$\mathcal{O}(\alpha^2)$ Next-to-Leading Photonic Corrections to Small-Angle Bhabha Scattering in the Structure Function Formalism$^*$

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Abstract

A general method for computing $\mathcal{O}(\alpha^2)$ and higher-order next-to-leading photonic corrections is presented and applied to the precision calculation of the small-angle Bhabha scattering cross section in the phase-space region of interest for the luminosity measurement at LEP. The formulation is based on a proper matching of exact $\mathcal{O}(\alpha)$ results with higher-order corrections in the Structure Function formalism. The results of the approach are analytically compared with theoretical calculations, both for $s$- and $t$-channel processes, available for simple Event Selections. Numerical predictions for realistic Event Selections are also provided and critically compared with the ones existing in the literature.

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1. Introduction

Small-angle Bhabha (SABH) Scattering is used at $e^+e^-$ colliders to measure the accelerator luminosity. Since the experimental error at present reached by the LEP Collaborations is better than 0.1% \[1\], in order to exploit such an experimental achievement it is mandatory to provide theoretical predictions for the cross section in the luminosity region at the same level of accuracy, or even better.

During 1995, within the Workshop on “Physics at LEP2” at CERN, the Working Group “Event Generators for Bhabha Scattering” took place \[2\]. The major task of the Working Group was to make an inventory of all the available Monte Carlo (MC) and non-MC codes for the computation of the SABH cross section, with the aim of reaching a deeper understanding of the SABH process. The main result of the Working Group was a substantial reduction of the theoretical error on the SABH cross section \[2, 3\] from 0.16% to 0.11%, but further improvements are demanded. In particular, at the present stage the dominant part of the theoretical error comes from the $O(\alpha^2 L)$ photonic corrections, where $L = \ln(Q^2/m^2)$ is the usual collinear logarithm, which for the time being are not fully under control for a realistic experimental Event Selection (ES).

The main tool by means of which such a result has been achieved, was the critical comparison of all the available theoretical formulations and the corresponding numerical codes \[4, 5, 6, 7, 8\], and the understanding of their differences, where present. Given the relevance of this subject, we considered important to improve our theoretical formulation presented in \[7\], with the goal of providing a contribution to a further reduction of the theoretical error. The aim of the present note is to describe the recent theoretical developments of our approach to the understanding of the SABH process, with particular emphasis on $O(\alpha^2 L)$ and higher-order next-to-leading photonic corrections. The results of the approach are analytically compared with theoretical calculations, both for s- and t-channel processes, available for simple ES’s. Numerical predictions for realistic ES’s are also provided and critically compared with the ones existing in the literature.

From now on, we will focus our attention on QED radiative corrections to the dominant part of the cross section in the small-angle regime, namely the cross section for t-channel photon exchange. The corrections due to the other non-dominant contributions, namely s-channel diagrams, $\gamma-Z$ interferences and so on, can be accounted for at the leading logarithmic (LL) level, as shown in \[7\]. We will start considering Bare ES’s. They are absolutely
unrealistic from the point of view of luminometry at LEP, so their interest is, in some sense, academical. In spite of this, they however represent a useful benching situation, since the only available calculation complete at the level of $\mathcal{O}(\alpha^2 L)$ corrections \cite{1} is conceived for such ES’s. The results concerning Calorimetric ES’s, the truly realistic ones, will be derived as special cases of the ones obtained for Bare ES’s: as a first step, final-state radiation will be switched off in the LL part of the result, consistently with the fact that, for an inclusive measurement, final-state mass “singularities” are absent; next, it will be taken into account by means of a proper definition of the final-state radiation factor.

2. Theoretical formulation

In order to settle down the basic notations, let us recall the form of the corrected cross section in the LL approximation within the Structure Function (SF) formalism \cite{9}:

$$
\sigma_{LL} = \int dx_1 dx_2 dy_1 dy_2 \int dI_{cm} D_{\beta_i}(x_1) D_{\beta_f}(x_2) \frac{d\sigma}{dI_{cm}} (I_{cm}(x_1, x_2, y_1, y_2)) D_{\beta_i}(y_1) D_{\beta_f}(y_2).
$$

(1)

In eq. (1), $D_{\beta_i}(x_{1,2})$ are the usual electron and positron structure functions, describing initial-state radiation, as can be found in \cite{7}; $D_{\beta_f}(y_{1,2})$ are the analogous ones for final-state radiation. In general, $\beta_i$ and $\beta_f$ are the radiation factors for initial- and final-state radiation, respectively; in the case of a Bare ES, one has $\beta_i = \beta_f = 2(\alpha/\pi) [\ln(-t/m^2) - 1]$. The case of a Calorimetric ES will be discussed later on. $d\sigma/dI_{cm}$ is the density to be QED corrected, in the centre of mass (cm) reference frame after initial state radiation. $I(x_1, x_2, y_1, y_2)$ represents a convenient set of independent variables, able to describe the configuration of the event corresponding to the kernel cross section, in presence of initial- and final-state radiation. It is understood that the event generated in the cm frame after initial state radiation is boosted along the beam-axis according to the amount of radiation lost longitudinally by the incoming particles, then modified by the effect of final-state radiation, and then accepted or rejected according to a given ES in the laboratory frame.

The $\mathcal{O}(\alpha)$ content of eq. (1) can be written as

$$
\sigma^{(\alpha)}_{LL} = \int dx \int dI_{cm} \frac{1 + x^2}{1 - x} \left\{ \frac{1}{4} \beta_i \left[ \frac{d\sigma}{dI_{cm}} (I_{cm}(x, 1, 1, 1)) \right] \right\}
$$
\[
\begin{align*}
\sigma_L &= \sigma_{LL} - \sigma_{LL}^{(\alpha)} + \sigma_{PT}^{(\alpha)}, \\
\sigma_F &= (1 + C_{NL}^H) \sigma_{LL}, \\
C_{NL}^H &= \frac{\sigma_{PT}^{(\alpha)} - \sigma_{LL}^{(\alpha)}}{\sigma_0} \equiv \frac{\sigma_{NL}^{(\alpha)}}{\sigma_0},
\end{align*}
\]

\(\sigma_0\) being the Born cross section; \(C_{NL}^H\) contains the non-log part of the \(O(\alpha)\) cross section, represented by \(\sigma_{NL}^{(\alpha)}\).

In order to clarify the physical content of eqs. (3) and (4), it is worth considering the perturbative expansions of their ingredients, which can be written as follows:
\[
\sigma_{LL} = \sigma_0 + \sigma_{LL}^{(\alpha)} + \sigma_{LL}^{(\alpha^2)} + \sigma_{LL}^{(\alpha^3)} + \ldots; \\
\sigma'_{LL|\alpha} = \sigma_0 + \sigma_{LL}^{(\alpha)}; \\
\sigma_{PT} = \sigma_0 + \sigma_{PT}^{(\alpha)}. 
\]

(5)

The dominant part of the difference between the additive cross section and the factorized one can be read off the perturbative expansions above:

\[
\sigma_A = \sigma_0 + \sigma_{PT}^{(\alpha)} + \sigma_{LL}^{(\alpha^2)} + \sigma_{LL}^{(\alpha^3)} + \mathcal{O}(\alpha^\Delta \mathcal{L}^\Delta); \\
\sigma_F = \sigma_A + C_{NL}^H \sigma_{LL}^{(\alpha)} + \mathcal{O}(\alpha^3 \mathcal{L}^\varepsilon). 
\]

(6)

The cross-section \(\sigma_{LL}^{(\alpha)}\) describes the LL universal part of the \(\mathcal{O}(\alpha)\) corrections, and \(C_{NL}^H\) the non-log (process dependent) part of the \(\mathcal{O}(\alpha)\) corrections. Therefore, the difference between the factorized cross-section \(\sigma_F\) and the additive cross-section \(\sigma_A\) is due to \(\mathcal{O}(\alpha^\varepsilon \mathcal{L})\) sub-leading corrections contained in \(\sigma_F\) and not in \(\sigma_A\). In particular, the term given by \(C_{NL}^H \sigma_{LL}^{(\alpha)}\) gives an approximation of the \(\mathcal{O}(\alpha^\varepsilon \mathcal{L})\) contributions to the cross section,

\[
\sigma^{(\alpha^2L)} \simeq C_{NL}^H \sigma_{LL}^{(\alpha)},
\]

(7)

since it is the direct product of the LL \(\mathcal{O}(\alpha \mathcal{L})\) cross section times the non-logarithmic non-universal \(\mathcal{O}(\alpha)\) correction \(C_{NL}^H\). A more detailed description of the \(\mathcal{O}(\alpha^\varepsilon \mathcal{L})\) correction would require a convolution of the non-universal non-logarithmic contribution, given by the difference between the exact \(\mathcal{O}(\alpha)\) cross section minus the LL \(\mathcal{O}(\alpha)\) one, with real+virtual radiation in the LL approximation. This can be achieved by applying the algorithm of eq. (2) in the following way:

\[
\sigma^{(\alpha^2L)} = \Sigma_{LL}^{(\alpha)} \left[ \frac{d\sigma^{(\alpha)}}{d\Omega} - \frac{d\sigma_{LL}^{(\alpha)}}{d\Omega} \right],
\]

(8)

where \(d\sigma/d\Omega\) are the differential distributions of the electron scattering angle, already integrated over the photonic phase-space. Equations (7) and (8) are, at a first sight, very different from one another. On the contrary, eq. (8) reduces to eq. (7) under the assumption that the ratios between the \(\mathcal{O}(\alpha)\) densities and the born one are smooth functions of the cm energy and scattering angle, which is a priori a reasonable assumption since the \(\mathcal{O}(\alpha)\) densities are infrared-dominated and hence almost factorized over the born one. Anyway, one of the
purposes of the present note is just to check the quality of the approximated description of eq. (7) and improve it by means of eq. (8). Some illustrative numerical results will be shown later on.

Equation (8) performs the $O(\alpha L)$ corrections to the $O(\alpha)$ non-logarithmic kernel, and hence provides $O(\alpha^e L)$ corrections, taking into account also convolution effects. Along the same way, one can improve it to take into account also the $O(\alpha^e L^{-\infty})$ corrections in the SF approach, by computing, instead of (8), its improved version, namely

$$
\sigma^{(a^n L^{n-1})} = \int dx_1 dx_2 dy_1 dy_2 \int dI_{cm} D_{\beta_i}(x_1) D_{\beta_f}(x_2) D_{\beta_f}(y_1) D_{\beta_f}(y_2) \\
\left[ \left( \frac{d\sigma^{(a)}}{dI_{cm}}(I_{cm}(x_1, x_2, y_1, y_2)) - \frac{d\sigma^{(a)}}{dI_{cm}}(I_{cm}(1, 1, 1)) \right) \\
- \left( \frac{d\sigma^{(a)}}{dI_{cm}}(I_{cm}(x_1, x_2, y_1, y_2)) - \frac{d\sigma^{(a)}}{dI_{cm}}(I_{cm}(1, 1, 1)) \right) \right]. \quad (9)
$$

Some comments are in order here. If one inserts into eq. (9) the $O(\alpha')$ contribution of the distributional expansion of the structure function, the result is automatically zero, confirming that eq. (4) contains contributions starting from $O(\alpha^e L)$. If one inserts in it the $O(\alpha)$ contribution of the distributional expansion of the structure function, one naturally recovers eq. (8). The higher-order contributions in the structure functions produce $O(\alpha^e L^{-\infty})$ corrections: they are complete at the LL level until $O(\alpha^e L^e)$, correct in the soft limit for all the higher-order next-to-leading contributions due to the fact that at present the structure functions employed take into account only up to $O(\alpha^e)$ hard-photon radiation. This, of course, is not a limitation in principle, and moreover has no practical influence for the usually adopted experimental cuts.

The discussion up to now, together with the definition of the various ingredients, allows the definition of a new, factorized cross section. In fact, it is easy to verify that the sum of the additive cross section of eq. (3) and the higher-order next-to-leading correction of eq. (9) defines the following factorized cross section:

$$
\sigma_A + \sigma^{(a^n L^{n-1})} = \sigma_F^{\text{new}},
$$

$$
\sigma_F^{\text{new}} = \int dx_1 dx_2 dy_1 dy_2 \int d\Omega_{\text{cm}} D_{\beta_i}(x_1) D_{\beta_f}(x_2) D_{\beta_f}(y_1) D_{\beta_f}(y_2) \\
\frac{d\sigma_0}{d\Omega_{\text{cm}}}(x_1, x_2, y_1, y_2) \left( 1 + C_{NL}^H(x_1, x_2, y_1, y_2) \right),
$$
\[ C^H_{NL}(x_1, x_2, y_1, y_2) = \frac{d\sigma^{(a)}_{ee}/d\Omega_{cm} - d\sigma^{(a)}_{LL}/d\Omega_{cm}}{d\sigma_0/d\Omega_{cm}}. \] (10)

The analytical identity between \( \sigma_A + \sigma^{(\alpha n L^{-1})} \) on the one hand, and \( \sigma_F^{\text{new}} \) on the other one can be seen as follows. \( \sigma_F^{\text{new}} \) is the sum of two integrals. The first one, involving only the born-approximation density, is nothing but \( \sigma_{LL} \). The second one, involving the product of the born-approximation density times the new \( C^H_{NL}(x_1, x_2, y_1, y_2) \), coincides with the full convolution of the difference between the exact \( \mathcal{O}(\alpha) \) density and the LL-approximated one. Let us focus the attention on this last integral. By inserting in it the \( \mathcal{O}(\alpha') \) distributional expansion of the structure functions and performing the integrations, one is left with \( \sigma_{PT}^{(a)} - \sigma_{LL}^{(a)} \) which, when summed up to \( \sigma_{LL} \), gives exactly \( \sigma_A \). The residual higher-order contributions to the second integral, by virtue of the fact that the structure functions are normalized to unity, recover exactly \( \sigma^{(\alpha n L^{-1})} \).

Equation (10) reproduces the result of eq. (4) under the assumption that \( C^H_{NL}(x_1, x_2, y_1, y_2) \) is a smooth function of its arguments, so that it can be extracted from the convolution integral and the densities appearing in its numerator and denominator can be replaced by the corresponding integrated cross sections. In this sense, eq. (10) is an improvement of eq. (4), that fully takes into account the angle and energy dependence of the non-logarithmic component of the exact \( \mathcal{O}(\alpha) \) correction, so that eq. (10) supersedes eq. (4) developed in ref. [7], which the published version of the FORTRAN code SAB-SPV [8] is based on. Moreover, from a technical point of view, the formulation described in [4] is worked out under the assumption that a specific cut on the final state fermions is applied, namely a cut on the individual energies of the fermions. The present formulation allows one, on the contrary, to impose arbitrary cuts on the final state fermions.

The formulation just described is conceived for applications to the SABH process. Nonetheless, it is completely general, and can be applied also to annihilation processes, once the non-logarithmic \( \mathcal{O}(\alpha) \) process-dependent correction \( (C^H_{NL}) \) is known.

It is worth mentioning that another source of \( \mathcal{O}(\alpha^2 L) \) corrections is represented by the production of additional light pairs; this contribution is at present well under control at the level of 0.03% [2, 3], and a further improvement will be necessary only when photonic corrections will be under control at the same level.

So far, only QED corrections to the dominant part of the small-angle Bhabha cross section, i.e. the \( t \)-channel photon exchange contribution, have
been considered. This is not a limitation of the approach: the very same algorithm of eq. (10) can be extended to the full Bhabha cross section in a straightforward way. Actually, from the practical point of view, at small scattering angle (20–60 mrad) it is necessary to take into account the residual born-approximation contributions plus their LL corrections, and this can be simply obtained by adding the residual born-approximation contributions to the integration kernel of eq. (11). Moreover, the only non-QED corrections relevant in the luminometry region are given by the photonic vacuum polarization, which also can be easily taken into account by using the running QED coupling constant.

### 3. Comparisons and numerical results

**Analytical comparisons** – The approach just described insures that the $\mathcal{O}(\alpha^k\mathcal{L})$ photonic contributions coming from an “external” collinear photon in association with an “internal” non-collinear one are automatically taken into account. This is not in principle the complete set of $\mathcal{O}(\alpha^k\mathcal{L})$ corrections: in this way, for instance, the truly irreducible two-loop corrections are missed, but they can be expected to give small contributions. Therefore, the $\mathcal{O}(\alpha^k\mathcal{L})$ corrections taken into account by this method represent the bulk of the complete set. This heuristic argument can be put on a firmer ground by comparing the results provided by this formulation with the ones obtained by means of complete $\mathcal{O}(\alpha^k\mathcal{L})$ calculations, already present in the literature for some specific ES’s. As a first step, we will compare our results with the ones available in analytical form, i.e. in the soft approximation limit; next we will consider comparisons beyond the soft approximation, for which only numerical results can be compared.

A first analytical result including $\mathcal{O}(\alpha^k\mathcal{L})$ corrections concerns $e^+e^-$ annihilation into $\mu^+\mu^-$ pairs, taking into account initial-state QED corrections [12]. In that paper an analytical formula is given, describing QED corrections for an ES where the total energy emitted by initial-state photons does not exceed $\Delta E$. If one works out eq. (11), namely by putting in this case $\beta_f = 0$, imposing the cut condition of [12], computing the proper $\mathcal{O}^{\mu\mu}_{NL}$ and truncating the result at the $\mathcal{O}(\alpha^k)$, one finds that:

- the $\mathcal{O}(\alpha)$ perturbative result is exactly recovered, by construction;

- all the infrared-singular terms, namely the ones containing $\ln^2 \varepsilon$ and $\ln \varepsilon$, where $\varepsilon = \Delta E/E$, are exactly recovered at the level of $\mathcal{O}(\alpha^k\mathcal{L}_f^\varepsilon)$, $\mathcal{O}(\alpha^k\mathcal{L}_f)$ and $\mathcal{O}(\alpha^k)$, where $L_s = \ln(s/m^2)$;
the difference between the two results starts at the level of \((\alpha/\pi)^2L_s\) times a constant;

in particular, such a difference reads

\[
\frac{\delta \sigma}{\sigma_0} \bigg|_{(\alpha^2L_s)} = \left(\frac{\alpha}{\pi}\right)^2 L_s \left[3\zeta(3) - \frac{3}{2}\zeta(2) + \frac{3}{16}\right], \quad (11)
\]

where \(\delta \sigma\) is the difference between the cross section of \([12]\) and the cross section of eq. (10). The difference numerically amounts to a relative deviation of about \(1.7 \times 10^{-4}\). The residual difference is at \(\mathcal{O}(\alpha^\varepsilon)\) times a constant and is numerically irrelevant.

It is worth noting that results including full \(\mathcal{O}(\alpha^\varepsilon\mathcal{L})\) corrections for the small-angle Bhabha scattering cross section are available for an academic trigger \([6]\) or under development for the most general case \([13]\). Also in this case it is possible to compare analytically the predictions of eq. (10) with the results shown in \([6]\) in the soft approximation. The comparison can be worked out along the same lines as in the previous case, with the only differences that the collinear logarithm is now \(L_t = \ln(-t/m^2)\), \(\beta_f = \beta_i = \beta_t\) and the cut condition imposed in \([6]\) requires that the energy of each photon does not exceed \(\Delta E\). The results of the comparison are the same as before up to the \(\mathcal{O}(\alpha^\varepsilon\mathcal{L}_{\|})\) corrections, namely the difference appears at the level of \((\alpha/\pi)^2L_t\) times a constant and reads

\[
\frac{\delta \sigma}{\sigma_0} \bigg|_{(\alpha^2L_t)} = 2 \left(\frac{\alpha}{\pi}\right)^2 L_t \left[3\zeta(3) - \frac{3}{2}\zeta(2) + \frac{3}{16}\right], \quad (12)
\]

where \(\delta \sigma\) is the difference between the cross section of \([6]\) in soft approximation and the cross section of eq. (10). This difference numerically amounts to a relative deviation of about \(2.2 \times 10^{-4}\), since the overall factor of two is compensated by the fact that \(L_t \simeq 2/3L_s\). In this case, also an additional difference appears, namely at the level of the infrared-sensitive truly \(\mathcal{O}(\alpha^\varepsilon)\) terms, which reads

\[
\frac{\delta \sigma}{\sigma_0} \bigg|_{(\alpha^2)} = -\left(\frac{\alpha}{\pi}\right)^2 \left[4 \ln^2 \varepsilon + 8 \ln \varepsilon\right], \quad (13)
\]

the residual difference being at \(\mathcal{O}(\alpha^\varepsilon)\) times a constant and hence numerically irrelevant. This last difference, being infrared-sensitive, can show up in the region of strong cuts.

Numerical results – Going beyond the soft approximation requires the discussion of numerical results. From now on, it is understood that all the numerical results, apart from the ones obtained by our new formulation, can be
The formulation described in this paper has been implemented in a new, so far unpublished, version of the code SABSPV [8].

Figure 1 shows the comparison between the results obtained by the present formulation of the problem and the results available in the literature [2] for a Bare ES. For the details concerning the ES, the reader is referred to [2]. The FORTRAN codes involved in the comparison are BHAGEN95 [4], BHLUMI [5], NLLBHA [6], OLDBIS+LUMLOG [5] and SABSPV [7, 8]. Again, the details concerning the codes and their underlying theoretical formulations can be better found in the original literature and in [2]. Figure 1 is drawn according to the convention adopted in [2], namely it shows the relative differences between the codes involved and a reference cross section, which in the present case is the cross section computed by BHLUMI. As a first comment, one can see that our new factorized solution corresponding to eq. (4) differs appreciably from the previous factorized one of ref. [7] due to the improved treatment of the final state radiation phase space. Both of these solutions, neglecting convolution effects at the level of $\mathcal{O}(\alpha^L)$ corrections, are at present obsolete. By looking at the figure, one can see that all the solutions shown group together essentially into two clusters. The first cluster of solutions is the one of the “additive” solutions, namely BHAGEN95, OLDBIS+LUMLOG and SABSPV in its additive version (see eq. (3)). All these solutions have the common feature that they miss the $\mathcal{O}(\alpha^L)$ contributions, since they do not fill the region of phase space characterized by the emission of one collinear and one acollinear photon. Their relative distance from the reference cross section is steadily around 0.10–0.15\%. The second cluster of solutions is the one of the “factorized” solutions, namely BHLUMI and SABSPV in its factorized versions (see eqs. (4) and (10)). All these solutions have the common feature that they include the bulk of the $\mathcal{O}(\alpha^L)$ contributions in some form. It is worth noting that all of them lie in the band $\sigma_{ref} \pm 0.1\%$. The comparison between the results of eqs. (4) and (10) is a measure of the degree of approximation of eq. (4) with respect to the new improved solution of eq. (10). The relative difference between them is maximum at $z_{min} = 0.1$, where it is roughly 0.1\% and can be attributed to convolution effects, and minimum at $z_{min} = 0.9$, where it amounts to around 0.02\% since the convolution effects switch off naturally. These differences should be compared with the corresponding relative difference with the cluster of additive solutions, which on the contrary is larger than 0.1\%. This means that the approximate solution of eq. (4) is, after all,

\footnote{Actually, the results shown in Fig. 1 have been obtained by means of the sum of eq. (3) and eq. (8), which anyway differs from eq. (10) starting from $\mathcal{O}(\alpha^3 L^c)$ corrections; these corrections are numerically irrelevant, as checked for the case under consideration, and shown later on for a realistic ES.}
a good approximation to eq. (10), which anyway supersedes it. It is worth stressing that the main difference between eq. (4) and (10) does not lie in the dynamical content, but rather in convolution and/or phase-space effects, that are correctly treated in (10) while approximated in (4). Note that the new improved solution and the reference cross section differ from one another by less than 0.03% over all the $z_{\text{min}}$ range. As far as the physical content is concerned, the results by NLLBHA are comparable with the factorized results by BHLUMI and SABSPV. In particular, also these results differ from the reference cross section by less than 0.1% over all the $z_{\text{min}}$ range and less than 0.06% for realistic $z_{\text{min}}$, namely $0.3 \leq z_{\text{min}} \leq 0.7$. The maximum difference between NLLBHA and BHLUMI/SABSPV is at $z_{\text{min}} = 0.9$, and could be traced back to the infrared-sensitive terms of eq. (13). This item deserves further investigation.

The formulation described above in eq. (10) is conceived for a Bare ES. Going to a Calorimetric ES requires a completely different treatment of final-state radiation in the LL part of the result. A first step can be simply dropping final-state radiation, which means putting $\beta_f = 0$ in all the structure function corrections, consistently with the fact that, for an inclusive measurement, final-state mass “singularities” are absent. An improvement with respect to this choice can be taking into account final-state radiation having defined a proper final-state radiation factor. This can be done by considering that adding the correction due to photons collinear to final-state leptons results into a modification of the $\beta$ factor \cite{14} according to

$$
\beta_t = 2(\alpha/\pi) \left[ \ln(-t/m^2) - 1 \right] \rightarrow \beta_\delta = 2(\alpha/\pi) \ln \left( \frac{-4t}{\delta^2 s} \right),
$$

where $\delta$ can be reasonably taken as the minimum aperture of the final-state cluster. The results obtained for a CALO2 ES (see \cite{2} for details) are shown in Fig. 2, where the same conventions as in Fig. 1 have been adopted. The only difference is that, in this case, the results by NLLBHA are absent and the numerical results relative to the present formulation have been produced first switching off final-state radiation (Fig. 2a), and then including its effect using in the structure functions $\beta_f = \beta_\delta$ with $\beta_\delta$ given by (14) (Fig. 2b). As in the case of the BARE ES, two clusters of solutions can be recognized. The first group of the additive solutions, due to the lack of $\mathcal{O}(\alpha^C \mathcal{L})$ contributions, differ from the reference cross section of about 0.10-0.15% for $0.1 \leq z_{\text{min}} \leq 0.7$ and of about 0.3-0.5% at the extreme value $z_{\text{min}} = 0.9$, where the differences can be expected to be enhanced by infrared-sensitive terms. The second cluster of the “factorized” solutions (BHLUMI and SABSPV in its factorized versions), which include the bulk of the $\mathcal{O}(\alpha^C \mathcal{L})$ corrections, is contained within a band $\sigma_{\text{ref}} \pm 0.1\%$, both without and with final-state radiation. In particular, for the
case of no final-state radiation (Fig. 2a), we show also the numerical results corresponding to the sum of eq. (3) with eq. (8), which, although in additive form, does contain $\mathcal{O}(\alpha^E \mathcal{L})$ contributions. As can be seen, the difference between the new factorized version of SABSPV of eq. (10) and the result given by the sum eq. (3) with eq. (8) is not appreciable, even at this level of precision, since the two prescriptions differ for sub-leading terms starting from $\mathcal{O}(\alpha^3 \mathcal{L}^E)$. The largest difference between the improved versions of SABSPV and the reference cross section is present at $z_{\text{min}} = 0.9$ in Fig. 2a, and it has to be ascribed to neglecting final-state radiation corrections in SABSPV, for this case. This interpretation is confirmed by the situation shown in Fig. 2b, where the predictions by SABSPV include the effect of final-state radiation. Actually, the new improved solution and the reference cross section differ, in this case, by less than 0.02% over the full $z_{\text{min}}$ range, similarly to the case of the BARE ES. As a general comment, it is worth noting that, wherever differences between solutions are present at the extreme value $z_{\text{min}} = 0.9$, these are due, as we explicitly shown independently of the ES, to differently treated infrared-sensitive contributions.

In conclusion, we have shown a new formulation for the calculation of photonic corrections to a generic kernel cross section. A method for computing the bulk of photonic $\mathcal{O}(\alpha^E \mathcal{L})$ corrections has been proposed, and analytically tested versus theoretical results present in the literature for simple ES’s. The algorithm has been applied to the precision calculation of the SABH cross section, relevant for the luminosity measurement at $e^+e^-$ colliders, and several numerical results have been shown and critically commented.

The present formulation, as applied to the SABH process, supersedes the one developed in [1], [2] and used in [2], [3] as one of the tools for the estimate of the theoretical error on the SABH process itself.

Following the strategy adopted in [3], [4], an appreciable reduction of such an error will only be reached by means of a critical comparison of all the available formulations of the problem, for a wide ensemble of ES’s. The present formulation is able to provide one of the ingredients for such an achievement, towards a theoretical error on luminosity well below 0.1%.

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Figure 1: Comparison of MC and non-MC results. The relative differences between the codes involved in the comparison and the cross section by BH-LUMI taken as a reference cross section are shown as functions of the cut $z_{\text{min}} = E_+E_-/E_{\text{beam}}^2$. $E_+,-$ are the energies deposited by the bare final state positron and electron, respectively. The details of the clustering algorithm (BARE1) are given in [2]. The centre of mass energy is $\sqrt{s} = 92.3$ GeV.
Figure 2: Comparison of Monte Carlo’s. The relative differences between the codes involved in the comparison and the cross section by BHLUMI taken as a reference cross section are shown as functions of the cut $z_{\text{min}} = E_+ E_- / E_{\text{beam}}^2$. $E_+, -$ are the energies deposited in the positron and electron clusters, respectively. The details of the clustering algorithm (CALO2) are given in [2]. The centre of mass energy is $\sqrt{s} = 92.3$ GeV.