Uniform behavior of families of Galois representations on Siegel modular forms and the Endoscopy Conjecture

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November 20, 2018

Abstract

We prove the following uniformity principle: if one of the Galois representations in the family attached to a genus two Siegel cusp form of weight \( k > 3 \), “semistable” and with multiplicity one, is reducible (for an odd prime \( p \)), then all the representations in the family are reducible. This, combined with Serre’s conjecture (which is now a theorem) gives a proof of the Endoscopy Conjecture.

1 Introduction

In this article, we will consider a genus two Siegel modular form \( f \) of level \( N \) and weight \( k > 3 \) (and multiplicity one) and the family of four dimensional symplectic Galois representations attached to it. We assume also that we are in a case where this family is “semistable”. In [D1], we have treated the level 1 case, giving conditions on \( f \) to ensure that these Galois representations have generically large image. In particular we have imposed an irreducibility condition on one characteristic polynomial of Frobenius (see [D1], condition

*AMS Mathematics Subject Classification: 11F80, 11F46. Keywords: Galois representations, Siegel modular forms.
Research supported by project MTM2006-04895, MECD, Spain
(4.8)) to obtain a large image result. Furthermore, with the same irreducibility condition, we showed in [D2], again for the level 1 case, that for every $p > 4k - 5$ the $p$-adic representations are irreducible. The only possible reducible case to be considered is the case of two 2-dimensional irreducible components having the same determinant (all other cases can not occur if $f$ is not of Saito-Kurokawa type, cf. [D1], [D2]), and from the results of [D2] this case can only happen if all characteristic polynomials are reducible, i.e., the 2-dimensional components will have coefficients in the same field that the 4-dimensional representations: the field $E$ generated by the eigenvalues of $f$.

In section 2 we will generalize the main results of [D1] and [D2] to the semistable case.

One of the consequences of Tate’s conjecture on the Siegel threefold is that reducibility for the Galois representations attached to $f$ must be a uniform property: if it is verified at one prime, then all the representations in the family are reducible. In this article, we will prove this uniformity principle:

**Theorem 1.1**: Let $f$ be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 3$ and level $N$, having multiplicity one, such that the attached Galois representations $\rho_{f,\lambda}$ are “semistable”. Suppose that for some odd prime $\ell_0 \nmid N$, $\lambda_0 \mid \ell_0$, the representation $\rho_{f,\lambda_0}$ is reducible. Then the representations $\rho_{f,\lambda}$ are reducible for every $\lambda$.

Moreover, if this happens, either $f$ is of Saito-Kurokawa type or $f$ is endoscopic.

After excluding the Saito-Kurokawa case, we will prove the result more generally for compatible families of geometric, pure and symplectic four-dimensional Galois representations which are “semistable”.

A previous version of this preprint dates from 2003, and since several papers using the results contained there have appeared since then, we prefer to present first (sections 2 and 3) the results contained in that early version, which constitute the “core” of this paper: this corresponds to the proof of “uniformity of reducibility” for “almost every” prime, and with the extra condition $\ell_0 > 4k - 5$. At the end of the paper (section 4) we will indicate how to (easily) remove this assumption on $\ell_0$. Finally, we will prove that
a standard combination of these results with Serre’s conjecture allows us to remove the “almost every” in the result and gives also the Endoscopy Conjecture, i.e., the modularity (up to twist) of the irreducible components.

What follows is a brief description of the tools that will appear in the proofs in the “core” part. We will use (as in [D2], section 4) as starting point Taylor’s recent results on the Fontaine-Mazur conjecture and the meromorphic continuation of $L$-functions for odd two-dimensional Galois representations (see [T2], [T3] and [T4]). Then, we will combine some of the results and techniques in [D1] (in particular the information about the description of the action of inertia obtained via $p$-adic Hodge theory) with Ribet’s results (see [R]) on two-dimensional semistable Galois representations (slightly generalized to higher weights), and finally Cebotarev density theorem, the fundamental theorem of Galois theory, and some group theory will suffice for the proof.

2 Preliminaries

As we already explained, the goal of sections 2 and 3 is to prove a theorem which is weaker than theorem 1.1, namely we will prove the following:

**Theorem 2.1** Let $f$ be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 3$ and level $N$, having multiplicity one, such that the attached Galois representations $\rho_{f,\lambda}$ are “semistable”. Suppose that for some prime $\ell_0 > 4k - 5$, $\ell_0 \nmid N$, $\lambda_0 | \ell_0$, the representation $\rho_{f,\lambda_0}$ is reducible. Then the representations $\rho_{f,\lambda}$ are reducible for almost every $\lambda$.

From now on we will make the following assumption: $f$ is a genus 2 level $N$ Siegel cuspidal Hecke eigenform of weight $k > 3$, having multiplicity one, and not of Saito-Kurokawa type (theorem 2.1 is trivial in the Saito-Kurokawa case, where by construction the Galois representations are reducible, with one 2-dimensional and two 1-dimensional components). Let $E = \mathbb{Q}(\{a_n\})$ be the field generated by its Hecke eigenvalues. Then, there is a compatible family of Galois representations constructed by Taylor [T1] and Weissauer [W2] verifying the following:
For any prime number $\ell$ and any extension $\lambda$ of $\ell$ to $E$ we have a continuous Galois representation

$$\rho_{f,\lambda} : G_\mathbb{Q} \to \text{GSp}(4, \overline{E}_\lambda)$$

unramified outside $\ell N$ and with characteristic polynomial of $\rho_{f,\lambda}(\text{Frob } p)$ equal to

$$\text{Pol}_p(x) = x^4 - a_p x^3 + (a_p^2 - a_p - p^{2k-4}) x^2 - a_p p^{2k-3} x + p^{4k-6}$$

for every $p \nmid \ell N$. If $\rho_{f,\lambda}$ is absolutely irreducible, then it is defined over $E_\lambda$.

In general, we cannot guarantee that the field of definition is $E_\lambda$, but the residual representation $\bar{\rho}_{f,\lambda}$ can be formally defined in any case (see [D1]) as a representation defined over the residue field of $\lambda$, $\mathbb{F}_\lambda$. Nevertheless, not knowing the field of definition of the representations that we will study is not a serious problem, we can work instead with the “field of coefficients” (*), i.e., the field generated by the coefficients of the characteristic polynomials $\text{Pol}_p(x)$, this field contains all the information we need.

The representations $\rho_{f,\lambda}$ are known to have the following properties (cf [W1], [W2], [D1]): they are pure (Ramanujan conjecture is satisfied) and if $\ell \nmid N$ they are crystalline with Hodge-Tate weights $\{0, k - 2, k - 1, 2k - 3\}$. This last property makes possible, via Fontaine-Laffaille theory, to obtain a precise description of the action of the inertia group at $\ell$ on the residual representation $\overline{\rho}_{f,\lambda}$: it acts through fundamental characters of level one or two, with exponents equal to the Hodge-Tate weights (see [D1] for more details).

We will need a further restriction: we want the representations to be “semistable” at every prime of $N$ (see the definition below).

Since we will not use the fact that our representations are modular, we can change to the more general setting of a family of four-dimensional symplectic Galois representations $\{\rho_\lambda\}$ with coefficients in a number field $E$ (not necessarily defined over $E_\lambda$, see (*)), $\det \rho_\lambda = \chi^{4k-6}$, which are pure, and such that there exists a finite set $S$ with, for every $\ell \notin S$, $\rho_\lambda$ unramified outside $\{\ell\} \cup S$, crystalline at $\ell$ with Hodge-Tate weights as above, and “semistable” at primes in $S$, i.e., verifying the following: $\rho_\lambda$ restricted to $I_q$ is a unipotent group for every $q \in S$.

For every $p \notin S$ we still denote $\text{Pol}_p(x)$ the characteristic polynomial of the
image of Frob $p$ and $a_p$ the trace of this image. The representations being symplectic, we have the standard factorization

$$Pol_p(x) = (x^2 - (a_p/2 + \sqrt{d_p})x + p^{2k-3})(x^2 - (a_p/2 - \sqrt{d_p})x + p^{2k-3}) \quad (2.1)$$

The results of generically large image and irreducibility proved in previous articles for the level 1 case (see [D1], theorem 4.2, and [D2], theorems 2.1 and 4.1) hold also in this generality:

**Theorem 2.2**: Let $\{\rho_\lambda\}$ be a family of Galois representations verifying the above properties, with $k > 3$. Assume that there is a prime $p \notin S$ such that

$$\sqrt{d_p} \notin E \quad (2.2)$$

where $d_p$ is defined by formula (2.1). Then for all but finitely many of the primes verifying

$$d_p \notin (\mathbb{F}_\lambda)^2$$

and, more generally, for all primes $\lambda$ in $E$ except at most for a set of Dirichlet density 0, the image of $\rho_\lambda$ is

$$A^k_\lambda = \{g \in \text{GSp}(4, \mathcal{O}_{E_\lambda}) : \det(g) \in (\mathbb{Q}_\ell^*)^{4k-6}\},$$

where $\mathcal{O}_{E_\lambda}$ denotes the ring of integers of $E_\lambda$.

Keeping condition (2.2) we also have: for every prime $\ell \geq 4k - 5, \ell \notin S, \lambda | \ell$, the representation $\rho_\lambda$ is absolutely irreducible.

Differences with the level 1 case:
The proof of the above results given in [D1] and [D2] extends automatically to the semistable case: recall that the determination of the images is done by considering the image of the residual mod $\lambda$ representations and eliminating all non-maximal proper subgroups of $\text{GSp}(4, \mathbb{F}_\lambda)$. When considering reducible cases (cf. [D1], sections 4.1 and 4.2) if we allow arbitrary ramification at a finite set $S$ then we have to allow the character appearing as one-dimensional component or determinant of a two-dimensional component of a reducible $\bar{\rho}_\lambda$ to ramify at $S$, but in the semistable case it is easy to see that this character will not ramify at primes in $S$. The same applies to the case of image equal to a group $G$ having a reducible index 2 normal subgroup $M$ (cf. [D1], section 4.4), the quadratic Galois character $G/M$ can not ramify at primes of $S$ if we assume semistability. Up to these easy remarks, all the
proof translates word by word to the semistable case.

Remark 1: Recall that condition (2.2) was introduced (cf. [D1]) specifically to deal with the case where the image of \( \overline{\rho}_\lambda \) is reducible, with two 2-dimensional irreducible components of the same determinant. All other cases of non-maximal image can be discarded, for almost every prime, without using condition (2.2).

Remark 2: In [D1], the large images result was proved (for the case of conductor 1) with an additional condition, called “untwisted”: this condition was imposed to eliminate the possibility that the projective residual image falls in a smaller symplectic group PGSp\((4, k')\), \( k' \) a proper subfield of \( k \), where \( k \) is the field generated by the traces of the residual representation. We have not included a similar condition in the above theorem because in the following lemma, we will explain that this condition is superfluous, i.e., that the case of smaller projective symplectic group can never happen if we assume semistability. In particular, this applies to level 1 Siegel cusp forms, so the condition “untwisted” can be removed from theorem 4.2 of [D1].

Lemma 2.3 : Let \( \{\rho_\lambda\} \) be a compatible families of Galois representations as above (in particular, a semistable family). Then for every prime \( q > 2k - 2 \), \( q \notin S \) and \( Q \) a prime in \( E \) dividing \( q \), if we call \( G \) the image of \( \overline{\rho}_Q \) and \( P(G) \) its projectivization, \( P(G) \) lies in PGSp\((4, k)\) if and only if \( G \) lies in GSp\((4, k)\), for every subfield \( k \) of \( \mathbb{F}_q \).

Proof: A similar result, for semistable two-dimensional representations, is lemma 2.4 in [R]. The proof given there translates word by word, once we have explained why in our case we also have an element \( c \) in the inertia group \( I_q \) such that \( \chi(c) \) is a generator of \( \mathbb{F}_q^* \) and the trace of \( \overline{\rho}_Q(c) \) is a non-zero element of \( \mathbb{F}_q \) (we know a priori, from the description of the action of \( I_q \), that this trace will be in \( \mathbb{F}_q \), what requires a proof is the fact that it is not 0).

We have given in [D1], proposition 3.1, a description of the action of \( I_q \) that applies in the current situation, because we are assuming that \( \rho_Q \) is symplectic and crystalline with Hodge-Tate weights \( \{0, k - 2, k - 1, 2k - 3\} \), and \( q > 2k - 2 \). Let \( \psi \) be a level 2 fundamental character, and take \( c \in I_q \) such that \( \psi(c) \) generates \( \mathbb{F}_{q^2}^* \). We have four possibilities for the trace of \( c \),
whose values are, after a suitable factorization:

\[(1 + \chi(c)^{k-1})(1 + \chi(c)^{k-2})
\]

\[(\psi(c)^{k-2} + \psi(c)^{(k-2)q})(\psi(c)^{k-1} + \psi(c)^{(k-1)q})
\]

\[(1 + \psi(c)^{(k-2)+(k-1)q})(1 + \psi(c)^{(k-1)+(k-2)q})
\]

\[(\psi(c)^{k-2} + \psi(c)^{(k-1)q})(\psi(c)^{k-1} + \psi(c)^{(k-2)q})
\]

In all cases, the inequality \( q > 2k - 2 \) implies that these traces are not 0.

### 3 Uniformity of reducibility

At this point, we can say that the validity or not of condition (2.2) at some prime \( p \not\in S \) determines the behavior of the family of representations \( \rho_\lambda \): If condition (2.2) is satisfied, then we have generically large image and irreducibility for every \( \ell \) sufficiently large compared with the weights.

What happens if condition (2.2) is not satisfied at any prime? This implies that the factorization (2.1) takes place over \( E \), i.e., that for every \( p \not\in S \), \( Pol_p(x) \) reduces over \( E \). The coefficients of all characteristic polynomials \( Pol_p(x) \) generate an order \( \mathcal{O} \) of \( E \), and if we restrict to primes \( \lambda \) not dividing the conductor of this order (we are neglecting only finitely many primes), we see that the field generated by the coefficients of the mod \( \lambda \) reduction of all the \( Pol_p(x) \) gives the whole \( \mathbb{F}_\lambda \). Thus, we see that for almost every prime, the failure of (2.2) implies that \( \bar{\rho}_\lambda \) has its image in \( GSp(4, \mathbb{F}_\lambda) \) and not in a smaller symplectic, but all characteristic polynomials reduce over \( \mathbb{F}_\lambda \): in this case the image can not be the whole symplectic group, because in the group \( GSp(4, \mathbb{F}_\lambda) \) most of the matrices have IRREDUCIBLE characteristic polynomial, and we know that for almost every prime only one possibility (see remark 1 after theorem 2.2 and lemma 2.3) remains:

**Lemma 3.1** : Let \( \{\rho_\lambda\} \) be as in the previous section, and assume that for every \( p \not\in S \), condition (2.2) is not satisfied. Then, for almost every prime \( \lambda \), the residual representation \( \bar{\rho}_\lambda \) is reducible with two 2-dimensional irreducible components of the same determinant.
3.1 A reducible member in the family: Residual consequences

From now on, assume that for a prime \( q > 4k - 5, q \notin S, Q \mid q \), the \( Q \)-adic representation \( \rho_Q \) is reducible, we know (using semistability and purity) that the only possible case is the case of two 2-dimensional irreducible components both with determinant \( \chi^{2k-3} \). Thus we have:

\[
\rho_Q \cong \sigma_{1,Q} \oplus \sigma_{2,Q} \tag{3.1}
\]

Since this representations is reducible, the last part of theorem 2.2 implies that condition (2.2) must fail at every prime. Therefore, \( \sigma_{1,Q} \) and \( \sigma_{2,Q} \) will also have coefficients in \( E \) and lemma 3.1 implies that for every prime \( \lambda \) in a cofinite set \( \Lambda \) of primes of \( E \), \( \bar{\rho}_\lambda \) will verify:

\[
\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}
\]

where \( \pi_{i,\lambda} \) is an irreducible two dimensional representation defined over \( F_\lambda \) having determinant \( \chi^{2k-3} \), for \( i = 1,2 \) and for every \( \lambda \in \Lambda \).

Moreover, we can determine the image of \( \bar{\rho}_\lambda \) for almost every prime in \( \Lambda \):

**Lemma 3.2**: Keep the above assumptions. For every prime \( \lambda \in \Lambda_2 \), a cofinite subset of \( \Lambda \), the image of \( \bar{\rho}_\lambda \) is a subgroup of \( \text{GSp}(4,F_\lambda) \) conjugated to \( M_\lambda = \{ A \times B \in \text{GL}(2,F_{1,\lambda}) \times \text{GL}(2,F_{2,\lambda}) : \text{det}(A) = \text{det}(B) \in \mathbb{F}^{2k-3}_2 \} \), where \( F_{1,\lambda}, F_{2,\lambda} \subseteq F_\lambda \) are the fields of coefficients of \( \pi_{1,\lambda} \) and \( \pi_{2,\lambda} \).

Proof: We have assumed that the representations \( \rho_\lambda \) have a finite ramification set \( S \) and they are semistable at every prime \( q \in S \). A fortiori, the same applies to their residual components \( \pi_{i,\lambda} \). Moreover, these two dimensional representations are irreducible for every \( \lambda \in \Lambda \). In a similar situation, Ribet has proved a large image result for \( \ell \geq 5 \), but he assumes that the action of \( I_\ell \), given by fundamental characters of level 1 or 2, has weights (i.e., exponents of the fundamental characters) 0 and 1. The main point of his proof is to exclude the dihedral case. In our case, using the information on the Hodge-Tate decomposition, we have this extra condition at \( I_\ell \) also verified by the twisted representation \( \pi_{i,\lambda} \otimes \chi^{-k+2} \) for, say, \( i = 2 \) (cf. [D1],[D2]). On the other hand, for \( \pi_{1,\lambda} \) Ribet’s result still holds if we restrict to primes \( \ell > 4k - 5 \), because the weights of the action of \( I_\ell \) being 0 and \( 2k - 3 \), the projectivization of the image if \( I_\ell \) gives a cyclic group of order \((\ell \pm 1)/\gcd(\ell \pm 1, 2k - 3) > 2 \), and this is all that you need to follow Ribet’s
argument. We also have a statement as lemma 2.3 for these two dimensional representations, again adapting lemma 2.4 in [R].

We conclude (cf. [R], theorem 2.5 and the remark after) that for \( \ell \) sufficiently large, the images of both irreducible components are conjugated to the subgroup of matrices in \( \text{GL}(2, \mathbb{F}_{i,\lambda}) \) with determinant in \( \mathbb{F}_\ell^{2k-3} \).

Finally, to prove that the image of \( \bar{\rho}_\lambda \) is as we want, it remains to show that the Galois fields corresponding to \( P(\pi_{1,\lambda}) \) and \( P(\pi_{2,\lambda}) \) are disjoint (\( P \) denotes projectivization). These fields having Galois groups isomorphic to the simple groups \( \text{PGL}(2, \mathbb{F}_{i,\lambda}) \) or \( \text{PSL}(2, \mathbb{F}_{i,\lambda}) \), they are either disjoint or equal: the second is not possible because the restriction of these two projective representations to \( I_\ell \) are different, and this proves the result.

### 3.2 A reducible member in the family: \( \lambda \)-adic consequences

In the decomposition (3.1) of \( \rho_Q \) it is clear that \( \sigma_{1,Q} \) has Hodge-Tate weights \( \{0, 2k-3\} \) and \( \sigma_{2,Q} \) has Hodge-Tate weights \( \{k-2, k-1\} \) (or viceversa).

Now, we invoke a result of Taylor (see [T2] and [T3], recall that \( q > 4k-5 \)) asserting that for a representation such as \( \sigma_{1,Q} \) it is possible to find a totally real number field \( F \) such that it is modular when restricted to this field, and therefore it agrees on \( F \) with the \( Q \)-adic motivic irreducible Galois representation (constructed by Blasius and Rogawski) attached to a Hilbert modular form \( h \). This implies that \( \sigma_{1,Q} \) appears in the cohomology of the restriction of scalars of the motive \( M_h \) associated to \( h \), and it can be checked from the fact that the \( Q \)-adic representation of the absolute Galois group of \( F \) attached to \( h \) has descended to a 2-dimensional representation of \( G_Q \), Cebotarev density theorem, and the fact that all modular Galois representations in the family \( \{\sigma_{h,\lambda}\} \) attached to \( h \) are known to be irreducible, that the whole family descends to a compatible family \( \{\sigma_{1,\lambda}\} \) of Galois representations of \( G_Q \) containing \( \sigma_{1,Q} \). To do this, one has to write the representation \( \sigma_{1,Q} \) as in the proof of theorem 6.6 in [T3], and define the representations \( \sigma_{1,\lambda} \) formally in the same way using the strongly compatible families associated to the base change of \( h \) to each \( E_i \) (recall that, for each \( i \), \( F/E_i \) is soluble, cf. [T3]).

Then, following an idea suggested to us by R. Taylor, one can check that the virtual representations \( \sigma_{1,\lambda} \) constructed this way are true Galois representations by applying the arguments of [T4], section 533.
It follows from the main result of [T3] that the family \( \{\sigma_1, \lambda\} \) is a strongly compatible family (cf. [T3] for the definition) of Galois representations. Strong compatibility proves the last steps of the following:

**Proposition 3.3** Let \( \rho_Q \) be as above, reducible as in (3.1), and let \( \sigma_{1,Q} \) be its irreducible component having Hodge-Tate weights \( \{0, 2k - 3\} \). Then, there exists a compatible family of Galois representations \( \{\sigma_1, \lambda\} \) containing \( \sigma_{1,Q} \), such that for every \( \ell \not\in S \), \( \lambda \mid \ell \), the representation \( \sigma_1, \lambda \) is unramified outside \( \{\ell\} \cup S \), is crystalline at \( \ell \) with Hodge-Tate weights \( \{0, 2k - 3\} \), and is semistable at every prime of \( S \). Of course, these representations are pure because \( \rho_Q \) is.

Recall that the representation \( \rho_\lambda \) being symplectic, for every \( g \in G_Q \) the roots of \( \rho_\lambda(g) \) come in reciprocal pairs: \( \{\alpha, \chi^{2k-3}(g)/\alpha, \beta, \chi^{2k-3}(g)/\beta\} \).

The following lemma is a first approach to compare the representations \( \sigma_{1,\lambda} \) and \( \rho_\lambda \):

**Lemma 3.4** For every \( \ell \not\in S \), \( \lambda \mid \ell \), and every \( g \in G_Q \), the roots of \( \sigma_{1,\lambda}(g) \) form a pair of reciprocal roots of those of \( \rho_\lambda(g) \).

Proof: From the compatibility of the families \( \{\sigma_{1,\lambda}\} \) and \( \{\rho_\lambda\} \) and the fact that \( \sigma_{1,Q} \) is a component of \( \rho_Q \) the lemma is obvious for the dense set of Frobenius elements at unramified places. Then, by continuity and Cebotarev the lemma follows for every element of \( G_Q \).

Recall that \( \Lambda_2 \) denotes the cofinite set of primes of \( E \) where lemma 3.2 is satisfied. We will shrink again this set by eliminating a finite set of primes, namely, those primes where the image of \( \sigma_{1,\lambda} \) fails to be maximal: in fact, if we call \( E' \subseteq E \) the field of coefficients of this family of representations and \( O' \) its ring of integers, using semistability and again the slight modification of the methods of [R] to higher weights (as we did before to obtain lemma 3.2) we see that for almost every prime \( \lambda \in E \) the image of \( \sigma_{1,\lambda} \) can be conjugated to the subgroup of \( \text{GL}(2, O'_\lambda) \) of matrices with determinant in \( \mathbb{Z}_{l}^{2k-3} \) (after proving the similar result for the residual representations, we apply a lemma of Serre in [S1] that shows that the \( \lambda \)-adic image is also large).

Remark: Here we need to know that the residual representations \( \bar{\sigma}_{1,\lambda} \) are almost all of them irreducible. This follows again from the good properties of the \( \lambda \)-adic family: purity, the fact that they are all crystalline with Hodge-Tate weights \( \{0, 2k - 3\} \) (and the uniform description of inertia that one gets.
from this), and semistability.

Thus, we exclude from Λ₂ the finite set of primes where the image of σ₁,λ fails to be maximal, and we obtain a cofinite set Λ₃ where the residual image of ρₐ is the full Mₐ and the image of σ₁,λ is maximal.

We want to extract more information from the relation derived in lemma 3.4. To start with, we work at the level of residual representations. Observe that the same relation proved in lemma 3.4 holds for the roots of the matrices in the image of the residual representations ¯ρₐ and ¯σ₁,λ:

Lemma 3.5: Let λ be a prime in Λ₃, then in the decomposition ¯ρₐ ∼= π₁,λ ⊕ π₂,λ we have π₁,λ ∼= ¯σ₁,λ.

Remark: Of course, we should write the above equality with πᵢ,λ for i = 1 or 2. But to fix notation, we will always call π₁,λ the component of ¯ρₐ where the inertia group at ℓ acts with weights 0 and 2k − 3 (as we did in section 3.1), this is a good way to distinguish the two components, and of course this is the only component that deserves being compared to ¯σ₁,λ.

Proof: Take λ ∈ Λ₃. Let L be the Galois field corresponding to ¯ρₐ, thus Gal(L/Q) ∼= Mₐ, and B the one corresponding to ¯σ₁,λ, thus if Fₐ is the residue field of Oₐ, and Uₐ = {A ∈ GL(2, Fₐ) : det(A) ∈ Fₐ²k−3}, Gal(B/Q) ∼= Uₐ.

We want to prove that B ⊆ L. Let M = L ∩ B, and consider an element z ∈ Gal(B/M). Let ¯z be a preimage of z in Gal(¯Q/M), that we can choose such that ¯ρₐ(¯z) = 1₁₄ (because it is trivial on M = L ∩ B). Then, the residual version of lemma 3.4 implies that 1 is a double root of the characteristic polynomial of ¯σ₁,λ(¯z). This implies that the group Gal(B/M) is unipotent, but this group is a normal subgroup of Gal(B/Q) ∼= Uₐ, and Uₐ has no non-trivial unipotent normal subgroup, thus B = M, i.e., B ⊆ L.

Then, we have a projection: φ : Gal(L/Q) → Gal(B/Q), that is to say, φ sends Mₐ onto Uₐ and thus ¯σ₁,λ is a quotient of ¯ρₐ.

Since ¯ρₐ ∼= π₁,λ ⊕ π₂,λ we conclude that ¯σ₁,λ ∼= πᵢ,λ with i = 1 or 2, and using the information on the Hodge-Tate decompositions we see that i = 1.

3.3 Proof of Theorem 2.1

We start by observing that part of the proof of lemma 3.5 can be translated to the λ-adic setting. Take λ ∈ Λ₃, and call L' the (infinite) Galois field corresponding to ρₐ and B' the one corresponding to σ₁,λ. Recall that
\[ \text{Gal}(B'/\mathbb{Q}) \text{ is isomorphic to the subgroup } U'_\lambda \text{ of } \text{GL}(2, \mathcal{O}'_\lambda) \text{ composed of matrices with determinant in } \mathbb{Z}^{2k-3}_2, \text{ and therefore again we have a group with no non-trivial unipotent subgroups, thus we conclude from lemma 3.4 as in the proof of lemma 3.3 that } B' \subseteq L' \text{ and that we have a projection: } \\
\phi' : \text{Gal}(L'/\mathbb{Q}) \rightarrow \text{Gal}(B'/\mathbb{Q}). \text{ We have } \phi' \circ \rho_\lambda = \sigma_{1,\lambda}. \text{ Let us consider the normal subgroup } \text{Gal}(L'/B') \text{ of } \text{Gal}(L'/\mathbb{Q}), \text{ i.e., we are considering the restriction } \rho_\lambda|_{\ker \phi'} \text{. The elements in this subgroup fix } B' \subseteq L', \text{ thus by lemma 3.4 we see that the corresponding matrices in } \text{GSp}(4, \mathcal{O}_\lambda) \text{ will have 1 as a double root.} \\
\text{On the other hand, we know that the residual representation } \bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda} \cong \bar{\sigma}_{1,\lambda} \oplus \pi_{2,\lambda} \text{ has maximal image } M_\lambda \text{ (see lemmas 3.2 and 3.5). Moreover, the representation } \sigma_{1,\lambda} \text{ being a “deformation” of } \pi_{1,\lambda} \text{ which is disjoint from } \pi_{2,\lambda} \text{ (in the sense established during the proof of lemma 3.2 i.e., up to the equality of determinants), we see that restricting to } \ker \phi' \text{ will only shrink the image of } \pi_{2,\lambda} \text{ by making the determinant trivial, in other words: the residual representation } \rho_\lambda|_{\ker \phi'} \text{ has image } \text{SL}(2, \mathbb{F}_{2,\lambda}) \oplus 1_2 \subseteq M_\lambda \text{ (3.2).} \\
\text{So, what do we know about } \rho_\lambda|_{\ker \phi'}? \text{ We have determined its residual image and we also know that all matrices in its image have 1 as a double root: this last property extends to the Zariski closure of the image, and using the information we have together with the list of possibilities for this Zariski closure given in [T1], we see that the image of } \rho_\lambda|_{\ker \phi'} \text{ must be contained in } \text{SL}(2, \mathcal{O}_\lambda) \oplus 1_2. \text{ If we call } \mathcal{O}'_\lambda \subseteq \mathcal{O}_\lambda \text{ the field generated by the traces of the image of } \rho_\lambda|_{\ker \phi'}, \text{ we can apply a lemma of Serre (cf. [S1]) \text{(and Carayol’s lemma for the assertion about the field of definition, cf. [C]) and conclude from (3.2) that the image of } \rho_\lambda|_{\ker \phi'} \text{ must be conjugated to } \text{SL}(2, \mathcal{O}'_\lambda) \oplus 1_2. \text{ Remark: ker } \phi' \text{ fixes } B' \text{ which is an infinite extension of } \mathbb{Q}, \text{ but Serre’s lemma can still be applied because } \text{G}_\mathbb{Q} \text{ is compact and the fixer of } B' \text{ is a closed subgroup.} \\
\text{Thus, we conclude that } \text{Image}(\rho_\lambda) \subseteq \text{GSp}(4, \tilde{E}_\lambda) \text{ contains a normal subgroup isomorphic to } \text{SL}(2, \mathcal{O}'_\lambda) \oplus 1_2, \text{ and the quotient by this subgroup gives } U'_\lambda. \text{ But it is easy to see that the normalizer of } \text{SL}(2, \mathcal{O}'_\lambda) \oplus 1_2 \text{ in } \text{GSp}(4, \tilde{E}_\lambda) \text{ is contained in the reducible group } \text{GL}(2, \tilde{E}_\lambda) \oplus \text{GL}(2, \tilde{E}_\lambda). \text{ Thus, } \rho_\lambda \text{ is reducible, for every } \lambda \in \Lambda_3, \text{ and } \sigma_{1,\lambda} \text{ is one of its two-dimensional irreducible components.} \]
4 From Theorem 2.1 to Theorem 1.1

Recall that the results and proofs given in previous sections date from 2003. The result of “existence of compatible families” that we proved in section 3.2 were extended in [D3], which is a “sequel” to this paper, where it was applied to prove some cases of the Fontaine-Mazur conjecture. Moreover, this result is key in the proof of Serre’s conjecture given in [D4], [KW1], [K], [D5] and [KW2].

In the case of Hodge-Tate weights \( \{0, 1\} \) the results of potential modularity of Taylor do not imply that the representation is motivic, but as observed in [D3] we still can apply the techniques explained in section 3.2 and prove existence of families. In this case, the natural restriction becomes \( \ell_0 > 2 \).

As in the previous sections, we assume that we are not in the Saito-Kurokawa case (thus the reducible case must be a case of two 2-dimensional irreducible components).

Thus, if \( \ell_0 \) is odd, \( \ell_0 \nmid N \), we consider the irreducible component \( \sigma_{2, \lambda_0} \) of \( \rho_{\lambda_0} \) having Hodge-Tate weights \( \{k - 2, k - 1\} \) and we apply existence of compatible families to \( \sigma_{2, \lambda_0} \otimes \chi^{2 - k} \) instead of \( \sigma_{1, \lambda_0} \).

The rest of the proof given in the previous sections extends word by word, except that \( \sigma_{2, \lambda_0} \) and \( \sigma_{1, \lambda_0} \) exchange roles. We conclude that theorem 2.1 is still true if we change the assumption \( \ell_0 > 4k - 5 \) by \( \ell_0 > 2 \).

We have shown that for almost every prime \( \lambda \) the representation \( \rho_{\lambda} \) in our compatible family is reducible, and one of its 2-dimensional irreducible components is \( \sigma_{2, \lambda} \), a representation that lies in a strongly compatible family. It is obvious (from the definition of compatibility) that the second components, even if they are a priori defined only for almost every prime, will also form a compatible family, let us call them \( \sigma_{1, \lambda} \). Moreover, from the formula:

\[
\rho_{\lambda} \cong \sigma_{1, \lambda} \oplus \sigma_{2, \lambda}
\]

we see not only the compatibility of the \( \sigma_{1, \lambda} \) but also that the representations \( \sigma_{1, \lambda} \) are crystalline if \( \ell \) is not in the ramification set \( S \), of Hodge-Tate weights \( \{0, 2k - 3\} \), with ramification set contained in \( S \) and semistable or unramified locally at primes of \( S \).

At this point, we apply an argument based on Serre’s conjecture, which is now a theorem (cf. [D5] and [KW2]). Serre proved in [S2] using his conjecture.
that compatible families of Galois representations as $\{\sigma_{2,\lambda} \otimes \chi^{2-k}\}$ or $\{\sigma_{1,\lambda}\}$ are modular. The fact that $\{\sigma_{1,\lambda}\}$ is a priori only defined for almost every $\lambda$ is irrelevant for the argument of Serre: he only needs residual modularity in infinitely many characteristics, not in ALL characteristics (it is a typical patching argument). The essential condition is that the family is compatible, with constant Hodge-Tate weights and uniformly bounded conductor.

We conclude that the families $\{\sigma_{2,\lambda} \otimes \chi^{2-k}\}$ and $\{\sigma_{1,\lambda}\}$ correspond to representations attached to classical modular forms, of weight 2 and $2k - 2$, respectively, and obviously this implies in particular that the family $\{\sigma_{1,\lambda}\}$ is also defined for EVERY prime $\lambda$. This concludes the proof of theorem 1.1.

Remark 1: Observe that in the particular case of a level 1 Siegel cusp-form, we conclude irreducibility for $p > 2$ if it is not of Saito-Kurokawa type. The reason is that one of the 2-dimensional components would give rise to a level 1 weight 2 classical modular form.

Remark 2: What we have shown is that our result of uniformity of reducibility, combined with Serre’s conjecture (now a theorem), implies the truth of the Endoscopy Conjecture in the semistable case.

Remark 3: Our result of “uniformity of reducibility” can be extended to the non-semistable case, with the same arguments. In fact, this have been done recently by Skinner and Urban (cf. [SU], section 3). The argument of Serre explained above also applies here, so also the Endoscopy Conjecture follows in this case.

5 Final Remarks

The Galois representations attached to a Siegel cusp form $f$ of level greater than one are known to verify the semistability condition when the ramified local components of (the automorphic representation corresponding to) $f$ are of certain particular types (for example, a Steinberg representation), as follows from recent works of Genestier-Tilouine and Genestier (cf. [GT]). Thus, the results in this article apply to these cases. We thank J. Tilouine for pointing out this fact to us.
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