Contraction groups and scales of automorphisms of totally disconnected locally compact groups

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Abstract

We study contraction groups for automorphisms of totally disconnected locally compact groups using the scale of the automorphism as a tool. The contraction group is shown to be unbounded when the inverse automorphism has non-trivial scale and this scale is shown to be the inverse value of the modular function on the closure of the contraction group at the automorphism. The closure of the contraction group is represented as acting on a homogenous tree and closed contraction groups are characterised.

1 Introduction

Interest in contraction groups and related concepts has been stimulated by applications in the theory of probability measures and random walks on groups and in representation theory.

In representation theory, contraction groups bring about the Mautner phenomenon. This is manifest in Wang’s examination [Wan84] of the phenomenon in \( p \)-adic Lie groups, though more in the background in Moore’s treatment of the Lie group case [Moo80]. To our knowledge the first to define and study contraction groups (in a slightly more general context than ours, see [MR76]) was Müller-Römer, who did so to study a representation theoretic question as well (the Wiener and Tauber property of a group algebra).

When studying semistable convolution semigroups of probability measures on locally compact groups, contraction groups arise naturally [HS88, DS91]. For this reason a lot of preliminary work on

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contraction groups was done by Siebert, and most of the known results are either derived or referenced in [Sie86]. Proposition 4.2 in [Sie86] reduces the study of locally compact contraction groups to a separate study of the connected and totally disconnected cases. The connected case is covered by Corollary 2.4 in loc.cit.. In some totally disconnected groups the contraction groups are closed and these are studied in [Sie89], see also Remark 3.33 below. We contribute to the full picture by treating general totally disconnected groups.

Just as Lie techniques are used to study contraction groups in the connected case, so the notions of scale and tidy subgroup (introduced in [Wil94]) are useful in the totally disconnected case. The relevant properties of tidy subgroups are summarized in section 2. Their connection to contraction subgroups and related concepts is explored in section 3. The basic properties are collected in subsection 3.1. This subsection culminates in our main result, Theorem 3.8, whose proof rests on the assumption that the group is metrizable.

Therefore, in the following sections we assume that all groups considered are totally disconnected locally compact metric unless explicitly stated otherwise.

From the reinterpretation of the scale function as the modular function restricted to various subgroups (Proposition 3.21) we infer that contraction groups of automorphisms whose inverse has non-trivial scale are unbounded (Corollary 3.24). We succeed in Proposition 3.32 to characterize closed contraction groups. Theorem 4.2 in Section 4 then represents contraction groups as groups of automorphisms of a homogeneous tree.

We stick to the following conventions: 0 is a natural number. The relations $\subseteq$, $\leq$, etc. always imply strict inclusion. Any automorphism of a topological group will be assumed to be a homeomorphism. By “$X$ is stable under $\alpha$” we mean $\alpha(X) = X$ whereas “$X$ is invariant under $\alpha$” means $\alpha(X) \subseteq X$. The modular function $\text{mod}_G$ of a locally compact topological group $G$ is defined by the equation $\mu(Mg) = \text{mod}_G(g)\mu(M)$ where $\mu$ is a left Haar measure on $G$. We use $e$ for the unit element of a group and 1 for the trivial group. The function $|\cdot|$ stands for the absolute value on complex or $p$-adic numbers, as the case may be.

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2 Basic facts about the scale function

Tidy subgroups for an automorphism \( \alpha \) of a totally disconnected locally compact group \( G \) provide a local description of \( \alpha \). For a subgroup \( V \) of \( G \) define subgroups \( V_+ \) \( V_- \) of \( V \) and \( V_{++} \) \( V_{--} \) of \( G \) by:

\[
V_\pm = \bigcap_{n \geq 0} \alpha^{\pm n}(V) \quad \text{and} \quad V_{\pm \pm} = \bigcup_{n \geq 0} \alpha^{\pm n}(V_{\pm}).
\]

The automorphism \( \alpha \) magnifies \( V_+ \) and shrinks \( V_- \), that is, \( \alpha(V_+) \geq V_+ \) \( \alpha^{-1}(V_-) \geq V_- \). We let \( V_0 \) denote the “neutral” part:

\[
V_0 := V_+ \cap V_- = \bigcap_{k \in \mathbb{Z}} \alpha^k(V) = \bigcap_{j \in \mathbb{N}} \alpha^j(V_-) = \bigcap_{j \in \mathbb{N}} \alpha^{-j}(V_+).
\]

It is stable under \( \alpha \).

**Definition 2.1** Let \( G \) be a totally disconnected locally compact group and \( \alpha \) an automorphism of \( G \). A compact-open subgroup \( V \) is called tidy for \( \alpha \) in \( G \) (or tidy for short if \( \alpha \) is understood) if it satisfies

\((T1)\) \( V = V_+V_- \) \( (= V_-V_+) \)

\((T2)\) \( V_- \) (and \( V_+ \)) are closed

The integer

\[
s_G(\alpha) := |\alpha(V_+) : V_+| = |\alpha(V) : V \cap \alpha(V)|
\]

is called the **scale of** \( \alpha \). The function \( s_G : G \to \mathbb{N} \) obtained by restricting attention to inner automorphisms will be called the **scale function** of \( G \).

The scale of an automorphism is well defined by Theorem 2 in [Wil94]. Observe that a subgroup \( V \) which is tidy for \( \alpha \) will be tidy for \( \alpha^{-1} \) as well.

Tidy subgroups exist. They may be found by running the following algorithm.

**Algorithm 2.2** (cf. proof of Theorem 3.1 in [Wil01])

[0] Choose a compact open subgroup \( O \leq G \).

[1] Let \( ^kO := \bigcap_{i=0}^k \alpha^i(O) \). We have \( ^kO_+ = O_+ \) and \( ^kO_- = \alpha^k(O_-) \). For some \( n \in \mathbb{N} \) (hence for all \( n' \geq n \)) the group \( ^nO \) satisfies \((T1)\).

Put \( O' := \cdot^nO \).

[2] Let \( L := \{ x \in G : \alpha^i(x) \in O' \text{ for almost all } i \in \mathbb{Z} \} \) and \( L := \overline{L} \).

[3] Form \( O^* := \{ x \in O' : \lim_{l \to \infty} \alpha^l(x) \in O' \forall l \in \mathbb{N} \} \) and define \( O'' := O^*L \).

The group \( O'' \) is tidy and we output \( O'' \).
Analyzing the above algorithm (cf. [Wil01, Theorem 3.1]) one arrives at the alternative characterization of the scale function as a minimal distortion value.

\[ s_G(\alpha) = \min \{|\alpha(O) : O \cap \alpha(O)| : O \text{ a compact, open subgroup}\} \]

The following simple criterion ensuring tidiness will be used repeatedly in the sequel.

**Example 2.3** A compact, open subgroup \( V \) satisfying \( V = V^- \) is tidy.

It follows immediately from the definition of the scale that \( \alpha \) has a tidy subgroup satisfying this criterion if and only if \( s(\alpha) = 1 \).

## 3 Contraction groups and parabolics

Let \( \alpha \) be an automorphism of a locally compact group \( G \). This section links the subgroups tidy for \( \alpha \) to two other subgroups having global dynamical definitions in terms of \( \alpha \).

### 3.1 Definitions and basic properties

We define the parabolic group and the contraction group of \( \alpha \) and begin to elucidate links between the various groups. Tidy subgroup methods show that the parabolic group is closed. They are also used in the proof of Theorem 3.8, which is fundamental for all of our results on contraction groups.

**Remark 3.1** If \( V \) is a compact, open subgroup of \( G \), then we have

\[ V^- \subseteq P_\alpha := \{x \in G : \{\alpha^n(x) : n \in \mathbb{N}\} \text{ is bounded}\} \]

and \( V^- = V \cap P_\alpha \) if \( V \) is tidy for \( \alpha \) (see Lemma 9 in [Wil94]).

The group \( P_\alpha \) is closed (Proposition 3 parts (iii) and (ii) in [Wil94] show this) and obviously contains the group

\[ U_\alpha := \{x \in G : \alpha^n(x) \xrightarrow{n \to \infty} e\} , \]

which need not be closed (see Example 3.13(2)).

**Notation 3.2** We call \( P_\alpha \) and \( U_\alpha \) respectively the **parabolic** subgroup and the **contraction** group associated to \( \alpha \). We also let \( M_\alpha := P_\alpha \cap P_{\alpha^{-1}} \) and call it the **Levi factor** attached to \( \alpha \). The
Levi factor is the set of all elements of the ambient group whose $\langle \alpha \rangle$-orbit is bounded. If the automorphism $\alpha$ is inner and is conjugation by $g$, we relax notation and write $P_g, U_g$ and $M_g$.

The term ‘parabolic group’ is suggested by Example 3.13 to follow. The name ‘contraction group’ however is standard, see [MR76, Wan84, Sie86, Sie89]. Results about contraction groups, notably in the case where $U_\alpha$ is closed, may be found in these papers. The next result is one such. We shall use it frequently.

**Lemma 3.3 ([Wan84], Proposition 2.1)** Let $G$ be a locally compact group and let $\alpha$ be an automorphism of $G$ such that $U_\alpha = G$. Then $\alpha$ is compactly contractive, that is, for any compact subset $C$ of $G$ and any neighborhood $O$ of $e$ we have $\alpha^n(C) \subseteq O$ for all $n \geq N(C, O)$.

Since $U_\alpha$ is $\alpha$-stable, this will imply that whenever the ambient group is locally compact and $U_\alpha$ is closed, the restriction of $\alpha$ to $U_\alpha$ is compactly contractive.

Since $\alpha$ shrinks $V_-$, one feels that there should be an even closer connection between $V_-$ and $U_\alpha$ than the one between $V_-$ and $P_\alpha$ displayed by Remark 3.1. Before establishing this, we note some elementary properties of the groups $U_\alpha, P_\alpha$ and $M_\alpha$:

- $U_\alpha^n = U_\alpha$ and $P_\alpha^n = P_\alpha$ for all $n \in \mathbb{N}$.
- $U_{id} = 1$ and $P_{id} = G$.
- $\beta(U_\alpha) = U_{\beta \alpha \beta^{-1}}$, $\beta(P_\alpha) = P_{\beta \alpha \beta^{-1}}$.

Further, as is plain from the definitions, when computing the contraction groups and parabolics inside a subgroup (stable under the automorphism in question) we get the intersections of the contraction groups and parabolics in the ambient group respectively with the subgroup.

Obviously $M_\alpha = M_{\alpha^{-1}}$ and $V_0 \leq V \cap M_\alpha$ for every compact open subgroup with equality if $V$ is tidy thanks to Remark 3.1.

**Proposition 3.4** Let $G$ be a locally compact group and let $\alpha$ be an automorphism. Then $U_\alpha$ is normal in $P_\alpha$, hence $U_\alpha M_\alpha \leq P_\alpha$.

**proof:** Let $x \in P_\alpha$, $u \in U_\alpha$ be given. By definition of $P_\alpha$ the set $\{\alpha^n(x) : n \in \mathbb{N}\}$, name it $K$, is compact. Given an open neighborhood $O$ of $e$, choose an open neighborhood $O'$ of $e$ satisfying $\bigcup_{k \in K} kO'k^{-1} \subseteq O$ ([HR79, II.4.9]). Then from $\alpha^n(u) \in O'$ for all $n \geq n_0$ we infer $\alpha^n(xux^{-1}) = \alpha^n(x)\alpha^n(u)\alpha^n(x)^{-1} \in \bigcup_{k \in K} kO'k^{-1} \subseteq O$, proving that indeed $xux^{-1} \in U_\alpha$. □

Theorem II.4.9 from [HR79] can also be used to prove the first claim in the following result, which is useful in computations.
3.1 Definitions and basic properties

**Lemma 3.5** Let \( G \) be a locally compact group and let \( d, v \in G \) be such that \( dv = vd \) and \( \langle v \rangle \) is bounded. Then \( U_{dv} = U_d \) and \( P_{dv} = P_d \). □

We note the following simple consequence.

**Corollary 3.6** Let \( G \) be a locally compact group and \( g \in G \) be given. Then either of \( U_g \neq 1 \) or \( P_g \neq G \) implies that \( \langle g \rangle \) is infinite cyclic. □

Next, we investigate the behavior of contraction groups and parabolics under quotient maps.

**Proposition 3.7** Let \( p: G \rightarrow \overline{G} \) be a homomorphism of totally disconnected locally compact metric groups which is an identification. Let \( \alpha \) be an automorphism of \( G \) leaving the kernel of \( p \) stable and thus inducing an automorphism \( \overline{\alpha} \) of \( \overline{G} \). Then \( p(U_{\alpha}) = U_{\overline{\alpha}} \).

Proposition 3.7 is obtained as a corollary of the next result taking \( H := \ker p \). We first introduce some notation. Let \( H \) be a subset of the topological group \( G \). Call a sequence of elements \( (x_n)_{n \in \mathbb{N}} \) in \( G \) convergent to \( e \) modulo \( H \), if for any neighborhood \( W \) of \( H \) there is an integer \( N_W \) such that \( W \) contains all terms of the subsequence \( (x_n)_{n \geq N_W} \) and write \( \lim_{n \in \mathbb{N}} x_n = e \mod H \) in this case. Let \( U_{\alpha/H} := \{ x \in G : \lim_{n \in \mathbb{N}} \alpha^n(x) = e \mod H \} \).

**Theorem 3.8** Let \( G \) be a totally disconnected locally compact metric group, \( \alpha \) an automorphism of \( G \) and \( H \) an \( \alpha \)-stable closed subgroup of \( G \). Then \( U_{\alpha/H} = U_{\alpha}H \).

The criterion provided by the following lemma is useful in the proof.

**Lemma 3.9** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in a locally compact group \( G \) and let \( H \) be a subset of \( G \). Then the following statements hold.

1. Let \( \{ x_n : n \in \mathbb{N} \} \) be bounded. If \( H \) contains each accumulation point of \( (x_n)_{n \in \mathbb{N}} \) then \( (x_n)_{n \in \mathbb{N}} \) converges to \( e \) modulo \( H \).

2. If \( (x_n)_{n \in \mathbb{N}} \) converges to \( e \) modulo \( H \), then each accumulation point of \( (x_n)_{n \in \mathbb{N}} \) is contained in \( \overline{H} \).

3. If \( \{ x_n : n \in \mathbb{N} \} \) is bounded and \( H \) is closed then \( (x_n)_{n \in \mathbb{N}} \) converges to \( e \) modulo \( H \) iff \( H \) contains each accumulation point of \( (x_n)_{n \in \mathbb{N}} \).
proof of lemma: (2) follows from the definitions, while (3) is implied by the statements (1) and (2). It remains to prove (1).

We argue by contradiction. Let \( W \) be a neighborhood of \( H \) such that its complement contains infinitely many elements of the set \( \{ x_n: n \in \mathbb{N} \} \). Since \( \{ x_n: n \in \mathbb{N} \} \) is bounded, the subsequence \( (x_{n_i})_{i\in\mathbb{N}} \) of \( (x_n)_{n\in\mathbb{N}} \) formed by the elements not in \( W \) has an accumulation point, which is in \( H \) by assumption. Then \( W \) must contain infinitely many elements of the set \( \{ x_n: i \in \mathbb{N} \} \) contradicting its definition.

\( \Box \)

proof of theorem: The right hand side is contained in the left hand side. To show the opposite inclusion, we need to show that whenever \( \alpha^n(x) \) converges to \( e \) modulo \( H \), then there is an element \( h \) of \( H \), such that \( \alpha^n(\alpha h) \) converges to \( e \).

Since \( G \) is metric, there is a decreasing sequence \( (O^{(i)})_{\infty}^{i=1} \) of compact open subgroups of \( G \) with trivial intersection. Put \( O^{(i)}(O^{(i)}) := G \). We use induction on \( i \in \mathbb{N} \) to show that there are sequences of elements \( (y_i)_{i\in\mathbb{N}} \) in \( G \) and natural numbers \( (N_i)_{i\in\mathbb{N}} \) such that

1. \( y_0 = x, \quad y_{i+1} \in y_i (O_0^{(i)} \cap H) \) for all \( i \in \mathbb{N} \),
2. \( \alpha^n(y_i) \in O^{(i)} \) for all \( n \geq N_i \),
3. \( \lim_{n\in\mathbb{N}} \alpha^n(y_i) = e \mod O_0^{(i)} \cap H \).

Putting \( y_0 := x \) and \( N_0 := 0 \) provides a basis for the induction. For the induction step we will use the following lemma.

Lemma 3.10 Let \( G \) be a totally disconnected locally compact group, \( \alpha \) an automorphism of \( G \), \( H \) an \( \alpha \)-stable closed subgroup of \( G \) and \( x \) an element of \( U_{\alpha/H} \). Then, for any compact open subgroup \( O \) of \( G \) there is an element \( h \) in \( H \) and a natural number \( N \) such that \( \alpha^n(xh) \) is contained in \( O \) for each \( n \geq N \). The sequence \( (\alpha^n(xh))_{n\in\mathbb{N}} \) converges to \( e \) modulo \( O_0 \cap H \).

proof of lemma: Applying step 1 of the Algorithm 2.2 to \( O \cap H \), we may assume that the intersection of \( O \) with \( H \) satisfies property (T1) with respect to \( \alpha \), hence \( O \cap H = (O \cap H) \cup O \cap H \). Using continuity of \( \alpha \), choose a compact open subgroup \( V \) of \( O \) such that \( \alpha(V) \subseteq O \). Then \( \alpha(V(O \cap H)) \subseteq O \alpha(O \cap H) \). Let \( N \) be such that \( \alpha^n(x) \) is contained in \( VH \) for all \( n \geq N \).

Choose an element \( h_0 \) in \( H \) such that \( \alpha^N(xh_0) \in V \). We will complete \( h_0 \) recursively to a sequence \( (h_i) \) of elements in \( h_0(O \cap H) \) such that

\[ \alpha^{N+j}(xh_i) \in V(O \cap H) \text{ for } 0 \leq j \leq i. \]

The recursion starts with \( h_0 \). Suppose then, that \( h_0 \) up to \( h_k \) have already been constructed to satisfy this condition.
Then, using $\alpha(O \cap H) \subseteq (O \cap H)\alpha((O \cap H)_+) \subseteq (O \cap H)$ we obtain

$$\alpha^{N+k+1}(xh) \in (O \cap H).$$

Choose $l_{k+1}$ in $(O \cap H)_+$ such that $\alpha^{N+k}(xh_{l_{k+1}}) \in O$ and put $h_{l_{k+1}} := h_k \alpha^{-N-k}(l_{k+1}^{-1})$. Then the element $h_{l_{k+1}}$ is in $h_0(O \cap H)$, and $\alpha^{N+k+1}(xh_{l_{k+1}}) \in O$. Further, since $xh_k$ is in the same $H$-coset as $x$, $\alpha^{N+k+1}(xh_{l_{k+1}}) \in V H$, by the definition of $N$.

Hence $\alpha^{N+k+1}(xh_{l_{k+1}}) \in O \cap V H = O \cap H$. For all natural numbers $i$ less than $k + 1$ we have $\alpha^{N+i}(xh_{l_{k+1}}) = \alpha^{N+i}(xh_k)\alpha^{-i}(l_{k+1}^{-1}) \in V(O \cap H)$ as well, showing the existence of our announced sequence.

The sequence $(xh_{l_{i}})_{i \in \mathbb{N}}$ is bounded, hence has an accumulation point $xh$ in $xH$. Then $\alpha^{N+i}(xh)$ is in $V(O \cap H)$ for any natural number $i$ because $V(O \cap H)$ is closed. In particular, it is in $O$, showing the first claim.

By the first claim, which we already proved, $\{\alpha^n(xh) : n \in \mathbb{N}\}$ is bounded. Continuity of $\alpha$ and $\alpha^{-1}$ imply that the set of accumulation points of $(\alpha^n(x))_{n \in \mathbb{N}}$ is an $\alpha$-stable subset of $O$. Hence each of these accumulation points belongs to $O_0$. On the other hand, $(\alpha^n(x))_{n \in \mathbb{N}}$ hence $(\alpha^n(xh))_{n \in \mathbb{N}}$ converges to $e$ modulo $H$. This can be reformulated using part 3 of Lemma 3.9 to read that each accumulation point of $(\alpha^n(xh))_{n \in \mathbb{N}}$ belongs to $H$. We conclude that each accumulation point of $(\alpha^n(xh))_{n \in \mathbb{N}}$ belongs to $H \cap O_0$. Applying part 3 of Lemma 3.9 once more we see that the sequence $(\alpha^n(xh))_{n \in \mathbb{N}}$ converges to $e$ modulo $O_0H \cap O_0$ of $H$, and we have established the second claim. The lemma is proved. □

Returning to the proof of the theorem, assume that the induction hypothesis has been established for $i$. We apply the lemma with $O_0(i) \cap H$ in place of $H$, $O^{i+1}$ in place of $O$ and $y_i$ in place of $x$. We deduce that there is an element $h_i \in O_0(i) \cap H$ and an integer $n_i$ such that $\alpha^n(y_i h_i)$ is contained in $O^{i+1}$ for each $n \geq n_i$. Furthermore the sequence $(\alpha^n(y_i h_i))_{n \in \mathbb{N}}$ converges to $e$ modulo $O_0^{i+1} \cap H$. Putting $y_{i+1} := y_i h_i$, this gives the induction statement for $i + 1$ proving that the statement holds for all positive integers.

Since $y_{i+1} \in y_i(O_0(i) \cap H)$ and $O^{i+1} \subseteq O^i$, $(y_i(O_0(i) \cap H))_{i=1}^\infty$ is a decreasing sequence of compact sets and $\bigcap_{i=1}^\infty y_i(O_0(i) \cap H)$ is a single point because $\bigcap_{i=1}^\infty O^i$ is trivial. Let $\{y\} = \bigcap_{i=1}^\infty y_i(O_0(i) \cap H)$. Then for every $n$

$$\alpha^n(y) \in \alpha^n(y_i(O_0(i) \cap H)) = \alpha^n(y_i)\alpha^n(O_0(i) \cap H).$$

Since $O_0(i) \cap H$ is $\alpha$-stable, this set equals $\alpha^n(y_i)(O_0(i) \cap H)$, which is contained in $O_0(i)$ for every $n \geq n_i$. Hence $\alpha^n(y)$ converges to $e$ as
n tends to $\infty$. Since $y_0 = x$, $y_{i+1} \in y_i H$ and $H$ is closed, $x^{-1} y \in H$ and so the proof is completed by setting $h := x^{-1} y$. \hfill \Box

Groups of the form $U_\alpha/H$ with $H$ compact arise naturally when studying (semi-)stable convolution semigroups of probability measures on the ambient group, see [HS88, DS91]. Theorem 3.8 above generalizes Theorem 2.4 in [DS91]. Theorem 3.1 in [HS88] covers the case where $H$ is a compact subgroup of a Lie group. We pose the question whether Theorem 3.8 can be generalised to include the non-metric case.

The analogue of the above Theorem for parabolic groups and their Levi factors does not hold. Indeed, there are discrete counterexamples.

**Example 3.11** Let $G$ be a finitely generated discrete group such that $G \neq [G, G] = Z(G)$ and $G/[G, G]$ is torsion free. For example, take $G$ to be the group of integral strict upper triangular matrices of rank 3. Take $L := [G, G]$ and let $g$ be some element of $G$. Since $G/L$ is abelian, the parabolic group (and its Levi factor) attached to $gL \in G/L$ is the whole group. We will show that $P_g$ surjects onto $P_{gL}$ only in the trivial case $g \in L$, providing the desired counterexample.

The assumption that $P_g$ surjects onto $P_{gL} = G$ implies that $G = P_g L = P_g Z(G) = P_g$. Hence every element of $G$ has only a finite number, $m(x)$ say, of $\langle g \rangle$-conjugates and we infer that $[g^{m(x)}, x] = e$. But $G$ is finitely generated, thus there is a positive $M$ such that $[g^M, x] = e$ holds for all $x$ in $G$. In other words, $g^M$ is in $Z(G)$. Since $G/Z(G) = G/[G, G]$ is torsion free, this implies that $g$ belongs to the center of $G$, that is, it belongs to $L$ as claimed.

Though Proposition 3.7 does not generalize to parabolic groups and Levi factors, the following weaker result is obvious.

**Proposition 3.12** Let $p: G \to \overline{G}$ be a perfect homomorphism of locally compact groups and let $\alpha$ be an automorphism of $G$ leaving the kernel of $p$ stable and thus inducing an automorphism $\overline{\alpha}$ of $\overline{G}$. Then $p^{-1}(P_\alpha) = P_\alpha$ and $p^{-1}(M_\alpha) = M_\alpha$. In particular $p(P_\alpha) = P_\overline{\alpha}$ and $p(M_\alpha) = M_\overline{\alpha}$.

We conclude the subsection with some examples illustrating these concepts.

**Example 3.13** The following examples present different types of behavior which can occur. The first and second of these should provide the reader with geometrical intuition on what is going on.
(1) Let \( k \) be a locally compact, totally disconnected field, e.g. the \( p \)-adic numbers \( \mathbb{Q}_p \). Let \( G = SL_n(k) \), equipped with the subspace topology in \( k^{n^2} \). Then \( G \) is a totally disconnected locally compact group.

In the case where \( G = SL_n(\mathbb{Q}_p) \), it follows from Lemma 3.5 that, when computing \( U_g \) and \( P_g \), we may suppose that \( g \) is semisimple. That means that, after conjugation, we may assume that \( g \) is diagonal over some finite extension field of \( \mathbb{Q}_p \) (also see Proposition 3.23). We may further assume, that the valuations of the diagonal entries are in decreasing order. It is easy to compute \( U_g \), \( P_g \) and \( M_g \) with this normalization and one finds that \( U_g \) is a closed (algebraic) normal subgroup of \( P_g \) consisting of unipotent matrices, and that \( P_g \) is the semidirect product of \( M_g \) and \( U_g \). If \( U_g \) is bounded, we have \( U_g = 1 \). If all eigenvalues have distinct absolute value, \( M_g \), \( P_g \) and \( U_g \) will be the groups of diagonal, upper and strictly upper triangular matrices respectively.

If \( G \) is any semisimple group, then essentially the same results hold for the group \( G(k) \): \( P_g \) is the group of rational points of a \( k \)-parabolic subgroup; \( U_g \) is the group of rational points of its unipotent radical; and \( M_g \) is the group of rational points of the centralizer of the unique \( k \)-split torus contained in \( P_g \cap P_g^{-1} \). (This is essentially contained in Lemma 2 of [Pra82]. One should note, that the hypothesis that \( G \) is almost \( k \)-simple is not used in its proof and that the hypothesis on the eigenvalues of \( \text{Ad}(g) \) is only needed to ensure that \( P_g \) is a proper subgroup.)

The scale function was computed for general and special linear groups over local skew fields (and some other linear groups) by Glöckner in [Glö98a]. This list includes the group \( SL_n(\mathbb{Q}_p) \) discussed above. In that group the subgroup \( \text{id} + M_n(p\mathbb{Z}_p) \), is shown to be tidy for diagonal \( g \). It turns out, that \( s(g) \) is the product of the absolute values of those eigenvalues of \( \text{Ad}(g) \) which have absolute value greater than or equal to 1. The scale function is computed for connected semisimple algebraic groups over arbitrary non-Archimedean fields in Proposition 3.23. If the characteristic of the field is 0 one can alternatively use Lie methods to compute the scale function as done in [Glö98b].

(2) Let \( T \) be the homogeneous tree of degree \( q + 1 \). Taking fixators of finite sets of vertices as basic neighborhoods of the identity induces a totally disconnected locally compact group topology on \( \text{Aut}(T) \). Each automorphism of \( T \) either has a fixed point, not necessarily a vertex (elliptic case) or a stable line, which is unique and is called the axis of \( g \) (hyperbolic case).

An elliptic element \( g \) is topologically periodic, hence has \( U_g = 1 \), \( P_g = \text{Aut}(T) = M_g \) and trivial scale. The stabilizer of any point...
fixed by \( g \) is a tidy subgroup since it contains \( g \). If \( g \) is hyperbolic, then it is easily shown that \( P_g \) is the stabilizer of the repelling end \( \epsilon_- \) of \( g \). An automorphism \( x \) is in \( U_g \) if and only if for each \( r > 0 \) there is a point \( p(x, r) \) on the axis of \( g \) such that all points within distance \( r \) of the ray \( [p(x, r), \epsilon_-] \) are fixed by \( x \). In this case, \( U_g \) is unbounded but not closed (the closure of \( U_g \) is the set of elliptic elements fixing \( \epsilon_- \)). The group \( M_g \) being the fixator of the two ends fixed by \( g \), we easily see that \( M_g \cap U_g \) is nontrivial. The scale of a hyperbolic element is \( q^{l(g)} \), where \( l(g) \) is the length of the translation induced by \( g \) on its axis. The fixator of a segment of length at least one on the axis of \( g \) is tidy for \( g \) (one can show using Lemma 3.31(3), that every tidy subgroup for \( g \) is essentially of that form). This example was treated in detail in section 3 of [Wil94].

(3) Let \( H \) be a totally disconnected locally compact group and \( O \) a compact, open subgroup. For example we may take both \( H \) and \( O \) equal to a finite group \( F \) carrying the discrete topology. The shift map \( i \mapsto i + 1 \) on \( \mathbb{Z} \) induces an automorphism \( \sigma \) of the restricted product \( G := \prod_{i \in \mathbb{Z}} H \mid O \). Any compact subset stable under the shift \( \sigma \) is contained in \( O^\mathbb{Z} = \prod_{i \in \mathbb{Z}} O \) and any open stable set contains \( O^\mathbb{Z} \). Therefore, \( O^\mathbb{Z} \) is the unique \( \sigma \)-stable compact, open subgroup of \( G \), and thus the only subgroup tidy for \( \sigma \). As a corollary, \( \sigma \) has scale 1.

The parabolic subgroup \( P_\sigma \) and Levi-factor \( M_\sigma \) are both equal to \( O^\mathbb{Z} \). The contraction group \( U_\sigma \) is the subgroup of all \( (x_i)_{i \in \mathbb{Z}} \in O^\mathbb{Z} \) such that \( x_i \xrightarrow{i \to -\infty} e \). This is a nontrivial and bounded group which is not closed.

3.2 The difference between shrinking and contracting

From now on we assume that the ambient group \( G \) is totally disconnected and \textbf{metrizable} unless explicitly stated otherwise. We establish the connection between that part, \( V_- \), of a compact, open subgroup \( V \) on which \( \alpha \) is shrinking and the corresponding contraction group \( U_\alpha \). The notion of contractivity is stronger than that of shrinking, as the first lemma demonstrates.

\textbf{Lemma 3.14} Let \( V \) be a compact, open subgroup of a totally disconnected locally compact group. Then \( U_\alpha \leq V_- \).

\textbf{proof:} Take \( v \in U_\alpha \). Then there is an \( N \) such that \( \alpha^n(v) \) belongs to (the \( e \)-neighborhood) \( V \) for every \( n \geq N \). Thus \( \alpha^N(v) \in V_- \) and \( v \in V_- \). \( \square \)
3.2 The difference between shrinking and contracting

Since $V_- \subseteq P_\alpha$ (see Remark 3.1), the following result is an immediate consequence of the lemma and Proposition 3.4.

**Corollary 3.15** Let $V$ be a compact, open subgroup of a totally disconnected locally compact group. Then $U_\alpha \trianglelefteq V_-, U_\alpha \cap V_- \trianglelefteq V_-$ and $U_\alpha \cap V_0 \trianglelefteq V_0$. □

We now make the interdependence between $U_\alpha$ and $V_-$ more precise. Clearly, $U_\alpha V_0 \trianglelefteq V_-$ and Lemma 3.3 implies that $V_- \subseteq U_\alpha$ only when $V_0 = 1$. In fact, $V_0$ is the ‘difference’ between $V_-$ and $U_\alpha$.

**Proposition 3.16** For a compact, open subgroup $V$, $V_- = U_\alpha V_0$.

**proof:** We are left to show $V_- \subseteq U_\alpha V_0$. For each $n \in \mathbb{N}$ and $v \in V$ we have $\alpha^n(v) \in \bigcap_{j=-N}^{N} \alpha^j(V)$ for all sufficiently large $n$. Since $\bigcap_{j=-N}^{N} \alpha^j(V)$ is compact and decreases to $V_0$ with $N$, $(\alpha^n(v))_{n \in \mathbb{N}}$ converges to $e$ modulo $V_0$ for all $v \in V_-$. The result follows from Theorem 3.8. □

The next result may be considered a global version of the above proposition.

**Corollary 3.17** $M_\alpha U_\alpha = P_\alpha$, in other words $U_\alpha/M_\alpha = P_\alpha$.

**proof:** Using Proposition 3.4 it suffices to show $P_\alpha \subseteq U_\alpha/M_\alpha$. Let $v$ be in $P_\alpha$. By definition of $P_\alpha$ the sequence $(\alpha^n(v))_{n \in \mathbb{N}}$ is bounded. Let $w$ be an accumulation point, i.e. a limit of a subsequence. Then for any $m \in \mathbb{Z}$ the point $\alpha^m(w)$ as well is a limit of a subsequence and belongs to the compact set $\{\alpha^n(v) : n \in \mathbb{N}\}$. This means $w \in M_\alpha$. Since the set $\{\alpha^n(v) : n \in \mathbb{N}\}$ is bounded and $M_\alpha$ is closed, we may apply part 3 of Lemma 3.9 to conclude that $v \in U_\alpha/M_\alpha$ as claimed. □

As might be expected, the topology of the quotient space $P_\alpha/U_\alpha$ is induced from $M_\alpha$.

**Lemma 3.18** The natural homomorphism $M_\alpha \to P_\alpha/U_\alpha$ is an identification.

**proof:** The homomorphism is continuous and surjective. To show that is open, it suffices to find an open subgroup of $M_\alpha$ such that the restriction of the homomorphism to this subgroup is open.

Let $V$ be a subgroup tidy for $\alpha$ in the ambient group. Then $V_0$ equals $V \cap M_\alpha$ and therefore is a compact open subgroup of $M_\alpha$. We first claim that the image $O$ of $V_0$ is open: The inverse image of $O$ is $V_0 U_\alpha \trianglelefteq P_\alpha$. The equations $V_0 U_\alpha = V_- \triangleright V_- = V \cap P_\alpha$ show
that \(V_0U_\alpha\) is an open hence closed subgroup of \(P_\alpha\). It follows that 
\(V_0U_\alpha\) must equal \(V_0U_\alpha\) and is therefore open. This proves that \(O\) is open as claimed.

By the open mapping theorem ([HR79, II.(5.29)]) the map \(V_0 \to O\) is necessarily open. Since \(O\) was shown to be open, this implies that the map \(V_0 \to P_\alpha/U_\alpha\) is open. □

Lemma 3.29 and Corollary 3.27 will show that the kernel of the homomorphism \(M_\alpha \to P_\alpha/U_\alpha\) is compact. It then follows that this homomorphism is a perfect map.

3.3 Reinterpretation of the scale function

We will investigate the links between contraction groups, parabolics and tidy subgroups further after giving alternative descriptions of the scale of an automorphism.

The next two lemmas are immediate consequences of Example 2.3 and Remark 3.1.

Lemma 3.19 Let \(V\) be a compact, open subgroup of a totally disconnected locally compact group tidy for the automorphism \(\alpha\). Then for all closed \(\alpha\)-stable subgroups \(H\) of \(P_\alpha\)

\[(V \cap H)_- = V_- \cap H = V \cap P_\alpha \cap H = V \cap H\]

is tidy for \(\alpha\) in \(H\). □

Lemma 3.20 Let \(V = V_-\) be a compact, open subgroup of the totally disconnected locally compact group \(H\) and let \(N \trianglelefteq H\) be stable under the automorphism \(\alpha\) of \(H\). Then the image \(q(V) \subseteq H/N\) under the canonical map satisfies

\[q(V_-) \subseteq q(V)_- \subseteq q(V) = q(V_-)\]

and hence is tidy for the induced automorphism \(\overline{\alpha}: H/N \to H/N\). □

The reason for the name 'scale' is that \(s(\alpha)\) is the factor by which \(\alpha\) scales up \(V_+\) when \(V\) is tidy. Thus \(s(\alpha)\) is just the value of the modular function at the restriction of \(\alpha\) to \(V_{++}\). The next result extends this interpretation of the scale.

Proposition 3.21 Let \(N \trianglelefteq H\) be a \(\alpha\)-stable closed subgroups of \(P_\alpha\) and let \(V\) be tidy for \(\alpha\) in the ambient group \(G\). Then writing \(q\) for the canonical map \(H \to H/N\) and \(\overline{\alpha}\), respectively \(\alpha_1\), for the induced automorphisms on \(H/N\) and \(N\), we have
3.3 Reinterpretation of the scale function

(1) \( s_H(\alpha^{-1}) = \Delta_H(\alpha^{-1}) \)
(2) \( s_H(\alpha^{-1}) = s_{H/N}(\alpha^{-1})s_N(\alpha^{-1}) \)
(3) \( s_G(\alpha^{-1}) = s_{P_{\alpha}}(\alpha^{-1}) = s_{V^-}(\alpha^{-1}) = s_{U_{\alpha}}(\alpha^{-1}) \).

**proof:**

“(1)”: By Lemma 3.19 we have \((V \cap H)_- = V \cap H\) and this group is tidy for \(\alpha\) in \(H\). Hence \(\Delta_H(\alpha^{-1}) = s_H(\alpha^{-1})\).

“(2)”: By Lemma 3.20 we have \(q(V \cap H)_- = q(V \cap H)\) and this group is tidy for \(\overline{\alpha}\) in \(H/N\). Hence \(\Delta_{H/N}(\overline{\alpha}^{-1}) = s_{H/N}(\overline{\alpha}^{-1})\). Additionally, applying (1) with \(H := N\) implies \(\Delta_N(\alpha^{-1}) = s_N(\alpha^{-1})\).

Since the modular function satisfies an equation of the form to be proved, substituting the values of the scale function for those of the modular function in that equation proves (2).

“(3)”: We have \(s_G(\alpha^{-1}) = |\alpha^{-1}(V_-) : V_-| = s_{P_{\alpha}}(\alpha^{-1})\) since \(V_- = V \cap P_{\alpha} = (V \cap P_{\alpha})_-\). The same argument works when \(P_{\alpha}\) is replaced by \(V_-\). Now (2) applied with \(H := P_{\alpha}\) and \(N := U_{\alpha}\) gives \(s_{P_{\alpha}}(\alpha^{-1}) = s_{P_{\alpha}/U_{\alpha}}(\overline{\alpha}^{-1})\), leaving us to prove \(s_{P_{\alpha}/U_{\alpha}}(\overline{\alpha}^{-1}) = 1\).

As the product of the scale of the restriction of an automorphism to a normal subgroup and the scale of the induced automorphism on the quotient always divides the scale of the automorphism by Proposition 4.7 in [Wil01], it is enough to show \(s_{M_{\alpha}}(\alpha^{-1}) = 1\), since \(P_{\alpha}/U_{\alpha}\) is a quotient of \(M_{\alpha}\) by Lemma 3.18. We compute

\[
s_{M_{\alpha}}(\alpha^{-1}) = |\alpha^{-1}(M_{\alpha} \cap V)^- : (M_{\alpha} \cap V)^-|
\]

Observing that

\[
M_{\alpha} \cap V = P_{\alpha} \cap P_{\alpha^{-1}} \cap V = V_+ \cap V_- = V_0,
\]

this implies \(s_{M_{\alpha}}(\alpha^{-1}) = |V_0 : V_0| = 1\), as had to be shown. \(\square\)

Combination of (3) and (1) above implies that \(s_G(g) = \Delta_{U_{\alpha}^{-1}}(g)\).

This enables us to compute the scale function of the group of rational points of a semisimple algebraic group \(G\) over a local field of positive characteristic. We start with the following lemma.

**Lemma 3.22** If \(\alpha\) and \(\beta\) are two commuting automorphisms then

(1) \(s(\alpha \beta) \leq s(\alpha)s(\beta)\),
(2) \(s(\beta) = 1 = s(\beta^{-1})\) implies \(s(\alpha \beta) = s(\alpha)\).

**proof:** As (2) follows from (1), it suffices to prove (1). We may suppose that \(\alpha\) and \(\beta\) are inner. Using Theorem 3.4 in [Wil], choose a compact open subgroup \(V\), which is tidy for \(\alpha\) and \(\beta\). The result follows from Proposition A.2 in [GW02]. \(\square\)
We now treat a slightly more general case than announced in Example 3.13.

**Proposition 3.23** Let $k$ be a nonarchimedean local field and let $G$ be a Zariski-connected reductive $k$-group. For any element $g$ of $G(k)$ its scale $s(g)$ equals the product of the absolute values of those eigenvalues of $\text{Ad}(g)$, whose valuation is greater than 1 (counted with their proper multiplicities).

**proof:** Observe first that the claim will be true for an element $g$ whenever it is true for a positive power of $g$. Likewise, using part (2) of Lemma 3.22, we see that we may replace $g$ by $g'$ whenever $g^{-1}g'$ is a compact element commuting with $g$. If $k$ has positive characteristic, some positive power of $g$ is semisimple. If the characteristic of $k$ is 0, $g$ has a Jordan-Chevalley decomposition $g = g_s g_u = g_u g_s$ with $g_s, g_u \in G(k)$ where $g_s$ is semisimple and $g_u$ is unipotent. Since unipotents are compact elements, we may replace $g$ by its semisimple part $g_s$. So, regardless of the characteristic, we may assume from the outset that $g$ is semisimple.

Let $S$ be the smallest torus of $G$ containing $g$, that is, let $S$ be the Zariski-closure of the group generated by $g$. From this characterization it is immediate that $S(k)$ is Zariski-dense in $S$. By the Galois-criterion $S$ is defined over $k$. Its largest split, respectively anisotropic, tori $S_d$ and $S_a$ are defined over $k$ as well.

Write $g$ as $g'a$, where $g'$ is in $S_d(k)$ and $a$ is in $S_a(k)$. By our introductory remarks we may suppose that $g = g'$ is in fact contained in a $k$-split torus.

We are going to use the Remarque after Corollaire 3.18 in [BT65]. Let $\sigma$ be the set of roots of $\Phi(S_d, G)$ with $|\sigma(g)| > 1$. It is a closed (even connected) subset and by [Bou68, Chapitre VI, §1, Proposition 22] there is an ordering on $\Phi(S_d, G)$ such that all the roots in $\sigma$ are positive. Hence $\sigma$ is unipotent. The set of $k$-rational points of the associated unipotent group $U_\sigma$ is equal to $U_{g^{-1}}$, hence $U_{g^{-1}}$ is already closed. This follows from the fact that Lie($G$) is the direct sum of Lie($U_\sigma$), Lie($U_{-\sigma}$) and the sum of the eigenspaces of $g$ to eigenvalues with valuation 1. In particular each eigenspace of $g$ to an eigenvalue with valuation greater than 1 is contained in Lie($U_\sigma$).

Arrange the elements $b_1, \ldots, b_m$ of $\sigma$ in increasing order and put $\sigma_i := \{b_1, \ldots, b_m\}$. The group $U_\sigma$ is defined and split over $k$. More precisely, $U_\sigma = U_{\sigma_1} \supseteq \cdots \supseteq U_{\sigma_m}$ is a filtration by unipotent $k$-groups whose successive quotients admit a structure of $k$-vector space such that any element $x$ of $S_d$ acts by scalar multiplication with $b_i(x)$ on $U_{\sigma_i}/U_{\sigma_{i+1}}$. It follows that $\Delta_{U_{g^{-1}}}(g) = \Delta_{U_\sigma(k)}(g) =$
\[ \prod_{i=1}^{m} b_i(g)^{d_i}, \text{ where } d_i = \dim(U_{\sigma_i}(k)/U_{\sigma_{i+1}}(k)). \] This is exactly what we claimed. \qed

As a further result, contraction groups are usually unbounded.

**Proposition 3.24** The following statements are equivalent:

1. \( s_G(\alpha^{-1}) = 1 \).
2. \( U_\alpha \) is bounded.
3. \( \forall O \text{ tidy for } \alpha : O \subseteq P_{\alpha^{-1}}. \quad (A) \quad U_\alpha \subseteq M_\alpha. \)
4. \( \exists O \text{ tidy for } \alpha : O \subseteq P_{\alpha^{-1}}. \quad (B) \quad M_\alpha = P_\alpha. \)
5. \( P_{\alpha^{-1}} \) is open. \quad (C) \quad \overline{U}_\alpha = U_0 := U_\alpha \cap \overline{U}_{\alpha^{-1}}.

**proof:** We will first prove equivalence of the statements in the left column. The proof that the characterizations in the right column are equivalent uses some results yet to come but are listed here for convenience of reference. These equivalences will not be used until later.

“(2) \Rightarrow (1)”: This is immediate combining parts (1) and (3) of Proposition 3.21 (with \( H := \overline{U}_\alpha \)). We prove that (1) implies (2) and (3). Take \( O \) tidy for \( \alpha \). The equality \( 1 = s_G(\alpha^{-1}) = |\alpha^{-1}(O) : \alpha^{-1}(O) \cap O| \) is equivalent to \( O \supseteq \alpha^{-1}(O) \) and to \( O_+ = O \). The first of these implies \( O_- \subseteq \bigcup_{i \in \mathbb{N}} \alpha^{-i}(O) \subseteq O \) hence that \( U_\alpha \subseteq O \) is bounded, giving (2). Further \( O = O_+ = O \cap P_{\alpha^{-1}} \) gives \( O \subseteq P_{\alpha^{-1}} \), hence (3) follows.

Obviously (3) \(\Rightarrow\) (4) \(\Rightarrow\) (5). Finally to prove that (5) implies (1) take \( O \) tidy for \( \alpha \) in \( P_{\alpha^{-1}} \). Since we assume that \( P_{\alpha^{-1}} \) is open, \( O \) is tidy for \( \alpha \) in the ambient group \( G \) as well and we get \( O_+ = O \cap P_{\alpha^{-1}} = O \) which is equivalent to \( 1 = s_G(\alpha^{-1}) \) as has already been seen.

We now prove equivalence of (2) and the statements in the right column. If we assume (2) then using \( \alpha \)-invariance of \( U_\alpha \) we get that \( \alpha^2(U_\alpha) = U_\alpha \) is bounded. The definition of \( M_\alpha \) then gives \( U_\alpha \subseteq M_\alpha \), that is (A).

The statements (A) and (B) are equivalent thanks to Corollary 3.17. For the remaining implications we need some results yet to be proved. If we assume (B) then since \( M_\alpha \) is closed \( \overline{U}_\alpha = \overline{U}_\alpha \cap M_\alpha \) which equals \( U_0 \) by Lemma 3.29 and we have derived (C). Assuming (C) we have \( U_\alpha \) contained in \( U_0 \), which is a compact group by Corollary 3.27 giving (2). The proof is complete. \qed

### 3.4 Small tidy subgroups

The contraction group, \( U_\alpha \), is closed if and only if there are arbitrarily small subgroups tidy for \( \alpha \). This and numerous other equivalences
are established below in Theorem 3.32. Careful examination of the tidying procedure, Algorithm 2.2, is required for the proofs.

**Corollary 3.25 (to Proposition 3.16)** If $V$ is a tidy subgroup produced by the Algorithm 2.2 from the compact, open subgroup $O$, then $V_- = O_-$. 

**proof:** We go through the steps of the Algorithm 2.2. Step 1 produces a subgroup $O'$ such that $O' = \alpha^n(O)$ hence

$$O' = O^- = O^\ast = O^- = O^-.$$

The group $L = \mathbb{Z}$ produced in step 2 of the algorithm equals $V_0 = O_0^\ast$ by Lemma 3.8 in [Wil01] and is easily seen, by its definition, to be contained in $O^-$. Hence, by Proposition 3.16,

$$V_- = U_\alpha V_0 = U_\alpha L \leq U_\alpha O^- = O^- = O^-.$$

To show the reverse inclusion we proceed to step 3 of the algorithm. Lemma 3.4 in [Wil01] implies

$$O_- = O^- = O^-.$$

From $O^\ast \leq V$ we get immediately $O^\ast \leq V$ and $O_- \leq V$ hence

$$O^- = O^- = O^- \leq V_- = V_-$$

since $V$ is tidy. We are done. 

As an immediate consequence, we have the following characterisation of the closure of $U_\alpha$.

**Theorem 3.26** \( \overline{U}_\alpha = \bigcap \{ V_- : V \text{ is tidy for } \alpha \} \).

**proof:** We already know $U_\alpha \subseteq V_-$ for every compact open subgroup $V$. Since $V_-$ is closed for $V$ tidy, the inclusion $\overline{U}_\alpha \subseteq \bigcap \{ V_- : V \text{ is tidy for } \alpha \}$ follows.

Now let $v \notin \overline{U}_\alpha$. We have to find a subgroup $V$ tidy for $\alpha$ such that $v \notin V_-$. There is a compact open subgroup $O$ such that $vO \cap \overline{U}_\alpha = \emptyset$. Then $vO \cap \overline{U}_\alpha O = \emptyset$ and it follows that

$$v \notin \overline{U}_\alpha O = \overline{O_-}.$$

Thanks to Corollary 3.25, $v \notin V_-$. where $V$ is the tidy subgroup constructed from $O$. We are done. 

One would expect the set of elements where $\alpha$ and $\alpha^{-1}$ are contracting to be small.
Corollary 3.27 The group $U_0$, defined as $\mathcal{U}_\alpha \cap \mathcal{U}_{\alpha^{-1}}$, is equal to

$$\bigcap \{V_0 : V \text{ is tidy for } \alpha\},$$

and hence to

$$\bigcap \{V : V \text{ is tidy for } \alpha\}$$
as well. In particular, it is compact.

proof: First observe that if $V$ is tidy, then so is $\alpha^n(V)$ for any integer $n$. Therefore

$$\bigcap \{V : V \text{ is tidy for } \alpha\} = \bigcap_{\text{tidy } n \in \mathbb{Z}} \alpha^n(V) = \bigcap \{V_0 : V \text{ is tidy for } \alpha\}.$$

This is a compact group. It remains to show that it equals $U_0$.

For this, note that, by Theorem 3.26, $U_0$ is the intersection of all pairs $V_-$ and $W_+$ where $V$ and $W$ run through all tidy subgroups.

Since $V_0 \subseteq V_-$ and $W_0 \subseteq W_+$, it follows that

$$\bigcap \{V_0 : V \text{ is tidy for } \alpha\} \subseteq U_0.$$The next lemma implies the reverse inclusion.

Lemma 3.28 If $V$ be a tidy subgroup of a totally disconnected locally compact group. Then $V_+ \cap V_- = V_0$.

proof: As already seen $V_0 = V_+ \cap V_- \subseteq V_+ \cap V_- \subseteq V_+ \cap V_- = V_0$ always. Conversely if $x \in V_+ \cap V_-$ then

$$x \in \alpha^k(V_+) \cap \alpha^l(V_-) \text{ for all } k \geq k_0 \geq l_0 \geq l.$$Now $\alpha^{-k}(x) \in V_+ \subseteq V \forall k \geq k_0$ and $\alpha^{-l}(x) \in V_- \subseteq V \forall l \leq l_0$. Therefore $x \in \mathcal{L}$. Assuming $V$ to be tidy gives $\mathcal{L} = V_0$, implying that $x \in V_0$. The proof of the corollary is complete. □ □

We will now turn to the question of existence of arbitrarily small tidy subgroups for an automorphism $\alpha$. Corollary 3.27 shows that $U_0$ is an obstruction to their existence. We attempt now to make this more precise.

Lemma 3.29 $\overline{\mathcal{U}}_\alpha \cap P_{\alpha^{-1}} = \overline{\mathcal{U}}_\alpha \cap M_\alpha = U_0.$
**proof:** The proof is an easy direct computation using Theorem 3.26 and that $V_0 = V \cap M_\alpha$ for tidy $V$:

\[
U_0 = \overline{U}_\alpha \cap \overline{U}_{\alpha^{-1}} \subseteq \overline{U}_\alpha \cap P_{\alpha^{-1}} = \overline{U}_\alpha \cap M_\alpha = \bigcap_{V \text{ tidy}} V_\alpha \cap M_\alpha = \\
= \bigcap_{V \text{ tidy}} (V_\alpha \cap M_\alpha) = \bigcap_{V \text{ tidy } \alpha \in \mathbb{N}} (\bigcup_{i \in \mathbb{N}} \alpha^{-i}(V \cap P_\alpha) \cap M_\alpha) = \\
= \bigcap_{V \text{ tidy } \alpha \in \mathbb{N}} \alpha^{-i}(V \cap M_\alpha) = \bigcap_{V \text{ tidy } \alpha \in \mathbb{N}} \alpha^{-i}(V_0) = \bigcap_{V \text{ tidy } \alpha \in \mathbb{N}} V_0 = U_0.
\]

As a Corollary we get a nice factorization of $\overline{U}_\alpha$.

**Corollary 3.30** $\overline{U}_\alpha = U_0 U_\alpha$, In other words $\overline{U}_\alpha = U_\alpha / U_0$.

**proof:** Let $\alpha_1$ be the restriction of $\alpha$ to $\overline{U}_\alpha$. Then $U_{\alpha_1} = U_\alpha$, $M_{\alpha_1} = U_0$ by Lemma 3.29 and $P_{\alpha_1} = \overline{U}_\alpha$ and the claim follows by Corollary 3.17.

After a further lemma we will be able to characterize the automorphisms with arbitrarily small tidy subgroups. Section 2 of [Wil] describes a tidying procedure which differs from Algorithm 2.2 by taking in step 3 the group $K := \overline{U}_\alpha \cap P_{\alpha^{-1}} \subseteq U_0$ instead of $L$ (constructed in step 2, which becomes superfluous). That is, we now put $O^* := \{x \in O': \|xk^{-1}\| \in O' K \forall k \in K\}$ and $O'^* := O^* K$. According to Lemma 3.3(1) in [Wil], $O^*$ is a compact open subgroup. It is smaller than $O$. We work with this algorithm in the proof of the lemma below.

It is proved in *loc.cit.*, that the outputs of the two algorithms are the same. One may wonder, whether the reason for this is that $K = U_0$. It follows from (2) and (1) below, that this is indeed the case.

**Lemma 3.31**

(1) $U_\alpha \cap P_{\alpha^{-1}} = U_\alpha \cap M_\alpha = U_\alpha \cap U_0$.

(2) $\overline{U}_\alpha \cap M_\alpha = U_0$; in particular, $U_0$ is the unique tidy subgroup for $\alpha$ restricted to $U_0$.

(3) A compact, open subgroup is tidy if and only if it satisfies (T1) and contains $U_0$.

**proof:** The first statement is easily derived from the descriptions of $U_0$ in Lemma 3.29.

From (1) we get that $\overline{U}_\alpha \cap M_\alpha$ equals the group $K$ introduced above. It follows from Lemma 3.29 that $K \subseteq U_0$, so assume we are given an element $x$ not in $K$. Since $K$ is closed, there is a compact
3.4 Small tidy subgroups

open subgroup $O$ such that $xO \cap K$ is empty. In other words $x$ does not belong to $KO$.

Applying the modified algorithm to $O$, we have $KO \supseteq KO^* = O''$ is a tidy subgroup, and $x \notin KO$ implies that $x$ is not in $U_0$ by Corollary 3.27. Thus $U_0 \subseteq K$ and therefore $U_0 = K = U_0 \cap M_0$ as claimed in (2).

To see the remaining part of sub-claim (2) we compute the group $U_0$ inside $U_0$: it is $U_0 \cap U_0 \cap U_0^{-1}$ by its definition, and this expression equals $U_0 \cap M_0 \cap U_0^{-1} \cap M_0 = U_0 \cap U_0 = U_0$ by what is already known. Corollary 3.27 implies then, that there can be no smaller tidy subgroups for the restriction of $\alpha$ to $U_0$ than $U_0$ itself.

We turn to (3). The conditions are clearly necessary for a subgroup $O$ to be tidy. If they are satisfied, the variant of the tidying-up procedure described above leaves the compact open subgroup unchanged. This shows that $O$ is tidy. $\square$

In all examples examined so far, $U_0$ is equal to $U_0 \cap U_0^{-1}$. We do not know whether this holds in general.

The existence of arbitrarily small tidy subgroups for a given automorphism has many equivalent reformulations.

**Theorem 3.32** The following conditions are equivalent:

1. $U_\alpha$ is closed.
2. $U_\alpha \cap M_\alpha = 1$.
3. $\overline{U_\alpha} \cap M_\alpha = 1$.
4. $U_0 = 1$.
5. For all compact, open $O \subseteq G$ there is a $k$ such that $kO$ is tidy.
6. $P_\alpha = M_\alpha \ltimes U_\alpha$ topologically.

$\text{(4')}$ There are arbitrarily small tidy subgroups for $\alpha$.

$\text{(6')} P_{\alpha^{-1}} = M_\alpha \ltimes U_{\alpha^{-1}}$ topologically.

$\text{(1')} U_{\alpha^{-1}}$ is closed.

$\text{(2')} U_{\alpha^{-1}} \cap M_\alpha = 1$.

$\text{(3')} \overline{U_{\alpha^{-1}}} \cap M_\alpha = 1$.

$\text{(4')} U_0 = 1$.

$\text{(5')} P_{\alpha^{-1}} = M_\alpha \ltimes U_{\alpha^{-1}}$ topologically.

$\text{(6')} P_{\alpha^{-1}} = M_\alpha \ltimes U_{\alpha^{-1}}$ topologically.

$\text{Proof:}$ "(5') $\iff$ (5)" Assume (5'). Run step 1 of the algorithm 2.2 on the given $O$ to obtain another compact, open subgroup of the stated form satisfying (T1) hence (T2) by assumption. Conversely assume that $O$ satisfies (T1). By assumption $O' := \bigcap_{i=0}^k \alpha^i(O)$ is tidy. As already used several times $O_+^i = O_+^{i+1}$ and $O_-^i = \alpha^i(O_-^i)$ hence $O_+^{i+1} = O'_+^i$ and $O_-^{i+1} = O'_-^i$ are closed since $O'$ is tidy. Hence $O$ is tidy and (T1) is shown to imply (T2).

Since the procedure applied to $O$ in (5) shrinks the compact, open subgroups in question, we have that (5) $\Rightarrow$ (4'), which evidently implies (4). Lemma 3.29 shows, that (4), (3) and (3') are all equivalent.
If we assume that $U_\alpha$ is not closed, then $1 \neq U_0$ by Corollary 3.30. Hence $M_\alpha \cap \overline{U_\alpha} \neq 1$ and we have shown that (3) implies (1).
By symmetry, (3’) implies (1’) as well.
Assuming (1) we find for any compact, open subgroup $V$ that $V_{-} = V_0 U_\alpha$ is closed, since $V_0$ is compact. Assuring $V$ satisfies (T1) we get from [Wil94, Lemma 3(b)] that $V_{++}$ is closed as well. This means that property (T2) is automatic once (T1) is known to hold, i.e. (5’). By symmetry again (1’) implies (5’). So far we have established equivalence of all conditions listed except (2), (2’), (6) and (6’).
It is trivial that (3) implies (2) and (3’) implies (2’). Lemma 3.31(2) gives that (2) implies (4). Symmetrically (2’) implies (4). Together with (4) $\iff$ (3), this leaves to prove the equivalence of (6) and (6’).
By symmetry it suffices to prove that (6) is equivalent to (2). Property (6) evidently implies (2). Assume (2) to conclude that $P_\alpha$ is the semidirect product of $M_\alpha$ and $U_\alpha$ as an abstract group. We may then apply Proposition 6.17 from [RD81]. Our claim is that (6) holds, which is statement (d) in that Proposition. It is shown there that it is equivalent to the statement (b) that the map $M_\alpha \to P_\alpha/U_\alpha$ obtained by restriction of the quotient map modulo the normal subgroup $U_\alpha$ is a topological isomorphism. Since we know that (2) implies (1), this map is the one considered in Lemma 3.18, where we showed that it is an identification in general. Under the assumption of (2) however, the kernel of this map is trivial and (b), thus (6), is established. We are done.

We now list some examples where the conditions of Theorem 3.32 hold.

**Remark 3.33** All contraction groups for general/inner automorphisms are closed in the following cases.

1. Groups with trivial contraction groups, among them
   (a) discrete groups, with respect to all and
   (b) SIN-groups with respect to inner automorphisms.
   (c) MAP-groups with respect to inner automorphisms. (The von-Neumann-kernel of a locally compact group contains all contraction groups of inner automorphisms: Let $\rho$ be a continuous finite-dimensional unitary representation of the group. Consider an element, $x$, say. Since $\rho(x)$ is unitary, the corresponding eigenvalues have absolute value 1. A trivial modification of Lemma II.(3.2) from [Mar89] shows, that $U_x$ fixes each eigenvector of $\rho(x)$. Since there is a basis of the representation space consisting of eigenvectors of $\rho(x)$, we are done.)
(d) nilpotent-by-compact groups. These are distal by [Ros79, Proposition 3]. This, by its definition (loc cit, introduction) easily implies that all contraction groups of inner automorphisms are trivial.

(2) Any (totally disconnected) locally compact group having an open subgroup satisfying the ascending chain condition on its closed subgroups. Thanks to ([Wan84, Lemma 3.2]) the group then satisfies the condition (1) of Theorem 3.32 with respect to any automorphism ([Wan84, Lemma 3.2]). This criterion applies to any $p$-adic Lie group ([Wan84, Theorem 3.5]), hence to any analytic group over any nonarchimedian field of characteristic 0.

(3) Any closed subgroup of a linear group over a local field with respect to inner automorphisms. By the introductory remark it is enough to prove this for the group $SL_n$, since we may embed any closed linear group therein as a closed subgroup. We have already seen in Example 3.13(1) that a contraction subgroup for an element of $SL_n$ is an algebraic subgroup, hence is closed.

Further examples with all contraction groups for inner automorphisms closed may be obtained from the above list by forming projective limits and restricted products.

**Remark 3.34**

(1) Note that item 1(c) above shows that all totally disconnected locally compact MAP-groups have trivial inner contraction groups hence scale function identically one (they are uniscalar) if they are metric. This is an improvement over the main result of [LR68].

(2) Let $G^+$ be the closed subgroup generated by all contraction groups of inner automorphisms. It is stable under any (bicontinuous) automorphism of $G$, hence is in particular normal. As a consequence of Proposition 3.7, for metric groups $G$, $G/G^+$ is uniscalar. The scale function should thus characterize best the groups satisfying $G = G^+$. Every non-uniscalar topologically simple group will belong to this class by Proposition 3.24, but no solvable group will because $G^+ \subsetneq [G, G]$. This last remark suggests that we define subgroups $G^{n+}$ by iterating the definition of $G^+$.

**4 The tree-representation Theorem**

We will show that, if $V$ is tidy for $\alpha$, then $V_-$ extended by the cyclic group generated by $\alpha$, that is, $V_- \rtimes \langle \alpha \rangle$, has a representation onto some closed subgroup of the automorphism group of a homogeneous
THE TREE-REPRESENTATION THEOREM

This representation can be used to analyse groups of this type, which are the simplest non-uniscalar groups.

Although this is not the approach we shall take, this representation is an instance of a general construction in the Bass-Serre theory of group actions on graphs which is described in [Ser80] and [DD89]. Since $\alpha : V_- \to \alpha(V_-)$ is an injection into $V_-$, the group $V_- \rtimes \langle \alpha \rangle$ is isomorphic to the HNN extension

$$V_- * t, \text{ where } t = \alpha^{-1} : \alpha(V_-) \to V_-.$$

The HNN extension is defined in [DD89, Example 3.5(v)] to be the fundamental group of the graph of groups at right. It is shown in [Ser80] and [DD89] how to represent HNN extensions on trees. For completeness, we shall describe the tree and the action of $V_- \rtimes \langle \alpha \rangle$ directly. Elements of $V_- \rtimes \langle \alpha \rangle$ will be denoted $\nu \alpha^m$.

The tree will be denoted by $T$. Its vertices are the left $V_-\text{-cosets}$ in $V_- \rtimes \langle \alpha \rangle$. Distinct vertices $xV_-$ and $yV_-$ are linked by a directed edge $xV_- \to yV_-$ if and only if

$$yV_- \subset xV_- \alpha = (x\alpha)\alpha^{-1}(V_-).$$

Equivalently, there is an edge from $\nu \alpha^m V_-$ to $w \alpha^n V_-$ if and only if $n = m + 1$ and $w \in \nu \alpha^m(V_-)$. Since $\alpha^{-1}(V_-)$ is the union of $s(\alpha^{-1})V_-$-cosets, there are $s(\alpha^{-1})$ out-edges from the vertex $xV_-$. Since $xV_- \subset ((x\alpha^{-1})\alpha)\alpha^{-1}(V_-)$ there is one edge into $xV_-\alpha$ from the vertex $(x\alpha^{-1})V_-$. Hence each vertex in $T$ has degree $s(\alpha^{-1}) + 1$.

We show next that $T$ is a tree when $\alpha$ has infinite order. For each $n \in \mathbb{Z}$, denote the vertex $\alpha^n V_-$ by $V^{(n)}$. Then $V^{(n)} = \alpha^n - \alpha(\nu V_\alpha) \subset \alpha^{-1}V_- \alpha$ and it follows from (1) that

$$\ldots V^{(-2)}, V^{(-1)}, V^{(0)}, V^{(1)}, V^{(2)}, \ldots$$

is an infinite path, call it $P$, in $T$. Consider a general vertex $\nu \alpha^m V_-$ in $T$, where $v \in \alpha^{-n}(V_-)$. Then $\nu \alpha^m V_- = \alpha^m w V_\alpha$, where $w = \alpha^{-m}(v) \subset \alpha^{-(m+n)}(V_-)$. If $m + n \leq 0$, it follows that $\nu \alpha^m V_- = V^{(m)}$. If $m + n > 0$, then $\nu \alpha^m V_-$ is descended by a path of length $m + n$ from $V^{(-n)}$. In either case we see that $\nu \alpha^m V_-$ is connected to $P$. Therefore $T$ is connected. To see that there are no circuits in $T$, observe first of all that, since each vertex has only one in-edge, any circuit without backtracking must be a directed path. However, the power of $\alpha$ in each coset $xV_-$ strictly increases in the direction
of any path in $T$ and so there can be no directed circuit unless $\alpha$ has finite order.

The action of $V_- \rtimes \langle \alpha \rangle$ on the vertices of $T$ is the usual action of the group on a space of left $V_-$-cosets. It is clear that this action preserves the adjacency relation defined in (1). Denote this action by $\rho: V_- \rtimes \langle \alpha \rangle \to \text{Aut}(T)$. Then $\rho$ is vertex and edge transitive on $T$ and so the quotient graph is a loop.

Denote the set of ends of $T$ by $\partial T$; the end of the path $P$ corresponding to $\{V_n(\nu)\}_{n=1}^\infty$ by $\infty$; and the other end of $P$ by $-\infty$. Then $\alpha$ acts as a translation of distance 1 on $P$ with $-\infty$ as the repelling end. Since every path in $P$ descends ultimately from $-\infty$ and it is the unique end of $T$ having this property, $-\infty$ is fixed by $\rho(V_- \rtimes \langle \alpha \rangle)$. This may also be shown directly by verifying that $v\alpha^mV(-r) = V^{(m-r)}$ provided that $r$ is sufficiently large that $\alpha^{r-m}(v) \in V_-$. Recalling that the automorphism group of $T$ is itself a totally disconnected locally compact group when equipped with the topology of uniform convergence on compact sets, we are in a position to state the tree-representation theorem.

**Theorem 4.1** Let $G$ be a totally disconnected locally compact group, $\alpha$ an automorphism of $G$ of infinite order and let $V$ be tidy for $\alpha$. Then $V_- \rtimes \langle \alpha \rangle \to \text{Aut}(T)$ is a continuous representation $\rho$ onto a closed subgroup of the automorphism group of a homogeneous tree $T$ of degree $s(\alpha^{-1}) + 1$.

1. The action of $V_- \rtimes \langle \alpha \rangle$ on $\text{Aut}(T)$: fixes an end, $-\infty$; is transitive on $\partial T \setminus \{-\infty\}$; and the quotient graph is a loop.
2. The stabiliser of each end in $\partial T \setminus \{-\infty\}$ is a conjugate of $V_0 \rtimes \langle \alpha \rangle$. The kernel of $\rho$ is the largest compact normal $\alpha$-stable subgroup of $V_-. $
3. The image of $V_-$ under $\rho$ is the set of elliptic elements in $\rho(V_- \rtimes \langle \alpha \rangle)$.

**proof:** Stabilisers of vertices $xV_-$ in $T$ form a subbasis for the topology on $\text{Aut}(T)$. Since $\rho^{-1}(\text{stab}(xV_-)) = xV_-, x^{-1}$, which is open in $V_- \rtimes \langle \alpha \rangle$, it follows that $\rho$ is continuous. The continuity of $\rho$ and the compactness of $V_-$ imply that $\rho(V_- \rtimes \langle \alpha \rangle) \cap \text{stab}(V(0)) = \rho(V_-)$ is compact and hence closed. Therefore $\rho(V_- \rtimes \langle \alpha \rangle)$ is closed by [HR79, II.(5.9)].

"(1)" It remains only to show that $\rho$ is transitive on $\partial T \setminus \{-\infty\}$ and, for this, it suffices to show that for each $\omega \in \partial T \setminus \{-\infty\}$ there is $v \in V_-$ such that $v.\infty = \omega$. Let $\{v_n\alpha^nV_\infty\}_{n=1}^\infty$ be a path converging to $\omega$. Since $(v_n, v_n+1\alpha^nV_-)$ is an edge, it follows from (1)
that \( v_{n+1}\alpha^{n+1}(V_-) \subset v_n\alpha^n(V_-) \) for each \( n \). Each of these sets is compact and so \( \bigcap_{n=1}^{\infty} v_n\alpha^n(V_-) \neq \emptyset \). Choose \( v \) in this intersection, so that \( v\alpha^n(V_-) = v_n\alpha^n(V_-) \) for each \( n \). Then

\[
v^{(n)} = \alpha^nV_- = \alpha^n(V_-)\alpha^n = v_n\alpha^n(V_-)\alpha^n = v_n\alpha^nV_-
\]

for each \( n \) and it follows that \( v \cdot \infty = \omega \).

“(2)”: For the first part, it suffices to show that the stabiliser of \( \infty \) equals \( V_0 \rtimes \langle \alpha \rangle \). It is clear that this group is contained in the stabiliser. Conversely, if \( x \) stabilises \( \infty \), then it leaves the path \( P \) invariant and in fact translates \( P \) by a distance, \( d \) say. Then \( x\alpha^{-d} \) fixes every vertex on \( P \). Hence \( x\alpha^{-d} \in \bigcap_{n \in \mathbb{Z}} \alpha^n(V_-) = V_0 \) and we have \( x \in V_0 \rtimes \langle \alpha \rangle \).

The kernel of \( \rho \) is the intersection of all the vertex stabilisers and is therefore a compact normal \( \alpha \)-stable subgroup of \( V_- \). If \( M \) is any such subgroup, then it fixes a point, \( p \) say, in \( T \) because it is compact. Since \( M \) is also normal in \( V_- \) and \( \alpha \)-invariant, it fixes every point in the \( V_- \rtimes \langle \alpha \rangle \)-orbit of \( p \). Since \( \rho \) is edge-transitive, it follows that \( M \) fixes every point in \( T \) and so \( M \) is contained in the kernel of \( \rho \).

“(3)”: Since the image of \( \rho \) fixes \( -\infty \), if \( g \in \text{im}(\rho) \) fixes a point \( p \), then it fixes every vertex on the path from \( -\infty \) to \( p \). Hence \( g \) fixes a point if and only if there is a \( k \in \mathbb{Z} \) such that \( g.V^{(k)} = V^{(k)} \). The set of elliptics therefore coincides with

\[
\bigcup_{k \in \mathbb{Z}} \text{stab}(V^{(k)}) \cap \text{im}(\rho) = \bigcup_{k \in \mathbb{Z}} \rho(\alpha^k(V_-)) = \rho(V_-).
\]

The representation \( \rho \) restricts to give a representation of \( \overline{U}_\alpha \rtimes \langle \alpha \rangle \) on \( T \). It follows from Theorem 3.16 that, for \( V_- \) metric, this is the same representation as obtained if Theorem 4.1 is applied with \( G \) equal to \( \overline{U}_\alpha \). Hence all the assertions of Theorem 4.1 hold with \( V_- \) replaced by \( \overline{U}_\alpha \). However, more can be said about the representation of \( \overline{U}_\alpha \rtimes \langle \alpha \rangle \).

**Theorem 4.2** Let \( G \) be a totally disconnected locally compact metric group and \( \alpha \) an automorphism of \( G \) of infinite order. Let \( \rho \) be the representation of \( \overline{U}_\alpha \rtimes \langle \alpha \rangle \) on the tree \( T \) as defined above.

1. The action of \( U_\alpha \) is transitive on \( \partial T \setminus \{-\infty\} \) and is simply transitive if and only if \( U_\alpha \) is closed.
2. If \( U_\alpha \) is closed, then \( \rho \) is a topological isomorphism onto its image.
proof: “(1)”: Theorem 4.1(1) shows that $\overline{\alpha}$ is transitive on the set $\partial T \setminus \{-\infty\}$. Since, by Corollary 3.30, $\overline{\alpha} = U_\alpha U_0$ and since $U_0$ is the stabiliser of $\infty$, we have
\[\partial T \setminus \{-\infty\} = \overline{U}_\alpha \cdot \infty = U_\alpha \cdot \infty.\]

The action is simply transitive if and only if $\text{stab}(\infty) = U_0 \cap U_\alpha$ is trivial. Since $U_0 \cap U_\alpha$ is dense in $U_0$, Theorem 3.32 shows that $U_0 \cap U_\alpha$ is trivial if and only if $U_\alpha$ is closed.

“(2)”: If $U_\alpha$ is closed, then $\alpha$ is compactly contractive. Hence the compact kernel of $\rho$ is trivial and $\rho$ is faithful. □

Subsection 3.2 examined the difference between the groups $V_-$ and $U_\alpha$ where $\alpha$ is respectively shrinking and contracting. Both groups act on $T$ with the $U_\alpha$-action being the restriction of the $V_-$-action. The next example shows that the difference between the groups may, or may not, be seen in the action on the tree.

Example 4.3
(1) Let $G$ be the group of upper triangular matrices in $SL_2(\mathbb{Q}_p)$. The contraction group with respect to the automorphism $\alpha$ given by inner conjugation by $\text{diag}(p, p^{-1})$ is the group of strict upper triangular matrices. The subgroup of upper triangular matrices with entries in $\mathbb{Z}_p$ is compact open in $G$ and satisfies $\alpha(V) \subseteq V$ hence is tidy by Example 2.3. The group $V_-$ is then the subgroup of elements in $G$ having diagonal entries in $\mathbb{Z}_p^*$. The tree representation $\rho$ of $V_-$ is seen to be the composite of the inclusion in $SL_2(\mathbb{Q}_p)$ with the representation of $SL_2(\mathbb{Q}_p)$ on its Bruhat-Tits tree. Since $V_-$ properly contains $U_\alpha$, $\rho$ distinguishes between these groups.

(2) On the other hand, if we are given a closed contraction group $U_\alpha$ we may construct a strictly larger group $G$, an extension of $\alpha$ to $G$ and a tidy subgroup $V$ for $\alpha$ acting on $G$ with $G = V_-$ such that the tree representation $\rho$ defined by $V_-$ satisfies $\rho(V_-) = \rho(U_\alpha)$ as follows. Take any nontrivial compact group $K$ and define $G$ by $G := K \times U_\alpha$. Let $\alpha$ act trivially on $K$. Choose a tidy subgroup $O$ for $\alpha$ in $U_\alpha$. Then $V := K \times O$ is tidy for $\alpha$ in $G$ and $V_- = K \times O_- = K \times U_\alpha = G$. Obviously $\rho(V_-) = \rho(U_\alpha)$.

Part (2) of the example has $V_-$ as the direct product of $U_\alpha$ and the kernel of $\rho$. That is not special to this example. The kernel of $\rho$ and $\overline{\rho}_\alpha$ are both closed normal subgroups of $V_-$ and so their product is always a subgroup and it is closed because $\ker \rho$ is compact. Group isomorphism theorems plus the fact that, if $U_\alpha$ is closed, then $U_\alpha \cap \ker \rho$ is trivial imply the following result.
Proposition 4.4 Let $V$ be tidy for the automorphism $\alpha$ of the totally disconnected locally compact metric group $G$.

(1) If $\rho(U_\alpha) = \rho(V_-)$, then $V_- = U_\alpha \ker \rho$.

(2) If, furthermore, $U_\alpha$ is closed, then $V_- = U_\alpha \times \ker \rho$. □

Remark 4.5 Essentially the same tree as constructed here has been associated with tidy subgroups and contraction groups elsewhere. In Theorem 3.4 of [Mö102] a tree construction is used to translate the definition of tidy subgroup into permutation group theoretic terms. The rooted tree constructed there is a branch of the tree constructed above. Also, the underlying tree constructed in Theorem 4.1, but not the group representation $\rho$, is already implicit in Proposition 3.7 in [Sie86], which studies the case where the contraction group is closed.

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