The Thermodynamic Approach to Whole-Life Insurance: A Method for Evaluation of Surrender Risk

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Abstract

We introduce a collective model for life insurance where the heterogeneity of each insured, including the health state, is modeled by a diffusion process. This model is influenced by concepts in statistical mechanics. Using the proposed framework, one can describe the total pay-off as a functional of the diffusion process, which can be used to derive a level premium that evaluates the risk of lapses due to the so-called adverse selection. Two numerically tractable models are presented to exemplify the flexibility of the proposed framework.

Keywords: Life insurance, Surrender Risk, Collective model, Feynman-Kac Formula

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1 Introduction

The risk of lapses, also referred to as surrender risk, is the instability associated with unexpected lapses of insurance contracts, which may result in a huge loss for the insurer. As insurance contracts are collective in nature, we can divide the cause of lapses into two categories: homogeneous and heterogeneous (among the policy holders). The former is basically due to macro-economic shocks — typically changes in interest rates, recessions, inflation, and so forth. The surrender risk arising from interest rate fluctuations, for example, can be evaluated using option-pricing type technologies (see e.g. [3]).
The latter, i.e. heterogeneous causes, can be further divided into two groups: economic and non-economic. A policyholder may surrender because she is unexpectedly short of money while the economy is, as a whole, healthy. Such a cause is classified as heterogeneous-economic (see e.g. [13]).

Among non-economic causes, demographic heterogeneity, which is variation in the force of mortality, has been a central issue both academically and practically, as it has been recognized as a (potential) cause of the so-called moral hazard or adverse selection. The risk from heterogeneity has been recognized in insurance (since the 19th century!), as is pointed out in the seminal paper by G.A. Akerlof [2], which made clear the role of the “asymmetry of information” in adverse selection. M. Rothschild and Nobel laureate J. Stiglitz proved the non-existence of an economic equilibrium under asymmetry in [16]. According to that paper, in a “rational world”, adverse selection leads to the non-existence of an equilibrium, implying potential instability of a life insurance contract.

Since then there have been many studies on life insurance, however most empirical studies do not seem to observe the predicted negative effects of adverse selection (see e.g. [6]). Some recent studies, like [12], suggest the effect of so-called advantageous selection: healthier people might be more risk averse. Thus many theoretical studies have been rather interested in modelling the effect of heterogeneity on the surrender risk without the hypothesis of rationality. One of the main streams can be found in direct modelling of the dependence of demographic heterogeneity and surrender inclination. In most cases, however, this is modeled by a simple stochastic model, a discrete-time, discrete-state Markov chain at best; see e.g. [5], [8], [18], and more recently [1], to name a few. By contrast, the present paper proposes a model using diffusion processes.

We will start in section 2.1 with introducing the basic framework of our model, without mortality or surrender risk, where the heterogeneity of each insured person is modeled by a multi-dimensional diffusion process aiming to describe the various causes discussed above. The introduced diffusion processes naturally define a probability measure, which describes the distribution of the “heterogeneity”, and we assume that it is approximated by its infinite-agent limit, which is analogous to a thermodynamic limit (Assumption 2.2 and Theorem 2.3). The “thermodynamic” procedure is the core of our framework. Then in section 2.2 we introduce the “lifetime” of an insured, modeled by the killing time of an associated diffusion process (Assumption 2.4). The thermodynamic probability measure is then compensated by the killing rate (Theorem 2.7). The basic framework with the lifetime is
used in section 2.3 to model the cash-flow of a whole-life insurance with a level premium. Its continuous-time version is presented in section 2.4, where, after taking the “thermodynamic limit”, the premium can be obtained by calculating the Laplace transform of expectations with respect to the heterogeneity distribution (see equation (16) in Theorem 2.9). The Laplace transform method is another core of our framework and enables computational tractability.

Then, in section 3, we add to the introduced model the surrender time, which is again modelled by a killing time (Assumption 3.1). Under the continuous-time thermodynamic model, the pay-off, which is the revenue minus expenditure (of the insurance company), is modelled by the Laplace transform of (27), after deriving the thermodynamic limit under the extended setting in Theorem 3.2.

In sections 4 and 5, we introduce two specific models where we can calculate the formula (27) analytically. In the former, we rely on the local constancy of the killing rates, which can however approximate fairly general rates. If the approximated rate is highly non-linear, then we need to resort to the numerical algorithm proposed in Theorem 4.2, which is among the mathematical contributions of the present paper. In contrast, the latter model gives us completely an analytical expression of (27) using the symmetry of the 2-dimensional Bessel process.

There are some studies analysing the surrender risk in the spirit of quantitative finance, such as [10] and more recently [4], which use continuous time processes similar to ours; however, these are not concerned with heterogeneity. They model the lapses by “jumps”, that is, exogenous events. Such an approach might be called a “reduced form approach”. From this point of view, our model can be understood as a structural version of, for example, the model proposed by O. Le Courtois and H. Nakagawa [10] (see Remark 3.4 in Section 3 below).

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2 Model without Surrender Risk

2.1 Basic Model

Let $X_t$ be a time-homogeneous diffusion process in $\mathbb{R}^d$,
\[ \{X_t : t \in \mathbb{R}_+\}, \{\mathbb{P}^x : x \in \mathbb{R}^n\}, \quad \mathbb{P}^x(X_0 = x) = 1. \quad (1) \]
Here, we consider $X$ to be a quantified personal profile such as health, economic state, or other conditions of a person, which depend on a time parameter $t \in \mathbb{R}_+$. The infinitesimal generator of $X$ will be denoted by $\mathcal{L}$.

**Assumption 2.1.** We assume that the transition probability of $X$ has a smooth density; that is, there exists a smooth function $q$ such that
\[ \mathbb{P}^x(X_t \in A) = \int_A q(t, x, y) \, dy, \quad (A \in \mathcal{B}(\mathbb{R}^d)). \quad (2) \]
We further assume that $\mathbb{P}^x(X_t \in A)$ is a continuous function in $x$ for any fixed $t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Our model for life insurance is as follows. Let $N \in \mathbb{N}$ and $\mathcal{I} = \{1, \ldots, N\}$ be the number and the set of the initial insured, respectively. The initial personal profile of each insured is $x^i \in \mathbb{R}^d$ for $i \in \mathcal{I}$. We assume that each of $X^i (i = 1, \ldots, N)$ is distributed as $\mathbb{P}^{x^i}$. We also assume that the initial profile of each insured takes values only in $N^{-1}\mathbb{Z}^d \equiv \{k/N : k \in \mathbb{Z}^d\}$

We define a probability measure $\mu_N$ on $\mathbb{R}^d$ whose support is $N^{-1}\mathbb{Z}^d$ by
\[ \mu_N(A) := \frac{1}{N} \sharp \{i \in \mathcal{I} : x^i \in A\} \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}^d). \quad (3) \]
We call it the *initial distribution measure*.

**Assumption 2.2.** We assume that there exists $f \geq 0$ with $\|f\|_1 = 1$ such that
\[ \lim_{N \to \infty} \int_{\mathbb{R}^d} h \left( \frac{x}{N} \right) \mu_N(dx) = \int_{\mathbb{R}^d} h(x)f(x)dx \quad (4) \]
for any bounded continuous function $h$.

Let a random counting measure $v^N$ be defined by
\[ v^N(t, A) := \frac{1}{N} \sharp \{i \in \mathcal{I} : X^i_t \in A\}, \quad (A \in \mathcal{B}(\mathbb{R}^d)). \]
This expresses the proportion of the insured whose profile is in $A$ at time $t$. 

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Theorem 2.3. Under Assumptions 2.1 and 2.2, we have that
\[ \lim_{N \to \infty} E[v^N(t, A)] = \int_A E[f(X^*_t)|X^*_0 = x] \, dx, \]
where \(X^*\) is the adjoint process of \(X\), that is, a diffusion process whose infinitesimal generator is \(L^*\), the adjoint operator of \(L\) with respect to \(L^2(\mathbb{R}^d, \text{Leb})\).

Proof. Observe that
\[
E[v^N(t, A)] = E \left[ \sum_{j \in I} \frac{1}{N} \mathbf{1}_{\{i \in I : X^i_t \in A\}}(j) \right] \\
= \frac{1}{N} \sum_{i \in I} \mathbb{P}(X^i_t \in A) = \frac{1}{N} \sum_{i \in I} \mathbb{P}^i(X_t \in A) \\
= \frac{1}{N} \sum_{k \in \mathbb{Z}^d} \mathbb{P}^k_*(X_t \in A) \mu_N(\{k/N\}) N \\
= \sum_{k \in \mathbb{Z}^d} \mathbb{P}^k_*(X_t \in A) \mu_N(\{k/N\}).
\]
Letting \(N \to \infty\), we get
\[
\lim_{N \to \infty} E[v^N(t, A)] = \int_{\mathbb{R}^d} \mathbb{P}^*(X_t \in A) f(x) \, dx
\]
by using (4) in Assumption 2.2. Moreover, this formula can be rewritten (where we can use Fubini’s theorem since both \(\mathbb{P}^x(X_t \in A)\) and \(f(x)\) are positive functions) as
\[
\lim_{N \to \infty} E[v^N(t, A)] = \int_{\mathbb{R}^d} \mathbb{P}^y(X_t \in A) f(y) \, dy \\
= \int \left( \int q(t, y, x) 1_A(x) \, dx \right) f(y) \, dy \\
= \int 1_A(x) \, dx \int q(t, y, x) f(y) \, dy.
\]
Therefore, \(A \mapsto u(t, A)\) is an absolutely continuous measure with respect to Lebesgue measure, and its density is given by
\[
u(t, x) := \int q(t, y, x) f(y) \, dy.
\]
Using the adjoint process (with respect to Lebesgue measure) \(X^*\), the function \(u\) can be expressed as
\[
u(t, x) = E[f(X^*_t)|X^*_0 = x] =: E^x[f(X^*_t)],
\]
where we abuse the notation \(E^x\) a bit. \(\square\)
2.2 Mortality Model

We introduce a mortality model, where the force of mortality is dependent only on the current personal profile. In the model, the insured do not surrender the policy.

Let \( \hat{X} \) be a killed process obtained from the Markov process defined with a random time \( \zeta \) as follows,

\[
\hat{X}_t = X_t 1_{\{\zeta > t\}} + \infty 1_{\{\zeta \leq t\}}. \tag{6}
\]

**Assumption 2.4.** We assume that

\[
P(\zeta > t \mid \sigma(X_s : s \leq t)) = e^{-\int_0^t V(X_s) ds} \tag{7}
\]

and

\[
\lim_{t \to \infty} P(\zeta > t \mid \sigma(X_s : s \leq t)) = 0 \tag{8}
\]

with a positive function \( V \).

**Remark 2.5.** The mortality in our model is consistent with the ones \[17\], and \[19\], \[20\] in demography.

Then, we know that (see e.g. \[15\], chapter III, Theorem 18.6]) the transition semigroup \( \hat{T}_t \) of \( \hat{X} \) is given by

\[
\hat{T}_t f(x) := E[f(\hat{X}_s) \mid X_0 = x] = E[f(X_s) e^{-\int_0^t V(X_s) ds} \mid X_0 = x], \quad (f \in C_0(\mathbb{R}^d)).
\]

By the Feynman-Kac formula (see e.g. \[15\], chapter III 19.]), the infinitesimal generator \( \hat{L} \) of the semigroup \( \hat{T} \) is given, using the infinitesimal generator \( L \) of \( T \), as follows

\[
\hat{L} f(x) = L f(x) - V(x) f(x), \quad (f \in D(L)),
\]

where the domain of \( L \) is, as usual, the space of such \( f \) that

\[
\lim_{t \to 0} \frac{\hat{T}_t f - f}{t}
\]

exists in \( C_0(\mathbb{R}^d) \). We will sometimes use the notation

\[
\hat{T}_t f(x) = E^x[f(\hat{X})] = E^x[f(X) e^{-\int_0^t V(X_s) ds}]
\]

for the purpose of clarifying both the starting point and the sample path which we are looking at, even though this may again be a bit of an abuse of notation.
Assumption 2.6. We assume that $\hat{T}_t$ has a density, that is, there exists a smooth function $q_V$ such that

$$
\hat{T}_t f(x) = \int_{\mathbb{R}^d} q_V(t, x, y) f(y) \, dy.
$$

Let the model size be $N$ as in the preceding section, and the initial distribution measure $\mu_N$ be as (3). Let a random counting measure $v^N$ be redefined by

$$
v^N(t, A) = \frac{1}{N} \sum_{i \in I} \mathbb{1}_{\{i \in I : \hat{X}_i^t \in A\}}, \quad (A \in \mathcal{B}(\mathbb{R}^d)).
$$

Theorem 2.7. Let $A \in \mathcal{B}(\mathbb{R}^d)$. Under Assumptions 2.6 and 2.2 we have that

$$
\lim_{N \to \infty} \mathbb{E}[v^N(t, A)] = \int_A \mathbb{E}[f(X^*_t)e^{-\int_0^t V(X^*_s)ds}|X^*_0 = x] \, dx. \quad (9)
$$

Proof. Let us calculate $\mathbb{E}[v^N(t, A)]$ as

$$
\mathbb{E}[v^N(t, A)] = \mathbb{E} \left[ \frac{1}{N} \sum_{j \in I} \mathbb{1}_{\{j \in I : \hat{X}^i_j \in A\}} \right]
$$

$$
= \frac{1}{N} \sum_{i \in I} \mathbb{P}(\hat{X}^i_t \in A) = \frac{1}{N} \sum_{i \in I} \mathbb{P}^{\hat{X}^i_t}(\hat{X}_t \in A)
$$

$$
= \sum_{k \in \mathbb{Z}^d} \mathbb{P}^{k/N}(\hat{X}_t \in A) \mu_N(\{k/N\}).
$$

Letting $N \to \infty$, we get the formula

$$
\lim_{N \to \infty} \mathbb{E}[v^N(t, A)] = \int_{\mathbb{R}^d} \mathbb{P}^{y}(\hat{X}_t \in A)(y) f(y) \, dy
$$

by using (4) in Assumption 2.2. Then, by the same procedure as we used in the proof of Theorem 2.3 we get (9).

2.3 Discrete-Time Level-Premium Insurance Model

We consider an insurance where both the premium and the insurance proceeds are paid discretely at time $t = 0, 1, \cdots$. Let $p$ and $A$ be the (level) premium and the sum insured, respectively. Then, the revenue of the insurance company $v_t(p)$ at each time $t$ is given by

$$
v_t(p) = p \cdot \mathbb{1}_{\{i \in I : \hat{X}^i_t \neq \infty\}} \quad (t = 0, 1, 2, \cdots).
$$
In our model, the expenditure of the insurance company $c_t$ at each time $t$ is given by
\[
c_0 = 0,
\]
\[
c_t = A \cdot \sharp \{ i \in I : \hat{X}_t^i = \infty, \hat{X}_{t-1}^i \neq \infty \} \quad (t = 1, 2, \ldots).
\]

Let the expected return $R_d$ of the discrete-time insurance model at time 0 be defined by
\[
R_d(N, p) := \sum_{t=0}^{\infty} e^{-rt} E[v_t(p) - c_t],
\]
and $p_d(N)$ be the solution of $R_d(N, p; N) = 0$. Note that the solution is unique and strictly positive since (10) is linear in $p$ and we have clearly
\[
\sum_{t=0}^{\infty} e^{-rt} \mathbb{P}(i \in I : \hat{X}_t^i \neq \infty) > 0
\]
and
\[
A \sum_{t=0}^{\infty} e^{-rt} E[\sharp \{ i \in I : \hat{X}_t^i = \infty, \hat{X}_{t-1}^i \neq \infty \}] > 0
\]
by (8) in Assumption 2.4.

**Theorem 2.8.** We have that
\[
p_d(N) = A \sum_{t=1}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^{\hat{X}_t^k}[e^{-\int_0^{t-1} V(X_s)ds} - e^{-\int_0^{t} V(X_s)ds}] \mu_N(\{k/N\})
\]
\[
= \frac{A \sum_{t=1}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^{\hat{X}_t^k}[e^{-\int_0^{t-1} V(X_s)ds} - e^{-\int_0^{t} V(X_s)ds}] \mu_N(\{k/N\})}{\sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^{\hat{X}_t^k}[e^{-\int_0^{t} V(X_s)ds}] \mu_N(\{k/N\})}.
\]

Moreover, $p_d(\infty)$ defined by
\[
p_d(\infty) := \lim_{N \to \infty} p_d(N)
\]
is given by
\[
p_d(\infty) = A \sum_{t=1}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}_{Y}^{y}[e^{-\int_0^{t-1} \tilde{V}(X_s)ds} f(X_{t-1}^y) - e^{-\int_0^{t} \tilde{V}(X_s)ds} f(X_{t}^y)]dy
\]
\[
= \frac{A \sum_{t=1}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}_{Y}^{y}[e^{-\int_0^{t-1} \tilde{V}(X_s)ds} f(X_{t-1}^y) - e^{-\int_0^{t} \tilde{V}(X_s)ds} f(X_{t}^y)]dy}{\sum_{t=0}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}_{Y}^{y}[e^{-\int_0^{t} \tilde{V}(X_s)ds} f(X_{t}^y)]dy},
\]
using the adjoint process.
Proof. We first obtain the expected return \( R_d(N, p) \) at time 0. The expected revenue at time 0 is

\[
\sum_{t=0}^{\infty} e^{-rt} \mathbb{E}[v_t(p)] = \sum_{t=0}^{\infty} e^{-rt} \mathbb{E} \left[ p \cdot 1_{\{i \in I : \hat{X}_i^t \neq \infty\}} \right]
\]

\[
= p \sum_{t=0}^{\infty} e^{-rt} \sum_{i \in I} P(\hat{X}_i^t \neq \infty)
\]

\[
= p \sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{P}_N(\hat{X}_t \neq \infty) \mu_N(\{k/N\}) N.
\]

Since \( \{\hat{X}_t \neq \infty\} = \{\zeta > t\} \), the expected revenue is now given by

\[
= p \sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{P}_N^\zeta(\zeta > t) \mu_N(\{k/N\}) N
\]

\[
= p \sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}_N^\zeta \left[ \mathbb{E}_N^\zeta \left[ 1_{\{\zeta > t-1\}} | \sigma(X_s : s \leq t) \right] \mu_N(\{k/N\}) N \right]
\]

\[
= p \sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}_N^\zeta \left[ e^{-\int_{t-1}^{t} V(X_s) ds} \mu_N(\{k/N\}) N \right] \tag{11}
\]

Next, we calculate the expected expenditure at time 0 as

\[
\mathbb{E}[c_t] = A \mathbb{E} \left[ 1_{\{i \in I : \hat{X}_i^t = \infty, \hat{X}_{i-1}^t \neq \infty\}} \right]
\]

\[
= A \sum_{i \in I} P(\hat{X}_i^t = \infty, \hat{X}_{i-1}^t \neq \infty)
\]

\[
= A \sum_{k \in \mathbb{Z}^d} \mathbb{P}_N(\hat{X}_t = \infty, \hat{X}_{t-1} \neq \infty) \mu_N(\{k/N\}) N.
\]

As above we can use expressions with \( \zeta \) instead, and the expected expenditure is now

\[
= A \sum_{k \in \mathbb{Z}^d} \mathbb{P}_N(t - 1 < \zeta \leq t) \mu_N(\{k/N\}) N
\]

\[
= A \sum_{k \in \mathbb{Z}^d} \left( \mathbb{E}_N^\zeta \left[ \mathbb{E}_N^\zeta \left[ 1_{\{\zeta > t-1\}} - 1_{\{\zeta > t\}} \right] \right] \mu_N(\{k/N\}) N \right)
\]

\[
= A \sum_{k \in \mathbb{Z}^d} \mathbb{E}_N^\zeta \left[ e^{-\int_{t-1}^{t} V(X_s) ds} - e^{-\int_{0}^{t} V(X_s) ds} \right] \mu_N(\{k/N\}) N. \tag{12}
\]

Using the formulas \(11\) and \(12\), the expected return \( R_d(N, p) \) is
calculated as follows:

\[ R_d(N, p) = \sum_{t=0}^{\infty} e^{-rt} \mathbb{E}[v_t(p) - c_t] \]

\[ = p \sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} E_N^k [e^{-\int_0^t V(X_s)ds} \mu_N(\{k/N\})N] \]

\[ - A \sum_{t=1}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} E_N^k [e^{-\int_0^{t-1} V(X_s)ds} - e^{-\int_0^t V(X_s)ds}] \mu_N(\{k/N\})N. \]

Thus, the premium \( p_d(N) \) is given by

\[ p_d(N) = \frac{A \sum_{t=1}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} E_N^k [e^{-\int_0^{t-1} V(X_s)ds} - e^{-\int_0^t V(X_s)ds}] \mu_N(\{k/N\})}{\sum_{t=0}^{\infty} e^{-rt} \sum_{k \in \mathbb{Z}^d} E_N^k [e^{-\int_0^{t} V(X_s)ds}] \mu_N(\{k/N\})}. \]

(13)

Letting \( N \to \infty \), which is possible since each term in (14) converges by Assumption 2.2, we get the following formula:

\[ p_d(\infty) = \frac{A \sum_{t=1}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^x [e^{-\int_0^{t-1} V(X_s)ds} - e^{-\int_0^t V(X_s)ds}] f(x) dx}{\sum_{t=0}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^x [e^{-\int_0^t V(X_s)ds}] f(x) dx}. \]

Moreover, this formula can be rewritten by using the adjoint process and the potential \( \tilde{V} \) of \( \mathcal{L}^* \) as follows:

\[ p_d(\infty) = \frac{A \sum_{t=1}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^y [e^{-\int_0^{t-1} \tilde{V}(X_s)ds} f(X_{t-1}^y) - e^{-\int_0^t \tilde{V}(X_s)ds} f(X_t^y)] dy}{\sum_{t=0}^{\infty} e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^y [e^{-\int_0^t \tilde{V}(X_s)ds} f(X_t^y)] dy}. \]

\[ \Box \]
2.4 Continuous-Time Level-Premium Insurance Model

We will consider a continuous-time payment model in this section. The revenue of the insurance company during \([0, t]\) is given by

\[
\int_0^t e^{-rs} v_t(p) \, ds,
\]

where

\[
v_t(p) = p \cdot 2\{i \in \mathcal{I} : \tilde{X}_i t \neq \infty\},
\]

while the expenditure of the insurance company during \([0, t]\) is given by

\[
C_t = \sum_{j \in \mathcal{I}} A e^{-r_C} 1_{\{i \in \mathcal{I}, \zeta_i < t\}}(j).
\]

Note that \(C_t\) is increasing and bounded so that \(c_\infty\) exists and is finite almost surely. In fact, it is bounded by \(AN\).

Let the expected return of the continuous time model at time 0 be defined by

\[
R_c(N, p) := \int_0^\infty e^{-rt} \mathbb{E}[v_t(p)] \, dt - \mathbb{E}[C_\infty],
\]

and \(p_c(N)\) be the solution of \(R_c(N, p) = 0\). Note that the solution is unique and strictly positive by the same reasoning as in Section 2.3.

**Theorem 2.9.** We have that

\[
p_c(N) = \frac{A \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^N_k [V(X_s) e^{-\int_0^t V(X_s) \, ds}] \mu_N(\{k/N\}) \, dt}{\int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^N_k [e^{-\int_0^t V(X_s) \, ds}] \mu_N(\{k/N\}) \, dt}.
\]

Moreover, \(p_c(\infty)\) defined by

\[
p_c(\infty) := \lim_{N \to \infty} p_c(N)
\]

is given by

\[
p_c(\infty) = \frac{A \int_0^\infty e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^y[f(X_t^*) \tilde{V}(X_t^*) e^{-\int_0^t \tilde{V}(X_s^*) \, ds}] \, dy \, dt}{\int_0^\infty e^{-rt} \int_{\mathbb{R}^d} \mathbb{E}^y[f(X_t^*) e^{-\int_0^t \tilde{V}(X_s^*) \, ds}] \, dy \, dt},
\]

where \(\tilde{V}\) is the potential of \(\mathcal{L}^\ast\).
Proof. We first obtain the expected return $R_c(N, p)$. Calculating $v_t(p)$ like we did in (11), we get

$$\int_0^\infty e^{-rt}E[v_t(p)]dt = p \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} E^N[e^{-\int_0^t V(X_s)ds}]\mu_N(\{k/N\})Ndt,$$

(17)

while the expected expenditure at time 0:

$$E[C_t] = E\left[ \sum_{j \in \mathcal{I}} A e^{-r\zeta_j} 1_{\{i \in \mathcal{I} : \zeta_i < t\}}(j) \right],$$

can be, by Lemma A.1 in Appendix A, rewritten as

$$A \int_0^t e^{-rs} \sum_{i \in \mathcal{I}} E[V(X_s)]e^{-\int_0^s V(X_u)du]ds}.$$

Then, as we did in (12), we get the following formula:

$$E[C_t] = A \int_0^t e^{-rs} \sum_{k \in \mathbb{Z}^d} E^N[V(X_t)e^{-\int_0^t V(X_u)du}]\mu_N(\{k/N\})Nds.$$

(18)

Using the formulas (17) and (18), the expected return $R_c(N, p)$ is obtained as

$$R_c(N, p) = \int_0^\infty e^{-rt}E[v_t(p)]dt - E[C_\infty]$$

$$= p \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} E^N[e^{-\int_0^t V(X_s)ds}]\mu_N(\{k/N\})Ndt$$

$$- A \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} E^N[V(X_t)e^{-\int_0^t V(X_u)du}]\mu_N(\{k/N\})Ndt.$$

(19)

Thus, the premium $p(N)$ is given by

$$p_c(N) = \frac{A \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} E^k[V(X_t)e^{-\int_0^t V(X_u)du}]\mu_N(\{k/N\})dt}{\int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} E^k[e^{-\int_0^t V(X_s)ds}]\mu_N(\{k/N\})dt}.$$

(20)

Then, by the same procedure as we used in the proof of Theorem 2.8, we get (16).
3 Model with Surrender Risk

3.1 Mortality Model with Surrender Risk

In this section we consider a model where insurers can surrender their policy by balancing their personal conditions with the premium. In addition to $\zeta$, we introduce a new random time $\xi(p)$, which is dependent on a parameter $p$, satisfying the following assumption.

**Assumption 3.1.** We assume that

1. 
   \[ P(\xi(p) > t \mid \sigma(X_s : s \leq t)) = e^{-\int_0^t D(p, X_s) \, ds} \]
   with a positive measurable function $D : [0, \infty) \times \mathbb{R}^d \ni (p, x) \mapsto D(p, x) \in \mathbb{R}_{>0}$, which is increasing in $p$ and decreasing in $x$.

2. The random times $\zeta$ and $\xi(p)$ are conditionally independent in the following sense:

   \[ P(\xi(p) > t, \zeta > t \mid \sigma(X_s : s \leq t)) = P(\xi(p) > t \mid \sigma(X_s : s \leq t)) P(\zeta > t \mid \sigma(X_s : s \leq t)) \]
   \[ = e^{-\int_0^t V(X_s) \, ds} - \int_0^t D(X_s, p) \, ds. \]  

3.2 Continuous-time Level-Premium Insurance Model with Surrender Risks

The revenue of the insurance company during $[0, t]$ is given by

\[ \int_0^t e^{-rs} v_s(p) \, ds, \]

where

\[ v_t(p) = p \cdot \mathbb{1}\{i \in I : \zeta^i > t, \xi^i(p) > t\}, \]  

while the expenditure of the insurance company during $[0, t]$ is given by

\[ C_t = A \sum_{j \in I} e^{-r\zeta^j} \mathbb{1}\{i \in I : \zeta^i \leq t, \xi^i(p) > t\} \mathbb{1}(j). \]

The expected return at time 0 is the same as the one in the previous section, namely,

\[ R_s(N, p) = \int_0^\infty e^{-rt} [v_t(p)] \, dt - \mathbb{E}[C_\infty], \]

where the subscript $s$ is put to indicate that it is the one with surrender risk. It should be noted that the expected return is no longer a linear function in $p$, and thus we may not have uniqueness of the solution $p$ for $R_c(N, p) = 0$. 

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Theorem 3.2. We have that
\[ R_s(N, p) = \int_0^\infty e^{-rt} \mathbb{E}[v_t(p)] dt - \mathbb{E}[C_\infty] \]
\[ = \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^k \left[ (p - AV(X_s)) e^{-\int_0^t V(X_s) + D(X_s, p) ds} \right] \mu_N({k/N}) N dt. \]

(24)

Proof. The proof is almost the same as for Theorem 2.9. First,
\[ \int_0^\infty e^{-rt} \mathbb{E}[v_t(p)] dt \]
\[ = \int_0^\infty e^{-rt} \mathbb{E} \left[ p \cdot \mathbb{1}_{\{i \in I : \xi^i > t, \xi^i(p) > t\}} \right] dt \]
\[ = p \int_0^\infty e^{-rt} \sum_{i \in I} \mathbb{P}(\xi^i > t, \xi^i(p) > t) dt \]
\[ = p \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{P}^k(\zeta > t, \xi^i(p) > t) \mu_N({k/N}) N dt \]
\[ = p \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^k \left[ \mathbb{E}[1_{\{\zeta > t, \xi(p) > t\}} | \sigma(X_s : s \leq t)] \right] \mu_N({k/N}) N dt \]

This formula can be rewritten, by (21), as
\[ = p \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^k \left[ e^{-\int_0^t (V(X_s) + D(X_s, p)) ds} \right] \mu_N({k/N}) N dt. \]

(25)

Next, the expected expenditure at time 0,
\[ \mathbb{E}[C_\infty] = \lim_{t \to \infty} \mathbb{E} \left[ A \sum_{j \in I} e^{-r\zeta^j} 1_{\{i \in I : \zeta^i < t, \xi^i(p) > t\}} (j) \right], \]
is obtained as
\[ A \int_0^t e^{-rs} \sum_{i \in I} \mathbb{P}(\zeta^i \in ds, \xi^i(p) > t) \]
\[ = A \int_0^t e^{-rs} \sum_{k \in \mathbb{Z}^d} \mathbb{P}^k(\zeta \in ds, \xi(p) > t) \mu_N({k/N}) N \]
\[ = A \int_0^t e^{-rs} \sum_{k \in \mathbb{Z}^d} \mathbb{E}^k \left[ \mathbb{E}^k \left[ 1_{\{\zeta \in ds, \xi(p) > t\}} | \sigma(X_u : u \leq t) \right] \right] \mu_N({k/N}) N \]
This expression can be shown, by (21) and Lemma A.1 in Appendix A, to be equal to

\[ A \int_0^\infty e^{-rt} \sum_{k \in \mathbb{Z}} E^x [V(X_t) e^{-\int_0^t (V(X_s) + D(X_s, p)) ds}] \mu_N(\{k/N\}) \mu dt. \]  

Taking the limit of (26) as \( t \to \infty \), and using the formula (25), we obtain (24).

We can define the virtual average expected return as

\[ \text{VAR}_s(p) := \lim_{N \to \infty} \frac{1}{N} R_s(N, p) = \int_0^\infty e^{-rt} \int_{\mathbb{R}^d} E^x [(p - AV(X_t)) e^{-\int_0^t V(X_s) + D(X_s, p) ds} f(x) dx dt \right. \]

\[ = \int_0^\infty e^{-rt} \int_{\mathbb{R}^d} E^y [(p - AV(X^*_t)) f(X^*_t) e^{-\int_0^t V(X^*_s) + D(X^*_s, p) ds} dy dt. \]  

The zero(s) of \( \text{VAR}(p) \) can be a good approximation of the zero(s) of \( R_s(N, p) \).

**Remark 3.3.** Since \( R_s(N, 0) < 0 \) and \( R_s \) is continuous in \( p \), the solution \( p^* \) of \( R_s(N, p^*) = 0 \) exists if and only if \( R_s(N, p) > 0 \) for some \( p \). An evident sufficient condition that ensures the existence of the solution is that \( V \) and \( D \) are bounded, since this implies \( \lim_{p \to \infty} R_s(N, p) = +\infty \).

**Remark 3.4.** In the paper by O. Le Courtois and H. Nakagawa [10], the number of surrenders is modeled by a counting process, and the expected number of the remaining participants at \( t \) is expressed by its intensity process, while in our framework it is given by \( \int f(x) E[e^{-\int_0^t D(X_s, p) ds}] dx \) using the “state process” \( X \).

4  A Model by Brownian Motion with Constant Drift

4.1 Description of the model and an expression of the expected return

In this section, we specifically assume that the personal condition process \( X_t \) is one dimensional Brownian motion with drift; that is,

\[ X_t = aW_t + bt, \]
where $W_t$ is standard Brownian motion, $a > 0$, $b \in \mathbb{R}$, so that

$$P^x(X_t \in A) = \int_A \frac{1}{\sqrt{2\pi a^2 t}} e^{-\frac{(y-x-bt)^2}{2a^2 t}} \, dy, \quad A \in \mathcal{B}(\mathbb{R}).$$

Clearly, Assumption 2.1 is satisfied.

Moreover, we assume that the killing rate functions $V(y)$ and $D(y,p)$ are step functions; that is,

$$V(y) = \sum_{i=1}^M \lambda_i 1_{(y_{i-1},y_i)}(y),$$

$$D(y,p) = \sum_{i=1}^M \mu_i(p) 1_{(y_{i-1},y_i)}(y),$$

$$(M \in \mathbb{N}, \lambda_i, \mu_i(p) \in \mathbb{R}, -\infty = y_0 < y_1 < \ldots < y_M = \infty).$$

The model is so designed that we can fit/calibrate any data to a certain extent provided that the personal condition is expressed by a real number.

In this case, the expected return $R_s(N,p)$ can be calculated, but only if we consider the inversion of a large scale matrix to be tractable. Below we first show how the expected return is calculated out of an inversion of a large matrix. Then, we propose a numerical scheme to reduce the burden.

We define $z_V$ and $z_1$ as follows: for $y \in \mathbb{R},$

$$z_V(y) := \int_0^\infty e^{-rt} E_y[V(X_t)e^{-\int_0^t(V(X_s)+D(X_s,p))ds}] \, dt,$$

and

$$z_1(y) := \int_0^\infty e^{-rt} E_y[e^{-\int_0^t(V(X_s)+D(X_s,p))ds}] \, dt.$$

Then we can express the expected return as

$$R_s(N,p) = \sum_{k=-\infty}^\infty N \mu_N \left( \frac{k}{N} \right) \left( pu_1 \left( \frac{k}{N} \right) - Au_V \left( \frac{k}{N} \right) \right).$$

**Lemma 4.1.** We have that

$$z_V(y) = \sum_{i=1}^M \left\{ C_i e^{\alpha_i y} + C_i e^{-\alpha_i y} + \gamma_i \right\} 1_{(y_{i-1} < y < y_i)},$$

and

$$z_1(y) = \sum_{i=1}^M \left\{ \tilde{C}_i e^{\alpha_i y} + \tilde{C}_i e^{-\alpha_i y} + \tilde{\gamma}_i \right\} 1_{(y_{i-1} < y < y_i)}.$$
where, for $i = 1, \cdots, M$,

$$
\alpha_{i,\pm} = \frac{-b \pm \sqrt{b^2 + 2a^2(\lambda_i + \mu_i(p) + r)}}{a^2},
$$

$$
\gamma_i = \frac{\lambda_i}{2(\lambda_i + \mu_i(p) + r)} = \lambda_i \tilde{\gamma}_i.
$$

The constants $C_{i,\pm}, \tilde{C}_{i,\pm} \in \mathbb{R}$ are given in the following way: $C_{1,-} = C_{M,+} = \tilde{C}_{1,-} = \tilde{C}_{M,+} = 0$, and

$$
\begin{align*}
\begin{bmatrix}
C_{2,-}, & \cdots, & C_{M,-}, & C_{1,+}, & \cdots, & C_{M-1,+}
\end{bmatrix} \in \mathbb{R}^{2M-2}
\end{align*}
$$

and

$$
\begin{align*}
\begin{bmatrix}
\tilde{C}_{2,-}, & \cdots, & \tilde{C}_{M,-}, & \tilde{C}_{1,+}, & \cdots, & \tilde{C}_{M-1,+}
\end{bmatrix} \in \mathbb{R}^{2M-2}
\end{align*}
$$

given as the unique solutions to the following equations:

$$
\begin{align*}
\begin{bmatrix}
-e^{\alpha_{i,1}^+}y + e^{-\alpha_{i,1}^-}tJ & e^{\alpha_{i,1}^-}y - e^{\alpha_{i,1}^+}y J \\
e^{-\alpha_{i,1}^-}y + e^{\alpha_{i,1}^+}y J \alpha_{i,1}^- & e^{\alpha_{i,1}^+}y - e^{\alpha_{i,1}^-}y J \alpha_{i,1}^+
\end{bmatrix} \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} \gamma_i \end{bmatrix},
\end{align*}
$$

(28)

and

$$
\begin{align*}
\begin{bmatrix}
-e^{\alpha_{i,2}^+}y + e^{-\alpha_{i,2}^-}tJ & e^{\alpha_{i,2}^-}y - e^{\alpha_{i,2}^+}y J \\
e^{-\alpha_{i,2}^-}y + e^{\alpha_{i,2}^+}y J \alpha_{i,2}^- & e^{\alpha_{i,2}^+}y - e^{\alpha_{i,2}^-}y J \alpha_{i,2}^+
\end{bmatrix} \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} \gamma_i \end{bmatrix},
\end{align*}
$$

(29)

where

$$
J = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0_{R^{M-2}} & I_{R^{M-2} \otimes R^{M-2}} & 0_{R^{M-2}}
\end{pmatrix} \in R^{M-1 \otimes R^{M-1}},
$$

$$
y := \text{diag}(y_1, \cdots, y_{M-1}) \in R^{M-1 \otimes R^{M-1}},
$$

$$
\alpha_{i,\pm} := \text{diag}(\alpha_{3,\pm}, \cdots, \alpha_{(2M-1),\pm}) \in R^{M-1 \otimes R^{M-1}},
$$

$$
\begin{bmatrix}
\gamma_2 - \gamma_1, & \cdots, & \gamma_M - \gamma_{M-1}
\end{bmatrix} \in R^{M-1},
$$

and

$$
\begin{bmatrix}
\tilde{\gamma}_2 - \tilde{\gamma}_1, & \cdots, & \tilde{\gamma}_M - \tilde{\gamma}_{M-1}
\end{bmatrix} \in R^{M-1}.
$$

Proof: Using the Feynman-Kac formula, we have that $z_V$ and $z_1$ satisfy

$$
-V(y) + rz_V(y) = \frac{1}{2}a^2 z_V''(y) + bz_V'(y) - (V(y) + D(y, p))z_V(y)
$$

(30)

$$
-1 + rz_1(y) = \frac{1}{2}a^2 z_1''(y) + bz_1'(y) - (V(y) + D(y, p))z_1(y)
$$

(31)

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for \( y \in \mathbb{R} \setminus \{y_1, \cdots, y_{M-1}\} \), respectively. Since \( z_1, z_V \in C^1 \) (see e.g. [1] Theorem 6.4.1),
\[
 z_s(y_i) = z_s(y_{i-1}), \quad z'_s(y_i) = z'_s(y_{i-1}), \quad (i = 1, 2, ..., M - 1)
\]
for \(* = V, 1\). Specifically,
\[
 C_{i,+} e^{\alpha_i+y_i} + C_{i,-} e^{\alpha_i-y_i} + \gamma_i
 = C_{i+1,+} e^{\alpha_{i+1}+y_i} + C_{i+1,-} e^{\alpha_{i+1}-y_i} + \gamma_{i+1}
\]
and
\[
 C_{i,+} \alpha_i e^{\alpha_i+y_i} + C_{i,-} \alpha_i e^{\alpha_i-y_i}
 = C_{i+1,+} \alpha_{i+1} e^{\alpha_{i+1}+y_i} + C_{i+1,-} \alpha_{i+1} e^{\alpha_{i+1}-y_i}
\]
for \( i = 1, 2, ..., M - 1 \), which is equivalent to the equation (28). Similarly the equations derived for \( \tilde{C}_{i,\pm} \), which is obtained by replacing \( \gamma_i \) with \( \tilde{\gamma}_i \), is equivalent to (29).

\[\square\]

### 4.2 A numerical scheme

As we remarked already, the inversion of the matrix
\[
\begin{pmatrix}
-e^{\alpha_1+y} + e^{-\alpha_1-y} y & e^{\alpha_1+y} - e^{\alpha_1+y} J

(-e^{\alpha_1+y} + e^{-\alpha_1-y} y) \alpha_+ & (e^{\alpha_1+y} - e^{\alpha_1+y} J) \alpha_+
\end{pmatrix}
\]
might become too heavy if \( M \) is large. We then propose a numerical scheme to solve the equations (28) and (29) which might work well when
\[
\delta := \min_{1 \leq i \leq M-2} (y_{i+1} - y_i)
\]
is very small.

For simplicity, we assume that \( y_1, y_2, \cdots, y_{M-1} \) are equally spaced:
\[
\delta = y_{i+1} - y_i, \quad i = 1, 2, \cdots, M - 1,
\]
where we can get an exact expression. First let us observe that
\[
\begin{pmatrix}
-e^{\alpha_1+y} + e^{-\alpha_1-y} J & e^{\alpha_1+y} - e^{\alpha_1+y} J

(-e^{\alpha_1+y} + e^{-\alpha_1-y} y) \alpha_+ & (e^{\alpha_1+y} - e^{\alpha_1+y} J) \alpha_+
\end{pmatrix}
= \begin{pmatrix}
I - \delta e^{\alpha_1+y} J & I - J e^{\alpha_1+y} \\
(I - \delta e^{\alpha_1+y} J) \alpha_+ & (I - J e^{\alpha_1+y} J) \alpha_+
\end{pmatrix}
\begin{pmatrix}
-e^{\alpha_1+y} & 0_{\mathbb{R}^{M-1} \otimes \mathbb{R}^{M-1}}

0_{\mathbb{R}^{M-1} \otimes \mathbb{R}^{M-1}} & e^{\alpha_1+y}
\end{pmatrix}
\]
\[
= L(\delta) \begin{pmatrix}
-e^{\alpha_1+y} & 0_{\mathbb{R}^{M-1} \otimes \mathbb{R}^{M-1}}

0_{\mathbb{R}^{M-1} \otimes \mathbb{R}^{M-1}} & e^{\alpha_1+y}
\end{pmatrix}.
\]
Since we have immediately
\[
\left( \begin{array}{cc}
-\alpha^+y & 0_{RM-1 \otimes RM-1} \\
0_{RM-1 \otimes RM-1} & \alpha^-y
\end{array} \right)^{-1} = \left( \begin{array}{cc}
-\alpha^+y & 0_{RM-1 \otimes RM-1} \\
0_{RM-1 \otimes RM-1} & \alpha^-y
\end{array} \right),
\]
we can concentrate on the inversion of $L(\delta)$.

Noting that
\[
L(\delta) = \left( \begin{array}{cc}
I - tJ & I - J \\
(I - tJ)\alpha^+ & (I - J)\alpha_+^-
\end{array} \right)
+ \sum_{k=1}^{\infty} \delta^k \frac{1}{k!} \left( \begin{array}{cc}
tJ(\alpha^+)^k & J(\alpha^-)^k \\
tJ(\alpha^+)^{k+1} & J(\alpha^-)^{k+1}
\end{array} \right),
\]
we propose the following scheme to invert the matrix $L(\delta)$, where we actually will need the inversion of
\[
L(0) = \left( \begin{array}{cc}
I - tJ & I - J \\
(I - tJ)\alpha^+ & (I - J)\alpha_+^-
\end{array} \right).
\]
For convenience, we denote
\[
\left( \begin{array}{cc}
tJ(\alpha^+)^k & J(\alpha^-)^k \\
tJ(\alpha^+)^{k+1} & J(\alpha^-)^{k+1}
\end{array} \right) =: L^{(k)}(0).
\]

**Theorem 4.2.** (i) Let $c \in \mathbb{R}^{2(M-1)}$ be fixed. Define $x_k$, $k = 0, 1, \ldots$ recursively by
\[
x_k = \begin{cases} 
L(0)^{-1}c & k = 0, \\
-L(0)^{-1} \sum_{j=0}^{k-1} \frac{1}{(k-j)!} L^{(k-j)}(0)x_j & k \geq 2.
\end{cases}
\]
Then $x := \sum_{k=0}^{\infty} \delta^k x_k$ satisfies $L(\delta)x = c$. (ii) In particular, $x - \sum_{k=0}^{n} x_k = O(\delta^{n+1})$ for each $n$. (iii) We have explicitly
\[
L(0)^{-1} = \left( \begin{array}{cc}
A_1(\alpha^-_+)(I - J)^{-1} & A_1(I - J)^{-1} \\
A_2(\alpha^-_+)(I - tJ)^{-1} & A_2(1 - tJ)^{-1}
\end{array} \right)
\]
where
\[
A_1 = \left( \begin{array}{cc}
(\alpha_{M,-} - \alpha_{1,+})^{-1}(\alpha_- - \alpha_+)^{-1}(\alpha_+ I - \alpha_{M,-} I) & 1_{RM-2} \\
(\alpha_{1,+} - \alpha_{M,-})^{-1} & t0_{RM-2}
\end{array} \right)
\]
and
\[
A_2 = \left( \begin{array}{cc}
t0_{RM-2} & (\alpha_{M,-} - \alpha_{1,+})^{-1} \\
(\alpha_- - \alpha_+)^{-1} & (\alpha_{M,-} - \alpha_{1,+})^{-1}(\alpha_+ I - \alpha_{1,+} I)_{RM-2}
\end{array} \right),
\]
with
\[
\alpha_\pm := \text{diag}(\alpha_{2,\pm}, \ldots, \alpha_{M-1,\pm}) \in \mathbb{R}^{M-2} \otimes \mathbb{R}^{M-2}.
\]
Proof. (i), (ii) For each $n \in \mathbb{N}$ define

$$L_n := \sum_{k=0}^{n} \frac{\delta^k}{k!} L^{(k)}(0).$$

Then,

$$L - L_n = O(\delta^{n+1}).$$

Since

$$L_n \sum_{k=0}^{n} x_k = O(\delta^{n+1}),$$

by a standard argument we have the assertion (ii), and hence (i).

(iii) Put

$$K := (I - J)^{-1}(I - tJ)$$

$$= \left( -t0_{R_{M-2} \otimes R_{M-2}} I_{R_{M-2}} \right).$$

Then we have that

$$L(0)^{-1} = \left( \begin{array}{cc} I - tJ & I - J \\ (I - tJ)_{\alpha_{+}} & (I - J)_{\alpha_{-}} \end{array} \right)^{-1}$$

$$= \left( \begin{array}{cc} (\alpha_{+} K - K \alpha_{+})^{-1} \alpha_{-} & (I - J)^{-1} \\ (\alpha_{+} K^{-1} - K^{-1} \alpha_{+})^{-1} \alpha_{+} & (I - J)^{-1} \end{array} \right).$$

We then see that

$$\alpha_{+} K - K \alpha_{-} = \left( \alpha_{1,+}, t0_{R_{M-2}} \right) \left( \begin{array}{ccc} \alpha_{+} & I_{R_{M-2}} \otimes R_{M-2} \\ 0_{R_{M-2}} & \alpha_{+} \end{array} \right) \left( \begin{array}{cc} t0_{R_{M-2}} & 1 \\ -I_{R_{M-2} \otimes R_{M-2}} & 1 \end{array} \right)$$

and

$$\alpha_{+} K - K^{-1} \alpha_{+} = \left( \alpha_{1,+} - \alpha_{M,-} \right) \left( \begin{array}{ccc} \alpha_{+} & I_{R_{M-2}} \otimes R_{M-2} \\ 0_{R_{M-2}} & \alpha_{+} \end{array} \right) \left( \begin{array}{cc} t0_{R_{M-2}} & 1 \\ -I_{R_{M-2} \otimes R_{M-2}} & 1 \end{array} \right)$$

$$= A_{-1}^{-1}$$

and

$$\alpha_{+} K - K^{-1} \alpha_{+} = \left( \begin{array}{cc} \alpha_{+} - \alpha_{M,-} \end{array} \right) \left( \begin{array}{cc} t0_{R_{M-2}} & 1 \\ -I_{R_{M-2} \otimes R_{M-2}} & 1 \end{array} \right)$$

$$= A_{-1}^{-1}.$$
5 A Model by the 2-Dimensional Squared Bessel Process

In this section, we specifically assume that the personal condition process $X_t$ is the 2-dimensional squared Bessel Process

$$dX_t = 2\sqrt{X_t}dW_t + 2dt \quad (a > 0).$$ (34)

We further assume that the initial condition distribution is approximated by the exponential distribution whose mean is $\frac{1}{\gamma}$; the limit density $f$ in Assumption 2.2 is given by

$$f(x) = \gamma e^{-\gamma x} \quad (\gamma > 0).$$

Moreover, we assume that the killing rate functions are $V(y)$ and $D(y, p)$

$$V(x) = mx + n,$$
$$D(x, p) = \varphi(p)x + \varphi(p),$$

$$(m > 0, \varphi(p) < 0, n, \varphi(p) \in \mathbb{R}).$$

We call this the 2-dimensional Squared Bessel model, 2SB model for short. The 2SB model, by nature, satisfies Assumptions 2.1, 2.2, and 3.1.

**Theorem 5.1.** The virtual average expected return in the 2SB model is explicitly calculated as:

$$\text{VAR}_s(p) = \frac{Am}{\lambda} + \left( \gamma p + \frac{Ame}{\lambda} - \gamma An \right) \frac{1}{\gamma \sqrt{2\lambda} + 2\lambda} \times \left( F \left( 1, \frac{1 + c}{2\sqrt{2\lambda}} - 2, \frac{1 + c}{2\sqrt{2\lambda}} - 1; -\frac{\gamma - \sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} \right) - \frac{1}{c + \sqrt{2\lambda}} \right),$$

where $F$ is the hypergeometric function,

$$c := r + n + \varphi(p),$$

and

$$\lambda := m + \varphi(p).$$

**Proof.** First note that

$$\text{VAR}_s(p) = \int_0^\infty e^{-rt}dt \int_0^\infty \gamma e^{-\gamma x}dx \times E[(p - A(mX_t + n))e^{-(m+\varphi(p))t} \int_0^t X_s ds] |X_0 = x]e^{-(n+\varphi(p))t}.$$
Now we see that we need to calculate
\[
\gamma \int_0^\infty e^{-ct} \int_0^\infty e^{-\gamma x} E[(aX_t + b)e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] \, dx \, dt
\]
for
\[
\lambda = m + \varphi(p),
\]
\[
c = r + n + \varphi(p),
\]
\[
a = -Am,
\]
and
\[
b = p - An,
\]
which can be further reduced to the calculation of
\[
I_\gamma(c, \lambda) := \int_0^\infty e^{-ct} \int_0^\infty e^{-\gamma x} E[e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] \, dx \, dt
\]
since
\[
\gamma \int_0^\infty e^{-ct} \int_0^\infty e^{-\gamma x} E[X_t e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] \, dx \, dt
\]
\[
= -\frac{\gamma}{\lambda} \int_0^\infty e^{-ct} \int_0^\infty e^{-\gamma x} \partial_t E[e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] \, dx \, dt
\]
\[
= -\frac{1}{\lambda} - \frac{c}{\lambda} \int_0^\infty e^{-ct} \int_0^\infty e^{-\gamma x} E[e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] \, dx \, dt.
\]
That is,
\[
\text{VAR}_s(p) = \frac{Am}{m + \varphi(p)} + \left( \gamma p + \frac{Am(r + n + \varphi(p))}{m + \varphi(p)} - \gamma An \right) I_\gamma (r + n + \varphi(p), m + \varphi(p)).
\]
It is well-known that (see e.g. Ikeda-Watanabe [7]) that
\[
E[e^{-\lambda \int_0^t X_s \, ds} | X_0 = x] = \frac{e^{-x\sqrt{2\lambda} \tanh \sqrt{2\lambda t}}}{\cosh \sqrt{2\lambda t}},
\]
therefore we have

\[ I = \int_0^\infty \frac{e^{-ct}}{(\gamma + \sqrt{2\lambda} \tanh \sqrt{2\lambda} t) \cosh \sqrt{2\lambda} t} \, dt \]

\[ = \int_0^\infty \frac{e^{-ct}}{\gamma \cosh \sqrt{2\lambda} t + \sqrt{2\lambda} \sinh \sqrt{2\lambda} t} \, dt \]

\[ = \int_0^\infty \frac{2}{\gamma + \sqrt{2\lambda}} \frac{e^{-(c+\sqrt{2\lambda})t}}{\gamma + \sqrt{2\lambda}} e^{-2\sqrt{2\lambda} t} \, dt \]

\[ = \int_0^\infty \frac{2e^{-(c+\sqrt{2\lambda})t}}{\gamma + \sqrt{2\lambda}} \sum_{j=1}^{\infty} \left( \frac{\gamma - \sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} e^{-2t} \right)^j \, dt \]

\[ = \frac{2}{\gamma + \sqrt{2\lambda}} \sum_{j=1}^{\infty} \left( \frac{\gamma - \sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} \right)^j \int_0^\infty e^{-(c+(2j+1)\sqrt{2\lambda})t} \, dt \]

\[ = \frac{2}{\gamma + \sqrt{2\lambda}} \sum_{j=1}^{\infty} \left( \frac{\gamma - \sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} \right)^j \frac{1}{c + (2j+1)\sqrt{2\lambda}} \]

\[ = \frac{1}{\gamma \sqrt{2\lambda} + 2\lambda} \times \left( F\left(1, \frac{1+c}{2\sqrt{2\lambda}} - 2, \frac{1+c}{2\sqrt{2\lambda}} - 1; \frac{\gamma - \sqrt{2\lambda}}{\gamma + \sqrt{2\lambda}} - \frac{1}{c + \sqrt{2\lambda}} \right) \right) \]

Note that here \((\cdot)_n\) is the Pochhammer symbol, that is,

\[ (x)_n = \prod_{k=0}^{n-1} (x + k) \]

for a complex number \(x\).

\[ \Box \]

6 Concluding Remark

The present paper proposed a totally new framework to evaluate the heterogeneous risks in whole-life insurance. We have employed a large-agent limit which is analogous to the thermodynamic one, and both the life-time and the surrender-time are modelled by the killing time of the diffusion process, while the cash-flow is evaluated by the Laplace transform. The two specific models have shown the potential of our framework.
Even though we have worked only on the determination of the level-premium, the proposed framework can be used for more general cases, including forward-looking models where a mean-field type approximation can work.

References

[1] Adams, C. J., Donnelly, C. and Macdonald, A. S. The impact of known breast cancer polygenes on critical illness insurance. *Scandinavian Actuarial Journal*, Volume 2015, Issue 2 141–171, 2015.

[2] Akerlof, G. A. The market for “lemons”: quality uncertainty and the market mechanism. *The Quarterly Journal of Economics*, pages 488–500, 1970

[3] Albizzati, M-O. and Geman, H. Interest Rate Risk Management and Valuation of the Surrender Option in Life Insurance Policies *The Journal of Risk and Insurance* Vol. 61, No. 4 (1994), pp. 616-637.

[4] Ballotta, L., Eberlein, E., Schmidt, T. and Zeineddine, R. Variable annuities in a Lévy-based hybrid model with surrender risk, *Quantitative Finance*, 20:5, (2020), 867-886,

[5] Bluhm, W. F. Cumulative antiselection theory, *Transactions of Society of actuaries*, 34 (1982).

[6] Cawley, J. and Philipson, T. An Empirical Examination of Barriers to Trade in Insurance” *American Economic Review*, 89 (4) (1999), 827-846.

[7] Ikeda, N. and Watanabe, S. *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, Kodansha-North Holland, 1989.

[8] Jones, B. L. A Model for Analyzing the Impact of Selective Lapsation on Mortality, *North American Actuarial Journal*, 2:1 (1998), 79–86.

[9] Karatzas, I. and Shreve, S. *Brownian Motion and Stochastic Calculus*, 2nd edition. Springer, 1991.

[10] Le Courtois, O. and Nakagawa, H., On surrender and default risks. Math. Finance 23 (1) (2013), 143–168.

[11] Loisel, S. and Milhaud, X. From deterministic to stochastic surrender risk models: Impact of correlation crises on economic capital, *European Journal of Operational Research*, Volume 214, Issue 2, 16 (2011), Pages 348-357.

[12] de Meza D, and Webb, D.C. Advantageous Selection in Insurance Markets *The RAND Journal of Economics* Vol. 32, No. 2 (2001), pp. 249–262.

[13] Milhaud, X., S. Loisel, and V. Maume-Deschamps, Surrender Triggers in Life Insurance: Classification and Risk Predictions, Bulletin Français d’Actuarial, 11(22), (2011):5-48,

[14] Gatzert, N., Hoermann, G. and Schmeiser, H. The Impact of the Secondary Market on Life Insurers’ Surrender Profits *The Journal of Risk and Insurance* Vol. 76, No. 4 (2009), pp. 887-908.
[15] Rogers, L.C.G. and Williams, D. *Diffusions, Markov Processes and Martingales Volume 1: Foundations* Cambridge 2000.

[16] Rothschild, M. and Stiglitz, J. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics*, Vo. 90, No.4 (1976) pages 629–649.

[17] Vaupel, J.W., Manton, K.G. and Stallard, E. The impact of heterogeneity in individual frailty on the dynamics of mortality. *Demography* 16, (1979) 439–454.

[18] Viswanathan, K.S., Lemaire, J., Withers, K., Armstrong, K., Baumritter, A., Hershey, J.C., Pauly, M.V. and Asch, D.A. Adverse Selection in Term Life Insurance Purchasing due to the BRCA 1/2 Genetic Test and Elastic Demand. The Journal of Risk and Insurance, Vol. 74, No.1, (2007), 65-86.

[19] Woodbury, M.A., and Manton, K.G. A Random-Walk Model of Human Mortality and Aging, *Theoretical Population Biology* 11 (1977), 37–48.

[20] Yashin, A.I., Manton, K.G., and Vaupel, J.W. Mortality and Aging in a Heterogeneous Population: A Stochastic Process Model with Observed and Unobserved Variables, *Theoretical Population Biology* 27, (1985) 154–75.

A Appendix

Lemma A.1. We have that

\[
E[e^{-r\zeta^i} 1_{\{\zeta^i \leq t\}}] = \int_0^t e^{-rs} E[V(X_s^i) e^{-\int_0^s V(X_u^i) du}] ds. \quad (35)
\]

Proof. Let \(G(u) = \mathbb{P}(\zeta^i > u)\) for \(u \geq 0\). Then, by taking the expectation of both sides of (35),

\[
G(s) = E[e^{-\int_0^s V(X_u^i) du}], \quad s \geq 0.
\]

Since \(e^{-\int_0^s V(X_u^i) du}\) is differentiable in \(s\) almost surely and uniformly bounded by 1, we see that \(G\) is also differentiable and

\[
G'(s) = -E[V(X_s^i) e^{-\int_0^s V(X_u^i) du}], \quad s > 0.
\]

Then, since

\[
E\left[e^{-r\zeta^i} 1_{\{\zeta^i \leq t\}}\right] = \int_0^t e^{-rs} \mathbb{P}(\zeta^i \in ds) = - \int_0^t e^{-rs} dG_s = - \int_0^t e^{-rs} G'(s) ds,
\]

we get (35).