The Mystery of the Shape Parameter

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Abstract. It’s well known that the multiquadrics \((-1)^{\lceil \beta \rceil} (c^2 + \|x\|^2)^\beta, \beta > 0\), and the inverse multiquadrics \((c^2 + \|x\|^2)^\beta, \beta < 0\), are very useful radial basis functions for generating approximating functions. However the optimal choice of the shape parameter \(c\) is unknown. This question has been perplexing many people for many years and is regarded as one of the most important questions in the theory of RBF. Hitherto there are very few theoretical works dealing with this topic. Instead, sporadic results of experiment can be seen in the literature. The purpose of this paper is to uncover its mystery.

keywords: radial basis function, multiquadric, shape parameter

1 Introduction

Before entering the core of our theory, we would like to make a clarification of the definition. We are not going to use the simple forms of the multiquadrics and inverse multiquadrics as mentioned in the abstract. Instead, we define

\[
h(x) := \Gamma \left( -\frac{\beta}{2} \right) (c^2 + |x|^2)^{\frac{\beta}{2}}, \quad \beta \in R \setminus 2N \geq 0, \quad c > 0
\]

(1)

where \(|x|\) is the Euclidean norm of \(x\) in \(R^n\), \(\Gamma\) is the classical gamma function and \(c, \beta\) are constants. The function \(h\) is called multiquadric or inverse multiquadric, respectively, depending on \(\beta > 0\) or \(\beta < 0\). The reason of our adopting \(\Gamma\) is that it will make the Fourier transform of \(h\) and our analytic work simpler.

If \(\beta > 0\), then \(h(x)\) will be conditionally positive definite \([\mathbb{S}]\) of order \(m = \lceil \frac{\beta}{2} \rceil \) where \(\lceil \frac{\beta}{2} \rceil\) denotes the smallest integer greater than or equal to \(\frac{\beta}{2}\). If \(\beta < 0\), \(h(x)\) is c.p.d.(conditionally positive definite) of order \(m = 0\). All these can be found in \([\mathbb{S}]\).

For any approximated function \(f\), our interpolating function will be of the form

\[
s(x) := \sum_{i=1}^{N} c_i h(x - x_i) + p(x)
\]

(2)
where \( p(x) \in P_{m-1} \), the space of polynomials of degree less than or equal to \( m-1 \) in \( R^n \), \( X = \{ x_1, \ldots, x_N \} \) is the set of centers (interpolation points). For \( m = 0 \), \( P_{m-1} := \{ 0 \} \). We require that \( s(\cdot) \) interpolate \( f(\cdot) \) at data points \( (x_1, f(x_1)), \ldots, (x_N, f(x_N)) \). Therefore a linear system of the form

\[
\sum_{i=1}^{N} c_i h(x_j - x_i) + \sum_{j=1}^{Q} b_j p_l(x_j) = f(x_j), \quad j = 1, \ldots, N \tag{3}
\]

and

\[
\sum_{i=1}^{N} c_i p_j(x_i) = 0, \quad j = 1, \ldots, Q \tag{4}
\]

where \( \{p_1, \ldots, p_Q\} \) is a basis of \( P_{m-1} \), has to be satisfied. Since \( h \) is c.p.d., this requirement will be theoretically satisfied \([8]\). However if \( c \) is very large, \( h \) will be numerically constant, making the linear system (3),(4) numerically unsolvable. Moreover, if \( c \) is very large, the coefficient matrix of the linear system will have a very large condition number, making the interpolating function \( f \) unreliable when \( f(x_1), \ldots, f(x_N) \) are not accurately evaluated, as pointed out by Madych in \([7]\).

Our approach for choosing \( c \) is based on Theorem 2.4 and Corollary 2.5 of \([4]\) which we will cite directly but make a slight modification to make them easier to understand.

Before introducing the main theorem, we need some fundamental definitions. Let \( \mathcal{D}(R^n) \) denote the space of complex-valued functions on \( R^n \) that are compactly supported and infinitely differentiable. For any integer \( m \geq 1 \), let

\[
\mathcal{D}_m = \{ \varphi \in \mathcal{D}(R^n) : \int x^\alpha \varphi(x) dx = 0 \text{ for all } |\alpha| < m \}.
\]

If \( m = 0 \), \( \mathcal{D}_m := \mathcal{D}(R^n) \).

**Definition 1.1** Let \( h \) be as in (1) and \( m = \max\{0, \lceil \frac{\beta}{2} \rceil \} \). We write \( f \in \mathcal{C}_{h,m}(R^n) \) if \( f \in C(R^n) \) and there is a constant \( c(f) \) such that for all \( \varphi \) in \( \mathcal{D}_m \),

\[
\left| \int f(x) \varphi(x) dx \right| \leq c(f) \left\{ \int \int h(x-y) \varphi(x) \overline{\varphi(y)} dx dy \right\}^{1/2}. \tag{5}
\]

If \( f \in \mathcal{C}_{h,m}(R^n) \), we let \( \|f\|_h \) to denote the smallest constant \( c(f) \) for which (5) is true.

The function space \( \mathcal{C}_{h,m}(R^n) \), abbreviated as \( \mathcal{C}_{h,m} \), is called native space whose characterization can be found in \([2, 3, 5, 6]\) and \([8]\).

**Definition 1.2** For \( n = 1, 2, 3, \ldots \), the sequence of integers \( \gamma_n \) is defined by \( \gamma_1 = 2 \) and \( \gamma_n = 2n(1 + \gamma_{n-1}) \) if \( n > 1 \).

**Definition 1.3** Let \( n \) and \( \beta \) be as in (1). The numbers \( \rho \) and \( \Delta_0 \) are defined as follows.

(a) Suppose \( \beta < n - 3 \). Let \( s = \lceil \frac{n-3-\beta}{2} \rceil \). Then

(i) if \( \beta < 0 \), \( \rho = \frac{3s}{s+1} \) and \( \Delta_0 = \frac{(2+s)(1+s)-3}{\rho^s} \);

(ii) if \( \beta > 0 \), \( \rho = 1 + \frac{2s}{2s+1} \) and \( \Delta_0 = \frac{(2m+2s)(2m+1+s)-(2m+3)}{\rho^{s+2}} \) where \( m = \lceil \frac{s}{2} \rceil \).

(b) Suppose \( n-3 \leq \beta < n-1 \). Then \( \rho = 1 \) and \( \Delta_0 = 1 \).
(c) Suppose $\beta \geq n - 1$. Let $s = -\lceil \frac{n-\beta-3}{2} \rceil$. Then
\[
\rho = 1 \text{ and } \Delta_0 = \frac{1}{(2m+2)(2m+1)\cdots(2m-s+3)} \text{ where } m = \lceil \frac{\beta}{2} \rceil.
\]

**Theorem 1.4** Let $h$ be defined as in (1) and $m = \max\{0, \lceil \frac{\beta}{2} \rceil \}$. Then given any positive number $b_0$, there are positive constants $\delta_0$ and $\lambda$, $0 < \lambda < 1$, which depend completely on $b_0$ and $h$ for which the following is true: For any cube $E$ in $\mathbb{R}^n$ of side length $b_0$, if $f \in C_{h,m}$ and $s$ is the map defined as in (2) which interpolates $f$ on a finite subset $X$ of $E$, then
\[
|f(x) - s(x)| \leq 2^{\frac{s+1+\beta}{n}} m^{\frac{1}{4n}} \sqrt{\pi} \alpha_n c^2 \sqrt{\Delta_0} \lambda^\frac{\beta}{2} \|f\|_h
\] (6)
holds for all $0 < \delta \leq \delta_0$ and all $x$ in $E$ provided that $\delta = d(E, X) := \sup_{y \in E} \inf_{x \in X} |y - x|$. Here, $\alpha_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, and $c$, $\Delta_0$ were defined in (1) and Definition 1.3 respectively. Moreover $\delta_0 = \frac{1}{6c\gamma_n(m+1)}$, and $\lambda = \left(\frac{2}{3}\right)^{\frac{m}{2n}}$ where
\[
C = \max \left\{ 2\rho' \sqrt{n} e^{2n\gamma_n}, \frac{2}{3b_0} \right\}, \quad \rho' = \frac{\rho}{c}.
\]
The integer $\gamma_n$ was defined in Definition 1.2, and $\|f\|_h$ is the $h$-norm of $f$ in $C_{h,m}$, as defined in Definition 1.1. The constant $\rho$ was defined in Definition 1.3.

**Remark:** Theorem 1.4 is cited directly from [4] with only a slight modification. The proof is very technical. The main contribution of the theorem is that it uncovers the mystery of $\lambda$ and $\delta_0$ whose values were unknown. Both numbers appear in the currently used exponential-type error bound for multiquadric interpolation which was only an existence theorem. Obviously the domain $E$ in Theorem 1.4 can be extended to a more general set $\Omega \subseteq \mathbb{R}^n$ which can be expressed as the union of rotations and translations of a fixed cube of side $b_0$.

In (6) it’s clearly seen that the error bound is greatly influenced by the shape parameter $c$. However, in order to present useful criteria for the choice of $c$, we still need a few theorems.

**Definition 1.5** For any $\sigma > 0$, the class of band-limited functions $f$ in $L^2(\mathbb{R}^n)$ is defined by
\[
B_\sigma = \{ f \in L^2(\mathbb{R}^n) : \hat{f}(\xi) = 0 \text{ if } |\xi| > \sigma \}
\]
where $\hat{f}$ denotes the Fourier transform of $f$.

**Theorem 1.6** Let $h$ be as in (1) with $\beta > 0$. Any function $f$ in $B_\sigma$ belongs to $C_{h,m}$ and
\[
\|f\|_h \leq \sqrt{m!} S(m, n) 2^{-n - \frac{1+\beta}{4} - \frac{1}{2} \frac{1+\beta+n}{4}} e^{\frac{\sigma}{c}} c^{\frac{1-\beta-n}{4}} \|f\|_{L^2(\mathbb{R}^n)}
\] (7)
where $c$, $\beta$ are as in (1) and $S(m, n)$ is a constant determined by $m$ and $n$.

**Proof.** We are going to show $B_\sigma \subseteq C_{h,m}$ by Corollary 3.3 and Theorem 5.2 of [6].

Let $m = \lceil \frac{\beta}{2} \rceil$. Then the requirement $\alpha_n = 0$ for all $|\alpha| = 2m$ in Corollary 3.3 of [6] is an immediate result of Theorem 5.2 of [6]. Since $B_\sigma \subseteq L^2(\mathbb{R}^n)$, any $f \in B_\sigma$ is a member of $S'$ where $S$ denotes the Schwarz space. Now, let $\rho(\xi)$ be the Borel measure mentioned in Corollary 3.3 of [6]. By (3.9) of [6], it suffices to show that
\[
\|f\|_h := \left\{ \sum_{|\alpha| = m} \left( \frac{m!}{\alpha!} \right) \| (D^\alpha f) \|_{L^2(\rho)}^2 \right\}^{1/2} < \infty
\]
for all $f \in B_\sigma$. We proceed as follows.

\[
\left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} (|D^\alpha f|)(\xi)^2 d\rho(\xi) \right\}^{1/2} = \left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} |m^\alpha f(\xi)|^2 d\rho(\xi) \right\}^{1/2} = \left\{ \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^n} \xi^{2m} |f(\xi)|^2 d\rho(\xi) \right\}^{1/2}
\]

\[
= (m!)^{1/2} \left\{ \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \xi^{2\alpha} |f(\xi)|^2 \frac{1}{(2\pi)^{2n}|\xi|^{2n} \hat{h}(\xi)} d\xi \right\}^{1/2} \quad \text{(by (3.8) and Theorem 5.2 of [6])}
\]

\[
= \frac{\sqrt{m!}}{(2\pi)^n} \left\{ \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{f}(\xi)|^2 \frac{1}{2^{1+\frac{n}{2}} (\frac{|\xi|}{\sqrt{2}})^{\frac{n}{2}} - \frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|)} d\xi \right\}^{1/2} \quad \text{(by Theorem 8.15 of [8])}
\]

\[
\leq \frac{\sqrt{m!}}{(2\pi)^n} \cdot 2^{-n+\frac{n}{2}+\frac{1}{4}} \left\{ S(m,n) \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \cdot \frac{|\xi|^\frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|)}{\sqrt{2^{1+\frac{n}{2}} (\frac{|\xi|}{\sqrt{2}})^{\frac{n}{2}} - \frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|) d\xi} \right\}^{1/2} \quad \text{where } S(m,n) \text{ denotes the number of terms in the “sum”}
\]

\[
\leq \frac{\sqrt{m!} S(m,n)}{(2\pi)^n} \cdot c^{\frac{n-\beta}{2}} \cdot \left\{ \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{|\xi|^\frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|)}{\sqrt{2^{1+\frac{n}{2}} (\frac{|\xi|}{\sqrt{2}})^{\frac{n}{2}} - \frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|) d\xi} \right\}^{1/2} \quad \text{(by Corollary 5.12 of [8])}
\]

\[
= \frac{\sqrt{m!} S(m,n)}{(2\pi)^n} \cdot c^{\frac{1-(n+\beta)}{2}} \left\{ \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{|\xi|^\frac{n}{2} \cdot e^{c|\xi|}}{\sqrt{2^{1+\frac{n}{2}} (\frac{|\xi|}{\sqrt{2}})^{\frac{n}{2}} - \frac{n}{2} \cdot \mathcal{K}_{\frac{n}{2}} (c|\xi|) d\xi} \right\}^{1/2}
\]

\[
= \sqrt{m!} S(m,n) \cdot 2^{-n+\frac{n}{2}+\frac{1}{4}} \cdot \pi^{-\frac{n}{2}} \cdot \frac{\sigma^{\frac{1}{2}} \cdot e^{\frac{\frac{n}{2}+\frac{1}{4}}}{\sigma} e^c}{\sqrt{2} \cdot e^{\frac{\frac{n}{2}+\frac{1}{4}}}{\sigma} e^c} \cdot \frac{1}{\|f\|_{L^2(\mathbb{R}^n)}} \leq \infty.
\]

Thus $B_\sigma \subseteq C_{h,m}$ and (7) follows.

If in (1) $\beta < 0$, the norm $\| \cdot \|_h$ is defined in a slightly different way. Hence we handle it separately and present the following theorem.

**Theorem 1.7** Let $h$ be as in (1) with $\beta < 0$ such that $n + \beta \geq 1$ or $n + \beta = -1$. Any function $f$ in $B_\sigma$ belongs to $C_{h,m}$ and satisfies

\[
\|f\|_h \leq 2^{-n+\frac{1}{2}} \pi^{-\frac{n}{2}} \sigma^{\frac{1}{2}} e^c \frac{1}{\|f\|_{L^2(\mathbb{R}^n)}}. \quad \text{(8)}
\]

**Proof:** If $\beta < 0$, then $h$ is conditionally positive definite of order $m = 0$ [8]. By Theorem 5.2 of [6], we know that in Corollary 3.3 of [6] $a_\gamma = 0$ for $|\gamma| = 2m$. Obviously any $f \in B_\sigma$ belongs to $S'$ since $f \in L^2(\mathbb{R}^n)$. In order to apply Corollary 3.3 of [6] to show that $B_\sigma \subseteq C_{h,m}$, it remains to show that $f \in B_\sigma$ implies $f \in L^2(\rho)$ where $d\rho(\xi) := r(\xi) d\xi = \frac{1}{(2\pi)^{2n} \hat{h}(\xi)} d\xi$ as stated in [6].
Now, let \( f \in B_\sigma \). By (3.9) of [8], it suffices to show that \( \|f\| = \|\hat{f}\|_{L^2(\rho)} < \infty \). We proceed as follows.

\[
\|\hat{f}\|_{L^2(\rho)} = \left\{ \int_{R^n} |\hat{f}(\xi)|^2 d\rho(\xi) \right\}^{1/2}
= \left\{ \int_{R^n} |\hat{f}(\xi)|^2 r(\xi) d\xi \right\}^{1/2}
= \left\{ \int_{R^n} |\hat{f}(\xi)|^2 (2\pi)^{-n} \frac{1}{w(-\xi)} d\xi \right\}^{1/2}
\]
where \( w(\cdot) = \hat{h}(\cdot) \) by Theorem 5.2 of [6].

\[
= \left\{ \int_{R^n} |\hat{f}(\xi)|^2 (2\pi)^{-2n-2^{-1}} \frac{1}{|\xi|^{\frac{n+\beta}{2}} \cdot \frac{1}{K_{\frac{n+\beta}{2}}(c|\xi|)}} d\xi \right\}^{1/2}
\]
by (8.7), p.109 of [8] if \( |n + \beta| \geq 1 \) by Corollary 5.12 of [8].

\[
\leq \left\{ 2^{1-\frac{\beta}{n} - 2n} \pi \cdot e^{-\frac{2n+\beta}{2}} \int_{R^n} |\hat{f}(\xi)|^2 |\xi|^{\frac{n+\beta}{2}} e^{c|\xi|} d\xi \right\}^{1/2}
\]

\[
\leq \left\{ 2^{1-\frac{1-\beta}{n} - 2n} \pi \cdot e^{-\frac{2n+\beta}{2} \sigma \cdot \frac{n+\beta+1}{2}} \int_{R^n} |\hat{f}(\xi)|^2 e^{c|\xi|} d\xi \right\}^{1/2}
\]

\[
\leq \left\{ 2^{1-\frac{1-\beta}{n} - 2n} \pi \cdot e^{-\frac{2n+\beta}{2} \sigma \cdot \frac{n+\beta+1}{2}} \int_{|\xi| \leq \sigma} |\hat{f}(\xi)|^2 d\xi \right\}^{1/2}
\]

\[
\leq 2^{1-\frac{1-\beta}{n} - n} \pi \cdot e^{-\frac{1}{4} \sigma \cdot \frac{n+\beta+1}{2}} e^{\frac{c}{4} \pi} \|f\|_{L^2(R^n)}
\]

\[
< \infty
\]

In the preceding deduction we put the restriction \( |n + \beta| \geq 1 \) on it. This is a drawback because the frequently seen case \( n = 1 \) and \( \beta = -1 \) is not covered. This gap will be discussed shortly.

**Corollary 1.8** For any positive number \( \sigma \) and any function \( f \in B_\sigma \), under the conditions of Theorem 1.4,

\[
|f(x) - s(x)| \leq \sqrt{m! S(m, n)(2\pi)^{-\frac{3p}{2}} \frac{\lambda^{\frac{n+\beta}{2}}}{\sqrt{n \sigma} \cdot \sqrt{m}} \cdot \sum_{k=0}^{\frac{1}{2}} \frac{\lambda^{\frac{n+\beta}{2}}}{\sqrt{m}} e^{\frac{c}{2} \lambda^{\frac{n+\beta}{2}}}} (\lambda)^{\frac{1}{2}} \|f\|_{L^2(R^n)}
\]

if \( n + \beta \geq 1 \) or \( n + \beta = -1 \). The number \( S(m, n) \) is determined by \( m \) and \( n \), and is one whenever \( \beta < 0 \). The other constants are as in Theorem 1.4.

### 2 How to choose \( c \)?—a more practical approach

In (9) the only things influenced by \( c \) are \( e^{\frac{1+\beta}{2} - n} \), \( e^{c|\xi|} \) and \( (\lambda)^{\frac{1}{2}} \). The number \( \delta \) is the famous fill distance of the data points, as explained in Corollary 2.5 of [4]. The constants \( \beta, \sigma \) are defined in (1) and Definition 1.5 respectively. Theoretically \( (\lambda)^{\frac{1}{2}} \) is very influential. However it contributes little to the error estimate in practice due to two reasons. First, \( \lambda \) is quite large and near one as shown in Theorem 1.4. Second, \( \delta \) can not be very small(near zero). As shown by Mady in
Let case 1 be as in (1) with $h$ is to the values of $\beta$, $n$, $\sigma$, $\delta$ whenever the preceding example all centers (interpolation points) lie in the cube of side 1. Therefore, whenever $\delta$ is too small, the condition number of the linear system (3), (4) will be very large, making its solution meaningless. Note that $\gamma_n \to \infty$ very fast as $n \to \infty$. The first examples are $\gamma_1 = 2$, $\gamma_2 = 12$, $\gamma_3 = 78$ and $\gamma_4 = 632$. Thus $\lambda$ reacts to the change of $c$ very slowly. The number $\lambda$ will become small enough to make $(\lambda)^{\frac{1}{2}}$ influential only when $c$ is extremely large. For example, if the dimension $n = 2$ and the domain cube $E$ has side length $b_0 = 1$, then

$$\lambda = \left(\frac{2}{3}\right)^\frac{m}{\sqrt{3}}$$

where

$$C = \max \left\{2 \rho' \sqrt{2} e^{4\gamma_2}, \frac{2}{3}\right\}, \quad \rho' = \frac{\rho}{c}, \quad \gamma_2 = 12.$$  

The number $\rho$ is equal to one or a bit larger than one, depending on $\beta$. Thus $C$ is usually very large and $\lambda$ is near one, unless $c$ is very large.

Another problem is that the function

$$h(x) := \Gamma\left(-\frac{\beta}{2}\right) (c^2 + |x|^2)^{\frac{\beta}{2}}$$

will in practice make the linear system (3), (4) numerically unsolvable whenever $c$ is too large. In our preceding example all centers (interpolation points) lie in the cube of side 1. Therefore $|x_i - x_j| \leq \sqrt{2}$. If our computer allows seven or fourteen significant digits, then $h(x_i - x_j)$ will become $\Gamma(-\frac{2}{3})(c^2)^{\frac{2}{3}}$ whenever $c \geq 10^4$ or $10^7$. In our example $\lambda > (\frac{2}{3})^\frac{1}{4}$ and is near one even when $c = 10^{12}$. Therefore $(\lambda)^{\frac{1}{2}}$ is only theoretically influential and can be ignored in practice. What’s in practice influential is $c \frac{\ln \lambda}{4} + c^{\frac{2}{3}}$.

In this section we assume that $\beta$, $b_0$ (the side length of the cube), and $\delta$ are fixed and $m = \max \left\{0, \left[\frac{\delta}{2}\right]\right\}$ as in Theorem 1.4. The optimal choice or suggested value of $c$ is presented according to the values of $\beta$, $n$, $\sigma$, $\delta$, and $b_0$.

**Case 1** $\beta + 1 - n \geq 0$

Let $f \in B_\sigma$ and $E$ be a cube in $R^n$ with side length $b_0$ as in Theorem 1.4. Let $h$ be as in (1) with $n + \beta \geq 1$ or $n + \beta = -1$. Let $0 < \delta < \frac{b_0}{4\gamma_n(m + 1)}$ be fixed in Theorem 1.4. Suppose $\beta + 1 - n \geq 0$. Then

(a) the optimal choice of $c$ is to let $c = 12\rho \sqrt{e^{2n\gamma_n} \gamma_n (m + 1) \delta}$ if $n \neq 1$ or $\beta \neq 1$, where $\rho$ and $\gamma_n$ were defined in Definition 1.3 and 1.2 respectively;

(b) the optimal choice of $c$, when $n = 1$ and $\beta = 1$, is as follows. Let $c_1 = 12\rho \sqrt{e^{2n\gamma_n} \gamma_n (m + 1) \delta}$, $c_0 = 3b_0 e^4$, and $\eta = \frac{1}{4\gamma_n(m + 1)}$. Then

(i) choose $c = 12\rho \sqrt{e^{2n\gamma_n} \gamma_n (m + 1) \delta}$ if $\frac{c}{2} + \eta < 0$;

(ii) if $\frac{c}{2} + \eta < 0$ and $c_1 < \frac{-c}{4(e^{-1} + \eta)} < c_0$, choose $c = c_1$ if $H(c_1) \leq H(c_0)$ and $c = c_0$ if $H(c_0) < H(c_1)$ where $H(c) := c^4 e^{\frac{c}{2}} \left(\frac{3}{8}\right)^{\frac{2}{3}}$;

(iii) choose $c = 12\rho \sqrt{e^{2n\gamma_n} \gamma_n (m + 1) \delta}$ if $\frac{c}{2} + \eta < 0$ and $c_0 \leq \frac{-1}{4(e^{-1} + \eta)}$;

(iv) choose $c = c_0$ if $\frac{c}{2} + \eta < 0$ and $\frac{-1}{4(e^{-1} + \eta)} \leq c_1$. 

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Let $\gamma$ naturally the minimal possible choice $c$ obtained by letting $C = \frac{2b}{3\delta}$ when $c = 3b_0e^4$. Increasing $c$ (i.e. $c > 3b_0e^4$) does not change $\delta_0$, but decreasing $c$ (i.e. $c < 3b_0e^4$) makes $\delta_0$ smaller. After $0 < \delta < \frac{b_0}{4\gamma_n(m+1)}$ is fixed, $c$ cannot be less than $12\rho\sqrt{ne^{2\gamma_n\gamma_n}(m+1)^2}$ because of the restriction $\delta \leq \delta_0$.

(a) follows from (9) obviously because $\lambda^\delta$ can be ignored and the only thing influenced by $c$ is $e^{\frac{\beta}{4} c_3 - c}$. The smaller $c$ is, the smaller the error bound (9) is.

(b) is more complicated. Although we previously mentioned that $\lambda^\delta$ can be ignored practically, there is an exception. When $n = 1$ and $\beta = 1$, $\lambda$ is not very large and $\delta$ is not required to be very small. Therefore the influence of $\lambda^\delta$ has to be taken into consideration.

For $n = 1$, $\beta = 1$, let

$$H(c) := e^{\frac{-\beta}{4} c_3} \cdot e^{\frac{\beta}{4} c_3}$$

$$= e^{\frac{\beta}{4} c_3} \left( \frac{2}{3} \right)^{\frac{1}{12} c_3}$$

$$= \begin{cases} \frac{c}{4} \cdot e^{\frac{\beta}{4} c_3} & \text{if } c \geq 3b_0e^4 \\ \frac{c}{4} \cdot e^{\frac{\beta}{4} c_3} \cdot e^{\eta c} & \text{if } c < 3b_0e^4. \end{cases}$$

$H(c)$ is increasing on $[3b_0e^4, \infty)$. For $c \in (0, 3b_0e^4)$,

$$H(c) = e^{\frac{\beta}{4} c_3} \left( \frac{2}{3} \right)^{\frac{1}{12} c_3}$$

$$= e^{\frac{\beta}{4} c_3} \cdot e^{\eta c} \text{ where } \eta := \frac{\ln \frac{2}{3}}{24e^2\delta} < 0$$

$$= e^{\frac{\beta}{4} c_3} \left( \frac{2}{3} \right)^{\frac{1}{12} c_3}.$$ Then

$$H'(c) = e^{-\frac{\beta}{4} c_3} \cdot e^{\frac{\beta}{4} c_3} \left[ \frac{1}{4} + c \left( \frac{\sigma}{2} + \eta \right) \right].$$

If $\frac{\beta}{2} + \eta \geq 0$, then $H'(c) > 0$ and $H$ is increasing on $(0, 3b_0e^4]$. In this situation the optimal $c$ is naturally the minimal possible choice $c = 12\rho\sqrt{ne^{2\gamma_n\gamma_n}(m+1)^2}$.

Assume $\frac{\beta}{2} + \eta < 0$. Then $H'(c) = 0$ if $c = \frac{\frac{\beta}{2} + \eta}{4} =: c^*$. $H'(c) < 0$ if $c > c^*$ and $H'(c) > 0$ if $c < c^*$. Our criteria (ii), (iii) and (iv) thus follow immediately.

Remark:(a) In Case1(b)(i) the error bound goes to infinity as $c \to \infty$. (b) The numbers $\rho$ and $\gamma_n$ are defined in Definition1.3 and 1.2 respectively.

Case2 $\beta + 1 - n < 0$ Let $f \in B_n$ and $E$ be a cube in $\mathbb{R}^n$ with side length $b_0$ as in Theorem1.4. Let $h$ be as in (1) with $n + \beta \geq 1$ or $n + \beta = -1$. If $0 < \delta < \frac{b_0}{4\gamma_n(m+1)}$ in Theorem1.4 is fixed and $\beta + 1 - n < 0$, the optimal choice of $c$ is to let $c = \max \left\{ \frac{n-\beta-1}{4\rho}, 12\rho\sqrt{ne^{2\gamma_n\gamma_n}(m+1)^2} \right\}$ where $m$, $\gamma_n$, and $\rho$ are defined as in Definition1.1, 1.2 and 1.3 respectively.

Reason: In (9) $(\lambda)^\delta$ can be ignored. The only things influenced by $c$ are $e^{\frac{\beta}{4} c_3}$ and $e^{\frac{\beta}{4} c_3}$. Let
\( g(c) := e^{\frac{b + 1 - n}{4} c^\frac{\beta - 1}{2}} e^{\frac{\beta + 1 - n}{2} c} \). Obviously \( g(c) \to \infty \) as \( c \to 0^+ \) or \( c \to \infty \). The minimum of \( g \) occurs when \( g'(c) = 0 \). Now

\[
g'(c) = \frac{\beta + 1 - n}{4} c^{\frac{\beta - 1}{2}} e^{\frac{\beta + 1 - n}{2} c} + \frac{\beta + 1 - n}{2} c^{\frac{\beta - 1}{2}} e^{\frac{\beta + 1 - n}{2} c}.
\]

In order to meet the requirement \( \delta \leq \delta_0 \) in Theorem 1.4, we set \( \delta < \frac{h}{\gamma_n(m+1)} \), which is the \( \delta_0 \) obtained by letting \( C = \frac{2}{\gamma_n} \). As \( c \) decreases, \( \delta_0 \) will also decrease. The acceptable domain of \( c \) is then \( [12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n(m+1) \delta, \infty) \). Therefore we choose the maximum of the two values as the optimal \( c \).

**Remark:** (a) Case 2 includes the well-known cases \( \beta = 1, n \geq 3 \). (b) Note that in Case 2, the upper bound of \( |f(x) - s(x)| \) goes to infinity if \( c \to \infty \) or \( c \to 0^+ \).

In both Case 1 and 2, we require \( |n + \beta| \geq 1 \). This is a drawback. The worst thing is that the frequently seen case \( \beta = -1, n = 1 \) is excluded. For \( \beta = -1 \), it’s just the inverse multiquadric \( h(x) = \frac{1}{\sqrt{c^2 + |x|^2}} \). For such \( h(x) \), Case 1 and 2 only deal with \( n \geq 2 \). If \( n = 1 \), we need special treatment which is absolutely nontrivial.

**Lemma 2.1** Let \( h \) be as in (1) with \( \beta = -1, n = 1 \). For any \( \sigma > 0 \), if \( f \in B_\sigma \), then \( f \in C_{h,m} \) and

\[
\|f\|_h \leq (2\pi)^{-n/2} \sqrt{\frac{1}{K_0(1)}} \int_{|\xi|<\frac{1}{c}} |\hat{f}(\xi)|^2 d\xi + \frac{1}{a_0} \int_{\frac{1}{c} < |\xi| \leq \sigma} |\hat{f}(\xi)|^2 \sqrt{c|\xi| |c|d\xi} \right)^{1/2}
\]

if \( \frac{1}{c} < \sigma \), where \( a_0 = \frac{\sqrt{\pi}}{2^{1/2}} \), and

\[
\|f\|_h \leq (2\pi)^{-n/2} \left\{ \frac{1}{K_0(1)} \int_{\frac{1}{c} \leq |\xi| \leq 1} |\hat{f}(\xi)|^2 d\xi \right\}^{1/2}
\]

if \( \frac{1}{c} \geq \sigma \).

Proof. By (3.9) of [10], \( \|f\|_h = \|\hat{f}\|_{L^2(\rho)} \) and

\[
\|\hat{f}\|_{L^2(\rho)} = \left\{ \int_{R^n} |\hat{f}(\xi)|^2 d\rho(\xi) \right\}^{1/2}
\]
\[
\left\{ \int_{R^n} |\hat{f}(\xi)|^2 r(\xi) d\xi \right\}^{1/2}
= \left\{ \int_{R^n} |\hat{f}(\xi)|^2 (2\pi)^{-2n} \cdot \frac{1}{w(-\xi)} d\xi \right\}^{1/2}
\]
where \( w(\cdot) = \hat{h}(\cdot) \) by Theorem 5.2 of [6].

\[
= \left\{ \int_{R^n} |\hat{f}(\xi)|^2 (2\pi)^{-2n} 2^{-\frac{1}{4}} \cdot \frac{1}{K_0(c|\xi|)} d\xi \right\}^{1/2}
\]
by (8.7) of [8] (p. 109).

\[
= \left(2\pi\right)^{-n} 2^{-\frac{1}{4}} \left\{ \int_{|\xi| \leq \frac{1}{c}} |\hat{f}(\xi)|^2 d\xi + \int_{|\xi| > \frac{1}{c}} \frac{|\hat{f}(\xi)|^2}{K_0(c|\xi|)} d\xi \right\}^{1/2}
\]
by Corollary 5.12 of [8] and p. 374 of [1] where \( a_0 = \frac{\sqrt{\pi}}{2\Gamma\left(\frac{1}{2}\right)}\sqrt{3} \).

Since \( \hat{f}(\xi) = 0 \) for \( |\xi| > \sigma \), our conclusion follows immediately.

\[\therefore\]

**Lemma 2.2** For any positive number \( \sigma \) and any function \( f \in B_\sigma \), under the conditions of Theorem 1.4 with \( \beta = -1 \) and \( n = 1 \),

\[
|f(x) - s(x)| \leq \sqrt{2\pi} \sqrt{\Delta_0(\lambda)^\frac{1}{2}} \left\{ \frac{A + B}{c} \right\}^{1/2}
\]

where \( A = \frac{1}{\lambda_0(1)} \int_{|\xi| \leq \frac{1}{c}} |\hat{f}(\xi)|^2 d\xi \), \( B = \frac{2\sqrt{\lambda_0(1)}}{\sqrt{\pi}} \int_{\frac{1}{c} < |\xi| \leq \sigma} |\hat{f}(\xi)|^2 \sqrt{c|\xi|} |e^{c|\xi|} d\xi \) if \( \frac{1}{c} \leq \sigma \) and \( B = 0 \) if \( \frac{1}{c} \geq \sigma \).

Proof. This is just an immediate result of Theorem 1.4 and Lemma 2.1.

\[\therefore\]

With Lemma 2.1 and 2.2 we can now begin to analyze the appropriate choice of \( c \).

**Case 3** \( n = 1, \beta = -1 \) Let \( f \in B_\sigma \) and \( E \) be a cube in \( R^n \) with side length \( b_0 \) as in Theorem 1.4. Let \( h \) be as in (1) with \( n = 1 \), \( \beta = -1 \). If \( 0 < \delta < \min\left\{ \frac{1}{2\lambda_0(1)}, \frac{b_0}{6} \right\} \) in Theorem 1.4 is fixed, under the conditions of Theorem 1.4, the suggested choice for \( c \) is to let \( c = \frac{1}{\sigma} \).

**Reason:** As pointed out in the beginning of this section, the number \((\lambda)^\frac{1}{2}\) in (12) can be essentially ignored. Hence the only thing in (12) influenced by changing \( c \) is \( \frac{A + B}{c} \). Note that if \( c \in (0, \frac{1}{\sigma}) \), then \( B = 0 \). We define \( H_1(c) = \frac{A}{c} \) if \( c \in (0, \frac{1}{\sigma}) \) and \( H_2(c) = \frac{A + B}{c} \) if \( c \in \left[ \frac{1}{\sigma}, \infty \right) \). Then

\[
H_1 \left( \frac{1}{\sigma} \right) = \sigma A = \frac{\sigma}{\lambda_0(1)} \int_{|\xi| \leq \sigma} |\hat{f}(\xi)|^2 d\xi = \frac{\sigma}{\lambda_0(1)} \|f\|_2^2 (R^n) = H_2 \left( \frac{1}{\sigma} \right)
\]

Now, for \( c \in (0, \frac{1}{\sigma}) \),

\[
\lim_{c \to 0^+} H_1(c) = \lim_{c \to 0^+} \frac{1}{c\lambda_0(1)} \int_{|\xi| \leq \frac{1}{c}} |\hat{f}(\xi)|^2 d\xi = \lim_{c \to 0^+} \frac{1}{c\lambda_0(1)} \|f\|_2^2 (R^n) = \infty.
\]
For $c \in [\frac{1}{e}, \infty)$,

$$
\lim_{c \to \infty} H_2(c) = \lim_{c \to \infty} \frac{1}{c} \left\{ \frac{1}{K_0(4)} \int_{|\xi| \leq \frac{1}{c}} |\hat{f}(\xi)|^2 d\xi + \frac{2\sqrt{3} \Gamma(\frac{1}{2})}{\sqrt{\pi}} \int_{\frac{1}{c} < |\xi| \leq \sigma} |\hat{f}(\xi)|^2 \sqrt{c |\xi|} d\xi \right\}
\leq \lim_{c \to \infty} \frac{1}{c} \left\{ \frac{\|f\|^2_{L^2(R^n)}}{K_0(1)} + \frac{2\sqrt{3} \Gamma(\frac{1}{2})}{\sqrt{\pi}} \cdot \sqrt{c e^{c\sigma}} \|f\|^2_{L^2(R^n)} \right\}
= \infty.
$$

By the structure of the integrands, we know that in fact $\lim_{c \to \infty} H_2(c) = \infty$, not just $\leq \infty$.

Therefore, the upper bound of $|f(x) - s(x)|$ in (12) goes to infinity as $c \to 0^+$ or $c \to \infty$. This phenomenon together with the agreement of $H_1$ and $H_2$ at $\frac{1}{e}$ suggests that $c = \frac{1}{e}$ is a safe and natural choice.

**Remark:** (a) In Theorem 1.4 we require that $\delta \leq \delta_0$ where $\delta_0 \to 0^+$ as $c \to 0^+$. If $\delta$ is fixed, $c$ cannot approach $0$ arbitrarily. Note that for $n = 1$, $\beta = -1$,

$$
\delta_0 = \left\{ \begin{array}{ll}
\frac{c}{3e^c} & \text{if } c \leq 3b_0 e^4 \\
\frac{1}{e} & \text{if } c \geq 3b_0 e^4.
\end{array} \right.
$$

The requirement $\delta \leq \frac{1}{3e^c}$ is equivalent to $24e^4\delta \leq c$. Therefore our exploration is in fact restricted in the range $c \in [24e^4\delta, 3b_0 e^4] \cup [3b_0 e^4, \infty) = [24e^4\delta, \infty)$. This is a drawback. However the assumption $\delta < \min \{ \frac{1}{24e^4\delta}, \frac{1}{3e^c} \}$ in Case3 guarantees that (i) $\delta \leq \delta_0$ for our final choice $c = \frac{1}{e}$, and $\delta \leq \delta_0$ for all $c \in [24e^4\delta, \infty)$ and (ii) $24e^4\delta \leq \frac{1}{e}$. If $c < 24e^4\delta$, then $\delta \leq \delta_0$ does not hold and (12) may not be true. Therefore in Case3 $c = \frac{1}{e}$ is just a suggested value, and we know that the smaller $\delta$ is, the more reliable our choice $c = \frac{1}{e}$ is because $24e^4\delta \to 0$ as $\delta \to 0$ and $\lim_{c \to 0^+} H_1(c) = \infty$.

(b) As in the case $n = 1$, $\beta = 1$(Case1), the number $\lambda$ may not be very large when $n = 1$, $\beta = -1$.

Therefore it seems that in Case3 the influence of $(\lambda)^6$ should be taken into consideration. However we put it into Case6 of subsection 3.1 because it’s meaningful and useful to see what will happen when $(\lambda)^6$ is ignored.

**Remark:** All the criteria provided in this section are based on Theorem 1.4 where we require that $\delta \leq \delta_0$. Note that $\delta_0 := \frac{1}{6\sqrt{e\gamma_n}}(m+1)$ is usually a very small number. Most time $C = \frac{2\sqrt{\sigma}}{\pi} 2^{8\gamma_n}$ where $\rho = 1$ or a bit greater than 1. For such $C$,

$$
\delta \leq \delta_0 \text{ iff } c \geq [12\rho \sqrt{\pi} e^{2\gamma_n} \gamma_n (m+1)] \delta.
$$

If $\delta = 0.125$, $n = 1$ and $m = 1$, then $\rho = 1$ and $c \geq 6e^4$. The lower bound of $c$ increases rapidly as $n$ increases. If $\delta = 0.125$, $n = 2$, $m = 1$, then $\rho = 1$ and

$$
c \geq 36\sqrt{2e^{48}} > 36\sqrt{2(2^{10})^4} > 36\sqrt{210^{12}}
$$

which will make the linear system (3),(4) numerically unsolvable because $c$ is too large. Since $\gamma_n \to \infty$ very fast as $n \to \infty$, the minimum requirement for $c$ will become extremely stringent for high dimensions. On the other hand, as is well known by RBF people, ill-conditioning will happen whenever the fill distance is small. Therefore the restriction $\delta \leq \delta_0$ is a big problem. Madych’s experiment [7] shows that when the domain size $b_0 = 8$, and $n = 1$, the condition number is already too large if $\delta = 0.125$. Of course $\delta_0$ can be increased by increasing $c$. However the linear system (3),(4) will become numerically unsolvable when $c$ is too large. Moreover, the condition number also grows...
when $c$ increases as Madych\cite{Madych} shows. Therefore, our criteria are still quite theoretical sometimes. A possible way of making things simpler is to decrease $\delta$ so that the choice $c = 12\rho \sqrt{n} e^{2n\gamma_n} \gamma_n (m + 1) \delta$ will be acceptable. Although small $\delta$ will result in a large condition number, this trouble can be coped with by the infinitely precise Mathematica, or by making $f(x_j)'s$ in (3) completely accurate.

3 How to choose $c$?—a more theoretical approach

In section 2 we took into consideration the problems of ill-conditioning and numerical unsolvability of the linear system. Therefore ($\lambda$)\textsuperscript{12} in (9) and (12) was not taken into consideration. Theoretically $\delta$ can be arbitrarily small and hence $\frac{1}{\delta}$ cannot be ignored. Moreover, $\lambda$ can be very small if the side length $b_0$ of the cube is very large and $c$ is very large. Consequently ($\lambda$)\textsuperscript{12} is theoretically very influential and highly depends on $c$. In this section the effect of ($\lambda$)\textsuperscript{12} is taken into consideration and more theoretical criteria are developed. We divide the section into two parts.

3.1 $b_0$ fixed

We first deal with the situation when the domain $E$ is a cube of fixed side length $b_0$.

Note that in Theorem 1.4

$$C = \max \left\{ 2\rho' \sqrt{n} e^{2n\gamma_n} \lambda, \frac{2}{3b_0} \right\}, \rho' = \frac{\rho}{c}.$$ 

Also, $2\rho' \sqrt{n} e^{2n\gamma_n} = \frac{2}{3b_0}$ if and only if $c = 3b_0 \rho \sqrt{n} e^{2n\gamma_n}$. Let $c_0 := 3b_0 \rho \sqrt{n} e^{2n\gamma_n}$. Then $C = 2\rho' \sqrt{n} e^{2n\gamma_n}$ if $c \in (0, c_0]$, and $C = \frac{2}{3b_0}$ if $c \in [c_0, \infty)$. For $c \in (0, c_0]$,

$$\lambda^{\frac{1}{2}} = \left( \frac{2}{3} \right)^{\frac{1}{2} - \gamma_n \delta}$$

$$= \left( \frac{2}{3} \right)^{\frac{1}{12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n \delta}}$$

$$= \left[ \left( \frac{2}{3} \right)^{\frac{1}{12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n \delta}} \right]^c$$

$$= e^{\eta(\delta) c}$$

where $\eta(\delta) < 0$ is some number depending on $\delta$. In fact $\eta(\delta) := \frac{\ln \left( \frac{2}{3} \right)}{12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n \delta}$. In (9), for $c \in (0, c_0]$,

the part influenced by $c$ is $e^{\frac{12 \beta - n}{4} - \eta(\delta) + \frac{2}{3} m}$.

Let’s define $H_1(c) := e^{\frac{12 \beta - n}{4} - \eta(\delta) + \frac{2}{3} m}$ for $c \in (0, c_0]$.

For $c \in [c_0, \infty)$, we define $H_2(c) := e^{\frac{12 \beta - n}{4} - \eta(\delta)}$ where $\lambda = \left( \frac{2}{3} \right)^{\frac{1}{12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n \delta}} = \left( \frac{2}{3} \right)^{\frac{1}{12 \rho \sqrt{n} e^{2n\gamma_n} \gamma_n \delta}}$.

Note that $C \geq \frac{2}{3b_0}$ if $c \in (0, c_0]$ and $C = \frac{2}{3b_0}$ if $c \in [c_0, \infty)$.

Now let

$$H(c) := \begin{cases} H_1(c) & \text{if } c \in (0, c_0] \\ H_2(c) & \text{if } c \in [c_0, \infty). \end{cases}$$

Then minimizing the error bound (9) is equivalent to minimizing $H(c)$.

The function $H(c)$ is continuous and satisfies $H_1(c_0) = H_2(c_0)$. Theoretically, $c^*$ is the optimal choice of $c$ if $H(c^*)$ is the minimum of $H$. 

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Case 1 \[1 + \beta - n \geq 0\] and \([\eta(\delta) + \frac{\sigma}{2} > 0]\) Let \(f \in B_\sigma\) and \(E\) be a cube in \(R^n\) with side length \(b_0\) as in Theorem 1.4. Let \(h\) be as in (1) with \(n + \beta \geq 1\) or \(n + \beta = -1\). For any fixed \(0 < \delta < \frac{b_0}{4\gamma_n(m+1)}\) in Theorem 1.4, if \(1 + \beta - n \geq 0\) and \(\eta(\delta) + \frac{\sigma}{2} > 0\) where \(\eta(\delta)\) is defined as in the beginning of 3.1, the smaller \(c\) is, the better it is. Since there is a requirement \(\delta \leq \delta_0\) in Theorem 1.4, \(c\) cannot be arbitrarily small. The optimal choice of \(c\) is then

\[
12\rho\sqrt{n\epsilon^{2n\gamma_n}\gamma_n(m+1)\delta}
\]

since \(\delta \leq \delta_0\) implies \(12\rho\sqrt{n\epsilon^{2n\gamma_n}\gamma_n(m+1)\delta} \leq c\) and \(H(c)\) is increasing.

Remark: (a) Usually \(\eta(\delta) \approx 0\) in practice since \(\delta\) cannot be very small. Hence \(\eta(\delta) + \frac{\sigma}{2} > 0\) usually holds. (b) Before \(c\) is given, \(\delta_0\) is unknown. The number \(\delta_0\) changes with \(c\) only when \(c \leq c_0 := 3b_0\rho\sqrt{n\epsilon^{2n\gamma_n}}\), and \(\delta_0 \to 0^+\) as \(c \to 0^+\). Therefore we require in advance that \(\delta < \frac{b_0}{4\gamma_n(m+1)}\) where \(\frac{b_0}{4\gamma_n(m+1)}\) is just the \(\delta_0\) when \(c = c_0\). The requirement \(\delta \leq \delta_0\) is a drawback and we have in fact investigated \(c\) only in the range \([12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta, \infty)\). However the smaller \(\delta\) is, the more meaningful our optimal choice of \(c\) will be.

Case 2 \[1 + \beta - n > 0\] and \([\eta(\delta) + \frac{\sigma}{2} < 0]\) Let \(f \in B_\sigma\) and \(E\) be a cube in \(R^n\) with side length \(b_0\) as in Theorem 1.4. Let \(h\) be as in (1) with \(n + \beta \geq 1\) or \(n + \beta = -1\). For any fixed \(0 < \delta < \frac{b_0}{4\gamma_n(m+1)}\) in Theorem 1.4, if \(1 + \beta - n > 0\) and \(\eta(\delta) + \frac{\sigma}{2} < 0\) where \(\eta(\delta)\) is defined as in the beginning of 3.1, the optimal \(c\) is chosen as follows.

Obviously \(H(c)\) is increasing on \([c_0, \infty)\). Therefore the minimum of \(H(c)\) happens in the interval \([12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta, c_0]\) under the assumption \(\delta < \frac{b_0}{4\gamma_n(m+1)}\). In this closed interval \(H(c) = H_1(c)\) and \(H'_1(c) = 0\) if \(c = \frac{n-\beta-1}{4\gamma+2\sigma} =: c^*\). Moreover, \(H'_1(c) > 0\) if \(c < c^*\) and \(H'_1(c) < 0\) if \(c > c^*\). Therefore,

(a) \(\min_{c \in [0, \infty)} H(c) = \min \{ H_1(c) : c = 12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta \text{ or } 3b_0\rho\sqrt{n\epsilon^{2n\gamma_n}} (= c_0) \} \text{ if } c^* \in [12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta, c_0]\),

(b) \(\min_{c \in [0, \infty)} H(c) = H_1(c_0)\) if \(c^* < 12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta\), and

(c) \(\min_{c \in [0, \infty)} H(c) = H_1(12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta)\) if \(c^* > c_0\).

The optimal choice of \(c\) is then where \(H(c)\) is minimal. In other words, let \(c = 12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta\) if \(c^* > c_0\), and \(c = c_0\) if \(c^* < 12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta\), and so on.

Remark: (a) Seemingly this criterion is complicated. However in practice \(\delta\) cannot be very small. Therefore \(\eta(\delta)\) is near zero and the situation \(\eta(\delta) + \frac{\sigma}{2} < 0\) rarely happens. (b) In this case although we wrote \(c \in (0, \infty)\) for simplicity, in fact we only investigated \(c \in [12\rho\sqrt{n\epsilon^{2n\gamma_n}}\gamma_n(m+1)\delta, \infty)\) because of the restriction \(\delta \leq \delta_0\). This kind of situation will appear again subsequently in the text.

Case 3 \[1 + \beta - n = 0\] and \([\eta(\delta) + \frac{\sigma}{2} < 0]\) Let \(f \in B_\sigma\) and \(E\) be a cube in \(R^n\) with side length \(b_0\) as in Theorem 1.4. Let \(h\) be as in (1) with \(n + \beta \geq 1\) or \(n + \beta = -1\). For any fixed \(0 < \delta < \frac{b_0}{4\gamma_n(m+1)}\) in Theorem 1.4, if \(1 + \beta - n = 0\) and \(\eta(\delta) + \frac{\sigma}{2} < 0\) where \(\eta(\delta)\) is defined as in the beginning of 3.1, the optimal choice of \(c\) is \(c = 3b_0\rho\sqrt{n\epsilon^{2n\gamma_n}}\). In other words, we choose \(c = c_0\).

Reason: By the definition of \(H(c)\), it’s obvious that in this situation \(H\) is decreasing if \(c \leq c_0\).
and increasing if \( c \geq c_0 \). Our criterion thus follows.

**Case 4** Let \( 1 + \beta - n < 0 \) and \( \eta + \frac{\sigma}{2} > 0 \) Let \( f \in B_\sigma \) and \( E \) be a cube in \( R^n \) with side length \( b_0 \) as in Theorem 1.4. Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( 0 < \delta < \frac{b_0}{4\gamma_n(m+1)} \) in Theorem 1.4, if \( 1 + \beta - n < 0 \) and \( \eta(\delta) + \frac{\sigma}{2} > 0 \) where \( \eta(\delta) \) is defined as in the beginning of 3.1, the optimal \( c \) is chosen as follows.

By the definition of \( H \), \( \lim_{c \to 0^+} H(c) = \lim_{c \to \infty} H(c) = \infty \).

\[
H_1'(c) = e^{\frac{1}{2} - \frac{\sigma}{4}} \cdot e^{(\eta + \frac{\sigma}{2})c} \cdot \left[ n - 1 - \beta \right] = 0 \text{ if } c = \frac{n - 1 - \beta}{4(\eta + \frac{\sigma}{2})} =: c_1.
\]

So

\[
\min_{c \in (0, c_0]} H(c) = \begin{cases} 
H_1(c_0) & \text{if } c_1 \geq c_0, \\
H_1(12\rho\sqrt{m e^{2n\gamma_n} \gamma_n(m+1)} \delta c_1) & \text{if } 12\rho\sqrt{m e^{2n\gamma_n} \gamma_n(m+1)} \delta c_1 < c_1 < c_0, \\
H_1(12\rho\sqrt{m e^{2n\gamma_n} \gamma_n(m+1)} \delta) & \text{if } 12\rho\sqrt{m e^{2n\gamma_n} \gamma_n(m+1)} \delta) < c_1 < c_0.
\end{cases}
\]

Now,

\[
H_2'(c) = e^{\frac{1}{2} - \frac{\sigma}{4}} \cdot e^{(\eta + \frac{\sigma}{2})c} \cdot \left[ \eta \right] = 0 \text{ if } c = \frac{n - 1 - \beta}{2\sigma} =: c_2.
\]

So

\[
\min_{c \in [c_0, \infty)} H(c) = \begin{cases} 
H_2(c_0) & \text{if } c_2 \leq c_0, \\
H_2(c_2) & \text{if } c_0 < c_2.
\end{cases}
\]

These give

\[
\min_{c \in (0, \infty)} H(c) = \min \left\{ \min_{c \in (0, c_0]} H(c), \min_{c \in [c_0, \infty)} H(c) \right\}.
\]

The optimal choice of \( c \) is where the minimum of \( H(c) \) happens.

**Case 5** Let \( f \in B_\sigma \) and \( E \) be a cube in \( R^n \) with side length \( b_0 \) as in Theorem 1.4. Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( 0 < \delta < \frac{b_0}{4\gamma_n(m+1)} \) in Theorem 1.4, if \( 1 + \beta - n < 0 \) and \( \eta(\delta) + \frac{\sigma}{2} < 0 \) where \( \eta(\delta) \) is defined as in the beginning of 3.1, the optimal \( c \) is (i) \( c = \frac{n - 1 - \beta}{2\sigma} \) if \( c_0 \leq \frac{n - 1 - \beta}{2\sigma} \) or (ii) \( c = c_0 \) if \( \frac{n - 1 - \beta}{2\sigma} < c_0 \).

**Reason:** In this case \( H(c) \) is decreasing on \([0, c_0]\). Then we have the following two cases. (i) Assume \( c_0 \leq \frac{n - 1 - \beta}{2\sigma} \). On \([c_0, \infty)\), \( H'(c) = 0 \) if and only if \( c = \frac{n - 1 - \beta}{2\sigma} \). \( H \) is decreasing on \([c_0, \frac{n - 1 - \beta}{2\sigma}]\) and increasing on \([\frac{n - 1 - \beta}{2\sigma}, \infty)\). (ii) Assume \( \frac{n - 1 - \beta}{2\sigma} < c_0 \). Then \( H'(c) > 0 \) on \([c_0, \infty)\). The optimal choice thus follows.

**Remark:** Up to now we always put the restriction \( n + \beta \geq 1 \) or \( n + \beta = -1 \) on the criteria. The frequently seen case \( n = 1, \beta = -1 \) still has to be treated.

We analyze this case as follows. Note that when \( n = 1, \beta = -1 \),

\[
C = \begin{cases} 
\frac{2e^4}{c} & \text{if } c \in (0, 3b_0e^4), \\
\frac{2}{3b_0} & \text{if } c \in [3b_0e^4, \infty).
\end{cases}
\]
There is a restriction $\delta \leq \delta_0 = \frac{\delta}{6(C - \gamma_n n^{1+})} = \frac{1}{12C}$. If $C = \frac{2}{3b_0}$, $\delta_0 = \frac{\delta}{8}$. For any fixed $\delta < \frac{\delta}{8}$, $\delta_0$ decreases as $c$ decreases in $(0, 3b_0e^4]$. In order to satisfy $\delta \leq \delta_0$, the minimal choice of $c$ is $24\delta e^4$.

In the domain $c \in [24\delta e^4, 3b_0e^4]$, $(\lambda)^+ \frac{1}{\delta}$ is a decreasing function of $c$, and

$$
(\lambda)^+ = \begin{cases}
\frac{\delta}{8} & \text{if } c = 24\delta e^4, \\
\left(\frac{2}{3}\right) \frac{b_0}{3^8} & \text{if } c = 3b_0e^4.
\end{cases}
$$

This fact together with $\lim_{c \to 0^+} H_1(c) = \infty$ in Case3 of section2 suggests that letting $c \to 0^+$ should be avoided and the smaller $\delta$ is, the more obvious this criterion is.

Now let’s investigate the behavior of the error bound when $c \in [3b_0e^4, \infty)$. In this case $C$, and hence $\lambda$, is fixed. Therefore $(\lambda)^+ \frac{1}{\delta}$ is independent of $c$. As in Case3 of section2, $\lim_{c \to \infty} H_2(c) = \infty$. Letting $c \to \infty$ should also be avoided.

In Case3 of section2 the suggested value of $c$ is $\frac{1}{3}$ when $(\lambda)^+ \frac{1}{\delta}$ is not taken into consideration. Now, as mentioned above, $(\lambda)^+ \frac{1}{\delta}$ is a decreasing function of $c$ on $[24\delta e^4, 3b_0e^4]$. If $3b_0e^4 < \frac{1}{3}$, we suggest the choice $c = \frac{1}{3}$. However if $\frac{1}{3} \leq 3b_0e^4$, the crucial part $H_2(c)$ of the error bound tends to be increasing for $c \in [\frac{1}{3}, 3b_0e^4]$, but $(\lambda)^+ \frac{1}{\delta}$ is decreasing for $c \in [\frac{1}{3}, 3b_0e^4]$. They offset each other. Note that if $\delta$ is very small, $(\lambda)^+ \frac{1}{\delta}$ will be influential. Therefore we suggest that $c = 3b_0e^4$ if $\delta$ is very small and $\frac{1}{3} \leq 3b_0e^4$, and $c = \frac{1}{\delta}$ if $\delta$ is not very small and $\frac{1}{3} \leq 3b_0e^4$. In order to make $\delta \leq \delta_0$ when $c = \frac{1}{\delta}$, we set $\delta \leq \frac{1}{24\delta e^4}$.

We summarize our discussion as follows.

**Case6** $n = 1$ and $\beta = -1$. Let $f \in B_\sigma$ and $E$ be a cube in $R^n$ with side length $b_0$ as in Theorem1.4. Let $h$ be as in (1) with $n = 1, \beta = -1$. For any fixed $0 < \delta < \min \left\{ \frac{1}{24\sigma^{n}}, \frac{\delta}{8} \right\}$ in Theorem1.4, the suggested value of $c$ is as follows.

(a) $c = \frac{1}{\sigma}$ if $3b_0e^4 < \frac{1}{3}$.

(b) $c = 3b_0e^4$ if $\frac{1}{3} \leq 3b_0e^4$ and $\delta$ is very small; $c = \frac{1}{\sigma}$ if $\frac{1}{3} \leq 3b_0e^4$ and $\delta$ is not very small.

**Remark:** In Case6 only suggested values, rather than optimal values, of $c$ were provided. The main reason is that in the crucial parts $H_1(c)$ and $H_2(c)$ of the error bound, as in Case3 of section2, the integrals involve the Fourier transform of the approximated function $f$ whose value cannot be controlled.

### 3.2 $b_0$ not fixed

Recall that in Theorem1.4 the crucial constant $C$ is defined by

$$
C = \max \left\{ \frac{2\rho}{\sqrt{n\sigma e^{2n\gamma_n}}, \frac{2}{3b_0}} \right\}, \rho' = \frac{\rho}{c}.
$$

The number $C$ is independent of $c$ whenever $c \in [3b_0\rho\sqrt{n\sigma e^{2n\gamma_n}}, \infty)$. Therefore increasing $c$ does not influence $(\lambda)^+ \frac{1}{\delta}$ in (6) and (9) whenever $c$ is large enough. However this situation will be totally changed if the side length $b_0$ of the cube is not fixed. In some domains any point is contained in a cube of side length $b_0$ where $b_0$ can be made arbitrarily large and the cube is still contained in the domain. For example,

$$
\Omega := \{ x = (x_1, \cdots, x_n) : x_i \geq 0, i = 1, \cdots, n \}$$
or \( \Omega = \mathbb{R}^n \). Madych calls such domains invariant under dilation in [7]. For such domains one can keep \( C = \frac{2\rho}{\sqrt{D_{2n\gamma_n}}} \) by increasing \( b_0 \) when necessary. (We never decrease \( b_0 \) because it will make the error bound worse.) Thus \( C \) and \((\lambda)\) are influenced by \( c \) even when \( c \) is very large. We consider this phenomenon to be only theoretically meaningful. Note that \( 2\rho' \sqrt{D_{2n\gamma_n}} = \frac{2\rho}{\sqrt{D_{2n\gamma_n}}} \) usually happens only when \( c \) is extremely large. In practice the linear equations (3),(4) will become numerically unsolvable and ill-conditioning will happen when \( c \) is too large. Most time \( C = \frac{2\rho}{\sqrt{D_{2n\gamma_n}}} \) automatically and we don’t need to increase \( b_0 \) to keep \( C = \frac{2\rho}{\sqrt{D_{2n\gamma_n}}} \).

The word domain should be further interpreted so that ambiguity can be avoided. In this paper the interpolated function \( f \) belongs to \( B_\sigma \) and therefore is defined on the entire \( \mathbb{R}^n \). However in practical problems often interpolation can be done in a subset \( \Omega \) of \( \mathbb{R}^n \). The centers (interpolation points) are in \( \Omega \) only. Here we call the set \( \Omega \) domain too.

In (9) the part influenced by \( c \) is then theoretically

\[
\frac{c^{\beta+1-n}}{c^{\frac{\beta}{2}}} \cdot \frac{e^{\eta^2}}{c^{\frac{\beta}{2}} (\lambda)^\frac{\beta}{2}} = \frac{c^{\beta+1-n}}{c^{\frac{\beta}{2}}} \cdot \frac{e^{\eta^2}}{c^{\frac{\beta}{2}} (\lambda)^\frac{\beta}{2}} \cdot \left( \frac{2}{3} \right)^{\frac{1}{12\rho \sqrt{D_{2n\gamma_n}}} n \gamma_n^\delta} \\
= \frac{c^{\beta+1-n}}{c^{\frac{\beta}{2}}} \left[ e^{\frac{\eta^2}{c^{\frac{\beta}{2}} (\lambda)^\frac{\beta}{2}}} \left( \frac{2}{3} \right)^{\frac{1}{12\rho \sqrt{D_{2n\gamma_n}}} n \gamma_n^\delta} \right]^c \\
= \frac{c^{\beta+1-n}}{c^{\frac{\beta}{2}}} \left[ H(\sigma, \delta) \right]^c
\]

where

\[
H(\sigma, \delta) := e^{\frac{\eta^2}{c^{\frac{\beta}{2}} (\lambda)^\frac{\beta}{2}}} \left( \frac{2}{3} \right)^{\frac{1}{12\rho \sqrt{D_{2n\gamma_n}}} n \gamma_n^\delta}.
\]

We analyze as follows.

**Case 1** \([\beta+1-n > 0] \) and \([H(\sigma, \delta) \geq 1]\) Let \( f \in B_\sigma \) and \( \Omega \subseteq \mathbb{R}^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \). Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( \delta > 0 \) in Theorem1.4, the optimal choice of \( c \) is to let \( c = \frac{12 \rho \gamma_n (m+1) \delta}{\sqrt{D_{2n\gamma_n}}} \) where \( m = \max \left\{ 0, \left\lfloor \frac{\beta}{2} \right\rfloor \right\} \) denotes the order of conditional positive definiteness of \( h \), and \( \gamma_n \) is defined as in Definition1.2.

**Reason:** In this case the error bound (9) tends to zero as \( c \to 0^+ \). However Theorem1.4 requires that \( \delta \leq \delta_0 \) where \( \delta_0 = \frac{12 \rho \sqrt{D_{2n\gamma_n}}} {\gamma_n (m+1)} \) because \( b_0 \) can be made arbitrarily large to keep \( C = \frac{2\rho}{\sqrt{D_{2n\gamma_n}}} \). Therefore the minimal possible choice of \( c \) is \( \frac{12 \rho \gamma_n (m+1) \delta}{\sqrt{D_{2n\gamma_n}}} \).

**Remark:** Now, for any \( \delta > 0 \), we can always choose \( c \) and \( b_0 \) making \( \delta \leq \delta_0 \) in Theorem1.4. In fact, for any \( \delta > 0 \), choose \( b_0 \) large enough such that \( \delta < \delta_0 := \frac{1}{6c \gamma_n (m+1)} \) where \( C := \frac{2 \rho}{\sqrt{D_{2n\gamma_n}}} \). Then The relation \( \delta \leq \delta_0 \) can be maintained by increasing \( b_0 \) and choosing \( c \geq \frac{12 \rho \gamma_n \sqrt{D_{2n\gamma_n}} (m+1) \delta}{\sqrt{D_{2n\gamma_n}}} \).

**Case 2** \([\beta+1-n < 0] \) and \([H(\sigma, \delta) > 1]\) Let \( f \in B_\sigma \) and \( \Omega \subseteq \mathbb{R}^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \). Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( \delta > 0 \) in Theorem1.4 the optimal choice of \( c \) is to let

\[
c = \begin{cases} 
  c_0 & \text{if } c^* := \frac{n-1-\beta}{4\eta} < c_0 \\
  c^* & \text{if } c_0 \leq c^* := \frac{n-1-\beta}{4\eta}
\end{cases}
\]
where \( c_0 := 12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta \) and \( \eta \) is a positive number defined by \( \eta := \frac{\sigma}{2} + \frac{\ln \frac{\delta}{\nu}}{12 \rho \sqrt{n e^{2n \gamma_n}} \gamma_n \delta} \).

**Reason:** It’s easily seen that in this case \( c \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c \rightarrow \infty \) both as \( c \rightarrow 0^+ \) and \( c \rightarrow \infty \). Since \( H(\sigma, \delta) > 1 \), there exists a positive number \( \eta \) such that \( H(\sigma, \delta) = \delta^{\eta} \). By the definition of \( H(\sigma, \delta) \) we easily get \( \eta = \frac{\sigma}{2} + \frac{\ln \frac{\delta}{\nu}}{12 \rho \sqrt{n e^{2n \gamma_n}} \gamma_n \delta} \). Since \( C = \frac{2\rho \sqrt{n e^{2n \gamma_n}}}{c} \) and there is a requirement \( \delta \leq \delta_0 = \frac{1}{12 \rho \sqrt{n e^{2n \gamma_n}} \gamma_n \delta} \) in Theorem 1.4, the minimal possible choice of \( c \) is \( c_0 := 12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta \).

Let \( G(c) := c \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c \). Then \( G'(c) = 0 \) iff \( c \equiv c^* := \frac{n - 1 - \beta}{4n} \). \( G'(c) < 0 \) if \( c < c^* \) and \( G'(c) > 0 \) if \( c > c^* \). In the domain \([c_0, \infty)\) the minimum of \( G(c) \) happens at \( c_0 \) if \( c^* < c_0 \), and at \( c^* \) if \( c^* \geq c_0 \). Our criterion thus follows.

**Remark:** The case \( \beta + 1 - n < 0 \) and \( H(\sigma, \delta) = 1 \) is trivial. The larger \( c \) is, the better it is.

**Case 3** \( \beta + 1 - n > 0 \) and \( |H(\sigma, \delta)| < 1 \): Let \( f \in B_r \) and \( \Omega \subseteq R^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \subseteq \Omega \). Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( \delta > 0 \) in Theorem 1.4, the larger \( c \) is, the better it is.

**Reason:** Obviously \( c \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c \rightarrow 0 \) both as \( c \rightarrow 0^+ \) and \( c \rightarrow \infty \) in this case. However as in Case 2 the requirement \( \delta \leq \delta_0 \) makes \( c \in [12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta, \infty) \).

**Case 4** \( \beta + 1 - n \leq 0 \) and \( |H(\sigma, \delta)| < 1 \): Let \( f \in B_r \) and \( \Omega \subseteq R^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \subseteq \Omega \). Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( \delta > 0 \) in Theorem 1.4, the larger \( c \) is, the better it is.

**Reason:** \( H(\sigma, \delta) < 1 \) implies \( H(\sigma, \delta) = \delta^\eta \) for some \( \eta < 0 \). Then \( c \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c \rightarrow 0 \) as \( c \rightarrow \infty \).

**Case 5** \( \beta + 1 - n = 0 \) and \( |H(\sigma, \delta)| > 1 \): Let \( f \in B_r \) and \( \Omega \subseteq R^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \subseteq \Omega \). Let \( h \) be as in (1) with \( n + \beta \geq 1 \) or \( n + \beta = -1 \). For any fixed \( \delta > 0 \) in Theorem 1.4, the optimal choice of \( c \) is to let \( c = 12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta \).

**Reason:** In this case \( \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c = |H(\sigma, \delta)|^c \) which increases as \( c \) increases. The requirement \( \delta \leq \delta_0 \) tells us that \( c \in [12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta, \infty) \). Hence we choose \( c = 12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta \).

**Remark:** In Case 5 it should be noticed that when \( c = 12\rho \sqrt{n e^{2n \gamma_n}} \gamma_n (m + 1) \delta \), \( c \frac{\beta + 1 - n}{2} \cdot |H(\sigma, \delta)|^c \rightarrow 1 \) and is not very small. The error bound (9) may not be very satisfactory. Also, in Case 4 and 5 we excluded the case \( H(\sigma, \delta) = 1 \) because it rarely happens.

**Case 6** \( n = 1 \) and \( \beta = -1 \): Let \( f \in B_r \) and \( \Omega \subseteq R^n \) such that for any \( x \in \Omega \) and any \( b_0 > 0 \) there is a cube \( E \) of side \( b_0 \) satisfying \( x \in E \subseteq \Omega \). Let \( h \) be as in (1) with \( n = 1 \) and \( \beta = -1 \). For any fixed \( 0 < \delta < \frac{1}{12 \rho \sqrt{n e^{2n \gamma_n}} \gamma_n} \) in Theorem 1.4, the suggested value of \( c \) is as follows.

Let \( \eta = \frac{\ln \frac{\delta}{\nu}}{12 \rho \sqrt{n e^{2n \gamma_n}} \gamma_n \delta} \). Then

(a) if \( \eta + \frac{\sigma}{2} < 0 \), the larger \( c \) is, the better it is;
(b) if $\eta + \frac{c}{2} > 0$, choose $c = \frac{1}{4(\eta + \frac{c}{2})}$ whenever $\frac{1}{4(\eta + \frac{c}{2})} \geq \frac{1}{\sigma}$ and choose $c = \frac{1}{\sigma}$ whenever $\frac{1}{4(\eta + \frac{c}{2})} < \frac{1}{\sigma}$.

**Reason:** The requirement $\delta \leq \delta_0$ leads to $c \in [12\rho \sqrt{\pi} e^{2n_\gamma \eta \gamma_0 \delta}, \infty)$. In Case3 of section2 when $(\lambda)^{\frac{1}{2}}$ was ignored, the suggested value of $c$ is $\frac{1}{\sigma}$. Hence we put here the restriction $12\rho \sqrt{\pi} e^{2n_\gamma \eta \gamma_0 \delta} < \frac{1}{\sigma}$, i.e. $0 < \delta < \frac{1}{12\rho \sqrt{\pi} e^{2n_\gamma \eta \gamma_0 \delta}}$.

Now the crucial part of (12) is $(\lambda)^{\frac{1}{2}} \cdot \{ \frac{A+B}{c} \}^{\frac{1}{2}}$ where

$$(\lambda)^{\frac{1}{2}} = \left( \frac{2}{3} \right)^{\frac{1}{6}} e^{\gamma \eta \frac{\pi}{2}} = \left( \frac{2}{3} \right)^{\frac{1}{6}} e^{12\rho \sqrt{\pi} e^{2n_\gamma \eta \gamma_0 \delta}} = (e^{\eta})^c = \frac{1}{e^{(\eta)c}}$$

and is decreasing for $c \in (0, \infty)$. By the reasoning of Case3 of section2, the suggested value of $c$ when $(\lambda)^{\frac{1}{2}}$ is ignored is $c = \frac{1}{\sigma}$. It’s obvious that $c \in (0, \frac{1}{\sigma})$ should be excluded when $(\lambda)^{\frac{1}{2}}$ is taken into consideration. The crucial part of (12) is then $(\lambda)^{\frac{1}{2}} \cdot \sqrt{H_2(c)}$ where $H_2(c) := \frac{A+B}{c}$ as in Case3 of section2. Now

$$(\lambda)^{\frac{1}{2}} \sqrt{H_2(c)} \leq e^{\eta c} \cdot \frac{1}{\sqrt{c}} \cdot \left[ \frac{1}{K_0(1)} \| f \|_2 + \frac{\sqrt{\eta}}{\sqrt{\pi}} \left( \frac{1}{2} \right) \sqrt{c} e^{\eta c} \| f \|_2 \right]^{1/2}$$

whose essential part is

$$G(c) := e^{\eta c} \cdot \frac{1}{\sqrt{c}} \cdot (\eta)^{\frac{1}{2}} \cdot e^{c \gamma \eta} \cdot \| f \|_2 = \frac{\sigma^{\frac{1}{4}}}{c^{\frac{1}{4}}} \cdot e^{(\eta + \frac{1}{2})c} \cdot \| f \|_2.$$ 

If $\eta + \frac{1}{2} < 0$, $G(c)$ is decreasing. If $\eta + \frac{1}{2} > 0$, on $[\frac{1}{\sigma}, \infty)$, $G'(c) = \sigma^{\frac{1}{4}} \cdot \frac{1}{c^{\frac{1}{2}}} \cdot e^{(\eta + \frac{1}{2})c} \cdot \| f \|_2$. It gives that $G'(c) = 0$ iff $c = c^* := \frac{1}{4(\eta + \frac{1}{2})}$. If $c^* < \frac{1}{\sigma}$, the suggested value of $c$ is naturally $\frac{1}{4(\eta + \frac{1}{2})}$, if $c^* < \frac{1}{\sigma}$, the suggested value of $c$ is $\frac{1}{\sigma}$ since $G(c)$ is increasing on $[\frac{1}{\sigma}, \infty)$.

**Remark:** In 3.2 we avoided letting the side length $b_0 \rightarrow 0$ because decreasing $b_0$ will only make the error bound worse.

As a final conclusion, we would like to point out an important fact. Although the approximated functions dealt with in this paper are required to be band-limited, it’s in fact not very restrictive. A lot of functions are numerically band-limited even though they are not theoretically band-limited. As long as the Fourier transform $\hat{f}(\xi)$ decays to zero very fast as $|\xi| \rightarrow \infty$, $\hat{f}$ will be numerically compactly supported and $f$ numerically band-limited. There are a multitude of such functions. For example, multiquadrics and Gaussians are themselves numerically band-limited.

In any case, non-band-limited functions are important and should be treated in a rigorous way. For this purpose we need an approach different from this paper. The results will appear in a forthcoming paper of the author.

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