Atiyah Classes and Todd Classes of Pullback dg Lie Algebroids Associated with Lie Pairs

Hsuan-Yi Liao

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan.
E-mail: hyliao@math.nthu.edu.tw

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Abstract: For a Lie algebroid $L$ and a Lie subalgebroid $A$, i.e. a Lie pair $(L, A)$, we study the Atiyah class and the Todd class of the pullback dg (i.e. differential graded) Lie algebroid $\pi^!L$ of $L$ along the bundle projection $\pi : A[1] \to M$ of the shifted vector bundle $A[1]$. Applying the homological perturbation lemma, we provide a new construction of Stiénon–Vitagliano–Xu’s contraction relating the cochain complex $(\Gamma(\pi^!L), Q)$ of sections of $\pi^!L$ to the Chevalley–Eilenberg complex $(\Gamma(\Lambda^\bullet A^\vee \otimes (L/A)), d_{\text{Bott}})$ of the Bott representation. Using this contraction, we construct two isomorphisms: the first identifies the cohomology of the cochain complex $(\Gamma(\pi^!L)^\vee \otimes \text{End}(\pi^!L), Q)$ with the Chevalley–Eilenberg cohomology $H^\bullet_{\text{CE}}(A, (L/A)^\vee \otimes \text{End}(L/A))$ arising from the Bott representation, while the second identifies the cohomologies $H^\bullet(\Gamma(\Lambda^\bullet (L/A)^\vee), Q)$ and $H^\bullet_{\text{CE}}(A, \Lambda (L/A)^\vee)$. We prove that this pair of isomorphisms identifies the Atiyah class and the Todd class of the dg Lie algebroid $\pi^!L$ with the Atiyah class and the Todd class of the Lie pair $(L, A)$, respectively.

Contents

1. Preliminaries ............................................... 705
   1.1 Connections for a Lie algebroid ....................... 705
   1.2 Atiyah class and Todd class of a Lie pair .............. 707
   1.3 Atiyah class and Todd class of a dg Lie algebroid ...... 708
2. Dg Lie Algebroid Associated with a Lie Pair ............... 711
   2.1 The pullback Lie algebroid $\pi^!L$ .................... 711
   2.2 Contractions induced by splittings .................... 712
   2.3 Dg Lie algebroid structures on $\pi^!L$ ............... 714

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3. Two Atiyah Classes Associated with a Lie Pair
   3.1 A $(\pi^1 L)$-connection on $\pi^1 L$ ........................................ 717
   3.2 The two Atiyah classes ......................................................... 719
   3.3 The two Todd classes .......................................................... 722
   3.4 Applications ........................................................................... 723
      3.4.1 Integrable distributions ...................................................... 723
      3.4.2 $\mathfrak{g}$-manifolds ......................................................... 724
A. Appendix: Contractions Over a dg Ring .................................... 725
   A.1 Homological perturbation lemma ............................................. 725
   A.2 Classical formulation of contractions ....................................... 727
   A.3 Hom spaces and tensor products of contractions ........................ 730

Introduction

In [1], Atiyah introduced a characteristic class, now known as the Atiyah class, to characterize the obstruction to the existence of holomorphic connections on a holomorphic vector bundle. Decades later, Kapranov [17] showed that the Atiyah class of a Kähler manifold $X$ induces an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,*}(T^1,0_X)$. Kapranov’s result was later shown to hold for all complex manifolds [21, 22]. The Atiyah class plays an important role in the construction of Rozansky–Witten invariants [17, 18]. In addition to Rozansky–Witten theory, Kontsevich [19] brought to light a deep relation between the Todd class of complex manifolds and the Duflo element of Lie algebras. See [23, 25] for a unified framework for deriving the Duflo–Kontsevich isomorphism for Lie algebras and Kontsevich’s isomorphism for complex manifolds [19].

The works of Kapranov [17] and Kontsevich [19] have led to many new developments in the theory of Atiyah classes. For instance, see [5, 7, 10, 16, 22, 33, 36]. In the present paper, we are particularly interested in Chen–Stiénon–Xu’s approach via Lie pairs [7] and Mehta–Stiénon–Xu’s approach via dg Lie algebroids [33]. By a Lie pair $(L, A)$, we mean a pair consisting of a Lie algebroid $L$ and a Lie subalgebroid $A$ of $L$ over a common base manifold $M$. In [7], Chen, Stiénon and Xu introduced the Atiyah class of a Lie pair $(L, A)$, which captures the obstruction to the existence of compatible $L$-connections on $L/A$ extending the Bott $A$-connection. Chen–Stiénon–Xu’s theory includes the Atiyah class of complex manifolds and the Molino class [35] of foliations as special cases. In a different direction, Mehta, Stiénon and Xu [33] introduced the Atiyah class of a dg vector bundle $E$ relative to a dg Lie algebroid $L$, which measures the obstruction to the existence of $L$-connections on $E$ which are compatible with the dg structure.

In fact, Mehta–Stiénon–Xu’s approach [33] is more general than Chen–Stiénon–Xu’s approach [7]. In [2], Batakidis and Voglaire constructed a dg manifold structure on $L[1] \oplus L/A$ — which was independently constructed by Stiénon and Xu [38] — by Fedosov’s iteration method and they proved that, for a matched pair of Lie algebroids, the Atiyah class of the Fedosov dg Lie algebroid $\mathcal{F} \to L[1] \oplus L/A$ can be identified with the Atiyah class of the Lie pair $(L, A)$. In [26, Section 1.7], Stiénon, Xu and the author obtained an analogous identification for arbitrary Lie pairs. In [8], Chen, Xiang and Xu constructed different quasi-isomorphisms for the Lie pairs $(T_M, F)$ arising from integrable distributions. They proved that the Atiyah class of the dg manifold $F[1]$ associated with a foliation corresponds to the Atiyah class of the Lie pair $(T_M, F)$ under a natural quasi-isomorphism. In the present paper, we prove a theorem analogous to Chen–Xiang–Xu’s theorem in full generality. Namely, for an arbitrary Lie pair $(L, A)$ over a manifold $M$, we investigate the pullback dg Lie algebroid $\mathcal{L} = \pi^1 L \to A[1]$ of
Atiyah Classes and Todd Classes

\( L \to M \) along the projection \( \pi : A[1] \to M \) whose associated cohomology \( H^\bullet(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L}), \mathcal{Q}) \) is isomorphic to the Chevalley–Eilenberg cohomology \( H_{CE}^\bullet(A, (L/A)^\vee \otimes \text{End}(L/A)) \), and we prove that the Atiyah class of the dg Lie algebroid \( \mathcal{L} \) is identified with the Atiyah class of the Lie pair \( (L, A) \) under this isomorphism. The dg Lie algebroid \( \mathcal{L} \to A[1] \) we consider here is much simpler than the Fedosov dg Lie algebroid \( \mathcal{F} \to L[1] \oplus L/A \).

Let \( (L, A) \) be a Lie pair, and \( B \) be the quotient vector bundle \( B = L/A \). In [37], Stiénon, Vitagliano and Xu studied the pullback Lie algebroid \( \mathcal{L} = \pi^! L \to A[1] \) and proved that \( \mathcal{L} \) is a dg Lie algebroid over the dg manifold \( (A[1], d_A) \), where \( d_A \) is the Chevalley–Eilenberg differential of the Lie algebroid \( A \). See [29, Section 4.2] for the definition of pullback Lie algebroids. Furthermore, by choosing a splitting of the short exact sequence of vector bundles

\[
0 \longrightarrow A \begin{array}{c} k \end{array} \begin{array}{c} i_A \end{array} \longrightarrow L \begin{array}{c} k \end{array} \begin{array}{c} p_A \end{array} \longrightarrow B \longrightarrow 0, \tag{1}
\]

Stiénon, Vitagliano and Xu proved the following

**Theorem A.** By choosing a splitting (1), one obtains the contraction data

\[
(\Gamma(\pi^* B), d^{Bott}) \xrightarrow{\cong} (\Gamma(\mathcal{L}), \mathcal{Q}) \tag{2}
\]

over the dg ring \((\Gamma(\Lambda^* A^\vee), d_A)\), where \( d^{Bott} \) is the Bott differential, and \( \mathcal{Q} \) is induced by the dg Lie algebroid structure of \( \mathcal{L} \).

Stiénon–Vitagliano–Xu’s method is computational and heavily based on explicit formulas. Here, we give a more conceptual proof of Theorem A relying on the homological perturbation lemma (Theorem A.3). It is immediate that a splitting of a short exact sequence of vector spaces (see (11)) induces a contraction data (see (12)). Applying this vector space construction fiberwisely to \( \mathcal{L} = T_{A[1]} \times_{TM} L \cong \pi^* A[1] \oplus \pi^* L \), from a splitting (1), we obtain a contraction data

\[
(\Gamma(\pi^* B), 0) \xrightarrow{i_B} \begin{array}{c} \cong \end{array} (\Gamma(\mathcal{L}), i_A) \xrightarrow{\cong} (\Gamma(\mathcal{L}), \tilde{i_A}) \tag{3}
\]

over \( \Gamma(\Lambda^* A^\vee) \). Then we perturb (3) by \( \mathcal{Q} - \tilde{i_A} \) and prove that the perturbed contraction coincides with Stiénon–Vitagliano–Xu’s contraction (2). See Proposition 2.2 and Theorem 2.7 for details.

In order to state our main theorem, we briefly review the Atiyah class and the Todd class of a Lie pair and of a dg Lie algebroid. Let \( (L, A) \) be a Lie pair, and \( \nabla \) be an \( L \)-connection on \( B = L/A \) extending the Bott connection. The curvature of \( \nabla \) induces a Chevalley–Eilenberg cocycle \( R^\nabla_{1,1} \in \Gamma(A^\vee \otimes B^\vee \otimes \text{End} B) \). The Atiyah class of the Lie pair \( (L, A) \) is the cohomology class \( \alpha_{L/A} = [R^\nabla_{1,1}] \in H^1_{CE}(A, B^\vee \otimes \text{End} B) \), which is independent of the choice of \( L \)-connection \( \nabla \). The Todd class of the Lie pair \( (L, A) \) is the cohomology class

\[
\text{Td}_{L/A} = \det \left( \frac{\alpha_{L/A}}{1 - e^{-\alpha_{L/A}}} \right) \in \bigoplus_{k=0}^{\infty} H^k_{CE}(A, \Lambda^k B^\vee). \]
Let $\mathcal{L} \to \mathcal{M}$ be a dg Lie algebroid equipped with the homological vector field $Q \in \mathfrak{X}(\mathcal{L})$, and let $\nabla$ be an $\mathcal{L}$-connection on $\mathcal{L}$. The Lie derivative $\text{At}_{\nabla} = L_Q(\nabla)$ of the connection $\nabla$ is a $Q$-cocycle in $\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L})$. Here, with a slight abuse of notation, $Q$ also represents the coboundary operator induced by the dg structure. The induced cohomology class $\alpha_L = [\text{At}_{\nabla}] \in H^1(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L}), \mathcal{Q})$ is independent of the choice of $\mathcal{L}$-connection $\nabla$ and is called the Atiyah class of the dg Lie algebroid $\mathcal{L}$. The Todd class of the dg Lie algebroid $\mathcal{L}$ is the cohomology class $\text{Td}_L = \text{Ber}(\frac{\alpha_L}{1 - e^{-\alpha_L}}) \in \prod_{k=0}^{\infty} H^k(\Gamma(\Lambda^k \mathcal{L}^\vee), \mathcal{Q})$, where $\text{Ber}$ denotes the Berezinian.

According to general algebraic constructions (Section A.3), the contraction (2) induces the contraction data

$$\left(\Gamma(\pi^*(B^\vee \otimes \text{End} B)), d\text{Bott}\right) \xrightarrow{\frac{T_1}{\Pi_2}} \left(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L}), \mathcal{Q}\right) \xhookrightarrow{\subset} H_2$$

and

$$\left(\Gamma(\pi^*(\Lambda B^\vee)), d\text{Bott}\right) \xrightarrow{T_\Lambda}{\Pi_\Lambda} \left(\Gamma(\Lambda \mathcal{L}^\vee), \mathcal{Q}\right) \xhookrightarrow{\subset} H_\Lambda,$$

where $\mathcal{L}$ is the pullback dg Lie algebroid $\pi^! L \to A[1]$. In particular, the projection maps $\Pi_2$ and $\Pi_\Lambda$ are quasi-isomorphisms. Our main theorem is the following

**Theorem B.** Given any Lie pair $(L, A)$, the isomorphisms

$$(\Pi_2)_* : H^1(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L}), \mathcal{Q}) \xrightarrow{\cong} H^1_{\text{CE}}(A, B^\vee \otimes \text{End} B),$$

$$(\Pi_\Lambda)_* : \prod_{k=0}^{\infty} H^k(\Gamma(\Lambda^k \mathcal{L}^\vee), \mathcal{Q}) \xrightarrow{\cong} \bigoplus_{k=0}^{\infty} H^k_{\text{CE}}(A, \Lambda^k B^\vee)$$

send the Atiyah class and the Todd class of the dg Lie algebroid $\mathcal{L} = \pi^! L$ to the Atiyah class and the Todd class of the Lie pair $(L, A)$, respectively:

$$(\Pi_2)_*(\alpha_L) = \alpha_{L/A},$$

$$(\Pi_\Lambda)_*(\text{Td}_L) = \text{Td}_{L/A}.$$

Let $L = T_M$ be the tangent bundle of a manifold $M$, and let $A = F \subset T_M$ be a Lie subalgebroid whose sections form an integrable distribution. The pullback dg Lie algebroid $\pi^! T_M$ can be identified with the dg Lie algebroid $T_{F[1]}$ of tangent bundle which is equipped with the Lie derivative $L_{dF}$ with respect to the Chevalley–Eilenberg differential $dF$ of $F$. In this case, the Atiyah class and the Todd class of the dg Lie algebroid $T_{F[1]}$ are exactly the Atiyah class and the Todd class of the dg manifold $F[1]$, respectively, and thus we recover Chen–Xiang–Xu’s theorems in [8] by Theorem B.
Notations and conventions  We fix a base field \( \mathbb{k} = \mathbb{R} \) or \( \mathbb{C} \) in this paper. The notation \( C^\infty (M) = C^\infty (M, \mathbb{k}) \) refers to the algebra of smooth functions on a manifold \( M \) valued in \( \mathbb{k} \), and \( T_M \) refers to \( T_M \otimes \mathbb{R} \mathbb{k} \) unless stated otherwise.

In this paper, graded means \( \mathbb{Z} \)-graded. We write \( \text{dg} \) for differential graded.

We say that a graded ring \( R \) is commutative if \( xy = (-1)^{|x||y|}yx \) for all homogeneous \( x, y \in R \).

When we use the notation \( |x| = k \), we mean \( x \) is a homogeneous element in a graded \( R \)-module \( V = \bigoplus_n V^n \) and the degree of \( x \) is \( k \), i.e. \( x \in V^k \). The notation \( V[i] \) refers to the \( R \)-module \( V \) with the shifted grading \( (V[i])^k = V^{i+k} \).

Let \( V, W, V', W' \) be graded modules over a graded ring \( R \). We denote by \( \text{Hom}_R^k (V, W) \) the space of \( R \)-linear maps from \( V \) to \( W \) of degree \( k \), and \( \text{Hom}_R (V, W) = \bigoplus_k \text{Hom}_R^k (V, W) \). Here, we say \( f : V \to W \) is \( R \)-linear of degree \( |f| \) if \( f (r \cdot x) = (-1)^{|r||f|} r \cdot f (x) \) for any homogeneous elements \( r \in R, x \in V \). Note that \( \text{Hom}_R^0 (V, W[i]) = \text{Hom}_R^i (V, W) \).

For \( f \in \text{Hom}_R^{n,l} (V, W) \) and \( g \in \text{Hom}_R^{l,k} (V', W') \), we denote by \( f \otimes g \) the map \( f \otimes g \in \text{Hom}_R^{n+l,k+l} (V \otimes V', W \otimes W') \) satisfying \( (f \otimes g)(v \otimes v') = (-1)^{|v||g|} f (v) \otimes g (v') \), \( \forall v \in V, v' \in V' \).

Let \( W \) be a graded module over a commutative graded ring \( R \). The graded exterior algebra \( \Lambda W \) generated by \( W \) over \( R \) is

\[
\Lambda W = \left( \bigoplus_{n=0}^{\infty} W \otimes_R \cdots \otimes_R W \right) \bigg/ \langle w_1 \otimes w_2 + (-1)^{|w_1||w_2|} w_2 \otimes w_1 \rangle,
\]

equipped with the product \( \wedge \) induced by the tensor product and the degree assignment

\[
|w_1 \wedge \cdots \wedge w_n| = |w_1| + \cdots + |w_n|.
\]

We denote by \( \Lambda^n W \) the image of \( W \otimes_R \cdots \otimes_R W \) in \( \Lambda W \) under the quotient map. It is well known that \( \Lambda^n W \) and \( S^n (W[-1])[|n|] \) are isomorphic as graded modules. By the symbol \( \Lambda^\cdot W \), we mean \( \Lambda^\cdot W = \bigoplus_k (\Lambda^k W)[-k] \) which is isomorphic to \( S (W[-1]) \).

We use the triple \((W, \delta; h)\) of big space, coboundary map on big space and homotopy operator to represent the contraction data

\[
(W, d) \xrightarrow{\delta} (W, \delta) \xrightarrow{h}.
\]

See Section A.2 for how one generates the whole contraction data from the triple \((W, \delta; h)\).

1. Preliminaries

1.1. Connections for a Lie algebroid  Let \( L \) be a Lie \( \mathbb{k} \)-algebroid over a smooth manifold \( M \) with the anchor map \( \rho : L \to T_M \). Let \( E \to M \) be a \( \mathbb{k} \)-vector bundle. An \( L \)-connection \( \nabla \) on \( E \) is a \( \mathbb{k} \)-bilinear map

\[
\nabla : \Gamma (L) \times \Gamma (E) \to \Gamma (E), (l, e) \mapsto \nabla_l e
\]

which satisfies the properties

\[
\nabla_{f \cdot l} e = f \cdot \nabla_l e,
\]
\[ \nabla_l (f \cdot e) = \rho(l)(f) \cdot e + f \nabla_l e, \]

for any \( l \in \Gamma(L) \), \( e \in \Gamma(E) \), and \( f \in C^\infty(M) \). A **representation** of \( L \) on \( E \) is a flat connection \( \nabla \) on \( E \), i.e. a \( L \)-connection \( \nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E) \) satisfying

\[
\nabla_{l_1} \nabla_{l_2} e - \nabla_{l_2} \nabla_{l_1} e - \nabla_{[l_1, l_2]} e = 0,
\]

for any \( l_1, l_2 \in \Gamma(L) \) and \( e \in \Gamma(E) \). A vector bundle equipped with a representation of the Lie algebroid \( L \) is called an \( L \)-**module**.

Let \( L \) be a Lie algebroid over a smooth manifold \( M \). The **Chevalley–Eilenberg differential** is the linear map

\[
d_L : \Gamma(\bigwedge^k L^\vee) \to \Gamma(\bigwedge^{k+1} L^\vee)
\]

defined by

\[
(d_L \omega)(l_0, \cdots, l_k) = \sum_{i=0}^k (-1)^i \rho(l_i)(\omega(l_0, \cdots, \hat{l}_i, \cdots, l_k)) \\
+ \sum_{i < j} (-1)^{i+j} \omega([l_i, l_j], l_0, \cdots, \hat{l}_i, \cdots, \hat{l}_j, \cdots, l_k)
\]

which makes the exterior algebra \( \bigoplus_{k=0}^\infty \Gamma(\bigwedge^k L^\vee) \) into a commutative dg algebra. Given an \( L \)-connection \( \nabla \) on a vector bundle \( E \to M \), the **covariant derivative** is the operator

\[
d_L^\nabla : \Gamma(\bigwedge^k L^\vee \otimes E) \to \Gamma(\bigwedge^{k+1} L^\vee \otimes E)
\]

which maps a section \( \omega \otimes e \in \Gamma(\bigwedge^k L^\vee \otimes E) \) to

\[
d_L^\nabla(\omega \otimes e) = d_L(\omega) \otimes e + \sum_{i=1}^{rk L} (v_i \wedge \omega) \otimes \nabla_{v_i} e,
\]

where \( v_1, \cdots, v_{rk L} \) is any local frame of the vector bundle \( L \), and \( v_1, \cdots, v_{rk L} \) is its dual frame. The flatness of connection \( \nabla \) is equivalent to that the covariant derivative \( d_L^\nabla \) is a coboundary map: \( d_L^\nabla \circ d_L^\nabla = 0 \).

**Example 1.1.** Let \( (L, A) \) be a **Lie pair**, i.e. a pair of a Lie algebroid \( L \to M \) and a Lie subalgebroid \( A \to M \) of \( L \). The **Bott connection** of \( A \) on the quotient bundle \( B = L/A \) is the flat connection

\[
\nabla_{Bott} : \Gamma(A) \times \Gamma(B) \to \Gamma(B)
\]

defined by

\[
\nabla_{Bott}^a (p_B(l)) = p_B([a, l]), \quad \forall a \in \Gamma(A), \ l \in \Gamma(L),
\]

where \( p_B : L \to B = L/A \) is the canonical projection. Its covariant derivative

\[
d_{Bott} : \Gamma(\bigwedge^* A^\vee \otimes B) \to \Gamma(\bigwedge^{*+1} A^\vee \otimes B)
\]

is called the **Bott differential**.
1.2. Atiyah class and Todd class of a Lie pair

A Lie pair \((L, A)\) consists of a Lie algebroid \(L\) and a Lie subalgebroid \(A\) of \(L\) over a common base manifold \(M\). The structure of Lie pairs arises from geometric problems naturally. A simple example is a pair of a Lie algebra and its Lie subalgebra. Such a pair is a Lie pair over a point. Another well-known example is from complex manifolds. If \(X\) is a complex manifold, then the pair \((T_X \otimes \mathbb{C}, T_X^{0,1})\) is a Lie pair. More generally, if \(F\) is any Lie subalgebroid of the tangent bundle \(T_M\), then the pair \((T_M, F)\) forms a Lie pair. Note that \(F\) can be considered as the tangent bundle of a regular foliation. In Section 3.4, we will also consider another type of Lie pairs which arise from \(g\)-manifolds (i.e. manifolds with Lie algebra actions).

Let \((L, A)\) be a Lie pair over a manifold \(M\), and \(B\) be the quotient vector bundle \(B = L/A\). We have the short exact sequence

\[
0 \longrightarrow A \overset{i_A}{\longrightarrow} L \overset{p_B}{\longrightarrow} B \longrightarrow 0
\]

of vector bundles. An \(L\)-connection \(\nabla\) on \(B\) is said to extend the Bott connection if

\[
\nabla_{i_A(a)}(p_B(l)) = \nabla^\text{Bott}_a(p_B(l)) = p_B([i_A(a), l]),
\]

for any \(a \in \Gamma(A), l \in \Gamma(L)\).

Let \(\nabla\) be an \(L\)-connection on \(B\) extending the Bott connection. The curvature of \(\nabla\) is the bundle map \(R^\nabla : \Lambda^2 L \to \text{End} B\) defined by

\[
R^\nabla(l_1, l_2) = \nabla_{l_1} \nabla_{l_2} - \nabla_{l_2} \nabla_{l_1} - [\nabla_{\{l_1, l_2\}}], \quad \forall l_1, l_2 \in \Gamma(L).
\]

Since the Bott connection is flat, the restriction of \(R^\nabla\) to \(\Lambda^2 A\) vanishes. Thus, the curvature \(R^\nabla\) induces a bundle map \(R^\nabla_{1,1} : A \otimes B \to \text{End} B\),

\[
R^\nabla_{1,1}(a, p_B(l)) = R^\nabla(a, l) = \nabla_a \nabla_l - \nabla_l \nabla_a - [a, l], \quad \forall a \in \Gamma(A), l \in \Gamma(L).
\]

**Remark 1.2.** An \(L\)-connection \(\nabla\) on a vector bundle \(E\) induces an \(L\)-connection on the vector bundle \(E^\nabla \otimes \text{End}(E) \cong \text{Hom}(E \otimes E, E)\) via the equation

\[
\nabla_l(\alpha(e_1 \otimes e_2)) = (\nabla_l(\alpha))(e_1 \otimes e_2) + \alpha((\nabla_l e_1) \otimes e_2) + \alpha(e_1 \otimes (\nabla_l e_2)),
\]

for any \(l \in \Gamma(L), e_1, e_2 \in \Gamma(E),\) and \(\alpha \in \Gamma(E^\nabla \otimes \text{End} E)\). If the given \(L\)-connection on \(E\) is flat, then the induced connection on \(E^\nabla \otimes \text{End} E\) is also flat.

**Proposition 1.3.** ([7]) The section \(R^\nabla_{1,1} \in \Gamma(A^\nabla \otimes B^\nabla \otimes \text{End} B)\) is a \(1\)-cocycle for the Lie algebroid \(A\) with values in the \(A\)-module \(B^\nabla \otimes \text{End} B\). Furthermore, the cohomology class \(\alpha_{L/A} \in H^1_{\text{CE}}(A, B^\nabla \otimes \text{End} B)\) of \(R^\nabla_{1,1}\) is independent of the choice of \(L\)-connection \(\nabla\) extending the Bott connection.

The cocycle \(R^\nabla_{1,1}\) is called the **Atiyah cocycle** of the Lie pair \((L, A)\) associated with the \(L\)-connection \(\nabla\). The induced cohomology class \(\alpha_{L/A} \in H^1_{\text{CE}}(A, B^\nabla \otimes \text{End} B)\) is called the **Atiyah class** of the Lie pair \((L, A)\).

The **Todd cocycle** of a Lie pair \((L, A)\) associated with an \(L\)-connection \(\nabla\) extending the Bott connection is the Chevalley–Eilenberg cocycle

\[
td^\nabla_{L/A} = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0}^{\infty} \Gamma(\Lambda^k A^\nabla \otimes \Lambda^k B^\nabla).
\]
The Todd class of a Lie pair \((L, A)\) is

\[
Td_{L/A} = \det \left( \frac{\alpha_{L/A}}{1 - e^{-\alpha_{L/A}}} \right) \in \bigoplus_{k=0}^\infty H^k_{CE}(A, A^k B^\vee).
\]

In the case of the Lie pair \((L, A) = (T_X \otimes \mathbb{C}, T^0_X)\) associated with a complex manifold \(X\), the Atiyah class and the Todd class of the Lie pair are, respectively, the classical Atiyah class of \(T_X\) and the classical Todd class of the complex manifold \(X\).

1.3. Atiyah class and Todd class of a dg Lie algebroid

A \((\mathbb{Z})\)-graded manifold \(M\) is a pair \((M, \mathcal{O}_M)\), where \(M\) is a smooth manifold, and \(\mathcal{O}_M\) is a sheaf of \(\mathbb{Z}\)-graded commutative \(\mathcal{O}_M\)-algebras over \(M\) such that there exist (i) a \(\mathbb{Z}\)-graded vector space \(V\), (ii) an open cover of \(M\), and (iii) an isomorphism of sheaves of graded \(\mathcal{O}_U\)-algebras \(\mathcal{O}_M|_U \cong \mathcal{O}_U \otimes S^V\) for every open set \(U\) of the cover. A \textbf{dg manifold} \((M, Q)\) is a graded manifold \(M\) endowed with a homological vector field \(Q\), i.e., a derivation \(Q\) of degree +1 of \(C^\infty(M) = \mathcal{O}_M(M)\) satisfying \([Q, Q]\) = 0. A \textbf{morphism} of dg manifolds \(\phi : (M, \mathcal{O}_M, Q) \to (M', \mathcal{O}_{M'}, Q')\) is a pair \(\phi = (\phi, \Phi)\), where \(\phi : M \to M'\) is a smooth map, and \(\Phi : \mathcal{O}_{M'} \to \phi^*_\mathcal{O}_M\) is a morphism of sheaves of graded \(\mathcal{O}_{M'}\)-algebras, such that \((\phi, \Phi)Q) \circ \Phi = \Phi \circ Q'.\) One also has the notion of morphisms of graded manifolds by regarding graded manifolds as dg manifolds with zero homological vector fields. See, for example, [4, 6, 31].

\begin{example}
Let \(A \to M\) be a vector bundle. Then \(A[1]\) is a graded manifold, and its function algebra is \(C^\infty(A[1]) \cong \Gamma(\Lambda^\bullet A^\vee)\). If \(A \to M\) is a Lie algebroid, then \(A[1]\), together with the Chevalley–Eilenberg differential \(d_A\), forms a dg manifold. According to Vaintrob [39], there is a bijection between the Lie algebroid structures on the vector bundle \(A \to M\) and the homological vector fields on the \(\mathbb{Z}\)-graded manifold \(A[1]\).
\end{example}

A \textbf{(graded) vector bundle} of rank \(\{k_i\}\) over a graded manifold \(\mathcal{M}\) is a graded manifold \(\mathcal{E}\) and a surjection \(\pi = (\pi, \Pi) : \mathcal{E} \to \mathcal{M}\) (i.e., \(\pi : \mathcal{E} \to M\) is surjective and \(\Pi : \mathcal{O}_M \to \pi^*_\mathcal{O}_\mathcal{E}\) is injective) equipped with an atlas of local trivializations \(\mathcal{E}|_{\pi^{-1}(U)} \cong \mathcal{M}|_U \times (\bigoplus_i \mathbb{R}^{k_i}[-i])\) such that the transition map between any two local trivializations is linear in the fiber coordinates. Given a graded vector bundle \(\pi : \mathcal{E} \to \mathcal{M}\), one can shift the degrees of fibers and obtain another graded vector bundle \(\pi[j] : \mathcal{E}[j] \to \mathcal{M}\). We will denote this degree-shifting functor by \([j]_\mathcal{M}\) when the base graded manifold is ambiguous. The section space \(\Gamma(\mathcal{E})\) of \(\pi : \mathcal{E} \to \mathcal{M}\) is defined to be \(\bigoplus_{j \in \mathbb{Z}} \Gamma^j(\mathcal{E})\), where \(\Gamma^j(\mathcal{M})\) consists of morphisms \(s : \mathcal{M} \to \mathcal{E}[j]\) such that \([\pi[j]] \circ s = \text{id}\_\mathcal{M}\).

A graded vector bundle \(\pi : \mathcal{E} \to \mathcal{M}\) is called a \textbf{dg vector bundle} if \(\mathcal{E}\) and \(\mathcal{M}\) are both dg manifolds, \(\pi\) is a morphism of dg manifolds, and the dg structure is compatible with the vector bundle structure in the following sense: the subset \(\Gamma(\mathcal{E}^\vee)\) of \(C^\infty(\mathcal{E})\), consisting of the fiberwise linear functions on \(\mathcal{E}\), remains stable under the homological vector field \(Q_\mathcal{E} \in \mathfrak{X}(\mathcal{E})\). It is well-known that the global sections of a dg vector bundle form a dg module over the dg algebra of functions on the base dg manifold. See [31–33] for further details. A more general concept of “dg fiber bundles” (also known as \(Q\)-bundles) and their relationship with gauge fields can be found in [20].

A \textbf{graded Lie algebroid} \(\mathcal{L} \to \mathcal{M}\) is a Lie algebroid object in the category of graded manifolds. More explicitly, it is a graded vector bundle \(\mathcal{L} \to \mathcal{M}\) together with a degree-zero bundle map \(\rho : \mathcal{L} \to T\mathcal{M}\) (the \textbf{anchor}) and a degree-zero Lie bracket \([-,-] : \Gamma(\mathcal{L}) \times \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})\) such that

\[
[X, fY] = \rho(X)(f) \cdot Y + (-1)^{|X||f|} f[X, Y],
\]
for any $X, Y \in \Gamma(\mathcal{L})$ and $f \in C^\infty(\mathcal{M})$. According to a well-known theorem of Văîntrob [39], the Chevalley–Eilenberg differential

$$d_L : \Gamma(\Lambda^\bullet \mathcal{L}^\vee) \to \Gamma(\Lambda^{\bullet+1} \mathcal{L}^\vee)$$

of the graded Lie algebroid $\mathcal{L} \to \mathcal{M}$ can be viewed as a homological vector field on $\mathcal{L}[1]$ so that $(\mathcal{L}[1], d_L)$ is a dg manifold.

Assume $\mathcal{L} \to \mathcal{M}$ is a dg vector bundle. Since the homological vector field $Q_L \in \mathcal{X}(\mathcal{L})$ preserves the fiberwise linear functions $\Gamma(\mathcal{L}^\vee)$ on $\mathcal{L}$, it induces a homological vector field $\tilde{Q}_L$ on $\mathcal{L}[1]$. The following definition is due to Mehta [32].

**Definition 1.5.** A **dg Lie algebroid** consists of a dg vector bundle $\mathcal{L} \to \mathcal{M}$ equipped with a pair of homological vector fields $Q_L$ and $Q_M$ on $\mathcal{L}$ and $\mathcal{M}$, respectively, and a graded Lie algebroid structure on the vector bundle $\mathcal{L} \to \mathcal{M}$ such that the Chevalley–Eilenberg differential $d_L$ of the graded Lie algebroid structure and the homological vector field $\tilde{Q}_L$ on $\mathcal{L}[1]$ induced by $Q_L$ — two derivations of the graded algebra $C^\infty(\mathcal{L}[1]) \cong \Gamma(\Lambda^\bullet \mathcal{L}^\vee)$ — commute:

$$[d_L, \tilde{Q}_L] = 0.$$

Note that, if $\mathcal{L} \to \mathcal{M}$ is a dg Lie algebroid, then the function algebra $C^\infty(\mathcal{L}[1])$ with the operators $d_L$ and $\tilde{Q}_L$ form a double complex. See [32, Section 4].

In the following, we describe a few ways to construct dg Lie algebroids.

**Example 1.6.** A fundamental example of dg Lie algebroid is the tangent bundle of a dg manifold. If $(\mathcal{M}, Q)$ is a dg manifold, then the Lie derivative $L_Q$ defines a dg structure on its tangent bundle $T\mathcal{M} \to \mathcal{M}$. This dg structure and the standard Lie bracket of vector fields form a dg Lie algebroid structure on $T\mathcal{M}$.

**Example 1.7.** For more sophisticated examples, one can consider double Lie algebroids [27,28]. According to [32] (also see [40]), from a double Lie algebroid

$$\begin{array}{ccc}
D & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & M,
\end{array}$$

one can construct two dg Lie algebroids: $D[1]_A \to B[1]$ and $D[1]_B \to A[1]$. These two dg Lie algebroids can be considered to be dual to each other.

**Example 1.8.** Let $A \to M$ be a Lie algebroid. The graded vector bundle $T_A[1]_A \to T_M[1]$ is naturally a dg Lie algebroid [32, Section 5.2]. If $A = \mathfrak{g}$ is a Lie algebra, then the double complex associated with the dg Lie algebroid $T_\mathfrak{g}[1]_\mathfrak{g} \to T_{\mathfrak{pt}}[1]$ is isomorphic to the Weil algebra $W(\mathfrak{g}) = \Lambda^\bullet \mathfrak{g}^\vee \otimes S(\mathfrak{g}^\vee[-2])$. See [32, Section 5.3]. If $A = \mathfrak{g} \ltimes M \to M$ is the action Lie algebroid, then the double complex associated with the dg Lie algebroid $T_A[1]_A \to T_M[1]$ is isomorphic to the BRST model of equivariant cohomology [32, Example 5.12].

**Example 1.9.** Let $(L, A)$ be a Lie pair over a manifold $M$, and $B = L/A$ be the quotient vector bundle. It is known [26,38] that $\mathcal{M} = L[1] \oplus B$ is a (formal) dg manifold, called a Fedosov dg manifold. Furthermore, the pullback $\mathcal{F} \to \mathcal{M}$ of $B \to M$ via the canonical projection $\mathcal{M} \to M$ is a dg Lie subalgebroid of the tangent dg Lie algebroid

$$T(\mathcal{F} \to \mathcal{M}) \cong T(\mathcal{F} \to \mathcal{M}) \oplus T(\mathcal{F} \to \mathcal{M}).$$
Proposition 1.10. Let \( \mathcal{L} \to \mathcal{M} \) be a graded Lie algebroid. A section \( s \) of degree +1 of \( \mathcal{L} \to \mathcal{M} \) satisfying \( [s, s] = 0 \) induces a dg Lie algebroid structure on \( \mathcal{L} \) with its induced differential on \( \Gamma(\mathcal{L}) \) being \( Q = [s, -] : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}) \).

A proof of Proposition 1.10 can be found in [37].

Let \( \mathcal{E} \to \mathcal{M} \) be a dg vector bundle, and let \( \mathcal{L} \to \mathcal{M} \) be a dg Lie algebroid with anchor \( \rho : \mathcal{L} \to T\mathcal{M} \). We denote both the induced differentials on \( \Gamma(\mathcal{E}) \) and \( \Gamma(\mathcal{L}) \) by \( Q \). An \( \mathcal{L} \)-connection on \( \mathcal{E} \to \mathcal{M} \) is a degree-preserving map \( \nabla : \Gamma(\mathcal{L}) \otimes \mathcal{E} \to \mathcal{E} \) defined by

\[
\nabla_{f, \lambda}(f \cdot \epsilon) = \rho(\lambda)(f) \cdot \epsilon + (-1)^{|f|} f \cdot \nabla_{\lambda} \epsilon,
\]

for \( f \in C^\infty(\mathcal{M}), \lambda \in \Gamma(\mathcal{L}), \) and \( \epsilon \in \Gamma(\mathcal{E}) \). Given a dg vector bundle \( \mathcal{E} \to \mathcal{M} \) and an \( \mathcal{L} \)-connection \( \nabla \) on it, we consider the bundle map \( \text{At}_\mathcal{E}^\nabla : \mathcal{L} \otimes \mathcal{E} \to \mathcal{E} \) defined by

\[
\text{At}_\mathcal{E}^\nabla(\lambda, \epsilon) = Q(\nabla_{\lambda} \epsilon) - \nabla_{Q(\lambda)} \epsilon - (-1)^{|\lambda|} \nabla_{\lambda}(Q(\epsilon)), \quad \forall \lambda \in \Gamma(\mathcal{L}), \epsilon \in \Gamma(\mathcal{E}).
\]

Proposition 1.11. ([33]) The bundle map \( \text{At}_\mathcal{E}^\nabla \) is a degree +1 section of \( \mathcal{L}^\vee \otimes \text{End} \mathcal{E} \) satisfying the cocycle equation: \( Q(\text{At}_\mathcal{E}^\nabla) = 0 \). The cohomology class \( \alpha_\mathcal{E} \in H^1(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{E}), Q) \) of \( \text{At}_\mathcal{E}^\nabla \) is independent of the choice of the \( \mathcal{L} \)-connection \( \nabla \).

The cocycle \( \text{At}_\mathcal{E}^\nabla \) is called the Atiyah cocycle associated with the \( \mathcal{L} \)-connection \( \nabla \).

The induced cohomology class \( \alpha_\mathcal{E} = [\text{At}_\mathcal{E}^\nabla] \in H^1(\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{E}), Q) \) is called the Atiyah class of the dg vector bundle \( \mathcal{E} \to \mathcal{M} \) relative to the dg Lie algebroid \( \mathcal{L} \to \mathcal{M} \). If \( \mathcal{E} = \mathcal{L} \), we say that \( \alpha_\mathcal{E} \) is the Atiyah class of the dg Lie algebroid \( \mathcal{L} \). If \( \mathcal{E} = \mathcal{L} = T\mathcal{M} \), we say that \( \alpha_{T\mathcal{M}} \) is the Atiyah class of the dg manifold \( \mathcal{M} \).

The Todd cocycle of a dg vector bundle \( \mathcal{E} \) associated with an \( \mathcal{L} \)-connection \( \nabla \) is the \( Q \)-cocycle

\[
\text{td}_\mathcal{E}^\nabla = \text{Ber} \left( \frac{\text{At}_\mathcal{E}^\nabla}{1 - e^{-\text{At}_\mathcal{E}^\nabla}} \right) \in \prod_{k=0}^\infty (\Gamma(\Lambda^k \mathcal{L}^\vee))^k,
\]

and the Todd class of a dg vector bundle \( \mathcal{E} \) relative to a dg Lie algebroid \( \mathcal{L} \) is

\[
\text{Td}_\mathcal{E} = \text{Ber} \left( \frac{\alpha_\mathcal{E}}{1 - e^{-\alpha_\mathcal{E}}} \right) \in \prod_{k=0}^\infty H^k(\Gamma(\Lambda^k \mathcal{L}^\vee), Q),
\]

where Ber denotes the Berezinian. It is well known that \( \text{Td}_\mathcal{E} \) can be expressed in terms of scalar Atiyah classes \( \frac{1}{k!}(\frac{1}{2\pi i})^k \text{str} \alpha^k_\mathcal{E} \in H^k(\Gamma(\Lambda^k \mathcal{L}^\vee), Q) \). Here \( \text{str} : \Lambda \mathcal{L}^\vee \otimes \text{End} \mathcal{E} \to \Lambda \mathcal{L}^\vee \) denotes the supertrace.
2. Dg Lie Algebroid Associated with a Lie Pair

Let \((L, A)\) be a Lie pair over a manifold \(M\), and let \(\pi_L : L \to M\) be the bundle projection. We denote by \(\pi : A[1] \to M\) the bundle projection of the shifted vector bundle \(A[1]\). In [37], Stiénon, Vitagliano and Xu investigated the pullback Lie algebroid \(\pi^!L\) of \(L \to M\) along \(\pi : A[1] \to M\). They proved that \(\pi^!L\) is equipped a dg Lie algebroid structure and constructed a contraction data

\[
\left( \Gamma (M, \Lambda^* A^\vee \otimes B), d^{\text{Bott}} \right) \xrightarrow{\rho_a} \left( \Gamma (A[1], \pi^!L), Q \right) \xrightarrow{\tau} \pi^!L,
\]

where \(B = L/A\), \(d^{\text{Bott}}\) is the Bott differential, and \(Q\) is induced by the dg Lie algebroid structure of \(\pi^!L\). Stiénon–Vitagliano–Xu’s method is computational and heavily based on explicit formulas. Here, we give an alternative construction of this contraction by the homological perturbation lemma.

2.1. The pullback Lie algebroid \(\pi^!L\)  

The pullback Lie algebroid (see [29, Section 4.2]) of \(L\) via \(\pi : A[1] \to M\) is the vector bundle

\[
\pi^!L := T_{A[1]} \times_T M L
\]

over \(A[1]\) with the graded Lie algebroid structure described in the next paragraph. Note that (i) the two maps for defining the fiber product are the tangent map \(\pi^* : T_{A[1]} \to T_M\) of \(\pi\) and the anchor \(\rho_L : L \to T_M\) of \(L\), (ii) \(\pi^!L\) is a vector bundle over \(A[1]\) whose bundle projection is the composition \(T_{A[1]} \times_T M L \to T_{A[1]} \to A[1]\), and (iii) a general section of \(\pi^!L \to A[1]\) is of the form

\[(X, v), \quad \forall X \in \mathcal{X}(A[1]), \ v \in \Gamma(\pi^*L), \quad (6)\]

satisfying the condition

\[
\pi_* X = \rho_L \circ v : A[1] \to T_M. \quad (7)
\]

In the equation (7), a vector field \(X\) on \(A[1]\) is regarded as a map \(X : A[1] \to T_{A[1]}\), and a section

\[
v \in \Gamma(\pi^*L) \cong C^\infty(A[1]) \otimes_{C^\infty(M)} \Gamma(L)
\]

is identified with a smooth map (not necessarily linear) \(v : A[1] \to L\) such that \(\pi_L \circ v = \pi\). Also note that the space \(\Gamma(\pi^*L)\) is generated by

\[
l \circ \pi, \quad l \in \Gamma(L),
\]

as a \(C^\infty(A[1])\)-module.

The pullback Lie algebroid \(\pi^!L\) is equipped with the anchor

\[
\rho : \pi^!L = T_{A[1]} \times_T M L \to T_{A[1]}, \ \rho(X, v) = X.
\]

The Lie bracket on \(\Gamma(\pi^!L)\) is characterized by the equation

\[
[(X, l \circ \pi), (X', l' \circ \pi)] := ([X, X'], [l, l'] \circ \pi),
\]
for $X, X' \in \mathfrak{X}(A[1])$ and $l, l' \in \Gamma(L)$. More explicitly, for $l_i, l'_j \in \Gamma(L)$ and $f_i, f'_j \in C^\infty(A[1])$, we have

$$
\left[(X, \sum_i f_i \otimes l_i \circ \pi), (X', \sum_j f'_j \otimes l'_j \circ \pi)\right] \\
= \left([X, X'], \sum_j X(f'_j) \otimes l'_j \circ \pi - \sum_i (-1)^{|X||f'_i|} X'(f_i) \otimes l_i \circ \pi + \sum_{i,j} (f_i f'_j) \otimes [l_i, l'_j] \circ \pi\right)
$$

in $\Gamma(\pi^1 L)$.

2.2. Contractions induced by splittings  By choosing a connection of $A$, one can decompose $T_A$ as the direct sum of a vertical subbundle $V \cong A \times_M A$ and a horizontal subbundle $H \cong A \times_M T_M$. Such a connection induces an isomorphism

$$
T_{A[1]} \cong A[1] \times_M (A[1] \oplus T_M).
$$

Thus, we have

$$
\pi^1 L = T_{A[1]} \times_M L \cong A[1] \times_M (A[1] \oplus L) \cong \pi^\ast(A[1]) \oplus \pi^\ast(L),
$$

where the last direct sum is a direct sum of vector bundles over $A[1]$. As a consequence, we have

$$
\Gamma(\pi^1 L) \cong \Gamma(\pi^\ast A[1]) \oplus \Gamma(\pi^\ast L)
$$

as $C^\infty(A[1])$-modules.

**Remark 2.1.** In (6), we describe a general section of $\pi^1 L \to A[1]$ by a pair $(X, v)$ of a vector field $X \in \mathfrak{X}(A[1])$ and a section $v \in \Gamma(\pi^1 L)$ which satisfies (7). By choosing a connection, one has a horizontal subbundle $H$ in $T_{A[1]}$ which is isomorphic to $\pi^\ast T_M$ via $\pi_\ast$. Let $\psi : \pi^\ast T_M \to H$ be the inverse of $\pi_\ast$. Then an element $(x, v) \in \Gamma(\pi^\ast A[1]) \oplus \Gamma(\pi^\ast L)$ corresponds to the pair $(x + \psi((\text{id} \otimes \rho_\ast)(v)), v) \in \Gamma(\pi^1 L)$, where $x \in \Gamma(\pi^\ast A[1])$ is identified with its associated vertical vector field on $A[1]$, $v \in \Gamma(\pi^\ast L) \cong C^\infty(A[1]) \otimes \Gamma(L)$, and $\text{id} \otimes \rho_L : C^\infty(A[1]) \otimes \Gamma(L) \to C^\infty(A[1]) \otimes \mathfrak{X}(M) \cong \Gamma(\pi^\ast T_M)$.

Note that if $W$ is a vector space, and if $V$ is a subspace of $W$, then the two-term complex $0 \to V \leftrightarrow W \to 0$ is homotopy equivalent to the quotient space $W/V$. A choice of splitting of

$$
0 \to V \leftrightarrow W \to W/V \to 0
$$

induces a homotopy inverse $i : W/V \to W$ of the quotient map $W \to W/V$ and a homotopy operator $p : W \to V$

![Diagram](\text{Diagram})

In the following, we apply this observation fiberwisely to the pullback bundles $\pi^\ast A \subset \pi^\ast L$ with proper degree-shifting, and we obtain a homotopy equivalence between $\Gamma(\pi^1 L)$
\[ \cong \Gamma(\pi^*A[1]) \oplus \Gamma(\pi^*L) \] with the differential induced by the inclusion map and \( \Gamma(\pi^*(L/A)) \cong \Gamma(\Lambda^\bullet A^\vee \otimes (L/A)) \) with the zero differential.

Let \( B \to M \) be the quotient vector bundle \( L/A \). Let \( i_A : A \hookrightarrow L \) be the inclusion map, and \( p_B : L \to B \) be the projection map. For simplicity, we also denote the induced inclusion \( i_A : \pi^*A \hookrightarrow \pi^*L \) and projection \( p_B : \pi^*L \to \pi^*B \) by the same notations. Let \( p_A : \pi^*L \to \pi^*A \) be a splitting of the short exact sequence

\[
0 \longrightarrow \pi^*A \overset{p_A}{\longrightarrow} \pi^*L \overset{i_A}{\longrightarrow} \pi^*B \longrightarrow 0,
\]

and let \( i_B : \pi^*B \to \pi^*L \) be the inclusion map such that \( p_B i_B = \text{id} \) and \( i_A p_A + i_B p_B = \text{id} \).

Let \( \tilde{i}_A : \Gamma(\pi^1L) \to \Gamma(\pi^1L) \) be the \( C^\infty(A[1]) \)-linear operator

\[ \tilde{i}_A(x, v) := (0, i_A(s(x))) \]

of degree +1, and let \( \tilde{p}_A : \Gamma(\pi^1L) \to \Gamma(\pi^1L) \) be the \( C^\infty(A[1]) \)-linear operator

\[ \tilde{p}_A(x, v) := (s(p_A(v)), 0) \]

of degree -1, where \((x, v) \in \Gamma(\pi^*A[1]) \oplus \Gamma(\pi^*L) \cong \Gamma(\pi^1L), s : \Gamma(\pi^*A[1]) \to \Gamma(\pi^*A) \) is the degree-shifting map of degree +1, and \( s : \Gamma(\pi^*A) \to \Gamma(\pi^*A[1]) \) is the degree-shifting map of degree -1. Since \( |\tilde{i}_A| = 1 \) and \( |\tilde{p}_A| = -1 \), one has

\[ \tilde{i}_A(f \cdot \lambda) = (-1)^{|f|} f \cdot \tilde{i}_A(\lambda), \quad \tilde{p}_A(f \cdot \lambda) = (-1)^{|f|} f \cdot \tilde{p}_A(\lambda), \]

for \( f \in C^\infty(A[1]) \) and \( \lambda \in \Gamma(\pi^1L) \). Also, note that the pairs \((\Gamma(\pi^1L), \tilde{i}_A) \) and \((\Gamma(\pi^*B), 0) \) are dg modules over \((C^\infty(A[1]), 0)\), and the projection map \( p_B : (\Gamma(\pi^1L), \tilde{i}_A) \to (\Gamma(\pi^*B), 0) \) forms a homotopy equivalence with the homotopy inverse \( i_B \) and the homotopy operator \( \tilde{p}_A \).

The following proposition is straightforward.

**Proposition 2.2.** The diagram

\[
\begin{array}{ccc}
\Gamma(\pi^*A[1]) & \xrightarrow{i_A} & \Gamma(\pi^*L) \\
\downarrow & & \downarrow p_B \\
0 & \xrightarrow{i_B} & \Gamma(\pi^*B) \\
\end{array}
\]

forms a contraction data over \((C^\infty(A[1]), 0)\).
2.3. Dg Lie algebroid structures on $\pi^1 L$  
Let $d_A : C^\infty(A[1]) \to C^\infty(A[1])$ be the Chevalley–Eilenberg differential. The following lemma was proved in [37].

**Lemma 2.3.** ([37]) If $\phi : A \to L$ is a Lie algebroid morphism, then the pair

$$s_\phi := (d_A, \tilde{\phi})$$

defines a section of $\pi^1 L \to A[1]$, following the description of sections as given in (6), where $\tilde{\phi} := \phi \circ s : A[1] \to A \to L$. This section satisfies the property

$$[s_\phi, s_\phi] = 0.$$  

In particular, the inclusion map $i_A : A \hookrightarrow L$ induces a section $s_{i_A} \in \Gamma(\pi^1 L)$ with the property $[s_{i_A}, s_{i_A}] = 0$. We denote

$$Q := [s_{i_A}, -] : \Gamma(\pi^1 L) \to \Gamma(\pi^1 L)$$

which is an operator of degree +1 such that $Q^2 = 0$. By Proposition 1.10, we have

**Proposition 2.4** ([37]). The pullback Lie algebroid $\pi^1 L$ is a dg Lie algebroid over the dg manifold $(A[1], d_A)$ whose global sections are equipped with the differential $Q$.

**Remark 2.5.** (Local formulas) Here, we choose a trivialization (10) and a splitting (13). Let $x^1, \ldots, x^n$ be a local coordinate system on $M$, and let $e_1, \ldots, e_r$ be a local frame of $A \to M$ that extends to a local frame $e_1, \ldots, e_{r+r'}$ of $L \to M$ so that $e_j = i_A p_b(e_j)$ for $j = r+1, \ldots, r+r'$. We also denote the induced local frame of $\pi^*L$ by the same notations, $e_1, \ldots, e_{r+r'}$.

In addition, let $\eta^1, \ldots, \eta^r$ be the corresponding local frame of $(A[1])^\vee \to M$, and let $\frac{\partial}{\partial \eta^r}, \ldots, \frac{\partial}{\partial \eta^r} \in \Gamma(T_A^{\text{vertical}}) \cong \Gamma(\pi^*A[1])$ be the corresponding local vertical vector fields. More explicitly, we choose $\eta^j$ and $\frac{\partial}{\partial \eta^j}$ so that

$$\frac{\partial}{\partial \eta^j}(\eta^i) = \delta^i_j, \quad \text{and} \quad \tilde{\rho}_A(e_j) = \frac{\partial}{\partial \eta^j},$$

for $i, j = 1, \ldots, r$.

Let $\rho_i = \rho_L(e_i) = \sum_{j=1}^{r} \rho_i^j(x) \frac{\partial}{\partial x^j}$ and $[e_i, e_j] = \sum_{k=1}^{r+r'} c_{ij}^k(x)e_k$. We have the local formula for the Chevalley–Eilenberg differential:

$$d_A = \sum_{i=1}^{r} \sum_{j=1}^{r} \rho_i^j \eta^i \frac{\partial}{\partial x^j} - \frac{1}{2} \sum_{i,j,k=1}^{r} c_{ij}^k \eta^i \eta^j \frac{\partial}{\partial \eta^k}, \quad (14)$$

where $\frac{\partial}{\partial x^j}$ are regarded as horizontal vector fields on $A[1]$ via (8). Furthermore, in $\Gamma(\pi^*A[1]) \oplus \Gamma(\pi^*L)$, we have

$$Q\left(\frac{\partial}{\partial \eta^j}\right) = \sum_{i,k=1}^{r} c_{ij}^k \eta^i \frac{\partial}{\partial \eta^k} + e_i, \quad (15)$$

$$Q(e_i) = \frac{1}{2} \sum_{i,j,k=1}^{r} \rho_i(c_{ij}^k) \eta^i \eta^j \frac{\partial}{\partial \eta^k} + \sum_{k=1}^{r+r'} \sum_{i=1}^{r} \eta^i c_{ij}^k e_k. \quad (16)$$
According to Proposition 2.2, we have the contraction $(\Gamma(\pi^1 L), \tilde{i}_A; \tilde{p}_A)$. Let

$$\partial = Q - \tilde{i}_A,$$

and

$$F^q(\pi^1 L) = \begin{cases} 
\Gamma(\pi^1 L) \cong \Gamma(\pi^* A[1]) \oplus \Gamma(\pi^* L), & \text{if } q \leq 0, \\
\Gamma(\pi^* A[1]), & \text{if } q = 1, \\
0, & \text{if } q \geq 2.
\end{cases}$$

It is clear that $F$ is an exhaustive complete filtration.

**Lemma 2.6.** The operator $\partial$ is a perturbation of $(\Gamma(\pi^1 L), \tilde{i}_A)$ over $\mathbb{k}$, and it satisfies the property

$$(\partial \tilde{p}_A)(F^q(\pi^1 L)) \subset F^{q+1}(\pi^1 L)$$

for all $q$.

**Proof.** Since $\tilde{i}_A$ and $\tilde{p}_A$ are $C^\infty(A[1])$-linear, the derivation property of $Q$ implies that

$$(\partial \tilde{p}_A)(f \cdot \lambda) = (-1)^{|f|} d_A(f) \cdot \tilde{p}_A(\lambda) + f \cdot (\partial \tilde{p}_A)(\lambda)$$

for $f \in C^\infty(A[1]), \lambda \in \Gamma(\pi^1 L)$. Due to this algebraic property of $\partial \tilde{p}_A$, it suffices to show that $(\partial \tilde{p}_A)(\frac{\partial}{\partial \eta^l}) \in F^2(\pi^1 L)$ and $(\partial \tilde{p}_A)(e_l) \in F^1(\pi^1 L)$:

$$(\partial \tilde{p}_A)(\frac{\partial}{\partial \eta^l}) = 0 \in F^2(\pi^1 L),$$

$$(\partial \tilde{p}_A)(e_l) = \partial \left( \sum_{i,k=1}^r c^i_k \eta^i \frac{\partial}{\partial \eta^k} \right) + e_l - e_l$$

$$= \sum_{i,k=1}^r c^i_k \eta^i \frac{\partial}{\partial \eta^k} \in \Gamma(\pi^* A[1]) = F^1(\pi^1 L).$$

This completes the proof. \qed

By Theorem A.3 and Corollary A.7, we have the following

**Theorem 2.7.** The operator $\partial = Q - \tilde{i}_A$ is a small perturbation of the contraction $(\Gamma(\pi^1 L), \tilde{i}_A; \tilde{p}_A)$ over $\mathbb{k}$. The perturbed contraction

$$(\Gamma(\pi^* B), d^{\text{Bott}}) \xrightarrow{\tau \rho_B} (\Gamma(\pi^1 L), \tilde{p}_A),$$

(17)

forms a contraction data over $(C^\infty(A[1]), d_A)$. Here, the coboundary operator $d^{\text{Bott}}$ is the Bott differential, $Q = [s_{i_A}, -]$, and

$$\tau = i_b - \tilde{p}_A \partial i_b = i_b - \tilde{p}_A Q i_b : \Gamma(\pi^* B) \hookrightarrow \Gamma(\pi^1 L).$$

(18)
Proof. Observe that

\[ \tilde{p}_A \partial \tilde{p}_A = 0, \quad \partial i_b = Q i_b, \quad p_b \partial = p_b Q, \quad p_b Q \tilde{p}_A = 0. \]

Thus, by Corollary A.7, the perturbed operators are

\[
(\tilde{p}_A)_{\partial} = \sum_{k=0}^{\infty} \tilde{p}_A (- \partial \tilde{p}_A)^k = \tilde{p}_A,
\]

\[
(i_b)_{\partial} = \sum_{k=0}^{\infty} (- \tilde{p}_A \partial)^k i_b = i_b - \tilde{p}_A Q i_b,
\]

\[
(p_b)_{\partial} = \sum_{k=0}^{\infty} p_b (- \partial \tilde{p}_A)^k = p_b - p_b Q \tilde{p}_A = p_b,
\]

\[
\delta_{\partial} = \sum_{k=0}^{\infty} p_b \partial (- \tilde{p}_A \partial)^k i_b = p_b Q i_b - p_b Q \tilde{p}_A Q i_b = p_b Q i_b,
\]

where \( \delta_{\partial} : \Gamma(\pi^* B) \to \Gamma(\pi^* B) \) is the perturbed differential defined as in Definition A.6.

By \( \tilde{p}_A i_b = 0 \), it can be easily shown that the perturbed inclusion \( \tau = (i_b)_{\partial} = i_b - \tilde{p}_A Q i_b \) is \( C^\infty(A[1]) \)-linear.

Let \( \tilde{e}_l = p_b(e_l), l = r + 1, \cdots, r + r' \). Since

\[
\langle e_i, d_{Bott}(\tilde{e}_l) \rangle = \nabla_{e_l}^{Bott} \tilde{e}_l = \sum_{k=r+1}^{r+r'} c_{l}^{k} \tilde{e}_{k}, \quad \forall \ i = 1, \cdots, r, \forall \ l = r + 1, \cdots, r + r',
\]

we have

\[
d_{Bott}(\tilde{e}_l) = \sum_{i=1}^{r} \sum_{k=r+1}^{r+r'} \eta^i c_{l}^{k} \tilde{e}_{k}.
\]

Furthermore,

\[
p_b Q i_b(\tilde{e}_l) = p_b Q(e_l)
\]

\[
= p_b \left( \frac{1}{2} \sum_{s=1}^{n} \sum_{j,k=1}^{r} \rho_i(c_{s}^{j})\eta^i \eta^j \partial \partial^k + \sum_{k=1}^{r} \sum_{i=1}^{r} \eta^i c_{l}^{k} e_k \right)
\]

\[
= \sum_{k=r+1}^{r+r'} \sum_{i=1}^{r} \eta^i c_{l}^{k} \tilde{e}_{k}
\]

\[
= d_{Bott}(\tilde{e}_l),
\]

for \( l = r + 1, \cdots, r + r' \). Since both the operators \( p_b Q i_b \) and \( d_{Bott} \) satisfy the equation

\[
D(f \cdot b) = d_A(f) \cdot b + (-1)^{|f|} f \cdot D(b),
\]

for \( f \in C^\infty(A[1]), b \in \Gamma(B), D = \tilde{p}_A Q i_b \) or \( d_{Bott} \), we conclude that \( \delta_{\partial} = d_{Bott}. \)

The contraction (17) coincides with Stiénon–Vitagliano–Xu’s contraction in [37].
3. Two Atiyah Classes Associated with a Lie Pair

Let \((L, A)\) be a Lie pair over a manifold \(M\), and let \(\nabla\) be an \(L\)-connection on \(B = L/A\) extending the Bott connection. We denote by \(R^{\nabla}_{1,1} \in \Gamma(A \otimes B \otimes \text{End } B)\) the Atiyah cocycle of the Lie pair \((L, A)\) associated with the connection \(\nabla\). Let \(L\) be the dg Lie algebroid \(\pi^1 L \to A[1]\), as described in Proposition 2.4. The contraction (17) induces a contraction \((\Gamma(L \otimes \text{End } L), Q; H^1_L)\), with the projection:

\[
\Pi^1_2 : \Gamma(L \otimes \text{End } L) \to \Gamma(\pi^* (B \otimes \text{End } B)) \cong \Gamma(A^* \otimes \text{Hom}(B \otimes B, B)),
\]

\[
\Pi^1_2(\Theta) = p_{\Theta} \circ \Theta \circ (\tau \otimes \tau),
\]

where \(\tau = i_\Theta - \rho_{\pi} Q i_\Theta : \Gamma(\pi^* B) \to \Gamma(L)\), and \(L \otimes \text{End } L\) is identified with \(\text{Hom}(L \otimes L, L)\). See Proposition A.9.

Our main theorem is the following

**Theorem 3.1.** There exists an \(L\)-connection \(\nabla^L\) on \(L\) with the property:

\[
\Pi^1_2(\text{At}^L_{\nabla^L}) = R^{\nabla}_{1,1},
\]

where \(\text{At}^L_{\nabla^L}\) is the Atiyah cocycle of the dg Lie algebroid \(L\) associated with \(\nabla^L\). In particular, the isomorphism

\[
(\Pi^1_2)_* : H^1(\Gamma(L \otimes \text{End } L), Q) \cong H^1_{\text{CE}}(A, B^\vee \otimes \text{End } B)
\]

induced by \(\Pi^1_2\) sends the Atiyah class of the dg Lie algebroid \(L = \pi^1 L\) to the Atiyah class of the Lie pair \((L, A)\).

We will prove Theorem 3.1 in Section 3.2.

3.1. A \((\pi^1 L)\)-connection on \(\pi^1 L\)  

Let \(\nabla : \Gamma(L) \times \Gamma(B) \to \Gamma(B)\) be an \(L\)-connection on \(B\) extending the Bott connection. By choosing a splitting (1), we further extend \(\nabla\) to an \(L\)-connection \(\tilde{\nabla}\) on \(A[1] \oplus L\). The connection \(\tilde{\nabla} : \Gamma(L) \times \Gamma(A[1] \oplus L) \to \Gamma(A[1] \oplus L)\) chosen in this way has the property

\[
\tilde{\nabla}_l(i_\Theta(b)) = i_\Theta(\nabla_l b), \quad \forall l \in \Gamma(L), b \in \Gamma(B).
\]

By choosing a connection, we have an isomorphism \(\Gamma(\pi^1 L) \cong \Gamma(\pi^* A[1] \oplus \Gamma(\pi^* L))\). See (10). We identify the graded vector bundle \(\pi^* A[1]\) with the vertical tangent bundle \(T^\text{vertical}_{A[1]}\), which is a graded Lie subalgebroid of \(T_{A[1]}\). Let \(\nabla^{A[1]}\) be a \(T^\text{vertical}_{A[1]}\)-connection on \(\pi^1 L\), and let

\[
\nabla^L : \Gamma(\pi^1 L) \times \Gamma(\pi^1 L) \to \Gamma(\pi^1 L)
\]

be the map

\[
\nabla^L_{(x, v)} \lambda = \nabla^L_{A[1]} \lambda + \nabla^{L_\lambda},
\]

for \((x, v) \in \Gamma(\pi^* A[1]) \oplus \Gamma(\pi^* L) \cong \Gamma(\pi^1 L), \lambda \in \Gamma(\pi^1 L)\), where

\[
\nabla^L : \Gamma(\pi^* L) \times \Gamma(\pi^*(A[1] \oplus L)) \to \Gamma(\pi^*(A[1] \oplus L)),
\]

\[
\nabla^L_{f \otimes l}(g \otimes (a, l')) = (\psi(f \otimes (l' + (fg) \otimes \tilde{\nabla}_l(a, l')))) \otimes (a, l'),
\]

\[
\nabla^L_{\rho_l}(g \otimes (a, l')) = (\psi(f \otimes (l')) \otimes (a, l' + (fg) \otimes \tilde{\nabla}_l(a, l'))).
\]
Lemma 3.2. The bilinear map $\nabla^L$ is an $L$-connection on $L$ with the following properties:

$$\nabla^L_{(0, l \circ \pi)} (a \circ \pi, l' \circ \pi) = (\nabla_l(a, l')) \circ \pi,$$

$$\nabla^L_{(a \circ \pi, 0)} (a' \circ \pi, 0) = 0,$$

for $a, a' \in \Gamma(A[1])$ and $l, l' \in \Gamma(L)$, where the pairs are elements of $\Gamma(\pi^*A[1]) \oplus \Gamma(\pi^*L) \cong \Gamma(\pi^1L)$. In particular,

$$\nabla^L_{(0, l \circ \pi)} (0, i_b(b) \circ \pi) = i_b(\nabla_b),$$

for $b \in \Gamma(B)$.

Proof. Since $\psi : \Gamma(\pi^*T_M) \to \Gamma(T_{A[1]}^{\text{horizontal}})$ is $C^\infty(A[1])$-linear, we have

$$\nabla^L_{f \circ \pi}(\lambda) = f \nabla^L_{\pi}(\lambda),$$

$$\nabla^L_{\pi}(f \lambda) = (\psi(\text{id} \otimes \rho_\pi)(\pi))(f) \cdot \lambda + f \nabla^L_{\pi}(\lambda),$$

for $f \in C^\infty(A[1]), v \in \Gamma(\pi^*L) \cong C^\infty(A[1]) \otimes \Gamma(L)$ and $\lambda \in \Gamma(\pi^1L)$. Note that, by Remark 2.1, we have

$$\rho(0, v) = \psi(\text{id} \otimes \rho_\pi)(v),$$

where $\rho : \Gamma(\pi^1L) \cong \Gamma(\pi^*A[1]) \oplus \pi^*L) \to \mathfrak{X}(A[1])$ is the anchor map. Thus, for $f \in C^\infty(A[1]), v \in \Gamma(\pi^*L), \lambda \in \Gamma(\pi^1L)$ and $x \in \Gamma(\pi^*A[1]) \cong \Gamma(T_{A[1]}^{\text{vertical}})$,

$$\nabla^L_{f \circ \pi}(\lambda) = f \nabla^L_{\pi}(\lambda),$$

$$\nabla^L_{\pi}(f \lambda) = (\rho(0, v))(f) \cdot \lambda + f \nabla^L_{\pi}(\lambda),$$

$$\nabla^L_{\pi}(f \lambda) = (\rho(x, 0))(f) \cdot \lambda + (-1)^{|f|} f \nabla^L_{\pi}(\lambda),$$

where the last equation is from the fact $\nabla^L_{\pi}A[1]$ is a $T_{A[1]}^{\text{vertical}}$-connection on $\pi^1L$. Therefore, $\nabla^L$ is an $L$-connection on $L$.

The first property of $\nabla^L$ follows directly from the definition. For the second property, note that both $(a \circ \pi, 0)$ and $(a' \circ \pi, 0)$ are of degree $-1$, and thus $|\nabla^L_{(a \circ \pi, 0)}(a' \circ \pi, 0)| = -2$. Nevertheless, the degree of each homogeneous element in $\Gamma(\pi^1L)$ is at least $-1$. Thus, $\nabla^L_{(a \circ \pi, 0)}(a' \circ \pi, 0) = 0$. □

In [37, Section 3.6], Stiénon, Vitagliano and Xu independently constructed an $L$-connection on $L$ for a different purpose. One also can use their connection for Theorem 3.1.
3.2. The two Atiyah classes

By Theorem 2.7 and Proposition A.9, we have the contraction

\[
\left( \Gamma(\pi^*(B^\vee \otimes End B)), d^{Bott} \right) \xrightarrow{T^1_2}{\Pi^1_2} \left( \Gamma(L^\vee \otimes End L), \mathcal{Q} \right) \subseteq H^1_2,
\]

where

\[
H^1_2 : \Gamma(L^\vee \otimes End L) \to \Gamma(L^\vee \otimes End L),
\]

\[
H^1_2(\Theta) = \tilde{p}_A \circ \Theta + (-1)^{[\Theta]} \sigma \circ \Theta \circ (\tilde{p}_A \otimes \text{id} + \sigma \otimes \tilde{p}_A),
\]

and

\[
\sigma = (i_B - \tilde{p}_A Q i_B) p_B = \text{id} - [\mathcal{Q}, \tilde{p}_A] : \Gamma(L) \to \Gamma(L).
\]

The small space \(\Gamma(\pi^*(B^\vee \otimes End B))\) of the contraction (19) is identified with \(\Gamma(\Lambda^\bullet A^\vee \otimes B^\vee \otimes End B)\), equipped with the Bott differential \(d^{Bott}\). The projection map is defined as

\[
\Pi^1_2 : \Gamma(L^\vee \otimes End L) \to \Gamma(\pi^*(B^\vee \otimes End B)), \Theta \mapsto p_B \circ \Theta \circ (\tau \otimes \tau).
\]

Here, \(\tau = i_B - \tilde{p}_A Q i_B : \Gamma(\pi^*B) \to \Gamma(L)\), as defined in Theorem 2.7.

Let \(R^1_{1,1} \in \Gamma(\Lambda^1 A^\vee \otimes B^\vee \otimes End B) \subset \Gamma(\pi^*(B^\vee \otimes End B))\) be the Atiyah cocycle of the Lie pair \((L, A)\) associated with an \(L\)-connection \(\nabla\) extending the Bott connection, and let \(At^\nabla_L \in \Gamma(L^\vee \otimes End L)\) be the Atiyah cocycle associated with the \(L\)-connection \(\nabla_L\) constructed in Lemma 3.2. Let \(\tilde{At}^\nabla_L \in \Gamma(\pi^*(B^\vee \otimes End B))\) be the image

\[
\tilde{At}^\nabla_L = \Pi^1_2(At^\nabla_L) = p_B \circ At^\nabla_L \circ (\tau \otimes \tau)
\]

of \(At^\nabla_L\) under the projection \(\Pi^1_2\). We will prove that \(\tilde{At}^\nabla_L = R^1_{1,1}\).

**Proof of Theorem 3.1.** Following the notations in Remark 2.5, we have

\[
\tilde{p}_A Q i_B(\tilde{e}_l) = \tilde{p}_A \left( \frac{1}{2} \sum_{i,j,k=1}^r \rho_i(c^k_{ij}) \eta^i \eta^j \frac{\partial}{\partial \eta^k} + \sum_{k=1}^r \sum_{i=1}^r \eta^i c^k_{il} e_k \right)
\]

\[
= - \sum_{k=1}^r \sum_{i=1}^r \eta^i c^k_{il} \frac{\partial}{\partial \eta^k},
\]

where \(\tilde{e}_l = p_B(e_l), l = r + 1, \ldots, r + r'.\) Thus,

\[
\tau(\tilde{e}_l) = \sigma(e_l) = e_l + \sum_{k=1}^r \sum_{i=1}^r \eta^i c^k_{il} \frac{\partial}{\partial \eta^k}, \quad \forall l = r + 1, \ldots, r + r'.
\]

Let \(\Gamma^k_{ij}, \frac{\Gamma^k_{ij}}{\lambda}, \frac{\Lambda^k_{ij}}{\lambda}, \frac{\Lambda^k_{ij}}{\lambda^k_{ij}}\) be the Christoffel symbols:

\[
\nabla^L_{e_l} e_j = \sum_{k=1}^{r+r'} \Gamma^k_{ij} e_k, \quad \nabla^L_{e_l} \frac{\partial}{\partial \eta^j} = \sum_{k=1}^{r+r'} \frac{\Gamma^k_{ij}}{\lambda} \frac{\partial}{\partial \eta^k}.
\]
For \( i, j, k \), it is straightforward to show that
\[
\frac{\partial}{\partial \eta^j} \Gamma^k_{ij} = \sum_{k=1}^r \Gamma^k_{ij} \frac{\partial}{\partial \eta^j} = 0, \quad \frac{\partial}{\partial \eta^j} e_j = \sum_{k=1}^r \Gamma^k_{ij} \frac{\partial}{\partial \eta^k}.
\]

Note that since \( \nabla^\mathcal{L}_{e_j} e_j = (\tilde{\nabla} e_j) \circ \pi \in \Gamma(\pi^*L) \), we do not have \( \frac{\partial}{\partial \eta^k} \)-terms in \( \nabla^\mathcal{L}_{e_j} e_j \).

Furthermore, it follows from Lemma 3.2 that
\[
\Gamma^k_{ij} = 0, \quad \forall i, j, k, \quad (21)
\]
\[
\Gamma^k_{ij} = 0, \quad \forall j \geq r + 1, k \leq r, \quad (22)
\]
\[
\Gamma^k_{ij} = c_{ij}^k, \quad \forall i \leq r, j, k \geq r + 1. \quad (23)
\]

For \( i = r + 1, \ldots, r + r' \), we have
\[
\tilde{A}_{\mathcal{L}}(\tilde{e}_i, \tilde{e}_j) = p_b A_{\mathcal{L}}(\sigma e_i, \sigma e_j)
\]
\[
= p_b A_{\mathcal{L}}(e_i + \sum_{s, t=1}^r \eta^s c_i^t \frac{\partial}{\partial \eta^t}, e_j + \sum_{u, v=1}^r \eta^u c_{uj}^v \frac{\partial}{\partial \eta^v})
\]
\[
= \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D},
\]
where
\[
\mathcal{A} = p_b A_{\mathcal{L}}(e_i, e_j),
\]
\[
\mathcal{B} = p_b A_{\mathcal{L}}(e_i, \sum_{s, t=1}^r \eta^s c_i^t \frac{\partial}{\partial \eta^t}),
\]
\[
\mathcal{C} = p_b A_{\mathcal{L}}(\sum_{s, t=1}^r \eta^s c_i^t \frac{\partial}{\partial \eta^t}, e_j),
\]
\[
\mathcal{D} = p_b A_{\mathcal{L}}(\sum_{s, t=1}^r \eta^s c_i^t \frac{\partial}{\partial \eta^t}, \sum_{u, v=1}^r \eta^u c_{uj}^v \frac{\partial}{\partial \eta^v}).
\]

Using (16), (22) and (23), one can show that
\[
\mathcal{A} = \sum_{p=1}^r \sum_{k=r+1}^{r+r'} \eta^p \rho_p (\Gamma_{ij}^k) \tilde{e}_k + \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p \Gamma_{ij}^k c_{pq}^k \tilde{e}_k - \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p c_{pi}^q \Gamma_{ij}^k \tilde{e}_k
\]
\[
- \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p c_{pi}^q \Gamma_{ij}^k \tilde{e}_k - \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p c_{pj}^q \Gamma_{ij}^k \tilde{e}_k - \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p c_{pi}^q \Gamma_{ij}^k \tilde{e}_k - \sum_{p=1}^r \sum_{q=k+1}^{r+r'} \eta^p c_{pj}^q \Gamma_{ij}^k \tilde{e}_k.
\]

By (15), (16) and (23), it is straightforward to show that
\[
p_b A_{\mathcal{L}}(e_i, \frac{\partial}{\partial \eta^j}) = - \sum_{k=r+1}^{r+r'} \Gamma_{ij}^k \tilde{e}_k, \quad \text{and} \quad p_b A_{\mathcal{L}}(\frac{\partial}{\partial \eta^j}, e_j) = 0.
\]

Thus,
\[
\mathcal{B} = \sum_{p, q=1}^r \sum_{k=r+1}^{r+r'} \eta^p c_{pj}^q \Gamma_{ij}^k \tilde{e}_k.
\]
\[ C = 0. \]

Since the degree of \( \text{At}^\nabla_C \left( \frac{\partial}{\partial \eta^r}, \frac{\partial}{\partial \eta^v} \right) \) is \(-1\), we have

\[
\text{At}^\nabla_C \left( \frac{\partial}{\partial \eta^r}, \frac{\partial}{\partial \eta^v} \right) \in \Gamma(\pi^* A[1]) \subset \ker(\rho_v),
\]

and thus

\[ D = 0. \]

Therefore,

\[
\tilde{\text{At}}^\nabla_C (\tilde{e}_i, \tilde{e}_j) = \mathcal{A} + \mathcal{B} + C + D
\]

For \( p = 1, \cdots, r \) and \( i, j = r + 1, \cdots, r + r' \), we have

\[
R^\nabla_{1,1}(e_p, \bar{e}_i) = \nabla_{e_p} \nabla_{e_i} \bar{e}_j - \nabla_{e_i} \nabla_{e_p} \bar{e}_j - \nabla_{(e_p, e_i)} \bar{e}_j
\]

Equivalently,

\[
R^\nabla_{1,1}(\bar{e}_i, \bar{e}_j) = \sum_{p=1}^{r} \sum_{k=r+1}^{r+r'} \eta^p \rho_p (\Gamma^k_{ij}) \bar{e}_k + \sum_{p=1}^{r} \sum_{q=k+1}^{r+r'} \eta^p \Gamma^q_{ij} c^k_{pq} \bar{e}_k - \sum_{p=1}^{r} \sum_{q=1}^{r+r'} \eta^p \rho_i (c^k_{pj}) \bar{e}_k
\]

By comparing the formulas of \( \tilde{\text{At}}^\nabla_C (\tilde{e}_i, \tilde{e}_j) \) and \( R^\nabla_{1,1}(\bar{e}_i, \bar{e}_j) \), we conclude that

\[
\Pi^1_2(\text{At}^\nabla_C) = \tilde{\text{At}}^\nabla_C = R^\nabla_{1,1}.
\]

The proof of Theorem 3.1 is complete. \( \square \)
3.3. The two Todd classes  

By Corollary A.11, the contraction (17) generates the contraction data

\[
\left( \Gamma \left( \pi^* (\Lambda B^\vee \otimes \text{End} B) \right), d^{Bott} \right) \xrightarrow{\hat{T}} \left( \Gamma (\Lambda L^\vee \otimes \text{End} \Lambda L), Q \right) \xrightarrow{\hat{h}} H_\Lambda
\]

whose inclusion map \( \hat{T} \) is an algebra morphism. Furthermore, we also have the contraction data

\[
\left( \Gamma \left( \pi^* (\Lambda B^\vee) \right), d^{Bott} \right) \xrightarrow{T_\Lambda} \left( \Gamma (\Lambda L^\vee), Q \right)_{\Lambda}
\]

by Lemma A.8 and Proposition A.10.

**Lemma 3.3.** The inclusion maps \( \hat{T} \) and \( T_\Lambda \) commute with the (super)traces:

\[
\Gamma \left( \pi^* (\Lambda B^\vee \otimes \text{End} B) \right) \xrightarrow{\hat{T}} \Gamma (\Lambda L^\vee \otimes \text{End} \Lambda L) \xrightarrow{\text{tr}} \Gamma \left( \pi^* (\Lambda B^\vee) \right) \xrightarrow{T_\Lambda} \Gamma (\Lambda L^\vee) \xrightarrow{\text{str}}
\]

**Proof.** Recall that for \( \omega \in \Gamma \left( \pi^* (\Lambda B^\vee) \right) \) and \( \Phi \in \Gamma (\pi^* (\text{End} B)) \), we have \( \hat{T} (\omega \otimes \Phi) = T_\Lambda (\omega) \otimes (\tau \Phi p_\Lambda) \), where \( \tau = i_\Lambda - \tilde{\rho}_\Lambda \partial i_\Lambda \). Since \( \text{im}(\tilde{\rho}_\Lambda \partial) \subset \Gamma (\pi^* A[1]) \subset \Gamma (L) \cong \Gamma (\pi^* A[1]) \oplus \Gamma (\pi^* L) \), the matrix representation of \( \tau \Phi p_\Lambda \) is of the form

\[
\begin{pmatrix}
i_\Lambda \Phi p_\Lambda & 0 \\
-\tilde{\rho}_\Lambda \partial i_\Lambda \Phi p_\Lambda & 0
\end{pmatrix},
\]

where the first row/column represents the even component \( \Gamma (\pi^* L) \), and the second row/column represents the odd component \( \Gamma (\pi^* A[1]) \). Thus,

\[
\text{str}(\tau \Phi p_\Lambda) = \text{str}(i_\Lambda \Phi p_\Lambda) = \text{tr}(\Phi).
\]

As a result, we have

\[
\text{str} \left( \hat{T} (\omega \otimes \Phi) \right) = T_\Lambda (\omega) \text{str}(\tau \Phi p_\Lambda) = T_\Lambda (\omega) \text{tr}(\Phi) = T_\Lambda (\text{tr}(\omega \otimes \Phi)).
\]

This completes the proof. \( \Box \)

**Theorem 3.4.** The projection map \( \Pi_\Lambda : \Gamma (\Lambda L^\vee) \to \Gamma (\pi^* \Lambda B^\vee) \) induces an isomorphism

\[
(\Pi_\Lambda)_* : H^* (\Gamma (\Lambda^k L^\vee), Q) \xrightarrow{\cong} H^*_{CE}(A, \Lambda^k B^\vee)
\]

for each \( k \), and

\[
(\Pi_\Lambda)_* (\text{Td}_L) = \text{Td}_{L/A}.
\]
Proof. Note that all the operators in the contraction (24) respect to $\Lambda^k$. Thus, one can decompose (24) to contractions for $\Lambda^k$, and the first assertion follows.

Equation (25) is equivalent to

$$(T\Lambda)_*(\text{Td}_{L/A}) = \text{Td}_L.$$ 

Since the Todd classes can be expressed in terms of scalar Atiyah classes, it suffices to show that

$$(T\Lambda)_*(\text{tr} \alpha^k_{L/A}) = \text{str} \alpha^k_L,$$

for each $k$. Since $\hat{T}_*$ is an algebra isomorphism, it follows from Lemma 3.3 and Theorem 3.1 that

$$(T\Lambda)_*(\text{tr} \alpha^k_{L/A}) = \text{str} (\hat{T}_*(\alpha^k_{L/A})) = \text{str} \alpha^k_L,$$

where $(T^1_\Lambda)_*$ is induced by the contraction (19).

3.4. Applications

3.4.1. Integrable distributions

Let $L = T_M$ be the tangent bundle of a manifold $M$. Let $A = F \subset T_M$ be a Lie subalgebroid whose sections form an integrable distribution. The pullback Lie algebroid

$$\pi^! T_M = T_F[1] \times_{T_M} T_M = T_F[1]$$

can be identified with the Lie algebroid $T_F[1]$. Furthermore, the dg structure on $\Gamma(T_F[1])$ is given by $[s_i F, -]$, where $s_i F = d_F \in \Gamma(T_F[1])$ is the Chevalley–Eilenberg differential. See Proposition 2.4 and Lemma 2.3. Thus, by Theorem 2.7, we have the contraction
data

$$(\Gamma(\Lambda^\bullet F^\lor \otimes \text{End} T_F[1]), L d_F) \xrightarrow{\tau} \left(\Gamma(T_F[1]), L d_F\right) \xrightarrow{\rho_A} \right.$$ (26)

where $B = T_M / F$. This contraction (26) coincides with Chen–Xiang–Xu’s contraction in [8, Lemma 2.2].

As a consequence of Theorem 3.1, we have the following

**Corollary 3.5.** The contraction (26) induces an isomorphism

$$H^1(\Gamma(T_F[1] \otimes \text{End} T_F[1]), L d_F) \xrightarrow{\cong} H^1_{CE}(F, B^\lor \otimes \text{End} B)$$

which sends the Atiyah class of the dg manifold $F[1]$ to the Atiyah class of the Lie pair $(F, T_M)$.

Similarly, by Theorem 3.4, we have

**Corollary 3.6.** The contraction (26) induces an isomorphism

$$\prod_{k=0}^\infty H^k(\Gamma(\Lambda^k T_F[1]), L d_F) \xrightarrow{\cong} \prod_{k=0}^\infty H^k_{CE}(F, \Lambda^k B^\lor)$$

which sends the Todd class of the dg manifold $F[1]$ to the Todd class of the Lie pair $(F, T_M)$.

We recover Chen–Xiang–Xu’s theorems in [8]. In particular, the Atiyah class and the Todd class of a complex manifold $X$ can be identified with the Atiyah class and the Todd class of the dg manifold $T^{0,1}_X[1]$, respectively. See [8, Theorem C].
3.4.2. $g$-manifolds Let $g$ be a finite-dimensional Lie algebra. A $g$-manifold is a smooth manifold $M$ together with a Lie algebra action, i.e. a morphism of Lie algebras $g \ni a \mapsto \dot{a} \in \mathfrak{X}(M)$. Given a $g$-manifold $M$, the action Lie algebroid $g \times M$ and the tangent bundle $T_M$ naturally form a matched pair of Lie algebroids [34]. Thus, we have a Lie pair $(L, A)$, where $L = (g \times M) \rightharpoonup T_M$ and $A = g \times M$. See [32] for its relation with BRST complexes.

More explicitly, the vector bundle $L$ is isomorphic to $(g \times M) \oplus T_M$ as vector bundles over $M$, and the anchor $\rho_L : \Gamma(L) \to \mathfrak{X}(M)$ is given by the formula $\rho_L(a + X) = \dot{a} + X$. The bracket $[-, -] : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ is determined by

$$[a, b] = [a, b]_g, \quad [X, Y] = [X, Y]_{\mathfrak{X}(M)}, \quad [a, X] = [\dot{a}, X].$$

Here $a, b \in g$ are identified with the corresponding constant functions in $C^\infty(M, g) \cong \Gamma(A)$, $X$ and $Y$ are vector fields on $M$, $[\cdot, -]_g$ is the Lie bracket of $g$, and $[\cdot, \cdot]_{\mathfrak{X}(M)}$ is the Lie bracket of vector fields.

Let $B$ be the quotient vector bundle $B = L / A \cong T_M$. In this case, the graded vector bundle $L = \pi^*L$ admits a natural Whitney sum decomposition over $A[1] \cong g[1] \times M$:

$$L \cong \pi^*A[1] \oplus \pi^*A \oplus \pi^*B,$$

where

$$\pi^*A[1] \cong (g[1] \times M) \times g[1], \quad \pi^*A \cong (g[1] \times M) \times g \quad \text{and} \quad \pi^*B \cong g[1] \times T_M.$$

Consequently, its space of sections admits the decomposition

$$\Gamma(L) \cong (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g[1]) \oplus (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g) \oplus (\Lambda^\bullet g^\vee \otimes \mathfrak{X}(M)).$$

Now we describe the contraction $(17)$ in this situation. By $(15)$ and $(16)$, a direct computation shows that the differential $Q$ acts on $\Gamma(L)$ as follows:

$$\Gamma(L) \cong (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g[1]) \oplus (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g) \oplus (\Lambda^\bullet g^\vee \otimes \mathfrak{X}(M))$$

$$\downarrow \quad \downarrow$$

$$\Gamma(L) \cong (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g[1]) \oplus (\Lambda^\bullet g^\vee \otimes C^\infty(M) \otimes g) \oplus (\Lambda^\bullet g^\vee \otimes \mathfrak{X}(M)).$$

The homotopy operator $\tilde{\rho}_s : \Gamma(L) \to \Gamma(L)$ is linear over $C^\infty(A[1]) \cong \Lambda^\bullet g^\vee \otimes C^\infty(M)$ and is determined by its values on the three components $\Gamma(\pi^*A[1])$, $\Gamma(\pi^*A)$ and $\Gamma(\pi^*B)$. The values of $\tilde{\rho}_s$ on the components $\Gamma(\pi^*A[1])$ and $\Gamma(\pi^*B)$ vanish. On the component $\Gamma(\pi^*A)$, it is given by extending the degree-shifting map $s : g \to g[1]$ via $C^\infty(A[1])$-linearity.

For the small complex $(\Gamma(\pi^*B), d_{\text{Bott}})$, it is clear that $\Gamma(A) \cong C^\infty(M, g)$ and $\Gamma(B) \cong \mathfrak{X}(M)$. The Bott connection $\nabla_{\text{Bott}} : C^\infty(M, g) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is determined by the formula

$$\nabla_{\text{Bott}}^a X = [\hat{a}, X], \quad \forall a \in g, \ X \in \mathfrak{X}(M),$$

where an element $a \in g$ is identified with the constant function with value $a$. The complex $(\Gamma(\pi^*B), d_{\text{Bott}})$ coincides with the Chevalley–Eilenberg complex $(\Lambda^\bullet g^\vee \otimes \mathfrak{X}(M), d_{\text{CE}})$ of the $g$-action $g \to \text{End} \mathfrak{X}(M)$, $a \mapsto L_{\hat{a}}$.

According to Theorem 2.7, the projection map $p_a : \Gamma(L) \to \Gamma(\pi^*B)$ of the contraction $(17)$ is the canonical projection onto $\Gamma(\pi^*B)$. Since the subset $0 \oplus 0 \oplus \Gamma(\pi^*B) \subset$
\( \Gamma(\mathcal{L}) \) is stable under \( \mathcal{Q} \), it follows from (18) that the map \( \tau \) coincides with the canonical inclusion in the case of \( g \)-manifolds. As a consequence, the projection maps in the contractions (19) and (24) are the canonical projections.

By Theorem 3.1 and Theorem 3.4, we have the following

**Corollary 3.7.** The canonical projections

\[
\Pi_1 : (\Gamma(\mathcal{L}^\vee \otimes \text{End} \mathcal{L}), \mathcal{Q}) \to (\Lambda^\bullet g^\vee \otimes \Gamma(T^\vee_M \otimes \text{End} T_M), d_{\text{CE}})
\]

and

\[
\Pi_\Lambda : (\Gamma(\Lambda^k \mathcal{L}^\vee), \mathcal{Q}) \to (\Lambda^\bullet g^\vee \otimes \Gamma(\Lambda^k T^\vee_M), d_{\text{CE}})
\]

are quasi-isomorphisms whose induced maps on cohomologies send the Atiyah class and the Todd class of the dg Lie algebroid \( L = \pi^! L \) to the Atiyah class and the Todd class of the Lie pair \((L, A) = (\mathfrak{g} \ltimes M) \ldots T_M, \mathfrak{g} \ltimes M\) respectively.

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### A. Appendix: Contractions Over a dg Ring

A contraction is an algebraic analogue of a deformation retract. The key proposition for us is the homological perturbation lemma which is an algebraic tool for perturbing a contraction to another contraction. See, for example, [11,14,30]. In this appendix, we summarize the necessary facts about contractions and the homological perturbation lemma.

We formulate contractions by characterizing the homotopy operator. In this formulation, one needs only a complex and a homotopy operator satisfying certain conditions, while in the classical formulation [14], one needs a small complex, a projection map and an inclusion map in addition. One can find an \( L_\infty \) version of our formulation in [3, Appendix B]. In Section A.2, we describe how one can generate the additional data in the usual definition of contraction data from the homotopy operator.

**A.1. Homological perturbation lemma** Let \( R = (R, d_R) \) be a commutative dg ring.

**Definition A.1.** A contraction over \( R \) is a triple \((W, \delta; h)\), where \((W, \delta)\) is a dg module over \( R \), and \( h \) is an \( R \)-linear operator \( h : W^p \to W^{p-1} \) of degree \(-1\) such that

\[
h^2 = 0 \quad \text{and} \quad h\delta h = h.
\]

The operator \( h \) is referred to as the homotopy operator of the contraction \((W, \delta; h)\). A perturbation \( \partial \) of \((W, \delta)\) over \( R \) is an \( R \)-linear operator \( \partial : W^p \to W^{p+1} \) of degree \(+1\) such that

\[
(\delta + \partial)^2 = 0.
\]
Note that if \( \partial \) is a perturbation of a dg module \((W, \delta)\) over \(R\), then \((W, \delta + \partial)\) is also a dg module over \(R\).

**Definition A.2.** We say a perturbation \( \partial \) is a **small perturbation** of a contraction \((W, \delta; h)\) if there exists a descending exhaustive (i.e. \( \cup q F^q(W) = W \)) complete (i.e. \( W = \lim_{\leftarrow q} W/F^q W \)) filtration

\[
F = \cdots \supset F^q W \supset F^{q+1} W \supset \cdots
\]

of the space \(W\) (*not* necessarily compatible with \(\delta\)) such that

\[
(\partial h)(F^q W) \subset F^{q+1} W, \quad \forall q.
\]

The following theorem is well-known.

**Theorem A.3** (Homological perturbation lemma). Let \( \partial \) be a small perturbation of a contraction \((W, \delta; h)\) over \(R\). Then (i) the operators \(\text{id} + h\partial\) and \(\text{id} + \partial h\) are invertible and their inverses are the convergent series

\[
(id + h\partial)^{-1} = \sum_{k=0}^{\infty} (-h\partial)^k \quad \text{and} \quad (id + \partial h)^{-1} = \sum_{k=0}^{\infty} (-\partial h)^k;
\]

(ii) the operator

\[
h_\partial := (id + h\partial)^{-1} h = h(id + \partial h)^{-1}
\]

(27)

is well-defined; and (iii) the triple \((W, \delta + \partial; h_\partial)\) forms a contraction over \(R\).

The contraction \((W, \delta + \partial; h_\partial)\) is referred to as the **perturbed contraction**.

**Proof.** The first two assertions are immediate. It follows from the equation (27) and \(h^2 = 0\) that

\[
h_\partial h_\partial = (id + h\partial)^{-1} hh(id + \partial h)^{-1} = 0.
\]

From \(h\delta h = h\), we obtain

\[
h(\delta + \partial)h = h + h\partial h = h(id + \partial h).
\]

It follows from the equation (27) that

\[
h_\partial(\delta + \partial)h_\partial = (id + h\partial)^{-1} h(\delta + \partial)h(id + \partial h)^{-1} = (id + h\partial)^{-1} h = h_\partial.
\]
A.2. Classical formulation of contractions  

Let \((V, d)\) and \((W, \delta)\) be two dg modules over a commutative dg ring \(R\). A contraction data is the data

\[(V, d) \xrightarrow{\tau}_\sigma (W, \delta) \xrightarrow{h} \]

where \(\tau : V \to W\) is an injective \(R\)-linear cochain map, \(\sigma : W \to V\) is a surjective \(R\)-linear cochain map, \(h : W \to W\) is an \(R\)-linear map of degree \(-1\), and

\[\sigma \tau = \text{id}_V, \quad \text{id}_W - \tau \sigma = h\delta + \delta h, \quad \sigma h = 0, \quad h\tau = 0, \quad hh = 0.\]

The space \(V\) is referred to as the small space of the contraction data. The maps \(\tau\) and \(\sigma\) will be referred to respectively as the injection (or the inclusion map) and the surjection (or the projection map) of the contraction. We refer the reader to [30] for the basic properties of contraction data.

Let \((W, \delta; h)\) be a contraction in the sense of Definition A.1. Since \(h\delta h = h, hh = 0,\) and \(\delta\delta = 0\), the operators \(\delta h, h\delta\) and \([\delta, h]\) are projection operators. Consider the subspace

\[V := \text{im}(\text{id} - [\delta, h]) = \ker[\delta, h] = \ker(\delta h) \cap \ker(h\delta)\]

of \(W\), let \(\tau : V \hookrightarrow W\) denote the inclusion of \(V\) into \(W\), and let \(\sigma : W \to V\) the surjection induced by the projection operator \(\varpi := \text{id} - [\delta, h]\). Note that, since \([\delta, h]\) is \(R\)-linear, \(V\) is an \(R\)-module. Furthermore, since \(\delta^2 = 0\), we have

\[\varpi \delta = (\text{id} - h\delta - \delta h)\delta = \delta - \delta h\delta = \delta (\text{id} - h\delta - \delta h) = \delta \varpi,\]

which shows that \(V = \text{im}(\varpi)\) is a subcomplex of \((W, \delta)\). The next lemma follows from a direct computation.

**Lemma A.4.** The data

\[(V, \delta|_V) \xrightarrow{\tau}_\sigma (W, \delta) \xrightarrow{h} \]

induced by \((W, \delta; h)\) forms a contraction data.

Let \(F\) be an exhaustive complete filtration on \(W\), and \(\partial\) be a perturbation of \((W, \delta)\) which satisfies the assumptions of Theorem A.3. Consequently, the operators \(\text{id} + \partial h\) and \(\text{id} + h\partial\) are invertible and we have a perturbed contraction \((W, \delta + \partial; h_\partial)\) with induced subcomplex

\[V_\partial := \text{im}(\text{id}_W - [\delta + \partial, h_\partial]) = \ker([\delta + \partial, h_\partial]) = \ker((\delta + \partial)h_\partial) \cap \ker(h_\partial(\delta + \partial)),\]

the image of the projection operator \(\varpi_\partial = \text{id} - [\delta + \partial, h_\partial]\).

Hence, we obtain the contraction data

\[(V_\partial, (\delta + \partial)|_{V_\partial}) \xrightarrow{\psi}_\varphi (W, \delta + \partial) \xrightarrow{h_\partial} \]

where \(\psi\) denotes the inclusion of \(V_\partial = \text{im}(\varpi_\partial)\) into \(W\) and \(\varphi : W \to V_\partial\) is the surjection induced by the projection operator \(\varpi_\partial\).
We note that
\[ (\text{id} + \partial h)^{-1} = \sum_{k=0}^{\infty} (-\partial h)^k = \text{id} - \partial h \delta \] (29)
and
\[ (\text{id} + h \partial)^{-1} = \sum_{k=0}^{\infty} (-h \partial)^k = \text{id} - h \delta. \] (30)

**Lemma A.5.** The following diagram is commutative:

\[
\begin{array}{ccc}
W & \xrightarrow{\sigma = \text{id} - h \delta - \delta h} & W \\
& \sigma_\partial = \text{id} - h \delta - (\partial + \delta) h \delta & \\
(id + h \partial)^{-1} & \xrightarrow{\text{id} - h \delta} & W \\
& \xleftarrow{\text{id} - (\partial + \delta) h \delta} & \\
W & \xleftarrow{\text{id} - h \delta} & W \\
& \sigma = \text{id} - h \delta - \delta h & \\
(id + h \partial)^{-1} & \xleftarrow{\text{id} - h \delta} & W \\
& \xleftarrow{\text{id} - h \delta} & \\
W & \xleftarrow{\delta \delta = \text{id} - h \delta - (\partial + \delta) h \delta} & W \\
\end{array}
\]

**Proof.** It follows immediately from \( \delta^2 = 0 \) that \((\text{id} - h \delta)(\text{id} - \delta h) = \sigma \).
Likewise, it follows immediately from \((\delta + \partial)^2 = 0\) that \((\text{id} - h \delta (\delta + \partial))(\text{id} - (\delta + \partial) h \delta) = \sigma \).

It follows from the equations (29) and (27) that
\[
\text{id} - (\delta + \partial) h \delta = (\text{id} - \partial h \delta) - \delta h \delta = (\text{id} + h \partial)^{-1} - \delta h (\text{id} + h \partial)^{-1} = (\text{id} - \delta h) (\text{id} + h \partial)^{-1}. \] (31)
Likewise, it follows from the equations (30) and (27) that
\[
\text{id} - h \delta (\delta + \partial) = (\text{id} - h \delta) - h \delta \delta = (\text{id} + h \partial)^{-1} - (\text{id} + h \partial)^{-1} h \delta = (\text{id} + h \partial)^{-1} (\text{id} - h \delta). \] (32)

The proof is complete. \(\square\)

In the diagram of Lemma A.5, all straight arrows are projection operators, while the two bended arrows are automorphisms of the graded \(R\)-module \(W\). Since
\[ (\text{id} + h \partial)^{-1} \circ \sigma \circ (\text{id} + h \partial)^{-1} = \sigma \partial, \]
the automorphism \((\text{id} + h \partial)^{-1}\) of the \(R\)-module \(W\) identifies the submodules \(V = \text{im}(\sigma)\) and \(V_\partial = \text{im}(\sigma \partial)\).

Hence, we obtain the commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\tau} & W \\
\downarrow & & \downarrow \\
W & \xleftarrow{\psi} & V_\partial \\
\downarrow & \xleftarrow{\psi} & \downarrow \\
W & \xleftarrow{\delta \delta = \text{id} - h \delta - (\partial + \delta) h \delta} & W \\
\end{array}
\]

\[ (\text{id} + h \partial)^{-1} \cong (\text{id} + h \partial)^{-1} \xleftarrow{\text{id} + h \partial} \cong (\text{id} + h \partial)^{-1} \]

\[ \sigma_\partial = \text{id} - h \delta - (\partial + \delta) h \delta \]

\[ \sigma = \text{id} - h \delta - \delta h \]

\[ \sigma \partial = \text{id} - h \delta - (\partial + \delta) h \delta \]
**Definition A.6.** The composition $V \xrightarrow{(\text{id} + h \partial)^{-1}} V_{\partial} \xrightarrow{\psi} W$ will be referred to as the **perturbed injection** and denoted by $\tau_{\partial}$. The composition $W \xrightarrow{\psi} V_{\partial} \xrightarrow{\text{id} + h \partial} V$ will be referred to as the **perturbed surjection** and denoted by $\sigma_{\partial}$. Since $V_{\partial}$ is stable under the differential $\delta + \partial$, the composition $(\text{id} + h \partial) \circ (\delta + \partial) \circ (\text{id} + h \partial)^{-1}$ stabilizes $V$ and determines a differential on $V$, which we will refer to as the **perturbed differential** and denote by $\delta_{\partial}$.

From (28) and (33), we obtain the contraction data

$$ (V, \delta_{\partial}) \xrightarrow{\tau_{\partial}} (W, \delta + \partial) \xrightarrow{h_{\partial}} $$

(34)

**Corollary A.7.** Under the assumptions of Theorem A.3, we have

$$ \tau_{\partial} = (\text{id} + h \partial)^{-1} \circ \tau = \sum_{k=0}^{\infty} (-h \partial)^{k} \tau $$

$$ \sigma_{\partial} = \sigma \circ (\text{id} + \partial h)^{-1} = \sum_{k=0}^{\infty} \sigma (-\partial h)^{k} $$

$$ \delta_{\partial} - \delta|_{V} = \sigma \circ \partial \circ (\text{id} + h \partial)^{-1} \circ \tau = \sum_{k=0}^{\infty} \sigma \partial (-h \partial)^{k} \tau. $$

**Proof.** The equations for $\tau_{\partial}$ and $\sigma_{\partial}$ follow immediately from the commutativity of the diagram (33).

According to Definition A.6, we have

$$ \tau_{\partial} = (\text{id} + h \partial)^{-1} \circ (\delta + \partial) \circ (\text{id} + h \partial)^{-1} \circ \tau. $$

Since $\sigma \tau = \text{id}_{V}$, it follows that

$$ \delta_{\partial} = \sigma \circ (\text{id} + h \partial) \circ (\delta + \partial) \circ (\text{id} + h \partial)^{-1} \circ \tau. $$

Since $\sigma h = 0$, the equation simplifies to

$$ \delta_{\partial} = \sigma \circ (\delta + \partial) \circ (\text{id} + h \partial)^{-1} \circ \tau $$

or, equivalently,

$$ \delta_{\partial} = \sigma \circ (\delta + \partial) \circ (\text{id} - h_{\partial} \partial) \circ \tau. $$

Since $h_{\partial} = h(\text{id} + \partial h)^{-1}$ and $\sigma h = 0$, we obtain

$$ \sigma \delta (\text{id} - h_{\partial} \partial) \tau = \delta|_{V} \sigma (\text{id} - h_{\partial} \partial) \tau = \delta|_{V} \sigma \tau = \delta|_{V}. $$

Therefore, we conclude that

$$ \delta_{\partial} - \delta|_{V} = \sigma \circ \partial \circ (\text{id} - h_{\partial} \partial) \circ \tau = \sigma \circ \partial \circ (\text{id} + h \partial)^{-1} \circ \tau. $$

$\Box$
A.3. Hom spaces and tensor products of contractions

In the study of Atiyah classes, it is necessary to construct a contraction with its big space being

\[ W^\vee \otimes_R \text{End}_R(W) \cong W^\vee \otimes_R W^\vee \otimes_R W \]

using a given contraction \((W, \delta; h)\) over a commutative dg ring \(R\). This construction is commonly known as the “tensor trick” in the literature. For example, contractions of \((m, n)\)-tensors can be found in [8]. Also see [9,30]. In this paper, we use the formulation of Hom spaces.

**Lemma A.8.** Let \((W, \delta; h)\) and \((W', \delta'; h')\) be contractions over a commutative dg ring \(R\). The triple

\[
(\text{Hom}_R(W, W'), D; H)
\]

with

\[
D(f) = \delta' \circ f - (-1)^{|f|} f \circ \delta,
\]
\[
H(f) = h' \circ f + (-1)^{|f|} (\text{id}_{W'} - [\delta', h']) \circ f \circ h
\]

is also a contraction over \(R\). Furthermore, if the contraction data associated with \((W, \delta; h)\) and \((W', \delta'; h')\) are

\[
(V, d) \xrightarrow{\tau} (W, \delta) \xrightarrow{\circ} h, \quad \text{and} \quad (V', d') \xrightarrow{\tau'} (W', \delta') \xrightarrow{\circ} h'
\]

respectively, then the contraction data associated with \((\text{Hom}_R(W, W'), D; H)\) is

\[
(\text{Hom}_R(V, V'), \tilde{D}) \xrightarrow{T} (\text{Hom}_R(W, W'), D) \xrightarrow{\circ} h
\]

with

\[
T(g) = \tau' \circ g \circ \sigma, \quad \Sigma(f) = \sigma' \circ f \circ \tau, \quad \tilde{D}(g) = d' \circ g - (-1)^{|g|} g \circ d,
\]

for \(g \in \text{Hom}_R(V, V')\) and \(f \in \text{Hom}_R(W, W')\).

Let \((W_i, \delta_i; h_i), i = 1, \cdots, n\), be contractions over \(R\). It follows from the usual tensor trick that the triple

\[
(W_1 \otimes_R \cdots \otimes_R W_n, D^n; H^n)
\]

with

\[
D^n = \sum_{i=1}^{n} \text{id}_{W_1} \otimes \cdots \otimes \text{id}_{W_{i-1}} \otimes \delta_i \otimes \text{id}_{W_{i+1}} \otimes \cdots \otimes \text{id}_{W_n}, \quad (35)
\]
\[
H^n = \sum_{i=1}^{n} (\text{id}_{W_1} - [\delta_1, h_1]) \otimes \cdots \otimes (\text{id}_{W_{i-1}} - [\delta_{i-1}, h_{i-1}]) \otimes h_i \otimes \text{id}_{W_{i+1}} \cdots \otimes \text{id}_{W_n}\]

(36)
is also a contraction over $R$. Furthermore, if the contraction data associated with $(W_i, \delta_i; h_i)$ is

$$(V_i, d_i) \xrightarrow{\tau_i} (W_i, \delta_i) \supset h_i,$$

then the contraction data of the tensor product is

$$(V_1 \otimes_R \cdots \otimes_R V_n, \tilde{D}^n) \xrightarrow{\tau^n} (W_1 \otimes_R \cdots \otimes_R W_n, D^n) \supset H^n,$$

where

$$\tau^n = \tau_1 \otimes \cdots \otimes \tau_n, \quad \Sigma^n = \sigma_1 \otimes \cdots \otimes \sigma_n,$$

$$\tilde{D}^n = \sum_{i=1}^n \text{id}_{V_1} \otimes \cdots \otimes \text{id}_{V_{i-1}} \otimes d_i \otimes \text{id}_{V_{i+1}} \otimes \cdots \otimes \text{id}_{V_n}.$$

By Lemma A.8, we have the following

**Proposition A.9.** Let $(W_i, \delta_i; h_i), i = 1, \cdots, n$, and $(\tilde{W}_j, \tilde{\delta}_j; \tilde{h}_j), j = 1, \cdots, m$, be contractions over $R$. Then the triple

$$(\text{Hom}_R(W_1 \otimes_R \cdots \otimes_R W_n, \tilde{W}_1 \otimes_R \cdots \otimes_R \tilde{W}_m), \ D^n_m; \ H^n_m)$$

is also a contraction over $R$, where $D^n_m$ and $H^n_m$ are determined by (35), (36) and Lemma A.8. Furthermore, if the contraction data associated with $(W_i, \delta_i; h_i)$ and $(\tilde{W}_j, \tilde{\delta}_j; \tilde{h}_j)$ are

$$(V_i, d_i) \xrightarrow{\tau_i} (W_i, \delta_i) \supset h_i \quad \text{and} \quad (\tilde{V}_j, \tilde{d}_j) \xrightarrow{\tilde{\tau}_j} (\tilde{W}_j, \tilde{\delta}_j) \supset \tilde{h}_j$$

respectively, then the diagram

$$(\overline{T}^m_n, \overline{D}^m_n) \xrightarrow{\tau^m_n} (\overline{T}^m_n, D^m_n) \supset H^m_n,$$

with

$$\overline{T}^m_n = \text{Hom}_R(V_1 \otimes_R \cdots \otimes_R V_n, \tilde{V}_1 \otimes_R \cdots \otimes_R \tilde{V}_m),$$

$$\overline{T}^m_n = \text{Hom}_R(W_1 \otimes_R \cdots \otimes_R W_n, \tilde{W}_1 \otimes_R \cdots \otimes_R \tilde{W}_m),$$

$$\overline{T}^m_n(g) = \tilde{T}^m \circ g \circ \Sigma^n, \quad \Sigma^n(f) = \tilde{\Sigma}^m \circ f \circ \tau^n.$$ is a contraction data.

Let $(W, \delta; h)$ be a contraction over $R$ whose associated contraction data is

$$(V, d) \xrightarrow{\tau} (W, \delta) \supset h.$$ By (35) and (36), one has the contraction $(\mathbb{T} W, D_T; H_{\mathbb{T}})$ of the tensor algebra $\mathbb{T} W$ generated by $W$ over $R$. Let $\Lambda W$ be the exterior algebra generated by $W$ over $R$, $D_{\Lambda}$ :
\( \Lambda W \to \Lambda W \) be the derivation generated by \( \delta \), and \( H_\Lambda \) be the operator defined by the composition

\[
H_\Lambda : \Lambda W \xrightarrow{\text{sym}} T^1 W \xrightarrow{H^1} T W \to \Lambda W,
\]

where \( \text{sym} : \Lambda W \to T^1 W \) is the map

\[
\text{sym}(w_1 \wedge \cdots \wedge w_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\sigma \cdot w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}
\]

\( S_n \) denotes the symmetric group on the set \( \{1, 2, \cdots, n\} \), and \( \chi_\sigma \) is the number (either \( +1 \) or \( -1 \)) satisfying the equation

\[
w_1 \wedge \cdots \wedge w_n = \chi_\sigma \cdot w_{\sigma(1)} \wedge \cdots \wedge w_{\sigma(n)}
\]

in \( \Lambda W \). In other words,

\[
H_\Lambda(w_1 \wedge \cdots \wedge w_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\left| w_{\sigma(1)} \right| + \cdots + \left| w_{\sigma(i-1)} \right| + \left| \delta, h \right|} (w_{\sigma(1)}) \\
\wedge \cdots \wedge (\text{id} - [\delta, h]) (w_{\sigma(i-1)}) \wedge h(w_{\sigma(i)}) \wedge w_{\sigma(i+1)} \\
\wedge \cdots \wedge w_{\sigma(n)}.
\]

**Proposition A.10.** The triple \((\Lambda W, D_\Lambda; H_\Lambda)\) is a contraction over \( R \) with the contraction data

\[
(\Lambda V, \tilde{D}_\Lambda) \xrightarrow{\tau_\Lambda} (\Lambda W, D_\Lambda) \xrightarrow{\Sigma_\Lambda} H_\Lambda,
\]

where \( \tilde{D}_\Lambda \) is the derivation generated by \( d \), and

\[
\tau_\Lambda(v_1 \wedge \cdots \wedge v_n) = \tau(v_1) \wedge \cdots \wedge \tau(v_n),
\]

\[
\Sigma_\Lambda(w_1 \wedge \cdots \wedge w_n) = \sigma(w_1) \wedge \cdots \wedge \sigma(w_n),
\]

for \( v_1, \cdots, v_n \in V \), \( w_1, \cdots, w_n \in W \).

We need the following contraction when studying the Todd classes.

**Corollary A.11.** The triple \((\Lambda W^\vee \otimes \text{End} W, \tilde{D}; \tilde{H})\) is a contraction over \( R \), where \( \tilde{D} \) and \( \tilde{H} \) are given by Lemma A.8 and Proposition A.10. Furthermore, the induced inclusion map

\[
\tilde{T} : \Lambda V^\vee \otimes \text{End} V \hookrightarrow \Lambda W^\vee \otimes \text{End} W
\]

is an algebra morphism.

**Proof.** The first assertion is clear. For the second one, since \( \sigma \circ \tau = \text{id}_V \), it follows from Lemma A.8 that \( \tau(g) \circ \tau(g') = \tau(g \circ g') \). Thus, by Proposition A.10, the tensor trick implies \( \tilde{T} = T_\Lambda^\vee \otimes \tilde{T} \) is an algebra morphism. \( \square \)

Note that since \( \tau \circ \sigma \neq \text{id}_W \), the projection map \( \Sigma : \text{End} W \to \text{End} V \) is not necessarily an algebra morphism.
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