Inseparability Criterion Using Higher-Order Schrödinger-Robertson Uncertainty Relation

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We formulate an inseparability criterion based on the recently derived generalized Schrödinger-Robertson uncertainty relation (SRUR) [Ivan et al. J. Phys. A: Math. Theor. 45, 195305 (2012)] together with the negativity of partial transpose (PT). This generalized SRUR systematically deals with two orthogonal quadrature amplitudes to higher-orders, so is relevant to characterize non-Gaussian quantum statistics. We first present a method that relies on the single-mode marginal distribution of two-mode fields under PT followed by beamsplitting operation. We then extend the SRUR to two-mode cases and develop a full two-mode version of inseparability criterion. We find that our formulation can be useful to detect entanglement of non-Gaussian states even when, e.g., the entropic criterion that also involves higher-order moments fails.

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I. INTRODUCTION

Uncertainty relation (UR) has played a fundamentally crucial role in characterizing quantum mechanics ever since its birth. UR can also be employed as a pivotal tool in quantum information tasks, e.g. entanglement detection. In particular, tests adopting matrices of moments (MMs) together with negativity of partial transposition (NPT) have been broadly used [1–8]. The principle behind such tests is that the negativity of the partially transposed density matrix of an entangled state also induces the negativity of its MM. As a matter of fact, provided that an infinite hierarchy is allowed, the converse is also true [6, 9], that is, if the MM of a partially transposed density matrix is negative then the state is NPT entangled. Criteria utilizing MM and NPT are particularly useful for continuous-variable (CV) states. For the discrete variable case, numerous measures and detection schemes for entanglement are well developed, which, however, is more challenging for the CV case except very special ones. This is because the Hilbert space of CV is infinite dimensional and thereby can have a more involved structure.

In this respect, Simon essentially derived a second-order MM for an inseparability criterion [1], which is a necessary and sufficient condition for Gaussian states [2] although it does not seem to be directly connected to NPT. An entangled Gaussian state does not need any higher-order MM to detect its inseparability. Since then, a number of other criteria have also been developed in order to address also non-Gaussian entangled states [3–8, 10–12]. Of these criteria, Shchukin and Vogel (SV)’s is recognized as a unified one that includes in its hierarchy all the MM inseparability criteria [6]. Later, the above MM inseparability criteria has been analyzed in terms of MM of normally ordered operators [10], which was also developed by SV aiming at a nonclassicality criterion [13].

In this article, in the same spirit of the above approaches—namely, based on MM and NPT—we also propose a unified inseparability criterion, especially focusing on the fourth- and even higher-order moments. Our study is based on a generalized Schrödinger-Robertson uncertainty relation (SRUR) recently derived by Ivan et al. [14]. This naturally includes the original SRUR in its hierarchy and extends it to arbitrary (higher) orders for a single-mode system. We first develop an inseparability criterion by applying this single-mode SRUR to a marginal distribution of a partially transposed bipartite system. We next extend the single-mode SRUR to a two-mode system and subsequently derive a full two-mode SRUR inseparability criterion. We show that our criterion is actually equivalent to SV’s, hence is a unified criterion and can detect any NPT entanglement. On the other hand, our formulation explicitly addresses two orthogonal quadrature amplitudes rather than the creation and the annihilation operators unlike other criteria [6, 10]. We also illustrate that our criterion can be more powerful in detecting certain classes of non-Gaussian entangled states than the entropy-based criterion [11] although the latter also addresses higher-order moments in a specific form.

Our approach as well as other ones based on MM is experimentally feasible and does not need a full tomography. Along with theoretical schemes [13, 15], experimental technology is being continuously developed. For instance, refer to a recent breakthrough about detecting higher-order moments of correlation [16, 17].

II. GENERALIZED SCHRÖDINGER-ROBERTSON UNCERTAINTY RELATION FOR SINGLE-MODE CASE

First, we briefly review the generalized Schrödinger-Robertson UR (SRUR) recently derived by Ivan et al. [14]. This generalized version involves moments higher than the second and naturally include the second-order ones in its hierarchy. We begin by noting that every uncertainty relation associated with moments relies on the positivity (precisely, positive semi-definiteness) of a density matrix $\rho$. For an arbitrary linear combination of quantum operators $\hat{F} = c_0\hat{1} + c_1\hat{f}_1 + c_2\hat{f}_2 + \cdots + c_n\hat{f}_n$ ($c_i$’s are c-numbers, $\hat{1}$ is the...
identity operator, and \(\hat{f}_i\)'s need not necessarily be hermitian), the mean value of the product of its adjoint and itself must be non-negative, i.e.,
\[
\langle \hat{F}^\dagger \hat{F} \rangle = \text{Tr} (\rho \hat{F}^\dagger \hat{F}) = c^\dagger M_\rho (c) \geq 0,
\]
where \(c = (c_0, c_1, c_2, \ldots, c_n)^T\).
\[
M_\hat{f}(\rho) = \begin{bmatrix}
\langle \hat{f}_0^\dagger \hat{f}_0 \rangle & \langle \hat{f}_0^\dagger \hat{f}_1 \rangle & \cdots & \langle \hat{f}_0^\dagger \hat{f}_n \rangle \\
\langle \hat{f}_1^\dagger \hat{f}_0 \rangle & \langle \hat{f}_1^\dagger \hat{f}_1 \rangle & \cdots & \langle \hat{f}_1^\dagger \hat{f}_n \rangle \\
\cdots & \cdots & \cdots & \cdots \\
\langle \hat{f}_n^\dagger \hat{f}_0 \rangle & \langle \hat{f}_n^\dagger \hat{f}_1 \rangle & \cdots & \langle \hat{f}_n^\dagger \hat{f}_n \rangle 
\end{bmatrix},
\]
and \(\hat{f} = (\hat{f}_0, \hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)\) with \(\hat{f}_0 = 1\). The inequality (1) must be satisfied for arbitrary \(c_i\)'s, which implies that the matrix of moments, \(M_\hat{f}\), should be positive semi-definite, i.e.,
\[
M_\hat{f}(\rho) \succeq 0.
\]
Note that the above inequality employing \(\hat{f} = (\hat{f}_1, \ldots, \hat{f}_n)\) can be made equivalent to \(M_\hat{f}(\rho) \succeq 0\) where \(\hat{f}^\dagger \equiv (\Delta \hat{f}_1, \ldots, \Delta \hat{f}_n)\) refers to a variance operator \(\Delta \hat{O} \equiv \hat{O} - \langle \hat{O} \rangle\) [10, 14]. Henceforth we will consider only the latter case to our aim.

In fact, the second-order SRUR of any hermitian operators \(\hat{A}\) and \(\hat{B}\) is equivalent to \(M_\hat{f} \succeq 0\) with \(\hat{f} = (\Delta \hat{A}, \Delta \hat{B})\), more specifically,
\[
\frac{\langle \Delta \hat{A}^\dagger \Delta \hat{A} \rangle}{\langle \Delta \hat{B}^\dagger \Delta \hat{A} \rangle} = \frac{\langle \Delta \hat{A}^\dagger \Delta \hat{B} \rangle}{\langle \Delta \hat{B}^\dagger \Delta \hat{B} \rangle} \geq 0.
\]
This inequality
\[
\langle \Delta \hat{A}^\dagger \Delta \hat{A} \rangle \langle \Delta \hat{B}^\dagger \Delta \hat{B} \rangle \geq |\langle \Delta \hat{A}^\dagger \Delta \hat{B} \rangle|^2
\]
is tighter than the Heisenberg UR (HUR)
\[
\langle \Delta \hat{A}^\dagger \Delta \hat{A} \rangle \langle \Delta \hat{B}^\dagger \Delta \hat{B} \rangle \geq \text{Im}^2 \langle \Delta \hat{A}^\dagger \Delta \hat{B} \rangle
\]
and can also be obtained if the real part of \(\langle \Delta \hat{A}^\dagger \Delta \hat{B} \rangle\) is not omitted when deriving HUR using the Cauchy-Schwartz inequality for operators.

Note that SRUR is invariant under the whole group \(\text{Sp}(2, R)\) of linear canonical transformations whereas HUR is so only under a restricted subset of \(\text{Sp}(2, R)\). The group \(\text{Sp}(2, R)\) consists of linear transformations for quadrature operators \(\hat{x}\) (position) and \(\hat{p}\) (momentum)
\[
\hat{X} = \begin{bmatrix}
\hat{x} \\
\hat{p}
\end{bmatrix} \rightarrow \hat{X}' = \Omega \begin{bmatrix}
\hat{x} \\
\hat{p}
\end{bmatrix},
\]
which preserves the canonical commutation relation
\[
[\hat{x}_i, \hat{x}_j] = i\Omega_{ij}, \quad \Omega = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]
In other words, \(S \in \text{Sp}(2, R)\) has the property of \(S \Omega S^T = \Omega\) (or \(S^\dagger \Omega S = \Omega\)). In two dimensional case \(\text{Sp}(2, R) = \text{SL}(2, F)\) \(\text{SL}(n, F)\): \(n\)-dimensional special linear group over a field \(F\)—hence, \(\text{Sp}(2, R)\) is a three-parameter group—though in general \(\text{Sp}(2n, R) \subset \text{SL}(2n, R)\). Each element of \(\text{Sp}(2, R)\) is directly related to its unitary representation \(\hat{U}(\hat{x})\) as
\[
\hat{U}(\hat{x}) = \begin{bmatrix}
\hat{x} \\
\hat{p}
\end{bmatrix}, \quad \hat{U}(S) = S \begin{bmatrix}
\hat{x} \\
\hat{p}
\end{bmatrix}.
\]
It also connects the density matrix with its Wigner function via Weyl-Wigner transform
\[
r' = \hat{U}(S) \rho \hat{U}(\hat{x}) \Longleftrightarrow W_{r'}(X) = W_\rho(S^{-1}X)
\]
where \(X = (x, p)\).

In this article, we just sketch the formulation of generalized SRUR following the steps taken by Ivan et al. [14]. Readers who are interested in more details including its group properties may refer to Refs. [10, 14]. We may start by extending the quadrature operators \(\hat{x}\) and \(\hat{p}\) to their higher-order forms such as \(\hat{x}^m \hat{p}^n\). For a systematic extension, a certain kind of ordering should be taken into account and Weyl ordering of \(\hat{x}\) and \(\hat{p}\) renders such higher-order products hermitian. Although the hermiticity of operators is not necessarily required, Weyl ordered “monomials” transform exactly the same as their classical variables do under linear canonical transformations. So the desired UR can also be transformed in a simple manner under such transformations. We define the Weyl-ordered monomial observable \(\hat{T}_{jm}(j = \frac{1}{2}, 1, \frac{3}{2}, \ldots, m = j, j - 1, \ldots, -j)\) as symmetrized homogeneous product of \(\hat{x}^m \hat{p}^n\) [14], that is,
\[
\langle \hat{T}_{jm} \rangle = \begin{bmatrix}
\hat{x} \\
\hat{p}
\end{bmatrix}, \quad \langle \hat{T}_{1m} \rangle = \frac{\xi^2}{\hat{p}^2} \langle \hat{p} \hat{x} \rangle, \quad \langle \hat{T}_{j2m} \rangle = \frac{\xi^3}{\hat{p}^3} \langle \hat{x} \hat{p} \rangle, \quad \ldots (11)
\]
Note that the angular momentum notation is used for \(\hat{T}_{jm}\) since its product form is determined by the Clebsch-Gordan (CG) coefficients of \(\text{SU}(2)\), i.e.,
\[
\hat{T}_{jm} \hat{T}_{jm'} = \sum_{j=|j-j'|}^{j+j'} C_{mm'm'}^{j+j'} \hat{T}_{jm+m}
\]
where \(C_{mm'm'}^{j+j'} = \langle jm, jm' \rangle \langle jm' \rangle \langle jm' \rangle \langle jm' \rangle\) is the CG coefficient of \(\text{SU}(2)\). The fact that \(\text{SU}(2)\) and \(\text{Sp}(2, R)\) have the same product law is due to the analytic continuation between \(\text{SU}(2)\) and \(\text{Sp}(2, R)\) in finite dimension. As can be seen in the above formula, the product \(\hat{T}_{jm} \hat{T}_{jm'}\) may not generally be hermitian, e.g.,
\[
\hat{T}_{jm} \hat{T}_{jm'} = \frac{1}{2m + m'} + i(2m - m')\hat{T}_{jm+m'}
\]
where \(\hat{T}_{jm} \hat{T}_{jm'} = \frac{1}{2m + m'} + i(m - m')\hat{T}_{jm+m'} - \frac{(-1)^m}{4(2 - \delta_{m,0})\delta_{m+m',0}}\hat{T}_{00}.
\]
The real and imaginary parts are discriminated when computing the corresponding covariance matrix, as will be clarified later. Given the Wigner function of a state, the mean value of $\hat{T}_{jm}$ can be easily calculated since it is already of Weyl-ordered form, i.e.,

$$\langle \hat{T}_{jm} \rangle_p = \int \int dx \, dp \, W_p(x, p) x^j p^m.$$  \hfill (14)

Now we are in a position to evaluate the MM in (2) with (hermitian) operators

$$\hat{\mathbf{f}} = (\Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \Delta \hat{T}_{11}, \Delta \hat{T}_{10}, \Delta \hat{T}_{1-1}, \cdots)$$ \hfill (15)

using formulas in Eqs. (11)-(14). One may use $\hat{\mathbf{f}}' = (\hat{T}_{00}, \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \hat{T}_{11}, \hat{T}_{10}, \hat{T}_{1-1}, \cdots)$ with $\hat{T}_{00} = \hat{1}$ whereby the dimension of MM increases by one. Note that $M_{\mathbf{f}}$ can turn into $M_{\hat{\mathbf{f}}}$ easily by the Schur complement of its $(1,1)$-entry. The corresponding MM

$$M_{\hat{\mathbf{f}}} = \langle (\Delta \hat{T}_{jm} \Delta \hat{T}_{j'm'}) \rangle$$ \hfill (16)

has its each entry as

$$M_{jm,j'm'} = V_{jm,j'm'} + \frac{i}{2} \Omega_{jm,j'm'},$$ \hfill (17)

where

$$V_{jm,j'm'} = \frac{1}{2} \langle (\hat{T}_{jm}, \hat{T}_{j'm'}) \rangle - \langle \hat{T}_{jm} \rangle \langle \hat{T}_{j'm'} \rangle,$$

$$\Omega_{jm,j'm'} = \frac{1}{2} \langle [\hat{T}_{jm}, \hat{T}_{j'm'}] \rangle.$$ \hfill (19)

Here, $j, j' = \frac{1}{2}, 1, \cdots, m = j, j', \cdots, -j$, and $m' = j', \cdots, -j'$ and the matrix $V$ ($\Omega$) is the real symmetric (imaginary anti-symmetric) part of $M_{\hat{\mathbf{f}}}$. Note that for $j = 1/2$, the matrix $(\Omega_{jm,j'm'})$ is nothing but $\Omega$ in (8).

Since the matrix in (16) is infinite dimensional, one should consider its finite truncated version for practical use, thereby producing a hierarchy of URs. The first one starts with $\hat{\mathbf{f}} = (\Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}) = (\Delta \hat{T}, \Delta \hat{p})$ and this observable set leads to the original SRUR, which we label as $(j = 1/2)$-th covariance matrix (CM)

$$M_{\frac{1}{2}}(\rho) \equiv \left[ \frac{\langle (\Delta \hat{\mathbf{f}})^2 \rangle}{\langle (\Delta \hat{\mathbf{f}})^2 \rangle} \right].$$ \hfill (20)

The next $(j = 1)$-th CM is constructed with

$$\hat{\mathbf{f}} = (\Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \Delta \hat{T}_{11}, \Delta \hat{T}_{10}, \Delta \hat{T}_{1-1})$$ \hfill (21)

and reads

$$M_{1}(\rho) \equiv \left[ \frac{M_{\frac{1}{2}}(\rho)}{M_{\frac{1}{2}}(\rho)} \frac{M_{\frac{1}{2}}(\rho)}{M_{\frac{1}{2}}(\rho)} \right].$$ \hfill (22)

where

$$M_{\frac{1}{2}}(\rho) = M_{\frac{1}{2}}(\rho) \equiv \left[ \frac{\langle (\Delta \hat{\mathbf{f}})^2 \rangle}{\langle (\Delta \hat{\mathbf{f}})^2 \rangle} \right].$$ \hfill (23)

Now if we label $M_{\hat{\mathbf{f}}}$ with $\hat{\mathbf{f}} = (\Delta \hat{T}_{\frac{1}{2} - \frac{1}{2}}, \cdots, \Delta \hat{T}_{j'})$ as $J$-th CM $M_J$, we can systematically extend the CM from $J$-th one to $(J + 1/2)$-th one, by adding the operators $\Delta \hat{T}_{j'+m}$ $(m = -J - \frac{1}{2}, \cdots, J + \frac{1}{2})$ as

$$M_{J+\frac{1}{2}} = \left[ \frac{M_J}{M_{J+\frac{1}{2}} \rho} \frac{M_{J+\frac{1}{2}}}{M_{J+\frac{1}{2}} \rho} \right].$$ \hfill (24)

Here $M_{J+\frac{1}{2}} = M'_{J+\frac{1}{2}} = \left[ \frac{M_J}{M_{J+\frac{1}{2}} \rho} \cdots M_{J+\frac{1}{2}} \rho \right]$ is the lower left $(2J + 2) \times N_J$ off-diagonal block matrix with $N_J = J(2J + 3)$. Equipped now with CM truncated up to $J$-th monomial observables, the desired hierarchy of SRUR can be phrased in the form of $N_J$-dimensional matrix as

$$M_J(\rho) = V_J(\rho) + \frac{i}{2} \Omega_J(\rho) \geq 0.$$ \hfill (25)

where $M_J = (M_{jm,j'm'})$ and $V_J$ and $\Omega_J$ are defined in the same way [14].

In order to check the nonnegativity of the above CM, one may use the Sylvester criterion [6, 9, 18, 19], which states that a hermitian matrix is nonnegative if and only if all its principal minors are nonnegative. Alternatively, one may adopt another simpler criterion, which states that a hermitian matrix is nonnegative if and only if one of its leading principal submatrix is positive and the corresponding Schur complement is nonnegative. In our case, this can be formulated as follows. After checking the positivity of $M_J$, one can proceed to check the nonnegativity of its Schur complement in the whole matrix of $M_{J+\frac{1}{2}}$ in (23), namely

$$M_{J+\frac{1}{2}} \equiv M_{J+\frac{1}{2}} \rho - M_{J+\frac{1}{2}} \rho M_J^{-1} M_{J+\frac{1}{2}} \rho \geq 0.$$ \hfill (26)

Here $M_J$ is the $N_J \times N_J$ leading principal submatrix $(J = \frac{1}{2}, 1, \frac{3}{2}, \cdots)$. In case that $M_J$ has zero eigenvalue(s), one can simply split it into its null space and the remaining invertible one and applying the above inequality by ignoring the null-space related block matrices. Or equivalently, and more simply, one can replace $M_{J+\frac{1}{2}}$ in the above by the Moore-Penrose inverse of $M_J$ [19].

Thus far, we have sketched the procedure to construct CM and to check its positivity in a systematic manner. The next part is devoted to mentioning its covariance property. We begin by noting that the span of $T_{jm}$ (of the same $j$) is invariant under the unitary operator $U(S)$ and that each transformed element is in that span, i.e.,

$$\hat{U}(S) T_{jm} \hat{U}(S) = \sum_{m'=-j}^{j} K_{jm}^{(j)}(S) T_{jm'}.$$ \hfill (27)

Comparing this with (9), we notice that $K^{(1/2)}(S) = S$. This $(2J + 1)$-dimensional $K^{(J)}(S)$ is the real (nonunitary) irreducible representation of $Sp(2, R)$ for $T_{jm}$ and can be obtained by the same transformation rule of classical monomial. For example, for arbitrary $S \in Sp(2, R)$, $K^{(j)}(S)$ can be obtained
by the transformation rule of the monomial set of \((x^2, xp, p^2)\),

\[
\begin{bmatrix}
x' \\
p' \\
\end{bmatrix} = S \begin{bmatrix}
x \\
p \\
\end{bmatrix} \rightarrow \begin{bmatrix}
x'^2 \\
p' x' \\
p'^2 \\
\end{bmatrix} = K^{(1)}(S) \begin{bmatrix}
x^2 \\
x p \\
p^2 \\
\end{bmatrix},
\]

(27)

which implies that

\[
S = \begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \rightarrow K^{(1)}(S) = \begin{bmatrix}
a^2 & 2ab & b^2 \\
ac & ad + bc & bd \\
e^2 & 2cd & d^2 \\
\end{bmatrix}
\]

(28)

and \(K^{(j)}(S)\) with higher \(J\) can be derived in a similar way.

Owing to the above covariance property, the resulting transformation rule of \(M_j\) becomes simple enough, that is,

\[
\rho \rightarrow \hat{U}(S)\rho \hat{U}(S)^\dagger \rightarrow M_j \rightarrow K_j(S)M_jK_j^T(S)
\]

(29)

where

\[
K_j(S) = K^{(j)}(S) \oplus K^{(1)}(S) \oplus \cdots \oplus K^{(j)}(S).
\]

(30)

For \(J = 1\) case, as an illustration, the relevant block matrices transform respectively as

\[
M_{\frac{1}{2}} \rightarrow S M_{\frac{1}{2}}S^T, \quad M_1 \rightarrow K^{(1)}(S)M_1K^{(1)}_S(S), M_{\frac{3}{2}} \rightarrow K^{(1)}(S)M_{\frac{3}{2}}K^{(1)}_S(S).
\]

III. INSEPARABILITY CRITERION USING SINGLE-MODE SCHRODINGER-ROBERTSON UNCERTAINTY RELATION

It is well known that partial transposition can map a bipartite inseparable state to a form not admissible as a legitimate quantum state. That is, partially transposed (PT) density matrix \(\rho^T\) of a bipartite entangled state \(\rho\) can possess a negative eigenvalue and this negativity is also passed on to its MM, \(M_2(\rho^T)\). Formally, its negativity is a necessary and sufficient condition for NPT [6, 9]. In our framework, in order to detect the entanglement of \(\rho\), we must show the condition \(M_2(\rho^T) < 0\) — or equivalently find its negative eigenvalue(s) — at a certain level of \(\epsilon\). However, as addressed in the next section, testing the positivity of two-mode SRUR is rather demanding. For instance, in order for \(M_2(\rho^T)\) to be probed up to the fourth-order \((J = 1)\), one should search the eigenvalues of a fourteen—or ten if one considers the Schur complement of \(M_4(\rho^T)\) [see (25)] —dimensional matrix (this will be clarified in the next section), which is practically burdensome.

A possible approach to bypass this issue is to use a single-mode marginal distribution of two-mode \(\rho^T\). One first changes quadrature variables to a new set by

\[
\begin{align*}
x_{\pm} &= \frac{x'_\pm + x''_\pm}{\sqrt{2}}, \\
p_{\pm} &= \frac{p'_{\pm} + p''_{\pm}}{\sqrt{2}},
\end{align*}
\]

(31)

where \(x'_j = \cos \theta_j x_j + \sin \theta_j p_j, p'_j = \cos \theta_j p_j - \sin \theta_j x_j\). By changing the arguments through the above, one can get a new Wigner function \(W(x_\pm, p_\pm, x_\mp, p_\mp)\) from the original Wigner function \(W(x_1, p_1, x_2, p_2)\) of \(\rho\), which essentially corresponds to a beam-splitting operation. With the commutation relation \([x_\pm, p_\pm] = i, W(x_\pm, p_\pm, x_\mp, p_\mp)\) can be considered the Wigner function associated with the two-mode quadratures \((x_\pm, p_\pm)\) and \((x_\mp, p_\mp)\). If one applies to it partial transposition [1, 11] by changing the sign of \(p'_{\pm}\) in (31), then

\[
W(x_\pm, p_\pm, x_\mp, p_\mp) \rightarrow W(x_\mp, p_\mp, x_\pm, p_\pm).
\]

(32)

If the original Wigner function \(W(x_1, p_1, x_2, p_2)\) describes a separable state, the resulting \(W(x_\pm, p_\pm, x_\mp, p_\mp)\) must also be a bona fide Wigner function. Therefore, its marginal single-mode distributions

\[
W(x_\pm, p_\pm) = \int dx_\mp dp_\mp W(x_\pm, p_\pm, x_\mp, p_\mp)
\]

(33)

are also legitimate Wigner functions. From this we can compute \(M_2(\rho^T)\) (hereafter, we denote \(\rho^T\) as the marginal PT density matrix) and check its nonnegativity as formulated in the previous section, i.e.,

\[
M_2(\rho^T) = V_2(\rho^T) + i \frac{1}{2} \Omega_2(\rho^T) \geq 0.
\]

(34)

Let us now apply the above SRUR to a dephased cat state

\[
\rho_{\text{cat}} = N[|\alpha, \alpha\rangle\langle\alpha, \alpha| + |\alpha, -\alpha\rangle\langle\alpha, -\alpha| - p(|\alpha, \alpha\rangle\langle\alpha, -\alpha| + |\alpha, -\alpha\rangle\langle\alpha, \alpha|)],
\]

(35)

where the amplitude \(\alpha\) is assumed to be real, \(0 \leq p \leq 1\) represents the degree of coherence, and \(N = 1/[2 - 2p \exp(-4a^2)]\) is the normalization factor. Note that \(\rho_{\text{cat}}\) is separable only when \(p = 0\) and its inseparability is not detected by the second-order criteria and that in Ref. [11] an entropic UR criterion is introduced aiming at its detection. That is,

\[
H[P(x_\pm)] + H[P(p_\pm)] \geq \ln(\pi e)
\]

(36)

where

\[
H[P(p)] = -\int dqP(q) \ln P(q)
\]

(37)

is the Shannon entropy for a probability distribution \(P(q)\). However, even this entropic criterion detects its inseparability only for large \(\alpha\) and \(p\) [see Fig. 1(a)] [11]. We now demonstrate that the fourth-order SRUR formulated here can detect the entanglement of \(\rho_{\text{cat}}\) for any values of \(\alpha\) and \(p\). As aforementioned, \(M_2(\rho^T) > 0\), which can be seen from its positive determinant \(\det(M_2) = 2a^2(2N - 1) > 0\). Hence, we proceed to check the next hierarchy, i.e., \(J = 1\) case

\[
M_{1_{\frac{1}{2}}} = M_{1_{\frac{1}{2}}} - M_{1_{\frac{3}{2}}}M_{1_{\frac{1}{2}}}^{-1}M_{1_{\frac{3}{2}}} \geq 0.
\]

(38)

If we choose the observables \(x_\pm, p_\pm\) and the parameters \(\theta_1 = \theta_2 = 0\) in (31), i.e., \(x_\pm = (x_1 - x_2)/\sqrt{2}\) and \(p_\pm = (p_1 + p_2)/\sqrt{2}\), we get \(M_{1_{\frac{3}{2}}} = 0\) and hence

\[
M_{1_{\frac{1}{2}}} = M_{1_{\frac{1}{2}}} = \frac{1}{2} \begin{bmatrix}
1 & i & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & i & -1 \\
-1 & -i & 1 + 2d & i(1 + 4d) \\
1 & -i(1 + 4d) & 1 + 8d(1 - 4N\alpha^2)
\end{bmatrix}
\]
where $d = \text{det}(M_1)$. This matrix has two positive and one negative eigenvalues and therefore is always negative for all $\alpha$ and $p$. In Fig. 1, we show how the sum of the entropy uncertainties in (36) and the determinant $\text{det}(M_{12}) = -8N\alpha^2d^2$. It is not necessary here but the negativity of (38) may also be optimized by introducing another local transformations. Since the local symplectic group $\text{Sp}(2,R) \otimes \text{Sp}(2,R)$, a subgroup of the full symplectic group $\text{Sp}(4,R)$, has six parameters, one can use another four parameters, i.e., another two local rotations along with two local squeezing actions after the two rotations related to $\theta_1$ and $\theta_2$ [20].

As can be seen from the figure, the fourth-order SRUR fully detects the inseparability of $\rho_{\text{cat}}$ whereas the entropic criterion does not. In view of this, it is worth emphasizing that although an entropic UR also involves higher-order moments of correlations in a certain form, it does not fully reflect the specific-order—fourth-order in this case—moments. This observation seems to be also supported by comparing the undetected region (the white region) in Fig. 1(a) and the region of relatively large negativity of $M_{12}$ in Fig. 1(b). We suppose that the entanglement of a dephased cat is coded relatively more in the fourth-order correlation when its size is not so large.

As another advantage over the entropic criterion, our method may provide an analytical result in algebraically simpler form. Even when viewed from the perspective of numerical cost, since the dimension of $M_{12}$ is as small as three, finding its eigenvalues does not require much effort compared to calculating the entropies. Moreover, let us remark further on a possible practical efficiency. According to Sylvester criterion mentioned in the previous section, we can search the negativity of $M_1(\rho^{\text{MF}})$ by choosing a certain partial number of rows (columns) and need not investigate the whole matrix. Indeed, in the above case, the principal submatrix of $M_{11}$ built by choosing the second and third rows/columns suffices to detect inseparability.

Finally, we mention that a dephased cat may be classified as an inseparable state whose entanglement is revealed by a fourth-order-quadrature-moment criterion but not by any lower-order one. In this sense, we might call a dephased cat as a fourth-order entangled state.

IV. GENERALIZED SCHRÖDINGER-ROBERTSON UNCERTAINTY RELATION FOR TWO-MODE CASE AND THE CORRESPONDING INSEPARABILITY CRITERION

In the previous sections, we have addressed the single-mode SRUR and used it for an inseparability criterion by applying it to a marginal PT state and checking its negativity. Despite its better performance over the entropic criterion, however, there can be states which are not detected by the criterion yet whose entanglement is still coded in the fourth-order correlation. Indeed, such states exist and will be introduced later. Thus, to conclusively determine whether or not the entanglement is due to a specific order correlation, we should have a full two-mode correlation criterion without marginalization.

With this motivation, we extend the single-mode SRUR derived by Ivan et al. in Sec. II to a two-mode case to be employed as an inseparability criterion. Apparently, it seems straightforward as we only need to include every observable of two modes up to the desired order. For instance, if the fourth order SRUR is to be addressed, it is necessary to construct the operator set $\hat{f}$ by all observables up to the second-order, namely, $\hat{f}^A_{j,A}, \hat{f}^B_{j,B}, \hat{f}^A_{j,A}, \hat{f}^B_{j,B}$, where $A$ and $B$ denote two distinct modes.

As one may readily appreciate, for a systematic extension, a specific ordering of observable operators is needed for an efficient construction of MM and more importantly for symplectic covariance. We explicitly give an ordering for generic observables $(\hat{f}^A_{j,A}, \hat{f}^B_{j,B})$ (37) and as before. $\hat{f}^A_{j,A}, \hat{f}^B_{j,B}$ comes before $\hat{f}^A_{j',A}, \hat{f}^B_{j',B}$ if and only if the first non-zero difference of $f^A_j + f^B_j - (f^A_{j'} + f^B_{j'})$, $j_A - j'_A$, $m_A - m'_A$, $m_B - m'_B$ is positive. In the case of the observables for $M_{f=1}$, for example, we have the following ordering

\[ \hat{f}^A_{11}, \hat{f}^A_{10}, \hat{f}^B_{11}, \hat{f}^B_{10}, \hat{f}^A_{11}, \hat{f}^B_{11}, \hat{f}^A_{10}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{11}, \hat{f}^B_{11}, \hat{f}^A_{10}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{11}, \hat{f}^B_{11}, \hat{f}^A_{10}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{11}, \hat{f}^B_{11}, \hat{f}^A_{10}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10}, \hat{f}^A_{01}, \hat{f}^B_{10} \] (39)

where “$\Delta$” notations are omitted for brevity. The remaining procedure of obtaining the two-mode CM $M_f(\rho)$ and checking its covariance property under symplectic transformations are straightforward as before. One can easily get the two-mode version of $K_f(S)$ in (30) for the full symplectic group $\text{Sp}(4,R)$.

Equipped with the full two-mode SRUR, the construction of the corresponding inseparability criterion is also straightforward. We have only to check the positivity of the two-mode CM $M_f(\rho^f)$ where $\rho^f$ is the PT density matrix of $\rho$ (this time a full two-mode one, not a marginal one). It is also necessary to consider the symplectic transformations which leave invariant
One can readily see that under Sp(2, R) ⊗ Sp(2, R), the aforementioned $K_f(S)$ reduces to

$$
K_f(S) = K_A^{(2)}(S) ⊕ K_B^{(2)}(S) ⊕ K_A^{(1)}(S) ⊕ \left[ K_A^{(4)}(S) ⊕ K_B^{(4)}(S) \right]

$$

$$
⊕ K_B^{(1)}(S) ⊕ \cdots ⊕ K_B^{(J)}(S).
$$

Unfortunately, however, inseparability criterion using this two-mode SRUR might be considered not practically useful since the size of relevant CM grows huge with dimension as $J$ increases. In more detail, in order to attain $4J$-th-order full SRUR in this hierarchy, one must compute $\frac{1}{6}(2J + 5)(2J^2 + 5J + 5)$ dimensional square matrix—e.g., $34 \times 34$ matrix for the case of the sixth-order SRUR ($J = 3/2$)—which is very demanding. Even worse, the dimension grows as $\sim J^4$ while that of the marginal one just as $\sim J^2$. One resolution to this issue is to use a principal submatrix, whose usage is justified in the previous section.

As mentioned in the early part of this section, there exist certain entangled states that are not detected by the marginal fourth-order SRUR criterion but detected by the full two-mode version. We illustrate this by introducing beam-splitter number states (BNSs) $\hat{B}[n, m]$ ($\hat{B}$: the unitary operator of a beam splitter, $[n, m]$: a two-mode number state). This class of states is scarcely detected by (38) but detected by its full two-mode version. Furthermore, as mentioned in the previous paragraph, the inseparability of BNSs can be practically detected by its submatrix: the four observables $\hat{J}^x_1, \hat{J}^x_2, \hat{J}^y_1, \hat{J}^y_2$ in (39) are sufficient to build a CM that detects the inseparability for any photon numbers $n$ and $m$. In more detail, the corresponding submatrix of $M_{1,1}$ has three positive and one negative eigenvalues and hence its negativity can again be checked by its determinant [See Fig. 2(a)]. Interestingly, in order to detect it, the author in Ref. [21] has resorted to a fourth-order HUR in the case of $n \neq m$ using the operators from $su(2)$ and $su(1,1)$ algebras [5, 7, 8] while, in the case of $n = m$, to an eighth-order HUR using their higher-order extended operators [22]. However, similarly as in the case of a dephased cat, we stress that the BNSs may be categorized as fourth-order entangled states.

In order to further illustrate the power of our two-mode approach, we here present another example of the fourth-order entangled states that are not detected by the marginal fourth-order criterion but by the full two-mode one. We consider a class of photon-number entangled state (PNEs), i.e., $|\Psi\rangle = \sum_n c_n |n, m\rangle$. The class of PNEs states has drawn much attention as it constitutes an important resource for continuous-variable quantum communication including both Gaussian (two-mode squeezed state) and non-Gaussian entangled states [23]. As an example, a truncated PNEs $|\Psi_2\rangle = c_0 |0, 0\rangle + c_1 |1, 1\rangle + c_2 |2, 2\rangle$ is here investigated where $c_0^2 + c_1^2 + c_2^2 = 1$. Like the case of BNSs, the inseparability can be detected by its submatrix; however, for this case, by using five observables $\hat{J}_1^x, \hat{J}_1^y, \hat{J}_2^x, \hat{J}_2^y$ (or $\hat{J}_1^x, \hat{J}_1^y, \hat{J}_1^z, \hat{J}_1^\theta, \hat{J}_1^\varphi$, $\hat{J}_2^x, \hat{J}_2^y, \hat{J}_2^z, \hat{J}_2^\theta, \hat{J}_2^\varphi$). Since the corresponding submatrix has four positive and one negative eigenvalues, its negativity can also be checked by its determinant. As can be seen in Fig. 2(b), the entanglement of the class of PNEs $|\Psi_2\rangle$ is again fully detected by our criterion.

![FIG. 2: (Color online) (a) Negative determinant of the submatrix of the fourth-order MM of PT beam-split number state $\hat{B}[n, m]$. From below, the lines denote the case of $n = 1, 2, 3, 4, 5$. (b) Determinant of the the submatrix of fourth-order MM of PT number-correlated state $c_0 |0, 0\rangle + c_1 |1, 1\rangle + c_2 |2, 2\rangle$. The contours are $-10, -20, -30, -40, -50$; the darker the region is, the large its absolute value is. For both cases, a negative value indicates that the entanglement of the corresponding state is detected.](image-url)
Note that the number of these necessary operators is equivalent to that of (39) and hence the two-mode fourth-order SRUR is equivalent to SV criterion employing 14 operators. In fact, they are equivalent up to any order since the number of independent quadrature operators $\hat{x}^2, \hat{p}^2, \hat{x}^2, \hat{p}^2$ is the same as that of the (independent) annihilation and creation operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ and the whole possible combinations of those operators are considered in the MM. However, it is worth stressing that the SRUR criterion is based on hermitian operators and transforms intuitively under local symplectic group $\text{Sp}(2, R) \otimes \text{Sp}(2, R)$.

Even though the case of $J = 1$ in (34) or equivalently (38) is obviously weaker than the above two full fourth-order criteria, it is arguably said to be more practical in terms of computational cost. For other fourth-order or even higher-order criteria, interested readers may refer to [10], wherein inseparability criteria are analyzed not only in terms of SV criterion but also in terms of SV nonclassicality criterion, which is based on a MM of normally ordered operators.

VI. SUMMARY

In this paper we have presented an inseparability criterion based on the recently derived generalized Schrödinger-Robertson uncertainty relation (SRUR) [14]. This generalized SRUR that involves two orthogonal quadratures to arbitrary high orders has a hierarchy naturally containing the original (second-order) SRUR as the lowest one. Employing the single-mode SRUR, we have first proposed a hierarchy of inseparability criterion using a marginal single-mode distribution of partially transposed state. This turns out to successfully detect the entanglement of a certain non-Gaussian continuous-variable (CV) state that is not fully detected by other second-order and entropic criteria. In particular, the entropic criterion also addresses higher-order correlations of a CV state in a specific form, however, the above example amounts to illustrating that the entropic criterion does not fully reveal step by step the specific-order moments of correlation wherein entanglement is coded.

To delve into the above issue more clearly, we have extended the single-mode SRUR to a two-mode one by introducing a systematic ordering for observable operators. Based on this two-mode SRUR, we have proposed an inseparability criterion which can fully detect the entanglement coded in a specific order of quadrature-variable correlations. We have also noted that the inseparability criterion based on two-mode generalization of SRUR is equivalent to Shchukin and Vogel’s criterion. They are both unified criteria based on moments together with the negativity of partial transposition. In principle, SV criterion may also employ not only non-Hermitian but also Hermitian operators. Our formulation directly employs Hermitian operators, a hierarchy of position and momentum operators, and its full sympletic invariance is more manifest.

It may be an interesting question whether there can be a class of inseparable states whose entanglement is detected by a sixth-order-moment criterion but not by any lower one, namely, so-called sixth-order entangled states—or, in general, $4J(>6)$-th-order entangled states. In this respect, we hope that our study could shed some light on unveiling the structure of entanglement and coming up with its useful classification. Furthermore, we expect that it would also be possible to classify by the same reasoning, i.e., in terms of moments, or a similar one nonclassical correlations such as nonlocality, steering and discord.

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