Characterization and Enumeration of Toroidal $K_{3,3}$-Subdivision-Free Graphs

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Abstract

We describe the structure of 2-connected non-planar toroidal graphs with no $K_{3,3}$-subdivisions, using an appropriate substitution of planar networks into the edges of certain graphs called toroidal cores. The structural result is based on a refinement of the algorithmic results for graphs containing a fixed $K_5$-subdivision in [A. Gagarin and W. Kocay, “Embedding graphs containing $K_5$-subdivisions”, Ars Combin. 64 (2002), 33-49]. It allows to recognize these graphs in linear-time and makes possible to enumerate labelled 2-connected toroidal graphs containing no $K_{3,3}$-subdivisions and having minimum vertex degree two or three by using an approach similar to [A. Gagarin, G. Labelle, and P. Leroux, "Counting labelled projective-planar graphs without a $K_{3,3}$-subdivision", submitted, arXiv:math.CO/0406140, (2004)].

1 Introduction

We use basic graph-theoretic terminology from Bondy and Murty [5] and Diestel [6], and deal with undirected simple graphs. Graph embeddings on a surface are important in VLSI design and in statistical mechanics. We are interested in non-planar graphs that can be embedded on the torus or on the projective plane. By Kuratowski’s theorem [13], a graph $G$ is non-planar if and only if it contains a subdivision of $K_5$ or $K_{3,3}$ (see Figure 1). In this paper we characterize (and enumerate) the 2-connected toroidal graphs with no $K_{3,3}$-subdivisions, following an analogous work for projective-planar graphs [9]. The next step in this research would be to characterize toroidal and projective-planar graphs containing a $K_{3,3}$-subdivision (with or without a $K_5$-subdivision).
We assume that $G$ is a 2-connected non-planar graph. A graph containing no $K_{3,3}$-subdivisions will be called $K_{3,3}$-subdivision-free. A general recursive decomposition of non-planar $K_{3,3}$-subdivision-free graphs is described in [16] and [12]. A local decomposition of non-planar graphs containing a $K_5$-subdivision of a special type is described in [7] and [8] (some $K_{3,3}$-subdivisions are allowed), that is used later in [8] to detect a projective-planar or toroidal graph. The results of [8] provide a toroidality criterion for graphs containing a given $K_5$-subdivision and avoiding certain $K_{3,3}$-subdivisions by examining the embeddings of $K_5$ on the torus. The torus is an orientable surface of genus one which can be represented as a rectangle with two pairs of opposite sides identified. The graph $K_5$ has six different embeddings on the torus shown in Figure 2. Notice that the hatched region of each of the embeddings $E_1$ and $E_2$ forms a single face $F$.

In [9] we prove the uniqueness of the decomposition of [8] for 2-connected non-planar projective-planar graphs with no $K_{3,3}$-subdivisions that gives a characterization of these graphs. In the present paper we state and prove an analogous structure theorem for the class $\mathcal{T}$ of 2-connected non-planar toroidal graphs with no $K_{3,3}$-subdivisions, involving certain “circular crowns” of $K_5\setminus e$ networks and substitution of strongly planar networks for edges. The structure theorem provides a practical algorithm to recognize the toroidal graphs with no $K_{3,3}$-subdivisions in linear-time. Here we use the structure theorem to
enumerate the labelled graphs in $T$ by using the counting techniques of [9] and [17] and improve known bounds for their number of edges. Finally, we enumerate the labelled graphs in $T$ having no vertex of degree two. Tables can be found at the end of the paper.

2 The structure theorem

A network is a connected graph $N$ with two distinguished vertices $a$ and $b$, such that the graph $N \cup ab$ is 2-connected. The vertices $a$ and $b$ are called the poles of $N$. The vertices of a network that are not poles are called internal. A network $N$ is strongly planar if the graph $N \cup ab$ is planar. We denote by $N_P$ the class of strongly planar networks.

The substitution of a network $N$ for an edge $e = uv$ is done in the following way: choose an arbitrary orientation, say $\vec{e} = \vec{uv}$ of the edge, identify the pole $a$ of $N$ with the vertex $u$ and $b$ with $v$, and disregard the orientation of $e$ and the poles $a$ and $b$. Note that both orientations of $e$ should be considered. It is assumed that the underlying set of $N$ is disjoint from $\{u, v\}$. The set of one or two resulting graphs is denoted by $e \uparrow N$.

More generally, given a graph $G_0$ with $k$ edges, $E = \{e_1, e_2, \ldots, e_k\}$, and a sequence $(N_1, N_2, \ldots, N_k)$ of disjoint networks, we define the composition $G_0 \uparrow (N_1, N_2, \ldots, N_k)$ as the set of graphs that can be obtained by substituting the network $N_j$ for the edge $e_j$ of $G_0$, $j = 1, 2, \ldots, k$. The graph $G_0$ is called the core, and the $N_i$’s are called the components of the resulting graphs. For a class of graphs $G$ and a class of networks $N$, we denote by $G \uparrow N$ the class of graphs obtained as compositions $G_0 \uparrow (N_1, N_2, \ldots, N_k)$ with $G_0 \in G$ and $N_i \in N$, $i = 1, 2, \ldots, k$. We say that the composition $G \uparrow N$ is canonical if for any graph $G \in G \uparrow N$, there is a unique core $G_0 \in G$ and unique (up to orientation) components $N_1, N_2, \ldots, N_k \in N$ that yield $G$.

In [9] we prove the uniqueness of the representation $K_5 \uparrow N_P$ for $K_{3,3}$-subdivision-free projective-planar graphs. This gives an example of a canonical composition.

Theorem 1 ([8, 9]) A 2-connected non-planar graph $G$ without a $K_{3,3}$-subdivision is projective-planar if and only if $G \in K_5 \uparrow N_P$. Moreover, the composition $K_5 \uparrow N_P$ is canonical.

Definition 1 Given two $K_5$-graphs, the graph obtained by identifying an edge of one of the $K_5$’s with an edge of the other is called an $M$-graph (see Figure 3a)), and, when the edge of identification is deleted, an $M^*$-graph (see Figure 3b)).

Definition 2 A network obtained from $K_5$ by removing the edge $ab$ between two poles is called a $K_5 \backslash e$-network. A circular crown is a graph obtained from a cycle $C_i$, $i \geq 3$, by substituting $K_5 \backslash e$-networks for some edges of $C_i$ in such a way that no pair of unsubstituted edges of $C_i$ are adjacent (see Figure 4).

Definition 3 A toroidal core is a graph $H$ which is isomorphic to either $K_5$, an $M$-graph, an $M^*$-graph, or a circular crown. We denote by $T_C$ the class of toroidal cores.
The main result of this paper is the following structure theorem. The proof is given in Section 4.

**Theorem 2** A 2-connected non-planar $K_{3,3}$-subdivision-free graph $G$ is toroidal if and only if $G \in T_C \uparrow N_P$. Moreover, the composition $T = T_C \uparrow N_P$ is canonical.

This theorem is used in Section 5 for the enumeration of labelled graphs in $T$. In the future we hope to use Theorem 2 to enumerate unlabelled graphs in $T$ as well.

### 3 Related known results

This section gives an overview of the structural results for toroidal graphs described in [8]. Following Diestel [6], a $K_5$-subdivision is denoted by $TK_5$. The vertices of degree 4 in $TK_5$ are the *corners* and the vertices of degree 2 are the *inner vertices* of $TK_5$. For a pair of corners $a$ and $b$, the path $P_{ab}$ between $a$ and $b$ with all other vertices inner vertices is called a *side* of the $K_5$-subdivision.

Let $G$ be a non-planar graph containing a fixed $K_5$-subdivision $TK_5$. A path $p$ in $G$ with one endpoint an inner vertex of $TK_5$, the other endpoint on a different side of $TK_5$, and all other vertices and edges in $G \backslash TK_5$, is called a *short cut* of the $K_5$-subdivision. A vertex $u \in G \backslash TK_5$ is called a 3-corner vertex with respect to $TK_5$ if $G \backslash TK_5$ contains internally disjoint paths connecting $u$ with at least three corners of the $K_5$-subdivision.

**Proposition 1** ([1, 7, 8]) Let $G$ be a non-planar graph with a $K_5$-subdivision $TK_5$ for which there is either a short cut or a 3-corner vertex. Then $G$ contains a $K_{3,3}$-subdivision.
Proposition 2 ([7, 8]) Let $G$ be a 2-connected graph with a $TK_5$ having no short cut or 3-corner vertex. Let $K$ denote the set of corners of $TK_5$. Then any connected component $C$ of $G \setminus K$ contains inner vertices of at most one side of $TK_5$ and $C$ is connected in $G$ to exactly two corners of $TK_5$.

Given a graph $G$ satisfying the hypothesis of Proposition 2, a side component of $TK_5$ is defined as the subgraph of $G$ induced by a pair of corners $a$ and $b$ in $K$ and the connected components of $G \setminus K$ which are connected to both $a$ and $b$ in $G$. Notice that side components of $G$ can contain $K_{3,3}$-subdivisions.

Corollary 1 ([7, 8]) For a 2-connected graph $G$ with a $TK_5$ having no short cut or 3-corner vertex, two side components of $TK_5$ in $G$ have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of $TK_5$.

Thus we see that a graph $G$ satisfying the hypothesis of Proposition 2 can be decomposed into side components corresponding to the sides of $TK_5$. Each side component $S$ contains exactly two corners $a$ and $b$ corresponding to a side of $TK_5$. If the edge $ab$ between the corners is not in $S$, we can add it to $S$ to obtain $S \cup ab$. Otherwise $S \cup ab = S$. We call $S \cup ab$ an augmented side component of $TK_5$. Side components of a subdivision of an $M$-graph are defined by analogy with the side components of a $K_5$-subdivision by considering pairs of adjacent vertices of the $M$-graph.

A planar side component $S$ of $TK_5$ in $G$ with two corners $a$ and $b$ is called cylindrical if the edge $ab \notin S$ and the augmented side component $S \cup ab$ is non-planar. Notice that a planar side component $S = S \setminus ab$ is embeddable in a cylindrical section of the torus. A cylindrical section is provided by the face $F$ of the embeddings $E_1$ and $E_2$ of $K_5$ on the torus shown in Figure 2. Toroidal graphs described in [8] can contain $K_{3,3}$-subdivisions because of a cylindrical side component $S$. An example of an embedding of the cylindrical side component $S = K_{3,3} \setminus e$ of a $TK_5$ on the torus is shown in Figure 6 where the graph $G$ of Figure 5 is embedded by completing the embedding $E_1$ of $K_5$ shown in Figure 2.

![Figure 5: A toroidal graph $G$ containing subdivisions of $K_{3,3}$ and of $K_5$.](image)

If a graph $G$ has no $K_{3,3}$-subdivisions, then Proposition 2 can be applied, in virtue of Proposition 1. In this case, a result of [8] can be summarized as follows.

Proposition 3 ([8]) A 2-connected non-planar $K_{3,3}$-subdivision-free graph $G$ containing a $K_5$-subdivision $TK_5$ is toroidal if and only if:
(i) all the augmented side components of $TK_5$ in $G$ are planar graphs, or
(ii) nine augmented side components of $TK_5$ in $G$ are planar, and the remaining side
component $S$ is cylindrical, or
(iii) $G$ contains a subdivision $TM$ of an $M$-graph, and all the augmented side compo-
nents of $TM$ in $G$ are planar.

Further analysis of the cylindrical side component $S$ of Proposition 3(ii) will provide
a proof of Theorem 2. Notice that graphs with 6 or more vertices satisfying Propositon
3 are not 3-connected. Therefore a 3-connected non-planar graph different from $K_5$ must
contain a $K_{3,3}$-subdivision (see also [1]).

4 Proof of the structure theorem

A side component $S$ having two corners $a$ and $b$ can be considered as a network. We use
the notation $\text{Int}(S)$ to denote the interior of $S$, that is the subgraph $\text{Int}(S) = S \setminus (\{a\} \cup \{b\})$
obtained by removing the two vertices $a$ and $b$. A network $S$ is called cylindrical if $ab \notin S$,
$S$ is a planar graph, but $S \cup ab$ is non-planar. Recall that a network $S$ is called strongly
planar if $S \cup ab$ is planar.

A block is a maximal 2-connected subgraph of a graph. A description of the block-
cutvertex tree decomposition of a connected graph can be found in [6]. We consider blocks
$G_i$ having two distinguished vertices $a_i$ and $b_i$. The distinguished vertices are called poles
of the block.

**Proposition 4** Let $G$ be a 2-connected non-planar toroidal $K_{3,3}$-subdivision-free graph
satisfying Proposition 3(ii) with the cylindrical side component $S$ having corners $a$ and $b$.
Then the block-cutvertex decomposition of $S$ forms a path of blocks $S_1, S_2, \ldots, S_k, k \geq 1$, as in Figure 7, and at least one of the blocks $S_1, S_2, \ldots, S_k, k \geq 1$, is a cylindrical network.
Moreover, every block $S_i, i = 1, 2, \ldots, k$, of $S$ is either a strongly planar network, or a
cylindrical network of the form $K_5 \setminus e \uparrow (N_1, N_2, \ldots, N_9)$, where $e = a_ib_i$ and the $N_j$'s are
strongly planar networks.

**Proof.** Since $G$ is 2-connected, each cut-vertex of $S$ belongs to exactly two blocks and
lies on the corresponding side $P_{ab}$ of $TK_5$. Therefore the blocks of $S$ form a path as in
Figure 7.
Therefore, \( S \) and \( N \) implies that at least one of the blocks \( S_i \), \( i = 1, 2, \ldots, k \), remains planar when the edge \( a_ib_i \) is added to \( S_i \). Then, clearly, \( S \cup ab \) remains planar as well. Hence the fact that \( S \) is cylindrical implies that at least one of the blocks \( S_i \), \( i = 1, 2, \ldots, k \), is itself a cylindrical network.

Suppose a block \( S_m \), \( 1 \leq m \leq k \), of \( S \) is cylindrical. Then, by Kuratowski’s theorem, \( S_m \cup a_ib_m \) contains a \( K_5 \)-subdivision \( T K'_5 \). Clearly, \( a_ib_m \in T K'_5 \), \( T K'_5 \) has no short-cut or 3-corner vertex in \( G \) and \( a_m \) and \( b_m \) are two corners of the \( T K'_5 \). The edge \( a_m b_m \) of \( T K'_5 \) can be replaced by a path \( P_{a_m b_m} \) in \( G \setminus \text{Int}(S_m) \) and we can decompose \( G \) into the side components of \( T K'_5 \).

Since \( G \) is toroidal and the side component \( G \setminus \text{Int}(S_m) \) of \( T K'_5 \) is cylindrical, all the other side components of \( T K'_5 \) in \( G \) must be strongly planar networks by Proposition 3(ii). Therefore \( S_m \) is a cylindrical network of the form \( K_5 \setminus e \cup (N_1, N_2, \ldots, N_9) \), with \( e = a_m b_m \) and \( N_j \in \mathcal{N}_P \), \( j = 1, 2, \ldots, 9 \).

Now we are ready to prove the structure Theorem 2 using Propositions 3 and 4.

**Proof of Theorem 2.** (Sufficiency) Suppose \( G \) is a graph in \( \mathcal{T}_C \uparrow \mathcal{N}_P \), i.e. \( G = H \uparrow (N_1, N_2, \ldots, N_k) \), where \( H \) is a toroidal core having \( k \) edges and \( N_i \)’s, \( i = 1, 2, \ldots, k \), are strongly planar networks. If \( H = K_5 \) or \( H = M \), then \( G \) can be decomposed into the side components of \( T K_5 \) or \( T M \) respectively and the augmented side components are planar graphs. Therefore, by Proposition 3(i) or 3(iii) respectively, \( G \) is toroidal \( K_{3,3} \)-subdivision-free.

If \( H = M^* \) or \( H \) is a circular crown, then we can choose a \( K_5 \setminus e \)-network \( N \) in \( H \) and find a path \( P_{ab} \) connecting \( a \) and \( b \) in the complementary part \( H \setminus \text{Int}(N) \). This determines a subdivision \( T K_5 \) in \( G \) such that nine augmented side components of \( T K_5 \) in \( G \) are planar, and the remaining side component \( S \) defined by the corners \( a \) and \( b \) of \( T K_5 \) is cylindrical. Therefore, by Proposition 3(ii), \( G \) is toroidal \( K_{3,3} \)-subdivision-free.

(Necessity and Uniqueness) Let \( G \) be a 2-connected non-planar \( K_{3,3} \)-subdivision-free toroidal graph \( G \). By Kuratowski’s theorem, \( G \) contains a \( K_5 \)-subdivision \( T K_5 \). Let us prove that \( G \in \mathcal{T}_C \uparrow \mathcal{N}_P \) by using Propositions 3 and 4. The fact that the composition \( H \uparrow \mathcal{N}_P \), \( H \in \mathcal{T}_C \), of \( G \) is canonical will follow from the uniqueness of the sets of corner vertices in Proposition 3.

Clearly, the sets of graphs corresponding to the cases (i), (ii) and (iii) of Proposition 3 are mutually disjoint. Suppose \( G \) contains a subdivision \( T K_5 \) or \( T M \) and all the augmented side components of \( T K_5 \) or \( T M \), respectively, in \( G \) are planar graphs as in Proposition 3(i, iii). Then \( G = K_5 \uparrow (N_1, N_2, \ldots, N_{19}) \) or \( G = M \uparrow (N_1, N_2, \ldots, N_{19}) \), respectively, \( K_5, M \in \mathcal{T}_C \) and all the \( N_j \)’s are in \( \mathcal{N}_P \). The uniqueness of the decomposition

![Figure 7: Block-cutvertex decomposition for the cylindrical side component $S$.](image)
in cases (i) and (iii) of Proposition 3 can be proved by analogy with Theorem 3 in [1]; the set of corners of the $K_5$-subdivision in Proposition 3(i) and the set of corners of the $M$-graph subdivision in Proposition 3(iii) are uniquely defined. This covers toroidal cores $K_5$ and the $M$-graph.

Suppose $S$ is the unique cylindrical side component of $TK_5$ in $G$ as in Proposition 3(ii). Notice that $G \setminus \text{Int}(S)$ itself is a cylindrical network of the form $K_5 \setminus e \uparrow \{N_1, N_2, \ldots, N_9\}$, where $e = ab$ and $N_j \in N_P$, $j = 1, 2, \ldots, 9$. By Proposition 4, the block-cutvertex decomposition of $S$ forms a path of blocks $S_1, S_2, \ldots, S_k, k \geq 1$, as in Figure 7, and at least one of the blocks $S_1, S_2, \ldots, S_k, k \geq 1$, is a cylindrical network. In this path we can regroup maximal series of consecutive strongly planar networks into single strongly planar networks so that at most one strongly planar network $N'$ is separating two cylindrical networks in the resulting path, and the poles of the strongly planar network $N'$ are uniquely defined by maximality. By Proposition 4, the cylindrical networks in the path are of the form $K_5 \setminus e \uparrow \{N_1, N_2, \ldots, N_9\}$, where $N_j \in N_P$, $j = 1, 2, \ldots, 9$, and the corners $a'$ and $b' = a'b$, are uniquely defined with respect to the corresponding $K_5$-subdivision $TK_5'$ in $G$. Therefore the unique set of corners completely defines a toroidal core $M^*$ or a circular crown $H$ having $k$ edges and the set of corresponding strongly planar networks $N_1, N_2, \ldots, N_k$, such that $G = M^* \uparrow \{N_1, N_2, \ldots, N_9\}$ or $G = H \uparrow \{N_1, N_2, \ldots, N_k\}$, respectively.

Theorems 1 and 2 imply that a projective-planar graph with no $K_{3,3}$-subdivisions is toroidal. However an arbitrary projective-planar graph can be non-toroidal. The characterizations of Theorems 1 and 2 can be used to detect projective-planar or toroidal graphs with no $K_{3,3}$-subdivisions in linear time. The implementation of this algorithm can be derived from [8] by using a breadth-first or depth-first search technique for the decomposition and by doing a linear-time planarity testing. The linear-time complexity follows from the linear-time complexity of the decomposition and from the fact that each vertex of the initial graph can appear in at most 7 different components.

A corollary to Euler’s formula for the plane says that a planar graph with $n \geq 3$ vertices can have at most $3n - 6$ edges (see, for example, [5] and [6]). Let us state this for 2-connected planar graphs with $n$ vertices and $m$ edges as follows:

$$m \leq \begin{cases} 3n - 5 & \text{if } n = 2 \\ 3n - 6 & \text{if } n \geq 3 \end{cases}.$$  

(1)

In fact, $m = 3n - 5 = 1$ if $n = 2$. The generalized Euler formula (see, for example, [15]) implies that a toroidal graph $G$ with $n$ vertices can have up to $3n$ edges. An arbitrary graph $G$ without a $K_{3,3}$-subdivision is known to have at most $3n - 5$ edges (see [11]). The following proposition shows that toroidal graphs with no $K_{3,3}$-subdivisions satisfy a stronger relation, which is analogous to planar graphs.

**Proposition 5** The number $m$ of edges of a non-planar $K_{3,3}$-subdivision-free toroidal $n$-vertex graph $G$ satisfies $m \leq 3n - 5$ if $n = 5$ or 8, and

$$m \leq 3n - 6, \text{ if } n \geq 6 \text{ and } n \neq 8.$$  

(2)
Proof. Clearly, toroidal graphs satisfying Theorem 2 also satisfy Proposition 3. By Proposition 3(i, ii), each side component $S_i$ of $TK_5$ in $G$, $i = 1, 2, \ldots, 10$, satisfies the condition (11) with $n = n_i$, the number of vertices, and $m = m_i$, the number of edges of $S_i$, $i = 1, 2, \ldots, 10$. Since each corner of $TK_5$ is in precisely 4 side components, we have $\sum_{i=1}^{10} n_i = n + 15$ and we obtain, by summing these 10 inequalities,

$$m = \sum_{i=1}^{10} m_i \leq \begin{cases} 3 \sum_{i=1}^{10} n_i - 50 = 3(n + 15) - 50 = 3n - 5 & \text{if } n = 5 \\ 3 \sum_{i=1}^{10} n_i - 51 = 3(n + 15) - 51 = 3n - 6 & \text{if } n \geq 6 \end{cases},$$

since $n = 5$ iff $n_i = 2$, $i = 1, 2, \ldots, 10$, and $n \geq 6$ if and only if at least one $n_j \geq 3$, $j = 1, 2, \ldots, 10$.

Similarly, by Proposition 3(iii), each side component $S_i$ of $TM$ in $G$, $i = 1, 2, \ldots, 19$, satisfies the condition (11) with $n = n_i$, the number of vertices, and $m = m_i$, the number of edges of $S_i$, $i = 1, 2, \ldots, 19$. Since 2 vertices of $TM$ are in precisely 7 side components, 6 vertices of $TM$ are in precisely 4 side components, and all the other vertices of $G$ are in a unique side component, we have $\sum_{i=1}^{19} n_i = n + 30$ and we obtain, by summing these 19 inequalities,

$$m = \sum_{i=1}^{19} m_i \leq \begin{cases} 3 \sum_{i=1}^{19} n_i - 95 = 3(n + 30) - 95 = 3n - 5 & \text{if } n = 8 \\ 3 \sum_{i=1}^{19} n_i - 96 = 3(n + 30) - 96 = 3n - 6 & \text{if } n \geq 9 \end{cases},$$

since $n = 8$ iff $n_i = 2$, $i = 1, 2, \ldots, 19$, and $n \geq 9$ if and only if at least one $n_j \geq 3$, $j = 1, 2, \ldots, 19$.

An analogous result for the projective-planar graphs can be found in [9]. Also note that Corollary 8.3.5 of [6] implies that graphs with no $K_5$-minors can have at most $3n - 6$ edges.

5 Counting labelled $K_{3,3}$-subdivision-free toroidal graphs

Now let us consider the question of the labelled enumeration of toroidal graphs with no $K_{3,3}$-subdivisions according to the numbers of vertices and edges. First, we review some basic notions and terminology of labelled enumeration together with the counting methods and technique used in [17, 9]. The reader should have some familiarity with exponential generating functions and their operations (addition, multiplication and composition). For example, see [2, 11, 14, or 18].

By a labelled graph, we mean a simple graph $G = (V, E)$ where the set of vertices $V = V(G)$ is itself the set of labels and the labelling function is the identity function. $V$ is called the underlying set of $G$. An edge $e$ of $G$ then consists of an unordered pair $e = uv$ of elements of $V$ and $E = E(G)$ denotes the set of edges of $G$. If $W$ is another
set and $\sigma : V \rightarrow W$ is a bijection, then any graph $G = (V, E)$ on $V$, can be transformed into a graph $G' = \sigma(G) = (W, \sigma(E))$, where $\sigma(E) = \{\sigma(e) = \sigma(u)\sigma(v) | e \in E\}$. We say that $G'$ is obtained from $G$ by vertex relabelling and that $\sigma$ is a graph isomorphism $G \cong G'$. An unlabelled graph is then seen as an isomorphism class $\gamma$ of labelled graphs. We write $\gamma = \gamma(G)$ if $\gamma$ is the isomorphism class of $G$. By the number of ways to label an unlabelled graph $\gamma(G)$, where $G = (V, E)$, we mean the number of distinct graphs $G'$ on the underlying set $V$ which are isomorphic to $G$. Recall that this number is given by $n!/|\text{Aut}(G)|$, where $n = |V|$ and $\text{Aut}(G)$ denotes the automorphism group of $G$.

A species of graphs is a class of labelled graphs which is closed under vertex relabellings. Thus any class $\mathcal{G}$ of unlabelled graphs gives rise to a species, also denoted by $\mathcal{G}$, by taking the set union of the isomorphism classes in $\mathcal{G}$. For any species $\mathcal{G}$ of graphs, we introduce its (exponential) generating function $\mathcal{G}(x, y)$ as the formal power series

$$\mathcal{G}(x, y) = \sum_{n \geq 0} g_n(y) \frac{x^n}{n!}, \quad \text{with} \quad g_n(y) = \sum_{m \geq 0} g_{n,m} y^m,$$

where $g_{n,m}$ is the number of graphs in $\mathcal{G}$ with $m$ edges over a given set of vertices $V_n$ of size $n$. Here $y$ is a formal variable which acts as an edge counter. For example, for the species $\mathcal{G} = K = \{K_n\}_{n \geq 0}$ of complete graphs, we have

$$K(x, y) = \sum_{n \geq 0} y^{(n)} x^n / n!,$$

while for the species $\mathcal{G} = \mathcal{G}_a$ of all simple graphs, we have $\mathcal{G}_a(x, y) = K(x, 1 + y)$.

A species of graphs is molecular if it contains only one isomorphism class. For a molecular species $\gamma = \gamma(G)$, where $G$ has $n$ vertices and $m$ edges, we have $\gamma(x, y) = \frac{y^m n!}{|\text{Aut}(G)|} \frac{x^n}{n!} = y^m x^n / |\text{Aut}(G)|$. For example,

$$K_5(x, y) = \frac{x^5 y^{10}}{5!}.$$

Also, for the graphs $M$ and $M^*$ described in Section 2, we have

$$M(x, y) = 280 \frac{x^8 y^{19}}{8!}, \quad M^*(x, y) = 280 \frac{x^8 y^{18}}{8!},$$

since $|\text{Aut}(M)| = |\text{Aut}(M^*)| = 144$.

For the enumeration of networks, we consider that the poles $a$ and $b$ are not labelled, or, in other words, that only the internal vertices form the underlying set. Hence the generating function of a class (or species) $\mathcal{N}$ of networks is defined by

$$\mathcal{N}(x, y) = \sum_{n \geq 0} \nu_n(y) \frac{x^n}{n!}, \quad \text{with} \quad \nu_n(y) = \sum_{m \geq 0} \nu_{n,m} y^m,$$

where $\nu_{n,m}$ is the number of networks in $\mathcal{N}$ with $m$ edges and a given set of internal vertices $V_n$ of size $n$. For example, we have

$$(K_5 \setminus e)(x, y) = \frac{x^3 y^{9}}{3!},$$
A species $\mathcal{N}$ of networks is called symmetric if for any $\mathcal{N}$-network $N$ (i.e. $N$ in $\mathcal{N}$), the opposite network $\tau \cdot N$, obtained by interchanging the poles $a$ and $b$, is also in $\mathcal{N}$. Examples of symmetric species of networks are the classes $\mathcal{N}_P$, of strongly planar networks, and $\mathcal{R}$, of series-parallel networks (see [17, 9]).

**Lemma 1** (T. Walsh [17, 9]) Let $\mathcal{G}$ be a species of graphs and $\mathcal{N}$ be a symmetric species of networks such that the composition $\mathcal{G} \uparrow \mathcal{N}$ is canonical. Then the following generating function identity holds:

$$ (\mathcal{G} \uparrow \mathcal{N})(x, y) = \mathcal{G}(x, \mathcal{N}(x, y)). \quad (9) $$

By Theorem 2 and Lemma 1, we have the following proposition.

**Proposition 6** The generating function $\mathcal{T}(x, y)$ of labelled non-planar $K_{3,3}$-subdivision-free toroidal graphs is given by

$$ \mathcal{T}(x, y) = (\mathcal{T}_C \uparrow \mathcal{N}_P)(x, y) = \mathcal{T}_C(x, \mathcal{N}_P(x, y)), \quad (10) $$

where $\mathcal{T}_C$ denotes the class of toroidal cores (see Definition 3).

Let $P$ denote the species of 2-connected planar graphs. Then the generating function of $\mathcal{N}_P$, the associated class of strongly planar networks, is given by

$$ \mathcal{N}_P(x, y) = (1 + y) \frac{2}{x^2 \partial y} P(x, y) - 1 \quad (11) $$

(see [17, 9]). Methods for computing the generating function $P(x, y)$ of labelled 2-connected planar graphs are described in [3] and [4]. Formula (11) can then be used to compute $\mathcal{N}_P(x, y)$. Therefore there remains only to compute the generating function $\mathcal{T}_C(x, y)$ for toroidal cores. Recall that $\mathcal{T}_C = K_5 + M + M^* + CC$, where $CC$ denotes the class of circular crowns. Circular crowns can be enumerated as follows using matching polynomials.

**Proposition 7** The mixed generating series $CC(x, y)$ of circular crowns is given by

$$ CC(x, y) = \frac{-12 x^4 y^9 + 12 x^5 y^{10} + x^5 y^{18} + 72 \ln(1 - \frac{x^4 y^9}{6} - \frac{x^5 y^{10}}{6})}{144}. \quad (12) $$

**Proof.** Recall that a matching $\mu$ of a finite graph $G$ is a set of disjoint edges of $G$. We define the matching polynomial of $G$ as

$$ M_G(y) = \sum_{\mu \in \mathcal{M}(G)} y^{\vert \mu \vert}, \quad (13) $$

where $\mathcal{M}(G)$ denotes the set of matchings of $G$. In particular, the matching polynomials $U_n(y)$ and $T_n(y)$ for paths and cycles of size $n$ are well known (see [10]). They are closely related to the Chebyshev polynomials. To be precise, let $P_n$ denote the path graph $(V, E)$
with \( V = [n] = \{1, 2, \ldots, n\} \) and \( E = \{\{i, i+1\} \mid i = 1, 2, \ldots, n-1\} \) and \( C_n \) denote the cycle graph with \( V = [n] \) and \( E = \{\{i, i+1(\text{mod} \ n)\} \mid i = 1, 2, \ldots, n\} \). Then we have

\[
U_n(y) = \sum_{\mu \in \mathcal{M}(P_n)} y^{\vert \mu \vert}, \quad T_n(y) = \sum_{\mu \in \mathcal{M}(C_n)} y^{\vert \mu \vert}. \tag{14}
\]

The dichotomy caused by the membership of the edge \( \{n - 1, n\} \) in the matchings of the path \( P_n \) leads to the recurrence relation

\[
U_n(y) = yU_{n-2}(y) + U_{n-1}(y), \tag{15}
\]

for \( n \geq 2 \), with \( U_0(y) = U_1(y) = 1 \). It follows that the ordinary generating function of the matching polynomials \( U_n(y) \) is rational. In fact, it is easily seen that

\[
\sum_{n \geq 0} U_n(y)x^n = \frac{1}{1 - x - yx^2}. \tag{16}
\]

Now, the dichotomy caused by the membership of the edge \( \{1, n\} \) in the matchings of the cycle \( C_n \) leads to the relation

\[
T_n(y) = yU_{n-2}(y) + U_{n}(y), \tag{17}
\]

for \( n \geq 3 \). It is then a simple matter, using (16) and (17) to compute their ordinary generating function, denoted by \( G(x, y) \). We find

\[
G(x, y) = \sum_{n \geq 3} T_n(y)x^n = \frac{x^3(1 + 3y + yx + 2y^2x)}{1 - x - yx^2}. \tag{18}
\]

In fact, we also need to consider the **homogeneous matchings polynomials**

\[
T_n(y, z) = z^nT_n\left(\frac{y}{z}\right) = \sum_{\mu \in \mathcal{M}(C_n)} y^{\vert \mu \vert}z^{n-\vert \mu \vert}, \tag{19}
\]

where the variable \( z \) marks the edges which are not selected in the matchings, whose generating function \( G(x, y, z) = \sum_{n \geq 3} T_n(y, z)x^n \) is given by

\[
G(x, y, z) = G(xz, \frac{y}{z}) = \frac{x^3z^2(z + 3y + xyz + 2xy^2)}{1 - xz - x^2yz}. \tag{20}
\]

We now introduce the species \( BC \) of pairs \((c, \mu)\), where \( c \) is a cycle of length \( n \geq 3 \) and \( \mu \) is a matching of \( c \), with weight \( y^{\vert \mu \vert}z^{n-\vert \mu \vert} \). Since there are \( \frac{(n-1)!}{2} \) non-oriented cycles on a set of size \( n \geq 3 \), and all these cycles admit the same homogeneous matching polynomial
\( T_n(y, z) \), the mixed generating function of labelled \( BC \)-structures is

\[
BC(x, y, z) = \sum_{n \geq 3} \frac{(n - 1)!}{2} T_n(y, z) \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq 3} T_n(y, z) \frac{x^n}{n} = \frac{1}{2} \int_0^x \frac{1}{t} G(t, y, z) \, dt
\]

\[
= -\frac{2xz + 2x^2yz + x^2z^2 + 2\ln(1 - xz - x^2yz)}{4}.
\]

(21)

Notice that in a circular crown, the unsubstituted edges are not adjacent, by definition, and hence form a matching of the underlying cycle, while the substituted edges are replaced by \( K_5 \setminus e \)-networks. We can thus write

\[
CC = BC \uparrow_z (K_5 \setminus e),
\]

(22)

where the notation \( \uparrow_z \) means that only the edges marked by \( z \) are replaced by \( K_5 \setminus e \)-networks. Moreover the decomposition (22) is canonical and we have

\[
CC(x, y) = BC(x, y, (K_5 \setminus e)(x, y)),
\]

(23)

which implies (12) using (8).

A substitution of the generating function \( N_P(x, y) \) of (11) counting the strongly planar networks for the variable \( y \) in (6), (5), and (12) gives the generating function for labelled 2-connected non-planar toroidal graphs with no \( K_{3,3} \)-subdivision, i.e.

\[
T(x, y) = K_5(x, N_P(x, y)) + M(x, N_P(x, y)) + M^*(x, N_P(x, y)) + CC(x, N_P(x, y)).
\]

(24)

Notice that the term \( K_5(x, N_P(x, y)) \) in (24) also enumerates non-planar 2-connected \( K_{3,3} \)-subdivision-free projective-planar graphs and that corresponding tables are given in [9]. Here we present the computational results just for labelled graphs in \( T \) that are not projective-planar. Numerical results are presented in Tables 1 and 2, where

\[
T(x, y) - K_5(x, N_P(x, y)) = \sum_{n \geq 8} \sum_m t_{n,m} x^n y^m / n! \text{ and } t_n = \sum_m t_{n,m} \text{ count labelled non-projective-planar graphs in } T.
\]

The homeomorphically irreducible non-projective-planar graphs in \( T \), i.e. the graphs having no vertex of degree two, can be counted by using several methods described in detail in Section 4 of [9]. We used the approach of Proposition 8 of [9] to obtain the numerical data presented in Tables 3 and 4 for labelled homeomorphically irreducible graphs in \( T \) that are not projective-planar.

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| $n$ | $m$ | $t_{n,m}$ | $n$ | $m$ | $t_{n,m}$ | $n$ | $m$ | $t_{n,m}$ |
|-----|-----|-----------|-----|-----|-----------|-----|-----|-----------|
| 8   | 18  | 280       | 13  | 23  | 1838008972800 | 15  | 25  | 5973529161600000 |
| 8   | 19  | 280       | 13  | 24  | 12383684913600 | 15  | 26  | 60679359861120000 |
| 9   | 19  | 50400     | 13  | 25  | 36576568828800 | 15  | 27  | 2806191247860000000 |
| 9   | 20  | 93240     | 13  | 26  | 61986597472800 | 15  | 28  | 785754392485600000 |
| 9   | 21  | 47880     | 13  | 27  | 66199273620480 | 15  | 29  | 14961423286129200000 |
| 10  | 20  | 5292000   | 13  | 28  | 46419992138520 | 15  | 30  | 2068477720590481200 |
| 10  | 21  | 13044400  | 13  | 29  | 22180672954440 | 15  | 31  | 2175937397296462800 |
| 10  | 22  | 15510600  | 13  | 30  | 7737403073400  | 15  | 32  | 18101289969034272000 |
| 10  | 23  | 5972400   | 13  | 31  | 2053743892200  | 15  | 33  | 122324124356652400 |
| 10  | 24  | 239400    | 13  | 32  | 348540192000   | 15  | 34  | 6731543806125138000 |
| 11  | 21  | 426888000 | 13  | 33  | 27935107200    | 15  | 35  | 293316324401310000 |
| 11  | 22  | 1700899200| 14  | 24  | 107217190080000| 15  | 36  | 962956642177530000 |
| 11  | 23  | 272404400 | 14  | 25  | 896474952172800| 15  | 37  | 2226049706380500000 |
| 11  | 24  | 213684240 | 14  | 26  | 335926561370400| 15  | 38  | 3218036781960000000 |
| 11  | 25  | 773295600 | 14  | 27  | 7460402644094400| 15  | 39  | 2182635655200000000 |
| 11  | 26  | 94386600  | 14  | 28  | 10948159170748800| 16  | 26  | 3225705747264000000 |
| 11  | 27  | 7900200   | 14  | 29  | 11253868616390400| 16  | 27  | 3914073529252560000 |
| 12  | 22  | 2945527200| 14  | 30  | 84676026060225600| 16  | 28  | 21877196871997440000 |
| 12  | 23  | 1555424600| 14  | 31  | 48569516960656000| 16  | 29  | 75157685291745220000 |
| 12  | 24  | 34841406000| 14  | 32  | 22222453326984000| 16  | 30  | 178928606393580650000 |
| 12  | 25  | 42429451600| 14  | 33  | 78518737337040000| 16  | 31  | 31623679286218835200 |
| 12  | 26  | 297599563800| 14  | 34  | 19720831801680000| 16  | 32  | 4354382548839420648000 |
| 12  | 27  | 118905448200| 14  | 35  | 3106454542200000| 16  | 33  | 4842535206589734384000 |
| 12  | 28  | 27683548200| 14  | 36  | 2294786894400000| 16  | 34  | 4455761048858447480000 |
| 12  | 29  | 4821201000 | 14  | 37  | 34198556200139638800 | 16  | 35  | 341985562001396388000 |
| 12  | 30  | 410810000 | 14  | 38  | 2168647220752412400000 | 16  | 36  | 34198556200139638800000 |
| 12  | 31  | 1110293723767938212500 | 16  | 37  | 444793568384905740000 |
| 12  | 32  | 133746538216030470000 | 16  | 38  | 4447935683849057400000 |
| 12  | 33  | 28310940294436880000 | 16  | 39  | 3753889677488000000 |
| 12  | 34  | 2341717849496000000 | 16  | 40  | 375388967748800000000 |

Table 1: The number of labelled non-planar non-projective-planar toroidal 2-connected graphs without a $K_{3,3}$-subdivision (having $n$ vertices and $m$ edges).
| $n$  | $t_n$         |
|------|--------------|
| 8    | 560          |
| 9    | 191520       |
| 10   | 42058800     |
| 11   | 7864256400   |
| 12   | 1407126890400|
| 13   | 257752421166240|
| 14   | 50607986220311520|
| 15   | 10995419195575214400|
| 16   | 2692773804667509763200|
| 17   | 747221542837742897724800|
| 18   | 233698171655650029030743040|
| 19   | 8147276505132560093387934080|
| 20   | 31268587126068905034073041062400|

Table 2: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs (having $n$ vertices).
| n   | m   | $t_{n,m}$ | n   | m   | $t_{n,m}$ | n   | m   | $t_{n,m}$ | n   | m   | $t_{n,m}$ |
|-----|-----|----------|-----|-----|----------|-----|-----|----------|-----|-----|----------|
| 8   | 18  | 230      | 14  | 26  | 605404800 | 16  | 29  | 5811886058000 |
| 8   | 19  | 230      | 14  | 27  | 2445751065600 | 16  | 30  | 621544891968000 |
| 9   | 19  | 6040     | 14  | 28  | 3601812854400 | 16  | 31  | 11935943091072000 |
| 10  | 20  | 2520     | 14  | 29  | 17840270448000 | 16  | 32  | 101350194001056000 |
| 10  | 22  | 22680    | 14  | 30  | 551333827044000 | 16  | 33  | 49937173276416000 |
| 10  | 23  | 46620    | 14  | 31  | 108994658572800 | 16  | 34  | 1611221546830896000 |
| 10  | 24  | 239400   | 14  | 32  | 131794531450000 | 16  | 35  | 3605404135132800000 |
| 11  | 23  | 10256400 | 15  | 28  | 1961511552000000 | 16  | 36  | 2967845927880834000 |
| 11  | 24  | 30492000 | 15  | 29  | 5753767219200000 | 16  | 37  | 1095216458946080000 |
| 11  | 25  | 1079416800 | 15  | 30  | 55718834371200000 | 16  | 38  | 2391904418904000000 |
| 11  | 26  | 3044487600 | 15  | 31  | 282795025312800000 | 16  | 39  | 1184982400602000000 |
| 12  | 24  | 1896048000 | 15  | 32  | 8936155496562800000 | 16  | 40  | 5738963267484144000 |
| 12  | 25  | 70794168000 | 15  | 33  | 389330165396957200000 | 16  | 41  | 11171878415312800000 |
| 12  | 26  | 30444876000 | 15  | 34  | 188861003030700000000 | 16  | 42  | 234171878419660000000 |
| 12  | 27  | 50806440000 | 15  | 35  | 766349001080320000000 | 16  | 43  | 573896326748414400000 |
| 12  | 28  | 208864860000 | 15  | 36  | 304448760000000000000 | 16  | 44  | 111718784153128000000 |
| 12  | 29  | 1079416800000 | 15  | 37  | 188861003030700000000 | 16  | 45  | 573896326748414400000 |
| 12  | 30  | 3044487600000 | 15  | 38  | 766349001080320000000 | 16  | 46  | 111718784153128000000 |
| 13  | 25  | 168648480000 | 15  | 39  | 282795025312800000000 | 16  | 47  | 573896326748414400000 |
| 13  | 26  | 228756280000 | 15  | 40  | 766349001080320000000 | 16  | 48  | 111718784153128000000 |
| 13  | 27  | 1266809544000 | 15  | 41  | 282795025312800000000 | 16  | 49  | 573896326748414400000 |
| 13  | 28  | 3826086294000 | 15  | 42  | 766349001080320000000 | 16  | 50  | 111718784153128000000 |

Table 3: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs with no vertex of degree 2 (having $n$ vertices and $m$ edges).
| $n$ | $t_n$         |
|-----|-------------|
| 8   | 560         |
| 9   | 5040        |
| 10  | 957600      |
| 11  | 123354000   |
| 12  | 16842764400 |
| 13  | 2764379217600 |
| 14  | 527554510282800 |
| 15  | 114387072403606000 |
| 16  | 2772856196888788000 |
| 17  | 74180318049678460000 |
| 18  | 2167306256125914230527200 |
| 19  | 685709965521372865035362400 |
| 20  | 233306923207078035272369412000 |

Table 4: The number of labelled non-planar non-projective-planar toroidal 2-connected $K_{3,3}$-subdivision-free graphs with no vertex of degree 2 (having $n$ vertices).