Schrödinger Invariance in Discrete Stochastic Systems

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Abstract

Local scale invariance for lattice models is studied using new realizations of the Schrödinger algebra. The two-point function is calculated and it turns out that the result can be reproduced from exact two-point correlation functions evaluated in the stationary state of several simple stochastic models.
1 Introduction

Among the most important concepts of physics is that of symmetries. Particularly in the context of critical phenomena, scale invariance and its consequences have been thoroughly studied for a long time. More recently, for isotropic, equilibrium critical points in two dimensions, conformal invariance has allowed for significant progress in the classification of the universality classes and the elucidation of their properties, see [1] and references therein.

Much less is known if the system under study is either strongly anisotropic or time-dependent, or both. Such systems will at criticality show an anisotropic scaling of, say, a correlation function $C(r, t)$ depending on a “space” coordinate $r$ (we take one space dimension throughout) and “time” coordinate $t$

$$C(\lambda r, \lambda^\theta t) = \lambda^{-2x/\theta} C(r, t)$$

where $\theta$ is the anisotropy or dynamical exponent and $x$ is a scaling dimension. One may ask whether this strongly anisotropic scale invariance can be sensibly generalized to include space-time dependent rescaling functions $\lambda(r, t)$ instead of merely taking $\lambda$ to be a constant. Indeed, at least for systems for which the equations of motion reduce in the continuum limit to the free Schrödinger or diffusion equation (and thus have $\theta = 2$), this seems to be possible. The group of scaling transformations is then the Schrödinger group, which is the maximal kinematic invariance group of the free Schrödinger equation [2]. The Schrödinger group can be obtained as the non-relativistic limit of the conformal group [3].

The Lie algebra of the Schrödinger group in $d = 1$ space dimension is spanned from the set 
{$X_{-1}, X_0, X_1, Y_{\frac{1}{2}}, Y_{\frac{-1}{2}}, M_0$} and the non-vanishing commutation relations are

$$[X_n, X_m] = (n - m)X_{n+m}$$
$$[X_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m}$$
$$[Y_{\frac{1}{2}}, Y_{\frac{-1}{2}}] = M_0$$

Note that $M_0$ commutes with the entire algebra and the Casimir operator is [4]

$$Q = (4M_0X_0 - 2\{Y_{\frac{-1}{2}}, Y_{\frac{1}{2}}\})^2 - 2\{2M_0X_{-1} - Y_{\frac{-1}{2}}^2, 2M_0X_{1} - Y_{\frac{1}{2}}^2\}$$
We shall denote the eigenvalues of $M_0$ by $-M$ and of $Q$ by $Q$. The projective unitary irreducible representations (PUIR) of (1.2) are classified [4]. There are four different types of PUIRs, but only one of them allows $M_0$ to be realized non-trivially. Of these, the PUIRs which have $Q \neq 0$ give rise to an infinite set of internal quantum numbers (spin does not arise in $d = 1$). We consider the PUIR with $Q = 0$ which is realized on (scalar) wave functions $\psi(r,t)$ which are solutions of the Schrödinger equation $(i\partial_t + 1/(2M)\partial_r^2)\psi(r,t) = 0$. It takes the form $\psi \rightarrow U\psi$, where

$$U\psi(r,t) = |-\gamma t + \alpha|^{-1/2} \exp\left\{ -\frac{iM}{\gamma t + \alpha} \left( \frac{\gamma r^2}{2} + \frac{\delta t - \beta v^2 - vr + (-\gamma t + \alpha)av}{\gamma t + \alpha} \right) \right\} \cdot \psi\left( r - (-\gamma t + \alpha)a - (\delta t - \beta)av, \frac{\delta t - \beta}{-\gamma t + \alpha} \right)$$

(1.4)

with $\alpha \delta - \beta \gamma = 1$ and $\alpha, \beta, \gamma, \delta, v$ and $a$ are constants. This realization was found by Niederer [2] as giving the maximal kinematic invariance group of the free Schrödinger equation and, as shown by Hagen [5], also arises when discussing the Schrödinger invariance of non-relativistic free field theory.

The Lie algebra generators are readily found from (1.4) and read

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n(n+1)}{4}M t^{n-1}r^2$$

$$Y_m = -t^{m+1/2}\partial_r - \left( m + \frac{1}{2} \right)M t^{m-1/2}$$

$$M_0 = -M$$

(1.5)

$X_{-1}$ and $Y_{-\frac{1}{2}}$ generate translations in time and space, respectively, $X_0$ generates global scale transformation, $Y_{\frac{1}{2}}$ is the generator of Galilei transformations and $X_1$ generates the special Schrödinger transformations. In order to implement Galilei invariance of the free Schrödinger equation, the scaling fields $\phi(r,t)$ of the theory will under a Galilei transformation pick up a complex phase characterized by the mass $M$ and thus $\phi$ has necessarily to be complex [6], as is well known. Thus the field $\phi(r,t)$ is characterized by its scaling dimension $x$ and its mass $M$, while its complex conjugate $\phi^*(r,t)$ is characterized by the pair $(x, -M)$. This structure is to be kept when going, via the continuation $M \rightarrow (2iD)^{-1}$, to a free diffusion equation with diffusion constant $D$. The requirement of covariance of the two-point function under the given Schrödinger transformation
determines the two-point function (written here in Euclidean time) completely [7]. One finds
\[<\phi_1(r_1,t_1)\phi_2^*(r_2,t_2)> = \Phi_0 \delta_{M_1,M_2} \delta_{x_1,x_2} (t_1 - t_2)^{-x_1} \exp \left[-\frac{M_1 (r_1 - r_2)^2}{2} \right] \] (1.6)
where \(\Phi_0\) is a normalization constant. Note that the exponential factor and the Bargmann superselection rule \(M_1 = M_2\) [8] follow from covariance under the Galilei sub-group spanned by \(\{X_-, Y_{\pm}^{\pm}, M_0\}\) alone. Several simple models can be shown to reproduce this result. Three-point functions and plane surfaces have been considered as well [7].

These results for the correlation functions do depend not so much on the algebraic structure but rather on the particular realization of the Schrödinger algebra used. For example, one could give another realization by using the space of wave functions of the harmonic oscillator rather than free particles as a carrier space [4, 9]. Here, we want to consider the consequences of another realization of (1.2). It differs from the free particle realization defined above in that the carrier space is the space of wave functions which are periodic in momentum space. We shall define this new realization in the next section and shall use it to obtain the two-point function by demanding covariance. Exact results for two-point functions directly obtained from some simple lattice models coincide with the expressions derived in section 2 and show that this realization can be viewed as evidence for a Schrödinger symmetry of lattice models.

2 Lattice realization of the Schrödinger algebra

We begin our discussion by noting that generators which satisfy the Schrödinger algebra (1.2) can be constructed from the generators \(\{P, X, E, T, M\}\) of a much simpler algebra \(A\) the generators of which satisfy the relations
\[
[ P , X ] = [ E , T ] = 1 \] (2.1)
and
\[
[ A , B ] = 0 \] (2.2)
for all other pairs of generators \(A, B \in \{P, X, E, T, M\}\). The generators \(X_{0,\pm}, Y_{\pm}^{\pm}, M_0\) defined as shown in table 1 satisfy indeed the commutation relations (1.2). Therefore any representation of the algebra \(A\) gives rise to a representation of the Schrödinger algebra. Clearly, \(P = \partial_r, X = r,\)
Table 1: Realization of the Schrödinger algebra in terms of generators of $\mathcal{A}$.

| generator | expression |
|-----------|------------|
| $X_{-1}$ | $-E$ |
| $X_0$    | $-TE - \frac{1}{2}XP$ |
| $X_1$    | $-T^2E - TXP - \frac{1}{2}MX^2$ |
| $Y_{\frac{1}{2}}$ | $-P$ |
| $Y_{\frac{3}{2}}$ | $-TP - MX$ |
| $M_0$    | $-M$ |

$E = \partial_t, T = t$ and $M = \mathcal{M}$ (where $r, t, \mathcal{M}$ are numbers) satisfy relations (2.1) and (2.2) and give rise to the "continuum" realization (1.5) of the Schrödinger algebra (see also table 2).

The point we wish to make is that instead of choosing $P = \partial_r$, one may represent $P$ by the symmetric difference operator

$$P = \frac{2}{a} \sinh \left( \frac{a}{2} \partial_r \right)$$

(2.3)

which is defined by its expansion in a power series). In order to preserve relations (2.1) one may now choose

$$X = \frac{1}{\cosh \frac{a}{2} \partial_r} r .$$

(2.4)

The generators $X_{0,\pm 1}, Y_{\pm \frac{1}{2}}, M_0$ defined in table 1, but now given by the expressions shown in table 2 still satisfy the commutation relations (1.2). The various operators are understood to be defined in terms of a power series expansion.

Some intuitive understanding of the role of the parameter $a$ comes from considering the generator $Y_{\frac{1}{2}}$ of “translations”

$$Y_{\frac{1}{2}} f(r) = -\frac{1}{a} \left( f(r + a/2) - f(r - a/2) \right) .$$

(2.5)

which, if we interpret $a$ as a lattice constant, may be understood as a discretized symmetric lattice derivative operator.
The Casimir operator of the Galilei subalgebra has the abstract form $C = ME - P^2/2$ and is therefore in the lattice realization given by

$$C = M\partial_t - \frac{1}{a^2} (\cosh (a\partial_x) - 1) .$$

(2.6)

Thus $C$ can be viewed as the continuum or lattice Schrödinger operator in the “continuum” or “lattice” realizations of table 2, respectively. We emphasize that the generators given in table 2 do not generate finite transformations but may still be viewed as generators of infinitesimal transformations of some Lie group. In this paper, however, we do not want to address the question of exponential maps of such generators.

We now turn to the two-point function

$$F = F(r_1, r_2; t_1, t_2) = <\phi_1(r_1, t_1)\phi_2^*(r_2, t_2)>$$

(2.7)

and study the consequences of covariance of $F$ under the “lattice” realization of (1.2). We will use the short-hand

$$\partial_a = \frac{\partial}{\partial t_a}; \quad D_a = \frac{\partial}{\partial r_a} .$$

(2.8)

In slight abuse of language we shall furthermore refer to the action of the operators $Y_m$ and $X_n$ as defined in table 2 as translation operator, Galilei operator etc. Two-particle operators are defined as the sum of two one-particle operators acting on the pair of coordinates $(r_1, t_1)$ and $(r_2, t_2)$ respectively.

Table 2: The “continuum” and “lattice” realizations of the Schrödinger algebra.
Demanding time translation invariance implies \( F = F(r_1, r_2; \tau) \) where \( \tau = t_1 - t_2 \). Invariance under \( M_0 \) gives the Bargmann superselection rule \( \mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M} \). Translation invariance, generated by \( Y_{-\frac{1}{2}} \), requires

\[
\left[ \sinh \left( \frac{a}{2} D_1 \right) + \sinh \left( \frac{a}{2} D_2 \right) \right] F(r_1, r_2; \tau) = 0 .
\] (2.9)

This equation can be solved by introducing the coordinates \( R = r_1 + r_2 \) and \( \rho = r_1 - r_2 \) and writing \( F = f(R, \rho; \tau) \). We get

\[
cosh \left( \frac{a}{2} \partial_{\rho} \right) \sinh \left( \frac{a}{2} \partial_R \right) f(R, \rho; \tau) = 0 .
\] (2.10)

This has the solutions

\[
f \left( R + \frac{a}{2}, \rho; \tau \right) - f \left( R - \frac{a}{2}, \rho; \tau \right) = 0
\] (2.11)

and \( f(R, \rho + a/2; \tau) = -f(R, \rho - a/2; \tau) \). Since we would like to recover in the limit \( a \to 0 \) the familiar continuum solution, we have to choose (2.11). The second choice would lead to \( F(r_1, r_2; t_1, t_2) = 0 \) for \( a \to 0 \) and we do not follow this possibility any further.

Next, we consider Galilei invariance

\[
\sum_{i=1}^{2} \left\{ \frac{2}{a} t_i \sinh \left( \frac{a}{2} D_i \right) + \epsilon_i \mathcal{M}_i \frac{1}{\cosh(a/2D_i)} r_i \right\} F = 0
\] (2.12)

where \( \epsilon_1 = -\epsilon_2 = 1 \). Using eq. (2.11) and expanding the cosh, we find

\[
\frac{1}{\cosh(a/2\partial_{\rho})} \left[ \frac{\tau}{a} \sinh (a\partial_{\rho}) + \mathcal{M} \rho \right] f \left( R + \frac{a}{2}, \rho; \tau \right) = 0 .
\] (2.13)

Consider now the equation \( (\cosh(a/2\partial_{\rho}))^{-1} h(\rho) = 0 \). We write \( h \) in the form \( h(\rho) = \sum_{k=0}^{\infty} h_k(\rho) a^k \).

Insertion then shows, after expanding the cosh, that order by order \( h_k(\rho) = 0 \) for all \( k \). We are thus left with

\[
\left[ \frac{\tau}{a} \sinh (a\partial_{\rho}) + \mathcal{M} \rho \right] f \left( R + \frac{a}{2}, \rho; \tau \right) = 0
\] (2.14)

This can be solved via Fourier transformation with the result

\[
f \left( R + \frac{a}{2}, \rho; \tau \right) = g \left( R + \frac{a}{2}, \tau \right) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \ e^{i(\tau/Ma^2)\cos k} e^{ik(\rho/a)} \right\} = g \left( R + \frac{a}{2}, \tau \right) \Phi(\rho/a, \tau/Ma^2)
\] (2.15)
If we interpret the \( r_i \) indeed to denote the sites of a lattice, \( \rho/a = n \) is an integer and \( \Phi(\rho/a, \tau/\mathcal{M}a^2) = I_n(\tau/\mathcal{M}a^2) \), where \( I_n \) is a modified Bessel function. We also note the following properties of \( \Phi \)

\[
\begin{align*}
\partial_{\tau} \Phi(\rho/a, \tau/\mathcal{M}a^2) &= \frac{1}{\mathcal{M}a^2} \cosh(a \partial_{\rho}) \Phi(\rho/a, \tau/\mathcal{M}a^2) \\
\tanh \left( \frac{a}{2} \partial_{\rho} \right) \left( \rho \Phi(\rho/a, \tau/\mathcal{M}a^2) \right) &= -\frac{2\tau}{\mathcal{M}a} \sinh^2 \left( \frac{a}{2} \partial_{\rho} \right) \Phi(\rho/a, \tau/\mathcal{M}a^2)
\end{align*}
\]

(2.16)

We now consider the scale transformations generated by \( X_0 \). Scale invariance requires that

\[
\sum_{i=1}^{2} \left\{ t_i \partial_i + \frac{1}{a} \frac{1}{\cosh(a/2 D_i)} r_i \sinh \left( \frac{a}{2} D_i \right) + \frac{1}{2} x_i \right\} F = 0
\]

(2.17)

where \( x_i \) are the scaling dimensions of the fields \( \phi_i \). Following the same lines as before and using that \( [\rho, f(a \partial_{\rho})] = -af'(a \partial_{\rho}) \) for any (differentiable) function \( f \), one obtains, with \( x = (x_1 + x_2)/2 \)

\[
\left[ \tau \partial_\tau + \frac{1}{a} \tanh \left( \frac{a}{2} \partial_{\rho} \right) \rho - \frac{1}{2} + x \right] g(R, \tau) \Phi(\rho/a, \tau/\mathcal{M}a^2) = 0
\]

(2.18)

Using (2.16), this can be reduced to

\[
\left[ \tau \partial_\tau + \frac{\tau}{\mathcal{M}a^2} + x - \frac{1}{2} \right] g(R, \tau) = 0
\]

(2.19)

with the solution

\[
g(R, \tau) = g_0(R) \tau^{-(x-1/2)} \exp \left( -\frac{\tau}{\mathcal{M}a^2} \right)
\]

(2.20)

where \( g_0(R) \) satisfies \( g_0(R + a/2) = g_0(R - a/2) \). Finally, we demand invariance under the special Schrödinger transformation generated by \( X_1 \):

\[
\sum_{i=1}^{2} \left\{ -t_i^2 \partial_i - \frac{2t_i}{a \cosh(a/2 D_i)} r_i \sinh(a/2 D_i) - \frac{\epsilon_i \mathcal{M}_i}{2} \left( \frac{1}{\cosh(a/2 D_i)} r_i \right)^2 - x_i t_i \right\} F = 0
\]

(2.21)

A straightforward but tedious calculation shows that the solution found so far is consistent, provided only that

\[
x_1 = x_2 = x
\]

(2.22)

The fields \( \phi_1 \) and \( \phi_2 \) must thus have the same scaling dimension \( x \) the value of which is not determined by Schrödinger invariance. Summarising, the final result is

\[
F(n, t) \equiv F(r_1, r_2; t_1, t_2) = g_0 \delta_{\mathcal{M}_1, \mathcal{M}_2} \delta_{x_1, x_2} t_1^{1/2 - x_1} e^{-t} I_n(t)
\]

(2.23)
with
\[ t = \frac{t_1 - t_2}{M_1 a^2} \quad \text{and} \quad n = \frac{r_1 - r_2}{a} \] (2.24)
where we assumed that both \( r_1 \) and \( r_2 \) are integer multiples of the “lattice” constant \( a \). As in continuous space, Schrödinger invariance on the lattice determines the two-point function completely up to the non-universal amplitude \( g_0 \), the non-universal constant mass \( M \) and the universal critical exponent \( x \). Note that if we require \( \phi_1 \) and \( \phi_2 \) to be locally conserved, i.e. \( \partial_t \sum_n F(n, t) = 0 \), then one has \( x = 1/2 \). If \( r_1 \) or \( r_2 \) are not integer multiples of \( a \), the Bessel function \( I_n(t) \) has to be replaced by the scaling function (2.15). We can consider \( g_0 \) as a normalization constant if both \( r_1 \) and \( r_2 \) are integer multiples of \( a \).

3 Exact two-particle correlations in lattice models

So far we have merely assumed the existence of physical systems in which lattice Schrödinger invariance holds and studied some of its consequences. Clearly it is desirable to have an example, even a very simple one, where correlation functions as predicted in the last section would be found. Such an example is many-particle lattice diffusion.

3.1 Symmetric, non-exclusive lattice diffusion

Consider a system of arbitrarily many particles hopping stochastically to their nearest neighbouring sites. We choose the probability of hopping from site \( i \) to site \( i \pm 1 \) proportional to the occupation number \( n_i \) of the site \( i \). For this stochastic process one may either write a master equation or, equivalently, a quantum Hamiltonian defining the time evolution of the system [10, 11, 12]. In this mapping the state of the system at some time \( t \) is given by (stochastic) occupation numbers \( \underline{n} = \{n_i\} \) where \( i \) is the number of a lattice site. Their dynamics are given by a Schrödinger equation for the probability distribution \( F(\underline{n}, t) \) with a quantum Hamiltonian \( H \)

\[ \partial_t |F(\underline{n}, t)\rangle = -H |F(\underline{n}, t)\rangle \] (3.1)

which for our very simple toy process is given by

\[ H = \frac{1}{2} \sum_i (a_i^\dagger - a_i^\dagger)(a_{i+1} - a_i) \] (3.2)
The operators \( a_i^{\dagger} \) and \( a_i \) satisfy bosonic commutation relations \([a_i, a_j^{\dagger}] = \delta_{i,j}\). Each state in the system is represented by a state vector \(|\mathbf{n}\rangle\) which together with the transposed vectors \langle\mathbf{n}|\) form an orthonormal basis. The creation operator \( a_i^{\dagger} \) for the stochastic process is then given by \( \delta_{n,n-1} \) acting on site \( i \), while the annihilation operator \( a_i \) is represented by \( n\delta_{n,n+1} \) (here \( \delta_{n,m} \) is the Kronecker symbol). In this notation one has \( |F(\mathbf{n},t)\rangle = \sum_n F(n,t) |n\rangle \). From the definition of the time evolution (3.1) we find that a state vector at time \( t \) is given in terms of the vector at time \( t = 0 \) by \( |F(n,t)\rangle = \exp (-Ht) |F(n,0)\rangle \). Time-dependent operators \( A(t) \) (in euclidean time) are defined by \( A(t) = \exp (Ht)A \exp (-Ht) \). Note that in one dimension this model is equivalent to an interface growth model with unrestricted height gradients [13].

\( H \) has a degenerate ground state with energy \( E_0 = 0 \), each sector with fixed particle number \( N \) contains such a vector which are the steady states of the system. Here we shall study vacuum expectation values

\[ F(r_1, r_2; t_1, t_2) = \langle 0 | a_{r_1}(t_1) a_{r_2}^{\dagger}(t_2) | 0 \rangle , \]

i.e. correlation functions in the vacuum state containing no particles. Here \( r_{1,2} \) denote sites on the 1D lattice with lattice constant \( a \). We furthermore assume \( t_1 \geq t_2 \). In the language of stochastic processes this is the conditional probability of finding the particle on site \( r_1 \) at time \( t_1 \) provided it was on site \( r_2 \) at time \( t_2 \). Two-point correlation functions in \( N \)-particle steady states can be reduced to such correlators [14].

In order to compute the correlation function (3.3) we first note (a) that since we are considering a steady state correlator, \( F \) depends only on \( r_1, r_2 \) and \( \tau = t_1 - t_2 \) and (b) that because of translational invariance, \( F \) depends only \( r = r_1 - r_2 \) and \( \tau \). Furthermore, operators \( a_i^{\dagger} \) and \( a_i \) satisfy the equations of motion

\[
\begin{align*}
\partial_t a_i^{\dagger} &= \left[ H, a_i^{\dagger} \right] = -\frac{1}{2} \left( a_{i+1}^{\dagger} + a_{i-1}^{\dagger} - 2 a_i^{\dagger} \right) \\
\partial_t a_i &= \left[ H, a_i \right] = \frac{1}{2} \left( a_{i+1} + a_{i-1} - 2 a_i \right) .
\end{align*}
\]

This is equivalent to \( Ca_i^{\dagger} = Ca_i = 0 \) where \( C \) is the lattice Schrödinger operator (2.9) with the eigenvalue \( \mathcal{M} = 1 \) of the mass operator \( M \) defined by \([N, a_i] = -Ma_i = -a_i \) and \([N, a_i^{\dagger}] = \mathcal{M} a_i^{\dagger} = a_i^{\dagger} \).
Eqs. (3.4) are easy to integrate and one finds

\[ F(r, \tau) \equiv F(r_1, r_2; t_1, t_2) = e^{-\tau} I_r(\tau) \]  

(3.5)

which is in full agreement with the prediction (2.23) with amplitude \( g_0 = 1 \), mass \( \mathcal{M} = 1 \), lattice constant \( a = 1 \) and critical exponent \( x = 1/2 \) obtained from Schrödinger invariance and current conservation which is implied in eqs. (3.4).

### 3.2 Driven non-exclusive lattice diffusion

Symmetric lattice diffusion as discussed above appears to be simplest many particle system which is invariant under the discretized Schrödinger symmetry introduced in section 2. One may also study its driven version where particles hop with different rates \( p_R = qn_i \) and \( p_L = q^{-1}n_i \) from site \( i \) to their right and left nearest neighbouring sites \( i \pm 1 \) respectively. In this case the Hamiltonian reads

\[ H = \frac{1}{2} \sum_i \left\{ q a_{i+1}^\dagger (a_{i+1} - a_i) + q^{-1} (a_{i+1}^\dagger - a_i^\dagger) a_{i+1} \right\} \]  

(3.6)

and the correlation function (3.3) is found by integrating the equations of motion for \( a_i \) and \( a_i^\dagger \) to be

\[ F(r, \tau) \equiv F(r_1, r_2; t_1, t_2) = e^{-b\tau - vr} e^{-\tau} I_r(\tau) \]  

(3.7)

with \( v = -\ln q \) and \( b = 2 \sinh^2(v/2) = \frac{1}{2}(q + q^{-1}) - 1 \). The same result can be obtained from yet another lattice realization of the Schrödinger algebra. To see this, we go back to the algebra \( \mathcal{A} \) (2.1). Rather than reusing the realization given in eqs. (2.3,2.4), we now take

\[ E = \partial_t + \mathcal{M}b \quad , \quad T = t \quad , \quad P = \frac{2}{a} \sinh \left( \frac{a}{2} (\partial_r + \mathcal{M}v) \right) \quad , \quad X = \frac{1}{\cosh(\frac{a}{2}(\partial_r + \mathcal{M}v))} r \]  

(3.8)

Using the definitions in table 1 this leads to a new realization of the Schrödinger operators and one may go again through the calculation of the two-point function. The result is

\[ F(n, t) = g_0 \delta_{\mathcal{M}_1, \mathcal{M}_2} \delta_{x_1, x_2} e^{-\mathcal{M}_1 bt - \mathcal{M}_1 \mathcal{N} t^{1/2} - x_1} e^{-t} I_n(t) \]  

(3.9)

with \( n \) and \( t \) defined as in (2.24). This is reproduced by the result (3.7) for asymmetric diffusion with \( g_0 = 1 \), \( a = 1 \), \( \mathcal{M} = 1 \) and \( x = 1/2 \). Note that in the continuum limit \( a \to 0 \) the correlation
function is proportional to \( \exp (-Mv^2t/2 - Mv\rho) \cdot t^{-1/2} \exp (-M\rho^2/(2t)) \). This is the correlation function of the undriven system, but observed in a frame of reference moving with constant velocity \( v \). On the lattice, eq. (3.9) is understood as follows. A Galilei transformation will bring us to the frame where particles in the mean are at rest. The effect of this Galilei transformation produces the result (3.9), and the condition \( b = 2a^{-2} \sinh^2(2av/2) \) takes care that the stationary state has indeed vanishing energy.

### 3.3 Exclusive symmetric lattice diffusion

We come back to lattice diffusion as defined in section 3.1 but demand that on each site there can be at most one particle [13]. Then the same expression (3.3) for the two-point correlation function (3.3) is found. The time evolution operator of this system is given by the spin 1/2 isotropic Heisenberg ferromagnet

\[
H = -\frac{1}{4} \sum_i \left( \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \sigma^z_i \sigma^z_{i+1} - 1 \right)
\]  

(3.10)

where \( \sigma^{x,y,z}_i \) are the Pauli matrices. This is an interesting example of a process where the time evolution operator is given in terms of operators satisfying a Hecke algebra [16]. The equations of motion for the creation and annihilation operators \( \sigma^\pm \), respectively, are nonlinear

\[
\partial_t \sigma^+_i = \frac{1}{2} \left( \sigma^+_i \sigma^+_i + \sigma^+_i - 2 \sigma^+_i \right) + \left( \sigma^-_{i+1} \sigma^+_i + \sigma^-_{i-1} \sigma^+_i \right) \sigma^+_i
\]  

(3.11)

and a similar expression for \( \sigma^-_i \). However, the nonlinear piece is proportional to the number operators at sites \( i \pm 1 \) and therefore vanishes in the expression (3.3). The linear piece is again the lattice Laplacian which accounts for the correlation functions being given by the same expressions as in the non-exclusive case.

We only showed here that the two-point function can be reproduced in the models studied and it is not immediately obvious that the same should be true for the higher correlators. But one might justify the consideration of Schrödinger invariance in this context by noting that Schrödinger invariance can be shown to hold also for at least certain types of non-linear Schrödinger equations, e.g. \( (i\partial_t + \partial_x^2 + u|\psi|^2)\psi = 0 \), see [17], [18] for more information. The equations of motion (3.11) are a discretized version of this equation. Furthermore, known results for a partially integrated
four-point function of the exclusion process [13], suggest that one might predict such correlators from an invariance of the kind discussed here, but adapted to systems with boundaries.

3.4 Other stochastic models

The same type of correlation functions can be reproduced from other systems as well. Consider the 1D kinetic Ising model with Glauber dynamics [19]. The connected spin-spin correlation function $< s_{r_1}(t_1)s_{r_2}(t_2) >_c$ of spins at the lattice sites $r_{1,2}$ at times $t_{1,2}$, when the system is in equilibrium at a temperature $T$, can be calculated exactly. If in the initial state there are no correlations between the spins, one has [19]

$$< s_{r_1}(t_1)s_{r_2}(t_2) >_c = e^{-\alpha t}I_r(\alpha \gamma t)$$

with $r = r_1 - r_2$, $t = |t_1 - t_2|$, $\alpha$ is a constant reaction rate, $\gamma = \tanh(2J/(k_B T))$ where $J$ is the Ising model coupling constant. For $\gamma = 1$ corresponding to the zero temperature static critical point of the one-dimensional Ising model this is obviously in agreement with the prediction eq. (2.23), upon identification of the non-universal parameters. But also the off-critical form of the correlation function ($\gamma \neq 1$) is in agreement with lattice Schrödinger invariance (3.9) with $v = 0$, $x = 1/2$ and choosing $b$ and $M$ appropriately. This corresponds to the invariance group of a Schrödinger equation in a constant potential, proportional to $b$.

Another example is provided by the stochastic adsorption-desorption process of dimers [20]. Consider a 1D lattice whose sites may be empty or occupied by at most one particle. A pair of empty nearest-neighbour sites may become occupied with rate $\epsilon$ (dimer adsorption), and a pair of occupied nearest-neighbor sites may become empty with the same rate $\epsilon$ (dimer desorption). A single particle may hop from an occupied site to an empty nearest neighbor site with a rate $h$. Let $n_r(t)$ denote the occupation number of site $r$ at time $t$. It can be shown that the system evolves towards a steady state, characterized by $< n_r >= 1/2$. In the steady state, the density-density correlation is on an infinite lattice [20]

$$< n_{r_1}(t)n_{r_2}(0) >= \frac{1}{4}e^{-2(\epsilon + h)t}I_r(2(h - \epsilon)t)$$

with $r = r_1 - r_2$. This correlation again reproduces lattice Schrödinger invariance (3.9), when choosing $b$ appropriately.
4 Conclusions

We have introduced new realizations of the Schrödinger algebra as describing dynamical symmetries of discrete lattice systems. The two-point correlation function is found from the requirement that it covariantly transforms under these realizations and includes the previously known result of the continuum as the special limiting case $a \to 0$. We find that the two-point function so obtained can be reproduced from the two-particle correlation function, calculated in the stationary state, of several simple stochastic models.

The construction of lattice realizations of continuous space-time symmetries using the algebra (2.1), (2.2) extends beyond the few cases studied here. Likewise, one may generalize this approach to higher dimensions. This will be reported elsewhere.

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