Symmetry and History Quantum Theory: An Analogue of Wigner’s Theorem

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May 1996

To appear in JMP

Abstract

The basic ingredients of the ‘consistent histories’ approach to quantum theory are a space $\mathcal{UP}$ of ‘history propositions’ and a space $\mathcal{D}$ of ‘decoherence functionals’. In this article we consider such history quantum theories in the case where $\mathcal{UP}$ is given by the set of projectors $\mathcal{P}(\mathcal{V})$ on some Hilbert space $\mathcal{V}$. We define the notion of a ‘physical symmetry of a history quantum theory’ (PSHQT) and specify such objects exhaustively with the aid of an analogue of Wigner’s theorem. In order to prove this theorem we investigate the structure of $\mathcal{D}$, define the notion of an ‘elementary decoherence functional’ and show that each decoherence functional can be expanded as a certain combination of these functionals. We call two history quantum theories that are related by a PSHQT ‘physically equivalent’ and show explicitly, in the case of history quantum mechanics, how this notion is compatible with one that has appeared previously.

PACS numbers: 03.65.Bz, 03.65.Ca, 03.65.Db

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1 Introduction

In this paper we discuss the mathematical aspects of a notion of ‘symmetry’ in a history quantum theory, such as the decoherent histories approach to quantum theory initiated by Griffiths [1], Omnès [2] and Gell–Mann and Hartle [3]. Given the major importance symmetries play in almost every physical theory, one would like to know what the counterpart of this concept is in theories who place the emphasis on ‘histories’ and ‘decoherence functionals’ rather than propositions and states at a fixed time point (as is done in standard quantum theory). This involves the problem of giving a meaning to the notion of two history theories being equivalent; when consistent sets can be called equivalent, etc.[4].

These matters are not settled yet, but I will show that, in case we adopt a particular notion of ‘symmetry’, it is possible to assign a well defined meaning to such concepts. In order to understand where these ideas fit into the structure of such history quantum theories, we have to rewrite the decoherent histories approach in a way that describes the ingredients of such a theory in a more transparent way.

The clarification of the structural content of history quantum theories (HQT) is due to C.J. Isham [5], who extracted the basic features of these theories in form of a set of axioms which determine the mathematical content of the framework of such theories. The aim is to place history quantum theories—as an entirely new approach to the problem of defining and constructing quantum theories—on an equally firm mathematical base as other, already existing approaches to quantum theories. The explanation of why the axioms take the particular form chosen is a very deep one and is partly motivated by problems arising in the area of quantum gravity, in particular the so called ‘problem of time’; it uses ideas of ‘quasi-temporal’ logic and much more. These matters have been discussed at some length by Isham [5], Isham and Linden [6] and Schreckenberg [7] and the reader is referred to those sources for a deeper appreciation of ‘history quantum theories’.

In this paper I will adopt a working, practical approach. I will mainly restrict myself to the case of the history version of finite–dimensional quantum mechanics. This will prove to be an ideal model to illustrate the concept of ‘symmetry’ introduced in this article.

Our discussion will be based on an investigation of how the mathematical structure of history quantum theories suggests a notion of ‘physical symmetries of history quantum theories’. It is striking—and indeed very satisfying—that the concept developed here is compatible with a definition and a result presented by Gell–Mann and Hartle in [4], even though this was not the original goal of the enterprise. I was only after completing the essential parts of the arguments here, that this relation was made explicit. This speaks on the one hand for the physical insight and arguments which led Gell–Mann and Hartle to the notion of ‘physical equivalence’, and, on the other hand, it shows the strength of the mathematical formalism developed by Isham [5] in order to capture the main ideas of the decoherent histories program in a precise manner. Because of this relation among
the results, the physical arguments presented in [4] can, to some extent, be regarded as physical arguments in favour of the notion developed here and, vice versa, the arguments presented here as a precise mathematical statement about such objects, which possess a very transparent description.

We will begin with an introduction to the formalism introduced in [5], recall the classification theorem for decoherence functionals proven in [8] and remind the reader of the content of Wigner’s theorem, which will be central to our investigation. In section 2 we reformulate the standard requirements for ‘physical symmetries’ given by Wigner in a way that is more suited to our problem in that it avoids some of the interpretative difficulties that arise when one is trying to induce a notion of symmetry in history quantum theories from symmetries defined at a single time point. We proceed by defining ‘physical symmetries of a history quantum theory’ (PSHQT) and show that a particular subset of PSHQT can be induced by unitary or anti–unitary operators $\hat{U}$ on $\mathcal{V}$, which I call ‘homogeneous symmetries’. We show that these symmetries possess a characterisation à la Wigner. We investigate the structure of the space $\mathcal{D}$ in some detail to show in section 3 that, in fact, PSHQT are in one–to–one correspondence with homogeneous symmetries and can thus be characterized by an analogue of Wigner’s theorem. We call two history quantum theories which are related by a physical symmetry of a history quantum theory physically equivalent. This expression first appeared in [4] and we show explicitly, for history quantum mechanics, how the notion of ‘physical equivalence’—as introduced in [4]—is naturally induced by a subset of the set of PSHQT. In the closing section 4 we mention some ways one could try to proceed in order to find a satisfactory physical interpretation of the symmetries considered in this article.

1.1 Decoherent Histories and History Quantum Theory

1.1.1 The algebraic structure for HQT

In the decoherent histories approach the two main ingredients are the so–called ‘histories’, namely sequences of Schrödinger picture projection operators $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$, with $t_1 < t_2 < \cdots < t_n$, defined on the single time Hilbert space $\mathcal{H}$, and ‘decoherence functionals’, namely a complex–valued functional $d$ of pairs of histories. For normal quantum mechanics the latter is given by:

$$d_{(H,\rho)}(\alpha, \beta) := \text{tr}_{\mathcal{H}}(\tilde{C}_\alpha \rho_{t_0} \tilde{C}_\beta),$$

(1.1)

where the ‘class’ operator $\tilde{C}_\alpha$ is defined to be

$$\tilde{C}_\alpha := \alpha_{t_1}(t_1)\alpha_{t_2}(t_2)\cdots\alpha_{t_n}(t_n)$$

(1.2)

with $\{\alpha_{t_i}(t_i) := e^{i\hat{K}(t_i-t_0)}\alpha_{t_i}e^{-i\hat{K}(t_i-t_0)}\}$ being the associated Heisenberg picture operators. For the ease of exposition, all histories $\alpha$ will be defined on $n$ arbitrary, but fixed, time–points.
This functional $d(\alpha, \beta)$ is an extension of the formula $d(\alpha, \alpha)$ in standard quantum mechanics for the joint probability of finding all the properties $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ with $t_1 < t_2 < \cdots < t_n$ in a time-ordered sequence of measurements. The aim is to determine with the aid of certain ‘consistency conditions’ on $d(\alpha, \beta)$ such histories, on which $d(\alpha, \alpha)$ defines a probability distribution.

The expression for the decoherence functional is rather messy: it is difficult to isolate the contribution of the histories $\alpha, \beta$ to the evaluation. Whereas before they were defined as sequences of Schrödinger operators they enter now as a product of Heisenberg operators. Hence the evolution operator should belong intrinsically to the decoherence functional as does the density matrix $\rho$.

The separation that I have in mind of the contribution to $d(H, \rho)(\alpha, \beta)$ of (i) the histories, and (ii) a part which encodes all the properties of the decoherence functional, is best illustrated by an analogous expression in standard quantum mechanics. Namely, the probability $p(x \in [a, b])$ of finding that the eigenvalue $x$ of an observable $X$ lies in the interval $[a, b], a, b \in \mathbb{R}$, given the state $\rho$ of the system, is evaluated as

$$p(x \in [a, b], \rho) = \text{tr}_H(P^{x}_{[a,b]} \rho).$$

One can immediately refer to the density operator to describe the contribution of the state and to the projection operator $P^{x}_{[a,b]}$—as the mathematical representation of the question asked—to describe the contribution of the corresponding observable to the value $p(x \in [a, b], \rho)$. This is, of course, due to Gleason’s theorem which establishes a one–to–one correspondence between states and density operators.

The appropriate rewriting of the expression for $d(H, \rho)(\alpha, \beta)$, which is described in detail in [8], relies on the mathematical identity

$$\text{tr}_H(A_1 A_2 \cdots A_m) = \text{tr}_{\otimes H^m}(A_1 \otimes \cdots \otimes A_m S)$$

which allows us to express the trace of a product of $m$ operators $\{A_m\}$ by means of the trace of a single operator $A_1 \otimes \cdots \otimes A_m$ on the $m$–fold tensor product space $V_m := \otimes H^m$ and a universal operator $S$.

Forgetting for a moment the $\rho$ in (1.1), $d(H, \rho)(\alpha, \beta)$ is given by the product of $2n$ Heisenberg operators. Using formula (1.4) we deduce that histories enter the decoherence functional in the following way:

$$\tilde{\alpha} = \alpha_{t_1}(t_1) \otimes \alpha_{t_2}(t_2) \otimes \cdots \otimes \alpha_{t_n}(t_n).$$

Heisenberg operators are just Schrödinger operators multiplied on the left and right by the evolution operators $U(t_i, t_0)$ and its inverse. Expression (1.4) shows that this dependence can be thrown onto the universal operator $S$, allowing us to represent histories by Schrödinger–picture operators

$$\alpha = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n} \in P(V_n)$$

3
which contribute to the value for $d_{(H,\rho)}(\alpha, \beta)$ through

$$d_{(H,\rho)}(\alpha, \beta) = \text{tr}_{\mathcal{V}_n \otimes \mathcal{V}_n}(\alpha \otimes \beta X_{(H,\rho)}), \quad (1.7)$$

for some operator $X_{(H,\rho)}$ defined on $\mathcal{V}_n \otimes \mathcal{V}_n$. The time–ordered strings of projection operators $(\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^n)$ are now represented by a homogeneous projection operator $\alpha_t^1 \otimes \alpha_t^2 \otimes \cdots \otimes \alpha_t^n$ on the $n$–fold tensor product space $\mathcal{V}_n = \otimes_{i=1}^n \mathcal{H}_i$ and one can easily see that this association is one–to–one. This motivates the definition of a space $\mathcal{UP}$ of ‘history propositions’ (also called ‘universal propositions’ or ‘propositions about the universe’), which is given by the set of all projection operators $\alpha \in \mathcal{P}(\mathcal{V}_n)$ on the $n$–fold tensor product space. By these means we have achieved the aim of separating the contribution of the histories to the value $d_{(H,\rho)}(\alpha, \beta)$, in that the operator $X_{(H,\rho)}$ encodes now all of the dynamical information and the initial conditions of the system under investigation.

One can also convince oneself that the decoherence functional $(1.7)$ satisfies the following properties:

- Hermiticity: $d(\alpha, \beta) = d(\beta, \alpha)^* \quad \forall \alpha, \beta \in \mathcal{P}(\mathcal{V}_n)$ \hspace{1cm} (1.8)
- Positivity: $d(\alpha, \alpha) \geq 0 \quad \forall \alpha \in \mathcal{P}(\mathcal{V}_n)$
- Additivity: $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$
- Normalisation: $d(1, 1) = 1$

which are the usual requirements for decoherence functionals in the consistent histories approach when expressed in this formalism. The operation ‘$\oplus$’ is given by the addition of two orthogonal projectors, i.e. history propositions, in $\mathcal{P}(\mathcal{V}_n)$.

This example seems to suggest that it might be worth trying to define a history quantum theory as a theory which has two main ingredients: A space of history propositions $\mathcal{UP}$ which, in this paper, will be the space of propositions $\mathcal{P}(\mathcal{V})$ onto a Hilbert space $\mathcal{V}$; and a decoherence functional $d \in \mathcal{D}$, where $\mathcal{D}$ denotes the space of all decoherence functionals, that is, all those functionals defined on $\mathcal{P}(\mathcal{V}) \times \mathcal{P}(\mathcal{V})$ which possess the properties $(1.8)$ mentioned above. Thus $\mathcal{V}$ does not necessarily have to be of the tensor–product form $\mathcal{V}_n$, and $d \in \mathcal{D}$ will in general not be of the form $d_{(H,\rho)}$. In this formalism, consistent sets of history propositions with respect to a $d \in \mathcal{D}$ correspond to certain partitions of the unit operator on $\mathcal{V}$ into mutually orthogonal projectors $\{\alpha_i\}_{i=1}^{\dim \mathcal{V}}$ such that

$$d(\alpha_i, \alpha_j) = \delta_{ij}d(\alpha_i, \alpha_i) \quad \forall i, j \in \{1, 2, \ldots, m\}. \quad (1.9)$$

The properties $(1.8)$ of $d \in \mathcal{D}$ ensure that the values $d(\alpha_i, \alpha_i)$ determine a probability distribution on the boolean algebra generated by the $\{\alpha_i\}_{i=1}^{\dim \mathcal{V}}$.

1.1.2 The Classification Theorem

We want to base our investigations of symmetries on the expression $(1.7)$ of the decoherence functional. But in order to do so, we must first be sure that it is not only a lucky
coincidence that we are able to cast this particular decoherence functional for the history version of quantum mechanics in the above form. The formalism for decoherent histories is not simply restricted to models with unitary evolution, inclusions of final density matrices and the like. Therefore, in order to formulate a notion of symmetry that is valid for all these cases, we have to find out whether every decoherence functional—i.e. every functional satisfying the properties (1.8) listed above—can be written in the form (1.7). The clear cut answer to this is given by the following theorem, see [8], which is valid for any theory in which the history propositions are given by projectors on a finite–dimensional Hilbert space $V$.

**Theorem** [8] If $\dim V > 2$, decoherence functionals $d$ are in one-to-one correspondence with operators $X = X_1 + iX_2$ on $V \otimes V$ according to the rule:

$$d(\alpha, \beta) = \text{tr}_{V \otimes V}(\alpha \otimes \beta X)$$

(1.10)

with the restriction that

\begin{align*}
  a) & \quad X^\dagger = MXM \quad \text{with} \quad M(|v \rangle \otimes |w \rangle) := |w \rangle \otimes |v \rangle, \quad \forall |v \rangle, |w \rangle \in V. \quad (1.11) \\
  b) & \quad \text{tr}_{V \otimes V}(\alpha \otimes \alpha X_1) \geq 0 \quad (1.12) \\
  c) & \quad \text{tr}_{V \otimes V}(X_1) = 1. \quad (1.13)
\end{align*}

The restrictions on the operator $X$ on $V \otimes V$ reflect the requirements (1.8). We denote by $\mathcal{X}_D$ the set of all such operators $X$. This theorem—which has been extended to arbitrary von Neumann algebras without factor of type II in [9]—is the cornerstone of the forthcoming investigation. It allows us to shift the investigation of the properties of decoherence functionals $d \in \mathcal{D}$, where $\mathcal{D}$ denotes the space of all decoherence functionals, to an analysis of the properties of the associated operator $X_d \in \mathcal{X}_D$, which, as we emphasize once again, carries all of the ‘dynamical’ content as well as the ‘initial conditions’ of the model under investigation.

It is important to understand the origin of these requirements. Condition (1.11) reflects the hermiticity requirement. The action of $M$ on $\alpha \otimes \beta$ is given by $M(\alpha \otimes \beta)M = (\beta \otimes \alpha)$. Equation (1.11) follows then from the condition

$$d(\alpha, \beta) = \text{tr}_{V \otimes V}(\alpha \otimes \beta X) = \text{tr}_{V \otimes V}(\beta \otimes \alpha X^\dagger) = d(\beta, \alpha)^*.$$ 

Condition (1.12) stems from the fact that, since $X^\dagger = MXM$ is equivalent to the pair of conditions

$$\begin{align*}
  X_1 &= MX_1 M \\
  X_2 &= -MX_2 M,
\end{align*}$$

(1.14)

it follows that

\begin{align*}
  \text{tr}_{V \otimes V}(\alpha \otimes \alpha X_2) &= -\text{tr}_{V \otimes V}(\alpha \otimes \alpha MX_2 M) \\
  &= -\text{tr}_{V \otimes V}(M(\alpha \otimes \alpha)MX_2) = -\text{tr}_{V \otimes V}(\alpha \otimes \alpha X_2).
\end{align*}
and so $\text{tr}_{V \otimes V}(\alpha \otimes \alpha X_2) = 0$ is implied already by the hermiticity requirement \([1.11]\).

It will be advantageous later to think of $d(\alpha, \beta)$ as the value of the complex–valued functional

$$\text{tr} : \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \times \mathcal{X}_\mathcal{D} \rightarrow \mathbb{C} \quad (\alpha \otimes \beta; X_d) \mapsto \text{tr}_{V \otimes V}(\alpha \otimes \beta X_d) \quad (1.16)$$

1.2 Wigner’s Theorem

In order to understand fully the importance of Wigner’s result it is crucial to distinguish between the notion of a symmetry and that of a physical symmetry.

**Definition** On a complex Hilbert space $\mathcal{H}$ a symmetry is a unitary or anti–unitary operator $U$. Thus it leaves invariant the modulus of the inner product of any pair of two vectors $|v\rangle, |w\rangle \in \mathcal{H}$, that is

$$|\langle v, w \rangle|^2 = |\langle Uv, Uw \rangle|^2, \quad \forall |v\rangle, |w\rangle \in \mathcal{H}. \quad (1.17)$$

This definition is only a mathematical one; it has, a priori, no motivation by physical arguments. Note also that it does not impose any further defining properties on $U$, such as commutativity with the Hamiltonian operator. Such requirements only enter at a much later stage, motivated by analogues of the assumption in classical mechanics, that a particular physical system evolves along the flowlines of a specific, Hamiltonian, vectorfield.

On the other hand, a physical system at a fixed moment of time is described in quantum mechanics by a state $\sigma \in \mathcal{S} : \mathcal{P}(\mathcal{H}) \rightarrow \mathbb{R}$, that is, a normalized ($\sigma(1) = 1$), positive–valued functional which is additive on disjoint projectors. The states are, via Gleason’s theorem, in one–to–one correspondence with density operators $\rho$ on $\mathcal{H}$ according to the rule

$$\sigma(\alpha, \rho) = \text{tr}_\mathcal{H}(\alpha \rho), \quad \forall \alpha \in \mathcal{P}(\mathcal{H}), \quad (1.18)$$

where $\rho$ is defined by the properties

$$\rho = \rho^\dagger, \quad \rho \geq 0 \quad \text{tr}_\mathcal{H}(\rho) = 1. \quad (1.19)$$

The set of all density operators is often denoted by $\mathcal{W}_\mathcal{S}$.

The physical assumption is now that all that matters are the relations among the states, which can be entirely described by means of their overlaps, that is the transition amplitudes $\text{tr}_\mathcal{H}(\rho_1 \rho_2)$.

In the finite–dimensional case the self–adjointness of $\rho$ implies that for every density operator there exists an orthonormal basis $\{|\psi_i\rangle\}$ such that

$$\rho = \sum r_i P_{|\psi_i\rangle}, \quad \sum r_i = 1 \quad r_i \geq 0. \quad (1.20)$$
Density operators of the form $P_{|\Psi_i\rangle} := |\Psi_i\rangle\langle\Psi_i| \in \mathcal{W}_{S^p}$ are called pure. Every density operator possesses an expansion in terms of pure density operators, also representing ‘rays’ of $\mathcal{H}$; instead of $\mathcal{W}_{S^p}$ we also use sometimes the notation $\mathcal{R}(\mathcal{H})$. Therefore, the invariance requirement on the transition amplitude between two arbitrary states $\rho_1, \rho_2$ can be reduced to the requirement that

$$\text{tr}_\mathcal{H}(P_{|\Psi_1\rangle} P_{|\Psi_2\rangle}) = \text{tr}_\mathcal{H}(P^\xi_{|\Psi_1\rangle} P^\xi_{|\Psi_2\rangle}),$$

(1.21)

for arbitrary one-dimensional pure density operators $P_{|\Psi_1\rangle}, P_{|\Psi_2\rangle}$ and an affine map $\xi : \mathcal{W}_{S^p} \to \mathcal{W}_{S^p}$, that is a map satisfying $\xi(\sum_i c_i P_{|\Psi_i\rangle}) = \sum_i c_i \xi(P_{|\Psi_i\rangle}) \equiv \sum_i c_i P^\xi_{|\Psi_i\rangle}$, $c_i \in \mathbb{C}$.

**Definition** A physical symmetry is an affine bijection $\xi : \mathcal{W}_{S^p} \to \mathcal{W}_{S^p}; P_{|\Psi\rangle} \mapsto P^\xi_{|\Psi\rangle}$, such that the transition amplitude between pure density operators remains invariant, i.e. that

$$\text{tr}_\mathcal{H}(P_{|\Psi_1\rangle} P_{|\Psi_2\rangle}) = \text{tr}_\mathcal{H}(P^\xi_{|\Psi_1\rangle} P^\xi_{|\Psi_2\rangle}), \quad \forall |\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}. \quad (1.22)$$

**Theorem** [10] (Wigner) Every symmetry induces a physical symmetry and, conversely, every one-to-one map $\xi : \mathcal{W}_{S^p} \to \mathcal{W}_{S^p}$ preserving orthogonality between rays is a physical symmetry and can be implemented by a unitary or anti-unitary operator $U$ on $\mathcal{H}$.

## 2 Symmetry and History Quantum Theory

The notion of symmetry discussed in the last section arose from discussing quantum mechanics at a single, fixed time point $t \in \mathbb{R}$ with a corresponding Hilbert space $\mathcal{H}$. At a single time point physics is described in terms of the pair $(\mathcal{S}, \mathcal{L})$, where $\mathcal{S}$ is the set of states and $\mathcal{L}$ is the lattice of projection operators on $\mathcal{H}$. In order to define physical symmetries in quantum mechanics—in the sense specified by Wigner—only the knowledge of one part of this pair was required, namely the knowledge of the properties of the set of states $\mathcal{S}$, via the map

$$\text{tr} : \mathcal{W}_{S^p} \times \mathcal{W}_{S^p} \to \mathbb{R} ; \quad (P_{|\Psi_1\rangle}; P_{|\Psi_2\rangle}) \mapsto \text{tr}_\mathcal{H}(P_{|\Psi_1\rangle} P_{|\Psi_2\rangle}). \quad (2.1)$$

The properties of the lattice of propositions $\mathcal{L} = \mathcal{P}(\mathcal{H})$ did not enter in full.

In order to arrive at a notion of symmetry for HQTs, recall that in a history quantum theory the pair $(\mathcal{UP}, \mathcal{D})$ can be seen as a formal analogue of the pair $(\mathcal{L}, \mathcal{S})$. Comparing the two expressions

$$\text{tr}_\mathcal{H}(\rho_1 \rho_2) \quad \text{and} \quad \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X_d) \quad (2.2)$$

reveals immediately the mathematical difference between them. In contrast to the quantum mechanical case at a single time point, in HQTs the map

$$\text{tr} : \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \times \mathcal{X}_D \to \mathbb{C} ; \quad (\alpha \otimes \beta) \times X_d \mapsto \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X_d) \quad (2.3)$$

would collapse to a single value.
intertwines the properties of $\mathcal{UP} = \mathcal{P}(\mathcal{V})$ and $\mathcal{XD}$. This is not surprising, since the classification theorem is more to be regarded as an analogue of Gleason’s theorem, which in a similar manner intertwines properties of $\mathcal{L}$ and $\mathcal{S}$.

The invariance requirement for the expression $\text{tr}_H(P_{|\Psi\rangle} P_{|\Phi\rangle})$ has a direct physical meaning. In HQTs the formal analogue of a density operator $\rho$ is an operator $X_d \in \mathcal{XD}$ so that the first guess for symmetries might be to look for transformations which leave ‘transition amplitudes between different $X_d$’ invariant. But such a requirement would be hard to interpret since HQT deal with ‘history propositions’ as entities in their own right. The theory is completely specified by choosing a particular decoherence functional, which is kept fixed throughout. Since the notion of ‘time’ in a specific history quantum theory is determined by the choice of the structure of the space of history propositions—for example, the nature of ‘time’ as a parameter $t \in \mathbb{R}$ in quantum mechanics is mirrored in the definition of $\mathcal{P}(\mathcal{V}_t) = \mathcal{P}(\otimes_{i=1}^n \mathcal{H}_t)$—and decoherence functionals associate numbers with these pairs of history propositions as an entity, a change of the decoherence functional must not occur.

How can we nonetheless use, at least at the mathematical level, the existing notion of a physical symmetry and later on Wigner’s theorem, to define a corresponding notion for HQT that does not suffer from the difficulty mentioned above? The main idea is to characterize the notion of a physical symmetry in a form which exploits the pairing (1.18) between density operators $\rho \in \mathcal{W}_S$ and propositions $\alpha \in \mathcal{P}(\mathcal{H})$ given by Gleason’s theorem.

### 2.1 Alternative Specification of Physical Symmetries

We start by neglecting entirely the considerations from which the expression

$$\text{tr}_H(P_{|\Psi\rangle} P_{|\Phi\rangle})$$

originally arose. Pure density operators belong trivially to the space of projection operators $\mathcal{P}(\mathcal{H})$ and therefore, instead of thinking of the map (2.1) as a pairing between states one can think of it as a map

$$\text{tr} : \{\mathcal{R}(\mathcal{H}) \subset \mathcal{P}(\mathcal{H})\} \times \mathcal{W}_{SP} \longrightarrow \mathbb{R}$$

$$P_{|\Psi\rangle} \times P_{|\Phi\rangle} \mapsto \text{tr}_H(P_{|\Psi\rangle} P_{|\Phi\rangle})$$

that establishes a pairing between a subset of the space of propositions and the set of pure states. Therefore, we see immediately that Wigner’s result can be read as follows: Wigner’s theorem determines all bijections

$$\xi : \mathcal{R}(\mathcal{H}) \times \mathcal{W}_{SP} \rightarrow \mathcal{R}(\mathcal{H}) \times \mathcal{W}_{SP}$$

$$(P_{|\Phi\rangle}, P_{|\Psi\rangle}) \mapsto (P^\xi_{|\Phi\rangle}, P^\xi_{|\Psi\rangle}),$$

that leave invariant the pairing

$$\text{tr}_H(P_{|\Psi\rangle} P_{|\Phi\rangle}) = \text{tr}_H(P^\xi_{|\Psi\rangle} P^\xi_{|\Phi\rangle}).$$
Now, when seen from this perspective it is natural to ask whether or not Wigner’s theorem specifies completely all affine one-to-one maps

\[ V : \mathcal{P}(\mathcal{H}) \times \mathcal{W}_S \rightarrow \mathcal{P}(\mathcal{H}) \times \mathcal{W}_S \]  
\[ (\alpha, \rho) \mapsto (\alpha^V, \rho^V) \]  

such that the pairing between propositions and density operators is left invariant for all \( \alpha \in \mathcal{P}(\mathcal{H}) \) and all \( \rho \in \mathcal{W}_S \), i.e.

\[ \text{tr}_\mathcal{H}(\alpha\rho) = \text{tr}_\mathcal{H}(\alpha^V\rho^V). \]  

The map is required to be affine since the space \( \mathcal{W}_S \) is a convex space. Convex combinations of elements of \( \mathcal{W}_S \) are again density operators.

Note that the question posed is not trivial: The space of projectors \( \mathcal{P}(\mathcal{H}) \) is a disjoint union of compact Grassmann manifolds and therefore allows for a much wider class of transformation than just unitary or anti-unitary operators \( U \) on \( \mathcal{H} \). The three conditions these maps have to satisfy are:

\begin{itemize}
  \item \( \star \) \( V : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}) \) \hspace{1cm} (2.10)
  \item \( \star \) \( V : \mathcal{W}_S \rightarrow \mathcal{W}_S \) \hspace{1cm} (2.11)
  \item \( \star \) \( \text{tr}_\mathcal{H}(\alpha\rho) = \text{tr}_\mathcal{H}(\alpha^V\rho^V) \) \hspace{1cm} (2.12)
\end{itemize}

If we consider transformations on \( \mathcal{P}(\mathcal{H}) \), the interesting transformations are given by transforming projectors of different dimensions to each other. So consider, for example, the transformation:

\[ G : \mathcal{P}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}) \]  
\[ \alpha \mapsto G[\alpha] \]  

whereby a particular one-dimensional projector \( P_\Phi \) is mapped into an \( m \)-dimensional one, \( G[P_\Phi] \), \( m > 1 \). Such a transformation might be bijective on \( \mathcal{P}(\mathcal{H}) \) and even obey requirement (2.12).

Now, regarding \( P_\Phi \) as a pure density operator, we see immediately that the trace of its image under \( G \) is \( \text{tr}_\mathcal{H}(G[P_\Phi]) = m \). Therefore, such a map does not comply with the requirement (2.13). Thus, only maps which map rays into rays are allowed and, therefore, all maps obeying the conditions (2.8, 2.9) are determined by Wigner’s theorem.

This reformulation of ‘physical symmetries’ in terms of the intersection of different sets of transformations fulfilling (2.10), (2.11) or (2.12) respectively possesses the advantage of never having to consider ‘transition amplitudes between states at a fixed moment of time’. Physical symmetries just preserve the intertwining between \( (\mathcal{L}, \mathcal{S}) \) via Gleason’s theorem.
by transforming $\mathcal{L}$ and $\mathcal{S}$ by the same transformation into itself.

This fact justifies trying to define symmetries of a history quantum theory by exact analogues of the requirements (2.10 – 2.12), i.e. by replacing

$$\{\mathcal{P}(\mathcal{H}); W_S; \operatorname{tr}_\mathcal{H}(\alpha\rho)\} \leftrightarrow \{\mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}); \mathcal{X}_\mathcal{D}; \operatorname{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta X_d)\}. \quad (2.14)$$

This notion of a ‘symmetry of a history quantum theory’ does not suffer from the interpretative difficulty mentioned above.

Through this choice, we will build in an invariance requirement for the values $d(\alpha, \beta)$ of the decoherence functional from the very start. Some physical arguments for such a choice can be found in [4, 11, 12]. I will discuss its relevance at various stages in this paper.

### 2.2 Definition and Proposition

**Definition** A physical symmetry of a history quantum theory (PSHQT) is any affine one-to-one map

$$Q : \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \times \mathcal{X}_\mathcal{D} \rightarrow \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \times \mathcal{X}_\mathcal{D} \quad (2.15)$$

that preserves the value of the pairing between history propositions and operators associated with decoherence functionals, i.e.

$$\operatorname{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta X_d) = \operatorname{tr}_{\mathcal{V}\otimes\mathcal{V}}([\alpha \otimes \beta]^{Q} X_d^{Q}). \quad (2.16)$$

We state once again the three requirements, such a map has to fulfill:

1. $Q : \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{V}) \otimes \mathcal{P}(\mathcal{V}) \quad (2.17)$
2. $Q : \mathcal{X}_\mathcal{D} \rightarrow \mathcal{X}_\mathcal{D} \quad (2.18)$
3. $\operatorname{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta X_d) = \operatorname{tr}_{\mathcal{V}\otimes\mathcal{V}}([\alpha \otimes \beta]^{Q} X_d^{Q}) \quad (2.19)$

Each condition separately determines a set of transformations, but only the intersection of these sets may be called a PSHQT. The word physical is chosen since this definition parallels the one for physical symmetries given by Wigner. Again, the history propositions $\alpha \in \mathcal{P}(\mathcal{V})$ and the decoherence functionals, represented by $X_d \in \mathcal{X}_\mathcal{D}$, are transformed together by the same transformation. We call two history quantum theories that are related by a physical symmetry of a history quantum theory physically equivalent. Furthermore, as will be shown later, this definition encompasses the notion of ‘physical equivalence’, first introduced and justified through physical arguments by Gell–Mann and Hartle in [4]. It is easy to see that the following Lemma holds.
Lemma The relation among two history quantum theories \((hqt_1, hqt_2)\) of being physically equivalent, denoted by \(hqt_1 \sim hqt_2\), is an equivalence–relation. Thus it is (i) reflexive: \(hqt_1 \sim hqt_1\), (ii) symmetric: \(hqt_1 \sim hqt_2 \Rightarrow hqt_2 \sim hqt_1\) and (iii) transitive: \((hqt_1 \sim hqt_2) \text{ and } (hqt_2 \sim hqt_3) \Rightarrow (hqt_1 \sim hqt_3)\).

There is an obvious class of transformations on \(V \otimes V\) that fulfills all three conditions (2.17 – 2.18) stated above:

**Definition** A homogeneous symmetry on \(V \otimes V\) is a unitary operator \(\hat{U} \otimes \hat{U}\) where \(\hat{U}\) may be a unitary or anti–unitary operator on \(V\).

**Lemma** Every homogeneous symmetry induces a PSHQT, i.e. \(\{HS\} \subset \{PSHQT\}\).

**Proof**

A homogeneous symmetry induces the maps

\[
\begin{align*}
\alpha \otimes \beta & \mapsto \hat{U} \otimes \hat{U} (\alpha \otimes \beta) \hat{U}^\dagger \otimes \hat{U}^\dagger \quad \text{(2.20)} \\
X_d & \mapsto \hat{U} \otimes \hat{U} X_d \hat{U}^\dagger \otimes \hat{U}^\dagger
\end{align*}
\]

so that for all \(\alpha \otimes \beta \in P(V) \otimes P(V)\) and all \(X_d \in \mathcal{X}_D\) it holds that

\[
d(\alpha, \beta) = \text{tr}_{V \otimes V}[(\hat{U} \alpha \hat{U}^\dagger \otimes \hat{U} \beta \hat{U}^\dagger)(\hat{U} \otimes \hat{U} X_d \hat{U}^\dagger \otimes \hat{U}^\dagger)]. \quad \text{(2.21)}
\]

One can easily check that \(\hat{U} \otimes \hat{U} X_d \hat{U}^\dagger \otimes \hat{U}^\dagger\) fulfills the defining properties for an operator \(X_d' \in \mathcal{X}_D\) given by the classification theorem. \(\square\)

Homogeneous symmetries possess a different characterisation, that can easily be derived from Wigner’s theorem for quantum mechanics. Recall the definition of the map \(M\) used in the classification theorem for decoherence functionals,

\[
M : \quad V \otimes V \to V \otimes V \quad \text{(2.22)}
\]

\[
|u\rangle \otimes |v\rangle \mapsto |v\rangle \otimes |u\rangle,
\]

for all \(|u\rangle, |v\rangle \in V \otimes V\). As a result, its action on projection operators of the form \(\alpha \otimes \beta\) is given by

\[
M(\alpha \otimes \beta)M = (\beta \otimes \alpha), \quad \forall \alpha \otimes \beta \in P(V) \otimes P(V). \quad \text{(2.23)}
\]

In particular, this holds true for \(\alpha, \beta \in \mathcal{R}(V)\), i.e. projection operators belonging to the space \(\mathcal{R}(V)\) of rays of \(V\).

Let \(\tau = \tau_1 \otimes \tau_2\) denote a map

\[
\tau : \mathcal{R}(V) \otimes \mathcal{R}(V) \to \mathcal{R}(V) \otimes \mathcal{R}(V) \quad \text{(2.24)}
\]

\[
P_{|\psi\rangle} \otimes P_{|\phi\rangle} \mapsto [P_{|\psi\rangle} \otimes P_{|\phi\rangle}]^\tau := P_{|\phi\rangle}^\tau_1 \otimes P_{|\psi\rangle}^\tau_2,
\]
where \( \tau_1, \tau_2 \) denote transformations on the space of pure density operators. A map \( \tau \) is said to \textit{commute} with \( M \), if \((M \circ \tau)(P_\psi \otimes P_\phi) = (\tau \circ M)(P_\psi \otimes P_\phi)\) for all elements \(P_\psi \otimes P_\phi \in \mathcal{R}(V) \otimes \mathcal{R}(V)\), written symbolically as \([\tau, M] = 0\).

**Definition** A homogeneous symmetry of a history quantum theory (HSHQT) is a one-to-one map \( \tau : \mathcal{R}(V) \otimes \mathcal{R}(V) \rightarrow \mathcal{R}(V) \otimes \mathcal{R}(V) \) that preserves the transition amplitude between two elements, i.e.

\[
\text{tr}_{V \otimes V}([P_\psi_1 \otimes P_\phi_1][P_\psi_2 \otimes P_\phi_2]) = \text{tr}_{V \otimes V}([P_\psi_1 \otimes P_\phi_1]^{\tau}[P_\psi_2 \otimes P_\phi_2]^{\tau}),
\]

and commutes with the map \( M \), that is \([\tau, M] = 0\).

**Proposition** Every homogeneous symmetry induces a HSHQT and, conversely, every one-to-one map \( \tau : \mathcal{R}(V) \otimes \mathcal{R}(V) \rightarrow \mathcal{R}(V) \otimes \mathcal{R}(V) \) that preserves orthogonality between the rays and commutes with \( M \) is a HSQHT and can be implemented by a unitary or anti-unitary operator \( \hat{U} \otimes \hat{V} \) on \( V \otimes V \). Symbolically,

\[
\{\text{HSHQT}\} \cong \{\text{HS}\}. \tag{2.25}
\]

**Proof**

The transition amplitude between two elements \([P_\psi_1 \otimes P_\phi_1], [P_\psi_2 \otimes P_\phi_2] \in \mathcal{R}(V) \otimes \mathcal{R}(V)\) is given by

\[
\text{tr}_{V \otimes V}([P_\psi_1 \otimes P_\phi_1][P_\psi_2 \otimes P_\phi_2]) = \text{tr}_{V}(P_\psi_1 P_\psi_2) \text{tr}_{V}(P_\phi_1 P_\phi_2). \tag{2.25}
\]

Therefore, by Wigner’s theorem, all transformations preserving orthogonality and the transition amplitude can be implemented by operators \( \hat{U} \otimes \hat{V} \), where \( \hat{U} \) and \( \hat{V} \) are either unitary or anti-unitary operators on \( V \). Requiring these transformations to commute with \( M \) concludes the proof.

**Remark:** It is important to understand why only transformations of the form \( \hat{U} \otimes \hat{U} \) are admitted, and not, for example, operators of the form

\[
\sum_i c_i \hat{U}_i \otimes \hat{U}_i, \quad \sum_i c_i = 1, \quad c_i \in \mathbb{R}. \tag{2.26}
\]

The reason is, that, starting from \( d(\alpha, \beta) \) only those transformations \( \tilde{T} : \alpha \otimes \beta \mapsto \tilde{T}(\alpha \otimes \beta) \tilde{T}^\dagger \) on \( V \otimes V \) are allowed for which

\[
\tilde{T}(\alpha \otimes \beta) \tilde{T}^\dagger = \alpha' \otimes \beta'. \tag{2.27}
\]

It is possible for these transformations only to write them as \( d(\alpha, \beta) \mapsto d'(\alpha', \beta') \).

We have therefore established the following relation among the three sets of transformations:

\[
\{\text{HSHQT}\} \cong \{\text{HS}\} \subset \{\text{PSHQT}\}. \tag{2.28}
\]
We argued that PSHQT determined by conditions (2.17 – 2.19) is an appropriate notion for symmetries of history quantum theories. What we want to show now is that all PSHQT are given by homogeneous symmetries of the form $\hat{U} \otimes \hat{U}$. In view of (2.28) it remains to be shown that $\{HS\} \supset \{PSHQT\}$.

2.3 The structure of $\mathcal{D}$

In order to show that all PSHQT can be characterized by means of rays in $\mathcal{R}(\mathcal{V}) \otimes \mathcal{R}(\mathcal{V})$ we have to discuss in more detail the structure of the space of decoherence functionals. Comparison with the case in standard quantum mechanics shows that what we now have to look for is a notion of ‘elementary decoherence functionals’, out of which all other decoherence functionals can be build by a certain superposition. The requirement (2.19) for PSHQT,

$$\text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X_d) = \text{tr}_{\mathcal{V} \otimes \mathcal{V}}([\alpha \otimes \beta]^Q X_d^Q),$$

can then be reduced to a requirement that has to hold only for all elementary decoherence functionals. We start our investigation with the following observation:

**Lemma** For any finite set $\{d^{(i)}\}_{i=1}^n, d^{(i)} \in \mathcal{D}$, it holds that $d := \sum_i r_i d^{(i)} \in \mathcal{D}, r_i \in \mathbb{R}$, provided that:

$$r_i \in \mathbb{R} \quad \forall i \in \{1, 2, \ldots, n\},$$

$$\sum_i r_i d^{(i)}(\alpha, \alpha) \geq 0 \quad \forall \alpha \in UP,$$

$$\sum_i r_i = 1.$$

These conditions reflect the requirements for $d$ of hermiticity, positivity and normalization. We call such superpositions of decoherence functionals a *weak convex combination* of decoherence functionals. All convex combinations are weak convex combinations but the converse is not true. For a convex combination it is required that $r_i \geq 0$; the second condition in (2.29) does not imply $r_i \geq 0$. It seems natural to look for so called ‘pure decoherence functionals’ which can not be written as weak convex combination of other decoherence functionals. An argument first given by N. Linden [13] shows that any decoherence functional can be written as the sum of two other decoherence functionals. Thus there can be no pure decoherence functionals. Nonetheless, in this context we are only interested in a convenient expansion of an arbitrary decoherence functional by what I will call *elementary decoherence functionals*. This will suffice to prove an analogue of Wigner’s theorem in the next section. I will show explicitly how these elementary decoherence functionals reflect Linden’s argument. The same notions apply *mutatis mutandis* for the associated operators $X_d \in \mathcal{X}_D$. 

13
By the classification theorem \([8]\) we know that for every decoherence functional \(d \in \mathcal{D}\) its associated operator \(X_d\) can be written as a sum of two self–adjoint operators \(X_d = X_1 + iX_2\) subject to the conditions

\[
X_1 = MX_1 M; \quad X_2 = -MX_2 M; \quad \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \alpha X_1) \geq 0; \quad \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(X_1) = 1. \tag{2.30}
\]

We seek an expansion for the real part \(X_1\) and the imaginary part \(X_2\) of \(X_d\) as a weak convex combination of decoherence functionals \(d^e\) that can not be written as a weak convex combination.

**Proposition** For each \(X = X_1 + iX_2 \in \mathcal{X}_\mathcal{D}\) there exist two ONB \(\{|e_i\}, \{|b_i\}\}\) on \(\mathcal{V}\) such that \(X\) can be written as:

\[
X = \sum_{i,j} \lambda_{ij} X_1^{(ij)} + i \sum_{l,m} \kappa_{lm} X_2^{[lm]} \tag{2.31}
\]

where

\[
X_1^{(ij)} = \frac{1}{2}(P_{e_i} \otimes P_{e_j} + P_{e_j} \otimes P_{e_i}); \tag{2.32}
\]

\[
\lambda_{ij} = \lambda_{ji}, \quad \sum_{i,j} \lambda_{ij} = 1, \quad \sum_{i,j} a_{ii} \lambda_{ij} a_{jj} \geq 0, \quad \lambda_{ij} \in \mathbb{R}
\]

and

\[
X_2^{[lm]} = \frac{1}{2}(P_{b_l} \otimes P_{b_m} - P_{b_m} \otimes P_{b_l}); \tag{2.33}
\]

\[
\kappa_{lm} = -\kappa_{ml}, \quad \kappa_{lm} \in \mathbb{R}.
\]

**Remark:** The positivity requirement \(\text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \alpha X_1) \geq 0\) gives rise to the condition \(\sum_{i,j} a_{ii} \lambda_{ij} a_{jj} \geq 0\) for an arbitrary projector \(\alpha = \sum_{i,j} a_{ij} |e_i\rangle \langle e_j|\) on \(\mathcal{V}\) when expanded in the basis \(\{|e_i\}, \{|b_i\}\}\).

**Proof**

The proof is a constructive one; it follows the proof of the classification theorem in \([8]\).

For each \(\alpha \in \mathcal{P}(\mathcal{V})\) define a function \(d_\alpha(\beta) : \mathcal{P}(\mathcal{V}) \to \mathbb{C}\) where \(d_\alpha(\beta) := d(\alpha, \beta)\). Let \(\Re d_\alpha\) and \(\Im d_\alpha\) denote the real and imaginary parts of \(d_\alpha\), so that

\[
d_\alpha(\beta) = \Re d_\alpha(\beta) + i\Im d_\alpha(\beta) \tag{2.34}
\]

with \(\Re d_\alpha(\beta) \in \mathbb{R}\) and \(\Im d_\alpha(\beta) \in \mathbb{R}\). We will develop the argument only for the real part \(\Re d_\alpha(\beta)\). The biadditivity condition on the \(d \in \mathcal{D}\) requires that \(\Re d_\alpha(\beta_1 \oplus \beta_2) = \Re d_\alpha(\beta_1) + \Re d_\alpha(\beta_2)\) for any orthogonal pair of projectors \(\beta_1, \beta_2\). Since \(d\) is assumed to be bounded, the same holds true for its real part \(\Re d_\alpha\). For any \(r \in \mathbb{R}\), the quantity

\[
\kappa_r(\beta) := r \dim(\beta) = r \text{tr}(\beta) \tag{2.35}
\]

is a real additive function of \(\beta\), and hence so are \(\Re d_\alpha + \kappa_r\) for any \(r \in \mathbb{R}\).
In [8] it was shown that there exists for each \( \alpha \in \mathcal{P}(V) \) two real numbers \( r_\alpha, \mu_\alpha \in \mathbb{R} \) such that there exists a density operator \( \rho^\mathbb{R}_\alpha \) on \( V \) for which it holds that

\[
\Re d_\alpha(\beta) = \text{tr}_V((\frac{1}{\mu_\alpha}\rho^\mathbb{R}_\alpha - r_\alpha)\beta) = \text{tr}_V(Y^\mathbb{R}_\alpha \beta)
\]

(2.36)

where \( Y^\mathbb{R}_\alpha := \frac{1}{\mu_\alpha}\rho^\mathbb{R}_\alpha - r_\alpha \). Since \( \rho^\mathbb{R}_\alpha \) is a density operator, there exists an orthonormal basis \( \{|e_i\}\}_{i=1}^{\text{dim}V} \) and positive numbers \( w_\alpha^i \in \mathbb{R} \) such that \( \rho^\mathbb{R}_\alpha = \sum_i w_\alpha^i P_{|e_i\rangle} \) and therefore

\[
Y^\mathbb{R}_\alpha = \sum_i (\frac{w_\alpha^i}{\mu_\alpha} - r_\alpha) P_{|e_i\rangle}.
\]

(2.37)

The additivity condition \( d(\alpha_1 \oplus \alpha_2, \beta) = d(\alpha_1, \beta) + d(\alpha_2, \beta) \) implies that

\[
\text{tr}_V(Y^\mathbb{R}_{\alpha_1 \oplus \alpha_2} \beta) = \text{tr}_V(Y^\mathbb{R}_{\alpha_1} \beta) + \text{tr}_V(Y^\mathbb{R}_{\alpha_2} \beta)
\]

(2.38)

which, since it is true for all \( \beta \in \mathcal{P}(V) \) (and hence for all operators on \( V \)), implies that the operator-valued map \( \alpha \mapsto Y^\mathbb{R}_\alpha \) is itself additive in the sense that

\[
Y^\mathbb{R}_{\alpha_1 \oplus \alpha_2} = Y^\mathbb{R}_{\alpha_1} + Y^\mathbb{R}_{\alpha_2}
\]

(2.39)

for all disjoint pairs of projectors \( \alpha_1, \alpha_2 \) on the Hilbert space \( V \).

Let \( \{|c_i\|_{i=1}^{\text{dim}V} \) be a orthonormal basis of \( V \); let \( \{|\langle c_i|\}_{i=1}^{\text{dim}V} \) denote its dual basis. Let \( \{B_{ij} := |c_i\langle c_j|; \quad i, j = 1, 2, \ldots, N\} \) be a vector-space basis for the operators on \( V \), so that the operators \( Y^\mathbb{R}_\alpha \) can be expanded as \( Y^\mathbb{R}_\alpha = \sum_{i,j=1}^{\text{dim}V} y^R_{ij}(\alpha) B_{ij} \). Then relation (2.39) shows that the complex expansion coefficients \( y^R_{ij}(\alpha), i, j = 1, 2, \ldots, \text{dim}V \) must satisfy the additivity condition:

\[
y^R_{ij}(\alpha_1 \oplus \alpha_2) = y^R_{ij}(\alpha_1) + y^R_{ij}(\alpha_2).
\]

(2.40)

Since \( Y^\mathbb{R}_\alpha \) is a bounded operator, its expansion coefficient functions \( \alpha \mapsto y^R_{ij}(\alpha) \forall \alpha \in \mathcal{P}(V) \) are bounded as well. It was shown [8] that there exists an operators \( \Lambda^R_{ij} \) on \( V \) such that:

\[
y_{ij}(\alpha) = \text{tr}_V(\alpha \Lambda^R_{ij}),
\]

(2.41)

and therefore:

\[
Y_\alpha = \sum_{i,j=1}^{N} \text{tr}_V(\alpha \Lambda^R_{ij}) B_{ij}.
\]

(2.42)

In particular,

\[
\Re d(\alpha, \beta) = \text{tr}_V \left\{ \sum_{i,j} \left( \text{tr}_V(\alpha \Lambda^R_{ij}) B_{ij} \beta \right) \right\} = \sum_{i,j} \text{tr}_V(\alpha \Lambda^R_{ij}) \text{tr}_V(B_{ij} \beta).
\]

(2.43)

We define an operator \( X^\mathbb{R} \) on \( V \otimes V \) by

\[
X^\mathbb{R} := \sum_{ij} \Lambda^R_{ij} \otimes B_{ij}
\]

(2.44)
for which it holds that $\Re d(\alpha, \beta) = \text{tr}_{\mathcal{V}} \text{tr} (\alpha \otimes \beta X^R)$.

From now on we choose the particular set of $\{|e_i\rangle\}$ of eigenvectors of the operator $\rho^R_\alpha$ as an orthonormal basis for $\mathcal{V}$, i.e. $B_{ij} = |e_i\rangle\langle e_j|$. As an operator on $\mathcal{V}$, $\Lambda^R_{ij}$ possesses an expansion

$$\Lambda^R_{ij} = \sum_{k,l} \lambda^R_{ij} B_{kl}, \quad (2.45)$$

so that

$$X^R = \sum_{i,j,k,l} \lambda^R_{ij} B_{kl} \otimes B_{ij}. \quad (2.46)$$

However, from equation (2.37) we see that, by using the basis $\{|e_i\rangle\}$, this sum reduces to

$$X^R = \sum_{i,k,l} \lambda^R_{ii} B_{kl} \otimes B_{ii}, \quad (2.47)$$

since only the $B_{ii} = P_{|e_i\rangle}$ appear in the expansion of $Y^R_\alpha$.

Remember now, that $X^R$ stands for the real part $X_1$ of $X_d$, the operator associated with a decoherence functional $d \in \mathcal{D}$. As such it has to fulfill that

$$X^R = MX^R M, \quad (2.48)$$

where $M$ was defined through the action $M(A \otimes B)M = (B \otimes A)$ for arbitrary operator $A, B$ on $\mathcal{V}$. This requirement is strong enough to reduce (2.47) to

$$X^R = \sum_{i,k} \lambda^R_{kk} B_{kk} \otimes B_{ii}. \quad (2.49)$$

Since $B_{ii} = |e_i\rangle\langle e_i| = P_{|e_i\rangle}$, we see that the real part $X^R \equiv X_1$ of the operator $X_d$ associated with a decoherence functional $d \in \mathcal{D}$ can be written as

$$X_1 \equiv X^R = \sum_{i,j} \lambda_{ij} P_{|e_i\rangle} \otimes P_{|e_j\rangle}, \quad (2.50)$$

where $\lambda_{ij} := \lambda^R_{jj}$. It is easy to see that these coefficient must obey

$$\lambda_{ij} = \lambda_{ji}, \quad \sum_{i,j} \lambda_{ij} = 1, \quad \sum_{i,j} a_{ij} \lambda_{ij} a_{jj} \geq 0, \quad (2.51)$$

which follow from the requirements of hermiticity, normalization and positivity. The $a_{ii} \in \mathbb{R}$ are expansion coefficients of an arbitrary projector $\alpha = \sum_{ij} a_{ij} |e_i\rangle\langle e_j|$ on $\mathcal{V}$ when expanded in the basis $\{|e_i\rangle\langle e_j|\}$.

Note that the operators $P_{|e_i\rangle} \otimes P_{|e_j\rangle}$ are not themselves operators associated with decoherence functionals. They do not obey the $X^1 = MXM$ requirement. By defining

$$X^{(ij)}_1 := \frac{1}{2}(P_{|e_i\rangle} \otimes P_{|e_j\rangle} + P_{|e_j\rangle} \otimes P_{|e_i\rangle}), \quad (2.52)$$
we see that $X^{(ij)}_1$ is an operator, that can be associated with a decoherence functional. This concludes the proof of the proposition for the real part.

By the same procedure we obtain the expansion (2.33) for the imaginary part $X_2$ of $X_d$ in terms of projectors $P(b_i)$ for a different orthonormal basis $\{|b_i\}\}$. This concludes the proof. \qed

Note that the imaginary part $X_2$ in itself is not an operator that can be associated with a $d \in D$. Thus, we have shown the following Corollary.

**Corollary** There exists a one-to-one correspondence between elementary decoherence functionals $d^e \in D$ and operators $X_{d^e} \in X_D$ which are given by the following expression:

$$X_{d^e}^{(ij)[lm]} = X^{(ij)}_1 + \kappa_{lm} X^{[lm]}_2, \quad \kappa_{lm} \in \mathbb{R},$$

where the operators $X^{(ij)}_1, X^{[lm]}_2$ are defined as above. Note that there is no sum over repeated indices.

We have thus shown that every decoherence functional can be written as a weak convex combination of elementary decoherence functionals. I now want to show why these $X_{d^e}$ must not be called ‘pure decoherence functionals’.

We show that every elementary decoherence functional can be written as the sum of two decoherence functionals as follows. Due to the fact that the imaginary part $X'_2$ of an operator $X' \in X_D$ associated with arbitrary decoherence functional $d' \in D$ can be added to the operator $X_{d^e}$ associated with an elementary decoherence functional $d^e \in X_D$ to produce a new decoherence functional, one can calculate that

$$X_{d^e} = \frac{1}{2}(X_{d^e} + iX'_2) + \frac{1}{2}(X_{d^e} - iX'_2).$$

Both terms $(X_{d^e} + iX'_2)$ and $(X_{d^e} - iX'_2)$ in this expression are proper decoherence functionals, even though $iX'_2$ in itself is not a decoherence functional. Thus, elementary decoherence functionals are not pure, but they still account for the simplest expansion of an arbitrary decoherence functional and this is all that is needed for the proof of the analogue of Wigner’s theorem.

We have now all the tools at hand to prove the following theorem.

### 3 An Analogue of Wigner’s Theorem

Recall that, up to this point, we know about the following relation between the sets of ‘homogeneous symmetries of a history quantum theory’, ‘homogeneous symmetries’ and
'physical symmetries of a history quantum theory':
\[
\{HSHQT\} \cong \{HS\} \subset \{PSHQT\}.
\] (3.1)

We are now going to show that the sets \(\{HS\}\) and \(\{PSHQT\}\) are identical.

### 3.1 The Theorem

**Theorem** There exist a one–to–one correspondence between homogeneous symmetries and physical symmetries of history quantum theories. Thus each PSHQT is given by an operator \(\hat{U} \otimes \hat{U}\) and induces a HSHQT; conversely, every one–to-one map \(\tau : \mathcal{R}(\mathcal{V}) \otimes \mathcal{R}(\mathcal{V}) \to \mathcal{R}(\mathcal{V}) \otimes \mathcal{R}(\mathcal{V})\) that preserves orthogonality between the rays and commutes with \(M\) can be implemented by a unitary operator \(\hat{U} \otimes \hat{U}\) on \(\mathcal{V} \otimes \mathcal{V}\), where \(\hat{U}\) may be unitary or anti–unitary.

**Proof**

We will first look for one–to–one maps which leave invariant the pairing between history propositions and decoherence functionals and map the set of rays into itself, and then restrict those transformations to homogeneous symmetries via the condition that they must map \(\mathcal{X}_D\) into itself.

We first consider the invariance requirement for the values of \(d(\alpha, \beta)\). This has to hold true for all decoherence functionals. In particular, for the functionals \(X^{(ij)}_1\) the relevant number is:

\[
2 \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X^{(ij)}_1) = \text{tr}_\mathcal{V}(\alpha P_{|e_i\rangle}) \text{tr}_\mathcal{V}(\beta P_{|e_j\rangle}) + \text{tr}_\mathcal{V}(\alpha P_{|e_j\rangle}) \text{tr}_\mathcal{V}(\beta P_{|e_i\rangle}),
\] (3.2)

for some ONB \(\{|e_i\rangle\}\) of \(\mathcal{V}\).

Recall that the requirement of the invariance of the imaginary part of a decoherence functional leads us to consider

\[
2 \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X^{(ij)}_2) = \text{tr}_\mathcal{V}(\alpha P_{|b_i\rangle}) \text{tr}_\mathcal{V}(\beta P_{|b_j\rangle}) - \text{tr}_\mathcal{V}(\alpha P_{|b_j\rangle}) \text{tr}_\mathcal{V}(\beta P_{|b_i\rangle}),
\] (3.3)

for some ONB \(\{|b_i\rangle\}\) of \(\mathcal{V}\).

Therefore, considering those decoherence functionals \(X_d\) for which \(|b_i\rangle = |e_i\rangle\) for all \(i \in \{1, 2, \ldots, \dim \mathcal{V}\}\), and \(\lambda_{ij} = \kappa_{ij}\) for all \(i, j \in \{1, 2, \ldots, \dim \mathcal{V}\}\) we see that the invariance requirement amounts to requiring that

\[
\text{tr}_\mathcal{V}(\alpha P_{|e_i\rangle}) \text{tr}_\mathcal{V}(\beta P_{|e_j\rangle})
\] (3.4)

should remain invariant under the appropriate transformations. At first we take the case when \(\alpha \otimes \beta \in \mathcal{R}(\mathcal{V}) \otimes \mathcal{R}(\mathcal{V})\). By Wigner’s theorem the transformations leaving (3.4) invariant are given by operators \(\hat{U} \otimes \hat{V}\) whereby \(\hat{U}\) and \(\hat{V}\) are unitary or anti–unitary operators on \(\mathcal{V}\). Even though these transformations leave the value of \(d(\alpha, \beta)\) invariant, they do not...
comply with the condition of mapping the set $\mathcal{X}_D$ into itself.

To see this, recall that an element $X_d \in \mathcal{X}_D$ is required to satisfy the condition $X_d^\dagger = MX_dM$, which is equivalent to $X_1^{(ij)} = MX_1^{(ij)}M$ and $X_2^{(ij)} = -MX_2^{(ij)}M$. Consider the following particular decoherence functional $X_1^{(ii)}$ under such a mapping:

$$\mathcal{X}_D \ni P|e_i\rangle \otimes P|e_i\rangle \mapsto \hat{P}_U|e_i\rangle \otimes \hat{P}_V|e_i\rangle.$$  \hfill (3.5)

Since

$$M[\hat{P}_U|e_i\rangle \otimes \hat{P}_V|e_i\rangle]M \neq \hat{P}_U|e_i\rangle \otimes \hat{P}_V|e_i\rangle,$$  \hfill (3.6)

we see that its image under the map $\hat{U} \otimes \hat{V}$ does not belong to $\mathcal{X}_D$. To comply with this requirement we need to require that $[\hat{U} \otimes \hat{V}, M] = 0$, i.e. consider only those operators of the form $\hat{U} \otimes \hat{V}$.

What is now left to show is that there can be no other transformations obeying conditions (2.17–2.19), even if we allow for arbitrary transformations $G = (G_0, G_0)$

$$G : \mathcal{P}(V) \otimes \mathcal{P}(V) \rightarrow \mathcal{P}(V) \otimes \mathcal{P}(V) \quad \alpha \otimes \beta \mapsto \alpha^{G_0} \otimes \beta^{G_0}.$$  \hfill (3.7)

The argument is much the same as in the standard quantum mechanical case: consider a transformation $G = (G_0, G_0)$ that maps a one-dimensional projector $\alpha \in \mathcal{P}(V)$ into $\alpha^{G_0} \in \mathcal{P}(V)$, an $m$-dimensional one, $m > 1$. Therefore,

$$\alpha \otimes \alpha \mapsto \alpha^{G_0} \otimes \alpha^{G_0}.$$  

It is easy to see that there exist a $d \in D$ such that $X_d = \alpha \otimes \alpha$, see also [14]. But since $\text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha^{G_0} \otimes \alpha^{G_0}) = m^2$, there exists no decoherence functional $d^{G_0} \in D$ for which $\alpha^{G_0} \otimes \alpha^{G_0}$ is the associated operator $X_d^\epsilon_\alpha$. This concludes the proof. \hfill $\square$

### 3.2 Discussion

The result of the theorem shows that every PSHQT can be induced by an unitary or anti–unitary operator $\hat{U}$ on $\mathcal{V}$ as follows:

$$\mathcal{U}\mathcal{P} \quad : \quad \alpha \mapsto \tilde{\alpha} := \hat{U}\alpha\hat{U}^\dagger,$$

$$\mathcal{D} \quad : \quad X_d \mapsto \tilde{X}_d := \tilde{X}_d := \hat{X}\hat{X} \equiv \hat{U} \otimes \hat{U} X^\dagger_\epsilon \otimes \hat{U}^\dagger.$$  \hfill (3.8)

As a consequence of this transformation the invariance

$$d(\alpha, \beta) = \tilde{d}(\tilde{\alpha}, \tilde{\beta})$$  \hfill (3.9)

for all $d \in D$ and all $\alpha, \beta \in \mathcal{P}(\mathcal{V})$ follows by the property of the trace.
By looking at the definition for a PSHQT from which this theorem arose, it seems rather unnecessary to proceed via the use of elementary decoherence functionals. We could have started immediately by looking for all transformations obeying condition (2.17); then restrict to those which map $\mathcal{X}_D$ into itself. But since it was not known to which extent $\mathcal{X}_D$ can accommodate more general transformations $G$ on $\mathcal{P}(\mathcal{V})$ it seems a sensible way to follow this hybrid path. In particular, we circumvented the problem of specifying all transformations that map $\mathcal{X}_D$ into itself.

A central requirement in the proof of the theorem was the invariance of the value $d(\alpha, \beta) \in \mathbb{C}$ for all pairs $\alpha, \beta \in \mathcal{P}(\mathcal{V})$. A closer inspection reveals that the existence of complex-valued functionals makes it possible to reduce the invariance requirement to the form (3.4). However, it is neither necessary to consider complex-valued functionals nor to require invariance for all pairs $(\alpha, \beta)$. We can investigate the possibility of softening the invariance requirement to hold only for the ‘diagonal’ values of $d \in \mathcal{D}$, i.e. $d(\alpha, \alpha)$. Then we are led to the condition that

$$
\text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \alpha X_1^{(ij)}) = \text{tr}_\mathcal{V}(\alpha P_{[e_i]}^{}) \text{tr}_\mathcal{V}(\alpha P_{[e_j]})
$$

has to remain invariant. In this case we see that we will end up with the same transformations $\hat{U} \otimes \hat{U}$ on $\mathcal{V} \otimes \mathcal{V}$ as requiring $d(\alpha, \beta)$ to remain invariant for all pairs $(\alpha, \beta)$. This is due to the fact that restricting to projectors of the form $\alpha \otimes \alpha$ can be formulated with the aid of the same operator $M$ which is used to formulate the defining property $X^\dagger = MXM$ for decoherence functionals. Therefore, the requirement on the diagonal part only is strong enough to enforce it onto the value of $d$ on any pair $(\alpha, \beta)$.

The particular feature of physical symmetries of history quantum theories is their property of being implemented by a unitary or anti-unitary operator $\hat{U}$ on $\mathcal{V}$. As a consequence, each partition of the unit operator in $\mathcal{V}$ into mutually orthogonal projectors, that is a set of projectors $\{\alpha_i\}$, such that

$$
\{\alpha_i\}_{i=1}^{m \leq \dim \mathcal{V}}; \quad \oplus_{i=1}^{m} \alpha_i = 1
$$

is mapped into another partition of unity. In particular, the cardinality of this set is preserved. Now, much emphasis in the decoherent histories approach is placed on finding consistent sets of history propositions with respect to a particular decoherence functional $d \in \mathcal{D}$. In the formalism used here, consistent sets are naturally associated with particular partitions of unity [14], namely those, for which it holds that

$$
d(\alpha_i, \alpha_j) = \delta_{ij}d(\alpha_i, \alpha_i) \quad \forall i, j \in \{1, 2, \ldots, m\}.
$$

We see therefore that a PSHQT will always map consistent sets into new consistent sets of the same cardinality. There has been some discussion [11] whether or not one should allow for transformations between consistent sets of different cardinality. We see that, at least in this context, this possibility is excluded if one agrees on the definition of PSHQT presented in this article.
3.3 Physical symmetries of history quantum mechanics

The main aim of this section is to show that the notion of ‘physical equivalence’, introduced in [4], is—when expressed in this formalism—a particular example of a physical symmetry of history quantum mechanics. It also serves the purpose of providing the explicit form of the decoherence functional for this theory.

Remember that, for standard quantum mechanics when looked at from the perspective of the history programme, the space of history propositions \( \alpha \in \mathcal{UP} \) is given by projectors \( \alpha \in \mathcal{P}(\mathcal{V}_n) = \mathcal{P}(\otimes_{i=1}^{n} \mathcal{H}_n) \). The particular decoherence functional is associated with an operator

\[
\tilde{X}_{(H,\rho_{t_0},\rho_{t_f})} = \frac{1}{\text{tr}_H(\rho_{t_0}\rho_{t_f}(t_f))} \tilde{X}_{(H,\rho_{t_0},\rho_{t_f})}
\]

on \( \mathcal{V}_n \otimes \mathcal{V}_n \), where I also have inserted a final density operator at time \( t_f \). When evaluated on homogeneous projectors \( \alpha_h = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n} \), the value of \( d_{(H,\rho_{t_0},\rho_{t_f})}(\alpha_h, \beta_h) \) is given by

\[
d_{(H,\rho_{t_0},\rho_{t_f})}(\alpha_h, \beta_h) = \frac{1}{\text{tr}_H(\rho_{t_0}\rho_{t_f}(t_f))} \text{tr}_{\mathcal{V}}(\alpha_h \otimes \beta_h \tilde{X}_{(H,\rho_{t_0},\rho_{t_f})})
\]

\[
= \frac{1}{\text{tr}_H(\rho_{t_0}\rho_{t_f}(t_f))} \text{tr}_H(\tilde{C}_{\alpha_h}^{\dagger} \rho_{t_0} \tilde{C}_{\beta_h} \rho_{t_f}(t_f)),
\]

which coincides with the form of the decoherence functional usually employed in the histories approach. But note—once again—that \( d_{(H,\rho_{t_0},\rho_{t_f})}(\alpha, \beta) \) is defined for all \( \alpha, \beta \in \mathcal{P}(\mathcal{V}) \). By following the procedure outlined in [8], one shows that the operator \( \tilde{X}_{(H,\rho_{t_0},\rho_{t_f})} \) is given by:

\[
\tilde{X}_{(H,\rho_{t_0},\rho_{t_f})} = \left[ U(t_1, t_0)^{\dagger} \rho_{t_0} U(t_1, t_0) \otimes U(t_2, t_1)^{\dagger} \otimes \cdots \otimes U(t_n, t_{n-1})^{\dagger} \right] \\
\otimes \left[ U(t_2, t_1) \otimes U(t_3, t_2) \otimes \cdots \otimes U(t_n, t_{n-1}) \otimes U(t_f, t_n) \rho_{t_f} U(t_f, t_n)^{\dagger} \right] \\
\times (R_{(n)} \otimes 1_{t_1} \otimes 1_{t_2} \otimes \cdots \otimes 1_{t_n}) \\
\times (S_{(2n)} \\
\times (R_{(n)} \otimes 1_{t_1} \otimes 1_{t_2} \otimes \cdots \otimes 1_{t_n}).
\]

The last three lines involve universal operators \( R_{(n)}, S_{(2n)} \) that arise by rewriting products of operators in terms of tensor–products [8]. Thus they are system independent. This operator is defined on \( \mathcal{V}_n \otimes \mathcal{V}_n \) and encodes the initial and final density operators as well as the dynamical evolution in form of the evolution operator \( U(t_i, t_{i-1}) \). This is the purest description of the content of the decoherence functional one can write down. It has a very transparent form.

Recall [4] that “two triples (\( \{C_\alpha\}, H, \rho \) and (\( \{\tilde{C}_\alpha\}, \tilde{H}, \tilde{\rho} \)) are called ‘physically equivalent’ if there are fields and conjugate momenta (\( \Phi(x), \pi(x) \)) and (\( \tilde{\Phi}(x), \tilde{\pi}(x) \)), respectively,
in which the triples’ histories, Hamiltonian, and initial condition take the same form.” As an example, the explicit transformation \((\alpha_i(t_i) \mapsto V \alpha_i(t_i)V^\dagger, H \mapsto VH^\dagger, \rho \mapsto V\rho V^\dagger)\) for a fixed unitary operator \(V\) on \(\mathcal{H}\) was shown to lead to physically equivalent triples. \textbf{Remark}: in order not to use the same symbol twice, I used the notation \(V\) instead of \(U\) as in [4] for the unitary operator; in the context of history quantum mechanics \(U(t_i, t_{i-1})\) already denotes the evolution operator.

First, notice that a transformation of the Heisenberg projection operators

\[
\alpha_i(t_i) \mapsto V \alpha_i(t_i)V^\dagger, \quad \forall i \in \{1, 2, \ldots, n\}
\]

(3.16)
is identical to the transformation

\[
\alpha_i(t_i) \mapsto VU(t_i, t_0)V^\dagger[V \alpha_iV^\dagger]VU(t_i, t_0)V^\dagger, \quad \forall i \in \{1, 2, \ldots, n\}
\]

(3.17)

that is a pair of transformations

\[
\alpha_i \mapsto V \alpha_i V^\dagger \quad \forall i \in \{1, 2, \ldots, n\}; \quad H \mapsto VH^\dagger,
\]

(3.18)

where the \(\alpha_i\) denote the Schrödinger projection operators. This is important since in the formalism used here only a string \(\alpha_h = \alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_n\) of \textit{Schrödinger projection operators} correspond to a homogeneous history proposition \(\alpha_h \in \mathcal{HP} = \mathcal{P}(\mathcal{V}_n)\). Now, defining the unitary operator \(\hat{U}_V := V \otimes V \otimes \cdots \otimes V, n\) times, remembering that \(\rho \mapsto V\rho V^\dagger\) and that the decoherence functional is given by equation (3.15), we see that the effect of this transformations on \(\hat{X}(H, \rho_{t_0}, \rho_{t_f})\) is given by \(\hat{X}(H, \rho_{t_0}, \rho_{t_f}) \mapsto (\hat{U}_V \otimes \hat{U}_V) \hat{X}(H, \rho_{t_0}, \rho_{t_f})(\hat{U}_V \otimes \hat{U}_V)\). Thus, this transformation between physically equivalent triples can be described as follows:

- The transformation of the history propositions \(\alpha_h \in \mathcal{P}(\mathcal{V}_n)\) as well as the operator \(X(H, \rho_{t_0}, \rho_{t_f})\) by a unitary \(\hat{U}_V \in B(\mathcal{V}_n)\) clearly leaves invariant the values of \(d(\alpha_h, \beta_h)\) since

\[
d(\alpha_h, \beta_h) = \text{tr}_{\mathcal{V}_n}(\hat{U}_V \alpha_h \hat{U}_V^\dagger \otimes \hat{U}_V \beta_h \hat{U}_V^\dagger)(\hat{U}_V \otimes \hat{U}_V)X(H, \rho_{t_0}, \rho_{t_f})(\hat{U}_V \otimes \hat{U}_V)\]

(3.19)

which is the definition of physical symmetry of a history quantum theory for which \(\mathcal{HP} = P(\mathcal{V}_n)\). Note that, in general, not all unitary operators \(\hat{U}\) on \(\mathcal{V}_n\) need to be of the form \(\hat{U}_V\) for a unitary operator \(V\) on the single-time Hilbert space \(\mathcal{H}\).

4 Summary and Outlook

In this article we proposed a notion of a ‘homogeneous symmetry’ (HS) and of a ‘physical symmetry of a history quantum theory’ (PSHQT). We proved an analogue of Wigner’s theorem which states that there exists a one–to–one correspondence between both HS and PSHQT. It has been shown that each PSHQT can be induced by a unitary or anti–unitary operator \(\hat{U}\) on \(\mathcal{V}\). History quantum theories that are related by a PSHQT are called ‘physically equivalent’ and we showed explicitly in the case of history quantum mechanics how
this notion encompasses the notion of physical equivalence introduced by Gell-Mann and Hartle in [4] in case one is dealing with a finite-dimensional, single-time Hilbert space \( \mathcal{H}_t \) at a finite number of time-points \((t_1, t_2, \ldots, t_n)\). An extension to infinite-dimensional \( \mathcal{H}_t \) as well as to a continuous range of time-points is clearly desirable since such spaces occur naturally in the context of continuous histories [15, 16].

In this article we also investigated the structure of the space of decoherence functionals; in particular, we defined the notion of an ‘elementary decoherence functional’ in terms of which every decoherence functional can be expanded. We showed that these decoherence functionals are not pure, an observation that agrees with a result by Linden [13] that there exist no pure decoherence functionals. These elementary decoherence functionals were employed in order to perform some proofs but have never been assigned any other status than a technical one. In particular, we never calculated ‘transition amplitudes between decoherence functionals’, something that entirely contradicts the spirit of history quantum theories. Do these elementary decoherence functionals possess any physical interpretation?

While the definition of symmetry presented here has very convenient properties, it does not treat consistent sets of history propositions in any way preferred to other elements \( \alpha \in \mathcal{P}(\mathcal{V}) \). But since these are the sets one ultimately wants to determine, it is reasonable to ask for a notion of symmetry which mirrors their importance. Reflecting a moment about the structure of consistent sets, one notices that this amounts to ask for an approach which places its emphasis on boolean subalgebras of the space \( \mathcal{P}(\mathcal{V}) \). Via the use of the consistency conditions on the values of the decoherence functional some of these boolean algebras—namely the ones associated with consistent sets—are selected. Within each of these algebras classical reasoning without running into logical paradoxes is possible, whereas reasoning about elements belonging to different consistent sets leads, in general, to inconsistencies in the use of the values \( d(\alpha, \alpha) \) of the decoherence functional as probabilities. The theory of ‘boolean manifolds’ [17], which seem to be the most appropriate objects to describe history quantum theories [18, 19], allows to describes the structure of \( \mathcal{P}(\mathcal{V}) \) in these terms and therefore, one is led to the problem of defining a transformation theory on boolean manifolds. This is a task for future research.

By proving a classification theorem for decoherence functionals—an analogue of Gleason’s theorem in the context of history quantum theories—and the analogue of Wigner’s theorem presented here, we have laid the mathematical foundations for an approach to quantum theory from the point of view of the history programme.

In history quantum theories the decoherence functional can be thought of as providing the ‘dynamical’ content of the theory. In standard quantum mechanics when investigated from the point of view of the history programme this is manifest in that the space of history propositions \( \mathcal{UP} \) is given by Schrödinger picture projection operators and thus represents the ‘kinematical’—as opposed to ‘dynamical’—ingredient of the theory. In contrast, the decoherence functional (3.15) contains the evolution operator and the initial
and final density operators and thus provides the ‘dynamical’ specification of the model under investigation. In a companion paper we will use the analogue of Wigner’s Theorem presented in this article to define and to investigate the properties of ‘symmetries of decoherence functionals’.

Acknowledgements
I would like to thank Dr. N. Linden and Professor Dr. C.J. Isham for useful comments.

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