Multiplicative Reweighting for Robust Neural Network Optimization

Noga Bar
Electrical Engineering
Tel Aviv University
nogabar@mail.tau.ac.il

Tomer Koren
Blavatnik School of Computer Science
and Google Research
Tel Aviv University
tkoren@tauex.tau.ac.il

Raja Giryes
Electrical Engineering
Tel Aviv University
raja@tauex.tau.ac.il

Abstract

Neural networks are widespread due to their powerful performance. However, they degrade in the presence of noisy labels at training time. Inspired by the setting of learning with expert advice, where multiplicative weight (MW) updates were recently shown to be robust to moderate data corruptions in expert advice, we propose to use MW for reweighting examples during neural networks optimization. We theoretically establish the convergence of our method when used with gradient descent and prove its advantages in 1d cases. We then validate our findings empirically for the general case by showing that MW improves the accuracy of neural networks in the presence of label noise on CIFAR-10, CIFAR-100 and Clothing1M. We also show the impact of our approach on adversarial robustness.

1 Introduction

Deep neural networks (DNNs) have gained massive popularity due to their success in a variety of applications. Large sets of accurately labeled data are required to train a DNN to achieve good prediction performance. Yet, such data is costly. To reduce costs, one may train a network using annotated datasets that were created with lesser efforts. These, however, contain label noise, which impedes the training process.

We study methods of mitigating the harmful effect of noise on training. Motivated by learning with expert advice, we employ Multiplicative Weight (MW) updates [46, 16] to promote robustness to noise. In online learning, it was shown theoretically that MW achieves optimal regret in a variety of scenarios, e.g., with losses drawn stochastically [58], adversarially [46, 16], or stochastic losses moderately corrupted by an adaptive adversary [3]. Thus, we suggest considering this technique in a new context of DNN training with noise, such as label noise in training data.

To employ MW for this purpose, we interpret the examples as experts and the predictions of the network as their advice. This leads to a loss function over the predictions, which facilitates using MW. Instead of learning by minimizing a uniform average of the example losses, as in regular stochastic gradient descent (SGD), we learn a weighted version of the losses, where examples do not affect the training process uniformly. Our proposed MW update rule for reweighting examples during training, which we denote as multiplicative reweighting (MR), alternates between SGD steps for optimizing the DNN’s parameters and MW updates for reweighting. Fig. 1 shows the weighting evolution during training, where noisy examples have notably lower weights than their ratio in the training data.
MR explicitly exploits the fact that in the presence of label noise, DNNs often start by fitting clean examples and eventually overfit to corrupted ones, as shown in previous studies [4, 106, 48]. MR is simple, generic and can fit easily in most training procedures. We establish the convergence of a simplified MR version under mild conditions and prove its efficacy for 1d input data.

Motivated by our theoretical findings, we show empirically that MR is beneficial for training using high-dimensional data with noisy labels. We demonstrate MR’s advantage using two popular DNN optimizers: SGD (with momentum) and Adam. We evaluate it both on artificial label noise (symmetric, pair-flip and instance dependent) in CIFAR-10 and CIFAR-100, and real noise in Clothing1M [94]. We compare to many noise robustness techniques [4, 98, 24, 50, 75] including sparsity-based ones [110, 34] and generic regularization strategies such as Mixup [104] and label-smoothing [80].

In the setting of learning with expert advice, MW is known to be optimal in the presence of worst-case adversarial loss. When used with neural networks, we show that MR leads to a more effective Lipschitz constant in the 1d case than GD. Note that a low Lipschitz constant has been shown to improve the robustness of neural networks to adversarial noise [29, 67]. This suggests that MR may be used to improve adversarial robustness. Hence, we also test MR for adversarial attacks. We show that MR can improve adversarial training, which is one of the best approaches to adversarial robustness. Specifically, we show MR’s advantage with Free Adversarial Training [71], TRADES [103], LAS [31] and MAIL [47].

Overall, we show that MR can fit into the deep learning optimization toolbox as an easy-to-use tool for improving general network robustness.

2 Related Work

Multiplicative weights. We are inspired by recent research on corruption-robust online learning, and specifically by new analyses of the multiplicative weight (MW) algorithm in the setting of learning with expert advice. Online learning is concerned with minimizing the regret which is the difference between cumulative loss of the learner’s actions and that of the best action in hindsight. In this context, MW [46, 16] is known to achieve optimal regret (of the form $\Theta(\sqrt{T \log N})$, where $T$ is the time horizon and $N$ is the number of experts), in a general setup where the losses are chosen by an adversary. Very recently, MW was shown to achieve a constant regret (i.e., independent of the time horizon) when the losses are i.i.d. [58], and that this bound deteriorates very moderately when some of the stochastic losses are corrupted by an adversary [3]. Analogous results have been established in related settings, e.g., in Multi-armed Bandits [52, 22, 112], Markov Decision Processes [53], and more. Note that the regret bound in the online learning setting with stochastic losses can be used to bound the associated expected optimization loss, which is the object of interest in supervised and stochastic learning settings. Ideas from online learning and regret minimization were used in other contexts in deep learning, e.g., for improving neural architecture search [61], online deep learning [69], robust loss functions design [2] and to induce compressible networks [7].

Noisy Labels. Many methods were proposed for learning with noisy labels [77]. It was suggested to adjust the network architecture [94, 19, 24, 89, 34], learn with a suitable loss function [18, 17, 57, 44, 23, 82, 106, 95, 83, 25, 14, 12, 87, 110, 49], employ semi-supervised techniques [85, 108, 62, 29, 48, 41] or choose a partial subset of the parameters in the network to update [5, 90]. Some works tried to analyze factors that impact the robustness to noise [37, 78, 55, 40, 13]. There are techniques that were not necessarily designed to training with noisy labels but significantly improve over the vanilla optimization. These include SAM [15], mixup [104] and label smoothing [80, 51].

A connection between label noise and adversarial robustness was studied. On the one hand, it was shown that label noise harms network robustness, where removing the noise is not sufficient to have
robustness [70]. On the other hand, adversarially robust training reduces vulnerability to noisy labels. Moreover, examples that are easy to attack are more likely to suffer from label noise [111].

**Loss Weighting in DNNs.** We focus on methods that use weighting to handle noisy labels. There are two main weighting approaches. One is loss weighting, where different classes affect the loss non-uniformly. This strategy was studied theoretically [60] and empirically [45, 64, 28, 93]. Note that the weighting of each class may be learned in an instance-dependent manner [92, 95, 96, 42].

The second approach is weighting examples, either locally within a mini-batch [30, 68] or globally, weighting all examples in the train set. Global weighting is usually done during the optimization of the DNN. These techniques include learning to weight with gradient methods [66, 75, 104], exploiting the variance and the DNN’s features and predictions [10, 97], employing graph NN and analysing the structural relations of inputs [102], using sample neighbourhood [21] or importance sampling for re-weighting the examples [50, 86], weighting the examples according to beta mixture model [4], exploiting conditional value-at-risk [76], and setting confidence for each example in training [47, 105].

We differ from existing works in using global weighting during optimization based on MW. We also theoretically establish the convergence of MR and demonstrate in 1d cases its possible benefits. Note that some of the above, unlike our method, use an additional clean set, for performing the weighting.

**Curriculum Learning** [6] or self-paced learning [32, 38] is another concept of ordering/weighting examples’ importance. It aims to improve the learning process by ordering carefully the input examples. Initially, the order was established using prior expert data where tasks that are perceived by humans as easy are given to the learning model first [6]. Later it was suggested to learn the ordering [14]. This concept is used to tackle label noise [33, 98, 54, 109, 91, 33, 109]. MR may be viewed as a curriculum learning method, where MW determines unsupervisedly the examples’ importance.

### 3 Training with Multiplicative Reweighting (MR)

In common classification training, we have $N$ training examples $x_i \in \mathbb{R}^d$ with labels $y_i \in \mathbb{R}$ ($1 \leq i \leq N$) sampled from an unknown distribution $P[X, Y]$, $\ell_i(\theta) = \ell(\theta; x_i, y_i)$ is the loss for the $i$'th example and $\theta$ is the net’s parameters.

Most DNNs are trained assigning the same importance to all samples in the training set. They usually use Empirical Risk Minimization (ERM) [72] treating all examples equally:

$$\theta^* = \arg \min_{\theta} \frac{1}{N} \sum_{i=1}^{N} \ell_i(\theta).$$

Yet, suggesting that not all examples are reliable with the same confidence, we use a weighted (non-uniformly averaged) loss function:

$$p^*, \theta^* = \arg \min_{p \in \Delta_N, p \leq \mu/N} \sum_{i=1}^{N} p_i \ell_i(\theta), \quad (1)$$

where $\Delta_N$ is the $N$ dimensional simplex and $p^*$ is supposed to match the reliability of the examples. Thus, finding $p^*$ requires learning the examples that are likely in $P[X, Y]$ without any extra data.

We can also view the problem of finding $p^*$ from a different perspective: The examples $x_i$ can be seen as experts, the predictions they produce by a forward pass as their advice and thus, by applying a loss function over the advice we can solve the problem under the setting of learning with expert advice. In the online learning setting, MW is a well-established algorithm known to produce optimal results in the context of expert advice when the true losses are stochastic and adversarial. This motivates us to use MW also with neural networks using our proposed perspective. The MW algorithm assigns weights to experts according to their cumulative loss, such that weights are negatively proportional to the losses. Hence, the learner gradually overlooks experts that perform poorly.

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**Algorithm 1 Multiplicative Reweighting (MR) with SGD**

**Input:** data $x_i, y_i$ for $1 \leq i \leq N$, $\alpha$-GD step size and $\eta$- MW step size, $B$-batch size, $\mu$ - maximal weight constrain.

**Initialize:** $w_{0,i} = 1$, network init: $\theta_0^{N/B}$

for $t = 1$ to $T$

Project: $p_t = \text{Project}(w_t)$, Algorithm 2

**SGD epoch:**

for $m = 1$ to $N/B$

$$\theta_t^0 = \theta_{t-1}^{N/B}$$

$$\theta_t^m = \theta_t^{m-1} - \alpha \left( \sum_{i \in B_t^m} p_{t,i} \nabla \ell_i(\theta_t^{m-1}) \right) \sum_{i \in B_t^m} p_{t,i}$$

end for

**MW:** $w_{t+1,i} = \exp \left( -\eta \sum_{s=1}^{t} \ell_i(\theta_s^{N/B}) \right)$

end for
Another motivation for using MW to improve network robustness is that in the presence of label noise, we know that DNNs optimized with SGD and cross entropy loss tend to learn first from clean examples and that at the end of the training process they overfit the corrupted data [4] [106] [48]. The corrupted examples are not typical of their class and suffer high loss values throughout a long period of training. The MW rule decays the weights of examples with high loss exponentially and treats them as bad experts. Thus, we expect an improved accuracy. Our method is not expected to harm the performance on the clean data, since in the general case, where there is no known label noise, one can train a network without hard or forgettable examples and achieve a good accuracy [79] [84].

MW outputs a distribution \( p \) over the experts and requires loss per expert as feedback. We suggest using the losses \( \ell_i(\theta) \) produced by the network’s parameters \( \theta \). A change in \( \theta \) yields different losses, which leads to a new MW update. In order to learn both \( \theta \) and \( p \) as in Eq. (1), we propose to alternate between SGD for optimizing the network parameters \( \theta \) and MW for calculating the distribution \( p \). We denote this approach multiplicative reweighting (MR), presented in detail in Algorithm 1.

To update \( \theta \), we use a full epoch with mini-batches of size \( B \) using the weighted loss Eq. (1) with \( p_t \), the distribution at epoch \( t \). The weighted loss leads to variations in the magnitude of updates over different mini-batches. To keep the learning rate stable across mini-batches, we normalize the update by the sum of weights \( \sum_{i \in B_m^t} p_{t,i} \) in the mini-batch \( B_m^t \), where \( m \) is the mini-batch index at epoch \( t \).

To update \( p_t \), we use the weights \( w_t \) that are calculated by the MW update using the DNN parameters of the previous epoch. We project \( w_t \) to a constrained domain described below to make \( p_t \) a valid distribution. The weights for \( t + 1 \) are updated according the entire available loss history. In the MW step, the step size \( \eta \) controls the non-uniformity of the distribution. Note that the MW update requires \( \ell_i(\theta) \) for \( 1 \leq i \leq N \), so an additional forward pass is performed at the end of each epoch.

To avoid learning degenerated probability where few examples have high probability, we limit the highest possible probability by adding to Eq. (1) a linear constraint \( p_i < \frac{\mu}{N}, \forall i, \mu \) is a given parameter. To maintain the linear constraint, we use a projection algorithm, which projects the probabilities that were derived from the recent MW update step into a limited domain in the simplex with non-degenerated probabilities. We present Algorithm 2 which details the procedure of projecting the raw weighting \( w_t \) to the constrained simplex with respect to the KL-divergence [88] [36]. First, the weights are normalized to be in the simplex. Next, all the excess mass of weights higher than \( \mu / N \) is redistributed according to the magnitude of the rest of the weights. This redistributes the excess mass in a way that changes the lowest value of the original KL-divergence of the distribution achieved by MW while maintaining the maximal weight constraint. The process is performed iteratively until all weights are below the chosen threshold. Its total complexity is \( O(N^2) \).

In Section 5 we explore the need for the projection and find that the weights are in the desired range without the projection and thus in practice it is not required. Practically, the network does not assign significantly lower losses to a small set of data during a long period of the training. We suspect that the reason for this is the training’s stochasticity, which induces high loss variability across epochs.

4 Theoretical Analysis

We now derive theoretical guarantees for MR. For the simplicity of analysis, we focus mainly on MR with full batch GD, but also demonstrate how to extend it to SGD. We start with MR with GD as described in Algorithm 5. We prove convergence to a critical point both for MR with GD and a simplified stochastic variant of it. To prove the advantage of MR with noisy data, we analyse two illustrative 1d examples. The first is logistic regression learned with GD. The second is linear regression using MR with a least squares loss, showing convergence to the optimal solution. Proofs and definitions of \( \beta \)-smoothness, \( G \)-lipschitz and \( B \)-bounded function are provided in Appendix C.

4.1 Convergence

It is known that if GD is applied on a \( \beta \)-smooth function, it converges to a critical point [8]. We analyze the full-batch version of MR and show that it converges to a critical point with the same assumptions. To demonstrate this, we provide an equivalent of the descent lemma, proving that in each step the weighted loss decreases, unless the parameters \( p_t, \theta_t \) are already at a critical point of the loss \( \sum_{i=1}^{N} p_{t,i} \ell_i(\theta_t) \).
Lemma 4.1 (Equivalence to Descent Lemma). For a $\beta$-smooth loss $\ell(\cdot)$, and $\theta_{t+1}, p_{t+1}$ updated as in GD+MR (supp. mat.) with GD step size of $\alpha = \frac{1}{\beta}$ and MW step size $\eta > 0$

$$\sum_{i=1}^{N} p_{t+1,i} \ell_i(\theta_{t+1}) - p_{t,i} \ell_i(\theta_t) \leq -\frac{1}{2\beta} \left( \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right)^2.$$ 

The lemma establishes that the loss of each two consecutive iterations in GD+MR is non-increasing. Using this result, we state formally and show convergence to a critical point in the next theorem.

Theorem 4.2 (Convergence). For a $\beta$-smooth loss $\ell(\cdot)$, and $\theta_{t+1}, p_{t+1}$ being updated as in GD+MR (supp. mat.) with GD step size of $\alpha = \frac{1}{\beta}$ and MR step size $\eta > 0$, we have that

$$\frac{1}{T} \sum_{t=1}^{T} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2 \leq \frac{2\beta}{T} \left( \frac{1}{N} \sum_{i=1}^{N} \ell_i(\theta_0) - \sum_{i=1}^{N} p_{t,i} \ell_i(\theta^*) \right),$$

where $p^*, \theta^* \in \arg \min_{p \in \Delta_N, \theta} \sum_{i=1}^{N} p_i \ell_i(\theta)$.

The theorem states that the average over the time horizon $T$ of the gradients of the weighted loss converges to 0. The convergence rate depends on constants, which are determined by the smoothness $\beta$, the starting point $\theta_0$ and the optimal parameters $p^*$, $\theta^*$. Using MR with GD may converge to a degenerate solution, where only one weight is non-zero. Thus, in the supp. mat., we also prove convergence when the distribution $p$ is constrained and each example has a minimal weight.

As the gradients’ average is bounded as Theorem 4.2 states, we can run GD+MR and achieve at a large enough time $t \in [T]$ any desired small gradient size. This essentially means that $\exists t \in [T]$ for which the parameters $\theta_t$ and $p_t$ are as close as we want to a critical point if such a point exists.

Corollary 4.3. Under the same conditions of Theorem 4.2 for $\bar{T} = O\left( \frac{\beta}{\eta} \left( \sum_{i=1}^{N} p_{0,i} \ell_i(\theta_0) - \sum_{i=1}^{N} p_{t,i} \ell_i(\theta^*) \right) \right)$ iterations of GD+MR (supp. mat.): 

$$\left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2 \leq \epsilon$$ for $t \geq \bar{T}$.

We now turn to analysing a stochastic version of the training, where instead of updating $\theta_{t+1}$ with respect to all examples, we sample one example $i_t$ each time according to the current distribution over the examples $p_t$. We consider updating $\theta_{t+1}$ based on the $i_t$ th example’s gradient, i.e., $\theta_{t+1} = \theta_t - \alpha \nabla \ell_{i_t}(\theta_t)$. Namely, we analyse a variant of the algorithm that replaces GD with a SGD variant that samples according to $p_t$ instead of sampling the examples uniformly.

Theorem 4.4. Assume that for $1 \leq i \leq N$: $\ell_i(\cdot)$ is $\beta$-smooth, G-lipschitz and B-bounded. Define $F(\theta, p) = \mathbb{E}_{i \sim p} \ell_i(\theta)$. Then when running MR with SGD that samples according to $i_t \sim p_t$ and uses $\alpha = \sqrt{\frac{2\beta}{G^2 T \Delta^2}}$, it holds that: 

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{i_1, \ldots, i_T} \left\| \nabla F(\theta_t, p_t) \right\|^2 \leq G \sqrt{\frac{2\beta H}{T}}.$$ 

Here we further use the common assumption that the loss is bounded and has bounded gradients. The above theorem bounds the average of the expectations of the gradients (instead of the actual gradients as in Theorem 4.2). The key idea in the proof is that the expectation of the gradient at time $t$ is an unbiased estimator of the average gradient $\sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t)$.

Our analysis above focused on worst-case convergence and we achieved a similar rate to the common GD and SGD algorithms under the same assumptions on the loss function [123].

4.2 Guarantees for 1d Case with Label Noise

Convergence to a critical point does not necessarily imply that the point is a good and robust solution. To motivate the use of MR for noisy and corrupted labels, we present 1d examples of linear and logistic regression, where our method shows advantages, both theoretically and empirically. We find that weighting the training examples limits the reliance on corrupt samples and the generalization to unseen clean data is proportional to the number of clean training examples.

Setup. Let $x_i, y_i, \bar{y}_i \in \mathbb{R}, 1 \leq i \leq N$, be the input samples and the clean and noisy labels. Let $x, y, \bar{y}$ be the data in vector forms, and $I_{cr} \subset [N]$ the set of indices where for $i \in I_{cr}$ it holds that $\bar{y}_i \neq y_i$. 


In both examples $|I_{cr}| = \sigma N$, where $\sigma = \frac{1}{2} - \Delta < \frac{1}{2}$ is the fraction of corrupted measurements and \( \Delta \) is the distance of the corruption from $\frac{1}{2}$. Let $L(\theta) = \frac{1}{N} \sum_{i=1}^{N} \ell(\theta; x_i, y_i)$ be the loss w.r.t. clean \( y \) and \( \theta_\infty \), the learned parameter when $t \to \infty$. We refer to $\tilde{y}$ as the labels seen by the learner.

**Classification with logistic regression.** Here $\tilde{y}_i, y_i \in \{ \pm 1 \}$, $1 \leq i \leq N$, are noisy and clean labels. Let the inputs be $x_i \in \{ \pm 1 \}$ such that the inputs are equal to the labels, $y_i = x_i$. The noise we have in the labels is flip noise, i.e., $\forall k \in I_{cr}: \tilde{y}_k = -y_k$ and $\forall i \notin I_{cr}: \tilde{y}_i = y_i$. We consider the non-stochastic case: both the input data $x$ and the label $y$ are known and fixed and the full batched GD is deterministic.

When running GD with clean labels $y$ we have $\theta_\infty \to \infty$, but running GD on the noisy version $\tilde{y}$ yields convergence to finite $\theta_\infty$. Infinite $\theta$ is desirable with logistic regression as the learner maximizes $|x_i \theta|$ to have high confidence prediction. Unlike GD, our MR variant does satisfy $\theta_\infty \to \infty$ as in the clean label case. This shows its advantage over GD.

We now prove this. In Lemma C.5 we prove that GD loss with clean labels can achieve any small loss when running GD with clean labels $y$. When using GD+MR (supp. mat.) with step sizes $\eta = 1$, $\mu \geq 2$ and $\theta_0 = 0$, then for any $\epsilon > 0$ exists $t \geq \max \left\{ \frac{-\log(\frac{1}{\Delta})}{\Delta} + 2, \frac{-\log(\exp(\epsilon) - \exp(-\epsilon))}{\exp(\epsilon) - \exp(-\epsilon)} \frac{2N}{\sigma} \right\}$ such that $L(\theta_t) \leq \epsilon$.

Note that $\theta^* = \theta_\infty$ is finite in contrast to the clean problem and encounters a constant positive loss. Now we will show the benefit of adding MW, which causes $\theta_\infty \to \infty$ as in the clean logistic problem.

**Theorem 4.6.** When using GD+MR (supp. mat.) with step sizes $\eta = 1$, $\mu \geq 2$ and $\theta_0 = 0$, then for any $\epsilon > 0$ exists $t \geq \max \left\{ \frac{-\log(\frac{1}{\Delta})}{\Delta} + 2, \frac{-\log(\exp(\epsilon) - \exp(-\epsilon))}{\exp(\epsilon) - \exp(-\epsilon)} \frac{2N}{\sigma} \right\}$ such that $L(\theta_t) \leq \epsilon$.

The main claim in the proof is $\sum_{t=1}^{\infty} (\theta_{t+1} - \theta_t) \to \infty$, which yields that $\theta_\infty$ is unbounded. This results in an asymptotic decrease of the loss to 0. The constraint $\mu > 2$ is due to $\sigma < \frac{1}{2}$. When the noise ratio $\sigma$ is known, the constraint can be relaxed to $\mu > \frac{1}{1-\sigma}$. Note that when $\Delta$ is very small, i.e, the corruption ratio is approaching $\frac{1}{2}$, it will take more optimization steps to reach a small loss $\epsilon$.

The above example clearly shows the benefit of adding MW: The loss of GD alone converges to a solution with a non-zero loss while adding MW makes the solution behave like GD with clean labels.

**Linear regression.** We show the benefits of MR when combined not only with a GD learning procedure, but also with least squares (LS) loss of a linear model. Here also $x_i = y_i \in \{ \pm 1 \}$, but the noisy “labels” are $\tilde{y} = y + \epsilon$, where $\epsilon_i = \epsilon$ for $k \in I_{cr}$ and $\epsilon_i = 0$ for $i \notin I_{cr}$. For example, if $x_1 = y_1 = -1$ and $\epsilon_1 = 2$ then $\tilde{y}_1 = 1$, which corresponds to label flipping. For a non-negative diagonal matrix $A$, we use $\sqrt{A}$ to denote applying an element-wise square-root on its entries.

As LS has a closed-form solution for linear regression, we use it and compare with it instead of GD. For MR, our learning procedure alternates between updating a Weighted Least Square (WLS) solution and updating the weights. The algorithm is detailed in Algorithm 4. The loss this algorithm encounters at each step should be weighted according to $p_t$ and thus, it uses LS solution of $x \sqrt{\mathbf{P}_t}$ and $\sqrt{\mathbf{P}_t} \tilde{y}$ to update $\theta_t$. In this case we do not limit the examples’ weights ($\mu = N$).

We show in the next theorem that in our setup, Algorithm 4 manages to eliminate all noisy examples and converges to the optimum of the clean problem $\theta^* = 1$.

**Theorem 4.7.** For Algorithm 4 we have that for $c > 0$ there exists $t > \frac{\log(\sigma + c)}{\eta\sigma^2} \geq 1$ such that $|\theta_t - 1| \leq c$, where $\theta_t$ is the learned parameter by the algorithm.

Note that the loss of the noisy LS solution with $\tilde{y}$ is $\frac{1}{2} \left( \sum_{j \in I_{cr}} x_j \epsilon_j \right)^2$. This implies that in many cases (e.g., label flipping), the LS solution yields a constant positive loss, while our reweighting method results in a zero-loss solution. When the corruption ratio approaches $\frac{1}{2}$, more optimization steps are required to achieve low loss. Motivated by [59, 65], we find the LS result another motivation for using MR for classification with noisy labels.

In Appendix D, we demonstrate our theoretical findings empirically, with noisy artificial data.
In this section we analyse the Lipschitzness of the MR solution in a 1d case. Previously, a low Lipschitz constant w.r.t. the input data of a network was shown to improve adversarial robustness and hence it is used as regularization for robust training [29][67]. In addition, low Lipschitz constant w.r.t. the parameters also yields robustness [104][11][27]. Specifically, a low Lipschitz constant yields high robustness against bounded \( \ell_2 \) norm attacks. This is due to the fact that the Lipschitz constant limits the change in the function (in \( \ell_2 \)) when small perturbations are applied, and adversarial attacks are defined as bounded and unobservable perturbations to the input which change the output.

We focus on a 1d case with logistic loss, \( \ell(x; \theta, y_i) = \log(1 + \exp(-x_i \theta y_i)) \), and \( y_i \in \{\pm 1\} \).

Since \( G \)-Lipschitzness is a general property of a function \( \ell(x; y, \theta) \) that bounds the \( \ell_2 \) gradient norm w.r.t. the input \( x \) over the whole domain, we focus on the effective weighted and average gradient \( \sum_{i=1}^{N} p_i \frac{\partial \ell(x; y_i, \theta)}{\partial x_i} \) for the MR case and \( \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \ell(x; y_i, \theta)}{\partial x_i} \) for the GD case. We present a lemma that analyses Algorithms 3 with a simpler MR update which does not encounter the history of the losses, i.e. \( p_i = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{N} \exp(-\eta \ell(x_j; y_j, \theta))} \).

**Lemma 4.8.** The MW update of \( p_i = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{N} \exp(-\eta \ell(x_j; y_j, \theta))} \) (w/o history) and \( y_i \in \{\pm 1\} \) holds that uniform average of the absolute derivatives is larger than the MR weighting of absolute derivatives,

\[
\frac{1}{N} \sum_{i=1}^{N} \left| \frac{\partial \ell(x_i; y_i, \theta)}{\partial x_i} \right| \geq \sum_{i=1}^{N} p_i \left| \frac{\partial \ell(x_i; y_i, \theta)}{\partial x_i} \right| .
\]

The lemma states that using an update of the weights without the loss history yields a lower effective Lipschitzness with MR than with uniform weighting. Thus, we suggest an assumption that for large enough \( \ell \) it holds that one can use an adjusted learning rate and a MW update without the history and get the same probabilities as MR with the loss history. To state the assumption formally, \( \tilde{p}_{t,i} = \tilde{p}_{t,i} \) where \( \tilde{p}_{t,i} = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{N} \exp(-\eta \ell(x_j; y_j, \theta))} \) (note the step size, \( \tilde{\eta} = \eta \ell \)). According to the assumption, we could use Lemma 4.8 and claim that the MR solution has better Lipschitz property and is therefore more robust to adversarial attacks. We demonstrate the validity of the assumption in Section 5.3.

**Lipschitzness simulations.** In order to demonstrate the results of the benefits of the Lipschitzness of MR we present a simulation. We use \( N = 1500 \) 1d input examples that are drawn randomly to be \( x_i \in \{-1, -1\} \) and \( y_i \in \{1, 1, -1\} \). We train GD and GD+MR with logistic loss. In Fig. 2a we present the effective absolute gradient w.r.t. the input and clearly show that the MR has lower effective Lipschitzness, as Lemma 4.8 suggests. Fig. 2b plots the statistical distance between \( \tilde{p}_{t,i} = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{N} \exp(-\eta \ell(x_j; y_j, \theta))} \) and \( \tilde{p}_{t,i} = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{N} \exp(-\eta \ell(x_j; y_j, \theta))} \), i.e. SD = \( \frac{1}{2} \| p_{t,i} - \tilde{p}_{t,i} \| \). In Fig. 2c it is clearly shown that our assumption is valid.

| Table 1. Results on CIFAR10/100 for different label noise ratio. **Bold** indicates best accuracies, while underlining highlights instances where MR improved performance. |
|-------|-------|-------|-------|-------|-------|-------|-------|
| Data  | Noise ratio | Base | Random weighting | Base + mixup | Base + smoothing | Base + smoothing | Arazo | Co-teaching |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| CIFAR10 | 0%    | 95.13 | 93.07 | 95.97 | 95.15 | **96.26** | 93.94 | 89.87 | 94.96 | 94.95 | 94.9 | 94.97 | 95.21 |
|       | 10%   | 90.83 | 88.52 | 92.75 | 91.32 | 93.21 | 92.85 | 88.16 | 87.86 | 94.32 | 94.58 | 94.63 | 94.39 | **94.66** |
|       | 30%   | 77.92 | 81.68 | 80.52 | 79.23 | 81.75 | 93.56 | 82.51 | 76.16 | 92.43 | 92.11 | 92.27 | 92.32 |
|       | 40%   | 69.22 | 78.77 | 72.29 | 71.08 | 83.87 | 92.77 | 78.44 | 71.82 | 92.75 | 92.37 | 92.36 | 92.62 |
| CIFAR100 | 0%    | 74.77 | 72.74 | 78.80 | 76.09 | **79.36** | 67.42 | 63.31 | 76.32 | 73.67 | 76.59 | 76.85 | 73.31 |
|       | 10%   | 63.59 | 66.97 | 72.35 | 69.95 | 74.42 | 70.77 | 62.23 | 70.72 | 77.04 | 73.92 | 72.13 | 73.51 |
|       | 30%   | 52.12 | 50.95 | 62.23 | 56.16 | 64.43 | 69.71 | 57.83 | 55.28 | 67.33 | 65.64 | 67.31 | 69.11 |
|       | 40%   | 48.37 | 40.47 | 58.46 | 51.51 | 59.55 | 66.42 | 53.44 | 45.42 | 57.55 | 67.75 | 60.02 | 69.95 |

**Table 2. Accuracy results on Clothing1M.** *for [4], we take the number from their paper (same setup). SR is the method in [110] |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Data  | Random Weighting | Base + SR | Base + mixup | Base + smoothing | Base + mixup * smoothing | Arazo | MR | MR + SR | MR + mixup | MR + smoothing | MR + mixup + smoothing |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| CIFAR10 | 68.94 | 69.96 | 70.17 | 70.22 | 69.04 | 70.65 | 71.01 | 70.69 | 70.35 | 69.98 | 71.12 | **71.18** |
5 Experiments

Motivated by the theoretical guarantees, we turn to show empirically that MR improves DNN robustness. We examine MR for training with noisy labels and in the presence of adversarial attacks at inference time. We use artificial random label noise with CIFAR-10 and CIFAR-100 and natural label noise with the large-scale Clothing1M dataset [94]. For adversarial attacks, we add MR on top of well-known robust training techniques and show improvement. Our code is attached to the paper.

5.1 Noisy Labels

Artificial label noise. We start by evaluating MR on CIFAR-10 and CIFAR-100. To examine our method in the presence of label noise, we changed each label independently with a constant uniform probability to a random one. Clearly, the change effects only the training set, while the test set is noise-free. We use ResNet-18 [26] as the classification network and a constant MR step size $\eta = 0.01$, which is found by a grid search; the full search results appear in Appendix E. We train the network for 200 epochs. An initial learning rate of $\alpha = 0.1$ is used and then multiplied by 0.1 at epochs 80 and 120. We use momentum with parameter 0.9, $l_2$ regularization with a factor of $5 \times 10^{-2}$ and $10^{-3}$ for CIFAR-10 and CIFAR-100, respectively, as well as random crop and random horizontal flip. We employed a single NVIDIA RTX-2080 GPU to undertake the learning processes.

The $\mu/N$ upper bound of the weights in MR is necessary to ensure theoretically that MR does not converge to degenerated $p$ vectors. We examine the effect of $\mu$ on the accuracy and the distribution $p$. In Fig. 5 we experiment with multiple values and observe that high $\mu$ values that do not limit the weighting do not affect the performance. Hence, in order to avoid the worst-case complexity of $O(N^2)$, we use unlimited probabilities (i.e., $\mu = N$), where we simply use $l_1$ normalization ($O(N)$ complexity). Thus, we can infer that in practice, MR implicitly exploits the large training set and the non-degenerate loss distribution that the DNN produces, and without the need for additional constraints, it does not converge to undesirable $p$.

We compare and combine our method with label smoothing [80] and mixup [104]. The label smoothing parameter is set to 0.1 and in mixup to $\alpha = 1$. These baselines are common tools for improving the robustness of networks - we use them specifically since they do not need an additional separate set of clean training examples as is required in many techniques for training with noisy labels. We also conducted experiments with WRN28-10 [100] and Adam optimizer [35] with a learning rate of 0.001. In addition we compare our method to random weighting (see [60]), where the weights are randomly distributed according to rectified Gaussian distribution, i.e., $w^{rd}_{t,i} = \max(0, z_{t,i})$, $z_i \sim \mathcal{N}(0, 1)$. We use as a baseline the unsupervised weighting scheme in [4].
Table 5. Robustness to adversarial attacks. **Bold** indicates best accuracies, while underlining highlights instances where MR improved performance.

| Method                  | CIFAR10 | CIFAR100 |
|-------------------------|---------|----------|
|                         | 0.01    | 0.02     | 0.05    | 0.01    | 0.02     | 0.05    | 0.01    | 0.02     | 0.05    |
| Natural Images           | PGD     | PGD      | FGSM    | FGSM    | PGD      | FGSM    | PGD     | FGSM     | FGSM    |
| Base                    | 94.02   | 51.96    | 51.88   | 48.83   | 71.31    | 71.40   | 51.88   | 48.83    | 71.31   |
| Random Weight.          | 92.57   | 51.96    | 51.88   | 48.83   | 71.31    | 71.40   | 51.88   | 48.83    | 71.31   |
| Base+MR                 | 94.71   | 52.47    | 53.86   | 51.44   | 71.05    | 71.45   | 51.44   | 48.42    | 71.05   |
| Base+mixup + MR         | 95.52   | 5.35     | 5.37    | 5.34    | 71.38    | 71.96   | 5.34    | 5.34     | 71.38   |
| Base+mixup + MR         | 95.52   | 5.35     | 5.37    | 5.34    | 71.38    | 71.96   | 5.34    | 5.34     | 71.38   |
| Free                    | 89.32   | 61.14    | 59.09   | 53.22   | 73.86    | 71.19   | 44.54   | 39.09    | 83.22   |
| Free+MR                 | 89.18   | 79.85    | 61.68   | 59.53   | 81.24    | 69.13   | 52.65   | 44.64    | 81.24   |
| Free+mixup              | 90.61   | 75.15    | 65.54   | 62.43   | 80.70    | 62.30   | 52.65   | 45.64    | 80.70   |
| Free+MR+mixup           | 89.4    | 63.44    | 69.72   | 66.66   | 84.47    | 76.63   | 52.65   | 45.64    | 81.24   |
| TRADES                  | 84.88   | 81.28    | 73.47   | 68.76   | 81.53    | 75.72   | 68.76   | 75.72    | 81.53   |
| TRADES + MR             | 83.4    | 78.66    | 59.32   | 58.74   | 73.41    | 67.83   | 58.74   | 67.83    | 73.41   |

which does not require an additional small clean training set. We also use Importance Reweighting [50] and Co-teaching [24] as baselines.

Table 1 summarizes the results and shows the superiority of using MR with respect to the SGD (Base) in the presence of label noise and that MR only slightly decreases accuracy for clean labels. Note that MR improves the accuracy when used with mixup and the training data contains noisy labels. This phenomenon is observed also with label smoothing and when using both label smoothing and mixup. Adding MR improves the results more significantly as the noise is more severe and the classification is harder, which is the case for CIFAR-100. In the majority of the scenarios, MR also outperforms [4], especially on CIFAR-100. In Appendix [3] attached to this paper, we provide an analysis of the statistical significance of the improvement achieved by incorporating MR.

The results with WRN28-10 and Adam optimizer for 40% label noise are found in Table 6. The results demonstrate that MR advantage remains when changing the architecture or the optimizer. Additional results with Adam. MR, mixup, label smoothing and methods requiring clean data [75, 28] also appear in Appendix [3]. Overall, we can say that MR is oblivious to changes in optimizer and architecture and enhances performance, independently of these variations when used for label noise.

Since MR can also be viewed as a curriculum learning method, we compare it with S2E [98], which changes the sampling probabilities of mini-batches. Additionally, we compared to a method named UNICON [34], MW-net [75] and Sparse Regularization (SR) [110]. For all of the methods the results are reported according to the settings described in their original papers. As for [34], when combined with MR, we use MR only for the warmup stage, with 10 and 30 epochs for CIFAR10 and CIFAR100, respectively. The results reported for UNICON with 20% noise level with CIFAR-10, differ from the original paper since we were not able to reproduce them. Note that MW-net requires an additional clean set of examples. We employ MR only on the part of MW-net that uses the noisy training set. Additional details appear in Appendix [3]. Overall, we can say that MR is oblivious to changes in optimizer and architecture and enhances performance, independently of these variations when used for label noise.

**Pair-flip and Instance-dependent label noise.** We applied MR in the presence of artificial yet more realistic label noise. Specifically, we employed pair-flip label noise [24], in which related classes are flipped, and instance-dependent label noise [92], where the labels are altered based on the image’s parts. The accuracy results for these two types and levels of noise are presented in Table 3. In all the tested cases, the use of MR consistently improves accuracy.

**Realistic noisy labels.** We examine MR on the Clothing1M [34] dataset, which includes 1M images of clothing products labeled by their keywords. The dataset includes 14 classes and is known to be noisy. We used the same hyperparameters as in [75] with ResNet-50 [26] pre-trained on ImageNet [12] and trained for an additional 10 epochs. We used $\eta = 0.01$ for MR step size and multiplied it by 1.5 after 2 and 6 epochs. Instead of using MR after each epoch, we employ it after half an epoch, since we found that 10 weighting updates are not significant enough. Table 3 details the results and clearly shows the improvement of MR over SGD with $\sim 1.5\%$ enhancement. Note that adding MR improves the results when used with the base optimizer, label smoothing and with the combination of mixup and label smoothing. Using SR [110] with MR is better than employing SR alone. Employing MR, mixup and label smoothing leads to a significant improvement of $\sim 2\%$ over SGD and even outperforms [4]. Thus, we find that MR is useful and favourable also with real label noise.
5.2 Adversarial Robustness

Another type of harmful noise to DNN is adversarial attacks. It has been shown that small perturbations to the input data (during inference) that are unrecognized visually can lead to an entirely different outcome in the network’s output [81]. A leading strategy among defense methods is adversarial training [20, 56], which adds adversarial examples to the training to improve robustness.

We have established a connection between MR and robustness to adversarial attacks using the relationship we show to a low Lipschitz constant in Section 4.3. In addition, a possible intuitive explanation for the connection between the two is that relying on examples with high confidence (low loss) during optimization can lead to improved adversarial robustness [47, 105].

To empirically assess the benefits of MR in this setting, we employ it with Free Adversarial Training [71] and TRADES [103], two well-established and efficient methods for robust training. We focus on these two due to their efficient use of resources. We evaluate MR with non-corrupted images (e.g., “Natural Images”) and against adversarial examples produced by the PGD [56] and FGSM [20] attacks with a bounded $\ell_\infty$ norm of $\epsilon \in \{0.01, 0.02, 8/255\}$. All experiment settings as well as additional tests (e.g., MR with Adam) appear in Appendix E.

Table 5 includes the results for base, mixup and robust training baselines (i.e. Free [71] and TRADES [103]). As can been seen from the results, adding MR weighting on top of robust training yields improvement in most tested scenarios, which is more significant with CIFAR-100. Adding MR to free training is more beneficial than adding mixup for CIFAR-100, while in CIFAR-10 the results are indefinite. As for base training, MR slightly improves robustness, but accuracy against strong attacks is poor. In addition, the accuracy of base training with both mixup and MR in all scenarios is worse than at least one of the standalone methods. When our method is added on top of the combination of robust training and mixup, we can see improvement with all the attacks except the ones with CIFAR-10 and $\epsilon = 0.02$, but we observe a slight degradation in accuracy with natural images. In most attacks the combinations with MR result in superior accuracy against adversarial examples.

Note that free training and its combination with MR and mixup do not perform as well as the regular networks on the natural data (without attacks). This problems also occurs with other robust training methods, but can be mitigated using additional unlabeled data in the training [9, 101].

We also apply MR with the additional adversarial training method LAS-AT [31] and MAIL [47]. Results appear in the Table 10. Combining MR with these methods yields enhanced robustness.

6 Limitations

One concern with using MR is its susceptibility to training with class imbalance, potentially resulting in assigning lower weights to examples from minority classes due to their higher loss. Future research may explore finding methodologies inspired by learning with expert advice, tailored to address class-imbalanced scenarios. Another limitation is that MR does not consistently improve performance across all tested cases and may not achieve state-of-the-art results in some instances. However, its simplicity and theoretical basis make it an appealing choice. Finally, employing MR involves increased computational overhead due to an additional forward pass (without gradient calculation) for all examples at the end of each epoch.

7 Conclusion

This work proposes the MR optimizer for improving neural network robustness. It is motivated by learning with expert advice and uses MW for reweighting examples along with the SGD DNN optimization. We proved theoretically the benefits of using this reweighting strategy for DNN training and showed through various experiments that MR improves DNN robustness for learning with label noise. We provided evidence that MR enhances robustness also to adversarial examples, showing improvement over leading adversarial approaches. We further suggest investigating the effect of using variants of MW, such as Follow the Regularized Leader and Online Mirror Descent [73] which involve different ways of decreasing the MW step size and may exhibit other robustness properties.

To summarize, we believe that the results shown in our work position MR as a useful optimizer in the DNN toolbox that can elevate its robustness along with other tools like mixup and label smoothing.
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Multiplicative Reweighting for Robust Neural Network Optimization: Supplementary Material

In this part we provide supplementary material to the main paper. First, we detail the projection algorithm to a constrained domain. Then, we present the proofs for the theorems that support the theoretical benefits of MR. Finally, we show additional experiments that include investigation of the learned weighting, utilizing MR with Adam and comparisons to other weighting methods. All equation and statements numbering in this document continue from where they stopped in the main paper, so we refer to their numbers as they appeared in the paper. Additionally, we use the same references numbering.

A Projection Algorithm

In Algorithm 2 we detail the algorithm of efficient projection to a constrained domain, where each probability has limited value of $\frac{\mu}{N}$. This algorithm was introduced in prior works in [88, 36]. The algorithm normalizes the weights into a probability and then iteratively distributes all the mass that exceeds the upper bound. The computation complexity is $O(N^2)$, where $N$ is the number of examples.

Algorithm 2 KL-projection to constrained domain [88, 36]

Input: unconstrained weighting $w_i$, maximal weighting ratio - $\mu$.
Output: distribution $p \in \Delta_N$ s.t. $p_i < \frac{\mu}{N}, \forall i = 1, ..., N$.
Normalize: $p_i = \frac{w_i}{\sum_{j=1}^{N} w_j}$
while $\max_{i=1,...,N} p_i > \frac{\mu}{N}$ do
   Excess mass: $r = \sum_{i=1}^{N} \max\{0, p_i - \frac{\mu}{N}\}$
   Redistributing: update $p_i$ s.t. $p_i < \frac{\mu}{N}$:
   $p_i \leftarrow p_i + r \cdot \frac{p_i}{\sum_{j: p_j < \frac{\mu}{N}} p_j}$
end while

B Algorithms Used for the Theoretical Analysis

Algorithm 3 Multiplicative Reweighting (MR) with Gradient Descent

Input: data: $x_i, y_i$, size $N$, $\alpha$- GD step size and $\eta$- MW step size.
Initialize: $w_{0,i} = 1$ and $\theta_0$ - the network initialization, $\mu$ - maximal weight constrain.
for $t = 1$ to $T$ do
   Project: $p_t = \text{Project}(w_t)$, Algorithm 2
   GD: $\theta_{t+1} = \theta_t - \alpha \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t)$
   MW: $w_{t+1,i} = \exp\left(-\eta \sum_{s=1}^{t+1} \ell_i(\theta_s)\right)$
end for

C Proofs

Here we present the proofs of the lemmas and theorems presented in the paper. For the convenience of the reader, we repeat the definitions of the lemmas and theorems that we prove.

C.1 Relevant Definitions

Prior to our proof we detail the definitions of the terms we use.
All examples are updated using an MW step followed by a projection. Thus, we get:

Algorithm 4 MR with LS loss (for the 1d example)

**Input:** data: $x$, $\tilde{y}$, and $\eta$ - MW step size.

**Initialize:** $w_{0,i} = 1$, $\theta_0 = 0$.

**for** $t = 1$ **to** $T$ **do**

- Normalize $P$: $p_{t,i} = \frac{w_{t,i}}{\sum_{j=1}^{N} w_{t,j}}$, update probability matrix: $P_t = \text{diag}(p_t)$
- **LS:** $\theta_{t+1} = x\sqrt{P_t} (\sqrt{P_t} x \sqrt{P_t})^T \sqrt{P_t} \tilde{y}$
- Calculate Loss: $\ell_{t+1,i} = \frac{1}{2} (\theta_{t+1} x_i - \tilde{y}_i)^2$
- **MW:** $w_{t+1,i} = \exp \left( -\eta \sum_{s=1}^{t+1} \tilde{\ell}_{s,i} \right)$

**end for**

**Lemma C.1.** For a probability update can be written formally as $\theta = \text{FTRL} (\text{Follow The Regularized Leader})$ algorithm with a constant step size $\alpha$.

**Proof.** Note that $\sum_{i=1}^{N} p_{t,i} \ell_i(\theta) = \beta$-smooth in $\theta$. Using $\beta$-smoothness and the GD step definition in Algorithm 3, we get:

$$\sum_{i=1}^{N} p_{t,i} \ell_i(\theta_{t+1}) \leq \sum_{i=1}^{N} p_{t,i} \ell_i(\theta_t) - \frac{1}{2\beta} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2 + \frac{\alpha^2 \beta}{2} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2.$$

Since $\alpha = \frac{1}{\beta}$, we get:

$$\sum_{i=1}^{N} p_{t,i} \ell_i(\theta_{t+1}) \leq \sum_{i=1}^{N} p_{t,i} \ell_i(\theta_t) - \frac{1}{2\beta} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2.$$

All that is left to be shown is that:

$$\sum_{i=1}^{N} p_{t+1,i} \ell_i(\theta_{t+1}) \leq \sum_{i=1}^{N} p_{t,i} \ell_i(\theta_{t+1}). \quad (2)$$

All examples are updated using an MW step followed by a projection. Thus, $p$ updates can be interpreted as FTRL (Follow The Regularized Leader) algorithm with a constant step size $\eta$ and the probability update can be written formally as

$$p_{t+1} = \arg \min_{\theta, w \in \Delta_N} \left\{ w \cdot \sum_{s=1}^{t+1} \ell(\theta_s) + \frac{1}{\eta} \sum_{s=1}^{N} w_i \ln(w_i) - w_i \right\},$$
Assume that for Theorem 4.2. For a \( \Delta \) is \( N \)-simplex, \( p_{t+1} \in \Delta_N \) is a vector and \( \ell(\theta) \) are treated as known fixed vectors.

Define a function \( R \) to be \( R(u) = \sum_{i=1}^{N} w_i \ln(w_i) - w_i \). Let \( \Phi(u) = \sum_{i=1}^{N} u_i \sum_{s=1}^{t} \ell_i(\theta_s) + \frac{1}{\beta} u_{i} \). \( \Phi \) is convex since it is a sum of linear functions and a negative Shannon entropy which are both convex. Note that \( \phi \) is also convex since it is linear. Using the convexity of \( \Phi \) and the fact that \( p_t \) is its minimum, we have that

\[ \Phi(p_{t+1}) - \Phi(p_t) \geq 0. \tag{3} \]

From the minimality of \( p_{t+1} \) w.r.t \( \Phi + \phi \), we have

\[ (\Phi + \phi)(p_t) - (\Phi + \phi)(p_{t+1}) \geq 0. \tag{4} \]

Following Eq. (3) and Eq. (4), we get

\[ \phi(p_t) - \phi(p_{t+1}) \geq \Phi(p_{t+1}) - \Phi(p_t) \geq 0. \]

Plugging the definitions of \( \phi \) and \( \Phi \) leads to Eq. (2), which completes the proof. \( \square \)

C.3 Proof of Theorem 4.2

**Theorem 4.2** For a \( \beta \)-smooth loss \( \ell(\cdot) \), and \( \theta_{t+1}, p_{t+1} \) being updated as in Algorithm 3 with GD step size of \( \alpha = \frac{1}{\beta} \) and MR step size \( \eta > 0 \), we have that

\[ \frac{1}{T} \sum_{t=1}^{T} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2 \leq \frac{2\beta}{T} \left( \frac{1}{N} \sum_{i=1}^{N} \ell_i(\theta_0) - \sum_{i=1}^{N} p^*_i \ell_i(\theta^*) \right), \]

where \( p^*, \theta^* = \arg \min_{p_i \in \Delta_N, \sum_{i=1}^{N} p_i \ell_i(\theta)} \sum_{i=1}^{N} p_i \ell_i(\theta) \).

**Proof.** From Lemma 4.1 we have that

\[ \sum_{i=1}^{N} p_{t+1,i} \ell_i(\theta_{t+1}) - \sum_{i=1}^{N} p_{t,i} \ell_i(\theta_t) \leq -\frac{1}{2\beta} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2. \]

Summing over \( t \), leads to

\[ \sum_{i=1}^{N} p_{T,i} \ell_i(\theta_T) - \sum_{i=1}^{N} p_{0,i} \ell_i(\theta_0) \leq -\frac{1}{2\beta} \sum_{t=1}^{T} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2. \]

Rearranging the above equation and using the minimality of \( p^*, \theta^* \), we get that

\[ \frac{1}{T} \sum_{t=1}^{T} \left\| \sum_{i=1}^{N} p_{t,i} \nabla \ell_i(\theta_t) \right\|^2 \leq \frac{2\beta}{T} \left( \sum_{i=1}^{N} p_{0,i} \ell_i(\theta_0) - \sum_{i=1}^{N} p^*_i \ell_i(\theta^*) \right). \]

\( \square \)

C.4 Proof of Theorem 4.4

**Theorem 4.4** Assume that for \( 1 \leq i \leq N : \ell_i(\cdot) \) is \( \beta \)-smooth, \( G \)-Lipshitz \( B \)-bounded. Define \( F(\theta, p) = \mathbb{E}_{i \sim p} \ell_i(\theta) \). Then when running MR with SGD that samples according to \( i_t \sim p_t \) and uses \( \alpha = \sqrt{\frac{2B}{\beta G^2 T}} \), it holds that:

\[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{i_1, \ldots, i_T} \left\| \nabla F(\theta_t, p_t) \right\|^2 \leq G \sqrt{\frac{2\beta B}{T}}. \]
We will take a closer look at the expression
\[ F(\theta_{t+1}, p_t) \leq F(\theta_t, p_t) - \alpha \nabla F(\theta_t, p_t)(-\nabla \ell_i, (\theta_t)) \]
\[ + \frac{\beta}{2} \alpha^2 \|\nabla \ell_i, (\theta_t)\|^2 \]
\[ \leq F(\theta_t, p_t) + \alpha \nabla F(\theta_t, p_t)\nabla \ell_i, (\theta_t) + \frac{\beta \alpha^2 G^2}{2}. \]

Notice that SGD with MW can still be seen as an FTRL algorithm with a constant step size, where the losses are fixed from the FTRL point of view. Thus, from Eq. (2), we have that \( F(\theta_{t+1}, p_t) \leq F(\theta_{t+1}, p_t) \) holds.

Taking expectation on both sides w.r.t \( i_1, \ldots, i_t \sim p_1, \ldots, p_t \) yields:
\[
\mathbb{E}_{i_1, \ldots, i_t} F(\theta_{t+1}, p_{t+1}) \leq \mathbb{E}_{i_1, \ldots, i_t} F(\theta_t, p_t) + \mathbb{E}_{i_1, \ldots, i_t} \left[ \nabla F(\theta_t, p_t)(-\alpha \nabla \ell_i, (\theta_t)) \right] + \frac{\beta}{2} \alpha^2 G^2.
\]

We will take a closer look at the expression \( \mathbb{E}_{i_1, \ldots, i_t} [\nabla F(\theta_t, p_t)(\nabla \ell_i, (\theta_t))] \). Using the law of total expectation, we have that
\[
\mathbb{E}_{i_1, \ldots, i_t} [\nabla F(\theta_t, p_t)(\nabla \ell_i, (\theta_t))] = \mathbb{E}_{i_1, \ldots, i_t} \mathbb{E}_{i_{t+1}} [\nabla F(\theta_t, p_t)(\nabla \ell_i, (\theta_t)) | i_1, \ldots, i_t]
\]
\[
= \mathbb{E}_{i_1, \ldots, i_{t-1}} \mathbb{E}_{i_t} [\nabla F(\theta_t, p_t) E_{i_t} [\nabla \ell_i, (\theta_t)] | i_1, \ldots, i_t]
\]
\[
= \mathbb{E}_{i_1, \ldots, i_{t-1}} [\|\nabla F(\theta_t, p_t)\|^2]
\]
\[
= \mathbb{E}_{i_1, \ldots, i_{t-1}} [\|\nabla F(\theta_t, p_t)\|^2].
\]

Plugging this into the earlier expression we get:
\[
\mathbb{E}_{i_1, \ldots, i_t} F(\theta_{t+1}, p_{t+1}) \leq \mathbb{E}_{i_1, \ldots, i_t} F(\theta_t, p_t) - \alpha \mathbb{E}_{i_1, \ldots, i_t} [\|\nabla F(\theta_t, p_t)\|^2]
\]
\[+ \frac{\beta}{2} \alpha^2 G^2.\]

Summing over \( t = 1, \ldots, T \) and averaging leads to
\[
\frac{1}{T} \mathbb{E} [\|\nabla F(\theta_{t+1}, p_{t+1})\|^2] \leq \frac{\mathbb{E} F(\theta_T, p_T) - \mathbb{E} F(\theta_1, p_1)}{\alpha T} + \frac{\beta}{2} \alpha G^2
\]
\[
\leq \frac{B}{\alpha T} + \beta G^2 \alpha.
\]

By setting \( \alpha = \sqrt{\frac{2B}{G^2 \beta T}} \), we get the desired bound. \( \square \)

### C.5 Logistic Regression Loss with Clean Labels

The above lemma implies that running GD with clean examples yields \( \theta_\infty \to \infty \) for minimizing the loss to 0.

**Lemma C.4.** When running GD starting with \( \theta_0 = 0 \), using the clean labels \( y \) and step size \( \alpha > 0 \), it holds that \( \forall \epsilon > 0 \) there exists \( t > \frac{-\log(\exp(\epsilon)-1)}{\alpha \exp(-\epsilon)(\exp(\epsilon)-1)} \) such that
\[ L(\theta_t) = \log(1 + \exp(-\theta_t)) < \epsilon. \]
Theorem 4.6. When running Algorithm 3 with the step sizes \( \theta \) we set \( \theta_{t+1} = \theta_t + \frac{\alpha}{1+\exp(\theta_t)} \). When we set \( \theta_0 = 0 \), we get a positive and increasing series.

By observing Eq. (5), our goal is to find \( \theta_t \in \mathbb{R}, t \geq 1 \) s.t. \( \theta_t \geq -\log(\exp(\epsilon) - 1) \). Note that for all \( t \) s.t \( \theta_t < -\log(\exp(\epsilon) - 1) \) it holds that

\[
\theta_{t+1} - \theta_t = \frac{\alpha}{1+\exp(\theta_t)} \geq \frac{\alpha}{1+\exp(-\log(\exp(\epsilon) - 1))} = \alpha \exp(-\epsilon)(\exp(\epsilon) - 1).
\]

Therefore, for \( t > \frac{-\log(\exp(\epsilon) - 1)}{\alpha \exp(-\epsilon)(\exp(\epsilon) - 1)} \), it must hold that \( L(\theta_t) \leq \epsilon \). \( \square \)

### C.6 Proof of Lemma 4.5

Lemma 4.5 When running GD starting with \( \theta_0 = 0 \), using the noisy labels \( \tilde{y} \) and step size \( \alpha = 1 \), the algorithm converges to \( \theta^* = \log(\frac{p_{\text{cr},t}}{p_{\text{cl},t}}) \). Formally, for any \( \epsilon > 0 \) and \( T = O(\frac{1}{\epsilon}) \) it holds that \( |\theta_t - \theta^*| \leq \epsilon \).

**Proof.** We will first show that the loss the learner observes is 1-smooth by bounding its second derivative:

\[
\frac{d^2L(\theta)}{d\theta^2} = \frac{\sigma}{(\exp(-\theta)+1)^2} + \frac{1-\sigma}{(\exp(\theta)+1)^2} \leq \sigma + 1 - \sigma = 1
\]

Therefore, it holds that \( \tilde{L}(\theta) \) is 1-smooth. Note that the observed loss is

\[
\tilde{L}(\theta) = \sigma \log(1 + \exp(\theta)) + (1-\sigma)\log(1 + \exp(-\theta))
\]

and its gradient is

\[
\nabla \tilde{L}(\theta) = \frac{\sigma}{1+\exp(-\theta)} - \frac{1-\sigma}{1+\exp(\theta)}
\]

For \( \theta^* = \log(\frac{p_{\text{cr},t}}{p_{\text{cl},t}}) \) it holds that \( \nabla L(\theta^*) = 0 \) and \( \nabla^2 \tilde{L}(\theta^*) > 0 \) so \( \theta^* \) achieves the minimum. Note that \( \theta^* \) is the only critical point. Since GD converges to a critical point when running it with \( \beta \)-smooth function and step size \( \alpha = \frac{1}{\beta} \) \( \Box \), then in this case GD converges to \( \theta^* \). \( \square \)

### C.7 Proof of Theorem 4.6

**Theorem 4.6** When running Algorithm 3 with the step sizes \( \eta = \alpha = 1, \mu \geq 2 \) and \( \theta_0 = 0 \), then for any \( \epsilon > 0 \) exists \( t \geq \max \left\{ \frac{-\log(\frac{1}{\Delta})}{2}, \frac{-\log(\exp(\epsilon) - 1)}{\exp(\epsilon) - 1} \frac{2N}{\sigma} \right\} \) such that \( L(\theta_t) \leq \epsilon \).

To prove that, we first show that \( \theta_t \) increases over time.

**Lemma C.5.** When running Algorithm 3 with the step sizes \( \eta = \alpha = 1, \mu \geq 2 \) and \( \theta_0 = 0 \), then \( \forall t \geq 0 \) it holds that \( \theta_{t+1} > \theta_t \).

**Proof.** Note that in each step \( t \geq 0 \) the losses of all corrupted examples is the same so they share the same probability which we will denote by \( p_{\text{cr},t} \). The same holds also for the clean examples’ probabilities, which we will denote by \( p_{\text{cl},t} \). Let \( \delta_t := \theta_{t+1} - \theta_t = \frac{(1-\sigma)p_{\text{cl},t}}{p_{\text{cl},t} + \sigma p_{\text{cr},t}} - \frac{\sigma p_{\text{cr},t}}{p_{\text{cl},t} + \sigma p_{\text{cr},t}} \). Lastly, denote by \( A_t = \frac{p_{\text{cl},t+1}}{p_{\text{cl},t}} \) and \( B_t = \frac{p_{\text{cr},t+1}}{p_{\text{cr},t}} \) the ratios between two consecutive clean and noisy probabilities.
The proof will hold by induction. For the base step it holds that \( \theta_1 = \Delta > 0 \). Next, assume that \( \forall s \leq t: \delta_s > 0 \). Then,

\[
\delta_{t+1} = \frac{(1 - \sigma)p_{cl,t} A_t}{1 + \exp(\theta_{t+1})} - \frac{\sigma p_{cr,t} B_t}{1 + \exp(-\theta_{t+1})}.
\]

Note that \( \frac{p_t}{A_t} = \exp(-\theta_{t+1}) \), and therefore

\[
\delta_{t+1} = A_t \left( \frac{(1 - \sigma)p_{cl,t}}{1 + \exp(\theta_{t+1})} - \frac{\sigma p_{cr,t} \exp(-\theta_{t+1})}{1 + \exp(-\theta_{t+1})} \right)
\]

\[= \frac{A_t}{1 + \exp(\theta_{t+1})} ((1 - \sigma)p_{cl,t} - \sigma p_{cr,t}). \tag{6}\]

Since for \( s \leq t \) it holds that \( \theta_s \geq 0 \), we have that \( \tilde{\ell}_{cr}(\theta_s) = \log(1 + \exp(\theta_s)) \geq \log(1 + \exp(-\theta_s)) = \tilde{\ell}_{cl}(\theta_s) \), which yields that \( p_{cr,t} \leq p_{cl,t} \). In addition, since all the \( (1-\sigma)N \geq \frac{N}{2} \) clean examples share the same weighting it must hold that \( p_{cl,t} \leq \frac{2s}{N} \leq \frac{N}{2} \). Thus, the upper bound does not limit our analysis. Therefore, all the terms in the above equation are strictly positive. \( \square \)

**Proof.** From the MW update rule it holds that \( p_{cr,t} = \exp \left( - \sum_{s=1}^{t} \theta_s \right) p_{cl,t} \). Using \( \sigma \leq \frac{1}{2} \) and Eq. (6), we have that

\[
\delta_{t+1} > \frac{A_t p_{cl,t} \sigma}{1 + \exp(\theta_{t+1})} \left( 1 - \exp \left( - \sum_{s=1}^{t} \theta_s \right) \right).
\]

Since \( \theta_1 = \Delta > 0 \) and by Lemma C.5 we get that \( \theta_t > \Delta \) for \( t > 1 \). In addition, since \( \theta_t \) is increasing (Lemma C.5), it holds that \( A_t \geq 1 \) for \( 1 \leq t \) so \( p_{cl,t} \geq p_{cl,1} = \frac{1}{N} \). So overall, we have that

\[
\delta_{t+1} \geq \frac{p_{cl,t+1} \sigma}{1 + \exp(\theta_{t+1})} \left( 1 - \exp(-\Delta(t - 1)) \right)
\]

\[
\geq \frac{\sigma}{N} \frac{1 - \exp(-\Delta(t - 1))}{1 + \exp(\theta_{t+1})}.
\]

For \( t \geq \frac{-\log \left( \frac{1}{\Delta} \right) + 1}{\Delta} \) it must hold that \( \delta_{t+1} > \frac{\sigma}{2N(1 + \exp(\theta_t))} \). Using the same argument as in the proof of ?? for the unbounded gradients, we get that for \( t \geq \max \left\{ \frac{-\log \left( \frac{1}{\Delta} \right) + 1}{\Delta}, \frac{-\log(\exp(\epsilon) - 1)}{\exp(-\epsilon)(\exp(\epsilon) - 1)} \frac{2N}{\sigma} \right\} \) the loss value satisfies \( L(\theta_t) \leq \epsilon \). \( \square \)

**C.8 Proof of Lemma C.6**

**Lemma C.6.** for \( i \notin I_{cr} \), we have that

\[
p_{1,i} \geq (N + N \sigma \exp(-2\eta \epsilon^2 \Delta) - 1)^{-1}.
\]

**Proof.** It holds that \( \tilde{y} = y + \epsilon, \theta^* = 1 \) is the LS solution with \( x^T \sqrt{P_1^{-1}} \) and \( \sqrt{P_2} y \). The least squares solution to the corrupted problem is \( \theta_1 = x^T P_1(y + \epsilon) = x^T P_1 y + x^T P_1 \epsilon = \theta^* + x^T P_1 \epsilon \).

Note that since \( L(\theta^*) = 0, x^T \theta^* - y = 0 \) the losses satisfy:

\[
i \notin I_{cr} : \tilde{\ell}_i(\theta_1) = \frac{1}{2} (x_i x^T P_1 \epsilon)^2 = \frac{1}{2N^2} (x \epsilon)^2,
\]

\[
k \in I_{cr} : \tilde{\ell}_k(\theta_1) = \frac{1}{2} (x_k x^T P_1 \epsilon - \epsilon_k)^2
\]

\[= \frac{1}{2N^2} (x \epsilon)^2 + \epsilon^2 + \frac{\epsilon_k}{N} x \epsilon.
\]

We want to show that the losses seen by the learner satisfy \( \tilde{\ell}_i(\theta_1) \leq \tilde{\ell}_k(\theta_1) \). This holds when:

\[
\epsilon^2 \geq 2 \left\lfloor \frac{\epsilon_k}{N} x \epsilon \right\rfloor.
\]
For $\sigma < \frac{1}{2}$, we have that

$$|2\frac{\epsilon_k}{N}x\epsilon| \leq \frac{2\epsilon \sigma N}{N} = 2\epsilon \sigma < \epsilon^2$$

Overall when $\sigma = \frac{1}{2} - \Delta$: $\tilde{\ell}_k(\theta_1) - \tilde{\ell}_i(\theta_1) \geq \epsilon^2 \Delta$. Then MW will assign lower weights to the corrupted examples.

Denote $A_1 = \frac{1}{N\tau}(x\epsilon)^2$. Then, we have

$$\forall i /\in I_{cr}: w_{1,i} = \exp(-\eta A_1)$$

$$\forall k \in I_{cr}: 0 \leq w_{1,k} \leq \exp(-\eta A_1 - \eta \epsilon^2 \Delta).$$

Looking at the non-corrupted examples’ probabilities leads to

$$p_t = \frac{\exp(-\eta A_1)}{(1 - \sigma)N \exp(-\eta A_1) + \sum_{j \in I_{cr}} w_{1,j}}$$

$$\geq \frac{\exp(-\eta A_1)}{(1 - \sigma)N \exp(-\eta A_1) + \sigma N \exp(-\eta A_1 - \eta \epsilon^2 \Delta)}$$

$$= \frac{1}{N} \cdot \frac{1}{1 + \sigma(\exp(-\eta \epsilon^2 \Delta) - 1)} > \frac{1}{N}$$

C.9 Proof of Theorem 4.7

Theorem 4.7. For Algorithm 4 we have that for $c > 0$ there exists $t > \frac{\ln\left(\frac{\epsilon c + 1 + \Delta}{\eta \epsilon^2 \Delta}\right)}{\eta \epsilon^2 \Delta}$ such that:

$$|\theta_t - 1| \leq c,$$

where $\theta_t$ is the learned parameter by the algorithm.

Theorem 4.7 proof relies on Lemma C.7 below, which lower bounds the probability of clean examples. Ideally, in our 1D setting, a single clean label is enough for learning the optimal $\theta^*$ with LS. Yet, the noise causes deviation from this solution. MR mitigates this problem, it assigns higher weights to clean data (as the lemma shows), which leads to a solution that is closer to the optimal one.

Lemma C.7. For any $t \in [T]$, $\forall i /\in I_{cr}$ it holds that $p_{t,i} \geq \frac{((1 - \sigma)N + \sigma N \exp(-\eta \epsilon^2 \Delta t))^{-1}}{1 - \sigma}.$

Proof. Proof by induction. The induction base is described in Lemma C.6. Assume that for $t$ it holds that $p_{t,i} \geq \frac{((1 - \sigma)N + \sigma N \exp(-\eta \epsilon^2 \Delta t))^{-1}}{1 - \sigma}.$ Hence,

$$\sum_{i \in I_{cr}} p_{t,i} \geq \frac{1 - \sigma}{1 + \sigma(\exp(-\eta \epsilon^2 \Delta t) - 1)} \geq 1 - \sigma.$$ (8)

Since we have a 1D normalized data with $L(\theta^*) = 0$, for any probability matrix, $P$ the LS solution is $\theta^*$, and hence we can use that $\theta_t = \theta^* + xP_t \epsilon$. Denote by $A_t = \frac{1}{2}(\sum_{j \in I_{cr}} p_{t,j}x_j \epsilon_j)^2$, then the loss term is:

$$k \in I_{cr}: \tilde{\ell}_k(\theta_t) = A_t + \frac{1}{2} \epsilon^2 - \epsilon_k \sum_{j \in I_{cr}} p_{t,j}x_j \epsilon_j,$$

$$i /\in I_{cr}: \tilde{\ell}_i(\theta_t) = A_t.$$

Hence (using Eq. (8)),

$$\forall s \in [t], k \in I_{cr}, i /\in I_{cr}:$$

$$\tilde{\ell}_k(\theta_s) - \tilde{\ell}_i(\theta_s) \geq \frac{1}{2} \epsilon^2 - \epsilon^2 \sum_{j \in I_{cr}} p_{s,j} \geq \epsilon^2 \Delta.$$
Finally, since for \( j \in I_c \) it holds that \( w_{t+1,j} \leq \exp\left( -\eta (\sum_{s=1}^{t+1} A_s) \right) \), we have that

\[
p_{t+1,i} = \frac{\exp\left(-\eta \sum_{s=1}^{t+1} A_s \right)}{(1 - \sigma)N \exp\left(-\eta \sum_{s=1}^{t+1} A_s \right) + \sum_{j \in I_c} w_{t+1,j}} \geq \frac{1}{(1 - \sigma)N + \sigma N \exp(-\eta \epsilon^2 \Delta (t + 1))}
\]

The lemma essentially states that the probability assigned to each clean example is increasing over time and goes to \( ((1 - \sigma)N)^{-1} \) and the sum of all clean probabilities goes to 1. We now show how this lemma yields that the error in each iteration can be bounded with term decreasing over time. Since \( \theta_t = xP_t(y + \epsilon) \), as we have seen above, we have that the true loss at time \( t \) for \( 1 \leq i \leq N \) obeys

\[
\ell_i(\theta_t) = \frac{1}{2} (x_i \theta_t - y_i)^2 = \frac{1}{2} (x_i xP_t(y + \epsilon) - y_i)^2
\]

\[
= \frac{1}{2} (xP_t y - 1 + xP_t \epsilon)^2 = \frac{1}{2} (xP_t \epsilon)^2 \leq \frac{\epsilon^2}{2} \left( \sum_{j \in I_c} p_{t,j} \right)^2
\]

Lemma C.7 yields that the sum of corrupted probabilities \( \sum_{j \in I_c} p_{t,j} \) decreases over time since the probabilities for each clean example increases (the sum of probabilities equals 1). Thus, the corrupted sum becomes closer to 0, which yields that the loss with respect to the true labels obeys \( L(\theta_t) \to 0 \), and we converge to 0 loss solution.

**Proof.** We know that for any distribution matrix \( P \) it holds that \( \theta^* = xPy \). Recall that \( \theta_t = xP_t(y + \epsilon) \). Therefore,

\[
|\theta_t - \theta^*| = |\theta^* - xP_t y - xP_t \epsilon|
\]

\[
= |xP_t \epsilon| = \left| \sum_{j=1}^{N} \epsilon_j p_{t,j} \right| = \epsilon \sum_{j \in I_c} p_{t,j}
\]

Using Lemma C.7 we get that \( \sum_{j \in I_c} p_{t,j} \leq \frac{\sigma \exp(-\eta \epsilon^2 \Delta t)}{1 + \sigma \exp(-\eta \epsilon^2 \Delta t) - 1} \leq \frac{\exp(-\eta \epsilon^2 \Delta t)}{1 + \Delta} \). Thus we have that,

\[
|\theta_t - \theta^*| \leq \epsilon \sum_{j \in I_c} p_{t,j} \leq \frac{\epsilon \exp(-\eta \epsilon^2 \Delta t)}{1 + \Delta},
\]

for \( t > \frac{\ln\left( \frac{\epsilon + 1}{\eta \epsilon^2 \Delta} \right)}{\eta \epsilon^2 \Delta} \) therefore it holds that \( |\theta^* - \theta_t| \leq c \)

**C.10 Proof of Lemma 4.8**

**Lemma 4.8** For an MW update of \( \theta_t = \frac{\exp(-\eta \ell(x_i; y_i, \theta))}{\sum_{j=1}^{\eta} \exp(-\eta \ell(x_j; y_j, \theta))} \) (without history) and \( y_i \in \{1, -1\} \) it holds that the uniform average of the absolute derivatives is larger than the MR weighting of absolute derivatives,

\[
\frac{1}{N} \sum_{i=1}^{N} \left| \frac{\partial \ell(x_i; y_i, \theta)}{\partial x_i} \right| \geq \sum_{i=1}^{N} p_i \left| \frac{\partial \ell(x_i; y_i, \theta)}{\partial x_i} \right|
\]

**Proof.** Recall the loss is \( \ell(x_i; y_i, \theta) = \log(1 + \exp(-x_i \theta y_i)) \), shortly we notate it as \( \ell_i \). The derivative is

\[
\frac{\partial \ell_i}{\partial x_i} = \frac{-\theta y_i \exp(-\theta y_i x_i)}{1 + \exp(-\theta y_i x_i)} = -\frac{\theta y_i}{\exp(x_i \theta y_i) + 1}
\]
Next, we establish the connection between the loss and the derivative. For \( y_i = 1 \) it holds
\[
\ell(x_i; y_i, \theta) = \log(1 + \exp(-x_i\theta)) \Rightarrow \exp(-x_i\theta) = \exp(\ell_i) - 1,
\]
and
\[
\frac{\partial \ell_i}{\partial x_i} = -\theta(1 - \exp(-\ell_i)).
\]
For \( y_i = -1 \) it holds
\[
\ell(x_i; y_i, \theta) = \log(1 + \exp(x_i\theta)) \Rightarrow \exp(x_i\theta) = \exp(\ell_i) - 1
\]
and
\[
\frac{\partial \ell_i}{\partial x_i} = \theta(1 - \exp(-\ell_i)).
\]
Overall, the absolute value of the derivative in both cases is \( |\theta|(1 - \exp(-\ell_i)) \) which is monotonically increasing in \( \ell_i \).

Assume w.l.o.g. that \( \ell_1 \) are sorted, i.e. \( \ell_1 \leq \ell_2 \leq \ldots \leq \ell_N \). Let \( k \) be an index such that for all \( i \geq k \) it holds \( p_i \leq \frac{1}{N} \). Note that this is possible due to the monotone weighting update rule
\[
p_i = \frac{\exp(-\eta \ell_i)}{\sum_{j=1}^{N} \exp(-\eta j)}.
\]
The \( k \)th to \( N \)th examples have the largest losses and the lowest weights.

Using the observation that higher loss induces higher absolute derivative value, we have that for \( i < k \) is holds that \( |\frac{\partial \ell_i}{\partial x_i}| \leq |\frac{\partial \ell_k}{\partial x_k}| \) and for \( i \geq k \) it holds \( |\frac{\partial \ell_i}{\partial x_i}| \geq |\frac{\partial \ell_k}{\partial x_k}| \). Then,
\[
\sum_{i=1}^{k-1} \left( \frac{1}{N} - p_i \right) \left| \frac{\partial \ell_i}{\partial x_i} \right| + \sum_{i=k}^{N} \left( \frac{1}{N} - p_i \right) \left| \frac{\partial \ell_i}{\partial x_i} \right|
\]
\[
\geq \sum_{i=1}^{k-1} \left( \frac{1}{N} - p_i \right) \left| \frac{\partial \ell_k}{\partial x_k} \right| + \sum_{i=k}^{N} \left( \frac{1}{N} - p_i \right) \left| \frac{\partial \ell_k}{\partial x_k} \right| = 0.
\]

\[\square\]

## D Theory Demonstration Details

We demonstrate empirically the theoretical results from Section 4.2. We randomly generated \( N = 1500 \) examples, where the values of \( x_i = y_i \in \{\pm 1\} \) are chosen uniformly. We used \( \epsilon = 1 \) for the linear regression and \( \sigma = 0.4 \) for the corruption ratio in both cases.

The results in Fig. 3 includes the logistic regression case and demonstrates clearly that the use of MR improves the learned parameter \( \theta \) since the loss decays almost similarly to the loss when learning with clean \( y \). The results illustrates Lemma 4.5, Lemma 4.6 and Theorem 4.7. Fig. 3 shows also an exponential decay of the loss when we learn a linear regression with MR, as stated in Theorem 4.7. We further simulate MR with GD (Algorithm 3) for linear regression, which we did not include in the theory above. We can observe the same convergence phenomena also there.

## E Additional Experiments and Details

Here we provide additional experiments to the ones presented in the paper. Specifically, we show the effect of MR on noisy examples’ probabilities, then we show the grid search performed to find the
Figure 4. Fraction of examples with noisy labels along weighting percentiles. Trained with CIFAR10 and (a), (b) are training with 40% and 20% noise levels respectively.

(a) Accuracy results when the weights are bounded. Training with multiple $\mu/N$ upper bounds values.

Figure 5. Upper bound effect.

only parameter of our method, the step size, i.e $\eta$. Then we experiment MR with Adam and list the results for training in the presence of noisy labels with mixup and label-smoothing. We further show the results for adversarial training with Adam with and without mixup.

Probabilities and loss with MR. Aiming to analyse the learned weighting we present in Fig. 2c the loss of the examples with the lowest and highest weights after training with 40% and 20% label noise with CIFAR-10. It is clear from the results that examples with low weights suffer from high loss while the heaviest examples incurred low loss.

In Fig. 4 we split the training examples to 100 percentiles according to their weights. Fig. 4 shows the fraction of noisy labels per percentile. It can be seen that indeed noisy labels get lower weights when trained with MR. Almost all the examples in the percentiles which are lower than the noise ratio are noisy.

Effect of upper bound. The $\mu/N$ upper bound of the weights is necessary to insure that MR does not converge to degenerated $p$ vectors. We examine the effect of $\mu$ of the accuracy and the weighting. In Fig. 5a are presented accuracy results of MR with different bounds on the weighting when trained on CIFAR10 with 40% noisy labels. One can see that the accuracy remain quite stable for high $\mu$ values, we see degradation in accuracy when $\mu \geq 1/0.8$ which forces the weighting to have non-zero weights for noisy examples. In Fig. 5b we plot the ratio of the learned weighting with $1/N$. It is shown that $\mu \in \{N, 1/0.2, 1/0.4\}$ the learned distribution is similar. Training with other $\mu$ values change the distribution and one can observe a fraction of the weighting have the maximal allowed value, as $\mu$ is closer to 1 there are more weighting with maximal values.

Experiments details with sparse regularization [110]. For the experiments with the addition of sparse regularization we use the same setting as in their original paper. We train 8-layers CNN for CIFAR-10 and ResNet34 for CIFAR-100 for 120 and 200 epochs, respectively. In addition,
Table 6. Results on CIFAR10 (left) and CIFAR100 (right) with 40% label noise, different optimizers and architecture. The best results within the same optimizer and architecture are in bold.

| Method      | WRN28-10 | ResNet18 | WRN28-10 | ResNet18 | WRN28-10 | ResNet18 |
|-------------|----------|----------|----------|----------|----------|----------|
| Base        | 75.58    | 69.22    | 63.84    | 72.4     | 55.3     | 48.3     |
| Base+MR     | 91.74    | 90.75    | 79.04    | 88.32    | 72.65    | 52.53    |

we use SGD with momentum 0.9 and cosine learning rate scheduler with 128 batch size. We employ \( \ell_2 \) regularization of \( 10^{-4} \) and \( 10^{-5} \) for CIFAR-10 and CIFAR-100 respectively with initial learning rate of 0.01. For the sparse regularization hyperparameters we use \((\tau, p, \lambda_0, \rho, r) = (0.5, 0.1, 1.1, 1.03, 1)\) for CIFAR-10 and \((\tau, p, \lambda_0, \rho, r) = (0.5, 0.01, 4, 1.02, 1)\) for CIFAR-100. For Clothing1M we use the hyperparameters that are utilized for training with WebVision [43] in the original paper, i.e \((\tau, p, \lambda_0, \rho, r) = (0.5, 0.01, 421.02, 1)\).

Experiments details with UNICON [34] We use PreAct ResNet18 [26], in their setting they employ a warmup stage where they train the network without their method, we employ MR in this stage with \( \eta = 0.01 \).

Experiments details with S2E [98] We use the same setting as described in the original paper, with CNN models that are fully detailed in the original paper for CIFAR10 and CIFAR100. We add MR on top of the of the S2E method and use the weighting of the examples within the batch for the NN’s update. At the end of each epoch we update the weighting. We use \( \eta = 0.01 \) for the MR step size.

Experiments details with MW-Net [75] We use WRN28-10 [100], in this setting we run for 40 epochs. We employ MR for the weighting of the noisy training data with \( \eta = 0.01 \).

Hyperparameter search. In Fig. 6 we present the hyperparameter search of the MW step size \( \eta \). We performed a grid search over a log scale and tried 4 different values of \( \eta \). We search with CIFAR-10 in the presence of 40% label noise. According the results we use \( \eta = 0.01 \) throughout all the experiments mentioned in the paper.

![Figure 6. Hyperparameter grid search of MW step size.](image)

Noisy Labels with Adam. We experiment MR when the optimizer, SGD+momentum, is replaced with Adam. Here we use \( \alpha = 0.5 \) for the mixup parameter for improved results. These results, presented in Table 7, and with Adam + WRN28-10 in Table 6. We show results with mixup, label smoothing, and other noise ratios for CIFAR-10 and CIFAR-100. Notice that the advantage of using MR increases with the noise level as this method is suitable for improving network robustness in the presence of noise. When compared to using only Adam with uniform weighting, the addition of MR always improves the results. Additionally, using MR with label smoothing improves results in the presence of noise. Using mixup and the combination of mixup and label smoothing with MR improves the results when the noise ratios are high for CIFAR-10, but for CIFAR-100 we see degradation in accuracy.

Additional comparison with label noise. We compared MR to two additional methods from [28, 75]. We use the same hyperparameters as mentioned in the original papers and replace the architecture to be ResNet-18 [26]. Unlike MR both baselines take advantage of a clean trusted data, we train the
Table 7. Results on CIFAR10 and CIFAR100 with different label noise ratio with Adam vs. MR combined with Adam.

| Dataset | Noise ratio | 0% | 10% | 30% | 40% | 0% | 10% | 30% | 40% |
|---------|-------------|----|-----|-----|-----|----|-----|-----|-----|
| CIFAR10 |             | Adam | Random Weight. | Adam+ mixup | Adam+ label smoothing | Adam+ mixup + label smoothing | MR | MR + mixup | MR + label smoothing | MR + mixup + label smoothing |
| CIFAR100|             | Adam | Random Weight. | Adam+ mixup | Adam+ label smoothing | Adam+ mixup + label smoothing | MR | MR + mixup | MR + label smoothing | MR + mixup + label smoothing |

Table 9. The results of significant tests (p-value) on CIFAR-10/100 with different noise levels.

| Data      | Noise ratio | Base | Smoothing | Mixup | Smoothing + Mixup |
|-----------|-------------|------|-----------|-------|-------------------|
| CIFAR10   | 20%         | 0.0000 | 0.0000 | 0.0002 | 0.0000 |
|           | 40%         | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| CIFAR100  | 20%         | 0.0096 | 0.0228 | 0.1998 | 0.3467 |
|           | 40%         | 0.0001 | 0.0159 | 0.0086 | 0.0100 |

Statistical significance tests. We conduct statistical tests to evaluate the significance of the improvement resulting from the addition of MR. We compare the performance of three baseline models: CE (Base), label smoothing, and mixup, to the same models when MR is included. To determine statistical significance, we calculate p-values using a two-independent-samples t-test based on three identical and independent experiments. The results are presented in Table 9 and in most the cases tested the improvement is significant with $p < 0.05$.

Hyperparameters for Adversarial Robustness. For the adversarial robustness experiments we use ResNet-34 [26]. For Free Adversarial Training [71] we train with $m = 4$ updates in each batch with CIFAR-10 and with $m = 6$ for CIFAR-100. We trained the network for 200 epochs with $l_2$ regularization of $10^{-3}$, momentum of 0.9, batch size of 256 and initial learning rate of 0.1 which is reduced on plateaus with a factor 0.9 and patience 3. We use MR step size of 0.01. For mixup the same hyperparameters as in artificial label noise are employed. We use an additional baseline, TRADES [103] with the same hyperparameters appear in the original paper: 10 internal optimization steps, 0.007 internal step-size and $1/\lambda = 6$. We train the network for 100 epochs for CIFAR-10 and 140 for CIFAR-100, weight decay of $2 \times 10^{-4}$, batch size of 128 and MR step 0.001. We apply an initial learning rate of 0.1 which is multiplied by 0.1 at epochs 75, 90 and 100.

We use Adversarial Robustness Toolbox [63] for the attacks. The PGD attack with $\epsilon \in \{0.01, 0.02, 8/255\}$ is produced with step sizes of 0.003, 0.005, 2/255, respectively, and maximal iteration of 100. For FGSM attack we employ step size of 0.001. The same attack batch size of 256 is applied.
Table 10. Robustness against adversarial attacks with WideResNet 34-10. All attacks are limited to $\epsilon = 8/255$.

| Method          | Natural Images | PGD 10 | PGD 20 | PGD 50 | CW 20 | AA  |
|-----------------|----------------|--------|--------|--------|-------|-----|
| LAS-AT          | 86.95          | 56.27  | 56.02  | 54.81  | 54.68 | 52.7|
| LAS-AT + MR     | 83.97          | 57.2   | 56.31  | 56.06  | 56.0  | 54.39|
| MAIL-TRADES     | 79.35          | 43.81  | 42.24  | 41.65  | 40.81 | 38.51|
| MAIL-TRADES + MR| 75.26          | 49.25  | 48.44  | 48.28  | 48.01 | 45.11|

Adversarial Robustness with LAS and MAIL. We also apply MR with the additional adversarial training method LAS-AT [31] and MAIL [47]. We use WRN 34-10 models and attacks that are bounded according to $l_\infty$ with $\epsilon = 8/255$ but differ in the number of iterations. It is shown in the Table 10 that using MR on top of this adversarial trainings improves the robustness of the networks. As seen also in some of the other baselines, accuracy with natural images decreases when MR is employed.

Adversarial Robustness with Adam. Table 11 shows the results when the original optimizer SGD + momentum is replaced with Adam for CIFAR-10 and CIFAR-100. We use $\alpha = 0.1$ for the mixup parameter, $\eta = 0.001$ MR step size and on each mini batch we perform $m = 4$ adversarial training steps since we found that those suits better for training with Adam. It can been seen that adding MR to adversarial training improves results also when the training is performed with Adam. With the change of optimizer, adding mixup to adversarial training degrades the results and adding MR to this combination does not mitigate the degradation. The results with Adam reflect the high sensitivity of adaptive methods against attacks with lower accuracy results compared to training with momentum.

Table 11. Adversarial robustness to attacks with CIFAR-10 with Adam vs. MR combined with Adam; Adv. stands for robust training [71].

| Method       | CIFAR10 | CIFAR100 | CIFAR100 |
|--------------|---------|----------|----------|
|               | Natural Images | PGD 0.01 | PGD 0.02 | FGSM 0.01 | FGSM 0.02 | FGSM 8/255 | Natural Images | PGD 0.01 | PGD 0.02 | FGSM 0.01 | FGSM 0.02 | FGSM 8/255 |
| Adam          | 91.84   | 11.86    | 5.53     | 5.21     | 37.9     | 21.88     | 12.59     | 68.81   | 14.26    | 11.26    | 10.75    | 25.86     | 20.99     | 18.31     |
| Adam+MR       | 91.52   | 13.87    | 5.39     | 5.73     | 42.98    | 24.97     | 15.49     | 69.23   | 13.67    | 10.32    | 9.85     | 26.78     | 20.17     | 14.74     |
| Adam+mixup    | 92.67   | 12.32    | 5.01     | 4.65     | 55.53    | 45.84     | 41.68     | 72.8    | 15.85    | 10.4     | 9.63     | 35.86     | 30.23     | 27.69     |
| Adam+mixup + MR| 90.95  | 11.07    | 5.92     | 5.31     | 38.65    | 33.01     | 36.01     | 65.91   | 10.40    | 8.65     | 8.46     | 28.42     | 20.82     | 17.85     |
| Adam+Adv.     | 84.19   | 74.92    | 55.31    | 32.47    | 78.61    | 62.96     | 47.91     | 59.12   | 48.26    | 32.71    | 19.21    | 56.72     | 38.48     | 28.72     |
| Adam+Adv + MR | 85.1    | 75.9     | 56.56    | 34.26    | 77.15    | 63.36     | 49.2      | 58.16   | 48.85    | 32.95    | 18.49    | 50.24     | 38.52     | 28.51     |
| Adam+Adv+mixup| 80.15   | 73.45    | 57.07    | 33.86    | 74.17    | 62.62     | 49.17     | 57.02   | 47.88    | 32.98    | 19.63    | 49.02     | 38.5      | 28.52     |
| Adam+Adv+MR+mixup| 84.62 | 75.05    | 56.9     | 31.19    | 76.56    | 61.88     | 46.38     | 57.25   | 49.51    | 35.16    | 21.79    | 50.83     | 39.44     | 38.7      |