Environment induced incoherent controllability

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We prove that the environment induced entanglement between two non interacting, two-dimensional quantum systems $S$ and $P$ can be used to control the dynamics of $S$ by means of the initial state of $P$. Using a simple, exactly solvable model, we show that both accessibility and controllability of $S$ can be achieved under suitable conditions on the interaction of $S$ and $P$ with the environment.

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INTRODUCTION

Control theoretical methods and concepts are playing an important role in the development of modern quantum mechanics and in particular of quantum information theory. Control theory ideas are used both in the analysis of quantum dynamics and generation of entanglement and in the development of algorithms for the control of quantum systems. Control theoretical methods and concepts are playing an important role in the development of modern quantum systems.

In this context, a fundamental question is to what extent it is possible to influence the dynamics of a quantum system by an external action. This problem is referred to as controllability. Methods of control which use tunable parameters in the Hamiltonian of the systems are referred to as coherent control methods. Motivated by several experimental scenarios, control techniques have recently been investigated where the control variable affects an auxiliary system which is then made interact with the quantum system to obtain control. More precisely, the system $S$ is allowed to interact with a second system $P$, called probe, and initially they are in an uncorrelated state $\rho_S \otimes \rho_P$. It is assumed that it is possible to modify the initial state of $P$ before the interaction, therefore in this case the controls enter the dynamics of $S$ through $\rho_P = \rho_P(u)$. This control method is referred to as incoherent control since it does not rely on modifications of the Hamiltonian of the system $S$.

Controllability and accessibility of $S$ in the incoherent control setting have been investigated under the assumption that the composite system $T = S + P$ is closed. In this case the dynamics is given by

$$\rho_S(t,u) = \text{Tr}_P(X(t)\rho_S \otimes \rho_P(u)X^\dagger(t)),$$

where $\text{Tr}_P$ is the partial trace over the degrees of freedom of $P$, $X(t) = e^{-iH_P t}$ is the unitary propagator and $H_T = H_S + H_P + H_{SP}$ is the Hamiltonian of $T$. The coupling between $S$ and $P$ is given by the interaction...
Hamiltonian $H_{SP}$. Necessary and sufficient condition for controllability and accessibility have been derived in the case of two-dimensional $S$ and $P$ \[17\], under the hypothesis that it is possible to obtain all the pure states of $\rho_P$ by an arbitrary choice of the control.

**Theorem 1** \[17\] The system $S$ evolving under $E$ is (incoherent) controllable if and only if there is a time $t$ at which the unitary evolution of the composite system $X(t)$ is locally equivalent to the SWAP operator.

Algebraic conditions of incoherent controllability equivalent to the ones expressed in Theorem 1 can be given by considering the Cartan decomposition \[18\] of $X(t)$,

$$X(t) = L_1(t)e^{at}L_2(t),$$

where $L_1(t), L_2(t)$ are local transformations, $a = c_x\sigma_x^S \otimes \sigma_x^P + c_y\sigma_y^S \otimes \sigma_y^P + c_z\sigma_z^S \otimes \sigma_z^P$ is an element of the Cartan subalgebra of $\mathfrak{su}(4)$, $c_i$ are real coefficients and $\sigma_i^{S,P}$ are the Pauli matrices in $S, P$ respectively.

Conditions for accessibility of $S$ can be expressed in terms the coefficients appearing in the Cartan decomposition.

**Theorem 2** \[17\] The system $S$ evolving under $E$ is accessible if and only if $c_i \neq 0$ for all $i = x, y, z$.

The assumption that $T$ is a closed system is valid only in first approximation. In general, there will be an external environment $E$ interacting with $T$ and thus affecting the controllability properties of $S$. Since in general there is no control on $E$, intuition suggests that the interaction between $E$ and $T$ is always a negative factor for the controllability properties of $S$, as it leads to a dissipative evolution for $T$. The main goal of this letter is to prove that this is not always true and that the interaction with $E$ can have a positive impact for the incoherent controllability of $S$ by $P$. In the following we shall consider a common model for the environment given by a large number of decoupled harmonic oscillators and show that for appropriate forms of the bath-system interaction we can have accessibility and controllability of a system which would otherwise be not controllable and not accessible as a closed system. Our research is related to the investigation in \[19\] where it was shown that the interaction with a common environment can generate entanglement for a couple of systems plunged in it. The case treated here is in essence the opposite of the one treated in \[17\]. In that paper system and probe were assumed interacting and no environment was present. In the case treated here, system and probe are assumed not directly interacting and their interaction is totally due to the presence of the environment.

### A MODEL OF TWO SYSTEMS INTERACTING THROUGH THE ENVIRONMENT

We consider the model of the environment described in \[20\]. $E$ given by a set of $N$ decoupled harmonic oscillators with Hamiltonian

$$H_E = \sum_{i=1}^{N} \hbar \omega_i \left( b_i^\dagger b_i + \frac{1}{2} \right)$$

where $b_i^\dagger, b_i$ are the creation and annihilation operators associated to the $i-$th oscillator and $\omega_i$ its angular frequency. This is the bosonic bath model as $N \to \infty$ and the considerations on controllability that will follow do not depend on $N$. We assume $H_T = 0$, that is the composite system of system and probe, $T = S + P$, has no free evolution. We assume a linear coupling between $E$ and $T$ depending on the positions of the oscillators,

$$H_{ET} = \sum_{i=1}^{N} A_T \otimes g_i(b_i + b_i^\dagger),$$

where $g_i$ is the coupling constant of the $i-$th oscillator and $A_T$ an arbitrary hermitian operator in the Hilbert space of $T$. The evolution of a state of $S$ is given by

$$\rho_S(t, u) = Tr_P Tr_E (X(t) \rho_S \otimes \rho_P (u) \otimes \rho_E X^\dagger(t))$$

where $X(t) = e^{-i[H_E + H_{ET}]t}$ and $S, P$ and $E$ are all initially decoupled. The environment is in the thermal state $\rho_E$. $A_T$ is a constant of motion since $[A_T, H_E + H_{ET}] = 0$, therefore it is possible to find the exact analytical expression of the dynamics. It is convenient to introduce the eigenvalues and eigenvectors of $A_T$, $A_T |\alpha_i\rangle = \alpha_i |\alpha_i\rangle$ for $i = 1, \ldots, 4$, therefore \[9\] becomes

$$\rho_S(t, u) = \sum_{i,j=1}^{4} Tr_P [\alpha_j] (\rho_S \otimes \rho_P(u))_{ij} \gamma_{ij}(t)$$

where we introduced the functions

$$\gamma_{ij}(t) = e^{-i(\alpha_i - \alpha_j)^2 f(t) + i(\alpha_i^2 - \alpha_j^2)} \varphi(t)$$

and

$$f(t) = \sum_{i=1}^{N} \left( \frac{g_i}{\hbar \omega_i} \right)^2 (1 + 2\bar{n}_i)(1 - \cos \omega_i t),$$

$$\varphi(t) = \sum_{i=1}^{N} \left( \frac{g_i}{\hbar \omega_i} \right)^2 (\omega_i t - \sin \omega_i t),$$

where $\bar{n}_i$ is the average thermal occupation number for the $i-$th oscillator \[20\].

To compute the partial trace in \[10\] we need to make some assumptions on the eigenvectors of $A_T$. In the study of the incoherent controllability for this system,
we find convenient to explore two opposite cases: either all the eigenvectors are factorized states in the Hilbert space of $S + P$, or they are maximally entangled states. By exploring these two extreme cases we will find examples of evolutions that are not accessible, accessible but not controllable or controllable. This last case will prove our claim that the environment induces incoherent controllability.

**CONTROLLABILITY AND ACCESSIBILITY PROPERTIES**

We first consider the case where the eigenvectors of $A_T$, $|\alpha_i\rangle$, are factorized states, i.e., $|\alpha_i\rangle = |\alpha_i^S\rangle \otimes |\alpha_i^P\rangle$ with $i = (k, l)$, $i = 1, \ldots, 4$ and $k, l = 1, 2$, and the sets $\{|\alpha_i^S\rangle, |\alpha^2\rangle\}$, $\{|\alpha^P\rangle, |\alpha^2_P\rangle\}$ are orthonormal bases in the Hilbert spaces of $S$ and $P$, respectively. In this case

$$Tr \rho P |\alpha_i\rangle \langle \alpha_j| = \delta_{in} |\alpha_i^S\rangle \langle \alpha_m^S|$$

and moreover

$$(\rho_S \otimes \rho_P(u))_{ij} = (\rho_S)_{km} (\rho_P(u))_{ln}$$

where $i = (k, l)$ and $j = (m, n)$. Thus equation (11) becomes

$$\rho_S(t, u) = \sum_{k,m=1}^{2} (\rho_S)_{km} |\alpha_k^S\rangle \langle \alpha_m^S| \cdot 2 \sum_{n=1}^{2} (\rho_P(u))_{nn} \gamma(k, n)(m, n)(t)$$

and initial states $\rho_S$ that are diagonal in the considered basis do not evolve. Examples of evolutions displaying this behavior are determined by interaction terms of the form $A_T = A_S + A_P$ or $A_T = A_S \otimes A_P$, where $A_S$ and $A_P$ are hermitian operators acting on the Hilbert spaces of $S$ and $P$. It follows that in these cases $S$ is neither accessible nor controllable, therefore a necessary condition for accessibility and controllability is that at least one eigenvector of $A_T$ is an entangled state in $S + P$.

We assume now that all the eigenvectors are maximally entangled states, i.e. Bell states

$$|\alpha_{1,2}\rangle = \frac{1}{\sqrt{2}}(|\alpha_1^S\rangle \otimes |\alpha_1^P\rangle \pm |\alpha_2^S\rangle \otimes |\alpha_2^P\rangle)$$

$$|\alpha_{3,4}\rangle = \frac{1}{\sqrt{2}}(|\alpha_1^S\rangle \otimes |\alpha_2^P\rangle \pm |\alpha_2^S\rangle \otimes |\alpha_1^P\rangle)$$

in suitable bases $\{|\alpha^S_i\rangle, |\alpha^S_j\rangle\}$ and $\{|\alpha^P_i\rangle, |\alpha^P_j\rangle\}$. It is convenient to use a coherence vector representation for the states in $S$ and $P$, that is

$$\rho_S = \frac{1}{2}(1 + \vec{s} \cdot \sigma^S), \quad \rho_P = \frac{1}{2}(1 + \vec{p} \cdot \sigma^P)$$

where $\vec{s}, \vec{p}$ are real vectors in the Bloch spheres of $S$ and $P$ and $\vec{p}^S, \vec{p}^P$ are the vectors of the Pauli matrices in $S$ and $P$. In this representation the dynamics (10) takes the form

$$\vec{s}(t, u) = A(t, \bar{s}_0) \vec{p}(u) + \vec{a}(t, \bar{s}_0)$$

where $A(t, \bar{s}_0)$ is the matrix

$$\frac{1}{2} \text{Im} \begin{pmatrix}
\gamma_{13-24}(t) & s_z \gamma_{13-24}(t) & s_y \gamma_{13+24}(t) \\
\gamma_{14-23}(t) & s_z \gamma_{14-23}(t) & -s_x \gamma_{23+14}(t) \\
-s_y \gamma_{12+34}(t) & s_x \gamma_{34-12}(t) & i \gamma_{12-34}(t)
\end{pmatrix}$$

and

$$\vec{a}(t, \bar{s}_0) = \frac{1}{2} \text{Re} \begin{pmatrix}
s_x \gamma_{13+24}(t) & s_y \gamma_{23+14}(t) \\
s_z \gamma_{12+34}(t) & s_z \gamma_{12-34}(t)
\end{pmatrix}.$$
\( \mathcal{R}(\rho_S, \hat{t}) = \mathcal{P}_S \) at some time \( \hat{t} \). This can be obtained by choosing \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) and \( \alpha_4 \neq 0 \). Since relations \[ \text{(22)} \] are satisfied, we have accessibility for all \( \alpha_4 \neq 0 \). Moreover, equations \[ \text{(19)} \] and \[ \text{(20)} \] simplify to

\[
A(t, \hat{s}_0) = \frac{1}{2} \begin{pmatrix}
1 - \gamma_r(t) & -s_z \gamma_i(t) & s_y \gamma_i(t) \\
-\gamma_i(t) & 1 - \gamma_r(t) & -s_x \gamma_i(t) \\
-s_y \gamma_i(t) & s_x \gamma_i(t) & 1 - \gamma_r(t)
\end{pmatrix}
\] (23)

and

\[
\hat{a}(t, \hat{s}_0) = \frac{1}{2}(1 + \gamma_r(t)) \begin{pmatrix}
s_x \\
s_y \\
s_z
\end{pmatrix}
\] (24)

where

\[
\gamma_r(t) = e^{-\alpha_4^2 f(t)} \cos(\alpha_4^2 \varphi(t)),
\]

\[
\gamma_i(t) = e^{-\alpha_4^2 f(t)} \sin(\alpha_4^2 \varphi(t)).
\] (25)

A sufficient condition for controllability is that \( A(\hat{t}, \hat{s}_0) = I \) and \( a(t, \hat{s}_0) = 0 \) at some time \( \hat{t} \). Therefore \( \gamma_r(\hat{t}) = -1 \) and \( \gamma_i(\hat{t}) = 0 \), that is

\[
\begin{cases}
\alpha_4^2 f(\hat{t}) = 0 \\
\alpha_4^2 \varphi(\hat{t}) = (2k_1 + 1)\pi, \ k_1 \in \mathbb{Z}
\end{cases}
\] (26)

The first condition in \[ \text{(26)} \] is satisfied if and only if \( \cos \omega_i \hat{t} = 1 \), that is \( \omega_i \hat{t} = 2k_2\pi \) for all \( i = 1, \ldots, N \), with \( k_2 \in \mathbb{Z} \). This condition can certainly be satisfied if the environment consists of a field in a cavity, for appropriate values of \( k_2 \). Finally, using the second equation with \( \sin \omega_i \hat{t} = 0 \) for all \( i = 1, \ldots, N \), we find a condition on the eigenvalue \( \alpha_4 \):

\[
\frac{1}{\alpha_4^2} = \frac{2}{2k_1 + 1} \sum_{i=1}^{N} k_2^2 \left( \frac{g_i}{k_{2i}} \right)^2.
\] (27)

with arbitrary \( k_1 \in \mathbb{Z} \). Therefore, controllability can be achieved for an appropriate combination of the parameters defining the dynamics of the bath (the frequencies \( \omega_i \)) and the parameters defining the interaction (the \( \alpha_j \)'s, \( j = 1, \ldots, 4 \)).

Condition \[ \text{(21)} \] is a rather strict request on the coefficient \( \alpha_4 \). However, if it is possible to change some parameters in the bath dynamics (e.g., the intensity of the electromagnetic field in a cavity) they could be tuned in order to realize an incoherently controllable system.

The crucial point to obtain controllability is that the interaction of the environment with the system \( T \) must have at least one entangled eigenvector. A physical example is given by two identical quantum dots, localized in different positions \( q_S \) and \( q_P \), in an electromagnetic cavity, with a non-dipole interaction with the electromagnetic field. The position degrees of freedom, not involved in \( A_T \), can be used to distinguish the probe from the system and then to perform the incoherent control protocol.

**Conclusions**

We have described a model of incoherent control of a system \( S \) by means of a probe \( P \), in the presence of a common environment \( E \). We have assumed that \( S \) and \( P \) do not evolve in absence of the environment, so their dynamics is only due to the interaction with \( E \). In this framework, we have proved that the induced correlations between \( S \) and \( P \) are, in some cases, rich enough to allow total control of \( S \) through \( P \). These results complement recent research on the creation of entanglement and suggests that further investigations of the control of a quantum system through its correlations with the environment will prove fruitful.

In this model, a necessary condition for accessibility and controllability is that the interaction of \( E \) with \( T = S + P \) is not a superposition of separate interactions. Otherwise, even if \( S \) and \( P \) become entangled, it is not possible to achieve accessibility or controllability of \( S \). It is still possible to drive \( S \) using \( P \), but this is a limited ability.

The dynamics considered in this work is non-Markovian, but this does not seem to be a fundamental assumption: in fact, it has been proved in \[ \text{(21)} \] that also Markovian dynamics can entangle initially uncorrelated systems.

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[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge, 2000

[2] Q. A. Turchette et al., Phys. Rev. Lett. 81, 3631 (1998)

[3] C. Cabrillo, J.I. Cirac, P. García-Fernández and P. Zoller, Phys. Rev. A 59, 1025 (1999)

[4] A. Sorensen and K. Mølmer, Phys. Rev. A 62, 022311 (2000)

[5] C.A. Sackett et al., Nature 404, 256 (2000)

[6] A. Beige et al., J. Mod. Opt. 47, 2583 (2000)

[7] J.P. Palao and R. Kosloff, Phys. Rev. Lett. 89, 188301 (2002)

[8] C.M. Tesch and R. de Vivie-Riedle, Phys. Rev. Lett. 89, 157901 (2002)

[9] J.J. García-Ripoll, P. Zoller and J.I. Cirac, Phys. Rev. Lett. 91, 157901 (2003)

[10] C. Rangan, A.M. Bloch, C. Monroe and P.H. Bucksbaum, Physical Review Letters 92, 113004 (2004)

[11] F. Albertini and D. D’Alessandro, IEEE Transactions on Automatic Control 48, 1399 (2003)

[12] C. Altafini, J. Math. Phys. 44, 2357 (2003)

[13] G. M. Huang, T. J. Tarn and J. W. Clark, J. Math. Phys. 24, 2608 (1983)

[14] V. Ramakrishna, M. V. Salapaka, M. Dahleh, H. Rabitz and A. Peirce, Phys. Rev. A 51, 960 (1995)

[15] R. Vilela Mendes and V.I. Manko, Phys. Rev. A 67, 053404 (2003)
[16] A. Mandilara and J. W. Clark, Phys. Rev. A 71, 013406 (2005)
[17] R. Romano and D. D’Alessandro, quant-ph/0510020
[18] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, 1978
[19] D. Braun, Phys. Rev. Lett. 89, 277901 (2002)
[20] D. Braun, F. Haake and W. T. Strunz, Phys. Rev. Lett. 86, 2913 (2001)
[21] F. Benatti, R. Floreanini and M. Piani, Phys. Rev. Lett. 91, 070402 (2003)