A characterization of positive normal functionals on the full operator algebra

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Abstract. Using the recent theory of Krein–von Neumann extensions for positive functionals we present several simple criteria to decide whether a given positive functional on the full operator algebra $B(H)$ is normal. We also characterize those functionals defined on the left ideal of finite rank operators that have a normal extension.

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The aim of this short note is to present a theoretical application of the generalized Krein–von Neumann extension, namely to offer a characterization of positive normal functionals on the full operator algebra. To begin with, let us fix our notations. Given a complex Hilbert space $H$, denote by $B(H)$ the full operator algebra, i.e., the $C^*$-algebra of continuous linear operators on $H$. The symbols $B_F(H), B_1(H), B_2(H)$ are referring to the ideals of continuous finite rank operators, trace class operators, and Hilbert–Schmidt operators, respectively. Recall that $B_2(H)$ is a complete Hilbert algebra with respect to the inner product

$$(X | Y)_2 = \text{Tr}(Y^* X) = \sum_{e \in \mathcal{E}} (X e | Y e), \quad X, Y \in B_2(H).$$

Here Tr refers to the the trace functional and $\mathcal{E}$ is an arbitrary orthonormal basis in $H$. Recall also that $B_1(H)$ is a Banach *-algebra under the norm $\|X\|_1 := \text{Tr}(|X|)$, and that $B_F(H)$ is dense in both $B_1(H)$ and $B_2(H)$, with respect to the norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. It is also known that $X \in B_1(H)$ holds if and only if $X$ is the product of two elements of $B_2(H)$. For the proofs and further basic properties of Hilbert-Schmidt and trace class operators we refer the reader to [1].

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Let $\mathcal{A}$ be a von Neumann algebra, that is a strongly closed $\ast$-subalgebra of $B(H)$ containing the identity. A bounded linear functional $f : \mathcal{A} \to \mathbb{C}$ is called normal if it is continuous in the ultraweak topology, that is $f$ belongs to the predual of $\mathcal{A}$. It is well known that the predual of $B(H)$ is $B_1(H)$, hence every normal functional can be represented by a trace class operator. We will use this property as the definition.

**Definition.** A linear functional $f : B(H) \to \mathbb{C}$ is called a normal functional if there exists a trace class operator $F$ such that

$$f(X) := \text{Tr}(XF) = \text{Tr}(FX), \quad X \in B(H).$$

Remark that such a functional is always continuous due to the inequality

$$|\text{Tr}(XF)| \leq \|F\|_1 \cdot \|X\|.$$

Our main tool is a canonical extension theorem for linear functionals which is analogous with the well-known operator extension theorem named after the pioneers of the 20th century operator theory M.G. Krein [2] and J. von Neumann [3]. For the details see Section 5 in [5], especially Theorem 5.6 and the subsequent comments. Let us recall the cited theorem:

**A Krein–von Neumann type extension.** Let $\mathcal{I}$ be a left ideal of the complex Banach $\ast$-algebra $\mathcal{A}$, and consider a linear functional $\varphi : \mathcal{I} \to \mathbb{C}$. The following statements are equivalent:

(a) There is a representable positive functional $\varphi^* : \mathcal{A} \to \mathbb{C}$ extending $\varphi$, which is minimal in the sense that

$$\varphi^*(x^*x) \leq \tilde{\varphi}(x^*x), \quad \text{holds for all } x \in \mathcal{A},$$

whenever $\tilde{\varphi} : \mathcal{A} \to \mathbb{C}$ is a representable extension of $\varphi$.

(b) There is a constant $C \geq 0$ such that $|\varphi(a)|^2 \leq C \cdot \varphi(a^*a)$ for all $a \in \mathcal{I}$.

We remark that the construction used in the proof of the above theorem is closely related to the one developed in [4] for Hilbert space operators. The main advantage of that construction is that we can compute the values of the smallest extension $\varphi^*$ on positive elements, namely

$$\varphi^*(x^*x) = \sup \left\{ |\varphi(x^*a)|^2 \mid a \in \mathcal{I}, \varphi(a^*a) \leq 1 \right\} \quad \text{for all } x \in \mathcal{A}. \quad (*)$$

The minimal extension $\varphi^*$ is called the Krein–von Neumann extension of $\varphi$.

The characterization we are going to prove is stated as follows.

**Main Theorem.** For a given positive functional $f : B(H) \to \mathbb{C}$ the following statements are equivalent:

(i) $f$ is normal.

(ii) There exists a normal positive functional $g$ such that $f \leq g$.

(iii) $f \leq g$ holds for any positive functional $g$ that agrees with $f$ on $B_F(H)$.

(iv) For any $X \in B(H)$ we have

$$f(X^*X) = \sup \{ |f(X^*A)|^2 \mid A \in B_F(H), f(A^*A) \leq 1 \}. \quad (***)$$

(v) $f(I) \leq \sup \{ |f(A)|^2 \mid A \in B_F(H), f(A^*A) \leq 1 \}$.
Proof. The proof is divided into three claims, which might be interesting on their own right. Before doing that we make some observations. For a given trace class operator $S$ let us denote by $f_S$ the normal functional defined by

$$f_S(X) := \text{Tr}(XS), \quad X \in B(H).$$

The map $S \mapsto f_S$ is order preserving between positive trace class operators and normal positive functionals. Indeed, if $S \geq 0$ then

$$f_S(A^*A) = \text{Tr}(A^*AS) = \|AS^{1/2}\|_2^2 \geq 0.$$ 

Conversely, if $f_S$ is a positive functional and $P_{(h)}$ denotes the orthogonal projection onto the subspace spanned by $h \in H$, we obtain $S \geq 0$ by

$$(Sh \mid h) = \text{Tr}(P_{(h)}S) = f_S(P_{(h)}^*P_{(h)}) \geq 0, \quad \text{for all } h \in H.$$ 

Our first two claims will prove that (i) and (iv) are equivalent.

**Claim 1.** Let $f$ be a normal positive functional and set $\phi := f|_{B_F(H)}$. Then $f$ is the smallest positive extension of $\phi$, i.e. $\phi^* = f$.

**Proof of Claim 1.** Since $f \geq 0$ is normal, there is a positive $S \in B_1(H)$ such that $f = f_S$. By assumption $\phi$ has a positive extension (namely $f$ itself is one), thus there exists also the Krein–von Neumann extension denoted by $\phi^*$. As $f_S - \phi^*$ is a positive functional due to the minimality of $\phi^*$, its norm is attained at identity $I$. Therefore it is enough to show that

$$\phi^*(I) \geq f_S(I) = \text{Tr}(S).$$

We know from (17) that

$$\phi^*(X^*X) = \sup\{|\phi(X^*A)|^2 \mid A \in B_F(H), \phi(A^*A) \leq 1\}$$

for any $X \in B(H)$. Choosing $A = \text{Tr}(S)^{-1/2}P$ for any projection $P$ with finite rank, we see that $\phi(A^*A) = \text{Tr}(S)^{-1} \text{Tr}(PS) \leq 1$, whence

$$\phi^*(I) \geq |\phi(A)|^2 = \frac{\text{Tr}(PS)^2}{\text{Tr}(S)}.$$ 

Taking supremum in $P$ on the right hand side we obtain $\phi^*(I) \geq \text{Tr}(S)$, which proves the claim.

**Claim 2.** The smallest positive extension of $\phi$, i.e. $(f|_{B_F(H)})^*$ is normal.

**Proof of Claim 2.** First observe that the restriction of $f$ to $B_2(H)$ defines a continuous linear functional on $B_2(H)$ with respect to the norm $\|\cdot\|_2$. Due to the Riesz representation theorem, there exists a unique representing operator $S \in B_2(H)$ such that

$$f(A) = (A \mid S)_2 = \text{Tr}(S^*A), \quad \text{for all } A \in B_2(H). \quad (***)$$

We are going to show that $S \in B_1(H)$. Indeed, let $\mathcal{E}$ be an orthonormal basis in $H$ and let $\mathcal{F}$ be any non-empty finite subset of $\mathcal{E}$. Denoting by $P_\mathcal{F}$ the orthogonal projection onto the subspace spanned by $\mathcal{F}$ we get

$$\sum_{e \in \mathcal{F}} (Se \mid e) = (P_\mathcal{F} \mid S)_2 = f(P_\mathcal{F}) \leq f(I).$$
Taking supremum in $\mathcal{F}$ we obtain that $S$ is in trace class. By Claim 1, the smallest positive extension $\varphi^*$ of $\varphi$ equals $f_S$ which is normal. This proves Claim 2.

Now, we are going to prove (ii)$\Rightarrow$(i).

Claim 3. If there exists a normal positive functional $g$ such that $f \leq g$ holds, then $f$ is normal as well.

Proof of Claim 3. Let $g$ be a normal positive functional dominating $f$, and let $T$ be a trace class operator such that $g = f_T$. According to Claim 2 it is enough to prove that $f = \varphi^*$. Since $h := f - \varphi^*$ is positive, this will follow by showing that $h(I) = 0$. We see from \[\text{[**]}\] that $h(A) = 0$ for any finite rank operator $A$. Consequently, as $h \leq f \leq f_T$, it follows that

$$h(I) = h(I - P) \leq f_T(I - P) = \text{Tr}(T) - \text{Tr}(TP),$$

for any finite rank projection $P$. Taking infimum in $P$ we obtain $h(I) = 0$ and therefore Claim 3 is established.

Completing the proof we mention all the missing trivial implications. Taking $g := f$, (i) implies (ii). As \[\text{[**]}\] means that $\varphi^* = f$, equivalence of (iii) and (iv) follows from the minimality of the Krein-von Neumann extension. Replacing $X$ with $I$ in \[\text{[**]}\] we obtain (v). Conversely, (v) implies (iv) as $\varphi^* \leq f$ and $f - \varphi^*$ attains its norm at $I$. \qed

Finally, we remark that the above proof contains a characterization of having normal extension for a functional defined on $B_F(H)$.

Corollary. Let $\varphi : B_F(H) \to \mathbb{C}$ be a linear functional. The following statements are equivalent to the existence of a normal extension.

(a) There is a $C \geq 0$ such that $|\varphi(A)|^2 \leq C \cdot \varphi(A^*A)$ for all $A \in B_F(H)$.

(b) There is a positive functional $f$ such that $f \big|_{B_F(H)} = \varphi$.

(c) There is an $F \in B_1(H)$ such that $\varphi(A) = \text{Tr}(FA)$ for all $A \in B_F(H)$.

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