Non-Cauchy surface foliations and their expected connection to quantum gravity theories

Merav Hadad
Department of Natural Sciences, The Open University of Israel P.O.B. 808, Raanana 43107, Israel
E-mail: meravha@openu.ac.il

Abstract. We argue that quantum gravity theories should involve constructing a quantum theory on non-Cauchy hypersurfaces and suggest that the hypersurface direction should be the same as the direction of the effective non-gravitational force field at a point. We start with a short review of works which support this idea and then we foliate spacetime along an effective non-gravitational force field direction. Next we discuss the implication of this foliation on the expected properties of quantum gravity. We also discuss the vagueness caused by constructing any quantum theory when using non-Cauchy foliation.

1. Introduction

Obtaining a gravitational theory from microscopic objects or quantum fields is extremely important and challenging. It is important because it is expected to help us understand problems of gravity at very high energy, such as the behavior of black holes and the origin of the universe. It is challenging because this kind of theory is as yet unknown. Attempts to describe gravity using microscopic objects, such as strings, loops or triangles, have not yet led us to the desired Einstein equations. Whereas attempts to treat the gravitational metric as simply another quantum field are problematic. They result in a theory that involves spin-2 massless fields and this kind of theory is not renormalizable in more than $2 + 1$ dimensions. Moreover, combining quantum mechanics with general relativity is also problematic since time plays a different role within these two frameworks: whereas general relativity treats time as a dynamical variable, quantum theories use the Hamiltonian formalism which causes time to act as an independent parameter through which states evolve. Moreover, attempting to use the Hamiltonian formalism in order to quantize the gravitational theory, leads to a non renormalizable theory and to the problem of time in the ADM formalism.

Instead of giving up the powerful Hamiltonian formalism when general relativity theories are concerned, we suggest to use this formalism differently. We suggest to consider the symmetry breaking caused by an effective non-gravitational force field, and to use its direction as an independent parameter through which states evolve. This mean that instead of singling out the direction of a time vector field in the Hamiltonian formalism, we single out the direction of the effective non-gravitational force.

As we will show in the next section this idea is supported by several works which all single out the direction of a non-gravitational force in order to obtain different aspects of quantum gravity. It is also in agreement with holography and Verlinde’s suggestion. Moreover, not only it uses the natural symmetry breaking caused by an effective non gravitational force field, it also eliminates the problem...
of time in the ADM formalism. Finally, this foliation enables us to obtain the conditions which are needed in order to derive an effective (2+1)D gravitational theory instead of (3+1)D theory, with can be renormalized in the quantum limit. However, since the direction of any force field is space-like, and not time-like this suggestion leads to vagueness regarding the basic concepts of relativistic quantum field theories. We suggest ways to overcome some of these vagueness in the last section.

This paper is organized as follows: we start with a review of works which support the idea that quantum gravity theories should involve constructing a quantum theory on non-Cauchy hypersurfaces and that the hypersurface direction at any point should be the same as the direction of the effective non-gravitational force field at this point. Then we foliate spacetime along this direction, and discuss the implication of this foliation on the expected properties of quantum gravity. Finally, we discuss the vagueness caused by constructing any quantum theory when using non-Cauchy foliation, and discuss the implications on quantum gravity.

2. Review on the relevance of non-Cauchy foliation to quantum gravity
The main idea of this paper is that in order to find the proper way to quantize the gravitational theory, we need to consider the symmetry breaking caused by an effective non-gravitational force field, and to use the direction of that field as the direction through which states evolve. This suggestion is supported by several examples which relate non-Cauchy surfaces to different aspects of quantum gravity.

Before listing these examples, we first mention three known properties relating quantum gravity to light-like hypersurfaces. The first is the holographic principle [1], first proposed by ’t Hooft [2], which states that in quantum gravity, the description of a volume of space can be encoded on a lower-dimensional boundary to the region. Next is the AdS/CFT [3] correspondence, which uses a non-perturbative formulation of string theory to obtain a realization of the holographic principle. And finally, the third is Verlinde’s suggestion [4] that gravity is an entropic force which arises from the statistical behavior of microscopic degrees of freedom encoded on a holographic screen. We discuss the way these are related to our suggestion and reinforce it, in the next section.

However, in these descriptions the holographic screen is a light-like surface and therefore a Cauchy surface. Conversely, treating any force direction as the direction which states “evolve”, involves non-Cauchy hypersurfaces. Fortunately, non-Cauchy hypersurfaces in general, and that resulting from singling out the direction of a non-gravitational force in particular, have also been found useful when dealing with aspects of quantum gravity. We will discuss in detail some such foliation.

2.1. The thermodynamics properties of gravity can also be derived by non-Cauchy foliation
The first example we discuss in details involves the surface density of space time degrees of freedom (DoF). These are expected to be observed by an accelerating observer in curved spacetime, i.e. whenever an external non-gravitational force field is introduced [5]. This DoF surface density was first derived by Padmanabhan for a static spacetime using thermodynamic considerations. We found that this density can also be constructed from specific canonical conjugate pairs as long as they are derived in a unique way. Here we briefly summaries the assumptions and conclusions, for the detailed calculation see [6].

To see this one starts by defining the direction of the space-like vector field in a stationary D-dimensional spacetime. We take accelerating detectors that have a $D$ velocity unit vector field $u^a$ and acceleration $a^a = u^a \nabla_b u^b \equiv \alpha n^a$ (where $n^a$ is a unit vector and $u^a n_a = 0$). Next one foliate spacetime with respect to the unit vector field $n_a$ by defining a $(D - 1)$- hyper-surface which is normal to $n_a$. The lapse function $M$ and shift vector $W_a$ satisfy $n_a = M n_a + W_a$ where $r^a \nabla_a r = 1$ and $r$ is constant on $\Sigma_{D-1}$. The $\Sigma_{D-1}$ hyper-surfaces metric $h_{ab}$ is given by $g_{ab} = h_{ab} + n_a n_b$. The extrinsic curvature
of the hyper-surfaces is given by $K_{ab} = -\frac{1}{2} \mathcal{L}_n h_{ab}$ where $\mathcal{L}_n$ is the Lie derivative along $n^a$. The $\Sigma_{D-2}$ hyper-surfaces metric $\sigma_{ab}$ is given by $h_{ab} = \sigma_{ab} - u_a u_b$. 

As was first noted by Brown [7] for generalized theories of gravity, the canonical conjugate variable of the extrinsic curvature $K_{ab}$ is $4\sqrt{h}n_a u_b U_0^{abcd}$. Where $U_0^{abcd}$ is an auxiliary variable, which equals $\frac{\partial \mathcal{L}^2}{\partial \sigma^{abcd}}$ when the equations of motion hold. We found in [6] that the relevant phase space for detectors with D-velocity $u^a$ at point $P$ can be identified by projection of the extrinsic curvature tensor and its canonical conjugate variable on the vector field $u^a$:

$$\left\{ K_{nm} u_n u_m, 4\sqrt{h} U_0^{abcd} n_a u_b u_c n_d \right\}$$ (1)

The gravitational density degrees of freedom detected by an accelerating detector with D-velocity $u^a$ at point $P$ is constructed from multiplying these special stationary canonically conjugate variables. Thus, using $K^{ab} u_b u_a = n^a u_a = a$, the gravitational $D - 2$ surface density of the spacetime DoF observed by an accelerating observer $\Delta n$ per unit time $\Delta t$ is

$$\frac{\Delta n}{\Delta t} = 4a \sqrt{h} U_0^{abcd} n_a u_b u_c n_d.$$ (2)

In order to derive $\Delta n$ one integrate this term during some natural period of time. In this case we can use the assumption that an accelerated observer detects her environment as an equilibrium system at temperature $T$ even in curved spacetime[9]. This kind of canonical ensemble gives thermal Greens functions which are periodic in the Euclidean time with period $\beta = 1/T$ [8]. Assuming this is also the case for fields which represent the spacetime degrees of freedom we integrate eq. (2) over the Euclidian period $\beta$ and find that the surface density of the spacetime degrees of freedom:

$$\Delta n = 4\beta \sqrt{h} U_0^{abcd} n_a u_b u_c n_d.$$ (3)

Next, using the assumption that the temperature $T = 1/\beta$ seen by an accelerated observer equals $Na/2\pi$ even in curved spacetime [9], and $\sqrt{h} = N\sqrt{\sigma}$ we deduce that the degrees of freedom surface density in a one period Euclidean time detected by an accelerating observer is:

$$\Delta n = 8\pi U_0^{abcd} \epsilon_{ab} \epsilon_{cd}.$$ (4)

(Where $\epsilon_{ab} = \frac{1}{2} (n_a u_b - n_b u_a)$ and $\epsilon_{ab} = \sqrt{\sigma} \epsilon_{ab}$). Up to factor of 4, this is precisely the expression for entropy density of spacetime found by Padmanabhan using the equipartition law for a static metric.

To conclude: we see that entropy density of spacetime can also be constructed from specific canonical conjugate pairs as long as they are derived by foliating spacetime with respect to the direction of the non-gravitational vector force field. That this aspect reinforces the importance of singling out a very unique spatial direction: the direction of a non-gravitational force.

2.2. The effects of string vibrations on the metric can also be derived by non-Cauchy foliation

The second example we discuss in details involves string theory excitations. String theory succeeds in describing a kind of quantum gravity, and with its aid one can identify the entropy of a BPS black hole as string theory excitations [11, 12]. Some of these vibrations create metrics with quantized conical singularities in the $r - y$ surface [13]. Thus we expect that a BPS black hole metric should be a kind of superposition of metrics, some of which have a conical singularity in the $r - y$ surface. This creates an uncertainty at the opening angle at the $r - y$ surface. It turns out that these singularities can also
be explained using the uncertainty principle, as long as the variables in the uncertainty principle are obtained in a unique way. Here we briefly summarizes the assumptions and conclusions for the detailed calculation see [14].

In order to see this we need to begin with the same foliation as before. Only now the $\Sigma_{D-1}$ hypersurfaces metric $h_{ab}$ is given by $g_{ab} = h_{ab} + (-1)^{s_u}u_au_b$, where $s_u = 0$ if $u_a$ is space-like and $s_u = 1$ if $u_a$ is time-like. We also define a $D - 2$ hypersurfaces defined by $t = const$ and $r = const$ are thus normal to the given vector $u_a$ and to $u_a$. The $D-2$ hypersurfaces metric $\sigma_{ab}$ is given by $h_{ab} = \sigma_{ab} + (-1)^{s_n}n_{a}n_b$, where $s_n = 0$ if $n_a$ is space-like and $s_n = 1$ if $n_a$ is time-like. Moreover, as was found out in [14], for the $D1D5$ black hole, instead of one unit normal vector $n_a$, one should deal with a several vectors. It is a special case where we can find $n (n < D - 1)$ unit vectors: $n(i)_a (i = 1...n)$ which are normal to each other and to the same unit vector $u_a$ and $n(i)_a \neq 0$. In this case the $D - n - 1$ metric $\tilde{\sigma}_{ab}$ is defined by $h_{ab} = \tilde{\sigma}_{ab} + \sum_{i=1}^{n}(-1)^{s_n}n_a n(b)_a n(b)_a$ and one obtains (see details in [14]):

$$\left[K^{ab}n_a n_b(x), \sqrt{\tilde{\sigma}}(0)U_0^{abcd}u_a n(i)_b n(i)_b u_d(\hat{x})\right] = h^D-1(x - \hat{x})$$

(5)

where $\sqrt{\tilde{\sigma}(0)} = \prod_{j \neq i} N_j \sqrt{-\tilde{\sigma}(-1)^{s_a + \sum_i s f}}$.

For a $D1D5$ black hole we examine the static metric

$$ds^2 = f(r)(-dt^2 + dy^2) + f(r)^{-1}(dr^2 + r^2d\Omega^2) + g(r)dz^2,$$

(6)

where $y$ and $z_i$ are compact. We will use the ‘naive’ geometry [12] where: $f(r) = (1 + \frac{Q_1}{r})^{-1/2}(1 + \frac{Q_2}{r})^{-1/2}$ and $g(r) = (1 + \frac{Q_1}{r})^{-1/2}(1 + \frac{Q_2}{r})^{-1/2}$.

Choosing the two vectors:

$$n_a(1) = (f(r)^{-1/2}, 0, 0, ..... 0)$$

$$n_a(2) = (0, f(r)^{-1/2}, 0, 0, ...... 0)$$

(7)

and define a $D - 3$ metric $\tilde{\sigma}_{ab}$ as $h_{ab} = \tilde{\sigma}_{ab} + (-1)^{s_1}n_1 n^1 n_2 n^2$, then

$$\left[K(x), \sqrt{\tilde{\sigma}}(0)U_0^{abcd}u_a n_1 n_2 u_d(\hat{x})\right] = h^D-1(x - \hat{x}).$$

(8)

$$\left[K_{yy}(x), \sqrt{\tilde{\sigma}}(0)U_0^{abcd}u_a n_1 n_2 u_d(\hat{x})\right] = h^D-1(x - \hat{x}).$$

(9)

where $K_{tt} \equiv K^{ab}n_1 n_1 u_a u_b$; $K_{yy} \equiv K^{ab}n_2 n_2 u_a u_b$ and $N = N_t = N_y$. Integrating over a closed $D - 1$ hypersurface we find

$$[\Theta_{r-t_E}, S_{W_r-t}] = h$$

(10)

$$[\Theta_{r-y}, S_{W_r-y}] = h$$

(11)

where

$$\Theta_{r-t_E} \equiv \oint ds_1 K_{tt} = \oint NdE K_{tt}$$

is the opening angle at the $r - t_E$ surface,

$$S_{W_r-t} \equiv \oint N d\gamma d^{D-3}x \sqrt{\sigma} u_a n_1 u_b U_0^{abcd}$$

$$\Theta_{r-y} \equiv \oint ds_2 K_{yy} = \oint NdE K_{yy}$$

is the opening angle at the $r - y$ surface,

$$S_{W_r-y} \equiv \oint N d\gamma d^{D-3}x \sqrt{\sigma} u_a n_2 u_b U_0^{abcd}.$$
Finally we get the following uncertainty relation on the horizon of a D1D5 black hole:

\[ \Delta \Theta_{r-t} \Delta S_W \geq \hbar \]  
(12)

\[ \Delta \Theta_{r-y} \Delta S_{Wr-y} \geq \hbar \]  
(13)

The uncertainty at the opening angle in the \( r - y \) surface for D1D5 black hole is in agreement with the fuzzball proposal, where it was found that some specific string vibrations form metrics with conical singularities at the \( r - y \) surface [12, 13, 15, 16].

To conclude: we see that these singularities can be explained not only by string vibrations but also from quantum properties: the uncertainty principle. Note that the variables in the commutation relations must be canonical conjugate pairs which are obtained by singling out the radial direction. The radial direction can be regarded as the direction of a non-gravitational force that causes observers to “stand steel” in these coordinate frame. Thus, these singularities, which according to string theory are expected in quantum gravity theories, are derived by the uncertainty principle only when singling out the non-gravitational force direction.

2.3. Several expected quantum black hole properties can also be derived by non-Cauchy foliation

Finally let's mention shortly several example regarding the benefit of the non-Cauchy surface foliation to the research of quantum black hole.

The first example comes long ago from the membrane paradigm [17]. The membrane models a black hole as a thin, classically radiating membrane vanishingly close to the black hole's event horizon. It turns out that this non-Cauchy surface is useful for visualizing and calculating the effects predicted by quantum mechanics for the exterior physics of black holes.

The second example involves the Wheeler-De Witt metric probability wave equation. Recently, in [18], foliation in the radial direction was used to obtain Wheeler-De Witt metric probability wave equation on the apparent horizon hypersurface of the Schwarzschild-de Sitter black hole. By solving this equation, the authors found that a quantized Schwarzschild-de Sitter black hole has a nonzero value for the mass in its ground state. This property of quantum black holes leads to stable black hole remnants, which is also an expected property of quantum gravity.

The last example we mention shortly for the benefit of non-Cauchy foliation to quantum gravity involves “holographic quantization”. The holographic quantization uses spatial foliation in order to quantize the gravitational fields for different backgrounds in Einstein theory. This is carried out by singling out one of the spatial directions in a flat background [19], and also singling out the radial direction for a Schwarzschild metric [20]. Moreover, other works [21] even suggest that the holographic quantization causes the (3+1)D Einstein gravity to become effectively reduced to (2+1)D after solving the Lagrangian analogues of the Hamiltonian and momentum constraints.

All these examples support the idea that quantum gravity theories should involve constructing a quantum theory on non-Cauchy hypersurfaces and that the hypersurface direction should be the same as the direction of the effective non-gravitational force field at a point. In the next section we foliate spacetime along this direction, and discuss the implication of this foliation on Einstein equations the expected properties of quantum gravity.

3. Expected advantages of non-Cauchy foliation to quantum gravity.

In order to discuss the implication of the foliation of spacetime along the direction of an effective non-gravitational force field to the expected properties of quantum gravity we need first to describe properly some terms. In this section we foliate spacetime along this direction and then discuss its
expected benefits regarding the problems of quantum gravity.

3.1. Foliate spacetime along the direction of an effective non-gravitational force field

In order to single out the direction of the gravitational force, we first define its direction. We define it as follows: for a given metric in a D-dimensional spacetime, we take accelerating detectors that have a $D$ velocity unit vector field $u^a$ and acceleration $a^a = u^b \nabla_b u^a \equiv a^a$ (where $v^a$ is a unit vector and $u^a n_a = 0$). Note also that this direction is a space-like vector field. Next, we foliate spacetime with respect to the unit vector field $n_a$ by defining a $(D-1)$ hyper-surface which is normal to $n_a$ and constant $r$ on $\Sigma_{D-1}$. The lapse function $M$ and shift vector $W_a$ satisfy $r_a = M n_a + W_a$ where $r^a \nabla_a r = 1$ and $r$ is constant on $\Sigma_r$. The $\Sigma_r$ hyper-surfaces metric $h_{ab}$ is given by $g_{ab} = h_{ab} + n_a n_b$. The extrinsic curvature of the hyper-surfaces is given by $K_{ab} = -\frac{1}{2} \mathcal{L}_n h_{ab}$ where $\mathcal{L}_n$ is the Lie derivative along $n^a$. The intrinsic curvature $R_{ab}^{(3)}$ is then given by the 2+1 Christoffel symbols: $\Gamma^k_{ab} = \frac{1}{2} h^{kl} \left( \frac{\partial h_{ak}}{\partial x^l} + \frac{\partial h_{al}}{\partial x^k} - \frac{\partial h_{ak}}{\partial x^l} \right)$ so that

$$R_{ab}^{(3)} = \frac{\partial^2 h_{ab}}{\partial x^2} - \frac{\partial h_{a}^{\ k} \partial h_{kb}^{\ l}}{\partial x^2} + \Gamma_{ab}^{k} \Gamma^{i}_{k} - \Gamma_{ai}^{\ k} \Gamma^{i}_{lb} .$$

Thus, instead of 3+1 Einstein equation,

$$R_{ab}^{(4)} = 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) ,$$

one finds [26] that the 3+1 Einstein equation, can be written in terms of the $\Sigma_r$ hyper-surfaces metric $h_{ab}$, it’s extrinsic curvature $K_{ab}$ and the intrinsic curvature $R_{ab}^{(3)} :$

$$R_{ab}^{(3)} + KK_{ab} - 2K_{ai}K^i_{b} - M^{-1} \left( \mathcal{L}_r K_{ab} + D_a D_b M \right) = 8\pi \left( S_{ab} - \frac{1}{2} (S - P) h_{ab} \right) ,$$

$$R^{(3)} + K^2 - K_{ab} K^{ab} = 16\pi P ,$$

$$D_b K^b_a - D_a K = 8\pi F_a .$$

where $D_a$ is the 2+1 covariant derivatives, $S_{ab} = h_{acd} h_{ad} T^{cd}$, $P = n_a n_b T^{ab}$ and $F_a = h_{acd} n_d T^{ab}$.

Note that whenever the extrinsic curvature does vanish, even only on the hypersurface $r_0$, this foliation leads to an interesting observation. Though a (2+1)D gravitational theory is believed to be a toy model for quantum gravity, this foliation suggests that a (3+1)D gravitational theory can be regarded as an “evolution” of a (2+1)D gravitational theory along a non-gravitational force direction. Thus, given the fact that a renormalized (2+1)D quantum gravity theory can be obtained, this leads to a (3+1)D quantum gravity originated from a (2+1)D renormalized theory. Whether or not this renormalized construction leads to an effectively renormalized gravitational theory on the (3+1)D is remain to be seen.

3.2. Implication of this foliation on the expected properties of quantum gravity

Now, let’s discuss the implication of this foliation on the expected properties of quantum gravity. We expect this foliation to contribute in three main topics. The first relates the equivalence principle and the insights of this equivalence in the quantum limit. The second is regarding the connection between quantum gravity and holography. Finally the third validates Verlinde’s suggestion which relates the origin of a third spatial dimension to the existence and direction of acceleration in a gravitational force. We expend on these issues.

1) The equivalence principle in the quantum limit. We argue that the foliation along the non-gravitational force, is related to the equivalence principle and can contribute impotent insights of this equivalence in the quantum limit. To see this, first note that any effective non-gravitational force causes
the observers to accelerate in a spatial direction and thus breaks the symmetry between the three spatial directions. As shown by the Unruh effect, this symmetry breaking is extremely important when dealing with quantum theories in flat space. Given the equivalence principle, we expect the same symmetry breaking to occur for accelerating observers in curved spacetime. Though it is extremely complicated (and maybe even impossible) to derive Unruh radiation in curved space, this means that accelerating detectors are expected to detect a kind of Unruh radiation \[25\] of a gravitational field as well. Moreover, as was shown in \[6\], singling out the effective non-gravitational field enables identification of the relevant canonical term and calculation of the expected surface density DoF obtained by acceleration observers. Thus singling out the effective non-gravitational force is expected to be a powerful tool in identifying and even eliminating the extra quantum gravitational fields obtained by accelerating observers in curved space.

2) The procedure reinforces the connection between quantum gravity and holography. We argue that this procedure reinforces the connection between quantum gravity and holography. This can be seen in Hamiltonian formalism, when singling out the effective non-gravitational force direction instead of a direction of time. In general the Hamiltonian formalism singles out the time direction and thus the time act as an independent parameter through which states evolve. We suggest to consider the symmetry breaking caused by an effective non-gravitational force field, and to use the direction of that field as the direction through which states evolve. This suggestion supports holography because using the Hamiltonian formalism in this case, leads to equations which determine the evolution of the gravitational fields along the effective non-gravitational force direction, i.e. along the space-like vector field \(r_a\). Moreover, they are determined not only by the Hamiltonian equation, but also by the "initial" condition on the unique non-Cauchy hyper-surface \(r = r_0\). Thus when singling out the effective non-gravitational force direction instead of a direction of time, we expect all the information needed to describe the gravitational field in the \(r \neq r_0\) is encoded on a non-Cauchy hyper-surface, as suggested by holography.

3) This procedure validates Verlinde’s suggestion. This procedure validates Verlinde’s suggestion which relates the origin of a third spatial dimension to the existence and direction of acceleration in a gravitational force. To clarify this, note that when we write the \(3+1\) Einstein equation, in terms of the \(\Sigma_r\) hyper-surfaces metric \(h_{ab}\), its extrinsic curvature \(K_{ab} = -\frac{1}{2}L_a h_{ab}\) and the intrinsic curvature \(K^{(3)}_{ab}\), we find that if on \(r = r_0\): \(K_{ab} = 0\), and also using gauge fixing in order to eliminate the term \(M^{-1}(L_a K_{ab} + D_a D_b M)\), then all the observers who are restricted to be on \(r = r_0\) are not accelerating. This is because \(a = n^a a_a = K^{ab} u_b u_a = 0\). Thus in this case gravity can be describe as a \((2 + 1)D\) theory. On the other hand, if on some \(r = r_0\): \(K_{ab} \neq 0\), then all the observers on the surface do accelerate and the \((3 + 1)D\) Einstein equation cannot be reduced to any kind of \((2 + 1)D\) Einstein equation. Thus this foliation reinforces Verlinde’s suggestion relating the direction of the acceleration direction to an extra \(3^{th}\) spatial dimension.

3.3. The implication of this foliation on the obstacles of quantum gravity

Next we discuss the implication of this foliation on the possibility of solving the obstacles of quantum gravity which are the problem of time in the ADM formalism and the renormalization problem.

Eliminate the problem of time in the ADM formalism. Our suggestion is strongly related to a problem with time in the ADM formalism \[3\]. This problem results from the conceptual difficulty that arises when attempting to combine quantum mechanics with general relativity. It arises from the
contrasting role of time within these two frameworks. In quantum theories time acts as an independent parameter through which states evolve, with the Hamiltonian operator acting as the generator of infinitesimal translations of quantum states through time. In contrast, general relativity treats time as a dynamical variable which interacts directly with matter. The problem of time occurs because in general relativity the Hamiltonian is a constraint that must vanish in vacuum. However, in any ordinary canonical theory, the Hamiltonian generates time translations. Therefore, we arrive at the conclusion that "nothing moves" ("there is no time") in general relativity. This so-called ‘frozen formalism’ caused much confusion when it was first discovered, since it seems to imply that nothing happens in a quantum theory of gravity.

The problem of time vanishes if, instead of the ordinary formulation which splits between one dimension of time and three dimensions of space, one splits between one dimension of space and the 2+1 dimensions of spacetime. This is because in this case, thou the new Hamiltonian vanish, the theory does still depend on the time coordinate even in the vacuum.

Solving the non-renormalization problem Moreover, if we consider a non-Cauchy foliation in the vacuum, then the vanishing of the new Hamiltonian turns into an advantage. This accrue because it leads to the possibility of describing the vacuum case as a renormalized (2+1)D quantum gravity theory. To see this, note that if instead of time direction, we foliate along one of the spatial directions, we arrive at the conclusion that, instead of "nothing changes along the time direction", it seems that "nothing changes along this spatial direction". In other words "there is no third spatial dimension" and the (3+1)D gravitational theory becomes effectively a (2+1)D theory. This is excellent from the point of view of quantum gravity since it is possible to quantize a (2+1)D gravitational theory.

4. Remarks on the expected causality problem due to the non-Cauchy foliation

This paper deals with the possibility of employing the Hamiltonian formalism in a unique way. Instead of singling out the direction of a time vector field, we single out the direction of the effective non-gravitational force. We expect this to be a powerful tool in order to obtain a quantum gravity theory that does not diverge in some cases, and to understand the origin of its divergence in others. However, since this involves singling out a spatial direction instead of a time direction, this also leads to vagueness regarding the basic concepts of relativistic quantum field theories.

Moreover, this is expected to suffer from problems such as causality, probability, conservation and unitarity and thus all these properties should be reexamined in this case. In order to learn how to overcome these difficulties we need to set aside the gravitational theory and focus on investigating known quantum field theories in flat spacetime. We need to investigate the possibility of obtaining relativistic quantum theories by singling out a spatial direction. This is expected to be useful since in this case we know what are the quantum properties that should be obtained and the fact that these theories do not have any of the expected problems.

In order to do that we repeat the known process of quantization, but instead of singling out time and thus Cauchy surface which can be defined as an equal time surface \( x_0 = 0 \), we use a non-Cauchy surface. A non-Cauchy surface can be defined by constant spatial coordinate surface \( x_1 = 0 \). In this case we expect that the quantization will be as follows. For a given Lagrangian density of a relativistic field theory \( \mathcal{L}(\phi(x), \partial_x \phi(x)) \), we define a new canonically conjugate momentum to the field variable \( \phi(x) \): \( \Pi_1(x) = \frac{\partial \mathcal{L}}{\partial \partial_x \phi(x)} \) where \( \phi' = \partial_x \phi \). The new Hamiltonian density will be \( \mathcal{H}_1 = \Pi_1(x) \phi' - \mathcal{L} \). Next we need to verify that the dynamical equation derived by Euler-Lagrange equation, can be written in the new Hamiltonian form as \( \phi(x)' = \{\phi(x), H_1\} \) and \( \Pi_1(x)' = \{\Pi_1(x), H_1\} \) (where \( H_1 = \int d^3x \mathcal{H}_1 = \int dx^0 dx^2 dx^3 \mathcal{H}_1 \)). For this purpose we need to assume equal \( x_1 \) bracket
relations between the field variable \( \phi(x) \) and the new conjugate momentum \( \Pi_1(x) \). However, if we use equal \( x_1 \) canonical Poisson bracket relations: \( \{ \phi(x), \phi(y) \}_x = 0 = \{ \Pi_1(x), \Pi_1(y) \}_x, \) and \( \{ \phi(x), \Pi_1(y) \}_x = \delta(x^0 - y^0)\delta(x^2 - y^2)\delta(x^3 - y^3) \) we get a non causal relation. Thus equal spatial coordinate bracket cannot be defined by the ordinary canonical Poisson brackets. We must identify correctly the equal \( x_1 \) classical bracket relations between the field variable \( \phi(x) \) and the new conjugate momentum \( \Pi_1(x) \), in order to find out whether the dynamical equation derived by Euler-Lagrange equation can be derived by using the Hamiltonian-like form. However, it seems that in order to identify correctly the equal \( x_1 \) classical bracket relations between the field variable \( \phi(x) \) and the new conjugate momentum \( \Pi_1(x) \) we need to have an extension of the canonical Poisson brackets in such a way that it will be causally defined even when foliating spacetime to non-Cauchy surfaces. Unfortunately, we do not have this kind of extension, and it may be that this kind of extension can not even be derived. For a scalar field, we suggested [27], a way to identify the equal \( x_1 \) canonical relation without using extended Poisson bracket and found that these relations are

\[
\{ \phi(x), \phi(y) \}_x = -i \int \frac{d^3k}{(2\pi)^3} e^{i(k \cdot (\vec{x} - \vec{y}))} \tag{14}
\]

\[
\{ \Pi_1(x), \Pi_1(y) \}_x = -i \int \frac{d^3k}{(2\pi)^3} e^{i(k \cdot (\vec{x} - \vec{y}))} \tag{15}
\]

\[
\{ \phi(x), \Pi_1(y) \}_x = 0. \tag{16}
\]

These relations are causal on the hypersurface \( x^1 = y^1 \) and it was found that the equation of motion derived by them and the new Hamiltonian leads to the expected equations of motion. This means that deriving a relativistic quantum gravity theory by singling out a spatial direction instead of a time direction is possible, at least for scalars in Minkowski space. However, in order to prove this generally one needs to extend this work to other cases such as different kind of fields, and hopefully in curved spacetime.

5. Summary

In this paper we suggest to use the Hamilton formalism in a unique way. We suggest to consider the symmetry breaking caused by an effective non-gravitational force field, and to use its direction as an independent parameter through which states evolve.

After a short introduction, we examined in the second section several works which support this idea. All these works, which comes from different areas in physics, single out the direction of a non-gravitational force in order to obtain different aspects of quantum gravity. We extended our discussion in two different areas: thermodynamics and string theory. In the third section we found that in some cases the \( (3+1)D \) gravitational theory can be regard as an “evolution” of a \( (2+1)D \) gravitational theory along a non-gravitational force direction. Then, given the fact that a renormalized \( (2+1)D \) quantum gravity theory can be obtained, this leads naturally to a \( (3+1)D \) quantum gravity which is originated from a renormalized theory. In the third section we also explained how this idea is in agreement with general expected concepts relating quantum gravity such as holography and Verlinde’s suggestion. Moreover, we discussed the option that this unique foliation is useful in order to eliminate the problem of time in the ADM formalism. Finally, in the forth section we discussed the causality problem expected from such non-Cauchy foliation. In this context we mention our latest work which solved the expected causality problem expected from such non-Cauchy foliation for scalars in Minkowski space.

To conclude, it seems that singling out the direction of an effective non-gravitational force field, instead of the direction of time, may leads to a \( (3+1)D \) quantum gravitational theory that can be regard as an “evolution” of a \( (2+1)D \) renormalized quantum gravitational theory along a non-gravitational
force direction. Whether or not this renormalized construction leads to an effectively renormalized gravitational theory on the (3+1)D is remain to be seen. However, even if this way of construction does not lead to a renormalized quantum gravitational field theory in the (3+1)D, this construction can be considered as a proof that our inability to renormalized the (3+1)D theory is related directly to the existence of non-gravitational forces in our everyday life.

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