Momentum and energy propagation in tapered granular chains

L. P. Machado · Alexandre Rosas · Katja Lindenberg

Abstract We study momentum and energy propagation in 1D tapered chains of spherical granules which interact according to a Hertz potential. In this work we apply the binary collision approximation, which is based on the assumption that transfer of energy along the chain occurs via a succession of two-particle collisions. Although the binary theory correctly captures the trends of increase or decrease of kinetic energy and momentum, the actual values of these quantities are not in good quantitative agreement with those obtained by numerically integrating the full equations of motion. To address this difficulty we have developed a mixed numerical/analytical correction algorithm to provide an improved estimate of the velocity of the particles during pulse propagation. With this corrected velocity we are in turn able to correctly predict the momentum and kinetic energy along the chain for several tapering configurations, specifically for forward linear, forward exponential, backward linear and backward exponential tapering.

Keywords granular chain · energy · momentum · binary collision approximation

PACS 45.70.-n · 46.40.Cd · 43.25.+y · 05.65.+b
1 Introduction

Pulse propagation through granular media has become a topic of intense theoretical, numerical, and experimental attention because it involves fundamental issues in nonlinear discrete systems, and because at the same time it is of major practical importance. When a granular chain of equal spherical granules without precompression and without gaps is hit at one end, a solitary wave travels down the chain essentially unchanged in time. This observation was first made by Nesterenko [1,2,3], who solved this nonlinear problem using an analytic continuum approximation and successfully compared his analytic solutions to numerical simulation results. Experimental verification followed soon after [4] and later in [5]. The way in which a pulse propagates along a chain is strongly dependent on the characteristics of the granules [6]. By modifying such a chain in various ways, e.g., by tapering, including large and small masses, adding impurities to the chain, and other such manipulations, it was soon realized that it is possible to control exactly how the pulse energy behaves as it propagates [7,8,9,10,11,12,13,14,15,16,17,18] or becomes trapped with slow leakage [19,20]. This control offers the possibility of designing granular media with carefully chosen shock absorption or signal propagation properties. Other interesting configurations leading to solitary wave train formation were explored [21,22]. The feasibility of optimizing granular configurations for clean reflection of a pulse from a wall was studied as a tool for nonintrusive or nondestructive sensors [16]. These and many other applications were discussed and reviewed in [23,24]. The effects of various forms of friction on pulse propagation were analyzed in [25,26,27], including the observation of entirely new waveforms as a result of momentum-conserving frictional forces.

On the theoretical front, a quantitatively accurate theory was first developed three decades ago for chains made up of identical spherical granules with no precompression and no gaps [12]. This approach was difficult to generalize to other granular configurations. As an alternative, we developed a theory based on a binary collision approximation (BCA) which has been generalized to tapered, and decorated chains in various initial states and even to chains with some random configurational elements [25,28,29,30,31,32]. This theory has been quantitatively accurate in many cases, and forms the basis for the discussion in this paper.

Experiments on granular chains has been restricted by the constraints of particular experimental setups. Analytic work can overcome such strictures and is helpful in expanding our fundamental understanding of these systems. The methods we have recently developed [28,29,30,31,32] have proved broadly applicable to chains of a variety of configurations. Our analytic BCA works remarkably well not only qualitatively but also quantitatively. Even though the BCA is restricted to conservative chains (elastic collisions), and real granular collisions are not entirely elastic, the BCA is nevertheless able to capture experimentally observed dynamics of chains of very hard spheres moving along very highly polished channels. This is the case with the experiments in the references cited above.
Why would one want to study chains of these particular configurations (such as tapered or decorated)? In addition to questions of theoretical interest, the reason is that such physical arrangements of granules provide the opportunity to optimize the shock absorption properties of the system, and it is hoped that what one learns in one dimension can guide the inquiries in more realistic higher dimensional systems. It is quite surprising how exquisitely sensitive the transmittal of energy and/or momentum along a chain can be to details of chain granule configuration. The characterization of energy and momentum propagation along a granular chain is in itself quite varied. For instance, one might ask how long it takes the maximum of a pulse to reach a given granule. Or one might wish to know how long the pulse maximum resides on a particular granule, or how much energy is focused on that granule during this time of residence. One might wish to know what happens to the width of a pulse as it travels along a chain, whether or not the pulse retains its integrity, and one might wish to predict whether or not the chain will fragment in the process. Particular quantities of interest in the assessment of shock absorption properties of these simplified one-dimensional granular systems are the momentum and energy with which a pulse that starts at one end of a granular chain would first hit a wall at the other end. In our earlier work we focused on the pulse velocity amplitude and on the time that the pulse spends on each granule before moving on to the next granule.

The BCA assumes that the traveling pulse resides on only two granules at a time, so that the many-body problem of a fully interacting chain is reduced to an iterated solution of successive two-body problems. The approximation is straightforwardly implemented for monodisperse [28] and tapered chains [29]. For decorated chains, that is, chains of large (possibly tapered) grains separated by one or more small granules, it is first necessary to replace the chain by an effective monodisperse or tapered one to which the approximation can then be applied [30][31]. The approximation relies on the assumption that the traveling pulse is very narrow, which holds for many granular geometries (including spherical granules). Within the binary collision approximation, we were able to make exceptionally good predictions for certain pulse properties. However, not every prediction of the BCA is equally quantitatively accurate. In particular, even though the change of the velocity of the particles, $\frac{dv_k}{dk}$, as a pulse moves along a chain of granules labeled by $k$ is predicted extremely accurately, the BCA consistently over-estimates the actual value $v_k$ of the velocity amplitude itself. The reason is that in reality the pulse does not reside precisely on only two granules at a time because the interactions are not hard-sphere.

In this work we focus on correcting the error made in the calculation of the velocities of the granules during pulse propagation along a chain of initially uncompressed spherical grains that just touch each other, which in turn affects our predictions for momentum and energy propagation. As we shall see, the corrections are mild but important. We introduce corrections that are partly analytical and partly numerical, and that provide a set of tools for further quantitative prediction.
Before launching into our presentation we clarify a point that might otherwise appear puzzling. As the energy pulse travels down our various chains, we observe that the momentum and energy appear not to be conserved. The momentum may grow or decay, and the energy may appear constant or in fact may also decay. This occurs because we are specifically focusing on the pulse that carries energy forward. At each collision event, there may also be a backscattering of granules that also carry energy and momentum, and it may also happen that some portion of the energy may lag behind in the grains behind the pulse. When this is taken into account, all is as it should be with energy and momentum conservation.

We organize our presentation as follows. In Sec. 2 we briefly recap the equations of motion that form the basis of all the work to follow. In Sec. 3 we present the results of our previous calculations for backward tapered chains and in Sec. 4 for forward tapered chains. In Sec. 5 we discuss the corrections that lead to more accurate predictions for energy and momentum propagation along tapered chains. Section 6 is a summary of our results.

2 Equations of motion

We consider a one-dimensional chain of spherical granules of density \( \rho \), labeled by index \( k \), with radius \( R_k' \) and mass \( M_k = \left( \frac{4}{3} \right) \pi \rho (R_k')^3 \). The equation of motion for the displacement \( y_k \) of the \( k \)th granule as a function of time \( \tau \) is

\[
M_k \frac{d^2 y_k}{d\tau^2} = a r_{k-1}' (y_{k-1} - y_k)^{3/2} \theta(y_{k-1} - y_k) - a r_k' (y_k - y_{k+1})^{3/2} \theta(y_k - y_{k+1}),
\]

with \( r_k' = \left[ 2(R_k' R_{k+1}'/(R_k' + R_{k+1}') \right]^{1/2} \) and \( a \) a constant determined by Young’s modulus and Poisson’s ratio. The Heaviside function \( \theta(y) \) ensures that the elastic interaction between grains only exists if they are in contact. Initially the granules just touch their neighbors in their equilibrium positions, that is, there is no pre compression and there are no gaps, and all but the leftmost particle are at rest. The initial velocity of the leftmost particle, \( k = 1 \), is \( V_1 \).

It is convenient to define scaled variables and parameters as follows,

\[
x_k = \frac{y_k}{R_1' \alpha^{2/5}}, \quad t = \frac{V_1 \tau}{R_1' \alpha^{2/5}}, \quad m_k = \frac{M_k}{M_1},
\]

with \( \alpha = M_1 V_1^2 / (a R_1')^{n+1/2} \).

Equation (1) can then be rewritten in the cleaner form, which we use for all subsequent analysis,

\[
m_k \frac{d^2 x_k}{dt^2} = r_{k-1}' (x_{k-1} - x_k)^{3/2} \theta(x_{k-1} - x_k) - r_k' (x_k - x_{k+1})^{3/2} \theta(x_k - x_{k+1}),
\]
with \( r_k = [2(R_k R_{k+1}/(R_k + R_{k+1}))^{1/2}]. \) The rescaled initial velocity is unity, \( v_1(t = 0) = 1, \) as is the mass \( m_1 = 1 \) and the radius \( R_1 \) of the first granule. The velocity of the \( k \)th granule in unscaled variables is simply \( V_1 \) times its velocity in the scaled variables. The equations of motion can be integrated numerically. All of our previous results are obtained beginning from this starting point.

In a series of papers [28,29,30,31] we went on to solve these equations of motion using an approximation that we called the Binary Collision Approximation (BCA). The approximation is based on the assumption that the transfer of energy along the chain occurs via a succession of two-particle collisions. Particle \( k = 1 \) of unit velocity collides with initially stationary particle \( k = 2, \) which then acquires velocity \( v_2 \) and collides with initially stationary particle \( k = 3, \) and so on. The velocities after each collision can easily be obtained from conservation of energy and momentum. This scheme is straightforward to implement if the chain is monodisperse or if it is tapered forward (decreasing radii) or backward (increasing radii). If the chain is “decorated” with small grains among large ones, then the decorated chain must first be replaced by a carefully constructed effective chain of the monodisperse or tapered variety to which the BCA can then be applied. Here we exhibit the final full analytic BCA result, which requires the application of a particular tapering protocol for further implementation in later sections. The velocity according to the BCA is

\[
v_k = \prod_{k'=1}^{k-1} \frac{2}{1 + \frac{m_{k'}}{m_k}}, \tag{5}
\]

Together with the mass \( m_k = (R_k/R_1)^3 \) this will then yield the pulse momentum \( P_k = m_k v_k \) and the pulse energy \( E_k = (1/2)m_k v_k^2. \)

As stated above, the BCA works very well for some quantities but is not quantitatively correct in its prediction of the pulse velocity. This quantitative discrepancy between approximate and exact results thus leads to errors in the prediction of the momentum and the energy of the pulse; the energy dependence on the square of the velocity of course magnifies the problem. We thus searched for a way to improve the theory. We call this improvement a “modified” theory. In the next two sections we simply present, without re-deriving, the results of the original BCA (in scaled variables) relevant to this discussion. The modified theory is then presented in Sec. 5.

3 Backward tapered chains

We start with a collection of the results obtained for backward tapered chains using the BCA. Backward tapered chains are alignments of granules of progressively increasing size and/or mass. We assume that the granules are all made of the same material, so that they have the same density and elastic properties. We study two different rules for the way that the radii \( R_k \) increase along the chain, whose granules are labeled by index \( k. \) In linearly tapered
chains, the radii increase arithmetically,
\[ R^{bl}(k) = 1 + S(k - 1), \] (6)
while exponential tapering is one in which the radii increase geometrically,
\[ R^{be}(k) = \frac{1}{(1 - q)^{k-1}}. \] (7)

\( S > 0 \) and \( 0 < q < 1 \) are the tapering parameters. Here, the radii of the particles are given in units of the radius of the leftmost granule (the impacting granule). The subscripts stand for “backward linear” and “backward exponential” respectively.

3.1 Linearly tapered chains

Using energy and momentum conservation, one can show that for linearly tapered chains the maximum velocity \( v^{bl}_u(k) \) of granule \( k \) as the pulse travels along the chain varies as [29]
\[ v^{bl}_u(k) \sim k^{-3/2} \] (8)
for large \( k \). The superscript indicates that these are results of the original unmodified BCA theory. On the other hand, since \( m \sim R^3 \) for any configuration,
\[ m^{bl}_u(k) \sim k^3. \] (9)
The masses of the granules are given in units of the mass of the leftmost particle. Therefore, the momentum and kinetic energy are predicted to have the asymptotic behavior
\[ P^{bl}_u(k) = m^{bl}_u(k)v^{bl}_u(k) \sim k^{3/2}, \] (10)
\[ E^{bl}_u(k) = \frac{1}{2}m^{bl}_u(k)[v^{bl}_u(k)]^2 \sim 1. \] (11)

It is worth mentioning that the exact form of \( v^{bl}_u \), and consequently of \( E^{bl}_u \) and \( P^{bl}_u \), can be obtained as a closed product expression if we use the expressions given in Sec. 2. However, for purposes of discussion it is here more instructive to focus on the asymptotic behaviour as given in Eqs. (8), (10) and (11). We stress that all the BCA data presented in this manuscript was obtained using the full analytic predictions of the theory and not just the asymptotic results. In Fig. 1 we show that for various values of the tapering parameter \( S \) the momentum growth follows the trend predicted by Eq. (10) – compare the slopes of the different sets of symbols with the slopes of the lines. The energy, in turn, exhibits the saturation behavior predicted by Eq. (11), as can be seen from Fig. 2.
Momentum as a function of granule number $k$ for backward linearly tapered chains for $S = 0.1$ to 0.9 from bottom to top, in steps of 0.4. The circles correspond to the numerical data, the + symbols represent the BCA and the solid line corresponds to an arbitrary line with the slope 3/2 predicted by the BCA.

Fig. 1

Kinetic energy as a function of $k$ for backward linearly tapered chains for $S = 0.1$ to 0.9 from top to bottom, in steps of 0.2. The circles correspond to the numerical data, the + symbols represent the BCA and the solid line is an arbitrary horizontal line indicating the asymptotic behavior predicted by the BCA.

Fig. 2

3.2 Exponentially tapered chains

For the backward exponentially tapered chain the velocity in the BCA and the mass behavior are respectively (see [29])

$$v_{be}^{u}(k) = A_{be}^{u}(q)e^{-k \ln A_{be}^{u}(q)},$$

$$m_{be}(k) = (1 - q)^{3(1-k)},$$

with

$$A_{be}^{u}(q) = \frac{1}{2} \left[ 1 + (1 - q)^{-3} \right]$$

(14)
Hence, the momentum in the BCA for all \( k \) (not just asymptotically) is given by

\[
P_{u_{bc}}(k) = \left(1 - \frac{3}{2}q + \frac{3}{2}q^2 - \frac{1}{2}q^3\right)^{1-k}.
\] (15)

This expression can be compared with the results of the numerical integration of the equations of motion as follows. If we take the natural logarithm of Eq. (15), we see that the BCA prediction for the momentum as a function of \( k \) is an exponential growth characterized by

\[
\mathcal{P}_{u_{bc}} \equiv \frac{\ln[P_{u_{bc}}(k) / k]}{k} \sim -\ln \left(1 - \frac{3}{2}q + \frac{3}{2}q^2 - \frac{1}{2}q^3\right).
\] (16)

An exponential increase of the momentum can indeed be seen in Fig. 3. Furthermore, the rate of increase is in good agreement with the BCA prediction. In Table II the first two columns compare the exponential rate of growth according to the BCA theory and the numerical results, respectively. Exceptional agreement up to the second or third decimal digits confirm the quality of the approximation.

![Figure 3](image_url)

**Fig. 3**: Momentum as a function of \( k \) for backward exponentially tapered chains for \( q = 0.01 \) to 0.08 from bottom to top, in steps of 0.01. The circles correspond to the numerical data and the + symbols represent the BCA.

We can similarly use the BCA to obtain the kinetic energy. The expression valid for all \( k \) may be written as

\[
E_{k_{bc}}(k) = \left(\frac{1}{2}\right)^{3-2k} \left[(1 - q)^{3/2} + (1 - q)^{-3/2}\right]^{2(1-k)}.
\] (17)
Thus, for large $k$ we have

$$E^{u}_{be} \equiv \frac{\ln[E^{u}_{be}(k)]}{k} \sim -2 \ln \frac{1}{2} \left[ (1 - q)^{3/2} + (1 - q)^{-3/2} \right].$$

(18)

We find that the BCA correctly predicts the exponential decay of the kinetic energy, as shown in Fig. 4, albeit not as accurately as the momentum prediction. In Table 2 we show the numerical comparison between the theory and the numerical integration of the equations of motion. The agreement is not as good as it is for the momentum because the kinetic energy depends on the square of the velocity, and the prediction of the BCA for the velocity is not accurate; hence the error in the velocity is amplified.

| $q$ | Backward BCA | Backward Numerical | Forward BCA | Forward Numerical |
|-----|--------------|-------------------|-------------|-----------------|
| 0.01 | 0.01496 | 0.01491 | -0.0150369 | -0.0151371 |
| 0.02 | 0.02986 | 0.02972 | -0.030145 | -0.0305032 |
| 0.03 | 0.04485 | 0.04440 | -0.0453208 | -0.0462143 |
| 0.04 | 0.05936 | 0.05896 | -0.0605606 | -0.0623547 |
| 0.05 | 0.07388 | 0.07340 | -0.0758609 | -0.0790053 |
| 0.06 | 0.08831 | 0.08773 | -0.0912182 | -0.0962407 |
| 0.07 | 0.10294 | 0.10194 | – | – |
| 0.08 | 0.11727 | 0.11605 | – | – |

Table 1 Comparison of the increase as characterized by $P^{u}_{be}$ and the decrease as characterized by $P^{u}_{fe}$ of the momentum as predicted by the BCA theory and the results obtained from the numerical integration of the equations of motion.

Fig. 4 Decay of the kinetic energy with $k$ for backward exponentially tapered chains for $q = 0.01$ to 0.08 from bottom to top, in steps of 0.01. The circles correspond to the numerical data and the + symbols represent the BCA.
It should be stressed that in spite of the velocity decrease (as a power law for the linearly tapered chain, exponentially for the exponentially tapered chain) the momentum increases for both backward tapered chains as a function of $k$. The energy, on the other hand, may decrease (exponential chain) or saturate (linear chain). With respect to the tapering parameters, we notice that the larger the values of $q$ or $S$, the larger is the momentum, but the smaller are the energy and velocity. Note that the momentum may be the important quantity to focus on when studying a shock absorber. Imagine our chain ending at a wall. If the interaction of the last particle of the chain with the wall is perfectly elastic, then the momentum traveling down the chain would be transferred to the wall. In this case the backward tapered chains would not be good shock absorbers; on the contrary, they magnify the shock. These statements would need to be appropriately modified if the interaction of the last particle with the wall is inelastic.

### 4 Forward tapered chains

For forward tapered chains, the granules progressively decrease in size. In linearly tapered chains the radii decrease arithmetically,

$$R_k^f = 1 - S(k - 1)$$  \hspace{1cm} (19)

where, as before, $S$ is the tapering parameter. This tapering imposes a limit on the number of granules that may compose the chain to avoid the absurd situation of granules with negative radii. Thus, the number $N$ of granules in the chain must obey the restriction

$$N < 1 + \frac{1}{S}. \hspace{1cm} (20)$$

For exponentially tapered chains the radii decrease geometrically,

$$R_k^e = \frac{1}{(1 + q)^{k-1}}. \hspace{1cm} (21)$$
For exponentially tapered chains a restriction on chain length is only imposed for practical reasons (not to have granules that are too small to deal with experimentally, and to avoid numerical errors in the analysis).

4.1 Linearly tapered chains

In this case, in the limit $k \ll 1 + 1/S$, the amplitude of the pulse velocity in the BCA goes as

$$v^u_{fl}(k) \simeq \left(1 - \frac{k}{1 + 1/S}\right)^{-3/2}, \quad (22)$$

and the mass is

$$m_{fl}(k) = \left[1 - S(k - 1)\right]^3 = (1 + S)^3 \left[1 - \frac{k}{1 + 1/S}\right]^3. \quad (23)$$

Hence the momentum and kinetic energy are predicted to be, respectively,

$$P^u_{fl}(k) \sim S^3 (1 + 1/S)^3 \left[1 - \frac{k}{1 + 1/S}\right]^{-3/2},$$

$$E^u_{fl}(k) \sim 1. \quad (24)$$

Here, as in the case of backward tapered chains, we present the asymptotic formulas for the velocity, momentum and energy, but one should keep in mind that closed formulas can be obtained [29]. In Fig. 5 we show that, for $k$ and $S$ not too large, the momentum indeed decreases linearly with increasing $k$. Furthermore, for sufficiently small $S$ the slope of the line is in fair agreement with the BCA predictions (see Table 3). The energy, shown in Fig. 6, is approximately constant for small $S$ and $k$ not too large.

4.2 Exponentially tapered chains

For forward exponentially tapered chains we simply map $q \to -q$ in the expressions for the backward tapered chains. Hence, the momentum in the BCA is given by

$$P_{fe}^u(k) = \left(1 + \frac{3}{2}q + \frac{3}{2}q^2 + \frac{1}{2}q^3\right)^{1-k}. \quad (26)$$

Thus, for forward tapered chains the BCA predicts an exponential decay of the momentum as opposed to the growth in the backward chains. This is observed in the numerical integration of the equations of motion, as can be seen in Fig. 7. The rate of decay is in good agreement with the BCA prediction (see Table 1).
For the kinetic energy, on the other hand, the BCA for large $k$ leads to

$$
\ln(E^u_{fe}(k)) \sim -2k \ln \frac{1}{2} \left[ (1 + q)^{3/2} + (1 + q)^{-3/2} \right].
$$

(27)

Thus, as for the backward tapered chains, the BCA predicts an exponential decay of the kinetic energy, which is corroborated by our data (see Fig. 8).
| \( q \) | Binary | Numerical |
|-------|--------|-----------|
| 0.001 | -0.0015 | -0.0010 |
| 0.002 | -0.0030 | -0.0020 |
| 0.003 | -0.0045 | -0.0030 |
| 0.004 | -0.0060 | -0.0039 |
| 0.005 | -0.0076 | -0.0047 |
| 0.006 | -0.0091 | -0.0056 |
| 0.007 | -0.0106 | -0.0064 |
| 0.008 | -0.0122 | -0.0071 |
| 0.009 | -0.0137 | -0.0078 |
| 0.01  | -0.0153 | -0.0085 |

Table 3  Coefficient of the linear decay of the momentum for forward exponentially tapered chains.

Fig. 7  Momentum as a function of \( k \) for forward exponentially tapered chains for \( q = 0.01 \) to 0.06 from top to bottom, in steps of 0.01. The circles correspond to the numerical data and the + symbols represent the BCA.

Once more, the agreement is not as good as for the momentum because the kinetic energy depends on the square of the velocity, and hence the error in the velocity is magnified (see the third and fourth columns of Table 2).

We end this section by assessing the behavior of the kinetic energy and momentum for the different tapering configurations. First we notice that the kinetic energy is almost constant (after a short transient) for both linearly tapered chains, while it decays exponentially for both of the exponentially tapered chains. Thus, if one wishes to distribute the energy more uniformly or minimize the energy first arriving at the end of the chain, one should choose an exponentially tapered (in either direction) chain. The momentum, on the other hand, increases for backward tapered chains, but decreases for forward
tapered chains. Therefore, if one wants to build a shock absorber, one should choose a forward tapered chain. Moreover, exponentially tapered chains are better candidates for this purpose since they lead to an exponential decrease of the momentum, in contrast with the linear decay in the linearly tapered chains.

5 Modified Binary Collision Approximation

Although the binary collision approximation correctly captures the trends of increase or decrease of the kinetic energy and momentum, the actual values of these quantities are not always in sufficient agreement with the results obtained from the numerical integration of the equations of motion to be of predictive value. In this section we propose a numerical rescaling of the velocity predicted by the BCA which leads to quantitatively satisfactory predictors for the kinetic energy and momentum. Our error estimates are in part, but only in part, numerical in the sense that the dependence of the corrections on the tapering parameters is given analytically. We thus propose corrections that can then be used for all appropriate situations once the tapering-parameter-independent coefficients and exponents have been determined numerically once and for all.

As noted above, the BCA successfully captures the qualitative behavior of the maximum velocity of the grains as the pulse moves along the chain. In proposing a correction that will improve the quantitative agreement with numerical simulations, we have therefore assumed that the functional dependence (exponential or power law) of the velocity $v_k$ on the grain position $k$
is correct as given by the theory, and that the quantitative differences arise because the constants that characterize these functional dependences require adjustment. We thus allow small adjustments of these constants, explicitly, for example, of the exponent of $k$ and the proportionality constant $a_u$ in Eq. (28) below. We modify $a_u^{bl}$ to a value $a_{bl}$ presumed to lie close to $a_u^{bl}$; similarly, the exponent of $k$ is perturbed by the small constant $\delta_{bl}$, cf. Eq. (29). We expect that similar corrections should work for other tapering choices once the BCA has been used to establish the $k$-dependence of the velocity.

In this section we will continue to denote all original BCA quantities by a superscript $u$ for unmodified and the corrected velocities by quantities without a superscript.

5.1 Backward tapered chains

5.1.1 Linearly backward tapered chains

For linearly backward tapered chains (indicated by a subscript $bl$), according to the BCA in the large $k$ limit the velocity of the granules increases as

$$v_u^{bl}(k) \simeq a_u^{bl} k^{-3/2}$$

[cf. Eq. (25)], where the constant $a_u^{bl}$ may depend on the tapering parameter $S$. We assume a modified form

$$v_{bl}(k) \simeq a_{bl} k^{-3/2 + \delta_{bl}},$$

where the constants $a_{bl}$ and $\delta_{bl}$ may again depend on the tapering parameter. If this is correct, then the ratio of these velocities should be

$$\frac{v_{bl}(k)}{v_u^{bl}(k)} \simeq \frac{a_{bl}}{a_u^{bl}} k^{\delta_{bl}}.$$  

(30)

For small $\delta_{bl}$ as measured by the condition $\delta_{bl} ln k \ll 1$, we can expand $k^{\delta_{bl}} = e^{\delta_{bl} ln k} \simeq 1 + \delta_{bl} ln k$. Consequently, except for a logarithmic correction, we expect the ratio of the velocities to be independent of granule number $k$,

$$v_{bl}(k)/v_u^{bl}(k) \simeq a_{bl}/a_u^{bl}.$$  

In Fig. 9 we show that this ratio is indeed almost constant, not only essentially independent of $k$ at sufficiently large granule number but also essentially independent of the tapering parameter $S$. As a working value we take the ratio to be around $2/3$. With this rescaling, we see excellent agreement of the velocity (Fig. 10), momentum (Fig. 1) and kinetic energy (Fig. 2) with the results of the numerical integration of the equations of motion. This rescaling can henceforth be used predictively for all linearly backward tapered chains:

$$v_{bl} \simeq \frac{2}{3} k^{-3/2}.$$  

(31)
Fig. 9 Ratios of the velocity from the modified BCA and from numerical integration data for backward linearly tapered chains for $S = 0.1$ to 0.9 from bottom to top at the right end of the figure, in steps of 0.2.

Fig. 10 Comparison between the modified BCA velocities (+ symbols) and those obtained from the numerical integration of the equations of motion (circles) in backward linearly tapered chains for $S = 0.1$ to 0.9 from top to bottom, in steps of 0.2.

5.2 Exponentially backward tapered chains

For the exponentially tapered chain, the velocity increases exponentially. We write the BCA and modified BCA velocities as

$$v_{be}^u(k) = a_{be}^u e^{\alpha_k^u k},$$

(32)
where \(a_{be}^u\) and \(\alpha_{be}^u\) are analytically known \(k\)-independent constants that depend on the tapering parameter \(q\), cf. Eq. (12), and \(a_{be}\) and \(\alpha_{be}\) are also assumed to be independent of \(k\). We assume \(|\alpha_{be} - \alpha_{be}^u|k \ll 1\) (a stronger restriction than the logarithmic one of the previous subsection). Hence, the ratio of the velocities is

\[
\frac{v_{be}(k)}{v_{be}^u(k)} = \frac{a_{be}}{a_{be}^u} e^{\alpha_{be} - \alpha_{be}^u|k} \simeq c_{be} + d_{be}k,
\]

where \(c_{be}\) and \(d_{be}\) are assumed to be \(k\)-independent. In other words, we expect a linear increase of the ratio of the velocities with \(k\) (the correction here is linear instead of logarithmic), as is indeed ascertained in Fig. 11. The coefficients in Eq. (34) do depend on the tapering parameter \(q\), as clearly seen in the figure. Note that the slope with granule number in the figure is mild, indicating that \(d_{be}\) is small for all \(q\).

**Fig. 11** Ratio of the modified BCA velocity over the velocity from the numerical integration in backward exponentially tapered chains for \(q = 0.01\) to \(0.08\) from bottom to top, in steps of \(0.01\).

For this approach to be predictive, it is necessary to parametrize the \(q\) dependence of the coefficients. We find that the dependence of \(c_{be}(q)\) and \(d_{be}(q)\) on \(q\) is well described by the power laws

\[
c_{be}(q) = c_{be}^0 q^{x_{be}} \quad \text{and} \quad d_{be}(q) = d_{be}^0 q^{y_{be}}.
\]

We can obtain \(c_{be}^0\) and \(d_{be}^0\) by a linear fit of

\[
\log c_{be}(q) = \log c_{be}^0 + x_{be} \log q.
\]
and similarly

$$\log d_{be}(q) = \log d_{be}^0 + y_{be} \log q.$$  \hfill (37)

We find the values

$$d_{be}^0 = 1.64, \quad x_{be} = 0.0259, \quad d_{be}^0 = 0.211, \quad y_{be} = 1.83.$$  \hfill (38)

In terms of these parameters the modified BCA can be written as

$$v_{be} = \frac{v_{be}^u}{1.64 q^{0.0259} + 0.211 q^{1.83} k^i}.$$  \hfill (39)

The next figure shows that there is now excellent agreement between the results of the modified BCA and those obtained by integrating the equations of motion. Specifically, we direct attention to Figs. (12) (velocity), (3) (momentum) and (4) (kinetic energy).

![Figure 12](image)

**Fig. 12** Comparison between the velocities predicted by the modified BCA (+ symbols), Eq. (39), and numerical data (circles) in backward exponentially tapered chains for $q = 0.01$ to 0.08 from top to bottom, in steps of 0.01.

5.3 Forward tapered chains

5.3.1 Linearly forward tapered chains

In this case, for $k \ll 1 + 1/S$ the BCA for the amplitude of the velocity is given in Eq. (22). Consequently, assuming that the simulation data can be described by $v_f^u(k) \sim (a_f^u + b_f^u k)^{-3/2}$ with constant $a_f^u$ and $b_f^u$, the leading term of the ratio $v_f(k)/v_f^u(k)$ is a constant, which we estimate as 0.680 (figure
not shown). Using this factor to modify the BCA result for the amplitude of the pulse velocity, we observe that the agreement with the numerical results improves greatly for the velocity (Fig. 13), the momentum (Fig. 5), and the energy (Fig. 6) except for the larger values of $S$, where slight deviations are noticeable.

![Graph](image)

**Fig. 13** Comparison between the velocities predicted by the modified BCA (+ symbols) and numerical data (circles) in forward linearly tapered chains for $S = 0.001$ to $0.009$ from bottom to top, in steps of 0.002.

Melo and collaborators [13] have presented interesting experimental results for forward linearly tapered chains (the authors claim that they use exponentially tapered chains, but the radii provided in their paper are closer to those of linearly tapered chains). They observed a linear decay of the momentum compatible with our findings for linear chains (see Fig. 5), showing that our theory captures the experimental observations.

### 5.3.2 Exponentially tapered chains

For the forward exponentially tapered chain we have rescaled the velocity with the same kind of function as in the backward case, cf. Eq. (39). As noted earlier [29], the forward tapering imposes a limitation on the length of the chain. Therefore, we limited our study to tapering parameters $q$ no larger than 0.06. We now find the values

$$c_{fe}^0 = 1.32, \quad x_{fe} = 0.0252, \quad d_{fe}^0 = 11.1, \quad y_{fe} = 2.57.$$  \hspace{1em} (40)

The scaled results are again in excellent agreement with the results of the numerical integration of the equations of motion for the velocity (Fig. 14).
The momentum and kinetic energy as functions of $k$ are shown in Fig. 7 and Fig. 8, respectively. The agreement is again seen to be excellent.

At this point, it is worth noticing that this kind of tapering has been considered in [7,10,12] using a hard sphere approximation. However, the exceptional agreement between the numerical simulations and the theory achieved with our formulation was not obtained within that approximation.

### 6 Brief Summary

Recognizing that our analytic binary collision approximation, while highly accurate for the prediction of a number of quantities that measure pulse propagation along tapered granular chains, is not as accurate for others, we set out to improve upon the approximation. In particular, we noted that the propagation of momentum and of energy, two quantities that are of particular interest in the design of shock absorbers, are among the latter. Since tapered chains are often analyzed with this particular application in mind, it is important to achieve analytic predictability for these physical quantities.

The binary collision approximation leads to analytic results and this is its strength, since it overcomes the resource limitations posed by numerical and experimental searches of optimal parameters. It was therefore our hope to provide an analytic modification to the approximation that could then be used for any tapering parameters. We succeeded to some extent. While we found it necessary to determine some of the coefficients in our corrective formulas numerically, these quantities do not depend on the tapering parameters and can therefore be adopted once and for all for predictive and optimization purposes.
This claim must of course be understood within the restrictions of the model: it is valid for Hertz potentials and for chains without precompression (although the extension to mildly precompressed chains is straightforward [32]). The theory is valid for small values of the tapering parameters $S$ and $q$, which is not a serious restriction since large values of these parameters would be experimentally difficult to implement. Our corrected formulas work extremely well in almost all cases, the weakest scenario being the kinetic energy in forward linearly tapered chains. However, we note that most of the literature that we are aware of, with the exception of the experiments of Melo et al. [13], focuses on exponentially tapered chains. In all of these our new results work exceedingly well.

We collect our results for convenience. We presented four kinds of tapered chains. The resulting corrected pulse velocity formulas are as follows.

- Backward linearly tapered chains
  \[ v_{bl}(k) = \frac{2}{3}k^{-3/2} \]  
  (41)

- Backward exponentially tapered chains with tapering parameter $q$,  
  \[ v_{be}(k) = \left\{ \left( \frac{1}{2} \right) \left[ 1 + (1 - q)^{-3} \right] \right\}^{1-k} \]  
  \[ \frac{1.64q^{0.0259} + 0.211q^{1.83}}{1.32q^{0.0252} + 1.1q^{2.57}} \] 
  (42)

- Forward linearly tapered chains
  \[ v_{fl}(k) = 0.680 \left( 1 - \frac{k}{1 + 1/S} \right)^{3/2} \] 
  (43)

- Forward exponentially tapered chains with tapering parameter $q$,  
  \[ v_{fe}(k) = \left\{ \left( \frac{1}{2} \right) \left[ 1 + (1 + q)^{-3} \right] \right\}^{1-k} \]  
  \[ \frac{1.64q^{0.0259} + 0.211q^{1.83}}{1.32q^{0.0252} + 1.1q^{2.57}} \] 
  (44)

We end by noting that the momentum transferred by the pulse is attenuated in forward tapered chains, making them good candidates for model studies of shock absorption. Moreover, the exponentially forward tapered chains are stronger contenders since the momentum decrease is exponential rather than linear (as in the linearly forward tapered chains). Conversely, backward tapered chains act as momentum focusers such as might be desirable in sensing tools, once again the exponential tapered chains being the stronger contenders. We stress that these are of course model systems that serve only to clarify how much more complex “real” shock absorbers or momentum focusing devices might be better understood. While one can imagine many other applications, at this point we make no claims other than the utility of these systems as ones that help in the understanding of the fundamental issues involved in energy propagation in granular matter.
Acknowledgments

Acknowledgment is made to the Donors of the American Chemical Society Petroleum Research Fund for partial support of this research (K.L.). A.R. acknowledges support from Bionanotec-CAPES and CNPq. L. P. M. acknowledges support by CAPES. The authors acknowledge helpful discussions with A. H. Romero and U. Harbola.

References

1. Nesterenko V F 1983 J Appl. Mech. Tech. Phys. 24 733
2. Nesterenko V F 1994 J. Phys. IV 4 C8-729
3. Nesterenko V F 2001 Dynamics of Heterogeneous Materials (Springer, New York)
4. Nesterenko V F 1985 J Appl. Mech. Tech. Phys. 26 405
5. Coste C, Falcon E and Fauve S 1997 Phys. Rev. E 56 6104
6. Hinch E J and Saint-Jean S 1999 Proc. R. Soc. London, Ser. A 455 3201
7. Sen S, Manciu F S and Manciu M 2001 Physica A 299 551
8. Wu D T 2002 Physica A 315 194
9. Nakagawa M, Agui J H, Wu D T and Extramiana 2003 Granular Matter 4 167
10. Doney R L and Sen S 2005 Phys. Rev. E 72 041304
11. Sokolow A, Pflanzer J M M M, Doney R L, Nakagawa M, Agui J H and Sen S 2005
    Appl. Phys. Lett. 87 254104
12. Doney R and Sen S 2006 Phys. Rev. Lett. 97 155502
13. Melo F, Job S, Santibanez F and Tapia F 2006 Phys. Rev. E 73 041305
14. Doney R L, Agui J H and Sen S 2009 J. Appl. Phys. 106 064905
15. Daraj C, Nesterenko V F, Sokolow A, Herbold E B and Jin S 2006 Phys. Rev. Lett. 96 058002
16. Job S, Melo F, Sokolow A and Sen S 2005 Phys. Rev. Lett. 94 178002
17. Hascoët E and Herrmann H J 2000 Eur. Phys. J. B 14 183
18. Hascoët E, Herrmann H J and Loreto V 1999 Phys. Rev, E 59 3202
19. Hong J 2005 Phys. Rev. Lett. 94 108901
20. Wang F J, Xia J H, Li Y D and Liu C S 2007 Phys. Rev. E 76 041305
21. Sokolow A, Bittle E G and Sen S 2007 EPL 77 24002
22. Job S, Melo F, Sokolow A, and Sen S 2007 Granular Matter 10 13
23. Sen S et al., 2003 Modern Challenges in Statistical Mechanics: Patterns, Noise and the
    Interplay of Nonlinearity and Complexity edited by Kenkre V M and Lindenberg K,
    AIP Conf. Proc. No.658 (AIP, Melville, NY, 2003)
24. Sen S, Hong J, Bang J, Avales E and Donery R 2008 Phys. Rep 462 21
25. Rosas A and Lindenberg K 2003 Phys. Rev. E 68 041304
26. Rosas A, Romero A H, Nesterenko V F and Lindenberg K 2007 Phys. Rev. Lett. 98
    164301
27. Rosas A, Romero A H, Nesterenko V F and Lindenberg K 2008 Phys. Rev. E 78 051303
28. Rosas A and Lindenberg K 2004 Phys. Rev. E 69 037601
29. Harbola U, Rosas A, Esposito M and Lindenberg K 2009 Phys. Rev. E 80 031303
30. Harbola U, Rosas A, Romero A U, Esposito M and Lindenberg K 2009 Phys. Rev. E 80
    051302
31. Harbola U, Rosas A, Romero A U, Esposito M and Lindenberg K 2010 Phys. Rev. E
    82 011306
32. Pinto I L D, Rosas A, Romero A U and Lindenberg K 2010 Phys. Rev. E 82 031308
33. The shape of the granules does not really matter as they are treated as point masses
    with an interaction determined by their radius of curvature and their mass. We talk about
    spherical granules for convenience, as is done in a great deal of the relevant literature.