A polynomial type Jost solution and spectrum of the self-adjoint quantum Dirac system

N Coskun and N Yokus
Department of Mathematics, Karamanoglu Mehmetbey University, Karaman, 70100, Turkey

E-mail: cannimet@kmu.edu.tr

Abstract. This paper is devoted to investigation of a polynomial type Jost solution for the self-adjoint quantum Dirac system. After examination of analytical features and asymptotic behaviour of the Jost solution, continuous and discrete spectrum and some properties of the eigenvalues of the operator \( L \) generated by the quantum difference system have been discussed.

1. Introduction
Spectral analysis of differential and difference operators is of great importance for the solutions of certain problems in many areas including engineering, economics, quantum mechanics and mathematical physics [1-5]. In this context, Dirac system of differential and discrete operators have been studied in [6-8].

In the last years, an important effort has been devoted to quantum calculus [9]. As a consequence of developments in quantum theory, quantum difference equations has been subject matter of various studies [10-12]. In particular, spectral analysis of quantum difference equations has been studied in [13-15]. However, the Dirac system of quantum difference equations including a polynomial type Jost solution has not been examined in the known literature yet.

In this paper, we assume \( q > 1 \) and use the notation
\[
q^n_{\mathbb{N}_0} := \left\{ q^n : n \in \mathbb{N}_0 \right\}
\]
where \( \mathbb{N}_0 \) indicates the set of nonnegative integers. The q-derivative of a function \( f : q^{\mathbb{N}} \rightarrow \mathbb{C} \) is defined by
\[
f^\Delta (t) := \frac{f(qt) - f(t)}{\mu(t)}, \quad \forall t \in q^{\mathbb{N}},
\]
where \( \mu(t) = (q - 1)t \) is the graininess function [10]. Hereafter, we will denote the Hilbert space \( l_2(q^{\mathbb{N}}, \mathbb{C}^2) \) including all sequences \( y = \{ y(t) \} = \left[ y^{(1)}(t) \right, y^{(2)}(t) \} \) with the inner product,
\[
\langle y, f \rangle_{(q^{\mathbb{N}}, \mathbb{C}^2)} := \sum_{t \in q^{\mathbb{N}}} y^{(1)}(t)f^{(1)}(t) + y^{(2)}(t)f^{(2)}(t), \quad y, f : q^{\mathbb{N}} \rightarrow \mathbb{C}
\]
and the norm
\[ \| y(t) \|_{L_2(q^n, \mathbb{C}^2)} = \left( \sum_{t \in q^n} \left[ y^{(1)}(t)^2 + y^{(2)}(t)^2 \right] \right)^{1/2} \text{ for } y: q^n \rightarrow \mathbb{C} \]
by \( L_2(q^n, \mathbb{C}^2) \).

Let us consider the system
\[
\begin{align*}
(y^{(2)}(t))' + p(t)y^{(1)}(t) &= \frac{\lambda}{\mu(t)} y^{(1)}(t), \\
-(y^{(1)}(t))' + r(t)y^{(2)}(t) &= \frac{\lambda}{\mu(t)} y^{(2)}(t),
\end{align*}
\]
for \[ t \in q^n \quad (1.1) \]
where \{p(t)\}_{t \in q^n}, \{r(t)\}_{t \in q^n} \text{ and } \{\mu(t)\}_{t \in q^n} \text{ are real sequences for all } t \in q^n \text{ and } \lambda \text{ is a spectral parameter.} \]

Note that, the system of equations (1.1) is quantum analogue of the well-known Dirac system
\[
\begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]
\([6], \text{Chapter 2}) \]. For this reason, the system (1.1) is called a quantum Dirac system. It is worth to point out here that the construction of the quantum Dirac system (1.1) is completely different from other studies \([7, 8]\) because of the quantum derivative of a function.

Let us define the operator \( L \) generated in \( L_2(q^n, \mathbb{C}^2) \) by the following system of quantum difference expression:
\[
(Ly)(t) = \begin{pmatrix} y^{(2)}(qt) - y^{(2)}(t) + p(t)\mu(t)y^{(1)}(t) \\ y^{(1)}(\frac{t}{q}) - y^{(1)}(t) + r(t)\mu(t)y^{(2)}(t) \end{pmatrix}
\]
where \{p(t)\}_{t \in q^n}, \{r(t)\}_{t \in q^n} \text{ and } \{\mu(t)\}_{t \in q^n} \text{ are real sequences.} \text{ It is clear that the system of equations (1.1) can be rewritten as}
\[
(Ly)(t) = \lambda y(t), \quad t \in q^n.
\]

The set up of this paper organized as follows: Section 2 is concerned with the investigation of the polynomial type Jost solution of the system (1.1) with the boundary condition
\[
y^{(1)}(1) = 0 \quad (1.2)
\]
and investigate analytic properties and asymptotic behavior of the Jost solution. Section 3 presents continuous and discrete spectrum and some properties of the eigenvalues of the boundary value problem BVP (1.1)-(1.2).

2. Jost solution and Jost function of (1.1)
Let the real sequences \{p(t)\}_{t \in q^n}, \{r(t)\}_{t \in q^n} \text{ and } \{\mu(t)\}_{t \in q^n} \text{ satisfy the condition}
\[
\sum_{t \in q^n} \frac{\ln t}{\ln q} \left( |p(t)\mu(t)| + |r(t)\mu(t)| \right) < \infty. \quad (2.1)
\]

**Theorem 2.1.** Under the condition (2.1), the system of equations (1.1) has unique solutions
\[f(t, z) = \left( \begin{array}{c} f^{(1)}(t, z) \\ f^{(2)}(t, z) \end{array} \right) = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \sum_{r \in q^{10}} A_{r} z^{2 \log_{r} t}, \quad (2.2)\]
\[f^{(1)}(1, z) = z + \sum_{r \in q^{0}} A_{r} z^{2 \log_{r} r+1} - i \sum_{r \in q^{0}} A_{r} z^{2 \log_{r} r}, \quad (2.3)\]

for \( \lambda = -iz - (iz)^{-1}, \ t \in q^{1} \) and \( |z| = 1 \) where
\[A_{r} = \begin{pmatrix} A_{r}^{11} & A_{r}^{12} \\ A_{r}^{21} & A_{r}^{22} \end{pmatrix}.\]

**Proof.** If we substitute the equation (2.2) in the system of equations (1.1) and take \( \lambda = -iz - (iz)^{-1} \) and \( |z| = 1 \), then we obtain
\[A_{rq}^{12} = - \sum_{s \in [q, \infty)} p(s) \mu(s) \mu(s),\]
\[A_{rq}^{11} = \sum_{s \in [q, \infty)} p(s) \mu(s) A_{sq}^{12},\]
\[A_{rq}^{22} = \sum_{s \in [q, \infty)} p(s) \mu(s) A_{sq}^{12},\]
\[A_{rq}^{21} = A_{rq}^{12} + p(t) \mu(t) A_{rq}^{11} + \sum_{s \in [q, \infty)} \left[ r(s) \mu(s) A_{sq}^{22} + p(s) \mu(s) A_{sq}^{11} \right],\]
\[A_{rq}^{12} = - \sum_{s \in [q, \infty)} \left[ p(s) \mu(s) A_{sq}^{11} + r(s) \mu(s) A_{sq}^{22} \right],\]
\[A_{rq}^{11} = - A_{rq}^{22} + \sum_{s \in [q, \infty)} \left[ p(s) \mu(s) A_{sq}^{12} - r(s) \mu(s) A_{sq}^{21} \right],\]
\[A_{rq}^{22} = - A_{rq}^{11} + \sum_{s \in [q, \infty)} \left[ p(s) \mu(s) A_{sq}^{12} - r(s \mu(s) A_{sq}^{21} \right],\]
\[A_{rq}^{21} = A_{rq}^{12} + \sum_{s \in [q, \infty)} \left[ p(s) \mu(s) A_{sq}^{11} + r(s) \mu(s) A_{sq}^{22} \right],\]

for \( r \geq q^{3} \)
\[A_{r}^{12} = A_{r}^{q} \frac{21}{q} - \sum_{s \in [q, \infty)} r(s) \mu(s) A_{r}^{22} + p(s) \mu(s) A_{r}^{11},\]
\[A_{r}^{11} = - A_{r}^{q} \frac{22}{q} + \sum_{s \in [q, \infty)} p(s) \mu(s) A_{r}^{12} - r(s) \mu(s) A_{r}^{21},\]
\[A_{r}^{22} = - A_{r}^{q} \frac{11}{q} + \sum_{s \in [q, \infty)} \left( p(s) \mu(s) A_{r}^{12} - r(s) \mu(s) A_{r}^{11} \right),\]
\[A_{r}^{21} = A_{r}^{q} \frac{12}{q} + \sum_{s \in [q, \infty)} \left( r(s) \mu(s) A_{r}^{22} + p(s) \mu(s) A_{r}^{11} \right).\]

Based on the condition (2.1), the series in the definition of \( A_{rq}^{ij} \) \( (i, j = 1, 2) \) are absolutely convergent.

Hence, \( A_{rq}^{ij} \) \( (i, j = 1, 2) \) can be uniquely determined by \( p(t), r(t) \) and \( \mu(t) \) \( t \in q^{1} \), i.e., the system (1.1) has the solution given by the equations (2.2) and (2.3). \( \Box \)

The solution \( f \) is called Jost solution of the system of equations (1.1). Using the inequalities for \( A_{rq}^{ij} \) \( (i, j = 1, 2) \) given in Theorem (2.1), we find
\[ |A_{ij}| \leq C \sum_{s|q, r|q^n} (|p(s)\mu(s)| + |r(s)\mu(s)|), \quad i, j = 1, 2, \quad (2.4) \]

by induction, where \( \left\lfloor \frac{\ln r}{2\ln q} \right\rfloor \) is the integer part of \( \frac{\ln r}{2\ln q} \), \( C > 0 \) is a constant and \( t, r \in q^n \).

It is clear from the equations (2.2), (2.3) and (2.4) that \( \begin{pmatrix} f^{(1)}(t, z) \\ f^{(2)}(t, z) \end{pmatrix} \) has analytic continuation from \( \{z : |z| = 1\} \) to \( D := \{z : |z| < 1\} \setminus \{0\} \).

**Theorem 2.2.** If the condition (2.1) holds then the Jost solution \( f \) satisfies the following asymptotic:

\[
\begin{pmatrix} f^{(1)}(t, z) \\ f^{(2)}(t, z) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + o(1) \begin{pmatrix} z \end{pmatrix}^{2\log_q r}, \quad z \in \overline{D} := \{z : |z| \leq 1\} \setminus \{0\}, \quad t \to \infty. \quad (2.5)
\]

**Proof.** It can be written from the equation (2.2) that

\[ f^{(1)}(t, z)z^{-2\log_q r^{i+1}} = 1 + \sum_{r \in q^n} A_{ir}^{11}z^{2\log_q r} - i \sum_{r \in q^n} A_{ir}^{12}z^{2\log_q r^{i+1}}. \quad (2.6) \]

From the equations (2.4) and (2.6), it is seen that

\[
\left| f^{(1)}(t, z)z^{-2\log_q r^{i+1}} \right| \leq 1 + \sum_{r \in q^n} |A_{ir}^{11}| + \sum_{r \in q^n} |A_{ir}^{12}| \\
\leq 1 + 2C \sum_{r \in q^n} \sum_{s|q, r|q^n} \left( |p(s)\mu(s)| + |r(s)\mu(s)| \right) \\
\leq 1 + 2C \sum_{r \in q^n} \sum_{s|q, r|q^n} \left( \frac{\ln s}{\ln q} - \frac{\ln t}{\ln q} \right) \left( |p(s)\mu(s)| + |r(s)\mu(s)| \right) \\
\leq 1 + 2C \sum_{s|q, r|q^n} \frac{\ln s}{\ln q} \left( |p(s)\mu(s)| + |r(s)\mu(s)| \right) \quad (2.7)
\]

where \( C > 0 \) is a constant. Then, we get from the equation (2.7) that

\[ f^{(1)}(t, z) = z^{2\log_q r^{i+1}}[1 + o(1)], \quad z \in \overline{D}, \quad t \to \infty. \quad (2.8) \]

In a similar way to equation (2.8), we have

\[ f^{(2)}(t, z) = -iz^{2\log_q r^{i+1}}[1 + o(1)], \quad z \in \overline{D}, \quad t \to \infty. \quad (2.9) \]

From equations (2.8) and (2.9), we obtain the equation (2.5). \( \square \)

### 3. Main Results

Now, we shall give some theorems and definitions to prove our main results. We begin with investigating continuous spectrum of the operator \( L \).

**Theorem 3.1.** Assume the condition (2.1) satisfies. Then \( \sigma_c(L) = [-2, 2] \), where \( \sigma_c(L) \) represents the continuous spectrum of the operator \( L \).
Proof. Let $L_0$ be the operator generated in $l_2(q^N, \mathbb{C}^2)$ by the following system of quantum difference expression:

$$
(l_0y)(t) = \begin{pmatrix} y^{(2)}(qt) - y^{(2)}(t) \\ y^{(1)}(\frac{t}{q}) - y^{(1)}(t) \end{pmatrix}
$$

with the boundary condition $y^{(1)}(1) = 0$. We also define the operator $L_1$ in $l_2(q^N, \mathbb{C}^2)$ by the following

$$
(l_1y)(t) = \begin{pmatrix} p(t)\mu(t)y^{(1)}(t) \\ r(t)\mu(t)y^{(2)}(t) \end{pmatrix}.
$$

One can easily observe that the operator $L_0$ is self-adjoint and $L_1$ is a compact operator [16]. Also, $L = L_0 + L_1$ holds. From the Weyl Theorem [17] of a compact perturbation, we find $\sigma_e(L) = \sigma_e(L_0) = [-2, 2]$.

Definition 3.1. The Wronskian of two solutions $y = \{y(t, z)\}_{t \in q^N}$ and $u = \{u(t, z)\}_{t \in q^N}$ of (1.1) is defined by $W[y, u](t) = y(t, z)u(qt, z) - y(qt, z)u(t, z)$ for $t \in q^N$.

Let $\varphi(z) = \varphi(\lambda) = \{\varphi(t, \lambda)\}, t \in q^N$ be the solution of the system of equations (1.1) subject to the initial conditions $\varphi^{(1)}(1, z) = 0, \varphi^{(2)}(q, z) = 1$.

Then from definition (3.1)

$$
W[f(z), \varphi(z)] = f^{(1)}(t, z)\varphi^{(2)}(qt, z) - f^{(2)}(qt, z)\varphi^{(1)}(t, z) = f^{(1)}(1, z),
$$

Since the operator $L$ is self-adjoint, the eigenvalues of $L$ are real valued. By using equation (3.1) and the definition of eigenvalues, we get

$$
\sigma_d(L) = \{\lambda : \lambda = -iz - (iz)^{-1}, iz \in (-1, 0) \cup (0, 1), f^{(1)}(1, z) = 0\},
$$

where $\sigma_d(L)$ symbolizes the eigenvalues of the operator $L$. 

Definition 3.2. The multiplicity of a zero of $f^{(1)}(1, z)$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of BVP (1.1), (1.2).

Theorem 3.2. Under the condition (2.1), the operator $L$ has a finite number of real eigenvalues in $D$.

Proof. In order to prove the theorem, it is necessary and sufficient to show that the function $f^{(1)}(1, z)$ has finite number of real zeros in $D$. The accumulation points of the zeros of the analytic function $f^{(1)}(1, z)$ can take real values $-i, 0$ and $i$. It is known that $L$ is self-adjoint bounded operator. So its eigenvalues is different from infinity. Assume $z_0 = 0$ is a zero of the function $f^{(1)}(1, z)$. But in this case the eigenvalue $\lambda$ is infinite. Therefore $z_0 = 0$ is not a zero of the function $f^{(1)}(1, z)$. Now, assume $z = \pm i$. In this case, $\lambda = \pm 2$ and $D$ is bounded. From Theorem (3.1), $\pm 2$ are elements of continuous spectrum of the operator $L$. From operator theory, discrete spectrum of a self-adjoint operator and continuous spectrum of this operator are distinct. Hence, the set of zeros of the function $f^{(1)}(1, z)$ in $D$ is finite from the Bolzano Weierstrass Theorem. It is obvious from (3.2) that the eigenvalues of the operator $L$ are real. This completes the proof. 

\[\square\]
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