Estimates of some integrals related to variations of smooth functions

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Abstract

Estimates of some integrals related to variations of smooth functions are presented.

1 Statement of the problem and the result

Let $Q$ be a closed cube in $\mathbb{R}^n$, and $Q_1$ be a neighbourhood of $Q$. Consider a twice differentiable function $\sigma: Q_1 \to \mathbb{R}$. Suppose that second partial derivatives of $\sigma$ are uniformly bounded, and $|\sigma| < 1$ in $Q$. Coordinates in $\mathbb{R}^n$ are $x_1, \ldots, x_n$. Denote $d\mu = dx_1 \ldots dx_n$ the volume element in $\mathbb{R}^n$.

Let $f: (0, 1] \to \mathbb{R}$ be a strictly decreasing positive smooth function such that $f(\xi)\sqrt{\xi}$ is bounded and $\int_0^{1} \frac{f(\xi)}{\sqrt{\xi}} d\xi$ converges.

For $a > 0$ denote

$$\tilde{\partial \sigma/\partial x}^a = \begin{cases} \partial \sigma/\partial x, & \text{if } |\sigma| < a, \\ 0, & \text{if } |\sigma| \geq a, \end{cases} \quad (1.1)$$

$$\Gamma_1(a) = \int_Q \|\tilde{\partial \sigma/\partial x}^a\| d\mu, \quad \Gamma_2(a) = \int_Q \left[\|\partial \sigma/\partial x\| - \|\tilde{\partial \sigma/\partial x}^a\|\right] f(|\sigma|) d\mu. \quad (1.2)$$

Integrals are understood in the sense of Riemann. The integrand in the last integral is defined by continuity at points $x$ where $\sigma = 0$: there it equals 0.
Proposition 1 For $0 < a < 1/2$

$$\Gamma_1(a) < C_1\sqrt{a}, \quad \Gamma_2(a) < C_2,$$

where $C_1, C_2$ are positive constants, i.e. values that do not depend on $a$.

This proposition with $f(\xi) = |\ln(\xi)|$ is contained in the thesis of the author [2]. It is presented here for the sake of reference. In [2] it is used to obtain estimates of probability of capture into resonance in systems with degeneracies.

2 Proof of Proposition 1

In this proof $c_i$ are positive constants, i.e. values that do not depend on $a$. Appearance such a constant in the proof means the assertion that such a constant exists.

2.1 Proof of existence of integrals

Divide $Q$ into cubes with a side $l > 0$. Denote $Q_1$ the union of cubes that contain points, where $||\partial \sigma/\partial x|| \leq l$; $Q_2 = Q \setminus Q_1$. Then $||\partial \sigma/\partial x|| < c_1l$ in $Q_1$. Because $||\partial \sigma/\partial x|| > l$ in $Q_2$, the set $\{x : \sigma(x) = a, x \in Q_2\}$ is a union of a finite number of smooth hypersurfaces. Therefore the function $\Phi = \|\partial \sigma/\partial x \|^a$ is discontinuous in $Q_2$ only on a finite number of smooth hypersurfaces. Thus this function is integrable in $Q_2$. Therefore there exists a partition of $Q_2$ into sets such that on each of these sets the oscillation of $\Phi$ does not exceed $l$. Consider the union of this partition and $Q_1$. This is a partition of $Q$. On each set of this partition the oscillation of $\Phi$ does not exceed $l + c_1l$. Because $l$ is arbitrarily small, $\Phi$ is integrable on $Q$. Thus $\Gamma_1(a)$ exists. Similarly, one can prove that $\Gamma_2(a)$ exists.
2.2 Reduction to the case of one variable

Let \( Q = \{(x_1, \ldots, x_n) : |x_j - x_j^0| \leq d, \ j = 1, \ldots, n\} \). We have

\[
\Gamma_1(a) = \int_Q \left\| \frac{\partial \sigma}{\partial x} \right\| d\mu = \int_Q \sqrt{\sum_{i=1}^{n} \left( \frac{\partial \sigma}{\partial x_i} \right)^2} d\mu \\
\leq \int_Q \left( \sum_{i=1}^{n} \left| \frac{\partial \sigma}{\partial x_i} \right| \right) d\mu = \sum_{i=1}^{n} \int_{Q_i} \prod_{j=1, j \neq i}^{n} dx_j \int_{x_i^0-d}^{x_i^0+d} \left| \frac{\partial \sigma}{\partial x_i} \right| dx_i,
\]

where \( Q_i = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : |x_j - x_j^0| \leq d, \ j \neq i\} \), and notation is similar to that in (1.1). Similarly, one can estimate \( \Gamma_2(a) \).

Lemma 2.1 Let a function \( \psi(z), 0 \leq z \leq d, \) be twice differentiable and \( |\psi(z)| < 1, \ |d^2\psi/dz^2| < c_1. \) Denote for numbers \( 0 \leq \alpha < \beta \leq 1 \)

\[
\alpha^\beta_{d\psi/dz} = \begin{cases} d\psi/dz, & \text{if } |\psi(z)| \in [\alpha, \beta), \\ 0, & \text{if } |\psi(z)| \notin [\alpha, \beta) \end{cases}
\]

(in particular, \( 0^\beta_{d\psi/dz} = d\psi/dz \), \( \beta^{d\psi/dz}_{1} = d\psi/dz - \alpha^{d\psi/dz}_{1} \)). Then

\[
\int_{0}^{d} |d\psi/dz| dz < c_2\sqrt{a}, \quad \int_{0}^{d} |\alpha^{d\psi/dz}_{1}| f(|\psi|) dz < c_3,
\]

where constants \( c_2 \) and \( c_3 \) can be expressed via \( c_1 \) and \( d \).

This lemma and the transformation of integrals above imply Proposition \( \Box \)

2.3 Proof of Lemma \( \Box \)

1° Estimate \( \int_{0}^{d} \alpha_{d\psi/dz}^{a} dz \). To this end we will use in a modified form a construction from \( [1] \). Let \( l > 0 \). Denote \( M \) the set of points in \([0, d]\), such that \( |d\psi/dz| \leq l \) in \( M \). Denote \( \Lambda = [0, d] \setminus M \). Introduce \( h = 0.5c_1^{-1}l \) and construct sets \( M_h \) and \( \Lambda_h \) as follows. Let \( 0 < h < \ldots <qh < d \leq (q + 1)h \).
Then $M_h$ is the union of those intervals $[(r - 1)h, rh], (r = 1, ..., q), [qh, d]$ that belong to $M$. Denote $\Lambda_h = [0, d] \setminus M_h$. Clearly,

$$\int_{M_h} |\tilde{d}\psi/\tilde{dz}| dz \leq dl.$$

**Lemma 2.2** $|d\psi/dz| > 0.5l^{-1}$ at points of $\Lambda_h$.

**Proof.** Let $z' \in \Lambda_h$. Then there exists $r$ such that $z' \in [(r - 1)h, rh]$ and not all points of this interval belong to $M$ (in the case if $r = q + 1$, not all points of the interval $[qh, d]$ belong to $M$). Then there exists $z'' \in \Lambda$ such that $|z' - z''| < h$. Then

$$|\left(\frac{d\psi}{dz}\right)_{z=z'}| > |\left(\frac{d\psi}{dz}\right)_{z=z''}|- c_1 h > l - c_1 0.5c_1^{-1}l = 0.5l$$

as required.

The set $\Lambda_h$ can be represented as a union of intervals $[\alpha_r, \beta_r], 1 \leq r \leq r_1 \leq q + 1 < dh^{-1} + 1$. On $[\alpha_r, \beta_r]$ the function $\psi$ is monotonous (because $|d\psi/dz| > 0.5l$). Therefore

\[
\int_{\alpha_r}^{\beta_r} |\tilde{d}\psi/\tilde{dz}| dz = |\int_{\alpha_r}^{\beta_r} (d\psi/dz) dz| \leq 2a, \\
\int_{\Lambda_h} |\tilde{d}\psi/\tilde{dz}| dz = \sum_{r=1}^{r_1} \int_{\alpha_r}^{\beta_r} |d\psi/dz| dz \leq 2r_1 a < 2(dh^{-1} + 1)a = 4dc_1l^{-1}a + 2a, \\
\int_{0}^{d} |\tilde{d}\psi/\tilde{dz}| dz = \int_{M_h} |d\psi/dz| dz + \int_{\Lambda_h} |d\psi/dz| dz < d(l + 4c_1l^{-1}a + 2a).
\]

Choose $l = \sqrt{a}$. Then we get the required estimate

$$\int_{0}^{d} |\tilde{d}\psi/\tilde{dz}| dz < d(l + 4c_1)\sqrt{a} + 2a < c_2 \sqrt{a}.$$
2°. Estimate $\Gamma = \int_0^d |\psi' d\psi/dz| f(|\psi|) dz$. Consider a partition $\{\kappa_r\}$ of the interval $[a, 1]$: 

$$a = \kappa_1 < \kappa_2 < \ldots < \kappa_{k+1} = 1.$$ 

Denote 

$$(\Delta \psi)_\nu = \int_0^d |\kappa_\nu d\psi/dz| |dz|$$ 

Then 

$$\Gamma = \sum_{\nu=1}^k \int_0^d |\kappa_\nu d\psi/dz| f(|\psi|) dz \leq \sum_{\nu=1}^k f(\kappa_\nu)(\Delta \psi)_\nu,$$ 

$$\int_0^d |\kappa_1 d\psi/dz| |dz| = \sum_{\nu=1}^r (\Delta \psi)_\nu.$$ 

Result of n. 1° implies that 

$$\int_0^d |\kappa_1 d\psi/dz| |dz| \leq \int_0^d |\kappa_{r+1} d\psi/dz| |dz| < c_2 \sqrt{\kappa_{r+1}}.$$ 

Therefore $\sum_{\nu=1}^r (\Delta \psi)_\nu < c_2 \sqrt{\kappa_{r+1}}$.

Denote $y = (y_1, \ldots, y_k)$, $Y(y) = \sum_{\nu=1}^k f(\kappa_\nu)y_\nu$, 

$$\mathcal{Y} = \{y : \sum_{\nu=1}^r y_\nu \leq c_2 \sqrt{\kappa_{r+1}}, y_\nu \geq 0 (r = 1, \ldots, k)\}.$$ 

**Lemma 2.3** $\sup_{y \in \mathcal{Y}} Y(y) = Y(y^0)$, where $y^0$ such that $\sum_{\nu=1}^r y^0_\nu = c_2 \sqrt{\kappa_{r+1}}$, $r = 1, \ldots, k$ (or, which is the same, $y^0_1 = c_2 \sqrt{\kappa_2}, y^0_\nu = c_2(\sqrt{\kappa_{r+1}} - \sqrt{\kappa_\nu}), \nu = 2, \ldots, k$).
Lemma 2.4 Denote $\theta$ the diameter of partition $\{\kappa_r\}$. Then $\lim_{\theta \to 0} Y(y^0) < c_3$.

Because $\Gamma \leq \sup_{y \in \mathcal{Y}} Y(y) = Y(y^0)$ and $\Gamma$ does not depend on a choice of partition $\{\kappa_r\}$, in the limit as $\theta \to 0$ we get $\Gamma < c_3$, as required.

3 Proofs of Lemmas 2.3 and 2.4

Proof of Lemma 2.3. Denote $\chi = c_2 \sqrt{\kappa_{r+1}}$. The considered supremum is attained at some point $y^0 \in \mathcal{Y}$. Suppose that $y^0_1 < \chi_1$. There exists an index $\nu_1$ ($1 < \nu_1 \leq k$) such that $y^0_{\nu_1} \neq 0$. Let $\nu_1 = 2$ for definiteness. Denote $\delta = \min\{\chi_1 - y^0_1, y^0_2\} > 0$, $y^0 = (y^0_1 + \delta, y^0_2 - \delta, y^0_3, \ldots, y^0_k)$. Clearly $y^0 \in \mathcal{Y}$.

Further,

$$Y(y^0) = \delta (f(\kappa_1) - f(\kappa_2)) + Y(y^0) > Y(y^0),$$

which contradicts to the definition of $y^0$. Therefore

$$y^0_1 = \chi_1 = c_2 \sqrt{\kappa_2} \quad \text{and} \quad \sum_{\nu=2}^k y^0_{\nu} \leq \chi_r - \chi_1 \quad (r = 2, \ldots, k)$$

Repeating this reasoning we find that

$$y^0_{\nu} = \chi_\nu - \chi_{\nu - 1} = c_2 (\sqrt{\kappa_{\nu + 1}} - \sqrt{\kappa_\nu}), \quad \nu = 2, \ldots, k.$$

Proof of Lemma 2.4.

$$y^0_1 = c_2 \sqrt{\kappa_2}, \quad y^0_{\nu} = c_2 (\sqrt{\kappa_{\nu + 1}} - \sqrt{\kappa_\nu}) = \frac{c_2 (\kappa_{\nu + 1} - \kappa_\nu)}{2 \sqrt{\kappa_\nu}},$$

where $\kappa_\nu < \kappa^*_\nu < \kappa_{\nu + 1}$ ($\nu = 2, \ldots, k$). Therefore

$$Y(y^0) = \sum_{\nu=1}^k y^0_{\nu} f(\kappa_{\nu}) = c_2 \left[ \sqrt{\kappa_2} f(\kappa_1) + \sum_{\nu=2}^k \frac{(\kappa_{\nu + 1} - \kappa_\nu) f(\kappa_{\nu})}{2 \sqrt{\kappa_\nu}} \right].$$

Notice that $\kappa_1 = a$, $\kappa_{k+1} = 1$. In the limit as $\theta \to 0$ we get

$$\lim_{\theta \to 0} Y(y^0) = c_2 \left[ \sqrt{a} f(a) + \frac{1}{2} \int_a^1 \frac{f(\kappa)}{\sqrt{\kappa}} d\kappa \right] < c_2 \left[ \sqrt{a} f(a) + \frac{1}{2} \int_0^1 \frac{f(\kappa)}{\sqrt{\kappa}} d\kappa \right] < c_3.$$
References

[1] Anosov D. V. Averaging in systems of ordinary differential equations with rapidly oscillating solutions. Izv. Akad. Nauk SSSR, Ser. Mat. 24, 721-742 (1960) (Russian).

[2] Neishtadt A. I. On some resonance problems in non-linear systems. Kandidat diss., Moscow Univ., Moscow (1975), 150 pp. (Russian)