PAIR CORRELATIONS OF APERIODIC INFLATION RULES
VIA RENORMALISATION: SOME INTERESTING EXAMPLES

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Abstract. This article presents, in an illustrative fashion, a first step towards an extension of the spectral theory of constant length substitutions. Our starting point is the general observation that the symbolic picture (as defined by the substitution rule) and its geometric counterpart with natural prototile sizes (as defined by the induced inflation rule) may differ considerably. On the geometric side, an aperiodic inflation system possesses a set of exact renormalisation relations for its pair correlation coefficients. Here, we derive these relations for some paradigmatic examples and infer various spectral consequences. In particular, we consider the Fibonacci chain, revisit the Thue–Morse and the Rudin–Shapiro system, and finally analyse a twisted extension of the silver mean chain with mixed singular spectrum.

1. Introduction

The spectral theory of primitive substitution rules and the symbolic dynamical systems defined by them has a long history and is well studied, compare [34] and references therein. Nevertheless, as soon as one leaves the realm of constant length substitutions, many open questions challenge our present level of understanding. This is partly due to the fact that the substitution system picture and the inflation tiling picture, where one works with natural prototile lengths, may differ considerably; compare [15, 35, 5, 8].

The majority of the dynamical systems literature on this subject deals with the symbolic case, where work by Dekking [17] and others, see [19, 20, 21, 29] and references therein, has led to a fairly complete understanding of the constant length case, which was recently extended in a systematic fashion in [13], including higher dimensions. In all these cases, the connection between the diffraction spectrum and the dynamical spectrum is rather well understood. The two notions are equivalent in the pure point case [28, 7, 9, 31], and recent progress for symbolic systems also establishes a complete equivalence via the inclusion of certain factors [10]. Therefore, at least algorithmically, any constant length substitution can be analysed completely as far as its spectrum is concerned; see [3, 6, 13] for recent developments.

The situation is less favourable outside this class. Even if one stays within the realm of primitive inflation rules, the determination of the dynamical spectrum is generally difficult. Here, the geometric counterpart studied in tiling theory has certain undeniable advantages. First, since such tilings are still of finite local complexity, the connection between the dynamical spectrum and the diffraction spectrum can still be used, if certain factors are included in the discussion [10]. For this reason, we employ an approach via the diffraction spectrum of the tiling spaces. Second, the geometric setting leads to the existence of a set of exact
renormalisation relations for the pair correlation coefficients of the system. We use the term 'exact’ to distinguish our approach from the widely used renormalisation schemes that are approximate or asymptotic in nature. This approach has not attracted much attention so far. In fact, we are not aware of any reference to this approach beyond the constant length case, though the principal idea has certainly been around for a while, and has been used, often in an approximate way, for several physical quantities such as electronic or transport properties; see [39] for an example and further references.

In this paper, we want to show the power of exact renormalisation relations for spectral properties, in the form of a first step via some illustrative examples. A more general approach will be presented in a forthcoming publication. It will be instrumental for our analysis that we formulate various aspects on the symbolic level, while the core of our analysis rests on the natural geometric realisation. To make the distinction as transparent as possible, we will speak of substitution rules on the symbolic side, but of inflation rule on its geometric counterpart, thus following the notation and terminology of the recent monograph [5]. Also, various results are briefly recalled from there. Rather than repeating the proofs, we provide precise references instead.

Below, we work along a number of examples, all of them with Pisot–Vijayaraghavan (PV) numbers as inflation multipliers. Since we do not demand the characteristic polynomial to be irreducible over \( \mathbb{Q} \), there is still enough freedom to encounter systems with mixed spectrum. In fact, all point sets that we encounter along the way will be linearly repetitive Meyer sets. An interesting linearly repetitive inflation point set with non-PV multiplier (hence not a Meyer set) will be discussed in detail in [2].

The paper is structured as follows. After recalling some facts about translation bounded measures on \( \mathbb{R} \) and their Fourier transforms in Section 2, we begin with a detailed analysis of the Fibonacci substitution and inflation in Section 3. This is both a paradigm of the theory and an instructive example along which we can develop our ideas as well as further notions, wherefore this is also the longest section. Here, we introduce a set of exact renormalisation equations for the general pair correlation coefficients and compare the findings with what is known from the model set description. Then, we use the new approach to derive an alternative proof of the pure point nature of the diffraction spectrum, and hence also of the dynamical spectrum via the known equivalence result for this case [28, 7, 31].

After that, in Section 4, we briefly revisit the classic Thue–Morse and Rudin–Shapiro sequences from the renormalisation point of view. The point of this exercise is to show that our approach, in the constant length setting where the symbolic and the geometric pictures coincide, is essentially equivalent to the traditional approach as described in [34] and recently extended in [13]. Still, there are several aspects of a more algebraic nature that seem to deserve further attention.

Finally, by imposing an involutory bar swap symmetry, we construct an extension of the silver mean chain with mixed spectrum in Section 5. The main point here is that such extensions are not restricted to the constant length case (where they are known from examples...
such as those of the previous section or many others as in [3, 4, 6]). In fact, as our example indicates, there is an abundance of interesting and completely natural primitive inflation rules that produce repetitive Meyer sets with mixed spectrum. In our opinion, this has hitherto been more or less neglected. For the explicit analysis, the exact renormalisation scheme is used to determine the spectral type by a somewhat subtle application of the Riemann–Lebesgue lemma, which leads to a singular spectrum of mixed type.

Overall, we believe that the geometric picture with its exact renormalisation relations for the pair correlation coefficients provides a powerful tool also for the general case of primitive inflation rules. This will be further developed in a forthcoming publication.

2. Preliminaries

A measure on \(\mathbb{R}\) is a continuous linear functional on the space \(C_c(\mathbb{R})\) of continuous functions with compact support. By the Riesz–Markov theorem, this specifies precisely the class of regular Borel measures on \(\mathbb{R}\). Note that these need not be finite measures, though all examples below will be translation bounded, meaning measures \(\mu\) such that the total variation measure \(|\mu|\) satisfies \(\sup_{t \in \mathbb{R}} |\mu|(t + K) < \infty\), for any compact \(K \subset \mathbb{R}\); see [26 and 5, Sec. 8.5] for more.

If \(\mu\) is a measure, its twisted counterpart \(\tilde{\mu}\) is defined via \(\tilde{\mu}(g) = \mu(\tilde{g})\) for \(g \in C_c(\mathbb{R})\), where \(\tilde{g}(x) := g(-x)\).

The convolution of two finite measures is denoted as \(\mu * \nu\), and the volume-averaged (or Eberlein) convolution of two translation bounded measures by

\[
\mu \ast \tilde{\nu} := \lim_{R \to \infty} \frac{\mu|_R \ast \nu|_R}{2R},
\]

provided the limit exists (we shall not consider any other case below). Here, \(\mu|_R\) denotes the restriction of \(\mu\) to the interval \((-R, R)\).

A measure \(\mu\) is called positive definite, if \(\mu(g \ast \tilde{g}) \geq 0\) holds for all \(g \in C_c(\mathbb{R})\). There are several possibilities to define the Fourier transform of a measure, provided it exists. This is a non-trivial issue, and part of our later analysis will rely on the existence. Here, we use a version of the Fourier transform that, for integrable functions on \(\mathbb{R}\), reduces to

\[
\hat{f}(k) = \int_{\mathbb{R}} e^{-2\pi ikx} f(x) \, dx.
\]

Any positive definite measure is Fourier transformable. The Fourier transform of a positive definite measure is a positive measure; see [14, Ch. I.4] or [5, Sec. 8.6] for details and [36, 14] for general background.

Lemma 1. Let \(\mu, \nu\) be translation bounded measures such that \(\mu \ast \tilde{\nu}\) as well as \(\mu \ast \tilde{\mu}\) and \(\nu \ast \tilde{\nu}\) exist. Then, \(\mu \ast \tilde{\nu}\) is a translation bounded and transformable measure, as is \(\tilde{\mu} \ast \nu\).

Proof. Observe first that \(\tilde{\mu} \ast \nu = \mu \ast \tilde{\nu}\), wherefore it suffices to prove the claim for \(\mu \ast \tilde{\nu}\). Now, as a variant of the (complex) polarisation identity, one verifies that

\[
(1) \quad \mu \ast \tilde{\nu} = \frac{1}{4} \sum_{\ell=1}^{4} i^{\ell} (\mu + i^{\ell} \nu) \ast (\mu + i^{\ell} \nu)^{\tilde{}}.
\]
where all measures on the right hand side exist due to our assumptions. Consequently, $\mu \star \tilde{\nu}$ is a complex linear combination of four positive definite measures, each of which is translation bounded and transformable, hence $\mu \star \tilde{\nu}$ is translation bounded and transformable as well. □

Let us now briefly recall an important result on measures with Meyer set support, which is due to Strungaru [41]. Let $\Lambda$ be a repetitive Meyer set, and assume that the hull of $\Lambda$ is uniquely ergodic under the translation action of $\mathbb{R}$. Then, the autocorrelation measure $\gamma$ of $\Lambda$ is well-defined and has a unique Eberlein decomposition as

$$\gamma = \gamma_s + \gamma_0$$

into a strongly almost periodic measure, whose Fourier transform exists and is a pure point measure, and a null-weakly almost periodic measure, whose Fourier transform is a continuous measure; see [23] for the underlying notions and results.

Moreover, both parts, $\gamma_s$ and $\gamma_0$, are supported in a model set with a closed, compact window. Given a cut and project scheme for $\Lambda$ according to [41], one can use the smallest such window that defines a model set cover of $\Lambda - \Lambda$. The same conclusion holds for the autocorrelation of a weighted extension. For our slightly more general situation, where we do not only consider measures of the form $\mu \star \tilde{\mu}$ but also products of the form $\mu \star \tilde{\nu}$ as in Lemma 1, one employs once more the complex polarisation identity (1) to obtain the corresponding decomposition result for the more general correlation measures that we need.

3. Example 1: The Fibonacci chain

One version of the ubiquitous Fibonacci substitution is given by the rule $\varrho$: $a \mapsto ab, b \mapsto a$. It is primitive, with substitution matrix

$$M = M_\varrho = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and characteristic polynomial $x^2 - x - 1$. This polynomial is irreducible over $\mathbb{Q}$, with roots $\tau$ and $\tau' = 1 - \tau$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio; see [5] for details and background.

For a geometric realisation, one reinterprets the letters as marked intervals in $\mathbb{R}$, with a point at their left endpoints. Choosing length $\tau$ for $a$ and length 1 for $b$, which emerges from the left Perron–Frobenius (PF) eigenvector of $M$, compare [5, Ch. 4] for background, one obtains the (geometric) Fibonacci inflation rule. Here, in one iteration step, each interval is stretched by a factor of $\tau$ and then dissected into intervals of the original size according to the symbolic rule, which is possible due to the eigenvector condition. By abuse of notation, we also call this inflation rule $\varrho$.

A bi-infinite fixed point is obtained via iterated inflation starting from a legal seed, $a|a$ say, where $|$ marks the reference point that is placed at 0 in the geometric realisation; see [5, Ch. 4] for the basic definitions. More precisely, the corresponding iteration sequence converges (in the local topology) towards a 2-cycle under the inflation. Each member of that cycle is then a fixed point under $\varrho^2$. It is a tiling of $\mathbb{R}$ with two prototiles, each of which carries a point
at its left endpoint. Consequently, one may equally well view it as a Delone set in \( \mathbb{R} \) that carries a colouring according to the types, \( a \) and \( b \). We shall use both pictures in parallel.

Let us note in passing that the two fixed points of \( g^2 \) are *proximal* in a strong sense. Indeed, they completely agree except for the first two positions on the left of the marker, where they alternate between \( ab \) and \( ba \) under the action of \( g \).

For concreteness, let us assume that we select the fixed point of \( g^2 \) with seed \( a|a \). Then, for the (coloured) Delone set \( A \) of its marker points, we obtain the relation

\[
A = A_a \cup A_b,
\]

where each of the three sets is a regular model set (or cut and project set) for the cut and project scheme (CPS), see [5, Sec. 7.2],

\[
\begin{array}{cccc}
\mathbb{R} & \xleftarrow{\pi} & \mathbb{R} \times \mathbb{R} & \xrightarrow{\pi_{\text{int}}} & \mathbb{R} \\
\text{dense} & \cup & \text{dense} & \cup & \text{dense} \\
\mathbb{Z}[\tau] & \xleftarrow{1-1} & \mathcal{L} & \xrightarrow{1-1} & \mathbb{Z}[\tau] \\
\| & \| & \| & \| & \|
\end{array}
\]

with \( \mathbb{Z}[\tau] \) being the ring of integers in the real quadratic field \( \mathbb{Q}(\sqrt{5}) \), while

\[
\mathcal{L} = \{ (x, x^*) \mid x \in \mathbb{Z}[\tau] \} \subset \mathbb{R}^2
\]

is the lattice that emerges from the ring \( \mathbb{Z}[\tau] \) via its Minkowski embedding; see [5, Fig. 3.3]. Here, \( \ast \) denotes algebraic conjugation in the quadratic field \( \mathbb{Q}(\sqrt{5}) \), as defined by \( \sqrt{5} \mapsto -\sqrt{5} \). This is the the \( \ast \)-map of the CPS of Eq. (2), with \( \tau^* = \frac{1}{\tau} = 1 - \tau \). Let us stress that the particular choice of the fixed point has no influence on the CPS — other choices would give the same setting; see [11] for a general result in this direction. Our choice of fixed point, however, does result in a specific window for the model set description of \( A \).

A model set in the CPS of Eq. (2) is a subset of direct space \( \mathbb{R} \) of the form

\[
\lambda(W) := \{ x \in \mathbb{Z}[\tau] \mid x^* \in W \},
\]

where the window \( W \subset \mathbb{R} \) is assumed to be a relatively compact set in internal space with non-empty interior. A model set is called regular when \( \partial W \) has zero measure in internal space. In particular, one finds [5, Ex. 7.3] for our selected fixed point with seed \( a|a \) that

\[
A_a = \lambda\left( (\tau - 2, \tau - 1) \right), \quad A_b = \lambda\left( (-1, \tau - 2) \right) \quad \text{and} \quad A = A_a \cup A_b = \lambda\left( (-1, \tau - 1) \right).
\]

The other fixed point of \( g^2 \), with seed \( b|a \), can be described similarly, with the windows now containing the left endpoint, but not the right one. Model sets are Meyer sets, which means that \( A \) is relatively dense while \( A - A \) is uniformly discrete; see [5, Prop. 7.5].

The natural *autocorrelation* \( \gamma_A \) of the model set \( A \) is a translation bounded pure point measure of the form \( \gamma_A = \sum_{z \in A - A} \eta(z) \delta_z \) with

\[
\eta(z) := \lim_{R \to \infty} \frac{\text{card}(A_R \cap (z + A_R))}{2R},
\]
where $\Lambda_R := \Lambda \cap (-R, R)$. The limit exists for all $z \in \mathbb{R}$, but is non-zero only for $z \in \Lambda - \Lambda$, the latter being a Delone set in this case (and, in fact, also a model set). Note that $\eta(-z) = \eta(z)$ holds for all $z \in \mathbb{R}$. For $z \in \mathbb{Q}(\sqrt{5})$, one obtains from [5] Prop. 9.8 the explicit formula

$$\eta(z) = \text{dens}(\Lambda) \frac{\text{vol}(W \cap (z^* + W))}{\text{vol}(W)} = \frac{1}{\sqrt{5}} \int_{\mathbb{R}} 1_W(y) 1_{z^* + W}(y) \, dy,$$

with $W = (-1, \tau - 1]$ as above and $\text{dens}(\Lambda) = \tau/\sqrt{5}$. Similar formulas emerge for the subsets $\Lambda_a$ and $\Lambda_b$. Since $\Lambda - \Lambda \subset \mathbb{Z}[\tau]$, all non-zero values of $\eta(z)$ are covered by formula (3).

The measure $\gamma_A$ is positive definite, and hence Fourier transformable. By the Bochner–Schwartz theorem, $\widehat{\gamma}_A$ is then a positive measure, called the diffraction measure [26, 5] of the set $\Lambda$. Here, it is a pure point measure of the form

$$\widehat{\gamma}_A = \sum_{k \in \mathcal{F}} I(k) \delta_k,$$

where $\mathcal{F} = \mathbb{Z}[\tau]/\sqrt{5} \subset \mathbb{Q}(\sqrt{5})$ and $\text{sinc}(x) := \frac{\sin(x)}{x}$; see [5] Sec. 9.4.1 for details. Note that $I(k) = 0$ for $k = m\tau$ with $m \in \mathbb{Z} \setminus \{0\}$, which are all points of $\mathcal{F}$ for which the intensity vanishes. Such points are known as the extinction points of the diffraction measure (or of the Fourier–Bohr spectrum); compare [5] Rem. 9.10 and Sec. 9.4.1.

The diffraction measure was calculated for the specific set $\Lambda$, but it is the same for all other members of the hull of $\Lambda$,

$$X(\Lambda) := \{t + \Lambda \mid t \in \mathbb{R}\},$$

where the closure is taken in the local topology as usual [5]. We thus know that the diffraction measure $\widehat{\gamma}_A$ is that of the hull as well, and hence that of the dynamical system $(X(\Lambda), \mathbb{R})$ with the canonical translation action of $\mathbb{R}$. This system is strictly ergodic [31, 38, 33], so it is minimal and possesses a unique invariant probability measure, $\mu_{\mathcal{F}}$ say. By the general equivalence theorem between pure point diffraction and dynamical spectra, see [37, 28, 7, 31] and references therein, we then also know that the dynamical system $(X(\Lambda), \mathbb{R}, \mu_{\mathcal{F}})$ has pure point spectrum. Here, the dynamical spectrum (in additive formulation) is given by

$$\mathcal{F} = \mathbb{Z}[\tau]/\sqrt{5},$$

which is the smallest additive subgroup of $\mathbb{R}$ that contains all points (wave numbers) $k$ with $I(k) > 0$; see [7, 30, 10] and references therein for further background.

Let us now refine the analysis a little, in that we not only look at the autocorrelation of $\Lambda$ as a point set, but as a coloured point set. To do so, let us define the pair correlation functions (or coefficients) $\nu_{\alpha\beta}(z)$ with $\alpha, \beta \in \{a, b\}$ as the relative frequency of two points in $\Lambda$ at distance $z$ subject to the condition that the left point is of type $\alpha$ and the right one of type $\beta$. The term ‘relative’ means that the frequency be determined per point of $\Lambda$ rather than per unit length. Clearly, $\nu_{\alpha\beta}(z) = 0$ for any $z \notin \Lambda - \Lambda$, and $\nu_{\alpha\beta}(0) = 0$ for $\alpha \neq \beta$ (as the type of a point is unique). In fact, the right PF eigenvector of the substitution matrix $M$ gives us the values $\nu_{aa}(0) = 1/\tau$ and $\nu_{bb} = 1/\tau^2$, which coincide with the relative frequencies of the letters $a$ and $b$ in the (symbolic) fixed point.
As follows once again from the model set description of \( \Lambda \) (or alternatively from the unique ergodicity of the hull under the translation action of \( \mathbb{R} \) via the ergodic theorem), the refined pair correlation coefficients exist. One has the general symmetry relation
\[
\nu_{\alpha\beta}(-z) = \nu_{\beta\alpha}(z),
\]
which can be verified directly from the definition of the coefficients as a mean by a simple calculation. Employing the above model set description, the coefficients are given by
\[
\nu_{\alpha\beta}(z) = \frac{\text{vol}(W_\alpha \cap (W_\beta - z^*))}{\text{vol}(W)}
\]
with \( W_\alpha = (\tau - 2, \tau - 1], W_b = (-1, \tau - 2] \) and \( W = W_a \cup W_b = (-1, \tau - 1] \). One also has the summatory relationship
\[
\eta(z) = \frac{\text{dens}(\Lambda)}{\text{dens}(\Lambda)} = \nu_{aa}(z) + \nu_{ab}(z) + \nu_{ba}(z) + \nu_{bb}(z),
\]
which is illustrated in Figure 1 as a relation in internal space, hence as a function of the variable \( z^* \). More precisely, looking at the right-hand side of Eq. (6) as a function of \( z^* \) for each of the choices \( \alpha, \beta \in \{a, b\} \) leads to four functions with a continuous extension. Two of them are the tent-shaped covariograms of \( W_a \) and \( W_b \); compare [5, Rem. 9.8]. They are shaded in Figure 1, one being shifted in vertical direction. The other two are trapezoidal functions. When adding them up as shown in the figure, one obtains the covariogram of the total window \( W \).

In view of the symmetry relation (5), it suffices to know the functions \( \nu_{\alpha\beta}(z) \) for non-negative \( z \in \Lambda - \Lambda \). Note that \( \Lambda - \Lambda = \Lambda' - \Lambda' \) holds for any \( \Lambda' \in \mathbb{X}(\Lambda) \), so that this Minkowski difference is constant on the hull. Now, the inflation structure tells us how the values of \( \nu_{\alpha\beta}(z) \) relate to the values on a scale that is reduced by a factor of \( \tau \).

**Proposition 2.** Consider the Fibonacci chain in the geometric setting with interval prototiles of lengths \( \tau \) and 1 as described above. Then, the pair correlation coefficients \( \nu_{\alpha\beta}(z) \), with
Let $\alpha, \beta \in \{a, b\}$, satisfy the exact renormalisation relations
\begin{align}
\nu_{aa}(z) &= \frac{1}{\tau}(\nu_{aa}(\frac{z}{\tau}) + \nu_{ab}(\frac{z}{\tau}) + \nu_{ba}(\frac{z}{\tau}) + \nu_{bb}(\frac{z}{\tau})) , \\
\nu_{ab}(z) &= \frac{1}{\tau}(\nu_{aa}(\frac{z}{\tau} - 1) + \nu_{ba}(\frac{z}{\tau} - 1)) , \\
\nu_{ba}(z) &= \frac{1}{\tau}(\nu_{aa}(\frac{z}{\tau} + 1) + \nu_{ab}(\frac{z}{\tau} + 1)) , \\
\nu_{bb}(z) &= \frac{1}{\tau}(\nu_{aa}(\frac{z}{\tau})),
\end{align}
with $z \in \mathbb{Z}[\tau]$ and $\nu_{\alpha\beta}(z) = 0$ whenever $z \notin \Lambda_\beta - \Lambda_\alpha$.

**Proof.** Since our inflation rule is aperiodic, we have local recognisability [34]. This means that, in any element of the hull, each tile lies inside a unique level-1 supertile that is identified by a local rule. Concretely, each patch of type $ab$ constitutes a supertile of type $a$, while each tile of type $a$ that is followed by another $a$ (to the right) stands for a supertile of type $b$. From now on, we simply say supertile, as no level higher than 1 will occur in this proof.

The distance $z$ of the left endpoints of two tiles is now given by the distance $z'$ of the left endpoints of the two supertiles containing them, plus a correction coming from the offsets of the tiles within their supertiles. We may have $z = z'$ (when both tiles are of the same type), but also $z = z' + \tau$ (when the left tile is of type $a$ and the right one is of $b$), or $z = z' - \tau$ (for the opposite order); see Figure 2 for an illustration.

In what follows, we first assume $z > 0$, and discuss the remaining cases afterwards. Due to the inflation structure, it is clear that the frequency of two supertiles of type $\alpha$ (left) and $\beta$ (right) at distance $z'$, determined relative to the tiles of the original size, is given by $\frac{1}{\tau} \nu_{\alpha\beta}(\frac{z'}{\tau})$. This follows from the simple observation that, for any $\Lambda' \in \mathbb{X}(\Lambda)$, the point set of the left endpoints of the supertiles is a set of the form $\tau \Lambda''$ for some $\Lambda'' \in \mathbb{X}(\Lambda)$.

Collecting all supertile distance frequencies that contribute to a given coefficient $\nu_{\alpha\beta}(z)$ then results in the claimed identities. For instance, $\nu_{bb}(z)$ equals the frequency of two type-$a$ supertiles at the same distance, counted relative to the original tile sizes, hence giving the last identity of (8), while $\nu_{aa}(z)$ has contributions from all pairs of supertiles at distance $z$.

The claim on the supports is obvious, and one can check that the set of equations properly extends to $z < 0$ via $\nu_{\alpha\beta}(-z) = \nu_{\beta\alpha}(z)$. Next, one can verify explicitly that the identities are also satisfied for $z = 0$, where they boil down to Eq. (9) below, because $\nu_{ab}(0) = \nu_{ba}(0) = 0$ as well as $\nu_{aa}(\pm 1) = \nu_{ba}(-1) = \nu_{ab}(1) = 0$ due to the geometry of the tiles. In particular, $\nu_{aa}(0)$ and $\nu_{ab}(0)$ are the relative prototile frequencies, in agreement with our setting. □

From the structure of the arguments, and recalling that $\tau^2 = \tau + 1$, it is immediately clear that all coefficients with arguments $z$ that satisfy $|z| > \tau^2$ are completely determined in a recursive manner from those with $|z| \leq \tau^2$, together with $\nu_{\alpha\beta}(z) = 0$ for $z \notin \Lambda - \Lambda$. This is the recursive part of the renormalisation relations. Likewise, the relations for all arguments with $|z| \leq \tau^2$ are closed, and provide finitely many linear equations to determine the $\nu_{\alpha\beta}(z) = 0$ for arguments in $[-\tau^2, \tau^2] \cap (\Lambda - \Lambda)$, which is the self-consistency part of the renormalisation relations. As mentioned in the Introduction, we use the term exact to demarcate the relations from similar (and widely used) concepts that are approximate or asymptotic in nature.
Let us begin with the latter, where we may further restrict ourselves to non-negative values of $z$ due to Eq. (5). Observe that

$$(A - A) \cap [0, \tau + 1] = \{0, 1, \tau, \tau + 1\},$$

which are thus the positions we have to consider first. Inserting the last relation of Eq. (8) into the first at $z = 0$ implies $\nu_{ab}(0) + \nu_{ba}(0) = 0$, which gives $\nu_{ab}(0) = \nu_{ba}(0) = 0$ via Eq. (5).

The first relation yields $\nu_{aa}(1) = 0$ because $1 \notin A - A$. This implies $\nu_{ab}(1) = 0$ via the third relation, which also gives $\nu_{aa}(-1) = \nu_{ba}(-1) = 0$ by symmetry. Since $\frac{1}{\tau} \notin A - A$, the last relation, at $z = 1$, gives $\nu_{bb}(1) = 0$, whence $\nu_{ba}(1)$ is the only correlation coefficient at 1 still to be accounted for.

The second relation implies $\nu_{ab}(\tau) = \frac{1}{\tau} \nu_{aa}(0)$, while the first gives $\nu_{aa}(\tau) = \frac{1}{\tau} \nu_{ba}(1)$. On the other hand, via the third relation, we see that $\nu_{ba}(1) = \frac{1}{\tau} (\nu_{aa}(\tau) + \nu_{ab}(\tau))$, and hence, via inserting the previous expressions for $\nu_{ba}(1)$ and $\nu_{ab}(\tau)$, also that

$$\nu_{aa}(\tau) = \frac{1}{\tau + 1} \left( \nu_{aa}(\tau) + \frac{1}{\tau} \nu_{aa}(0) \right).$$

This, in turn, gives $\nu_{aa}(\tau) = \frac{1}{\tau} \nu_{aa}(0)$ and thus also $\nu_{ba}(1) = \frac{1}{\tau} \nu_{aa}(0)$, so that all values at $z = 1$ are now determined. Since $2 \notin A - A$ and $\nu_{aa}(1) = 0$, we get $\nu_{ba}(\tau) = \nu_{bb}(\tau) = 0$, so that also all values at $z = \tau$ are accounted for. The values $\nu_{aa}(\tau + 1), \nu_{ab}(\tau + 1)$ and $\nu_{bb}(\tau + 1)$ now follow recursively from Eq. (8). Finally, the third relation gives

$$\nu_{ba}(\tau + 1) = \frac{1}{\tau} (\nu_{aa}(\tau + 1) + \nu_{ab}(\tau + 1))$$

and thus fixes the value $\nu_{ba}(\tau + 1)$. We have determined all coefficients with arguments inside $[-\tau^2, \tau^2]$.

Note that, at $z = 0$, the renormalisation relations reduce to the eigenvector equation

$$(9) \quad M \begin{pmatrix} \nu_{aa}(0) \\ \nu_{bb}(0) \end{pmatrix} = \tau \cdot \begin{pmatrix} \nu_{aa}(0) \\ \nu_{bb}(0) \end{pmatrix}$$
with the substitution matrix $M$ from above. Since the eigenspace of the PF eigenvalue $\tau$ is one-dimensional, the solution of Eq. (8) depends on a single number only, in line with what we derived from the fourth relation. In summary, we have the following result.

**Lemma 3.** The renormalisation relations of Eq. (8), subject to the symmetry relation of Eq. (5) and the condition that all coefficients $\nu_{\alpha\beta}(z)$ vanish for any $z \notin \Lambda - \Lambda$, have a unique solution once the value of $\nu_{aa}(0)$ is given.

**Proof.** The unique determination of all coefficients $\nu_{\alpha\beta}(z)$, as a function of $\nu_{aa}(0)$, for $z \in \Lambda - \Lambda$ with $|z| \leq \tau + 1$ was derived above. For all remaining $z \in \Lambda - \Lambda$, the recursive structure of Eq. (8) then provides unique values for the coefficients under the assumptions made. □

**Remark 1.** Note that we could have used a stronger restriction of the support, by defining one for each coefficient $\nu_{\alpha\beta}(z)$ separately, namely as $S_{\alpha\beta} = \Lambda - \Lambda$, which once again would give the same set for any element of the hull. Indeed, the set $\Lambda - \Lambda$ is a common superset of these individual supports. Nevertheless, this does not make any difference here, as the solution even with the larger support is the same, in the sense that the coefficients vanish at the additional points. Let us also recall [34, 5] that strict ergodicity of our dynamical system implies that $\nu_{\alpha\beta}(z) > 0$ if and only if $z \in S_{\alpha\beta}$.

Let us mention that $\Lambda - \Lambda$ is itself a model set for the CPS in Eq. (2), where one has $\Lambda - \Lambda = \Lambda ([-\tau, \tau])$. Later, we shall need to work with model sets with compact windows, such as $\Lambda ([-\tau, \tau])$. The previous lemma can be extended as follows.

**Proposition 4.** The linear renormalisation relations of Eq. (8), subject to the symmetry relation of Eq. (5) and the condition that all coefficients $\nu_{\alpha\beta}(z)$ vanish for any $z \notin \Lambda - \Lambda$, has a one-dimensional solution space. In other words, we get a unique solution once the value of $\nu_{aa}(0)$ is given.

**Proof.** Observe first that $\Lambda ([-\tau, \tau]) \setminus (\Lambda - \Lambda) = \{ \pm \frac{1}{\tau} \}$, which are two points inside the interval $[-\tau^2, \tau^2]$. Due to the symmetry relations, we only need to consider what the renormalisation relations of Eq. (8) impose at $z = \frac{1}{\tau}$. One finds $\nu_{bb}(\frac{1}{\tau}) = 0$ from the fourth relation, $\nu_{ba}(\frac{1}{\tau}) = 0$ from the third and $\nu_{aa}(\frac{2}{\tau}) = 0$ from the first, because the arguments on the right hand sides are then outside the set $\Lambda ([-\tau, \tau])$. Furthermore, one obtains

$$\nu_{ab}(\frac{1}{\tau}) = \frac{1}{\tau} (\nu_{aa}(\frac{1}{\tau}) + \nu_{ba}(\frac{1}{\tau})) = \frac{1}{\tau} \nu_{ab}(\frac{1}{\tau})$$

by the previous identities and the symmetry relation. Clearly, this implies $\nu_{ab}(\frac{1}{\tau}) = 0$, which brings us back to the situation of Lemma 2 and the claim is proved. □

**Remark 2.** Both in Lemma 2 and in Proposition 4 we started from conditions that are satisfied by the pair correlation coefficients of the Fibonacci chain. In particular, the symmetry condition (5) was imposed. Interestingly, one can dispense with it as follows. If one considers the relations (5) under the sole condition that all coefficients $\nu_{\alpha\beta}(z)$ vanish for any point $z \notin \Lambda - \Lambda$, but with no further assumption on the symmetry, the solution space is still one-dimensional. This can be proved explicitly by a slight extension of our calculations. Clearly,
any solution can uniquely be written as the sum of a function that is symmetric under exchanging the indices and simultaneously inverting the argument with another function that is anti-symmetric under this operation. Since we already know a symmetric solution, we may conclude that the only anti-symmetric solution to Eq. (8) is the trivial one in this example.

To proceed, let us introduce the measures

\[ \nu_{\alpha\beta} := \sum_{z \in \Lambda - \Lambda} \nu_{\alpha\beta}(z) \delta_z. \]

These are translation bounded pure point measures with \( \nu_{\alpha\beta}(\{z\}) = \nu_{\alpha\beta}(z) \). Defining the invertible continuous function \( f \) by \( f(x) = \tau x \) and its action \( f.\mu \) on a measure \( \mu \) as usual by \( (f.\mu)(g) := \mu(g \circ f) \), one finds \( f.\delta_x = \delta_{f(x)} \). With this, a short calculation shows that one can rewrite the recursion relations of Eq. (8) in measure-value form as

\[
\begin{align*}
\nu_{aa} &= \frac{1}{\tau} (f.\nu_{aa} + f.\nu_{ab} + f.\nu_{ba} + f.\nu_{bb}) \\
\nu_{ab} &= \frac{1}{\tau} \delta_{-\tau} \ast (f.\nu_{aa} + f.\nu_{ba}) \\
\nu_{ba} &= \frac{1}{\tau} \delta_{\tau} \ast (f.\nu_{aa} + f.\nu_{ab}) \\
\nu_{bb} &= \frac{1}{\tau} (f.\nu_{aa})
\end{align*}
\]

(10)

where it is understood that the support of the measures on the left hand sides is contained in \( \Lambda - \Lambda \), which is uniformly discrete. Let us note in passing that Eq. (10) can also be written as a matrix convolution identity.

**Corollary 5.** Consider the renormalisation relations of Eq. (10), subject to the symmetry conditions \( \tilde{\nu}_{\alpha\beta} = \nu_{\beta\alpha} \) and the requirement that each measure \( \nu_{\alpha\beta} \) is a pure point measure with support in \( \Lambda - \Lambda \). Then, there is a non-trivial solution of this system of equations, which is unique once the initial condition \( \nu_{aa}(\{0\}) \) is specified. The same conclusion holds if the support is allowed to be \( \Lambda([-\tau, \tau]) \).

**Proof.** Note first that, under the restriction to pure point measures, the condition \( \tilde{\nu}_{\alpha\beta} = \nu_{\beta\alpha} \) is just the reformulation of the symmetry relations from Eq. (5) in this setting. The claim now follows from Lemma 3 and Proposition 4. \( \square \)

Now, observing the identity

\[ \nu_{\alpha\beta} = \frac{\delta_{\alpha\alpha} \otimes \delta_{\beta\beta}}{\text{dens}(\Lambda)}, \]

we know from Lemma 3 that all measures in Eq. (10) are transformable, where \( \tilde{\nu}_{\alpha\beta} = \nu_{\beta\alpha} \) leads to \( \tilde{\nu}_{\alpha\beta} = \tilde{\nu}_{\beta\alpha} \). Since \( \hat{f.\mu} = \frac{1}{\tau} f^{-1}.\hat{\mu} \), a Fourier transform of the relations in Eq. (10) gives

\[
\begin{align*}
\tilde{\nu}_{aa} &= \frac{1}{\tau} f^{-1}(\tilde{\nu}_{aa} + \tilde{\nu}_{ab} + \tilde{\nu}_{ba} + \tilde{\nu}_{bb}) \\
\tilde{\nu}_{ab} &= \frac{1}{\tau} e^{-2\pi i \tau} f^{-1}(\tilde{\nu}_{aa} + \tilde{\nu}_{ba}) \\
\tilde{\nu}_{ba} &= \frac{1}{\tau} e^{2\pi i \tau} f^{-1}(\tilde{\nu}_{aa} + \tilde{\nu}_{ab}) \\
\tilde{\nu}_{bb} &= \frac{1}{\tau} f^{-1}\tilde{\nu}_{bb}
\end{align*}
\]

(11)
Let us check the consistency of Eq. (11) with the model set description that is available here. From the latter, we know that all \( \nu_{\alpha\beta} \) are pure point measures [5]. In particular, we have
\[
\nu_{\alpha\beta} = \sum_{k \in \mathcal{F}} I_{\alpha\beta}(k) \delta_k \quad \text{with Fourier module} \quad \mathcal{F} = \mathbb{Z}[\tau]/\sqrt{5} \quad \text{and intensities} \quad I_{\alpha\beta}(k) = \nu_{\alpha\beta}(\{k\}).
\]

An explicit calculation on the basis of [5, Sec. 9.4.1], adjusted to our use of relative frequencies, results in the relations
\[
\begin{align*}
I_{aa}(k) &= \frac{1}{\tau^2} \text{sinc}(\pi k^*)^2 \\
I_{ab}(k) &= \frac{1}{\tau} e^{-\pi i k^*} \text{sinc}(\pi k^*) \text{sinc}(\frac{\pi k}{\tau}) \\
I_{ba}(k) &= \frac{1}{\tau} e^{\pi i k^*} \text{sinc}(\pi k^*) \text{sinc}(\frac{2\pi k}{\tau}) \\
I_{bb}(k) &= \frac{1}{\tau^2} \text{sinc}^2(\frac{2\pi k}{\tau})^2
\end{align*}
\]
(12)
where \( I_{a\beta}(k) = \overline{I_{\beta\alpha}(k)} = I_{\beta\alpha}(-k) \). Here, we have \( I_{aa}(0) = (\nu_{aa}(0))^2 \) and \( I_{bb}(0) = (\nu_{bb}(0))^2 \), together with \( \sum_{\alpha,\beta} I_{\alpha\beta}(0) = 1 \) in line with our frequency normalisation. Note also that the 2\( \times \)2-matrix \( \mathcal{I}(k) := (I_{a\beta}(k)) \), at any fixed \( k \), is only Hermitian, not real in general. In fact, the term ‘intensity’ is only justified for the index pairs \( aa \) and \( bb \), where one has, up to a factor \( (\text{dens}(\Lambda))^2 \), the diffraction intensities of the Dirac combs \( \delta_{4a} \) and \( \delta_{4b} \), respectively. Still, the matrix \( \mathcal{I}(k) \) is positive semi-definite, with \( \det(\mathcal{I}(k)) = 0 \) for all \( k \in \mathbb{R} \).

**Proposition 6**. For all \( k \in \mathcal{F} \), where \( \mathcal{F} = \mathbb{Z}[\tau]/\sqrt{5} \) is the additive spectrum from Eq. (1), the intensity functions of Eq. (12) satisfy the relations
\[
\begin{align*}
I_{aa}(k) &= \frac{1}{\tau^2} \left( I_{aa}(\tau k) + I_{ab}(\tau k) + I_{ba}(\tau k) + I_{bb}(\tau k) \right), \\
I_{ab}(k) &= \frac{1}{\tau} e^{-2\pi i k} \left( I_{aa}(\tau k) + I_{ba}(\tau k) \right), \\
I_{ba}(k) &= \frac{1}{\tau} e^{2\pi i k} \left( I_{aa}(\tau k) + I_{ab}(\tau k) \right), \quad \text{and} \\
I_{bb}(k) &= \frac{1}{\tau^2} I_{aa}(\tau k).
\end{align*}
\]

**Proof.** The last relation is obvious from Eq. (12), as \( (\tau k)^* = -k^*/\tau \) and \( I_{aa}(k) \) is symmetric under \( k \mapsto -k \). For the other three, one needs some less obvious calculations. To do so, one has to use the fact that \( k \in \mathcal{F} \) implies \( k + k^* \in \mathbb{Z} \), so that
\[
e^{-2\pi i k} = e^{2\pi i (\tau k)^*} = e^{-2\pi i k^*/\tau}.
\]
The remaining explicit steps are now standard, and hence left to the reader. \( \square \)

Let us mention in passing that Proposition 6 also entails the scaling relation
\[
\det(\mathcal{I}(\tau k)) = \tau^4 \det(\mathcal{I}(k))
\]
for the intensity matrix. Since the intensities are clearly bounded, this is only compatible with \( \det(\mathcal{I}(k)) = 0 \), as calculated earlier from Eq. (12).

Looking again at Eq. (11) one realises that it can also be written in matrix form as
\[
\begin{pmatrix}
\hat{\nu}_{aa} \\
\hat{\nu}_{ab} \\
\hat{\nu}_{ba} \\
\hat{\nu}_{bb}
\end{pmatrix}
= \frac{1}{\tau^2} \mathbb{A}(\tau)
\begin{pmatrix}
f^{-1}\hat{\nu}_{aa} \\
f^{-1}\hat{\nu}_{ab} \\
f^{-1}\hat{\nu}_{ba} \\
f^{-1}\hat{\nu}_{bb}
\end{pmatrix}
\]
(13)
or $\tilde{\nu} = \tau^{-2} A(.) (f^{-1} \tilde{\nu})$ for short, with the matrix function

$$A(k) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
e^{-2\pi i r k} & 0 & e^{-2\pi i r k} & 0 \\
e^{2\pi i r k} & e^{2\pi i r k} & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} = B(k) \otimes \overline{B(k)}.$$

Here, $\otimes$ denotes the Kronecker product (the representation of the tensor product in the standard lexicographic choice of basis), and $B(k)$ is the matrix function

$$B(k) = \begin{pmatrix}
1 & 1 \\
e^{2\pi i r k} & 0
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} + e^{2\pi i r k} \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},$$

where $B(0) = M$ is the substitution matrix of the Fibonacci rule from above. Below, we will refer to $B(k)$ as the Fourier matrix of the inflation. The name is chosen to reflect the fact that $B(k)$ is the matrix of phase factors that emerge from the relative shifts of tiles within their level-1 supertiles.

**Lemma 7.** The matrix family $B_\varepsilon := \{ B(k) \mid 0 \leq k < \varepsilon \}$ is $\mathbb{C}$-irreducible for any $\varepsilon > 0$.

**Proof.** Let $\varepsilon > 0$ be fixed. Irreducibility of $B_\varepsilon$ means that the only simultaneous invariant subspaces of the entire family are the trivial spaces, $\{0\}$ and $\mathbb{C}^2$. The algebra generated by $B_\varepsilon$ does not depend on $\varepsilon$, and equals the (complex) algebra generated by the two matrices

$$D_0 = \begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} \quad \text{and} \quad D_\tau = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},$$

as is immediate from the representation in Eq. (14). It is routine to check (via products and linear combinations) that these two matrices generate the ring $\text{Mat}(2, \mathbb{C})$ with unit group $\text{GL}(2, \mathbb{C})$, which is clearly irreducible as a matrix group. \hfill $\square$

In fact, as we shall see in our later examples, the complex algebra generated by the (generalised) digit matrices $D_0$ and $D_\tau$ contains important information about the inflation $\varrho$; compare [42] for a justification of our terminology. In view of the meaning of the $D$-matrices, we call this algebra the inflation displacement algebra of $\varrho$, or IDA for short. The proof of Lemma 7 then also shows the following result.

**Corollary 8.** The IDA of the Fibonacci inflation rule $\varrho = \varrho_F$ is the full matrix algebra $\text{Mat}(2, \mathbb{C})$, and thus irreducible. \hfill $\square$

Let us go back to Eq. (13). Each entry in the measure vector $\tilde{\nu}$ has a unique decomposition

$$\tilde{\nu}_{\alpha \beta} = (\tilde{\nu}_{\alpha \beta})_{\text{pp}} + (\tilde{\nu}_{\alpha \beta})_{\text{cont}}$$

into its pure point (pp) and continuous (cont) part, where the supporting sets $F_{\alpha \beta}$ of the pure point parts are (at most) countable sets, while the continuous parts are concentrated on their
complements. Defining $F = \bigcup_{\alpha,\beta} F_{\alpha,\beta}$, we see that $F$ is still (at most) a countable set and that we can now write

$$\left(\tilde{\nu}_{\alpha,\beta}\right)_{\text{pp}} = \tilde{\nu}_{\alpha,\beta}\big|_F \quad \text{and} \quad \left(\tilde{\nu}_{\alpha,\beta}\right)_{\text{cont}} = \tilde{\nu}_{\alpha,\beta}\big|_{F^c}$$

with $F^c := \mathbb{R} \setminus F$, simultaneously for all $\alpha, \beta$. In particular, we then have a clear meaning of the decomposition $\tilde{\nu} = (\tilde{\nu})_{\text{pp}} + (\tilde{\nu})_{\text{cont}}$.

**Proposition 9.** Let $\tilde{\nu}$ be a solution of Eq. (13). Then, the pure point part $(\tilde{\nu})_{\text{pp}}$ and the continuous part $(\tilde{\nu})_{\text{cont}}$ satisfy Eq. (13) separately.

**Proof.** The pure point and continuous parts are mutually orthogonal in the measure-theoretic sense; compare [5, Prop. 8.4]. Since $A(k)$ is analytic in $k$ and $f^{-1}$ just induces a rescaling, it is clear that $(A(.)\left(f^{-1}\tilde{\nu}\right))_{\text{pp}} = A(.)\left(f^{-1}\tilde{\nu}\right)_{\text{pp}}$, and analogously for the continuous parts. With the supporting set $F$ from above, which we may assume to be invariant under the function $f$ without loss of generality, the evaluation of $\tilde{\nu}$ on any Borel set $B$ can thus be split into two terms via $B = (B \cap F) \cup (B \cap F^c)$, from which the claim follows by standard arguments because the pure point and the continuous components cannot mix. □

Now, the pure point and continuous parts are transformable measures [23], and their (inverse) Fourier transforms provide the unique decomposition

$$\nu = (\nu)_s + (\nu)_0$$

of $\nu$ into its strongly almost periodic $(s)$ and null weakly almost periodic $(0)$ parts, to be read componentwise as before. Since Fourier transform is invertible on transformable measures, these parts must then separately satisfy the renormalisation relations of Eq. (10). Note that the parts still also satisfy the symmetry relation, but it is not clear what the supporting sets of the parts are. This is caused by $\Lambda - \Lambda$ not being a group, wherefore we initially get such a decomposition only within $\mathbb{R}$. To improve the situation, we need a smaller covering object of $\Lambda - \Lambda$ with good harmonic properties to continue.

At this point, we do not refer to the (known) model set description recalled earlier, but rather proceed by employing Strungaru’s result from Section 2. This tells us that both parts, $(\nu)_s$ and $(\nu)_0$, still are pure point measures with support in $\mathcal{R} \left([-\tau, \tau]\right)$. This has the following strong consequence, which crucially builds on Proposition 4 in a non-trivial way.

**Theorem 10.** Let $\nu$ be the unique solution of Eq. (11) according to Proposition 4, with initial condition $\nu_{aa} (\{0\}) = \frac{1}{\tau}$, say. If $\nu = (\nu)_s + (\nu)_0$ is the decomposition from Eq. (15), one has $(\nu)_0 = 0$, which means that all measures $\tilde{\nu}_{\alpha,\beta}$ are pure point measures.

**Proof.** From Proposition 4 we know that the solution of Eq. (11) is unique under the constraints formulated in this proposition. These contraints are met by both parts from Eq. (15) separately, which are both supported on $\mathcal{R} \left([-\tau, \tau]\right)$ in the CPS of Eq. (2) by Strungaru’s theorem [41]. However, we do not know how the initial condition is split between the two
parts. However, the uniqueness of the solution means that the two parts are either proportional to one another (which is controlled by the initial condition at 0), or one part is trivial. We are clearly in the latter case here, because a non-zero measure cannot be strongly almost periodic and null-weakly almost periodic at the same time; compare [23, 41].

Since we know that the Fourier transform of $\nu_{aa}$ must have a Dirac measure with positive weight at 0 (in fact, we even know that the pure point part has a relatively dense support by a result due to Strungaru [40]), we have $(\nu)_0 \neq 0$, hence $(\nu)_0 = 0$ and $\hat{\nu}$ is a vector of pure point measures as claimed. □

This provides a (partly) independent proof of the pure point nature for the diffraction measure of the Fibonacci inflation rule. It is only partly independent in the sense that it needs Strungaru’s theorem for Eq. (15) as input, and this theorem still relies, to some extent, on the CPS in the background. Our approach may be viewed as some explicit way to prove that the Fibonacci dynamical system is an a.e. 1-1 cover of its maximal equicontinuous (or Kronecker) factor. Note that this approach only provides the nature of the diffraction measure, but not its explicit form, which was described earlier via the projection formalism.

4. Example 2: Thue–Morse and Rudin–Shapiro

As another example, let us take a look at the classic sequences of Thue–Morse and Rudin–Shapiro. Since both are examples of constant length substitutions, the symbolic and the geometric pictures coincide. This results in a significant simplification in the sense that one can derive recursion relations directly for the autocorrelation coefficients, as explained in detail in [5, Chs. 10.1 and 10.2]. Nevertheless, it is instructive to also take a look at the proper analogues of Eq. (8), which is actually in line with the general treatment in [34, 13]. The common feature is an additional symmetry that shows up as an involution on the alphabet, which consists of an even number of letters. We call it a bar swap symmetry and use an adjusted alphabet to highlight its action. For other examples with a bar swap symmetry, we refer to [3, 6] and references therein.

4.1. Thue–Morse. In the light of our general comment, we use the alphabet $\mathcal{A} = \{a, \bar{a}\}$ and the substitution $a \mapsto aa$, $\bar{a} \mapsto \bar{a}a$. The alphabet as well as the substitution rule is invariant under the bar swap involution $a \leftrightarrow \bar{a}$, which we call $P$. Let us now assume that the letters represent intervals of length 1, so that the coincidence of the symbolic and the geometric picture is compatible with the embedding of $\mathbb{Z}$ in $\mathbb{R}$. Note that is suffices to study the $\mathbb{Z}$-action in this case, as the $\mathbb{R}$-action emerges from a standard suspension [18].

In this setting, and for each $\alpha, \beta \in \mathcal{A}$, the correlation coefficient $\nu_{\alpha\beta}$ has support inside $\mathbb{Z}$, and the analogue of Eq. (8) can compactly be written as

$$2\nu_{\alpha\beta}(z) = \nu_{\alpha\beta}(\frac{z}{2}) + \nu_{\alpha\beta}(\frac{z-1}{2}) + \nu_{\alpha\beta}(\frac{z+1}{2}) + \nu_{\alpha\beta}(\frac{z}{2}),$$

with the understanding that a coefficient always vanishes when the argument is not an integer. It is clear that this set of four equations is invariant under a bar swap. More precisely, applying
$P$ to all first indices just permutes the four equations, and the same is true when $P$ is applied to all second indices, or to all indices.

Before we analyse Eq. (16), let us consider the autocorrelation coefficients of the TM system. Via standard arguments, compare [5, Rem. 10.3], one needs two sets for the decomposition of a general TM chain (resp. its autocorrelation) with weights $h_a$ and $h_b$. We thus define

$$\eta_{\pm}(z) := (\nu_{aa}(z) + \nu_{ab}(z)) \pm (\nu_{ba}(z) + \nu_{aa}(z))$$

and observe that the relations (16) then lead to the decoupled relations

$$\eta_{\pm}(z) = \eta_{\pm}(\frac{z}{2}) \pm \frac{1}{2} \left( \eta_{\pm}(\frac{z-1}{2}) + \eta_{\pm}(\frac{z+1}{2}) \right),$$

which is a consequence of the bar swap symmetry. Since $\eta_{\pm}$ are functions on $\mathbb{Z}$, we now distinguish even and odd $z$. This allows to rewrite the last recursion as

$$\eta_{\pm}(2n) = \eta_{\pm}(n),$$

$$\eta_{\pm}(2n + 1) = \pm \frac{1}{2} (\eta_{\pm}(n) + \eta_{\pm}(n + 1)),$$

with $n \in \mathbb{Z}$.

It is easy to check inductively that Eq. (18) implies $\eta_{\pm}(n) = \eta_{\pm}(0)$ for all $n \in \mathbb{Z}$, so that $\sum_{n \in \mathbb{Z}} \eta_{\pm}(n) \delta_n = \eta_{\pm}(0) \delta_{\mathbb{Z}}$. Moreover, one finds $\eta_{\pm}(\pm 1) = -\frac{1}{2} \eta_{\pm}(0)$, so that the coefficients $\eta_{\pm}(n)$ are the autocorrelation coefficients of the balanced Thue–Morse chain as described in [5, Sec. 10.1]. Alternatively, they can be viewed as the Fourier coefficients of the standard choice of the maximal spectral measure in the orthocomplement of the pure point sector, in line with the original treatment in [27]; see also [34].

Let us return to the original relations in Eq. (16). Here, the interesting situation arises that the solution space does depend on the support. This is an important difference to the Fibonacci example in Section 3. For a precise formulation, we select a fixed point of the inflation, with seed $a|a$ say, and define the decomposition $\mathbb{Z} = \Lambda_a \cup \Lambda_b$ with $\Lambda_\alpha$ denoting all integers that are the left endpoint of an interval of type $\alpha$. Then, analogously to before, $S_{\alpha\beta} = A_\beta - A_\alpha$ is the support of the relative pair correlation coefficient $\nu_{\alpha\beta}$ of the TM system. Note that this support is the same for the entire hull, even though we defined it via a selected fixed point, as a consequence of strict ergodicity of the dynamical system.

**Proposition 11.** The renormalisation relations (16), viewed as relations between functions $\nu_{\alpha\beta} : \mathbb{Z} \to \mathbb{R}$ with $\alpha, \beta \in \mathcal{A}$, possesses a two-dimensional solution space.

Moreover, under the additional restriction that $\text{supp}(\nu_{\alpha\beta}) = S_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{A}$, the solution space is only one-dimensional.

**Proof.** Here, the finitely many self-consistency equations are those with arguments in $[-1, 1]$. As before, they fully determine the dimension of the solution space, because all other equations are of a purely recursive nature and determine the remaining coefficients uniquely. Our claim is now a straight-forward calculation, which can be left to the reader. □
Let us explore the difference to the Fibonacci inflation in more algebraic terms. Here, the Fourier matrix $B$ reads

$$B(k) = \begin{pmatrix} 1 & e^{2\pi i k} \\ e^{2\pi i k} & 1 \end{pmatrix} = 1 + e^{2\pi i k} J$$

with $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This implies that the IDA is generated by the commuting digit matrices

$$D_0 = 1 \quad \text{and} \quad D_1 = J.$$

As a consequence, the IDA is reducible and certainly not the full matrix algebra Mat($2, \mathbb{C}$). Indeed, the eigenspaces of $J$, namely $\mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, are non-trivial invariant subspaces of the IDA. Consequently, also the Kronecker product

$$A(k) = B(k) \otimes B(k) = \begin{pmatrix} 1 & e^{-2\pi i k} & e^{2\pi i k} & 1 \\ e^{-2\pi i k} & 1 & 1 & e^{2\pi i k} \\ e^{2\pi i k} & 1 & 1 & e^{-2\pi i k} \\ 1 & e^{2\pi i k} & e^{-2\pi i k} & 1 \end{pmatrix}$$

has a $k$-independent eigenbasis. The two eigenvectors $(1, 1, 1, 1)^T$ and $(1, -1, -1, 1)^T$ correspond to the two solutions $\eta_{\pm}$ from Eq. (17). In fact, each of these eigenvectors is the Kronecker product of one of the eigenvectors of $B(k)$ with itself. The (symmetrised) mixed Kronecker products, one eigenvector times the other, do not play a role here. Even though these vectors span an invariant subspace, too, the off-diagonal sectors of the tensor product alone cannot lead to a positive measure. This is possible only in combination with the diagonal parts, which span invariant subspaces already by themselves \[34\] [13].

4.2. Rudin–Shapiro. For this example, we use the alphabet $A = \{a, b, \bar{a}, \bar{b}\}$ together with the substitution rule

$$\varrho_{RS}: \quad a \mapsto ab, \quad b \mapsto a\bar{b}, \quad \bar{a} \mapsto \bar{a}b, \quad \bar{b} \mapsto \bar{a}\bar{b}.$$

This is equivalent to the formulation used in \[5\] Sec. 4.7.1 via the identifications $a \equiv 0, \bar{a} \equiv 3, b \equiv 2$ and $\bar{b} \equiv 1$. As before, both the alphabet and the substitution rule are invariant under the complete bar swap $a \leftrightarrow \bar{a}, b \leftrightarrow \bar{b}$, again called $P$. So, we have

$$P(A) = A \quad \text{and} \quad \varrho_{RS} \circ P = P \circ \varrho_{RS},$$

which has similar consequences as in the previous example. In fact, we have a little more in this example: One can also check that

$$R \circ \varrho_{RS} = E \circ \varrho_{RS} \circ E,$$

where $R$ is the permutation $a \mapsto b \mapsto \bar{a} \mapsto \bar{b} \mapsto a$ of order 4, with $R^2 = P$, and $E$ is the letter exchange involution defined by $a \leftrightarrow b, \bar{a} \leftrightarrow \bar{b}$. Since $R \circ E$ is another involution, the group generated by $R$ and $E$ is the dihedral group of order 8.

The Fourier matrix for $\varrho_{RS}$ reads

$$B(k) = D_0 + e^{2\pi i k} D_1,$$
with the (non-commuting) generating digit matrices

\[ D_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \]

As one can easily check, \( \ker(D_0) \cap \ker(D_1) = \mathbb{C}(1, -1, 1, -1)^4 \), which is a non-trivial invariant subspace. The IDA generated by \( D_0 \) and \( D_1 \) is thus reducible, but the two digit matrices cannot be diagonalised simultaneously. However, one has the block diagonal form

\[ T_P D_0 T_P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_P D_1 T_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \]

where \( T_P = \frac{1}{\sqrt{2}} \left( \frac{1}{2} \right) \otimes 1_2 \) is an involution that is induced by the bar swap map \( P \).

The renormalisation relations for the pair correlation coefficients read

\[ 2 \nu_{aa}(z) = \nu_{aa} \left( \frac{z}{2} \right) + \nu_{ab} \left( \frac{z}{2} \right) + \nu_{ba} \left( \frac{z}{2} \right) + \nu_{bb} \left( \frac{z}{2} \right), \]

\[ 2 \nu_{ab}(z) = \nu_{aa} \left( \frac{z-1}{2} \right) + \nu_{ab} \left( \frac{z-1}{2} \right) + \nu_{ba} \left( \frac{z+1}{2} \right) + \nu_{bb} \left( \frac{z+1}{2} \right), \]

\[ 2 \nu_{ba}(z) = \nu_{aa} \left( \frac{z+1}{2} \right) + \nu_{ab} \left( \frac{z+1}{2} \right) + \nu_{ba} \left( \frac{z+1}{2} \right) + \nu_{bb} \left( \frac{z+1}{2} \right), \]

\[ 2 \nu_{bb}(z) = \nu_{aa} \left( \frac{z}{2} \right) + \nu_{ab} \left( \frac{z}{2} \right) + \nu_{ba} \left( \frac{z}{2} \right) + \nu_{bb} \left( \frac{z}{2} \right). \]

Here, each line actually represents four equations, the other three being obtained by simultaneously applying \( P \) to all first, to all second, or to all indices at once. Let \( S_{\alpha \beta} \) again denote the supports of the coefficient functions, defined in complete analogy to above. Then, with essentially the same arguments as in Proposition 11 one finds the following result.

**Proposition 12.** The 16 renormalisation relations specified by Eq. (19), viewed as relations between functions \( \nu_{\alpha \beta} : \mathbb{Z} \to \mathbb{R} \) with \( \alpha, \beta \in \mathcal{A} \), possesses a two-dimensional solution space.

Moreover, under the additional restriction that \( \text{supp}(\nu_{\alpha \beta}) = S_{\alpha \beta} \) for all \( \alpha, \beta \in \mathcal{A} \), the solution space is only one-dimensional. \( \square \)

The next step, as in the previous section, consists in the analysis of the Kronecker product

\[ A(k) = B(k) \otimes \overline{B(k)} = D_0 \otimes D_0 + D_1 \otimes D_1 + e^{2\pi i k} D_1 \otimes D_0 + e^{-2\pi i k} D_0 \otimes D_1 \]

with respect to the splitting into even and odd sectors under \( x \otimes y \mapsto \overline{y} \otimes \overline{x} \), viewed as real vector spaces, and with respect to further invariant subspaces, which exist as a result of Proposition 12. As we have seen above, with the change of basis induced by \( T_P \), \( B(k) \) can be brought to block-diagonal form, \( B(k) = B_+ \otimes B_-(k) \), which acts on a direct sum \( V_+ \oplus V_- \), both of whose summands are separately left invariant. As in the Thue–Morse case, only the tensors in \( V_+ \otimes V_+ \) and \( V_- \otimes V_- \) can lead to positive measures. The off-diagonal parts in the tensor product alone cannot lead to a positive measure. As \( B_+(k) \) has rank one, the corresponding Kronecker product \( B_+(k) \otimes \overline{B_+(k)} \) also has rank one, and is symmetric.
On the subspace $V_+ \otimes V_+$, $A(k)$ acts with an eigenvalue $2 + 2\cos(2\pi k)$, which results in the pure point part of the spectrum of the Rudin–Shapiro sequence, just as in the Thue–Morse case. On the subspace $V_-$, on the other hand, $B_-(k)$ generates the full matrix algebra, and its tensor product algebra, generated by $B_-(k) \otimes B_-(k)$, is irreducible, too, if confined to the symmetric sector of $V_- \otimes V_-$. This irreducible invariant sector carries the other spectral component, which is known to be absolutely continuous; compare [34, 13].

5. Example 3: Twisted silver mean

The Thue–Morse and Rudin–Shapiro sequences from the last section are two examples of structures with a mixed spectrum. Many more, also higher-dimensional ones, can be found in [19, 20, 5, 3, 4, 6, 13]. Most of them have two features in common, which they share with the Thue–Morse and Rudin–Shapiro sequences: They are based on constant-length substitutions, where the letters (which can be identified with the prototiles here) come in pairs, such that prototiles within a pair are geometrically equal, but still differ in their type. We denote this, as before, by the presence or absence of a bar.

To expand on this, let $A$ denote the set of prototiles. The key feature then is that the map $P: A \rightarrow A$ which changes (or swaps) the bar status of all prototiles simultaneously, so $\alpha \mapsto \bar{\alpha}$ for all $\alpha \in A$ with $\bar{\bar{\alpha}} = \alpha$, commutes with the inflation rule, and is thus a symmetry. Clearly, the map $P$ is an involution and has an obvious extension to arbitrary finite patches of tiles, and then also to the hull defined by a primitive inflation rule for $A$. By slight abuse of notation, we always denote the bar swap by $P$. The map $P$ is the key to constructing examples with mixed spectrum, and we shall see in this section that it is not confined to constant length substitutions.

5.1. General Setup. Before constructing particular examples, let us analyse the general situation. For ease of exposition, we assume a one-dimensional tiling, though the entire construction works in higher dimensions as well. Let $\sigma$ be a primitive inflation rule on a prototile set $A$ with bar swap symmetry, so that

$$P(A) = A \quad \text{and} \quad \sigma \circ P = P \circ \sigma$$

hold for the map $P$ defined above. Let $\Omega$ be the hull that is generated by $\sigma$; see [5] for background. We will now construct a globally 2-1 factor map $\varphi$ from the dynamical system $(\Omega, \mathbb{R})$ to a factor dynamical system $(\Omega', \mathbb{R})$, which identifies tilings related by a bar swap, so $\varphi(\omega) = \varphi(P(\omega))$ for all $\omega \in \Omega$. By construction, the map $\varphi$ commutes with the translation action, so that we may suppress the latter without danger of confusion.

To construct such a factor map explicitly, we first rewrite the inflation $\sigma$ in terms of collared tiles. The latter consist of pairs of tiles $t_1t_2$, where $t_1$ is the actual tile and $t_2$ is a (one-sided) collar of $t_1$. This is seen as a label attached to $t_1$ that specifies the type of the right neighbour tile of $t_1$. Obviously, there is an induced inflation rule $\tilde{\sigma}$ on the collared tiles. If $\sigma(t_1) = u_1 \ldots u_k$, and $\sigma(t_1t_2) = u_1 \ldots u_{\ell}$, the inflation of the collared tile $t_1t_2$ consists of the sequence of collared tiles $u_iu_{i+1}$, where $i$ runs from 1 to $k$. The new inflation is still primitive,
and defines a unique hull, which we call \( \tilde{\Omega} \). Clearly, collaring is a local operation, which has a local inverse (the forgetful map wiping out the collars), and it commutes with the inflation. In other words, the collared and the uncollared inflations, \( \tilde{\sigma} \) and \( \sigma \), define mutually locally derivable (MLD) hulls, which are thus topologically conjugate; compare [5, Sec. 5.2].

We now observe that also \( \tilde{\sigma} \) has a bar swap symmetry, which simultaneously swaps the bar status of the tile and its collar. We can now consider the factor map \( \varphi \) that is induced by identifying collared tiles related by a bar swap. Clearly, \( \varphi \) also identifies pairs of global tilings related by a bar swap, so \( \varphi(\tilde{\omega}) = \varphi(P(\tilde{\omega})) \) for all \( \tilde{\omega} \in \tilde{\Omega} \). Since \( \Omega \simeq \tilde{\Omega} \), our procedure induces a unique mapping from \( \Omega \) to \( \Omega' := \varphi(\tilde{\Omega}) \), which we simply call \( \varphi \) again.

Let \( \omega' \in \Omega' \) now be a tiling in the image of \( \varphi \), given by a bi-infinite sequence of tiles \( t_i \), where \( i \in \mathbb{Z} \). The collared tile \( t_0t_1 \) has exactly two possible preimages — let us chose one of them. The neighbouring collared tile, \( t_1t_2 \), also has two preimages, but as we have already chosen a preimage of the collared tile \( t_0t_1 \), and thus of \( t_1 \), there is only one choice left. Continuing like this, we see that, once we have chosen a preimage of the collared tile \( t_0t_1 \), the lifts of all other tiles to the right is fixed, and analogously to the left as well. Consequently, \( \omega' \) has precisely two preimages, and the mapping \( \varphi: \Omega \to \Omega' := \varphi(\tilde{\Omega}) \), which we simply call \( \varphi \) again.

Wiping out all bars from the tiles of the original inflation \( \sigma \) also induces a factor map, but this one need not be globally 2-1. Let \( \Omega'' \) denote the image under this map and note that the latter must also be a factor of \( \Omega' \), so we actually have a sequence of factor maps

\[
\Omega \xrightarrow{\varphi_{\text{2-1}}} \Omega' \xrightarrow{\varphi'_{\text{1-1 a.e.}}} \Omega''
\]

where \( \Omega' \) is obtained by identifying collared tiles that are related by a bar swap, while \( \Omega'' \) is obtained by identifying original tiles related by a bar swap. The second map \( \varphi' \) is 1-1 a.e., because the composition of \( \varphi \) and \( \varphi' \) is a.e. 2-1. Almost all tilings in \( \Omega'' \) consist of a single, infinite order supertile, and these have exactly two preimages in \( \Omega \), which differ by a bar swap. Only tilings consisting of two adjacent infinite order supertiles may have more than two preimages, but these are of measure zero. \( \Omega' \) and \( \Omega'' \) may coincide (in the Thue–Morse case they do), but in general they are different. Note that all systems under consideration here are strictly ergodic.

Each of the translation dynamical systems has a maximal equicontinuous factor (MEF), also known as their Kronecker factor, so that the above sequence of factor maps can be completed as follows,

\[
\Omega \xrightarrow{\varphi_{\text{2-1}}} \Omega' \xrightarrow{\varphi'_{\text{1-1 a.e.}}} \Omega''
\]

(20)

\[
\Omega_{\text{MEF}} \xrightarrow{\psi} \Omega'_{\text{MEF}} \xrightarrow{\varphi'_{\text{1-1 a.e.}}} \Omega''_{\text{MEF}}
\]

It is well known (compare [12]) that a tiling dynamical system has pure point dynamical spectrum if and only if the factor map to the underlying MEF is 1-1 almost everywhere. Let us now assume that \( \Omega' \), and thus also \( \Omega'' \) by standard results [8], has pure point spectrum,
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so that both \( \xi' \) and \( \xi'' \) are 1-1 a.e. As \( \varphi' \) is 1-1 a.e., the MEFs \( \Omega_M' \) and \( \Omega_M'' \) are equal. There still remain two possibilities, however. The map \( \psi \) is a group homomorphism, and can thus be either 1-1 or 2-1. If \( \psi \) is 1-1, this implies that \( \xi \) is 2-1 a.e., so that \( \Omega \) must have mixed spectrum, whereas, if \( \psi \) is 2-1, \( \xi \) is 1-1 a.e., and \( \Omega \) has pure point spectrum.

In order to compare \( \Omega_M' \) and \( \Omega_M'' \), we need to compare their respective modules of eventual return vectors. Recall that \( r \) is a return vector of a tiling dynamical system \( \Omega \) when there exist two tiles \( t_1 \) and \( t_2 \) of the same type in some tiling \( \omega \in \Omega \) such that the distance of their left endpoints is \( r \). The module of return vectors, \( R_\Omega \), is the \( \mathbb{Z} \)-span of all return vectors. This is a finitely generated submodule of the \( \mathbb{Z} \)-module generated by all tile lengths, \( T_\Omega \). The (additive) pure point spectrum of \( \Omega \) now consists of all those \( k \in \mathbb{R} \) such that, for any return vector \( r \), one has \( e^{2\pi i \lambda k r} \to 1 \) as \( n \to \infty \) by \[38\]. This pure point spectrum can only be non-trivial if the inflation factor (or multiplier) \( \lambda \) is a PV number, which we assume.

The quantity of interest now is the \( \mathbb{Z} \)-module of eventual return vectors, given by

\[
\mathcal{M}_\Omega = \langle \{ x \in T_\Omega \mid \lambda^n x \in R_\Omega, \text{ for some } n \in \mathbb{N} \} \rangle_\mathbb{Z}.
\]

Clearly, \( \Omega \) and \( \Omega' \) have the same MEF if and only if \( \mathcal{M}_\Omega = \mathcal{M}_\Omega' \). For constant length substitutions, the module \( \mathcal{M}_\Omega \) is known as the height lattice; compare \[20\]. If \( \mathcal{M}_\Omega = h \mathbb{Z} \), with \( |h| > 1 \), the substitution is said to have non-trivial height \( |h| \). What matters in our context, however, is not whether any height is non-trivial, but whether \( \Omega \) and \( \Omega' \) have the same height.

5.2. Extending the silver mean inflation. Let us now look at concrete examples. Our goal is to construct almost 2-1 extensions of the silver mean inflation \[5, \text{Ex. 4.5}\]

\[
\sigma_{sm} : \quad a \mapsto ab, \quad b \mapsto a,
\]

in such a way that we gain a bar swap symmetry. The scaling factor (or inflation multiplier) of \( \sigma_{sm} \) is \( \lambda = 1 + \sqrt{2} \), and the natural tile lengths for \( a \) (long) and \( b \) (short) are \( \lambda \) and 1, respectively. Here, \( \lambda \) is a Pisot unit, and it is well known that \( \sigma \) generates tilings with pure point spectrum; see \[5, \text{Chs. 7 and 9}\] for details.

We now add a barred version of each tile, thus giving the new prototile set \( A = \{ a, b, \bar{a}, \bar{b} \} \), and introduce an inflation which is primitive and commutes with the bar swap involution \( P \). Furthermore, this inflation shall reduce to \( \sigma \) under the identification of \( a \) with \( \bar{a} \) and \( b \) with \( \bar{b} \). A first attempt of such an inflation could be

\[
\tilde{\sigma} : \quad a \mapsto \bar{a}b, \quad b \mapsto a, \quad \bar{a} \mapsto \bar{a}a, \quad \bar{b} \mapsto \bar{a}.
\]

It is easy to see that, for any element of the hull \( \Omega \) defined by \( \tilde{\sigma} \), exactly every second tile carries a bar. As multiplication by \( \lambda \) induces a isomorphism on \( T_\Omega \), this means that the module of eventual return vectors \( \mathcal{M}_\Omega \) is an index-2 submodule of \( T_\Omega \), whereas \( \mathcal{M}_\Omega' = T_\Omega \). The multiplicity of the map \( \psi \) in diagram \[20\] is thus 2, and the map \( \xi \) is 1-1 a.e. in this case, so that \( \Omega' \) still has pure point spectrum.
Our next attempt is the inflation $\tilde{\sigma}$, given by

$$\tilde{\sigma} : \ a \rightarrow ab\bar{a}, \ b \rightarrow \bar{a}, \ \bar{a} \rightarrow \bar{a}b, \ \bar{b} \rightarrow a.$$ 

This inflation is primitive, commutes with $P$, and it is easy to see that $R_\Omega = M_\Omega = T_\Omega$, so that $\tilde{\sigma}$ must have mixed spectrum. We call it the *twisted silver mean* (TSM) inflation. It is instructive to have a closer look at $\Omega'$. For that purpose, we rewrite the inflation in terms of collared tiles:

$$\sigma_1 : \ A \mapsto CD\bar{B}, \ B \mapsto CD\bar{A}, \ C \mapsto CD\bar{A}, \ D \mapsto \bar{A},$$

together with $\bar{\alpha} \mapsto P(\sigma_1(\alpha))$ for all $\alpha \in \{A, B, C, D\}$, where $A = aa, \ B = a\bar{a}, \ C = ab,$ and $D = b\bar{a}$. There are three variants of the long tile, and one short tile. However, since $\sigma_1(B) = \sigma_1(C)$, we can actually merge the two prototiles $B$ and $C$, and then rename the old $D$ as the new $C$, so that we arrive at

$$\sigma_2 : \ A \mapsto BCD\bar{B}, \ B \mapsto BCD\bar{A}, \ C \mapsto \bar{A},$$

once again with $\bar{\alpha} \mapsto P(\sigma_2(\alpha))$. The hulls of $\sigma_1$ and $\sigma_2$ are MLD, so that we can stick to the latter. If we wipe out the bars in $\sigma_2$, we obtain the inflation for $\Omega'$, which has pure point spectrum in this case, and actually is a model set by standard arguments [9], because $\Omega'$ is an a.e. 1-1 extension of the original silver mean hull.

For each prototile type, when combining a tile with its barred version, there is an associated window in the CPS

$$\mathbb{R} \xleftarrow{\pi} \mathbb{R} \times \mathbb{R} \xrightarrow{\pi_{\text{int}}} \mathbb{R} \quad \text{dense} \ \cup \ \cup \ \cup \ \text{dense}$$

and

$$\mathbb{Z}[[\sqrt{2}]] \xleftarrow{1-1} \mathcal{L} \xrightarrow{1-1} \mathbb{Z}[[\sqrt{2}]]$$

When we distinguish according to the bars, however, we do not have a model set. Nevertheless, for any fixed tiling $\omega$, we can still take the set of left endpoints of all tiles of a given type (which is a subset of $\mathbb{Z}[\lambda] = \mathbb{Z}[[\sqrt{2}]]$ in our setting), and see what the closure of the image under the $*\text{-map}$ of this set is. In this way, we can determine a covering window for each tile type. Of course, we cannot expect these covering windows for the different tile types to be disjoint. The result is shown in the Fig. [3].

We see that a barred and an unbarred tile always share the same covering window. Of course, every left endpoint of a tile is either the endpoint of a barred or an unbarred tile, but not both. So, the identical covering windows emerge as the closure of disjoint point sets. This means that one cannot determine the bar status of a tile by looking at its internal space coordinate. The bars represent a sort of chemical modulation of the original silver mean tiling, where the modulation is completely independent of internal space coordinates.

Another interesting point is that the hull $\Omega''$ of the original silver mean tiling is indeed different from $\Omega'$ as generated by the inflation (21). The latter has two long tiles, not one,
and is not MLD to $\Omega''$. To see this, let us look at the structure of the windows. In suitable units, the total window is an interval of length $1 + \lambda = 2 + \sqrt{2}$, which is split into three pieces of length 1, 1, and $\sqrt{2}$. The first bit belongs to the short tile, the second to one of the long tiles, and the third part is fractally split between the two long tiles. It is first assigned to one of them, but is then split in the proportion $1 : \sqrt{2} : 1$, and the middle part is assigned to the other long tile, where it is again split into three pieces, the middle one of which is assigned to the first long tile, and so on. Although all interval boundaries showing up in this process are contained in $\mathbb{Z}[\lambda^*] = \mathbb{Z}[\sqrt{2}]$, there is an accumulation point of interval boundaries in the middle of the subwindow of length $\sqrt{2}$, which is not contained in $\mathbb{Z}[\sqrt{2}]$. This point represents an extra singular cut in the CPS. Two tilings correspond to this point, both of which project to the same silver mean tiling.

This degeneracy also shows up in the cohomology. Using the Anderson–Putnam method \[1\], it is routine to compute $H_1(\Omega) = \mathbb{Z}^3$ and $H_1(\Omega') = \mathbb{Z}^2$, which is in line with \[22, \text{Thm. 5.1}\]:

The extra dimension comes from the extra $\mathbb{Z}[\sqrt{2}]$-orbit of singular points.

5.3. Spectral Structure. From the last section, we know that the TSM tiling dynamical system has a mixed spectrum. So, in addition to the well-known pure point part, there must be a continuous part in the spectrum. Here, we want to investigate what the nature of this continuous part is.

We have seen that the dynamical system of the twisted silver mean tiling is an almost 2-1 extension of the dynamical system of the original silver mean tiling. Due to the bar swap symmetry, the Hilbert space of square-integrable functions on the hull, $L^2(\Omega_{tsm})$, is a tensor product, and can be split into an even and an odd sector under the bar swap symmetry:

\[
L^2(\Omega_{tsm}) = L^2(\Omega_{sm}) \otimes \mathbb{C}^2 = \left( L^2(\Omega_{sm}) \otimes \chi_+ \right) \oplus \left( L^2(\Omega_{sm}) \otimes \chi_- \right) =: \mathcal{H}_+ \oplus \mathcal{H}_-,
\]

where $\chi_{\pm}$ is the even/odd character of the bar swap $P$. The factor map induces a corresponding map on the Hilbert spaces. It sends the first summand isomorphically to $L^2(\Omega_{sm})$, and has the second summand as its kernel. The translation group acts on this Hilbert space via a unitary representation, and leaves both sectors separately invariant.
Lemma 13. The spectral measure of the translation action on $\Omega_{\text{tsm}}$, confined to either of the two sectors $H_+$ and $H_-$, is spectrally pure, so it has only one non-vanishing component in its Lebesgue decomposition. In particular, it is a pure point measure on $H_+$, and a continuous one on $H_-$, where the latter is either purely singular continuous or purely absolutely continuous.

Proof. This is a consequence of known results on index-2 extensions of irrational rotations; compare [34, 25]. On the symbolic side, our system is almost everywhere 1 : 1 over the irrational rotation defined by the silver number. There, the claim follows from [34, Cor. 3.6]; see also [25]. Since this purity law is a Hilbert space result, it carries over to the shift space. Moreover, the same result remains true if one goes to the continuous translation action of $\mathbb{R}$ as obtained by a standard suspension; see [16, 18] for background.

Our geometric setting can be viewed as a suspension with two heights. Since the inflation multiplier is a PV unit, the symbolic and the geometric dynamical systems are conjugate by [15, Thm. 3.1]. The two spectral types are thus the same, and our claim follows.

Lemma 13 still leaves two possibilities for the spectral type in the odd sector. In order to discriminate between the two, we look at the asymptotic behaviour of the correlation functions $\nu_{\alpha\beta}$, in the same way as has been done for the Thue–Morse substitution. Because of the purity result of Lemma 13, we can in fact take any combination of correlation functions which is odd under the bar swap. A simple such combination is the auto-correlation of a structure where the left end points of the unbarred tiles are decorated with a weight +1, and those of the barred tiles with a weight −1. So, let $\Lambda$ be the set of all left endpoints of tiles of a tiling $\omega$, and set $w(x) = \pm 1$, if $x \in \Lambda$ and $x$ is the left endpoint of an unbarred (barred) tile. With $\Lambda_R = \Lambda \cap (-R, R)$, the relevant autocorrelation coefficients then are

$$\nu_{\text{tsm}}(z) = \lim_{R \to \infty} \frac{1}{|\Lambda_R|} \sum_{x,y \in \Lambda_R, x-y=z} w(x)w(y).$$

Note that we have used here the same relative normalisation as for the correlation functions of Eq. (8). In practice, instead of computing the limit (24), it is more convenient to partition $\Lambda$ into certain patches, which all have well-defined relative frequencies, and add up the contributions of these patches, weighted by their frequencies.

We now have to determine the decay or non-decay of $\nu_{\text{tsm}}(z)$ as $z \to \infty$.

Lemma 14. One has $\lim_{n \to \infty} \nu_{\text{tsm}}(z_n) = 1 - \sqrt{2}$, where $z_n = (1 + \lambda)\lambda^n$.

Proof. Before we do any concrete computations, let us sketch the strategy of the proof. The contributions to $\nu_{\text{tsm}}(1 + \lambda)$ come from pairs of tiles where the left endpoint of the second tile $t_2$ is located, by a shift of $1 + \lambda$, to the right of that of tile $t_1$. For any such pair, $t_2$ is the second neighbour to the right of $t_1$. We can therefore determine all possible triples of three consecutive tiles in the tiling, and add up their contributions to the correlation at distance $1 + \lambda$, weighted with the relative frequency of each triple. The relative frequencies can be determined by Perron–Frobenius theory [34, 5]. We regard the triples as doubly right-collared tiles, and determine the inflation matrix on the induced inflation on these collared
The possible triples of consecutive (super-)tiles in the TSM tiling, along with their inflations and relative frequencies. For each triple given, there is also a bar-swapped version, which contributes the same to the correlation. As given, the relative frequencies add up to $\frac{1}{2}$.

| no. | triple | inflated triple | rel. frequency |
|-----|--------|-----------------|----------------|
| 1   | $aaa$  | $aab$ $aba$ $aab$ | $x = -1 + \frac{3\sqrt{2}}{4}$ |
| 2   | $a\bar{a}$ | $ab\bar{a}$ $a$ | $x = -1 + \frac{3\sqrt{2}}{4}$ |
| 3   | $ab\bar{a}$ | $ab\bar{a}$ $\bar{a}b$ | $y = \frac{1}{2} - \sqrt{2}$ |
| 4   | $aab$  | $ab\bar{a}$ $\bar{a}b$ | $z = \frac{3}{2} - \sqrt{2}$ |
| 5   | $b\bar{a}a$ | $\bar{a}b\bar{a}$ $ab\bar{a}$ | $x = -1 + \frac{3\sqrt{2}}{4}$ |
| 6   | $b\bar{a}\bar{a}$ | $\bar{a}b\bar{a}$ $\bar{a}ba$ | $z = \frac{3}{2} - \sqrt{2}$ |

Table 1. Possible triples of consecutive (super-)tiles in the TSM tiling, along with their inflations and relative frequencies. For each triple given, there is also a bar-swapped version, which contributes the same to the correlation. As given, the relative frequencies add up to $\frac{1}{2}$.

tiles. The relative frequencies are then given by the components of the right Perron–Frobenius eigenvector of that matrix.

For the correlation at distance $(1 + \lambda)\lambda^n$, we do the same with triples of supertiles of order $n$. These have the same relative frequencies as the triples of tiles. The pairs of tiles contributing to the correlation then consist of a left tile in the left supertile of the triple, and a corresponding right tile at distance $(1 + \lambda)\lambda^n$, which is contained in one of the other two (collar) supertiles. Since the underlying silver mean tiling has pure point spectrum, the density of tiles $t_1$ not having a corresponding tile $t_2$ with the same geometry at distance $(1 + \lambda)\lambda^n$ asymptotically vanishes as $n \to \infty$. This will become evident from the explicit computations below.

To this end, we have to look for pairs of tiles which either match (have the same geometry and bar status) or anti-match (have the same geometry, but opposite bar status). Specifically, if $p_1$ and $p_2$ are two patches of tiles with the same support, we denote by $n_{\pm}(p_1, p_2)$ the number of tiles in $p_1$ that have a (anti-)matching partner in $p_2$ at the same position. We are then interested in the asymptotic overlap

$$c(p_1, p_2) = \lim_{n \to \infty} \tilde{c}(\tilde{\sigma}^n(p_1), \tilde{\sigma}^n(p_2)),$$

where

$$\tilde{c}(p_1, p_2) = \frac{n_+(p_1, p_2) - n_-(p_1, p_2)}{n_+(p_1, p_2) + n_-(p_1, p_2)}.$$

The possible triples of tiles are given in Table 1 along with their first inflations and relative frequencies. The subpatches that need to be compared are underlined.

Let us now discuss how the different triples contribute. The first two triples together contribute $c(ab\bar{a}, b\bar{a}a) + c(ab\bar{a}, barba)$. We see that the contributions of the first two tiles of each triple cancel. What remains is the anti-match of the third tiles. Asymptotically, we get
an anti-match on a fraction $\lambda^{-1}$ of the total patch length, which has to be weighted with the relative frequency of the triples. The third and the fifth triple each contribute an anti-match on the whole length of the patch, whereas the sixth triple contributes a match. Somewhat more complicated is the contribution $c(ab, b\bar{a})$ of the fourth triple. We get an anti-match due to the third tile on a fraction $\lambda^{-1}$ of the patch, plus $c(ab, b\bar{a})$. Inflating the latter once, we obtain

$$c(ab, b\bar{a}) = \lambda^{-1}(c(b\bar{a}, ab) - \sqrt{2})$$

and, in a similar way,

$$c(b\bar{a}, ab) = \lambda^{-1}(c(\bar{a}b, ab) + \sqrt{2})$$

These equations can be solved for $c(ab, b\bar{a})$, which gives $c(ab, b\bar{a}) = 1 - \sqrt{2}$. Putting everything together, we get

$$\lim_{n \to \infty} \nu_{\text{tsm}}(z_n) = \frac{\lambda (-2x\lambda^{-1} - y - z \frac{\lambda+(1+\lambda)(1-\sqrt{2})}{2\lambda+1}) + z - x}{\lambda(2x + y + z) + x + y} = 1 - \sqrt{2},$$

where the relative frequencies $x$, $y$ and $z$ have been taken from Table 1.

\textbf{Theorem 15.} The twisted silver mean dynamical system has two spectral components, both of which are singular. The dynamical spectrum from the even sector under the bar swap is pure point, whereas that from the odd sector is purely singular continuous.

\textbf{Proof.} By Lemma 13, the dynamical spectrum in the even sector is pure point, whereas in the odd sector it is continuous and of pure spectral type. By Lemma 14, the correlation measure $\sum_{z \in \Lambda - \Lambda} \nu_{\text{tsm}}(z) \delta_z$ does not decay to zero towards infinity, so that, by the Riemann–Lebesgue lemma and the fact that $\Lambda - \Lambda$ is uniformly discrete for a Pisot inflation, its Fourier transform must be singular. This implies that there must be a singular continuous component in the diffraction spectrum, and hence in the dynamical spectrum of the odd sector. As that spectrum is of pure type, it must be purely singular continuous.

\textbf{Remark 3.} Starting from a tiling with pure point spectrum, we constructed another tiling with a mixed spectrum by splitting each tile into a barred and an unbarred variant, and modified the inflation such that a bar swap symmetry results. Tilings with mixed spectrum can also be obtained if each tile of a pure point tiling is split into $k$ copies, and the inflation is modified such that a permutation symmetry acting on these $k$ copies results. However, a spectral purity result for the continuous spectrum sector, as in Lemma 13, can only be expected if $k = 2$.

\textbf{Remark 4.} The set $A_+$ of left endpoints of all un-barred tiles of a twisted silver mean tiling is a Meyer set (it is a relatively dense subset of a model set), which is linearly repetitive (it
is a component of a primitive inflation tiling), and which has mixed spectrum of singular type. This shows that there are highly ordered Meyer sets with mixed spectrum. In line with a result of Sinai, they have entropy 0, in contrast to Meyer sets that can be thought of as model sets with an (a posteriori) thinning disorder of positive entropy. Such cases would have a mixed spectrum with a non-trivial pure point part and some absolutely continuous component. One example is provided by sets of the form \(2\mathbb{Z} \cup \Lambda\) where \(\Lambda\) is a Bernoulli subset of \(2\mathbb{Z} + 1\) obtained by coin flipping, compare [5, Sec. 11.2], and similar constructions on the basis of arbitrary model sets. Another example is provided by the random noble means inflation rules discussed in [24, 32].

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