Riemann-Roch-Hirzebruch theorem and Topological Quantum Mechanics

Boris Feigin, Andrey Losev, and Boris Shoikhet

Abstract

In the present paper we discuss an independent on the Grothendieck-Sato isomorphism approach to the Riemann-Roch-Hirzebruch formula for an arbitrary differential operator. Instead of the Grothendieck-Sato isomorphism, we use the Topological Quantum Mechanics (more or less equivalent to the well-known constructions with the Massey operations from [KS], [P], [Me]). The statement that the Massey operations can "produce" the integral in some set-up, has an independent from the RRH theorem interest.

We finish the paper by some open questions arising when the main construction is applied to the cyclic homology (instead of the Hochschild homology).
Introduction

Let $X$ be an $n$-dimensional compact complex manifold, $E$ be a holomorphic vector bundle over $X$. Consider the sheaf $\mathcal{E}$ of holomorphic sections of the bundle $E$. Denote by $\text{Diff}(\mathcal{E})$ the algebra of global holomorphic differential operators acting in the sheaf $\mathcal{E}$.

Each $D \in \text{Diff}(\mathcal{E})$ acts on $H^i(X, \mathcal{E}), i \geq 1$. Define the super-trace of $D$ as

$$\text{str}(D) := \sum_{n=0}^{\infty} (-1)^i \text{Tr}|_{H^i(X, \mathcal{E})}(D)$$

As each space $H^i(X, \mathcal{E})$ is finite-dimensional because $X$ is compact, and $H^i(X, \mathcal{E}) = 0$ for $i > n$, the super-trace $\text{str}(D)$ is well-defined.

It is clear that $\text{str} : \text{Diff}(\mathcal{E}) \to \mathbb{C}$ is a trace on the algebra $\text{Diff}(\mathcal{E})$, that is, $\text{str}([D_1, D_2]) = 0$ for any $D_1, D_2 \in \text{Diff}(\mathcal{E})$ where $[D_1, D_2] = D_1 \cdot D_2 - D_2 \cdot D_1$.

When $D$ is the identity differential operator in the sheaf $\mathcal{E}$, $D = \text{Id}$, the super-trace $\text{tr}$ is equal to the Euler characteristic $\chi(\mathcal{E}) = \sum_{i \geq 0} (-1)^i \text{dim} H^i(X, \mathcal{E})$ of the sheaf $\mathcal{E}$. Then the Riemann-Roch-Hirzebruch theorem expresses $\chi(\mathcal{E})$ via topological invariants of the manifold $X$ and of the bundle $E$. Namely, $\chi(\mathcal{E}) = \int_X Td(T_X) \cdot \text{ch}(E)$ where $Td$ and $\text{ch}$ are the Todd genus and the Chern character, respectively.

In the present paper we find a formula for $\text{str}(D)$ for an arbitrary $D$ using

1) a technical assumption that $\chi(\mathcal{E}) \neq 0$,

2) the Riemann-Roch-Hirzebruch theorem itself.

To be honest we should warn the reader that the formula for $\text{str}(D)$ involving explicitly the Hochschild cocycle from [FFSh] will not appear in this paper. Instead of it, we deal with another formula, see (9) below, which is equivalent to the formula for $\text{str}(D)$ from the Hochschild cocycle.

We decided to write this paper by several reasons. At first, the proof here uses absolutely different from the existing approaches ideas and maybe even more suitable for further generalisations (for higher algebraic K-theory, etc). At second, the integral here appears as a consequence of a very general construction of the Topological Quantum Mechanics (Massey operations in homotopical algebra). We have in mind the existence of other applications of this scheme. The integral will be replaced by something different, but from our point of view, will be given by an analogous construction. This algebraic
(physical) definition of the integral and how it works in the framework of
the RRH theorem was one of our main motivations to write this paper.

The main idea

Let us outline the main idea of our approach here.

As the super-trace \( \text{str}(D) \) is a trace on the associative algebra \( \text{Dif}(E) \),
it is natural to consider it as a linear functional on 0-th Hochschild homology \( \text{HH}_0(\text{Dif}(E)) \).
The Hochschild homology \( \text{HH}_\bullet(\text{Dif}(E)) \) looks very big,
and we have no way to compute it. In particular, the associative algebra
\( \text{Dif}(E) \) can admit many other traces besides \( \text{str} \).
We want to do several things to improve the situation, namely, to redefine the Hochschild homology \( \text{HH}_\bullet(\text{Dif}(E)) \) such that they will become computable and nice. As usual,
we have two ways to do that. First, we can replace the algebra \( \text{Dif}(E) \) to
a natural dg algebra \( \text{Dif}^\bullet(E) \) such that \( \text{H}^0(\text{Dif}^\bullet(E)) \simeq \text{Dif}(E) \). After that,
we can hope that the Hochschild (and any other) homology \( \text{HH}_\bullet(\text{Dif}^\bullet(E)) \)
is more computable. Second, we can replace the concept of the Hochschild chain complex using some completions. In the definition of the Hochschild homology we meant before, the Hochschild chain of an algebra \( A \) are elements from the (algebraic) tensor products \( A^\otimes k \). We can consider completed tensor products, and the homology can change after that (and it is really the case).

In the first step, we set \( \text{Dif}^\bullet(E) \) be the Dolbeault complex of the sheaf \( \widehat{\text{Dif}}(E) \) of holomorphic differential operators in the sheaf \( E \),

\[
\text{Dif}^\bullet(E) = \Gamma_X(\text{Dolb}^\bullet(X, \widehat{\text{Dif}}(E)))
\]

where

\[
\text{Dolb}^\bullet(X, \widehat{\text{Dif}}(E)) := \text{Dolb}^\bullet(X, \mathcal{O}) \otimes_{\mathcal{O}} \widehat{\text{Dif}}(E)
\]

(Here \( \text{Dolb}^\bullet(X, \mathcal{O}) \) is the \( \bar{\partial} \)-resolution of the structure sheaf on the manifold \( X \).) Still now, the algebraic Hochschild homology \( \text{HH}_\bullet(\text{Dif}^\bullet(E)) \) do not seem to be regular.

In Section 3.2 we prove the following theorem (see Theorem 3.2):

Theorem. There exists a completion of all tensor products \( \text{Dif}^\bullet(E)^\otimes k \), \( k \geq 1 \), such that for the corresponding Hochschild homology \( \widehat{\text{HH}}^\bullet(\text{Dif}^\bullet(E)) \) one has:

\[
\widehat{\text{HH}}^{-k}(\text{Dif}^\bullet(E)) = \text{H}^{2n-k}(X, \mathbb{C})
\]

where \( n = \dim_{\mathbb{C}}(X) \). We use the convention (very natural from the point of view of homological algebra) that the Hochschild chains of a usual algebra \( A \) (concentrated in degree 0) are negatively graded \( \deg(A^{\otimes (k+1)}) = -k \).
We are interesting basically in traces, therefore, in 0-th Hochschild homology. Clearly we have a map

\[ \vartheta_1 : \text{HH}_0(\text{Dif}^*(\mathcal{E})) \to \widehat{\text{HH}}_0(\text{Dif}^*(\mathcal{E})) \]  

(5)

because we have the inclusion of the algebraic tensor powers of an algebra to completed tensor powers, and this map is compatible with the differential. Besides of it, we have the map

\[ \vartheta_0 : \text{HH}_0(\text{Dif}(\mathcal{E})) \to \text{HH}_0(\text{Dif}^*(\mathcal{E})) \]  

(6)

induced by the imbedding of algebras \( i : \text{Dif}(\mathcal{E}) \hookrightarrow \text{Dif}^*(\mathcal{E}) \). Denote by \( \vartheta \) the composition \( \vartheta = \vartheta_1 \circ \vartheta_0 \). Then we have the map

\[ \vartheta : \text{HH}_0(\text{Dif}(\mathcal{E})) \to \widehat{\text{HH}}_0(\text{Dif}^*(\mathcal{E})) \simeq H^{2n}(X) \]  

(7)

(the last isomorphism is stated by the Theorem above, the explicit form of this isomorphism will be specified later). As any global holomorphic differential operator \( D \in \text{Dif}(\mathcal{E}) \) defines an element in \( \text{HH}_0(\text{Dif}(\mathcal{E})) \), the map \( \vartheta \) gives a map

\[ D \mapsto [D] \in H^{2n}(X) \]  

(8)

Our subject in this paper is to describe the last map, that is, to compute \( [D] \) for any holomorphic differential operator \( D \in \text{Dif}(\mathcal{E}) \).

**Conjecture 1.** Let \( X \) be a compact manifold. Then

\[ \int_{\langle X \rangle} [D] = \text{str}(D) \]  

(9)

Here we prove the Conjecture using the technical assumption that \( \chi(\mathcal{E}) \neq 0 \). In the proof here we use the Riemann-Roch-Hirzebruch Theorem.

**Main Theorem.** The Conjecture above is true when \( \chi(\mathcal{E}) \neq 0 \).

Our main step in the proof of the Main Theorem is to lift the super-trace \( \text{str} \) first to \( \text{HH}_0(\text{Dif}^*(\mathcal{E})) \), and second to \( \text{HH}_0(\text{Dif}^*(\mathcal{E})) \). Denote the last linear functional by \( \text{str} \). Then we have two linear functionals on \( H^{2n}(X) \), the first is the integral, and the second is \( \text{str} \). As \( H^{2n}(X) \) is 1-dimensional, to prove that they coincide it is enough to have an element \( D \in \text{Dif}(\mathcal{E}) \) such that the values of the both functionals on \( D \) coincide and are non-zero. We take \( D = \text{Id} \). Then we need to compute \([\text{Id}]\). We do it in [FFSh], this the only place where we refer to [FFSh]. We prove there that \([\text{Id}] = Td(T_X) \cdot \text{ch}(\mathcal{E}).\) Then we apply the RRH Theorem to prove that the two linear functionals
on $H^{2n}(X)$ coincide on $D = \text{Id}$. When this value is 0, we cannot deduce that the two linear functionals coincide, it just means that $D = \text{Id}$ belongs to the kernel of the map $\vartheta$. To be more precise, let us reformulate the Conjecture above:

**Conjecture 2.** The map $\widehat{\text{str}} : H^{2n}(X) \to \mathbb{C}$ coincides with the integral map.

It is clear that the first Conjecture follows from the second one. Let us notice that Conjecture 2 by its formulation is completely independent on the RRH theorem. In our limping approach to this Conjecture we use the RRH theorem to prove it in the case when $\chi(E) \neq 0$. In the case $\chi(E) = 0$ Conjecture 2 still remains to be open.

In this approach, the most difficulty (and actually the subject of this paper) is to lift the super-trace $\text{str}$ defined on $HH_0(\text{Dif}(E))$ to a map $\widehat{\text{str}}$ defined on $\widehat{HH}_0(\text{Dif}^\bullet(E))$. Actually this is the only difficulty. To do it, we should "work on the level of complexes". We do it using the general formalism of Topological Quantum Mechanics, introduced by one of us (A.S.L) with Vyacheslav Lysov [Lys]. This formalism is a nice way to reformulate the well-known construction with the Massey operations (see [KS], [P], [Me]). The statement that the Massey operations can give the analytic integral, looks to be new (the statement of Conjecture 2 we prove in the particular case when $\chi(E) \neq 0$). It is very interesting to find possible generalizations of this construction, as well as to find a more direct (independent on the RRH Theorem) approach to the Conjecture 2.
1 Hochschild homology and super-traces

Recall that the Hochschild homology of an associative algebra $A$ is the (co)homology of the Hochschild chain complex $\text{Hoch}_\bullet(A)$. This is the complex

$$\cdots \to \text{Hoch}_2(A) \xrightarrow{d_{\text{Hoch}}} \text{Hoch}_1(A) \xrightarrow{d_{\text{Hoch}}} \text{Hoch}_0(A) \to 0$$

where $\text{Hoch}_k(A) = A^{\otimes (k+1)}$, and the differential $d_{\text{Hoch}}: A^{\otimes (k+1)} \to A^{\otimes k}$ is defined as follows:

$$d_{\text{Hoch}}(a_0 \otimes a_1 \otimes \cdots \otimes a_k) =$$

$$a_0a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_0 \otimes (a_1a_2) \otimes \cdots \otimes a_k + \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots 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In particular, in the simplest case \( k = 2 \) one has:

\[
\Theta_1([D_1, D_2]) + \Theta_2(\bar{\partial}D_1 \otimes D_2) + \Theta_2(D_1 \otimes \bar{\partial}D_2) = 0 \tag{12}
\]

where \( D_1, D_2 \) are \( C^\infty \)-differential operators acting in the sheaf \( \mathcal{E} \). When \( D_1, D_2 \) are holomorphic, \( \bar{\partial}D_i = 0 \), we obtain the usual trace condition \( \Theta_1([D_1, D_2]) = 0 \).

Denote by \( K^\bullet = \Gamma_X(\text{Dolb}^\bullet(X, \mathcal{E})) \) the Dolbeault complex of the sheaf \( \mathcal{E} \). Let us suppose we have a decomposition of complexes \( K^\bullet = K^\bullet_0 \oplus K^\bullet_1 \) where \( K^\bullet_0 \simeq H^\bullet(K^\bullet), d_{K^\bullet_0} = 0, \) and \( H^\bullet(K^\bullet_1) = 0 \). For any \( C^\infty \) differential operator \( \mathcal{D} \) define \( \Theta_1(\mathcal{D}) \) as

\[
\Theta_1(\mathcal{D}) := \text{str}(\Pi_{K^\bullet_0} \mathcal{D} \Pi_{K^\bullet_0}) \tag{13}
\]

where \( \Pi_{K^\bullet_0}, \Pi_{K^\bullet_1} \) are the projectors of \( K^\bullet \) to \( K^\bullet_0, K^\bullet_1 \), correspondingly, and the super-trace here is the super-trace of the endomorphism of \( K^\bullet_0 \).

It is clear that for a holomorphic \( \mathcal{D}, \Theta_1(\mathcal{D}) \) coincides with the usual super-trace. For non-holomorphic differential operators, \( \Theta_1([D_1, D_2]) \neq 0 \), and the problem is to find \( \Theta_2 \) such that equation (12) holds.

There is a theory due to Andrey Losev and Vyacheslav Lysov [Lys], called topological quantum mechanics, which allows us to construct all \( \Theta_k, k \geq 1 \).

2 The Topological Quantum Mechanics

2.1 The general set-up

Suppose \( \mathcal{A}^\bullet \) is a dg associative algebra, and \( M^\bullet \) is a dg module over it. It means in particular that there is a map

\[
\rho: \mathcal{A}^\bullet \otimes M^\bullet \rightarrow M^\bullet
\]

which is a map of complexes. Denote by \( d_\mathcal{A} \) the differential in \( \mathcal{A}^\bullet \), and by \( Q \) the differential in \( M^\bullet \). Denote by \( \rho(a), a \in \mathcal{A}^\bullet \), the corresponding endomorphism of \( M^\bullet \). Then

\[
\rho(d_\mathcal{A}(a)) = \{Q, \rho(a)\} \tag{14}
\]

Suppose also that \( M^\bullet \) has finite-dimensional cohomology.

Our main example is the case \( \mathcal{A}^\bullet = \text{Dif}^\bullet(\mathcal{E}), M^\bullet = K^\bullet \) in the notations of previous Sections.
Choose a decomposition $M^* = M^*_0 \oplus M^*_1$ where $M^*_0 \simeq H^*(M^*)$, $d_{M^*_0} = 0$, and $H^*(M^*_1) = 0$. The topological quantum mechanics constructs an $A_\infty$-morphism $\mathcal{F} : A^* \to \text{End} M^*_0$. That is, it constructs a collection of maps 

$$\mathcal{F}_k : A^* \otimes^k \to \text{End} M^*_0[1-k], \quad k \geq 1$$

satisfying the following relations:

\[
\begin{align*}
[F_{k-1}((a_1 \cdot a_2) \otimes a_3 \otimes \cdots \otimes a_k) & \mp F_{k-1}(a_1 \otimes (a_2 \cdot a_3) \otimes \cdots \otimes a_k) \pm \cdots \\
& \mp F_{k-1}(a_1 \otimes a_2 \otimes \cdots \otimes (a_{k-1} \cdot a_k)) \pm \\
& \pm \sum_{\ell=1}^{k-1} F_\ell(a_1 \otimes \cdots \otimes a_\ell) \circ F_{k-\ell}(a_{\ell+1} \otimes \cdots \otimes a_k) = 0,
\end{align*}
\]

$k \geq 1$ \hspace{1cm} (16)

Here in the last term $\circ$ denotes the composition in $\text{End}(M^*_0)$. We construct such an $A_\infty$-morphism below in this Section.

Remark. In [KS], [P], [Me] the authors consider the following situation. Let $B^* = B^*_0 \oplus B^*_1$ where $H^*(B^*_0) \simeq H^*(B^*)$ and $B^*_1$ is acyclic. Notice that we do not suppose that the differential in $B^*_0$ is zero and do not suppose that $B^*_0$ is a subalgebra. Then in the papers cited above the authors construct an $A_\infty$-algebra structure on $B^*_0$ in these assumptions such that this $A_\infty$-algebra is $A_\infty$-isomorphic to $B^*$. It is more or less a construction of higher Massey operations. The relation with our situation is as follows: set $B^*_0 = A^* \oplus M^*[1-1]$ where the product on $A^*$ is the initial product, the product of any two elements in $M^*[1-1]$ is zero, and the product of an element in $A^*$ with an element in $M^*[1-1]$ is the module action. Then $B^*$ is an associative dg algebra. Set $B^*_0 = A^* \oplus M^*_0[1-1]$ and $B^*_1 = M^*_1[1-1]$. It is clear that we are in the assumptions above. Then in the papers cited above the authors construct an $A_\infty$-algebra structure on $A^* \oplus M^*_0[1-1]$ such that the product restricted to $A^*$ is the initial product, and the higher products containing more than 1 element from $M^*_0$ are 0. It is clear that this structure is exactly the same that an $A_\infty$-morphism $\mathcal{F} : A \to \text{End} M^*_0$. We are going to present an exposition of this construction independent on the papers cited above.
2.2 Construction of $F_k$

Let $a_1, \ldots, a_k \in \mathcal{A}^*$, $\tau_1 < \tau_2 < \cdots < \tau_k$ be real numbers. Choose a homotopy $\kappa: M^\bullet \to M_1^\bullet[-1]$ such that $\{Q, \kappa\} = \Pi M_1^\bullet$ where $\Pi M_1^\bullet$ is the projection to $M_1^\bullet$. Define the spaces

$$Conf_k = \{\tau_1 < \cdots < \tau_k, \ \tau_i \in \mathbb{R}\}$$

and

$$C_k = Conf_k / G^{(1)}$$

where $G^{(1)}$ is the 1-dimensional group of shifts $\tau_i \mapsto \tau_i + c$, $i = 1, \ldots, k$.

Set

$$\Omega_{a_1, \ldots, a_k} = \Pi M_1^\bullet \circ \rho(a_k) \circ \exp[-d(\tau_k - \tau_{k-1})\kappa - (\tau_k - \tau_{k-1})\Pi M_1^\bullet] \circ \rho(a_{k-1}) \circ \exp[-d(\tau_{k-1} - \tau_{k-2})\kappa - (\tau_{k-1} - \tau_{k-2})\Pi M_1^\bullet] \circ \rho(a_{k-2}) \circ \cdots \circ \exp[-d(\tau_2 - \tau_1)\kappa - (\tau_2 - \tau_1)\Pi M_1^\bullet] \circ \rho(a_1) \circ \Pi M_0^\bullet \in \End^\bullet(M_0^\bullet) \otimes \Omega^\bullet(C_{k+1})$$

(a_1, \ldots, a_k \in \mathcal{A}^*) \tag{17}$$

which is a non-homogeneous differential form with the top component of degree $k - 1$ on the space $C_k$ with values in $\End(M_0^\bullet)$.

Denote $d_A \Omega_{a_1, \ldots, a_k} = \sum_{i=1}^k \Omega_{a_1, \ldots, a_i a_{i+1}, \ldots, a_k}$. Denote by $d_r$ the de Rham differential in the space $C_k$.

**Lemma.**

$$(d_A - d_r)\Omega_{a_1, \ldots, a_k} = 0 \tag{18}$$

**Proof.** We use the identity for $t \in \mathbb{R}$

$$dt\kappa + t\Pi M_1^\bullet = \{Q + dt, t\kappa\} \tag{19}$$

In particular,

$$\{Q + dt, \exp[dt\kappa + t\Pi M_1^\bullet]\} = 0 \tag{20}$$

We have:

$$d_A \Omega_{a_1, \ldots, a_k} =$$

$$\Pi M_1^\bullet \circ \rho(d_A(a_k)) \circ \exp[-d(\tau_k - \tau_{k-1})\kappa - (\tau_k - \tau_{k-1})\Pi M_1^\bullet] \circ \rho(a_{k-1}) \circ \exp[-d(\tau_{k-1} - \tau_{k-2})\kappa - (\tau_{k-1} - \tau_{k-2})\Pi M_1^\bullet] \circ \rho(a_{k-2}) \circ \cdots \circ \exp[-d(\tau_2 - \tau_1)\kappa - (\tau_2 - \tau_1)\Pi M_1^\bullet] \circ \rho(a_1) \circ \Pi M_0^\bullet + \cdots$$

$$\Pi M_0^\bullet \circ \{Q, \rho(a_k)\} \circ \exp[-d(\tau_k - \tau_{k-1})\kappa - (\tau_k - \tau_{k-1})\Pi M_1^\bullet] \circ \rho(a_{k-1}) \circ \exp[-d(\tau_{k-1} - \tau_{k-2})\kappa - (\tau_{k-1} - \tau_{k-2})\Pi M_1^\bullet] \circ \rho(a_{k-2}) \circ \cdots \circ \exp[-d(\tau_2 - \tau_1)\kappa - (\tau_2 - \tau_1)\Pi M_1^\bullet] \circ \rho(a_1) \circ \Pi M_0^\bullet + \cdots \tag{21}$$
Next, the r.h.s. of (21) is
\[
- \Pi_{M_0} \circ \rho(a_k) \circ \{Q, \exp[-d(\tau_k - \tau_{k-1})\kappa - (\tau_k - \tau_{k-1})\Pi_{M_0^\bullet}] \circ \rho(a_{k-1}) \circ \\
\exp[-d(\tau_{k-1} - \tau_{k-2})\kappa - (\tau_{k-1} - \tau_{k-2})\Pi_{M_0^\bullet}] \circ \rho(a_{k-2}) \circ \\
\cdots \circ \exp[-d(\tau_2 - \tau_1)\kappa - (\tau_2 - \tau_1)\Pi_{M_0^\bullet}] \circ \rho(a_1) \circ \Pi_{M_0^\bullet} + \ldots
\]
because $Q$ acts by 0 on $\text{End}(M_0^\bullet)$ and commutes with $\Pi_{M_0^\bullet}$. Now the claim of Lemma follows from (20).

This Lemma means that the differential form $\Omega_{a_1, \ldots, a_k}$ is "almost closed", and we can apply to it the Stokes formula as usually in topological field theories to get some algebraic identities (see, for instance, [K]).

We set
\[
F_k(a_1 \otimes \cdots \otimes a_k) := \int_{C_k} \Omega_{a_1, \ldots, a_k} \in \text{End}(M_0^\bullet)
\]  

The space $C_k$ is a smooth manifold of dimension $k - 1$, and only the top degree component (of degree $k - 1$) of the non-homogeneous differential form $\Omega_{a_1, \ldots, a_k}$ contributes. A priori this integral could diverge because the space $C_k$ is not compact. However, it will be clear from the sequel, that the integral absolutely converges.

### 2.3 The Stokes formula

Here we prove the following result:

**Theorem.** The maps $F_k: \mathcal{A}^\otimes k \to \text{End}(M_0^\bullet)[1-k], k \geq 1,$ satisfy the following relation:

\[
[F_{k-1}(a_1 \cdot a_2) \otimes a_3 \otimes \cdots \otimes a_k] \pm F_{k-1}(a_1 \otimes (a_2 \cdot a_3) \otimes \cdots \otimes a_k) \pm \ldots \\
\pm F_k(\bar{\partial}(a_1 \otimes a_2 \otimes \cdots \otimes (a_{k-1} \cdot a_k))) \pm \\
\pm \sum_{\ell=1}^{k-1} F_\ell(a_1 \otimes \cdots \otimes a_\ell) \circ F_{k-\ell}(a_{\ell+1} \otimes \cdots \otimes a_k) = 0,
\]

$k \geq 1$

(Here in the last term $\circ$ denotes the composition in $\text{End}(M_0^\bullet)$.)
Proof. First, we consider a compactification $\overline{C_k}$ of the manifold $C_k$ for any $k \geq 1$. This compactification is a manifold with corners stratified by locally-closed strata isomorphic to products $C_{i_1} \times C_{i_2} \times \cdots \times C_{i_\ell}$. Let us first describe this compactification informally. When two points in $Conf_k$ move close to each other, we want to have a stratum of codimension 1. When $\ell$ points move close to each other we want to have a stratum of codimension $\ell - 1$. Next, when the distance between two neighbour points tends to $\infty$, say the distance between the $\ell$'th and the $(\ell + 1)$'st points, then we want to have the stratum $C_\ell \times C_{k-\ell}$. Next, for higher degenerations the strata are labeled by trees, and so on.

There exists a nice explicit construction of $\overline{C_k}$, the cube. Consider the cube generated by $k - 1$ ortogonal vectors of length 1. The $i$'th vector has the length $\tau_{i+1} - \tau_i$. We compactify the ray $\{\tau_{i+1} - \tau_i \mid \tau_{i+1} > \tau_i\}$ by a point at the infinity. Then it is clear that this compactification obeys all properties listed above.

It is clear that the differential form $\Omega_{a_1,\ldots,a_k}$ constructed above can be extended to a smooth closed differential form on the cube $\overline{C_k}$. This proves, in particular, that the integrals $\int_{C_k} \Omega_{a_1,\ldots,a_k}$ absolutely converge.

Now we apply the Stokes formula to get (24). We have from (18)

$$0 = \int_{\partial C_k} \Omega_{a_1,\ldots,a_k} - \int_{C_k} dA \Omega_{a_1,\ldots,a_k} \quad (25)$$

Recall that $\Omega_{a_1,\ldots,a_k}$ is a non-homogeneous form with the top component of degree $k - 1$. We consider in details now the first summand in the r.h.s. of (25). We can restrict ourselves only in homogeneous component of degrees $k - 1$ and $k - 2$ in $\Omega_{a_1,\ldots,a_k}$. In the first summand in the r.h.s of (25) only the boundary strata of codimension 1 in $\partial C_k$ contributes. These boundary strata are exactly the faces of codimension 1 of the cube.

The boundary strata corresponding to these faces are of the two types, S1 and S2 below.

S1) Two points in $Conf_k$ move close to each other. The boundary stratum is $C_{k-1}$.

S2) The points are divided by two groups, $\{p_1,\ldots,p_\ell\}$ and $\{p_{\ell+1},\ldots,p_k\}$, and the distance between the groups tends to infinity. The boundary stratum is $C_\ell \times C_{k-\ell}$.

Geometrically, let the cube we consider be the unit cube in $\mathbb{R}^{k-1}$. Then it has two ”distinguished” vertices, namely, $A_0 = (0,0,\ldots,0)$ and $A_\infty = (1,1,\ldots,1)$. (Here the point 1 on each ray $\{\tau_{i+1} - \tau_i \mid \tau_{i+1} > \tau_i\}$ is the image of the ”point” $\tau_{i+1} - \tau_i = \infty$ under the compactification we consider). Then each face of codimension 1 of the cube contains either the point $A_0$ or the
point $A_{\infty}$. The case S1) above describes exactly the faces which contain the point $A_0$, while the case S2) describes exactly the faces which contain the point $A_{\infty}$.

It is clear that the other degenerations have higher codimension and are corresponded to the faces of the cube of higher codimension.

Now we want to compute the integral over $\partial C_k$. We have:

$$
\int_{\partial C_k} \Omega_{a_1,\ldots,a_k} = \int_{\partial S_1 C_k} \Omega_{a_1,\ldots,a_k} + \int_{\partial S_2 C_k} \Omega_{a_1,\ldots,a_k} \quad (26)
$$

We need to compute the restrictions $\Omega_{a_1,\ldots,a_k}\mid_{\partial S_1}$ and $\Omega_{a_1,\ldots,a_k}\mid_{\partial S_2}$.

2.3.1

Lemma.

(i) $\Omega_{a_1,\ldots,a_k}\mid_{\partial S_1,i} = \pm \Omega_{a_1,\ldots,a_i,a_{i+1},\ldots,a_k}$ as a differential form on the space $C_{k-1}$. Here we denote by $S_1,i$ the boundary stratum of the type S1 corresponding to the case when the $i$-th point moves close to the $i+1$-st, $1 \leq i \leq k-1$.

(ii) $\Omega_{a_1,\ldots,a_k}\mid_{\partial S_2,\ell} = \pm \Omega_{a_1,\ldots,a_{\ell}} \boxtimes \Omega_{a_{\ell+1},\ldots,a_k}$ as a differential form on the space $C_{\ell} \times C_{k-\ell}$. Here $\boxtimes$ denotes the external product of forms on the product of spaces, and the composition of the forms with values in $\text{End}M_0^*$. 

Proof. We prove (i) and (ii) separately.

For the proof of (i) denote $t_i = \tau_{i+1} - \tau_i$, $i \leq k-1$. We should analyze the quantity $\exp(-dt_i \kappa - t_i \Pi_{M_i^*})$ when $t_i \to 0$. Then $dt_i$ is identically 0, and the whole exponent is equal to 1. Now the claim (i) of Lemma follows from the identity $\rho(a_{i+1}) \cdot \rho(a_i) = \rho(a_{i+1} \cdot a_i)$ (which holds because $M^*$ is a dg module over the dg algebra $A^*$).

For the proof of (ii), we should consider the case when $t_i \to \infty$. Then $dt_i$ is also 0 identically, because $t_i = \infty$ is a point of the contified space. Now the claim follows from the identity

$$
\exp(-\infty \Pi_{M_i^*}) = \Pi_{M_{0}^*} \quad (27)
$$

It is clear: the exponent maps to 0 any element of $M_i^*$, and for $m \in M_{0}^*$ all summands in the power series expansion of the exponent $\exp(-\infty \Pi_{M_i^*})(m)$ are 0 except for the first which is the identity operator. The equation (27), and the Lemma are proven.
Theorem 2.3 is proven.

We have proved that the collection of maps \( \{ F_k, \ k \geq 1 \} \) define an \( A_\infty \)-morphism \( F: A^\bullet \to \text{End} M_0^\bullet \).

**Remark.** Notice that we never used in the course of the proof that \( \Pi_{M_1} \) is the projector. Namely, we can axiomatize the situation as follows. We have a decomposition \( M^\bullet = M_0^\bullet \oplus M_1^\bullet \), \( Q \) is the differential in \( M^\bullet \), and both \( M_0^\bullet \), \( M_1^\bullet \) are subcomplexes. Let \( \Pi_{M_0} \) be the projector to \( M_0 \). Choose a homotopy \( \kappa: M^\bullet \to M_1^\bullet[-1] \) such that \( \{ Q, \kappa \} = \Delta \) where \( \Delta: M^\bullet \to M^\bullet \) is an operator such that:

(i) \( \Delta | M_0 = 0 \),

(ii) \( \Delta(M_1) \subset M_1 \),

(iii) \( \exp(-t\Delta)|_{M_1} = 0 \) when \( t \to \infty \).

The conditions (i)-(iii) together mean that the exponent \( \exp(-\infty \Delta) \) is the projector \( \Pi_{M_0} \). In other words, we can replace the projector \( \Pi_{M_1} \) by the operator \( \Delta \) above with the weaker properties. Namely, we replace the property \( \Pi_{M_1}|M_1 = \text{Id} \) of the projector by the weaker property (iii) of the operator \( \Delta \). We use this Remark later in Section 3.1.2.

### 2.4 Applications to Hochschild and cyclic homology, and to supertraces

We prove here the following statement:

**Lemma.** Any \( A_\infty \)-morphism of associative dg algebras algebras \( F: A^\bullet \to B^\bullet \) induces a map of the Hochschild complexes \( F_{\text{Hoch}}: \text{Hoch}^*(A^\bullet) \to \text{Hoch}^*(B^\bullet) \), and a map of the cyclic complexes \( F_{\text{Cycl}}: \text{Cycl}^*(A^\bullet) \to \text{Cycl}^*(B^\bullet) \).

**Proof.** First construct the map \( F_{\text{Hoch}}: \text{Hoch}^*(A^\bullet) \to \text{Hoch}^*(B^\bullet) \). Let \( \Omega \in A^\bullet \otimes^k \). Define the set \( \Xi_k \) of all ordered partitions \( \xi \) of \( k \), \( \xi = (k_1, k_2, \ldots, k_\ell) \), \( k = k_1 + k_2 + \cdots + k_\ell \). Suppose \( \Omega = a_1 \otimes a_2 \otimes \cdots \otimes a_k \).
Define

\[ F_{\text{Hoch}}(a_1 \otimes \cdots \otimes a_k) = \sum_{\xi \in \Xi} \{ [F_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes F_{k_\ell}(a_{k_{\ell-1}}+1 \otimes \cdots \otimes a_k)] \]

\[+ \sum_{s=2} \sum_{\xi,s,\Omega} (-1)^{n(\xi,s,\Omega)} [F_{k_\ell}(a_{k-\ell+s+1} \otimes \cdots \otimes a_k \otimes a_1 \otimes \cdots \otimes a_{s-1}) \otimes \cdots \otimes F_{k_{\ell-1}}(a_{s+k_{\ell-1}+1} \otimes \cdots \otimes a_{s+k_{\ell-2}+k_{\ell-2}-1})] \]  

(28)

In the formula above \( F_{k_1}, \ldots, F_{k_\ell} \) are the components of the \( A_\infty \)-morphism \( F \), and the sign \((-1)^{n(\xi,s,\Omega)}\) is defined as follows. If \( \deg a_i = 0 \) for all \( i = 1, \ldots, k \), \( n(\xi,s,\Omega) = (k-\ell+s-1)(k-\ell-s+1) \), that is, the sign is equal to the sign of the permutation \( (1,2,\ldots,k) \mapsto (k-\ell+s,\ldots,k,1,2,\ldots,k-k+\ell+s-1) \). In the general case, we should take into account the degrees of \( a_i \)’s.

The reader can easily check that the map \( F_{\text{Hoch}} \) defines in fact a map of the Hochschild complexes. Moreover, it also defines a map of the cyclic complexes, so we can set \( F_{\text{Cycl}} = F_{\text{Hoch}} \). \( \square \)

Next, if we have any trace \( \Upsilon : \mathbb{H}^0(\operatorname{Hoch}(B^*)) \to \mathbb{C} \), then the composition \( \Upsilon_{\text{new}} = \Upsilon \circ F_{\text{Hoch}} \) gives us a trace \( \mathbb{H}^0(\operatorname{Hoch}(A^*)) \to \mathbb{C} \).

3 Applications to the algebra \( A^* = \text{Dif}^*(\mathcal{E}) \)

3.1 The trace on the algebra \( \text{Dif}^*(\mathcal{E}) \) before the completion

We do not know what is the algebraic (not completed) Hochschild homology \( \mathbb{H}^*(\operatorname{Hoch}(\text{Dif}(\mathcal{E}))) \). Nevertheless, the theory developed in Section 2 allows to construct a trace \( \text{str}^* : \mathbb{H}^0(\operatorname{Hoch}(\text{Dif}(\mathcal{E}))) \to \mathbb{C} \). Let us describe this trace (in more details than before).

At first, we have an \( A_\infty \)-morphism \( F : \text{Dif}^*(\mathcal{E}) \to \operatorname{End}(K_0^*) \) where \( K^* = \Gamma_X(\text{Dolb}^*(X,\mathcal{E})) \). In our assumptions that \( X \) is compact the cohomology \( H^*(K^*) = K_0^* \) is finite-dimensional. We have the corresponding map

\[ F_{\text{Hoch}} : \operatorname{Hoch}(\text{Dif}(\mathcal{E})) \to \operatorname{Hoch}(\operatorname{End}K_0^*) \]  

(29)

(see Section 2.4). In particular, we have the map

\[ F_{\text{Hoch}}^0 : \mathbb{H}^0(\operatorname{Hoch}(\text{Dif}(\mathcal{E}))) \to \mathbb{H}^0(\operatorname{Hoch}(\operatorname{End}K_0^*)) \]  

(30)
Therefore, to construct the super-trace \( \text{str}^\bullet : \mathbb{H}^0(\text{Hoch}_\bullet(\text{Dif}^\bullet(\mathcal{E}))) \to \mathbb{C} \), we need to construct a trace \( \mathbb{H}^0(\text{Hoch}_\bullet(\text{End}K^0_\bullet)) \to \mathbb{C} \).

As \( K^0_\bullet = H^\bullet K^\bullet \) are finite-dimensional, we have a very simple situation: the Hochschild homology of the algebra \( A^\bullet \) of endomorphisms of a \( \mathbb{Z} \)-graded finite-dimensional vector space \( V^\bullet \). We can use the following statement:

**Lemma.** In the assumptions above, the Hochschild homology \( \mathbb{H}^i(\text{Hoch}_\bullet(\mathcal{A}^\bullet)) \) is \( \mathbb{C} \) for \( i = 0 \), and is 0 for \( i \neq 0 \). We can describe a canonical (nonzero) super-trace \( \text{str}_{\text{can}} : \mathbb{H}^0(\text{Hoch}_\bullet(\mathcal{A}^\bullet)) \to \mathbb{C} \) as follows: it is nonzero on \( [\mathcal{A}^\bullet \otimes k]^k \) only for \( k = 1 \), and for an endomorphism of degree 0 \( m \in \mathcal{A}^0 \), the super-trace is

\[
\text{str}_{\text{can}}(m) = \text{Tr}|_{V^{\text{even}}}(m) - \text{Tr}|_{V^{\text{odd}}}(m)
\]

(31)

It is a standard fact.

Now we want to pull-back the super-trace \( \text{str}_{\text{can}} \) by the \( A_\infty \)-morphism \( \mathcal{F} \). This pull-back was described in Section 2.4 above. The only specification here is that the trace \( \text{str}_{\text{can}} \) we start with is nonzero on \( [\mathcal{A}^\bullet \otimes k]^k \) only for \( k = 1 \). It means that in the notations of Section 2.4 we should consider the partitions \( (k_1, \ldots, k_\ell) \), \( k = k_1 + \cdots + k_\ell \) only with \( \ell = 1 \). Thus, we have proved

3.1.1

**Lemma.** Let \( \mathcal{F}_k, k \geq 1 \) be the Taylor components of the \( A_\infty \)-morphism \( \mathcal{F}: \text{Dif}^\bullet(\mathcal{E}) \to \text{End}(K^0_\bullet) \). Then the supertrace \( \Upsilon = \mathcal{F}_k^{*}(\text{str}_{\text{can}}) \) can be described as follows: its value \( \Upsilon(a) \) for \( a \in [\text{Dif}^\bullet(\mathcal{E}) \otimes k]^{k-1} \) is

\[
\Upsilon(a) = \sum_{s=0}^{k-1} \text{str}_{\text{can}}(\mathcal{F}_k(C^s(a)))
\]

(32)

where \( C \) is the cyclic shift defined on \( \mathcal{A}^\bullet \otimes k \) as

\[
C(a_1 \otimes \cdots \otimes a_k) = (-1)^{(\deg a_k + 1)(\deg a_1 + \cdots + \deg a_{k-1} + k-1)} a_k \otimes a_1 \otimes \cdots \otimes a_{k-1}
\]

(33)

**Proof.** It follows from Lemma 2.4.

3.1.2 Topological Quantum Mechanics and Hodge theory

Here we specify the formula (32) for the trace even more, taking in the account a Kahler metric on the manifold \( X \) and an Hermitian metric in the
bundle $E$, and the corresponding Hodge theory. For simplicity we consider here only the case $E = \mathbb{C}$.

Now let $\mathcal{A} = \text{Diff}^\bullet(\mathcal{O})$ and $M^\bullet = \Gamma_X(\text{Dolb}^\bullet(X, \mathcal{O}))$, $Q = \bar{\partial}$. Suppose a Kahler metric on $X$ is chosen. Denote by $\Delta$ the $\bar{\partial}$-Laplacian. As before, we suppose that $X$ is compact. Set

$$M^0 = \text{Ker} \Delta = \text{harmonic forms}$$
$$M^1 = \text{Im}(\bar{\partial}) \oplus \text{Im}(\bar{\partial}^*)$$

(34)

According to the classical Hodge theory, $M^\bullet = M^0 \oplus M^1$, the differential on $M^0$ is 0, and $M^1$ is acyclic. Choose the homotopy $\kappa: M^\bullet \to M^1[-1]$ as $\bar{\partial}^*$. Then $\{Q, \kappa\} = \Delta$, the $\bar{\partial}$-Laplacian. So we replace $\Pi_{M^1}$ by the Laplacian, that is, we are in the situation of the Remark 2.3.1. Only what we need to check is the equation (iii) in this Remark, namely,

$$\exp(-t\Delta)(m_1) \to 0 \text{ when } t \to +\infty \text{ for } \forall m_1 \in M^1$$

(35)

To prove (35) notice that the Laplacian $\Delta$ on any compact Kahler manifold has a discrete non-negative spectrum. The space $M^0$ is exactly the 0 eigenspace. Let $\lambda_1$ be the smallest positive eigenvalue of $\Delta$. Then

$$\| \exp(-t\Delta)(m_1) \| \leq \exp(-t\lambda_1) \cdot \| m_1 \|$$

(36)

It is clear that for a fixed $m_1$, the r.h.s. of (36) tends to 0 when $t \to +\infty$.

In the next Sections we extend the super-trace $\Upsilon$ to some completion of the Hochschild chain complex $\text{Hoch}^\bullet(\text{Dif}^\bullet(\mathcal{E}))$. The difficulty here is that for this completed complex we have not any map to the Hochschild complex of the algebra $\text{End}K^\bullet$. Therefore, we can not anymore construct our supertrace $\hat{\text{str}}$ by the pull-back of some trace on the algebra of endomorphisms. The only what remains is to look at the formula (32) and to try to extend it to the completed complex. Then this extension will be a trace (a linear functional on 0-th cohomology of the completed Hochschild complex) automatically.

3.2 The completed Hochschild complex $\text{Hoch}^\bullet(\text{Dif}^\bullet(\mathcal{E}))$

We are going to describe here a compactification of the Hochschild complex $\text{Hoch}^\bullet(\text{Dif}^\bullet(\mathcal{E}))$ of the algebra $\text{Dif}^\bullet(\mathcal{E})$ which gives the ”right” homology. We start with a simpler question: Let $M$ be a $C^\infty$-manifold (of dimension $n$, or a dg manifold), what is the right definition of the Hochschild homology of the algebra $A = C^\infty(M)$? By the right definition we mean such a definition that
the Hochschild-Kostant-Rosenberg theorem holds, namely, the homology $HH_i(A)$ are isomorphic to $i$–th smooth differential forms on $M$, $\Omega^i(M)$. This is well known that there are several such definitions of $Hoch_\bullet(A)$. Recall here all of them:

1. $A \otimes^k = C^\infty(M^k)$
2. $A \otimes^k = \text{germs}_\Delta C^\infty(M^k)$
3. $A \otimes^k = \text{jets}_\Delta C^\infty(M^k)$

(here $\Delta : M \hookrightarrow M^k$ is the diagonal). In particular, the naive algebraic definition $A \otimes^k$ as the usual algebraic tensor power of $A$ does not work in the $C^\infty$-case. It works when $M = V$ is a vector space, and $A$ is the algebra of polynomial functions on $V$.

We have the following

**Lemma 1.** In all three definitions above the Hochschild homology $HH_i(A) = \Omega^i(M)$, the $C^\infty$ differential forms. \hfill \Box

In the present paper we use the first definition from the list above basically because we want to define a completion. Notice that from the point of view of proofs the second and the third definitions are simpler because we can use the sheaf theory in the cohomology computations.

Now we pass to differential operators in the $C^\infty$ sense (in trivial bundle).

**Lemma 2.** Let $M$ be a $C^\infty$-manifold, $\dim_{\mathbb{R}} M = n$, and let $A = \text{Dif}(M)$ be the algebra of $C^\infty$ differential operators on $M$. Define $A \otimes^k = \text{Dif}(M^k)$. Then the Hochschild homology $HH_{-i}(\text{Dif}(M)) = H^{2n-i}(M)$.

**Proof.** One way to prove this Lemma is to localize Lemma 1 in the sense of the Tsygan formality (see [Sh]). Another way is to prove somehow (maybe again using the Tsygan formality) the local statement for $M = V$ and then to use the simplicial methods. Here the main difficulty is that we should extend the Eilenberg-Zilber theorem for the completed tensor products. Anyway, if we fix the isomorphism locally, we get a canonical global isomorphism. \hfill \Box

**Corollary.** Let $M$ be a $C^\infty$-manifold, $A = \mathfrak{gl}_N(\text{Dif}(M))$, and $A \otimes^k = \mathfrak{gl}_N \otimes \text{Dif}(M^k)$. Then again $HH_{-i}(A) = H^{2n-i}(M)$.

**Proof.** Use the Kunneth formula for the Hochschild homology. \hfill \Box

Our main result here is
Theorem. Let $X$ be a complex manifold, $\dim_{\mathbb{C}} M = n$, $E$ be a holomorphic vector bundle over it, $\mathcal{E}$ be the corresponding sheaf of holomorphic sections. Define now the $k$-th tensor power $\text{Diff}^\bullet(\mathcal{E})^\otimes k$ as the $\text{Diff}(\mathcal{E}^\otimes k) \otimes_{C^\infty(X)^\otimes k} C^\infty(T^{0,1}[1],X)^\otimes k$ where $\text{Diff}$ stands for $C^\infty$-differential operators. Then the corresponding chain Hochschild complex has the homology $\hat{\text{HH}}_{-i}(\text{Diff}^\bullet(\mathcal{E})) = H_{2n-i}(X)$. There exists a canonical (depending on $E$) isomorphism $\vartheta_E : \hat{\text{HH}}_{-i}(\text{Diff}^\bullet(\mathcal{E})) \to H_{2n-i}(X)$.

Proof. We use simplicial methods and the Eilenberg-Zilber theorem (actually, a "completed" version of it). For the local computation we use the Corollary above. It gives us automatically the map $\vartheta_E$. □

Remark. We used here the first definition (in the listing at the beginning of Section 3.2) of the completed Hochschild homology basically because of aesthetic reasons. We mean that this definition more than others two is associated with a completion. If we use the third definition we could escape the simplicial methods and a completion of the Eilenberg-Zilber theorem. Instead of it, we can use elementary sheaf methods. We mean that in this (the third) definition we can consider $A^\otimes k$ as a sheaf on $X$ (not on $X \times \cdots \times X$). Then we can simply globalize the local computation, for which we need to have the Hochschild-Kostant-Rosenberg theorem.

3.3 The Integral conjecture

3.3.1 The trace $\hat{\text{str}}$

First of all, we prove the following

Theorem. The formula (32) (in the explicit homotopy of Section 3.1.2) for the super-trace $\Upsilon = F^*_{\text{Hoch}}(\text{str}_{\text{can}})$ on the algebra $\text{Diff}^\bullet(\mathcal{E})$ (which is nonzero only on $[\text{Diff}^\bullet(\mathcal{E})^\otimes k]^{k-1}, k \geq 1)$ can be continued to the completed tensor power $[\text{Diff}^\bullet(\mathcal{E})^\otimes k]^{k-1}$ defined in Section 3.2, and then it defines a trace $\hat{\text{str}}$ on the completed Hochschild chain complex of the algebra $\text{Diff}^\bullet(\mathcal{E})$.

Proof. Let $D = D_k \otimes \cdots \otimes D_1 \in [\text{Diff}^\bullet(\mathcal{E})^\otimes k]^{k-1}$ be a typical indecomposable chain before the completion. We can rewrite the formula (32) for $\Upsilon(D)$ as follows. First, we can associate with $D$ canonically an object on $X^{\times k}$, namely, an element of $\text{Diff}(\mathcal{E}^\otimes k) \otimes_{C^\infty(X)^\otimes k} C^\infty(T^{0,1}[1],X)^\otimes k$. Denote this element by $D$. We want to rewrite our formula for $\Upsilon(D)$ to make it sense for an arbitrary ("completed") $D \in \text{Diff}(\mathcal{E}^\otimes k) \otimes_{C^\infty(X)^\otimes k} C^\infty(T^{0,1}[1],X)^\otimes k$, not only for $D$ equal to the tensor product of $k$ factors. It is sufficiently to do it.
for $\mathcal{F}_k(D)$ because of the formula (32). As in Section 3.1.2, we introduce a Kahler metric on $X$ and consider the $\bar{\partial}$-Laplacian $\Delta$, and the corresponding Hodge theory. Then we can rewrite formula (17) as

$$
\Omega_D = \Pi_{K^\bullet_0} \circ D_k \circ \exp[-d(\tau_k - \tau_{k-1})\bar{\partial}^* - (\tau_k - \tau_{k-1})\Delta] \circ D_{k-1} \circ \exp[-d(\tau_{k-1} - \tau_{k-2})\bar{\partial}^* - (\tau_{k-1} - \tau_{k-2})\Delta] \circ D_{k-2} \circ \cdots \circ \exp[-d(\tau_2 - \tau_1)\bar{\partial}^* - (\tau_2 - \tau_1)\Delta] \circ D_1 \circ \Pi_{K^\bullet_0} \in \text{End}^\bullet(K^\bullet_0) \otimes \Omega^\bullet(C_{k+1})
$$

(37)

Now introduce

$$
\Phi = \exp[-d(\tau_k - \tau_{k-1})\bar{\partial}^* - (\tau_k - \tau_{k-1})\Delta]_{k-1} \circ \cdots \circ \exp[-d(\tau_2 - \tau_1)\bar{\partial}^* - (\tau_2 - \tau_1)\Delta]_1
$$

(38)

(we consider $\Phi$ as an object on $X \times X \times \cdots \times X$, the lower index denotes the variable at which the operator acts, and $\circ$ denotes the composition of operators). It is clear that $\Phi$ is not a differential operator. Nevertheless, the action of it on the $k$-th tensor power of the Dolbeault complex $K^\bullet \otimes_k k = \Gamma_X(\text{Dolb}^\bullet(X,E)) \otimes_k k$ is well-defined. Moreover, as operators acting on different variables commute, we can rewrite (37) as

$$
\Omega_D = \Pi_{K^\bullet_0} \circ m(\Phi \circ (D_k \otimes \cdots \otimes D_1)) \circ \Pi_{K^\bullet_0}
$$

(39)

where $m$ is the restriction to the diagonal $X \hookrightarrow X \times \cdots \times X$, or, in other words, the product. The last formula clearly makes sense for an arbitrary $D \in \text{Dif}(\mathcal{E}^\otimes k) \otimes \text{C}^\infty(X) \otimes_k \text{C}^\infty(T^{0,1}[1] X) \otimes_k$, and defines a trace on the completed Hochschild complex.

**Remark.** We used in this proof that $\Delta$ is a positive operator with a discrete spectrum.

### 3.3.2 The integral conjecture and the Riemann-Roch-Hirzebruch theorem

In Theorem 3.3.1 we constructed a trace $\Upsilon: \text{End}^\bullet(\text{Hoch}^\text{•}(\text{Dif}^\bullet(\mathcal{E}))) \to \mathbb{C}$, and in Theorem 3.2 we proved that $\text{H}^0(\text{Hoch}^\bullet(\text{Dif}^\bullet(\mathcal{E}))) \simeq \text{H}^{2n}(X)$ (the last isomorphism is $\vartheta_E$). These two results give us a map

$$
\Upsilon: \text{H}^{2n}(X) \to \mathbb{C}
$$

(40)

**The Integral Conjecture.** The map $\Upsilon$ is the integral over the fundamental cycle of the (compact) manifold $X$. 

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This statement looks quite mysterious: we constructed the integral starting from absolutely different things. We can prove this Conjecture only in the case when $\chi(\mathcal{E}) \neq 0$, and using the RRH theorem. Moreover, we need the following

**Lemma.** Consider the map $\vartheta_E: \hat{\Pi}_0(\text{Dif}^\bullet(\mathcal{E})) \to H^{2n}(X)$ from Theorem 3.2. Let $\text{Id}$ be the identity differential operator. Then $\vartheta(\text{Id}) = [Td(T_X) \cdot ch(E)]_{2n}$.

**Proof.** This statement is proven, in a form, in many places. See in particular [FT], [NT1], [FFSh].

Now the following statement is almost a tautology:

**Theorem.** The Integral Conjecture is true when $\chi(\mathcal{E}) \neq 0$.

**Proof.** It follows from Lemma above and from the RRH theorem that if $\chi(\mathcal{E}) \neq 0$ the identity differential operator $\text{Id}$ does not belong to the kernel of the map $\vartheta_E$. Then the image $\vartheta_E(\text{Id})$ is a non-zero element in the 1-dimensional space $H^{2n}(X)$. By the construction,

$$\text{str}(\mathcal{D}) = \Im(\vartheta_E(\mathcal{D}))$$

(41)

For any holomorphic differential operator $\mathcal{D}$ in $\mathcal{E}$. Applying this to $\mathcal{D} = \text{Id}$ and using the RRH theorem obtain $\Im(\vartheta_E(\text{Id})) = \int_X (\vartheta_E(\text{Id}))$. Then we conclude that $\Im = \int_X$ because $\vartheta_E(\text{Id}) \neq 0$ and because the space $H^{2n}(X)$ is 1-dimensional.

We have

**Corollary.** Suppose that $\chi(\mathcal{E}) \neq 0$. Then for any holomorphic differential operator $\mathcal{D} \in \text{Dif}(\mathcal{E})$ one has

$$\text{str}(\mathcal{D}) = \int_X (\vartheta_E(\mathcal{D}))$$

(42)

The last formula can be rewritten explicitly using the Hochschild cocycle from [FFSh] and formal geometry.

The reader could notice that here the Integral Conjecture has the primary interest for us, while the RRH theorem and formula (42) play an auxiliary role. It would be very interesting to find a direct approach to the
Integral Conjecture, and, in particular, to prove it without the assumption $\chi(\mathcal{E}) \neq 0$.

In the next Section we consider the analog of the Integral Conjecture in the case of cyclic homology. Here we have not any RRH theorem, and even the complete formulation of the conjecture still remains to be open.

4 The case of cyclic homology

Consider the cyclic homology instead of the Hochschild homology in all constructions above.

The $A_{\infty}$-morphism $\mathcal{F} : \text{Dif}^\bullet(\mathcal{E}) \to K^\bullet_0$ induces the map $\mathcal{F}_{\text{Cycl}} : \text{Cycl}^\bullet_{\text{Dif}}(\mathcal{E}) \to \text{Cycl}^\bullet_{\text{End}K_0}$ (see Section 2.4).

We have the following

Lemma 1. Let $A^\bullet$ be the algebra of endomorphisms of a finite-dimensional graded (with zero differential) vector space $V^\bullet$. Then the cyclic homology $\mathbb{H}^i(\text{Cycl}^\bullet(A^\bullet))$ is equal to $\mathbb{C}$ for an even not-positive $i$ and is equal to 0 otherwise. One has a canonical functional $\text{Tr}_{2i+1} : \mathbb{H}^{-2i}(\text{Cycl}^\bullet(A^\bullet)) \to \mathbb{C}$ given by the formula

$$\text{Tr}_{2i+1}(A_1 \otimes A_2 \otimes \cdots \otimes A_{2i+1}) = \text{Tr}\left( \sum_{\sigma \in \Sigma_{2i+1}} (-1)^{\sharp(\sigma)} A_{\sigma(1)} \cdots A_{\sigma(2i+1)} \right)$$

(43)

The advantage comparably with the case of the Hochschild homology is that here we have "many" traces, namely, a trace in each not-positive even degree, for the algebra $A^\bullet$ of endomorphisms, while in the Hochschild case we have such a trace only in degree 0.

We can take the pull-back $\Upsilon_{2i} = \mathcal{F}_{\text{Cycl}}^\ast(\text{Tr}_{2i+1})$. This is a "higher trace", that is, a linear functional

$$\Upsilon_{2i} : \mathbb{H}^{-2i}(\text{Cycl}^\bullet_{\text{Dif}}(\mathcal{E})) \to \mathbb{C}$$

Now it is interesting to compute the cyclic homology $\mathbb{H}^\bullet(\text{Cycl}^\bullet_{\text{Dif}}(\mathcal{E}))$. As in the Hochschild case, the answer is not interesting before the completion. There exists a completion of the cyclic complex $\text{Cycl}^\bullet_{\text{Dif}}(\mathcal{E})$ analogous to the completion of the Hochschild complex described in Section 3.2. Denote the completed cyclic complex by $\text{Cycl}_{\text{End}}^\bullet_{\text{Dif}}(\mathcal{E})$. We have
Lemma 2. The cohomology \( H^{-i}(\widehat{\text{Cycl}}(\text{Dif}^\bullet(\mathcal{E}))) \) is equal to \( H^{2n-i}(X) \oplus H^{2n-i+2}(X) \oplus H^{2n-i+4}(X) \oplus \ldots. \)

In particular, \( H^0(\widehat{\text{Cycl}}(\text{Dif}^\bullet(\mathcal{E}))) \simeq H^{2n}(X) \) and \( H^{-2}(\widehat{\text{Cycl}}(\text{Dif}^\bullet(\mathcal{E}))) \simeq H^{2n}(X) \oplus H^{2n-2}(X) \).

We get a map

\[ \mathcal{S}_{-2} : H^{2n}(X) \oplus H^{2n-2}(X) \to \mathbb{C} \quad (44) \]

What is this map? The construction depends on the bundle \( E \) and therefore the map \( \mathcal{S}_{-2} \) also could depend. Conjecturally, its component \( \mathcal{S}^{-2}_{2n} : H^{2n}(X) \to \mathbb{C} \) is the integral (at least, up to a constant depending only on the dimensions). But we have no tools in the moment to describe the map \( \mathcal{S}^{-2}_{2n} : H^{2n-2}(X) \to \mathbb{C} \). It could be 0 but could be not. The only what we can say is the following:

Let \( \mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3 \) be a cyclic cycle, \( \mathcal{D}_i \) be holomorphic differential operators in \( \mathcal{E} \). Then

\[ \text{str} \left( \sum_{\sigma \in \Sigma_3} (-1)^{2\sigma} \mathcal{D}_{\sigma(1)} \circ \mathcal{D}_{\sigma(2)} \mathcal{D}_{\sigma(3)} \right) = \mathcal{S}^{-2}_{2n}([\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3]) \quad (45) \]

where \( [\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3] \in H^{2n-2}(X) \) be the corresponding class. This formula follows directly from the definitions.

We can compute \( [\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \mathcal{D}_3] \) in one particular case, namely, we can compute \( [\text{Id} \otimes \text{Id} \otimes \text{Id}] \). It was proven in Nest-Tsygan papers that \( [\text{Id} \otimes \text{Id} \otimes \text{Id}] = [Td(T_X) \cdot ch(E)]_{2n-2} \) and \( [\text{Id}] = [Td(T_X) \cdot ch(E)]_{2n} \). Therefore, if one believes that \( \mathcal{S}^{2n}_{2} \) is the integral, \( \mathcal{S}^{-2}_{2n}([Td(T_X) \cdot ch(E)]_{2n-2}) \) should be 0 by the RRH theorem. In general, we expect that \( \mathcal{S}^{-2}_{2n-2} \) is the integral over some combination in codimension 2 of the Poincare dual to the Chern classes of the bundle \( E \), and of the natural bundles associated with \( X \).

It would be very interesting to understand better this topic.

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Independent University of Moscow,
11 Bolshoj Vlas'evskij pereulok, 121002 Moscow, RUSSIA
e-mail: feigin@mccme.ru

Moscow Institute for Theoretical and Experimental Physics (ITEP),
25 Bolshaja Cheremushkinskaja ulica, Moscow RUSSIA
e-mail: losev@mail.itep.ru

Department of Mathematics, ETH Zurich,
CH-8092 Zurich SWITZERLAND
e-mail: borya@math.ethz.ch