Bifurcation Analysis and $H_{\infty}$ Control of a Stochastic Competition Model with Time Delay

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Abstract In this paper, we study a stochastic competition model with time delay and harvesting. We first simplify it through the stochastic center manifold reduction principle and stochastic averaging method as a one-dimensional Markov diffusion process. Singular boundary theory and an invariant measure are applied to analyze stochastic stability and bifurcation. The T-S fuzzy model of the system is constructed, and the $H_{\infty}$ fuzzy controller is designed to eliminate the bifurcation phenomenon through a linear matrix inequality approach. Numerical simulation is used to demonstrate our results.

Keywords Stochastic competition system · Stage-structured · Hopf bifurcation · Discrete time-delay · T-S fuzzy system · $H_{\infty}$ control

1 Model description

In nature, competition among populations is widespread because of limited natural resources. Early studies mainly considered deterministic models [1, 2]. However, the population must
be disturbed by realistic environmental noise [3, 4], which is important in the study of bio-
mathematical models, and research of stochastic models is inevitable. Individual organisms
experience a growth process. Young individuals have a weaker ability to cope with envi-
ronmental disturbances, predators, and competitors’ survival pressure. The survival ability
of adult individuals is strong, and they are able to conceive the next generation. Hence, the
stage-structured model is popular among scholars [5–8]. Because the impact of competition
between populations is not immediate, it is necessary to consider the time delay [7–9]. We
discuss the following stage-structured competition model with time-delay:

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_2 - a_{12}x_1^2 - sx_1, \\
\dot{x}_2 &= a_{21}x_1 - a_{22}x_2^2 - d_1x_2x_3(t - \tau_1) - \beta x_2, \\
\dot{x}_3 &= x_3(r(1 - \frac{x_3}{k_3}) - d_2x_2(t - \tau_2) - qE),
\end{align*}
\]

(1)

\(x_i\) is the density of the \(i\)th species, \(i = 1, 2, 3\), where \(x_1, x_2\) are respectively the juveniles and
adults of one of two species. \(a_{11}\) is the birth rate of juveniles and \(a_{21}\) is the transformation
rate from juveniles to adults. \(a_{12}, a_{22}\) denote interspecific competitive coefficients of \(x_1\) and
\(x_2\). Considering that \(x_1\) is young and not competitive, we assume that only \(x_2\) and \(x_3\) are
competitive, and \(d_1\) and \(d_2\) are the rates of loss of \(x_2\) and \(x_3\) populations in competition. \(r\)
and \(k_3\) are respectively the intrinsic growth rate and environmental capacity of species \(x_3\).
The sum of the death and the conversion rate of the juveniles \(x_1\) and the sum of the death
rate of the adults \(x_2\) are expressed by \(s\) and \(\beta\), respectively. \(q\) is the catchability coefficient
of species \(x_3\). \(E\) denotes the effort used to harvest population \(x_3\). \(\tau_1, \tau_2\) is the time delay in
competition. All the parameters are positive constants.

Considering environmental noise [7] such as lightning, rain, and drought, we have the
following stochastic differential system:

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_2 - a_{12}x_1^2 - sx_1 + \sigma_1 x_1 \xi_1(t), \\
\dot{x}_2 &= a_{21}x_1 - a_{22}x_2^2 - d_1x_2x_3(t - \tau_1) - \beta x_2 + \sigma_1 x_2 \xi_1(t), \\
\dot{x}_3 &= x_3(r(1 - \frac{x_3}{k_3}) - d_2x_2(t - \tau_2) - qE) + \sigma_2 x_3 \xi_2(t),
\end{align*}
\]

(2)
where \( \xi_i(t), i = 1, 2 \), is independent white noise with intensity \( \sigma_i^2 \).

Throughout this article, we make the following assumption.

**Assumption 1** Because of limited environmental supply and interspecific and intraspecific constraints, there must be environmental capacity \( k_i \) of species \( x_i \).

## 2 Stochastic stability

\( P(x_i^*, x_2^*, x_3^*) \) is the positive equilibrium point of the deterministic system (2). Let \( y_1 = x_1 - x_1^* \), \( y_2 = x_2 - x_2^* \), \( y_3 = x_3 - x_3^* \). Then

\[
\begin{aligned}
\dot{y}_1 &= b_{11}y_1 + b_{12}y_2 - a_{12}y_1^2 + \sigma_1(y_1 + x_1^*)\xi_1(t), \\
\dot{y}_2 &= b_{21}y_1 + b_{22}y_2 + b_{23}y_3(t - \tau_1) - a_{22}y_2^2 - d_{12}y_2y_3(t - \tau_1) + \sigma_2(y_2 + x_2^*)\xi_2(t), \\
\dot{y}_3 &= b_{32}y_2(t - \tau_2) + b_{33}y_3 - \frac{r}{k_3}y_3^2 - d_{23}y_2(t - \tau_2)y_3 + \sigma_3(y_3 + x_3^*)\xi_3(t),
\end{aligned}
\]

where

\[
\begin{align*}
b_{11} &= -(2a_{12}x_1^* + s), & b_{12} &= a_{11}, & b_{21} &= a_{21}, & b_{22} &= -(2a_{22}x_2^* + d_1x_1^* + \beta), \\
\text{and } b_{23} &= -d_1x_3^*, & b_{32} &= -d_2x_2^*, & b_{33} &= -2\frac{r}{k_3}x_3^2 - d_1x_2^* - qE,
\end{align*}
\]

For simplicity, writing system (3) in the following matrix form:

\[
\dot{y}(t) = Ay(t) + B_1y(t - \tau_1) + B_2y(t - \tau_2) + F(y(t), y(t - \tau_1), y(t - \tau_2), \xi_1(t), \xi_2(t)),
\]

For simplicity, we write system (3) in matrix form:

\[
\begin{align*}
A &= \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{23} & 0 \\ 0 & 0 & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_{32} & 0 \end{bmatrix}, \\
F &= \begin{bmatrix} -a_{12}y_1^2 + \sigma_1(y_1 + x_1^*)\xi_1(t) \\ -a_{22}y_2^2 - d_{12}y_2y_3(t - \tau_1) + \sigma_2(y_2 + x_2^*)\xi_2(t) \\ -\frac{r}{k_3}y_3^2 - d_{23}y_2(t - \tau_2)y_3 + \sigma_3(y_3 + x_3^*)\xi_3(t) \end{bmatrix}.
\end{align*}
\]
The characteristic equation of linear system \( \dot{y}(t) = Ay(t) + B_1 y(t - \tau_1) + B_2 y(t - \tau_2) \) of system (4) is

\[
p(\lambda) = \lambda^3 + p_1 \lambda^2 + \lambda^p_3 \lambda e^{-\lambda(\tau_1 + \tau_2)} + p_4 e^{-\lambda(\tau_1 + \tau_2)} + p_5 = 0,
\]

where \( p_1 = -(b_{11} + b_{22} + b_{32}), p_2 = b_{11} b_{33} + (b_{11} + b_{22}) b_{32} - b_{12} b_{21}, p_3 = -b_{23} b_{32}, p_4 = b_{11} b_{23} b_{32}, \)

\( p_5 = b_{12} b_{23} b_{32} - b_{11} b_{22} b_{32} \) equation (5) has no zero characteristic root when \( p_3 + p_4 + p_5 \neq 0 \). Thus, the trivial equilibrium \( y = 0 \) becomes unstable only when equation (5) has at least a pair of purely imaginary roots \( \lambda = \pm i \omega \) (\( i \) is the imaginary unit), at which Hopf bifurcation occurs. Substituting \( \lambda = i \omega \) in equation (5), one can obtain that

\[
p(i \omega) = G(i \omega, \tau_1, \tau_2) + iM(i \omega, \tau_1, \tau_2) = 0,
\]

\[
G = -p_1 \omega^2 + p_3 \omega \sin \omega(\tau_1 + \tau_2) + p_4 \cos \omega(\tau_1 + \tau_2) + p_5 = 0,
\]

\[
M = -\omega^3 + p_2 \omega + p_3 \omega \cos \omega(\tau_1 + \tau_2) - p_4 \sin \omega(\tau_1 + \tau_2) = 0.
\]

While studying the critical infinite-dimensional problem on a two-dimensional center manifold, we express the delay equation as an abstract evolution equation on complete probability space \( C([-\tau, 0], R^3) \):

\[
\dot{y}(t) = Dy(t) + F(t, y(t), \xi_1(t), \xi_2(t)),
\]

where \( y_0(\theta) = y(t + \theta), -\tau < \theta < 0, \) and \( D \) is a linear operator for the critical case, expressed for the initial condition \( u(\theta) \in C \) by

\[
Du(\theta) = \begin{cases} 
\frac{du(\theta)}{d\theta}, & \theta \in [-\tau, 0) \\
A u(0) + B_1 u(-\tau_1) + B_2 u(-\tau_2), & \theta = 0
\end{cases}
\]

The nonlinear operator \( F \) has the form

\[
F = \begin{cases} 
0, & \theta \in [-\tau, 0) \\
F(0, u(-t), \xi_1(0), \xi_2(0)), & \theta = 0
\end{cases}
\]
Similarly, the continuously differentiable function \( v \) is on the adjoint space \( \hat{C}([0, \tau], R^3) \) with dual operator

\[
D^* v(\theta) = \begin{cases} 
\frac{du(\theta)}{d\theta}, & \theta \in [0, \tau) \\
A^* u(0) + B_1^* u(\tau_1) + B_2^* u(\tau_2), & \theta = 0
\end{cases}
\]

The case \( p(\lambda) = 0 \) has a single pair of purely imaginary eigenvalues \( \lambda = \pm i\omega \) through Hopf bifurcation. \( C \) can be split into two subspaces as \( C = P_\lambda \oplus Q_\lambda \), where \( P_\lambda \) is a two-dimensional space spanned by the eigenvectors of the operator \( D \) corresponding to the eigenvalues \( \pm i\omega \), and \( Q_\lambda \) is the infinite-dimensional subspace corresponding to remaining eigenvalues of \( p(\lambda) = 0 \). Then, for \( u \in C, v \in \hat{C} \), we can define a bilinear operator,

\[
(u, v) = (v(0), u(0)) - b_{23} \int_{-\tau_1}^{0} v(\zeta + \theta) u(\zeta) d\zeta - b_{32} \int_{-\tau_2}^{0} v(\zeta + \theta) u(\zeta) d\zeta
\]

(8)

Corresponding to the critical characteristic root \( i\omega \), the complex eigenvector \( q(\theta) \in C \) satisfies

\[
\frac{dq(\theta)}{d\theta} = i\omega q(\theta), \quad \theta \in [-\tau, 0),
\]

(9)

The general solution of equation (9) is \( q(\theta) = \tilde{e} e^{i\omega \theta} \). From the boundary condition equation (7b), we find the following matrix equation:

\[
\begin{bmatrix}
  b_{11} - i\omega & b_{12} & 0 \\
  b_{21} & b_{22} - i\omega & b_{23} e^{-i\omega \tau_1} \\
  0 & b_{32} e^{-i\omega \tau_2} & b_{33} - i\omega
\end{bmatrix} \tilde{e} = 0
\]

Let \( \tilde{e} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)^T \). We choose \( \tilde{e}_1 = 1 \); then \( \tilde{e}_2 \), \( \tilde{e}_3 \) can be determined uniquely, i.e.,

\[
\tilde{e} = \left[ \frac{i\omega - b_{11}}{b_{12}} \right. b_2 b_3 + \frac{b_1}{b_{12}} b_3 e^{-i\omega \tau_1} \left. \right] e^{i\omega \tau_1}
\]

Thus, the real bases in \( P \subset C \) and \( \hat{P} \subset \hat{C} \) are respectively as follows:

\[
\Phi(\theta) = (\phi_1, \phi_2) = \begin{bmatrix}
  \cos \omega \theta & \sin \omega \theta \\
  -b_{11} \cos \omega \theta - b_{22} \sin \omega \theta & b_{12} \omega \cos \omega \theta - b_{11} \sin \omega \theta \\
  H_1 \cos \omega (\theta + \tau_1) - H_2 \sin \omega (\theta + \tau_1) & H_1 \sin \omega (\theta + \tau_1) + H_2 \cos \omega (\theta + \tau_1)
\end{bmatrix}
\]

(10)
\[ \Psi(s) = (\psi_1, \psi_2) = \begin{bmatrix} 
L_1 \cos \omega (\tau_2 - s) + L_2 \sin \omega (\tau_2 - s) & L_2 \cos \omega (\tau_2 - s) - L_1 \sin \omega (\tau_2 - s) \\
-b_{13} \cos \omega (\tau_2 - s) + \omega \sin \omega (\tau_2 - s) & b_{13} \cos \omega (\tau_2 - s) - \omega \sin \omega (\tau_2 - s) \\
\cos \omega s & -\sin \omega s 
\end{bmatrix}, \]

where \( H_1 = -\frac{b_{12} b_{21} - b_{11} b_{23} + \omega^2}{b_{12} b_{23}}, \) \( H_2 = -\frac{(b_{11} + b_{22}) \omega}{b_{12} b_{23}}, \) \( L_1 = \frac{\omega^2 - b_{11} b_{23}}{b_{12} b_{23}}, \) \( L_2 = -\frac{(b_{11} + b_{22}) \omega}{b_{12} b_{23}}, \)

the inner product matrix \( (\Psi(s), \Phi(\theta)) = (\psi_j(s), \phi_k(\theta)), j, k = 1, 2 \) has the following expression:

\[ (\Psi(s), \Phi(\theta)) = \begin{bmatrix} 
\psi_T(s) \\
\psi_T(s) 
\end{bmatrix} \begin{bmatrix} 
\phi_1(\theta) & \phi_2(\theta) 
\end{bmatrix} = \begin{bmatrix} 
(\psi_T(s) \phi_1(\theta)) (\psi_T(s) \phi_2(\theta)) \\
(\psi_T(s) \phi_1(\theta)) (\psi_T(s) \phi_2(\theta)) 
\end{bmatrix}, \]

\[ (\psi_T(s) \phi_1(\theta)) = (L_1 \cos \omega (\tau_2 - s) + L_2 \sin \omega (\tau_2 - s)) \cos \omega \theta + \frac{b_{13} \cos \omega (\tau_2 - s) + \omega \sin \omega (\tau_2 - s)}{b_{32}} - \cos \omega s (H_1 \cos \omega (\tau_1 + \theta) - H_2 \sin \omega (\tau_1 + \theta)), \]

\[ (\psi_T(s) \phi_2(\theta)) = (L_1 \cos \omega (\tau_2 - s) + L_2 \sin \omega (\tau_2 - s)) \sin \omega \theta + \frac{b_{13} \cos \omega (\tau_2 - s) + \omega \sin \omega (\tau_2 - s)}{b_{32}} - \cos \omega s (H_1 \cos \omega (\tau_1 + \theta) + H_2 \sin \omega (\tau_1 + \theta)), \]

\[ (\psi_T(s) \phi_1(\theta)) = (L_2 \cos \omega (\tau_2 - s) - L_1 \sin \omega (\tau_2 - s)) \cos \omega \theta + \frac{\omega \cos \omega (\tau_2 - s) - b_{13} \sin \omega (\tau_2 - s)}{b_{32}} + \sin \omega s (H_1 \cos \omega (\tau_1 + \theta) - H_2 \sin \omega (\tau_1 + \theta)), \]

\[ (\psi_T(s) \phi_2(\theta)) = (L_2 \cos \omega (\tau_2 - s) - L_1 \sin \omega (\tau_2 - s)) \sin \omega \theta + \frac{\omega \cos \omega (\tau_2 - s) - b_{13} \sin \omega (\tau_2 - s)}{b_{32}} + \sin \omega s (H_1 \cos \omega (\tau_1 + \theta) + H_2 \sin \omega (\tau_1 + \theta)), \]

The substitution of the elements of \( (\Psi, \Phi) \) in the bilinear relation (8) yields the nonsingular matrix

\[ (\Psi(s), \Phi(\theta))_{nx} = \begin{bmatrix} 
\psi_1, \phi_1 \\
\psi_1, \phi_2 \\
\psi_2, \phi_1 \\
\psi_2, \phi_2 
\end{bmatrix} = \begin{bmatrix} 
N_1 & N_2 \\
N_3 & N_4 
\end{bmatrix} \]

Then the basis \( \Psi \in \mathcal{C} \) is normalized to

\[ \Psi(s) = (\Psi(s))^{\dagger} \begin{bmatrix} 
\bar{\psi}_{11} \\
\bar{\psi}_{12} \\
\bar{\psi}_{21} \\
\bar{\psi}_{22} \\
\bar{\psi}_{31} \\
\bar{\psi}_{32} 
\end{bmatrix} \]
Thus, \( u = u^P + u^Q = \Phi(\Psi, u) + u^Q = \Phi z + u^Q \) (13)

which implies that the projection of \( u \) on the center manifold is \( \Phi z \). Then, applying (13) and (6) results in

\[
(\Psi, \Phi z + u^Q) = (\Psi, D(\Phi z + u^Q)) + (\Psi, F(\Phi z + u^Q))
\]

Thus,

\[
(\Psi, \Phi) \dot{z} = (\Psi, D\Phi) z + (\Psi, F(\Phi z + u^Q))
\]
Finally, we obtain the equation of the stochastic center manifold:

\[
\dot{z} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} z + \mathcal{N}(z, \xi_1, \xi_2)
\]

where \(\mathcal{N}(z, \xi_1, \xi_2)\) represents the nonlinear terms contributed from the original system to the stochastic center manifold.

By the change of variable \(y^\Phi_t(\theta) = \Phi_t(\theta)z(t) + y^\Psi_t(\theta)\) with \(z(t) = (\tilde{\Psi}(s), \Phi^\psi_t(\theta))\), a first-order approximation for \(\theta = -\tau_1, -\tau_2\) is as follows:

\[
\begin{bmatrix}
y_1(t - \tau_1) \\
y_2(t - \tau_1) \\
y_3(t - \tau_1)
\end{bmatrix} = \Phi_\tau(-\tau_1) z(t) = \begin{bmatrix}
z_1 \cos \omega \tau_1 - z_2 \sin \omega \tau_1 \\
z_1 \cos \omega \tau_1 - \frac{z_2(\cos \omega \tau_1 - \cos \omega \tau_2) + z_1(\cos \omega \tau_1 + \cos \omega \tau_2)}{b_{12}} \\
z_1 \cos \omega \tau_1 + z_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1(t - \tau_2) \\
y_2(t - \tau_2) \\
y_3(t - \tau_2)
\end{bmatrix} = \Phi_\tau(-\tau_2) z(t)
\]

The lowest-order nonlinear terms of the center manifold needed to determine the solutions are \(\mathcal{N}(z) = \tilde{\Psi}^T(0) F(\Phi_{\xi_1}, \Phi_{\xi_2})\). Therefore, one can have the stochastic center manifold

\[
\begin{align*}
\dot{z}_1 &= -\omega z_2 + e_{10}z_1 + e_{100}z_2 + e_{11}z_1^2 + e_{12}z_1z_2 + e_{13}z_2^2 + \sigma_1(e_{14}z_1 + e_{15}z_2 + e_{16})\dot{\xi}_1(t) \\
&\quad + \sigma_2(e_{17}z_1 + e_{18}z_2 + e_{19})\dot{\xi}_2(t), \\
\dot{z}_2 &= \omega z_1 + e_{20}z_1 + e_{200}z_2 + e_{21}z_1^2 + e_{22}z_1z_2 + e_{23}z_2^2 + \sigma_1(e_{24}z_1 + e_{25}z_2 + e_{26})\dot{\xi}_1(t) \\
&\quad + \sigma_2(e_{27}z_1 + e_{28}z_2 + e_{29})\dot{\xi}_2(t), \\
\end{align*}
\]

(14)
Setting the coordinate transformation $z_1 = r \cos \theta$, $z_2 = r \sin \theta$ for system (14),

\[
\dot{r}(t) = r e_{10} \cos^2 \theta + r(e_{101} + e_{20}) \sin \theta \cos \theta + r e_{200} \sin^2 \theta + r^2 (e_{13} \cos^3 \theta + e_{21} \sin^3 \theta) \\
+ (e_{11} + e_{22}) \sin^2 \theta \cos \theta + (e_{12} + e_{23}) \sin \theta \cos^2 \theta + \sigma_1 (e_{16} \cos \theta + e_{26} \sin \theta) \\
+ r(e_{15} \cos^2 \theta + (e_{25} + e_{14}) \sin \theta \cos \theta + e_{24} \sin^2 \theta)) \xi_1(t) \\
+ \sigma_2 (e_{19} \cos \theta + e_{29} \sin \theta + r(e_{18} \cos^2 \theta + (e_{28} + e_{17}) \sin \theta \cos \theta + e_{27} \sin^2 \theta)) \xi_2(t)
\]

\[
\dot{\theta}(t) = \omega + e_{21} \cos^2 \theta + (e_{20} + e_{200}) \sin \theta \cos \theta + e_{200} \sin^2 \theta + r(e_{23} \cos^3 \theta - e_{11} e_{21} \sin^3 \theta) \\
+ (e_{21} - e_{12}) \sin^2 \theta \cos \theta + (e_{22} - e_{13}) \sin \theta \cos \theta + \sigma_1 (\frac{1}{r} (e_{26} \cos \theta - e_{16} \sin \theta) \\
+ e_{25} \cos^3 \theta + (e_{24} - e_{15}) \sin \theta \cos \theta - e_{14} \sin^2 \theta)) \xi_1(t) + \sigma_2 (\frac{1}{r} (e_{29} \cos \theta - e_{19} \sin \theta) \\
+ e_{28} \cos^2 \theta + (e_{27} - e_{18}) \sin \theta \cos \theta - e_{17} \sin^2 \theta)) \xi_2(t)
\]

(15)

According to the Khasminskii limit theorem [10, 11], if $\sigma_1, \sigma_2$ are small enough, then the response process \{r(t), \theta(t)\} weakly converges to the two-dimensional Markov diffusion process. Through the stochastic averaging method [10], we obtain Itô's stochastic differential equation (19), and

\[
\begin{cases}
    dr = m_r dt + \sigma_{11} dW_r + \sigma_{12} dW_\theta, \\
    d\theta = m_\theta dt + \sigma_{21} dW_r + \sigma_{22} dW_\theta,
\end{cases}
\]

(16)
where $W_r, W_0$, respectively, are the independent and standard Wiener processes. $\begin{bmatrix} m_r \\ m_o \end{bmatrix}$ is a drift vector, and $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is a diffusion coefficient matrix. Set the parameters as follows:

$$
m_r = \frac{r}{2}(e_{10} + e_{20}) + \frac{\sigma_1^2}{8}r(5e_{15}^2 + 5e_{24}^2 + 3(e_{14} + e_{25})^2 - 2e_{15}e_{24})
\quad + \frac{\sigma_2^2}{8}r(5e_{18}^2 + 5e_{27}^2 + 3(e_{17} + e_{28})^2 - 2e_{18}e_{27}) + \frac{r}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)).
$$

$$
\sigma_{11} = \frac{1}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)) + \frac{\sigma_1^2}{8}(3e_{15}^2 + 2e_{15}e_{24} + (e_{14} + e_{25})^2 + 3e_{24}^2)
\quad + \frac{\sigma_2^2}{8}(3e_{18}^2 + 2e_{18}e_{27} + (e_{17} + e_{28})^2 + 3e_{27}^2).
$$

$$
\sigma_{12} = -\frac{r}{4}(\sigma_1^2(e_{25} - e_{14})(e_{24} + e_{15}) + \sigma_2^2(e_{28} - e_{17})(e_{18} + e_{27})).
$$

When $\sigma_{12}^2 = \sigma_{21}^2 = 0$, i.e., $e_{25} - e_{14} = 0$ or $e_{28} - e_{17} = 0$, $r(t)$ is a one-dimensional Markov diffusion process. Thus, we have

$$
dr = [(\mu_1 + \frac{\mu_1}{8})r + \frac{\mu_2}{r}]dr + (\mu_2 + \frac{\mu_1}{8}r^2)^{1/2}dW_r,
$$

where

$$
\mu_1 = \frac{1}{2}(e_{10} + e_{20}), \quad \mu_2 = \omega + \frac{1}{2}(e_{20} + e_{10}),
$$

$$
\mu_1 = \sigma_1^2(5e_{15}^2 + 5e_{24}^2 + 3(e_{14} + e_{25})^2 - 2e_{15}e_{24}) + \sigma_2^2(5e_{18}^2 + 5e_{27}^2 + 3(e_{17} + e_{28})^2 - 2e_{18}e_{27}),
$$

$$
\mu_2 = \frac{1}{2}(\sigma_1^2(e_{16}^2 + e_{26}^2) + \sigma_2^2(e_{19}^2 + e_{29}^2)).
$$

$$
\mu_3 = \sigma_1^2(3e_{15}^2 + 2e_{15}e_{24} + (e_{14} + e_{25})^2 + 3e_{24}^2) + \sigma_2^2(3e_{18}^2 + 2e_{18}e_{27} + (e_{17} + e_{28})^2 + 3e_{27}^2).
$$

Next, the stability of the system is analyzed according to singular boundary theory. In fact, $\mu_2$ is not equal to 0. One can find $r = 0$ is a nonsingular boundary of the system (17) if $\mu_2 + \frac{\mu_1}{8}r^2 \neq 0$ at $r = 0$. Through calculation, we can find that $r = 0$ is a regular boundary. If $r = +\infty$, $m_r = \infty$, then $m_r = +\infty, r = +\infty$ is the second singular boundary.

$$
\alpha_r = 2, \quad \beta_r = 1,
$$

$$
c_r = \lim_{r \to +\infty} \frac{2m_r^{\alpha_r - \beta_r}}{\sigma_{11}^2} = \lim_{r \to +\infty} \frac{2[(\mu_1 + \frac{\mu_1}{8})r + \frac{\mu_2}{r}]r}{\mu_2 + \frac{\mu_1}{8}r^2} = -\frac{2(8\mu_10 + \mu_1)}{\mu_3}.
$$
So, if \( c_r > -1 \), i.e., \( \frac{8\mu_{10} + \mu_1}{\mu_3} < \frac{1}{2} \), then the boundary \( r = +\infty \) is exclusively natural;
if \( c_r < -1 \), i.e., \( \frac{8\mu_{10} + \mu_1}{\mu_3} > \frac{1}{2} \), then the boundary \( r = +\infty \) is attractively natural;
if \( c_r = -1 \), i.e., \( \frac{8\mu_{10} + \mu_1}{\mu_3} = \frac{1}{2} \), then the boundary \( r = +\infty \) is strictly natural.

**Remark 1** If \( c_r < -1 \), then the boundary \( r = +\infty \) is attractively natural, i.e., the system is unstable at the equilibrium point, and *Hopf* bifurcation is possible.

### 3 Hopf bifurcation

Based on (17), we can obtain the FPK equation as

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r} \left\{ \left[ (\mu_{10} + \frac{\mu_1}{8}) r + \frac{\mu_2}{r} \right] p(r) \right\} + \frac{1}{2} \frac{\partial^2}{\partial r^2} \left\{ \left( \mu_2 + \frac{\mu_3}{8} r^2 \right) p(r) \right\}
\]

with the initial value condition

\[
p(r,t|\tau_0,t_0) \to \delta(r \to \tau_0), t \to t_0.
\]

where \( p(r,t|\tau_0,t_0) \) is the transition probability density of diffusion process \( r(t) \). The invariant measure of \( r(t) \) is the steady-state probability density \( p_{st}(r) \), which is the solution of the degenerate system, as follows:

\[
-\frac{\partial}{\partial t} \left\{ \left[ (\mu_{10} + \frac{\mu_1}{8}) r + \frac{\mu_2}{r} \right] p(r) \right\} + \frac{1}{2} \frac{\partial^2}{\partial r^2} \left\{ \left( \mu_2 + \frac{\mu_3}{8} r^2 \right) p(r) \right\} = 0.
\]

Solving the above equation, one can obtain

\[
p_{st}(r) = 4 \sqrt{\frac{7}{\pi}} 2^{-3h} \mu_2^{-2h} \left( \frac{\mu_3}{\mu_2} \right)^{1/2} \Gamma(2-h) \Gamma(\frac{1}{2} - h))^{-1} r^2 (\mu_3 r^2 + 8\mu_2)^{h-2},
\]

where \( h = \frac{8\mu_{10} + \mu_1}{\mu_3} \), \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \). We now calculate the most possible amplitude \( r^* \) of system (17), i.e., \( p_{st}(r) \) has a maximum value at \( r^* \). So, we have

\[
\frac{d p_{st}(r)}{dr} \bigg|_{r=r^*} = 0, \quad \frac{d^2 p_{st}(r)}{dr^2} \bigg|_{r=r^*} < 0,
\]
and the solution is \( r = 0 \) or \( r = \tilde{r} = \sqrt{\frac{-8\mu_2}{8\mu_1 - \mu_3}} \). The relations
\[
\frac{d^2 p_\alpha(r)}{dr^2} \bigg|_{r=0} = 2^{7+3}(8\mu_1 + \mu_1 - \mu_3)\mu^{-1} \frac{2+(8\mu_1 + \mu_1 - \mu_3)\mu^{-1}}{\mu_2^2} > 0,
\]
\[
\frac{d^2 p_\alpha(r)}{dr^2} \bigg|_{r=\tilde{r}} = \frac{(8\mu_1 + \mu_1 - \mu_3)^2(8\mu_2 - \frac{8\mu_2\mu_1}{8\mu_1 + \mu_1 - \mu_3})}{-16(8\mu_1 + \mu_1 - \mu_3)} < 0,
\]
show that \( r = \tilde{r} \). \( p_\alpha(r) \) is minimum at \( r = 0 \). Stochastic system (17) is almost unstable at \( r = 0 \). Hence, system (2) may show Hopf bifurcation at \( r = \tilde{r} \). Then
\[
x_1^2 + x_2^2 + x_3^2 = \frac{-8\mu_2}{8\mu_1 + \mu_1 - \mu_3}, (r = \tilde{r}).
\]

4 \( H_\infty \) control

Hopf bifurcation indicates the instability of the system, which is not what we want to see. T-S fuzzy control [12] approximates or represents a global nonlinear system model using several local linear system models, and solves the control problem of the whole local nonlinear system by means of analysis and control of the linear system. \( H_\infty \) control based on T-S fuzzy control [13–16] is widely used in various fields. To eliminate this bifurcation phenomenon, the application of T-S fuzzy \( H_\infty \) control to solve the problem of nonlinear biological systems will be a general trend in the field of biological control [17, 18].

Next, introducing the state feedback control for system (3), one can obtain the controlled system (19):

\[
\begin{align*}
\dot{y}_1 &= b_{11}y_1 + b_{12}y_2 - a_{12}y_2^2 + \sigma_1(y_1 + x_1^2)\xi_1(t), \\
\dot{y}_2 &= b_{21}y_1 + b_{22}y_2 + b_{23}y_3(t - \tau_1) - a_{22}y_2^2 - d_{11}y_2y_3(t - \tau_1) + \sigma_1(y_2 + x_2^2)\xi_1(t), \\
\dot{y}_3 &= b_{32}y_2(t - \tau_2) + b_{33}y_3 - \frac{k^2}{\tau_3}y_3^2 - d_{21}y_2(t - \tau_2)y_3 + u(t) + \sigma_2(y_3 + x_3^2)\xi_2(t),
\end{align*}
\]

(19)

where \( u(t) \) is the control variable.

We suppose that \( y_i(t) \in [-k, k], i = 1, 2, 3 \). A system of fuzzy equations is given as follows, which can describe system (19) when \( y_i(t) \in [-k, k], i = 1, 2, 3 \):

**Rule 1:** If \( y_1(t) \) is \( M_{11} \), \( y_2(t) \) is \( M_{21} \), and \( y_3(t) \) is \( M_{31} \), then
\[
\dot{y}(t) = A_{11}y(t) + A_{12}y(t - \tau_1) + A_{21}y(t - \tau_2) + B_{11}\xi_1(t) + B_{21}\xi_2(t) + C_1u(t).
\]
Rule 2: if \( y_1(t) \) is \( M_{12} \), \( y_2(t) \) is \( M_{22} \), \( y_3(t) \) is \( M_{32} \), then

\[
\dot{y}(t) = A_2 y(t) + A_{12} y(t - \tau_1) + A_{22} y(t - \tau_2) + B_{12} \bar{\xi}(t) + B_{22} \bar{\xi}(t) + C_2 u(t),
\]

where

\[
y(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T, \quad \bar{\xi}(t) = \begin{bmatrix} \xi_1(t) & \xi_2(t) \end{bmatrix}^T,
\]

\[
M_{11} = \frac{1}{2}(1 - y_1(t)/k_1), \quad M_{21} = \frac{1}{2}(1 - y_2(t)/k_2), \quad M_{31} = \frac{1}{2}(1 - y_3(t)/k_3),
\]

\[
M_{12} = \frac{1}{2}(1 + y_1(t)/k_1), \quad M_{22} = \frac{1}{2}(1 + y_2(t)/k_2), \quad M_{32} = \frac{1}{2}(1 + y_3(t)/k_3),
\]

\[
A_1 = \begin{bmatrix} b_{11} + a_{12} k_1 & b_{12} & 0 \\ b_{21} & b_{22} + a_{22} k_2 & 0 \\ 0 & 0 & b_{33} + r \end{bmatrix}, \quad A_2 = \begin{bmatrix} b_{11} - a_{12} k_1 & b_{12} & 0 \\ b_{21} & b_{22} - a_{22} k_2 & 0 \\ 0 & 0 & b_{33} - r \end{bmatrix},
\]

\[
A_{11} = \begin{bmatrix} 0 & 0 & b_{23} + d_1 k_2 \\ 0 & 0 & 0 \\ 0 & d_2 k_3 + b_{32} 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{32} - d_1 k_2 \end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 -d_2 k_3 + b_{32} 0 \end{bmatrix}, \quad B_1 = B_{11} = B_{12} = \begin{bmatrix} \sigma_1 x_1^* & 0 \\ 0 & \sigma_1 x_2^* \\ 0 & 0 \end{bmatrix},
\]

\[
B_2 = B_{21} = B_{22} = \begin{bmatrix} \sigma_1 0 \\ 0 & \sigma_1 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}, \quad C = C_1 = C_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[h_i(y(t))\] are the membership functions for \( y(t) \in M_{ij}(i, j = 1, 2)\), \( \bar{h}_i(y(t)) = \prod_{j=1}^{3} M_{ij}(y_j(t)) \geq 0\), \( h_i(y(t)) = \bar{h}_i(y(t))/\sum_{j=1}^{3} \bar{h}_j(y(t))\), and \( \sum_{i=1}^{3} h_i(y(t)) = 1\). The global model can be described as:

\[
\dot{y}(t) = \sum_{i=1}^{3} h_i(y(t)) \left[ A_{1} y(t) + A_{12} y(t - \tau_1) + A_{22} y(t - \tau_2) + B_1 \bar{\xi}(t) + B_2 \bar{\xi}(t) + C u(t) \right] \tag{20}
\]
Making a transformation $y(t - \tau_1) = y(t) - \int_{-\tau_1}^{0} \dot{y}(t + \alpha) d\alpha$ from the Newton-Leibniz formula, system (20) is equivalent to

$$\dot{y}(t) = \sum_{i=1}^{2} h_i(y(t)) \left[ \tilde{A}_i y(t) - A_{1i} \int_{-\tau_1}^{0} \dot{y}(t + \alpha) d\alpha - A_{2i} \int_{-\tau_2}^{0} \dot{y}(t + \alpha) d\alpha + B_1 \xi(t) + B_2 \bar{\xi}(t) + Cu(t) \right]$$

(21)

where $\tilde{A}_i = A_i + A_{1i} + A_{2i}$.

**Lemma 1** [19] For any $u, v \in \mathbb{R}^n$, and any matrix $Z > 0$ with appropriate dimensions, the following inequality holds:

$$u^T v + v^T u \leq u^T Z^{-1} u + v^T Z v.$$ 

**Theorem 1** For a given $\gamma > 0$, if there exist matrices $Q > 0, P_i > 0, R_i > 0, M_i > 0$ which satisfy the linear matrix inequalities

$$\begin{bmatrix}
\tilde{A}_i Q + Q \tilde{A}_j^T + C M_j + M_j^T C & 0 & 0 & 0 & B_1 & B_2 & \frac{A_{1i} + A_{1j}}{2} & \frac{A_{2i} + A_{2j}}{2} \\
* & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -R_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \tau_1 P_1 + \tau_2 P_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & -\frac{1}{\tau_1} P_1 & 0 \\
* & * & * & * & * & * & * & -\frac{1}{\tau_2} P_2 \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
\end{bmatrix}$$
where * indicates a symmetric term, then \(u(t) = \sum_{j=1}^{2} h_j(y(t))K_jy(t)\) makes the \(H_\infty\) norm of system (23) less than \(\gamma\). The closed-loop system is quadratically stable, where \(P = Q^{-1}\) and \(S_i = R_i^{-1}\). By Lemma 1, let \(u = 2y^T(t)PA_{11}, \ v = \hat{y}(t + \alpha), \ Z = P_1\). Then

\[
-\sum_{j=1}^{2} \sum_{i=1}^{2} \int_{-\tau_1}^{0} 2y^T(t)PA_{11}\hat{y}(t + \alpha)d\alpha \leq \sum_{j=1}^{2} \sum_{i=1}^{2} \tau_1y^T(t)P\frac{A_{11} + A_{1j}}{2}P_1^{-1}(A_{11} + A_{1j})^T P_{1}y(t)
+ \int_{-\tau_1}^{0} \hat{y}^T(t + \alpha)P_1\hat{y}(t + \alpha)d\alpha
\]
where the Lyapunov function is given from the above equations by

\[
    V(t) = \int_{t_1}^{t} \left( y^T(t) S_1 y(t) - y^T(t - \tau_1) S_1 y(t - \tau_1) + \tau_1 y^T(t) P_1 y(t) - \int_{t_1 - \tau_1}^{t} y^T(t + \alpha) P_1 y(t + \alpha) d\alpha \right) dt.
\]

Then the derivative of the Lyapunov function is from the above equations by

\[
    \dot{V}(t) \leq \sum_{j=1}^{\delta} \sum_{i=1}^{\delta} h_j h_i R_{ij} \eta^T(t) + \sum_{j=1}^{\delta} \sum_{i=1}^{\delta} h_j h_i Q_{ij} \eta^T(t) - z^T z + \gamma^2 \xi^T(t) \xi(t).
\]

where

\[
    R_{ij} = P(A_1 + CK_j) + (A_1 + CK_j) P + S_1 + S_2 + \tau_1 P A_1 + A_1 P_1^{-1} (A_1 + A_1)^T P + \tau_2 P A_2 + A_2 P_2^{-1} (A_2 + A_2)^T P.
\]
Thus, \( Q_{ij} < 0 \) implies \( \| z \|^2 \leq \gamma^2 \| \xi \|^2 \). Therefore, when \( Q < 0 \), the system (23) is quadratically stable with an \( H_\infty \) norm bound \( \gamma \). Applying the Schur complement theorem, \( Q_{ij} < 0 \) can be transformed to Eq. (22).

**Remark 2** The fuzzy controller is designed to represent the government’s management of resource development. The bifurcation phenomenon in biological system can be eliminated by state feedback control to ensure the stability of the system. With the purpose of maintain sustainable development and market stability, managers can take certain measures in daily life, such as, adjusting the tax rate, controlling catches, reduction of environmental pollution and so on.

### 5 Numerical analysis

We select some parameters to fully reflect the relationship among the steady-state probability density and position of stochastic Hopf bifurcation with the value of \( \mu_2 \) in Section (3), and further demonstrate the effect of the controller in Section (4), with the following parameters:

\[
\begin{align*}
\dot{x}_1 &= 0.15x_2 - 0.04x_1^2 - 0.6x_1 + 0.5x_1\xi_1(t), \\
\dot{x}_2 &= 0.5x_1 - 0.001x_2^2 - 0.003x_2x_3 - 0.12x_2 + 0.5x_2\xi_1(t), \\
\dot{x}_3 &= x_3(1.68(1 - \frac{x_3}{k_3}) - 0.024x_2 - 1.5 + 1) + 0.7x_3\xi_2(t).
\end{align*}
\]

We know by calculation that when \( \mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_3 = 0.7710 \), the four curves in Fig. 1 correspond to cond 1, 2, 3, 4, respectively, in Table 1. It can be seen from
Table 1 The probabilities and the positions of the Hopf bifurcation occurrence.

| Condition | Parameters | $r = \tilde{r}$ | $P_{\tilde{r}}(\tilde{r})$ |
|-----------|------------|-----------------|---------------------------|
| Cond1     | $\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0040, \mu_3 = 0.7710$ | 0.0059 | 7.6800 |
| Cond2     | $\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0095, \mu_3 = 0.7710$ | 0.01441 | 4.9850 |
| Cond3     | $\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.0255, \mu_3 = 0.7710$ | 0.3792 | 3.042 |
| Cond4     | $\mu_{10} = -0.6720, \mu_1 = 2.2717, \mu_2 = 0.1635, \mu_3 = 0.7710$ | 1.786 | 1.201 |

Fig. 1 The steady-state probability density and position of stochastic Hopf bifurcation at $\mu_{10} = -0.6720, \mu_1 = 2.27170, \mu_2 = 0.0040, 0.0095, 0.0255, 0.1635, \mu_3 = 0.7710$.

Fig. 1 that the position of stochastic Hopf bifurcation increases with the increase of $\mu_2$, while the steady-state probability density of stochastic Hopf bifurcation decreases.

The $H_\infty$ controller is designed for system (19), so that the $H_\infty$ norm of the closed-loop system (19) is less than $\gamma = 0.01$. Using LMI toolbox in MATLAB, the parameters satisfying Theorem 1 can be obtained as follows:

$$P = \begin{bmatrix} -28.9455 & -44.8728 & 0.2033 \\ -44.8728 & -59.6085 & 1.0517 \\ 0.2033 & 1.0517 & 12.0072 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0.2227 & 0 & 0 \\ 0 & 0.2058 & 0 \\ 0 & 0 & 0.2232 \end{bmatrix}.$$
Fig. 2 Responses of $y_1(t)$, $y_2(t)$, $y_3(t)$, $z(t)$ in system (19).

$$P_2 = \begin{bmatrix} 0.6404 & 0 & 0 \\ 0 & 0.7156 & 0 \\ 0 & 0 & 0.6402 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 3.2168 & 0.1595 & -0.0245 \\ 0.1595 & 3.3274 & 0.0165 \\ -0.0245 & 0.0165 & 3.2893 \end{bmatrix}.$$

$$S_2 = \begin{bmatrix} 3.2168 & 0.1595 & -0.0245 \\ 0.1595 & 3.3274 & 0.0165 \\ -0.0245 & 0.0165 & 3.2893 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -0.0049 & 0.0027 & -0.0650 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0094 & 0.0065 & -0.0115 \end{bmatrix}.$$

Fig. 2 shows that the output and state variables of the closed-loop system (19) gradually tend to zero over time. This fully demonstrates the feasibility of the controller.

6 Conclusion

This paper studied the Hopf bifurcation phenomenon of a stochastic competition population with stage structure under the influence of white noise. By using the stochastic center
manifold and stochastic average method to reduce dimension, and through stability analysis and bifurcation position research, we can take $\mu_2$ as a bifurcation parameter, and the position of stochastic Hopf bifurcation will increase with the increase of $\mu_2$, while the steady-state probability density of stochastic Hopf bifurcation will decrease. To eliminate the bifurcation phenomenon, a fuzzy state feedback controller was designed using the T-S fuzzy system control method to ensure the stability of the closed-loop system.

Conflict of Interest

The authors declare that they have no conflict of interest.

Founding

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References

1. M. L. Zeeman. Extinction in Competitive Lotka-Volterra Systems[J]. Proceedings of the American Mathematical Society, 1995, 123(1):87-96.
2. F. Chen. Global asymptotic stability in n-species non-autonomous Lotka-Volterra competitive systems with infinite delays and feedback control[J]. Applied Mathematics and Computation, 2005, 170(2):1452-1468.
3. R. Rudnicki. Long-time behaviour of a stochastic prey-predator model[J]. Stochastic Process. Appl. 108 (2003) 93-107.
4. C. Zhu, G. Yin. On competitive Lotka-Volterra model in random environments[J]. Journal of Mathematical Analysis and Applications, 2009, 357(1):154-170.
5. R. Kon, Y. Saito, Y. Takeuchi. Permanence of single-species stage-structured models[J]. Journal of Mathematical Biology, 2004, 48(5):515-528.
6. X. Song, L. Chen. Optimal harvesting and stability for a two-species competitive system with stage structure[J]. Mathematical Biosciences, 2001, 170(2):0-186.
7. Y. Zhang, Q. Zhang. Dynamic behavior in a delayed stage-structured population model with stochastic fluctuation and harvesting[J]. Nonlinear Dynamics, 2011, 66(1-2):231-245.
8. Z. Huang, Q. Yang. J. Cao. Complex dynamics in a stochastic internal HIV model[J]. Chaos Solitons Fractals, 2011, 44(11):954-963.
9. Z. Huang. The Study of Dynamic Properties of Stochastic Differential Systems with Delay[D]. South China University of Technology, 2011.
10. D. Huang, H. Wang. J. Feng, et al. Hopf bifurcation of the stochastic model on HAB nonlinear stochastic dynamics[J]. Chaos, Solitons Fractals, 2006, 27(4):1072-1079.
11. E. Knobloch, K. A. Wiesenfeld. Bifurcations in fluctuating systems: The center-manifold approach[J]. Journal of Statistical Physics, 1983, 33(3):611-637.
12. B. Zhu. Analysis and Control for a Kind of T-S Fuzzy Descriptor System[D]. Northeastern University, 2006.
13. Y. Sun, J. Xu, C. Chen, et al. Fuzzy $H_\infty$ robust control for magnetic levitation system of maglev vehicles based on T-S fuzzy model: Design and experiments[J]. Journal of Intelligent and Fuzzy Systems, 2018:1-12.
14. X. Han, Y. Ma. Sampled-data Robust $H_\infty$ Control for T-S Fuzzy Time-delay Systems with State Quantization[J]. International Journal of Control, Automation and Systems, 2019, 17(1):46-56.
15. B. Wu, X. Chang, X. Zhao. Fuzzy $H_\infty$ Output Feedback Control for Nonlinear NCSs with Quantization and Stochastic Communication Protocol[J]. IEEE Transactions on Fuzzy Systems, 2020.
16. H. Y, T. W, H. Z, et al. Event-triggered $H_\infty$ control for uncertain networked TCS fuzzy systems with time delay[J]. Neurocomputing, 2015, 157:273-279.
17. S. Xing. Analysis and Control of a Class of Singular Stochastic Systems with Itô-Type[D]. Northeastern University, 2015.
18. Y. Zhang, Q. Zhang, T. Zhang. $H_\infty$ control of generalized bio-economic Systems[J]. Journal of North-eastern University (Natural Science), 2011(10):1369-1373.
19. B. Chen, X. Liu. Delay-Dependent Robust Control for T-S Fuzzy Systems With Time Delay[J]. IEEE Transactions on Fuzzy Systems, 2005, 13(4):544-556.