WELL-POSEDNESS OF THE FREE BOUNDARY PROBLEM IN INCOMPRESSIBLE ELASTODYNAMICS

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ABSTRACT. In this paper, we prove the local well-posedness of the free boundary problem in incompressible elastodynamics under a natural stability condition, which ensures that the evolution equation describing the free boundary is strictly hyperbolic. Our result gives a rigorous confirmation that the elasticity has a stabilizing effect on the Rayleigh-Taylor instability.

1. INTRODUCTION

1.1. Presentation of the problem. In this paper, we consider the incompressible inviscid flow in 3-D elastodynamics:

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\partial_t (\rho u) + u \cdot \nabla (\rho u) + \nabla p &= \mathrm{div} (\rho F F^\top), \\
\partial_t F + u \cdot \nabla F &= \nabla u F, \\
\mathrm{div} u &= 0,
\end{align*}
\]

where \( \rho \) is the density of fluids, \( u(t, x) = (u_1, u_2, u_3) \) denotes the fluid velocity, \( p(t, x) \) is the pressure, \( F(t, x) = (F_{ij})_{3 \times 3} \) is the deformation tensor, \( F^\top = (F_{ji})_{3 \times 3} \) denotes the transpose of the matrix \( F \), \( F F^\top \) is the Cauchy-Green tensor in the case of neo-Hookean elastic materials, \( (\nabla u)_{ij} = \partial_j u_i \), \( (\nabla u F)_{ij} = \sum_{k=1}^3 F_{kj} \partial_k u_i \), \( (\mathrm{div} F) = \sum_{j=1}^3 \partial_j F_{ji} \), \( (\mathrm{div} F F^\top )_i = \sum_{j,k=1}^3 \partial_j (F_{ik} F_{jk}) \).

We study the solution of (1.1) which are smooth on each side of a smooth interface \( \Gamma(t) \) in a domain \( \Omega \). Precisely, for simplicity, we let

\[
\begin{align*}
\Omega &= T^2 \times [-1, 1] \subset \mathbb{R}^3, \\
\Gamma(t) &= \{ x \in \Omega : x_3 = f(t, x'), x' = (x_1, x_2) \in T^2 \}, \\
\Omega^\pm_t &= \{ x \in \Omega : x_3 \geq f(t, x'), x' = (x_1, x_2) \in T^2 \}, \\
Q^\pm_T &= \bigcup_{t \in (0, T)} \{ t \} \times \Omega^\pm_t.
\end{align*}
\]

We consider that \( \rho_{\Omega^\pm_t} = \rho^\pm \) are two constants, and

\[
u^\pm := u|_{\Omega^\pm_t}, \quad F^\pm := F|_{\Omega^\pm_t}, \quad p^\pm := p|_{\Omega^\pm_t},
\]

are smooth in \( Q^\pm_T \) and satisfy

\[
\begin{align*}
\rho^\pm (\partial_t u^\pm + u^\pm \cdot \nabla u^\pm) + \nabla p^\pm &= \rho^\pm \sum_{j=1}^3 (F^\pm_{ij} \cdot \nabla) F^\pm_{ij} \quad \text{in} \quad Q^\pm_T, \\
\mathrm{div} u^\pm &= 0, \quad \mathrm{div} F^\pm = 0 \quad \text{in} \quad Q^\pm_T, \\
\partial_j F^\pm_j + u^\pm \cdot \nabla u^\pm_j &= (F^\pm_{ij} \cdot \nabla) u^\pm_j \quad \text{in} \quad Q^\pm_T,
\end{align*}
\]

with the boundary conditions on the moving interface \( \Gamma_t \):

\[
[p]^\text{def} = p^+ - p^- = 0, \quad u^\pm \cdot n = V(t, x), \quad F^\pm_j \cdot n = 0.
\]
Here $\mathbf{F}_j^\pm = (F_{1j}^\pm, F_{2j}^\pm, F_{3j}^\pm)$, $\mathbf{n}$ is the outward unit normal to $\partial \Omega_t^-$, and $V(t, x)$ is the normal velocity of $\Gamma_t$. On the artificial boundary $\Gamma_t^\pm = T^2 \times \{ \pm 1 \}$, we impose the following boundary conditions on $(\mathbf{u}^\pm, \mathbf{F}^\pm)$:

\begin{equation}
(1.4) \\
\begin{array}{ll}
\mathbf{u}_3^\pm = 0, & F_{3j}^\pm = 0 \quad \text{on } \Gamma_t^\pm.
\end{array}
\end{equation}

The system (1.2) is supplemented with the initial data

\begin{equation}
(1.5) \\
\begin{array}{ll}
\mathbf{u}^\pm(0, x) = \mathbf{u}_0^\pm(x), & \mathbf{F}^\pm(0, x) = \mathbf{F}_0^\pm \quad \text{in } \Omega_0^\pm,
\end{array}
\end{equation}

where the initial data satisfies

\begin{equation}
(1.6) \\
\begin{array}{ll}
\text{div} \mathbf{u}_0^\pm = 0, & \text{div} \mathbf{F}_{j,0}^\pm = 0 \quad \text{in } \Omega_0^\pm, \\
\mathbf{u}_0^+ \cdot \mathbf{n}_0 = \mathbf{u}_0^- \cdot \mathbf{n}_0, & \mathbf{F}_{j,0}^+ \cdot \mathbf{n}_0 = \mathbf{F}_{j,0}^- \cdot \mathbf{n}_0 \quad \text{on } \Gamma_0.
\end{array}
\end{equation}

The system (1.2)-(1.5) is called the vortex sheet problem for incompressible elastodynamics. One of main goals in this paper is to study the local well-posedness of this system under some suitable stability conditions imposed on the initial data.

In our setting, the boundary condition on $\Gamma_t$ in (1.2) is transformed into

\begin{equation}
[p] = 0, \quad \mathbf{u}^\pm \cdot \mathbf{N} = \partial_t f, \quad \mathbf{F}_j^\pm \cdot \mathbf{N} = 0 \quad \text{on } \Gamma_t,
\end{equation}

where $\mathbf{N} = (-\partial_1 f, -\partial_2 f, 1)$ and $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$.

Let us remark that the divergence free restriction on $\mathbf{F}_j^\pm$ is automatically satisfied if $\text{div} \mathbf{F}_{j,0}^\pm = 0$. Indeed, if we apply the divergence operator to the third equation of (1.2), we will deduce the following transport equation

\begin{equation}
\partial_t \text{div} \mathbf{F}_j^\pm + \mathbf{u}^\pm \cdot \nabla \text{div} \mathbf{F}_j^\pm = 0.
\end{equation}

Similar argument can be also applied to yield that $\mathbf{F}_j^\pm \cdot \mathbf{N} = 0$ if $\mathbf{F}_{j,0}^\pm \cdot \mathbf{N}_0 = 0$.

For a special case $\rho_+ = 0$, the problem reduces to another type of free boundary problem for idea incompressible elastodynamics, that is,

\begin{equation}
(1.7) \\
\begin{array}{ll}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \sum_{j=1}^3 \mathbf{F}_j \cdot \nabla \mathbf{F}_j \quad & \text{in } \Omega_t^-, \\
\text{div} \mathbf{u} = 0, & \text{div} \mathbf{F}^\top = 0 \quad \text{in } \Omega_t^-, \\
\partial_t \mathbf{F}_j + \mathbf{u} \cdot \nabla \mathbf{F}_j = \mathbf{F}_j \cdot \nabla \mathbf{u} \quad & \text{in } \Omega_t^-,
\end{array}
\end{equation}

where $\mathbf{F}_j = (F_{1j}, F_{2j}, F_{3j})$ and $\Omega_t^-$ and $\Gamma_t$ are defined as above. On the free boundary $\Gamma_t$, the boundary conditions are given by

\begin{equation}
(1.8) \\
p = 0, \quad \mathbf{u} \cdot \mathbf{N} = \partial_t f, \quad \mathbf{N} \cdot \mathbf{F}_j = 0 \quad \text{on } \Gamma_t,
\end{equation}

while on the bottom boundary $\Gamma^-$, it holds that

\begin{equation}
(1.9) \\
\mathbf{u}_3 = 0, \quad F_{3j} = 0 \quad \text{on } \Gamma^-.
\end{equation}

This system is supplemented with initial data

\begin{equation}
(1.10) \\
\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \mathbf{F}(0, x) = \mathbf{F}_0 \quad \text{in } \Omega_0^-.
\end{equation}
1.2. Background. For incompressible inviscid flow, the Kelvin-Helmholtz instability has been known for over a century [12]. It was well known that the surface tension can stabilize the Kelvin-Helmholtz and Rayleigh-Taylor instability [2, 6, 17]. Syrovatskij [20] and Axford [3] found that the magnetic field has a stabilization effect on the Kelvin-Helmholtz instability. Recently, there are many important works devoted to confirming this stabilizing mechanism. For the current-vortex sheet problem, we refer to [23, 24, 4, 27] for compressible case and [13, 22, 7, 18] for incompressible case. For plasma-vacuum problem, we refer to [25, 21] for incompressible case and [14, 19] for incompressible case. We also refer to some related works [10, 9, 8] on the incompressible plasma-vacuum problem.

For the incompressible elastodynamics, there are several recent progress on the free boundary problems in incompressible elastodynamics, Chen-Hu-Wang [5] analyzed the linearized stability and proved the stabilization effect of elasticity on vortex sheets. In [26], Trakhinin proved the well-posedness of the one-fluid free boundary problem in compressible elastodynamics under the condition that there are two columns of the $3 \times 3$ deformation tensor which are non-collinear at each point of the initial surface. For the incompressible case, Hao-Wang [11] proved a priori estimates for solutions in Sobolev spaces under the Rayleigh-Taylor sign condition. The aim of this paper is to show the local well-posedness for both two free boundary problems in incompressible elastodynamics under a natural stability condition by using the method developed in [18]. The basic idea is to derive an evolution equation describing the deformation of the interface and prove the stability under a suitable stability condition. This idea is very effective to study the free boundary problems of the incompressible Euler equations [28, 29, 30, 16].

1.3. Main results. To ensure the stability of the system (1.2)-(1.5) and the system (1.7)-(1.10), certain stability conditions are required. In this paper, we assume the following stability condition for (1.2)-(1.5):

\begin{equation}
\inf_{\Gamma_2} \inf_{\xi \in \mathbb{S}^2, \xi \cdot \mathbf{N} = 0} \left\{ (\rho^+ + \rho^-) \left[ \rho^+(\xi \cdot \mathbf{F}^+)^2 + \rho^-(\xi \cdot \mathbf{F}^-)^2 \right] - \rho^+ \rho^- (\xi \cdot [\mathbf{u}])^2 \right\} > 0,
\end{equation}

where $\xi \cdot \mathbf{F} = (\xi, F_{ij})_{1 \leq j \leq 3}$ and $[\mathbf{u}] = \frac{1}{2}(\mathbf{u}^+ - \mathbf{u}^-)$ on $\Gamma_f$. (1.11) is equivalent to

\begin{equation}
\Lambda(\mathbf{F}^\pm, \mathbf{v}) \overset{\text{def}}{=} \inf_{x \in \Omega, \varphi^1 + \varphi^2 = 1} \sum_{j=1}^{3} \left( \frac{\rho^+}{\rho^+ + \rho^-} (F_{ij}^+ \varphi_1 + F_{ij}^+ \varphi_2)^2 + \frac{\rho^-}{\rho^+ + \rho^-} (F_{ij}^- \varphi_1 + F_{ij}^- \varphi_2)^2 \right)
- (v_1 \varphi_1 + v_2 \varphi_2)^2 > 0,
\end{equation}

where $(v_1, v_2, v_3) = \sqrt{\frac{\rho^+}{\rho^+ + \rho^-}} [\mathbf{u}]$. Our first main result is stated as follows.

**Theorem 1.1.** Let $s \geq 3$ be an integer and assume that

\[ f_0 \in H^{s + \frac{1}{2}}(\mathbb{T}^2), \quad \mathbf{u}^\pm_0, \mathbf{F}^\pm_0 \in H^s(\Omega^\pm_0). \]

Furthermore we assume that there exists $c_0 > 0$ so that

1. $-(1 - 2c_0) \leq f_0 \leq (1 - 2c_0)$;
2. $\Lambda(\mathbf{F}^+_0, \mathbf{v}_0) \geq 2c_0$.

Then there exists $T > 0$ such that the system (1.2) admits a unique solution $(f, \mathbf{u}, \mathbf{F})$ in $[0, T]$ satisfying

1. $f \in L^\infty([0, T), H^{s + \frac{1}{2}}(\mathbb{T}^2))$;
2. \( u^\pm, F^\pm \in L^\infty(0, T; H^s(\Omega^\pm_T)) \);
3. \( -(1-c_0) \leq f \leq (1-c_0) \);
4. \( \Lambda(F^\pm, v) \geq c_0 \).

When \( \rho^+ = 0 \), the stability condition (1.11) reduces to \((\xi \cdot F^-)^2 > 0\) on \( \Gamma^s \) for any \( \xi \in S^2 \) with \( \xi \cdot N = 0 \), which is equivalent to \( \text{rank}(F) = 2 \). Therefore, as a corollary of Theorem 1.1, we have the following result which concerns the well-posedness for the system (1.7)-(1.10).

**Theorem 1.2.** Let \( s \geq 3 \) be an integer and assume that

\[
 f_0 \in H^{s+\frac{3}{2}}(\mathbb{T}^2), \quad u_0, F_0 \in H^s(\Omega^\pm_0).
\]

Furthermore we assume that there exists \( c_0 > 0 \) so that

1. \( -(1-2c_0) \leq f_0 \leq (1-2c_0) \);
2. \( \text{rank}(F_0) = 2 \) on \( \Gamma_0 \).

Then there exists \( T > 0 \) such that the system (1.7) admits a unique solution \((f, u, F)\) in \([0, T]\) satisfying

1. \( f \in L^\infty([0, T), H^{s+\frac{3}{2}}(\mathbb{T}^2)) \);
2. \( u, F \in L^\infty(0, T; H^s(\Omega^\pm_T)) \);
3. \( -(1-c_0) \leq f \leq (1-c_0) \);
4. \( \text{rank}(F) = 2 \) on \( \Gamma_t \).

**Remark 1.3.** We remark that the assumption \( \text{rank}(F_0) = 2 \) on \( \Gamma_0 \) is weaker than the assumption proposed by Trakhinin [26], which says that among the three vectors \( F_1, F_2 \) and \( F_3 \) there are two which are non-collinear at each point of \( \Gamma_0 \). There is another type of stability condition which would also ensure the existence of solutions to the system (1.7)-(1.10):

\[
(1.13) \quad -N \cdot \nabla p > 0, \quad \text{on } \Gamma_t,
\]

see [11] for a priori estimates results. One would also be interested in studying the well-posedness under the following mixed type of stability condition:

\[
(1.14) \quad \{x \in \Gamma_t : \text{rank}(F(x)) = 2\} \cup \{x \in \Gamma_t : -N \cdot \nabla p > 0\} = \Gamma_t.
\]

These cases can also be handled in this framework, which will be left in a forthcoming work.

**Remark 1.4.** Our method could be applied to 2-D case, which in particular means that the elasticity has a stabilization effect on the Rayleigh-Taylor instability. Indeed, for the 2-D case, we have that \( F^\pm \) are \( 2 \times 2 \) matrices, and \( F^\pm_j \cdot N = 0, [u] \cdot N = 0 \), which implies that \( F^\pm_j, [u] \) are collinear to each other. Therefore, the stability condition (1.11) for (1.2)-(1.5) reduces to

\[
(1.15) \quad (\rho^+ + \rho^-)\left[\rho^+|F^+|^2 + \rho^-|F^-|^2\right] - \rho^+\rho^-|[u]|^2 > 0,
\]

and for the system (1.7)-(1.10), the stability condition \( \text{rank}(F) = 2 \) reduces to

\[
(1.16) \quad |F| > 0.
\]

The solutions can be constructed in a similar way as Theorem 1.1 and 1.2.

The rest of this paper is organized as follows. In Section 2, we will introduce the reference domain, harmonic coordinate, and the Dirichlet-Neumann operator. In Section 3, we reformulate the system into a new formulation. In Section 4, we present the uniform estimates for the linearized system. In Section 5, we prove the existence and uniqueness of the solution. In Section 6, we present a sketch of the proof of Theorem 1.2.
2. Reference domain, harmonic coordinate and Dirichlet-Neumann Operator

For free boundary problems, as the domain of the fluid is changing with time \( t \), we always draw the moving domain back to a fixed domain which is called reference domain [18].

Let \( \Gamma_* \) be a fixed graph given by

\[
\Gamma_* = \{(y_1, y_2, y_3) : y_3 = f_*(y_1, y_2)\}.
\]

The reference domain \( \Omega_*^{\pm} \) is given by

\[
\Omega_*^+ = \mathbb{T}^2 \times (-1, 1), \quad \Omega_*^- = \{y \in \Omega_* | y_3 \leq f_*(y_1, y_2)\}.
\]

We will look for the free boundary which lies in a neighborhood of the reference domain. As a result, we define

\[
\Upsilon(\delta, k) \overset{\text{def}}{=} \left\{ f \in H^k(\mathbb{T}^2) : \|f - f_*\|_{H^k(\mathbb{T}^2)} \leq \delta \right\}.
\]

For \( f \in \Upsilon(\delta, k) \), we can define the graph \( \Gamma_f \) by

\[
\Gamma_f \overset{\text{def}}{=} \left\{ x \in \Omega_t | x_3 = f(t, x'), \int_{\mathbb{T}^2} f(t, x') dx' = 0 \right\}.
\]

The graph \( \Gamma_f \) separates \( \Omega_t \) into two parts:

\[
\Omega^+_f = \left\{ x \in \Omega_t | x_3 > f(t, x') \right\}, \quad \Omega^-_f = \left\{ x \in \Omega_t | x_3 < f(t, x') \right\}.
\]

Let \( N_f = (N_1, N_2, N_3) \) be the outward normal vector of \( \Omega^-_f \) where

\[
N_f \triangleq (-\partial_1 f, -\partial_2 f, 1), \quad n_f \triangleq N_f / \sqrt{1 + |\nabla f|^2}.
\]

Then we need to find the draw back maps. For this purpose, we introduce the harmonic coordinate. Given \( f \in \Upsilon(\delta, k) \), we define a map \( \Phi^\pm_f : \Omega_*^+ \to \Omega^+_f \) by harmonic extension:

\[
\begin{aligned}
\Delta_\gamma \Phi^\pm_f &= 0, & y \in \Omega_*^+, \\
\Phi^\pm_f(y', f_*(y')) &= (y', f(y')), & y' \in \mathbb{T}^2, \\
\Phi^\pm_f(y', \pm 1) &= (y', \pm 1), & y' \in \mathbb{T}^2.
\end{aligned}
\]

Given \( \Gamma_* \), there exists \( \delta_0 = \delta_0(\|f_*\|_{W^{1, \infty}}) > 0 \) so that \( \Phi^\pm_f \) is a bijection when \( \delta \leq \delta_0 \). Then we can define an inverse map \( \Phi^{-1\pm}_f : \Omega^+_f \to \Omega_*^\pm \) such that

\[
\Phi^{-1\pm}_f \circ \Phi^\pm_f = \Phi^\pm_f \circ \Phi^{-1\pm}_f = \text{Id}.
\]

The following properties come from [18].

**Lemma 2.1.** Let \( f \in \Upsilon(\delta_0, s - \frac{1}{2}) \) for \( s \geq 3 \). Then there exists a constant \( C \) depending only on \( \delta_0 \) and \( \|f_*\|_{H^{s+\frac{1}{2}}} \) so that

1. If \( u \in H^\sigma(\Omega^+_f) \) for \( \sigma \in [0, s] \), then
   \[
   \|u \circ \Phi^\pm_f\|_{H^\sigma(\Omega^+_f)} \leq C\|u\|_{H^\sigma(\Omega^+_f)}.
   \]

2. If \( u \in H^\sigma(\Omega_*^+) \) for \( \sigma \in [0, s] \), then
   \[
   \|u \circ \Phi^{-1\pm}_f\|_{H^\sigma(\Omega^+_f)} \leq C\|u\|_{H^\sigma(\Omega_*^+)}.\]

3. If \( u, v \in H^\sigma(\Omega_*^+) \) for \( \sigma \in [2, s] \), then
   \[
   \|uv\|_{H^\sigma(\Omega^+_f)} \leq C\|u\|_{H^\sigma(\Omega^+_f)}\|v\|_{H^\sigma(\Omega^+_f)}.
   \]
We will use the Dirichlet-Neumann operator, which maps the Dirichlet boundary value of a harmonic function to its Neumann boundary value. That is to say, for any \( g(x') = g(x_1, x_2) \in H^k(T^2) \), we denote by \( \mathcal{H}^\pm_f g \) the harmonic extension to \( \Omega^\pm_f \):
\[
\begin{cases}
\Delta \mathcal{H}^\pm_f g = 0, & x \in \Omega^\pm_f, \\
(\mathcal{H}^\pm_f g)(x', f(x')) = g(x'), & x' \in T^2, \\
\partial_3 \mathcal{H}^\pm_f g(x', \pm 1) = 0, & x' \in T^2.
\end{cases}
\]

Then the Dirichlet-Neumann operator is defined by
\[
\mathcal{N}^\pm_f g \overset{\text{def}}{=} N_f \cdot (\nabla \mathcal{H}^\pm_f g)|_{\Gamma_f}.
\]

We will use the following properties from [1, 18].

**Lemma 2.2.** It holds that

1. \( \mathcal{N}^\pm_f \) is a self-adjoint operator:
\[
(\mathcal{N}^\pm_f \psi, \phi) = (\psi, \mathcal{N}^\pm_f \phi), \quad \forall \phi, \psi \in H^{1/2}(T^2);
\]

2. \( \mathcal{N}^\pm_f \) is a positive operator:
\[
(\mathcal{N}^\pm_f \phi, \phi) = \|\nabla \mathcal{H}^\pm_f \phi\|^2_{L^2(\Omega_f)} \geq 0, \quad \forall \phi \in H^{1/2}(T^2);
\]

Especially, if \( \int_{T^2} \phi(x')dx' = 0 \), there exists \( c > 0 \) depending on \( c_0, \|f\|_{W^{1, \infty}} \) such that
\[
(\mathcal{N}^\pm_f \phi, \phi) \geq c\|\mathcal{H}^\pm_f \phi\|^2_{H^{1/2}(\Omega_f)} \geq c\|\phi\|^2_{H^{1/2}}.
\]

3. \( \mathcal{N}^\pm_f \) is a bijection from \( H^{k+1}_0(T^2) \) to \( H^k_0(T^2) \) for \( k \geq 0 \), where
\[
H^k_0(T^2) \overset{\text{def}}{=} H^k(T^2) \cap \{ \phi \in L^2(T^2) : \int_{T^2} \phi(x')dx' = 0 \}.
\]

We will use \( x = (x_1, x_2, x_3) \) or \( y = (y_1, y_2, y_3) \) to denote the coordinates in the fluid region, and use \( x' = (x_1, x_2) \) or \( y' = (y_1, y_2) \) to denote the natural coordinates on the interface or on the top/bottom boundary. In addition, we will use the Einstein summation notation where a summation from 1 to 2 is implied over repeated index, while a summation from 1 to 3 over repeated index will be explicitly figured out by the symbol \( \sum \) (i.e. \( a_ib_i = a_1b_1 + a_2b_2, \sum_{i=1}^3 a_ib_i = a_1b_1 + a_2b_2 + a_3b_3 \)).

For a function \( g : \Omega \to \mathbb{R} \), we denote \( \nabla g = (\partial_1g, \partial_2g, \partial_3g) \), and for a function \( \eta : T^2 \to \mathbb{R} \), \( \nabla \eta = (\partial_1\eta, \partial_2\eta) \). For a function \( g : \Omega^+_f \to \mathbb{R} \), we can define its trace on \( \Gamma_f \), which is denoted by \( \underline{g}(x') \). Thus, for \( i = 1, 2, \)
\[
\partial_i \underline{g}(x') = \partial_i g(x', f(x')) + \partial_3 g(x', f(x'))\partial_i f(x').
\]

We denote by \( \| \cdot \|_{H^s(\Omega)} \) the Sobolev norm in \( \Omega \), and by \( \| \cdot \|_{H^s} \) the Sobolev norm in \( T^2 \).

### 3. Reformulation of the problem

In this section, we derive a new system which is equivalent to the original system (1.2)-(1.5). The system consists of the evolution equations of the following quantities:

- The height function of the interface: \( f \);
- The scaled normal velocity on the interface: \( \theta = u^\pm \cdot N_f \);
- The curl part of velocity and deformation tensor in the fluid region: \( \omega^\pm = \nabla \times u^\pm \), \( G^\pm_f = \nabla \times F^\pm_f \);
Lemma 3.1. \cite{18} For \( u = u^\pm, F_j^\pm \), we have
\[
(u \cdot \nabla u) \cdot N_f - \partial_3 u_j N_j (u \cdot N_f) \bigg|_{x_3 = f(t,x')}
= u_i \partial_1 (u_j N_j) + u_3 \partial_2 (u_j N_j) + \sum_{i,j=1,2} w_i w_j \partial_i \partial_j f.
\]

Combining the first equation of (1.2) and Lemma 3.1 (recall \( F_j^\pm \cdot N_f = 0 \) on \( \Gamma_f \)), one can obtain
\[
\partial_t \theta = (\partial_1 u^+ + \partial_3 u^+ \partial_3 f) \cdot N_f + u^+ \cdot \partial_t N_f \bigg|_{x_3 = f(t,x')}
= (-u^+ \cdot \nabla u^+ + \sum_{j=1}^{3} (F_j \cdot \nabla) F_j - \nabla p^+ + \partial_3 u^+ \partial_3 f) \cdot N_f
- u^+ \cdot (\partial_1 \partial_3 f, \partial_3 f, 0) \bigg|_{x_3 = f(t,x')}
= (- (u^+ \cdot \nabla) u^+ + \partial_3 u^+ (u^+ \cdot N_f)) \cdot N_f + \sum_{j=1}^{3} (F_j \cdot \nabla) F_j \cdot N_f
- N_f \cdot \nabla p^+ - u^+ \cdot (\partial_1 \theta, \partial_2 \theta, 0) \bigg|_{x_3 = f(t,x')}
\]
\[
= -2(u_i^+ \partial_1 \theta + u_3^+ \partial_2 \theta) - \frac{1}{\rho^+} N \cdot \nabla p^+ - \sum_{s,r=1}^{2} u_i^+ u_i^+ \partial_s \partial_r f + \sum_{j=1}^{3} \sum_{s,r=1}^{2} F_{ij}^+ F_{jr}^+ \partial_s \partial_r f,
\]
and similarly,
\[
\partial_t \theta = -2(u_i^- \partial_1 \theta + u_3^- \partial_2 \theta) - \frac{1}{\rho^-} N \cdot \nabla p^- - \sum_{s,r=1}^{2} u_i^- u_i^- \partial_s \partial_r f + \sum_{j=1}^{3} \sum_{s,r=1}^{2} F_{ij}^- F_{jr}^- \partial_s \partial_r f.
\]

Taking the divergence to the first equation of (1.2), we get
\[
\Delta p^\pm = \rho^\pm \left( \sum_{j=1}^{3} \text{tr}(\nabla F_j^\pm)^2 - \text{tr}(\nabla u^\pm)^2 \right).
\]

Recall that \( p^\pm = p^\pm \big|_{\Gamma_f} \) and \( H_j^\pm \) is the harmonic extension from \( \Gamma_f \) to \( \Omega_f^\pm \). Then for the pressure \( p^\pm \), we have the following important representation:
\[
p^\pm = H_j^\pm u^\pm + \rho^\pm p u^\pm, u^\pm - \rho^\pm \sum_{j=1}^{3} p F_j^\pm F_j^\pm.
\]
where $p_{u_1,u_2}$ is the solution of elliptic equation

\[
\begin{aligned}
\Delta p_{u_1^\pm,u_2^\pm} &= -\text{tr}(\nabla u_1^\pm \nabla u_2^\pm) & \text{in } \Omega_f^\pm, \\
p_{u_1^\pm,u_2^\pm} &= 0 & \text{on } \Gamma_f, \\
e_3 \cdot \nabla p_{u_1^\pm,u_2^\pm} &= 0 & \text{on } \Gamma^\pm.
\end{aligned}
\]

(3.7)

Thus, from (3.4) and (3.5), we have on $\Gamma_f$

\[
\frac{1}{\rho^+}N_f^+ \cdot \nabla H_f^+ p^+ - \frac{1}{\rho^-}N_f^- \cdot \nabla H_f^- p^- = -g^+ + g^-.
\]

From the definition of DN operator, one has

\[
(3.8)
\]

\[-\frac{1}{\rho^+}N_f^+ p^+ - \frac{1}{\rho^-}N_f^- p^- = -g^+ + g^-.
\]

As $p^+ - p^- = 0$ on $\Gamma_f$, we have

\[
p^+ = \tilde{N}_f^{-1}(g^+ - g^-),
\]

where

\[
\tilde{N}_f = \frac{1}{\rho^+}N_f^+ + \frac{1}{\rho^-}N_f^-.
\]

In addition, we can write

\[
N_f^+ = (\frac{1}{\rho^+} + \frac{1}{\rho^-})^{-1}(\tilde{N}_f + \frac{1}{\rho^-}(N_f^+ - N_f^-)),
\]

\[
N_f^- = (\frac{1}{\rho^+} + \frac{1}{\rho^-})^{-1}(\tilde{N}_f - \frac{1}{\rho^-}(N_f^+ - N_f^-)),
\]

which implies

\[
\frac{1}{\rho^+}N_f^+ \tilde{N}_f^{-1} g^- + \frac{1}{\rho^-}N_f^- \tilde{N}_f^{-1} g^+ = \frac{\rho^+ g^+ + \rho^- g^-}{\rho^+ + \rho^-} - \frac{1}{\rho^+ + \rho^-}(N_f^+ - N_f^-) \tilde{N}_f^{-1} (g^+ - g^-).
\]

Consequently, we can obtain

\[
\partial_1 \theta = \frac{1}{\rho^+}N_f^+ p^+ - g^+ = \frac{1}{\rho^+}N_f^+ \tilde{N}_f^{-1} (g^+ - g^-) - g^+ = -\frac{1}{\rho^+}N_f^+ \tilde{N}_f^{-1} g^- - \frac{1}{\rho^-}N_f^- \tilde{N}_f^{-1} g^+ = -\frac{\rho^+ g^+ + \rho^- g^-}{\rho^+ + \rho^-} + \frac{1}{\rho^+ + \rho^-}(N_f^+ - N_f^-) \tilde{N}_f^{-1} (g^+ - g^-) = -\frac{2}{\rho^+ + \rho^-}((\rho^+ u_1^+ + \rho^- u_1^-) \partial_1 \theta + (\rho^+ u_2^+ + \rho^- u_2^-) \partial_2 \theta).
\]
As in [18], we derive the evolution of tangential parts of $u$ and $F$

\begin{align}
\frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^2 \left( \rho^+ u_{s}^+ u_{r}^+ - \rho^- \sum_{j=1}^3 E_{sj}^+ F_{rj}^+ \right) \partial_s \partial_r f \\
- \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^2 \left( \rho^- u_{s}^- u_{r}^- - \rho^+ \sum_{j=1}^3 E_{sj}^- F_{rj}^- \right) \partial_s \partial_r f \\
+ \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{N}_f^{-1} \mathcal{P} \left( \sum_{s,r=1}^2 (u_{s}^+ u_{r}^+ - \sum_{j=1}^3 E_{sj}^+ F_{rj}^+) \partial_s \partial_r f \right) \\
- \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{N}_f^{-1} \mathcal{P} \left( \sum_{s,r=1}^2 (u_{s}^- u_{r}^- - \sum_{j=1}^3 E_{sj}^- F_{rj}^-) \partial_s \partial_r f \right) \\
+ \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{N}_f^{-1} \mathcal{P} \left( (u_{1}^- - u_{1}^+) \partial_1 \theta + (u_{2}^- - u_{2}^+) \partial_2 \theta \right) \\
+ \frac{1}{\rho^+ + \rho^-} N_f \cdot \nabla (\rho^+ p_{u^+,u^+} - \rho^+ \sum_{j=1}^3 p_{F_j^+} F_j^+) \\
+ \frac{1}{\rho^+ + \rho^-} N_f \cdot \nabla (\rho^- p_{u^-,u^-} - \rho^- \sum_{j=1}^3 p_{F_j^-} F_j^-) \\
- \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \tilde{N}_f^{-1} \mathcal{P} N_f \cdot \nabla \left( \sum_{j=1}^3 p_{F_j^+} F_j^+ - \sum_{j=1}^3 p_{F_j^-} F_j^- \right),
\end{align}

(3.9)

Here $\mathcal{P} : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is a projection operator defined by

$$\mathcal{P} g = g - \langle g \rangle$$

with $\langle g \rangle \overset{\text{def}}{=} \int_{\mathbb{T}^2} g dx'$. We can apply the operator $\mathcal{P}$ to some of the terms in (3.9) for the same reasons as in [18], because it does not change the formulation of this system owing to $\mathcal{P} g^\pm = g^\pm$.

3.2. Equations for the vorticity and the curl of deformation tensor. We derive the equations for

$$\omega^\pm = \nabla \times u^\pm, \quad G_j^\pm = \nabla \times F_j^\pm.$$

It is direct to obtain from (1.2) that $(\omega^\pm, G_j^\pm)$ satisfies

(3.10)

$$\begin{cases}
\partial_t \omega^\pm + u^\pm \cdot \nabla \omega^\pm - \sum_{j=1}^3 F_j^\pm \cdot \nabla G_j^\pm = \omega^\pm \cdot \nabla u^\pm - \sum_{j=1}^3 G_j^\pm \cdot \nabla F_j^\pm & \text{in } \Omega_t; \\
\partial_t G_j^\pm + u^\pm \cdot \nabla G_j^\pm - F_j^\pm \cdot \nabla \omega^\pm = G_j^\pm \cdot \nabla u^\pm - \omega^\pm \cdot \nabla F_j^\pm - 2 \sum_{i=1}^3 \nabla u_i^\pm \times \nabla F_{ij}^\pm & \text{in } \Omega_t.
\end{cases}$$

3.3. The evolution of tangential parts of $u$ and $F_j$ on top and bottom boundaries.

As in [18], we derive the evolution of

$$\beta_i^\pm = \int_{\mathbb{T}^2} u_i^\pm(t,x', \pm1) dx', \quad \gamma_{ij}^\pm(t) = \int_{\mathbb{T}^2} F_{ij}^\pm(t,x', \pm1) dx' \quad \text{for } i=1,2 \text{ and } j=1,2,3.$$
As \( u^\pm_3(t, x', \pm 1) \equiv 0 \), we deduce that for \( i = 1, 2 \)
\[
\partial_t u^\pm_i + u^\pm_j \partial_j u^\pm_i - \sum_{j=1}^3 F^\pm_{sj} \partial_s F^\pm_{ij} - \partial_i p^\pm = 0 \quad \text{on } \Gamma^\pm.
\]
Consequently, one has
\[
\partial_t \beta^\pm_i + \int_{\Gamma^\pm} (u^\pm_k \partial_k u^\pm_i - \sum_{s,j=1}^3 F^\pm_{sj} \partial_s F^\pm_{ij}) \, dx' = 0,
\]
or equivalently
\[
\beta^\pm_i(t) = \beta^\pm_i(0) - \int_0^t \int_{\Gamma^\pm} (u^\pm_k \partial_k u^\pm_i - \sum_{s,j=1}^3 F^\pm_{sj} \partial_s F^\pm_{ij}) \, dx' \, d\tau.
\]
Similarly, we have
\[
\gamma^\pm_{ij}(t) = \gamma^\pm_{ij}(0) - \int_0^t \int_{\Gamma^\pm} (u^\pm_k \partial_k F^\pm_{ij} - F^\pm_{sj} \partial_s u^\pm_i) \, dx' \, d\tau.
\]

3.4. Solvability conditions of Div-Curl system. To recover the divergence-free velocity field or deformation tensor field from its curl part, we solve the following div-curl system:
\[
\begin{align*}
\text{curl } u^\pm &= \omega^\pm, & \text{div } u^\pm &= g^\pm & \text{in } & \Omega_f^\pm, \\
u^\pm \cdot N_f &= \theta^\pm & \text{on } & \Gamma_f, \\
u^\pm \cdot e_3 &= 0 & \text{on } & \Gamma^\pm, \\
\int_{\Gamma^\pm} u^\pm_i \, dx' &= \beta^\pm_i (i = 1, 2).
\end{align*}
\]
(3.11)
The solvability of the above system was obtained in [18] under the following compatibility conditions:

C1. \( \text{div } \omega^\pm = 0 \) in \( \Omega_f^\pm \),

C2. \( \int_{\Gamma^\pm} \omega^\pm_3 \, dx' = 0 \),

C3. \( \int_{\Gamma_f} \theta \, dx' = \mp \int_{\Gamma^\pm} g^\pm \, dx' \),

and the main result are stated in Proposition A.1.

4. Uniform estimates for the linearized system

In this section, we will present the uniform energy estimates for the linearized system around given functions \( (f, u^\pm, F^\pm) \). We assume that there exists \( T > 0 \) for any \( t \in [0, T] \):

\[
\begin{align*}
\| (u^\pm, F^\pm)(t) \|_{L^\infty(\Gamma_f)} &\leq L_0, \\
\| f(t) \|_{H^{s+\frac{1}{2}}(\mathbb{T}^2)} + \| \partial_t f(t) \|_{H^{s-\frac{1}{2}}(\mathbb{T}^2)} + \| u^\pm(t) \|_{H^s(\Omega_f^\pm)} + \| F^\pm(t) \|_{H^s(\Omega_f^\pm)} &\leq L_1, \\
\| (\partial_t u^\pm, \partial_t h^\pm)(t) \|_{L^\infty(\Gamma_f)} &\leq L_2, \\
\| f(t) - f_s \|_{H^{s-\frac{1}{2}}} &\leq \delta_0, \\
1 - c_0 &\leq f(t, x') \leq (1 + c_0), \\
\Lambda(F^\pm, v)(t) &\geq c_0.
\end{align*}
\]
(4.1) - (4.6)

Together with

\[
\begin{align*}
div u^\pm &= \text{div } F^\pm_j = 0 & \text{in } & \Omega_f^\pm, \\
F^\pm_j \cdot N_f &= 0 & \text{on } & \Gamma_f, \\
\partial_t f &= u^\pm_n \cdot N_f & \text{on } & \Gamma_f, \\
u^\pm_j &= F^\pm_{3j} = 0 & \text{on } & \Gamma^\pm.
\end{align*}
\]
(4.7)
Here $L_0, L_1, L_2, c_0, \delta_0$ are positive constants.

4.1. The linearized system for the height function of the interface. For the system (3.2) and (3.9), we introduce the following linearized system:

$$
\begin{align*}
\frac{\partial_t \bar{f}}{\partial_t \bar{\theta}} &= \bar{\theta}; \\
\frac{\partial_t \bar{\theta}}{\partial t} &= -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u_i^+ + \rho^- w_i^-) \partial_{i} \bar{\theta} + (\rho^+ u_i^+ + \rho^- w_i^-) \partial_i \bar{\theta} \right) \\
&- \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^+ u_i^+ u_j^+ - \rho^- u_i^- u_j^- \right) \partial_s \partial_r \bar{\theta} \\
&- \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^- u_i^- w_j^- - \rho^+ \partial_i \bar{\theta} \right) \partial_s \partial_r \bar{\theta} + g
\end{align*}
$$

(4.8)

where

$$
(4.9) \quad g = \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \bar{N}_f^{-1} \left( \sum_{s,r=1}^{2} \left( u_i^+ u_j^+ - u_i^- u_j^- \right) + \sum_{j=1}^{3} \left( E_{ij}^+ - E_{ij}^- \right) \right) \partial_s \partial_r \bar{f}
$$

and

$$
\begin{align*}
&+ \frac{1}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \bar{N}_f^{-1} \left( \sum_{j=1}^{3} p_{F_j^+} - p_{F_j^-} \right) \\
&+ \frac{\rho^+}{\rho^+ + \rho^-} N_f \cdot \nabla (p_{u^+} u^- - \sum_{j=1}^{3} p_{F_j^+} - p_{F_j^-}) \\
&- \frac{\rho^+}{\rho^+ + \rho^-} (N_f^+ - N_f^-) \bar{N}_f^{-1} \left( \sum_{j=1}^{3} p_{F_j^+} - p_{F_j^-} \right) - \rho^- N_f \cdot \nabla (p_{u^+} u^- - \sum_{j=1}^{3} p_{F_j^+} - p_{F_j^-}) \\
&\triangleq g_1 + g_2 + g_3 + g_4.
\end{align*}
$$

Here we need to be careful that $\int_{\mathbb{T}_2} \bar{\theta} dx'$ may not equal to 0.

Remark 4.1. Let $D_t = \partial_t + w_1 \partial_1 + w_2 \partial_2$. Thus, we have

$$
D_t^2 \bar{f} = \sum_{s,r=1}^{2} \left( -v_s v_r + \frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^{3} \left( E_{ij}^+ - E_{ij}^- \right) \right) \partial_s \partial_r \bar{f} + \text{low order terms}.
$$

The principal symbol of the operator on the right-hand side is

$$
(\partial_t \bar{\theta})^2 - \left( \frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^{3} \left( E_{ij}^+ - E_{ij}^- \right) \right)^2 = \left( \frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^{3} \left( E_{ij}^+ - E_{ij}^- \right) \right)^2.
$$

The negativity of this symbol is ensured by the stability condition (4.6)(see (1.12)). Therefore, $\bar{f}$ satisfies a strictly hyperbolic equation, and thus the system should be linearly well-posed.

Define the energy functional $E_s$ as

$$
E_s(\partial_t \bar{f}, \bar{f}) \triangleq \left\| (\partial_t + w_i \partial_i) \langle \nabla \rangle^{s-\frac{\lambda}{2}} \bar{f} \right\|_{L^2}^2 - \left\| v_i \partial_i \langle \nabla \rangle^{s-\frac{\lambda}{2}} \bar{f} \right\|_{L^2}^2
$$

(4.11)

$$
+ \frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^{3} \left\| E_{ij}^+ \partial_i \langle \nabla \rangle^{s-\frac{\lambda}{2}} \bar{f} \right\|_{L^2}^2 + \frac{\rho^-}{\rho^+ + \rho^-} \sum_{j=1}^{3} \left\| E_{ij}^- \partial_i \langle \nabla \rangle^{s-\frac{\lambda}{2}} \bar{f} \right\|_{L^2}^2,
$$

where $\langle \nabla \rangle^s f = \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \hat{f})$ and

$$
w_i = \frac{1}{\rho^+ + \rho^-} (\rho^+ u_i^+ + \rho^- u_i^-), \quad v_i = \frac{\sqrt{\rho^+ \rho^-}}{\rho^+ + \rho^-} (u_i^+ - u_i^-).
$$
Obviously, there exists $C(L_0) > 0$ so that
\[(4.12) \quad E_s(\partial_t \bar{f}, \bar{f}) \leq C(L_0) \left( \| \partial_t \bar{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \bar{f} \|_{H^{s+\frac{1}{2}}}^2 \right).\]
In addition, we deduce from the stability condition (4.6) that there exists $C(c_0, L_0)$ so that
\[(4.13) \quad \| \partial_t \bar{f} \|_{H^{s-\frac{1}{2}}}^2 + \| \bar{f} \|_{H^{s+\frac{1}{2}}}^2 \leq C(c_0, L_0) \left\{ E_s(\partial_t \bar{f}, \bar{f}) + \| \partial_t \bar{f} \|_{L^2}^2 + \| \bar{f} \|_{L^2}^2 \right\}.\]

Firstly, we have the estimate of $g$ defined by (4.9).

**Lemma 4.2.** It holds that
\[\|g\|_{H^{s-\frac{1}{2}}} \leq C(L_1).\]

**Proof.** The proof is similar to Lemma 6.2 in [18]. By using Proposition A.3 and Proposition A.4, we have
\[\|g_1\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \left( \| u^+_j u^-_j - E^+_j E^-_j \| \right) \leq C(L_1) \||u^+_j, E^\pm_j\|_{H^{s-\frac{1}{2}}} \| f \|_{H^{s+\frac{1}{2}}} \leq C(L_1),\]
\[\|g_2\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \| u^\pm \|_{H^{s-\frac{1}{2}}} \| \theta \|_{H^{s+\frac{1}{2}}} \leq C(L_1),\]
\[\|g_3, g_4\|_{H^{s-\frac{1}{2}}} \leq C(L_1) \left( \| \nabla (P_{a^-} u^- - \sum_{j=1}^3 p F_j^- \bar{F}_j^-) \|_{H^{s+\frac{1}{2}}} + \| \nabla (P_{a^+} u^+ - \sum_{j=1}^3 p F_j^+ \bar{F}_j^+) \|_{H^{s-\frac{1}{2}}} \right)\]
\[\leq C(L_1) \left( \| \nabla (P_{a^-} u^- - \sum_{j=1}^3 p F_j^- \bar{F}_j^-) \|_{H^{s}(\Omega^-)} + \| \nabla (P_{a^+} u^+ - \sum_{j=1}^3 p F_j^+ \bar{F}_j^+) \|_{H^{s}(\Omega^+)} \right)\]
\[\leq C(L_1) \| (u^\pm, F^\pm) \|_{H^{s}(\Omega^\pm)} \leq C(L_1).\]
The proof is finished. \(\square\)

Then we have the following estimate.

**Proposition 4.3.** Assume that $g \in L^\infty(0, T; H^{s-\frac{1}{2}}(\mathbb{T}^2))$. Given the initial data $(\bar{\theta}_0, \bar{f}_0) \in H^{s-\frac{1}{2}} \times H^{s+\frac{1}{2}}(\mathbb{T}^2)$, there exists a unique solution $(\bar{f}, \bar{\theta}) \in C([0, T]; H^{s+\frac{1}{2}} \times H^{s-\frac{1}{2}}(\mathbb{T}^2))$ to the system (4.8) so that
\[
\sup_{t \in [0, T]} \left( \| \partial_t \bar{f}(t) \|_{H^{s-\frac{1}{2}}}^2 + \| \bar{f}(t) \|_{H^{s+\frac{1}{2}}}^2 \right) \leq C(c_0, L_0) \left( \| \bar{\theta}_0 \|_{H^{s-\frac{1}{2}}}^2 + \| \bar{f}_0 \|_{H^{s+\frac{1}{2}}}^2 + \int_0^T \| g(\tau) \|_{H^{s+\frac{1}{2}}} \, d\tau \right) e^{C(c_0, L_1, L_2) T}.
\]

**Proof.** It suffices to prove the uniform estimates.

From the equation (4.8), we obtain
\[\partial_t^2 \bar{f} = -2 (w_1 \partial_t \bar{\theta} + w_2 \partial_2 \bar{\theta}) + \sum_{s, t = 1, 2} (-w_3 w_t - v_s v_t) \partial_s \partial_t \bar{f}\]
\[+ \sum_{s, r = 1}^2 \left( \frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^3 E^+_j E^+_j + \frac{\rho^-}{\rho^+ + \rho^-} \sum_{j=1}^3 E^-_j E^-_j \right) \partial_s \partial_r \bar{f} + g,
\]
which yields that
\[\frac{1}{2} \frac{d}{dt} \| (\partial_t + w_1 \partial_t) (\nabla)^{s-\frac{1}{2}} \bar{f} \|^2_{L^2(\mathbb{T}^2)}
\]

From Lemma A.2, one has
\[
\langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, (\nabla)^{s-\frac{1}{2}} \partial_i^2 \bar{f} + w_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) + \partial_t w_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f}) \rangle
\]
\[
= \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, (\nabla)^{s-\frac{1}{2}} (-2w_i \partial_i \partial_t \bar{f} + \sum_{s,r=1}^2 (-w_s w_r - v_s v_r) \partial_s \partial_r \bar{f})
\]
\[
+ \sum_{s,r=1}^2 \left( \frac{\rho^+}{\rho^+ + \rho} \sum_{j=1}^3 E_{sj}^+ E_{rj}^+ + \frac{\rho^-}{\rho^- + \rho} \sum_{j=1}^3 E_{sj}^- E_{rj}^- \right) \partial_s \partial_r \bar{f}
\]
\[
+ \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, (\nabla)^{s-\frac{1}{2}} (2w_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f}) + \partial_t w_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \bar{f})
\]
\[
= \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, -w_i \partial_i (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f})
\]
\[
+ \sum_{s,r=1}^2 \left( \frac{\rho^+}{\rho^+ + \rho} \sum_{j=1}^3 E_{sj}^+ E_{rj}^+ + \frac{\rho^-}{\rho^- + \rho} \sum_{j=1}^3 E_{sj}^- E_{rj}^- \right) \partial_s \partial_r \bar{f}
\]
\[
+ 2 \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, [w_i, (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f})
\]
\[
+ \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, [w_i, (\langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \bar{f})
\]
\[
+ \langle \partial_t + w_i \partial_i \rangle (\nabla)^{s-\frac{1}{2}} \bar{f}, \bar{f}
\]
\[
= \partial_t \mathcal{I}_1 + \cdots \mathcal{I}_5.
\]

From Lemma A.2, one has
\[
\mathcal{I}_3 \leq 2 \| (\partial_t + w_i \partial_i) (\nabla)^{s-\frac{1}{2}} \bar{f} \|_{L^2} \| [w_i, (\langle \nabla \rangle^{s-\frac{1}{2}}] \partial_t \partial_i \bar{f} \|_{L^2}
\]
\[
\leq C \mathcal{E}_s (\partial_t \bar{f}, \bar{f}) \frac{1}{2} \| w \|_{H^{\frac{s}{2}}} \| \partial_t \bar{f} \|_{H^{\frac{s}{2}}},
\]
and
\[
\mathcal{I}_4 \leq C \mathcal{E}_s (\partial_t \bar{f}, \bar{f}) \frac{1}{2} \left( \| w \|_{H^{\frac{s}{2}}}^2 + \| v \|_{H^{\frac{s}{2}}}^2 + \| \mathbf{E}^\pm \|_{H^{\frac{s}{2}}}^2 \right) \| \bar{f} \|_{H^{\frac{s}{2}}},
\]
In addition, it holds that
\[
\mathcal{I}_5 \leq \mathcal{E}_s (\partial_t \bar{f}, \bar{f}) \frac{1}{2} \left( \| \mathbf{g} \|_{H^{\frac{s}{2}}} + \| \partial_t w \|_{L^\infty} \| \bar{f} \|_{H^{\frac{s}{2}}} \right).
\]
It follows from integration by parts that
\[
\langle \partial_t (\nabla)^{s-\frac{1}{2}} \bar{f}, -w_i \partial_i (\nabla)^{s-\frac{1}{2}} \partial_t \bar{f} \rangle \leq \| w_i \|_{L^\infty} \| \partial_t \bar{f} \|_{H^{\frac{s}{2}}}^2,
\]
\[
\langle w_i \partial_i (\nabla)^{s-\frac{1}{2}} \bar{f}, -w_i \partial_i (\nabla)^{s-\frac{1}{2}} \partial_t \bar{f} \rangle + \frac{1}{2 \| \partial_t \|_{L^1}} \| w_i \partial_i (\nabla)^{s-\frac{1}{2}} \partial_t \bar{f} \|_{L^2}^2
\]
\[
= \langle w_i \partial_i (\nabla)^{s-\frac{1}{2}} \bar{f}, \partial_t w_i \partial_i (\nabla)^{s-\frac{1}{2}} \bar{f} \rangle \leq \| w \|_{L^\infty} \| \partial_t w \|_{L^\infty} \| \bar{f} \|_{H^{\frac{s}{2}}}^2,
\]
which implies
\[
(4.14) \quad \mathcal{I}_1 \leq -\frac{1}{2} \frac{d}{dt} \| w_i \partial_i (\nabla)^{s-\frac{1}{2}} \bar{f} \|_{L^2}^2 + (1 + \| w \|_{W^{1,\infty}} + \| \partial_t w \|_{L^\infty})^2 \left( \| \bar{f} \|_{H^{\frac{s}{2}}}^2 + \| \partial_t \bar{f} \|_{H^{\frac{s}{2}}}^2 \right).
\]
To estimate $I_2$, we can derive
\[
\left\langle \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f}, - w_i w_j \partial_i \partial_j \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \right\rangle - \frac{1}{2} \frac{d}{dt} \| w_i \partial_i \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[= - \left\langle w_i \partial_i \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f}, \partial_i w_j \partial_j \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \right\rangle + \left\langle \langle \nabla \rangle^{s-\frac{1}{2}} \partial_t \tilde{f}, \partial_t (w_i w_j) \partial_j \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \right\rangle
\]
\[\leq \|w\|_{L^\infty} (\|\partial_t w\|_{L^\infty} + \|\nabla w\|_{L^\infty}) \left(\|\tilde{f}\|_{H^{s+\frac{1}{2}}}^2 + \|\partial_t \tilde{f}\|_{H^{s-\frac{1}{2}}}^2\right),
\]
and similarly
\[
\left\langle w_k \partial_k \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f}, - w_i w_j \partial_i \partial_j \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \right\rangle \leq C \|w\|_{L^\infty} \|\nabla w\|_{L^\infty} \|\tilde{f}\|_{H^{s+\frac{1}{2}}}^2,
\]
as well as
\[
\left\langle \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} + w_k \partial_k \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f}, \left(\frac{\rho^+}{\rho^+ + \rho^-} \sum_{j=1}^3 E_{ij}^+ E_{ij}^+ + \frac{\rho^-}{\rho^+ + \rho^-} \sum_{j=1}^3 E_{ij}^- E_{ij}^+ \right) \partial_t \partial_j \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \right\rangle
\]
\[\leq - \frac{1}{2} \frac{\rho^\pm}{\rho^+ + \rho^-} \frac{d}{dt} \sum_{j=1}^3 \| E_{ij}^\pm \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[+ C \| \mathbf{E}^\pm \|_{L^\infty} \left(\| \partial_t \mathbf{E}^\pm \|_{L^\infty} + \| \nabla \mathbf{E}^\pm \|_{L^\infty} \right) \left(\| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2\right).
\]
Therefore, we obtain
\[
I_2 \leq \frac{1}{2} \frac{d}{dt} \| w_i \partial_i \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| v_i \partial_i \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[- \frac{1}{2} \frac{\rho^\pm}{\rho^+ + \rho^-} \frac{d}{dt} \sum_{j=1}^3 \| E_{ij}^\pm \partial_t \langle \nabla \rangle^{s-\frac{1}{2}} \tilde{f} \|_{L^2}^2
\]
\[+ C \left(1 + \| (\mathbf{u}^\pm, \mathbf{F}^\pm) \|_{W^{1,\infty}} + \| \partial_t (\mathbf{u}^\pm, \mathbf{F}^\pm) \|_{L^\infty}\right) \left(\| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2\right).
\]
Combining the estimates of $I_1, \ldots, I_5$ together yields that
\[
\frac{d}{dt} E_s (\partial_t \tilde{f}, \tilde{f}) \leq \| \mathbf{g} \|_{H^{s-\frac{1}{2}}}^2
\]
\[+ C(L_0) \left(1 + \| (\mathbf{u}^\pm, \mathbf{F}^\pm) \|_{H^{s-\frac{1}{2}}} + \| \partial_t (\mathbf{u}^\pm, \mathbf{F}^\pm) \|_{L^\infty}\right) \left(\| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2\right).
\]
On the other hand, it is easy to show that
\[
\frac{d}{dt} (\| \partial_t \tilde{f} \|_{L^2}^2 + \| \tilde{f} \|_{L^2}^2) \leq C(L_0) \left(\| \tilde{f} \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \tilde{f} \|_{H^{s-\frac{1}{2}}}^2\right) + \| \mathbf{g} \|_{L^2}^2.
\]
Let $\mathcal{E}(t) \triangleq \| \tilde{f}(t) \|_{H^{s+\frac{1}{2}}}^2 + \| \partial_t \tilde{f}(t) \|_{H^{s-\frac{1}{2}}}^2$. It follows from (14.13) that
\[
\mathcal{E}(t) \leq C(c_0, L_0) \left(\| \tilde{f}_0 \|_{H^{s-\frac{1}{2}}}^2 + \| \tilde{f}_0 \|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \| \mathbf{g}(\tau) \|_{H^{s-\frac{1}{2}}}^2 d\tau \right.
\]
\[+ \int_0^t \left(1 + \| (\mathbf{u}^\pm, \mathbf{F}^\pm)(\tau) \|_{H^{s-\frac{1}{2}}} + \| \partial_t (\mathbf{u}^\pm, \mathbf{F}^\pm) (\tau) \|_{L^\infty}\right)^3 \mathcal{E}(\tau) d\tau \right),
\]
which together with Lemma 2.1 gives
\[
\mathcal{E}(t) \leq C(c_0, L_0) \left(\| \tilde{f}_0 \|_{H^{s-\frac{1}{2}}}^2 + \| \tilde{f}_0 \|_{H^{s+\frac{1}{2}}}^2 + \int_0^t \| \mathbf{g}(\tau) \|_{H^{s-\frac{1}{2}}}^2 d\tau + C(L_1, L_2) \int_0^t \mathcal{E}(\tau) d\tau \right).
\]
By using Grönwall’s inequality, we conclude the desired estimate. \(\square\)
4.2. The linearized system of \((\omega^\pm, G^\pm)\). For the vorticity system (3.10), we introduce the following linearized system:

\[
\begin{aligned}
\begin{cases}
\partial_t \omega^\pm + u^\pm \cdot \nabla \omega^\pm - \sum_{j=1}^{3} \mathbf{F}_j^\pm \cdot \nabla G_j^\pm = \omega^\pm \cdot \nabla u^\pm - \sum_{j=1}^{3} G_j^\pm \cdot \nabla F_j^\pm, \\
\partial_t G_j^\pm + u^\pm \cdot \nabla G_j^\pm - F_j^\pm \cdot \nabla \omega^\pm = G_j^\pm \cdot \nabla u^\pm - \bar{\omega}^\pm \cdot \nabla F_j^\pm - 2 \sum_{s=1}^{3} \nabla u_s^\pm \times \nabla F_{sj}^\pm, \\
\omega^\pm(0, x) = \bar{\omega}_0^\pm, \quad G_j^\pm(0, x) = \bar{G}_j^\pm .
\end{cases}
\end{aligned}
\]

We first assume the existence of solutions to (4.15). Then it holds the following estimate.

Proposition 4.4. It holds that

\[
\begin{aligned}
\sup_{t \in [0,T]} \|\bar{\omega}^\pm(t)\|^2_{H^{s-1}(\Omega_f^\pm)} + \sum_{j=1}^{3} \|G_j^\pm(t)\|^2_{H^{s-1}(\Omega_f^\pm)} \\
\leq \left( 1 + \|\bar{\omega}_0^\pm\|^2_{H^{s-1}(\Omega_f^\pm)} + \sum_{j=1}^{3} \|\bar{G}_j^\pm\|^2_{H^{s-1}(\Omega_f^\pm)} \right) e^{C(L_1)T}.
\end{aligned}
\]

Proof. Using \(\partial_t f = u^\pm \cdot \mathbf{N}_f\) and integrating by parts, we obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega_f^\pm} |\nabla^{s-1} \bar{\omega}^\pm(t, x)|^2 + \sum_{j=1}^{3} |\nabla^{s-1} G_j^\pm(t, x)|^2 dx \\
= \int_{\Omega_f^\pm} \nabla^{s-1} \bar{\omega}^\pm \cdot \nabla^{s-1} \partial_t \omega^\pm + \sum_{j=1}^{3} \nabla^{s-1} G_j^\pm \cdot \nabla^{s-1} \partial_t G_j^\pm dx \\
+ \frac{1}{2} \int_{\Gamma_f} (|\nabla^{s-1} \bar{\omega}^\pm|^2 + \sum_{j=1}^{3} |\nabla^{s-1} G_j^\pm|^2)(u^\pm \cdot \mathbf{n}) d\sigma.
\end{aligned}
\]

From (4.15) and the fact that \(\mathbf{F}_j^\pm \cdot \mathbf{N}_f = 0\), we can derive

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega_f^\pm} |\nabla^{s-1} \bar{\omega}^\pm(t, x)|^2 + \sum_{j=1}^{3} |\nabla^{s-1} G_j^\pm(t, x)|^2 dx \\
\leq \int_{\Omega_f^\pm} \nabla^{s-1} \bar{\omega}^\pm \cdot \nabla^{s-1}[-u^\pm \cdot \nabla \bar{\omega}^\pm] + \sum_{j=1}^{3} \nabla^{s-1} G_j^\pm \cdot \nabla^{s-1}[-u^\pm \cdot \nabla \bar{G}_j^\pm] dx \\
+ \int_{\Omega_f^\pm} \nabla^{s-1} \bar{\omega}^\pm \cdot \nabla^{s-1}[\sum_{j=1}^{3} \mathbf{F}_j^\pm \cdot \nabla \bar{G}_j^\pm] + \sum_{j=1}^{3} \nabla^{s-1} \bar{\omega}^\pm \cdot \nabla^{s-1}[\mathbf{F}_j^\pm \cdot \nabla \bar{G}_j^\pm] dx \\
+ \frac{1}{2} \int_{\Gamma_f} (|\nabla^{s-1} \bar{\omega}^\pm|^2 + \sum_{j=1}^{3} |\nabla^{s-1} G_j^\pm|^2)(u^\pm \cdot \mathbf{n}) d\sigma \\
+ C(L_1) \left( 1 + \|\bar{\omega}^\pm(t)\|^2_{H^{s-1}(\Omega_f^\pm)} + \sum_{j=1}^{3} \|G_j^\pm(t)\|^2_{H^{s-1}(\Omega_f^\pm)} \right) \\
\leq \frac{1}{2} \int_{\Omega_f^\pm} -u^\pm \cdot \nabla(\nabla^{s-1} \bar{\omega}^\pm)^2 + \sum_{j=1}^{3} \nabla^{s-1} G_j^\pm|^2 dx
\end{aligned}
\]
where \( \tilde{\omega} \) and \( \tilde{G}_j \) satisfy (4.1) and (4.6). Given the initial data \((\omega_0^\pm, G_{j,0}^\pm) = (0,0), (\mathbf{g}_0^\pm, \mathbf{g}_j^\pm) \in L^1([0,T]; H^{s-1}(\Omega_f^\pm)) \), there exists a unique solution \((\omega^\pm, G_j^\pm) \in C([0,T]; H^{s-1}(\Omega_f^\pm)) \times H^{s-1}(\Omega_f^\pm)) \) to the system (4.15) satisfying the following estimate

\[
\sup_{t \in [0,T]} \left( \|\omega^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)} + \sum_{j=1}^{3} \|G_j^\pm(t)\|_{H^{s-1}(\Omega_f^\pm)} \right) \leq C(L_1,T) \|(\mathbf{g}_0^\pm, \mathbf{g}_j^\pm)\|_{L^1([0,T]; H^{s-1}(\Omega_f^\pm))}.
\]

Proof. Let \( \mathbf{W}^\pm = (\omega^\pm, G^\pm) \). We rewrite the system as

\[
L(\mathbf{W}^\pm) = \mathbf{g}^\pm.
\]

We define the flow map \( X^\pm(t,\cdot) \) as

\[
\frac{dX^\pm(t,\tilde{\omega})}{dt} = \mathbf{u}^\pm(t, X^\pm(t,\tilde{\omega})), \quad (t, \tilde{\omega}) \in [0,T] \times \Omega_f^\pm,
\]

with \( t = \tilde{t} \). Now we write \((t, x) \in Q_f^\pm \) and \((\tilde{t}, \tilde{x}) \in [0,T] \times \Omega_f^\pm \). Then we rewrite \( L \) in the new coordinate as

\[
\tilde{L}(\tilde{\mathbf{W}}^\pm) = \partial_{\tilde{t}} \tilde{\mathbf{W}}^\pm + \tilde{M}(\tilde{\mathbf{W}}^\pm) = \tilde{\mathbf{g}}^\pm,
\]

where \( \tilde{\mathbf{W}}^\pm(\tilde{t}, \tilde{x}) = \mathbf{W}^\pm(\tilde{t}, X^\pm(\tilde{t}, \tilde{x})), \tilde{\mathbf{g}}^\pm(\tilde{t}, \tilde{x}) = \mathbf{g}^\pm(\tilde{t}, X^\pm(\tilde{t}, \tilde{x})), \) and \( \tilde{M} \) is given by

\[
\tilde{M}(\tilde{\mathbf{W}}^\pm) = \begin{pmatrix}
- \sum_{j=1}^{3} \left( \tilde{F}_j^\pm(\tilde{t}, \tilde{x}) : \frac{\partial X^{s-1}(\tilde{t}, \cdot)}{\partial x} \right) \tilde{G}_j^\pm(\tilde{t}, \tilde{x}) \\
- \tilde{F}_1^\pm(\tilde{t}, \tilde{x}) : \frac{\partial X^{s-1}(\tilde{t}, \cdot)}{\partial x} \tilde{\omega}^\pm(\tilde{t}, \tilde{x}) \\
- \tilde{F}_2^\pm(\tilde{t}, \tilde{x}) : \frac{\partial X^{s-1}(\tilde{t}, \cdot)}{\partial x} \tilde{\omega}^\pm(\tilde{t}, \tilde{x}) \\
- \tilde{F}_3^\pm(\tilde{t}, \tilde{x}) : \frac{\partial X^{s-1}(\tilde{t}, \cdot)}{\partial x} \tilde{\omega}^\pm(\tilde{t}, \tilde{x})
\end{pmatrix}.
\]
We define
\[ D = \{ (\tilde{v}^+, \tilde{v}^-, \tilde{v}_1^+, \tilde{v}_2^+, \tilde{v}_3^+) \in C^\infty([0, T] \times \Omega_{j_0}^+) | \tilde{v}_i^+(T, \tilde{x}) = 0 \}. \]

Then \( W^\pm \) solves (4.17) if and only if for every \( \tilde{v}^\pm \in D \),
\[ \int_0^T \int_{\Omega_{j_0}^+} \tilde{L}(\tilde{W}^\pm) \cdot \tilde{v}^\pm \, dx \, dt = \int_0^T \int_{\Omega_{j_0}^+} \tilde{g}^\pm \cdot \tilde{v}^\pm \, dx \, dt. \]

Thanks to \( \text{div} \tilde{F}_j^\pm = 0, \tilde{F}_j^\pm \cdot \tilde{N}_f = 0 \) on \( \Gamma_f \), using the flow map \( X^\pm(t, \cdot) \), it is easy to show that \( W^\pm \) solves (4.17) if and only if for every \( \tilde{v}^\pm \in D \),
\[ \int_0^T \int_{\Omega_{j_0}^+} \tilde{W}^\pm \cdot L^*(\tilde{v}^\pm) \, dx \, dt = \int_0^T \int_{\Omega_{j_0}^+} \tilde{g}^\pm \cdot \tilde{v}^\pm \, dx \, dt, \]
where \( L^* \) denotes the dual of \( L \), i.e.,
\[ L^*(\tilde{v}) = - \begin{pmatrix} \partial_t \tilde{v}_0^+ - \sum_{j=1}^3 \tilde{F}_j^+, \frac{\partial X^+}{\partial x} \cdot \nabla \tilde{v}_j^+ \\ \partial_t \tilde{v}_1^+ - \tilde{F}_1^+, \frac{\partial X^+}{\partial x} \cdot \nabla \tilde{v}_0^+ \\ \partial_t \tilde{v}_2^+ - \tilde{F}_2^+, \frac{\partial X^+}{\partial x} \cdot \nabla \tilde{v}_0^+ \\ \partial_t \tilde{v}_3^+ - \tilde{F}_3^+, \frac{\partial X^+}{\partial x} \cdot \nabla \tilde{v}_0^+ \end{pmatrix}. \]

We denote
\[ L^*(\tilde{v}^\pm) = \tilde{V}^\pm. \]

It is easy to show that
\[ \sup_{t \in [0, T]} \| \tilde{V}^\pm \|_{L^2(\Omega_{j_0}^+)}(t) \leq C(L_1) \| \tilde{V}^\pm \|_{L^1([0, T]; L^2(\Omega_{j_0}^+))}. \]

Hence, the operator \( L^* \) is a bijection from \( D \) to \( L^*(D) \). Let \( N_0 \) be its inverse. By Hahn-Banach theorem, we can extend \( N_0 \) (denoted by \( N \)) to the space \( L^1([0, T]; L^2(\Omega_{j_0}^+)) \):
\[ N : L^1([0, T]; L^2(\Omega_{j_0}^+)) \rightarrow C([0, T]; L^2(\Omega_{j_0}^+)), \quad \tilde{V}^\pm \rightarrow \tilde{V}^\pm. \]

We denote by \( N^* \) the dual of \( N \):
\[ N^* : \mathcal{M}([0, T]; L^2(\Omega_{j_0}^+)) \rightarrow L^\infty([0, T]; L^2(\Omega_{j_0}^+)), \quad \tilde{g}^\pm \rightarrow \tilde{W}^\pm. \]

Then for \( \tilde{g}^\pm \in L^1([0, T]; L^2(\Omega_{j_0}^+)) \), \( \tilde{W}^\pm = N^*(\tilde{g}^\pm) \) satisfies (4.18) and
\[ \| \tilde{W}^\pm \|_{L^\infty([0, T]; L^2(\Omega_{j_0}^+))} \leq C(L_1) \| \tilde{g}^\pm \|_{L^1([0, T]; L^2(\Omega_{j_0}^+))}. \]

This proves the existence of the solution.

The regularity of the solution could be proved by using standard difference quotient method. The uniqueness is obvious. \( \square \)

Now we consider the system (4.17) with nonzero initial data
\[ \omega^\pm(0, x) = \omega_0^\pm, \quad G^\pm_j(0, x) = G^\pm_{j_0}, \]
where \( (\omega_0^+, G_{j_0}^+) \in H^{s-1}(\Omega_{j_0}^+) \times H^{s-1}(\Omega_{j_0}^+) \). Let \( \tilde{W}^\pm = W^\pm - (\omega_0^+, G_{j_0}^+) \). Then the problem is reduced to the case of zero initial data with \( g^\pm \) replace by \( g^\pm - M(\omega_0^+, G_{j_0}^+) \). From Lemma 4.5, we know that the solution \( \tilde{W}^\pm \) exists but with the loss of regularity. To recover the
desired regularity, we may first mollify the initial data, and then use the following uniform estimate for smooth solutions:

\[
\sup_{t \in [0, T]} \left(||\omega^\pm(t)||^2_{H^{s-1}(\Omega^\pm_T)} + \sum_{j=1}^{3} ||G^\pm_j(t)||^2_{H^{s-1}(\Omega^\pm_T)} \right)
\leq C(L_1, T) \left( ||g^\pm||_{L^2([0, T]; H^{s-1}(\Omega^\pm_T))} + ||\omega^\pm_0||^2_{H^{s-1}(\Omega^\pm_T)} + \sum_{j=1}^{3} ||G^\pm_{j,0}||^2_{H^{s-1}(\Omega^\pm_T)} \right).
\]

Thus, we can conclude the following proposition.

**Proposition 4.6.** Assume that \( f, u^\pm, F^\pm \) satisfy (4.1)-(4.6). Given the initial data \((\omega^\pm_0, G^\pm_{j,0}) \in H^{s-1}(\Omega^\pm_{f_0}) \times H^{s-1}(\Omega^\pm_{f_0})\), there exists a unique solution \((\omega^\pm, G^\pm_j) \in C([0, T]; H^{s-1}(\Omega^\pm_T) \times H^{s-1}(\Omega^\pm_T))\) to the system (4.15) satisfying the estimate (4.16).

For the solutions to (4.15), we also have

**Lemma 4.7.** It holds that

\[
\frac{d}{dt} \int_{\Gamma^\pm} \omega^\pm_3 dx' = 0, \quad \frac{d}{dt} \int_{\Gamma^\pm} G^\pm_{3j} dx' = 0.
\]

**Proof.** These are direct consequences of (4.15) and (4.7). From the fact that \( \partial_t u^\pm_i = \partial_t F^\pm_{3j} = 0 \) \((i = 1, 2)\) on \( \Gamma^\pm \), we have

\[
\frac{d}{dt} \int_{\Gamma^+} \omega^\pm_3 dx' = \int_{\Gamma^+} (-u^+_1 \partial_1 \omega^+_3 - u^+_2 \partial_2 \omega^+_3 + \partial_3 \omega^+_3) dx' + \sum_{j=1}^{3} (F^+_i \partial_1 G^+_3 + F^+_2 \partial_1 G^+_3 - G^+_3 \partial_3 F^+_3) dx'
\]

\[
= \int_{\Gamma^+} (\partial_1 u^+_1 + \partial_2 u^+_2 + \partial_3 u^+_3) \omega^+_3 dx' - \sum_{j=1}^{3} (\partial_1 F^+_i + \partial_2 F^+_2 + \partial_3 F^+_3) G^+_3 dx'
\]

\[
= 0.
\]

Similarly, it holds that

\[
\frac{d}{dt} \int_{\Gamma^+} G^+_3 dx' = -2 \int_{\Gamma^+} \sum_i (\partial_1 u^+_i \partial_2 F^+_i + \partial_2 u^+_i \partial_1 F^+_i) dx'
\]

\[
= 2 \int_{\Gamma^+} \sum_i (u^+_i \partial_1 \partial_2 F^+_i + u^+_i \partial_2 \partial_1 F^+_i) dx'
\]

\[
= 0.
\]

The proof for \( \omega^\pm_3, G^\pm_{3j} \) is similar. \( \square \)

5. CONSTRUCTION AND CONTRACTION OF THE ITERATION MAP

We assume that

\[
f_0 \in H^{s+\frac{1}{2}}(\mathbb{T}^2), \quad u^\pm_0, F^\pm_0 \in H^s(\Omega^\pm_{f_0}).
\]

In addition, we assume that there exists \( c_0 > 0 \) such that
1. \(-(1 - 2c_0) \leq f_0(x') \leq (1 - 2c_0)\);
2. \(\Lambda(F_0^\pm, v) \geq 2c_0\).

Let \(f_\pm = f_0\), and \(\Omega^\pm = \Omega^0_0\) be the reference region. We take the initial data \((f_I, (\partial_t f)_I, \omega^\pm_{*I}, G^\pm_{*I}, \beta^\pm_{*I}, \gamma^\pm_{*I})\) for the equivalent system as follows

\[
\begin{align*}
  f_I &= f_0, \quad (\partial_t f)_I = u^0_0(x', f_0(x')) \cdot (-\partial_1 f_0, -\partial_2 f_0, 1), \\
  \omega^\pm_{*I} &= \text{curl} \, u^0_0, \quad G^\pm_{*I} = \text{curl} \, F_0^\pm, \\
  \beta^\pm_{*I} &= \int_{T^2} u^0_0(x', \pm 1) \, dx', \quad \gamma^\pm_{*I} = \int_{T^2} F^\pm_0(x', \pm 1) \, dx',
\end{align*}
\]

which satisfy

\[
\|f_I\|_{H^{3/2} + 1} + \|(\omega^\pm_{*I}, G^\pm_{*I})\|_{H^{s-1}(\Omega^\pm)} + \|(\partial_t f)_I\|_{H^{3/2} + 1} + |\beta^\pm_{*I}| + |\gamma^\pm_{*I}| \leq M_0
\]

for some \(M_0 > 0\). Then we define the following functional space.

**Definition 5.1.** Given two positive constants \(M_1, M_2 > 0\) with \(M_1 > 2M_0\), we define the space \(\mathcal{X} = \mathcal{X}(T, M_1, M_2)\) be the collection of \((f, \omega^\pm, G^\pm, \beta^\pm, \gamma^\pm)\), which satisfies

\[
\begin{align*}
  \sup_{t \in [0, T]} \|f(t, \cdot) - f_\pm(t)\|_{H^{3/2}} &\leq \delta_0, \\
  \sup_{t \in [0, T]} \left( \|f(t)\|_{H^{3/2} + 1} + \|\partial_t f(t)\|_{H^{3/2} + 1} + \|(\omega^\pm, G^\pm)(t)\|_{H^{s-1}(\Omega^\pm)} + |\beta^\pm(t)| + |\gamma^\pm(t)| \right) &\leq M_1, \\
  \sup_{t \in [0, T]} \left( \|f(t)\|_{H^{3/2}} + \|(\partial_t f, \partial_t G^\pm)\|_{H^{s-2}(\Omega^\pm)} + |\partial_t \beta^\pm| + |\partial_t \gamma^\pm| \right) &\leq M_2,
\end{align*}
\]

**5.1. Recover the bulk region, velocity and deformation tensor field.** Recall

\[
\Omega^+_f = \{x \in \Omega | x_3 > f(t, x')\}, \quad \Omega^-_f = \{x \in \Omega | x_3 < f(t, x')\},
\]

and the harmonic coordinate map \(\Phi^\pm_f : \Omega^\pm_0 \rightarrow \Omega^\pm_f\). Define

\[
\hat{\omega}^\pm \triangleq P^\text{div}_f (\omega^\pm \circ \Phi^{-1}_f), \quad \hat{G}^\pm \triangleq P^\text{div}_f (G^\pm \circ \Phi^{-1}_f),
\]

where \(P^\text{div}_f\) is an project operator which maps a vector field \(\Omega^\pm_0\) to its divergence-free part. More precisely, \(P^\text{div}_f \omega^\pm = \omega^\pm - \nabla \phi^\pm\) with

\[
\begin{align*}
  \Delta \phi^\pm &= \text{div} \omega^\pm \quad \text{in} \ \Omega^\pm_f, \\
  \partial_3 \phi^\pm &= 0 \quad \text{on} \ \Gamma^\pm_f, \\
  \phi^\pm &= 0 \quad \text{on} \ \Gamma_f.
\end{align*}
\]

Obviously, we have \(\text{div} P^\text{div}_f \omega^\pm = 0\) in \(\Omega^\pm_f\), and \(e_3 \cdot P^\text{div}_f \omega^\pm = \omega^\pm_3\) on \(\Gamma^\pm_f\). Thus, \(P^\text{div}_f \omega^\pm\) satisfies conditions (C1) and (C2) on \(\Omega^\pm_f\). Following the same arguments, so does \(P^\text{div}_f G^\pm\).
Moreover, we have
\begin{align}
&\|\tilde{\omega}^\pm, \tilde{G}^\pm\|_{H^{s-1}(\Omega_f^\pm)} \leq C(M_1), \\
&\|\partial_t \tilde{\omega}^\pm, \partial_t \tilde{G}^\pm\|_{H^{s-2}(\Omega_f^\pm)} \leq C(M_1, M_2).
\end{align}
Then we define $u^\pm$ and $F^\pm$ as the solution of the following system
\begin{equation}
\begin{cases}
\text{curl } u^\pm = \tilde{\omega}^\pm, \quad \text{div } u^\pm = 0 & \text{in } \Omega_f^\pm, \\
u^\pm \cdot N_f = \partial_t f & \text{on } \Gamma_f, \\
u^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} u_t dx' = \beta^\pm_i (i = 1, 2) & \text{on } \Gamma^\pm,
\end{cases}
\end{equation}
and
\begin{equation}
\begin{cases}
\text{curl } F_j^\pm = \tilde{G}_j^\pm, \quad \text{div } F_j^\pm = 0 & \text{in } \Omega_f^\pm, \\
F_j^\pm \cdot N_f = 0 & \text{on } \Gamma_f, \\
F_j^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} F_{ij} dx' = \gamma^\pm_j (i = 1, 2) & \text{on } \Gamma^\pm.
\end{cases}
\end{equation}
From Proposition A.1 and (5.2), we deduce that
\begin{align}
&\|u^\pm\|_{H^s(\Omega_f^\pm)} \leq C(M_1) (\|\tilde{\omega}^\pm\|_{H^{s-1}(\Omega_f^\pm)} + \|\partial_t f\|_{H^{s-\frac{1}{2}}} + |\beta^\pm_1| + |\beta^\pm_2|) \leq C(M_1), \\
&\|F_j^\pm\|_{H^s(\Omega_f^\pm)} \leq C(M_1) (\|\tilde{G}^\pm_j\|_{H^{s-1}(\Omega_f^\pm)} + |\gamma^\pm_1| + |\gamma^\pm_2|) \leq C(M_1).
\end{align}
Moreover, there holds
\begin{equation}
u^\pm(0) = u^\pm_0, \quad F^\pm(0) = F^\pm_0.
\end{equation}
From the fact that
\begin{equation}
\partial_t (u^\pm \cdot N_f) = \partial_t u^\pm \cdot N_f + u^\pm \cdot \partial_t N_f = (\partial_t u^\pm + \partial_3 u^\pm \partial_t f) \cdot N_f + u^\pm \cdot \partial_t N_f
\end{equation}
on $\Gamma_f$, one can easily deduce that $\partial_t u^\pm$ satisfies
\begin{equation}
\begin{cases}
\text{curl } \partial_t u^\pm = \partial_t \tilde{\omega}^\pm, \quad \text{div } \partial_t u^\pm = 0 & \text{in } \Omega_f^\pm, \\
\partial_t u^\pm \cdot N_f = \partial_t f - \partial_3 f \partial_t u^\pm \cdot N_f + u^\pm_1 \partial_1 \partial_t f + u^\pm_2 \partial_2 \partial_t f & \text{on } \Gamma_f, \\
\partial_t u^\pm \cdot e_3 = 0, \quad \int_{\Gamma^\pm} \partial_t u_t^\pm dx = \partial_t \beta^\pm_i (i = 1, 2) & \text{on } \Gamma^\pm.
\end{cases}
\end{equation}
By Proposition A.1 again and (5.3), we get
\begin{equation}
\|\partial_t u^\pm\|_{H^{s-1}(\Omega_f^\pm)} \leq C(M_1, M_2),
\end{equation}
which implies
\begin{equation}
\|u^\pm(t)\|_{L^\infty(\Gamma_f)} \leq \|u^\pm_0\|_{L^\infty(\Gamma_f)} + \int_0^t \|\partial_t u^\pm\|_{L^\infty(\Gamma_f)} dt \leq \frac{M_0}{2} + TC(M_1, M_2).
\end{equation}
Applying similar arguments, we can show that
\begin{equation}
\|\partial_t F^\pm_j(t)\|_{H^{s-1}(\Omega_f^\pm)} \leq C(M_1, M_2),
\end{equation}
\begin{equation}
\|F^\pm_j(t)\|_{L^\infty(\Gamma_f)} \leq \frac{M_0}{2} + TC(M_1, M_2).
\end{equation}
Moreover, we have
\begin{equation}
\|f(t) - f_0\|_{L^\infty} \leq \|f(t) - f_0\|_{H^{s-\frac{1}{2}}} \leq T\|\partial_t f\|_{H^{s-\frac{1}{2}}} \leq TM_1,
\end{equation}
\begin{equation}
|\Lambda(F^\pm, v) - \Lambda(F^\pm_0, v_0)| \leq TC(\|\partial_t u^\pm\|_{L^\infty(\Gamma_f)} + \|\partial_t F^\pm\|_{L^\infty(\Gamma_f)}) \leq TC(M_1, M_2).
\end{equation}
Choose $T$ small enough such that

$$TM_1 \leq \min\{\delta_0, c_0\}, \quad TC(M_1) + TC(M_1, M_2) \leq \frac{M_0}{2}, \quad TC(M_1, M_2) \leq c_0,$$

and $L_0 = M_0, L_1 = M_1, L_2 = C(M_1, M_2)$. Then we can obtain that for any $t \in [0, T]$:

- $-(1 - c_0) \leq f(t, x') \leq (1 - c_0)$;
- $\Lambda(F, \nu)(t) \geq c_0$;
- $\|u^\pm, F^\pm\|_{L^\infty(\Gamma_t)} \leq L_0$;
- $\|f(t) - f_s\|_{H^{s-\frac{1}{2}}} \leq \delta_0$;
- $\|f(t)\|_{H^{s+\frac{1}{2}}} + \|\partial_t f(t)\|_{H^{s-\frac{1}{2}}} + \|u^\pm(t)\|_{H^s(\Omega_t^\pm)} + \|F^\pm(t)\|_{H^s(\Omega_t^\pm)} \leq L_1$;
- $\|\partial_t u^\pm, \partial_t F^\pm\|_{L^\infty(\Gamma_t)} \leq L_2$.

5.2. **Define the iteration map.** Given $(f, u^\pm, F^\pm)$ and define initial data as follows:

$$(\bar{f}_1(0), \bar{\omega}(0), \bar{G}^\pm(0)) = (f_0, (\partial_t f)_I, \bar{\omega}_{i1}^\pm, \bar{\xi}_{i1}^\pm).$$

We can solve $\bar{f}_1$ and $\bar{\omega}^\pm, \bar{G}^\pm$ by the linearized system (4.8) and (4.15). We define

$$\bar{\omega}_+^\pm = \omega_+^\pm \circ \Phi_f^\pm, \quad \bar{G}_+^\pm = G_+^\pm \circ \Phi_f^\pm,$$

$$\bar{\beta}_i^\pm(t) = \beta_i^\pm(0) - \int_0^t \int_{\Gamma_t^\pm} u_s^\pm \partial_s u_i^\pm - \sum_{j=1}^3 F_s^\pm \partial_s F_{ij}^\pm dx'd\tau,$$

$$\bar{\gamma}_{ij}^\pm(t) = \gamma_{ij}^\pm(0) - \int_0^t \int_{\Gamma_t^\pm} u_s^\pm \partial_s F_{ij}^\pm - F_s^\pm \partial_s F_{ij}^\pm dx'd\tau.$$

Then we have the iteration map $\mathcal{F}$ as follows:

$$\mathcal{F}(f, \omega_+^\pm, G_+^\pm, \beta_i^\pm, \gamma_{ij}^\pm) \overset{\text{def}}{=} (\bar{f}, \bar{\omega}_+^\pm, \bar{G}_+^\pm, \bar{\beta}_i^\pm, \bar{\gamma}_{ij}^\pm).$$

To ensure $\langle \bar{f} \rangle = \langle f_0 \rangle$ and $\int_{\Gamma_t^\pm} \partial_t \bar{f}(t, x') dx' = 0$ for $t \in [0, T]$, $\bar{f}$ in the above equation is given by

$$\bar{f}(t, x') = \bar{f}_1(t, x') - \langle \bar{f}_1 \rangle + \langle f_0 \rangle.$$

**Proposition 5.2.** There exist $M_1, M_2, T > 0$ depending on $c_0, \delta_0, M_0$ so that $\mathcal{F}$ is a map from $\mathcal{X}(T, M_1, M_2)$ to itself.

**Proof.** We know that the initial conditions are automatically satisfied according to the Definition 5.1. From Proposition 4.3 and Proposition 4.4, we have

$$\sup_{t \in [0, T]} \left( \|\bar{f}(t)\|_{H^{s+\frac{1}{2}}} + \|\partial_t \bar{f}(t)\|_{H^{s-\frac{1}{2}}} + \|\bar{\omega}_+^\pm(t)\|_{H^{s-2}(\Omega_t^\pm)} + \|\bar{G}_+^\pm(t)\|_{H^{s-1}(\Omega_t^\pm)} \right) \leq C(c_0, M_0)e^{C(M_1, M_2)T}.$$

From the equation (4.8), (4.15), we deduce that

$$\sup_{t \in [0, T]} \left( \|\partial_t^2 \bar{f}(t)\|_{H^{s-\frac{1}{2}}} + \|\partial_t \bar{\omega}_+^\pm, \partial_t \bar{G}_+^\pm\|_{H^{s-2}(\Omega_t^\pm)} \right) \leq C(M_1).$$

Obviously, we have

$$|\bar{\beta}_i^\pm(t)| + |\bar{\gamma}_{ij}^\pm(t)| \leq M_0 + TC(M_1),$$

$$|\partial_t \bar{\beta}_i^\pm(t)| + |\partial_t \bar{\gamma}_{ij}^\pm(t)| \leq C(M_1),$$

$$\|\bar{f}(t) - f_s\|_{H^{s-\frac{1}{2}}} \leq \int_0^t \|\partial_t \bar{f}(\tau)\|_{H^{s-\frac{1}{2}}} d\tau.$$
We firstly take $M_2 = C(M_1)$ and then take $M_1$ large enough so that
\begin{equation}
C(c_0, M_0) < M_1/2.
\end{equation}
Next, we take $T$ small enough which only depends only on $c_0, \delta_0, M_0$ so that all other conditions in Definition 5.1 are satisfied.

5.3. Contraction of the iteration map. Now we prove the contraction of the iteration map $F$. Let $(f^A, \omega^+_A, G^+_A, \beta^+_i, \gamma^+_i), (f^B, \omega^+_B, G^+_B, \beta^+_i, \gamma^+_i) \in X(T, M_1, M_2)$, and $(f^C, \omega^+_C, G^+_C, \beta^+_i, \gamma^+_i) = F(f^C, \omega^+_C, G^+_C, \beta^+_i, \gamma^+_i)$ for $C = A, B$. In addition, we use $g^D$ to denote the difference $g^A - g^B$. For instance, $f^D = f^A - f^B, \omega^+D = \omega^+_A - \omega^+_B$.

**Proposition 5.3.** There exists $T > 0$ depending on $c_0, \delta_0, M_0$ so that
\begin{equation}
E^{D} \triangleq \sup_{t \in [0, T]} \left( \|f^D(t)\|_{H^s} + \|\partial_t f^D(t)\|_{H^{s-\frac{1}{2}}} + \|\omega^D(t)\|_{H^{s-2}(\Omega^\pm)} + \|G^D(t)\|_{H^{s-2}(\Omega^\pm)} + |\beta^D(t)| + |\gamma^D(t)| \right)
\end{equation}
\begin{equation}
\leq \frac{1}{2} \sup_{t \in [0, T]} \left( \|f^D(t)\|_{H^s} + \|\partial_t f^D(t)\|_{H^{s-\frac{1}{2}}} + \|\omega^D(t)\|_{H^{s-2}(\Omega^\pm)} + \|G^D(t)\|_{H^{s-2}(\Omega^\pm)} + |\beta^D(t)| + |\gamma^D(t)| \right) \triangleq E^{D}.
\end{equation}

**Proof.** Firstly, we have following elliptic estimate
\begin{equation}
\|\Phi^\pm_{f^A} - \Phi^\pm_{f^B}\|_{H^{s-1}(\Omega^\pm)} \leq C(M_1)\|f^A - f^B\|_{H^{s-\frac{1}{2}}} \leq CE^D.
\end{equation}
We cannot estimate the difference between $u^A$ and $u^B$ directly, since they are defined on different regions. For this end, we introduce for $C = A, B$,
\begin{equation}
u^+_C = u^+_C \circ \Phi^\pm_{f^C}, \quad F^+_j = F^+_j \circ \Phi^\pm_{f^C}.
\end{equation}

Now we show that
\begin{equation}
\|u^+_C\|_{H^{s-1}(\Omega^\pm)} + \|F^+_j\|_{H^{s-1}(\Omega^\pm)} \leq CE^D
\end{equation}
We introduce
\begin{equation}
\begin{aligned}
curl_C v^+_s &= (\curl(v^+_s \circ (\Phi^\pm_{f^C})^{-1})) \circ \Phi^\pm_{f^C}, \\
\div_C v^+_s &= (\div(v^+_s \circ (\Phi^\pm_{f^C})^{-1})) \circ \Phi^\pm_{f^C},
\end{aligned}
\end{equation}
for vector field $v^+_s$ defined on $\Omega^\pm_s$. Then it holds for $C = A, B$ that
\begin{equation}
\begin{aligned}
\curl_C u^+_C &= \omega^+_C \quad \text{in } \Omega^\pm_s, \\
\div_C u^+_C &= 0 \quad \text{in } \Omega^\pm_s, \\
u^+_C \cdot N^C_f &= \partial_t f^C \quad \text{on } \Gamma^\pm_s, \\
u^+_C \cdot e_3 &= 0, \quad \int_{\Gamma^+} u^+_C \cdot dx' = \beta^+_i \quad \text{on } \Gamma^\pm.
\end{aligned}
\end{equation}
Thus, we can deduce
\begin{equation}
\begin{aligned}
\curl_{A^C} u^+_C &= \omega^+_C + (\curl_{B^C} - \curl_{A^C}) u^+_C \quad \text{in } \Omega^\pm_s, \\
\div_{A^C} u^+_C &= (\div_{B^C} - \div_{A^C}) u^+_C \quad \text{in } \Omega^\pm_s, \\
u^+_C \cdot N^C_f &= \partial_t f^C + u^+_C \cdot (N^C_f - N^C_A) \quad \text{on } \Gamma^\pm_s, \\
u^+_C \cdot e_3 &= 0, \quad \int_{\Gamma^+} u^+_C \cdot dx' = \beta^+_i \quad \text{on } \Gamma^\pm.
\end{aligned}
\end{equation}
It is direct to obtain
\begin{equation}
\|(\curl_{B^C} - \curl_{A^C}) u^+_C\|_{H^{s-2}(\Omega^\pm)} \leq C\|\Phi^\pm_{f^A} - \Phi^\pm_{f^B}\|_{H^{s-1}(\Omega^\pm)}
\end{equation}
\[ \leq C\|f^D\|_{H^{s-\frac{1}{2}}} \leq CE^D, \]

and similarly,
\[ \|(\text{div}_B - \text{div}_A)u^+_s\|_{H^{s-2}(\Omega^+_s)} \leq CE^D, \]
\[ \|u^+_s \cdot (N_{f^B} - N_{f^B})\|_{H^{s-\frac{3}{2}}} \leq CE^D. \]

Then applying Proposition A.1 yields that
\[ \|u^+_s\|_{H^{s-1}(\Omega^+_s)} \leq C \left( \|u^+_s\|_{H^{s-2}(\Omega^+_s)} + \|\partial_t f^D\|_{H^{s-1} + E^D} \right) \leq CE^D. \]

Similarly, we have
\[ \|F_s^D\|_{H^{s-1}(\Omega^+_s)} \leq CE^D. \]

Recall that
\[ \partial_t \tilde{f}_1^D = \tilde{\theta}^D, \]
\[ \partial_t \tilde{\theta}^D = -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u^+_s + \rho^- u^-_s)\partial_t \tilde{\theta}^D + (\rho^+ u^+_s + \rho^- u^-_s)\partial_2 \tilde{\theta}^D \right) \]
\[ - \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^+ u^+_s u^+_r - \rho^+ \sum_{j=1}^{3} E^A_{sj} E^A_{rj} \right) \partial_s \partial_r \tilde{f}_1^B \]
\[ - \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^+ u^+_s u^+_r - \rho^+ \sum_{j=1}^{3} E^A_{sj} E^A_{rj} \right) \partial_s \partial_r \tilde{f}_1^B \]
\[ - \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^+ u^+_s u^+_r - \rho^+ \sum_{j=1}^{3} E^A_{sj} E^A_{rj} \right) \partial_s \partial_r \tilde{f}_1^B \]
\[ + g^A - g^B, \]

and for \( C = A, B, \)
\[ g^C = \frac{1}{\rho^+ + \rho^-} (N_{f^C}^+ - N_{f^C}^-) \tilde{N}_{f^C}^{-1} \left( \sum_{s,r=1}^{2} (u^+_s u^+_r + C) - \sum_{j=1}^{3} E^C_{s,j} E^C_{r,j} \right) \]
\[ - \sum_{s,r=1}^{2} (u^+_s u^+_r - C) + \sum_{j=1}^{3} E^C_{s,j} E^C_{r,j} \partial_s \partial_r f^C \)
where \( v_{x} \) and \( v_{\bar{x}} \) are the \( x \)-component and \( \bar{x} \)-component of \( v \), respectively.

Recalling the fact that

\[
\rho \frac{\partial v}{\partial t} + (\rho v + p)_{\bar{x}} - \nabla \cdot \bar{T} = \rho \sum_{j=1}^{3} p_{ij}^{c} C_{ij}^{c},
\]

we can show that

\[
\rho \frac{\partial v}{\partial t} + (\rho v + p)_{\bar{x}} - \nabla \cdot \bar{T} = \rho \sum_{j=1}^{3} p_{ij}^{c} C_{ij}^{c}.
\]

Similar to the proof of Lemma 4.2, we can show that

\[
H^{\bar{x}}(x_1, x_2) \text{ is the trace of } v \text{ on } \Gamma \text{ which interpreted as } v(x_1, x_2, f^C(x_1, x_2)).
\]

Similar to the proof of Proposition 4.4, it can be verified that

\[
\frac{d}{dt} \left( \bar{E}^D_{f, t} \right) + \| \bar{f}_{+}^{D} \|_{L^2}^{2} + \| \partial_t \bar{f}_{+}^{D} \|_{L^2}^{2} \leq C (E^D + \bar{E}^D_{f})
\]

where

\[
\bar{E}^D_{f} = \sup_{t \in [0, T]} \left( \| \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - \frac{1}{2}} + \| \partial_t \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - \frac{1}{2}} \right).
\]

Recalling the fact that

\[
\| \bar{f}_{+}^{D} \|_{H^{\bar{x}} - \frac{1}{2}} + \| \partial_t \bar{f}_{+}^{D} \|_{H^{\bar{x}} - \frac{1}{2}} \leq C \left( \bar{E}^D_{f, t} + \| \bar{f}_{+}^{D} \|_{L^2}^{2} + \| \partial_t \bar{f}_{+}^{D} \|_{L^2}^{2} \right),
\]

we obtain

\[
\sup_{t \in [0, T]} \left( \| \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - 1} + \| \partial_t \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - \frac{1}{2}} \right) \leq C (e^{CT} - 1) E^D,
\]

which induces

\[
\sup_{t \in [0, T]} \left( \| \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - 2} + \| \partial_t \bar{f}_{+}^{D} (t) \|_{H^{\bar{x}} - \frac{1}{2}} \right) \leq C (e^{CT} - 1) E^D.
\]

Similarly to the proof of Proposition 4.4, it can be verified that

\[
\sup_{t \in [0, T]} \left( \| \bar{\omega}_{+}^{D} (t) \|_{H^{\bar{x}} - 2} + \| \bar{G}_{+}^{D} \|_{H^{\bar{x}} - 2} \right) \leq C (e^{CT} - 1) E^D.
\]

From the equation

\[
\bar{\beta}_{+}^{C} (t) = \bar{\beta}_{+}^{C} (0) - \int_{0}^{t} \int_{T_{+}^{x}} u^{+}_{x} C \partial_{s} u^{+}_{x} C - \sum_{j=1}^{3} F_{s}^{+} C \partial_{s} F_{ij}^{+} C \, dx' \, dt,
\]
we have
\begin{equation}
(5.18) \quad |\bar{\beta}_i^D(t)| \leq |\beta_{ij}^D| + TCE^D.
\end{equation}

It is similar to show that
\begin{equation}
(5.19) \quad |\bar{\gamma}_{ij}^D(t)| \leq |\gamma_{ij}^D| + TCE^D.
\end{equation}

Thus, thanks to (5.13) and (5.16)–(5.19), we can conclude that
\[ E^D \leq C(e^{CT} - 1 + T)E^D. \]

Taking \( T \) small enough depending on \( c_0, \delta_0, M_0 \), we obtain the proof of the proposition. \( \square \)

5.4. The limit system. It follows from Proposition 5.2 and Proposition 5.3 that there exists a unique fixed point \((f, \omega^\pm, \mathbf{G}^\pm, \beta^\pm, \gamma^\pm)\) of the map \( \mathcal{F} \) in \( \mathcal{X}(T, M_1, M_2) \). In addition, from the construction of \( \mathcal{F} \), we have that \((f, \omega^\pm, \mathbf{G}^\pm, \beta^\pm, \gamma^\pm) = (f, \omega^\pm \circ \Phi_f^{-1}, \mathbf{G}^\pm \circ \Phi_f^{-1}, \beta^\pm, \gamma^\pm)\) satisfies
\begin{equation}
(5.20) \quad \partial_t f = \mathcal{P} \theta,
\end{equation}
\[ \partial_t \theta = -\frac{2}{\rho^+ + \rho^-} \left( (\rho^+ u^+_1 + \rho^- u^-_1) \partial_1 \theta + (\rho^+ u^+_2 + \rho^- u^-_2) \partial_2 \theta \right) \]
\[ - \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^+ u^+_s u^+_r - \rho^- \sum_{j=1}^{3} E^+_{s,j} E^+_{r,j} \partial_s \partial_r f \right) \]
\[ - \frac{1}{\rho^+ + \rho^-} \sum_{s,r=1}^{2} \left( \rho^- u^-_s u^-_r - \rho^- \sum_{j=1}^{3} E^-_{s,j} E^-_{r,j} \partial_s \partial_r f \right) \]
\[ + \frac{1}{\rho^+ + \rho^-} \left( N_f^+ - N_f^- \right) \tilde{N}_f^{-1} \mathcal{P} \left( \sum_{s,r=1}^{2} \left( u^+_s u^+_r - \rho^- \sum_{j=1}^{3} E^+_{s,j} E^+_{r,j} \right) \partial_s \partial_r f \right) \]
\[ - \frac{1}{\rho^+ + \rho^-} \left( N_f^+ - N_f^- \right) \tilde{N}_f^{-1} \mathcal{P} \left( \sum_{s,r=1}^{2} \left( u^-_s u^-_r - \rho^- \sum_{j=1}^{3} E^-_{s,j} E^-_{r,j} \right) \partial_s \partial_r f \right) \]
\[ + \frac{1}{\rho^+ + \rho^-} \left( N_f^+ - N_f^- \right) \tilde{N}_f^{-1} \mathcal{P} \left( (u^+_1 - u^-_1) \partial_1 \theta + (u^+_2 - u^-_2) \partial_2 \theta \right) \]
\[ + \frac{1}{\rho^+ + \rho^-} N_f \cdot \nabla (\rho^+ p_{u^+, u^+} - \rho^- \sum_{j=1}^{3} p_{E^+_j, E^+_j}) \]
\[ + \frac{1}{\rho^+ + \rho^-} N_f \cdot \nabla (\rho^- p_{u^-, u^-} - \rho^- \sum_{j=1}^{3} p_{E^-_j, E^-_j}) \]
\begin{equation}
(5.21) \quad - \frac{1}{\rho^+ + \rho^-} \left( N_f^+ - N_f^- \right) \tilde{N}_f^{-1} \mathcal{P} N_f \cdot \nabla (p_{u^+, u^+} - \rho^- \sum_{j=1}^{3} p_{E^+_j, E^+_j} - p_{u^-, u^-} + \rho^+ \sum_{j=1}^{3} p_{E^-_j, E^-_j}),
\end{equation}
where \((u^\pm, F^\pm)\) solves the div-curl system

\[
\begin{aligned}
\text{curl } u^\pm &= P_j^{\text{div}} \omega^\pm, \quad \text{div } u^\pm = 0 \quad \text{in } \Omega_j^\pm, \\
u^\pm \cdot N_f &= \delta_i f \quad \text{on } \Gamma_f, \\
u_3^\pm &= 0 \quad \text{on } \Gamma^\pm, \\
\int_{\Gamma^+} u_i^+ dx' &= \beta_i^+, \\
\partial_t \beta_i^+ &= -\int_{\Gamma^\pm} (u_j^\pm \partial_j u_i^\pm - \sum_{j=1}^3 F_{sj}^\pm \partial_s F_{ij}^\pm) dx',
\end{aligned}
\]

and

\[
\begin{aligned}
\text{curl } F_j^\pm &= P_j^{\text{div}} G_j^\pm, \quad \text{div } F_j^\pm = 0 \quad \text{in } \Omega_j^\pm, \\
F_j^\pm \cdot N_f &= 0 \quad \text{on } \Gamma_f, \\
F_{3j}^\pm &= 0 \quad \text{on } \Gamma^\pm, \\
\int_{\Gamma^+} F_j^+ dx' &= \gamma_{ij}^+, \\
\partial_t \gamma_{ij}^+ &= -\int_{\Gamma^\pm} (u_i^\pm \partial_s F_{ij}^\pm - F_{sj}^\pm \partial_s u_i^\pm) dx'.
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t \omega^\pm + u^\pm \cdot \nabla \omega^\pm - \sum_{i=1}^3 F_i^\pm \cdot \nabla G_i^\pm &= \omega^\pm \cdot \nabla u^\pm - \sum_{i=1}^3 G_i^\pm \cdot \nabla F_i^\pm, \\
\partial_t G_j^\pm + u^\pm \cdot \nabla G_j^\pm - F_j^\pm \cdot \nabla \omega^\pm &= G_j^\pm \cdot \nabla u^\pm - \omega^\pm \cdot \nabla F_j^\pm - 2 \sum_{s=1}^3 \nabla u_s^\pm \times \nabla F_{sj}^\pm.
\end{aligned}
\]

Here we recall that \(p_{u_i^\pm, u_2^\pm}\) in (5.21) is defined by

\[
\begin{aligned}
\Delta p_{u_i^\pm, u_2^\pm} &= -\text{tr}(\nabla u_i^\pm \nabla u_2^\pm) \quad \text{in } \Omega_f^\pm, \\
p_{u_i^\pm, u_2^\pm} &= 0 \quad \text{on } \Gamma_f, \\
e_3 \cdot \nabla F_{u_i^\pm, u_2^\pm} &= 0 \quad \text{on } \Gamma^\pm.
\end{aligned}
\]

To finish the proof of Theorem 1.1, we need to show that the limit system (5.20)-(5.25) is equivalent to the original system (1.2)-(1.5). We introduce the pressure \(p^\pm\) of the fluid by

\[
p^\pm = H_j^\pm p_j^\pm + \rho^\pm p_{u_i^\pm, u_2^\pm} - \rho^\pm \sum_{j=1}^3 p_{F_j^\pm, F_j^\pm},
\]

where

\[
p_j^\pm = \tilde{p}_j^\pm = \tilde{N}^{-1}_j p(g^+ - g^-)
\]

with

\[
g^\pm = 2(u_1^\pm \partial_1 \theta + u_2^\pm \partial_2 \theta) + N \cdot \nabla (p_{u_i^\pm, u_2^\pm} - p_{h_i^\pm, h_2^\pm}) + \sum_{i,j=1}^2 \sum_{l=1}^2 (u_i^\pm u_j^\pm - \sum_{l=1}^3 F_{ij}^\pm F_{il}^\pm) \partial_i \partial_j f.
\]

The key idea to prove the consistence is to show that

\[
\begin{aligned}
\text{div } w^\pm &= 0, \quad \text{curl } w^\pm = 0 \quad \text{in } \Omega_f^\pm, \\
w^\pm \cdot N_f &= 0 \quad \text{on } \Gamma_f, \\
w_3^\pm &= 0 \quad \text{on } \Gamma^\pm, \quad \int_{\Gamma^\pm} w_i^\pm dx' = 0 (i = 1, 2).
\end{aligned}
\]
for
\[ w^\pm = \partial_t u^\pm + u^\pm \cdot \nabla u^\pm - \sum_{j=1}^{3} F_j^\pm \cdot \nabla F_j^\pm + \nabla p^\pm, \]
or
\[ w^\pm = \partial_t F_j + u \cdot \nabla F_j - F_j \cdot \nabla u, \quad j = 1, 2, 3. \]
The proof of (5.26) can be accomplished by following [18, Section 9] line by line, so we omit the details here.

6. Proof of Theorem 1.2

In this section, we consider the system (1.7)-(1.10). Since the proof of Theorem 1.2 is quite analogous to the proof of Theorem 1.1, we only present main steps which are different from the problem (1.2)-(1.5).

We first note that the stability condition \( \text{rank}(F) = 2 \) is equivalent to (1.11) with \( \rho^+ = 0 \) and \( F^- = F \), which further implies that there exists \( c_0 > 0 \) such that
\[
\Lambda(F) \overset{\text{def}}{=} \inf_{x \in \Gamma, \varphi_1, \varphi_2} \sum_{j=1}^{3} (F_{1j} \varphi_1 + F_{2j} \varphi_2)^2 \geq c_0.
\]
Following the derivation of (3.4), one can deduce that
\[
\partial_t f = \theta,
\]
\[
\partial_t \theta = -2(u_1 \partial_1 \theta + u_2 \partial_2 \theta) - \frac{1}{\rho} \mathbf{N} \cdot \nabla p - \sum_{s,r=1}^{2} u_s u_r \partial_s \partial_r f + \sum_{j=1}^{3} \sum_{s,r=1}^{2} E_{sj} E_{rj} \partial_s \partial_r f.
\]
with \( p = \sum_{j=1}^{3} p_{F_j,F_j} - p_{u,u}. \)

By the stability condition (6.1), we obtain
\[
E_s(\partial_t f, f) \overset{\text{def}}{=} \left\| (\partial_t + u_i \partial_i) (\nabla)^{s-\frac{1}{2}} f \right\|_{L^2}^2 + \sum_{j=1}^{3} \left\| E_{sj} \partial_t (\nabla)^{s-\frac{1}{2}} f \right\|_{L^2}^2 \geq \left\| (\partial_t + u_i \partial_i) (\nabla)^{s-\frac{1}{2}} f \right\|_{L^2}^2 + c_0 \sum_{i=1}^{2} \left\| \partial_t (\nabla)^{s-\frac{1}{2}} f \right\|_{L^2}^2.
\]
Consequently, it holds that
\[
\| \partial_t f \|_{H^{s-\frac{1}{2}}}^2 + \| f \|_{H^{s+\frac{1}{2}}}^2 \leq C(c_0, L_0) \left\{ E_s(\partial_t f, f) + \| \partial_t f \|_{L^2}^2 + \| f \|_{L^2}^2 \right\},
\]
which is actually (4.13). Then, the remain parts of the proof can follow the proof of Theorem 1.1 step by step.

Appendix A

A.1. Div-Curl system. From Section 5 of [18], we know that for each div-curl system
\[
\begin{aligned}
\text{curl} u &= \omega, & \text{div} u &= g & \text{in} & & \Omega^+,
\text{u \cdot N}_f &= \vartheta & \text{on} & & \Gamma_f,
\text{u \cdot e}_3 &= 0, & \int_{T^2} u_i dx' &= \alpha_i (i = 1, 2) & \text{on} & & \Gamma^+.
\end{aligned}
\]
with \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s \geq 2 \) and satisfying
\[
-(1-c_0) \leq f \leq (1-c_0),
\]
have a unique solution.

**Proposition A.1.** Let \( \sigma \in [2,s] \) be an integer. Given \( \omega, g \in H^{\sigma-1}(\Omega_f^+), \vartheta \in H^{\sigma-\frac{1}{2}}(\Gamma_f) \) with the compatibility condition:
\[
\int_{\Omega_f^+} g dx = \int_{\Gamma_f} \vartheta ds,
\]
and \( \omega \) satisfies
\[
div \omega = 0 \quad \text{in} \quad \Omega_f^+, \quad \int_{\Gamma^+} \omega_\nu ds' = 0,
\]
Then there exists a unique \( u \in H^\sigma(\Omega_f^+) \) of the div-curl system (A.1) so that
\[
\|u\|_{H^\sigma(\Omega_f^+)} \leq C(c_0, \|f\|_{H^{s+\frac{1}{2}}}) \left( \|\omega\|_{H^{\sigma-1}(\Omega_f^+)} + \|g\|_{H^{\sigma-1}(\Omega_f^+)} + \|\vartheta\|_{H^{\sigma-\frac{1}{2}}(\Gamma_f)} + |\alpha_1| + |\alpha_2| \right).
\]

**A.2. Commutator estimate.**

**Lemma A.2.** If \( s > 1 + \frac{d}{2} \), then we have
\[
\|[a, \langle \nabla \rangle^s] u\|_{L^2} \leq C\|a\|_{H^s} \|u\|_{H^{s-1}}.
\]

**A.3. Sobolev estimates of DN operator.**

**Proposition A.3.** If \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then it holds that for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\|N_f^+ \psi\|_{H^\sigma} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^{s+1}}.
\]
Moreover, it holds that for any \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\|(N_f^+ - N_f^-) \psi\|_{H^\sigma} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^s},
\]
where \( K_{s+\frac{1}{2},f} \) is a constant depending on \( c_0 \) and \( \|f\|_{H^{s}} \).

**Proposition A.4.** If \( f \in H^{s+\frac{1}{2}}(\mathbb{T}^2) \) for \( s > \frac{5}{2} \), then it holds that for any \( \sigma \in \left[ -\frac{1}{2}, s - \frac{1}{2} \right] \),
\[
\|G_f^\pm \psi\|_{H^{\sigma+1}} \leq K_{s+\frac{1}{2},f} \|\psi\|_{H^\sigma},
\]
where \( G_f^\pm \triangleq (N_f^\pm)^{-1} \).

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