THE REGULARITY LIFTING METHODS FOR NONNEGATIVE
SOLUTIONS OF LANE-EMDEN SYSTEM

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(Communicated by Enrico Valdinoci)

Abstract. In this paper, we focus on the regularity of nonnegative solutions
of Lane-Emden system
\[
\begin{cases}
-\Delta u = v^p \\
-\Delta v = u^q
\end{cases}
\text{ in } \mathbb{R}^n.
\]
By means of Kelvin transform, we turn this problem into estimating the local
integrability of \((\bar{u}, \bar{v})\). Assume that \((\bar{u}, \bar{v})\) possesses some initial local integra-
bility beforehand.
\[(\bar{u}, \bar{v}) \in L^{r_0}_{loc}(\mathbb{R}^n) \times L^{s_0}_{loc}(\mathbb{R}^n)\]
for any suitable \(r_0\) and \(s_0\) under specified conditions. Then through a regularity
lifting method by contracting operators, we prove that
\[(\bar{u}, \bar{v}) \in L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n)\]
for \(r\) and \(s\) sufficiently large under twice regularity lifting if needed. Furth-
more, we lift the regularity of solutions to
\[L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n).\]
We believe that these new methods employed in this paper can be widely
applied to study a variety of other problems with different spaces and linear or
nonlinear problems.

1. Introduction. In this paper, we mainly focus on nonnegative solutions of Lane-
Emden system
\[
\begin{cases}
-\Delta u = v^p \\
-\Delta v = u^q
\end{cases}
\text{ in } \mathbb{R}^n,
\]
where \(p > 1, q > 1\).

The well-known Lane-Emden system is closely related to the weighed Hardy-
Littlewood-Sobolev (WHLS) inequality (see [17])
\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{s}} \, dx dy \right| \leq C(\alpha, \beta, s, \lambda, n) \|f\|_r \|g\|_s,
\]
where
\[1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}\text{ and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2.\]
Maximize the functional

\[ J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha} |x-y|^{\lambda} |y|^{\beta}} \, dx \, dy \]

to obtain the best constant in the WHLS inequality (1.2). Under constraints \( \|f\|_r = 1 \) and \( \|g\|_s = 1 \), the corresponding Euler-Lagrange equations are

\[
\begin{align*}
\lambda_1 r f^{r-1}(x) &= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{g(y)}{|y|^{\beta} |x-y|^{-\lambda}} \, dy \\
\lambda_2 s g^{s-1}(x) &= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{\alpha} |x-y|^{-\lambda}} \, dy
\end{align*}
\]

in \( \mathbb{R}^n \), \hspace{1cm} (1.3)

where \( f \geq 0 \), \( g \geq 0 \) and \( \lambda_1 r = \lambda_2 s = J(f, g) \). Let \( u = c_1 f^{r-1}, v = c_2 g^{s-1}, \) \( \frac{1}{q+1} = 1 - \frac{1}{r}, \frac{1}{p+1} = 1 - \frac{1}{s} \) and \( pq \neq 0 \), \( p, q > 0 \), and select proper constants \( c_1 \) and \( c_2 \), the system (1.3) becomes Euler-Lagrange system of the WHLS

\[
\begin{align*}
u(x) &= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^p(y)}{|y|^q |x-y|^{\lambda}} \, dy \\
v(x) &= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u^q(y)}{|y|^p |x-y|^{\lambda}} \, dy
\end{align*}
\]

in \( \mathbb{R}^n \), \hspace{1cm} (1.4)

where \( u, v \) are nonnegative and

\[
\begin{align*}
0 \leq \alpha + \beta &\leq n - \lambda, \\
\alpha &< \frac{1}{p+1} < \frac{\lambda + \alpha}{n}, \\
\frac{1}{q+1} + \frac{1}{q+1} &\leq \frac{\lambda + \alpha + \beta}{n}.
\end{align*}
\]

Furthermore, in the case \( \alpha = 0, \beta = 0 \) and \( \lambda = n - 2 \), the system (1.4) becomes much more simple and reduces to

\[
\begin{align*}
u(x) &= \int_{\mathbb{R}^n} \frac{v^p(y)}{|x-y|^{n-\lambda}} \, dy \\
v(x) &= \int_{\mathbb{R}^n} \frac{u^q(y)}{|x-y|^{n-\lambda}} \, dy
\end{align*}
\]

in \( \mathbb{R}^n \), \hspace{1cm} (1.5)

where

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}. \hspace{1cm} (1.6)
\]

And the system (1.5) is the integral form of Lane-Emden system (1.1) (see Proposition 1).

Similar to the single Lane-Emden equation, Lane-Emden system can be normally divided into three different cases according to the value of \( p \) and \( q \). In fact, the system (1.5) above under the condition (1.6) actually belongs to the critical case since \( (p, q) \) lies on Sobolev hyperbola (see [11, 12]):

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}.
\]

It is called the supercritical case if \( (p, q) \) is above Sobolev hyperbola, that is,

\[
\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}.
\]
and the subcritical case when below it, that is,
\[ \frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}. \]

It is known that Lane-Emden system admits some nonnegative classical solutions in \( \mathbb{R}^n \) under the supercritical or critical case (see [16]). Chen and Li [1] as well as Mitidieri [11] indicate that Lane-Emden system admits no trivial solutions with finite energy under subcritical case. Then the well-known Lane-Emden conjecture (see [3, 18]) states that the system (1.1) admits no nonnegative classical solutions in the subcritical case.

Lane-Emden conjecture was proved to be true for radial solutions in all dimensions (see [12]). As for non-radial solutions, the conjecture was solved if the dimension \( n \leq 2 \) (see [13, 21]). This issue becomes more complex when \( n \geq 3 \). Under \((p, q)\) being subcritical, respectively, and
\[ p, q \leq \frac{n+2}{n-2}, \quad (p, q) \neq \left( \frac{n+2}{n-2}, \frac{n+2}{n-2} \right). \]
Figueiredo and Felmer [7], Reichel and Zou [15] proved the conjecture, and Ma and Chen [10] generalized it to some more integral systems.

Then for \( n = 3 \), this issue was solved by Serrin and Zou [19] with an additional condition that \((u, v)\) has at most polynomial growth at infinity and Poláčik, Quittner and Souplet [14] removed this condition so that the conjecture was proved to be true if \( n = 3 \). If \( n = 4 \), the conjecture was also solved by Souplet [20] with a new region of non-existence found.

Hence the research on Lane-Emden conjecture in a part of subcritical case with \( n \geq 5 \) becomes more attractive. In fact, the non-existence in Lane-Emden conjecture is closely related to the regularity of solutions in the exterior area. Cheng, Huang and Li [5] state that if there exists a positive number \( s \) with
\[ n - s\beta < 1, \quad \beta = \frac{2(q+1)}{pq-1} \]
such that
\[ \int_{B_R} v^s \leq CR^{n-s\beta}, \]
the conjecture is shown to be true. This sufficient condition depicts the behaviour of solutions at infinity, which enlightens us to focus on the integrability of solutions in the exterior area.

Inspired by [5], we pay attention to the integrability of solutions of the system (1.1) in the exterior area, and apply Kelvin transform (see Definition 2.2) to translate the integrability of solutions \((u, v)\) of the system (1.1) in the exterior domain into a local one. By direct calculation, Lane-Emden system (1.1) becomes
\[ \left\{ \begin{array}{l}
-\Delta \bar{u} = |x|^{-a} \bar{v}^p \\
-\Delta \bar{v} = |x|^b \bar{u}^q
\end{array} \right. \quad \text{in } \mathbb{R}^n. \quad (1.7) \]
While Kelvin transform only makes sense when \(|x| \neq 0\), the system (1.7) is well-defined in \( \mathbb{R}^n \) since
\( (\bar{u}, \bar{v}) \in L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\mathbb{R}^n) \), according to Theorem 2.3. The exponents of \(|x|\) in the system (1.7) are exactly positive and negative, respectively, on account of the value of \( p \) and \( q \). Hence we put an extra minus sign before \( a \) ensuring that both \( a \) and \( b \) are positive.
Hereafter, we focus on the local integrability of $(\bar{u}, \bar{v})$. For convenience, we still denote $(\bar{u}, \bar{v})$ by $(u, v)$ and assume $0 < a < 2, b > 0$. Hence we obtain the differential equation system
\[
\begin{align*}
-\Delta u &= |x|^{-a} v^p \quad \text{in } \mathbb{R}^n; \\
-\Delta v &= |x|^b u^q \quad \text{in } \mathbb{R}^n;
\end{align*}
\] (1.8)
with its equivalent integral form (see Proposition 1) as
\[
\begin{align*}
&\begin{cases}
  u(x) = \int_{\mathbb{R}^n} \frac{|y|^{-a} v^p(y)}{|x-y|^{n-2}} dy \\
v(x) = \int_{\mathbb{R}^n} \frac{|y|^b u^q(y)}{|x-y|^{n-2}} dy
\end{cases} \quad \text{in } \mathbb{R}^n. \quad (1.9)
\end{align*}
\]
And the system (1.9) is equivalent to
\[
\begin{align*}
&\begin{cases}
  u(x) = \int_0^\infty \frac{\int_{B_t(x)} |y|^{-a} v^p(y) dy dt}{t^{n-2}} \frac{1}{t} \\
v(x) = \int_0^\infty \frac{\int_{B_t(x)} |y|^b u^q(y) dy dt}{t^{n-2}} \frac{1}{t}
\end{cases} \quad \text{in } \mathbb{R}^n; \quad (1.10)
\end{align*}
\]
by introducing parameter $t$ and changing the sequence of integration (see Proposition 2). After this transform, the system (1.10) becomes a special case of the fully nonlinear integral system involving Wolff potentials [9]:
\[
\begin{align*}
&\begin{cases}
  u(x) = W_{\beta, \gamma}(f)(x) \\
v(x) = W_{\beta, \gamma}(g)(x)
\end{cases} \quad x \in \mathbb{R}^n; \quad (1.11)
\end{align*}
\]
where
\[
W_{\beta, \gamma}(f)(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} f(y) dy}{t^{n-\beta \gamma}} \right]^{\frac{1}{\gamma}} dt.
\]
It can be verified that Wolff potential becomes the well-known Newton potential if $\beta = 1, \gamma = 2$. Furthermore, when $f(x) = |x|^{-a} v^p(x), g(x) = |x|^b u^q(x)$, the system (1.11) corresponds to the integral system (1.10).

Additionally, Wolff potential corresponds to Riesz potential which is also closely related to the HLS inequality (2) if $\beta = \frac{\alpha}{2}, \gamma = 2$. There is also a series of studies on the relation between Wolff potentials and the corresponding nonlinear differential equations. Hence we are able to study the system (1.10) by the integral estimate on Wolff potential.

Based on the integral system (1.10), a regularity lifting method by contracting operators is introduced in order to lift the regularity and integrability of nonnegative solutions. Furthermore, we combine the regularity lifting method with the Kelvin transform together and lift the integrability in the exterior area. Through this new idea, the regularity for nonnegative solutions of Lane-Emden system is lifted to a higher specified space and the results of the paper are as follows.

**Theorem 1.1 (Regularity Lifting).** Let $(u, v) \in L_{loc}^{r_0}(\mathbb{R}^n) \times L_{loc}^{s_0}(\mathbb{R}^n)$ be a pair of nonnegative solutions for the system (1.8) under $0 < a < 2, b > 0, p > 1, q > 1, n \geq 4,
and
\[
\begin{align*}
\frac{1}{r_0} &< 1 - \frac{2}{n}, \\
\frac{1}{s_0} &< 1 - \frac{2}{n}, \\
\frac{q}{r_0} &< \frac{1}{2}, \\
\frac{p}{s_0} &< \frac{2 - a}{n}.
\end{align*}
\] (1.12)

Then
(i) At least one of \(u\) and \(v\) belongs to \(L^t_{loc}(\mathbb{R}^n)\) for any \(t\) sufficiently large.
(ii) To be more precise, we conclude that
\[
\begin{align*}
u &\in L^r_{loc}(\mathbb{R}^n), \quad \text{if } \frac{2}{n} - \frac{q - 1}{r_0} < 0; \\
v &\in L^s_{loc}(\mathbb{R}^n), \quad \text{if } \frac{a - 2}{n} + \frac{p - 1}{s_0} > 0; \\
(u,v) &\in L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n), \quad \text{if } \frac{a - 2}{n} + \frac{p - 1}{s_0} \leq 0 < \frac{2}{n} - \frac{q - 1}{r_0};
\end{align*}
\]
for \(r\) and \(s\) sufficiently large.
(iii) Furthermore, \((u,v) \in L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n)\) when \((\frac{1}{r}, \frac{1}{s})\) lies in the square region, that is,
\[
\left(\frac{1}{r}, \frac{1}{s}\right) \in \left(\max\left\{0, \frac{q - 1}{s_0} + \frac{a - 2}{n}\right\}, \min\left\{1 - \frac{q - 1}{r_0}, 1 - \frac{2}{n}\right\}\right) \\
\times \left(\max\left\{0, \frac{q - 1}{r_0} - \frac{2}{n}\right\}, \min\left\{\frac{n - a}{n} - \frac{p - 1}{s_0}, 1 - \frac{2}{n}\right\}\right).
\]

(iv) Repeat the regularity lifting again, we obtain
\[
(u,v) \in L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n)
\]
for \(r\) and \(s\) sufficiently large.

**Theorem 1.2.** Under the same conditions as in the Regularity Lifting Theorem 1.1 above, \(u,v \in L^\infty(\mathbb{R}^n)\).

2. Preliminaries. In this section, some necessary theorems, lemmas, propositions and other preparatory work will be listed.

**Definition 2.1** (Weak Derivatives[6]). Suppose \(u,v \in L^1_{loc}(U), U \subseteq \mathbb{R}^n\) is open, and \(\alpha\) is a multi-index. \(v\) is the \(\alpha^{th}\)-weak partial derivative of \(u\) provided
\[
\int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx
\]
for all \(\phi \in C^\infty_c(U)\), written
\[D^\alpha u = v.\]

**Definition 2.2** (Kelvin Transform). Kelvin transform with respect to solutions \((u,v)\) is defined as
\[
\begin{align*}
\bar{u} &= \frac{1}{|x|^{n-2}} u \left(\frac{x}{|x|^2}\right) \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\
\bar{v} &= \frac{1}{|x|^{n-2}} v \left(\frac{x}{|x|^2}\right).
\end{align*}
\]
Theorem 2.3 (Bocher theorem[8]). Let $v(x) \in L^1_{loc}(B_1(0)\setminus\{0\})$ be a nonnegative solution in $\mathbb{R}^n$ ($n \geq 2$) to
$$-\Delta v(x) + c(x)v(x) = f(x) \geq 0 \text{ on } B_1(0)\setminus\{0\}$$
for some $f(x) \in L^1_{loc}(B_1(0)\setminus\{0\})$ and $c(x)$ is bounded from above, then
(i) $v(x), f(x) \in L^1_{loc}(B_1(0))$.
(ii) $-\Delta v(x) + c(x)v(x) = f(x) + a\delta_0 \text{ on } B_1(0)$,
for some constant $a \geq 0$.

Remark 1. Suppose $(u,v)$ is a pair of nonnegative solutions of system (1.8) and $u,v \in L^1_{loc}(\mathbb{R}^n)$. Then after Kelvin transform, $\bar{u}, \bar{v} \in L^1_{loc}(\mathbb{R}^n)$ according to Theorem 2.3.

Lemma 2.4 (Regularity Lifting[4]). Let $T$ be a contraction mapping from $X$ into itself and from $Y$ into itself. Assume $f \in X$, and $g \in Z := X \cap Y$ s.t $f = Tf + g$ in $X$. Then $f \in Z$. More importantly, $f \in Y$.

Proof. Step 1. Let
$$\sqrt{\|f\|^2_X + \|f\|^2_Y}$$
be a norm on $Z$. Firstly, we show that
$$T : Z \to Z$$
is a contraction mapping. Since $T$ is a contraction on both $X$ and $Y$, there exist constants $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$ such that
$$\|Th_1 - Th_2\|_X \leq \theta_1 \|h_1 - h_2\|_X, \forall h_1, h_2 \in X,$$
and
$$\|Th_1 - Th_2\|_Y \leq \theta_2 \|h_1 - h_2\|_Y, \forall h_1, h_2 \in Y.$$Let $\theta = \max\{\theta_1, \theta_2\}$. Then for any $h_1, h_2 \in Z$, it follows
$$\|Th_1 - Th_2\|_Z = \sqrt{\|Th_1 - Th_2\|_X^2 + \|Th_1 - Th_2\|_Y^2} \leq \sqrt{\theta_1^2 \|h_1 - h_2\|_X^2 + \theta_2^2 \|h_1 - h_2\|_Y^2} \leq \theta \|h_1 - h_2\|_Z.$$Step 2. Since $T$ is a contraction mapping on $Z$, it is feasible to find a solution $h \in Z$ such that $h = Th + g$ provided $g \in Z$. Since $T$ is also a contraction mapping on $X$, the solution of the equation $f = Tf + g$ must be unique in $X$. Therefore $f = h \in Z$ since both $h$ and $f$ are solutions in $X$.

Remark 2. In practice, we may choose $X$ and $Y$ to be Sobolev spaces. At first $f$ belongs to a lower regularity space
$$X = L^r_{loc}(\mathbb{R}^n) \text{ or } X = L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n).$$Through Lemma 2.4 above, $f$ can be lifted to a higher regularity space
$$Y = L^r_{loc}(\mathbb{R}^n) \text{ or } Y = L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n)$$with larger $r$ and $s$. Hence Lemma 2.4 enlightens us on lifting the regularity of the solutions for Lane-Emden system.
Remark 3. For linear mappings, contraction mappings are equivalent to shrinking mappings. If $T$ is a linear mapping from $X$ into $Y$, $T$ is a contraction mapping if and only if there exists a constant $\theta$, such that $0 < \theta < 1$, $$\|Tf\|_Y \leq \theta \|f\|_X, \forall f \in X.$$ 

Proof. Let $T$ be a linear shrinking mapping from $X$ to $Y$. It follows that $$f - g \in X, \forall f, g \in X.$$ Hence $$\|T(f - g)\|_Y \leq \|f - g\|_X$$ since $T$ is a shrinking mapping.

To the opposite, each of contraction mappings is a shrinking mapping (let $g = 0$). And obviously comes the equivalence. \hfill \square

Remark 4. However, it is very difficult to prove to be a contraction mapping for a nonlinear operator in some given spaces, or even impossible sometimes. And there is a more general method applied to nonlinear operators (see [4]).

Lemma 2.5 (Hardy-Littlewood-Sobolev Inequality[2]). Let $$0 < \lambda < n, s, r > 1$$ under $$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2.$$ Assume $f \in L^r(\mathbb{R}^n), g \in L^s(\mathbb{R}^n)$. Then $$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x - y|^{-\lambda} g(y) dx dy \leq C(n, s, \lambda) \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)},$$ where $$C(n, s, \lambda) = \frac{n|B^n|^\lambda/n}{(n - \lambda)rs} \left[ \left( \frac{n}{1 - n/r} \right)^{\frac{1}{r}} \left( \frac{n}{1 - 1/s} \right)^{\frac{1}{s}} \right]$$ and $|B^n|$ represents the volume of the unit ball in $\mathbb{R}^n$.

Corollary 1. Let $$h(x) = \int_{\mathbb{R}^n} |x - y|^{n - n} g(y) dy.$$ Assume $g \in L^{\frac{n\alpha}{n - \alpha}}(\mathbb{R}^n)$. Then $$\|h\|_{L^t(\mathbb{R}^n)} \leq C(n, t, \alpha) \|g\|_{L^{\frac{n\alpha}{n - \alpha}}(\mathbb{R}^n)}$$ for any $t > \frac{n}{\alpha} - \frac{n}{\alpha}$.

Proof. Define $$Tg(x) := \int_{\mathbb{R}^n} |x - y|^{n - n} g(y) dy.$$ According to Lemma 2.5, we obtain $$\langle f, Tg \rangle = \langle Tf, g \rangle \leq C(n, s, \alpha) \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)},$$ where $\langle \cdot, \cdot \rangle$ represents the $L^2$ inner product. Then $$\|h\|_{L^t(\mathbb{R}^n)} = \|Tg\|_{L^t(\mathbb{R}^n)} = \sup_{\|f\|_{L^2} = 1} \langle f, Tg \rangle \leq C(n, s, \alpha) \|g\|_{L^s(\mathbb{R}^n)}$$
under
\[
\begin{cases}
\frac{1}{r} + \frac{1}{t} = 1, \\
\frac{1}{r} + \frac{1}{s} + \frac{n - \alpha}{n} = 2.
\end{cases}
\]

Here we denote \( \| f \|_{L_r^r(\mathbb{R}^n)} \) by \( \| f \|_r. \) After basic calculation, we have
\[
s = \frac{nt}{n + \alpha t}.
\]

**Remark 5.** Corollary 1 is actually the equivalent form of the HLS inequality and plays an important role in our problem.

**Corollary 2.**
\[
\| W_{\beta, \gamma}(g) \|_{L^s(\mathbb{R}^n)} \leq C \| g \|_{L^{\frac{1}{r}}(\mathbb{R}^n)}
\]
for any \( s > \frac{n}{n - \beta \gamma} \) and \( \frac{1}{s} = \frac{1}{r} - \frac{\beta \gamma}{n}. \)

**Proof.** It is known for us that there is a relationship between the Wolff potential and Riesz potential, that is,
\[
C_1 \| W_{\beta, \gamma}(g) \|_{L^s(\mathbb{R}^n)} \leq \| I_{\beta \gamma}(g) \|_{L^{\frac{1}{s}}(\mathbb{R}^n)} \leq C_2 \| W_{\beta, \gamma}(g) \|_{L^s(\mathbb{R}^n)}
\]
where \( s > \gamma - 1 \) and
\[
I_\alpha(f)(x) = \int_0^\infty \int_{B_t(x)} f(y) dy \frac{dt}{t} = C \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) dy.
\]
Together with Corollary 1, we obtain
\[
r = \frac{n \gamma - 1}{n + \beta \gamma}
\]
or
\[
\frac{\gamma - 1}{s} = \frac{1}{r} - \frac{\beta \gamma}{n}.
\]

**Remark 6.** If \( \beta = 1 \) and \( \gamma = 2 \) in Corollary 2,
\[
W_{1, 2}(f)(x) = I_{2}(f)(x).
\]
Hence
\[
\left\| \int_0^\infty \int_{B_t(x)} f(y) dy \frac{dt}{t^{n-2}} \right\|_{L^s(\mathbb{R}^n)} \leq C \| f \|_{L^r(\mathbb{R}^n)}
\]
for any \( s > \frac{n}{n - 2} \) and \( \frac{1}{s} = \frac{1}{r} - \frac{2}{n}.

**Lemma 2.6** (Hopf’s Lemma[6]). Assume that
\[
u \in C^2(U) \cap C(\bar{U})
\]
and
\[
c \equiv 0 \text{ in } U
\]
where \( c \equiv c(x) \) is the coefficient function of \( u \) in the differential operator \( L. \) Suppose further
\[
Lu \leq 0 \text{ in } U,
\]
and there exists a point \( x^0 \in \partial U \) such that
\[
u(x^0) > u(x), \text{ for all } x \in U.
\]
Assume finally that $U$ satisfies the interior ball condition at $x^0$, that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

(i) Then
\[
\frac{\partial u}{\partial \nu}(x^0) > 0,
\]
where $\nu$ is the outer unit normal to $B$ at $x^0$.

(ii) If $c \geq 0$ in $U$,

the same conclusion holds provided
\[
u(x^0) \geq 0.
\]

**Proposition 1.** The differential equation system (1.8) and the integral equation system (1.9) are equivalent.

**Proof.** Assume $(u, v)$ is a pair of nonnegative solutions of the differential equation system (1.8). Let $\delta(x)$ be the Dirac delta function, $\phi(x)$ be the Green’s function of the following boundary value problem
\[
\begin{cases}
-\Delta \phi = \delta(x) & x \in B_r(0), \\
\phi = 0 & x \in \partial B_r(0).
\end{cases}
\]

By Lemma 2.6, one can be verified that the outward normal derivative
\[
\frac{\partial \phi}{\partial \nu} \leq 0, \quad x \in \partial B_r(0).
\]

Multiply the first equation of the system (1.8) by $\phi(x)$ and integrate it over $B_r(0)$. Then we have
\[
\int_{B_r(0)} \nu^a v^p(x) \phi(x) dx = u(0) + \int_{\partial B_r(0)} u \frac{\partial \phi}{\partial \nu} dS \leq u(0)
\]
(2.2)
after integrating by parts twice and the condition (2.1). Let $r \to \infty$, we derive
\[
\phi(x) \to \frac{C}{|x|^{n-2}}.
\]

Hence
\[
C \int_{\mathbb{R}^n} |x|^{-a} v^p(x) \frac{1}{|x|^{n-2}} dx \leq u(0).
\]

Through the convexity argument and a simple calculation with the fact that
\[
\frac{\partial \phi}{\partial \nu} \leq \frac{C}{r^{n-1}}, \quad x \in \partial B_r(0),
\]
there exists a sequence $\{r_m\}_{m=1}^{\infty}$ such that
\[
\begin{cases}
r_m \to +\infty, \quad \text{as} \quad m \to \infty, \\
\int_{\partial B_{r_m}(0)} u \frac{\partial \phi}{\partial \nu} dS \to 0;
\end{cases}
\]

Applying Lebesgue convergence theorem to (2.2) and taking the limit $m \to \infty$ along $\{r_m\}_{m=1}^{\infty}$, we obtain
\[
C \int_{B_r(0)} |y|^{-a} v^p(y) \frac{C}{|y|^{n-2}} dy = u(0).
\]
By means of a translation, it is obvious that the first equation of the system (1.8) is satisfied if we multiply \((u, v)\) by a constant. Similarly for the second equation of the system (1.8), thus the solutions of system (1.8) satisfy the integral system (1.9).

To check the sufficiency, the conclusion can be drew immediately if we apply \(-\triangle\) to the both side of two equations of the system (1.9).

Remark 7. The differential equation system (1.1) and the integral equation system (1.5), that is, two forms of Lane-Emden system without Kelvin transform, are also equivalent. And the proof is similar to Proposition 1.

Proposition 2. Integral equation systems (1.9) and (1.10) are equivalent.

Proof. From the first equation of the system (1.10) and change the sequence of integration, we have

\[
  u(x) = \int_0^\infty \int_{B_t(x)} |y|^{-a} v^p(y) dy \frac{dt}{t^{n-2}} = \int_{R^n} \int_{|x-y|}^\infty \frac{dy}{t^{n-1}} |y|^{-a} v^p(y) dy
\]

Similarly for \(v(x)\), and it is proved that \((u, v)\) multiplied by a constant is a pair of solutions of the system (1.9) and vice versa. 

3. Regularity Lifting. In this section, we will prove Theorem 1.1 and Theorem 1.2. For convenience, we denote \(\|\cdot\|_{L^t_{loc}(R^n)}\) by \(\|\cdot\|_{L^t_{loc}}\).

3.1. Proof of Theorem 1.1. Step 1. Let

\[
  v_A(x) := \begin{cases} 
    v(x), & v(x) > A \text{ or } |x| > A, \\
    0, & \text{else};
  \end{cases}
\]

and

\[
  v_B(x) := v(x) - v_A(x).
\]

Similarly for \(u(x)\), let

\[
  u_A(x) := \begin{cases} 
    u(x), & u(x) > A \text{ or } |x| > A, \\
    0, & \text{else};
  \end{cases}
\]

and

\[
  u_B(x) := u(x) - u_A(x)
\]

with constant \(A\) to be determined. Define

\[
  T_1 g := \int_0^\infty \int_{B_t(x)} |y|^{-a} v^p_A(y) g(y) dy \frac{dt}{t^{n-2}}, \quad T_2 f := \int_0^\infty \int_{B_t(x)} |y|^{-a} v^p_B(y) f(y) dy \frac{dt}{t^{n-2}}
\]

and

\[
  F := \int_0^\infty \int_{B_t(x)} |y|^{-a} u^q_B(y) dy \frac{dt}{t^{n-2}} \quad \text{in } R^n,
\]

\[
  G := \int_0^\infty \int_{B_t(x)} |y|^{-a} u^q_B(y) dy \frac{dt}{t^{n-2}} \quad \text{in } R^n.
\]

After defining mappings \(T_1\) and \(T_2\), we can write a mapping \(T\) as

\[
  T(f, g) := (T_1 g, T_2 f)
\]
and construct an equation as
\[(f, g) = T(f, g) + (F, G).\]  

(3.1)

On account of \((u, v)\) being a pair of solutions of the system (1.8) or (1.10) (see Proposition 1 and Proposition 2), \((u, v)\) should be also the solution of the equation (3.1) since there is no intersection between the supports of \(u_A\) and of \(u_B\) (same for \(v_A\) and \(v_B\)). Due to this construction of the operator \(T\), we need to prove \(T\) is a contraction mapping and estimate \(F\) and \(G\).

Step 2. Now we show that \(T_1\) and \(T_2\) are both contractions. By Corollary 2 and generalized H"older inequality,
\[
\|T_1 g\|_{L^r_{loc}} \leq C \left\| y^{-a} v^{p-1}_A g \right\|_{L^{\frac{n}{n-2}}_{loc}} \leq C \left\| y^{-a} \right\|_{L^{r_1}_{loc}} \left\| v^{p-1}_A \right\|_{L^{r_2}_{loc}} \| g \|_{L^r_{loc}}
\]

(3.2)

under the following conditions:
\[
\frac{1}{r} < 1 - \frac{2}{n}, \quad (3.3)
\]
\[
p - 1 \frac{1}{s_1} + \frac{1}{s} = \frac{1}{r} + \frac{2}{n}. \quad (3.4)
\]

Similarly for \(T_2\),
\[
\|T_2 f\|_{L^s_{loc}} \leq C \left\| y^b u^{q-1}_A f \right\|_{L^{\frac{n}{n-2}}_{loc}} \leq C \left\| y^b \right\|_{L^{s_1}_{loc}} \left\| u^{q-1}_A \right\|_{L^{s_2}_{loc}} \| f \|_{L^s_{loc}}
\]

(3.5)

under the following conditions:
\[
\frac{1}{s} < 1 - \frac{2}{n}, \quad (3.6)
\]
\[
q - 1 \frac{1}{r_1} + \frac{1}{r} = \frac{1}{s} + \frac{2}{n}. \quad (3.7)
\]

It follows that
\[T_1 : L^s_{loc}(\mathbb{R}^n) \to L^r_{loc}(\mathbb{R}^n)\]

and
\[T_2 : L^r_{loc}(\mathbb{R}^n) \to L^s_{loc}(\mathbb{R}^n)\]

are contractions if \(A\) is large enough and
\[as_1 < n. \quad (3.8)\]

To be more rigorous, \(\|u_A\|_{L^{s_1}_{loc}}\) and \(\|v_A\|_{L^{r_1}_{loc}}\) can be as small as we expect for large \(A\). On the other hand, the condition (3.8) ensures that the item \(\|y^{-a}\|_{L^{r_1}_{loc}}\) in the right hand side of (3.2) is less than infinity. Hence there exist constants \(\theta_1\) and \(\theta_2\) such that \(0 < \theta_1, \theta_2 < 1\),
\[
\|T_1 g\|_{L^{r_1}_{loc}} \leq \theta_1 \| g \|_{L^{r_1}_{loc}}, \quad \|T_2 f\|_{L^{s_1}_{loc}} \leq \theta_2 \| f \|_{L^{s_1}_{loc}}
\]

by inequalities (3.2) and (3.5). From Remark 3, we find that \(T_1, T_2\) are both contractions and we conclude that \(T\) is a contraction mapping from \(L^r_{loc}(\mathbb{R}^n) \times L^s_{loc}(\mathbb{R}^n)\) into itself under conditions (3.3), (3.4), (3.6), (3.7) and (3.8).
Step 3. In this step, we estimate $F$ and $G$. By Corollary 2 and Hölder inequality,
\[
\|F\|_{L^r_{\text{loc}}} \leq C \|y^{-a} u_B\|_{\frac{nr}{n+a}} \leq C \|y^{-a}\|_{L^r_{\text{loc}}} \|v_B\|_{L^r_{\text{loc}}}
\]
under the condition (3.3) and
\[
as'_1 < n, \quad \frac{1}{s_1} + \frac{1}{r'} = \frac{1}{r} + \frac{2}{n}. \quad (3.9)
\]
Indeed, the condition (3.9) implies $\frac{1}{s_1} > \frac{a}{n}$. And from (3.10) we need
\[
\frac{1}{r} + \frac{2}{n} > \frac{a}{n},
\]
that is,
\[
\frac{1}{r} > \frac{a - 2}{n}
\]
which holds naturally when $0 < a < 2$. Hence there exist $s'_1$ and $s'_2$ satisfying (3.9) and (3.10).

Similarly for $G$,
\[
\|G\|_{L^s_{\text{loc}}} \leq C \|y^b u_B\|_{\frac{ns}{n+b}} \leq C \|y^b\|_{L^s_{\text{loc}}} \|u_B\|_{L^s_{\text{loc}}}
\]
under the condition (3.6) and
\[
\frac{1}{r'_1} + \frac{1}{r'_2} = \frac{1}{s} + \frac{2}{n}. \quad (3.11)
\]
Since there is no other restrictions on $r'_1$ and $r'_2$, there obviously exist $r'_1$ and $r'_2$ satisfying (3.11). Together with the boundedness of $u_B(x)$ and $v_B(x)$ we have finished the estimation of $F$ and $G$. Indeed, $u_B(x)$ and $v_B(x)$ and their supports are both bounded by the construction.

So far we have concluded that
\[
\begin{align*}
T & \text{ is a contraction mapping from } L^r_{\text{loc}}(\mathbb{R}^n) \times L^s_{\text{loc}}(\mathbb{R}^n) \text{ into itself}, \\
(F, G) & \in L^r_{\text{loc}}(\mathbb{R}^n) \times L^s_{\text{loc}}(\mathbb{R}^n);
\end{align*}
\]
under conditions (3.3), (3.4), (3.6), (3.7) and (3.8), that is,
\[
\begin{align*}
\frac{1}{r} & < 1 - \frac{2}{n}, \\
\frac{p - 1}{s_0} + \frac{1}{s_1} + \frac{1}{s} = \frac{1}{r} + \frac{2}{n}, \\
\frac{1}{s} & < 1 - \frac{2}{n}, \\
\frac{q - 1}{r_0} + \frac{1}{r_1} + \frac{1}{r} = \frac{1}{s} + \frac{2}{n}, \\
as_1 & < n.
\end{align*}
\]

Step 4. It would be more intuitive to comprehend conditions (3.3), (3.4), (3.6), (3.7) and (3.8) if we consider a plane with $\frac{1}{r}$ as its horizontal co-ordinate and $\frac{1}{s}$ as its vertical co-ordinate. Notice that conditions (3.4) and (3.7) are possible to satisfy only if they present a same line in the $(\frac{1}{r}, \frac{1}{s})$ ordinate system. Therefore, rewrite (3.4) and (3.7) as
\[
\frac{1}{r} - \frac{1}{s} = \frac{p - 1}{s_0} + \frac{1}{s_1} - \frac{2}{n}, \quad (3.12)
\]
\[
\frac{1}{r} - \frac{1}{s} = - \left( \frac{q - 1}{r_0} + \frac{1}{r_1} - \frac{2}{n} \right).
\]  
(3.13)

And horizontal intercepts of the line (3.12) and the line (3.13) should be equal. Hence

\[
\frac{q - 1}{r_0} + \frac{p - 1}{s_0} + \frac{1}{r_1} + \frac{1}{s_1} = \frac{4}{n}.
\]  
(3.14)

Indeed, the condition (3.8) implies \(\frac{1}{s_1} > \frac{a}{n}\). To ensure that there exist such \(s_1\) and \(r_1\), we need

\[
\frac{q - 1}{r_0} + \frac{p - 1}{s_0} < \frac{4 - a}{n}
\]  
(3.15)

which holds under the condition (1.12). Meanwhile, the condition (3.14) also shows that the value of \((\frac{1}{r_1}, \frac{1}{s_1})\) can be taken as \((\frac{1}{r_0}, \frac{1}{s_0})\) when we further seek the value range of \((\frac{1}{r_1}, \frac{1}{s_1})\).

Indeed, applying the relation (3.14), we have

\[
\frac{1}{r_1} + \frac{1}{s_1} = \frac{4}{n} - \frac{p - 1}{s_0} - \frac{q - 1}{r_0}.
\]

Analogously to the previous analysis, regarded \((\frac{1}{r_1}, \frac{1}{s_1})\) as a point in the corresponding plane, it falls on the line segment between \((0, \frac{4}{n} - \frac{p - 1}{s_0} - \frac{q - 1}{r_0}, \frac{a}{n})\) and \((\frac{4 - a}{n} - \frac{p - 1}{s_0} - \frac{q - 1}{r_0}, \frac{a}{n})\) since the value of \(s_1\) follows the condition (3.8). If we put \((\frac{1}{r_1}, \frac{1}{s_1}) = (\frac{1}{r_0}, \frac{1}{s_0})\) in (3.12) and (3.13). Then we need

\[
\begin{align*}
\frac{1}{r_0} & = \frac{p}{s_0} + \frac{1}{s_1} - \frac{2}{n}, \\
\frac{1}{s_0} & = \frac{q}{r_0} + \frac{1}{r_1} - \frac{2}{n},
\end{align*}
\]

or

\[
\begin{align*}
\frac{1}{r_0} & = \frac{1}{s_0} - \frac{q}{r_0} + \frac{2}{n}, \\
\frac{1}{r_1} & = \frac{1}{s_0} - \frac{p}{s_0} + \frac{2}{n},
\end{align*}
\]

which satisfies the condition (3.14). More importantly, the latter two inequalities of the condition (1.12) ensure that \((\frac{1}{r_1}, \frac{1}{s_1})\) indeed falls on the line segment mentioned above since these two inequalities imply

\[
\begin{align*}
0 < \frac{1}{r_1} & = \frac{1}{s_0} - \frac{q}{r_0} + \frac{2}{n} < \frac{4 - a}{n} - \frac{p - 1}{s_0} - \frac{q - 1}{r_0}, \\
\frac{a}{n} < \frac{1}{s_1} & = \frac{1}{r_0} - \frac{p}{s_0} + \frac{2}{n} < \frac{4 - a}{n} - \frac{p - 1}{s_0} - \frac{q - 1}{r_0}.
\end{align*}
\]

From the previous discussion, we conclude that

\[
\begin{align*}
T & \text{ is a contraction mapping from } L_{loc}^r(\mathbb{R}^n) \times L_{loc}^s(\mathbb{R}^n) \text{ into itself,} \\
(F,G) & \in L_{loc}^r(\mathbb{R}^n) \times L_{loc}^s(\mathbb{R}^n).
\end{align*}
\]
under conditions
\[
\begin{aligned}
&1 < 1 - \frac{2}{n}, \\
&\frac{1}{r_0} < 1 - \frac{2}{n}, \\
&\frac{1}{s_0} < 1 - \frac{2}{n}, \\
&\frac{q}{r_0} - \frac{1}{n} < \frac{2}{n}, \\
&\frac{p}{s_0} - \frac{1}{r_0} < \frac{2 - a}{n}.
\end{aligned}
\]

It is indeed the condition (1.12) with respect to \((r_0, s_0)\), which is also the required initial integrability of \((u, v)\). And the condition (3.15) can be derived from the last two inequalities.

By means of Lemma 2.4, under taking
\[
X = L_{loc}^{r_0}(\mathbb{R}^n) \times L_{loc}^{s_0}(\mathbb{R}^n), \quad Y = L_{loc}^{r}(\mathbb{R}^n) \times L_{loc}^{s}(\mathbb{R}^n)
\]
and
\[
Z := X \cap Y,
\]
the regularity of \((u, v)\) can be lifted from \(X\) to \(Z\). Hence \((u, v) \in Y\) under specified conditions from the previous analysis, that is,
\[(u, v) \in L_{loc}^{r}(\mathbb{R}^n) \times L_{loc}^{s}(\mathbb{R}^n)\]
for any \((\frac{1}{r}, \frac{1}{s})\) satisfying (3.3), (3.4), (3.6), (3.7) and (3.8) under the specified initial integrability of \((u, v)\).

Next we should verify that the admissible set of \((r, s)\) or \((\frac{1}{r}, \frac{1}{s})\) cannot be an empty set. From conditions (3.3) and (3.6), the admissible set of \((r, s)\) is not an empty set if and only if the line (3.12) or (3.13) exactly go through the region
\[
\begin{aligned}
0 < \frac{1}{r} < 1 - \frac{2}{n}, \\
0 < \frac{1}{s} < 1 - \frac{2}{n}.
\end{aligned}
\]

It is equivalent to confirm
\[
\begin{aligned}
\frac{2}{n} - 1 < - \left( \frac{q - 1}{r_0} + \frac{1}{r_1} - \frac{2}{n} \right) < 1 - \frac{2}{n}, \\
\frac{2}{s} - 1 < \left( \frac{p - 1}{s_0} + \frac{1}{s_1} - \frac{2}{n} \right) < 1 - \frac{2}{n},
\end{aligned}
\]
that is,
\[
\begin{aligned}
\frac{4}{n} - 1 < \frac{q - 1}{r_0} + \frac{1}{r_1} < 1, \\
\frac{4}{n} - 1 < \frac{p - 1}{s_0} + \frac{1}{s_1} < 1;
\end{aligned}
\]
which holds naturally under the condition (3.14) if \(n \geq 4\). (i) holds since the admissible set of \((r, s)\) or \((\frac{1}{r}, \frac{1}{s})\) must includes either large \(r\) or large \(s\).

Step 5. In this step, we further consider the range of \((r, s)\) or \((\frac{1}{r}, \frac{1}{s})\). Since the line (3.12) and (3.13) are the same one in the \((\frac{1}{r}, \frac{1}{s})\) ordinate system under the condition (3.14), it is feasible to search the value range of \((\frac{1}{r}, \frac{1}{s})\) by the horizontal intercept of the line (3.12) or (3.13). From the specific form of the line (3.12) or (3.13), we find twice the horizontal intercept of it can be expressed as
\[
\frac{p - 1}{s_0} - \frac{q - 1}{r_0} + \frac{1}{s_1} - \frac{1}{r_1}.
\]
Hence it is of vital significance to consider the item \( \frac{1}{r_1} - \frac{1}{r_1} \). It has been already proved that \( \left( \frac{1}{r_1}, \frac{1}{s_1} \right) \), a point in the corresponding plane, falls on the line segment between \((0, \frac{4}{n} - \frac{p-1}{s_0} - \frac{q-1}{r_0})\) and \(\left( \frac{4-a}{n} - \frac{p-1}{s_0} - \frac{q-1}{r_0}, \frac{a}{n} \right)\). Then from the endpoints of the segment we find

\[
\frac{1}{s_1} - \frac{1}{r_1} \in \left( \frac{2a - 4}{n} + \frac{p-1}{s_0} + \frac{q-1}{r_0}, \frac{4}{n} - \frac{p-1}{s_0} - \frac{q-1}{r_0} \right)
\]

and the value range of the horizontal intercept

\[
\frac{1}{2} \left( \frac{p-1}{s_0} - \frac{q-1}{r_0} + \frac{1}{s_1} - \frac{1}{r_1} \right) \in \left( \frac{a-2}{n} + \frac{p-1}{s_0}, \frac{2}{n} - \frac{q-1}{r_0} \right) := I.
\]

From the condition (3.15), we obtain

\[
\frac{a-2}{n} + \frac{p-1}{s_0} < \frac{2}{n} - \frac{q-1}{r_0}.
\]

And the interval

\[
I \subset \left( \frac{2}{n} - 1, 1 - \frac{2}{n} \right)
\]

under \( n \geq 4 \) from the previous analysis. Hence for any \( \left( \frac{1}{r}, \frac{1}{s} \right) \) lying in the strip region between the following two lines

\[
\begin{align*}
\frac{1}{r} - \frac{1}{s} &= \frac{2}{n} - \frac{q-1}{r_0}, \\
\frac{1}{r} - \frac{1}{s} &= \frac{a-2}{n} + \frac{p-1}{s_0};
\end{align*}
\]

under \( 0 < \frac{1}{r} < 1 - \frac{2}{n}, 0 < \frac{1}{s} < 1 - \frac{2}{n} \). If we further observe the relationship between 0 and the interval \( I \), it can be concluded that

\[
\begin{align*}
&u \in L^r_{\text{loc}}(\mathbb{R}^n), \quad \text{if} \quad \frac{2}{n} - \frac{q-1}{r_0} < 0; \\
v \in L^s_{\text{loc}}(\mathbb{R}^n), \quad \text{if} \quad \frac{a-2}{n} + \frac{p-1}{s_0} > 0; \\
(u, v) \in L^r_{\text{loc}}(\mathbb{R}^n) \times L^s_{\text{loc}}(\mathbb{R}^n), \quad \text{if} \quad \frac{a-2}{n} + \frac{p-1}{s_0} \leq 0 \leq \frac{2}{n} - \frac{q-1}{r_0};
\end{align*}
\]

for \( r \) and \( s \) sufficiently large. Hence (ii) is proved.

Then by interpolations, \( u \) belongs to \( L^r_{\text{loc}}(\mathbb{R}^n) \) regardless of where \( v \) belongs to and vice versa. Hence the admissible set of \( \left( \frac{1}{r}, \frac{1}{s} \right) \) can be extended to a larger square generated by the strip region above. After simple calculation,

\[
(u, v) \in L^r_{\text{loc}}(\mathbb{R}^n) \times L^s_{\text{loc}}(\mathbb{R}^n)
\]

under

\[
\left( \frac{1}{r}, \frac{1}{s} \right) \in \left( \max \left\{ 0, \frac{p-1}{s_0} + \frac{a-2}{n} \right\}, \min \left\{ 1 - \frac{q-1}{r_0}, 1 - \frac{2}{n} \right\} \right)
\times \left( \max \left\{ 0, \frac{q-1}{r_0} - \frac{2}{n} \right\}, \min \left\{ \frac{n-a}{n} - \frac{p-1}{s_0}, 1 - \frac{2}{n} \right\} \right).
\]

Hence (iii) is proved. And since

\[
\frac{q-1}{r_0} + \frac{p-1}{s_0} < \frac{4-a}{n},
\]

that is,

\[
\frac{q-1}{r_0} - \frac{2}{n} + \frac{p-1}{s_0} + \frac{a-2}{n} < 0,
\]
it is impossible for items $\frac{q-1}{r_0} - \frac{2}{n}$ and $\frac{p-1}{s_0} + \frac{a-2}{n}$ to be larger than zero at the same time. Consequently, it also gives that at least one of $u$ and $v$ belongs to $L_{loc}^r(\mathbb{R}^n)$ for any $t$ sufficiently large.

Step 6. So far it has been concluded that for any $(r_0, s_0)$ under the condition (1.12), the regularity of solutions $(u, v)$ can be lifted to spaces under the certain range in conclusion (iii). Choose specified $(r_1, s_1)$ to be

$$
\begin{align*}
\frac{1}{r_1} &= \frac{1}{s_0} - \frac{q}{r_0} + \frac{2}{n}, \\
\frac{1}{s_1} &= \frac{1}{r_0} - \frac{p}{s_0} + \frac{2}{n},
\end{align*}
$$

as in Step 3, which ensures that $(r_0, s_0)$ lies on the line (3.12) or (3.13) and the horizontal intercept of this line definitely belong to the interval $I$. Moreover, as is shown in Figure 1 and due to the condition (1.12), there exists $(r, s)$ such that

$$
\begin{align*}
\frac{1}{r} &< \frac{2}{nq}, \\
\frac{1}{s} &< \frac{2 - a}{np}.
\end{align*}
$$

![Figure 1. Required initial integrability](image)

The shading in Figure 1 shows the required value range of $\left(\frac{1}{r_0}, \frac{1}{s_0}\right)$, or the initial integrability of $(u, v)$ and red lines (Line 1) with positive slopes represent the last two inequalities in (1.12). Another red line represents the condition (3.15). Green lines (Line 3) and blue lines (Line 2) represents lines in (3.16) if we put

$$
\left(\frac{1}{r_0}, \frac{1}{s_0}\right) = \left(\frac{2}{nq}, 0\right)
$$

and

$$
\left(\frac{1}{r_0}, \frac{1}{s_0}\right) = \left(0, \frac{2 - a}{np}\right),
$$

respectively. If we regard the specified $(r, s)$ above as the initial integrability of solutions and repeat the regularity lifting process again, we obtain

$$(u, v) \in L_{loc}^r(\mathbb{R}^n) \times L_{loc}^s(\mathbb{R}^n)$$
for $r$ and $s$ sufficiently large, the specified $(r, s)$ mentioned above satisfies

$$\begin{cases}
\frac{1}{r_0} < \frac{2}{nq} < \frac{2}{n(q-1)}, \\
\frac{1}{s_0} < \frac{2-a}{np} < \frac{2-a}{n(p-1)},
\end{cases}$$

that is,

$$\frac{a-2}{n} + \frac{p-1}{s_0} \leq 0 \leq \frac{2}{n} - \frac{q-1}{r_0},$$

which is the third situation in the conclusion (ii). Hence (iv) is proved. Now we have lifted the regularity of $(u, v)$ and thus completed the proof of Theorem 1.1.

### 3.2. Proof of Theorem 1.2

Based on Theorem 1.1 above, we attempt to further lift the regularity of nonnegative solutions for the system (1.8) to $L^\infty(\mathbb{R}^n)$.

From the form of the system (1.10), we write $u(x)$ as

$$u(x) = \int_0^1 \int_{B(t,x)} |y|^{-a} v^p(y) dy \, dt + \int_1^\infty \int_{B(t,x)} |y|^{-a} v^p(y) dy \, dt := I_1 + I_2.$$

Next we consider $I_1$ and $I_2$, respectively.

For $I_1$, estimate the integral inside primarily. Applying Hölder inequality under large $m_2$,

$$\frac{1}{m_1} + \frac{1}{m_2} = 1$$

and

$$1 < m_1 < \frac{n}{a},$$

we obtain

$$\int_{B_1(x)} |y|^{-a} v^p(y) dy \leq \left( \int_{B_1(x)} |y|^{-am_1} dy \right)^{\frac{1}{m_1}} \left( \int_{B_1(x)} v^{pm_2}(y) dy \right)^{\frac{1}{m_2}} \leq C_1^{2n},$$

where $C$ represents different constants in different circumstances. Hence

$$I_1 \leq \int_0^1 \frac{C_1^{2n}}{t^{n-1}} \, dt \leq C_1. \tag{3.17}$$

For $I_2$, denote $\delta \equiv |x - z| < 1$, then

$$I_2 \leq \int_1^\infty \int_{B_{t+\delta}(x)} |y|^{-a} v^p(y) dy \, dt \frac{(1 + \delta)^{n-1}}{(t + \delta)^{n-2}} \leq C_2 \int_1^\infty \int_{B_{t+\delta}(x)} |y|^{-a} v^p(y) dy ds \leq C_2 u(z). \tag{3.18}$$

Combining (3.17) with (3.18), we obtain

$$u(x) \leq C_1 + C_2 u(z). \tag{3.19}$$

Integrate on both sides of (3.19) with respect to $z$ over $B_\delta(x)$, we conclude

$$\int_{B_\delta(x)} u(x) \, dz \leq C + \int_{B_\delta(x)} u(z) \, dz \leq C + C \int_{B_\delta(x)} u^\prime(z) \, dz \leq C,$$

where $r$ is large enough such that $u \in L^r_{loc}(\mathbb{R}^n)$. Notice that constants $C$ are independent of $x$, therefore

$$u(x) \leq C.$$
for almost all $x \in \mathbb{R}^n$. Hence
\[ u \in L^\infty(\mathbb{R}^n). \]
Similarly for $v(x)$. Thus
\[ u, v \in L^\infty(\mathbb{R}^n). \]
Now we completed the proof of Theorem 1.2.

**Acknowledgments.** The author would like to thank professor Congming Li and Leyun Wu for their interest in this work and for many helpful discussions.

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Received June 2020; revised January 2021.

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