Complex Manifolds
Research Article

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Regularization of closed positive currents and intersection theory

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Abstract: We prove the existence of a closed regularization of the integration current associated to an effective analytic cycle, with a bounded negative part. By means of the King formula, we are reduced to regularize a closed differential form with $L^1_{\text{loc}}$ coefficients, which by extension has a test value on any positive current with the same support as the cycle. As a consequence, the restriction of a closed positive current to a closed analytic submanifold is well defined as a closed positive current. Lastly, given a closed smooth differential $(q', q')$-form on a closed analytic submanifold, we prove the existence of a closed $(q', q')$-current having a restriction equal to that differential form. After blowing up we deal with the case of a hypersurface and then the extension current is obtained as a solution of a linear differential equation of order 1.

Keywords: Chern class, Green operator, MacPherson graph construction, Modification, Positive current, Residue current

MSC: 14C17, 32C30, 32J25

1 Introduction

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ and let $T$ be a closed positive current in $X$ of bidegree $(q, q)$. We prove (see Proposition 9.2) the following result.

Proposition. There are a constant $C \geq 0$ and a sequence $T_l$ of $C^\infty$ closed differential $(q, q)$-forms in $X$, weakly converging in $X$ to $T$ and satisfying $T_l \geq -C \omega^q$ in $X$ for any $l$.

This theorem is a version in any codimension $q$ of Demailly’s regularization theorem (see [11]) known only for $q = 1$. Recall the statement of Demailly’s theorem in its most precise form.

Let $u$ be a $C^\infty$ positive differential $(1, 1)$-form on $X$ such that $\frac{1}{2\pi} \Theta_T X + u \otimes \text{id}_{TX} \geq 0$ in the sense of Griffiths in $X$, where $\Theta_T X \in C^\infty_1(X, T^*X \otimes TX)$ is the Chern curvature form. For $T$ of bidegree $(1, 1)$, we can write $T = \lim T_l$ weakly in $X$ with $(T_l)$ a sequence of $C^\infty$ closed differential $(1, 1)$-forms in $X$ such that $T_l \geq -\lambda_l u$ where $(\lambda_l)$ is a decreasing sequence of continuous functions in $X$ satisfying $\lambda_l(x) \to v(T, x)$ for every $x \in X$, with $v(T, x)$ the Lelong number of $T$ at $x$. In particular, $\lambda_l u$ converges to 0 weakly in $X$.

Recall also the regularization theorem of Dinh-Sibony (see [13]) which applies to any $q$ but only claims the existence of a closed regularization with a negative part that is bounded in the $L^1_{\text{loc}}$ sense.

This theorem asserts precisely the existence of a constant $C$ dependent of $\omega$ such that all $T$ in $X$ can be written $T = \lim (T_l^+ - T_l^-)$ weakly in $X$, where $T_l^+$, $T_l^-$ are some closed positive differential $(q, q)$-forms of class $C^\infty$.

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on $X$ satisfying $\int_X \| T^+_T \| \leq C \int_X \| T \|$ and $\int_X \| T^-_T \| \leq C \int_X \| T \|$. Here, we denote by $\| \gamma \|$ the mass measure of $T$. A canonical decomposition $T = T^+ - T^-$ with $T^+, T^-$ closed positive is then obtained.

We prove by a direct method (see Propositions 2.2 and 5.1) the following result, similar to that of Dinh-Sibony.

**Proposition.** There is a sequence $(T_l)$ of $C^\infty$ differential $(q, q)$-forms closed in $X$ weakly converging in $X$ to $T$ and such that $T_l \geq -u_l$ with $(u_l)$ a sequence of $C^\infty$ differential $(q, q)$-forms in $X$ converging to $0$ for the $L^1_{\text{loc}}$ topology in $X$.

The existence of the sequence $T_l$ in the first above Proposition is obtained here by a biduality argument and $T_l$ has no explicit construction.

On the other hand we give an explicit construction of a closed regularization in the sense of Dinh-Sibony of the integration current associated with an effective algebraic cycle $\sum_j m_j Z_j$, using a Green form of this cycle (see [22, 24]). The Green form is obtained from a locally free projective resolution of the ideal sheaves, following Bismut-Bost-Gillet-Soulé (see [4, 6]), but here without any assumption of smoothness and without any hypothesis of compatibility between the Hermitian metrics.

Specifically we assume $X$ projective and we consider $Z$ an analytic subset of codimension $q$ of $X$ which is the locus of zeros of a holomorphic section $s$ of a Hermitian holomorphic vector bundle $E$ above $X$. To calculate a Green form of the codimension $q$ cycle $\sum_j m_j Z_j$ associated to $s$, we denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaves generated by the components of $s$ and we suppose given a projective resolution

$$0 \to \mathcal{O}(F_n) \xrightarrow{g_n} \mathcal{O}(F_{n-1}) \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_2} \mathcal{O}(F_1) \xrightarrow{g_1} \mathcal{O}(E^*) \xrightarrow{g_0} \mathcal{I} \xrightarrow{0} 0$$

with holomorphic vector bundles $F_i$ defined in $X$, equipped with Hermitian metrics $h_{0,i}$ of class $C^\infty$ in $X$.

We get explicitly a differential form $\Gamma$ with coefficients of class $C^\infty$ in $X \sim Z$ and $L^1_{\text{loc}}$ in $X$ such that in the sense of currents on $X$, we have

$$\sum_j m_j [Z_j] - \ddc \Gamma = c_q(\Theta_E) + (-1)^q - 1 \times \text{component of bidegree (q, q)} \cdot \prod_{1 \leq i \leq n} c(\Theta_{F_i})(-1)\cdot q - 1, \quad (1)$$

with $c_q(\Theta_E)$ the $q$th Chern form of the curvature form $\Theta_E$ of $(E, h)$ and with $c(\Theta_{F_i}) = \sum_k c_k(\Theta_{F_i})$ the total Chern form of $\Theta_{F_i}$, for Hermitian metrics $h$ and $h_i$ naturally derived from that of $E$ and $h_{0,i}$.

We proceed here directly from the formula of King

$$\sum_j m_j [Z_j] = (\ddc \log \| s \|)^q - \| \mathcal{I} \|_{\sim \mathcal{Z}} (\ddc \log \| s \|)^q.$$

Applying the Bott-Chern calculations of transgression forms for the Chern classes involved in the exact sequence $0 \to C_Z \to E \to E / C_Z \to 0$ in $X \sim Z$, we first make explicit a differential form $\psi_1$ with coefficients of class $C^\infty$ in $X \sim Z$ and $L^1_{\text{loc}}$ in $X$ such that in $X$ in the sense of currents, we have

$$\sum_j m_j [Z_j] = c_q(\Theta_E) - c_q(\Theta_{E / C_Z}) + \ddc \psi_1.$$

A generalization of the Poincaré-Lelong formula was also obtained by Andersson (see [1, 2]).

To express $c_q(\Theta_{E / C_Z})$ we again argue on $X \sim Z$ by breaking the resolution in short exact sequences of holomorphic vector bundles. The local integrability of $\Gamma$ is obtained as a result of the construction of MacPherson of the graph (see [3]), which gives the existence on the blow up of $X$ along $Z$ of an exact complex of holomorphic vector bundles $G_i$, extending the one given on $X \sim Z$.

In conclusion and as an application, we obtain (see Proposition 8.3) the following result.

**Proposition.** There is an explicit sequence of differential forms $\tilde{W}_l$ of class $C^\infty$, closed in $X$, converging weakly in $X$ to $\sum_j m_j [Z_j]$ and satisfying $\tilde{W}_l \geq -\tilde{U}_l$ where $\tilde{U}_l$ is of class $C^\infty$ in $X$ and converges to $0$ for the $L^1_{\text{loc}}$ topology in $X$.

When $Z$ is smooth, it is further shown that every differential $(q', q')$-form of class $C^\infty$ that is closed in $Z$ extends as a closed $(q', q')$-current in $X$. This allows for any manifold $X$ to extend formula (1) by setting

$$[Z] = \theta_q + \ddc \psi_1 \quad (2)$$
with $\theta_q$ a closed $(q, q)$-current satisfying $\{\theta_q \mid Z\} = c_q(N_XZ)$ and with $\psi_1$ a differential $(q-1, q-1)$-form explicitly calculated (see Proposition 7.1).

2 Closed regularization with negative part converging weakly towards 0

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ and let $T$ be a closed positive current in $X$ of bidegree $(q, q)$. We give here a closed regularization of $T$ obtained using the Green operator of $(X, \omega)$ (see [5, 16, 23]).

**Proposition 2.1.** There is a linear operator

$$N: \{\text{current of bidegree } (q, q) \text{ in } X\} \rightarrow \{\text{current of bidegree } (q, q) \text{ in } X\},$$

that is continuous for the weak topology and such that $\theta - N(\theta)$ is closed for every current $\theta$ of bidegree $(q, q)$ in $X$ and $N(\theta) = 0$ if $\theta$ is closed. Also, if $\theta$ is of class $C^\infty$ in $X$ then $N(\theta)$ is also of class $C^\infty$.

Proof. We can write $\theta = h + \overline{\partial}A(\overline{\theta}) + A(\overline{\theta})$ with harmonic $h$,

$$A = \overline{\partial}^*G = G\overline{\partial}^*$$

where $G$ is the Green operator. With $\Delta$ the $\overline{\partial}$-Laplacian associated with $\omega$, then $I - \Delta G$ is the orthogonal projection on the space of harmonic differential $(q, q)$-forms on $X$. Moreover $\partial G = G\partial$ and $\overline{\partial}G = G\overline{\partial}$. See [16, 23] for the construction of $G$ and for the continuity of $G$ for the weak topology of currents and the $L^1_{\text{loc}}$ topology.

We set $\alpha = A(\theta)$, then we can write $\alpha = h' + \partial\partial^*A' + A'(\partial\partial^*A)$ with harmonic $h'$,

$$A' = \partial^*G = G\partial^*.$$

Since $\overline{\partial}$ commutes with $\partial^*$, then $\overline{\partial}$ commutes with $A'$ and we have $\theta = h + \overline{\partial}\partial^*A' + N(\theta)$ with $N(\theta) = A'(\overline{\partial}\partial\alpha) + A(\overline{\partial}\theta)$. Then $\theta - N(\theta) = h + \overline{\partial}\partial^*A(A(\theta))$ is closed in $X$.

More we have $\partial\partial^*A = \partial\partial^*A = \partial\partial - \partial A(\overline{\partial}\theta) = \partial\partial - A(\overline{\partial}\overline{\partial}\theta)$ since $\partial$ commutes with $A$. Thus $N(\theta) = A'(\partial\partial^*A) - A'(\partial\partial^*A)$ satisfies $N(\theta) = 0$ if $\theta$ is closed. \hfill $\square$

**Proposition 2.2.** There is a sequence $(T_1)$ of closed $C^\infty$ differential $(q, q)$-forms on $X$ that weakly converge to $T$ in $X$ and satisfy $T_1 \geq -u_1$, where $(u_1)$ is a sequence of $C^\infty$ differential $(q, q)$-forms on $X$ that weakly converge to 0.

Proof. $(U_\alpha)$ is a finite covering of $X$ with open sets of coordinate maps and $(\lambda_\alpha)$ is a $C^\infty$ partition of the unit subordinate to $(U_\alpha)$. With $\chi_\varepsilon \geq 0$ of class $C^\infty$ approximating the Dirac mass at 0 in $\mathbb{C}^n$, we set

$$\theta_\varepsilon = \sum_\alpha \lambda_\alpha (T_{U_\alpha} \ast \chi_\varepsilon)$$

which is a positive differential form of class $C^\infty$ in $X$ weakly converging in $X$ to $T$. So $\theta_\varepsilon - N(\theta_\varepsilon)$ is of class $C^\infty$ closed in $X$, weakly converges in $X$ to $T - N(T) = T$ and is $\geq -N(\theta_\varepsilon)$ with $N(\theta_\varepsilon)$ weakly converging in $X$ to $N(T) = 0$.

Note that $N(\theta_\varepsilon)$ is smooth but not necessarily positive nor closed. \hfill $\square$

3 Closed regularization constructed from the formula of King

We will give a closed regularization of the current of integration built using an approximation formula of [18].

Let $Z$ be an analytical subset of codimension $q$ of $X$, let $I \subset O_X$ be coherent ideal subsheaves such that $\text{supp} (O_X/I) = Z$, let $(Z_j)$ be the family of irreducible analytic components of $Z$ of codimension exactly $q$ and
let $m_j \in \mathbb{N}^*$ be the generic multiplicity of $I$ along $Z_j$. So
\[
\sum_j m_j[Z_j] = \lim_{\epsilon \to 0^+} \frac{(q + 1)\epsilon}{2^q(\rho + \epsilon)} (\ddc \log(\rho + \epsilon))^q
\]
with a function $\rho \in C^\infty$ in $X$ always $\geq 0$ such that $\rho^{-1}(0) = Z$ built using a method of [9] to be reminded now.

Let $(U_\alpha)$ be an open finite covering of $X$ and let $f_\alpha : U_\alpha \to \mathbb{C}^{N_\alpha}$ be a holomorphic map satisfying: for every $x \in U_\alpha$, the germs in $x$ of the components of $f_\alpha$ generate the ideal $I_x$ and therefore $f_\alpha^{-1}(0) = Z \cap U_\alpha$. We denote by $H_\alpha(x)$ a positive definite Hermitian $N_\alpha \times N_\alpha$ matrix dependent in the $C^\infty$ manner of $x \in U_\alpha$. We set
\[
\psi_\alpha = f_\alpha^* H_\alpha \text{f}_\alpha
\]
which is a function $C^\infty$ in $U_\alpha$ always $\geq 0$ with $\psi_\alpha^{-1}(0) = Z \cap U_\alpha$. With $(\lambda_\alpha)$ a $C^\infty$ partition of the unit subordinate to $(U_\alpha)$, taking
\[
\rho = \sum_\alpha \lambda_\alpha^2 \psi_\alpha,
\]
there is a constant $C \geq 0$ such that $\ddc \log(\rho + \epsilon) + C \omega \geq 0$ for all $\epsilon > 0$.

**Proposition 3.1.** We have the approximation formula of the integration current
\[
\sum_j m_j[Z_j] = \lim_{\epsilon \to 0^+} \frac{(q + 1)\epsilon}{2^q(\rho + \epsilon)} (\ddc \log(\rho + \epsilon) + C \omega)^q.
\]

**Proof.** Using the binomial theorem. Just look for $k$ with $1 \leq k \leq q$, we have
\[
\frac{\epsilon}{\rho + \epsilon} (\ddc \log(\rho + \epsilon))^q \to 0
\]
weakly in $X$ when $\epsilon \to 0^+$. By the extension theorem of Skoda-El Mir, the Monge-Ampère operator $(\ddc \log \rho)^q - k$ is a differential form with $L^1_{\text{loc}}$ coefficients in $X$. So $\frac{\rho}{\rho + \epsilon}(\ddc \log(\rho + \epsilon))^q - k$ weakly converges in $X$ to $(\ddc \log \rho)^q - k$ when $\epsilon \to 0^+$. \hfill \Box

We set $V_\epsilon = \frac{q + 1}{2^q}(\ddc \log(\rho + \epsilon) + C \omega)^q$ so
\[
\sum_j m_j[Z_j] = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\rho + \epsilon} V_\epsilon
\]
with $V_\epsilon \geq 0$ of class $C^\infty$. We apply the Proposition 2.1 taking $\theta = \frac{\epsilon}{\rho + \epsilon} V_\epsilon$. We set $U_\epsilon = N(\theta)$ and $W_\epsilon = \theta - N(\theta)$, so $W_\epsilon$ is a differential form of class $C^\infty$, closed in $X$. When $\epsilon \to 0^+$, $\theta$ weakly converges in $X$ to $\sum_j m_j[Z_j]$ which is closed and therefore $U_\epsilon$ weakly converges in $X$ to $0$. Then $W_\epsilon$ weakly converges in $X$ to $\sum_j m_j[Z_j]$. More $W_\epsilon \geq -U_\epsilon$.

We have
\[
W_\epsilon = h_\epsilon + \overline{\partial} \partial \overline{\partial} \lambda(A(\frac{\epsilon}{\rho + \epsilon} V_\epsilon))
\]
with harmonic $h_\epsilon$ convergent when $\epsilon \to 0^+$ in the space of differential forms of class $C^\infty$ of bidegree $(q, q)$ in $X$.

As $\frac{\epsilon}{\rho + \epsilon} V_\epsilon$ is positive and bounded by mass, by extracting a sequence, we can assume $\frac{\epsilon}{\rho + \epsilon} V_\epsilon$ weakly converging to a positive current, thus a current with measure coefficients. Because of the orders of singularity in $G$ and $A = \overline{\partial}G$, then $G(\frac{\epsilon}{\rho + \epsilon} V_\epsilon)$ converges for the $L^1_{\text{loc}}$ topology and $A(\frac{\epsilon}{\rho + \epsilon} V_\epsilon)$ too. This implies that $A'(A(\frac{\epsilon}{\rho + \epsilon} V_\epsilon))$ converges for the $L^1_{\text{loc}}$ topology.

Finally
\[
U_\epsilon = N(\frac{\epsilon}{\rho + \epsilon} V_\epsilon) = N(V_\epsilon - \frac{\rho}{\rho + \epsilon} V_\epsilon) = -N(\frac{\rho}{\rho + \epsilon} V_\epsilon)
\]
since $V_\epsilon$ is closed and therefore $N(V_\epsilon) = 0$. But $\frac{\rho}{\rho + \epsilon} V_\epsilon$ converges for the $L^1_{\text{loc}}$ topology since $\rho(\frac{1}{2}\ddc \log \rho)^q = \rho(\sum_j m_j[Z_j] + (\frac{1}{2}\ddc \log \rho)^q_{X \setminus Z})$ and $\rho(1) = 0$.
Moreover $\partial(\frac{\rho}{\rho + \epsilon} V_e) = \{ \frac{\partial \rho}{\rho + \epsilon} - \frac{\rho \partial \rho}{(\rho + \epsilon)^2} \} \wedge V_e$ converges for the $L^1_{\text{loc}}$ topology. In effect $\frac{\partial \rho}{\rho + \epsilon} \wedge V_e$ converges for the $L^1_{\text{loc}}$ topology since

$$\frac{\partial \rho}{\rho + \epsilon} \wedge \left( \frac{1}{2} \partial^c \log \rho \right)^q = \frac{\partial \rho}{\rho + \epsilon} \wedge \left( \sum_j m_j [Z_j] + \left( \frac{1}{2} \partial^c \log \rho \right)^q \right)_{X \setminus Z} = \frac{\partial \rho}{\rho + \epsilon} \wedge \left( \frac{1}{2} \partial^c \log \rho \right)^q_{X \setminus Z}$$

and since $\frac{\partial \rho}{\rho + \epsilon} \wedge \left( \frac{1}{2} \partial^c \log \rho \right)^q_{X \setminus Z} = \partial \log \rho \wedge \left( \frac{1}{2} \partial^c \log \rho \right)^q_{X \setminus Z}$ is with $L^1_{\text{loc}}$ coefficients in $X$, as being the direct image by the blow up $\mu$ of $X$ of center $Z$ of a differential form with $L^1_{\text{loc}}$ coefficients in $\tilde{X}$. Also $\frac{\partial \rho}{\rho + \epsilon} \wedge V_e$ converges for the $L^1_{\text{loc}}$ topology.

In the same manner $\tilde{\partial}(\frac{\rho}{\rho + \epsilon} V_e) = \left( \frac{\partial \rho}{\rho + \epsilon} - \frac{2 \partial \rho \wedge \tilde{\partial} \rho}{(\rho + \epsilon)^2} - \frac{\rho \partial \rho \wedge \tilde{\partial} \rho}{(\rho + \epsilon)^2} \right) \wedge V_e$ converges for the $L^1_{\text{loc}}$ topology, thanks to $\tilde{\partial} \log \rho = \frac{\partial \rho}{\rho} - \partial \log \rho \cap \tilde{\partial} \log \rho$.

Consequently $-U_e = N(\frac{\rho}{\rho + \epsilon} V_e) = A' \tilde{\partial}(\frac{\rho}{\rho + \epsilon} V_e) = A'(\frac{\rho}{\rho + \epsilon} V_e) + A(\tilde{\partial}(\frac{\rho}{\rho + \epsilon} V_e))$ converges for the $L^1_{\text{loc}}$ topology to 0.

Note the expression

$$h_e = \frac{\epsilon}{\rho + \epsilon} V_e - \Delta G \left( \frac{\epsilon}{\rho + \epsilon} V_e \right)$$

for the orthogonal projection of $\theta = \frac{\rho}{\rho + \epsilon} V_e$ on the space of harmonic differential $(q, q)$-forms and note that cohomologically $\sum_j m_j [Z_j] = \lim_{\epsilon \to 0} \{ h_e \}$.

4 Restriction of a closed positive current to a closed complex submanifold

Suppose $Z$ is a closed complex submanifold of codimension $q$ in $X$ and $T$ is a closed positive current in $X$ of bidimension $(p', q')$. To define the restriction $T_{|Z}$, we have to define the intersection $T \wedge [Z]$ under the formula $i_* (T_{|Z}) = T \wedge [Z]$ where $i : Z \to X$ is the canonical injection.

**Proposition 4.1.** With the notation of Proposition 3.1, there is a sequence $\epsilon_j \to 0^+$ such that the positive not necessarily closed currents

$$T \wedge \left( \frac{q+1}{2q} \delta (\log (\rho + \epsilon_j) + C \omega)^q \right)$$

converge towards a closed positive current which is $T \wedge [Z]$.

**Proof.** We use that

$$T \wedge \left( \frac{q+1}{2q} \delta (\log (\rho + \epsilon) + C \omega)^q \right) \leq T \wedge \left( \frac{q+1}{2q} \delta (\log (\rho + \epsilon) + C \omega)^q \right)$$

and the total mass $\int_X T \wedge \left( \frac{q+1}{2q} \delta (\log (\rho + \epsilon) + C \omega)^q \wedge \omega^{p'-q} = \int_X \{ T \} \left( \frac{q+1}{2q} C \omega \right)^{p'}$ is constant.

To see that $T \wedge [Z]$ is closed, we use that

$$\frac{\epsilon}{(\rho + \epsilon)^2} \delta \rho \wedge (\delta (\log (\rho + \epsilon))^q \rangle \to 0$$

when $\epsilon \to 0$ for $0 \leq j \leq q$. In effect $\lim_{\epsilon \to 0} T \wedge \left( \frac{\epsilon}{\rho + \epsilon} \delta (\log (\rho + \epsilon))^q \right)$ has support in $Z$ and is written $i_* R$. Thus $\lim_{\epsilon \to 0} T \wedge \left( \frac{\epsilon}{\rho + \epsilon} \delta \rho \wedge (\delta (\log (\rho + \epsilon))^q \right) = \lim_{\epsilon \to 0} (i_* R \wedge \delta \rho) = 0$ since $\rho | Z = 0$. Also a restriction $T_{|Z}$ can be expressed by means of $\mu^* T$ which is closed when $T$ is closed. 

A priori the limit $T \wedge [Z]$ in the Proposition 4.1 depends on the sequence $\epsilon_j \to 0^+$. 

5 Closed regularization with negative part converging to 0 for the $L^1_{\text{loc}}$ topology

We now use the writing
\[ T = \Theta + \dd^c S \]
with $\Theta$ a $C^\infty$ differential $(q, q)$-form closed in $X$ and $S$ a $(q - 1, q - 1)$-current in $X$. This is obtained from Proposition 2.1 which gives that $T - \overline{\partial}A'(A(T))$ is of class $C^\infty$ since $T$ is closed.

**Proposition 5.1.** There is a sequence $(T_l)$ of $C^\infty$ differential $(q, q)$-forms closed in $X$ weakly converging in $X$ to $T$ and such that $T_l \geq -u_l$ with $(u_l)$ a sequence of $C^\infty$ differential $(q, q)$-forms in $X$ converging to $0$ for the $L^1_{\text{loc}}$ topology in $X$.

**Proof.** Since $T$ is positive thus with measure coefficients, $A'(A(T))$ as $S$ is with $L^1_{\text{loc}}$ coefficients. But $\overline{\partial}A'(A(T)) = A'(A(T)) = A'(T - h)$ since $\overline{\partial}T = 0$. As a consequence $\overline{\partial}S$ is with $L^1_{\text{loc}}$ coefficients and since $S$ is real, $\partial S$ too.

In other words, one can choose $S$ with $L^1_{\text{loc}}$ coefficients, with $\partial S$ and $\overline{\partial}S$ with $L^1_{\text{loc}}$ coefficients. We set
\[
\tilde{T}_\epsilon = \Theta + \dd^c \left( \sum_\alpha \lambda_\alpha \left( (S_{U_\alpha}^*) \ast \chi_\epsilon \right) \right)
\]
which is a differential $(q, q)$-form of class $C^\infty$ closed in $X$ weakly converging in $X$ to $T$ when $\epsilon \to 0^+$. We have
\[
\tilde{T}_\epsilon = \Theta + \sum_\alpha \{ \lambda_\alpha \left( \left( (\dd^c S_{U_\alpha}^*) \ast \chi_\epsilon \right) + \left( \dd^c \lambda_\alpha \right) \wedge \left( \left( (S_{U_\alpha}^*) \ast \chi_\epsilon \right) \wedge \left( \dd^c \lambda_\alpha \right) \right) \right) \}
\]
then using that $\dd^c S = T - \Theta$ we have $\tilde{T}_\epsilon = \sum_\alpha \lambda_\alpha \left( (T_{U_\alpha}^*) \ast \chi_\epsilon \right) - \tilde{u}_\epsilon$ with
\[
-\tilde{u}_\epsilon = \Theta - \sum_\alpha \lambda_\alpha \left( \left( \Theta_{U_\alpha} \ast \chi_\epsilon \right) + \sum_\alpha \left( \dd^c \lambda_\alpha \right) \wedge \left( \left( (S_{U_\alpha}^*) \ast \chi_\epsilon \right) \wedge \left( \dd^c \lambda_\alpha \right) \right) \right).
\]
We have $\tilde{T}_\epsilon \geq -\tilde{u}_\epsilon$ with $\tilde{u}_\epsilon$ which converges for the $L^1_{\text{loc}}$ topology necessarily to $0$.

Note that $\tilde{u}_\epsilon$ is smooth but not necessarily positive nor closed.

Since $\tilde{u}_\epsilon \to 0$ for the $L^1_{\text{loc}}$ topology when $\epsilon \to 0^+$, there are $\epsilon_l \to 0^+$ and $g \geq 0$ and $L^1_{\text{loc}}$ in $X$ such that $\| u_{\epsilon_l} \| \leq g$ for all $l$ (see the theorem of Fischer-Riesz in [12], Proposition 13.11.4 (ii)). The regularization obtained is therefore of the same type of the Dinh-Sibony regularization (see [13]).

6 Closed extension of a closed current defined on a closed complex submanifold

Suppose $Z$ is a closed complex submanifold of codimension $q$ in $X$ and $\theta$ is a $C^\infty$ differential form closed in $Z$ of bidegree $(q', q')$.

Let $\mu : \tilde{X} \to X$ be the blow up of center $Z$ and let $T$ be a closed current of bidegree $(q', q')$ in $X$. We define the restriction of $T$ to $Z$ by
\[
T|_Z = (\mu|_H)_* \{ (\mu^* T)|_H \wedge c_1(\Theta_{O_P(N_{X/Z}(1))})^{q'-1} \}
\]
by denoting by $H \subset \tilde{X}$ the exceptional divisor.

We are looking for $T$ such that
\[
T|_Z = \theta
\]
and it will be so when $(\mu^* T)|_H = \alpha$ with $\alpha = (\mu_1|_H)^* \theta$ which is closed in $H$. We will set $T = \mu_* S$ with $S$ a current closed in $\tilde{X}$ satisfying $S|_H = \alpha$. This equality returns to
\[
S \wedge [H] = j_* \alpha
\]
with $j : H \to \tilde{X}$ the canonical injection.
Proposition 6.1. With \( \sigma \in H^0(\tilde{X}, \mathcal{O}(H)) \) satisfying \( \sigma^{-1}(0) = H \), we can write \( [H] = \gamma D\sigma \wedge D\sigma^* \) with \( \gamma \) a measure in \( \tilde{X} \) with support \( H \) and \( j_*\alpha = \beta \gamma \wedge D\sigma \wedge D\sigma^* \) with \( \beta \) a \( C^\infty \) differential form of bidegree \( (q', q') \) in \( \tilde{X} \).

\[ [H] = C\delta(z_n)dz_n \wedge d\sigma_n. \]

But \( D\sigma = dz_n \otimes \epsilon + z_n D\epsilon \) and \( D\sigma^* = d\epsilon - z_n D\epsilon^* \) and therefore \( dz_n \wedge d\epsilon = (dz_n \otimes \epsilon) \wedge (d\epsilon \otimes \epsilon^*) = (D\epsilon - z_n D\epsilon) \wedge (D\epsilon^* - Dz_n D\epsilon^*) = D\epsilon \wedge D\epsilon^* \) modulo terms containing \( z_n \) or \( \epsilon_n \).

For the other formula, \( \alpha \) is \( C^\infty \) closed in \( H \) and there is \( \beta \) a differential form \( C^\infty \) in \( \tilde{X} \) not necessarily closed satisfying \( \alpha = \beta|_H \). So \( j_*\alpha = j_*\beta|_H = \beta \wedge [H] \).

The aim is therefore of finding \( S \) closed in \( \tilde{X} \) satisfying

\[ j_*\alpha = S \wedge [H] \leftrightarrow \beta \gamma \wedge D\sigma \wedge D\sigma^* = S \gamma \wedge D\sigma \wedge D\sigma^*. \]

But we know that \( d\beta|_H = 0 \leftrightarrow d\beta = \sigma \otimes U + V \otimes \sigma^* + D\sigma \otimes W + T \otimes D\sigma^* \) for some \( U, V, W, T \). We will set

\[ S = \beta + \sigma \otimes u + v \otimes \sigma^* + D\sigma \otimes w + t \otimes D\sigma^* \]

with \( u, v, w, t \) to determine so that \( dS = 0 \). This is equivalent to

\[ -(\sigma \otimes U + V \otimes \sigma^* + D\sigma \otimes W + T \otimes D\sigma^*) = d(\sigma \otimes u + v \otimes \sigma^* + D\sigma \otimes w + t \otimes D\sigma^*) \]

Then we can assume that \( U = V^* \) and \( W = T^* \) and a solution \( S \) will be real if \( u = v \) and \( w = -t \).

The differential equation of \( \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) + (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* = 0. \)

Suppose \( \Theta \) and \( \beta \) real, so \( \sigma \otimes U + V \otimes \sigma^* + D\sigma \otimes W + T \otimes D\sigma^* = U^* \otimes \sigma^* + \sigma \otimes V^* + W^* \otimes D\sigma^* + D\sigma \otimes T^* \) is closed in \( C^\infty \) in \( \tilde{X} \).

The existence of a solution to the equation \( (3) \) therefore causes the necessary condition

\[ \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) + (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* \]

so the \( \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) + (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* \)

Thus \( u, v, w, t \) must satisfy the linear differential equation of order 1

\[ \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) + (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* = 0. \]

(3)

Moreover

\[ (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* = (Du + \Theta w + U) \otimes \sigma^* + (u - Dw + W) \otimes D\sigma^* \]

is the adjoint of \( \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) \) and its differential is the adjoint of \( \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) \) which is

\[ (Du + \Theta w + U) \otimes \sigma^* - (U^* - Dw^*) \otimes D\sigma^* = (DV - \Theta T) \otimes \sigma^* + (DT - V) \otimes D\sigma^*. \]

The existence of a solution to the equation \( (3) \) therefore causes the necessary condition

\[ \sigma \otimes (Du + \Theta w + U) + D\sigma \otimes (u - Dw + W) + (Dv + \Theta t + V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* = 0. \]

(4)

But we know that \( \sigma \otimes U + V \otimes \sigma^* + D\sigma \otimes W + T \otimes D\sigma^* \) is exact thus closed. Thus it is known that

\[ 0 = D\sigma \otimes U + \sigma \otimes Du + D^2 \sigma \otimes W - D\sigma \otimes Dw + Dv \otimes \sigma^* - V \otimes D\sigma^* + DT \otimes D\sigma^* + T \otimes D^2 \sigma^* \]

\[ = \sigma \otimes (Du + \Theta w + U) + (Dv - \Theta t - V) \otimes \sigma^* + (v + Dt + T) \otimes D\sigma^* \]

i.e. that equation \( (4) \) is satisfied. \( S \) is first constructed in \( \tilde{X} \setminus H \) i.e. in \( \sigma \neq 0 \) and then extends as a closed current in \( \tilde{X} \).

Proposition 6.2. There is a current \( T \) closed in \( X \) of bidegree \( (q', q') \) such that \( T|_Z = \Theta \).
7 Formula of King

The restriction $\pi : H = P(N_X Z) \to Z$ of the blow up $\mu$ in the exceptional divisor $H$ is a fibration and $\mu|_H = i \circ \pi$ where $i : Z \to X$ is the canonical injection. For $y \in \tilde{X}$ the differential $d\mu(y) : T_y \tilde{X} \to T_{\mu(y)}X$ satisfies $d\mu(y)(T_y X) \subset T_{\mu(y)}Z$ if $y \in H$. Hence there is an induced map $(N_X Z)_x \to (N_X Z)_y$ where the image is $O_{P(N_X Z)}(-1)$. So $N_X Z = O_{P(N_X Z)}(-1) = O_H(-1)$ then $O(H)|_H = N_X Z = O_H(-1)$.

With $g = \text{codim}_X Z$ we have

$$[Z] = \mu_*([H] \wedge c_1(\Theta_{O_H(1)})^{q-1})$$

with $[H] = c_1(\Theta_{O_H(1)}) + \dd c \log ||\sigma||$ where $\sigma \in H^0(\tilde{X}, O(H))$ satisfies $\sigma^{-1}(0) = H$. So $-c_1(\Theta_{O_H(1)}) = \dd c \log ||\sigma||$ in $\tilde{X} \setminus H$ then

$$[Z] = \mu_*([H] \wedge c_1(\Theta_{O_H(-1)})^{q-1}) = \mu_*([H] \wedge (-c_1(\Theta_{O_H(1)})^{q-1})$$

$$= \dd c \mu_*((\dd c \log ||\sigma||)(\dd c \log ||\sigma||)^{q-1}) - \mu_*((\dd c \log ||\sigma||)^q)$$

$$= \dd c(\mu_*((\dd c \log ||\sigma||)(\dd c \log ||\sigma||)^{q-1}) - (\dd c \mu_*((\dd c \log ||\sigma||))^{q})$$

hence the formula of King

$$[Z] = \dd c(\log ||\mu_*\sigma||)(\dd c \log ||\mu_*\sigma||)^{q-1}) - (\dd c \log ||\mu_*\sigma||)^q$$

with

$$\mu_*\sigma = (\mu|_{\tilde{X} \setminus H})^{-1\ast}\sigma \in H^0(X \setminus Z, (\mu|_{\tilde{X} \setminus H})^{-1\ast}(O(H)|_{\tilde{X} \setminus H})).$$

But $O(-H) = \mu^\ast(\mathcal{L}_Z)$ and then $\mu_*O(H)) = \mu^\ast\mu^\ast(\mathcal{L}_Z)$ extends the holomorphic vector bundle $(\mu|_{\tilde{X} \setminus H})^{-1\ast}(O(H)|_{\tilde{X} \setminus H})$ defined in $X \setminus Z$.

We will now express the term $-(\dd c \log ||\mu_*\sigma||)^q$ considering the exact sequence

$$0 \to O_H(-1) \to \pi^* N_X Z \to Q \to 0$$

above $H$ and with $Q$ the quotient vector bundle. As $Q$ has rank $q - 1$, we have $c_q(Q) = 0$. But $c(Q) = \pi^*\pi(N_X Z) \land c(O_H(-1)) = \sum_j \pi^*\pi_j(N_X Z) \land \sum_k (-c_1(O_H(-1)))^k$ and therefore $c_q(Q) = \sum_{0 \leq k \leq q} \pi^*\pi_{q-k}(N_X Z) \land c_1(O_H(1))^k$ i.e

$$0 = c_1(O_H(1))^q + \sum_{1 \leq k \leq q - 1} \pi^*\pi_{q-k}(N_X Z) \land c_1(O_H(1))^k + \pi^*\pi_q(N_X Z).$$

Since $O_H(1) = O(-H)|_H$, with $\xi = -c_1(O(H))$ it comes

$$0 = \xi^q + \sum_{1 \leq k \leq q - 1} \pi^*\pi_{q-k}(N_X Z) \land \xi^k + \pi^*\pi_q(N_X Z).$$

For $0 \leq k \leq q - 1$, using Proposition 6.2, we can write

$$c_{q-k}(N_X Z) = \{\theta_{q-k}|_Z\}$$

(5)

where $\theta_{q-k}$ is a closed current in $X$ of bidgree $(q - k, q - k)$. Since $i \circ \pi = \mu \circ j$ we have $\pi^*c_{q-k}(N_X Z) = \{\pi^*\pi^\ast\theta_{q-k}\} = \{j^\ast\mu^\ast\theta_{q-k}\} = \{(\mu^\ast\theta_{q-k})|_H\}$. So

$$0 = \xi^q_H + \sum_{1 \leq k \leq q - 1} \{(\mu^\ast\theta_{q-k} \wedge (\xi^k))(\mu^\ast\theta|_H)\} + \{(\mu^\ast\theta_q)|_H\}$$

then

$$\xi^q + \sum_{1 \leq k \leq q - 1} \{(\mu^\ast\theta_{q-k} \wedge (\xi^k)) + \{(\mu^\ast\theta_q)\} = \{(\mu^\ast\varphi)\}$$

with $\varphi$ a closed current in $X$ of bidgree $(q, q)$ which satisfies $\{\varphi|_Z\} = 0$. Replacing $\theta_q$ by $\theta_q - \varphi$, we can assume

$$\xi^q + \sum_{1 \leq k \leq q - 1} \{(\mu^\ast\theta_{q-k} \wedge (\xi^k)) + \{(\mu^\ast\theta_q)\} = 0.$$
With \( \alpha = (-c_1(\Theta_{\mathcal{O}(H)}))^q + \sum_{1 \leq k \leq q-1} \mu^* \theta_{q-k} \wedge (-c_1(\Theta_{\mathcal{O}(H)}))^k + \mu^* \theta_q \), then there is \( w \) such that \( \alpha = df \) in \( \bar{X} \).

Let \( m \) be an integer such that \( \mu_*(\|o\|^m w) \) is \( C^\infty \) in \( X \) and 0 on \( Z \), so

\[
\mu_* w = \frac{1}{\|\mu_* o \|^m} \left( \mu_* o + \mu_* b \otimes \mu_*(-) + D(\mu_* o) \otimes \mu_* c + \mu_* d \otimes D(\mu_* o) \right).
\]

Since \( -c_1(\Theta_{\mathcal{O}(H)}) = df \log \|o\| \) in \( \bar{X} \setminus H \), it finally comes the relation

\[
(dd^c \log \|\mu_* o \|)_X \otimes Z = - \sum_{1 \leq k \leq q-1} \theta_{q-k} \wedge (dd^c \log \|\mu_* o \|^k - \theta_q)
\]

\[
+ dd^c \left\{ \|\mu_* o \|^m \left( \mu_* o + \mu_* b \otimes \mu_*(-) + D(\mu_* o) \otimes \mu_* c + \mu_* d \otimes D(\mu_* o) \right) \right\}.
\]

**Proposition 7.1.** With \( \theta_{q-k} \) the currents closed in \( X \) defined by (5), we can write

\[
[Z] = (dd^c \log \|\mu_* o \|)_X \otimes Z = - \sum_{1 \leq k \leq q-1} \theta_{q-k} \wedge (dd^c \log \|\mu_* o \|^k + \theta_q)
\]

\[
- dd^c \left\{ \|\mu_* o \|^m \left( \mu_* o + \mu_* b \otimes \mu_*(-) + D(\mu_* o) \otimes \mu_* c + \mu_* d \otimes D(\mu_* o) \right) \right\}
\]

with differential forms \( a, c \) with values in \( \mathcal{O}(-H) \) and \( b, d \) with values in \( \mathcal{O}(H) \).

### 8 Closed regularization obtained from a locally free resolution of the ideal sheaves

Suppose the analytic subset \( Z \) of \( X \) of codimension \( q = n-p \) is written \( Z = s^{-1}(0) \) with \( s \in H^0(X, E) \) a nontrivial holomorphic section of \( E \to X \) a Hermitian holomorphic vector bundle of rank \( N \). Let \( \mathcal{I} \) be the image of the morphism \( \mathcal{O}(E^*) \to \mathcal{O}_X \) of sheaves of \( \mathcal{O}_X \)-modules induced by \( s \).

King’s formula (see [17, 20]) expresses here that the differential form \( (\log \|s\|)(dd^c \log \|s\|)^{q-1} \) which is \( C^\infty \) in \( X \setminus Z \) is with locally integrable coefficients in \( X \) and that

\[
\sum_j m_j [Z_j] = dd^c \left\{ (\log \|s\|)(dd^c \log \|s\|)^{q-1} \right\} - \mathbb{I}_{X \setminus Z} (dd^c \log \|s\|)^q
\]

where \( (Z_j) \) still refers to the family of irreducible analytic components of \( Z \) of dimension \( p \) exactly and \( m_j \in \mathbb{N}^* \) is the generic multiplicity of \( \mathcal{I} \) along \( Z_j \).

The current \( dd^c \left\{ (\log \|s\|)(dd^c \log \|s\|)^{q-1} \right\} \) is also denoted by \( (dd^c \log \|s\|)^q \) and can be obtained from a Monge-Ampère operator in the sense of \([10, 14]\).

Now we will express the term \( -(dd^c \log \|s\|)^q \) of the equation (6) following the usual method of Bott-Chern (see [7, 8]) that we will remember.

Consider an exact sequence \( 0 \to L \to E \to Q \to 0 \) of holomorphic vector bundles on \( X \) with \( \text{rk } L = 1 \). The \( C^\infty \) Hermitian metric on \( E \) induces metrics on \( L \) and \( Q \). Denote by \( \Theta_E, \Theta_L, \Theta_Q \) the \((1,1)\)-forms of curvature of the Chern connections respectively on \( E, L \) and \( Q \).

Denoting by \( c(\Theta_E) = \sum_k c_k(\Theta_E) \) the total Chern form associated to \( \Theta_E \), we have to make explicit a solution \( \varphi \) of class \( C^\infty \) in \( X \) of the equation

\[
c(\Theta_E) - c(\Theta_L)c(\Theta_Q) = -dd^c \varphi
\]

where we did not include here the sign \( \wedge \). To define \( \varphi \) we use the following notations: \( \text{Hom}(E, E) = E \otimes E^* \) injects itself into the exterior algebra \( \wedge (E \otimes E^*) \) and the total Chern of \( \Theta_E \) is then written as \( c(\Theta_E) = (1 + \frac{i}{2\pi} \Theta_E)^N \) identifying \( \wedge^N E \otimes \wedge^N E^* \) with \( \mathbb{C} \) using \( I_E^N \), so \( c_k(\Theta_E) = \binom{N}{k} I_E^{N-k} (\frac{i}{2\pi} \Theta_E)^k \).

With \( v \) a holomorphic local frame of \( L, \psi \in E^* \) the adjoint, we set \( \sigma = \frac{|v|^2}{\|v\|^2} \) and \( \alpha = \frac{Dv \psi^*}{\|v\|^2} \) where \( D \) means the Chern connection on \( E \). We can then take

\[
\varphi = \frac{N}{2} \sum_{1 \leq j \leq N-1} \frac{1}{j} \sigma \left( (I_E + \frac{i}{2\pi} \Theta_E)^j - I_E \right) (I_E + \frac{i}{2\pi} \Theta_E + \frac{i}{2\pi} \psi) \left( (I_E + \frac{i}{2\pi} \Theta_E)^{N-j-1} \right).
\]
We have also \( c(\Theta_Q) = c(\Theta_L)^{-1} c(\Theta_E) + \ddc^s (c(\Theta_L)^{-1} \varphi) \) and since \( c(\Theta_L) = 1 + c_1(\Theta_L) \), we have
\[
-(c_1(\Theta_L))^q = c_q(\Theta_E) - c_q(\Theta_Q) + \sum_{1 \leq k \leq q-1} (-c_1(\Theta_L))^k \wedge c_{q-k}(\Theta_E) + \ddc \psi_0
\]
where \( \psi_0 = \sum_{0 \leq k \leq q-1} (-c_1(\Theta_L))^k \wedge \varphi_{q-k-1} \) with \( \varphi_{q-k-1} \) the component of bidegree \((q-k-1, q-k-1)\) of \( \varphi \).

Let \( \mu : \hat{X} \to X \) be the blow up of \( X \) along \( T \) and let \( H \) be the exceptional divisor. We take \( L \) the line sub-bundle of \( \mu^* E \) such that \( \mu^* T = O(L^*) \) (see [21]).

Let us apply the above to the exact sequence
\[
0 \to L \to \mu^* E \to \mu^* E/L \to 0
\]
on \( \hat{X} \) and take the direct images by \( \mu \). Since \( c_1(\Theta_L) = -\ddc \log \| \mu^* s \| \) in \( \hat{X} \sim H \) we have
\[
-(\ddc \log \| s \|)^q_{X \sim Z} = c_q(\Theta_E) - c_q(\Theta_E/\mathbb{C}) + \sum_{1 \leq k \leq q-1} (\ddc \log \| s \|)^k \wedge c_{q-k}(\Theta_E) + \ddc \psi
\]
where \( \psi = \sum_{0 \leq k \leq q-1} (\ddc \log \| s \|)^k \wedge \varphi_{q-k-1} \) and \( \varphi \) is given by (6) with \( \sigma = \frac{s_1^*}{\| s \|} \) and \( \alpha = \frac{D_1 D_2 s^*}{\| s \|^2} \). Being the direct image by the blow up \( \mu \) of a differential form \( \mathcal{C}^\infty \) in \( \hat{X} \), \( \psi \) as \( c_q(\Theta_E/\mathbb{C}) \) is with \( L^1_{\text{loc}} \) coefficients in \( X \).

Then, since in \( X \sim Z \)
\[
(\ddc \log \| s \|)^k = \ddc \left\{ -\frac{1}{2(k-1)} \left( \frac{\ddc^s \| s \|}{2\| s \|^2} \right)^{k-1} \right\} = \ddc ((\log \| s \|) (\ddc \log \| s \|))^{k-1}
\]
for \( k \geq 2 \), we have
\[
-(\ddc \log \| s \|)^q_{X \sim Z} = c_q(\Theta_E) - c_q(\Theta_E/\mathbb{C}) + \ddc \psi'
\]
with
\[
\psi' = \sum_{1 \leq k \leq q-1} (\log \| s \|) (\ddc \log \| s \|)^k \wedge c_{q-k}(\Theta_E) + \psi.
\]
Then the following result generalizes the Poincaré-Lelong formula, a generalization is also due to Andersson (see [1, 2]). The differential forms \( c_q(\Theta_E/\mathbb{C}) \) and \( \psi' \) are \( \mathcal{C}^\infty \) in \( X \sim Z \), are with \( L^1_{\text{loc}} \) coefficients in \( X \) and we have in \( X \) the equality between currents
\[
\sum_j m_j [Z_j] = c_q(\Theta_E) - c_q(\Theta_E/\mathbb{C}) + \ddc \psi_1
\]
with \( \psi_1 = \psi' + (\log \| s \|) (\ddc \log \| s \|)^q_{\sim Z} \).

Assume now \( X \) projective, equipped with an ample line bundle \( B \). There are then \( k \in \mathbb{N} \) and sections \( s_1, \ldots, s_N \in H^0(X, B^k) \) such that \( Z = s_1^{-1}(0) \cap \ldots \cap s_N^{-1}(0) \). We can then take \( E = (B^k)^{\oplus N} \) and \( s = (s_1, \ldots, s_N) \).

Now calculate as Bismut-Bost-Gillet-Soulé (see [4, 6, 22]) a Green form of \( Z \) in \( X \) assuming given an exact sequence of sheaves of \( \mathcal{O}_X \)-modules
\[
0 \to \mathcal{O}(F_n) \xrightarrow{g_n} \mathcal{O}(F_{n-1}) \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_2} \mathcal{O}(F_1) \xrightarrow{g_1} \mathcal{O}(E^*) \xrightarrow{g_0} \mathcal{O}(T) \to 0
\]
with each \( F_i \) a holomorphic vector bundle over \( X \), with a Hermitian metric \( h_{0,j} \) of class \( \mathcal{C}^\infty \) in \( X \). Denote by \( h_{0,0} \) the Hermitian metric induced on \( E^* \) by that of \( E \).

In \( X \sim Z \), we have an exact sequence of holomorphic vector bundles
\[
0 \to F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_2} F_1 \xrightarrow{g_1} E^* \xrightarrow{g_0} \mathbb{C} \to 0,
\]
we can break into short exact sequences of holomorphic vector bundles
\[
0 \to K_0 \to E^* \to \mathbb{C} \to 0,
\]
for \(1 \leq i \leq n-2\),

\[
0 \longrightarrow K_i \longrightarrow F_i \xrightarrow{g_i} K_{i-1} \longrightarrow 0
\]

with \(K_i\) defined in \(X \sim Z\). We set \(F_0 = E^*, F_{n-1} = \mathbb{C}\). For \(0 \leq i \leq n\), \(F_i\) is immersed in \(F_j \oplus F_{i-1}\) identifying \((F_j)_x\) with the graph of \(g_j(x)\) for \(x \in X\) and we denote by \(h_i\) the metric induced in \(F_j\).

We will now still apply the Bott-Chern calculations in the case of an exact sequence

\[
0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0
\]

but this without any assumption on \(\text{rk} \ L\). We can then take

\[
\varphi = N\sigma \int_0^1 \{(I_E + \frac{i}{2\pi} \Theta_t)N^{-1} - (I_E + \frac{i}{2\pi} \Theta_0)^{N-1}\} \frac{dt}{t}
\]

with

\[
\Theta_t = \Theta_E + (1-t)(D'\sigma'\sigma - D'\sigma) - (1-t)^2(D'\sigma D'\sigma + D'\sigma D'\sigma)
\]

and \(\sigma : E \rightarrow E\) the orthogonal projection onto \(L\).

We can write in \(X \sim Z\) the equalities

\[
c(\Theta_{K_{i-1},q_{i-1}}) = c(\Theta_{K_i,h_i})^{-1}c(\Theta_{F_i})(1 + \text{dd}^c(c(\Theta_{F_i})^{-1}\Phi_i))
\]

for \(1 \leq i \leq n-2\) and

\[
c(\Theta_{K_{n-2},q_{n-2}}) = c(\Theta_{F_n,h_{n-1}})^{-1}c(\Theta_{F_{n-1}})(1 + \text{dd}^c(c(\Theta_{F_{n-1}})^{-1}\Phi_{n-1}))
\]

noting \(q_i\) the quotient metric on \(K_i = F_{i+1}/K_{i+1}\) and \(\Phi_i\) the differential form \(\varphi\) of (9) for the exact sequence

\[
0 \longrightarrow K_i \longrightarrow F_i \longrightarrow K_{i-1} \longrightarrow 0.
\]

Since more

\[
c(\Theta_{K_i,h_i}) - c(\Theta_{K_i,q_i}) = -\text{dd}^c\Lambda_i
\]

where

\[
\Lambda_i = \frac{\text{rk} \ K_i}{2} \int_0^1 \text{L}_{K_i,M_i}(I_{K_i} + \frac{i}{2\pi} \Theta_{K_i,M_i}) \frac{\text{rk} K_i}{2} dt
\]

with \(M_i\) the Hermitian metric \((1-t)q_i + th_i\) on \(K_i\) and \(\text{L}_{K_i,M_i}\) the endomorphism of \(K_i\) for writing the Hermitian form \(h_i - q_i\) with \(M_i\) (see [7], page 83), we have

\[
c(\Theta_{K_0}) = \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{-1} \prod_{1 \leq i \leq n-1} \left(1 + \text{dd}^c(c(\Theta_{F_i})^{-1}\Phi_i)\right)^{-1} \prod_{0 \leq i \leq n-1} \left(1 + \text{dd}^c(c(\Theta_{K_i,h_i})^{-1}\Lambda_i)\right)^{-1},
\]

with \(K_{n-1} = F_n\) and \(c(\Theta_{K_{n-1},q_{n-1}}) = c(\Theta_{K_n})\), the metric \(q_{n-1}\) being obtained from the exact sequence \(0 \rightarrow F_n \rightarrow K_{n-1} \rightarrow 0\) and with \(c(\Theta_{K_0}) = c(\Theta_{K_0,h_0})\), the metric \(h_0\) being obtained from the exact sequence \(0 \rightarrow K_0 \rightarrow E^s \rightarrow \mathbb{C} \rightarrow 0\) which amounts to the exact sequence \(0 \rightarrow \mathbb{C} \rightarrow E \rightarrow E/\mathbb{C} \rightarrow 0\).

We equip \(E\) with the Hermitian metric \(h\) dual of the Hermitian metric \(h_0\) on \(F_0 = E^s\) and \(E/\mathbb{C}\) with the Hermitian metric induced by \(h\). So

\[
c_q(\Theta_{K_0}) = (-1)^q c_q(\Theta_{E/\mathbb{C}^s}, h)
\]

and we get a differential \((q-1,q-1)\)-form \(\Psi\) of class \(C^\infty\) in \(X \sim Z\) such that in \(X \sim Z\), we have

\[
(-1)^q c_q(\Theta_{E/\mathbb{C}^s}, h) - \text{dd}^c\Psi = \text{component of bidegree} (q,q) \text{ of } \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{-1}\text{dd}^c\ Psi.
\]

To prove that \(\Psi\) is with \(L^1_{loc}\) coefficients in \(X\) and that the equality holds in \(X\) in the sense of currents, MacPherson’s construction using the graph (see [3] page 120, [15]) gives the following result.
Proposition 8.1. Let still $\tilde{X} \to X$ be the blow up of $X$ along $Z$ with $H = \mu^{-1}(Z)$ the exceptional divisor. There is an exact sequence

$$0 \to G_n \gamma_n \to G_{n-1} \gamma_{n-1} \to \cdots \to G_1 \gamma_1 \to G_0 \gamma_0 \to \mathbb{C} \to 0$$

of holomorphic vector bundles on $\tilde{X}$ satisfying for $0 \leq i \leq n$,

(i) $G_i$ is a holomorphic vector subbundle of $\mu^* F_i \oplus \mu^* F_{i-1}$,

(ii) $G_i|\tilde{X} \sim H$ is the graph of $\mu^* F_i \xrightarrow{i} \mu^* F_{i-1}$ and is isomorphic to $(\mu^* F_i)|\tilde{X} \sim H$.

(iii) with $G_{-1} = \mathbb{C}$, the morphism $G_i \xrightarrow{\gamma_i} G_{i-1}$ over $\tilde{X}$ extends the morphism $\mu^* F_i \xrightarrow{i} \mu^* F_{i-1}$ over $\tilde{X} \sim H$.

Proof. Suppose $X$ connected and let $N_i$ be the rank of $F_i$. So for $x \in X \sim Z$ the graph of $g_i(x)$ is a vector subspace of dimension $N_i$ of $(F_i \oplus F_{i-1})_X$. Consider the Grassmann bundles $G(N_i, F_i \oplus F_{i-1}) \to X$ and the fiber product

$$G = G(N_i, F_i \oplus F_{i-1}) \times_X \cdots \times_X G(N_0, F_0 \oplus F_{-1}) \xrightarrow{\pi} X.$$  

This gives a holomorphic map $\epsilon : X \sim Z \to G$. Note $\xi_i$ the tautological vector bundle over $G(N_i, F_i \oplus F_{i-1})$ and $pr_1^*\xi_i$ its inverse image over $G$ which is a holomorphic vector sub-bundle of rank $N_i$ of $\pi^*(F_i \oplus F_{i-1})$.

Since for $x \in X \sim Z$, $(pr_1^*\xi_i)_{\epsilon(x)} \simeq (\pi^* F_i)_{\epsilon(x)}$ and the complex of the $F_i$ is exact in $X \sim Z$ so we have

$$\sum_{0 \leq i \leq n} (-1)^i (pr_1^*\xi_i)_{\epsilon(X \sim Z)} = \mathbb{C}$$

in the $K$-theory of $\epsilon(X \sim Z)$.

According to [3], page 122, with $\bar{\epsilon}(X \sim Z)$ the closure of $\epsilon(X \sim Z)$ in $G$, then

$$\mu : \bar{\epsilon}(X \sim Z) \xrightarrow{\pi} X$$

is the blow up of $X$ along $Z$ and in the $K$-theory of $\tilde{X}$, we still have $\sum_{0 \leq i \leq n} (-1)^i (pr_1^*\xi_i)_{\tilde{X}} = \mathbb{C}$.

Note $G_i = (pr_1^*\xi_i)_{\tilde{X}}$ for $0 \leq i \leq n$ and $G_{-1} = \mathbb{C}$. Over $\tilde{X} \sim H$, the morphism $\mu^* F_i \xrightarrow{i} \mu^* F_{i-1}$ with the isomorphisms $\mu^* F_i \simeq G_i$ and $\mu^* F_{i-1} \simeq G_{i-1}$ is nothing other than the restriction

$$(\mu^* F_i \oplus \mu^* F_{i-1}) \simeq G_i \xrightarrow{pr_2} G_{i-1} \simeq (\mu^* F_{i-1} \oplus \mu^* F_{i-2})$$

of the projection $pr_2 : \mu^* F_i \oplus \mu^* F_{i-1} \to \mu^* F_{i-1}$. Over $\tilde{X}$, we still have $pr_2(G_i) \subset G_{i-1}$ and therefore $\gamma_i : G_i \xrightarrow{\gamma_i} G_{i-1}$ extends $\mu^* F_i \xrightarrow{i} \mu^* F_{i-1}$. Finally the resulting complex $G_i$ is exact over $\tilde{X}$. \hfill $\Box$

When $Z$ is singular, $\tilde{X}$ can be singular and we reduce to the case when $\tilde{X}$ is smooth considering a modification of $\tilde{X}$ which is a desingularization of $\tilde{X}$. The exact complex of holomorphic vector bundles on $\tilde{X}$ remains exact when the inverse images over the desingularization are taken.

Let’s break down the exact sequence

$$0 \to G_n \gamma_n \to G_{n-1} \gamma_{n-1} \to \cdots \to G_1 \gamma_1 \to G_0 \gamma_0 \to \mathbb{C} \to 0$$

into short exact sequences of holomorphic vector bundles

$$0 \to S_0 \to G_0 \gamma_0 \to \mathbb{C} \to 0,$$

$$0 \to S_i \to G_i \gamma_i \to S_{i-1} \to 0$$

for $1 \leq i \leq n - 2$,

$$0 \to G_n \gamma_n \to G_{n-1} \gamma_{n-1} \to \mathbb{C} \to 0$$

with $S_i$ defined in $\tilde{X}$.

We equip $G_i \subset \mu^* F_i \oplus \mu^* F_{i-1}$ with the Hermitian metric induced by the initial Hermitian metric of $\mu^* F_i \oplus \mu^* F_{i-1}$.
We finally obtain the following result which explicitly gives a Green form of the algebraic cycle $C_	ext{class}$ since $C_	ext{coefficients of class}$ the differential forms $\text{Proposition 8.2}$. for any Hermitian metric on $G$ $Q$ the coherent sheaf $\mathcal{O}$ with $c$.

Note also that if one uses only the existence of an exact sequence $q_X$ being dense in $H$ with the property (iii) of Proposition 8.1, then one has for any Hermitian metric on $G_1$, instead of the previous equality $c_k(\Theta G_1) = \mu^*c_k(\Theta F_1)$, the equality $c_k(\Theta G_1) = \mu^*c_k(\Theta F_1) + \dd c \cdot A_{j,k} + j_* R_{l,k}$.

So in $\tilde{X} \sim H$ we have short exact sequences

$$0 \longrightarrow \mu^* K_0 \longrightarrow \mu^* F_0 \longrightarrow \mathbb{C} \longrightarrow 0,$$

$$0 \longrightarrow \mu^* K_i \longrightarrow \mu^* F_i \longrightarrow \mu^* K_{i-1} \longrightarrow 0$$

for $1 \leq i \leq n - 2$,

$$0 \longrightarrow \mu^* F_n \longrightarrow \mu^* F_{n-1} \longrightarrow \mu^* K_{n-2} \longrightarrow 0$$

and the Hermitian metric on $\mu^* F_i$ induced by $h_i$ and that obtained by the isomorphism with $G_i$ are the same.

So for $0 \leq k \leq n$ we have the equality between Chern forms

$$c_k(\Theta G_i)_{\tilde{X} \sim H} = c_k(\Theta \mu^* F_i)_{\tilde{X} \sim H}$$

then $\tilde{X} \sim H$ being dense in $\tilde{X}$, we even have $c_k(\Theta G_i) = c_k(\Theta \mu^* F_i)$.

To express $\Psi$, write $\Phi_i = \mu_* \Phi_i$ with $\Phi_i$ of class $C^\infty$ in $\tilde{X}$ satisfying

$$\mu^* c(\Theta F_i) - c(\Theta S_{i-1}, i_{\tilde{H}})c(\Theta S_{i-1}, i_{\tilde{H}}) = - \dd c \Phi_i$$

and given by the formula (9) written for the exact sequence $0 \longrightarrow S_i \longrightarrow G_i \longrightarrow 0$ with $\tilde{H}$ (respectively $\tilde{H}_i$) the metric restriction on $S_i$ (respectively quotient on $S_{i-1}$). On the other hand, write $\Lambda_i = \mu_* \Lambda_i$ with $\Lambda_i$ of class $C^\infty$ in $\tilde{X}$ satisfying

$$c(\Theta S_{i-1}, i_{\tilde{H}}) - c(\Theta S_i, i_{\tilde{H}}) = - \dd c \Lambda_i$$

given as in formula (10). Thus we have $\Psi = \mu_* \Psi$ with $\Psi$ of class $C^\infty$ in $\tilde{X}$ satisfying

$$c_q(\Theta S_0) - \dd \Psi = \text{component of bidegree } (q, q) \text{ of } \mu^* \prod_{1 \leq i \leq n} c(\Theta F_i)^{(-1)^{i-1}}.$$
where $A_{i,k}$ is a differential $(k - 1, k - 1)$-form with coefficients $L^1_{loc}$ in $\tilde{X}$, $j : H \to \tilde{X}$ is the canonical injection and $R_{i,k}$ is a $(k - 1, k - 1)$-current closed in $H$. The formula (13) becomes

$$\mu^* c(\Theta_F) - c(\Theta_{S_i/j}) = -\text{d} \text{d}^c (\Phi_i + \sum_k A_{i,k}) - \sum_k j_* R_{i,k}$$

so that $\mu^* c(\Theta_F)$ is replaced by $\mu^* c(\Theta_F) + \sum_k j_* R_{i,k}$ and Proposition 8.2 does not generalize a priori because in order to calculate $\Psi$ from the formula generalizing the formula (11), we must make products of currents.

Let as in the proof of Proposition 5.1 $(U_\alpha)$ be a finite covering of $X$ with open sets of coordinate maps and let $(\lambda_\alpha)$ be a $C^\infty$ partition of the unit subordinate to $(U_\alpha)$. With $\chi_\epsilon \geq 0$ of class $C^\infty$ approximating the Dirac mass at 0 in $\mathbb{C}^n$, we set

$$\Gamma_\epsilon = \sum_\alpha \lambda_\alpha (\Gamma_{|U_\alpha}) * \chi_\epsilon$$

which is a differential form of class $C^\infty$ in $X$, weakly converging in $X$ to $\Gamma$. Set

$$\tilde{W}_\epsilon = \text{d} \text{d}^c \Gamma_\epsilon + c_q(\Theta_E) + (-1)^q - 1 \times \text{component of bidegree } (q, q) \text{ of } \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{(-1)^q-1}.$$ 

In conclusion, we have the following result.

**Proposition 8.3.** The differential forms $\tilde{W}_\epsilon$ are of class $C^\infty$ closed in $X$, weakly converge in $X$ to $\sum_j m_j [Z_j]$ and satisfy $\tilde{W}_\epsilon \geq -\tilde{U}_\epsilon$ where the $\tilde{U}_\epsilon$ are of class $C^\infty$ in $X$ and converge to 0 for the $L^1_{loc}$ topology, when $\epsilon \to 0^+$.

**Proof.** See the proof of Proposition 5.1.

In particular since $\tilde{U}_\epsilon \to 0$ for the $L^1_{loc}$ topology when $\epsilon \to 0^+$, there are a sequence $\epsilon_l \to 0^+$ and a function $g \geq 0$ and $L^1_{loc}$ in $X$ such that $\|\tilde{U}_{\epsilon_l}\| \leq g$ for all $l$.

## 9 Closed regularization with bounded negative part

Using the formula of King, we will show for the current $\sum_j m_j [Z_j]$ of integration the existence of a closed regularization with bounded negative part.

First log $\|s\|$ is a quasi-plurisubharmonic function in $X$ because $\text{d} \text{d}^c \log \|s\| + \left\{ \frac{i}{2\pi} \Theta_E (s) , s \right\} \geq 0$ where $\Theta_E \in C^\infty (X, E \otimes E^*)$ is the curvature of $E$, that satisfies $i \Theta_E$ Hermitian. We choose a differential $(1, 1)$-form $C^\infty$ closed positive in $X$ such that $\|s\|^2 u \geq \left\{ \frac{i}{2\pi} \Theta_E (s) , s \right\}$. For $\epsilon > 0$ we have by the Cauchy-Schwarz inequality

$$\frac{i}{2\pi} \log (\|s\|^2 + \epsilon) \geq \frac{i}{2\pi} \log \left( \frac{\|s\|^2 + \epsilon}{\|s\|^2} \right) \geq 0$$

(14)

so

$$\frac{i}{2\pi} \log (\|s\|^2 + \epsilon) \geq \frac{\epsilon u}{\|s\|^2 + \epsilon} \geq 0.
$$

King’s formula is written as

$$\sum_j m_j [Z_j] = (\text{d} \text{d}^c \log \|s\| + u)^q - (\text{d} \text{d}^c \log \|s\| + u)^q \big|_{X \setminus Z}$$

since $(\text{d} \text{d}^c \log \|s\|)^k = (\text{d} \text{d}^c \log \|s\|^k)^k \big|_{X \setminus Z}$ for $0 \leq k < q$.

We have the regularization

$$\sum_j m_j [Z_j] = \lim_{\epsilon \to 0} T_\epsilon$$

with $T_\epsilon = (\frac{i}{2\pi} \log (\|s\|^2 + \epsilon) + u)^q - R_\epsilon$ where $R_\epsilon$ is real $C^\infty$ closed, weakly converging in $X$ to $(\text{d} \text{d}^c \log \|s\| + u)^q \big|_{X \setminus Z}$.

**Proposition 9.1.** There is $R_\epsilon$ such that $\lim_{\epsilon \to 0} \int_X R_\epsilon \wedge \theta$ exists in $\mathbb{R}$ for every current $\theta \geq 0$ in $X$ with supp $\theta \subset \bigcup_j Z_j$. 

Proof. If \( \theta \) is a \((n - q, n - q)\)-current in \( X \), according to Poly, \( \mu^* \theta \) exists as a current in \( \bar{X} \), in the sense that \( \mu^* \theta \) is a current such that \( \mu_\ast \mu^* \theta = \theta \) (see [19]). But \( \theta \rightarrow \mu^* \theta \) is not weakly continuous. By Proposition 4.1 if \( \theta \geq 0 \) is closed, then \( \mu^* \theta \) is \( \geq 0 \) and closed.

If \( \theta = d\psi \) with \( \psi \) of class \( C^\infty \), then \( \mu^* \theta = d\mu^* \psi \). If \( \theta = d\psi \) with \( \psi \) a current, since \( \mu : \bar{X} \setminus H \rightarrow X \setminus Z \) is a submersion, \( (\mu^* d\psi - d\mu^* \psi)_{|\bar{X} \setminus H} = 0 \). So \( \mu^* \theta - d\mu^* \psi \) has support in \( H \) then

\[
\mu^* \theta = d\mu^* \psi + j_* \nu
\]

with \( j : H \rightarrow \bar{X} \) the canonical injection. So \( \theta = \mu_\ast \mu^* \theta = d\psi + \mu_\ast (j_* \nu) \) therefore \( \mu_\ast (j_* \nu) = 0 \).

For \( \theta \) not necessarily exact, \( \mu^* \theta \) will be replaced by \( \mu^* \theta - j_* \nu \) where \( \nu \) satisfies \( \mu_\ast (j_* \nu) = 0 \) and \( \nu = 0 \) if \( \theta = d\psi \). This will still be an inverse image of \( \theta \) by \( \mu \), in the sense that its direct image by \( \mu \) will be equal to \( \theta \).

We have the formula

\[
(dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta = \mu_\ast ((-\xi + \mu^* u)^q \wedge \mu^* \theta)
\]

claiming that \( \mu_\ast ((-\xi + \mu^* u)^q \wedge \mu^* \theta) \) is a current in \( X \) extending the current \( (dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta \) defined in \( X \setminus Z \).

If \( \theta \) is \( C^\infty \) in \( X \) then \( \int_X (-\xi + \mu^* u)^q \wedge \mu^* \theta = \int_X (dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta, \) so if \( \theta \) is any current, the linear form

\[
\theta \mapsto \int_X (-\xi + \mu^* u)^q \wedge \mu^* \theta
\]

extends \( (dd^c \log \|s\| + u)^q_{|X \setminus Z} \). So we set

\[
L(\theta) = \int_X (-\xi + \mu^* u)^q \wedge (\mu^* \theta - j_* \nu)\theta \wedge \theta
\]

which still satisfies \( L(\theta) = \int_X (dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta \) if \( \theta \) is \( C^\infty \), because then \( \nu = 0 \). Moreover we have \( L(\theta) = 0 \) if \( \theta = d\psi \) with \( \psi \) a current.

If there is \( R_\epsilon \) closed of class \( C^\infty \) satisfying \( \lim_{\epsilon \to 0} \int_X R_\epsilon \wedge \theta = L(\theta) \) for every current \( \theta \geq 0 \) with \( supp \theta \subset \cup_j Z_j \) and for every \( \theta \) smooth, then in particular \( L(\theta_0 - \theta_1) = 0 \) when \( \theta_0 \geq 0 \) and \( \theta_1 \geq 0 \) are such that \( supp \theta_1 \subset \cup_j Z_j \) and \( \theta_0 - \theta_1 = d\psi \) with \( \psi \) a current. This necessary condition is satisfied.

Thus one concludes that there is \( R_\epsilon \) closed \( C^\infty \) satisfying \( \lim_{\epsilon \to 0} \int_X R_\epsilon \wedge \theta = \int_X (-\xi + \mu^* u)^q \wedge \mu^* \theta \in \mathbb{R} \) for every current \( \theta \geq 0 \) with \( supp \theta \subset \cup_j Z_j \) and weakly converging to \( (dd^c \log \|s\| + u)^q_{|X \setminus Z} \).

This does not necessarily mean the convergence of the integral \( \int_X (dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta \) since the \( R_\epsilon \wedge \theta \) do not necessarily converge in \( X \) to the product \((dd^c \log \|s\| + u)^q_{|X \setminus Z} \wedge \theta \) of currents.

We have

\[
\int_X (dd^c \log \|s\|)^q \wedge \theta = \int_X \left( -\frac{1}{2(q - 1)} \left( \frac{dd^c \log \|s\|}{2\|s\|^2} \right)^{q - 1} \right) \wedge \theta = \int_X \left( -\frac{1}{2(q - 1)} \left( \frac{dd^c \log \|s\|}{2\|s\|^2} \right) \right)^{q - 1} \wedge dd^c \theta
\]

and this last integral can be defined by means of the Hörmander-Lojasiewicz division theorem, when \( \theta \) is a current. Thus by changing the first term in \( T_\epsilon \), we can assume that \( \lim_{\epsilon \to 0} \int_X T_\epsilon \wedge \theta \) exists in \( \mathbb{R} \) for every current \( \theta \geq 0 \) in \( X \) with \( supp \theta \subset \cup_j Z_j \).

With \( \omega \) a Kähler metric in \( X \), set \( C(\theta) = sup_{\epsilon > 0} (-\frac{1}{\epsilon} I_{X, \omega, \epsilon} \cdot \theta) \). If there is a sequence of smooth differential forms \( \theta_j \geq 0 \) such that \( \lim_j C(\theta_j) = +\infty \), replacing \( \theta_j \) by \((f_X \omega^q \wedge \theta_j)_{\epsilon}^{-1} \theta_j \) which is bounded by mass and using an argument of double limit, it can be assumed \( \theta_j \) converging to a current \( \theta \) \geq 0 \) such that \( \lim_{\epsilon \to 0} \int_X T_\epsilon \wedge \theta \) is at most \( -\infty \) and such that \( supp \theta \subset \cup_j Z_j \) by an application of the Lebesgue-Nikodym theorem. This is a contradiction and there exists a constant \( C \geq 0 \) such that \(-\int_X T_\epsilon \wedge \theta \leq C \int_X \omega^q \wedge \theta \) for every smooth differential form \( \theta \geq 0 \) . Finally \( T_\epsilon \geq -C \omega^q \).

Proposition 9.2. There exists a sequence \( T_l \) of closed smooth differential \((q, q)\)-forms in \( X \) that weakly converge in \( X \) towards \( \sum_j m_j [Z_j] \) and satisfy \( T_l \geq -C \omega^q \) for all \( l \), where \( C \) is a certain constant \( \geq 0 \).
Remark. Since the restriction of \([Z]\) in \(X \sim Z\) is 0, we can write \([Z] = \lim_{\epsilon \to 0} T_\epsilon\) with \(T_\epsilon\) smooth closed in \(X\) such that

(i) \(\int_X T_\epsilon \wedge \theta\) converges for all positive current \(\theta\) in \(X\) with \(\text{supp} \, \theta \subset Z\),

(ii) \(\int_X T_\epsilon \wedge \theta\) converges to 0 for all positive \(\theta\) in \(X\) with compact support in \(X \sim Z\).

As a consequence, \(T_\epsilon|_{\Omega}\) converges to 0 in the space of smooth differential \((q,q)\)-forms in \(\Omega\), for all relatively compact open subset \(\Omega \subset X \sim Z\).

Now for \(T\) a closed positive current of bidegree \((q,q)\) in \(X\), using the formula

\[
T = \text{pr}_{1*}(\left[D_X\right] \wedge \text{pr}_{2*}^2 T)
\]

with \(D_X\) the diagonal in \(X \times X\), we can write \(T = \lim_{\epsilon \to 0} T_\epsilon\) with \(T_\epsilon\) smooth closed in \(X\) satisfying \(T_\epsilon \geq -C \omega^q\) for all \(\epsilon > 0\), where \(C\) is a certain constant \(\geq 0\). Moreover if \(T\) is smooth on an open subset \(U \subset X\), then \(T_\epsilon|_{\Omega}\) converges to \(T|_{\Omega}\) in the space of smooth differential \((q,q)\)-forms in \(\Omega\), for all relatively compact open subset \(\Omega \subset U\).

For the proof, we write \([D_X]\) = \(\lim_{\epsilon \to 0} \Delta_\epsilon\) with \(\Delta_\epsilon\) smooth closed in \(X \times X\) such that \(\Delta_\epsilon \geq -C_0(\text{pr}_1^* \omega + \text{pr}_2^* \omega)^n\) for all \(\epsilon > 0\), with \(C_0 \geq 0\). We set

\[
T_\epsilon = \text{pr}_{1*}(\Delta_\epsilon \wedge \text{pr}_{2*}^2 T)
\]

which is smooth closed in \(X\) and weakly converges in \(X\) towards \(T\). Moreover

\[
T_\epsilon \geq -C_0 \text{pr}_{1*}(\left(\text{pr}_1^* \omega + \text{pr}_2^* \omega\right)^n \wedge \text{pr}_{2*}^2 T)
\]

and using the binomial theorem we have

\[
\text{pr}_{1*}\left(\left(\text{pr}_1^* \omega + \text{pr}_2^* \omega\right)^n \wedge \text{pr}_{2*}^2 T\right) = \sum_{k=0}^n C_n^k \text{pr}_{1*}\left(\text{pr}_1^* \omega^k \wedge \text{pr}_2^* \omega^{n-k} \wedge \text{pr}_{2*}^2 T\right) = \sum_{k=0}^n C_n^k \omega^k \wedge \text{pr}_{1*}\text{pr}_2^*(\omega^{n-k} \wedge T).
\]

Since \(\omega^{n-k} \wedge T\) is a current on \(X\), we can take \(n - k + q \leq n\) by considering the bidegree. Since the fibers of \(\text{pr}_1\) are of dimension \(n\), we can take \(n \leq n - k + q\). In such a way, we are reduced to take \(k = q\), therefore we conclude that

\[
T_\epsilon \geq -C_0 C_n^q \omega^q \text{pr}_{1*}\text{pr}_2^*(\omega^{n-q} \wedge T)
\]

where the constant \(\text{pr}_{1*}\text{pr}_2^*(\omega^{n-q} \wedge T)\) is the volume of \(T\) with respect to \(\omega\).

On the other hand, let \(\chi\) be a smooth function on \(X\) with compact support in \(U\), equal to 1 on an open neighbourhood of \(\overline{\Omega}\). We write

\[
T_\epsilon = \text{pr}_{1*}(\Delta_\epsilon \wedge \text{pr}_2^2(\chi T)) + \text{pr}_{1*}(\Delta_\epsilon \wedge \text{pr}_2^2((1 - \chi) T))
\]

and we use the hypocontinuity of the convolution of a distribution by a smooth function with compact support. Since \(\text{supp} \,(\chi T) \subset U\), \(\chi T\) is smooth in \(X\) thus \(\text{pr}_{1*}(\Delta_\epsilon \wedge \text{pr}_2^2(\chi T))\) converges to \(\chi T\) in the space of smooth differential \((q,q)\)-forms in \(X\) and \((\chi T)|_{\Omega} = T|_{\Omega}\). For the second term, we use that if \(x \in \overline{\Omega}\) and \(y \in \text{supp} \, ((1 - \chi) T)\), then \(y \not\in \{x = 1\}\), thus \((x, y) \not\in D_X\). Since \(\Delta_\epsilon\) converges to 0 in the space of smooth differential forms on every relatively compact open subset \(\subset (X \times X) \sim D_X\), this second term converges to 0 in the space of smooth differential forms on \(\Omega\).

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