The Lie algebra structure on the first Hochschild cohomology group of a monomial algebra

Claudia Strametz

Département de Mathématiques, Université de Montpellier 2,
F-34095 Montpellier cedex 5, France
E-mail: strametz@math.univ-montp2.fr

Abstract

We study the Lie algebra structure of the first Hochschild cohomology group of a finite dimensional monomial algebra Λ, in terms of the combinatorics of its quiver, in any characteristic. This allows us also to examine the identity component of the algebraic group of outer automorphisms of Λ in characteristic zero. Criteria for the (semi-)simplicity, the solvability, the reductivity, the commutativity and the nilpotency are given.

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1 Introduction

The Hochschild cohomology $H^k(\Lambda, \Lambda)$ of any associative algebra Λ over a field $k$ has the structure of a Gerstenhaber algebra (see [3]). In particular, the first Hochschild cohomology group $H^1(\Lambda, \Lambda)$ is a Lie algebra, a fact which can be verified directly. Note that in the finite dimensional case in characteristic 0 this Lie algebra can be regarded as the Lie algebra of the algebraic group of outer automorphisms $Out(\Lambda) = Aut(\Lambda)/Inn(\Lambda)$ of the algebra Λ. It has been treated by Guil-Asensio and Saorín in [8]. Huisgen-Zimmermann and Saorín proved in [11] that the identity component $Out(\Lambda)^o$ of the outer automorphism group of Λ is invariant under derived equivalence and Keller [13] showed that Hochschild cohomology is preserved under derived equivalence as a graded (super) Lie algebra. Consequently the Lie algebra $H^1(\Lambda, \Lambda)$ is invariant under derived equivalence.

The purpose of this paper is to study the Lie algebra structure of $H^1(\Lambda, \Lambda)$ in the case of finite dimensional monomial algebras without any restriction on the characteristic of the field $k$ using only algebraic methods. The relationship between $H^1(\Lambda, \Lambda)$ and $Out(\Lambda)$ allows us to transfer the results obtained in this way to the identity component $Out(\Lambda)^o$ of the algebraic group $Out(\Lambda)$ in characteristic
0. Thus we give a different proof of Guil-Asensio and Saorín’s criterion for the solvability of $\text{Out}(\Lambda)^{\circ}$ and we generalize some results they obtained using algebraic group theory and methods in algebraic geometry.

This paper is organized in the following way: in the first section we will use the minimal projective resolution of a monomial algebra $\Lambda$ (as a $\Lambda$-bimodule) given by Bardzell in [1] to get a handy description of the Lie algebra $H^1(\Lambda, \Lambda)$ in terms of parallel paths. The purpose of the second section is to link this description to Guil-Asensio and Saorín’s work on the algebraic group of outer automorphisms of monomial algebras (see [3]). In section three we carry out the study of the Lie algebra $H^1(\Lambda, \Lambda)$. In particular we give criteria for the (semi-)simplicity, the solvability, the reductivity, the commutativity and the nilpotency of this Lie algebra and consequently of the connected algebraic group $\text{Out}(\Lambda)$.

We denote by $\Lambda$ a finite dimensional monomial $k$-algebra, that is a finite dimensional $k$-algebra which is isomorphic to a quotient of a path algebra $kQ/I$ where the two-sided ideal $I$ of $kQ$ is generated by a set $Z$ of paths of length $\geq 2$. We shall assume that $Z$ is minimal, i.e. no proper subpath of a path in $Z$ is again in $Z$. Let $B$ be the set of paths of $Q$ which do not contain any element of $Z$ as a subpath. It is clear that the (classes modulo $I = (Z)$) elements of $B$ form a basis of $\Lambda$. We shall denote by $B_n$ the subset $Q_n \cap B$ of $B$ formed by the paths of length $n$.

Let $E \simeq kQ_0$ be the separable subalgebra of $\Lambda$ generated by the (classes modulo $I$ of the) vertices of $Q$. We have a Wedderburn-Malcev decomposition $\Lambda = E \oplus r$ where $r$ denotes the Jacobson radical of $\Lambda$.

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2 Projective resolutions and the Lie bracket

The Hochschild cohomology $H^*(\Lambda, \Lambda) = \text{Ext}^*_\Lambda(\Lambda, \Lambda)$ of a $k$-algebra $\Lambda$ can be computed using different projective resolutions of $\Lambda$ over its enveloping algebra $\Lambda^e = \Lambda \otimes_k \Lambda^{op}$. The standard resolution $\mathcal{P}_{\text{Hoch}}$ is

$$
\cdots \to \Lambda \otimes_k x_1 x_2 x_3 \cdots \otimes_k x_n \to \cdots \to \Lambda \otimes_k \Lambda \xrightarrow{\delta} \Lambda \to 0
$$

where $\delta(x_1 \otimes_k x_2) = x_1 x_2$ and

$$
\delta(x_1 \otimes_k \cdots \otimes_k x_n) = \sum_{i=1}^{n-1} (-1)^{i+1} x_1 \otimes_k \cdots \otimes_k x_i x_{i+1} \otimes_k \cdots \otimes_k x_n
$$

for $x_1, \ldots, x_n \in \Lambda$. Applying the functor $\text{Hom}_{\Lambda^e}(\Lambda, \Lambda)$ to $\mathcal{P}_{\text{Hoch}}$ and identifying $\text{Hom}_{\Lambda^e}(\Lambda \otimes_k \Lambda \otimes_k \ldots \otimes_k \Lambda, \Lambda)$ with $\text{Hom}_k(\Lambda^e \otimes_k, \Lambda)$ for all $n \in \mathbb{N}$, yields the cochain
complex $C_{\text{Hoch}}$ defined by Hochschild:

$$0 \to \Lambda \xrightarrow{d_0} \text{Hom}_k(\Lambda, \Lambda) \to \cdots \to \text{Hom}_k(\Lambda^\otimes_k, \Lambda) \xrightarrow{d_n} \text{Hom}_k(\Lambda^\otimes_{k+1}, \Lambda) \to \cdots$$

where $(d_0a)(x) = ax - xa$ for all $a, x \in \Lambda$ and

$$(d_n f)(x_1 \otimes_k \cdots \otimes_k x_{n+1}) = \sum_{i=1}^{n} (-1)^i f(x_1 \otimes_k \cdots \otimes_k x_{i-1} \otimes_k x_i x_{i+1} \otimes_k \cdots \otimes_k x_{n+1}) + (-1)^{n+1} f(x_1 \otimes_k \cdots \otimes_k x_n x_{n+1})$$

for all $f \in \text{Hom}_k(\Lambda^\otimes_k, \Lambda)$, $n \in \mathbb{N}$, and $x_1, \ldots, x_{n+1} \in \Lambda$. In 1962 the structure of a Gerstenhaber algebra was introduced on the Hochschild cohomology $H^*(\Lambda, \Lambda)$ by Gerstenhaber in [5]. In particular the first cohomology group $H^1(\Lambda, \Lambda)$ which is the quotient of the derivations modulo the inner derivations of $\Lambda$ is a Lie algebra whose bracket is induced by the Lie bracket on $\text{Hom}_k(\Lambda, \Lambda)$

$$[f, g] = f \circ g - g \circ f$$

where $f, g \in \text{Hom}_k(\Lambda, \Lambda)$. The 1-coboundaries, i.e. the derivations $\text{Der}_k(\Lambda) = \text{Ker} d_1 = \{ f \in \text{Hom}_k(\Lambda, \Lambda) \mid f(ab) = af(b) + f(a)b \ \forall a, b \in \Lambda \}$ form a Lie subalgebra of $\text{Hom}_k(\Lambda, \Lambda)$ and the 1-cocycles, i.e. the inner derivations $\text{Ad}_k(\Lambda) = \text{Im} d_0 = \{ f \in \text{Hom}_k(\Lambda, \Lambda) \mid \exists a \in \Lambda \text{ such that } f(x) = ax - xa \ \forall x \in \Lambda \}$ an ideal of this Lie algebra.

In order to compute the first Hochschild cohomology group $H^1(\Lambda, \Lambda)$ of the finite dimensional monomial algebra $\Lambda$, we shall use the minimal projective resolution of the $\Lambda$-bimodule $\Lambda$ given by Bardzell in [7]. The part of this resolution $\mathcal{P}_{\text{min}}$ in which we are interested is the following:

$$
\cdots \to \Lambda \otimes_E kZ \otimes_E \Lambda \xrightarrow{\delta_1} \Lambda \otimes_E kQ_1 \otimes_E \Lambda \xrightarrow{\delta_0} \Lambda \otimes_E \Lambda \xrightarrow{\pi} \Lambda \to 0
$$

where the $\Lambda$-bimodule morphisms are given by

$$\pi(\lambda \otimes_E \mu) = \lambda \mu$$

$$\delta_0(\lambda \otimes_E a \otimes_E \mu) = \lambda a \otimes_E \mu - \lambda \otimes_E a \mu$$

$$\delta_1(\lambda \otimes_E p \otimes_E \mu) = \sum_{d=1}^{n} \lambda a_{n-d+1} \otimes_E a_d \otimes_E a_{d-1} \cdots a_1 \mu$$

for all $\lambda, \mu \in \Lambda$, $a, a_n, \ldots, a_1 \in Q_1$ and $p = a_n \cdots a_1 \in Z$ (with the conventions $a_{n+1} = t(a_n)$ and $a_0 = s(a_1)$).

**Remark 2.1** Our description of $\mathcal{P}_{\text{min}}$ is equivalent to Bardzell’s, because if $X$ denotes a set of paths and if $kX$ is the corresponding $E$-bimodule, then the map

$$\otimes_{\gamma \in X} \Lambda \ell(\gamma) \otimes_k s(\gamma) \Lambda \to \Lambda \otimes_E kX \otimes_E \Lambda$$

which is given by $\lambda \ell(\gamma) \otimes_k s(\gamma) \mu \mapsto \lambda \otimes_E \gamma \otimes_E \mu$, where $\gamma \in X$ and $\lambda, \mu \in \Lambda$, is clearly a $\Lambda$-bimodule isomorphism. Notate that the $\Lambda$-bimodules $\Lambda \otimes_E \Lambda$ and $\Lambda \otimes_E E \otimes_E \Lambda \simeq \Lambda \otimes_E kQ_0 \otimes_E \Lambda$ are isomorphic.

**Lemma 2.2** Let $M$ be an $E$-bimodule and $T$ a $\Lambda$-bimodule. Then the vector space $\text{Hom}_\Lambda^*(\Lambda \otimes_E M \otimes_E \Lambda, T)$ is isomorphic to $\text{Hom}_{E^*}(M, T)$.

**Proof:** The linear morphism $\text{Hom}_\Lambda^*(\Lambda \otimes_E M \otimes_E \Lambda, T) \to \text{Hom}_{E^*}(M, T)$ which sends a $\Lambda$-bimodule morphism $f : \Lambda \otimes_E M \otimes_E \Lambda \to T$ to the $E$-bimodule morphism $M \to T$ given by $m \mapsto f(1 \otimes_E m \otimes_E 1_{\Lambda})$ and the linear morphism $\text{Hom}_{E^*}(M, T) \to \text{Hom}_\Lambda^*(\Lambda \otimes_E M \otimes_E \Lambda, T)$ which associates to $g : M \to T$ the element $\Lambda \otimes_E M \otimes_E \Lambda \to T$, $\lambda \otimes_E m \otimes_E \mu \mapsto \lambda \ell(g(m)) \mu$, are inverse to each other.

q.e.d. 2.2
Applying the functor $\text{Hom}_{\Lambda^e}(\cdot, \Lambda)$ to $P_{\text{min}}$ and using the preceding Lemma yields the cochain complex $C_{\text{min}}$

\[0 \to \text{Hom}_{E^e}(kQ_0, \Lambda) \xrightarrow{\delta_0^e} \text{Hom}_{E^e}(kQ_1, \Lambda) \xrightarrow{\delta_1^e} \text{Hom}_{E^e}(kZ, \Lambda) \to \cdots\]

where the coboundaries $\delta_0^e$ and $\delta_1^e$ are given by

\[(\delta_0^e f)(a) = af(s(a)) - f(t(a))a \quad \text{and} \quad (\delta_1^e g)(p) = \sum_{d=1}^{n} a_n \cdots a_{d+1}g(a_d)a_{d-1} \cdots a_1\]

where $f \in \text{Hom}_{E^e}(kQ_0, \Lambda)$, $a, a_n, \ldots, a_1 \in Q_1$, $g \in \text{Hom}_{E^e}(kQ_1, \Lambda)$ and $p = a_n \cdots a_1 \in Z$.

As it is useful to interpret $H^1(\Lambda, \Lambda)$ in terms of parallel paths we introduce now the following notion: two paths $\varepsilon, \gamma$ of $Q$ are called parallel if $s(\varepsilon) = s(\gamma)$ and $t(\varepsilon) = t(\gamma)$. If $X$ and $Y$ are sets of paths of $Q$, the set $X//Y$ of parallel paths is formed by the couples $(\varepsilon, \gamma) \in X \times Y$ such that $\varepsilon$ and $\gamma$ are parallel paths. For instance, $Q_0//Q_n$ is the set of oriented cycles of $Q$ of length $n$.

**Lemma 2.3** Let $X$ and $Y$ be sets of paths of $Q$ and let $kX$ and $kY$ be the corresponding $E$-bimodules. Then the vector spaces $k(X//Y)$ and $\text{Hom}_{E^e}(kX, kY)$ are isomorphic.

**Proof:** Define a linear morphism $k(X//Y) \to \text{Hom}_{E^e}(kX, kY)$ by sending $(\varepsilon, \gamma) \in X//Y$ to the elementary map which associates $\gamma$ to $\varepsilon$ and 0 to any other path of $X$. If $f : kX \to kY$ is an $E$-bimodule morphism, we have for every path $\varepsilon \in X$ that $f(\varepsilon) = f(t(\varepsilon)s(\varepsilon)) = t(\varepsilon)f(\varepsilon)s(\varepsilon)$ and thus $f(\varepsilon) = \sum_{(\varepsilon, \gamma) \in X//Y} \lambda(\varepsilon, \gamma)\lambda$, where $\lambda(\varepsilon, \gamma) \in k$. This allows us to define a linear morphism $\text{Hom}_{E^e}(kX, kY) \to k(X//Y)$ which assigns to $f : kX \to kY$ the element $\sum_{(\varepsilon, \gamma) \in X//Y} \lambda(\varepsilon, \gamma)\lambda$. Obviously, the two morphisms are inverse to each other.

q.e.d. 2.3

Using the following notations we will rewrite the coboundaries:

**Notations 2.4**

(i) Let $\varepsilon$ be a path in $Q$ and $(a, \gamma) \in Q_1//B$. We denote by $\varepsilon^{(a, \gamma)}$ the sum of all nonzero paths (i.e. paths in $B$) obtained by replacing one appearance of the arrow $a$ in $\varepsilon$ by the path $\gamma$. If the path $\varepsilon$ does not contain the arrow $a$ or if every replacement of a in $\varepsilon$ by $\gamma$ is not a path in $B$, we set $\varepsilon^{(a, \gamma)} = 0$. Suppose that $\varepsilon^{(a, \gamma)} = \sum_{i=1}^{n} \varepsilon_i$, where $\varepsilon_i \in B$ and let $\eta$ be a path of $B$ parallel to $\varepsilon$. By abuse of language we denote by $\eta, \varepsilon^{(a, \gamma)}$ the sum $\sum_{i=1}^{n} \eta, \varepsilon_i$ in $k(Z//B)$ (with the convention that $\eta, \varepsilon^{(a, \gamma)} = 0$ if $\varepsilon^{(a, \gamma)} = 0$).

(ii) The function $\chi_B : \prod_{n \in \mathbb{N}} Q_n \to \{0, 1\}$ denotes the indicator function which associates 1 to each path $\gamma \in B$ and 0 to $\gamma \not\in B$.

(iii) If $X$ is a set of paths of $Q$ and $e$ a vertex of $Q$, the set $eX$ is formed by the paths of $X$ with source vertex $e$. In the same way $e^X$ denotes the set of all paths of $X$ with terminus vertex $e$.

**Remark 2.5**

If $(a, \gamma) \in Q_1//(B - Q_0)$, then all the nonzero summands of $\varepsilon^{(a, \gamma)}$ are distinct.

If we carry out the identification suggested in the preceding Lemma, we obtain:
Proposition 2.6
The beginning of the cochain complex $C_{\min}$ can be characterized in the following way

$$0 \to k(Q_0//B) \xrightarrow{\psi_0} k(Q_1//B) \xrightarrow{\psi_1} k(Z//B) \xrightarrow{\psi_2} \cdots$$

where the maps are given by

$$\psi_0 : k(Q_0//B) \to k(Q_1//B)$$

$$(e, \gamma) \mapsto \sum_{a \in Q_1} \chi_B(a\gamma)(a, a\gamma) - \sum_{a \in Q_1} \chi_B(\gamma a)(a, a\gamma)$$

$$\psi_1 : k(Q_1//B) \to k(Z//B)$$

$$(a, \gamma) \mapsto \sum_{p \in Z} (p, p(a, \gamma))$$

In particular, we have $H^1(\Lambda, \Lambda) \simeq \text{Ker } \psi_1/\text{Im } \psi_0$.

The verifications are straightforward.

Theorem 2.7 The bracket

$$[(a, \gamma), (b, \varepsilon)] = (b, \varepsilon^{(a, \gamma)}) - (a, \gamma^{(b, \varepsilon)})$$

for all $(a, \gamma), (b, \varepsilon) \in Q_1//B$ induces a Lie algebra structure on $\text{Ker } \psi_1/\text{Im } \psi_0$ such that $H^1(\Lambda, \Lambda)$ and $\text{Ker } \psi_1/\text{Im } \psi_0$ are isomorphic Lie algebras.

Proof: As $\mathcal{P}_{\text{Hoch}}$ and $\mathcal{P}_{\text{min}}$ are projective resolutions of the $\Lambda$-bimodule $\Lambda$, there exist, thanks to the Comparison Theorem, chain maps $\omega : \mathcal{P}_{\text{Hoch}} \to \mathcal{P}_{\text{min}}$ and $\zeta : \mathcal{P}_{\text{min}} \to \mathcal{P}_{\text{Hoch}}$. Let us choose these $\Lambda$-bimodule morphisms such that

$$\omega_0 : \Lambda \otimes_E \Lambda \to \Lambda \otimes_k \Lambda$$

$$\omega_1 : \Lambda \otimes_k \Lambda \otimes_k \Lambda \to \Lambda \otimes_E kQ_1 \otimes_E \Lambda$$

$$\omega_1 : \Lambda \otimes_k \Lambda \otimes k \Lambda \to \Lambda \otimes_k \Lambda$$

where $\lambda, \mu \in \Lambda, a, a_1, \ldots, a_3 \in Q_1$ and $a_0, a_1, \ldots a_3 \in B$. As a chain map between projective resolutions is unique up to chain homotopy equivalence, $\omega$ and $\zeta$ are unique up to chain homotopy equivalence and such that $\omega \circ \zeta$ is homotopic to $\text{id } \mathcal{P}_{\text{min}}$ and $\zeta \circ \omega$ is homotopic to $\text{id } \mathcal{P}_{\text{Hoch}}$. Therefore, the cochain maps $\text{Hom}_{\Lambda^e}(\omega, \Lambda)$ and $\text{Hom}_{\Lambda^e}(\zeta, \Lambda)$ are such that $\text{Hom}_{\Lambda^e}(\omega, \Lambda) \circ \text{Hom}_{\Lambda^e}(\zeta, \Lambda)$ is homotopic to $\text{id } \mathcal{P}_{\text{min}}$ and $\text{Hom}_{\Lambda^e}(\omega, \Lambda) \circ \text{Hom}_{\Lambda^e}(\zeta, \Lambda)$ is homotopic to $\text{id } \mathcal{P}_{\text{Hoch}}$. Hence these maps induce inverse linear isomorphisms at the cohomology level. Taking into account the identifications above we obtain that

$$\varpi_1 : k(Q_1//B) \to \text{Hom}_k(\Lambda, \Lambda)$$

$$(a, \gamma) \mapsto \Lambda \to \Lambda$$

$$\varpi_1 : \text{Hom}_k(\Lambda, \Lambda) \to k(Q_1//B)$$

$$f : \Lambda \to \Lambda \quad \varepsilon \mapsto \sum_{\gamma \in B} \lambda_{\varepsilon, \gamma}$$

$$\sum_{a \in Q_1} \sum_{(a, \gamma) \in Q_1//B} \lambda_{a, \gamma}(a, \gamma)$$

induce inverse linear isomorphisms between $H^1(\Lambda, \Lambda) = \text{Ker } d_1/\text{Im } d_0$ and $H^1(\Lambda, \Lambda) = \text{Ker } \psi_1/\text{Im } \psi_0$. This allows us to transfer the Lie algebra structure of $\text{Ker } d_1/\text{Im } d_0$ to $\text{Ker } \psi_1/\text{Im } \psi_0$. Define on $k(Q_1//B)$ the bracket

$$[(a, \gamma), (b, \varepsilon)] := \varpi_1(\varpi_1(a, \gamma), \varpi_1(b, \varepsilon)) = (b, \varepsilon^{(a, \gamma)}) - (a, \gamma^{(b, \varepsilon)})$$
for all \((a, \gamma), (b, \varepsilon) \in Q_1//B\). It is easy to check that this is a Lie bracket. As \(\overline{\omega} : C_{\text{Hoch}} \to C_{\text{min}}\) and \(\overline{\zeta} : C_{\text{min}} \to C_{\text{Hoch}}\) are maps of complexes, we conclude from the fact that \(\text{Ker} d_1 = \text{Der}_k(\Lambda)\) is a Lie subalgebra of \(\text{Hom}_k(\Lambda, \Lambda)\), that \(\text{Ker} \psi_1\) is a Lie subalgebra of \(k(Q_1//B)\). In the same way we deduce from the fact that \(\text{Im} \psi_0 = \text{Ad}_k(\Lambda)\) is a Lie ideal of \(\text{Ker} d_1\), that \(\text{Im} \psi_0\) is a Lie ideal of \(\text{Ker} \psi_1\). By construction the quotient Lie algebra \(H^1(\Lambda, \Lambda) = \text{Ker} \psi_1/\text{Im} \psi_0\) is isomorphic to the Lie algebra \(H^1(\Lambda, \Lambda) = \text{Ker} d_1/\text{Im} d_0\).

\(\text{q.e.d. 2.7}\)

It is remarkable that Bardzell’s minimal projective resolution and the resolution considered by Gerstenhaber and Schack in \([\text{B}]\) using \(E\)-relative Hochschild cohomology yield the same 1-coboundaries and 1-cocycles as the following Proposition and Corollary show:

**Proposition 2.8** The Lie algebras \(\text{Ker} \psi_1\) (with the bracket described in the preceding Theorem) and \(\text{Der}_{E^*}(\Lambda)\) (with the canonical bracket) are isomorphic.

**Proof:** Let \(\sigma : \Lambda \to \Lambda\) be an element of \(\text{Der}_{E^*}(\Lambda) = \text{Der}_k(\Lambda) \cap \text{Hom}_{E^*}(\Lambda, \Lambda)\). Since \(\sigma\) is an \(E\)-bimodule morphism, the image of an arrow \(a\) is a linear combination of paths of \(B\) parallel to \(a\). Suppose \(\sigma(a) = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)\) for \(a \in Q_1\) with \(\lambda(a, \gamma) \in k\). The fact that \(\sigma\) is a well-defined derivation implies that we have for every path \(p = p_1 \ldots p_{l+1} \in Z\) of length \(l\)

\[0 = \sigma(p) = \sigma(p_l \ldots p_1) = \sum_{i=1}^l p_l \ldots p_{i+1} \sigma(p_i) p_{i-1} \ldots p_1 = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma) p^{(a, \gamma)}\]

Thus we have in \(k(Z//B)\) the equality \(0 = \sum_{p \in Z} \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(p, p^{(a, \gamma)})\) which is the same as to say that \(\sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(a, \gamma)\) is an element in \(\text{Ker} \psi_1\). This enables us to define the linear function

\[\xi : \text{Der}_{E^*}(\Lambda) \to \text{Ker} \psi_1, \quad \sigma \mapsto \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(a, \gamma)\]

On the other hand we can associate to every \(x = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(a, \gamma)\) in \(\text{Ker} \psi_1\) an \(E\)-bimodule morphism \(\sigma_x : \Lambda \to \Lambda\) by setting \(\sigma_x(e) := 0\) for every vertex \(e \in Q_0\) and \(\sigma_x(c) := \sum_{(c, \gamma) \in Q_1//B} \lambda(c, \gamma)\) for every arrow \(c \in Q_1\). To make \(\sigma_x\) a derivation we extend \(\sigma_x\) to paths \(\varepsilon = b_m \ldots b_1\) of length \(m \geq 2\) by the formula

\[\sigma_x(\varepsilon) = \sigma_x(b_m \ldots b_1) = \sum_{j=1}^m b_m \ldots b_{j+1} \sigma_x(b_j) b_{j-1} \ldots b_1 = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma) \varepsilon^{(a, \gamma)}\]

Thanks to \(0 = \psi_1(x) = \sum_{p \in Z} \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(p, p^{(a, \gamma)})\) we have for every \(p \in Z\) the relation \(0 = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(p, p^{(a, \gamma)}) = \sigma_x(p)\) which shows that \(\sigma_x\) is well-defined. This allows us to define the linear function

\[\zeta : \text{Ker} \psi_1 \to \text{Der}_{E^*}(\Lambda), \quad x \mapsto \sigma_x\]

It is clear that \(\xi\) and \(\zeta\) are inverse to each other which proves that \(\text{Der}_{E^*}(\Lambda)\) and \(\text{Ker} \psi_1\) are isomorphic \(k\)-vector spaces. In order to check that \(\zeta\) is a Lie algebra morphism let us fix \(x = \sum_{(a, \gamma) \in Q_1//B} \lambda(a, \gamma)(a, \gamma)\) and \(y = \sum_{(b, \varepsilon) \in Q_1//B} \mu(b, \varepsilon)(b, \varepsilon)\)
in $\text{Ker } \psi_1$. For every arrow $c \in Q_1$ we have

$$\left[\sigma_x, \sigma_y\right](c) = (\sigma_x \circ \sigma_y - \sigma_y \circ \sigma_x)(c)$$

$$= \sum_{(c,e) \in Q_1/B} \mu_{(c,e)}(\varepsilon) - \sum_{(c,\gamma) \in Q_1/B} \lambda_{(c,\gamma)}(\gamma)$$

$$= \sum_{(c,e),(a,\gamma) \in Q_1/B} \mu_{(c,e)} \lambda_{(a,\gamma)}(\varepsilon) - \sum_{(c,\gamma),(b,\epsilon) \in Q_1/B} \lambda_{(c,\gamma)}(b,\epsilon)$$

$$= \sigma_z(c)$$

where $z = \sum_{(a,\gamma),(b,\epsilon) \in Q_1/B} \lambda_{(a,\gamma)}(b,\epsilon)((b,\epsilon)^{(a,\gamma)} - (a,\epsilon))$. Theorem 2.7 shows that $z$ is equal to the element $[x, y]$ of $\text{Ker } \psi_1$. Therefore $\zeta$ is a Lie algebra morphism and $\text{Der } E^v(\Lambda)$ and $\text{Ker } \psi_1$ are isomorphic Lie algebras.

$q.e.d. \ 2.8$

**Corollary 2.9** The Lie ideal $\text{Im } \psi_0$ of $\text{Ker } \psi_1$ and the Lie ideal $\text{Ad } E^v(\Lambda)$ of $\text{Der } E^v(\Lambda)$ are isomorphic.

**Proof:** Let $y = \sum_{\gamma \in B} \lambda_{\gamma} \gamma \in \Lambda$ be such that $\text{ad } (y)$ is an $E$-bimodule morphism. Then we have for every vertex $e \in Q_0$

$$0 = ey - ye = e \text{ad } (y)(e) = ye - ey = \sum_{\gamma \in B} \lambda_{\gamma}(\delta_{\kappa(\gamma),e} - \delta_{\iota(\gamma),e})\gamma$$

This shows that every path $\gamma \in B$ satisfying $\lambda_{\gamma} \neq 0$ is a cycle and so $y = \sum_{(e,\gamma) \in Q_0/B} \lambda_{\gamma} \gamma$. For every arrow $a \in Q_1$ we have

$$\text{ad } (y)(a) = ya - ay = \sum_{(e,\gamma) \in Q_0/B} \lambda_{\gamma} a \gamma - \sum_{(e,\gamma) \in Q_1/B} \lambda_{\gamma} a \gamma = \sum_{(e,\gamma) \in Q_0/B} \lambda_{\gamma}(a \gamma - a \gamma)$$

We deduce from this that the image of $\text{ad } (y) \in \text{Ad } E^v(\Lambda)$ by $\xi$ is

$$\xi(\text{ad } (y)) = \sum_{(e,\gamma) \in Q_0/B} \lambda_{\gamma}(\sum_{a \in Q_1/B} (a, \gamma) - \sum_{a \in Q_1/B} (a, a \gamma)) = - \sum_{(e,\gamma) \in Q_0/B} \lambda_{\gamma} \psi_0(e, \gamma)$$

and thus we obtain that the image of the Lie algebra $\text{Ad } E^v(\Lambda)$ by the isomorphism of Lie algebras $\xi$ is $\text{Im } \psi_0$.

$q.e.d. \ 2.9$

**Remark 2.10** In general, the minimal projective resolution and the resolution considered by Gerstenhaber and Schack do not yield the same $n$-coboundaries and $n$-cocycles: for example in the case of hereditary algebras, i.e. $\Lambda = \emptyset$, we have for $n = 2$ that $\text{Ker } \psi_2 = 0$, because $k(\mathbb{Z}/B) = 0$, but the multiplication $m \in \text{Hom } E^v(\Lambda \otimes E \Lambda, \Lambda)$ is a nontrivial $2$-coboundary as well as a $2$-cocycle of the complex yielded by Gerstenhaber and Schack’s resolution.

### 3 The algebraic group of outer automorphisms of a monomial algebra

In this section, we assume that the characteristic of the field $k$ is $0$. Our aim is to relate the description of the Lie algebra $H^1(\Lambda, \Lambda) = \text{Ker } \psi_1/\text{Im } \psi_0$ obtained in
the preceding section to the algebraic groups which appear in Guil-Asensio and Saorín’s study of the outer automorphisms. Denote by $\text{Aut}(A)$ the algebraic group of all $k$-algebra automorphisms of a finite dimensional $k$-algebra $A$. The group of inner automorphisms $\text{Inn}(A)$ of $A$ is the image of the morphism of algebraic groups $\varphi : A^* \to \text{Aut}(A)$ given by $a \mapsto t_a = \text{conjugation by } a$. Thus $\text{Inn}(A)$ is a closed normal and connected subgroup of $\text{Aut}(A)$. The algebraic group of outer automorphisms $\text{Out}(A)$ is defined as the quotient $\text{Aut}(A)/\text{Inn}(A)$. Its identity component $\text{Out}(A)^\circ$ is $\text{Aut}(A)^\circ/\text{Inn}(A)$. Note that if $A$ is a basic algebra, for instance the monomial algebra $\Lambda$, then the group $\text{Out}(A)$ is isomorphic to the Picard group $\text{Pic}_k(A)$, that is the group of (isomorphism types of) Morita autoequivalences of the category of left $A$-modules (see [15], p. 1860).

**Proposition 3.1** Let $k$ be a field of characteristic $0$ and $A$ a finite dimensional $k$-algebra. The derivations $\text{Der}_k(A)$ form the Lie algebra of the algebraic group $\text{Aut}(A)$ and the inner derivations $\text{Ad}_k(A)$ form the Lie algebra of the algebraic group $\text{Inn}(A)$. The Lie algebra $H^1(A, A)$ can be regarded as the Lie algebra of the algebraic group $\text{Out}(A)$ or as the Lie algebra of its identity component $\text{Out}(A)^\circ$.

**Proof:** In [12] 13.2 it is shown that $\text{Der}_k(A)$ is the Lie algebra of $\text{Aut}(A)$. The differential $d\varphi : A \to \text{Der}_k(A)$ of the above-mentioned morphism $\varphi$ is given by $a \mapsto \text{ad}(a) = \text{inner derivation of } a$. Since the field $k$ has characteristic $0$, the Lie algebra of the image of a morphism of algebraic groups is the image of the differential of this morphism. The construction of the quotient $\text{Aut}(A)/\text{Inn}(A)$ implies that the Lie algebra of the algebraic group $\text{Aut}(A)/\text{Inn}(A) = \text{Out}(A)$ is $\mathcal{L}(\text{Aut}(A))/\mathcal{L}(\text{Inn}(A)) = \text{Der}_k(A)/\text{Ad}_k(A) = H^1(A, A)$ (see [12] 11.5 and 12).

$q.e.d.$ 3.1

The automorphism group of any finite dimensional algebra was studied by Pollack in [3]. Guil-Asensio and Saorín worked on the outer automorphisms of any finite dimensional algebra (see [8]). The case of the finite dimensional monomial algebras was treated by them in their paper [4]. We are going to follow their notations. Define the algebraic group

$$H_\Lambda := \{ \sigma \in \text{Aut}(\Lambda) \mid \sigma(e) = e \forall e \in Q_0 \}$$

and denote by $H_\Lambda^\circ$ its identity component. According to [15] and to Proposition 1.1 of [4] we have

$$\text{Out}(\Lambda)^\circ \cong H_\Lambda^\circ/H_\Lambda \cap \text{Inn}(\Lambda)$$

**Proposition 3.2** Let $k$ be a field of characteristic $0$ and $\Lambda$ a finite dimensional monomial algebra. The Lie algebra of the algebraic group $H_\Lambda$ is the Lie algebra $\text{Der}_{E^\circ}(\Lambda) \cong \text{Ker } \psi_1$ and $\text{Ad}_{E^\circ}(\Lambda) \cong \text{Inn } \psi_0$ is the Lie algebra of $H_\Lambda \cap \text{Inn}(\Lambda)$.

**Proof:** For every vertex $e \in Q_0$ we write $G_e := \{ \sigma \in \text{GL}(\Lambda) \mid \sigma(e) = e \}$ and $G_e := \{ \sigma \in \text{GL}(\Lambda) \mid \sigma(e) = 0 \}$. Thus we get the equality $H_\Lambda = \text{Aut}(\Lambda) \cap \bigcap_{e \in Q_0} G_e$. Since we have assumed char $k = 0$, paragraph 13.2 of [12] shows that

$$\mathcal{L}(H_\Lambda) = \mathcal{L}(\text{Aut}(\Lambda)) \cap \bigcap_{e \in Q_0} \mathcal{L}(G_e) = \text{Der}_k(\Lambda) \cap \bigcap_{e \in Q_0} G_e = \{ \sigma \in \text{Der}_k(\Lambda) \mid \sigma(e) = 0 \forall e \in Q_0 \} = \text{Der}_{E^\circ}(\Lambda)$$

and

$$\mathcal{L}(H_\Lambda \cap \text{Inn}(\Lambda)) = \mathcal{L}(H_\Lambda) \cap \mathcal{L}(\text{Inn}(\Lambda)) = \text{Der}_{E^\circ}(\Lambda) \cap \text{Ad}_k(\Lambda) = \text{Ad}_{E^\circ}(\Lambda)$$

$q.e.d.$ 3.2
The closed unipotent and connected subgroup
\[ \hat{E} := \varphi(E^*) = \{ t_a \in \text{Inn}(\Lambda) \mid a \in E^* \} \]
of \( H_\Lambda \) is isomorphic as a group to the group of acyclic characters \( \text{Ch}(Q,k) \) appearing in Guil-Asensio and Saorín’s paper (see definition 7 in \[ \text{[8]} \)). The group
\[ \text{Inn}^*(\Lambda) := \varphi(1+r) = \{ t_a \in \text{Inn}(\Lambda) \mid \exists x \in r \text{ such that } a = 1 + x \} \]
is a closed unipotent and connected subgroup of \( \text{Inn}(\Lambda) \). We have \( H_\Lambda \cap \text{Inn}(\Lambda) = (H_\Lambda \cap \text{Inn}^*(\Lambda)) \times \hat{E} \). Define the algebraic groups
\[ U_\Lambda := \{ \sigma \in \text{Aut}(\Lambda) \mid \sigma(a) \equiv a \mod r^2 \forall a \in Q_1 \} \quad \text{and} \quad V_\Lambda := \{ \sigma \in H_\Lambda \mid \sigma(a) = \sum_{(a,b) \in Q_1/\{Q_1\}} \lambda_{(a,b)} b \forall a \in Q_1 \} \]
According to Proposition 1.1 of \[ \text{[9]} \) we have \( H_\Lambda^\circ = (H_\Lambda \cap U_\Lambda) \rtimes V_\Lambda^\circ \) and therefore \( \text{Out}(\Lambda)^\circ \) can be described as follows:

**Proposition 3.3** Let \( k \) be a field of characteristic 0 and \( \Lambda \) a finite dimensional monomial \( k \)-algebra. Then the identity component \( \text{Out}(\Lambda)^\circ \) of the algebraic group of outer automorphisms is the semidirect product
\[ \text{Out}(\Lambda)^\circ = \frac{H_\Lambda \cap U_\Lambda}{H_\Lambda \cap \text{Inn}^*(\Lambda)} \rtimes \frac{V_\Lambda^\circ}{E} \]
Since there is an inclusion preserving \( 1-1 \) correspondence between the collection of closed connected subgroups of \( \text{Out}(\Lambda)^\circ \) (resp. \( H_\Lambda^\circ \)) and the collection of their Lie algebras, regarded as subalgebras of \( H^1(\ Lambda, \Lambda) = \text{Ker} \psi_1/\text{Im} \psi_0 \) (resp. \( \text{Ker} \psi_1 \)) we are interested in identifying the subalgebras of \( H^1(\ Lambda, \Lambda) = \text{Ker} \psi_1/\text{Im} \psi_0 \) (resp. \( \text{Ker} \psi_1 \)) corresponding to the algebraic groups \( \frac{H_\Lambda \cap U_\Lambda}{H_\Lambda \cap \text{Inn}^*(\Lambda)} \) and \( \frac{V_\Lambda^\circ}{E} \) (resp. \( H_\Lambda \cap U_\Lambda, H_\Lambda \cap \text{Inn}^*(\Lambda), V_\Lambda^\circ \) and \( \hat{E} \)). We have the following dictionary:

**Proposition 3.4** Let \( k \) be a field of characteristic 0 and \( \Lambda \) a finite dimensional monomial \( k \)-algebra.

(i) The Lie algebra of the closed connected normal subgroup \( H_\Lambda \cap U_\Lambda \) of \( H_\Lambda^\circ \) is the Lie ideal \( k(Q_1/\{B = (Q_0 \cup Q_1) \}) \cap \text{Ker} \psi_1 \) of the Lie algebra \( \text{Ker} \psi_1 \).

(ii) The Lie algebra of the closed connected subgroup \( V_\Lambda^\circ \) of \( H_\Lambda^\circ \) is the Lie subalgebra \( k(Q_1/\{Q_1\}) \cap \text{Ker} \psi_1 \) of the Lie algebra \( \text{Ker} \psi_1 \).

(iii) The Lie algebra of the closed connected normal subgroup \( H_\Lambda \cap \text{Inn}^*(\Lambda) \) of \( H_\Lambda \cap \text{Inn}(\Lambda) \) is the Lie ideal of \( \text{Im} \psi_0 \) generated by the elements \( \sum_{a \in Q_1} \{e, a^\gamma\} - \sum_{a \in Q_1} \{a, a^\gamma\} \) where \( (e, \gamma) \in Q_0/\{B = Q_0\} \).

(iv) The Lie algebra of the closed connected subgroup \( \hat{E} \) of \( H_\Lambda \cap \text{Inn}(\Lambda) \) is the Lie subalgebra of \( \text{Im} \psi_0 \) generated by the elements \( \sum_{a \in Q_1} \{a, a\} - \sum_{a \in Q_1} \{a, a\} \) where \( e \in Q_0 \).

**Proof:** The derivation of the morphism of algebraic groups
\[ \varepsilon_\Lambda : \quad \text{Aut}(\Lambda) \longrightarrow \text{GL}(r/r^2) \cong \text{GL}(kQ_1) \quad \sigma \longrightarrow r/r^2 \rightarrow r/r^2, \quad x \mod r^2 \rightarrow \sigma(x) \mod r^2 \]
is given by
\[ d\varepsilon_\Lambda : \quad \text{Der}_k(\Lambda) \longrightarrow \text{gl}(r/r^2) \cong \text{gl}(kQ_1) \quad \sigma \longrightarrow r/r^2 \rightarrow r/r^2, \quad x \mod r^2 \rightarrow \sigma(x) \mod r^2 \]
Recall that we have shown in Proposition 1.8 that

\[
\xi : \begin{array}{ccc}
\sigma : & \Lambda & \rightarrow \\
& a \in Q_1 & \mapsto \sum_{(a,\gamma) \in Q_1//B} \lambda_{a,\gamma}(a,\gamma)
\end{array} \rightarrow \ker \psi_1
\]

is a Lie algebra isomorphism.

(i) Since \( U_\Lambda \) is the kernel of \( \varepsilon_\Lambda \) we obtain, thanks to the assumption \( \text{char } k = 0 \),

\[
\mathcal{L}(H_\Lambda \cap U_\Lambda) = \mathcal{L}(H_\Lambda) \cap \ker d\varepsilon_\Lambda = \{ \sigma \in \text{Der } E^*(\Lambda) \mid \sigma(a) \in r^2 \ \forall a \in Q_1 \}
\]

The fact \( (\mathcal{L}(H_\Lambda \cap U_\Lambda)) = k(Q_1//B - (Q_0 \cup Q_1)) \cap \ker \psi_1 \) finishes the proof.

(ii) From \( V_\Lambda \simeq \text{Im } \varepsilon_\Lambda |_{H_\Lambda} \) (see Lemma 22 in \[8\]) and \( \text{char } k = 0 \) we deduce

\[
\mathcal{L}(V_\Lambda) = \mathcal{L}(V_\Lambda^\circ) = \text{Im } d\varepsilon_\Lambda |_{\mathcal{L}(H_\Lambda)} = \{ \sigma \in \text{Der } E^*(\Lambda) \mid \sigma(a) = \sum_{(a,b) \in Q_1//Q_1} \lambda_{a,b}(a,b) \ \forall a \in Q_1 \}
\]

The equality \( \xi(\mathcal{L}(V_\Lambda^\circ)) = k(Q_1//Q_1) \cap \ker \psi_1 \) shows that we are done.

(iii) We have

\[
\mathcal{L}(H_\Lambda \cap \text{Im }^*(\Lambda)) = \{ \text{ad } (1 + x) \in \text{Ad } E^*(\Lambda) \mid x \in r \} = \{ \text{ad } x \in \text{Ad } E^*(\Lambda) \mid x \in r \}
\]

Let \( x \) be an element of the radical \( r \) such that \text{ad } (x) \) is an \( E \)-bimodule morphism. The proof of Corollary \[29\] shows that \( x \) is a linear combination of oriented cycles of length \( \geq 1 \), so \( x = \sum_{(e,\gamma) \in Q_0//B - Q_0} \lambda_{e,\gamma} \gamma \) with \( \gamma \in k \). Since \( \xi(\text{ad } (x)) = \sum_{(e,\gamma) \in Q_0//B - Q_0} \lambda_{e,\gamma} \gamma = \sum_{a \in Q_1} (a,\gamma a) - \sum_{a \in Q_1} (a,a\gamma) \), we are done.

(iv) It is obvious that the Lie algebra of \( \hat{E} \) is generated by the inner derivations \text{ad } (e) where \( e \in Q_0 \). Since \( \xi(\text{ad } (e)) = \sum_{a \in Q_1} (a,a) - \sum_{a \in Q_1} (a,a) \) the proof is finished.

\[ \text{q.e.d. 3.4} \]

Using the notations which will be introduced at the beginning of the next section we get:

**Corollary 3.5** Let \( k \) be a field of characteristic 0 and \( \Lambda \) a finite dimensional monomial \( k \)-algebra. The Lie algebra of the closed connected normal subgroup \( H_\Lambda \cap \text{Im }^*(\Lambda) \) of the connected algebraic group \( \text{Out } (\Lambda)^0 \) is the Lie ideal \( \oplus_{i \geq 1} L_i \) of \( H^1(\Lambda,\Lambda) \) and the Lie algebra of the closed connected subgroup \( V_\Lambda^\circ \) of \( \text{Out } (\Lambda)^0 \) is the Lie subalgebra \( L_0 \) of \( H^1(\Lambda,\Lambda) \).

### 4 The Lie algebra \( H^1(\Lambda,\Lambda) \) of a monomial algebra

Since Hochschild cohomology is additive and since its Lie algebra structure follows this additive decomposition we will assume henceforth that the quiver \( Q \) is connected. For the study of the Lie algebra \( H^1(\Lambda,\Lambda) \) of the monomial algebra \( \Lambda = kQ//Z \) we will use the description which we obtained in Theorem \[2.7\]. For every element \( x \in \ker \psi_1 \) we will also denote its class in \( H^1(\Lambda,\Lambda) = \ker \psi_1/\text{Im } \psi_0 \) by \( x \).

If \( (a,\gamma) \in Q_1//B_n \) and \( (b,\varepsilon) \in Q_1//B_m \), the formula we have obtained shows that \( [(a,\gamma),(b,\varepsilon)] \) is an element of \( k(Q_1//B_{n+m-1}) \). Thus, we have a graduation on the Lie algebra \( k(Q_1//B) = \oplus_{i \in \mathbb{N}} k(Q_1//B_i) \) by considering that the elements of \( k(Q_1//B_i) \) have degree \( i - 1 \) for all \( i \in \mathbb{N} \). It is clear that the Lie subalgebra \( \ker \psi_1 \)}
of $k(Q_1//B)$ preserves this graduation and that $\text{Im } \psi_0$ is a graded ideal. Therefore, the Lie algebra $H^1(\Lambda, \Lambda) = \text{Ker } \psi_1//\text{Im } \psi_0$ has also a graduation. If we set

\[ L_{-1} := k(Q_1//Q_0) \cap \text{Ker } \psi_1 \]

\[ L_0 := k(Q_1//Q_1) \cap \text{Ker } \psi_1/(\sum_{a \in Q_1} (a, a) - \sum_{a \in Q_1} (a, a) | e \in Q_0) \quad \text{and} \]

\[ L_i := k(Q_1//B_{i+1}) \cap \text{Ker } \psi_1/(\sum_{a \in Q_1} (a, \gamma a) - \sum_{a \in Q_1} (a, a\gamma) | (e, \gamma) \in Q_0//Q_1) \]

for all $i \geq 1$, $i \in \mathbb{N}$, we obtain $H^1(\Lambda, \Lambda) = \oplus_{i \geq -1} L_i$ and $[L_i, L_j] \subset L_{i+j}$ for all $i, j \geq -1$ where $L_{-2} = 0$.

**Lemma 4.1** $L_{-1}$ equals 0 if and only if there exists for every loop $(a, e) \in Q_1//Q_0$ a path $p$ in $Z$ such that $p(a, e) \neq 0$.

**Proof:** Clear, since $\psi_1(a, e) = \sum_{p \in Z} (p, p(a, e))$ for every loop $(a, e) \in Q_1//Q_0$.

q.e.d. 4.1

We study first the subalgebra $L_{-1}$ of $H^1(\Lambda, \Lambda)$.

**Proposition 4.2** Each of the following conditions implies $L_{-1} = 0$:

(i) The quiver $Q$ does not have a loop.

(ii) For every loop $(a, e) \in Q_1//Q_0$ of $Q$ the characteristic of the field $k$ does not divide the integer $m \geq 2$ for which $a^m \in Z$ and $a^{m-1} \in B$.

(iii) The characteristic of the field $k$ is equal to 0.

(iv) $\Lambda$ is a truncated quiver algebra $kQ//\langle Q_m \rangle$ for a quiver $Q$ different from the loop and for $m \geq 2$.

(v) $Q$ is the loop quiver and $\Lambda = kQ//\langle Q_m \rangle$ is a truncated quiver algebra such that the characteristic of $k$ does not divide $m \geq 2$.

**Proof:** (i) : clear

(ii) : Let $(a, e) \in Q_1//Q_0$ be a loop of $Q$. Since $\Lambda$ is finite dimensional, there exists an integer $m \geq 2$ such that $p := a^m \in Z$ and $a^{m-1} \in B$. If the characteristic of $k$ does not divide $m$, then $p^{(a, e)} = ma^{m-1}$ is different from 0 and so $ma^{m-1} \not\in \langle Z \rangle$ i.e. $p^{(a, e)} \neq 0$.

(iii) : Clear, because (iii) implies (ii).

(iv) : Using (ii) we can suppose that char $k$ divides $m$. Let $(a, e) \in Q_1//Q_0$ be a loop of the quiver $Q$. By assumption the connected quiver itself is not a loop, therefore there exists an arrow $b \in Q_1$ different from $a$ such that $ba^{m-1}$ or $a^{m-1}b \in Z = Q_m$.

The fact that char $k$ divides $m$ implies that the characteristic of the field $k$ does not divide $m - 1$ and thus $0 \neq (m - 1)ba^{m-2} \not\in \langle Z \rangle = \langle Q_m \rangle$ or $0 \neq (m - 1)a^{m-2}b \not\in \langle Z \rangle = \langle Q_m \rangle$.

(v) : Clear, because (v) implies (ii).

q.e.d. 4.2

Before we study the case where $L_{-1}$ equals zero we consider the following exceptional case:
Proposition 4.3 Let $Q$ be the loop having a vertex $e$ and an arrow $a$. Let $Z = \{a^m\}$ be such that the characteristic $p$ of the field $k$ divides the integer $m \geq 2$ and let $\Lambda = kQ/\langle Z \rangle$ be the monomial algebra associated to $Z$. The following conditions are equivalent:

(i) The Lie algebra $H^1(\Lambda, \Lambda)$ is simple.

(ii) The Lie algebra $H^1(\Lambda, \Lambda)$ is semisimple.

(iii) The integer $m \geq 2$ is equal to the characteristic $p$ of the field $k$ and $p > 2$.

(iv) $H^1(\Lambda, \Lambda)$ is isomorphic to the Witt Lie algebra $W(1, 1) := \text{Der}(k[X]/(X^p))$ and $p > 2$.

Proof: (i) $\Rightarrow$ (ii) : trivial

(ii) $\Rightarrow$ (iii) : The assumption $\text{char} k = p$ implies $[L_{-1}, L_{p-1}] = 0$ and thus, $L_{p-1} \oplus \cdots \oplus L_{m-2}$ is a solvable Lie ideal of the Lie algebra $\text{H}^1(\Lambda, \Lambda) = \bigoplus_{i=1}^{m-2} L_i$. Since $\text{H}^1(\Lambda, \Lambda)$ is semisimple, we deduce $m = p$. We have $p > 2$, because there is no semisimple Lie algebra of dimension 2.

(iii) $\Rightarrow$ (iv) : Since $\Lambda$ is isomorphic to the commutative algebra $k[X]/(X^m)$, the Lie algebra $\text{H}^1(\Lambda, \Lambda)$ is isomorphic to the Lie algebra of derivations $\text{Der}(k[X]/(X^m))$.

(iv) $\Rightarrow$ (i) : If $\text{char} k > 2$, then the Witt Lie algebra $W(1, 1) := \text{Der}(k[X]/(X^m))$ is one of the non classical simple Lie algebras.

$q.e.d. 4.3$

Proposition 4.3 shows that the case where $L_{-1}$ is different from zero is quite exceptional. We will assume henceforth that $L_{-1} = 0$. In that case $\bigoplus_{i \geq 1} L_i$ is a solvable Lie ideal of $\text{H}^1(\Lambda, \Lambda)$ since $\text{H}^1(\Lambda, \Lambda)$ is finite dimensional. It is obvious that $L_0$ is a Lie subalgebra of $\text{H}^1(\Lambda, \Lambda)$ whose bracket is

$$[(a, c), (b, d)] = \delta_{a,d}(b, c) - \delta_{b,c}(a, d)$$

for all $(a, c), (b, d) \in L_0$. It follows that we have $\text{Rad} \text{H}^1(\Lambda, \Lambda) = \text{Rad} L_0 \oplus \bigoplus_{i \geq 1} L_i$ and $\text{H}^1(\Lambda, \Lambda)/\text{Rad} \text{H}^1(\Lambda, \Lambda) = L_0/\text{Rad} L_0$ where $\text{Rad} \text{H}^1(\Lambda, \Lambda)$ (resp. $\text{Rad} L_0$) denotes the radical of $\text{H}^1(\Lambda, \Lambda)$ (resp. $L_0$). As a consequence the study of the Lie algebra $\text{H}^1(\Lambda, \Lambda)$ can be often reduced to the study of the Lie algebra $L_0$.

We recall a few definitions introduced by Guil-Asensio and Saorín (see 2.3 in [8] and 25 in [9]) for the convenience of the reader which will be useful in the following.

Definition 4.4 Let $(a, b)$ be a couple of parallel arrows. We shall say that the ideal $\langle Z \rangle$ of the algebra $kQ$ is $(a, b)$-saturated, if for every path $p$ of $Z$ we have $p^{(b,a)} = 0$. This is denoted by $a \leq_Z b$. The ideal $\langle Z \rangle$ is called completely saturated if it is $(a, b)$-saturated for all $(a, b) \in Q_1/\langle Q_1 \rangle$.

Remarks 4.5

(i) For every class of parallel arrows $\overline{a}$ the relation $\leq_{\langle Z \rangle}$ on $\overline{a}$ is reflexive and transitive.

(ii) For $p \in Z$ and $(a, b) \in Q_1/\langle Q_1 \rangle$ we have $p^{(b,a)} = 0$ if and only if each term of the sum is zero (see [8, 9]), i.e. each replacement of one appearance of $b$ in $p$ by $a$ is a path in $Z$. 
The hereditary algebra where $Z = \emptyset$ and the truncated algebras where $Z = Q_m$, $m \geq 2$, are examples of monomial algebras whose ideal $(Z)$ is completely saturated. Locatelli studied the Hochschild cohomology of truncated algebras in [12] and Cibils treated the case of radical square zero algebras (see [3]). It is clear that the relation of parallelism is an equivalence relation on $Q_1$. Denote by $Q_1//\approx$ its set of equivalence classes. We shall call a class of parallel arrows non trivial if it contains at least two arrows. It is easy to check that we have on every class of parallel arrows $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in Q_1//\approx$ the following equivalence relation

$$\alpha_i \approx \alpha_j := \alpha_i \leq_{(Z)} \alpha_j \text{ and } \alpha_j \leq_{(Z)} \alpha_i$$

We denote by $\pi//\approx$ its set of equivalence classes. For $R, S \in \pi//\approx$ we write $R \leq_{(Z)} S$ if there exist arrows $a \in R$ and $b \in S$ such that $a \leq_{(Z)} b$ which is equivalent to saying that $a \leq_{(Z)} b$ for all arrows $a \in R$ and $b \in S$. Note that the relation $\leq_{(Z)}$ on $\pi//\approx$ is an order relation.

The notion of saturation can be reformulated in terms of the Lie algebra structure:

**Remark 4.6** Let $(a, b)$ be a couple of parallel arrows. The ideal $(Z)$ is $(a,b)$-saturated if and only if $(b,a)$ is an element of the Lie algebra $L_0$. Thus the parallel arrows $a$ and $b$ are equivalent if and only if $(a, b) \in L_0$ and $(b, a) \in L_0$. The ideal $(Z)$ is completely saturated if and only if every two parallel arrows are equivalent.

**Proposition 4.7**

A basis $B$ of the Lie algebra $L_0$ is given by the union of the following sets:

(i) all the couples $(a, b) \in L_0$ such that the parallel arrows $a$ and $b$ are different

(ii) for every class of parallel arrows $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in Q_1//\approx$, all the elements $(\alpha_i, \alpha_i) \in L_0$ such that $i < n$

(iii) $|Q_1//\approx| - |L_0| + 1$ linearly independent elements $(c, c) \in L_0$ different from those in (ii).

**Proof:** An element $\sum_{(a,b)\in Q_1//Q_1} \lambda_{(a,b)}(a,b) \in k(Q_1//Q_1)$, with $\lambda_{(a,b)} \in k$, is contained in $\ker \psi_1$ if and only if

$$0 = \sum_{(a,b)\in Q_1//Q_1} \lambda_{(a,b)} \psi_1(a,b) = \sum_{(a,b)\in Q_1//Q_1} \lambda_{(a,b)} \sum_{p \in Z} \lambda_{(a,b)}(p, p^{(a,b)})$$

First we recall that for a couple of distinct arrows $(a, b)$, the non zero terms of the sum defining $p^{(a,b)}$ are all distinct (see [23], forming a subset of the basis $Z//B$). Second we check the following: if $a \neq b$ and if $(p, p^{(a,b)}) = (q, q^{(c,d)})$ then $p = q$ and if $p^{(a,b)} \neq 0$ then $(a, b) = (c, d)$. Indeed let $p = p_n \ldots p_1$ be a path of $Z$ of length $l$ and consider the summand $p_n \ldots p_{i+1}b p_{i-1} \ldots p_1 \in B$ of $p^{(a,b)}$ obtained by replacing the arrow $p_i = a$ by $b \neq a$. The fact $(p, p^{(a,b)}) = (p, p^{(c,d)}) \neq 0$ implies that there exists $j \in \{1, \ldots, n\}$ such that $p_n \ldots p_{j+1}b p_{j-1} \ldots p_{i+1}b p_{i-1} \ldots p_1 = p_n \ldots p_{j+1}q p_{j-1} \ldots p_{i+1}q p_{i-1} \ldots p_1$. Since $a \neq b$, we have $i = j$ as well as $a = d$ and $c = b$. This shows that if $a \neq b$ then the non zero elements $(p, p^{(a,b)})$ of $k(Z//B)$ are linearly independent. Therefore we can choose a basis of $k(Q_1//Q_1) \cap \ker \psi_1$ in $Q_1//Q_1$: take all elements $(a, a) \in Q_1//Q_1$ and all elements $(a, b) \in Q_1//Q_1$, $a \neq b$, such that the ideal $(Z)$ is $(b, a)$-saturated. We have for every vertex $e \in Q_0$ in $L_0 = k(Q_1//Q_1) \cap \ker \psi_1/k(Q_1//Q_1) \cap \im \psi_0$ the relation $\sum_{e \in Q_1 \cap \ker \psi_1} (e, a) - \sum_{e \in Q_1 \cap \ker \psi_1} (a, e) = 0$. Since $|Q_0| - 1$ of those $|Q_0|$ relations are linearly independent, we see that $\dim k(Q_1//Q_1) \cap \im \psi_0 = |Q_1| - |Q_0| + 1$. Thus we get a basis of $L_0$ if we impose the conditions (ii) and (iii).
Let $\mathcal{B}$ be a basis of $L_0$ as described in the preceding Proposition.

**Remark 4.8** If the quiver $Q$ has a (non oriented or oriented) cycle, then there exists at least one class of parallel arrows $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$, $n \geq 1$, such that $\mathcal{B}$ contains all the elements $(\alpha_i, \alpha_j) \in L_0$, $1 \leq i \leq n$.

For every class of parallel arrows $\overline{\alpha} \in Q_1//\llbracket$ we denote by $L_0^{\overline{\alpha}}$ the Lie ideal of $L_0$ generated by the elements $(a, b) \in \mathcal{B}$ such that $a, b \in \overline{\alpha}$. Obviously the Lie algebra $L_0$ is the product of these Lie algebras:

$$L_0 = \prod_{\overline{\alpha} \in Q_1//\llbracket} L_0^{\overline{\alpha}}$$

To study the radical of this Lie algebra we need the following Lemma:

**Lemma 4.9** Let $J \neq 0$ be a Lie ideal of $L_0^{\overline{\alpha}}$ generated by elements $(\alpha_i, \alpha_j) \in L_0^{\overline{\alpha}}$ such that $(\alpha_j, \alpha_i) \not\in L_0^{\overline{\alpha}}$. Then $\dim_k [J, J] < \dim_k J$.

**Proof:** Let $(\alpha_i, \alpha_j)$ be any element of $J$. If $(\alpha_i, \alpha_j) \not\in [J, J]$ then nothing is to show. So suppose $(\alpha_i, \alpha_j) \in [J, J]$. This is the case if and only if there exists an arrow $\alpha_i \in \overline{\alpha}$, $\alpha_i \neq \alpha_j$, such that $(\alpha_i, \alpha_j) \in J$ and $(\alpha_i, \alpha_j) \in J$. If $(\alpha_i, \alpha_j) \not\in [J, J]$ then we are done. If not we start again. Suppose that we have $(\alpha_i, \alpha_j) \in [J, J]$. Thus there exists an arrow $\alpha_k \in \overline{\alpha} \setminus \{\alpha_i, \alpha_j\}$ such that $(\alpha_i, \alpha_k) \in J$ and $(\alpha_k, \alpha_i) \in J$. The fact $(\alpha_i, \alpha_j) \in J$ implies $(\alpha_j, \alpha_i) \not\in J$ and so $\alpha_k \neq \alpha_j$. If $(\alpha_i, \alpha_k) \not\in [J, J]$ then we are done. If not we start again. Since $\overline{\alpha}$ is a finite set, this procedure stops after a finite number of steps.

**Theorem 4.10** The radical of the Lie algebra $L_0$ is generated as a k-vector space by the following elements of $L_0$: for every class of parallel arrows $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$

- $\sum_{\alpha_i \in S} (\alpha_i, \alpha_i)$ for every equivalence class $S \in \overline{\alpha} / \approx$
- $(\alpha_i, \alpha_j)$, $i \neq j$, if $(\alpha_i, \alpha_j) \in L_0$ and $(\alpha_j, \alpha_i) \not\in L_0$
- $(\alpha_{i_1}, \alpha_{i_2})$, $i_1 \neq i_2$, $(\alpha_{i_1}, \alpha_{i_1})$ and $(\alpha_{i_1}, \alpha_{i_1})$ for all $S = \{\alpha_{i_1}, \alpha_{i_2}\} \in \overline{\alpha} / \approx$ if $\text{char} k = 2$

**Proof:** Let $I$ be the k-vector space generated by the above-described elements. If we define $I_\overline{\alpha} := I \cap L_0^{\overline{\alpha}}$ we obtain $I = \prod_{\overline{\alpha} \in Q_1//\llbracket} I_\overline{\alpha}$. Let $\overline{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ be a class of parallel arrows of $Q$. It is easy to check using the definition of the bracket on $L_0$, that $I_\overline{\alpha}$ is a Lie ideal of $L_0^{\overline{\alpha}}$ and that $I^{(3)}_\overline{\alpha} := [I^{(2)}_\overline{\alpha}, I^{(2)}_\overline{\alpha}]$ only contains elements $(\alpha_i, \alpha_j) \in L_0^{\overline{\alpha}}$ such that $(\alpha_i, \alpha_j) \not\in L_0^{\overline{\alpha}}$. From the fact that $\dim_k I_\overline{\alpha} < \infty$ we deduce successively using the preceding Lemma that there exists an $l \in \mathbb{N}$ such that $I^{(l)}_\overline{\alpha} := [I^{(l-1)}_\overline{\alpha}, I^{(l-1)}_\overline{\alpha}] = 0$. Thus $I_\overline{\alpha}$ is a solvable Lie ideal of $L_0^{\overline{\alpha}}$ and we are done if we show that $L_0^{\overline{\alpha}} / I_\overline{\alpha}$ is a semisimple Lie algebra or 0. From the relations in $L_0^{\overline{\alpha}} / I_\overline{\alpha}$ follows that we obtain a basis of $L_0^{\overline{\alpha}} / I_\overline{\alpha}$ by taking all elements $(\alpha_i, \alpha_j)$, where $\alpha_i \neq \alpha_j$ and $\alpha_i, \alpha_j \in S$, and $|S| - 1$ elements $(\alpha_i, \alpha_i)$ where $\alpha_i \in S$ for every $S \in \overline{\alpha} / \approx$, with $|S| > 2$ in case $\text{char} k = 2$. For every $S \in \overline{\alpha} / \approx$ we denote by $S_{L_0^{\overline{\alpha}} / I_\overline{\alpha}}$ the Lie ideal generated by the elements $(\alpha_i, \alpha_j) \in L_0^{\overline{\alpha}}$ such that $\alpha_i, \alpha_j \in S$. Then we have $L_0^{\overline{\alpha}} / I_\overline{\alpha} = \prod_{S \in \overline{\alpha} / \approx} S_{L_0^{\overline{\alpha}} / I_\overline{\alpha}}$. Let $S = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ be an equivalence class in $\overline{\alpha} / \approx$, with $m \geq 2$ if $\text{char} k \neq 2$ and $m \neq 2$ if $\text{char} k = 2$. The elements $(\alpha_{i_p}, \alpha_{i_q})$, $i_p \neq i_q$, and $(\alpha_{i_p}, \alpha_{i_p})$, $i_p \neq i_m$, form a basis of $S_{L_0^{\overline{\alpha}} / I_\overline{\alpha}}$ and we have the relation...
\[ \sum_{\alpha_i \in S} (\alpha_{i_p}, \alpha_{i_p}) = 0. \] As the bracket on \( S \mathcal{L}_{0}/I \) is given by \([ (\alpha_{i_p}, \alpha_{i_p}), (\alpha_{i_k}, \alpha_{i_l}) ] = \delta_{i_p,i_k} (\alpha_{i_p}, \alpha_{i_l}) \) we see that \( S \mathcal{L}_{0}/I \rightarrow \mathfrak{gl}(m, k) := \mathfrak{gl}(m, k)/k \) given by \( (\alpha_{i_p}, \alpha_{i_k}) \rightarrow e_{pq} \mod k \) is a Lie algebra isomorphism. Since \( \mathfrak{gl}(m, k) \) is a semisimple Lie algebra if \( m \geq 2 \) when \( \text{char } k \neq 2 \) and if \( m > 2 \) when \( \text{char } k = 2 \), we conclude that \( L \mathcal{L}_{0}/I \) is a semisimple Lie algebra.

q.e.d. 4.10

**Corollary 4.11** The semisimple Lie algebra \( H^1(\Lambda, \Lambda)/\text{Rad } H^1(\Lambda, \Lambda) = L_0/\text{Rad } L_0 \) is the product of Lie algebras having a factor \( \mathfrak{gl}(|S|, k) := \mathfrak{gl}(|S|, k)/k \) for every equivalence class \( S \in \pi/ \approx \) of a class of parallel arrows \( \pi/ \) such that \( |\pi| \geq 2 \) if \( \text{char } k \neq 2 \) and \( |\pi| > 2 \) if \( \text{char } k = 2 \).

**Corollary 4.12** Let \( Q \) be a connected quiver and \( \Lambda = kQ/\langle Z \rangle \) a finite dimensional monomial algebra such that \( L_{-1} = k(Q_0/\langle Q_1 \rangle \cap \text{Ker } \psi_1 = 0. \) The following conditions are equivalent:

(i) The Lie algebra \( H^1(\Lambda, \Lambda) \) is solvable.

(ii) Every equivalence class \( S \) of a class of parallel arrows \( \pi/ \approx \) of \( Q \) contains one and only one arrow if the characteristic of the field \( k \) is not 2. In the case \( \text{char } k = 2 \) we have \( |\pi| \leq 2 \) for all \( S \in \pi/ \approx, \pi/ \approx \in Q_1/ \).

Since in characteristic 0 the Lie algebra of a connected algebraic group is solvable if and only if the connected algebraic group is solvable (as a group), using Proposition 4.2 we infer the following result also proved by Guil-Asensio and Saorín (see Corollary 2.22 in [9]):

**Corollary 4.13** Let \( k \) be a field of characteristic 0, \( Q \) a connected quiver and \( \Lambda = kQ/\langle Z \rangle \) a finite dimensional monomial algebra. Then the identity component of the algebraic group of outer automorphisms \( \text{Out } (\Lambda)^\circ \) is solvable if and only if every two parallel arrows of \( Q \) are not equivalent.

**Definition 4.14** Let \( \overrightarrow{Q} \) be the subquiver of \( Q \) obtained by taking a representative for every class of parallel arrows of \( Q \).

**Theorem 4.15** Let \( Q \) be a connected quiver and \( \Lambda = kQ/\langle Z \rangle \) a finite dimensional monomial algebra such that \( L_{-1} = k(Q_0/\langle Q_1 \rangle \cap \text{Ker } \psi_1 = 0. \) The following conditions are equivalent:

(i) The Lie algebra \( H^1(\Lambda, \Lambda) \) is semisimple.

(ii) The quiver \( \overrightarrow{Q} \) is a tree, \( Q \) has at least one non trivial class of parallel arrows and the ideal \( \langle Z \rangle \) is completely saturated. If the characteristic of the field \( k \) is 2, then \( Q \) does not have a class of parallel arrows containing exactly two arrows.

(iii) \( H^1(\Lambda, \Lambda) \) is isomorphic to the non trivial product of Lie algebras

\[ \prod_{\pi/ \approx \in Q_1/} \mathfrak{gl}(|\pi|, k) \]

where \( |\pi| \neq 2 \) if the characteristic of \( k \) is equal to 2.
Proof: Let $B$ be a basis of $L_0$ as described in Proposition 4.14.

(i) $\Rightarrow$ (ii) : If $Q$ contained a (non oriented or oriented) cycle, there would exist a class $\bar{\alpha} \in Q_1//$ of parallel arrows such that $(\alpha, \alpha) \in B$ for all $\alpha \in \bar{\alpha}$ (see 4.8) and so $k(\sum_{\alpha \in \bar{\alpha}}(\alpha, \alpha))$ would be a nontrivial abelian ideal of $L_0$ which contradicts the fact that $0 = \text{Rad} H^1(\Lambda, \Lambda) = \text{Rad} L_0 \oplus \bigoplus_{i \geq 1} L_i$. To insure that $H^1(\Lambda, \Lambda) = L_0 \neq 0$ it is necessary that $Q$ contains at least one nontrivial class of parallel arrows. Suppose that $(Z)$ is not completely saturated. According to Remark 4.16 there exist two parallel arrows which are not equivalent. This means that there exists a class of parallel arrows $\bar{\alpha} \in Q_1//$ such that $\bar{\alpha} / \approx$ contains at least two elements, say $R$ and $S$. The properties of the basis $B$ show that either $\sum_{\alpha \in R}(a, a) \neq 0$ or $\sum_{\alpha \in S}(a, a) \neq 0$ which contradicts the fact that $\text{Rad} L_0 \subset \text{Rad} H^1(\Lambda, \Lambda) = 0$. The last statement is an immediate consequence of Theorem 4.10.

(ii) $\Rightarrow$ (iii) : Let $Q$ and $Z$ satisfy the given conditions. From the fact that $Q$ is a tree, we deduce $\bigoplus_{i \geq 1} L_i = 0$ and so $H^1(\Lambda, \Lambda) = L_0$. Since $(Z)$ is completely saturated we have $L_0 = k(Q_1//Q_1)/(\sum_{\alpha \in Q_1}(a, a) - \sum_{\alpha \in Q_1}(a, a) | e \in Q_0)$. This and the fact that $Q$ does not have an equivalence class containing two arrows if $\text{char } k = 2$, show that its radical is generated by the elements of type $\sum_{\alpha \in \bar{\alpha}}(\alpha, \alpha)$, $\bar{\alpha} \in Q_1//$. (see Theorem 4.10). These elements being equal to zero we see that $\text{Rad} H^1(\Lambda, \Lambda) = 0$ and Corollary 4.11 yields the result.

(iii) $\Rightarrow$ (i) : Clear in view of the following Remark.

$q.e.d. 4.15$

Remark 4.16 The Lie algebra $\mathfrak{pgl}(n, k) := \mathfrak{gl}(n, k)/k1$, $n \geq 2$, is isomorphic to the classical simple Lie algebra $\mathfrak{sl}(n, k)$ of $n \times n$-matrices having trace zero if the characteristic of the field $k$ does not divide $n$. If $\text{char } k$ divides $n$ and $n \neq 2$, then $\mathfrak{pgl}(n, k)$ is a semisimple algebra without being a direct product of simple Lie algebras.

Corollary 4.17 Let $Q$ be a connected quiver and $\Lambda = kQ/(Z)$ a finite dimensional monomial algebra such that $L_{-1} = k(Q_0//Q_1) \cap \text{Ker } \psi_1 = 0$. The following conditions are equivalent:

(i) The Lie algebra $H^1(\Lambda, \Lambda)$ is simple.

(ii) The quiver $\overline{Q}$ is a tree, the quiver $Q$ has exactly one class of parallel arrows $\bar{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ such that $n \geq 2$ and the characteristic of the field $k$ does not divide $n$. The ideal $(Z)$ is completely saturated.

(iii) There exists an integer $n \geq 2$ such that the characteristic of the field $k$ does not divide $n$ and $H^1(\Lambda, \Lambda)$ is isomorphic to the Lie algebra $\mathfrak{sl}(n, k)$.

Since the first Hochschild cohomology group is the Lie algebra of the algebraic group of outer automorphisms in characteristic 0 (see Proposition 4.14), we get the following Corollary (see Corollary 4.9 in [1] for a partial result on hereditary algebras):

Corollary 4.18 Let $k$ be a field of characteristic 0, Q a connected quiver and $\Lambda = kQ/(Z)$ a finite dimensional monomial algebra. The following conditions are equivalent:

(i) The algebraic group $\text{Out } (\Lambda)^c$ is semisimple.

(ii) The quiver $\overline{Q}$ is a tree, the quiver $Q$ has at least one non trivial class of parallel arrows and the ideal $(Z)$ is completely saturated.
(iii) The algebraic group \( \text{Out}(\Lambda)^\circ \) is isomorphic to the non trivial product of algebraic groups \( \prod_{\pi \in Q_{i,1}}/\text{PGL}(|\pi|,k) \).

Proof: In characteristic 0 a connected algebraic group is semisimple if and only if its Lie algebra is semisimple (see [2] 13.5).

(i) \( \Leftrightarrow \) (ii): This equivalence follows immediately from the preceding Theorem and Proposition 3.4.

(ii) \( \Rightarrow \) (iii): Let \( Q \) and \( Z \) satisfy the given condition. According to [3] we know that \( \text{Out}(\Lambda)^\circ = \frac{H_{\text{ad}U_{\Lambda}}}{H_{\text{ad}U_{\Lambda}}(\Lambda)} \times \frac{V_1}{E} \). Since \( \mathcal{O} \) is a tree, there is no path in \( \mathcal{B}' := B - Q_1 \cup Q_1 \) of length \( \geq 2 \) which is parallel to an arrow. From the fact that we have for every element \( \sigma \in H_{\Lambda} \cap U_{\Lambda} \) that \( \sigma(a) = a + \sum_{(a,\gamma) \in Q_1/B} \lambda(a,\gamma)^\gamma \) for all \( a \in Q_1 \) we deduce that \( H_{\Lambda} \cap U_{\Lambda} = \{\text{id}_\Lambda\} \). As the ideal \( \gen{Z} \) is completely saturated by assumption, any two parallel arrows are equivalent. Therefore the main Theorem 2.20 of the article [9] implies \( V_1^\Lambda \simeq \prod_{\pi \in Q_{i,1}}/\text{GL}(|\pi|,k) \). Since the quiver \( Q \) is assumed to be a tree, we have \( \hat{E} = \text{Ch}(Q,k) = \prod_{\pi \in Q_{i,1}}/k^*I_{|\pi|} \) thanks to example 8 in [3]. Hence we obtain

\[
\text{Out}(\Lambda)^\circ = \frac{H_{\text{ad}U_{\Lambda}}}{H_{\text{ad}U_{\Lambda}}(\Lambda)} \times \frac{V_1^\Lambda}{E} \simeq \prod_{\pi \in Q_{i,1}}/\text{GL}(|\pi|,k) = \prod_{\pi \in Q_{i,1}}/\text{PGL}(|\pi|,k)
\]

(iii) \( \Rightarrow \) (i): The semisimplicity of the Lie algebra \( \mathcal{L}(\text{Out}(\Lambda)^\circ) = \prod_{\pi \in Q_{i,1}}/\text{pGL}(|\pi|,k) \) implies the semisimplicity of the connected algebraic group \( \text{Out}(\Lambda)^\circ \).

\[\text{q.e.d. 4.18}\]

The fact that in characteristic 0 the Lie algebra of a connected algebraic group is simple if and only if the connected algebraic group is almost simple, yields the following result:

**Corollary 4.19** Let \( k \) be a field of characteristic 0, \( Q \) a connected quiver and \( \Lambda = kQ/\gen{Z} \) a finite dimensional monomial algebra. The following conditions are equivalent:

(i) The algebraic group \( \text{Out}(\Lambda)^\circ \) is almost simple.

(ii) The quiver \( \overline{\mathcal{O}} \) is a tree, the quiver \( Q \) has exactly one class of parallel arrows \( \overline{\sigma} = \{\sigma_1, \ldots, \sigma_n\} \) of order \( n \geq 2 \) and the ideal \( \gen{Z} \) is completely saturated.

(iii) There exists an integer \( n \geq 2 \) such that the algebraic group \( \text{Out}(\Lambda)^\circ \) is isomorphic to the algebraic group \( \text{PGL}(n,k) \).

In order to get criteria for the commutativity and the reductivity of the Lie algebra \( H^1(\Lambda,\Lambda) \) we need to study its center. Therefore we introduce the following definitions: for every class of parallel arrows \( \overline{\sigma} \) of \( Q \) we call a set \( C \subset \overline{\sigma} \) connected, if for every two arrows \( \sigma_1 \) and \( \sigma_r \) of \( C \) there exist arrows \( \sigma_2, \ldots, \sigma_{r-1} \) in \( C \) such that we have \( \sigma_i \leq \gen{Z} \sigma_{i+1} \) or \( \sigma_{i+1} \leq \gen{Z} \sigma_i \) for all \( i \in \{1, \ldots, r-1\} \). A connected set \( C \subset \overline{\sigma} \) is called a connected component of \( \overline{\sigma} \) if it is maximal for the connection, i.e. for every arrow \( \beta \in \overline{\sigma} - C \) there is no arrow \( \alpha \in C \) such that \( \alpha \leq \gen{Z} \beta \) or \( \beta \leq \gen{Z} \alpha \). Clearly the connected components of a class of parallel arrows \( \overline{\sigma} \) form a partition of \( \overline{\sigma} \).

**Lemma 4.20** Let \( Q \) be a connected quiver and \( \Lambda = kQ/\gen{Z} \) a finite dimensional monomial algebra.

(i) The center \( Z(L_0) \) of the Lie algebra \( L_0 \) is generated by the elements \( \sum_{a \in C}(a,a) \) where \( C \) denotes a connected component of a class of parallel arrows of \( Q \).
(ii) If the field $k$ has characteristic 0 or if the quiver $Q$ does not have an oriented cycle, then the center $Z(H^1(\Lambda, \Lambda))$ of the Lie algebra $H^1(\Lambda, \Lambda)$ is contained in the center $Z(L_0)$ of $L_0$.

**Proof:** (i) : For every element $(a, b) \in Q_1//Q_1$, $a \neq b$, we have $[(a, b), (a, a)] = (a, b)$ so that $Z(L_0)$ is contained in the abelian Lie subalgebra of $L_0$ generated by the elements $(a, a)$, $a \in Q_1$. For every linear combination $\sum_{a \in Q_1} \lambda_a(a, a)$, $\lambda_a \in k$, we have $\sum_{a \in Q_1} \lambda_a(a, a), (b, c) = (-\lambda_b + \lambda_c)(b, c)$ for all $(b, c) \in L_0$. Therefore $\sum_{a \in Q_1} \lambda_a(a, a)$ is contained in $Z(L_0)$ if and only if we have $\lambda_b = \lambda_c$ for all arrows $b, c$ such that $c \geq (\gamma) b$. This shows that $Z(L_0)$ is generated by the elements $\sum_{a \in C}(a, a)$.

(ii) : In both cases we have $L_{-1} = 0$ according to Proposition 4.2. If the characteristic of $k$ is 0, then we have for every element $0 \neq \sum_{(a, \gamma) \in Q_1//B_{i+1}} \lambda_{(a, \gamma)}(a, \gamma) \in L_i$, $\lambda_{(a, \gamma)} \in k$, $i \geq 1$, that

$$[\sum_{(a, \gamma) \in Q_1//B_{i+1}} \lambda_{(a, \gamma)}(a, \gamma), \sum_{b \in Q_1} (b, b)] = -i \sum_{(a, \gamma) \in Q_1//B_{i+1}} \lambda_{(a, \gamma)}(a, \gamma) \neq 0$$

This shows $Z(H^1(\Lambda, \Lambda)) \subset Z(L_0)$. We assume now that $Q$ does not have an oriented cycle. Then every path $\gamma \in B$ of length $\geq 2$ parallel to an arrow $a$ cannot contain $a$ and so $[(a, \gamma), (a, a)] = (a, \gamma)$. Taking into account the fact $H^1(\Lambda, \Lambda) = L_0 \oplus \bigoplus_{i \geq 1} k(Q_1//B_{i+1}) \cap \text{Ker} \psi_1$ yields $Z(H^1(\Lambda, \Lambda)) \subset Z(L_0)$.

q.e.d. 4.20

Recall that a Lie algebra is called reductive if its radical and its center are equal.

**Proposition 4.21** Let $Q$ be a connected quiver and $\Lambda = kQ//Z$ a finite dimensional monomial algebra. If $Q$ does not have an oriented cycle or if the field $k$ has characteristic 0, then the following conditions are equivalent:

(i) The Lie algebra $H^1(\Lambda, \Lambda)$ is reductive.

(ii) The Lie ideal $\bigoplus_{i \geq 1} L_i$ is equal to 0 and the relation $\leq _{Z}$ is symmetric on every class of parallel arrows. If the characteristic of $k$ is 2, then there exists no equivalence class $S \in \overline{\pi}/ \approx$ containing exactly two arrows of a class of parallel arrows $\overline{\pi}$.

**Proof:** (i) $\Rightarrow$ (ii) : Let be $\text{Rad} H^1(\Lambda, \Lambda) = Z(H^1(\Lambda, \Lambda))$. If $Q$ does not have an oriented cycle or if $\text{char} k = 0$, then $L_{-1} = 0$ and $\text{Rad} H^1(\Lambda, \Lambda) = \text{Rad} L_0 \oplus \bigoplus_{i \geq 1} L_i$.

The preceding Lemma shows that the center of $H^1(\Lambda, \Lambda)$ is included in $L_0$ and thus $\text{Rad} H^1(\Lambda, \Lambda) \subset L_0$. This implies $\bigoplus_{i \geq 1} L_i = 0$ and therefore $H^1(\Lambda, \Lambda) = L_0$.

In view of Theorem 4.10 and the preceding Lemma, the assumption $\text{Rad} L_0 = \text{Rad} H^1(\Lambda, \Lambda) = Z(H^1(\Lambda, \Lambda)) = Z(L_0)$ implies that there is no element $(a, b) \in L_0$ such that $(b, a) \notin L_0$, i.e. the relation $\leq _{Z}$ is symmetric. Furthermore there exists no equivalence class $S \in \overline{\pi}/ \approx$, $\overline{\pi} \in Q_1//$, containing exactly two elements if $\text{char} k = 2$.

(ii) $\Rightarrow$ (i) : The assumptions imply $L_{-1} = 0$. From $\bigoplus_{i \geq 1} L_i = 0$ we deduce $H^1(\Lambda, \Lambda) = L_0$. Theorem 4.10 and the preceding Lemma show that $\text{Rad} H^1(\Lambda, \Lambda) = \text{Rad} L_0 = Z(L_0) = Z(H^1(\Lambda, \Lambda))$.

q.e.d. 4.21

Since in characteristic 0 a connected algebraic group is reductive if and only if its Lie algebra is reductive, we deduce immediately using Proposition 3.3 and Corollary 3.7.
Corollary 4.22 Let \( k \) be a field of characteristic 0, \( Q \) a connected quiver and \( \Lambda = kQ/\langle Z \rangle \) a finite dimensional monomial algebra. Then the identity component of the algebraic group of the outer automorphisms \( \text{Out}(\Lambda) \) is reductive if and only if the relation \( \subseteq \langle Z \rangle \) is symmetric on every class of parallel arrows of \( Q \) and if \( H^1(C_0) = 0 \).

Define a sequence of ideals of a Lie algebra \( L \) by setting \( C^0L := L(= L^{(0)}) \), \( C^1L := [L, L](= L^{(1)}) \), \( C^2L := [L, C^1L] \), \( \ldots \), \( C^nL := [L, C^{n-1}L] \). A Lie algebra \( L \) is called nilpotent if there exists a nonnegative integer \( m \) such that \( C^mL = 0 \). The integer \( m \) such that \( C^mL = 0 \) and \( C^{m-1}L \neq 0 \) is called the nilindex of \( L \). A Lie algebra \( L \) is called filiform if it is nilpotent of maximal nilindex \( m \) that is \( \dim C^kL = m - k \) for \( 1 \leq k \leq m \). The fact \( L^{(n)} \subset C^nL \) for all \( n \in \mathbb{N} \) implies that nilpotent algebras are solvable.

Proposition 4.23 Let \( Q \) be a connected quiver and \( \Lambda = kQ/\langle Z \rangle \) a finite dimensional monomial algebra. If \( Q \) does not have an oriented cycle or if \( \text{char } k = 0 \), then the following conditions are equivalent:

(i) The Lie algebra \( H^1(\Lambda, \Lambda) \) is abelian.

(ii) The Lie algebra \( H^1(\Lambda, \Lambda) \) is filiform.

(iii) The Lie algebra \( H^1(\Lambda, \Lambda) \) is nilpotent.

(iv) \( \bigoplus_{i \geq 1} L_i = 0 \) and there exist no parallel arrows \( a \neq b \) satisfying \( a \subseteq \langle Z \rangle \) \( b \).

(v) The Lie algebra \( H^1(\Lambda, \Lambda) \) is generated by the elements \( (a, a) \) of \( L_0 \).

(vi) The dimension of the Lie algebra \( H^1(\Lambda, \Lambda) \) equals the Euler characteristic \( |Q_1| - |Q_0| - 1 \).

Proof: (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are obvious.

(iii) \( \Rightarrow \) (iv) : If there were two parallel arrows \( a \neq b \) such that \( a \subseteq \langle Z \rangle \) \( b \), we would have \( (b, a) \in L_0 \) and \( [(a, a), (b, a)] = (b, a) \), and so \( (a, a) \in \mathcal{C}^0 \) \( \Lambda \) for all \( n \in \mathbb{N} \), contrary to the assumption. If \( Q \) does not have an oriented cycle, then there exists for every element \( 0 \neq \sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a) \in L_i, i \geq 1 \), an arrow \( b \in Q_1 \) such that \( \sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a), b, b) = \sum_{(b, b) \in Q_1/\langle B_i \rangle} \lambda(b, b)(b, b) = 0 \). Thus we have \( 0 = \sum_{(b, b) \in Q_1/\langle B_i \rangle} \lambda(b, b)(b, b) \in \mathcal{C}^1 \) \( \Lambda \) for all \( n \in \mathbb{N} \). Since the Lie algebra \( H^1(\Lambda, \Lambda) \) is assumed to be nilpotent, it follows that \( \bigoplus_{i \geq 1} L_i = 0 \). In case \( \text{char } k = 0 \) we have for every element \( \sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a) \in L_i, i \geq 1 \),

\[
\sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a), b, b) = -i \sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a) \neq 0
\]

and thus \( \sum_{(a, a) \in Q_1/\langle B_i \rangle} \lambda(a, a)(a, a) \in \mathcal{C}^n \) \( \Lambda \) for all \( n \in \mathbb{N} \) which implies \( \bigoplus_{i \geq 1} L_i = 0 \).

(iv) \( \Rightarrow \) (v) : By assumption \( H^1(\Lambda, \Lambda) = L_0 \) and \( L_0 \) does not contain an element \( (a, b), a \neq b \).

(v) \( \Rightarrow \) (vi) : We deduce from the given condition that \( \dim \ker \psi_1 = |Q_1| \). Since the dimension of \( \text{Im } \psi_0 = \langle \sum_{a \in Q_1} e(a, a) - \sum_{a \in Q_1} e(a, a) \mid e \in Q_0 \rangle \) is equal to \( |Q_0| - 1 \) we obtain \( \dim H^1(\Lambda, \Lambda) = \dim \ker \psi_1 - \dim \text{Im } \psi_0 = |Q_1| - |Q_0| + 1 \).

(vi) \( \Rightarrow \) (i) : From \( \dim \text{Im } \psi_0 = |Q_0| - 1 \) and \( \dim H^1(\Lambda, \Lambda) = |Q_1| - |Q_0| + 1 \) it follows that \( \dim \ker \psi_1 = |Q_1| \) and thus \( \ker \psi_1 \) is generated by the elements \( (a, a) \in Q_1/\langle Q_1 \rangle \). Since the bracket of the Lie algebra \( k(Q_1/\langle Q_1 \rangle) \) is such that \( [(a, a), (b, b)] = 0 \) for all \( a, b \in Q_1 \), we see that \( H^1(\Lambda, \Lambda) \) is an abelian Lie algebra.
The following Corollary is clear thanks to Proposition 3.1, because in characteristic 0 a connected algebraic group is abelian (resp. nilpotent) if and only if its Lie algebra is abelian (resp. nilpotent) (see 12 13.4 and 10.5). It generalizes Guil-Asensio and Saorín’s criterion on the commutativity of the algebraic group $V_\Lambda = \varepsilon_\Lambda(G_\Lambda)$ (see Corollary 2.23 in [12]).

**Corollary 4.24** Let $k$ be a field of characteristic 0 and $\Lambda = kQ/\langle Z \rangle$ a finite dimensional monomial algebra. The following conditions are equivalent:

(i) The algebraic group $\text{Out} (\Lambda)^c$ is abelian.

(ii) The algebraic group $\text{Out} (\Lambda)^c$ is nilpotent.

(iii) For every path $\gamma \in B$ parallel to an arrow $a \neq \gamma$ there exists a path $p$ of $Z$ such that $p^{(a,\gamma)} \neq 0$.

5 Application to group algebras where the group admits a normal cyclic Sylow $p$-subgroup

The starting point for the following application is the paragraph ‘Représentations modulaires des groupes finis’ in Gabriel’s article [3] (see also [11] p.75). Let $k$ be an algebraically closed field and $G$ a finite group. If the characteristic of $k$ does not divide the order of $G$, then Maschke’s Theorem states that the group algebra $kG$ is semisimple. Since $k$ is supposed to be algebraically closed, this is equivalent to saying that the group algebra $kG$ is separable and thus $\text{H}^1(kG,kG) = 0$. So henceforth let $k$ be a field of characteristic $p$ dividing the order of $G$. Let $n = p^aq$ be the order of $G$ with $q$ prime to $p$. We are interested in this section in the particular case where $G$ contains only one cyclic Sylow $p$-subgroup $S$, necessarily normal in $G$. Then $S$ is a normal Hall subgroup of $G$ and Schur’s splitting theorem (see [10] 9.3.6) states that there exists a supplement $K$ of $S$ in $G$, i.e. a subgroup such that $S \cap K = \{1\}$ and $SK = G$. It is unique up to a group isomorphism. If $\sigma$ is a generator of $S$, the action of $K$ on $S$ by conjugation is given by a formula of the type

$$x\sigma x^{-1} = \sigma^{\chi(x)}$$

with $x \in K$ and $\chi(x) \in (\mathbb{Z}/p^a\mathbb{Z})^\times$. The group $G$ is the semidirect product of $S$ and $K$. For every $kK$-module $N$ we denote by $\chi N$ the underlying $k$-vector space of $N$ equipped with a new action $*$ of $K$ such that we have

$$x * m := \chi(x)x \cdot m$$

for all $x \in K$ and $m \in N$. If the $kK$-module $N$ belongs to the isomorphism class $e$, we denote by $\chi_e$ the isomorphism class of the $kK$-module $\chi N$. That which enables us to study the Lie algebra $H^1(kG,kG)$ is the association of a quiver $Q$ to the group $G$ in the following way: the set $Q_0$ of vertices of $Q$ consists in the set of isomorphism classes of simple $kK$-modules and for every vertex $e$ we take an arrow $e \rightarrow \chi e$. The quiver $Q$ is a disjoint union of crowns i.e. of oriented cycles. Gabriel proved in [11] that the category of $kG$-modules is equivalent to the category of $kQ/(Q_{p^a})$-modules. In other words the $k$-algebras $kG$ and $kQ/(Q_{p^a})$ are Morita equivalent. Note that in general they are not isomorphic. A necessary and sufficient condition for the existence of an algebra isomorphism between $kG$ and $kQ/(Q_{p^a})$ is the commutativity of the group $K$. Since the Hochschild cohomology $H^*(\Lambda, \Lambda)$ is for every $k$-algebra $\Lambda$ Morita invariant as a Gerstenhaber algebra (see [11])

$q.e.d. 4.23$
Theorem 5.1  Let $k$ be an algebraically closed field and $G$ a finite group such that the characteristic $p > 0$ of $k$ divides the order of $G$. Let $n = p^a q$ be the order of $G$ with $q$ prime to $p$. Suppose that $G$ contains only one Sylow $p$-subgroup which in addition is cyclic. The following conditions are equivalent:

(i) The Lie algebra $H^1(kG, kG)$ is semisimple.

(ii) The group $G$ is the direct product of a group $K$ of order $q$ and of a cyclic group $C_p$ of order $p$. The characteristic $p$ of the field $k$ is different from 2.

(iii) $H^1(kG, kG)$ is a product of Witt algebras $W(1,1) := \text{Der}(k[X]/(X^p))$ and the characteristic $p$ of the field $k$ is different from 2.

Proof: Let $Q$ be the above-described quiver associated to the group $G$.

(i) $\Rightarrow$ (ii): It is clear that the Lie algebra $H^1(kG, kG)$ is semisimple if and only if the Lie algebra $H^1(kC/\langle C_{p^a}\rangle, kC/\langle C_{p^a}\rangle)$ is semisimple for every crown $C$ of the quiver $Q$. From Proposition 4.2 and Theorem 4.15 we deduce that $Q$ does not have a crown of length $\geq 2$, i.e. $Q$ is a disjoint union of loops. Therefore we have $\chi(x) = 1$ for all $x \in G$ which implies that $G$ is the direct product of the Sylow $p$-subgroup $S$ and its supplement $K$. Proposition 4.3 shows that for every loop $C$ of the quiver $Q$ the Lie algebra $H(kC/\langle C_{p^a}\rangle, kC/\langle C_{p^a}\rangle)$ is semisimple if and only if $p^a = p > 2$.

(ii) $\Rightarrow$ (iii): Let $G = K \times C_p$ and so $\chi = 1$. Therefore $Q$ is a disjoint union of loops. According to Proposition 4.3 we have for every loop $C$ of the quiver $Q$ that $H^1(kC/\langle C_{p^a}\rangle, kC/\langle C_{p^a}\rangle)$ is isomorphic to the Witt Lie algebra $W(1,1)$.

(iii) $\Rightarrow$ (i): This is clear, because the Witt algebra is one of the nonclassical simple Lie algebras if the characteristic $p > 0$ of $k$ is different from 2.

q.e.d. 5.1

Corollary 5.2  Let $k$ be an algebraically closed field and $G$ a finite group such that the characteristic $p > 0$ of $k$ divides the order of $G$. Let $n = p^aq$ be the order of $G$ with $q$ prime to $p$. Suppose that $G$ contains only one Sylow $p$-subgroup which in addition is cyclic. The following conditions are equivalent:

(i) The Lie algebra $H^1(kG, kG)$ is simple.

(ii) The group $G$ is cyclic of order $p$ and the characteristic $p$ of the field $k$ is different from 2.

(iii) $H^1(kG, kG)$ is isomorphic to the Witt Lie algebra $W(1,1) := \text{Der}(k[X]/(X^p))$ and the characteristic $p$ of the field $k$ is different from 2.

Proof: Let $Q$ be the above-described quiver associated to the group $G$.

(i) $\Rightarrow$ (ii): Let the Lie algebra $H^1(kG, kG)$ be simple and thus semisimple. The preceding Theorem shows that $G$ is the direct product of a group $K$ of order $q$ and of a cyclic group $C_p$ of order $p$, where $q$ and $p$ are prime. Therefore we have $\chi = 1$ which implies that the quiver $Q$ is a disjoint union of loops. From the fact...
that \( H^1(kG, kG) = \prod H^1(kC/\langle C_p \rangle, kC/\langle C_p \rangle) \) is a simple Lie algebra it follows that \( Q \) has only one loop. Since the field \( k \) is algebraically closed and since \( p \) does not divide the order \( q \) of the group \( K \), the number of isomorphism classes of simple \( kK \)-modules, which is the number of vertices of \( Q \), is equal to the number of conjugation classes of the group \( K \). Thanks to the fact that the loop \( Q \) has only one vertex we obtain that \( K \) has only one conjugation class and so \( K = \{1\} \).

(ii) \( \Rightarrow \) (iii) : Since the quiver \( Q \) associated to the cyclic group \( C_p \) is the loop, Proposition 4.3 shows that \( H^1(kG, kG) \) is isomorphic to the Witt Lie algebra.

(iii) \( \Rightarrow \) (i) : clear

\[ q.e.d. \ 5.2 \]

References

[1] M.J. Bardzell, The alternating syzygy behavior of monomial algebras, \textit{J. Algebra} \textbf{188}, No. 1 (1997), 69-89.

[2] C. Cibils, Hochschild cohomology algebra of radical square zero algebras, Algebras and modules II: Eighth International Conference on Representations of Algebras,II (Geiranger, 1996), 93-101, \textit{CMS Conf.Proc. 24}, Amer. Math. Soc. Providence, RI., 1998.

[3] P. Gabriel, Problèmes actuels de théorie des représentations, \textit{Enseign Math. (2)} \textbf{20} (1974), 323-332.

[4] P. Gabriel, A.V. Roiter, Representations of finite-dimensional algebras, with a chapter by B. Keller, \textit{Encyclopaedia of Math. Sci. 73}, Algebra VIII, Springer, Berlin, 1992.

[5] M. Gerstenhaber, The cohomology structure of an associative ring, \textit{Ann. of Math.} \textbf{78}, No.2 (1963), 267-288.

[6] M. Gerstenhaber, S.D. Schack, Algebraic cohomology and deformation theory, in “Deformation theory of algebras and structures and applications” (M. Hazewinkel and M. Gerstenhaber, Eds.), NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, Vol. 247, pp. 11-264, Kluwer Academic Publishers, Dodrecht/Boston, 1988.

[7] M. Gerstenhaber, S.D. Schack, Relative Hochschild cohomology, rigid algebras, and the Bockstein, \textit{J. Pure Appl. Algebra} \textbf{43}, No. 1 (1986), 53-74.

[8] F. Guil-Asensio, M. Saorín, The group of outer automorphisms and the Picard group of an algebra, \textit{Algebr. Represent. Theory} \textbf{2}, No. 4 (1999), 313-330.

[9] F. Guil-Asensio, M. Saorín, The automorphism group and the Picard group of a monomial algebra, \textit{Comm. Algebra} \textbf{27}, No. 2 (1999), 857-887.

[10] G. Hochschild, On the cohomology groups of an associative algebra, \textit{Ann. of Math.} \textbf{46}, No. 2 (1946), 58-67.

[11] B. Huisgen-Zimmermann, M. Saorín, Geometry of chain complexes and outer automorphisms under derived equivalence, \textit{Trans. Amer. Math. Soc.} \textbf{353}, No. 12 (2001), 4757-4777.

[12] J.E. Humphreys, “Linear algebraic groups”, \textit{Graduate Texts in Math.} Vol. 21, Springer, Berlin, 1975.
[13] B. Keller, Hochschild cohomology and derived Picard groups, Lecture at
the international workshop “Representation Theory around the Channel” in
Amiens, France, 31 August to 1 September 2001.

[14] A.C. Locateli, Hochschild cohomology of truncated quiver algebras, Comm.
Algebra 27, No. 2 (1999), 645-664.

[15] R.D. Pollack, Algebras and their automorphism groups, Comm. Algebra
17, No. 8 (1989), 1843-1866.

[16] W.R. Scott, “Group theory”, Prentice-Hall, Inc., Englewood Cliffs, New
Jersey, 1964.

[17] C.A. Weibel, “An introduction to homological algebra”, Cambridge Studies
in Advanced Mathematics Vol. 38, Cambridge University Press, Cambridge,
1994.