THE BETA FUNCTIONS OF A SCALAR THEORY COUPLED TO GRAVITY

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Abstract
We study a scalar field theory coupled to gravity on a flat background, below Planck’s energy. Einstein’s theory is treated as an effective field theory. Within the context of Wilson’s renormalization group, we compute gravitational corrections to the beta functions and the anomalous dimension of the scalar field, taking into account threshold effects.
1. Introduction

Our present understanding of the four known interactions is based on the Standard Model of particle physics and on General Relativity. With the exception of astrophysical phenomena, there seem to be few instances where the full machinery of these theories need to be applied together. In particular, for all the situations which are within experimental reach, the mutual influence between gravitational and particle phenomena is very weak. On one hand, the gravitational fields produced by the elementary particles is exceedingly weak; on the other, the contribution of gravitons to quantum amplitudes are suppressed by powers of momentum over the Planck mass, and therefore are negligible at presently available energies.

In grand unified theories, however, one has to discuss effects occurring at energies which are quite close to the Planck scale. In these situations it is obviously important to be able to estimate the effects due to the gravitational field. As a first step in this direction, we will investigate here the influence of gravity on the renormalization group of a scalar field.

The renormalization group describes the change in the effective action as some external scale \( k \) is varied. The physical meaning of this parameter \( k \) can vary from one problem to another, but in all cases it acts effectively like an infrared cutoff. This point of view originated with work of Wilson and others [1] and has undergone considerable development recently [2]. The particular implementation of these ideas that we shall use here is the so called average effective action, which has been studied, in the case of scalar fields, in [3]. In this paper we will consider the average effective action of a scalar field coupled to gravity, at energies below Planck’s energy. We will assume that in this regime gravity can be described by Einstein’s theory, treated as an effective field theory with a cutoff at the Planck scale. We make two simplifying assumptions: Newton’s constant is assumed not to run, and higher derivative terms are neglected. Both these assumptions will probably be violated as one approaches the Planck energy. A physical picture of the transition from the sub-planckian to the super-planckian regime is given in [4].

This paper is organized as follows. In sections 2 and 3 we define the average effective action for a scalar field, paying due attention to the overall normalization, and compute the beta functions of some of its parameters. In sections 4 and 5 we compute the gravitational corrections to the beta functions of the scalar fields. In section 6 we put together our results and draw our conclusions.

2. Average effective action for a scalar field

In order to illustrate the general method in a simpler setting and also to establish some formulae that will be useful later, we will discuss first the case of a self-interacting scalar field \( \phi \) in the spontaneously broken phase in flat space. The aim of this section is to give the definition of the average effective action \( \Gamma_k \), an action describing accurately the physics of the system at momentum scale \( k \). In particular, we will be interested in the average effective potential \( V_k(\phi) \) and in the “average wave function renormalization” \( Z_k(\phi) \), which are defined by the expansion of \( \Gamma_k \) in powers of derivatives of \( \phi \):

\[
\Gamma_k(\phi) = \int d^4x \left[ V_k(\phi) + \frac{1}{2} Z_k(\phi) \partial_\mu \phi \partial^\mu \phi + \ldots \right].
\]

(2.1)

In the next section we will write the renormalization group equation for some of the parameters appearing in \( V_k \) and \( Z_k \).

Let \( \Gamma(\phi) \) be the usual effective action, which is obtained by means of a Legendre transformation, using the background field method. At one loop it is given by

\[
\Gamma(\phi) = \frac{1}{2} \ln \left( \frac{\text{Det} \mathcal{O}(\phi)}{\text{Det} \mathcal{O}(\phi_0)} \right),
\]

(2.2)

where \( \mathcal{O}(\phi) \) is the operator describing the propagation of small fluctuations around the background \( \phi \), and \( \phi_0 \) is a fixed constant field, that one may conveniently choose as the minimum of the effective potential. The denominator in (2.2) gives the correct overall normalization of the effective action. It does not depend on \( \phi \) and therefore is often disregarded, but it will play a role in what follows.

The determinants in (2.2) can be expanded perturbatively as sums of one loop graphs. Each of these involves a single integration over momenta. We assume that both integrals have been regularized by introducing some ultraviolet cutoff \( \Lambda \). This UV cutoff will not appear explicitly in what follows.
The average effective action $\Gamma_k$ is obtained from $\Gamma$ by introducing some kind of infrared cutoff at momentum $k$. For simplicity one can think at first of a sharp cutoff. There are various ways of dealing with the determinant in the denominator, leading to different definitions of $\Gamma_k$.

For example, if we do not introduce the IR cutoff in the denominator and choose $\phi_0$ to be the minimum of the effective potential $V_0$ at scale $k = 0$, we get an effective potential $V_k$ which satisfies $V_0(\phi_0) = 0$ and whose value at the minimum is a decreasing function of $k$. (Basically this is because one is removing the zero point energy of the modes with $|q| < k$.) The effective potential $V_k(\phi)$ defined in this way cannot be interpreted as the energy density of a translationally invariant vacuum. Instead, it represents the energy density of the system when it is enclosed in a box of size $L = 2\pi/k$. The modes with momentum less than $k$ are removed from the spectrum, so that for very large $k$ the energy density decreases like $k^4$. This is the well known Casimir effect [5]. This definition of the effective action $\Gamma_k$ is therefore relevant in the description of inhomogeneous processes, for example bubble nucleation.

In this paper we will be interested in another definition. We will put the IR cutoff also in the denominator and interpret $\phi_0$ as the minimum of the average potential $V_k$ (from here on we will use the notation $\phi_k$ instead of $\phi_0$, reserving the notation $\phi_0$ for the minimum of the usual effective potential $V = V_0$). With this definition, the minimum of the average potential is $V_k(\phi_k) = 0$, independently of $k$. Physically, one can think that the system has been enclosed in a finite box of side $L > 2\pi/k$ and the limit $L \to \infty$ has been taken, with $k$ fixed. This definition is relevant if we are interested in phenomena which occur in spacetime regions of size $\sim k^{-1}$, but the system is not physically confined therein.

Note that the result of the two procedures differs only in the value of the potential at the minimum. Therefore, it is only in the presence of gravity that the distinction becomes important. We also note that with our definition of the average action the one loop renormalization group equation becomes exact [6]. Even though we will not write down the exact renormalization group in what follows, we will keep in mind that with our definition of the average action the renormalization group equation has a validity that goes beyond the one loop approximation.

The use of a sharp cutoff has certain disadvantages, so we will follow [3] and define the average effective action in the following way. Consider the function

$$P_k(q^2) = \frac{q^2}{1 - f_k^2(q^2)}, \quad (2.3)$$

where $f_k(q^2) = \exp\left(-a\frac{q^2}{b}\right)$, for some constants $a$, $b$. The function $P_k(q^2)$ approaches exponentially the function $q^2$ for $|q| > k$, but tends to $k^2$ (when $b = 1$) or diverges (when $b > 1$) for $q^2 \to 0$. The average effective action $\Gamma_k$ is obtained from the ordinary effective action $\Gamma$ by replacing in the loop integrals the momentum variable $q^\mu$ by $\sqrt{P_k(q^2)}q^\mu$, where $q^\mu = q^\nu/|q|$. The effect is that the propagation of the modes with momenta smaller than $k$ is suppressed. In the limit $b \to \infty$ the function $f_k$ becomes a step function and the modes with $|q| < k$ do not propagate at all. \footnote{If we were to replace $q^2$ by $P_k(q^2)$ in the numerator of (2.2), leaving the denominator alone, one would have $V_0(\phi_0) = 0$ for $k = 0$, but the minimum of the potential would be an increasing function of $k$, because the contribution of the modes with $|q| < k$ is enhanced. The physical meaning of this procedure is not clear.}

Our definition of the effective action is then

$$\Gamma_k = \frac{1}{2} \ln \left(\frac{\text{Det}_k \mathcal{O}(\phi)}{\text{Det}_k \mathcal{O}(\phi_k)}\right), \quad (2.4)$$

where $\phi_k$ is the minimum of $V_k$ and $\text{Det}_k$ is the determinant with the momentum integration modified as described above. With this definition the minimum of the potential is zero for all scales:

$$V_k(\phi_k) = 0. \quad (2.5)$$

On the other hand, the derivatives of $V_k$ with respect to $\phi$ are not affected by the denominator and therefore all the results which were previously obtained for scalar theories (in the absence of gravity) [3] remain valid. For a related discussion of the role of the minimum of the potential in the renormalization group, see [7].
Einstein’s theory is a gauge theory and the definition of the average effective action involves some complications which are not present in the pure scalar case, since an IR cutoff will generally break diffeomorphism invariance. In the case of gauge theories, this point has been discussed in [8,9]. We will not discuss these issues here: we shall follow the approach of [9], where it is shown that using the background field method one can preserve gauge invariance with respect to background gauge transformations.

3. Flow equations

It is obviously impossible to follow the renormalization group flow of all the parameters appearing in the effective action $\Gamma_k$, so we will concentrate our attention on the first few terms in the Taylor expansion of $V_k$ and $Z_k$. We parametrize the action by the position of the minimum of the potential $\phi_k$, the quartic coupling at the minimum $\lambda_k$, and the wave function renormalization at the minimum $\bar{Z}_k$:

$$V'_k(\phi_k) = 0 \quad , \quad \lambda_k = V''_k(\phi_k) \quad , \quad \bar{Z}_k = Z_k(\phi_k) ,$$

where a prime denotes derivative with respect to $\phi^2$. The average action (2.1) is thus approximated by assuming

$$V_k(\phi) = \frac{1}{2} \lambda_k (\phi^2 - \phi^2_k)^2 , \quad Z_k(\phi) = \bar{Z}_k$$

and neglecting all other terms. We will follow the flow of the parameters $\phi_k$, $\lambda_k$ and $\bar{Z}_k$.

The renormalization group describes the change in the effective action as one integrates away fluctuations of the fields with decreasing momentum. The effective action at the scale $k$ is used as classical action in the functional integral giving the effective action at a lower scale. We will therefore assume that the classical action entering in the definition of the path integral has the form (2.1), with classical potential $V(\phi) = \frac{1}{2} \lambda(\phi^2 - \phi^2_{\text{min}})^2$ and $Z(\phi) = \bar{Z}$, a constant. We ignore terms with higher derivatives of the fields, and higher powers of $\phi$ in $V$ and in $Z$. After taking the derivative with respect to $k$ we replace the classical parameters $\phi_{\text{min}}, \lambda$ and $\bar{Z}$ by their running counterparts $\phi_k$, $\lambda_k$ and $\bar{Z}_k$. This is the “renormalization group improvement”.

By taking the derivative of (3.1) with respect to $k$ we get

$$k \frac{\partial \phi^2_k}{\partial k} = - \frac{1}{\lambda_k} \left( k \frac{\partial V'_k}{\partial k} \right) \bigg|_{\phi = \phi_k} \equiv k^2 \gamma(k) ,$$

$$k \frac{\partial \lambda_k}{\partial k} = \left( k \frac{\partial V''_k}{\partial k} \right) \bigg|_{\phi = \phi_k} \equiv \beta(k) ,$$

$$k \frac{\partial \ln \bar{Z}_k}{\partial k} = \frac{k}{\bar{Z}_k} \left( \frac{\partial \bar{Z}_k}{\partial k} \right) \bigg|_{\phi = \phi_k} \equiv \eta(k) .$$

In (3.3b) and (3.3c) we are neglecting terms $V''_k(\phi_k)k^2 \frac{\partial^2 \phi}{\partial k^2}$ and $Z'_k(\phi_k)k^2 \frac{\partial^2 \phi}{\partial k^2}$, which take into account the variation in the point of definition of $\lambda_k$ and $\bar{Z}_k$. This is in accordance with the approximations (3.2).

In order to obtain explicit expressions for the beta functions $k^2 \gamma$ and $\beta$ and the anomalous dimension $\eta$, we have to write first the expressions for $V_k$ and $Z_k$. If we expand the action around a classical field $\phi_{\text{cl}}$, the small fluctuation operator has the form

$$\mathcal{O}(\phi_{\text{cl}}) = - \bar{Z} \partial^2 + 6 \lambda \phi_{\text{cl}}^2 - 2 \lambda \phi_{\text{min}}^2 .$$

In order to calculate the effective potential one chooses a constant classical field $\phi_{\text{cl}} = \bar{\phi}$; then

$$V_k(\bar{\phi}) = \frac{1}{\Omega} \bar{\Omega}_k(\bar{\phi}) = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \ln \left( \frac{\bar{Z} P_k(q^2) + 6 \lambda \bar{\phi}^2 - 2 \lambda \phi_{\text{min}}^2}{Z P_k(q^2) + 4 \lambda \phi_{\text{min}}^2} \right) ,$$

where $\Omega$ is the spacetime volume. The beta functions which are defined in the r.h.s. of (3.3a,b) can be obtained by deriving (3.5) and then replacing the classical parameters $\phi_{\text{min}}, \lambda$ and $\bar{Z}$ by their running
can be obtained by computing the effective action for a nonconstant background. We choose

\[ \gamma(k) = \frac{1}{32\pi^2} \frac{1}{k^2} \int dx \frac{6\bar{Z}_k}{(Z_k P_k + 4\lambda_k \bar{\phi}_k^2)^2} \frac{\partial P_k(x)}{\partial k} = \frac{3}{16\pi^2} \frac{1}{Z_k} I_{-2} \left( \frac{4\lambda_k \bar{\phi}_k^2}{Z_k} \right), \]  

(3.6a)

\[ \beta(k) = \frac{1}{32\pi^2} \int dx \frac{72\bar{Z}_k^2 \lambda_k^2}{(Z_k P_k(x) + 4\lambda_k \bar{\phi}_k^2)^3} \frac{\partial P_k}{\partial k} = \frac{9}{4\pi^2} \frac{\lambda_k^2}{Z_k^2} I_{-3} \left( \frac{4\lambda_k \bar{\phi}_k^2}{Z_k} \right), \]  

(3.6b)

where \( x = |q|^2, P_k = P_k(x) \) and

\[ k^{2(n+3)} I_n(w) = \int dx (P_k + w)^n \frac{\partial P_k}{\partial k}. \]  

(3.7)

These integrals are related to the integrals \( L_n^4(w) \) used in [3,9] by \( I_n(w) = L_n^4(w)/(n+1) \). Since \( k \frac{\partial P_k}{\partial k} \) is peaked at \( k^2 \) and goes to zero exponentially for \( x \to \infty \) and as a power for \( x \to 0 \), these integrals are automatically UV and IR convergent.

Let us now derive the expression for the anomalous dimension \( \eta \). The wave function renormalization can be obtained by computing the effective action for a nonconstant background. We choose

\[ \phi_{cl}(x) = \phi + \epsilon \cos(p \cdot x), \]  

(3.8)

with \( \phi \) constant. The wave function renormalization constant is

\[ Z_k(\bar{\phi}) = \lim_{\epsilon \to 0} \lim_{\epsilon \to 0} \frac{2}{\Omega} \frac{\partial}{\partial p^2} \frac{\partial^2}{\partial \epsilon^2} \Gamma_k(\phi_{cl}). \]  

(3.9)

The small fluctuation operator (3.4) is represented in momentum space by the kernel

\[ M(q,q') = M_0(q,q') + \epsilon M_1(q,q') + \epsilon^2 M_2(q,q'), \]  

(3.10)

where

\[ M_0(q,q') = \bar{Z}q'^2 + 6\lambda\bar{\phi}^2 - 2\lambda\phi_{\text{min}}^2 \delta(q + q'), \]
\[ M_1(q,q') = 6\bar{\lambda}(\delta(q+q'+p) + \delta(q+q'-p)) , \]
\[ M_2(q,q') = 3\lambda \delta(q+q') + \frac{3}{2} \lambda \delta(q+q'+2p) + \delta(q+q'-2p) . \]  

(3.11)

We have

\[ \ln \text{Det} \mathcal{O} = \text{Tr} \ln M = \text{Tr} \ln M_0 + \epsilon \text{Tr} M_0^{-1} M_1 + \epsilon^2 \left( \text{Tr} M_0^{-1} M_2 - \frac{1}{2} \text{Tr} M_0^{-1} M_1 M_0^{-1} M_1 \right) + \ldots \]  

(3.12)

where \( \text{Tr} \) denotes the functional trace. The determinants appearing in (2.4) are obtained by replacing \( \text{Tr} \) with \( \text{Tr}_k \), a functional trace in which the momentum integrations are modified as described in the previous section. The first term in (3.12) then reproduces the potential (3.5). The term of order \( \epsilon \) and the first term of order \( \epsilon^2 \) are zero. The remaining term gives, after some manipulations,

\[ Z_k(\bar{\phi}) = -72\lambda^2 \bar{\phi}^2 \lim_{\nu \to 0} \frac{\partial}{\partial \nu} \int d^4q \frac{1}{(2\pi)^4} \frac{(ZP_k(q') + 6\lambda\phi^2 - 2\lambda\phi_{\text{min}}^2)(Z(\sqrt{\nu}q + p)^2 + 6\lambda\phi^2 - 2\lambda\phi_{\text{min}}^2)}{((ZP_k(q') + 6\lambda\phi^2 - 2\lambda\phi_{\text{min}}^2)^2} , \]  

(3.13)

We have used the fact that upon symmetric integration, the integral in the first line reduces to a function of \( p^2 \) only. One can then use \( \frac{\partial}{\partial \nu} = \frac{1}{\lambda} \frac{\partial}{\partial p^2} \frac{\partial}{\partial \nu} \). This definition of \( Z_k \) differs from the one used in [3], where \( P_k((q + p)^2) \) is used instead of \( (\sqrt{P_k}q + p)^2 \). It simplifies the calculations considerably and leads practically to the same final results. This simplification will be important in Section 5.
According to (3.3c), the anomalous dimension is obtained by deriving (3.13) with respect to \( k \), replacing \( \phi_{\text{min}}, \lambda \), and \( \tilde{Z} \) by their running counterparts \( \phi_k, \lambda_k \) and \( \tilde{Z}_k \), and setting also \( \phi = \phi_k \):

\[
\eta(k) = -\frac{72\lambda_k \tilde{Z}_k \phi_k^4}{\pi^2} \int dx \frac{1}{(Z_k P_k(x) + 4\lambda_k \phi_k^2)\phi} \frac{\partial P_k}{\partial k} = -\frac{72\lambda_k \phi_k^4}{\pi^2 \tilde{Z}_k^2 k^4 I_{-5}} \left( 4\lambda_k \phi_k^2 \right). \tag{3.14}
\]

It is convenient to use the rescaled field variables \( \bar{\phi}(x) = \sqrt{Z_k} \phi(x) \) and the rescaled coupling constant \( \bar{\lambda}_k = \tilde{Z}_k^{-2} \lambda_k \). The equations (3.3) can be rewritten

\[
k \frac{\partial \bar{\phi}_k^2}{\partial k} = \eta_k \bar{\phi}_k^2 + \frac{3}{16\pi^2} k^2 I_{-2} \left( 4\bar{\lambda}_k \bar{\phi}_k^2 \right), \tag{3.15a}
\]
\[
k \frac{\partial \bar{\lambda}_k}{\partial k} = -2\eta \bar{\lambda}_k + \frac{9}{4\pi^2} \bar{\lambda}_k^2 I_{-3} \left( 4\bar{\lambda}_k \bar{\phi}_k^2 \right), \tag{3.15b}
\]
\[
k \frac{\partial \ln \tilde{Z}_k}{\partial k} = -\frac{72\lambda_k \phi_k^4}{\pi^2 k^4} I_{-5} \left( 4\lambda_k \phi_k^2 \right). \tag{3.15c}
\]

Note that \( \tilde{Z}_k \) does not appear in the r.h.s. anymore.

It is not possible to give a solution of the resulting system of p.d.e.’s in closed form. However, analytic solutions can be obtained for \( k^2 \) very large or very small. Let us consider first the anomalous dimension \( \eta \). Since the function \( k^2 \partial I_{-2}/\partial k \) is peaked at \( q \approx k \) and goes to zero exponentially for large \( q \), and as a power for small \( q \), the main contribution to the integrals (3.6) and (3.14) comes from the region \( q \approx k \), where \( P_k(q^2) \approx k^2 \). In this region, for \( k^2 \gg \phi_k^2 \), we can neglect \( \phi_k^2 \) with respect to \( P_k \). One can therefore approximately evaluate \( I_{-5} \) at \( \phi_k = 0 \), so the integral gives only a numerical coefficient. We see that \( \eta = -\frac{72\lambda_k^2 I_{-5}(0) \phi_k^4}{128\pi^2 \phi_k^2} \ll 1 \). In the case \( k^2 \ll \phi_k^2 \), by a similar reasoning, we can neglect the term \( P_k \) with respect to \( \bar{\phi}_k \) in the denominator. In this regime \( \eta = -\frac{9\lambda_k(0)}{128\pi^2 \phi_k^2} \ll 1 \). So in both regimes the anomalous dimension \( \eta \) is small. This is in accordance with the analysis of the exact RG given in [6].

Let us now consider the equations (3.15a,b). Assume again that for large \( k^2 \) the mass terms in the denominators can be neglected with respect to factors \( P_k \). Neglecting also the anomalous dimension, the equations (3.15) then reduce to the following:

\[
k \frac{\partial \bar{\phi}_k^2}{\partial k} = \frac{3I_{-2}(0)}{16\pi^2} k^2, \tag{3.16a}
\]
\[
k \frac{\partial \bar{\lambda}_k}{\partial k} = \frac{9I_{-3}(0)}{4\pi^2} \bar{\lambda}_k^2, \tag{3.16b}
\]

Using that \( I_{-3}(0) = 1 \), independently of \( a \) and \( b \), (3.16) gives the usual logarithmic running of the quartic coupling at high energies, whereas \( \phi_k \) scales like \( k \), as dimensional arguments would suggest.

This last result seems to invalidate the approximation \( k^2 \gg \phi_k^2 \) and to cast doubt on the consistency of these results. However, if we write \( \phi_k^2 = c k^2 \), with \( c \) a constant, and insert in (3.6,14), it is easy to see that the conclusions of the previous analysis are confirmed; only the numerical coefficients appearing in (3.16) would be modified.

Let us now consider the opposite limit: \( k^2 \ll \phi_k^2 \). This is the limit \( k \to 0 \), when \( \phi_k \neq 0 \). The beta functions become \( \gamma(k) = \frac{3a}{256\pi^2} k \phi_k^2 \ll 1 \) and \( \beta(k) = \frac{9a}{256\pi^2} \frac{k^6}{\lambda_k \phi_k^2} \ll 1 \). In equations (3.15a,b) the anomalous dimension terms are of the same order as the other terms. We get

\[
k \frac{\partial \bar{\phi}_k^2}{\partial k} = -\frac{15I_0(0)}{256\pi^2} \frac{1}{\lambda_k^2 \phi_k^4} k^6, \tag{3.17a}
\]
\[
k \frac{\partial \bar{\lambda}_k}{\partial k} = \frac{45I_0(0)}{256\pi^2} \frac{1}{\lambda_k \phi_k^6} k^6. \tag{3.17b}
\]
The linearized Euclidean action is a quadratic form which can be written, after Fourier transforming (we decompose the tensor $h$
\[\tilde{\phi}_k^2 = \phi_0^2 \left[ 1 - \frac{5I_0}{512\pi^2} \frac{1}{\lambda_0^2} \phi_0^2 \right], \quad (3.18a)\]
\[\tilde{\lambda}_k = \tilde{\lambda}_0 \left[ 1 + \frac{15I_0}{512\pi^2} \frac{1}{\lambda_0^2} \phi_0^2 \right]. \quad (3.18b)\]

4. The effect of gravity on the beta functions

Let us now turn the gravitational field on. We add to the action the Einstein-Hilbert term, so the total classical action becomes
\[S(\phi, g) = \int d^4x \sqrt{g} \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) + \kappa R \right], \quad (4.1)\]
where $\kappa = (1/16\pi G)$ and $V(\phi) = \frac{1}{2} \lambda (\phi^2 - \phi_{\min}^2)$. We treat the metric as a quantum field, but we will not take into account the running of Newton's constant.

We expand the fields $\phi$ and $g_{\mu \nu}$:
\[\phi = \phi_{cl} + \delta \phi; \quad g_{\mu \nu} = \delta_{\mu \nu} + \frac{1}{\sqrt{\kappa}} h_{\mu \nu}. \quad (4.2)\]

The linearized Euclidean action is a quadratic form which can be written, after Fourier transforming (we use $\partial_\mu \rightarrow iq_\mu$)
\[S^{(2)}(\psi, \omega, \sigma; \phi) = \frac{1}{2} \int d^4q \int d^4q' \sum_{A, B} \Phi_A(q) \cdot \mathcal{O}_{AB} \cdot \Phi_B(q') , \quad (4.3)\]
where the indices $A, B$ label the two types of fields $\Phi_1 = h_{\mu \nu}$, $\Phi_2 = \delta \phi$ and the dots stand for contraction over the tensor indices. When written out explicitly in terms of the components of the fields, $\mathcal{O}$ is a $11 \times 11$ matrix. When $V = 0$, this linearized action is invariant under linearized gauge transformations. Let $x^\mu = x^\mu - v^\mu$ be an infinitesimal coordinate transformation. The variations of the fields are
\[\delta g_{\mu \nu} = \partial_\mu v_\nu + \partial_\nu v_\mu, \quad \delta \phi = 0. \quad (4.4)\]

There follows that the fields
\[h_{\mu \nu} = i(q_\mu v_\nu + q_\nu v_\mu), \quad \delta \phi = 0 \quad (4.5)\]
are null vectors for the operator $\mathcal{O}$. We choose the following gauge-fixing term:
\[S_{GF} = \frac{1}{2\alpha} \int d^4x \partial_\mu h^\mu \partial^\sigma h_{\sigma \nu}. \quad (4.6)\]

In this gauge the ghost determinant is field independent, so it will be neglected in what follows.

To compute the one-loop effective action one now needs to calculate the functional determinant of the operator $\mathcal{O}$ appearing in the previous formulas and (3.2). It is convenient to use the method of the spin projector operators. Choose a coordinate system such that $x^L$ is in the direction of the momentum and $x^i$ are transverse coordinates. In these coordinates the momentum has components $q^\mu = (|q|, 0, 0, 0)$. One can decompose the tensor $h_{\mu \nu}$ into the fields $h_{LL}$, $h_{L(i)}$, $h = \sum_k h_{kk}$ and $h_{ij} = h_{ij} - \frac{1}{2} \delta_{ij} h$, carrying spin and parity $0^+, 1^-, 0^+$ and $2^+$ respectively. The field $\delta \phi$ obviously has spin-parity $0^+$. A complete set of spin projectors for this system is listed in the Appendix.

The total linearized quadratic action, including the gauge-fixing, ghost and potential terms, can be rewritten as
\[S^{(2)} = \frac{1}{2} \int d^4q \, \Phi_A(-q) \cdot a_{ij}^{AB}(J^\mu) \cdot P_{ij}^{AB}(J^\nu) \cdot \Phi_B(q), \quad (4.7)\]
where \(a_{ij}^{AB}(J^P)\) are coefficient matrices, representing the inverse propagators of each set of fields with definite spin and parity. They are

\[
a(2^+) = -\frac{1}{2} \left( q^2 + \frac{V}{\kappa} \right),
\]

\[
a(1^-) = \frac{1}{2} \left( \frac{1}{\alpha} q^2 - \frac{V}{\kappa} \right),
\]

\[
a(0^+) = \begin{bmatrix}
q^2 + \frac{1}{4\kappa} V & \frac{\sqrt{3}}{4\kappa} V & \sqrt{\frac{3}{\kappa}} \phi V' \\
\frac{\sqrt{3}}{4\kappa} V & \frac{1}{\alpha} q^2 - \frac{1}{4\kappa} V & \frac{\sqrt{\frac{3}{\kappa}}}{V} \phi V' \\
\sqrt{\frac{2}{\kappa}} \phi V' & \frac{1}{\sqrt{\kappa}} \phi V' & Z q^2 + 2V' + 4\phi^2 V''
\end{bmatrix}.
\]

The matrix elements of \(a(0^+)\) refer to the fields \(h_{LL}, h\) and \(\delta \phi\), in this order. Taking into account the multiplicity of these contributions, the one-loop effective potential is now

\[
V_k(\phi) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left[ 5 \ln \left( \frac{a_k(2^+)(\phi)}{a_k(2^+)(\phi_k)} \right) + 3 \ln \left( \frac{a_k(1^-)(\phi)}{a_k(1^-)(\phi_k)} \right) + \ln \left( \frac{\det a_k(0^+)(\phi)}{\det a_k(0^+)(\phi_k)} \right) \right],
\]

where the modified inverse propagators \(a_k\) are obtained from the \(a_k\)'s given in (4.8) by replacing \(q^2\) with \(P_k(q^2)\).

Proceeding as in the previous section we find for the beta functions

\[
\gamma(k) = \frac{1}{32\pi^2} \frac{1}{k^2} \int dxx \frac{6\bar{Z}_k}{(Z_k P_k(x) + 4\lambda_k \phi_k^2)^2} \frac{\partial P_k}{\partial k},
\]

\[
\beta(k) = \frac{1}{32\pi^2} \int dxx \frac{1}{4\kappa P_k(x)^2(Z_k P_k(x) + 4\lambda_k \phi_k^2)^3} \left[(13\alpha - 21)\bar{Z}_k^2 \lambda_k P_k(x)^3 + 4\bar{Z}_k \lambda_k^2 (72\kappa + (43\alpha - 51)\bar{Z}_k \phi_k^2) P_k(x)^2 + 720(\alpha - 1)\bar{Z}_k \lambda_k^3 \phi_k^2 P_k(x) + 960(\alpha - 1)\lambda_k^4 \phi_k^6 \right] \frac{\partial P_k}{\partial k}.
\]

Note that (4.10a) is identical to (3.6a) and the term containing \(\kappa\) in the numerator of (4.10b) reproduces the beta function of the pure scalar theory, (3.6b).

5. The effect of gravity on the wave function renormalization

We have to compute the wave function renormalization constant \(\bar{Z}_k\) in the presence of gravitons. It is given again by (3.9), with the effective action now including the effect of graviton loops. The calculation begins with the expansion of the classical action (4.1) around the background (4.2), with \(\phi_{cl}\) now given by (3.8). The linearized action reads

\[
S^{(2)} = \frac{1}{2} \int dx \left\{ h_{\mu\nu} \left[ \left( \delta_{\mu\nu} \partial_{\rho} \partial_{\sigma} - \frac{1}{2} \delta_{\mu\nu} \delta_{\rho\sigma} \partial^2 - \delta^{\rho\sigma} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \delta_{\mu\rho} \delta_{\nu\sigma} \partial^2 \right) \right] \\
+ \left( \frac{Z}{2\kappa} (\partial \phi_{cl})^2 + \frac{V}{\kappa} \right) \left[ \frac{1}{4} \left( \delta_{\mu\rho} \delta_{\nu\sigma} - \frac{1}{2} \delta_{\mu\nu} \delta_{\rho\sigma} \right) - \frac{Z}{2\kappa} \delta_{\mu\rho} \partial_{\phi_{cl}} \partial_{\sigma} \phi_{cl} + \frac{Z}{\kappa} \delta_{\mu\rho} \phi_{cl} \partial_{\sigma} \phi_{cl} \right] h_{\rho\sigma} \\
+ h_{\mu\nu} \left[ Z \delta_{\mu\rho} \partial_{\phi_{cl}} \partial_{\rho} - 2Z \delta_{\mu\rho} \phi_{cl} \partial_{\rho} + \delta_{\mu\rho} \frac{dV}{d\phi} \right] \delta_{\phi_{cl}} \\
+ \delta_{\phi} \left[ -Z \partial^2 + \frac{d^2V}{d\phi^2} \right] \delta_{\phi_{cl}} \right\}
\]

Next one has to use the Taylor expansion of \(V, \frac{dV}{d\phi}\) and \(\frac{d^2V}{d\phi^2}\) around \(\phi_{cl}\), keeping terms up to order \(\epsilon^2\) (this involves derivatives of \(V\) up to fourth order). Upon Fourier transforming, the result can be recast in the form
The operator $\mathcal{O}$ contains again terms up to second order in $\epsilon$, and can be written as in (3.10), where $M_0$, $M_1$ and $M_2$ are now matrices in the space of the fields $h_{\mu\nu}$ and $\delta \phi$. The coefficients $M$ are conveniently displayed as matrices of the form

$$M = \begin{bmatrix} M_{\mu\nu} \rho\sigma & M_{\mu\nu} \rho'\sigma' \\ M_{\mu\nu} \rho'\sigma & M_{\mu\nu} \rho\sigma' \end{bmatrix},$$

where the dots label the entries that correspond to $\delta \phi$. It is convenient to write $M_0(q, q') = \delta(q + q') \tilde{M}_0(q')$, with

$$\tilde{M}_0 = a(2^+) P(2^+) + a(-1^+) P(1^-) + a_{ij}(0^+) P_{ij}(0^+) ,$$

where $i, j = 1, 2, 3$ refer to $h_{LL}$, $h$ and $\delta \phi$. The matrix structure (5.2) is carried by the projectors. Not that (5.3) is the kernel which appears in (4.7). The advantage of this way of writing is that $M_0^{-1}(q', q'') = \delta(q' + q'') \tilde{M}_0(q')^{-1}$, where

$$\tilde{M}_0^{-1} = a(2^+)^{-1} P(2^+) + a(-1)^{-1} P(1^-) + a_{ij}(0^+)^{-1} P_{ij}(0^+) ,$$

$a^{-1}$ being the inverses of the matrices given in (4.8). We can write

$$M_1(q, q') = M_1^+(q') \delta(q + q' + p) + M_1^-(q') \delta(q + q' - p) ,$$

where

$$M_1^+(q')_{\mu\nu} \rho\sigma = \frac{1}{2\kappa} \frac{dV}{d\delta \phi} \left( \frac{1}{4} \delta_{\mu\nu} \delta_{\rho\sigma} - \frac{1}{2} \delta_{\rho\mu} \delta_{\nu\sigma} \right),$$

$$M_1^+(q')_{\mu\nu} \rho'\sigma' = \frac{1}{2\sqrt{\kappa}} \left( Zp_{\mu} q_{\nu}' - \frac{1}{2} Z \delta_{\mu\nu} p \cdot q' + \frac{1}{2} d^2 V \delta_{\mu\nu} \right),$$

$$M_1^+(q') \rho\sigma = \frac{1}{2\sqrt{\kappa}} \left( -Zp_{\rho} p^\sigma + \frac{1}{2} Z \delta_{\rho\sigma} p^2 - Z p^\rho q_{\sigma} + \frac{1}{2} \delta_{\rho\sigma} p \cdot q' + \frac{1}{2} d^2 V \delta_{\rho\sigma} \right),$$

$$M_1^+(q') \rho'\sigma' = \frac{1}{2 d\delta \phi^3},$$

and

$$M_1^-(q')_{\mu\nu} \rho\sigma = \frac{1}{2\kappa} \frac{dV}{d\delta \phi} \left( \frac{1}{4} \delta_{\mu\nu} \delta_{\rho\sigma} - \frac{1}{2} \delta_{\rho\mu} \delta_{\nu\sigma} \right),$$

$$M_1^-(q')_{\mu\nu} \rho'\sigma' = \frac{1}{2\sqrt{\kappa}} \left( -Zp_{\mu} q_{\nu}' - \frac{1}{2} Z \delta_{\mu\nu} p \cdot q' - \frac{1}{2} d^2 V \delta_{\mu\nu} \right),$$

$$M_1^-(q') \rho\sigma = \frac{1}{2\sqrt{\kappa}} \left( -Zp_{\rho} p^\sigma + \frac{1}{2} Z \delta_{\rho\sigma} p^2 + Z p^\rho q_{\sigma} - \frac{1}{2} \delta_{\rho\sigma} p \cdot q' - \frac{1}{2} d^2 V \delta_{\rho\sigma} \right),$$

$$M_1^-(q') \rho'\sigma' = \frac{1}{2 d\delta \phi^3},$$

All derivatives of $V$ are evaluated at $\tilde{\phi}$. Finally

$$M_2(q, q') = \tilde{M}_2(q') \delta(q + q') + \tilde{M}_2(q') (\delta(q + q' + 2p) + \delta(q + q' - 2p)) ,$$

where

$$\tilde{M}_2(q')_{\mu\nu} \rho\sigma = \frac{1}{2\kappa} \left( Z \delta_{\mu\nu} p^\rho p^\sigma - \frac{1}{2} Z \delta_{\mu\nu} p^\rho p^\sigma \right) + \frac{1}{4} \left( Z p^2 + \frac{d^2 V}{d\delta \phi^2} \right) \left( \frac{1}{4} \delta_{\mu\nu} \delta_{\rho\sigma} - \frac{1}{2} \delta_{\rho\mu} \delta_{\nu\sigma} \right),$$

$$\tilde{M}_2(q')_{\mu\nu} \rho'\sigma' = \frac{1}{2 d\delta \phi^3},$$

$$\tilde{M}_2(q') \rho\sigma = \frac{1}{2 d\delta \phi^3},$$

$$\tilde{M}_2(q') \rho'\sigma' = \frac{1}{2 d\delta \phi^3},$$

and

$$\tilde{M}_2(q') \rho^\mu \rho' \sigma' = \frac{1}{8 \sqrt{\kappa} \delta \phi^3} \delta_{\mu\nu} \rho\sigma.$$
As in section 3, \( \text{Tr} M^{-1}_0 M_1 = 0 \) and \( \text{Tr} M^{-1}_0 M_2 = 0 \). (The explicit form of \( \tilde{M}_2 \) is not needed to obtain the latter result). The remaining term entering in \( Z_k \) is

\[
\text{Tr} M^{-1}_0 M_1 M^{-1}_0 M_1 = 2\Omega \int \frac{d^4q}{(2\pi)^4} \text{tr} \left( M^{-1}_1(q)\tilde{M}^{-1}_0(q)M^+_1(q-p)\tilde{M}^{-1}_0(q-p) \right),
\]

where \( \text{tr} \) denotes the trace over the matrix indices in the sense of (5.2). There remains to take the second derivative of this expression with respect to \( p^\mu \) and evaluate at \( p^\mu = 0 \). This is the most tedious part of the calculation. It is made slightly easier by considering only \( \phi = \phi_{\text{min}} \). Then \( V = V' = 0 \) and the matrix (4.8c) becomes diagonal. Furthermore in (5.6-7) one can put \( \frac{dV}{d\phi} = 0 \), \( \frac{d^2V}{d\phi^2} = 4\lambda \phi^2_{\text{min}} \), \( \frac{d^3V}{d\phi^3} = 12\lambda \phi_{\text{min}} \), \( \frac{d^4V}{d\phi^4} = 12\lambda \).

The final result is

\[
Z_k(\phi_{\text{min}}) = -\frac{1}{4\kappa} \int \frac{d^4q}{(2\pi)^4} \left[ \lambda \frac{Z^2\phi^2_{\text{min}}}{ZP_k(q^2) + 4\lambda \phi^2_{\text{min}}} \left( \frac{13\alpha + 3}{8} \frac{Z\phi^2_{\text{min}}}{P_k(q^2)^2} - 4(3 + \alpha) \frac{\lambda^2 \phi^6_{\text{min}}}{P_k(q^2)^2} \right) + \frac{\alpha - 3}{2} \right.
\]

\[
\left. + \frac{32(3 + \alpha)\lambda^3 Z^4 \phi^6_{\text{min}}}{(ZP_k(q^2) + 4\lambda \phi^2_{\text{min}})^2} - \frac{1152\lambda^5 Z^6 \phi^4_{\text{min}}}{(ZP_k(q^2) + 4\lambda \phi^2_{\text{min}})^3} \right] \cdot
\]

Note that the last term reproduces the result (3.13) of a pure scalar theory. The remaining terms are all of order \( 1/\kappa \). The corresponding anomalous dimension is

\[
\eta(k) = \frac{1}{64\pi^2k} \int dx \left[ \frac{1}{8P_k(x)^3(ZP_k(x) + 4\lambda \phi^2_k)^2} \right. \left( 13\alpha + 3 \right) Z^5 P_k(x)^6 + 12(19\alpha + 9)\lambda_k Z^4 \phi^2_k P_k(x)^5
\]

\[
+ 48(31\alpha + 21)\lambda^2_k Z^3 \phi^4_k P_k(x)^4 + 64(49\alpha - 9)Z_k\phi^3_k - 576\lambda_k^3 Z_k \phi^4_k P_k(x)^3
\]

\[
- 2560(\alpha + 9)\lambda^4_k Z_k \phi^6_k P_k(x)^2 - 2048(7\alpha + 27)\lambda^6_k \phi^{10}_k P_k(x) - 16384(\alpha + 3)\lambda^6 \bar{Z}^{-1}_k \phi^{12}_k \right] \frac{\partial P_k}{\partial k}.
\]

The term containing \( \kappa \) in the numerator reproduces the anomalous dimension of the pure scalar theory, (3.14).

6. Discussion

The renormalization group equation for the average effective action of the scalar fields, taking into account the graviton contribution, is given by (3.3), with the beta functions (4.10) and anomalous dimension (5.12). Because there are now two mass scales in this problem, the behaviour is more complicated than in the pure scalar case: intermediate mass scales appear.

Let us discuss first the anomalous dimension. There are four relevant mass scales: \( \phi^{4/3}_k \ll \phi^2_k ^{1/3} \ll \kappa \), dividing the energy range from zero to the Planck mass in four domains.

For \( \phi^{4/3} \ll \kappa \), the first term in the numerator and in the denominator of (5.12) dominate. For \( \phi^2 \ll \kappa \), the term containing \( \kappa \) in the numerator dominates over the first term, which is of order \( P_k^6 \approx k^{12} \). For \( \phi^{8/3} \ll \kappa \), the term containing \( \kappa \) in the numerator dominates over the last term, which is of order \( \phi^{12} \), while the \( \phi^2 \) term dominates in the denominator. Finally, for \( k^2 \ll \phi^{8/3} \kappa^{-1/3} \) the terms with the highest power of \( \phi_k \) dominate.

\[
\phi^{4/3}_k \ll k^2 \ll \kappa \quad \eta = \frac{(13\alpha + 3)}{512\pi^2} L_2(0) \frac{k^2}{\kappa} \ll 1,
\]

\[
\phi^2_k \ll k^2 \ll \phi^{4/3}_k \ll \kappa \quad \eta = -\frac{72\lambda^3}{\pi^2} L_5(0) \frac{\phi^4_k}{k^4} \ll 1,
\]

\[
\phi^{8/3}_k \ll k^2 \ll \phi^2_k \quad \eta = -\frac{9}{8\lambda^2_k \pi^2} L_6(0) \frac{k^6}{\phi^6_k} \ll 1,
\]

\[
k^2 \ll \phi^{8/3}_k \ll \kappa \quad \eta = -\frac{(\alpha + 3)\lambda_k}{2\pi^2} L_3(0) \frac{\phi^2_k}{\kappa} \ll 1.
\]
In all cases, the anomalous dimension is small.

We have already observed that the function $\gamma$ describing the running of the v.e.v. of the scalar field is not modified by the presence of gravity. The running of the v.e.v. of the field is given again by (3.16a) for $k^2 \gg \phi_k^2$ and by (3.17a) for $k^2 \ll \phi_k^2$.

In discussing the function $\beta(k)$ one has to distinguish three regimes, separated by the scales $\phi_k^3 \kappa^{-1/2} \ll \phi_k^2 \ll \kappa$. For $\phi_k^2 \ll k^2 \ll \kappa$ the dominant terms in (4.10b) are the ones with the highest power of $P_k$. Expanding (4.10b) to first order in $k^2/\kappa$ we get

$$\beta(k) = \frac{9\lambda_k^2}{4\pi^2 Z_k^2} + \frac{(13\alpha - 21)}{128\pi^2} I_{-2(0)}(\lambda_k) k^2 \kappa^{-1}.$$  \hspace{1cm} (6.2)

The second term is of the same order as the anomalous dimension (6.1a). Putting together in (3.15b) we find

$$k \frac{\partial \lambda_k}{\partial k} = \frac{9\lambda_k^2}{4\pi^2} + \frac{(13\alpha - 45)}{256\pi^2} I_{-2(0)}(\lambda_k) k^2 \kappa^{-1}.$$ \hspace{1cm} (6.3)

This seems to show that when the energy approaches the Planck energy, the coupling begins to run much faster than logarithmically. This conclusion should be taken with some care, however, since at energies comparable to Planck’s energy the validity of Einstein’s theory as an effective theory becomes questionable. We expect the coupling constant to run again logarithmically above the Planck scale, but with a different coefficient that will depend on the details of the “new physics” that one encounters in this regime. The power–like behaviour indicated by (6.3) is probably limited to the threshold region.

For $\phi_k^3 \kappa^{-1/2} \ll k^2 \ll \phi_k^2$ the term containing $\kappa$ in the numerator is the dominant one. We find

$$\beta(k) = \frac{9\tilde{Z}_k}{256\pi^2} I_0(0) \frac{1}{\lambda_k \phi_k^6} \ll 1.$$ \hspace{1cm} (6.4)

Just below the mass threshold of the scalar particles, at $k^2 = \phi_0^2$, this is of the same order as the anomalous dimension (6.1c) and

$$k \frac{\partial \lambda_k}{\partial k} = \frac{785}{256\pi^2} I_0(0) \frac{1}{\lambda_k \phi_k^6}.$$ \hspace{1cm} (6.5)

Finally, for $k^2 \ll \phi_k^3 \kappa^{-1/2}$ the terms with the highest power of $\phi_k$ dominate and

$$\beta(k) = \frac{15(\alpha - 1)}{128\pi^2} I_{-2(0)}(\lambda_k) k^2 \kappa^{-1} \ll 1.$$ \hspace{1cm} (6.6)

This is much smaller than the anomalous dimension (6.1d), so in the extreme infrared the running of the coupling constant is dominated by $\eta$. Since $\phi_k \approx \tilde{\phi}_0$,

$$k \frac{\partial \tilde{\lambda}_k}{\partial k} = \frac{2(\alpha + 3)}{2\pi^2} \frac{\tilde{\phi}_0^2}{\kappa} \lambda_k^2 \ll 1.$$ \hspace{1cm} (6.7)

The solution has the form

$$\lambda_k^2 = \frac{\tilde{\phi}_0^2}{1 - \frac{2(\alpha + 3)}{2\pi^2} \frac{\phi_0^2}{\kappa} \lambda_k^2 \ln \left( \frac{k}{\phi_0} \right)}.$$ \hspace{1cm} (6.8)

and therefore $\lambda_k$ tends (very slowly) to zero as $k$ goes to zero. This should be contrasted with equation (3.18b) of the pure scalar theory, where $\lambda_k$ tends to a constant. The different behaviour is due to the presence of a massless particle, the graviton.

To summarize our results, we have found that the v.e.v. runs quadratically and the coupling constant runs logarithmically above the mass of the scalar particles, while the running is suppressed at energies much lower than the mass of the scalar. The details of the residual running at low energies seem to differ above and below a certain mass scale $\phi_k^3 \kappa^{-1/4}$. If the scalar has a mass in the electroweak range, this scale is of the order of 10 MeV, while if the scalar has a mass in the GUT range, this scale is of the order of $10^{19}$ GeV.
While probably of little practical significance, the appearance of this additional scale is theoretically quite intriguing.

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Appendix: Spin-projector operators
For completeness, we list here the explicit expressions of the spin-projector operators $P^{AB}(J^P)$ that appear in (4.7):

\[ P^{hh}(2^+)_{\rho\sigma}^{\alpha\beta} = T^{(\alpha \sigma \beta)} = \frac{1}{3}T^{\rho\sigma}T^{\alpha\beta}, \]
\[ P^{hh}(1^-)_{\rho\sigma}^{\alpha\beta} = 2T^{(\alpha L^\beta)}_{(\rho L^\sigma)}, \]
\[ P^{hh11}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{3}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{hh12}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{hh21}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{hh22}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{hh31}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{hh32}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]
\[ P^{h\phi33}(0^+)_{\rho\sigma}^{\alpha\beta} = \frac{1}{\sqrt{3}}T^{\rho\sigma}L^{\alpha\beta}, \]

where
\[ \hat{q}^\mu = q^\mu / \sqrt{q^2}, \quad L^\nu = \hat{q}^\mu \hat{q}^\nu, \quad T^{\nu}_{\mu} = \delta^{\nu}_{\mu} - L^{\nu}_{\mu}. \]

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