Weakly commensurable groups, with applications to differential geometry

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(joint work with Gopal Prasad)

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1 Geometric introduction
   - Isospectral and length-commensurable manifolds
   - Hyperbolic manifolds

2 Weakly commensurable arithmetic groups
   - Definition of weak commensurability
   - Arithmetic groups
   - Results on weak commensurability

3 Back to geometry
   - Length-commensurability vs. weak commensurability
   - Some results
[1] G. Prasad, A.S. Rapinchuk, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*, Publ. math. IHES 109(2009), 113-184.

[2] — , —, *Local-global principles for embedding of fields with involution into simple algebras with involution*, Comment. Math. Helv. 85(2010), 583-645.

[3] — , —, *On the fields generated by the lengths of closed geodesics in locally symmetric spaces*, arXiv:1110.0141.

**Survey:**

[4] — , —, *Number-theoretic techniques in the theory of Lie groups and differential geometry*, 4th International Congress of Chinese Mathematicians, AMS/IP Stud. Adv. Math. 48, AMS 2010, pp. 231-250.
Outline

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$M_1$ and $M_2$ are \textbf{commensurable} if they have a common \textbf{finite-sheeted} cover:
Question: Are $M_1$ and $M_2$ necessarily isometric (commensurable) if
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Example: Let $M_1$ and $M_2$ be spheres of radii $r_1$ and $r_2$. Then $L(M_i) = \{2\pi r_i\}$, so $L(M_1) = L(M_2) \Rightarrow M_1 \& M_2$ are isometric.
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Even though there are examples of noncommensurable isospectral manifolds (Lubotzky et al.), it appears that commensurability is the property that one may be able to establish in various situations.
Conditions (1) & (2) are not invariant under passing to a commensurable manifold, while condition (3) - length-commensurability \((\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2))\) - is.
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Our project: Understand consequences of length-commensurability.
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Conditions (1), (2) and (3) are related:

- For Riemann surfaces: \(E(M_1) = E(M_2) \iff L(M_1) = L(M_2)\)
- For any compact locally symmetric spaces:
  \[E(M_1) = E(M_2) \Rightarrow L(M_1) = L(M_2).\]
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- For Riemann surfaces: \(\mathcal{E}(M_1) = \mathcal{E}(M_2) \iff \mathcal{L}(M_1) = \mathcal{L}(M_2)\)
- For any compact locally symmetric spaces:
  \[\mathcal{E}(M_1) = \mathcal{E}(M_2) \implies \mathcal{L}(M_1) = \mathcal{L}(M_2)\].

So, results for length-commensurable locally symmetric spaces imply results for isospectral spaces.
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In particular:

- we know when length-commensurability $\Rightarrow$ commensurability (answer depends on Lie type of isometry group)

- locally symmetric spaces length-commensurable to a given arithmetically defined locally symmetric space form finitely many commensurability classes.
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Isometry group of $\mathbb{H}^d$ is $G = \text{PSO}(d,1)$.

**Arithmetically defined hyperbolic $d$-manifold** is $M = \mathbb{H}^d/\Gamma$, where $\Gamma$ is an *arithmetical* subgroup of $G$. 
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**Theorem.** Let $M_1$ and $M_2$ be *arithmetically defined* hyperbolic $d$-manifolds.
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(1) Suppose $d$ is even or $\equiv 3 \pmod{4}$.

If $M_1$ and $M_2$ are not commensurable then after a possible interchange of $M_1$ and $M_2$, there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio $\lambda_1/\lambda_2$ is transcendental.

In particular, $M_1$ and $M_2$ are not length-commensurable.
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**Further question:** Suppose $M_1$ and $M_2$ are not length-commensurable. How different are $L(M_1)$ and $L(M_2)$?
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Under minor additional conditions we prove the following:

Let $\mathcal{F}_i$ be subfield of $\mathbb{R}$ generated by $L(M_i)$. Then $\mathcal{F}_1 \mathcal{F}_2$ has infinite transcendence degree over $\mathcal{F}_1$ or $\mathcal{F}_2$. 
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So, $L(M_1)$ and $L(M_2)$ are very much different.
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So, \( L(M_1) \) and \( L(M_2) \) are very much different.

(We have similar results for complex and quaternionic hyperbolic spaces.)
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Weak commensurability

Let $G_1$ and $G_2$ be two semi-simple groups over a field $F$ of characteristic zero.

- Semi-simple $g_i \in G_i(F)$ ($i = 1, 2$) are weakly commensurable if there exist maximal $F$-tori $T_i \subset G_i$ such that $g_i \in T_i(F)$ and for some $\chi_i \in X(T_i)$ (defined over $\overline{F}$) we have
  $$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$  

- (Zariski-dense) subgroups $\Gamma_i \subset G_i(F)$ are weakly commensurable if every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.
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Recall: given an $F$-torus $T \subset \text{GL}_n$, an element $t \in T(F)$, and a character $\chi \in X(T)$, the character value

$$\chi(t) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $t$ (i.e. $t$ is conjugate to $\text{diag}(\lambda_1, \ldots, \lambda_n)$), and $a_1, \ldots, a_n \in \mathbb{Z}$.
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Pick matrix realizations \( G_i \subset \text{GL}_{n_i} \) for \( i = 1, 2 \).
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Let \( g_1 \in G_1(F) \) and \( g_2 \in G_2(F) \) be semi-simple elements with eigenvalues

\[
\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}.
\]
Then $g_1$ and $g_2$ are **weakly commensurable** if

$$
\chi_1(g_1) = \lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} = \chi_2(g_2) \neq 1
$$

for some $a_1, \ldots, a_{n_1}$ and $b_1, \ldots, b_{n_2} \in \mathbb{Z}$. 
Example

Let

\[
A = \begin{pmatrix}
2 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 1/6 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
6 & 0 & 0 \\
0 & 1 & 0 \\
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\end{pmatrix} \in SL_3(\mathbb{C}).
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Then $A$ and $B$ are weakly commensurable because

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However, no powers $A^m$ and $B^n$ ($m, n \neq 0$) are conjugate.
Main question: What can one say about Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ ($i = 1, 2$) given that they are weakly commensurable?
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More specifically, under what conditions are \( \Gamma_1 \) and \( \Gamma_2 \) necessarily commensurable?
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Recall: subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of a group $G$ are commensurable if

$$[\mathcal{H}_i : \mathcal{H}_1 \cap \mathcal{H}_2] < \infty \quad \text{for} \quad i = 1, 2.$$
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$\Gamma_1$ and $\Gamma_2$ are commensurable up to an $F$-isomorphism between $G_1$ and $G_2$ if there exists an $F$-isomorphism $\sigma : G_1 \to G_2$ such that $\sigma(\Gamma_1)$ and $\Gamma_2$ are commensurable in usual sense.
Algebraic Perspective

**General framework:** Characterization of linear groups in terms of spectra of its elements.
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**Complex Representations of Finite Groups:**

Let $\Gamma$ be a finite group,

$$\rho_i: \Gamma \to GL_{n_i}(\mathbb{C}) \quad (i = 1, 2)$$

be representations. Then

$$\rho_1 \simeq \rho_2 \iff \chi_{\rho_1}(g) = \chi_{\rho_2}(g) \quad \forall g \in \Gamma,$$

where $\chi_{\rho_i}(g) = \text{tr} \rho_i(g) = \sum \lambda_j$ (\(\lambda_1, \ldots, \lambda_{n_i}\) eigenvalues of \(\rho_i(g)\)).
Algebraic perspective

- **Data** afforded by **weak commensurability** is more **convoluted** than **data** afforded by character of a group representation: when computing

  \[ \chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n} \]

  one can use **arbitrary** integer weights \(a_1, \ldots, a_n\). **So**, weak commensurability appears to be more **difficult** to analyze.

- **Example.** Let \(\Gamma \subset SL_n(\mathbb{C})\) be a neat Zariski-dense subgroup. For \(d > 0\), let

  \[ \Gamma^{(d)} = \langle \gamma^d \mid \gamma \in \Gamma \rangle. \]

  Then any \(\Gamma^{(d)} \subset \Delta \subset \Gamma\) is **weakly commensurable** to \(\Gamma\).
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Then any \( \Gamma^{(d)} \subset \Delta \subset \Gamma \) is weakly commensurable to \( \Gamma \). So, one needs to limit attention to some special subgroups in order to generate meaningful results.
Weakly commensurable arithmetic groups

Definition of weak commensurability

Geometric perspective

- Weak commensurability (of fundamental groups) adequately reflects length-commensurability of locally symmetric spaces.

- Let $G = SL_2$. Corresponding symmetric space:
  
  \[ SO_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) = \mathbb{H} \]  
  (upper half-plane)

- Any (compact) Riemann surface of genus $> 1$ is of the form
  
  \[ M = \mathbb{H}/\Gamma \]

  where $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup (with torsion-free image in $PSL_2(\mathbb{R})$).
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We will demonstrate this for Riemann surfaces - for now.

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- Any **closed geodesic** \( c \) in \( M \) corresponds to a **semi-simple** \( \gamma \in \Gamma \), i.e. \( c = c_\gamma \).

- It has **length**
  
  \[ \ell(c_\gamma) = \left(\frac{1}{n_\gamma}\right) \cdot \log t_\gamma \]

  where \( t_\gamma \) is **eigenvalue** of \( \pm \gamma \) which is \( > 1 \),

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If $M_i = \mathbb{H}/\Gamma_i$ ($i = 1, 2$) are length-commensurable then:

- for any nontrivial semi-simple $\gamma_1 \in \Gamma_1$ there exists a nontrivial semi-simple $\gamma_2 \in \Gamma_2$ such that

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So,

\[ t_{\gamma_1}^{n_1} = t_{\gamma_2}^{n_2} \]
This means that

\[ \chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1 \]

where \( \chi_i \) is the character of the maximal \( \mathbb{R} \)-torus \( T_i \subset \text{SL}_2 \) corresponding to \( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{n_i} \).
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It follows that

$\Gamma_1$ and $\Gamma_2$ are weakly commensurable.
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Philosophy: An arithmetic group is a group that “looks like” $SL_n(\mathbb{Z})$. 
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More precisely: Let $G \subset GL_n$ be an algebraic $\mathbb{Q}$-group. Set

$$G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z}).$$

Subgroups of $G(F)$, where $F/\mathbb{Q}$, commensurable with $G(\mathbb{Z})$ are called arithmetic.
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More generally: For a number field $K$ and a set $S$ of places of $K$, containing all archimedean ones, $\mathcal{O}(S)$ denotes ring of $S$-integers.

E.g.: If $K = \mathbb{Q}$ and $S = \{\infty, 2\}$ then $\mathcal{O}(S) = \mathbb{Z}[1/2]$. 
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More generally: For a number field $K$ and a set $S$ of places of $K$, containing all archimedean ones, $\mathcal{O}(S)$ denotes ring of $S$-integers.

Given an algebraic $K$-group $G \subset \text{GL}_n$, set $G(\mathcal{O}(S)) = G \cap \text{GL}_n(\mathcal{O}(S))$; subgroups of $G(F)$ ($F/K$) commensurable with $G(\mathcal{O}(S))$ are $(K,S)$-arithmetic.
What is an arithmetic subgroup of an algebraic group which is NOT defined over a number field?

E.g.: What is an arithmetic subgroup of $G(R)$ where $G = \text{SO}_3(f)$ and $f = x^2 + e \cdot y^2 - \pi \cdot z^2$?

We define arithmetic subgroups of $G(F)$ in terms of forms of $G$ over subfields of $F$ that are number fields.

We can consider rational quadratic forms $R$-equivalent to $f$: $f_1 = x^2 + y^2 - 3z^2$ or $f_2 = x^2 + 2y^2 - 7z^2$. 
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Then $SO_3(f_i) \simeq SO_3(f)$ over $\mathbb{R}$, and

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One can also consider \( K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R} \) and \( f_3 = x^2 + y^2 - \sqrt{2}z^2 \). Then

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One can further replace integers by $S$-integers, etc.
Definition of arithmeticity

Definition. Let $G$ be an absolutely almost simple algebraic group over a field $F$, $\text{char } F = 0$, and $\pi: G \to \overline{G}$ be isogeny onto adjoint group.

1. a number field $K$ with a fixed embedding $K \hookrightarrow F$;
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Then subgroups $\Gamma \subset G(F)$ such that $\pi(\Gamma)$ is commensurable with $\mathcal{G}(\mathcal{O}_K(S))$ are called $(\mathcal{G}, K, S)$-arithmetic.
Convention:  $S$ does not contain nonarchimedean $v$ such that $G$ is $K_v$-anisotropic.
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Convention: S does not contain nonarchimedean v such that G is K_v-anisotropic.

We do NOT fix an F-isomorphism $F \mathcal{G} \simeq \overline{G}$ in n° 3; by varying it we obtain a class of groups invariant under F-automorphisms.

Proposition. Let $G_1$ and $G_2$ be connected absolutely almost simple algebraic groups defined over a field F, (char F = 0), and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K_i, S_i)$-arithmetic group $(i = 1, 2)$.

Then $\Gamma_1$ and $\Gamma_2$ are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if

- $K_1 = K_2 =: K$;
- $S_1 = S_2$;
- $G_1$ and $G_2$ are K-isomorphic.
In above example, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are \textit{pairwise noncommensurable}.
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Recall: \( f_1 = x^2 + y^2 - 3z^2, \; f_2 = x^2 + 2y^2 - 7z^2, \; f_3 = x^2 + y^2 - \sqrt{2}z^2 \).

\( \bullet \) \( \Gamma_1 \) and \( \Gamma_2 \) are **NOT** commensurable b/c the corresponding \( \mathbb{Q} \)-forms \( G_1 = \text{SO}_3(f_1) \) and \( G_2 = \text{SO}_3(f_2) \) are **NOT** isomorphic over \( \mathbb{Q} \).
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• \( \Gamma_3 \) is **NOT** commensurable to either \( \Gamma_1 \) or \( \Gamma_2 \) b/c they have **different fields of definition:**

\[
\mathbb{Q}(\sqrt{2}) \text{ for } \Gamma_3, \quad \text{and } \mathbb{Q} \text{ for } \Gamma_1 \text{ and } \Gamma_2.
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Results of Prasad-R. and follow-up results Garibaldi, Garibaldi-R. provide a (virtually) complete analysis of weak commensurability for arithmetic groups.

In particular:

- we know when weak commensurability $\Rightarrow$ commensurability (answer depends on Lie type of algebraic group)

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Theorem 1. Let $G_1$ and $G_2$ be two connected absolutely almost simple algebraic groups defined over a field $F$ of characteristic zero.
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If there exist finitely generated Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ ($i=1,2$) that are weakly commensurable then

either $G_1$ and $G_2$ have the same Killing-Cartan type, or

one of them is of type $B_n$ and the other is of type $C_n$ ($n \geq 3$).
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either $G_1$ and $G_2$ have the same Killing-Cartan type, or one of them is of type $B_n$ and the other is of type $C_n$ ($n \geq 3$).

Note: groups of types $B_n$ and $C_n$ can indeed contain Zariski-dense weakly commensurable subgroups.
**Theorem 2.** Let \( \Gamma_i \subset G_i(F) \) be a Zariski-dense \((G_i, K_i, S_i)\)-arithmetic subgroup for \( i = 1, 2 \).

If \( \Gamma_1 \) and \( \Gamma_2 \) are weakly commensurable then \( K_1 = K_2 \) and \( S_1 = S_2 \).
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The forms $G_1$ and $G_2$ may NOT be $K$-isomorphic in general, but we have the following.
**Theorem 2.** Let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K_i, S_i)$-arithmetic subgroup for $i = 1, 2$.

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**Theorem 3.** Let $G_1$ and $G_2$ be of the same type different from $A_n$, $D_{2n+1}$ with $n > 1$, and $E_6$, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i, K, S)$-arithmetic subgroup.

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For types $A_n$, $D_{2n+1}$ ($n > 1$) and $E_6$ we have counterexamples.
Theorem 4. Let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense $(K,S)$-arithmetic subgroup. Then the set of Zariski-dense $(K,S)$-arithmetic subgroups $\Gamma_2 \subset G_2(F)$ that are weakly commensurable to $\Gamma_1$, is a union of finitely many commensurability classes.
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**Theorem 5.** Let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(G_i,K,S)$-arithmetic subgroup for $i = 1, 2$. If $\Gamma_1$ and $\Gamma_2$ are weakly commensurable then $\text{rk}_K G_1 = \text{rk}_K G_2$; in particular, if $G_1$ is $K$-isotropic then so is $G_2$. 
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Notations

- $G$ a connected absolutely (almost) simple algebraic group over $\mathbb{R}$; $\mathcal{G} = G(\mathbb{R})$
- $\mathcal{K}$ a maximal compact subgroup of $\mathcal{G}$; $\mathfrak{K} = \mathcal{K}\backslash\mathcal{G}$ associated symmetric space, $\text{rk}\, \mathfrak{K} = \text{rk}_{\mathbb{R}} G$
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Given $G_1, G_2, \Gamma_i \subset G_i := G_i(\mathbb{R})$ etc. as above, we will denote corresponding locally symmetric spaces by $\mathfrak{x}_{\Gamma_i}$. 
Fact. Assume that $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are of finite volume. If $X_{\Gamma_1}$ and $X_{\Gamma_2}$ are length-commensurable then (under minor technical assumptions) $\Gamma_1$ and $\Gamma_2$ are weakly commensurable.

- in rank one case - on the result of Gel’fond and Schneider (1934): if $\alpha$ and $\beta$ are algebraic numbers $\neq 0, 1$, then $\frac{\log \alpha}{\log \beta}$ is either rational or transcendental.

- in higher rank case - on the following Conjecture (Shanuel) If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of field generated by $z_1, \ldots, z_n; e^{z_1}, \ldots, e^{z_n}$ is $\geq n$.
Fact. Assume that $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ are of \textit{finite volume}. If $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ are \textit{length-commensurable} then (under minor technical assumptions) $\Gamma_1$ and $\Gamma_2$ are \textit{weakly commensurable}.

The proof relies:

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   - Length-commensurability vs. weak commensurability
   - Some results
Theorem 6. Let $\mathfrak{X}_{\Gamma_1}$ be an arithmetically defined locally symmetric space.

- The set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ that are length-commensurable to $\mathfrak{X}_{\Gamma_1}$, is a union of finitely many commensurability classes.

- It consists of a single commensurability class if $G_1$ and $G_2$ have the same type different from $A_n$, $D_{2n+1}$ with $n > 1$ and $E_6$. 
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Theorem 7. Let $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume, and assume that one of the spaces is arithmetically defined.

If $\mathcal{X}_{\Gamma_1}$ and $\mathcal{X}_{\Gamma_2}$ are length-commensurable then compactness of one of the spaces implies compactness of the other.
Theorem 8. Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces.

If $\mathfrak{X}_{\Gamma_1}$ is arithmetically defined then so is $\mathfrak{X}_{\Gamma_2}$. 
Theorem 8. Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces.

If $\mathfrak{X}_{\Gamma_1}$ is arithmetically defined then so is $\mathfrak{X}_{\Gamma_2}$.

Theorem 9. Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces, and assume that at least one of the spaces is arithmetically defined.

Then $G_1 = G_2 =: G$.

Moreover, unless $G$ is of type $A_n$, $D_{2n+1}$ ($n > 1$) or $E_6$, spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.