CRITICAL TRAVELING WAVE SOLUTIONS FOR A VACCINATION MODEL WITH GENERAL INCIDENCE

YU YANG*
School of Statistics and Mathematics
Shanghai Lixin University of Accounting and Finance
Shanghai 201209, China

JINLING ZHOU
Department of Mathematics, Zhejiang International Studies University
Hangzhou 310023, China

CHENG-HSIUNG HSU
Department of Mathematics, National Central University, Zhongli District
Taoyuan City 32001, Taiwan

(Communicated by Emeric Bouin)

Abstract. This paper is concerned with the existence of traveling wave solutions for a vaccination model with general incidence. The existence or non-existence of traveling wave solutions for the model with specific incidence were proved recently when the wave speed is greater or smaller than a critical speed respectively. However, the existence of critical traveling wave solutions (with critical wave speed) was still open. In this paper, applying the Schauder’s fixed point theorem via a pair of upper- and lower-solutions of the system, we show that the general vaccination model admits positive critical traveling wave solutions which connect the disease-free and endemic equilibria. Our result not only gives an affirmative answer to the open problem given in the previous specific work, but also to the model with general incidence. Furthermore, we extend our result to some nonlocal version of the considered model.

1. Introduction. This paper is concerned with the existence of critical traveling wave solutions for a vaccination model with general incidence. Recently, Xu et al. [11] considered the following system:

\[
\begin{align*}
S_t &= d_1 \Delta S - (\mu + \gamma)S + \Lambda - \frac{\sigma \beta_1 I}{1 + I} S, \\
V_t &= d_2 \Delta V - \mu V + \gamma S - \frac{\sigma \beta_2 I}{1 + I} V, \\
I_t &= d_3 \Delta I - \mu I + \frac{\sigma I}{1 + I} (\beta_1 S + \beta_2 V),
\end{align*}
\]

(1)

2020 Mathematics Subject Classification. Primary: 35C07, 35K57; Secondary: 92B05.

Key words and phrases. Traveling wave, general incidence, critical wave speed, upper- and lower-solutions, Schauder’s fixed point theorem.

The third author was partially supported by the MOST (Grant No. 107-2115-M-008-009- MY3) and NCTS of Taiwan.

* Corresponding author: Yu Yang.
where $S(x,t)$, $V(x,t)$ and $I(x,t)$ represent the densities of susceptible, vaccinated and infective individuals at position $x \in \mathbb{R}$ and time $t > 0$, respectively. The constant $d_i > 0$ ($i = 1, 2, 3$) means the diffusion coefficient, while other parameters are assumed to be non-negative. For the biological meanings of the parameters, we refer to [11, 15] and their references. In [11], the authors studied the global stability and existence of traveling wave solutions of system (1). From their results, we know that system (1) admits positive traveling wave solutions connecting disease-free equilibrium and endemic equilibrium provided the wave speed is greater than a critical speed $c^* > 0$ and the basic reproduction number is greater than one. Furthermore, such traveling wave solutions do not exist when the wave speeds are smaller than $c^*$.

Motivated by the above work, Zhou et al. [15] extended system (1) to the following nonlocal dispersal system with general incidence:

$$
\begin{cases}
S_t = d_1 [J_{\rho_1} * S - S] + \Lambda - f(I)S - (\mu + \gamma)S, \\
V_t = d_2 [J_{\rho_2} * V - V] + \gamma S - g(I)V - \mu V, \\
I_t = d_3 [J_{\rho_3} * I - I] + f(I)S + g(I)V - \mu I,
\end{cases}
$$

(2)

for $x \in \mathbb{R}$ and $t > 0$, where $J_{\rho_i}(x) := \rho_i^{-1}J(x/\rho_i)$ ($i = 1, 2, 3$) are $\rho_i$-parametrized dispersal kernels of the species for some kernel function $J(\cdot)$ satisfying the following assumption:

(J) $J(x) \in C^1(\mathbb{R})$ is a positive, even and Lipschitz continuous function with compact support and

$$
\int_{-\infty}^{+\infty} J(x)dx = 1.
$$

Throughout this paper, we assume that $f(\cdot)$ and $g(\cdot)$ satisfy the following conditions:

(A1) $f(\cdot)$, $g(\cdot) \in C^2(\mathbb{R})$ with $f(0) = g(0) = 0$,

$$
f(I), g(I) > 0, f'(I), g'(I) > 0 \text{ and } f''(I), g''(I) < 0 \text{ for all } I > 0.
$$

(A2) $S_0 f_\infty + V_0 g_\infty < \mu < S_0 f'(0) + V_0 g'(0)$, where \( \lim_{I \to +\infty} f(I)/I = f_\infty \) and

$$
\lim_{I \to +\infty} g(I)/I = g_\infty.
$$

Noting that (A1) prevents the functions $f(\cdot)$ and $g(\cdot)$ to be linear and therefore the assumptions do not include the classical SIR model. Recently, Zhou et al. [15] proved the existence of traveling wave solutions of (2) when the wave speed is greater than a critical speed $c^*$ (see (8)) by using the Schauder’s fixed point theorem. Moreover, they also studied the boundary asymptotic behaviour of traveling wave solutions at $+\infty$ by using an appropriate Lyapunov function. In addition, using two-sided Laplace transform, they showed the non-existence of traveling wave solutions when the wave speed is smaller than $c^*$. However, the existence of critical traveling wave solutions (with wave speed $c^*$) was open in Xu et al. [11] and Zhou et al. [15].

Motivated by the works of [11, 15], in this article we consider the exact traveling wave solutions of the following reaction-diffusion system with general incidence:

$$
\begin{cases}
S_t = d_1 \Delta S + \Lambda - Sf(I) - (\mu + \gamma)S, \\
V_t = d_2 \Delta V + \gamma S - Vg(I) - \mu V, \\
I_t = d_3 \Delta I + Sf(I) + Vg(I) - \mu I,
\end{cases}
$$

(3)
Biologically, the critical traveling wave solution represent the epidemic wave with the minimal speed, which play an important role in the study of epidemic spread, see e.g., Li et al. [6] and the cited references. In contrast to the non-critical traveling wave solutions, the existence problem of critical traveling wave solutions is more difficult in various models. In past years, using limiting arguments or Schauder’s fixed point theorem, there were some literature devoting to investigate the existence of critical traveling wave solutions of different models, see e.g., [2, 3, 7, 8, 10, 13, 14]. Applying the limiting arguments, it is sufficient to consider the convergence problem for a sequence of traveling wave solutions with wave speed greater than the critical speed. However, the convergence problem is non-trivial. Motivated by the literature [13, 15], in this work we establish a pair of upper- and lower-solutions of system (3) and apply the Schauder’s fixed point theorem to prove the existence of critical traveling wave solutions of (1) with general incidence.

To this end, we further give the following assumption:

(A3) There exist $k_1, k_2 \geq 0$ such that 
$$f(I) \geq f'(0)I - k_1 I^2, \quad g(I) \geq g'(0)I - k_2 I^2,$$
for all $I > 0$.

One can easily verify that system (1) satisfies (A1)–(A3). Before stating our main result, we first recall the existence of equilibria of system (3) as follows. Let $S_0$ and $V_0$ be the constants given by

$$S_0 = \frac{\Lambda}{\mu + \gamma} \quad \text{and} \quad V_0 = \frac{\gamma \Lambda}{\mu (\mu + \gamma)}.$$

It is clear that $E_0 := (S_0, V_0, 0)$ is a disease-free equilibrium of system (3). Moreover, from [15], the basic reproduction number of system (3) is defined by

$$R_0 := \frac{S_0 f'(0)}{\mu} + \frac{V_0 g'(0)}{\mu}.$$

According to [15, Proposition 1], we know that

- If $R_0 < 1$, system (3) has a unique disease-free equilibrium $E_0 = (S_0, V_0, 0)$.
- If $R_0 > 1$, system (3) admits a unique endemic equilibrium $E^* = (S^*, V^*, I^*)$ for some $I^* > 0$ with

$$S^* := \frac{\Lambda}{\mu + \gamma + f(I^*)} \quad \text{and} \quad V^* := \frac{\gamma \Lambda}{(\mu + g(I^*))(\mu + \gamma + f(I^*)}).$$

Then we have the following main result.

**Theorem 1.1.** Assume (A1)–(A3) and $R_0 > 1$. Let $\xi = x + c^* t$, then system (3) admits a positive traveling wave solution $(S(x, t), V(x, t), I(x, t)) = (S(\xi), V(\xi), I(\xi))$ which satisfies

$$\lim_{\xi \to -\infty} (S(\xi), V(\xi), I(\xi)) = E_0 \quad \text{and} \quad \lim_{\xi \to +\infty} (S(\xi), V(\xi), I(\xi)) = E^*.$$  \hspace{1cm} (4)

Here we remark that our result not only gives an affirmative answer to the open problem given in [11], but also to the general system (3).

The rest of this paper is organized as follows. In Section 2, we provide some preliminary results of system (3). In Section 3, inspired by the construction of [15], we establish a different pair of upper- and lower-solutions of system (3). Then, applying the Schauder’s fixed point theorem together with the upper- and lower-solutions of system (3) and constructing Lyapunov function, we prove the statement of Theorem 1.1 in Section 4.
2. Preliminaries. A traveling wave solution \((S(x, t), V(x, t), I(x, t))\) of system (3) means that
\[
(S(x, t), V(x, t), I(x, t)) = (S(x + ct), V(x + ct), I(x + ct)) \tag{5}
\]
for some \(c \in \mathbb{R}\). Let us set the moving coordinate \(\xi := x + ct\). By (5), the profile equations of system (3) have the form:
\[
\begin{cases}
cS' = d_1S'' + \lambda - Sf(I) - (\mu + \gamma)S, \\
cV' = d_2V'' + \gamma S - Vg(I) - \mu V, \\
cI' = d_3I'' + \gamma Sf(I) + Vg(I) - \mu I.
\end{cases} \tag{6}
\]
According to [15], the characteristic function for the third equation of (6) linearized around \(E_0\) is
\[
\Delta(\lambda, c) := d_3\lambda^2 - c\lambda + S_0f'(0) + V_0g'(0) - \mu. \tag{7}
\]
The function \(\lambda^* := 2\sqrt{d_3(S_0f'(0) + V_0g'(0) - \mu)}\) gives the critical wave speed.

As stated in introduction section, system (3) is a special case as system (2). Therefore, the existence and non-existence of non-critical traveling wave solutions (with \(c \neq c^*\)) of system (3) follows directly from [15, Theorems 3.1 and 4.1]. We state the results in the following theorem.

**Theorem 2.1.** Assume (A1)–(A2) and \(R_0 > 1\).

1. For any \(c > c^*\), system (6) admits a positive traveling wave solution \((S(\xi), V(\xi), I(\xi))\) satisfying the condition (4).

2. For any \(0 < c < c^*\), system (6) has no nontrivial bounded positive solution satisfying the condition (4).

Thus our goal is to prove the existence of positive solutions of system (6) satisfying the condition (4) when \(c = c^*\).

3. Upper- and lower-solutions. To prove the main theorem, we first construct a pair of upper- and lower-solutions of system (6) in this section, cf. [1, 3, 4, 9, 13]. Inspired by the construction of [15], we define the following functions:
\[
S(\xi) := S_0, \text{ for } \xi \in \mathbb{R}, S(\xi) := \max \left\{ S_0 - \frac{1}{\alpha_1}e^{\alpha_1\xi}, B \right\} = \begin{cases} S_0 - \frac{1}{\alpha_1}e^{\alpha_1\xi}, & \xi < \xi_2, \\
B, & \xi \geq \xi_2, \end{cases} \tag{9}
\]
\[
V(\xi) := V_0, \text{ for } \xi \in \mathbb{R}, V(\xi) := \max \left\{ V_0 - \frac{1}{\theta_1}e^{\theta_1\xi}, 0 \right\} = \begin{cases} V_0 - \frac{1}{\theta_1}e^{\theta_1\xi}, & \xi < \xi_3, \\
0, & \xi \geq \xi_3, \end{cases} \tag{10}
\]
\[
I(\xi) := \min \left\{ -L_1\xi e^{\lambda\xi}, A \right\} = \begin{cases} -L_1\xi e^{\lambda\xi}, & \xi < \xi_1, \\
A, & \xi \geq \xi_1, \end{cases} \tag{11}
\]
\[
I(\xi) := \max \left\{ [-L_1\xi - L_2(-\xi)^{1/2}]e^{\lambda\xi}, 0 \right\} = \begin{cases} [-L_1\xi - L_2(-\xi)^{1/2}]e^{\lambda\xi}, & \xi < \xi_1, \\
0, & \xi \geq \xi_1, \end{cases} \tag{12}
\]
where $\alpha_1, \theta_1, B, L_2$ are positive constants to be determined later,
\[
\xi_1 := -\frac{1}{\lambda^*}, \quad L_1 = e^\lambda A, \quad \xi_2 := \frac{1}{\alpha_1} \ln(\alpha_1(S_0 - B)), \quad \xi_3 := \frac{1}{\theta_1} \ln(\theta_1 V_0), \quad \xi_4 := -\left(\frac{L_2}{L_1}\right)^2.
\]
(13)

\[S_0 > B\] and $A$ is a positive constant satisfying
\[S_0 f(A) + V_0 g(A) = \mu A.\]
(14)

Then we have the following lemmas.

**Lemma 3.1.** Assume (A1)-(A3). The functions $S(\xi), V(\xi)$ and $T(\xi)$ satisfy the following inequalities:
\[
\begin{align*}
  d_1 S''(\xi) + \Lambda - S(\xi)f(T(\xi)) - (\mu + \gamma)S(\xi) - c^*S'(\xi) &\leq 0, \quad \forall \xi \in \mathbb{R}, \\
  d_2 V''(\xi) + \gamma S(\xi) - V(\xi)g(T(\xi)) - \mu V(\xi) - c^*V'(\xi) &\leq 0, \quad \forall \xi \in \mathbb{R}, \\
  d_3 T''(\xi) + S(\xi)f(T(\xi)) + V(\xi)g(T(\xi)) - \mu T(\xi) - c^*T'(\xi) &\leq 0, \quad \forall \xi \in \mathbb{R} \setminus \{\xi_1\}.
\end{align*}
\]
(15)

**Proof.** From (9) and (10), we have $S(\xi) = S_0$ and $V(\xi) = V_0$ for $\xi \in \mathbb{R}$. Hence the first two inequalities of (15) hold obviously.

To prove the third inequality of (15), we first assume that $\xi > \xi_1$. Due to (11), $T(\xi) = A$. Then the equality (14) implies that
\[d_3 T''(\xi) + S(\xi)f(T(\xi)) + V(\xi)g(T(\xi)) - \mu T(\xi) - c^*T'(\xi) = S_0 f(A) + V_0 g(A) - \mu A \leq 0.
\]

Next, we assume $\xi < \xi_1$. According to (A1), we know that $f(I) \leq f'(0)I$ and $g(I) \leq g'(0)I$. Since $T(\xi) = -L_1 e^\lambda \xi$ for $\xi < \xi_1$, it follows that
\[
\begin{align*}
  d_3 T''(\xi) + S(\xi)f(T(\xi)) + V(\xi)g(T(\xi)) - \mu T(\xi) - c^*T'(\xi) &\leq d_3 T''(\xi) + (S_0 f'(0) + V_0 g'(0))T(\xi) - \mu T(\xi) - c^*T'(\xi) \\
  &\leq d_3(-L_1 e^\lambda \xi) f'(0) + (S_0 f'(0) + V_0 g'(0) - \mu)(-L_1 e^\lambda \xi) - c^*(-L_1 e^\lambda \xi)' \\
  &= (c^* - 2d_3 e^\lambda \xi)L_1 e^\lambda \xi - \Delta(\lambda^*, c^*)L_1 e^\lambda \xi = 0.
\end{align*}
\]

The proof is complete.

**Lemma 3.2.** Assume (A1)-(A3). Let $B < \Lambda/(f'(0)A + \mu + \gamma)$. There exists a sufficiently small $\alpha_1 \in (0, \min\{c^*/d_1, \lambda^*\})$ such that
\[
\begin{align*}
  d_1 S''(\xi) + \Lambda - S(\xi)f(T(\xi)) - (\mu + \gamma)S(\xi) - c^*S'(\xi) &\geq 0, \quad \forall \xi \neq \xi_2.
\end{align*}
\]

**Proof.** By the definition of $S(\xi)$, we may choose $\alpha_1 > 0$ being sufficiently small such that $\xi_2 < \min\{0, \xi_1\}$. If $\xi > \xi_2$, we have $S(\xi) = B$ and $f(T(\xi)) \leq f'(0)T(\xi) \leq f'(0)A$ Then the assumption of $B$ implies that
\[
\begin{align*}
  -d_1 S''(\xi) - \Lambda - S(\xi)f(T(\xi)) + (\mu + \gamma)S(\xi) + c^*S'(\xi) &\leq -\Lambda + B(f'(0)A + \mu + \gamma) \\
  &\leq 0.
\end{align*}
\]

On the other hand, when $\xi < \xi_2$, we have
\[
S(\xi) = S_0 - \frac{1}{\alpha_1} e^{\alpha_1 \xi} \text{ and } T(\xi) = -L_1 e^\lambda \xi.
\]
By elementary computations, we can obtain
\[
d_1S''(\xi) + \Lambda - S(\xi)f(\overline{T}(\xi) s - (\mu + \gamma)S(\xi) - c^*s
\]
\[
\geq d_1S''(\xi) + \Lambda - f'(0)S(\xi)\overline{T}(\xi) - (\mu + \gamma)S(\xi) - c^*sS''(\xi)
\]
\[
= \Lambda - d_1\alpha_1 e^{\alpha_1\xi} - S_0 f'(0) (-L_1 \xi e^{\lambda^*\xi}) + f'(0) \frac{1}{\alpha_1} e^{\alpha_1\xi} (-L_1 \xi e^{\lambda^*\xi}) -
\]
\[
(\mu + \gamma)S_0 + (\mu + \gamma) \frac{1}{\alpha_1} e^{\alpha_1\xi} + c^* e^{\alpha_1\xi}
\]
\[
\geq -d_1\alpha_1 e^{\alpha_1\xi} - S_0 f'(0) (-L_1 \xi e^{\lambda^*\xi}) + c^* e^{\alpha_1\xi}
\]
\[
eq c^* e^{\alpha_1\xi} (c^* - d_1\alpha_1 + L_1 \xi S_0 f'(0) e^{(\lambda^*-\alpha_1)\xi}) \geq 0,
\]
when $c^* > d_1\alpha_1$ and $\alpha_1 \in (0, \lambda^*)$ is sufficiently small. Here we use the fact $\xi e^{(\lambda^*-\alpha_1)\xi} \to 0$ as $\alpha_1 \to 0$ (i.e., $\xi \to -\infty$). The proof is complete.

Lemma 3.3. Assume (A1)-(A3). There exists a sufficiently small $\theta_1 \in (0, \min\{\alpha_1, \lambda^*, c^*/d_2\})$ such that
\[
d_2V''(\xi) + \gamma S(\xi) - V(\xi)g(\overline{T}(\xi)) - \mu V(\xi) - c^*V'(\xi) \geq 0, \text{ for all } \xi \neq \xi_3.
\]
Proof. By the definition of $V(\xi)$, we may choose $\theta_1 > 0$ being sufficiently small such that $\xi_3 < \xi_2$. If $\xi > \xi_3$, we have $V(\xi) = 0$. Hence the inequality of this lemma holds obviously.

Next, we consider $\xi < \xi_3$. Clearly,
\[
S(\xi) = S_0 - \frac{1}{\alpha_1} e^{\alpha_1\xi}, \quad V(\xi) = V_0 - \frac{1}{\theta_1} e^{\theta_1\xi} \quad \text{and} \quad \overline{T}(\xi) = -L_1 \xi e^{\lambda^*\xi}.
\]
By elementary computations, we can derive
\[
d_2V''(\xi) + \gamma S(\xi) - V(\xi)g(\overline{T}(\xi)) - \mu V(\xi) - c^*V'(\xi)
\]
\[
\geq d_2V''(\xi) + \gamma S(\xi) - g'(0)V(\xi)\overline{T}(\xi) - \mu V(\xi) - c^*V'(\xi)
\]
\[
= \gamma S_0 - d_2\theta_1 e^{\theta_1\xi} - \frac{\gamma}{\alpha_1} e^{\alpha_1\xi} - g'(0)V_0 (-L_1 \xi e^{\lambda^*\xi}) + g'(0) \frac{1}{\theta_1} e^{\theta_1\xi} (-L_1 \xi e^{\lambda^*\xi}) -
\]
\[
\mu V_0 + \frac{\mu}{\theta_1} e^{\theta_1\xi} + c^* e^{\theta_1\xi}
\]
\[
\geq -d_2\theta_1 e^{\theta_1\xi} - \frac{\gamma}{\alpha_1} e^{\alpha_1\xi} - g'(0)V_0 (-L_1 \xi e^{\lambda^*\xi}) + c^* e^{\theta_1\xi}
\]
\[
eq e^{\theta_1\xi} (c^* - d_2\theta_1 + L_1 g'(0)V_0 e^{(\lambda^*-\theta_1)\xi} - \frac{\gamma}{\alpha_1} e^{(\alpha_1-\theta_1)\xi} \geq 0,
\]
when $c^* > d_2\theta_1$ and $\theta_1 \in (0, \min\{\alpha_1, \lambda^*\})$ is sufficiently small. Here we use the fact $L_1 g'(0)V_0 e^{(\lambda^*-\theta_1)\xi} - \frac{\gamma}{\alpha_1} e^{(\alpha_1-\theta_1)\xi} \to 0$ as $\theta_1 \to 0$ (i.e., $\xi \to -\infty$).

The proof is complete.

Lemma 3.4. Assume (A1)-(A3). There exists a sufficiently large $L_2 > 0$ such that
\[
d_3L''(\xi) + S(\xi)f(I(\xi)) + V(\xi)g(I(\xi)) - \mu I(\xi) - c^*I'(\xi) \geq 0, \text{ for all } \xi \neq \xi_4.
\]
Proof. By the definition of $I(\xi)$, we may choose $L_2 > 0$ being sufficiently large such that $\xi_4 < \xi_3$. If $\xi > \xi_4$, we have $I(\xi) = 0$. Hence the inequality of this lemma holds obviously.
Next, we consider $\xi < \xi_4$. Clearly, by (9)-(12), we have
\[ \mathcal{S}(\xi) = S_0 - \frac{1}{\alpha_1} e^{\alpha_1 \xi}, \quad V(\xi) = V_0 - \frac{1}{\theta_1} e^{\theta_1 \xi}, \]
\[ \bar{I}(\xi) = [-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} \leq \bar{T}(\xi) = -L_1 \xi e^{\lambda \xi}. \]
Then, by (A3), we can derive
\[ \mathcal{S}(\xi)f(I(\xi)) \geq \mathcal{S}(\xi)J(I(\xi))(f'(0) - k_1 I(\xi)) \]
\[ = f'(0)\mathcal{S}(\xi)J(I(\xi)) - k_1 \mathcal{S}(\xi)J^2(I(\xi)) \geq f'(0)\mathcal{S}(\xi)J(I(\xi)) - k_1 S_0 J^2(\xi) \]
\[ \geq f'(0)(S_0 - \frac{1}{\alpha_1} e^{\alpha_1 \xi})[-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} - k_1 S_0 L_1^2 \xi^2 e^{2\lambda \xi} \]
\[ \geq f'(0)S_0[-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} + f'(0)\frac{1}{\alpha_1} L_1 \xi e^{(\lambda + \alpha_1) \xi} - \]
\[ k_1 S_0 L_1^2 \xi^2 e^{2\lambda \xi}. \]
Similarly, one can also obtain
\[ V(\xi)g(I(\xi)) \geq V(\xi)J(I(\xi))(g'(0) - k_2 J(\xi)) \]
\[ \geq g'(0)V_0[-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} + g'(0)\frac{1}{\theta_1} L_1 \xi e^{(\lambda + \theta_1) \xi} - \]
\[ k_2 V_0 L_1^2 \xi e^{2\lambda \xi}. \]
Let $\Delta_\lambda(\lambda, c)$ be the derivative of $\Delta(\lambda, c)$ with respect to $\lambda$. Noting that $\Delta(\lambda^*, c^*) = 0$. Then it follows that
\[ d_3 L''(\xi) + \mathcal{S}(\xi)f(I(\xi)) + V(\xi)g(I(\xi)) - \mu L(\xi) - c^* I'(\xi) \]
\[ \geq 2 d_3 \lambda^*[-L_1 + \frac{1}{2} L_2 (-\xi)^{-1/2}] e^{\lambda^* \xi} + \frac{1}{4} d_3 L_2 (-\xi)^{-3/2} e^{\lambda^* \xi} + \]
\[ d_3 (\lambda^*)^2[-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda^* \xi} + \]
\[ (S_0 f'(0) + V_0 g'(0))[-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} + \]
\[ f'(0)\frac{1}{\alpha_1} L_1 \xi e^{(\lambda + \alpha_1) \xi} + g'(0)\frac{1}{\theta_1} L_1 \xi e^{(\lambda + \theta_1) \xi} - \]
\[ (k_1 S_0 + k_2 V_0) L_1^2 \xi^2 e^{2\lambda \xi} - c^* \lambda^* [-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} - \]
\[ e^\xi [-L_1 \xi + \frac{1}{2} L_2 (-\xi)^{1/2}] e^{\lambda \xi} - \mu [-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} \]
\[ = [-L_1 \xi - L_2 (-\xi)^{1/2}] e^{\lambda \xi} \Delta(\lambda^*, c^*) + [-L_1 + \frac{1}{2} L_2 (-\xi)^{-1/2}] e^{\lambda \xi} \Delta_\lambda(\lambda^*, c^*) + \]
\[ \frac{1}{4} d_3 L_2 (-\xi)^{-3/2} e^{\lambda \xi} + f'(0)\frac{1}{\alpha_1} L_1 \xi e^{(\lambda + \alpha_1) \xi} + g'(0)\frac{1}{\theta_1} L_1 \xi e^{(\lambda + \theta_1) \xi} - \]
\[ (k_1 S_0 + k_2 V_0) L_1^2 \xi e^{2\lambda \xi} \]
\[ = e^{\lambda \xi} \left[ \frac{1}{4} d_3 L_2 (-\xi)^{-3/2} + f'(0)\frac{1}{\alpha_1} L_1 \xi e^{\alpha_1 \xi} + g'(0)\frac{1}{\theta_1} L_1 \xi e^{\theta_1 \xi} - \right] \]
\[ (k_1 S_0 + k_2 V_0) L_1^2 \xi e^{2\lambda \xi} \]
\[ = e^{\lambda \xi} (-\xi)^{-3/2} \frac{1}{4} d_3 L_2 - f'(0)\frac{L_1}{\alpha_1} (-\xi)^{5/2} e^{\alpha_1 \xi} - g'(0)\frac{L_1}{\theta_1} (-\xi)^{5/2} e^{\theta_1 \xi} - \]
\[ (k_1 S_0 + k_2 V_0) L_1^2 (-\xi)^{7/2} e^{\lambda \xi} \geq 0, \]
when \( L_2 \) is sufficient large (by (13), one has \( \xi \to -\infty \)) and
\[
\lim_{\xi \to -\infty} \left[ f'(0) \frac{L_1}{\alpha_1} (-\xi)^{5/2} e^{\alpha_1 \xi} + g'(0) \frac{L_1}{\beta_1} (-\xi)^{5/2} e^{\alpha_1 \xi} + (k_1 S_0 + k_2 V_0) L_1^2 (-\xi)^{-5/2} e^{\lambda \xi} \right] = 0.
\]
The proof is complete. \( \square \)

Summing up the results of Lemmas 3.1–3.4, the functions \((\mathcal{S}(\xi), \mathcal{V}(\xi), \mathcal{I}(\xi))\) and \((\mathcal{S}(\xi), \mathcal{V}(\xi), \mathcal{I}(\xi))\) constitute a pair of upper- and lower-solutions of system (6), respectively. Here we remark that the construction of upper- and lower-solutions of system (6) is different to those of [11, 15].

4. Proof of Theorem 1.1. Using the upper- and lower-solutions of system (6), and following the method given in [3, 11, 13], we are ready to prove the main result by applying the Schauder’s fixed point theorem.

First, we rewrite system (6) as
\[
\begin{align*}
d_1 S''(\xi) - c^* S'(\xi) - \beta_1 S(\xi) + H_1(S, V, I)(\xi) &= 0, \\
d_2 V''(\xi) - c^* V'(\xi) - \beta_2 V(\xi) + H_2(S, V, I)(\xi) &= 0, \\
d_3 I''(\xi) - c^* I'(\xi) - \beta_3 I(\xi) + H_3(S, V, I)(\xi) &= 0,
\end{align*}
\]
where \( \beta_1 > \mu + \gamma + f(A), \beta_2 > \mu + g(A), \beta_3 > \mu \) and
\[
\begin{align*}
H_1(S, V, I)(\xi) &= \Lambda - S(\xi) f(I(\xi)) + (\beta_1 - (\mu + \gamma)) S(\xi), \\
H_2(S, V, I)(\xi) &= \gamma S(\xi) - V(\xi) g(I(\xi)) + (\beta_2 - \mu) V(\xi), \\
H_3(S, V, I)(\xi) &= S(\xi) f(I(\xi)) + V(\xi) g(I(\xi)) + (\beta_3 - \mu) I(\xi).
\end{align*}
\]
Furthermore, we denote by \( \lambda_{i}^\pm \) \((i = 1, 2, 3)\) the roots of equation \( d_i \lambda^2 - c^* \lambda - \beta_i = 0 \).
It is clear that
\[
\lambda_{i}^\pm = \frac{c^* \pm \sqrt{(c^*)^2 + 4d_i \beta_i}}{2d_i}, \quad i = 1, 2, 3.
\]
Then we choose \( 0 < \mu < \min\{-\lambda_{1}^-, -\lambda_{2}^-, -\lambda_{3}^-\} \) and define the function space \( \Gamma \) by
\[
\Gamma := \{(S(\cdot), V(\cdot), I(\cdot)) \in B_\mu(\mathbb{R}, \mathbb{R}^3) | \mathcal{S}(\xi) \leq S(\xi) \leq \overline{\mathcal{S}}(\xi), \mathcal{V}(\xi) \leq V(\xi) \leq \overline{\mathcal{V}}(\xi), \mathcal{I}(\xi) \leq I(\xi) \leq \overline{\mathcal{I}}(\xi)\},
\]
where \( B_\mu(\mathbb{R}, \mathbb{R}^3) \) is a functions space given by
\[
B_\mu(\mathbb{R}, \mathbb{R}^3) := \{ \Phi = (\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot)) \in C(\mathbb{R}, \mathbb{R}^3) | \sup_{\xi \in \mathbb{R}} |\phi_i(\xi)| e^{-\mu |\xi|} < \infty, \text{ for } i = 1, 2, 3 \}.
\]
In addition, we equip the space \( B_\mu(\mathbb{R}, \mathbb{R}^3) \) with the norm
\[
\| \Phi \|_\mu := \max \{ \sup_{\xi \in \mathbb{R}} |\phi_i(\xi)| e^{-\mu |\xi|} | i = 1, 2, 3 \}.
\]
Then one can find that \( \Gamma \) is nonempty, closed and convex and \( B_\mu(\mathbb{R}, \mathbb{R}^3) \) is a Banach space with the norm \( \| \cdot \|_\mu \). According to (16), we define the operator
\[
F(S, V, I)(\xi) = (F_1(S, V, I)(\xi), F_2(S, V, I)(\xi), F_3(S, V, I)(\xi)) : \Gamma \to C(\mathbb{R}, \mathbb{R}^3)
\]
by

\[ F_i(S, V, I)(\xi) := \frac{1}{\Lambda_i} \int_{-\infty}^{\xi} e^{\lambda_i^+(\xi-\eta)} H_i(S, V, I)(\eta) d\eta + \frac{1}{\Lambda_i} \int_{\xi}^{\infty} e^{\lambda_i^+(\xi-\eta)} H_i(S, V, I)(\eta) d\eta, \]

for \( i = 1, 2, 3 \), where

\[ \Lambda_i := d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{(c^*)^2 + 4d_i\beta_i}, \ i = 1, 2, 3. \]

According to the variation of constant formula of ODE system, any solution of system (16) is a fixed point of the operator \( F(S, V, I)(\xi) \), and vice versa.

In order to find a fixed point of \( F(S, V, I)(\xi) \) by using the Schauder’s fixed point theorem, we first prove that \( F(S, V, I)(\xi) \) is a continuous and compact operator from \( \Gamma \) to \( \Gamma \).

**Lemma 4.1.** The operator \( F(S, V, I)(\xi) \) maps \( \Gamma \) to \( \Gamma \).

**Proof.** Let \((S(\cdot), V(\cdot), I(\cdot)) \in \Gamma \) for all \( \xi \in \mathbb{R} \). For convenience, we denote

\[ D_i[\phi](\xi) := -d_i\phi''(\xi) + c^*\phi'(\xi) + \beta_i\phi(\xi), \ for \ i = 1, 2, 3, \]

for any twice differentiable function \( \phi(\xi) \). By definition of \( F_1(S, V, I)(\xi) \), we have

\[ F_1(S, V, I)(\xi) = \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+(\xi-\eta)} H_1(S, V, I)(\eta) d\eta + \frac{1}{\Lambda_1} \int_{\xi}^{\infty} e^{\lambda_1^+(\xi-\eta)} H_1(S, V, I)(\eta) d\eta \]

\[ \leq \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+(\xi-\eta)} (A + (\beta_1 - (\mu + \gamma))S_0) d\eta + \frac{1}{\Lambda_1} \int_{\xi}^{\infty} e^{\lambda_1^+(\xi-\eta)} (A + (\beta_1 - (\mu + \gamma))S_0) d\eta \]

\[ = \frac{\beta_1 S_0}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+(\xi-\eta)} d\eta + \frac{\beta_1 S_0}{\Lambda_1} \int_{\xi}^{\infty} e^{\lambda_1^+(\xi-\eta)} d\eta = S_0 = \mathcal{S}(\xi). \]

In addition, from (16), we can obtain

\[ F_1(S, V, I, \mathcal{T})(\xi) = \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+(\xi-\eta)} H_1(S, V, I, \mathcal{T})(\eta) d\eta + \frac{1}{\Lambda_1} \int_{\xi}^{\infty} e^{\lambda_1^+(\xi-\eta)} H_1(S, V, I, \mathcal{T})(\eta) d\eta \]

\[ \geq \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+(\xi-\eta)} D_1[S](\eta) d\eta + \frac{1}{\Lambda_1} \int_{\xi}^{\infty} e^{\lambda_1^+(\xi-\eta)} D_1[S](\eta) d\eta. \quad (17) \]
By (17) and elementary computations, for \( \xi < \xi_2 \), we have

\[
F_1(S, V, \bar{T})(\xi) \geq \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+ (\xi - \eta)} D_1[S](\eta) d\eta + \frac{1}{\Lambda_1} \int_{\xi}^{\xi_2} e^{\lambda_1^+ (\xi - \eta)} D_1[S](\eta) d\eta \\
\geq \frac{1}{\Lambda_1} \int_{-\infty}^{\xi} e^{\lambda_1^+ (\xi - \eta)} (\beta_1 S_0 + \sigma_1 (d_1 \alpha_1^2 - c^* \alpha_1 - \beta_1) e^{\alpha_1^+ \eta}) d\eta + \\
\frac{1}{\Lambda_1} \int_{\xi}^{\xi_2} e^{\lambda_1^+ (\xi - \eta)} (\beta_1 S_0 + \sigma_1 (d_1 \alpha_1^2 - c^* \alpha_1 - \beta_1) e^{\alpha_1^+ \eta}) d\eta \\
= S_0 - \frac{1}{\alpha_1} e^{\alpha_1 \xi} + \frac{S_0 d_1 \alpha_1}{\Lambda_1} e^{\lambda_1^+ (\xi - \xi_2)} \geq S_0 - \frac{1}{\alpha_1} e^{\alpha_1 \xi} = \bar{S}(\xi).
\]

When \( \xi > \xi_2 \), we have \( \bar{S}(\xi) = B \). Then (17) gives \( F_1(S, V, \bar{T})(\xi) \geq B = \bar{S}(\xi) \) and

\[
\bar{S}(\xi) \leq F_1(S, V, I)(\xi) \leq \bar{S}(\xi), \text{ for all } \xi. \tag{18}
\]

Now we consider \( F_2(S, V, I)(\xi) \). It is easy to see that

\[
F_2(S, V, I)(\xi) = \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} H_2[S, V, I](\eta) d\eta + \\
\frac{1}{\Lambda_2} \int_{\xi}^{\infty} e^{\lambda_2^+ (\xi - \eta)} H_2[S, V, I](\eta) d\eta \\
\leq \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} D_2[V](\eta) d\eta + \frac{1}{\Lambda_2} \int_{\xi}^{\infty} e^{\lambda_2^+ (\xi - \eta)} D_2[V](\eta) d\eta \\
= \frac{\beta_2 V_0}{\Lambda_2} \left( \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} d\eta + \int_{\xi}^{\infty} e^{\lambda_2^+ (\xi - \eta)} d\eta \right) = V_0 = V(\xi).
\]

If \( \xi > \xi_3 \), we have \( V(\xi) = 0 \), which implies

\[
F_2(S, V, I)(\xi) \geq \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} D_2[V](\eta) d\eta + \frac{1}{\Lambda_2} \int_{\xi}^{\infty} e^{\lambda_2^+ (\xi - \eta)} D_2[V](\eta) d\eta \\
= 0 = V(\xi).
\]

When \( \xi < \xi_3 \), elementary computations give

\[
F_2(S, V, \bar{T})(\xi) = \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} H_2(S, V, \bar{T})(\eta) d\eta + \\
\frac{1}{\Lambda_2} \int_{\xi}^{\infty} e^{\lambda_2^+ (\xi - \eta)} H_2(S, V, \bar{T})(\eta) d\eta \\
\geq \frac{1}{\Lambda_2} \int_{-\infty}^{\xi} e^{\lambda_2^+ (\xi - \eta)} (\beta_2 V_0 + \frac{1}{\theta_1} (d_2 \theta_1^2 - c^* \theta_1 - \beta_2) e^{\theta_1 \eta}) d\eta + \\
\frac{1}{\Lambda_2} \int_{\xi}^{\xi_3} e^{\lambda_2^+ (\xi - \eta)} (\beta_2 V_0 + \frac{1}{\theta_1} (d_2 \theta_1^2 - c^* \theta_1 - \beta_2) e^{\theta_1 \eta}) d\eta
\]
If $\xi < \xi_1$, then elementary computations give

$$\frac{\beta_2 V_0}{\Lambda_2} e^{\lambda_2^+ \xi} \int_{-\infty}^{\xi} e^{-\lambda_2^- \eta} d\eta + \frac{1}{\theta_1 \Lambda_2} (d_2 \theta_1^2 - c^* \theta_1 - \beta_2) e^{\lambda_2^+ \xi} \int_{-\infty}^{\xi} e^{(\theta_1 - \lambda_2^-) \eta} d\eta + \frac{\beta_2 V_0}{\Lambda_2} e^{\lambda_2^+ \xi} \int_{\xi}^{\xi_1} e^{-\lambda_2^- \eta} d\eta + \frac{1}{\theta_1 \Lambda_2} (d_2 \theta_1^2 - c^* \theta_1 - \beta_2) e^{\lambda_2^+ \xi} \int_{\xi}^{\xi_1} e^{(\theta_1 - \lambda_2^-) \eta} d\eta + V_0 - \frac{1}{\theta_1} e^{\theta_1 \xi} + \frac{V_0 d_2 \theta_1}{\Lambda_2} e^{\lambda_2^+ (\xi - \xi_1)} \geq V_0 - \frac{1}{\theta_1} e^{\theta_1 \xi} = V(\xi).$$

One can get that

$$\tilde{V}(\xi) \leq F_2(S, V, I)(\xi) \leq \tilde{V}(\xi), \text{ for all } \xi. \quad (19)$$

Similarly, we consider $F_3(S, V, I)(\xi)$. When $\xi > \xi_1$, $\tilde{T}(\xi) = A$. Then it follows that

$$F_3(\mathcal{S}, \mathcal{V}, \mathcal{T})(\xi) = \frac{1}{\Lambda_3} \int_{-\infty}^{\xi} e^{\lambda_3^+ (\xi - \eta)} H_3(\mathcal{S}, \mathcal{V}, \mathcal{T})(\eta) d\eta + \frac{1}{\Lambda_3} \int_{\xi}^{\infty} e^{\lambda_3^+ (\xi - \eta)} H_3(\mathcal{S}, \mathcal{V}, \mathcal{T})(\eta) d\eta \leq \frac{1}{\Lambda_3} \int_{-\infty}^{\xi} e^{\lambda_3^+ (\xi - \eta)} D_3[\mathcal{T}](\eta) d\eta + \frac{1}{\Lambda_3} \int_{\xi}^{\infty} e^{\lambda_3^+ (\xi - \eta)} D_3[\mathcal{T}](\eta) d\eta = \frac{\beta_3 A}{\Lambda_3} \left( \int_{-\infty}^{\xi} e^{\lambda_3^- (\xi - \eta)} d\eta + \int_{\xi}^{\infty} e^{\lambda_3^- (\xi - \eta)} d\eta \right) = A = \tilde{T}(\xi).$$

If $\xi < \xi_1$, $\tilde{T}(\xi) = -L_1 \xi e^{\lambda_3^+ \xi}$. Then elementary computations give

$$F_3(\mathcal{S}, \mathcal{V}, \mathcal{T})(\xi) \leq \frac{1}{\Lambda_3} \int_{-\infty}^{\xi} e^{\lambda_3^+ (\xi - \eta)} (d_3 (\lambda^*)^2 - c^* \lambda^* - \beta_3) L_1 \eta e^{\lambda_3^- \eta} d\eta + \frac{1}{\Lambda_3} \int_{\xi}^{\xi_1} e^{\lambda_3^+ (\xi - \eta)} (d_3 (\lambda^*)^2 - c^* \lambda^* - \beta_3) L_1 \eta e^{\lambda_3^- \eta} d\eta + \frac{1}{\Lambda_3} \int_{\xi}^{\xi_1} e^{\lambda_3^+ (\xi - \eta)} \beta_3 A d\eta = \frac{1}{\Lambda_3} L_1 (d_3 (\lambda^*)^2 - c^* \lambda^* - \beta_3) e^{\lambda_3^- \xi} \int_{-\infty}^{\xi} \eta e^{(\lambda^* - \lambda_3^-) \eta} d\eta + \frac{1}{\Lambda_3} L_1 (d_3 (\lambda^*)^2 - c^* \lambda^* - \beta_3) e^{\lambda_3^+ \xi} \int_{\xi}^{\xi_1} \eta e^{(\lambda^* - \lambda_3^+) \eta} d\eta + \frac{\beta_3 A}{\Lambda_3} e^{\lambda_3^+ \xi} \int_{\xi_1}^{\infty} e^{-\lambda_3^- \eta} d\eta = -L_1 \xi e^{\lambda_3^+ \xi} = \tilde{T}(\xi).$$
On the other hand, for $\xi > \xi_4$, $\overline{I}(\xi) = 0$. Thus,

$$
F_3(S, V, I)(\xi) = \frac{1}{A_3} \int_{-\infty}^{\xi} e^{\lambda^*_\xi(\xi-\eta)}H_3(S, V, I)(\eta)d\eta
$$

$$
+ \frac{1}{A_3} \int_{\xi}^{\infty} e^{\lambda^*_\xi(\xi-\eta)}H_3(S, V, I)(\eta)d\eta
$$

$$
\geq \frac{1}{A_3} \int_{-\infty}^{\xi} e^{\lambda^*_\xi(\xi-\eta)}D_3[I](\eta)d\eta + \frac{1}{A_3} \int_{\xi}^{\infty} e^{\lambda^*_\xi(\xi-\eta)}D_3[I](\eta)d\eta = 0 = I(\xi).
$$

For $\xi < \xi_4$, $\overline{I}(\xi) = [L_1\xi - L_2(-\xi)^{1/2}]e^{\lambda^*_\xi}$. By elementary computations, one has

$$
F_3(S, V, I)(\xi) \geq \frac{1}{A_3} \int_{-\infty}^{\xi} e^{\lambda^*_\xi(\xi-\eta)}D_3[I](\eta)d\eta + \frac{1}{A_3} \int_{\xi}^{\infty} e^{\lambda^*_\xi(\xi-\eta)}D_3[I](\eta)d\eta
$$

$$
= \frac{d_3(\lambda^*)^2 - c^*\lambda^* - \beta_3}{A_3} \int_{-\infty}^{\xi} e^{\lambda^*_\xi(\xi-\eta)}(L_1\eta)e^{\lambda^*_\eta} + L_2(-\eta)^{1/2}e^{\lambda^*_\eta})d\eta -
$$

$$
\frac{1}{4A_3} d_3 \int_{-\infty}^{\xi} e^{\lambda^*_\xi(\xi-\eta)}L_2(-\eta)^{-3/2}e^{\lambda^*_\eta}d\eta -
$$

$$
\frac{1}{4A_3} d_3 \int_{\xi}^{\infty} e^{\lambda^*_\xi(\xi-\eta)}L_2(-\eta)^{-3/2}e^{\lambda^*_\eta}d\eta +
$$

$$
\frac{d_3(\lambda^*)^2 - c^*\lambda^* - \beta_3}{A_3} \int_{\xi}^{\infty} e^{\lambda^*_\xi(\xi-\eta)}(L_1\eta)e^{\lambda^*_\eta} + L_2(-\eta)^{1/2}e^{\lambda^*_\eta})d\eta
$$

$$
= [-L_1\xi - L_2(-\xi)^{1/2}]e^{\lambda^*_\xi} + \frac{L_1d_3}{2A_3} e^{\lambda^*_\xi + \lambda^*_\eta(\xi-\xi_4)} +
$$

$$
\geq [-L_1\xi - L_2(-\xi)^{1/2}]e^{\lambda^*_\xi} = \overline{I}(\xi).
$$

We conclude that

$$
\overline{I}(\xi) \leq F_3(S, V, I)(\xi) \leq \overline{I}(\xi), \quad \text{for all } \xi.
$$

According to (18)–(20), we conclude that $F(S, V, I)(\xi)$ maps $\Gamma$ to $\Gamma$. This completes the proof. 

Next, we show that $F(S, V, I)(\xi)$ is a continuous operator in $\Gamma$.

**Lemma 4.2.** The operator $F(S, V, I)(\xi)$ is continuous with respect to $\| \cdot \|_{\mu}$ in $B_{\mu}(\mathbb{R}, \mathbb{R}^3)$.

**Proof.** Let $\Phi_1(\xi) = (S_1(\xi), V_1(\xi), I_1(\xi))$, $\Phi_2(\xi) = (S_2(\xi), V_2(\xi), I_2(\xi)) \in \Gamma$. By (A1), we have

$$
| - S_1(\xi)f(I_1(\xi)) + S_2(\xi)f(I_2(\xi)) | \leq | S_1(\xi)f(I_1(\xi)) - f(I_2(\xi)) | +
$$

$$
| f(I_2(\xi))(S_1(\xi) - S_2(\xi)) |
$$

$$
\leq S_0f(0)|I_1(\xi) - I_2(\xi)| + f(A)|S_1(\xi) - S_2(\xi)|
$$

and

$$
| V_1(\xi)g(I_1(\xi)) - V_2(\xi)g(I_2(\xi)) | \leq | V_1(\xi)g(I_1(\xi)) - g(I_2(\xi)) | +
$$

$$
| g(I_2(\xi))(V_1(\xi) - V_2(\xi)) |
$$

$$
\leq V_0g(0)|I_1(\xi) - I_2(\xi)| + g(A)|V_1(\xi) - V_2(\xi)|.
$$
Then it follows that

\[
|F_1(\Phi_1)(\xi) - F_1(\Phi_2)(\xi)|e^{-\mu|\xi|} \\
\leq \frac{e^{-\mu|\xi|}}{A_1} \int_{-\infty}^{\xi} e^{\lambda_1^{\top} (t-\eta)} |H_1(\Phi_1)(\eta) - H_1(\Phi_2)(\eta)| d\eta + \\
\frac{e^{-\mu|\xi|}}{A_1} \int_{\xi}^{\infty} e^{\lambda_1^{\top} (t-\eta)} |H_2(\Phi_1)(\eta) - H_2(\Phi_2)(\eta)| d\eta \\
= \frac{e^{-\mu|\xi|}}{A_1} \int_{-\infty}^{\xi} e^{\lambda_1^{\top} (t-\eta)} |S_1(\eta) f(I_1(\eta)) + S_2(\eta) f(I_2(\eta))| \\
+ (\beta_1 - (\mu + \gamma)) |S_1(\eta) - S_2(\eta)| d\eta + \\
\frac{e^{-\mu|\xi|}}{A_1} \int_{\xi}^{\infty} e^{\lambda_1^{\top} (t-\eta)} |S_1(\eta) f(I_1(\eta)) + S_2(\eta) f(I_2(\eta))| \\
+ (\beta_1 - (\mu + \gamma)) |S_1(\eta) - S_2(\eta)| d\eta \\
\leq \frac{e^{-\mu|\xi|}}{A_1} \int_{-\infty}^{\xi} e^{\mu |\eta|} e^{\lambda_1^{\top} (t-\eta)} ((\beta_1 + f(A) - (\mu + \gamma)) |S_1(\eta) - S_2(\eta)| e^{-\mu |\eta|} + \\
S_0 f'(0) |I_1(\eta) - I_2(\eta)| e^{-\mu |\eta|} d\eta + \\
\frac{e^{-\mu|\xi|}}{A_1} \int_{\xi}^{\infty} e^{\mu |\eta|} e^{\lambda_1^{\top} (t-\eta)} ((\beta_1 + f(A) - (\mu + \gamma)) |S_1(\eta) - S_2(\eta)| e^{-\mu |\eta|} + \\
S_0 f'(0) |I_1(\eta) - I_2(\eta)| e^{-\mu |\eta|} d\eta \\
\leq \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_2 + \mu} \right) \sup_{\xi \in \mathbb{R}} |S_1 - S_2| e^{-\mu |\xi|} + \\
\frac{S_0 f'(0)}{A_1} \sup_{\xi \in \mathbb{R}} |I_1 - I_2| e^{-\mu |\xi|} \\
\leq K_1 ||\Phi_1(\xi) - \Phi_2(\xi)||_\mu, \quad (21)
\]

for some $K_1 > 0$. Furthermore, we also obtain

\[
|F_2(\Phi_1)(\xi) - F_2(\Phi_2)(\xi)|e^{-\mu|\xi|} \\
\leq \frac{e^{-\mu|\xi|}}{A_2} \int_{-\infty}^{\xi} e^{\lambda_2^{\top} (t-\eta)} |H_2(\Phi_1)(\eta) - H_2(\Phi_2)(\eta)| d\eta + \\
\frac{e^{-\mu|\xi|}}{A_2} \int_{\xi}^{\infty} e^{\lambda_2^{\top} (t-\eta)} |H_1(\Phi_1)(\eta) - H_1(\Phi_2)(\eta)| d\eta \\
= \frac{e^{-\mu|\xi|}}{A_2} \int_{-\infty}^{\xi} e^{\lambda_2^{\top} (t-\eta)} |\gamma S_1(\eta) - \gamma S_2(\eta) - V_1(\eta) g(I_1(\eta)) + V_2(\eta) g(I_2(\eta))| \\
+ (\beta_2 - \mu) |V_1(\eta) - V_2(\eta)| d\eta + \\
\frac{e^{-\mu|\xi|}}{A_2} \int_{\xi}^{\infty} e^{\lambda_2^{\top} (t-\eta)} |\gamma S_1(\eta) - \gamma S_2(\eta) - V_1(\eta) g(I_1(\eta)) + V_2(\eta) g(I_2(\eta))| \\
+ (\beta_2 - \mu) |V_1(\eta) - V_2(\eta)| d\eta \\
\leq \frac{e^{-\mu|\xi|}}{A_2} \int_{-\infty}^{\xi} e^{\mu |\eta|} e^{\lambda_2^{\top} (t-\eta)} (|\gamma |S_1(\eta) - S_2(\eta)| e^{-\mu |\eta|} + V_0 g'(0) |I_1(\eta) - I_2(\eta)| e^{-\mu |\eta|} + \\
(\beta_2 + g(A) - \mu) |V_1(\eta) - V_2(\eta)| e^{-\mu |\eta|} d\eta + \\
\frac{e^{-\mu|\xi|}}{A_2} \int_{\xi}^{\infty} e^{\mu |\eta|} e^{\lambda_2^{\top} (t-\eta)} (|\gamma |S_1(\eta) - S_2(\eta)| e^{-\mu |\eta|} + V_0 g'(0) |I_1(\eta) - I_2(\eta)| e^{-\mu |\eta|} + \\
(\beta_2 + g(A) - \mu) |V_1(\eta) - V_2(\eta)| e^{-\mu |\eta|} d\eta \\
\leq \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_2 + \mu} \right) \sup_{\xi \in \mathbb{R}} |S_1 - S_2| e^{-\mu |\xi|} + \frac{V_0 g'(0)}{A_2} \sup_{\xi \in \mathbb{R}} |I_1 - I_2| e^{-\mu |\xi|} \\
+ \frac{\beta_2 + g(A) - \mu}{A_2} \sup_{\xi \in \mathbb{R}} |V_1 - V_2| e^{-\mu |\xi|} \\
\leq K_2 ||\Phi_1(\xi) - \Phi_2(\xi)||_\mu, \quad (22)
\]

for some $K_2 > 0$. 
Similarly, we can obtain
\[
|F_3(\Phi_1)(\xi) - F_3(\Phi_2)(\xi)| e^{-\mu|\xi|} \\
\leq \frac{e^{-\mu|\xi|}}{\lambda_3} \int_\xi^\infty e^{\lambda_3^+ (\xi - \eta)} \left| H_3(S_1, V_1, I_1)(\eta) - H_3(S_2, V_2, I_2)(\eta) \right| d\eta + \\
\frac{e^{-\mu|\xi|}}{\lambda_3} \int_\xi^\infty e^{\lambda_3^+ (\xi - \eta)} \left| H_3(S_1, V_1, I_1)(\eta) - H_3(S_2, V_2, I_2)(\eta) \right| d\eta
\]
for some \( K_3 > 0 \).
Combining the inequalities (21)–(23), we know the operator \( F(\cdot) \) is continuous with respect to the norm \( |\cdot|_\mu \) in \( B_\mu(\mathbb{R}, \mathbb{R}^3) \). The proof is complete. \( \square \)

Furthermore, we show that \( F(S, V, I)(\xi) \) is a compact operator in \( \Gamma \).

**Lemma 4.3.** The operator \( F(S, V, I)(\xi) \) is compact with respect to \( \| \cdot \|_\mu \) in \( B_\mu(\mathbb{R}, \mathbb{R}^3) \).

**Proof.** For any \( (S(\xi), V(\xi), I(\xi)) \in \Gamma \), we have
\[
|F'_1(S, V, I)(\xi)| = \left| \frac{\lambda_1^+}{\Lambda_1} \int_\xi^\infty e^{\lambda_1^+ (\xi - \eta)} H_1(S, V, I)(\eta) d\eta + \int_\xi^\infty e^{\lambda_1^+ (\xi - \eta)} H_1(S, V, I)(\eta) d\eta \right|
\leq - \frac{\Lambda + (\beta_1 - (\mu + \gamma)) S_0}{\Lambda_1} \left( \frac{\lambda_1^+}{\Lambda_1} \int_\xi^\infty e^{\lambda_1^+ (\xi - \eta)} d\eta - \lambda_1^+ \int_\xi^\infty e^{\lambda_1^+ (\xi - \eta)} d\eta \right)
= \frac{2(\Lambda + (\beta_1 - (\mu + \gamma)) S_0)}{\Lambda_1}, \quad (24)
\]
\[
|F'_2(S, V, I)(\xi)| = \left| \frac{\lambda_2^+}{\Lambda_2} \int_\xi^\infty e^{\lambda_2^+ (\xi - \eta)} H_2(S, V, I)(\eta) d\eta + \int_\xi^\infty e^{\lambda_2^+ (\xi - \eta)} H_2(S, V, I)(\eta) d\eta \right|
\leq - \frac{\gamma S_0 + (\beta_2 - \mu) V_0}{\Lambda_2} \left( \frac{\lambda_2^+}{\Lambda_2} \int_\xi^\infty e^{\lambda_2^+ (\xi - \eta)} d\eta - \lambda_2^+ \int_\xi^\infty e^{\lambda_2^+ (\xi - \eta)} d\eta \right)
= \frac{2(\gamma S_0 + (\beta_2 - \mu) V_0)}{\Lambda_2}, \quad (25)
\]
Let us define the Lyapunov function

$$L(\xi) := e^{\xi} L_1(\xi) + d_1 S'(\xi) \left( \frac{S^*}{S(\xi)} - 1 \right) + d_2 V'(\xi) \left( \frac{V^*}{V(\xi)} - 1 \right) + d_3 I'(\xi) \left( \frac{I^*}{I(\xi)} - 1 \right).$$

where

$$L_1(\xi) := S(\xi) - S^* - S^* \ln \frac{S(\xi)}{S^*} + V(\xi) - V^* - V^* \ln \frac{V(\xi)}{V^*} + I(\xi) - I^* - I^* \ln \frac{I(\xi)}{I^*}.$$ 

It is clear that $L$ is well defined. Notice that

$$\Lambda = S^* f(I^* + (\mu + \gamma)S^*) + \gamma S^* V^* + S^* f(I^*) + V^* g(I^*) = \mu I^*.$$
Then, elementary computations give that
\[
\frac{dL}{d\xi} = c^* \frac{dL_1}{d\xi} + d_1 S''(\xi) \left( \frac{S^*}{S(\xi)} - 1 \right) + d_1 S'(\xi) \left( - \frac{S^* S'(\xi)}{S^2(\xi)} \right) + \\
d_2 V''(\xi) \left( \frac{V^*}{V(\xi)} - 1 \right) + d_2 V'(\xi) \left( - \frac{V^* V'(\xi)}{V^2(\xi)} \right) + \\
d_3 I''(\xi) \left( \frac{I^*}{I(\xi)} - 1 \right) + d_1 I'(\xi) \left( - \frac{I^* I'(\xi)}{I^2(\xi)} \right)
\]
\[
= - \mu S^* \chi \left( \frac{S(\xi)}{S'} \right) - (\mu + \gamma) S^* \chi \left( \frac{S^*}{S(\xi)} \right) - \mu V^* \chi \left( \frac{V(\xi)}{V^*} \right) - \gamma S^* \chi \left( \frac{S(\xi)}{S'} \right) - \\
S^* f(I^*) \left( \chi \left( \frac{S^*}{S(\xi)} \right) + \chi \left( \frac{I^* S(\xi)}{I(\xi)} \right) \right) - V^* g(I^*) \chi \left( \frac{I^* V(\xi)}{I(\xi)} \right) - \\
S^* f(I^*) \left( \frac{f(I(\xi))}{f(I^*)} - \frac{I(\xi)}{I^*} + \ln \frac{f(I(\xi))}{f(I^*)} \right) + V^* g(I^*) \left( \frac{g(I(\xi))}{g(I^*)} - \frac{I(\xi)}{I^*} + \\
\ln \frac{g(I(\xi))}{g(I^*)} \right) - d_1 S^* \left( \frac{S'(\xi)}{S(\xi)} \right)^2 - d_2 V^* \left( \frac{V'(\xi)}{V^*} \right)^2 - d_3 I^* \left( \frac{I'(\xi)}{I(\xi)} \right)^2,
\] (27)
where \( \chi(x) = x - 1 - \ln x \).

Moreover, by (A1) and the inequality \( \ln x \leq x - 1 \), we can obtain
\[
\frac{f(I(\xi))}{f(I^*)} - \frac{I(\xi)}{I^*} + \ln \left( \frac{f(I(\xi))}{f(I^*)} \right) \leq \left( \frac{f(I(\xi))}{f(I^*)} - \frac{I(\xi)}{I^*} \right) \left( 1 - \frac{f(I^*)}{f(I(\xi))} \right) \leq 0,
\] (28)
\[
\frac{g(I(\xi))}{g(I^*)} - \frac{I(\xi)}{I^*} + \ln \left( \frac{g(I(\xi))}{g(I^*)} \right) \leq \left( \frac{g(I(\xi))}{g(I^*)} - \frac{I(\xi)}{I^*} \right) \left( 1 - \frac{g(I^*)}{g(I(\xi))} \right) \leq 0.
\] (29)

Then it follows from (27)–(29) that \( \frac{dL}{d\xi} \leq 0 \) and \( \frac{dL}{d\xi} = 0 \) if and only if
\[
S(\xi) = S^*, \ V(\xi) = V^*, \ I(\xi) = I^*, \ S'(\xi) = 0, \ V'(\xi) = 0, \ I'(\xi) = 0.
\]

Therefore, we can conclude that \( (S(\xi), V(\xi), I(\xi)) \to E^* \) as \( \xi \to +\infty \). The proof of the main theorem is complete.

**Remark 1.** (1) Applying Fourier transform, Taylor’s formula and taking the first order approximation (cf. [5]), one can see that system (3) is a special case as the model (2). Therefore, the existence and non-existence results of non-critical traveling wave solutions for system (2) also hold for system (3).

(2) Following the similar arguments given in previous sections, we can extend our main result to the non-local system (2) but replace the nonlocal term \([J_{p_1} \ast I - I]\) for the I-th equation by \( \Delta I \), i.e.,
\[
\begin{align*}
S_t &= d_1 [J_{p_1} \ast S - S] + \Lambda - f(I)S - (\mu + \gamma)S, \\
V_t &= d_2 [J_{p_2} \ast V - V] + \gamma S - g(I)V - \mu V, \\
I_t &= d_3 \Delta I + f(I)S + g(I)V - \mu I.
\end{align*}
\] (30)

In fact, from the construction of the upper- and lower-solutions, we know that equation (7) plays a crucial role. This gives the reason why we replace the nonlocal term \([J_{p_3} \ast I - I]\) for the I-th equation by \( \Delta I \) in (30). However, due to the nonlocal term \([J_{p_3} \ast I - I]\) of system (2), the construction of upper- and lower-solutions of (2) when \( c = c^* \) becomes more difficult. The existence of critical traveling wave solutions of system (2) is still open.
Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and suggestions which have led to an improvement of the presentation.

REFERENCES

[1] Y.-S. Chen and J.-S. Guo, Traveling wave solutions for a three-species predator-prey model with two aborigine preys, Japan J. Indust. Appl. Math., (2020).

[2] A. Ducrot, J.-S. Guo, G. Lin and S. X. Pan, The spreading speed and the minimal wave speed of a predator-prey system with nonlocal dispersal, Z. Angew. Math. Phys., 70 (2019), 25 pp.

[3] S.-C. Fu, Traveling waves for a diffusive SIR model with delay, J. Math. Anal. Appl., 435 (2016), 20–37.

[4] J.-S. Guo, K. I. Nakamura, T. Ogiwara and C.-C. Wu, Traveling wave solutions for a predator-prey system with two predators and one prey, Nonlinear Anal. RWA, 54 (2020), 103111, 13pp.

[5] L. I. Ignat and J. D. Rossi, A nonlocal convection-diffusion equation, J. Funct. Anal., 251 (2007), 399–437.

[6] Y. Li, W.-T. Li and G. Lin, Traveling waves of a delayed diffusive SIR epidemic model, Commun. Pure Appl. Anal., 14 (2015), 1001–1022.

[7] J. D. Wei, J. B. Zhou, Z. L. Zhen and L. X. Tian, Super-critical and critical traveling waves in a two-component lattice dynamical model with discrete delay, Appl. Math. Comput., 363 (2019), 124621.

[8] J. D. Wei, J. B. Zhou, Z. L. Zhen and L. Tian, Super-critical and critical traveling waves in a three-component delayed disease system with mixed diffusion, J. Comput. Appl. Math., 367 (2020), 112451, 15pp.

[9] J. D. Wei, J. B. Zhou, W. X. Chen, Z. L. Zhen and L. X. Tian Traveling waves in a nonlocal dispersal epidemic model with spatio-temporal delay, Commun. Pure. Appl. Anal., 19 (2020), 2853–2886.

[10] C. F. Wu, Y. Yang, Q. Y. Zhao, Y. L. Tian and Z. T. Xu, Epidemic waves of a spatial SIR model in combination with random dispersal and non-local dispersal, Appl. Math. Comput., 313 (2017), 122–143.

[11] Z. T. Xu, Y. Q. Xu and Y. H. Huang, Stability and traveling waves of a vaccination model with nonlinear incidence, Comput. Math. Appl., 75 (2018), 561–581.

[12] L. Zhao and Z.-C. Wang, Traveling wave fronts in a diffusive epidemic model with multiple parallel infectious stages, IMA J. Appl. Math., 81 (2016), 795–823.

[13] J. B. Zhou, L. Y. Song, J. D. Wei and H. M. Xu, Critical traveling waves in a diffusive disease model, J. Math. Anal. Appl., 476 (2019), 522–538.

[14] J. B. Zhou, L. Y. Song and J. D. Wei, Mixed types of waves in a discrete diffusive epidemic model with nonlinear incidence and time delay, J. Differ. Equ., 268 (2020), 4491–4524.

[15] J. L. Zhou, Y. Yang and C.-H. Hsu, Traveling waves for a nonlocal dispersal vaccination model with general incidence, Discrete Contin. Dyn. Syst. Ser. B, 25 (2020), 1469–1495.

Received July 2020; revised January 2021.

E-mail address: yangyu@lixin.edu
E-mail address: jzhou@amss.ac.cn
E-mail address: chhsu@math.ncu.edu.tw