On the nonchaotic nature of monotone dynamical systems

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Abstract. Two common types of dynamics, chaotic and monotone, are compared. It is shown that monotone maps in strongly ordered spaces do not have chaotic attractors.

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1. Introduction

This article reviews and contrasts two protean classes of dynamical systems frequently used in applied fields, chaotic and monotone.

1.1. Terminology

"Space" means a topological space with a metric denoted by $d$. The distance from a point $y$ to a set $Q$ is

$\text{dist}(y, Q) := \inf\{d(y, q) : q \in Q\}.$

Maps are assumed continuous.

A dynamical system on a space $S$ — the set of states — is a a family $F = \{F_t\}$ of maps between subsets of $S$, called the dynamic. The parameter $t$, representing time, varies over an appropriate set of real numbers. An initial state $x$ evolves to $F_tx$ at $t \geq 0$, so that $F_t(F_{s}x) = F_{s+t}x$, and $F_0$ is the identity map on $S$. Every map $g : S \to S$ defines a dynamical system on $S$ whose dynamic is the set of iterates $\{g^j\}$ where $j$ runs over the integers $\geq 0$, or all the integers when $g$ is a homeomorphism of $S$. When $F$ is clear from the context, $F_t(x)$ may be denoted by $x(t)$.

Let $T : Y \to Y$ denote a map. The orbit of $y \in Y$ under $T$ is the set

$\{y, Ty, T^2y, \ldots \} \subset Y.$

If $T^k y = y$ for some $k \geq 1$ then $y$ and its orbit are periodic.

A nonempty set $A \subset Y$ attracts $y \in Y$ if

$\text{dist}(y, \{F_t\}A) \to 0$ as $t \to \infty$. 

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\[ \lim_{k \to \infty} \text{dist}(T^k y, A) = 0. \]  

We call \( A \) an attractor provided:

- \( A \) is compact and nonempty,
- \( TA \subset A \),
- \( A \) attracts every point in some neighborhood of \( A \).

It follows that the set of points attracted to \( A \) is an open neighborhood \( N \) of \( A \) such that \( A \subset TN \subset N \).

We call \( A \) a global attractor if it attracts every point of \( Y \).

1.2. Chaotic dynamics

The hallmark of a chaotic dynamical system is this:

**Sensitivity to Initial Conditions:** There exists \( \delta > 0 \) with the following property: If \( x, y \) are distinct initial states, there exists a time \( t > 0 \) such that the corresponding future states satisfy:

\[ d(x(t), y(t)) > \delta. \]

When such a dynamical system is used to model the evolution of a natural system, (e.g., the atmosphere, an economy, an ecology, a disease), accurate long-term predictions are not possible. This was discovered by the meteorologist Edward Lorenz in his seminal 1963 article, “Deterministic Non-periodic Flow” [26]. After drastically simplifying standard equations for fluid flow, Lorenz arrived at the system of differential equations:

\[
\begin{align*}
\dot{x} &= 10(y - z), \\
\dot{y} &= 28x - y - xz, \\
\dot{z} &= xy - (8/3)z.
\end{align*}
\]

Despite the simple algebraic form of these equations, Lorenz found a disturbing feature in his extensive computations of solutions:

“...two states differing by imperceptible amounts may eventually evolve into two considerably different states. If, then, there is any error whatever in observing the present state— and in any real system such errors seem inevitable— an acceptable prediction of an instantaneous state in the distant future may well be impossible.”

This unexpected problem for applied dynamics inspired a great many publications analyzing it from various points of view. The first mathematical proof of Lorenz’s numerical discovery, computer-assisted but rigorous, is due to W. Tucker [41]. For interesting discussions of this work, see I. Stewart [36] and M. Viana [42].

R. Devaney [7, p.324] validated Lorenz’s conclusions dynamically by constructing a Poincaré (or “first return”) map \( T: C \to C \) for Lorenz’s differential equations (2), such that:
• \( C \subset \mathbb{R}^3 \) is an affine open 2-cell, transverse to the solution curves of (2).

• \( T \) has a chaotic global attractor.

His proof is based on estimates derived from (2) by simple algebra. R. Williams [43] showed that Lorenz’s attractor is structurally stable, which means that all sufficiently small perturbations preserve the essential features of Lorenz’s equation.

The term “chaos” is used in many ways in mathematics. In a widely accepted definition, R. Devaney [6] called a map \( T : A \to A \) to be chaotic if it satisfies three axioms:

**Sensitivity to Initial Conditions:** There exists \( \delta > 0 \) such that if \( x, y \in A \) are distinct, then \( d(T^k x, T^k y) \geq \delta \) for some \( k > 0 \).

**Dense Periodic Points:** Every nonempty open subset of \( Y \) contains a periodic point.

**Topological Transitivity:** If \( U, V \subset A \) are nonempty open sets, \( T^k U \cap T^k V \neq \emptyset \) for some \( k \geq 0 \).

Topological Transitivity holds if some orbit is dense, and the converse is proved in the Results section.

In order to avoid trivialities, I add a fourth axiom:

**Nonfiniteness:** \( A \) is not a finite orbit.

When all four axioms are satisfied, \( A \) and \( T \) are chaotic.

### 1.3. Remarks

• P. Touhey [38] found a remarkably simple condition equivalent to Devaney’s definition:

**Sharing of Periodic Orbits:** Every pair of nonempty open sets have a periodic point in common.

• Sensitivity to Initial Conditions follows from Devaney’s other two axioms (Silverman [29]), so it is independent of the metric.

• In most applications \( Y \) is a complete, separable metric space. In this case the Baire Category Theorem can be used to show that Topological Transitivity is equivalent to existence of dense orbit (stated in Devaney [6]). A proof for \( Y \) compact is given in the Results section.

For some other approaches to chaos, see [4, 11, 22, 23, 37, 43].

Our main result, Theorem 2 below, shows that:

**Monotone maps in strongly ordered spaces do not have chaotic attractors.**

*Suggested by Devaney [8].
1.4. Monotone dynamics

The state space of a monotone dynamical system is a space $X$ endowed with a (partial) order denoted by $\preceq$. The set $\{(x, y) \in X \times X : x \preceq y\}$ is assumed to be closed. A map $T$ between ordered spaces is monotone provided

$$x \preceq y \implies Tx \preceq Ty.$$ 

We use the standard notation:

$$x \prec y \iff x \preceq y, x \neq y.$$ 

If $A$ and $B$ are sets, $A < B \iff a < b, \ (a \in A, b \in B).$

$$a < B \iff \{a\} < B.$$ 

In the main result $X$ is strongly ordered:

If $W \subset X$ is an open neighborhood of $x$, there are nonempty open sets $U, V \subset W$ such that $U < x < V$.

**Examples.**

- Euclidean space $\mathbb{R}^n$ is strongly ordered by the classical vector order:

  $$x \preceq y \iff x_j \leq y_j, \ (j = 1, \ldots, n). \quad (3)$$

- Many Banach spaces of continuous real-valued functions on a space $S$ are strongly ordered by the functional order:

  $$f_1 \preceq f_2 \iff f_1 x \leq f_2 x, \ (x \in S).$$

In many dynamical models of natural systems the state space reflects the relative size of states—density, population, etc.

Scientific fields are often modeled by a dynamical system whose state space $S \subset \mathbb{R}^n$ has an order that reflects the relative “size” of states—density, population, etc. Each coordinate

A typical example of a monotone dynamical system is one that models an ecology of $n$ species that is biologists call commensual: an increase in the growth rate of any species tends to increase the sizes of the others. The state space is the positive orthant $\mathbb{R}_n^+$ with the vector order; coordinate $x_i$ is a measure of the size (e.g., the density) of population $i$, and dynamic is defined by a set of ordinary differential equations

$$\frac{dx_i}{dt} = x_i G_i(x_1, \ldots, x_n), \ (i = 1, \ldots, n). \quad (4)$$

Monotonicity is established by assuming

$$i \neq j \implies \frac{\partial G_i}{\partial x_j} \geq 0. \quad (5)$$

If the species reproduce only once a year, the ecology can be modeled by a map $T: \mathbb{R}_n^+ \to \mathbb{R}_n^+$. The dynamic $\{T^k\}$, parametrized by integers $k \geq 0$, is monotone if the partial derivatives of $T$ are nonnegative.
An opposite kind of ecology—for example, sheep and wolves—is modeled by a competitive system of differential equations (4), which instead of (5) satisfies:

\[ i \neq j \implies \frac{\partial G_i}{\partial x_j} \leq 0. \]  

(6)

Here an increase in the growth rate of one species tends to decrease the sizes of others. The theory of cooperative systems can be used to analyze competitive systems by “time reversal”— choosing \(-t\) as the time parameter. This produces a cooperative system.

While cooperative systems cannot have chaotic attractors, by Theorem 2, many competitive systems do have them. They can be constructed using a theorem of S. Smale [30]:

Let \( F \) be a dynamical system on the \((n - 1)\)-simplex

\[ \Delta := \{ x \in \mathbb{R}^n_+ : \Sigma^n_1 x_i = 1 \}, \]

determined by a continuously differentiable vector field. Then \( F \) extends to a dynamical system \( \hat{F} \) in \( \mathbb{R}^n_+ \) determined by a suitable \( G \) in (4), and has \( \Delta \) as a global attractor.

It follows that if \( A \subset \Delta \) is a chaotic global attractor for \( F \), then \( A \) is also a chaotic global attractor for \( \hat{F} \).

Monotone dynamical systems often permit reliable predictions of long-term behavior. In many cases it can be proved that typical trajectories tend toward fixed points or periodic orbits. See for example references [2, 3, 5, 9, 10, 12, 13, 15–17, 19, 20, 24, 25, 28, 31, 34, 44]. The recent survey by H. Smith [35] has an extensive bibliography.

Monotonicity and chaos play quite different roles in dynamical models:

Monotonicity is easily verified for many standard classes of systems, and in many cases standard theory gives accurate predictions of long-term behavior.

Chaos is quite difficult to either prove or disprove, and it makes accurate long-term prediction impossible. But it is unavoidable: as Lorenz discovered, simple models of real systems exhibit chaos.

Results

Proposition 1. If \( Y \) is a compact metric space and \( H : Y \to Y \) is topologically transitive, then \( H \) has a dense orbit.\(^\dagger\)

Proof. \( Y \) is covered by a family \( U_1, U_2, \ldots \) of open sets whose diameters go to 0 as \( i \) goes to \( \infty \). Using Topological Transitivity repeatedly, one finds a sequence \( K_i \) of compact sets satisfying:

\[ U_1 \supset K_1 \supset K_2 \supset \cdots \]  

(7)

\(^\dagger\)Devaney [6] points out that it suffices for \( Y \) to be any separable, complete metric space.
and
\[ U_i \cap T^i(K_i) \text{ is not empty.} \]  
(8)

By compactness of the \( K_i \) there exists \( p \in \bigcap_i K_i \), whence (7) and (8) show that the orbit of \( p \) is dense.

\[ \square \]

**Theorem 1.** Assume \( T : X \to X \) is a monotone map in a strongly ordered space, and \( A \) is an attractor for \( T \). Then \( T|A \) is not topologically transitive, hence \( A \) is not chaotic.

**Proof.** Assume per contra that \( T|A \) is topologically transitive. Then Proposition 1, with \( H = T|A \), implies:

(i) *Some orbit is dense in \( A \)*,

which implies:

(ii) *No point is isolated in \( A \).*

To reach a contradiction we rely on a *deus ex machina*, Corollary 6.4 of Hirsch [12]:

*If no point is isolated in \( A \), then no finite union of orbits is dense in \( A \).*

Therefore (i) is false, proving that \( T|A \) is topologically transitive.

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**References**

[1] J. Banks et al., J. Brooks, G. Cairns, G. Davis & P. Stacey, *On Devaney’s definition of chaos*, American Math. Monthly 99 (1992), 332–334.

[2] M. Akian, S. Gaubert & Bas Lemmens, *Stability and convergence in discrete convex monotone dynamical systems*, arXiv: 1003.5346v1 (2010).

[3] Bas Lemmens, O. van Gaans, H. van Imhoff. *Monotone dynamical systems with dense periodic points*, J. Differential Equations 265 (1918), no. 11, 5709–5715,

[4] C. Good & S. Macfas, *What is topological about topological dynamics?*, Discrete Contin. Dyn. Syst. 38 (2018), no. 3, 1007–1031.

[5] P. De Leenheer, *The puzzle of partial migration*, J. Theroretical Biology 412 (2017), 172–185.

[6] R. Devaney, “An introduction to chaotic dynamical systems,” Benjamin/Cummings, Menlo Park, CA, (1986).
[7] R. Devaney, M. Hirsch & S. Smale, “Differential Equations, Dynamical Systems & an Introduction to Chaos,” Elsevier Academic Press 2004.

[8] R. Devaney, personal communication (2019).

[9] G. Dirr et al., Separable Lyapunov functions for monotone systems: constructions and limitations, Disc. & Cont. Dyn. Systems, Series B 20 (2015), 2497–2526.

[10] G. Enciso & E. Sontag, Global attractivity, i/o monotone small-gain theorems, and biological delay systems, Disc. & Cont. Dyn. Systems 14 (2006), 249–578.

[11] J. Guckenheimer & R. Williams, Structural stability of Lorenz attractors, Publ. Math. IHES 50 (1979), 59-72.

[12] M. Hirsch, Attractors for discrete-time monotone dynamical systems in strongly ordered spaces. Geometry and Topology: Lecture Notes in Mathematics 1167, 141–153. J. Alexander, J.Harer, editors. Springer-Verlag, New York, 1985.

[13] M. Hirsch, Systems of differential equations which are competitive or cooperative. I: limit sets. SIAM J. Appl. Math. 13 (1982), 167–179.

[14] M. Hirsch, Systems of differential equations which are competitive or cooperative, II: convergence almost everywhere, SIAM J. Math. Anal., 16, (1985) 423–439.

[15] M. Hirsch, Stability and convergence in strongly monotone dynamical systems, J. reine und angewandte Mathematik 383 (1988), 1–53.

[16] M. Hirsch, Systems of differential equations which are competitive or cooperative, III: competing species, Nonlinearity 1, (1988) 51–71.

[17] M. Hirsch, Differential equations and convergence almost everywhere in strongly monotone semiflows, Contemp. Math. 17, (1983) 267–285.

[18] M. Hirsch, The dynamical systems approach to differential equations, Bull. Amer. Math. Soc. 11 (1984), 1–64.

[19] M. Hirsch, Monotone dynamical systems with polyhedral order cones and dense periodic points, AIMS Mathematics 2 (2017), 24–27. Also in: http://arxiv.org/abs/1611.09251

[20] M. Hirsch & H. Smith, Monotone Dynamical Systems, “Handbook of Differential Equations,” volume 2, chapter 4. A. Cañada, P. Drábek & A. Fonda, editors. Elsevier North Holland 2005.

[21] S.-B. Hsu, H. Smith & P. Waltman, Dynamics of competition in the unstirred chemostat, Canadian Appl. Math. Quart. 2 (1994), 461–483.

[22] A. Kolesov & N. Rozov, On the definition of chaos, Uspekhi Mat. Nauk 64 (2009), 125–172; translation 744. Russian Math. Surveys 64 (2009), no. 4, 701–744.
[23] M. Kulczycki, *Noncontinuous maps and Devaney’s chaos*, Regular & Chaotic Dyn. **13** (2008), no. 2, 81–84.

[24] A. Lajmanovich & J. Yorke, *A deterministic model for gonorrhea in a nonhomogeneous population*, Math. Biosciences **28** (1976), 221–236.

[25] A. Landsberg & E. Friedman, *Dynamical Effects of Partial Orderings in Physical Systems*, Physical Review E **54** (1996), 3135–3141.

[26] E. Lorenz, *Deterministic Non-periodic Flow*, J. Atmos. Sci. **20** (1963), 130–141.

[27] J. Mallet-Paret & H. Smith, *The Poincaré-Bendixson theorem for monotone cyclic feedback systems*, J. Dynamics and Diff. Equations **2** (1990), 367–421.

[28] C. Potsche, *Order-preserving nonautonomous discrete dynamics: Attractors and entire solutions* Positivity **19** (2015), 547–576.

[29] S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain J. Math. **22** (1992), 332–334

[30] S. Smale, *On the differential equations of species in competition*, J. Math. Biol. **3** (1976), 5–7.

[31] J. Selgrade, *Asymptotic behavior of solutions to single loop positive feedback systems*, J. Diff. Eqns. **38** (1980), 80–103.

[32] J. Smillie, *Competitive and cooperative tridiagonal systems of differential equations*, SIAM J. Math. Anal. **15** (1984), 530–534.

[33] H. Smith, *Periodic competitive differential equations and the discrete dynamics of competitive systems*, J. Diff. Eqns. **64** (1986), 165–194.

[34] H. Smith, “Monotone Dynamical Systems: an introduction to the theory of competitive and cooperative systems,” Amer. Math. Soc. Surveys and Monographs **41**, 1995.

[35] H. Smith, *Monotone dynamical systems: reflections on new advances & applications*, Disc. & Cont. Dyn. Systems Series B **37** (2017), no. 1, 485–504.

[36] I. Stewart, *The Lorenz attractor exists*, Nature **406** (2000), 948–949.

[37] V. To, *A note on Devaney’s definition of chaos*, J. Dyn. Syst. Geom. Theor. **2** (2004), 23–26.

[38] P. Touhey, *Yet another definition of chaos*, Amer. Math. Monthly **104** (1997), no. 5, 411–414.

[39] P. Touhey, *Chaos: the evolution of a definition*, Irish Math. Soc. Bull. No. 40 (1998), 60–70.

[40] P. Touhey, *Persistent properties of chaos*, J. Differ. Equations Appl. **6** (2000), no. 3, 249–256.

[41] W. Tucker, *The Lorenz attractor exists*, C. R. Acad. Sci. Paris, t. 328, Série I (1999), 1197–1202.
[42] M. Viana, *What's new on Lorenz strange attractors?* Math. Intelligencer 22 (2000), 6-19.

[43] R. Williams, *The structure of Lorenz attractors*, Publ. Math. IHES 50 (1979), 73–99.

[44] X.-Q. Zhao, *Global attractivity and stability for discrete strongly monotone dynamical systems with applications to biology*, Technical Report No. 94005. Beijing: Inst. Appl. Math., Ac. Sinica (1994).