Measures of noncompactness in the space of regulated functions on an unbounded interval

Szymon Dudek¹ · Leszek Olszowy¹

Received: 13 May 2022 / Accepted: 21 July 2022 © The Author(s) 2022

Abstract
In this paper, we formulate a criterion for relative compactness in the space of regulated functions on an unbounded interval and not necessarily bounded. Next we construct measure of noncompactness in this space and investigate its properties. The presented measure is simpler and more convenient to use than all known so far in space of regulated functions on an unbounded interval. Moreover, we show the applicability of the measure of noncompactness in proving the existence of solutions of some Volterra type integral equation.

Keywords Space of regulated functions · Criterion of relative compactness · Fréchet space · Measure of noncompactness

Mathematics Subject Classification 47H30 · 46E400

1 Introduction

Fixed point theorems are one of the main tools used to study of solvability various kinds of equations. There are many fixed point theorems which use measures of noncompactness. In this approach, very important is to choice the suitable space and define the effective and convenient measure. A detailed discussion of the application of measures of noncompactness in various spaces in the study of the solvability of various types of integral equations can be found in [11]. One of these spaces is the space of regulated functions (consisting of functions having

¹ Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, Powstańców Warszawy 8, 35-959 Rzeszow, Poland
one-side limits at every point). It obviously contains the space of continuous functions, as well as the space of functions of bounded variation. This space is naturally used in theory of measure differential equations (MDEs, for short), also known as differential equations driven by measures, which arise in many areas of applied mathematics such as nonsmooth mechanics, game theory. In the last years, there have investigated several variants of this space introducing in them the structure of Banach space (see [2, 4–7, 10, 12]).

However, when considering regulated functions on an unbounded interval $\mathbb{R}_+$, it is better to locate the considerations in the Fréchet space $G(\mathbb{R}_+, E)$ than in the Banach space $BG(\mathbb{R}_+, E)$. In the case of the space $BG(\mathbb{R}_+, E)$ all known and convenient to use measures of noncompactness have the disadvantage that they do not capture all relatively compact sets, that is, the family of those sets on which these measures are zero is significantly smaller than the family of all relatively compact sets. It is quite different in the case of the space $G(\mathbb{R}_+, E)$. As we will show later in this paper, the Fréchet space structure in $G(\mathbb{R}_+, E)$ allows for the formulation of an elegant compactness criterion. The measure of noncompactness constructed by this criterion does not have this disadvantage as before, i.e. the family of sets on which this measure is zero is exactly equal to the family of all relatively compact sets in $G(\mathbb{R}_+, E)$. An even more important feature of the measure in $G(\mathbb{R}_+, E)$ constructed in this work is that it is simpler, because it is only two-component, while the aforementioned measures of noncompactness in $BG(\mathbb{R}_+, E)$ have three components and when using such measures to investigate the solvability various kinds of equations require stronger assumptions, enforced by the presence of these three components.

The paper is organized as follow. The second section is devoted to recalling some notions, facts and theorems. In the third section, we introduce in the space $G(\mathbb{R}_+, E)$ the structure of Fréchet space (using the family of pseudonorm). Moreover, we provide the compactness criterion and using it we define the new measure of noncompactness which is more convenient than usually used in the space $BG(\mathbb{R}_+, E)$. Additionally, we study properties of this measure of noncompactness and we provide the example showing its applicability.

## 2 Notation, definitions and auxiliary facts

In this section, we recall some facts which are needed further on.

Assume that $E$ is a real Banach space with the norm $\| \cdot \|$ and the zero element $\theta$. Denote by $B_E(x, r)$ the closed ball centered at $x$ and with radius $r$. The symbol $B_E(r)$ stands for the ball $B_E(\theta, r)$. For $X$ being a nonempty subset of $E$ we write $\overline{X}, \text{conv}X, \text{Conv}X$ to denote the closure, convex hull and the convex closure of a set $X$, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of $E$ and by $\mathfrak{K}_E$ its subfamily consisting of all relatively compact sets.
We accept the following definition of the notion of a measure of noncompactness [1].

**Definition 2.1** A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+ := [0, \infty)$ is said to be a measure of noncompactness in a Banach space $E$ if it satisfies the following conditions:

1° The family $\ker \mu := \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{M}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3° $\mu(\text{Conv}X) = \mu(X)$.

4° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

5° If $\{X_n\}$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the intersection $X_\infty := \cap_{n=1}^{\infty} X_n$ is nonempty.

In the sequel, we will use measures of noncompactness having some additional properties. Namely, a measure $\mu$ is said to be sublinear if it satisfies the following two conditions:

6° $\mu(\lambda X) = |\lambda| \mu(X), \; X \subset \mathcal{M}_E, \; \lambda \in \mathbb{R}$.

7° $\mu(X + Y) \leq \mu(X) + \mu(Y), \; X, Y \subset \mathcal{M}_E$.

We consider also weak maximum property

8° $\mu(X \cup \{y\}) = \mu(X), \; X \subset \mathcal{M}_E, \; y \in E$.

A sublinear measure of noncompactness $\mu$ satisfying the condition (maximum property), i.e.

9° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}, \; X, Y \subset \mathcal{M}_E$

and such that $\ker \mu = \mathcal{M}_E$ is said to be regular.

For a given $X \subset \mathcal{M}_E$, we denote by $\beta(X)$ the so-called Hausdorff measure of noncompactness of $X$. This quantity is defined by the formula

$$\beta(X) := \{r > 0 : X \text{ has a finite } r-\text{net in } E\}$$

and it is an example of the regular measure of noncompactness in $E$.

Let $J$ denote the interval, bounded $J = [0, T]$ or unbounded $J = \mathbb{R}_+ := [0, \infty)$. We use the following notation which will be needed further on: if $J = [0, T]$, then we put $J^+ := [0, T)$ and $J^- := (0, T]$, but if we take $J = \mathbb{R}_+$, then $J^+ := [0, \infty)$ and $J^- := (0, \infty)$.

Now we recall some facts concerning regulated functions.
Definition 2.2 A function \( x : J \to E \), where \( E \) is a Banach space, is said to be a regulated function if for every \( t \in J^+ \) the right-sided limit \( x(t^+) := \lim_{s \to t^+} x(s) \) exists and for every \( t \in J^- \) the left-sided limit \( x(t^-) := \lim_{s \to t^-} x(s) \) exists. Denote by \( G(J, E) \) the space consisting of all regulated functions defined on the interval \( J \) with values in a Banach space \( E \). Obviously, \( G(J, E) \) is a linear space.

In the case of \( J = [0, T] \), every function \( x \in G([0, T], E) \) is bounded. Hence, \( G([0, T], E) \) can be equipped with the classical supremum norm \( \|x\|_T := \sup\{ \|x(t)\| : t \in J \} \). It is easy to show that \( G([0, T], E) \) is a Banach space.

Compactness criteria and measures of noncompactness in the space \( G([0, T], E) \) were investigated in several research papers (see [2, 4, 5, 10]). We recall below the concept of a equiregulated subset of the space \( G([0, T], E) \).

Definition 2.3 We will say that the set \( X \subset G([0, T], E) \) is equiregulated on the interval \( J = [0, T] \) if the following two conditions are satisfied:

\[
\forall \epsilon > 0 \exists \delta > 0 \forall x \in X \forall t, t_0 \in (t-\delta, t] \cap [0, T] \|x(t_0) - x(t)\| \leq \epsilon,
\]

\[
\forall t \in [0, T] \forall \epsilon > 0 \exists \delta > 0 \forall x \in X \forall t_1, t_2 \in (t, t+\delta) \cap [0, T] \|x(t_2) - x(t_1)\| \leq \epsilon.
\]

Now we are going to recall the compactness criterion in \( G([0, T], E) \) formulated by Fraňková [10], (see also [2, 4, 5]).

Theorem 2.4 A nonempty subset \( X \subset G([0, T], E) \) is relatively compact in \( G([0, T], E) \) if and only if \( X \) is equiregulated on the interval \( [0, T] \) and the sets \( X(t) \) are relatively compact in \( E \) for \( t \in [0, T] \).

Let us recall the construction of regular measure of noncompactness in the space \( G([0, T], E) \) (see [2, 4, 5, 9, 15]). Therefore, let us take a set \( X \in \mathcal{M}_{G([0, T], E)} \). For \( x \in X \) and \( \epsilon > 0 \) let us denote the following quantities:

\[
\omega_T^-(x, t, \epsilon) := \sup\{ \|x(t_2) - x(t_1)\| : t_1, t_2 \in (t - \epsilon, t) \cap [0, T] \}, \quad t \in (0, T],
\]

\[
\omega_T^+(x, t, \epsilon) := \sup\{ \|x(t_2) - x(t_1)\| : t_1, t_2 \in (t, t + \epsilon) \cap [0, T] \}, \quad t \in [0, T).
\]

The above quantities can be interpreted as left-hand and right-hand sided moduli of convergence of the function \( x \) at the point \( t \). Further, let us put:
$\omega^-(X, t, \epsilon) := \sup \{\omega^-(x, t, \epsilon) : x \in X\}, \quad t \in (0, T],$

$\omega^+(X, t, \epsilon) := \sup \{\omega^+(x, t, \epsilon) : x \in X\}, \quad t \in [0, T],$ 

$\omega^-(X, t) := \lim_{\epsilon \to 0^+} \omega^-(X, t, \epsilon), \quad t \in (0, T],$

$\omega^+(X, t) := \lim_{\epsilon \to 0^+} \omega^+(X, t, \epsilon), \quad t \in [0, T),$

$\omega^-(X) := \sup_{t \in (0, T]} \omega^-(X, t),$ 

$\omega^+(X) := \sup_{t \in (0, T]} \omega^+(X, t),$ 

$\overline{\beta}(X) := \sup_{t \in [0, T]} \beta(X(t)).$

It is clear that $X \subset G([0, T], E)$ is equiregulated if and only if $\max\{\omega^-(X), \omega^+(X)\} = 0.$

Finally, let us define quantity

$$\mu_T(X) := \max \{\omega^-(X), \omega^+(X)\} + \overline{\beta}(X). \quad (2.1)$$

**Theorem 2.5** [15] The function $\mu_T$ given by formula (2.1) satisfies conditions $1^\circ - 8^\circ$ in the space $G([0, T], E)$ and moreover $\ker \mu = \mathfrak{R}_{G([0,T],E)}.$

### 3 Measures of noncompactness in $G(\mathbb{R}_+, E)$

Now we define the space mentioned in the title of this paper. Denote by $G(\mathbb{R}_+, E)$ the space consisting of all regulated functions defined on the interval $\mathbb{R}_+$ with values in a Banach space $E.$ There is no natural norm in this space, which would provide the structure of Banach space. Therefore, we introduce the notion $\|x\|_T := \sup \{\|x(t)\| : t \in [0, T]\}$ for $T > 0$ and we define in $G(\mathbb{R}_+, E)$ the sequence of pseudonorms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}.$ The space $G(\mathbb{R}_+, E)$ becomes a Fréchet space furnished with the distance

$$d(x, y) := \sum_{n=1}^\infty 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}, \quad x, y \in G(\mathbb{R}_+, E).$$

The above defined metric implies the following property:

(A) a sequence $\{x_k\} \subset G(\mathbb{R}_+, E)$ is convergent to $x \in G(\mathbb{R}_+, E)$ if and only if $\{x_k\}$ is uniformly convergent to $x$ on bounded subsets of $\mathbb{R}_+,$ i.e. $\lim_{k \to \infty} \|x_k - x\|_n \to 0$ for $n = 1, 2, ...$

We will say that the set $X \subset G(\mathbb{R}_+, E)$ is **bounded** if $\|X\|_n < \infty$ for $n = 1, 2, ...$

Obviously every relative compact set in $G(\mathbb{R}_+, E)$ is also bounded.
Definition 3.1 We will say that the set \( X \subset G(\mathbb{R}_+, E) \) is \textit{equiregulated} in \( G(\mathbb{R}_+, E) \) if for each \( T > 0 \) the set \( X|_{[0, T]} \) is equiregulated in \( G([0, T], E) \).

Now, we can formulate the following compactness criterion for considered space.

Theorem 3.2 Let \( E \) be a real Banach space. Then a nonempty subset \( X \subset G(\mathbb{R}_+, E) \) is relatively compact iff the following two conditions are satisfied:

(a) The set \( X(t) \) is relatively compact in \( E \) for each \( t \in \mathbb{R}_+ \).
(b) For each \( T > 0 \) the set \( X|_{[0, T]} \) is equiregulated.

Proof \( (\Rightarrow) \) In view of relative compactness of the set \( X \) in the space \( G(\mathbb{R}_+, E) \) and condition (A), we infer that the set \( X|_{[0, T]} \) is also relatively compact in the space \( G([0, T], E) \) for arbitrary \( T > 0 \). Hence, keeping in mind Theorem 2.3, we get the conditions \( (a) \) and \( (b) \).

\( (\Leftarrow) \) In what follows we fix \( X \subset G(\mathbb{R}_+, E) \) which satisfies the conditions \( (a) \) and \( (b) \), and let us consider an arbitrary sequence \( \{x_n\} \subset X \). Conditions \( (a) \) and \( (b) \) connected with Theorem 2.4 imply that the sequence \( \{x_n\} \) has a subsequence \( \{x_{1,n}\} \) converging on \([0, 1]\) with respect to the pseudonorm \( \| \cdot \|_1 \) to some regulated function defined on \([0, 1]\). Similarly, there exists a subsequence \( \{x_{2,n}\} \) of the sequence \( \{x_{1,n}\} \) converging on \([0, 2]\) with respect to the pseudonorm \( \| \cdot \|_2 \) to some regulated function defined on \([0, 2]\). Therefore, we obtain the sequence of subsequences \( \{x_{i,n}\} \), \( i = 1, 2, ... \) converging on the interval \([0, i]\) with respect to the pseudonorm \( \| \cdot \|_i \) to some regulated function defined on \( \mathbb{R}_+ \). Next, putting \( z_n := x_{n,n}, \ n = 1, 2, ... \) we get the sequence \( \{z_n\} \), uniformly convergent on bounded intervals to some regulated function \( z : \mathbb{R}_+ \to E \). In virtue of the condition (A) the sequence \( \{z_n\} \) is convergent in the space \( G(\mathbb{R}_+, E) \) and it implies that \( X \) is relatively compact in \( G(\mathbb{R}_+, E) \).

Using the above proved compactness criterion, we are going to introduce the measure of noncompactness in the space \( G(\mathbb{R}_+, E) \). However, in contrast to the Banach space \( E \) where the measure of noncompactness is single function \( \mu : \mathcal{M}_E \to \mathbb{R}_+ \) (satisfying Definition 2.1), in the case of Fréchet space \( F \), it is more convenient to introduce the sequence of functions \( \{\mu_n\} \) playing the role of measure of noncompactness (often called a family of measures of noncompactness). Let us provide the appropriate notions. The family of all nonempty and bounded subsets of \( F \) will be denoted by \( \mathcal{M}_F \) while its subfamily consisting of all relatively compact sets is denoted by \( \mathcal{K}_F \).

Now we can present the definition of the family of measures of noncompactness (see [14]).

Definition 3.3 A family of functions \( \{\mu_n\}_{n \in \mathbb{N}} \) where \( \mu_n : \mathcal{M}_F \to [0, \infty) \), is said to be a \textit{measure of noncompactness} in real Fréchet space \( F \) if it satisfies the following conditions:
The measure of noncompactness in the space of regulated functions...  

\[(P_1)\] The family \(\ker\{\mu_n\} := \{X \in \mathcal{M}_F : \mu_n(X) = 0 \text{ for } n \in \mathbb{N}\}\) is nonempty and \(\ker\{\mu_n\} \subseteq \mathcal{M}_F.\)

\[(P_2)\] \(X \subseteq Y \Rightarrow \mu_n(X) \leq \mu_n(Y)\) for \(n \in \mathbb{N}.\)

\[(P_3)\] \(\mu_n(\text{Conv}X) = \mu_n(X)\) for \(n \in \mathbb{N}.\)

\[(P_4)\] \(\mu_n(\lambda X + (1 - \lambda)Y) \leq \lambda \mu_n(X) + (1 - \lambda)\mu_n(Y)\) for \(\lambda \in [0, 1],\) \(n \in \mathbb{N}.\)

\[(P_5)\] If \(\{X_i\}\) is a sequence of closed sets from \(\mathcal{M}_F\) such that \(X_{i+1} \subseteq X_i (i = 1, 2, \ldots)\) and if \(\lim_{i \to \infty} \mu_n(X_i) = 0\) for each \(n \in \mathbb{N},\) then the intersection set \(X_\infty := \bigcap_{i=1}^\infty X_i\) is nonempty.

For any fixed \(n \in \mathbb{N},\) we introduce the mapping \(\mu_n : \mathcal{M}_{G(R^+, E)} \to \mathbb{R}_+\) defined by the formula

\[
\mu_n(X) := \max\{\omega_n^-(X), \omega_n^+(X)\} + \beta_n(X), \quad X \in \mathcal{M}_{G(R^+, E)}. \tag{3.1}
\]

The main properties of the family of function \(\{\mu_n\}\) are contained in the below given theorem.

**Theorem 3.4** The family \(\{\mu_n\}_{n \in \mathbb{N}}\) of functions \(\mu_n : \mathcal{M}_{G(R^+, E)} \to \mathbb{R}_+\) \(n \in \mathbb{N}\) given by the formula (3.1) is a measure of noncompactness in the Fréchet space \(G(R^+, E).\) Moreover, \(\ker\{\mu_n\} = \mathcal{N}_{G(R^+, E)}\) and it satisfies the following conditions

\[(P_6)\] \(\mu_n(\lambda X) = |\lambda| \mu_n(X)\) for \(\lambda \in \mathbb{R},\) \(n \in \mathbb{N},\)

\[(P_7)\] \(\mu_n(X + Y) \leq \mu_n(X) + \mu_n(Y)\) for \(X, Y \in \mathcal{M}_E,\) \(n \in \mathbb{N},\)

\[(P_8)\] \(\mu_n(X \cup \{y\}) = \mu_n(X)\) for \(X \in \mathcal{M}_E,\) \(y \in E, n \in \mathbb{N}.\)

**Proof** If \(X \subseteq G(R^+, E)\) is nonempty, then in virtue of Theorem 3.2, we obtain that \(X\) is relatively compact if and only if \(\mu_n(X) = 0\) for \(n \in \mathbb{N}.\) Hence, \(\ker\{\mu_n\} = \mathcal{N}_{G(R^+, E)}\) and \((P_1)\) is proved.

Let us note that properties \((P_2)-(P_4)\) and \((P_6)-(P_8)\) follow immediately from theorem 2.5. Now we show that \((P_3)\) is satisfied. Let the sequence \(\{X_i\}\) is as in the condition \((P_5).\) Choose arbitrarily \(x_i \in X_i\) and let us define the sets \(Y_i := \{x_i : k \geq i\}, i = 1, 2, \ldots\) Obviously \(Y_{i+1} = \{x_{i+1}\} \cup Y_i\) and then, in virtue of \((P_8)\) we derive \(\mu_n(Y_{i+1}) = \mu_n(Y_i)\) for \(n, i \in \mathbb{N}.\) Since \(\lim_{i \to \infty} \mu_n(X_i) = 0,\) then \(\mu_n(Y_i) = 0\) for \(n, i \in \mathbb{N}.\) Hence, we obtain \(Y_i \in \ker\{\mu_n\} = \mathcal{N}_{G(R^+, E)}.\) This fact and closedness of \(Y_i\) means that \(Y_i\) is compact and nonempty for \(i \in \mathbb{N}.\) Since sequence \(\{Y_i\}\) is decreasing, we get \(\emptyset \neq \bigcap_{i=1}^\infty Y_i \subseteq \bigcap_{i=1}^\infty X_i.\)

It can be shown (using simple examples) that the sequence \(\{\mu_n\}\) does not satisfy the strong maximum property, i.e. there exists the sets \(X, Y \in \mathcal{M}_{G(R^+, E)}\) such that \(\mu_n(X \cup Y) \neq \max\{\mu_n(X), \mu_n(Y)\}, n \in \mathbb{N}.\)

**Remark 3.5** Let us notice that the family \(\{\mu_n\}_{n \in \mathbb{N}}\) given above is simpler than the measure of noncompactness appearing in the the paper [9]. Namely, the measure
of noncompactness (3.1) is composed of only two components. Hence, this measure requires weaker assumptions when we applying fixed point theorems than other measures.

4 An applications

At the beginning of this section, we start with recall some necessary facts.

**Lemma 4.1** [13] If \( \{x_n\}_{n=1}^\infty \subset L^1([0, T], E) \) is uniformly integrable (i.e. there is \( h \in L^1([0, T], \mathbb{R}_+) \) such that \( \|x_n(t)\| \leq h(t) \) for a.e. \( t \in [0, T] \) and for all \( n \in \mathbb{N} \)) then the function \( [0, T] \ni t \mapsto \beta(\{x_n(t)\}_{n=1}^\infty) \) is measurable and

\[
\beta\left(\left\{ \int_0^t x_n(s)ds \right\}_{n=1}^\infty\right) \leq 2 \int_0^t \beta(\{x_n(s)\}_{n=1}^\infty)ds, \quad t \in [0, T].
\]

where \( \beta \) is the Hausdorff measure of noncompactness.

**Lemma 4.2** [3] Let \( E \) be a Banach space. If \( X \subset E \) is a bounded set, then there exist a sequence \( \{x_n\}_{n=1}^\infty \subset X \) such that \( \beta(X) \leq 2\beta(\{x_n\}_{n=1}^\infty) \).

**Lemma 4.3** If \( q \in L^1([0, 1], \mathbb{R}), m \in \mathbb{N}, \) then

\[
\int_0^t q(s)\int_0^{s_1}\int_0^{s_{m-2}}\int_0^{s_{m-1}}q(s_{m-1})ds_{m-1}...ds_1ds = \frac{\left(\int_0^t q(s)ds \right)^m}{m!}, \quad t \in [0, T].
\]

**Proof** We omit the simple inductive proof that uses integration by substitution for absolutely continuous function.

Now we recall some generalization of Darbo fixed point theorem.

**Theorem 4.4** [8] Assume that \( \Omega \) is a nonempty, bounded, convex and closed subset of the Fréchet space \( F \) and the mapping \( V : \Omega \to \Omega \) is continuous. For an arbitrary set \( X \subset \Omega \) let us put

\[
\tilde{V}^1(X) := V(\text{Conv}(X)), \quad \tilde{V}^n(X) := V(\text{Conv}(\tilde{V}^{n-1}(X))), \quad n = 2, 3, ...
\]

If there exist a sequence \( \{k_n\}_{n=1}^\infty \subset [0, 1) \) and sequence of natural numbers \( \{m_n\}_{n=1}^\infty \) such that

\[
\mu_n(\tilde{V}^{m_n}(X)) \leq k_n\mu_n(X) \quad \text{for} \quad X \subset \Omega, \quad n \in \mathbb{N}
\]

where \( \{\mu_n\}_{n \in \mathbb{N}} \) is the family of measures of noncompactness in Fréchet space \( F \), then \( V \) has at least one fixed point in the set \( \Omega \).
Now we will show the applicability of the family of measures of noncompactness constructed in the previous section in the Frechet space $G(\mathbb{R}_+, E)$. We will study the following Volterra type integral equation

$$x(t) = g(t) \int_0^t v(t, s, x(s))\,ds, \quad t \geq 0. \tag{4.1}$$

We will impose the following conditions on the functions appearing in this equation.

\( (H_1) \) The function $g : \mathbb{R}_+ \to \mathbb{R}$ is regulated, i.e. $g \in G(\mathbb{R}_+, \mathbb{R})$.

\( (H_2) \) The mapping $v : \{(t, s, x) : 0 \leq s \leq t, x \in E\} \to E$ is such that the function $[0, t] \ni s \mapsto v(t, s, x(s)) \in E$ is integrable for every $x \in G(\mathbb{R}_+, E)$ and $t \geq 0$.

\( (H_3) \) There exist nondecreasing functions $p_1, p_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|v(t, s, x)\| \leq p_1(t)p_2(s)\|x\| \quad \text{for } 0 \leq s \leq t, x \in E.$$

\( (H_4) \) There are nondecreasing functions $\psi, \phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0^+} \psi(\varepsilon) = 0$, such that

$$\|v(t_2, s, x) - v(t_1, s, x)\| \leq \psi(|t_2 - t_1|)\phi(\|x\|), \quad \text{for } 0 \leq s \leq t_1 \leq t_2, x \in E.$$

\( (H_5) \) There exists a function $q \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ such that $\beta(v(t, s, X)) \leq q(s)\beta(X)$ for $0 \leq s \leq t$ and bounded $X \subset E$.

Now we formulate our existence theorem as:

**Theorem 4.5** Under the assumptions $(H_1) - (H_4)$ Eq. (4.1) has at least one solution in $G(\mathbb{R}_+, E)$.

**Proof** First, we define operator $V$ such that for any $x \in G(\mathbb{R}_+, E)$

$$(Vx)(t) := g(t) \int_0^t v(t, s, x(s))\,ds.$$

The assumptions $(H_1)$ and $(H_2)$ guarantee that $Vx \in G(\mathbb{R}_+, E)$. Moreover we put

$$\overline{g}(t) := \sup_{s \in [0, t]} |g(s)|.$$

Next we define a function
\[ r(t) := \overline{g}(t)p_1(t)e^{\int_0^t \overline{g}(s)p_1(s)p_2(s)ds}, \quad t \geq 0. \]

It is easy to check that above defined \( r \) satisfies the following inequality

\[ r(t) \geq \overline{g}(t)p_1(t) \int_0^t p_2(s)r(s)ds. \quad (4.2) \]

Next let us consider the set

\[ \Omega := \{ x \in G(\mathbb{R}_+, E) : \|x(t)\| \leq r(t), \ t \geq 0 \}. \]

The set \( \Omega \) is bounded, convex and closed. In virtue of \((H_3)\) and \((4.2)\) we obtain

\[ \| (Vx)(t) \| \leq \overline{g}(t) \int_0^t p_1(t)p_2(s)r(s)ds = \overline{g}(t)p_1(t) \int_0^t p_2(s)r(s)ds \leq r(t) \]

for \( x \in G(\mathbb{R}_+, E) \). It shows that \( V : \Omega \to \Omega \). We omit standard proof of continuity of operator \( V \).

Let us put \( \overline{r}(t) := \sup_{s \in [0,t]} r(s) \).

Next let us fix \( X \subset \Omega, x \in X, n \in \mathbb{N}, \varepsilon > 0, t \in [0,n), t_1, t_2 \in (t, t + \varepsilon) \cap [0,n], t_1 < t_2 \). Using \((H_3), (H_4)\) and previously established notions, we derive

\[
\| (Vx)(t_2) - (Vx)(t_1) \| \leq |g(t_2)| \int_{t_1}^{t_2} \| v(t_2, s, x(s)) \| ds \\
+ |g(t_2)| \int_{0}^{t_1} \| v(t_2, s, x(s)) - v(t_1, s, x(s)) \| ds + |g(t_2) - g(t_1)| \int_{0}^{t_1} \| v(t_1, s, x(s)) \| ds \\
\leq \overline{g}(n)\overline{r}(n) \int_{t_1}^{t_2} p_1(n)p_2(s)ds + \overline{g}(n) \int_{0}^{n} \psi(\varepsilon)\phi(\overline{r}(n))ds + \omega_n^+(g, t, \varepsilon)\overline{r}(n) \int_{0}^{n} p_1(n)p_2(s)ds.
\]

Hence, we get

\[
\omega_n^+(VX, t, \varepsilon) \leq \overline{g}(n)\overline{r}(n)p_1(n) \sup \left\{ \int_{t_1}^{t_2} p_2(s)ds : t_1, t_2 \in [0,n], 0 \leq t_2 - t_1 \leq \varepsilon \right\} \\
+ \overline{g}(n)\phi(\overline{r}(n)) \int_{0}^{n} \psi(\varepsilon)ds + \omega_n^+(g, t, \varepsilon)\overline{r}(n) \int_{0}^{n} p_1(n)p_2(s)ds.
\]

Keeping in mind \((H_1), (H_4)\) and integrability of \( p_2 \), we obtain (under \( \varepsilon \to 0^+ \)) the following equalities \( \omega_n^+(X, t) = 0, \omega_n^+(X) = 0 \). Analogously, we have \( \omega_n^-(X) = 0 \) and consequently.
Let us fix nonempty set $X \subset \Omega$. In view of Lemma 4.2, there exists the sequence \( \{x_n\} \subset \text{Conv}X \), such that $\beta(\text{Conv}X) \leq 2\beta(\{x_n\})_{n=1}^{\infty}$. This fact, the assumptions $(H_3)$, $(H_5)$ and Lemma 4.1 yield

$$\max \{\omega_n^-(X), \omega_n^+(X)\} = 0. \quad (4.3)$$

Hence we get

$$\beta(\tilde{V}^1(X)(t)) = |g(t)|\beta\left(\int_0^t v(t, s, \text{Conv}X(s))ds\right) \leq 2\overline{g}(t)\beta\left(\int_0^t v(t, s, \{x_n(s)\}_{n=1}^{\infty})ds\right) \leq 4\overline{g}(t)\int_0^t \beta\left(v(t, s, \{x_n(s)\}_{n=1}^{\infty})\right)ds$$

$$\leq 4\overline{g}(t)\int_0^t q(s)\beta(\{x_n(s)\}_{n=1}^{\infty})ds \leq 4\overline{g}(t)\int_0^t q(s)\beta(X(s))ds.$$ 

Continuing this reasoning we obtain

$$\beta(\tilde{V}^m(X)(t)) \leq (4\overline{g}(t))^m\int_0^t q(s)\int_0^s q(s_1)\ldots q(s_{m-1})\beta(X(s_{m-1}))ds_{m-1}...ds_1ds.$$ 

Then, in virtue of (3.1) we have

$$\overline{\beta}_n(\tilde{V}^m(X)) \leq (4\overline{g}(n))^m\int_0^n q(s)\int_0^s q(s_1)\ldots q(s_{m-1})ds_{m-1}...ds_1ds \cdot \overline{\beta}_n(X)$$

and applying Lemma 4.3 we get
\[
\bar{\beta}_n(\tilde{V}^m(X)) \leq \left( \frac{4g(n) \int_0^n q(t)dt}{m!} \right)^m \bar{\beta}_n(X).
\]
Since \( \lim_{m \to \infty} \left( \frac{4g(n) \int_0^n q(t)dt}{m!} \right)^m = 0 \), then for any fixed \( n \in \mathbb{N} \) there exists \( m_n \in \mathbb{N} \) such that
\[
k_n \defeq \frac{\left( \frac{4g(n) \int_0^n q(t)dt}{m_n!} \right)^m}{m_n!} < 1
\]
Keeping in mind (2.1) and (4.3) we obtain \( \mu_n(\tilde{V}^{m_n}(X)) \leq k_n \mu_n(X) \). This fact and Theorem 4.4 complete the proof. \( \square \)

**Remark 4.6** Let us notice that many existence theorems, proved using the classical version of Darbo theorem, need some inconvenient assumption (existence of a solution of some artificial inequalities). Using theorem 4.4, we do not have this problem. The assumption of this type is not necessary.

**Acknowledgements** This work was completed with the support of our TeX-pert.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Banaś, J., Goebel, K.: Measure of Noncompactness in Banach Spaces. Lecture Notes in Pure and Applied Math, vol. 60. Marcel Dekker, New York (1980)
2. Banaś, J., Zając, T.: On a measure of noncompactness in the space of regulated functions and its applications. Adv. Nonlinear Anal. (2019). https://doi.org/10.1515/anona-2018-0024
3. Botche, D.: Multivalued perturbation of m-accretive differential inclusions. Isr. J. Math. 108, 109–138 (1998)
4. Cichoń, K., Cichoń, M., Metwali, M.A.: On some parameters in the space of regulated functions and their applications. Carpath. J. Math. 34(1), 17–30 (2018)
5. Cichoń, K., Cichoń, M., Satco, B.: On regulated functions. Fasc. Math. (2018). https://doi.org/10.1515/fasmath-2018-0003
6. Cichoń, K., Cichoń, M., Satco, B.: Measure differential inclusions through principles in the space of regulated functions. Mediterr. J. Math. 15, 148 (2018). https://doi.org/10.1007/s00009-018-1192-y1660-5446/18/040001-19
7. Drewnowski, L.: On Banach spaces of regulated functions. Comment. Math. 57(2), 153–169 (2017)
8. Dudek, S.: Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations. Appl. Anal. Discrete Math. 11, 340–357 (2017)
9. Dudek, S., Olszowy, L.: Measures of noncompactness and superposition operator in the space of regulated functions on an unbounded interval. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 114–168 (2020)
10. Fraňková, D.: Regulated functions. Math. Bohem. **116**, 20–59 (1991)
11. Gabeleh, M., Malkowsky, E., Mursaleen, M., Rakočević, V.: A new survey of measures of noncompactness and their applications. Axioms **11**, 299 (2022). [https://doi.org/10.3390/axioms11060299](https://doi.org/10.3390/axioms11060299)
12. Michalak, A.: On superposition operators in spaces of regular and of bounded variation functions. Z. Anal. Anwend. **35**, 285–308 (2016)
13. Mönch, H.: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. Nonlinear Anal. TMA **4**, 985–999 (1980)
14. Olszowy, L.: Fixed point theorems in the Fréchet space $\mathcal{C}([\mathbb{R}_+\to \mathbb{R}_+])$ and functional integral equations on an unbounded interval. Appl. Math. Comput. **218**, 9066–9074 (2012)
15. Olszowy, L.: Measures of noncompactness in the space of regulated functions. J. Math. Anal. Appl. **476**, 860–874 (2019)