THE DISTRIBUTION OF $p$-TORSION IN DEGREE $p$ CYCLIC FIELDS

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Abstract. We compute all the moments of the $p$-torsion in the first step of a filtration of the class group defined by Gerth [15] for cyclic fields of degree $p$, unconditionally for $p = 3$ and under GRH in general. We show that it satisfies a distribution which Gerth conjectured as an extension of the Cohen-Lenstra-Martinet conjectures. In the $p = 3$ case this gives the distribution of the 3-torsion of the class group modulo the Galois invariant part. We follow the strategy used by Fouvry and Klüners in their proof of the distribution of the 4-torsion in quadratic fields [13].

1. Introduction

Let $K$ be a number field of degree $n$. Let $Cl_K$ denote the class group and $Cl_{K,p}$ denote the $p$-part. Let $S$ be the set of finite abelian $p$-groups. We are interested in the question: what is the probability of any $A \in S$ occurring as $Cl_{K,p}$ for $K$ of degree $n$? The Cohen-Lenstra heuristics propose an answer to this question in the form of a probability distribution on $S$.

We make the question more precise as follows. Let $D_K$ denote the discriminant of $K$. For any $X$ define

$$S_X^\pm(A) = \frac{\{K \mid 0 < \pm D_K < X, Cl_{K,p} \cong A\}}{\sum_{K, 0 < \pm D_K < X} 1}.$$ 

The probability of $A$ occurring as $Cl_{K,p}$ is $\lim_{X \to \infty} S_X(A)$. In general this is not known to exist. Cohen and Lenstra conjectured [6] that it does and proposed a distribution on $S$ which should equal this quantity. For $s \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ let

$$\eta_s(p) = \prod_{i=1}^s \left(1 - \frac{1}{p^i}\right).$$

One can show [6, 18] that

$$\sum_{G \in S} \frac{1}{|\text{Aut}G|} = \frac{1}{\eta_\infty(p)} < \infty.$$ 

Then for any $A \in S$ and $u \geq 0$ define

$$\mu_u(A) = \frac{\eta_\infty(p)}{|\text{Aut}A| |A|^u}.$$ 

This defines a probability measure on $S$, called the Cohen-Lenstra distribution. They originally considered the case $n = 2$, $p \neq 2$ and considered complex quadratic and real quadratic fields separately.
Conjecture 1.1 (Cohen-Lenstra). For $A \in S$

$$\mu_0 (A) = \lim_{X \to \infty} S_X^-(A),$$
$$\mu_1 (A) = \lim_{X \to \infty} S_X^+(A).$$

These conjectures were extended to higher degree number fields by Cohen and Martinet [7] again for $p \nmid n$.

No cases of these conjectures are known in full strength, though there has been much recent work on the subject. In the setting of number fields there are results giving the average size of the class group or subgroup thereof. There is the classical result of Davenport-Heilbronn [9] and Datskovsky-Wright [8] for the average size of 3-torsion of quadratic fields. Bhargava-Varma [4] compute the average 2 torsion of cubic fields. There are also partial results for 8 and 16 torsion of quadratic fields due to Milovic [21, 22] and Smith [23]. Below we will discuss in more detail the work of Fouvry and Klüners [13] on 4 torsion of quadratic fields. There are also non-abelian versions which have been studied by Alberts [1] and Bhargava [3].

The conjectures have also been studied in the setting of function fields which provides additional tools such as moduli schemes. Some results here are the work of Ellenberg-Venkatesh-Westerland [11], Boston-Wood [5] and Wood [24].

The original conjectures ignored the case when $p$ divides the degree of the number fields. Gerth extended them to $p$-torsion in degree $p$ cyclic fields by proposing a distribution for a certain subgroup of $Cl_K [p]$ [14, 15, 16]. For the case $p = 2$ and $n = 2$ Gerth proposed the conjectures should hold in their original form, but with $Cl_K^2$ instead of $Cl_K$. This was proved by Fouvry and Klüners [13]. To state their result, let $rk_4 (Cl_K) = rk_2 (Cl_K^2)$ and for any $k \in \mathbb{Z}_{\geq 1}$ let

$$M_k^+ (2) = \lim_{X \to \infty} \frac{\sum_{K: 0 < D_K < X} 2^{krk_4(Cl_K)}}{\sum_{K: 0 < D_K < X} 1}.$$  

Define $N (k, p)$ to be the number of subspaces of $\mathbb{F}_p^k$.

Theorem 1.2 (Fouvry-Klüners). For every $k \in \mathbb{Z}_{\geq 1}$

$$M_k^- (2) = N (k, 2),$$
$$M_k^+ (2) = N (k + 1, 2) - N (k, 2).$$

By a separate result [12] Fouvry and Klüners deduce that these moments are enough to determine a distribution.

Theorem 1.3 (Fouvry-Klüners). The density of complex quadratic fields $K$ with $rk_4 (Cl_K) = s$ is

$$\frac{\eta_\infty^2 (2)}{\eta_s^2 (2) 2^{s^2}},$$

and the density of real quadratic fields with $rk_4 (Cl_K) = s$ is

$$\frac{\eta_\infty (2)}{\eta_s (2) \eta_{s+1} (2) 2^{s(s+1)}}.$$
Gerth conjectured a distribution for a certain subgroup of $Cl_K[p]$ of cyclic $p$ fields for all $p$. To state it we first define some notation. Let $K$ be a cyclic field of degree $p$ with Galois group $G = \langle \sigma \rangle$. Let $\varphi = 1 - \sigma$ act on $Cl_K[p]$. It can be shown there is a filtration

$$Cl_K[p]^G = \ker \varphi \subseteq \ker \varphi^2 \subseteq \cdots \subseteq \ker \varphi^{p-1} = Cl_K[p].$$

Then Gerth conjectured a distribution for the $p$-rank of $\varphi(\ker \varphi^2)$. Notice that for $p = 3$ we have $\ker \varphi^2 = Cl_K[3]$ and so the above filtration implies $\varphi(\ker \varphi^2) \cong Cl_K[3]/Cl_K[3]^G$. We prove the following theorem which verifies Gerth’s conjecture for $p = 3$:

**Theorem 1.4.** The density of cyclic cubic fields with $\text{rk}_3 \left( Cl_K[3]/Cl_K[3]^G \right) = s$ is

$$\frac{\eta_\infty (3)}{\eta_s (3) \eta_{s+1} (3) 3^{s(s+1)}}.$$

We can extend this to all $p$ under the assumption of GRH:

**Theorem 1.5.** Assume GRH for Artin $L$-functions. The density of degree $p$ cyclic fields with $\text{rk}_p (\varphi (\ker \varphi^2)) = s$ is

$$\frac{\eta_\infty (p)}{\eta_s (p) \eta_{s+1} (p) p^{s(s+1)}}.$$

Before continuing we make some remarks about $Cl_K[p]^G$. It is the part of $Cl_K[p]$ corresponding by class field theory to the genus field of $K$, that is the maximal unramified extension of $K$ which is abelian over $\mathbb{Q}$. It can be shown $|Cl_K[p]^G| = p^{r-1}$ where $r$ is the number of primes ramified in $K$ and that the average of $\text{rk}_p \left( Cl_K[p]^G \right)$ is $\infty$. In the case $p = 2$ removing this part corresponds to replacing 2-rank by 4-rank.

We deduce Theorems 1.4 and 1.5 from the following theorem together with [12]. Define

$$M_k (p) = \lim_{X \to \infty} \frac{\sum_{K,D_K<X} P^k_{p}\left(\varphi (\ker \varphi^2)\right)}{\sum_{K,D_K<X} 1}.$$

**Theorem 1.6.** Let $k \in \mathbb{Z}_{\geq 1}$. Then unconditionally for $p = 3$ and under the assumption of GRH for Artin $L$-functions for $p > 3$ we have

$$M_k (p) = N (k + 1, p) - N (k, p).$$

The proof of Theorem 1.6 follows the strategy of Fouvry and Klünners. For any degree $p$ cyclic field $K$ we express $|Cl_K[p]|$ using a sum of idele class characters, and then sum over all degree $p$ cyclic fields of discriminant up to $X$. We then study the asymptotics of this expression using techniques from analytic number theory. In the $p = 3$ case we require several versions of a large seive inequality for cubic characters to bound the error term. We prove one such version as well as applying several others from the literature, due to Heath-Brown [19], Baier-Young [2] and Iwaniec-Kowalski [20]. The reason for assuming GRH in the general case is that certain versions of the large seive are not yet available for order $p$ characters. In particular we lack analogues
of Propositions 5.3 and 5.4. This is the only obstacle to an unconditional proof for all \( p \).

Finally we remark briefly about an equivalent formulation of the Cohen-Lenstra conjectures which is commonly used. The distribution \( \mu_u \) is characterized by the fact [11] that for all \( A \in S \)

\[
\mathbb{E}_{G \sim \mu_u} (|\text{Sur} (G, A)|) = \sum_{G \in S} \mu_u (G) \cdot |\text{Sur} (G, A)| = \frac{1}{|A|^r}.
\]

This is often called the \( A \)-moment of \( \mu_u \) and computing it only for certain \( A \) can still provide information about the distribution of elements in \( Cl_K \).

It is clear that \(|\text{Hom} \left( G, (\mathbb{Z}/p\mathbb{Z})^k \right)\| = p^{rk_p(G)}\). Furthermore \(|\text{Hom} \left( G, (\mathbb{Z}/p\mathbb{Z})^k \right)\| = \sum_{i=0}^{k} n \cdot \left( k, i, p \right) \cdot \text{Sur} \left( G, (\mathbb{Z}/p\mathbb{Z})^i \right)\) where \( n \cdot \left( k, i, p \right) \) is the number of \( i \)-dimensional subspaces of \( \mathbb{F}_p^k \). Hence theorem [1.6] can be rephrased as computing the \( A \) moments in the above sense for all the groups \( A = (\mathbb{Z}/3\mathbb{Z})^k \).

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## 2. Counting \( p \)-torsion in degree \( p \) cyclic fields

Let \( K \) be a degree \( p \) Galois extension of \( \mathbb{Q} \) with Galois group \( G = \langle \sigma \rangle \). Let \( \varphi = 1 - \sigma \).

There is a filtration

\[
Cl_K[p]^G = \ker \varphi \subseteq \ker \varphi^2 \subseteq \cdots \subseteq \ker \varphi^{p-1} = Cl_K[p].
\]

From this we can write down the exact sequence

\[
1 \rightarrow Cl_K[p]^G \rightarrow \ker \varphi^2 \rightarrow \varphi \left( \ker \varphi^2 \right) \rightarrow 1
\]

so that \( |\ker \varphi^2| = |Cl_K^G| |\varphi(\ker \varphi^2)| \). We will consider \( \varphi \) acting on \( \ker \varphi^2 \) and throughout the section write \( \text{im} \varphi = \varphi(\ker \varphi^2) \).

Denote by \( N \) the norm map \( N_{K/\mathbb{Q}} \) (both on ideals and elements of \( K \)). Let \( J \) be the group of fractional ideals of \( K \). Furthermore let \( P_1, \ldots, P_r \) be the ramified primes of \( K \), and let \( B = \{ P_1^{e_1} \cdots P_r^{e_r} \mid e_i = 0, 1, \ldots, p-1 \} \).

**Proposition 2.1.** Let \( r \) be the number of primes ramified in \( K \). Then \( |Cl_K^G| = p^{r-1} \).

**Proof.** Let \( \phi : (\mathbb{Z}/p\mathbb{Z})^r \rightarrow Cl_K^G \) be the homomorphism \( (A_1, \ldots, A_r) \mapsto [\prod_i P_i^{a_i}] \). We will show that the \( P_i \) generate \( Cl_K^G \) which will imply that \( \phi \) is surjective. Suppose \( [I] \in Cl_K^G \), so that \( (\alpha) I = \sigma (I) \) for some \( \alpha \in K^\times \). Taking norm gives \( N (\alpha) = 1 \) so we can pick \( \alpha \) such that \( N\alpha = 1 \) in \( K \). By Hilbert’s Theorem 90 there is some \( \beta \in K \) such that \( \alpha = (1 - \sigma) \beta \). Thus we get the equality of ideals \( (\beta) I = \sigma ((\beta) I) \). This implies \( (\beta) I \) is supported only on ramified and rational primes. Thus \( [I] \in \text{im} \phi \) and hence \( \phi \) is surjective.

It remains to show that \( \ker \phi \) is non-empty. Let \( h : \mathcal{O}_K^\times/\pm 1 \rightarrow \mathcal{O}_K^\times/\pm 1 \) be defined by \( h(u) = u/\sigma(u) \). First we show \( |\text{coker} h| = p \). Consider the map \( \log_K : K \rightarrow \prod_i \mathbb{R}_{\sigma_i} \).
By Dirichlet’s unit theorem \( \log_K \) gives an isomorphism between \( \mathcal{O}_K^\times / \pm 1 \) and a lattice of rank \( p - 1 \) contained in the trace 0 hyperplane. Let \( V \) be this lattice. Then \( 1 - \sigma \) acts as a linear map on \( V \) of rank \( p \). Thus the index of \( (1 - \sigma) \log_K (\mathcal{O}_K^\times / \pm 1) = \log_K ((1 - \sigma) (\mathcal{O}_K^\times / \pm 1)) \) in \( \log_K (\mathcal{O}_K^\times / \pm 1) \) is \( p \). Thus \( |\text{coker} h| = p \).

Let \( u \in \mathcal{O}_K^\times \) represent a non-trivial element of \( \text{coker} h \). Since \( u \in \mathcal{O}_K^\times \) we have \( N_{K/Q} u = 1 \) so by Hilbert’s Theorem 90 \( u = \sigma (x) / x \) for some \( x \in K \). Since \( u \) is non-trivial in \( \text{coker} h \) this implies \( x \notin \mathcal{O}_K^\times \). So consider \( I = (x) \) which is not the unit ideal. Since \( \sigma (x) / x \in \mathcal{O}_K^\times \) this implies \( I \) is supported only on ramified primes and rational primes. If \( x \) is only supported on rational primes then \( u = \sigma (x) / x = 1 \) which contradicts the fact that \( u \) represent a non-trivial element of \( \text{coker} h \). This shows that \( \ker \phi \) is non-empty, which proves what we wanted. \( \square \)

Next we give another description of \( \text{im} \varphi \).

**Lemma 2.2.** Consider \( N \) acting on \( J \) the group of fractional ideals of \( K \). Then

\[
\ker N/\text{im} \varphi = 1.
\]

*Proof.* It is clear that \( \text{im} \ (1 - \sigma) \subset \ker N. \) Suppose \( NJ = 1 \) for some ideal \( I \in J \). Then \( I \) can only be supported on split primes. Any set of \( p \) split primes \( Q_1 \alpha_1 Q_2 \alpha_2 \cdots Q_p \alpha_p \) in the decomposition of \( I \) contributes \( \mathcal{O}^\times Q \Sigma \alpha_i \) in \( NI \), which implies that \( \Sigma \alpha_i = 0 \). Then

\[
Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_p^{\alpha_p} = \left( Q_1^{\alpha_1} Q_2^{\alpha_2+\alpha_2} \cdots Q_{p-1}^{\alpha_{p-1}} \right)^{1-\sigma}.
\]

Applying this to all primes in \( I \) shows \( I \in \text{im} (1 - \sigma) \). \( \square \)

**Lemma 2.3.** For any class \( b \in \text{Cl}_K^G \) we have

\[
b \in \text{im} \varphi \iff N b = N \alpha
\]

for some \( \alpha \in K \) (note this condition is independent of the ideal representing \( b \)).

*Proof.* Suppose first that \( b \in \text{im} \varphi \). So for some \( a \in \text{Cl}_K \), we have \( b = (1 - \sigma) (a) = a/\sigma (a) \). Let \( a \) also denote some representative such that \( \alpha b = a/\sigma (a) \) in the group of ideals. Taking norm of this gives \( N b = N \alpha^{-1} \) which proves one direction.

Now suppose that \( N b = N \alpha \) for some \( \alpha \in K \). Hence \( b = (\alpha) I \) for some ideal \( I \in \ker N \). By Lemma 2.2 we have \( I = \text{im} (1 - \sigma) a \) for some ideal \( a \in \ker (1 - \sigma)^2 \).

It remains to show \( a \in \text{Cl}_K \). Since \( b = a^{1-\sigma} \) and \( b \in \text{Cl}_K^G \) this implies \( a^{1-\sigma} = a^{\sigma - \sigma^2} \), and so \( aa^{\sigma^2} = a^\sigma a^\sigma \). Multiplying both sides by \( a^{(p-2)\sigma} \) gives

\[
(a^\sigma)^p = a^{1+(p-2)\sigma+\sigma^2} = a^{(1-\sigma)^2} = 1
\]

Thus \( a \in \text{Cl}_K \) as required. \( \square \)

Let \( D_K \) denote the discriminant of \( K \). Then the discriminant is of the form \( D_K = (p_1 \cdots p_r)^{p-1} \) where each factor is a prime congruent to \( 1 \) mod \( p \) or equal to \( p^2 \) and they
are distinct. Now let
\[
D = \begin{cases} 
  D_K & \text{if } p \nmid D_K \\
  D_K/p^2 & \text{if } p \mid D_K.
\end{cases}
\]

Let \(b \in Cl_{K^G}\) and \(b = N\beta\). The class \(b\) has a representative which lies in \(B\) which implies \(b \mid D\). By Lemma 2.3 we want to count the number of classes \(b \in Cl_{K^G}\) such that \(N\beta = N\alpha\) for some \(\alpha \in K\). This is the number of divisors \(b\) of \(D\) such that \(b = N\alpha\) for some \(\alpha \in K\), divided by \(p\), since \(p\) different divisors will come from the same class in \(Cl_{K^G}\).

Thus we have shown

**Proposition 2.4.** With the above notation
\[
|\text{im}\varphi| = \frac{1}{p} \{ b \mid D \mid b = N(\alpha) \text{ for some } \alpha \in K^\times \}.
\]

### 3. The \(p\)-torsion as a character sum

Let \(K/\mathbb{Q}\) be a degree \(p\) cyclic extension. Then the discriminant is of the form 
\(D_K = (p_1 \cdots p_r)^{p-1}\) where each factor is a prime congruent to 1 mod \(p\) or equal to \(p^2\) and they are distinct. Conversely every integer of this form is a discriminant of a degree \(p\) cyclic field. Each such extension corresponds to a character \(\chi\) of \(C_Q \cong \prod_l \mathbb{Z}_l^\times\) the idele class group of \(\mathbb{Q}\) with \(\ker \chi = NC_K\) an index \(p\) subgroup of \(C_Q\). That is, a character \(\chi: (1 + p\mathbb{Z}_p) \times \prod_{l \mid D_K, l \neq p} \mathbb{F}_l^\times \rightarrow \mu_p\) where the \((1 + p\mathbb{Z}_p)\) factor appears only when \(p \mid D_K\). The character \(\chi\) is non-trivial on each factor. Furthermore it factors into local components \(\chi = \prod_{l \mid D_K} \chi_l\).

The goal of this section will be to prove

**Theorem 3.1.** For each degree \(p\) cyclic field \(K\) let \(\sigma_K\) denote a generator of the Galois group and \(D_K\) the discriminant. Then

\[
(3.1) \quad \sum_{K, D_K < X^{p-1}} |\text{im}(1 - \sigma_K)| = \frac{1}{(p-1)p} \sum_{D < X} \mu^2(D) \frac{1}{p^{v(D)}} \sum_{D_0 \cdots D_{p-2}} \sum_{\chi_1} \prod_{v \mid D_{v_1p+v_2}} \chi_1 \left( \prod_{u} D_{u_1p+u_2}^{\Phi(u,v)} \right)
\]

where on the right the first sum is over square-free integers whose prime factors are congruent to 1 mod \(p\) or equal to \(p\) and the second sum is over all such factorizations of \(D\) and the third sum is over all tuples of order \(p\) characters \((\chi_p)_{p|D}\). The products are over \(u, v \in \mathbb{Z}/p^2\mathbb{Z}\), and \(\Phi(u_1p + u_2, v_1p + v_2) = u_1(v_2 - u_1)\).

**Proof.** First fix \(K\). We start with Proposition 2.3. Let \(D\) be as defined at the end of the previous section. Since \(K\) is cyclic, by the Hasse norm theorem \(b\) is a global norm if and only if \(b\) is a local norm everywhere:
\[
b = N\alpha \text{ for some } \alpha \in K \iff b = N\alpha_p \text{ for some } \alpha_p \in K_p, \text{ for all } p.
\]

Here \(K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p\). So we want to detect when \(b \mid D\) is a norm in \(K_p \cong K \otimes_{\mathbb{Q}} \mathbb{Q}_p\) for all \(p \mid D\).
It is a standard fact that $b$ is a local norm at $l$ if and only if the idele $(1, \ldots, b, \ldots, 1) \in C_Q$ is a global norm from $C_K$ (here $b$ is in the coordinate corresponding to $l$). Under the identification $C_Q \cong \prod Z_l^\times$ this is the idele $i_{b,l} = \left( \frac{1}{p}, \ldots, \frac{b}{p}, \ldots, \frac{1}{p} \right)$, where $i$ is maximal such that $l^i \mid b$. Recall that $NC_K = \ker \chi$. Hence $b$ is a local norm at $l$ if and only if $i_{b,l}$ is in the kernel of the character $\chi$ as described above:

$$\chi (i_{b,l}) = \chi_l \left( \frac{b}{p^i} \right) \prod_{q \neq l} \chi_q \left( \frac{1}{l^i} \right) = 1.$$  

We write $\chi (b, l) = \chi (i_{b,l})$. Hence we will use the following expression to detect when $b$ is a local norm:

$$\left( \frac{1 + \chi + \cdots + \chi^{p-1}}{p} \right) (b, l) = \begin{cases} 1 & \text{if } b \text{ is a norm at } l \\ 0 & \text{else} \end{cases}.$$  

Note that $D$ is a $p-1$ power. Write a divisor of $D$ as $b_1 b_2^2 \cdots b_{p-1}^{p-1}$ where the $b_i$ are square free. Thus

$$p | \im (1 - \sigma_K) | = \sum_{b_1 b_2^2 \cdots b_{p-1}^{p-1} | D} \{ b_1 b_2^2 \cdots b_{p-1}^{p-1} \text{ is a global norm} \}$$

In the following we will let $B = b_1 b_2^2 \cdots b_{p-1}^{p-1}$. Using the above we get that this is equal to

$$\sum_{D = (b_0 \cdots b_{p-1})^{p-1}} \prod_{l \mid D} \left( \frac{1 + \chi + \cdots + \chi^{p-1}}{p} \right) (B, l)$$

$$= \frac{1}{p^{\omega(D)}} \sum_{D = (b_0 \cdots b_{p-1})^{p-1}} \prod_{l \mid b_i} \left( 1 + \cdots + \chi^{p-1} \right) (B, l)$$

Further expanding gives

$$\frac{1}{p^{\omega(D)}} \sum_{D = (b_0 \cdots b_{p-1})^{p-1}} \prod_{i=0}^{p-1} \left( \sum_{b_i = D_{i0} D_{i1} \cdots D_{ip-1} j=0}^{p-1} \prod_{D_{ip+j} \mid D} \chi (B^j, l) \right).$$

By definition of $\chi (B^j, l)$ this can be rewritten as

$$\frac{1}{p^{\omega(D)}} \sum_{D_0 \cdots D_{p-1}} \prod_{\substack{i=0 \ldots p-1 \text{ \& not } D_{ip+j} \mid D_q \quad q \neq l}} \chi_q (\frac{1}{l^i}) \chi_l (\frac{B^j}{l^i}).$$

where the first sum is over all such factorizations into coprime integers of $D^{1/p-1}$. Further rearranging:
\[
\frac{1}{p^{\omega(D)}} \sum_{D_{0\cdots D_{p^2-1}}} p^{-1} p^{-1} \left[ \prod_{q\mid D_{D_{ip+j}}} \chi_q \left( \prod_{l\mid D_{ip+j}} \frac{1}{ij} \right) \right] \left[ \prod_{q\mid D_{D_{ip+j}}} \chi_q \left( \frac{1}{ij} \right) \right] \left[ \prod_{q\mid D_{ip+j}} \chi_l \left( \frac{B^j}{ij} \right) \right] \\
= \frac{1}{p^{\omega(D)}} \sum_{D_{0\cdots D_{p^2-1}}} p^{-1} p^{-1} \left[ \prod_{q\mid D_{D_{ip+j}}} \chi_q \left( \prod_{l\mid D_{ip+j}/q} \frac{1}{ij} \right) \right] \left[ \prod_{q\mid D_{D_{ip+j}}} \chi_q \left( \prod_{l\mid D_{ip+j}/q} \frac{q^j}{D_{ip+j}} \right) \right] \left[ \prod_{q\mid D_{ip+j}} \chi_l \left( \frac{B^j}{q^j} \right) \right] .
\]

Now let \( \overline{D}_j = \prod_{i=0}^{p-1} D_{ip+j} \) and \( \overline{D} = \prod_{i,j} D_{ij} \). Then the above is

\[
\frac{1}{p^{\omega(D)}} \sum_{D_{0\cdots D_{p^2-1}}} p^{-1} \prod_{l\mid \overline{D}_j} \chi_l \left( \frac{B^j}{\overline{D}} \right) .
\]

From the definition of \( B \) we have
\[
B = \prod_{i=0}^{p-1} (D_{ip} D_{ip+1} \cdots D_{ip+p-1})^i
\]

hence the exponent of \( D_{u1p+u2} \) in \( B_{\overline{D}} \) is
\[
u_1 \nu_2 - u_1 u_2 = u_1 (\nu_2 - u_2) .
\]

Let \( u = u_1 p + u_2 \) and \( v = v_1 p + v_2 \). Define a map on \( \mathbb{Z}/p^2 \mathbb{Z} \)
\[
\Phi (u, v) = u_1 (v_2 - u_2) .
\]

Thus we conclude that
\[
(3.2) \quad |\text{im} (1 - \sigma_K)| = \frac{1}{p^{\omega(D)}} \sum_{D_{0\cdots D_{p^2-1}}} \prod_{v\mid D_{u1p+v2}} \chi_l \left( \prod_{u} D_u^{\Phi(u,v)} \right) .
\]

Summing over all characters corresponds to summing over all degree \( p \) cyclic fields of discriminant \( D \) but overcounts by a factor of \( p - 1 \) since for a fixed discriminant \( D \) the characters \( \prod_{l\mid D} \chi_l \) and \( \prod_{l\mid D} \chi_l^j \) for \( 0 \leq j \leq p - 1 \) correspond to the same field (indeed the above expression does not change if we replace each \( \chi_l \) with \( \chi_l^j \) simultaneously). Thus we have shown

\[
\sum_{K, D_K = D} |\text{im} (1 - \sigma_K)| = \frac{1}{(p - 1) p^{\omega(D)}} \sum_{D_{0\cdots D_{p^2-1}}} \sum_{\chi_1, \chi_2} \prod_{v\mid D_{u1p+v2}} \chi_l \left( \prod_{u} D_u^{\Phi(u,v)} \right) .
\]

where the second sum on the right is over all tuples of characters \( (\chi_p)_{\mid D} \). Since we are interested in computing the average over all degree \( p \) Galois fields we sum over these (up to \( X^{p-1} \)) to get
\[ \sum_{K, D_K < X^{p-1}} |\text{im} (1 - \sigma_K)| = \frac{1}{(p-1)p} \sum_{D < X} \mu^2(D) \frac{1}{p^{\omega(D)}} \sum_{D_0 \cdots D_{p^2-1}} \prod_{\ell \mid D_v} \chi_{\ell} \left( \prod_u D_u^{\Phi(u,v)} \right). \]

which proves Theorem 3.1. □

4. AN EXPRESSION FOR THE kTH MOMENT

Define
\[ S_k(X) = \sum_{K, D_K < X^{p-1}} |\text{im} (1 - \sigma_K)|^k. \]

We want to generalize Theorem 3.1 to obtain a similar expression for \( S_k(X) \).

We follow the same method as Fouvry and Klüners, to write the \( k \) factorizations of \( D \) as
\[ D = \prod_{u_1} D^{(1)}_{u_1} \cdots = \prod_{u_k} D^{(k)}_{u_k} \]
where each index \( u_i \in \mathbb{Z}/p^2\mathbb{Z} \) (note this differs from the notation in the previous section). From this we obtain a further factorization of each \( D^{(l)}_{u_l} \) by
\[ D^{(l)}_{u_l} = \prod_{1 \leq i \leq k; i \neq l} \gcd(D^{(1)}_{u_1}, \ldots, D^{(k)}_{u_k}). \]

Define
\[ D_{u_1, \ldots, u_k} = \gcd(D^{(1)}_{u_1}, \ldots, D^{(k)}_{u_k}). \]

Hence taking 3.2 to the \( k \)th power we get
\[ \frac{1}{p^k \cdot p^{\omega(D)}} \sum \cdots \sum \prod_{(v_1, \ldots, v_k) \mid D_{v_1, \ldots, v_k}} \prod_{(u_1, \ldots, u_k)} \chi_{\ell} \left( \prod_{(u_1, \ldots, u_k)} D^{\Phi(u_1, v_1) + \cdots + \Phi(u_k, v_k)}_{u_1, \ldots, u_k} \right) \]
where there are \( k \) sums and each is over all factorizations of \( D \). To simplify notation we let \( u = (u_1, \ldots, u_k) \), \( v = (v_1, \ldots, v_k) \), and \( \Phi_k(u, v) = \sum_i \Phi(u_i, v_i) \). Then the expression becomes
\[ \frac{1}{p^k \cdot p^{\omega(D)}} \sum \prod_{(D_u, v) \mid D_v} \chi_{\ell} \left( \prod_{(u)} D^{\Phi_k(u,v)}_u \right) \]
where the sum is over \( p^{2k} \)-tuples of integers with \( \prod_u D_u = D \).

Now we sum over all characters to get
\[ S_k(X) = \frac{1}{(p-1) \cdot p^k \cdot p^{\omega(D)}} \sum \prod_{(D_u, v) \mid D_v} \chi_{\ell} \left( \prod_{(u)} D^{\Phi_k(u,v)}_u \right). \]
Finally we sum over all $D < X$ such that $D^{p-1}$ is a discriminant of a degree $p$ cyclic field which changes the above sum to be over $p^{2k}$-tuples of integers whose prime factors are congruent to 1 mod $p$ or equal to $p$ and $\prod_u D_u < X$.

Thus we have proven

**Proposition 4.1.** For any $k$,

$$
\sum_{K,D_K < X^{p-1}} |\text{im} (1 - \sigma_K)|^k = \frac{1}{(p-1) \cdot p^k} \sum_{(D_u)} \sum_{(\chi_l)} \frac{\mu^2(\prod D_u)}{p^{\omega(\prod D_u)}} \prod_v \prod \chi_l \left( \prod_u D_u^{\Phi_k(u,v)} \right)
$$

where the first sum on the right is over $p^{2k}$-tuples of integers which are coprime and whose prime factors are congruent to 1 mod $p$ or equal to $p$ and the second sum is over all tuples of order $p$ characters $(\chi_p)_{\prod D_u}$ and $u$ and $v$ run over $(\mathbb{Z}/p^2\mathbb{Z})^k$.

Note we can alternately view the indices $u$ in $(\mathbb{F}_p \times \mathbb{F}_p)^k$. We will adopt this notation in Section 7.

The goal will now be to separate this expression into a main term and an error term where the error term is $o(X)$. The significance of $X$ is that it is asymptotically the number of degree $p$ cyclic fields of discriminant bounded by $X^{p-1}$.

5. Analytics Tools

We list the analytic results that will be needed in the sequel. The first two we take directly from [13].

**Lemma 5.1.** There exists an absolute constant $B_0$, such that for every $X \geq 3$ and every $l \geq 0$ we have

$$
|\{n \leq X \mid \omega(n) = l, \mu^2(n) = 1\}| \leq B_0 \frac{X}{\log X} \frac{(\log \log X + B_0)^l}{l!}.
$$

**Lemma 5.2.** Let $\gamma \in \mathbb{R}$ with $\gamma > 0$. Then we have

$$
\sum_{X - Y < n < X} \gamma^{\omega(n)} \ll Y \left( \log X \right)^{\gamma - 1}
$$

for $2 \leq X \exp \left(-\sqrt{\log X} \right) \leq Y \leq X$.

Let $\mathcal{O} = \mathbb{Z}[\zeta_3]$, the ring of integers of the quadratic extension $\mathbb{Q}(\zeta_3)$. Let $\left( \frac{x}{y} \right)_3$ denote the cubic residue symbol for $x, y \in \mathcal{O}$ coprime.

We will need the following results for estimating bilinear sums. They are all versions of the large sieve inequality. The first two containing the $(MN)^\epsilon$-type factor will be used when $M$ and $N$ are close together, and the latter two which do not contain this factor will be used when $M$ and $N$ are far apart. The first is Theorem 2 from [19].
Proposition 5.3. Let $c_n$ be a sequence of complex numbers indexed by elements of $\mathcal{O}$. Then for any $\epsilon > 0$

$$
\sum_{N(n) \leq M} \left| \sum_{N(m) \leq N} \mu^2 (N(n) N(m)) c_n \left( \frac{n}{m} \right) \right|^2 \ll \epsilon \left( M + N + (MN)^{2/3} \right) (MN) \epsilon \sum_n |c_n|^2
$$

where the sums are over elements of $\mathcal{O}$ congruent to 1 mod 3.

Next we have a version for cubic Dirichlet characters and sums over integers, which is Theorem 1.4 from [2].

Proposition 5.4. Let $c_n$ be a sequence of complex numbers. Then for any $\epsilon > 0$

$$
\sum_{Q < q < 2Q \chi \mod q} \left| \sum_{M < m < 2M} a_m \mu^2 (m) \chi (m) \right|^2 \ll \epsilon \left( Q^{11/9} + Q^{2/3} M \right) (QM) \epsilon \sum_m |a_m|^2
$$

where the $\chi$ are primitive Dirichlet characters satisfying $\chi^3 = 1$.

The next version is from [20] and applies to all Dirichlet characters.

Proposition 5.5. Let $c_n$ be a sequence of complex numbers. Then for any $\epsilon > 0$

$$
\sum_{q < Q \chi \mod q} \left| \sum_{M < m < 2M} a_m \mu^2 (m) \chi (m) \right|^2 \ll \epsilon \left( Q^2 + M \right) \sum_m |a_m|^2
$$

where the $\chi$ are primitive Dirichlet characters.

The next is also from [20]. In their terminology a set of $\alpha_r = (\alpha_{r,1}, \ldots, \alpha_{r,k}) \in \mathbb{R}^k$ is $\delta$-spaced if $\max_i |\alpha_{r,i} - \alpha_{r',i}| \geq \delta$ for all $r \neq r'$.

Proposition 5.6. Let $d \geq 1$ and $\delta > 0$ and let $\alpha_r = (\alpha_{r,1}, \ldots, \alpha_{r,d})$ be $\delta$-spaced points in $\mathbb{R}^d / \mathbb{Z}^d$ and $a_n$ a sequence in $\mathbb{C}$ indexed by $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ with $1 \leq n_i \leq N$. Then

$$
\sum_r \sum_n |a_n \exp (2\pi i (n \cdot \alpha_r))|^2 \ll \left( \delta^{-d} + N^d \right) |a|.
$$

Finally we will need a generalized version of Siegel-Walfisz for character sums from [17]. We state a slightly weaker simplified version here.

Proposition 5.7. Let $A, \epsilon > 0$. Let $K$ be Galois of degree $n$ and let $\chi$ be a finite Hecke character of $K$ with conductor $f_\chi$. Then there exists a positive constant $c = c(A, \epsilon)$, not depending on $K$ or $\chi$ such that

$$
\sum_{\substack{N(p) \leq x, (p, f_\chi) = 1}} \chi (p) = O \left( dx \log^2 x \exp \left( -cn (\log x)^{1/2} / d \right) \right)
$$

where $d = n^3 |D_K N (f_\chi)|^e c^{-n}$. The implied constant does not depend on $K$ or $\chi$.
6. Bounding The Error Term

We start with the expression for \( \sum_{K,D_K<D} |\text{im}(1 - \sigma_K)|^k \) which we derived in Proposition 4.1.

\[
S_k(X) = \frac{1}{(p - 1) \cdot p^k} \sum_{(D_u)} \sum_{(\chi_l)} \frac{\mu^2(\prod D_u)}{p^{k\omega(\prod D_u)}} \prod_{v|D_u} \chi_l \left( \prod_{v} D^\Phi_{k}(u,v) \right)
\]

On the right we are summing over \( p^{2k} \)-tuples of integers with \( \prod D_u < X \) whose primes factors are congruent to 1 mod \( p \) or equal to \( p \).

Fix \( k \in \mathbb{Z}_{\geq 1} \) and let \( \Delta = 1 + \log^{-1(p-1)p^k} X \). Define \( A \) to be a tuple \( (A_i)_{i=0}^{p^{2k}} \) of variables with each \( A_i \) corresponding to \( D_i \), and each \( A_i = \Delta^j \) for some \( j \geq 0 \). We can partition \( S_k(X) \) according to the various \( A \), by letting \( S_k(X, A) \) be the above sum but now restricted to tuples \( (D_i) \) for which \( A_i \leq D_i \leq \Delta A_i \) and \( \prod D_i < X \). Hence

\[
S_k(X) = \sum_{A} S_k(X, A).
\]

Note that if \( \Delta = 1 + \log^{-1(p-1)p^k} X \) then there are \( O \left( \log X \right)^2 \left( 1 + (p-1)p^k \right) \) possible \( A \) with \( S_k(X, A) \) not empty. This is since there are \( O \left( \log X \right) \left( 1 + (p-1)p^k \right) \) choices for each \( 1 \leq A_i \leq X \).

We now consider certain families of the \( A \) which make a negligible contribution to the sum and hence can be removed. These will be the same as the four families from [13], section 5.4. In the case of the first two the argument is identical. For completeness we reproduce their arguments here. The proofs involving the third and fourth families require modification.

First we reduce the sum to terms where all of the \( D_u \) satisfy \( \omega(D_u) \leq \Omega \), where we define \( \Omega = e p^{2k} (\log \log X + B_0) \). We restate the argument from [13].

Let \( S_0 \) be the terms in (6.1) where not all of the \( D_u \) satisfy \( \omega(D_u) \leq \Omega \). Let \( n = \prod D_u \). Then

\[
S_0 \ll \sum_{n < X, \omega(n) > \Omega} \mu^2(n) p^{k\omega(n)}.
\]
Then splitting the sum up by number of prime factors and applying Lemma 5.1 we get the bound

\[
\sum_{n < X, \omega(n) > \Omega} \mu^2(n) p^{k\omega(n)} \ll \sum_{l \geq \Omega} \frac{X}{\log X} p^{k\ell} \frac{(\log \log X + B_0)^l}{l!} \\
\ll \frac{X}{\log X} \sum_{l \geq \Omega} \left( \frac{p^k (\log \log X + B_0)}{l/e} \right)^l \\
\ll \frac{X}{\log X} \sum_{l \geq \Omega} \left( \frac{1}{p^k} \right)^l \\
\ll X/\log X
\]

where in the last inequality we are using \( l/e \geq p^{2k} (\log \log X + B_0) \). Thus we can assume in the remainder that all variables \( D_u \) satisfy \( \omega(D_u) \leq \Omega \). We will only need this fact to bound family 4.

6.1. The first family. Note that it is possible to have an \( A \) for which \( S_k(X, A) \) is not empty, but

\[
(6.2) \quad \prod_i \Delta A_i > X
\]

making the restriction \( D < X \) necessary. By Lemma 5.2 we have

\[
\sum_{A \text{ satisfies } (6.2)} S_k(X, A) \ll \sum_{\Delta^{-p^{2k}} X \leq D \leq X} p^{2k\omega(D)} (p - 1)^{\omega(D)} \left( \frac{1}{p^k} \right)^{k\omega(D)} \\
= \sum_{\Delta^{-p^{2k}} X \leq D \leq X} ((p - 1) \cdot p^k)^{\omega(D)} \\
\ll \left( 1 - \Delta^{-p^{2k}} \right) X (\log X)^{(p-1)p^k-1}
\]

Using that \( (1 + x)^\alpha = 1 + \alpha x + O(x^2) \) for \( x \to 0 \), we get \( \Delta^{-p^{2k}} = \left( 1 + \log^{-2(p-1)p^k} X \right)^{-p^{2k}} = 1 - p^{2k} \log^{-2(p-1)p^k} X + O \left( \log^{-2(p-1)p^k} X \right) \). This gives the bound

\[
\ll \left( p^{2k} \log^{-2(p-1)p^k} X + O \left( \log^{-2(p-1)p^k} X \right) \right) X (\log X)^{(p-1)p^k-1} \\
\ll X/\log X.
\]

6.2. The second family. In the remaining sections we will need the following quantities. Let

\[
X^\dagger = \log^{4(1+p^{2k}(1+(p-1)p^k))} X \\
X^\ddagger = \exp(\log^n X)
\]
for some small $\eta$. Consider the $A$ which satisfy

\[(6.3)\]

at most $p^{k-1}$ variables $A_i > X^\dagger$.

Let $r$ be the number of variables greater than $X^\dagger$. We have

$$
\sum_{A \text{ satisfies } (6.3)} S_k (X, A) \ll \sum_{r=0}^{p^{k-1}} \sum_{m < (X^\dagger)^{p^{2k-r}}} \mu^2 (m) \tau_{p^{2k-r}} (m) \left( \frac{p-1}{p^k} \right) \omega(m) \\
\times \sum_{n < X/m} \mu^2 (n) \tau_r (n) \left( \frac{p-1}{p^k} \right) \omega(n) \\
= \sum_{r=0}^{p^{k-1}} \sum_{m < (X^\dagger)^{p^{2k-r}}} \mu^2 (m) \left( p^{2k} - r \right) \omega(m) \left( \frac{p-1}{p^k} \right) \omega(m) \\
\times \sum_{n < X/m} \mu^2 (n) \left( \frac{p-1}{p^k} \right) r \omega(n)
$$

Then we get using Lemma 5.2

$$
\ll \sum_{r=0}^{p^{k-1}} \sum_{m < (X^\dagger)^{p^{2k-r}}} \frac{\left( (p-1) \cdot p^k \right) \omega(m)}{m} (X/m) (\log X)^{(p-1)r/p^k-1} \\
\ll X (\log X)^{\eta(p-1) p^{k-1}/p} .
$$

6.3. **The third family, the case $p = 3$.** For the third and fourth families we will assume $p = 3$ and bound the error term unconditionally. Afterwards we will handle the case of general $p$ case under the assumption of GRH.

Before continuing we use cubic reciprocity to rewrite the expression we have been working with so far. Let $\zeta$ be a cube root of unity. We recall some facts about the field $\mathbb{Q} (\zeta)$ and the cubic residue symbol which can be found in [2]. The ring of integers of $\mathbb{Q} (\zeta_3)$ is $\mathcal{O} = \mathbb{Z} [\zeta]$. It is a principle ideal domain and every ideal $(n) \subset \mathcal{O}$ with $(n, 3) = 1$ has a unique generator $n$ which satisfies $n \equiv 1 \mod 3$. The only prime which ramifies is $3 = (1 - \zeta)^2$. The primes of $\mathbb{Z}$ which split in $\mathcal{O}$ are exactly the ones congruent to $1 \mod 3$ in $\mathbb{Z}$. We can choose a basis $\{1, \zeta\}$ for $\mathcal{O}$, so that every element can be written as $a + b\zeta$, and then $N (a + b\zeta) = a^2 + b^2 - ab$. Using this it can be shown that $|\{a + b\zeta \in \mathcal{O} \mid N (a + b\zeta) \leq A\}| \ll A$ (both $a$ and $b$ are $O (A^{1/2})$).

In the remainder of the paper all summations over elements of $\mathcal{O}$ will be restricted to those which are products of split primes and which are congruent to $1 \mod 3$ (by the above these can be viewed as summations over integral ideals of $\mathcal{O}$).

Denote by $\left( \frac{a}{m} \right)_3$ the cubic residue symbol, defined for $(m) \neq (1 - \zeta)$ and $n$ coprime to $m$. For $l \neq 3$ the characters $\chi_l$ are cubic dirichlet characters and hence for $n \in \mathbb{Z}$ we have $\chi_l (n) = \left( \frac{n}{\pi} \right)_3$ for some $\pi$ with $N (\pi) = l$. 
We define some terminology which will be used in the remainder of the paper. We define indices \( u \) and \( v \) to be linked if \( \Phi_k (u,v) \neq 0 \). Otherwise we say they are unlinked.

Consider the third family which consists of those \( A \) such that there are two linked indices \( A_u \) and \( A_v \), and

\[
(6.4) \quad A_u, A_v > X^*. 
\]

Fix such an \( A \) and two linked indices \( u, v \). We consider two cases: case 1 will be when both \( \Phi_k (u,v) \) and \( \Phi_k (v,u) \) are nonzero in \([6.1]\) and case 2 will be when only one of these is nonzero.

Case 1: Both \( \Phi_k (u,v) \) and \( \Phi_k (v,u) \) are nonzero.

**Lemma 6.1.** For any linked indices \( u \) and \( v \) with \( \Phi_k (u,v) \) and \( \Phi_k (v,u) \) both nonzero,

\[
(6.5) \quad S_k (X, A) \ll \sum_{D_u, w \neq u, v} \sum_{d_u, d_w \in \mathcal{O}} a (d_u) a (d_w) \left( \frac{d_u}{d_w} \right)_3 
\]

with \( |a (d_i)| \leq 1 \), where the summations over elements of \( d_i \in \mathcal{O} \) congruent to 1 mod 3 which are products of split primes, \( \mu^2 (N (d_i)) = 1 \) and \( N (d_i) \leq \Delta A_i \).

**Proof.** Fix two indices \( u', v' \) such that \( \Phi_k (u', v') \neq 0 \) and \( \Phi_k (u', v') \neq 0 \). In the following we write \( D = \prod_w D_w \) and \( D' = D_u D_v \). Then from \([6.1]\) we get

\[
S_k (X, A) = \frac{1}{2 \cdot 3^k} \sum_{D_w, w \neq u, v} \frac{\mu^2 (D)}{3^{k \omega (D)}} \prod_y \prod_{l \mid D_y} \chi_l \left( \prod_z D_z ^{\Phi_k (z, y)} \right) 
\]

\[
= \frac{1}{2 \cdot 3^k} \sum_{D_w, w \neq u, v} \sum_{(\chi_l), l \mid D'/D} 1 \prod_{D_u, D_v} \prod_{(\chi_l), l \mid D'} \frac{\mu^2 (D)}{3^{k \omega (D'/D')}} \prod_z D_z ^{\Phi_k (z, y)} 
\]

\[
\times \prod_y \prod_{l \mid D_y} \chi_l \left( \prod_z D_z ^{\Phi_k (z, y)} \right) 
\]

We can split this sum into 3 pieces according to whether \( 3 \mid D_u \), \( 3 \mid D_v \) or \( 3 \mid D'/D' \) and apply the next argument to each case separately. We illustrate the case \( 3 \mid D_v \), the others being handled similarly.

In the inner sum over \( D_u, D_v \) we can replace the \( \chi_l \) with cubic residue symbols:

\[
\sum_{D_u, D_v} \frac{\mu^2 (D)}{3^{k \omega (D'/D')}} \prod_y \prod_{l \mid D_y} \chi_l \left( \prod_z D_z ^{\Phi_k (z, y)} \right) 
\]

\[
\ll \sum_{d_u, d_v} b (d_u) b (d_v) \left( \frac{D_u}{d_u} \right)_3 \left( \frac{D_v}{d_v} \right)_3 \left( \frac{D_v/3}{d_u} \right)_3 
\]

\[
(6.6) \quad = \sum_{d_u, d_v} a (d_u) a (d_v) \left( \frac{D_u}{d_u} \right)_3 \left( \frac{D_v/3}{d_v} \right)_3 
\]
where in the second line $D_u, D_v$ now denote $N(d_u), N(d_v)$ and

$$b(d_u) = \frac{\mu^2(D)}{3^{k_\infty(D)}} \prod_{y \neq u, v \mid D_y} \chi_1(D_u^{\Phi_k(u,y)}) \left( \frac{3^{\Phi_k(u,v)} \prod_{y \neq u,v} D_v^{\Phi_k(y,u)}}{d_u} \right)^3$$

and similarly for $b(d_v)$ (which will contain the factor $\chi_3(D_u)$). The elements $d_v, d_u$ in the summation satisfy $A_v \leq N(d_v) \leq \Delta A_v$ and $A_u \leq N(d_u) \leq \Delta A_u$ and $d_u \equiv 1 \mod 3$ and $d_v/(1 - \zeta) \equiv 1 \mod 3$. We can instead sum over $d_v$ such that $A_v/3 \leq N(d_v) \leq \Delta A_v/3$ and $d_v \equiv 1 \mod 3$. Note $\Phi_k$ is either 1 or 2, and squaring a cubic character is the same as conjugating it. Since we are summing over a set of characters closed under conjugation removing $\Phi_k$ from the exponent permutes the coefficients.

Furthermore by cubic reciprocity, for any $D_u = N(d_u)$ and $D_v = N(d_v)$ not divisible by 3 we have

$$\left( \frac{D_u}{d_v} \right)_3 \left( \frac{D_v}{d_u} \right)_3 = \left( \frac{d_u}{d_v} \right)_3 \left( \frac{\overline{d_u}}{d_v} \right)_3 \left( \frac{d_v}{d_u} \right)_3 \left( \frac{\overline{d_v}}{d_u} \right)_3$$

$$= \left( \frac{d_u}{d_v} \right)_3^2 \left( \frac{\overline{d_u}}{d_v} \right)_3 \left( \frac{d_v}{d_u} \right)_3 \left( \frac{\overline{d_v}}{d_u} \right)_3$$

$$= \left( \frac{d_u}{d_v} \right)_3 \left( \frac{\overline{d_u}}{d_v} \right)_3.$$

This proves the lemma. \qed

We will follow the standard strategy of bounding bilinear sums using Cauchy-Schwarz followed by a large sieve type bound. By Cauchy-Schwarz we have

$$\left| \sum_{d_v \in \mathcal{O}} \sum_{d_u \in \mathcal{O}} a(d_u) a(d_v) \left( \frac{d_u}{d_v} \right)_3 \right| \ll A_v^{1/2} \left( \sum_{d_v \in \mathcal{O}} \left| \sum_{d_u \in \mathcal{O}} a(d_u) \left( \frac{d_u}{d_v} \right)_3 \right| \right)^{1/2}$$

Now we focus on bounding the summation on the right hand side. For $r, n \in \mathcal{O}$, with $(r, n) = 1$, $n \equiv 1 \mod 3$ and $\mu^2(N(n)) = 1$ define

$$g(r, n) = \sum_{d \mid (modn)} \left( \frac{d}{n} \right)_3 \tilde{\epsilon}(rd/n)$$

to be the generalized cubic gauss sum where $\tilde{\epsilon}(z) = \exp(2\pi i (z + \overline{z}))$. The following facts can be found in [2]. It satisfies the property

$$g(rs, n) = \overline{\left( \frac{s}{n} \right)_3} g(r, n).$$

We will write $g(n) = g(1, n)$. 


Thus the above sum becomes
\[
\sum_{d_v} \left| \sum_{d_u} a (d_u) \left( \frac{d_u}{d_v} \right) \right|^2 \leq \sum_{d_v} \frac{1}{|g(d_v)|^2} \left| \sum_{d_u} a (d_u) g (d_u, d_v) \right|^2 \\
\leq \sum_{d_v} \frac{1}{|g(d_v)|^2} \left| \sum_{d_{\text{mod}d_v}} \left( \frac{d}{d_v} \right) \sum_{d_u} a (d_u) \check{e} (dd_u/d_v) \right|^2 \\
\leq \sum_{d_v} \frac{1}{|g(d_v)|^2} \left| \sum_{d_{\text{mod}d_v}} \chi (d) \sum_{d_u} a (d_u) \check{e} (dd_u/d_v) \right|^2 \\
\leq \sum_{d_v} \sum_{d_{\text{mod}d_v}} \left| \sum_{d_u} a (d_u) \check{e} (dd_u/d_v) \right|^2
\]

where the summation \( \sum_{d_{\text{mod}d_v}}^* \) is over primitive characters of \( (\mathcal{O}/d_v)^{\times} \) and in the last line we are opening the square and using orthogonality of characters. Using our previously chosen basis \( \{1, \zeta\} \) we can rewrite a summation over elements of \( \mathcal{O} \) as one over \( \mathbb{Z}^2 \). We obtain the bound
\[
\sum_{d_v} \sum_{d_{\text{mod}d_v}} \left| \sum_{d_u} a (d_u) \check{e} (dd_u/d_v) \right|^2 \ll \sum_{d_v} \sum_{d_{\text{mod}d_v}} \left| \sum_{s_1, s_2} a (s_1, s_2) \check{e} (dd_u/d_v) \right|^2
\]

where \( s_i \in \mathbb{Z} \) and \( s_i \ll A_v^{1/2} \).

Write \( d/d_v = (d_1, d_2) \) and \( d_u = (s_1, s_2) \). Then \( dd_u/d_v = (d_1s_1 - d_2s_2, d_1s_2 + d_2s_1 - d_2s_2) \) and
\[
\text{tr} (dd_u/d_v) = s_1 (2d_1 - d_2) + s_2 (-d_1 - d_2) \\
= (s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2).
\]

Hence we can rewrite the right hand side of the above inequality as
\[
\sum_{d_v} \sum_{d_{\text{mod}d_v}} \left| \sum_{s_1, s_2} a (s_1, s_2) \exp (2\pi i (s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2)) \right|^2
\]

We want to apply a multivariable large sieve inequality of Proposition 5.6 so we will first show that the set \( S \) of \( (2d_1 - d_2, d_1 - d_2) \) (where \( d/d_v \) runs over the above summation) is \( 1/A_v \)-spaced. Note that for any \( a/c, b/c \in \mathbb{Q} \) we have \( |a/c - b/c| \geq 1/c \). Hence the spacing of a set is determined by the denominators of the coordinates of its elements.

First note that the set of \( (d_1, d_2) \) obtained from \( d/d_v \) is distinct in \( (\mathbb{Q}/\mathbb{Z})^2 \). If not this implies \( d/d_v = d'/d_v' + n \) for some \( n \in \mathcal{O} \), so \( dd_v' = dd_v' + nd_v d_v' \) and noting that \( (d, d_v) = (d', d_v') = 1 \) and considering divisors of both sides gives a contradiction (recall \( \mathcal{O} \) is a UFD). Furthermore \( (d_1, d_2) \mapsto (2d_1 - d_2, d_1 - d_2) \) is a linear map which is
invertible over \(\mathbb{Z}\), hence the elements of \(S\) are all distinct. We can write
\[
\frac{d}{d_v} = \frac{dd_v}{N(d_v)} = \frac{a}{N(d_v)} + \frac{b}{N(d_v)} \zeta
\]
for some \(a, b \in \mathbb{Z}\). Since \(N(d_v) \leq A_v\) it follows that \(S\) is \(1/A_v\)-spaced as required. Thus by Proposition 5.6 we get
\[
\sum_{d_v \in O} \left| \sum_{d(v) \in O} a(s_1, s_2) \exp(2\pi i(s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2)) \right|^2 \ll (A_v^2 + A_u) A_u.
\]
Returning to the expression we started with we get
\[
\sum_{d_v \in O} \left| \sum_{d(v) \in O} a(d_u) a(d_v) \left(\frac{d_u}{d_v}\right) \right| \ll A_{1/2}^2 ((A_v^2 + A_u) A_u)^{1/2}
\]
(6.8)
\[
= A_v A_u \left(\frac{A_v}{A_u} + \frac{1}{A_v}\right)^{1/2}.
\]
By symmetry we can also bound this by \(A_v A_u \left(\frac{A_v}{A_u} + \frac{1}{A_v}\right)^{1/2}\).

By symmetry we can let \(A_v \leq A_u\). First suppose \(A_v^2 < A_u\). By the above bound (6.8) we get
\[
S_k (X, A) \ll \sum_{D_w, w \neq u, v} A_v A_u \left(\frac{A_v}{A_u} + \frac{1}{A_v}\right)^{1/2}
\]
\[
= X \left(\frac{1}{A_u^{1/2}} + \frac{1}{A_v}\right)^{1/2}
\]
\[
= X/ \log^{1+9^k(1+2k)} X.
\]
Now suppose \(A_v \leq A_u^2\). Then by Proposition 5.3 we get the bound, for any \(\epsilon > 0\),
\[
S_k (X, A) \ll \sum_{D_w, w \neq u, v} A_v^{1/2} \left(\left(\frac{A_u + A_v + (A_u A_v)^{2/3}}{A_v}\right) (A_u A_v)^{\epsilon} A_u\right)^{1/2}
\]
\[
\ll X \left(\frac{1}{A_v} + \frac{1}{A_u} + \frac{1}{(A_u A_v)^{1/3}}\right) (A_u A_v)^{\epsilon} A_u \right)^{1/2}
\]
\[
\ll X \left(\frac{1}{X^{1/2}}\right)^{1/2}
\]
\[
= X/ \log^{1+9^k(1+2k)} X.
\]

Case 2: Only one of \(\Phi_k (u, v)\) and \(\Phi_k (v, u)\) is nonzero. Without loss of generality assume \(\Phi_k (u, v)\) is nonzero.
**Lemma 6.2.** For any linked indices $u$ and $v$ with $\Phi_k(u, v)$ and $\Phi_k(v, u)$ both nonzero,

$$S_k(X, A) \leq \sum_{D_{w, w \neq u, v}} \left| \sum_{d_v \in \mathcal{O}} \sum_{D_w \in \mathbb{Z}} a(D_w) a(D_u) \left( \frac{D_u}{d_v} \right)_3 \right|^2,$$

with $|a(d_i)| \leq 1$, where the summations over elements of $d_i \in \mathcal{O}$ which are products of split primes, $\mu^2(N(d_i)) = 1$ and $N(d_i) \leq \Delta A_i$ and $D_i \in \mathbb{Z}$ a product of primes congruent to $1 \mod 3$, $\mu^2(D_i) = 1$ and $D_i \leq \Delta A_i$.

**Proof.** This is a less involved version of the proof of Lemma 6.1. □

The above expression is no longer symmetric in $u$ and $v$ hence we must consider several subcases. In the case when the variables $A_u$ and $A_v$ are close together, specifically $A_u < A_v < A_u^2$ or $A_v < A_u < A_v^2$, we apply Theorem 5.4. For example if $A_u < A_v < A_u^2$ we get

$$S_k(X, A) \leq \sum_{D_{w, w \neq u, v}} A_u^{1/2} \left( \sum_{D_u} \sum_{D_v} a(D_u) \left( \frac{D_u}{d_v} \right)_3 \right)^2 \ll \sum_{D_{w, w \neq u, v}} A_v^{1/2} \left( (A_u^{11/9} + A_v^{2/3} A_u) (A_u A_v)' A_u \right)^{1/2} \ll X \left( \frac{A_u^{2/9} + 1}{A_v^{1/3}} (A_u A_v)' \right)^{1/2} \ll X/\log(1+9k(1+2\cdot3^k))^{1/2} X$$

Now suppose $A_u^2 < A_v$. Define a cubic Hecke character

$$\psi : (d_v) \mapsto \left( \frac{D_u}{d_v} \right)_3$$

of modulus $9D_u$.

As in case 1 define the Gauss sum

$$g(r, n) = \sum_{d \in (\mathcal{O}/9D_u)^\times} \psi(d) \bar{\epsilon}(rd/n)$$

which satisfies for $(r, 9D_u) = 1$

$$g(r, D_u) = \overline{\psi(r)} g(1, D_u).$$

As in case 1 we get

$$\sum_{D_u} \left| \sum_{d_v} a(D_v) \left( \frac{D_u}{d_v} \right)_3 \right|^2 \ll \sum_{D_u} \sum_{d \in (\mathcal{O}/9D_u)^\times} \left| \sum_{s_1, s_2} a(s_1, s_2) \bar{\epsilon}(dd_v/D_u) \right|^2$$

where $s_i \in \mathbb{Z}$ and $s_i \ll A_v^{1/2}$. 


Write \( d/D_u = (d_1, d_2) \) and \( d_v/D_u = (d_1 s_1 - d_2 s_2, d_1 s_2 + d_2 s_1 - d_2 s_2) \) and
\[
\text{tr}(dd_v/D_u) = s_1 (2d_1 - d_2) + s_2 (-d_1 - d_2) = (s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2).
\]
Hence we can rewrite the right hand side of the above inequality as
\[
\sum_{D_u} \sum_{d \in (O/9D_u)^k} \left| \sum_{s_1, s_2} a(s_1, s_2) \exp \left( 2\pi i (s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2) \right) \right|^2
\]
By the same argument as before the set \( S \) of \((2d_1 - d_2, d_1 - d_2)\) (where \( d/D_u \) runs over the above summation) is distinct and \( 1/A_u \)-spaced since we can write \( \frac{d}{D_u} = \frac{a}{D_u} + \frac{b}{D_u} \zeta \) for some \( a, b \in \mathbb{Z} \). Thus by Proposition 5.6 we get
\[
\sum_{D_u} \sum_{d \in (O/9D_u)^k} \left| \sum_{s_1, s_2} a(s_1, s_2) \exp \left( 2\pi i (s_1, s_2) \cdot (2d_1 - d_2, d_1 - d_2) \right) \right|^2 \ll (A_u^2 + A_v) A_v.
\]
Returning to the expression we started with we get
\[
\left| \sum_{D_u} \sum_{d_v \in O} a(D_u) a(D_v) \left( \frac{D_u}{d_v} \right) \right| \ll A_u^{1/2} \left( (A_u^2 + A_v) A_v \right)^{1/2}
\]
By the above bound we get
\[
S_k(X, A) \ll \sum_{D_u, w \neq u, v} A_u A_v \left( \frac{A_u}{A_v} + \frac{1}{A_u} \right)^{1/2}
\]
Finally summing over all such \( A \) we get
\[
\sum_{A \text{ satisfies } (6.4)} S_k(X, A) \ll X/\log X.
\]
6.4. **The fourth family, the case** \( p = 3 \). Now consider the fourth family which consists of those \( A \) which are not in the third family and such that there are two linked indices \( A_u \) and \( A_v \) and

\[
(6.10) \quad A_u > X^\dagger, 2 \leq A_v < X^\dagger.
\]

Note the condition \( 2 \leq A_v < X^\dagger \) is forced by the assumption that \( A \) is not in the third family. We in fact consider the collection of all indices \( v \) which satisfy the above condition. Denote this set by \( S \).

**Lemma 6.3.** For \( A, u \) and \( S \) as defined above we have

\[
S_k (X, A) \ll \sum_{D_v, v \neq u \ (\chi_1)_l | D_v} \sum_{\omega (\prod_{w \neq u} D_w)} \frac{1}{3^\omega (\prod_{w \neq u} D_w)} \left| \sum_{d_u \in \mathcal{O}} \mu^2 (D) \psi (d_u) \right|
\]

where we denote \( D_u = N (d_u) \) and \( D = \prod_{w \neq u} D_w \) and the summation is over \( d_u \equiv 1 \mod 3 \) which are products of split primes and \( A_u \leq N (d_u) \leq \Delta A_u \) and \( D_i \in \mathbb{Z} \) a product of primes congruent to \( 1 \mod 3 \) and \( A_v \leq D_v \leq \Delta A_v \). Furthermore \( \psi \) is a Hecke character given by

\[
\psi (d_u) = \left( \frac{\prod_{l \neq u} D_{v (u, v)}^{\Phi_l (v, u)} d_u}{d_v} \right)^3 \prod_{3 \in S \ | | D_v} \chi_l (D_{u (u, v)}) \chi_3 (D_u)^i.
\]

**Proof.** The proof is similar to that of Lemma 6.1. As in the previous subsection we consider separately the cases when \( 3 \mid D_u \) and \( 3 \nmid D_u \). In the first case we replace \( d_u \) by \( d_u / (1 - \zeta) \) and sum in the range \( A_u / 3 \leq N (d_u) \leq \Delta A_u / 3 \). The factor \( \chi_3 \) in the definition of \( \psi \) appears only when \( 3 \mid D_v \) for some \( v \in S \). \( \square \)

We remark that the \( v \in S \) by assumption satisfy \( A_v < X^\dagger \). The modulus of \( \psi \) is \( f_\psi = 9 \prod_{v \in S} D_v \) and \( N (f_\psi) \leq 81 (X^\dagger)^{2.96} \). Note the \( \mu^2 \) factor means the non-zero terms of the summation above come from \( d_u \) whose prime factors are all split.

Now looking at the inner sum

\[
\sum_{d_u} \mu^2 (D) \psi (d_u) = \sum_{l=0}^{\Omega} \frac{1}{3^l} \sum_{\pi_1, \ldots, \pi_{l-1}} \psi (\pi_1 \cdots \pi_{l-1})
\]

\[
\times \sum_{\pi_l} \mu^2 \left( \prod_{w \neq u} D_w N (\pi_1 \cdots \pi_l) \right) \psi (\pi_l)
\]

where \( (\pi_i) \) are split primes with \( N (\pi_i) \leq \Delta A_u \) and \( A_u \leq N (\pi_l) \leq \Delta A_u / N (\pi_1 \cdots \pi_{l-1}) \). Then again looking at the inner sum

\[
\sum_{\pi_l} \mu^2 \left( \prod_{w \neq u} D_w N (\pi_1 \cdots \pi_l) \right) \psi (\pi_l) \ll \sum_{\pi_l (\pi_l, f_\psi) = 1} \psi (\pi_l) + \Omega + A_u^{1/2}
\]
where the \( \Omega \) term comes from removing \( \mu^2 \) and the \( A_u^{1/2} \) term comes from including inert primes in the sum—recall that in our original summation we only included primes above \( l \) such that \( l \equiv 1 \mod 3 \) and \( l = 3 \). These are exactly the split primes. The number of inert primes \( l \) in \( \mathcal{O} \) with \( l^2 = Nl < A_u \) is \( A_u^{1/2} \). Now we apply Proposition \ref{prop:split_primes} with \( f_\psi \) and \( x = \Delta A_u / N (\pi_1 \cdots \pi_{l-1}) \) to get

\[
\sum_{\pi_i \mid (\pi_i, f_\psi) = 1} \psi(\pi_i) \ll \frac{N (f_\psi)^e x (\log x)^2}{\exp \left( c^{4} (\log x)^{1/2} / 3^{2+\varepsilon} N (f_\psi)^e \right)}
\ll \frac{N (f_\psi)^e x (\log x)^2}{\exp \left( c^{4} (\log x)^{1/2} / 3^{2+5\varepsilon} (X^\dagger)^{2.9k_e} \right)}
\ll X^{\dagger 2.9k_e} \frac{A_u}{N (\pi_1 \cdots \pi_{l-1})} \frac{(\log x)^2}{\exp \left( c^{4} (\log X)^{\eta/4-2.9k_e} / 3^{2+5\varepsilon} \right)}
\]

where in the second inequality we use \( N (f_\psi)^e \leq 81 (X^\dagger)^{2.9k_e} \) which implies \( \exp \left( -1/N (f_\psi)^e \right) \ll \exp \left( -1/81^e (X^\dagger)^{2.9k_e} \right) \). In the third inequality we use

\[
x \geq N (\pi_i)
\geq A_u^{1/4}
\geq \exp \left( \log^{\eta/2} X \right)
\]

and that \( X^\dagger \) is some fixed power of \( \log X \). Summing over \( \pi_1 \cdots \pi_{l-1} \) gives the bound

\[
\sum_{\pi_1, \ldots, \pi_{l-1}} \frac{1}{N (\pi_1 \cdots \pi_{l-1})} \leq \log A_u.
\]

Finally we use \( A_v^e < X^\dagger \) for all \( v \in S \) and end up with

\[
\ll A_u \frac{(\log X)^{4(1+9k(1+2.3^k))}+3}{\exp \left( c^{4} (\log X)^{\eta/4-2.9k_e} / 3^{2+5\varepsilon} \right)}
\]

and summing over the remaining variables \( D_v \) gives

\[
S_k (X, A) \ll X \frac{(\log X)^{4(1+9k(1+2.3^k))}+3}{\exp \left( c^{4} (\log X)^{\eta/4-2.9k_e} / 3^{2+5\varepsilon} \right)}
\]

Then summing over all \( A \)

\[
\sum_{A \text{ satisfies \ref{eq:conditions}}} S_k (X, A) \ll X \frac{(\log X)^{4(1+9k(1+2.3^k))}+3+9k(1+2.3^k)}{\exp \left( c^{4} (\log X)^{\eta/4-2.9k_e} / 3^{2+5\varepsilon} \right)}
= o(X).
\]
6.5. **The third and fourth families for all** \( p \). Assume GRH for Artin \( L \)-functions. The missing ingredients required to extend our result to general \( p \) unconditionally are analogs of Proposition 5.3 and 5.4. That is, we cannot deal with the case in family 3 when \( A_u \) and \( A_v \) are close together. We will instead give a proof assuming GRH. The following argument replaces the sections containing families 3 and 4 for \( p = 3 \) above.

Suppose \( A \) does not belong to families 1 and 2, that is \( A \) does not satisfy (6.2) and (6.3). In particular there are at least \( p^{k-1} + 1 \) variables \( A_w \) which satisfy \( A_w > X^4 \). Let \( A_u \) be the largest of these. Let \( S \) be the set of indices linked with \( u \) and suppose it is not empty.

Let \( \zeta = e^{2\pi i/p} \) and let \( \mathcal{O} = \mathbb{Z} [\zeta] \) the ring of integers of \( \mathbb{Q} (\zeta) \) which is a degree \( p - 1 \) extension of \( \mathbb{Q} \).

Define

\[
\left[ \frac{A}{B} \right]_p = \prod_{\ell \mid B} \left( \chi_{\ell} + \cdots + \chi_{\ell}^{p-1} \right) \left( \frac{A}{\ell} \right). 
\]

Then we have

\[
S_k (X, A) \ll \sum_{D_w, w \neq u} \left| \sum_{D_u} \mu^2 (D) \left( \frac{D_u}{c_u} \right)_p \left[ \frac{C_u}{D_u} \right]_p \right|
\]

where \( C_u = \prod_w D_w^{e_w} \) for \( e_w \leq p - 1 \) and some choice of \( c_u \in \mathcal{O} \) with \( N (c_u) = \prod_{w \in S} D_w \). The range of summation is \( A_i \leq D_i \leq \Delta A_i \) and divisibility by \( p \) is handled as in the \( p = 3 \) cases above.

Note that

\[
\chi : D_u \rightarrow \left( \frac{D_u}{c_u} \right)_p 
\]

is a Dirichlet character of modulus \( N (c_u) \). Let \( K = \mathbb{Q} (\zeta, \sqrt{C_u}) \) and identify \( \phi : \text{Gal} (K/\mathbb{Q} (\zeta)) \rightarrow \mu_p \) where \( \mu_p \) is the group of roots of unity and define a representation of \( \text{Gal} (K/\mathbb{Q} (\zeta)) \)

\[
\tau (g) = \begin{pmatrix} \phi (g) \\ \vdots \\ \phi (g)^{p-1} \end{pmatrix}.
\]

Let \( \rho \) be the induction of \( \tau \) to \( \text{Gal} (K/\mathbb{Q}) \). Then since \( D_u \) is a product of primes \( q \equiv 1 (p) \) we have

\[
\left[ \frac{C_u}{D_u} \right]_p = \prod_{q \mid D_u} \text{tr} \rho (F_q) \left( \frac{q}{C_u} \right)_p / p^{\omega (D_u)}.
\]

Let \( M \) be the degree \( p \) cyclic field corresponding to the character \( \chi \) and let \( L = KM \). Let \( \sigma \) be the representation of \( \text{Gal} (L/\mathbb{Q}) \) given by

\[
\sigma (q) = \left( \frac{q}{c_u} \right)_p \otimes \rho (q).
\]
Now consider the $L$-function

$$L(s, \sigma) = \prod_q \det \left( I - \frac{\sigma(F_q)}{q^{-s}} \right)^{-1}$$

where the product is over primes congruent to 1 mod 3 and not dividing $pN(c_u) = \prod_{w \in S} D_w$. Similarly define the function

$$L(s) = \prod_q \left( 1 + \frac{\text{tr} \sigma(F_q)}{pq^{-s}} \right).$$

Then one can show that

$$L(s) = L(s, \sigma)^{1/p} F(s)$$

where $F(s) = \prod_q (1 + O(1/q^{2s}))$ is absolutely convergent for $s > 1/2$.

By assumption of GRH $L(s, \sigma)$ has no zeros to the right of $1/2$ and hence $L(s, \sigma)^{1/p}$ has no poles. Then by a standard argument (see for instance [10]) we get

$$\sum_{d < x} \mu^2(pN(c_u) d) \frac{\prod_{q|d} \text{tr} \rho(F_q)}{p^{\omega(d)}} = \int_{2-iT}^{2+iT} L(s) x^s \frac{ds}{s} + O \left( \frac{x^2}{T \log x} \right)$$

$$\ll x^{1/2+\epsilon} \int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} \frac{|L(s, \sigma)^{1/p}| ds}{|s|}$$

$$+ O \left( \frac{x^2 (T \cdot N(c_u))^\epsilon}{T} \right) + O \left( \frac{x^2}{T \log x} \right)$$

$$\ll x^{1/2+\epsilon} (T \cdot N(c_u))^\epsilon \int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} \frac{1}{|s|} ds$$

$$+ O \left( \frac{x^2 (T \cdot N(c_u))^\epsilon}{T} \right) + O \left( \frac{x^2}{T \log x} \right)$$

$$\ll x^{1/2+\epsilon} N(c_u)^\epsilon T^\epsilon + O \left( \frac{x^2 (T \cdot N(c_u))^\epsilon}{T} \right) + O \left( \frac{x^2}{T \log x} \right)$$

where the last line follows by setting $T = x^3$. Then we bound the inner sum in (6.11) as

$$\sum_{D_u} \mu^2(D) \left( \frac{D_u}{c_u} \right)_p [C_u]_{D_u} \ll \sum_{D_u} \mu^2(D) \frac{\prod_{q|d} \text{tr} \rho(F_q)}{p^{\omega(D_u)}}$$

$$\ll A_u^{1/2+\epsilon} N(c_u)^\epsilon.$$

Note that

$$N(c_u)^\epsilon = \left( \prod_{w \in S} D_w \right)^\epsilon$$

$$\leq D_u^{p^2 k \epsilon}.$$
Then summing over all the remaining $D_w$ we get

$$S_k(X, A) \leq \frac{X}{A_w^{1/4}} = \frac{X}{X^{1/4}} = o(X).$$

This argument shows that we can remove all $A$ in which there is a variable larger than $X^\downarrow$ and linked with any other $A_w > 1$. This is equivalent to removing the $A$ which belong to families 3 or 4.

We summarize the results of this section in the following theorem

**Theorem 6.4.** Let $\sum_A S_k(X, A)$ denote a summation over all tuples $A$ which do not belong to any of the 4 families, that is they do not satisfy any of (6.2), (6.3), (6.4), (6.10). Then

$$S_k(X) = \sum_A' S_k(X, A) + o(X).$$

7. Computing the $k$-th Moment

We will use the notation $N(k) = N(k, p)$ which we recall is the number of vector subspaces of $\mathbb{F}_p^k$. We now want to prove Theorem 1.6. We will do this by proving:

**Theorem 7.1.**

$$S_k(X) = p^{-k} (N(k + 1) - N(k)) \sum_{n < X} (p - 1)^{\omega(n)-1} + o(X).$$

Note $S_k(X) = \sum_{K, D_K < X} \left| \text{im}(1 - \sigma_K) \right|^k$ is a sum over discriminants up to $X^{p-1}$. Furthermore $\sum_{n < X} \mu^2(n) (p - 1)^{\omega(n)-1} + o(X)$ is the number of degree $p$ cyclic fields with discriminant up to $X^{p-1}$, which is also equal to $cX + o(X)$. Thus it follows immediately from combining these facts with the above theorem that

$$\lim_{X \to \infty} \frac{\sum_{K, D_K < X} \left| \text{im}(1 - \sigma_K) \right|^k}{\sum_{K, D_K < X} 1} = \frac{N(k + 1) - N(k)}{p^k}.$$

We start by proving some facts about maximal unlinked sets of indices. It will turn out that for each $A$ in the sum in Theorem 6.4 all the indices $u$ with $A_u > 1$ form a maximal unlinked set.

For $k = 1$ write each index $u = u_1 p + u_2 \in \mathbb{Z}/p^2 \mathbb{Z}$ in base $p$ as $u = (u_1, u_2)$. Recall that for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ we defined

$$\Phi(u, v) = (u_1)(v_2 - u_2).$$

Now if we represent each index as $u = (u_1, \ldots, u_k)$ as $(u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{k1}, u_{k2}) \in \mathbb{F}_p^{2k}$ then

$$\Phi_k(u, v) = \sum_{i=1}^k (u_{i1})(v_{i2} - u_{i2}).$$

Next we show that translating a maximal unlinked set to the origin in $\mathbb{F}_p^{2k}$ yields a subspace.
Lemma 7.2. Let $\mathcal{U}$ be a maximal unlinked set and let $a \in \mathcal{U}$. Let $V = \mathcal{U} - a$. Then $V \subset \mathbb{F}_{p}^{2k}$ is a subspace.

Proof. Let $u, v \in \mathcal{U}$. We need to show that $(u - a) + (v - a) + a = u + v - a \in \mathcal{U}$. Since $\mathcal{U}$ is maximal we show $u + v - a$ is unlinked with every element of $\mathcal{U}$. Let $w \in \mathcal{U}$. We have

$$
\Phi_k (u + v - a, w) = \sum_{i=1}^{k} (u_{i1} + v_{i1} - a_{i1}) (w_{i2} - u_{i2} - v_{i2} + a_{i2})
$$

$$
= \sum_{i=1}^{k} (u_{i1} + v_{i1} - a_{i1}) ((a_{i2} - u_{i2}) + (w_{i2} - v_{i2}))
$$

$$
= \sum_{i=1}^{k} v_{i1} (a_{i2} - u_{i2}) + u_{i1} (w_{i2} - v_{i2}) - a_{i1} (w_{i2} - v_{i2})
$$

$$
= 0
$$

since for instance $v_{i1} (-u_{i2} + a_{i2}) = -v_{i1} (u_{i2} - v_{i2}) + v_{i1} (-v_{i2} + a_{i2})$ and $u, v, w, a$ are all unlinked. Similarly

$$
\Phi_k (w, u + v - a) = \sum_{i=1}^{k} (w_{i1}) (u_{i2} + v_{i2} - a_{i2} - w_{i2})
$$

$$
= 0.
$$

This proves the lemma. $\square$

It is easy to see that the $V$ in the above lemma does not depend on the choice of $a$. Let $p : \mathbb{F}_{p}^{2k} \rightarrow \mathbb{F}_{p}^{k}$ be the projection onto the even coordinates, that is

$$
p (u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{k1}, u_{k2}) = (u_{12}, u_{22}, \ldots, u_{k2})
$$

and let $q$ be the projection onto the odd coordinates. Let $V_1 = \ker p$ and let $V_2 = \ker q$. Then $\mathbb{F}_{p}^{2k} = V_1 \oplus V_2$. Next we prove

Lemma 7.3. Let $V$ be a subspace of $\mathbb{F}_{p}^{2k}$. Then there is a maximal unlinked set $\mathcal{U}$ such that $\mathcal{U} = V + a$ if and only if $V = p (V)^{\perp} \oplus p (V)$ where $p (V)^{\perp}$ is taken in $\mathbb{F}_{p}^{k}$.

Proof. First assume $\mathcal{U}$ is unlinked, not necessarily maximal. Suppose that $\mathcal{U} = V + a$. Then for any $w \in V$ we have $\mathcal{U} = \mathcal{U} + w$. Let $u' \in V$ and let $u = u' + a$, so $u \in \mathcal{U}$.
Note $a \in \mathcal{U}$. Then we have
\[
0 = \Phi_k(u + w, a + w) = \sum_{i=1}^{k} (u_{i1} + w_{i1})(a_{i2} + w_{i2} - u_{i2} - w_{i2})
\]
\[
= \Phi_k(u, a) + \sum_{i=1}^{k} w_{i1}(a_{i2} - u_{i2})
\]
\[
= -\sum_{i=1}^{k} w_{i1}u'_{i2}.
\]

Since $w, u' \in V$ were arbitrary this shows $q(V) \in p(V)\perp$. Thus $V \subset p(V)\perp \oplus p(V)$.

Now suppose $V \subset p(V)\perp \oplus p(V)$. Let $a = 0$. Let $\mathcal{U} = V$. For $v, w \in V$
\[
\Phi_k(v, w) = \sum_{i=1}^{k} v_{i1}(w_{i2} - v_{i2}) = 0
\]
by the assumption and since $w - v \in V$. Hence $\mathcal{U}$ is unlinked.

Now if $V$ satisfies the above then clearly $p(V)\perp \oplus p(V)$ also does so if $\mathcal{U}$ is maximal then we have equality $V = p(V)\perp \oplus p(V)$.

Conversely note that equality $V = p(V)\perp \oplus p(V)$ implies $\dim V = k$. If $\mathcal{U}$ is not maximal then let $\mathcal{U}'$ be a maximal unlinked set containing it. By Lemma 7.2 and the first part there is a subspace $V' = \mathcal{U}' - a$ which will contain $V$ and such that $V' = p(V')\perp \oplus p(V')$. Hence $\dim V' = k$ also and we must have $V' = V$ so $\mathcal{U} = \mathcal{U}'$ is maximal.

Hence the maximal unlinked sets are all obtained as translates of subspaces which satisfy the conditions of Lemma 7.3. Note for such subspaces $V$ that $\dim V = \dim W \oplus p(V) = k$ and hence every maximal unlinked set has size $p^k$.

With this we can rewrite $S_k(X)$ in a form closer to Theorem 7.1.

**Proposition 7.4.** Let $U$ be the number of maximal unlinked sets. Then
\[
\sum_{K,D_K < X^{p^{-1}}} |\im (1 - \sigma_K)|^k = \left(\frac{U}{p^k}\right) \sum_{n < X} (p - 1)^{\omega(n)-1} + o(X)
\]

**Proof.** From the work above it is easy to show that the largest possible intersection of two maximal unlinked sets has size $p^{k-1}$. Hence a set of $p^{k-1} + 1$ unlinked variables determines a unique maximal unlinked set. Thus each family $\mathbf{A}$ in Theorem 6.4 corresponds to a unique maximal unlinked set. We can partition $\sum' \mathbf{A} S(X, \mathbf{A})$ according to these maximal unlinked sets, and additionally adding back the previous error term consisting of sums $S(X, \mathbf{A})$ which have at most $p^{k-1}$ variables with $A_i > X^\perp$ at the cost of an error $X/\log X$ we get
\[
\sum_{A} S(X, A) = \frac{U}{p^{k}(p-1)} \sum_{0 \leq \prod_{j=0}^{k} n_{j} \leq X} \mu^{2} \left( \prod_{j=0}^{k} n_{j} \right) \left( \frac{p-1}{p} \right)^{\omega \left( \prod_{j=0}^{k} n_{j} \right)} + o(X)
\]

\[
= \frac{U}{p^{k}} \sum_{n < X} n^{2} (p-1)^{\omega(n)-1} + o(X).
\]

The final step of the proof will be the next proposition. Define \( n(k, r) \) to be the number of \( r \)-dimensional subspaces of \( \mathbb{F}_{p}^{k} \). We will need two properties of this function which can be found in Lemmas 1 and 3 from [13]:

**Lemma 7.5.** The function \( n(k, r) \) satisfies

\[
n(k, r) = n(k, k - r),
\]

\[
\sum_{r=0}^{k} p^{r} n(k, r) = \mathcal{N}(k+1) - \mathcal{N}(k).
\]

**Proposition 7.6.** The number of maximal unlinked sets \( U \) is

\[
U = \mathcal{N}(k+1) - \mathcal{N}(k).
\]

**Proof.** By an argument similar to the proof of Lemma 7.3 we can show that if \( U \) is maximal unlinked and \( U = V + a' \) then \( U + a \) is maximal unlinked if and only if \( q(a) \in p(V)^{\perp} \). Hence given any \( k \)-dimensional subspace \( V \subset \mathbb{F}_{p}^{2k} \) there are

\[
p^{k} \left( p^{\dim p(V)^{\perp}} \right)
\]

vectors which translate \( V \) to a maximal unlinked set. However since translating by \( a_{1} \) and \( a_{2} \) gives the same set if and only if \( a_{1} \) and \( a_{2} \) are in the same coset of \( V \) this implies that there are \( \left( p^{\dim p(V)^{\perp}} \right) \) distinct maximal unlinked sets that can be obtained from \( V \).

Now let \( S \) be the set of \( k \)-dimensional subspaces \( V \subset \mathbb{F}_{p}^{2k} \) which satisfy Lemma 7.3. We compute the size of this set. Fix some subspace \( V_{0} \subset \mathbb{F}_{p}^{k} \) with \( \dim V_{0} = r \) and suppose \( V \) satisfies \( p(V) = V_{0} \). So \( \dim p(V)^{\perp} = k - r \). We want \( V \subset p(V)^{\perp} \oplus p(V) = V_{0}^{\perp} \oplus V_{0} \) but both sides have dimension \( k \) and hence there is a unique \( V \) with \( p(V) = V_{0} \) and satisfying the condition of Lemma 7.3. Hence the number of \( V \in |S| \) with \( \dim p(V) = r \) is \( n(k, r) \).

Thus we have

\[
U = \sum_{V \in S} p^{\dim p(V)^{\perp}}
\]

\[
= \sum_{r=0}^{k} p^{r} n(k, r)
\]

\[
= \mathcal{N}(k+1) - \mathcal{N}(k)
\]
by Lemma 7.5.

Thus we have computed

\[ S_k(X) = p^{-k}(N'(k + 1) - N'(k)) \sum_{n < X} (p - 1)^{\omega(n) - 1} + o(X) \]

which proves Theorem 7.1. As remarked at the beginning of the section it follows that

\[ \lim_{X \to \infty} \frac{\sum_{K,D_K < X} |\text{im}(1 - \sigma_K)|^k}{\sum_{K,D_K < X} 1} = p^{-k}(N'(k + 1) - N'(k)). \]

By the results of [12] Theorems 1.3 and 1.5 follow immediately.

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