The characteristic polynomials of hypergraphs and cospectral hypergraphs

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Abstract
There are abundant conclusions about the characteristic polynomials of graphs. Due to the lack of simple formulas for calculating the characteristic polynomials of hypergraphs, the characteristic polynomials of hypergraphs have few research results. In this paper, we give the iterating formula of the characteristic polynomials of k-uniform hypergraphs with pendent edges using the Poisson formula of resultants. The characteristic polynomials of k-uniform hyperstar, k-uniform hyperpath with length 3 and two non-isomorphic cospectral k-uniform hypergraphs are given. And we point out that if two graphs are cospectral, their power hypergraphs are not necessarily cospectral.

Keywords: k-uniform hypergraph, Poisson formula, Characteristic polynomial, Cospectral hypergraph.
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1. Introduction
For a positive integer n, let \([n] = \{1, \ldots, n\}\). An order k dimension n tensor \(T = (t_{i_1\cdots i_k}) \in \mathbb{C}^{n \times \cdots \times n}\) is a multidimensional array with \(n^k\) entries, where \(i_j \in [n], j = 1, \ldots, k\). When \(k = 1\), \(T\) is a column vector of dimension \(n\). When \(k = 2\), \(T\) is an \(n \times n\) matrix.

In 2005, the concept of eigenvalues of tensors was introduced by Qi [13] and Lim [11], independently. For \(T = (t_{i_1\cdots i_k}) \in \mathbb{C}^{n \times \cdots \times n}\) and \(x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n\), \(Tx^{k-1}\) is a vector in \(\mathbb{C}^n\) whose \(i\)-th component is

\[
(Tx^{k-1})_i = \sum_{i_2 \cdots i_k} t_{i_1i_2\cdots i_k} x_{i_2} \cdots x_{i_k}.
\]

A number \(\lambda \in \mathbb{C}\) is called an eigenvalue of \(T\) if there exists a nonzero vector \(x \in \mathbb{C}^n\) such that \(Tx^{k-1} = \lambda x^{[k-1]}\), where \(x^{[k-1]} = (x_1^{k-1}, \ldots, x_n^{k-1})^T\). In this case, \(x\) is an eigenvector of \(T\) associated with \(\lambda\).
The characteristic polynomial of $T$, denoted by $\phi_T(\lambda)$, is the resultant of the system of polynomials $\{F_1, \ldots, F_n\}$, where $F_i = (\lambda x_0^{[k-1]} - T x_0^{k-1})_{i, \in [n]}$.

$\phi_T(\lambda)$ is a monic polynomial in $\lambda$ of degree $n(k-1)^{n-1}$ whose roots are exactly the eigenvalues of $T$, the (algebraic) multiplicity of an eigenvalue is taken to be the multiplicity as a root of $\phi_T(\lambda)$. The lack of a simple formula for the characteristic polynomial makes many questions about tensors much more difficult than the same questions for matrices [3].

A hypergraph $H = (V(H), E(H))$ is called $k$-uniform if each edge of $H$ contains exactly $k$ distinct vertices. In 2012, Cooper and Dutle [2] investigated spectral properties of $k$-uniform via adjacency tensor. The adjacency tensor of a $k$-uniform hypergraph $H$ with $n$ vertices, denoted by $A_H$, is an order $k$ dimension $n$ tensor with entries

$$a_{i_1, i_2, \ldots, i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(H), \\ 0, & \text{otherwise}. \end{cases}$$

The characteristic polynomial of $A_H$ is called the characteristic polynomial of $H$.

In this paper, we research the characteristic polynomials of hypergraphs using the Poisson formula of resultants. “Poisson formula” can be found in any of [1, 6, 9]. Cox, Little, and O’Shea [1] gave the following statement of the Poisson formula.

**Theorem 1.1.** [1] Let $F_0, F_1, \ldots, F_n$ be homogeneous polynomials of respective degrees $d_0, \ldots, d_n$ in $K[x_0, \ldots, x_n]$ where $K$ is an algebraically closed field. For $0 \leq i \leq n$, let $\overline{F}_i$ be the homogeneous polynomial in $K[x_0, \ldots, x_n]$ obtained by substituting $x_0 = 0$ in $F_i$, and let $f_i$ be the polynomial in $K[x_0, \ldots, x_n]$ obtained by substituting $x_0 = 1$ in $F_i$. Let $V$ be the set of simultaneous zeros of the system of polynomials $\{f_1, f_2, \ldots, f_n\}$, that is, $V$ is the affine variety defined by the polynomials. If $\text{Res}(\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n) \neq 0$, then $V$ is a zero-dimensional variety (a finite set of points), and

$$\text{Res}(F_0, F_1, \ldots, F_n) = \text{Res}(\overline{F}_1, \ldots, \overline{F}_n)^{d_0} \prod_{p \in V} f_0(p)^{m(p)},$$

where $m(p)$ is the multiplicity of a point $p \in V$.

When $\text{Res}(F_0, F_1, \ldots, F_n)$ is the characteristic polynomial of $H$, the field will be $K = \overline{\mathbb{C}[\lambda]}$, the algebraic closure of the (fraction field of) polynomials in an indeterminant $\lambda$ with complex coefficients. The condition $\text{Res}(\overline{F}_1, \ldots, \overline{F}_n) \neq 0$ always holds [3]. By Bezout’s theorem, the variety $V$ referred to above has most $\prod_{i=1}^n \deg(f_i)$ points in it, and in our applications we exhibit all points in $V$.

The characteristic polynomials of graphs are important research objects in theory of graph spectra [8,16]. In 1974, Schwenk [14] gave the iterating formula of the characteristic polynomials of graphs with pendent edges. This results is helpful to research the characteristic polynomial of tree [12,15]. In [3], Cooper and Dutle gave the characteristic polynomial of 3-uniform hyperstar, and mentioned that the technique in [3] can not give the characteristic polynomial of
$k$-uniform hyperstar. Inspired by the above, we discuss the characteristic polynomial of $k$-uniform hypergraphs.

In this paper, we give the iterating formula of the characteristic polynomials of $k$-uniform hypergraphs with pendant edges. And we give the characteristic polynomials of $k$-uniform hyperstar, which solves the problem Cooper mentioned in [3]. The characteristic polynomial of $k$-uniform hyperpath with length 3 and two non-isomorphic cospectral $k$-uniform hypergraphs are given. We also point out that if two graphs are cospectral, their power hypergraphs are not necessarily cospectral.

2. The characteristic polynomials of $k$-uniform hypergraphs

2.1. The characteristic polynomial of $k$-uniform hyperstar

In 2015, Cooper and Dutle [3] gave the characteristic polynomial of 3-uniform hyperstar using the Poisson formula. In this section, we give the characteristic polynomials of $k$-uniform hyperstar.

Let $G = (V(G), E(G))$ be an ordinary graph (i.e. 2-uniform hypergraph). For any $k \geq 3$, the $k$-th power of $G$ denoted by $G^{(k)} = (V(G^{(k)}), E(G^{(k)}))$, is a $k$-uniform hypergraph with $E(G^{(k)}) = \{ e \cup \{ i_e, 1, \ldots, i_e, k-2 \} | e \in E(G) \}$, $V(G^{(k)}) = V(G) \cup \{ i_e,j \mid e \in E(G), j \in [k-2] \}$. The $k$-th power of star is called $k$-uniform hyperstar. We use $S_m^{(k)}$ denote the $k$-uniform hyperstar with $m$ edges.

Cooper and Dutle gave the characteristic polynomial of $S_1^{(k)}$ as follows (see [2], Theorem 4.3).

Lemma 2.1. [2] The characteristic polynomial of the $k$-uniform hyperstar with one edge $S_1^{(k)}$ is

$$\phi_{S_1^{(k)}}(\lambda) = \lambda^{k(k-1)^{k-1}} - (\lambda^k - 1)^{k^{k-2}}.$$ 

Let $A$ be the adjacency tensor of $S_1^{(k)}$, $x = (x_0, x_1, \ldots, x_k) \in \mathbb{R}$, the homogeneous polynomial $F_i = F_i(x_0, \ldots, x_k) = (\lambda x^{k-1} - Ax^{k-2})$, $i = 0, 1, \ldots, k-1$. Then

$$F_0 = \lambda x_0^{k-1} - x_1 x_2 x_3 \cdots x_{k-1}$$
$$F_1 = \lambda x_1^{k-1} - x_0 x_2 x_3 \cdots x_{k-1}$$
$$F_2 = \lambda x_2^{k-1} - x_0 x_1 x_3 \cdots x_{k-1}$$
$$\vdots$$
$$F_{k-1} = \lambda x_{k-1}^{k-1} - x_0 x_1 x_2 \cdots x_{k-2}$$

$$\phi_{S_1^{(k)}}(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{k-1}).$$

Let $f_i = F_i(1, x_1, \ldots, x_{k-1})$ for all $i \in [k-1]$, i.e.,

$$f_0 = \lambda x_0^{k-1} - x_2 x_3 \cdots x_{k-1}$$
$$f_1 = \lambda x_1^{k-1} - x_1 x_3 \cdots x_{k-1}$$
$$\vdots$$
$$f_{k-1} = \lambda x_{k-1}^{k-1} - x_1 x_2 \cdots x_{k-2}$$

$$\phi_{S_1^{(k)}}(\lambda) = \text{Res}(f_0, f_1, \ldots, f_{k-1}).$$
By the characteristic polynomial of $S_1^{(k)}$ and Poisson formula, we have the following result which will help us give the characteristic polynomials of $k$-uniform hypergraphs.

**Theorem 2.2.** Let $\mathcal{V}$ be the affine variety defined by

$$\left\{ \begin{array}{l}
f_1 = \lambda x_1^{k-1} - x_2 x_3 \cdots x_{k-1} \\
f_2 = \lambda x_2^{k-1} - x_1 x_3 \cdots x_{k-1} \\
\vdots \\
f_{k-1} = \lambda x_{k-1}^{k-1} - x_1 x_2 \cdots x_{k-2}
\end{array} \right.$$  

Then for $p \in \mathcal{V}$, $\sum_{p \neq 0} m(p) = k^{k-2}$, $m(0) = (k-1)^{k-1} - k^{k-2}$.

**Proof.** Let $A$ be the adjacency tensor of $S_1^{(k)}$, the homogeneous polynomial $F_i = F_i(x_0, \ldots, x_{k-1}) = (\lambda x_i^{k-1} - A x_i^{k-1})$, $i = 0, 1, \ldots, k-1$, where $x = (x_0, x_1, \ldots, x_{k-1})^T$. Then $\phi_{S_1^{(k)}}(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{k-1})$ and $f_i = F_i(1, x_1, \ldots, x_{k-1})$ for all $i \in [k-1]$.

Let $\overline{F}_i = F_i(0, x_1, \ldots, x_{k-1})$. Then $\overline{F}_i = \lambda x_i^{k-1}$ for all $i \in [k-1]$, $\text{Res}\{\overline{F}_1, \ldots, \overline{F}_{k-1}\}$ is the characteristic polynomial of the order $k$, dimension $k-1$ tensor with all zero entries. This clearly has $0$ as its only eigenvalue, and so

$$\text{Res}\{\overline{F}_1, \ldots, \overline{F}_{k-1}\} = \lambda^{(k-1)^{k-1}}.$$

Since $\text{Res}\{\overline{F}_1, \ldots, \overline{F}_{k-1}\} \neq 0$, form Theorem 1.1, we know that,

$$\phi_{S_1^{(k)}}(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{k-1})$$

$$= \text{Res}(\overline{F}_1, \ldots, \overline{F}_{k-1})^{k-1} \prod_{p \in \mathcal{V}} f_0(p)^{m(p)}$$

$$= \lambda^{(k-1)^k} \prod_{p \in \mathcal{V}} f_0(p)^{m(p)}. \quad (1)$$

Let $p = (p_1, p_2, \ldots, p_{k-1})^T \in \mathcal{V}$. Then we have $p_i f_i(p) = \lambda p_i^k - p_1 p_2 \cdots p_{k-1} = 0$, $\lambda p_i^k = p_1 p_2 \cdots p_{k-1}$, for all $i \in [k-1]$. Then $\lambda p_1^k = \lambda p_2^k = \cdots = \lambda p_{k-1}^k$, $\lambda^{k-1} p_1^{k-1} p_2^k \cdots p_{k-1}^k = p_1^{k-1} p_2^k \cdots p_{k-1}^k$. Thus

$$p_1^{k-1} p_2^{k-1} \cdots p_{k-1}^{k-1} (\lambda^{k-1} p_1 p_2 \cdots p_{k-1} - 1) = 0.$$

We have $p_1 p_2 \cdots p_{k-1} = 0$ or $1 / \lambda^{k-1}$, when $p = 0$, $p_1 p_2 \cdots p_{k-1} = 0$. When $p \neq 0$, from $\lambda p_1^k = \lambda p_2^k = \cdots = \lambda p_{k-1}^k$, we have $p_1 p_2 \cdots p_{k-1} \neq 0$. Thus $p_1 p_2 \cdots p_{k-1} = \lambda^{k-1}$. Then

$$p_1 p_2 \cdots p_{k-1} = \begin{cases} 0 \\ \frac{1}{\lambda^{k-1}} \end{cases}, \quad p = 0, p \neq 0. \quad (2)$$

Hence

$$f_0(p) = \lambda - p_1 p_2 \cdots p_{k-1} = \begin{cases} \lambda \\ \lambda - \frac{1}{\lambda^{k-1}} \end{cases}, \quad p = 0, p \neq 0.$$
From Eq. (1), we know that
\[
\phi_{S_1^{(k)}}(\lambda) = \lambda^{(k-1)k} \prod_{p \in \mathcal{V}} f_0(p)^{m(p)} \\
= \lambda^{(k-1)k} \prod_{p=0} \lambda^{m(p)} \prod_{p \neq 0} (\lambda - \frac{1}{\lambda^{k-1}})^{m(p)} \\
= \lambda^{(k-1)k} \lambda^{m(0)} (\lambda - \frac{1}{\lambda^{k-1}}) \sum_{p \neq 0} m(p) \\
= \lambda^{(k-1)k + m(0) - (k-1)} \sum_{p \neq 0} m(p) (\lambda^{k} - 1) \sum_{p \neq 0} m(p).
\]

From Lemma 2.1, we have
\[
\phi_{S_1^{(k)}}(\lambda) = \lambda^{k(k-1)k-1} (\lambda^{k} - 1)^{k-2},
\]
then \(\sum_{p \neq 0} m(p) = k^{k-2}, \ m(0) = (k - 1)^{k-1} - k^{k-2}. \)

By Theorem 2.2 and Poisson formula, we get the the characteristic polynomial of \(S_m^{(k)}\) as follows.

**Theorem 2.3.** The characteristic polynomial of the \(k\)-uniform hyperstar with \(m\) edges \(S_m^{(k)}\) is
\[
\phi_{S_m^{(k)}}(\lambda) = \lambda^{(m-1)(k-1)m(k-1)+1} \prod_{r=0}^{m} (\lambda^{k} - r)^{(m)}((k-1)^{k-1} - k^{k-2})^{m-r}.
\]

**Proof.** Label the \((m-1)(k-1)+1\) vertices of \(S_m^{(k)}\) as \(v_0\) for the vertex whose degree is more than one. Let the edges of \(S_m^{(k)}\) be \(\{v_0, v_{i,1}, v_{i,2}, \ldots, v_{i,k-1}\}\) for all \(i \in [m], \)
\[
F_0 = \lambda x_0^{k-1} - \sum_{i=1}^{m} x_{i,1} x_{i,2} \cdots x_{i,k-1},
\]
and for each \(i \in [m], \)
\[
\left\{ \begin{array}{l}
F_{i,1} = \lambda x_{i,1}^{k-1} - x_0 x_{i,2} \cdots x_{i,k-1} \\
F_{i,2} = \lambda x_{i,2}^{k-1} - x_0 x_{i,1} x_{i,3} \cdots x_{i,k-1} \\
\vdots \\
F_{i,k-1} = \lambda x_{i,k-1}^{k-1} - x_0 x_{i,1} x_{i,2} \cdots x_{i,k-2} 
\end{array} \right.
\]
Then
\[
\phi_{S_m^{(k)}}(\lambda) = \text{Res}(F_0, F_{1,1}, \ldots, F_{1,k-1}, F_{2,1}, \ldots, F_{2,k-1}, \ldots, F_{m,1}, \ldots, F_{m,k-1}).
\]
Note that \(F_{i,j} = \lambda x_{i,j}^{k-1} \) for all \(i \in [m]\) and \(j \in [k-1] \). Then
\[
\text{Res}(F_{1,1}, \ldots, F_{1,k-1}, \ldots, F_{m,1}, \ldots, F_{m,k-1}) = \lambda^{m(k-1)^{m(k-1)}}.
\]
Let the system of polynomials

\[ \phi_s^{(k)}(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{i,k-1}, F_{i+1}, \ldots, F_{m,k-1}) \]

be the affine variety defined by

\[ \{ (f_0, f_1, \ldots, f_{i,k-1}) : f_0 = 0 \} \]

for all \( i \in [m] \), and consider the system of polynomials

\[
\begin{align*}
  f_{i,1} &= \lambda x_{i,1}^{k-1} - x_{i,2} \cdots x_{i,k-1} \\
  f_{i,2} &= \lambda x_{i,2}^{k-1} - x_{i,1} x_{i,3} \cdots x_{i,k-1} \\
  &\vdots \\
  f_{i,k-1} &= \lambda x_{i,k-1}^{k-1} - x_{i,1} x_{i,2} \cdots x_{i,k-2}
\end{align*}
\]

Let \( V_i \) be the affine variety defined by \( \{ f_{i,1}, f_{i,2}, \ldots, f_{i,k-1} \} \), \( p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,k-1})^T \in V_i \), for all \( i \in [m] \). Similar to Eq. (2), we know that

\[
p_{i,1} p_{i,2} \cdots p_{i,k-1} = \begin{cases} 
  0 & p_i = 0, \\
  \frac{1}{\lambda^{p_i}} & p_i \neq 0.
\end{cases}
\]

By Theorem 2.2, we have for \( p_i \in V_i \), \( \sum_{p_i \neq 0} m(p_i) = k^{k-2} \), \( m(0) = (k - 1)^{k-1} - k^{k-2} \), \( |V_i| = (k - 1)^{k-1} \). Let \( p = (p_{1,1}, p_{2,1}, \ldots, p_{m,1}, p_{m,2}, \ldots, p_{m,k-1})^T \). \( V \) is the affine variety defined by \( \{ f_{1,1}, \ldots, f_{1,k-1}, f_{m,1}, \ldots, f_{m,k-1} \} \), so \( p \in V \). Since \( |V_i| = (k - 1)^{k-1} \), for all \( i \in [m] \), we can build \( (k - 1)^{m(k-1)} \) points \( p \in V \), through \( p_1, p_2, \ldots, p_m \). By Bezout’s theorem, the variety \( V \) has at most \( \prod_{i=1}^m \prod_{j=1}^{k-1} \deg(f_{i,j}) = (k - 1)^{m(k-1)} \) points. Hence all points of the variety are given by the above construction.

Next we note that \( f_0(p) = \lambda - \sum_{i=1}^m p_{i,1} p_{i,2} \cdots p_{i,k-1} \). Let \( r = |\{ i \mid p_i \neq 0 \}| \), by Eq. (4), we have \( f_0(p) = \frac{x^r - \lambda}{x^r - \lambda} \). For a fixed \( r \), there are \( \binom{m}{r} (k - 1)^r ((k - 1)^{k-1} - k^{k-2})^{m-r} \) such points \( p \in V \). We get

\[
\prod_{p \in V} f_0(p)^{m(p)} = \prod_{r=0}^m \left( \frac{\lambda^r - \lambda}{\lambda^{k-1}} \right)^{\binom{m}{r} (k - 1)^r ((k - 1)^{k-1} - k^{k-2})^{m-r}} = \lambda^{-(k - 1)^{m(k-1)+1}} \prod_{r=0}^m \left( \frac{\lambda^r - \lambda}{\lambda^{k-1}} \right)^{\binom{m}{r} (k - 1)^r ((k - 1)^{k-1} - k^{k-2})^{m-r}},
\]

substituting this expression into (3) gives the claimed results.
2.2. The characteristic polynomials of hypergraphs with pendent edges

In 1974, Schwenk [14] gave the calculation formula of the characteristic polynomials of graphs with pendent edges as follows.

**Theorem 2.4.** [14] Let \( G_v \) denote the graph obtained from \( G \) by adding a pendent edge at the vertex \( v \), \( G - v \) denote the graph obtained from \( G \) by removing \( v \) together with all edges incident to \( v \). Then

\[
\phi_{G_v}(\lambda) = \lambda \phi_G(\lambda) - \phi_{G - v}(\lambda).
\]

In this section, we extend this result to \( k \)-uniform hypergraphs using Poisson formula. Next we introduce some necessary lemmas and notations.

**Lemma 2.5.** [1, Page 97 and 102] Let \( F_0, F_1, \ldots, F_n \) be homogeneous polynomials of respective degrees \( d_0, \ldots, d_n \) in \( K[x_0, \ldots, x_n] \) where \( K \) is an algebraically closed field. For \( 0 \leq i \leq n \), let \( \bar{F}_i \) be the homogeneous polynomial in \( K[x_0, \ldots, x_n] \) obtained by substituting \( x_n = 0 \) in \( F_i \). Then

1. \( \text{Res}(F_0, F_1, \ldots, F_n) = \lambda^{d_0 \cdots d_n - 1} \text{Res}(F_0, \bar{F}_1, \ldots, \bar{F}_n) \),
2. \( \text{Res}(F_0, F_1, \ldots, F_{n-1}, x_n^d) = \text{Res}(F_0, \bar{F}_1, \ldots, \bar{F}_{n-1})^d \).

Let \( M(x) = (x - g_1(x))(x - g_2(x)) \cdots (x - g_t(x)) \),

\[
\Gamma(M, \frac{1}{x^{k-1}}) = (x - g_1(x) - \frac{1}{x^{k-1}})(x - g_2(x) - \frac{1}{x^{k-1}}) \cdots (x - g_t(x) - \frac{1}{x^{k-1}}).
\]

We call \( \Gamma(M, \frac{1}{x^{k-1}}) \) is “\( \frac{1}{x^{k-1}} \)-translational fraction” of \( M(x) \).

We give the characteristic polynomials of \( k \)-uniform hypergraphs with pendent edges as follows.

**Theorem 2.6.** Let \( H \) be a \( k \)-uniform hypergraph with \( n \) vertices, \( H_v \) denote the \( k \)-uniform hypergraph obtained from \( H \) by adding a pendent edge at the vertex \( v \), \( H - v \) denote the \( k \)-uniform hypergraph obtained from \( H \) by removing \( v \) together with all edges incident to \( v \). Then

\[
\phi_{H_v}(\lambda) = \lambda^{(k-1)n + k - 1} \phi_{H - v}(\lambda)^{(k-1)k} M(\lambda)^{(k-1)^2 - k^2} \Gamma(M, \frac{1}{\lambda^{k-1}})^{k^2 - 2}, \tag{5}
\]

where \( M(\lambda) = \frac{\phi_H(\lambda)}{\phi_{H - v}(\lambda)^{k-1}}, \Gamma(M, \frac{1}{\lambda^{k-1}}) \) is \( \frac{1}{\lambda^{k-1}} \)-translational fraction of \( M(\lambda) \).

**Proof.** Let \( V(H) = \{v_0, v_1, \ldots, v_{n-1}\} \), \( x = \{x_0, x_1, \ldots, x_{n-1}\} \). For an edge \( e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \) of \( H \), let \( x^e \) denote the monomial \( x_{i_1}x_{i_2} \cdots x_{i_k} \),

\[
x^e \{v_j\} = x_{i_1}x_{i_2} \cdots x_{i_{j-1}}x_{i_{j+1}} \cdots x_{i_k}.
\]

Let \( A \) be the adjacency tensor of \( H \),

\[
F_i = (\lambda x_i^{k-1} - A x_i)^i = \lambda x_i^{k-1} - \sum_{\substack{e \in E(H) \atop v_i \in e}} x^e \{v_i\}, i = 0, 1, \ldots, n - 1,
\]
then \( \phi_H(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{n-1}) \). Let \( f_i = F_i(1, x_1, \ldots, x_{n-1}) \), for all \( i \in [k-1] \). From Theorem 1.1, we have

\[
\text{Res}(F_0, F_1, \ldots, F_{n-1}) = \text{Res}(F_1^{\ast}, \ldots, F_{n-1}^{\ast})^{k-1} \prod_{p \in V_1} f_0(p)^{m_1(p)}, \tag{6}
\]

where \( V_1 \) is the affine variety defined by \( \{ f_1, f_2, \ldots, f_{n-1} \} \), \( m_1(p) \) is the multiplicity of a point \( p \in V_1 \), \( F_i = F_i(0, x_1, x_2, \ldots, x_{n-1}) \), \( i \in [n-1] \).

Let \( A' \) be the adjacency tensor of \( H - v_0 \), \( y = (x_1, x_2, \ldots, x_{n-1})^T \), then \( F_i = (\lambda y^{k-1} - A' y) \), \( i \in [n] \). Hence \( \text{Res}(F_1^{\ast}, \ldots, F_{n-1}^{\ast}) = \phi_{H - v_0}(\lambda) \). From Eq. (6),

\[
\phi_H(\lambda) = \text{Res}(F_0, F_1, \ldots, F_{n-1})
= \text{Res}(F_1^{\ast}, \ldots, F_{n-1}^{\ast})^{k-1} \prod_{p \in V_1} f_0(p)^{m_1(p)}
= \phi_{H - v_0}(\lambda)^{k-1} \prod_{p \in V_1} (\lambda - \sum_{e \in E(H) \atop v_0 \in e} p^{e \setminus \{v_0\}})^{m_1(p)}. \tag{7}
\]

Next we consider the relationship between \( \phi_{H - v_0}(\lambda) \) and \( \phi_{H - v_0}(\lambda) \). Let the pendent edge of \( H - v_0 \) be \( e' = \{ v_0, v_n, v_{n+1}, \ldots, v_{n+k-2} \} \); \( F_0 = \lambda x_n^{k-1} - \sum_{e \in E(H) \atop v_0 \in e} x^{e \setminus \{v_0\}} - x^{e' \setminus \{v_0\}} \), \( F_j = \lambda x_j^{k-1} - x^{e' \setminus \{v_0\}} \), \( j = n, n+1, \ldots, n+k-2 \). Then

\[
\phi_{H - v_0}(\lambda) = \text{Res}(F_0, F_1, F_2, \ldots, F_{n-1}, F_{n}, \ldots, F_{n+k-2}).
\]

By Theorem 1.1, we have

\[
\phi_{H - v_0}(\lambda) = \text{Res}(F_1, F_2, \ldots, F_{n-1}, F_n, \ldots, F_{n+k-2})^{k-1} \prod_{q \in V} f_0(q)^{m(q)}, \tag{8}
\]

where \( V \) is the affine variety defined by \( \{ f_1, f_2, \ldots, f_{n-1}, f_n, \ldots, f_{n+k-2} \} \), \( m(q) \) is the multiplicity of a point \( q \in V \).

Note that \( F_j = \lambda x_j^{k-1} \), \( j = n, \ldots, n+k-2 \), so

\[
\text{Res}(F_1, F_2, \ldots, F_{n-1}, F_n, \ldots, F_{n+k-2}) = \text{Res}(F_1, F_2, \ldots, F_{n-1}, \lambda x_n^{k-1}, \ldots, \lambda x_{n+k-2}^{k-1}).
\]

From Lemma 2.5 (1), we have

\[
\text{Res}(F_1, F_2, \ldots, F_{n-1}, \lambda x_n^{k-1}, \ldots, \lambda x_{n+k-2}^{k-1})
= \lambda^{(k-1)^{n+k-2}} \text{Res}(F_1, F_2, \ldots, F_{n-1}, x_n^{k-1}, \ldots, x_{n+k-2}^{k-1}). \tag{9}
\]

Since \( F_i \) is unchanged, when substitute \( x_j = 0 \) in \( F_i \), for each \( i \in [n-1] \), \( j \in [n, n+1, \ldots, n+k-2] \), from Lemma 2.5 (2), we have

\[
\text{Res}(F_1, F_2, \ldots, F_{n-1}, x_n^{k-1}, \ldots, x_{n+k-2}^{k-1}) = \text{Res}(F_1, F_2, \ldots, F_{n-1})^{(k-1)^{k-1}}. \tag{10}
\]
Hence
\[
\text{Res}(F_1, F_2, \ldots, F_{n-1}, F_n, \ldots, F_{n+k-2}) = \text{Res}(F_1, F_2, \ldots, F_{n-1}, \lambda x_n^{k-1}, \ldots, \lambda x_{n+k-2}^{k-1}) = \lambda^{-1}x_n^{k-1} \text{Res}(F_1, F_2, \ldots, F_{n-1})^{(k-1)^{k-2}} = \lambda^{-1}(k-1)^{k-2} \phi_{H-v_0}(\lambda)^{(k-1)^{k-1}}.
\]

By Eq. (8), we have
\[
\phi_{H-v_0}(\lambda) = \lambda^{-1}(k-1)^{k-1} \phi_{H-v_0}(\lambda)^{(k-1)^{k-1}} \prod_{q \in V} f_0(q)^{m_q(q)}.
\]  
where \( V \) be the affine variety defined by \( \{ f_1, f_2, \ldots, f_{n-1}, f_n, \ldots, f_{n+k-2} \} \). Let \( V_2 \) be the variety defined by \( \{ f_n, \ldots, f_{n+k-2} \} \), then \( V = V_1 \times V_2 \). For \( r = (r_n, r_{n+1}, \ldots, r_{n+k-2})^T \in V_2 \), let \( m_2(r) \) be the multiplicity of a point \( r \in V_2 \).

From Lemma 2.2, we know that
\[
p^{e_{v_0}}(v_0) = r_n \cdots r_{n+k-2} = \begin{cases} 0, & \text{if } r = 0, \\ 1, & \text{if } r \neq 0, \end{cases}
\]
and \( m_2(0) = (k-1)^{k-1} - k^{k-2}, \sum_{r \neq 0} m_2(r) = k^{k-2} \). Hence
\[
\prod_{q \in V} f_0(q)^{m_q(q)}
\]
\[
= \prod_{q \in V} (\lambda - \sum_{e \in E(H)} q^{e_{v_0}} - q^{e_{v_0}})^{m_q(q)}
\]
\[
= \prod_{p \in V_1, r \in V_2} (\lambda - \sum_{e \in E(H)} p^{e_{v_0}} - p^{e_{v_0}})^{m_1(p)m_2(r)}
\]
\[
= \prod_{p \in V_1, r \neq 0} (\lambda - \sum_{e \in E(H)} p^{e_{v_0}})^{m_1(p)(k-1)^{k-1} - k^{k-2}} \prod_{p \in V_1, r \neq 0} (\lambda - \sum_{e \in E(H)} p^{e_{v_0}} - \frac{1}{\lambda^{k-1}})^{m_1(p)k^{k-2}}
\]

From Eq. (7), we know that
\[
\prod_{p \in V_1, r = 0} (\lambda - \sum_{e \in E(H)} p^{e_{v_0}})^{m_1(p)} = \frac{\phi_H(\lambda)}{\phi_{H-v_0}(\lambda)^{k-1}} = M(\lambda),
\]
so
\[
\prod_{p \in V_1, r \neq 0} (\lambda - \sum_{e \in E(H)} p^{e_{v_0}} - \frac{1}{\lambda^{k-1}})^{m_1(p)} = \Gamma(M, \frac{1}{\lambda^{k-1}}).
\]

Then
\[
\prod_{q \in V} f_0(q) = M(\lambda)^{(k-1)^{k-1} - k^{k-2}} \Gamma(M, \frac{1}{\lambda^{k-1}})^{k^{k-2}}.
\]
substituting this expression into (11) gives the claimed results. \(\Box\)
In Theorem 2.6, when $H$ is a graph with $n$ vertices,

$$M(\lambda) = \frac{\phi_H(\lambda)}{\phi_H(v)(\lambda)} = \lambda - \frac{\phi_H(\lambda) - \lambda \phi_H(v)(\lambda)}{\phi_H(v)(\lambda)},$$

$$\Gamma(M, \frac{1}{\lambda}) = \lambda - \frac{\phi_H(\lambda) - \lambda \phi_H(v)(\lambda)}{\phi_H(v)(\lambda)} - \frac{1}{\lambda} = \frac{\phi_H(\lambda)}{\phi_H(v)(\lambda)} - \frac{1}{\lambda}.$$

So

$$\phi_{H,v}(\lambda) = \lambda \phi_H(v)(\lambda) \left( \frac{\phi_H(\lambda)}{\phi_H(v)(\lambda)} - \frac{1}{\lambda} \right) = \lambda \phi_H(\lambda) - \phi_H(v)(\lambda).$$

In this case, Theorem 2.6 is in keeping with Theorem 2.4.

2.3. The characteristic polynomial of hyperpath with length 3

We call the $k$-th power of path is called $k$-uniform hyperpath. $k$-uniform hyperpath with length $m$ is denoted by $P_m^{(k)}$. Since $P_1^{(k)} = S_1^{(k)}$, $P_2^{(k)} = S_2^{(k)}$, we have introduced the characteristic polynomial of $P_1^{(k)}$ and $P_2^{(k)}$. Next we give the characteristic polynomial of $P_3^{(k)}$.

**Theorem 2.7.** The characteristic polynomial of $k$-uniform hyperpath with length 3 $P_3^{(k)}$ is

$$\phi_{P_3^{(k)}}(\lambda) = \lambda^a (\lambda^k - 1)^b (\lambda^k - 2)^c (\lambda^{2k} - 3\lambda^k + 1)^d,$$

where

$$\begin{align*}
a &= (3k-2)(k-1)^3(k-1) - 3k^k-1(k-1)^2(k-1) + 2k^{2k-3}(k-1)^{k-1} - k^{3k-5}, \\
b &= 3k^k-2(k-1)^2k-2 - 4k^{2k-4}(k-1)^{k-1} + k^{3k-6}, \\
c &= 2k^{2k-3}(k-1)^{k-1} - 2k^{3k-6}, \\
d &= k^{3k-6}.
\end{align*}$$

**Proof.** Let $v$ be a vertex with degree 1 of $P_2^{(k)}$, then $P_3^{(k)} = P_3^{(k)} \cdot P_2^{(k)} - v$ is the $k$-uniform hypergraph with $k-2$ isolated vertices and a single edge. Let $E(P_2^{(k)} - v) = \{\{v_1, v_2, \ldots, v_{2k-2}\}\}$, the edge of $P_3^{(k)} - v$ be $e = \{v_1, v_2, \ldots, v_k\}$. Denote $F_i = \lambda x_i^{k-1} - x_i^{k-(v_i)}$, $i = 1, 2, \ldots, k$, $F_j = \lambda x_j^{k-1}$, $j = k+1, \ldots, 2k-2$. Then

$$\phi_{P_2^{(k)} - v}(\lambda) = \text{Res}(F_1, F_2, \ldots, F_k, F_{k+1}, \ldots, F_{2k-2})$$

$$= \text{Res}(F_1, F_2, \ldots, F_k, \lambda x_{k+1}^{k-1}, \ldots, \lambda x_{2k-2}^{k-1}).$$

From Lemma 2.5, we know that

$$\text{Res}(F_1, F_2, \ldots, F_k, \lambda x_{k+1}^{k-1}, \ldots, \lambda x_{2k-2}^{k-1}) = \lambda^{(k-1)^2k-3(k-2)} \text{Res}(F_1, F_2, \ldots, F_k, \lambda x_{k+1}^{k-1}, \ldots, \lambda x_{2k-2}^{k-1})$$

$$= \lambda^{(k-1)^2k-3(k-2)} \phi_{P_2^{(k)}}(\lambda)^{(k-1)^{k-2}}.$$
By Lemma 2.1, we have
\[ \phi_{p_2^{(k)}}(\lambda) = \lambda^k(k-1)k^{k-1} - k^{k-1}(\lambda^k - 1)^{k^{k-2}}, \]
so
\[ \phi_{p_2^{(k)}-v}(\lambda) = \lambda^{2(k-2)(k-1)2^{k-3} - k^{k-1}(k-1)^{k-2}}(\lambda^k - 1)^{k^{k-2}(k-1)^{k-2}}. \] (12)

By iterating formula (5), we have
\[ \phi_{p_3^{(k)}}(\lambda) = \lambda^{(k-1)2^{k-2}} \phi_{p_2^{(k)}-v}(\lambda)^{k(1-k)^{k-2}} \Gamma(M, \frac{1}{\lambda^{k-1}})^{k^{k-2}}, \] (13)
where \( M(\lambda) = \frac{\phi_{p_2^{(k)}-v}(\lambda)}{\phi_{p_2^{(k)}}(\lambda)} \), \( \Gamma(M, \frac{1}{\lambda^{k-1}}) \) is the translational fraction of \( M(\lambda) \).

From Theorem 2.3, we have
\[ \phi_{p_2^{(k)}}(\lambda) = \lambda^{(2k-1)(k-1)2^{k-2} - k^{k-1}(k-1)^{k-1} + k^{2k-3}^k} - 2k^{2k-4}(\lambda^k - 1)^{2k^{k-2}}(k-1)^{k-1} - 2k^{2k-4} \]
So
\[ M(\lambda) = \frac{\phi_{p_2^{(k)}}(\lambda)}{\phi_{p_2^{(k)}-v}(\lambda)}^{k^{k-1}} = \lambda^{(2k-1)(k-1)2^{k-2} - k^{k-1}(k-1)^{k-1} + k^{2k-3}^k} - 2k^{2k-4}(\lambda^k - 1)^{k^{k-2}(k-1)^{k-1}} \]
\[ = \lambda^{(k-1)2^{k-2} - k^{k-1}(k-1)^{k-1} + k^{2k-3}^k} - 2k^{2k-4}(\lambda - \frac{1}{\lambda^{k-1}})^{k^{k-2}(k-1)^{k-1} - k^{k-2}} \]
(14)
Hence
\[ \Gamma(M, \frac{1}{\lambda^{k-1}}) = (\lambda - \frac{1}{\lambda^{k-1}})^{k^{k-2}(k-1)^{k-1} - k^{k-2}} (\lambda - \frac{\lambda - 2}{\lambda^{k-1}})^{k^{k-2}(k-1)^{k-1} - k^{k-2}}. \] (15)

Substituting Eq.(12),(14),(15) into Eq.(13) gives the claimed results.

3. Cospectral hypergraphs

A fundamental question in the theory of graph spectra is “What kinds of graphs are determined by the spectrum?” The problem dates back to more than
50 years and originates from chemistry. In 1956, Guthard and Primas \[7\] raised the question in a paper that relates the theory of graph spectra to Hückel’s theory from chemistry. At that time it was believed that every graph is determined by the spectrum, until one year later Collatz and Sinogowitz \[4\] gave two non-isomorphic cospectral graphs (see Figure 1). In this section, we give two two non-isomorphic cospectral hypergraphs.

![Fig.1. Two non-isomorphic cospectral graphs](image)

In 1973, Schwenk gave a method of constructing cospectral trees in [15]. Let \(T\) be the graph shown in Figure 2. Since \(T - v\) and \(T - v'\) are isomorphic, \(\phi_{T - v} (\lambda) = \phi_{T - v'} (\lambda)\). From Theorem 2.4, we know that \(\phi_{T_v} (\lambda) = \phi_{T_{v'}} (\lambda)\). Since \(T_v\) and \(T_{v'}\) are not isomorphic, \(T_v\) and \(T_{v'}\) are two non-isomorphic cospectral graphs.

![Fig.2. T](image)

Inspired by this, we give two non-isomorphic cospectral hypergraphs. Let \(T^{(k)}\) be the \(k\)-th power of \(T\). \(T\) is shown in Figure 2. Since \(T^{(k)} - v\) and \(T^{(k)} - v'\) are isomorphic, \(\phi_{T^{(k)} - v} (\lambda) = \phi_{T^{(k)} - v'} (\lambda)\). From Theorem 2.6, we know that \(\phi_{T^{(k)}_v} (\lambda) = \phi_{T^{(k)}_{v'}} (\lambda)\). Since \(T^{(k)}_v\) and \(T^{(k)}_{v'}\) are not isomorphic, \(T^{(k)}_v\) and \(T^{(k)}_{v'}\) are two non-isomorphic cospectral \(k\)-uniform hypergraphs.

Let \(G\) is a graph. In [18], the author gives the numerical relation between the spectral radii of \(G\) and \(G^{(k)}\). Then we know that if two graphs are cospectral, their power hypergraphs have same spectral radii. Next we think about "If two
graphs are cospectral, are their power hypergraphs cospectral?"

A vertex with degree one is called a core vertex. For a $k$-uniform hypergraph $H$, if $e \in E(H)$ is an edge contains core vertices, then we use $H - e$ to denote a $k$-uniform sub-hypergraph of $H$ obtained by deleting the edge $e$ and all core vertices in $e$.

**Lemma 3.1.** [18] Let $H$ be a $k$-uniform hypergraph, and let $e \in E(H)$ be an edge contains at least two core vertices. If $\lambda$ is an eigenvalue of $H - e$, then $\lambda$ is an eigenvalue of $H$.

Let $L^{(k)}$ and $(L')^{(k)}$ be the $k$-th power of $L$ and $L'$, respectively. $L$ and $L'$ are the graphs shown in Figure 1. We know that $L$ and $L'$ are cospectral. Next we show that when $k \geq 4$, $L^{(k)}$ and $(L')^{(k)}$ are not cospectral.

For any $e \in E(L^{(k)})$, $L^{(k)} - e$ consists of $P_3^{(k)}$ and a isolated point. From Theorem 2.7, we know that $\lambda = \left(\frac{3 + \sqrt{5}}{2}\right)^\frac{k}{2}$ is an eigenvalue of $P_3^{(k)}$. Then $\lambda = \left(\frac{3 + \sqrt{5}}{2}\right)^\frac{k}{2}$ is an eigenvalue of $L^{(k)} - e$. Since when $k \geq 4$, all edges of $L^{(k)}$ contain at least two core vertices, from Lemma 3.1, we know that all eigenvalues of $L^{(k)} - e$ is eigenvalues of $L^{(k)}$. So $\lambda = \left(\frac{3 + \sqrt{5}}{2}\right)^\frac{k}{2}$ is an eigenvalue of $L^{(k)}$.

Since $(L')^{(k)} = S_4^{(k)}$, from Theorem 2.3, $\lambda = \left(\frac{3 + \sqrt{5}}{2}\right)^\frac{k}{2}$ is not an eigenvalue of $(L')^{(k)}$. So $L^{(k)}$ and $(L')^{(k)}$ are not cospectral. Hence we have the following proposition.

**Proposition 3.2.** If two graphs are cospectral, their power hypergraphs are not necessarily cospectral.

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