IDENTITIES INVOLVING FROBENIUS-EULER POLYNOMIALS
ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM

ABSTRACT. In this paper we consider non-linear differential equations which are closely related to the generating functions of Frobenius-Euler polynomials. From our non-linear differential equations, we derive some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order.

1. INTRODUCTION

Let $u \in \mathbb{C}$ with $u \neq 1$. Then the Frobenius-Euler polynomials are defined by generating function as follows:

\[ F_u(t, x) = \frac{1 - u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x \mid u) \frac{t^n}{n!}, \quad (\text{see [1,2]}). \]

In the special case, $x = 0$, $H_n(0 \mid u) = H_n(u)$ are called the $n$-th Frobenius-Euler numbers (see [2]).

Thus, by (1) and (2), we get the recurrence relation for $H_n(u)$ as follows:

\[ H_0(u) = 1, \quad (H(u) + 1)^n - H_n(u) = \begin{cases} 1 - u & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \]

with the usual convention about replacing $H(u)^n$ by $H_n(u)$ (see [2,10,12]).

The Bernoulli and Euler polynomials can be defined by

\[ \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \]

In the special case, $x = 0$, $B_n(0) = B_n$ are the $n$-th Bernoulli numbers and $E_n(0) = E_n$ are the $n$-th Euler numbers.

The formula for a product of two Bernoulli polynomials are given by

\[ B_m(x)B_n(x) = \sum_{r=0}^{\infty} \binom{m}{2r} n + \binom{n}{2r} m \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \frac{m!}{(m+n)!} B_{m+n}, \]

where $m + n \geq 2$ and $\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m(m-1)\cdots(m-n+1)}{n!}$ (see [1,3]).
From (1), we note that \( H_n(x \mid -1) = E_n(x) \). In [10], Nielson also obtained similar formulas for \( E_n(x)E_m(x) \) and \( E_m(x)B_n(x) \).

In view point of (4), Carlitz have considered the following identities for the Frobenius-Euler polynomials as follows:

\[
H_m(x \mid \alpha)H_n(x \mid \beta) = H_{m+n}(x \mid \alpha\beta)\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} + \frac{\alpha(1-\beta)}{1-\alpha\beta} \sum_{r=0}^{m} \binom{m}{r} H_r(\alpha)H_{m+n-r}(x \mid \alpha\beta) + \frac{\beta(1-\beta)}{1-\alpha\beta} \sum_{s=0}^{n} \binom{n}{s} H_s(\beta)H_{m+n-s}(x \mid \alpha\beta),
\]

where \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 1, \beta \neq 1 \) and \( \alpha\beta \neq 1 \) (see [2]).

In particular, if \( \alpha \neq 1 \) and \( \alpha\beta = 1 \), then

\[
H_m(x \mid \alpha)H_n(x \mid \alpha^{-1}) = - (1-\alpha) \sum_{r=1}^{m} \binom{m}{r} H_r(\alpha) \frac{B_{m+n-r+1}(x)}{m+n-r+1} - (1-\alpha^{-1}) \sum_{s=1}^{n} \binom{n}{s} H_s(\alpha^{-1}) \frac{B_{m+n-s+1}(x)}{m+n-s+1} + (-1)^{n+1} \frac{m!n!}{(m+n+1)!} (1-\alpha)H_{m+n+1}(\alpha).
\]

For \( r \in \mathbb{N} \), the \( n \)-th Frobenius-Euler polynomials of order \( r \) are defined by generating function as follows:

\[
F^r_u(t, x) = \frac{F_u(t, x) \times F_u(t, x) \times \cdots \times F_u(t, x)}{\text{r-times}} = \left( \frac{1-u}{e^t-u} \right) \times \left( \frac{1-u}{e^t-u} \right) \times \cdots \times \left( \frac{1-u}{e^t-u} \right) e^{xt}
\]

\[
= \sum_{n=0}^{\infty} \frac{H_n^{(r)}(x \mid u)}{n!} t^n \quad \text{for } u \in \mathbb{C} \text{ with } u \neq 1.
\]

In the special case, \( x = 0 \), \( H_n^{(r)}(0 \mid u) = H_n^{(r)}(u) \) are called the \( n \)-th Frobenius-Euler numbers of order \( r \) (see [1-15]).

In this paper we derive non-linear differential equations from (1) and we study the solutions of non-linear differential equations. Finally, we give some new and interesting identities and formulae for the Frobenius-Euler polynomials of higher order by using our non-linear differential equations.

2. Computation of sums of the products of Frobenius-Euler numbers and polynomials

In this section we assume that

\[
F = F(t) = \frac{1}{e^t-u}, \quad \text{and} \quad F^N(t, x) = F \times \cdots \times F e^{xt} \quad \text{for } N \in \mathbb{N}.
\]

Thus, by (7), we get

\[
F^{(1)} = \frac{dF(t)}{dt} = -\frac{e^t}{(e^t-u)^2} = -\frac{1}{e^t-u} + \frac{u}{(e^t-u)^2} = -F + uF^2.
\]
By (8), we get
\[ F^{(1)}(t, x) = (F^{(1)}(t)e^{tx} = -F(t, x) + uF^2(t, x), \]
and \[ F^{(1)} + F = uF^2. \]

Let us consider the derivative of (8) with respect to \( t \) as follows:
\[ 2uF F' = F'' + F'. \]

Thus, by (10) and (8), we get
\[ 2u^2F^3 - 2uF^2 = F'' + F'. \]

From (11), we note that
\[ 2u^2F^3 = F^{(2)} + 3F' + 2F, \quad \text{where} \quad F^{(2)} = \frac{d^2 F}{dt^2}. \]

Thus, by the derivative of (12) with respect to \( t \), we get
\[ 2u^23F^2F' = F^{(3)} + 3F^{(2)} + 2F^{(1)}, \quad \text{and} \quad F^{(1)} = uF^2 - F. \]

By (13), we see that
\[ 3u^3F^4 = F^{(3)} + 6F^{(2)} + 11F^{(1)} + 6F. \]

Thus, from (14), we have
\[ 3u^4F^4(t, x) = F^{(3)}(t, x) + 6F^{(2)}(t, x) + 11F^{(1)}(t, x) + 6F(t, x). \]

Continuing this process, we set
\[ (N - 1)!u^{N-1}F^N = \sum_{k=0}^{N-1} a_k(N)F^{(k)}, \]

where \( F^{(k)} = \frac{d^k F}{dt^k} \) and \( N \in \mathbb{N} \).

Now we try to find the coefficient \( a_k(N) \) in (15). From the derivative of (15) with respect to \( t \), we have
\[ N!u^{N-1}F^{N-1}F^{(1)} = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)} = \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \]

By (8), we easily get
\[ N!u^{N-1}F^{N-1}F^{(1)} = N!u^{N-1}F^{N-1}(uF^2 - F) = N!u^NF^{N+1} - N!u^{N-1}F^N. \]

From (16) and (17), we can derive the following equation (18):
\[ N!u^NF^{N+1} = N(N - 1)!u^{N-1}F^N + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)} \]
\[ = N \sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^{N} a_{k-1}(N)F^{(k)}. \]

In (15), replacing \( N \) by \( N + 1 \), we have
\[ N!u^{N}F^{N+1} = \sum_{k=0}^{N} a_k(N + 1)F^{(k)}. \]
By (18) and (19), we get
\[ N \sum_{k=0}^{N} a_k (N + 1) F^{(k)} = N! u^N F^{N+1} \]
\[ = N \sum_{k=0}^{N-1} a_k (N) F^{(k)} + \sum_{k=1}^{N} a_{k-1} (N) F^{(k)}. \]

By comparing coefficients on the both sides of (20), we obtain the following equations:
\[ N a_0 (N) = a_0 (N + 1), \quad a_N (N + 1) = a_{N-1} (N). \]

For \( 1 \leq k \leq n - 1 \), we have
\[ a_k (N + 1) = Na_k (N) + a_{k-1} (N), \]
where \( a_k (N) = 0 \) for \( k \geq N \) or \( k < 0 \). From (21), we note that
\[ a_0 (N + 1) = N a_0 (N) = N (N - 1) a_0 (N - 1) = \cdots = N (N - 1) \cdots 2 a_0 (2). \]

By (8) and (15), we get
\[ F + F' = u F^2 = \sum_{k=0}^{1} a_k (2) F^{(k)} = a_0 (2) F + a_1 (2) F^{(1)}. \]

By comparing coefficients on the both sides of (24), we get
\[ a_0 (2) = 1, \quad \text{and} \quad a_1 (2) = 1. \]

From (23) and (25), we have \( a_0 (N) = (N - 1)! \). By the second term of (21), we see that
\[ a_N (N + 1) = a_{N-1} (N) = a_{N-2} (N - 1) = \cdots = a_1 (2) = 1. \]

Finally, we derive the value of \( a_k (N) \) in (15) from (22).

Let us consider the following two variable function with variables \( s, t \):
\[ g(t, s) = \sum_{N \geq 1} \sum_{0 \leq k \leq N - 1} a_k (N) \frac{t^N}{N!} s^k, \quad \text{where} \quad |t| < 1. \]

By (22) and (27), we get
\[ \sum_{N \geq 1} \sum_{0 \leq k \leq N - 1} a_{k+1} (N + 1) \frac{t^N}{N!} s^k \]
\[ = \sum_{N \geq 1} \sum_{0 \leq k \leq N - 1} N a_{k+1} (N + 1) \frac{t^N}{N!} s^k + \sum_{N \geq 1} \sum_{0 \leq k \leq N - 1} a_k (N) \frac{t^N}{N!} s^k \]
\[ = \sum_{N \geq 1} \sum_{0 \leq k \leq N - 1} N a_{k+1} (N) \frac{t^N}{N!} s^k + g(t, s). \]
It is not difficult to show that

\[(29) \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} Na_k(N) \frac{t^N}{N!} s^k = \frac{1}{s} \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} Na_k(N) \frac{t^N}{N!} s^{k+1} \]

\[= \frac{1}{s} \sum_{N \geq 1} \sum_{0 \leq k \leq N} a_k(N) \frac{t^N}{(N-1)!} s^k = \frac{1}{s} \sum_{N \geq 1} \left( \sum_{0 \leq k \leq N} a_k(N) \frac{t^N s^k}{(N-1)!} - \frac{a_0(N)t^N}{(N-1)!} \right) \]

\[= \frac{1}{s} \sum_{N \geq 1} \left( \sum_{0 \leq k \leq N} a_k(N) \frac{t^N}{(N-1)!} s^k - t^N \right) = t \sum_{N \geq 1} \sum_{0 \leq k \leq N} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} - \frac{1}{1-t} \]

From (28) and (29), we can derive the following equation:

\[(30) \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1}(N+1) \frac{t^N s^k}{N!} = \frac{1}{s} \left( g'(t, s) - \frac{1}{1-t} \right) + g(t, s). \]

The left hand side of (13) is:

\[(31) \sum_{N \geq 2} \sum_{0 \leq k \leq N-2} a_{k+1}(N) \frac{t^{N-1} s^k}{(N-1)!} = \frac{1}{s} \left( \sum_{N \geq 2} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} \right) \]

\[= \frac{1}{s} \left( \sum_{N \geq 2} \left( \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} \right) - a_0(N) \frac{t^{N-1}}{(N-1)!} \right) \]

\[= \frac{1}{s} \left( \sum_{N \geq 2} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} - \frac{t}{1-t} \right) \]

\[= \frac{1}{s} \left( \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} - a_0(1) - \frac{t}{1-t} \right) \]

By (30) and (31), we get

\[(32) g(t, s) + \frac{1}{s} \left( g'(t, s) - \frac{1}{1-t} \right) = \frac{1}{s} \left( g'(t, s) - \frac{1}{1-t} \right). \]

Thus, by (32), we easily see that

\[(33) 0 = g(t, s) + \frac{t-1}{s} g'(t, s) + \frac{1-t}{s(1-t)} = g(t, s) + \frac{t-1}{s} g'(t, s) + \frac{1}{s}. \]

By (33), we get

\[(34) g(t, s) + \frac{t-1}{s} g'(t, s) = -\frac{1}{s}. \]

To solve (34), we consider the solution of the following homogeneous differential equation:

\[(35) 0 = g(t, s) + \frac{t-1}{s} g'(t, s). \]
Thus, by (35), we get

(36) \[ -g(t, s) = \frac{t - 1}{s} g'(t, s). \]

By (33), we get

(37) \[ \frac{g'(t, s)}{g(t, s)} = \frac{s}{1 - t}. \]

From (37), we have the following equation:

(38) \[ \log g(t, s) = -s \log(1 - t) + C. \]

By (38), we see that

(39) \[ g(t, s) = e^{-s \log(1 - t)} \lambda \quad \text{where} \quad \lambda = e^C. \]

By using the variant of constant, we set

(40) \[ \lambda = \lambda(t, s). \]

From (39) and (40), we note that

(41) \[ g'(t, s) = \frac{dg(t, s)}{dt} = \lambda'(t, s) e^{-s \log(1 - t)} + \frac{\lambda(t, s)e^{-s \log(1 - t)}}{1 - t} s \]

where \( \lambda'(t, s) = \frac{d\lambda(t, s)}{dt}. \)

By multiplying \( \frac{t - 1}{s} \) on both sides in (41), we get

(42) \[ \frac{t - 1}{s} g'(t, s) + g(t, s) = \lambda' \frac{t - 1}{s} e^{-s \log(1 - t)}. \]

From (34) and (42), we get

(43) \[ -\frac{1}{s} = \lambda' \frac{t - 1}{s} e^{-s \log(1 - t)}. \]

Thus, by (43), we get

(44) \[ \lambda' = \lambda'(t, s) = (1 - t)^{s-1}. \]

If we take indefinite integral on both sides of (44), we get

(45) \[ \lambda = \int \lambda' dt = \int (1 - t)^{s-1} dt = -\frac{1}{s}(1 - t)^s + C_1, \]

where \( C_1 \) is constant.

By (39) and (45), we easily see that

(46) \[ g(t, s) = e^{-s \log(1 - t)} \left( -\frac{1}{s}(1 - t)^s + C_1 \right). \]

Let us take \( t = 0 \) in (46). Then, by (27) and (46), we get

(47) \[ 0 = -\frac{1}{s} + C_1, \quad C_1 = \frac{1}{s}. \]
Thus, by (46) and (47), we have

\[ g(t, s) = e^{-s \log(1-t)} \left( \frac{1}{s} - \frac{1}{s} (1-t)^s \right) = \frac{1}{s} (1-t)^{-s} \left( 1 - (1-t)^s \right) \]

\[ = \frac{(1-t)^{-s} - 1}{s} = \frac{1}{s} \left( e^{-s \log(1-t)} - 1 \right). \]

From (48) and Taylor expansion, we can derive the following equation (49):

\[ g(t, s) = \frac{1}{s} \sum_{n \geq 1} \frac{s^n}{n!} \left( - \log(1-t) \right)^n = \sum_{n \geq 1} \frac{1}{n!} \left( \sum_{l_1=1}^{\infty} \frac{t^{l_1}}{l_1} \right)^n \]

\[ = \sum_{n \geq 1} \frac{s^{n-1}}{n!} \sum_{N \geq n} \left( \sum_{l_1=1}^{\infty} \frac{1}{l_1} \right) \frac{1}{l_1 l_2 \cdots l_n} t^N. \]

Thus, by (49), we get

\[ g(t, s) = \sum_{k \geq 0} \frac{s^k}{(k+1)!} \sum_{N \geq k+1} \left( \sum_{l_1, \ldots, l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} \right) t^N \]

\[ = \sum_{N \geq 1} \left( \sum_{0 \leq k \leq N-1} \frac{N!}{(k+1)!} \sum_{l_1, \ldots, l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} \right) \frac{t^N}{N!} s^k. \]

From (27) and (50), we can derive the following equation (51):

\[ a_k(N) = \frac{N!}{(k+1)!} \sum_{l_1, \ldots, l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}}. \]

Therefore, by (15) and (51), we obtain the following theorem.

**Theorem 1.** For \( u \in \mathbb{C} \) with \( u \neq 1 \), and \( N \in \mathbb{N} \), let us consider the following non-linear differential equation with respect to \( t \):

\[ F^N(t) = \frac{N}{u N!} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1, \ldots, l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} F^{(k)}(t), \]

where \( F^{(k)}(t) = \frac{d^k F(t)}{dt^k} \) and \( F^N(t) = F(t) \times \cdots \times F(t) \). Then \( F(t) = \frac{1}{e^{u}} \) is a solution of (52).

Let us define \( F^{(k)}(t, x) = F^{(k)}(t)e^{tx} \). Then we obtain the following corollary.

**Corollary 2.** For \( N \in \mathbb{N} \), we set

\[ F^N(t, x) = \frac{N}{u N!} \sum_{k=0}^{N} \frac{1}{(k+1)!} \sum_{l_1, \ldots, l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} F^{(k)}(t, x). \]

Then \( e^{tx} \) is a solution of (53).
From (1) and (6), we note that

\[
\frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!},
\]

and

\[
\left( \frac{1 - u}{e^t - u} \right) \times \left( \frac{1 - u}{e^t - u} \right) \times \cdots \times \left( \frac{1 - u}{e^t - u} \right) = \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!},
\]

where \( H_n^{(N)}(u) \) are called the \( n \)-th Frobenius-Euler numbers of order \( N \).

By (7) and (54), we get

\[
F_N(t) = \left( \frac{1}{e^t - u} \right) \times \left( \frac{1}{e^t - u} \right) \times \cdots \times \left( \frac{1}{e^t - u} \right)
\]

\[
= \frac{1}{(1 - u)^N} \sum_{l=0}^{\infty} H_l^{(N)}(u) \frac{t^l}{l!},
\]

and

\[
F(t) = \left( \frac{1}{e^t - u} \right) \left( \frac{1}{1 - u} \right) = \frac{1}{1 - u} \sum_{l=0}^{\infty} H_l(u) \frac{t^l}{l!}.
\]

From (55), we note that

\[
F^{(k)}(t) = \frac{d^k F(t)}{dt^k} = \sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!}.
\]

Therefore, by (52), (55) and (56), we obtain the following theorem.

**Theorem 3.** For \( N \in \mathbb{N} \), \( n \in \mathbb{Z}_+ \), we have

\[
H_n^{(N)}(u) = N \left( \frac{1 - u}{u} \right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1 + \cdots + l_{k+1} = N} \frac{H_{n+k}(u)}{l_1 l_2 \cdots l_{k+1}}.
\]
From (55), we can derive the following equation:
\[
\sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} = \left( \frac{1 - u}{e^t - u} \right) \times \left( \frac{1 - u}{e^t - u} \right) \times \cdots \times \left( \frac{1 - u}{e^t - u} \right)
\]
\[
= \left( \sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{t^{l_1}}{l_1!} \right) \times \cdots \times \left( \sum_{l_N=0}^{\infty} H_{l_N}(u) \frac{t^{l_N}}{l_N!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l_1+\cdots+l_N=n} \binom{N}{l_1, \ldots, l_N} \cdot H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u) \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \ldots, l_N} \cdot H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u) \right) \frac{t^n}{n!}
\]
Therefore, by (57), we obtain the following corollary.

**Corollary 4.** For \( N \in \mathbb{N}, n \in \mathbb{Z}_+ \), we have
\[
\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \ldots, l_N} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u)
\]
\[
= N \left( \frac{1 - u}{u} \right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \binom{n}{l_1, \ldots, l_{k+1}} \frac{H_{n+k}(u)}{l_1! l_2! \cdots l_{k+1}!}.
\]

By (53), we obtain the following corollary.

**Corollary 5.** For \( N \in \mathbb{N}, n \in \mathbb{Z}_+ \), we have
\[
H_n^{(N)}(x|u)
\]
\[
= N \left( \frac{1 - u}{u} \right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \binom{n}{l_1, \ldots, l_{k+1}} \frac{H_{n+k}(u) x^{n-m}}{l_1! l_2! \cdots l_{k+1}!}
\]
From (6), we note that
\[
\sum_{n=0}^{\infty} H_n^{(N)}(x|u) \frac{t^n}{n!} = \left( \frac{1 - u}{e^t - u} \right) \times \left( \frac{1 - u}{e^t - u} \right) \times \cdots \times \left( \frac{1 - u}{e^t - u} \right) e^{xt}
\]
\[
= \left( \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l^{(N)}(u) \right) \frac{t^n}{n!}
\]
By comparing coefficients on both sides of (58), we get
\[
H_n^{(N)}(x|u) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l^{(N)}(u).
\]
By the definition of notation, we get
\[ F^{(k)}(t, x) = F^{(k)}(t)e^{tx} = \left( \sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_{l+k}(u)x^{n-l} \right) \frac{t^n}{n!}. \]

From (6), we note that
\[ \sum_{n=0}^{\infty} H^{(N)}_n(x|u) \frac{t^n}{n!} = \left( \frac{1 - u}{e^t - u} \right)^N e^{xt} \]

\[ = \left( \sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{t^{l_1}}{l_1!} \right) \times \cdots \times \left( \sum_{l_N=0}^{\infty} H_{l_N}(u) \frac{t^{l_N}}{l_N!} \right) \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{l_1 + \cdots + l_N + m = n} \frac{H_{l_1}(u)H_{l_2}(u) \cdots H_{l_N}(u)}{l_1! \cdots l_N! m!} x^m \right) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{l_1 + \cdots + l_N + m = n} \binom{n}{l_1, \ldots, l_N, m} H_{l_1}(u) \cdots H_{l_N}(u)x^m \right) \frac{t^n}{n!}. \]

By comparing coefficients on both sides of (58), we get
\[ H^{(N)}_n(x|u) = \sum_{l_1 + \cdots + l_N + m = n} \binom{n}{l_1, \ldots, l_N, m} H_{l_1}(u) \cdots H_{l_N}(u)x^m. \]

REFERENCES

[1] L. Carlitz, *The product of two Eulerian polynomials*, Mathematics Magazine 23(1959), 247-260.
[2] L. Carlitz, *The product of two Eulerian polynomials*, Mathematics Magazine 36(1963), 37-41.
[3] L. Carlitz, *Note on the integral of the product of several Bernoulli polynomials*, J. London Math. Soc. 34(1959), 361-363.
[4] I. N. Cangul, Y. Simsek, *A note on interpolation functions of the Frobenious-Euler numbers*, Application of mathematics in technical and natural sciences, 59-67, AIP Conf. Proc., 1301, Amer. Inst. Phys., Melville, NY, 2010.
[5] K.-W. Hwang, D.V. Dolgy, T. Kim, S.H. Lee, *On the higher-order q-Euler numbers and polynomials with weight alpha*, Discrete Dyn. Nat. Soc. 2011(2011), Art. ID 354329, 12 pp.
[6] L. C. Jang, *On multiple generalized w-Genocchi polynomials and their applications*, Math. Probl. Eng. 2010, Art. ID 316870, 8 pp.
[7] T. Kim, *New approach to q-Euler polynomials of higher order*, Russ. J. Math. Phys. 17 (2010), no. 2, 218-225.
[8] T. Kim, Some identities on the $q$-Euler polynomials of higher order and $q$-Stirling numbers by the fermionic $p$-adic integral on $\mathbb{Z}_p$, Russ. J. Math. Phys. 16 (2009), 484-491.

[9] T. Kim, $q$-generalized Euler numbers and polynomials, Russ. J. Math. Phys. 13 (2006), no. 3, 293-298.

[10] N. Nielsen, Traite elementaire des nombres de Bernoulli, Paris, 1923.

[11] H. Ozden, I. N. Cangul, Y. Simsek, Multivariate interpolation functions of higher-order $q$-Euler numbers and their applications, Abstr. Appl. Anal. 2008(2008), Art. ID. 390857, 16 pages.

[12] H. Ozden, I. N. Cangul, Y. Simsek, Remarks on sum of products of $(h, q)$-twisted Euler polynomials and numbers, J. Inequal. Appl. 2008, Art. ID 816129, 8 pp.

[13] C. S. Ryoo, Some identities of the twisted $q$-Euler numbers and polynomials associated with $q$-Bernstein polynomials, Proc. Jangjeon Math. Soc. 14 (2011), 239–348.

[14] Y. Simsek, O. Yurekli, V. Kurt, On interpolation functions of the twisted generalized Frobenius-Euler numbers, Adv. Stud. Contemp. Math. 15 (2007), 187-194.

[15] Y. Simsek, Special functions related to Dedekind-type $DC$-sums and their applications, Russ. J. Math. Phys. 17 (2010), 495-508.

Taekyun Kim
Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea, E-mail: tkkim@kw.ac.kr