GOODNESS-OF-FIT TESTS FOR COPULAS OF MULTIVARIATE TIME SERIES

BRUNO RÉMILLARD

Abstract. The asymptotic behaviour of the empirical copula constructed from residuals of stochastic volatility models is studied. It is shown that if the stochastic volatility matrix is diagonal, then the empirical copula process behaves like if the parameters were known, a remarkable property. However, that is not true if the stochastic volatility is genuinely non-diagonal. Applications for goodness-of-fit and structural change of the dependence between innovations are discussed.

1. Introduction

In many financial applications, e.g., pricing of options on multiple assets or exchange rates, multiname credit derivatives, portfolio management and risk management, it is necessary to model the dependence between different assets. That can be done simply by using copulas, which are distribution functions of multivariate uniform variables.

It has been shown, e.g., Embrechts et al. (2002) and Berrada et al. (2006), that the choice of the copula is of paramount importance since it can lead to significant differences in pricing. The same is true for measures of risk. From an economic point of view, it is also pertinent to try to find the kind of dependence linking several economic series. One could be interested for example in modeling the dependence between several exchange rates with respect to the US currency, or to show that there is a strong dependence between an exchange rate and the value of a commodity. Using the recent economic context in Greece, one could also be interested in modeling the dependence between exchange rates (Euro vs USD) and bond values. Actuaries also have to model dependence between pairs or multivariate vectors of data.

Motivated by actuarial, economic and financial applications, the problem of the choice of the copula to model dependence between data correctly is quite recent and has been tackling mainly for serially independent observations. See, e.g., Genest et al. (2009) for a comparison and review of the consistent goodness-of-fit tests that can be used in that context.
However, in economic and financial applications, there is almost always serial dependence and when looking for the choice of a copula family, the serial dependence problem is either ignored, i.e., the data are not “filtered” to remove serial dependence, as in Dobrić and Schmid (2005, 2007) and Kole et al. (2007), or the data are “filtered” but the potential inferential problems of using these transformed data are not taken into account. For example, Panchenko (2005) uses a goodness-of-fit test on “filtered” data (residuals of GARCH models in his case), without proving that his proposed methodology works for residuals. However he mentioned in passing that working with residuals could destroy the asymptotic properties of his test. A similar situation appears in Breymann et al. (2003) where both the problem of working with residuals and the problem of the estimation of the copula parameters are ignored. The same criticisms can be addressed to van den Goorbergh et al. (2005) and Patton (2006).

It seems that the first paper addressing rigorously the problems raised by the use of residuals in estimation and goodness-of-fit of copulas is Chen and Fan (2006). An unpublished document (Chen et al., 2005) also circulated some time ago proposing a kernel-based test using residuals of GARCH models. However the proof of their main result is missing and the technical report never got published. Using a multivariate GARCH-like model with diagonal innovation matrix, Chen and Fan (2006) showed the remarkable result that estimating the copula parameters using the rank-based pseudo-likelihood method of Genest et al. (1995) and Shih and Louis (1995) with the ranks of the residuals instead of the (non-observable) ranks of innovations, leads to the same asymptotic distribution. In particular, the limiting distribution of the estimation of the copula parameters does not depend on the unknown parameters used to estimate the conditional means and the conditional variances. That property is crucial if one wants to develop goodness-of-fit tests for the copula family of the innovations. In Chen and Fan (2006), the authors also propose ways of selecting copulas based on pseudo-likelihood ratio tests. However, their comparison test is not a goodness-of-fit test in the sense that one could select a model which is inadequate though better than the other proposed models.

In the present paper, goodness-of-fit tests are proposed, with the related statistics being functions of empirical processes, since tests based on empirical processes are generally consistent and more powerful than other classes of test statistics, including likelihood ratio tests. One extends the results of Chen and Fan (2006) by proving that under similar technical assumptions, the empirical copula process has the same limiting distribution as if one would have started with the innovations instead of the residuals. Other methods of estimation of the parameters of the copula families, not considered in Chen and Fan (2006), also share the same properties. For example, the asymptotic behaviour of Kendall’s tau, Spearman’s rho, van der Waerden and Blomqvist’s coefficients are exactly the same as with serially independent observations. An immediate consequence is that all tools developed recently for the serially independent case remain valid for the residuals. In particular, one can use the 1-level and 2-level parametric bootstrap of Genest and Rémillard (2008) to estimate p-values of tests statistics, if the estimator is regular (Genest and Rémillard, 2008). That is the case for the usual estimators like pseudo likelihood estimators and
moment estimators. Such properties are in sharp contrast with the ones using consecutive residuals of a single time series for testing serial independence (Ghoudi and Rémillard, 2010), where the limiting copula process does depend on parameters, even in simple ARMA models. It is also shown that when the volatility matrix is genuinely non-diagonal, then all these nice properties stop holding true. The estimation of the copula parameters and the limiting empirical copula depend on the conditional mean and conditional variance parameters. It is the case for general BEKK models, used in Patton (2006) and Dias and Embrechts (2009).

In what follows, one starts, in Section 2, by describing the model and discussing parameter estimation for copulas. Tests statistics based on the empirical copula process and the Rosenblatt’s transform are then proposed in Sections 3 and 4 respectively, together with implementations of the parametric bootstrap. The main result for testing goodness-of-fit using the empirical copula process is given in Proposition 5, while its analog for using Rosenblatt’s transform is given in Proposition 6. Change-point problems are discussed in Section 5, either for univariate series or copulas, while an example of application using some data of Chen and Fan (2006) is treated in Section 6. The main results on the convergence of the empirical processes are stated and proved in the Appendix.

2. Model and estimation

Following Chen and Fan (2006), one considers a stochastic volatility model for a multivariate time series $X_i$, i.e., for $i \geq 1$ and $j = 1, \ldots, d$,

$$X_{ji} = \mu_{ji}(\theta) + h_{ji}(\theta)^{1/2} \varepsilon_{ji},$$

or in vector form

$$X_i = \mu_i(\theta) + \sigma_i(\theta) \varepsilon_i,$$

where the innovations $\varepsilon_i = (\varepsilon_{1i}, \ldots, \varepsilon_{di})^\top$ are i.i.d. with continuous distribution function $K$, $\sigma_i = \text{diag}\{h_{1i}^{1/2}, \ldots, h_{di}^{1/2}\}$, and where $\mu_i$, $\sigma_i$ are $F_{i-1}$-measurable and independent of $\varepsilon_i$. Here $F_{i-1}$ contains information from the past and possible information from exogenous variables. That model, studied in Chen and Fan (2006), contains as a particular case, BEKK models (Engle and Kroner, 1995) with diagonal conditional volatility matrix. Note that in many applications, univariate stochastic volatility models are fitted separately to each time series $(X_{ji})_{i=1}^n$, $j = 1, \ldots, d$.

Given an estimation $\theta_n$ of $\theta$, compute the residuals $e_{i,n} = (\varepsilon_{1i,n}, \ldots, \varepsilon_{di,n})^\top$, where

$$e_{i,n} = \sigma_i^{-1}(\theta_n) \{X_i - \mu_i(\theta_n)\}.$$

Since the distribution function $K$ is continuous, there exists a unique copula $C$ (Sklar, 1959) so that for all $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$,

$$K(x) = C(F(x)) = F_1(x_1), \ldots, F_d(x_d))^\top,$$

where $F_1, \ldots, F_d$ are the marginal distribution functions of $K$, i.e., $F_j$ is the distribution function of $\varepsilon_{ji}$. Setting $U_i = F(\varepsilon_i)$, one gets that $U_i$ has distribution $C$, denoted by $U_i \sim C$, $i = 1, \ldots, n$. 

Electronic copy available at: http://ssrn.com/abstract=1729982
Since the copula is independent of the margins, it is generally suggested\(^1\) to remove their effect by replacing \(e_i\) with the associated rank vector

\[
U_{i,n} = (U_{1i,n}, \ldots, U_{di,n})^T, \quad U_{ji,n} = \text{Rank}(e_{ji,n})/(n + 1),
\]

with \(\text{Rank}(e_{ji,n})\) being the rank of \(e_{ji,n}\) amongst \(e_{j1,n}, \ldots, e_{j1,n}, j = 1, \ldots, d\). That can also be written as \(U_{i,n} = F_n(e_{i,n})\), where \(F_n(x) = (F_{1n}(x_1), \ldots, F_{dn}(x_d))^T\), and

\[
F_{jn}(s, x_j) = \frac{1}{n + 1} \sum_{i=1}^{[ns]} 1(e_{ji,n} \leq x_j), \quad j = 1, \ldots, d, \quad (s, x) \in [0, 1] \times \mathbb{R}^d. \tag{2}
\]

The main results on the paper are deduced from the asymptotic behaviour (see Appendix B) of the partial-sum empirical process

\[
K_n(s, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \{1(e_{i,n} \leq x) - K(x)\}, \quad s \in [0, 1], x \in \mathbb{R}^d. \tag{3}
\]

The reason for introducing partial sums in (2) and (3) will become clear in Section 5 when one studies detection of change-points.

In the next two sections one will study tests of goodness-of-fit for parametric copula families, i.e., one wants to test the null hypothesis

\[
H_0 : C \in \mathcal{C} = \{C_\phi; \phi \in \mathcal{O}\},
\]

for some parametric family of copula \(\mathcal{C}\). Typical families are of the meta-elliptic type (Gaussian and Student copula) and Archimedean copulas (Clayton, Frank, Gumbel). As proposed in Dias and Embrechts (2004), Chen and Fan (2006) and Patton (2006), one could also consider mixtures of such copulas. These families are listed in Appendix D, together with their parameters.

It is assumed that \(K\) and \(F_1, \ldots, F_d\) have continuous densities \(h, f_1, \ldots, f_d\) respectively. As a result, the copula \(C\) has a density \(c\) which satisfies

\[
h(x) = c\{F(x)\} \prod_{j=1}^d f_j(x_j).
\]

Under \(H_0\), each copula \(C_\phi\) is assumed to admit a density \(c_\phi\) satisfying hypotheses B1–B3 described in Appendix A, and depends on the parameter \(\phi\) which must be estimated.

In what follows one lists some estimators of copula parameters and one also studies some of their asymptotic properties.

---

\(^1\)Another method proposed by Xu (1996) is to find parametric estimators for each margin, so it becomes a fully parametric problem with two-stage estimation. On the negative side, it is less accurate than a fully parametric estimation and errors on the margins will reflect in the estimation of the parameters of the copula.
2.1. Estimation of copula parameters.

2.1.1. Pseudo likelihood estimators. In Chen and Fan (2006), it is shown that under smoothness conditions (conditions D, C, and N in their article), the pseudo maximum likelihood estimator

$$\phi_n = \arg \max_{\phi} \left\{ \sum_{i=1}^{n} \log c_\phi(U_{i,n}) \right\}$$

is asymptotically Gaussian with covariance matrix depending only on $c_\phi$. Therefore, the asymptotic behaviour does not depend on the estimation of the parameter $\theta$ required for the evaluation of the residuals! In fact, it has the same representation as the estimator studied by Genest et al. (1995) in the serially independent case, i.e., if the parameter $\theta$ was known. More precisely, one has

$$\Phi_n = \sqrt{n}(\phi_n - \phi) = J^{-1}(\mathbb{W}_n - Z_n) + o_P(1),$$

where $\mathbb{W}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{c}(U_{i})}{c(U_{i})}$, $Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{d} \sum_{i=1}^{n} Q_j(U_{ji})$, $Q_j(u_{j}) = \int_{(0,1)^d} \frac{\hat{c}(v)^\top \partial c(v)}{c(v)} \{1(u_j \leq v_j) - v_j\} dv$, and where $J$ is the Fisher’s information matrix $\int_{(0,1)^d} \frac{\hat{c}(u)^\top c(u)}{c(u)} du$. Here $\hat{c}$ is the row vector given by the gradient of $c_\phi$ with respect to $\phi$. Note that $\left( \begin{array}{c} \mathbb{W}_n \\ Z_n \end{array} \right)$ converges in law to $\left( \begin{array}{c} \mathbb{W} \\ Z \end{array} \right) \sim N(0, \Sigma)$, with $\Sigma = \left( \begin{array}{cc} J & 0 \\ 0 & J \end{array} \right)$. It follows that $\Phi_n$ converges in law to $\Phi \sim N(0, J^{-1} + J^{-1} J J^{-1})$. Note also that $E(\Phi \mathbb{W}^\top) = I$, i.e., $\phi_n$ is a regular estimator for $\phi$ in the sense of Genest and Rémillard (2008), where it is shown that regular estimators are essential for the validity of the parametric bootstrap procedure.

2.1.2. Two-stage estimators. In addition to pseudo likelihood estimators, one may consider a two-stage estimator. That is, suppose that $\phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)$. Decompose also $\mathbb{W}_n$ and $Z_n$ accordingly. Suppose that $\phi_{1,n}$ is an estimator of $\phi_1$ that is regular in the sense that $\Phi_{1,n} = \sqrt{n}(\phi_{1,n} - \phi_1)$ converges in law to $\Phi_1 \sim N(0, \Sigma_1)$ and $E(\Phi_1 \mathbb{W}_1^\top) = I$, $E(\Phi_1 \mathbb{W}_2^\top) = 0$. Now define $\phi_{2,n}$ as the pseudo-likelihood estimator of the reduced log-likelihood viz

$$\phi_{2,n} = \arg \max_{\phi_{2} \in O_2} \left\{ \sum_{i=1}^{n} \log c_{\phi_{1,n},\phi_2}(U_{i,n}) \right\}.$$
since \((\Phi, W)\) is a centered Gaussian vector with \(E(\Phi W^\top) = I\). Two-stage estimation is often used for meta-elliptical copulas which depend on a correlation matrix \(\rho\) and possibly other parameters. It is known that \(\rho\) can be expressed in terms of functions of Kendall’s tau, playing the role of \(\phi_1\), while the remaining parameters are defined as \(\phi_2\). In fact, \(\tau_{jk} = \tau(U_{ji}, U_{ki}) = \frac{2}{\pi} \arcsin(\rho_{jk})\) (Fang et al., 2002). For example, in the Student copula case, \(\phi_2\) would be the degrees of freedom.

It follows from Proposition 1 in the next section that if \(\tau_{jk,n}\) is the empirical Kendall’s tau for the pairs \((U_{ji,n}, U_{ki,n})\), \(i = 1, \ldots, n\), then for all \(1 \leq j < k \leq d\),

\[
\sqrt{n}(\tau_{jk,n} - \tau_{jk}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ 8C^{(j,k)}(U_{ji}, U_{ki}) - 4U_{ji} - 4U_{ki} + 2 - 2\tau_{jk} \right\} + o_P(1),
\]

converge to centered Gaussian variables \(R_{K}^{jk}\), where \(C^{(j,k)}\) is the copula of \((U_{ji}, U_{ki})\). Setting \(\rho_{jk,n} = \sin(\pi \tau_{jk,n}/2) + o_P(1)^2\), it follows that

\[
R_{jk,n} = \sqrt{n}(\rho_{jk,n} - \rho_{kj}) + o_P(1) = \frac{\pi}{2}(1 - \rho_{jk}^2)^{1/2}\sqrt{n}(\tau_{jk,n} - \tau_{jk}) + o_P(1)
\]

and \((R_{jk,n})_{1 \leq j < k \leq d}\) converges in law to \((R_{jk})_{1 \leq j < k \leq d}\), with \(R_{jk} = \frac{\pi}{2}(1 - \rho_{jk}^2)^{1/2}R_{K}^{jk}\). Note that

\[
E \left( R_{jk}^K W^\top \right) = 8 \int_{[0,1]^2} \hat{C}^{(j,k)}(u_j, u_k) dC^{(j,k)}(u_j, u_k) = \partial \phi \tau_{jk}.
\]

Hence

\[
E \left( R_{jk} W^\top \right) = (1 - \rho_{jk}^2)^{1/2}\partial \phi \{ \arcsin(\rho_{jk}) \} = \partial \phi \rho_{jk}.
\]

As a result, if \(\phi_1\) is the vector of components \(\rho_{jk}\) with \(1 \leq j < k \leq d\), and \(\phi_2\) does not depend on \(\rho\), then \(E(\Phi_1 W_1^\top) = I\) and \(E(\Phi_1 W_1^\top) = 0\). This shows that two-stage estimators are regular for meta-elliptic copulas families (defined in Appendix D.2).

Many copula families have parameters linked to rank-based measures of dependence. The most common Archimedean families (Clayton, Frank and Gumbel) can all be indexed by Kendall’s tau (see Appendix D.1, Table 1), Gaussian copula has van der Waerden correlation matrix as parameters (see Appendix D.2) and the Plackett copula can be indexed by Spearman’s rho (Nelsen, 2006). The estimation of these parameters when using the ranks of residuals is treated next.

2.2. Asymptotic behaviour of some rank-based dependence measures. In this section one investigates the asymptotic behaviour of four well-known rank-based dependence measures: Kendall’s tau, Spearman’s rho, van der Waerden and Blomqvist’s coefficients. The main result is that these measures behave asymptotically like the ones computed from innovations, extending the results of Chen and Fan (2006). The proofs depend on the asymptotic behaviour of the empirical copula process and they are given in Appendix C.

\(^2\)If \(d > 2\), one could have to slightly modify the vector with components \(\sin(\pi \tau_{jk,n}/2)\) in order to make \((\rho_n)\) a correlation matrix.
2.2.1. **Kendall’s tau.** \( \tau_{jk,n} \), the empirical Kendall’s coefficient for the pairs \((e_{ji,n}, e_{ki,n})\), \(i = 1, \ldots, n\), is defined by

\[
\tau_{jk,n} = \frac{2}{n(n - 1)} \left( \text{number of concordant pairs} - \text{number of discordant pairs} \right),
\]

where the pairs \((e_{ji,n}, e_{ki,n})\) and \((e_{jl,n}, e_{kl,n})\) are concordant if \((e_{ji,n} - e_{jl,n})(e_{ki,n} - e_{kl,n}) > 0, i \neq l\). Otherwise, they are said to be discordant. Its theoretical counterpart can be written as

\[
\tau_{jk} = 4 \int_0^1 \int_0^1 C^{(j,k)}(u_j, u_k) dC^{(j,k)}(u_j, u_k) - 1,
\]

with values in \([-1, 1]\) and with value 0 under independence. Let \(\tilde{\tau}_{jk,n}\) be Kendall’s tau calculated with the pairs of innovations \((\varepsilon_{ji}, \varepsilon_{ki}), i = 1, \ldots, n\).

**Proposition 1.** Under assumptions (A1)-(A6), for all \(1 \leq j < k \leq d\), \(\sqrt{n}(\tau_{jk,n} - \tilde{\tau}_{jk,n}) = o_P(1)\) and

\[
\sqrt{n}(\tau_{jk,n} - \tau_{jk}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ 8C^{(j,k)}(U_{ji}, U_{ki}) - 8U_{ji} - 8U_{ki} + 6 - 2\tau_{jk} \right\} + o_P(1).
\]

converge to centered Gaussian variables \(\mathcal{R}^K_{jk}\), with

\[
E\left( \mathcal{R}^K_{jk} \mathbb{W}^\top \right) = 8 \int_0^1 \int_0^1 \dot{C}^{(j,k)}(u_j, u_k) dC^{(j,k)}(u_j, u_k) = \partial_\phi(\tau_{jk}).
\]

2.2.2. **Spearman’s rho.** Spearman’s empirical coefficient \(\rho_{jk,n}^S\) is the correlation coefficient of the pairs \((U_{ji,n}, U_{ki,n})\), \(i = 1, \ldots, n\), while its theoretical counterpart \(\rho_{jk}^S\) is

\[
\text{Cor}(U_{ji,n}, U_{ki,n}) = 12\text{Cov}(U_{ji,n}, U_{ki,n}) = 12 \int_0^1 \int_0^1 \left\{ C^{(j,k)}(u_j, u_k) - u_ju_k \right\} du_j du_k.
\]

It has values in \([-1, 1]\) and has value 0 under independence. Further let \(\tilde{\rho}_{jk,n}^S\) be Spearman’s rho calculated with the pairs \((U_{ji,n}, U_{ki,n})\), \(i = 1, \ldots, n\).

**Proposition 2.** Under assumptions (A1)-(A6), \(\sqrt{n}(\rho_{jk,n}^S - \tilde{\rho}_{jk,n}^S) = o_P(1)\) and

\[
\sqrt{n}(\rho_{jk,n}^S - \rho_{jk}^S) = \frac{12}{\sqrt{n}} \sum_{i=1}^n \left\{ 12(U_{ji} - 1/2)(U_{ki} - 1/2) - \rho_{jk}^S 
\right. \\
+ \left. 6(U_{ji} - 1/2)^2 + 6(U_{ki} - 1/2)^2 - 1 \right\} + o_P(1).
\]

converge to centered Gaussian variables \(\mathcal{R}^S_{jk}\) with

\[
E\left( \mathcal{R}^S_{jk} \mathbb{W}^\top \right) = 12 \int_0^1 \int_0^1 \dot{C}^{(j,k)}(u_j, u_k) du_j du_k = \partial_\phi(\rho_{jk}^S).
\]

2.2.3. **van der Waerden’s coefficient.** Let \(\mathcal{N}\) and \(\mathcal{N}^{-1}\) be respectively the distribution function and the quantile function of the standard Gaussian distribution. Then the van der Waerden’s empirical coefficient \(\rho_{jk,n}^W\) is the correlation coefficient of the pairs

\[
\text{Cor}(\mathcal{N}(U_{ji,n}), \mathcal{N}(U_{ki,n})) = 8 \int_0^1 \int_0^1 C^{(j,k)}(u_j, u_k) dC^{(j,k)}(u_j, u_k).
\]
Proposition 4. \( \rho_{jk} \) is defined by

\[
\text{Cor}(Z_{ji}, Z_{ki}) = E(Z_{ji}Z_{ki}) = \int_0^1 \int_0^1 \{ C_{ji,k}(u_j, u_k) - u_j u_k \} d\mathcal{N}^{-1}(u_j) d\mathcal{N}^{-1}(u_k),
\]

with \( Z_{ji} = \mathcal{N}^{-1}(U_{ji}) \). It has values in \([-1, 1]\) and has value 0 under independence. Further let \( \hat{\rho}_{jk}^W \) be van der Waerden’s coefficient calculated with the pairs \( (U_{ji}, U_{ki}) \), \( i = 1, \ldots, n \).

Proposition 3. Under assumptions (A1)-(A6), \( \sqrt{n} \left( \rho_{jk,n}^W - \hat{\rho}_{jk,n}^W \right) = o_P(1) \) and

\[
\sqrt{n} \left( \rho_{jk,n}^W - \rho_{jk}^W \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ Z_{ji}Z_{ki} - \rho_{jk}^W - \kappa_{jk}(Z_{ji}) - \kappa_{kj}(Z_{ki}) \right\} + o_P(1)
\]

converge to \( R_{jk}^W \) centered Gaussian variables, with

\[
E \left( R_{jk}^W \right) = 12 \int_0^1 \int_0^1 \hat{C}^{(j,k)}(u_j, u_k) d\mathcal{N}^{-1}(u_j) d\mathcal{N}^{-1}(u_k) = \partial \phi \left( \rho_{jk}^W \right),
\]

where

\[
\kappa_{jk}(z_j) = \int_{\mathbb{R}} \{ 1(z_j \leq x) - \mathcal{N}(x)E(Z_{k1}|Z_{j1} = x) \} dx
\]

and

\[
\kappa_{kj}(z_k) = \int_{\mathbb{R}} \{ 1(z_k \leq y) - \mathcal{N}(y) \} E(Z_{j1}|Z_{k1} = y) dy.
\]

2.2.4. Blomqvist’s coefficient. Blomqvist’s empirical coefficient \( \rho_{jk,n}^B \) is defined as

\[
\rho_{jk,n}^B = \frac{4}{n} \sum_{i=1}^n 1(U_{ji,n} \leq 1/2, U_{ki,n} \leq 1/2) - 1.
\]

Its theoretical counterpart \( \rho_{jk}^B = 4P(U_{ji} \leq 1/2, U_{ki} \leq 1/2) - 1 \), has values in \([-1, 1]\) with value zero under independence. Further let \( \hat{\rho}_{jk,n}^B \) be Blomqvist’s coefficient calculated with the pairs \( (U_{ji}, U_{ki}) \), \( i = 1, \ldots, n \).

Proposition 4. Under assumptions (A1)-(A6), \( \sqrt{n} \left( \rho_{jk,n}^B - \hat{\rho}_{jk,n}^B \right) = o_P(1) \) and

\[
\sqrt{n} \left( \rho_{jk,n}^B - \rho_{jk}^B \right) = \frac{4}{\sqrt{n}} \sum_{i=1}^n \left\{ 1(U_{ji} \leq 1/2, U_{ki} \leq 1/2) - C^{(j,k)}(1/2, 1/2) \right\}
\]

\[
- \left\{ 1(U_{ji} \leq 1/2) - 1/2 \right\} \partial_{u_j} C^{(j,k)}(1/2, 1/2)
\]

\[
- \left\{ 1(U_{ki} \leq 1/2) - 1/2 \right\} \partial_{u_k} C^{(j,k)}(1/2, 1/2)
\]

converge to centered Gaussian variables \( R_{jk}^B \) with

\[
E \left( R_{jk}^B \right) = 4\hat{C}^{(j,k)}(1/2, 1/2) = \partial \phi \left( \rho_{jk}^B \right).
\]
3. Inference procedure using the empirical copula

Tests of goodness-of-fit can be designed by computing some kind of distance between the empirical copula \( C_n \), defined by

\[
C_n(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(U_{i,n} \leq u), \quad u \in [0,1]^d,
\]

and the “best” representative \( C_{\phi_n} \) of the parametric family \( C \), since \( C_n \) is a non-parametric estimator of \( C_{\phi_n} \) of the true copula \( C \). Here, it is assumed that \( \phi_n \) is a rank-based estimator of \( \phi \), i.e., \( \phi_n = T_n(U_{1,n}, \ldots, U_{n,n}) \), for some deterministic function \( T_n(u_1, \ldots, u_n) \).

For example, for testing \( H_0 \), one could use the Cramér-von Mises type statistic based on the process \( A_n = \sqrt{n}(C_n - C_{\phi_n}) \) viz.

\[
S_n = \int_{[0,1]^d} A_n^2(u) dC_n(u) = \sum_{i=1}^{n} \left\{ C_n(U_{i,n}) - C_{\phi_n}(U_{i,n}) \right\}^2.
\]

According to Genest et al. (2009), \( S_n \) is one of the best statistic constructed from \( A_n \) for an omnibus test\(^3\), and is much more powerful and easier to compute than the Kolmogorov-Smirnov type statistic \( \|A_n\| = \sup_{u \in [0,1]^d} |A_n(u)| \). That is why the later is ignored in the present paper.

To be able to state the convergence result for \( S_n \), one needs to introduce auxiliary empirical processes. For any \( s \in [0,1] \) and \( x \in \mathbb{R}^d \), set

\[
\alpha_n(s,u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \left\{ \prod_{j=1}^{d} \mathbf{1}(U_{ji} \leq u_j) - C(u) \right\},
\]

and \( \beta_{j,n}(s,u_j) = \alpha_n(s,1,\ldots,1,u_j,1,\ldots,1) = \frac{1}{n} \sum_{i=1}^{[ns]} \{ \mathbf{1}(U_{ji} \leq u_j) - u_j \}, j = 1, \ldots, d. \)

It is well known (Bickel and Wichura, 1971) that \( \alpha_n \sim \alpha^d \), where \( \alpha \) is a \( C \)-Kiefer process, i.e., \( \alpha \) is a continuous centered Gaussian process with \( \text{Cov} \{ \alpha(s,u), \alpha(t,v) \} = (s \wedge t) \{ C(u \wedge v) - C(u)C(v) \} \), \( s \in [0,1] \) and \( u, v \in [0,1]^d \). Here \( (u \wedge v)_j = \min(u_j, v_j) \), \( j = 1, \ldots, d. \)

For convenience, set

\[
\hat{C}(s,u) = \alpha(s,u) - s \sum_{j=1}^{d} \beta_j(1,u_j) \partial_{u_j} C(u), \quad (s,u) \in [0,1] \times [0,1]^d.
\]

Recall that \( \hat{C}(-, \cdot) \) is the limit of the empirical copula process constructed from innovations; see, e.g., Gänßler and Stute (1987), Fermanian et al. (2004), Tsukahara

\(^3\)Of course, if the parametric family under \( H_1 \) is specified, then one can find better tests statistics than \( S_n \), see, e.g., Berg and Quessy (2009).

\(^4\)Convergence of processes means convergence with respect to the Skorohod topology for the space of cadlag processes, and is denoted by \( \rightsquigarrow \). The processes studied here are indexed by \( [0,1] \times [0,1]^d \), \( [0,1] \times [0,\infty, +\infty]^d \), or product of these spaces. Note that random vectors belong to these spaces, being constant random functions.
(2005). The process $\hat{C}$, that could be called the Kiefer copula process, will be important in Section 5.

It follows from Corollary 1 that the empirical copula process $C_n (1, u) = \sqrt{n} (C_n(u) - C(u))$ converges to $\hat{C}(1, u)$, which does not depend on the parameters of the conditional mean and conditional volatility. As a result, the limiting distribution of $A_n$ also shares that property, depending only on $\phi$ and $\Phi$ under $H_0$. The basic result for testing goodness-of-fit using the empirical copula process is stated next.

As in Genest and Rémillard (2008), assume, for identifiability purposes, that for every $\delta > 0$,

$$\inf \left\{ \sup_{u \in [0,1]^d} |C_{\phi}(u) - C_{\phi_0}(u)| : \phi \in O \text{ and } |\phi - \phi_0| > \delta \right\} > 0.$$  

Furthermore, the mapping $\phi \mapsto C_{\phi}$ is assumed to be Fréchet differentiable with derivative $\dot{C}$, i.e., for all $\phi_0 \in O$,

$$\lim_{h \to 0} \sup_{u \in [0,1]^d} \frac{|C_{\phi_0+h}(u) - C_{\phi_0}(u) - \dot{C}(u) h|}{\|h\|} = 0. \quad (8)$$

Before stating the main result of the section, one needs to extend the notion of regularity of $\phi_n$ as defined in Genest and Rémillard (2008). One says that $\phi_n$ is regular for $\phi$ if $(\alpha_n, \Phi_n) \rightsquigarrow (\alpha, \Phi)$ where the latter is centered Gaussian with $E(\Phi \Phi^T) = I$. Note that the estimators described in Section 2.1 (under the additional assumptions of Chen and Fan (2006)) and Section 2.2 (under assumptions (A1)–(A6), stated in Appendix B) are all regular.

**Proposition 5.** Under assumptions (A1)–(A6), if $\phi_n$ is regular for $\phi$, then $S_n$ converges in law to $S = \int_{[0,1]^d} A^2(u) dC(u)$, where $A = \hat{C} - \dot{C}^T \Phi$.

In fact, if $\psi$ is a continuous function on the space $C([0,1])$, then the statistic $T_n = \psi(A_n)$ converges in law to $T = \psi(A)$. Moreover, the parametric bootstrap algorithm described next or the two-level parametric bootstrap proposed in Genest et al. (2009) can be used to estimate $P$-values of $S_n$ or $T_n$.

### 3.1. Parametric bootstrap for $S_n$.

The following procedure leads to an approximate $P$-value for the test based on $S_n$. The adaptations required for any other function of $A_n$ are obvious. It can be used only if there is an explicit expression for $C_{\phi}$. Otherwise, 2-level parametric bootstrap must be used.

1. Compute $C_n$ as defined in (5) and estimate $\phi$ with $\phi_n = T_n (U_{1,n}, \ldots, U_{n,n})$.
2. Compute the value of $S_n$, as defined by (6).
3. For some large integer $N$, repeat the following steps for every $k \in \{1, \ldots, N\}$:
   a. Generate a random sample $Y_{1,n}^{(k)}, \ldots, Y_{n,n}^{(k)}$ from distribution $C_{\phi_n}$ and compute the pseudo-observations $U_{i,n}^{(k)} = R_{i,n}^{(k)}/(n + 1)$, where $R_{1,n}^{(k)}, \ldots, R_{n,n}^{(k)}$ are the associated rank vectors of $Y_{1,n}^{(k)}, \ldots, Y_{n,n}^{(k)}$. 

(b) Set
\[ C_n^{(k)}(u) = \frac{1}{n} \sum_{i=1}^{n} 1\left(U_{i,n}^{(k)} \leq u\right), \quad u \in [0, 1]^d \]
and estimate \( \phi \) by \( \phi_n^{(k)} = T_n\left(U_{1,n}^{(k)}, \ldots, U_{n,n}^{(k)}\right) \).

(c) Set
\[ S_n^{(k)} = \sum_{i=1}^{n} \left\{ C_n^{(k)}\left(U_{i,n}^{(k)}\right) - C_n^{(k)}\left(U_{i,n}^{(k)}\right)\right\}^2. \]

An approximate \( P \)-value for the test is then given by \( \sum_{k=1}^{N} 1\left(S_n^{(k)} > S_n\right) / N \).

Remark 1. Some authors, e.g., Kole et al. (2007), proposed tests statistics of the Anderson-Darling type, dividing \( A_n(u) \) by \( \sqrt{C_n(u)(1 - C_n(u))} \), and then integrating or taking the supremum. As argued in Genest et al. (2009) and Ghoudi and Rémillard (2010), one should be very careful with these tests and in fact avoid them totally since the denominator only makes sense in the univariate case when parameters are not estimated. In the present context, the limiting distribution of such weighted processes has not been proven and in fact, Ghoudi and Rémillard (2010) gave an example where the limiting variance of the weighted process is infinite.

4. Inference procedure using Rosenblatt’s transform

Based on recent results of Genest et al. (2009), one might also propose to use goodness-of-fit tests constructed from the Rosenblatt’s transform (Rosenblatt, 1952). In their study, such tests were among the most powerful omnibus tests.

Recall that the Rosenblatt’s mapping of a \( d \)-dimensional copula \( C \) is the mapping \( R \) from \((0, 1)^d \rightarrow (0, 1)^d\) so that \( u = (u_1, \ldots, u_d) \mapsto R(u) = (e_1, \ldots, e_d) \) with \( e_1 = u_1 \) and
\[ e_i = \frac{\partial^{i-1}C(u_1, \ldots, u_i, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}} / \frac{\partial^{i-1}C(u_1, \ldots, u_{i-1}, 1, \ldots, 1)}{\partial u_1 \cdots \partial u_{i-1}}, \quad i = 2, \ldots, d. \]
Rosenblatt’s transforms for Archimedean copulas and meta-elliptic copulas are quite easy to compute for any dimension; see, e.g., Rémillard et al. (2010). The usefulness of Rosenblatt’s transform lies in the following properties (Rosenblatt, 1952): Suppose that \( V \sim C_\perp \), where \( C_\perp \) is the independence copula, which is equivalent to say that \( V \) is uniformly distributed over \((0, 1)^d\). Then \( R(U) \sim C_\perp \) if and only if \( U \sim C \). In addition, \( R^{-1}(V) \sim C \). Since \( U = R^{-1}(V) \) can be computed in a recursive way, this is particularly useful for simulation purposes.

It follows that the null hypothesis \( H_0 : C \in \mathcal{C} = \{C_\phi; \phi \in \mathcal{O}\} \) can be written in terms of Rosenblatt’s transforms viz.
\[ H_0 : R \in \{R_\phi; \phi \in \mathcal{O}\}. \]

Using an idea of Breymann et al. (2003), extending previous ideas of Durbin (1973) and Diebold et al. (1998), one can build tests of goodness-of-fit by comparing the
empirical distribution function of \( E_{i,n} = R_{\phi_n}(U_{i,n}), \ i = 1, \ldots, n, \) with \( C_\perp, \) since under \( H_0, \ E_{i,n} \) should have approximately distribution \( C_\perp. \) More precisely, set

\[
\mathbb{D}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ I(E_{i,n} \leq u) - C_\perp(u) \right\}, \quad u \in [0, 1]^d,
\]

and define

\[
S_{n}^{(B)} = \int_{[0,1]^d} \mathbb{D}_n^2(u) du = \frac{n}{3^d} - \frac{1}{2^{d-1}} \sum_{i=1}^{n} \prod_{k=1}^{d} (1 - E_{ki,n}^2) + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{d} (1 - E_{ki,n} \lor E_{kj,n}),
\]

where \( a \lor b = \max(a, b). \)

To define regular estimators in that setting, one needs to define

\[
\mathbb{B}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ I(E_{i} \leq u) - C_\perp(u) \right\}, \quad u \in [0, 1]^d.
\]

It is easy to check that check that \( (\mathbb{B}_n, \mathbb{W}_n) \rightsquigarrow (\mathbb{B}, \mathbb{W}), \) where the joint law is Gaussian, and \( \mathbb{B} \) is a \( C_\perp-\)Brownian bridge. Now, when using Rosenblatt’s transforms, one says that \( \phi_n \) is regular for \( \phi \) if \( (\mathbb{B}_n, \mathbb{W}_n, \Phi_n) \rightsquigarrow (\mathbb{B}, \mathbb{W}, \Phi) \) where the latter is centered Gaussian with \( E(\Phi \mathbb{W}^\top) = I. \) Again, all estimators described in Section 2.1 (under the additional assumptions of Chen and Fan (2006)) and Section 2.2 (under assumptions (A1)–(A6)) are all regular.

As in the case of copula processes studied in the previous section, in order to prove the following result, one must assume that \( R_{\phi} \) is Fréchet differentiable, i.e.,

\[
\lim_{h \to 0} \sup_{u \in [0,1]^d} \frac{|R_{\phi_n+h}(u) - R_{\phi_n}(u) - \hat{\mathcal{R}}(u) h|}{\| h \|} = 0.
\]

One also has to assume that \( \mathcal{R} \) is continuously differentiable with respect to \( u \in (0, 1). \) One can now state the main result of the section.

**Proposition 6.** Under assumptions (A1)–(A6), if \( \phi_n \) is regular for \( \phi \), then \( S_{n}^{(B)} \) converges in law to \( S_{n}^{(B)} = \int_{[0,1]^d} \mathbb{D}_n^2(u) du, \) where \( \mathbb{D} \) is a continuous centered Gaussian process depending only on \( C_\phi \) and \( \Phi. \)

In fact, if \( \psi \) is a continuous function on the space \( C([0,1]) \), then the statistic \( T_n = \psi(\mathbb{D}_n) \) converges in law to \( T = \psi(\mathbb{D}). \) Moreover, the parametric bootstrap algorithm described next in Section 4.1 can be used to estimate \( P \)-values of \( S_{n}^{(B)} \) or \( T_n. \)

The expression for \( \mathbb{D} \) is given in Theorem 2 of Appendix B.

**Remark 2.** Set \( \tilde{U}_{i,n} = R_i/(n+1), \) where \( R_1, \ldots, R_n \) are the associated rank vectors of \( U_1, \ldots, U_n, \) and let \( \tilde{E}_{i,n} = R_{\phi_n}(\tilde{U}_i), \) where \( \tilde{\phi}_n \) is the estimation of \( \phi \) calculated
with \( \mathbf{U}_{i,n} = \mathbf{R}_i/(n+1) \), \( i = 1, \ldots, n \). Then, it follows from Theorem 2 that \( \hat{D}_n \overset{D}{\rightarrow} D \), where
\[
\hat{D}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ I(\mathbf{E}_{i,n} \leq u) - C_{\perp}(u) \}, \quad u \in [0,1]^d.
\] (13)

4.1. A parametric bootstrap for \( S_{n}^{(B)} \). The following algorithm is described in terms of statistic \( S_{n}^{(B)} \) but can be applied easily to any statistic of the form \( T_n = \psi(D_n) \).

1. Estimate \( \phi \) by \( \phi_n = \mathcal{T}_n(\mathbf{U}_{1,n}, \ldots, \mathbf{U}_{n,n}) \), compute \( D_n \) and \( S_{n}^{(B)} \) according to formulas (10) and (11).
2. For some large integer \( N \), repeat the following steps for every \( k \in \{1, \ldots, N\} \):
   a. Generate a random sample \( \mathbf{Y}^{(k)}_{1,n}, \ldots, \mathbf{Y}^{(k)}_{n,n} \) from distribution \( C_{\phi_n} \) and compute the pseudo-observations \( \mathbf{U}_{i,n}^{(k)} = \mathbf{R}_{i,n}^{(k)}/(n+1) \), where \( \mathbf{R}^{(k)}_{1,n}, \ldots, \mathbf{R}^{(k)}_{n,n} \) are the associated rank vectors of \( \mathbf{Y}^{(k)}_{1,n}, \ldots, \mathbf{Y}^{(k)}_{n,n} \).
   b. Estimate \( \phi \) by \( \phi_n^{(k)} = \mathcal{T}_n(\mathbf{U}^{(k)}_{1,n}, \ldots, \mathbf{U}^{(k)}_{n,n}) \), and compute and compute \( \mathbf{E}_{i,n}^{(k)} = \mathcal{R}_{\phi_n^{(k)}}(\mathbf{U}^{(k)}_{i,n}) \), \( i \in \{1, \ldots, n\} \).
   c. Let
   \[
   D_{n}^{(k)}(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ I(\mathbf{E}_{i,n}^{(k)} \leq u) - C_{\perp}(u) \}, \quad u \in [0,1]^d
   \]
   and set
   \[
   S_{n,k} = \int_{[0,1]^d} \{ D_{n}^{(k)}(u) \}^2 \, du.
   \]
   An approximate \( P \)-value for the test is then given by \( \sum_{k=1}^{N} I\left(S_{n,k}^{(B)} > S_{n}^{(B)}\right)/N \).

5. Change-point problems

In this section, one describes non-parametric tests for detecting change-points. First, inspired by Ghoudi and Rémillard (2010), detection of change-point for univariate series is tackled. Next, one proposes a new test for change-point detection for the copula, provided there is no change-point in the marginal distributions.

5.1. Detection of change-point for univariate series. Detection of change-point for the univariate series \( \varepsilon_{ji} \) can be based on the process
\[
A_{j,n}(s, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \{ I(\varepsilon_{ji,n} \leq x_j) - F_{j,n}(1, x_j) \} = F_{j,n}(s, x_j) - \frac{[ns]}{n} F_{j,n}(1, x_j),
\]
for if \( F_{j,n}^{(s)} \) denotes the empirical distribution function of the \( \{ \varepsilon_{ji,n} \}_{i=1}^{[ns]} \) and \( \tilde{F}_{j,n}^{(1-s)} \) denotes the empirical distribution function of the \( \{ \varepsilon_{ji,n} \}_{i=[ns]+1}^{n} \), then under the null hypothesis that \( F_j \) is the distribution function of \( \varepsilon_{ji} \) for all \( i = 1, \ldots, n \), \( \varepsilon_{ji} \), i.e., there is no change-point, then
\[
\sqrt{n} \left\{ F_{j,n}^{(s)}(x_j) - \tilde{F}_{j,n}^{(1-s)}(x_j) \right\} = \frac{[ns]}{n} A_{j,n}(s, x_j).
\]
Under assumptions (A1)–(A6), \(A_{j,n} \sim A_{j}\), where
\[
A_{j}(s,x_j) = F_j(s,x_j) - sF_j(1,x_j) = \beta_j\{s,F_j(x_j)\} - s\beta_j\{1,F_j(x_j)\} = K_j\{s,F_j(x_j)\}.
\]
The latter shows that the limiting distribution of the statistics
\[
T^{(KS)}_{j,n} = \sup_{s \in [0,1]} \sup_{x_j \in \mathbb{R}} |A_{j,n}(s,x_j)| \quad \text{and} \quad T^{(CVM)}_{j,n} = \int_0^1 \int_0^1 \{K_j(s,u)\}^2 dsdF_{j,n}(x_j)
\]
are distribution free, converging respectively to
\[
T^{(KS)}_j = \sup_{s \in [0,1]} |K_j(s,u)| \quad \text{and} \quad T^{(CVM)}_j = \int_0^1 \int_0^1 \{K_j(s,u)\}^2 dsdu.
\]
That result extends the one obtained in Ghoudi and Rémillard (2010) for residuals of ARMA processes. Note that \(K_j\) is a continuous centered Gaussian process with covariance given by
\[
\text{Cov}\{K_j(s,u),K_j(t,v)\} = \{\min(s,t) - st\} \{\min(u,v) - uv\}.
\]
As remarked in Ghoudi and Rémillard (2010), that process appears as the limit of many other processes used in tests of change-point (Picard, 1985, Carlstein, 1988) and tests of independence (Blum et al., 1961, Ghoudi et al., 2001). Furthermore, tables for the limiting distribution of \(T^{(KS)}_{j,n}\) and \(T^{(CVM)}_{j,n}\) are given in Ghoudi et al. (2001) Table IV page 206 and Table I page 204 respectively. In case the sample size considered is not available in these tables, it is suggested in Ghoudi and Rémillard (2010) to use the simulations since \(K_j\) also appear as the limit of \(\hat{K}_{j,n}(s,u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lceil ns \rceil} \left\{1 \left( \frac{R_i}{n} \leq u \right) - \frac{\lfloor nu \rfloor}{n} \right\}\), \(s,u \in [0,1]\), where \(R_i\) is the rank of \(U_i\), amongst the i.i.d. uniform variables \(U_1, \ldots, U_n\).

5.2. Detection of change-point for copulas. Suppose now that the null hypotheses that there is no change-point in the marginal distributions are all accepted.

Next, if one is interested in possible change-points in the dependence structure, one could do something similar to the previous section. That is the methodology proposed next. Previous work on structural change for copulas include the parametric change-point approach of Dias and Embrechts (2004, 2009), a filtering/non-parametric methodology proposed by Harvey (2010) and parametric/kernel-based approach proposed by Guégan and Zhang (2010). In the latter, to perform the test, the authors have to select a family for their so-called “static” copula. Their test is based on kernel estimates. Here, in contrast, one starts by performing a non-parametric change-point test. If the null hypothesis is accepted, then one may try to select a static copula. Furthermore, Guégan and Zhang (2010) work with residuals, without ever proving that the methodology is valid. In Harvey (2010), no residuals are used. The methodology is based on time-varying quantiles and some kind of filtering technique. It would be interesting to compare the approach proposed here to the one proposed in Harvey (2010).

Let’s now describe the proposed methodology, which is closely related to the test of equality between two copulas proposed by Rémillard and Scaillet (2009). First, it is easy to check that if \(C^{(s)}_n\) denotes the empirical distribution function of the first \([ns]\) pseudo-observations \(U_{i,n}\) and \(\bar{C}^{(1-s)}_n\) denotes the empirical distribution function
of the remaining \( n - \lfloor ns \rfloor \) pseudo-observations, then under the null hypothesis that there is no change-point of the dependence structure, one has

\[
\sqrt{n} \{ C_n(s) - \check{C}_n^{(1-s)}(u) \} = \frac{n^2}{\lfloor ns \rfloor (n - \lfloor ns \rfloor)} \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{ 1(U_{i,n} \leq u) - C_n(u) \}.
\]

Setting

\[
C_n(s, u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{ 1(U_{i,n} \leq u) - C(u) \},
\]

one gets that

\[
G_n(s, u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{ 1(U_{i,n} \leq u) - C_n(u) \} = C_n(s, u) - \frac{\lfloor ns \rfloor}{n} C_n(1, u).
\]

Therefore change-point tests can be based on \( G_n \). For example, one could define

\[
T_n = \max_{1 \leq k \leq n} \max_{1 \leq i \leq n} | G_n(k/n, U_{i,n}) |
\]

and reject the null hypothesis for large values of \( T_n \). The limiting distribution of \( T_n \) and \( G_n \) is given next.

**Proposition 7.** Under Assumptions (A1)–(A6), \( C_n \sim C \), \( G_n \sim G \) and \( T_n \sim T \), where

\[
C(s, u) = \alpha(s, u) - s \sum_{j=1}^d \partial_{u_j} C(u) \beta_j(1, u_j),
\]

\[
G(s, u) = C(s, u) - s C(1, u) = \alpha(s, u) - s \alpha(1, u),
\]

and \( T = \sup_{s \in [0,1], u \in [0,1]^d} | G(s, u) |. \)

Even if the law of \( G \) depends on the unknown copula \( C \), it is easy to simulate independent copies, using a multiplier method adapted from Scaillet (2005) and Rémillard and Scaillet (2009). That technique is described next.

5.2.1. **Multipliers method for \( T_n \).** The following procedure leads to an approximate \( P \)-value for the test based on \( T_n \). The adaptations required for any other function of \( G_n \) are obvious.

1. Compute \( T_n \) as defined in (14).

2. For some large integer \( N \), repeat the following steps for every \( k \in \{1, \ldots, N\} \):
   (a) Generate a random sample \( \xi_{i,k} \sim N(0,1), i = 1, \ldots, n \).
   (b) For \((s, u) \in [0,1]^{d+1}\), set \( \alpha_n^{(k)}(s, u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_{i,k} \{ 1(U_{i,n} \leq u) - C_n(u) \} \) and \( \check{G}_n^{(k)}(s, u) = \alpha_n^{(k)}(s, u) - \frac{\lfloor ns \rfloor}{n} \alpha_n^{(k)}(1, u) \).
   (c) Evaluate \( T_n^{(k)} = \max_{1 \leq j \leq n} \max_{1 \leq i \leq n} | \check{G}_n^{(k)}(j/n, U_{i,n}) | \).

An approximate \( P \)-value for the test is then given by \( \sum_{k=1}^{N} \mathbf{1} \left( S_n^{(k)} > S_n \right) / N \).
Remark 3. Using Theorem 1, a non-parametric change-point for the innovations $\varepsilon_i$ can be based on $K_n(s,x) - sK_n(1,x) = \sqrt{n} \sum_{i=1}^{\lceil ns \rceil} \{1(e_{i,n} \leq x) - K_n(x)\}$. Because of the form of the limiting distribution, one has to use multipliers technique to generate asymptotically independent copies. See Rémillard (2010) for details.

6. Example

In order to be able to make comparisons with Chen and Fan (2006), one of their data set is used, namely the Deutsche Mark/US and Japanese Yen/US exchanges rates, from April 28, 1988 to Dec 31, 1998. AR(3)-GARCH(1,1) and AR(1)-GARCH(1,1) models were fitted on the 2684 log-returns.

For such a large sample size, one must be sure that there is no structural change-point. To that end, univariate change-point were performed first on the standardized residuals and the null hypothesis was accepted each time. Then, the copula change-point test was performed, leading once again to the acceptance of the null hypothesis, since the p-value was estimated to be 33%, using $N = 100$ replications.

Next, the usual standard copula models (Gaussian, Student, Clayton, Frank, Gumbel) were checked for goodness-of-fit. In each case the null hypothesis was rejected since the p-value was estimated to be 0 (using N=100 replications). That shows the limitations of the model selection methodology proposed by Chen and Fan (2006). It can only be used to rank models, even if none is adequate, which is the case here.

Having rejected the standard copula models, one tried to fit a mixture of two Gaussian copulas. Similar models were proposed by Dias and Embrechts (2004), Chen and Fan (2006) and Patton (2006). The null hypothesis is accepted with a 84% p-value ($S_n^{(B)} = 0.0183$), calculated from $N = 100$ replications. The parameters of the two Gaussian copulas are $\hat{\rho}_1 = 0.8205$, $\hat{\rho}_2 = 0.3749$, and $\hat{\pi}_1 = 0.4017$, $\hat{\pi}_2 = 0.5983$.

7. Conclusion

The asymptotic behaviour of the empirical copula constructed from residuals of stochastic volatility models was studied. It was shown that if the stochastic volatility matrix is diagonal, then the empirical copula process behaves like if the parameters were known. That remarkable property makes it possible to construct consistent tests of goodness-of-fit for the copula of innovations. Tests of structural change in the dependence structure were also proposed.

Appendix A. Smoothness conditions for parametric bootstrap

Following Genest and Rémillard (2008), assume that the family of densities $c_\phi$ satisfies

(B1) The density $c_\phi$ of $C_\phi$ admits first and second order derivatives with respect to all components of $\phi$. The gradient (row) vector with respect to $\phi$ is denoted $\hat{c}_\phi$, and the Hessian matrix is represented by $\hat{\hat{c}}_\phi$.

(B2) For arbitrary $u \in (0,1)^d$ and every $\phi_0 \in \mathcal{O}$, the mappings $\phi \mapsto \hat{c}_\phi(u)/c_\phi(u)$ and $\phi \mapsto \hat{\hat{c}}_\phi(u)/c_\phi(u)$ are continuous at $\phi_0$. 
measurable, and such that for any $k$,
the partial-sum empirical process
Further set
\[ \Theta_n = n^{1/2} (\theta_n - \theta). \]
The goal is to study the asymptotic behaviour of
the partial-sum empirical process $K_n$ defined by \( \theta \).
The following assumptions are needed in order to prove
the convergence of $K_n$.

Let $d_{i,n} = e_i - e_{i,n} - (\gamma_{0i} \Theta_n + \sum_{k=1}^d \gamma_{ki} \gamma_{1ki} \Theta_n) / \sqrt{n}$, where $\gamma_{0i}$ and $\gamma_{1i}$ are $\mathcal{F}_{i-1}$-measurable, and such that for any $j = 1, \ldots, d$, and any $x \in \mathbb{R}^d$,
\[ (A1) \quad \Gamma_{0,n}(s) = \frac{1}{n} \sum_{i=1}^{[ns]} \gamma_{0i} \xrightarrow{p} s \Gamma_0, \quad \Gamma_{1k,n}(s) = \frac{1}{n} \sum_{i=1}^{[ns]} \gamma_{1ki} \xrightarrow{p} s \Gamma_{1k}, \text{ uniformly in} \]
\[ s \in [0, 1], \text{ where } \Gamma_0 \text{ and } \Gamma_{1k} \text{ are deterministic, } k = 1, \ldots, d. \]

(A2) \[ \frac{1}{n} \sum_{i=1}^n E \left( \| \gamma_{0i} \|^k \right) \text{ and } \frac{1}{n} \sum_{i=1}^n E \left( \| \gamma_{1ji} \|^k \right) \text{ are bounded, for } k = 1, 2. \]

(A3) There exists a sequence of positive terms $r_i > 0$ so that $\sum_{i \geq 1} r_i < \infty$ and such that the sequence $\max_{1 \leq i \leq n} \| d_{i,n} \| / r_i$ is tight.

(A4) $\max_{1 \leq i \leq n} \| \gamma_{0i} \| / \sqrt{n} = o_P(1)$ and $\max_{1 \leq i \leq n} \| \epsilon_{ji} \| \gamma_{1ji} \| / \sqrt{n} = o_P(1)$.

(A5) $(\alpha_n, \Theta_n) \rightsquigarrow (\alpha, \Theta)$ in $\mathcal{D}([0, \infty]^d) \times \mathbb{R}^p$.

(A6) $\partial_x K(x)$ and $x_j \partial_{x_j} K(x)$ are bounded and continuous on $\mathbb{R}^d = [-\infty, +\infty]^d$. 

(A7) Suppose that for all \( k \neq j \), \( f_j(x_j)E\{|\varepsilon_{k1}\mathbf{1}(\varepsilon_1 \leq x)|\varepsilon_{j1} = x_j\} \) and \( x_j f_j(x_j)E\{|\varepsilon_{k1}\mathbf{1}(\varepsilon_1 \leq x)|\varepsilon_{j1} = x_j\} \) are bounded and continuous on \( \mathbb{R}^d \).

**Remark 4.** Note that (A1) and (A2) are trivially satisfied if the sequences \( \gamma_{0k} \) and \( \gamma_{1k} \) are stationary, ergodic and square integrable. Also, if \( \frac{1}{n} \sum_{i=1}^{n} \gamma_{0k} \xrightarrow{Pr} \Gamma_0 \), \( \frac{1}{n} \sum_{i=1}^{n} \gamma_{1k} \xrightarrow{Pr} \Gamma_{1k} \), and \( \frac{1}{n} \sum_{i=1}^{n} E(\|\gamma_{0k}\|^2) \) converge, then (A1) and (A2) are satisfied.

Set \( \tilde{K}_n(s, x) = \alpha_n\{s, F(x)\} \) and \( \tilde{K}(s, x) = \alpha\{s, F(x)\} \). One can now prove the main theorem.

**Theorem 1.** Under assumptions (A1)–(A7), \( \mathbb{K}_n \xrightarrow{\mathcal{L}} \mathbb{K} \), with

\[
\mathbb{K}(s, x) = \mathbb{K}(s, x) + s \nabla K(x) \Gamma_0 \Theta + s \sum_{j=1}^{d} \sum_{k=1}^{d} G_{jk}(x)(\Gamma_{1k} \Theta)_j,
\]

where \( G_{jk}(x) = f_j(x_j)E\{\varepsilon_{k1}\mathbf{1}(\varepsilon_1 \leq x)|\varepsilon_{j1} = x_j\} \). In particular, \( G_{jj}(x) = x_j \partial_{x_j} K(x) \).

Furthermore, for all \( j = 1, \ldots, d \), \( \mathbb{F}_{j,n} \xrightarrow{\mathcal{L}} \mathbb{F}_j \), where

\[
\mathbb{F}_j(s, x_j) = \beta_j\{s, F_j(x_j)\} + s f_j(x_j)\{(\Gamma_0 \Theta)_j + x_j(\Gamma_{1j} \Theta)_j\} + s \sum_{k \neq j} f_j(x_j)E|\varepsilon_{k1}|\varepsilon_{j1} = x_j(\Gamma_{1k} \Theta)_j.
\]

If \( \sigma \) is diagonal, (A7) is not needed for the convergence of \( \mathbb{K}_n \). In that case,

\[
\mathbb{K}(s, x) = \mathbb{K}(s, x) + s \nabla K(x) \Gamma_0 \Theta + s \sum_{j=1}^{d} G_{jj}(x)(\Gamma_{1j} \Theta)_j.
\]

**Remark 5.** \( (\Gamma_{1k} \theta)_j = 0 \) for all \( \theta \) and all \( j \neq k \) if and only if \( (\Gamma_{1k} \theta)_j = 0 \) for all \( l \) and all \( j \neq k \). That can occur for example if

\[
\{\sigma_l(\theta)\}_{jk} = \{\sigma_l(\theta)\}_{kk}(A_l)_{jk}, \text{ with } (A_l)_{jj} = 1.
\]

In that case \( A_l \) must be known since it is parameter free. This is true in particular if \( \sigma_i \) is diagonal, in which case \( A_l \) is the identity matrix.

It then follows from (15) that

\[
\varepsilon_{ji} - e_{ji,n} = \frac{1}{\sigma_{jji}(\theta_n)} \sum_{k=1}^{d} (A_i^{-1})_{jk} \{\mu_{ki}(\theta_n) - \mu_{ki}(\theta)\} + \left\{ \frac{\sigma_{jji}(\theta_n) - \sigma_{jji}(\theta)}{\sigma_{jji}(\theta_n)} \right\} \varepsilon_{ji}
\]

Setting \( H_i \) to be the diagonal matrix with \( (H_i)_{jj} = (\sigma_i)_{jj}, j = 1, \ldots, d \), then one can rewrite the model as

\[
X_i = \mu_i + A_i H_i \varepsilon_i,
\]

so \( Y_i = A_i^{-1}X_i = A_i^{-1} \mu_i + H_i \varepsilon_i \). Since \( A_i \) is known, this model is a simple rescaling of a model with diagonal volatility, and as such has little interest. So if the model cannot be transformed into a diagonal one, the limiting empirical copula process is not parameter free.
Corollary 1. Under assumptions (A1)–(A6), if the volatility matrix is diagonal then \( C = \beta K \) for any \( \beta \).

where \( \beta_j(s, u_j) = \alpha(s, \ldots, 1, u_j, 1, \ldots, 1) \). If the volatility matrix is not diagonal, then

\[
C(s, u) = \hat{C}(s, u) + s \sum_{j \neq k} \tilde{G}_{jk}(u)(\Gamma_{1k}\Theta)_j,
\]

where \( \tilde{G}_{jk}(u) = E \{ \varepsilon_{k1} 1(U_1 \leq u) | U_{j1} = u_j \} \), where \( U_{j1} = F_j(\varepsilon_{j1}) \).

Corollary 1 follows directly from Theorem 1, using Genest et al. (2007)[Proposition A.1].

Proof. Set \( S_d = \{1, \ldots, d\} \). Further set

\[
\mu_{j,n}(s, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \left\{ 1(e_{ji,n} \leq x_j) - 1(\varepsilon_{ji} \leq x_j) \right\} \prod_{k \neq j} 1(\varepsilon_{ki} \leq x_k)
\]

and

\[
\mu_{A,n}(s, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \prod_{j \in A} \left\{ 1(e_{ji,n} \leq x_j) - 1(\varepsilon_{ji} \leq x_j) \right\} \prod_{k \in A^c} 1(\varepsilon_{ki} \leq x_k),
\]

for any \( A \subset S_d \). Using the multinomial formula, one has

\[
\mathbb{K}_n(s, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \left[ \sum_{A \subset S_d, j \in A} \prod_{j \in A} \left\{ 1(e_{ji,n} \leq x_j) - 1(\varepsilon_{ji} \leq x_j) \right\} \prod_{j \notin A} 1(\varepsilon_{ji,n} \leq x_j) - K(x) \right]
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \left\{ \prod_{j=1}^{d} 1(\varepsilon_{ji} \leq x_j) - K(x) \right\}
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \sum_{j=1}^{d} 1(e_{ji,n} \leq x_j) \prod_{k \neq j} 1(\varepsilon_{ki} \leq x_k)
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \sum_{|A| > 1} \prod_{j \in A} 1(e_{ji,n} \leq x_j) - 1(\varepsilon_{ji} \leq x_j) \prod_{k \in A^c} 1(\varepsilon_{ki} \leq x_k)
\]

\[
= \mathbb{K}_n(s, x) + \sum_{j=1}^{d} \mu_{j,n}(s, x) + \sum_{|A| > 1} \mu_{A,n}(s, x).
\]

To prove the theorem, it suffices to show that for any \( 1 \leq j \leq d \), uniformly in \( (s, x) \), \( \mu_{j,n}(s, x) \) converges in probability to \( s \partial_{x_j} K(x)(\Gamma_0\Theta)_j + s \sum_{k=1}^{d} G_{jk}(x)(\Gamma_{1k}\Theta)_j \), and that for any \( |A| > 1 \), \( \mu_{A,n}(s, x) \) converges in probability to zero. These proofs
will be done for $j = 1$ and $A \supset \{1, 2\}$, the other cases being similar. Also suppose that $\sigma$ is diagonal. The general case is similar.

For simplicity, set $\gamma_{0j} = \gamma_{0i}$, for all $j = 1, \ldots, d$.

Let $\delta \in (0, 1)$ be given. From (A2), (A3) and (A5), one can find $M > 0$ such that if $n$ is large enough, then $P(B_{M, n}) > 1 - \delta$, where

$$B_{M, n} = \left\{ \|\Theta_n\| \leq M \right\} \cap \{ d_{j_i, n} \leq M r_i \} \cap \{ \frac{1}{n} \sum_{i=1}^{n} \|\gamma_{kji}\| \leq M \}.$$ 

Because the closed ball of radius $M$ is compact, it can be covered by finitely many balls of radius $\lambda \in (0, 1)$.

Further set $C_{\lambda, n} = \{ \max_{1 \leq i \leq n}(\|\gamma_{0i}\| + |\varepsilon_i|\|\gamma_{1i}\|)/\sqrt{n} \leq \lambda \}$. By (A4), $P(C_{\lambda}) \geq 1 - \delta$ if $n$ is large enough. On $B_{M, n} \cap C_{\lambda, n} \cap \{ \|\Theta_n - \zeta\| < \lambda \}$, one has

$$\varepsilon_{1i} - e_{1i, n} \leq M r_i + \frac{(\gamma_{1i}\zeta_1 + \lambda\|\gamma_{0i}\| + \lambda|\varepsilon_i|\|\gamma_{0i}\|)}{\sqrt{n}},$$

$$\geq -M r_i + \frac{(\gamma_{1i}\zeta_1 - \lambda\|\gamma_{0i}\| - \lambda|\varepsilon_i|\|\gamma_{0i}\|)}{\sqrt{n}}.$$ 

Set $y_{i, n} = M r_i + ((\gamma_{0i}\zeta_1 + \lambda\|\gamma_{0i}\|)/\sqrt{n}, z_{i, n} = (\gamma_{1i}\zeta_1)/\sqrt{n}$ and $w_{i, n} = \lambda\|\gamma_{1i}\|/\sqrt{n}$. Further set $a_{i, n} = M r_i + c\|\gamma_{0i}\|/\sqrt{n} + c|\varepsilon_i|\|\gamma_{1i}\|/\sqrt{n}$, where $c = 1 + \|\zeta\|$. It follows that

$$1(e_{1i, n} \leq x_1) \leq 1(\varepsilon_{1i} \leq x_1 + y_{i, n} + \varepsilon_i z_{i, n} + |\varepsilon_i| w_{i, n})$$

$$\leq 1(\varepsilon_{1i} \leq x_1 + a_{i, n}),$$

and

$$\left|1(e_{1i, n} \leq x_1) - 1(\varepsilon_{1i} \leq x_1)\right| \leq 1(x_1 - a_{i, n} < \varepsilon_{1i} \leq x_1 + a_{i, n}),$$

if $i \geq i_1$, for some $i_1 \geq i_0$, since $r_i \to 0$.

As a result, for any $A \supset \{1, 2\}$,

$$|\mu_{A, n}(s, x)| \leq \frac{i_1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} 1(x_1 - a_{i, n} < \varepsilon_{1i} \leq x_1 + a_{i, n}) 1(x_2 - 2c\lambda < \varepsilon_{2i} \leq x_2 + 2c\lambda).$$

Set $\kappa = (\kappa_1, \kappa_2, \kappa_3)$, $\eta_{k, n}(\kappa) = \kappa r_i + \{(\gamma_{0i}\kappa_2) + \varepsilon_{1i}(\gamma_{1i}\kappa_2) + \kappa_3\|\gamma_{0i}\| + \kappa_3|\varepsilon_i|\|\gamma_{1i}\| / \sqrt{n},$ and set

$$\mu_{1, n}(s, x; \kappa) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ 1(\varepsilon_{1i} \leq x_1 + \eta_{i, n}(\kappa)\right\} - 1(\varepsilon_{1i} \leq x_1) \right\} \prod_{k=2}^{d} 1(\varepsilon_{ki} \leq x_k).$$

Further set $\mu_{12, n}(s, x_1, x_2; \kappa) = \mu_{1, n}(s, x_1, x_2, \infty, \ldots, \infty; \kappa)$.

Then

$$\mu_{1, n}(s, x; -M, \theta, -\lambda) \leq \mu_{1, n}(s, x) \leq \mu_{1, n}(s, x; M, \theta, \lambda).$$
and

\[ |μ_A,n(s, x)| \leq \frac{i_1}{\sqrt{n}} + \tilde{μ}_{12,n}(s, x_1, x_2 + 2cλ; M, 0, cλ) \]

\[ -\tilde{μ}_{12,n}(s, x_1, x_2 - 2cλ; M, 0, cλ) \]

\[ -\tilde{μ}_{12,n}(s, x_1, x_2 + 2cλ; -M, 0, -cλ) \]

\[ +\tilde{μ}_{12,n}(s, x_1, x_2 - 2cλ; -M, 0, -cλ). \]

Next, note that \( P(ε_i \leq x_1 + η_i,n(κ), ε_2i \leq z_1, \ldots, ε_di \leq z_{d-1}|F_{i-1}) \) is given by

\[ K\left\{ \frac{(x_1 + κ_1r_i + (γ_0iρκ_2)1/√n + κ_3\|γ_0iρ\|/√n)^+}{1 - (γ_1iκ_2)1/√n - κ_3\|γ_1iκ\|/√n}, z \right\} - K(0, z) \]

\[ +K\left\{ -\frac{(x_1 + κ_1r_i + (γ_0iρκ_2)1/√n + κ_3\|γ_0iρ\|/√n)^-}{1 - (γ_1iκ_2)1/√n + κ_3\|γ_1iκ\|/√n}, z \right\}. \]

Next, set \( Γ_1(x) = \partial_{x_1}K(x)(Γ_{01} + x_1Γ_{11}) \) and define

\[ \tilde{Γ}_{1,n}(s, x; κ) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} [P(ε_i \leq x_1 + η_i,n(κ), ε_2i \leq x_2, \ldots, ε_di \leq x_d|F_{i-1}) - K(x)] \]

It follows from (A1), (A2) and (A6) that on \( B_{M,n}, \)

\[ \sup_{x \in \mathbb{R}^d} \left| \tilde{Γ}_{1,n}(s, x; κ) - sk_2Γ_1(x) \right| \]

can be made arbitrarily small with large probability. The final step is to show that

\( \tilde{μ}_{1,n}(s, x; κ) = μ_{1,n}(s, x; κ) - \tilde{Γ}_{1,n}(s, x) \) can be made arbitrarily small by choosing \( κ_3 \) small enough. The proof is similar to the proof of Lemmas 7.1-7.2 in Ghoudi and Remillard (2004). Suppose \( 1/2 < ν < 1 \) and set \( N_n = [n^ν] \). Then, set \( y_k = F_i^{-1}(k/N_n), 1 \leq k < N_n \). Further set \( y_0 = -∞ \) and \( y_{N_n} = +∞ \). Now, if \( y_k \leq x_1 < y_{k+1} \), and \( z = (x_2, \ldots, x_d) \). First, note that one can cover \( \mathbb{R}^d \) by a finite number \( N_n \times J \) of intervals of the form \([a, b] = [y_k, y_{k+1}) \times [u_l, v_l], for which \( 0 \leq K(y_{k+1}, z) - K(y_k, z) \leq F_1(y_{k+1}) - F_1(y_k) = 1/N_n \). Set

\[ U_{i,n}(x) = 1(\{ε_i ≤ x_1 + η_i,n(κ)\} - 1(ε_i ≤ x_1) \prod_{j=2}^d 1(ε_j ≤ x_j), \]

and set \( V_{i,n}(x) = E\{U_{i,n}(x)|F_{i-1}\}. One cannot directly with \( U_{i,n} - V_{i,n} \). Better bounds are obtained by decomposing \( U_{i,n} \) and \( V_{i,n} \) as follows: Set

\[ U_{i,n}^+(x) = \left[ 1(\{ε_i ≤ x_1 + η_i,n^+(κ)\}) - 1(ε_i ≤ x_1) \right] \prod_{j=2}^d 1(ε_j ≤ x_j), \]

and

\[ U_{i,n}^-(x) = \left[ 1(ε_i ≤ x_1) - 1(ε_i ≤ x_1 + η_i,n^-(κ)) \right] \prod_{j=2}^d 1(ε_j ≤ x_j). \]
Similarly, set $V_{i,n}^+(x) = E\{U_{i,n}^+(x) | \mathcal{F}_{i-1}\}$ and define

$$\tilde{\mu}_{1,n}^+(s, x; \kappa) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \{U_{i,n}^+(x) - V_{i,n}^+(x)\}.$$ 

Then $U_{i,n} - V_{i,n} = U_{i,n}^+ - V_{i,n}^+ = \{U_{i,n}^- - V_{i,n}^+\}$, so $\tilde{\mu}_{1,n} = \tilde{\mu}_{1,n}^+ - \tilde{\mu}_{1,n}^-$. To complete the proof, it is enough to show that $\tilde{\mu}_{1,n}$ can be made arbitrarily small. Only the proof for the + part is given, the other one being similar.

Now, for $x \in [y_k, y_{k+1}) \times [u_l, u_{l+1})$, observe that

$$U_{i,n}^+(x) \leq U_{i,n}^+(y_{k+1}, v_l) + 1(y_k < \varepsilon \leq y_{k+1})$$

and

$$U_{i,n}^+(x) \geq U_{i,n}^+(y_k, u_l) - 1(y_k < \varepsilon \leq y_{k+1}).$$

Taking expectations over the last two inequalities and summing over $i$ yield the following bound:

$$\sup_{s \in [0,1]} \sup_{x \in [y_k, y_{k+1}) \times [u_l, u_{l+1})} |\tilde{\mu}_{1,n}^+(s, x; \kappa)|$$

$$\leq \sup_{s \in [0,1]} \max_{x \in [y_k, y_{k+1}) \times [u_l, u_{l+1})} \{ |\tilde{\mu}_{1,n}^+(s, y_{k+1}, v_l; \kappa)|, |\tilde{\mu}_{1,n}^+(s, y_k, u_l; \kappa)| \} + 2\frac{\sqrt{n}}{N_n}$$

$$+ \sup_{s \in [0,1]} |\beta_{1,n}(s, y_{k+1}) - \beta_{1,n}(s, y_k)| + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{V_{i,n}^+(y_{k+1}, v_l) - V_{i,n}^+(y_k, u_l)\}.$$ 

Next $\xi_{i,n} = U_{i,n}^+ - V_{i,n}^+$ is a martingale difference such that $|\xi_{i,n}| \leq 2$ and $E(\xi_{i,n}^2 | \mathcal{F}_{i-1}) = V_{i,n}^+(1 - V_{i,n}^+)$, so $V_{i,n}^+$ is a martingale. As a result, from the maximum inequality for martingales, one gets,

$$P \left\{ \sup_{s \in [0,1]} \max_{x \in [y_k, y_{k+1}) \times [u_l, u_{l+1})} |\tilde{\mu}_{1,n}^+(y_k, u_l; \kappa)| > \lambda_0 \right\} \leq (N_n \times j) \lambda_0^{-4} \sup_{x \in \mathbb{R}^d} E \left[ \{\tilde{\mu}_{1,n}^+(1, x; \kappa)\}^4 \right]$$

which is bounded by

$$c(N_n \times j) \lambda_0^{-4} \sup_{x \in \mathbb{R}^d} \left[ \frac{16}{n} + \frac{1}{n^2} E \left\{ \left( \sum_{i=1}^{n} V_{i,n}^+(x) \right)^2 \right\} \right],$$

for some universal constant $c$. Using (A3) and (A6), the latter is $O(N_n/n)$, proving that $\sup_{s \in [0,1]} \max_{1 \leq k \leq N_n} \max_{1 \leq l \leq j} |\tilde{\mu}_{1,n}^+(y_k, u_l; \kappa)|$ converges in probability to zero. Similarly, $\sup_{s \in [0,1]} \max_{1 \leq k \leq N_n} \max_{1 \leq l \leq j} |\tilde{\mu}_{1,n}^+(y_k, v_l; \kappa)|$ also converges in probability to zero. Next,

$$\sup_{s \in [0,1]} \max_{1 \leq k \leq N_n} |\beta_{1,n}(s, y_{k+1}) - \beta_{1,n}(s, y_k)| = \sup_{s \in [0,1]} \left| \tilde{\beta}_{1,n}(s, (k + 1)/N_n) - \tilde{\beta}_{1,n}(s, k/N_n) \right|$$

converges in probability to zero, where $\tilde{\beta}_{1,n}$ is the empirical Kiefer process constructed from uniform variables.
Finally, from (A1) and (A2), one may conclude that 
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ V_{i,n}^{+}(y_{k+1}, v_{i}) - V_{i,n}^{+}(y_{k}, u_{i}) \}
\]
is bounded, for some constants \( c_1, \ldots, c_4 \) depending on \( \|f_1\|_{\infty} \) and \( \|g_1\|_{\infty} \), by
\[
c_1|\kappa_1| \frac{\sum_{i=1}^{n} r_i}{n} + c_2 \max_{0 \leq k < N_n} \max_{1 \leq l \leq J} |f_1(y_{k+1}, v_{i}) - f_1(y_{k}, u_{i})|
+c_3 \max_{0 \leq k < N_n} \max_{1 \leq l \leq J} |g_1(y_{k+1}, v_{i}) - g_1(y_{k}, u_{i})| + c_4|\kappa_3|.
\]
That can be made as small as necessary, provided \( n \) is large, \( \kappa_3 \) is small and the mesh of the covering is small enough.

\[\Box\]

The following theorem gives the result of the convergence of the empirical process based on Rosenblatt’s transformation.

Set \( \hat{U}_{i,n} = R_i/(n + 1) \), where \( R_1, \ldots, R_n \) are the associated rank vectors of \( U_1, \ldots, U_n \), and let \( \hat{E}_{i,n} = R_{\hat{\phi}_n}(\hat{U}_i) \), where \( \hat{\phi}_n \) is the estimation of \( \phi \) calculated with \( \hat{U}_{i,n} = R_i/(n + 1) \), \( i = 1, \ldots, n \), and define
\[
\tilde{D}_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{1(\hat{E}_{i,n} \leq u) - C_\perp(u)\}, \quad u \in [0, 1]^d. \quad (16)
\]

**Theorem 2.** Under Assumptions (A1)–(A6), if the volatility matrix is diagonal and if \( (\phi_n) \) is regular for \( \phi \), then \( D_n \rightarrow \tilde{D}_n \) and \( \tilde{D}_n \rightarrow \tilde{D} \), where
\[
D(u) = B(u) - \kappa(u) - \gamma(u)^{\top} \Phi,
\]
where \( B \) is a \( C_\perp \)-Brownian bridge, \( E\{B(u)W\} = \gamma(u) \), \( E\{\kappa(u)\W\} = 0 \), and
\[
\kappa(u) = \sum_{j=1}^{d} \sum_{k=1}^{j} E \left\{ (1(\hat{E} \leq u) \beta_j(1, \hat{U}_k) \partial \theta_k \mathcal{R}^{(j)}(\hat{U}) | \hat{E} = u_{j}) \right\},
\]
where \( \hat{U} \sim C = C_\phi \) and \( \hat{E} = \mathcal{R}(\hat{U}) \), with \( \hat{U} \) independent of all other observations.

**Proof.** First, note that \( B_n(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{1\{\mathcal{R}(U_i) \leq u\} - C_\perp(u)\} \sim B \), where \( B \) is a \( C_\perp \)-Brownian bridge. Next, set \( \hat{H}_n(x) = \mathcal{R}_{\hat{\phi}_n} \{ \hat{F}_n(x) \} \) and \( H(x) = \mathcal{R}\{F(x)\} \), where \( \phi_n = \mathcal{T}_n(U_{1,n}, \ldots, U_{n,n}) \) and \( \hat{\phi}_n = \mathcal{T}_n(\hat{F}_n(U_1), \ldots, \hat{F}_n(U_n)) \). Then \( \hat{V}_{i,n} = \hat{H}_n(\epsilon_i) \) and \( \hat{V}_i = H(\epsilon_i) \), for all \( i = 1, \ldots, n \). Since \( \sqrt{n}(\hat{\phi}_n - \phi) \sim \Phi \), using the results in Sections 2.1–2.2, and since \( \hat{H}_n = \sqrt{n}(\hat{H}_n - H) \sim \mathcal{H}_n \), where, for any \( j = 1, \ldots, d \),
\[
\mathcal{H}^{(j)}(x) = \Phi^\top \mathcal{R}^{(j)}\{F(x)\} + \sum_{k=1}^{j} \partial \theta_k \mathcal{R}^{(j)}\{F(x)\} \beta_k \{1, F_k(u_k)\},
\]
it follows from the results in Ghoudi and Rémillard (2004) that \( \tilde{D}_n \sim \tilde{D} \), where
\[
\tilde{D}(u) = B(u) - \kappa(u) - \Phi^\top \left\{ \sum_{j=1}^{d} \gamma_j(u) \right\}, \quad u \in [0, 1]^d.
\]
with
\[ \gamma_j(u) = E \left\{ \bar{R}^{(j)}(\bar{U})1(\bar{E} \leq u) | \bar{E}_j = u_j \right\}, \quad j = 1, \ldots, d. \quad (17) \]

Next, it follows from Rémillard (2010)[Lemma 1] that \( \gamma = \sum_{j=1}^{d} \gamma_j \), so \( \bar{D} = \mathcal{B} - \kappa - \Phi^T \gamma \). Hence \( E \{ \mathcal{B}(u) \mathcal{W} \} = \gamma(u) \), as claimed. Next, since we already know that for any \( j = 1, \ldots, d, E \{ \beta_j(1, u_j) \mathcal{W} \} = 0 \), proving that \( E \{ \kappa(u) \mathcal{W} \} = 0 \), for all \( u \in [0, 1]^d \).

As a result,
\[ E \{ \bar{D}(u) \mathcal{W} \} = \gamma(u) - E (\Phi^T \mathcal{W} \gamma) = 0, \]

for all \( u \in [0, 1]^d \), since any \( \phi_n \) in Sections 2.1–2.2 is a regular estimator of \( \phi \), so \( E (\Phi^T \mathcal{W}) = I \). It then follows from Genest and Rémillard (2008) that the parametric bootstrap work for \( \mathcal{D}_n \). To complete the proof, it only remains to show that \( \mathcal{D}_n - \mathcal{D}_n \nabla \to 0 \). To that end, note that \( \mathbf{V}_{i,n} = H_n(e_{i,n}) \), where \( H_n = \mathcal{R}_{\phi_n} \circ \mathbf{F}_n \), so if \( \mathbb{H}_n = \sqrt{n}(H_n - H) \), then \( \mathbb{H}_n \nabla \to \mathbb{H} \), where, for all \( j = 1, \ldots, d, \)
\[ \mathbb{H}^{(j)}(x) = \Phi^T \mathcal{R}^{(j)} \{ \mathbf{F}(x) \} + \sum_{k=1}^{j} \partial_{u_k} \mathcal{R}^{(j)} \{ \mathbf{F}(x) \} \mathbf{F}_k \{ 1, F_k(u_k) \} \]
\[ = \hat{\mathbb{H}}^{(j)} + \sum_{k=1}^{j} \partial_{u_k} \mathcal{R}^{(j)} \{ \mathbf{F}(x) \} f_k(x_k) \{ (\Gamma_0 \Theta)_k + x_k(\Gamma_{1k} \Theta)_k \}. \]

Next,
\[ V_{ji} - V_{ji,n} = -\frac{\mathbb{H}^{(j)}(e_{i,n})}{\sqrt{n}} + H^{(j)}(e_i) - H^{(j)}(e_{i,n}) \]
\[ = -\frac{\mathbb{H}^{(j)}(e_{i,n})}{\sqrt{n}} + \sum_{k=1}^{j} \partial_{u_k} \mathcal{R}^{(j)}(U_i) \left\{ d_{ki,n} + (\gamma_0, \Theta)_n + \sum_{l=1}^{d} \varepsilon_{li}(\gamma_{1l}, \Theta)_k / \sqrt{n} \right\} \]
\[ i=1, \ldots, n. \] It then follows from the proof of Theorem 1, the tightness of \( \mathbb{H} \) and Ghoudi and Rémillard (2004)[Lemma 5.1] that \( \mathcal{D}_n - \mathcal{D}_n \nabla \to 0. \]

\[ \square \]

APPENDIX C. OTHER PROOFS

Before starting the proofs, one states a lemma that is quite useful in some proofs.

\textbf{Lemma 1.} Suppose that \( C \) and \( D \) and distribution functions on \([0, 1]^2\), so that \( C \) is continuous and \( D \) has mean 1/2 for each marginal distribution. Then
\[ \int D(u, v) dC(u, v) = \int C(u, v) dD(u, v). \]

\textbf{C.1. Proof of Propositions 1–4.} To prove Proposition 1, note that
\[ \int C_n^{(j,k)} dC_n^{(j,k)} = \int \left\{ C_n^{(j,k)} - C^{(j,k)} \right\} dC_n^{(j,k)} + \int C^{(j,k)} dC_n^{(j,k)} \]
\[ = \int \left\{ C_n^{(j,k)} - C^{(j,k)} \right\} dC_n^{(j,k)} + \int C^{(j,k)} dC^{(j,k)}, \]
using Lemma 1, since \( C_n^{(j,k)} \) and \( C^{(j,k)} \) satisfy the assumptions. Then
\[
\sqrt{n}(\tau_{jk,n} - \tau_{jk}) = 4 \sqrt{n} \left\{ \int C_n^{(j,k)}dC_n^{(j,k)} - 4 \int C^{(j,k)}dC^{(j,k)} \right\} + o_P(1)
\]
\[
= 4 \int C_n^{(j,k)}dC_n^{(j,k)} + 4 \int C_n^{(j,k)}dC^{(j,k)} + o_P(1)
\]
\[
= 8 \int C_n^{(j,k)}dC_n^{(j,k)} + o_P(1).
\]
Similarly,
\[
\sqrt{n}(\tilde{\tau}_{jk,n} - \tau_{jk}) = 8 \int \tilde{C}_n^{(j,k)}dC_n^{(j,k)} + o_P(1).
\]
By Corollary 1, \( C_n = \tilde{C}_n + o_P(1) \sim \tilde{C} \), proving that \( R_{jk,n}^K \) converges to \( 8 \int C^{(j,k)}dC_n^{(j,k)} \).

Next, it is easy to check that
\[
\hat{\mathcal{C}}_n(\nu) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ 1(U_i \leq u) - C(u) - \sum_{j=1}^{d} 1(U_{ji} \leq u_j)\partial u_j C(u) \right\}
\]
converges to \( \hat{\mathcal{C}} \). As a result, for any \( 1 \leq j < k \leq d \),
\[
8 \int \hat{\mathcal{C}}_n dC = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ 8C^{(j,k)}(U_{ji}, U_{ki}) - 4U_{ji} - 4U_{ki} + 2 - 2\tau_{jk} \right\}
\]
converges to \( R_{jk}^K \). To compute the covariance between \( R_{jk}^K \) and \( W \), note that \( E \{ \hat{\mathcal{C}}(u)W \} = \hat{\mathcal{C}}(u) \), and the latter is 0 if all but one \( u_j \) is 1. As a result,
\[
E \left(R_{jk}^K W \right) = 8 \int \hat{C}^{(j,k)}dC^{(j,k)} = \partial \phi \{ \tau_{jk} \},
\]
using integration by parts, since \( \tau_{jk} = 4 \int C^{(j,k)}dC^{(j,k)} - 1. \) The proof of Propositions 2–4 is similar. It is sufficient to note that for the three estimators, one has
\[
\sqrt{n}(\rho_{jk,n} - \rho_{jk}) = \sqrt{n} \left[ \int \{ L(u_j) - \bar{L} \} \{ L(u_k) - \bar{L} \} dC_n^{(j,k)}(u_j, u_k) \right.
\]
\[
- \int \{ L(u_j) - \bar{L} \} \{ L(u_k) - \bar{L} \} dC^{(j,k)}(u_j, u_k) \right] + o_P(1)
\]
\[
= \int C_n(j,k)\{ J(x), J(y) \} dxdy + o_P(1),
\]
for an appropriate distribution function \( J \) with left-continuous inverse \( L \).\(^5\)

According to Genest and Rémillard (2004) and Corollary 1, the latter converges to
\[
\int C(j,k)\{ J(x), J(y) \} dxdy. = \int \hat{\mathcal{C}}(j,k)\{ J(x), J(y) \} dxdy.
\]

\(^5\)\( J = \mathcal{N} \) for van der Waerden, \( J \) is the distribution of the uniform over \([0, \sqrt{12}] \) for Spearman’s rho while \( J \) is the distribution function of the discrete random variable taking values 0 and 2 with \( p = 1/2 \) for Blomqvist’s coefficient.
The representations come from the convergence of \( \hat{C}_n \) to \( C \). The proof of the covariance with \( W \) can be dealt similarly to the one involving Kendall’s tau.

C.2. Proof of Proposition 5. The convergence of \( C_n = \sqrt{n}(C_n - C) \) follows from Corollary 1 and the joint convergence of \((\alpha_n, \Phi_n)\) follows from the representation of \( \alpha_n \) and the estimators of Sections 2.1–2.2. Using the smoothness of \( c_\phi \), it follows that \( \dot{C} = \partial_\phi C_\phi \) is continuous and under \( H_0 \),

\[
A_n = \sqrt{n}(C_n - C_\phi_n) = C_n - \sqrt{n}(C_\phi_n - C_\phi) = C_n - \dot{C}^T \Phi_n.
\]

As a result, \( A_n \xrightarrow{d} A = C - \dot{C}^T \Phi = \tilde{C} - \dot{C}^T \Phi \).

Following Genest and Rémillard (2008), the parametric bootstrap approach will work since \( E (\Phi W^\top) = I \), as shown in Sections 2.1–2.2.

C.3. Proof of Proposition 7. Note that \( C_n(s, u) = \mathbb{K}_n\{s, F_n^{-1}(u)\} + \frac{\lfloor ns \rfloor}{n} \sqrt{n} \left[ K\{F_n^{-1}(u)\} - C(u) \right] \).

It then follows from Theorem 1 and the proof of Genest et al. (2007) [Proposition A.1] that

\[
C_n(s, u) \xrightarrow{d} \alpha(s, u) - s \sum_{j=1}^d \partial u_j C(u) \beta_j(1, u_j).
\]

As a result, \( G_n \xrightarrow{d} G \), with

\[
G(s, u) = C(s, u) - sC(1, u) = \alpha(s, u) - s\alpha(1, u).
\]

To show that the multipliers method works, it suffices to note that conditionally on \( U_{1,n}, \ldots, U_{n,n} \), the finite dimensional distributions of \( \alpha_n^{(k)} \) converge to those of \( \alpha^{(k)} \), an independent copy of \( \alpha \) by construction. Next the tightness of \( \alpha_n^{(k)} \) follows from the tightness of \( \frac{C_n(u)}{\sqrt{n}} \sum_{i=1}^n \xi_{i,k} \) and the tightness of \( \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} 1(U_{i,n} \leq u) \), using Bickel and Wichura (1971) and the convergence in probability of \( \frac{1}{n} \sum_{i=1}^{\lfloor ns \rfloor} 1(U_{i,n} \leq u) \) to the continuous distribution function \( sK(u) \) at each \((s, u) \in [0, 1]^{d+1}\). As a result, \( (T_n, T_n^{(1)}, \ldots, T_n^{(N)}) \) converges to \((T, T^{(1)}, \ldots, T^{(N)})\), independent and identically distributed. Hence, if \( N \) is large enough, \( \frac{1}{N} \sum_{k=1}^N 1 \left( T_n^{(k)} > T_n \right) \) is an approximate \( P \)-value for \( T_n \).

**Appendix D. Copula families**

In this section, one describes the copulas families used in the paper, beginning with two major families, the Archimedean copulas and the meta-elliptic copulas.
D.1. **Archimedean copulas.** Archimedean copulas were first defined by Genest and MacKay (1986). A copula \( C \) is said to be Archimedean (with generator \( \phi \)) when it can be expressed in the form

\[
C(u_1, \ldots, u_d) = \phi^{-1} \left\{ \phi(u_1) + \cdots + \phi(u_d) \right\},
\]

for all \( u \in (0, 1)^d \) so that \( \sum_{j=1}^{d} \phi(u_j) < \phi(0^+) \), where \( \phi: (0, 1] \rightarrow [0, \infty) \), is a bijection such that \( \phi(1) = 0 \) and

\[
(-1)^i \frac{d^i}{dx^i} \phi^{-1}(x) > 0, \quad 1 \leq i \leq d.
\]

The generator is unique up to a constant. If the generator yields a copula for any \( d \geq 2 \), then \( \phi^{-1} \) is necessarily the Laplace transform of a non-negative random variable \( \xi \) (Marshall and Olkin, 1988), i.e., \( \phi^{-1}(s) = E\left(e^{-s\xi}\right) \), for all \( s \geq 0 \). Table 1 gives the generators for three well-known Archimedean copulas: Clayton, Frank, and Gumbel-Hougaard families. These three classes share the interesting property that the copula exists for any dimension, for the values of parameters listed in the table. See Joe (1997) and Nelsen (2006) for further examples on copulas.

| Family         | \( \phi(t) \) | Range of \( \theta \) | Kendall’s \( \tau \) |
|----------------|----------------|------------------------|----------------------|
| Clayton        | \( (t^{-\theta} - 1)/\theta \) | \( (0, \infty) \) | \( \theta/(\theta + 2) \) |
| Frank          | \(- \log \left( \frac{1 - \theta^t}{1 - \theta} \right) \) | \( (0, 1) \) | \( \frac{\log(\theta)^2 + 4 \log(\theta) + 4d \log(\theta)}{\log(\theta)^2} \) |
| Gumbel-Hougaard | \( |\log t|^{1/\theta} \) | \( (0, 1) \) | \( 1 - \theta \) |

Here, \( \text{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt \) stands for the dilog function.

For the Clayton family of parameter \( \theta \in (0, \infty) \), the associated \( \xi \) has Gamma distribution with parameters \((1/\theta, 1)\) since \( E\left(e^{-s\xi}\right) = (1 + s)^{-1/\theta} \). For the Frank family with parameter \( \theta \in (0, 1) \), the associated \( \xi \) is discrete and has a logarithmic series distribution given by \( P(\xi = k) = \frac{1}{\log(1/\theta)} \frac{(1-\theta)^k}{k}, \quad k = 1, 2, \ldots, \) since

\[
E\left(e^{-s\xi}\right) = \log \left\{ 1 - (1-\theta)e^{-s} \right\} / \log(\theta) = \frac{1}{\log(1/\theta)} \sum_{k=1}^{\infty} (1-\theta)^k e^{-ks}.
\]

Finally, for the Gumbel-Hougaard family with parameter \( \theta \in (0, 1) \), \( \xi \) has a positive stable distribution of parameter \( \theta \), since \( E\left(e^{-s\xi}\right) = e^{-s\theta} \).

For formulas giving densities and Rosenblatt’s transforms for these three families, see, e.g., Rémillard et al. (2010).
D.2. **Meta-elliptic copulas.** Meta-elliptic copulas are simply copulas associated with elliptic distributions through relation (1), the most popular in applications being the Student copula and the (now infamous) Gaussian copula. Recall that a vector $Y$ has an elliptic distribution with generator $g$ and parameters $\mu$ and (positive definite symmetric matrix) $\Sigma$, denoted $Y \sim \mathcal{E}(g, \mu, \Sigma)$, if its density $h$ is given by

$$h(y) = \frac{1}{|\Sigma|^{1/2}} g \left\{ (y - \mu)^\top \Sigma^{-1} (y - \mu) \right\}, \quad y \in \mathbb{R}^d,$$

where

$$\frac{\pi^{d/2} r^{(d-2)/2} g(r)}{\Gamma(d/2)}$$

is the density of $\xi = (Y - \mu)^\top \Sigma^{-1} (Y - \mu)$. It is easy to check that the underlying copula depends only on $g$ and $R$, $R$ being the correlation matrix associated with $\Sigma$.

Here are some general families of elliptic distributions.

### Table 2. Generators of some $d$-dimensional elliptic distributions.

| Family         | Generator                                                                 |
|----------------|---------------------------------------------------------------------------|
| Gaussian       | $g(r) = \frac{1}{(2\pi)^{d/2}} e^{-r/2}$                                  |
| Pearson type II| $g(r) = \frac{\Gamma(\alpha + d/2)}{\pi^{d/2} \Gamma(\alpha)} (1 - r)^{\alpha - 1}$, where $0 < r < 1$ and $\alpha > 0$ |
| Pearson type VII| $g(r) = \frac{\Gamma(\alpha + d/2)}{(\pi \nu)^{d/2} \Gamma(\alpha)} (1 + r/\nu)^{-\alpha - d/2}$, where $\alpha, \nu > 0$ |

**Remark 6.** The case $\alpha = \nu/2$ for the Pearson type VII corresponds to the multivariate Student, while if $\alpha = 1/2$ and $\nu = 1$, it corresponds to the multivariate Cauchy distribution. However, since $\nu$ is a scaling parameter for that family, the meta-elliptic copula only depends on $\alpha$, so any meta-elliptic copula in that family is necessarily a Student copula with parameters $(R, 2\alpha)$. In particular, the Cauchy copula is the Student copula with 1 degree of freedom.

Suppose that $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{E}(g, 0, \Sigma)$, where $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$.

It is easy to check that $X_1 \sim \mathcal{E}(g_1, 0, R_{11})$, where

$$g_1(r) = \int_{\mathbb{R}^{d_2}} g(\|x_2\|^2 + r) \, dx_2 = \frac{2\pi^{d_2/2}}{\Gamma(d_2/2)} \int_0^{\infty} s^{d_2-1} g(s^2 + r) \, ds.$$  (19)
As a consequence, the density of any marginal distribution of a $d$-dimensional elliptic distribution with generator $g$ and parameters $(0, R)$ is

$$f(x) = \frac{\pi^{(d-1)/2}}{\Gamma(d/2)} \int_0^\infty s^{(d-3)/2} g(s + x^2) ds, \quad x \in \mathbb{R}. \quad (20)$$

For example, if $g$ is the generator of the $d$-dimensional Pearson type VII with parameters $(\alpha, \nu)$, then $g_i$ is the generator of the $d_i$-dimensional Pearson type VII with parameters $(\alpha, \nu), i = 1, 2$. One can also show that if $g$ is the generator of the $d$-dimensional Pearson type II with parameter $\alpha$, then $g_i$ is the generator of the $d_i$-dimensional Pearson type II with parameter $\alpha + d_{3-i}, i = 1, 2$. In particular, the marginal distributions of a Pearson type VII has density

$$f(x) = \frac{\Gamma(\alpha + 1/2)}{(\pi \nu)^{1/2} \Gamma(\alpha)} (1 + x^2/\nu)^{-\alpha - 1/2}.$$

Note that formula (19) is particularly useful for computing Rosenblatt’s transforms.

D.3. Other copula families. As proposed in Dias and Embrechts (2004), one could also consider mixtures of copulas, i.e., consider families of the form

$$C_{\theta, \pi} = \sum_{k=1}^m \pi_k C_{\theta_k}^{(k)},$$

with $\sum_{k=1}^m \pi_k = 1, \pi_k > 0$, and $\theta = (\theta_1, \ldots, \theta_m)$. Copulas $C_{\theta_k}^{(k)}$ may be part of the same family; for example, one could consider a mixture of Gaussian copulas. One could also take a mixture of different families, e.g. a mixture of Clayton and Gumbel copulas.

Other families considered recently in applications include hierarchical copulas (Savu and Trede, 2006), (McNeil, 2008), and copula vines (Bedford and Cooke, 2002), (Aas et al., 2009).

References

Aas, K., Czado, C., Frigessi, A., and Bakken, H. (2009). Pair-copula constructions of multiple dependence. *Insurance Math. Econom.*, 44(2):182–198.

Bedford, T. and Cooke, R. M. (2002). Vines—a new graphical model for dependent random variables. *Ann. Statist.*, 30(4):1031–1068.

Berg, D. and Quessy, J.-F. (2009). Local power analyses of goodness-of-fit tests for copulas. *Scand. J. Stat.*, 36(3):389–412.

Berrada, T., Dupuis, D. J., Jacquier, E., Papageorgiou, N., and Rémillard, B. (2006). Credit migration and derivatives pricing using copulas. *J. Comput. Fin.*, 10:43–68.

Bickel, P. J. and Wichura, M. J. (1971). Convergence criteria for multiparameter stochastic processes and some applications. *Ann. Math. Statist.*, 42:1656–1670.

Blum, J. R., Kiefer, J., and Rosenblatt, M. (1961). Distribution free test of independence based on the sample distribution function. *Ann. Math. Statist.*, 32:485–498.

Breymann, W., Dias, A., and Embrechts, P. (2003). Dependence structures for multivariate high-frequency data in finance. *Quant. Finance*, 3:1–14.
Carlstein, E. (1988). Nonparametric change-point estimation. *Ann. Statist.*, 16(1):188–197.

Chen, X. and Fan, Y. (2006). Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. *Journal of Econometrics*, 135(1-2):125 – 154.

Chen, X., Fan, Y., and Patton, A. (2005). Simple tests for models of dependence between multiple financial time series, with applications to u.s. equity returns and exchange rates. Technical Report 483, Financial Markets Group, London School of Economics.

Dias, A. and Embrechts, P. (2004). Dynamic copula models for multivariate high-frequency data in finance. Technical report, ETH Zürich.

Dias, A. and Embrechts, P. (2009). Testing for structural changes in exchange rates dependence beyond linear correlation. *European Journal of Finance*, 15(7):619–637.

Diebold, F. X., Gunther, T. A., and Tay, A. S. (1998). Evaluating density forecasts with applications to financial risk management. *International Economic Review*, 39(4):863–883.

Dobrić, J. and Schmid, F. (2005). Testing goodness of fit for parametric families of copulas: Application to financial data. *Comm. Statist. Simulation Comput.*, 34:1053–1068.

Dobrić, J. and Schmid, F. (2007). A goodness of fit test for copulas based on Rosenblatt’s transformation. *Comput. Statist. Data Anal.*, 51:4633–4642.

Durbin, J. (1973). Weak convergence of the sample distribution function when parameters are estimated. *Ann. Statist.*, 1(2):279–290.

Embrechts, P., McNeil, A. J., and Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls. In *Risk management: value at risk and beyond (Cambridge, 1998)*, pages 176–223. Cambridge Univ. Press, Cambridge.

Engle, R. F. and Kroner, K. F. (1995). Multivariate simultaneous generalized ARCH. *Econometric Theory*, 11:122–150.

Fang, H.-B., Fang, K.-T., and Kotz, S. (2002). The meta-elliptical distributions with given marginals. *J. Multivariate Anal.*, 82(1):1–16.

Fermanian, J.-D., Radulović, D., and Wegkamp, M. J. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10:847–860.

Gänßler, P. and Stute, W. (1987). *Seminar on Empirical Processes*, volume 9 of *DMV Seminar*. Birkhäuser Verlag, Basel.

Genest, C., Ghoudi, K., and Rémillard, B. (2007). Rank-based extensions of the Brock Dechert Scheinkman test for serial dependence. *J. Amer. Statist. Assoc.*, 102:1363–1376.

Genest, C., Ghoudi, K., and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82:543–552.

Genest, C. and MacKay, R. J. (1986). Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. *The Canadian Journal of Statistics*, 14(2):145–159.
Genest, C. and Rémillard, B. (2004). Tests of independence or randomness based on the empirical copula process. *Test*, 13:335–369.
Genest, C. and Rémillard, B. (2008). Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. *Ann. Inst. H. Poincaré Sect. B*, 44:1096–1127.
Genest, C., Rémillard, B., and Beaudoin, D. (2009). Omnibus goodness-of-fit tests for copulas: A review and a power study. *Insurance Math. Econom.*, 44:199–213.
Ghoudi, K., Kulperger, R. J., and Rémillard, B. (2001). A nonparametric test of serial independence for time series and residuals. *J. Multivariate Anal.*, 79:191–218.
Ghoudi, K. and Rémillard, B. (2004). Empirical processes based on pseudo-observations. II. The multivariate case. In *Asymptotic Methods in Stochastics*, volume 44 of *Fields Inst. Commun.*, pages 381–406. Amer. Math. Soc., Providence, RI.
Ghoudi, K. and Rémillard, B. (2010). Diagnostic tests for innovations of ARMA models using empirical processes of residuals. Technical Report G-2010-23, Gerad.
Guégan, D. and Zhang, J. (2010). Change analysis of a dynamic copula for measuring dependence in multivariate financial data. *Quant. Finance*, 10(4):421–430.
Harvey, A. (2010). Tracking a changing copula. *Journal of Empirical Finance*, 17(3):485–500.
Joe, H. (1997). *Multivariate models and dependence concepts*, volume 73 of *Monographs on Statistics and Applied Probability*. Chapman & Hall, London.
Kole, E., Koedijk, K., and Verbeek, M. (2007). Selecting copulas for risk management. *Journal of Banking & Finance*, 31(8):2405–2423.
Marshall, A. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83:834–841.
McNeil, A. J. (2008). Sampling nested Archimedean copulas. *Journal of Statistical Computation and Simulation*, 78(6):567–581.
Nelsen, R. B. (2006). *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag, New York, second edition.
Panchenko, V. (2005). Goodness-of-fit test for copulas. *Phys. A*, 355:176–182.
Patton, A. J. (2006). Modelling asymmetric exchange rate dependence. *International Economic Review*, 47(2):527–556.
Picard, D. (1985). Testing and estimating change-points in time series. *Adv. in Appl. Probab.*, 17(4):841–867.
Rémillard, B. (2010). Validity of the parametric bootstrap for goodness-of-fit testing in dynamic models. Technical report.
Rémillard, B., Papageorgiou, N., and Soustra, F. (2010). Dynamic copulas. Technical Report G-2010-18, Gerad.
Rémillard, B. and Scaillet, O. (2009). Testing for equality between two copulas. *J. Multivariate Anal.*, 100:377–386.
Rosenblatt, M. (1952). Remarks on a multivariate transformation. *Ann. Math. Stat.*, 23:470–472.
Savu, C. and Trede, M. (2006). Hierarchical Archimedean copulas. Technical report, University of Münster.
Scaillet, O. (2005). A Kolmogorov-Smirnov type test for positive quadrant dependence. *Canad. J. Statist.*, 33(3):415–427.

Shih, J. H. and Louis, T. A. (1995). Inferences on the association parameter in copula models for bivariate survival data. *Biometrics*, 51:1384–1399.

Sklar, M. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231.

Tsukahara, H. (2005). Semiparametric estimation in copula models. *Canad. J. Statist.*, 33(3):357–375.

van den Goorbergh, R., Genest, C., and Werker, B. (2005). Bivariate option pricing using dynamic copula models. *Insurance: Mathematics and Economics*, 37:101–114.

Xu, J. (1996). *Statistical Modelling and Inference for Multivariate and Longitudinal Discrete Response Data*. PhD thesis, University of British Columbia.

GERAD and DEPARTMENT OF MANAGEMENT SCIENCES, HEC MONTRÉAL, 3000 CHEMIN DE LA CÔTE SAINTE-CATHERINE, MONTRÉAL (QUÉBEC), CANADA H3T 2A7