New Exact Solutions for Generalized (3+1) Shallow Water-Like (SWL) Equation

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Abstract

In this study, we use the improved Bernoulli sub-equation function method for exact solutions to the generalized (3+1) shallow water-like (SWL) equation. Some new solutions are successfully constructed. We carried out all the computations and the graphics plot in this paper by Wolfram Mathematica.

Keywords: Generalized (3+1) shallow water-like (SWL) equation, Improved Bernoulli sub-equation function method, Exact solution

AMS 2010 codes: 35Q35, 37N10

1 Introduction

In various fields of physical sciences, nonlinear evolution equations (NLEEs) and their exact solutions are important for non-linear phenomena. In this paper, generalized (3 + 1) shallow water-like (SWL) equation [1, 2] which is one of these equations will be discussed and new solutions will be examined.

\[ u_{xxxx} + 3u_xu_{xy} + 3u_xu_{xy} - u_{yt} - u_{xz} = 0. \]  

(1)

There are some studies on this equation. Rational solutions and lump solutions are obtained for equation(1) by Zhang et al. [1] and Grammian and Pfaffian solutions are obtained by Tang et al. [2]. Also, this equation solved by Tian and Gao [3] via the tanh method, by Zayed [4] via the \(G'/G\) expansion method. Lump-type solutions and their interaction solutions are generated by Sadat [5]. In this context, various papers were presented to the literature [6–23]. The organization of this paper is as follows: firstly, we give the methodology of the improved Bernoulli sub-equation function method. Then we apply this method to the SWL equation for finding new exact solutions. At last, we give a conclusion.
2 Material and Method

In this part, we use the improved Bernoulli sub-equation function method [24–28] for solutions eq. (1).

Step 1. Let’s consider the following partial differential equation:

\[ P(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, \ldots) = 0. \]  

(2)

and take the wave transformation

\[ u(x, y, z, t) = U(\xi), \xi = x + ky + mz - wt, \]  

(3)

where \( k, m \) and \( w \) are nonzero constants. Substituting Eq. (2) into Eq. (3), we obtain the following nonlinear ordinary differential equation:

\[ N = (U, U', U'', U''' , \ldots) = 0. \]  

(4)

Step 2. Considering trial equation of solution in Eq. (4), it can be written as following:

\[ U(\xi) = \sum_{i=0}^{n} a_i F_i(\xi) \sum_{j=0}^{m} b_j F_j(\xi) \]  

(5)

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for as following:

\[ F' = \alpha F + \beta F^M, \alpha, \beta \neq 0, M \in R - 0, 1 \]  

(6)

where \( F \) is Bernoulli differential polynomial. Substituting Eq. (5-b6) into Eq. (4), it converts an equations of polynomial \( \sigma(F) \) as following:

\[ \sigma(F) = \rho_s F^s + \ldots + \rho_1 F + \rho_0 = 0 \]  

(7)

According to the balance principle, we can determine the relationship between \( n, m \) and \( M \).

Step 3. The coefficients of \( \sigma(F) \) all be zero will yield us an algebraic system of equations;

\[ \rho_i = 0, i = 0, \ldots, s \]  

(8)

Solving this system of equation, we reach the values of \( a_0, \ldots, a_n \) and \( b_0, \ldots, b_m \). Step 4. When we solve nonlinear Bernoulli differential equation Eq. (6), we obtain the following two situations according to \( \alpha \) and \( \beta \):

\[ F(\xi) = \left[ -\frac{\beta}{\alpha} +\frac{E}{e^{\alpha(M-1)\xi}} \right] \frac{1}{1+\nu}, \alpha \neq \beta \]  

(9)

\[ F(\xi) = \left[ \frac{(E-1)+(E+1)tanh(\frac{\alpha(1-M)\xi}{2})}{1-tanh(\frac{\alpha(1-M)\xi}{2})} \right] \frac{1}{1+\nu}, \alpha = \beta, E \in R. \]  

(10)

3 Findings

In this section, application of the improved Bernoulli sub-equation function method to SWL equation is presented. Using the wave transformation on Eq. (1)

\[ u(x, y, z, t) = U(\xi), \xi = x + ky + mz - wt, \]  

(11)

we get the following nonlinear ordinary differential equation:

\[ kU^{(4)} + 6kU'U'' + (kw-m)U'' = 0. \]  

(12)
Integrating the equation in (12), we get

\[ kU''' + 3k[U']^2 + (kw - m)U' = 0. \] (13)

Finally, if we write \( V \) instead of \( U' \), the equation (13) becomes a second order nonlinear ordinary differential equation:

\[ kV'' + 3kV^2 + (kw - m)V = 0. \] (14)

Balancing Eq. (14) by considering the highest derivative \( V'' \) and the highest power \( V^2 \), we obtain

\[ n + 2 = 2M + m. \]

Choosing \( M = 2, m = 1 \), gives \( n = 3 \). Thus, the trial solution to Eq. (1) takes the following form:

\[ U(\xi) = \frac{a_0 + a_1 F(\xi) + a_2 F^2(\xi) + a_3 F^3(\xi)}{b_0 + b_1 F(\xi)}, \] (15)

where \( F' = \alpha F + \beta F^2, \alpha, \beta \neq 0 \). Substituting Eq. (15), its second derivative and power along with \( F' = \alpha F + \beta F^2, \alpha, \beta \neq 0 \), into Eq. (14), yields a polynomial in \( F \). Solving the system of the algebraic equations, yields the values of the parameter involved. Substituting the obtained values of the parameters into Eq. (15), yields the solutions to Eq. (1). We can find following coefficients:

**Case 1**

\[ a_0 = -\frac{(1 + k)wb_1}{3k}, a_1 = \frac{2w^{3/2}b_0}{\sqrt{k / (1 + k)}}, a_2 = -2w^2 b_0 + \frac{2w^{3/2}b_1}{\sqrt{k / (1 + k)}}, a_3 = -2w^2 b_1, \sigma = -\frac{i\sqrt{w}}{\sqrt{k / (1 + k)}}, \] (16)

**Case 2**

\[ a_0 = 0, a_1 = 0, a_2 = \frac{2iw^{3/2}b_1}{\sqrt{k / (1 + k)}}, a_3 = -2w^2 b_1, b_0 = 0, \sigma = -\frac{i\sqrt{w}}{\sqrt{k / (1 + k)}}, \] (17)

**Case 3**

\[ a_0 = 0, a_1 = -\frac{mb_1}{3k}, a_2 = -\frac{4m^{3/2}b_1}{k^{3/2}}, a_3 = -\frac{8m^2 b_1}{k^2}, b_0 = 0, w = \frac{2m}{k}, \sigma = \frac{\sqrt{m}}{\sqrt{k}}; \] (18)

**Case 4**

\[ a_0 = 0, a_1 = \frac{-kw b_1}{3k}, a_2 = -\frac{2iw \sqrt{m - kw b_1}}{\sqrt{k}}, a_3 = -2w^2 b_1, b_0 = 0, \sigma = \frac{i\sqrt{m - kw}}{\sqrt{k}}; \] (19)

Choosing the suitable values of parameters, we performed the numerical simulations of the obtained solutions for (16,17) case by plotting their 2D and 3D.

### 4 Result and Discussion

In this article, new solutions are obtained for the SWL equation using the IBSEFM method. We have seen that the results we obtained are new solutions when we compare them with previous ones. The results may be useful to explain the physical effects of various nonlinear models in non-linear sciences. IBSEFM is a powerful and efficient mathematical tool that can be used to process various nonlinear mathematical models.
Fig. 1 The 3D and 2D surfaces of the solution for (16) for suitable values

Fig. 2 The 3D and 2D surfaces of the solution for (17) for suitable values

References

[1] Y. Zhang, H. Dong, X. Zhang, and H. Yang, Rational solutions and lump solutions to the generalized (3+1)-dimensional Shallow Water-like equation, Computers and Mathematics with Applications, vol. 73, no. 2, pp. 246–252, 2017.

[2] Y.-N. Tang, W.-X. Ma, and W. Xu, “Grammian and Pfaffian solutions as well as Pfaffianization for a (3+1)-dimensional generalized shallow water equation,” Chinese Physics B, vol. 21, no. 7, 2012.

[3] B. Tian and Y.-T. Gao, "Beyond travelling waves: A new algorithm for solving nonlinear evolution equations," Computer Physics Communications, vol. 95, no. 2-3, pp. 139-142, 1996.

[4] E. Zayed, "travelling wave solutions for higher dimensional nonlinear evaluation equations using $G'/G$ expansion method," Journal of Applied Mathematics and Informatics, vol. 28, no. 1, 2, pp. 383-395, 2010.

[5] R. Sadat, M. Kassem, and Wen-Xiu Ma, "Abundant Lump-Type Solutions and Interaction Solutions for a Nonlinear (3+1) Dimensional Model," Advances in Mathematical Physics, vol. 2018, Article ID 9178480, 8 pages, 2018. https://doi.org/10.1155/2018/9178480.

[6] Akin, L. 2017. "Some Weighted Martingale Inequalities On Rearrangement Invariant Quasi-Banach Function Spaces", MSU Journal of Science, 5(2), 483-486.

[7] Akin, L. and Zeren, Y. 2017. Approximation To Generalized Taylor Derivatives By Integral Operator Families, MSU Journal of Science, 5(2), 421-423.

[8] Baskonus, H.M. and Bulut, H. 2015. "An Effective Scheme for Solving Some Nonlinear Partial Differential Equation Arising In Nonlinear Physics", Open Physics, 13, 1, 2807289.

[9] Baskonus, H.M., Sulaiman, T.A. and Bulut, H. 2017. "New Solitary Wave Solutions to the (2+1)-Dimensional Calogero-Bogoyavlenskii-Schiff and the Kadomtsev-Petviashvili Hierarchy Equations", Indian Journal of Physics, 91, 10, 1237-1243.

[10] Baskonus, H.M., Bulut, H. and Atas, S.S. 2018. "Contour Surfaces in the (2+1)-dimensional Sine-Poisson Model", International Journal of Innovative Engineering Applications, 2(2), 44-49.

[11] Modanli, M. 2018. "Two numerical methods for fractional partial differential equation with nonlocal boundary value problem", Advances in Difference Equations 2018:333. https://doi.org/10.1186/s13662-018-1789-2
[12] Modanlı, M. and Akgül, A. 2017. "Numerical solution of fractional telegraph differential equations by theta-method", Eur. Phys. J. Special Topics 226, 3693-3703.

[13] Polat, N. and Pişkin, E. 2015. "Existence and asymptotic behavior of solution of Cauchy problem for the damped sixth-order Boussinesq equation", Acta Mathematicae Applicatae Sinica, English Series, 31(3) 735-746.

[14] Polat, N. and Pişkin, E. 2012. "Asymptotic behavior of solution of Cauchy problem for the generalized damped multi-dimensional Boussinesq equation", Applied Mathematics Letters, 25 1871-1874.

[15] Pişkin, E. and Polat, N. 2014. "Existence, global nonexistence and asymptotic behavior of solutions for Cauchy problem of a multidimensional generalized damped Boussinesq-type equation", Turkish Journal of Mathematics, 38: 706-727.

[16] Pişkin, E. 2013. "Blow up of solutions for the Cauchy problem of the damped sixth-order Boussinesq equation", Theoretical Mathematics and Applications, vol. 4, no. 3 61-71.

[17] Dusunceli, F. and Celik, E. 2018. "Numerical Solution For High-Order Linear Complex Differential Equations with Variable Coefficients", Numerical Methods for Partial Differential Equations, 34(5), 1645-1658. DOI: 10.1002/num.22222.

[18] Dusunceli, F. and Celik, E. 2017. "Numerical Solution for High-Order Linear Complex Differential Equations By Hermite Polynomials", Iğdır University Journal of the Institute of Science and Technology, 7(4): 189-201.

[19] Dusunceli, F. and Celik, E. 2017. "Fibonacci matrix Polynomial Method For Linear Complex Differential Equations", Asian Journal of Mathematics and Computer Research, 15(3): 229-238.

[20] Dusunceli, F. and Celik, E. 2015. "An Effective Tool: Numerical Solutions by Legendre Polynomials for High-Order Linear Complex Differential Equations", British Journal of Applied Science and Technology, 8(4): 348-355.

[21] Ilhan, O.A., Esen, A., Bulut, H. and Baskonus, H.M. 2019. "Singular Solitons in the Pseudo-parabolic Model Arising in Nonlinear Surface Waves", Results in Physics, 12, 1712-1715.

[22] Baskonus, H.M.2019. "Complex Soliton Solutions to the Gilson-Pickering Model", Axioms, 8(1), 18.

[23] Cattani, C., Sulaiman, T.A., Baskonus, H.M. and Bulut, H. 2018. "On the soliton solutions to the Nizhnik-Novikov-Veselov and the Drinfel'd-Sokolov systems", Optical and Quantum Electronics, 50(3), 138.

[24] Baskonus, H.M. and Bulut H. 2015. "On the Complex Structures of Kundu-Eckhaus Equation via Improved Bernoulli Sub-Equation Function Method", Waves in Random and Complex Media, 25,4, 720-728.

[25] Bulut, H., Yel, G. and Baskonus, H.M. 2016. An Application Of Improved Bernoulli Sub-Equation Function Method To The Nonlinear Time-Fractional Burgers Equation, Turkish Journal of Mathematics and Computer Science, 5, 1-17.

[26] Dusunceli, F. 2018. "Solutions for the Drinfeld-Sokolov Equation Using an IBSEFM Method" MSU Journal Of Science, 6,1,505-510.

[27] Dusunceli, F. 2019. "New Exponential and Complex Traveling Wave Solutions to the Konopelchenko-Dubrovsky Model," Advances in Mathematical Physics, vol. 2019, Article ID 7801247, 9 pages. https://doi.org/10.1155/2019/7801247.

[28] Dusunceli,F. 2019. "New Exact Solutions for the (3 + 1) Dimensional B-type Kadomtsev-Petviashvili Equation". Journal of science and Technology, 12 (1), 463-468., DOI: 10.18185/erzifbed.493777
