A retraction theorem for topological fundamental groups with application to the Hawaiian earring

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Abstract

A characterization of regular topological fundamental groups yields a ‘no retraction theorem’ for spaces constructed in similar fashion to the Hawaiian earring.

1 Introduction

Analysis of algebraic properties of fundamental groups \( \pi_1(X) \) and \( \pi_1(Y) \) often leads to the conclusion that \( X \) cannot be embedded as a retract of \( Y \). This paper illustrates the possibility to reach the same conclusion by analyzing topological properties of these groups.

Theorem 2 offers a characterization of those spaces \( Y \), whose fundamental group \( \pi_1(Y) \), topologized in a natural way, is a \( T_1 \) space: The space \( \pi_1(Y) \) is \( T_1 \) if and only if each retraction of \( Y \) induces an embedding between topological fundamental groups.

Recent work of the author [2] shows that the topological fundamental group of a space \( X \) constructed in similar fashion to the Hawaiian earring is not a Baire space.

Consequently (Theorem 5), \( X \) cannot be embedded as a retract of a space \( Y \) whose topological fundamental group is completely metrizable. For example \( X \) cannot be embedded as a retract of the countable product of locally simply connected spaces.
2 Definitions and preliminaries

Suppose $X$ is a metrizable space and $p \in X$. Let $C_p(X) = \{f : [0, 1] \to X \mid f \text{ is continuous and } f(0) = f(1) = p\}$. Endow $C_p(X)$ with the topology of uniform convergence.

The topological fundamental group $\pi_1(X, p)$ is the set of path components of $C_p(X)$ endowed with the quotient topology under the canonical surjection $q : C_p(X) \to \pi_1(X, p)$ satisfying $q(f) = q(g)$ if and only if $f$ and $g$ belong to the same path component of $C_p(X)$.

Thus a set $U \subset \pi_1(X, p)$ is open in $\pi_1(X, p)$ if and only if $q^{-1}(U)$ is open in $\pi_1(X, p)$.

Remark 1 The space $\pi_1(X, p)$ is a topological group under concatenation of paths. (Proposition 3.1[1]). A map $f : X \to Y$ determines a continuous homomorphism $f^* : \pi_1(X, p) \to \pi_1(Y, f(p))$ via $f^*([\alpha]) = [f(\alpha)]$ (Proposition 3.3[1]).

Let $[p] \in \pi_1(X, p)$ denote the trivial element. Thus $[p]$ is the path component of the constant path in $C_p(X)$.

If $X \subset Y$ then $X$ is a retract of $Y$ if there exists a map $f : Y \to X$ such that $f_X = id_X$. The space $X$ is $T_1$ if each one point subset of $X$ is closed. The $T_1$ space $X$ is completely regular if for each closed set $A$ and each point $x^* \notin A$ there exists a map $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(x^*) = 1$. The topological space $X$ is completely metrizable if $X$ admits a complete metric compatible with its topology.

A metric space $(X, d)$ is similar to the Hawaiian earring provided all of the following hold. Suppose $p \in X$ and $X = \cup_{n=1}^{\infty} Y_n$ with $Y_n \cap Y_m = \{p\}$ whenever $n \neq m$. Assume the space $Y_n$ is path connected, locally simply connected at $p$, and $Y_n$ is not simply connected. Assume also that $\lim_{n \to \infty} diam(Y_n) = 0$. For example if $Y_n$ is a simple closed curve then $X$ is the familiar Hawaiian earring.

3 A characterization of $T_1$ topological fundamental groups

The metrizable spaces with $T_1$ topological fundamental groups are precisely the spaces whose retracts induce embeddings between fundamental groups.
**Theorem 2** Suppose $Y$ is a metrizable space. The following are equivalent.

1. The trivial element of $\pi_1(Y, p)$ is closed in $\pi_1(Y, p)$.
2. $\pi_1(Y, p)$ is a $T_1$ space.
3. $\pi_1(Y, p)$ is completely regular.
4. Whenever $X$ is a retract of $Y$ and $j^* : \pi_1(X, p) \to \pi_1(Y, p)$ is the monomorphism induced by inclusion, then $j^*$ is an embedding onto a closed subgroup of $\pi_1(Y, p)$.

**Proof.** The equivalence of 1 2 and 3 follows from elementary facts about topological groups (ex. 6 p. 145, ex. 5 p237[3]).

1 $\Rightarrow$ 4. Suppose $r : Y \to X$ is a retraction. Let $q : C_p(X) \to \pi_1(X, p)$ and $Q : C_p(Y) \to \pi_1(Y, p)$ denote the canonical quotient maps. Suppose $A \subset \pi_1(X, p)$ is closed. Let $B = \phi(A) \subset \pi_1(Y, p)$. To prove $B$ is closed it suffices to prove $Q^{-1}(B)$ is closed in $C_p(Y)$. Suppose $g \in Q^{-1}(B)$. Let $g = \lim g_n$ with $Q(g_n) \in B$. Note $r(g_n) \to r(g)$. Since $Q(g_n) \in B$ there exists $f_n \in C_p(X)$ path homotopic in $Y$ to $g_n$ such that $f_n \in q^{-1}(A)$. Thus $r(g_n)$ and $r(f_n) = f_n$ are path homotopic in $X$. Thus $r(g_n) \in q^{-1}(A)$. Since $X$ is closed in $Y$, $C_p(X)$ is closed in $C_p(Y)$. Since $q^{-1}(A)$ is closed in $C_p(X)$, and since $C_p(X)$ is closed in $C_p(Y)$ it must be that $r(g) \in q^{-1}(A)$. Note $g_n * r(g_n)$ is homotopically trivial and converges to $g * r(g)$. Since $[p]$ is closed in $\pi_1(Y, p)$ it follows that the path component of the constant map is a closed subspace of $C_p(Y)$. Thus $g * r(g)$ must be homotopically trivial in $Y$. Hence $g$ and $r(g)$ are path homotopic in $Y$. Thus $g \in Q^{-1}(B)$. Hence $B$ is closed. Therefore $\phi$ is a closed map and hence an embedding.

4 $\Rightarrow$ 1. Note the one point space $X = \{p\}$ is a retract of $Y$. Thus $j^*(\pi_1(X, p))$ is a closed subspace of $\pi_1(Y, p)$.

**Remark 3** Theorem[2] does not apply to all spaces $Y$. The harmonic archipalego, explored in detail in [3], provides an example of a compact path connected metric space $Y$ such that $\pi_1(Y, p)$ is not a $T_1$ space.

4 Application: A “no retraction” theorem for the Hawaiian earring

**Corollary 4** Suppose each of $Z$ and $Y$ are metrizable. Suppose $\pi_1(Z, p)$ is not completely metrizable, and suppose $\pi_1(Y, p)$ is completely metrizable.
Then $Z$ cannot be embedded as a retract of $Y$.

**Proof.** Since $\pi_1(Y, p)$ is metrizable, the one point subsets of $\pi_1(Y, p)$ are closed subspaces. In particular if $X$ is a retract of $Y$ then by Theorem 2 $\pi_1(X, p)$ is homeomorphic to a closed subspace of $\pi_1(Y, p)$. Thus $\pi_1(X, p)$ completely metrizable. Hence $Z$ cannot be a retract of $Y$. ■

**Theorem 5** Suppose $X$ is similar to the Hawaiian earring, suppose $Y$ is metrizable and suppose $\pi_1(Y, p)$ is completely metrizable. Then $X$ cannot be embedded as a retract of $Y$.

**Proof.** The main result of [2] is that $\pi_1(X, p)$ is not a Baire space. Hence $\pi_1(X, p)$ is not completely metrizable. ■

**Example 6** Suppose $Y = Z_1 \times Z_2 \times \ldots$ where each $Z_n$ has the homotopy type of a bouquet of $n$ loops. Then $Y$ shares some properties with the Hawaiian earring. For example $Y$ is not locally contractible, for each $Z_n$ is a retract of $Y$, and $\pi_1(Y)$ is uncountable. However, (Proposition 5.2 [4]) $\pi_1(Y)$ is canonically isomorphic and homeomorphic to the product $\pi_1(Z_1) \times \pi_1(Z_2)\ldots$ and hence completely metrizable (since each factor has the discrete topology.) Thus $Y$ has no retract similar to the Hawaiian earring.

**References**

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[4] Munkres, James R., Topology: a first course. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.