Nonlocal wave turbulence in the Charney-Hasegawa-Mima equation: a short review

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Rossby wave turbulence, as modelled by the Charney-Hasegawa-Mima (CHM) equation, is nonlocal in scale. As a result, the formal stationary Kolmogorov-Zakharov solutions of the Rossby wave kinetic equation, which describe local cascades, are not valid. Rather the solution of the kinetic equation is dominated by interactions between the large and small scales. This suggests an alternative analytic approach based on an expansion of the collision integral in a small parameter obtained from scale separation. This expansion approximates the integral collision operator in the kinetic equation by anisotropic diffusion operators in wavenumber space as first shown in a series of papers by Balk, Nazarenko and Zakharov in the early 1990’s. In this note we summarise the foundations of this theory and provide the technical details which were absent from the original papers.

I. INTRODUCTION TO WEAK WAVE TURBULENCE IN THE CHARNEY-HASEGAWA-MIMA EQUATION

In the limit of weak nonlinearity, the statistical evolution of CHM turbulence can be described by the theory of weak wave turbulence. See [1, 2] for a review of the theory. The principal result of this theory is the fact that that the wave spectrum, $n_k$, evolves according to the wave equation:

\[
\frac{\partial n_k}{\partial t} = 4\pi \int \left| V_{\mathbf{q} \mathbf{r}}^k \right|^2 \delta(\mathbf{k} - \mathbf{q} - \mathbf{r}) \delta(\omega_k - \omega_\mathbf{q} - \omega_\mathbf{r}) \times \left[ n_\mathbf{q} n_\mathbf{r} - n_k n_\mathbf{q} \operatorname{sgn}(\omega_k \omega_\mathbf{r}) - n_k n_\mathbf{r} \operatorname{sgn}(\omega_k \omega_\mathbf{q}) \right] d\mathbf{q} d\mathbf{r},
\]

where

\[
\omega_k = -\frac{\beta \rho^2 k_1}{1 + \rho^2 k^2}
\]

and

\[
V_{\mathbf{q} \mathbf{r}}^k = \frac{i}{2} \rho^2 \sqrt{\beta |k_q|} \left( \frac{q_2}{1 + \rho^2 q^2} + \frac{r_2}{1 + \rho^2 r^2} - \frac{\rho^2 k_2}{1 + \rho^2 k^2} \right)
\]

Throughout these notes we shall use boldface text to denote vectors in $\mathbb{R}^2$, regular text to denote their magnitudes and subscripts to denote their components. Thus, for example, $\mathbf{k} = (k_1, k_2)$ and $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$. The nonlinear interaction coefficient, $V_{\mathbf{q} \mathbf{r}}^k$, has several symmetries which will be useful later:

\[
V_{(k_1, k_2)}^{(q_1, q_2)}(r_1, r_2) = V_{(r_1, r_2)}^{(k_1, k_2)}(q_1, q_2) = -V_{(k_1, k_2)}^{(r_1, r_2)}(q_1, q_2) = -V_{(r_1, r_2)}^{(q_1, q_2)}(-q_1, -q_2)
\]

\[
= -V_{(q_1, q_2)}^{(r_1, r_2)}(-r_1, -r_2) = V_{(q_1, q_2)}^{(-k_1, -k_2)}(r_1, r_2)
\]

\[
= V_{(-k_1, -k_2)}^{(r_1, r_2)}(-q_1, -q_2) = V_{(-q_1, -q_2)}^{(r_1, r_2)}(-r_1, -r_2).
\]

According to Eq. (1), exchange of energy is only possible among triads of modes, $(\mathbf{k}, \mathbf{q}, \mathbf{r})$, which satisfy the resonance conditions:

\[
\mathbf{k} = \mathbf{q} + \mathbf{r}
\]

\[
\omega_\mathbf{k} = \omega_\mathbf{q} + \omega_\mathbf{r}.
\]

For each given $\mathbf{k}$, the modes $\mathbf{q}$ permitted to interact with $\mathbf{k}$ lie on a one-dimensional curve in the $(k_1, k_2)$ plane known as the resonant manifold of the mode $\mathbf{k}$ (the third member of the triad is given by $\mathbf{r} = \mathbf{k} - \mathbf{q}$). The resonant manifolds typically have the shape shown in Fig. 1.

The kinetic equation, Eq. (1), becomes scale invariant only in the limits, $k \gg \rho^{-1}$ and $k \ll \rho^{-1}$. In these limits, it is possible to formally find stationary Kolmogorov-Zakharov (KZ) solutions of Eq. (1) describing turbulent cascades of energy and enstrophy. The inverse cascade spectrum, describing the transfer of energy from small scales to large, turns out to be non-local meaning that the RHS of Eq. (1) diverges on this spectrum so that it is not a consistent stationary solution. It is worth...
stressing that the fact that the “usual” KZ spectrum is not realisable does not necessarily invalidate the use of the kinetic equation itself. Rather, the non-locality of the KZ solution leads one to expect that the transfer of energy from small scales to large in the CHM equation proceeds by a direct interaction between the large and small scales rather than via a scale-local cascade process. This is what is meant by non-local turbulence. In this section, we provide a detailed description of this process. This analysis was originally presented in [3, 4] and is reproduced here in slightly expanded form for the sake of clarity and completeness.

The x-component of the phase speed of a wave having wavenumber \( k \) is

\[
    c_k = \frac{\omega_k}{k_1} = -\frac{\beta \rho^2}{1 + \rho^2 k^2}.
\] (6)

Notice that all waves have a phase velocity to the left. Furthermore for large scale waves, \( k \ll \rho^{-1}, \) \( c_k \rightarrow -\beta \rho^2 = -v^* \) and all waves travel at the same speed, known as the drift velocity:

\[
    v^* = \beta \rho^2.
\] (7)

We are interested in describing a scenario in which small scale turbulence evolves by direct interaction with large scale waves. Since the large scale waves are all travelling at the drift velocity in the x-direction, it is natural to work in a frame moving in the x-direction with the drift velocity, \(-\beta \rho^2\). In this frame, using Eq. (6), the phase speed of mode \( k \) changes to:

\[
    c_k \rightarrow c_k + \beta \rho^2 = \frac{\beta \rho^4 k^2}{1 + \rho^2 k^2}.
\]

The frequencies are Doppler-shifted accordingly:

\[
    \omega_k = c_k k_1 \rightarrow \beta \rho^2 k_1 + \omega_k.
\] (8)

We shall use \( \Omega_k \) to denote the frequency in the moving frame:

\[
    \Omega_k = \rho^2 k_1 + \omega_k = \frac{\beta \rho^4 k^2}{1 + \rho^2 k^2}.
\] (9)

In the moving frame the kinetic equation, Eq. (1), becomes

\[
    \frac{\partial n_k}{\partial t} = 4\pi \int V_k q r r^2 \delta(k - q - r) \delta(\omega_k - \omega_q - \omega_r) \times [n_q n_r - n_k n_q sgn(\omega_k \omega_r) - n_k n_r sgn(\omega_k \omega_q)] d\Omega q d\Omega r,
\] (10)

Note from Eq. (8) that the change from \( \omega_k \rightarrow \Omega_k \) does not change the resonant manifolds, Eqs. (5), due to the presence of the momentum delta function, \( \delta (k - k_1 - k_2) \).

Let us now consider small scale turbulence with typical wave-vector \( k \). From Fig. 1, we can see that the resonant manifold containing the set of wave-vectors, \( k_1 \), permitted to interact with \( k \) typically intersects the \( k_1 = 0 \) axis in two places. Setting \( k_1 = 0 \) in Eqs. (5) and doing some calculations shows that these points are

\[
    P_1 = (0, 0) \quad \text{and} \quad P_2 = (0, -2k_2).
\] (11) (12)

If we are interested in non-local interaction between wavenumbers in the neighbourhood of \( k \) and zonal flows, then the points \( P_1 \) and \( P_2 \) will give the dominant contributions. Interaction with the point \( P_1 \) always describes interaction with a large scale zonal flow. The wave number of \( P_2 \), on the other hand, is comparable to the \( k_2 \) component of the small scale turbulence. Interaction with the point \( P_2 \) can correspond to interaction with a large or small scale zonal flow depending on whether the small scale turbulence has an appreciable \( k_2 \) component or not. We focus on the interaction with large scale zonal flows first.

II. NONLOCAL INTERACTION WITH LARGE SCALE ZONAL FLOWS

Let us introduce a large scale reference scale, \( K \ll \rho^{-1} \), and after integrating out \( r \) from Eq. (10), split the right hand side of the kinetic equation as

\[
    \frac{\partial n_k}{\partial t} = \int_{q < K} \text{Coll} [n_k, n_q, k, q] \, dq \quad \text{and} \quad + \int_{q \geq K} \text{Coll} [n_k, n_q, k, q] \, dq,
\] (13)

where the collision integral is:

\[
    \text{Coll} [n_k, n_q, k, q] = 4\pi |V_k q r|^2 \delta(\Omega_k - \Omega_q - \Omega_k - q) \times [n_q n_r - n_k n_q sgn(\omega_k \omega_r) - n_k n_r sgn(\omega_k \omega_q)] d\Omega k d\Omega q
\] (14)

If we are interested in the dynamics of the small scales, \( k \gg K \), then the assumption of nonlocality means that we can neglect the second term in Eq. (13) compared to the first. Further simplifications can be made in the first term using the fact that \( k \gg q \) everywhere in the integrand and assuming that \( n_k \ll n_q \) everywhere in the integrand. This latter inequality means that we are describing a situation where small scale turbulence is interacting with an intense large scale zonal flow. This discussion does not tell us how this large scale zonal flow was generated in the first place.

As \( q \rightarrow 0 \), \( sgn(\omega_k \omega_q) = 1 \) and we can approximate the \( n_k \) dependence in the collision integral as follow:

\[
    [n_q n_k - n_k n_q sgn(\omega_k \omega_q) - n_k n_q sgn(\omega_k \omega_q)] \rightarrow [n_k - n_k] n_q
\]

where we neglect the term \( n_k n_k \) describing small-scale small-scale interactions on the basis that it is \( O(n_k^2) \) as
\( \mathbf{q} \to 0 \) and much smaller than the terms describing small-scale large-scale interactions. In other words, we assume that \( n_k \ll n_q \). We therefore approximate Eq. (10) by

\[
\frac{\partial n_k}{\partial t} = \int_{q<q_K} F(k, \mathbf{q}) \, d\mathbf{q} \tag{15}\]

where

\[
F(k, \mathbf{q}) = 4\pi \left| V_{k-q}^4 k \right|^2 \delta(\Omega_k - \Omega_q - \Omega_{k-q}) \times n_q \, (n_{k-q} - n_k). \tag{16}\]

Using the symmetries of \( V_{k-q}^4 k \) and \( \Omega_k \) it can be shown by direct computation that

\[
F(k, \mathbf{q}) = -F(k - \mathbf{q}, \mathbf{q}). \tag{17}\]

Thus we can then write:

\[
\frac{\partial n_k}{\partial t} = \frac{1}{2} \int_{q<q_K} F(k, \mathbf{q}) \, d\mathbf{q} - \int_{q<q_K} F(k - \mathbf{q}, -\mathbf{q}) \, d\mathbf{q} \\
= \frac{1}{2} \int_{q<q_K} (F(k, \mathbf{q}) - F(k + \mathbf{q}, \mathbf{q})) \, d\mathbf{q} \\
= -\frac{1}{2} \int_{q<q_K} d\mathbf{q} \, \mathbf{q} \cdot \nabla_k F(k, \mathbf{q}), \tag{18}\]

where in the final step we have Taylor expanded \( F(k + \mathbf{q}, \mathbf{q}) \) with respect to \( \mathbf{q} \) in the first argument and neglected terms of \( O(q^2) \).

We should now Taylor expand \( F(k, \mathbf{q}) \) with respect to \( \mathbf{q} \) in Eq. (18). To do so, we first Taylor expand the argument of the \( \delta \)-function. Taylor expanding \( \Omega_{k-q} \) with respect to \( \mathbf{q} \) gives:

\[
\Omega_{k-q} = \Omega_k - \mathbf{q} \cdot \nabla_k \Omega_k + O(q^2)
\]

as \( \mathbf{q} \to 0 \), where \( \nabla_k = (\partial_{k_1}, \partial_{k_2}) \) is the gradient operator in \( k \)-space. Thus the difference \( \Omega_k - \Omega_{k-q} \) behaves as \( \mathbf{q} \cdot \nabla_k \Omega_k \) as \( \mathbf{q} \to 0 \). On the other hand, the remaining term \( \Omega_q \) behaves as \( q^2 \mathbf{q} \) as \( \mathbf{q} \to 0 \) as can be seen directly from Eq. (9). Thus the latter can be neglected compared with the former in the limit \( \mathbf{q} \to 0 \) and we can replace

\[
\delta(\Omega_k - \Omega_q - \Omega_{k-q}) \to \delta(\mathbf{q} \cdot \nabla_k \Omega_k)
\]

in Eq. (14). Therefore \( F(k, \mathbf{q}) \) in Eq. (18) can be approximated as

\[
F(k, \mathbf{q}) \approx 4\pi \left| V_{k-q}^4 k \right|^2 \delta(\mathbf{q} \cdot \nabla_k \Omega_k)(n_{k-q} - n_k) \approx -4\pi \left| V_{k-q}^4 k \right|^2 \delta(\mathbf{q} \cdot \nabla_k \Omega_k) q \cdot \nabla_k n_k, \tag{19}\]

where we have used the fact that \( n_{k-q} - n_k = -\mathbf{q} \cdot \nabla_k n_k + O(q^2) \) as \( \mathbf{q} \to 0 \). Combining Eq. (19) with Eq. (18), the kinetic equation for the small scales can be written as an anisotropic diffusion equation in \( k \)-space:

\[
\frac{\partial n_k}{\partial t} = \frac{\partial}{\partial k_i} S_{ij}(q, k_2) \frac{\partial n_k}{\partial k_j} \tag{20}\]

where repeated component subscripts are summed over. The diffusion tensor is

\[
S_{ij}(k) = 2\pi \int_{q<q_K} d\mathbf{q} \left| V_{q-k}^4 q \right|^2 \delta(\mathbf{q} \cdot \nabla_k \Omega_k) q_i q_j n_q \tag{21}\]

Let us now suppose that the large scales are principally supported at scales \( q \ll K \). The reference wavenumber, \( K \), can then be extended to infinity in Eq. (21) and we can perform the integration with respect to \( q_i \):

\[
S_{ij}(k) = 2\pi \int_{-\infty}^{\infty} dq_1 dq_2 \left| V_{q-k}^4 q \right|^2 \delta(q_1 \frac{\partial \Omega}{\partial k_1} + q_2 \frac{\partial \Omega}{\partial k_2}) q_i q_j n_q \]

\[
= 2\pi \int_{-\infty}^{\infty} dq_1 dq_2 \left| V_{q-k}^4 q \right|^2 \left| \frac{\partial \Omega}{\partial k_1} \right|^{-1} \delta(q_1 + \theta k q_2) q_i q_j n_q \]

\[
= 2\pi \left| \frac{\partial \Omega}{\partial k_1} \right|^{-1} \int_{-\infty}^{\infty} dq_2 \left[ \left| V_{q-k}^4 q \right|^2 q_i q_j n_q \right]_{q_1=\theta k q_2} \]

where we have introduced the scalar function

\[
\tilde{S}(k) = 2\pi \left( \frac{\partial \Omega}{\partial k_1} \right)^{-2} \int_{-\infty}^{\infty} dq_2 \left[ \left| V_{q-k}^4 q \right|^2 n_q \right]_{q_1=\theta k q_2}. \tag{24}\]

The matrix in Eq. (23) is singular. It has eigenvalues 0 and \( 1 + \theta_k^2 \) with corresponding eigenvectors \((\theta_k^{-1}, 1)\) and \((-\theta_k, 1)\). This suggests that at any point in \( k \)-space, diffusion only occurs in the direction \((-\theta_k, 1)\) corresponding
to the positive eigenvalue. Eq. (20) should then describe diffusion along one-dimensional curves in \( k \)-space. In order to show this and to find these curves, we need to find a change of variables which reduces Eq. (20) to a 1-dimensional diffusion equation.

The general structure of Eq. (20) is

\[
\frac{\partial n_k}{\partial t} = \left( \frac{\partial}{\partial k_1}, \frac{\partial}{\partial k_2} \right) S(k) \left( \frac{\partial}{\partial k_1}, \frac{\partial}{\partial k_2} \right) n_k
\]

(25)

where our notation tacitly assumes that Eq. (26) is used to express all \( k \) dependence in \( S(k) \) and \( n_k \) in terms of \( \kappa_1 \) and \( \kappa_2 \). \( J \) is the Jacobian matrix of the change of variables:

\[
J = \begin{pmatrix}
\frac{\partial n_1}{\partial k_1} & \frac{\partial n_1}{\partial k_2} \\
\frac{\partial n_2}{\partial k_1} & \frac{\partial n_2}{\partial k_2}
\end{pmatrix}.
\]

(28)

For a typical anisotropic diffusion equation for which the diffusion tensor would have two positive eigenvalues, the maximum simplification would be obtained if new coordinates could be found so that \( J S(k) J^T \) is diagonal. In this case, since the diffusion tensor is singular, we can do better and reduce the equation to a one-dimensional diffusion equation. By direct computation we observe that

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \kappa_1}{\partial k_1} - \theta_k \\
1
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^T
= \begin{pmatrix}
(b-a \theta_k)^2 & (b-a \theta_k)(d-c \theta_k) \\
(b-a \theta_k)(d-c \theta_k) & (d-c \theta_k)^2
\end{pmatrix}.
\]

Using this we see that

\[
J \begin{pmatrix}
\frac{\partial \kappa_1}{\partial k_1} - \theta_k \\
1
\end{pmatrix} J^T = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

provided that our new coordinates satisfy the equations

\[
\frac{\partial \kappa_1}{\partial k_2} - \theta_k \frac{\partial \kappa_1}{\partial k_1} = 1,
\frac{\partial \kappa_2}{\partial k_1} - \theta_k \frac{\partial \kappa_2}{\partial k_1} = 0.
\]

The first of these equations can be easily solved by inspection. The second is also obvious once one recalls the definition of \( \theta_k \) from Eq. (22). We obtain:

\[
\kappa_1(k_1, k_2) = k_2
\]

\[
\kappa_2(k_1, k_2) = \Omega(k_1, k_2).
\]

(29)

Under a change of coordinates,

\[
k_1 \to \kappa_1(k_1, k_2)
\]

\[
k_2 \to \kappa_2(k_1, k_2),
\]

(26)

Eq. (25) transforms to [5]

\[
\frac{\partial n_k}{\partial t} = \det J \left( \frac{\partial}{\partial \kappa_1}, \frac{\partial}{\partial \kappa_2} \right) \det J^{-1} J S(k) J^T \left( \frac{\partial}{\partial \kappa_1}, \frac{\partial}{\partial \kappa_2} \right) n_k.
\]

(27)

FIG. 2:

The level sets of the function \( \Omega_k \) given by Eq. (9).

For this change of variables, \( \det J = \frac{\partial \Omega_k}{\partial k_1} \) so that in the \((\kappa_1, \kappa_2)\) plane, the diffusion equation Eq. (25) takes the form

\[
\frac{\partial n_k}{\partial t} = \frac{\partial \Omega_k}{\partial k_1} \frac{\partial}{\partial k_1} S(k) \frac{\partial n_k}{\partial \kappa_1}.
\]

(30)

In the \((\kappa_1, \kappa_2)\) plane Eq. (30) describes one dimensional diffusion in the \( \kappa_1 \) direction with \( \kappa_2 \) constant. Using Eqs. (29) to translate this back into the \((k_1, k_2)\) plane, Eq. (25) therefore describes one dimensional diffusion in the \( k_2 \) direction along lines of constant \( \Omega_k \). Illustrative examples of the curves \( \Omega_k = \text{constant} \) are plotted in Fig. 2.

III. NONLOCAL INTERACTION WITH SMALL SCALE ZONAL FLOWS

In this section we perform the analogous calculation assuming that the evolution is dominated by interaction
with the point \( P_2 = (0, 2k_2) \) corresponding to small scale zonal flows. The results of this section were first presented in [6].

Following the same reasoning as before, we introduce a reference scale, \( K \), satisfying \( K \ll k \) and \( K \ll 1/\rho \), and neglect the contributions to the collision integral coming from outside of a ball of radius \( K \) of the point \( \mathbf{p} = (0, 2k_2) \). We thus approximate Eq. (10) by

\[
\frac{\partial n_k}{\partial t} = \int_{|\mathbf{q} - \mathbf{p}| < K} d\mathbf{q} F(k, \mathbf{q})
\]

Taylor expanding the first argument about \( \mathbf{q} = \mathbf{p} \) gives

\[
\frac{\partial n_k}{\partial t} = -\int_{|\mathbf{q} - \mathbf{p}| < K} d\mathbf{q} \left[ F(k - p, -\mathbf{q}) + (\mathbf{q} - \mathbf{p}) \cdot \nabla_{\mathbf{q}} F(k - \mathbf{q}, \mathbf{q})\right]_{\mathbf{q} = \mathbf{p}} + O(|\mathbf{p} - \mathbf{q}|^2)
\]

\[
\approx -\int_{|\mathbf{q} - \mathbf{p}| < K} d\mathbf{q} \left[ F(k - \mathbf{p}, -\mathbf{q}) - (q_1, q_2 - 2k_2) \cdot \left( \partial_{k_1} F(k, -\mathbf{q}), \partial_{k_2} F(k, -\mathbf{q}) \right) \right]
\]

where we have introduced the shorthand notation

\[
\tilde{k} = k - p = (k_1, -k_2).
\]

We now expand \( F(k, -\mathbf{q}) \) about \( \mathbf{q} = \mathbf{p} \). Since \( V_{\mathbf{q} \cdot \mathbf{k} - \mathbf{q} \cdot \mathbf{k}} \) and \( n_{\mathbf{q}} \) both vary rapidly near \( \mathbf{q} = \mathbf{p} \), we should only expand the \( n_{k+\mathbf{q}} - n_k \) and \( \delta(\Omega_k - \Omega_{-\mathbf{q}} - \Omega_{k+\mathbf{q}}) \) terms. Let us look at these two terms in turn.

\[
n_{k+\mathbf{q}} - n_k = n_{k+p} - n_k + (\mathbf{q} - \mathbf{p}) \cdot \nabla_{\mathbf{q}} n(\tilde{\mathbf{k}} + \mathbf{q})\bigg|_{\mathbf{q} = \mathbf{p}} + O(|\mathbf{p} - \mathbf{q}|^2)
\]

\[
\approx n_{k} - n_{k} + (q_1, q_2 - 2k_2) \cdot (\partial_{k_1} n_k, \partial_{k_2} n_k).
\]

Note that, unlike the expansion of the collision integral about the point \((0, 0)\) corresponding to large scale zonal flows detailed in Sec. II, the leading order term in the Taylor expansion of \( F(\mathbf{k}, -\mathbf{q}) \) about \( \mathbf{q} = \mathbf{p} \) is not necessarily zero. It vanishes only if the spectrum is symmetric about the \( k_1 \) axis (i.e. \( n(k_1, k_2) = n(k_1, -k_2) \)). We shall return to this point below. Next, let us look at the argument of the \( \delta \)-function near \( \mathbf{q} = \mathbf{p} \).

\[
\Omega_k - \Omega_{-\mathbf{q}} - \Omega_{k+\mathbf{q}} = \Omega_k - \Omega_{-\mathbf{p}} - \Omega_k + q_1 \left[ \frac{\partial \Omega}{\partial q_1}(-q_1', -q_2') + \frac{\partial \Omega}{\partial q_1}(k_1 + q_1', -k_2 + q_2') \right]_{(q_1', q_2') = (0, 2k_2)}
\]

\[
+ (q_2 - 2k_2) \left[ \frac{\partial \Omega}{\partial q_2}(-q_1', -q_2') + \frac{\partial \Omega}{\partial q_2}(k_1 + q_1', -k_2 + q_2') \right]_{(q_1', q_2') = (0, 2k_2)} + O(|\mathbf{p} - \mathbf{q}|^2)
\]

\[
\approx -\frac{\partial \Omega_k}{\partial k_2} (q_2 - 2k_2 + \xi_k q_1),
\]

where

\[
\xi_k = \frac{3(k_1^2 - k_2^2) + \rho^2(k_1^4 + 6k_1^2k_2^2 - 3k_2^4)}{2k_1k_2(1 + 4\rho^2k_2^2)}.
\]

The various derivatives have been computed from Eq. (9). For example:

\[
\left[ \frac{\partial \Omega}{\partial q_2}(k_1 + q_1', -k_2 + q_2') \right]_{(q_1', q_2') = (0, 2k_2)} = \frac{\partial \Omega_k}{\partial k_2} = -\frac{2\beta \rho^3 k_1 k_2}{(1 + \rho^2 k_2^2)^2},
\]

and so forth. The leading order term is zero owing to the fact that \( \mathbf{k}, \mathbf{p} \) and \( \tilde{\mathbf{k}} \) are resonant. Putting this together, the Taylor expansion of the \( \delta \)-function is

\[
\delta(\Omega_k - \Omega_{-\mathbf{q}} - \Omega_{k+\mathbf{q}}) \approx \frac{\partial \Omega_k}{\partial k_2}^{-1} \delta(q_2 - 2k_2 + \xi_k q_1).
\]
From Eqs. (32), (34) and (37) we see that the leading order term in the kinetic equation coming from nonlocal interaction with small scale zonal flows is:

\[
\frac{\partial n_k}{\partial t} = Y_k \left[ n(\mathbf{k}_1, -\mathbf{k}_2) - n(\mathbf{k}_1, \mathbf{k}_2) \right], \tag{38}
\]

where

\[
Y_k = 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \int_{|\mathbf{p} - \mathbf{q}_1| < K} d\mathbf{q}_1 d\mathbf{q}_2 \left| V_{-\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 \delta(q_2 - 2k_2 + \xi_k q_1) n(q_1, q_2)
\]

\[
= 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \int_{|\mathbf{p} - \mathbf{q}_1| < K} d\mathbf{q}_1 d\mathbf{q}_2 \left| -V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 \delta(q_2 - 2k_2 - \xi_k q_1) n(-q_1, q_2)
\]

\[
= 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \int_{|\mathbf{q}_1| < K} \left| V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 n_{\mathbf{q}_1} \delta\left(q_2 - 2k_2\right) d\mathbf{q}_1.
\] \tag{39}

In the intermediate step, we have used the symmetries, Eq. (4), of \(V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k}\) and relabelled the integration variable \(q_1 \rightarrow -q_1\) to bring the interaction coefficient to a neater form. At the final step the delta function has been used to integrate out \(q_2\) to leading order. We have also assumed that the spectrum is symmetric about the \(k_2\) axis although this is an inessential point. Eq. (38) tells us that the leading order effect of interactions with small scale zonal flows is to cause the spectrum to relax to a state which is symmetric about the \(k_1\) axis, a point which was first made in [3, 4].

If we wish to observe any redistribution of spectral energy density due to the interactions with small scale zonal flows, we need to consider the higher order terms in Eq. (32). The first order term vanishes after integration over \(\mathbf{q}\) since the integrand is an odd function of \(\mathbf{p} - \mathbf{q}\). The next contribution is therefore the second order one. From Eqs. (32) and (34) the second order contribution can again be presented as an anisotropic diffusion equation in \(\mathbf{k}\)-space:

\[
\frac{\partial n_k}{\partial t} = \frac{\partial}{\partial k_i} B_{ij}(\mathbf{k}) \frac{\partial n_k}{\partial k_j},
\] \tag{40}

where the diffusion tensor is given by the matrix

\[
B(\mathbf{k}) = 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \int_{|\mathbf{p} - \mathbf{q}_1| < K} d\mathbf{q}_1 d\mathbf{q}_2 \left| V_{-\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 \delta(q_2 - 2k_2 + \xi_k q_1) n(q_1, q_2) \left( q_1^2 (q_2 - 2k_2^2) q_1 (q_2 - 2k_2^2) \right)
\]

\[
= 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \int_{|\mathbf{p} - \mathbf{q}_1| < K} d\mathbf{q}_1 d\mathbf{q}_2 \left| -V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 \delta(q_2 - 2k_2 - \xi_k q_1) n(-q_1, q_2) \left( -q_1^2 (q_2 - 2k_2^2) q_1 (q_2 - 2k_2^2) \right)
\]

\[
= 4\pi \left| \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right|^{-1} \left( \begin{array}{l} 1 - \xi_k \\ \xi_k^2 \end{array} \right) \int_{|\mathbf{q}_1| < K} \left| V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 n(-q_1, q_2) q_1^2 \delta\left(q_2 - 2k_2\right) d\mathbf{q}_1
\]

\[
= \left( \frac{\partial \Omega_k}{\partial \mathbf{q}_2} \right)^{-1} \hat{B}(\mathbf{k}) \left( \begin{array}{l} 1 - \xi_k \\ \xi_k^2 \end{array} \right)
\] \tag{41}

where we have performed the same manipulations as those used to arrive at Eq. (39) above and defined the scalar quantity

\[
\hat{B}(\mathbf{k}) = 4\pi \int_{|\mathbf{q}_1| < K} \left| V_{\mathbf{q}_1 \mathbf{q}_2} \mathbf{k} \right|^2 n(-q_1, q_2) q_1^2 \delta\left(q_2 - 2k_2\right) d\mathbf{q}_1.
\] \tag{42}

The diffusion tensor, \(B(\mathbf{k})\), is again singular indicating that Eq. (40) should be reducible to a one-dimensional diffusion equation by an appropriate change of variables. Following the same procedure as in Sec. II, we introduce new variables

\[
k_1 \rightarrow \kappa_1(k_1, k_2)
\]

\[
k_2 \rightarrow \kappa_2(k_1, k_2),
\] \tag{43}

whose Jacobian, \(J\), given by

\[
J = \left( \begin{array}{cc} \frac{\partial \kappa_1}{\partial k_1} & \frac{\partial \kappa_1}{\partial k_2} \\ \frac{\partial \kappa_2}{\partial k_1} & \frac{\partial \kappa_2}{\partial k_2} \end{array} \right).
\]
should diagonalise the matrix \( \begin{pmatrix} 1 & -\xi_k \\ -\xi_k & \xi_k^2 \end{pmatrix} \) appearing in the diffusion tensor, Eq. (41). Observing that
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\xi_k \\ -\xi_k & \xi_k^2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} (a - b \xi_k)^2 & (a - b \xi_k)(c - d \xi_k) \\ (a - b \xi_k)(c - d \xi_k) & (c - d \xi_k)^2 \end{pmatrix},
\]
we see that
\[
J \begin{pmatrix} 1 & -\xi_k \\ -\xi_k & \xi_k^2 \end{pmatrix} J^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
provided that our new coordinates satisfy the equations
\[
\frac{\partial \kappa_1}{\partial k_1} - \xi_k \frac{\partial \kappa_1}{\partial k_2} = 1, \\
\frac{\partial \kappa_2}{\partial k_1} - \xi_k \frac{\partial \kappa_2}{\partial k_2} = 0.
\]
The first equation can be easily solved by inspection while the second requires a little more effort using the method of characteristics. One obtains
\[
\kappa_1(k) = k_1, \\
\kappa_2(k) = Z_k = \arctan \left( \frac{k_2 + \sqrt{3} k_1}{\rho k^2} \right) - \arctan \left( \frac{k_2 - \sqrt{3} k_1}{\rho k^2} \right) - 2\sqrt{3} \rho k_1 \frac{1}{1 + \rho^2 k^2}
\]
(45)
With this done, \( \det J = \frac{\partial Z_k}{\partial k_2} \), so that, according to Eq. (27), the diffusion equation, Eq. (40), transforms into
\[
\frac{\partial n_\kappa}{\partial t} = \frac{\partial Z_k}{\partial k_2} \frac{\partial}{\partial \kappa_2} \left[ \left( \frac{\partial Z_k}{\partial k_2} \right)^{-1} \left( \frac{\partial \Omega_k}{\partial k_2} \right)^{-1} B(k) \frac{\partial n_\kappa}{\partial \kappa_1} \right].
\]
(46)
where \( \tilde{B}(k) \) is given by Eq. (42). In the \((\kappa_1, \kappa_2)\) plane Eq. (46) describes one dimensional diffusion in the \( \kappa_1 \) direction with \( \kappa_2 \) constant. Translating back into the \((k_1, k_2)\) plane, using Eqs. (44) and (45), Eq. (46) therefore describes one-dimensional diffusion in the \( k_1 \) direction along lines of constant \( Z_k \). Illustrative examples of the curves \( Z_k \) = constant are plotted in Fig. 3.

Interestingly \( Z_k \) defined in Eq. (45), is closely related to the third invariant (in additional to the energy and enstrophy) of the full kinetic equation, Eq. (1), discovered in [7] (see also [8, 9]). From Fig. 3 we note that the diffusion due to interaction with small scale zonal flows tends to transfer the wave action to large scales, a process which was investigated numerically in [10].

**IV. CONCLUSIONS**

In this note we have summarised the derivation of the nonlocal kinetic equation for Rossby wave turbulence modelled by the wave kinetic equation obtained from the Charney-Hasegawa-Mima equation. For small scale turbulence there are two possible sources of nonlocal interactions in \( k \)-space. The first is with large scale zonal flows with spectral support around the point \( P_1 = (0, 0) \). The second is with small scale zonal flows with spectral support around the point \( P_2 = (0, 2k_2) \). In both cases, the redistribution of small-scale wave action in \( k \)-space is described at leading order by an anisotropic diffusion equation in \( k \)-space. In both cases, the diffusion tensor is singular meaning that, via an appropriate change of variables, it can be shown that the diffusion occurs along one-dimensional curves in \( k \)-space. We showed how to obtain these curves explicitly. In the case of nonlocal interaction with the point \( P_1 \), the curves are open and tend to transport wave action to \( k_2 = \pm \infty \). In the case of nonlocal interaction with the point \( P_2 \), the curves are closed and tend to transport wave action towards the origin. Recent numerical investigations have demonstrated the presence of these diffusive mechanisms in simulations of the original CHM equation [11].

[1] V. Zakharov, V. Ll_ov, and G. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992).
[2] A. Newell, S. Nazarenko, and L. Biven, Physica D 152-
FIG. 3:
The level sets of the function $Z_k$ given by Eq. (45).