Semiparametric Slepian-Bangs Formula for Complex Elliptically Symmetric Distributions

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Abstract

This letter aims at deriving a Semiparametric Slepian-Bangs (SSB) formula for Complex Elliptically Symmetric (CES) distributed data vectors. The Semiparametric Cramér-Rao Bound (SCRB), related to the proposed SSB formula, provides a lower bound on the Mean Square Error (MSE) of any robust estimator of a parameter vector parameterizing the mean vector and the scatter matrix of the given CES-distributed vector in the presence of an unknown, nuisance, density generator.

Index Terms

Semiparametric model, Slepian-Bangs formula, Complex Elliptically Symmetric distributions, Semiparametric Cramér-Rao Bound.

I. INTRODUCTION

Introduced by Slepian and Bangs in [1] and [2], the celebrated Slepian-Bangs (SB) formula has been extensively used for many years in array Signal Processing (SP) applications. The SB formula is a compact expression of the Fisher Information Matrix (FIM) for parameter estimation under a Gaussian data model [3, Appendix 3C]. Specifically, let $\theta \in \Theta \subset \mathbb{R}^d$ be a $d$-dimensional, deterministic parameter vector and let $\mathbb{C}^N \ni z \sim \mathcal{CN}(\mu(\theta), \Sigma(\theta))$ be a possibly complex, Gaussian-distributed, random vector (also called snapshot), representing the available

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observation. Then the SB formula provides us with a closed-form expression for the FIM for the estimation of \( \theta \in \Theta \).

Due to its central role in many practical applications, including Direction of Arrival (DOA) estimation, the SB formula has been the subject of active research. In particular, it has been generalized to non-Gaussian and mismatched \footnote{The reader that is not familiar with the semiparametric theory may have a look at the books [12] and [13] or to the wide statistical literature available on this topic and partially collected in the reference lists of [9], [10] and [11].} estimation frameworks. Specifically, in [5], Besson and Abramovich proposed a generalization of the classical, SB formula to Complex Elliptically Symmetric (CES) distributed data [6]. On the other hand, Richmond and Horowitz in [7] showed an extension of the classical, Gaussian-based, SB formula to estimation problems under model misspecification. The natural follow-on of the works [5] and [7] has been proposed in [8], where SB-type formulas, that encompass the ones previously obtained in [5] and [7] as special cases, have been derived for parameters estimation problems involving CES distributed data under model misspecification.

The aim of this letter is to take a step forward to the generalization of the SB formula for robust estimation in a CES distribution framework. Building upon our previous works [9], [10] and [11], we propose a Semiparametric SB (SSB) formula that provides us with a compact expression of the inverse of the Semiparametric\footnote{The reader that is not familiar with the semiparametric theory may have a look at the books [12] and [13] or to the wide statistical literature available on this topic and partially collected in the reference lists of [9], [10] and [11].} Cramér-Rao Bound (SCRB) ([10], [11]), i.e. the Semiparametric FIM (SFIM), for the robust estimation of \( \theta \in \Theta \) in CES distributed data. Specifically, let \( \mathbb{C}^N \ni z \sim CES_N(\mu(\theta), \Sigma(\theta), h) \) be a CES-distributed random vector parameterized by \( \theta \in \Theta \), then the SCRB related to the proposed SSB formula provides a lower bound on the Mean Square Error (MSE) of any robust estimator of \( \theta \) in the presence of an unknown, nuisance density generator \( h \).

Remark: Throughout this letter, the unknown parameter vector \( \theta \in \Theta \) is considered to be real-valued. This assumption does not represent a limitation, since we can always maps a complex vector in a real one simply by stacking its real and the imaginary parts. Moreover, Wirtinger calculus (see e.g. [14], [15], [16], [17], [18], [19], [20]) may be exploited to obtain the proposed SSB formula directly for the complex case.

Notation: Italics indicates scalar quantities \( (a, A) \), lower case and upper case boldface indicate column vectors \( (\mathbf{a}) \) and matrices \( (\mathbf{A}) \), respectively. Each entry of a matrix \( \mathbf{A} \) is indicated as \( a_{i,j} \equiv [\mathbf{A}]_{i,j} \). Sometimes, we indicate a vector-valued function as \( \mathbf{a} \equiv \mathbf{a}(x) \). The superscripts \( *, 1 \),
T and H indicate the complex conjugate, the transpose and the Hermitian operators, respectively. Moreover, 
\[ A^{-T} \triangleq (A^{-1})^T = (A^T)^{-1} \text{ and } A^{-H} \triangleq (A^{-1})^H = (A^H)^{-1}. \]
Let \( A(\theta) \) be a matrix (or possibly a vector or even a scalar) function of the vector \( \theta \), then \( A_0 \triangleq A(\theta_0) \) while \( A_0^j \triangleq \frac{\partial A(\theta)}{\partial \theta_j} \big|_{\theta = \theta_0} \) and \( A_{ij}^0 \triangleq \frac{\partial^2 A(\theta)}{\partial \theta_i \partial \theta_j} \big|_{\theta = \theta_0} \), where \( \theta_0 \) is a particular (or “true”) value of \( \theta \). \( I_N \) defines the \( N \times N \) identity matrix. Finally, for random variables or vectors, the notation \( =_d \) stands for "has the same distribution as".

II. Preliminaries

Let \( \mathbb{C}^N \ni z \sim CES_N(\mu_0, \Sigma_0, h_0) \) be a CES-distributed random vector whose mean value \( \mu_0 \triangleq \mu(\theta_0) \) and scatter matrix \( \Sigma_0 \triangleq \Sigma(\theta_0) \) are parameterized by a \( d \)-dimensional parameter vector \( \theta \in \Theta \subset \mathbb{R}^d \) to be estimated. The true density generator \( h_0 \), belonging to some suitable function space \( G \), is left unspecified since it represents an unknown, infinite-dimensional nuisance parameter whose estimate is not strictly required. We assume here that \( \Sigma(\theta) \) is a full rank, positive definite, Hermitian matrix for any possible value of \( \theta \in \Theta \). Consequently, as shown in [21], [22, Ch. 3], [6] and [23, Ch. 4], the “true” pdf of \( z \), i.e. the pdf characterized by the true density generator \( h_0 \) and evaluated at the true parameter vector \( \theta_0 \), can be expressed as:

\[
p_Z(z|\theta_0, h_0) = |\Sigma_0|^{-1}h_0((z - \mu_0)^H\Sigma_0^{-1}(z - \mu_0)). \tag{1}
\]

According to the notation introduced in [9], [10] and [11], in the sequel, we indicate with \( p_0(z) \triangleq p_Z(z|\theta_0, h_0) \) and \( E_0\{\cdot\} \) the true pdf of the CES-distributed random vector \( z \) and the expectation operator w.r.t. it, respectively.

The assumption of positive definiteness of \( \Sigma_0 \) guarantees the existence of its principal Hermitian square root \( \Sigma_0^{1/2} \), such that \( \Sigma_0^{H/2}\Sigma_0^{1/2} = \Sigma_0^{1/2}\Sigma_0^{H/2} = \Sigma_0 \). Then, using the Stochastic Representation Theorem (see e.g. [6, Theorem 3]) we have that:

\[
z - \mu_0 =_d \sqrt{Q}\Sigma_0^{1/2}u \tag{2}
\]

where \( u \sim U(CS^N) \) is an \( N \)-dimensional complex random vector uniformly distributed on the unit sphere with \( N - 1 \) topological dimension while the positive random variable \( Q =_d Q_0 = (z - \mu_0)^H\Sigma_0^{-1}(z - \mu_0) \) is called second-order modular variate whose pdf is given by:

\[
p_Q(q) = \pi^N\Gamma(N)^{-1}q^{N-1}h_0(q), \tag{3}
\]

\(^2\)The definition of the pdf for CES distributed random vectors given here differs from the one proposed in [6, eq. 16] due to the fact that the normalizing constant \( c_{N,h} \) in [6, eq. 16] has been absorbed in the density generator \( h_0 \).
where $\Gamma(\cdot)$ is the Gamma function. In order to avoid the scale ambiguity problem between the scatter matrix and the density generator [6, Sec. C], we impose a constraint on the functional form of $h_0$. Specifically, following the same procedure adopted in [8], we assume that $h_0 \in G$ is parameterized in order to satisfy the following constraint:

$$E\{Q\} = \pi^N \Gamma(N)^{-1} \int_0^{+\infty} q^{N-1} h_0(q) dq = N. \quad (4)$$

As a consequence, the scatter matrix $\Sigma_0$ equates the covariance matrix of $z$, i.e. $\Sigma_0 = E_0\{(z - \mu_0)(z - \mu_0)^H\}$ [6]. For further reference, we define the set $\hat{G} \subset G$ as the set of all the density generators satisfying the constraint in (4).

We now focus our attention on the semiparametric group nature of the family of all the pdfs, say $P_{\theta,h}$, of an (absolutely continuous) CES-distributed random vector $z \sim CES_N(\mu(\theta), \Sigma(\theta), h)$ with $\theta \in \Theta$ and $h \in G$. Following the discussion provided in Secs. 4.2 and 4.3 of [12] and recalled in our recent works [10] and [11], let us firstly introduce the group $\mathcal{A}$ of affine transformations:

$$\mathcal{A} \ni \alpha_{\theta} : \mathbb{C}^N \rightarrow \mathbb{C}^N, \forall \theta \in \Theta$$

$$\mathbb{C}^N \ni w \mapsto \alpha_{\theta}(w) = \mu(\theta) + \Sigma(\theta)^{1/2} w. \quad (5)$$

Then, as shown in [12, Sec. 4.2, Lemma 2], the model $P_{\theta,h}$ can be considered as a semiparametric group model generated by $\mathcal{A}$. Consequently, it can be explicitly expressed as:

$$P_{\theta,h} = \{ p_Z | p_Z(z|\theta,h) = |\Sigma(\theta)|^{-1/2} h(\|\alpha_{\theta}^{-1}(z)\|^2), \theta \in \Theta, h \in \hat{G} \}, \quad (6)$$

where $\alpha_{\theta}^{-1}(\cdot) = \Sigma(\theta)^{-1/2}(\cdot - \mu(\theta))$ is the inverse transformation of $\alpha_{\theta} \in \mathcal{A}$ and $\|\cdot\|$ indicates the Euclidean norm. Under regularity conditions on the mapping $\theta \rightarrow (\mu(\theta), \Sigma(\theta))$ discussed in [12, Sec. 4.2, pp. 92, Assumptions (iii), (iv), (v)], we can exploit the peculiar properties of the semiparametric group models (see [12, Sec. 4.2] for the general case and [10] for the RES distribution case) to evaluate the Semiparametric FIM $\hat{I}(\theta_0|h_0)$ for the estimation of $\theta_0 \in \Theta$ in the presence of the nuisance (and unknown) density generator $h_0$.

III. THE SINGLE SNAPSHOT CASE

Let us start with the case in which we have only one snapshot sampled from an unspecified CES distribution, i.e. $z \sim CES_N(\mu_0, \Sigma_0, h_0)$. As discussed in the relevant statistical literature (see, among the others, [13, Theorem 4.1], [24], [25] and [12, Sec. 3.4]) and recalled in [9],
[10, Theorem IV.1] and in [11, Sec. III], the SFIM for the estimation of $\theta_0 \in \Theta$ is defined as $\breve{I}(\theta_0|h_0) \triangleq E_0\{s_{\theta_0}^{'H} s_{\theta_0}^{'}\}$ where the semiparametric efficient score vector $s_{\theta_0} \equiv s_{\theta_0}(z)$ is given by:

$$s_{\theta_0} \triangleq s_{\theta_0} - \Pi(s_{\theta_0}|T_{h_0}),$$  \hspace{1cm} (7)

where $s_{\theta_0}$ is the score vector evaluated at the true parameter vector $\theta_0$ and $\Pi(s_{\theta_0}|T_{h_0})$ is the orthogonal projection of $s_{\theta_0}$ on the semiparametric nuisance tangent space of $T_{h_0}$ in (6) evaluated at the true density generator $h_0$.

A. Evaluation of the semiparametric efficient score vector $\breve{s}_{\theta_0}$

Let us start with the calculation of the score function $s_{\theta_0}$. Following the same calculation of [8, Sec. 3.1] and [5, Sec. III], each entry of $s_{\theta_0}$ can be easily evaluated as:

$$[s_{\theta_0}]_i \triangleq \frac{\partial \ln p_Z(z; \theta)}{\partial \theta_i} \bigg|_{\theta = \theta_0} = \text{tr}(P_i^0) + \psi_0(Q_0) \frac{\partial Q_0}{\partial \theta_i}$$ \hspace{1cm} (8)

where

$$\psi_0(t) = d \ln h_0(t)/dt,$$

and $P_i^0 \triangleq \Sigma_i^{-1/2} \Sigma_0^{1/2} \Sigma_i^{-1/2}$. Moreover, from [8, eq. (22)] and [5, eq. (8)], we have:

$$\frac{\partial Q_0}{\partial \theta_i} = -2\text{Re} \left[ (z - \mu_0)^H \Sigma_i^{-1} \mu_i^0 \right] - (z - \mu_0)^H S_i^0 (z - \mu_0),$$ \hspace{1cm} (10)

where, according to the notation previously introduced, $\mu_i^0 \triangleq \frac{\partial \mu_i}{\partial \theta_i}$ and $S_i^0 = \Sigma_i^{-1} \Sigma_i \Sigma_0^{-1}$. By collecting the previous results, the entries of the score vector can be expressed as:

$$[s_{\theta_0}]_i = \text{tr} (P_i^0) - \psi_0(Q_0) \left[ 2\text{Re} \left[ (z - \mu_0)^H \Sigma_i^{-1} \mu_i^0 \right] + (z - \mu_0)^H S_i^0 (z - \mu_0) \right], \hspace{1cm} i = 1, \ldots, d. \hspace{1cm} (11)$$

Using the Stochastic Representation given in (2), Eq. (11) can be rewritten as:

$$[s_{\theta_0}]_i = -\psi_0(Q) \left( 2\sqrt{Q} \text{Re} \left[ u^H \Sigma_i^{-1/2} \mu_i^0 \right] + Q u^H \Sigma_i^{-1/2} S_i^0 u \right) + \text{tr} (P_i^0)$$

$$= -\psi_0(Q) \left( 2\sqrt{Q} \text{Re} \left[ u^H \Sigma_i^{-1/2} \mu_i^0 \right] + Q u^H P_i^0 u \right) + \text{tr} (P_i^0), \hspace{1cm} i = 1, \ldots, d. \hspace{1cm} (12)$$

The orthogonal projection $\Pi(s_{\theta_0}|T_{h_0})$ can be obtained by following exactly the same procedure discussed in [10, Sec. IV.B]. For the sake of conciseness, here we report only the final result. Specifically:

$$[\Pi(s_{\theta_0}|T_{h_0})]_i = E_{0|\sqrt{Q}}\{[s_{\theta_0}]_i | \sqrt{Q} \}$$

$$= \text{tr} (P_i^0) - 2\sqrt{Q} \psi_0(Q) \text{Re} \left[ E\{u^H \Sigma_i^{-1/2} \mu_i^0 \} - Q \psi_0(Q) \text{tr} (P_i^0 E\{u^H u\}) \right]$$

$$= \text{tr} (P_i^0) - N^{-1} Q \psi_0(Q) \text{tr} (P_i^0), \hspace{1cm} i = 1, \ldots, d. \hspace{1cm} (13)$$
Finally, by substituting (12) and (13) in (7), we get explicit expressions for the \(d\) entries of the semiparametric efficient score vector \(\bar{s}_{\theta_0}\) as:

\[
[\bar{s}_{\theta_0}]_i = d \psi_0(\mathcal{Q}) \left( N^{-1} \mathcal{Q} \text{tr}(P^0_i) - 2\sqrt{\mathcal{Q}} \text{Re} \left[ u^H \Sigma_0^{-1/2} \mu_i^0 \right] - \mathcal{Q} u^H P^0_i u \right) = d \psi_0(\mathcal{Q}) \left( N^{-1} \mathcal{Q} \text{tr}(P^0_i) - \sqrt{\mathcal{Q}} u^H \Sigma_0^{-1/2} \mu_i^0 - \sqrt{\mathcal{Q}} (\mu_i^0)^H \Sigma_0^{-1/2} u - \mathcal{Q} u^H P^0_i u \right) \quad i = 1, \ldots, d. \tag{14}
\]

### B. Evaluation of Semiparametric Fisher Information Matrix (SFIM) \(\bar{I}(\theta_0|h_0)\)

As mentioned before, the SFIM for the estimation of \(\theta_0\) in the presence of the unknown, infinite-dimensional, nuisance parameter \(h_0\) is given by \(\bar{I}(\theta_0|h_0) \triangleq E_0 \{ \bar{s}_{\theta_0} \bar{s}_{\theta_0}^H \}\). In the sequel, a sketch of the calculation required to obtain an explicit expression for each entry of \(\bar{I}(\theta_0|h_0)\), that represents the SSB formula, is reported. Note that this procedure strongly relies on the calculation already developed in [8].

Let us start by defining the vector \(\mathbf{t} \triangleq \Sigma_0^{-1/2} \mathbf{u}\) and then, substituting \(\mathbf{t}\) in (14), we get:

\[
[\bar{s}_{\theta_0}]_i = d \psi_0(\mathcal{Q}) \left( N^{-1} \mathcal{Q} \text{tr}(P^0_i) - \sqrt{\mathcal{Q}} \mathbf{t}^H \mu_i^0 - \sqrt{\mathcal{Q}} (\mu_i^0)^H \mathbf{t} - \mathcal{Q} \mathbf{t}^H \Sigma_0^0 \mathbf{t} \right). \tag{15}
\]

The next step consists in evaluating the products:

\[
[\bar{s}_{\theta_0}]_i[\bar{s}_{\theta_0}]_j^* = \psi(\mathcal{Q})^2 \left[ N^{-2} \mathcal{Q}^2 \text{tr}(P^0_i) \text{tr}(P^0_j) - N^{-1} \mathcal{Q}^2 \left( \text{tr}(P^0_i) \mathbf{t}^H \Sigma_0^0 \mathbf{t} + \text{tr}(P^0_j) \mathbf{t}^H \Sigma_0^0 \mathbf{t} \right) \times + \right. \\
\left. + \mathcal{Q} \mathbf{t}^H \mu_i^0 \mathbf{t} \mu_j^0 + \mathcal{Q} \mathbf{t}^H \mu_i^0 (\mu_j^0)^H \mathbf{t} + \\
+ \mathcal{Q} (\mu_i^0)^H \mathbf{t} \mu_j^0 + \mathcal{Q} (\mu_i^0)^H \mathbf{t} (\mu_j^0)^H \mathbf{t} + \\
\left. + \mathcal{Q}^2 \mathbf{t}^H \Sigma_0^0 \mathbf{t} \mathbf{t}^H \Sigma_0^0 \mathbf{t} \right] \quad i, j = 1, \ldots, d. \tag{16}
\]

Finally, by taking the expectation w.r.t. the true pdf \(p_0(z)\) in (1) and by using the relations derived in (B.4)-(B.10) of [8, Appendix B], it is easy to verify that each entry of the SFIM \(\bar{I}(\theta_0|h_0)\) can be expressed as:

\[
[I(\theta_0|h_0)]_{ij} = \frac{2E_0 \{ \mathcal{Q} \psi_0(\mathcal{Q})^2 \}}{N} \text{Re}[(\mu_i^0)^H \Sigma_0^{-1} \mu_j^0] + \\
+ \frac{E_0 \{ \mathcal{Q}^2 \psi_0(\mathcal{Q})^2 \}}{N(N+1)} \left[ \text{tr}(\Sigma_0^{-1} \Sigma_i^0 \Sigma_0^{-1} \Sigma_j^0) - N^{-1} \text{tr}(\Sigma_0^{-1} \Sigma_i^0) \text{tr}(\Sigma_0^{-1} \Sigma_j^0) \right] \quad i, j = 1, \ldots, d, \tag{17}
\]

that represent the SSB formula for a single CES-distributed snapshot.
C. A compact expression for $\bar{I}(\theta_0|h_0)$

Using the well-known properties of the Kronecker product $\otimes$ and of the standard vectorization operator $\text{vec}$ (see e.g. [26], [27]), we now show how to rewrite the SFIM in (17) in a more compact and easy-to-use form. Let us define the two Jacobian matrices of the mean vector $\mu(\theta)$ and of the scatter matrix $\Sigma(\theta)$ as $M_0 = \nabla_\theta \mu(\theta_0) \in \mathbb{C}^{N \times d}$ and $V_0 = \nabla_\theta \text{vec}(\Sigma(\theta_0)) \in \mathbb{C}^{N^2 \times d}$, respectively. Note that both $M_0$ and $V_0$ are evaluated at the true parameter vector $\theta_0$. Then, the $\bar{I}(\theta_0|h_0)$ can be written in a compact Gramian form as:

$$
\bar{I}(\theta_0|h_0) = 2 \frac{E\{Q\psi_0(Q)^2\}}{N} \Re[(\Sigma_0^{-1/2}M_0)^H(\Sigma_0^{-1/2}M_0)] + \frac{E\{Q^2\psi_0(Q)^2\}}{N(N+1)}(T^{1/2}V_0)^H(T^{1/2}V_0)
$$

where the matrices $T^{1/2}$ and $\Pi_{\text{vec}(I_N)}^\perp$ are:

$$
T^{1/2} = \Pi_{\text{vec}(I_N)}^\perp(\Sigma_0^{-T/2} \otimes \Sigma_0^{-1/2}),
$$

$$
\Pi_{\text{vec}(I_N)}^\perp = I_{N^2} - N^{-1}\text{vec}(I_N)\vec{(I_N)^T}.
$$

The matrix $\Pi_{\text{vec}(I_N)}^\perp$ is the orthogonal projection matrix on the orthogonal complement of $\text{span}(\text{vec}(I_N))$.

Then, by using the property of $\otimes$ given, e.g., in [28, eq. 6] and the fact that an orthogonal projection matrix is idempotent, we have that:

$$
T \triangleq \Sigma_0^{-T} \otimes \Sigma_0^{-1} - N^{-1}\text{vec}(\Sigma_0^{-1})\text{vec}(\Sigma_0^{-1})^H.
$$

It is worth noticing that the compact expression of $\bar{I}(\theta_0|h_0)$ obtained in (18) encompasses as special cases the expressions of the SFIM for the scatter matrix estimation derived in [10, eq. 56] for the RES case and in [11, eq. 25] for the CES case. To clarify this point, let us consider the scatter matrix estimation problem under the assumption of a perfectly known mean vector. If we further assume that the scatter matrix $\Sigma_0$ is a real (symmetric) matrix, then the unknown parameter vector can be recast as $\theta_0 = \text{vecs}(\Sigma_0)$, where the $\text{vecs}$ operator maps the symmetric $N \times N$ matrix $\Sigma_0$ to an $N(N+1)/2$-dimensional vector containing the elements of the lower triangular sub-matrix of $\Sigma_0$. This definition of $\theta_0$ implies that the Jacobian matrix of the mean vector $M_0$ is nil while the Jacobian matrix of the scatter matrix is given by $V_0 = \nabla_{\text{vec}(\Sigma_0)}\text{vec}(\Sigma_0) = D_N$, where $D_N$ is the so-called duplication matrix and the last equality follows from [27, Lemma 3.8]. Finally, by substituting the derived expressions for the two
Jacobian matrices in (18), we immediately obtain the expression of the SFIM for the (real) scatter matrix estimation problem, already derived in [10, eq. 56]. The more general equivalence between [11, eq. 25] and the SSB formula presented here in (18) can easily be obtained by using the Wirtinger calculus.

IV. THE MULTIPLE SNAPSHOT CASE

In this section, we provide two extensions of the single-snapshot SB formula derived in Sec. III to two multi-snapshot scenarios. Before starting with the derivation, a comment is in order. The classical multiple-snapshot scenario considered in array SP applications is characterized by the availability of \( L \) independent, CES-distributed, data vectors \( z_l \sim CES_N(z_l; \mu_l(\theta_0), \Sigma_0(\theta_0), h_0) \) sharing the same scatter matrix but with a possibly different mean vector from snapshot to snapshot. Due to the possible variation of the mean vectors, the available data \( \{z_l\}_{l=1}^L \) are not identically distributed. Then the semiparametric framework presented in [12] and exploited to obtain the results provided in this letter and in our previous works ([9], [10] and [11]) cannot be applied straightforwardly. The extension of the classical semiparametric theory to the non-i.i.d. (independent and identical distributed) case is a well established topic (see e.g. [29] for a summary of main works in this filed or the seminal paper [30]), but falls outside the scope of this letter. Consequently, we left the extension of the SSB formula in (18) to the general, non-i.i.d. multiple snapshot case for future work, while here we focus our attention on two less general models which, however, are still relevant in practice.

A. Semiparametric Bangs formula

Assume to have a set of \( L \) i.i.d. CES-distributed random vectors \( \{z_l\}_{l=1}^L \) sampled from \( CES_N(z_l; \mu, \Sigma_0, h_0) \), where the mean vector is assumed to be constant and perfectly known while the scatter matrix \( \Sigma_0 \equiv \Sigma(\theta_0) \) is constant with respect to \( l \) and it is parameterized by the unknown parameter vector \( \theta_0 \in \Theta \). Note that this model is the one used in array SP applications to model the received data under the assumption of a stochastic signal model (see e.g. [28]). Since the data are i.i.d. random vectors, the multiple-snapshot extension of the SB formula in (18) is trivial. Let us define the multi-snapshot SFIM as \( \bar{I}_L(\theta_0|h_0) \triangleq E_0\{s_{\theta_0}(\{z_l\}_{l=1}^L)s_{\theta_0}(\{z_l\}_{l=1}^L)^H\} \), then, from (18), we have:

\[
\bar{I}_L(\theta_0|h_0) = L\frac{E\{Q^2\psi_0(Q)^2\}}{N(N+1)}V_0^HTV_0, \tag{22}
\]
where the function $\psi_0$ is already defined in (9), and matrices $V_0$ and $T$ have been defined in Sec. III-C.

**B. SSB formula for the Elliptical Vector (EV) Model**

The so-called Elliptical Vector model has been already used in [22], [31], [7] and in [5]. Specifically, in [7] the EV model has been exploited to derive the *misspecified* SB formula under the mismatched Gaussian assumption. The basic idea behind the EV model is to consider the $L$ sub-vector of $LN$-dimensional, CES-distributed random vectors, where $L$ is the number of available snapshots. More formally, suppose to have an $LN$-dimensional CES-distributed random vector $\mathbb{C}^{LN} \ni z \triangleq \left[ z_1^T, \ldots, z_L^T \right]^T \sim CES_{LN}(z; \mu_0, \Sigma_0, h_0)$ whose mean vector and scatter matrix are defined as:

$$
\mu_0 \triangleq \left[ \mu_1(\theta_0)^T, \ldots, \mu_L(\theta_0)^T \right]^T \equiv \left[ \mu_{1,0}^T, \ldots, \mu_{L,0}^T \right]^T \in \mathbb{C}^{LN},
$$

(23)

$$
\Sigma_0 \triangleq I_L \otimes \Omega(\theta_0) \equiv I_L \otimes \Omega_0 \in \mathbb{C}^{LN \times LN}.
$$

(24)

Under these assumptions on $z$, we can use [21, Lemma 3.5] and [6, Theorem 2] to derive some useful properties of the sub-vectors $\{z_l\}_{l=1}^L$. Specifically, for each $l$, $z_l \sim CES_N(z_l; \mu_{l,0}, \Sigma_0, \tilde{h}_0)$ is an $N$-dimensional CES-distributed random vector with mean vector $\mu_{l,0}$, scatter matrix $\Sigma_0$ and “marginal” density generator $\tilde{h}_0$ that is related to $h_0$ by the integral equation given in [22, eq. 3.89]. It is important to note that, even if in general the functional form of $h_0$ is different from the one of its “marginal” counterpart $\tilde{h}_0$, the vector $z$ and all its sub-vectors $\{z_l\}_{l=1}^L$ share the same characteristic generator [6, Theorem 2]. From [21, Lemma 3.5], $z_l$ admits the following stochastic representation: $z_l - \mu_{l,0} = d \sqrt{Q_l} \Omega_0^{1/2} u_l, \forall l = 1, \ldots, L$, where $u_l \sim U(\mathbb{C}^N)$ is independent of $Q_l$. Furthermore, $Q_l = d \beta Q$ where $\beta \sim Beta(N, N(L-1))$ is a Beta-distributed random variable, independent of $Q$ that is the second-order modular variate of $z$. The derivation of the SSB formula for the EV model can be easily obtained by substituting the expressions of $\mu_0$ and $\Sigma_0$, given in (23) and (24), in the SSB formula already derived in (17). Finally, by using the properties of the Kronecker product, we get:

$$
[I_L(\theta_0|g_0)]_{i,j} = \frac{2E\{Q\psi_0(Q)^2\}}{LN} \sum_{l=1}^L \Re \left[ (\mu_{i,l}^0)^H \Sigma_0^{-1} \mu_{j,l}^0 \right] + \frac{E\{Q^2\psi_0(Q)^2\}}{N(LN+1)} \left[ \tr(\Omega_0^{-1} \Sigma_0^{-1} \Omega_0^{-1}) - N^{-1} \tr(\Omega_0^{-1} \Omega_0^{-1} \tr(\Omega_0^{-1} \Omega_0^{-1})) \right].
$$

(25)

Clearly, this expression of the SFIM for the SV model can be rewritten in a compact Gramian form, by following the same procedure used in Sec. III-C.
V. Conclusions

In this letter, the Semiparametric Slepian-Bangs (SSB) formula for vector parameter estimation in CES-distributed data was proposed by considering the density generator, that characterizes the actual data distribution, as an unknown, infinite-dimensional, nuisance parameter. Future work will explore the possibility to further extend the derived SSB formula to a set of non-i.i.d snapshots. Moreover, the exploitation of the obtained results to classical array signal processing applications, such as the DOA estimation, will be investigated.

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