Amortised Resource Analysis and Typed Polynomial Interpretations
(extended version)*

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We introduce a novel resource analysis for typed term rewrite systems based on a potential-based type system. This type system gives rise to polynomial bounds on the innermost runtime complexity. We relate the thus obtained amortised resource analysis to polynomial interpretations and obtain the perhaps surprising result that whenever a rewrite system $\mathcal{R}$ can be well-typed, then there exists a polynomial interpretation that orients $\mathcal{R}$. For this we adequately adapt the standard notion of polynomial interpretations to the typed setting.

\textit{Key words:} Term Rewriting, Types, Amortised Resource Analysis, Complexity of Rewriting, Polynomial Interpretations

1 Introduction

In recent years there have been several approaches to the automated analysis of the complexity of programs. Mostly these approaches have been developed independently in different communities and use a variety of different, not easily comparable techniques.

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Without hope for completeness, we mention work by Albert et al. \cite{1} that underlies COSTA, an automated tool for the resource analysis of Java programs. Related work, targeting C programs, has been reported by Alias et al. \cite{2}. In Zuleger et al. \cite{22} further approaches for the runtime complexity analysis of C programs is reported, incorporated into LOOPUS. Noschinski et al. \cite{17} study runtime complexity analysis of rewrite systems, which has been incorporated in AProVE. Finally, the RaML prototype \cite{11} provides an automated potential-based resource analysis for various resource bounds of functional programs and \( \mathcal{T}Cf \) \cite{4} is one of the most powerful tools for complexity analysis of rewrite systems.

Despite the abundance in the literature almost no comparison results are known that relate the sophisticated methods developed. Indeed a precise comparison often proves difficult. For example, on the surface there is an obvious connection between the decomposition techniques established by Gulwani and Zuleger in \cite{8} and recent advances on this topic in the complexity analysis of rewrite systems, cf. \cite{3}. However, when investigated in detail, precise comparison results are difficult to obtain. We exemplify the situation with a simple example that will also serve as running example throughout the paper.

**Example 1.1.** Consider the following term rewrite system (TRS for short) \( \mathcal{R}_\text{que} \), encoding a variant of an example by Okasaki \cite{18, Section 5.2}.

\[
\begin{align*}
1: & \text{chk}(\text{que}(\text{nil}, r)) \rightarrow \text{que}(\text{rev}(r), \text{nil}) \\
2: & \text{chk}(\text{que}(x \not\in xs, r)) \rightarrow \text{que}(x \not\in x, r) \\
3: & \text{tl}(\text{que}(x \not\in f, r)) \rightarrow \text{chk}(\text{que}(f, r)) \\
4: & \text{snoc}(\text{que}(f, r), x) \rightarrow \text{chk}(\text{que}(f, x \not\in r)) \\
5: & \text{rev}(x \not\in xs, ys) \rightarrow \text{rev}(x, x \not\in ys) \\
6: & \text{enq}(s(n)) \rightarrow \text{snoc}(\text{enq}(n), n) \\
7: & \text{enq}(0) \rightarrow \text{que}(\text{nil}, \text{nil}) \\
8: & \text{rev}(\text{nil}, ys) \rightarrow ys \\
9: & \text{rev}(xs) \rightarrow \text{rev}(xs, \text{nil}) \\
10: & \text{hd}(\text{que}(x \not\in f, r)) \rightarrow x \\
11: & \text{hd}(\text{que}(\text{nil}, r)) \rightarrow \text{err}_\text{head} \\
12: & \text{tl}(\text{que}(\text{nil}, r)) \rightarrow \text{err}_\text{tail}
\end{align*}
\]

\( \mathcal{R}_\text{que} \) encodes an efficient implementation of a queue in functional programming. A queue is represented as a pair of two lists \( \text{que}(f, r) \), encoding the initial part \( f \) and the reversal of the remainder \( r \). Invariant of the algorithm is that the first list never becomes empty, which is achieved by reversing \( r \) if necessary. Should the invariant ever be violated, an exception (\text{err}_\text{head} or \text{err}_\text{tail}) is raised.

We exemplify the physicist’s method of amortised analysis \cite{19}. We assign to every queue \( \text{que}(f, r) \) the length of \( r \) as potential. Then the amortised cost for each operation is constant, as the costly reversal operation is only executed if the potential can pay for the operation, compare \cite{18}. Thus, based on an amortised analysis, we deduce the optimal linear runtime complexity for \( \mathcal{R} \).

On the other hand let us attempt an application of the interpretation method to this example. Termination proofs by interpretations are well-established and can be traced back to work by Turing \cite{21}. We note that \( \mathcal{R}_\text{que} \) is polynomially terminating. Moreover, it is rather straightforward to restrict so-called polynomial interpretations \cite{5} suitably so that compatibility of a TRS \( \mathcal{R} \) induces polynomial runtime complexity, cf. \cite{6}. Such
polynomial interpretations are called restricted. However, it turns out that no restricted polynomial interpretation can exist that is compatible with \( R_{\text{que}} \). The reasoning is simple. The constraints induced by \( R_{\text{que}} \) imply that the function \( \text{snoc} \) has to be interpreted by a linear polynomial. Thus an exponential interpretation is required for enqueuing (\( \text{enq} \)). Looking more closely at the different proofs, we observe the following. While in the amortised analysis the potential of a queue \( \text{que}(f, r) \) depends only on the remainder \( r \), the interpretation of \( \text{que} \) has to be monotone in both arguments by definition. This difference induces that \( \text{snoc} \) is assigned a strongly linear potential in the amortised analysis, while only a linear interpretation is possible for \( \text{snoc} \).

Still it is possible to precisely relate amortised analysis to polynomial interpretations if we base our investigation on many-sorted (or typed) TRSs and make suitable use of the concept of annotated types originally introduced in [14]. More precisely, we establish the following results. We establish a novel runtime complexity analysis for typed constructor rewrite systems. This complexity analysis is based on a potential-based amortised analysis incorporated into a type system. From the annotated type of a term its derivation height with respect to innermost rewriting can be read off (see Theorem 3.1). The correctness proof of the obtained bound rests on a suitable big-step semantics for rewrite systems, decorated with counters for the derivation height of the evaluated terms. We complement this big-step semantics with a similar decorated small-step semantics and prove equivalence between these semantics. Furthermore we strengthen our first result by a similar soundness result based on the small-step semantics (see Theorem 4.1). Exploiting the small-step semantics we prove our main result that from the well-typing of a TRS \( R \) we can read off a typed polynomial interpretation that orients \( R \) (see Theorem 5.1).

While the type system exhibited is inspired by Hoffmann et al. [13] we generalise their use of annotated types to arbitrary (data) types. Furthermore the introduced small-step semantics (and our main result) directly establish that any well-typed TRS is terminating, thus circumventing the notion of partial big-step semantics introduced in [12]. Our main result can be condensed into the following observations. The physicist’s method of amortised analysis conceptually amounts to the interpretation method if we allow for the following changes:

- Every term bears a potential, not only constructor terms.
- Polynomial interpretations are defined over annotated types.
- The standard compatibility constraint is weakened to orientability, that is, all ground instances of a rule strictly decrease.

Our study is purely theoretic, and we have not (yet) attempted an implementation of the provided techniques. However, automation appears straightforward. Furthermore we have restricted our study to typed (constructor) TRSs. In the conclusion we sketch application of the established results to innermost runtime complexity analysis of untyped TRSs.

This paper is structured as follows. In the next section we cover some basics and introduce a big-step operational semantics for typed TRSs. In Section 3 we clarify our
and $V$ (by upper-case Greek letters. Let $\Gamma$ be a context and let $t: \sigma \in \Gamma$ be a term. The typing relation $A \vdash t : A$ expresses that based on context $\Gamma$, $t$ has type $A$ (with respect to the signature $F$). The typing rules that define the typing relation are given in Figure 2, where we forget the annotations. In the sequel we sometimes make use of an abbreviated notation for sequences of types $\vec{A} = A_1, \ldots, A_n$ and terms $\vec{t} := t_1, \ldots, t_n$.

A typed rewrite rule is a pair $l \to r$ of terms, such that (i) the type of $l$ and $r$ coincides, (ii) $rt(l) \in D$, and (iii) $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. An $S$-typed term rewrite system

2 Typed Term Rewrite Systems

Let $C$ denote a finite, non-empty set of constructor symbols and $D$ a finite set of defined function symbols. Let $S$ be a finite set of (data) types. A family $(X_A)_{A \in S}$ of sets is called $S$-typed and denotes as $X$. Let $\mathcal{V}$ denote an $S$-typed set of variables, such that the $\mathcal{V}_s$ are pairwise disjoint. In the following, variables will be denoted by $x, y, z, \ldots$, possibly extended by subscripts.

Following [16], a type declaration is of form $[A_1 \times \cdots \times A_n] \to C$, where $A_i$ and $C$ are types. Type declarations serve as input-output specifications for function symbols. We write $A$ instead of $[] \to A$. A signature $F$ (with respect to the set of types $S$) is a mapping from $C \cup D$ to type declarations. We often write $f: [A_1 \times \cdots \times A_n] \to C$ if $F(f) = [A_1 \times \cdots \times A_n] \to C$ and refer to a type declaration as a type, if no confusion can arise. We define the $S$-typed set of terms $T(D \cup C, \mathcal{V})$ (or $T$ for short): (i) for each $A \in S$: $\mathcal{V}_A \subseteq T_A$, (ii) for $f \in C \cup D$ such that $F(f) = [A_1, \ldots, A_n] \to A$ and $t_i \in T_{A_i}$, we have $f(t_1, \ldots, t_n) \in T_A$. Type assertions are denoted $t : C$. Terms of type $A$ will sometimes be referred to as instances of $A$: a term of list type, is simply called a list. If $t \in T(C, \emptyset)$ then $t$ is called a ground constructor term or a value. The set of values is denoted $T(C)$. The ($S$-typed) set of variables of a term $t$ is denoted $\mathcal{V}ar(t)$. The root of $t$ is denoted $rt(t)$ and the size of $t$, that is the number of symbols in $t$, is denoted as $|t|$. In the following, terms are denoted by $s, t, u, \ldots$, possibly extended by subscripts. Furthermore, we use $v$ (possibly indexed) to denote values.

A substitution $\sigma$ is a mapping from variables to terms that respects types. Substitutions are denoted as sets of assignments: $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$. We write $\text{dom}(\sigma)$ (rg($\sigma$)) to denote the domain (range) of $\sigma$; $\mathcal{V}rg(\sigma) := \mathcal{V}ar(\text{rg}(\sigma))$. Let $\sigma$ be a substitution and $V$ be a set of variables; $\sigma \upharpoonright V$ denotes the restriction of the domain of $\sigma$ to $V$. The substitution $\sigma$ is called a restriction of a substitution $\tau$ if $\tau \upharpoonright \text{dom}(\sigma) = \sigma$. Vice versa, $\tau$ is called extension of $\sigma$. Let $\sigma, \tau$ be substitutions such that $\text{dom}(\sigma) \cap \text{dom}(\tau) = \emptyset$. Then we denote the (disjoint) union of $\sigma$ and $\tau$ as $\sigma \uplus \tau$. We call a substitution $\sigma$ normalised if all terms in the range of $\sigma$ are values. In the following, all considered substitutions will be normalised.

A typing context is a mapping from variables $\mathcal{V}$ to types. Type contexts are denoted by upper-case Greek letters. Let $\Gamma$ be a context and let $t$ be a term. The typing relation $\Gamma \vdash t : A$ expresses that based on context $\Gamma$, $t$ has type $A$ (with respect to the signature $F$). The typing rules that define the typing relation are given in Figure 2, where we forget the annotations. In the sequel we sometimes make use of an abbreviated notation for sequences of types $\vec{A} = A_1, \ldots, A_n$ and terms $\vec{t} := t_1, \ldots, t_n$.

A typed rewrite rule is a pair $l \to r$ of terms, such that (i) the type of $l$ and $r$ coincides, (ii) $rt(l) \in D$, and (iii) $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. An $S$-typed term rewrite system
\[
\begin{align*}
\sigma \vdash x & \Rightarrow v \\
x \sigma = v \\
\sigma \vdash_0 x & \Rightarrow v \\
c \in \mathcal{C} \quad x_1 \sigma = v_1 \quad \cdots \quad x_n \sigma = v_n \\
\sigma \vdash_0 c(x_1, \ldots, x_n) & \Rightarrow c(v_1, \ldots, v_n)
\end{align*}
\]

\[
f(l_1, \ldots, l_n) \rightarrow r \in \mathcal{R} \quad \exists \tau \forall i: x_i \sigma = l_i \tau \\
\sigma \vdash_\tau \tau \vdash_\tau^m r \Rightarrow v \\
f(x_1, \ldots, x_n) \Rightarrow v
\]

all \( x_i \) are fresh

\[
\sigma \vdash_\rho \vdash_\rho^m f(x_1, \ldots, x_n) \Rightarrow v \\
\sigma \vdash_\rho \vdash_\rho^m t_1 \Rightarrow v_1 \quad \cdots \quad \sigma \vdash_\rho \vdash_\rho^m t_n \Rightarrow v_n \\
\sum_{i=0}^n m_i
\]

Here \( \rho := \{ x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \} \). Recall that \( \sigma, \tau, \) and \( \rho \) are normalised.

Figure 1: Operational Big-Step Semantics

(\textit{TRS for short}) over the signature \( \mathcal{F} \) is a finite set of typed rewrite rules. We define the \textit{innermost rewrite relation} \( \rightarrow_\mathcal{R} \) for typed TRSs \( \mathcal{R} \). For terms \( s \) and \( t \), \( s \rightarrow_\mathcal{R} t \) holds, if there exists a context \( C \), a normalised substitution \( \sigma \) and a rewrite rule \( l \rightarrow r \in \mathcal{R} \) such that \( s = C[l\sigma], \; t = C[r\sigma] \) and \( s, \; t \) are well-typed. In the sequel we are only concerned with \textit{innermost} rewriting. A TRS is \textit{orthogonal} if it is left-linear and non-overlapping \([5, 20]\). A TRS is \textit{completely defined} if all ground normal-forms are values. These notions naturally extend to typed TRS. In particular, note that an orthogonal typed TRS is confluent.

\textbf{Definition 2.1.} We define the \textit{runtime complexity} (with respect to \( \mathcal{R} \)) as follows:

\[
\text{rc}(n) := \max\{dh(t, \rightarrow) \mid t \text{ is basic and } |t| \leq n\}
\]

where a term \( t = f(t_1, \ldots, t_k) \) is called \textit{basic} if \( f \) is defined, and the terms \( t_i \) are only built over constructors and variables.

We study \textit{typed constructor} TRSs \( \mathcal{R} \), that is, for each rule \( f(l_1, \ldots, l_n) \rightarrow r \), the \( l_i \) are constructor terms. Furthermore, we restrict to \textit{completely defined} and \textit{orthogonal} systems. These restrictions are natural in the context of functional programming. If no confusion can arise from this, we simply call \( \mathcal{R} \) a TRS. \( \mathcal{F} \) denotes the signature underlying \( \mathcal{R} \). In the sequel, \( \mathcal{R} \) and \( \mathcal{F} \) are kept fixed.

\textbf{Example 2.1} (continued from Example 1.1). Consider the TRS \( \mathcal{R}_{\text{que}} \) and let \( S = \{ \text{Nat}, \text{List}, \text{Q} \} \), where \( \text{Nat}, \text{List}, \) and \( \text{Q} \) represent the type of natural numbers, lists over over natural number, and queues respectively. Then \( \mathcal{R}_{\text{que}} \) is an \( S \)-typed TRSs over signature \( \mathcal{F} \), where the signature of some constructors is as follows:

\[
\begin{align*}
0 &: \text{Nat} \\
s &: \text{Nat} \rightarrow \text{Nat} \\
\text{que} &: \text{List} \times \text{List} \rightarrow \text{Q} \\
nil &: \text{List} \\
\text{\#} &: \text{Nat} \times \text{List} \rightarrow \text{List}
\end{align*}
\]
In order to exemplify the type declaration of defined function symbols, consider

\[ \text{snoc}: [Q \times \text{Nat}] \rightarrow Q. \]

As \( R \) is completely defined any derivation ends in a value. On the other hand, as \( R \) is non-overlapping any innermost derivation is determined modulo the order in which parallel redexes are contracted. This allows us to recast innermost rewriting into an operational big-step semantics instrumented with resource counters, cf. Figure 1. The semantics closely resembles similar definitions given in the literature on amortised resource analysis (see for example [15, 12, 10]).

Let \( \sigma \) be a (normalised) substitution and let \( f(x_1, \ldots, x_n) \) be a term. It follows from the definitions that \( \sigma \vdash^m f(x_1, \ldots, x_n) \Rightarrow^* v \) iff \( \sigma \vdash f(x_1, \ldots, x_n) \Rightarrow v \). More precisely we have the following proposition.

**Proposition 2.1.** Let \( f \) be a defined function symbol of arity \( n \) and \( \sigma \) a substitution. Then \( \sigma \vdash^m f(x_1, \ldots, x_n) \Rightarrow v \) holds iff \( \text{dh}(f(x_1, \ldots, x_n), \vdash R) = m \) holds.

**Proof.** In proof of the direction from left to right, we show the stronger statement that \( \sigma \vdash^m t \Rightarrow v \) implies \( \text{dh}(t\sigma, \vdash R) = m \) by induction on the size of the proof of the judgement \( \sigma \vdash^m f(x_1, \ldots, x_n) \Rightarrow v \). For the opposite direction, we show that if \( \text{dh}(t\sigma, \vdash R) = m \), then \( \sigma \vdash^m t \Rightarrow v \) by induction on the length of the derivation \( D: t\sigma \vdash^* v \). \( \square \)

The next (technical) lemma follows by a straightforward inductive argument.

**Lemma 2.1.** Let \( t \) be a term, let \( v \) be a value and let \( \sigma \) be a substitution. If \( \sigma \vdash^m t \Rightarrow v \) and if \( \sigma' \) is an extension of \( \sigma \), then \( \sigma' \vdash^m t \Rightarrow v \). Furthermore the sizes of the derivations of the corresponding judgements are the same.

### 3 Annotated Types

Let \( S \) be a set of types. We call a type \( A \in S \) **annotated**, if \( A \) is decorated with resource annotation. These annotations will allow us to read off the potential of a well-typed term \( t \) from the annotations.

**Definition 3.1.** Let \( S \) be a set of types. An **annotated type** \( A^\vec{p} \), is a pair consisting of a type \( A \in S \) and a vector \( \vec{p} = (p_1, \ldots, p_k) \) over non-negative rational numbers, typically natural numbers. The vector \( \vec{p} \) is called **resource annotation**.

Resource annotations are denoted by \( \vec{p}, \vec{q}, \vec{u}, \vec{v}, \ldots \), possibly extended by subscripts and we write \( A \) for the set of such annotations. For resource annotations \( (p) \) of length 1 we write \( p \). The empty annotation \( () \) is written 0. We will see that a resource annotation does not change its meaning if zeroes are appended at the end, so, conceptually, we can identify \( () \) with \( (0) \). If \( \vec{p} = (p_1, \ldots, p_k) \) we write \( k = |\vec{p}| \) and \( \max \vec{p} = \max_i p_i \). We define the notations \( \vec{p} \leq \vec{q} \) and \( \vec{p} + \vec{q} \) and \( \lambda \vec{p} \) for \( \lambda \geq 0 \) component-wise, filling up with 0s if
needed. So, for example \((1, 2) \leq (3, 4, 5)\) and \((1, 2) + (3, 4, 5) = (4, 6, 5)\). Furthermore, we recall the additive shift \(\uparrow\) given by
\[
\uparrow(p) := (p_1 + p_2, p_2 + p_3, \ldots, p_{k-1} + p_k, p_k).
\]
We also define the interleaving \(\vec{p} \|\| \vec{q}\) by \((p_1, q_1, p_2, q_2, \ldots, p_k, q_k)\) where, as before the shorter of the two vectors is padded with 0s. Finally, we use the notation \(\triangleleft p = p_1\) for the first entry of an annotation vector.

If no confusion can arise, we refer to annotated types simply as types. In contrast to Hoffmann et al. [13, 9], we generalise the concept of annotated types to arbitrary (data) types. In [13] only list types, in [9] list and tree types have been annotated.

**Definition 3.2.** Let \(\mathcal{F}\) be a signature. Suppose \(\mathcal{F}(f) = [A_1 \times \cdots \times A_n] \rightarrow C\), such that the \(A_i\) \((i = 1, \ldots, n)\) and \(C\) are types. Consider the annotated types \(A_i^{\vec{v}_i}\) and \(A^{\vec{v}}\). Then an **annotated type declaration** for \(f\) is a type declaration over annotated types, decorated with a number \(p\):
\[
[A_1^{\vec{v}_1} \times \cdots \times A_n^{\vec{v}_n}] \xrightarrow{p} C^{\vec{v}}.
\]
The set of annotated type declarations is denoted as \(\mathcal{F}_{\text{pol}}\).

We write \(A^0\) instead of \([\nabla] \xrightarrow{0} A^0\). We lift signatures to **annotated signatures** \(\mathcal{F} : C \cup \mathcal{D} \rightarrow (\mathcal{P}(\mathcal{F}_{\text{pol}}) \setminus \emptyset)\) by mapping a function symbol to a non-empty set of annotated type declarations. Hence for any \(f \in C \cup \mathcal{D}\) we allow multiple types. If \(f\) has result type \(C\), then for each annotation \(C^{\vec{q}}\) there should exist exactly one declaration of the form \([A_1^{\vec{v}_1} \times \cdots \times A_n^{\vec{v}_n}] \xrightarrow{p} C^{\vec{q}}\) in \(\mathcal{F}(f)\). Moreover, constructor annotations are to satisfy the **superposition principle**: If a constructor \(c\) admits the annotations \([A_1^{\vec{v}_1} \times \cdots \times A_n^{\vec{v}_n}] \xrightarrow{p} C^{\vec{q}}\) and \([A_1^{\vec{v}_1} \times \cdots \times A_n^{\vec{v}_n}] \xrightarrow{p'} C^{\vec{q'}}\) then it also has the annotations \([A_1^{\lambda \vec{v}_1} \times \cdots \times A_n^{\lambda \vec{v}_n}] \xrightarrow{\lambda p} C^{\lambda \vec{q}}\) \((\lambda > 0)\) and \([A_1^{\vec{v}_1 + \bar{v}} \times \cdots \times A_n^{\vec{v}_n + \bar{v}}] \xrightarrow{\bar{p} + \bar{p'}} C^{\bar{q} + \bar{q'}}\).

Note that, in view of superposition and uniqueness, the annotations of a given constructor are uniquely determined once we fix the annotated types for result annotations of the form \((0, \ldots, 0, 1)\) (remember the implicit filling up with 0s). An annotated signature \(\mathcal{F}\) is simply called signature, where we sometimes write \(f : [A_1 \times \cdots \times A_n] \xrightarrow{\uparrow} C\) instead of \([A_1 \times \cdots \times A_n] \xrightarrow{\uparrow} C \in \mathcal{F}(f)\).

**Example 3.1** (continued from Example 2.1). In order to extend \(\mathcal{F}\) to an annotated signature we can set
\[
\begin{align*}
\mathcal{F}(0) & := \{\text{Nat}^{\vec{p}} \mid \vec{p} \in A\} & \mathcal{F}(\text{Nat}) & := \{[\text{Nat}^{\vec{q}}] \xrightarrow{\triangleleft \vec{p}} \text{Nat}^{\vec{p}} \mid \vec{p} \in A\} \\
\mathcal{F}(\text{nil}) & := \{\text{List}^{\vec{p}} \mid \vec{p} \in A\} & \mathcal{F}(\text{List}) & := \{[\text{Nat}^0 \times \text{List}^{\vec{q}}] \xrightarrow{\triangleleft \vec{p}} \text{List}^{\vec{p}} \mid \vec{p} \in A\} \\
\mathcal{F}(\text{Que}) & := \{[\text{List}^{\vec{p}} \times \text{List}^{\vec{q}}] \xrightarrow{\downarrow \vec{p} \vec{q}} \text{Q}^{\vec{p} \vec{q}} \mid \vec{p}, \vec{q} \in A\}
\end{align*}
\]
In particular, we have the typings \(\sharp : [\text{Nat}^0 \times \text{List}^{\vec{q}}] \xrightarrow{\sharp} \text{List}^{\vec{q}}\) and \(\sharp : [\text{Nat}^0 \times \text{List}^{(10, 7)}] \xrightarrow{3} \text{List}^{(3, 7)}\) and que : \([\text{List}^1 \times \text{List}^3] \xrightarrow{0} \text{Q}(1, 3)\).

We omit annotations for the defined symbols and refer to Example 3.3 for a complete signature with different constructor annotations.
The next definition introduces the notion of the potential of a value.

**Definition 3.3.** Let \( v = c(v_1, \ldots, v_n) \in T(C) \) and let \([A_1 \times \cdots \times A_n] \stackrel{l}{\Rightarrow} C \in F(c)\). Then the potential of \( v \) is defined inductively as

\[
\Phi(v; C) := p + \Phi(v_1; A_1) + \cdots + \Phi(v_n; A_n).
\]

Note that by assumption the declaration in \( F(c) \) is unique.

**Example 3.2** (continued from Example 3.1). It is easy to see that for any term \( t \) of type \( \text{Nat}^0 \), we have \( \Phi(t; \text{Nat}^0) = 0 \) and \( \Phi(t; \text{Nat}^1) = \lambda t \).

If \( l \) is a list then \( \Phi(l; \text{List}^{p,q}) = p \cdot |l| + q \cdot \left(\frac{|l|}{2}\right) \), where \( |l| \) denotes the length of \( l \), that is the number of \( \varepsilon \) in \( l \). Let \( |l| = \ell \). We proceed by induction on \( \ell \). Let \( \ell = 0 \). Then \( \Phi(\text{nil}; \text{List}^{p,q}) = 0 \) as required. Suppose \( \ell = \ell' + 1 \):

\[
\Phi(n \# \ell'; \text{List}^{p,q}) = p + \Phi(n; \text{Nat}^0) + \Phi(l'; \text{List}^{p+q,q})
\]

\[
= p + (p + q) \cdot \ell' + q \cdot \left(\frac{\ell'}{2}\right)
\]

\[
= p \cdot \ell + q \cdot \left[\frac{\ell' + 1}{2}\right] + p \cdot \ell + q \cdot \left(\frac{\ell}{2}\right).
\]

More generally, we have \( \Phi(l; \text{List}^p) = \sum_{i} p_i (\frac{|l|}{i}) \). Finally, if \( \text{que}(l, k) \) has type \( Q \) then \( \Phi(\text{que}(l, k); Q^{p\bar{p}q}) = \Phi(l; \text{List}^p) + \Phi(k; \text{List}\bar{q}) \).

The sharing relation \( \gamma(A^\bar{p} | A^p_1, A^p_2) \) holds if \( A = A_1 = A_2 \) and \( p_1 + p_2 = \bar{p} \). The subtype relation is defined as follows: \( A^\bar{p} <: B^\bar{q} \), if \( A = B \) and \( p \geq q \).

**Lemma 3.1.** If \( \gamma(A^\bar{p} | A^p_1, A^p_2) \) then \( \Phi(v; A^\bar{p}) = \Phi(v; A^p_1) + \Phi(v; A^p_2) \) holds for any value of type \( A \). If \( A^\bar{p} <: B^\bar{q} \) then \( \Phi(v; A^\bar{p}) \geq \Phi(v; B^\bar{q}) \) again for any \( v : A \).

**Proof.** The proof of the first claim is by induction on the structure of \( v \). We note that by superposition together with uniqueness the additivity property propagates to the argument types. For example, if we have the annotations \( s : [\text{Nat}^2] \rightarrow [\text{Nat}^3] \) and \( s : [\text{Nat}^4] \rightarrow [\text{Nat}^5] \) and \( s : [\text{Nat}^7] \rightarrow [\text{Nat}^9] \) then we can conclude \( x = 6 \), \( y = 8 \), for this annotation must be present by superposition and there can only be one by uniqueness.

The second claim follows from the first one and nonnegativity of potentials.

The set of typing rules for TRSs are given in Figure 2. Observe that the type system employs the assumption that \( R \) is left-linear. In a nutshell, the method works as follows: Let \( \Gamma \) be a typing context and let us consider the typing judgement \( \Gamma \vdash t : A \) derivable from the type rules. Then \( p \) is an upper-bound to the amortised cost required for reducing \( t \) to a value. The derivation height of \( t\sigma \) (with respect to innermost rewriting) is bound by the difference in the potential before and after the evaluation plus \( p \). Thus if the sum of the potential of the arguments of \( t\sigma \) is in \( O(n^k) \), where \( n \) is the size of the arguments, then the runtime complexity of \( R \) lies in \( O(n^k) \).

Recall that any rewrite rule \( l \rightarrow r \in R \) can be written as \( f(l_1, \ldots, l_n) \rightarrow r \) with \( l_i \in T(C, V) \). We introduce well-typed TRSs.
for all \( \sigma \) such that \( \Phi(\sigma) \) on the length of \( \Xi \).

**Proof.** Let \( \Pi \) be the proof deriving \( \Phi(\sigma) \). We state and prove our first soundness result.

Let for all \( \sigma \) be well-typed according to \( \Gamma \) if for all \( x \in \text{dom}(\Gamma) \) \( x\sigma \) is of type \( \Gamma(x) \). We extend Definition 3.3 to substitutions \( \sigma \) and typing contexts \( \Gamma \). Suppose \( \sigma \) is well-typed with respect to \( \Gamma \). Then 
\[ \Phi(\sigma; \Gamma) := \sum_{x \in \text{dom}(\Gamma)} \Phi(x\sigma; \Gamma(x)) \]
We state and prove our first soundness result.

**Theorem 3.1.** Let \( \mathcal{R} \) and \( \sigma \) be well-typed. Suppose \( \Gamma \vdash \sigma \vdash t \vdash A \) and \( \sum_{i=1}^{n} k_i \). Then \( \Phi(\sigma; \Gamma) \) is well-typed.

**Proof.** Let \( \Pi \) be the proof deriving \( \sigma \vdash t \vdash v \) and let \( \Xi \) be the proof of \( \Phi(\sigma; \Gamma) \). The proof of the theorem proceeds by main-induction on the length of \( \Pi \) and by side-induction on the length of \( \Xi \).

1. Suppose \( \Pi \) has the form
\[
\frac{x \sigma = v}{\sigma \vdash t \Rightarrow v},
\]
such that \( t = x \) and \( v = x\sigma \). Wlog. \( \Xi \) is of form \( x \vdash A \vdash x : A \). Then 
\[ \Phi(\sigma; \Gamma) = \Phi(x\sigma : A) \]
and the theorem follows.
2. Suppose \( \Pi \) has the form
\[
c \in C \quad x_1 \sigma = v_1 \quad \ldots \quad x_n \sigma = v_n
\]
\[
\sigma \vdash^m c(x_1, \ldots, x_n) \Rightarrow c(v_1, \ldots, v_n),
\]
such that \( t = c(x_1, \ldots, x_n) \) and \( v = c(v_1, \ldots, v_n) \). Further wlog. we suppose that \( \Xi \) ends in the following judgement:
\[
x_1: A_1^{u_1}, \ldots, x_n: A_n^{u_n} \vdash^m c(x_1, \ldots, x_n) : C^{u}. \]
Then we have \([\vdash]^m A_1^{u_1} \times \cdots \times A_n^{u_n}] \vdash^m \Xi \in \mathcal{F}(c)\) and thus:
\[
\Phi(\sigma; \Gamma) + p = p + \sum_{i=1}^{n} \Phi(\sigma; \mathcal{A}_i^{u_i}) = p + \sum_{i=1}^{n} \Phi(v_i; \mathcal{A}_i^{u_i}) = \Phi(c(v_1, \ldots, v_n); C^{u}),
\]
from which the theorem follows.

3. Suppose \( \Pi \) ends in the following rule:
\[
\exists f(l_1, \ldots, l_n) \rightarrow r \in \mathcal{R} \quad \exists \tau \forall i: x_i \sigma = l_i \tau \quad \sigma \vdash^m r \Rightarrow v \quad \sigma \vdash^{m+1} f(x_1, \ldots, x_n) \Rightarrow v
\]
Then \( t = f(x_1, \ldots, x_n) \) and \( f(x_1, \ldots, x_n) \sigma = f(l_1, \ldots, l_n) \tau \). Suppose \( \text{Var}(f(l)) = \{y_1, \ldots, y_k\} \) and let \( \text{Var}(l_i) = \{y_{i1}, \ldots, y_{ik}\} \) for \( i \in \{1, \ldots, n\} \). As \( \mathcal{R} \) is left-linear we have \( \text{Var}(f(l_1, \ldots, l_n)) = \bigcup_{i=1}^{n} \text{Var}(l_i) \). We set \( \Gamma = x_1: A_1, \ldots, x_n: A_n \). By the assumption \( \vdash^m t: A \) and well-typedness of \( \mathcal{R} \) we obtain
\[
\Phi(\sigma; \Gamma) = \Phi(\sigma; \mathcal{A}) = \sum_{i=1}^{n} \Phi(\mathcal{A}_i) = \sum_{i=1}^{n} (k_i + \Phi(y_{i1} \tau; B_{i1}) + \cdots + \Phi(y_{ik} \tau; B_{ik}))
\]
\[
= \Phi(\sigma \cup \tau; \Delta) + \sum_{i=1}^{n} k_i.
\]
Here the first equality follows by an inspection on the case for the constructors. In sum, we obtain
\[
\Phi(\sigma; \Gamma) - \Phi(v; C) + p = \Phi(\sigma \cup \tau; \Delta) + \sum_{i=1}^{n} k_i - \Phi(v; C) + p \geq m + 1,
\]
from which the theorem follows.
4. Suppose the last rule in $\Pi$ has the form
\[
\sigma \triangleright \rho \vdash f(x_1, \ldots, x_n) \Rightarrow v \quad \sigma \vdash t_1 \Rightarrow v_1 \quad \cdots \quad \sigma \vdash t_n \Rightarrow v_n \quad m = \sum_{i=0}^{n} m_i
\]
\[
\sigma \vdash f(t_1, \ldots, t_n) \Rightarrow v.
\]
We can assume that $t$ is linear, compare the case employing the share operator. Hence the last rule in the type inference $\Xi$ is of the following form.
\[
=\Delta
\]
\[
y_1: A_1, \ldots, y_n: A_n \vdash p_0 : f(\overrightarrow{y}) : C \\
\Gamma_1 \vdash p_1 : t_1 : A_1 \quad \cdots \quad \Gamma_n \vdash p_n : t_n : A_n \quad p = \sum_{i=0}^{n} p_i
\]
\[
\Gamma_1, \ldots, \Gamma_n \vdash f(t_1, \ldots, t_n) : C.
\]
By induction hypothesis: $\Phi(\sigma; \Gamma_i) - \Phi(v_i : A_i) + p_i \geq m_i$ for all $i = 1, \ldots, n$. Hence
\[
\sum_{i=1}^{n} \Phi(\sigma; \Gamma_i) - \sum_{i=1}^{n} \Phi(v_i : A_i) + \sum_{i=1}^{n} p_i \geq \sum_{i=1}^{n} m_i.
\] (2)
Again by induction hypothesis we obtain:
\[
\Phi(\sigma \triangleright \rho; \Delta) - \Phi(v : C) + p_0 \geq m_0.
\] (3)
Now $\Phi(\sigma; \Gamma) = \sum_{i=1}^{n} \Phi(\sigma; \Gamma_i)$ and $\Phi(\sigma \triangleright \rho; \Delta) = \sum_{i=1}^{n} \Phi(v_i : A_i)$. Due to (2) and (3), we obtain
\[
\Phi(\sigma; \Gamma) + \sum_{i=0}^{n} p_i = \sum_{i=1}^{n} \Phi(\sigma; \Gamma_i) + \sum_{i=1}^{n} p_i + p_0
\]
\[
\geq \sum_{i=1}^{n} \Phi(v_i : A_i) + \sum_{i=1}^{n} m_i + p_0 \geq \Phi(v : C) + \sum_{i=0}^{n} m_i,
\]
and thus $\Phi(\sigma; \Gamma) \geq \Phi(v : C) + p \geq m$.

5. Suppose $\Xi$ is of form
\[
\Gamma \vdash p : t : C \quad p' \geq p
\]
\[
\Gamma \vdash p' : t : C.
\]
By side-induction on $\Gamma \vdash p : t : C$ together with $\sigma \vdash t \Rightarrow v$ we conclude $\Phi(\sigma; \Gamma) = \Phi(v : A) + p \geq m$. Then the theorem follows from the assumption $p' \geq p$.

6. Suppose $\Xi$ is of form
\[
\Gamma \vdash p : t : C
\]
\[
\Gamma, x : A \vdash t : C.
\]
We conclude by side-induction together with $\sigma \vdash t \Rightarrow v$ we conclude $\Phi(\sigma; \Gamma) = \Phi(v : A) + p \geq m$. Clearly $\Phi(\sigma; \Gamma, x : A) \geq \Phi(\sigma; \Gamma)$ and the theorem follows.
7. Suppose $\Xi$ is of form
\[
\frac{\Gamma, x: A_1, y: A_2 \vdash t[x, y]: C \quad \gamma(A_1, A_2)}{\Gamma, z: A \vdash t[z, z]: C}
\]
By assumption $\sigma \vdash^m t[z, z] \Rightarrow v$; let $\rho := \sigma \uplus \{ x \mapsto z\sigma, y \mapsto z\sigma \}$. As $\sigma \vdash^m t[x, y] \Rightarrow v$ by definition. From the side-induction on $\Gamma, x: A_1, y: A_2 \vdash t[x, y]: C$ and $\rho \vdash^m t[x, y] \Rightarrow v$ we conclude that
\[
\Phi(\rho; \Gamma, x: A_1, y: A_2) - \Phi(v; C + p \geq m).
\]
The theorem follows as by definition of $\rho$ and Lemma 3.1 we obtain
\[
\Phi(\sigma; \Gamma, z: A) = \Phi(\rho; \Gamma, x: A_1, y: A_2).
\]

8. Suppose $\Xi$ is of form
\[
\frac{\Gamma, x: B \vdash t: C \quad A <: B}{\Gamma, x: A \vdash t: C}
\]
By assumption $\sigma \vdash^m t \Rightarrow v$ and by induction hypothesis $\Phi(\sigma; \Gamma, x: B) - \Phi(v; A) + p \geq m$. By definition of the subtype relation $\Phi(x\sigma; A) \geq \Phi(x\sigma; B)$. Hence the theorem follows.

9. Suppose $\Xi$ is of form
\[
\frac{\Gamma \vdash^p t: D \quad D <: C}{\Gamma \vdash^p t: C}
\]
The case follows similarly to the sub-case before by induction hypothesis. From this the theorem follows.

The second assertion of the theorem follows from the first together with the assumption that every defined symbol in $\mathcal{F}$ is well-typed and Proposition 2.1.

Example 3.3 (continued from Example 1.1). Consider the TRS $\mathcal{R}_{\text{que}}$ from Example 1.1. We detail the signature $\mathcal{F}$, starting with the constructor symbols.

\[
\begin{align*}
0 & : \text{Nat}\quad s : \text{Nat} \rightarrow \text{Nat} \\
\text{err\_head} & : \text{Nat} \\
\text{nil} : \text{List}^q & : \text{Nat}^0 \times \text{List}^q \\
\text{enq} & : \text{Nat} \\
\text{rev\_tail} & : \text{Nat}^0 \times \text{List}^q \\
\text{rev} & : \text{List}^q \rightarrow \text{List}^q \\
\text{snoc} & : \text{List}^q \rightarrow \text{Nat} \\
\end{align*}
\]
where $p, q \in \mathbb{N}$. Furthermore we make use of the following types for defined symbols.

\[
\begin{align*}
\text{chk} & : [\text{Nat}^0]^3 \rightarrow \text{Nat}^0 \\
\text{tl} & : [\text{Nat}^0]^4 \rightarrow \text{Nat}^0 \\
\text{hd} & : \text{Nat}^0 \rightarrow \text{Nat}^0 \\
\text{rev'} & : [\text{List}^q \times \text{List}^q]^1 \rightarrow \text{List}^q \\
\text{rev} & : [\text{List}^q \times \text{List}^q]^2 \rightarrow \text{List}^q \\
\text{snoc} & : [\text{Nat}^0]^5 \rightarrow [\text{Nat}^0]^1
\end{align*}
\]
Let $F$ denote the induced signature. Based on the above definitions it is not difficult to verify that $R_{\text{que}}$ is well-typed w.r.t. $F$. We show that $\text{enq}$ is well-typed. Consider rule 6. First, we observe that 6 resource units become available for the recursive call, as $n: \text{Nat}^6 \vdash \text{enq}(s(n)) : \text{Nat}^6$ is derivable. Second, we have the following partial type derivation; missing parts are easy to fill in.

\[\begin{array}{c}
q: \text{Q}^{(0,1)}, m: \text{Nat}^0 \vdash \text{snoc}(q, m): \text{Q}^{(0,1)}, n_1: \text{Nat}^6 \vdash \text{enq}(n_1): \text{Q}^{(0,1)} \\
n_1: \text{Nat}^6, n_2: \text{Nat}^0 \vdash \text{snoc}(\text{enq}(n_1), n_2): \text{Q}^{(0,1)} \\
n: \text{Nat}^6 \vdash \text{snoc}(\text{enq}(n), n): \text{Q}^{(0,1)} \\
n_2: \text{Nat}^0 \vdash 0 n_2: \text{Nat}^0
\end{array}\]

Considering rule 7, it is easy to see that $n: \text{Nat}^6 \vdash \text{que}(\text{nil}, \text{nil}): \text{Q}^{(0,1)}$ is derivable. Thus $\text{enq}$ is well-typed and we conclude optimal linear runtime complexity of $R_{\text{que}}$.

**Polynomial bounds** Note that if the type annotations are chosen such that for each type $A$ we have $\Phi(v: A) \in O(n^k)$ for $n = |v|$ then $rc_R(n) \in O(n^{k+1})$ as well. The following proposition gives a sufficient condition as to when this is the case and in particular subsumes the type system in [13].

**Theorem 3.2.** Suppose that for each constructor $c$ with $[A^i_{\vec{v}} \times \cdots \times A^n_{\vec{v}}] \overset{c}{\to} C_{\vec{w}} \in F(c)$, there exists $\vec{r}_i \in A$ such that $\vec{v}_i \leq \vec{w} + \vec{r}_i$ where $\max|\vec{r}_i| \leq \max|\vec{w}| =: r$ and $p \leq r$ with $|\vec{r}_i| < |\vec{w}| = k$. Then $\Phi(v: C_{\vec{w}}) \leq r^k|v|^k$.

**Proof.** The proof is by induction on the size of $v$. Note that, if $k = 0$ then $\Phi(v: C_{\vec{w}}) = 0$. This follows by superposition and uniqueness. Otherwise, we have

\[\Phi(c(v_1, \ldots, v_n): C_{\vec{w}}) \leq r + \Phi(v_1: A^i_{\vec{v}_1} + \vec{r}_1) + \cdots + \Phi(v_n: A^n_{\vec{v}_n} + \vec{r}_n) \leq r(1 + |v_1|^k + |v_1|^{k-1} + \cdots + |v_n|^k + |v_n|^{k-1}) \leq r(1 + |v_1| + \cdots + |v_n|)^k = r|v|^k \, .\]

Here we employ Lemma **3.1** to conclude for all $i = 1, \ldots, n$: 

\[\Phi(v_i: A^i_{\vec{v}_i} + \vec{r}_i) = \Phi(v_i: A^i_{\vec{w}}) + \Phi(v_i: A^i_{\vec{r}_i}) \, .\]

Based on this observation we apply induction hypothesis to obtain the second line. Furthermore in the last line we employ the multinomial theorem. \qed

We note that our running example satisfies the premise to the proposition. In concrete cases more precise bounds than those given by Theorem **3.2** can be computed as has been done in Example **3.2**. The next example clarifies that potentials are not restricted to polynomials.

**Example 3.4.** Consider that we annotate the constructors for natural numbers as $0: \text{Nat}^{\vec{f}}$ and $s: [\text{Nat}^{2\vec{f}}] \overset{\vec{f}}{\to} \text{Nat}^{\vec{f}}$. We then have, for example, $\Phi(t: \text{Nat}^1) = 2^i + 1 - 1$.  

13
\[ x\sigma = v \]
\[ \frac{}{\vdash^0 (x, \sigma) \rightarrow (v, \sigma)} \]
\[ c \in C \quad x_1\sigma = v_1 \cdots x_n\sigma = v_n \]
\[ \frac{}{\vdash^0 (c(x_1, \ldots, x_n), \sigma) \rightarrow (c(v_1, \ldots, v_n), \sigma)} \]

\[ \forall i: v_i \text{ is a value } \]
\[ \rho = \{ x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \} \quad f \text{ is defined and all } x_i \text{ are fresh} \]
\[ \frac{}{\vdash^0 (f(v_1, \ldots, v_n), \sigma) \rightarrow (f(x_1, \ldots, x_n), \sigma \uplus \rho)} \]
\[ f(l_1, \ldots, l_n) \rightarrow r \in \mathcal{R} \quad \forall i: x_i\sigma = l_i\tau \]
\[ \frac{}{\vdash (f(x_1, \ldots, x_n), \sigma) \rightarrow (r, \sigma \uplus \tau)} \]
\[ \frac{}{\vdash (t_i, \sigma) \rightarrow (u, \sigma')} \]
\[ \frac{}{\vdash (f(\ldots, t_i, \ldots), \sigma) \rightarrow (f(\ldots, u, \ldots), \sigma')} \]

Note that the substitutions \( \sigma, \sigma', \tau, \) and \( \rho \) are normalised.

Figure 3: Operational Small-Step Semantics

As mentioned in the introduction, foundational issues are our main concern. However, the potential-based method detailed above seems susceptible to automation. One conceives the resource annotations as variables and encodes the constraints of the typing rules in Figure 2 over these resource variables.

4 Small-Step Semantics

The big-step semantics, the type system, and Theorem 3.1 provide an amortised resource analysis for typed TRSs that yields polynomial bounds. However, Theorem 3.1 is not directly applicable, if we want to link this analysis to the interpretation method. We recast the method and present a small-step semantics, which will be used in our second soundness results (Theorem 4.1 below), cf. Figure 3. As the big-step semantics, the small-step semantics is decorated with counters for the derivation height of the evaluated terms.

Suppose \( \vdash (s, \sigma) \rightarrow (t, \sigma') \) holds for terms \( s, t \) and substitutions \( \sigma, \sigma' \). An inspection of the rules shows that \( \sigma' \) is an extension of \( \sigma \). Moreover we have the following fact.

**Lemma 4.1.** Let \( s, t \) be terms, let \( \sigma \) be a normalised substitution such that \( \text{Var}(s) \subseteq \text{dom}(\sigma) \) and suppose \( \vdash (s, \sigma) \rightarrow (t, \sigma') \). Then \( \sigma' \) extends \( \sigma \) and \( s\sigma = s\sigma' \).

**Proof.** The first assertion follows by induction on the relation \( \vdash (s, \sigma) \rightarrow (t, \sigma') \). Now suppose \( \sigma = \sigma' \upharpoonright \text{dom}(\sigma) \). Then \( s\sigma = s(\sigma' \upharpoonright \text{dom}(\sigma)) = s\sigma' \).

The transitive closure of the judgement \( \vdash^m (s, \sigma) \rightarrow (t, \tau) \) is defined as follows:

1. \( \vdash^m (s, \sigma) \rightarrow (t, \tau) \) if \( \vdash^m (s, \sigma) \rightarrow (t, \tau) \)
2. \( \vdash^{m_1+m_2} (s, \sigma) \rightarrow (u, \rho) \) if \( \vdash^{m_1} (s, \sigma) \rightarrow (t, \tau) \) and \( \vdash^{m_2} (t, \tau) \rightarrow (u, \rho) \).

14
The next lemma proves the equivalence of big-step and small-step semantics.

**Lemma 4.2.** Let $\sigma$ be a normalised substitution, let $t$ be a term, $\text{Var}(t) \subseteq \text{dom}(\sigma)$, and let $v$ be a value. Then $\sigma \vdash t \Rightarrow v$ if and only if $\vdash (t,\sigma) \Rightarrow (v,\sigma')$, where $\sigma'$ is an extension of $\sigma$.

**Proof.** First we prove the direction from left to right.

1. Suppose $\Pi$ has the form:

   \[
   \sigma \uplus \exists \sigma = v
   \]

   such that $t = x$ and $v = x\sigma$. Hence we obtain $\vdash (x,\sigma) \Rightarrow (v,\sigma)$.

2. Suppose $\Pi$ has the form:

   \[
   c \in C \quad x_1\sigma = v_1 \quad \ldots \quad x_n\sigma = v_n
   \]

   such that $t = c(x_1,\ldots,x_n)$ and $v = c(v_1,\ldots,v_n)$. Again, we directly obtain $\vdash (t,\sigma) \Rightarrow (v,\sigma)$.

3. Suppose the last rule in $\Pi$ if of form:

   \[
   f(l_1,\ldots,l_n) \Rightarrow r \in R \quad \forall i: x_i\sigma = l_i\tau \quad \sigma \uplus \tau \vdash (l_i\tau,\sigma) \Rightarrow (r,\sigma \uplus \tau)
   \]

   where $t = f(x_1,\ldots,x_n)$. By hypothesis there exists an extension $\sigma'$ of $\sigma \uplus \tau$ such that $\vdash (r,\sigma \uplus \tau) \Rightarrow (v,\sigma')$. Furthermore, we have $\vdash (t,\sigma) \Rightarrow (v,\sigma')$. By definition $\text{dom}(\sigma) \cap \text{dom}(\tau) = \emptyset$. Hence $\sigma'$ is an extension of $\sigma$.

4. Finally, suppose the last rule in $\Pi$ has the form

   \[
   \sigma \uplus \rho \vdash m_0 f(x_1,\ldots,x_n) \Rightarrow v \quad \sigma \vdash m_1 t_1 \Rightarrow v_1 \quad \ldots \quad \sigma \vdash m_n t_n \Rightarrow v_n \quad m = \sum_{i=0}^{n} m_i
   \]

   where $t = f(t_1,\ldots,t_n)$. By induction hypothesis (and repeated use of Lemma 2.1), we have for all $i = 1,\ldots,n$: $\vdash (t_{i-1},\sigma_{i-1}) \Rightarrow (v_{i-1},\sigma_i)$, where we set $\sigma_0 = \sigma$ and note that all $\sigma_i$ are extensions of $\sigma$. As $\vdash (v_0,\ldots,v_n,\sigma_n) \Rightarrow (f(x_1,\ldots,x_n),\sigma_n \uplus \rho)$ we obtain:

   \[
   \vdash m_0 (f(t_1,\ldots,t_n),\sigma) \Rightarrow (f(x_1,\ldots,x_n),\sigma_n \uplus \rho).
   \]

   Furthermore, by Lemma 2.1 and the induction hypothesis there exists a substitution $\sigma'$ such that

   \[
   \vdash m_0 (f(x_1,\ldots,x_n),\sigma_n \uplus \rho) \Rightarrow (v,\sigma'),
   \]

   where $\sigma'$ extends $\sigma_n \uplus \rho$ (and thus also $\sigma$ as $\text{dom}(\sigma_n) \cap \text{dom}(\rho) = \emptyset$). From (4) and (5) we obtain $\vdash (t,\sigma) \Rightarrow (v,\sigma')$. 

15
This establishes the direction from left to right. Now we consider the direction from right to left. The proof of the first reduction \[ m \langle t, \sigma \rangle \rightarrow \langle u, \sigma'' \rangle \] in \( D \) is denoted as \( \Xi \).

1. Suppose \( \Xi \) has either of the following forms
\[
\begin{align*}
x\sigma &= v & x_1\sigma &= v_1 & \ldots & x_n\sigma &= v_n
\end{align*}
\]
Then the lemma follows trivially.

2. Suppose \( \Xi \) has the form
\[
\forall i: v_i \text{ is a value } \rho = \{ x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \} \quad f \text{ is defined and all } x_i \text{ are fresh}
\]
We apply the induction hypothesis to conclude \( \sigma \uplus \rho \vdash_m f(x_1, \ldots, x_n) \Rightarrow v \). Moreover, we observe that \( \sigma \vdash_0 v_i \Rightarrow v_i \) holds for all \( i = 1, \ldots, n \). (This follows by a straightforward inductive argument.) From this we derive \( \sigma \vdash_0 f(v_1, \ldots, v_n) \Rightarrow v \) as follows:
\[
\begin{align*}
\sigma \uplus \rho \vdash_m f(x_1, \ldots, x_n) &\Rightarrow v \\
\sigma \vdash_0 v_1 &\Rightarrow v_1 \\
\ldots \\
\sigma \vdash_0 v_n &\Rightarrow v_n
\end{align*}
\]
3. Suppose \( \Xi \) has the form
\[
f(t_1, \ldots, t_n) \rightarrow r \in R \quad \forall i: x_i\sigma = l_i\tau
\]
\[
\vdash_m \langle f(x_1, \ldots, x_n), \sigma \rangle \rightarrow \langle r, \sigma \uplus \tau \rangle
\]
such that \( \sigma' \) is an extension of \( \sigma \uplus \tau \). By induction hypothesis we conclude \( \sigma \uplus \tau \vdash_{m'} r \Rightarrow v \). In conjunction with an application of the rule
\[
\begin{align*}
f(t_1, \ldots, t_n) &\rightarrow r \in R \quad \forall i: x_i\sigma = l_i\tau \\
\sigma \vdash_{m'+1} f(x_1, \ldots, x_n) &\Rightarrow v
\end{align*}
\]
we derive \( \sigma \vdash_{m'+1} f(x_1, \ldots, x_n) \Rightarrow v \) as required.

4. Suppose \( \Xi \) has the form
\[
\vdash_m \langle t_i, \sigma \rangle \rightarrow \langle u, \sigma'' \rangle
\]
\[
\vdash_m \langle f(\ldots, t_i, \ldots), \sigma \rangle \rightarrow \langle f(\ldots, u, \ldots), \sigma'' \rangle
\]
such that \( \sigma' \) is an extension of \( \sigma'' \). Then by induction hypothesis we obtain: \( \sigma'' \vdash_{m'} f(\ldots, u, \ldots) \Rightarrow v \). Furthermore by induction hypothesis we have \( \sigma \vdash_{m'} t_i \Rightarrow v_1 \).
5. Suppose the initial sequence of $D$ is based on the following reductions, where $m = \sum_{i=1}^{n} m_i + m'$.

$$
\begin{align*}
\vdash^{m_1} & \langle f(t_1, \ldots, t_n), \sigma \rangle \to \langle f(v_1, \ldots, t_n), \sigma_1 \rangle \\
\vdash^{m_n} & \langle f(v_1, \ldots, t_n), \sigma \rangle \to \langle f(v_1, \ldots, v_n), \sigma_n \rangle \\
\vdash^0 & \langle f(v_1, t_2, \ldots, t_n), \sigma_n \rangle \to \langle f(x_1, \ldots, x_n), \sigma_n \cup \rho \rangle.
\end{align*}
$$

We apply induction hypothesis on $\vdash^{m'} \langle f(x_1, \ldots, x_n), \sigma' \cup \rho \rangle \to \langle v, \sigma' \rangle$ and conclude: $\sigma' \cup \rho \vdash^{m'} f(x_1, \ldots, x_n) \Rightarrow v$. Again by induction hypothesis and inspection of the corresponding proofs, we obtain $\sigma_{i-1} \vdash^{m_i} t_i \Rightarrow v_i$ for all $i = 1, \ldots, n$. (We set $\sigma_0 := \sigma$.) Due to Lemma 4.1 we have $t_i \sigma_i = t_i \sigma$. Thus, for all $i$, $\sigma \vdash^{m_i} t_i \Rightarrow v_i$.

Note that $\text{dom}(\sigma_n) \cap \text{dom}(\rho) = \emptyset$. Hence, from $\sigma_n \cup \rho \vdash^{m'} f(x_1, \ldots, x_n) \Rightarrow v$ we obtain $\sigma \cup \rho \vdash^{m'} t \Rightarrow v$ follows.

We extend the notion of potential (cf. Definition 3.3) to ground terms.

**Definition 4.1.** Let $t = f(t_1, \ldots, t_n) \in T(D \cup C)$ and let $[A_1 \times \cdots \times A_n] \xrightarrow{p} C q \in F(f)$. Then the potential of $t$ is defined as follows:

$$
\Phi(t; C) := (p - q) + \Phi(t_1; A_1) + \cdots + \Phi(t_n; A_n).
$$

Note that by assumption the declaration in $F(f)$ is unique.

**Example 4.1** (continued from Example 3.3). Recall the types of $\text{que}$ and $\text{chk}$. Let $q = \text{que}(f, r)$ be a queue. We obtain $\Phi(\text{chk}(q); Q^{0,1}) = 3 + \Phi(q; Q^{0,1}) = 3 + \Phi(f; \text{List}^0) + \Phi(r; \text{List}^1) = 3 + |r|$.

**Lemma 4.3.** Let $\mathcal{R}$ and $\sigma$ be well-typed. Suppose $\Gamma \vdash^E t: A$. Then we have $\Phi(\sigma; \Gamma) + p \geq \Phi(t; A)$.

**Proof.** Let $\Xi$ denote the proof of $\Gamma \vdash^E t: A$.

1. Let $t = x$ and thus wlog. $\Xi$ is of form

$$
\dfrac{
x: A \vdash^0 x: A.
}{\text{Then } \Phi(\sigma; \Gamma) = \Phi(x \sigma; A) = \Phi(t \sigma; A), \text{ from which the lemma follows.}}
$$

2. Let $t = f(x_1, \ldots, x_n)$ where $f \in C \cup D$. Thus wlog. $\Xi$ is of form

$$
\begin{align*}
f & \in C \cup D \quad \quad [A_1^{u_1} \times \cdots \times A_n^{u_n}] \xrightarrow{p} C^\varnothing \in F(f) \\
x_1: A_1^{u_1}, \ldots, x_n: A_n^{u_n} & \vdash^{p} f(x_1, \ldots, x_n): C^\varnothing.
\end{align*}
$$

17
Hence we obtain
\[
\Phi(\sigma; \Gamma) + p = \sum_{i=1}^{n} \Phi(x_i \sigma; A_i^p) + p = \Phi(t \sigma; C^p),
\]
and the lemma follows.

3. Suppose \( t = f(t_1, \ldots, t_n) \), such that \( t \not\in V \) and \( f \in C \cup D \). Thus \( \Xi \) is of form
\[
\begin{array}{c}
\xrightarrow{\rho} \Delta \\
\xrightarrow{p_0} f(x_1, \ldots, x_n): A \\
\xrightarrow{p_1} t_1: A_1 \\
\xrightarrow{\vdots} \xrightarrow{\Gamma_i} \\
\xrightarrow{\Gamma_n} t_n: A_n \\
\xrightarrow{\Gamma_1, \ldots, \Gamma_0} f(t_1, \ldots, t_n): A
\end{array}
\]
where \( p = \sum_{i=0}^{n} p_i \). Then by induction hypothesis we have \( \Phi(\sigma; \Gamma_i) + p_i \geq \Phi(t_i \sigma; A_i) \) for all \( i = 1, \ldots, n \). Hence \( \sum_{i=1}^{n} \Phi(\sigma; \Gamma_i) + p_0 \geq \sum_{i=1}^{n} \Phi(t_i \sigma; A_i) \).

Let \( \rho := \{ x_1 \mapsto t_1 \sigma, \ldots, x_n \mapsto t_n \sigma \} \). Again by induction hypothesis we have \( \Phi(\rho; \Delta) + p_0 \geq \Phi(f(x_1, \ldots, x_n) \rho; A) \). Note that \( f(x_1, \ldots, x_n) \rho = t \sigma \) and \( x_i \rho = t_i \sigma \) by construction. We obtain
\[
\Phi(\sigma; \Gamma) + p_0 \geq \Phi(t \sigma; A).
\]

4. Suppose \( \Xi \) is of form:
\[
\begin{array}{c}
\xrightarrow{\rho} t: C \\
\xrightarrow{p'} p' \geq p
\end{array}
\]
By induction hypothesis, we have \( \Phi(\sigma; \Gamma) + p \geq \Phi(t \sigma; A) \). Then the lemma follows from the assumption \( p' \geq p \).

5. Suppose \( \Xi \) ends with one of the following structural rules
\[
\begin{align*}
\xrightarrow{\rho} t: C & \quad \xrightarrow{\rho} t: C \\
\xrightarrow{p} t[x, y]: C & \quad \xrightarrow{\rho} t[z, z]: C
\end{align*}
\]
We only consider the second rule, as the first alternatives follows trivially. Let \( \rho := \sigma \uplus \{ x \mapsto z \sigma, y \mapsto z \sigma \} \); by induction hypothesis, we have \( \Phi(\sigma; \Gamma, x: A_1, y: A_2) + p \geq \Phi(t[x, y] \rho; A) \). By definition of \( \rho \) and Lemma 3.1 we obtain
\[
\Phi(\sigma; \Gamma, z: A) = \Phi(\rho; \Gamma, x: A_1, y: A_2).
\]
Hence \( \Phi(\sigma; \Gamma, z: A) + p \geq \Phi(t[z, z] \sigma; A) \) follows from \( t[x, y] \rho = t[z, z] \sigma \).
6. Suppose $\Xi$ ends either in a sub- or in a supertyping rule:

$$
\frac{\Gamma, x:B \vdash p \cdot t:C}{\Gamma, x:A \vdash p \cdot t:C} \quad \frac{\Gamma \vdash p \cdot t:D \quad D <: C}{\Gamma \vdash p \cdot t:C}
$$

Consider the second rule. We have to show that $\Phi(\sigma; \Gamma) + p \geq \Phi(t\sigma; C)$. This follows from induction hypothesis, which yields $\Phi(\sigma; \Gamma) + p \geq \Phi(t\sigma; D)$ as $\Phi(t\sigma; D) \geq \Phi(t\sigma; C)$ by definition of the subtyping relation. The argument for the first rule is similar. This concludes the inductive argument.

We obtain our second soundness result.

**Theorem 4.1.** Let $R$ and $\sigma$ be well-typed. Suppose $\Gamma \vdash p \cdot t:A$ and $\nu \cdot (t, \sigma) \rightarrow (u, \sigma')$. Then $\Phi(\sigma; \Gamma) - \Phi(u\sigma'; A) + p \geq m$. Thus if for all ground basic terms $t$ and types $A$: $\Phi(t; A) \in O(n^k)$, where $n = |t|$, then $rc_R(n) \in O(n^k)$.

**Proof.** Let $\Pi$ be the proof of the judgement $\nu \cdot (t, \sigma) \rightarrow (u, \sigma')$ and let $\Xi$ denote the proof of $\Gamma \vdash p \cdot t:A$. The proof proceeds by main-induction on the length of $\Pi$ and by side-induction on the length of $\Xi$. We focus on some interesting cases.

1. Suppose $\Pi$ has the form

$$
\frac{x\sigma = u}{\nu \cdot \langle x, \sigma \rangle \rightarrow \langle u, \sigma \rangle },
$$

such that $t = x$ and $u = x\sigma$. As $\sigma$ is normalised $u$ is a value. Wlog, we can assume that $\Xi$ is of form $x:A \vdash p \cdot x:A$. It suffices to show $\Phi(\sigma; \Gamma) \geq \Phi(u\sigma; A)$, which follows from Lemma 4.3 as $x\sigma = u = u\sigma$.

2. Suppose $\Pi$ has the form

$$
\frac{x_1\sigma = u_1 \quad \cdots \quad x_n\sigma = u_n}{\nu \cdot \langle c(x_1, \ldots, x_n), \sigma \rangle \rightarrow \langle c(u_1, \ldots, u_n), \sigma \rangle },
$$

such that $t = c(x_1, \ldots, x_n)$ and $u = c(x_1\sigma, \ldots, x_n\sigma)$, which again is a value. Further let $\Xi$ end in the judgement:

$$
x_1:A_{v_1}, \ldots, x_n:A_{u_n} \vdash p \cdot c(x_1, \ldots, x_n):C^\sigma.
$$

Let $\Gamma = x_1:A_{v_1}, \ldots, x_n:A_{u_n}$; by Lemma 4.3 we have $\Phi(\sigma; \Gamma) + p \geq \Phi(t\sigma; A) = \Phi(u\sigma; A)$ as $t\sigma = u = u\sigma$.

3. Suppose $\Pi$ has the form

$$
\forall i: v_i \text{ is a value} \quad \rho = \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\} \quad f \text{ is defined and all } x_i \text{ are fresh}
\frac{\nu \cdot (f(v_1, \ldots, v_n), \sigma) \rightarrow (f(x_1, \ldots, x_n), \sigma \uplus \rho)}
$$

19
5. Suppose the last rule in \( \Pi \) has the form
\[
\tau = \tau \in \text{dom}(\sigma) \cap \text{dom}(\rho) = \varnothing. \quad \text{By Lemma 4.3 we have } \Phi(\sigma: \Gamma) + p \geq \Phi(\tau: A). \quad \text{Then the theorem follows as } t\sigma = f(x_1, \ldots, x_n)(\sigma \uplus \rho) \text{ from above.}
\]

4. Suppose \( \Pi \) has the form
\[
f(l_1, \ldots, l_n) \rightarrow r \in R \quad \forall i; \quad x_i \sigma = l_i \tau
\]
Then \( t = f(x_1, \ldots, x_n) \) and \( f(x_1, \ldots, x_n)\sigma = f(l_1, \ldots, l_n)\tau \). Suppose \( \text{Var}(f(l_i)) = \{y_1, \ldots, y_{k_i}\} \) and let \( \text{Var}(l_i) = \{y_{i1}, \ldots, y_{i d_i}\} \) for \( i \in \{1, \ldots, n\} \). As \( R \) is left-linear we have \( \text{Var}(f(l_1, \ldots, l_n)) = \bigcup_{i=1}^{n} \text{Var}(l_i) \). We set \( \Gamma = x_1: A_1, \ldots, x_n: A_n \). By the assumption \( \Gamma \vdash t: A \) and well-typedness of \( R \) we obtain
\[
\begin{array}{c}
y_1: B_1, \ldots, y_{\ell}: B_{\ell} \vdash \sum_{i=1}^{n} k_i \rightarrow r \vdash C
\end{array}
\]
as in (1). We have
\[
\Phi(\sigma: \Gamma) + p = \sum_{i=1}^{n} \Phi(x_i \sigma: A_i) + p
\]
\[
= \sum_{i=1}^{n} (k_i + \Phi(\tau_{i1} \cap B_{i1}) + \cdots + \Phi(\tau_{i d_i} \cap B_{d_i})) + p
\]
\[
= \Phi(\tau: \Delta) + \sum_{i=1}^{n} k_i + (p - 1) + 1
\]
\[
\geq \Phi(\tau: C) + 1 \geq \Phi(\sigma \uplus \tau: C) + 1.
\]

Here the first equality follows by an inspection on the cases for the constructors and \( \Phi(\tau: \Delta) + \sum_{i=1}^{n} k_i + (p - 1) \geq \Phi(\tau: C) \) follows due to Lemma 4.3 and (6). Furthermore note that \( \tau \tau = r(\sigma \uplus \tau) \), as \( \text{dom}(\sigma) \cap \text{dom}(\tau) = \varnothing. \)

5. Suppose the last rule in \( \Pi \) has the form
\[
\begin{array}{c}
f(t_1, \ldots, t_n) \rightarrow \langle u, \sigma' \rangle \quad \Gamma_1 \vdash f(t_1, \ldots, t_n), \sigma \rightarrow \langle f(u, \ldots, t_n), \sigma' \rangle.
\end{array}
\]
Wlog. the last rule in the type inference \( \Xi \) is of the following form, where we can assume that every variable occurs at most once in \( f(t_1, \ldots, t_n) \).
\[
\begin{array}{c}
\Delta \vdash x_1: A_1, \ldots, x_n: A_n \quad \vdash f(\vec{x}) \quad CT_1 \vdash t_1: A_1 \quad \cdots \quad \Gamma_n \vdash t_n: A_n \quad p = \sum_{i=0}^{n} p_i
\end{array}
\]
\[
\begin{array}{c}
\Gamma_1, \ldots, \Gamma_n \vdash f(t_1, \ldots, t_n): C
\end{array}
\]
By induction hypothesis on $\Gamma, t_1: A_1$ we obtain (i) $\Phi(\sigma: \Gamma_1) - \Phi(u\sigma': A_1) + p_1 \geq 1$ and $n - 1$ applications of Lemma 4.3 yield (ii) $\Phi(\sigma: \Gamma_i) + p_i \geq \Phi(t_i \sigma: A_i)$ for all $i = 2, \ldots, n$. We set $\rho := \{x_1 \mapsto u\sigma', x_2 \mapsto t_2 \sigma, \ldots, x_n \mapsto t_n \sigma\}$. Another application of Lemma 4.3 on $\Delta \vdash t_1, \ldots, t_n: C$ yields (iii) $\Phi(\rho: \Delta) + p_0 \geq \Phi(f(x_1 \rho, x_2 \rho, \ldots, x_n \rho): C)$. Finally, we observe $\Phi(\sigma: \Gamma) = \sum_{i=1}^n \Phi(\sigma: \Gamma_i)$. The theorem follows by combining the equations in (i)–(iii).

6. Suppose $\Xi$ is of form:

$$\Gamma \vdash t: C \quad p' \geq p$$

By side-induction on $\Gamma \vdash t: C$ and $m \langle t, \sigma \rangle \rightarrow \langle u, \sigma' \rangle$ we conclude that $\Phi(\sigma: \Gamma) - \Phi(u\sigma': A) + p \geq m$. Then the theorem follows from the assumption $p' \geq p$.

7. Suppose $\Xi$ is of form:

$$\Gamma, x: A \vdash t: C$$

We conclude by side-induction that $\Phi(\sigma: \Gamma) - \Phi(u\sigma': A + p) \geq m$. As $\Phi(\sigma: \Gamma, x: A) \geq \Phi(\sigma: \Gamma)$ the theorem follows.

8. Suppose $\Xi$ is of form:

$$\Gamma, x: A_1, y: A_2 \vdash t[x, y]: C \quad \gamma(A|A_1, A_2)
\Gamma, z: A \vdash t[z, z]: C$$

By assumption $m \langle t[z, z], \sigma \rangle \rightarrow \langle u, \sigma' \rangle$; let $\rho := \sigma \uplus \{x \mapsto z \sigma, y \mapsto z \sigma\}$. By side-induction on $\Gamma, x: A_1, y: A_2 \vdash t[x, y]: C$ and $m \langle t[x, y], \rho \rangle \rightarrow \langle u, \sigma' \rangle$ we conclude that for all $\Phi(\rho: \Gamma, x: A_1, y: A_2) - \Phi(u\sigma': A) + p \geq m$. By definition of $\rho$ and Lemma 3.1, we obtain $\Phi(\sigma: \Gamma, z: A) = \Phi(\rho: \Gamma, x: A_1, y: A_2)$, from which the theorem follows.

9. Suppose $\Xi$ ends either in a sub- or in a supertyping rule:

$$\Gamma, x: B \vdash t: C \quad A <: B \quad \Gamma \vdash t: D \quad D <: C$$

Consider the first rule. By assumption $m \langle t, \sigma \rangle \rightarrow \langle u, \sigma' \rangle$ and by definition $\Phi(\sigma: \Gamma, x: A) \geq \Phi(\sigma: \Gamma, x: B)$. Thus the theorem follows by side-induction hypothesis.

\[\square\]
5 Typed Polynomial Interpretations

We adapt the concept of polynomial interpretation to typed TRSs. For that we suppose a mapping \([ \cdot ]\) that assigns to every annotated type \(C\) a subset of the natural numbers, whose elements are ordered with \(>\) in the standard way. The set \([ [C] ]\) is called the interpretation of \(C\).

**Definition 5.1.** An interpretation \(\gamma\) of function symbols is a mapping from function symbols and types to functions over \(\mathbb{N}\). Consider a function symbol \(f\) and an annotated type \(C\) such that \(\mathcal{F}(f) \ni [A_1 \times \cdots \times A_n] \xrightarrow{p} C\). Then the interpretation \(\gamma(f, C) : [A_1] \times \cdots \times [A_n] \rightarrow [C]\) of \(f\) is defined as follows:

\[
\gamma(f, C)(x_1, \ldots, x_n) := x_1 + \cdots + x_n + p.
\]

Note that by assumption the declaration in \(\mathcal{F}(f)\) is unique and thus \(\gamma(f, C)\) is unique.

Interpretations of function symbols naturally extend to interpretation on ground terms.

\[
\llbracket f(t_1, \ldots, t_n) : C \rrbracket := \gamma(f, C)([t_1 : A_1]^\gamma, \ldots, [t_n : A_n]^\gamma).
\]

Let \(\mathcal{R}\) be a well-typed and let the interpretation \(\gamma\) of function symbols in \(\mathcal{F}\) be induced by the well-typing of \(\mathcal{R}\). Then by construction \([t : A]^\gamma = \Phi(t : A)\).

**Example 5.1** (continued from Example 3.3). Based on Definition 5.1 we obtain the following definitions of the interpretation of function symbols \(\gamma\). We start with the constructor symbols.

\[
\begin{align*}
\gamma(0, \text{Nat}^p) &= 0 \\
\gamma(s, \text{Nat}^p)(x) &= x + p \\
\gamma(\text{err}\_\text{head}, \text{Nat}^p) &= 0 \\
\gamma(\text{nil}, \text{List}^q) &= 0 \\
\gamma(\sharp, \text{List}^q)(x, y) &= x + y + q \\
\gamma(\text{que}(Q^{(0,1)}))(x, y) &= x + y,
\end{align*}
\]

where \(p, q \in \mathbb{N}\). Similarly the definition of \(\gamma\) for defined symbols follows from the signature detailed in Example 3.3. It is not difficult to see that for any rule \(l \rightarrow r \in \mathcal{R}_{\text{que}}\) and any substitution \(\sigma\), we obtain \([l\sigma]^\gamma > [r\sigma]^\gamma\). We show this for rule 1.

\[
\llbracket \text{chk}(\text{que}(\text{nil}, r\sigma)) : Q^{(0,1)} \rrbracket^\gamma = [r\sigma : \text{List}^1]^\gamma + 3 > 0 = [\text{rev}(r\sigma) : \text{List}^0]^\gamma + [\text{nil} : \text{List}^1]^\gamma = [\text{que}(\text{rev}(r\sigma), \text{nil}) : Q^{(0,1)}]^\gamma.
\]

Orientability of \(\mathcal{R}_{\text{que}}\) with the above given interpretation implies the optimal linear innermost runtime complexity.

We lift the standard order \(>\) on the interpretation domain \(\mathbb{N}\) to an order on terms as follows. Let \(s\) and \(t\) be terms of type \(A\). Then \(s > t\) if for all well-typed substitutions \(\sigma\) we have \([s\sigma : A]^\gamma > [t\sigma : A]^\gamma\).
Theorem 5.1. Let $\mathcal{R}$ be well-typed, constructor TRS over signature $\mathcal{F}$ and let the interpretation of function symbols $\gamma$ be induced by the type system. Then $l > r$ for any rule $l \rightarrow r \in \mathcal{R}$. Thus if for all ground basic terms $t$ and types $A$: $\llbracket t : A \rrbracket^\gamma \in O(n^k)$, where $n = |t|$, then $\text{rc}\mathcal{R}(n) \in O(n^k)$.

Proof. Let $l = f(l_1, \ldots, l_n)$ and let $x_1, \ldots, x_n$ be fresh variables. Suppose further $\mathcal{F}(f) \ni [A_1 \times \cdots \times A_n] \xrightarrow{p} C$. As $\mathcal{R}$ is well-typed we have

$$\vdash_{\mathcal{F}} x_1 : A_1, \ldots, x_n : A_n \vdash_p f(x_1, \ldots, x_n) : C,$$

for $p \in \mathbb{N}$.

Now suppose that $\tau$ denotes any well-typed substitution for the rule $l \rightarrow r$. It is standard way, we extend $\tau$ to a well-typed substitution $\sigma$ such that $l\tau = f(x_1, \ldots, x_n)$. By definition of the small-step semantics, we obtain

$$\vdash (f(x_1, \ldots, x_n), \sigma) \rightarrow (r, \sigma \uplus \tau).$$

Then by Lemma 4.1 $\Phi(\sigma; \Gamma) + p > \Phi(r(\sigma \uplus \tau); C)$ and by definitions, we have:

$$\Phi(l\tau; C) = \Phi(f(x_1\sigma, \ldots, x_n\sigma); C) = \sum_{i=1}^{n} \Phi(x_i\sigma; A_i) + p = \Phi(\sigma; \Gamma) + p.$$

Furthermore, observe that $r(\sigma \uplus \tau) = r\tau$ as $\text{dom}(\sigma) \cap \text{dom}(\tau) = \emptyset$. In sum, we obtain $\Phi(l\tau; C) > \Phi(r\tau; C)$, from which we conclude $\llbracket l\tau; C \rrbracket^\gamma \gamma > \llbracket r\tau; C \rrbracket^\gamma$. As $\tau$ was chosen arbitrarily, we obtain $\mathcal{R} \subseteq >$.

We say that an interpretation orients a typed TRS $\mathcal{R}$, if $\mathcal{R} \subseteq >$. As an immediate consequence of the theorem, we obtain the following corollary.

Corollary 5.1. Let $\mathcal{R}$ be a well-typed and constructor TRS. Then there exists a typed polynomial interpretation over $\mathbb{N}$ that orients $\mathcal{R}$.

At the end of Section 3 we have remarked on the automatability of the obtained amortised analysis. Observe that Theorem 5.1 gives rise to a conceptually quite different implementation. Instead of encoding the constraints of the typing rules in Figure 2 one directly encode the orientability constraints for each rule, cf. [7].

6 Conclusion

This paper is concerned with the connection between amortised resource analysis, originally introduced for functional programs, and polynomial interpretations, which are frequently used in complexity and termination analysis of rewrite systems.

In order to study this connection we established a novel resource analysis for typed term rewrite systems based on a potential-based type system. This type system gives rise to polynomial bounds for innermost runtime complexity. A key observation is that

23
the classical notion of potential can be altered so that not only values but any term can be assigned a potential. I.e. the potential function $\Phi$ is conceivable as an interpretation. Based on this observation we have shown that well-typedness of a TRSs $\mathcal{R}$ induces a typed polynomial interpretation orienting $\mathcal{R}$.

Apart from clarifying the connection between amortised resource analysis and polynomial interpretation our results seems to induce two new methods for the innermost runtime complexity of typed TRSs as indicated above.

We emphasise that these methods are not restricted to typed TRSs, as our cost model gives rise to a persistent property. Here a property is persistent if, for any typed TRS $\mathcal{R}$ the property holds iff it holds for the corresponding untyped TRS $\mathcal{R}'$. While termination is in general not persistent [20], it is not difficult to see that the runtime complexity is a persistent property. This is due to the restricted set of starting terms. Thus it seems that the proposed techniques directly give rise to novel methods of automated innermost runtime complexity analysis.

In future work we will clarify whether the established results extend to the multivariate amortised resource analysis presented in [10]. Furthermore, we will strive for automation to assess the viability of the established methods.

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