Abstract

We give in this paper which is the third in a series of four a theory of covariant derivatives of representatives of multivector and extensor fields on an arbitrary open set $U \subset M$, based on the geometric and extensor calculus on an arbitrary smooth manifold $M$. This is done by introducing the notion of a connection extensor field $\gamma$ defining a parallelism structure on $U \subset M$, which represents in a well defined way the action on $U$ of the restriction there of some given connection $\nabla$ defined on $M$. Also we give a novel and intrinsic presentation (i.e., one that does not depend on a chosen orthonormal moving frame) of the torsion and curvature fields of Cartan’s theory. Two kinds of Cartan’s connection operator fields are identified, and both appear in the intrinsic Cartan’s structure equations satisfied by the Cartan’s torsion and curvature extensor fields. We introduce moreover a metrical extensor $g$ in $U$ corresponding to the restriction there of given metric tensor $g$ defined on $M$ and also introduce the concept a geometric structure $(U, \gamma, g)$ for $U \subset M$ and study metric compatibility of covariant derivatives induced by the connection extensor $\gamma$. This permits the presentation of the concept of gauge (deformed) derivatives which satisfy noticeable properties useful in differential geometry and geometrical theories of the gravitational field. Several derivatives operators in metric and geometrical structures, like ordinary and covariant Hodge coderivatives and some duality identities are exhibit.
1 Introduction

This is the third paper in a series of four, where we continue our exposition of how to use Clifford and extensor algebras methods in the study of the differential geometry of an of a n-dimensional smooth manifold $M$ of arbitrary topology, supporting a metric field $g$ (of signature $(p, q)$, $p + q = n$) and an arbitrary connection $\nabla$. It has three main sections (2, 3 and 4), besides the introduction, conclusions and an Appendix. Section 2 is dedicated to the description of the theory of covariant derivatives of representatives of multivector and extensor fields in our formalism. In Section 2.1 we recall how choosing, like in [2] a chart $(U_o, \phi_o)$ of a given atlas of $M$ and the associated canonical vector space $U_o$, we can represent the effects of the restriction $\nabla|_U$ on $U$ of any connection on $M$ defining a parallelism structure by a connection extensor field $\gamma$ on $U \subset U_o$. 

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In Sections 2.3 and 2.4 we introduce the concepts of $a$-directional \emph{covariant} derivatives (which represents $\nabla_a$ on $U \subset U_o$) of the representatives of multivector and extensor fields, respectively, and prove the main properties satisfied by these objects. In Section 2.5 we give a thoughtful study of the so-called \emph{symmetric parallelism structures}, where among others we present and prove a \emph{Bianchi-like identity} and give an intrinsic Cartan theory of the torsion and curvature extensor fields. \emph{Cartan’s connections of the first and second kind} are identified, and both appear on our version of Cartan’s structure equations. We emphasize the novelty of our approach to Cartan theory, namely that it does not depend on any \emph{chosen} orthonormal moving frame, hence, the name \emph{intrinsic} used above. In the Appendix some examples are worked in detail in order to show how some of the concepts developed in Section 2 are related to standard ones which deals with the same subject.

In Section 3 using previous results ([1, 2] and Section 2) we first introduce (Section 3.1) a metric structure on a smooth manifold through the concept of a metric extensor field $g$ associated to a given metric tensor $g$. Christoffel operators\textsuperscript{1} and the associated Levi-Civita connection field for $U \subset U_o \subset M$ are introduced and their properties are given. The structure of the Levi-Civita connection field is shown to consist of two pieces, a $g$-symmetric and a $g$-skewsymmetric parts which have deep geometric and algebraic meanings. In Section 3.2 we study metric compatible covariant derivatives. There, the relationship between the connection extensor fields on $U$ and covariant derivatives corresponding to \emph{deformed} (metric compatible) geometrical structures $(U, \gamma, g)$ are given and analyzed. The crucial result of this section is Theorem 2 which relates pair of deformed covariant derivative operators associated to different deformed metrics, and is thus an important tool for correct approximations in geometric theories of the gravitational field where some metric is supposed to be a ‘small’ deformation of some other.

In Section 4 we continue the development of our theory of multivector and extensor calculus on smooth manifolds, introducing in Section 4.1 the concept of ordinary Hodge coderivatives, duality identities, and Hodge coderivative identities. Then in Section 4.2 we analyzed deeply the concept of a Levi-Civita connection in our formalism and the remarkable concept of \emph{gauge} deformed derivatives. Several important formulas that appear in the Lagrangian formulation of the theory of multivector and extensor fields on smooth manifolds are obtained\textsuperscript{2}. In Section 4.3 we introduce the concept of \emph{covariant} Hodge coderivative. We study in details how all these important concepts are related and how they can be utilized to clarify several issues in geometrical theories of the gravitational field.

We emphasize, as already done in [1], that the methods introduced in this paper are completely general and applies to the study of the geometry of any arbitrary smooth manifold of arbitrary topology. And indeed, suppose that the

\textsuperscript{1}These objects generalize the Christofell symbols of the standard formalism that are defined only for vector fields of a coordinate basis.

\textsuperscript{2}In [2] we gave a preliminary presentation of the Lagrangian theory of multivector and extensor fields on Minkowski spacetime. See also related developments in [3].
canonical vector space $\mathcal{U}_o$ associated to the chart $(U_o, \phi_o)$ of a given atlas is not enough to perform calculations involving a region $V$ outside $U_o$. In this case, all we need to do is to choose another chart $(U_1, \phi_1)$ of the atlas with coordinates $\{x^\mu\}$ and such that $V \subset U_1$ and construct a geometrical algebra associated to the canonical space $\mathcal{U}_1$ determined by $(U_1, \phi_1)$. Of course, if the manifold is toroidal then an unique chart may cover it and life will be simpler, in the sense that some global questions can also be discussed with only the introduction of the canonical geometrical algebra of a unique canonical space.

2 Covariant Derivatives of Multivector and Extensor Fields

2.1 Representation of a Parallelism Structure on $U$

Let $M$ be a $n$-dimensional smooth manifold. Then, as already recalled in [1] for any point $o \in M$ there exists a local coordinate system $(U_o, \phi_o)$ with coordinates $\{x^\mu\}$.

As in [1], let $\mathcal{U}_o$ be the canonical vector space for $(U_o, \phi_o)$, and $U$ be an open subset of $U_o$. We denote the ring (with identity) of smooth scalar fields on $U$, the module of (representatives) of smooth vector fields on $U$ and the module of (representatives) of smooth multivector fields on $U$ respectively by $S(U)$, $\mathcal{V}(U)$ and $\mathcal{M}(U)$. The set of (representatives) of smooth $k$-vector fields on $U$ is denoted by $\mathcal{M}^k(U)$. The module of smooth $k$-extensor fields on $U$ is denoted by $k\text{-}\text{ext} (\mathcal{M}^1_o(U), \ldots, \mathcal{M}^k_o(U); \mathcal{M}^o(U))$.

Any smooth vector elementary 2-extensor field on $U$ is said to be a connection field on $U$. As we will see in what follows it represents the effect of the restriction on $U$ of a connection $\nabla$ defined in $M$. A general connection field will be denoted by $\gamma$, i.e., $\gamma : U \rightarrow 2\text{-}\text{ext}^1(U_o)$. The smoothness of such $\gamma$ means that for all $a, b \in \mathcal{V}(U)$ the vector field defined by $U \ni p \mapsto \gamma_p(a(p), b(p)) \in \mathcal{U}_o$ is itself smooth.

The open set $U$ equipped with such a connection field $\gamma$, namely $(U, \gamma)$, will be said to be a parallelism structure on $U$ representing there the action of a given connection $\nabla$ defined in $M$.

Remark 1 Please, take notice that, of course, as defined $\gamma$ cannot be extended to all $M$. However, this does not reduce in any way its theoretical importance and as a tool for performing easily very sophisticated calculations.

Let us take $a \in \mathcal{U}_o$. A smooth $(1, 1)$-extensor field on $U$, namely $\gamma_a$, defined as $U \ni p \mapsto \gamma_a|_p \in \text{ext}^1_1(U_o)$ such that for all $b \in \mathcal{U}_o$

$$\gamma_a|_p(b) = \gamma_p(a, b),$$

will be called an $a$-directional connection field on $U$, of course, associated to $(U, \gamma)$.
We emphasize that the $(1,1)$-extensor character and the smoothness of $\gamma_a$ are immediate consequences of the vector elementary $2$-extensor character and the smoothness of $\gamma$.

A smooth $(1,2)$-extensor field on $U$, namely $\Omega$, defined as $U \ni p \mapsto \Omega(p) \in \text{ext}^2_1(U_\circ)$ such that for all $a \in U_\circ$

$$\Omega(p)(a) = \frac{1}{2} \text{biv}[^{\gamma_a}(p)],$$

will be called (for reason that will become clear in what follows) a gauge connection field on $U$.

From the definition of $\text{biv}[^t]$ (see [2]), taking any pair of reciprocal frame fields on $U$, say $\{e_\mu\}, \{e^\mu\}$, and using Eq.(1), we can write Eq.(2) as

$$\Omega(p)(a) = \frac{1}{2} \gamma(p)(a, e^\mu(p)) \wedge e_\mu(p) = \frac{1}{2} \gamma(p)(a, e_\mu(p)) \wedge e^\mu(p).$$

So, we see that the $(1,2)$-extensor character and the smoothness of $\Omega$ are easily deduced from the vector elementary $2$-extensor character and the smoothness of $\gamma$.

Let us take any pair of reciprocal frame fields on $U$, say $\{e_\mu\}, \{e^\mu\}$, i.e.,

$$e_\mu \cdot e^\nu = \delta^\nu_\mu.$$ Let $\Gamma_a$ be the generalized (extensor field) of $\gamma_a$ (see [2]), i.e., $\Gamma_a$ defined as $U \ni p \mapsto \Gamma_a(p) \in \text{ext}(U_\circ)$, is a smooth extensor field on $U$ such that for all $X \in \bigwedge U_\circ$

$$\Gamma_a(p)(X) = \gamma_a(p)(e^\mu(p)) \wedge (e_\mu(p) \lrcorner X) = \gamma_a(p)(e_\mu(p)) \wedge (e^\mu(p) \lrcorner X).$$

It is easily seen that (for a given $X$) the multivector appearing on the right side of Eq.(4) does not depend on the choice of the reciprocal frame fields. The extensor character and the smoothness of $\Gamma_a$ follows from the $(1,1)$-extensor character and the smoothness of $\gamma_a$.

We will usually omit the letter $p$ in writing the definitions given by Eq.(1), Eq.(2) and Eq.(4), and other equations using extensor fields. No confusion should arise with this standard practice. Eq.(4) might be also written in the succinct form $\Gamma_a(X) = \gamma_a(\partial_b) \wedge (b \lrcorner X)$.

We end this section by presenting some of the basic properties satisfied by $\Gamma_a$.

i. $\Gamma_a$ is grade-preserving, i.e.,

if $X \in \mathcal{M}^k(U)$, then $\Gamma_a(X) \in \mathcal{M}^k(U)$.

ii. For any $X \in \mathcal{M}(U)$,

$$\Gamma_a(\hat{X}) = \Gamma_a(\widetilde{X}),$$
$$\Gamma_a(\tilde{X}) = \Gamma_a(X),$$
$$\Gamma_a(X) = \Gamma_a(\bar{X}).$$

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iii. For any \( f \in \mathcal{S}(U) \), \( b \in \mathcal{V}(U) \) and \( X, Y \in \mathcal{M}(U) \),
\[
\begin{align*}
\Gamma_a(f) &= 0, \\
\Gamma_a(b) &= \gamma_a(b), \quad (9) \\
\Gamma_a(X \wedge Y) &= \Gamma_a(X) \wedge Y + X \wedge \Gamma_a(Y). \quad (11)
\end{align*}
\]

iv. The adjoint of \( \Gamma_a \), namely \( \Gamma^\dagger_a \), is the generalized of the adjoint of \( \gamma_a \), namely \( \gamma^\dagger_a \), i.e.,
\[
\Gamma^\dagger_a(X) = \gamma^\dagger_a(\partial_b) \wedge (b \ll X). 
\]

v. The symmetric (skew-symmetric) part of \( \Gamma_a \), namely \( \Gamma_{a\pm} \), is the generalized of the symmetric (skew-symmetric) part of \( \gamma_a \), namely \( \gamma_{a\pm} = \frac{1}{2}(\gamma_a \pm \gamma^\dagger_a) \), i.e.,
\[
\Gamma_{a\pm}(X) = \gamma_{a\pm}(\partial_b) \wedge (b \ll X). 
\]

vi. \( \Gamma_{a-} \) can be factorized by a remarkable formula which only involves \( \Omega \). It is
\[
\Gamma_{a-}(X) = \Omega(a) \times X. 
\]

vii. For any \( X, Y \in \mathcal{M}(U) \) it holds
\[
\Gamma_{a-}(X \ast Y) = \Gamma_{a-}(X) \ast Y + X \ast \Gamma_{a-}(Y), 
\]
where \( \ast \) means any suitable product of smooth multivector fields, either \( \wedge \), \( \cdot \), \( \ll \) or \( (b\text{-Clifford product}) \), see [2].

2.2 \( a \)-Directional Covariant Derivatives of Multivector Fields

Given a parallelism structure \( (U, \gamma) \), let us take \( a \in U_0 \). Then, associated to \( (U, \gamma) \) we can introduce two \( a \)-directional covariant derivative operators \( (a\text{-DCDO’s}) \), namely \( \nabla^+_a \) and \( \nabla^-_a \), which act on the module of smooth multivector fields on \( U \).

They are defined by \( \nabla^\pm_a : \mathcal{M}(U) \to \mathcal{M}(U) \) such that
\[
\begin{align*}
\nabla^+_a X(p) &= a \cdot \partial_o X(p) + \Gamma_a|^+_a(X(p)) \quad (16) \\
\nabla^-_a X(p) &= a \cdot \partial_o X(p) - \Gamma_a|^-_a(X(p)) \quad (17)
\end{align*}
\]
where \( a \cdot \partial_o \) is the canonical \( a\text{-DODO} \) as defined in [1].

We emphasize that each of \( \nabla^+_a \) and \( \nabla^-_a \) satisfies \textit{indeed} the fundamental properties which a well-defined covariant derivative is expected to have. This is trivial to verify whenever we take into account the well-known properties of \( a \cdot \partial_o \) (see [1]), and the properties of \( \Gamma_a \) given by Eq.(5), Eqs.(9), (10), and (11), and Eq.(12).

As usual we will write Eq.(16) and Eq.(17) by omitting \( p \) when no confusion arises.
The smooth multivector fields on $U$, namely $\nabla^+_a X$ and $\nabla^-_a X$, will be respectively called the plus and the minus $a$-directional covariant derivatives of $X$.

We summarize some of the most important properties for the pair of $a$-DCDO’s $\nabla^+_a$ and $\nabla^-_a$.

i. $\nabla^+_a$ and $\nabla^-_a$ are grade-preserving operators on $\mathcal{M}(U)$, i.e.,

\[ \text{if } X \in \mathcal{M}^k(U), \text{ then } \nabla^\pm_a X \in \mathcal{M}^k(U). \]  
(18)

ii. For all $X \in \mathcal{M}(U)$, and for any $\alpha, \alpha' \in \mathbb{R}$ and $a, a' \in U_o$ we have

\[ \nabla^\pm_a (\alpha a + \alpha' a') X = \alpha \nabla^+_a X + \alpha' \nabla^-_a X. \]  
(19)

iii. For all $f \in S(U)$ and $X, Y \in \mathcal{M}(U)$ we have

\[ \nabla^\pm_a f X = a \cdot \partial_o f, \]
\[ \nabla^+_a (X + Y) = \nabla^+_a X + \nabla^+_a Y, \]
\[ \nabla^-_a (f X) = (a \cdot \partial_o f) X + f(\nabla^\pm_a X). \]  
(20), (21), (22)

iv. For all $X, Y \in \mathcal{M}(U)$ we have

\[ \nabla^\pm_a (X \wedge Y) = (\nabla^+_a X) \wedge Y + X \wedge (\nabla^-_a Y). \]  
(23)

v. For all $X, Y \in \mathcal{M}(U)$ we have

\[ (\nabla^+_a X) \cdot Y + X \cdot (\nabla^-_a Y) = a \cdot \partial_o (X \cdot Y). \]  
(24)

It should be noticed that $(\nabla^+_a, \nabla^-_a)$ as defined by Eq.(16) and Eq.(17) is the unique pair of $a$-DCDO’s associated to $(U, \gamma)$ which satisfies the remarkable property given by Eq.(24).

We emphasize here that the $a$-DODO $a \cdot \partial_o$ acting on $\mathcal{M}(U)$, is also a well-defined $a$-DCDO. In this particular case, the connection field $\gamma$ is identically zero and the plus and minus $a$-DCDO’s are equal to each other, and both of them coincide with $a \cdot \partial_o$.

We introduce yet another well-defined $a$-DCDO which acts also on the module of smooth multivector fields on $U$, namely $\nabla^0_a$.

It is defined by

\[ \nabla^0_a X = \frac{1}{2}(\nabla^+_a X + \nabla^-_a X). \]  
(25)

But, by using Eqs.(16) and (17), and Eq.(14), we might write else

\[ \nabla^0_a X = a \cdot \partial_o X + \Omega(a) \times X. \]  
(26)

The $a$-DCDO $\nabla^0_a$ satisfies the same properties which hold for each one of the $a$-DCDO’s $\nabla^+_a$ and $\nabla^-_a$. But, it has also an additional remarkable property

\[ (\nabla^0_a X) \cdot Y + X \cdot (\nabla^0_a Y) = a \cdot \partial_o (X \cdot Y). \]  
(27)

Moreover, it satisfies a Leibnitz-like rule for any suitable product of smooth multivector fields, i.e.,

\[ \nabla^0_a (X \ast Y) = (\nabla^0_a X) \ast Y + X \ast (\nabla^0_a Y). \]  
(28)
2.2.1 Connection Operators

Associated to any parallelism structure \((U, \gamma)\) we can introduce two remarkable operators which map 2-uples of smooth vector fields to smooth vector fields.

They are defined by \(\Gamma^\pm : \mathcal{V}(U) \times \mathcal{V}(U) \rightarrow \mathcal{V}(U)\) such that

\[
\Gamma^\pm(a, b) = \nabla^\pm_a b,
\]

and will be called the connection operators of \((U, \gamma)\).

We summarize the basic properties of them.

i. For all \(f \in S(U)\), and \(a, a', b, b' \in \mathcal{V}(U)\), we have

\[
\Gamma^\pm(a + a', b) = \Gamma^\pm(a, b) + \Gamma^\pm(a', b),
\]

\[
\Gamma^\pm(a, b + b') = \Gamma^\pm(a, b) + \Gamma^\pm(a, b'),
\]

\[
\Gamma^\pm(fa, b) = f\Gamma^\pm(a, b),
\]

\[
\Gamma^\pm(a, fb) = (a \cdot \partial_a f)b + f\Gamma^\pm(a, b).
\]

As we can observe both connection operators satisfy the linearity property only with respect to the first smooth vector field variable. Thus, connection operators are not extensor fields, because no linear in the second argument.

ii. For all \(a, b, c \in \mathcal{V}(U)\), we have

\[
\Gamma^+(a, b) \cdot c + b \cdot \Gamma^-(a, c) = a \cdot \partial_a (b \cdot c),
\]

which is an immediate consequence of Eq.\((24)\).

2.2.2 Deformation of Covariant Derivatives

Let \((\nabla^+_a, \nabla^-_a)\) be any pair of \(a\)-DCDO’s and \(\lambda\) a non-singular smooth \((1, 1)\)-extensor field on \(U\). We define the deformation of these covariant derivatives as the pair \((\lambda \nabla^+_a, \lambda \nabla^-_a)\) by

\[
\lambda \nabla^+_a X = \Lambda(\nabla^+_a \Lambda^{-1}(X)),
\]

\[
\lambda \nabla^-_a X = \Lambda^\dagger(\nabla^-_a \Lambda^{-1}(X)),
\]

where \(\Lambda\) is the extended\(^4\) of \(\lambda\), is a well-defined pair of \(a\)-DCDO’s, since it satisfies as it is trivial to verify the fundamental properties given by Eqs.\((20)\), \((21)\) and \((22)\), Eq.\((23)\) and Eq.\((24)\). For instance,

\[
\lambda \nabla^+_a f = \Lambda(\nabla^+_a \Lambda^{-1}(f)) = a \cdot \partial_a f.
\]

\(^4\)Recall that \(\lambda^* = (\lambda^{-1})^\dagger = (\lambda^\dagger)^{-1}\), and \(\Lambda^{-1} = (\Lambda)^{-1} = (\lambda^{-1})\) and \(\Lambda^\dagger = (\Lambda)^\dagger = (\lambda^\dagger)\), see [2].
We verify now that the definitions given by Eq. (35) and Eq. (36) also satisfy a property analogous to the one given by Eq. (24). Indeed, we have
\[(\lambda \nabla_a^+ X) \cdot Y + X \cdot (\lambda \nabla_a^- Y) = (\nabla_a^+ \lambda^{-1}(X)) \cdot A(X) + \lambda^{-1}(X) \cdot (\nabla_a^- \lambda Y)\]
\[= a \cdot \partial_o(\lambda^{-1}(X) \cdot \lambda Y),\]
\[= a \cdot \partial_o(X \cdot Y).\]

(38)

### 2.3 \(a\)-Directional Covariant Derivatives of Extensor Fields

The three \(a\)-DCDO’s \(\nabla_a^+\), \(\nabla_a^-\) and \(\nabla_a^0\) which act on \(\mathcal{M}(U)\) can be extended in order to act on the module of smooth \(k\)-extensor fields on \(U\). For any \(t \in k\)-ext(\(\mathcal{M}_1^k(U), \ldots, \mathcal{M}_\sigma^k(U); \mathcal{M}_\sigma(U)\)), we can define exactly \(3^{k+1}\) covariant derivatives, namely \(\nabla_a^{\sigma_1 \cdots \sigma_k \sigma} t \in k\)-ext(\(\mathcal{M}_1^k(U), \ldots, \mathcal{M}_\sigma^k(U); \mathcal{M}_\sigma(U)\)), where each of \(\sigma_1, \ldots, \sigma_k, \sigma\) is being used to denote either (+), (−) or (0). They are given by the following definition.

For all \(X_1 \in \mathcal{M}_1^1(U), \ldots, X_k \in \mathcal{M}_\sigma^k(U), X \in \mathcal{M}_\sigma(U)\)

\[(\nabla_a^{\sigma_1 \cdots \sigma_k} t)_p(X_1(p), \ldots, X_k(p)) \cdot X(p)
= a \cdot \partial_o(t_p)(\ldots) \cdot X(p) - t_p(\nabla_a^{\sigma_1} X_1(p), \ldots) \cdot X(p)
- \cdots - t_p(\ldots, \nabla_a^{\sigma_k} X_k(p)) \cdot X(p) - t_p(\ldots) \cdot \nabla_a^{\sigma} X(p),\]

(39)

for each \(p \in U\).

As usual when no confusion arises we will write Eq. (39) by omitting \(p\).

We call the reader’s attention that each one of the \(\nabla_a^{\sigma_1 \cdots \sigma_k} t\) defined by Eq. (39) is in fact a smooth \(k\)-extensor field. Its \(k\)-extensor character and smoothness can be easily deduced from the respective properties of \(t\). We note also that in the first term on the right side of Eq. (39), \(a \cdot \partial_o\) refers to the canonical \(a\)-DODO as was defined in [1].

We notice that any smooth (1, 1)-extensor field on \(U\), say \(t\), has just \(3^{1+1} = 9\) covariant derivatives. For instance, four important covariant derivatives of such \(t\) are given by

\[(\nabla_a^{+} t)(X_1) \cdot X = a \cdot \partial_o(t(X_1) \cdot X)
- t(\nabla_a^{+} X_1) \cdot X - t(X_1) \cdot \nabla_a^{+} X,\]

(40)

\[(\nabla_a^{-} t)(X_1) \cdot X = a \cdot \partial_o(t(X_1) \cdot X)
- t(\nabla_a^{-} X_1) \cdot X - t(X_1) \cdot \nabla_a^{-} X,\]

(41)

\[(\nabla_a^{-} t)(X_1) \cdot X = a \cdot \partial_o(t(X_1) \cdot X)
- t(\nabla_a^{-} X_1) \cdot X - t(X_1) \cdot \nabla_a^{-} X,\]

(42)

\[(\nabla_a^{-} t)(X_1) \cdot X = a \cdot \partial_o(t(X_1) \cdot X)
- t(\nabla_a^{-} X_1) \cdot X - t(X_1) \cdot \nabla_a^{-} X,\]

(43)

where \(X_1 \in \mathcal{M}_1^1(U)\) and \(X \in \mathcal{M}_\sigma(U)\),

We present now some of the basic properties satisfied by these \(a\)-directional covariant derivatives of smooth \(k\)-extensor fields.
Note the inversion between $\sigma_1$ and $\sigma$ into the $a$-DCDO’s above. As we can see, the three $a$-DCDO’s $\nabla_+^a$, $\nabla_-^a$ and $\nabla_0^a$ commute indeed with the adjoint operator $^\dagger$.

The proof of the above result is as follows. Let us take $X_1 \in \mathcal{M}_1^a(U)$ and $X \in \mathcal{M}^o(U)$. By recalling the fundamental property of the adjoint operator $^\dagger$, and in accordance with Eq. (49), we can write

$$\left(\nabla_a^{\sigma}\cdot t\right)(X) = \left(\nabla_a^{\sigma}\cdot t\right)(X_1) \cdot X$$

\begin{align*}
&= a \cdot \partial_0(t(X_1) \cdot X) - t(\nabla_a^{\sigma}X_1) \cdot X - t(X_1) \cdot \nabla_a^{\sigma}X \\
&= a \cdot \partial_0(t(X_1) \cdot X) - t(X_1) \cdot \nabla_a^{\sigma}X_1 - t(\nabla_a^{\sigma}X) \cdot X_1, \\
&= (\nabla_a^{\sigma}\cdot t)(X) \cdot X_1.
\end{align*}

Hence, by non-degeneracy of scalar product, the expected result immediately follows.

iii. For all $t \in 1$-$\text{ext}(\mathcal{M}_1^a(U), \mathcal{M}^o(U))$, it holds

$$\left(\nabla_a^{\sigma}\cdot t\right)(X_1) = \nabla_a^{\sigma}(t(X_1)) - t(\nabla_a^{\sigma}X_1),$$  \hspace{1cm} (47)

$$\left(\nabla_a^{\sigma}\cdot t\right)(X_1) = \nabla_a^{\sigma}(t(X_1)) - t(\nabla_a^{\sigma}X_1),$$  \hspace{1cm} (48)

$$\left(\nabla_a^{\sigma}\cdot t\right)(X_1) = \nabla_a^{\sigma}(t(X_1)) - t(\nabla_a^{\sigma}X_1),$$  \hspace{1cm} (49)

$$\nabla_a^{\sigma}(t(X_1)) = \nabla_a^{\sigma}(t(X_1)) - t(\nabla_a^{\sigma}X_1).$$  \hspace{1cm} (50)

We prove here only Eq. (47). Let us take $X_1 \in \mathcal{M}_1^a(U)$ and $X \in \mathcal{M}^o(U)$. In accordance with Eq. (49) and by recalling Eq. (24), we have

\begin{align*}
\left(\nabla_a^{\sigma}\cdot t\right)(X_1) \cdot X &= a \cdot \partial_0(t(X_1) \cdot X) - t(\nabla_a^{\sigma}X_1) \cdot X - t(X_1) \cdot \nabla_a^{\sigma}X \\
&= \nabla_a^{\sigma}(t(X_1)) \cdot X + t(X_1) \cdot \nabla_a^{\sigma}X \\
&\quad - t(\nabla_a^{\sigma}X_1) \cdot X - t(X_1) \cdot \nabla_a^{\sigma}X \\
&= (\nabla_a^{\sigma}(t(X_1)) - t(\nabla_a^{\sigma}X_1)) \cdot X.
\end{align*}

Hence, by non-degeneracy of scalar product, it follows what was to be proved.

2.4 Torsion and Curvature Fields

Let $(U, \gamma)$ be a parallelism structure on $U$, representing there some connection $\nabla$ defined in $M$. The smooth vector elementary 2-exform field on $U$, namely, $\tau$ such that for all $a, b \in \mathcal{V}(U)$

$$\tau(a, b) = \nabla_a^{\tau}b - \nabla_b^{\tau}a - [a, b],$$  \hspace{1cm} (51)
i.e.,
\[ \tau(a, b) = \gamma_a(b) - \gamma_b(a), \]  
(52)
is called the torsion field of \((U, \gamma)\).

The smooth vector elementary 3-extensor field on \(U\), namely \(\rho\), such that for all \(a, b, c \in V(U)\)
\[ \rho(a, b, c) = [\nabla^+_a \cdot \nabla^+_b]c - \nabla^+_{[a,b]}c, \]  
(53)
i.e.,
\[ \rho(a, b, c) = (a \cdot \partial_o \gamma_b)(c) - (b \cdot \partial_o \gamma_a)(c) + [\gamma_a, \gamma_b](c) - \gamma_{[a,b]}(c), \]  
(54)
will be called the curvature field of \((U, \gamma)\).

It should be emphasized that the curvature field \(\rho\) is skew-symmetric in the first and second variables, i.e.,
\[ \rho(a, b, c) = -\rho(b, a, c). \]  
(55)

### 2.4.1 Symmetric Parallelism Structures

A parallelism structure \((U, \gamma)\) is said to be symmetric if and only if
\[ \gamma_a(b) = \gamma_b(a). \]  
(56)
As we can easily be verified this condition is completely equivalent to
\[ \nabla^+_a b - \nabla^+_b a = [a, b], \]  
(57)
for all \(a, b \in V(U)\).

So, taking into account Eq. (51) and Eq. (52) we have that a parallelism structure is symmetric if and only if it is torsionless, i.e.,
\[ \tau(a, b) = 0. \]  
(58)

We now present and prove some basic properties of a symmetric parallelism structure.

1. The curvature field \(\rho\) satisfies the cyclic property
\[ \rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = 0. \]  
(59)

The proof is as follows. Recalling Eq. (53) we can write
\[ \rho(a, b, c) = \nabla^+_a \nabla^+_b c - \nabla^+_b \nabla^+_a c - \nabla^+_{[a,b]} c, \]  
(60)
\[ \rho(b, c, a) = \nabla^+_b \nabla^+_c a - \nabla^+_c \nabla^+_b a - \nabla^+_{[b,c]} a, \]  
(61)
\[ \rho(c, a, b) = \nabla^+_c \nabla^+_a b - \nabla^+_a \nabla^+_c b - \nabla^+_{[c,a]} b. \]  
(62)
By adding Eqs. (60), (61) and (62), wherever by taking into account Eq. (57), we get
\[
\rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = \nabla^+_d (\nabla^+_a c - \nabla^+_c b) + \nabla^+_b (\nabla^+_c a - \nabla^+_a c) + \nabla^+_d (\nabla^+_a b - \nabla^+_b a) - \nabla^+_{[a,b]} c - \nabla^+_{[b,c]} a - \nabla^+_{[c,a]} b,
\]
\[
= [a, [b, c]] + [b, [c, a]] + [c, [a, b]].
\] (63)

Hence, by recalling the so-called Jacobi’s identity for the Lie product of smooth vector fields [1], the expected result immediate follows.

ii. The curvature field \( \rho \) satisfies the so-called Bianchi’s identity, i.e.,
\[
(\nabla^+_a \nabla^+_b - \rho)(a, b, c) + (\nabla^+_a \nabla^+_c - \rho)(b, d, c) + (\nabla^+_b \nabla^+_d - \rho)(d, a, c) = 0.
\] (64)

The proof of Eq. (64) is as follows. Let us take \( a, b, c, d, w \in \mathcal{V}(U) \). Taking into account Eq. (59), and using Eq. (24), we have
\[
(\nabla^+_a \nabla^+_b - \rho)(a, b, c) \cdot w = d \cdot \partial_a (\rho(a, b, c) \cdot w) - \rho(\nabla^+_a a, b, c) \cdot w - \rho(a, \nabla^+_b b, c) \cdot w - \rho(a, b, \nabla^+_c c) \cdot w - \rho(a, b, c) \cdot \nabla^- w,
\]
i.e.,
\[
(\nabla^+_a \nabla^+_b - \rho)(a, b, c) = \nabla^+_a \rho(a, b, c) - \rho(\nabla^+_a a, b, c) - \rho(a, \nabla^+_b b, c) - \rho(a, b, \nabla^+_c c).
\] (65)

By cycling the letters \( a, b, d \) into Eq. (65), we get
\[
(\nabla^+_a \nabla^+_b - \rho)(b, d, c) = \nabla^+_a \rho(b, d, c) - \rho(\nabla^+_a b, d, c) - \rho(b, \nabla^+_d d, c),
\]
\[
(\nabla^+_b \nabla^+_d - \rho)(d, a, c) = \nabla^+_b \rho(d, a, c) - \rho(\nabla^+_b d, a, c) - \rho(d, \nabla^+_a a, c),
\]
\[
(\nabla^+_d \nabla^+_a - \rho)(a, b, c) = \nabla^+_d \rho(a, b, c) - \rho(\nabla^+_d a, b, c) - \rho(a, b, \nabla^+_c c).
\] (66)

Now, by adding Eqs. (65), (66), and (67), wherever by using Eq. (55) and Eq. (57), we get
\[
(\nabla^+_a \nabla^+_b - \rho)(a, b, c) + (\nabla^+_a \nabla^+_b - \rho)(b, d, c) + (\nabla^+_b \nabla^+_d - \rho)(d, a, c) = \nabla^+_a \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) - \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) - \rho(a, b, \nabla^+_c c) - \rho(b, d, \nabla^+_a c) - \rho(d, a, \nabla^+_c c).
\] (68)

But, in agreement with Eq. (63), we can write
\[
\nabla^+_a \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) = \nabla^+_a \rho(a, b, c) + \nabla^+_a \rho(b, d, c) + \nabla^+_b \rho(d, a, c) - \rho([a, b], d, c) - \rho([b, d], a, c) - \rho([d, a], b, c) - \rho(a, b, \nabla^+_c c) - \rho(b, d, \nabla^+_a c) - \rho(d, a, \nabla^+_c c).
\]
and

\[
-\rho ([a, b], d, c) - \rho ([b, d], a, c) - \rho ([d, a], b, c) \\
= -\nabla_{[a, b]}^+ \nabla_d^+ c - \nabla_{[b, d]}^+ \nabla_a^+ c - \nabla_{[d, a]}^+ \nabla_b^+ c \\
+ \nabla_d^+ \nabla_{[a, b]}^+ c + \nabla_a^+ \nabla_{[b, d]}^+ c + \nabla_b^+ \nabla_{[d, a]}^+ c \\
+ \nabla_{[a, b], d}^+ c + \nabla_{[b, d], a}^+ c + \nabla_{[d, a], b}^+ c.
\]

i.e., by recalling the Jacobi’s identity,

\[
-\rho ([a, b], d, c) - \rho ([b, d], a, c) - \rho ([d, a], b, c) \\
= -\nabla_{[a, b]}^+ \nabla_d^+ c - \nabla_{[b, d]}^+ \nabla_a^+ c - \nabla_{[d, a]}^+ \nabla_b^+ c \\
+ \nabla_d^+ \nabla_{[a, b]}^+ c + \nabla_a^+ \nabla_{[b, d]}^+ c + \nabla_b^+ \nabla_{[d, a]}^+ c.
\] (70)

Now, by adding Eqs. (69) and (70), and using Eq. (53), we get

\[
\nabla_d^+ \rho (a, b, c) + \nabla_a^+ \rho (b, d, c) + \nabla_b^+ \rho (d, a, c) \\
- \rho ([a, b], d, c) - \rho ([b, d], a, c) - \rho ([d, a], b, c) \\
= \rho (a, b, \nabla_d^+ c) + \rho (b, d, \nabla_a^+ c) + \rho (d, a, \nabla_b^+ c). 
\] (71)

Finally, putting Eq. (71) into Eq. (68), the expected result follows.

2.4.2 Cartan Fields

The smooth \((1, 2)\)-extensor field on \(U\), namely \(\Theta\), defined by

\[
\Theta (c) = \frac{1}{2} \partial_a \wedge \partial_b \tau (a, b) \cdot c
\] (72)

will be called the Cartan torsion field of \((U, \gamma)\).

We should notice that such \(\Theta\) contains all of the geometric information which is just contained in \(\tau\). Indeed, Eq. (72) can be inverted in such a way that given any \(\Theta\), there is an unique \(\tau\) that verifies Eq. (72). We have, indeed that

\[
\tau (a, b) = \partial_c (a \wedge b) \cdot \Theta (c).
\] (73)

The smooth bivector elementary 2-extensor field on \(U\), namely \(\Omega\), which is defined by

\[
\Omega (c, d) = \frac{1}{2} \partial_a \wedge \partial_b \rho (a, b, c) \cdot d
\] (74)

will be called the Cartan curvature field of \((U, \gamma)\).

Since Eq. (74) can be inverted, by giving \(\rho\) in terms of \(\Omega\), we see that such \(\Omega\) contains the same geometric information as \(\rho\). The inversion is realized by

\[
\rho (a, b, c) = \partial_d (a \wedge b) \cdot \Omega (c, d).
\] (75)
2.4.3 Cartan’s Structure Equations

Associated to any parallelism structure $(U, \gamma)$ we can introduce two noticeable operators which map 2-uples of smooth vector fields to smooth vector fields. They are:

(a) The mapping $\gamma^+ : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)$ defined by

$$\gamma^+(b, c) = \partial_o(\nabla^+_o b) \cdot c$$

which will be called the Cartan connection operator of first kind of $(U, \gamma)$.

(b) The mapping $\gamma^- : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)$ defined by

$$\gamma^-(b, c) = \partial_s b \cdot (\nabla^-_o c)$$

which will be called the Cartan connection operator of second kind of $(U, \gamma)$.

We summarize some of the basic properties which are satisfied by the Cartan operators.

i. For all $f \in \mathcal{S}(U)$, and $b, b', c, c' \in \mathcal{V}(U)$, we have

$$\gamma^+(b + b', c) = \gamma^+(b, c) + \gamma^+(b', c),$$

$$\gamma^+(b, c + c') = \gamma^+(b, c) + \gamma^+(b, c').$$

$$\gamma^+(fb, c) = (\partial_o f) b \cdot c + f \gamma^+(b, c),$$

$$\gamma^+(b, fc) = f \gamma^+(b, c).$$

ii. For all $f \in \mathcal{S}(U)$, and $b, b', c, c' \in \mathcal{V}(U)$, we have

$$\gamma^-(b + b', c) = \gamma^-(b, c) + \gamma^-(b', c),$$

$$\gamma^-(b, c + c') = \gamma^-(b, c) + \gamma^-(b, c').$$

$$\gamma^-(fb, c) = f \gamma^-(b, c),$$

$$\gamma^-(b, fc) = (\partial_o f) b \cdot c + f \gamma^-(b, c).$$

We have that the Cartan operator of first kind has the linearity property with respect to the second variable, and the Cartan operator of second kind is linear with respect to the first variable.

iii. For any $a, b \in \mathcal{V}(U)$,

$$\gamma^+(b, c) + \gamma^-(b, c) = \partial_o(b \cdot c).$$

It is an immediate consequence of Eq.(24).

First Cartan’s Structure Equation

For any $c \in \mathcal{V}(U)$ it holds

$$\Theta(c) = \partial_o \wedge c + \partial_s \wedge \gamma^-(s, c),$$

where $\partial_o$ is the Hestenes derivative operator introduced in [1].
We prove Eq. (87) as follows. Using Eq. (51) we can write
\[
\Theta(c) = \frac{1}{2} \partial_a \wedge \partial_b (\nabla_a^+ b - \nabla_b^+ a - [a, b]) \cdot c,
\]
\[
= \partial_a \wedge \partial_b (\nabla_a^+ b - a \cdot \partial_b b) \cdot c. \tag{88}
\]

A straightforward calculation then yields
\[
\partial_a \wedge \partial_b (\nabla_a^+ b) \cdot c = \partial_a \wedge \partial_b \gamma^+ (b, c) \cdot a
\]
\[
= \partial_b \wedge \partial_a (\gamma^- (b, c) \cdot a - a \cdot \partial_a (b \cdot c)),
\]
\[
= \partial_b \wedge \gamma^- (b, c) - \partial_b \wedge \partial_a (b \cdot c), \tag{89}
\]

and
\[
-\partial_a \wedge \partial_b (a \cdot \partial_b b) \cdot c
\]
\[
= \partial_a \wedge \partial_b (b \cdot (a \cdot \partial_b c) - a \cdot \partial_b (b \cdot c))
\]
\[
= \partial_a \wedge (a \cdot \partial_c c) + \partial_b \wedge \partial_a a \cdot \partial_b (b \cdot c),
\]
\[
= \partial_a \wedge c + \partial_b \wedge \partial_0 (b \cdot c). \tag{90}
\]

Thus, by putting Eq. (89) and Eq. (90) into Eq. (88), we get the expected result.

**Second Cartan’s Structure Equation**

For any \(c, d \in \mathcal{V}(U)\) it holds
\[
\Omega(c, d) = \partial_a \wedge \gamma^+ (c, d) + \gamma^+ (c, \partial_b) \wedge \gamma^- (s, d). \tag{91}
\]

To prove Eq. (91) we use Eq. (53) and write
\[
\Omega(c, d) = \frac{1}{2} \partial_a \wedge \partial_b ([\nabla_a^+ \cdot \nabla_b^+] c - \nabla_{[a, b]}^+ c) \cdot d,
\]
\[
= \partial_a \wedge \partial_b (\nabla_a^+ (\nabla_b^+ c) - \nabla_{[a, b]}^+ c) \cdot d. \tag{92}
\]

But, by taking a pair of reciprocal frame fields \(\{e_a\}, \{e^\sigma\}\) we can write
\[
\nabla_a^+ (\nabla_b^+ c) \cdot d = \nabla_a^+ (\gamma^+ (c, e^\sigma) \cdot b e_a) \cdot d
\]
\[
= a \cdot \partial_a (\gamma^+ (c, e^\sigma) \cdot b e_a) \cdot d + \gamma^+ (c, e^\sigma) \cdot b \nabla_a^+ e_a \cdot d
\]
\[
= a \cdot \partial_a (\gamma^+ (c, e^\sigma) \cdot b e_a) \cdot d + \gamma^+ (c, e^\sigma) \cdot b \gamma^+ (e_a, d) \cdot a,
\]
\[
= a \cdot \partial_a \gamma^+ (c, e^\sigma) \cdot b e_a \cdot d + \gamma^+ (c, d) \cdot (a \cdot \partial_a b)
\]
\[
+ \gamma^+ (c, e^\sigma) \cdot b \gamma^+ (e_a, d) \cdot a. \tag{93}
\]

Now, the first term into Eq. (92), by using Eq. (93) and the well-known identity \(\partial_a \wedge (f X) = (\partial_a f) \wedge X + f \partial_a \wedge X\), where \(f \in \mathcal{S}(U)\) and \(X \in \mathcal{M}(U)\), can be
written
\[
\partial_a \wedge \partial_b \nabla^+_a (∇^+_b c) \cdot d = \partial_a \wedge \gamma^+(c, e^\sigma)(e_\sigma \cdot d) + \partial_a \wedge \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_\sigma b) \\
- \gamma^+(c, e^\sigma) \wedge \gamma^+(e_\sigma, d), \\
= \partial_a \wedge \gamma^+(c, e^\sigma)(e_\sigma \cdot d) \\
- \gamma^+(c, e^\sigma) \wedge (\partial_\sigma(e_\sigma \cdot d) - \gamma^-(e_\sigma, d)) \\
+ \partial_a \wedge \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_\sigma b), \\
= \partial_a \wedge \gamma^+(c, d) \\
+ \gamma^+(c, e^\sigma) \wedge \gamma^-(e_\sigma, d) \\
+ \partial_a \wedge \partial_b \gamma^+(c, d) \cdot (a \cdot \partial_\sigma b).
\] (94)

It is also
\[
- (\nabla^+_a \partial_\sigma c) \cdot d = -\gamma^+(c, d) \cdot (a \cdot \partial_\sigma b).
\] (95)

Finally, by putting Eq. (94) and Eq. (95) into Eq. (92), we get the expected result.

2.5 Metric Structure

Let \( U \) be an open subset\(^4\) of \( U_o \). Any symmetric and non-degenerate smooth \((1, 1)\)-extensor field on \( U \), namely \( g \), will be said to be a metric field on \( U \). It is quite obvious that it is the extensor representative on \( U \) of a given admissible metric tensor \( g \) defined in \( M \). This means that \( g : U_o \to ext^1_1(U_o) \) satisfies \( g(p) = g^+(p) \) and \( \det[g] \neq 0 \), for each \( p \in U \), and for all \( v \in \mathcal{V}(U) \) the vector field defined by \( U_o \ni p \mapsto g(p)(v(p)) \) belongs to \( \mathcal{V}(U) \), see \cite{1}.

The open set \( U \) equipped with such a metric field \( g \), namely \((U, g)\), will be said to be a metric structure on \( U \).

The existence of a metric field on \( U \) makes possible the introduction of three kinds of metric products of smooth multivector fields on \( U \). These are: (a) the \( g \)-scalar product of \( X, Y \in \mathcal{M}(U) \), namely \( X \cdot g Y \in \mathcal{S}(U) \); (b) the left and right \( g \)-contracted products of \( X, Y \in \mathcal{M}(U) \), namely \( X \circ g Y \in \mathcal{M}(U) \) and \( X \circ g Y \in \mathcal{M}(U) \).

These products are defined by
\[
(X \cdot g Y)(p) = g_p(X(p)) \cdot Y(p) \\
(X \circ g Y)(p) = g_p(X(p)) \cdot Y(p) \\
(X \circ g Y)(p) = X(p) \cdot g_p(Y(p)), \text{ for each } p \in U.
\] (96), (97), (98)

Note that in the above formulas \( g \) is the extended (extensor field) of \( g \) \cite{2}.

We recall that the \( g \)-Clifford product of \( X, Y \in \mathcal{M}(U) \), namely \( X \cdot g Y \in \mathcal{M}(U) \), is defined by the following axioms.

\(^4\)In this paper we will use the nomenclature and notations just used in \cite{1,2}.
For all \( f \in \mathcal{S}(U), \ b \in \mathcal{V}(U) \) and \( X, Y, Z \in \mathcal{M}(U) \)

\[
\begin{align*}
  f g X &= X f = f X \quad \text{(scalar multiplication on } \mathcal{M}(U)\text{).} \quad (99) \\
  g b X &= b g X + b \wedge X, \quad (100) \\
  g X b &= X g b + X \wedge b. \quad (101) \\
  g (Y, Z) &= (X, Y) g Z. \quad (102)
\end{align*}
\]

\( \mathcal{M}(U) \) equipped with each one of the products \((\_ g \_)(\_ g \_))\) is a non-associative algebra induced by the respective \(b\)-interior algebra of multivectors. They are called the \(g\)-\textit{interior algebras of smooth multivector fields}.

\( \mathcal{M}(U) \) equipped with \((g)\) is an associative algebra fundamentally induced by the \(b\)-\textit{Clifford algebra of multivectors}. It is called the \(g\)-\textit{Clifford algebra of smooth multivector fields}.

### 2.6 Christoffel Operators

Given a metric structure \((U, g)\) we can introduce the following two operators which map 3-uples of smooth vector fields into smooth scalar fields.

(a) The mapping \([\ , \ , ] : \mathcal{V}(U) \times \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{S}(U)\) defined by

\[
[a, b, c] = \frac{1}{2} \left( a \cdot \partial_\omega (b \cdot c) + b \cdot \partial_\omega (c \cdot a) - c \cdot \partial_\omega (a \cdot b) + c \cdot [a, b] + b \cdot [c, a] - a \cdot [b, c] \right) \quad (103)
\]

which is called the \textit{Christoffel operator of first kind}. Of course, it is associated to \((U, g)\).

(b) The mapping \(\{\} : \mathcal{V}(U) \times \mathcal{V}(U) \to \mathcal{S}(U)\) defined by

\[
\left\{ \begin{array}{c}
  c \\
  a, b
\end{array} \right\} = [a, b, g^{-1}(c)] \quad (104)
\]

which is called the \textit{Christoffel operator of second kind}.

**Remark 2** Before we proceed we remark that the Christoffel operators of first and second kinds generalize the well known Christofell symbols of the standard formalism used in the differential geometry that is defined only for the vector fields of a coordinate basis.

We summarize now some of the basic properties of the Christoffel operator of first kind.
i. For all \( f \in \mathcal{S}(U) \), and \( a, a', b, b', c, c' \in \mathcal{V}(U) \),

\[
[a + a', b, c] = [a, b, c] + [a', b, c], \quad (105)
\]
\[
[fa, b, c] = f[a, b, c]. \quad (106)
\]
\[
[a, b + b', c] = [a, b, c] + [a, b', c], \quad (107)
\]
\[
[a, fb, c] = f[a, b, c] + (a \cdot \partial_o f)b \cdot c. \quad (108)
\]
\[
[a, b, c + c'] = [a, b, c] + [a, b, c'], \quad (109)
\]
\[
[a, b, fc] = f[a, b, c]. \quad (110)
\]

These equations say that the Christoffel operator of the first kind has the linearity property which respect to the first and third smooth vector field variables.

ii. For all \( a, b, c \in \mathcal{V}(U) \),

\[
[a, b, c] + [b, a, c] = a \cdot \partial_o (b \cdot c) + b \cdot \partial_o (c \cdot a) - c \cdot \partial_o (a \cdot b)
\]
\[
+ b \cdot [c, a] - a \cdot [b, c], \quad (111)
\]
\[
[a, b, c] - [b, a, c] = c \cdot [a, b]. \quad (112)
\]
\[
[a, b, c] + [a, c, b] = a \cdot \partial_o (b \cdot c), \quad (113)
\]
\[
[a, b, c] - [a, c, b] = b \cdot \partial_o (c \cdot a) - c \cdot \partial_o (a \cdot b)
\]
\[
+ c \cdot [a, b] + b \cdot [c, a] - a \cdot [b, c]. \quad (114)
\]
\[
[a, b, c] + [c, b, a] = b \cdot \partial_o (c \cdot a) + c \cdot [a, b] - a \cdot [b, c], \quad (115)
\]
\[
[a, b, c] - [c, b, a] = a \cdot \partial_o (b \cdot c) - c \cdot \partial_o (a \cdot b) + b \cdot [c, a]. \quad (116)
\]

### 2.7 Levi-Civita Connection Field

We now introduce a remarkable decomposition of the Christoffel operator of first kind.

**Proposition.** There exists a smooth \((1, 2)\)-extensor field on \( U \), namely \( \omega_0 \), such that the Christoffel operator of first kind can be written as

\[
[a, b, c] = \left( a \cdot \partial_o b + \frac{1}{2}g^{-1} \circ (a \cdot \partial_o g)(b) + \omega_0(a) \times b \right) \cdot c. \quad (117)
\]

Such \( \omega_0 \) is given by

\[
\omega_0(a) = -\frac{1}{4}g^{-1}(\partial_b \wedge \partial_c)a \cdot ((b \cdot \partial_o g)(c) - (c \cdot \partial_o g)(b)). \quad (118)
\]

**Proof**

By using \( a \cdot \partial_o (X \cdot Y) = (a \cdot \partial_o X) \cdot Y + X \cdot (a \cdot \partial_o Y) + (a \cdot \partial_o g)(X) \cdot Y \), for all \( X, Y \in \mathcal{M}(U) \), we have

\[
a \cdot \partial_o (b \cdot c) = (a \cdot \partial_o b) \cdot c + b \cdot (a \cdot \partial_o c) + (a \cdot \partial_o g)(b) \cdot c, \quad (119)
\]
\]
and, by cycling the letters $a, b$ and $c$, we get
\[ b \cdot \partial_a (c \cdot a) = (b \cdot \partial_a c) \cdot a + c \cdot (b \cdot \partial_a a) + (b \cdot \partial_a g)(c) \cdot a, \quad (120) \]
\[ -c \cdot \partial_a (a \cdot b) = - (c \cdot \partial_a a) \cdot b - a \cdot (c \cdot \partial_a b) - (c \cdot \partial_a g)(a) \cdot b. \quad (121) \]

A straightforward calculation yields
\[
\begin{align*}
\partial_g [a, b] &= c \cdot (a \cdot \partial_g b) - c \cdot (b \cdot \partial_g a), \\
\partial_g [b, c] &= b \cdot (c \cdot \partial_g a) - b \cdot (a \cdot \partial_g c), \\
\partial_g [c, a] &= - a \cdot (b \cdot \partial_g c) + a \cdot (c \cdot \partial_g b).
\end{align*}
\]

Now, by adding Eqs. (129), (120), (121) and Eqs. (122), (123), (124) we get
\[ 2[a, b, c] = (a \cdot \partial_g b) \cdot c + (a \cdot \partial_g g)(b) \cdot c + (b \cdot \partial_g g)(c) \cdot a - (c \cdot \partial_g g)(a) \cdot b, \]

hence, by taking into account the symmetry property $(n \cdot \partial_g g)\hat{\leftrightarrow} = n \cdot \partial_g g$, it follows
\[ [a, b, c] = (a \cdot \partial_g b) \cdot c + \frac{1}{2} g^{-1} (a \cdot (b \cdot \partial_g g)(c) - (c \cdot \partial_g g)(b)). \quad (125) \]

On another side, a straightforward calculation yields
\[
\omega_0(a) \times b = - g(b) \cdot \omega_0(a)
\]
\[
\begin{align*}
&= \frac{1}{4} g(b) g^{-1} (\partial_p \wedge \partial_q) a \cdot ((p \cdot \partial_g g)(q) - (q \cdot \partial_g g)(p)) \\
&= \frac{1}{4} g(b) g^{-1} (\partial_q) a \cdot ((p \cdot \partial_g g)(q) - (q \cdot \partial_g g)(p)) \\
&= \frac{1}{2} g^{-1} (\partial_q) a \cdot ((b \cdot \partial_g g)(q) - (q \cdot \partial_g g)(b)),
\end{align*}
\]

hence, it follows
\[ (\omega_0(a) \times b) \cdot c = \frac{1}{2} a \cdot ((b \cdot \partial_g g)(c) - (c \cdot \partial_g g)(b)). \quad (126) \]

Finally, putting Eq. (126) into Eq. (125), we get the required result. ■

The smooth vector elementary 2-extensor field on $U$, namely $\lambda$, defined by
\[ \lambda(a, b) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_g g)(b) + \omega_0(a) \times b \quad (127) \]
is a well-defined connection field on $U$. It will be called the Levi-Civita connection field on $U$. The open set $U$ endowed with $\lambda$, namely $(U, \lambda)$, will be said to be
a Levi-Civita parallelism structure on $U$. It is clear that a particular $(U, \lambda)$ which is $g$-antisymmetric may in our formalism be the description in $U$ of the restriction on $U$ of the usual Levi-Civita connection $D$ of $g$ on $M$.

The $a$-DCDO’s associated to $(U, \lambda)$, namely $D^+_a$ and $D^-_a$, are said to be Levi-Civita $a$-DCDO’s. They are fundamentally defined by $D^\pm_a : \mathcal{M}(U) \to \mathcal{M}(U)$ such that

\[ D^+_a X = a \cdot \partial_o X + \Lambda_a(X), \quad (128) \]
\[ D^-_a X = a \cdot \partial_o X - \Lambda^*_a(X), \quad (129) \]

where $\Lambda_a$ is the generalized (extensor field) of $\lambda_a$.

It should be noted that such an $a$-DCDO $D^+_a$ satisfies the fundamental property

\[ (D^+_a b) \cdot c = \left\{ \begin{array}{l} c \quad \text{if } a, b, c \in V(U). \end{array} \right. \]

Eq.(130) follows immediately from Eq.(117) once we change $c$ for $g^{-1}(c)$ and take into account the definitions given by Eq.(104), Eq.(127) and Eq.(128).

We present now two remarkable properties of $\omega_0$.

**i.** For all $a, b, c \in V(U)$ we have the cyclic property

\[ \omega_0(a) \times b \cdot c + \omega_0(b) \times c \cdot a + \omega_0(c) \times a \cdot b = 0. \]

(131)

We show the cyclic property by recalling Eq.(126) used in the proof of Eq.(117). Indeed, we can write

\[ \omega_0(a) \times b \cdot c = \frac{1}{2} g(a) \cdot ((b \cdot \partial_o g)(c) - (c \cdot \partial_o g)(b)), \]

and, by cycling the letters $a, b$ and $c$, we get

\[ \omega_0(b) \times c \cdot a = \frac{1}{2} g(b) \cdot ((c \cdot \partial_o g)(a) - (a \cdot \partial_o g)(c)), \]

\[ \omega_0(c) \times a \cdot b = \frac{1}{2} g(c) \cdot ((a \cdot \partial_o g)(b) - (b \cdot \partial_o g)(a)). \]

(134)

Now, by adding Eqs.(132), (133) and (134) we get

\[ \omega_0(a) \times b \cdot c + \omega_0(b) \times c \cdot a + \omega_0(c) \times a \cdot b \]
\[ = \frac{1}{2} g(a) \cdot (b \cdot \partial_o g)(c) - c \cdot (b \cdot \partial_o g)(a)) + \frac{1}{2} g(b) \cdot (c \cdot \partial_o g)(a) - a \cdot (c \cdot \partial_o g)(b)) + \frac{1}{2} (c \cdot (a \cdot \partial_o g)(b) - b \cdot (a \cdot \partial_o g)(c)). \]

**5**We recall from [3] that the generalized of $t \in \text{ext}^1(\mathcal{U}_a)$ is $T \in \text{ext}(\mathcal{U}_a)$ given by $T(X) = t(\partial_a) \wedge (n \lrcorner X)$. 20
Then, by taking into account the symmetry property \((n \cdot \partial_o g)\dagger = n \cdot \partial_o g\), the expected result immediately follows.

ii. \(\omega_0\) is just the \(g\)-gauge connection field associated to \((U, \lambda)\), i.e.,

\[
\omega_0(a) = \frac{1}{2} \text{biv}_g[\lambda_a].
\]

The proof of Eq.(135) uses the noticeable formulas: \((a \cdot \partial_o t) \circ t^{-1} + t \circ (a \cdot \partial_o t^{-1}) = 0\), for all non-singular smooth \((1, 1)\)-extensor field \(t\), \((a \cdot \partial_o g^{-1})\dagger = a \cdot \partial_o g^{-1}, \partial_n \wedge (s(n)) = 0\), for all symmetric \(s \in \text{ext}^1(U_o)\), and \(\partial_n \wedge (B \times n) = -2B\), where \(B \in \bigwedge^2 U_o\). A straightforward calculation\(^6\) allows us to get

\[
\text{biv}_g[\lambda_a] = -\partial_n \wedge (\lambda_a \circ g^{-1}(n))
\]

\[
= -\frac{1}{2} \partial_n \wedge g^{-1} \circ (a \cdot \partial_o g) \circ g^{-1}(n) - \partial_n \wedge (\omega(a) \times g^{-1}(n))
\]

\[
= \frac{1}{2} \partial_n \wedge (a \cdot \partial_o g^{-1})(n) - \partial_n \wedge (\omega(a) \times n),
\]

\[
= 0 + 2\omega_0(a).
\]

We present now three remarkable properties of the Levi-Civita connection field \(\lambda\) on \(U\).

i. \(\lambda\) is symmetric with respect to the interchanging of vector variables, i.e.,

\[
\lambda(a, b) = \lambda(b, a).
\]

To show this, let us take \(a, b, c \in \mathcal{V}(U)\). Then,

\[
\lambda(a, b) \cdot c = \frac{1}{2} (a \cdot \partial_o g)(b) \cdot c + \omega_0(a) \times g^{-1}(n) \cdot b \cdot c,
\]

and interchanging the letters \(a\) and \(b\) we have

\[
\lambda(b, a) \cdot c = \frac{1}{2} (b \cdot \partial_o g)(a) \cdot c + \omega_0(b) \times g^{-1}(n) \cdot a \cdot c.
\]

Now, subtracting Eq.(138) from Eq.(137), and taking into account Eq.(131) used in the proof of Eq.(131), we get

\[
(\lambda(a, b) - \lambda(b, a)) \cdot c = \omega_0(c) \times g^{-1}(n) \cdot a \cdot b + \omega_0(a) \times g^{-1}(n) \cdot b \cdot c - \omega_0(b) \times g^{-1}(n) \cdot a \cdot c.
\]

Then, by recalling the multivector identity \(B \times v \cdot w = -B \times w \cdot v\), where \(B \in \bigwedge^2 U_o\) and \(v, w \in U_o\), and Eq.(131)\(^7\), the required result immediately follows by the non-degeneracy of the \(g\)-scalar product.

\(^6\)Recall that \(\text{biv}_g[t] = -\partial_n \wedge (t(n))\) and \(\text{biv}_g[t \circ g^{-1}] = \text{biv}_g[t] \circ g^{-1}\).
ii. The $g$-symmetric and $g$-skew symmetric parts of $\lambda_a$, namely $\lambda_{a\pm(\cdot)} = \frac{1}{2}(\lambda_a \pm \lambda_a^{(\cdot)})$, are given by

\begin{align*}
\lambda_{a+(\cdot)}(b) & = \frac{1}{2} g^{-1} \circ (a \cdot \partial_a)g(b), \\
\lambda_{a-(\cdot)}(b) & = \omega_0(a) \times b.
\end{align*}

To show these important formulas we calculate first the metric adjoint of $\lambda_a$, namely $\lambda_a^\dagger$, by using the fundamental property \[2\] of the metric adjoint operator $\dagger$. By recalling the symmetry property $(a \cdot \partial_a)g \dagger = a \cdot \partial_a g$ and the multivector identity $B \times v \cdot w = -B \times w \cdot v$, where $B \in \bigwedge^2 U_o$ and $v, w \in U_o$, we can write that

\begin{align*}
\lambda_a^\dagger(b) \cdot c &= b \cdot \lambda_a(c) \\
&= \frac{1}{2} b \cdot (a \cdot \partial_a)g(c) - \omega_0(a) \times b \cdot c \\
&= (\frac{1}{2} g^{-1} \circ (a \cdot \partial_a)g(b) - \omega_0(a) \times b) \cdot c,
\end{align*}

hence, by the non-degeneracy of the $g$-scalar product, it follows that

$$\lambda_a^\dagger(b) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_a)g(b) - \omega_0(a) \times b.$$ 

Now, we can get that

$$\lambda_a(b) + \lambda_a^\dagger(b) = g^{-1} \circ (a \cdot \partial_a)g(b),$$

$$\lambda_a(b) - \lambda_a^\dagger(b) = 2\omega_0(a) \times b.$$ 

iii. The generalized of $\lambda_a$, namely $\Lambda_a$, is given by the following formula

$$\Lambda_a(X) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_a)g(X) + \omega_0(a) \times X,$$ 

where $g$ and $g^{-1}$ are extended of $g$ and extended of $g^{-1}$, respectively.

The above property is an immediate consequence of using the noticeable formulas: $\mathbf{t}^{-1} \circ (a \cdot \partial_{\mathbf{t}})g(\mathbf{t}) \wedge (n \wedge X) = \mathbf{t}^{-1} \circ (a \cdot \partial_{\mathbf{t}})\mathbf{t}(X)$, for all non-singular smooth $(1, 1)$-extensor field $\mathbf{t}$, and $(B \times \partial_n)g(\wedge (n \wedge X) = B \times X$, where $B \in \bigwedge^2 U_o$ and $X \in \bigwedge U_o$.

From the above, we suspect (and this is indeed the case, see next section) that a a particular $\lambda_a$ which is metric compatible and symmetric may represent in $U$ the effects of the restriction in $U$ of the Levi-Civita connection of $g$ on $M$.
3 Covariant Derivatives and Metric Compatibility

3.1 Geometric Structure

A parallelism structure \((U, \gamma)\) is said to be compatible with a metric structure \((U, g)\) if and only if

\[
\gamma_{a+(g)} = \frac{1}{2} g^{-1} \circ (a \cdot \partial_og),
\]

i.e., \(g \circ \gamma_a + \gamma^b_a \circ g = a \cdot \partial_og.\)

Sometimes for abuse of language we will say that \(\gamma\) is metric compatible (or \(g\)-compatible, for short).

3.1.1 Geometrical Structure in \(U\)

The open set \(U\) equipped with a connection field \(\gamma\) and a metric extensor field \(g\), namely \((U, \gamma, g)\), such that \((U, \gamma)\) is compatible with \((U, g)\), will be said to be a geometric structure on \(U\).

The Levi-Civita parallelism structure \((U, \lambda)\), according to Eq. (139), is compatible with the metric structure \((U, g)\), i.e., \(\lambda\) is \(g\)-compatible. It immediately follows that \((U, \lambda, g)\) is a well-defined geometric structure on \(U\).

Using the above results we have in any geometric structure \((U, \gamma, g)\) an important result.

**Theorem 1.** There exists a smooth \((1, 2)\)-extensor field on \(U\), namely \(\omega\), such that

\[
\gamma_{a+(g)}(b) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_og)(b) + \omega(a) \times g(b).
\]

We give now three properties involving \(\gamma_a\) and \(\omega\).

i. The \(g\)-symmetric and \(g\)-skew-symmetric parts of \(\gamma_a\), namely \(\gamma_{a\pm(g)} = \frac{1}{2}(\gamma_a \pm \gamma^b_a)\), are given by

\[
\gamma_{a+(g)}(b) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_og)(b), \quad \gamma_{a-(g)}(b) = \omega(a) \times g(b).
\]

ii. The generalized of \(\gamma_a\), namely \(\Gamma_a\), is given by

\[
\Gamma_a(X) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_og)(X) + \omega(a) \times g(X).
\]

iii. \(\omega\) is just the \(g\)-gauge connection field associated to \((U, \gamma)\), i.e.,

\[
\omega(a) = \frac{1}{2} biv[\gamma_a].
\]
### 3.2 Metric Compatibility

The pair of \(a\)-DCDO’s associated to \((U, \gamma)\), namely \((\nabla_a^+, \nabla_a^-)\), is said to be metric compatible (or \(g\)-compatible, for short) if and only if

\[
\nabla_a^{++} g = 0, \tag{148}
\]

or equivalently,

\[
\nabla_a^- g^{-1} = 0. \tag{149}
\]

We emphasize that Eq. (148) and Eq. (149) are completely equivalent to each other. It follows from the remarkable formula \((\nabla_a^{++} t) \circ t^{-1} + t \circ (\nabla_a^- t) = 0\), valid for all non-singular smooth \((1, 1)\)-extensor field \(\tau\).

\((U, \gamma)\) is compatible with \((U, g)\) if and only if \((\nabla_a^+, \nabla_a^-)\) is metric compatible. Indeed, let us take \(b \in \mathcal{V}(U)\), we have that

\[
(\nabla_a^{++} g)(b) = \nabla_a^- g(b) - g(\nabla_a^+ b) = a \cdot \partial_o g(b) - \gamma_a^1 \circ g(b) - g(a \cdot \partial_o b) - g \circ \gamma_a(b),
\]

\[
= (a \cdot \partial_o g)(b) - g \circ \gamma_a(b) - \gamma_a^1 \circ g(b). \tag{150}
\]

Now, if \(\gamma\) is \(g\)-compatible, by using Eq. (148) into Eq. (150), it follows that \((\nabla_a^{++} g)(b) = 0\), i.e., \((\nabla_a^+, \nabla_a^-)\) is \(g\)-compatible. And, if \((\nabla_a^+, \nabla_a^-)\) is \(g\)-compatible, by using Eq. (148) into Eq. (150), we get that \(g \circ \gamma_a(b) + \gamma_a^1 \circ g(b) = (a \cdot \partial_o g)(b)\), i.e., \(\gamma\) is \(g\)-compatible.

We now present some basic properties which are satisfied by a \(g\)-compatible pair of \(a\)-DCDO’s, namely \((D_a^+, D_a^-)\).

1. For any \((D_a^+, D_a^-)\) we have

\[
D_a^{++} g = 0, \tag{151}
\]

\[
D_a^- g^{-1} = 0. \tag{152}
\]

where \(g\) and \(g^{-1}\) are the so-called extended of \(g\) and \(g^{-1}\), respectively.

In order to prove the first statement we only need to check that for all \(f \in \mathcal{S}(U)\) and \(b_1, \ldots, b_k \in \mathcal{V}(U)\)

\[
(D_a^{++} g)(f) = 0 \text{ and } (D_a^{++} g)(b_1 \wedge \ldots \wedge b_k) = 0.
\]

But, by using the fundamental property \(g(f) = f\), we get

\[
(D_a^{++} g)(f) = D_a^- g(f) - g(D_a^+ f) = D_a^- f - g(a \cdot \partial_o f) = a \cdot \partial_o f - a \cdot \partial_o f = 0.
\]

And, by using the fundamental property \(g(b_1 \wedge \ldots \wedge b_k) = g(b_1) \wedge \ldots \wedge g(b_k)\)

we get

\[
(D_a^{++} g)(b_1 \wedge \ldots \wedge b_k)
\]

\[
= D_a^- g(b_1 \wedge \ldots \wedge b_k) - g(D_a^+ (b_1 \wedge \ldots \wedge b_k))
\]

\[
= D_a^- g(b_1) \wedge \ldots \wedge g(b_k) + \cdots + g(b_1) \wedge \cdots D_a^- g(b_k)
\]

\[
- g(D_a^+ b_1) \wedge \ldots \wedge g(b_k) - \cdots - g(b_1) \wedge \ldots g(D_a^+ b_k)
\]

\[
= (D_a^{++} g)(b_1) \wedge \ldots g(b_k) + \cdots + g(b_1) \wedge \ldots (D_a^{++} g)(b_k) = 0,
\]

\[
\text{where } g \text{ and } g^{-1}\text{ are the so-called extended of } g \text{ and } g^{-1}, \text{ respectively.}
\]

In order to prove the first statement we only need to check that for all \(f \in \mathcal{S}(U)\) and \(b_1, \ldots, b_k \in \mathcal{V}(U)\)

\[
(D_a^{++} g)(f) = 0 \text{ and } (D_a^{++} g)(b_1 \wedge \ldots \wedge b_k) = 0.
\]

But, by using the fundamental property \(g(f) = f\), we get

\[
(D_a^{++} g)(f) = D_a^- g(f) - g(D_a^+ f) = D_a^- f - g(a \cdot \partial_o f) = a \cdot \partial_o f - a \cdot \partial_o f = 0.
\]

And, by using the fundamental property \(g(b_1 \wedge \ldots \wedge b_k) = g(b_1) \wedge \ldots \wedge g(b_k)\)

we get

\[
(D_a^{++} g)(b_1 \wedge \ldots \wedge b_k)
\]

\[
= D_a^- g(b_1 \wedge \ldots \wedge b_k) - g(D_a^+ (b_1 \wedge \ldots \wedge b_k))
\]

\[
= D_a^- g(b_1) \wedge \ldots \wedge g(b_k) + \cdots + g(b_1) \wedge \cdots D_a^- g(b_k)
\]

\[
- g(D_a^+ b_1) \wedge \ldots \wedge g(b_k) - \cdots - g(b_1) \wedge \ldots g(D_a^+ b_k)
\]

\[
= (D_a^{++} g)(b_1) \wedge \ldots g(b_k) + \cdots + g(b_1) \wedge \ldots (D_a^{++} g)(b_k) = 0,
\]
since $D_a^+ g = 0$.

The second statement can be proved analogously.

ii. $(D_a^+, D_a^-)$ satisfies the fundamental property

$$D_a^- g(X) = g(D_a^+ X).$$

(153)

Indeed, by Eq. (151) we have that for all $X \in \mathcal{M}(U)$

$$(D_a^+ g)(X) = 0,$$

$$D_a^- g(X) - g(D_a^+ X) = 0.$$

iii. Ricci-like theorems. Let $X, Y$ be a smooth multivector fields. Then,

$$a \cdot \partial_o (X \cdot gY) = (D_a^+ X) \cdot g Y + X \cdot (D_a^+ Y),$$

(154)

$$a \cdot \partial_o (X \cdot g^{-1} Y) = (D_a^- X) \cdot g^{-1} Y + X \cdot (D_a^- Y).$$

(155)

The proof is as follows. As we know, $(D_a^+, D_a^-)$ must satisfy the fundamental property

$$(D_a^+ X) \cdot Y + X \cdot (D_a^- Y) = a \cdot \partial_o (X \cdot Y).$$

Then, by substituting $Y$ for $g(Y)$ and using Eq. (153), the first statement follows immediately. To prove get the second statement it is enough to substitute $Y$ for $g^{-1}(Y)$ and once again use Eq. (153).

iii. $(D_a^+, D_a^-)$ satisfies Leibnitz-like rules for all of the $g$ and $g^{-1}$ suitable products, namely $\cdot$ and $\cdot^{-1}$, of smooth multivector fields

$$D_a^+ (X \cdot Y) = (D_a^+ X) \cdot g Y + X \cdot (D_a^+ Y),$$

(156)

$$D_a^- (X \cdot g^{-1} Y) = (D_a^- X) \cdot g^{-1} Y + X \cdot (D_a^- Y).$$

(157)

We prove only the first statement. The other proof is similar.

Firstly, if $\cdot$ is just $(\cdot)$, then Eq. (156) is nothing more than the Leibniz rule for the exterior product of smooth multivector fields i.e.,

$$D_a^+ (X \wedge Y) = (D_a^+ X) \wedge Y + X \wedge (D_a^+ Y).$$

(158)

As we know it is true.

Secondly, if $\cdot$ is $(\cdot)$, then $D_a^+(X \cdot Y) = a \cdot \partial_o (X \cdot Y)$ and it follows that Eq. (156) is nothing more than the Ricci-like theorem for $D_a^+$.

In order to prove Eq. (156), whenever $\cdot$ is either $(\cdot)$ or $(\cdot)$, we use the identities

$$(X, Y) \cdot Z = Y \cdot (\bar{X} \wedge Z)$$

and

$$(X, Y) \cdot Z = X \cdot (Z \wedge \bar{Y}),$$

for all $X, Y, Z \in \mathcal{M}(U)$.

Here, as in (2) $\cdot$ means any product, either $(\cdot), (\cdot), (\cdot, \cdot)$ or $(g$-Clifford product). Analogously for $g^{-1}$. 

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and Eq. (154) and Eq. (158). For instance, for the left $g$-contracted product we can indeed write
\[
a \cdot \partial_o((X \cdot Y) \cdot Z) = a \cdot \partial_o(Y \cdot (\tilde{X} \wedge Z))
\]
and then
\[
(D^+_a(X \cdot Y)) \cdot Z + (X \cdot Y) \cdot (D^+_a Z)
= (D^+_a Y) \cdot (\tilde{X} \wedge Z) + Y \cdot ((D^+_a \tilde{X}) \wedge Z) + Y \cdot (\tilde{X} \wedge (D^+_a Z))
\]
Also,
\[
(D^+_a(X \cdot Y)) \cdot Z = ((D^+_a X)_g Y + X \cdot (D^+_a Y)) \cdot Z.
\]
Hence, by the non-degeneracy of the $g$-scalar product, the Leibniz rule for $(\cdot)$ immediately follows, i.e.,
\[
D^+_a(X \cdot Y) = (D^+_a f)_g X + X \cdot (D^+_a Y).
\]
In order to prove Eq. (159) whenever $\ast$ means $(\cdot)$, we only need to check that for all $f \in \mathcal{S}(U)$ and $b_1, \ldots, b_k \in \mathcal{V}(U)$
\[
D^+_a(f \cdot X) = (D^+_a f)_g X + f \cdot (D^+_a X),
\]
\[
D^+_a(b_1 \ldots b_k \cdot X) = (D^+_a (b_1 \ldots b_k))_g X + b_1 \ldots b_k \cdot (D^+_a X).
\]
The verification of Eq. (160) is trivial.
To verify Eq. (161) we will use complete induction over the $k$ smooth vector fields $b_1, \ldots, b_k$.
Let us take $b \in \mathcal{V}(U)$, by using Eq. (159) and Eq. (158), we have
\[
D^+_a(b \cdot X) = (D^+_a b)_g X + b \cdot (D^+_a X)
= (D^+_a b)_g X + b \cdot (D^+_a X) + (D^+_a + b)_g X + b \cdot (D^+_a X),
\]
\[
D^+_a(b \cdot X) = (D^+_a b)_g X + b \cdot (D^+_a X).
\]
Now, let us $b_1, \ldots, b_k, b_{k+1} \in \mathcal{V}(U)$. By using twice the inductive hypothesis
and Eq. (162) we can write
\[ D_a^+(b_1, \ldots, b_k, b_{k+1}) X \]
\[ = (D_a^+(b_1, \ldots, b_k) b_{k+1} + b_1, \ldots, b_k (D_a^+(b_{k+1}) X) \]
\[ + b_1, \ldots, b_k b_{k+1} (D_a^+ X) \]
\[ = ((D_a^+(b_1, \ldots, b_k) b_{k+1}) + b_1, \ldots, b_k (D_a^+ b_{k+1})) X \]
\[ + b_1, \ldots, b_k b_{k+1} (D_a^+ X) \]
\[ = (D_a^+(b_1, \ldots, b_k, b_{k+1}) X + b_1, \ldots, b_k b_{k+1} (D_a^+ X). \]

From the theory of extensors developed in (2) it immediately follows that for any metric \( g \), there is a \textit{non-singular} smooth \((1, 1)\)-extensor field \( h \) such that
\[ g = h^1 \circ \eta \circ h, \tag{163} \]
where \( \eta \) is an \textit{pseudo-orthogonal metric field} with the same signature as \( g \). Such \( h \) will be said to be a \textit{gauge metric field} for \( g \).

We now ask if there is any noticeable relationship between a \textit{g}-compatible pair of \textit{a-DCDO}'s and a \( \eta \)-compatible pair of \textit{a-DCDO}'s. The answer is YES.

\textbf{Theorem 2.} Let \( h \) be a gauge metric field for \( g \). For any \( g \)-compatible pair of \textit{a-DCDO}'s, namely \((g D_a^+, g D_a^-)\), there exists an unique \( \eta \)-compatible pair of \textit{a-DCDO}'s, namely \((\eta D_a^+, \eta D_a^-)\), such that\(^8\)
\[ h(g D_a^+ X) = \eta D_a^+ h(X), \tag{164} \]
\[ h^*(g D_a^- X) = \eta D_a^- h^*(X). \tag{165} \]

And reciprocally, given any \( \eta \)-compatible pair of \textit{a-DCDO}'s, say \((\eta D_a^+, \eta D_a^-)\), there is an unique \( g \)-compatible pair of \textit{a-DCDO}'s, say \((g D_a^+, g D_a^-)\), such that the above formulas are satisfied.

\textbf{Proof}

Given \((g D_a^+, g D_a^-)\), since \( h \) is a non-singular smooth \((1, 1)\)-extensor field, we can indeed construct a well-defined pair of \textit{a-DCDO}'s, namely \((h D_a^+, h D_a^-)\), by the following formulas
\[ h D_a^+ X = h(g D_a^+ h^{-1}(X)) \text{ and } h D_a^- X = h^*(g D_a^+ h^1(X)). \]
As defined above, \((h D_a^+, h D_a^-)\) is the \textit{h-deformation} of \((g D_a^+, g D_a^-)\).

But, it is obvious that \( h D_a^+ \) and \( h D_a^- \) as defined above satisfy in fact Eq. (164) and Eq. (165), i.e.,
\[ h(g D_a^+ X) = h D_a^+ h(X) \text{ and } h^*(g D_a^- X) = \eta D_a^- h^*(X). \]

\(^8\)Recall that \( h^* = (h^{-1})^t = (h^1)^{-1} \) and \( h^{-1} = (h)^{-1} = (h^1)^{-1} \) and \( h^1 = (h^1)^1 = (h^1). \)
In order to check the $\eta$-compatibility of $(\eta D^+_a, \eta D^-_a)$, we can write

$$(\eta D^+_a \eta)(b) = \eta D^-_a \eta(b) = \eta \eta(D^+_a b)$$

$$= h^*(\eta D^-_a h \circ \eta(b)) = \eta h \circ (\eta D^+_a h^{-1}(b))$$

$$= h^*(\eta D^-_a h \circ \eta h^{-1}(b) - h \circ \eta h(D^+_a h^{-1}(b)))$$

$$= h^*(\eta D^-_a g(h^{-1}(b)) - g(D^+_a h^{-1}(b)))$$

$$= h^*(\eta D^+_a g(h^{-1}(b)).$$

This implies that $\eta D^+_a \eta = h^* \circ (\eta D^+_a g) \circ h^{-1}$. Then, since $\eta D^+_a g = 0$, it follows that $\eta D^+_a \eta = 0$, i.e., $(\eta D^+_a, \eta D^-_a)$ is $\eta$-compatible.

Now, if there exists another $\eta$-compatible pair $(\eta D'^+_a, \eta D'^-_a)$, which satisfies Eq. (164) and Eq. (165), i.e.,

$$\eta D'^-_a = \eta D^+_a \circ h^{-1}(X)$$

$$\eta D'^-_a \eta(X) = \eta D^+_a \eta(X).$$

Then, by substituting $X$ for $h^{-1}(X)$ in the first one and $X$ for $h^1(X)$ in the second one, it follows that $\eta D'^-_a = \eta D^+_a$ and $\eta D'^-_a = \eta D^+_a$.

So the existence and uniqueness are proved. Such a $\eta$-compatible pair of $\eta$-DCDO’s satisfying Eq. (164) and Eq. (165) is just the $h$-deformation of the $g$-compatible pair of $\eta$-DCDO’s.

By following analogous steps we can also prove that such a $g$-compatible pair of $\eta$-DCDO’s satisfying Eq. (164) and Eq. (165) is just the $h^{-1}$-deformation of the $\eta$-compatible pair of $\eta$-DCDO’s.

In the fourth paper of this series we study the relation between the curvature and torsion tensors of a pair $(\eta D^+_a, \eta D^-_a)$ and its deformation $(g D^+_a, g D^-_a)$.

4 Derivative Operators in Metric and Geometric Structures

4.1 Ordinary Hodge Coderivatives

Let $U$ be an open subset of $U_\mu$, and let $(U, g)$ be a metric structure on $U$. Let us take any pair of reciprocal frame fields on $U$, say $\{e_\mu\}, \{e^\nu\}$, i.e., $e_\mu \cdot e^\nu = \delta^\nu_{\mu}$. In particular, the fiducial frame field, namely $\{b_\mu\}$, due to its orthonormality, i.e., $b_\mu \cdot b_\nu = \delta_{\mu \nu}$, is a self-reciprocal frame field, i.e., $b^\mu = b_\mu$. We note also that $\{b_\mu\}$ is an ordinarily constant frame field on $U$, i.e.,

$$a \cdot \partial_\mu b_\mu = 0, \text{ for each } \mu = 1, \ldots, n. \quad (166)$$

Associated to $\{e_\mu\}, \{e^\nu\}$, the smooth pseudoscalar field on $U$, namely $\tau$, defined by

$$\tau = \sqrt{e_\Lambda \cdot e^\Lambda}, \quad (167)$$

where $e_\Lambda = e_1 \wedge \ldots \wedge e_n \in \mathcal{M}^n(U)$ and $e^\Lambda = e^1 \wedge \ldots \wedge e^n \in \mathcal{M}^n(U)$, is said to be the standard volume pseudoscalar field for the local coordinate system $(U_\sigma, \phi_\sigma)$. 28
Such \(\tau \in \mathcal{M}^n(U)\) has the fundamental property
\[
\tau \cdot \tau = \tau \cdot \bar{\tau} = \tau \bar{\tau} = 1.
\] (168)

It follows from the obvious result \(e_\lambda \cdot e^\lambda = 1\).

From Eq. (168) we can get an expansion formula for smooth pseudoscalar fields on \(U\), i.e.,
\[
I = (I \cdot \tau)\tau.
\] (169)

In particular, because of the obvious properties \(b_\lambda \cdot b_\lambda = 1\) and \(b^\wedge = b_\lambda\), the standard volume pseudoscalar field associated to \(\{b_\mu\}\) is just \(b_\lambda\). It will be called the **canonical volume pseudoscalar field** for \((U_o, \phi_o)\).

We emphasize that \(b_\lambda\) is an *ordinarily constant* smooth pseudoscalar field on \(U\), i.e.,
\[
a \cdot \partial_\alpha b_\lambda = 0.
\] (170)

Eq. (170) can be proved by using Eq. (166) and the Leibniz rule for the exterior product of smooth multivector fields.

By using Eq. (170) and the general Leibniz rule for \(a \cdot \partial_\alpha\) we can deduce the remarkable property
\[
a \cdot \partial_\alpha (b_\lambda \ast X) = b_\lambda \ast (a \cdot \partial_\alpha X),
\] (171)

where \(\ast\) means either exterior product or any canonical product of smooth multivector fields.

On the other hand we have that all \((\{e_\mu\}, \{e^\mu\})\) must be necessarily an **extensor-deformation** of \(\{b_\mu\}\). This statement means that there exists a non-singular smooth \((1,1)\)-extensor field on \(U\), say \(\varepsilon\), such that
\[
e_\mu = \varepsilon(b_\mu),
\] (172)
\[
e^\mu = \varepsilon^*(b_\mu), \text{ for each } \mu = 1, \ldots, n.
\] (173)

Then, by putting Eq. (172) and Eq. (173) into Eq. (167), we have that
\[
\tau = (\varepsilon(b_\lambda) \cdot \varepsilon(b_\lambda))^{1/2} \varepsilon^*(b_\lambda) = (\det^2 [\varepsilon] b_\lambda \cdot b_\lambda)^{1/2} \det^{-1} [\varepsilon] b_\lambda = \text{sgn}(\det[\varepsilon]) b_\lambda,
\]
i.e.,
\[
\tau = \pm b_\lambda.
\] (174)

From Eq. (170) and Eq. (174), by taking into account Eq. (174), we have two remarkable properties
\[
a \cdot \partial_\alpha \tau = 0,
\] (175)
\[
a \cdot \partial_\alpha (\tau \ast X) = \tau \ast (a \cdot \partial_\alpha X), \text{ for all } X \in \mathcal{M}(U).
\] (176)

Associated to \((\{e_\mu\}, \{e^\mu\})\), the smooth pseudoscalar field on \(U\), namely \(\tau^g\), defined by
\[
\tau^g = \sqrt{\varepsilon_\lambda \cdot \varepsilon_\lambda} e^\lambda = \sqrt{|\det[g]|} \tau,
\] (177)
will be said to be a metric volume pseudoscalar field for \((U_o, \phi_o)\).

Such \(\tau \in \mathcal{M}^n(U)\) satisfies the basic property

\[
\tau \cdot g^{-1} \tau = \tau \cdot g^{-1} \tau = -1.
\]

(178)

In order to prove it we should recall that \(\text{sgn}(\det[g]) = (-1)^q\), where \(q\) is the number of negative eigenvalues of \(g\).

An expansion formula for smooth pseudoscalar fields on \(U\) can be also obtained from Eq. (178), i.e.,

\[
I = (-1)^q (I \cdot \tau)_{g^{-1}}.
\]

(179)

Associated to \(\tau\), the smooth extensor field on \(U\), namely \(*\), defined by

\[
* X = \bar{X} \cdot \tau = \bar{X} \tau,
\]

(180)

will be called the standard Hodge extensor field on \(U\).

Such \(*\) is non-singular and its inverse \(*^{-1}\) is given by

\[
*^{-1} X = \tau \cdot \bar{X} = \tau \bar{X}.
\]

(181)

By using a property analogous to that result given by Eq. (176) we can easily prove that the standard Hodge extensor field is ordinarily constant, i.e.,

\[
a \cdot \partial_o * = 0.
\]

(182)

Also,

\[
a \cdot \partial_g *^{-1} = 0.
\]

(183)

Associated to \(\tau\), the smooth extensor field on \(U\), namely \(*\), defined by

\[
* X = \bar{X} \cdot \tau = \bar{X} \tau = \sqrt{\det[g]} g^{-1} (\bar{X}) \cdot \tau = \sqrt{\det[g]} g^{-1} (\bar{X}) \tau,
\]

(184)

will be called the metric Hodge extensor field on \(U\). Of course, it is associated to the metric structure \((U, g)\).

Such \(*\) is also non-singular and its inverse, namely \(*^{-1}\), is given by

\[
*^{-1} X = (-1)^q \tau \cdot g^{-1} \bar{X} = (-1)^q \tau \cdot g^{-1} \bar{X} = (-1)^q g^{-1} \sqrt{\det[g]} \tau^{-1} \bar{X} = (-1)^q g^{-1} \sqrt{\det[g]} \tau^{-1} \bar{X}.
\]

(185)

4.2 Duality Identities

We present in this subsection two interesting and useful formulas which relate the ordinary curl \(\partial_o \wedge\) to the ordinary contracted divergence \(\partial_o \cdot\).
For all \( X \in \mathcal{M}(U) \) it holds
\[
\tau(\partial_o \wedge X) = (-1)^{n+1} \partial_o,\tau(X). \quad (186)
\]

To prove Eq.\( (186) \) we will use the so-called duality identity \( I(a \wedge Y) = (-1)^{n+1} a,\tau(Y) \), where \( a \in \mathcal{U}_o \), \( I \in \bigwedge^n \mathcal{U}_o \) and \( Y \in \bigwedge \mathcal{U}_o \). By recalling the known identities \( \partial_a \wedge (a \cdot \partial_o X) = \partial_a \wedge X \) and \( \partial_o,\tau(a \cdot \partial_o X) = \partial_o,\tau X \), and using Eq.\( (176) \), we get
\[
\tau(\partial_o \wedge X) = \tau(\partial_a \wedge (a \cdot \partial_o X)) = (-1)^{n+1} \partial_o,\tau(a \cdot \partial_o X),
\]
\[
= (-1)^{n+1} \partial_o,\tau(a \cdot \partial_o (\tau X)) = (-1)^{n+1} \partial_o,\tau(X).
\]

ii. For all \( X \in \mathcal{M}(U) \) it holds
\[
\tau(\partial_o \wedge X) = (-1)^{n+1} \partial_o,\tau(X). \quad (187)
\]

We prove \( (187) \) using the identity \( I^{-1} Y = \det^{-1} [g] g(\tau Y) \) and Eq.\( (186) \).

### 4.3 Hodge Duality Identities

Now, we present two noticeable identities which relate the curl \( \partial_o \wedge \) to the contracted divergence \( \partial_o,\tau \) involving the standard and the metric Hodge extensor fields \( \ast \) and \( \ast_g \).

i. For all \( X \in \mathcal{M}(U) \) it holds
\[
\ast^{-1}(\partial_o \wedge (\ast X)) = -\partial_o,\tau X. \quad (188)
\]

To show Eq.\( (188) \) we will use the duality identity given by Eq.\( (186) \). By using Eq.\( (181) \) and Eq.\( (180) \), and recalling the identities \( \partial_o \wedge Y = \partial_o \wedge Y \) and \( \partial_o \wedge X = \partial_o \wedge X \), and the obvious property \( \tau \tau = (-1)^n \), we get
\[
\ast^{-1}(\partial_o \wedge (\ast X)) = \tau(\partial_o \wedge (X \tau)) = \tau(\partial_o \wedge (\tau X)),
\]
\[
= (-1)^{n+1} \partial_o,\tau X = -\partial_o,\tau X.
\]

ii. For all \( X \in \mathcal{M}(U) \) it holds
\[
\ast^{-1}(\partial_o \wedge (\ast X)) = -\frac{1}{\sqrt{|\det[g]|}} g(\partial_o,\tau X). \quad (189)
\]

This follows from appropriate use of the duality identity given by Eq.\( (187) \).

Indeed, a straightforward calculation using Eq.\( (185) \) and Eq.\( (184) \) allows us to

---

\(^9\)In order to prove it we should use the following identities: \( I^{-1} Y = \det^{-1} [g] g(\tau Y) \), \( X \tau^{-1}(Y) = \mathcal{L}^{-1}(\mathcal{L}(X),Y) \) and \( \mathcal{L}^{-1}(I) = \det^{-1} [I] \), and so forth.
\[ \ast^{-1}(\partial_\alpha \wedge (\ast X)) = (-1)^q \sqrt{|\det g|} \tau g^{-1} (\partial_\alpha \wedge (X)) \]
\[ = (-1)^q \sqrt{|\det g|} \tau g^{-1} \partial_\alpha \wedge (\sqrt{|\det g|} \tau g^{-1}(X)) \]
\[ = (-1)^q \sqrt{|\det g|} \frac{(-1)^{n+1}}{\det g} g(\partial_\alpha, (\sqrt{|\det g|} \tau g^{-1}(X))) , \]
\[ = - \frac{1}{\sqrt{|\det g|}} g(\partial_\alpha, (\sqrt{|\det g|} g^{-1}(X))). \]

### 4.4 Ordinary Hodge Coderivative Operators

We introduce the so-called standard Hodge derivative operator \( \delta : \mathcal{M}(U) \rightarrow \mathcal{M}(U) \) such that
\[ \delta X = \ast^{-1}(\partial_\alpha \wedge (\ast X)). \]  
Its basic property is
\[ \delta X = - \partial_\alpha \wedge X. \]  
It immediately follows from the Hodge duality identity given by Eq. (188).

We introduce the so-called metric Hodge coderivative operator \( \delta : \mathcal{M}(U) \rightarrow \mathcal{M}(U) \) such that
\[ \delta g^X = \ast g^{-1}(\partial_\alpha \wedge (\ast X)). \]  
It satisfies the basic property
\[ \delta g^X = - \frac{1}{\sqrt{|\det g|}} g(\partial_\alpha, (\sqrt{|\det g|} g^{-1}(X))), \]  
which is an immediate consequence of the Hodge duality identity given by Eq. (189).

### 4.5 Levi-Civita Geometric Structure

We recall from Section 2 that the Levi-Civita connection field is the smooth vector elementary 2-extensor field on \( U \), namely \( \lambda \), defined by
\[ \lambda(a, b) = \frac{1}{2} g^{-1} \circ (a \cdot \partial_\alpha g)(b) + \omega_0(a) \times b, \]  
where \( \omega_0 \) is the smooth \((1, 2)\)-extensor field on \( U \) given by
\[ \omega_0(a) = - \frac{1}{4} g^{-1}(\partial_b \wedge \partial_c) a \cdot ((b \cdot \partial_\alpha g)(c) - (c \cdot \partial_\alpha g)(b)). \]  
Such \( \omega_0 \) satisfies
\[ \omega_0(a) \times b \cdot c = \frac{1}{2} a \cdot ((b \cdot \partial_\alpha g)(c) - (c \cdot \partial_\alpha g)(b)). \]
The open set $U$ endowed with $\lambda$ and $g$, namely $(U, \lambda, g)$, is a geometric structure on $U$, a statement that means that Levi-Civita parallelism structure $(U, \lambda)$ is compatible with the metric structure $(U, g)$. Or equivalently, the pair of $a$-DCDO’s associated to $(U, \lambda)$, namely $(D_a^+, D_a^-)$, is $g$-compatible.

The Levi-Civita $a$-DCDO’s $D_a^+$ and $D_a^-$ are defined by

$$D_a^+ X = a \cdot \partial_a X + \Lambda_a(X), \quad (197)$$

$$D_a^- X = a \cdot \partial_a X - \Lambda_a^\perp(X). \quad (198)$$

Note that $\Lambda_a$ is the so-called generalized of $\lambda_a$. The latter is the so-called $a$-directional connection field associated to $\lambda$, given by $\lambda_a(b) = \lambda(a, b)$.

We present now two pairs of noticeable properties of $\lambda$.

i. The scalar divergence of $\lambda_a(b)$ with respect to the first variable, namely $\partial_a \cdot \lambda_a(b)$, and the curl of $\lambda_a^\perp(b)$ with respect to the first variable, namely $\partial_a \wedge \lambda_a^\perp(b)$, are given by

$$\partial_a \cdot \lambda_a(b) = \frac{1}{\sqrt{|\det[g]|}} b \cdot \partial_a \sqrt{|\det[g]|}, \quad (199)$$

$$\partial_a \wedge \lambda_a^\perp(b) = 0. \quad (200)$$

In order to prove the first result we will use the formula $\tau^* (\partial_a) \cdot (a \cdot \partial_a \tau)(n) = \det^{-1} |\tau| a \cdot \partial_a \det[\tau]$, valid for all non-singular smooth $(1, 1)$-extensor field $\tau$. By using the symmetry property $(g^{-1})^\dagger = g^{-1}$ and Eq. (196), we can write

$$\partial_a \cdot \lambda_a(b) = \frac{1}{2} \partial_a \cdot (g^{-1} \circ (a \cdot \partial_a g)(b) + \partial_a \cdot (\omega_0(a) \times b)$$

$$= \frac{1}{2} g^{-1} (\partial_a) \cdot (a \cdot \partial_a g)(b) + \omega_0(g^{-1}(\partial_a)) \times b \cdot a$$

$$= \frac{1}{2} g^{-1} (\partial_a) \cdot (b \cdot \partial_a g)(a) = \frac{1}{2} \frac{1}{\det[g]} b \cdot \partial_a \det[g].$$

Then, recalling the identity $(f)^{-1} b \cdot \partial_a f = 2(|f|)^{-1/2} b \cdot \partial_a (|f|)^{1/2}$ valid for all non-zero $f \in S(U)$, the expected result immediately follows.

To prove the second result we only need to take into account the symmetry property $\lambda_a(b) = \lambda_b(a)$. We have

$$\partial_a \wedge \lambda_a^\perp(b) = \partial_a \wedge \partial_a (n \cdot \lambda_a^\perp(b)) = \partial_a \wedge \partial_a (\lambda_a(n) \cdot b) = 0.$$

ii. The left contracted divergence of $\lambda_a(X)$ with respect to $a$, namely $\partial_a \wedge \lambda_a(X)$, and the curl of $\lambda_a^\perp(X)$ with respect to $a$, namely $\partial_a \wedge \lambda_a^\perp(X)$, are given by

$$\partial_a \wedge \lambda_a(X) = \frac{1}{\sqrt{|\det[g]|}} (\partial_a \sqrt{|\det[g]|}) \wedge X, \quad (201)$$

$$\partial_a \wedge \lambda_a^\perp(X) = 0. \quad (202)$$
In order to prove Eq. (201), we will use the multivector identities \( v, X \subset Y \) = \( (v, X) \wedge Y + \hat{X} \wedge (v, Y) \) and \( X, Y \subset Z \) = \( (X \wedge Y) \wedge Z \), where \( v \in \mathcal{M} \) and \( X, Y, Z \in \bigwedge \mathcal{U} \). A straightforward calculation using Eq. (199) allows us to get
\[
\partial_n \Lambda_n(X) = \partial_n \cdot \lambda_n(\partial_n) (\Lambda_n X) - \lambda_n(\partial_n) \wedge (\partial_n \Lambda_n X))
\]
\[
= \frac{1}{\sqrt{|\det[g]|}} \partial_n \cdot \lambda_n(\partial_n) (\Lambda_n X) - \lambda_n(\partial_n) \wedge (\partial_n X),
\]
\[
= \frac{1}{\sqrt{|\det[g]|}} \partial_n(n \cdot \lambda_n \sqrt{|\det[g]|}) \Lambda_n X - \lambda_n(\partial_n) \wedge ((a \wedge n) \Lambda_n X).
\]
Then, recalling the identity \( \partial_n(n \cdot \partial_n Y) = \partial_n Y \) and the symmetry property \( \lambda_n(b) = \lambda_n(a) \), we get the required result.

The proof of the second property follows immediately by using Eq. (200), and recalling that the adjoint of generalized is equal to the generalized of adjoint, i.e., \( \Lambda_n^\dagger(X) = \lambda_n^\dagger(\partial_n) \wedge (n \Lambda_n X) \).

4.6 Levi-Civita Derivatives

We introduce now the canonical covariant divergence operator, \( D^+ \triangledown : \mathcal{M}(U) \rightarrow \mathcal{M}(U) \) such that
\[
D^+ \triangledown X = \partial_n \wedge (D_a^+ X),
\]
i.e., \( D^+ \triangledown X = e^\mu \wedge (D_a^+ X) = e^\mu \wedge (D_a^+ X) \), where \( \{e_\mu\} \) is any pair of canonical reciprocal frame fields on \( U \).

Its basic property is
\[
D^+ \triangledown X = \frac{1}{\sqrt{|\det[g]|}} \partial_n \wedge (\sqrt{|\det[g]|}) X.
\]
Indeed, using the identity \( \partial_n \wedge (a \cdot \partial_n X) = \partial_n \wedge X \) and Eq. (201) into the definition given by Eq. (197), we get
\[
D^+ \triangledown X = \partial_n \wedge (a \cdot \partial_n X) + \partial_n \wedge \Lambda_n(X),
\]
\[
= \partial_n X + \frac{1}{\sqrt{|\det[g]|}} \partial_n \wedge (\sqrt{|\det[g]|}) \Lambda_n X.
\]
So, by recalling the identity \( \partial_n \wedge (f Y) = (\partial_n f) \wedge X + f(\partial_n X) \), for all \( f \in \mathcal{S}(U) \) and \( Y \in \mathcal{M}(U) \), we can get the proof for this remarkable property.

The so-called metric covariant divergence, covariant curl and metric covariant gradient operators, namely \( D^+ \triangledown, D^- \wedge \) and \( D^- g^{-1} \), map also smooth multivector fields to smooth multivector fields and are defined by
\[
D^+ \triangledown g^{-1} X = \partial_a \wedge (D_a^+ X) = g^{-1}(\partial_a) \wedge (D_a^+ X),
\]
\[
D^- \wedge g^{-1} X = \partial_a \wedge (D_a^- X),
\]
\[
D^- g^{-1} X = \partial_a \wedge (D_a^- X).
\]
These operators are related by
\[ D_{g^{-1}}^{-} X = D_{g^{-1}}^{-} \cdot Y + D_{g^{-1}}^{-} \cdot X. \] (208)

The basic properties of the metric covariant divergence are
\[ D_{g^{-1}}^{-} \cdot Y = g(D^{-} g^{-1}(X)), \] (209)
\[ D_{g^{-1}}^{-} \cdot X = \frac{1}{\sqrt{|\det g|}} g(\partial_o \cdot (\sqrt{|\det g|} g^{-1}(X))). \] (210)

Eq. (209) follows from the fundamental property \( D_{g}(g(X)) = g(D_{a}^{+}(X)) \) which holds for any \( g \)-compatible pair of \( a \)-\( DCDO \)'s, by using the identity \( X_{\cdot} = (\cdot)^{T}(X)_{\cdot} \). Eq. (210) is deduced by using Eq. (209) and Eq. (204).

A remarkable property which follows from Eq. (210) is
\[ D_{g}^{-} \cdot Y = 0. \] (211)
In order to prove it we should use the known identity \( \partial_o \cdot (\partial_o \cdot Y) = 0 \).

By comparing Eq. (193) and Eq. (210) we get
\[ D_{g^{-1}}^{-} \cdot X = -\delta X. \] (212)

The basic property for the covariant curl is
\[ D_{g^{-1}}^{-} \cdot X = \partial_o \cdot X. \] (213)
It immediately follows by using the identity \( \partial_o \cdot (a \cdot \partial_o X) = \partial_o \cdot X \) and Eq. (202) into the definition given by Eq. (198).

The four derivative-like operators defined by Eq. (203), Eq. (205), Eq. (206) and Eq. (207) will be called the Levi-Civita derivatives. They are involved in three useful identities which are used in the Lagrangian theory of multivector fields (34). These are,
\[
(\partial_o \cdot X)_{g^{-1}} \cdot Y + X_{g^{-1}} \cdot (D_{g^{-1}}^{-} \cdot Y) = \frac{1}{\sqrt{|\det g|}} \partial_o \cdot (\sqrt{|\det g|} \partial_n (n \cdot X)_{g^{-1}} \cdot Y),
\] (214)
\[
(D_{g^{-1}}^{-} \cdot Y)_{g^{-1}} \cdot X + X_{g^{-1}} \cdot (\partial_o \cdot Y) = \frac{1}{\sqrt{|\det g|}} \partial_o \cdot (\sqrt{|\det g|} \partial_n (n \cdot X_{g^{-1}} \cdot Y),
\] (215)
\[
(D_{g^{-1}}^{-} \cdot X)_{g^{-1}} \cdot Y + X_{g^{-1}} \cdot (D_{g^{-1}}^{-} \cdot Y) = \frac{1}{\sqrt{|\det g|}} \partial_o \cdot (\sqrt{|\det g|} \partial_n (n \cdot X)_{g^{-1}} \cdot Y). \] (216)
4.7 Gauge (Deformed) Derivatives

Let \( h \) be a gauge metric field for \( g \). This statement, as we already know means that there is a smooth \((1, 1)\)-extensor field \( h \) such that \( g = h^\dagger \circ \eta \circ h \), where \( \eta \) is an orthogonal metric field with the same signature as \( g \). As we know, associated to a Levi-Civita \( a\)-DCDO’s, namely \((D^+_a, D^-_a)\), there must be an unique \( \eta \)-compatible pair of \( a\)-DCDO’s, namely \((D^+_a, D^-_a)\), given by the following formulas

\[
D^+_a X = h(D^+_a h^{-1}(X)),
\]
\[
D^-_{a^*} X = h^*(D^+_a h^*(X)).
\]

These equations say that \((D^+_a, D^-_a)\) is the \( h \)-deformation of \((D^+_a, D^-_a)\). So, \( D^+_a \) and \( D^-_{a^*} \) will be called the gauge (deformed) covariant derivatives associated to \( D^+_a \) and \( D^-_a \).

We present here two noticeable properties of \( D^+_a \).

i. For all \( a, b, c \in \mathcal{V}(U) \), it holds

\[
(D^+_a b) \cdot c = [h(a), h^{-1}(b), h^{-1}(c)].
\]

To prove Eq.\((219)\) we use Eq.\((217)\), the identity \((D^+_a b) \cdot c = \sum c a b \), and the definition of the Christoffel operator of second kind, i.e., \([c a b] = [a, b, g^{-1}(c)]\).

Indeed, we can write

\[
(D^+_a b) \cdot c = D^+_a (h^{-1}(b)) \cdot h^\dagger \circ \eta(c)
\]

\[
= \left\{ \begin{array}{l}
  h^\dagger \circ \eta(c) \\
  h(a), h^{-1}(b)
\end{array} \right\} = [h(a), h^{-1}(b), g^{-1} \circ h^\dagger \circ \eta(c)],
\]

and recalling that \( g^{-1} = h^{-1} \circ \eta \circ h^* \), the required result immediately follows.

ii. There exists a smooth \((1, 2)\)-extensor field on \( U_\partial \), namely \( \Omega_0 \), such that

\[
D^+_a X = a \cdot \partial_\alpha X + \Omega_0(a) \times X.
\]

Such \( \Omega_0 \) is given by

\[
\Omega_0(a) = -\frac{1}{2} \eta(\partial_b \wedge \partial_c)[a, h^{-1}(b), h^{-1}(c)].
\]

The proof of this important result is as follows. First we prove a particular case of the above property, i.e.,

\[
D^+_a b = a \cdot \partial_\alpha b + \Omega_0(a) \times b.
\]

To do so, we will use the following properties of the Christoffel operator of first kind \([a, b, c] - [a, c, b] = 2[a, b, c] - a \cdot \partial_\alpha (b \cdot c) \), and \([a, b + b', c] = [a, b, c] + \]

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[a, b’, c] and [a, fb, c] = f[a, b, c] + (a \cdot \partial_a f)b \cdot c. A straightforward calculation permit us to write

\[ \Omega_o(a) \times b \cdot c = (b \wedge c) \cdot \Omega_o(a) = \frac{1}{2}(b \wedge c) \cdot (\partial_p \wedge \partial_q)[a, h^{-1}(p), h^{-1}(q)] \]

\[ = \frac{1}{2} \det \begin{bmatrix} b \cdot \partial_p & b \cdot \partial_q \\ c \cdot \partial_p & c \cdot \partial_q \end{bmatrix} [a, h^{-1}(p), h^{-1}(q)] \]

\[ = \frac{1}{2} b \cdot \partial_p c \cdot \partial[a, h^{-1}(p), h^{-1}(q)] - b \cdot \partial_q c \cdot \partial_p[a, h^{-1}(p), h^{-1}(q)] \]

\[ = \frac{1}{2} b \cdot \partial_p c \cdot \partial_q([a, h^{-1}(p), h^{-1}(q)] - [a, h^{-1}(q), h^{-1}(p)]) \]

\[ = \frac{1}{2} b \cdot \partial_p c \cdot \partial_q([2[a, h^{-1}(p), h^{-1}(q)] - a \cdot \partial_o(h^{-1}(p)) \cdot h^{-1}(q))] \]

\[ = b \cdot \partial_p c \cdot \partial_q(p \cdot b[\eta(a, h^{-1}(b)), h^{-1}(c)] + (a \cdot \partial_o)p \cdot q) \]

\[ = b \cdot b[\eta(a, h^{-1}(b), h^{-1}(c)) + (a \cdot \partial_o)b \cdot q - (a \cdot \partial_o)b \cdot q] \]

\[ = [a, h^{-1}(b), h^{-1}(c)] - (a \cdot \partial_o)b \cdot q \]

hence, by using Eq. (211), the particular case given by Eq. (222) immediately follows. Now, we can prove the general case of the above property.

As we can see, the \( a \)-directional connection field on \( U \) for \( D^+_a, D^-_a \) is given by \( b \mapsto \Omega_o(a) \times b \). Then, its generalized (extensor field) must be given by \( X \mapsto \Omega_o(a) \times \partial_b \wedge (b \wedge X) \). But, by recalling the noticeable identity \( (B \times \partial_b)(b \wedge X) = B \times X \), where \( B \in \bigwedge^2 U_o \) and \( X \in \bigwedge U_o \), we find that it can be written as \( X \mapsto \Omega_o(a) \times X \).

We introduce now the gauge covariant divergence, gauge covariant curl and gauge covariant gradient operators, namely \( D^-_{\partial_o} \), \( D^- \wedge \) and \( D^- \). They all map smooth multivector fields to smooth multivector fields and are defined by

\[ D^-_{\partial_o} X = h^* (\partial_o)(D^-_{h^{-1}} X), \quad (223) \]

\[ D^- \wedge X = h^*(\partial_o) \wedge (D^-_{h^{-1}} X), \quad (224) \]

\[ D^- X = h^*(\partial_o) (D^-_{h^{-1}} X). \quad (225) \]

It is obvious that the relationship among them is given by

\[ D^-_{\partial_o} X = D^-_{\partial_o} X + D^- \wedge X. \quad (226) \]

Their basic properties are given by the following golden formulas

\[ h^*(D^-_{\partial_o} X) = D^-_{h^{-1}} h^*(X) \]

\[ h^*(D^- \wedge X) = D^- \wedge h^*(X) \]

\[ h^*(D^- X) = D^- h^*(X). \]

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These formulas can be proved by using the golden formula deduced in [2],
\[ h^*(X \ast Y) = h^*(X) \ast h^*(Y), \]
where \( X, Y \in \bigwedge^k \mathcal{U}, \) and \( \ast \) means either exterior product or any \( g^{-1} \)-product of smooth multivector fields, and analogously for \( \eta \). It is also necessary to take into account the master formulas \( g = h^1 \circ \eta \circ h \) and \( g^{-1} = h^{-1} \circ \eta \circ h^* \), and the relationship between \( D^+ \) and \( D^- \), i.e.,
\[ \eta(D^+ a(X)) = D^- \eta \circ h^{-1} \circ \eta(X). \]

From Eq.(210) and Eq.(213) by using Eq.(227) and Eq.(228) we find the interesting identities
\[ D^- \eta X = \frac{1}{\text{det}[h]} \eta \circ h(\partial_o(\text{det}[h]h^{-1} \circ \eta(X))), \]
\[ D^- \wedge X = h^*(\partial_o \wedge h^1(X)). \]

### 4.8 Covariant Hodge Coderivative

Let \((U, \gamma, g)\) be a geometric structure on \( U \), and let us denote by \((D^+_a, D^-_a)\) the \( g \)-compatible pair of \( a \)-DCDO’s associated to \((U, \gamma, g)\). We will present two noticeable properties which are satisfied by the second \( a \)-DCDO.

**i.** \( \tau \) is a covariantly constant smooth pseudoscalar field on \( U \), i.e.,
\[ D^-_a \tau = 0. \]

To show Eq.(232) we will use the basic property and the expansion formula given by Eq.(178) and Eq.(179). From the Ricci-like theorem for \( D^-_a \), we have that
\[ (D^-_a \tau) \cdot g^{-1} + \tau \cdot g^{-1} (D^-_a \tau) = 0, \]
i.e., \( (D^-_a \tau) \cdot g^{-1} = 0 \). Then, \( D^-_a \tau = (-1)^g((D^-_a \tau) \cdot g^{-1}) \tau = 0 \).

**ii.** For all \( X \in \mathcal{M}(U) \) it holds
\[ D^-_a (\tau \ast g^{-1} X) = \tau \ast g^{-1} D^-_a(X), \]
where \( \ast \) means either exterior product or any \( g^{-1} \)-product of smooth multivector fields.

This result follows at once using Eq.(232) and the general Leibniz rule for \( D^-_a \).

We introduce now the **metric covariant divergence**, **covariant curl** and **metric covariant gradient operators**, namely \( D^-_a \), \( D^- \wedge \) and \( D^-_a \), which map smooth multivector fields to smooth multivector fields. They are defined by
\[ D^-_a \cdot g^{-1} X = \partial_a \cdot g^{-1} (D^-_a X), \]
\[ D^- \wedge X = \partial_a \wedge (D^-_a X), \]
\[ D^-_a \bigwedge g^{-1} X = \partial_a \bigwedge g^{-1} (D^-_a X). \]
Indeed, using Eq. (185) and Eq. (184), and recalling the identities

defined by

which follows trivially from Eq. (239).

iii. For all \( X \in \mathcal{M}(U) \) it holds

\[
\mathbf{t}_{g^{-1}}(\mathcal{D}^- \wedge X) = (-1)^{n+1}\mathcal{D}^-_{g^{-1}\mathbf{t}}(X).
\]  

(238)

To show Eq. (238) we will use the \( g^{-1} \)-duality identity \( I_{g^{-1}}(a \wedge Y) = (-1)^{n+1}a_{g^{-1}}(I_{g^{-1}}Y) \), where \( a \in \mathcal{U}_o, I \in \bigwedge^n \mathcal{U}_o \), and \( Y \in \bigwedge \mathcal{U}_o \). A straightforward calculation by taking into account Eq. (239) gives

\[
\tau_{g^{-1}}(\mathcal{D}^- \wedge X) = (-1)^{n+1}\partial_a_{g^{-1}}(\tau_{g^{-1}}(\mathcal{D}^- X))
\]

\[
= (-1)^{n+1}\partial_a_{g^{-1}}(\mathcal{D}^- (\tau_{g^{-1}} X)),
\]

\[
= (-1)^{n+1}\mathcal{D}^-_{g^{-1}\tau}(\tau_{g^{-1}} X).
\]

iv. For all \( X \in \mathcal{M}(U) \) it holds

\[
\ast^{-1}_g(\mathcal{D}^- \wedge (\ast X)) = -\mathcal{D}^-_{g^{-1}\tau_X}.
\]  

(239)

This follows by appropriate use of the duality identity given by Eq. (238). Indeed, using Eq. (186) and Eq. (184), and recalling the identities \( \mathcal{D}^- \wedge X = \mathcal{D}^- \wedge X \) and \( XY = YX \), and the obvious property \( \tau_{g^{-1}} = (-1)^{n+q} \), we get

\[
\ast^{-1}_g(D^- \wedge (\ast X)) = (-1)^q \tau_{g^{-1}}(\mathcal{D}^- \wedge \hat{X} \tau_{g^{-1}})
\]

\[
= (-1)^q \tau_{g^{-1}}(\mathcal{D}^- \wedge (\hat{X} \tau_{g^{-1}}))
\]

\[
= (-1)^q (-1)^{n+1} \mathcal{D}^-_{g^{-1}\tau}(\tau_{g^{-1}} \tau_{g^{-1}} X),
\]

\[
= -\mathcal{D}^-_{g^{-1}\tau_X}.
\]

We introduce now the **covariant Hodge coderivative operator**, denoted \( \Delta_g \), and defined by

\[
\Delta_g : \mathcal{M}(U) \to \mathcal{M}(U),
\]

\[
\Delta_g X = \ast^{-1}_g(\mathcal{D}^- \wedge (\ast \hat{X})),
\]

(240)

Its basic property is

\[
\mathcal{D}^- g^{-1} \tau_X = -\Delta_g X,
\]

which follows trivially from Eq. (239).
In this paper we studied how the geometrical calculus of multivector and extensor fields can be successfully applied to the study of the differential geometry of an arbitrary smooth manifold \( M \) equipped with a general connection \( \nabla \) and with a metric tensor \( g \). In Section 2 we introduced the notion of a parallelism structure on \( U \subset M \), i.e., a pair \((U, \gamma)\) where \( \gamma \) is a connection extensor field on \( U \subset M \) (representing there the restriction in \( U \) of some connection defined on \( M \)), and the theory of \( a \)-directional covariant derivatives of the representatives of multivector and extensor fields on \( U \), analyzing the main properties satisfied by these objects. We also give in Section 2 a novel and intrinsic presentation (i.e., one that does not depend on a chosen orthonormal moving frame) of Cartan theory of the torsion and curvature fields and of Cartan’s structure equations. We next introduce a metric metric structure \((U, g)\) in \( U \subset M \) and corresponding Christoffel operators\(^{10}\) and the Levi-Civita connection field. In Section 3 we studied in details the metric compatibility of covariant derivatives and proved several important formulas that are useful for applications of the formalism. In Section 4 several operator derivatives in geometric structures \((U, \gamma, g)\), like ordinary and covariant Hodge coderivatives, are presented. The noticeable concept of gauge (deformed) derivative is introduced and a crucial result (Theorem 2) involving pairs of distinct metric compatible derivatives is proved. In Section 4 we proved several useful duality identities. The relation between all the concepts introduced has been carefully investigated.

6 Appendix

Let \( U \) be an open subset of \( U_o \), and let \((U, \phi)\) and \((U, \phi')\) be two local coordinate systems on \( U \) compatibles with \((U_o, \phi_o)\). As we know there must be two pairs of reciprocal frame fields on \( U \). The covariant and contravariant frame fields \( \{b_\alpha \cdot \partial x_o\} \) and \( \{\partial_\alpha x^o\} \) associated to \((U, \phi)\), and those ones \( \{b_\mu \cdot \partial' x_o\} \) and \( \{\partial_\alpha x'^o\} \) associated to \((U, \phi')\).

A.1 The \( n^3 \) smooth scalar fields on \( U \), namely \( \Gamma_{\alpha \beta}^{\gamma} \), defined by

\[
\Gamma_{\alpha \beta}^{\gamma} = \Gamma^+(b_\alpha \cdot \partial x_o, b_\beta \cdot \partial x_o) \cdot \partial_\alpha x^\gamma
\]

(A1)

correspond to the classically so-called coefficients of connection, of course, associated to \((U, \phi)\). The coefficients of connection associated to \((U, \phi')\) are given by

\[
\Gamma_{\mu \nu}^{\lambda} = \Gamma^+(b_\mu \cdot \partial' x_o, b_\nu \cdot \partial' x_o) \cdot \partial_\alpha x^\lambda
\]

(A2)

We check next what is the law of transformation between them. We will employ the simplified notations: \( b_\mu \cdot \partial' x_o = \frac{\partial x^\sigma}{\partial x'^\mu} b_\sigma \) and \( \partial_\alpha x^\sigma = \frac{\partial x^\alpha}{\partial x'^\sigma} \), etc.

Then, by recalling the expansion formulas for smooth vector fields, \( v = (v \cdot b_\mu \cdot \partial' x_o) \cdot \partial_\alpha x^\lambda \), these objects generalize the Christoffell symbols of the standard formalism that are defined for coordinate basis vectors.
\[ \partial_x x^\alpha \partial_x x_\alpha \text{ and } v = (v \cdot b \cdot \partial_x) \partial_{x^\gamma}, \text{ using Eqs. (30), (31), (32) and (33), and recalling the remarkable identity } (b \cdot \partial_x) \cdot \partial X = b \cdot \partial X, \text{ where } X \in \mathcal{M}(U) \text{ (see [1])}, \]

we can write

\[ \Gamma^\gamma_{\mu \nu} = \Gamma^\gamma_{\mu \nu} \left( \frac{\partial x^\alpha}{\partial x^\mu} b_\alpha \cdot \partial_x b_\beta \cdot \partial_x \right) \cdot \frac{\partial x^\gamma}{\partial x^\nu} \partial_x x_\gamma \]

\[ = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\gamma} \Gamma^\gamma_{\alpha \beta} \]

\[ + \frac{\partial x^\alpha}{\partial x^\mu} (b_\alpha \cdot \partial_x) \cdot \partial \left( \frac{\partial x^\gamma}{\partial x^\nu} \partial_x b_\beta \cdot \partial_x v_\nu \right) \]

\[ = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\gamma} \Gamma^\gamma_{\alpha \beta} + \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\gamma} \Gamma^\gamma_{\alpha \beta} \]

\[ = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\gamma} \Gamma^\gamma_{\alpha \beta} + \frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\nu} \frac{\partial x^\gamma}{\partial x^\gamma} \frac{\partial x^\gamma}{\partial x^\gamma}, \quad \text{(A3)} \]

It is the well-known law of transformation for the classical coefficients of connection associated to each of the coordinate systems \{x^\mu\} and \{x'^\mu\}.

**A.2** Given a smooth vector field on \( U \), say \( v \), the covariant and contravariant components of \( v \) with respect to \( (U, \phi) \) and \( (U, \phi') \) are respectively given by

\[ v_\alpha = v \cdot (b_\alpha \cdot \partial_x), \quad \text{(A4)} \]

\[ v^\alpha = v \cdot \partial_x, \quad \text{(A5)} \]

\[ v'^\alpha = v \cdot (b'_\alpha \cdot \partial'_x), \quad \text{(A6)} \]

\[ v'^\alpha = v \cdot \partial'_x. \quad \text{(A7)} \]

We check the relationship between the covariant components of \( v \) with respect to each of the coordinate systems \{x^\mu\} and \{x'^\mu\}. By recalling the expansion formula \( w = (w \cdot \partial_x b) b \cdot \partial_x \), we have

\[ v'_\alpha = v \cdot (b'_\alpha \cdot \partial'_x \cdot \partial_x b_\beta \cdot \partial_x), \]

\[ v'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} v_\beta. \quad \text{(A8)} \]

It is the expected law of transformation for the covariant components of \( v \) associated to each of \{x^\mu\} and \{x'^\mu\}.

Analogously, by using the expansion formula \( w = (w \cdot b_\beta \cdot \partial_x) \cdot \partial_x x^\beta, \) we can get the classical law of transformation for the contravariant components of \( v \), i.e.,

\[ v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} v_\beta. \quad \text{(A9)} \]

**A.3** We see next which is the meaning of the classical covariant derivatives of the contravariant and covariant components of a smooth vector field. By using
the expansion formula \( v = v^\alpha b_\alpha \cdot \partial x_\alpha \), and Eqs. (21) and (22), and recalling once again the remarkable identity \((b_\mu \cdot \partial x_\alpha) \cdot \partial_o X = b_\mu \cdot \partial X\), where \( X \in \mathcal{M}(U) \), we can write

\[
(\nabla_{b_\nu, \partial x_\alpha}^+ v) \cdot \partial_o x^\lambda = (b_\mu \cdot \partial x_\alpha) (b_\alpha \cdot \partial x_\alpha \cdot \partial_o x^\lambda) \\
+ v^\alpha (\nabla_{b_\nu, \partial x_\alpha}^+ b_\alpha \cdot \partial x_\alpha) \cdot \partial_o x^\lambda \\
= \frac{\partial v^\alpha}{\partial x^\mu} \delta^\lambda_\alpha + v^\alpha \Gamma^\lambda_\mu \nu (b_\mu \cdot \partial x_\alpha) \cdot \partial_o x^\lambda \\
= \frac{\partial v^\lambda}{\partial x^\mu} + \Gamma^\lambda_\mu \nu v^\alpha,
\]

i.e.,

\[
(\nabla_{b_\nu, \partial x_\alpha}^+ v) \cdot \partial_o x^\lambda = v^\lambda_\mu. \quad (A10)
\]

By using the expansion formula \( v = v_\alpha \partial_o x^\alpha \), Eqs. (21) and (22), and Eq. (34), we get

\[
(\nabla_{b_\nu, \partial x_\alpha}^- v) \cdot b_\nu \cdot \partial x_\alpha = \frac{\partial v_\nu}{\partial x^\mu} - \Gamma^\alpha_\mu \nu v_\alpha,
\]

i.e.,

\[
(\nabla_{b_\nu, \partial x_\alpha}^- v) \cdot b_\nu \cdot \partial x_\alpha = v_\nu_\mu. \quad (A11)
\]

**A.4** Now, for instance, let us take a smooth \((1,1)\)-extensor field on \( U \), say \( t \). The covariant and contravariant components of \( t \) with respect to \((U, \phi)\) and \((U, \phi')\) are respectively defined to be

\[
t_{\mu \nu} = t(b_\mu \cdot \partial x_\alpha) \cdot b_\nu \cdot \partial x_\alpha, \quad (A12)
\]
\[
t^{\mu \nu} = t(\partial_\alpha x^\mu) \cdot \partial_\alpha x^\nu, \quad (A13)
\]
\[
t^{\mu \nu'} = t(b_\mu \cdot \partial' x_\alpha) \cdot b_\nu \cdot \partial' x_\alpha, \quad (A14)
\]
\[
t^{\mu' \nu'} = t(\partial_\alpha x^{\mu'}) \cdot \partial_\alpha x^{\nu'}. \quad (A15)
\]

It is also possible to introduce two mixed components of \( t \) with respect to \((U, \phi)\) and \((U, \phi')\). They are defined by

\[
t^{\mu \nu'} = t(b_\mu \cdot \partial x_\alpha) \cdot \partial_\alpha x^{\nu'}, \quad (A16)
\]
\[
t^{\mu' \nu} = t(\partial_\alpha x^{\mu'}) \cdot b_\nu \cdot \partial x_\alpha, \quad (A17)
\]
\[
t^{\mu \nu'} = t(b_\mu \cdot \partial' x_\alpha) \cdot \partial_\alpha x^{\nu'}, \quad (A18)
\]
\[
t^{\mu' \nu} = t(\partial_\alpha x^{\mu'}) \cdot b_\nu \cdot \partial' x_\alpha. \quad (A19)
\]

We will try to check the law of transformation for the covariant component...
of \( t \). We can write
\[
t_{\mu'\nu'} = t(b_\mu \cdot \partial' x_\alpha) \cdot b_\nu \cdot \partial' x_\beta,
\]
\[
= t\left( \frac{\partial x_\alpha}{\partial x_\mu} b_\alpha \cdot \partial x_\beta \right),
\]
\[
= \frac{\partial x_\alpha}{\partial x_\mu} \frac{\partial x_\beta}{\partial x_\nu'} t(b_\alpha \cdot \partial x_\beta),
\]
\[
t_{\mu'\nu'} = \frac{\partial x_\alpha}{\partial x_\mu} \frac{\partial x_\beta}{\partial x_\nu'} t_{\alpha\beta}.
\]
(A20)

This is the classical law of transformation for the covariant components of a smooth 2-tensor field from \( \{x^\mu\} \) to \( \{x'^\mu\} \).

By following similar steps we can get the laws of transformation for the contravariant and mixed components of \( t \). We have
\[
t_{\mu'\nu'} = \frac{\partial x_\mu'}{\partial x_\mu} \frac{\partial x_\nu'}{\partial x_\nu} t_{\alpha\beta},
\]
(A21)
\[
t_{\mu'\nu'} = \frac{\partial x_\mu'}{\partial x_\mu} \frac{\partial x_\nu'}{\partial x_\nu} t_{\alpha\beta},
\]
(A22)
\[
t_{\mu'\nu'} = \frac{\partial x_\mu'}{\partial x_\mu} \frac{\partial x_\nu'}{\partial x_\nu} t_{\alpha\beta}.
\]
(A23)

They perfectly agree with the classical laws of transformation for the respective contravariant and mixed components of a smooth 2-tensor field from \( \{x'^\mu\} \) to \( \{x^\mu\} \).

A.5 We now obtain the relation between the covariant derivatives of \( t \) and the classical concepts of covariant derivatives of the covariant, contravariant and mixed components of a smooth 2-tensor field. For instance, we have
\[
(\nabla_{b_\mu \partial x_\alpha}^+ t)(b_\alpha \cdot \partial x_\beta) = b_\mu \cdot \partial x_\alpha \cdot \partial\left( t(b_\alpha \cdot \partial x_\beta) \right) - t(\nabla_{b_\mu \partial x_\alpha}^+ b_\alpha \cdot \partial x_\beta) - t(\nabla_{b_\mu \partial x_\alpha}^+ b_\alpha \cdot \partial x_\beta)
\]
\[
= \frac{\partial t_{\alpha\beta}}{\partial x_\mu} - t(\Gamma^+(b_\mu \cdot \partial x_\alpha, b_\alpha \cdot \partial x_\tau) \cdot \partial_{\alpha} \partial_{\tau} b_\sigma \cdot \partial x_\beta) - t(b_\alpha \cdot \partial x_\beta) \cdot \Gamma^+(b_\mu \cdot \partial x_\alpha, b_\beta \cdot \partial x_\tau) \cdot \partial_{\alpha} \partial_{\tau} b_\sigma \cdot \partial x_\beta
\]
\[
= \frac{\partial t_{\alpha\beta}}{\partial x_\mu} - \Gamma^+_{\mu\alpha} t_{\sigma\beta} - \Gamma^+_{\nu\beta} t_{\alpha\tau},
\]

i.e.,
\[
(\nabla_{b_\mu \partial x_\alpha}^+ t)(b_\alpha \cdot \partial x_\beta) = t_{\alpha\beta} \cdot b_\mu.
\]
(A24)
We can also write

\[
\begin{align*}
(\nabla_{b_\alpha \partial x_\sigma}^- t)(b_\alpha \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta \\
+ \partial t_\alpha^\beta \mu & \cdot \partial x_\sigma \\
= b_\mu \cdot \partial x_\sigma \cdot \partial_\sigma t(b_\alpha \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta - t(\nabla_{b_\alpha \partial x_\sigma}^- b_\alpha \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta \\
- t(b_\alpha \cdot \partial x_\sigma) \cdot \nabla_{b_\alpha \partial x_\sigma}^- \partial_\sigma x^\beta \\
\quad = \partial t_\alpha^\beta \mu \cdot \partial x_\sigma - t(\Gamma^+ (b_\mu \cdot \partial x_\sigma, b_\alpha \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta b_\sigma \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta \\
& - t(b_\alpha \cdot \partial x_\sigma) \cdot \Gamma^- (b_\mu \cdot \partial x_\sigma, \partial_\sigma x^\beta) \cdot b_\sigma \cdot \partial x_\sigma \partial_\tau x^\tau. \\
\quad = \frac{\partial t_\alpha^\beta \mu}{\partial x_\sigma} - \Gamma^\sigma_{\mu \alpha} t_\sigma^\beta + \Gamma^\beta_{\mu \alpha} t_\alpha^\tau,
\end{align*}
\]
i.e.,

\[
(\nabla_{b_\alpha \partial x_\sigma}^- t)(b_\alpha \cdot \partial x_\sigma) \cdot \partial_\sigma x^\beta = t_\alpha^\beta \mu.
\] (A25)

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