Optimal portfolio of an investor in a financial market

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Abstract. An investor seeks to diversify assets and optimal portfolio, which provide the maximum expected returns at a given level of risk. Optimal portfolio problems of an investor with logarithmic utility have been studied. However, there is scarce information on other utility functions, such as power utility function, which captures the concept of diversification of portfolios. This study was therefore designed to consider the general expected utility of an investor in the financial market. Ito’s integral where extended by the lofty properties of forward integral to diversify the investor’s portfolio. A filtration was built and used as a set of information for the investor. A semimartingale was used to enlarge the investor’s information. Matlab Mathematical software was used to compute the investors varying rates of return.

Keywords: Power utility function; Diversification; Itô’s-integral; Semimartingale

1. Introduction

The present economic situation in the globe means that reliance on a single source of income can no longer satisfy the needs of an average middle-class family. Thus, the need to explore multiple streams of income is on the front burner of many homes. However, the possible means of achieving multiple streams of income, but better means is to invest in an asset. In economics, we know that man's needs are insatiable; thus, man always seeks means to increase his expected financial returns. A person who seeks to put finances into the acquisition of an asset in order to get higher returns is called an investor. Generally, investors are classified into three categories, namely: risk-neutral, risk-averse, and risk-seeking. Risk-averse investor's appetite is zero; on the contrary, a risk-seeking investor takes more risk for more promising return and risk-neutral is neither of the former mentioned [1-6]. Investors' attitude to risk determines his investment preference; for instance, risk-averse would prefer to deposit his money in a bank account, invest in bonds, or put his money into a fixed deposit. Also, the expected returns of an investor are determined by his attitude to risk. The expected returns of an investor with more risk appetite surpass risk averse. However, risk averse investor has a chance of getting his expected returns through diversification. Investment policies and management are crucial practices for portfolio optimization [7-23].

Risks occur as a result of uncertainty in data. Risk is also seen as the likelihood of actual future returns varying from the expected returns. However, it is equally the likelihood of actual cash flows differ from the estimated one [21]. Risk and expected returns are an intrinsic part of investments and are treated concurrently. An investor resorts to diversification so as to spread investment and reduce risk. A diversified portfolio is a more robust investment option [11]. Diversification aims to reduce the unsystematic risk in an investment portfolio, which occurs as a result of mismanagement, poor forecasting accuracy, or wrongful planning process and decision making. Diversification helps to
reduce the volatility of portfolio performance. This is because the diversified portfolio is more robust, with less variation in expected return.

1.1 Methodology
In our methodology, the Itô's integral where extended by the lofty properties of forward integral to diversify the investor’s portfolio. A filtration was built and used as a set of information for the investor. A semimartingale was used to enlarge the investors' information. Matlab Mathematical software was used to compute the investors varying rates of volatility. The models derived were:

\[
U'(S_{\beta_1Y_1+y\sigma}(T))S_{\beta_1Y_1+y\sigma}(T)\mathbb{E}(y) = S_{\beta_1Y_1+y\sigma}(T)\mathbb{E}(y)
\]

where \(U'(x) = \frac{dU(x)}{dx}\) is satisfied if:

\[
\sup_{x \in (-\delta, \delta)} \left\{ E\left[ S_{\beta_1Y_1+y\sigma}(T)\mathbb{E}(y)\right] | p \right\} < \infty \text{ for some } p > 1
\]

\[
0 < E \left[ U'(S_{\beta_1Y_1+y\sigma}(T))S_{\beta_1Y_1+y\sigma}(T)\right] < \infty \text{ and } S_{\beta_1Y_1+y\sigma}(T) = S_{\beta_1Y_1+y\sigma}(T)N_{\beta_1Y_1}(y),
\]

where \(N_{\beta_1Y_1}(y) := S_0 \exp \int_0^T [\theta(s) - \lambda(s) - \omega^2(s)\beta_1(s)y_1(s)] ds + \int_0^T \omega(s)dD(s) \) and coefficients \(\lambda(t), \theta(t), \omega(t)\) are bounded with \(\mathbb{A}\) bounded, then there exist \(\delta > 0\) and \(\gamma e(-\delta, \delta)\), where \(D(t)\) is Brownian motion(representing risky asset fluctuations price), on a filtered space of probability \((\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t \geq 0}, p)\) and coefficients \(\lambda(t), \theta(t), \omega(t)\) are \(\mathbb{N} = \{\mathbb{F}_t\}_{t \geq 0}\) adapted with \(\mathbb{N} \subseteq \mathbb{F}_t\) \(\forall \ [0, T]\) and \(T > 0\) a fixed final time. The Itô’s integral is \(\mathbb{F} = \{\mathbb{N}_t\}_{t \geq 0}\) adapted. The forward integral showed that when an investor buys units amount \(\alpha\) at a random time \(\tau_1\) and hold until a certain time \(\tau_2\): \(\tau_1 < \tau_2 < T\), and eventually sells them at a subsequent time, the profits realized would be \(\propto D(\tau_2) - \propto D(\tau_1)\) expressed as the portfolios forward integration \(\alpha(t) = \propto I(\tau_1, \tau_2)\), \(\tau \in [0, T]\) with respect to \(D(t)\) that is,

\[
\int_0^\tau \sigma(t) d^-D(t) = \lim_{\Delta \rightarrow 0} \sum_j \sigma(t_j) \times \Delta D(t_j) = \int_{\tau_1}^{\tau_2} d\hat{D}(t) = \propto D(\tau_2) - \propto D(\tau_1).
\]

The filtration

\[
\mathbb{N} = \{\mathbb{N}_t\}_{t \geq 0}
\]

outlined the information available for the investor. The semimartingale integral

\[
\int_0^\tau \sigma(t) d^-D(t) = \int_0^\tau \sigma(t) d^-D
\]

gives a decomposition

\[
D(t) = \hat{D}(t) + A(t), \ 0 \leq t \leq T
\]

where

\[
\int_0^\tau \sigma(t) d^-D(t) = \int_0^\tau \sigma(t) d\hat{D}(t) + \int_0^\tau \sigma(t) dA(t)
\]

for \(\mathbb{N}_t = \mathbb{F}_t \lor \alpha D(T_0)\); \(0 \leq t \leq T\) that is, \(\mathbb{N}_t\) is the result created by \(\mathbb{F}_t\) and the final value \(D(T_0)\), where \(\hat{D}(t)\) is a \(\mathbb{N}_t\) -Brownian motion and \(A(t)\) is a continuous \(\mathbb{N}_t\) -adapted finite variation process. The varying rates of volatility \(\sigma = 1, 0.5, S_0 = 100, \mu = 1\), revealed that the expected return is more
when volatility $\sigma = 1$, thereby yielding optimal portfolio. The optimal portfolio of an investor was established using power utility function and showed higher investors return as the investor diversified his investment. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$ of $D(\tau)$. An investor’s business logistics is $\sigma(\tau) = \mathbb{I}(\tau_1, \tau_2)(\tau)$, $\tau \in [0, T]$ considered as a buy-and-hold strategy with respect to $D(\tau)$ then
\[
\int_0^T \sigma(\tau) dD(t) = \sigma(\tau) \int_0^T dD(t)
\]
(1)
where $\sigma$, $D(t)$ is adapted to $\mathcal{N}_t \supseteq \mathcal{F}_t$, such that $D(t)$ is a semimartingale which decomposes
\[
D(t) = \hat{D}(t) + A(t), \quad 0 \leq t \leq T
\]
(2)
Where $\hat{D}(t)$ is a $\mathcal{N}_t -$ Brownian motion and $A(t)$ is a continuous $\mathcal{N}_t -$ adapted finite variation process. If $A(t)$ has the form
\[
A(t) = \int_0^t \alpha(s) ds,
\]
(3)
if (2) holds, we define
\[
\int_0^T \sigma(\tau) dD(t) = \int_0^T \sigma(\tau) d\hat{D}(t)
\]
Section 1 presents the introduction and methodology as well as relevant mathematical preliminaries for this study, brief literature review, the lofty properties of forward integral, also known as anticipating integral couple with its techniques and the implications of using this model were presented in Section 2. In Section 3, the variational method of investors’ optimal portfolios were presented. Finally, in Section 4, we present our conclusions.

2. Brief Literature review

The choice of a portfolio is a classical issue in mathematical finance, where the principal intention of any investor is to seek for the best quantity of money to invest in the risky assets for a benefit. The establishment of an optimal portfolio constitutes some difficulties for market partakers since they operate in an unpredictable environment.

The authors in [18] initiated the mean-variance (MV) model, developed the choice of portfolio problem as an optimisation problem, that is made up of minimizing the variance (an investors risk measure) of the final value for a wished amount of expected return. He considered the highest expected return with a minimum variance in return, which is not always the case. The authors in [19], observed that the mean-variance optimised portfolios are difficult to comprehend and made it difficult for investors to practice easily. However, in the actual sense, His formulation fails to consider the investors consumption.
Another interesting area of study in the sense of portfolio optimisation is concerned with the utility theory as well as expected utility maximization, where the likes, preferences and desires of investors are expressed by the utility function. In this sense, investors goal is to maximize the expected value of the utility function. The general continuous-time issue of optimal consumption and choice guideline of portfolio was considered first by [20], and established the concept for dynamic portfolio selection under unpredictable scenario. His concept of dynamic programming inspired a partial differential equations (PDE) of nonlinear which is complex and a general issue considering the process controlling the volatility; this appearance makes it challenging to determine the best approach for the portfolio and ideal consumption. However, [20] clearly proffer a PDE based solution under a consistent risky asset volatility. Up to date, researchers have developed and sustained huge interest on portfolio optimisation problems with stochastic volatility. [12] resolved the long standing issue of both the consumption and the choice of portfolio simultaneously, where the driven stochastic volatility is related to the diffusion process. In contrast [13], obtained clear results for portfolios exhibiting log-optimal in a complete market with semimartingale specification of the price process. [15] considered arbitrage-free model where relevant results for the problem of optimal investment was established. [10], achieved more promising result for an incomplete market driven by the CIR model [8], [2], established a time-varying volatility model of a stock market through the process of regime switching method and a constant interaction influence. However, an issue synonymous with that of [20] was resolved by [7] and [23]. [17] introduced a clear solution to the problem choice of portfolio dynamics, where the risky assets return is controlled by a ‘‘quadratic process”, which takes the form of Markovian diffusion process as well as a consistent relative risk aversion (CRRA) of investors utility function. [9], numerically solved the Merton model with a robust finite difference method. The authors in [9] admitted that the condition of convergence was looked upon, and the method proposed needed to be recast for the time-discretisation to converge. The revisiting of the portfolio choice problem and optimal consumption of an investor with access to a risk-free asset and with a consistent expected return and stochastic volatility. His intention was double; first, he determined a detailed portfolio dynamics solution as well as the issue of selection, when the returns of the risky assets are volatile and controlled by priority process of Ornstein-Uhlenbeck, for an investor under (CRRA). Secondly, he computed several numerical tests with the aid of the obtained solution to be able to resolve the optimal amount that is sensitive as well as the consumption and several variables in the model. The risky asset expected return, investors risk-averse, the force of mean-reverting, the long-term mean together with the diffusion coefficient of the stochastic influence of Brownian motion was also considered. The authors in [8] considered a logarithmic utility, where the optimal size of money on stock absolutely relies on the model parameter. The optimal portfolio with logarithmic Utility disregards the differences between the present value and influence of the future of an economy; as a result, it is myopic. Because the size of wealth owned in stock appears to be the same throughout the periods of time, even at the random change of variables of the market, furthermore, power utility functions would consider future investment opportunities. For example, if the current interest rate of the risky assets is presumed higher than the future, an investor might be receptive to the risky assets investment to take advantage of its unpredictable drop in the future. In this respect, we look at the general expected utility function, which maximises an investor’s portfolio. Our model is built from [16] and [8] whose aim was to optimize the logarithmic utility from the final amount, we are using the tool of forward integral to optimise the general expected utility of an investor whose intention is to diversify investment with the information he or she has. Next, we discuss some important applications of forward integral in a financial market.

3 Specification of the optimal portfolio of investors

(i) A non-risky asset(e.g., a Bank account),where the price \( L_0(\tau) e^{[0,\xi]} \) per unit at time \( \tau \) is

\[
dL_0(\tau) = \lambda L_0(\tau) d\tau, \quad L_0(0)=1 \text{ and}
\]
(ii) A risky asset (e.g., a stock), where the price $L_i(\tau), \tau \in [0, \xi]$ per unit at time $\tau$ is also
\[ dL_i(\tau) = \theta(\tau)L_i(\tau)d\sigma + \omega(\tau)L_i(\tau)d^R D(\tau) \]
where $L_i(0) > 0, \theta(\tau), \omega(\tau)$ are constant coefficients and $\omega(\tau), \tau \in [0, \xi]$ are deterministic functions on the assumption that $E\left[ \int_0^\xi \left( |\theta(\tau)| + |\omega(\tau)| + \omega^2(\tau) \right) d\tau < \infty \right]$ where $\omega(\tau)$ is a gla d.

Let $(v_0(\tau), v_1(\tau)), \tau \in [0, \xi]$ denotes portfolios. Then, its value at time $\tau$ is represented as
\[ X(\tau) = v_0(\tau)L_0(\tau) + v_1(\tau)L_1(\tau). \]
However, it is self-financing assuming
\[ dX(\tau) = v_0(\tau)dL_0(\tau) + v_1(\tau)dL_1(\tau) \]
from $X(\tau) = v_0(\tau)L_0(\tau) + v_1(\tau)L_1(\tau)$ we make $v_0(\tau)$ the subject thus
\[ v_0(\tau) = \frac{X(\tau) - v_1(\tau)L_1(\tau)}{L_0(\tau)} \]substituting in our self-financing equation of
\[ dX(\tau) = v_0(\tau)L_0(\tau) + v_1(\tau)L_1(\tau) \]
and using \(dL_0(\tau) = \theta L_0(\tau)d\sigma, L_0 = 1\) and \(dL_1(\tau) = \omega L_1(\tau)d^R D(\tau), \quad L_1(0) > 0\) then
\[ dX(\tau) = (X(\tau) - v_1(\tau)L_1(\tau)) \frac{dL_0(\tau)}{L_0(\tau)} + v_1(\tau)dL_1(\tau) = \lambda(\tau) X(\tau)d\sigma + v_1(\tau)L_1(\tau)(\theta - \lambda(\tau))d\sigma + \omega(\tau)d^R D(\tau) \]
Since $\omega(\tau) = 0$ for a.a. written as
\[ dX(\tau) = \lambda(\tau) X(\tau)d\sigma + \omega(\tau)v_1(\tau)L_1(\tau)(\alpha(\tau)d\sigma + d^R D(\tau)) \]
where $\alpha = \frac{\theta - \lambda(\tau)}{\omega(\tau)}$ set $v_1(\tau) = \beta_1(\tau)$ the part payment or fraction of money invested in stock.

Then an investor has desirable return when he diversifies his investment on a portfolio that is, where $\omega(\tau)(\beta_1(\tau)\gamma_1(\tau))$ is a gla d, $N$ – adapted and forward integrable stochastic process such that
\[ \int_0^\xi \left( |\theta(\tau)| + |\omega(\tau)| + \omega^2(\tau) \right) |\beta_1(\tau)\gamma_1(\tau)| + \omega^2(\tau) \beta_1^2(\tau) \gamma_1^2(\tau) d\tau < \infty \] a.s. holds, then the wealth
\[ X(\tau) = X_{\beta(\tau)\gamma(\tau)} \] of an investor at $\tau$ would satisfy
\[ d^R X(\tau) = \frac{dX(\tau)}{X_0} + \omega(\tau)\beta_1(\tau)\gamma_1(\tau)d^R D(\tau) \]
By Itô
\[ X(\tau) = \exp \left[ \int_0^\xi \left( \frac{\lambda(\tau) + \theta(\tau) - \lambda(\tau))\beta_1(\tau)\gamma_1(\tau)}{2} \right) d\sigma + \int_0^\xi \omega(\tau)\beta_1(\tau)\gamma_1(\tau) d^R D(\tau) \]
Considering $\beta^*_1(\tau)\gamma^*_1(\tau) \in \mathbb{A}_n$ in the sense that
\[ \sup_{\beta(\tau)\gamma(\tau)\in \mathbb{A}_n} H \left[ \eta \left( X_{\beta(\tau)\gamma(\tau)} \right) \right] = H \left[ \eta \left( X_{\beta^*_1(\tau)\gamma^*_1(\tau)} \right) \right] \]
Definition 3.1: We define the family of admissible portfolios as $\mathcal{A}_N$ and:

i. $\beta_{j\gamma_1}\in\mathcal{A}_N$ is $c(a,gl,a,d)$ and $N$

ii. For all $\beta_{j\gamma_1}\in\mathcal{A}_N$, $H\left[\int_0^\xi [\theta(t)-\lambda(t)]\left|\beta_1(t)\gamma_1(t)\right|+\omega^2(t)(\beta_1(t)\gamma_1(t))^2\right]dt<\infty$

iii. $\beta_{j\gamma_1}\in\mathcal{A}_N$, then $(\beta_{j\gamma_1})\omega$ is forward integrable and $c(a,gl,a,d)$ with respect to the Brownian motion $D$

iv. For all $\beta_{j\gamma_1}\in\mathcal{A}_N$, we have $0<H\left[\eta'\left(X_{\beta(j\gamma_1)}(\xi)\right)X_{\beta(j\gamma_1)}(\xi)\right]<\infty$. Where

$v'(x) = \frac{d}{dx} \eta(x)$

v. For all $\beta_{j\gamma_1},\omega\in\mathcal{A}_N$, there exists $\nu>0$, with $\sigma$ bounded, then

$\beta_{j\gamma_1}+j\omega\in\mathcal{A}_N$. For all $j\in(-\nu,\nu)$, the expression:

$\eta'\left(X_{\beta(j\gamma_1)}(\xi)\right)X_{\beta(j\gamma_1)}(\xi)\left|N_{\beta(j\gamma_1)}(\xi)\right|_{j\in(-\nu,\nu)}$ is uniformly integrable, where

vi. $N_{\beta_{j\gamma_1}}(\tau) = \int_0^{\xi} \left[\theta(s)-\lambda(s)-\omega^2(s)\beta_{j\gamma_1}(s)\right]ds+\int_0^{\xi} \omega(s)dD(s), \tau\in[0,\xi]$

vii. A buy-hold sell strategy $\sigma$, that is $\sigma(t) = \alpha I(\tau,t+f)\left(t\right), \tau\in[0,\xi]$ with $0\leq\tau<t+f\leq\xi$ and $\alpha$ is $N_t$-measurable, belonging to $\mathcal{A}_N$. Then the portfolio $\beta_{j\gamma_1}\in\mathcal{A}_N$ is optimal if $H\left[\eta\left(X_{\beta(j\gamma_1)}(\xi)\right)\right] = H\left[\eta\left(X_{\alpha}(\xi)\right)\right]$ for all $\omega\in\mathcal{A}_N$ bounded and $j\in(-\nu,\nu)$, with $\nu>0$, given in (5).

Definition 3.2: Assume $\sigma$ is a forward integrable stochastic process, and $N$ a random variable. Then the product $N\sigma$ is forward integrable stochastic process and

$$\int_0^{\xi} N\sigma(t)dD(t) = N\int_0^{\xi}\sigma(t)dD(t)$$

(6)

where $\sigma = X(t)\nu(t)$ such that $\nu_1 = \beta_{j\gamma_1}$.

Theorem 3.1: A stochastic process $N_{\beta_{j\gamma_1}}(\tau)$, for $\tau\in[0,\xi]$ is a $\left(N_t,Q_{\beta_{j\gamma_1}}(\tau)\right)$ martingale that is a martingale with respect to the filtration $N$ under the measure $Q_{\beta_{j\gamma_1}}(\tau)$ if it is optimal.

Proof: Firstly, supposing $\beta_{j\gamma_1}$ is optimal then for all $\sigma\in\mathcal{A}_N$ bounded implies

$$0 = \frac{d}{dj}H\left[\eta\left(X_{\beta(j\gamma_1)}(\xi)\right)\right]_{j=0}$$

$$0 = H[\eta'\left(X_{\beta_{j\gamma_1}}(\xi)\right)X_{\beta_{j\gamma_1}}(\xi)]\int_0^{\xi} \sigma(s)(\theta-\lambda(s)-\omega^2(s)\beta_{j\gamma_1}(s)]ds+\int_0^{\xi} \sigma(s)\omega(s)dD(s)$$

(7)
Now fix $\tau, f : 0 \leq \tau < \tau + f \leq \xi$ and choose $\sigma(s) = \alpha I(\tau, \tau + f)(\tau), \tau \in [0, \xi]$.

Where $\alpha$ is an arbitrary bounded and $\mathcal{N}_\tau$ - measurable random variable.1

(7) becomes

$$0 = H[\eta'(X_{\beta \gamma}(\xi)) X_{\beta \gamma}(\xi) \int_{0}^{\tau + f} \sigma(s)(\theta - \lambda(s) - \omega^2(s)\beta \gamma(s))ds + \int_{0}^{\tau + f} \omega(s)dD(s)\alpha] (8)$$

Since this holds for all $\alpha$, we conclude that

$$H[F_{\beta \gamma}(\xi)(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau))] = 0 \quad \text{where} \quad F_{\beta \gamma}(\xi) = \frac{\eta'(X_{\beta \gamma}(\xi))}{H[\eta'(X_{\beta \gamma}(\xi)) X_{\beta \gamma}(\xi) + \int_{0}^{\tau + f} \sigma(s)(\theta - \lambda(s) - \omega^2(s)\beta \gamma(s))ds + \int_{0}^{\tau + f} \omega(s)dD(s)\alpha]}$$

and

$$\left( N_{\beta \gamma}(\tau) \right) = \exp \left[ - \int_{0}^{\tau} \theta - \lambda(s) - \omega^2(s)\beta \gamma(s)ds + \int_{0}^{\tau} \omega(s)dD(s), \tau \in [0, \xi] \right] (9)$$

That is,

$$H[F_{\beta \gamma}(\xi)\left( N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau) \right)] = 0 \quad \text{by the application of Bayes Theorem,} \quad H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau$$

(10) becomes

$$0 = H[F_{\beta \gamma}(\xi)|\mathcal{N}_\tau] H[F_{\beta \gamma}(\xi)(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau))] \mathcal{N}_\tau] (11)$$

Since $N_{\beta \gamma}(\tau)$ is $\mathcal{N}_\tau$- adapted, this gives $H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau$.

Hence $N_{\beta \gamma}(\tau)$ is an $(\mathcal{N}_\tau, Q_{\beta \gamma})$ - martingale. Let the probability measure $Q_{\beta \gamma}$ on $\mathcal{N}_\tau$ be $dQ_{\beta \gamma} = F_{\beta \gamma}(\xi)dm$. and set $H_{Q_{\beta \gamma}}(\tau)$ to represent $Q_{\beta \gamma}$ expectation.

Then $H[F_{\beta \gamma}(\xi)(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau))] \mathcal{N}_\tau = 0$ written as

$$H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau = 0 \quad \text{by} \quad H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau = 0 \quad \text{by} \quad H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau = 0$$

(12)

Secondly, assuming $N_{\beta \gamma}(\tau)$ is a $(\mathcal{N}_\tau, Q_{\beta \gamma})$ - martingale, then (12) for all $\tau, f$ therefore $0 \leq \tau < \tau + f \leq \xi$. Equivalently, $H_{Q_{\beta \gamma}}(N_{\beta \gamma}(\tau + f) - N_{\beta \gamma}(\tau)] \mathcal{N}_\tau = 0$ for all $\alpha$ bounded $\mathcal{N}_\tau$ - measurable. Thus (8) holds. Also taking linear combination, it is valid for all step processes $\sigma \in \mathcal{A}_\mathcal{N}$ of $\alpha$ gla d. Ref. assumption (1) and (5) of definition (3.1) (7) holds with boundedness of All $\sigma \in \mathcal{A}_\mathcal{N}$.

Since $j \rightarrow H[\eta(X_{\beta \gamma}(\xi))]$, $j \in (-v, v)$ maximum is obtained at $j = 0$. Thus, $0 = \frac{d}{dj} H[\eta(X_{\beta \gamma}(\xi))]|j = 0$

**Theorem 3.2** Assuming $\beta \gamma, \epsilon \in \mathcal{A}_\mathcal{N}$ is optimal for the problem (5) then

$$g(j) = H[\eta(X_{\beta \gamma}(\xi))], \quad j \in (-v, v), \text{ is concave for each } \sigma \in \mathcal{A}_\mathcal{N} \text{ and } N_{\beta \gamma}(\tau), \tau \in [0, \xi] \text{ is a } (\mathcal{N}_\tau, Q_{\beta \gamma}) - \text{martingal.}$$
Proof: the proof of theorem 3.1 establishes the optimality, we show that it is concave

\[ U(\tau) \): \( v_1 \rightarrow \lambda(\tau) + (\vartheta(\tau, v) - \lambda(\tau))v - \frac{1}{2}\omega^2(\tau)v^2 \]

\[ \lambda(\tau) + \vartheta(\tau, v)v - \lambda(\tau)v - \frac{1}{2}\omega^2(\tau)v^2 \]

\[ \vartheta(\tau, v) \times 1 + v \times \vartheta'(\tau, v) - \lambda(\tau) \times 1 + v \times 0 - \frac{1}{2}\omega^2(\tau) \times 2v + 0 \times v^2 \]

\[ \vartheta(\tau, v) \times 1 + \vartheta'(\tau, v)v - \lambda(\tau) - \omega^2(\tau)v \]

Upon differentiation, we have

\[ \vartheta'(\tau, v) + \vartheta''(\tau, v)v + \vartheta'(\tau, v) \times 1 - \omega(\tau) \]

\[ \vartheta''(\tau, v)v + 2\vartheta'(\tau, v) - \omega^2(\tau) \leq 0. \]

Theorem 3.3 \( \beta_1 \gamma \epsilon \mathcal{A}_1 \) is optimal for (5) only if

\[ \zeta(s) := H_{\mathcal{D}_t} \left[ \frac{dm}{d\mathcal{D}_t} | \mathcal{N}_s \right] = \left( H \left[ F_{\beta_1 \gamma \epsilon \mathcal{A}_1} \left( \xi \right) | \mathcal{N}_s \right] \right)^{-1}, \text{ } s \epsilon [0, \xi] \text{ is a } (\mathbb{N}, m) - \text{martingale} \]

And

\[ \zeta(\tau) = \int_0^\tau \omega(s) d^* \mathcal{D}(s), \text{ } \mathcal{D}(t) \text{ is a } (\mathbb{N}, m) - \text{semimartingale} [5]. \]

Example 3.1

\[ U(x) = \frac{x^{\gamma^*}}{\gamma} \text{ where } \gamma^* = \vartheta - \lambda, \text{ and } \gamma = 1 - c \text{ then, the optimal portfolio is } \]

\[ v(t) = \frac{\vartheta - \lambda}{(1-c)\omega^2} \]

\[ v(t) = \frac{0.09 - 0.04}{(1-0.8) \times (0.04)} \]

At all times \( t \) where \( t < 1 \), \( v(t) = \frac{0.09 - 0.04}{0.2 \times (0.04)} = \frac{0.05}{0.008} = 6.25 \)

when \( \vartheta = 9, \omega = 20\%, \lambda = 4\%, c = 0.8 \text{ 25\% of his money be invested in the risk asset } S, \text{ and 75\% on the risk-free asset.} \)

Example 3.2

Given \( \eta(x) = \frac{1}{h} x^h, \text{ } x > 0 \), where \( h \epsilon (0,1) \), we have

\[ \eta'(X_{\beta_1 \gamma \epsilon \mathcal{A}_1} \left( \xi \right)) X_{\beta_1 \gamma \epsilon \mathcal{A}_1} \left( \xi \right) | \mathcal{M}(h) = X_{\beta_1 \gamma \epsilon \mathcal{A}_1} \left( \xi \right) \mathcal{M}(h) \text{ and condition (4)in our definition 3.1 is satisfied if } \sup_{h \rightarrow \infty} H \left[ \left( X_{\beta_1 \gamma \epsilon \mathcal{A}_1} \left( \xi \right) | \mathcal{M}(h) \right)^p \right] < \infty \text{ for } \hat{p} > 0, \text{ then set } \]
\[ X_{\gamma_{1,n+1}}(\xi) = X_{\gamma_{1,n+1}}(\xi)N(h) \]

from the Holder’s inequality,

\[
H\left(\left(X_{\gamma_{1,n+1}}(\xi)\right)^{\beta}\right) \leq \left(H\left(\left(X_{\gamma_{1,n+1}}(\xi)\right)^{\beta}\right)\right)^{\frac{1}{\beta}} \left(H\left(N(h)^{\beta}\right)\right)^{\frac{1}{\beta}} \left(H\left(M(h)^{\beta}\right)\right)^{\frac{1}{\beta}}
\]

Where \(\tilde{a}_1, \tilde{a}_2 : \frac{1}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} = 1\) and \(\tilde{b}_1, \tilde{b}_2 : \frac{1}{\tilde{b}_1} + \frac{1}{\tilde{b}_2} = 1\) choosing \(\tilde{a}_1 = \frac{2}{2-p}, \tilde{a}_2 = \frac{2}{\hat{p}}\) and also

\[
\tilde{b}_1 = \frac{2 - \hat{p}}{\hat{p}}, \tilde{b}_2 = \frac{2 - \hat{p}}{2 - \hat{p} - \hat{p}}\]

for some \(\hat{p} \notin \left(1, \frac{1}{h+1}\right)\) hence,

\[
H\left(\left(X_{\gamma_{1,n+1}}(\xi)\right)^{\beta}\right) \leq \left(H\left(\left(X_{\gamma_{1,n+1}}(\xi)\right)^{\beta}\right)\right)^{\frac{1}{\beta}} \left(H\left(N(h)^{2\beta}\right)\right)^{\frac{2}{2\beta}} \left(H\left(M(h)^{2\beta}\right)\right)^{\frac{2}{2\beta}}
\]

Supposing \(N_{\gamma_{1,n}}(\xi)\) in (9) satisfies \(\left(N_{\gamma_{1,n}}(\xi)^{2}\right) < \infty\) then item (4) and (5) in definition (3.1) is valid if

\[
\sup_{h \in (-\delta, \delta)} H\left(\left(N(h)^{\frac{2\hat{p}}{2 - \hat{p} - \hat{p}}}\right)\right) < \infty\]

and also

\[
Hexp\left[k\int_{0}^{\xi}(|\beta_{\gamma_{1}}(\tau)| + |\beta_{\gamma_{1}}(\tau)|d\tau)\right] + \int_{0}^{\xi} \beta_{\gamma_{1}}(\tau) \omega(\tau) dD(\tau) < \infty
\]

Figure 1: The graph of an investor when volatility is 0.5 and drift is 1.

Figure 1 represents the low return rate of investors due to the inability to take risks capable of generating huge returns. It is obvious this investor is at the losing end.
Figure 2: The graph of an investor when volatility is 0.8 and drift is 1.

Figure 2 shows the trend of an investor whose risk appetite is low; this indicates that the investor is not prone to risk, and the part where the investment return outgrew the risk volatility is meager. Such an investor goes to the market with the motive of no huge return mentality.

Figure 3: The graph of an investor when volatility is 1 and drift is 1

Figure 3 shows the result of a sensitive investor who resorts to taking more risks by diversifying with a huge rate of return. The trajectory of the return increases significantly and out weight the risk a sensitive investor has undergone.
Figure 4: This graph indicates the varying rate of risk-averse investor

Figure 4 indicates the varying rate of risk-averse investors behaviour on the risky assets, the proportion of investment on the risky assets at some period is on the increase due to low cost of the transaction and positive turn out, at some point in the interval, he refrains from further investing due to high volatility of risky assets. The graph indicates a huge investment in the risky assets and lower risk-aversion also at a lower level of risk-aversion; the proportion held in stock is relatively high to compensate for the decrease in the proportion held in the bond. Similarly, it is equally observed that at a higher level of risk-aversion, the proportion held in stock witnessed a drastic decrease due to higher payoff on bond.

Figure 5: An investor’s return in investing \(100(2p - 1)\%\) of his capital.

Figure 5 shows the trend of investor’s possibility of gaining promising return when \(\hat{\zeta} = 0.5\) \(\alpha = 0\) thus if \(\hat{\zeta} = \frac{1}{2}\) the investor gains confidence in investing \(100(2p - 1)\%\) of his capital that is, when
0 ≤ \hat{\zeta} ≤ \frac{1}{2} \text{ then } E\left[U\left(X\right)\right] \text{ is the greatest value when } \alpha = 0, \text{ that is, when no investment is made by the investor. But when } \hat{\zeta} = 0.6, \alpha = 0.2, \text{ similarly when } \hat{\zeta} = 0.7, \alpha = 0.4, \text{ when } \hat{\zeta} = 0.8 \alpha = 0.6, \text{ when } \hat{\zeta} = 0.9, \alpha = 0.8, \text{ the trajectory of the graph increases showing the investor’s possibility of investing due to a promising positive return.}

4 Conclusions

Generally, maximizing the general expected utility of an investor in a financial market was studied. An investor seeks diversification so as to spread the risk of loss. A diversified portfolio or group of assets has a smoother risk behavior [11]. The optimal portfolio with logarithmic utility fails to consider a thorough assessment of the present from the future values of economic influences. Therefore, it is short-sighted. However, power utility considers future investment opportunities. For instance, if the dynamics of the current rate of interest of the risky assets would probably be higher than the future, an investor may consider investing in the risky assets in the advantage of a potential increase in price now as a result of its speculated drop in the future. An indispensable concept of "buy-low", sell-high strategy was used in this study, for instance, if you owned above 40% of your money in the risky assets, you could transfer some into riskless assets, and if you owned above 60% on the riskless assets, you move some of the capital to risk assets. The significance of this is to know the best size of money held in the available assets at each trading period. Furthermore, it sounds wise to shuffle your capital between the two assets, no matter the size. Our Empirical results show that when volatility \( \sigma = 1, \ t = 1, \ S_0 = 100, \ \mu = 1 \), the expected return in investment is more than when \( \sigma = 0.5 \). Moreover, if \( \tau = 0.05 \), and \( \mu = 0.08 \ \sigma = 0.04 \) then an investor should consider putting 75% of its money on the risky assets. Our contributions to knowledge are enumerated below:

i. The optimal portfolio of an investor was established using a power utility function and showed higher investors return as the investor diversifies his investment.

ii. The extension of the Itô’s integral by forward integral with its lofty properties was used to diversify the investor’s portfolio.

iii. A filtration was built as a set of information for the investor, while a semimartingale was applied to enlarge the investors’ information.

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