The Brauer group of modified supergroup algebras

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Abstract

The computation of the Brauer group $BM$ of modified supergroup algebras is performed, yielding, in particular, the computation of the Brauer group of all finite-dimensional triangular Hopf algebras when the base field is algebraically closed and of characteristic zero. The results are compared with the computation of lazy cohomology and with Yinhuo Zhang’s exact sequence. As an example, we compute explicitly the Brauer group and lazy cohomology for modified supergroup algebras with (extensions of) Weyl groups of irreducible root systems as a group datum and their standard representation as a representation datum.

Introduction

Finite-dimensional triangular Hopf algebras over an algebraically closed base field of characteristic zero have been completely described in [1,18,19,20]. They can all be reduced, by Drinfeld twists (the construction dual to a cocycle twist), to a particular class of pointed Hopf algebras, the so-called modified supergroup algebras. Such algebras are directly constructed starting from a finite group $G$, a central involution $u$ of $G$, and a representation $V$ of $G$ on which $u$ acts as $-1$. The $R$-matrix can always be reduced to $R_{u} = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)$.

In [6] and [7] several group invariants for finite-dimensional Hopf algebras have been defined. In particular, the Brauer group $BM(k,H,R)$ of a quasitriangular Hopf algebra $(H,R)$ has been defined as the Brauer group of the braided monoidal category of left $H$-modules. It has been shown in [8] as a consequence of

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of results in [38] that this Brauer group is invariant under Drinfeld twists of \((H, R)\) and this fact can be used in order to replace \((H, R)\) with a new pair \((H', R')\) that is easier to deal with. In particular, this property has been used to replace the \(R\)-matrix \(R\) with a simpler one. In the last few years several explicit examples have been computed for triangular or quasitriangular Hopf algebras. All of these Hopf algebras admitted several distinct quasitriangular structures and most of them admitted at least one triangular structure: they were in fact modified supergroup algebras. The first explicit computation was performed in [39] where the Brauer group of Sweedler Hopf algebra \(H_4\) is determined when the triangular structure is \(R_\mu\). This computation was generalized to all possible (quasi)triangular structures of \(H_4\) in [8]. In terms of modified supergroup algebras, \(H_4\) corresponds to the data \(G = \mathbb{Z}_2\) and \(V\) its non-trivial irreducible representation. In [9] the computation of \(BM\) is generalized to the case in which \(G\) is the cyclic group \(\langle g \rangle\) of order \(2\nu\) for \(\nu\) an odd integer and \(V\) is the irreducible representation on which \(g\) again acts as \(-1\). In [10] the first case of a non-abelian group was considered and in [11] the case in which \(G = \mathbb{Z}_2\) and \(V\) is given by \(n\)-copies of the non-trivial irreducible representation of \(G\) was treated, for all triangular structures. In all those cases important roles were played by the classical Brauer group, by the Brauer-Wall group, and by lazy cohomology as defined in [35] (with the name central cohomology) and studied in [3] and [16]. It is therefore natural to generalize the approach used in the aforementioned papers in order to deal with all \(G\) and \(V\) at the same time, for \(R\) triangular, obtaining in particular the computation of the Brauer group of all finite-dimensional triangular Hopf algebras over an algebraically closed field of characteristic zero. This construction should also give an indication on how to compute the Brauer group of quasitriangular Hopf algebras admitting also a triangular structure.

We list here the content of the paper. First, the case in which \(V = 0\) is dealt with. In this case the Brauer group of the group algebra \(k[G]\) with \(R\)-matrix \(R_\mu\) is

- isomorphic to \(Br(k) \times H^2(G, k^-)\) if \(\mu = 1\);

- a central extension of \(Br(k)\) by \(H^2_\chi(G, k^-)\) if \(\mu \neq 1\) and \(U = \langle \mu \rangle\) is not a direct summand of \(G\). Here, for a group \(K\), the symbol \(H^2_\chi(K, k^-)\) denotes an abelian group coinciding with \(H^2(K, k^-)\) as a set but with a modified group structure given in Section 2.1 Conditions on \(K\) or \(k\) for \(H^2_\chi(K, k^-)\) to be isomorphic to \(H^2(K, k^-)\) are given in Proposition 2.3. For instance, the two groups coincide when \(-1\) is a square in \(k\).

- a central extension of \(Br(k)\) by \(Q(k, G)\), which is itself a central extension of \(H^2_\chi(G, k^-)\) by \(\mathbb{Z}_2\), if \(\mu \neq 1\) and \(U = \langle \mu \rangle\) is a direct summand of \(G\).
Thanks to the results in [20] we obtain:

**Theorem** Let \((H, R)\) be a semi-simple and cosemi-simple triangular Hopf algebra over \(k = \bar{k}\) with \(\gcd(\text{char}(k), \dim H) = 1\). Then

\[
BM(k, H, R) = \begin{cases} 
H^2(G, k) \times \mathbb{Z}_2 & \text{if } u \neq 1 \text{ and } G \cong \langle u \rangle \times G/\langle u \rangle; \\
H^2(G, k) & \text{otherwise}
\end{cases}
\]

where \(G, u\) and \(V = 0\) are the corresponding data for \((H, R)\).

The more general case in which \(V \neq 0\) is dealt with by showing that the Brauer group in this case is the direct product of the Brauer group of \(k[G]\) and the group of \(G\)-invariant symmetric bilinear forms on \(V^*\). In this case \(R = R_u\) with \(u \neq 1\).

By [19, Proposition 2.6] we obtain:

**Theorem** Let \((H, R)\) be a finite-dimensional triangular Hopf algebra over \(k = \bar{k}\) with \(\text{char}(k) = 0\). Then

\[
BM(k, H, R) = \begin{cases} 
\mathbb{Z}_2 \times H^2(G, k) \times S^2(V^*)^G & \text{if } u \neq 1 \text{ and } G \cong U \times G/U \\
H^2(G, k) \times S^2(V^*)^G & \text{otherwise}
\end{cases}
\]

where \(G, u, V, U = \langle u \rangle\) are the corresponding data for \((H, R)\).

As in the classical case, a link is expected between a second “cohomology group” for the Hopf algebra \(H\) and its full Brauer group, that is, the Brauer group of the Drinfeld double \(D(H)\) of \(H\). It is also expected, and evidence was given by all computations made so far, that this cohomology group could be given by the second lazy cohomology group \(H^2_L(H)\). We adapt the analysis we made for the Brauer group in order to compute \(H^2_L(H)\) for modified supergroup algebras and compare it to \(BM(k, H, R_u)\). If the representation datum \(V\) is trivial then \(H = k[G]\) and \(H^2_L(H)\) is the usual cohomology in degree two of the group datum \(G\). If the representation datum \(V\) is non-trivial then \(H^2_L(H)\) is isomorphic to the direct product of \(S^2(V^*)^G\) and the second cohomology group of \(G/U\). Thus, the linear part of lazy cohomology is a direct summand of \(BM(k, H, R_u)\), while for the group cohomology component the situation is more involved. This phenomenon is not completely new because already when \(H = k[\mathbb{Z}_2]\) the cohomology of \(\mathbb{Z}_2\) with trivial coefficients can be seen as a subquotient but not as a subgroup of \(BM(k, H, R_u) \cong BW(k)\), the Brauer-Wall group of the base field \(k\). These relations however cannot be generalized directly to all finite-dimensional triangular Hopf algebras because \(H^2_L(H)\) does not seem to be invariant under Drinfeld twists but only under cocycle twists. However, an analysis of the cases in the literature where \(H^2_L(H)\) has been computed shows that, for instance, for all finite-dimensional triangular Hopf algebras \(H\) over \(k = \bar{k}\) of characteristic zero, for which the \(G\)-action is faithful, \(H^2_L(H^*)\) is a direct summand of \(BM(k, H, R)\).
We relate the computation of $BM$ with the exact sequence in [40] which generalizes sequences in [2], [15], [37]. It involves the Brauer group $BC(k, H, R) \cong BM(k, H^*, R)$ of a dual quasitriangular Hopf algebra $(H, R)$ and a group of quantum commutative biGalois objects. The results contained in this paper should facilitate the interpretation of the results in [40] and their comparison should serve as an indication for handling the quasitriangular case.

In order to illustrate the preceding results, we deal explicitly with the case of the modified supergroup algebras over $\mathbb{C}$ with (an extension of) the Weyl group corresponding to an irreducible root system $\Phi$ as a group datum and its standard representation as a representation datum. Here, explicit computations are simpler because the representation $V$ of $G$ is faithful and irreducible, the classical Schur multiplier of all such groups is known, and, due to the particular presentation of $G$ as a Coxeter group, most of the different cases occurring in the general case coincide.

1 Preliminaries

In this Section we shall introduce the basic notions that we shall use in this paper. Unless otherwise stated, $H$ shall denote a finite-dimensional Hopf algebra over the base field $k$ with product $m$, coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and antipode $S$. We shall always assume that $\gcd(\dim H, \text{char}(k)) = 1$. Unadorned tensor product will be intended to be over $k$. The symbol $u$ shall always denote a central element of a group $G$ for which $u^2 = 1$ and $U$ shall denote the subgroup of $G$ generated by $u$.

1.1 Modified supergroup algebras

In this subsection we shall recall the notion of a modified supergroup algebra and its importance within the theory of finite-dimensional triangular Hopf algebras.

Definition 1.1 A finite dimensional triangular Hopf algebra $H$ with $R$-matrix $R$ is called a modified supergroup algebra if there exist:

- a finite group $G$,
- a central element $u$ of $G$ with $u^2 = 1$,
- a linear representation of $G$ on a finite-dimensional vector space $V$ on which $u$ acts as $-1$,

such that: $H \cong k[G] \ltimes \wedge V$ as an algebra; the elements in $G$ are grouplike; the elements in $V$ are $(u, 1)$-primitive; $R = R_u = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)$. 

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This name is motivated by the fact that such an Hopf algebra is obtained from a cocommutative Hopf superalgebra through a suitable modification (see [30] where it is called bozonisation, or [18], [19]) and by the description, in [27, Theorem 3.3], of all finite-dimensional cocommutative Hopf superalgebras over $\mathbb{C}$.

Modified supergroup algebras are the model for finite-dimensional triangular Hopf algebras, at least over “good fields”.

**Theorem 1.2** ([11, Theorem 5.1.1], [20, Theorem 4.3]) Every finite-dimensional triangular Hopf algebra over an algebraically closed field of characteristic zero is the Drinfeld twist of a modified supergroup algebra.

For the well-known notion of a Drinfeld twist the reader is referred to [11, Section 2.4]. If $H$ satisfies the conditions of Theorem 1.2 then it is the twist of $k[G] \ltimes V$ for some $G$ with central $u$, such that $u^2 = 1$ and for some representation $V$ on which $V$ acts as $-1$. We shall call $G$ the group datum and $V$ the representation datum corresponding to $H$.

**Example 1.3** If $G = \mathbb{Z}_2$ then $k[G] \ltimes V$ is isomorphic to the Hopf algebra usually denoted by $E(n)$ with $n = \dim(V)$. We recall that $E(n)$ is generated by $c$ and $x_i$ for $i = 1, \ldots, n$ with relations $c^2 = 1, cx_i + x_ic = 0, x_ix_j + x_jx_i = 0, \forall i, j$, coproduct $\Delta(c) = c \otimes c, \Delta(x_i) = 1 \otimes x_i + x_i \otimes c$ and antipode $S(c) = c, S(x_j) = cx_j$. We shall use the isomorphism given by $c \mapsto u$ and $x_i \mapsto uv_i$, where $v_1, \ldots, v_n$ is a basis of $V$. It is well-known that $E(n)$ has a family of triangular $R$-matrices parametrized by $S^2(V^*)$, that is, by symmetric $n \times n$ matrices with coefficients in $k$ (see [11, Proposition 2.1] and [32]). The family is given as follows. For the symmetric matrix $A = (a_{ij})$ and for the $s$-tuples $P, F$ of increasing elements in $\{1, \ldots, n\}$ we define $|P| = |F| = s$ and $v_P$ as the product of the $v_j$’s whose index belongs to $P$, taken in increasing order. A map $\eta$ from the elements of the $s$-uple $P$ to the elements of the $s$-uple $F$ determines an element of the symmetric group $S_s$ which we identify with $\eta$. We denote then by $\text{sign}(\eta)$ the sign of $\eta$. If $P$ is empty, i.e., if $s = 0$ we take $F$ to be empty and $\eta$ to have sign equal to $1$. Finally $a_{P,\eta(F)}$ denotes the product $a_{p_1,f_{\eta(1)}} \cdots a_{p_s,f_{\eta(s)}}$. If $P$ is empty we define $a_{P,\eta(F)} := 1$. Then the $R$-matrix corresponding to $A$ is

$$R_A = \frac{1}{2} \sum_P (-1)^{|P|(|P|-1)/2} \sum_{F, |F| = |P|, \eta \in S_s} \text{sign}(\eta) a_{P,\eta(F)} (v_P \otimes v_F$$

$$+ u v_P \otimes v_F + (-1)^{|P|} v_P \otimes uv_F - (-1)^{|P|} v_P \otimes uv_F).$$

It is also well-known that there exists a Hopf algebra isomorphism $\phi : E(n) \rightarrow E(n)^*$ given by $\phi(1) = \varepsilon, \phi(v_j) = v_j^* \text{ and } \phi(u) = 1^* - u^*$. This is not the
case for a general modified supergroup algebra. By self-duality, $E(n)$ has a family of dual triangular structures parametrized by symmetric matrices. The family is given by:

$$r_A = \sum_P (-1)^{|P|(|P|-1)/2} \sum_{F: |F|=|P|, \eta \in S_{|P|}} \text{sign}(\eta) a_{P,\eta}(F)((v_P)^* \otimes (v_F)^* + (v_P)^* \otimes (uv_F)^*) + (-1)^{|P|}(uv_P)^* \otimes (uv_F)^*).$$

The pair $(E(n), R_A)$ is twist equivalent to the pair $(E(n), R_0) = (E(n), R_u)$ for every $A$.

1.2 The Brauer group $BM(k, H, R)$

In this Section we shall recall the construction of the Brauer group of the category $H M$ of left modules for a quasitriangular Hopf algebra (see [6], [34], [38] for further details). Let $(H, R)$ be a finite-dimensional quasitriangular Hopf algebra over the base field $k$. Let us write $R = \sum R_{(1)} \otimes R_{(2)}$. We recall that $H M$ is a braided monoidal category with braiding given, for all $H$-modules $M, N$ and for all $m \in M$ and $n \in N$, by

$$\psi_{MN} : M \otimes N \to N \otimes M, m \otimes n \mapsto \sum (R_{(2)} \cdot n) \otimes (R_{(1)} \cdot m)$$

for all $m \in M, n \in N$.

Given two $H$-module algebras $A$ and $B$ their braided product $A \sharp B$ is the $H$-module algebra with $A \otimes B$ as underlying $H$-module and multiplication given by

$$(a \sharp x)(b \sharp y) = a \psi_{BA}(x \otimes b)y = \sum a(R_{(2)} \cdot b)\sharp(R_{(1)} \cdot x)y,$$

for all $a, b \in A, x, y \in B$. Given a $H$-module algebra $A$ its $H$-opposite algebra $\overline{A}$ is the $H$-module algebra equal to $A$ as an $H$-module and with multiplication given by $ab = m_A \psi_{AA}(a \otimes b) = \sum (R_{(2)} \cdot b)(R_{(1)} \cdot a)$ for all $a, b \in A$.

The endomorphism algebra $\text{End}(M)$ of a finite-dimensional $H$-module $M$ is an $H$-module algebra with action

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m).$$

(1.1)

Its opposite algebra $\text{End}(M)^{\text{op}}$ is also a left $H$-module algebra once it is equipped with the $H$-action:

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m).$$
The following two maps are $H$-module algebra maps:

\[
F_1 : A \sharp A \to \text{End}(A), \quad F_1(a \sharp b)(c) = \sum a(R(2) \cdot c)(R(1) \cdot b),
\]

\[
F_2 : \overline{A} \to \text{End}(A)^{\text{op}}, \quad F_2(a \sharp b)(c) = \sum (R(2) \cdot a)(R(1) \cdot c)b.
\]

We shall say that a finite-dimensional $H$-module algebra $A$ is $H$-Azumaya (or $(H, R)$-Azumaya if more $R$-matrices are involved) if $F_1$ and $F_2$ are isomorphisms.

The set of isomorphism classes of $H$-Azumaya algebras is equipped with the equivalence relation: “$A \sim B$ if there exist finite dimensional $H$-modules $M, N$ such that $A \sharp \text{End}(M) \cong B \sharp \text{End}(N)$ as $H$-module algebras”. The quotient set $BM(k, H, R)$ inherits a natural group structure where the multiplication is induced by $\sharp$, the inverse of a class represented by $A$ is represented by $A$ and the neutral element is represented by $\text{End}(M)$ for a finite dimensional $H$-module $M$. If $(H, R)$ is triangular then $BM(k, H, R)$ is abelian.

The Brauer group $BM(k, H, R)$ embeds naturally into $BM(k, D(H), \mathcal{R})$, that is, into the Brauer group of the Drinfeld double of $H$. This group is also called the full Brauer group of $H$ and it is usually denoted by $BQ(k, H)$.

For a dual quasitriangular Hopf algebra $(H', r)$ we can perform the construction dual to the construction of $BM$ obtaining the Brauer group $BC(k, H', r)$ of the category $\mathcal{M}^H$ of right $H$-comodules. In particular, if $(H, R)$ is a quasitriangular Hopf algebra then $BM(k, H, R) \cong BC(k, H^*, R)$.

We shall often make use of the following result, which is a consequence of the functoriality of the Brauer group (see [34], [38]):

**Theorem 1.4** ([38, Dual of Proposition 3.1]) The Brauer group $BM(k, H, R)$ is invariant under Drinfeld twists of $(H, R)$.

We end this Section recalling a few notions on the second lazy cohomology group of a Hopf algebra, as defined in [35] and studied in [3] and [16].

We recall that a left 2-cocycle for a Hopf algebra $H$ is a normalized convolution invertible element $\sigma$ in $(H \otimes H)^*$ such that

\[
\sum \sigma(a(1), b(1)) \sigma(a(2)b(2), c) = \sum \sigma(b(1), c(1)) \sigma(a, b(2)c(2))
\]

for every $a, b, c \in H$ and that a right 2-cocycle is a normalized convolution invertible element in $(H \otimes H)^*$ such that

\[
\sum \sigma(a(1)b(1), c) \sigma(a(2), b(2)) = \sum \sigma(a, b(1)c(1)) \sigma(b(2), c(2))
\]

for every $a, b, c \in H$. A lazy cocycle is a left 2-cocycle such that

\[
\sum \sigma(a(1), b(1)) a(2)b(2) = \sum \sigma(a(2), b(2)) a(1)b(1)
\]
Lemma describes of lazy cocycles forms a group under convolution which we shall denote by $Z^2_L(H)$ ([12 Page 227], [3, Lemma 1.2]).

A left 2-cocycle $\sigma$ in $(H \otimes H)^*$ is called a left coboundary if

$$\sigma(a,b) = \partial(\gamma)(a,b) = \sum \gamma(a_{(1)})\gamma(b_{(1)})\gamma^{-1}(a_{(2)}b_{(2)})$$

for some convolution invertible $\gamma \in H^*$. The set of left coboundaries will be denoted by $B^2(H)$. Similarly, a right cocycle $\omega$ is called a right coboundary if

$$\omega(a,b) = \sum \gamma(a_{(1)}b_{(1)})\gamma^{-1}(a_{(2)})\gamma^{-1}(b_{(2)})$$

for some invertible $\gamma \in H^*$. If $\gamma$ is lazy, that is, if $\gamma$ is central in $H^*$, then we shall say that $\sigma = \partial(\gamma)$ is a lazy coboundary. The set of lazy coboundaries forms a central subgroup of $Z^2_L(H)$ which we shall denote by $B^2_L(H)$. The factor group $Z^2_L(H)/B^2_L(H)$ is called the second lazy cohomology group of $H$ and it is denoted by $H^2_L(H)$.

One should observe that $B^2(H) \cap Z^2_L(H) \neq B^2_L(H)$ in general. The factor group $B^2(H) \cap Z^2_L(H)/B^2_L(H)$ shall play a role in the following sections.

A Hopf algebra automorphism $\psi$ is called coinner if it is of the form $\psi(a) = \text{ad}(\gamma) := \sum \gamma^{-1}(a_{(1)})a_{(2)}\gamma(a_{(3)})$ for some grouplike element $\gamma \in H^*$. The group of coinner automorphism of $H$ is denoted by $\text{CoInn}(H)$.

A Hopf algebra automorphism $\psi$ is called cointernal if it is of the form $\psi(a) = \text{ad}(\gamma) := \sum \gamma^{-1}(a_{(1)})a_{(2)}\gamma(a_{(3)})$ for some convolution invertible element $\gamma \in H^*$. The group of cointernal automorphism of $H$ is denoted by $\text{CoInt}(H)$. It is proved in [3 Lemma 1.11] that $\text{ad}(\gamma)$ is a Hopf algebra automorphism if and only if $\partial(\gamma) \in Z^2_L(H)$.

**Lemma 1.5** The groups $B^2(H) \cap Z^2_L(H)/B^2_L(H)$ and $\text{CoInt}(H)/\text{CoInn}(H)$ are isomorphic.

**Proof:** The proof follows from the proof of [3 Lemma 1.12]. Indeed, the assignment $\partial(\gamma) \mapsto \text{ad}(\gamma) \circ \text{CoInn}(H)$ is a well-defined surjective group morphism $B^2(H) \cap Z^2_L(H) \to \text{CoInt}(H)/\text{CoInn}(H)$. Its kernel consists of those $\partial(\gamma)$ for which $\gamma = \gamma_1 * \gamma_2$ with $\gamma_1$ a lazy cochain and $\gamma_2$ an algebra morphism $H \to k$. In other words, the kernel is $B^2_L(H)$.

We shall denote the quotient $\text{CoInt}(H)/\text{CoInn}(H)$ by $K(H)$. The following Lemma describes $K(H)$ when $H$ is a modified supergroup algebra.
Lemma 1.6  Let $H$ be the modified supergroup algebra corresponding to the group $G$, the representation $V \neq 0$, and the central element $u$. Then $\text{CoInt}(H) \cong \mathbb{Z}_2$, its non-trivial element is conjugation by $u$ and $K(H) = 1$ if and only if there exists a morphism $\chi: G \rightarrow k'$ with $\chi(u) = -1$.

Proof: Let $\text{ad}(\gamma) \in \text{CoInt}(H)$ with $\gamma(1) = 1$. Then $\text{ad}(\gamma)(g) = g$ for every $g \in H$. Besides, $\text{ad}(\gamma)(gh) = g\text{ad}(\gamma)(h)$ and $\text{ad}(\gamma)(hg) = \text{ad}(\gamma)(hg)$ for every $h \in H$ and every $g \in G$. When $g = u$ this implies that the $\mathbb{Z}_2$-grading induced by conjugation by $u$ must be preserved by $\text{ad}(\gamma)$. Therefore, since for every $v \in V$ we have $\text{ad}(\gamma)(v) = \gamma^{-1}(u)\gamma(v)(u - 1) + \gamma^{-1}(u)v$, we necessarily have $\gamma^{-1}(u)\gamma(v)(u - 1) = 0$ so $\gamma(v) = 0$ and $\text{ad}(\gamma)(v) = v\gamma^{-1}(u)$ for every $v$.

Similarly, $\text{ad}(\gamma)(uv) = \gamma((uv)(1 - u) + \gamma(u)uv$ must be odd whence $\gamma(uv) = 0$ and $\text{ad}(\gamma)(uv) = uv\gamma(u)$. On the other hand, $\text{ad}(\gamma)(uv) = u\text{ad}(\gamma)(v)$ implies that $\gamma(u) = \eta$ with $\eta \in \{\pm 1\}$ and necessarily $\text{ad}(\gamma)(gv_1 \cdots v_n) = \eta^n gv_1 \cdots v_n$, that is, $\text{CoInt}(H) \subset \{\text{id, conjugation by } u\}$. If we choose representatives $\bar{g}$ for the classes of $G/U$ with $1$ representing $U$ we can define $\gamma(\bar{g}) = 1$, $\gamma(u\bar{g}) = -1$, and $\gamma(h) = 0$ if $h \not\in k[G]$. Then, conjugation by $u$ is indeed $\text{ad}(\gamma)$ so that $\text{CoInt}(H) \cong \mathbb{Z}_2$ and we have the first statement. Besides, $K(H)$ is trivial if and only if conjugation by $u$ is $\text{ad}(\chi)$ for some algebra morphism $\chi: H \rightarrow k$. This happens if and only if there exists an algebra morphism $\chi: H \rightarrow k$ with $\chi(u) = -1$. \hfill \Box

2  The Brauer group $BM(k, k[G], R_u)$

Let $(H, R)$ be a finite dimensional semisimple co-semisimple triangular Hopf algebra. By [18] if $k$ is algebraically closed $(H, R)$ is twist equivalent to $(k[G], R_u)$ for some finite group $G$ and some central element $u \in G$ with $u^2 = 1$. For this reason, we shall devote this Section to the computation of $BM(k, k[G], R_u)$.

If $u = 1$, then $R_u = 1 \otimes 1$ so by [29] Theorem 1.12] we have: $BM(k, k[G], 1 \otimes 1) \cong Br(k) \times H^2(G, k^\times)$. If $u \neq 1$ the role of $H^2(G, k^\times)$ will be played by the group which is the subject of next subsection.

2.1  The group $H^2_k(G, k^\times)$

For a finite group $K$ let $Z^2(K, k^\times)$ (resp. $B^2(K, k^\times)$, resp. $H^2(K, k^\times)$) denote the group of 2-cocycles of $K$ (resp. the group of 2 coboundaries of $K$, resp. the second cohomology group of $K$ with trivial coefficients).

For any pair of finite groups $L$ and $K$, let $P(L, K; k^\times)$ be the group of pairings
of $L$ and $K$ in $k^*$, i.e., the group of maps $\beta: L \times K \to k^*$ such that

$$\beta(ab, c) = \beta(a, c)\beta(b, c) \text{ and } \beta(a, cd) = \beta(a, c)\beta(a, d)$$

(2.1)

for every $a, b \in L$ and every $c, d \in K$. It is well-known that $P(L, K; k^*) \cong \Hom(K, \Hom(L, k^*)) \cong \Hom(L, \Hom(K, k^*))$.

Let $G, u \neq 1$, $U$ be as in the previous section. The assignment $\theta: Z^2(G, k^*) \to P(G/U; k^*)$ defined by $\theta(\sigma)(g, u^t) = \sigma(g, u^t)\sigma^{-1}(u^t, g)$ for every $g \in G$ and $t \in \{0, 1\}$, is a group morphism inducing a group morphism

$$\theta: H^2(G, k^*) \to \Hom(G, \Hom(\mathbb{Z}_2, k^*)) \cong \Hom(G, \mathbb{Z}_2)$$

(see [24, Lemma 2.2.6]). Hence, each cohomology class $\bar{\sigma}$ of $G$ determines a $\mathbb{Z}_2$-grading on $k[G]$: the element $g$ is even if $\theta(\sigma)(g, u) = 1$ and odd if $\theta(\sigma)(g, u) = -1$. For all such gradings $u$ is always even. By (2.1) with $b = u^t, c = u^s$, we also see that the image of $\theta$ lies in $P(G/U, U; k^*) \cong \Hom(G/U, \mathbb{Z}_2)$.

We shall denote by $|g|_\sigma$ the degree induced by $\sigma$ of the element $g$, with values in $\{0, 1\}$. We point out that for $\sigma, \omega \in Z^2(G, k^*)$ we have

$$(-1)^{|\sigma||\omega|} = \theta(\sigma \ast \omega)(g, u) = \theta(\sigma)(g, u)\theta(\omega)(g, u) = (-1)^{|\sigma|+|\omega|}.$$  

(2.2)

Let $Reg^2(G, k^*)$ denote the group of 2-cocycles on $G$.

**Proposition 2.1** The assignment $\sharp: Z^2(G, k^*) \times Z^2(G, k^*) \to Reg^2(G, k^*)$ given by $(\sigma \sharp \omega)(g, h) := (\sigma \ast \omega)(g, h)(-1)^{|\sigma||\omega|}$ for every $\sigma, \omega \in Z^2(G, k^*)$ and every $g, h \in G$, endows $Z^2(G, k^*)$ of a group structure inducing an abelian group structure on $H^2(G, k^*)$.

**Proof:** Let $\sigma, \omega \in Z^2(G, k^*)$. Then $(\sigma \sharp \omega)$ is a 2-cocycle if and only if

$$(-1)^{|\sigma||\omega|+|\sigma||\omega|+|\sigma||\omega|} = (-1)^{|\sigma||\omega|+|\sigma||\omega|+|\sigma||\omega|}.$$  

(2.3)

for every $g, h, l \in G$. The above equality holds because $|.|_\omega$ and $|.|_\sigma$ are $\mathbb{Z}_2$-gradings, hence $\sharp$ defines an operation on $Z^2(G, k^*)$.

Let $\sigma, \omega, \beta \in Z^2(G, k^*)$. Then, for every $g, h \in G$ we have:

$$((\sigma \sharp \omega) \sharp \beta)(g, h) = ((\sigma \ast \omega) \sharp \beta)(g, h)(-1)^{|\sigma||\omega|} = ((\sigma \ast \omega) \ast \beta)(g, h)(-1)^{|\sigma||\omega|+|\sigma||\omega|+|\sigma||\omega|} = ((\sigma \ast (\omega \ast \beta))(g, h)(-1)^{|\sigma||\omega|+|\sigma||\omega|+|\sigma||\omega|+|\sigma||\omega|} = ((\sigma \ast (\omega \sharp \beta))(g, h)(-1)^{|\sigma||\omega|+|\sigma||\omega|} = (\sigma \ast (\omega \sharp \beta))(g, h)$$

so $\sharp$ is associative.
The trivial cocycle $\varepsilon \times \varepsilon$ induces the trivial grading on $k[G]$. Hence,

$$(\sigma^{♯} (\varepsilon \times \varepsilon)) = (\varepsilon \times \varepsilon) = \sigma = ((\varepsilon \times \varepsilon) * \sigma) = ((\varepsilon \times \varepsilon)^{♯} \sigma)$$

and $\varepsilon \times \varepsilon$ is a neutral element for $(Z^2(G, k^{♯}), \oplus)$. Let $\sigma \in Z^2(G, k^{♯})$. Then the map $\sigma^{♯}: G \times G \to k^{♯}$ defined by $\sigma^{♯}(g, h) := \sigma^{-1}(g, h)(-1)^{|g|_{σ}|h|_{σ}}$ lies in $Z^2(G, k^{♯})$ because $\sigma^{-1}$ holds when $σ = \omega$ and because $σ^{-1}$ is a 2-cocycle. Since $\text{Im}(\phi) \subset \text{Hom}(G, \mathbb{Z}_2)$ we have $|g|_{σ} = |g|_{σ^{-1}}$ so $σ^{♯} \sigma = σ^{♯} σ = ε \times ε$ and $(Z^2(G, k^{♯}), \oplus)$ is a group.

The coboundaries induce the trivial grading because $u$ is central, hence the group product of an element in $Z^2(G, k^{♯})$ with an element in $B^2(G, k^{♯})$ coincides with the usual product. Therefore, $B^2(G, k^{♯})$ is a central subgroup of $(Z^2(G, k^{♯}), \oplus)$ and $H^2(G, k^{♯})$ inherits the group structure $\oplus$.

Let now $σ, σ' \in Z^2(G, k^{♯})$. Then a priori $σ^{♯} σ' \neq σ'^{♯} σ$. However, if we define $γ: G \to k^{♯}$ by $γ(σ) := (-1)^{|g|_{σ}|h|_{σ}}$ then $γ(σ) γ(h) γ^{-1}(gh) = (-1)^{|g|_{σ}|h|_{σ} + |g|_{σ} |h|_{σ}}$ so $σ^{♯} σ' = σ^{♯} σ = ε \times ε$ and $(H^2(G, k^{♯}), \oplus)$ is abelian.

From now on we shall denote the group $(H^2(G, k^{♯}), \oplus)$ by $H^2(G, k^{♯})$.

**Example 2.2** Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $x = u$ and $y$. Then $H^2(G, k^{♯}) \cong k^{♯}/(k^{♯})^2 \times k^{♯}/(k^{♯})^2 \times \mathbb{Z}_2$. Here, the first (respectively, second) copy of $k^{♯}/(k^{♯})^2$ is represented by cocycles $σ_a$ (respectively $ω_a$) such that $σ_a(x, x) = a$ (respectively $ω_a(y, y) = a$) and trivial elsewhere, for every $a \in k^{♯}$. The non-trivial class of the copy of $\mathbb{Z}_2$ is represented by the cocycle $λ(x^a y^b, x^c y^d) = (-1)^{tm}$. Then $σ_a$ and $ω_a$ induce the trivial grading for every $a$ while $λ$ induces a grading for which $x$ is even and $y$ is odd. Then, $(λ^a λ)(x^a y^b, x^c y^d) = (-1)^{sm}$ so $λ^a λ = ω_{-1}$. If $-1$ is a square in $k$ then $H^2(G, k^{♯}) \cong H^2_k(G, k^{♯})$. If $-1$ is not a square in $k$ then $H^2(G, k^{♯}) \not\cong H^2_k(G, k^{♯})$ because there exists an element in $H^2_k(G, k^{♯})$ whose order is different from 2. We shall see that this example is indeed a key example, in order to see whether the two group structures on the cohomology groups are isomorphic.

Next we shall investigate some properties of $H^2_k(G, k^{♯})$ that will be needed in the sequel. Let $π: G \to G/U$ be the natural projection. We recall that composition with $π \times π$ induces the inflation morphism $\text{Infl}: H^2(G/U, k^{♯}) \to H^2(G, k^{♯})$. The image of $\text{Infl}$ consists of cohomology classes that can be represented by doubly $U$-invariant cocycles. Composition with $π \times π$ induces also an inflation morphism $H^2(G/U, k^{♯}) \to H^2_k(G, k^{♯})$, which we again shall denote by $\text{Infl}$, because doubly $U$-invariant cocycles induce a trivial grading. The kernel and the image of $\text{Infl}$ coincide respectively with the kernel and the image of the classical inflation map.
The classical Hochschild-Serre exact sequence reads:

\[
1 \rightarrow \text{Hom}(G/U, k') \xrightarrow{\sigma \pi} \text{Hom}(G, k') \xrightarrow{\text{Res}} \text{Hom}(U, k') \xrightarrow{T} H^2(G/U, k') \xrightarrow{\text{Infl}} H^2(G, k').
\]

Here \(\text{Res}\) denotes the usual restriction and \(T: \text{Hom}(U, k') \to H^2(G/U, k')\) is the transgression morphism defined as follows: if \(\phi\) is a section \(G/U \to G\) then \(T(f)(g, h) = f(\phi(gU)\phi(hU)\phi^{-1}(ghU))\) for every \(f \in \text{Hom}(U, k')\) and every \(g, h \in G\). It follows that \(\text{Infl}\) is injective if and only if \(T\) is trivial if and only if \(\text{Im}(\text{Res}) \cong \text{Hom}(U, k') \cong \mathbb{Z}_2\) if and only if there exists a group morphism \(\chi: G \to k'\) such that \(\chi(u) = -1\). Thus, the same statement holds for \(H^2_c(G, k')\).

We recall that by [24, Theorem 2.2.7], there exists an exact sequence

\[
H^2(G/U, k') \xrightarrow{\text{Infl}} H^2(G, k') \xrightarrow{\text{res} \times \theta} H^2(U, k') \times \text{Hom}(G/U, \mathbb{Z}_2) \tag{2.4}
\]

where \(\text{res}\) is the morphism induced by restriction on cocycles.

If \(G \cong U \times G/U\) then \(\text{Infl}\) is injective and \(\text{res} \times \theta\) is surjective and split, hence

\[
H^2(G, k') \cong H^2(U, k') \times H^2(G/U, k') \times \text{Hom}(G/U, \mathbb{Z}_2). \tag{2.5}
\]

Thus, as a set, \(H^2_c(G, k') \cong H^2(U, k') \times H^2(G/U, k') \times \text{Hom}(G/U, \mathbb{Z}_2)\). Besides, an analysis of the embedding of \(H^2(G/U, k') \times H^2(U, k')\) shows that this is a central subgroup of \(H^2_c(G, k')\). In particular, the product on \(H^2_c(G, k')\) is given by the rule:

\[
(\sigma_U, \sigma_{G/U}, \chi)(\omega_U, \omega_{G/U}, \chi') = (\sigma_U \ast \omega_U, \sigma_{G/U} \ast \omega_{G/U} \ast c_{\chi, \chi'}, \chi \chi') \tag{2.6}
\]

where \(c_{\chi, \chi'}(gU, hU) := (-1)^{\chi(g)} \chi'(h)\).

If \(G \not\cong U \times G/U\) sequence (2.4) still yields an exact sequence for \(H^2_c(G, k')\):

\[
1 \rightarrow \text{Hom}(U, k') \xrightarrow{T} H^2(G/U, k') \xrightarrow{\text{Infl}} H^2(U, k') \xrightarrow{\text{res} \times \theta} H^2(G, k') \times \text{Hom}(G/U, \mathbb{Z}_2).
\]

Let us also observe that in this case \(\text{Hom}(G/U, \mathbb{Z}_2) \cong \text{Hom}(G, \mathbb{Z}_2)\) since there is no group morphism \(G \to \mathbb{Z}_2 \cong U\) taking value \(-1\) on \(u\).

The following Proposition lists some conditions on \(k\) or on \(G\) ensuring that \(H^2(G, k') \cong H^2_c(G, k')\). This allows the reader to compare the results contained in Section 2.2 with those in [14] and [15]. By construction, symmetric cocycles induce the trivial grading so \(\text{Ext}(G, k')\) can be seen as a subgroup of \(H^2_c(G, k')\).

In particular, if \(G\) is cyclic, \(H^2_c(G, k') = H^2(G, k')\). We shall consider some further cases.
Proposition 2.3 Let $G$, $u$, and $k$ be as before. Then $H^2(G, k') \cong H^2_{\ast}(G, k')$ in the following cases:

1. $-1$ is a square in $k$;

2. $G$ is nilpotent, its 2-Sylow $S_2$ is abelian and $u$ is contained in a direct product of cyclic 2-groups none of which is $\mathbb{Z}_2$;

3. $G$ is nilpotent, its 2-Sylow $S_2$ is abelian and $S_2 \cong \prod_i \mathbb{Z}_{n_i}$ with $n_i = 2^{e_i}$ and $e_i \geq 2$ for $i \geq 2$.

Proof: Let $t(k')$ denote the torsion part of $k'$. By [24 Lemma 2.3.19, Theorem 2.3.21] we have $H^2(G, k') \cong H^2(G, t(k')) \times H^2(G, k'/t(k'))$ where all classes in the second summand can be represented by symmetric cocycles. Hence, $H^2(G, k'/t(k'))$ is a subgroup of $H^2_{\ast}(G, k')$ and for all its elements $\bar{\sigma}$ and every element $\bar{\omega} \in H^2_{\ast}(G, k')$ we have $\bar{\sigma} \bar{\omega} = \bar{\sigma + \omega}$. Moreover, for every pair of cocycles $\omega$ and $\sigma$ in $Z^2(G, t(k'))$, their product $\sigma \bar{\omega}$ and their inverses still take values in $t(k')$, so $H_{\ast}^2(G, t(k'))$ is a subgroup of $H_{\ast}^2(G, k')$ and $H_{\ast}^2(G, k') \cong H_{\ast}^2(G, t(k')) \times H^2(G, k'/t(k'))$.

Let us assume that $-1 = \zeta^2$ for some $\zeta \in k$. For every morphism $\chi: G/U \to \mathbb{Z}_2$ we define $\mu: k[G] \to k'$ by

$$
\mu(g) = \begin{cases} 
1 & \text{if } \chi(g) = 1 \\
\zeta & \text{if } \chi(g) = -1 
\end{cases}
$$

and we put $|g|_\chi = 0$ if $\chi(g) = 1$ and $|g|_\chi = 1$ if $\chi(g) = -1$. Then we have:

$$
\partial(\mu)(g, h) = \mu(g)\mu(h)\mu^{-1}(gh) = \zeta|g|_\chi \zeta|h|_\chi (\zeta^{-1})|gh|_\chi = \begin{cases} 
1 & \text{if } \chi(g) = 1 \text{ or } \chi(h) = 1 \\
\zeta^2 & \text{if } \chi(g) = \chi(h) = -1 \\
(-1)^{\chi(g)\chi(h)} & \text{otherwise}
\end{cases}
$$

Thus, for every cocycle $\sigma$, the products $\sigma \ast \sigma$ and $\sigma \ast \sigma$ are cohomologous. It follows that, for every class $\bar{\sigma}$, the order of $\bar{\sigma}$ is the same in $H^2_{\ast}(G, k')$ and in $H^2(G, k')$. Since $H^2_{\ast}(G, t(k'))$ is a finite abelian group of the same order as $H^2(G, t(k'))$ and since for every $m$ the number of elements of order $m$ in both groups coincide, we have the statement in case 1.

Let us now assume that $G \cong \prod_p S_p$ is the direct product of its Sylow subgroups. By [24 Corollary 2.3.15], $H^2(G, k') \cong \prod_p H^2(S_p, k')$. The cocycle $\sigma$ representing an element in $H^2(S_p, k')$ with $p$ odd induces a trivial grading so
\[ \sigma^* \omega = \sigma \ast \omega \] for every \( \omega \in Z^2(G, k^\ast) \). Besides, the \( \ast \)-product of two cocycles representing elements in \( H^2(S_2, k^\ast) \) is represented by a 2-cocycle on \( S_2 \times S_2 \) because \( \theta(\sigma)(h, u) = 1 \) for every \( h \in \prod_{p \neq 2} S_p \). Therefore, we have a group isomorphism

\[
H^2_\ast(G, k^\ast) \cong H^2_\ast(S_2, k^\ast) \times \prod_{p \neq 2} H^2(S_p, k^\ast)
\]

and it is enough to show that \( H^2_\ast(S_2, k^\ast) \cong H^2_\ast(S_2, k^\ast) \). Let us assume that we are not in case 1 and that \( S_2 \) is abelian, so that \( S_2 \cong \prod_j \mathbb{Z}_{n_j} \) with \( n_j = 2^{e_j} \) and \( e_t \leq e_{t+1} \). Then [24] Theorem 2.3.13 implies that

\[
H^2(S_2, k^\ast) \cong \prod_j k^\ast /(k^\ast)^{n_j} \times \prod_{i \leq j} P(\mathbb{Z}_{n_i}, \mathbb{Z}_{n_j}; k^\ast).
\]

The elements in \( k^\ast /(k^\ast)^{n_j} \) are represented by symmetric cocycles for every \( j \), hence these groups are also subgroups of \( H^2_\ast(S_2, k^\ast) \). Since \(-1\) is not a square \( P(\mathbb{Z}_{n_i}, \mathbb{Z}_{n_j}; k^\ast) \cong \text{Hom}(\mathbb{Z}_{n_i}, \text{Hom}(\mathbb{Z}_{n_j}, \mathbb{Z}_2)) \cong \text{Hom}(\mathbb{Z}_{n_i}, \mathbb{Z}_2) \cong \mathbb{Z}_2 \). If we put \( \mathbb{Z}_{n_i} \cong \langle v_i \rangle \) for every \( i \), the non-trivial element in \( P(\mathbb{Z}_{n_i}, \mathbb{Z}_{n_j}; k^\ast) \) is represented by \( \beta_{ij}(v_i^e, v_j^f) = (-1)^{st} \). The corresponding cohomology class is represented by the cocycle \( \sigma_{ij} \) on \( \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_j} \) such that \( \sigma_{ij}(v_i^e v_j^f, v_i^e v_j^f) = \beta_{ij}(v_i^e, v_j^f) \) and trivial on the remaining summands. Let us determine \( \theta(\sigma_{ij}) \). The element \( u \) can be written as \( \prod_i v_i^{l_i} \) with \( l_i \in \{0, 2^{e_i-1}\} \). For every \( g = v_1^{s_1} \cdots v_m^{s_m} \) we have

\[
\sigma_{ij}(v_1^{s_1} \cdots v_m^{s_m}, u)\sigma_{ij}^{-1}(u, v_1^{s_1} \cdots v_m^{s_m}) = (-1)^{s_i l_j + l_i s_j}.
\]

If \( u \) is as in case 2 or if no summand isomorphic to \( \mathbb{Z}_2 \) occurs in \( S_2 \) then \( l_t \) is even for every \( t \), the grading is always trivial and we have the statement.

Let us then assume that \( e_1 = 1 \), \( l_1 = 1 \) and that there is one summand isomorphic to \( \mathbb{Z}_2 \) in \( S_2 \). Then \( \sigma_{jk} \ast \sigma_{rs} = \sigma_{jk} \ast \sigma_{rs} \) if \( r, s, j, k \neq 1 \) while we have

\[
\theta(\sigma_{1j})(v_1^{s_1} \cdots v_m^{s_m}, u) = (-1)^{s_i}.
\]

Thus, for \( j \leq t \) we have \( (\sigma_{1j} \ast \sigma_{1t}) = (\sigma_{1j} \ast \sigma_{1t}) \ast c_{jt} \) where

\[
c_{jt}(v_1^{s_1} \cdots v_m^{s_m}, v_1^{p_1} \cdots v_m^{p_m}) = (-1)^{s_j p_t}.
\]

When \( j \neq t \), the cocycle \( c_{jt} \) is \( \sigma_{jt} \) introduced before. If \( j = t \) and \( n_j = 2^{e_j} \) with \( e_j \geq 2 \) then \( c_{jj} \) is a coboundary. Indeed, the cochain \( \mu \) given by

\[
\mu \left( \prod_{b=1}^m v_b^{a_b} \right) = (-1)^{\binom{e_j}{2}}
\]
Proof: If \( k \) is a \( k \)-algebra, Lemma 2.5 is applicable, whose elements have all order \( \leq 2 \). Hence, it must be isomorphic to \( \prod_{i< j} P(\mathbb{Z}_{n_i}, \mathbb{Z}_{n_j}; k') \) with the usual product. Thus, \( H^2_k(S_2, k') \cong \prod_j k'/(k')^{n_j} \times \prod_i \mathbb{Z}_2 \) and we have the statement in case 3. \( \Box \\

Corollary 2.4 \ If \( k = \bar{k} \) then \( H^2_k(G, k') \cong H^2(G, k') \).

\( \Box \)

2.2 The computation of \( BM(k, k[G], R_u) \)

In this Section we shall compute \( BM(k, k[G], R_u) \) when \( u \neq 1 \).

The pull-back along the embedding \( i: k[U] \to k[G] \) induces a group morphism \( i^*: BM(k, k[G], R_u) \to BM(k, k[U], R_u) \cong BW(k) \). As a set, \( BW(k) \cong Br(k) \times k/(k')^2 \times \mathbb{Z}_2 \). The elements of \( Br(k) \) are represented by central simple algebras with trivial \( \mathbb{Z}_2 \)-grading. The non-trivial element of \( \mathbb{Z}_2 \) is represented by the algebra \( C(1) \) generated by the odd element \( x \) with relation \( x^2 = 1 \). The elements of \( k/(k')^2 \) are represented by algebras \( C(1)xC(-\alpha) \) with \( \alpha \in k/(k')^2 \), where \( C(-\alpha) \) is generated by the odd element \( y \) with the relation \( y^2 = -\alpha \) and with \( \mathfrak{z} \) as in Section 1.2. More precisely, to the class of a \( \mathbb{Z}_2 \)-graded central simple algebra \( A \) one associates: \( ([A], 1, 0) \) if its odd part \( A_1 = 0 \); \( ([A], f(u)^2, 0) \) if \( A \) is central simple and \( f: k[U] \to A \) is the map realizing the \( \mathbb{Z}_2 \)-action; \( ([A_0], \delta(A), 1) \) if \( A \) is not central simple, \( A_0 \) is its even part, \( \delta(A) = z^2 \) where \( z \in Z(A), z \notin k \) and \( z^2 \in k \) (2.8 Theorem V.3.10)). Let us analyze the image of \( i^* \).

Lemma 2.5 The following assertions are equivalent:

1. The morphism \( i^* \) is surjective;

2. The subgroup \( U \) is a direct summand of \( G \);

3. The morphism \( i^* \) is surjective and split.

Proof: If \( i^* \) is surjective then \( C(1) \) lies in the image of \( i^* \), that is, there exists a \( k[G] \)-module algebra \( A \) which is \( \mathbb{Z}_2 \)-graded central simple and isomorphic, as a \( k[U] \)-module algebra, to \( C(1)xEnd(P) \) for some \( k[U] \)-module \( P \), with \( U \)-action on \( End(P) \) induced by the \( U \)-action on \( P \) as in (1.1). Since the \( U \)-action on \( End(P) \) is strongly inner, \( C(1)xEnd(P) \cong C(1) \otimes End(P) \) as a \( k[U] \)-module algebra. An isomorphism is given by \( a \otimes F \mapsto a \otimes f(u)^n F \) if \( f \) is the algebra morphism \( k[U] \to End(P) \) realizing the \( U \)-action (2.8 Theorem IV.2.7). The centre of \( A \) is a \( k[G] \)-submodule algebra. Indeed, if \( z \in Z(A) \) then for every \( y \in A \) and every \( g \in G \)

\[
(g.z)y = (g.z)(g.g^{-1}.y) = g.(z(g^{-1}.y)) = g.((g^{-1}.y)z) = y(g.z).
\]
Besides, $Z(A)$ is isomorphic to $C(1)$ as a $k[U]$-module algebra because $A \cong C(1) \otimes \operatorname{End}(P)$ as an algebra. Thus, if $i^*$ is surjective then we can lift the $U$-action on $C(1)$ to a $G$-action. If this is possible, for every $g \in G$ we have $g.x = a_g x + b_g$ for some scalars $a_g, b_g$ and $ug.x = gu.x = -g.x$ implies that $b_g = 0$ for every $g$. In this case, $g.x = \chi(g)x$ for some algebra morphism $\chi: k[G] \to k$ with $\chi(u) = -1$. Besides, since $x^2 = 1$, the $k[G]$-module algebra condition for $g.x^2$ implies that $\chi(g)^2 = 1$. Hence if $i^*$ is surjective there exists a group morphism $\chi: G \to \mathbb{Z}_2 \cong \{\pm 1\}$ with $\chi(u) = -1$. Then $G \cong U \times G/U$ so 1 implies 2.

If 2 holds, the pull-back along the projection map $G \to U$ induces a group morphism splitting $i^*$ at the Brauer group level, so 2 implies 1 and 3. 

\[\square\]

**Remark 2.6** Let us observe that the proof of Lemma 2.5 shows also that $[C(\alpha)] \in \operatorname{Im}(i^*)$ if and only if $U$ is a direct summand of $G$.

Let us denote by $\operatorname{res}: H^2(G, k') \to H^2(U, k') \cong k'/(k')^2$ the morphism induced by the restriction map. We recall that $BW(k)$ is a central extension of $Br(k)$ by a group $Q(k)$ which is $k'/(k')^2 \times \mathbb{Z}_2$ as a set, with multiplication rule $(\bar{\alpha}, (-1)^e)(\bar{\beta}, (-1)^f) = (\alpha\beta(-1)^{e+f}, (-1)^{e+f})$ with $e, f \in \{\pm 1\}$. In particular, the central extension of $Br(k)$ by $\operatorname{res}(H^2(G, k))$ is a subgroup of $BW(k)$, which we shall denote by $Br(k) \ast \operatorname{res}(H^2(G, k'))$.

**Proposition 2.7** If $U$ is not a direct summand of $G$ the image of the map $i^*$ is $Br(k) \ast \operatorname{res}(H^2(G, k'))$.

**Proof:** Any central simple algebra $B$ with trivial $G$-action represents an element $[B]$ in $BM(k, k[G], R_u)$ whose image under $i^*$ is exactly $[B] \in Br(k)$. Hence $Br(k)$ lies in the image of $i^*$. It follows from the proof of Lemma 2.5 and Remark 2.6 that if $U$ is not a direct summand of $G$, the classes $[C(1)]$ and $[C(\alpha)]$ do not lie in $\operatorname{Im}(i^*)$. The proof of the proposition follows from the knowledge of the multiplication rules in $BW(k)$ (23 Theorem V.3.9) if we show that $[C(1) \sharp C(-\alpha)] \in \operatorname{Im}(i^*)$ if and only if $\overline{\alpha} \in \operatorname{Im}(\operatorname{res}) \subset k'/(k')^2$.

Let $[C(1) \sharp C(-\alpha)] \in \operatorname{Im}(i^*)$. Then for some $k[G]$-module algebra $B$ which is $\mathbb{Z}_2$-graded central simple, $B \cong C(1) \sharp C(-\alpha) \otimes \operatorname{End}(P)$ as $k[U]$-module algebras. As above, $B \cong (C(1) \sharp C(-\alpha)) \otimes \operatorname{End}(P)$, so $B$ is central simple (and in fact, a matrix algebra). Then the $k[G]$-action on $B$ is necessarily inner and if $f: k[G] \to B$ is the map realizing the action, $f(g)f(h) = f(gh)c(g, h)$ for some 2-cocycle $c$ for $G$. The restriction of $c$ to $U \times U$ is determined by $c(u, u) = \beta$. It is not hard to see that the (cohomology) class of $\operatorname{res}(c)$ does not depend on the choice of the representative of $[C(1) \sharp C(-\alpha)] \in BW(k)$. Let us choose then $C(1) \sharp C(-\alpha)$. The element realizing, by conjugation, the $u$-action on $C(1) \sharp C(-\alpha)$ is $x \sharp y$, where
The even part of Proposition 2.8.

$x^2 = 1$ and $y^2 = -\alpha$ are odd generators of $C(1)$ and $C(-\alpha)$, respectively. Then \((x^2y)^2 = \alpha\) is in the same class modulo \((k')^2\) as \(\beta\). Therefore, if \([C(1)\sharp C(-\alpha)] \in \text{Im}(\iota^*)\) then \(\bar{\alpha} \equiv \text{res}(c(u, u)) \mod (k')^2\) for some \(c \in \mathbb{Z}^2(G, k')\).

Conversely, let \(\bar{\alpha} \in k'/\!(k')^2\) lie in the image of \(\text{res}\). Then there exists a 2-cocycle \(c\) such that \(c(u, u) = \alpha\). We may construct as in [29 Page 567] the \(k[G]\)-module algebra \(A^c\) with underlying algebra \(\text{End}(k[G])\). We recall that the \(G\)-action on \(F \in A^c\) is given by \(g.F = e^{-1}(g^{-1}, g)f_g \circ F \circ f_{g^{-1}}\) where \(f_g(h) = c(g, h)gh\). If we prove that \(A^c\) is \(R_u\)-Azumaya, then \(i^*(\!\left[ A^c \right]\!)\) corresponds to the triple \((\![k], c(u, u), 0\!\)) \(\i.e., i^*(\!\left[ A^c \right]\!) = \![C(1)\sharp C(-\alpha)]\!\).

The algebra \(A^c\) will be \(R_u\)-Azumaya if and only if it is \(\mathbb{Z}_2\)-graded central simple.

This happens if and only if its \(\mathbb{Z}_2\)-graded centre is trivial because \(A^c\) is simple. The \(u\)-action on \(A^c\) is given by \(\!(u.F) := \frac{1}{c(u, u)}f_u \circ F \circ f_u\).

Let \(\mu: G/U \to G\) be a section for \(G \to G/U\), with \(\mu(U) = u\). A basis for the even part \(A^c_0\) of \(A^c\) is given by the elements

\[ F_{g,h} := h \otimes g^* + \frac{c(u,h)}{c(u,g)}hu \otimes (gu)^* \]

for \(g \in G\) and \(h \in \mu(G/U)\). A basis for the odd part \(A^c_1\) of \(A^c\) is given by the elements

\[ L_{g,h} := h \otimes g^* - \frac{c(u,h)}{c(u,g)}hu \otimes (gu)^* \]

for \(g \in G\) and \(h \in \mu(G/U)\).

Let \(z\) be an element of the graded centre, with homogeneous summands \(z_0 = \sum_{g,h} a_{g,h}F_{g,h}\) and \(z_1 = \sum b_{g,h}L_{g,h}\).

Commutation of \(z_0\) with \(F_{r,s}\) evaluated in \(r \in \mu(G/U)\) shows that \(a_{g,h} = 0\) unless \(h \in \{g, gu\}\) and that

\[ a_{rr} = a_{ss} \quad \text{and} \quad \frac{a_{su,s}}{c(u, su)} = \frac{a_{ru,r}}{c(u, ru)} \quad \forall r, s \in \mu(G/U) \]

so \(z_0 = a_{u,u}\text{Id} + a_{1,u,1,u} \sum_{g \in \mu(G/U)} c(u, gu)F_{gu,g}\). For \(r, s \in \mu(G/U)\) we have \(0 = z_0L_{rs} - L_{sr}z_0 = 2a_{1,u}c(u,s)su\), hence \(z_0\) is a multiple of the identity. Commutation of \(z_1\) with \(F_{r,s}\) shows that \(b_{g,h} = 0\) unless \(g \in \{h, uh\}\), that \(b_{ss} = b_{rr}\) and that \(b_{su,s} = -b_{ru,r}\). Commutation of \(z_1\) with \(F_{ru,s}\) shows that \(b_{ss} = -b_{rr}\) and that \(b_{su,s} = b_{ru,r}c(u,ru)\). Thus, \(z_1 = 0\) and we have the statement.

**Proposition 2.8** If \(G \not\cong G/U \times U\) each element in \(BM(k, k[G], R_u)\) can be represented by a central simple algebra. If \(G \cong G/U \times G/U\) each element in \(BM(k, k[G], R_u)\) can be represented by a product of the form \(B\sharp C(1)^a\) with \(B\) central simple and \(a = 0, 1\). In particular, the elements that can be represented by central simple algebras form a subgroup of \(BM(k, k[G], R_u)\).
Proof: Let $[M]$ be an element in $BM(k, k[G], R_u)$. The braiding induced by $R_u$ shows that an algebra is $(k[G], R_u)$-Azumaya if and only if it is $\mathbb{Z}_2$-graded central simple. If $U$ is not a direct summand of $G$ then $i^*([M]) = [A_\sharp C(1)\sharp C(-\alpha)]$ in $BW(k)$ for some central simple algebra $A$ with trivial action and for some scalar $\alpha$. Thus, for some $k[U]$-module $P$ we have $k[U]$-module algebra isomorphisms

$$M \cong \text{End}(P)\sharp A_\sharp C(1)\sharp C(-\alpha)$$
$$\cong \text{End}(P) \otimes (A_\sharp C(1)\sharp C(-\alpha))$$
$$\cong \text{End}(P) \otimes A \otimes (C(1)\sharp C(-\alpha))$$

so $M$ is central simple.

If $U$ is a direct summand of $G$ then $i^*([M]) = [A_\sharp C(1)\sharp C(-\alpha)\sharp C(1)^a]$ in $BW(k)$ for some central simple algebra $A$ with trivial action, some scalar $\alpha$ and some $a \in \{0, 1\}$. Thus, for some $k[U]$-module $P$ we have $k[U]$-module algebra isomorphisms

$$M \cong \text{End}(P)\sharp A_\sharp C(1)\sharp C(-\alpha)\sharp C(1)^a$$
$$\cong (\text{End}(P) \otimes A \otimes (C(1)\sharp C(-\alpha)))\sharp C(1)^a$$

so $M$ is of the required form. The elements represented by a central simple algebra are those with $a = 0$. They form a subgroup, namely the preimage of the subgroup of $BW(k)$ consisting of triples $([A], \alpha, 0)$ (for the multiplication rules in $BW(k)$, see [28, Theorem V.3.9(2)]).

We are ready to state the main results of this Section.

**Theorem 2.9** If $G \not\cong G/U \times U$ there is a short exact sequence

$$1 \longrightarrow Br(k) \longrightarrow BM(k, k[G], R_u) \longrightarrow H^2_k(G, k^\ast) \longrightarrow 1.$$ 

If $-1$ is a square in $k$ then $H^2_k(G, k^\ast)$ can be replaced by $H^2(G, k^\ast)$.

**Proof:** By Proposition 2.8 each element in $BM(k, k[G], R_u)$ can be represented by a central simple algebra $B$. The action of $G$ is inner and if $f: k[G] \rightarrow B$ is a convolution invertible map for which $g.b = f(g)b f(g)^{-1}$ then any other such map is of the form $f * \gamma$ with $\gamma$ a convolution invertible map $k[G] \rightarrow k$. Therefore the subalgebra generated by $f(g)$ for all $g \in G$ is well-defined and since $B$ is central, $f(g)f(h) = f(gh)c(g, h)$ for some $c \in Z^2(G, k^\ast)$. A different choice of the map $f$ would yield a cohomologous cocycle, so we may associate a cohomology class to $B$. If we consider a different representative for $[B]$, say, $B_\sharp \text{End}(P)$ for some $k[G]$-module $P$, then we have an algebra map $f': k[G] \rightarrow \text{End}(P)$ realizing the $k[G]$-action on $\text{End}(P)$ and

$$g.(b^\sharp F) = g.b^\sharp g.F = f(g)b f^{-1}(g)^\sharp f'(g) \circ F \circ (f'(g))^{-1}.$$
Since $f'$ is an algebra map, each $f'(g)$ is even with respect to the $\mathbb{Z}_2$-grading induced on $\text{End}(P)$ by the action of $u$. Then for every $g, h \in G$ the inverse of $\bar{f}(h)\sharp f'(g)$ in $B\sharp \text{End}(P)$ is $f(h)^{-1} \sharp f'(g)^{-1}$. Besides, since

$$f(g)f(u) = f(gu)c(g, u) = f(ug)c(u, g)\theta(c)(g, u) = f(u)f(g)\theta(c)(g, u)$$

the parity of $g$ with respect to the $\mathbb{Z}_2$-grading induced by $c$ coincides with the parity of $f(g)$ with respect to the $\mathbb{Z}_2$-grading induced by the $u$-action on $B$ and the same holds for $f'$, $\text{End}(P)$.

Since the $u$-action is strongly inner, $B\sharp \text{End}(P) \cong B \otimes \text{End}(P)$ as an algebra, hence it is central simple and the action is again inner.

Then, for homogeneous $b \in B$ and $f \in \text{End}(P)$ we have:

$$\begin{align*}
(f(g)\sharp f'(gu|g|e))(b\sharp F) &= (f^{-1}(g)\sharp f'(gu|g|e))^{-1}(b\sharp F) \\
&= (f(g)b\sharp f'(gu|g|e))F \cdot (f^{-1}(g)\sharp f'(gu|g|e))^{-1} \\
&= (-1)^{|F||g|e}bF \cdot (f^{-1}(g)\sharp f'(gu|g|e)) \circ f' \circ (f(g)\sharp f'(gu|g|e))^{-1} \\
&= (-1)^{|F||g|e}bF \cdot (f(g)\sharp f'(gu|g|e)) - \theta \circ f \circ (f(g)\sharp f'(gu|g|e))^{-1}
\end{align*}$$

so the map $\tilde{f} : k[G] \to B\sharp \text{End}(P)$ given by $g \mapsto f(g)\sharp f'(gu|g|e)$ realizes the $G$-action. Then

$$\tilde{f}(g)\tilde{f}(h) = ((f(g)\sharp f'(gu|g|e))(f(h)\sharp f'(hu|h|e)))^{-1}$$

Thus, the cohomology class of $c$ does not depend on the choice of the representative of $[B]$ and we have a well-defined map $BM(k, k[G], R_a) \to H^2(G, k')$.

Let $[B]$ and $[C] \in BM(k, k[G], R_a)$ with maps $f$ and $s$ inducing the action and with corresponding cocycles $d$ and $e$. We wish to relate the cohomology class corresponding to $B\sharp C$ with $\bar{d}$ and $\bar{e}$.

The degree of $f(g)$ (resp. $s(g)$) in $B$ (resp. $C$) for the grading induced by the $u$-action coincides with the degree of $g$ induced by the cocycle $d$ (resp. $e$) through the map $\theta$. It is not hard to verify that

$$(f(g)f(u)|g|e\sharp s(g)s(u)|g|a)^{-1} = (-1)^{|g|e|g|a}f^{-1}(u)|g|e f^{-1}(g)|g|a s^{-1}(u)|g|a s^{-1}(g).$$
Let $b \in B$, $c \in C$ be homogeneous. Then

$$(f(g)f(u)|g|s(gs(u)|g|d) \times (b\circ c)((-1)|g|=|g|d f^{-1}(u)|g|=f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= (-1)|g|d f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= (-1)|g|d f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= (f(g)b) f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= g.bCG.$$  

hence the map $\bar{f}: k[G] \to B\sharp C$ given by $\bar{f}(g) = (f(g)f(u)|g|s(gs(u)|g|d))$ realizes the $G$-action on $B\sharp C$. The computation

$$\bar{f}(g)\bar{f}(h) = (f(g)f(u)|g|s(gs(u)|g|d)) (f(h)f(u)|g|s(u)|g|d)$$

$$= (-1)|g|d f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= (-1)|g|d f^{-1}(g)_{s}(s(u)|g|d) - 1(g))$$

$$= (dCG)(g,h)\bar{f}(gh)$$

shows that the assignment $\zeta: BM(k,[G],R_u) \to H^2_{\sharp}(G,k)$ defined by $[B] \mapsto \bar{d}$ is a group morphism. The construction of $\bar{C}$ in the proof of Proposition 2.7 shows that $\zeta$ is surjective. The kernel of the morphism $\zeta$ contains $Br(k)$ because every central simple algebra with trivial $G$-action represents an element in the kernel of $\zeta$. Let $[B] \in \text{Ker}(\zeta)$. Then $B$ is central simple, with strongly inner $G$-action. Let us consider the algebra $B\sharp_{\text{triv}}^{op}$, which is isomorphic to $B^{op}$ as an algebra, and with trivial $G$-action. Then, as algebras,

$$B\sharp_{\text{triv}}^{op} \cong B \otimes B\sharp_{\text{triv}}^{op} \cong B \otimes B^{op} \cong \text{End}(B).$$

Besides, if $f$ is the algebra map realizing the $G$-action on $B$, then $g \mapsto (b\circ b') = (f(g)f)(f(g)_{s}(s(u)|g|d))$ so $B\sharp B^{op}_{\text{triv}}$ is a matrix algebra with strongly inner $k[G]$-action. Thus, $[B] = [B^{op}_{\text{triv}}]^{-1} = [B_{\text{triv}}]$ can be represented by a central simple algebra with trivial action and $\text{Ker}(\zeta) = Br(k)$. The last statement follows from Corollary 2.4.

\[\square\]

**Theorem 2.10** If $G \cong U \times G/U$ there is a short exact sequence

$$1 \to Br(k) \to BM(k,[G],R_u) \to Q(k,G) \to 1$$

where $Q(k,G)$ is the group with underlying set

$$H^2_{\sharp}(G,k') \times \mathbb{Z}_2 \cong H^2(G/U,k') \times \text{Hom}(G/U,\mathbb{Z}_2) \times k'/k)^2 \times \mathbb{Z}_2$$
and with multiplication rule
\[
(\sigma_{G/U}, \chi, t, (−1)^s)(\omega_{G/U}, \chi', s, (−1)^f)
= (\sigma_{G/U} * \omega_{G/U} * c_{\chi \chi'}, \chi \chi', st(−1)^{e+f}, (−1)^{e+f})
\]
with \(c_{\chi \chi'}(g, h) = (−1)^{x(g)x'(h)}\).

If \((-1)\) is a square in \(k\) we may replace \(Q(k, G)\) by \(H^2(G, k') \times \mathbb{Z}_2\).

**Proof:** By Proposition 2.8 each class of \(BM(k, k[G], R_u)\) is represented by some product \(B^2C(1)\) with \(B\) central simple and \(a \in \{0, 1\}\). As in the previous case we may associate a cohomology class of \(G\) to each class represented by a central simple algebra. Let us define the map
\[
ζ: BM(k, k[G], R_u) \to Q(k, G)
[B^2C(1)^a] \mapsto (σ_B, (−1)^a)
\]
where \(σ_B\) is the cohomology class associated with \(B\). This is a well-defined map because the cohomology class, centrality and simplicity of an algebra do not depend on the choice of the representative.

The map \(ζ\) is a group morphism. Indeed, if two classes and are represented by a central simple algebra then the image lies in \(H^2(G, k') \times 1\) and the proof is as in Theorem 2.9. If we have \(B\) and \(C^2C(1)\), with \(B\) and \(C\) central simple then \(B^2C^2C(1)\) is not central simple. Then
\[
ζ([B^2C^2C(1)]) = (σ_{B^2C}, −1) = (σ_B * σ_C, −1) = ζ([B])ζ([C^2C(1)])
\]
follows as in Theorem 2.9. If we are given \(B^2C(1)\) and \(C^2C(1)\) with \(B\) and \(C\) central simple, then \(A := B^2C(1)C^2C(1) \cong B^2C^2C(1)\) is central simple by 2.9. If \(f\) is the map realizing the action on \(B^2C\) then \(f(u)x(y)\) with \(x, y\) homogeneous generators of \(C(1)\) realizes the \(u\)-action on \(A\), while \(f(g)x\) realizes the \(g\)-action on \(A\) for every \(g \in G/U\). Therefore if \(σ_{B^2C} = (σ_{G/U}, χ, α)\) then \(σ_A = (σ_{G/U}, χ, −α)\). Hence \(ζ([B^2C(1)]C^2C(1)]) = ζ([B^2C(1)])ζ([C^2C(1)])\) and \(ζ\) is a group morphism. As is Theorem 2.9 it is not hard to show that \(Ker(ζ) \cong Br(k)\). Let \((σ, (−1)^α) \in Q(k, G)\). Then \(A^αC(1)^e\) represents an element in \(BM(k, k[G], R_u)\) whose image under \(ζ\) is \((σ, (−1)^ε)\).

Hence \(ζ\) is bijective and we have the first statement. The second statement follows from Corollary 2.4. □

**Remark 2.11** If \(G \cong U \times G/U\) then \(BM(k, k[G], R_u)\) can also be given as a central extension of \(Br(k)\) with a group isomorphic to \(H^2(G/U, k') \times H^2(U, k') \times Hom(G, \mathbb{Z}_2)\) as a set and with multiplication rule
\[
(σ_U, σ_{G/U}, χ)(ω_U, ω_{G/U}, χ') = (σ_U * ω_U, σ_{G/U} * ω_{G/U} * c_{χ \chi'}, χ \chi') \quad (2.7)
\]
where \(c_{χ \chi'}(g, h) := (−1)^{x(g)x'(h)}\).
In Section 5 we shall compare the result in this and the following sections with classical exact sequences involving Brauer goups and groups of Galois objects.

It follows from [18, Theorem 5.1, Theorem 5.2, Corollary 6.2] that all semi-simple and cosemi-simple triangular Hopf algebras over an algebraically closed field are Drinfeld twists of group algebras with $R = 1 \otimes 1$ or $R = R_u$ for some central element $u$ of order 2 in $G$. This enables us to state the following theorem.

**Theorem 2.12** Let $(H, R)$ be a semi-simple and cosemi-simple triangular Hopf algebra over $k = \bar{k}$. Then

$$BM(k, H, R) = \begin{cases} H^2(G, k') \times \mathbb{Z}_2 & \text{if } u \neq 1, \ G \cong U \times G/U; \\ H^2(G, k') & \text{otherwise} \end{cases}$$

where $G$ and $U = \langle u \rangle$ are the group data corresponding to $H$.

### 2.3 The multiplication rules

So far we have described $BM(k, k[G], R_u)$ with $u \neq 1$ as a central extension of the classical Brauer group $Br(k)$ of the base field $k$. Our next aim is the explicit description of the multiplication rules for $BM(k, k[G], R_u)$.

We shall deal with the cases $G \not\cong U \times G/U$ and $G = U \times G/U$ separately.

- **Let $G \not\cong U \times G/U$.** In this case each element in $BM(k, k[G], R_u)$ can be written as a product of a central simple algebra with trivial $G$-action and an algebra of the form $A^\sigma$. We will first describe how to do so. Let $[A] \in BM(k, k[G], R_u)$, let $\zeta$ be as in Theorem 2.9 and let $\zeta([A]) = \sigma$. Then $[A^\sigma_A A^\sigma] \in \text{Ker}(\zeta) = Br(k)$ and we can represent $[A^\sigma_A A^\sigma]$ by $(A^\sigma_A A^\sigma)_{\text{triv}}$ with the same underlying algebra and trivial $G$-action. Therefore, $[A] = [(A^\sigma_A A^\sigma)_{\text{triv}}] [A^\sigma]$ and we need only to describe the classical Brauer group class of $(A^\sigma_A A^\sigma)_{\text{triv}}$. We recall that $A^\sigma$ is a matrix algebra and it is graded central simple, so the same holds for its graded opposite. Let us observe that if conjugation by $f_u$ induces the $u$-action on $A^\sigma$, then the $u$-action on its graded opposite can again be induced by $f_u$, and $f_u^2 = \sigma(u, u)$ in both cases. If $\sigma(u, u)$ is a square, i.e., if the $U$-action is strongly inner, then $A^\sigma \cong A \otimes \mathbb{Z}_2$ so the Brauer group class of this element is the class of $A$. If $\sigma(u, u)$ is not a square $28$ Theorem IV.3.8 part (4) for $D = F$ implies that $A^\sigma$ is isomorphic to the tensor product of a matrix algebra with trivial $\mathbb{Z}_2$-grading and the quaternion algebra $\langle -\sigma(u, u), 1 \rangle$ generated by odd $x$ and $y$ subject to the relations $x^2 = -\sigma(u, u)$, $y^2 = 1$ and $xy + yx = 0$. The graded opposite $A^\sigma$ is thus isomorphic to the tensor product of a matrix algebra with
trivial $\mathbb{Z}_2$-grading and the quaternion algebra $\langle \frac{\sigma(u,u), -1}{k} \rangle$, generated by odd $x$ and $y$ subject to the relations $x^2 = \sigma(u,u)$, $y^2 = -1$ and $xy + yx = 0$. When we refer to an algebra $B$ with trivial $G$-action, we shall denote it by $B_{\text{triv}}$.

Summarizing we have:

$$[A] = \begin{cases} [A_{\text{triv}}][A^\sigma] & \text{if } \sigma(u,u) \in (k^\times)^2 \\ ([A_{\sharp}(\frac{\sigma(u,u), -1}{k}))_{\text{triv}}][A^\sigma] & \text{if } \sigma(u,u) \not\in (k^\times)^2. \end{cases}$$

Next we would like to determine the product of these elements. The only case needing an analysis is the product $[A^\sigma][A^\omega]$. We have:

$$[A^\sigma][A^\omega] = \left( [A_{\sharp}^\sigma][A^\omega_{\sharp}(\frac{\sigma\omega(u,u), -1}{k})]_{\text{triv}} \right) \left( [A^\sigma\omega] \right)$$

where for the last two equalities we have used the computations in [28, Lemma V.3.2].

- Let $G \cong U \times G/U$. In this case, if $[A] \in BM(k, k[G], R_u)$ then either $A$ is a central simple algebra, and it can be treated as in the previous case, or $A \cong B_{\sharp}C(1)$, with $B$ central simple and trivial $G/U$-action on $C(1)$. In this case we also have $[A_{\sharp}C(1)] = [B_{\sharp}C(1)] [C(1)] = [B]$. Let $A$ be not central simple and let $\zeta$ be as in Theorem 2.10. Then $\zeta([A]) = (\sigma, -1)$ for some $\sigma$ and an argument similar to the one in the previous case applied to $A_{\sharp}C(1)$ gives $[A] = \left( (A_{\sharp}C(1)_{\sharp}(\frac{-\sigma(u,u), 1}{k}))_{\text{triv}} \right) [A^\sigma][C(1)]$. The multiplication rules follow from those in the previous case once we understand the product for $[C(1)][C(1)] = [C(1)]^2$. The cocycle $\sigma$ associated with this element can be chosen to be trivial everywhere except from $\sigma(u,u) = -1$. Then we have

$$[C(1)]^2 = \left( \left( \left( \frac{1, 1}{k} \right)_{\text{triv}} \left( \frac{-1, -1}{k} \right) \right) \right) [A^\sigma] = [A^\sigma].$$

The reader is invited to compare these rules with those in [28, Theorem V.3.9], being alert that not all representatives in this paper have been chosen as in [28]. For instance, the computation of the square of the triple in [28] corresponding to $C(1)$ yields $\left( \frac{1}{k} \right)_{\text{triv}} \left( \frac{1}{k} \right)$. The first term is a matrix algebra with trivial action while the second term lies in the same class as our $A^\sigma$.
3 The Brauer group of \((k[G] \ltimes \wedge V, R_u)\)

Let \((H, R) = (k[G] \ltimes \wedge V, R_u)\) be as in Section \([1.1]\). Let us denote by \(\rho(g)\) the matrix performing the action of the element \(g\) by conjugation on \(V\). Let \(v_1, \ldots, v_n\) denote a basis for \(V\). We choose to write \(g.v_i = \sum \rho(g)_{ji}v_j\) so that \(\rho\) is a group morphism.

The triangular Hopf subalgebra \((k[G], R_u)\) is also a quotient of \(H\). The corresponding inclusion and projection maps \(\iota\) and \(\pi\) are both quasitriangular. Therefore, the pull-back \(\iota^*\) along \(\iota\) induces a surjective and split map \(BM(k, H, R_u) \rightarrow BM(k, k[G], R_u)\) which we still denote by \(\iota^*\). Since the Brauer group of a triangular Hopf algebra is abelian, \(BM(k, H, R_u) \cong BM(k, k[G], R_u) \times \text{Ker}(\iota^*)\). We shall denote the rest of this Section to the computation of \(\text{Ker}(\iota^*)\).

The Hopf subalgebra generated by \(u\) and by the \((u, 1)\)-primitive elements in \(V\) is isomorphic, as a triangular Hopf algebra, to the Hopf algebra \(E(n)\) in Example \([1.3]\) with \(R\)-matrix \(R_u = R_0\). The corresponding inclusion map \(i_E\) is a quasitriangular map and the pull-back along \(i_E\) defines a group morphism \(BM(k, H, R_u) \rightarrow BM(k, E(n), R_u)\) which we shall denote by \(i_E^*\). The group \(BM(k, E(n), R_u)\) has been computed in \([11]\) through an analysis of the kernel of the split group epimorphism \(j^*: BM(k, E(n), R_u) \rightarrow BW(k)\) induced by the restriction of the \(E(n)\)-action to a \(U\)-action. One has an isomorphism \(\chi: \text{Ker}(j^*) \rightarrow S^2(V^*)\) so \(BM(k, E(n), R_u) \cong BW(k) \times S^2(V^*)\).

The restriction of the morphism \(i_E^*\) to \(\text{Ker}(\iota^*)\) has image in the Kernel of \(j^*\) because \([A] \in \text{Ker}(\iota^*)\) if and only if \(A\) is an endomorphism algebra with strongly inner \(G\)-action and \([B] \in \text{Ker}(j^*)\) if and only if \(B\) is an endomorphism algebra with strongly inner \(U\)-action. Thus we have the following commutative diagram.

\[
\begin{array}{ccc}
1 \rightarrow & \text{Ker}(\iota^*) & \longrightarrow & BM(k, H, R_u) & \xrightarrow{\iota^*} & BM(k, k[G], R_u) \rightarrow 1 \\
\downarrow i_E^* & & & \downarrow i_E^* & & \downarrow \iota^* \\
1 \rightarrow & \text{Ker}(j^*) & \longrightarrow & BM(k, E(n), R_u) & \xrightarrow{j^*} & BW(k) \rightarrow 1 \\
\cong & \downarrow \chi & & & & \\
& & & S^2(V^*) & & \\
\end{array}
\]

We shall denote the composition \(\chi \circ i_E^*: \text{Ker}(\iota^*) \rightarrow S^2(V^*)\) by \(\delta\).

Let us recall how the map \(\chi: \text{Ker}(j^*) \rightarrow S^2(V^*)\) was defined in \([11]\). If \(A\) represents an element in \(\text{Ker}(j^*)\), it is an endomorphism algebra. The \(E(n)\)-action is thus inner and the \(U\)-action is strongly inner. The subalgebra of \(A\) generated by the images of \(E(n)\) under a map \(f\) realizing the action, satisfies the relations \(f(h)f(l) = \sum f(h_{(1)}l_{(1)})c(h_{(2)}, l_{(2)})\) for some 2-cocycle \(c\). We can make sure that
Replacing and the new cocycle \( \sigma \in g, h \) µ
If an algebra morphism. Besides, we have hence of construction for \( f \gamma \varepsilon \) v
Proof: Let us choose \( f \) so that its restriction to \( E(n) \) gives the privileged cocycle chosen in \([11]\). Let us denote the corresponding cocycle on \( H \) by \( \omega \). The restriction of \( \omega \) to \( k[G] \otimes k[G] \) is a coboundary \( \partial(\gamma) \). Let us extend \( \gamma \) trivially outside \( k[G] \)
Replacing \( f \) by \( f * \gamma^{-1} \) changes \( \omega \) into \( \omega^{\gamma} = (\gamma^{-1} \circ m) * \omega * (\gamma \otimes \gamma) \) and by construction \( \omega^{\gamma}(g, h) = 1 \) for every \( g, h \in G \). For \( g = h = u \) this shows that \( \gamma(u) = \pm 1 \). The cocycle \( \omega^{\gamma} \) satisfies:
\[
\omega^{\gamma}(u, u) = 1; \quad \omega^{\gamma}(u, v_i) = \omega^{\gamma}(v_i, u) = 0; \quad \omega^{\gamma}(u^a v_i, u^b v_j) = \omega(u^a v_i, u^b v_j).
\]
If \( f * \gamma^{-1} \) does not satisfy the second condition, we replace \( f * \gamma^{-1} \) by \( \tilde{f} = f * \gamma^{-1} \mu \) with \( \mu: H \to k \) given by \( \mu = \varepsilon + \sum_i \omega^{\gamma}(g, v_i)(gv_i)^* \). Then \( \tilde{f}(g)\tilde{f}(v) = f(gv) \) and the new cocycle \( \sigma \) is such that \( \sigma(g, v) = 0 \) for every \( g \in G \) and \( v \in V \).
Since \( \mu = \varepsilon \) on \( k[G] \) the restriction of \( f \) to \( k[G] \) coincides with \( f * \gamma^{-1} \) which is an algebra morphism. Besides, we have \( \tilde{f}(v)\tilde{f}(u) = \tilde{f}(vu) \), as it was for \( f \) and for \( f * \gamma^{-1} \) because \( \omega(u, v_i) = 0 \). The right cocycle condition gives, for every \( g, h \in G \) and every \( v \in V \):
\[
\sigma(g, hv)\sigma(h, 1) + \sigma(g, hu)\sigma(h, v) = \sigma(g, h)\sigma(gh, v)
\]
hence \( \sigma(g, hv) = 0 \). It follows that for every \( h, g \in G \),
\[
\tilde{f}(h)\tilde{f}(v_i)\tilde{f}(g) = \tilde{f}(hv_i)\tilde{f}(g) = \tilde{f}(hv, g)\sigma(h, g) + \tilde{f}(hug)\sigma(hv_i, g) = \tilde{f}(hgg^{-1}v_i g) + \tilde{f}(hug)\sigma(hv_i, g) = \sum \rho(g^{-1})_{ji}\tilde{f}(hgv_j) + \tilde{f}(hug)\sigma(hv_i, g) = \sum \rho(g^{-1})_{ji}\tilde{f}(hg)\tilde{f}(v_j) + \tilde{f}(hug)\sigma(hv_i, g).
\]
We observe that \( f(v) \) and \( f(u) \) skew-commute because \( u \) and \( v \) do so. Hence, the first term skew-commutes with \( f(u) \) and so does each summand in the last term of the chain of equalities, except from \( f(hug)\sigma(hvi,g) \), which must be zero. Since \( f(hug) \) is invertible, \( \sigma(hvi,g) = 0 \) and we have the statement. \( \square \)

Let us observe that the above Lemma is a generalization of the procedure \([11, \text{Lemma } 3.3]\). In particular, it is compatible with the choice of the realizing map used for the construction of \( \chi \) because the applied modifications do not change the value of the cocycle on the pairs \((v_i, v_j)\).

**Lemma 3.2** The group morphism \( \delta \) is injective.

**Proof:** Since \( \delta = \chi \circ i_E^* \) and \( \chi \) is injective, \( \text{Ker}(\delta) = \text{Ker}(i_E^*) \cap \text{Ker}(i_E^*) \). Hence, if \( A \) represents an element in \( \text{Ker}(\delta) \) and \( f \) is a realizing map for the action on \( A \), we may always choose \( f \) so that its restriction to \( k[G] \) is an algebra morphism and such that \( \sigma(u, v_i) = \sigma(v_i, u) = 0 \) for every \( i \). Since \([A] \in \text{Ker}(i_E^*) \), the restriction of \( \sigma \) to \( E(n) \times E(n) \) is a right coboundary \( \mu \circ m \ast (\mu^{-1} \otimes \mu^{-1}) \) for some cochain \( \mu \). Since \( \sigma(u, u) = 1 \) we have \( \mu(u) = \pm 1 \). We can always make sure that \( \mu(u) = 1 \).

Indeed, if \( \mu(u) = -1 \), we can replace \( \mu \) with \( \mu \ast (1^* - u^*) \). This is possible because \( 1^* - u^* \) is a lazy cochain and an algebra morphism and, by \([3, \text{Lemma } 1.6] \), \((\partial(\mu \ast (1^* - u^*))^{-1} = (\partial(\mu))^{-1} \). Let us extend \( \mu \) by \( \varepsilon \) outside \( E(n) \). If we replace \( f \) by \( \tilde{f} = f \ast \mu \), the new realizing map \( \tilde{f} \) is an algebra morphism when restricted to both \( k[G] \) and \( E(n) \). To the cocycle \( \sigma^\mu \) we can apply the procedure used in Lemma \([3, \text{Lemma } 3.1] \) to obtain \( \sigma^\mu(g, hv) = \sigma^\mu(gv, h) = 0 \). Such a procedure can be iterated for increasing \( l = |P| \) using \( \gamma_l := \varepsilon + \sum_{g \in G, |P| = 1} \sigma^{l-1}(g, v_P)(gv_P)^* \).

At each step we do not modify the realizing map on the part of the filtration of \( H \) given by \( \bigoplus_{j \leq l} k[G] \rhd \wedge^l V \) corresponding to \( t < l \). Therefore the \( l \)-th step ensures that \( \sigma(g, hv_P) = 0 \) for \( |P| \leq l \). Iterating it up to \( l = n \) ensures that there exists a realizing map \( \tilde{f} \) for which \( \sigma(hv_i, g) = 0 \) and \( \tilde{f}(gv_P) = \tilde{f}(g) \tilde{f}(v_P) \) for every \( g, h \in G \), every \( i \) and every \( P \). Besides, the restriction of such an \( \tilde{f} \) to \( k[G] \) and \( E(n) \) is still a morphism thus \( \tilde{f} \) is defined on \( H \) exactly by \( \tilde{f}(gv_P) = \tilde{f}(g) \tilde{f}(v_P) \). Since the defining algebra relations of \( H \) are either those in \( E(n) \), those in \( k[G] \) or those determined by \( v_i g = g(v_i g) = \sum_j \rho(g^{-1})_{ji} g v_j \), all of them are preserved by \( \tilde{f} \) because

\[
\tilde{f}(v_i) \tilde{f}(g) = \tilde{f}(v_i g) = \sum_j \rho(g^{-1})_{ji} \tilde{f}(v_j) = \sum_j \rho(g^{-1})_{ji} \tilde{f}(g) \tilde{f}(v_j)
\]

where for the first equality we have used that \( \sigma(v_i, g) = 0 \). Therefore \( \tilde{f} \) is an algebra morphism, \( A \) is a matrix algebra with strongly inner \( H \)-action, \([A] = 1 \) and \( \delta \) is injective. \( \square \)
Lemma 3.3 The Kernel of $\iota^*$ is isomorphic to $S^2(V^*)^G$, i.e., to the group of symmetric matrices that are $G$-invariant with respect to the right action $\Sigma.g = \rho(g)^t\Sigma\rho(g)$ for every $g \in G$.

Proof: Let $\Sigma$ be the symmetric matrix with coefficients $\sigma(v_i, v_j)$ determined as in [1] and let $f$ be chosen as in Lemma 3.1, so that $g.f(v_i) = f(g)(v_i)f(g)^{-1} = f(gv_i g^{-1}) = f(g.v_i)$. Then for every $g \in G$ and $i,j \in \{1, \ldots, n\}$

$$0 = f(g)(f(v_i v_j + v_j v_i))(f(g)^{-1}$$

$$= f(g)(f(v_i)f(v_j) + f(v_j)f(v_i) - 2\Sigma_{ij})f(g)^{-1}$$

$$= (g.f(v_i))(g.f(v_j)) + (g.f(v_j))(g.f(v_i)) - 2\Sigma_{ij}$$

$$= (f(g.v_i))(f(g.v_j)) + (f(g.v_j))(f(g.v_i)) - 2\Sigma_{ij}$$

$$= f(g.v_i v_j + v_j v_i) + 2\sigma(g.v_i, g.v_j) - 2\Sigma_{ij}$$

$$= 2(\rho(g)^t\Sigma\rho(g))_{ij} - 2\Sigma_{ij}$$

hence $\Sigma \in S^2(V^*)^G$. Conversely, if $\Sigma \in S^2(V^*)^G$, let us consider $\omega_{\Sigma} := r_0 * r_{-\Sigma}$ where $r_0$ and $r_{-\Sigma}$ are as in Example 1.3. By [3] Example 1.4, $\omega_{\Sigma}$ is a lazy cocycle for $E(n)$. We have, with notation as in Example 1.3

$$\omega_{\Sigma} = \sum_P (-1)^{|P|(|P|-1)/2} F_{|F|} = |P| \sum_{P \in S_{|P|}} \text{sign}(\eta)(\Sigma_{P,\eta(F)})(v_P^*) \otimes (v_F^*)$$

$$+ (uv_P^*) \otimes (v_P^*) + (-1)^{|P|}(v_P^*) \otimes (uv_F^*) + (-1)^{|P|}(uv_P^*) \otimes (uv_F^*) .$$

By direct computation

$$\omega_{\Sigma}(u^a v_P, u^{b+m} v_Q) = (-1)^{|P|} \omega_{\Sigma}(v_P, u^m v_Q)$$

(3.1)

$$\omega_{\Sigma}(v_P, v_Q) = \begin{cases} 0 & \text{if } |P| \neq |Q| \\ (-1)^{|P|/2}\det_{PQ}(\Sigma) & \text{if } |P| = |Q| \end{cases}$$

where $\det_{PQ}$ denotes the minor corresponding to the rows indexed by elements in $P$ and columns indexed by the elements in $Q$. A straightforward computation shows that

$$\omega_{\Sigma}(g.v_P, g.v_Q) = \begin{cases} 0 & \text{if } |P| \neq |Q| \\ (-1)^{|P|/2}\det_{PQ}(\rho(g)^t\Sigma\rho(g)) & \text{if } |P| = |Q| \end{cases}$$

and $G$-invariance of $\Sigma$ implies that $\omega_{\Sigma}(g.a, g.b) = \omega_{\Sigma}(a, b)$ for every $a, b \in E(n)$. Thus, $\omega_{\Sigma}$ is a $G$-invariant lazy cocycle on $k[U] \ltimes V \cong E(n)$ with $\omega_{\Sigma}(v_i, v_j) = \Sigma_{ij}$. Let us define:

$$\lambda(gv, hw) := \omega_{\Sigma}(h^{-1}v, w) \quad \forall v, w \in \wedge E(n), g, h \in G .$$

(3.2)
Then $\lambda$ is well-defined thanks to (4.1), it coincides with $\omega_\Sigma$ on $E(n) \otimes E(n)$, and it is a 2-cocycle on $k[G] \ltimes \wedge V$. Indeed, for every $a, b, c \in \wedge V, g, h, l \in G$ we have:

$$\sum \lambda((ga)_{(1)}, (hb)_{(1)})(ga)_{(2)}(hb)_{(2)} = \sum \omega_\Sigma(h^{-1}.a_{(1)}, b_{(1)})\omega_\Sigma(l^{-1}.((h^{-1}.a_{(2)})b_{(2)}), c)$$

$$= \sum \omega_\Sigma(h^{-1}.a_{(1)}, b_{(1)})\omega_\Sigma((h^{-1}.a_{(2)})b_{(2)}, l.c)$$

$$= \sum \omega_\Sigma(b_{(1)}, l.c_{(1)})\omega_\Sigma((h^{-1}.a), b_{(2)}l.c_{(2)})$$

$$= \sum \omega_\Sigma(l^{-1}.b_{(1)}, c_{(1)})\omega_\Sigma(l^{-1}.((h^{-1}.a), (l^{-1}.b_{(2)})c_{(2)})$$

$$= \sum \lambda(b_{(1)}, lc_{(1)})\lambda(a, hl(l^{-1}.b_{(2)})c_{(2)})$$

$$= \sum \lambda((hb)_{(1)}, (lc)_{(1)})\lambda(ga, (hb)_{(2)}(lc)_{(2)})$$

hence, since $\lambda(1, hw) = \omega_\Sigma(1, w) = \varepsilon(w) = \omega_\Sigma(w, 1) = \lambda(w, 1)$, $\lambda$ is a 2-cocycle for $k[G] \ltimes \wedge V$. Besides, $\lambda$ is lazy because for every $a, b \in \wedge V, g, h \in G$ we have:

$$\sum \lambda((ga)_{(1)}, (hb)_{(1)})(ga)_{(2)}(hb)_{(2)} = \sum \omega_\Sigma(h^{-1}.a_{(1)}, b_{(1)})gh(h^{-1}.a_{(2)})b_{(2)}$$

$$= \sum \omega_\Sigma(h^{-1}.a_{(2)}, b_{(2)})gh(h^{-1}.a_{(1)})b_{(1)}$$

$$= \sum \lambda(a_{(2)}, hb_{(2)})gh(h^{-1}.a_{(1)})b_{(1)}$$

$$= \sum \lambda((ga)_{(2)}, (hb)_{(2)})(ga)_{(1)}(hb)_{(1)}.$$ 

If we construct a $(k[G] \ltimes \wedge V, R_u)$-Azumaya algebra $A^\lambda$ associated to $\lambda$ as in [11] Lemma 4.8], then the $G$-action is strongly inner because $\lambda(g, h) = 1$ for every $g, h \in G$, so $[A^\lambda] \in \text{Ker}(i^*)$. The restriction of $\lambda$ to $(k[U] \ltimes \wedge V)^{\otimes 2}$ is a 2-cocycle with $\lambda(u, u) = 1, \lambda(u, v_i) = \lambda(v_i, u) = 0$ and $\lambda(v_i, v_j) = \omega_\Sigma(v_i, v_j) = \Sigma_{ij}$ (that is, its restriction to $E(n)$ is lazy cohomologous to the privileged $\sigma$ with $\sigma(v_i, v_j) = \Sigma_{ij}$ in [11]). Hence $\delta(\text{Ker}(i^*)) = S^2(V^*)^G$ and we have the statement. 

**Remark 3.4** The reader is invited to compare formula (3.2) with [16] Formula (4.20)) where a different language for an analogous construction is used.

Combining all results in this Section we have the following theorems.

**Theorem 3.5** Let $(k[G] \ltimes \wedge V, R_u)$ be a modified supergroup algebra. Then

$$BM(k, k[G] \ltimes \wedge V, R_u) \cong BM(k, k[G], R_u) \times S^2(V^*)^G.$$ 

**Theorem 3.6** Let $(H, R)$ be a finite-dimensional triangular Hopf algebra over $k = \overline{k}$ with $\text{char}(k) = 0$. Then

$$BM(k, H, R) = \begin{cases} H^2(G, k^*) \times \mathbb{Z}_2 \times S^2(V^*)^G & \text{if } u \neq 1 \text{ and } G \cong U \times G/U \\ H^2(G, k^*) \times S^2(V^*)^G & \text{otherwise} \end{cases}$$

where $G, u, U = \langle u \rangle, V$ are the corresponding data for $(H, R)$.
The multiplication rules in Section 2.A together with the relation \([A^\sigma][A^\omega] = [A^{\sigma \ast \omega}]\) for representatives \(\sigma\) and \(\omega\) of elements in \(H^2_k(E(n))\), provide the multiplication rules for \(BM(k, k[G] \ltimes \land V, R_u)\). In this case, if \(\delta([A^\sigma]) = \Sigma\) and \(\delta([A^\omega]) = \Omega\) then \(\delta([A^{\sigma \ast \omega}]) = \Sigma + \Omega\).

4 Brauer groups and lazy cohomology

In this Section we shall apply some arguments similar to those in [16] in order to compute \(H^2_k(H)\) for \(H = k[G] \ltimes \land V\). We shall avoid the language of Yetter-Drinfeld modules. Indeed, in this particular case not so much ado is needed because \(\land V\) and \(E(n)\) are not too different from each other. When \(V = 0\) we have \(H = k[G]\) and lazy cohomology coincides with usual group cohomology. We shall assume that \(V \neq 0\).

Let \(\sigma\) be a 2-cocycle for \(H\). Then its restrictions \(\sigma_E\) and \(\sigma_G\) to \(E(n)^{\otimes 2}\) and to \(k[G]^{\otimes 2}\), respectively, are both 2-cocycles. If \(\sigma\) is lazy then \(\sigma_E\) is lazy. If \(\sigma\) is a coboundary then \(\sigma_E\) will be a coboundary, too. If \(\sigma\) is a lazy 2-cocycle for \(H\), then the lazy condition applied to \(g\) and \(hv\) (respectively, \(hv\) and \(g\)) shows that \(\sigma_G(g, hu) = \sigma_G(g, h)\) and \(\sigma_G(hu, g) = \sigma_G(h, g)\) for every \(g, h \in G\). Therefore \(\sigma_G\) is doubly \(U\)-invariant and it determines a cocycle \(\sigma_{G/U}\) for \(G/U\). Besides, if \(\sigma\) is a lazy coboundary, then \(\sigma = \partial(\gamma)\) for some \(\gamma\) with \(ad(\gamma) = \text{id}\). Then \(ad(\gamma)(gv) = gv\) implies that \(\gamma(gu) = \gamma(g)\), so \(\sigma_{G/U}\) is a coboundary for \(G/U\). Thus, the assignment \(\sigma \mapsto (\sigma_E, \sigma_{G/U})\) induces a group morphism \(\text{Bires}: H^2_k(H) \to H^2_k(E(n)) \times H^2(G/U, \mathbb{C})\). The rest of this section will be devoted to the analysis of this morphism.

**Lemma 4.1** The morphism \(\text{Bires}\) is injective.

**Proof:** For every lazy 2-cocycle \(\sigma\) for \(H\) we consider the biceleft extension \(k_\mu \ast H\) with cleaving map \(f\). We shall show that if \(\sigma\) represents an element in \(\ker(\text{Bires})\), then we may replace \(f\) with an algebra map. This will imply that \(\sigma\) is a coboundary. Then we shall show that we can choose it to be a lazy coboundary. Since \(\sigma_{G/U}\) is a coboundary \(\partial(\gamma)\), the cochain which is trivial outside \(k[G]\) and defined by \(\gamma\pi\) on \(k[G]\), with \(\pi\) the natural projection on \(G/U\), is lazy. Hence we might as well replace \(\sigma\) by \(\sigma^{(\gamma\pi)^{-1}} = ((\gamma\pi)^{-1} \otimes (\gamma\pi)^{-1}) \ast \sigma \ast (\gamma\pi) \circ m\) without changing its restriction \(\sigma_E\) to \(E(n)^{\otimes 2}\). Therefore we may always assume that the restriction of \(\sigma\) to \(G\) is trivial so that the restriction of the cleaving map \(f\) to \(k[G]\) is an algebra map. Besides \(\sigma_E = \partial(\mu)\) for some lazy cochain \(\mu\). Let us observe that, in particular, \(\mu(1) = 1\). If we extend \(\mu\) to \(H\) by \(\mu(gv) = \varepsilon(gv)\) if \(g \in G, g \notin U\), then \(\mu\) is lazy for \(H\). If we replace \(\sigma\) by \(\sigma^{(m)^{-1}} = (\mu^{-1} \otimes \mu^{-1}) \ast \sigma \ast m\) we have \(\sigma_G = \varepsilon \otimes \varepsilon\) and \(\sigma_E = \varepsilon \otimes \varepsilon\). Hence, we may always assume that the restrictions of the cleaving
map \( f \) to \( k[G] \) and to \( E(n) \) are algebra maps. Arguments similar to those used to prove Lemma 3.2 show that \( \sigma \) is cohomologous to a 2-cocycle for which \( f \) is an algebra map, i.e., \( \sigma \in Z^2_{\lambda}(H) \cap B^2(H) \). By Lemma 1.3, \( \sigma = \partial(\gamma) \) with \( \gamma \) either lazy or \( \text{ad} (\gamma) \) equal to conjugation by \( u \). In the latter case, the proof of Lemma 1.3 shows that \( \text{ad} (\gamma) = \text{ad}(\zeta) \) for some \( \zeta \) trivial outside \( k[G] \). Therefore \( \gamma \circ \zeta^{-1} \) is lazy and \( \sigma \) is lazy cohomologous to \( \partial(\zeta) \). By hypothesis \( \text{Bires}(\bar{\sigma}) = \text{Bires}(\partial(\zeta)) = 1 \) so the restriction of \( \partial(\zeta) \) to \( G \) is a coboundary for \( G/U \). Thus, we may choose \( \zeta \) to be \( U \)-invariant on \( G \). This implies that \( \zeta \) is lazy for \( H \), whence the statement. \( \square \)

**Remark 4.2** We could also replace \( \text{Bires} \) by a morphism \( H^2_{\lambda}(H) \to S^2(V^*)^G \times \text{Infl}(H^2(G/U, k^\cdot)) \). The kernel of this morphism would be isomorphic to \( K(H) \). This should explain why \( K(H) \cong \text{Ker}(\text{Infl}) \) (see Section 2.1).

**Lemma 4.3** The image of \( \text{Bires} \) is \( S^2(V^*)^G \times H^2(G/U, k^\cdot) \).

**Proof:** The subgroup \( 1 \times H^2(G/U, k^\cdot) \) of \( S^2(V^*) \times H^2(G/U, k^\cdot) \) is contained in \( \text{Im}(\text{Bires}) \). Indeed, any cocycle \( \sigma \) on \( G/U \) can be extended to a lazy cocycle on \( H \) by: \( \sigma(gvp, hvQ) := \text{Infl}(\sigma)(g, h)\delta_P\theta \).

Besides, it follows from the construction of the cocycle \( \lambda \) in the proof of Lemma 3.2 that the subgroup \( S^2(V^*)^G \times 1 \) of \( S^2(V^*) \times H^2(G/U, k^\cdot) \) is also contained in \( \text{Im}(\text{Bires}) \). On the other hand, if \( \sigma \) is a lazy cocycle for \( H \) then \( \sigma_E \) is lazy cohomologous to a 2-cocycle \( c \) with \( c(v_i, v_j) \) symmetric. If \( \gamma \) is the \( E(n) \)-lazy cochain for which \( \gamma_E = c \), then \( \gamma(u) = 1 \) and we may extend \( \gamma \) on \( H \) to a lazy cochain by \( \gamma(g) = 1 \) for every \( g \in G \) and \( \gamma(t) = 0 \) if \( t \notin k[G] \cup E(n) \). Replacing \( \sigma \) by \( \sigma' \) we obtain \( \sigma(v_i, v_j) = \Sigma_{ij} \) with \( \Sigma \) symmetric. The lazyness condition on \( g \) and \( hv \) (\( h \) and \( g \), respectively) shows that we necessarily have \( \sigma(g, hv_1) = 0 = \sigma(hv_1, g) \) for every \( g, h \in G \) and for every \( i \) and a computation similar to the one in the proof of Lemma 3.2 applied to \( k^\cdot \alpha \) shows that \( \Sigma \in S^2(V^*)^G \). Thus

\[
S^2(V^*)^G \times 1 \subset \text{Im}(\text{Bires}) \subset S^2(V^*)^G \times H^2(G/U, k^\cdot).
\] (4.1)

The above inclusions yield the statement. \( \square \)

We have proved the following Theorem.

**Theorem 4.4** Let \( H = k[G] \rtimes \wedge V \). Then \( H^2_{\lambda}(H) \cong S^2(V^*)^G \times H^2(G/U, k^\cdot) \).
Remark 4.5 Combining the approach in [16] with the above description it should be possible to compute the lazy cohomology group for Radford biproducts, at least when the acting Hopf algebra is cocommutative. This is the subject of a joint forthcoming paper.

The computations above show that, even if there is a relation between lazy cohomology and the Brauer group $BM$, in general lazy cohomology is not a direct summand of $BM$, nor always a quotient of it. In fact, even when $G = \mathbb{Z}_2$ and $V = 0$, $H^2_L(k[G])$ is not a subgroup of $BM(k, k[G], R_0) \cong BW(k)$, nor of $BQ(k, k[G]) \cong BD(k, k[\mathbb{Z}_2])$, computed in [17]. On the other hand, the linear summand of lazy cohomology is always a direct summand of the Brauer group.

5 Comparison with well-known exact sequences

In this Section we compare our results with some known exact sequences involving Brauer groups and groups of Galois objects. This comparison involves lengthy but not enlightening computations, so we will skip the details whenever possible.

Let for the moment $G$ be a finite abelian group. If $k$ has enough roots of unity then $G \cong G^*$ and

$$BM(k, k[G], R) \cong BC(k, k[G]^*, R) \cong BC(k, k[G^*], R) \cong B_R(k, G^*),$$

where $B_R(k, G^*)$ denotes the Brauer group of $G^*$-graded Azumaya algebras as in [14], [15] and [26]. The first isomorphism is induced by the usual duality functor and the last one depends on the fact that a $G^*$-comodule algebra $A$ a $G^*$-graded by $x = \sum x_\chi \in G^*\chi \chi \in G^*$ if $\rho(x) = \sum x_\chi \chi \chi \otimes \chi$ and vice versa.

In [14], [15] an exact sequence

$$1 \rightarrow Br(k) \rightarrow B_R(k, G) \rightarrow \pi \rightarrow Galz_R(k, G)$$

where $Galz_R(k, G)$ is a suitable group of $G$-graded Galois objects is described. The rightmost arrow is surjective when $k$ is nice enough (e.g., when $k$ is a field, which is our case). In [31] a cohomological interpretation of the group of equivalence classes of Galois extensions with normal basis is given. In our terms, the map $B_R(k, G^*) \rightarrow Galz_R(k, G^*)$ associates to the class $[A]$ the equivalence class of the centralizer, in a suitable representative $A$, of the $G$-invariant subalgebra of $A$. The representative has to be chosen $G^*$-fully graded, in the terminology of [15]. We are concerned with the images of our classes $[A^\sigma]$ and $[C(1)]$. It is not hard to verify that $A^\sigma$ is fully graded and that $\pi([A^\sigma])$ is the class of the twisted group algebra $k_\sigma[G]$ with associated cocycle $\sigma$, i.e., the subalgebra generated by
\( f_h = f(h) \) with \( h \in G \). Since \( \rho(f_h) = \sum_{g \in G} \frac{1}{\sigma(g,h)^{-1}} f_g \circ f_h \circ f_{g^{-1}} \otimes g^* \), a direct
computation shows that \( f_h \) has degree \( \chi_h \) where \( \chi_h(g) = \sigma(g,h)\sigma^{-1}(h,g) \). By
[15] Page 311 the right \( G^* \)-action on this representative of \( \pi([A^\sigma]) \) is given by
\( (f_h \hookrightarrow \chi) = \chi(h)R_u(\chi, \chi_h)f_h = \chi(u)\chi(h) f_h \).

When \( R = R_u \) and \( G \cong U \times G/U \) the class \([C(1)]\) is an element in \( B_R(k, G^*) \).
Its representative \( C(1) \) is not fully graded because \( (G/U)^* \) acts trivially on it. Once
we replace \( C(1) \) by \( C(1) \in \text{End}(k[G^*]) \) we see that \( \pi([C(1)]) \) is the isomorphism
class of \( k[G^*] \cong k[U^*] \otimes k[(G/U)^*] \) with the usual \( \mathbb{Z}_2 \)-action on \( k[U^*] \cong C(1) \)
and the regular \( (G/U)^* \)-action on \( k[(G/U)^*] \). The grading of \( k[U^*] \) is the non-
trivial \( \mathbb{Z}_2 \)-grading while \( k[(G/U)^*] \) is trivially graded.

The exact sequence in [15] has been generalized in [2] to the case of commutative
cocommutative Hopf algebras with trivial \( R \)-matrices and in [37] to commutative
cocommutative Hopf algebras with \( R \) a bipairing on \( H^* \otimes H^* \). The most
general case is dealt with in [40] where an exact sequence
\[
1 \longrightarrow Br(k) \longrightarrow BC(k, H, R) \longrightarrow \pi \text{Gal}(H_R) \]  
\[(5.1)\]
is constructed for \((H, R)\) a dual quasitriangular Hopf algebra. Here \( H_R \) is the
braided Hopf algebra of [30] Theorem 7.4.1] and \( \text{Gal}(H_R) \) is a group of quantum
commutative biGalois objects for \((H_R)^*\). In [13] it is shown that \( \text{Gal}(H_R) \) is
invariant under cocycle twist. The sequence \((5.1)\) has relevant theoretical meaning
but \( \text{Gal}(H_R) \) might be very hard to compute. It is possible that the analysis of the
relations between \((5.1)\) and our computations in the triangular case will give an
indication on how to handle \( BM(k, H, R) \) in the general quasitriangular case.

If \((H, R)\) is dual triangular and, for instance, \( \mathcal{K} = k \), then \((H, R)\) is the Doi
twist of \((k[G] \ltimes \wedge V)^*, R_u\) for some \( G, V, u \). Zhang’s sequence becomes:
\[
1 \longrightarrow Br(k) \longrightarrow BM(k, k[G] \ltimes \wedge V, R_u) \longrightarrow \pi \psi \text{Gal}(H_R). \]  
\[(5.2)\]
Here we have used that \([A] \mapsto [A^{op}]\) defines a group isomorphism
\[
\psi: BM(k, k[G] \ltimes \wedge V, R_u) \rightarrow BC(k, (k[G] \ltimes \wedge V)^*, R_u) \]
stemming from the monoidal functor
\[
(\mathcal{D}, \tau_{UV}, \text{id}): (H \cdot M, \otimes, k, \text{id}, \text{id}, \text{id}) \rightarrow (M^H, \otimes^{rev}, k, \text{id}, \text{id}, \text{id}) \]
where \( \mathcal{D} \) is the usual duality functor from the category \( H \cdot M \) of left \( H^* \)-modules
to the category \( M^H \) of right \( H \)-comodules and \( \otimes^{rev} \) denotes the opposite tensor
product. If \((H^*, R) = (k[G] \ltimes \wedge V, R_u)\), by [30] Theorem 7.4.2] we have
\( H_R^* \cong k[G] \ltimes \wedge V \) as an algebra, with the (super cocommutative) Hopf superalgebra
structure given on generators by \( \Delta(g) = g \otimes g \) and \( \Delta(v) = v \otimes 1 + 1 \otimes v \). We
shall denote this Hopf superalgebra by $A$. An element in $\text{Ga}l(\mathcal{H}_R)$ is determined by an $H$-Yetter-Drinfeld module structure and an $A$-bimodule structure. The unit element in $\text{Ga}l(\mathcal{H}_R)$ is represented by $H^*$, with actions and coactions defined by [40] Formulas (3), (6), (11)).

The map $\pi$ is given as follows. For a class in $BC(k,H,R_u)$ we consider a representative $A$ which is Galois. This is always possible by [40] Corollary 4.2. Then, as an algebra, $\pi([A]) = C_A(A_0)$, the centralizer of the $H$-coinvariants. The Yetter-Drinfeld module structure on $C_A(A_0)$ is given by the restrictions of the Miyashita-Ulbrich-Van Oystaeyen (MUVO) action → (see [6] §2.3)) of $H$ and of the $H$-coaction $\rho$ on $A$. The $A$-bimodule structure on $C_A(A_0)$ is given by $\rho_1$ and $\rho_2$ which are dual to actions $\to$ and $\leftarrow$. These actions are modifications through the $-r$-matrix ([40] Formulas (3),(6))) and through $\rho$ of the $H$-action $\to$.

We aim at the description of the image of the classes $[A^\sigma]$ when $H^* = k[G]$ and $R = 1 \otimes 1, R_u$ and when $H^* = k[g]^\ast \wedge V$ and $R = R_u$, and the image of the class $[C(1)]$ when $H^* = k[G]$ or $H^* = k[G]^\ast \wedge V; R = R_u$ and $G \cong U \times G/U$.

Let us first assume that $V = 0$. Then $A = H^* = k[G]$ is a genuine co-commutative Hopf algebra with $G$ not necessarily abelian. Its dual $k[G]^\ast$ is the span of $\nu_h = h^\ast$ for $h \in G$ with product $\nu_h \nu_g = \delta_{h,g} \nu_h$. The unit element in $\text{Ga}l(\mathcal{H}_R)$ is represented by $k[G]$ with the regular left and right coactions $\rho_1$ and $\rho_2$ for $A; k[G]^\ast$-action given by $\nu_h \cdot g = \delta_{h,g} \nu_h$ and $k[G]^\ast$-coaction given by $\rho(g) = \sum_{h \in G} hgh^{-1} \otimes \nu_h$.

If $R = 1 \otimes 1, R_u$ then the set $H^2(G,k^\ast)$ occurs in $BM(k,k[G],R)$. The image of $[A^\sigma]$ can be computed as follows. The class of $A^\sigma$ (with product $\circ$) in $BM(k,k[G],R)$ is mapped by $\psi$ to the class of $[(A^\sigma)^{\text{op}}]$. The coinvariants in $(A^\sigma)^{\text{op}}$ correspond to the centralizer of the induced subalgebra, that is, of the algebra generated by the $f_g$’s in $A^\sigma$. As an algebra, $\pi(\psi([A^\sigma]))$ is the equivalence class of the algebra $(k[A^\sigma]^\ast)^{\text{op}}$ with product $\bullet$. The $k[G]^\ast$-comodule structure is given, for both choices of $R$, by:

$$\rho(f_h) = \sum_{g \in G} \frac{\sigma(g,h)\sigma(gh,g^{-1})}{\sigma(g,g^{-1})} f_{ghg^{-1}} \otimes \nu_g.$$  

The MUVO action is, for both choices of $R$, as follows. If $\beta$ denotes the natural map $A \otimes A_0 A \to A \otimes k[G]^\ast$ and if $(1 \otimes \nu_h) = \beta(\sum F_i(h) \otimes A_0 f_i(h))$, then

$$\delta_{gh} = \sum F_i(h) \bullet f_0(h)(f_{21}(h), g)$$
$$= \sum \frac{1}{\sigma(g,g^{-1})} F_i(h) \bullet (f_0 \circ f_i(h) \circ f_{g^{-1}})$$
$$= \sum \frac{1}{\sigma(g,g^{-1})} F_i(h) \bullet f_{g^{-1}} \bullet f_i(h) \bullet f_g$$
$$= (\nu_h \mapsto f_{g^{-1}}) \bullet f_{g^{-1}}.$$
Thus \( v_h \rightarrow f_g = \delta_{k,g^{-1}}f_g \).

If \( R = 1 \otimes 1 \) then the \( A \)-actions \( \rightarrow \) and \( \leftarrow \) coincide with the \( \rightarrow \) action. The \( A \)-comodule structures are:

\[
p_1(f_g) = f_g \otimes g^{-1}; \quad p_2(f_g) = g^{-1} \otimes f_g.
\]

If \( R = R_u \) with \( u \neq 1 \) then

\[
v_h \rightarrow f_g = f_g \leftarrow v_h = \begin{cases} \delta_{h,g^{-1}} f_g & \text{if } |g|_\sigma = 0; \\ \delta_{h,ug^{-1}} f_g & \text{if } |g|_\sigma = 1. \end{cases}
\]

The \( A \)-comodule structures are:

\[
p_1(f_g) = \begin{cases} f_g \otimes g^{-1} & \text{if } |g|_\sigma = 0; \\ f_g \otimes ug^{-1} & \text{if } |g|_\sigma = 1. \end{cases} \quad p_2(f_g) = \begin{cases} g^{-1} \otimes f_g & \text{if } |g|_\sigma = 0; \\ ug^{-1} \otimes f_g & \text{if } |g|_\sigma = 1. \end{cases}
\]

The assignment \( f_g \mapsto f_{g^{-1}} \) determines an algebra isomorphism \( k_\sigma\left[G\right]^{\text{op}} \cong k_{\sigma'}[G] \) with \( \sigma'(g,h) = \sigma(h^{-1},g^{-1}) \). Therefore we may write: \( \pi([A^g]) \cong k_{\sigma'}[G] \) as an algebra, with: comodule structure

\[
\rho(f_{g'}) = \sum_{g \in G} \sigma'(g,g^{-1}) f_{ggh^{-1}}' \otimes v_g;
\]

action \( v_h \rightarrow f_g' = \delta_{h,g} f_g' \); and \( A \)-comodule structures

\[
p_1(f_g') = f_g' \otimes g; \quad p_2(f_g') = g \otimes f_g'
\]

if \( R = 1 \otimes 1 \); and

\[
p_1(f_g') = \begin{cases} f_g' \otimes g & \text{if } |g|_\sigma = 0; \\ f_g' \otimes ug & \text{if } |g|_\sigma = 1. \end{cases} \quad p_2(f_g') = \begin{cases} g \otimes f_g' & \text{if } |g|_\sigma = 0; \\ ug \otimes f_g' & \text{if } |g|_\sigma = 1. \end{cases}
\]

if \( R = R_u \) with \( u \neq 1 \). In both cases, if \( k' = (k')^2 \) then by [24, Lemma 3.6] we may choose \( \sigma \) in its cohomology class so that \( \sigma' = \sigma^{-1} \).

When \( R = R_u \) and \( G \cong U \times G/U \), we analyze the image of \( [C(1)] \). If we take \( A = C(1)^{\text{End}(k[G]^*)} \) as a representative of this class in \( BC(k,k[G]^*,R_u) \), its image \( \pi([A]) \) is as follows. Let \( g \mapsto f_g \) for \( g \in G \) denote the map realizing the (strongly inner) \( k[G] \)-algebra action on \( \text{End}(k[G]^*)^{\text{op}} \cong D^{-1}(\text{End}(k[G]^*)) \).

The algebra \( \pi([A]) = C_A(A_0) \) is generated by the elements \( 1 \otimes f_h \) for \( h \in G/U \) and \( x \otimes f_u \). Since the usual isomorphism \( C(1)^{\text{End}(k[G]^*)^{\text{op}}} \cong C(1) \otimes \text{End}(k[G]^*)^{\text{op}} \) maps \( x \otimes f_u \) to \( x \otimes 1 \), the element \( x \otimes f_u \) is central in \( A \). Then, by [6]...
Lemma 2.3.1 (a)] the MUVO action on it is trivial. Let now $v_g = g^* \in k[G]^*$. Then the $k[G]^*$-comodule structure is given by:

$$\rho(x^*f_u) = (x^*f_u) \otimes \sum_{h \in G/U} (v_h - v_{hu})$$

$$\rho(1^*f_h) = \sum_{t \in G} (1^*f_{tgt^{-1}}) \otimes v_t.$$ 

A computation similar to the case of $[A^\sigma]$ shows that $v_h \mapsto (1^*f_g) = \delta_{h,g^{-1}}(1^*f_g)$ for $g \in G/U$. An analysis of the structure of the actions $\rightarrow$ and $\leftarrow$ in [40 Formulas (3), (6)] on the elements of $k[G/U]$ shows that they both coincide with the MUVO-action, while

$$(x^*f_u) \leftrightarrow v_h = \delta_{h,u}(x^*f_u) = v_h \wedge (x^*f_u).$$

Thus, the corresponding comodule maps are

$$\rho_1(x^*f_u) = (x^*f_u) \otimes u; \quad \rho_2(x^*f_u) = u \otimes (x^*f_u);$$

$$\rho_1(1^*f_h) = (1^*f_h) \otimes h^{-1}; \quad \rho_2(1^*f_h) = h^{-1} \otimes (1^*f_h).$$

The assignment $x^*f_u \mapsto u$ and $1^*f_h \mapsto h^{-1}$ for $h \in G/U$ determines an algebra isomorphism $\pi\psi([A]) \rightarrow k[G]$. Through this isomorphism $\pi\psi([A]) \cong k[G]$ with coregular $k[G]$-comodule structures $\rho_1$ and $\rho_2$ and with $k[G]^*$-comodule structure $\rho$ given by:

$$\rho(u) = u \otimes \sum_{h \in G/U} (v_h - v_{hu}), \quad \rho(h) = \sum_{g \in G} ghg^{-1} \otimes v_g$$

for $h \in G/U$. The $k[G]^*$-action is given by $v_h \mapsto g = \delta_{h,g}g$ for $g \in G/U$ and $v_h \mapsto u = \delta_{1,h}u$, for every $h \in G$.

Let us now assume that $V \neq 0$. Then necessarily $R = R_u$ with $u \neq 1$ and $A = k[G] \ltimes V$ is a Hopf superalgebra.

We are concerned with the image of $[A^\sigma]$. We shall separately deal with the cases: $\sigma$ is lazy, trivial on $k[G]$; $\sigma$ is any cocycle for $G$, trivial outside $k[G]$. In the first case $A^\sigma$ is isomorphic to $\text{End}(H^*)$ as an algebra and $\pi\psi([A^\sigma])$ is the opposite of the induced subalgebra of $A^\sigma$, i.e., the algebra generated by the $f_h$’s as before, for $h \in H^*$. Let us denote by $\circ$ the product in $A^\sigma$ and by $\bullet$ its opposite product. The $H$-comodule structure on $\pi\psi(A^\sigma)$ is given by

$$\rho(f_h) = \sum_{a \in H^*} f_{a_{(1)}} \circ f_h \circ f_{a_{(2)}^{-1}} \otimes a^*$$

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where the expression \( f_a^{-1} \) stands for the convolution inverse of \( f_a \). It can be proved as in the previous cases that the MUVO action on \( f_g \) for \( g \in G \) is given by \( h^* \mapsto f_g = \delta_{h^{-1},g} f_g \), for \( h^* \in H \) dual to \( h \). For the elements \( f_v \) with \( v \in V \) we have, if 
\[
\beta(\sum F_i(a) \otimes f_i(a)) = 1 \otimes a^* \text{ for } a^* \in H:\n\]
\[
\delta_{a,uv} = \sum F_i(a) \cdot f_0(a)(f_{i+1}(a),uv) = \sum F_i(a) \cdot (uv,f_i(a)) = \sum F_i(a) \cdot (f_{uv}(a) \circ f_i(a)) + \sum F_i(a) \cdot (f_i(a) \circ f_{u^{-1}}) = \sum F_i(a) \cdot f_u \cdot f_i(a) \cdot f_{uv} + \sum F_i(a) \cdot f_v \cdot f_i(a) = (a^* \mapsto f_u) \cdot f_{uv} + a^* \mapsto f_v = -\delta_{a,uv} f_v + a^* \mapsto f_v
\]
where for the fourth equality we have used the formula for \( f_h^{-1} \) in [11] Lemma 4.8 and for the last equality we have used our knowledge of the MUVO action on \( f_g \) for \( g \in G \). Thus, \( a^* \mapsto f_v = \delta_{a,uv} + \delta_{a,uf_v} \). A direct computation using that: \( \Delta(a^*) \) is a linear combination of elements \( r^* \otimes s^* \) with \( rs = a \) and that \( S \) preserves the filtration induced by powers of \( V \) shows that for every \( a \in H^* \), every \( g \in G \), and every \( v \in V \):
\[
a^* \mapsto f_g \ = f_g \otimes -a^* = \delta_{g^{-1},a} f_g; \quad a^* \mapsto f_v \ = f_v \otimes -a^* = \delta_{a,uv} + \delta_{a,1} f_v.
\]
The \( A \)-bicomodule structure is, for \( g \in G \) and \( v \in V \):
\[
\rho_1(f_g) = f_g \otimes g^{-1}; \quad \rho_1(f_v) = 1 \otimes uv + f_v \otimes 1; \quad \rho_2(f_g) = g^{-1} \otimes f_g; \quad \rho_2(f_v) = uv \otimes 1 + 1 \otimes f_v.
\]
We observe that \( \pi \psi([A^\sigma]) \) is isomorphic, as an algebra, to \( (k^*_\sigma H^*)^{op} \). The assignment \( f_h \mapsto f_{h^{-1}}^{\sigma'}(h) \) determines an algebra isomorphism \( (k^*_\sigma H^*)^{op} \rightarrow H^* \otimes k \), with right cocycle \( \sigma' \) given by \( \sigma'(a,b) = \sigma(S(a),S(b)) \). Since \( \sigma \) is lazy, \( \sigma' \) is lazy as well. Through this identification, \( \pi \psi([A^\sigma]) \) is \( k^*_\sigma H^* \) with Yetter-Drinfeld module structure
\[
\rho(f_g) = \sum a \in H^* \sigma^{-1}(Sa(3),a(4)) f_{a(2)} f_{h^{-1}(a(1))} \otimes a^*; \quad a^* \mapsto f_g = \delta_{a,g} f_g; \quad a^* \mapsto f_{uv} = \delta_{a,uv} + \delta_{a,1} f_{uv}
\]
and with \( A \)-bicomodule structure:
\[
\rho_1(f_g) = f_g \otimes g; \quad \rho_1(f_{uv}) = f_{uv} \otimes 1 + 1 \otimes uv; \quad \rho_2(f_g) = g \otimes f_g; \quad \rho_2(f_{uv}) = uv \otimes 1 + 1 \otimes f_{uv}.
\]
Let us observe that if \( \sigma \) is the privileged cocycle in [11] corresponding to the symmetric matrix \( \Sigma \) then \( \sigma' \) is the privileged cocycle corresponding to \( -\Sigma \), i.e., it represents the inverse cohomology class.
Let now $\sigma$ be a cocycle for $k[G]$ and trivial elsewhere. Since $(A^\sigma)^{op} = (\text{End}(k[G]))^{op}$ is not Galois, we consider the representative $(A^\sigma)^{op} \# \text{End}(H^{op})$ where $\text{End}(H^{op})$ is as in \([40, \text{Corollary 4.2}].\) This corresponds to the representative $A \cong ((A^\sigma)^{op} \# \text{End}(H^{op}))^{op}$ of the class $[A^\sigma]$ in $BM$. The usual flip map $\tau$ determines an isomorphism $A \cong \text{End}(H^{op})^{op} \# A^\sigma$, and $\text{End}(H^{op})^{op}$ has a strongly inner $H^*$-action. Thus, $A \cong A^\sigma \otimes \text{End}(H^{op})$ is a matrix algebra, hence the action is inner. The elements $F_h$ for $h \in H^*$ realizing the action are subject to the relations: $F_h \cdot F_k = \sum F_{h_{(1)}k_{(1)}} \sigma(h_{(2)}, k_{(2)})$. One can show that if $f_h$ (respectively, $f'_h$) for $h \in H^*$ realize the action on $A^\sigma$ (respectively, $\text{End}(H^*)$), then $F_g = f_g \# f'_g \otimes g$ and $F_v = 1 \# f'_v$. The image of $[A]$ through $\pi\psi$ is, as in the previous cases, isomorphic as an algebra to the subalgebra generated by the $F_h$'s, with opposite product. The coaction $\rho$ is the restriction of the coaction to this subalgebra; the MUVO action is given by $a^* \rightarrow F_g = \delta_{a,g}^{-1} F_g$ and $a^* \rightarrow F_v = \delta_{a,u,v} + \delta_{a,u} F_v$. The $A$-bicomodule structure behaves on $F_g$, for $g \in G$, as in the case $V = 0$ and $R = R_u$, and it behaves on $F_v$, for $v \in V$, as in the case $V \neq 0$ and $\sigma$ lazy, trivial on $k[G]$. The assignment $F_h \mapsto F_{S(h)}$ determines an algebra isomorphism from $\pi\psi([A]) \rightarrow k_{a'} H^*$, where $\sigma'$, the actions and the coactions are as in the previous cases. In particular, if $\sigma$ is lazy, i.e., if it is doubly $U$-invariant, then $|g|_{\sigma} = 0$ for every $g \in G$ and we obtain the same type of formulas that we obtained for $\sigma$ lazy and trivial on $k[G]$.

If $G \cong U \times G/U$ then we take the representative $A = C(1) \# \text{End}(H^{op})$ of the class in $BC$, with product $\circ$. It corresponds in $BM$, to the representative $A^{op} = (C(1) \# \text{End}(H^{op}))^{op}$ which is isomorphic through $\tau$, as a module algebra, to the algebra $B = \text{End}(H^{op})^{op} C(1) \cong \text{End}(H^{op})^{op} \otimes C(1)$. Let us denote by $f_h$ for $h \in H^*$ the elements realizing the $H^*$-action on $\text{End}(H^{op})^{op}$, with product $\circ$. It is not hard to verify that $A_0 = B_0$ is generated by the elements of the form: $F^{\pm 1}$ with $F$ commuting with $f_h$ for every $h \in H^*$, and $F^{\pm x}$ with $F' \equiv x$ commuting with $f_g$ for $g \in G/U$ and skew-commuting with $f_u$ and $f_v$. Thus, $\pi\psi([A])$ is generated by the elements $1 \# f_g$, for $g \in G/U$ and $x \# f_u$ in $A$. The element $x \# f_u$ is central in $A$ because it corresponds to the element $1 \otimes x$ through the above algebra isomorphisms. By \([6, \text{Lemma 2.3.1 (a)}]\), $a^* \rightarrow (x \# f_u) = \delta_{a,1} (x \# f_u)$. As in the case with $V = 0$, for every $g \in G/U$ and every $a \in H^*$ we have $a^* \rightarrow (1 \# f_g) = \delta_{a,g^{-1}} (1 \# f_g)$. The MUVO-action on the elements of the form $1 \# f_v$ is slightly more complicated to compute. Let $B(\sum F_i(a) \otimes A_0 f_i(a)) = 1 \otimes a^*$, and let $f_i(a) = 1 \# s_i(a) + x \# t_i(a)$. A computation similar to the previous cases, using that $\rho(x \# 1) = x \# 1 \otimes \chi$ with $\chi = \sum_{g \in G/U} (g^* - (ug^*)^*)$, that the $H^*$-action is strongly inner on the opposite of $\text{End}(H^{op})$, and that if $M$ and $N$ are $H$-comodules with $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ and $\rho_N(n) = \sum n_{(0)} \otimes n_{(1)}$ then $\rho_M(m \otimes n) =$
\[\sum m(0) \otimes n(0) \otimes n(1)m(1) \text{ gives} \]
\[\delta_{a,uv} = a^* \rightarrow (1 \sharp f_v) + \sum F_i(a) \bullet (1 \sharp s_i(a)) \bullet (1 \sharp f_{uv}) - \sum F_i(a) \bullet (1 \sharp f_u) \bullet (x \sharp t_i(a)) \bullet (1 \sharp f_{uv}).\]

A computation similar to the previous ones shows that
\[\delta_{a,u} = \sum F_i(a) \bullet f_i0(a)\langle f_i1(a), u \rangle \]
\[= \sum F_i(a) \bullet (1 \sharp f_u) \bullet (1 \sharp s_i(a)) \bullet (1 \sharp f_u) - \sum F_i(a) \bullet (1 \sharp f_u) \bullet (x \sharp t_i(a)) \bullet (1 \sharp f_u)\]
so, since \((1 \sharp f_{uv}) = 1 \sharp (f_u \circ f_v) = -(1 \sharp f_u) \bullet (1 \sharp f_v),\) we have
\[a^* \rightarrow (1 \sharp f_v) = \delta_{a,uv} + \delta_{a,u}(1 \sharp f_v).\]
The actions \(\rightarrow\) and \(\leftarrow\) are as follows:
\[a^* \rightarrow (x \sharp f_u) = \delta_{a,a}(x \sharp f_u) = (x \sharp f_u) \leftarrow a^*;\]
\[a^* \rightarrow (1 \sharp f_g) = \delta_{a,g^{-1}}(1 \sharp f_u) = (1 \sharp f_u) \leftarrow a^*;\]
where we used that \(\Delta(a^*)\) is a linear combination of terms of the form \(r^* \otimes s^*\) with \(rs = a;\) that \(S(a^*)\) is a linear combination of terms of the form \(r^*\) with \(r \in k[G] \ltimes \wedge^1 V\) if \(a \in k[G] \ltimes \wedge^1 V;\) that if \(a \in k[G]\) the formulas are as in the case of \(V = 0.\) Similarly, for elements in \(V\) we have:
\[a^* \rightarrow (1 \sharp f_v) = \delta_{a,uv} + \delta_{a,1}(1 \sharp f_v) = (1 \sharp f_v) \leftarrow a^*.\]
The assignment \((x \sharp f_u) \mapsto u; (1 \sharp f_h) \mapsto h^{-1}\) and \((1 \sharp f_v) \mapsto uv\) determines an algebra isomorphism \(\xi: (k[U] \otimes (k[G/U] \ltimes \wedge V))^\text{op} \rightarrow k[U] \otimes (k[G/U] \ltimes \wedge V).\)
Then \(\pi_\psi([C(1)]) \cong k[G] \ltimes \wedge(k_1 \otimes V)\) with \(k_1\) the 1-dimensional \(G\)-module corresponding to the character \(\chi,\) generalizing \([40, \text{Theorem 5.7, ii}].\) The Galois object structure is determined by the \(H\)-coaction:
\[\rho(u) = u \otimes \chi; \quad \rho(gv_P) = \sum_{a \in H^*} \left(\sum a_{(2)}(gv_P)S^{-1}(a_{(1)})\right) \otimes a^*;\]
the \(H^*\)-action:
\[a^* \rightharpoonup g = \delta_{a,g}g; \quad a^* \rightharpoonup u = \delta_{a,u};\]
\[a^* \rightharpoonup uv = \delta_{a,uv} + \delta_{a,uv};\]
for every \(a^* \in H, g \in G/U,\) and \(v \in V;\) and the \(\mathcal{A}\)-coactions:
\[\rho_1(v) = 1 \otimes v + v \otimes 1, \quad \rho_2(v) = 1 \otimes v + v \otimes 1;\]
\[\rho_1(g) = g \otimes g, \quad \rho_2(g) = g \otimes g, \quad \forall g \in G.\]
6 Example: Weyl groups

In this Section we shall explicitly compute $H_L^2(H)$ and $BM(k, H, R)$ for a particular family of modified supergroup algebras. We shall assume that $k = \mathbb{C}$. Let $\Phi$ be an irreducible root system and let $W(\Phi)$ be the corresponding Weyl group. Let $V = h^*$ be its (complexified) natural representation (or its dual), obtained extending the action on $\Phi$ by linearity. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a fixed fundamental system of simple roots and let $s_1, \ldots, s_n$ be the corresponding reflections generating $W(\Phi)$ as a Coxeter group. For further details the reader is referred to [4] and [22].

It is well-known that $V$ is faithful and irreducible ([22 Corollary 5.5, Corollary 6.2, Proposition 6.3]). By [22 Proposition 6.3, Lemma and Theorem 6.4] the non-zero, real, symmetric, $W(\Phi)$-invariant matrices are all equal up to a scalar. Since the matrices of the representation corresponding to the basis $\{\alpha_1, \ldots, \alpha_n\}$ are real, the same property holds for non-zero, complex, symmetric, $W(\Phi)$-invariant matrices and we have $S^2(V^*)^W(\Phi) \cong \mathbb{C}$.

Let $w_0$ be the longest element in $W(\Phi)$ ([22 Section 1.8]). If $\Phi$ is of type $A_1$, $B_n$, $D_n$ $(n$ even), $E_7, E_8, F_4, G_2$ ([22 Corollary 3.19]), then $w_0$ acts as $-1$ and we shall put $G(\Phi) = W(\Phi)$ and $u = w_0$. In this case, by [22 Corollary 3.19] for any ordering $i_1, \ldots, i_n$ of $\{1, \ldots, n\}$ we have $u = w_0 = (s_{i_1} \cdots s_{i_n})^2$ where $h$ is the Coxeter number of $W(\Phi)$ (see [22 Table on page 80]). The presentation of $W(\Phi)$ ([4 Chapitre 4.1.3, Chapitre VI.4.1 Théorème 1]) shows that $U = \langle u \rangle$ is a direct summand of $G(\Phi)$ if and only if there exists a group morphism $\chi: G(\Phi) \to k$ with $\chi(u) = -1$. This happens exactly when $\Phi$ is of type $A_1$, $B_n$ for $n$ odd, $E_7$ and $G_2$.

If $\Phi$ is of type $A_n$ $(n \geq 2)$, $D_n$ $(n$ odd) or $E_6$, then $w_0$ does not act as $-1$ on $V$. In this case there exists an automorphism $\vartheta$ of order 2 of the Dynkin diagram corresponding to $\Phi$ such that, if we extend linearly the action of $\vartheta$ on $V = \text{span}\{\alpha_1, \ldots, \alpha_n\}$, the composition of the actions of $\vartheta$ and $w_0$ is $-1$. The subgroup $\text{Aut}(\Phi)$ of $\text{GL}(V)$ leaving $\Phi$ invariant is the semi-direct product of its normal subgroup $W(\Phi)$ and the subgroup of automorphisms of the Dynkin diagram ([21 §III.9.2, §III.12.2]). So, when $w_0 \neq -1$ on $V$ we define $G(\Phi)$ as the subgroup of $\text{Aut}(\Phi)$ generated by $\vartheta$ and $W(\Phi)$. Then $G(\Phi)$ is the semi-direct product of $W(\Phi)$ and $\langle \vartheta \rangle$. The representation $V$, viewed as a $G(\Phi)$-representation is still irreducible and faithful. For if $\vartheta w = \vartheta w_0 w_0 w$ acted trivially on $V$ then $w_0 w$ would act as $-1$. Since there is no element in $W(\Phi)$ different from $w_0$ mapping all positive roots to negative roots, this is possible only when $w = 1$ and $w_0$ acts as $-1$, which is not the case. The structure of $G(\Phi)$ is determined by the action of $\vartheta$ on $W(\Phi)$ that is $\vartheta \cdot w = \vartheta w_0 w_0 w_0 w_0 \vartheta = w_0 w_0 w_0$. In particular, $\vartheta$ and $w_0$ commute and $\vartheta w_0$ is a central involution which we shall denote by $u$. Since $G(\Phi)$ is also generated by the normal subgroup $W(\Phi)$ and the central subgroup $U = \langle u \rangle$, we have $G(\Phi) \cong W(\Phi) \times U$.  

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Let $H(\Phi) := k[G(\Phi)] \times \wedge V$ with $G(\Phi), V, u$ as before. We recall that in the $A_1$ case we obtain Sweedler’s Hopf algebra $H_4$. By Theorem 4.4 we have:

$$H^2_L(H(\Phi)) \cong H^2(G(\Phi)/U, \mathbb{C}) \times \mathbb{C}$$

that is

$$H^2_L(H(\Phi)) \cong \begin{cases} H^2(W(\Phi)/U, \mathbb{C}) \times \mathbb{C} & \text{for } A_1; B_n, n \geq 2; D_{2m}, m \geq 2; \\
H^2(W(\Phi), \mathbb{C}) \times \mathbb{C} & \text{for } A_n, n \geq 2; D_{2m+1}, m \geq 2; E_6. \\
\end{cases}$$

We shall compute these groups explicitly. The Schur multipliers for irreducible Coxeter groups have been determined in [24]. For the Weyl groups we have:

$$H^2(W(\Phi), \mathbb{C}) \cong \begin{cases} 1 & \text{for } A_1; A_2; \\
\mathbb{Z}_2 & \text{for } A_n, n \geq 3; B_2; E_6; E_7; E_8; G_2; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } B_3; D_n, n \geq 5; F_4; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } B_n, n \geq 4; D_4 \\
\end{cases}$$

so we know $H^2_L(H(\Phi))$ when $w_0 \neq -1$ on $V$, i.e., when $G(\Phi) \cong W(\Phi) \times U$.

Let $G(\Phi) = W(\Phi)$. If $\Phi = A_1$ then $W(\Phi) = U$ and we have nothing to prove. If $\Phi$ is of type $B_n$ for odd $n$, $E_7$ or $G_2$ then $U$ is a direct summand of $W(\Phi)$ and formula (2.5) allows us to deduce $H^2(W(\Phi)/U, \mathbb{C})$ from the knowledge of $H^2(W(\Phi), \mathbb{C})$ and the analysis of $\text{Hom}(W(\Phi)/U, \mathbb{Z}_2)$. In these cases we always have $\text{Hom}(W(\Phi), \mathbb{Z}_2)/\text{Hom}(W(\Phi)/U, \mathbb{Z}_2) \cong \mathbb{Z}_2$ with $\text{Hom}(W(\Phi)/U, \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $\Phi = B_{2m+1}; G_2$ and $\text{Hom}(W(E_7)/U, \mathbb{Z}_2) \cong 1$.

If $\Phi$ is of type $B_n$ for $n$ even, $D_n$ for $n$ even, $E_8$ or $F_4$ then $U$ is not a direct summand of $W(\Phi)$ but sequence (2.4) yields:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(W(\Phi)/U, \mathbb{C}) \longrightarrow H^2(W(\Phi), \mathbb{C}) \longrightarrow \text{Hom}(W(\Phi), \mathbb{Z}_2).$$

(6.1)

Let $\Phi = B_2$. The exact sequence (6.1) becomes:

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\tau} H^2(W(\Phi)/U, \mathbb{C}) \xrightarrow{\text{Infl}} \mathbb{Z}_2 \xrightarrow{\theta} \mathbb{Z}_2 \times \mathbb{Z}_2.$$

We would like to describe the image of $\theta$. Let $P$ denote a lift of a projective representation corresponding to a cocycle $\sigma$ and let $T_i := P(s_i)$ for every $i$. Then $P(w_0)$ is a scalar multiple of $T_1T_2T_1T_2$ and centrality of $w_0$ ensures that

$$P(w_0)T_i = T_iP(w_0)\sigma(s_i, w_0)\sigma^{-1}(w_0, s_i) = T_iP(w_0)\theta(\sigma)(s_i, u).$$

The relations among the $T_i$’s are described in [25, Table 7.1]. If $\sigma$ represents the non-trivial class in $H^2(W(\Phi), \mathbb{C})$ we find $\theta(\sigma)(s_i, u) = -1$ for $i = 1, 2$. It follows that $\text{Ker}(\theta) = \text{Im}(\text{Infl}) = 1$ so $H^2(W(\Phi)/U, \mathbb{C}) \cong \mathbb{Z}_2$.  

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Let $\Phi$ be $B_{2m}$ for $m \geq 2$. The exact sequence (6.1) becomes:

\[ 1 \rightarrow \mathbb{Z}_2 \xrightarrow{T} H^2(W(B_{2m})/U, \mathbb{C}) \xrightarrow{\text{Inf}} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\theta} \mathbb{Z}_2 \times \mathbb{Z}_2. \]

As before, we would like to understand the map $\theta$. Let $P$ denote the lift of a projective representation corresponding to a cocycle $\sigma$ and let $T_i := P(s_i)$ for every $i$. Then $P(w_0)$ is a scalar multiple of $(T_1 \cdots T_{2m})^{2m}$ and we have $P(w_0)T_i = T_i P(w_0) \theta(\sigma)(s_i, u)$. By the relations in [25 Table 7.1] we see that $\theta$ maps the cocycle corresponding to $(a, b, c)$ with $a, b, c \in \{\pm 1\}$ to the morphism $W(\Phi) \rightarrow \mathbb{Z}_2$ mapping each generator to $c$. In other words, the only non-trivial element in $\text{Im}(\theta)$ is $\epsilon(w) = (-1)^{\ell(w)}$ where $\ell$ denotes the length function on a Coxeter group. Hence $H^2(W(B_{2m})/U, \mathbb{C})/\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We would like to find out whether $H^2(W(B_{2m})/U, \mathbb{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $H^2(W(B_{2m})/U, \mathbb{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

As in the proof of [25 Lemma 7.2.8, Lemma 7.2.9] if $\sigma$ is a cocycle for $W(B_{2m})/U$ and if $T_i$ denotes the image of $s_i$ for a projective representation corresponding to $\sigma$, then the following relations hold, with $n = 2m$:

\[
\begin{align*}
T_i^2 &= 1 \quad 1 \leq i \leq n, \\
(T_i T_j)^2 &= a \quad 1 \leq i < j \leq n, \\
(T_i T_j)^3 &= 1 \quad 1 \leq i \leq n - 2, \\
(T_{n-i} T_n)^4 &= c, \\
(T_{n-i} T_n)^2 &= b \quad 1 \leq i \leq n - 2, \\
(T_{n-i} T_n)^3 &= 1 \quad 1 \leq i \leq n - 2,
\end{align*}
\]

where $a, b, c \in \{\pm 1\}$ and $\lambda \in \mathbb{C}$. We know that $T_i (T_1 \cdots T_n)^n = c (T_1 \cdots T_n)^n T_i$ by the previous computation. Since $(T_1 \cdots T_n)^n$ and $(T_1 T_3 \cdots T_{n-1} T_2 \cdots T_n)^n$ are both lifts of $w_0$, they must differ only by a sign because the relations among the $T_i$’s are the Coxeter ones up to a sign. Thus, necessarily $c = 1$. Besides, since $w^2 = w_0^2 = 1$ in $W(B_{2m})$, then for its lift there must hold: $(T_1 T_3 \cdots T_{n-1} T_2 \cdots T_n)^{2n} = \lambda^2 = \pm 1$. If $\psi$ and $\phi$ are two distinct homomorphisms $W(B_{2m}) \rightarrow \mathbb{Z}_2 = \{0, 1\}$ then $\sigma(w U, w U') = (-1)^{\phi(w) \psi(w')}$ is a 2-cocycle on $W(B_{2m})/U$ corresponding to $(a, b, \lambda) = (1, -1, 1)$, as one can check computing $\sigma(r, s) \sigma^{-1}(s, r)$ as in [25 Lemma 7.2.8]. The standard representation for $W(B_{2m})$ is a projective representation for $W(B_{2m})/U$ corresponding to $(a, b, \lambda) = (1, 1, -1)$. If we prove that a triple $(-1, 1, \pm 1)$ is also represented, then $H^2(W(B_{2m})/U, \mathbb{C}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ because it has four elements of order 2. By §6 §7, up to a different notation, the representations corresponding to $(a, b, c) = (-1, 1, 1)$ are those with $T_n = \pm 1$ and correspond to the projective, nonlinear representations of the symmetric group $S_{2m}$, i.e., linear representations of its double cover $\tilde{S}_{2m}$ generated by the $s_i$’s and $-1$. This means that the relations become, with $n = 2m$:

\[
\begin{align*}
T_i^2 &= 1, \quad 1 \leq i \leq n - 1, \\
(T_i T_{i+1})^3 &= 1 \quad 1 \leq i \leq n - 2, \\
(T_i T_j)^2 &= -1, \quad 1 \leq i < j + 1 \leq n, \\
(T_i T_3 \cdots T_{n-1} T_2 \cdots T_{n-2})^n &= \lambda.
\end{align*}
\]

It is well-known that $w_0' = (s_1 s_3 \cdots s_{n-1} s_2 s_4 \cdots s_{n-2})^n$ is the longest element of $W(A_{n-1}) = S_{2m}$. Since $w_0'$ is an involution and since the relations for the $T_i$’s are
the Coxeter ones up to a sign we have \((T_1 T_3 \cdots T_{n-1} T_2 T_4 \cdots T_{n-2})^n = \pm 1\) and the statement.

Let \(\Phi\) be \(D_4\). The exact sequence (6.1) becomes:

\[
1 \longrightarrow \mathbb{Z}_2 \stackrel{T}{\longrightarrow} H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2.
\]

In this case \(\Theta\) is surjective and it maps \((a, b, c)\) with \(a, b, c \in \{\pm 1\}\) to \(abc\). Therefore \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\). We would like to find out whether \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) or \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\).

As in the previous case if \(\sigma\) is a cocycle for \(W(D_4)/U\) and if \(T_i\) denotes the image of \(s_i\) for the lift of a projective representation corresponding to \(\sigma\), then the following relations hold:

\[
T_i^2 = 1 \quad 1 \leq i \leq 4 \quad (T_1 T_3)^3 = 1 \quad i \neq 2
\]

with \(a, b, c \in \{\pm 1\}\) and \(\lambda \in \mathbb{C}^\ast\). Since \(T_i (T_1 T_3 T_1 T_3 T_1 T_3 T_1 T_3)^3 = abc(T_1 T_3 T_1 T_3 T_1 T_3 T_1 T_3)^3 T_i\) we have \(c = ab\). A direct computation using the relations shows that \((T_1 T_3 T_1 T_3 T_1 T_3 T_1 T_3)^6 = 1\), hence \(\lambda = \pm 1\), and \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\). Here again, the image of the transgression map is achieved by the standard representation of \(W(D_4)\).

Let \(\Phi\) be \(D_{2m}\) for \(m \geq 3\). The exact sequence (6.1) becomes:

\[
1 \longrightarrow \mathbb{Z}_2 \stackrel{T}{\longrightarrow} H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2.
\]

Using the relations in (25) Table 7.1] a direct computation shows that \(\Theta\) maps \((a, b)\) to \(b\) so \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2\). We shall see that, in analogy with the previous cases \(H^2(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{C}^\ast) = \mathbb{Z}_2 \times \mathbb{Z}_2\). If \(\sigma\) is a cocycle for \(W(D_{2m})/U\) and if \(t_i\) denotes the image of \(s_i\) for the lift of a projective representation corresponding to \(\sigma\), then the following relations hold, with \(n = 2m\):

\[
(t_i)^2 = 1 \quad 1 \leq i \leq n; \quad (t_i t_{i+1})^3 = 1 \quad 1 \leq i \leq n - 1;
\]

where \(a, b \in \{\pm 1\}\) and \(\lambda \in \mathbb{C}^\ast\). Besides, since \(t_i (t_1 \cdots t_n)^{n-1} = b(t_1 \cdots t_n)^{n-1} t_i\) we necessarily have \(b = 1\). By [25 Appendix] all projective representations of \(W(2m)\) are restrictions of projective representations of \(W(B_{2m})\) with \(t_i = T_i\) for \(1 \leq i \leq n - 1\) and \(t_n = T_n T_{n-1} T_n\). In particular, the representation corresponding to \((a, b) = (-1, 1)\) can be built as the restriction of a representation of \(W(B_{2m})\) corresponding to \((a, b, c) = (-1, 1, 1)\). We have seen that such representations come from projective representations of \(S_{2m}\) by letting \(T_n = \pm 1\), that is, \(t_n = \pm 1\)
2.3 we have: Next we would like to determine is non-trivial because it corresponds to the pair \( \theta \). Hence \( \operatorname{Im} (\operatorname{Infl}) \cong \mathbb{Z}_2 \) and we have an exact sequence

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1.
\]

Let \( \Phi = F_4 \). The exact sequence (6.1) becomes:

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(W(F_4)/U, \mathbb{C}) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1.
\]

The relations in [25, Table 7.1] show that \( \theta \) maps the pair \((a, b)\) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) to \((b, b)\), that is, the only non-trivial element in the image of \( \theta \) is, as before, \( \epsilon(w) = (-1)^{\ell(w)} \). Hence \( \operatorname{Im} (\operatorname{Infl}) \cong \mathbb{Z}_2 \) and we have an exact sequence

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(W(F_4)/U, \mathbb{C}) \longrightarrow \mathbb{Z}_2 \longrightarrow 1.
\]

Let \( \chi: W(F_4) \rightarrow \mathbb{Z}_2 = \{0, 1\} \) be the morphism mapping \( s_1 \) and \( s_2 \) to 1 and \( s_3 \) and \( s_4 \) to 0. The map \( c(x, y) = (-1)^{\ell(x)\ell(y)} \) is a 2-cocycle on \( W(F_4) \). Since \( \operatorname{Hom}(W(F_4)/U, \mathbb{C}) \cong \operatorname{Hom}(W(F_4), \mathbb{C}) \), the cocycle \( c \) is also a cocycle for \( W(F_4)/U \). Besides, a direct computation using [25, Table 7.1] shows that \( \operatorname{Infl}(c) \) is non-trivial because it corresponds to the pair \((-1, 1)\). Since \( c^2 = 1 \), the class \( \bar{c} \) determines a section \( \mathbb{Z}_2 \rightarrow H^2(W(F_4)/U, \mathbb{C}) \) splitting \( \operatorname{Infl} \).

Let \( \Phi = E_8 \). The exact sequence (6.1) becomes:

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow H^2(W(E_8)/U, \mathbb{C}) \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 1.
\]

Then \( \theta(a) = a \), that is, the non-trivial element in \( H^2(W(E_8)/U, \mathbb{C}) \) maps to \( \epsilon(w) = (-1)^{\ell(w)} \). Hence \( \theta \) is injective, \( \operatorname{Infl} \) is trivial and \( H^2(W(E_8)/U, \mathbb{C}) \cong \mathbb{Z}_2 \).

We have reached the following result:

\[
H^2_I(H(\Phi)) \cong \begin{cases} 
\mathbb{C} & \text{for } A_1; A_2; G_2; \\
\mathbb{Z}_2 \times \mathbb{C} & \text{for } A_n, n \geq 3; B_2; B_3; E_6; E_7; E_8; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } B_{2m+1}, m \geq 2; D_n, n \geq 5; F_4; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } B_{2m}, m \geq 2; D_4.
\end{cases}
\]

Next we would like to determine \( BM(\mathbb{C}, H, R_u) \). By Theorem 3.6 and Proposition 2.3 we have:

\[
BM(\mathbb{C}, H(\Phi), R_u) \cong \begin{cases} 
\mathbb{Z}_2 \times H^2(W(\Phi), \mathbb{C}) \times \mathbb{C} & \text{for } A_1; B_{2m+1}; E_7; G_2 \\
\mathbb{Z}_2 \times H^2(G(\Phi), \mathbb{C}) \times \mathbb{C} & \text{for } A_n, n \geq 2; D_{2m+1}; E_6 \\
H^2(W(\Phi), \mathbb{C}) \times \mathbb{C} & \text{for } B_{2m}; D_{2m}; E_8; F_4.
\end{cases}
\]

Let \( \Phi = A_n, n \geq 2; D_n, n \text{ odd}, E_6 \). Then \( G(\Phi) \cong W(\Phi) \times U \) and (2.5) yields

\[
H^2(G(\Phi), k) \cong H^2(W(\Phi), k) \times \mathbb{Z}_2.
\]
We can conclude with:

\[
BM(\mathbb{C}, H, R_u) \cong \begin{cases} 
\mathbb{Z}_2 \times \mathbb{C} & \text{for } A_1; A_2; B_2; E_8; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } A_n, n \geq 3; D_{2m}, m \geq 3; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } B_3; B_{2m}, m \geq 2; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } D_4; D_{2m+1}, m \geq 2; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C} & \text{for } B_{2m+1}, m \geq 2.
\end{cases}
\]

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