leq 1 \}.\]

The Reeb dynamics on \( \partial E \) is very explicit and easy to determine (cf. [GH18, §2.1]). Specifically, there are exactly \( n \) non-degenerate, simple, closed Reeb orbits given by

\[\gamma_i = \partial E \cap C_i \quad \text{for} \quad i = 1, \ldots, n.\]

Here \( C_i \subset C^n \) is the \( i \)th complex axis in \( C^n \). Every Reeb orbit \( \gamma \) is an iterate of one of these orbits. The action and Conley-Zehnder index of an orbit \( \gamma = \gamma_i^j \) is given by

\[A(\gamma) = j \cdot a_i \quad \text{and} \quad CZ(\gamma) = n - 1 + 2 \cdot \#\{\text{orbits } \eta \text{ of } \partial E : A(\eta) \leq A(\gamma)\}.\]

Note that the Conley-Zehnder index is well-defined without reference to a trivialization, since \( c_1(S^3, \xi_{\text{std}}) = 0 \) and \( \tau_1(S^3) = 0 \).

**Lemma 2.17.** The contact homology algebra \( CH(S^{2n-1}, \xi_{\text{std}}) \) is isomorphic to a graded-symmetric algebra freely generated by generators \( x_k \) of grading \( -2n - 2 - 2(k - 1) \) for each \( k \geq 1 \), namely,

\[CH(S^{2n-1}, \xi_{\text{std}}) \cong \text{Sym}_* [x_k : k \geq 1].\]

**Proof.** Consider \( S^{2n-1} \) equipped with the contact form induced as the boundary of an irrational ellipsoid \( E(a_1, \ldots, a_n) \). The (cohomological) SFT grading of a closed Reeb orbit \( \gamma \) is given by

\[|\gamma| = -2n - 2 - 2 \cdot \#\{\text{orbits } \eta \text{ of } \partial E : A(\eta) \leq A(\gamma)\}.\]

In particular, the SFT grading is even for all generators of \( A(\partial E) \) and the differential is trivial. \( \square \)

In §5, we will require the following property of cobordism maps given by inclusion.

**Lemma 2.18.** Let \( E(a) \subset E(b) \) be two ellipsoids and consider the map \( \Phi : A(\partial E(a)) \to A(\partial E(b)) \) corresponding to the cobordism \( E(b) \setminus E(a) \). Then, \( \Phi \) is an isomorphism on the chain level and its inverse is word-length non-decreasing.
Proof. The exact cobordism $E(b) \setminus E(a)$ is isomorphic to a cylindrical cobordism, and so the map

$$\Phi : A(\partial E(a)) \to A(\partial E(b))$$

is a quasi-isomorphism inducing the natural isomorphism on homology. Since the differentials of $A(\partial E(a))$ and $A(\partial E(b))$ are trivial, $\Phi$ is in fact an isomorphism of dg-algebras. Moreover, by Theorem 2.11(c)-(d), the inverse $\Phi^{-1}$ is the cobordism map induced by $E(b) \setminus E(c \cdot a)$ for any $c > 0$ sufficiently small.

It remains to show that cobordism maps between ellipsoids are word-length non-decreasing. Since cobordism maps are algebra maps, this is equivalent to having a non-constant generator being mapped to the constants. In the case of the ellipsoids this is impossible, since the constants have grading zero, the non-constant generators lie in positive degrees and the cylindrical cobordism maps preserve the $\mathbb{Z}$-grading. □

3. Spectral Gaps

In this section, we discuss contact homology spectral gaps and related structures, including abstract constraints, constrained cobordism maps and spectral invariants.

3.1. Abstract Constraints. An abstract constraint provides a purely homological tool for tracking the ways in which a holomorphic curve can be tangent to (or asymptotic to) a set of points in a symplectic cobordism. Rigorously, we have the following definition.

Definition 3.1. An abstract constraint $P$ in contact homology with $m$ points, dimension $n$ and codimension $\text{codim}(P) = k$ is a degree $k$ map

$$P : \bigotimes_{i=1}^{m} CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}[k],$$

or equivalently, a cohomology class $P \in CH(\bigcup_{i=1}^{m} S^{2n-1})$ of grading $\text{codim}(P) = k$.

Example 3.2 (Empty Constraint). The empty constraint $P_{\emptyset}$ is the codimension 0 map given by

$$P_{\emptyset}(1) = 1 \quad \text{and} \quad P_{\emptyset}(x) = 0 \quad \text{if} \quad |x| > 0.$$ 

Alternatively, $P_{\emptyset}$ is the unique $\mathbb{Z}$-graded algebra map $CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}$.

Example 3.3 (Tangency Constraints). It follows from Lemma 2.17 that

$$\ker(P_{\emptyset})/\ker(P_{\emptyset})^2 \cong \bigoplus_{k=0}^\infty \mathbb{Q}[2n - 2 + 2k].$$

In particular, there is (up to multiplication by a non-zero constant) a unique surjective map

$$\Pi_k : \ker(P_{\emptyset})/\ker(P_{\emptyset})^2 \to \mathbb{Q}[2n - 2 + 2k] \quad \text{for each} \quad k \geq 0.$$ 

Therefore, we have an abstract constraint (well-defined up to multiplication by a constant)

$$P_k : CH(S^{2n-1}, \xi_{\text{std}}) = \mathbb{Q} \oplus \ker(P_{\emptyset}) \to \ker(P_{\emptyset})^2 \to \ker(P_{\emptyset})/\ker(P_{\emptyset})^2 \xrightarrow{\Pi_k} \mathbb{Q}[2n - 2 + 2k].$$

Remark 3.4. As observed by Siegel [Sie19, §5.5], the constraint $P_k$ coincides with the map acquired by counting genus 0 holomorphic curves in a star-shaped domain, with one positive puncture, passing through a point $p$ and tangent to a local divisor $D$ through $p$ to order $k$.

In §5, we will use a version of this fact in the proof of our main results. We also give an alternate argument for a weak version of this correspondence (for certain curves counted in constrained cobordism maps and counts of cylinders passing through a point) in §4.4.
Example 3.5 (Dual Constraints). Let $E$ be an irrational ellipsoid and let $\Xi$ be an orbit tuple on $\partial E$. There is a dual abstract constraint

$$P_\Xi : CH(\partial E) \simeq CH(S^{2n-1}, \xi_{\text{std}}) \to \mathbb{Q}[\|\Xi\|]$$

determined by $\Xi$. This is defined in the usual way, with

$$P_\Xi(x_\Gamma) = 1 \text{ if } \Gamma = \Xi \quad \text{and} \quad P_\Xi(x_\Gamma) = 0 \quad \text{otherwise}$$

As a special case, the abstract constraint $P_k$ in Example 3.3 coincides with $P_\gamma$ where $\gamma$ is the unique closed orbit of $E$ with $|\gamma| = 2n - 2 + 2k$.

3.2. Constrained Cobordism Maps. By using abstract constraints, we can formulate a generalization of the cobordism maps in contact homology, which morally counts curves satisfying a number of tangency constraints.

Definition 3.6. Let $- : Y \to \bar{Y}$ be a connected exact cobordism and let $\%$ be an abstract constraint. The $\%$-constrained cobordism map

$$\Phi_{X,P} : A(Y_+) \to A(Y_-)[\text{codim}(P)]$$

is the filtered chain map of degree $\text{codim}(P)$ constructed by the following procedure. Choose a star-shaped domain $W_i$ for $i = 1, \ldots, m$ and an embedding

$$\iota : W = W_1 \cup \cdots \cup W_m \to \text{int}(X).$$

Since $W_i$ are star-shaped domains, $\iota$ is automatically weakly exact (see §2.2.1). Thus we may assume (after deformation) that $\iota$ is exact and $X \setminus W$ is an exact cobordism. Also, choose a cochain $\pi_{\partial W,P}$ representing $P$, i.e., a chain map

$$\pi_{\partial W,P} : A(\partial W) \cong A(\bigcup_m S^{2n-1}) \to \mathbb{Q}[\text{codim}(P)]$$

which is in the cohomology class $P$. Then, we define $\Phi_{X,P}$ to be the composition

$$A(Y_+) \xrightarrow{\Phi_{X,W}} A(Y_- \cup \partial W) = A(Y_-) \otimes A(\partial W) \xrightarrow{\text{Id} \otimes \pi_{\partial W,P}} A(Y_-) \otimes \mathbb{Q}[\text{codim}(P)].$$

Definition 3.7. Let $P$ be an abstract tangency constraint and $Y$ be a connected, closed contact manifold. The $U$-map

$$U_P : A(Y) \to A(Y)[\text{codim}(P)].$$

is the $P$-constrained cobordism map $\Phi_{X,P}$ where $X = [0, 1] \times Y$ is the trivial cobordism.

![Figure 5](image-url) A cartoon of (a) an abstract constraint $P$, visualized as a weighting of the orbits on the sphere and (b) the curves counted in constrained cobordism maps, asymptotic to the orbits with non-zero weights under $P$.

Remark 3.8. The constrained cobordism maps $\Phi_{X,P}$ are not dg-algebra maps in general. However, some constrained cobordism maps are compatible with the algebra structure in other ways (see Lemma 3.12).

Lemma 3.9. The cobordism maps $\Phi_{X,P}$ are well-defined up to filtered chain homotopy.
Proof. Choose a disjoint union $W = W_1 \cup \cdots \cup W_m$ of star-shaped domains $U_i$, an embedding $W \subset X$, Floer data $(J, \theta)$ on $X \setminus W$ and a cochain representative $\pi_{\partial W, p}$ of $P$. We adopt the notation
\[
\Phi_{X, p, W, j, \theta} := (\text{Id}_{Y_\text{n}} \otimes \pi_{\partial W, p}) \circ \Phi_{X, j, \theta}.
\]
Since chain homotopy is a closed relation under composition, $\Phi_{X, p, W, j, \theta}$ is independent of the choice of $\pi_{\partial W, p}$ up to filtered chain homotopy. Note that we are viewing $\pi_{\partial W, p}$ as a filtered map by equipping $\mathbb{Q}[\text{codim}(P)]$ with the trivial filtration.

To show independence of $W$ and $(j, \theta)$ up to filtered homotopy, we start by considering two special cases and then move on to address the general case.

**Case 1.** Let $\iota : [0,1] \times W \to X$ be a family of symplectic embeddings. This family $\iota$ induces a family of exact symplectic cobordisms
\[
X \setminus \iota(W) : Y_+ \to Y_- \cup \partial W
\]
By Theorem 2.11(b), the induced cobordism maps of $X \setminus \iota(W)$ and $X \setminus \iota(W)$ are homotopic (for any choices of Floer data). Since filtered homotopy is a closed relation under composition with filtered chain maps, we see that
\[
\Phi_{X, p, \iota(W), j, \theta} = (\text{Id}_{Y_\text{n}} \otimes \pi_{\partial W, p}) \circ \Phi_{X, \iota(W), j, \theta} \simeq (\text{Id}_{Y_\text{n}} \otimes \pi_{\partial W, p}) \circ \Phi_{X, \iota(W), j', \theta'} = \Phi_{X, p, \iota(W), j', \theta'}
\]

**Case 2.** Let $V = V_1 \cup \cdots \cup V_m$ be a collection of star-shaped domains with inclusions $V_i \subset W_i$ that are strictly exact, i.e., that intertwine the Liouville forms. Let $W \setminus V : \partial W \to \partial V$ be the difference cobordism. Consider the cobordism map
\[
\Phi_{W \setminus V, j, \varphi} : A(\partial W) \to A(\partial V)
\]
for some choice of Floer data $(I, \varphi)$ on $W \setminus V$. We may choose the cochains $\pi_{\partial W, p}$ and $\pi_{\partial V, p}$ so that
\[
\pi_{\partial W, p} \simeq \pi_{\partial V, p} \circ \Phi_{W \setminus V, j, \varphi}.
\]
On the other hand, by Theorem 2.11(c), we have
\[
\Phi_{X \setminus V, j, \theta} \simeq \Phi_{(Y_+ \times [0,1]) \cup W \setminus V} \circ \Phi_{X \setminus W, j', \theta'} = (\text{Id}_{Y_\text{n}} \otimes \Phi_{W \setminus V, j, \varphi}) \circ \Phi_{X \setminus W, j', \theta'}
\]
for appropriate choices of Floer data $(J, \theta)$ on $X \setminus V$ and $(J', \theta')$ on $X \setminus W$. Therefore,
\[
\Phi_{X, p, V, j', \theta'} = (\text{Id}_{Y_\text{n}} \otimes \pi_{\partial V, p}) \circ \Phi_{X \setminus V, j', \theta'} \simeq (\text{Id}_{Y_\text{n}} \otimes \Phi_{W \setminus V, j, \varphi}) \circ (\text{Id}_{Y_\text{n}} \otimes \Phi_{X \setminus W, j, \theta}) \circ \Phi_{X \setminus W, j', \theta'} \simeq (\text{Id}_{Y_\text{n}} \otimes \pi_{\partial V, p}) \circ \Phi_{X, p, V, j, \theta} = \Phi_{X, p, V, j, \theta}.
\]

**General Case.** Let $W = W_1 \cup \cdots \cup W_m$ and $V = V_1 \cup \cdots \cup V_m$ be any two choices of $m$ disjoint star-shaped domains in $X$. We may choose star-shaped domains $E_i$ (e.g. sufficiently small ellipsoids) that include into $E_i \subset W_i$ and $E_i \subset V_i$, and such that the embeddings
\[
E_i \to W_i \to X \quad \text{and} \quad E_i \to V_i \to X
\]
are homotopic through a homotopy of embeddings $\iota : [0,1] \times E_i \to X$. Thus, by Cases 1 and 2, $\Phi_{X, p, W, j, \theta}$ and $\Phi_{X, p, V, j', \theta'}$ are filtered chain homotopic (for any Floer data).

As mentioned above, an abstract constraint $P$ is an assignment of numerical weights to the Reeb orbits on the boundary of a star-shaped domain $W$. Constrained cobordism maps are acquired by deleting $W$ from a cobordism $X$ and counting curves with ends on $\partial W$ with non-zero weights.

Let us make this intuition precise in a specific case. Fix a connected exact cobordism $X : Y_+ \to Y_-$ between contact manifolds with non-degenerate contact forms. Let $E \subset X$ be an embedded irrational ellipsoid and let $\Xi$ be an orbit tuple in $\partial E$. Consider the exact cobordism
\[
X \setminus E : Y_+ \to Y_- \cup \partial E
\]
Finally, fix an orbit $\gamma$ of $Y_+$ and an orbit tuple $\Gamma$ of $Y_-$. The following lemma is immediate from Construction 2.9, Remark 2.10 and Definition 3.6.

**Lemma 3.10.** Let $P_\Xi$ be the dual constraint to $\Xi$ (see Example 3.5). Let $J$ be a compatible almost complex structure on $J$ such that

$$\mathcal{M}_{0,A}(X \setminus E; \gamma, \Gamma \cup \Xi)$$

is regular and compact in the BEHWZ topology for each homology class $A \in S(X \setminus E; \gamma, \Gamma \cup \Xi)$. Then the $x_\Gamma$-coefficient of $\Phi_{X,P_\Xi}(x_\gamma)$ is given by

$$\langle x_\Gamma, \Phi_{X,P_\Xi}(x_\gamma) \rangle := \frac{1}{\mu_{\gamma} \cdot \kappa_{\Gamma}} \cdot \sum_A \mathcal{M}_{0,A}(X \setminus E; \gamma, \Gamma \cup \Xi)$$

Here the sum is over all classes with $\text{ind}(0, A) = 0$ and $\#$ denotes an oriented point count.

Constrained cobordism maps satisfy a number of useful (and expected) axioms presented in the following lemma.

**Lemma 3.11.** The constrained cobordism maps $\Phi_{X,P}$ satisfy the following properties.

A. (Functoriality) If $X : Y \to Y'$ and $X' : Y' \to Y''$ are two exact cobordisms, and $P$ and $Q$ are two tangency constraints, then

$$\Phi_{X \times X', P \otimes Q} \simeq \Phi_{X,P} \circ \Phi_{X',Q} \quad \text{and} \quad U_Q \circ \Phi_{X,P} = \Phi_{X,P \otimes Q} = \Phi_{X,P} \circ U_Q.$$  

B. (Additivity) If $X : Y \to Y'$ is an exact cobordism, and $P$ and $Q$ are two tangency constraints of the same dimension, then

$$\Phi_{X,P+Q} \simeq \Phi_{X,P} + \Phi_{X,Q}.$$  

C. (Empty Constraint) Let $P_{\emptyset}$ be the empty constraint. Then

$$\Phi_{X,P_{\emptyset}} = \Phi_X \quad \text{and} \quad U_{P_{\emptyset}} = \text{Id}.$$  

**Proof.** It suffices to prove these properties for $\Phi_{X,P}$, as the $U$-maps are a special case.

**Axiom A.** Choose collections of disjoint star-shaped domains $W = W_1 \cup \cdots \cup W_m \subset X$ and $V = V_1 \cup \cdots \cup V_n \subset X'$. Up to deformation of exact cobordisms, we may write

$$(X \circ X') \setminus (W \cup V) = \left( (X \setminus W) \cup ([0,1] \times \partial V) \right) \circ (X' \setminus V).$$

Therefore, by Theorem 2.11(c) we have (up to filtered chain homotopy) the equivalence

$$\Phi_{(X \circ X') \setminus (W \cup V)} \simeq \Phi_{(X \setminus W) \cup (\partial V \times [0,1])} \circ \Phi_{X' \setminus V} \simeq (\Phi_{X \setminus W} \otimes \text{Id}_{\partial V}) \circ \Phi_{X' \setminus V}.$$  

We may choose the cochains representing $P$, $Q$ and $P \otimes Q$ to satisfy

$$\pi_{\partial(W \cup V)} \circ \Phi_{P \otimes Q} = \pi_{\partial W} \otimes \pi_{\partial V} \circ \Phi_{P \otimes Q}.$$  

This implies the desired composition property, by the following calculation.

$$\Phi_{X \times X', P \otimes Q} = (\text{Id}_{\gamma'} \otimes \pi_{\partial(W \cup V)} \circ \Phi_{(X \times X') \setminus (W \cup V)})$$

$$\simeq (\text{Id}_{\gamma'} \otimes \pi_{\partial W} \circ \pi_{\partial V} \circ \Phi_{X \setminus W} \otimes \text{Id}_{\partial V}) \circ \Phi_{X' \setminus V}$$

$$= \pi_{\partial W} \circ \Phi_{X \setminus W} \circ \text{Id}_{\gamma'} \otimes \pi_{\partial V} \circ \Phi_{X' \setminus V} \circ \Phi_{X \setminus W} + \text{Id}_{\gamma'} \otimes \pi_{\partial V} \circ \Phi_{X' \setminus V}.$$  

**Axiom B.** Choose a collection of disjoint star-shaped domains $W = W_1 \cup \cdots \cup W_m \subset X$. We may choose the cochain representatives of $P$, $Q$, and $P + Q$ so that

$$\pi_{\partial W} \circ \Phi_{P + Q} = \pi_{\partial W} \circ \Phi_{P} + \pi_{\partial W} \circ \Phi_{Q}.$$  

Thus, we can calculate that

$$\Phi_{X,P+Q} = (\text{Id}_{\gamma'} \otimes (\pi_{\partial W} + \pi_{\partial W}, \circ \Phi_{X \setminus W})$$

$$= (\text{Id}_{\gamma'} \otimes \pi_{\partial W} \circ \text{Id}_{\gamma'} \otimes \text{Id}_{\partial W} \circ \Phi_{X \setminus W}) \circ \Phi_{X \setminus W} = \Phi_{X,P} \circ \Phi_{X,Q}.$$
Axiom C. Any star-shaped domain $W$ determines a unital, $\mathbb{Z}$-graded dg-algebra map

$$\Phi_W : CH(\partial W) \to CH(\mathcal{Z}) = \mathbb{Q}.$$ 

The cohomology class of this map is unique, and equal to $P_{\mathcal{Z}}$. Therefore,

$$\Phi_{X,P_{\mathcal{Z}}} = (\text{Id} \otimes \Phi_W) \circ \Phi_{X|W} = \Phi_X.$$

The constrained cobordism maps $\Phi_{X,P_k}$ with respect to the tangency constraints $P_k$ defined in Example 3.3 satisfy, in addition, a chain level Leibniz rule.

**Lemma 3.12.** The constrained cobordism map $\Phi_{X,P_k}$ of the tangency constraints $P_k$ satisfies

$$\Phi_{X,P_k}(xy) = \Phi_{X,P_k}(x) \cdot \Phi_X(y) + \Phi_X(x) \cdot \Phi_{X,P_k}(y).$$

As a special case, the $U$-maps $U_{P_k}$ satisfy the Leibniz rule.

$$U_{P_k}(xy) = U_{P_k}(x) \cdot y + x \cdot U_{P_k}(y).$$

**Proof.** Let $W \subset X$ be an embedded irrational ellipsoid. Let $y_j$ denote the generators of $CH(\partial W)$. We first note that the map $\Phi_{X|W}$ in Definition 3.6 satisfies

$$\Phi_{X|W}(z) = \Phi_X(z) \otimes 1 + \sum_{k=0}^{\infty} \Phi_{X,P_k}(z) \otimes y_{k+1} + r$$

where $r$ is a sum of factors $x \otimes y$ where $y$ is a monomial of word length $\geq 2$. This follows from Definition 3.6, the fact that $P_k$ is the dual constraint to the $(k+1)$-th generator $y_{k+1}$ (see Example 3.5) and the fact that $P_{\mathcal{Z}}$ is dual to $1 \in CH(\partial W)$. Since $\Phi_{X|W}$ is an algebra map, we thus have

$$\Phi_{X|W}(zz') = \Phi_X(zz') \otimes 1 + \sum_{k=1}^{\infty} (\Phi_{X,P_k}(z)\Phi_X(z') + \Phi_X(z)\Phi_{X,P_k}(z')) \otimes y_{k+1} + r'$$

where $r'$ is a remainder term of the same form as $r$. On the other hand, the map $\Pi = \text{Id} \otimes \pi_{\partial W,P_k}$ from Definition 3.6 is given by

$$\Pi(x \otimes y_1 \ldots y_{kn}) = \begin{cases} x & \text{if } m=1, k_1 = k+1 \\ 0 & \text{otherwise} \end{cases}$$

By Definition 3.6, we have $\Phi_{X,P_k} = \Pi \circ \Phi_{X|W}$. Thus it follows from (3.4) and (3.5) that

$$\Phi_{X,P_k}(zz') = \Pi((\Phi_{X,P_k}(z)\Phi_X(z') + \Phi_X(z)\Phi_{X,P_k}(z')) \otimes y_{k+1}) = \Phi_{X,P_k}(z)\Phi_X(z') + \Phi_X(z)\Phi_{X,P_k}(z')$$

which is the desired Leibniz rule.

**□**

### 3.3. Spectral Invariants

We now recall the definitions and properties of spectral invariants and capacities in the setting of contact homology.

**Definition 3.13.** The contact homology spectral invariant $s_\sigma(Y, \alpha)$ of a closed contact manifold $Y$ with contact form $\alpha$ and a class $\sigma \in CH(Y)$ is given by

$$s_\sigma(Y, \alpha) := \mathcal{A}(\sigma) = \min\{ L : \sigma \in CH^L(Y) \subset CH(Y) \} \in (0, \infty).$$

**Definition 3.14.** The contact homology capacity $c_p(W)$ of a Liouville domain $(W, \lambda)$ and an abstract constraint $P$ is given by

$$c_p(W) := \inf\{ s_\sigma(\partial W, \lambda|_{\partial W}) : \Phi_{W,P}(\sigma) \neq 0 \}.$$

Note that here we are viewing $W$ as an exact cobordism from $\partial W$ to $\partial \mathcal{Z}$.

**Remark 3.15.** If $(Y, \alpha)$ is a closed contact manifold where $\alpha$ is non-degenerate, (3.6) is equivalent to the minimum action of a cycle in the dg-algebra $A(Y)$ representing $\sigma$,

$$s_\sigma(Y, \alpha) = \min\{ A(x) : x \in A(Y) \text{ satisfying } \partial x = 0 \text{ and } [x] = \sigma \}.$$

**Theorem 3.16.** The contact homology spectral invariants $s_\sigma$ satisfy the following properties.
A. (Conformality) If \((Y, \alpha)\) is a contact manifold with contact form \(\alpha\) and \(a > 0\) is a constant, then
\[
s_a(Y, a \cdot \alpha) = a \cdot s_a(Y, \alpha).
\]

B. (Cobordism Map) If \(X : (Y_+, \alpha_+) \to (Y_-, \alpha_-)\) is an exact symplectic cobordism, \(P\) is an abstract constraint and \(W \subset X\) is a (weakly) exactly embedded Liouville domain, then
\[
s_{\Phi_{X, P}(\alpha)}(Y_-, \alpha_-) + c_P(W) \leq s_\alpha(Y_+, \alpha_+).
\]

C. (U-Map) If \((Y, \alpha)\) is a contact manifold with contact form \(\alpha\) and \(P\) is an abstract constraint, then
\[
s_{\U_P(\alpha)}(Y, \alpha) \leq s_\alpha(Y, \alpha).
\]

D. (Monotonicity) Let \(f : Y \to [0, \infty)\) be a smooth non-negative function on \(Y\). Then,
\[
s_\alpha(Y, \alpha) \leq s_\alpha(Y, e^f \cdot \alpha).
\]

Moreover, \(s_\alpha(Y, -)\) is continuous in the \(C^0\)-topology on contact forms.

E. (Reeb Orbits) For each class \(\sigma \in CH(Y)\), there is a Reeb orbit tuple \(\Gamma\) such that
\[
s_\sigma(Y, \alpha) = \mathcal{A}(\Gamma).
\]

Furthermore, if \(\alpha\) is non-degenerate and \(\sigma\) has grading \(|\sigma|\), then
\[
|\Gamma| = |\sigma|.
\]

**Proof.** We demonstrate each of these axioms individually.

**Axiom A.** This follows immediately from the definition and the fact that the contact homology groups of \(CH(Y)\) with respect to \(\alpha\) and \(a \cdot \alpha\) are canonically identified with action filtrations differing by the scaling factor \(a\).

**Axiom B.** By the functoriality property of constrained cobordism maps stated in Lemma 3.11, \(\Phi_{X, P}\) can be written as the composition
\[
CH(Y_+) \xrightarrow{\Phi_{X, W}} CH(Y_-) \otimes CH(\partial W) \xrightarrow{\Id \otimes \Phi_{W, P}} CH(Y_-) \otimes CH(\partial [\mathrm{codim}(P)]) = CH(Y_-)[\mathrm{codim}(P)].
\]

Choose basis \(x_i\) of \(CH(Y_-)\) of pure action filtration, that is, \(x_i\) is a linear combination of generators which have the same action. Then, for some set of elements \(y_i \in CH(\partial W)\), all but finitely many of which vanish), we may write
\[
\Phi_{X, W}(\sigma) = \sum_i x_i \otimes y_i \quad \text{and} \quad \Phi_{X, P}(\sigma) = \sum_i \Phi_{W, P}(y_i) \cdot x_i.
\]

Since \(x_i\) is a basis of pure filtration, we know that
\[
\mathcal{A}(\Phi_{X, W}(\sigma)) = \mathcal{A}(\sum_i x_i \otimes y_i) = \max\{\mathcal{A}(x_i \otimes y_i) : y_i \neq 0\}.
\]

(3.7)

\[
\mathcal{A}(\Phi_{X, P}(\sigma)) = \mathcal{A}\left(\sum_i \Phi_{W, P}(y_i) \cdot x_i\right) = \max\{\mathcal{A}(x_i) : \Phi_{W, P}(y_i) \neq 0\}.
\]

(3.8)

Let \(m\) be the index such that \(\mathcal{A}(x_m) = \mathcal{A}(\Phi_{X, P}(\sigma))\) and \(\Phi_{W, P}(y_m) \neq 0\). By Definitions 3.13 and 3.14, we have
\[
s_{\Phi_{X, P}(\sigma)}(Y_-, \alpha_-) = \mathcal{A}(\Phi_{X, P}(\sigma)) = \mathcal{A}(x_m) \quad \text{and} \quad c_P(W) \leq \mathcal{A}(y_m).
\]

(3.9)

On the other hand, by (3.7) and the monotonicity of the action filtration under cobordism maps, we know that
\[
\mathcal{A}(x_m) + \mathcal{A}(y_m) = \mathcal{A}(x_m \otimes y_m) \leq \mathcal{A}(\Phi_{X, W}(\sigma)) \leq \mathcal{A}(\sigma) = s_\alpha(Y_+, \alpha_+).
\]

(3.10)

The constrained monotonicity property follows immediately from (3.9) and (3.10).
**Axiom C.** Fix $e > 0$ and consider the cobordism $X : (Y, e^e \cdot \alpha) \to (Y, \alpha)$ given by $X = [0, e]_r \times Y$ equipped with the standard Liouville form $\lambda = e^e \lambda$. The $U$-map $U_p$ is the constrained map

$$\Phi_{X, p} : CH(Y) \to CH(Y).$$

By the usual monotonicity and scaling axioms, we therefore know that

$$s_{U_p(a)}(Y, \alpha) \leq \lim_{\epsilon \to 0} s_\sigma(Y, e^e \cdot \alpha) = s_\sigma(Y, \alpha).$$

**Axiom D.** To prove monotonicity, assume that $f : Y \to [0, \infty)$ and choose $e > 0$. Consider the cobordism $X : (Y, e^f \cdot \alpha) \to (Y, e^{-e} \cdot \alpha)$ given by

$$X := \{(r, y) : 0 \leq r \leq f(y)\} \subset \mathbb{R}_r \times Y \quad \text{with Liouville form} \quad \lambda = e^e \lambda$$

The cobordism map and scaling axioms imply that $e^{-1} \cdot s_\sigma(Y, \alpha) \leq s_\sigma(Y, e^f \cdot \alpha)$. Thus, we take $e \to 0$ to acquire the monotonicity inequality.

To deduce continuity, let $\alpha_i = f_i \cdot \alpha$ be a sequence of contact forms that $C^0$ converges to $\alpha$. Then, $f_i \to 1$ and so there exists a sequence of constants $C_i > 0$ such that

$$C_i \to 1 \quad \text{as} \quad i \to \infty \quad \text{and} \quad C_i > f_i > C_i^{-1}.$$

By the cobordism map and scaling axioms, we see that

$$C_i^{-1} \cdot s_\sigma(Y, \alpha) = s_\sigma(Y, C_i^{-1} \cdot \alpha) \leq s_\sigma(Y, f_i \cdot \alpha) \leq s_\sigma(Y, C_i \cdot \alpha) = C_i \cdot s_\sigma(Y, \alpha).$$

By taking the limit as $i \to \infty$, we see that $s_\sigma(Y, \alpha_i) \to s_\sigma(Y, \alpha)$.

**Axiom E.** Assume that $(Y, \alpha)$ is non-degenerate and let $\sigma \in CH(Y)$. Then there exists a cycle $x$ representing $\sigma$ such that

$$x = \sum_{\Gamma} c_\Gamma \cdot x_\Gamma \quad \text{with} \quad A(x) = \max\{x_\Gamma : c_\Gamma \neq 0\} = s_\sigma(Y, \alpha).$$

If $\sigma$ has pure homological grading $|\sigma|$, then we can assume that $c_\Gamma = 0$ for $|\Gamma| \neq |\sigma|$. Let $\Gamma$ be the maximal action orbit tuple with $c_\Gamma \neq 0$, then

$$s_\sigma(Y, \alpha) = A(\Gamma) \quad \text{and} \quad |\Gamma| = |\sigma|$$

as desired.

If $\alpha$ is degenerate, we can take a sequence of non-degenerate contact forms $\alpha_i$ that $C^\infty$ converges to $\alpha$. Then, the corresponding orbit tuples $\Gamma_i$ have bounded total action and thus converge to an orbit tuple $\Gamma$ of $\alpha$ as $i \to \infty$. This convergence can be seen via arguments similar to those in the proof of Theorem 12 (3.4.1). Since $s_\sigma$ is continuous in the $C^\infty$-topology on contact forms, this implies that

$$s_\sigma(Y, \alpha) = \lim_{i \to \infty} s_\sigma(Y, \alpha_i) = \lim_{i \to \infty} A(\Gamma_i) = A(\Gamma). \quad \Box$$

**Theorem 3.17.** The contact homology capacities $c_p$ satisfy the following properties.

A. (Conformality) If $(W, \lambda)$ is a Liouville domain and $a > 0$ is a constant, then

$$c_p(W, a \cdot \lambda) = a \cdot c_p(W, \lambda).$$

B. (Monotonicity) If $(W, \lambda) \to (V, \mu)$ is a (weakly) exact embedding of Liouville domains, then

$$c_p(W, \lambda) \leq c_p(V, \mu).$$

C. (Tensor Product) If $P$ and $Q$ are two abstract constraints, then

$$c_p(W, \lambda) \leq c_p \otimes Q(W, \lambda).$$
D. (Reeb Orbits) If $P$ is an abstract constraint, then there is a tuple of Reeb orbits $\Gamma$ of $\partial W$ such that
\[ c_P(W, \lambda) = \mathcal{A}(\Gamma). \]

Furthermore, if $\lambda|_{\partial W}$ is non-degenerate, then
\[ |\Gamma| = \text{codim}(P) \mod 2m \quad \text{where} \quad m := \min |c_1(W) \cdot A|. \]

Proof. Axioms A, B, and D are proven by approaches that are essentially identical to the analogous properties (respectively A, D and E) in Theorem 3.16. For Axiom C, we note that
\[ \Phi_{W, P \otimes Q} = \Phi_{W, P} \circ U_Q. \]

Therefore, if $\Phi_{W, P \otimes Q}(\sigma) \neq 0$, we have $\Phi_{W, P}(U_Q(\sigma)) \neq 0$. We thus acquire the inequality
\[ c_P(W, \lambda) \leq \min \{ A(\tau) : \Phi_{W, P}(\tau) \neq 0 \} \leq \min \{ A(U_Q(\sigma)) : \Phi_{W, P \otimes Q}(\sigma) \neq 0 \} \]
\[ \leq \min \{ A(\sigma) : \Phi_{W, P \otimes Q}(\sigma) \neq 0 \} = c_{P \otimes Q}(W, \lambda). \]

3.4. Spectral Gap. We are now ready to introduce the contact homology spectral gap.

Definition 3.18. Let $Y$ be a closed contact manifold with contact form $\alpha$ and let $\sigma \in CH(Y)$ be a contact homology class. The spectral gap of $(Y, \alpha)$ in class $\sigma$ is defined to be
\[ \text{gap}_\sigma(Y, \alpha) := \inf \left\{ \frac{s_\sigma(\sigma, \alpha) - s_{U_P}(\sigma)(Y, \alpha)}{c_P(B^{2n})} : \text{$P$ is an abstract constraint} \right\}. \]

The (total) spectral gap of the contact manifold $(Y, \alpha)$ is given by
\[ \text{gap}(Y, \alpha) := \inf_{\sigma \in CH(Y)} \text{gap}_\sigma(Y, \alpha). \]

Theorem 3.19 (Theorem 3). Let $(Y, \alpha)$ be a closed contact manifold with contact form such that
\[ \text{gap}(Y, \alpha) = 0. \]

Then, $(Y, \alpha)$ satisfies the strong closing property, namely, for every non-zero $f : Y \to [0, \infty)$ there exists $t \in [0, 1]$ such that $(1 + tf)\alpha$ has a closed Reeb orbit passing through the support of $f$. Moreover, if $\text{gap}_\sigma(Y, \alpha) = 0$ for some $\sigma \in CH(Y)$, then the period of this orbit is bounded by $s_\sigma(Y, \alpha)$.

Proof. Let $f : Y \to [0, \infty)$ and fix $L \in [0, \infty]$. Assume, if possible, that for all $t \in [0, 1]$, the contact form $(1 + tf)\alpha$ does not have a periodic Reeb orbit of action up to $L$ through the support of $f$. In this case, the action spectrum of $(1 + tf)\alpha$ up to $L$ remains the same as $t$ varies. Note that the action spectrum of $\alpha$ is a measure zero set. As observed in [Iri15, Lemma 2.2], this is a consequence of the fact that the critical values of the contact action functional are contained in the critical values of a smooth function on a finite dimensional manifold, which can be constructed by adapting the proof of [Sch00, Lemma 3.8] from the Hamiltonian setting. So, the continuity (in $t$) of the spectral invariants, stated in Theorem 3.16, guarantees that
\[ s_\sigma(Y, (1 + tf)\alpha) = s_\sigma(Y, \alpha), \]
for all $t \in [0, 1]$ and $\sigma \in CH(Y)$ such that $s_\sigma(Y, \alpha) \leq T$. We will show that the cobordism property gives a positive lower bound for the spectral gap of such contact homology classes.

Fix $\varepsilon > 0$ small and let $X$ be the cobordism from $(Y, e^\varepsilon(1 + f)\alpha)$ to $(Y, \alpha)$ given by
\[ X := [0, \varepsilon] \times Y \cup \{ (\varepsilon + \log(1 + t \cdot f(y), y)) : t \in [0, 1] \} \subset \tilde{Y}. \]

There exists a number $r = r(f, Y, \alpha)$ such that the ball $B^2r$ of radius $r$ embeds into $X$. By the cobordism property of spectral invariants stated in Theorem 3.16, for any homology class $\sigma \in CH(Y)$ and an abstract constraint $P$ it holds that
\[ s_\sigma(Y, e^\varepsilon(1 + f)\alpha) - s_{\Phi_{X, P}(\sigma)}(Y, \alpha) \geq c_P(B^{2n}(r)). \]
By the conformality property of the capacities stated in Theorem 3.17, we have $c_p(B^{2n}(r)) = r^2 \cdot c_p(B^{2n})$. Rearranging the above inequality we obtain

$$r^2 \leq \frac{s_\sigma(Y, e^\varepsilon(1 + f)\alpha) - s_\Phi_{\sigma,Y}(Y, \alpha)}{c_p(B^{2n})}.$$  \hfill (3.14)  

We will show that the latter lower bound contradicts the vanishing of the spectral gap. Let $X' := [0, \varepsilon/2] \times Y$ be a trivial cobordism and decompose $X$ as $X = X' \cup (X \setminus X')$. By the functoriality property of the constrained cobordism map we have

$$\Phi_{X,p} \cong \Phi_{X',p} \circ \Phi_{X',X'} = U_p \circ \Phi_{X,X'}.$$  

Combining this with inequality (3.14) and equation (3.13), and using again the properties of spectral invariants, we conclude that for every homology class $\sigma \in CH(Y)$ such that $s_\sigma(Y, \alpha) \leq T$,  

$$r^2 \leq \frac{s_\sigma(Y, e^\varepsilon(1 + f)\alpha) - s_\Phi_{\sigma,Y}(Y, \alpha)}{c_p(B^{2n})} = \frac{e^\varepsilon s_\sigma(Y, (1 + f)\alpha) - s_\Phi_{\sigma,Y}(Y, \alpha)}{c_p(B^{2n})} \leq \frac{s_\sigma(Y, \alpha) - s_\Phi_{\sigma,Y}(Y, \alpha)}{c_p(B^{2n})} + \frac{r^2}{2},$$

where the last inequality holds when we take $\varepsilon \leq \log \left(1 + \frac{r^2}{2s_\Phi_{\sigma,Y}(Y, \alpha)}\right)$. Applying this estimate for every abstract constraint $P$, we get a positive lower bound for the spectral gap,

$$\frac{r^2}{2} \leq \inf_P \frac{s_\sigma(Y, \alpha) - s_\Phi_{\sigma,Y}(Y, \alpha)}{c_p(B^{2n})} = \text{gap}_\sigma(Y, \alpha).$$  \hfill (3.15)

To prove the first assertion of the theorem, take $T = \infty$. Then the above lower bound for $\text{gap}_\sigma(Y, \alpha)$ for all $\sigma$ implies that $\text{gap}(Y, \alpha)$ is positive, in contradiction with the hypothesis. To prove the second part of the theorem, suppose $\text{gap}_\sigma(Y, \alpha) = 0$ for some $\sigma \in CH(Y)$ and take $T = s_\sigma(Y, \alpha)$. Then (3.15) yields a contradiction. \hfill \Box  

Remark 3.20 (Semi-continuity of the spectral gap). The spectral gap is upper semi-continuous, that is, for every sequence of contact forms $\alpha_i$ converging to $\alpha$ and for every class $\sigma \in CH(Y)$ we have

$$\limsup_{i \to \infty} \text{gap}_\sigma(Y, \alpha_i) \leq \text{gap}_\sigma(Y, \alpha),$$

$$\limsup_{i \to \infty} \text{gap}(Y, \alpha_i) \leq \text{gap}(Y, \alpha).$$

This is due to the fact that the spectral gaps are defined as an infimum over continuous functions.

In fact, for a fixed class $\sigma$, the $\sigma$-spectral gap is continuous. More generally, under some conditions on the rate of convergence of $\alpha_i$ to $\alpha$, the spectral gap of $\alpha$ can be bounded from above by spectral gaps of $\alpha_i$. In order to give a more precise statement of this fact we adapt the following notation.

Notation 3.21. Let $\alpha$ and $\alpha'$ be two contact forms on $Y$. We write $\alpha \leq \alpha'$ if there exists an exact cobordism from $(Y, \alpha')$ to $(Y, \alpha)$.

Proposition 3.22. Let $(Y, \alpha)$ be a contact manifold and suppose there exist sequences $\{\alpha_i\}$ of contact forms, $\{\sigma_i\}$ of contact homology classes, and $\{\varepsilon_i\}$ of positive numbers, such that:

(i) $\alpha_i \leq \alpha \leq (1 + \varepsilon_i)\alpha_i$, and

(ii) $\varepsilon_i \cdot s_{\sigma_i}(Y, \alpha_i) \to 0$.  

Then, $\text{gap}(Y, \alpha) \leq \liminf_{i \to \infty} \text{gap}_{\sigma_i}(Y, \alpha_i)$. In particular, if $\text{gap}_{\sigma_i}(Y, \alpha_i) \to 0$, then $\text{gap}(Y, \alpha) = 0$.  

Proof. Fix $\delta > 0$. For each $i$, let $P_i$ be an abstract constraint such that

$$s_{\partial_i}(Y, \alpha_i) - s_{U_{P_i}(\alpha_i)}(Y, \alpha_i) < \text{gap}_{\partial_i}(Y, \alpha_i) + \delta. \tag{3.16}$$

By our assumption that $\alpha_i \leq \alpha \leq (1 + \epsilon_i)\alpha_i$ we have the following commutative diagram of filtered homologies

$$\begin{array}{ccc}
CH(Y, (1 + \epsilon_i)\alpha_i) & \xrightarrow{U^{(1+\epsilon_i)\alpha_i}_{P_i}} & CH(Y, (1 + \epsilon_i)\alpha_i) \\
\Phi_1 & & \Phi_1 \\
\downarrow & & \downarrow \\
CH(Y, \alpha) & \xrightarrow{U^{\alpha_i}_{P_i}} & CH(Y, \alpha) \\
\Phi_2 & & \Phi_2 \\
\downarrow & & \downarrow \\
CH(Y, \alpha_i) & \xrightarrow{U^{\alpha_i}_{P_i}} & CH(Y, \alpha_i).
\end{array} \tag{3.17}$$

Here, in order to distinguish between the $U$-maps on contact homologies filtered by the contact forms $\alpha$ and $\alpha_i$, we adapt the notations $U^{\alpha_i}_{P_i}$ and $U^{\alpha_i}_{P_i}$ respectively. By definition, for $\alpha_i \in CH(Y, (1 + \epsilon_i)\alpha_i)$ we have

$$\text{gap}(Y, \alpha) \leq s_{\partial_i}(Y, \alpha) - s_{U^{\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha) = s_{\Phi_i(\alpha_i)}(Y, \alpha) - s_{\Phi_i U^{(1+\epsilon_i)\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha),$$

and

$$\leq s_{\partial_i}(Y, (1 + \epsilon_i)\alpha_i) - s_{\Phi_i U^{(1+\epsilon_i)\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha_i),$$

where the last inequality follows from the cobordism property of spectral invariants, stated in Theorem 3.16. By Theorem 2.11, the composition $\Phi_2 \Phi_1 : CH(Y, (1 + \epsilon_i)\alpha_i) \to CH(Y, \alpha_i)$ of maps between the contact homologies is the identity map with filtration rescaled by $1/(1 + \epsilon_i)$. Using the conformality property of spectral invariants (Theorem 3.16) we obtain

$$\text{gap}(Y, \alpha) \leq s_{\partial_i}(Y, (1 + \epsilon_i)\alpha_i) - s_{U^{\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha_i) = (1 + \epsilon_i)s_{\alpha_i}(Y, \alpha_i) - s_{U^{\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha_i)$$

$$= (s_{\alpha_i}(Y, \alpha_i) - s_{U^{\alpha_i}_{P_i}(\alpha_i)}(Y, \alpha_i)) + \epsilon_i \cdot s_{\alpha_i}(Y, \alpha_i)$$

$$\leq \text{gap}_{\alpha_i}(Y, \alpha_i) + \delta + \epsilon_i \cdot s_{\alpha_i}(Y, \alpha_i),$$

where the last inequality follows from our choice of the abstract constraint $P_i$. Since our $\delta$ can be taken to be arbitrarily small, we conclude that if the product $\epsilon_i \cdot s_{\alpha_i}(Y, \alpha_i)$ converges to zero, then $\text{gap}(Y, \alpha) \leq \lim_{i \to \infty} \text{gap}_{\alpha_i}(Y, \alpha_i).$ \hfill $\square$

3.4.1. Detecting periodicity via spectral gaps. Theorem 12 from the introduction states that if there exists a homology class $\sigma \in CH(Y, \xi)$ such that $\text{gap}_{\sigma}(Y, \alpha) = 0$, then the Reeb flow of $\alpha$ is periodic. The following proof is similar to an argument from ECH that was pointed out to us by Oliver Edtmair.

Proof of Theorem 12. A classical theorem of Wadsley [Wad75] implies that if all Reeb orbits are closed then the flow is periodic. Therefore, it is sufficient to show that there is a periodic orbit passing through every point of $Y$.

Fix a point $y \in Y$, let $\{V_n\}_{n=1}^{\infty}$ be open sets in $Y$ such that $\cap_n V_n = \{y\}$. Let $f_n : Y \to [0, \infty)$ be a smooth non-zero function supported in $V_n$, such that $\|f_n\|_{C^3} \leq \frac{1}{n}$. Recall our assumption that $\text{gap}_{\sigma}(Y, \alpha) = 0$ and denote $T := \sigma_g(Y, \alpha)$. By Theorem 3.19, there exists $t_n \in [0, 1]$ such that the contact form $\alpha_n := (1 + t_n f_n)\alpha$ has a periodic orbit $\gamma_n$ of period $T_n \leq T$ passing through the support of $f_n$. By extracting a subsequence we may assume that $T_n$ converge to some $T_* \leq T$. We
think of \( \gamma_n \) as a map \( \mathbb{R} \to Y \), and denote \( \gamma_n(\cdot) := \gamma_n(T_n \cdot \cdot) : S^1 \cong \mathbb{R}/\mathbb{Z} \to Y \). Denote by \( R_{\alpha_n} \) and \( R_\alpha \) the Reeb vector fields of \( \alpha_n \) and \( \alpha \), respectively. The sequence \( \gamma_n \) is equicontinuous since the derivatives \( \gamma_n' = T_n \cdot R_{\alpha_n} \) are uniformly bounded. Therefore, we can apply Arzelà-Ascoli theorem and conclude that there is a subsequence \( \{ \gamma_{n_k} \} \) that converges to a limit \( \gamma : S^1 \to Y \). Clearly, \( \gamma \) passes through \( y \) since \( \gamma_{n_k} \) passes through \( V_n \). Let us show that \( \gamma \) is differentiable and that its derivative is \( T_\ast \cdot R_\alpha \). Fix \( t \in S^1 \) and consider a chart in \( Y \) around \( \gamma(t) \). For large enough \( n \), \( \gamma_n(t) \) lies in this chart and satisfies

\[
\gamma_n(t + \epsilon) - \gamma_n(t) = \int_0^\epsilon T_n \cdot R_{\alpha_n} \circ \gamma_n(t + \tau) d\tau,
\]

for small enough \( \epsilon \). Taking the limit when \( n \to \infty \) over this equation, we obtain

\[
\gamma(t + \epsilon) - \gamma(t) = \int_0^\epsilon T_\ast \cdot R_\alpha \circ \gamma(t + \tau) d\tau
\]

for every \( \epsilon > 0 \) small enough. Therefore, \( \gamma \) is indeed differentiable, its derivative is \( T_\ast \cdot R_\alpha \), and its reparametrization \( \gamma(t) := \gamma(t/T_\ast) \) is a periodic orbit of \( R_\alpha \) that passes through \( y \), as required. \( \square \)

4. Analysis of the constrained moduli spaces

In this section, we analyze the compactified moduli spaces of holomorphic cylinders between select orbits of a Morse-Bott contact form. We prove that, under some conditions, these moduli spaces are cut-out transversely and consist of a single point. Before formally stating the main result for this section, let us fix the setting and present the relevant definitions and notations.

**Setup 4.1.** Throughout this section we fix the following structural assumptions.

(i) A contact manifold \((Y, \xi)\) of dimension \(2n - 1\), for \( n \geq 3 \).

(ii) A contact form \( \alpha \) on \( Y \) such that the flow \( \varphi_t \) of the corresponding Reeb vector field \( R \) is periodic.

(iii) An almost complex structure \( J \) on \( \xi \) and a Riemannian metric \( g_J(-,-) = \alpha \otimes \alpha + d\alpha(-, J-) \) on \( Y \).

(iv) A Morse-Bott function \( f : Y \to \mathbb{R} \) that is \( \varphi_t \)-invariant and whose critical manifolds are 1-dimensional. Note that the \( \varphi_t \)-invariance of \( f \) implies that the critical manifolds are disjoint unions of periodic Reeb orbits. For such \( f \), we let:

- \( X_f \) be the vector field defined by \( \alpha(X_f) \equiv 0 \) and \( d\alpha(-, X_f) = df(-); \) and
- \( \nabla_J f \) be the vector field defined by \( g_J(-, \nabla_J f) = df(-) \). Note that \( \nabla_J f = -X_f \).

(v) Given \( f \), the \( \epsilon > 0 \) we define perturbations of \( \alpha, R \) and \( J \) as follows.

- The perturbation of \( \alpha \) is \( \alpha_\epsilon := e^{\epsilon f} \alpha \).
- The Reeb vector field of \( \alpha_\epsilon \) is \( R_\epsilon = e^{-\epsilon f}(R - \epsilon X_f) \).
- Let \( J_\epsilon \) be the \( \mathbb{R} \)-invariant almost complex structure on \( \hat{Y} = \mathbb{R}_\epsilon \times Y \) satisfying:
  * \( J_\epsilon(\partial_\epsilon) = R_\epsilon \),
  * \( J_\epsilon \) preserves the bundle \( \xi \), and
  * the restriction of \( J_\epsilon \) to \( \xi \) is equal to \( J \).

(vi) Let \( \gamma_+ \) and \( \gamma_- \) be circles of global maxima and minima of \( f \), respectively. Considering \( \gamma_\pm \) as Reeb orbits of \( \alpha_\epsilon \), assume that their periods are the same and coincide\(^1\) with the minimal period of the flow \( \varphi_t \).

(vii) Let \( z \in Y \) be a point in the intersection of the unstable manifold of \( \gamma_+ \) and the stable manifold of \( \gamma_- \).

\(^1\)For the arguments in Sections 4.1-4.3, it is enough to assume that the periods of \( \gamma_\pm \) are the same and \textit{divide} the minimal period of the flow \( \varphi_t \). The assumption that the periods \( \gamma_\pm \) coincide with the minimal period of the flow is used only in Section 4.4 for regularity purposes. See Remark 5.8
(viii) Consider a sequence of nested manifolds $Y_3 \subset \cdots \subset Y_{2n-1} = Y$, such that for each $j$:

(a) $\dim(Y_{2j-1}) = 2j - 1$ and $H_2(Y_{2j-1}) = 0$;
(b) $Y_{2j-1}$ is invariant under the Reeb flow $\varphi_t$;
(c) $(Y_{2j-1}, \xi \cap TY_{2j-1})$ is a contact manifold and $\alpha|_{TY_{2j-1}}$ is a contact form for it. Moreover, $\xi \cap TY_{2j-1}$ is $J$-invariant;
(d) $\nabla f$ is tangent to $Y_{2j-1}$ and $f|_{Y_{2j-1}}$ is Morse-Bott. In particular, $Y_{2j-1}$ is invariant under the gradient flow of $f$;
(e) Along any critical circle $\gamma$ of $f$, other than $\gamma_+$ which lies in $Y_{2n-3}$, the restriction of the Hessian of $f$ to the symplectic orthogonal of $\xi \cap TY_{2j-1}$ in $\xi \cap TY_{2j+1}$ is positive definite\(^2\);
(f) $z \in Y_3 \subset \cdots \subset Y_{2n-1}$.

We call such a sequence $\{Y_{2j-1}\}_{j=2}^n$ a contact flag. This is illustrated in Figure 4.

![Figure 6](image.png)

**Figure 6.** An illustration of a contact flag, as defined in Setup 4.1. The lavender sphere $Y_7$, the pink disk $Y_5$ and the purple circle $Y_3$ represent consecutive nested sub-manifolds in the flag. The orbits $\gamma_+$ and $\gamma_-$ lie in $Y_3$, and are connected by a Morse flow line in green.

**Example 4.2.** The example of interest of this setting is when $Y = \partial E$ is the boundary of an ellipsoid with rationally dependent entries. The contact flag is given by intersecting $\partial E$ with complex linear subspaces: $Y_{2j-1} = \partial E \cap (C^{j-1} \times \{0\}^{n-j} \times C)$. In Section 5.1 we study this example in detail and explain how it fits into Setup 4.1.

The main purpose of this section is to prove the following claim.

**Proposition 4.3.** Consider Setup 4.1. The compactified moduli space\(^3\)

$$\overline{M}(Y, J_f, \gamma_+, \gamma_-; z)$$

of $J_f$-holomorphic cylinders in $\hat{Y}$ between $\gamma_-$ and $\gamma_+$ containing $(0, z)$ consists of a single point which is transversely cut-out.

**Remark 4.4.** The single point in the moduli space $\overline{M}(Y, J_f, \gamma_+, \gamma_-; z)$ is in fact a certain lift of the gradient flow line of $f$ from $\gamma_-$ to $\gamma_+$ that passes through $z$. This lift is described in section 4.1 below.

---

\(^2\)In particular, this means that $\gamma_+$ is the only maximum of $f$ in $Y_{2n-3}$. This assumption will be used for computing certain Conley-Zehnder indices of an $f$-perturbation of the Reeb flow.

\(^3\)See section 2.2.11 for the definition of this moduli space.
Remark 4.5. The contact form $\alpha_f := e^f \alpha$ is non-degenerate in a finite action window when $\varepsilon$ is small enough. That is, fixing $T_\varepsilon > 0$, there exists $\varepsilon > 0$ such that every periodic orbit of $\alpha_f$ with action less than $T_\varepsilon$ is non-degenerate. These periodic orbits lie over the critical circles of $f$, and thus correspond to pairs $(S, p)$ of a Morse–Bott family $S$ of orbits of $\alpha$ and a circle $p$ of $f$-critical points. The Conley-Zehnder index of such an orbit can be calculated from the Robbin-Salamon index of the family $S$, its dimension, and a Morse-type index of $p$, as shown in the following lemma.

Lemma 4.6. The Conley-Zehnder index of $\gamma = (S, p)$ is given by

$$
\text{CZ}(S, p) = \text{RS}(S) + \frac{1}{2} \dim(S) - \text{ind}_{\text{Morse}}^S(f; p),
$$

where $\text{ind}_{\text{Morse}}^S(f; p)$ is the number of negative eigenvalues of the restriction of the hessian of $f$ to the tangent space to the image of $S$ in $Y$.

Proof. To see this, write the linearized Reeb flow of $\alpha_f$ as a composition

$$
dq\phi_t = (dq^{f}_{t} \circ dq^{-1}_{t} \circ d\varphi_{t}).
$$

As explained in [Gut14], the composition is homotopic to the concatenation of paths. By the additivity and homotopy invariance of the Conley-Zehnder index (Proposition 2.2), the index of the orbit $(S, p)$ with respect to $\alpha_f$ is equal to the sum of the Robbin-Solomon index of the family $S$ with respect to $\alpha$ and the CZ-index of the path $dq : (dq^{-1}_{t} \circ dq^{f}_{t}) \circ d\varphi_{T}$, where $T$ is the period of $S$. Let us compute the latter. First, notice that for every $t \leq T$, the image $\varphi_{t}(\gamma(0))$ lies in a small neighborhood of the point $\varphi_{T}(\gamma(0))$, since the flows $\varphi_{t}$ and $\varphi^{f}_{t}$ differ by a small time reparametrization on critical circles of $f$. Identifying a neighborhood of $\varphi_{T}\gamma(0)$ with its Darboux chart, the path $dq$ solves the ODE $dq = J_0 \text{Hess}(\varepsilon f)\gamma(0)\varepsilon dq$. When $\varepsilon$ is sufficiently small, the path $\Phi(t)$ crosses the Maslov cycle only at the origin and the crossing form is $\Gamma(\gamma(0)) := -Hess(\varepsilon f)\gamma(0)|_{\ker(dq_{0} - Id)}$. The kernel of $dq_{0} - Id = dq_{T} - Id$ coincides with the tangent space to the image of $S$ in $Y$ by our assumption that $S$ is a Morse–Bott family. Since the hessian of $f$ is degenerate only in the Reeb direction, the signature of this crossing form is by definition the number of positive eigenvalues minus the number of negative eigenvalues, and hence coincides $\dim(S) - 2 \text{ind}_{\text{Morse}}^{S}(f; p)$. This shows that (4.1) holds. \qed

The rest of this section is dedicated to proving the above proposition. In subsection 4.1 we describe an element of $\overline{M}(Y, J_{f}; \gamma_{\pm}, \gamma_{\pm}; z)$. In subsection 4.2 we prove that there are no other elements in this moduli space. In subsection 4.3 we show that this moduli space is cut-out transversely.

4.1 Lifting flow lines to holomorphic cylinders. In this section we lift a gradient flow line of the Morse-Bott function $f$ in $Y$ to $J_{f}$-holomorphic curve in the symplectization $\hat{Y}$. Let $\eta : \mathbb{R} \to Y$ be a gradient flow line of $f$, that is, $\eta(s) = \nabla_{f} f(\eta(s))$, such that $\lim_{s \to \pm \infty} \eta(s)$ is contained in $\gamma_{\pm}$, respectively. Further, assume that $\eta(0) = z$. Such a gradient flow line exists by our choice of the point $z$ stated in Setup 4.1 and is clearly unique. Denoting by $T > 0$ the common period of $\gamma_{\pm}$ with respect to the contact form $\alpha$, we can define a $J_{f}$ holomorphic map $u_{\eta} : \mathbb{R} \times S^{1} \to \hat{Y}$ by

$$
u_{\eta}(s, t) := (a(s), \varphi_{T\cdot t}(\eta(\varepsilon Ts))),
$$

where $a(s)$ is defined by the ODE

$$
\dot{a}(s) = T \cdot e^{\varepsilon f(\eta(\varepsilon Ts))}, \quad a(0) = 0.
$$

Note that $u_{\eta}$ is indeed a $J_{f}$-holomorphic curve that limits to $\gamma_{\pm}$ at the ends, and contains $(0, z)$ in its image, since $\nu_{\eta}(0, 0) = (0, \eta(0))$. Therefore, $u_{\eta}$ lies in the moduli space $\overline{M}(Y, J_{f}; \gamma_{\pm}, \gamma_{\pm}; z)$. 
4.2. Uniqueness. In this section we show that the moduli space $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ has no elements other than the lift $u_\eta$ constructed above.

**Proposition 4.7.** Consider Setup 4.1. The only point in the moduli space $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ is the curve $u_\eta$ constructed in Section 4.1.

The proof of Proposition 4.7 uses the intersection theories in [Sie11, MS19]. We will show inductively that any element of $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ is contained in the symplectizations of all of the submanifolds $Y_3 \subset \cdots \subset Y_{2n-1} = Y$. Then it will follow from a result in [Sie11] that this element is in fact contained in the image of $u_\eta$. The structure of this section is as follows. First, we show that the buildings in $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ consist only of cylinders, then we give an overview of the holomorphic intersection theory of Moreno-Siefring, and finally we explain how to use it to show that $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ consists of only one element.

4.2.1. Ruling out non-cylindrical buildings. Our first step towards proving that $u_\eta$ is the unique element in the moduli space $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ is to show that all buildings in $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ consist only of cylinders.

**Lemma 4.8.** For $\varepsilon > 0$ sufficiently small, each building in $\mathcal{M}(Y, J_f; \gamma_+, \gamma_-; z)$ consists solely of $J_f$-holomorphic cylinders.

**Proof.** Fix a building $\mathfrak{u} \in \mathcal{M}(\gamma_+, \gamma_-; J_f)$ with $k \geq 1$ levels. Let $\Gamma^1, \Gamma^2, \ldots, \Gamma^{k+1}$ be the associated limits and notice that $\Gamma^1 = (\gamma_-)$ and $\Gamma^{k+1} = (\gamma_+)$.

We start by finding a sequence of orbits $\{y^j \in \Gamma^j\}$ of non-decreasing actions:

$$\int_{y^1} \alpha_f \leq \int_{y^2} \alpha_f \leq \ldots \leq \int_{y^{k+1}} \alpha_f.$$ 

This sequence is constructed inductively as follows. Pick $\gamma^1 = \gamma_-$. Given $\gamma^j$, there exists a unique connected $J_f$-holomorphic curve $u^j$ in $\mathfrak{u}$ such that $\gamma^j$ is one of its negative ends. Recall that every connected component of a level in $\mathfrak{u}$ has a single positive end, as explained in section 2.2.11. We take $\gamma^{j+1}$ to be the unique positive end of $u^j$. See Figure 7 for an illustration.

![Figure 7. An illustration of a holomorphic building $\mathfrak{u}$, and the sequence of orbits $\{y^j\}$.](image)

We will now show that the connected components $u^j$ must be all cylinders. This will conclude the proof, since the positive end of $u^{k+1}$ is $\gamma_+$. Let $\gamma^j_1 := \gamma^j, \gamma^j_2, \ldots, \gamma^j_{N_j}$ be the negative ends of $u^j$. We need to show that $N_j = 1$ for all $j$. For any given punctured $J_f$-holomorphic curve in $\hat{Y}$, the total integral of $\alpha_f$ over the positive limits is greater than or equal to the total integral of $\alpha_f$ over the negative limits. Applying this to $u^j$, we find that

$$\int_{y^{j+1}} \alpha_f \geq \sum_{i=1}^{N_j} \int_{y^j_i} \alpha_f \geq \int_{y^j} \alpha_f + (N_j - 1) \cdot T_{\text{min}},$$

where $T_{\text{min}}$ is the minimum period of $J_f$.
where $T_{\text{min}}$ is the minimal period of a periodic orbit of $\alpha_f$. In particular, the $\alpha_f$-integrals are non-decreasing along the sequence $\{\gamma^j\}$. The $\alpha_f$ integrals of the first and last orbits in this sequence are

$$\int_{\gamma^j} \alpha_f = e^{\varepsilon f(\gamma^j)} T$$

where $T > 0$ is the common period of $\gamma^j$ under the Reeb flow of $\alpha$. Combining this with (4.3) we find that for all $j$,

$$e^{\varepsilon f(\gamma^j)} T = \int_{\gamma^j_+} \alpha_f \geq \int_{\gamma^j + 1} \alpha_f \tag{4.3}$$

$$\geq \int_{\gamma^j_-} \alpha_f + (N_j - 1) \cdot T_{\text{min}}$$

$$= e^{\varepsilon f(\gamma_-^j)} T + (N_j - 1) \cdot T_{\text{min}}.$$

Rearranging the above, we obtain

$$(N_j - 1) \cdot T_{\text{min}} \leq (e^{\varepsilon f(\gamma^j_+)} - e^{\varepsilon f(\gamma_-^j)}) T. \tag{4.4}$$

The minimal action $T_{\text{min}}$ of a Reeb orbit of $\alpha_f = e^{\varepsilon f}$ is uniformly bounded (in $\varepsilon$) below by a constant depending only on $\alpha$, as long as $\varepsilon$ is taken to be smaller than some constant depending only on $f$. This implies that the inequality (4.4) can only hold if $N_j - 1 = 0$ for all $j \in \{1, \ldots, k + 1\}$. We thus conclude that $u^j$ are all cylinders, and therefore $\overline{p}$ is composed solely of cylinders as well. \qed

4.2.2. An overview of holomorphic intersection theory. Our main tool in proving the uniqueness result stated in Proposition 4.7 is the holomorphic intersection theory developed in [Sie11, MS19]. We give here an overview of this theory, following [MS19] and adapted to our case.

Let $(Y, \xi, \alpha)$ be a closed contact manifold with a contact form and denote the Reeb vector field by $R$. Let $J$ be an almost complex structure on $\hat{Y}$ and consider a codimension-2 submanifold $Z$ of $\hat{Y}$ such that:

- there exist closed codim-2 submanifolds $Z^\pm \subset Y$ such that $Z$ is asymptotically cylindrical over $Z^\pm$ (see Definition 4.9 below),
- the sub-bundles $\xi_{Z^\pm} := \xi \cap TZ^\pm$ are $J$-invariant,
- $(Z^\pm, \xi_{Z^\pm})$ are contact manifolds and $\alpha|_{TZ^\pm}$ are contact forms on them, and
- $Z$ is invariant under the flow of $R$.

The contact structure $\xi$ splits along $Z^\pm$ into a pair of symplectic vector bundles

$$(\xi|_{Z^\pm}, d\alpha) \simeq (\xi_{Z^\pm}, \omega_Z) \oplus (\xi_N, \omega_N)$$

where $\xi_{Z^\pm} = TZ^\pm \cap \xi$ as mentioned above, $\xi_N$ is the symplectic complement of $\xi_Z$ in $\xi$, and $\omega_Z$, $\omega_N$ are the restrictions of $d\alpha$ to $\xi_Z, \xi_N$ respectively.

**Definition 4.9.** The hypersurface $Z \subset \hat{Y}$ is asymptotically cylindrical over $Z^\pm \subset Y$ if there exists $s$-families of sections of $\xi_N$, i.e. $\zeta_+ : [R', \infty) \to \Gamma(\xi_N|_{Z^+})$ and $\zeta_- : (-\infty, -R'] \to \Gamma(\xi_N|_{Z_-})$ such that

$$Z \cap ([R', \infty) \times Y) = \bigcup_{(r,p) \in [R, \infty) \times Z^+} \text{exp}_{(r,p)}(\zeta_+(r)(p)),$$

$$Z \cap ((-\infty, R'] \times Y) = \bigcup_{(r,p) \in (-\infty, R'] \times Z_-} \text{exp}_{(r,p)}(\zeta_-(r)(p)).$$
Remark 4.10. We will consider two simple cases of asymptotically cylindrical hypersurfaces. The first is a symplectization of a contact submanifold, namely $Z := \hat{Z}_+$. The second case is when $Y$ is 3-dimensional and $Z$ is a pseudoholomorphic cylinder. It follows from [MS19, Theorem 2.2] that any pseudoholomorphic cylinder is asymptotically cylindrical over its ends (see also Definition 4.12 below).

Given a $J$-holomorphic cylinder $u$ in $\hat{Y}$ with non-degenerate ends $\gamma^u_+$, the works [MS19, Sie] define a “holomorphic intersection number” $u \ast Z$ of $u$ with the manifold $Z$ and prove a “positivity of intersection” property (see Theorem 4.13). The holomorphic intersection number is defined as a sum of a “relative intersection number” and a contribution from the Conley-Zehnder index of the ends in the normal contact direction $\xi_N$.

**Definition 4.11** (Holomorphic intersection number, [MS19, Sie]). Let $Y, Z, u$ and $\gamma^u_+$ be as above. Fix a trivialization of $\xi_N$ along the orbits of $R$ that lie in $Z$ (if there are any) and collectively denote it by $\tau$.

- (Relative intersection number). Let $u^\tau$ be a deformation of $u$ as described in Definition 4.12 below. Roughly speaking, this deformation uses a $\tau$-constant section of $\xi_N$ to push any ends of $u$ that lie in $Z$ off of it. In particular, if the ends of $u$ do not lie in $Z$ then $u^\tau = u$.

Define the relative intersection number of $u$ with $Z$ to be

$$i^\tau(u, Z) := u^\tau \cdot Z \in \mathbb{Z},$$

namely, the standard transverse intersection number of the deformation $u^\tau$ with the hypersurface $Z$.

- (Normal Conley–Zehnder index). Let $\gamma$ be a non-degenerate Reeb orbit which lies in $Z_+$. Its normal Conley–Zehnder index $CZ^N_\tau(\gamma)$ is the Conley–Zehnder index of the path of $\mathbb{R} \times Z$ symplectic matrices defined by applying the trivialization $\tau$ to the projection of the linearized Reeb flow to $\xi_N$. For an orbit $\gamma$ that does not lie in $Z$, we define $CZ^N_\tau(\gamma) := 0$.

The holomorphic intersection number of $u$ and $Z$ is defined to be

$$u \ast Z := i^\tau(u, Z) + [CZ^N_\tau(\gamma^u_+)/2] - [CZ^N_\tau(\gamma^u_-)/2].$$

The deformation $u^\tau$ required for the definition of the relative intersection number (4.5) is constructed as follows.

**Definition 4.12** (Deformation $u^\tau$ of $u$). We write down the deformation at the positive end, the deformation at the negative end is analogous with the appropriate changes in sign. If $\gamma^u_-$ is not contained in $Z_+$, then we do not deform $u$ near the positive end. Suppose $\gamma^u_+$ is contained in $Z_+$ and extend the trivialization $\tau$ to a trivialization $\tau : U \times \mathbb{R}^2 \to \xi_N|_U$ on some open neighborhood $U \subset Z_+$ of $\gamma^u_+$. Then, [MS19, Theorem 2.2] states that the map $u$ may be written as

$$(s, t) \mapsto \exp_{u_T(s, t)}(u_N(s, t)), $$

where:

- $(s, t) \in [R, +\infty) \times S^1$ for $R \gg 1$ large enough,
- $u_T(s, t)$ lies in $\mathbb{R} \times U \subset \mathbb{R} \times Z_+$,
- $u_N$ is a smooth section of $u_T^*\pi^*\xi_N$, where $\pi$ is the projection $\mathbb{R} \times Z_+ \to Z_+$.

We then perturb the map $u$ by replacing the above parametrization of $u$ near $+\infty$ by the map

$$(s, t) \mapsto \exp_{u_T(s, t)}(u_N(s, t) + \beta(|s|)\pi(u_T(s, t), e))$$

where $\beta : [0, \infty) \to [0, 1]$ is a smooth cut-off function equal to 0 for $s < R + 1$ and equal to 1 for $s > R + 2$, and $0 \neq e \in \mathbb{R}^2$. 

The following theorem is a restriction of Theorem 2.5 from [MS19] adapted to our notations.

**Theorem 4.13** ([MS19, Theorem 2.5]). Let $Y, Z, u$ be as above, and assume further that the image of $u$ is not contained in $Z$. Then, $u \ast Z \geq 0$ and it is equal to 0 if and only if the image of $u$ does not intersect $Z$.

4.2.3. **Proof of uniqueness.** In this section we use the intersection theory reviewed above to prove Proposition 4.7. We continue with Setup 4.1. Our first step is to show that the relative intersection number of any $\mathbb{1}$-holomorphic cylinder $D$ in $\hat{Y}_{2i+1}$ is zero. We will later apply this lemma to $Z = \hat{Y}_{2i-1}$ when $i > 1$ and to $Z = im(u_\eta)$ when $i = 1$.

**Lemma 4.14.** Let $u$ be a $\mathbb{1}$-holomorphic cylinder in $\hat{Y}_{2i+1}$, and let $Z \subset \hat{Y}_{2i+1}$ be a codimension-2 asymptotically cylindrical hypersurface that is invariant under $\varphi_t$. Then, $\iota^*(u, Z) = 0$.

**Proof.** Let $u^\tau$ be the deformation of $u$ as in Definition 4.12, and denote by $\gamma^\mu_{\pm}$ the ends after the deformation. Let $\ell : \mathbb{R} \to Y_{2i+1}$ be a curve that connects $\gamma^\mu_{\pm}$, namely $\lim_{s \to \pm \infty} \ell(s) \in \gamma^\mu_{\pm}$. Since $Y_{2i+1} \setminus Z$ is connected, we can choose the path $\ell$ to not intersect $Z$. Consider the cylinder

$$
\sigma : \mathbb{R} \times S^1 \to \hat{Y}_{2i+1}, \quad \sigma(s, t) := (s, \varphi_t \ell(s)) \in \mathbb{R} \times Y_{2i+1} = \hat{Y}_{2i+1}.
$$

Since $\ell$ does not intersect the $\varphi_t$-invariant submanifold $Z$, its orbit under this action does not intersect it as well. Therefore, the cylinder $\sigma$ is disjoint from $Z$ (see Figure 8). Moreover, our assumption that $H_2(Y_{2i+1}) = 0$ guarantees that the union of $u^\tau$ and $\sigma$ is null-homologous. Consider the compactification $\overline{Y}$ of $\hat{Y}$ into a manifold with boundary, diffeomorphic to $[0, 1] \times Y$. By the homology invariance of the standard algebraic intersection number (e.g., [Bre13, Part VI, Section 11]), the intersection number of $u \# \sigma$ with the compactification of $Z$ vanishes. Thus,

$$
\iota^*(u, Z) = u^\tau \cdot Z = \sigma \cdot Z = 0.
$$

**Figure 8.** An illustration of the deformation $u^\tau$ of $u$, whose ends are disjoint from the hypersurface, and the cylinder $\sigma$ that has the same ends and does not intersect $\hat{Y}_{2n-3}$.

The next lemma concerns the normal Conley-Zehnder indices of the periodic orbits of the $f$-perturbed Reeb flow $\varphi^f_t$. This will be useful for computing the holomorphic intersection number of a $\mathbb{1}$-holomorphic cylinder with the submanifolds in the contact flag.

**Lemma 4.15.** Take $Y = Y_{2i+1}$ and $Z = \hat{Y}_{2i-1}$ for $i \in \{2, \ldots, n-1\}$. Let $\gamma$ be a periodic orbit of the perturbed contact form $\alpha_f$ that lies in $Y_{2i-1}$. There exists a trivialization $\tau$ of $\xi_N$ such that the normal CZ index of $\gamma$ is

$$
\text{CZ}_\tau^N(\gamma) = \begin{cases} 
+1 & \text{if } \gamma = \gamma_+, \\
-1 & \text{if } \gamma \neq \gamma_+.
\end{cases}
$$
Proof. Let \( \tau \) be any trivialization of \( \xi_N \) along \( \gamma \) that is invariant under the periodic Reeb flow \( \varphi_t \), namely, \( \tau : \mathbb{R}^2 \times \gamma \to \xi_N \) satisfying
\[
(4.8) \quad d\varphi_t \circ \tau(\cdot, x) = \tau(\cdot, \varphi_t(x))
\]
for all \( x \in \gamma \). In this trivialization the linearized flow of \( R_f \) is given by
\[
\Phi(t) := \tau(\cdot, \gamma(t))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)) = \tau(\cdot, \varphi_{\epsilon f}(\gamma(0)))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)).
\]
Since \( \gamma \) is a periodic orbit of \( R_f \), it is a critical circle of \( f \). By the definition of \( R_f \) stated in (v), it is proportional to \( R \) wherever \( X_f \) vanishes. Therefore,
\[
\Phi(t) = \tau(\cdot, \varphi_{\epsilon f}(\gamma(0)))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)) = (4.8) \quad \tau(\cdot, \gamma(0))^{-1} \circ (d\varphi_{\epsilon f}(\gamma(0)))^{-1} \circ d\varphi_t^f \circ \tau(\cdot, \gamma(0)).
\]
Denoting \( \phi_t := (\varphi_{\epsilon f}(\gamma(0)))^{-1} \circ d\varphi_t^f \), the linearization \( d\phi_t \) is conjugate to \( \Phi(t) \) by \( \tau(\cdot, \gamma(0)) \). Identifying a neighborhood of \( \gamma(0) \) with its Darboux chart, the path \( d\phi_t \) of matrices solves the ODE \( d\phi_t = -\mathcal{J}_0 \cdot \text{Hess}(\epsilon f) d\phi_t \). When \( \epsilon \) is sufficiently small, the path \( \Phi(t) \) crosses the Maslov cycle only at the origin and the crossing form is \( \Gamma(\gamma(0)) := -\text{Hess}(\epsilon f)(\gamma(0))_{\xi N} \). By assumption (viii)(e) from Setup 4.1, the Hessian of \( f \) is positive definite on \( \xi_N \) unless \( \gamma = \gamma_+ \), in which case it is negative definite. Therefore, the normal Conley-Zehnder index of \( \gamma \) is \( \frac{1}{2} \cdot (+2) = +1 \) if \( \gamma = \gamma_+ \) and \( \frac{1}{2} \cdot (-2) = -1 \) otherwise.

We are now ready to prove that, under the assumptions of Setup 4.1, the only point in the compactified moduli space \( \overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) \) is \( u_\eta \).

Proof of Proposition 4.7. Let \( \overline{u} = (u_1, \ldots, u_k) \in \overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) \) be a building of \( J_f \)-holomorphic curves. Lemma 4.8 above states that the components \( u_j \) of \( \overline{u} \) are all cylinders. Therefore, we may apply Lemma 4.14 and conclude that \( t^p(u_j, \hat{Y}_{2n-3}) = 0 \) for all \( j \). Denoting the negative and positive ends of \( u_j \) by \( \gamma_{j-1} \) and \( \gamma_j \) respectively, we notice that \( \gamma_{j-1} \neq \gamma_j \) for all \( j \). This is due to the fact that \( \gamma_+ \) is a global maximum for \( f \) and that the \( \alpha_f \)-action is decreasing along \( J_f \)-holomorphic curves. Therefore, Lemma 4.15 asserts that, for a properly chosen trivialization \( \tau \), \( C\mathcal{Z}_t^N(\gamma_{j-1}) = -1 \) and \( C\mathcal{Z}_t^N(\gamma_j) \leq 1 \) for all \( j \). It follows that the holomorphic intersection numbers of each piece with the hypersurface \( Z = \hat{Y}_{2n-3} \) is non-positive. Indeed,
\[
u_j \in \hat{Y}_{2n-3} := t^p(u_j, \hat{Y}_{2n-3}) + [C\mathcal{Z}_t^N(\gamma_j)/2] - [C\mathcal{Z}_t^N(\gamma_{j-1})/2]
= 0 + [C\mathcal{Z}_t^N(\gamma_j)/2] - [-1/2] = [C\mathcal{Z}_t^N(\gamma_j)/2] \leq 0.
\]

Theorem 4.13 (which is a special case of [MS19, Theorem 2.5]) states that, in this case, each \( u_j \) is either disjoint from \( \hat{Y}_{2n-3} \) or contained in it. By definition of \( \overline{\mathcal{M}}(Y, J_f; \gamma_+, \gamma_-; z) \), the image of \( \overline{u} \) contains the point \((0, z) \in \hat{Y}_3 \subset \cdots \subset \hat{Y}_{2n-3} \) Denoting by \( u_{j_0} \) the component of \( \overline{u} \) whose image contains \((0, z) \), we find that \( u_{j_0} \) is not disjoint from \( \hat{Y}_{2n-3} \), and thus is contained in it.

We now consider the curve \( u_{j_0} \) in \( \hat{Y}_{2n-3} \). Arguing as above, we see that its holomorphic intersection number with the hypersurface \( \hat{Y}_{2n-5} \) is again non-positive. Since its image intersects \( \hat{Y}_3 \subset \cdots \hat{Y}_{2n-5} \) we conclude that \( u_{j_0} \subset \hat{Y}_{2n-5} \). Continuing by induction we conclude that the image of \( u_{j_0} \) is contained in \( \hat{Y}_3 \).

To finish the proof, notice that the above arguments imply that the holomorphic intersection number of \( u_{j_0} \) and \( u_\eta \) in \( \hat{Y}_3 \) is non-positive. Since these two \( J_f \)-holomorphic curves intersect at \((0, z) \), it follows from Theorem 4.13 that \( u_{j_0} = u_\eta \). In particular, their ends agree: \( \gamma_j = \gamma_+ \) and \( \gamma_{j-1} = \gamma_- \). Recalling that \( \gamma_+ \) and \( \gamma_- \) are global maxima and minima for \( f \), it follows from action
The proof of Proposition 4.7 uses positivity of intersection to rule out the illustrated scenario, in which the piece of the building that intersects the hypersurface is not contained in it.

considerations that the only ends that could coincide with them are \( \gamma_k \) and \( \gamma_0 \) respectively, we conclude that the building \( \overline{u} \) consists of a single cylinder, \( u_{j_0} = u_\eta \), which concludes the proof. \( \square \)

4.3. Transversality. The aim of this section is to show that the \( J_f \)-holomorphic curve \( u_\eta \), described in Section 4.1, is cut-out transversely.

4.3.1. Local transversality of the moduli space without a point constraint. We start by showing that the moduli space without the point constraint is locally transversely cut-out at \( u_\eta \). The notion of a moduli space being locally transversely cut-out at a curve appears in [Par15, Section 2.11]; we give a brief overview here for convenience.

Definition 4.16 ([Par15, Section 2.11]). Let \( u \) be a \( J_f \)-holomorphic curve in \( \hat{\mathbf{Y}} \).

- For \( k \geq 0 \) and \( \delta < 1 \), define the associated weighted Sobolev space

\[
W^{k,2,\delta}(u^*\hat{T}\hat{\mathbf{Y}}) = \left\{ f \mid \mu \cdot f \in W^{k,2}(u^*T\hat{\mathbf{Y}}) \right\}, \quad \|f\|_{k,2,\delta} := \| \mu \cdot f \|_{k,2}
\]

where \( \mu : \mathbb{R} \times S^1 \rightarrow \mathbb{R}_{>0} \) equals 1 away from the ends and equals \( e^{\delta|s|} \) near the ends.

- The linearized operator\(^4\)

\[
D_u : W^{k,2,\delta}(u^*\hat{T}\hat{\mathbf{Y}}) \rightarrow W^{k-1,2,\delta}(u^*T\hat{\mathbf{Y}} \otimes \Omega^{0,1}(\mathbb{R} \times S^1))
\]

of \( u \) is defined as follows. Given any symmetric connection \( \nabla \) on \( T\hat{\mathbf{Y}} \), the linearized operator takes the form [Wen15, Section 2.1]

\[
D_u w = \nabla w + J_f(u) \circ \nabla w \circ j + (\nabla w J_f) \cdot du \circ j
\]

- A \( J_f \)-holomorphic curve \( u \) is transversely cut-out if the linearized operator \( D_u \) is surjective.

Proposition 4.17. Consider Setup 4.1. The linearized Cauchy-Riemann operator at \( u_\eta \) is surjective.

Our strategy for proving Proposition 4.17 is to show that the linearized operator splits into a direct sum of real-linear Cauchy–Riemann operators on complex line bundles. Then, we apply the automatic transversality results from [Wen10] to deduce surjectivity.

Recall the contact flag \( \mathcal{Y}_3 \subset \cdots \subset \mathcal{Y}_{2n-1} = \hat{\mathbf{Y}} \). It defines a splitting

\[
\xi = \xi_3 \oplus \cdots \oplus \xi_{2n-1}
\]

of the contact distribution into two-dimensional sub-bundles as follows. The bundle \( \xi_3 \) is defined to be the contact structure of \( \mathcal{Y}_3 \). For \( i > 1 \), the bundle \( \xi_{2i+1} \) is defined to be the symplectic

\(^4\)The domain of the linearized operator considered by Pardon in [Par15] is slightly bigger. Since our aim here is to prove surjectivity this does not cause any issue.
complement of \( T Y_{2i-1} \) in the contact structure of \( Y_{2i+1} \). Denoting by \( V \) the span of \( \partial_r \) and \( R \), we have a splitting
\[
T \hat{Y}_{2i+1} = V \oplus \left( \bigoplus_{k=1}^{i} \xi_{2k+1} \right), \quad i = 1, \ldots, n - 1.
\]

Moreover, it follows from the assumptions in Setup 4.1 that \( J_f \) preserves this splitting. For any \( i = 1, \ldots, n - 1 \), we write \( D_i \) for the linearized operator of \( u_\eta \), considered as a \( J_f \)-holomorphic curve in \( \hat{Y}_{2i+1} \). We will also denote the full linearized operator \( D_{n-1} \) by \( D \).

**Lemma 4.18** (Decomposing the linearized operator). For any \( 2 \leq i \leq n + 1 \) there exists a Fredholm real-linear Cauchy–Riemann operator \( L_i \) on the bundle \( \xi_{2i+1} \) such that the operator \( D_i \) has the matrix form
\[
\begin{pmatrix}
D_{i-1} & 0 \\
0 & L_i
\end{pmatrix}
\]

with respect to the splitting \( T \hat{Y}_{2i+1} = T \hat{Y}_{2i-1} \oplus \xi_{2i+1} \).

The linearized operator can be written in terms of a symmetric connection on \( T \hat{Y} \), as in (4.11). It will be convenient, for the purposes of computation, to use the Levi–Civita connection of a metric which is well-adapted to the contact flag.

**Lemma 4.19** (Choice of connection). There exist symmetric connections \( \nabla^i \) on \( T \hat{Y}_{2i+1} \), such that for any \( v \) tangent to the hypersurface \( Y_{2i-1} \), the operator \( \nabla^i_v \) can be written as
\[
(4.13) \quad \nabla^i_v = \begin{pmatrix} \nabla^{i-1}_v & 0 \\ 0 & \nabla'_v \end{pmatrix}
\]

with respect to the splitting \( T \hat{Y}_{2i+1} = T \hat{Y}_{2i-1} \oplus \xi_{2i+1} \).

**Proof.** Let \( h_1 \) be the metric on \( Y_3 \) equal to the one induced by \( J_f \). Define for each \( i \in \{2, \ldots, n - 1\} \) a metric \( h_i \) on \( Y_{2i+1} \) which satisfies the following conditions:

- The submanifold \( Y_{2i-1} \) is totally geodesic with respect to \( h_i \).
- The splitting \( T Y_{2i+1}|_{Y_{2i-1}} = T Y_{2i-1} \oplus \xi_{2i+1} \)

is \( h_i \)-orthogonal.

The metrics \( h_i \) can be constructed inductively as follows. Fix a tubular neighborhood of \( Y_{2i-1} \) in \( Y_{2i+1} \), determined by \( \xi_{2i+1} \). Given \( h_{i-1} \) on \( Y_{2i-1} \), define \( h_i \) to be cylindrical in this tubular neighborhood and extend it outwards.

For each \( i \in \{2, \ldots, n - 1\} \), let \( \hat{h}_i \) denote the cylindrical metric over \( h_i \) on the cylinder \( \hat{Y}_{2i+1} \). Let \( \nabla^i \) denote the Levi–Civita connection on \( \hat{Y}_{2i+1} \) with respect to \( \hat{h}_i \). By definition, \( \nabla^i \) is symmetric. The splitting (4.13) follows from the fact that \( \hat{Y}_{2i-1} \) is totally geodesic with respect to \( \hat{h}_i \) and hence the \( \hat{h}_i \)-parallel transport preserves the orthogonal direct sum decomposition \( T Y_{2i+1}|_{Y_{2i-1}} = T Y_{2i-1} \oplus \xi_{2i+1} \). \( \square \)

The next lemma states that the asymptotic operators of \( D \), the linearized operator of \( u_\eta \) in \( \hat{Y} \), respect the direct sum decomposition induced by the contact flag, and that the pieces in the contact direction are non-degenerate.

**Lemma 4.20.** The asymptotic operators \( A_\pm \) of \( D \) respect the splitting (4.12). Moreover, their restrictions to \( \gamma_{\pm}^* \xi_{2i+1} \) are non-degenerate.

**Proof.** Consider the connections from Lemma 4.19, and write \( V := \nabla^{n-1} \) for the connection on the total manifold \( \hat{Y} \). Following [Wen10], the asymptotic operators \( A_\pm \) are defined by
\[
\lim_{s \to \pm \infty} D(\partial_s) = V_s - A_s.
\]
Evaluating (4.11) as \( s \to \pm \infty \) we obtain
\[
\lim_{s \to \pm \infty} D(\partial_s) = \lim_{s \to \pm \infty} V_s + I_s \nabla + (V_I) \partial_u = \nabla_s + T(I_s \nabla + T(V_I) \cdot R_f = \nabla_s + T(I_s \nabla + T(V_I) R_f - T(I_s \nabla R_f).
\]
Notice that \( I_s \nabla R_f = \partial_s \) is covariantly constant, since the metric defining \( \nabla \) is cylindrical. Therefore the asymptotic operators are
\[
A_s = -T \cdot (I_s \nabla - \nabla R_f) + T(V_I) R_f = -T \cdot (I_s \nabla R_f).
\]
\[
A_s = -T \cdot (I_s \nabla - \nabla R_f) + T(V_I) R_f.
\]
Since \( I_s \) and the flow of \( R_f \) preserve the splitting, the operator \( I_s \nabla R_f \) does as well. Denoting by \( A_s^n \) the restrictions of \( A_s \) to \( \gamma_s^\pm \xi_{2s+1} \), it remains to show that they are non-degenerate. Notice that the operators \( A_s^n \) coincide with the restriction to \( \gamma_s^\pm \xi_{2s+1} \) of the linearization of the flow \( q_s^n \) of \( R_f \). Let \( \tau_s \) be covariantly constant along \( \gamma_s^\pm \) that are invariant under the periodic Reeb flow \( q_s^n \), as in 4.8. As shown in the proof of Lemma 4.15, the linearization of the flow \( q_s^n \) is conjugate to a path of matrices \( d\phi_t \) that solves the ODE \( d\phi_t = -J_0 \cdot \text{Hess}(\epsilon f) d\phi_t \). Note that the Hessian of \( f \) is degenerate in the direction of the Reeb vector field \( R \), since \( f \) is \( R \)-invariant. However, by the Morse-Bott condition, the restriction of the Hessian to \( \xi_{2s+1} \), is non-degenerate. When \( \epsilon \) is sufficiently small, this implies that \( \{d\phi_t\}_{t=0}^T \) is non-degenerate as well. □

We are now ready to show that the linearized operator splits into a direct sum.

**Proof of Lemma 4.18.** Let \( V^i \) be the connection from Lemma 4.19, and write \( D_i \) as
\[
D_i = \underbrace{V^i + I_s \circ V^i \circ j + (V_I) \cdot du_j \circ j}_i.
\]
The splitting of the first part, \((i)\), follows immediately from the splitting of \( V^i \) and the fact that \( I_s \) preserves the decomposition. Indeed, since \( \partial_s u_q \) and \( \partial_t u_q \) lie in the subspace \( T\hat{Y}_{2s+1} \), Lemma 4.19 guarantees that \( V^i_{\partial_s u_q} = V^i_{\partial_t u_q} \oplus V^i_{\partial_s u_q} \) and the same for \( \partial_t u_q \). Therefore,
\[
(i) = \begin{pmatrix} V^{i-1} + I_s (u_q) V^{i-1} \circ j & 0 \\ 0 & V^i + I_s (u_q) V^i \circ j \end{pmatrix}.
\]
Let us rewrite the second part of \( D_i \) applied to \( \partial_s \) (the computation for \( \partial_t \) is completely analogous):
\[
(ii)(\partial_s) = (V^i I_s) \cdot du_j \circ j \partial_s = (V^i I_s) \cdot \partial_t u_q = V^i (I_s R_f) = T \cdot (V^i (-\partial_s) - I_s V^i R_f) = T \cdot (0 - I_s V^i R_f) = -T(I_s V^i R_f).
\]
\[
(ii)(\partial_s) = V^i (I_s R_f) = T \cdot (0 - I_s V^i R_f) = -T(I_s V^i R_f).
\]
where, in the second and third lines, we used the fact that \( \partial_t u_q = T \cdot R_f \) by construction, and that \( R_f = -I_s \partial_s \) respectively. In the last row we used the fact that the metric \( \hat{h}^i \) is cylindrical in the \( r \)-direction, which implies that the vector field \( \partial_s \) is covariantly constant. We now evaluate \((ii)(\partial_s) = -T(I_s V^i R_f) \) on a tangent vector field \( v \) and split into cases with respect to the decomposition \( T\hat{Y}_{2s+1} \mid \hat{Y}_{2s-1} = T\hat{Y}_{2s-1} \oplus \hat{Y}_{2s+1} \):

- \( v \in \Gamma(u_q^* T\hat{Y}_{2s-1}) \): Using Lemma 4.19 and the fact that \( R_f \) is tangent to \( \hat{Y}_{2s-1} \) as well, we obtain \( V^i_{\partial_s} R_f = V^i_{\partial_s} R_f \). Therefore \((ii)(\partial_s) v = -T(I_s V^i_{\partial_s}) R_f \).
Proof of Proposition 4.17.

Let \( v \in \Gamma(u^*_\eta \xi_{2i+1}) \): Extend \( v \) into a vector field \( \bar{v} \) defined on a neighborhood of the image of \( u_\eta \) in \( \hat{Y}_{2i+1} \). Then, along the hypersurface \( \hat{Y}_{2i-1} \) we have

\[
(i) (\partial_i) v = -T J_f \nabla^i \tilde{R}_f = -T J_f \left( \nabla^i \tilde{R}_f + [\tilde{v}, R_f] \right)
= T J_f \left( [R_f, \tilde{v}] - \nabla^i \tilde{R}_f \right)
= T J_f \left( [R_f, v] - \nabla^i \tilde{R}_f \right)

\text{Lemma 4.19}
= T J_f \left( [R_f, v] - \nabla'^i \tilde{R}_f \right) = J_f [\partial_i u_\eta, v] - J_f \nabla'^i u_\eta v
= J_f [du_\eta \circ j(\partial_i), v] - J_f \nabla' \circ j(\partial_i)(v).
\]

Above, we used the fact that the connection \( \nabla^i \) is symmetric and that \( R_f \) is tangent to the hypersurface \( \hat{Y}_{2i-1} \).

Overall we conclude that the second part of \( D_i \) decomposes as

\[
(ii) = \left( \begin{array}{cc}
(\nabla^{i-1}) f \cdot du_\eta \circ j & 0 \\
0 & J_f [du_\eta \circ j, -] - J_f \nabla' \circ j
\end{array} \right).
\]

Summing \( i \) and \( ii \) we obtain a decomposition of \( D_i \):

\[
D_i = \left( \begin{array}{cc}
D_{i-1} & 0 \\
0 & \nabla' + J_f [du_\eta \circ j, -]
\end{array} \right).
\]

A simple computation shows that \( L_i := \nabla' + J_f [du_\eta \circ j, -] \) is a Cauchy-Riemann operator. Moreover, since \( \partial_i u_\eta = TR_f \), the asymptotic operators of \( L_i \) are indeed the restrictions of \( A_{\pm} \) to \( \gamma^*_\pm \xi_{2i+1} \). By Lemma 4.20, they are non-degenerate. This implies that \( L_i \) is a Fredholm operator (e.g. \([Wen10\), Section 2.1]), and thus concludes the proof. \( \square \)

Having established the splitting of the linearized operators \( D_i \), we are ready to prove surjectivity.

Proof of Proposition 4.17. We will prove by induction that the Cauchy-Riemann operators \( D_i \) are surjective for all \( i \). This will use the decomposition from Lemma 4.18, as well as automatic transversality \([Wen10\] for \( D_1 \) and \( L_i \). Starting with the base case of the induction, \( D_1 \) is the linearized operator at the (non-constant) curve \( u_\eta \) inside the 4-dimensional manifold \( \hat{Y}_3 \). Theorem 1 from \([Wen10\] states that\(^5\) in a 4-dimensional manifold, an immersed pseudoholomorphic curve is regular if

\[
(4.16) \quad \text{ind}(u) > \frac{1}{2} (\text{ind}(u) - 2 + 2g + \#\Gamma_0 + \#\Pi_0(\partial\Sigma)),
\]

where:

- \( \Sigma \) is the domain of the curve,
- \( g \) is the genus of \( \Sigma \),
- \( \Gamma_0 \) is the set of ends of \( u \) with even CZ index.

In our case, \( \Sigma = \mathbb{R} \times S^1 \), and hence \( g = 0 \) and \( \#\Pi_0(\Sigma) = 0 \). Moreover, \( \#\Gamma_0 \leq 2 \) since \( u_\eta \) has only two ends. Therefore, condition \((4.16)\) holds if \( \text{ind}(D_1) > 0 \). The latter inequality obviously holds in our case (in fact, \( \text{ind}(D_1) = \dim(\hat{Y}_3) - 2 = 2 \)).

Having established the base case of the induction we move on to the induction step. We assume that \( D_{i-1} \) is surjective for some \( i \) and show surjectivity of \( D_i \). By the decomposition of \( D_i \) given in Lemma 4.18, this is equivalent to showing that the real-linear Cauchy-Riemann operator \( L_i \) is surjective. This is an operator on sections of a complex line bundle. Proposition 2.2

\(^5\)We only state a special case of Wendl’s theorem, adapted to our notations.
from [Wen10] states that given a complex line bundle $E \to \Sigma$, a Cauchy-Riemann Fredholm operator $L: W^{1,2}(E) \to L^2(E \otimes \Omega^{0,1} (\mathbb{R} \times S^1))$ is surjective if

$$\text{ind}(L) \geq 0 \quad \text{and} \quad \text{ind}(L) > \frac{1}{2}(\text{ind}(L) - 2 + 2g + \#\Gamma_0 + \#\pi_0(\partial \Sigma)),$$

where:

- $g$ is the genus of $\Sigma$,
- $\Gamma_0$ is the set of ends for which the CZ index of the asymptotic operator is even.

As before, in our case $g = 0$ and $\#\pi_0(\partial \Sigma) = 0$. Therefore the second condition in (4.17) amounts to $\text{ind}(L) > -2 + \#\Gamma_0$. Since the domain $\Sigma$ in our case is a cylinder, $\#\Gamma_0 \leq 2$, and we are left with the requirement $\text{ind}(L) > 0$. A simple computation shows that $\text{ind}(L_i) = 2$ for all $i$. This essentially follows from the direct sum decomposition given in Lemma 4.18, the additivity of the Fredholm index, and the fact that $\text{ind}(D_i) = 2i$. We therefore conclude that $L_i$ is surjective, and this completes the proof. \qed

**Remark 4.21.** In the above proof we showed that $L_i$ is surjective as an operator between the spaces $W^{1,p}$ and $L^p$. Note that this implies surjectivity of the corresponding operator between $W^{1,2}$ and $W^{k-1,2}$, by elliptic regularity. Indeed, the cokernel of $L_i$ is the kernel of its formal adjoint operator. Applying elliptic regularity to the formal adjoint, we find that its kernel is independent of $k$ and $p$.

4.3.2. The evaluation map is a submersion. Let $z$, $\eta$, $u_\eta$ be as given in Section 4.1. We showed in Proposition 4.17 that the linearized operator

$$D: W^{k,2,\beta}(u_\eta \ast T \hat{Y}) \to W^{k-1,2,\beta} (u \ast T \hat{Y} \otimes \Omega^{0,1} (\mathbb{R} \times S^1))$$

is surjective. This implies that the moduli space $\mathcal{M}(\hat{Y}, I_f; \gamma_+, \gamma_-)$ of $I_f$-holomorphic cylinders near $u_\eta$ admits the structure of a smooth manifold of dimension $\text{ind}(D)$ near $u_\eta$, with tangent space at $u_\eta$ identified with $\ker(D) \oplus \mathbb{R} \cdot \partial_r$. The extra factor comes from the fact that we are not quotienting by the translation action; recall that the operator $D$ controls the deformations of the quotient moduli space $\mathcal{M}(Y, I_f; \gamma_+, \gamma_-) = \mathcal{M}(\hat{Y}, I_f; \gamma_+, \gamma_-)/\mathbb{R}$.

Our ultimate goal is to show that the point-constrained moduli space $\mathcal{M}(Y, I_f; \gamma_+, \gamma_-; z)$ is transversely cut-out at $u_\eta$. The point constrained moduli space is defined as the inverse image $ev^{-1}(0, z)$, where $ev$ is the evaluation map

$$ev: \mathcal{M}(\hat{Y}, I_f; \gamma_+, \gamma_-) \times \mathbb{R}^s \times S^1 \to \hat{Y}, \quad (u, s, t) \mapsto u(s, t).$$

It is transversely cut-out at $u_\eta$ if and only if $ev$ is a submersion at the point $(u_\eta, 0, 0)$. The linearization of $ev$ at $(u_\eta, 0, 0)$, which we write as $D_{ev}$, is the linear map $D_{ev}: \ker(D) \oplus \mathbb{R} \oplus T_{(0,0)}(\mathbb{R} \times S^1) \to T_{(0,z)}\hat{Y}$ defined by

$$\langle V, \partial_r, c \rangle \mapsto V(0,0) + c \cdot \partial_r + (\partial_r u_\eta)(0,0).$$

In order to show that the moduli space $\mathcal{M}(Y, I_f; \gamma_+, \gamma_-; z)$ is cut-out transversely at $u_\eta$, it remains to prove the following proposition.

**Proposition 4.22.** The map $D_{ev}$ is surjective.

The proof of Proposition 4.22 uses the surjectivity of the linearized evaluation map on Morse flow lines, as stated in the following lemma.

---

\[\text{Again, we state a special case adapted to our notations.}\]
Lemma 4.23. Let Morse($\gamma_+, \gamma_-$) be the space of Morse flow lines from $\gamma_+$ to $\gamma_-$. Then it is a smooth manifold and, for every $x \in$ Morse($\gamma_+, \gamma_-$) the map

$$D_{ev}^{\text{Morse}} : T_x \text{Morse}(\gamma_+, \gamma_-) \to T_x(0)Y,$$

defined by $V \mapsto V(0)$
is surjective.

**Proof.** The space Morse($\gamma_+, \gamma_-$) is composed of all smooth maps $x : \mathbb{R} \to Y$ such that $\dot{x}(s) = \nabla f(x(s))$, which limit to points in $\gamma_+$ and $\gamma_-$ respectively as $s \to \pm \infty$. Consider the natural evaluation map

$$\text{ev}_{\text{Morse}} : \text{Morse}(\gamma_+, \gamma_-) \to Y$$
sending a flow line $x$ to the point $x(0)$. The image of Morse($\gamma_+, \gamma_-)$ under this evaluation map is the intersection of the unstable manifold of $\gamma_+$ and the stable manifold of $\gamma_-$. This intersection is a smooth open submanifold of $Y$. This follows from the fact that the unstable manifold of $\gamma_+$ and the stable manifold of $\gamma_-$ are both smooth open submanifolds of maximal dimension in $Y$, so they must intersect transversely. By existence and uniqueness of ODEs, this implies that the space Morse($\gamma_+, \gamma_-)$ is a smooth manifold and that the evaluation map is a diffeomorphism onto an open subset of $Y$.

The tangent space $T_x \text{Morse}(\gamma_+, \gamma_-)$ at any flow line $x$ is a $2n - 1$-dimensional vector space of smooth vector fields $V$ along $x$ satisfying the linearized Morse flow line equation

$$\nabla V - \text{Hess}_f(V) = 0,$$

where $\nabla$ denotes the Levi–Civita connection of the metric $g_\|_t$ and $\text{Hess}_f(f)$ is the Hessian of $f$ with respect to this metric. The linearization of evaluation map at $x$ is the operator

$$D_{ev}^{\text{Morse}} : T_x \text{Morse}(\gamma_+, \gamma_-) \to T_x(0)Y, \quad V \mapsto V(0).$$

Since $\text{ev}_{\text{Morse}}$ is a diffeomorphism, its linearization is surjective. More explicitly, by existence of solutions to ODEs, for any point $x'_0$ in a neighborhood of $x(0)$, there is a flow line $x'$ such that $x'(0) = x'_0$.

**Proof of Proposition 4.22.** There is a natural parameterization of a set of cylinders near $u_\eta$ in $\mathcal{M}(\hat{Y}, I_f; \gamma_+, \gamma_-)$ with a neighborhood of $\eta$ in the space of Morse flow lines from $\gamma_+$ to $\gamma_-$. Lemma 4.23 states that the corresponding evaluation map, which sends a flow line to its image at the point $0 \in \mathbb{R}$, is a submersion. We will show that the linearization of this evaluation map at $\eta$ factors through $D_{ev}$.

In Section 4.1 we explained that every Morse flow line $x$ can be lifted to a $I_f$ holomorphic curve given by

$$u_x(s, t) := (a(s), \varphi_T T(x(\epsilon Ts))), \quad \text{where} \quad \dot{a}(s) = T \cdot e^{\epsilon f(x(\epsilon Ts))}, \quad a(0) = 0.$$

Consider the lift map $L : \text{Morse}(\gamma_+, \gamma_-) \to \mathcal{M}(\gamma_+, \gamma_-; I_f)$ that sends a flow line $x$ to its lift $u_x$.

The space $\mathcal{M}(\hat{Y}, I_f; \gamma_+, \gamma_-)$ is cut out transversely near $u_\eta$ by Proposition 4.17. Therefore we can consider the linearization of $L$ at $\eta$,

$$T_\eta L : T_\eta \text{Morse}(\gamma_+, \gamma_-) \to \ker(D) \oplus \mathbb{R},$$

which sends a vector field $V \in T_\eta \text{Morse}(\gamma_+, \gamma_-)$ to the vector field

$$T_\eta V(s, t) := \int_0^t \epsilon df(V(\epsilon T s')) \cdot \dot{a}(s') ds' \cdot \partial_r + d\varphi_T T \cdot V(\epsilon Ts)$$
in $\ker(D)$. Denoting by $D\Pi$ the projection $T_{(0, z)} \hat{Y} \to T_2 Y$, we claim that

$$D_{ev}^{\text{Morse}} = D\Pi \circ D_{ev} \circ T_\eta L.$$
Indeed, given $V \in T_\eta \text{Morse}(\gamma_+, \gamma_-)$, the operator $D^{\text{Morse}}_{\eta \text{ev}}$ sends it to $V(0)$. On the other hand, since the coefficient of $\partial_t$ in $T_\eta V$ vanishes when $s = 0$, the projection $D\Pi \circ D_{\text{ev}}$ sends $T_\eta V(0,0)$ to $V(0)$ as well.

Now we use (4.19) to conclude the proposition. By Lemma 4.23, the operator $D^{\text{Morse}}_{\text{ev}}$ is surjective. This implies that the map $D\Pi \circ D_{\text{ev}}$ must be surjective, so by definition of $D\Pi$ we deduce

$$\operatorname{Span}(\partial_t) + \operatorname{Im}(D_{\text{ev}}) = \operatorname{Im}(D\Pi \circ D_{\text{ev}}) = T_{(0,z)} \hat{Y}.$$  

Recalling (4.18), we see that $\operatorname{Span}(\partial_t) \subset \operatorname{Im}(D_{\text{ev}})$. Therefore, $\operatorname{Im}(D_{\text{ev}}) = \operatorname{Span}(\partial_t) + \operatorname{Im}(D_{\text{ev}}) = T_{(0,z)} \hat{Y}$, i.e., $D_{\text{ev}}$ is surjective. □

4.4. Relation to abstract constraints. Thus far in §4, we have counted points in moduli spaces of cylinders with a point constraint. To conclude this section, we relate this count to a point count in the moduli spaces involved in the $U$-maps constructed in §3.2.

We continue to work with the objects and notation described in Setup 4.1.

Let $X : Y \to Y$ be the trivial cobordism $[-\delta, \delta] \times Y$ equipped with the Liouville form

$$\lambda = e^{\delta} \alpha_f = e^{\delta + \epsilon f} \alpha$$

and let $E \subset X$ be an embedded, irrational ellipsoid that is the image of an embedding

$$t : E(b_1, \ldots, b_n) \to X$$

such that $t(0) = 0 \times z \in [-\delta, \delta] \times Y$

The boundary $\partial E$ has a shortest simple orbit $\kappa$. Let $J'$ be a compatible almost complex structure on $X \setminus E$ and fix the shorthand notation

$$\mathcal{M}_{X \setminus E} := \mathcal{M}_{0,\mathbf{A}}(X \setminus E; J'; \gamma_+\gamma_-, \Gamma_0), \quad \text{where} \quad \Gamma_0 = (\gamma_-, \kappa).$$

This is the moduli space of genus 0 $J$-holomorphic curves in the cobordism $X \setminus E : Y \to Y \cup \partial E$ asymptotic to $\gamma_+$ at $+\infty$ and $\gamma_- \cup \kappa$ at $-\infty$. Moreover, let $J''$ be an almost complex structure on $E$ and fix the shorthand notation

$$\mathcal{M}_E := \ev^{-1}(0) \subset \mathcal{M}_{0,\mathbf{A},1}(E; J''; \kappa, \varnothing).$$

This is the moduli space of $J''$-holomorphic planes in $E$ that pass through $0 \in E$ and that are asymptotic at $+\infty$ to $\kappa$. The goal for the rest of the section is to prove the following result.

**Proposition 4.24.** Assume that the period of $\gamma_\pm$ is equal to the minimal period of the Reeb flow of $\alpha$. There exists a choice of complex structures $J'$ on $X \setminus E$ and $J''$ on $E$ such that the moduli spaces

$$\mathcal{M}_{X \setminus E} \quad \text{and} \quad \mathcal{M}_E$$

are transversely cut out, 0-dimensional and equal to their compactifications (i.e. there are no buildings in their compactifications). Furthermore, they satisfy

$$\# \mathcal{M}(Y, J_f; \gamma_+; \gamma_-; z) = \# \mathcal{M}_{X \setminus E} \cdot \# \mathcal{M}_E \mod 2$$  \hspace{1cm} (4.20)

**Remark 4.25.** In fact, the equality (4.20) should be true over $\mathbb{Q}$ for any choice of complex structures, as a virtual count of points in the framework of Pardon [Par15, Par16]. This version of this result is proven by Siegel [Sie19], using $J$-holomorphic cascades and the Morse-Bott formalism.

In the spirit of the other results of this paper and for the sake of completeness, we provide a proof that uses only transversely cut out holomorphic curves and avoids Morse-Bott theory.

Our strategy to prove Proposition 4.24 is quite standard. First, we choose a complex structure satisfying a number of regularity hypotheses for somewhere injective curves. Second, we show the desired compactness results for $\mathcal{M}_{X \setminus E}$ and $\mathcal{M}_E$ for the chosen complex structures. Last, we use a parametric moduli space to construct a topological cobordism

$$\tilde{\mathcal{M}} \quad \text{from} \quad \mathcal{M}(Y, J_f; \gamma_+; \gamma_-; z) \quad \text{to} \quad \mathcal{M}_{X \setminus E} \times \mathcal{M}_E.$$
A possible building in the moduli space $\mathcal{M}_{X\setminus E \cup E}$. We will show in Lemma 4.27 that (generically) these buildings must be much simpler than this.

The existence of such a cobordism proves the desired result.

To begin the argument, we fix almost complex structures such that the relevant moduli spaces of somewhere injective curves are sufficiently regular. We are permitted to do this by standard generic transversality results (reviewed in §2.2.10, see Proposition 2.5).

**Setup 4.26.** For the rest of the section, fix compatible almost complex structures $J'$ on $X \setminus E$ and $J''$ on $E$ satisfying the following properties:

(i) the moduli spaces of finite energy, somewhere injective curves 

$$
\mathcal{M}^i_{b,A,m}(X \setminus E, J'; \Gamma_+, \Gamma_-) \quad \text{and} \quad \mathcal{M}^i_{b,h,j}(E, J'', \Xi_+, \Xi_-)
$$

are regular,

(ii) the evaluation maps 

$$
ev : \mathcal{M}^i_{b,h,1}(E, J'', \Xi_+, \Xi_-) \to \hat{\mathcal{X}}
$$

are transverse to $z$.

Given $J'$ and $J''$ as in Setup 4.26, there is a compactified moduli space that we denote by $\overline{\mathcal{M}}_{X\setminus E \cup E}$ consisting of genus 0 buildings $\bar{u}$ with the following levels.

- A sequence of levels $u^+_1, \ldots, u^+_k$ in $\hat{Y}$.
- A single level $u$ in the cobordism $X \setminus E$ of genus 0 with one positive end.
- A sequence of levels $u^-_1, \ldots, u^-_l$ in $\hat{Y}$.
- A sequence of levels $\nu_1, \ldots, \nu_m$ in $\partial E$.
- A single level $\nu$ in $\hat{E}$.

where the positive ends and negative ends of adjacent levels match in the usual way. A depiction of a possible building $\bar{u}$ is depicted in Figure 10.

We now show that (under our hypotheses) the only buildings appearing in $\overline{\mathcal{M}}_{X\setminus E \cup E}$ are those of the simplest possible form. We will require the following lemma.

**Lemma 4.27.** Consider $\epsilon > 0$ from Setup 4.1 and let $J'$ as in Setup 4.26. Let $\zeta_+$ and $\zeta_-$ be orbits of $Y$ satisfying

$$
A(\gamma_+) \geq A(\zeta_+) \geq A(\zeta_-) \geq A(\gamma_-)
$$
Finally, let $u$ be a finite energy, connected $1'$-holomorphic map in $\tilde{X} \setminus E$ of genus 0 such that

$$(4.22) \quad u \to \zeta_+ \; \text{at} \; +\infty \quad \text{and} \quad u \to \zeta_- \cup \Xi \; \text{at} \; -\infty$$

where $\Xi$ is a non-empty orbit sequence in $Y \cup \partial E$. Then for sufficiently small $\varepsilon$, we must have

$$\zeta_+ = \gamma_+ \quad \zeta_- = \gamma_- \quad \text{and} \quad \Xi = \kappa$$

**Proof.** First, factor $u$ as a branched cover $\psi$ and a somewhere injective map $v$,

$$u : \Sigma' \xrightarrow{\psi} \Sigma \xrightarrow{v} \hat{\Upsilon}.$$ 

Here $\nu$ has a positive and negative ends asymptotic to orbits

$$\beta_\pm \text{ covered by } \zeta_\pm \quad \text{and} \quad \Xi' \text{ covered by } \Xi.$$ 

It suffices to show that $\beta_\pm = \gamma_\pm$ and $\Xi' = \kappa$. Note that this will imply that $u = \nu$.

First, note that $\nu$ is genus 0. Indeed, we can compactify $\psi$ to a branched cover $S^2 \to \hat{\Sigma}$ where $\hat{\Sigma}$ is a closed surface given by compactifying $\Sigma$ along the punctures. Any closed surface with a non-trivial branched cover from $S^2$ is genus 0.

Next, note that since $\gamma_+$ and $\gamma_-$ are orbits corresponding to perturbations of Morse-Bott orbits of the same period and using the action bounds (4.21), we have

$$A(\beta_+) - A(\beta_-) = \frac{A(\zeta_+) - A(\zeta_-)}{\deg(\psi)} \leq \frac{A(\gamma_+) - A(\gamma_-)}{\deg(\psi)} = O(\varepsilon).$$

Thus, the orbit set $\Xi'$ cannot contain any orbits in $Y$ for small $\varepsilon$, otherwise the action of the orbit set $\beta_- \cup \Xi'$ would be larger than that of $\beta_+$. Moreover, any pair of orbits of $\alpha_f$ satisfying

$$A(\beta_+) - A(\beta_-) \leq O(\varepsilon) \quad \text{and} \quad A(\beta_-) \leq A(\beta_+),$$

must be perturbations of Morse-Bott orbits in the same Morse-Bott family $S_L$ for sufficiently small $\varepsilon$. In particular, $\beta_\pm$ correspond to pairs $(S_L, p_\pm)$, where $S_L \subset S_T$ and $p_\pm$ are critical circles of $f$. By Lemma 4.6, the Conley-Zehnder indices are given by

$$CZ(\beta_+) = RS(S_L) - \dim(S_L)/2 + \text{ind}_{Morse}^S(f; p_+)$$

$$CZ(\beta_-) = RS(S_L) - \dim(S_L)/2 + \text{ind}_{Morse}^S(f; p_-)$$

where $\text{ind}_{Morse}^S(f; p_\pm)$ denotes the Morse-Bott index of the critical circle $p_\pm$ in the tangent directions to $S_L$. In particular,

$$CZ(\beta_+) - CZ(\beta_-) \leq 2n - 2,$$

with equality only if $S_L = S_T$ (where $T$ is the minimal period of the Reeb flow of $\alpha$), and $\beta_\pm = \gamma_\pm$.

Finally, note that since $H_2(Y) = 0$, the Fredholm index is given by

$$\text{ind}(\nu) = CZ(\beta_+) - CZ(\beta_-) - |\Xi'| \leq 2n - 2 - |\Xi'|.$$ 

By the calculation in Example 2.16, we know that

$$|\Xi'| = n - 3 + \sum_{\gamma \in \Xi} CZ(\gamma) \geq 2n - 2,$$

since every orbit $\gamma$ on the boundary of an ellipsoid had $CZ(\gamma) \geq n + 1$. Moreover, equality holds if and only if $\Xi'$ is the length 1 sequence of the minimum action orbit $\kappa$. Overall we conclude that

$$\text{ind}(\nu) \leq 0$$

with equality if and only if

$$u = \nu, \quad \beta_+ = \gamma_+, \quad \beta_- = \gamma_- \quad \text{and} \quad \Gamma = \kappa.$$ 

By hypothesis, every somewhere injective curve is non-negative index. Therefore $\text{ind}(V) = 0$ and thus it coincides with $u$, and their ends coincide with $\gamma_+$ and $\gamma_- \cup \kappa$ as required.
Lemma 4.28. Let $\epsilon > 0$ in Setup 4.1 be sufficiently small and choose $J', J''$ as in Setup 4.26. Then

(i) The moduli spaces $\mathcal{M}_{X, E}$ and $\mathcal{M}_E$ are compact, regular and 0-dimensional.

(ii) We have an equality of moduli spaces

$$\overline{\mathcal{M}}_{X, E, E} = \mathcal{M}_{X, E} \times \mathcal{M}_E$$

Proof. The regularity and dimension of both moduli spaces follows from our choice of $J'$ and $J''$. We now argue for the compactness of $\mathcal{M}_{X, E}$ and $\mathcal{M}_E$, and the equality (ii).

First, to show that $\mathcal{M}_{X, E}$ is compact, consider an arbitrary building

$$\tilde{u} \in \overline{\mathcal{M}}(y_+, y_- \cup \kappa; J') = \overline{\mathcal{M}}_{0, A}(X \setminus E, J'; y_+, y_-)$$

Denote by $u$ the level of $\tilde{u}$ in $X \setminus E$. Then, $u$ is genus 0 and has one positive end asymptotic to an orbit $\xi_+$ with $A(\xi_+) \leq A(y_+)$, a negative end $\xi_-$ satisfying $A(\xi_-) \geq A(y_-)$ and some other negative ends $\Xi$. Thus by Lemma 4.27, we have

$$\xi_+ = y_+, \quad \xi_- = y_-, \quad \Xi = \kappa.$$  

This implies that $\tilde{u} = u$ and so $\mathcal{M}_{X, E}$ is compact.

Next, to show that $\mathcal{M}_E$ is compact, let $\tilde{v}$ be a building in the compactification $\overline{\mathcal{M}}(\kappa; z, J'')$, then there is a bottom level

$$\nu : \Sigma \to \hat{E} \quad \text{with} \quad z \in \text{Im}(\nu)$$

Since the whole building is genus 0, so is $\nu$, and $\nu$ has no negative ends since $E$ is a cobordism to the empty set. Moreover, the action of the positive ends $\Xi_+$ must be less than $A(\kappa)$. Since $\kappa$ has the minimum action over all orbit sets, we must have $\Xi_+ = \kappa$, so

$$\nu \in \mathcal{M}_E \quad \text{and} \quad \nu = \tilde{v}$$

Finally, to show (ii), let $\tilde{u}$ be a building in the compactified moduli space $\overline{\mathcal{M}}_{X, E, E}$. Then the level $u$ in $X \setminus E$ satisfying the hypotheses of Lemma 4.27, and so we have

$$u \in \mathcal{M}_{X, E}$$

The level $\nu$ of $\tilde{u}$ in $E$ must thus have positive ends of action bounded by the actions of the negative end $\kappa$ of $u$. This is only possible if

$$\nu \in \mathcal{M}_E$$

The only remaining levels in $\tilde{u}$ must be trivial for action reasons, so $\tilde{u}$ consists only of the levels $u$ and $\nu$. This proves the result. $\square$

Given a choice of $J'$ and $J''$, there is a family $J_{[1, \infty)}$ of almost complex structures on $X$,

$$J'_{#R}J'' \quad \text{for} \quad R \in [1, \infty).$$

This family is acquired by identifying $X$ with the space

$$X \simeq X \setminus E \cup_{\partial E} [0, R] \times \partial E \cup_{\partial E} E$$

and setting $J'_{#R}J''$ be equal to $J'$ on $X \setminus E$, $J''$ on $E$ and letting $J'_{#R}J''$ be translation invariant on $[0, R] \times \partial E$ in the $[0, R]$-direction. This family has associated parametric moduli space of genus 0 curves from $y_+$ to $y_-$, i.e. the space

$$\mathcal{M}_{0, A, 1}(X, J_{[1, \infty)}; y_+, y_-).$$

This map has an evaluation map $ev$ to $\hat{X}$, and we now need to consider the parametric moduli space of cylinders passing through $z$. We adopt the shorthand notation

$$\widetilde{\mathcal{M}}_{[1, \infty)} := ev^{-1}(z) \subset \mathcal{M}_{0, A, 1}(X, J_{[1, \infty)}; y_+, y_-).$$
See §2.2.9 for a discussion of parametric moduli spaces. By the appropriate version of SFT compactness (cf. [BEH⁺03, Prop. 10.6]), given a sequence of elements

$$(R_i, u_i) \in \hat{\mathcal{M}}_{[1, \infty)} \quad \text{with} \quad R_i \to \infty$$

there is a limiting building

$$\hat{u} \in \hat{\mathcal{M}}_{X_{\infty}} \overset{\text{Lemma 4.28}}{=} \mathcal{M}_{X_{\infty}} \times \mathcal{M}_E$$

that $u_i$ converges to with respect to the BEHWZ topology in [BEH⁺03]. Since $\mathcal{M}_{X_{\infty}}$ and $\mathcal{M}_E$ consist entirely of regular curves (by Setup 4.26), there is a gluing map

$$\text{glue} : \mathcal{M}_{X_{\infty}} \times \mathcal{M}_E \times (R, \infty) \to \hat{\mathcal{M}}_{[1, \infty)}$$

that is a homeomorphism for sufficiently large $R$. The construction of this gluing map is carried out, for example, in [Par15, §5].

**Lemma 4.29.** There exists a $[0, \infty)$-family of compatible almost complex structures on $X$, denoted

$$I_{[0,\infty)} = \{I_R\}_{R \in [0, \infty)}$$

that has the following properties.

(i) $I_R = I_f$ for $R$ near $0$ and $I_R = I'_\# R I_f$ for $R$ sufficiently large.

(ii) The parametric moduli space

$$\hat{\mathcal{M}}_{[0, \infty)} := \text{ev}^{-1}(z) \subset \mathcal{M}_{0, A, 1}(X, I_{[0, \infty)}; \gamma^+, \gamma^-)$$

is a 1-manifold with boundary $\mathcal{M}(Y, I_f; \gamma^+, \gamma^-; z)$.

(iii) The natural projection map to the parameter space

$$\pi : \hat{\mathcal{M}}_{[0, \infty)} \to [0, \infty)$$

is proper.

**Proof.** Let $\{I_R\}_{R \in [0, \infty)}$ be a 1-parameter family of compatible almost complex structures on $X$ satisfying (i). We use $\mathcal{M}_R$ to fiber of $\pi$ at $R$, i.e.

$$\mathcal{M}_R := \{u : (R, u) \in \hat{\mathcal{M}}_{[0, \infty)}\}$$

We also let $\mathcal{M}^i_R$ denote the subset of somewhere injective curves in $\mathcal{M}_R$ and $\overline{\mathcal{M}}_R$ denote the BEHWZ compactification. All of the holomorphic curves in the moduli spaces

$$\overline{\mathcal{M}}(Y, I_f; \gamma^+, \gamma^-; z) \quad \text{and} \quad \mathcal{M}_R \text{ for large } R > 0$$

consist entirely of regular curves (and no buildings) by construction of $I'$ and $I''$, and Proposition 4.3. As regularity is an open condition, we automatically know that $\mathcal{M}_R$ is compact and regular (in the usual sense, not parametrically) outside of $[a, b] \subset [0, \infty)$ for some $0 < a < b < \infty$. Using Proposition 2.5, we perturb $\{I_R\}_{R \in [0, \infty)}$ on $(a - \delta, b + \delta)$ for small $\delta$ so that the moduli spaces

$$\mathcal{M}^i_{g, A, m}(X, I_{[a, b]}; \Gamma^+, \Gamma^-)$$

are parametrically regular and the natural evaluation maps

$$\text{ev} : \mathcal{M}^i_{g, A, 1}(X, I_{[a, b]}; \Gamma^+, \Gamma^-) \to \hat{X}$$

are transverse to $z$. We now claim that under these hypotheses, we have

$$\text{(4.23) } \mathcal{M}^i_R = \mathcal{M}_R = \overline{\mathcal{M}}_R \text{ for any } R \in [a, b].$$

This implies that the space $\hat{\mathcal{M}}_{[0, \infty)}$ is a compact 1-manifold, implying (ii) and (iii).

---

Note that the gluing result in [Par15] is much more general than the one required here. In particular, we only require the gluing map for a specific stratum of the moduli space denoted $\mathcal{M}_{IV, \text{reg}}$ in [Par15].
To prove (4.23), we consider a $f_R$-holomorphic building 
\[ \bar{u} \in \overline{M}_R. \]
Let $u$ be the level of $\bar{u}$ that contains $z$ and let $v$ be the underlying simple curve. By identical reasoning to Lemma 4.27, $v$ must be a holomorphic cylinder in $\hat{X}$ of area $O(\varepsilon)$. For small $\varepsilon$, $v$ must therefore be asymptotic to closed orbits $\beta_{\pm}$ that are Morse perturbations of orbits in the same Morse-Bott family $S_L$. In particular
\[ \text{ind}(v) \leq |\beta_{+}| - |\beta_{-}| = CZ(\beta_{+}) - CZ(\beta_{-}) \leq 2n - 2, \]
with equality if and only if
\[ (4.24) \quad v = u, \quad \beta_{+} = \gamma_{+} \quad \text{and} \quad \beta_{-} = \gamma_{-}. \]
Thus consider the case where $\text{ind}(v) < 2n - 2$. Then the parametric moduli space containing $v$ with a marked point added,
\[ \mathcal{M}_{0,1}^i(X, J_{[a,b]}; \beta_{+}, \beta_{-}), \]
is a manifold of dimension less than $2n$. By the Sard-Smale theorem, the image of the (smooth) evaluation map
\[ ev : \mathcal{M}_{0,1}^i(X, J_{[a,b]}; \beta_{+}, \beta_{-}) \to \hat{X} \]
is of the first category (i.e. a countable union of nowhere dense sets) and has dense complement. In particular, after pulling back $J_{[a,b]}$ by a $[a,b]$-family of small symplectomorphisms supported near $z$, we can assume that
\[ z \notin \text{ev}(\mathcal{M}_{0,1}^i(X, J_{[a,b]}; \beta_{+}, \beta_{-})) \]
for any $\beta_{+}, \beta_{-}$ satisfying the hypotheses above. In particular, after modifying $J_{[a,b]}$, we may assume that $v$ does not contain $z$ unless $\text{ind}(v) = 2n - 2$, which implies (4.24). This concludes the proof. \hfill \Box

We can now conclude this section with a proof of Proposition 4.24.

Proof. The compactness, transversality and dimension of the moduli spaces
\[ \mathcal{M}_{X;E} \quad \text{and} \quad \mathcal{M}_{E} \]
are demonstrated in Lemma 4.28. To prove (4.20), we simply note that for a choice of $f_{[0,x]}$ as in Lemma 4.29, the parametric moduli space
\[ \mathcal{M} := \pi^{-1}([0,R]) \subset \mathcal{M}_{(0,x);y} \]
is a cobordism from $\mathcal{M}(Y, J_{f}; \gamma_{+}, \gamma_{-}; z)$ to $\mathcal{M}_{X;E} \times \mathcal{M}_{E}$ as long as $a > 0$ is close to 0 and $b$ is sufficiently large. \hfill \Box

5. Vanishing spectral gap for ellipsoids

Our main goal for this section is to prove Theorem 6, which states that the strong closing property holds for all ellipsoids. We will show that the spectral gap vanishes for ellipsoids and therefore, the strong closing property will follow from Theorem 3.19.

Following the strategy discussed in §1.3, we proceed in two steps. First, in Section 5.1 we consider ellipsoids $E(m) = E(m_1, \ldots, m_n)$ where $m_j \in \mathbb{N}$. The Reeb flow is periodic on the boundary $\partial E(m)$, so we may apply the results of Section 4. Specifically, we use Proposition 4.3 and Proposition 4.24 to show that certain coefficients of the map $U_{P_0}$, defined in Example 3.3, are non-zero and deduce that the spectral gap of a certain class vanishes for such ellipsoids. Second, in Section 5.2, we use the vanishing of spectral gaps for integer ellipsoids to show that the total spectral gap vanishes for irrational ellipsoids as well.
Remark 5.4. We fix the following notation for the rest of the section. Given an \( n \)-tuple of positive numbers \( a := (a_1, \ldots, a_n) \), consider the ellipsoid \( E(a) \).

- We denote by \( \{M_k\}_{k \in \mathbb{N}} \) or \( \{M^n_k\}_{k \in \mathbb{N}} \), the sequence obtained by reordering the union \( a_1 \cdot \mathbb{N} \cup \cdots \cup a_n \cdot \mathbb{N} \) to be a non-decreasing sequence with repetitions.
- We let \( \gamma_k \) or \( \gamma^n_k \) denote the \( k \)-th periodic Reeb orbit ordered by action. Here we introduce Morse-Bott perturbations if the Reeb flow on \( \partial E(a) \) is degenerate.
- We let \( x_k \) denote the generator of \( CH(\partial E(a)) \) corresponding to \( \gamma_k \) (see Example 2.16). Note that
  \[ |x_k| = 2n - 2 + k \quad \text{and} \quad \mathcal{A}(\gamma_k) = M^n_k. \]

5.1. Rational Ellipsoids. Consider an integer ellipsoid \( E(m) := E(m_1, \ldots, m_n) \) where \( m_j \in \mathbb{N} \) for \( j = 1, \ldots, n \). Denote the least common multiple of \( m_1, \ldots, m_n \) by
  \[ T := \text{lcm}(m_1, \ldots, m_n). \]

The Reeb flow on \( \partial E(m) \) is periodic of period \( T \). Since \( m_j \) divides \( T \) for all \( j = 1, \ldots, n \), the common multiple \( T \) appears in the sequence \( \{M^n_k\} \) exactly \( n \) times. We denote the first index in which \( T \) appears in \( M^n_k \) by
  \[ k_T := \min\{k \in \mathbb{N} : M^n_k = T\}. \]

The main goal of this subsection is to prove the following result.

Theorem 5.2. Let \( E(m) \) be an integer ellipsoid as above, and let \( T := \text{lcm}(m_1, \ldots, m_n) \). Then
  \[ \langle U_{P_0} x_{k_T + n - 1}, x_{k_T} \rangle \neq 0. \]

Here \( P_0 \) is the tangency abstract constraint from Example 3.3.

As mentioned above, the least common multiple \( T \) of \( m_1, \ldots, m_n \) occurs with multiplicity \( n \) in the sequence \( M^n_k \). Therefore, the action \( M_{k_T + n - 1} \) of \( x_{k_T + n - 1} \) and the action \( M_{k_T} \) of \( x_{k_T} \) are both equal to \( T \). This implies the following corollary of Theorem 5.2.

Corollary 5.3. (Theorem 7) Let \( E(m) \) be an integer ellipsoid. Then
  \[ s_{\|E \|}(\partial E, \lambda|_{\partial E}) = s_{\|E \|}(\partial E, \lambda|_{\partial E}) = T \quad \text{where} \quad U = U_{P_0} \text{ and } \sigma = x_{k_T + n - 1}. \]

Remark 5.4 (Generalizations). Theorem 5.2 and Corollary 5.3 can be generalized in two directions.

(i) If \( E(a) \) is a rational ellipsoid, i.e. an ellipsoid where
  \[ a_i/a_j \in \mathbb{Q} \text{ for any } i, j, \quad \text{or equivalently} \quad c \cdot a = m \in \mathbb{Z}^n \text{ for some } c > 0. \]

In this case, the contact form on \( \partial E(a) \) is simply a scaling of the one on \( \partial E(m) \). Moreover, there is an equivalence of filtered groups
  \[ CH^L(\partial E(a)) = CH^L(\partial E(c \cdot a)) \]
  that commutes with all (constrained) cobordism maps. Thus Theorem 5.2 and Corollary 5.3 generalize immediately to this case.

(ii) Any positive integer multiple \( q \) of the period \( T \) similarly corresponds to \( n \) classes
  \[ x_{k_1}, \ldots, x_{k_q + n - 1} \in CH(\partial E(m)) \]
  of the same action. Our proof of Theorem 5.2 almost generalizes to show that
  \[ \langle U_{P_0} x_{k_T + n - 1}, x_{k_T} \rangle \neq 0. \]

However, the proof uses Proposition 4.24, which we only show for the case of \( q = T \). We believe that the discussion in [Sie19, \S 5.5], rigorously carried out using the VFC methods
of [Par15], should suffice to generalize the moduli space correspondence in Proposition 4.24, but we do not carry out this analysis here.

**Proof of Theorem 5.2.** The result follows from combining Proposition 4.3 and Proposition 4.24, which together assert that the moduli space count corresponding to the coefficient of $x_{k^2}$ in $U_{P_0, x_{k^2} + \mathbb{R}^{n-1}}$ is non-zero.

**Step 1 - Setup.** We start by showing that our setting satisfies the assumptions of Propositions 4.3 and 4.24, which are stated in Setup 4.1. Let $(Y, \alpha)$ be the boundary of the integer ellipsoid $E := E(m)$ with its standard contact form. The Reeb flow associated with this contact form is given by

$$(z_1, \ldots, z_n) \mapsto (e^{2\pi i / m_1} z_1, \ldots, e^{2\pi i / n_1} z_n)$$

and has period $T$. Consider the function $f : Y \to \mathbb{R}$ given by the restriction to $\partial E$ of the map

$$(\tilde{f} : \mathbb{C}^n \to \mathbb{C}^n, \quad z = (z_1, \ldots, z_n) \mapsto \frac{\sum_{j=1}^n (j-1)|z_j|^2}{\sum_{j=1}^n |z_j|^2}.$$  \label{eq:5.1}

The function $f$ is invariant under the $\mathbb{T}^n$-action on $\mathbb{C}^n$ and thus is invariant under the Reeb flow. Its critical circles are precisely the intersections of $\partial E$ with the complex axes

$$\gamma_i := \partial E \cap C_i = \left\{ (0, \ldots, \frac{m_i}{m} e^{2\pi i / m_1}, 0, \ldots, 0) : t \in [0, m_i] \right\}.$$  \label{eq:5.2}

The function $f$ is Morse–Bott. Indeed, the restriction of its Hessian to the normal to $\gamma_i$ is non-degenerate, can be seen as follows. Along $\gamma_i$ the Hessian of the map $\tilde{f}$ is given by

$$\text{Hess}(\tilde{f})|_{\gamma_i} = \frac{\pi}{m_i} \cdot \text{diag}(1 - i, 1 - i, 2 - i, 2 - i, \ldots, n - i, n - i),$$

which is degenerate only on the $i$-th $\mathbb{C}$ factor. The tangent space to $\partial E$ along $\gamma_i$ is given by

$$T\partial E|_{\gamma_i} = C^{i-1} \oplus \langle R|_{\gamma_i} \rangle \oplus \mathbb{C}^{n-i},$$

where $R$ is the Reeb vector field. Therefore, the Hessian of $f$ along $\gamma_i$ is degenerate only in the Reeb direction which lies in the tangent space of $\gamma_i$, so $f$ is Morse-Bott with index

$$\text{ind}(f, \gamma_i) = 2(i - 1)$$

at the critical circle $\gamma_i$.

The orbits $\gamma_+$ and $\gamma_-$ are the period $T$ iterate of $\gamma_1$ and $\gamma_n$, i.e.

$$\gamma_+ := \gamma_1^{T/m_1} \quad \text{and} \quad \gamma_- := \gamma_n^{T/m_n}.$$  \label{eq:5.3}

The contact flag is given by the following sequence $Y_3 \subset \cdots \subset Y_{2n-1} = Y$ of nested ellipsoids:

$$Y_{2j-1} := \partial E \cap (\mathbb{C}^{j-1} \times \{0\}^{n-j} \times \mathbb{C}) \cup \{0\}^{n-j} \times \mathbb{C}, \quad j = 2, \ldots, n.$$  \label{eq:5.4}

Since $Y_{2j-1} \simeq \partial E(a_1, \ldots, a_{j-1}, a_n) \subset C^J$, we have $H_2(Y_{2j-1}) = 0$. Moreover, $Y_{2j-1}$ is invariant under the periodic Reeb flow and the gradient flow of $f$. The intersection $\xi_{2j-1} := \xi \cap TY_{2j-1}$ is a $J$-invariant contact structure on $Y_{2j-1}$. The formula (5.2) implies that the restriction of $f$ to $Y_{2j-1}$ is also Morse–Bott. In addition, for $i$ and $j$ such that $\gamma_i \subset Y_{2j-1}$ (that is, either $i = n$ or $i < j$), the restriction of $\text{Hess}(f)|_{\gamma_i}$ to the symplectic orthogonal of $\xi_{2j-1}$ in $\xi_{2j+1}$ is equal to $\text{diag}(j - i, j - i)$, which is positive definite unless $i = n$.

Finally, take $z$ be any point in $Y_3 \backslash \{\gamma_- \cup \gamma_+\}$. Since $\gamma_+$ and $\gamma_-$ are the only critical circles in $Y_3$, $z$ lies in the intersection of the stable manifold of $\gamma_-$ and the unstable manifold of $\gamma_+$ which concludes the assumptions stated in Setup 4.1.

**Step 2 - Applying §4.** Now we apply the results of §4 to show that the coefficient $\langle U_{P_0, x_{k^2} + \mathbb{R}^{n-1}}, x_{k^2} \rangle$ does not vanish. Consider the contact form $\alpha_f$ on $Y$ and the trivial exact cobordism

$$X = [-\delta, \delta] \times Y : (Y, e^\delta \alpha_f) \to (Y, e^{-\delta} \alpha_f).$$
We let $W \subset X$ be an embedded irrational ellipsoid with minimal Reeb orbit $\kappa$. By Proposition 4.3 and 4.24, for any $\varepsilon > 0$ sufficiently small as in Setup 4.1, we can choose a compatible complex structure $J$ on $X \setminus W$ and $\delta > 0$ (as in §4.4) such that the moduli space

$$\overline{M}_{0,A}(\gamma_+, \gamma_- \cup \kappa) = \mathcal{M}_{0,A}(\gamma_+, \gamma_- \cup \kappa)$$

is regular, contains no buildings and has point count

$$\#\overline{M}_{0,A}(\gamma_+, \gamma_- \cup \kappa) = \#\overline{\mathcal{M}}(Y, J; \gamma_+, \gamma_-; z) = 1 \mod 2.$$

Here $A \in S(X \setminus W; \gamma_+; \gamma_- \cup \kappa)$ is the unique homology class from $\gamma_+$ to $\gamma_- \cup \kappa$ in $X \setminus W$. On the other hand, the abstract constraint $P_0$ is the dual constraint (see Example 3.5) of the generator $\kappa$ of $CH(W)$. Thus by Lemma 3.10, we know that

$$\langle U_{P_0} x_{k_T+n-1}, x_{k_T} \rangle = 1 \mod 2.$$

In particular, $\langle U_{P_0} x_{k_T+n-1}, x_{k_T} \rangle$ is non-zero. □

5.2. Irrational Ellipsoids. We now use the non-vanishing of coefficients of $U_{P_0}$ for integer ellipsoids, established in Theorem 5.2, to conclude that the spectral gap vanishes for all ellipsoids.

**Theorem 5.5.** For any ellipsoid $E(a) := E(a_1, \ldots, a_n)$, we have $\text{gap}(\partial E(a)) = 0$.

We begin by giving a very simple proof using the rational ellipsoid case (Corollary 5.3 and Remark 5.4) and Proposition 3.22. We start with the following approximation property for ellipsoids. This uses Dirichlet’s approximation theorem.

**Lemma 5.6.** Let $E(a)$ be any ellipsoid and fix $\varepsilon > 0$. There exists an ellipsoid $E(r) = E(r_1, \ldots, r_n)$ with rationally dependent entries such that

$$E(r) \subseteq E(a) \subset \sqrt{1 + \frac{\varepsilon}{T}} E(r)$$

Here $T$ denotes the period of the Reeb flow on $\partial E(a)$.

**Proof.** We apply the simultaneous version of Dirichlet’s approximation theorem to the sequence

$$\frac{1}{a_1}, \ldots, \frac{1}{a_n}.$$

This result states that, for any natural number $N$, there exist integers $q_1, \ldots, q_n$, and $T'$ such that $T' \leq N$ and for all $i = 1, \ldots, n$,

$$\left| \frac{1}{a_i} - \frac{q_i}{T'} \right| \leq \frac{1}{T'} \cdot \frac{1}{N^{1/n}}, \quad \text{or equivalently,} \quad |a_i q_i - T'| \leq \frac{a_i}{N^{1/n}}.$$

For $\varepsilon > 0$ fixed as in the lemma statement, choose $N$ so that

$$N > \left(2 \frac{\max\{a_1, \ldots, a_n\}}{\varepsilon} \right)^n.$$

Under this hypothesis on $N$, we have the following bound for each $i$,

$$|a_i q_i - T'| < \varepsilon/2.$$ (5.4)

Denote $T := T' - \varepsilon/2$ and $r_i := \frac{T}{q_i}$, and consider

$$E(r) := E(r_1, \ldots, r_n) = E \left( \frac{T}{q_1}, \ldots, \frac{T}{q_n} \right).$$

Clearly, the entries of $E(r)$ are rationally dependent. To check that $E(r) \subseteq E(a)$, we note that

$$a_i \left( \frac{T'}{q_i} - \frac{\varepsilon}{2q_i} \right) = \frac{T' - \varepsilon/2}{q_i} = r_i.$$
To check the second inclusion in (5.3), we note that
\[ a_i \left( \frac{5.4}{q_i} \right) = \frac{T' + \varepsilon/2}{q_i} = \frac{T + \varepsilon}{q_i} = (1 + \frac{\varepsilon}{T}) \cdot \frac{T}{q_i} = (1 + \frac{\varepsilon}{T}) \cdot r_i. \]
Therefore, we find that
\[ E(a) \subseteq \sqrt{1 + \frac{\varepsilon}{T} \cdot E(r) = E \left( 1 + \frac{\varepsilon}{T} \cdot r \right)} \]
Note that \( T \) as above is a positive integer satisfying \( T/r_i = q_i \) for all \( i = 1, \ldots, n \), but it is not necessarily the smallest positive integer with this property (i.e. the period of the Reeb flow on \( \partial E(r) \)). However, for any \( T_0 < T \) we have
\[ \sqrt{1 + \frac{\varepsilon}{T_0} \cdot E(r)} \subseteq \sqrt{1 + \frac{\varepsilon}{T} \cdot E(r)} \]
and so the assertion of the lemma still holds. \[ \square \]

Having established a good enough approximation of a general ellipsoid by a rational one, we are ready to prove that the spectral gap vanishes for all ellipsoids.

**Proof of Theorem 5.5.** Let \( E(a) \) be any ellipsoid. By Lemma 5.6, for each \( \varepsilon > 0 \) there exists an ellipsoid \( E(r(\varepsilon)) \) with rationally dependent entries \( r_i(\varepsilon) \) such that
\[ E(r(\varepsilon)) \subseteq E(a) \subseteq \sqrt{1 + \frac{\varepsilon}{T(\varepsilon)} \cdot E(r(\varepsilon))} \]
Here \( T(\varepsilon) \) is the period of the Reeb flow on \( \partial E(r(\varepsilon)) \). This implies that the contact forms satisfy
\[ \lambda_{\text{std}}|\partial E(r(\varepsilon))| \leq \lambda_{\text{std}}|\partial E(a)| \leq (1 + \frac{\varepsilon}{T}) \cdot \lambda_{\text{std}}|\partial E(r)|. \]
Here the contact forms on \( \partial E(r(\varepsilon)) \) and \( \partial E(a) \) are identified with contact forms on the sphere, and the order on contact forms is as in Notation 3.21. By Corollary 5.3 and Remark 5.4, there is a class
\[ \sigma(\varepsilon) \in \text{CH}(\partial E(r(\varepsilon))) \quad \text{with} \quad s_{U_{P_0}\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = s_{\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = T(\varepsilon) \]
In particular, the spectral gap of \( \partial E(r(\varepsilon)) \) satisfies
\[ \text{gap}_{\sigma(\varepsilon)}(\partial E(r(\varepsilon))) = 0 \]
The result is now an immediate consequence of Proposition 3.22, which asserts that
\[ \text{gap}(E(a)) \leq \lim_{\varepsilon \to 0} \text{gap}_{\sigma(\varepsilon)}(E(r(\varepsilon))) = 0 \]
\[ \square \]

5.3. **Structure Of The \( U \)-Map.** In [Iri22], Irie stated a second conjecture, which claims that certain sequence of coefficients of the \( U_{P_0} \) map do not vanish for ellipsoids. Our methods suffice to prove a related (but weaker) structure result of this type.

**Theorem 5.7.** Let \( E(a) \) be an irrational ellipsoid. There exists a sequence \( k(i) \xrightarrow[i \to \infty]{} \infty \) such that
\[ (5.5) \quad \{ U_{P_0} x_{k(i)} + n - 1, x_{k(i)} \} \neq 0 \quad \text{and} \quad \partial s_{x_{k(i)} + n - 1}(\partial E(a)) - s_{x_{k(i)}}(\partial E(a)) \xrightarrow[i \to \infty]{} 0. \]
This statement can compared to [Iri22, Conj. 5.1]. Theorem 5.7 implies Theorem 5.5 in the irrational case, and also uses the approximation of irrational ellipsoids by rational ones, given in Lemma 5.6.

\[ ^8 \]However, we warn the reader that there is a sizeable difference in our notation from [Iri22].
Proof. We will show that, for any \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that
\[
\langle U_{p_0} x_{k+n-1}, x_k \rangle \neq 0 \quad \text{and} \quad s_{x_{k+n-1}}(E(a)) - s_{x_k}(E(a)) \leq \varepsilon.
\]
Since \( E(a) \) is irrational, the spectral invariants \( s_{x_k}(\hat{E}(a)) = M_k^a \) are strictly increasing in \( k \).
Therefore, for any fixed \( k \), the difference in (5.6) is non-zero. This implies that we can construct a sequence of distinct \( k(i) \) diverging to \( \infty \) satisfying (5.5).

Thus, fix \( \varepsilon > 0 \). Let \( E(r) \) be the ellipsoid with rationally dependent entries from Lemma 5.6, and let \( T \) be the period of the Reeb flow on \( \hat{E}(r) \). Our proof consists of three steps. First, we study the cobordism map
\[
\Phi : A(\hat{E}((1 + \frac{\varepsilon}{T}) \cdot r)) \to A(\hat{E}(a)) \quad \text{of the cobordism map} \quad X = E((1 + \frac{\varepsilon}{T}) \cdot r) \setminus E(a)
\]
Next, we use the information we derive on \( \Phi \) to compare the \( U_{p_0} \)-maps for the two ellipsoids, apply Theorem 5.2 to \( E((1 + \frac{\varepsilon}{T}) \cdot r)) \) and deduce the non-vanishing of a certain coefficient in the \( U_{p_0} \)-map of \( E(a) \). Finally, we compare the actions of the relevant orbits in both ellipsoids and conclude that the difference in (5.6) is bounded \( \varepsilon \).

We let \( \{y_k\}_k \) denote the generators of \( A(\hat{E}(r)) \). Note that \( A(\hat{E}((1 + \frac{\varepsilon}{T})r)) = A(\hat{E}(r)) \) where the action is scaled by \( 1 + \frac{\varepsilon}{T} \). We will only use \( A(y_k) \) to denote the action with respect to \( \hat{E}(r) \). We also let \( k_T \) be the first index in which \( T \) appears in the sequence \( \{M_k^a\}_k \).

**Step 1 - Cobordism Map.** The goal of this step is to show that the image of \( y_{k_T+n-1} \) under \( \Phi \) does not contain \( x_1 \cdot x_{k_T} \) for sufficiently small \( \varepsilon \). That is
\[
\langle \Phi(y_{k_T+n-1}), x_1 \cdot x_{k_T} \rangle = 0 \quad \text{for sufficiently small} \quad \varepsilon > 0.
\]
Suppose otherwise. Then since the map \( \Phi \) decreases action and \( A(x_1) = M_1^a = a_1 \), we have
\[
(1 + \frac{\varepsilon}{T}) \cdot M_{k_T+n-1}^a = (1 + \frac{\varepsilon}{T}) \cdot A(y_{k_T+n-1}) \geq A(x_1) + A(x_{k_T}) = a_1 + M_{k_T}^a.
\]
On the other hand, by the definition of \( k_T \) and Lemma 5.6, we have
\[
M_{k_T+n-1}^a = M_{k_T}^a = T \quad \text{and} \quad M_k^a \leq M_k^a \leq (1 + \frac{\varepsilon}{T}) \cdot M_k^a \quad \text{for all} \quad k.
\]
We can then calculate that
\[
(1 + \frac{\varepsilon}{T}) \cdot M_{k_T+n-1}^a - M_{k_T}^a = (1 + \frac{\varepsilon}{T}) \cdot T - M_{k_T}^a \leq (1 + \frac{\varepsilon}{T}) \cdot T - M_{k_T}^a = (1 + \frac{\varepsilon}{T}) \cdot T - T = \varepsilon.
\]
When \( \varepsilon \) is smaller than \( a_1 \), this contradicts (5.8) and thus proves (5.7).

**Step 2 - \( U \)-map.** The goal of this step is to prove that
\[
\langle U_{p_0} (x_{k_T+n-1}, x_{k_T}) \rangle \neq 0.
\]
To prove this, we consider the commutative diagram
\[
\begin{array}{ccc}
A(\hat{E}((1 + \frac{\varepsilon}{T}) \cdot r)) & \xrightarrow{U_{p_0}} & A(\hat{E}((1 + \frac{\varepsilon}{T}) \cdot r)) \\
\downarrow & & \downarrow \\
A(\hat{E}(a)) & \xrightarrow{U_{p_0}} & A(\hat{E}(a))
\end{array}
\]
We need three observations about the terms in this diagram. First, by Theorem 5.2 and Remark 5.4, we know that that
\[
c_1 := \langle U_{p_0} (y_{k_T+n-1}, x_{k_T}) \rangle \neq 0.
\]
Moreover, since \( \Phi \) is a \( \mathbb{Z} \)-graded isomorphism of algebras that decreases word length (by Lemma 2.18), we must have
\[
c_2 := \langle \Phi(y_k), x_k \rangle \neq 0,
\]
Finally, by (5.7) we know that $\Phi(y_{kT+n-1})$ has a zero $x_1 \cdot x_{kT}$-coefficient. In other words
\begin{equation}
\langle \Phi(y_{kT+n-1}), x_1 \cdot x_{kT} \rangle = 0.
\end{equation}
Combining the equations (5.10-5.11), we can thus conclude that
\begin{equation*}
\langle U_{P_0} \circ \Phi(y_{kT+n-1}), x_{kT} \rangle = \langle \Phi \circ U_{P_0}(y_{kT+n-1}), x_{kT} \rangle = c_1 \cdot \langle \Phi(y_{kT}), x_{kT} \rangle = c_1 \cdot c_2 \neq 0.
\end{equation*}
We now claim that (5.12) implies the desired formula (5.9). Indeed, consider the element $h := \langle \Phi(y_{kT+n-1}), x_{kT+n-1} \rangle - \Phi(y_{kT+n-1})$.

Since $\Phi$ preserves grading and respects word length, $h$ is a linear combination of monomials $x_{i_1} \ldots x_{i_k}$ of length $k \geq 2$. Furthermore, $\langle h, x_1 \cdot x_{kT} \rangle = 0$ by (5.12). Since $U_{P_0}$ satisfies the Leibniz rule (Lemma 3.12), the only terms of word-length greater than 1 that can be mapped to $x_{kT}$ are of the form $x_j \cdot x_{kT}$, for $j$ such that $U_{P_0}(x_j)$ is a multiple of the unit. Since the $U_{P_0}$ decreases degree by $2n - 2$, the only elements that could be mapped to a constant have to be of degree $2n - 2$ and hence a multiple of $x_1$. Over all, $x_1 \cdot x_{kT}$ is the only term of word-length greater than 1 that can be mapped to $x_{kT}$. As a consequence, we conclude that $\langle U_{P_0}(h), x_{kT} \rangle = 0$. Therefore,
\begin{align*}
\langle \Phi(y_{kT+n-1}), x_{kT+n-1} \rangle \cdot \langle U_{P_0}(x_{kT+n-1}), x_{kT} \rangle &= \left( \langle U_{P_0} \circ \Phi(y_{kT+n-1}), x_{kT} \rangle + \langle U_{P_0}(h), x_{kT} \rangle \right) \\
&= \langle U_{P_0} \circ \Phi(y_{kT+n-1}), x_{kT} \rangle \neq 0.
\end{align*}
Since $\langle \Phi(y_{kT+n-1}), x_{kT+n-1} \rangle \neq 0$, this implies (5.9).

**Step 3 - action comparison.** Now choose $k = k_T$. We proved the $U$-map formula in (5.6) in Step 2, and we conclude by showing that the difference in (5.6) is bounded by $\varepsilon$. First, note that by definition
\begin{equation}
s_{x_{k+n-1}}(\partial E(a)) - s_{x_k}(\partial E(a)) = M^a_{k+n-1} - M^b_k.
\end{equation}
Recalling that $M^r_k \leq M^a_k \leq (1 + \frac{\varepsilon}{T}) \cdot M^r_k$ for all $k$, we have
\begin{equation*}
s_{x_{k+n-1}}(\partial E(a)) - s_{x_k}(\partial E(a)) \leq (1 + \frac{\varepsilon}{T})M^r_{k+n-1} - M^r_k = (1 + \frac{\varepsilon}{T}) \cdot T - T = \varepsilon.
\end{equation*}
This proves the desired bound, and concludes the proof. \[\square\]

**Remark 5.8** (Generalizations). Theorem 5.7 can be extended to rational ellipsoids given a more general correspondence result than Proposition 4.24, such as in [Sie19]. This is due to the fact that our assumption that the period of the orbits $\gamma_{\pm}$ is equal to the minimal period of the flow, rather than is divisible by this minimal period, was used only in Section 4.4. Given a more general correspondence result, one would be able to prove Theorem 5.2 for any $T$ that is divisible by $\text{lcm}(m_1, \ldots, m_n)$, and hence conclude the assertion of Theorem 5.7 for rational ellipsoids.

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