REPRESENTATIONS OF BRAID GROUPS VIA CONJUGATION ACTIONS ON CONGRUENCE SUBGROUPS

KEVIN P. KNUDSON

Abstract. We construct two families of representations of the braid group $B_n$ by considering conjugation actions on congruence subgroups of $GL_{n-1}(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}])$. Many of these representations are shown to be faithful.

1. Introduction

Denote by $B_n$ the braid group on $n$ strings. The purpose of this note is to construct two families of representations

$\rho_n(\alpha) : B_n \rightarrow SL_{n(n-2)}(\mathbb{C})$

and

$\mu_n(\alpha, \beta) : B_n \rightarrow SL_{N}(\mathbb{C}), \ N = \binom{n}{2} - 1,$

where $\alpha$ and $\beta$ are nonzero complex numbers. If $\alpha, \beta$ lie in some subfield $F$ of $\mathbb{C}$, then the representations are defined over $F$ (in fact, over $\mathbb{Z}[\alpha^{\pm 1}, \beta^{\pm 1}]$).

The starting point for $\rho_n(\alpha)$ is the reduced Burau representation $\beta_n : B_n \rightarrow GL_{n-1}(\mathbb{Z}[t, t^{-1}])$. For each $i \geq 1$, set

$K^i(\alpha) = \{ A \in SL_{n-1}(\mathbb{C}[t, t^{-1}]) : A \equiv I \mod (t - \alpha)^i \}.$

The sequence $\{K^i(\alpha)\}_{i \geq 1}$ is a central series in $K(\alpha) = K^1(\alpha)$ (i.e., $K^{i+j}(\alpha) \supseteq [K^i(\alpha), K^j(\alpha)]$). Moreover, the graded quotients satisfy

$K^i(\alpha)/K^{i+1}(\alpha) \cong sl_{n-1}(\mathbb{C}).$

The conjugation action of $GL_{n-1}(\mathbb{C}[t, t^{-1}])$ on $K(\alpha)$ induces a homomorphism

$f_n(\alpha) : GL_{n-1}(\mathbb{C}[t, t^{-1}]) \rightarrow Aut(K(\alpha)/K^2(\alpha)) \cong GL_{n(n-2)}(\mathbb{C})$

(note that $sl_{n-1}(\mathbb{C})$ is a vector space of dimension $(n-1)^2 - 1 = n(n-2)$).

We define

$\rho_n(\alpha) = f_n(\alpha) \circ \beta_n.$

The kernel of $\rho_n(1)$ is easily described: it is the subgroup $P_n$ of pure braids. The map $\rho_n(1)$ is thus a representation of the symmetric group $\Sigma_n.$

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Theorem 2.2. Suppose \( n \geq 4 \). Denote by \( V \) the standard \((n-1)\)-dimensional representation of \( \Sigma_n \) and by \( W \) the representation corresponding to the partition \( n-2, 2 \). Then
\[
\rho_n(1) \cong V \oplus \bigwedge^2 V \oplus W.
\]
Also, \( \rho_3(1) \cong V \oplus \bigwedge^2 V \).

When \( \alpha \neq 1 \), however, the image of \( \rho_n(\alpha) \) is infinite. Denote by \( \Gamma_{n(n-2)}(\alpha) \) the subgroup of \( SL_{n(n-2)}(\mathbb{Z}[\alpha^{\pm 1}]) \) consisting of matrices congruent to the identity modulo \((\alpha - 1)\). Then we have the following result.

Proposition 2.1. The image of \( P_n \) under \( \rho_n(\alpha) \) lies in \( \Gamma_{n(n-2)}(\alpha) \).

For the representations \( \mu_n(\alpha, \beta) \), we begin with the Lawrence–Krammer–Bigelow representation
\[
\kappa_n : B_n \to GL_{(n/2)}(\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]).
\]
For each \( i \geq 1 \), define a subgroup \( L^i(\alpha, \beta) \) by
\[
L^i(\alpha, \beta) = \{ A \in SL_{(n/2)}(\mathbb{C}[t^{\pm 1}, q^{\pm 1}]) : A \equiv I \mod (t - \alpha, q - \beta)^i \}.
\]
The first graded quotient satisfies
\[
L(\alpha, \beta)/L^2(\alpha, \beta) \cong \mathfrak{sl}_{(n/2)}(\mathbb{C}) \times \mathfrak{sl}_{(n/2)}(\mathbb{C}).
\]
The conjugation action of \( GL_{(n/2)}(\mathbb{C}[t^{\pm 1}, q^{\pm 1}]) \) on \( L(\alpha, \beta)/L^2(\alpha, \beta) \) is diagonal; that is, if \((v, w) \in L(\alpha, \beta)/L^2(\alpha, \beta)\) then a matrix \( A \) acts as
\[
A : (v, w) \mapsto (AvA^{-1}, AwA^{-1}).
\]
Let \( N = (\frac{n}{2})^2 - 1 \) and let \( g_n(\alpha, \beta) \) be the homomorphism
\[
g_n(\alpha, \beta) : GL_{(n/2)}(\mathbb{C}[t^{\pm 1}, q^{\pm 1}]) \to GL_N(\mathbb{C})
\]

obtained by restricting the conjugation action to the first factor of \( \mathfrak{sl}_{(n/2)}(\mathbb{C}) \). Define \( \mu_n(\alpha, \beta) \) to be the composite
\[
\mu_n(\alpha, \beta) = g_n(\alpha, \beta) \circ \kappa_n.
\]

The maps \( \mu_n(\alpha, \beta) \) are more complicated than the \( \rho_n(\alpha) \) mostly because the Burau matrices \( \beta_n(\sigma_i) \) of the braid generators are block diagonal, while the matrices \( \kappa_n(\sigma_i) \) are not. We content ourselves to analyze a few special cases. For example, the kernel of \( \mu_n(1, 1) \) is \( P_n \) and so \( \mu_n(1, 1) \) is a representation of \( \Sigma_n \). We give the decomposition of \( \mu_n(1, 1) \) for \( n = 3, 4, 5 \).

In Section 4 we discuss the faithfulness of the maps \( \rho_n(\alpha) \) and \( \mu_n(\alpha, \beta) \). Of course, none of them is faithful since the center of \( B_n \) lies in the kernel of each. Moreover, since \( \beta_n \) is not faithful for \( n \geq 5 \), there must be additional elements in the kernel of \( \rho_n(\alpha) \). However, the map \( \kappa_n \) is faithful for all \( n \), and this allows us to deduce the following result.

Theorem 4.4. If \( \alpha \) and \( \beta \) are algebraically independent, then the kernel of the representation \( \mu_n(\alpha, \beta) \) is precisely the center of \( B_n \).
In the particular case of $B_4$, it is known \[4\] that $\beta_4$ is faithful if and only if the matrices $\beta_4(\sigma_3\sigma_1^{-1})$ and $\beta_4(\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})$ generate a free group of rank $2$. We have the following result, which motivated the study of the $\rho_n(\alpha)$ in the first place.

**Theorem 4.1.** There is a positive integer $M$ such that for any $m \geq M$, the matrices $\beta_4(\sigma_3\sigma_1^{-1})^m$ and $\beta_4(\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})^m$ generate a free group of rank $2$.

This was proved first by S. Moran \[8\], who showed also that one can take $M = 3$. It was our hope that passing to the map $\rho_4(\alpha)$ would allow us to show that $M = 1$ is possible. We show in Section 4 that the method of proof fails for $M = 1, 2$.

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2. **The representations $\rho_n(\alpha)$**

Recall the reduced Burau representation $\beta_n : B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$ defined as follows. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the standard generators of $B_n$ and set

$$
\beta_n(\sigma_1) = \begin{bmatrix}
-t & 1 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}, \quad \beta_n(\sigma_r) = \begin{bmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & 0 \\
& & t & -t \\
& & 0 & 1
\end{bmatrix}
$$

where the row containing $t - t 1$ in the matrix for $\sigma_r$ is the $r$th row, $1 < r < n$. The map $\beta_n$ is not faithful for $n \geq 5$ \[2\]. Let $\alpha \in \mathbb{C}^\times$ and for each $i \geq 1$, define

$$K^i(\alpha) = \{ A \in SL_{n-1}(\mathbb{C}[t, t^{-1}]) : A \equiv I \mod (t - \alpha)^i \}.$$  

One checks easily that $K^\bullet(\alpha)$ is a descending central series in $K(\alpha) = K^1(\alpha)$. If $A \in K^i(\alpha)$, we may write $A = I + (t - \alpha)^i Y \mod (t - \alpha)^{i+1}$, where $Y$ is a matrix with entries in $\mathbb{C}$. Since $\det A = 1$, we have $\text{trace}(Y) = 0$. Define a map

$$\pi : K^i(\alpha) \longrightarrow sl_{n-1}(\mathbb{C})$$

by $\pi(A) = Y$. This is easily seen to be a surjective homomorphism and the kernel of $\pi$ is the subgroup $K^{i+1}(\alpha)$. Thus, $\pi$ induces an isomorphism

$$K^i(\alpha)/K^{i+1}(\alpha) \cong sl_{n-1}(\mathbb{C}).$$
Now let \( i = 1 \). Consider the following matrices in \( K(\alpha) \):

\[
A_{ij} = I + (t - \alpha)e_{ij}, \quad i \neq j, \quad 1 \leq i, j \leq n - 1
\]

\[
B_{ii} = I + (t - \alpha)e_{ii} - (t - \alpha)e_{i+1,i+1} - (t - \alpha)e_{i,i+1} + (t - \alpha)e_{i+1,i},
\]

\[
1 \leq i \leq n - 2
\]

where \( e_{ij} \) is the matrix having 1 in the \( i, j \) position and zeros elsewhere. As a basis of \( K(\alpha)/K^2(\alpha) \), we choose the matrices \( A_{ij}, 1 \leq i, j \leq n - 1 \) and \( A_{ii} = B_{ii}A_{i+1,i+1}^{-1}A_{i+1,i}, 1 \leq i \leq n - 2 \). Note that under the map \( \pi : K(\alpha) \rightarrow \mathfrak{sl}_{n-1}(\mathbb{C}) \), we have

\[
\pi(A_{ij}) = e_{ij}, \quad \text{and} \quad \pi(A_{ii}) = e_{ii} - e_{i+1,i+1}.
\]

The group \( GL_{n-1}(\mathbb{C}[t, t^{-1}]) \) acts on \( K(\alpha) \) via conjugation and hence acts on the quotient \( K(\alpha)/K^2(\alpha) \). Denote by \( f_n(\alpha) \) the map

\[
f_n(\alpha) : GL_{n-1}(\mathbb{C}[t, t^{-1}]) \rightarrow \text{Aut}(K(\alpha)/K^2(\alpha)) \cong GL_{n(n-2)}(\mathbb{C})
\]

(note that \( \dim \mathfrak{sl}_{n-1}(\mathbb{C}) = (n - 1)^2 - 1 = n(n - 2) \)). Restricting this action to the \( \beta_n(\sigma_i) \) gives a map

\[
\rho_n(\alpha) : B_n \rightarrow GL_{n(n-2)}(\mathbb{C})
\]

(that is, \( \rho_n(\alpha) = f_n(\alpha) \circ \beta_n \)). The action of each \( \sigma_i \) on \( K(\alpha)/K^2(\alpha) \) is given by the following formulæ:

\[
\sigma_1 : \quad A_{ij} \mapsto A_{ij} \quad 3 \leq i \leq n - 1
\]

\[
A_{ii} \mapsto A_{ii} \quad 3 \leq i \leq n - 2
\]

\[
A_{1j} \mapsto -\alpha A_{1j} \quad 3 \leq j \leq n - 1
\]

\[
A_{2j} \mapsto A_{1j} + A_{2j} \quad 3 \leq j \leq n - 1
\]

\[
A_{11} \mapsto -\frac{1}{\alpha}A_{11} + \frac{1}{\alpha}A_{22} \quad 3 \leq i \leq n - 1
\]

\[
A_{12} \mapsto -\alpha A_{12}
\]

\[
A_{21} \mapsto \frac{1}{\alpha}A_{12} - \frac{1}{\alpha}A_{21} - \frac{1}{\alpha}A_{11}
\]

\[
A_{11} \mapsto A_{11} - 2A_{12}
\]

\[
A_{22} \mapsto A_{12} + A_{22}
\]
and for $2 \leq k \leq n - 2$ and $i, j \neq k - 1, k, k + 1$

\[
\sigma_k : \begin{align*}
A_{ij} &\mapsto A_{ij} \\
A_{ii} &\mapsto A_{ii} \\
A_{i,k-1} &\mapsto A_{i,k-1} \\
A_{jk} &\mapsto A_{i,k-1} - \frac{1}{\alpha} A_{ik} + \frac{1}{\alpha} A_{i,k+1} \\
A_{i,k+1} &\mapsto A_{i,k+1} \\
A_{k-1,j} &\mapsto A_{k-1,j} + \alpha A_{kj} \\
A_{k,j} &\mapsto -\alpha A_{kj} \\
A_{k+1,j} &\mapsto A_{k+1,j} + A_{k+1,k-1} \\
A_{k-1,k} &\mapsto A_{k-1,k-1} + -\frac{1}{\alpha} A_{k-1,k} + \frac{1}{\alpha} A_{k,k+1} + \frac{1}{\alpha} A_{k,k-1} + A_{k+1} \quad (k \geq 3) \\
A_{k,k+1} &\mapsto A_{k,k+1} - \alpha A_{k,k-1} + 2A_{k,k+1} \\
A_{k+1,k+1} &\mapsto A_{k+1,k+1} + \alpha A_{k,k+1} \\
A_{k+1,k} &\mapsto A_{k+1,k} + A_{k+1,k-1} \\
A_{k-1,k} &\mapsto A_{k-1,k-1} + -\frac{1}{\alpha} A_{k-1,k} + \frac{1}{\alpha} A_{k,k+1} + \frac{1}{\alpha} A_{k,k-1} + A_{k+1,k} \quad (k \leq n - 3)
\end{align*}
\]

and finally

\[
\sigma_{n-1} : \begin{align*}
A_{ij} &\mapsto A_{ij} \\
A_{ii} &\mapsto A_{ii} \\
A_{n-2,j} &\mapsto A_{n-2,j} + \alpha A_{n-1,j} \\
A_{n-1,j} &\mapsto -\alpha A_{n-1,j} \\
A_{i,n-1} &\mapsto A_{i,n} - \frac{1}{\alpha} A_{i,n-1} \\
A_{n-1,n-1} &\mapsto A_{n-1,n-1} - \frac{1}{\alpha} A_{n-1,n-2} + A_{n-1,n-2} + A_{n-2,n-2} \\
A_{n-3,n-3} &\mapsto A_{n-3,n-3} - \alpha A_{n-1,n-2} \\
A_{n-2,n-2} &\mapsto 2\alpha A_{n-1,n-2} + A_{n-2,n-2}
\end{align*}
\]

Consider the particular case $n = 3$. We have $\dim \rho_3(\alpha) = 3$. The 3-dimensional representations of $B_3$ were classified by Tuba and Wenzl [10]—they are characterized uniquely by the eigenvalues of the matrices for $\sigma_1$ and $\sigma_2$. Using the basis $e_1 = -A_{12}, e_2 = A_{11}, e_3 = -\alpha A_{21}$, the matrices of $\rho_3(\alpha)(\sigma_1)$ and $\rho_3(\alpha)(\sigma_2)$ are

\[
\rho_3(\alpha)(\sigma_1) = \begin{bmatrix}
-\alpha & 2 & 1 \\
0 & 1 & 1 \\
0 & 0 & -\frac{1}{\alpha}
\end{bmatrix}, \quad \rho_3(\alpha)(\sigma_2) = \begin{bmatrix}
-\frac{1}{\alpha} & 0 & 0 \\
-1 & 1 & 0 \\
1 & -2 & -\alpha
\end{bmatrix}.
\]

These correspond to the matrices in [10] with $\lambda_1 = -\alpha, \lambda_2 = 1$ and $\lambda_3 = -\frac{1}{\alpha}$. We have thus found a naturally occurring one-parameter family of the abstract representations defined in [10].
Observe that the map \( \rho_n(\alpha) \) is defined over the ring \( \mathbb{Z}[\alpha, \alpha^{-1}] \). Denote by \( \Gamma_{n(n-2)}(\alpha) \) the subgroup of \( GL_{n(n-2)}(\mathbb{Z}[\alpha, \alpha^{-1}]) \) consisting of matrices congruent to \( I \) modulo \( \alpha - 1 \). Let \( P_n \) be the subgroup of pure braids.

**Proposition 2.1.** The image of \( P_n \) under \( \rho_n(\alpha) \) lies in \( \Gamma_{n(n-2)}(\alpha) \).

**Proof.** The group \( P_n \) is generated by the \( \sigma^2_i \) along with some conjugates of these. Since the \( \sigma_i \) are all conjugate in \( B_n \), so are the \( \sigma^2_i \). Thus it suffices to show that \( \rho_n(\alpha)(\sigma^2_i) \in \Gamma_{n(n-2)}(\alpha) \). But this is easy:

\[
\begin{align*}
\sigma^2_1 : & \quad A_{ij} \mapsto A_{ij} \quad 3 \leq i, j \leq n - 1 \\
& \quad A_{ii} \mapsto A_{ii} \quad 3 \leq i \leq n - 2 \\
& \quad A_{1,j} \mapsto (1 + 2(\alpha - 1) + (\alpha - 1)^2)A_{1,j} \quad 2 \leq j \leq n - 1 \\
& \quad A_{2,j} \mapsto (1 - \alpha)A_{1,j} + A_{2,j} \quad 3 \leq j \leq n - 1 \\
& \quad A_{i,1} \mapsto (1 + \frac{1-\alpha}{\alpha} + \frac{1-\alpha}{\alpha^2})A_{i,1} + \frac{\alpha - 1}{\alpha^2}A_{i,2} \quad 3 \leq i \leq n - 1 \\
& \quad A_{i,2} \mapsto A_{i,2} \quad 3 \leq i \leq n - 1 \\
& \quad A_{21} \mapsto -\frac{1-\alpha}{\alpha}A_{12} + (1 + \frac{1-\alpha}{\alpha} + \frac{1-\alpha}{\alpha^2})A_{21} + \frac{1-\alpha}{\alpha^2}A_{11} \quad 11 \mapsto A_{11} - 2(1 - \alpha)A_{12} \\
& \quad A_{12} \mapsto (1 - \alpha)A_{12} + A_{22} \\
\end{align*}
\]

In particular, if \( \alpha = 1 \), then \( P_n \subseteq \ker \rho_n(1) \). (This also follows from the fact that \( \beta_n(P_n) \subseteq K(1) \) and hence the \( \beta_n(\sigma^2_i) \) act trivially on \( K(1)/K^2(1) \).) The reverse inclusion also holds since the action of any \( \rho_n(1)(\sigma_i) \) is the same as the action of \( \beta_n(\sigma_i) \) evaluated at \( t = 1 \), and this collection of matrices is a faithful representation of the symmetric group \( \Sigma_n \). For example, when \( n = 3 \) we obtain (using the basis \( e_1 = A_{12}, e_2 = A_{21}, e_3 = A_{11} - A_{12} + A_{21} \))

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0 \\
\end{bmatrix}
\]

The character of this is

\[
\chi_{\rho_3}(1) = \begin{bmatrix}
1 & (12) & (123) \\
3 & -1 & 0 \\
\end{bmatrix}
\]

If \( V \) denotes the standard 2-dimensional representation of \( \Sigma_3 \), then \( \chi_{\rho_3(1)} = \chi_V + \chi_{V^2} \). Thus, \( \rho_3(1) \cong V \oplus V^2 \).

For \( n = 4 \) we have

\[
\begin{bmatrix}
1 & (12) & (123) & (1234) & (12)(34) \\
8 & 0 & -1 & 0 & 0 \\
\end{bmatrix}
\]

Denote by \( W \) the representation corresponding to the partition 2,2. Then an easy check shows that \( \chi_{\rho_4} = \chi_V + \chi_{V^2} + \chi_W \).

The character of \( \rho_5(1) \) is

\[
\begin{bmatrix}
1 & (12) & (123) & (1234) & (12345) & (12)(34) & (12)(34)(345) \\
15 & 3 & 0 & -1 & 0 & -1 & 0 \\
\end{bmatrix}
\]
Theorem 2.2. Suppose $n \geq 4$. Denote by $V$ the standard $(n-1)$-dimensional representation of $\Sigma_n$ and by $W$ the representation corresponding to the partition $n-2, 2$. Then
\[ \rho_n(1) \cong V \oplus \bigwedge^2 V \oplus W. \]

Proof. First note that $\dim V = n - 1$, $\dim \bigwedge^2 V = \frac{(n-1)(n-2)}{2}$ and $\dim W = \frac{n(n-3)}{2}$ (5, p. 50). Thus,
\[ (n - 1) + \frac{(n-1)(n-2)}{2} + \frac{n(n-3)}{2} = n(n-2) \]
so that $\dim \rho_n(1) = \dim(V \oplus \bigwedge^2 V \oplus W)$. Now if $C_{\hat{x}}$ denotes the conjugacy class in $\Sigma_n$ consisting of cycles with $i_1$ 1-cycles, $i_2$ 2-cycles, etc., then
\[ (\chi_V + \chi_{\bigwedge^2 V} + \chi_W)(C_{\hat{x}}) = i_1(i_1 - 2) \]
(this follows from [5], 4.15, p. 51). Direct calculation shows that
\[ \chi_{\rho_n(1)}((12)) = (n - 2)(n - 4) = i_1(i_1 - 2) \]
\[ \chi_{\rho_n(1)}((123)) = (n - 3)(n - 5) = i_1(i_1 - 2) \]
\[ \chi_{\rho_n(1)}((12)(n-1,n)) = (n - 4)(n - 6) = i_1(i_1 - 2), \]
etc. Thus, $\chi_{\rho_n(1)} = \chi_V + \chi_{\bigwedge^2 V} + \chi_W$ and since $V, \bigwedge^2 V,$ and $W$ are distinct irreducible representations, the result follows. \qed

3. The representations $\mu_n(\alpha, \beta)$

Let $R = \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ be the ring of Laurent polynomials in $t, q$ and let $A$ be the free $R$-module
\[ A = \bigoplus_{1 \leq i < j \leq n} R_{x_{ij}}. \]
The Lawrence–Krammer–Bigelow representation
\[ \kappa_n : B_n \rightarrow GL(n, \mathbb{C}) \]
is defined by
\[ \kappa_n(\sigma_k) : x_{k,k+1} \mapsto tq^2x_{k,k+1} \]
\[ x_{ik} \mapsto (1 - q)x_{ik} + qx_{i,k+1} \quad i < k \]
\[ x_{i,k+1} \mapsto x_{ik} + tq^{k-i+1}(q-1)x_{k,k+1} \quad i < k \]
\[ x_{kj} \mapsto tq(q-1)x_{k,k+1} + qx_{k+1,j} \quad k + 1 < j \]
\[ x_{k+1,j} \mapsto x_{kj} + (1-q)x_{k+1,j} \quad k + 1 < j \]
\[ x_{ij} \mapsto x_{ij} \quad i < j < k \text{ or } k + 1 < i < j \]
\[ x_{ij} \mapsto x_{ij} + tq^{k-i}(q-1)^2x_{k,k+1} \quad i < k < k + 1 < j. \]
The map $\kappa_n$ is faithful for all $n \geq 4$. This is the only known faithful representation of $B_n$, $n \geq 4$.

For each $i \geq 1$ define, for $\alpha, \beta \in \mathbb{C}^\times$,
\[ L^i(\alpha, \beta) = \{ A \in SL(n, \mathbb{C}) \mid A \equiv I \mod (t - \alpha, q - \beta)^i \} \]
where \( A_1, A_q \) are \( \binom{n}{2} \times \binom{n}{2} \) matrices over \( \mathbb{C} \) and \( X \equiv 0 \mod (t - \alpha, q - \beta)^2 \). Again, the condition \( \det A = 1 \) forces \( \operatorname{tr}(A_t) = 0 = \operatorname{tr}(A_q) \). Define a map

\[
\pi : L(\alpha, \beta) \rightarrow \mathfrak{sl}(\binom{n}{2})(\mathbb{C}) \times \mathfrak{sl}(\binom{n}{2})(\mathbb{C})
\]

by \( \pi(A) = (A_t, A_q) \). This is a surjective group homomorphism with kernel \( L^2(\alpha, \beta) \).

If \( Z \in GL(\binom{n}{2})(\mathbb{C}[t^{\pm 1}, q^{\pm 1}]) \), we may write \( Z = Z_0Z_1 \), where \( Z_0 \in GL(\binom{n}{2})(\mathbb{C}) \) and \( Z_1 \in L(\alpha, \beta) \). Then \( Z_1 \) acts trivially on \( L(\alpha, \beta)/L^2(\alpha, \beta) \) and so \( Z \) acts on \( L(\alpha, \beta)/L^2(\alpha, \beta) \) by

\[
Z : (A_t, A_q) \mapsto (Z_0A_tZ_0^{-1}, Z_0A_qZ_0^{-1});
\]

that is, the action is diagonal. Consider the action on the first factor (call it \( L \)):

\[
g_n(\alpha, \beta) : GL(\binom{n}{2})(\mathbb{C}[t^{\pm 1}, q^{\pm 1}]) \rightarrow \operatorname{Aut}(L) \cong GL_N(\mathbb{C})
\]

where \( N = \binom{n}{2} - 1 \). Restricting this to the image of \( B_n \) under \( \kappa_n \) yields a map

\[
\mu_n(\alpha, \beta) : B_n \rightarrow GL_N(\mathbb{C})
\]

(i.e., \( \mu_n(\alpha, \beta) = g_n(\alpha, \beta) \circ \kappa_n \)).

We shall not write down a formula for the \( \mu_n(\alpha, \beta) \) in general. Indeed, the following formula for \( \mu_3(\alpha, \beta) \) shows that the general case is hopelessly complicated. Using the basis described in (1) and (2) below, the matrix of \( \mu_3(\alpha, \beta)(\sigma_1) \) is

\[
\begin{bmatrix}
\alpha \beta (\beta - 1) & \alpha \beta^2 - \frac{\alpha \beta}{2} (\beta - 1)^3 & \alpha \beta (\beta - 1) & 0 & 0 & 0 & -2 \alpha \beta (\beta - 1)^2 & \alpha (\beta - 1)^2 \\
\alpha \beta & 0 & -\frac{\alpha \beta}{2} (\beta - 1)^2 & 0 & 0 & 0 & -2 \alpha (\beta - 1) & \alpha (\beta - 1) \\
0 & 0 & 0 & 0 & \frac{\alpha \beta}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha \beta}{2} & 0 & 0 & 0 \\
0 & 0 & -\frac{\alpha \beta}{2} (\beta - 1)^2 & \beta & \frac{\beta (\beta - 1)^3}{\beta^2} & -\frac{\beta - 1}{\beta^2}; & 1 - \beta & 2(\beta - 1) \\
0 & 0 & 1 - \frac{1}{\beta} & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 - \frac{1}{\beta} & 0 & \frac{1}{\beta} & 0 & 1 - \beta & 1 - 1 \\
\end{bmatrix}
\]

and that of \( \mu_3(\alpha, \beta)(\sigma_2) \) is

\[
\begin{bmatrix}
-\frac{\beta - 1}{\beta^2} & \frac{\beta - 1}{\beta^3} & \frac{1}{\beta^2} & -\frac{(\beta - 1)^2}{\beta^2} & 0 & 0 & -2 \beta (\beta - 1) & \beta - 1 \\
0 & -\beta - \beta (\beta - 1) & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta & -\beta (\beta - 1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\beta - 1}{\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\beta (\beta - 1)^3}{\beta^2} & 0 & \frac{\beta (\beta - 1)^3}{\beta^2} & 0 & \alpha \beta (\beta - 1) & -\alpha \beta (\beta - 1)^2 & 2 \alpha \beta^2 (\beta - 1) \\
0 & 0 & -\alpha \beta (\beta - 1) & 0 & -\alpha \beta (\beta - 1)^2 & 0 & \alpha \beta (\beta - 1) & -\alpha \beta (\beta - 1)^2 & 2 \alpha \beta (\beta - 1)^2 \\
1 - \beta & (\beta - 1)^2 & 0 & 0 & 1 - \beta & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 - \beta & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Note however that $\mu_n(\alpha, \beta)$ is defined over $\mathbb{Z}[\alpha^{\pm 1}, \beta^{\pm 1}]$ and we have the following result. Denote by $\Gamma_N(\alpha, \beta)$ the subgroup of $GL_N(\mathbb{Z}[\alpha^{\pm 1}, \beta^{\pm 1}])$ consisting of those matrices that are congruent to $I$ modulo $(\alpha - 1, \beta - 1)$.

**Proposition 3.1.** The image of $P_n$ under $\mu_n(\alpha, \beta)$ lies in $\Gamma_N(\alpha, \beta)$.

**Proof.** It suffices to check this for the $\mu_n(\alpha, \beta)(\sigma_k^2)$. First note the following formula for $\kappa_n(\sigma_k^2)$:

$$
\begin{align*}
&x_{k,k+1} \mapsto t^2q^4x_{k,k+1} \\
x_{ik} \mapsto (1 + (q - 1) + (q - 1)^2)x_{ik} + (1 - q)qx_{i,k+1} + tq^{k-i+1}(q - 1)x_{k,k+1} \\
x_{i,k+1} \mapsto (1 - q)x_{ik} + (1 + (q - 1))x_{i,k+1} + t^2q^{k-i+3}(q - 1)x_{k,k+1} \\
x_{kj} \mapsto t^2q^2(q - 1)x_{k,k+1} + (1 + (q - 1))x_{kj} + q(q - 1)x_{k+1,j} \\
x_{k+1,j} \mapsto tq(q - 1)x_{k,k+1} + (1 + (q - 1) + (q - 1)^2)x_{k+1,j} + (1 - q)x_{kj} \\
x_{ij} \mapsto x_{ij} + tq^{k-i}(q - 1)^2(1 + tq^2)x_{k,k+1}
\end{align*}
$$

where the ranges on $i, j$ are $(i < k), (i < k), (k + 1 < j), (k + 1 < j), (i < j < k) or (k + 1 < i < j)$, and $(i < k < k + 1 < j)$, respectively. Note that $t^2q^4 \equiv 1 + 2(t - 1) + 4(q - 1) \mod (t - 1, q - 1)^2$. Thus, after evaluating $\kappa_n(\sigma_k^2)$ at $t = \alpha, q = \beta$, we may write

$$
\kappa_n(\sigma_k^2) = X_kY_k,
$$

where $X_k \in L(\alpha, \beta)$ and $Y_k \equiv I \mod (\alpha - 1, \beta - 1)$. The action of $\kappa_n(\sigma_k^2)$ on $L$ is via conjugation by $Y_k$. Write

$$
Y_k = I + (\alpha - 1)Y_\alpha + (\beta - 1)Y_\beta + Z
$$

where $Z \equiv 0 \mod (\alpha - 1, \beta - 1)^2$. Then if $A = I + (t - \alpha)A_t + X \ (X \equiv 0 \mod (t - \alpha, q - \beta)^2)$, we have

$$
\begin{align*}
Y_kAY_k^{-1} & = (I + (\alpha - 1)Y_\alpha + (\beta - 1)Y_\beta + Z)(I + (t - \alpha)A_t + X)(I - (\alpha - 1)Y_\alpha - (\beta - 1)Y_\beta + U) \\
& \equiv I + (t - \alpha)A_t + (\alpha - 1)(t - \alpha)Y_\alpha + (\beta - 1)(t - \alpha)Y_\beta + U \mod (t - \alpha, q - \beta)^2 \\
& \equiv A \mod (\alpha - 1, \beta - 1).
\end{align*}
$$

Thus, $\kappa_n(\sigma_k^2)$ acts as the identity on $L$ modulo $(\alpha - 1, \beta - 1)$; that is, $\mu_n(\alpha, \beta)(\sigma_k^2) \in \Gamma_N(\alpha, \beta)$.

**Corollary 3.2.** The kernel of $\mu_n(1, 1)$ is the subgroup $P_n$ and so $\mu_n(1, 1)$ is a representation of $\Sigma_n$.
\( t = 1, q = 1 \), one obtains the permutation matrix

\[
\tau_k : \begin{align*}
x_{k,k+1} & \mapsto x_{k,k+1} \\
x_{ik} & \mapsto x_{i,k} & i < k \\
x_{i,k+1} & \mapsto x_{ik} & i < k \\
x_{kj} & \mapsto x_{k+1,j} & k + 1 < j \\
x_{k+1,j} & \mapsto x_{kj} & k + 1 < j \\
x_{ij} & \mapsto x_{ij} & i < j < k, k + 1 < i < j, i < k < k + 1 < j.
\end{align*}
\]

Order the basis of \( A = \bigoplus_{1 \leq i < j \leq n} Rx_{ij} \) as

\[
e_1 = x_{12}, e_2 = x_{13}, \ldots, e_{n-1} = x_{1n}, e_n = x_{23}, \ldots, e_{n/2} = x_{n-1,n}
\]

and consider \( \tau_k \) as a permutation of the set \( \{1, 2, \ldots, \binom{n}{2}\} \). The action of \( \mu_n(1,1) \) may then be described as

\[
\mu_n(1,1)(\sigma_k) : \begin{cases}
A_{ij} & \mapsto A_{\tau_k(i),\tau_k(j)} \\
A_{ii} & \mapsto \begin{cases}
\sum_{\ell = \tau_k(i)}^{\tau_k(i+1)-1} A_{\ell\ell} & \tau_k(i) < \tau_k(i+1) \\
- \sum_{\ell = \tau_k(i+1)}^{\tau_k(i)-1} A_{\ell\ell} & \tau_k(i+1) < \tau_k(i).
\end{cases}
\end{cases}
\]

The characters of \( \mu_n(1,1) \) for \( n = 3, 4, 5 \) are as follows:

\[
\begin{array}{c|ccc}
\chi_{\mu_3(1,1)} & (1) & (12) & (123) \\
\hline
8 & 3 & 0 & -1 \\
\chi_{\mu_4(1,1)} & (1) & (12) & (123) & (124) & (12)(34) \\
\hline
35 & 3 & -1 & 3 & -1 \\
\chi_{\mu_5(1,1)} & (1) & (12) & (123) & (1234) & (12345) & (12)(34) & (12)(345) \\
\hline
99 & 15 & 0 & -1 & -1 & 3 & 0 \\
\end{array}
\]

A straightforward calculation then shows the following.

\[
\begin{align*}
\mu_3(1,1) & \cong (\text{alt}) \oplus (\text{triv}) \oplus V^{\oplus 3} \\
\mu_4(1,1) & \cong (\text{alt}) \oplus (\text{triv})^{\oplus 2} \oplus (\bigwedge^2 V)^{\oplus 3} \oplus W^{\oplus 4} \oplus V^{\oplus 5} \\
\mu_5(1,1) & \cong (V \otimes (\text{alt})) \oplus (\text{triv})^{\oplus 2} \oplus (W \otimes (\text{alt}))^{\oplus 3} \oplus (\bigwedge^2 V)^{\oplus 4} \oplus W^{\oplus 6} \oplus V^{\oplus 6}
\end{align*}
\]

(recall that \( V \) is the standard \((n-1)\)-dimensional representation and \( W \) is the representation corresponding to the partition \( n-2, 2 \)).

Note that the \( A_{ii}, 1 \leq i \leq \binom{n}{2} - 1 \) form a \( \Sigma_n \)-submodule of \( L \).

**Proposition 3.3.** The submodule \( U \) spanned by the \( A_{ii}, 1 \leq i \leq \binom{n}{2} - 1 \), is isomorphic to \( V \oplus W \).
Proof. Recall that \( W \) corresponds to the partition \( n-2, 2 \); we have \( \dim W = \frac{n(n-3)}{2} \). Then

\[
\dim V + \dim W = (n - 1) + \frac{n(n-3)}{2} = \frac{n^2 - n - 2}{2} = \binom{n}{2} - 1 = \dim U.
\]

Note that \( \tau_1 = (2, n)(3, n+1) \cdots (n-1, 2n-3) \) and so

\[
\chi_U(12) = \text{tr}(\tau_1) = \sum_{j=2n-3}^{\binom{n}{2}-1} 1 = \left( \binom{n}{2} - 1 \right) - (2n - 3) + 1 = (n - 3) + \frac{1}{2}(n - 3)(n - 4) = (\chi_V + \chi_W)(12).
\]

The values of \( \chi_U \) for other conjugacy classes are obtained similarly. \( \square \)

4. Faithfulness

It is clear that none of the representations \( \rho_n(\alpha), \mu_n(\alpha, \beta) \) is faithful. Indeed, the center of \( B_n \) lies in the kernel of each \( \rho_n(\alpha) \) and \( \mu_n(\alpha, \beta) \).

Is there more in the kernel? The answer is certainly yes for \( \rho_n(\alpha), n \geq 5 \) since \( \beta_n \) is not faithful.

4.1. Associated graded algebras. One approach is to study the map on associated graded algebras. Let \( \alpha = -1 \). Then the image of \( P_n \) under \( \rho_n(-1) \) lies in the subgroup

\[
\Gamma_{n(n-2)}(-1) = \{ A \in SL_{n-2}(\mathbb{Z}) : A \equiv I \mod 2 \}.
\]

The lower central series of \( \Gamma_{n(n-2)}(-1) \) is well-understood via the work of Bass–Milnor–Serre [1]; the \( i \)th term of the lower central series is

\[
\Gamma^i_{n(n-2)}(-1) = \{ A \in \Gamma_{n(n-2)}(-1) : A \equiv I \mod 2^i \}
\]

and the graded quotients satisfy

\[
\Gamma^i/\Gamma^{i+1} \cong \mathfrak{sl}_{n(n-2)}(\mathbb{F}_2).
\]

The structure of \( \text{Gr}^*P_n \) is known thanks to the work of Kohno [6]. Each graded quotient \( \Gamma^i P_n/\Gamma^{i+1} P_n \) is free abelian with rank \( \varphi_i(n) \) given by the
formula
\[ \prod_{i=1}^{\infty} (1 - t^i)^{\varphi_i(n)} = \prod_{j=1}^{n-1} (1 - jt). \]

Consider the map of associated graded algebras
\[ \text{Gr}^\bullet \rho_n(-1) : \text{Gr}^\bullet P_n \to \text{Gr}^\bullet \Gamma_{n(n-2)}(-1). \]

Let us examine first the case \( n = 3 \). Here, using the basis \( e_1 = -A_{12}, e_2 = A_{11} \) and \( e_3 = A_{21} \), we have
\[
\rho_3(-1)(\sigma_1) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_3(-1)(\sigma_2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.
\]

Denote the generators of \( P_3 \) by
\[ B_{12} = \sigma_1^2, \quad B_{13} = \sigma_2 \sigma_1^2 \sigma_2^{-1}, \quad B_{23} = \sigma_2^2. \]

Then the map
\[ \text{Gr}^1 \rho_3(-1) : H_1(P_3; \mathbb{Z}) \to H_1(\Gamma_3(-1); \mathbb{Z}) \]
is the map \( \mathbb{Z}\{B_{12}, B_{13}, B_{23}\} \to \mathfrak{sl}(\mathbb{F}_2) \)
\[
B_{12} \mapsto e_{23}, \quad B_{13} \mapsto e_{21} + e_{23}, \quad B_{23} \mapsto e_{21}.
\]

Note that in \( \mathfrak{sl}(\mathbb{F}_2) \), \([e_{21}, e_{23}] = 0 \) and so the image of \( \text{Gr}^\bullet \rho_3(-1) \) is simply the submodule of \( \text{Gr}^\bullet \Gamma_3(-1) \) spanned by \( e_{12}, e_{23} \in \text{Gr}^1 \Gamma_3(-1) \); that is
\[ \text{Gr}^i \rho_3(-1) : \Gamma^i P_n/\Gamma^{i+1} P_n \to \Gamma^i_3(-1)/\Gamma^{i+1}_3(-1) \]
is the zero map for \( i \geq 2 \). In particular, this tells us that if \( x \in \Gamma^i P_3 \), \( i \geq 2 \), then \( \rho_3(-1)(x) \) is congruent to the identity matrix modulo \( 2^{i+1} \) instead of \( 2^i \).

By contrast, for \( n \geq 4 \) the map \( \text{Gr}^\bullet P_n \to \text{Gr}^\bullet \Gamma_{n(n-2)}(-1) \) is highly non-trivial. Moreover, for \( i \) large, the map
\[ (\Gamma^i P_n/\Gamma^{i+1} P_n) \otimes \mathbb{F}_2 \to \Gamma^i_{n(n-2)}(-1)/\Gamma^{i+1}_{n(n-2)}(-1) \]
cannot be injective (the rank of the domain is greater than \( (n(n-2))^2 - 1 \) for \( i \) large). This gives a method for searching for elements in the kernel of \( \beta_n \)—find an element in the kernel of \( \text{Gr}^i \rho_n(-1) \), lift it to \( P_n \), and compute its Burau matrix. Of course, this is terribly inefficient.
4.2. The case of $B_2$. Let us examine the maps $\rho_4(\alpha)$ in greater detail. According to Theorem 3.19 of [3], $\beta_4$ is faithful if and only if the matrices

$$x = \beta_4(\sigma_3\sigma_1^{-1}) \quad \text{and} \quad y = \beta_4(\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})$$

generate a rank 2 free subgroup of $GL_3(\mathbb{Z}[t, t^{-1}])$. In turn, this will hold if and only if the matrices

$$X(\alpha) = \rho_4(\alpha)(\sigma_3\sigma_1^{-1}) \quad \text{and} \quad Y(\alpha) = \rho_4(\alpha)(\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1})$$

($\alpha \neq 1$) generate a rank 2 free subgroup of $SL_8(\mathbb{C})$. We do not have a proof of this, but we do have the following.

**Theorem 4.1.** There is a positive integer $M$ such that for all $m \geq M$, the group generated by $X(\alpha)^m$ and $Y(\alpha)^m$ is free of rank 2.

**Proof.** We use Proposition 3.12 of [9]. We first establish notation. If $g \in GL(V)$, where $V$ is a complex vector space, write its characteristic polynomial as $\prod_{i=1}^{n}(t - \alpha_i)$ and let $\Omega = \{\alpha_i : \alpha_i = \max_j(||\alpha_j||)\}$. Define polynomials $f_1(t)$ and $f_2(t)$ by

$$f_1(t) = \prod_{\alpha \in \Omega} (t - \alpha) \quad \text{and} \quad f_2(t) = \prod_{\alpha \notin \Omega} (t - \alpha).$$

Let $A(g)$ be the subspace of $\mathbb{P}_V$ (the projective space associated to $V$) corresponding to the kernel of $f_1(g)$ and let $A'(g)$ be that corresponding to the kernel of $f_2(g)$.

To prove the theorem, we need only show

(3) \quad $A(X(\alpha)), A(X(\alpha)^{-1}), A(Y(\alpha)), A(Y(\alpha)^{-1})$ are points

(4) \quad $A(X(\alpha)) \cup A(X(\alpha)^{-1}) \subset \mathbb{P}_V - A'(Y(\alpha))$ and $A(Y(\alpha)) \cup A(Y(\alpha)^{-1}) \subset \mathbb{P}_V - A'(X(\alpha))$.

The characteristic polynomial of each of $X(\alpha), X(\alpha)^{-1}, Y(\alpha), Y(\alpha)^{-1}$ is

$$f(t) = (t - 1)^2(t - 1/\alpha^2)(t - \alpha^2)(t + \alpha^2),$$

Assume that $||\alpha|| > 1$. Then $f_1(t) = t - \alpha^2$ and $f_2(t) = f(t)/f_1(t)$. An easy calculation shows the following.

1. \quad $A(X(\alpha)) \leftrightarrow \text{span}\{-\alpha + 1, A_{31} + A_{32}\}$
2. \quad $A(X(\alpha)^{-1}) \leftrightarrow \text{span}\{A_{12} - \frac{1+\alpha}{\alpha}A_{13}\}$
3. \quad $A(Y(\alpha)) \leftrightarrow \text{span}\{-\alpha^2A_{21} + A_{23} - \alpha^2A_{31} - A_{32} - A_{22}\}$
4. \quad $A(Y(\alpha)^{-1}) \leftrightarrow \text{span}\{-\frac{1}{\alpha}A_{12} - A_{13} + \alpha A_{21} - \alpha A_{23} + A_{11}\}$
5. \quad $A'(X(\alpha)) \leftrightarrow \text{span}\{A_{23}, A_{32}, A_{22}, A_{12}, A_{13}, A_{11}, \frac{\alpha+1}{\alpha}A_{21} + A_{31}\}$
6. \quad $A'(X(\alpha)^{-1}) \leftrightarrow \text{span}\{A_{32}, A_{22}, A_{12}, A_{11}, A_{31}, A_{21}, A_{13} + (\alpha + 1)A_{23}\}$
7. \quad $A'(Y(\alpha)) \leftrightarrow \text{span}\{-A_{31} + A_{22}, A_{23}, 2A_{31} + A_{11}, A_{12} + \alpha A_{31}, A_{13}, A_{21} - \frac{1}{\alpha}A_{31}, -\alpha A_{31} + A_{22}\}$
8. \quad $A'(Y(\alpha)^{-1}) \leftrightarrow \text{span}\{A_{21}, A_{12} - \alpha^2A_{23}, -2A_{23} + A_{22}, A_{23} + A_{32}, A_{31}, A_{13} - \alpha^2A_{23}, A_{23} + A_{11}\}$.

It is easy to check that conditions (3) and (4) hold. \qed
Remark 4.2. That the matrices $\beta_4(\sigma_3\sigma_1^{-1})^m$ and $\beta_4(\sigma_2\sigma_3^{-1}\sigma_2^{-1})^m$ generate a free group was proved by S. Moran [8] using the same technique over the field $\mathbb{C}(t)$. It was our hope that passing to the matrices $X(\alpha), Y(\alpha)$ would allow us to take $M = 1$. This is not the case however. Indeed, denote by $v_1$ the basis vector of $A(X(\alpha))$ and by $v_2$ the basis vector of $A(Y(\alpha))$. Then it is easy to see that

$$||Yv_1 - v_2|| \geq 1 \quad \text{and} \quad ||Y^2v_1 - v_2|| \geq 1$$

so that no neighborhood of $A(X(\alpha))$ in $\mathbb{P}_V$ can be taken into a small neighborhood of $A(Y(\alpha))$. Proposition 1.1 of [9] therefore does not apply to $\langle X(\alpha), Y(\alpha) \rangle$.

4.3. Faithfulness of $\mu_n(\alpha, \beta)$. The map $\kappa_n : B_n \to GL_v(Z[t^{\pm 1}, q^{\pm 1}])$ is faithful for all $n$. In particular, if $\alpha$ and $\beta$ are algebraically independent complex numbers, then the map induced by the homomorphism $t \mapsto \alpha$, $q \mapsto \beta$ yields a faithful representation

$$B_n \to GL_v(Z[\alpha^{\pm 1}, \beta^{\pm 1}]).$$

Thus, $B_n \cap L(\alpha, \beta) = \{1\}$ in this case.

Recall that $\mu_n(\alpha, \beta)$ is the composition of $\kappa_n$ with the map

$$g_n(\alpha, \beta) : GL_v(Z[t^{\pm 1}, q^{\pm 1}]) \to Aut(L).$$

We note the following.

Lemma 4.3. Denote by $Z$ the center of $GL_v(Z[t^{\pm 1}, q^{\pm 1}])$. Then the kernel of $g_n(\alpha, \beta)$ is the subgroup $Z \cdot L(\alpha, \beta)$.

Proof. It is clear that the kernel contains $Z \cdot L(\alpha, \beta)$. For the reverse inclusion, note that any $X \in GL_v(Z[t^{\pm 1}, q^{\pm 1}])$ can be written as $X = X_0X_1$ where $X_0 \in GL_v(Z[t^{\pm 1}, q^{\pm 1}])$ and $X_1 \in L(\alpha, \beta)$. The action of $X$ on $L$ is then given by conjugation by $X_0$. By considering the action on the basis of $L$, if $X_0$ acts trivially, we see that $X_0$ must be a diagonal matrix with all entries equal; that is, $X_0 \in Z$. \qed

As $\kappa_n$ is faithful, we may identify $B_n$ with its image in $GL_v(Z[t^{\pm 1}, q^{\pm 1}])$. Then if $\alpha$ and $\beta$ are algebraically independent, we see that the kernel of $\mu_n(\alpha, \beta)$ is $B_n \cap Z$. We have thus proved the following result.

Theorem 4.4. If $\alpha$ and $\beta$ are algebraically independent, then the kernel of $\mu_n(\alpha, \beta)$ is precisely the center of $B_n$. \qed

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Department of Mathematics and Statistics, Mississippi State University, P.O. Drawer MA, Mississippi State, MS 39762

E-mail address: knudson@math.msstate.edu

URL: http://www2.msstate.edu/∼kk116/