Stratification parameters and dispersion of internal solitary waves

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Abstract. A theoretical model of internal solitary waves of large amplitude in a weakly stratified fluid is considered. It is assumed that the background density profile depends linearly or exponentially on the fluid depth. It is demonstrated that inverse problem on determining fine-scale structure of the density profile by a known curve of amplitude dispersion is reduced to solving a linear Fredholm integral equation of the first kind having special form of the kernel. The one-to-one correspondence between the density coefficient and the dispersion function is established in the case of analytical stratification.

1. Introduction
We study in this paper the relationship between the density stratification of a non-homogeneous fluid and kinematic characteristics of nonlinear internal waves. The attention is focused on analysis of the possibility to restore the density profile by the known curves of amplitude dispersion for internal solitary waves. This interest is stimulated by the problems of remote sensing internal waves with straight thermohaline measurements. At present, these operations employ a set of experimental oceanographic methods which include the observations from the ships and long-term buoy stations, monitoring satellite data and measurements by deep research devices [1]. The problem on determining the density of vertically stratified environment by the known curves of the phase velocity of linear dispersive waves has already been considered [2,3]. These studies used the theory of inverse spectral problem based on the method of integral equations. We propose to consider mathematical model of a strongly nonlinear soliton-type waves in order to formulate the inverse problem on the density stratification. Numerous observation data [4] show that the trains of internal solitary waves form regularly by the interaction of tidal currents with continental slope. Spatial amplitudes of these nonlinear disturbances are comparable with total depth of sea water on the shelf [5,6]. Therefore the asymptotic analysis which involves non-linear equations of dispersive long waves [7] seems to be suitable for the modelling. This method uses asymptotic expansion on powers of the small Boussinesq parameter which characterizes vertical gradient of the fluid density. By those circumstances, it is supposed that the fluid density disturbs slightly the linear- or exponential profile, and there are no hypotheses on small amplitude of the waves. As a result, the inverse problem is reduced to solving the Fredholm integral equation of the first kind which couples the density profile with the function describing amplitude dispersion of solitary wave.
2. Basic equations

We consider two-dimensional flow of an ideal incompressible non-homogeneous fluid in a horizontal layer bounded by a flat bottom $y = 0$ and rigid lid $y = h$. The flow is stationary in the frame of the reference associated with the solitary wave. Therefore, the stationary Euler equations are used here as a basic mathematical model for velocity field $u = (u,v)$, density $\rho$ and pressure $p$:

$$u_x + v_y = 0, \quad u \rho_x + v \rho_y = 0, \quad uu_x + vv_y + \rho^{-1} p_x = 0, \quad uv_x + vv_y + \rho^{-1} p_y = -g$$

(1)

where $g$ is the gravity acceleration. The boundary conditions at the bottom and the lid require the equality $v = 0$ which should be satisfied at $y = 0$ and $y = h$. Furthermore, the velocity vector $u(x, y)$ should tend to the constant vector $u_0 = (u_0, 0)$ as $|x| \rightarrow \infty$. Here $u_0$ is the wave speed with respect to quiescent fluid. Similarly, the fluid density $\rho(x, y)$ should tend to its undisturbed profile $\rho_\infty(y)$. It is well known [8] that the introduction of a stream function $\psi$ by the formulae $u = \psi_y, \ v = -\psi_x$ leads after eliminating the pressure $p$ from the system (1) to single scalar PDE equation of the second order. This is the Dubreil-Jacotin — Long equation

$$\rho(\psi)(\psi_x + \psi_{yy}) + \rho'(\psi)\left(gy - \frac{y \psi}{u_0} + \frac{1}{2}(\psi_x^2 + \psi_y^2 - u_0^2)\right) = 0.$$  

(2)

The coefficient $\rho(\psi)$ depends here on the stream function due to the formula $\rho(\psi) = \rho_\infty(\psi/u_0)$. Boundary conditions on rigid walls and radiation condition at infinity have the form

$$\psi(x, 0) = 0, \quad \psi(x, h) = u_0 h; \quad \psi(x, y) \rightarrow u_0 y \quad (x \rightarrow -\infty).$$

(3)

The direct problem on solitary wave requires a solution $\psi$ of equations (2), (3) which differs from uniform flow $\psi_\infty(y) = u_0 y$. By that, the dependence of density $\rho_\infty(y)$ on $y$ is assumed to be known far upstream. In this context, the problem is formulated as a nonlinear eigenvalue problem with spectral parameter $u_0$ which determines the speed of solitary wave. Let us note that the equation (2) admits a variational formulation with the Lagrangian

$$L = -\frac{1}{2} \int_{u_0y} \rho(\psi)(\psi_x^2 + \psi_y^2 - u_0^2) + g \int_{u_0y} (\rho(\chi) - \rho(\psi))d(\chi).$$

(4)

The undisturbed upstream flow is characterized by two dimensionless parameters, i.e. the Boussinesq parameter $\sigma$ and the parameter $\lambda$ which is a square of inverse densimetric Froude number,

$$\sigma = \frac{N_0^2 h_0}{\pi g}, \quad \lambda = \frac{\sigma g h_0}{\pi u_0^2}.$$  

Here the constant $N_0$ characterizes the buoyancy frequency $N$ defined by the formula $N^2(y) = -g \rho_\infty(y)/\rho_\infty(y)$. Thus, the parameter $\sigma$ is an appropriate small parameter appearing in the case of weak stratification. According to the well-known ideas about the thermohaline stratification of sea water [7,9–11], the fluid density can be modeled at the equilibrium state by the equation

$$\rho_\infty(y) = \rho_\infty(0) \left(1 - \sigma \rho_s(\pi y/h) - \sigma^2 \rho_l(\pi y/h, \sigma)\right)$$

(5)

where $\rho_s$ and $\rho_l$ are the density of salt and fresh water, respectively.
where the function $\rho_1$ defines the fine-scale structure of the density profile. The scaling factor $\pi$ is entered here in order to simplify the form of trigonometric eigenfunctions related to the normal modes of internal waves.

Note that there is a relatively small number of background density profiles $\rho_*$ which can be realized for sea water by natural conditions. They include the linear- and the exponential profiles, and also the layered stratification with piecewise constant density can be observed, as well as all these profiles can be combined. In contrast, the fine-scale density $\rho_1$ appears to be much more diverse. However, the typical time of its evolution can be clearly longer than the temporal periods of internal waves in a coastal zones. Therefore we can simulate this stratification by stationary profile having the dimensionless form

$$\rho(y, \sigma) = 1 - \sigma y - \sigma^2 \rho_0(y) + O(\sigma^3) \quad (5)$$

with $0 \leq y \leq \pi$. The function $\rho_0(y) = \rho_1(y, 0)$ characterizes here the fine-scale stratification at the leading order on $\sigma$. The class of the profiles (5) include the linear density $\rho = 1 - \sigma y$ and the exponential stratification $\rho = \exp(-\sigma y)$.

3. Direct problem on solitary waves in a weakly stratified fluid

Following [7], we consider here the long-wave approximation having the ratio of the vertical- and horizontal scales of the order $\sqrt{\sigma}$. It can be shown that this asymptotic order follows immediately from the dispersion relation for the modes of long waves corresponding to the density profile (5). Using the dimensionless variables with “slow” horizontal variable $x \to \sqrt{\sigma} x$ in equations (2), (3), we can consider only the terms up to the order $O(\sigma)$ inclusively,

$$\sigma \psi_{xx} + \psi_{yy} + \lambda(\psi - y) = \sigma \left( \psi \psi_{yy} + (y - \psi)\rho'_0(\psi) + \frac{1}{2}(\psi^2 - 1) \right) + O(\sigma^2), \quad (6)$$

$$\psi(x, y) = 0, \quad \psi(x, \pi) = \pi; \quad \psi(x, y) = y \quad (x \to \infty).$$

We look for the solution in the form

$$\psi(x, 0) = y + v_0(x, y) + \sigma v_1(x, y) + ..., \quad (7)$$

and the parameter $\lambda$ should be also presented by the power series $\lambda = \lambda_0 + \sigma \lambda_1 + ...$ with small parameter $\sigma$. Thus, we obtain from the equation (6) the recursive set of equations for the coefficients $v_i \ (i = 1, 2, ...)$

$$v_{yy} + \lambda_0 v_i = f_i \quad (0 < y < \pi),$$

$$v_i(x, 0) = v_i(x, \pi) = 0,$$

where the right-hand terms $f_0$ and $f_1$ have the form

$$f_0 = 0, \quad f_1 = -v_{0xx} + (y + v_0)v_{0yy} - \lambda_0(y + v_0)v_0 + v_{0y} + \frac{1}{2} v_{0y}^2 - \lambda_1 v_0.$$

The equations with $i = 0$ result $\lambda_0 = m^2 \ (m = 1, 2, 3, ...).$ Thus, the 1-mode wave solution has the form $v_0(x, y) = a(x) \sin y$ while the function $a(x)$ should be determined from the compatibility conditions for a system of subsequent equations with $i = 1$. Namely, the orthogonality condition with the eigenfunction of the first mode should be fulfilled for the right-hand side term $f_1$ as follows:

$$\int_0^\pi f_1(x, y) \sin y dy = 0.$$
As a result, this gives the nonlinear ordinary differential equation

\[ \frac{d^2a}{dx^2} + F'(a) = 0, \]

where the nonlinearity \( F(a) = s(a) + \frac{1}{2} \lambda_1 a^2 \) is determined by the function

\[ s(a) = \frac{2}{\pi} \int_0^\infty y^+ a \sin y \int y (\rho_0(y + b \sin y) - \rho_0(\psi))d\psi dy + \frac{\pi}{4} a^2 + \frac{2}{3\pi} a^3. \]

Note that the specific form of this function inherits the similar structure of the Lagrangian (4) for the Dubreil-Jacotin — Long equation. The function \( s(a) \) has the double root at \( a = 0 \), therefore the function \( F(a) \) also has a double root here. As a consequence, the solitary wave-solution is obtained when the value \( \lambda_1 \) belongs to the range of the function

\[ \lambda_1(b) = -\frac{2s(b)}{b^2}, \]

and the corresponding value of \( b \neq 0 \) should be a simple root of the function \( F \). By that, the unknown function \( a(x) \) is implicitly determined by the quadrature

\[ x = \pm \int_a^b \frac{d\alpha}{\alpha \sqrt{\lambda_1(\alpha) - \lambda_1(b)}}. \]

The parameter \( b = a(0) \) characterizes the velocity magnitude in the middle part of the solitary wave when the wave crest places at \( x = 0 \), and the value \( \lambda_1 = \lambda_1(b) \) determines the wave speed by the formula

\[ u_0^2 = \frac{\sigma gh_0}{\pi(1 + \sigma \lambda_1(b))} \]

with the accuracy \( O(\sigma^3) \). According to this formula, the function \( \lambda_1(b) \) presents the amplitude dispersion function of solitary wave.

The perturbation procedure with small density slope described above has been developed first in the original paper [7] which treated the case of linear stratification. In this case, the equation (7) has the non-linearity defined by a cubic polynomial \( F(a) \), as well as the classical Korteweg – de Vries model has, and similar situation occurs by an exponential stratification. The including higher-order terms to the density profile can change significantly the structure of equation (7). In particular, the perturbation of the fine-scale stratification leads to the solution like a smooth bore ("kink") and plateau-shape solitary waves ("kink-antikink" solutions). In this concern, an approximate model was proposed in the paper [9] to describe solitary wave with attached vortex zone. Another limit forms of internal solitary waves of finite amplitude such as bores and table-top waves were studied in detail in [10, 11] where the bifurcation of internal waves was associated with monotonic behavior of dispersion function \( \lambda_1(b) \). In particular, the solitary wave bifurcates to the plateau-shape wave near the amplitude values \( b \) being the local minima of the function \( \lambda_1(b) \) which correspond to local maxima of the wave propagation speed \( u_0 \).
4. Inverse problem on determining the fine-scale stratification
 Returning to the equations (8) and (9), we note that these formulae provide an analytical relation between the coefficient $\rho_0(y)$, that specifies the fine-scale stratification, and the dispersion function $\lambda_1(b)$. These linear relations allow to formulate the inverse problem of reconstructing the fluid density by known dispersion characteristics of nonlinear waves. Let us consider the integral term from the formula (8) having the form

$$I(b) = \frac{2}{\pi} \int_{0}^{\pi} \int_{y}^{y+b \sin y} (\rho_0(y + b \sin y) - \rho_0(\psi)) d\psi dy$$

(10)

Partial integration of the variable $y$ reduces this function $I(b)$ to the form that contains only single integrals:

$$I(b) = \frac{2}{\pi} \int_{0}^{\pi} [b \sin y + y(1 + b \cos y)]\rho_0(y + b \sin y) dy - \frac{2}{\pi} \int_{0}^{\pi} y\rho_0(y) dy.$$

Further we change the first of these integrals by replacing $y$ with a new integration variable

$$\psi = y + b \sin y.$$ 

(11)

In fact, this is the von Mises transformation which changes the roles of the vertical independent variable $y$ and the stream function $\psi$. This transformation sets $y = y(\psi, b)$ as an implicit function of $\psi$ and $b$. It is uniquely provided by the condition $\psi_y = 1 + b \cos y > 0$ which has a simple sense that there are no return flows in the middle section of a solitary wave. This requirement imposes a natural limit $|b| < 1$ on amplitude parameter $b$. Taking into account the definition (9) of the function $\lambda_1(b)$ and the above-formulated transformation of the function $I(b)$, we obtain from the formula (8) the equation

$$\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(2y(\psi, b))}{1 + b \cos y(\psi, b)} \rho_0(\psi) d\psi = \lambda_1(b) + \frac{\pi}{2} + \frac{4b^3}{3\pi}.$$ 

(12)

This relation is a linear Fredholm integral equation of the first kind with respect to the unknown function $\rho_0$ if the function $\lambda_1(b)$ is given. The kernel of integral operator (12) defined implicitly through the formula (11) has no singularities in view of analyticity of the function $y(\psi, b)$ by $0 \leq y \leq \pi$, $|b| < 1$. The Fredholm equations of this type are typical for inverse problems of mathematical physics which are generally ill-posed. However, there are effective regularizing algorithms for solving these equations. By that, the main topic is the providing sufficiently broad classes of existence and uniqueness of the solutions. Using the representation of the function $s(b)$ with the double integral (10), we can easily check that the dispersion function $\lambda_1(b)$ can be presented by the power series

$$\lambda_1(b) = \sum_{i=0}^{\infty} l_i b^i.$$ 

(13)

if the function $\rho_0(y)$ has the form of a power series

$$\rho_0(y) = \sum_{i=1}^{\infty} r_i y^i.$$ 

(14)
The coefficients \( r = (r_1, \ldots, r_n, \ldots)^T \) and \( l = (l_0, \ldots, l_{n-1}, \ldots)^T \) of these series are coupled by the linear equation \( l = A r + d \) with infinite vector \( d = (-\pi/2, -4/3\pi, 0, \ldots, 0, \ldots)^T \) and infinite matrix \( A = \|A_{ij}\| \). A more detailed analysis suggests that the matrix \( A \) is the upper triangular matrix having the coefficients as follows:

\[
A_{n-i+1,n} = -\frac{4}{\pi} \frac{n-i+1}{n-i+2} C_n^{i-1} \int_0^\pi y^{i-1} \sin^{n-i+2} y dy.
\]

In particular, the diagonal elements of the matrix \( A \) have the form

\[
A_{nn} = -\frac{8}{\pi} \frac{n}{n+1} \frac{n!!}{(n+1)!!} \frac{\pi}{2} \quad (n \text{ is odd}),
\]

\[
A_{nn} = -\frac{8}{\pi} \frac{n}{n+1} \frac{n!!}{(n+1)!!} \quad (n \text{ is even}).
\]

These diagonal elements \( A_{nn} \) do not vanish for all integer \( n \geq 1 \). Thus, there is the one-to-one correspondence between the coefficients \( l_i \) and \( r_k \) which is given by the relation

\[
\begin{pmatrix}
  l_0 \\
  l_1 \\
  l_2 \\
  l_3 \\
  \vdots
\end{pmatrix}
= \begin{pmatrix}
  -1 & -\pi & -\pi^2 + 3/2 & -\pi^3 + 3\pi & \cdots \\
  -32/9\pi & -16/3 & -32\pi/3 + 1280/27\pi & \cdots \\
  0 & -9/8 & -9\pi/4 & \cdots \\
  0 & 0 & -256/75\pi & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  r_1 \\
  r_2 \\
  r_3 \\
  r_4 \\
  \vdots
\end{pmatrix}
+ \begin{pmatrix}
  -\pi/2 \\
  -4/3\pi \\
  \vdots \\
  \vdots \\
  \vdots
\end{pmatrix}.
\]

One can conclude under these assumptions that the fine-scale density profile (14) is uniquely determined by the function (13) of amplitude dispersion of internal solitary waves, at least in the class of formal power series.

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