Chiral models in dilaton–Maxwell gravity

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December 26, 2021

Abstract

We study symmetry properties of the Einstein–Maxwell theory non-minimally coupled to the dilaton field. We consider a static case with pure electric (magnetic) Maxwell field and show that the resulting system becomes a nonlinear $\sigma$-model which possesses a chiral representation. We construct the corresponding chiral matrix and establish a representation which is related to the pair of Ernst–like potentials. These potentials are used for separation of the symmetry group into the gauge and nongauge (charging) sectors. New variables, which linearize the action of charging symmetries, are also established; a solution generation technique based on the use of charging symmetries is formulated. This technique is used for generation of the electrically (magnetically) charged dilatonic fields from the static General Relativity ones.
1 Introduction

It is well known fact that the Kaluza–Klein, supergravity and perturbative (super)string theories provide a large number of the effective four–dimensional gravity models \[1\], \[2\]. These models describe various interacting scalar, vector and tensor fields coupled to gravity. In the simplest case of the single scalar and gauge fields the corresponding action reads

\[ S = \int d^4x |g|^{1/2} \left[ -\frac{1}{4} R + 2 (\partial\phi)^2 - e^{-2\alpha\phi} F^2 \right], \]  

(1.1)

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). Here \(\alpha = \sqrt{3}\) in the case of the five–dimensional Kaluza–Klein theory compactified on a spatial–like circle and \(\alpha = 1\) for the naturally (i.e., with zero moduli) truncated low–energy heterotic string theory without axion field as well as for the N=2, D=4 supergravity. Other values of the parameter \(\alpha\) can arise in the framework of less convenient truncations of multidimensional or (super)string theories compactified to four dimensions, so the study of the theory (1.1) is interesting for applications.

In \[3\] it was shown that this theory in the stationary case becomes a nonlinear \(\sigma\)–model. It was also shown that this \(\sigma\)–model possesses a symmetry group which contains a gauge part for the arbitrary value of \(\alpha\). However, only in the Kaluza–Klein theory case it arises also a sector of the nongauge symmetries, which consists of the Ehlers–Harrison type transformations \[4\]. The presence of this nongauge and nonlinear sector is closely related to the appearance of integrable properties which arise after the following reduction to two dimensions (i.e. for the stationary and axisymmetric fields). In fact, for integrable systems it exists an infinite set of the conserving quantities; much of powerful and perfect methods can be applied to such systems (see \[5\] for the inverse scattering transform technique and \[6\] for the Bachlund transformation).

Below we consider the static case, i.e. the case when all fields do not depend on a time and the four–dimensional line element can be parametrized as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f dt^2 - f^{-1} h_{ij} dx^i dx^j = f dt^2 - f^{-1} dl^2. \]  

(1.2)

Moreover, we will study pure electric or pure magnetic configurations (i.e. we will suppose that \(A_t \neq 0, A_i = 0\) or \(A_t = 0, A_i \neq 0\)). As it was shown in
in both cases the theory (1.1) leads to three–dimensional model which can be described by the action

$$3S = \int d^3x h^\frac{1}{2} [-3R + 3\mathcal{L}],$$  

(1.3)

where $3R$ is a Ricci scalar constructed using $h_{ij}$ and

$$3\mathcal{L} = 2 (\nabla \phi)^2 + \frac{1}{2} f^{-2} (\nabla f)^2 - f^{-1} e^{\pm2\alpha\phi} (\nabla w)^2.$$  

(1.4)

In the “matter” Lagrangian (1.4) the signs “−” and “+” are related to the electric and magnetic cases correspondingly; for the electric case

$$w = \sqrt{2}A_t,$$  

(1.5)

whereas for the magnetic one

$$\nabla w = \sqrt{2}e^{-2\alpha\phi} f \nabla \times \vec{A}.$$  

(1.6)

Thus, $w$ is the electric (magnetic) potential; so it is a scalar (pseudoscalar) field from the three–dimensional point of view.

In the next section we construct several representations of the theory (1.3)–(1.4) to clarify its symmetry properties in the case of $\alpha \neq 0$. It will be shown that for the arbitrary value of $\alpha$ one obtains a chiral model which is closely related to the one appearing in the framework of pure (nonstatic) General Relativity [7].

In [3] it was shown that the complete three–dimensional theory, corresponding to (1.1) (i.e. the extension of Eq. (1.4) to the case of $g_{ti} \neq 0$ and both $A_t \neq 0, A_i \neq 0$), in the string theory case has not the nongauge symmetries and does not become integrable in two dimensions. However, the including of the axion field into the action (1.1) in a way predicting by the heterotic string theory provides a “right” correction of the theory (1.3)–(1.4): the resulting model possesses a reach symmetry structure in three and two dimensions [8].

### 2 Gauge and charging symmetries

First of all, from Eq. (1.4) it follows that the discrete transformation

$$\phi \rightarrow -\phi$$  

(2.1)
maps the electric system into the magnetic one and overwise. In fact, this
discrete symmetry is the “part” of the continuous electric–magnetic dual-
ity, which exists for the complete (i.e., electric–magnetic) effective three-
dimensional theory. To establish other symmetries let us introduce new
functional variables $F$, $\Phi$ and $W$ accordingly to the formulae

$$F = f e^{\mp 2 \alpha \phi}, \quad e^{2 \Phi} = f^\pm e^{2 \phi}, \quad W = \sqrt{1 + \alpha^2} w. \quad (2.2)$$

In terms of these variables

$$3L = \frac{3L}{1 + \alpha^2}, \quad \text{where}$$

$$3L = 2 (\nabla \Phi)^2 + \frac{1}{2} F^{-2} (\nabla F)^2 - 2 F^{-1} (\nabla W)^2. \quad (2.3)$$

One can see that $3L$ is the sum of two noncoupled Lagrangians; the uniquely
symmetry of the first one $3L_1 = 2 (\nabla \Phi)^2$ is

$$\Phi \to \Phi + \lambda_0, \quad (2.4)$$

where $\lambda_0$ is the arbitrary real parameter. To establish the symmetry group
of the second Lagrangian $3L_2 = \frac{1}{2} F^{-2} (\nabla F)^2 - 2 F^{-1} (\nabla W)^2$ it is useful to
introduce the functions

$$E_\pm = F^{\frac{1}{2}} \pm W; \quad (2.5)$$

in terms of them

$$3L_2 = 8 \frac{\nabla E_+ \nabla E_-}{(E_+ + E_-)^2}. \quad (2.6)$$

From this form of $3L_2$ it immediately follows that

$$E_\pm \to e^{\lambda_1} E_\pm \quad \text{(scaling)} \quad \text{and} \quad E_\pm \to E_\pm \pm \lambda_2 \quad \text{(shift)} \quad (2.7)$$

are the symmetries for the arbitrary real parameters $\lambda_1$ and $\lambda_2$. Then, the
map

$$E_\pm \to E_\mp^{-1} \quad (2.8)$$
is the important discrete symmetry of Eq. (2.6), this symmetry generalizes
the corresponding transformation established by Kramer and Neugebauer in
[9] for the stationary General Relativity (see also [10] for the complete, i.e.,
with the nonzero moduli, heterotic string theory case). It is easy to see that
Eq. (2.8) maps the scaling transformation into itself, whereas shift becomes
the Ehlers–like symmetry [11]
\[ E_{\pm}^{-1} \to E_{\pm}^{-1} \pm \lambda_3 \] (Ehlers) \hspace{1cm} (2.9)
(here \( \lambda_3 \) is the arbitrary real parameter). Thus, we obtain three one-
parametric symmetry transformations for the system (2.6); one can prove
that their generators form an algebra of the group SL(2, R).

Let us now introduce the following 2 \times 2 matrix
\[ M = F^{-\frac{1}{2}} \begin{pmatrix} 1 & W \\ W & W^2 - F \end{pmatrix}, \] \hspace{1cm} (2.10)
then
\[ ^3L_2 = \text{Tr} \left( J_M^2 \right) \text{ with } J_M = \nabla M M^{-1}. \] \hspace{1cm} (2.11)
The matrix \( M \) is a symmetric matrix of the signature + −; its determinant is
equal to −1. The general symmetry transformation preserving these prop-
erties is \( M \to C^T MC \) with \( \det C = 1 \), so \( C \in SL(2, R) \) as it was noted before.
One can prove that the matrix \( \mathcal{M} = e^\Phi M \) provides a chiral representation
of the whole \( ^3L \). Actually, in view of Eqs. (2.3) and (2.11) one immediately ob-
tains that \( ^3L = \text{Tr} J_M^2 \), where \( J_M = \nabla \mathcal{M} \mathcal{M}^{-1} \). Then the general symmetry
transformation reads
\[ \mathcal{M} \to C^T \mathcal{M} \mathcal{C} \] \hspace{1cm} (2.12)
with \( C \in GL(2, R) \). Here the additional \( U(1) \) transformation exactly cor-
responds to the one of Eq. (2.4); this \( U(1) \) transformation moves the \( \Phi \)–
asymptotics. It is not difficult to prove that the parametrization of \( C \) in
terms of the before introduced parameters \( \lambda_\mu, \mu = 0, ..., 3 \) reads:
\[ C = e^{\frac{\lambda_0 - \lambda_1}{2}} \begin{pmatrix} 1 & \lambda_2 \\ \lambda_3 & e^{\lambda_1} + \lambda_2 \lambda_3 \end{pmatrix}. \] \hspace{1cm} (2.13)
Let us now establish symmetries which preserve asymptotical flatness property of the field configurations, i.e., the charging symmetries. Thus, we consider fields with the spatial asymptotics \( f_{\infty} = 1 \) and \( \phi_{\infty} = w_{\infty} = 0 \), i.e., we suppose that \( \Phi_{\infty} = 0 \) and \( (E_{\pm})_{\infty} = 1 \). It is easy to see that for the corresponding transformations \( \lambda_0 = 0 \). Then, after some algebra one obtains that the remaining \( SL(2, R) \) transformations do not change asymptotics only if \( \lambda_2 = \lambda_3 \equiv \lambda \) and \( e^{\lambda_1} = 1 - \lambda^2 \). In this case

\[
C = \frac{1}{\sqrt{1 - \lambda^2}} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}.
\]

(2.14)

The corresponding transformations of the Ernst potentials read:

\[
E_{\pm} \rightarrow \frac{E_{\pm} \pm \lambda}{1 \pm \lambda E_{\pm}}.
\]

(2.15)

From Eq. (2.14) it follows that the charging symmetry transformation is the boost with the velocity parameter \( \zeta/2 = \text{Arcth} \lambda \). It is possible to linearize it by the choose of new appropriate field variables. Using the evident analogy of the Lagrangian (2.6) with the one of stationary General Relativity (see [7] for the details) it is natural to take

\[
Z_{\pm} = \frac{1 - E_{\pm}}{1 + E_{\pm}};
\]

(2.16)

then

\[
Z_{\pm} \rightarrow e^{\mp \zeta} Z_{\pm}.
\]

(2.17)

These formulae together with the condition \( \Phi = \text{inv} \) completely define the action of charging symmetry on the potentials. All the remaining symmetries move field asymptotics and form a gauge sector of the complete symmetry group. One can also consider the action of charging symmetry transformation on the charges, which we define accordingly the expansions \( f \rightarrow 1 - 2M_{gr}/r, \phi \rightarrow D/r, A_t \rightarrow Q_e/r \) (we consider the electric case for definiteness). Then, as it can easily be checked, the combination \( D - \alpha M_{gr} \) remains invariant, whereas the quantities \( M_{gr} + \alpha D \mp \sqrt{1 + \alpha^2 Q_e} \) transform exactly as the potentials \( Z_{\pm} \).
3 Solution generation technique

The variables $Z_{\pm}$ together with $\Phi$ form the most natural set of potentials for the asymptotically flat field configurations. The procedure for generation of the asymptotically flat solutions from the known ones becomes trivial in terms of these variables. For example, if one starts from the pure General Relativity with the static line element (1.2), and performs the charging symmetry transformation, one obtains the following solution:

$$ds^2 = \frac{f}{H^{1+\alpha^2}} dt^2 - \frac{H^{1+\alpha^2}}{f} dl^2,$$

$$\phi = \frac{\alpha}{1+\alpha^2} \ln H, \quad A_t = \frac{\sinh \zeta}{2\sqrt{1+\alpha^2}} \frac{1-f}{H}, \quad (3.1)$$

where $H = [1 + f + (1 - f) \cosh \zeta]/2$. Thus, the transformation preserving asymptotical flatness (we suppose that $f_{\infty} = 1$) generates the nontrivial electric and dilaton potentials (for obtaining of the magnetic solution one must apply the map (2.1) and replace $A_t$ to $\vec{A}$, see Eq. (3.10) below). In particular, from the massive solutions one obtains the electrically (magnetically) charged ones; so these transformations are actually charging transformations [12] (see also [10] for the heterotic string theory case).

Now let us consider the action of the full group of symmetry transformations on the set of divergent–free currents uniquely related to the problem. First of all, the equations of motion can be written as

$$\nabla J_0 = \nabla J_1 = \nabla J_2 = 0, \quad (3.2)$$

where

$$J_0 = \nabla \Phi, \quad J_1 = F^{-1} \nabla \left( F - W^2 \right), \quad J_2 = F^{-1} \nabla W. \quad (3.3)$$

It is easy to prove that these currents together with

$$J_3 = F^{-1} \left[ \left( F + W^2 \right) \nabla W - W \nabla F \right] \quad (3.4)$$
form the matrix current $J_M$. Actually, the straightforward calculation leads to

$$J_M = \begin{pmatrix} J_0 - J_1/2 & -J_2 \\ J_3 & J_0 + J_1/2 \end{pmatrix};$$

(3.5)

so from the equation $\nabla J_M = 0$ it follows that $\nabla J_3 = 0$. In fact, this last equation is the algebraic consequence of Eqs. (3.2); however the current $J_3$ is actually independent on the currents $J_0$, $J_1$ and $J_2$. These four algebraically independent currents linearly transform under the group of $GL(2, R)$ symmetries, because from Eq. (2.12) it follows that

$$J_M \rightarrow C^T J_M \left(C^T\right)^{-1}.$$  

(3.6)

Now let us consider the charging symmetry transformation with the matrix $C$ written in $\zeta$–terms. Then, after some algebraical analysis one leads to the following result: two current combinations

$$J_\pm = J_2 + J_3 \pm J_1$$  

(3.7)

transform exactly as $Z_\pm$ (see Eq. (2.17)),

$$J_\pm \rightarrow e^{\mp \zeta} J_\pm,$$  

(3.8)

whereas $J_0$ and $J_2 - J_3$ ones remain invariant. Let us also note that

$$\nabla \times \vec A = \frac{J_2}{\sqrt{1 + \alpha^2}}.$$  

(3.9)

(see Eqs. (1.6), (2.2) and (3.3)), so the calculation of $\vec A$ for transformed magnetic solution is equivalent to the calculation of the transformed current $J_2$. Then, for the magnetic variant of the above formulated generation procedure (see Eqs. (3.1) and remember the map (2.1)) one obtains that

$$\vec A = -\frac{\sinh \zeta}{2\sqrt{1 + \alpha^2}} \vec \Lambda.$$  

(3.10)

Here $\nabla \times \vec A = f^{-1} \nabla f$ is the single nonvanishing current of the original static solution of General Relativity. For example, if one starts from the Schwarzschild solution, i.e., if one takes $f = 1 - 2m/r$ and $dl^2 = dr^2 + r(r - 2m)(d\theta^2 + \sin^2 \theta d\phi^2)$, then the only nonzero $\vec \Lambda$–component is $\Lambda_\phi = -2m \cos \theta$. This result is more than natural in the framework of the magnetically charged dilatonic black hole physics, as well as all the formulae (3.1), see [13].
4 Discussion

In this work we have established the full symmetry group of the static electric and magnetic sectors of the four–dimensional dilaton–Maxwell gravity. From this symmetry group we have extracted a subgroup which preserves an asymptotical flatness property and have established the action of this subgroup on the potentials and currents. We have found the special potential and current combinations ($Z_\pm$ and $J_\pm$) which extremely simplify the action of charging symmetries and, moreover, transform in the same way, see Eqs. (2.17) and (3.8). It was also shown that the remaining potential and current degrees of freedom form invariants of the charging symmetry transformations. Using the developed formalism we have constructed the charging symmetry invariant extension of the static Einstein fields to the static electric and magnetic dilaton–Maxwell gravity. The established formulae (3.1) and (3.10) can be used, for example, in the black–hole physics after the substitution of the concrete values of $f$ and $h_{ij}$.

The effective theory (1.3)–(1.4), as a model possessing a chiral representation, becomes completely integrable after the following reduction to two dimensions (for example, in the axisymmetric case). This means, in particular, an appearance of the infinite–dimensional symmetry group and the infinite number of the divergent–free currents. This symmetry group, which is the analogy of the General relativity Geroch group [14], can be obtained using the spatial localization of the global transformation (2.12) in a way similar to one established for the principal chiral fields in [15]. We hope to perform the corresponding analysis in the forthcoming publication.

Acknowledgements

This work was supported by RFBR grant № 00 02 17135.

References

[1] M. Cvetic, D. Yuom, Phys. Rev. Lett. 75 (1995) 4165; J.M. Overduin, P.S. Wesson, Phys. Rept. 283 (1997) 303.

[2] E. Kiritsis, Introduction to Superstring Theory, CERN–TH/97-218, hep-th/9709002.
[3] D.V. Galtsov, A.A. Garcia, O.V. Kechkin, Class. Quant. Grav. 12 (1995) 2887.

[4] D. Maison, Gen. Rel. Grav., 10 (1979) 717.

[5] V.A. Belinsky, V.E. Zakharov, Sov. Phys. JETP 48 (1978) 985; Sov. Phys. JETP 50 (1979) 1.

[6] B.K. Harrison, J. Math. Phys. 24 (1983) 2178.

[7] O.V. Kechkin, Gen. Rel. Grav. (1999) 1075; O.V. Kechkin, M.V. Yurova, J. Math. Phys. 39 (1998) 5446.

[8] D.V. Galtsov, O.V. Kechkin, Phys. Rev. D50 (1994) 7394.

[9] D. Kramer, G. Neugebauer, H. Stephani, Forschr. Physik 20 (1972) 1; D. Kramer, H. Stephani, M. MacCallum, E. Herlt, Exact Solutions of the Einstein Field Equations, Deutcher Verlag der Wissenschaften, Berlin, 1980.

[10] A. Herrera–Aguilar, O.V. Kechkin, Phys. Rev. D59 (1999) 124006.

[11] J. Ehlers, in “Les Theories de la Gravitation” (CNRS, Paris, 1959).

[12] W. Kinnersley, J. Math. Phys. 18 (1977) 1529.

[13] M. Gurses, E. Sermutlu, Class. Quant. Grav. 12 (1995) 2799; G.W. Gibbons, D.L. Wiltshire, Annals Phys. 167 (1987) 393.

[14] R. Geroch, J. Math. Phys. 13 (1972) 394; W. Kinnersley and D.M. Chitre, Phys. Rev. Lett. 40 (1978) 1608.

[15] Hou Bo–yo, Ge Mo–lin, Wu Young–shi, Phys. Rev. D24 (1981) 2238.