The stochastic modeling of the short-time variations of the galactic cosmic rays

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Abstract. We present the stochastic model of the galactic cosmic ray (GCR) particles transport in the heliosphere. The model is created based on the numerical solution of the Parker transport equation (PTE) describing the non-stationary transport of charged particles in the turbulent medium. We present the numerical schemes for the strong order integration of the set of the stochastic differential equations (SDEs) corresponding to the non-stationary PTE. Among the employed methods are the strong order Euler-Maruyama, Milstein and stochastic Runge-Kutta methods. We perform the selection of the method resulting in the highest agreement of the model of the 27-day variation of the GCR intensity with the experimental observations.

1. Introduction
The stochastic differential equations (SDEs), in conjunction with various Monte Carlo technics, are broadly used in many fields. Nowadays, in the era of high-performance computing, efficient parallel algorithms to realize large-scale simulations are crucial in solving realistic problems inspired by physical sciences and engineering. One of these problems is the non-stationary stochastic motion of charged particles in three-dimensional space, described by diffusion equation of Fokker-Planck type. The main statistical characteristic is the representation of the solution of the Fokker-Planck equation (FPE) as a probability density function. Unfortunately, most of the problems existing in nature are unsolvable analytically, so numerical methods must be applied. The difficulty of the numerical solution of this type of equations increases with the problem dimension. The reason is the instability of the numerical schemes like finite-differences and finite-volume in the higher dimensions. To ensure the scheme stability and convergence the density of numerical grid must be improved, with the computational complexity enlargement, at the same time. To overcome this problem, the stochastic methods can be applied (e.g. [1], [2]). In this approach, the individual particle motion is described as a Markov stochastic process, and the system evolves probabilistically. Employment of probabilistic description with Monte Carlo simulations allow reducing the solution of the partial differential equation (PDE) describing the analyzed phenomena to the integration of SDEs.

The aim of this paper is threefold: (1) to compose a reliable mathematical model of the galactic cosmic ray (GCR) particle transport in the heliosphere using the SDEs; (2) to solve Parker transport equation (PTE) by applying to the SDEs the strong order integration Euler-Maruyama, Milstein and stochastic Runge-Kutta numerical methods and then, (3) to employ the developed methods to simulate the 27-day variation of the GCR intensity being in an agreement...
with the experimental data.

We employ the stochastic methodology to model the GCR transport in the heliosphere. Modulation of the GCR is a result of an action of four primary processes: convection by the solar wind, diffusion on irregularities of the heliospheric magnetic field (HMF), particles drifts in the non-uniform magnetic field and adiabatic cooling. Transport of the GCR particles in the heliosphere can be described by the Parker transport equation [3]:

\[ \frac{\partial f}{\partial t} = \nabla \cdot (K_{ij} \cdot \nabla f) - (\vec{v}_d + \vec{U}) \cdot \nabla f + \frac{R}{3} (\nabla \cdot \vec{U}) \frac{\partial f}{\partial R}, \]  

(1)

where \( f = f(\vec{r}, R, t) \) is an omnidirectional distribution function of three spatial coordinates \( \vec{r} = (r, \theta, \phi) \), particles rigidity \( R \) and time \( t \); \( \vec{U} \) is the solar wind velocity, \( \vec{v}_d \) the drift velocity, and \( K_{ij} \) is the symmetric part of the diffusion tensor of the GCR particles.

This paper is an extension of our previous results presented in [4, 5]. We have presented [4] that the GCR transport can be effectively modeled based on the solution of the set of SDEs corresponding to the PTE (Eq. 1) by Euler-Maruyama method. In [5] we have increased the accuracy of the SDEs solution by applying the higher order methods i.e. Milstein and stochastic Runge-Kutta. In this paper, we employ the higher order methods i.e. Milstein and stochastic Runge-Kutta to simulate the short-time variations of the GCR intensity. We perform the selection of the numerical integration method resulting in the highest agreement of the short-time variations of the GCR intensity with the experimental observations.

2. Stochastic differential equation corresponding to the Parker transport equation

The PTE (Eq. 1) in the 3-D heliocentric spherical coordinate system \((r, \theta, \phi)\) written as time-backward FPE diffusion equation has the form:

\[ \frac{\partial f}{\partial t} = A_1 \frac{\partial^2 f}{\partial r^2} + A_2 \frac{\partial^2 f}{\partial \theta^2} + A_3 \frac{\partial^2 f}{\partial \phi^2} + A_4 \frac{\partial^2 f}{\partial r \partial \theta} + A_5 \frac{\partial^2 f}{\partial r \partial \phi} + A_6 \frac{\partial^2 f}{\partial \theta \partial \phi} + A_7 \frac{\partial f}{\partial r} + A_8 \frac{\partial f}{\partial \theta} + A_9 \frac{\partial f}{\partial \phi} + A_{10} \frac{\partial f}{\partial R}, \]  

(2)

the coefficients \( A_1,...,A_{10} \) are presented in detail in [4]. Depending on the choice of the numerical approximation the corresponding to Eq. 2 set of SDEs has a form:

\[ dr = \underbrace{A_7 \cdot dt + B_{21} \cdot dW_r + \frac{1}{2} B_{21} \frac{\partial B_{21}}{\partial r} (dW_r^2 - dt)}_{\text{Euler-Maruyama}} + \Phi_1 \]  

\[ \underbrace{\text{Milstein}}_{\text{Stochastic Runge-Kutta}} \]

\[ d\theta = \underbrace{A_8 \cdot dt + B_{212} \cdot dW_r + B_{22} \cdot dW_\theta + \frac{1}{2} B_{212} \frac{\partial B_{212}}{\partial r} (dW_r^2 - dt) + \frac{1}{2} B_{22} \frac{\partial B_{22}}{\partial \theta} (dW_\theta^2 - dt)}_{\text{Euler-Maruyama}} + \Phi_2 \]  

\[ \underbrace{\text{Milstein}}_{\text{Stochastic Runge-Kutta}} \]

\[ d\phi = \underbrace{A_9 \cdot dt + B_{31} \cdot dW_r + B_{32} \cdot dW_\theta + B_{33} \cdot dW_\phi + \Phi_3 + \Phi_4}_{\text{Euler-Maruyama}} \]  

\[ \underbrace{\text{Milstein}}_{\text{Stochastic Runge-Kutta}} \]

\[ dR = A_{10} \cdot dt. \]

The \( B_{ij} \), \((i,j = r, \theta, \phi)\), is the lower triangular matrix presented in [4], and the coefficients \( \Phi_1, \Phi_2, \Phi_3 \) and \( \Phi_4 \) are presented in detail in [5].
3. The simulations results and discussion

The code for the numerical solution of the set of Eqs. 3 is realized in MatLab environment. The code is easy to parallelize versus the number of simulated pseudoparticles using Matlab Parallel Toolbox. This approach falls from the assumption that any random process is independent of the other realization, accordingly each pseudoparticle’s trajectory is independent on another. We performed the simulations applying all three numerical methods given by Eqs. 3. The trajectory of \( n = 2000 \) pseudoparticles was traced backward in time in the spherical heliocentric coordinate system. The pseudoparticles were initialized in the point representing the Earth’s orbit (i.e. \( r = 1 \) AU, \( \theta = 90^\circ \), \( \varphi = 180^\circ \)) and traced backward in time until crossing the heliosphere boundary assumed at 100 AU.

The value of the particle distribution function \( f(\vec{r}, R) \) for the starting point was found as an average of \( f_{\text{LIS}}(R) \) value for pseudoparticles characteristics at the entry positions, \( f(\vec{r}, R) = \frac{1}{N} \sum_{n=1}^{N} f_{\text{LIS}}(R) \), where \( f_{\text{LIS}}(R) \) is the cosmic ray local interstellar spectrum (LIS) taken as in [6] for rigidity \( R \) of the \( n^{\text{th}} \) particle at the entrance point.

To solve Eqs. 3 in spherical coordinates system we used the boundary conditions of form [7]: \( \varphi_i < 0 \rightarrow \varphi_i = \varphi_i + 2\pi; \varphi_i > 2\pi \rightarrow \varphi_i = \varphi_i - 2\pi; \theta_i < 0 \rightarrow \theta_i = -\theta_i \) & \( \varphi_i = \varphi_i \pm \pi \) and \( \theta_i > \pi \rightarrow \theta_i = 2\pi - \theta_i \) & \( \varphi_i = \varphi_i \pm \pi \). The reflecting boundary is considered as the inner radial boundary, \( \frac{\partial f}{\partial r} = 0 \) at \( r = 0.001 \) AU. An empty heliosphere constitutes an initial condition, \( f_i(0.01 \text{AU} < r < 100 \text{AU}; \theta, \varphi, R, 0) = 0 \), as was shown in [8].

In the case, when we know the analytical solution of the equation that we want to solve, it is easy to compare the efficiency of each applied numerical method. However, in the case of GCR particles transport in the heliosphere, we do not have such a possibility. In general, using the higher order numerical method should result in the increase of the accuracy of the solution. Particularly, the smaller number of simulated particles for higher order method should give better or comparable results to those provided by a weaker method employing more pseudoparticles. Thus, we have calculated the values of the particle distribution function, \( f(\vec{r}, R) \) and its standard deviations for models employing all presented numerical methods (Eqs. 3) with number of simulated particles increasing from \( n = 500 \) to \( n = 5500 \). We have analyzed the changes of the standard deviations of the particle distribution function \( f(\vec{r}, R) \) calculated for the Euler-Maruyama, Milstein and stochastic Runge-Kutta methods vs. the number of simulated pseudoparticles with the rigidity of 10 GV. Results of calculations are presented in Fig. 1. Fig. 1 shows that the standard deviation of the numerical solution of the Eqs. 3 is the largest for Euler-
Figure 2. Temporal changes of the observed daily solar wind velocity (points) and dashed curve represent its approximation – sum of three harmonics (27, 14 and 9 days) wave during 7 September – 3 October 2007.

Figure 3. The expected 27-day variation of the GCR intensity at the Earth orbit for the particles with rigidity of 10 GV in comparison with the GCR intensity registered by Oulu neutron monitor during 7 September - 3 October 2007.

Maruyama method and the lowest for stochastic Runge-Kutta. So, by applying the higher order method (especially stochastic Runge-Kutta), the statistical accuracy of the numerical solution is significantly increased. The obtained results suggest that all applied numerical methods give reliable results. Performed tests proved that the Milstein and stochastic Runge-Kutta methods are more stable and return the same results of differential spectra when we decrease the number of simulated particles by a factor of three.

4. Model of the 27-day variation of the GCR intensity

The recurrence of the GCR intensity connected with the solar rotation period (\(\sim 27\) days) is commonly called the 27-day variation, although the duration can slightly differ. The 27-day variation of the GCR intensity is connected with the heliolongitudinal asymmetry of the electromagnetic conditions in the heliosphere. The recent minimum of solar activity between solar cycles No. 23 and 24 was quite exceptional. Recurrent variations connected with corotating structures (\(\sim 27\) days), at the end of 2007 and almost for the whole year 2008 were clearly established in all solar wind and interplanetary parameters. Accordingly, the 27-day variation of cosmic ray intensity was clearly visible in a variety of cosmic ray counts of neutron monitors (e.g., [9, 10]) and space probes (e.g., [11]).

We present the model of the 27-day variation of the GCR intensity considering a particular period of the solar rotation starting at 2007.09.07. As it was stated by [9, 12], the 27-day variation of the GCR intensity in the minimum epochs of solar activity is preferentially related to the heliolongitudinal asymmetry of the solar wind velocity. In the model, we apply approximation of in situ measurements of the solar wind speed as the source of the 27-day variation of the GCR intensity approximated by the formula: 

\[
U = U_0(1 - 0.31\sin(\varphi + 61.00) + 0.06\sin(2\varphi + 0.82) - 0.10\sin(3\varphi - 1.04)),
\]

where \(U_0 = 400\, \text{km/s}\). Fig. 2 presents temporal changes of the observed daily solar wind velocity (points) and dashed curve represents its approximation being the sum of three harmonics (27, 14 and 9 days) wave during 7 September – 3 October 2007.

The diffusion coefficient \(K_\parallel\) of cosmic ray particles has the form: 

\[
K_\parallel = K_0 \cdot K(r) \cdot K(R),
\]

where \(K_0 = 10^{22}\, \text{cm}^2/\text{s}\), \(K(r) = 1 + 0.5 \cdot (r/1 \text{AU})\) and \(K(R) = R^{0.5}\).

The expected changes of the 27-day variation of the GCR intensity for the rigidity of 10 GV in comparison with the profile of the daily GCR intensity recorded by Oulu neutron monitor during 7 September - 3 October 2007 presents Fig. 3. Fig. 3 compares the results of the solutions of the PTE with SDEs by the Euler-Maruyama, Milstein and stochastic Runge-Kutta
methods. One can see that model utilizing all presented methods can reconstruct the 27-day wave of the GCR intensity at the Earth orbit (1 AU, \( \theta = 90^\circ \)). However, in some periods, one can see that the model results do not coincide one to one with experiment. The model indeed does not describe all heliolongitudinal variations in the GCR registration. The reason is the assumed approximation. We incorporated in the model of the 27-day variation only the approximation of the first three harmonics of the solar wind velocity (Fig. 2). Comparing the results of the SDEs solutions by Euler-Maruyama, Milstein and stochastic Runge-Kutta methods, one can see that the stochastic Runge-Kutta method reproduced the data with the highest statistical precision. The Euler-Maruyama and Milstein methods give almost the same results in the scope of the accuracy. However, short period fluctuations of the GCR intensity could not be completely reproduced by the proposed model.

Presented model results (Fig. 3) show an acceptable agreement between experimental data of Oulu neutron monitor and modeling results dealing mainly with the shape and average amplitude of the 27-day variation of the GCR intensity. It is hardly expected that modeling could completely explain any short period fluctuation of the GCR intensity. We demonstrated that proposed approach, with an implementation of the solar wind velocity in situ measurements from one hand, and application of the higher order numerical methods from the other, makes the proposed model more realistic and statistically significant. Moreover, the model of the 27-day variation obtained based on the solution of the SDEs allows to reflect the stochastic character of the GCR particles distribution in the heliosphere. The presented model visualizes the pseudoparticle trajectory throughout the 3D heliosphere, which is not possible based on the solution of the Parker transport equation by e.g. the finite difference method [10, 12].

5. Conclusions

(i) We presented the solution of the Parker transport equation using a numerical solution of the set of stochastic differential equations driven by a Wiener process with the strong order Euler-Maruyama, Milstein, and stochastic Runge-Kutta methods. We showed that application of the higher order methods (especially stochastic Runge-Kutta) significantly increased the statistical accuracy of the numerical solution.

(ii) The SDEs were integrated backward in time in the heliocentric spherical coordinates applying the full 3D anisotropic diffusion tensor. The models obtained based on the solution of the SDEs reflect the stochastic character of the pseudoparticles distribution in the heliosphere. Additionally, this technique allows observing the statistically possible changes of the pseudoparticles trajectories, which is not feasible based on the solution of the Parker transport equation by e.g. finite difference methods.

(iii) We presented the model of the 27-day variation of the GCR intensity obtained based on the stochastic approach to the solution of the Parker transport equation. The modeling results are in good agreement with the experimental data.

6. Acknowledgments

This work is supported by The Polish National Science Centre grant awarded by decision number DEC-2012/07/D/ST6/02488. We thank the principal investigators of Oulu neutron monitor and OMNIweb for the ability to use their data.

7. References

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