Partial Differentiation, Differentiation and Continuity on \(n\)-Dimensional Real Normed Linear Spaces

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Summary. In this article, we aim to prove the characterization of differentiation by means of partial differentiation for vector-valued functions on \(n\)-dimensional real normed linear spaces (refer to [15] and [16]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [7], [1], [3], [4], [5], [17], [11], [13], [6], [9], [14], [10], [8], [12], and [18].

One can prove the following propositions:

1. Let \(n\), \(i\) be elements of \(\mathbb{N}\), \(q\) be an element of \(\mathbb{R}^n\), and \(p\) be a point of \(E_n\). If \(i \in \text{Seg } n\) and \(q = p\), then \(|p_i| \leq |q|\).
2. For every real number \(x\) and for every element \(v_1\) of \(\langle E^1, \| \cdot \| \rangle\) such that \(v_1 = \langle x \rangle\) holds \(\|v_1\| = |x|\).
3. Let \(n\) be a non empty element of \(\mathbb{N}\), \(x\) be a point of \(\langle E^n, \| \cdot \| \rangle\), and \(i\) be an element of \(\mathbb{N}\). If \(1 \leq i \leq n\), then \(\|(\text{Proj}(i, n))(x)\| \leq \|x\|\).
(4) For every nonempty element $n$ of $\mathbb{N}$ and for every element $x$ of $\langle \mathcal{L}^n, \| \cdot \| \rangle$ and for every element $i$ of $\mathbb{N}$ holds $\|(\text{Proj}(i, n))(x)\| = |(\text{proj}(i, n))(x)|$.

(5) Let $n$ be a nonempty element of $\mathbb{N}$, $x$ be an element of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $|(\text{proj}(i, n))(x)| \leq |x|$.

(6) Let $m$, $n$ be nonempty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then $\text{Proj}(i, n)$ is a bounded linear operator from $\langle \mathcal{E}^n, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and $(\text{BdLinOpsNorm}((\mathcal{E}^n, \| \cdot \|), (\mathcal{E}^1, \| \cdot \|))(\text{Proj}(i, n))) \leq 1$.

(7) Let $m$, $n$ be nonempty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then

(i) $\text{Proj}(i, n) \cdot s$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and

(ii) $(\text{BdLinOpsNorm}(\langle \mathcal{E}^m, \| \cdot \| \rangle, (\mathcal{E}^1, \| \cdot \|))(\text{Proj}(i, n)) \cdot s) \leq (\text{BdLinOpsNorm}(\langle \mathcal{E}^n, \| \cdot \| \rangle, (\mathcal{E}^1, \| \cdot \|))(\text{Proj}(i, n)) \cdot (\text{BdLinOpsNorm}(\langle \mathcal{E}^m, \| \cdot \| \rangle, (\mathcal{E}^n, \| \cdot \|))(s))$.

(8) For every nonempty element $n$ of $\mathbb{N}$ and for every element $i$ of $\mathbb{N}$ holds $\text{Proj}(i, n)$ is homogeneous.

(9) Let $n$ be a nonempty element of $\mathbb{N}$, $x$ be an element of $\mathcal{R}^n$, $r$ be a real number, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(r \cdot x) = r \cdot (\text{proj}(i, n))(x)$.

(10) Let $n$ be a nonempty element of $\mathbb{N}$, $x$, $y$ be elements of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(x + y) = (\text{proj}(i, n))(x) + (\text{proj}(i, n))(y)$.

(11) Let $n$ be a nonempty element of $\mathbb{N}$, $x$, $y$ be points of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. Then $(\text{Proj}(i, n))(x - y) = (\text{Proj}(i, n))(x) - (\text{Proj}(i, n))(y)$.

(12) Let $n$ be a nonempty element of $\mathbb{N}$, $x$, $y$ be elements of $\mathcal{R}^n$, and $i$ be an element of $\mathbb{N}$. Then $(\text{proj}(i, n))(x - y) = (\text{proj}(i, n))(x) - (\text{proj}(i, n))(y)$.

(13) Let $m$, $n$ be nonempty elements of $\mathbb{N}$, $s$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, $i$ be an element of $\mathbb{N}$, and $s_1$ be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$. If $s_1 = \text{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$, then $\|s_1\| \leq \|s\|$.

(14) Let $m$, $n$ be nonempty elements of $\mathbb{N}$, $s$, $t$ be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, $s_1$, $t_1$ be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, and $i$ be an element of $\mathbb{N}$. If $s_1 = \text{Proj}(i, n) \cdot s$ and $t_1 = \text{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$, then $\|s_1 - t_1\| \leq \|s - t\|$.

(15) Let $K$ be a real number, $n$ be an element of $\mathbb{N}$, and $s$ be an element of $\mathcal{R}^n$. Suppose that for every element $i$ of $\mathbb{N}$ such that $1 \leq i \leq n$ holds
\[ |s(i)| \leq K. \text{ Then } |s| \leq n \cdot K. \]

(16) Let \( K \) be a real number, \( m \) be a non empty element of \( \mathbb{N} \), and \( s \) be an element of \( \langle E^n, \| \cdot \| \rangle \). Suppose that for every element \( i \) of \( \mathbb{N} \) such that \( 1 \leq i \leq n \) holds \( \|(\text{Proj}(i, n))(s)\| \leq K \). Then \( \|s\| \leq n \cdot K \).

(17) Let \( K \) be a real number, \( m \) be a non empty element of \( \mathbb{N} \), and \( s \) be an element of \( \mathcal{R}^n \). Suppose that for every element \( i \) of \( \mathbb{N} \) such that \( 1 \leq i \leq n \) holds \( \langle\text{proj}(i, n)\rangle(s) \leq K \). Then \( |s| \leq n \cdot K \).

(18) Let \( m, n \) be non empty elements of \( \mathbb{N} \), \( s \) be a point of the real norm space of bounded linear operators from \( \langle E^m, \| \cdot \| \rangle \) into \( \langle E^n, \| \cdot \| \rangle \), and \( K \) be a real number. Suppose that for every element \( i \) of \( \mathbb{N} \) and for every point \( s_1 \) of the real norm space of bounded linear operators from \( \langle E^m, \| \cdot \| \rangle \) into \( \langle E^i, \| \cdot \| \rangle \) such that \( s_1 = \text{Proj}(i, n) \cdot s \) and \( 1 \leq i \leq n \) holds \( \|s_1\| \leq K \). Then \( \|s\| \leq n \cdot K \).

(19) Let \( m, n \) be non empty elements of \( \mathbb{N} \), \( s, t \) be points of the real norm space of bounded linear operators from \( \langle E^m, \| \cdot \| \rangle \) into \( \langle E^n, \| \cdot \| \rangle \), and \( K \) be a real number. Suppose that for every element \( i \) of \( \mathbb{N} \) and for all points \( s_1, t_1 \) of the real norm space of bounded linear operators from \( \langle E^m, \| \cdot \| \rangle \) into \( \langle E^1, \| \cdot \| \rangle \) such that \( s_1 = \text{Proj}(i, n) \cdot s \) and \( t_1 = \text{Proj}(i, n) \cdot t \) and \( 1 \leq i \leq n \) holds \( \|s_1 - t_1\| \leq K \). Then \( \|s - t\| \leq n \cdot K \).

(20) Let \( m, n \) be non empty elements of \( \mathbb{N} \), \( f \) be a partial function from \( \langle E^m, \| \cdot \| \rangle \) to \( \langle E^n, \| \cdot \| \rangle \), \( X \) be a subset of \( \langle E^m, \| \cdot \| \rangle \), and \( i \) be an element of \( \mathbb{N} \). Suppose \( 1 \leq i \leq m \) and \( X \) is open. Then the following statements are equivalent

(i) \( f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( f^i|X \) is continuous on \( X \),
(ii) for every element \( j \) of \( \mathbb{N} \) such that \( 1 \leq j \leq n \) holds \( \text{Proj}(j, n) \cdot f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( \text{Proj}(j, n) \cdot f^i|X \) is continuous on \( X \).

(21) Let \( m, n \) be non empty elements of \( \mathbb{N} \), \( f \) be a partial function from \( \langle E^m, \| \cdot \| \rangle \) to \( \langle E^n, \| \cdot \| \rangle \), and \( X \) be a subset of \( \langle E^m, \| \cdot \| \rangle \). Suppose \( X \) is open. Then \( f \) is differentiable on \( X \) and \( f^1|X \) is continuous on \( X \) if and only if for every element \( j \) of \( \mathbb{N} \) such that \( 1 \leq j \leq n \) holds \( \text{Proj}(j, n) \cdot f \) is differentiable on \( X \) and \( (\text{Proj}(j, n) \cdot f)^1|X \) is continuous on \( X \).

(22) Let \( m, n \) be non empty elements of \( \mathbb{N} \), \( f \) be a partial function from \( \langle E^m, \| \cdot \| \rangle \) to \( \langle E^n, \| \cdot \| \rangle \), and \( X \) be a subset of \( \langle E^m, \| \cdot \| \rangle \). Suppose \( X \) is open. Then for every element \( i \) of \( \mathbb{N} \) such that \( 1 \leq i \leq m \) holds \( f \) is partially differentiable on \( X \) w.r.t. \( i \) and \( f^i|X \) is continuous on \( X \) if and only if \( f \) is differentiable on \( X \) and \( f^1|X \) is continuous on \( X \).

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