Unstable Schwarzschild-Tangherlini black holes in fourth-order gravity

Yun Soo Myung

Institute of Basic Sciences and Department of Computer Simulation, Inje University
Gimhae 621-749, Korea

Abstract

We study the stability of Schwarzschild-Tangherlini (ST) black holes in fourth-order gravity which provides a higher dimensional linearized massive equation. The linearized Ricci tensor perturbations are employed to exhibit unstable modes featuring the Gregory-Laflamme (GL) instability of higher dimensional black strings, in comparison to the stable ST black holes in Einstein gravity. It turns out that the GL instability of the ST black holes in the fourth-order gravity originates from the massiveness, but not a nature of fourth-order derivative theories giving ghost states.

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*E-mail address: ysmymung@inje.ac.kr
1 Introduction

Babichev and Fabbri have shown that the Schwarzschild black holes in the dRGT massive gravity [1,2] do not exist [3]. This was done mainly by comparing the linearized massive equation with the four-dimensional linearized equation around a five-dimensional black string which indicates the Gregory-Laflamme (GL) instability of $l = 0$ mode [4]. In addition, the authors [5] have confirmed this result by considering the Schwarzschild-de Sitter black hole and extending $l = 0$ mode to generic modes of $l \neq 0$.

On the other hand, it is well known that the fourth-order gravity provides a massive gravity with ghosts [6,7]. We have recently shown that the Schwarzschild black hole in fourth-order gravity with $\alpha = -3\beta$ (Einstein-Weyl gravity) is unstable against the linearized-Ricci tensor perturbation [8]. This was shown by comparing the linearized massive equation for Ricci tensor with the metric-perturbation equation around the five-dimensional black string. Furthermore, we have studied the stability of Schwarzschild-AdS black hole in Einstein-Weyl gravity which was known to be stable against the metric perturbations [9]. It turned out that solving the linearized-Einstein tensor equation exhibits unstable modes featuring the GL instability of a five-dimensional AdS black string [10]. These results are meaningful because they ensure that the instability of the black hole in the Einstein-Weyl gravity is due to the massiveness, but not a feature of fourth-order derivative theory giving ghost states. Also, the mechanism of GL instability plays the important role in testing the stability of a black hole in a massive gravity. This is clearly the case that the GL instability could be mapped to unstable modes for a black hole in massive gravity theories [11]. Importantly, applying the GL instability to black holes in fourth-order gravity changes the stability of the black hole [12,9] drastically into the instability [8,10].

In this work, we wish to investigate the stability of the Schwarzschild-Tangherlini (ST) black holes (higher dimensional Schwarzschild black holes) [13] in fourth-order gravity with $\alpha = \frac{4(1-D)}{D}\beta$. This will be based on the GL instability of higher dimensional black strings. It is known that the ST black holes are dynamically stable against all metric perturbations of scalar, vector, and tensor in Einstein gravity [14]. The unstable mode of metric perturbations for black string whose intersection is the ST black hole, is only the $l = 0$ mode of scalar perturbation [15]. If the ST black holes are unstable against the linearized-Ricci tensor perturbation, it will confirm that the instability of the black hole in fourth-order gravity is due to the massiveness, but not a nature of fourth-order derivative theory giving.
ghost states.

2 Linearized fourth-order gravity

We start with the fourth-order gravity action \[9\]

\[
S_{FO} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left[ R + \alpha R_{\mu
u} R^{\mu\nu} + \beta R^2 \right]
\]

(1)

with two parameters \(\alpha\) and \(\beta\). Here the Gauss-Bonnet term is excluded because (1) admits solutions of the Einstein gravity including the ST black holes. From (1), the Einstein equation is derived to be

\[
G_{\mu\nu} + E_{\mu\nu} = 0,
\]

(2)

where the Einstein tensor is given by

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}
\]

(3)

and \(E_{\mu\nu}\) takes the form

\[
E_{\mu\nu} = 2\alpha \left( R_{\mu\rho\sigma\nu} R^{\rho\sigma} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) + 2\beta R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right)
\]

\[
+ \alpha \left( \nabla^2 R_{\mu\nu} + \frac{1}{2} \nabla^2 R g_{\mu\nu} - \nabla_\mu \nabla_\nu R \right) + 2\beta \left( g_{\mu\nu} \nabla^2 R - \nabla_\mu \nabla_\nu R \right).
\]

(4)

Eq.(2) allows a \(D\)-dimensional ST black hole solution \[13\]

\[
d_s^2_{ST} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{D-2}^2
\]

(5)

with the metric function

\[
V(r) = 1 - \left( \frac{r_0}{r} \right)^{D-3}.
\]

(6)

Hereafter we denote the background quantities with the “overbar”. In this case, the background spacetimes is given by the Ricci-flat as

\[
\bar{R}_{\mu\nu} = 0, \quad \bar{R}_{\rho\sigma\mu\nu} \neq 0.
\]

(7)

We usually introduce the metric perturbation around the ST black hole to perform the stability analysis

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}.
\]

(8)
Then, the linearized Einstein equation is given by

$$\delta G_{\mu \nu} + \alpha \left[ \bar{\nabla}^2 \delta G_{\mu \nu} + 2 \bar{R}_{\rho \mu \sigma \nu} \delta G^{\rho \sigma} \right] + (\alpha + 2 \beta) \left[ - \bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{g}_{\mu \nu} \bar{\nabla}^2 \right] \delta R = 0, \quad (9)$$

where the linearized Einstein tensor, Ricci tensor, and Ricci scalar are given by

$$\delta G_{\mu \nu} = \delta R_{\mu \nu} - \frac{1}{2} \delta R \bar{g}_{\mu \nu}, \quad (10)$$

$$\delta R_{\mu \nu} = \frac{1}{2} \left( \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu \rho} + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\mu \rho} - \bar{\nabla}^2 h_{\mu \nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right), \quad (11)$$

$$\delta R = \bar{\nabla}^\mu \bar{\nabla}_\nu h_{\mu \nu} - \bar{\nabla}^2 h. \quad (12)$$

with $h = h^\rho \rho$. It is not easy to solve the linearized equation (9) directly because it is a coupled second-order equation for $\delta G_{\mu \nu}$ and $\delta R$. Thus, it would be better to decouple $\delta R$ from (9). Taking the trace of (9) leads to

$$\left[ (D \alpha + 4(D - 1) \beta) \bar{\nabla}^2 - (D - 2) \right] \delta R = 0, \quad (13)$$

which indicates that the $D$-dimensional D’Alembertian operator disappears if one chooses

$$\alpha = \frac{4(1 - D)}{D} \beta. \quad (14)$$

In this case, the linearized Ricci scalar is constrained to vanish

$$\delta R = 0 \quad (15)$$

which implies that $\delta G_{\mu \nu} \to \delta R_{\mu \nu}$. Substituting this into (9) leads to the equation for the linearized Ricci tensor only

$$\left[ \bar{\nabla}^2 - \frac{D}{4(D - 1) \beta} \right] \delta R_{\mu \nu} + 2 \bar{R}_{\rho \mu \sigma \nu} \delta R^{\rho \sigma} = 0 \quad (16)$$

If we do not require the condition of (14) which eliminates a massive spin-0 (scalar graviton), we could not go any further process. In all dimensions, (14) enables us to write the Lagrangian together with an auxiliary field in the Fierz-Pauli form [16]. For $D = 3$, it is a new massive gravity [17] and the case of $D = 4$ corresponds to the Einstein-Weyl gravity [18]. For $D > 4$, it gives us a critical gravity on $\text{AdS}_D$ spacetimes [19].

After choosing the transverse-traceless gauge (TTG)

$$\bar{\nabla}^\mu h_{\mu \nu} = 0 \text{ and } h = 0, \quad (17)$$
the linearized Ricci tensor takes the form

$$\delta R_{\mu\nu} = \frac{1}{2} \Delta h_{\mu\nu}$$  \hspace{1cm} (18)$$

with the Lichnerowicz operator

$$\Delta h_{\mu\nu} = -\nabla^2 h_{\mu\nu} - 2 \bar{R}_{\rho\mu\sigma\nu} h^{\rho\sigma}.$$  \hspace{1cm} (19)$$

Eq. (16) could be expressed as a fourth-order differential equation \[9\]

$$\Delta (\Delta + M_{D}^2) h_{\mu\nu} = 0$$  \hspace{1cm} (20)$$

which may imply a massless spin-2 equation

$$\Delta h_{\mu\nu}^m = 0$$  \hspace{1cm} (21)$$

and a massive spin-2 equation

$$\Delta (\Delta + M_{D}^2) h_{\mu\nu}^M = 0$$  \hspace{1cm} (22)$$

with the \(D\)-dimensional mass

$$M_{D}^2 = \frac{D}{4(D-1)\beta} = \frac{-1}{\alpha}.$$  \hspace{1cm} (23)$$

On the background of the ST black hole \[5\], we rewrite Eq. (16) as a second-order equation for the linearized Ricci tensor

$$\nabla^2 \delta R_{\mu\nu} + 2 \bar{R}_{\rho\mu\sigma\nu} \delta R^{\rho\sigma} = M_{D}^2 \delta R_{\mu\nu}.$$  \hspace{1cm} (24)$$

Similarly, Eq. (21) is expressed as a linearized massless equation

$$\nabla^2 h_{\mu\nu}^m + 2 \bar{R}_{\rho\mu\sigma\nu} h_{\rho\sigma}^m = 0.$$  \hspace{1cm} (25)$$

and Eq. (22) takes the linearized massive equation

$$\nabla^2 h_{\mu\nu}^M + 2 \bar{R}_{\rho\mu\sigma\nu} h_{\rho\sigma}^M = M_{D}^2 h_{\mu\nu}^M.$$  \hspace{1cm} (26)$$

At this stage, we wish to point out a difference between (24) and (26). The former equation is a second-order equation for the linearized Ricci tensor, whereas the latter is a suggesting second-order equation from the fourth-order equation (20) for the metric perturbation. It is known that the introduction of fourth-order derivative terms give rise to ghost-like
massive graviton [9], which may automatically imply instability of a black hole even if a solution exists [10]. Hence, even though (21) and (22) were used as a linearized massless [massive] equation on the background of Schwarzschild black hole [9], their validity is not yet proved because they seem to be free from ghost. Splitting (20) into two second-order equations (21) and (22) is dangerous because the ‘−’ sign in the front of (22) disappears. A ghost state arises from this sign. Because of a missing of ‘−’, one may argue that Eq. (22) by itself does not represent a correctly linearized equation for studying the stability of the black hole in the fourth-order gravity. However, the overall ‘−’ sign in (22) does not make any difference unless an external source will be put on the right-hand side of (20). Hence, the fourth-order gravity does not automatically imply the instability of the black hole even if one uses (26). Importantly, if one uses (24) instead of (26), one might avoid the ghost issue because (24) is a genuine second-order equation. This is the reason why we will take the Ricci tensor perturbation.

3 Instability of ST Black holes

Let us briefly review the GL stability analysis of (D + 1)-dimensional black strings. This can be represented by a matrix form

\[
\begin{pmatrix}
    h_{\mu \nu} & h_{\mu z} \\
    h_{z \nu} & h_{zz}
\end{pmatrix}
\]

around the (D + 1)-dimensional black string [4]

\[
ds_{BS}^2 = ds_{ST}^2 + dz^2
\]  

with the ST element of \( ds_{ST}^2 \) in [5].

Choosing \( h_{\mu z} = h_{zz} = 0 \), a D-dimensional \( s(l = 0) \)-mode takes the form

\[
h_{D \mu \nu} = e^{\Omega t} e^{ikz} \begin{pmatrix}
    H^H(r) & H^{tr}(r) & 0 & 0 & \ldots \\
    H^{tr}(r) & H^{rr}(r) & 0 & 0 & \ldots \\
    0 & 0 & K(r) & 0 & \ldots \\
    0 & 0 & 0 & K(r) \frac{1}{\sin^2 \theta} & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(29)
The metric perturbation $h^D_{\mu\nu}$ satisfies the massive spin-2 equation

$$\nabla^2 h^D_{\mu\nu} + 2\bar{R}_{\rho\sigma\nu\mu}h^D_{\rho\sigma} = k^2 h^D_{\mu\nu}$$

(30)

together with the TTG of $\nabla^\mu h^D_{\mu\nu} = 0$ and $h^D = 0$. For $k^2 \neq 0$, eliminating all but $H^{tr}$, Eq. (30) reduces to a second-order equation for $H^{tr}$

$$A(r; r_0, D, \Omega^2, k^2) \frac{d^2}{dr^2} H^{tr} + B \frac{d}{dr} H^{tr} + C H^{tr} = 0.$$  

(31)

The two boundary conditions are required: a normalizable solution at infinity is

$$H^{tr}_\infty \sim e^{-\sqrt{\Omega^2 + k^2} r},$$  

(32)

while the solution near the horizon behaves as

$$H^{tr}_{r_0} \sim \frac{1}{(r - r_0)^{1 - \frac{\Omega r_0}{r_0}}}.  

(33)

This is a one-parameter shooting problem with a shooting parameter $\Omega > 0$. Solving this problem numerically to search for $\Omega$ and $k$ shows unstable modes for each $D$-dimensions (see Fig.1 in Ref.[4]). Especially for $e^{\frac{\Omega r_0}{r_0}} e^{i \frac{k r_0}{r_0}}$ setting, there exists a critical non-zero wave number $k_c$ where for $k < k_c(k > k_c)$, the black string is unstable (stable) against the metric perturbations. There is an unstable (stable) mode for any wavelength larger (smaller) than the critical wavelength $\lambda_{GL} = 2\pi r_0/k_c$: $\lambda > \lambda_{GL}(\lambda < \lambda_{GL})$. The critical wave number $k_c$ depending on $D$-dimensions is given by [11]

$$(\begin{array}{c|cccccccccccccc}
D & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
k_c(= r_0 M_D^2) & 0.88 & 1.24 & 1.60 & 1.86 & 2.08 & 2.30 & 2.50 & 2.69 & 2.87 & 3.03 & 3.18 \\
\end{array})$$

(34)

For a massive gravity theory in the Minkowski background, there is correspondence between linearized Ricci tensor $\delta R_{\mu\nu}$ [11] and Ricci spinor $\Phi_{ABCD}$ when using the Newman-Penrose formalism [20]. Here the null real tetrad is necessary to specify polarization modes of massive graviton, as the massive gravity requires null complex tetrad to specify six polarization modes [21, 22]. This implies that in fourth-order gravity with $\alpha = \frac{4(1-D)}{D} \beta$, one may take linearized Ricci tensor $\delta R_{\mu\nu}$ [8], instead of the metric perturbation $h_{\mu\nu}$ in Einstein gravity.

In the above GL analysis, it is obvious that the obtained unstable mode is not a gauge artifact but a genuine physical mode. This is because imposing the TTG condition on a
symmetric tensor \( h^D_{\mu\nu} \) leads to \( D(D+1)/2 - (D+1) = (D+1)(D-2)/2 \) DOF. Considering the s-mode instability, these \( (D+1)(D-2)/2 \) DOF reduces to a single DOF of \( H^{tr} \). Up to now, we did not take into account DOF of \( \delta R_{\mu\nu} \) as physical modes. Here we could not choose a gauge condition like the TTG (17) directly for a linearized Ricci tensor \( \delta R_{\mu\nu} \). Instead, the linearized version of the Bianchi identity

\[
\bar{\nabla}_{[\mu} \delta R_{\nu\rho]} = 0 \tag{35}
\]

implies a relation for \( \delta R_{\mu\nu} \) and \( \delta R \) when contracting (35) as

\[
2\bar{\nabla}^\mu \delta R_{\mu\nu} - \bar{\nabla}_\mu \delta R = 0, \tag{36}
\]

leading to the well-known Bianchi identity \( \bar{\nabla}^\mu \delta G_{\mu\nu} = 0 \) the linearized Einstein tensor. Considering \( \delta R = 0 \) (15), the contracted Bianchi identity (36) reduces to

\[
\bar{\nabla}^\mu \delta R_{\mu\nu} = 0 \tag{37}
\]

which plays a role of the transverse condition. Taking into account (37) together with the traceless condition (15) leads to DOF for \( \delta R_{\mu\nu} \) as

\[
\frac{D(D+1)}{2} - (D+1) = \frac{(D+1)(D-2)}{2} \tag{38}
\]

which is exactly the same DOF for a metric tensor \( h^D_{\mu\nu} \).

At this stage, we emphasize again that (24) is considered as the second-order equation with respect to \( \delta R_{\mu\nu} \), but not the fourth-order equation (20) for \( h_{\mu\nu} \). Hence, we propose \( \delta R_{\mu\nu} \) as physical observables on the ST black hole background. Similarly, we find the same equation (24) when substituting \( h^D_{\mu\nu} \) and \( k^2 \) into \( \delta R_{\mu\nu} \) and \( M_D^2 \) in (30). Also, we impose (37) and (15) to find the s-mode instability of \( \delta R_{\mu\nu} \). Accordingly, the relevant equation for a physical mode of \( \delta R^{tr} \) takes the same form

\[
A(r; r_0, \Omega^2, M_D^2 ) \frac{d^2}{dr^2} \delta R^{tr} + B \frac{d}{dr} \delta R^{tr} + C \delta R^{tr} = 0 \tag{39}
\]

which shows unstable modes for

\[
0 < M_D < M_D^c = \frac{r_0 M_D^c}{r_0} = \frac{k^c}{r_0} \tag{40}
\]

with the D-dimensional mass

\[
M_D = \sqrt{\frac{D}{4(D-1)\beta}}. \tag{41}
\]
and the $D$-dimensional critical mass $M^D_{c}$ (34). Especially, even if one uses (26) as a linearized massive equation [9], our conclusion remains unchanged because (26) and (24) are the same equation for different tensors. It turned out that for $M^2_{D} = 0$, the ST black holes are stable against the metric perturbations in Einstein gravity when using (25) [14].

Consequently, the instability arises from the massiveness ($M^D_{D} > 0$) but not from a feature of the fourth-order equation which gives the $-$ sign (ghost=-negative norm state) when splitting it into two second-order equations. This means that the ST black holes in fourth-order gravity with $\alpha = \frac{4(1-D)}{D} \beta$ do not exist and/or they do not form in the gravitational collapse.

Finally, we could not carry out the stability of the ST black holes in fourth-order gravity with arbitrary $\alpha$ and $\beta$ because the linearized equation (9) is a coupled equation for $\delta G_{\mu\nu}$ and $\delta R$, leading to a fourth-order equation for $\delta \tilde{R}_{\mu\nu} = \delta R_{\mu\nu} - \delta R g_{\mu\nu}/4$ [7].

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