GEODESIC DISTANCE FOR RIGHT INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE Diffeomorphism Group. II

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Abstract. The geodesic distance vanishes on the group \( \text{Diff}_c(M) \) of compactly supported diffeomorphisms of a Riemannian manifold \( M \) of bounded geometry, for the right invariant weak Riemannian metric which is induced by the Sobolev metric \( H^s \) of order \( 0 \leq s < \frac{1}{2} \) on the Lie algebra \( \mathfrak{X}_c(M) \) of vector fields with compact support.

1. Introduction

In the article [1] we studied right invariant metrics on the group \( \text{Diff}_c(M) \) of compactly supported diffeomorphisms of a manifold \( M \), which are induced by the Sobolev metric \( H^s \) of order \( s \) on the Lie algebra \( \mathfrak{X}_c(M) \) of vector fields with compact support. We showed that for \( M = S^1 \) the geodesic distance on \( \text{Diff}(S^1) \) vanishes if and only if \( s \leq \frac{1}{2} \). For other manifolds, we showed that the geodesic distance on \( \text{Diff}_c(M) \) vanishes for \( M = \mathbb{R} \times \mathbb{N}, s < \frac{1}{2} \) and for \( M = S^1 \times \mathbb{N}, s \leq \frac{1}{2} \), with \( \mathbb{N} \) being a compact Riemannian manifold.

Now we are able to complement this result by: The geodesic distance vanishes on \( \text{Diff}_c(M) \) for any Riemannian manifold \( M \) of bounded geometry, if \( 0 \leq s < \frac{1}{2} \).

We believe that this result holds also for \( s = \frac{1}{2} \), but we were able to overcome the technical difficulties only for the manifold \( M = S^1 \), in [1]. We also believe that it is true for the regular groups \( \text{Diff}_{H^\infty}(\mathbb{R}^n) \) and \( \text{Diff}_S(\mathbb{R}^n) \) as treated in [8], and for all Virasoro groups, where we could prove it only for \( s = 0 \) in [2].

In Section 2, we review the definitions for Sobolev norms of fractional orders on diffeomorphism groups as presented in [1] and extend them to diffeomorphism groups of manifolds of bounded geometry. Section 3 is devoted to the main result.

2. Sobolev metrics \( H^s \) with \( s \in \mathbb{R} \)

2.1. Sobolev metrics \( H^s \) on \( \mathbb{R}^n \). For \( s \geq 0 \) the Sobolev \( H^s \)-norm of an \( \mathbb{R}^n \)-valued function \( f \) on \( \mathbb{R}^n \) is defined as

\[
\|f\|_{H^s(\mathbb{R}^n)}^2 = \|F^{-1}(1 + |\xi|^2)^{\frac{s}{2}}Ff\|_{L^2(\mathbb{R}^n)}^2,
\]

where \( F \) is the Fourier transform

\[
Ff(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) \, dx,
\]

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and $\xi$ is the independent variable in the frequency domain. An equivalent norm is given by
\begin{equation}
\|f\|_{\mathcal{F}^2(R^n)}^2 = \|f\|_{L^2(R^n)}^2 + \|\xi^m \mathcal{F} f\|_{L^2(R^n)}^2.
\end{equation}

The fact that both norms are equivalent is based on the inequality
\begin{equation}
\frac{1}{C} (1 + \sum_j |\xi_j|^s) \leq (1 + \sum_j |\xi_j|^s)^2 \leq C (1 + \sum_j |\xi_j|^s),
\end{equation}
holding for some constant $C$. For $s > 1$ this says that all $\ell^s$-norms on $R^{n+1}$ are equivalent. But the inequality is true also for $0 < s < 1$, even though the expression does not define a norm on $R^{n+1}$. Using any of these norms we obtain the Sobolev spaces with non-integral $s$
\begin{equation}
H^s(R^n) = \{ f \in L^2(R^n) : \|f\|_{H^s(R^n)} < \infty \}.
\end{equation}

We will use the second version of the norm in the proof of the theorem, since it will make calculations easier.

2.2. Sobolev metrics for Riemannian manifolds of bounded geometry.

Following [13, Section 7.2.1] we will now introduce the spaces $H^s(M)$ on a manifold $M$. If $M$ is not compact we equip $M$ with a Riemannian metric $g$ of bounded geometry which exists by [5]. This means that

- (I) The injectivity radius of $(M, g)$ is positive.
- (B$\infty$) Each iterated covariant derivative of the curvature is uniformly $g$-bounded: $\|\nabla^i R\|_g < C_i$ for $i = 0, 1, 2, \ldots$.

The following is a compilation of special cases of results collected in [3, Chapter 1], who treats Sobolev spaces only for integral order.

**Proposition** ([6], [10], [4]). If $(M, g)$ satisfies (I) and (B$\infty$) then the following holds:

1. $(M, g)$ is complete.
2. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there is a countable cover of $M$ by geodesic balls $B_\varepsilon(x_\alpha)$ such that the cover of $M$ by the balls $B_\varepsilon(x_\alpha)$ is still uniformly locally finite.
3. Moreover, there exists a partition of unity $1 = \sum_\alpha \rho_\alpha$ on $M$ such that $\rho_\alpha \geq 0$, $\rho_\alpha \in C^\infty(M)$, $\text{supp}(\rho_\alpha) \subset B_{2\varepsilon}(x_\alpha)$, and $|D^2_u \rho_\alpha| < C_\beta$ where $u$ are normal (Riemann exponential) coordinates in $B_{2\varepsilon}(x_\alpha)$.
4. In each $B_{2\varepsilon}(x_\alpha)$, in normal coordinates, we have $|D^2_u g_{ij}| < C'_\beta$, $|D^3_u g^{ij}| < C''_\beta$, and $|D^3_u \Gamma^m_{ij}| < C'''_\beta$, where all constants are independent of $\alpha$.

We can now define the $H^s$-norm of a function $f$ on $M$:
\begin{equation}
\|f\|_{H^s(M, g)} = \sum_{\alpha=0}^{\infty} \|\rho_\alpha f \circ \exp_{x_\alpha}\|^2_{H^s(R^n)} = \sum_{\alpha=0}^{\infty} \|\mathcal{F}^{-1}(1 + |\xi|^s)^{\frac{1}{2}} \mathcal{F}((\rho_\alpha f) \circ \exp_{x_\alpha})\|_{L^2(R^n)}^2.
\end{equation}

If $M$ is compact the sum is finite. Changing the charts or the partition of unity leads to equivalent norms by the proposition above, see [13, Theorem 7.2.3]. For integer $s$ we get norms which are equivalent to the Sobolev norms treated in [3,
Chapter 2. The norms depend on the choice of the Riemann metric \( g \). This
dependence is worked out in detail in [3].

For vector fields we use the trivialization of the tangent bundle that is induced
by the coordinate charts and define the norm in each coordinate as above. This
leads to a (up to equivalence) well-defined \( H^s \)-norm on the Lie algebra \( \mathfrak{X}_c(M) \).

2.3. Sobolev metrics on \( \text{Diff}_c(M) \). A positive definite weak inner product on
\( \mathfrak{X}_c(M) \) can be extended to a right-invariant weak Riemannian metric on \( \text{Diff}_c(M) \).

In detail, given \( \varphi \in \text{Diff}_c(M) \) and \( X,Y \in T\varphi \text{Diff}_c(M) \) we define
\[
G^s_\varphi(X,Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_{H^s(M)}.
\]

We are interested solely in questions of vanishing and non-vanishing of geodesic
distance. These properties are invariant under changes to equivalent inner products,
since equivalent inner products on the Lie algebra
\[
1 \leq \langle X,Y \rangle_1 \leq \langle X,Y \rangle_2 \leq C \langle X,Y \rangle_1
\]
imply that the geodesic distances will be equivalent metrics
\[
1 \frac{1}{C} \text{dist}_1(\varphi,\psi) \leq \text{dist}_2(\varphi,\psi) \leq C \text{dist}_1(\varphi,\psi).
\]

Therefore the ambiguity – dependence on the charts and the partition of unity – in
the definition of the \( H^s \)-norm is of no concern to us.

3. Vanishing geodesic distance

3.1. Theorem (Vanishing geodesic distance). The Sobolev metric of order \( s \) induces
vanishing geodesic distance on \( \text{Diff}_c(M) \) if:

\( 0 \leq s < \frac{1}{2} \) and \( M \) is any Riemannian manifold of bounded geometry.

This means that any two diffeomorphisms in the same connected component of
\( \text{Diff}_c(M) \) can be connected by a path of arbitrarily short \( G^s \)-length.

In the proof of the theorem we shall make use of the following lemma from [1].

3.2. Lemma ([1, Lemma 3.2]). Let \( \varphi \in \text{Diff}_c(\mathbb{R}) \) be a diffeomorphism satisfying
\( \varphi(x) \geq x \) and let \( T > 0 \) be fixed. Then for each \( 0 \leq s < \frac{1}{2} \) and \( \varepsilon > 0 \) there exists a
time dependent vector field \( u^\varepsilon_R(t,x) \) of the form
\[
u^\varepsilon_R(t,x) = 1_{[f^\varepsilon(t),g^\varepsilon(t)]} \ast G_\varepsilon(x),
\]
with \( f,g \in C^\infty([0,T]) \), such that its flow \( \varphi^\varepsilon(t,x) \) satisfies – independently of \( \varepsilon \) –
the properties \( \varphi^\varepsilon(0,x) = x \), \( \varphi^\varepsilon(T,x) = \varphi(x) \) and whose \( H^s \)-length is smaller than
\( \varepsilon \), i.e.,
\[
\text{Len}(\varphi^\varepsilon) = \int_0^T \|u^\varepsilon_R(t,\cdot)\|_{H^s} dt \leq C \|f^\varepsilon - g^\varepsilon\|_\infty \leq \varepsilon.
\]

Furthermore \( \{ t : f^\varepsilon(t) < g^\varepsilon(t) \} \subseteq \text{supp}(\varphi) \) and there exists a limit function
\( h \in C^\infty([0,T]) \), such that \( f^\varepsilon \to h \) and \( g^\varepsilon \to h \) for \( \varepsilon \to 0 \) and the convergence is
uniform in \( t \).

Here, \( G_\varepsilon(x) = \frac{1}{\varepsilon} G_1(\frac{x}{\varepsilon}) \) is a smoothing kernel, defined via a smooth bump function
\( G_1 \) with compact support.
Proof of Theorem 3.1. Consider the connected component \( \text{Diff}_0(M) \) of Id, i.e. those diffeomorphisms of \( \text{Diff}_c(M) \), for which there exists at least one path, joining them to the identity. Denote by \( \text{Diff}_c(M)^{L=0} \) the set of all diffeomorphisms \( \varphi \) that can be reached from the identity by curves of arbitrarily short length, i.e., for each \( \varepsilon > 0 \) there exists a curve from Id to \( \varphi \) with length smaller than \( \varepsilon \).

Claim A. \( \text{Diff}_c(M)^{L=0} \) is a normal subgroup of \( \text{Diff}_0(M) \).

Claim B. \( \text{Diff}_c(M)^{L=0} \) is a non-trivial subgroup of \( \text{Diff}_0(M) \).

By [12] or [7], the group \( \text{Diff}_0(M) \) is simple. Thus claims A and B imply \( \text{Diff}_c(M)^{L=0} = \text{Diff}_0(M) \), which proves the theorem.

The proof of claim A can be found in [1, Theorem 3.1] and works without change in the case of \( M \) being an arbitrary manifold and hence we will not repeat it here. It remains to show that \( \text{Diff}_c(M)^{L=0} \) contains a diffeomorphism \( \varphi \neq \text{Id} \).

We shall first prove claim B for \( M = \mathbb{R}^n \) and then show how to extend the arguments to arbitrary manifolds. Choose a diffeomorphism \( \varphi \in \text{Diff}_c(\mathbb{R}) \) with \( \varphi(x) \geq x \) and \( \text{supp}(\varphi(x)) \subseteq [1, \infty) \). Then let

\[
\varphi_R^\varepsilon(t,x) := \mathbb{I}_{[f(t),g(t)]} * G_\varepsilon(x)
\]

be the family of vector fields constructed in Lemma 3.2, whose flows at time \( T \) equal \( \varphi_R \). We extend the vector field \( u_R^\varepsilon \) to a vector field \( u_R^\varepsilon_n \) on \( \mathbb{R}^n \) via

\[
u_R^\varepsilon_n(t,x_1,\ldots,x_n) := \left( u_R^\varepsilon(t,|x|), 0,\ldots, 0 \right).
\]

The flow of this vector field is given by

\[
\varphi_R^\varepsilon_n(t,x_1,\ldots,x_n) = (\varphi_R^\varepsilon(t,|x|),x_2,\ldots,x_n),
\]

where \( \varphi_R^\varepsilon \) is the flow of \( u_R^\varepsilon \). In particular we see that at time \( t = T \)

\[
\varphi_R^\varepsilon_n(t,x_1,\ldots,x_n) = (\varphi_R(|x|),x_2,\ldots,x_n),
\]

the flow is independent of \( \varepsilon \). So it remains to show that for the length of the path \( \varphi_R^\varepsilon_n(t,\cdot) \) we have

\[
\text{Len}(\varphi_R^\varepsilon_n) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

We can estimate the length of this path via

\[
\text{Len}(\varphi_R^\varepsilon_n)^2 = \left( \int_0^T \|u_R^\varepsilon_n(t,\cdot)\|_{H^s(\mathbb{R}^n)} dt \right)^2 \leq T \int_0^T \|u_R^\varepsilon_n(t,\cdot)\|_{H^s(\mathbb{R}^n)}^2 dt
\]

\[
= T \int_0^T \|u_R^\varepsilon(t,\cdot)\|_{H^s(\mathbb{R}^n)}^2 dt = T \int_0^T \|\mathbb{I}_{[f(t),g(t)]} * G_\varepsilon(|x|)\|_{H^s(\mathbb{R}^n)}^2 dt
\]

\[
\leq C(G_1,T) \int_0^T \|\mathbb{I}_{[f(t),g(t)]}(\cdot)\|_{H^s(\mathbb{R}^n)}^2 dt,
\]

where the last estimate follows from

\[
\|\mathbb{I}_{[f(t),g(t)]} * G_\varepsilon(|x|)\|_{H^s(\mathbb{R}^n)}^2 =
\]

\[
= \int_{\mathbb{R}^n} (1 + |\xi|^2) \left[ \mathcal{F} \left( \mathbb{I}_{[f(t),g(t)]}(\cdot) \right) (\xi) \right]^2 \left[ \mathcal{F} \left( G_\varepsilon(|\cdot|) \right) (\xi) \right]^2 d\xi
\]

\[
= \int_{\mathbb{R}^n} (1 + |\xi|^2) \left[ \mathcal{F} \left( \mathbb{I}_{[f(t),g(t)]}(\cdot) \right) (\xi) \right]^2 \left[ \mathcal{F} \left( G_1(|\cdot|) \right) (\varepsilon\xi) \right]^2 d\xi
\]

\[
\leq \|\mathcal{F}G_1(|\cdot|)\|_{L^\infty}^2 \cdot \|\mathbb{I}_{[f(t),g(t)]}(\cdot)\|_{H^s(\mathbb{R}^n)}^2.
\]
The above integral is non-zero only for those
in the same way by setting
Hence it is sufficient to show that
3.2 and our assumptions on \( \phi \)
This gives us
The \( \infty \int \) above integral using the identity \([9, (10.22.57)]\),
will omit the dependence of the vector field \( 1 \)
coordinates, this becomes
We will only consider the term involving
The \( H \)
\( \int \)
Fourier-transform of a radially symmetric function \( v(\cdot) \) is again radially symmetric and given by the following formula, see \([11, \text{Theorem 3.3}]\),
This integral can be evaluated explicitly using the following integral identity for
Bessel functions from \([9, (10.22.1)]\)
To compute the \( H^s \)-norm of \( \mathbb{I}_{[f^\varepsilon(t), g^\varepsilon(t)]}(\cdot) \) we first Fourier-transform it. The Fourier-transform of a radially symmetric function \( v(\cdot) \) is \( L^1(\mathbb{R}^n) \) is again radially symmetric and given by the following formula, see \([11, \text{Theorem 3.3}]\),
with \( J_{n/2-1} \) denoting the Bessel function of order \( \frac{n}{2} - 1 \). To simplify notation we will omit the dependence of the vector field \( \mathbb{I}_{[f^\varepsilon(t), g^\varepsilon(t)]}(\cdot) \) on \( t \) and \( \varepsilon \). Changing coordinates, this becomes
This integral can be evaluated explicitly using the following integral identity for Bessel functions from \([9, (10.22.1)]\)
This gives us
The \( H^s \)-norm of \( \mathbb{I}_{[f^\varepsilon,t,g^\varepsilon]}(\cdot) \) is given by
We will only consider the term involving \( |\xi|^{2s} \), since the \( L^2 \)-term can be estimated in the same way by setting \( s = 0 \). Transforming to polar coordinates we obtain
The above integral is non-zero only for those \( t \), where \( f^\varepsilon(t) \neq g^\varepsilon(t) \). From Lemma 3.2 and our assumptions on \( \phi_R \) we know that
Thus both \( f^\varepsilon(t) \) and \( g^\varepsilon(t) \) are different and away from 0 and we can evaluate the above integral using the identity \([9, (10.22.57)]\),
\[
\int_0^\infty \frac{J_\mu(at) J_\nu(at)}{t^\lambda} \, dt = \frac{\left( \frac{\lambda}{2} \right) \Gamma \left( \frac{\lambda}{2} + \frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2} \right) \Gamma \left( \lambda \right)}{2 \Gamma \left( \frac{\lambda}{2} + \frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\lambda}{2} + \frac{\mu}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\lambda}{2} + \frac{\nu}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2} \right) \Gamma \left( \lambda + \frac{\mu}{2} + \frac{\nu}{2} + 1 \right)}
\]
which holds for $\Re(\mu + \nu + 1) > \Re \lambda > 0$ and the identity [9, (10.22.56)],

\[
\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda} \, dt = \frac{a^\mu \Gamma\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}\right)}{2^\lambda \mu^\mu \nu^\nu \lambda^\lambda} \Gamma\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}\right) \, F\left(\frac{\nu}{2} + \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}, \frac{\mu}{2} - \frac{\lambda}{2} + \frac{1}{2}; \mu + 1; \frac{a^2}{2}\right),
\]

which holds for $0 < a < b$ and $\Re(\mu + \nu + 1) > \Re \lambda > -1$. Here $F(a; b; c; d)$ is the regularized hypergeometric function. Using these identities with $\lambda = 1 - 2s$, $\mu = \nu = \frac{n}{2}$, $a = 2\pi f$ and $b = 2\pi g$ we obtain

\[
\int_0^\infty r^{2s-1} J_{n/2}(2\pi fr)^2 \, dr = \frac{1}{2} \left(\frac{\pi f}{g}\right)^{-2s} \Gamma\left(\frac{n}{2} + s\right) \Gamma(1-2s) \Gamma(1-s)^2 \Gamma\left(\frac{n}{2} + 1 - s\right)
\]

and

\[
\int_0^\infty r^{2s-1} J_{n/2}(2\pi fr) J_{n/2}(2\pi gr) \, dr = -\frac{1}{2} \left(\frac{\pi g}{f}\right)^{-2s} \left(\frac{f}{g}\right)^{n/2} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(1-s)} \Gamma\left(\frac{n}{2} + s, s\right) \frac{\Gamma\left(\frac{n}{2} + 1 + \frac{1}{2}; \frac{f^2}{g^2}\right)}{\Gamma(1-s)}
\]

Putting it together results in

\[
\int_{\mathbb{R}^n} |\xi|^{2s} (F(1, f, g)(|\cdot|)(\xi))^2 \, d\xi = \Vol(S^{n-1}) \left(\frac{f^{-2s} + g^{-2s}}{2^{2s}} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(1-s)^2} \Gamma\left(\frac{n}{2} + 1 - s\right) - \frac{g^{-2s} f^{n/2}}{2^{n/2}} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(1-s)} \Gamma\left(\frac{n}{2} + s, s\right) \frac{\Gamma\left(\frac{n}{2} + 1 + \frac{1}{2}; \frac{f^2}{g^2}\right)}{\Gamma(1-s)}\right).
\]

In the limit $\varepsilon \to 0$ we know from Lemma 3.2 that $f^\varepsilon(t) \to h(t)$ and $g^\varepsilon(t) \to h(t)$ uniformly in $t$ on $[0, T]$ and hence $f^\varepsilon(t) \to 1$. For the regularized hypergeometric function $F(a; b; c; d)$ at $d = 1$ we have the identity [9, (15.4.20)]

\[
F(a; b; c; 1) = \frac{\Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)},
\]

for $\Re(c - a - b) > 0$. Applying the identity with $a = \frac{n}{2} + s$, $b = s$ and $c = \frac{n}{2} + 1$ we get

\[
F\left(\frac{n}{2} + s, s; \frac{n}{2} + 1 + \frac{1}{2}; \frac{f^2}{g^2}\right) = \frac{\Gamma(1-2s)}{\Gamma(1-s) \Gamma\left(\frac{n}{2} + 1 - s\right)}.
\]

Using the continuity of the hypergeometric function it follows that

\[
\int_{\mathbb{R}^n} |\xi|^{2s} (F(1, f, g)(|\cdot|)(\xi))^2 \, d\xi \to 0,
\]

as $\varepsilon \to 0$ and the convergence is uniform in $t$. This concludes the proof that

\[
\|f^\varepsilon(t), g^\varepsilon(t)\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in} \quad t,
\]

and hence we have established claim B for $\text{Diff}_c(\mathbb{R}^n)$.

To prove this result for an arbitrary manifold $M$ of bounded geometry we choose a partition of unity $(\tau_j)$ such that $\tau_0 \equiv 1$ on some open subset $U \subset M$, where normal coordinates centred at $x_0 \in M$ are defined. If $\varphi_\mathbb{R}$ is chosen with sufficiently
small support, then the vector field \( u^\varepsilon_M \) has support in \( \exp_{x_0}(U) \) and we can define the vector field \( u^\varepsilon_M := (\exp_{x_0}^{-1})^*u^\varepsilon_R \) on \( M \). This vector field generates a path \( \varphi^\varepsilon_M(t, \cdot) \in \text{Diff}_0(M) \) with an endpoint \( \varphi^\varepsilon_M(T, \cdot) = \varphi_M(\cdot) \) that doesn’t depend on \( \varepsilon \) with arbitrarily small \( H^s \)-length since

\[
\text{Len}(\varphi^\varepsilon_M) \leq C_1(\tau) \int_0^T \| u^\varepsilon_M \|_{H^s(M, \tau)} \, dt = C_1(\tau) \int_0^T \| \exp_{x_0}^*(\tau_0. u^\varepsilon_M) \|_{H^s(\mathbb{R}^n)} \, dt
\]

Thus we can reduce the case of arbitrary manifolds to \( \mathbb{R}^n \) and this concludes the proof. \( \square \)

References

[1] Martin Bauer, Martins Bruveris, Philipp Harms, and Peter W. Michor. Geodesic distance for right invariant sobolev metrics of fractional order on the diffeomorphism group. To appear in Ann. Global Anal. Geom., 2012.

[2] Martin Bauer, Martins Bruveris, Philipp Harms, and Peter W. Michor. Vanishing geodesic distance for the Riemannian metric with geodesic equation the KdV-equation. Ann. Global Anal. Geom., 41(4):461–472, 2012.

[3] J. Eichhorn. Global analysis on open manifolds. Nova Science Publishers Inc., New York, 2007.

[4] Jürgen Eichhorn. The boundedness of connection coefficients and their derivatives. Math. Nachr., 152:145–158, 1991.

[5] R. E. Greene. Complete metrics of bounded curvature on noncompact manifolds. Arch. Math. (Basel), 31(1):89–95, 1978/79.

[6] Yu. A. Kordyukov. \( L^p \)-theory of elliptic differential operators on manifolds of bounded geometry. Acta Appl. Math., 23(3):223–260, 1991.

[7] J. N. Mather. Commutators of diffeomorphisms. Comment. Math. Helv., 49:512–528, 1974.

[8] Peter W. Michor and David Mumford. A zoo of diffeomorphism groups on \( \mathbb{R}^n \). 2012. In preparation.

[9] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. NIST handbook of mathematical functions. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010.

[10] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. Astérisque, (207):5, 35–108, 1992. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).

[11] Elias M. Stein and Guido Weiss. Introduction to Fourier analysis on Euclidean spaces. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[12] W. Thurston. Foliations and groups of diffeomorphisms. Bull. Amer. Math. Soc., 80:304–307, 1974.

[13] Hans Triebel. Theory of function spaces. II, volume 84 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 1992.

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