Estimating the boundary of the region of attraction of Lotka–Volterra system with time delays

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ABSTRACT
This paper considers the local stability problem and estimates the region of attraction (RA) of the positive equilibrium of Lotka-Volterra (L-V) competitive system with time-delays. Based on the stability theory and the quadratic system theory, by choosing some less conservative integral inequalities and appropriate Lyapunov-Krasovskii (L-K) functional, a local stability condition is obtained by means of linear matrix inequalities (LMI) and the estimate of RA of the positive equilibrium is first discussed. Furthermore, the corresponding optimization problem for the estimate of RA is given. Numerical simulations show that the proposed stability condition in this paper is less conservative as compared with the most existing ones and the method of estimating RA is effective.

1. Introduction
In this paper, the following two-species competitive Lotka-Volterra (L-V) type system with discrete delays is considered:

\[
\begin{align*}
    x(t) &= x(t)[b_1 - a_{11}x(t - \tau_{11}) - a_{12}y(t - \tau_{12})], \\
    y(t) &= y(t)[b_2 - a_{21}x(t - \tau_{21}) - a_{22}y(t - \tau_{22})],
\end{align*}
\]

where \( x(t) \) and \( y(t) \) are densities of population at time \( t \), respectively; \( b_i, a_{ij}(i, j = 1, 2) \) are positive constants.

The initial condition of system (1) is given as

\[
\begin{align*}
    x(s) &= \phi_1(s), -\tau \leq s \leq 0, \phi_1'(s) > 0, \\
    y(s) &= \phi_2(s), -\tau \leq s \leq 0, \phi_2'(s) > 0,
\end{align*}
\]

where \( \tau = \max_{1 \leq i \leq 2} \{\tau_i\} \), and \( \phi_1(s), \phi_2(s) \) are assumed to be continuous.

If \( a_{11}/a_{21} > b_1/b_2 > a_{12}/a_{22} \), by denoting \( x^* = (x^*, y^*) \) with \( x^* = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \ y^* = \frac{b_2a_{11} - b_1a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \), it is seen that \( x^*, y^* \in (0, 1) \). The four equilibrium points of system (1) are (0, 0), (0, \( b_2/a_{22} \)), (\( b_1/a_{11}, 0 \)), and \( z^* \) respectively. Define \( z(t) = (x(t), y(t)) \) as all the positive solutions of system (1). The point \( z^* \) is the only positive equilibrium point. The equilibrium point \( z^* \) indicates the coexistence of the two species.

It is well known that the foundation of two-species competitive L-V type system (1) was laid by Lotka and Volterra sixty years ago (Lotka, 1956; Volterra, 1931). During the past several decades, L-V type systems have received much attention. The most basic questions arising from this system are competitive species’ persistence, attractivity, extinctions, global (or local) asymptotic behaviour and other dynamical behaviours (Ahmad & Stamova, 2015; Chen et al., 2012; Jiang & Liang, 2020; Kuang, 1996; Lai & Fang, 2020; Ma et al., 2019; Park, 2005; Sun & Meng, 2007; Teng & Yu, 2000; Wang et al., 1995; Zhao et al., 2014; Zhen & Ma, 2002; Zhu & Liu, 2017).

There are considerable works on the study of the global and local asymptotical behaviours of L-V type systems. However, few studies have discussed the estimate of the region of attraction (RA) or the edge of chaotic attractors in L-V systems (Lai & Fang, 2020). In Lai and Fang (2020), a topological approach for plotting the boundary of the region of asymptotic stability of L-V predator-prey system was proposed. But the systems considered by Lai and Fang (2020) don’t contain time delay. To the best of our knowledge, the estimate of the RA in L-V system with time delays has not been considered due to probably the mathematical complexity.

Two-species competitive L-V type system is a quadratic system. Quadratic systems play an important role in the modelling of a wide class of nonlinear processes (electrical, robotic, biological, etc.). For such systems, it is of mandatory importance not only to determine whether the origin of the state space is locally asymptotically stable but also to ensure that the operative range is included into the convergence region of the equilibrium. Over the years, several papers have focused on the estimate of...
the RA of the zero-equilibrium point of quadratic system (Caldeira et al., 2018; Chen et al., 2013; Chesi et al., 2005; Chiang et al., 1988; Genesio & Vicino, 1984; Li-Ya et al., 2019; Merola et al., 2017; Tesi et al., 1996; Xu et al., 2020). In Genesio and Vicino (1984), a Lyapunov-based procedure is proposed to compute an ellipsoidal estimate of the RA of a quadratic system. Considering this method is computationally heavy, an estimate of the RA based on topological considerations was provided in Chiang et al. (1988) or based on linear matrix inequalities (LMIs) feasibility problem (Chesi et al., 2005; Tesi et al., 1996).

The estimate of the RA of time-delay systems is tackled by means of a two-steps procedure: (i) choice of a Lyapunov–Krasovskii (L-K) functional, which proves local asymptotic stability of the equilibrium of system (Qian, Li, Chen, et al., 2020; Qian, Xing, and Fei, 2020); (ii) computation of the estimate of the RA associated to that particular L-K functional. The integral terms are common in the derivative of L-K functionals, and the approximation methods are used to replace the integral terms with some more effective expressions (Dong et al., 2019). So, the choice of the optimal L-K functional and the less conservative inequality are important and may severely affect the conservativeness of the estimate.

In this paper, our main contributions are given as follows: (1) we study local stability and estimate the RA of the positive equilibrium of L-V competitive system with time-delays based on quadratic system theory; (2) By choosing an appropriate L-K functionals and using less conservative inequalities, we develop LMI to ensure the local stability and estimate the RA of the positive equilibrium point with a guaranteed region of stability inside some polytopic region of the state-space.

The paper is organized as follows: In Section 2, the problem we deal with is precisely stated and some preliminary notation is provided. Section 3 proposes some preliminaries. In Section 4, we derive the local stability conditions and estimate the RA of the positive equilibrium $x^*$. Finally, simulations are given to illustrate the effectiveness of the obtained results.

**Notation.** The superscript $^T$ is the transpose of a matrix. The matrix $P > 0 (P \geq 0)$ denotes that $P$ is positive definite (positive semi-definite). $|| \cdot ||$ denote the 2-norm of a vector. $\lambda(R)_{M}$ is the maximum eigenvalue value of matrix $R$. The symmetric terms in a matrix are denoted by $\ast$.

### 2. Problem formulation

Letting $u(t) \triangleq x(t) - x^*, \nu(t) \triangleq y(t) - y^*$, system (1) can be written as

$$\begin{align*}
\dot{u}(t) &= [u(t) + x^*] - a_{11}u(t - \tau_{11}) - a_{12}v(t - \tau_{12}), \\
\dot{v}(t) &= [v(t) + y^*] - a_{21}u(t - \tau_{21}) - a_{22}v(t - \tau_{22}).
\end{align*}$$

(3)

Denote $e(t) \overset{\Delta}{=} (u^T(t), v^T(t))^T$ and rewrite (3) in the following matrix form:

$$\dot{e}(t) = -\sum_{i,j=1}^{2} [A_{ij} + B_{ij}(e)]e(t - \tau_{ij}),$$

(4)

where

$$A_{11} = \begin{pmatrix} a_{11}x^* & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & a_{12}x^* \\ 0 & 0 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 0 & 0 \\ a_{21}y^* & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 0 \\ 0 & a_{22}y^* \end{pmatrix},$$

$$B_{1j}(e) = \begin{pmatrix} e^T A_{1j}/x^* \\ e^T B_{1j}/y^* \end{pmatrix}, \quad j = 1, 2,$$

with $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The initial condition $e(s) = [\phi_1^T(s), \phi_2^T(s)]^T, s \in [-\tau, 0]$ associated with the system (4) is defined by initial condition (2).

In this paper, the initial condition $e(s) (s \in [-\tau, 0])$ of the system (4) is assumed to belong to a set of the form

$$X_\mu = \left\{ e(s) \in C^1[-\tau, 0] : \max_{s \in [-\tau, 0]} ||e(s)|| \leq \rho_1, \max_{s \in [-\tau, 0]} ||\dot{e}(s)|| \leq \rho_2 \right\}.$$  

(5)

where $\rho_1 > 0, \rho_2 > 0$ are scalars to be maximized.

We introduce the square

$$\chi = [-\tilde{e}_1, \tilde{e}_1] \times [-\tilde{e}_2, \tilde{e}_2],$$

(6)

which can be equivalently written as follows:

$$\chi = Co(v_1, v_2, v_3, v_4) = \{ e \in \mathbb{R}^2 : |h_i^T e| \leq \tilde{e}_i, i = 1, 2 \},$$

(7)

where ‘Co’ denote the convex hull and the corresponding vertices $v_i (i = 1, 2, 3, 4)$ are given as follows:

$$v_1 = \begin{bmatrix} -\tilde{e}_1 & \tilde{e}_2 \end{bmatrix}^T, \quad v_2 = \begin{bmatrix} -\tilde{e}_1 & \tilde{e}_2 \end{bmatrix}^T,$$

$$v_3 = \begin{bmatrix} \tilde{e}_1 & -\tilde{e}_2 \end{bmatrix}^T, \quad v_4 = \begin{bmatrix} \tilde{e}_1 & \tilde{e}_2 \end{bmatrix}^T,$$

$$h_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad h_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T.$$
3. Preliminaries

Next, we will introduce the following integral inequalities, which are important in obtaining our main results.

**Lemma 3.1:** (Park et al., 2015): For a given $n \times n$ matrix $R > 0$, two scalars $a$ and $b$ satisfying $b > a$ and a differentiable vector function $\omega(t) \in \mathbb{R}^n$, the following integral inequalities hold:

1. $$(b - a) J_a \omega^T(t) R \omega(s) ds \geq \left( J_a \omega(s) ds \right)^T R \left( J_a \omega(s) ds \right) + 3 \Theta_2 R \Theta_2,$$
2. $$(b - a) J_a \omega^T(t) R \omega(s) ds \geq \left( J_a \omega(s) ds \right)^T R \left( J_a \omega(s) ds \right) + 3 \Theta_2 R \Theta_1 + 5 \Theta_2 R \Theta_2,$$
3. $$\frac{(b-a)^2}{2} J_a \omega^T(t) R \omega(s) ds \geq \left( J_a \omega(s) ds \right)^T R \left( J_a \omega(s) ds \right) + 8 \Theta_2 R \Theta_3,$$

where

$$\Theta_1 = \int_a^b \omega(s) ds - \frac{2}{b-a} \int_a^b \int_a^b \omega(s) ds d\theta,$$
$$\Theta_2 = \int_a^b \omega(s) ds - \frac{6}{b-a} \int_a^b \int_a^b \omega(s) ds d\theta$$
$$+ \frac{12}{(b-a)^2} \int_a^b \int_a^b \int_a^b \omega(s) ds d\theta d\lambda,$$
$$\Theta_3 = \int_a^b \int_a^b \omega(s) ds d\theta - \frac{3}{b-a} \int_a^b \int_a^b \int_a^b \omega(s) ds d\theta d\lambda.$$

**Lemma 3.2:** (Qian, Li, Zhao et al., 2020): For a given $n \times n$ matrix $Z > 0$, two scalars $a$ and $b$ satisfying $b > a$ and a differentiable vector function $\omega(t) \in \mathbb{R}^n$, the following two inequalities hold:

1. $$(b - a) J_a \omega^T(t) Z \omega(s) ds \geq \left( J_a \omega(s) ds \right)^T Z \left( J_a \omega(s) ds \right),$$
2. $$\frac{(b-a)^2}{2} J_a \omega^T(t) Z \omega(s) ds d\theta \geq \left( J_a \omega(s) ds \right)^T Z \left( J_a \omega(s) ds \right) d\theta.$$

4. Main results

In this section, we will first use L-K stability theory and the quadratic systems theory to derive the local asymptotic stability conditions of the equilibrium point $z^\ast$. For presentation convenience, we denote that

$$\alpha_{ij}(t) \triangleq \int_{t-\tau_j}^t e(s) ds, \quad \alpha_{ij}'(t) \triangleq \int_{t-\tau_j}^{t-\tau_k} e(s) ds,$$
$$\beta_{ij}(t) \triangleq \int_{t-\tau_j}^t \int_{t+\theta}^t e(s) ds d\theta,$$
$$\beta_{ij}'(t) \triangleq \int_{t-\tau_j}^t \int_{t+\theta}^{t-\tau_k} e(s) ds d\theta.$$
\[ \Phi_{23} = \begin{bmatrix} \Theta & 0_{20 \times 2} \end{bmatrix}, \]
\[ \Phi_{33} = -720 \text{diag}(\tau_{11}^{-1} R_{11}, \ldots, \tau_{22}^{-4} R_{22}, \tau_{112}^{-4} R_{112}, \ldots, \tau_{2122}^{-4} R_{2122}), \]
\[ \Sigma = -Z + \sum_{i,j=1}^{2} 6 \tau_{ij} R_{ij}, \]
\[ \Xi^{11} = \text{diag}(\Xi_{11}, \ldots, \Xi_{22}, \Xi_{1111}, \ldots, \Xi_{2222}), \]
\[ \Xi^{12} = \text{diag}(\Xi_{11}, \ldots, \Xi_{22}, \Xi_{1111}, \ldots, \Xi_{2222}), \]
\[ \Xi^{22} = \text{diag}(\Xi_{11}, \ldots, \Xi_{22}, \Xi_{1111}, \ldots, \Xi_{2222}), \]
\[ \Theta = [\Sigma_{11} \ldots \Sigma_{22} \Sigma_{11}^{12} \ldots \Sigma_{22}^{12}] , \]
\[ \Sigma_{11}^{12} = -R_{11}^{2} - R_{12}^{2} - R_{21}^{2} - R_{22}^{2}, \]
\[ \mu = -T_{2} - T_{2}^{2} + \sum_{i=1}^{2} \tau_{ij}^{2} R_{ij} \]
\[ + \sum_{i=1}^{2} \sum_{j=1}^{2} (\tau_{ij} - \tau_{kl})^{2} R_{ij} R_{kl} \]
\[ \Sigma_{ij} = 6 R_{ij}, \]
\[ \Sigma_{ijkl} = 6 R_{ijkl}, \]
\[ \gamma_{1} = \sum_{i=1}^{2} \left( Q_{ij} - 9 R_{ij} \right), \]
\[ \gamma_{11} = -Q_{11} - \sum_{k=1}^{2} \tau_{11k} Q_{11k} - 9 R_{11} - \sum_{k=1}^{2} 9 R_{11k}, \]
\[ \gamma_{22} = -Q_{22} + \sum_{k=1}^{2} \tau_{22k} Q_{22k} - 9 R_{22} - \sum_{k=1}^{2} 9 R_{22k}, \]
\[ \gamma_{ij} = -Q_{ij} + \sum_{k=1}^{2} \tau_{ijk} Q_{ijk} - \sum_{k=1}^{2} \tau_{ijk} Q_{ijk} - 9 R_{ij} \]
\[ - \sum_{k=1}^{2} 9 R_{ijk}^{2}, \]
\[ 10i + j < 10k + \tilde{j} < 10k + i, \; (i, \tilde{j}) = (1, 2) \text{ or } (2, 1), \]
\[ \tilde{R}_{ij}(v_{m}) = 3 R_{ij} - T_{1} \left[ A_{ij} + B_{ij}(v_{m}) \right], \]
\[ \tilde{R} = -24 \begin{bmatrix} \tau_{11}^{-1} R_{11} & \tau_{12}^{-1} R_{12} & \tau_{21}^{-1} R_{21} & \tau_{22}^{-1} R_{22} \end{bmatrix}, \]
\[ \tilde{R} = 36 \text{diag}(\tau_{11} R_{11}, \tau_{12} R_{12}, \tau_{21} R_{21}, \tau_{22} R_{22}), \]
\[ \tilde{R} = \begin{bmatrix} 3 \tau_{11}^{-1} R_{11}^{2} & 3 \tau_{12}^{-1} R_{12}^{2} & 3 \tau_{21}^{-1} R_{21}^{2} & 3 \tau_{22}^{-1} R_{22}^{2} \\
 2 \tau_{11}^{-1} R_{11}^{2} & 0 & 0 & 0 \\
 0 & 2 \tau_{12}^{-1} R_{12}^{2} & 0 & 0 \\
 0 & 0 & 2 \tau_{21}^{-1} R_{21}^{2} & 0 \\
 0 & 0 & 0 & 2 \tau_{22}^{-1} R_{22}^{2} \end{bmatrix}, \]
\[ \tilde{\lambda} = 60 \begin{bmatrix} \tau_{11}^{-2} R_{11} & \tau_{12}^{-2} R_{12} & \tau_{21}^{-2} R_{21} & \tau_{22}^{-2} R_{22} \end{bmatrix}, \]
\[ \tilde{\lambda} = -60 \text{diag}(\tau_{11}^{-2} R_{11}, \tau_{12}^{-2} R_{12}, \tau_{21}^{-2} R_{21}, \tau_{22}^{-2} R_{22}), \]
\[ \Theta = 360 \text{diag}(\tau_{11}^{-3} R_{11}, \tau_{12}^{-3} R_{12}, \tau_{21}^{-3} R_{21}, \tau_{22}^{-3} R_{22}), \]
\[ \mu = -T_{2} - T_{2}^{2} + \sum_{i=1}^{2} \tau_{ij}^{2} R_{ij} \]
\[ + \sum_{i=1}^{2} \sum_{j=1}^{2} (\tau_{ij} - \tau_{kl})^{2} R_{ij} R_{kl} \]
\[ \Sigma_{ij} = 6 R_{ij}, \]
\[ \Sigma_{ijkl} = 6 R_{ijkl}, \]
\[ \gamma_{1} = \sum_{i=1}^{2} \left( Q_{ij} - 9 R_{ij} \right), \]
\[ \gamma_{11} = -Q_{11} - \sum_{k=1}^{2} \tau_{11k} Q_{11k} - 9 R_{11} - \sum_{k=1}^{2} 9 R_{11k}, \]
\[ \gamma_{22} = -Q_{22} + \sum_{k=1}^{2} \tau_{22k} Q_{22k} - 9 R_{22} - \sum_{k=1}^{2} 9 R_{22k}, \]
\[ \gamma_{ij} = -Q_{ij} + \sum_{k=1}^{2} \tau_{ijk} Q_{ijk} - \sum_{k=1}^{2} \tau_{ijk} Q_{ijk} - 9 R_{ij} \]
\[ - \sum_{k=1}^{2} 9 R_{ijk}^{2}, \]
\[ 10i + j < 10k + \tilde{j} < 10k + i, \; (i, \tilde{j}) = (1, 2) \text{ or } (2, 1), \]
\[ \tilde{R}_{ij}(v_{m}) = 3 R_{ij} - T_{1} \left[ A_{ij} + B_{ij}(v_{m}) \right], \]
\[ \tilde{R} = -24 \begin{bmatrix} \tau_{11}^{-1} R_{11} & \tau_{12}^{-1} R_{12} & \tau_{21}^{-1} R_{21} & \tau_{22}^{-1} R_{22} \end{bmatrix}, \]
\[ \tilde{R} = 36 \text{diag}(\tau_{11} R_{11}, \tau_{12} R_{12}, \tau_{21} R_{21}, \tau_{22} R_{22}), \]
\[ \tilde{R} = \begin{bmatrix} 3 \tau_{11}^{-1} R_{11}^{2} & 3 \tau_{12}^{-1} R_{12}^{2} & 3 \tau_{21}^{-1} R_{21}^{2} & 3 \tau_{22}^{-1} R_{22}^{2} \\
 2 \tau_{11}^{-1} R_{11}^{2} & 0 & 0 & 0 \\
 0 & 2 \tau_{12}^{-1} R_{12}^{2} & 0 & 0 \\
 0 & 0 & 2 \tau_{21}^{-1} R_{21}^{2} & 0 \\
 0 & 0 & 0 & 2 \tau_{22}^{-1} R_{22}^{2} \end{bmatrix}, \]
\[ \tilde{\lambda} = 60 \begin{bmatrix} \tau_{11}^{-2} R_{11} & \tau_{12}^{-2} R_{12} & \tau_{21}^{-2} R_{21} & \tau_{22}^{-2} R_{22} \end{bmatrix}, \]
\[ \tilde{\lambda} = -60 \text{diag}(\tau_{11}^{-2} R_{11}, \tau_{12}^{-2} R_{12}, \tau_{21}^{-2} R_{21}, \tau_{22}^{-2} R_{22}), \]
\[ \Theta = 360 \text{diag}(\tau_{11}^{-3} R_{11}, \tau_{12}^{-3} R_{12}, \tau_{21}^{-3} R_{21}, \tau_{22}^{-3} R_{22}), \]
\[ \mu = -T_{2} - T_{2}^{2} + \sum_{i=1}^{2} \tau_{ij}^{2} R_{ij} \]

**Proof:** Choose the following augmented L-K functional:

\[ V(t) = V_{1}(t) + V_{2}(t) + V_{3}(t) \]
where

\[ V_1(t) = \eta^T(t) P \eta(t) + \sum_{i,j=1}^{2} \int_{t-\tau_i}^{t} e^T(s) Q_{ij} e(s) ds \]
\[ + \sum_{i,j=1}^{2} \tau_{ij} \int_{t-\tau_i}^{t} e^T(s) R_{ij} \dot{e}(s) ds + \tau_{ij} \int_{t-\tau_i}^{t} e^T(s) R_{ij} \dot{e}(s) ds \theta, \]

\[ V_2(t) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \tau_{ijkl} \int_{t-\tau_i}^{t} e^T(s) R_{ijkl}^l e(s) ds, \]

\[ V_3(t) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \tau_{ijkl} \int_{t-\tau_i}^{t} e^T(s) R_{ijkl}^l e(s) ds, \]

with

\[ \eta(t) = \left[ \begin{array}{cccc}
\eta_1(t) & \alpha_1(t) & \alpha_1(t) & \beta_1(t) \\
\eta_2(t) & \alpha_2(t) & \alpha_2(t) & \beta_2(t) \\
\eta_3(t) & \alpha_3(t) & \alpha_3(t) & \beta_3(t) \\
\end{array} \right]. \]

Through some direct calculations, one can obtain that

\[ \dot{V}(t) = V_1(t) + V_2(t) + V_3(t) \quad (12) \]

where

\[ \dot{V}_1(t) = 2 \eta^T(t) P \dot{\eta}(t) + e^T(t) \sum_{i,j=1}^{2} Q_{ij} e(t) \]
\[ - \sum_{i,j=1}^{2} \tau_{ij} \int_{t-\tau_i}^{t} e^T(s) R_{ij} \dot{e}(s) ds, \]

\[ \dot{V}_2(t) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \tau_{ijkl} \left[ \int_{t-\tau_i}^{t} e^T(s) R_{ijkl}^l \dot{e}(s) ds \right], \]

\[ \dot{V}_3(t) = \sum_{i,j=1}^{2} \sum_{k,l=1}^{2} \tau_{ijkl} \left[ \int_{t-\tau_i}^{t} \int_{t-\tau_i}^{t} e^T(s) R_{ijkl}^l \dot{e}(s) ds \right], \]

Using the inequality (2) of Lemma 3.1, it is seen that

\[ - \tau_{ij} \int_{t-\tau_i}^{t} e^T(s) R_{ij} \dot{e}(s) ds \leq - \left[ \begin{array}{c}
\xi^{(1)}_{ij}(t) \\
\xi^{(2)}_{ij}(t) \\
\xi^{(3)}_{ij}(t) \\
\end{array} \right]^T \times \left[ \begin{array}{ccc}
R_{ij} & 0 & 0 \\
0 & 3R_{ij} & 0 \\
0 & 0 & 5R_{ij} \\
\end{array} \right] \times \left[ \begin{array}{c}
\xi^{(1)}_{ij}(t) \\
\xi^{(2)}_{ij}(t) \\
\xi^{(3)}_{ij}(t) \\
\end{array} \right], \quad (13)\]

\[ - \tau_{ijkl} \int_{t-\tau_i}^{t} e^T(s) R_{ijkl}^l \dot{e}(s) ds \leq - \left[ \begin{array}{c}
\xi^{(1)}_{ijkl}(t) \\
\xi^{(2)}_{ijkl}(t) \\
\xi^{(3)}_{ijkl}(t) \\
\end{array} \right]^T \times \left[ \begin{array}{ccc}
R_{ijkl}^l & 0 & 0 \\
0 & 3R_{ijkl}^l & 0 \\
0 & 0 & 5R_{ijkl}^l \\
\end{array} \right] \times \left[ \begin{array}{c}
\xi^{(1)}_{ijkl}(t) \\
\xi^{(2)}_{ijkl}(t) \\
\xi^{(3)}_{ijkl}(t) \\
\end{array} \right], \quad (14)\]

where

\[ \xi^{(1)}_{ij}(t) = e(t) - e(t-\tau_i), \]
\[ \xi^{(2)}_{ij}(t) = e(t) + e(t-\tau_i) - 2\tau_{ij}^{-1} \alpha_i(t), \]
\[ \xi^{(3)}_{ij}(t) = e(t) - e(t-\tau_i) + 6\tau_{ij}^{-1} \alpha_i(t) - 12\tau_{ij}^{-2} \beta_i(t), \]
\[ \xi^{(1)}_{ijkl}(t) = e(t-\tau_k) - e(t-\tau_i), \]
\[ \xi^{(2)}_{ijkl}(t) = e(t-\tau_i) + e(t-\tau_k) - 2\tau_{ijkl}^{-1} \alpha_i(t), \]
\[ \xi^{(3)}_{ijkl}(t) = e(t-\tau_k) - e(t-\tau_i) + 6\tau_{ijkl}^{-1} \alpha_i(t) - 12\tau_{ijkl}^{-2} \beta_i(t). \]

For any matrix \( T_1, T_2 \in \mathbb{R}^{2 \times 2} \), it follows that the following equation is true:

\[ 2[e^T(t) T_1 + \bar{e}^T(t) T_2] \]
\[ \times - \left\{ \sum_{i,j=1}^{2} [A_{ij} + B_{ij}(e(t)) \dot{e}(t-\tau_i) - \dot{e}(t)] \right\} = 0. \quad (15)\]

Adding the left side of (15) to \( \dot{V}(t) \) and using (13) and (14) yields

\[ \dot{V}(t) \leq \zeta^T(t) [\Phi(e) + \text{sym}(F_1^T P F_2)] \zeta(t) \quad (16) \]

where

\[ \zeta(t) = \left[ e^T(t) \bar{e}^T(t) \alpha^T(t) \bar{\alpha}^T(t) \beta(t) \bar{\beta}^T(t) \bar{e}^T(t) \right]^T \]
\[ \Phi(e) = \left[ \begin{array}{cccc}
\Phi_{11}(e) & \Phi_{12} & \Phi_{13}(e) \\
* & \Phi_{22} & \Phi_{23} \\
* & * & \Phi_{33} \\
\end{array} \right], \]
\[ \Phi_{13}(e) = \left[ \begin{array}{cc}
\bar{A} & 0_{2 \times 12} \\
\bar{A} & \Pi(e) \\
\end{array} \right], \]
\[ \Phi_{11}(e) = \left[ \begin{array}{cccc}
\gamma_1 & \bar{\gamma}_{11}(e) & \bar{\gamma}_{12}(e) & \bar{\gamma}_{21}(e) & \bar{\gamma}_{22}(e) \\
* & \gamma_1 & 3R_{11} & 3R_{12} & 3R_{21} & 3R_{22} \\
* & * & \gamma_1 & 3R_{11} & 3R_{12} & 3R_{21} \\
* & * & * & \gamma_1 & 3R_{11} & 3R_{12} \\
* & * & * & * & \gamma_1 & 3R_{11} \\
\end{array} \right]. \]
For all initial state \( e(t) \in X_e \), satisfying \( V(0) \leq 1 \), it is seen from (18) and (20) that all trajectories \( e(t) \) are contained in the set \( \varepsilon(Z, 1) \). Noting (17) and (21), it is concluded that the system (4) is asymptotically stable for all \( e(s) \) satisfying \( V(0) \leq 1 \). This completes the proof.

**Remark 4.1:** Let \( Q_{ij}^{kl}, R_{ij}^{kl} = 0 \), the interconnected terms related to four time delays in the functional (11) have been ignored, then the dimension of matrix \( P \) in Theorem 4.1 will be greatly reduced to 18 × 18. The LMI (8) will be turned into

\[
\begin{bmatrix}
\Phi_{11}(v_m) & \Phi_{12}'(v_m) \\
\Phi_{21}'(v_m) & \Phi_{13}(v_m)
\end{bmatrix}
+ \text{sym}(F_1'PF_2') < 0, \quad m = 1, 2, 3, 4,
\]

where

\[
F_1' = \begin{bmatrix}
I & 0_{2\times 8} & 0_{2\times 16} & 0 \\
0_{16\times 2} & 0_{16\times 8} & I_{16\times 16} & 0_{16\times 2}
\end{bmatrix},
\]

\[
\Phi_{12}' = \begin{bmatrix}
\hat{R}_T & \hat{R}_T' \\
\hat{R}_T' & \hat{R}_T
\end{bmatrix}, \quad \Phi_{13}(v_m) = \begin{bmatrix}
\bar{\Lambda} & -T_I \\
\bar{\Lambda} & \bar{\Pi}(v_m)
\end{bmatrix},
\]

\[
F_2' = \begin{bmatrix}
0 & 0_{2\times 8} & 0_{2\times 8} & 0_{8\times 8} \\
\Gamma_1 & -I_{8\times 8} & 0_{8\times 8} & 0_{8\times 8} \\
\Gamma_3 & 0_{8\times 8} & -I_{8\times 8} & 0_{8\times 8}
\end{bmatrix},
\]

\[
\Xi = \begin{bmatrix}
\Xi^{11} & \Xi^{12} \\
* & \Xi^{22}
\end{bmatrix},
\]

\[
\Phi_{22}' = -192 \text{diag}(\tau_{11}^{-2}\tau_{11,2}, \cdots, \tau_{22}^{-2}\tau_{22}),
\]

\[
\Phi_{23}' = \begin{bmatrix}
\Theta' & 0_{8\times 2}
\end{bmatrix},
\]

\[
\Phi_{33}' = -720 \text{diag}(\tau_{11}^{-4}\tau_{11,2}, \cdots, \tau_{22}^{-4}\tau_{22}, -\mu / 720),
\]

\[
\Theta' = 360 \text{diag}(\tau_{11}^{-3}\tau_{11,2}, \cdots, \tau_{22}^{-3}\tau_{22}).
\]

When the time delays are varying, the system we concern is as follows:

\[
\begin{cases}
\dot{x}(t) = x(t)(b_1 - a_1x(t - \tau_{11}(t)) - a_{12}y(t - \tau_{12}(t))), \\
\dot{y}(t) = y(t)(b_2 + a_{21}x(t - \tau_{21}(t)) - a_{22}y(t - \tau_{22}(t))),
\end{cases}
\]

where \( 0 \leq \tau_{ij}(t) \leq d_{ij}, i_j(t) \leq h_j \) with \( d_{ij} > 0, h_j > 0(i, j = 1, 2) \).

The initial condition of system (22) is given as

\[
\begin{cases}
x(s) = \phi_1'(s), -\tau \leq s \leq 0, \phi_1'(s) > 0, \\
y(s) = \phi_2'(s), -\tau \leq s \leq 0, \phi_2'(s) > 0,
\end{cases}
\]

where \( \tau = \max_{1 \leq i, j \leq 2} \tau_{ij}(t) \), and \( \phi_1'(s), \phi_2'(s) \) are assumed to be continuous.

Similar to the previous operation, we can rewrite (22) in the following matrix form:

\[
\dot{e}(t) = -\sum_{i,j=1}^2 [A_{ij} + B_{ij}(e)]e(t - \tau_{ij}(t)),
\]
denote $\tilde{a}_{ij}(t) \triangleq \int_{t-\tau_j(t)}^t e(s)ds$, then we can obtain the following Corollary 4.1.

**Corollary 4.1:** Let the scalars $\tilde{e}_1 > 0, \tilde{e}_2 > 0$ and $d_{ij} > 0, \tau_j \geq 0, i,j = 1,2$ be given. The system (23) is locally asymptotically stable, if there exist $10 \times 10$ dimensional symmetric matrix $\tilde{P}$, and $2 \times 2$ dimensional matrices $\tilde{Q}_{ij} > 0, \tilde{R}_{ij} > 0, i,j = 1,2, \tilde{Z} > 0, \tilde{T}_1, \tilde{T}_2$, such that the following LMs hold:

\[
\begin{bmatrix}
\tilde{\Phi}_{11}(\nu_m) & \tilde{\Phi}_{12}(\nu_m) \\
\ast & \tilde{\Phi}_{22}
\end{bmatrix} + \text{sym}(\tilde{F}_1^T\tilde{P}\tilde{F}_2) < 0, \ m = 1,2,3,4,
\]

**Proof:** Choose the following augmented L-K functional:

\[
\tilde{V}(t) = \tilde{\eta}^T(t)\tilde{P}\tilde{\eta}(t) + \sum_{i,j=1}^{2} \int_{t-\tau_j(t)}^t e^T(s)\tilde{Q}_{ij}e(s)ds \\
+ \sum_{i,j=1}^{2} d_{ij} \int_{t-\tau_j(t)}^t e^T(s)\tilde{R}_{ij}\dot{e}(s)dsdt, \quad (27)
\]

with

\[
\tilde{\eta}(t) = \begin{bmatrix} e^T(t) & \tilde{a}_{11}^T(t) & \tilde{a}_{12}^T(t) & \tilde{a}_{21}^T(t) & \tilde{a}_{22}^T(t) \end{bmatrix}^T.
\]

Through some direct calculations, one can obtain that

\[
\dot{\tilde{V}}(t) \leq 2\tilde{\eta}^T(t)\tilde{P}\dot{\tilde{\eta}}(t) + e^T(t)\sum_{i,j=1}^{2} \tilde{Q}_{ij}e(t) \\
- \sum_{i,j=1}^{2} (1-h_{ij})e^T(t-\tau_j(t))\tilde{Q}_{ij}e(t-\tau_j(t)) \\
+ e^T(t)\sum_{i,j=1}^{2} d_{ij}^2\tilde{R}_{ij}\dot{e}(t) \\
- \sum_{i,j=1}^{2} d_{ij} \int_{t-\tau_j(t)}^t e^T(s)\tilde{R}_{ij}\dot{e}(s)ds,
\]

Using the inequality (1) of Lemma 3.1, it is seen that

\[
- d_{ij} \int_{t-\tau_j(t)}^t e^T(s)\tilde{R}_{ij}\dot{e}(s)ds \\
\leq - \begin{bmatrix} \tilde{\xi}_{ij}^{(1)}(t) \tilde{\xi}_{ij}^{(2)}(t) \end{bmatrix}^T \begin{bmatrix} \tilde{R}_{ij} & 0 \\ 0 & 3\tilde{R}_{ij} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_{ij}^{(1)}(t) \\ \tilde{\xi}_{ij}^{(2)}(t) \end{bmatrix} \quad (28)
\]

where

\[
\tilde{\xi}_{ij}^{(1)}(t) = e(t) - e(t-\tau_j(t)), \\
\tilde{\xi}_{ij}^{(2)}(t) = e(t) + e(t-\tau_j(t)) - 2d_{ij}^{-1}\tilde{a}_{ij}(t),
\]

For any matrix $T_1, T_2 \in \mathbb{R}^{2 \times 2}$, it follows that the following equation is true:

\[
2[e^T(t)T_1 + e^T(t)T_2] \\
\times \left\{ - \sum_{i,j=1}^{2} [A_{ij} + B_{ij}(e)]e(t-\tau_j(t)) - \dot{e}(t) \right\} = 0. \quad (29)
\]

Adding the left side of (29) to $\dot{V}(t)$ and using (28) yields

\[
\dot{V}(t) \leq \tilde{\zeta}^T(t)[\tilde{\Phi}(e) + \text{sym}(\tilde{F}_1^T\tilde{P}\tilde{F}_2)]\tilde{\zeta}(t) \quad (30)
\]
where

\[
\tilde{\zeta}(t) = \begin{bmatrix} e^T(t) & e^T(t - \tau_{11}(t)) & \cdots & e^T(t - \tau_{22}(t)) \\
\end{bmatrix}^T \\
\hat{\Phi}(e) = \begin{bmatrix} \hat{\Phi}_{11}(e) & \hat{\Phi}_{12}(e) \\
\hat{\Phi}_{21}(e) & \hat{\Phi}_{22}(e) \\
\end{bmatrix}, \\
\hat{\Phi}_{11}(e) = \begin{bmatrix} \gamma_1 & \gamma_1 & 0 & 0 & 0 \\
\gamma_1 & \gamma_1 & 0 & 0 & 0 \\
\gamma_1 & \gamma_1 & \gamma_1 & 0 & 0 \\
\gamma_1 & \gamma_1 & \gamma_1 & \gamma_1 & 0 \\
\end{bmatrix},
\]

with

\[
\hat{R}_j(e) = -2\hat{R}_j - T_1[A_j + B_j(e)],
\]

Note that \(\hat{\Phi}(e)\) is affine with respect to the states \(e_1(t)\) and \(e_2(t)\). Hence, if the LMIs in (24) are satisfied, the inequality \(\hat{\Phi}(e) < 0\) can be ensured on \(\chi\). Then, we have

\[
\dot{V}(t) < 0 \quad (31)
\]
on the square \(\chi\), which implies that

\[
V(t) \leq V(0), \quad t \geq 0. \quad (32)
\]

On the other hand, using the integral inequalities (1) and (2) in Lemma 3.2, one can obtain the inequality

\[
V(t) \geq \eta^T(t) \tilde{\Phi}(\eta) + \sum_{i,j = 1}^{2} d_{ij}^{-1}(\tilde{\alpha}_{ij}(t)) \tilde{R}_{ij}(t) + \\
2d_{ij}^2 \tau_{ij}^{-2}(t)[\tau_{ij}(t)e(t) - \tilde{\alpha}_{ij}(t)]^T \hat{R}_{ij} \\
\times [\tau_{ij}(t)e(t) - \tilde{\alpha}_{ij}(t)] + \\
= \eta^T(t) \left( \hat{\Phi} + \begin{bmatrix} \hat{\Sigma} \\
\end{bmatrix} \right) \eta(t) \
\] (33)

where \(\hat{\Sigma} = \sum_{i,j = 1}^{2} 2d_{ij}\hat{R}_{ij}\).

Using (25) and noting (33), it follows that

\[
V(t) \geq e^T(t) \tilde{\Phi}(e(t)), \quad (34)
\]

which shows that the L-K functional \(V(t)\) is positive definite.

In addition, it can be inferred from (26) that

\[
\varepsilon(\tilde{Z}, 1) \subseteq \chi. \quad (35)
\]

For all initial state \(e(s) \in \mathcal{X}_p\) satisfying \(V(0) \leq 1\), it is seen from (32) and (34) that all trajectories \(e(t)\) are contained in the set \(\varepsilon(\tilde{Z}, 1)\). Noting (31) and (35), it is concluded that the system (22) is asymptotically stable for all \(e(s)\) satisfying \(V(0) \leq 1\). This completes the proof.

**Remark 4.2:** Because of the difficulties in tackling the varying time-delays, in above proof, we use some simple inequalities to obtain the results. In (28), we use (1) in Lemma in place of (2) in lemma 3.1. In (33), we use Jensen inequalities in place of Wirtinger inequalities.

Now, we will consider the estimate of RA of the positive equilibrium the region of asymptotic. In this end, we introduce the following inequalities:

\[
\begin{align*}
\mathcal{P} &\leq \text{diag}(\Lambda_0, \Lambda_{11}, \cdots, \Lambda_{22}, \Lambda_{11}^{22}, \cdots, \bar{\Lambda}_{11}, \cdots, \bar{\Lambda}_{22}, \bar{\Lambda}_{11}^{22}, \cdots, \bar{\Lambda}_{22}^{22}), \\
\Lambda_0 &
\leq p_{0l}, \quad \Lambda_{ij} \leq p_{yjl}, \quad \Lambda_{ij} \leq \bar{p}_{yjl}, \quad Q_{ij} \leq q_{ijl}, \quad R_{ij} \leq r_{ijl}, \\
\Lambda_{ij}^{kl} &\leq p_{ijl}^{kl}, \quad \Lambda_{ij} \leq \bar{p}_{ijl}^{kl}, \quad Q_{ij}^{kl} \leq q_{ijl}^{kl}, \quad R_{ij}^{kl} \leq r_{ijl}^{kl}, \\
&\leq \hat{\lambda}_1^{-1} \max_{s \in [-\tau, 0]} ||e(s)||^2 + \gamma_2 \max_{s \in [-\tau, 0]} ||\hat{e}(s)||^2
\end{align*}
\]

where

\[
\begin{align*}
\Lambda_0 > 0, \Lambda_{ij} > 0, &\quad \Lambda_{ij}^{kl} > 0, \quad \bar{\Lambda}_{ij} > 0, \\
\bar{\Lambda}_{ij}^{kl} > 0, &\quad p_{0l} > 0, p_{yjl} > 0, p_{ijl}^{kl} > 0, \quad \bar{p}_{yjl} \\
&\quad > 0, \quad \bar{p}_{ijl}^{kl} > 0, q_{ijl} > 0, r_{ijl} > 0, q_{ijl}^{kl} > 0, \quad r_{ijl}^{kl} > 0.
\end{align*}
\]

Using (36) and the Jensen inequalities, one can obtain

\[
V(0) \leq \hat{\lambda}_1 \max_{s \in [-\tau, 0]} ||e(s)||^2 + \lambda_2 \max_{s \in [-\tau, 0]} ||\hat{e}(s)||^2
\]

where

\[
\begin{align*}
\lambda_1 &\equiv \lambda_M(\Lambda_0) + \sum_{i,j = 1}^{2} \tau_{ij}[\tau_{ij} \lambda_M(\Lambda_{ij}) \\
&\quad + (\tau_{ij}^3/4) \lambda_M(\bar{\Lambda}_{ij}) + \lambda_M(Q_{ij})] + \\
&\quad \sum_{i,j = 1}^{2} \sum_{k,l = 1}^{2} \tau_{ijl}^2 \lambda_M(\Lambda_{ij}^{kl}) + \lambda_M(Q_{ij}^{kl})) \\
&\quad + (\Lambda_{ij} + \tau_{kl})^2/4) \lambda_M(\Lambda_{ij}^{kl}) + \lambda_M(Q_{ij}^{kl}))], \\
\lambda_2 &\equiv \sum_{i,j = 1}^{2} (\tau_{ij}^3/2) \lambda_M(R_{ij}) \\
&\quad + \sum_{i,j = 1}^{2} \sum_{k,l = 1}^{2} [(\tau_{ijl}^3/2) \lambda_M(R_{ij}^{kl})], \\
\hat{\lambda}_1 &\equiv \rho_0 + \sum_{i,j = 1}^{2} \tau_{ij}[\tau_{ij} \rho_{yjl} + 0.25 \tau_{ij}^3 \rho_{yjl} + q_{ij}] \\
&\quad + \sum_{i,j = 1}^{2} \sum_{k,l = 1}^{2} \tau_{ijl} \rho_{yjl}^{kl} + 0.25 (\tau_{ij} + \tau_{kl}) \rho_{yjl}^{kl} + q_{ij}^{kl} \\
&\quad + \sum_{i,j = 1}^{2} \sum_{k,l = 1}^{2} \tau_{ijl} \rho_{yjl}^{kl} + 0.25 (\tau_{ij} + \tau_{kl}) \rho_{yjl}^{kl} + q_{ij}^{kl} \\
\hat{\lambda}_2 &\equiv \sum_{i,j = 1}^{2} 0.5 \tau_{ij}^3 r_{ij} + \sum_{i,j = 1}^{2} 0.5 [\tau_{ijl}^2 (\tau_{ij} + \tau_{kl}) r_{ij}^{kl}].
\end{align*}
\]
Then, the maximization of the estimate of the region of asymptotic stability in Theorem 4.1 can be formulated as

\[
\begin{align*}
\text{OP.1.} & \quad \min_{P,Q,p_i,q_i,\lambda_1,\lambda_2} \tilde{\lambda}_1 + \tilde{\lambda}_2, \\
\text{s.t.} & \quad \text{LMIs (8) \text{ -- (10) and (36) hold.}}
\end{align*}
\]

By solving the optimization problem OP.1, one can obtain the scalars \(\lambda_1\) and \(\lambda_2\). Recalling that all initial condition \(e(s)\) satisfy \(V(0) \leq 1\), from (37), we can require that the initial condition \(e(s)\) satisfy the following relationship:

\[
\lambda_1 \max_{s \in [-\tau,0]} ||\phi(s)||^2 + \lambda_2 \max_{s \in [-\tau,0]} ||\phi(s)||^2 \leq 1. \quad (38)
\]

5. Numerical simulations

In this section, we will consider the numerical example given in Zhen and Ma (2002), Park (2005), Sun and Meng (2007) and Dong et al. (2019) to illustrate the effectiveness and the sharpness of our results. In Zhen and Ma (2002), Park (2005) and Sun and Meng (2007), \(b_1 = b_2 = a_11 = a_{22} = 1\) and \(a_{12} = a_{21} = 0.5\). It is easy to verify that the system (1) has one positive equilibrium point \((2/3, 2/3)\).

When \(\tau_j = \tau\), the upper bounds \(\tau\) for local asymptotic stability obtained by the existing results and the Theorem 4.1 in this paper are listed in Table 1.

Table 1 shows that the result in this paper can propose larger delay bound than those in Zhen and Ma (2002), Park (2005), Sun and Meng (2007) and Dong et al. (2019).

For the equilibrium point \((0, 0)\) of the system (4), by solving the optimization problem OP.1 with \(\tau_j = 0.6\) \((i,j = 1, 2)\), \(\bar{e}_1 = 0.66\), \(\bar{e}_2 = 0.57\), we can obtain \(\lambda_1 = 5.669\), \(\lambda_2 = 0.8017\). Then, we obtain the local stability condition of system (4) is the initial function \(e(s)\) should satisfy the following relationship as shown in Figure 1:

\[
5.669||e(s)||^2 + 0.8017||\dot{e}(s)||^2 \leq 1. \quad (39)
\]

For example, if \(e(s)\) is constant, then we estimate the RA of the equilibrium point \((0, 0)\) of system (4) should satisfy

\[
\phi_1^2(s) + \phi_2^2(s) \leq 0.176. \quad (40)
\]

Choosing the initial condition \((\phi_1'(s), \phi_2'(s))^T = (0.5 \cos t, 0.4)^T\), we obtain the state responses of system (1) as Figure 2.

**Table 1. Maximum admissible ranges of the time delay \(\tau\).**

| Zhen and Ma (2002) | Park (2005) | Sun and Meng (2007) | Dong et al. (2019) | Theorem 4.1 |
|-------------------|-------------|---------------------|-------------------|-------------|
| 0 < \(\tau\) < 0.34 | 0 < \(\tau\) < 0.88 | 0 < \(\tau\) < 1.4307 | 0 < \(\tau\) < 1.5674 | 0 < \(\tau\) < 1.5804 |

6. Conclusions

In this paper, local stability analyses and the estimate of the RA of the L-V competitive system with time-delays are conducted firstly based on quadratic system theory. Some less conservative stability conditions have been obtained by constructing an appropriate L-K functional and choosing less conservative inequalities. On the other hand, the boundary of the RA of the system is plotted and maximized by solving the optimization problem OP.1 in the framework of LMIs. Finally, simulation results have been given to demonstrate the effectiveness of the obtained conditions. However, Wirtinger-based inequalities are not the most effective, and simulations show that the boundary of the RA estimation in this paper is conservative. This is our future research.

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