Periodic solutions for a non-monotone family of delayed differential equations with applications to Nicholson systems

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Abstract
For a family of \( n \)-dimensional periodic delay differential equations which encompasses a broad set of models used in structured population dynamics, the existence of a positive periodic solution is obtained under very mild conditions. The proof uses the Schauder fixed point theorem and relies on the permanence of the system. A general criterion for the existence of a positive periodic solution for Nicholson’s blowflies periodic systems (with both distributed and discrete time-varying delays) is derived as a simple application of our main result, generalizing the few existing results concerning multi-dimensional Nicholson models. In the case of a Nicholson system with discrete delays all multiples of the period, the global attractivity of the positive periodic solution is further analyzed, improving results in recent literature.

Keywords: delay differential equation; periodic Nicholson system; positive periodic solution; Schauder fixed point theorem; permanence.

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1. Introduction
In recent years, the question of the existence of periodic solutions for periodic delay differential equations (DDEs) has attracted the interest of many researchers, and a plethora of positive answers has been provided by using a variety of methods. To a large extent, the techniques used in the literature apply to a specific equation only, while other ones apply to a very particular class of DDEs, with emphasis on scalar models. For some classical models from mathematical biology, the available existence results require a very restrictive set of assumptions, which are not easily verifiable, much less extendable to other families of DDEs.

The main purpose of this paper is to investigate the existence of a positive periodic solution for a broad class of periodic and in general non-monotone \( n \)-dimensional DDEs which encompasses a large number of population models with patch structure. DDEs with patch structure have extensive applications in population dynamics, where the patch structure accounts for situations of heterogeneous environments due to several aspects, or in disease and epidemic models with different classes for cells or individuals, with transition among the...
classes. In particular, the study of periodic models is especially significant, as they reflect
periodical variations of the weather or seasonality of the habitat in general, so the quest for
positive periodic solutions for such models becomes quite relevant.

In this paper, we consider a family of periodic delayed population models with patch
structure and multiple time-varying delays of the form

\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} b_{ik}(s, x_i(s)) \, ds \eta_{ik}(t, s), \quad i = 1, \ldots, n,
\]

where all the coefficients and delay functions are assumed to be continuous, non-negative
and periodic on \( t \), with a common period \( \omega > 0 \), and \( \eta_{ik}(t, s) \) are bounded, nondecreasing
on \( s \), locally integrable and \( \omega \)-periodic on \( t \). Some additional conditions on the coefficients
\( d_i(t), a_{ij}(t), \beta_{ik}(t) \) and on the nonlinearities \( b_{ik}(t, x) \) will be assumed. Special attention will be
given to the study of (1.1) with \( \eta_{ik}(t, s) = H_{t-\tau_{ik}(t)}(s) \), where \( H_t(s) \) is the Heaviside function
\( H_t(s) = 0 \) if \( s \leq t \), \( H_t(s) = 1 \) if \( s > t \). In this case, we obtain a system with discrete delays
of the form

\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) h_{ik}(t, x_i(t-\tau_{ik}(t))), \quad i = 1, \ldots, n.
\]

Many important delayed non-autonomous models from mathematical biology can be written
in the form (1.1), see e.g. [15, 20, 25]. In Section 2, a descriptive set of hypotheses, as
well as a brief biological interpretation of the model, will be given.

The present paper is a continuation of the research recently conducted by Faria, Obaya
and Sanz in [7], where the asymptotic behavior of solutions for non-autonomous systems (1.2)
was carefully analyzed, and sufficient conditions for either the permanence or extinction of
all population given. A permanence result was established in [7] for a generic system (1.2)
with all the coefficients and delays given by non-negative, continuous, bounded functions (not
necessarily periodic), under very mild and optimal sufficient conditions. As we shall see, the
permanence result in [7] can be easily extended to systems (1.1). Here, the leading ideas are,
on one hand, to interpret (1.1) as the result of adding bounded delayed perturbations to a
linear homogeneous ordinary differential equation (ODE) and, on the other hand, to use the
uniform persistence of (1.1); under the same hypotheses for permanence, by applying the
Schauder fixed point theorem we further show that at least one positive period solution must
exist. Our results are achieved under a very weak set of assumptions and have significant
applications.

Among them, and as an important illustration, we have in mind to apply our results to
Nicholson systems with time-dependent, either distributed or discrete, delays, of the forms

\[
x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \gamma_{ik}(s)\gamma_i(s) e^{-c_{ik}(s)\gamma_i(s)} \, ds, \quad i = 1, \ldots, n,
\]
with harvesting terms, we refer also to [28, 29, 31].

solutions in the case concerns periodic Nicholson systems, results concerning the existence of positive periodic solutions for general non-monotone periodic DDEs. In what delays (1.3) with \( n \) be virtually non-existent for the situation of (1.4) with \( n > 1 \).

models, with priority in autonomous systems. For other criteria of existence of periodic solutions for scalar periodic DDEs with no time delay \( N(0) \), see [12, 13, 14, 15, 16].

in a number of papers. The simplest version of (1.6) is when the delay is a multiple of the period, in which case the \( N \)-periodic solution (1.5) is established in a few papers, see [18, 26], and see also [4, 10, 19, 27, 30] and references therein.

The Nicholson’s blowflies equation

\[
N'(t) = -dN(t) + \beta N(t - \tau)e^{-aN(t-\tau)} \quad (d, \beta, a, \tau > 0) \tag{1.5}
\]

was introduced by Gurney et al. in 1980 [11], and its biological impact was immediately apparent, as the proposed model agreed with Nicholson’s experimental data on the Australian sheep blowfly (see e.g. [21]). Since then, an immense literature concerning Nicholson’s equation, generalizations, related models and applications to real world problems has been produced. The periodic version of (1.5), given by

\[
N'(t) = -d(t)N(t) + \beta(t)N(t - \tau(t))e^{-a(t)N(t-\tau(t))}, \tag{1.6}
\]

with \( d(t), \beta(t), \tau(t), a(t) \) positive, \( \omega \)-periodic continuous functions \( (\omega > 0) \), has been studied in a number of papers. The simplest version of (1.6) is when the delay is a multiple of the period, in which case the \( \omega \)-periodic solutions of (1.6) are exactly the \( \omega \)-periodic solutions of the equation with no time delay \( N'(t) = -d(t)N(t) + \beta(t)N(t)e^{-a(t)N(t)} \). For the particular situation of \( \tau(t) = m\omega \) \( (m \in \mathbb{N}) \) and \( a(t) \equiv a > 0 \), Saker and Agarwal [23] showed that there is a positive \( \omega \)-periodic solution \( N^*(t) \) of (1.6) if \( \min_{t \in [0, \omega]} \beta(t) > \max_{t \in [0, \omega]} d(t) \), and gave some additional conditions for its global attractivity. See also [16] for a refinement of the result in [23]. A significant breakthrough was later achieved by Chen [3], who used the continuation theorem of coincidence degree to establish the existence of a positive \( \omega \)-periodic solution of (1.6) under much more general conditions. More recently, an elegant unifying method, based on the continuation theorem, was proposed by Amster and Idels [1] to show the existence of positive periodic solutions for a general class of scalar periodic DDEs with the form \( x'(t) = f(a(t), x(t), x(t) \pm \lambda b(t)g(x_t)) \) (\( \lambda > 0 \) a parameter). Their results apply to the case of a Nicholson scalar equation with distributed delay, as well as to other important biological models. For other criteria of existence of periodic solutions for scalar periodic DDEs within the class \( x'(t) = f(a(t), x(t), x(t) \pm \lambda b(t)g(x_t)) \) and based on several fixed point methods, see [1, 10, 14, 27, 30] and references therein.

Only recently has some attention been given to multi-dimensional versions of Nicholson models, with priority in autonomous Nicholson systems [2, 6, 8, 17]. For \( n > 1 \), very little is known about positive periodic solutions for general non-monotone periodic DDEs. In what concerns periodic Nicholson systems, results concerning the existence of positive periodic solutions in the case \( n = 2 \) have been established in a few papers, see [18, 26], and seem to be virtually non-existent for the situation of (1.4) with \( n > 2 \), or for the case of distributed delays (1.3) with \( n \geq 2 \). For related results for periodic or almost-periodic Nicholson systems with harvesting terms, we refer also to [28, 29, 31].
In spite of the variety of methods and tools that have been proposed, to the best of our knowledge, there is no general result in the literature concerning the existence of positive periodic solutions for periodic \(n\)-dimensional Nicholson systems (1.3) or (1.4). Surprisingly, here a general criterion is obtained as an immediate consequence of our Theorem 3.1 established for a far more general framework. The results in [16] and [19] are recovered by our general criterion, when applied to the scalar version of (1.4). We however believe that sharper results are to be expected, under more natural restrictions involving the average integrals of the coefficients over the interval \([0, \omega]\), as in [3, 26] for \(n = 1\) – rather than the pointwise values of such coefficients, as in the results presented here. This will be the subject of future research.

The contents of the remainder of the paper are now summarized. Section 2 is a section of preliminaries, where a set of assumptions is introduced and the general criterion for permanence in [7] extended to the family of DDEs (1.1). The main result of the paper, Theorem 3.1, is given in Section 3: in the case of periodic systems (1.1) we show that the sufficient conditions for permanence are enough to guarantee the existence of at least one positive \(\omega\)-periodic solution. The result for periodic \(n\)-dimensional Nicholson systems (1.3) is deduced as a particular case. In Section 4, we consider (1.4) with all the delays multiple of the period, and give sufficient conditions for the global attractivity of the positive periodic solution. Our results extend the ones in [16, 19] and improve some criteria in [1, 26]. The situation of systems with autonomous coefficients is also considered and the global asymptotic stability of a positive equilibrium deduced under optimal conditions, generalizing the results in [2, 8]. Although emphasis is given to periodic Nicholson systems, other relevant population models satisfy the hypotheses imposed here; see Sections 2 and 3 for examples.

2. Preliminaries

We start by introducing some standard notation. For \(\tau \geq 0\), set \(C := C([-\tau, 0]; \mathbb{R}^n)\) to be the Banach space endowed with the norm \(\|\phi\| = \max_{\theta \in [-\tau, 0]} |\phi(\theta)|\), where \(\cdot\) is a fixed norm in \(\mathbb{R}^n\). We shall also use \(|A|\) to denote the (operator) norm of an \(n \times n\) matrix \(A\) with constant entries. A vector \(v \in \mathbb{R}^n\) is identified in \(C\) with the constant function \(\psi(s) = v\) for \(-\tau \leq s \leq 0\).

A DDE in \(C\) takes the general form

\[
x'(t) = f(t, x_t),
\]

(2.1)

where \(f : \Omega \subset \mathbb{R} \times C \to \mathbb{R}^n\) and \(x_t\) denotes the restriction of a solution \(x(t)\) to the time interval \([t - \tau, t]\), i.e., \(x_t \in C\) is given by \(x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0\). Take \(\Omega = [\alpha, \infty) \times D\) with \(\alpha \in \mathbb{R}\) and \(D \subset C\), and suppose that \(f\) is continuous and regular enough so that the initial value problem is well-posed, in the sense that for each \((\sigma, \phi) \in [\alpha, \infty) \times D\) there exists a unique solution of the problem \(x'(t) = f(t, x_t), x_\sigma = \phi\), defined on a maximal interval of existence: in this situation, this solution is denoted by \(x(t, \sigma, \phi)\) in \(\mathbb{R}_+^n\) or \(x_t(\sigma, \phi)\) in \(C\). Whenever necessary, the more explanatory notation \(x(t, \sigma, \phi, f)\) is used.

We designate by \(C^+\) the cone of nonnegative functions in \(C\), \(C^+ = C([-\tau, 0]; [0, \infty)^n)\), and by \(int C^+\) its interior. In \(C\), \(\leq\) denotes the usual partial order generated by \(C^+: \phi \leq \psi\).
if and only if $\psi - \phi \in C^+$. The relations $\geq$ and $\gg$ are defined in the obvious way; thus, we write $\psi \geq 0$ for $\psi \in C^+$ and $\psi \gg 0$ for $\psi \in int C^+$. The situation of no delays ($\tau = 0$) is included in our setting, in which case $C$ is identified with $\mathbb{R}^n$ and $C^+ \cap \mathbb{R}^n_+: = [0, \infty)^n$.

For simplicity, here we say that (2.1) (or $f$) is cooperative if it satisfies Smith’s quasi-monotone condition, given by (see [24])

$$ \Psi \text{ (Q) for } \phi, \psi \in D, \phi \leq \psi \text{ and } \phi_i(0) = \psi_i(0), \text{ then } f_i(t, \phi) \leq f_i(t, \psi), \ i = 1, \ldots, n, t \geq \alpha. $$

Condition (Q) allows comparison of solutions between two related DDEs $x'(t) = f(t, x_t)$ and $x'(t) = g(t, x_t)$: if $f \leq g$ on $[\alpha, \infty) \times D$ and either $f$ or $g$ is cooperative, then, for $\sigma \geq \alpha$ and $\phi, \psi \in D$ with $\phi \leq \psi$, we have $x(t, \sigma, \phi, f) \leq x(t, \sigma, \psi, g)$ for $t \geq \sigma$ whenever the solutions are defined (see [24]). In particular, (Q) guarantees the monotonicity of solutions of (2.1) relative to initial data.

Consider a family of non-autonomous systems (1.1), and further suppose that $b_{ik}(t, 0) = 0$ for $t \in \mathbb{R}$ and $b_{ik}$ have partial derivatives with respect to the second variable at $x = 0^+$, given by $\frac{\partial b_{ik}}{\partial x}(t, 0) = \gamma_{ik}(t)$, for all $i, k$. In this way, system (1.1) takes the general form

$$ x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \gamma_{ik}(s) h_{ik}(s, x_i(s)) d_s \eta_{ik}(t, s), \ i = 1, \ldots, n, \tag{2.2} $$

where all the coefficients, kernels and delay functions are supposed to be continuous, bounded and nonnegative. As a special case of (2.2), we shall consider systems with time-dependent discrete delays in the nonlinear terms, written in the form

$$ x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) h_{ik}(t, x_i(t - \tau_{ik}(t))), \ i = 1, \ldots, n. \tag{2.3} $$

Systems (2.2) and (2.3) are considered as abstract DDEs in the phase space $C = C([-\tau, 0]; \mathbb{R}^n)$, where

$$ \tau = \sup \{ \tau_{ik}(t) : t \geq 0, i = 1, \ldots, n, k = 1, \ldots, m \}. $$

For future reference, $\mathbb{R}^n$ is supposed to be equipped with the supremum norm $|x| = \max_{1 \leq i \leq n} |x_i|$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that (2.2) is obtained by adding a delayed perturbation $M(t, x_t)$ to the linear ODE

$$ x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t), \ i = 1, \ldots, n, \tag{2.4} $$

with $M(t, x_t)$ of the form $M(t, x_t) = (M_1(t, x_{1,t}), \ldots, M_n(t, x_{n,t}))$; for (2.2) each component $M_i(t, \phi_i)$ is given by

$$ M_i(t, \phi_i) = \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \gamma_{ik}(s) h_{ik}(s, \phi_i(s - t)) d_s \eta_{ik}(t, s), \tag{2.5} $$
whereas for (2.3) the components $M_i(t, \phi_i)$ read as

$$M_i(t, \phi_i) = \sum_{k=1}^{m} \beta_{ik}(t)h_{ik}(t, x(t - \tau_{ik}(t))),$$

(2.6)

for $t \in \mathbb{R}, \phi_i \in C([-\tau_0, 0] ; \mathbb{R}), i = 1, \ldots, n$.

Typically, system (2.2) can be used to model the population growth of either a single or multiple species structured into $n$ classes or patches, with migration among them: $x_i(t)$ denotes the density of the $i$th-species population, $a_{ij}(t)$ is the dispersal rate of the population migrating from class $j$ to class $i$, $d_i(t)$ is the coefficient of instantaneous loss for class $i$ (which integrates both the death rate and the migration coefficients referring to the individuals that leave class $i$ to move to other classes), and $M_i(t, \phi_i)$ is the birth function for class $i$. Following the general approach in the literature – though not always justifiable from a biological viewpoint [5] – multiple time-varying delays have been incorporated in the birth contribution.

Throughout the paper, hypotheses will be taken from the following set of conditions:

(H0) the functions $d_i, a_{ij}, \beta_{ik}, \gamma_{ik}, h_{ik} (\cdot, x) (x \geq 0), \eta_{ik} (\cdot, s) (s \in \mathbb{R})$ and $\tau_{ik}$ are $\omega$-periodic $(\omega > 0)$ on $t \in \mathbb{R}$;

(H1) $d_i, a_{ij} : \mathbb{R} \rightarrow \mathbb{R} (j \neq i)$ are continuous, with $a_{ij}(t) \geq 0, i \neq j, d_i(t) > 0$ for $t \in \mathbb{R}$ and $i, j \in \{1, \ldots, n\}$;

(H2) there exist a vector $u = (u_1, \ldots, u_n) \geq 0$ and $t_0 \in \mathbb{R}$ such that $d_i(t)u_i \geq \sum_{j=1, j\neq i}^{n} a_{ij}(t)u_j$ for $t \in \mathbb{R}$, with $d_i(t_0)u_i > \sum_{j=1, j\neq i}^{n} a_{ij}(t_0)u_j, i \in \{1, \ldots, n\}$;

(H3) $\tau_{ik}, \beta_{ik}, \gamma_{ik} : \mathbb{R} \rightarrow [0, \infty)$ are continuous, $\eta_{ik} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded, with $\eta_{ik}(t, s)$ nondecreasing on $s$ and locally integrable on $t$, and

$$\beta_i(t) := \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{ik}(t)}^{t} \gamma_{ik}(s) d_{ik}(t, s) > 0,$$

(2.7)

for $i \in \{1, \ldots, n\}, k \in \{1, \ldots, m\}$;

(H4) $h_{ik} : \mathbb{R} \times [0, \infty) \rightarrow [0, \infty)$ are bounded, continuous and locally Lipschitzian in $x$, with $h_{ik}(t, 0) = 0$ for $t \in \mathbb{R}$ and

$$h_{ik}(t, x) \geq h_{ik}^{-} (x), \quad t \in \mathbb{R}, x \geq 0, k = 1, \ldots, m,$$

where $h_{ik}^{-} : [0, \infty) \rightarrow [0, \infty)$ is continuous on $[0, \infty)$, continuously differentiable in a right neighborhood of 0, with $h_{ik}^{-} (0) = 0, (h_{ik}^{-})'(0) = 1$ and $h_{ik}^{-} (x) > 0$ for $x > 0, i \in \{1, \ldots, n\}$.

Assumptions (H1)-(H4), together with either (H0) or the boundedness of all functions in their domains, guarantee the existence and uniqueness of solutions for the initial value problems of (2.2) with $x_\sigma = \phi \in C^+$, defined for $t \geq \sigma$ [13]. For (2.3), $\beta_i(t)$ in (2.7) reads
simply as $\beta_i(t) = \sum_{k=1}^{m} \beta_{ik}(t), i = 1, \ldots, n$. Typically, in (H4) we take $h_i^-(x) = \min\{h_{ik}(t, x) : t \in [0, \omega], 1 \leq k \leq m\}$.

Hereafter, we designate by $A(t), B(t), D(t), M(t)$ the $\omega$-periodic $n \times n$ matrices defined on $\mathbb{R}$ and given by

\[
D(t) = \text{diag}(d_1(t), \ldots, d_n(t)), \quad A(t) = [a_{ij}(t)]
\]
\[
B(t) = \text{diag}(\beta_1(t), \ldots, \beta_n(t)), \quad M(t) = B(t) + A(t) - D(t),
\]

where $a_{ii}(t) \equiv 0, 1 \leq i \leq n$. In the literature, $M(t)$ is often called the community matrix for (2.2). In biological terms, (H1)-(H3) are quite natural conditions for periodic structured population models; for a discussion see [3], also for further references. Jointly with (H0)-(H4), we shall also consider the following assumption:

(H5) there exists $v = (v_1, \ldots, v_n) \gg 0$ such that $M(t)v \gg 0$ for $t \in [0, \omega]$.

Besides [1,3], where the nonlinearities are of Ricker-type, other useful population models satisfying the above hypothesis (H4) can be considered. Among them, models (2.2) with $h_{ik}(t, x) = xe^{-c_{ik}(t)x^\alpha} (\alpha > 0)$ or with nonlinearities of Mackey-Glass type

\[
h_{ik}(t, x) = \frac{x}{1 + c_{ik}(t)x^\alpha} \quad (\alpha \geq 1),
\]

where $c_{ik}(t)$ are continuous, positive and bounded, satisfy (H4). See Section 3 for an illustrative example.

Motivated by its biological interpretation, only nonnegative solutions of (2.2) are meaningful, and therefore admissible. Here, initial conditions are taken in $C_0$, where

\[
C_0 = \{\phi \in C^+ : \phi(0) \gg 0\}.
\]

The notions of uniform persistence and permanence given below (see e.g. [1,3]) will always refer to the choice of $C_0$ as the set of admissible initial conditions, given, as convention, at the instant of time $t = 0$: i.e., initial conditions read as $x_0 = \phi \in C_0$; of course, one can replace $[0, \infty)$ by any time interval $[\alpha, \infty), \alpha \in \mathbb{R}$.

**Definition 2.1.** A DDE $x'(t) = f(t, x_t)$ is said to be uniformly persistent (in $C_0$) if all solutions $x(t, 0, \phi)$ with $\phi \in C_0$ are defined on $[0, \infty)$ and there is $m > 0$ such that $\lim \inf x_i(t, 0, \phi) \geq m$ for all $1 \leq i \leq n, \phi \in C_0$. The DDE $x'(t) = f(t, x_t)$ is said to be permanent (in $C_0$) if it is dissipative and uniformly persistent; in other words, all solutions $x(t, 0, \phi), \phi \in C_0$, are defined on $[0, \infty)$ and there are positive constants $m, L$ such that, given any $\phi \in C_0$, there exists $t_0 = t_0(\phi)$ for which

\[
m \leq x_i(t, 0, \phi) \leq L \quad \text{for} \quad t \geq t_0, \quad i = 1, \ldots, n.
\]

In general, the nonlinearities in (2.2) are non-monotone in $x$, thus monotone techniques do not apply directly. Nevertheless, for the case of systems with discrete delays (2.3), in [2] the authors considered convenient auxiliary cooperative systems, and exploited results from the theory of monotone DDEs as in [24], to deduce the global asymptotic behavior of solutions. For periodic systems (2.2), some consequences and generalizations of results in [2] are given in the following theorem:
**Theorem 2.1** (i) If (H0)-(H2) are satisfied, then the \( \omega \)-periodic linear homogeneous system (2.2) is cooperative and exponentially asymptotically stable.

(ii) If (H0)-(H4) are satisfied, all solutions of (2.2) with initial conditions in \( C_0 \) are defined and strictly positive on \([0, \infty)\); moreover, (2.2) is dissipative (in \( C_0 \)).

(iii) If (H0)-(H5) are satisfied, then (2.2) is permanent (in \( C_0 \)).

**Proof.** The assertions in (i) and (ii) are immediate consequences of Theorems 2.1 and 2.3 in [1]. For the proof of (iii), below we adapt the arguments for the proof of [1, Theorem 3.3], omitting however details.

After the scaling of variables \( \hat{x}_j(t) = x_j(t)/v_j \), system (2.2) reads as

\[
\hat{x}_i'(t) = -d_i(t)\hat{x}_i(t) + \sum_{j=1, j \neq i}^{n} \hat{a}_{ij}(t)\hat{x}_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_k(t)}^{t} \gamma_{ik}(s)\hat{h}_{ik}(s, \hat{x}_i(s)) d_s \eta_{ik}(t, s), \quad i = 1, \ldots, n,
\]

where \( \hat{a}_{ij}(t) = v_i^{-1}a_{ij}(t)v_j, j \neq i \), and \( \hat{h}_{ik}(t, x) = v_i^{-1}h_{ik}(t, v_i x) \). The matrix \( D(t) - [\hat{a}_{ij}(t)] \) still satisfies (H2). In this way, and dropping the hats for simplicity, we consider the original system (2.2), but suppose that (H5) holds with \( \omega = 1 := (1, \ldots, 1) \). Thus, there exist constants \( \eta_i > 0 (i = 1, \ldots, n) \) such that

\[
\beta_i(t) \geq d_i(t) - \sum_{j \neq i} a_{ij}(t) + \eta_i, \quad t \in \mathbb{R}.
\]

On the other hand, \( d_i(t) - \sum_{j \neq i} a_{ij}(t) \leq \overline{d}_i := \max_{t \in [0, \omega]} d_i(t) \), and with \( 1 < \alpha_i < 1 + \eta_i/\overline{d}_i \) we obtain

\[
\alpha_i^{-1} \beta_i(t) - d_i(t) + \sum_{j \neq i} a_{ij}(t) > 0, \quad \text{for} \quad t \in \mathbb{R}, i = 1, \ldots, n. \tag{2.11}
\]

From the dissipativeness of the system asserted in (ii), and for \( h_i^- \) as in (H4), we can choose \( L > m > 0 \) such that the uniform estimate

\[
\lim_{t \to \infty} \sup_{x \in C_0} x_i(t, 0, \phi) < L \quad \text{for} \quad \phi \in C_0, \quad i = 1, \ldots, n, \tag{2.12}
\]

holds and \( h_i^-(m) = \min_{x \in [m, L]} h_i^-(x) \), with \( (h_i^-)'(x) > 0 \) and \( \alpha_i^{-1} x < h_i^-(x) \) for \( x \in (0, m] \) and all \( i \).

Consider the auxiliary cooperative system

\[
x_i'(t) = -\hat{d}_i(t)x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_k(t)}^{t} \gamma_{ik}(s)H_i(x_i(s)) d_s \eta_{ik}(t, s) =: F_i(t, x_1), \quad i = 1, \ldots, n, \tag{2.13}
\]

where \( H_i(x) = h_i^-(x) \) if \( 0 \leq x \leq m \), \( H_i(x) = h_i^-(m) \) if \( x \geq m \).
For \( x(t) \) a positive solution of (2.2), for \( t > 0 \) sufficiently large and \( 1 \leq i \leq n \), we have \( x_i(t) \leq L \) and \( h_{ik}(t,x_i(t)) \geq H_i(x_i(t)) \). Therefore, if (2.13) is uniformly persistent, then (2.2) is uniformly persistent as well [24].

Now, we consider any solution \( x(t) = x(t,\sigma,\phi,F) \) of (2.13) with \( x_0 = \phi \in C_0 \) and \( \sigma \in \mathbb{R} \). We claim that there is \( T = T(\sigma,\phi) \geq \sigma \) such that

\[
x_i(t) \geq m \quad \text{for} \quad t \geq T, 1 \leq i \leq n.
\]

We first prove that if \( \min \{ x_j(t) : 1 \leq j \leq n, t \in [T,T + \tau] \} \geq m \) for some \( T \geq \sigma \), then \( x_j(t) \geq m \) for all \( t \geq T \) and \( j = 1,\ldots,n \).

For simplicity of exposition, take \( T = \sigma = 0 \). Assume that \( x_j(t) \geq m \) for \( t \in [0,\tau] \) and \( j = 1,\ldots,n \). Let \( t_0 \in [\tau,2\tau] \) and \( i \in \{ 1,\ldots,n \} \) be such that \( x_i(t_0) = \min \{ x_j(t) : 1 \leq j \leq n, t \in [\tau,2\tau] \} \). We have

\[
0 \geq x'_i(t_0) = -d_i(t_0)x_i(t_0) + \sum_{j \neq i} a_{ij}(t_0)x_j(t_0) + \sum_{k=1}^{m} \beta_{ik}(t_0) \int_{t_0}^{t_0+\tau_k(t_0)} \gamma_{ik}(s)H_i(x_i(s))ds \eta_{ik}(t_0,s).
\]

Suppose that \( x_i(t_0) < m \). For \( k = 1,\ldots,m \) and \( s \in [t_0-\tau_k(t_0),t_0] \subset [0,t_0] \), we have \( x_i(s) \geq x_i(t_0) \), hence \( H_i(x_i(s)) \geq H_i(x_i(t_0)) \). From the definition of \( \beta_i(t) \) in (2.7), we obtain

\[
0 \geq \left( -d_i(t_0) + \sum_{j=1}^{n} a_{ij}(t_0) \right) x_i(t_0) + \beta_i(t_0)H_i(x_i(t_0))
\]

\[
\geq \left( -d_i(t_0) + \sum_{j=1}^{n} a_{ij}(t_0) + \alpha_i^{-1}\beta_i(t_0) \right) x_i(t_0) > 0,
\]

which is not possible. Thus, \( x_i(t_0) \geq m \). This implies that \( x_j(t) \geq m \) on \([0,2\tau]\) for all \( j = 1,\ldots,n \). By iteration, we obtain the same lower bound \( m \) on \([0,\infty)\), which proves (2.14).

Next, we need to show that there exists an interval of length \( \tau \) where the minima of all components \( x_i(t) \) are larger to or equal to \( m \). The proof follows by adapting slightly the arguments in [2], so we do not include it here. \( \square \)

**Remark 2.1** For systems (2.3) with all the coefficients and delay functions continuous and bounded, if we replace (H2) and (H5) by slightly stronger assumptions, the claims in the above theorem remain valid without assuming that the coefficients and delay functions are periodic. See [7] for details, as well as for supplementary results. See also [22], for the uniform persistence of a Nicholson almost-periodic system with one constant delay in each equation.

### 3. Existence of a positive periodic solution

In the case of periodic systems, we now show that the criterion for uniform persistence in Theorem (2.1)(iii) also provides a criterion for the existence of a positive \( \omega \)-periodic solution. We start with some algebraic definitions.
Definition 3.1. Let $A = [a_{ij}]$ be a square matrix. We say that $A$ is nonnegative, and write $A \geq 0$, if all its entries are nonnegative. For $A$ with nonpositive off-diagonal entries (i.e., $a_{ij} \leq 0$ for $i \neq j$), $A$ is said to be a non-singular M-matrix or a matrix of class $K$ if all its eigenvalues have positive real parts.

We remark that some authors use simply the term M-matrix to designate a non-singular M-matrix. There are many alternative equivalent definitions of non-singular M-matrices, see e.g. [3]. Namely, for a square matrix $A$ with nonpositive off-diagonal entries, the following conditions are equivalent: (i) $A$ is a non-singular M-matrix; (ii) there exists a vector $u \succ 0$ such that $Au \succ 0$; (iii) $A$ is non-singular and $A^{-1} \geq 0$.

Theorem 3.1 Assume (H0)-(H5). Then (2.2) has a positive $\omega$-periodic solution.

Proof. The proof will be divided in several steps.

(i) From Theorem 2.1(i), the linear homogeneous ODE (2.4) is exponentially asymptotically stable. Let $K \geq 1, \alpha > 0$ be such that $|X(t)X^{-1}(s)| \leq Ke^{-\alpha(t-s)}$ for $t \geq s$, where $X(t)$ is the fundamental matrix solution of (2.4) with $X(0) = I$. For $y_0 \in \mathbb{R}^n$, the solution of (2.4) with initial condition $y(s) = y_0$ is given by $X(t)X^{-1}(s)y_0$. It was observed in Section 2 that (2.4) is cooperative, hence its solutions are monotone relative to the order in $\mathbb{R}^n$, i.e., $y(t, s, x_0) \leq y(t, s, y_0)$ if $x_0 \leq y_0$. Moreover, for $s \in \mathbb{R}, y_0 \in \mathbb{R}^n$, a solution $y(t) = y(t, s, y_0)$ of (2.4) satisfies $y(t)^{\prime}(t) \geq -d_i(t)y_i(t), i = 1, \ldots, n$, and therefore $y(t, s, y_0) \geq 0$ for $t \geq s$ whenever $y_0 \geq 0$, with $y_i(t, s, y_0) = (X(t)X^{-1}(s)y_0)_i > 0$ if $y_0i > 0$, for any $1 \leq i \leq n$. For $t \geq s$, we derive that $X(t)X^{-1}(s) \geq 0, i.e.,$ all entries of the matrices $X(t)X^{-1}(s)$ are nonnegative (and that their diagonal entries are positive). For the monodromy matrix $C = X(\omega)$, we have $C = X^{-1}(t)X(\omega + t)$ for $t \in \mathbb{R}$. The matrices $T(t) := X(\omega + t)X^{-1}(t), t \in \mathbb{R}$, are nonnegative, $\omega$-periodic and similar to $C$. Since all the characteristic multipliers of (2.4) have moduli less than one, the spectral radius $\rho(T(t))$ of $T(t)$ is less than one. Consequently, $I - T(t)$ is a nonsingular M-matrix, and has inverse $(I - T(t))^{-1} \geq 0$.

(ii) From (H5), there exists $v = (v_1, \ldots, v_n) \succ 0$ such that

$$\eta_i := \min_{t \in [0, \omega]} \left( \beta_i(t)v_i - d_i(t)v_i + \sum_{j \neq i} a_{ij}(t)v_j \right) > 0, \quad i = 1, \ldots, n. \quad (3.1)$$

As before, we effect the scaling of variables $\hat{x}_j(t) = x_j(t)/v_j$ in (2.2), and obtain a new system of the form (2.2) where $\hat{a}_{ij}(t) = v_j^{-1}a_{ij}(t)v_j, j \neq i$, and $\hat{h}_{ik}(t, x) = v_j^{-1}h_{ik}(t, x)$. Hence, and without loss of generality, we may consider (2.2) and take $v = (1, \ldots, 1) = 1$ in (H5). As in the proof of Theorem 2.1(iii) we deduce that there are constants $\alpha_i > 1$ such that (2.11) is satisfied.

Theorem 2.1(iii) implies that (2.2) is permanent. Consider uniform lower and upper bounds $m, L$ for all positive solutions of (2.2), as in the uniform estimates (2.10). As a result of (H4), we can choose $L > m > 0$, with $m$ sufficiently small such that $h_i^{-}(m) = \min_{x \in [m, L]} h_i^{-}(x)$, with $h_i^{-}(x)$ increasing on $[0, m]$ and

$$\alpha_i^{-1}x < h_i^{-}(x) \quad \text{for} \quad x \in [0, m], \quad i = 1, \ldots, n.$$
Thus, we look for a fixed point \( \phi \in C_t \) because 

\[
\phi(t + \omega) = \phi(t) \quad \text{for all } t, t + \omega \in [-\tau, 0],
\]

we write \( \hat{\phi} \) for the \( \omega \)-periodic function defined in \( \mathbb{R} \) which coincides with \( \phi \) on \([-\tau, 0] \). Denote by \( C_{\omega}, C_{\omega}^+ \) the sets of \( \omega \)-periodic continuous functions \( \phi : \mathbb{R} \to \mathbb{R}^n \), respectively \( \phi : \mathbb{R} \to \mathbb{R}^n_+ \), which can be identified as subsets of \( C, C^+ \), respectively, with the same topology.

Now, suppose that \( x(t) = x(t, -\tau, \phi) \) is a solution of (2.2), with initial condition \( x_{-\tau} = \phi \in C_\omega \). By the variation of constants formula for ODEs,

\[
x(t) = X(t)X^{-1}(t_0)x(t_0) + X(t)\left( \int_{t_0}^t X^{-1}(s)M(s, x_s)\,ds \right) \quad (t, t_0 \geq -\tau),
\]

where \( M(t, \phi) = (M_1(t, \phi_1), \ldots, M_n(t, \phi_n)) \) is given by (2.5) and, as before, \( x_s = x|_{[\alpha, \tau]} \) for \( s \geq -\tau \). Clearly, \( x(t) \) is \( \omega \)-periodic if and only if \( x_{\omega} = x_0 \). From (3.2), \( x(\omega + \theta) = x(\theta) \) for \( \theta \in [-\tau, 0] \) if and only if

\[
x(\theta) = X(\omega + \theta)X^{-1}(\theta)x(\theta) + X(\omega + \theta)\int_{\theta}^{\omega + \theta} X^{-1}(s)M(s, x_s)\,ds,
\]

for \( \theta \in [-\tau, 0] \). This is equivalent to saying that \( x(t) \) is a fixed point of the operator \( F : C_\omega \to C \) defined by

\[
(F\phi)(\theta) = \left( I - T(\theta) \right)^{-1} \left( X(\omega + \theta)\int_{\theta}^{\omega + \theta} X^{-1}(s)M(s, \phi_s)\,ds \right), \quad \phi \in C_\omega, \theta \in [-\tau, 0].
\]

Thus, we look for a fixed point \( \phi \in \text{int}(C_{\omega}^+) \) of the operator \( F \).

(iv) The aim is to apply the Schauder fixed point theorem to the operator \( F \) in an appropriate subset of \( \text{int}(C_{\omega}^+) \).

We first claim that \( F\phi \in C_{\omega}^+ \) whenever \( \phi \in C_{\omega}^+ \). Let \( \phi \in C_\omega \), and set

\[
G(t; \phi) := X(\omega + t)\int_t^{\omega + t} X^{-1}(s)M(s, \phi_s)\,ds, \quad t \in \mathbb{R}.
\]

We have

\[
G(\omega + t; \phi) = \int_t^{\omega + t} X(2\omega + t)X^{-1}(\omega + s)M(\omega + s, \phi_{\omega + s})\,ds = G(t; \phi), \quad t \in \mathbb{R},
\]

because \( \phi(t) \) and \( t \mapsto M(t, \psi) \) are \( \omega \)-periodic and \( X(2\omega + t)X^{-1}(\omega + s) = X(\omega + t)X^{-1}(s) \). From step (i), \( T(t) \) is also \( \omega \)-periodic, hence \( F\phi \in C_\omega \). Since \( X(\omega + t)X^{-1}(s) \geq 0, (I - T(\theta)) \geq 0 \), we further derive that \( F\phi \geq 0 \) for \( \phi \in C_{\omega}^+ \).

From the continuity of \( (I - T(\theta))^{-1} \), there exists \( c = \max_{\theta \in [-\theta, 0]} \| (I - T(\theta))^{-1} \| \), thus

\[
\| F\phi \| \leq cK \beta^0 \beta^0 \frac{1}{\alpha}(1 - e^{-\alpha\omega}),
\]

where \( L^0, \beta^0 \) are such that \( h_{ik}(t, x) \leq L^0 \) and \( \beta_i(t) \leq \beta^0 \) for all \( i, k \) and \( t \in \mathbb{R}, x \geq 0 \). Therefore, \( F \) transforms \( C_{\omega}^+ \) into a bounded set of \( C_{\omega}^+ \). Choose \( R \geq L \), for \( L \) as in (2.10), such that \( F(C_{\omega}^+) \subset [0, R]\mathbb{I}_{\omega} \), where \( [0, R]\mathbb{I}_{\omega} := \{ \phi \in C_{\omega}^+ : \phi_i \leq R, 1 \leq i \leq n \} \).
We now prove that $\mathcal{F}(C^+_\omega)$ is equicontinuous in $C^+_\omega$. For $\phi \in C^+_\omega$, consider $G(t;\phi)$ as in (3.5), and $t_1, t_2 \in [-\omega, 0]$. We have

$$|(\mathcal{F}\phi)(t_1) - (\mathcal{F}\phi)(t_2)| \leq c|G(t_1;\phi) - G(t_2;\phi)| + \left| (I - T(t_1))^{-1} - (I - T(t_2))^{-1} \right| G(t_2;\phi). \quad (3.6)$$

Observe that

$$\left| \int_t^{\omega + t} X^{-1}(s)M(s, \phi_s) \, ds \right| \leq K \beta_0 L^0 \frac{1}{\alpha} (e^{\omega} - 1) =: C_1,$$

and

$$|G(t;\phi)| \leq K \beta_0 L^0 \frac{1}{\alpha} (1 - e^{-\omega}), \quad \forall \phi \in C^+_\omega, \forall t \in [-\omega, 0],$$

for all $\phi \in C^+_\omega$, $t_1, t_2 \in [-\omega, 0]$. Inserting these estimates in (3.6), we conclude that the family $\mathcal{F}(C^+_\omega)$ is equicontinuous. By Ascoli-Arzelà theorem, $\mathcal{F}(C^+_\omega)$ is relatively compact in $C^+_\omega$.

Next, we claim that

$$\mathcal{F}(m1, \infty)_\omega \subset [m1, \infty)_\omega,$$ \hfill (3.7)

where $[m1, \infty)_\omega := \{ \phi \in C^+_\omega : \phi_i \geq m, 1 \leq i \leq n \}$. Note that all solutions $x(t) = x(t, \sigma, \phi)$ of (2.2) (with $\phi \in C_0$) satisfy $m \leq x_i(t) \leq R$ for $t$ sufficiently large and $1 \leq i \leq n$.

Take $\phi \in C_\omega$ with $\phi(s) \geq m1$ for $s \in \mathbb{R}$. From step (ii), we have $h_i(s, \phi(s)) \geq h^+ \phi_i(s) \geq \alpha_i^{-1} \alpha_i m$ for all $i, k$ and $s \in \mathbb{R}$. For $M_i$ defined in (2.5), we obtain $M_i(s, \phi_i, s) \geq \beta_i(s) \alpha_i^{-1} m$, and (2.11) yields $M_i(s, \phi_i, s) \geq m |d_i(s) - \sum_{j \neq i} a_{ij}(s)|$, $i = 1, \ldots, n$, $s \in \mathbb{R}$. Since $X(\omega + \theta)X^{-1}(s) \geq 0$, we deduce that

$$X(\omega + \theta) \int_0^{\omega + \theta} X^{-1}(s)M(s, \phi_s) \, ds \geq mX(\omega + \theta) \int_0^{\omega + \theta} X^{-1}(s)[D(s) - A(s)] \, ds. \quad (3.8)$$

The differentiation of the identity $I = X^{-1}(s)X(s)$ leads to $\frac{d}{ds} \left( X^{-1}(s) \right) = X^{-1}(s)[D(s) - A(s)]$. From (3.8), we derive

$$X(\omega + \theta) \int_0^{\omega + \theta} X^{-1}(s)M(s, \phi_s) \, ds \geq m[I - T(\theta)] \, 1,$$

and finally from (3.4) obtain

$$(\mathcal{F}\phi)(\theta) \geq m \left[ I - T(\theta) \right]^{-1} \left[ I - T(\theta) \right] \, 1 = m1,$$
which proves the claim \(3.7\).

Consider the convex, closed bounded subset \([m1, R1]_\omega := \{\phi \in C_\omega : m1 \leq \phi \leq R1\}\) of \(C_\omega\). Applying Schauder’s fixed point theorem to the restriction (still denoted by \(\mathcal{F}\)) \(\mathcal{F} : [m1, R1]_\omega \rightarrow [m1, R1]_\omega\), we conclude that there exists a fixed point \(\phi^* \in [m1, R1]_\omega\). From (ii), \(\phi^*(t)\) is an \(\omega\)-periodic solution of \((2.2)\). The proof is complete. \(\Box\)

A general criterion concerning the existence of a positive periodic solution for periodic \(n\)-dimensional Nicholson systems is trivially obtained as a consequence of Theorem 3.1.

**Theorem 3.2** Consider \((1.3)\) where all the functions \(d_i(t), a_{ij}(t), \beta_{ik}(t), \gamma_{ik}(t), c_{ik}(t), \tau_{ik}(t)\) satisfy \((H0)-(H3)\) and \((H5)\). Then there exists (at least) one positive \(\omega\)-periodic solution of \((1.3)\). A similar result holds for \((1.4)\), with \(\beta_i(t)\) in \((2.7)\) replaced by \(\beta_i(t) = \sum_{k=1}^m \beta_{ik}(t)\).

As a by-product, Theorem 3.3 also provides conditions for the existence of a positive equilibrium for systems with autonomous coefficients.

**Theorem 3.3** Consider the system

\[
x'_i(t) = -d_ix_i(t) + \sum_{j=1,j \neq i}^n a_{ij}x_j(t) + \sum_{k=1}^m \beta_{ik}h_{ik}(x_i(t - \tau_{ik}(t))), \quad i = 1, \ldots, n, t \geq 0,
\]

where \(d_i > 0, a_{ij} \geq 0, \beta_{ik} \geq 0\) with \(\beta_i := \sum_{k=1}^m \beta_{ik} > 0, \tau_{ik} : [0, \infty) \rightarrow [0, \infty)\) are continuous and uniformly bounded from above by some \(\tau > 0\), and

\((H4^*)\) \(h_{ik} : [0, \infty) \rightarrow [0, \infty)\) are bounded, locally Lipschitz continuous on \([0, \infty)\) and continuously differentiable on a right neighborhood of 0, with \(h_{ik}(0) = 0, h'_{ik}(0) = 1\) and \(h_{ik}(x) > 0\) for \(x > 0\),

for all \(i, j = 1, \ldots, n, k = 1, \ldots, m\). Define the \(n \times n\) matrices

\[
A = [a_{ij}], \quad B = \text{diag}(\beta_1, \ldots, \beta_n), \quad D = \text{diag}(d_1, \ldots, d_n), \quad M = B - D + A,
\]

where \(a_{ii} := 0 (1 \leq i \leq n)\). Assume that: (i) \(D - A\) is a non-singular \(M\)-matrix; (ii) \(Mv \gg 0\) for some vector \(v \gg 0\). Then \((3.9)\) has a positive equilibrium.

**Proof.** For \(D, A\) as in \((3.10)\), hypotheses \((H1)\), \((H2)\) are satisfied, thus the linear autonomous ODE \(x' = -(D - A)x\) is exponentially asymptotically stable. Together with \((3.9)\), consider its associated ODE without delays:

\[
x'_i(t) = -d_ix_i(t) + \sum_{j=1,j \neq i}^n a_{ij}x_j(t) + \sum_{k=1}^m \beta_{ik}h_{ik}(x_i(t)), \quad i = 1, \ldots, n, \ t \geq 0.
\]

Systems \((3.9)\) and \((3.11)\) have the same equilibria. We apply Theorem 3.1 to \((3.11)\), noticing that in this case \(\omega = 0, \tau = 0\) and \(C = \mathbb{R}^n\), and deduce the existence of a positive equilibrium. \(\Box\)
Remark 3.1 Consider the case of autonomous ODEs \( x' = f(x) \) with \( f : \mathbb{R}^n \to \mathbb{R}^n \) a \( C^1 \) function. From the work of Hofbauer [14], it follows that if \( x' = f(x) \) is dissipative and the nonnegative cone \([0, \infty)^n\) is forward invariant for its flow, then there exists a saturated equilibrium \( x^* \geq 0 \) (see [14] for a definition), which may however lay on the border of \([0, \infty)^n\). Supplementary results can be found in [12, 14]. On the other hand, if in addition \( f(0) = 0 \) and the system \( x' = f(x) \) is uniformly persistent in \([0, \infty)^n \setminus \{0\}\), obviously there are no nonnegative equilibria \( x^* \) besides the trivial one, hence a positive equilibrium must exist. For the ODE (3.11), Theorem 3.3 asserts the existence of such an equilibrium without demanding the \( C^1 \)-smoothness of the vector field, though; in fact, from our assumptions the vector field in (3.11) is simply locally Lipschitzian (in order to guarantee the uniqueness of solutions) and continuously differentiable in a vicinity of \( 0^+ \). Moreover, for the case of autonomous Nicholson systems (1.4) (thus with constant coefficients and delays) with \( c_{ik}(t) \equiv 1 \), the existence of a positive equilibrium was established in [8] exactly under the conditions in Theorem 3.3.

From Theorem 3.2, we recover or improve some results in the literature.

Corollary 3.1 Consider the equation

\[
x'(t) = -d(t)x(t) + \sum_{k=1}^{m} \beta_k(t)h_k(t, x(t - \tau_k(t))),
\]

(3.12)

where the functions \( d(t), \beta_k(t), \tau_k(t) \) are continuous, non-negative and \( \omega \)-periodic, with \( d(t) > 0 \) for \( t \in \mathbb{R} \), and \( h_k(t, x) \) satisfy (H4). If

\[
\sum_{k=1}^{m} \beta_k(t) > d(t), \quad t \in [0, \omega],
\]

(3.13)

then there exists a positive \( \omega \)-periodic solution of (3.12).

Corollary 3.2 Consider the periodic Nicholson’s equations with distribute delays

\[
x'(t) = -d(t)x(t) + \sum_{k=1}^{m} \beta_k(t) \int_{t-\tau_k(t)}^{t} \gamma_k(s)x(s)e^{-c_k(s)x(s)} ds,
\]

(3.14)

where \( d(t), c_k(t) > 0, \beta_k(t), \gamma_k(t), \tau_k(t) \geq 0 \) are continuous and \( \omega \)-periodic. If

\[
\sum_{k=1}^{m} \left( \beta_k(t) \int_{t-\tau_k(t)}^{t} \gamma_k(s) ds \right) > d(t), \quad t \in [0, \omega],
\]

then (3.14) has a positive \( \omega \)-periodic solution. In particular, for the equation

\[
x'(t) = -d(t)x(t) + \beta(t) \int_{t-\tau(t)}^{t} \gamma(s)x(s)e^{-c(s)x(s)} ds,
\]

(3.15)

there is a positive \( \omega \)-periodic solution if \( \beta(t) \int_{t-\tau(t)}^{t} \gamma(s) ds > d(t), \ t \in [0, \omega]. \)
proved the existence of a positive \( \omega \) of a positive Corollary 3.1. On the other hand, Corollary 3.2 improves the result in \[1\], where the existence fixed point theorem on cones. It is clear that the result in \[16\] follows as a particular case of Corollary 3.3. Consider the planar system given by

\[
\begin{align*}
x'_1(t) &= -d_1(t)x_1(t) + a_1(t)x_2(t) + \sum_{k=1}^{m_1} \beta_{1k}(t)h_{1k}(t, x_1(t - \tau_{1k}(t))) \\
x'_2(t) &= -d_2(t)x_2(t) + a_2(t)x_1(t) + \sum_{k=1}^{m_2} \beta_{2k}(t)h_{2k}(t, x_2(t - \tau_{2k}(t)))
\end{align*}
\] (3.17)

where \( d(t) > 0, c_k(t) > 0, \beta_k(t) \geq 0, \tau_k(t) \geq 0 \) are continuous and \( \omega \)-periodic, Li and Du \[16\] proved the existence of a positive \( \omega \)-periodic solution if (3.13) holds, by using the Krasnoselskii fixed point theorem on cones. It is clear that the result in \[16\] follows as a particular case of Corollary 3.1. On the other hand, Corollary 3.2 improves the result in \[1\], where the existence of a positive \( \omega \)-periodic solution for (3.15) was obtained under the stronger condition

\[
\min_{t \in [0, \omega]} \gamma(t) > \max_{t \in [0, \omega]} \frac{d(t)}{\tau(t)\beta(t)}.
\]

For \( n = 2 \), the hypotheses (H2), (H5) are also easily verifiable in practice. For illustration, we state here a criterion for systems with discrete time-varying delays.

**Corollary 3.3** Consider the planar system given by

\[
\begin{align*}
x'_1(t) &= -d_1(t)x_1(t) + a_1(t)x_2(t) + \sum_{k=1}^{m_1} \beta_{1k}(t)h_{1k}(t, x_1(t - \tau_{1k}(t))) \\
x'_2(t) &= -d_2(t)x_2(t) + a_2(t)x_1(t) + \sum_{k=1}^{m_2} \beta_{2k}(t)h_{2k}(t, x_2(t - \tau_{2k}(t)))
\end{align*}
\] (3.17)

where \( m_1, m_2 \in \mathbb{N} \), \( d_i(t), a_i(t), \beta_{ik}(t), t \mapsto h_{ik}(t, x) \ (x \geq 0), \tau_{ik}(t) \) are continuous, nonnegative and \( \omega \)-periodic, with \( d_i(t), a_i(t) \) and \( \beta_i(t) := \sum_{k=1}^{m_i} \beta_{ik}(t) \) strictly positive for \( t \in [0, \omega] \), and \( h_{ik} \) satisfy (H4), \( i = 1, 2, k = 1, \ldots, m_i \). In addition, suppose that:

(i) \( \min_{t \in [0, \omega]} \frac{d_1(t)}{a_1(t)} > \max_{t \in [0, \omega]} \frac{a_2(t)}{d_2(t)} \);

(ii) there exist constants \( u_1, u_2 > 0 \) such that

\[
u_1(\beta_1(t) - d_1(t)) + u_2a_1(t) > 0, \quad u_2(\beta_2(t) - d_2(t)) + u_1a_2(t) > 0, \quad t \in [0, \omega].\]

Then (3.17) has at least one positive \( \omega \)-periodic solution.

**Proof.** From condition (i), choose \( v_2 \) with \( \max_{t \in [0, \omega]} \frac{a_2(t)}{d_2(t)} < v_2 < \min_{t \in [0, \omega]} \frac{d_1(t)}{a_1(t)} \). With \( v = (1, v_2) \), we have

\[
\begin{bmatrix}
d_1(t) & -a_1(t) \\
-a_2(t) & d_2(t)
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
v_2(t)
\end{bmatrix}
\geq
0,
\quad t \in [0, \omega],
\]

thus (H2) is satisfied. On the other hand, (ii) is hypothesis (H5) for the case \( n = 2 \), and the result follows from Theorem 3.1. \( \square \)
Remark 3.3 Liu [18] considered the planar Nicholson system
\[
x'_1(t) = -d_1(t)x_1(t) + a_1(t)x_2(t) + \sum_{k=1}^{m_1} \beta_{1k}(t)x_1(t - \tau_{1k}(t))e^{-c_{1k}(t)x_1(t - \tau_{1k}(t))}
\]
\[
x'_2(t) = -d_2(t)x_2(t) + a_2(t)x_1(t) + \sum_{k=1}^{m_2} \beta_{2k}(t)x_2(t - \tau_{2k}(t))e^{-c_{2k}(t)x_1(t - \tau_{2k}(t))}
\]
with all coefficients and delay functions $\omega$-periodic, continuous and positive. By constructing a suitable Lyapunov functional, Liu obtained the existence (and uniqueness) of a positive $\omega$-periodic solution by imposing some other rather restrictive constraints. Among these additional conditions, it was assumed that (cf. [18, Theorem 2.1])
\[
\min_{t \in [0, \omega]} \left( \sum_{k=1}^{m_i} \beta_{ik}(t) - d_i(t) \right) > 0, \quad \max_{t \in [0, \omega]} a_i(t) + e^{-2} \sum_{k=1}^{m_i} \max_{t \in [0, \omega]} \beta_{ik}(t) < \min_{t \in [0, \omega]} d_i(t), \quad i = 1, 2,
\]
which are assumptions stronger than (i),(ii) in Corollary 3.3.

Remark 3.4 As observed in the Introduction, our results are not optimal, and better criteria involving the average of the periodic coefficients $d_i(t), a_{ij}(t), \beta_i(t)$ in (2.22) are desirable. In fact, even for the case of $n = 1$ with one discrete delay, our method does not allow to recover the criterion of Chen [2], who establish the existence of a positive $\omega$-periodic solution of (1.6) under the conditions
\[
\bar{\beta} > \bar{d}\exp(2\omega \bar{d}), \quad \text{if } \tau(t) \text{ is } \omega-\text{periodic}
\]
\[
\bar{\beta} > \bar{d}, \quad \text{if } \tau(t) = m\omega
\]
where $\bar{\beta} := \frac{1}{\omega} \int_0^\omega \beta(t) \, dt, \bar{d} := \frac{1}{\omega} \int_0^\omega d(t) \, dt,$ and $m$ is some positive integer. Another limitation of our approach is that it cannot be applied directly when there exist some $i \in \{1, \ldots, n\}, t_0 \in [0, \omega]$ such that either $d_i(t_0) = 0$ or $\beta_i(t_0)$ (see [11] for an example).

Example 3.1 Consider the $\pi$-periodic planar system of Mackey-Glass type
\[
x'_1(t) = -(\epsilon_1 + \sin^2 t)x_1(t) + |\cos(2t)|x_2(t) + \frac{\left( \delta_1 + \cos^2 t \right)x_1(t - \sin^2 t)}{1 + e^{-2\sin^2 t}x_1(t - \sin^2 t)}
\]
\[
x'_2(t) = -(\epsilon_2 + \cos^2 t)x_2(t) + |\cos(2t)|x_1(t) + \frac{\left( \delta_2 + \sin^2 t \right)x_2(t - \cos^2 t)}{1 + (2 + \cos(2t))x_2(t - \cos^2 t)}
\]
with $\epsilon_i, \delta_i > 0$ for $i = 1, 2$ and $\alpha, \beta \geq 1$. The nonlinearities have the form (2.9). For (3.19) and with the notation in (2.8),
\[
D(t) - A(t) = \begin{bmatrix} \epsilon_1 + \sin^2 t & -|\cos(2t)| \\ -|\cos(2t)| & \epsilon_2 + \cos^2 t \end{bmatrix},
\]
\[
M(t) = \begin{bmatrix} \left( \delta_1 + \cos^2 t \right) - (\epsilon_1 + \sin^2 t) & |\cos(2t)| \\ |\cos(2t)| & \left( \delta_2 + \sin^2 t \right) - (\epsilon_2 + \cos^2 t) \end{bmatrix}.
\]
In addition, suppose that $\epsilon_1 \epsilon_2 \geq 1$, $\delta_1 > \epsilon_1$, $\delta_2 > \epsilon_2$. Note that

$$M(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \delta_1 - \epsilon_1 + \cos(2t) + |\cos(2t)| \\ \delta_2 - \epsilon_2 - \cos(2t) + |\cos(2t)| \end{bmatrix}.$$  

Clearly, (H5) is satisfied with $u = (1, 1)$. Next, take $v_2 > 0$ such that $\epsilon_2^{-1} \leq v_2 \leq \epsilon_1$. With $v = (1, v_2)$, we obtain

$$[D(t) - A(t)]v \geq \begin{bmatrix} \sin^2 t \\ \epsilon_2^{-1} \cos^2 t \end{bmatrix} \quad \text{for } t \in [0, \pi].$$

Thus, assumption (H2) holds. From Theorem 3.1, we conclude that (3.19) has a $\pi$-periodic positive solution.

**Example 3.2** Consider the planar Nicholson system

$$x_1'(t) = -(\epsilon_1 + \cos^2 t)x_1(t) + a_{12}e^{-2+\sin^2 t}x_2(t) + e^{\cos^2 t} \int_{t-\beta_1e^{-\cos^2 t+1}}^{t} x_1(s)e^{-(1+|\sin s|)x_1(s)}ds$$

$$x_2'(t) = -(\epsilon_2 + \sin^2 t)x_2(t) + a_{21}e^{\cos^2 t}x_1(t) + e^{\sin^2 t} \int_{t-\beta_2e^{-\sin^2 t+1}}^{t} x_2(s)e^{-e^2(t)x_2(s)}ds,$$

where $a_{12}, a_{21}, \epsilon_1, \beta_i > 0$ for $i = 1, 2$. The functions $\beta_i(t)$ in (2.7) are given by

$$\beta_1(t) = e^{\cos^2 t} \int_{t-\beta_1e^{-\cos^2 t+1}}^{t} ds = \beta_1 + e^{\cos^2 t}, \quad \beta_2(t) = e^{\sin^2 t} \int_{t-\beta_2e^{-\sin^2 t+1}}^{t} ds = \beta_2 + e^{\sin^2 t},$$

for $t \in \mathbb{R}$. For $y = \cos^2 t$ and matrices defined as in (2.8), we have

$$D(t) - A(t) = \begin{bmatrix} \epsilon_1 + y & -a_{12}e^{-(y+1)} \\ -a_{21}e^y & \epsilon_2 + 1 - y \end{bmatrix}, M(t) = \begin{bmatrix} \beta_1 + e^y & 0 \\ 0 & \beta_2 + e^{1-y} \end{bmatrix} + A(t) - D(t).$$

Suppose that

$$\epsilon_1 \epsilon_2 + \min\{\epsilon_1, \epsilon_2\} \geq a_{12} a_{21}. \quad (3.21)$$

In this case, $\epsilon_2 + \min\{\epsilon_1, \epsilon_2\} \geq a_{12} a_{21}$ for $y \in [0, 1]$, thus one can choose a constant $\eta$ such that $\epsilon_2 + \min\{\epsilon_1, \epsilon_2\} \geq \eta \leq (\epsilon_1 + y)a_{21}^{-1}$ for $y \in [0, 1]$. With $v = (1, \eta y)$, we have $[D(t) - A(t)]v \geq 0, t \in \mathbb{R}$ and $[D(t) - A(t)]v \neq 0$. Furthermore, for $u_1, u_2 > 0$ we have

$$M(t) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \geq \begin{bmatrix} u_1(\beta_1 - \epsilon_1 + 1) + u_2 a_{12}e^{-(y+1)} \\ u_1 a_{21}e^y + u_2(\beta_2 - \epsilon_2 + 1) \end{bmatrix}.$$

Assume also that

either $\beta_i - \epsilon_i + 1 \geq 0$ for some $i \in \{1, 2\}$ or $e^2(\epsilon_1 - \beta_1 - 1)(\epsilon_2 - \beta_2 - 1) < a_{12} a_{21}. \quad (3.22)\] 

If either $\beta_i - \epsilon_i + 1 \geq 0$ or $\beta_2 - \epsilon_2 + 1 \geq 0$, one finds $u \gg 0$ such that $M(t)u \gg 0, t \in \mathbb{R}$; if $\beta_i - \epsilon_i + 1 < 0$ for $i = 1, 2$ and $e^2(\epsilon_1 - \beta_1 - 1)(\epsilon_2 - \beta_2 - 1) < a_{12} a_{21}$, then $M(t)u \gg 0, t \in \mathbb{R}$ with $u = (1, u_2)$ and $u_2$ chosen so that

$$e^2(\epsilon_1 - \beta_1 - 1)a_{12}^{-1} < u_2 < a_{21}(\epsilon_2 - \beta_2 - 1)^{-1}.$$  

From Theorem 3.2, (3.21), (3.22) imply that there is a positive $\pi$-periodic solution of (3.20).
4. An application to periodic Nicholson systems

For a general system (2.2), it is important to establish conditions for the global attractivity of the positive $\omega$-periodic solution, whose existence was shown in Theorem 3.1. In order to obtain this global asymptotic behavior of solutions, it is clear that additional constraints depending strongly on the particular shape of the nonlinearities $h_{ik}$ should be imposed. In this section, we analyze this situation in the case of a Nicholson system (1.4) with constant discrete delays all multiple of the period. For simplicity, we only consider one delay in each equation of the system, but straightforward changes allow to consider several delays (all multiple of the period) as in (1.4).

Consider the periodic Nicholson’s system

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t) + \beta_i(t)x_i(t - m_i\omega)e^{-c_i(t)x_i(t - m_i\omega)}, \quad i = 1, \ldots, n, \quad (4.1)$$

where $m_i \in \mathbb{N}$, $\omega > 0$, $d_i(t)$, $a_{ij}(t)$, $\beta_i(t)$, $c_i(t)$ are continuous and $\omega$-periodic, with $d_i(t), \beta_i(t), c_i(t)$ positive and $a_{ij}(t)$ nonnegative, for all $i, j$.

We start with an auxiliary lemma regarding the particular nonlinearity $h(x) = xe^{-x}$.

**Lemma 4.1** For any $x \in (0, 2)$, define $G_x : [0, \infty) \to \mathbb{R}$ by

$$G_x(y) = \begin{cases} \frac{h(y) - h(x)}{y - x}, & y \neq x \\ (1 - x)e^{-x}, & y = x \end{cases}$$

where $h(x) = xe^{-x}, x \geq 0$. Then, for each $m \in (0, 1)$ there is $\delta(x) := \max_{y \geq m} |G_x(y)| < e^{-x}$.

**Proof.** Fix $x \in (0, 2)$, and consider $G_x$ defined as above. Note that $G_x(x) = h'(x)$. It was shown in [3] that $|h(y) - h(z)| < e^{-\beta}|y - z|$ for all $y > 0$ and $z \in (0, 2]$. Since $G_x$ is continuous and $G(\infty) = 0$, for any $m \in (0, 1)$ there exists $\delta(x) := \max_{y \geq m} |G_x(y)|$. But $\delta(x) < e^{-x}$ because $|G_x(y)| < e^{-x}$ for $y \neq x$ and $|G_x(x)| = |1 - x|e^{-x} < e^{-x}$. \□

Next, we denote

$$c_i^- := \min_{t \in [0, \omega]} c_i(t), \quad c_i^+ := \max_{t \in [0, \omega]} c_i(t), \quad i = 1, \ldots, n.$$

**Lemma 4.2** For some positive vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, suppose that

$$\alpha_i(v) := \min_{t \in [0, \omega]} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_{j \neq i} a_{ij}(t)v_j} > 1, \quad 1 \leq i \leq n, \quad (4.2)$$

and define

$$\gamma_i(v) := \max_{t \in [0, \omega]} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_{j \neq i} a_{ij}(t)v_j}, \quad 1 \leq i \leq n. \quad (4.3)$$

A positive $\omega$-periodic solution $x^*$ of (4.1) (whose existence is given in Theorem 3.2) satisfies

$$\frac{x_i^*(t)}{v_i} \leq \max_{1 \leq j \leq n} \frac{\log \gamma_j(v)}{v_j c_j^-}, \quad t \in [0, \omega], \quad i = 1, \ldots, n.$$
Theorem 4.1. For which implies $t \to \infty$ (4.2) holds with $v_i$ to insert the weights $v'_i$s in the final estimates.

Proof. As before, suppose that (4.2) holds with $v_i$ such that there is a vector $v = (v_1, \ldots, v_n)$ such that $\max_{t \in [0,\omega]} |x^*(t)| = x^*_i(t_0)$, we have

$$0 = -d_i(t_0)x^*_i(t_0) + \sum_{j \neq i} a_{ij}(t_0)x^*_j(t_0) + \beta_i(t_0)x^*_i(t_0)e^{\gamma_i(t_0)x^*_i(t_0)}$$

which implies $e^{\gamma_i(t_0)x^*_i(t_0)} \leq \gamma_i(t_0)$, thus $x^*_i(t_0) \leq \log \gamma_i(t_0)/(v_i c_i^-)$, and the result follows. $\square$

Theorem 4.1 For (4.1), suppose that there is a vector $v = (v_1, \ldots, v_n) \gg 0$ such that

$$\alpha_i(v) := \min_{t \in [0,\omega]} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_{j \neq i} a_{ij}(t)v_j} > 1$$

$$\gamma_i(v) := \max_{t \in [0,\omega]} \frac{\beta_i(t)v_i}{d_i(t)v_i - \sum_{j \neq i} a_{ij}(t)v_j} < e^{\gamma_0(v)}, \quad 1 \leq i \leq n, \quad (4.5)$$

where $c_0(v) = \min_{1 \leq i \leq n}(v_i c_i^-)$, $e^0(v) = \max_{1 \leq i \leq n}(v_i c_i^+)$. Then there exists a unique positive $\omega$-periodic solution $x^*(t)$, which is a global attractor of all other positive solutions of (4.1); that is, $x(t) - x^*(t) \to 0$ as $t \to \infty$ for any solution $x(t) = x(t, 0, \varphi)$ of (4.1) with initial condition $\varphi \in C_0$.

Proof. As before, suppose that (4.2) holds with $v = 1$, and replace $c_i(t)$ by $v_i c_i(t)$, so that (4.1) reads as

$$x'_i(t) = -d_i(t)x_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t) + \frac{\beta_i(t)}{v_i c_i(t)} h(v_i c_i(t)x_i(t - m_i \omega)), \quad i = 1, \ldots, n, \quad (4.6)$$

where $h(x) = xe^{-x}$ as in Lemma 4.1.

We have $\alpha_i(v) > 1$, $\gamma_i(v) < e^{2c_0(v)/e^0(v)}$. From Theorem 3.1 and Lemma 4.2, there is a positive $\omega$-periodic solution $x^*(t)$ of (4.6) whose components satisfy $0 < v_i c_i(t)x^*_i(t) \leq c^0(v)x^*_i(t) \leq e^0(v) \max_i \left( \frac{\log \gamma_i(v)}{v_i c_i^-} \right) < 2$ for $t \in [0,\omega]$. Effecting the change of variables $y_i(t) =
\[
\frac{x_i(t)}{x_i^*(t)} - 1 \text{ and using (4.9), (4.10) becomes}
\]
\[
y'_i(t) = \frac{1}{x_i^*(t)} \left\{ -d_i^*(t)y_i(t) + \sum_{j \neq i} a_{ij}(t)x_j^*(t)y_j(t) + \frac{\beta_i(t)}{v_i c_i(t)} \left[ h(v_i c_i(t)x_i^*(t)(1 + y_i(t - m_i \omega))) - h(v_i c_i(t)x_i^*(t)) \right] \right\}, \tag{4.7}
\]
where
\[
d_i^*(t) = \sum_{j \neq i} a_{ij}(t)x_j^*(t) + \beta_i(t)x_i^*(t)e^{-v_i c_i(t)x_i^*(t)}.
\]

Let \( y(t) = (y_1(t), \ldots, y_n(t)) \) be any solution of (4.7) with initial condition \( y_0 \geq -1, y(0) > -1 \). Define \( -z_i = \liminf_{t \to \infty} y(t), u_i = \limsup_{t \to \infty} y(t), \) and \( u = \max_i u_i, z = \max_i z_i \). From the uniform persistence of \( (4.7), x_i^*(t)(1 + y_i(t)) \geq m, t \geq 0, \) for some \( m \in (0, 1) \), and \(-1 < -z_i \leq u_i < \infty \).

It is sufficient to show that \( \max(u, z) = 0 \). Suppose that \( \max(u, z) = u > 0 \) (the situation \( \max(u, z) = z \) is treated in a similar way). Choose \( i \) such that \( u = u_i \) and take a sequence \( t_k \to \infty \) with \( y_i(t_k) \to u, y'_i(t_k) \to 0 \). Let \( \varepsilon > 0 \) be small. From (4.7) and Lemma 4.1 for large \( k \) we get
\[
y'_i(t_k) \leq \frac{1}{x_i^*(t_k)} \left\{ -d_i^*(t_k) + \sum_{j \neq i} a_{ij}(t_k)x_j^*(t_k) \right\} y_i(t_k) + \beta_i(t_k)x_i^*(t_k)\delta(v_i c_i(t_k)x_i^*(t_k))y_i(t_k - m_i \omega) + O(\varepsilon) \tag{4.8}
\]
\[
= \beta_i(t_k)\left[-e^{-v_i c_i(t_k)x_i^*(t_k)}y_i(t_k) + \delta(v_i c_i(t_k)x_i^*(t_k))y_i(t_k - m_i \omega)\right] + O(\varepsilon).
\]

For some subsequence of \( (t_k) \), still denoted by \( (t_k) \), \( \lim_{k} v_i c_i(t_k)x_i^*(t_k) = \xi \in (0, 2), \lim_{k} \beta_i(t_k) = b > 0, \lim_{k} y_i(t_k - m_i \omega) = w \in [-z, u] \). The estimate (4.8) leads to
\[
0 \leq b(-e^{-\xi}u + \delta(\xi)|w|) \leq b(-e^{-\xi} + \delta(\xi))u
\]
which is not possible because \( \delta(\xi) < e^{-\xi} \) for any \( \xi \in (0, 2) \). Thus \( u = 0 \). \( \square \)

Several important consequences can be deduced from Theorem 4.1.

**Corollary 4.1** Consider the classic periodic Nicholson’s equation with a delay multiple of the period:
\[
x'(t) = -d(t)x(t) + \beta(t)x(t - m \omega)e^{-c(t)x(t - m \omega)}, \tag{4.9}
\]
where \( \omega > 0, d(t), \beta(t), c(t) \) are continuous, positive and \( \omega \)-periodic functions and \( m \in \mathbb{N} \). Set \( \min_{t \in [0, \omega]} c(t) = c^-, \max_{t \in [0, \omega]} c(t) = c^+, \) and suppose that
\[
1 < \frac{\beta(t)}{d(t)} < e^{2c^-/c^+}, \quad t \in [0, \omega].
\]
Then there exists a unique positive \( \omega \)-periodic solution \( x^*(t) \), which is a global attractor of all other positive solutions of (4.9). In particular, if \( c(t) \equiv c > 0, \) the global attractivity of \( x^*(t) \) holds true if \( 1 < \beta(t)/d(t) < e^2, t \in [0, \omega] \).
Corollary 4.2 Consider the autonomous Nicholson’s system
\begin{equation}
  x_i'(t) = -d_i x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij} x_j(t) + \sum_{k=1}^{m} \beta_{ik} x_i(t - \tau_{ik}) e^{-c_i x_i(t-\tau_{ik})}, \quad i = 1, \ldots, n, \quad t \geq 0,
\end{equation}
where \( d_i > 0 \), \( c_i > 0 \), \( a_{ij} \geq 0 (j \neq i) \), \( \tau_{ik} \geq 0 \), \( \beta_{ik} \geq 0 \) with \( \beta_i := \sum_{k=1}^{m} \beta_{ik} > 0 \) for all \( i, j, k \).

If there exists a vector \( v > 0 \) such that
\begin{equation}
  1 < \gamma_i(v) < e^{\frac{2 \min_j (v_j)}{\max_j (v_j)}}, \quad \gamma_i(v) = \frac{\beta_i v_i}{d_i v_i - \sum_{j \neq i} a_{ij} v_j}, \quad i = 1, \ldots, n,
\end{equation}
then there exists a unique positive equilibrium which is a global attractor of all positive solutions of \((4.10)\).

Proof. The proof follows as the proof of Theorem 4.1 with the positive \( \omega \)-periodic solution \( x^*(t) \) replaced by the positive equilibrium \( x^* \). \qed

Corollary 4.3 Consider the autonomous Nicholson’s system
\begin{equation}
  x_i'(t) = -d_i x_i(t) + \sum_{j=1, j \neq i}^{n} a_{ij} x_j(t) + \sum_{k=1}^{m} \beta_{ik} x_i(t - \tau_{ik}) e^{-x_i(t-\tau_{ik})}, \quad i = 1, \ldots, n, \quad t \geq 0,
\end{equation}
where \( d_i > 0 \), \( a_{ij} \geq 0 (j \neq i) \), \( \tau_{ik} \geq 0 \), \( \beta_{ik} \geq 0 \) with \( \beta_i := \sum_{k=1}^{m} \beta_{ik} > 0 \) for all \( i, j, k \). If there exists a vector \( v > 0 \) such that
\begin{equation}
  1 < \gamma_i(v) < e^{\frac{2 \min_j (v_j)}{\max_j (v_j)}}, \quad i = 1, \ldots, n,
\end{equation}
where \( \gamma_i(v) \) are defined in \((4.11)\), then there exists a unique positive equilibrium which is a global attractor of all positive solutions of \((4.12)\). In particular, this is the case if
\begin{equation}
  1 < \gamma_i < e^2 \quad \text{for} \quad \gamma_i := \frac{\beta_i}{d_i - \sum_{j \neq i} a_{ij}}, \quad i = 1, \ldots, n.
\end{equation}

Remark 4.1 For the particular situation \((4.12)\), the result in Corollary 4.3 was proven in [8] under the hypothesis \( 1 < \gamma_i \leq e^2 \), \( 1 \leq i \leq n \), for \( \gamma_i = \gamma_i(1) \) as in (1.14). To obtain the result for \( \max_i \gamma_i = e^2 \), the proof however uses results on \( \omega \)-limit sets for autonomous DDEs, which do not carry out for (4.14), much less for more general periodic systems (1.4). On the other hand, adapting the proof in [8], it is now apparent that Corollary 4.2 is valid with
\begin{equation}
  1 < \frac{\beta_i v_i}{a_i v_i - \sum_{j \neq i} a_{ij} v_j} \leq \exp \left( \frac{2 \min_j (v_j)}{\max_j (v_j)} \right), \quad i = 1, \ldots, n,
\end{equation}
which improves the criterion in [8].

Example 4.1 Consider the 2-dimensional \( \omega \)-periodic Nicholson system with one single discrete delay given by
\begin{align}
  x_1'(t) & = -a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t - \tau) e^{-x_1(t-\tau)}, \\
  x_2'(t) & = -a_2(t)x_2(t) + b_2(t)x_2(t) + c_2(t)x_2(t - \tau) e^{-x_2(t-\tau)},
\end{align}
where \( a_i(t), b_i(t), c_i(t) \ (i = 1, 2) \) are positive, continuous and \( \omega \)-periodic functions, and suppose that \( \tau = m\omega \) for some \( m \in \mathbb{N} \). Applying Theorem 4.1, we derive that (4.15) has a globally attractive positive \( \omega \)-periodic solution if there exist positive constants \( v_1, v_2 \) such that
\[
1 < \frac{c_1(t)v_1}{a_1(t)v_1 - b_1(t)v_2} < e^2 \quad \text{and} \quad 1 < \frac{c_2(t)v_2}{a_2(t)v_2 - b_2(t)v_1} < e^2, \quad t \in [0, \omega].
\]
In particular, this assertion is valid if
\[
1 < \frac{c_i(t)}{a_i(t) - b_i(t)} < e^2, \quad t \in [0, \omega], i = 1, 2.
\]
(4.16)

A similar result holds with \( \tau, \omega \) rationally dependent.

We now compare this criterion with the one in [26]. Recently, Troib [26] used the continuation theorem of coincidence degree to show the existence of a positive periodic solution \( x^*(t) \) for (4.15) under the following constraints:
\[
2D_i \min\{e^{A_1}, e^{A_2}\} < A_i \leq 4D_i \max\{e^{A_1}, e^{A_2}\}, \quad 2C_i > e^{A_1}A_i, \quad i = 1, 2,
\]
(4.17)
where \( A_i = 2\omega \pi_i, B_i = \omega \tau_i, C_i = \omega \tau_i, D_i = \max\{B_i, C_i\}, i = 1, 2, \) and the notation \( \overline{f} = \frac{1}{\omega} \int_0^\omega f(t) \, dt \) is used for an \( \omega \)-periodic function. We observe however that for the particular case of the scalar periodic Nicholson equation (1.6) with \( \tau(t) \equiv \tau \) and \( c(t) \equiv 1 \), the criterion in [26] does not apply, since the conditions (4.17) would read as \( \beta > \overline{d} e^{2\omega \beta}, \gamma e^{2\omega \gamma} < \overline{d} < 2\gamma e^{2\omega \gamma}, \) and the set of functions \( \beta, d \) satisfying these conditions is empty. By using a suitable Lyapunov functional, in [26] the author further obtained the global asymptotic stability of \( x^*(t) \) under the additional restrictions
\[
\min_{t \in \mathbb{R}} c_i(t) > e^{M_i} \max_{t \in \mathbb{R}} a_i(t), \quad \max_{t \in \mathbb{R}} c_i(t) < \left( \min_{t \in \mathbb{R}} a_i(t) - \max_{t \in \mathbb{R}} b_i(t) \right) e^2, \quad i = 1, 2,
\]
for \( M_i \geq \limsup_{t \to \infty} x_i(t), i = 1, 2, \) for all solutions \( (x_1(t), x_2(t)) \) of (4.15), a hypothesis much stronger than the assumptions (4.16).

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