VARIATIONAL PRINCIPLES FOR T-ENTROPY,
THE SPECTRAL POTENTIAL OF TRANSFER OPERATOR,
AND ENTROPY STATISTIC THEOREM
ARE EQUIVALENT

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For any transfer operator we establish the equivalence of variational principles for
$t$-entropy, the spectral potential and entropy statistic theorem and give new proofs
for all these statements.

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For any transfer operator we will establish the equivalence of the next statements:
‘variational principle for $t$-entropy’ $\iff$ ‘variational principle for the spectral potential’ $\iff$
‘entropy statistic theorem’.

1 Variational principles for $t$-entropy and
the spectral potential of transfer operators

Let us start with recalling the main objects and notions in question.

1 We will call $C$ a base algebra if it is a self-adjoint part of a certain commutative $C^*$-
algebra with an identity $1$. This means that there exists a commutative $C^*$-algebra $B$
with an identity $1$ such that

$$C = \{ b \in B \mid b^* = b \}.$$

As is known the Gelfand transform establishes an isomorphism between $C$ and the
algebra $C(X)$ of continuous real-valued functions on a Hausdorff compact space $X$, which
is the maximal ideal space of the algebra $C$. Throughout the article we identify $C$ with
$C(X)$ mentioned above.

The next known result (see, for example, [3]) establishes a correspondence between
endomorphisms of base algebras and dynamical systems.
Theorem 2 If $\delta : C \to C$ is an endomorphism of a base algebra $C$ then there exists an open-closed subset $Y \subset X$ and a continuous mapping $\alpha : Y \to X$ (both $Y$ and $\alpha$ are uniquely defined) such that

$$[\delta f](x) = \chi_Y(x)f(\alpha(x)), \quad f \in C, \quad x \in X,$$

where $\chi_Y$ is the index function of $Y$. In particular, if $\delta(1) = 1$ then $Y = X$ and

$$[\delta f](x) = f(\alpha(x)). \quad (1)$$

Remark 3 It is clear that any endomorphism of a $C^*$-algebra $B$ is completely defined by its restriction onto the self-adjoint part $C$ of $B$ and on the other hand any endomorphism of $C$ extends uniquely up to an endomorphism of $B$. Therefore the correspondence between endomorphisms and dynamical systems presented in the theorem can be equally described in terms of endomorphisms of $B$.

4 In what follows the pair $(C, \delta)$, where $C$ is a base algebra and $\delta$ is its certain endomorphism such that $\delta(1) = 1$, will be called a $C^*$-dynamical system, and the pair $(X, \alpha)$ described in Theorem 2 will be called the dynamical system corresponding to $(C, \delta)$. The algebra $C$ will be also called the base algebra of the dynamical system $(X, \alpha)$.

Throughout the paper notation $C$, $\delta$, $X$, $\alpha$ will denote the objects introduced above and we will use either of them (say $\delta$ or $\alpha$) for convenience reasons (once $\alpha$ is chosen then $\delta$ is defined uniquely by (1) and vice versa).

Definition 5 Let $(C, \delta)$ be a $C^*$-dynamical system. A linear operator $A : C \to C$ is called a transfer operator, if it possesses the following two properties

a) $A$ is positive (it maps nonnegative elements of $C$ into nonnegative ones);

b) it satisfies the homological identity

$$A((\delta f)g) = fAg \quad \text{for all} \quad f, g \in C.$$

If in addition this operator maps 1 into 1 we will call it a conditional expectation operator.

Remark 6 Any transfer operator $A : C \to C$ can be naturally extended up to a transfer operator on $B = C + iC$ by means of the formula

$$A(f + ig) = Af + iAg.$$

On the other hand, given any transfer operator on $B$, its restriction to $C$ (which is well defined in view of property a) of Definition 4) is also a transfer operator. Therefore transfer operators can be equivalently introduced as by means of $C^*$-algebra $B$ so also by means of its self-adjoint part — the base algebra $C$. We prefer to exploit the base algebra since in what follows we use the Legendre transform which is an essentially real-valued object.

7 Let $(C, \delta)$ be a $C^*$-dynamical system and $(X, \alpha)$ be the corresponding dynamical system. We denote by $M(C)$ the set of all positive normalized linear functionals on $C$ (which take nonnegative values on nonnegative elements and are equal to 1 on the unit). Since we are identifying $C$ and $C(X)$, the Riesz theorem implies that the set $M(C)$ can be identified
with the set of all regular Borel probability measures on $X$ and the identification is established by means of the formula

$$\mu[\varphi] = \int_X \varphi \, d\mu, \quad \varphi \in \mathcal{C} = C(X),$$

where $\mu$ in the right-hand part is a measure on $X$ assigned to the functional $\mu \in M(\mathcal{C})$ in the left-hand part. That is why with a slight abuse of language we will call elements of $M(\mathcal{C})$ measures.

A measure $\mu \in M(\mathcal{C})$ is called $\delta$-invariant if for each $f \in \mathcal{C}$ we have $\mu[f] = \mu[\delta f]$. The set of all $\delta$-invariant measures from $M(\mathcal{C})$ will be denoted by $M_\delta(\mathcal{C})$. Clearly, in terms of the dynamical system $(X, \alpha)$ the condition $\mu[f] = \mu[\delta f]$ is equivalent to the condition $\mu[f] = \mu[f \circ \alpha]$, $f \in C(X)$. Therefore $M_\delta(\mathcal{C})$ can be identified with the set of all $\alpha$-invariant regular Borel probability measures on $X$.

Let $A : \mathcal{C} \to \mathcal{C}$ be a fixed transfer operator for a $C^*$-dynamical system $(\mathcal{C}, \delta)$. In what follows we consider the family of operators $A_\varphi : \mathcal{C} \to \mathcal{C}$, defined by means of the formula $A_\varphi f := A(e^{\varphi} f)$. Evidently, all the operators in this family are transfer operators for $(\mathcal{C}, \delta)$ as well.

Let $\lambda(\varphi)$ be the logarithm of the spectral radius of $A_\varphi$:

$$\lambda(\varphi) = \lim_{n \to \infty} \frac{1}{n} \ln \|A^n_\varphi 1\|, \quad \varphi \in \mathcal{C}. \quad (2)$$

The functional $\lambda(\varphi)$ will be called the spectral potential of transfer operator $A$.

By a partition of unity in the algebra $\mathcal{C}$ we mean any finite set $D = \{g_1, \ldots, g_k\}$ consisting of nonnegative elements $g_i \in \mathcal{C}$ satisfying the identity $g_1 + \cdots + g_k = 1$.

Definition of $t$-entropy $\tau(\mu)$ is given in the following way:

$$\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{\tau_n(\mu)}{n}, \quad \tau_n(\mu) := \inf_D \tau_n(\mu, D), \quad (3)$$

$$\tau_n(\mu, D) := \sup_{m \in M(\mathcal{C})} \sum_{g \in D} \mu[g] \ln \frac{\mu[A^n g]}{\mu[g]}, \quad \mu \in M_\delta(\mathcal{C}). \quad (4)$$

The infimum in (3) is taken over all the partitions of unity $D$ in the algebra $\mathcal{C}$.

If we have $\mu[g] = 0$ for a certain $g \in D$, then we set the corresponding summand in (4) to be zero independently of the value $m[A^n g]$. And if there exists an element $g \in D$ such that $A^n g = 0$ and simultaneously $\mu[g] > 0$, then we set $\tau(\mu) = -\infty$.

**Theorem 9 (variational principle for $t$-entropy)** Let $(\mathcal{C}, \delta)$ be a $C^*$-dynamical system, $A : \mathcal{C} \to \mathcal{C}$ be a certain transfer operator for $(\mathcal{C}, \delta)$, and $A_\varphi = A(e^{\varphi} \cdot)$ for all $\varphi \in \mathcal{C}$. Then the following equality takes place:

$$\tau(\mu) = \inf_{\varphi \in \mathcal{C}} (\lambda(\varphi) - \mu[\varphi]), \quad \mu \in M_\delta(\mathcal{C}) \quad (5)$$

(where $M_\delta(\mathcal{C})$ is the set of all positive normalized $\delta$-invariant linear functionals on $\mathcal{C}$).
Theorem 10  For each linear functional $\mu$ on $\mathcal{C}$ that does not belong to $M_\delta(\mathcal{C})$ the following equality takes place:

$$\inf_{\varphi \in \mathcal{C}} (\lambda(\varphi) - \mu[\varphi]) = -\infty. \quad (6)$$

These two theorems show that it is natural to put

$$\tau(\mu) = -\infty \quad \text{for all } \mu \in \mathcal{C}^* \setminus M_\delta(\mathcal{C}). \quad (7)$$

Then formulae (5) and (6) are united into one:

$$\tau(\mu) = \inf_{\varphi \in \mathcal{C}} (\lambda(\varphi) - \mu[\varphi]), \quad \mu \in \mathcal{C}^*. \quad (8)$$

The proof of Theorems 9 and 10 will be implemented in a number of steps.

Lemma 11  For any $\varphi \in \mathcal{C}$ and $\mu \in M_\delta(\mathcal{C})$ one has

$$\lambda(\varphi) \geq \mu[\varphi] + \tau(\mu). \quad (9)$$

Proof. Let us show that for any $\varphi \in \mathcal{C}$, $\mu \in M_\delta(\mathcal{C})$, $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a partition of unity $D$ such that

$$\varepsilon + \ln \|A^n_\varphi\| = \varepsilon + \ln \|A^n_\varphi 1\| \geq \varepsilon + \ln m[A^n_\varphi] = \varepsilon + \ln \sum_{g \in D} m[A^n_{\varphi, g}]. \quad (10)$$

Once this is done then by arbitrariness of $\varepsilon > 0$ inequality (9) follows from (10) by taking infimum with respect to $D$, $n$.

So it is enough to verify (10).

Let us introduce the notation

$$S_n\varphi := \varphi + \delta \varphi + \cdots + \delta^{n-1} \varphi, \quad \varphi \in \mathcal{C}.$$ 

Applying $n$ times the homological identity to the operator $A^n_\varphi = (Ae^\varphi)^n$, we obtain

$$A^n_\varphi f = A(e^\varphi A(e^\varphi \cdots A(e^\varphi f)) \cdots) = A^n(e^{S_n\varphi} f). \quad (11)$$

For arbitrary numbers $n \in \mathbb{N}$ and $\varepsilon > 0$ we choose a partition of unity $D$ such that on the support of each function $g \in D$ the oscillation of function $S_n\varphi$ does not exceed $\varepsilon$. This $D$ is in fact the desired partition.

Set

$$S_n\varphi(g) := \sup\{S_n\varphi(x) \mid g(x) \neq 0\},$$

where we identify $g$ with the corresponding function in $C(X)$ (cf. 1 and 4).

Equality (11) and concavity of the logarithm function imply the following inequalities for all functionals $m \in M(\mathcal{C})$ and $\mu \in M_\delta(\mathcal{C})$:

$$\varepsilon + \ln \|A^n_\varphi\| = \varepsilon + \ln \|A^n_\varphi 1\| \geq \varepsilon + \ln m[A^n_\varphi] = \varepsilon + \ln \sum_{g \in D} m[A^n_{\varphi, g}].$$

$$\geq \ln \sum_{g \in D} e^{S_n\varphi(g)} m[A^n_g] \geq \ln \sum_{\mu[g] \neq 0} \mu[g] e^{S_n\varphi(g)} m[A^n_g].$$
\[\geq \sum_{\mu[g] \neq 0} \mu[g] \ln \frac{e^{S_n \varphi(g) m[A^n g]}}{\mu[g]} = \sum_{\mu[g] \neq 0} \mu[g] S_n \varphi(g) + \sum_{\mu[g] \neq 0} \mu[g] \ln \frac{m[A^n g]}{\mu[g]}\]

\[\geq \sum_{\mu[g] \neq 0} \mu[g] S_n \varphi + \sum_{\mu[g] \neq 0} \mu[g] \ln \frac{m[A^n g]}{\mu[g]} = \mu[S_n \varphi] + \sum_{g \in D} \mu[g] \ln \frac{m[A^n g]}{\mu[g]} .\]

Passing in these inequalities to the supremum over \(m \in M(\mathcal{C})\) and taking into account (11), one obtains the inequality

\[\varepsilon + \ln \|A^n\| \geq \mu[S_n \varphi] + \tau_n(\mu, D) = n\mu[\varphi] + \tau_n(\mu, D),\]

which implies (10). \(\square\)

To finish the proof of Theorem 9 we need two more lemmas.

In the next lemma the notation \(\lambda(\varphi, A)\) has the same meaning as \(\lambda(\varphi)\) and \(\lambda(n \varphi, A^n)\) denotes the logarithm of spectral radius of the operator \(A^n(e^{n \varphi} \cdot)\).

**Lemma 12** The following inequality takes place:

\[n \lambda(\varphi, A) \leq \lambda(n \varphi, A^n), \quad n \in \mathbb{N} .\] (12)

**Proof.** Note that for any natural \(k\) one has

\[\exp\{S_{nk} \varphi\} = \exp\left\{\sum_{i=0}^{n-1} \sum_{j=0}^{k-1} \delta^{i+nj} (\varphi)\right\} = \prod_{i=0}^{n-1} \exp\left\{\sum_{j=0}^{k-1} \delta^{i+nj} (\varphi)\right\} .\] (13)

Let \(c = \|\varphi\|\). Bearing in mind observations (11), (13), and exploiting Hölder inequality in the form

\[\nu[\psi_1 \cdots \psi_n] \leq \prod_{i=1}^{n} \nu[|\psi_i|^n]^{1/n} ,\]

where \(\nu[\cdot] = m[A^{n(k+1)}(\cdot)]\) and \(m \in M(\mathcal{C})\), one obtains

\[e^{-nc} m[A^{n(k+1)} \mathbf{1} ] = e^{-nc} m[A^{n(k+1)} (e^{S_{nk} \varphi} \mathbf{1})] \leq m[A^{n(k+1)} (e^{S_{nk} \varphi} \mathbf{1})] = \nu \left[ \prod_{i=0}^{n-1} \exp\left\{\sum_{j=0}^{k-1} \delta^{i+nj} (\varphi)\right\} \right]^{1/n} \]

\[= \prod_{i=0}^{n-1} m[A^{n-i} A^k A^i \left(\exp\left\{\sum_{j=0}^{k-1} \delta^{i+nj} (n \varphi)\right\}\mathbf{1}\right)]^{1/n} \]

\[= \prod_{i=0}^{n-1} m[A^{n-i} (e^{n \varphi})^k (A^i \mathbf{1})]^{1/n} \leq \prod_{i=0}^{n-1} \left\| A^{n-i} (e^{n \varphi})^k (A^i \mathbf{1}) \right\|^{1/n} \]

\[\leq \|A^n\| \left\| (e^{n \varphi})^k \right\| .\]

Noting that

\[\|A^{n(k+1)}\| = \sup_{m \in M(\mathcal{C})} m[A^{n(k+1)} \mathbf{1}] ,\]

\[\|e^{n \varphi}\|^k \]
we deduce that relations just obtained imply
\[ e^{-nc} \| A_{n}^{(k+1)} \| \leq \| A \|^n \|(A^n e^{nc})^k \|, \]
and therefore
\[ -nc + \ln \| A_{n}^{(k+1)} \| \leq n \ln \| A \| + \ln \|(A^n e^{nc})^k \|. \]
Dividing the latter inequality by \( k \) and turning \( k \to \infty \) one gets (12). □

Let us fix a measure \( \mu \in M_{\delta}(\mathcal{C}) \), natural number \( n \) and partition of unity \( D \) in \( \mathcal{C} \). For these objects there exists a sequence of measures \( m_k \in M(\mathcal{C}) \) on which the supremum in (11) is attained. One may choose a subsequence \( m_{k_i} \) of this sequence such that the following limits do exist simultaneously:
\[ \lim_{i \to \infty} m_{k_i}[A^n(g)] =: C_n(\mu, g, D), \quad g \in D. \] (14)
Then by construction one has
\[ \tau_n(\mu, D) = \sum_{g \in D} \mu[g] \ln \frac{C_n(\mu, g, D)}{\mu[g]}. \] (15)

**Lemma 13** If \( \tau_n(\mu, D) > -\infty \) then
\[ \sup_{m \in M(\mathcal{C}), \mu[g] > 0} \sum_{g \in D, \mu[g] > 0} \mu[g] \frac{m[A^n g]}{C_n(\mu, g, D)} = 1. \] (16)

**Proof.** The finiteness of \( \tau_n(\mu, D) \) and (15) imply that \( C_n(\mu, g, D) > 0 \) whenever \( \mu[g] > 0 \). For each \( m \in M(\mathcal{C}) \) let us consider the function
\[ \eta(t) = \sum_{g \in D, \mu[g] > 0} \mu[g] \ln \frac{(1 - t)C_n(\mu, g, D) + tm[A^n g]}{\mu[g]}, \quad t \in [0, 1]. \]
By definition of the numbers \( C_n(\mu, g, D) \) this function attains its maximal value equal to \( \tau_n(\mu, D) \) at \( t = 0 \). Therefore its derivative at \( t = 0 \)
\[ \frac{d\eta(t)}{dt} \bigg|_{t=0} = \sum_{g \in D, \mu[g] > 0} \mu[g] \frac{m[A^n g]}{C_n(\mu, g, D)} - 1 \]
is nonpositive and so the left hand part in (16) does not exceed its right hand part.

The equality in (16) is attained on the sequence of measures \( m_{k_i} \) from (14). □

Now we can finish the proof of Theorem 9.

Let us fix an arbitrary measure \( \mu \in M_{\delta}(\mathcal{C}) \), natural number \( n \) and partition of unity \( D \) in \( \mathcal{C} \).

Suppose at first that there exists an element \( g \in D \) satisfying the inequality \( \mu[g] > 0 \) and equality \( m[A^n g] = 0 \) for all \( m \in M(\mathcal{C}) \). Then by definition one has \( \tau_n(\mu, D) = -\infty \) and therefore \( \tau(\mu) = -\infty \). Thus in this case equality (15) takes the form
\[ -\infty = \inf_{\varphi \in \mathcal{C}} (\lambda(\varphi) - \mu[\varphi]). \] (17)
Let us verify it.

Consider the family of elements $\varphi_t = tg/n$, where $t \in \mathbb{R}$. Inequalities $0 \leq g \leq 1$ and the Lagrange theorem imply that

$$e^{n\varphi_t} = e^{tg} \leq 1 + e^tg.$$

Therefore for each measure $m \in M(C)$ one has

$$m[A^n(e^{n\varphi_t}1)] \leq m[A^n(1 + e^tg)] = m[A^n1] + e^tm[A^ng] = m[A^n1] \leq \|A^n\|.$$ 

Thus $\|A^n e^{n\varphi_t}\| \leq \|A^n\|$. Applying Lemma [12] we obtain the following estimate

$$n\lambda(\varphi_t) = n\lambda(\varphi_t, A) \leq \lambda(n\varphi_t, A^n) \leq \ln\|A^n e^{n\varphi_t}\| \leq \ln\|A^n\|.$$ 

On the other hand,

$$\mu[\varphi_t] = \mu[tg/n] = t\mu[g/n] \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$ 

And therefore $\lambda(\varphi_t) - \mu[\varphi_t] \rightarrow -\infty$ when $t \rightarrow +\infty$. So equality [17] is verified.

It remains to consider the situation when for each element $g \in D$ satisfying the condition $\mu[g] > 0$ there exists a measure $m_g \in M(C)$ such that $m_g[A^ng] > 0$. Taking the measure $m := |D|^{-1} \sum_g m_g$ one obtains that

$$m[A^ng] > 0 \quad \text{as soon as} \quad \mu[g] > 0.$$ 

Therefore $\tau_n(\mu, D) > -\infty$. Note also that finiteness of $\tau_n(\mu, D)$ along with [15] implies that the condition $\mu[g] > 0$ automatically implies the inequality $C_n(\mu, g, D) > 0$.

Now let us define the family of elements

$$\varphi_\varepsilon := \frac{1}{n} \ln \left\{ \sum_{\mu[g] \geq 0} \frac{\mu[g]}{C_n(\mu, g, D)} g + \sum_{\mu[g] = 0} \varepsilon g \right\}, \quad \varepsilon > 0. \quad (18)$$

For any $m \in M(C)$ one has

$$m[A^n(e^{n\varphi_\varepsilon}1)] = m\left[A^n\left( \sum_{\mu[g] > 0} \frac{\mu[g]}{C_n(\mu, g, D)} g \right) + \sum_{\mu[g] = 0} \varepsilon g \right]$$

$$= \sum_{\mu[g] > 0} \mu[g] \frac{m[A^ng]}{C_n(\mu, g, D)} + \varepsilon m\left[ A^n\left( \sum_{\mu[g] = 0} g \right) \right] \leq 1 + \varepsilon\|A^n\|$$

(where in the final inequality we exploited Lemma [13]).

Therefore,

$$\|A^n e^{n\varphi_\varepsilon}\| \leq 1 + \varepsilon\|A^n\|.$$ 

This along with Lemma [12] implies the estimate

$$n\lambda(\varphi_\varepsilon) \leq \lambda(n\varphi_\varepsilon, A^n) \leq \ln\|A^n e^{n\varphi_\varepsilon}\| \leq \ln(1 + \varepsilon\|A^n\|) \leq \varepsilon\|A^n\|. \quad (19)$$
On the other hand, applying concavity of logarithm and (15) one obtains

$$\mu[n\varphi] = \mu\left[\ln\left\{\sum_{\mu[g] > 0} \frac{\mu[g]}{C_n(\mu, g, D)}g + \sum_{\mu[g] = 0} \varepsilon g\right\}\right] = -\tau_n(\mu, D).$$

(20)

Combining (20) and (19) we get

$$\frac{\tau_n(\mu, D)}{n} \geq -\mu[\varphi] \geq -\mu[\varphi] + \left(\lambda(\varphi) - \frac{\varepsilon\|A^n\|}{n}\right),$$

and therefore

$$\frac{\tau_n(\mu, D)}{n} + \frac{\varepsilon\|A^n\|}{n} \geq \lambda(\varphi) - \mu[\varphi] \geq \inf_{\varphi \in \mathcal{C}} \left(\lambda(\varphi) - \mu[\varphi]\right).$$

This inequality along with arbitrariness of $\varepsilon$, $n$, $D$ and definition (3) of $\tau(\mu)$ implies the inequality

$$\tau(\mu) \geq \inf_{\varphi \in \mathcal{C}} \left(\lambda(\varphi) - \mu[\varphi]\right).$$

Together with inequality (9) this proves (5). $\square$

Note now that Theorem 10 is a straightforward corollary of the next observation.

**Lemma 14** If a linear functional $\mu$ on $\mathcal{C}$ possesses the property

$$\inf_{\varphi \in \mathcal{C}} \left(\lambda(\varphi) - \mu[\varphi]\right) > -\infty,$$

(21)

then $\mu \in M_3(\mathcal{C})$. In particular, this is true for every subgradient of the function $\lambda(\varphi)$.

This lemma can be proven absolutely in the same way as the corresponding result (Lemma 7) in [4]. The proof is based on the following properties of the functional $\lambda(\varphi)$.

**Lemma 15** The spectral potential $\lambda(\varphi)$ possesses the following properties:

1. $\lambda(\varphi) \geq \psi$, then $\lambda(\varphi) \geq \lambda(\psi)$ (monotonicity);
2. $\lambda(\varphi + t) = \lambda(\varphi) + t$ for all $t \in \mathbb{R}$ (additive homogeneity);
3. $|\lambda(\varphi) - \lambda(\psi)| \leq \|\varphi - \psi\|$ (Lipschitz condition);
4. $\lambda((1 - t)\varphi + t\psi) \leq (1 - t)\lambda(\varphi) + t\lambda(\psi)$ for $t \in [0, 1]$ (convexity);
5. $\lambda(\varphi + \delta\psi) = \lambda(\varphi + \psi)$ (strong $\delta$-invariance).

This lemma is proven in [1], [2].

**Remark 16** By Lemma 15 the functional $\lambda(\varphi)$ is convex and continuous. Theorems 9 and 10 in essence state that the functional $-\tau(\mu)$ is the Legendre transform of $\lambda(\varphi)$. This automatically implies that $t$-entropy $\tau(\mu)$ is concave and upper semicontinuous (in the $^*$-weak topology) on the dual space to $\mathcal{C}$. In [1] concavity and upper semicontinuity of $t$-entropy were proven independently and in an essentially more complicated way.
Finally we observe that variational principle for the spectral potential can be easily derived from Theorem 9 and Lemmas 15, 14.

**Theorem 17 (variational principle for the spectral potential)** For each $\varphi \in C$ the following equality takes place:

$$\lambda(\varphi) = \max_{\mu \in M_\delta(C)} \left( \tau(\mu) + \mu[\varphi] \right).$$  \hfill (22)

*Proof.* By Lemma 15 the functional $\lambda(\varphi)$ is convex and continuous. Thus at each point $\varphi_0$ there exists at least one subgradient $\mu$ for $\lambda(\varphi)$. By Lemma 14 this subgradient belongs to $M_\delta(C)$. By Theorem 9 and definition of a subgradient we have

$$\tau(\mu) = \inf_{\varphi \in C} (\lambda(\varphi) - \mu[\varphi]) = \lambda(\varphi_0) - \mu[\varphi_0].$$

Therefore $\lambda(\varphi_0) = \tau(\mu) + \mu[\varphi_0]$. Combining this equality with (9) one obtains (22). \qed

## 2 Entropy Statistic Theorem

Entropy statistic theorem is naturally formulated in terms of the dynamical system $(X, \alpha)$ corresponding to $(C, \delta)$ (see [14]).

Here by $M(X)$ we denote the set of all Borel probability measures on $X$. Let $x$ be an arbitrary point of $X$. The *empirical measures* $\delta_{x,n} \in M(X)$ are defined by the formula

$$\delta_{x,n}(f) := \frac{f(x) + f(\alpha(x)) + \cdots + f(\alpha^{n-1}(x))}{n} = \frac{1}{n} S_n f(x), \quad f \in C(X).$$  \hfill (23)

Evidently, the measure $\delta_{x,n}$ is supported on the trajectory of the point $x$ of length $n$.

We endow the set $M(X)$ with the *-weak topology of the dual space to $C(X)$. Given a measure $\mu \in M(X)$ and its certain neighborhood $O(\mu)$ we define the sequence of sets $X_n(O(\mu))$ as follows:

$$X_n(O(\mu)) := \{ x \in X \mid \delta_{x,n} \in O(\mu) \}. \hfill (24)$$

**Theorem 18 (entropy statistic theorem)** Let $A: C(X) \to C(X)$ be a certain transfer operator for $(X, \alpha)$. Then for any measure $\mu \in M(X)$ and any number $\varepsilon > 0$ there exist a neighborhood $O(\mu)$ in the *-weak topology, a (large enough) number $C(\varepsilon, \mu)$ and a sequence of functions $\chi_n \in C(X)$ majorizing the index functions of the sets $X_n(O(\mu))$ such that for all $n$ the following estimate holds

$$\| A^n \chi_n \| \leq C(\varepsilon, \mu) e^{n (\tau(\mu) + \varepsilon)}. \hfill (25)$$

If $\tau(\mu) = -\infty$ then the number $\tau(\mu) + \varepsilon$ in (25) should be replaced by $-1/\varepsilon$.

*Proof.* By the variational principle for $t$-entropy there exists $\varphi \in C(X)$ such that

$$\lambda(\varphi) - \mu[\varphi] < \tau(\mu) + \varepsilon/3$$

(or $\lambda(\varphi) - \mu[\varphi] < -1/\varepsilon - \varepsilon/3$ in the case when $\tau(\mu) = -\infty$). Let us set

$$O(\mu) := \{ \nu \in M(X) \mid \lambda(\varphi) - \nu[\varphi] < \tau(\mu) + \varepsilon/3 \}.$$
Then
$$X_n(O(\mu)) = \{ x \in X \mid S_n\varphi(x) = n\delta_{x,\varphi} > n(\lambda(\varphi) - \tau(\mu) - \varepsilon/3) \}.$$ Let
$$Y_n := \{ x \in X \mid S_n\varphi(x) = \delta_{x,\varphi} \leq n(\lambda(\varphi) - \tau(\mu) - \varepsilon/2) \}.$$ Then by Uhryson’s Lemma there exist continuous functions $\chi_n$ such that
$$0 \leq \chi_n \leq 1, \quad \chi_n(X_n(O(\mu))) = 1 \quad \text{and} \quad \chi_n(Y_n) = 0.$$ Clearly, $\chi_n$ majorizes the index function of $X_n(O(\mu)).$

Take a constant $C(\varepsilon, \mu)$ so large that
$$\|A^n\varphi\| \leq C(\varepsilon, \mu) e^{n(\lambda(\varphi)+\varepsilon/2)}, \quad n \in \mathbb{N}.$$ Now (25) follows from the calculation
$$C(\varepsilon, \mu)e^{n(\lambda(\varphi)+\varepsilon/2)}\|1\| \geq \|A^n\varphi 1\| \geq \|A^n\chi_n\|
= \|A^n(e^{S_n\varphi}\chi_n)\| \geq e^{n(\lambda(\varphi)-\tau(\mu)-\varepsilon/2)}\|A^n\chi_n\|. \quad \Box$$

To summarize the material presented we recall that in [1] the next chain of statements for transfer operators has been proven: ‘entropy statistic theorem’ $\Rightarrow$ ‘variational principle for the spectral potential’ $\Rightarrow$ ‘variational principle for $t$-entropy’, where each step is rather nontrivial. In this article we obtained the inverted chain: ‘variational principle for $t$-entropy’ $\Rightarrow$ ‘variational principle for the spectral potential’ $\Rightarrow$ ‘entropy statistic theorem’. Thus we established equivalence: ‘variational principle for $t$-entropy’ $\Leftrightarrow$ ‘variational principle for the spectral potential’ $\Leftrightarrow$ ‘entropy statistic theorem’.

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