Low-dimensional behavior of generalized Kuramoto model

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1 Introduction

Synchronization is a phenomenon representing the collective behavior in populations of interacting units. An essential contribution in modeling collective dynamics was introduced by Kuramoto [1,2]. The original Kuramoto model assumes a solid interaction between dynamical units. Even though it describes the collective phenomena, it misses the interplay between structure and dynamics. However, the generalized model, inspired by natural systems, takes into account the correlation between structure and dynamics by considering connections as a function of the intrinsic dynamics of units. This kind of coupling is characteristic of some natural and synthetic systems, such as power grids and social interactions. Of particular interest in this work is the collective phenomena in a system of two groups with opposing dynamics. For example, some neurons are excitatory in neuronal populations, and some are inhibitory. In the social systems, some individuals are for and some are against a party or an ideology. In addition to the continuous phase transition to the synchronized state, our numerical simulations show that the generalized Kuramoto model with bimodal frequency distribution leads to a two-step transition and a first-order, or discontinuous synchronization transition, i.e., an abrupt and irreversible phase transition to the synchronized state, called explosive synchronization (ES). Despite its high-dimensional nature, the system shows a phase transition in low
dimensions. Different approaches have been used for studying low-dimensional dynamics in the classical Kuramoto model [3–5]. We apply the Ott–Antonsen ansatz [6] to drive a low-dimensional description of the dynamics in the generalized Kuramoto model. In a fully connected network, using a mean-field approach, we obtain a set of characteristic equations governing the system’s dynamics and bifurcation boundaries, where the character of the phase transition changes. This paves the road to studying how the system’s parameters affect the dynamics and the transition to the synchronized state. We investigate the effect of frequency distribution on the evolution of the order parameter and phase transition as a function of the coupling strength. We then compare the results with the critical synchronization transition obtained from numerical simulations.

2 The frequency weighted Kuramoto model

The generalized Kuramoto model studies an ensemble of \( N \) coupled limit-cycle oscillators with interactions weighted by the absolute value of their intrinsic frequencies. The phase of the \( i \)’th oscillator is given by \( \theta_i \), and its evolution is governed by:

\[
\frac{d\theta_i}{dt} = \omega_i + \frac{\lambda|\omega|}{k_i} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i),
\]

\( i = 1, 2, \ldots, N. \) \hspace{1cm} (1)

Here, \( \omega_i \) is the natural frequency of the \( i \)th oscillator, which is drawn from a frequency distribution. The connectivity pattern is given by \( A_{ij} \), representing the topology of the interacting network. \( A_{ij} = 1 \) for connected nodes, and \( A_{ij} = 0 \) otherwise. Therefore, \( k_i = \sum_j A_{ij} \) is the degree of node \( i \). \( \lambda \) stands for the strength of coupling. The significant characteristic of this model is the frequency-weighted coupling, which causes a positive correlation between the coupling strength of oscillators and the absolute value of their natural frequencies, providing a heterogeneous interaction between units. The complex order parameter, as a measure of synchrony, can be defined as:

\[
Z = \int_{0}^{2\pi} e^{i\theta} \left( \frac{1}{N} \sum_{j} \delta(\theta - \theta_j) \right) d\theta.
\]

2.1 Continuum limit

We consider a fully connected network and use a mean-field approach. In the limit of the infinite number of oscillators, \( N \to \infty \), for each natural frequency \( \omega \), there is a continuum of oscillators distributed around the unit circle. We define this distribution with \( \rho(\theta, \omega, t) \), a \( 2\pi \) periodic function of \( \theta \), so that \( \rho(\theta, \omega, t) d\theta \) is the fraction of oscillators with natural frequency \( \omega \), whose phase at time \( t \) is between \( \theta \) and \( \theta + d\theta \). Since the frequencies are taken from the distribution \( g(\omega) \), it follows \( \int_{0}^{2\pi} \rho(\theta, \omega, t) d\theta = g(\omega) \).

Given that \( \int_{-\infty}^{\infty} g(\omega) d\omega = 1 \), we can write the normalization condition as \( \int_{-\infty}^{\infty} \int_{0}^{2\pi} \rho(\theta, \omega, t) d\theta d\omega = 1 \). Having conservation of oscillators of frequency \( \omega \), the evolution of \( \rho \) is governed by the continuity equation:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial \theta} = 0.
\]

The velocity in this continuous representation can be stated as:

\[
v(\theta, \omega, t) = \omega + \lambda|\omega| \int_{-\infty}^{\infty} \int_{0}^{2\pi} \rho(\theta', \omega, t) \times \sin(\theta' - \theta) d\theta' d\omega,
\]

or alternatively as:

\[
v(\theta, \omega, t) = \omega + \lambda|\omega| \text{Im}(Ze^{-i\theta}).
\]

With \( Z \), complex order parameter, defined with the following nonlinear integro-partial-differential equation:

\[
Z(t) = \int_{-\infty}^{+\infty} \int_{0}^{2\pi} e^{i\theta'} \rho(\theta', \omega, t) d\theta' d\omega,
\]

which is the continuous version of Eq. (2). In the incoherent state, which corresponds to the case of small coupling strength, each oscillator has its own intrinsic dynamics. So the population is distributed around unit circle: \( \rho_{0}(\theta, \omega, t) = \frac{g(\omega)}{2\pi}. \) When the system starts to synchronize, this solution becomes unstable. In other words, at this point, the incoherent solution loses its stability. One can find the critical coupling in which the system switches to the synchronized state. To do so, we add a perturbation, \( \eta \), to the incoherent density of oscillators:
\[ \rho(\theta, \omega, t) = g(\omega) \left[ \frac{1}{2\pi} + \eta(\theta, \omega, t) \right]. \]  

where \( \eta \) is a real, 2\( \pi \)-periodic function. The normalization condition requires that \( \int_0^{2\pi} \eta(\theta, \omega, t) d\theta = 0 \). We expand \( \eta \) in its Fourier modes \( \eta(\theta, \omega, t) = \sum_n \eta_n(\omega, t)e^{in\theta} \) with \( \eta_n(\omega, t) = \alpha(\omega, t)^n \), as considered by Ott–Antonsen \([6]\). \( |\alpha(\omega, t)| \leq 1 \) to avoid divergence of series. With a family of distribution functions, this ansatz naturally provides information about the synchronized and non-synchronized dynamics. By plugging this ansatz into Eq. (7), the density can be written as:

\[ \rho(\theta, \omega, t) = g(\omega) \left[ \frac{1}{2\pi} + \sum_{n=1}^{\infty} (\alpha(\omega, t)e^{i\theta})^n \right]. \]

Therefore, by inserting \( \rho \) from Eq. (7), and velocity from Eq. (4), and arranging powers of \( e^{in\theta} \) for each \( n \), we can study the evolution of each mode. For the first modes which are the coefficients of \( e^{i\theta}, e^{-i\theta} \) we get:

\[ \frac{d\alpha}{dt} + \frac{\lambda}{2} |\omega| (-Z* + \alpha^2 Z) + i\alpha \omega = 0, \]

and

\[ \frac{d\alpha^*}{dt} + \frac{\lambda}{2} |\omega| (-Z + \alpha^2 Z*) - i\alpha^* \omega = 0. \]

The approach so far can be used for any frequency distribution. As an example, we consider bimodal Lorentzian distribution with four poles at \( \omega = \pm \omega_0 \pm i\Delta \). Where \( \omega_0 \) is the center of the distribution, and \( \Delta \) is the standard deviation which defines the heterogeneity of the distribution. The distribution in partial fraction form can be written as:

\[ g(\omega) = \frac{1}{4\pi i} \left[ \frac{1}{(\omega - \omega_0) - i\Delta} - \frac{1}{(\omega - \omega_0) + i\Delta} \right. \\
\left. + \frac{1}{(\omega + \omega_0) - i\Delta} - \frac{1}{(\omega + \omega_0) + i\Delta} \right]. \]

To make sure that \( g(\omega) \) is concave in the center and therefore can be considered bimodal, it has to fulfill the condition \( \omega_0/\Delta > 1/\sqrt{3} \). Otherwise, the tails of the two modes merge and they form a unimodal distribution. We consider \( \alpha(\omega, t) \) to be continuous in the complex \( \omega \) plane. Evaluating \( Z^* = \int_{-\infty}^{\infty} \alpha(\omega, t) g(\omega) d\omega \), by deforming the integration from the real \( \omega \) axis to \( Im(\omega) \to -\infty \), the order parameter becomes:

\[ Z = Z_1 + Z_2, \]

where \( Z_1(t) = \alpha^*(-\omega_0 - i\Delta, t) \) and \( Z_2(t) = \alpha^*(+\omega_0 - i\Delta, t) \). We use Eq. (10) to study the evolution of the first mode of the perturbation around the poles of the distribution. Note that since \( g(\omega) \) is an even function, the continuity equation i.e. Eq. (3) and Eqs. (9, 10) are invariant under the reflection \( (\theta(\omega) \to -\theta(-\omega)) \). Because of this O(2) symmetry, there are two independent solutions. By plugging the values of \( Z_1, Z_2 \) and the corresponding value for \( \omega \) into Eq. (10), the system reduces to a set of two coupled complex ODEs:

\[ \frac{dZ_1}{dt} = \frac{\lambda|\omega_0 - i\Delta|}{4}(Z_1 + Z_2 - Z_1^2(Z_1^* + Z_2^*)) + iZ_1(-\omega_0 - i\Delta), \]

\[ \frac{dZ_2}{dt} = \frac{\lambda|\omega_0 + i\Delta|}{4}(Z_1 + Z_2 - Z_2^2(Z_1^* + Z_2^*)) + iZ_2(+\omega_0 - i\Delta). \]

This set of equations provides information about how the dynamics evolve around these two poles. We will study the linear stability and flow field in the polar coordinate in the following.

### 2.1.1 Linear stability

We analyze the linear stability of the incoherent state by perturbing the system around the fixed points. By adding perturbations \( (Z_1 \to Z_1 + \delta Z_1 \) and \( Z_2 \to Z_2 + \delta Z_2 \) to Eq. (12), and keeping the linear terms, the eigenvalues of the two-dimensional system become:

\[ \Lambda_{1,2} = \frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{4} + \Delta \pm \sqrt{\frac{\lambda^2(\omega_0^2 + \Delta^2)}{16} - \omega_0^2}. \]

For the second square root to be real, it has to fulfill the condition \( \frac{\lambda^2(\omega_0^2 + \Delta^2)}{16} \geq \omega_0^2 \), which sets out the bifurcation boundary of the system at

\[ \Delta = \omega_0 \sqrt{\frac{16 - \lambda^2}{\lambda^2}}. \]
For \( \Delta < \omega_0 \sqrt{\frac{16 - \lambda^2}{\Delta^2}} \), i.e., below the solid line, the last term in \( \Lambda_{1,2} \) is imaginary; Therefore, the real part is always positive. As a result, zero is an unstable node. While for \( \Delta \geq \omega_0 \sqrt{\frac{16 - \lambda^2}{\Delta^2}} \) the last term in Eq. (13) is real and both eigenvalues are purely real. Note that even though one eigenvalue is always positive, the other one can be negative, which leads to saddle node behavior in Fig. 1 above the solid line. In the next section, by exploring the dynamics of the flow field in these two regions, we study the characteristic of the fixed points.

2.1.2 Flow field in polar coordinate

We represent complex order parameters in polar coordinate and a phase difference between two dynamical clusters as:

\[
Z_1 = R_1 e^{i\phi_1}, \quad Z_2 = R_2 e^{i\phi_2}, \quad \psi = \phi_2 - \phi_1 \tag{15}
\]

Plugging them in Eqs. (12) and comparing real parts on both sides of the equation, we get a differential equation for \( R \), which is the radius of the deviation from the random state. By comparing imaginary parts, we get a differential equation for each cluster’s phase evolution, namely \( \dot{\phi} \). Using the expressions obtained for \( \dot{\phi}_2 \) and \( \phi_1 \) we can find an expression for \( \dot{\psi} \). So, we have a set of three differential equations for the amplitude and phase difference of the clusters.

\[
\dot{R} = \frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{4} \left[ (1 - R_1^2)(R_1 + R_2 \cos \psi) \right] + R_1 \Delta,
\]

\[
\dot{\psi} = -\frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{4} \left[ \frac{R_1^2 + R_2^2 + 2R_1 R_2 \Delta}{R_1 R_2} \right] \sin \psi + 2\omega_0. \tag{16}
\]

We look for solutions that satisfy the symmetry condition \( R_1(t) = R_2(t) = R(t) \). Considering this condition, the dimension of the system reduces to two.

\[
\dot{R} = \frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{4} \left[ (1 - R^2)(R + R \cos \psi) \right] + R \Delta,
\]

\[
\dot{\psi} = -\frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{4} \left[ \frac{R^2 + R^2 + 2R^2 \Delta}{RR} \right] \sin \psi + 2\omega_0. \tag{17}
\]

By defining: \( q = R^2 \), then \( \dot{q} = 2\dot{R} R \), and multiplying both sides of the equation for \( \dot{R} \) with \( 2R \), the first equation becomes:

\[
2R\dot{R} = R^2 \frac{\lambda \sqrt{\omega_0^2 + \Delta^2}}{2} \left[ (1 - R^2)(1 + \cos \psi) \right] + 2R^2 \Delta. \tag{18}
\]
We end up with a two-dimensional phase plane:

\[
\dot{q} = q \sqrt{\frac{\omega_0^2 + \Delta^2}{2}} - \lambda \left[ (1 - q) + (1 - q) \cos \psi \right] + 2q \Delta,
\]

\[
\dot{\psi} = -\lambda \sqrt{\frac{\omega_0^2 + \Delta^2}{2}} [1 + q] \sin \psi + 2\omega_0. \tag{19}
\]

We use equations in (19) to visualize dynamical flow in the \(q - \psi\) plane. In Fig. 1, the colored region shows the parameter values \((\Delta, \omega_0)\) that have the same type of dynamics, same for the white region. Zero is a saddle-node in shadowed regions, while it is an unstable fixed point in the white regions. Different colors represent different values of \(\lambda\). Green stands for \(\lambda = 2.1\), blue for the case of \(\lambda = 2.4\), and red for \(\lambda = 3\). There is a good match between the solid line, obtained from Eq. (14), and the boundary of the shaded region obtained from flow field characteristic using Eq. (19). The little mismatch is due to the higher-order terms we have ignored in linearization.

Using the set of equations in (19), the \(q = 0\) case which corresponds to the incoherent state, gives rise to \(\dot{q} = 0, \dot{\psi} = -\lambda \sqrt{\omega_0^2 + \Delta^2} \sin \psi + 2\omega_0\). If in addition to \(q = 0, \dot{q} = 0, \dot{\psi} = 0\), then we have \(\psi = \sin^{-1} \left[ \frac{4\omega_0}{\lambda \sqrt{\omega_0^2 + \Delta^2}} \right]\). In addition to the trivial incoherent state, that is \(q = 0\), the other fixed points of the system occurs when \(q \neq 0, \dot{q} = 0, \dot{\psi} = 0\). Putting these conditions in Eq. (19), we can find two trigonometric equations:

\[
\cos \psi = \left[ \frac{-4\Delta}{\lambda \sqrt{\omega_0^2 + \Delta^2}} - (1 - q) \right] \frac{1}{1 - q},
\]

\[
\sin \psi = \frac{4\omega_0}{\lambda [1 + q] \sqrt{\omega_0^2 + \Delta^2}}. \tag{20}
\]

Using the trigonometric relation \(\sin^2 x + \cos^2 x = 1\), one obtains a third order polynomial equation for fixed points of \(q\). The characteristic equation of this polynomial:

\[
\delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2,
\]

with \(a, b, c, d\) being the coefficients of polynomial, provides information about its roots. From that we can find the bifurcation in the system.

\[
\delta = 4 \left( -\frac{\Delta \sqrt{\omega_0^2 + \Delta^2}}{2} \right) (\omega_0^2 + \Delta^2)
\]

\[
\left[ - (\omega_0^2 + \Delta^2)^2 + 12(- \frac{\Delta \sqrt{\omega_0^2 + \Delta^2}}{2})^2 \right], \tag{21}
\]

which changes sign when:

\[
\frac{\omega_0}{\Delta} = \sqrt{3\lambda^2 - 1}. \tag{22}
\]

Meaning that Eq. (22) is the bifurcation line of the system, and by crossing this line, the dynamics of the system change. We plot it in Fig. 3, comparing it with the results obtained from numeric simulations. In the next section, considering flow field and numerical simulation, we will explain how the dynamics change by crossing this line.

### 2.2 Numerical approach

By simulating the generalized Kuramoto model, Eq. (1), for a fully connected network and a bimodal Lorentzian distribution of frequencies, Eq. (11), we investigate the effect of the correlation between coupling and intrinsic dynamics of each node in the transition to the synchronized state. By fixing the mean of the distributions at \(+2, -2\), we investigate the impact of the width of the distribution, i.e., \(\Delta\), which reflects the heterogeneity of the dynamics on the transition to the synchronized state.

We compute the stationary value of order parameter, namely \(Z\), as defined in Eq. (2), by progressively changing the coupling strength in an adiabatic way and recording the average value at a time window in the steady-state. The forward path is obtained by increasing the coupling strength, using the outcome of the previous step as the initial condition for the next. We perform the same progress for backward continuation by decreasing the coupling strength. We use a fully connected network of \(N = 1000\) oscillators in all cases. The initial phases are taken randomly between 0 and \(2\pi\). In Fig. 2, we report the results for different values of \(\Delta\). For narrow distributions, i.e., \(\Delta < 1\) in Fig. 2a,
Synchronization on a complete graph with the natural frequencies taken from a bimodal Lorentz distribution with peaks at $\omega_0 = 2$ and $\omega_0 = -2$. Each panel corresponds to a different distribution width parameter $\Delta$, with (a) $\Delta = 0.1$, (b) $\Delta = 1$, (c) $\Delta = 1.6$, (d) $\Delta = 2$.

The system has a two-step transition to the synchronized state. The transition from asynchronous to the intermediate state, which happens for $\lambda \in [0, 0.2]$, follows a semi-continuous transition referred to as Bellerophon phase [7]; In Ref [7], by fixing the width of the distribution, they show that the closer the center frequency to zero, the larger the size of the hysteresis region. In the intermediate step, the oscillators are phase-locked into two clusters. Inside each cluster, the oscillators are synchronized with each other, one centered around $\omega_0$ and the other around $-\omega_0$, but the two clusters are not in sync with each other, as can be seen from the blocks in the correlation matrix in Figs. 6 and 7. Instantaneous frequencies, on the other hand, are not locked but rather distributed shown in Fig. 5a. This region remains stable even with an increase in the coupling strength. At the final transition, i.e., $\lambda \approx 2$, these two clusters join, and the whole population becomes synchronized as shown in Fig. 5e.

By simulating for different values of $\Delta$, we have observed that even though the transition to the full synchrony always happens at $\lambda = 2$, the critical value for the first transition, that is to the intermediate state, depends on the width of the distribution. The larger the width of distribution, the larger the critical value for transition; hence smaller intermediate state. By getting closer to $\Delta = 1$, the size of this intermediate region shrinks, and around $\Delta = 1$, the system shows a continuous transition to the synchronized state, as shown in Fig. 2b, similar to the classical Kuramoto model. The reason is that for small values of $\Delta$, two clusters are far apart and focused around their natural frequencies. The narrower the distribution, the larger coupling is required to combine both clusters.

By further increasing the width of the distribution, we observe a first-order transition with hysteresis to the synchronized state for $\Delta > 1$ as shown in Fig. 2c,d. The existence of hysteresis is indicative of memory in the system. A first-order transition to the synchronized state with a hysteresis loop is a sign of explosive synchronization. As the width of the distribution, i.e., $\Delta$, increases, because of the heterogeneity in the population, a larger coupling constant is required to harmonize the population. Hence the transition happens in larger values of $\lambda$, causing the system to have a larger bistable region shown in Fig. 2d. The exact value of the critical coupling obtained in Ref. [8] for transition to the synchronized state is compatible with our results.

There are also other scenarios leading to explosive synchronization [10–25]. For uniform frequency distribution also, explosive transition to synchrony happens [9]. Recently, experimental evidence of explosive synchronization in electrical, chemical, and neuronal systems has been reported [26,27].
Fig. 3  Stability diagram. The blue line is the critical value of the forward transition to the synchronized state. The green line shows the critical point of the backward path. Both are obtained from numerical simulation. The region between the blue and green lines is the bistable region. The magenta line is the bifurcation line which is obtained from Eq. (22). For all the cases $\omega_0 = 2$. For (a), (c): $\Delta = 0.5$, and $\lambda = 0.5, 0.97$, respectively. For (b), (d), (e): $\Delta = 0.8$, $\lambda = 0.64, 1.05, 1.52$, respectively.

2.2.1 Bifurcation in parameter space

In Fig. 3, we summarize the critical transition points for different values of parameters of the frequency distribution ($\omega_0, \Delta$). We plot the critical value of the transition, $\lambda_c$, in the parameter space. To do so, we simulate the system for values of $\omega_0$ and $\Delta$, the same procedure as Fig. 2. For simplicity, we fix $\omega_0 = 2$ and obtain the critical value of the transition $\lambda_c$ for different values of $\Delta$. Connecting these values gives the solid blue line for the critical value of the transition in the forward path and the green lines for the critical point of transition in the backward path. The light green shows the initial step of the transition, which leads to the intermediate region. The dark green shows the final step in the transition to the synchronized state. The region between them is the region of the standing wave. The region between the blue and green lines corresponds to the hysteresis behavior in which the system shows a bistable dynamic for $\Delta > 1$. These bifurcation lines separate different dynamical regimes, namely synchronized, bistable, standing wave, and incoherent states.

We use Eq. (19) again to plot the flow field in the $q - \psi$ plane for a given value of parameters. The parameters correspond to the value at the center of the circle for each flow field plot. We observe different types of flow field characteristics in different regions of the parameter space. In all cases, zero is an unstable node, which means that the unsynchronized state is not stable. In Fig. 3a, b, c, there is an unstable nonzero node in the second quarter, which indicates the instability of the synchronized state. In addition, there exists a stable limit cycle centered at zero. It implies the fact that there are two giant dynamical clusters around the centers of the distributions and the phase difference between them continues to increase. We have observed the same type of dynamics until close to the magenta line, which is obtained from Eq. (22) and defines the boundaries for the separation of different types of transitions. Close to this line, the zero-centered limit cycle disappears, and a stable fixed point emerges, Fig. 3d. A stable fixed point reflects a stable synchronized state. This region corresponds to the bistable dynamics. In the lower part, shown in Fig. 3e, there exists a stable fixed point for which $\dot{q} = 0, \dot{\psi} = 0$ indicating a stable synchronized state.

Due to frequency weighted coupling, this parameter space separation in dynamics is different from when the same scenario is applied to the Kuramoto model with constant coupling [28]. Our result is also compatible with analytic results obtained for the exact critical coupling of the transition [8], in case there is hysteresis. We did simulations for bimodal Gaussian distribution as well and observed similar behavior. In periodic systems, the delay affects the transition to synchrony in several ways [29]; delay can also induce hysteresis behavior [30]. The effect of delay in the generalized model remains to be explored.
3 Conclusion

In conclusion, a low-dimensional description of the generalized Kuramoto model has been studied analytically and checked numerically. We have demonstrated that the Ott–Antonsen ansatz is valid for the generalized Kuramoto model. It gives a reasonable estimate of the separation of the dynamics in the parameter space. Except for the bistable region, all the attractors of the infinite-dimensional system lie in the low-dimensional manifold obtained by this ansatz. Our numeric simulations show that for bimodal frequency distribution, the transition to the synchronized state can be a two-step transition, a continuous transition, or an explosive transition with hysteresis, depending on the width of the distribution.

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Author contributions KAS and SA designed the research. SA performed the numerical simulation and theoretical analysis and wrote the paper with input from KAS. Both authors analyzed the results and approved the manuscript.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: Frequency distribution

To shed more light on the system’s dynamics, in Figs. 4 and 5, we plot the steady-state average frequency distribution, \( P(\langle \dot{\theta} \rangle) \), in the network for forward and backward transitions. To understand the difference between explosive and two-step transition, we compare the frequency distribution for two values of \( \Delta \). The first row corresponds to \( \Delta = 0.8 \) and the second row to \( \Delta = 2 \). We look at the frequency distribution in the system at different values of \( \lambda \). Given that for small coupling strength, frequencies are symmetric, as shown in Fig. 4a, b, the overall frequency distribution has an average around zero; until becoming asymmetric at (c). This is the point where the frequency distribution loses its symmetry. By further increasing the coupling strength, one cluster becomes stronger than the other. Finally, at (d), \( \lambda = 2.04 \), one cluster takes over. While in the \( \Delta = 2 \) case, the distribution remains symmetric until a critical point in Fig. 4k. A similar dynamic also happens for the backward transition (Fig. 5).

Appendix B: Correlation matrix

To study the collective dynamics in more detail, we look at the correlation matrix. The local phase configurations of the system can be determined by the correlation matrix \( D \), which is defined as:

\[
D_{ij} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{t_i}^{t_i + \Delta t} \cos(\theta_i(t) - \theta_j(t)) dt.
\] (B1)

where \( t_i \) is the time needed for reaching a stationary state. The correlation matrix element \( D_{ij} \) is a measure of coherency between the pair of oscillators at \( i \) and \( j \) positions and takes a value in the interval \( 1 \leq D \leq -1 \). \( D_{ij} = 1 \) when there is a full synchrony between nodes \( i \) and \( j \) (\( \theta_i = \theta_j \)), and \( D_{ij} = -1 \) when they are in the anti-phase state (\( \theta_i = \theta_j \pm \pi \)). Here, we used \( 5 \times 10^5 \) time steps for the stationary time \( t_s \) and \( 3 \times 10^4 \) time steps for the averaging window \( \Delta t \).

We calculate the time-averaged correlation matrices (\( D_{ij} \)) in Fig. 6. To compare the dynamics of the two-step transition, which happens for \( \Delta < 1 \), and the explosive transition, happening for \( \Delta > 1 \), we look at the correlation matrices of \( \Delta = 0.8, 2 \). The first row in Fig. 6 shows the results of \( \Delta = 0.8 \), and the second row corresponds to the case of \( \Delta = 2 \). Each row shows how the correlation between the oscillators changes by a progressive increase in the coupling constant. There is an obvious difference between the nature of transition for \( \Delta < 1 \) and \( \Delta > 1 \). For the narrow distribution, the first row in Figs. 4, 6 the system first goes into a two-cluster state, and then these clusters get closer to being synchronized. For very small \( \lambda \), (a), there is no correlation in the system the frequency distribution being symmetric. By increase in \( \lambda \), two clusters appear in the system (Fig. 6b–c), the oscillators inside each cluster are synchronized but not in sync with the other cluster. So, the increase in the coupling strength...
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Fig. 4 Probability density function of the frequency $P(\langle \dot{\theta} \rangle)$ in forward transition. The first row corresponds to the case of $\Delta = 0.8$, and second row for the last row $\Delta = 2$. Each column corresponds to a single value of $\lambda$. (a), (g): $\lambda = 0.02$, (b), (h): $\lambda = 1.8$, (c), (i): $\lambda = 1.98$, (d), (j): $\lambda = 2.04$, (e), (k): $\lambda = 2.98$, (f), (l): $\lambda = 3$

Fig. 5 Probability density function of the frequency $P(\langle \dot{\theta} \rangle)$ in backward transition. The first row corresponds to the case of $\Delta = 0.8$, and the second row to $\Delta = 2$. Each column corresponds to a single value of $\lambda$. (a), (g): $\lambda = 3$, (b), (h): $\lambda = 2.1$, (c), (i): $\lambda = 2.08$, (d), (j): $\lambda = 2$, (e), (k): $\lambda = 1.8$, (f), (l): $\lambda = 1$

does not affect the order parameter. Further increase in the coupling constant, $\lambda$, makes the frequency distribution asymmetric, and more oscillators from one cluster join the other. The probability density, $P(D_{ij})$, shifts toward zero. On the other hand, for the wide distribution (second row in Fig. 6), the system stays totally out of sync up to a threshold point (Fig. 6k) ($\lambda = 2.98$) and does an explosive transition to the synchronized state (Fig. 6l) ($\lambda = 3$) as shown in Fig. 2d.

To see the difference between the dynamics in forward and backward transition, we present the frequency distribution and correlation matrix in Figs. 5, 7, respec-
Fig. 6 Correlation matrix \((D_{ij})\) in forward transition. The first row corresponds to the case of \(\Delta = 0.8\), and the second row to \(\Delta = 2\). Each column corresponds to a single value of \(\lambda\). (a), (g): \(\lambda = 0.02\), (b), (h): \(\lambda = 1.8\), (c), (i): \(\lambda = 1.98\), (d), (j): \(\lambda = 2.04\), (e), (k): \(\lambda = 2.98\), (f), (l): \(\lambda = 3\)

Fig. 7 Correlation matrix \((D_{ij})\) in backward transition. The first row corresponds to the case of \(\Delta = 0.8\), and the second row to \(\Delta = 2\). Each column corresponds to a single value of \(\lambda\). (a), (g): \(\lambda = 3\), (b), (h): \(\lambda = 2.1\), (c), (i): \(\lambda = 2.08\), (d), (j): \(\lambda = 2\), (e), (k): \(\lambda = 1.8\), (f), (l): \(\lambda = 1\)

respectively, similar to the forward plots (Figs. 4, 6). The first row corresponds to the case of \(\Delta = 0.8\), and the second row \(\Delta = 2\). For \(\Delta = 0.8\), we observe three types of dynamics. For \(\lambda \geq 2.08\), (a–c), the system is in the high-order parameter state, with a large population of oscillators being in a frequency-locked state, which shows itself as an asymmetric distribution in frequency (Fig. 5). By increasing the coupling constant, the system goes to a full unsynchronized state (Fig. 5d). For \(\Delta = 2\), on the other hand, there is no intermediate state, and the system shows a sudden transition to the unsynchronized state.

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