EXISTENCE OF POSITIVE SOLUTIONS OF SCHröDINGER EQUATIONS WITH VANISHING POTENTIALS

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Abstract. We prove the existence of at least one positive solution for a Schrödinger equation in $\mathbb{R}^N$ of type

$$-\Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N$$

with a vanishing potential at infinity and subcritical nonlinearity $f$. Our hypotheses allow us to consider examples of nonlinearities which do not verify the Ambrosetti-Rabinowitz condition, neither monotonicity conditions for the function $f(x, s)/s$. Our argument requires new estimates in order to prove the boundedness of a Cerami sequence.

1. Introduction. In the present paper, we prove the existence of at least one positive solution of the equation

$$\begin{cases}
-\Delta u + V(x)u = f(x, u), & x \in \mathbb{R}^N \\
u > 0 \\
u \in H^1(\mathbb{R}^N),
\end{cases}$$

for $N \geq 3$ and assuming that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory nonnegative function and $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative potential. Here, we assume that $f$ is superlinear at the origin and at infinity and has subcritical growth. Also, we consider cases where the potential $V : \mathbb{R}^N \to \mathbb{R}$ can vanish at infinity.

Recently several authors studied equation (1) using variational techniques such as fractional Sobolev spaces, reduction methods, generalized mountain pass theorem, dual variational formulation, generalized fountain theorem and generalized linking.

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theorems, see for instance [3, 6, 12, 13, 14, 16], where these authors consider several kinds of behaviour for the potential $V : \mathbb{R}^N \to \mathbb{R}$ and for the nonlinearity $f$.

Equation (1) appears in various applications, such as chemotaxis, population genetics, chemical reactor theory and also the solutions of this class of problems are related to the existence of standing wave solutions $\psi(x, t) = \exp\left(-\frac{iEt}{\varepsilon}\right)\psi(x)$ for nonlinear Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + (V(x) + E)\psi - K(x)|\psi|^{-1}g(|\psi|)\psi, \quad x \in \mathbb{R}^N,$$

(2)

where $\varepsilon > 0$, $E \in \mathbb{R}$ and $v$ is a real function. Equation (2) is one of the main objects of the quantum physics, since it appears in problems which involve nonlinear optics, plasma physics and condensed matter physics. We notice that $\psi$ satisfies (2) if, and only if, the function $v(x)$ satisfies equation (1) with $f(u) = g(|u|)u$.

An interesting class of problems associated with (1) is the zero mass case. This case happens when the potential $V : \mathbb{R}^N \to \mathbb{R}$ vanishes at infinity, which means, $\lim_{|x| \to +\infty} V(x) = 0$.

In [1], the authors define that $(V, K) \in \mathcal{K}$ if the following conditions hold:

(I): $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

(II): If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq R$, for all $n$ and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \quad (K_1)$$

(III): One of the below conditions occurs:

$$\frac{K}{V} \in L^\infty(\mathbb{R}^N) \quad (K_2)$$

or there is $p \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$, such that

$$\frac{K(x)}{|V(x)|^{\frac{2^*-2}{2}}} \to 0 \quad \text{as } |x| \to +\infty. \quad (K_3)$$

The main advantage on considering such hypothesis relies on the fact that the space $D^1_V(\mathbb{R}^N)$ endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx$$

is compactly embedded into the weighted Lebesgue space $L^p_K(\mathbb{R}^N)$, for all $p \in (2, 2^*)$. (see [1, Proposition 2.1]). We notice that this set of conditions generalizes the $(V, K)$ condition stated in [2].

More recently, in [9], the author introduces many different conditions in order to prove that the embedding described above is compact for $p \in [2, 2^*)$. See [9, Theorem 4.1].

In order to obtain a nontrivial solution of equation (1) in the zero mass case, previous results always consider such equation in the following form

$$- \Delta u + V(x)u = K(x)f(u), \quad (3)$$

where this function $K : \mathbb{R}^N \to \mathbb{R}$ is used to obtain some compactness results, see [1, 2, 3, 4, 9] and the references therein for instance.
The novelty here is that we will introduce hypotheses on the nonlinearities that allows us to consider examples which do not verify the condition of Ambrosetti-Rabinowitz or certain monotonicity conditions. Also, our approach enables us to include examples which can not be treated as equation (3). Note that we use variational methods involving Cerami sequences where the most difficult part in our argument is to prove that such sequences are bounded.

For instance, we can consider

$$f(x, s) = \frac{a(x, s)}{1 + |x|^q} s \ln(1 + s),$$

where $a : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing function, such that $0 \leq a(x, s) \leq 1$ and $a(x, s) \to 1$ as $s \to \infty$ uniformly in $x$ and $q > N$. An example of such function is $a(x, s) = 2 \pi \arctan ((1 + |x|)s)$. This function $f$ does not verify the classical Ambrosetti-Rabinowitz condition. Another example that verifies our hypotheses is

$$f(x, s) = \frac{b(x, s)}{1 + |x|} (s + 1)^{p-1},$$

where $0 < b_1 < b(x, s) < b_2$ with $b_1$, $b_2$ constants and $2 < p < 2^*$. Notice that $\frac{f(x, s)}{s}$ may not be monotone in $s$ according to the shape of $b(x, s)$. We notice that both examples of nonlinearities can not be written as in (3). It should be noted that the results we will use to handle compactness are mainly included in [9, Theorem 4.1].

Since we are interested in obtaining positive solutions to problem (1), we assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ verifies $f(x, s) > 0$ for $s > 0$ and $f(x, s) = 0$ for $s \leq 0$.

Firstly we introduce our hypotheses on the functional $V$:

$(V_1)$ $V(x) > 0$ for all $x \in \mathbb{R}^N$ is measurable in $\mathbb{R}^N$ and $V(x) \to 0$ almost everywhere (a.e.) as $|x| \to +\infty$;

$(V_2)$ There is a positive measurable function $K : \mathbb{R}^N \to \mathbb{R}$ such that $K \in L^\infty(\mathbb{R}^N)$ and $\omega(x) := K(x)V^{-1}(x) > 0$, satisfies

$$\omega(x) \to 0$$

a.e. as $|x| \to +\infty$.

We notice that if hypotheses $(V_1)$ and $(V_2)$ hold, then we may use some compactness results of [9].

Concerning the function $f$ we assume the following conditions:

$(f_1)$

$$\limsup_{s \to 0} \frac{f(x, s)}{K(x)s} = 0,$$

uniformly for $x \in \mathbb{R}^N$, where $K$ is the function given in hypothesis $(V_2)$;

$(f_2)$

$$\limsup_{s \to +\infty} \frac{f(x, s)}{K(x)s^{p-1}} < +\infty$$

uniformly for $x \in \mathbb{R}^N$, with $p \in (2, 2^*)$;

$(f_3)$

$$\lim_{s \to +\infty} \frac{f(x, s)}{K(x)s} = +\infty,$$

uniformly for $x \in \mathbb{R}^N$;
(f_4) For \( \hat{F}(x,s) := \frac{1}{2} f(x,s)s - F(x,s) \), where \( F(x,s) = \int_0^s f(x,t)dt \), there exist constants \( \tau > \frac{N}{2} \), \( c_0 > 0 \) and \( R_0 > 0 \) such that for all \( s \geq R_0 \),
\[
\left( \frac{f(x,s)}{s} \right)^\tau \leq c_0 K^{\tau-1}(x) \hat{F}(x,s)
\]
and for any \( s_0 < +\infty \), there exists a function \( \psi \in L^1(\mathbb{R}^N) \) (not necessarily positive) such that
\[
\hat{F}(x,s) \geq \psi(x)
\]
for every \( 0 \leq s \leq s_0 \) and \( x \in \mathbb{R}^N \);

(f_5) For all \( 0 < a < b < +\infty \), we have
\[
\limsup_{|x| \to +\infty} \frac{f(x,s)}{sV(x)} = \limsup_{|x| \to +\infty} \frac{f(x,s)}{K(x)} \frac{K(x)}{sV(x)} = 0,
\]
uniformly for \( s \in [a,b] \).

Remark 1. Condition (f_4) will be essential in the proof of the boundedness of the Cerami sequence (see Lemma 3.3 below). To the best of our knowledge, a similar condition to (f_4) was firstly introduced in [7] for a Schrödinger equation in the case of a periodic potential and superlinear nonlinearities. Since we are assuming a different type of superlinearity given by hypothesis (f_3), it is natural to consider a modified condition such as (f_4). We also notice that since \( K \in L^\infty(\mathbb{R}^N) \), it follows that hypothesis (f_4) implies that hypothesis (N4) in [7] holds. To the best of our knowledge, our work is the first to consider an hypothesis such as (f_4) in the context of Schrödinger equations with vanishing potentials.

Remark 2. Our choice for conditions similar to the ones presented in [9, Theorem 4.1] is based on the fact that the compactness result presented in [9] holds for \( p = 2 \) which is essential in the proof of Lemma 3.3 below. For more results concerning these type of embeddings we refer the reader to [2, 4].

Remark 3. Hypotheses (V_2) and (f_5) provide the following relation between the nonlinearity \( f \) and the potential \( V \):
For all \( 0 < a < b < +\infty \), we have
\[
\limsup_{|x| \to +\infty} \frac{f(x,s)}{sV(x)} = \limsup_{|x| \to +\infty} \frac{f(x,s)}{K(x)} \frac{K(x)}{sV(x)} = 0,
\]
uniformly for \( s \in [a,b] \).

Our main result is

**Theorem 1.1.** Suppose that \( V \) satisfies hypotheses (V_1)-(V_2) and \( f \) satisfies (f_1)-(f_5). Then, problem (1) possesses at least one nonnegative solution \( u \). If in addition we suppose that the potential \( V \) is bounded then, problem (1) possesses at least one positive solution \( u \).

Note that in [9, Theorem 1.2], in order to obtain the existence of a positive solution, the author imposes the following assumption: There exists \( \eta \geq 1 \) such that \( \hat{F}(s) \leq \eta \hat{F}(t) \) for all \( 0 \leq s \leq t \), which is a type of monotonicity in \( \hat{F} \) and implies that \( \hat{F} \geq 0 \) for all \( s \geq 0 \). We notice that our assumption in \( \hat{F} \) allows us to produce examples where \( \hat{F}(x,s) < 0 \) for some \( s \in \mathbb{R} \) even in the case when \( f(x,s) = K(x)g(s) \).
The main idea for the proof of Theorem 1.1 is to use variational methods. In order to prove that the functional associated with equation (1) has at least one positive critical point, we first obtain a Cerami sequence of this functional showing that it possesses a mountain pass structure, then we prove that the sequence is bounded, which is the most difficult part of the present paper since, for example, we are not assuming conditions such as Ambrosetti-Rabinowitz, neither monotonicity of $\frac{f(x,s)}{s}$. After that, we prove some convergence results and finally we prove that the Cerami sequence converges to a nontrivial critical point of the functional.

The rest of this paper is organized in the following way: In section 2 we gather some preliminary results and present our variational setting involving Cerami sequences. In section 3, we show that the Cerami sequence is bounded. Finally, in section 4 we prove Theorem 1.1 and present examples of our main result.

2. Preliminary results. In this section we present the main tools in order to prove Theorem 1.1. In [9], the author introduces some conditions in order to prove that the embedding

$$D^1_V(\mathbb{R}^N) \hookrightarrow L^q_K(\mathbb{R}^N)$$

is compact for all $q \in [2, 2^*)$, where

$$D^1_V(\mathbb{R}^N) = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \, dx < +\infty \right\}$$

is endowed with the norm

$$\|u\|^2_{D^1_V(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx$$

and

$$L^q_K(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} ; u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q \, dx < \infty \right\},$$

for some $q > 1$ is the weighted Lebesgue space, where as usual

$$D^{1,2}(\mathbb{R}^N) = \{ u : \mathbb{R}^N \to \mathbb{R} ; u \text{ is measurable and } u \in L^{2^*}(\mathbb{R}^N), |\nabla u| \in L^2(\mathbb{R}^N) \}.$$ See [9, Theorem 4.1]. We notice that under the hypotheses of Theorem 1.1 we can apply Theorem 4.1 of [9] to show that the embedding (4) is compact for $q \in [2, 2^*)$.

Since we are working with the subcritical case we shall work with a smaller space than $D^1_V(\mathbb{R}^N)$. This space is

$$H^1_V(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \, dx < +\infty \right\}.$$ We denote the norm in $H^1_V(\mathbb{R}^N)$ by

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx.$$

The energy functional associated to (1) is given by

$$J(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx - \int_{\mathbb{R}^N} F(x,u) \, dx$$

defined on $E := H^1_V(\mathbb{R}^N)$, where $F(x,t) = \int_0^t f(x,s) \, ds$ is the primitive of $f$. 
Remark 4. By hypotheses \((f_1), (f_2)\) and \((f_5)\) it follows that
\[
f(x, s) \leq K(x)(\varepsilon s + c_\varepsilon s^{p-1}),
\]
where \(\varepsilon > 0\) is sufficiently small and \(c_\varepsilon > 0\) is sufficiently large, for \(p \in (2, 2^*)\).

Remark 5. It follows from Remark 4 that \(f(x, s)\) is bounded in compact sets of \(\mathbb{R}^N \times [0, +\infty)\).

For the convenience of the reader we state here a direct consequence of [9, Theorem 4.1]

**Proposition 1.** Let \(2 \leq t < 2^*\) and \(V, \omega : \mathbb{R}^N \to \mathbb{R}\) be two measurable functions such that \(V(x), \omega(x) > 0\) for \(x \in \mathbb{R}^N\), \(K \in L^r(\Omega)\), some \(r \in \left(\frac{2^*}{2^* - t}, \infty\right)\) and for each subset \(\Omega\) of \(\mathbb{R}^N\) having finite measure \(|\Omega| < \infty\), and such that \(|\{x \in \mathbb{R}^N ; \omega(x) \geq c\}| < \infty\). Then, the embedding \(D_{V}^{1}(\mathbb{R}^N) \hookrightarrow L_{K}^{1}(\mathbb{R}^N)\) is compact.

We notice that the embedding
\[
H_{V}^{1}(\mathbb{R}^N) \hookrightarrow D_{V}^{1}(\mathbb{R}^N)
\]
is continuous and therefore we have the following result, which is a direct consequence of (6) and Proposition 1.

**Corollary 1.** Let \(V\) satisfies hypotheses \((V_1)\) and \((V_2)\). Then the embedding \(H_{V}^{1}(\mathbb{R}^N) \hookrightarrow L_{K}^{q}(\mathbb{R}^N)\) is compact for \(q \in [2, 2^*)\)

Notice that by Remark 4 and the embeddings described above and by standard Dominated Convergence Theorem arguments we have that the functional \(J\) is well defined on \(E\) and \(J \in C^1(E, \mathbb{R})\) with Fréchet derivative given by
\[
J'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} f(x,u)vdx
\]
for any \(u, v \in E\). Therefore, positive solutions of (1) correspond to positive critical points of \(J\) on \(E\).

3. Boundedness of the Cerami sequence. In this section, we obtain a Cerami sequence for the functional (5) and show that this sequence is bounded. Finally, we prove some convergence results which will be essential in the proof of Theorem 1.1.

Let \((X, \| \cdot \|)\) be a real Banach space with dual space \((X^*, \| \cdot \|_*)\), \(I \in C^1(X, \mathbb{R})\) and \(c \in \mathbb{R}\). We call a sequence \(\{x_n\} \subset X\) a Cerami sequence at level \(c\) and denote \((C)_c\) for short, if \(I(x_n) \to c\) and \((1 + \|x_n\|)\|I'(x_n)\|_* \to 0\) as \(n \to \infty\) and we say that \(I\) satisfies the Cerami condition if every \((C)_c\) sequence has a strongly convergent subsequence in \(X\).

The following result is a version of the mountain pass theorem for \((C)_c\) sequences. See [15]. This result states that the mountain pass geometry is sufficient to obtain a \((C)_c\) sequence.

**Theorem 3.1.** Let \(X\) be a real Banach space. Suppose that \(I \in C^1(X, \mathbb{R})\) satisfies \(I(0) = 0\) and
\[
(I_1): \text{ there exist } \rho, \alpha > 0 \text{ such that } I(u) \geq \alpha > 0 \text{ for all } \|u\| = \rho,
\]
\[
(I_2): \text{ there exists } x \in X \text{ with } \|x\| > \rho \text{ such that } I(x) \leq 0.
\]
Then $I$ possesses a $(C)_c$ sequence at level
$$
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
$$
with
$$
\Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = 0 \text{ and } I(\gamma(1)) \leq 0, \text{ where } \gamma(1) = x \}. $

In the next result we prove that the functional $J$ possesses the mountain pass geometry.

**Lemma 3.2.** Suppose that hypotheses $(V_1) - (V_2)$, $(f_1) - (f_3)$ hold. Then the functional $J$ satisfies $(I_1)$ and $(I_2)$.

**Proof.** Firstly, notice that $J(0) = 0$. By Remark 4 and Proposition 1 it follows that
$$
\mathcal{J}(x,u)dx \leq \int_{\mathbb{R}^N} K(x) \left( \frac{\varepsilon}{2} |u|^2 + \frac{C_s}{p} |u|^p \right) dx \leq C_1 \varepsilon |u|^2 + C_2 |u|^p \tag{8}
$$
Thus,
$$
J(u) = \frac{1}{2} |u|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geq \left( \frac{1}{2} - C_1 \varepsilon \right) |u|^2 - C_2 |u|^p \geq \alpha > 0,
$$
for $\|u\| = \rho$ sufficiently small.

Now, we show that there is a $e \in E$ such that $\|e\| > \rho$ and $J(e) < 0$. Hypothesis $(f_3)$ implies that for each $M > 0$, there exists $s_M > 0$ such that $F(x,s) \geq MK(x)s^2$, for $s > s_M$, for every $x \in B(x_0, \delta)$ for some $x_0 \in \mathbb{R}^N$ and $\delta > 0$ with
$$
\int_{B(x_0,\delta)} K(x)dx > 0.
$$
Let $\phi(x) \equiv 1$ in $B(x_0, \delta)$, $\phi(x) \geq 0$, for all $x \in \mathbb{R}^N$ and $\phi \in C^\infty_0(\mathbb{R}^N)$. We claim that there is $R_0 > 0$ such that, for any $R > R_0$, we have $J(R\phi) < 0$. If that is the case, we take $e = R\phi$ with $R > 0$ large enough.

For any $R \geq s_M$, we have
$$
J(R\phi) = \frac{1}{2} R^2 |\phi|^2 - \int_{\mathbb{R}^N} F(x,R\phi)dx
\leq \frac{1}{2} R^2 |\phi|^2 - \int_{B(x_0,\delta)} F(x,R)dx
\leq \frac{1}{2} R^2 |\phi|^2 - \int_{B(x_0,\delta)} MK(x)R^2 dx.
$$
Thus,
$$
J(R\phi) = \frac{1}{2} R^2 |\phi|^2 - \int_{\mathbb{R}^N} F(x,R\phi)dx
\leq \frac{1}{2} R^2 |\phi|^2 - M \int_{B(x_0,\delta)} K(x)dx R^2
\leq \frac{1}{2} |\phi|^2 - M \int_{B(x_0,\delta)} K(x)dx < 0.
$$
Therefore, taking $M$ sufficiently large, we obtain
$$
\frac{1}{2} |\phi|^2 - M \int_{B(x_0,\delta)} K(x)dx < 0
$$
which means that for $R > R_0(s_M)$, $J(R\phi) < 0$ and this concludes the proof. \qed
In the following lemma, we prove that the \((C)_c\)-sequence obtained in the previous result is bounded in \(E\).

**Lemma 3.3.** If \(\{u_n\} \subset E\) is a \((C)_c\)-sequence of \(J\), then \(\{u_n\}\) is bounded in \(E\).

**Proof.** Suppose that \(\{u_n\} \subset E\) is a \((C)_c\)-sequence of \(J\), i.e., \(J(u_n) \to c\) and \((1 + \|u_n\|)\|J'(u_n)\|_{E^*} \to 0\) as \(n \to \infty\). Then,

\[
J(u_n) \to c, \quad J'(u_n) u_n \to 0
\]

as \(n \to \infty\). By (9), for \(n\) sufficiently large,

\[
c + o_n(1) = J(u_n) - \frac{1}{2} J'(u_n) u_n = \int_{\mathbb{R}^N} \hat{F}(x,u_n) dx.
\]

We may assume, by contradiction, that for some subsequence, still denoted by \(\{u_n\}\), \(\|u_n\| \to \infty\). We set

\[
w_n = \frac{u_n}{\|u_n\|}.
\]

Then \(\{w_n\}\) is bounded in \(E\) with \(\|w_n\| = 1\) and up to a subsequence we can assume that

\[
w_n \rightharpoonup w
\]

in \(E\) and by Corollary 1

\[
w_n(x) \to w(x)
\]
a.e. in \(\mathbb{R}^N\).

Notice that

\[
J'(u_n) u_n = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) dx - \int_{\mathbb{R}^N} f(x,u_n) u_n dx
\]

\[
= \|u_n\|^2 - \int_{\mathbb{R}^N} f(x,u_n) u_n dx.
\]

Thus,

\[
\frac{J'(u_n) u_n}{\|u_n\|^2} = 1 - \int_{\mathbb{R}^N} \frac{f(x,u_n) u_n}{\|u_n\|^2} dx.
\]

Therefore,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x,u_n) u_n}{\|u_n\|^2} dx = 1.
\]

Let \(0 \leq a < b \leq +\infty\) and we define

\[
A_n(a,b) = \{ x \in \mathbb{R}^N ; a \leq u_n(x) < b \}.
\]

It follows from (10) that for \(n\) sufficiently large we have

\[
c + o_n(1) = \int_{A_n(a,a)} \hat{F}(x,u_n) dx + \int_{A_n(a,b)} \hat{F}(x,u_n) dx + \int_{A_n(b,\infty)} \hat{F}(x,u_n) dx
\]

Let \(\varepsilon > 0\) be arbitrary and \(C_3 > 0\) be such that \(\|w\|^2_{L^2_k(\mathbb{R}^N)} \leq C_3 \|w\|^2\) for each \(w \in H^1_k(\mathbb{R}^N)\). By \((f_1)\), there exists \(a = a_\varepsilon > 0\) such that \(|f(x,s)| \leq \frac{\varepsilon}{3C_3} K(x)s\) for
each \( |s| \leq a \). Then, for any \( n \in \mathbb{N} \), we have
\[
\int_{A_n(0,a)} \frac{f(x, u_n) u_n}{\|u_n\|^2} \, dx \\
\leq \frac{\varepsilon}{3C_3} \int_{A_n(0,a)} K(x)|u_n| |u_n| \, dx \\
\leq \frac{\varepsilon}{3C_3} \int_{A_n(0,a)} K(x)|w_n| |w_n| \, dx \\
\leq \frac{\varepsilon}{3C_3} C_3 \|w_n\| \|w_n\| = \frac{\varepsilon}{3}
\]
for all \( n \in \mathbb{N} \).

Now we deal with the set \( A_n(b, +\infty) \). Firstly, we notice that by hypotheses \((f_3), (f_4)\) and \((12)\),
\[
\tau = \int_{A_n(b, +\infty)} \hat{F}(x, u_n) \, dx \geq \int_{A_n(b, +\infty)} K(x) \left( \frac{f(x, u_n)}{K(x) u_n} \right)^\tau \, dx \\
\geq c_0 \inf_{s \geq b} \left( \frac{f(x, s)}{s \|K\|_{L^\infty(\mathbb{R}^N)}} \right)^\tau \int_{A_n(b, +\infty)} K(x) \, dx
\]
where the first inequality follows from hypothesis \((f_4)\). Therefore,
\[
\int_{A_n(b, +\infty)} K(x) \, dx \to 0 \tag{13}
\]
as \( b \to +\infty \) uniformly in \( n \).

Hence, we obtain for any \( s \in (1, 2^*) \)
\[
\int_{A_n(b, +\infty)} K(x) |w_n|^s \, dx \\
= \int_{A_n(b, +\infty)} K \frac{s^*}{s^* - s} (x) \hat{F}^\frac{s}{s^*} (x) |w_n|^s \, dx \tag{14}
\]
\[
\leq \left( \int_{A_n(b, +\infty)} K(x) \, dx \right)^{\frac{2^* - s}{2^* - s^*}} \left( \int_{A_n(b, +\infty)} K(x) |w_n|^{2^*} \, dx \right)^{\frac{s}{2^*}} \\
\leq C \left( \int_{A_n(b, +\infty)} K(x) \, dx \right)^{\frac{2^* - s}{2^* - s^*}} \to 0
\]
as \( b \to \infty \) uniformly in \( n \). Thus, by hypothesis \((f_4), (12)\) and \((14)\) it follows that
\[
\int_{A_n(b, +\infty)} f(x, u_n) \frac{u_n}{\|u_n\|^2} \, dx \\
= \int_{A_n(b, +\infty)} f(x, u_n) \frac{|w_n|^2}{u_n} \, dx \\
\leq \int_{A_n(b, +\infty)} K \frac{s-1}{s} (x) \hat{F}^{\frac{s}{2}} (x, u_n) w_n w_n \, dx \\
\leq \left( \int_{A_n(b, +\infty)} K(x) |w_n|^{\frac{2^*}{s-1}} \, dx \right)^{\frac{s-1}{2^*}} \left( \int_{A_n(b, +\infty)} K(x) |w_n|^{\frac{2^*}{s}} \, dx \right)^{\frac{s-1}{2^*}}
\[
\left( \int_{A_n(b, +\infty)} F(x, u_n)dx \right)^{\frac{1}{2}} < \frac{\varepsilon}{3}.
\]

Finally, we deal with the set \( A_n(a, b) \), where \( a \) and \( b \) were chosen in the previous steps. By Remark 3, there is \( R > 0 \) sufficiently large such that
\[
\int_{A_n(a,b)} \frac{f(x, u_n)u_n}{\|u_n\|^2} dx = \left( \int_{A_n(a,b)} \frac{f(x, u_n)}{\|u_n\|^2} V(x)|u_n|^2 dx \right) \leq \frac{\varepsilon R \int_{|x|>R} V(x)|u_n|^2 dx}{6} \leq \frac{\varepsilon}{6} \|u_n\| = \frac{\varepsilon}{6}.
\]

Since \( f(x,s) \) is bounded in compact sets of \( \mathbb{R}^N \times [0, +\infty) \), we have that there exists a constant \( C = C_{a,b,R} > 0 \) such that
\[
\int_{A_n(a,b)} \frac{f(x, u_n)u_n}{\|u_n\|^2} dx \leq \frac{C}{\|u_n\|^2} \leq \frac{\varepsilon}{6}
\]
for \( n \) sufficiently large since we are assuming that \( \|u_n\| \to +\infty \) as \( n \to +\infty \).

Gathering all these informations, we obtain
\[
\int_{\mathbb{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^2} dx < \varepsilon < 1
\]
and this is a contradiction with (11). Therefore, \( \{u_n\} \) is bounded in \( E \) and the lemma is proved.

**Lemma 3.4.** Let \( V \) satisfies hypotheses (V1) – (V2), \( f \) satisfies (f1) – (f2) and \( \{u_n\} \subset E \) be a bounded sequence. If \( u_n \) is such that \( u_n \rightharpoonup u \) in \( E \), then
\[
\int_{\mathbb{R}^N} f(x, u_n)u_n dx \to \int_{\mathbb{R}^N} f(x, u)udx
\]
as \( n \to +\infty \).

**Proof.** As noticed in Remark 4,
\[
f(x,t) \leq K(x)(\varepsilon t + c_t t^{p-1})
\]
for all \( \varepsilon > 0 \) with some sufficiently large \( c_t > 0 \). Since we are assuming that \( u_n \rightharpoonup u \) in \( E \) it follows from Corollary 1 that
\[
\int_{\mathbb{R}^N} f(x, u_n)|u_n - u| dx
\]
\[
\leq \int_{\mathbb{R}^N} K(x)(\varepsilon u_n + c_t u_n^{p-1})|u_n - u| dx
\]
\[
\leq \varepsilon \|u_n\|_{L^2_K(\mathbb{R}^N)} \|u_n - u\|_{L^2_K(\mathbb{R}^N)} + c_t \|u_n\|^{p-1}_{L^p_K(\mathbb{R}^N)} \|u_n - u\|_{L^p_K(\mathbb{R}^N)} \to 0
\]
as \( n \to +\infty \). Also, by Corollary 1 we have that \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \) and \( u_n \to u \) in \( L^t_K(\mathbb{R}^N) \) for all \( t \in [2, 2^*) \), which means that
\[
\int_{\mathbb{R}^N} [K^{\frac{1}{t}}(x)u_n - K^{\frac{1}{t}}(x)u]^t dx \to 0
\]
as \( n \to +\infty \). Thus, by the reciprocal of the dominated convergence theorem (see [5, Theorem 4.9]) we have that up to a subsequence, there exists a function \( h \in L^t(\mathbb{R}^N) \)
for all $t \in [2,2^*)$ such that $|K^\frac{1}{2}(x)u_n(x)| \leq h(x)$. Thus,

$$|f(x,u_n)| \leq \varepsilon K(x)u_n + c_\varepsilon K(x)u_n^{p-1}u = \varepsilon K^\frac{1}{2}(x)u_n K^\frac{1}{2}(x)u + c_\varepsilon (K(x)u_n)^{\frac{p-1}{2}} K^\frac{1}{2}(x)u \leq (\varepsilon h_1(x)K^\frac{1}{2}(x)u + c_\varepsilon h_2^{p-1}(x)K^\frac{1}{2}(x)u) \in L^1(\mathbb{R}^N),$$

where $h_1 \in L^2(\mathbb{R}^N)$ and $h_2 \in L^p(\mathbb{R}^N)$ were obtained by the reciprocal of the dominated convergence theorem. Thus, by the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^N} f(x,u_n)u_n dx \rightarrow \int_{\mathbb{R}^N} f(x,u)u dx \quad (17)$$
as $n \rightarrow \infty$.

Gathering (16) and (17) we obtain

$$\int_{\mathbb{R}^N} f(x,u_n)u_n dx \rightarrow \int_{\mathbb{R}^N} f(x,u)u dx \quad (18)$$
as $n \rightarrow \infty$ and the lemma is proved.

4. **Proof of Theorem 1.1.** In this section we prove Theorem 1.1. We denote by $\{u_n\}$ the Cerami sequence obtained in Lemma 3.2, i.e,

$$J(u_n) \rightarrow c \quad \text{and} \quad (1 + \|u_n\|)||J'(u_n)|| \rightarrow 0.$$

It follows from Lemma 3.3 that $\{u_n\}$ is bounded and thus up to a subsequence, we can assume that there is $u \in E$ such that $u_n \rightharpoonup u$ in $E$.

Since $J'(u_n)u_n = o_N(1)$, we have

$$\lim_{n \rightarrow +\infty} \|u_n\|^2 = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} f(x,u_n)u_n dx$$

and thus by Lemma 3.4 we obtain

$$\lim_{n \rightarrow +\infty} \|u_n\|^2 = \int_{\mathbb{R}^N} f(x,u)u dx. \quad (19)$$

Also, since $J'(u_n)u = o_N(1)$, we obtain

$$\|u\|^2 = \int_{\mathbb{R}^N} f(x,u)u dx. \quad (20)$$

Hence, by (19) and (20),

$$\lim_{n \rightarrow +\infty} \|u_n\|^2 = \|u\|^2$$

which shows that

$$u_n \rightarrow u \quad \text{in} \quad E.$$

Therefore, $u$ is a nontrivial solution of problem (1) with $J(u) = c$. Aplying $u^-(x) = \max\{-u(x),0\}$ as a test function in the weak formulation of problem (1) we conclude that $u$ is a nonnegative solution of (1). Besides that, if we assume that the potential $V$ in (1) is bounded, we obtain as a direct consequence of [8, Theorem 8.18] that $u > 0$ which completes the proof.

**Example 1.** Consider the following equation

$$\left\{ -\Delta u + \left( \frac{1}{1+|x|} \right)^\alpha u = \frac{a(x,u)}{1+|x|}u^{p-1}, \quad x \in \mathbb{R}^N, \right\} \quad (21)$$

where $0 < \alpha < 1$, $p \in (2,2^*)$ and the function $a : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is such that $0 < a_1 < a(x,s) < a_2$ with $\frac{a_1}{2} - \frac{a_2}{p} > 0$. One can take a function $K : \mathbb{R}^N \rightarrow \mathbb{R}$
given by $K(x) = \frac{1}{1 + |x|^q}$. In this case, hypotheses $(V_1) - (V_2)$ are verified. Also, it is easy to check that hypotheses $(f_1) - (f_5)$ are verified, where hypothesis $(f_4)$ is verified with $\tau = \frac{p}{p - 2}$. Therefore, (20) possesses a positive solution. Notice that if the function $a$ oscillates or decreases, then the function $f(x, s) = \frac{a(x, s)}{(1 + |x|)(s^+)^{p - 1}}$ does not verify any monotonicity condition.

**Example 2.** Consider the equation

$$\begin{cases}
-\Delta u + \left(\frac{1}{1 + |x|^q}\right) u = \frac{b(x, u)}{1 + |x|^q} u \ln(1 + |u|), & x \in \mathbb{R}^N
\end{cases}$$

where $q > N$, $0 < \beta < q$, $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $0 \leq b(x, s) \leq 1$ and $b(x, s) \to 1$ as $s \to \infty$ uniformly in $x \in \mathbb{R}^N$. Considering the function $K : \mathbb{R}^N \to \mathbb{R}$ given by $K(x) = \frac{1}{1 + |x|^q}$, we see that hypotheses $(V_1) - (V_2)$ are verified. Also, one can notice that $f(x, s) = \frac{b(x, s)}{1 + |x|^q} s \ln(1 + |s|)$ verifies hypotheses $(f_1) - (f_3)$ and $(f_5)$. In addition, we notice that the function $h : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$ given by $h(x, s) = s \ln(1 + s) \frac{1}{1 + |s|^q}$ verifies hypothesis $(f_4)$ with $K(x) = \frac{1}{1 + |x|^q}$ for every $\tau > \frac{N}{2}$ and since $b(x, s) \leq 1$ it follows that function $f$ verifies hypothesis $(f_4)$.

Therefore, we have that all hypotheses of Theorem 1 are satisfied and it follows that problem (21) possesses a positive solution. One may notice that the function $f(x, s) = \frac{b(x, s)}{1 + |x|^q} s \ln(1 + |s|)$ does not verify the classical Ambrosetti-Rabinowitz condition.

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