Intrinsic Construction of Lyapunov Functions on Riemannian Manifold

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Abstract

For systems evolving on a Riemannian manifold, we propose converse Lyapunov theorems for asymptotic and exponential stability. The novelty of the proposed approach is that it does not rely on local Euclidean coordinate, and is thus valid on a wider domain than the immediate vicinity of the considered equilibrium. We also show how the constructions can be useful for robustness analysis.

I Introduction

Since the foundation of modern control theory, Lyapunov’s second method has been among the most important tools for the investigations of stability of control systems. The success of this method owes partially to the fact that it is applicable for a wide class of dynamical systems, including continuous and discrete time systems, stochastic systems, systems evolving on manifolds, distributed parameter systems, delay systems, etc. For each scenario, the key procedure of the Lyapunov’s second method is the seeking of a Lyapunov function. Before searching for such a function, we may ask whether it exists and, if yes, how to construct it. These joint are known as the converse Lyapunov problem. A thorough review of the problem regarding continuous time systems evolving on Euclidean space can be found in [1], [2]. For more recent developments, we refer the reader to a survey paper [3] and the references therein. The importance of the converse problem lies not only in its theoretic interests but also its help to the analysis of some other control problems. Typical applications of converse theorem can be found in robust stability [4] and input-to-state stability analysis [5].

In this paper, we study the converse Lyapunov problem for systems evolving on Riemannian manifolds, for which not much attention has been drawn upon. It is true that most control problems are studied in Euclidean spaces; the reason seems to be that in most cases, stability is discussed in local sense, so the manifold can be seen as Euclidean in a local coordinate. On the one hand, this does simplify the problem, on the other hand, this may at the same time preclude the utilization of the underlying rich structure of the manifold, for example, to design coordinate-free control laws. In contrast to this situation, a whole field of research is devoted to geometric tool for system analysis and design, see for example, [6], [7], [8]. By “geometric” we mean coordinate-free or intrinsic objects. However, in terms of converse Lyapunov problems, theories in parallel with those in the classical Euclidean space still need to be established.

We list some previous works related to the problem that is going to be treated in this paper, from which we will see that several issues need to be settled. In 2010, F. Pait et al. [9] suggested to use the quadratic distance defined therein as a Lyapunov function. However, no explicit form of such a function was not derived. In [10], the authors constructed semiconcave control Lyapunov functions for general manifolds, by relying on some constructions in local coordinates instead of the intrinsic metric on the manifold. Besides, the constructed Lyapunov function is not smooth. Later, F. Taringoo et al. systematically studied the problem on Riemannian manifold [11]. For example, locally asymptotically stable and exponentially stable systems on Riemannian manifolds were proved to admit certain Lyapunov functions. The theorems are proved by “pulling back” the Lyapunov function constructed in Euclidean space to the manifold. The problem of this procedure is that it relies on the local coordinate of the manifold hence the result is local, and further analysis on whether the construction can be extended to the region of attraction are needed.

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This paper aims at solving the issues mentioned in the previous paragraph in this paper. Coordinate-free constructions will be given, and in the proof we do not resort to the existing theories in Euclidean space. The Lyapunov functions to be constructed will be seen to hold exactly the same meaning as that in Euclidean space. The two crucial tools in the proof of our result is the modified version of Lipschitz continuity and the first variation formula of arc length.

The paper is organized as follows. In Section II, we review some basic definitions from Riemannian geometry and stability notions on Riemannian manifolds. In Section III, the main theorems are proved, namely the construction of Lyapunov function for GAS and GES systems in a coordinate-free manner. Section IV includes some further discussions and extensions of the main theorem, particularly, an application to input-to-state stability is given.

**Notation 1.** Throughout this paper, we adopt the following notations. \( \mathcal{X} \): the riemannian manifold; \( \Gamma(\mathcal{X}) \): the set of smooth vector fields on \( \mathcal{X} \); \( T_x\mathcal{X} \): the tangent space at \( x \); \( \langle v_x, u_x \rangle \): the Riemannian product of \( v_x, u_x \in T_x\mathcal{X} \); \( \nabla \): Levi-Civita connection; \( \frac{d}{dt} \): the Covariant derivative; \( \ell(c) \): the length of the curve \( c \); \( d(x, y) \): the Riemannian distance between \( x \) and \( y \); \( L_f V(x) \): the Lie derivative of \( V(x) \) along the flow generated by \( f(x) \); \( P^t_p \): the parallel transport from \( p \) to \( q \); \( \phi: \) the push forward of a diffeomorphism \( \phi: \mathcal{M} \to \mathcal{N} \); \( \phi^* \): the pull back of a smooth map \( \phi: \mathcal{M} \to \mathcal{N} \); \( | \cdot |_\infty \): the infinity norm; \( \mathbb{R}_+ \): the set of non negative numbers; \( B^c_x \): the open ball with radius \( c \) centered at \( x \); \( \phi(t; t_0; x_0) \): the flow of a system with initial condition \( (t_0, x_0) \).

## II Preliminaries

### II.1 Riemannian manifolds

In this section, we summarize a few important tools that we will use repeatedly from Riemannian geometry and the stability notions on Riemannian manifold. A standard treatment of Riemannian geometry can be found in [12].

On a Riemannian manifold, connections can be defined, among which there exists an important one called the Levi-Civita connection.

**Theorem 1 (Levi-Civita).** Given a Riemannian manifold \( \mathcal{X} \), there exists a unique affine connection on \( \mathcal{X} \), such that

a. \( \nabla \) is symmetric, namely, \( \nabla_X Y - \nabla_Y X = [X, Y] \) for all \( X, Y \in \Gamma(\mathcal{X}) \).

b. \( \nabla \) is compatible with the Riemannian metric, namely,

\[
X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \quad X, Y, Z \in \Gamma(\mathcal{X})
\]

From (1) we get the following important equality: \( \frac{d}{dt} \langle V, W \rangle = \langle \nabla^t V, W \rangle + \langle V, \nabla^t W \rangle \) where \( \nabla^t \) are vector fields along a given curve parametrized by \( t \). Throughout this paper, we use \( \nabla \) to denote the underlying Levi-Civita connection for the Riemannian manifold \( \mathcal{X} \).

The Levi-Civita connection can be used to describe the geodesic curve, which will serve as the crucial object in our paper. Loosely speaking, given two points \( x, y \in \mathcal{X} \), if there exists a \( C^2 \) curve \( \gamma \) joining \( x \) to \( y \) such that the length of \( \gamma \) is minimized under small variations, then \( \gamma \) should verify \( \nabla_{\gamma'(s)} \gamma'(s) = 0 \) for all \( s \in I \). In this case \( \gamma \) is called a geodesic. Given \( (x, v) \in T_x\mathcal{X} \), there exists (locally) a unique geodesic \( \gamma: (-\delta, \delta) \to \gamma(0) = v \), where \( \delta \) may depend on \( v \).

Once the Riemannian metric is given, the Riemannian distance \( d(x, y) \) between two points \( x \) and \( y \) can be defined which makes \( \mathcal{X} \) into a metric space. According to Hopf-Rinow theorem, this metric space is complete for complete Riemannian manifold. Throughout this paper, we always assume the Riemannian manifold to be complete.

We also mention an important theorem which will be used repeatedly throughout this paper.

**Theorem 2 (First variation of arc length).** Let \( \gamma: [a, b] \to \mathcal{X} \) be a differentiable curve and \( F \) a variation of \( \gamma \). Denote \( \ell(c) \) the length of curve \( c \). Then

\[
\left. \frac{d}{dt} \right|_{t=0} \ell(F(t, \cdot)) = \frac{1}{\|DF/\partial s\|} \left[ \frac{\partial F}{\partial s} \right]_{s=b} - \int_a^b \left[ \frac{\partial F}{\partial t} \nabla_{\partial F/\partial s} \frac{\partial F}{\partial s} \right]_{t=0} \]
An equilibrium point $x_*$ for $\mathcal{X}$ is

1. (locally) uniformly stable (US) if there exists a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$, independent of $t_0$, such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq \alpha(d(x_0, x_*)), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}; \]

2. (locally) uniformly asymptotically stable (UAS) if there exists a class $\mathcal{KL}$ function $\beta$ and a positive constant $c$, independent of $t_0$, such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq \beta(d(x_0, x_*), t-t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}. \]

3. (locally) exponentially stable (LES) if there exists three positive constants $K$, $\lambda$ and $c$ such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq Ke^{-\lambda(t-t_0)}d(x_0, x_*), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}. \]

4. uniformly globally asymptotically stable (UGAS) if (5) is satisfied for any $x_0$; uniformly exponentially asymptotically (UGES) if (5) is satisfied for all $x_0$.

Remark 1. In [8], [11], stability definitions are given by the $\varepsilon$-$\delta$ language. However, it’s not hard to show that the two ways are equivalent, see for example [11]. Similar to the Euclidean case, the comparison function will simplify the stability analysis.

A Lyapunov candidate $V : \mathbb{R}_+ \times U \to \mathbb{R}_+$ is a locally positive definite function about the equilibrium $x_*$, where $U$ is an open neighborhood of $x_*$, namely, $V(t, x) \geq 0$ for all $t \in \mathbb{R}_+$ and $x \in U$ and $V(t, x) = 0$ if and only if $x = x_*$. On manifold, the partial derivative of a function is normally not a coordinate-free notion. And in control systems, we will deal with time varying vector fields, so we need the concept of timed Lie derivative.

Definition 2. [Timed Lie derivative] Consider the time invariant system

\[
\frac{dx(t)}{dt} = f(s(t), x(t))
\]

with initial condition

\[ x(t_0) = x_0, \quad s(t_0) = t_0. \]

The Lie derivative of $V$ with respect to (3) is defined as the Lie derivative of $V$ with respect to (6), and is denoted $L_f V$ more precisely,

\[ L_f V(t, x) = : L_f V(t, x) \]

where $f$ is the augmented vector field $(f(s, x), 1)$. II.2 Stability notions on Riemannian manifolds

Consider the following dynamical system evolving on $\mathcal{X}$:

\[
\dot{x} = f(t, x)
\]

where $f(t, x)$ is a time varying $C^1$ vector field on $\mathcal{X}$. An equilibrium point $x_0$ is such that $\phi(t; t_0, x_0) = x_0$ for all $t \geq t_0$, where $\phi$ is the flow of $f$. Various stability concepts can be given similarly as in Euclidean space by replacing the norm by the Riemannian distance $d$ on $\mathcal{X}$.

Definition 1. An equilibrium $x_*$ for $\mathcal{X}$ is

1. (locally) uniformly stable (US) if there exists a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$, independent of $t_0$, such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq \alpha(d(x_0, x_*)), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}; \]

2. (locally) uniformly asymptotically stable (UAS) if there exists a class $\mathcal{KL}$ function $\beta$ and a positive constant $c$, independent of $t_0$, such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq \beta(d(x_0, x_*), t-t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}. \]

3. (locally) exponentially stable (LES) if there exists three positive constants $K$, $\lambda$ and $c$ such that
   
   \[ d(\phi(t; t_0, x_0), x_*) \leq Ke^{-\lambda(t-t_0)}d(x_0, x_*), \quad \forall t \geq t_0 \geq 0, \quad \forall x_0 \in B^c_{x_*}. \]

4. uniformly globally asymptotically stable (UGAS) if (5) is satisfied for any $x_0$; uniformly exponentially asymptotically (UGES) if (5) is satisfied for all $x_0$.
**Remark 2.** $L_f V$ is well defined since it's the usual Lie derivative of a time-invariant function with respect to a time-invariant vector field. Since the flow of $\dot{f}$ is $(t, \phi_f(t; t_0, x_0))$. In coordinates, at point $(t_0, x_0)$, it reads

$$L_f V(t_0, x_0) = \lim_{t \to t_0} \frac{V(t, \phi_f(t; t_0, x_0)) - V(t_0, x_0)}{t - t_0}$$

$$= \frac{\partial V}{\partial t}(t_0, x_0) + \frac{\partial V}{\partial x}(t_0, x_0)f(t_0, x_0),$$

which coincides with the time derivative of $V$ along (3).

We are now in position to give the Lyapunov stability theorem on Riemannian manifolds.

**Theorem 3.** Let $x_*$ be an equilibrium point of the system (3) and $D$ be an open connected neighborhood of $x_*$. Let $V : \mathbb{R}^+ \times D \to \mathbb{R}^+$ be a Lyapunov candidate such that

$$W_1(d(x, x_*)) \leq V(t, x) \leq W_2(d(x, x_*)), \forall t \geq 0, x \in D, \tag{8}$$

then $x_*$ is uniformly stable if

$$\mathcal{L}_f V(t, x) \leq 0;$$

it is uniformly asymptotically stable if

$$\mathcal{L}_f V(t, x) \leq -W_3(d(x, x_*)), \tag{9}$$

where $W_i$ are class $C^k$ functions. If $W_i(r) = c_i r^p$, where $c_i > 0$, for $i = 1, 2, 3$ and $p > 0$, then (8) and (9) together imply exponentially stability.

This theorem can be proved by repeating the procedures used in Euclidean space by noticing that

$$\frac{d}{dt} V(t, \phi(t; t_0, x_0)) = \mathcal{L}_f V(t, x).$$

**III Converse theorem on Riemannian manifold**

Though the converse theorem can be done locally, in order to simplify the analysis and computation, and to streamline our idea, in the sequel, we will assume global stability. The proof can be easily extended to local case, by replacing “globally exponentially stable” with “exponentially stable with region of attraction $U$ where $U$ is an invariant set.”. We start with exponential stability.

Recall that, in the proof of converse theorems, there is a key assumption: the global Lipschitz condition. In $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is said be Lipschitz continuous if there exists a constant $L$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

where $|\cdot|$ is the Euclidean norm. On Riemannian manifold, if $f$ is a vector field, $f(x)$ and $f(y)$ will live in different tangent spaces, so it’s not possible to compare them directly. In [11], the authors considered the tangent map

$$Tf : T\mathcal{X} \to TT\mathcal{X}.$$ 

At every point $x \in \mathcal{X}$, $T_x f$ is a linear operator. The authors assume this operator to be uniformly bounded and claim that if

$$|T_x f(t, x)(X)|_e \leq c_2 |T_x f(t, x)(X)|_g, \tag{10}$$

when

$$|X|_e \leq c_2 |X|_g, \forall X \in T_x \mathcal{X} \tag{11}$$

where $|\cdot|_e$ and $|\cdot|_g$ stand for the Euclidean and Riemannian metric respectively. However, $T_x f(t, x)(X)$ lives in $T_{f(x)} T\mathcal{X}$ so its Riemannian norm needs to be defined. There exist canonical Riemannian metrics on the second order tangent bundle, such as the Sasaki metric, we remark that however, even if $|T_x f(t, x)(X)|_g$ is replaced by a Riemannian metric on $TT\mathcal{X}$, the implication from(11) to (10) is not clear.

Instead of defining a metric on $TT\mathcal{X}$ and studying the tangent map, we consider the Riemannian version of Lipschitz continuity. This definition can be found for example in [13] Chapter II.3. Intuitively, we transport two tangent vectors into a same tangent space so that we can make the comparison between them.
Definition 3 (Parallel transport). Let \( c: I \to X \) be a differentiable curve in \( X \) and \( V_0 \) a vector tangent to \( X \) at \( c(t_0) \). Then there exists a unique parallel vector field \( V \) along \( c \), i.e. \( \nabla_{\gamma'(t)}V(t) = 0 \) such that \( V(t_0) = V_0 \). We call \( V(t) \) the parallel transport of \( V(t_0) \) along \( c \).

For a complete Riemannian manifold, given \( x, y \in X \), there exists a minimizing geodesic curve \( \gamma: [0, 1] \to X \) joining \( x \) to \( y \). Given \( V \in T_xX \), let \( V(t) \) be the parallel transport of \( V \) along \( \gamma \), then we denote \( P_\gamma V =: V(1) \in T_yX \), i.e. we transport the vector \( V \) in \( T_xX \) to \( T_yX \).

Definition 4. A vector field \( X \) on \( X \) is said to be globally Lipschitz continuous on \( X \), if there exists a constant \( L > 0 \) such that for \( p, q \in X \) and all \( \gamma \) geodesic joining \( p \) to \( q \), there holds
\[
|P_\gamma X(p) - X(q)| \leq Ld(p,q)
\]
where \(| \cdot |\) is the norm induced by the Riemannian metric.

It can be easily shown that if \( |\nabla_{c'(0)}X| \leq L \) for all \( c(t) \) such that \( |c'(0)| = 1 \), then \( X \) is Lipschitz continuous with constant \( L \). Since the Levi-Civita connection \( \nabla \) is an affine connection, we have \( \nabla_vV = |v|\nabla_{|v|v}V \), consequently, \( |\nabla_{c'(0)}X| \leq L \) is equivalent to \( |\nabla_vV| \leq L|v| \) for all \( v \). We will see in the following that in Riemannian manifold, it’s the covariant derivative rather than the tangent map which takes into play.

Lemma 1. Given that the system is globally Lipschitz continuous with constant \( L \), then there holds the following estimation
\[
d(x_1, x_2)e^{-L(\tau-t)} \leq d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2)) \leq d(x_1, x_2)e^{L(\tau-t)}, \forall \tau \geq t, \forall x \in X. \tag{12}
\]

Proof. Suppose that \( x_1 \) and \( x_2 \) is joined by normalized geodesic \( \gamma: [0, \hat{s}] \to X \), with \( \gamma(0) = x_1 \) and \( \gamma(\hat{s}) = x_2 \), where \( \hat{s} \) is the length of \( \gamma \). Then the map \( F(t,s) = \phi(t,t_0, \gamma(s)) \) defines a variation of \( \gamma \). By the first variation formula of arc length, we have
\[
\frac{d}{d\tau} \bigg|_{\tau=t} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2)) \bigg| = \left( \frac{\partial \phi(t,t_0, \gamma(s))}{\partial s}, \frac{\partial \phi(t,t_0, \gamma(s))}{\partial t} \right) \bigg|_{s=\hat{s}, \tau=t}
\]
\[
= \langle \gamma'(\hat{s}), f(x_2) \rangle - \langle \gamma'(0), f(x_1) \rangle
\]
\[
= \langle P^x_{\gamma(\hat{s})} \gamma'(\hat{s}), P^x_{\gamma(0)} f(x_2) \rangle - \langle \gamma'(0), f(x_1) \rangle
\]
\[
= \langle \gamma'(0), P^x_{\gamma(0)} f(x_2) - f(x_1) \rangle,
\]
where the third equality follows from the inner product preserving property of the parallel transport operator. Since \( \gamma \) is normalized, by the Lipschitz continuity, we have
\[
\left\| \frac{d}{d\tau} \bigg|_{\tau=t} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2)) \right\| \leq Ld(x_1, x_2).
\]

Using the semi-group property of the flow, for any \( s > t \), we have
\[
\frac{d}{d\tau} \bigg|_{\tau=s} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2))
\]
\[
= \frac{d}{d\tau} \bigg|_{\tau=s} d(\phi(\tau; s, \phi(s; t, x_1)), \phi(\tau; s, \phi(s; t, x_2))),
\]
hence
\[
\left\| \frac{d}{d\tau} \bigg|_{\tau=s} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2)) \right\| \leq Ld(\phi(s; t, x_1), \phi(s; t, x_2)).
\]

Or equivalently,
\[
-Ld(\phi(\tau; t, x_1), \phi(\tau; t, x_2)) \leq \frac{d}{d\tau} d(\phi(\tau; t, x_1)), \phi(\tau; t, x_2)) \leq Ld(\phi(\tau; t, x_1), \phi(\tau; t, x_2)),
\]
from which we get (12). \( \square \)
**Theorem 4.** Assume that $f(\cdot, x)$ is globally Lipschitz (with constant $L$). Let $x_*$ be a UGES equilibrium point of the system (3) on the $\mathcal{X}$. Then there exists a Lyapunov candidate $V(t, x)$ verifying the following properties:

1. There exist two positive constants $c_1$ and $c_2$, such that
   \[
   c_1 d(x, x_*) \leq V(t, x) \leq c_2 d(x, x_*) \quad \forall x \in \mathcal{X}. \tag{14}
   \]

2. The Lie derivative of $V(t, x)$ in the sense of Definition 2 along the system satisfies
   \[
   \mathcal{L}_f V(t, x) \leq -c_3 V(t, x) \tag{15}
   \]
   where $c_3$ is a positive constant.

3. If $d(\cdot, x_*) : \mathcal{X} \to \mathbb{R}$ is class $C^1$. Then for every $t$, the differential of $V(t, x)$, $dV(t, x) \in T^* \mathcal{X}$ is uniformly bounded on $T^* \mathcal{X}$:
   \[
   |dV(t, x)| \leq c_4 \tag{16}
   \]
   where $c_4$ is a positive constant independent of $t$ and $x$.

**Proof.** **Item 1:** Consider the function
   \[
   V(t, x) = \int_{t}^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau. \tag{17}
   \]
   Setting $x_2 = x_*$ in Lemma 1, we get the following estimate by the fact that $x_*$ is an equilibrium point:
   \[
   d(x, x_*)e^{-L(\tau-t)} \leq d(\phi(\tau; t, x), x_*) \leq d(x, x_*)e^{L(\tau-t)}, \quad \forall \tau \geq t, \forall x \in X. \tag{18}
   \]
   Thus the defined function (17) admits the following bounds:
   \[
   V(t, x) = \int_{t}^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau \geq \int_{t}^{t+\delta} d(x, x_*)e^{-L(\tau-t)} d\tau = \frac{1 - e^{-L\delta}}{L} d(x, x_*),
   \]
   and
   \[
   V(t, x) = \int_{t}^{t+\delta} d(\phi(\tau; t, x), x_*) d\tau \leq \int_{t}^{t+\delta} Ke^{-\lambda(\tau-t)} d(x, x_*) d\tau = \frac{K(1 - e^{-\lambda\delta})}{L} d(x, x_*).
   \]
   Thus we can find two positive constants $c_1, c_2$ such that
   \[
   c_1 d(x, x_*) \leq V(t, x) \leq c_2 d(x, x_*) \quad \forall x \in \mathcal{X}. \tag{19}
   \]

**Item 2:** In order to estimate the evolution of $V(t, x)$ along the system, we again utilize the semi-group property:
   \[
   V(s, \phi(s; t, x)) = \int_{s}^{s+\delta} d(\phi(\tau; s, \phi(s; t, x)), x_*) d\tau = \int_{s}^{s+\delta} d(\phi(\tau; t, x), x_*) d\tau.
   \]
   Therefore
   \[
   \mathcal{L}_f V(t, x) = \frac{d}{ds} \bigg|_{s=t} V(s, \phi(s; t, x)) = d(\phi(t+\delta; t, x), x_*) - d(x, x_*)
   \]
   \[
   \leq -(1 - Ke^{-\lambda\delta}) d(x, x_*)
   \]
   \[
   = -K' d(x, x_*) \quad \forall t, \forall x \in \mathcal{X}, \tag{20}
   \]
   where $\delta$ is chosen such that $K' > 0$. By (19),
   \[
   \mathcal{L}_f V(t, x) \leq -K' d(x, x_*) \leq \frac{K'}{c_2} V(t, x).
   \]
Now $c_3$ can be set as $c_3 = K'/c_2$. **Item 3:** Denote $h_t(x) = V(t,x)$, then for any $v \in T_c X$,
\[
dh_t(v) = \left. \frac{d}{ds} \right|_{s=0} h_t(c(s))
\]
where $c : [-\varepsilon, \varepsilon] \to X$ is a smooth curve with $c'(0) = v$. So
\[
dh_t(v) = \int_t^{t+\delta} \frac{d}{ds} \left. \langle \phi(\tau,t,c(s)), x_* \rangle d\tau
\]
By first variation formula,
\[
\left. \frac{d}{ds} \right|_{s=0} \langle \phi(\tau,t,c(s)), x_* \rangle = \langle \phi(\tau,t,x)_*, v, \gamma'(1) \rangle,
\]
where $\phi(\tau,t,x)_* v$ is the push forward of the vector $c'(0)$ by the map $x \mapsto \phi(\tau,t,x)$ and $\gamma$ is the normalized geodesic joining $x_*$ to $\phi(\tau,t,x)$. Hence
\[
\left\| \left. \frac{d}{ds} \right|_{s=0} \langle \phi(\tau,t,c(s)), x_* \rangle \right\|^2 \leq \langle \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle.
\]
Now we estimate the term on the right hand side.
\[
\frac{1}{d\tau} \langle \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle = \left. \frac{D}{d\tau} \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \right\rangle = \langle \nabla f(\phi(t,t,x)), \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle
\]
where we have used the symmetry of the Levi-Civita connection, i.e.
\[
\nabla_X Y - \nabla_Y X = [X,Y].
\]
However,
\[
[f(\phi(t,t,x)), \phi(\tau,t,x)_* v] = L f(\phi(t,t,x)) \phi(\tau,t,x)_* v = \frac{d}{d\tau} \phi^* v = \frac{d}{d\tau} v = 0.
\]
Therefore
\[
\frac{1}{d\tau} \langle \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle = \langle \nabla_\phi(\tau,t,x)_* v, f(\phi(\tau,t,x)), \phi(\tau,t,x)_* v \rangle \leq L \langle \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle
\]
where we have used that fact that $|\nabla_v f| \leq L|v|$. So
\[
\langle \phi(\tau,t,x)_*, v, \phi(\tau,t,x)_* v \rangle \leq e^{2L(t-\tau)}|v|^2
\]
or
\[
|\phi(\tau,t,x)_* v| \leq e^{L(t-\tau)}|v|, \forall \tau \geq t.
\]
\[
dh_t(v) \leq \int_t^{t+\delta} e^{L(t-\tau)}|v| d\tau = \frac{e^{L\delta} - 1}{L}|v|.
\]
So we have obtained
\[
|d_x V(t,x)(v)| \leq c_4 |v|
\]
or
\[
|d_x V(t,x)| \leq c_4, \forall x \in X,
\]
where $c_4 = (e^{L\gamma} - 1)/L$. □
Remark 3. In Item 3, we have asked \( d(x, x_*) \) to be differentiable. This is however, not guaranteed in general. For example, in Euclidean space, \( d(x, x_*) = |x - x_*| \), which is not differentiable at the point \( x_* \). A possible way to resolve this problem is to consider the following Lyapunov candidate:

\[
V(t, x) = \int_t^{t+\gamma} d(\phi(\tau; t, x), x_*)^p d\tau
\]

where \( p \geq 1 \). In the Euclidean case, we set \( p = 2 \), and \( d(x, x_*)^2 = |x - x_*|^2 \) is differentiable. In effect, it can be easily verified (23) can still serve as a Lyapunov function. The proof can be carried out in exactly the same way as Theorem 4. However, when \( p \neq 1 \), the claims of Item 1 and 3 should change accordingly. For example, the estimate of \( V(t, x) \) becomes

\[
c_1 d(x, x_*)^p \leq V(t, x) \leq c_2 d(x, x_*)^p
\]

and \( dV \) satisfies

\[
|dV(t, x)| \leq c_4 d(x, x_*)^{p-1}.
\]

Remark 4. In contrast to the proof in [11], all the proof here is coordinate-free. So if the system has an invariant set \( U \) as region of attraction, then a Lyapunov function can be naturally defined everywhere on \( U \). Moreover, the Lyapunov function can be constructed such that \( V(t, x) \to \infty \) when \( x \) approaches the boundary of \( U \).

Remark 5. In Euclidean space, the Lyapunov candidate becomes

\[
V(t, x) = \int_t^{t+\gamma} |\phi(\tau; t, x)|^2 d\tau
\]

by setting \( p = 2 \) in (23), reducing to the standard construction, see [5].

IV Discussions and applications

Theorem 4 requires the system to be globally stable. However, such requirement is not essential. In fact, all the procedures of the proof can be done locally in the same manner. Hence we can obtain local version of converse theorems.

The extension to asymptotically stability is also not difficult. Following [5], we just need to modify the Lyapunov candidate to

\[
V(t, x) = \int_t^\infty G(d(\phi(\tau; t, x), x_*)) d\tau
\]

where \( G \) is constructed from the following Massera’s lemma.

Lemma 2 (Massera). Let \( g : \mathbb{R}_+ \to \mathbb{R} \) be a positive, continuous, strictly decreasing function with \( g(t) \to 0 \) as \( t \to \infty \). Let \( h : \mathbb{R}_+ \to \mathbb{R} \) be a positive, continuous, non decreasing function. Then, there exists a function \( G(t) \) such that

1. \( G \) and its derivative \( G' \) are class \( \mathcal{K} \) functions defined for all \( t \geq 0 \);
2. For any continuous function \( u(t) \) that satisfies \( 0 \leq u(t) \leq g(t) \) for all \( t \geq 0 \), there exist positive constants \( k_1 \) and \( k_2 \), independent of \( u \), such that

\[
\int_0^\infty G(u(t)) dt \leq k_1; \quad \int_0^\infty G'(u(t)) h(t) dt \leq k_2.
\]

The rest of the proof can be done similarly as that of Theorem 4. Hence we have the following theorem.

Theorem 5. Assume that \( f(\cdot, x) \) is globally Lipschitz (with constant \( L \)) in the sense of 4. Let \( x_* \) be a UGAS equilibrium point of the system (3) on the \( \mathcal{X} \), i.e.

\[
d(\phi(t; t_0, x_0), x_*) \leq \beta(d(x_0, x_*), t - t_0), \quad \forall t \geq t_0, x_0 \in \mathcal{X}
\]

for a class \( \mathcal{KL} \) function \( \beta \). Then there exists a Lyapunov candidate \( V \) verifying the following two properties:
1. There exist a $C^1$ function $V$, such that
\[ \alpha_1(d(x,x_*)) \leq V(t,x) \leq \alpha_2(d(x,x_*)), \forall x \in X. \]

2. The Lie derivative of $V(t,x)$ along the system satisfies
\[ L_f V(t,x) \leq -\alpha_3(V(t,x)) \]

3. If $d(\cdot,x_*) : X \to \mathbb{R}$ is class $C^1$. Then for every $t$, the differential of $V(t,x)$, $dV(t,x) \in T^*X$ is uniformly bounded on $T^*X$:
\[ |dV(t,x)| \leq \alpha_4(V(t,x)) \tag{24} \]
where $\alpha_i$, $i = 1, 2, 3, 4$ are class $K$ functions.

As application, we show that Theorem 4 can be applied to prove the input-to-state stability (ISS) of a class of systems. The classical form of this theorem can be found in [5].

**Corollary 1.** Consider the control system
\[ \dot{x} = f(t,x,u) \tag{25} \]
on Riemannian manifold $X$, where $f$ is $C^1$ and globally Lipschitz in $x$. Additionally, we assume $f$ is globally Lipschitz in $u$ with constant $L$, i.e.
\[ |f(t,x,u) - f(t,x,0)| \leq L|u|. \]

If the unforced system $\dot{x} = f(t,x,0)$ is UGES with respect to equilibrium point $x = 0$, then the system (25) is ISS.

**Proof.** By Theorem (4), a Lyapunov function $V(t,x)$ verifying the three conditions can be constructed for the unforced system $\dot{x} = f(t,x,0)$ when $\delta$ is large enough. Rewrite
\[ f(t,x,u) = f_1 + f_2 \]
where
\[ f_1 = f(t,x,0) \]
\[ f_2 = f(t,x,u) - f(t,x,0). \]

By assumption, $L_f V \leq -c_3 V$. The Lie derivative of $V(t,x)$ with respect to (25) reads
\[
L_f V = L_f V = \frac{\partial V}{\partial t} + L_f V = \frac{\partial V}{\partial t} + dV(f_1 + f_2)
= L_f V + dV[f(t,x,u) - f(t,x,0)]
\leq -c_3 V + c_4|f(t,x,u) - f(t,x,0)|
\leq -c_3 V + c_4 L|u|_{\infty}.
\]

Now invoking standard arguments from ISS theory, we conclude that the system (25) is ISS.

**V Concluding remarks**

We have proved the converse theorems on Riemannian manifolds, in a coordinate-free way. The constructed Lyapunov functions and the line of proofs share a lot in common with that in Euclidean space. This may suggests that there is no essential difference of Lyapunov stabilities between Riemannian manifolds and Euclidean space in regardless of the global topology. Further studies may include the application of the results and the extension to Finsler manifolds.
VI ACKNOWLEDGEMENT

We thank Dr. Antoine Chaillet for his instructions and fruitful discussions during the preparation of this manuscript.

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