Stability of detonation profiles in the ZND limit

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Abstract

Confirming a conjecture of Lyng–Raoofi–Texier–Zumbrun, we show that stability of strong detonation waves in the ZND, or small-viscosity, limit is equivalent to stability of the limiting ZND detonation together with stability of the viscous profile associated with the component Neumann shock. More, on bounded frequencies the nonstable eigenvalues of the viscous detonation wave converge to those of the limiting ZND detonation, while on frequencies of order one over viscosity, they converge to one over viscosity times those of the associated viscous Neumann shock. This yields immediately a number of examples of instability and Hopf bifurcation of reacting Navier–Stokes detonations through the extensive numerical studies of ZND stability in the detonation literature.

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1 Introduction

In one-dimensional, Lagrangian coordinates, the reactive Navier–Stokes (rNS) equations modeling reacting flow for a one-step reaction may be written in abstract form as

\[ u_t + f(u)_x = \varepsilon(B(u)u)_x + kq\varphi(u)z, \]
\[ z_t = \varepsilon(C(u,z)z)_x - k\varphi(u)z, \]

where \( u, f, q \in \mathbb{R}^n, B \in \mathbb{R}^{n \times n}, z, k, C, \varphi \in \mathbb{R} \), and \( k, \varepsilon > 0 \). Here, \( u \) comprises the gas-dynamical variables of specific volume, particle velocity, and total energy; \( z \) measures mass fraction of unburned reactant or, more generally, “progress” of a single reaction involving multiple reactants; \( \varphi(u) \) is an “ignition function”, monotone increasing in temperature, and usually assumed for fixed density to be zero below a certain ignition temperature and positive above; \( q \) comprises quantities produced in reaction, in particular heat released; and \( k \) corresponds to reaction rate. Coefficients \( B \) and \( C \) model transport effects of, respectively, viscosity and heat conduction, and species diffusion, and \( \varepsilon \) measures relative size of transport vs. reaction coefficients, typically quite small.

A right-going viscous strong detonation wave is a smooth traveling-wave solution

\[ (u, z)(x,t) = (\bar{u}, \bar{z})(x - st), \quad \lim_{x \to \pm \infty} (\bar{u}, \bar{z})(x) = (u_\pm, z_\pm) \]

5 Region I: \( |\lambda| \leq C \)

5.1 “Slow”, or “reaction” zone, \( \bar{x} \leq -M \)

5.2 “Fast”, or “Neumann shock” zone, \( \bar{x} \geq -M \)

5.2.1 Variation in \( \varepsilon \)

5.3 Convergence to \( D_{ZND} \)

6 Region II: \( C/\varepsilon \geq |\lambda| \geq C >> 1 \)

6.1 Fast zone \( \bar{x} \geq -M \)

6.2 Slow zone \( \bar{x} \leq -M \)

6.2.1 Case a. \( C/\varepsilon \geq |\lambda| \geq 1/C\varepsilon, C > 0 \) arbitrary

6.2.2 Case b. \( 1/C\varepsilon \geq |\lambda| \geq C >> 1 \)

6.3 Convergence to \( D_{NS} \)

7 Region III: \( |\lambda| \geq C/\varepsilon, C >> 1 \)

A Asymptotic ODE theory

A.1 The conjugation lemma

A.2 The convergence lemma

A.3 The tracking lemma
of solutions of (1.1) with speed $s > 0$ connecting a burned state on the left to an unburned state on the right,

\[(1.3)\]
\[z_- = 0, \quad z_+ = 1,\]

with necessarily

\[(1.4)\]
\[
\phi(u_-) > 0, \quad \phi(u_+) = \phi'(u_+) = 0,
\]

and satisfying the extreme Lax characteristic conditions

\[(1.5)\]
\[
a^-_1 < \cdots < a^-_{n-1} < s < a^-_n, \quad a^+_1 < \cdots < a^+_n < s,
\]

where $a^\pm_j$ denote the eigenvalues of $df(u^\pm)$, ordered by increasing real part. Left-going viscous detonation waves satisfy symmetric conditions obtained by reflection, $x \to -x$.

Multi-step reactions may be modeled by the same equations with vectorial reaction variable $z \in \mathbb{R}^m$, and coefficients $q, C, \varphi, k$ modified accordingly. Likewise, the functions $f, \phi, B$ may be modified to depend, more realistically, also on $z$, reflecting the different chemical makeup of the gas after reaction, with no essential change at a mathematical level. For further discussion, see, e.g., [CF, FD, GS, Z1, LyZ1, LyZ2, LR TZ, TZ4, HuZ2].

A standard simplification in detonation theory is to neglect the small constant $\varepsilon$ and consider instead the formal $\varepsilon = 0$ limit, or Zeldovich–von Neumann-Doering (ZND) model

\[(1.6)\]
\[
\begin{align*}
  u_t + f(u)_x &= kq\varphi(u)z, \\
  z_t &= -k\varphi(u)z.
\end{align*}
\]

Indeed, there is by now a tremendous body of literature on this model; see for example [CF, FD, FW, Er1, Er2, LS, BMR, AT, AlT, KS] and references therein. The corresponding object to a viscous detonation wave for the (ZND) model is a right-going strong ZND detonation $\bar{u}^0$ of form (1.2)–(1.4) satisfying (1.6), smooth except at a single shock discontinuity at (without loss of generality) $x = 0$, known as a “Neumann shock” [CF, M, GS, LyZ1, LyZ2], where $u$ jumps from $u_*$ to $u_+$ as $x$ crosses zero from left to right, with $z \equiv 1$. We have the intuitive picture [CF] of a shock, or “reaction spike”, compressing a quiescent mixture and heating it to ignition point, followed by a slow “reaction tail” in which the reaction proceeds until all reactant is burned, while, meanwhile, $u$ varies from $u_*$ to $u_-$. A ZND detonation profile is determined implicitly [CF, HuZ2] by the property, obtained by integrating the traveling-wave ODE

\[(1.7)\]
\[
\begin{align*}
  -su' + f(u)' &= kq\varphi(u)z, \\
  -sz' &= -k\varphi(u)z
\end{align*}
\]

and adding $q$ times the second equation to the first, that $-s(u + qz) + f(u) \equiv$ constant, which, together with $z = 1$ for $x \geq 0$ and $z(-\infty) = 0$ implies that

\[(1.8)\]
\[
- su_+ + f(u_+) = - su_* + f(u_*) = -s(u_- - q) + f(u_-),
\]

\[3\]
giving a unique \( u_- \) and profile \( \tilde{u}^0, x \leq 0 \), for each Neumann shock \((u_+, u_-)\) of speed \( s \), so long as \( df(u) - sI \) remains invertible for all \( 0 \leq z \leq 1 \) along the curve determined by

\[
- s(u + qz) + f(u) \equiv -s(u_+) + f(u_+).
\]

For, solving (1.9) for \( u = u(z) \) by the Implicit Function Theorem then yields the profile on \( x \leq 0 \) by solution of the second equation \( z' = (-k/s)\phi(u(z))z \), a scalar equation with nonvanishing righthand side, so long as \( u \) remains in the region for which \( \phi(u) > 0 \).

A natural question is the relation between the formally limiting (ZND) equations and the behavior of the full (rNS) equations as \( \varepsilon \to 0 \). At the level of existence of detonation profiles, this was investigated by Majda [M] for a simplified model with \( u, z \in \mathbb{R}^1 \) and \( C \equiv 0 \) using direct, planar phase portrait analysis, and extended to the physical (rNS) model by Gardner [G] using Conley index techniques and Gasser–Szmolyan [GS] by geometric singular perturbation theory. More recently, Williams [W] has revisited the existence problem using more quantitative singular perturbation methods, generating detailed matched asymptotic expansions to all orders. In each case, with varying levels of detail, the result is that for each strong detonation profile \((\tilde{u}^0, \tilde{z}^0)\) of the (ZND) model with physical choice of \( f \), there exists a family \((\tilde{u}^\varepsilon, \tilde{z}^\varepsilon)\) of strong detonation profiles converging away from \( x = 0 \) as \( \varepsilon \to 0 \) to \((\tilde{u}^0, \tilde{z}^0)\), and near \( x = 0 \) to a viscous shock profile for the associated Neumann shock, in microscopic variables \( \tilde{x} = x/\varepsilon \).

In the present paper, using singular perturbation/asymptotic Evans function techniques developed in [PZ, HLZ, CHNZ, BHZ, HLyZ1, HLyZ2, OZ1, Z3], we investigate the stability of profiles \((\tilde{u}^\varepsilon, \tilde{z}^\varepsilon)\) in the ZND limit \( \varepsilon \to 0 \) for a class of models (1.1) including both the Majda model [M] studied in [M, L, RV, LyZ2, LRTZ] and the physical (rNS) equations studied in [G, GS, W, Z1, LyZ1, JLW, TZ4]. Our conclusion, confirming a conjecture of [LR TZ], is that (linear and nonlinear) stability in the ZND limit is equivalent to viscous stability of the component Neumann shock profile together with hyperbolic stability of the associated ZND detonation.

Together with the results of [HLyZ1] verifying viscous stability of ideal gas shocks, this gives a rigorous connection between viscous stability of (rNS) detonations and inviscid stability of the associated (ZND) detonations, yielding immediately a number of stability and bifurcation results through the extensive (ZND) literature. Specialized to the Majda model, it recovers the sole previous result, due to Roquejoffre and Vila [RV].

### 1.1 Assumptions

Loosely following [Z1, Z2, MaZ3, MaZ4, LRTZ, TZ4], we make the assumptions:

\[ \text{(H0)} \quad f, B, \phi, C \in C^2. \]

\[ \text{(H1)} \quad \text{The eigenvalues of } df(u) \text{ are real, distinct, and different from } s, \text{ for all } u \text{ near the image of ZND profile } \tilde{u}^0, \text{ in particular for } u = u_-, u_+, u_\pm. \]

---

1 Strictly speaking, a variant [L] with nonzero \( z \)-diffusion \( C > 0 \); however, the extension to the original Majda model is straightforward, substituting the weighted norm analysis of Sattinger [Sa] for the pointwise analysis of [LRTZ] in order to conclude nonlinear stability.
(H2) \[ B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \] with \( \Re b, \Re C \geq \theta > 0 \), and \( df = \begin{pmatrix} df_{11} & df_{12} \\ df_{21} & df_{22} \end{pmatrix} \), \( df_{11} \) and \( df_{12} \) constant, with the eigenvalues of \( df_{11} \) real, semisimple, and of one sign relative to \( s \), for all \( u \) under consideration (i.e., near a given detonation profile).

(H3) \[ \Re \sigma \left( -i \xi df(u) - \xi^2 B(u) \right) \leq -\frac{\theta \xi^2}{1+\xi^2}, \theta > 0, \] for all \( \xi \in \mathbb{R} \), and for all \( u \) near the image of ZND profile \( \bar{u}^0 \), in particular for \( u = u_-, u_+, u_+ \).

Remark 1.1. By block upper-triangular structure, we obtain from (H3) also

\[
\Re \sigma \left( -i \xi \begin{pmatrix} df(u) & 0 \\ 0 & 0 \end{pmatrix} - \xi^2 \begin{pmatrix} B(u) & 0 \\ 0 & C(u,z) \end{pmatrix} + \begin{pmatrix} 0 & qk\phi(u) \\ 0 & -k\phi(u) \end{pmatrix} \right) \leq -\frac{\theta \xi^2}{1+\xi^2},
\]

\( \theta > 0, \) for all \( \xi \in \mathbb{R} \), and for all \( u \) near the image of ZND profile \( \bar{u}^0 \), in particular for \( u = u_-, u_+, u_+ \), an assumption in the nonlinear stability/bifurcation analysis of \( \{TZ\} \).

Regarding connecting profiles, we make the further assumptions:

(P1) There exists a ZND profile \( u(x,t) = \bar{u}^0(x-st) \) of (1.6), smooth for \( x \geq 0 \), with \( \bar{u}^0(0^-) = u_* \) and \( \bar{u}^0(0^+) = u_+ \), that is transversal in the sense that \( df(\bar{u}^0) - sI \) is invertible for all \( x \), so that the profile is locally unique by (1.8).

(P2) There exists a viscous Neumann shock profile

(1.11) \[ u(x,t) = \bar{u} \left( \frac{x-st}{\varepsilon} \right), \quad \lim_{x \to -\infty} \dot{u}(x) = u_*, \quad \lim_{x \to +\infty} \dot{u}(x) = u_+ \]

of the associated nonreacting Navier-Stokes equations \( u_t + f(u)_x = \varepsilon (B(u)u)_x \), i.e., a connection between \( u_*, u_+ \) of the traveling-wave ODE \( B(\dot{u})\dot{u}' = f(\dot{u}) - f(u_+) - s(\dot{u} - u_+) \), that is transversal in the sense that \( df(\bar{u}) - s \) (constant by assumption (H2)) is invertible.

(P3) For \( \delta > 0 \) fixed and \( \varepsilon > 0 \) sufficiently small, there exist viscous detonation profiles \( \bar{u}^\varepsilon \) of (1.1), (1.2) satisfying for some \( C, \theta > 0 \), and \( 0 \leq k \leq 2 \),

\[
(1.12) \quad |\partial_x^k \left( (\bar{u}^\varepsilon(x), \bar{z}^\varepsilon) - (\bar{u}^0, \bar{z}^0) \right) \right| \leq C \varepsilon e^{-\theta |x|} \quad \text{for } x \leq -\delta,
\]

\[
(1.13) \quad |\partial_x^k \left( (\bar{u}^\varepsilon(x), \bar{z}^\varepsilon) - (\dot{u}(x/\varepsilon), 1) \right) \right| \leq C \varepsilon + C \varepsilon^{1-k} e^{-\theta |x|/\varepsilon} \quad \text{for } -\delta \leq x \leq 0,
\]

and

\[
(1.14) \quad |\partial_x^k \left( (\bar{u}^\varepsilon(x), \bar{z}^\varepsilon) - (\dot{u}(x/\varepsilon), 1) \right) \right| \leq C \varepsilon^{1-k} e^{-\theta |x|/\varepsilon} \quad \text{for } x \geq 0.
\]

\[ \text{This implies that the traveling wave ODE is nondegenerate type; the profile is then necessarily transversal by the extreme shock assumption \( \{M\} \) \{MaZ\}.} \]
Example 1.1. The physical single-species reactive compressible Navier–Stokes equations, in Lagrangian coordinates, are [Ch] [TZ4]

\[
\begin{align*}
\partial_t \tau - \partial_x u &= 0, \\
\partial_t u + \partial_x p &= \partial_x (\nu \tau^{-1} \partial_x u), \\
\partial_t E + \partial_x (pu) &= \partial_x (q d \tau^{-2} \partial_x z + \kappa \tau^{-1} \partial_x T + \nu \tau^{-1} u \partial_x u), \\
\partial_z + k \phi(T)z &= \partial_x (d \tau^{-2} \partial_x z),
\end{align*}
\]

where $\tau > 0$ denotes specific volume, $u$ velocity, $E = e + \frac{1}{2} u^2 + qz > 0$ total specific energy, $e > 0$ specific internal energy, and $0 \leq z \leq 1$ mass fraction of the reactant. Here, $\nu > 0$ is a viscosity coefficient, $\kappa > 0$ and $d > 0$ are respectively coefficients of heat conduction and species diffusion, $k > 0$ represents the rate of the reaction, and $q$ is the heat release parameter, with $q > 0$ corresponding to an exothermic reaction and $q < 0$ to an endothermic reaction. Finally, $T = T(\tau, e, z) > 0$ represents temperature and $p = p(\tau, e, z)$ pressure.

Under the standard assumptions of a reaction-independent ideal gas equation of state, $p = \Gamma \tau^{-1} e$, $T = c^{-1} e$, where $c > 0$ is the specific heat constant and $\Gamma$ is the Gruneisen constant, and a smooth ignition function $\phi$ vanishing identically for $T \leq T_i$ and strictly positive for $T > T_i$, it is shown in [MaZ3, TZ4] that each of (H0)–(H3) are satisfied. Likewise, (P1)–(P3) have been verified in [HuZ2, W] in this case.

Remark 1.2. We expect that (P3) can be shown by an argument like that of [W] to be a general consequence of (H0)–(H3) and (P1)–(P2) and not an independent assumption. Note that (P3) typically requires more regularity than we assume in (H0).

Remark 1.3. In (H1), it is enough that eigenvalues be semisimple. In the present, spectral, analysis, we use only that eigenvalues are real and distinct from $s$ (used to separate decaying and growing slow modes in the analysis of Section 6.2.2). In the linearized and nonlinear stability analysis for viscous detonations, one may relax the strict hyperbolicity assumption of [TZ4] to semisimplicity, as discussed for the viscous shock case in [MaZ4, Z1].

1.2 Main results

Recall that, associated with the linearized eigenvalue problems for ZND and rNS detonations are the Evans–Lopatinski determinant $D_{ZND}$ and the Evans determinant $D_{rNS}$, each analytic on $\Re \lambda \geq -\eta < 0$, with zeros corresponding to normal modes of the respective linear problems; see Sections 2 and 3 for precise definitions. Likewise, there is an Evans determinant $D_{NS}$ associated with the linearized eigenvalue problem for the associated viscous Neumann shock of the nonreacting Navier–Stokes equations (NS); see Section 5.

Weak Evans–Lopatinski stability of ZND detonations is defined as nonvanishing of $D_{ZND}$ on $\Re \lambda > 0$ and strong Evans–Lopatinski stability as nonvanishing on $\Re \lambda \geq 0$ except for a simple zero at $\lambda = 0$ [Er1, Er2, Z1, JLW]. Similarly, weak Evans stability of rNS detonations is defined as nonvanishing of $D_{rNS}$ on $\Re \lambda > 0$ and strong Evans stability as nonvanishing on $\Re \lambda \geq 0$ except for a simple zero at $\lambda = 0$ [Z1, LyZ1, LyZ2, JLW, LRTZ1, TZ4].
Likewise, weak Evans stability of the associated viscous Neumann shock is defined as nonvanishing of $D_{rNS}$ on $\Re \lambda > 0$ and strong stability as nonvanishing on $\Re \lambda \geq 0$ except for a simple zero at $\lambda = 0$: equivalently, nonvanishing of $\frac{D_{NS}(\lambda)}{\lambda}$ on $\Re \lambda \geq 0$ [MaZ3, Z1, Z2, Z4].

The following result established in [LRTZ, TZ4] equates strong Evans stability with linear and nonlinear stability of rNS detonations. A corresponding result holds for the component viscous Neumann shock [MaZ4, Z2, R, HR, HRZ, A2, Z4].

**Proposition 1.4 ([TZ4]).** Given (H0)–(H3), a viscous detonation profile (1.2) of (1.1) is $L^1 \cap L^p \to L^p$ linearly orbitally stable, $p \geq 1$, if and only if it is strongly Evans stable, in which case it is $L^1 \cap H^3 \to L^p \cap H^3$ asymptotically orbitally stable, for $p > 1$, with

\[
|\text{((\bar{u}, \bar{z}))(\cdot, t) - (\bar{u}, \bar{z})(\cdot - st - \alpha(t)))|_L^p \leq C|((\bar{u}_0, \bar{z}_0) - (\bar{u}, \bar{z})|_{L^1 u H^3}(1 + t)^{-\frac{1}{2} \left(1 - \frac{1}{p}\right)},
\]

(1.16)

\[
|\alpha(t)| \leq C|\bar{U}_0^\varepsilon - \bar{U}|_{L^1 u H^3},
\]

\[
|\dot{\alpha}(t)| \leq C|\bar{U}_0^\varepsilon - \bar{U}|_{L^1 u H^3}(1 + t)^{-\frac{1}{2}}
\]

for some $\alpha(\cdot)$, where $(\bar{u}, \bar{z})$ is the solution of (1.1) with initial data $(\bar{u}_0, \bar{z}_0)$.

**Proof.** This was established in Theorem 1.12, [TZ4], for the case described in example 1.1. However, the proof relied only on (H0)–(H3) and the consequent (1.10), hence extends to the general case. \qed

Our main theorem is the following result linking Evans stability of (rNS) profiles $(\bar{u}^\varepsilon, \bar{z}^\varepsilon)$ in the limit as $\varepsilon \to 0$ with Evans–Lopatinski stability of the limiting (ZND) profile $(\bar{u}^0, \bar{z}^0)$.

**Theorem 1.5.** Assuming (H0)–(H3), (P1)–(P3), weak Evans stability of $(\bar{u}^\varepsilon, \bar{z}^\varepsilon)$ for all $\varepsilon > 0$ sufficiently small implies weak Evans stability of the viscous profile of the component Neumann shock together with weak Lopatinski stability of the limiting ZND detonation $(\bar{u}^0, \bar{z}^0)$, while strong Evans stability of $(\bar{u}^\varepsilon, \bar{z}^\varepsilon)$ for all $\varepsilon > 0$ sufficiently small is implied by strong Evans stability of the viscous profile of the component Neumann shock together with strong Lopatinski stability of the limiting ZND detonation $(\bar{u}^0, \bar{z}^0)$.

More precisely, (i) For $\varepsilon > 0$ and $\eta > 0$ sufficiently small, there are no zeros of $D_{rNS}$ on $\Re \lambda \geq -\eta$ for $|\lambda| \geq C/\varepsilon$, $C$ sufficiently large. (ii) For $C \leq |\lambda| \leq C/\varepsilon$, $C$ sufficiently large, on $\Re \lambda > -\eta$ for $\eta, \varepsilon > 0$ sufficiently small, $\varepsilon$ times each zero of $D_{rNS}^\varepsilon$ converges to a zero of $\frac{D_{NS}(\lambda)}{\lambda}$ on $\Re \lambda \geq 0$; moreover, each zero of $D_{NS}$ on $\Re \lambda > 0$ is the limit of $\varepsilon$ times a zero of $D_{rNS}$ on $\Re \lambda > 0$, for $C \leq |\lambda| \leq C/\varepsilon$. (iii) For $|\lambda| \leq C_0$, $C_0$ arbitrary, on $\Re \lambda \geq -\eta < 0$, the set of zeros of $D_{NS}^\varepsilon$ converges as $\varepsilon \to 0$ to the set of zeros of $D_{ZND}$, for any sufficiently small $\eta > 0$ such that $D_{ZND}$ does not vanish for $\Re \lambda = -\eta$ and $|\lambda| \leq C_0$.

**Proof.** Assertions (i)–(iii) are established in Proposition 7.1, Corollary 6.2, and Corollary 5.3, whence the remaining assertions follow by definition of weak and strong stability of the various waves. \qed

**Remark 1.6.** Assuming Evans stability of the associated viscous Neumann shock, we recover from (ii)–(iii), taking $C_0 \to \infty$ and $\varepsilon \to 0$, the somewhat delicate result of [Er1, Er2] that $D_{ZND}$ does not vanish on $\Re \lambda \geq 0$ for $|\lambda|$ sufficiently large.
Remark 1.7. Recall [JLW] that, for fixed $\varepsilon$, low-frequency Evans stability, defined as strong Evans stability for $|\lambda| \leq c_0$, some $c_0 > 0$, is equivalent for either ZND or rNS detonations to the simpler condition of “Chapman–Jouget” stability; see [JLW] for further details. Thus, low-frequency stability of ZND waves is necessary for Evans stability of rNS detonations, along with weak Evans–Lopatinski stability as stated in Theorem 1.5.

1.3 Discussion and open problems

Together, Proposition 1.4 and Theorem 1.5 give a rigorous connection between Evans–Lopatinski stability of (ZND) detonations and nonlinear stability of nearby (rNS) detonations for $\varepsilon > 0$ sufficiently small, giving a satisfying mathematical validation of physical conclusions made through the extensive ZND stability studies in detonation literature. By contrast, to our knowledge there is no analog of Proposition 1.4 for the (ZND) equations themselves, and indeed the physical meaning of Evans–Lopatinski stability in that context in terms of nonlinear stability or well-posedness is unclear; see [JLW] for further discussion.

More, convergence of zeros of $D_{rNS}^\varepsilon$ to those of $D_{ZND}$ implies that finer phenomena such as Hopf bifurcation are inherited in the ZND limit as well as stability. This is perhaps more important, as it is well-known that ZND detonations are frequently unstable, bifurcating to pulsating and cellular fronts. See [TZ2, TZ3, TZ4, SS, BeS Z] for a rigorous discussion of such bifurcations in the context of (rNS).

Existence, stability, and bifurcation of detonations away from the ZND limit are important open problems. Such general situations appear to require numerical investigation, as is standard in the combustion literature even for the simpler (ZND) model; see, e.g., [LS, HuZ2] and references therein. Treatment of the ZND limit in multi-dimensions is another important open problem. In particular, the implications for nearby (rNS) profiles of high-frequency ZND instabilities pointed out in [Er3] is an intriguing mathematical puzzle; see [JLW] for further discussion.

We note that one-dimensional stability in the ZND limit was previously established in [RV] for detonation profiles of Majda’s model. Our results both recover and illuminate this prior result, since the associated Neumann shock profile, because it is scalar, is in this case always stable (see, e.g., [Sa]), as is the limiting ZND detonation. Related singular perturbation results for systems of conservation laws may be found in [PZ, HLZ, CHNZ, HLYZ1, HLYZ2, BHZ] and (for multi-wave patterns) in [OZ1, Z3]. In particular, we rely heavily on the basic methods of analysis developed in [PZ, HLZ, Z3, and BHZ].

2 The Evans–Lopatinski determinant for (ZND)

We begin by recalling the linearized stability theory for ZND detonations following [Er1, Z1, JLW, HuZ2]. Shifting to coordinates $\tilde{x} = x - st$ moving with the background Neumann shock, write (1.6) as

$$W_t + F(W)_x = R(W),$$

(2.1)
where

\[(2.2)\]

\[W := \begin{pmatrix} u \\ z \end{pmatrix}, \quad F := \begin{pmatrix} f(u) - su \\ -sz \end{pmatrix}, \quad R := \begin{pmatrix} qkz\phi(u) \\ -kz\phi(u) \end{pmatrix} .\]

To investigate solutions in the vicinity of a discontinuous detonation profile, we postulate existence of a single shock discontinuity at location \(X(t)\), and reduce to a fixed-boundary problem by the change of variables \(x \rightarrow x - X(t)\). In the new coordinates, the problem becomes

\[(2.3)\]

\[W_t + (F(W) - X'(t)W)_x = R(W), \quad x \neq 0,\]

with jump condition

\[(2.4)\]

\[X'(t)[W] - [F(W)] = 0,\]

\([h(x,t)] := h(0^+, t) - h(0^-, t)\) as usual denoting jump across the discontinuity at \(x = 0\).

### 2.1 Linearization

In moving coordinates, \(\bar{W}^0\) is a standing detonation, hence \((\bar{W}^0, \bar{X}) = (\bar{W}^0, 0)\) is a steady solution of \((2.3)-(2.4)\). Linearizing \((2.3)-(2.4)\) about \((\bar{W}^0, 0)\), we obtain the linearized equations

\[(2.5)\]

\[(W_t - X'(t)(\bar{W}^0)'(x)) + (AW)_x = EW,\]

\[(2.6)\]

\[X'(t)[\bar{W}^0] - [AW] = 0, \quad x = 0,\]

where \(A := (\partial/\partial W)F, \quad E := (\partial/\partial W)R\).

### 2.2 Reduction to homogeneous form

As pointed out in [JLW], it is convenient for the stability analysis to eliminate the front from the interior equation \((2.5)\). Therefore, we reverse the original transformation to linear order by the change of dependent variables

\[(2.7)\]

\[W \rightarrow W - X(t)(\bar{W}^0)'(x),\]

motivated by the calculation \(W(x - X(t), t)) - W(x, t) \sim X(t)W_x(x, t) \sim X(t)(\bar{W}^0)'(x)\), approximating to linear order the original, nonlinear transformation. Substituting \((2.7)\)

in \((2.5)-(2.6)\), and noting that \(x\)-differentiation of the steady profile equation \(F(\bar{W}^0)_x = R(\bar{W}^0)\) gives \((A(\bar{W}^0)(\bar{W}^0)'(x))_x = E(\bar{W}^0)(\bar{W}^0)'(x)\), we obtain modified, homogeneous interior equations

\[(2.8)\]

\[W_t + (AW)_x = EW\]

agreeing with those that would be obtained by a naive calculation without consideration of the front, together with the modified jump condition

\[(2.9)\]

\[X'(t)[\bar{W}^0] - [A(W + X(t)(\bar{W}^0)')] = 0\]

correctly accounting for front dynamics.
2.3 The stability determinant

Seeking normal mode solutions $W(x,t) = e^{\lambda t}W(x)$, $X(t) = e^{\lambda t}X$, $W$ bounded, of the linearized equations (2.8)–(2.9), we are led to the generalized eigenvalue equations

(2.10) \[(AW)' = (-\lambda I + E)W, \quad x \neq 0,\]

\[X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [AW] = 0,\]

where “’” denotes $d/dx$. or, setting $Z := AW$, to

(2.11) \[Z' = GZ, \quad x \neq 0,\]

(2.12) \[X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [Z] = 0,\]

with

(2.13) \[G := (-\lambda I + E)A^{-1},\]

where we are implicitly using the fact that $A$ is invertible, by (P1) and $s > 0$.

Lemma 2.1 \([\text{Er1, Er2, JL W}]\). On $\mathbb{R}\lambda > 0$, the limiting $(n+1) \times (n+1)$ coefficient matrices $G_\pm := \lim_{z \to \pm\infty} G(z)$ have unstable subspaces of fixed rank: full rank $n+1$ for $G_+$ and rank $n$ for $G_-$. Moreover, these subspaces extend analytically to $\mathbb{R}\lambda \leq -\eta < 0$.

Proof. Straightforward calculation using upper-triangular form of $G_\pm \[\text{Er1, Er2, Z1, JL W}\]$.

Corollary 2.2 \([\text{Z1, JL W}]\). On $\mathbb{R}\lambda > 0$, the only bounded solution of (2.11) for $x > 0$ is the trivial solution $W \equiv 0$. For $x < 0$, the bounded solutions consist of an $(n)$-dimensional manifold \(\text{Span}\{Z_1^+,...,Z_n^+\}(\lambda,x)\) of exponentially decaying solutions, analytic in $\lambda$ and tangent as $x \to -\infty$ to the subspace of exponentially decaying solutions of the limiting, constant-coefficient equations $Z' = G_-Z$; moreover, this manifold extends analytically to $\mathbb{R}\lambda \leq -\eta < 0$.

Proof. The first observation is immediate, using the fact that $G$ is constant for $x > 0$. The second follows from standard asymptotic ODE theory, using the conjugation lemma of \([\text{McZ1}]\) (see Lemma A.1 Appendix A) together with the fact that $G$ decays exponentially to its end state as $x \to -\infty$.

Definition 2.3. We define the Evans–Lopatinski determinant

(2.14) \[D_{ZN\text{D}}(\lambda) := \det (Z_1^-(\lambda,0), \cdots, Z_n^-(\lambda,0), \lambda[\bar{W}^0] - [A(\bar{W}^0)']) = \det (Z_1^+(\lambda,0), \cdots, Z_n^+(\lambda,0), \lambda[\bar{W}^0] + A(\bar{W}^0)'(0^-)),\]

where $Z_j^-(\lambda,x)$ are as in Corollary 2.2.

The function $D_{ZN\text{D}}$ is exactly the stability function derived in a different form by Erpenbeck \([\text{Er1, Er2}]\). The formulation (2.14) is of the standard form arising in the simpler context of (nonreactive) shock stability \([\text{Er4}]\). Evidently (by (2.12) combined with Corollary 2.2), $\lambda$ is a generalized eigenvalue/normal mode for $\mathbb{R}\lambda \geq 0$ if and only if $D_{ZN\text{D}}(\lambda) = 0$. 


3 The Evans determinant for (rNS)

3.1 Linearization

Linearizing (1.1) about \((\bar{u}, \bar{z})\) in moving coordinates yields linearized eigenvalue equations

\[
\lambda W + (\tilde{A}W)_x = EW + (\tilde{\varepsilon}B\tilde{W}_x)_x,
\]

where \(W = \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}\), \(\tilde{W}^\varepsilon = \begin{pmatrix} \tilde{u}^\varepsilon \\ z^\varepsilon \end{pmatrix}\), \(\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & b \\ 0 & C \end{pmatrix}\) \((\tilde{W}^\varepsilon)\), and

\[
\tilde{A}v := dF(\tilde{W}^\varepsilon)v - \varepsilon(d\tilde{B}(\tilde{W}^\varepsilon)v)\tilde{W}^\varepsilon_x, \quad (\tilde{A}_{11}, \tilde{A}_{12}) = (df_{11} - s, (df_{12}, 0)) \equiv constant,
\]

\(E, F\) as in (2.8).

3.2 Expression as a first-order system

Setting \(\tilde{x} = x/\varepsilon\) and

\[
W = \begin{pmatrix} Y \\ W_2 \end{pmatrix} := \begin{pmatrix} \tilde{A}W - \varepsilon\tilde{B}W_x \\ W_2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad W_2 := \begin{pmatrix} u_2 \\ z \end{pmatrix},
\]

we may write (3.1) as a first-order system

\[
\dot{W} = G^\varepsilon(\lambda, x)W,
\]

where

\[
G^\varepsilon := \begin{pmatrix} \varepsilon(E - \lambda)\tilde{A}_{11}^{-1} & 0 & -\varepsilon(E - \lambda)\tilde{A}_{11}^{-1}\tilde{A}_{12} \\ 0 & \varepsilon(E - \lambda) \\ \tilde{b}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1} & -\tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}) \end{pmatrix}
\]

and \(\dot{\cdot}\) denotes \(d/d\tilde{x}\). Here, we are using implicitly the facts that \(\tilde{A}_{11}\) and \(\tilde{b}\) are invertible, by (3.2), (P2), and (H2).

We have the following analogs of Lemma 2.1 and Corollary 2.2 which follow in essentially the same way; see [LRTZ] [TZ4] for further details.

Lemma 3.1 ([Z1] [TZ4]). On \(\mathbb{R}\lambda > 0\), the limiting \((n + 1 + r) \times (n + 1 + r)\) coefficient matrices \(G_\pm := \lim_{z \to \pm\infty} G(z),\) \(r = \dim u_2 + 1\), have stable subspaces of fixed, equal rank \(r\). Moreover, these subspaces extend analytically to \(\mathbb{R}\lambda \leq -\eta < 0\).

Corollary 3.2 ([Z1] [TZ4]). On \(\mathbb{R}\lambda > 0\), the bounded solutions of (3.1) on \(x \leq 0\) consist of an \((n + 1)\)-dimensional manifold \(\text{Span}\{W_1^\pm, \ldots, W_{n+1}^\pm\}(\lambda, x)\) of exponentially decaying solutions in the backward \(x\) direction, analytic in \(\lambda\) and tangent as \(x \to -\infty\) to the subspace.
of exponentially decaying solutions in the backward \( x \) direction of the limiting, constant-coefficient equations \( W' = G \cdot W \); the bounded solutions of (3.1) on \( x \geq 0 \) consist of an \((r - 1)\)-dimensional manifold \( \text{Span}\{W_{n+2}^+, \ldots, W_{n+1+r}^+\}(\lambda, x) \) of exponentially decaying solutions, analytic in \( \lambda \) and tangent as \( x \to +\infty \) to the subspace of exponentially decaying solutions of the limiting, constant-coefficient equations \( W' = G \cdot W \); Moreover, these manifolds extend analytically to \( \Re \lambda \leq -\eta < 0 \).

3.3 The stability determinant

**Definition 3.3.** We define the Evans function

(3.5) \[ D^\varepsilon_{rNS}(\lambda) := \det (W_1^-, \ldots, W_{n+1}^-, W_{n+2}^+, \ldots, W_{n+1+r}^+)(\lambda, 0), \]

where \( W_j^\pm(\lambda, x) \) are as in Corollary 3.2.

4 Fast vs. slow coordinates

Note that the coordinate transformation to stretched, or “fast” variables

(4.1) \[ (\tilde{x}, \tilde{t}, \tilde{\lambda}) = (x/\varepsilon, t/\varepsilon, \varepsilon\lambda), \]

changes equations (1.1) to

(4.2) \[ \begin{align*}
    u_{\tilde{t}} + f(u)_{\tilde{x}} &= (B(u)u_{\tilde{x}})_{\tilde{x}} + \varepsilon kq\varphi(u)z, \\
    z_{\tilde{t}} &= (C(u, z)z_{\tilde{x}})_{\tilde{x}} - \varepsilon k\varphi(u)z,
\end{align*} \]

shifting the small parameter \( \varepsilon \) from diffusion to reaction terms. The computations to follow may be thought of as alternating between (original) slow variables and fast variables in our analysis, as convenient for different regions in \( \lambda \) and \( x \). In particular, first-order equations (3.3)–(3.4) may be recognized as the first-order linearization of fast equations (4.2).

5 Region I: \( |\lambda| \leq C \)

We first study the critical “ZND” region \( |\lambda| \leq C \), or \( \varepsilon/C \leq |\tilde{\lambda}| \leq C\varepsilon \), where behavior of the (rNS) Evans function \( D_{rNS} \) is governed by that of the Evans–Lopatinski determinant \( D_{ZND} \). Setting \( M >> 1 \) to be a large constant to be determined later, we study separately the zones \( \tilde{x} = x/\varepsilon \leq -M \) and \( \tilde{x} = x/\varepsilon \geq -M \), on which the profile \( \tilde{W}^\varepsilon \) is dominated respectively by the (ZND) profile \( \tilde{W}^0 \) and the viscous shock profile \( \tilde{W} \), as described in (P3).

5.1 “Slow”, or “reaction” zone, \( \tilde{x} \leq -M \)

Note that \( \tilde{A} \), hence \( \tilde{N} := \tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12}) \) is invertible for \( x \leq -\delta \), by (3.2) and (P1)–(P2). Setting \( \mathcal{W} = T \mathcal{Z} \), where \( T := \begin{pmatrix} I & 0 \\ -N^{-1}\ell & I \end{pmatrix} \) and \( \ell := (\tilde{b}^{-1}\tilde{A}_{21}\tilde{A}_{11}^{-1}, -\tilde{b}^{-1}) \),

\[ -N^{-1}\ell = -(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})^{-1}(\tilde{A}_{21}\tilde{A}_{11}^{-1}, -I), \]
and noting that $\tilde{N}, \tilde{\ell} \sim \tilde{W}\varepsilon$ on $\tilde{x} \leq -M$, by (P3), we transform (5.3)–(5.4) to

$$
(5.1) \quad \dot{Z} = \mathcal{H}^\varepsilon Z,
$$

$$
(5.2) \quad \mathcal{H}^\varepsilon = T^{-1}G^\varepsilon T - T^{-1}\dot{T} = \begin{pmatrix} \varepsilon (E - \lambda) \tilde{A}^{-1} & \varepsilon m \\ 0 & N + O(\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ O(\varepsilon + |\tilde{W}|) & 0 \end{pmatrix},
$$

where $m := (E - \lambda) \begin{pmatrix} -\tilde{A}^{-1} \tilde{A}_{12} \\ I \end{pmatrix}$, and terms $O(\cdot)$ are smooth and analytic in $\lambda$. Here, $N$, by (1.3) and the relation between viscous and inviscid shock structure [MaZ3], has one positive eigenvalue and $r - 1$ negative eigenvalues, where $r = \dim u_2 + 1$.

By a further transformation $Z = SY, S = \begin{pmatrix} I & -\varepsilon N^{-1}m \\ 0 & I \end{pmatrix}$, we transform to $\dot{Y} = \mathcal{K}^\varepsilon Y$, where

$$
(5.3) \quad \mathcal{K}^\varepsilon = S^{-1}\mathcal{H}^\varepsilon S - S^{-1}\dot{S} = \begin{pmatrix} \varepsilon (E - \lambda) \tilde{A}^{-1} & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2 + \varepsilon|\tilde{W}|) & O(\varepsilon^2 + \varepsilon|\tilde{W}|) \\ O(\varepsilon + |\tilde{W}|) & O(\varepsilon + |\tilde{W}|) \end{pmatrix}.
$$

By a further transformation $\begin{pmatrix} I \\ 0 \\ r \end{pmatrix}$ if necessary, we may take without loss of generality $N$ block-diagonal, $N = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$, with $\Re N_1 \geq \eta > 0$ and $\Re N_2 \leq -\eta < 0$, both with a uniform spectral gap from block $\varepsilon (E - \lambda) \tilde{A}^{-1} \sim \varepsilon$.

Applying the asymptotic ODE results of Lemma A.7 with $\eta \sim 1, \delta = O(\varepsilon + |\tilde{W}|)$, and $|\Theta| \leq C$, and of Remark A.9.1 with $\eta \sim 1, \delta := \varepsilon(\varepsilon + |\tilde{W}|)$, $|\Theta_{12}| + |\Theta_{11}| \leq C, |\Theta_{21}| + |\Theta_{22}| \leq C/\varepsilon$, we find that there is a change of coordinates $Z = SU, U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, defined on $\tilde{x} \leq -M$,

$$
(5.4) \quad S^{-1} = \begin{pmatrix} I & -\Phi_1 \\ -\Phi_2 & I \end{pmatrix} = \begin{pmatrix} I & \varepsilon O(\varepsilon + |\tilde{W}|) \\ O(\varepsilon + |\tilde{W}|) & I \end{pmatrix} = \begin{pmatrix} I & o(\varepsilon) \\ o(\varepsilon) & I \end{pmatrix} = \begin{pmatrix} I & o(1) \\ o(1) & I \end{pmatrix} = \begin{pmatrix} I & o(1) \\ o(1) & I \end{pmatrix},
$$

close to the identity converting (5.1) into three decoupled equations

$$
(5.5) \quad \dot{Z}_1 = \varepsilon (E - \lambda) \tilde{A}^{-1} Z_1 + O(\varepsilon^2 + \varepsilon|\tilde{W}|) Z_1; \quad U_2 \equiv 0, U_3 \equiv 0,
$$

$$
\dot{Z}_2 = (N_1 + o(1)) Z_2; \quad U_1 \equiv 0, U_3 \equiv 0,
$$

$$
\dot{Z}_3 = (N_2 + o(1)) Z_3; \quad U_1 \equiv 0, U_2 \equiv 0.
$$

Focusing on the “slow” $Z_1$ mode, changing coordinates from $\tilde{x}$ back to $x = \varepsilon \tilde{x}$, we obtain

$$
(5.6) \quad Z'_1 = (E - \lambda) \tilde{A}^{-1} Z_1 + O(\varepsilon + |\tilde{W}|(x/\varepsilon)) Z_1,
$$

13
where \(\d/dx\) denotes \(d/dx\), and all coefficients converge uniformly as \(Ce^{-\eta|x|}\), \(\eta > 0\) to their limits as \(x \to -\infty\). Applying the convergence lemma, Lemma A.4 (Appendix A.2), for \(x \leq -M\), together with Remark A.5 (A.16), we find that there is a further coordinate change \(Z_1 = PZ, P = I + O(\varepsilon + e^{-\eta M}) = I + o(1)\) for \(|x| \leq 2M\), taking the decaying/growing modes of (5.6) to those of \(Z' = (E - \lambda)A^{-1}Z\), which may be recognized as exactly the interior equation (2.11),(2.13) associated with the Evans–Lopatinski development.

Tracing back through our coordinate transformations, we find that the \(n\) slowly-decaying modes \(W_{-1}, \ldots, W_{-n}\) as \(\tilde{x} \to -\infty\) are given at \(\tilde{x} = -M\) by

\[(5.7)\]
\[W_j^- = (I + o(1)) \begin{pmatrix} Z_j^- \\ \ast \end{pmatrix}, \quad j = 1, \ldots, n,\]

where \(Z_j^-\) are as in (2.14), while the single fast-decaying mode as \(\tilde{x} \to -\infty\) is given at \(\tilde{x} = -M\) by

\[(5.8)\]
\[W_{-n+1}^+ = c(-M) \begin{pmatrix} o(1) \varepsilon v \\ (I + o(1))v \end{pmatrix},\]

where \(c(\tilde{x})\) is exponentially decaying as \(\tilde{x} \to -\infty\) and \(v\) lies in the unique stable eigendirection of \(N(-M), N := \tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})\). Here, \(o(1)\) depends on \(M\), going to zero as \(M \to \infty\) and \(\varepsilon \to 0\) at the same time.

**Remark 5.1.** At \(\varepsilon = 0\), the single fast-decaying mode reduces to the translational zero eigenfunction of the associated viscous Neumann shock (necessarily fast-decaying), and so we may refine (5.8) to

\[(5.9)\]
\[W_{-n+1}^- = \begin{pmatrix} o(1) \varepsilon \hat{W}^0_2 \\ (I + o(1))\hat{W}^0_2 \end{pmatrix}, \quad \hat{W}^0_2 = \begin{pmatrix} \hat{u}^{\ast} \\ \hat{z} \end{pmatrix}.\]

### 5.2 “Fast”, or “Neumann shock” zone, \(\tilde{x} \geq -M\)

Applying the convergence lemma, Lemma A.3 for \(\tilde{x} \geq -M\), using the asymptotics of (P3), together with Remark A.5 (A.16), we find that there is a change of coordinates \(\tilde{P}_+^\varepsilon\) with \(\tilde{P}_+^\varepsilon = I + O(\varepsilon)\) for \(|\tilde{x}| \leq M\), such that \(W := \tilde{P}_+^\varepsilon V\) converts (3.3)–(3.4) to the same equations with \(\varepsilon \equiv 0\), i.e., the shock eigenvalue system at \(\lambda = 0\), hence, by inspection, the decaying modes \(V_{n+2}^+, \ldots, V_{n+1+r}^+\) at \(+\infty\), necessarily fast-decaying, by the Lax characteristic assumption (1.5) (see [MaZ3]), are by inspection (see [Z1] for similar calculations) of form

\[(5.10)\]
\[V_j^+ = \begin{pmatrix} 0 \\ v_j(x) \end{pmatrix},\]
\(v_j\) independent, hence for \(|x| \leq M\)

\[(5.11) \quad W_j^+ = (I + O(\varepsilon)) \begin{pmatrix} 0 \\ v_j \end{pmatrix}, \quad j = n + 2, n + 1 + r.\]

Here the constant \(O(\cdot)\) depends on (growing exponentially with) the fixed constant \(M\), since we are shifting \(\tilde{x} \rightarrow \tilde{x} + M\) in order to apply the convergence lemma. Evaluating at \(\tilde{x} = -M\), we have

\[(5.12) \quad W_j^+(-M) = (I + O(\varepsilon)) \begin{pmatrix} 0 \\ v_j(-M) \end{pmatrix}, \quad j = n + 2, \ldots, n + 1 + r.\]

### 5.2.1 Variation in \(\varepsilon\)

At this point, gathering information, and noting, by Abel’s formula, that the Wronskian \(D_{\varepsilon NS}(\lambda)\) evaluated at \(\tilde{x} = 0\) is equal to a nonzero constant times the same Wronskian evaluated at the point \(\tilde{x} = -M\) at which we have information about solutions from both sides, we have that \(D_{\varepsilon NS}(\lambda)\) is proportional by a nonzero constant, analytic in \(\lambda, \varepsilon\), to

\[(5.13) \quad \det \left( \begin{array}{cc} (I + o(1)) \begin{pmatrix} Z_1^- \star \\ \vdots \\ Z_{n+1}^- \end{pmatrix}, & (I + o(1)) \begin{pmatrix} 0 \\ v_{n+2} \\ \vdots \\ v_{n+1+r} \end{pmatrix} \end{array} \right) = O(\varepsilon),\]

since up to \(O(\varepsilon)\) there are only \(n\) nonzero entries \(Z_j^-\) in the \((n + 1)\)-dimensional \(Y\) block. Here, the constants \(O(\varepsilon)\) and \(o(1)\) by construction are analytic in \(\lambda, \varepsilon\) as well.

This reflects the fact that at \(\varepsilon = 0\) there is a solution \(W = \partial_{\tilde{x}} \tilde{W} = \begin{pmatrix} \partial_{\tilde{x}} u_0 \\ 0 \end{pmatrix}\) decaying at both \(\pm\infty\) of \(\mathbb{R}\), corresponding to the translational eigenmode of the linearized equations about the associated viscous Neumann shock; see Remark 5.1. To extract the next-order behavior, we compute the first variation of this special mode with respect to \(\varepsilon\) at \(\varepsilon = 0\), by an argument similar to those used in [GZ, ZS, LyZ1, LyZ2] to compute the first variation with respect to \(\lambda\) at \(\lambda = 0\).

Specifically, recalling that \(v_{n+1}, \ldots, v_{n+1+r}\) are a basis in \(C^r\), we may perform a column operation using the final \(r\) columns to cancel the entry \(v_{n+1}\) in the \((n + 2)\)nd column in determinant \(5.13\), to obtain

\[(5.14) \quad D_{\varepsilon NS}(\lambda) = \psi(\lambda, \varepsilon)(I + o(1)) \varepsilon \det \begin{pmatrix} v_{n+2} & \ldots & v_{n+1+r} \end{pmatrix} \det \begin{pmatrix} Z_1^- & \ldots & Z_n^- & -Y_\varepsilon \end{pmatrix}
\]

hence

\[(5.15) \quad \frac{D_{\varepsilon NS}(\lambda)}{\varepsilon \Psi(\lambda, \varepsilon)} = \det \begin{pmatrix} Z_1^- & \ldots & Z_n^- & -Y_\varepsilon \end{pmatrix} + o(1),\]

15
as $\varepsilon \to 0$, $o(1) \to 0$ as $M \to \infty$, where $\psi$ and $\Psi := \psi \det \left( v_{n+2} \ldots v_{n+1+r} \right)$ are nonvanishing factors analytic in $\lambda$, $\varepsilon$, and

\begin{equation}
(5.16) \quad Y_\varepsilon := \partial_\varepsilon Y^*|_{\varepsilon=0} = \partial_\varepsilon \mathcal{W}^*|_{\varepsilon=0},
\end{equation}

where $\mathcal{W}^*(\varepsilon, \lambda, x)$ is the (necessarily fast-) decaying solution of (5.11) at $x \to +\infty$ defined by

\begin{equation}
(5.17) \quad \mathcal{W} = \hat{P}_+(\lambda, x) \begin{pmatrix} 0 \\ v_{n+1} \end{pmatrix},
\end{equation}

where $\hat{P}_+(\lambda, x)$ is the conjugating transformation described above, and

\begin{equation}
W^*|_{\varepsilon=0} = \partial_\varepsilon \hat{W} = \left( \partial_\varepsilon \hat{u} \right).
\end{equation}

Writing (3.1) in $\bar{x}$ coordinates as

\begin{equation}
(5.18) \quad \varepsilon(E - \lambda)W = (\bar{A}^\varepsilon W - \bar{B}^\varepsilon W_{\bar{x}})_{\bar{x}} = \bar{Y},
\end{equation}

and differentiating (3.1) with respect to $\varepsilon$, we obtain the variational equations

\begin{equation}
(5.19) \quad (E - \lambda)W^*|_{\varepsilon=0} - (\partial_\varepsilon \bar{A}^\varepsilon W^* - \partial_\varepsilon \bar{B}^\varepsilon W^*_{\bar{x}})|_{\varepsilon=0} = \partial_\varepsilon \bar{Y},
\end{equation}

or

\begin{equation}
E \hat{W}_{\bar{x}} - (\lambda \hat{W} + \partial_\varepsilon \bar{A}^\varepsilon \hat{W}_{\bar{x}} - \partial_\varepsilon \bar{B}^\varepsilon \hat{W}_{\bar{x}\bar{x}})_{\bar{x}} = \hat{Y}_\varepsilon.
\end{equation}

Finally, recall that differentiating the traveling-wave ODE

\begin{equation}
(\bar{A} \bar{W}^\varepsilon)_{\bar{x}} = \varepsilon E \bar{W} + (\bar{B} \bar{W}^\varepsilon)_{\bar{x}}
\end{equation}

with respect to $\bar{x}$ yields

\begin{equation}
(5.20) \quad E \bar{W}^\varepsilon_{\bar{x}} = \varepsilon^{-1}(\bar{A} \bar{W}^\varepsilon - \bar{B} \bar{W}^\varepsilon_{\bar{x}\bar{x}})_{\bar{x}} = (\bar{A} \bar{W}^\varepsilon_{\bar{x}} - \varepsilon \bar{B} \bar{W}^\varepsilon_{\bar{x}\bar{x}})_{\bar{x}}.
\end{equation}

Together with the estimates

\begin{equation}
(5.21) \quad |\bar{W}^\varepsilon_{\bar{x}} - \hat{W}_{\bar{x}}| \leq C \varepsilon e^{-\eta|\bar{x}|},
\end{equation}

$\eta > 0$, for $x \geq -M$ and

\begin{equation}
(5.22) \quad |\hat{W}_{\bar{x}}(-M)|, |\hat{W}_{\bar{x}\bar{x}}(-M)| = o(1)
\end{equation}

coming from asymptotics (P2), (P3), we find, combining (5.19), (5.20), (5.21), and (5.22) that

\begin{equation}
(5.23) \quad (Y_\varepsilon)_{\bar{x}} = E(\bar{W}_{\bar{x}} - \hat{W}_{\bar{x}}) + (\bar{A} \bar{W}^\varepsilon_{\bar{x}} - \varepsilon \bar{B} \bar{W}^\varepsilon_{\bar{x}\bar{x}})_{\bar{x}} - (\lambda \hat{W} + \partial_\varepsilon \bar{A}^\varepsilon \hat{W}_{\bar{x}} - \partial_\varepsilon \bar{B}^\varepsilon \hat{W}_{\bar{x}\bar{x}})_{\bar{x}}
\end{equation}

\begin{equation}
= (\bar{A} \bar{W}^\varepsilon_{\bar{x}} - \lambda \hat{W} - \varepsilon \bar{B} \bar{W}^\varepsilon_{\bar{x}\bar{x}} - \partial_\varepsilon \bar{A} \bar{W}^\varepsilon_{\bar{x}} + \partial_\varepsilon \bar{B} \bar{W}^\varepsilon_{\bar{x}\bar{x}})_{\bar{x}} + O(\varepsilon e^{-\eta|\bar{x}|}).
\end{equation}
Integrating (5.23) in $\tilde{x}$ from $\tilde{x} = -M$ to $\tilde{x} = +\infty$ thus yields

$$-Y_\varepsilon(-M) = (-\tilde{A}W_\varepsilon^x + \lambda \tilde{W} + \varepsilon \tilde{B}W_\varepsilon^x + \partial_\varepsilon \tilde{A}^x \tilde{W} - \partial_\varepsilon \tilde{B}^x \tilde{W}_{xx})|_{\tilde{x} = -M} + O(\varepsilon), \tag{5.24}$$

From asymptotics (P3), we find that

$$\tilde{A}W_\varepsilon^x - \varepsilon \tilde{B}W_{xx} = \tilde{A}W_0^x + (\tilde{A}W_x - \varepsilon \tilde{B}W_{xx}) + o(1). \tag{5.25}$$

Recalling (by differentiation in $x$ of the traveling-wave ODE for the viscous Neumann shock profile) that

$$\tilde{A}W_x - \varepsilon \tilde{B}W_{xx} \equiv 0,$$

we find, substituting (5.25) into (5.24) and discarding lower-order terms and vanishing boundary terms at $+\infty$,

$$-Y_\varepsilon(-M) = \lambda [\tilde{W}^0] - [A(\tilde{W}^0)'(0)] + o(1), \tag{5.26}$$

where $\tilde{W}^0$ as in (P1) denotes the associated ZND profile and $[\cdot]$ the jump in values across its Neumann shock discontinuity.

### 5.3 Convergence to $D_{ZN D}$

**Proposition 5.2.** Assuming (H0)–(H3), (P1)–(P3), for $|\lambda| \leq C$ and $\Re \lambda \geq -\eta$, $\eta > 0$ sufficiently small, $\frac{D^\varepsilon_{NS}(\lambda)}{\varepsilon \Psi(\lambda,\varepsilon)}$ converges uniformly as $\varepsilon \to 0$ to $D_{ZN D}(\lambda)$, where $\Psi(\cdot, \cdot)$ is a nonvanishing factor that is analytic in $\lambda$.

Proof. Choosing monotone sequences $M_j$ (increasing) and $\varepsilon_j$ (decreasing) such that $o(1) \leq 1/j$ in (5.26) and (5.15), define $\Psi(\lambda, \varepsilon)$ to be equal to the function $\Psi(\lambda, \varepsilon)$ in (5.15) that is associated with $M_j$, where $j$ is the maximum integer such that $\varepsilon \leq \varepsilon_j$. Then, $\Psi$ is analytic in $\lambda$ by construction, and, combining (5.15), (5.26), and the definition of $\varepsilon_j$, $M_j$, we have

$$\left| \frac{D^\varepsilon_{NS}(\lambda)}{\varepsilon \Psi(\lambda, \varepsilon)} - D_{ZN D}(\lambda) \right| \leq C/j \to 0 \text{ as } \varepsilon \to 0.$$  

\[\square\]

**Corollary 5.3.** Assuming (H0)–(H3), (P1)–(P3), for $|\lambda| \leq C$ and $\Re \lambda \geq -\eta$, $\eta > 0$, the set of zeros of $D^\varepsilon_{NS}$ converges as $\varepsilon \to 0$ to the set of zeros of $D_{ZN D}$, for any sufficiently small $\eta > 0$ such that $D_{ZN D}$ does not vanish for $\Re \lambda = -\eta$ and $|\lambda| \leq C$.

Proof. Noting that zeros of $D^\varepsilon_{NS}$ agree with zeros of $\frac{D^\varepsilon_{NS}(\lambda)}{\varepsilon \Psi(\lambda, \varepsilon)}$, we obtain the result by properties of uniform limits of analytic functions.

\[\square\]
6 Region II: $C/\varepsilon \geq |\lambda| \geq C >> 1$

We next consider the “Neumann shock” region $C\varepsilon \leq \tilde{\lambda} \leq C$, $C >> 1$, in which behavior of $D_{rNS}$ is dominated by that of the Evans function $D_{NS}$ of the associated viscous Neumann shock. Here, $D_{NS}$ is defined similarly as in Definition 3.3 of $D_{rNS}$ as

$$D_N(\lambda) := \det (\hat{W}_1, \ldots, \hat{W}_{n+1}, \hat{W}_{n+2}^+, \ldots, \hat{W}_{n+1+r}^+) (\lambda, 0),$$

where $\hat{W}_j^\pm$ are decaying modes at $\pm \infty$ of the linearized eigenvalue equation

$$\tilde{\lambda} W + (\hat{A}W)_{\tilde{x}} = (\hat{B}W)_{\tilde{x}}$$

about $\hat{W}$, written as a first order system

$$\dot{\hat{W}} = \hat{G}(\tilde{x})\hat{W}, \quad \hat{G} := \left( \begin{array}{ccc} -\tilde{\lambda} \hat{A}_{11}^{-1} & 0 & -\tilde{\lambda} \hat{A}_{11}^{-1} \hat{A}_{12} \\ 0 & 0 & -\tilde{\lambda} \\ \hat{b}^{-1} \hat{A}_{21} \hat{A}_{11}^{-1} & -\hat{b}^{-1} & \hat{b}^{-1} (\hat{A}_{22} - \hat{A}_{21} \hat{A}_{11}^{-1} \hat{A}_{12}) \end{array} \right),$$

with $\hat{W} = \left( \begin{array}{c} \hat{Y} \\ \hat{W} \end{array} \right) = \left( \begin{array}{c} \hat{A}W - \hat{B}\hat{W} \\ \hat{W} \end{array} \right)$, $\hat{B} := \hat{B}(\hat{W})$, and $\hat{A} \upsilon := dF(\hat{W})\upsilon - (d\hat{B}(\hat{W})\upsilon)\hat{W}_{\tilde{x}}$, where $F$ is as in (2.8). For further details, see, e.g., [Z1, Z2].

6.1 Fast zone $\tilde{x} \geq -M$

Noting that $\tilde{\lambda} = \varepsilon \lambda$ is bounded by assumption, we have by (3.4) and (P3)

$$\hat{G}^\varepsilon = \left( \begin{array}{ccc} \varepsilon E - \tilde{\lambda} \hat{A}_{11}^{-1} & 0 & -\varepsilon E - \tilde{\lambda} \hat{A}_{11}^{-1} \hat{A}_{12} \\ 0 & 0 & \varepsilon E - \tilde{\lambda} \\ \hat{b}^{-1} \hat{A}_{21} \hat{A}_{11}^{-1} & -\hat{b}^{-1} & \hat{b}^{-1} (\hat{A}_{22} - \hat{A}_{21} \hat{A}_{11}^{-1} \hat{A}_{12}) \end{array} \right) = \hat{G} + O(\varepsilon e^{-|x|}),$$

where $\hat{G}$ is bounded and converges at uniform exponential rate to its limits at $\pm \infty$. Applying the convergence lemma, Lemma A.4 for $\tilde{x} \geq -M$, together with Remark A.5 (A.14), similarly as in Section 5.2 but now treating $\tilde{\lambda}$ as a fixed parameter, we find that there is a change of coordinates $\hat{P}_+^\varepsilon$ with $\hat{P}_+^\varepsilon = I + O(\varepsilon)$ for $\tilde{x} \geq -M$, such that $W := \hat{P}_+^\varepsilon \hat{W}$ converts (3.3)-(3.4) to the viscous shock system (6.3). Evaluating at $\tilde{x} = -M$, we thus have $W_j^+(\lambda, -M) = (I + O(\varepsilon))\hat{W}_j^+(\lambda, -M)$, or, by the assumption that $|\tilde{\lambda}| >> \varepsilon$,

$$(6.5) \quad W_j^+(\lambda, -M) = (I + o(\varepsilon))\hat{W}_j^+(\lambda, -M), \quad j = n + 2, \ldots, n + 1 + r.$$

6.2 Slow zone $\tilde{x} \leq -M$

6.2.1 Case a. $C/\varepsilon \geq |\lambda| \geq 1/C\varepsilon$, $C > 0$ arbitrary

We first treat the easier case $1/C\varepsilon \leq |\lambda| \leq C/\varepsilon$, or $C^{-1} \leq |\tilde{\lambda}| \leq C$, for arbitrary $C > 0$. From (H3), it follows that $\hat{G}(\lambda, -\infty)$ has no pure imaginary eigenvalues for $\Re \tilde{\lambda} > 0$ and
\( \lambda \neq 0 \). By continuity, the same holds for \( G(\lambda, \tilde{x}) \) for \( \tilde{x} \leq -M \), \( \Re \tilde{\lambda} > 0 \) and \( \tilde{\lambda} \) bounded and bounded away from zero.

By the assumption \( 1/C \leq |\tilde{\lambda}| \leq C \), therefore, the stable and unstable subspaces of \( G \) have a uniform spectral gap for all \( \tilde{x} \leq -M \), hence, by standard matrix perturbation theory \([K, ZH, Z5, GMWZ5]\), there exist smooth transformations \( T(G) \) such that

\[
T^{-1}GT = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad \Re M_1 \geq \eta > 0, \quad \Re M_2 \leq -\eta < 0.
\]

Making the change of variables \( W = TZ \), we thus obtain

\[
\dot{Z} = (T^{-1}GT - T^{-1}\dot{T})Z = \left( \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} + o(1) \right)Z
\]

for \( \tilde{x} \leq -M \). Applying the tracking lemma, Lemma \([A.7]\), we find that the manifold of solutions of (3.3)–(3.4) decaying at \(-\infty\) is within angle \( o(1) \) of the unstable subspace of \( G(\lambda, \tilde{x}) \) at each \( \tilde{x} \leq -M \), in particular for \( \tilde{x} = -M \).

But, by the same reasoning, the manifold of decaying solutions of (6.3) at \( \tilde{x} = -M \) is also within angle \( o(1) \) of the unstable subspace of \( \hat{G} \), which, by continuity in \( \varepsilon/\)uniform spectral gap is within angle \( O(\varepsilon) = o(1) = o(\tilde{\lambda}) \) of the unstable subspace of \( G \). From this, and (6.5), it follows that, up to a normalizing factor \( \hat{\Psi}(\varepsilon, \tilde{\lambda}) \) that may be taken analytic in \( \tilde{\lambda} \),

\[
(6.6) \quad \frac{D_{rNS}(\lambda)}{\hat{\Psi}(M, \varepsilon, \lambda)} = D_{NS}(\tilde{\lambda}) + o(\tilde{\lambda})
\]

on \( \Re \lambda > -\eta, C^{-1} \leq |\tilde{\lambda}| \leq C \), uniformly as \( M \to \infty \), and \( \varepsilon \to 0 \), for \( C > 0 \) arbitrary.

**6.2.2 Case b.** \( 1/C \varepsilon \geq |\lambda| \geq C >> 1 \)

Finally, we treat the more delicate case \( C \leq |\lambda| \leq 1/C \varepsilon \), or \( C \varepsilon \leq |\tilde{\lambda}| \leq C^{-1} \), with \( C >> 1 \). Proceeding as in Section 5.1 by the series of coordinate transformations (5.1)–(5.6), but taking account of the different order of \( \tilde{\lambda} \) in this case, in particular, noting that \( \varepsilon m \) in (5.2) is now \( O(\tilde{\lambda}) \) and not \( O(\varepsilon) \) as before, we obtain

\[
(6.7) \quad \mathcal{K} = \begin{pmatrix} (\varepsilon E - \tilde{\lambda})\tilde{A}^{-1} + \tilde{\lambda}^2 \beta & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} O(\varepsilon|\tilde{\lambda}| + |\tilde{\lambda}||\tilde{W}^{\varepsilon}|) & O(\varepsilon|\tilde{\lambda}| + |\tilde{\lambda}||\tilde{W}^{\varepsilon}|) \\ O(\varepsilon + |\tilde{W}^{\varepsilon}|) & O(\varepsilon + |\tilde{W}^{\varepsilon}|) \end{pmatrix}
\]

in place of (5.3), where \( \beta \) is the “frozen-coefficient corrector” obtained by dropping terms involving \( \tilde{x} \)-derivatives of the transformations involved,

\[
(6.8) \quad S^{-1} = \begin{pmatrix} I & -\Phi_1 \\ -\Phi_2 & I \end{pmatrix} = \begin{pmatrix} I & O(\varepsilon + |\tilde{W}^{\varepsilon}|) \\ O(\varepsilon + |\tilde{W}^{\varepsilon}|) & I \end{pmatrix} \begin{pmatrix} I & o(1) \\ o(1) & I \end{pmatrix} = \begin{pmatrix} I & o(1) \\ o(1) & I \end{pmatrix}
\]

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for \( \tilde{x} \geq -M \) in place of (5.4), and

\[
W_{n+1}^- = c(-M) \left( \frac{o(1)}{v} \right) (I + o(1))v
\]

in place of (5.8), where \( v \) lies in the unique stable eigendirection of \( N(-M) \), with

\[
N := \tilde{b}^{-1}(\tilde{A}_{22} - \tilde{A}_{21}A_{11}^{-1}\tilde{A}_{12}).
\]

By the same reasoning, applied to the viscous shock equations (6.3), we have

\[
\dot{W}_{n+1}^- = c(-M) \left( \frac{o(1)}{\hat{v}} \right) (I + o(1))\hat{v},
\]

where \( \hat{v} \) lies in the same unique eigendirection. and \( \dot{W}_{n+1}^- \) is the single fast-decaying mode of (6.3) at \(-\infty\). Comparing, we thus have, for some normalizing factor \( \psi(M, \varepsilon, \lambda) \), analytic in \( \lambda \),

\[
W_{n+1}^-(M) = (I + o(|\lambda|))\psi(M, \varepsilon, \lambda)\dot{W}_{n+1}^-(M).
\]

We now turn to the description of the remaining, slow-decaying, modes \( W_1^-, \ldots, W_n^- \). Focusing on the region \( \tilde{x} \geq -C_2|\log|\tilde{\lambda}||, C_2 >> 1 \), we find that the slow, first, equation of (5.5) becomes

\[
\dot{Z_1} = \left( (\varepsilon E - \tilde{\lambda}) \hat{A}^{-1} + \beta \hat{\lambda}^2 \right) Z_1 + O(\varepsilon + |\tilde{W}(x/\varepsilon)|)|\hat{\lambda}|Z_1
\]

\[
= \left( (\varepsilon E - \tilde{\lambda}) \hat{A}^{-1} + \beta \hat{\lambda}^2 \right) Z_1 + o(1)|\hat{\lambda}|^2Z_1,
\]

where \(( (\varepsilon E - \tilde{\lambda}) \hat{A}^{-1} + \beta \hat{\lambda}^2 \) agrees to second order in \( \tilde{\lambda} << 1 \) with the restriction of \( G \) to its slow subspace, which, recall, is the direct sum of \( n \) slow-decaying modes as \( \tilde{x} \to -\infty \) and a single slow-growing mode as \( \tilde{x} \to -\infty \). Here, \( o(1) \to 0 \) as \( C \to \infty \) and \( \varepsilon \to 0 \).

Computing explicitly, and noting that \( \tilde{z} \sim e^{-\theta|z|} \), we have

\[
\varepsilon E = \varepsilon \begin{pmatrix} qk\tilde{\phi}(\tilde{u})\tilde{z} & qk\tilde{\phi}(\tilde{u}) \\ -k\tilde{\phi}(\tilde{u})\tilde{z} & -k\tilde{\phi}(\tilde{u}) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & qk\tilde{\phi}(\tilde{u}) \\ 0 & -k\tilde{\phi}(\tilde{u}) \end{pmatrix} + \alpha(x, \tilde{\lambda}),
\]

where \( \alpha = O(\varepsilon e^{-\theta|z|}) \) is both \( o(\tilde{\lambda}) \) and uniformly bounded in \( L^1(\tilde{x}) \). Thus, up to \( O(\alpha) + o(|\tilde{\lambda}|^2) \) the eigenvalues of \(( (\varepsilon E - \tilde{\lambda}) \hat{A}^{-1} + \beta \hat{\lambda}^2 \) agree with the growth rates \( \mu_j \) of slow modes \( e^{\mu_j\tilde{x}}W_j \), \( W_j = \) constant, of the nonreacting frozen-coefficient operator

\[
\hat{\lambda}I + \begin{pmatrix} df(u) - sI & 0 \\ 0 & -s \end{pmatrix} \partial_{\tilde{x}} + \begin{pmatrix} 0 & 0 \\ C(u, z) \end{pmatrix} \partial_{\tilde{x}}^2 + \begin{pmatrix} 0 & qk\tilde{\phi}(\tilde{u}) \\ 0 & -k\tilde{\phi}(\tilde{u}) \end{pmatrix}
\]

obtained by neglecting \( O(\partial_{\tilde{x}}\tilde{W}(\varepsilon)) \) derivative terms and setting \( \tilde{z} \) to zero.

A standard low-frequency matrix perturbation computation [Z1, LyZ1, LyZ2, TZ4, LRTZ] shows that the modes \( W_j \) of (6.12) are analytic in \( \tilde{\lambda} \) and \( \varepsilon \), lying approximately...
in the eigendirections of $\tilde{A}$, with the associated growing (i.e., negative real part) eigenvalue $\mu_{n+1}$ separated in modulus by order $|\tilde{\lambda}|$ from the decaying ones, moreover, by (1.10), the growing eigenvalue is separated in real part by order $\sim |\tilde{\lambda}|^2$ from associated decaying (positive real part) eigenvalues.

From this, and the fact that the entire coefficient matrix is order $\tilde{\lambda}$, it follows that there exists a smooth matrix-valued function $Q(\tilde{W}^\varepsilon)$ with

$$Q^{-1}((\varepsilon E - \tilde{\lambda})\tilde{A}^{-1} + \beta \tilde{\lambda}^2)Q = \begin{pmatrix} m_1 + \alpha_1 & 0 \\ 0 & m_2 + \alpha_2 \end{pmatrix},$$

(6.13) $\Re m_1 \geq \tilde{\eta}|\tilde{\lambda}|^2 > 0$, $\Re m_2 \leq -\tilde{\eta}|\tilde{\lambda}|^2 < 0$, $|\alpha_j|_{L^1} \leq C$, $\tilde{\eta} = $ constant.

Making the change of coordinates $Z_1 = Qz$, and noting that $Q' = O(\tilde{W}^\varepsilon)$, we thus obtain

$$\dot{z} = \left(Q^{-1}((\varepsilon E - \tilde{\lambda})\tilde{A}^{-1} + \beta \tilde{\lambda}^2)Q - Q^{-1}\dot{Q}\right)z = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} z + (o(|\tilde{\lambda}|^2) + O(\tilde{W}^\varepsilon))z,$$

with gap condition (6.13).

Since $|\tilde{W}^\varepsilon| = O(\varepsilon) = o(\tilde{\lambda})$, and $m_j$ are spectrally separated by modulus $\sim |\tilde{\lambda}|$, there is a further smooth coordinate change $z = R\dot{y}$ with $R = I + o(1)$ converting the equations to

$$\dot{\tilde{y}} = \begin{pmatrix} m_1 + \tilde{\alpha}_1 & 0 \\ 0 & m_2 + \tilde{\alpha}_2 \end{pmatrix} \tilde{y} + (o(|\tilde{\lambda}|^2) + \beta)\tilde{y},$$

$\tilde{\alpha}_j = \alpha_j + O(\tilde{W}^\varepsilon)$, $\beta = o(\tilde{W}^\varepsilon)$, where $|\tilde{\alpha}_j|_{L^1} \leq C$, $|\beta|_{L^1(\tilde{x})} = o(1)$, and $m_j$ satisfy (6.13).

That is, we have an equation of form (A.27) with $\delta = o(|\tilde{\lambda}|^2)$ and $\eta \sim |\tilde{\lambda}|^2$, so that $\delta/\eta = o(1)$.

Applying the tracking lemma, Lemma A.7, as generalized in (A.28), Remark A.10 at $\tilde{x} = -C|\log |\tilde{\lambda}||$, and untangling coordinate changes, we thus find that the slow-decaying modes $W^\varepsilon_j$, $j = 1, \ldots, n$ lie within angle $o(1)$ of the stable subspace of $G(-C|\log |\tilde{\lambda}||)$, which in turn lies within angle $o(1)$ of the stable subspace of $G(\tilde{\lambda}, -\infty)$ and (by a repetition of the same argument), analytic multiples of the slow-decaying modes $\tilde{W}^\varepsilon_j$, $j = 1, \ldots, n$, at $\tilde{x} = -M$.

Finally, going back to the original equation (6.11), and noting that $\tilde{Z}_1 = O(|\tilde{\lambda}|)Z_1$, we find that the change in $Z_1$ in evolving from $\tilde{x} = -C|\log |\tilde{\lambda}||$ to $\tilde{x} = -M$ is order

$$(e^{[\log |\tilde{\lambda}|| - 1]|Z_1(-C|\log |\tilde{\lambda}||) \sim |\tilde{\lambda}||\log |\tilde{\lambda}||Z_1(-C|\log |\tilde{\lambda}||) = o(1)|Z_1(-C|\log |\tilde{\lambda}||) ||\tilde{X} = -M also the slow-decaying modes $W^-_j$, $j = 1, \ldots, n$ lie within angle $o(1)$ of analytic multiples of the slow-decaying modes $\tilde{W}^-_j$ at $\tilde{x} = -M$. Collecting facts, we have

(6.14) $W^-_j(-M) = (I + o(1))\psi_j(M, C_2, \varepsilon, \tilde{\lambda})\tilde{W}^-_j(-M)$, $j = 1, \ldots, n$, 

\footnote{This does not require strict hyperbolicity of $df$, but only $\det(df - s) \neq 0$; see Remark 1.3}
where $\psi_j$ are nonvanishing and analytic in $\lambda$. (Here, we are using also the fact that, by (6.7)–(6.8), the manifold of all slow modes, both growing and decaying, stays angle $o(1)$ close to the slow subspace of $G(\lambda, \tilde{x})$ for all $\tilde{x} \geq -M$.)

Finally, collecting estimates (6.5), (6.10), and (6.14), we see that fast modes $W^\pm_j$, up to nonvanishing analytic factor $\psi_j$, are given by $(I + o(\lambda))$ times the corresponding fast modes $\hat{W}^\pm_j$ of the associated shock stability problem, while slow modes $W^-_j$ are given by $(I + o(1))$ times the corresponding slow modes $\hat{W}^-_j$. Recalling, similarly as in estimate (5.13), that at $\tilde{\lambda} = 0$, the fast mode $\hat{W}^-_{n+1}$ is a linear combination of the fast modes $\hat{W}^+_j$, so that at $\tilde{\lambda}$ it remains within angle $O(\lambda)$ of their combination, we find, applying a column operation cancelling $\hat{W}^-_{n+1}$ to order $\lambda$ and factoring out $\lambda$ from that column, that we reduce the (rNS) determinant (3.5) to

$$
\left(\lambda(1 + o(1))(\Pi_j \hat{\psi}_j)D_{NS}(\lambda)\right)
$$

The key observation here is that because only fast modes are involved in the vanishing of $D_{NS}$ at $\lambda = 0$, only fast modes must be estimated to the sharper relative error $o(\lambda)$ in order to obtain the result (6.15), with $o(1)$ tolerance sufficing for slow modes.

Thus, we obtain again an estimate $\frac{D_{NS}(\lambda)}{\hat{\Psi}(M, C, \varepsilon, \lambda)} = D_{NS}(\lambda) + o(\lambda)$ on $\Re \lambda > -\eta, C \varepsilon \leq |\lambda| \leq C^{-1}$, where $o(1) \to 0$ uniformly as $M \to \infty, C \to \infty$, and $\varepsilon \to 0$, with $\hat{\Psi}(\cdot, \cdot)$ a nonvanishing factor that is analytic in $\lambda$. Note that we are using here the assumption that $C >> 1$, which was not needed in case a.

### 6.3 Convergence to $D_{NS}$

**Proposition 6.1.** Assuming (H0)–(H3), (P1)–(P3), For $C/\varepsilon \geq |\lambda| \geq C >> 1$, $\Re \lambda > -\eta, C \varepsilon \leq |\lambda| \leq C^{-1}$, $D_{NS}(\varepsilon \lambda)$ converges uniformly as $\varepsilon \to 0$ to $\frac{D_{NS}(\lambda)}{\hat{\Psi}(M, C, \varepsilon, \lambda)}$, where $\hat{\Psi}(\cdot, \cdot)$ is a nonvanishing factor that is analytic in $\lambda$.

**Proof.** From (6.6), (6.15), we have in either case a or b that

$$
\left| D_{rNS}(\lambda) \right| \hat{\Psi}(M, C, \varepsilon, \lambda) = D_{NS}(\varepsilon \lambda) + o(1)\varepsilon \lambda
$$
on $\Re \lambda > -\eta$, uniformly as $\varepsilon \to 0$. where $o(1) \to 0$ uniformly as $M \to \infty, C \to \infty$, and $\varepsilon \to 0$. Choosing monotone increasing sequences $C_j$, $M_j$ and a monotone decreasing sequence $\varepsilon_j$ such that $o(1) \leq 1/j$ in (6.10), define $\hat{\Psi}(\lambda, \varepsilon)$ to be equal to the function $\hat{\Psi}(\lambda, \varepsilon)$ in (6.16) that is associated with $C_j, M_j$, where $j$ is the maximum integer such that $\varepsilon \leq \varepsilon_j$. Then, $\hat{\Psi}$ is analytic in $\lambda$ by construction, and, combining (6.15) with the definition of $\varepsilon_j, M_j, C_j$, we have $\left| \frac{D_{rNS}(\lambda)}{\varepsilon \hat{\Psi}(\lambda, \varepsilon)} - \frac{D_{NS}(\varepsilon \lambda)}{\varepsilon \lambda} \right| \leq C/j \to 0$ as $\varepsilon \to 0$, giving the result.

**Corollary 6.2.** Assuming (H0)–(H3), (P1)–(P3), for $C \leq |\lambda| \leq C/\varepsilon, C$ sufficiently large, on $\Re \lambda > -\eta$ for $\eta, \varepsilon > 0$ sufficiently small, $\varepsilon$ times each zero of $D_{rNS}^\varepsilon$ converges to a zero
of \( \frac{D_{NS}(\lambda)}{\lambda} \) on \( \Re \lambda \geq 0 \); moreover, each zero of \( D_{NS} \) on \( \Re \lambda > 0 \), for \( C \leq |\lambda| \leq C/\varepsilon \).

**Proof.** Noting that zeros of \( D_{rNS}^\varepsilon \) agree with zeros of \( \frac{D_{NS}^\varepsilon(\lambda)}{\varepsilon \Phi(\lambda, \varepsilon)} \), we obtain the result by Proposition 6.1 and properties of uniform limits of analytic functions. \( \square \)

7 Region III: \( |\lambda| \geq C/\varepsilon, \ C >> 1 \)

Finally, we consider the straightforward “hyperbolic–parabolic” region \( |\lambda| \geq C/\varepsilon, \ C >> 1, \) or \( |\tilde{\lambda}| >> 1, \) on which zeros of \( D_{rNS} \) are prohibited stable by basic hyperbolic–parabolic structure/well-posedness of the underlying problem (1.1).

**Proposition 7.1.** \( D_{rNS} \) does not vanish for \( |\lambda| \geq C/\varepsilon, \ C >> 1, \) \( \Re \lambda \geq 0 \).

**Proof.** In fast coordinates \( \tilde{x} \), \( |\tilde{\lambda}| >> 1 \), this follows by the same high-frequency analysis used in [MaZ3] to treat the viscous shock case, based on the tracking/reduction lemma, Lemma A.7. See Proposition 5.2, [MaZ3], or Proposition 4.33, [Z2]. \( \square \)

A Asymptotic ODE theory

A.1 The conjugation lemma

Consider a general first-order system

\[
W' = A^p(x, \lambda)W
\]

with asymptotic limits \( A^p_\pm \) as \( x \to \pm \infty \), where \( p \in \mathbb{R}^m \) denote model parameters.

**Lemma A.1 ([MeZ1] [PZ]).** Suppose for fixed \( \theta > 0 \) and \( C > 0 \) that

\[
|A^p - A^p_\pm(x, \lambda)| \leq Ce^{-\theta|x|}
\]

for \( x \geq 0 \) uniformly for \( (\lambda, p) \) in a neighborhood of \( (\lambda_0, p_0) \) and that \( A \) varies analytically in \( \lambda \) and smoothly (resp. continuously) in \( p \) as a function into \( L^\infty(x) \). Then, there exist in a neighborhood of \( (\lambda_0, p_0) \) invertible linear transformations \( P^p_\pm(x, \lambda) = I + \Theta^p_\pm(x, \lambda) \) defined on \( x \geq 0 \) and \( x \leq 0 \), respectively, analytic in \( \lambda \) and smooth (resp. continuous) in \( p \) as functions into \( L^\infty[0, \pm \infty) \), such that

\[
|\Theta^p_\pm| \leq C_1 e^{-\theta|x|} \quad \text{for} \ x \geq 0,
\]

for any \( 0 < \tilde{\theta} < \theta \), some \( C_1 = C_1(\tilde{\theta}, \theta) > 0 \), and the change of coordinates \( W =: P^p_\pm Z \) reduces (A.1) to the constant-coefficient limiting systems

\[
Z' = A^p_\pm Z \quad \text{for} \ x \geq 0.
\]
Proof. The conjugators $P^p_\pm$ are constructed by a fixed point argument \cite{McZ1} as the solution of an integral equation corresponding to the homological equation
\[ P' = A^p P - A^p_\pm P. \]
The exponential decay (A.2) is needed to make the integral equation contractive with respect to $L^\infty[M, +\infty]$ for $M$ sufficiently large. Continuity of $P_\pm$ with respect to $p$ (resp. analyticity with respect to $\lambda$) then follow by continuous (resp. analytic) dependence on parameters of fixed point solutions. Here, we are using also the fact that (A.2) plus continuity of $A^p$ from $p \to L^\infty$ together imply continuity of $e^{\tilde{\theta}|x|}(A^p - A^p_\pm)$ from $p$ into $L^\infty[0, \pm\infty)$ for any $0 < \tilde{\theta} < \theta$, in order to obtain the needed continuity from $p \to L^\infty$ of the fixed point mapping. See also \cite{PZ, GMWZ3}.

Remark A.2. In the case that $A$ is block diagonal or triangular, the conjugators $P_\pm$ may evidently be taken block diagonal or triangular as well, by carrying out the same fixed-point argument on the invariant subspace of (A.5) consisting of matrices with this special form. This can be of use in problems with multiple scales; see, for example, Thm 1.16, \cite{BHZ}.

Definition A.3 (Abstract Evans function). Suppose that on the interior of a set $\Omega$ in $\lambda, p$, the dimensions of the stable and unstable subspaces of $A^p_\pm(\lambda)$ remain constant, and agree at $\pm\infty$ ("consistent splitting" \cite{AGJ}), and that these subspaces have analytic bases $R_\pm^\pm(x)$ extending continuously to boundary points of $\Omega$. Then, the Evans function is defined on $\Omega$ as
\[ D^p(\lambda) := \det(P^+_1 R^+_1, P^+_2 R^+_2, P^-_1 R^-_1, P^-_2 R^-_2)|_{x=0}, \]
where $P^\pm_\pm$ are as in Lemma A.1.

A.2 The convergence lemma
Consider a family of first-order equations
\[ W' = A^p(x, \lambda)W \]
indexed by a parameter $p$, and satisfying exponential convergence condition (A.2) uniformly in $p$. Suppose further that
\[ |(A^p - A^p_\pm) - (A^0 - A^0_\pm)| \leq C|p|e^{-\theta|x|}, \quad \theta > 0 \]
and
\[ |(A^p - A^0_\pm)| \leq C|p|. \]

Lemma A.4 (\cite{PZ, BHZ}). Assuming (A.2) and (A.8)-(A.9), for $|p|$ sufficiently small, there exist invertible linear transformations $P^p_\pm(x, \lambda) = I + \Theta^p_\pm(x, \lambda)$ and $P^0_\pm(x, \lambda) = I + \Theta^0_\pm(x, \lambda)$
defined on \( x \geq 0 \) and \( x \leq 0 \), respectively, analytic in \( \lambda \) as functions into \( L^\infty[0,\pm\infty) \), such that

\[
|y^p - y^p_0| \leq C_1|p|e^{-\hat{\theta}|x|} \quad \text{for } x \geq 0,
\]

for any \( 0 < \hat{\theta} < \theta \), some \( C_1 = C_1(\hat{\theta}, \theta) > 0 \), and the change of coordinates \( W =: P^p_\pm Z \) reduces \((A.7)\) to the constant-coefficient limiting systems

\[
(A.11) \quad Z' = A^p_\pm(\lambda)Z \quad \text{for } x \geq 0.
\]

**Proof.** Applying the conjugating transformation \( W \to (P^p_0)^{-1}W \) for the \( p = 0 \) equations, we may reduce to the case that \( A^0 \) is constant, and \( P^0_+ \equiv I \), noting that the estimate \((A.8)\) persists under well-conditioned coordinate changes \( W = QZ, Q(\pm\infty) = I \), transforming to

\[
|Q^{-1}A^p_0Q - Q^{-1}Q' - A^p_\pm) - (Q^{-1}A^0Q - Q^{-1}A^0)\pm)\rangle | + |Q^{-1}(A^p - A^0)\pm Q - (A^p - A^0)\pm\rangle |, \]

where

\[
(A.13) \quad |Q^{-1}(A^p - A^0)\pm Q - (A^p - A^0)\pm| = O(|Q - I||(A^p - A^0)\pm| = O(e^{-\theta|x|})|p|.
\]

In this case, \((A.8)\) becomes just

\[
|A^p - A^p_\pm| \leq C_1|p|e^{-\hat{\theta}|x|},
\]

and we obtain directly from the conjugation lemma, Lemma \( A.1 \), the estimate

\[
|P^p_+ - P^p_0| = |P^p_+ - I| \leq CC_1|p|e^{-\hat{\theta}|x|}
\]

for \( x > 0 \), and similarly for \( x < 0 \), verifying the result.

**Remark A.5.** In the case \( A^p_\pm \equiv \text{constant} \), or, equivalently, for which \((A.8)\) is replaced by

\[
|A^p - A^0| \leq C_1|p|e^{-\hat{\theta}|x|},
\]

we find that the change of coordinates \( W = \tilde{P}^p_\pm Z, \tilde{P}^p_\pm := (P^0)^{-1}P^p_\pm \), converts \((A.7)\) to \( Z' = A^0 \pm Z \), where \( \tilde{P}^p_\pm = I + \Theta^p_\pm \) with

\[
(A.14) \quad |\Theta^p_\pm| \leq CC_1|p|e^{-\hat{\theta}|x|} \quad \text{for } x \geq 0.
\]

That is, we may conjugate not only to constant-coefficient equations, but also to exponentially convergent variable-coefficient equations, with sharp rate \((A.14)\).

In the general case \( A^p \neq A^0 \pm \), we may still conjugate \((A.7)\) to \( Z' = A^0 \pm Z \) by the change of coordinates \( W = \tilde{P}^p_\pm Z, \tilde{P}^p_\pm := (P^0)^{-1}Q^p_\pm P^p_\pm \), where \( Q^p_\pm \) defined by

\[
(A.15) \quad Q' = A^p_\pm Q - Q A^0_\pm
\]

conjugates the constant-coefficient equation \( Y' = A^0 \pm Y \) to \( X' = A^0_\pm X \), obtaining bounds

\[
(A.16) \quad \tilde{P}^p_\pm = I + \Theta^p_\pm, \quad |\Theta^p_\pm| \leq CC_1|p| \quad \text{for } |x| \leq C
\]

valid for finite values of \( x \). However, in general, \( Q^p_\pm \) grow without bound as \( x \to \pm\infty \).
Remark A.6. As observed in [PZ], provided that the stable/unstable subspaces of $A_+^p/A_-^p$ converge to those of $A_0^+\alpha A_0^-$, as typically holds given (A.9) - in particular, this holds by standard matrix perturbation theory [K] if the stable and unstable eigenvalues of $A_0^\pm$ are spectrally separated - (A.10) gives immediately uniform convergence of the Evans functions $D^p$ to $D^0$ on compact sets of $\Omega$, by definition (A.3).

A.3 The tracking lemma

Consider an approximately block-diagonal system

\[(A.17)\quad W' = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} (x,p) + \delta(x,p)\Theta(x,p)W,\]

where $\Theta$ is a uniformly bounded matrix, $\delta(x)$ scalar, and $p$ a vector of parameters, satisfying a pointwise spectral gap condition

\[(A.18)\quad \min \sigma(\Re M_1^\pm) - \max \sigma(\Re M_2^\pm) \geq \eta(x) \text{ for all } x.\]

(Here as usual $\Re N := (1/2)(N + N^*)$ denotes the “real”, or symmetric part of $N$.) Then, we have the following tracking/reduction lemma of [MaZ3, PZ].

Lemma A.7 ([MaZ3, PZ]). Consider a system (A.17) under the gap assumption (A.18), with $\Theta^\pm$ uniformly bounded and $\eta \in L^1_{\text{loc}}$. If $\sup(\delta/\eta)(x)$ is sufficiently small, then there exist (unique) linear transformations $\Phi_1(x,p)$ and $\Phi_2(x,p)$, possessing the same regularity with respect to $p$ as do coefficients $M_j$ and $\delta\Theta$, for which the graphs $\{(Z_1, \Phi_2 Z_1)\}$ and $\{(\Phi_1 Z_2, Z_2)\}$ are invariant under the flow of (A.17), and satisfy

\[(A.19)\quad \sup |\Phi_1|, \sup |\Phi_2| \leq C \sup(\delta/\eta)\]

and

\[(A.20)\quad |\Phi_1^\pm(x)| \leq C \int_x^{+\infty} e^{\int_y^x \eta(z)dz} \delta(y)dy, \quad |\Phi_1^\pm(x)| \leq C \int_{-\infty}^x e^{\int_y^0 -\eta(z)dz} \delta(y)dy.\]

Proof. By the change of coordinates $x \to \tilde{x}$, $\delta \to \tilde{\delta} := \delta/\eta$ with $d\tilde{x}/dx = \eta(x)$, we may reduce to the case $\eta \equiv \text{constant} = 1$ treated in [MaZ3]. Dropping tildes and setting $\Phi_2 := \psi_2\psi_1^{-1}$, where $(\psi_1^1, \psi_2^1)^t$ satisfies (A.17), we find after a brief calculation that $\Phi_2$ satisfies

\[(A.21)\quad \Phi_2' = (M_2\Phi_2 - \Phi_2 M_1) + \delta Q(\Phi_2),\]

where $Q$ is the quadratic matrix polynomial $Q(\Phi) := \Theta_{21} + \Theta_{22}\Phi - \Phi\Theta_{11} + \Phi\Theta_{12}\Phi$. Viewed as a vector equation, this has the form

\[(A.22)\quad \Phi_2' = M\Phi_2 + \delta Q(\Phi_2).\]
with linear operator $\mathcal{M}\Phi := M_2\Phi - \Phi M_1$. Note that a basis of solutions of the decoupled equation $\Phi' = \mathcal{M}\Phi$ may be obtained as the tensor product $\Phi = \phi\tilde{\phi}$ of bases of solutions of $\phi' = M_2\phi$ and $\tilde{\phi}' = -M_1^*\tilde{\phi}$, whence we obtain from \text{(A.18)}

\begin{equation}
    e^{\mathcal{M}z} \leq Ce^{-\eta z}, \quad \text{for } z > 0,
\end{equation}

or uniform exponentially decay in the forward direction.

Thus, assuming only that $\Phi_2$ is bounded at $-\infty$, we obtain by Duhamel’s principle the integral fixed-point equation

\begin{equation}
    \Phi_2(x) = T\Phi_2(x) := \int_{-\infty}^{x} e^{\mathcal{M}(x-y)}\delta(y)Q(\Phi_2)(y)\,dy.
\end{equation}

Using \text{(A.23)}, we find that $T$ is a contraction of order $O(\delta/\eta)$, hence \text{(A.24)} determines a unique solution for $\delta/\eta$ sufficiently small, which, moreover, is order $\delta/\eta$ as claimed. Finally, substituting $Q(\Phi) = O(1 + |\Phi|^2) = O(1)$ in \text{(A.24)}, we obtain

\begin{equation*}
    |\Phi_2(x)| \leq C \int_{-\infty}^{x} e^{\eta(x-y)}\delta(y)\,dy
\end{equation*}

in $\tilde{x}$ coordinates, or, in the original $x$-coordinates, \text{(A.20)}. A symmetric argument establishes existence of $\Phi_1$ with the asserted bounds. Regularity with respect to parameters is inherited as usual through the fixed-point construction via the Implicit Function Theorem. \hfill \Box

\textbf{Remark A.8.} For $\eta$ constant and $\delta$ decaying at exponential rate strictly slower than $e^{-\eta x}$ as $x \to +\infty$, we find from \text{(A.20)} that $\Phi_2(x)$ decays like $\delta/\eta$ as $x \to +\infty$, while if $\delta(x)$ merely decays monotonically as $x \to -\infty$, we find that $\Phi_2(x)$ decays like $(\delta/\eta)$ as $x \to -\infty$, and symmetrically for $\Phi_1$.

\textbf{Remark A.9.} 1. A closer look at the proof of Lemma \text{A.7} shows that, in the approximately block lower-triangular case, $\delta\Theta_{21}$ not necessarily small, there exists a block-triangularizing transformation $\Phi_1 = O(\sup|\delta/\eta|) << 1$, under the much less restrictive conditions

\begin{equation*}
    \sup(|\delta/\eta|||\Theta_{11}| + |\Theta_{22}|) < 1 \quad \text{and} \quad \sup(|\delta/\eta|||\Theta_{21}|) << \frac{1}{\sup|\delta/\eta|}.
\end{equation*}

2. Similarly, in the standard, approximately block-diagonal case, an examination of the proof shows that bounds \text{(A.19)} may be sharpened to

\begin{equation}
    \sup |\Phi_1| \leq C \sup(\delta/\eta)\left(\sup |\Theta_{12}| + \sup(\delta/\eta)\sup(|\Theta_{11}| + |\Theta_{22}|) + \sup(\delta/\eta)^2 \sup|\Theta_{21}|ight),
\end{equation}

\begin{equation}
    \sup |\Phi_2| \leq C \sup(\delta/\eta)\left(\sup |\Theta_{21}| + \sup(\delta/\eta)\sup(|\Theta_{11}| + |\Theta_{22}|) + \sup(\delta/\eta)^2 \sup|\Theta_{12}|ight).
\end{equation}

\textbf{Remark A.10.} An important observation of \textit{MaZ3, PZ} is that hypothesis \text{(A.18)} of Lemma \text{A.7} may be weakened to

\begin{equation}
    \min \sigma(\Re M^e_1) - \max \sigma(\Re M^e_2) \geq \eta(x) + \alpha(x, p)
\end{equation}
with no change in the conclusions, for any $\alpha$ satisfying a uniform $L^1$ bound $|\alpha(\cdot, p)|_{L^1} \leq C_1$. (Substitute $e^{Mx} \leq Ce^{C_1 e^{-\eta z}}$ for [A.23], with no other change in the proof.) More generally, (A.27) may be replaced in Lemma A.7 by

(A.27) \[ W' = \begin{pmatrix} M_1 + \alpha_1 & 0 \\ 0 & M_2 + \alpha_2 \end{pmatrix} (x, p) + (\delta \Theta + \beta)(x, p) W, \]

where $|\alpha_j|_{L^1}$ are bounded and $|\beta|_{L^1}$ is sufficiently small, with the conclusion that

(A.28) \[ \sup |\Phi_1|, \sup |\Phi_2| \leq C (\sup(\delta/\eta) + |\beta|_{L^1}). \]

(The additional term $\int_{-\infty}^{x} e^{M(x-y)} Q_{\beta}(\Phi_2)(y) dy$, $Q_{\beta}(\Phi_2) := \beta_{21} + \beta_{22} \Phi - \Phi \beta_{11} + \Phi \beta_{12} \Phi$, now appearing in the righthand side of [A.24] is contractive for $|\beta|_{L^1}$ small.) These allow us to neglect commutator terms in some of the more delicate applications of tracking: for example, the high-frequency analysis of [MaZ3], or the analysis of case IIb in Section 6.2.2.

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