Geometric Phase and Chiral Anomaly in Path Integral Formulation

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Abstract

All the geometric phases, adiabatic and non-adiabatic, are formulated in a unified manner in the second quantized path integral formulation. The exact hidden local symmetry inherent in the Schrödinger equation defines the holonomy. All the geometric phases are shown to be topologically trivial. The geometric phases are briefly compared to the chiral anomaly which is naturally formulated in the path integral.

1 Second quantization

To analyze various geometric phases in a unified manner [1]-[8], we start with an arbitrary complete basis set

$$\int d^3x v^*_n(t, \vec{x}) v_m(t, \vec{x}) = \delta_{nm}$$

and expand the field variable as $$\psi(t, \vec{x}) = \sum_n b_n(t) v(t, \vec{x})$$. The action

$$S = \int_0^T dt d^3x [\psi^*(t, \vec{x}) i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) - \psi^*(t, \vec{x}) \hat{H}(\hat{p}, \hat{x}, X(t)) \psi(t, \vec{x})]$$ (1)

with background variables $$X(t) = (X_1(t), X_2(t), ..)$$ then becomes

$$S = \int_0^T dt \{ \sum_n b_n^*(t) i\hbar \partial_t b_n(t) - H_{eff} \}$$ (2)

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with the effective Hamiltonian in the second quantized version

\[
\hat{H}_{eff}(t) = \sum_{n,m} \hat{b}_n(t)\left[\int d^3x v_n^*(t, \vec{x}) \hat{H}(\hat{p}, \hat{x}, X(t)) v_m(t, \vec{x}) \right. \\
- \int d^3x v_n^*(t, \vec{x}) i\hbar \frac{\partial}{\partial t} v_m(t, \vec{x}) \left. \right] \hat{b}_m(t),
\]  \tag{3}

and $[\hat{b}_n(t), \hat{b}_m^\dagger(t)]_\mp = \delta_{n,m}$, but statistics is not important in our application.

We use fermions (Grassmann numbers) in the path integral.

The Schrödinger picture $\hat{H}_{eff}(t)$ is defined by replacing $\hat{b}_n(t)$ by $\hat{b}_n(0)$ in $\hat{H}_{eff}(t)$, and the evolution operator is given by \cite{10, 11}

\[
\langle m | T^* \exp\left\{ -\frac{i}{\hbar} \int_0^t \hat{H}_{eff}(t) dt \right\} | n \rangle = \langle m(t) | T^* \exp\left\{ -\frac{i}{\hbar} \int_0^t \hat{H}(\hat{p}, \hat{x}, X(t)) dt \right\} | n(0) \rangle
\]

with time ordering symbol $T^*$. On the left-hand side $| n \rangle = \hat{b}_n^\dagger(0)|0\rangle$ and on the right-hand side $\langle \vec{x}|n(t)\rangle = v_n(t, \vec{x})$.

The Schrödinger probability amplitude with $\psi_n(0, \vec{x}) = v_n(0, \vec{x})$ is defined by \cite{11}

\[
\psi_n(t, \vec{x}) = \langle 0|\psi(t, \vec{x})\hat{b}_n^\dagger(0)|0\rangle \\
= \sum_m v_m(t, \vec{x}) \langle m | T^* \exp\left\{ -\frac{i}{\hbar} \int_0^t \hat{H}_{eff}(t) dt \right\} | n \rangle \tag{4}
\]

and the path integral representation is given by

\[
\langle m | T^* \exp\left\{ -\frac{i}{\hbar} \int_0^t \hat{H}_{eff}(t) dt \right\} | n \rangle = \int \prod_n Db_n Db_n^* \phi_n^*(b_n^*(t)) \\
\times \exp\left\{ \frac{i}{\hbar} \int_0^t dt [b_n^*(t) i\hbar \partial_t b_n(t) - H_{eff}(t)] \phi_n(b_n^*(0)) \right\} \tag{5}
\]

with suitable wave functions $\phi_n^*(b_n^*(t))$ and $\phi_n(b_n^*(0))$ in the holomorphic representation \cite{9}. The general geometric terms automatically appear in the second term of the exact $H_{eff}(t)$ (3) and thus the naive holomorphic wave functions are sufficient. This means that the analysis of geometric phases is reduced to a simple functional analysis in the second quantized path integral.

If one uses a specific basis $\hat{H}(\hat{p}, \hat{x}, X(t)) v_n(\vec{x}; X(t)) = E_n(X(t)) v_n(\vec{x}; X(t))$ and assumes ”diagonal dominance” in the effective Hamiltonian in (5), we have the adiabatic formula

\[
\psi_n(t, \vec{x}) \simeq v_n(\vec{x}; X(t)) \exp\left\{ -\frac{i}{\hbar} \int_0^t [E_n(X(t)) - v_n^* i\hbar \partial_t v_n] dt \right\} \tag{6}
\]
which shows that the adiabatic approximation is equivalent to the approximate diagonalization of $H_{\text{eff}}$, and thus the geometric phases are dynamical \[10\] \[11\].

2 Hidden local gauge symmetry

Since $\psi(t, \vec{x}) = \sum_n b_n(t) v_n(t, \vec{x})$, we have an exact local symmetry \[11\]

$$
\begin{align*}
    v_n(t, \vec{x}) &\to v'_n(t, \vec{x}) = e^{i\alpha_n(t)} v_n(t, \vec{x}), \\
    b_n(t) &\to b'_n(t) = e^{-i\alpha_n(t)} b_n(t), \quad n = 1, 2, 3, \ldots
\end{align*}
$$

(7)

This symmetry means an arbitrariness in the choice of the coordinates in the functional space. The exact Schrödinger amplitude $\psi_n(t, \vec{x}) = \langle 0 | \hat{\psi}(t, \vec{x}) \hat{b}_n(0) | 0 \rangle$ is transformed under this substitution rule as

$$
\psi'_n(t, \vec{x}) = e^{i\alpha_n(0)} \psi_n(t, \vec{x})
$$

for any $t$. Namely, we have the ray representation with a constant phase change, and we have an enormous hitherto unrecognized exact hidden symmetry behind the ray representation.

The combination $\psi_n(0, \vec{x})^* \psi_n(T, \vec{x})$ thus becomes manifestly gauge invariant. For example, for adiabatic approximation (6) we have

$$
\psi_n(0, \vec{x})^* \psi_n(T, \vec{x}) = v_n(0, \vec{x}; X(0))^* v_n(T, \vec{x}; X(T))
$$

$$
\times \exp\left\{ -\frac{i}{\hbar} \int_0^T [E_n(X(t)) - v_n^* i\hbar \frac{\partial}{\partial t} v_n] dt \right\}
$$

(8)

If one chooses a specific gauge $v_n(T, \vec{x}; X(T)) = v_n(0, \vec{x}; X(0))$, the prefactor $v_n(0, \vec{x}; X(0))^* v_n(T, \vec{x}; X(T))$ is real and positive. The factor in the exponential then represents the entire gauge invariant phase.

Parallel transport and holonomy

The parallel transport of $v_n(t, \vec{x})$ is defined by

$$
\int d^3 x v_n^\dagger(t, \vec{x}) \frac{\partial}{\partial t} v_n(t, \vec{x}) = 0
$$

which follows from $\int d^3 x v_n^\dagger(t, \vec{x}) v_n(t + \delta t, \vec{x}) = \text{real and positive}$, and

$$
\int d^3 x v_n^\dagger(t + \delta t, \vec{x}) v_n(t + \delta t, \vec{x}) = \int d^3 x v_n^\dagger(t, \vec{x}) v_n(t, \vec{x})
$$

By using the hidden
local gauge $\bar{v}_n(t, \vec{x}) = e^{i\alpha_n(t)} v_n(t, \vec{x})$ for general $v_n(t, \vec{x})$, one may impose the condition
\[
\int d^3x \bar{v}_n^\dagger(t, \vec{x}) \frac{\partial}{\partial t} \bar{v}_n(t, \vec{x}) = 0
\]
which gives
\[
\bar{v}_n(t, \vec{x}) = \exp[i \int_0^t dt' \int d^3x v_n^\dagger(t', \vec{x}) i \partial_{t'} v_n(t', \vec{x})] v_n(t, \vec{x}).
\]
The holonomy for a cyclic motion is then defined by
\[
\bar{v}_n^\dagger(0, \vec{x}) \bar{v}_n(T, \vec{x}) = v_n^\dagger(0, \vec{x}) v_n(T, \vec{x}) \exp\{ -i \int_0^T dt \int d^3x \bar{v}_1^\dagger(t, \vec{x}) H_{\text{eff}}(t, \vec{x}) \}
\]
This holonomy of basis vectors, not Schrödinger amplitude, determines all the geometric phases in the second quantized formulation [13].

3 Non-adiabatic phase

(i) Cyclic evolution:
The cyclic evolution is defined by $\psi(T, \vec{x}) = e^{i\phi} \bar{\psi}(0, \vec{x})$ or by
\[
\psi(t, \vec{x}) = e^{i\phi(t)} \bar{\psi}(t, \vec{x}), \quad \bar{\psi}(T, \vec{x}) = \bar{\psi}(0, \vec{x})
\]
with $\phi(T) = \phi$, $\phi(0) = 0$. If one chooses the first element of the arbitrary basis set $\{v_n(t, \vec{x})\}$ such that $v_1(t, \vec{x}) = \bar{\psi}(t, \vec{x})$, one has diagonal $H_{\text{eff}}(t)$ and
\[
\psi(t, \vec{x}) = v_1(t, \vec{x}) \exp\{ -i \int_0^t dt \int d^3x v_1^\dagger(t, \vec{x}) H v_1(t, \vec{x}) \}
\]
\[
- \int_0^t dt \int d^3x v_1^\dagger(t, \vec{x}) i \hbar \partial_{t'} v_1(t, \vec{x}) \}
\]
in (4). Under the hidden local symmetry, we have $\psi(t, \vec{x}) \to e^{i\alpha_1(0)} \psi(t, \vec{x})$ and the gauge invariant quantity
\[
\psi_1^\dagger(0, \vec{x}) \psi(T, \vec{x}) = v_1^\dagger(0, \vec{x}) v_1(T, \vec{x}) \exp\{ -i \int_0^T dt \int d^3x [v_1^\dagger(t, \vec{x}) H v_1(t, \vec{x}) \]
\]
\[
- v_1^\dagger(t, \vec{x}) i \hbar \partial_{t'} v_1(t, \vec{x}) \}
\]
If one chooses $v_1(0, \vec{x}) = v_1(T, \vec{x})$, $v_1^*(0, \vec{x})v_1(T, \vec{x})$ becomes real and positive, and the factor \[\beta = \oint dt \int d^3x v_1^*(t, \vec{x}) \frac{\partial}{\partial t} v_1(t, \vec{x})\] (15) gives the non-adiabatic phase\[5\].

Note that the so-called “projective Hilbert space” and the transformation

$$\psi(t, \vec{x}) \rightarrow e^{i\omega(t)}\psi(t, \vec{x}),$$

which is not the symmetry of the Schrödinger equation, is not used in our formulation. Also, the holonomy of the basis vector, not the Schrödinger amplitude, determines the non-adiabatic phase. Our derivation of the non-adiabatic phase (15), which works in the path integral (5) also, is quite different from that in \[5\].

(ii) Non-cyclic evolution:

It is shown that any exact solution of the Schrödinger equation is written in the form \[13\],

$$\psi_k(\vec{x}, t) = v_k(\vec{x}, t) \exp\left\{-i \frac{\hbar}{\hbar} \int_0^t \int d^3x [v_k^\dagger(\vec{x}, t) \hat{H}(t)v_k(\vec{x}, t) - v_k^\dagger(\vec{x}, t)i\hbar \frac{\partial}{\partial t} v_k(\vec{x}, t)]\right\}$$

(16)

if one suitably chooses the basis set \{v_k(\vec{x}, t)\}, though the periodicity is generally lost, $v_k(\vec{x}, 0) \neq v_k(\vec{x}, T)$. $\psi_k(t, \vec{x})$ is transformed as $\psi_k(t, \vec{x}) \rightarrow e^{i\alpha_k(t)}\psi_k(t, \vec{x})$ under the hidden local symmetry, and

$$\int d^3x v_k^\dagger(0, \vec{x}) \psi_k(T, \vec{x}) = \int d^3x v_k^\dagger(0, \vec{x}) v_k(T, \vec{x})$$

$$\times \exp\left\{-i \frac{\hbar}{\hbar} \int_0^T dt d^3x [v_k^\dagger(t, \vec{x}) \hat{H}(t)v_k(t, \vec{x}) - v_k^\dagger(t, \vec{x})i\hbar \frac{\partial}{\partial t} v_k(t, \vec{x})]\right\}$$

(17)

is thus manifestly gauge invariant. By choosing a suitable hidden symmetry $v_k(t, \vec{x}) \rightarrow e^{i\alpha_k(t)}v_k(t, \vec{x})$, one can make $\int d^3x v_k^\dagger(0, \vec{x}) v_k(T, \vec{x})$ real and positive. Then the exponential factor defines the non-cyclic non-adiabatic phase\[6\]. The present definition also works in the path integral (5).

It is shown that geometric phases for mixed states \[7\ \\ 8\] are similarly formulated in the second quantized formulation \[13\].
4 Exactly solvable example

We study the model

\[
\hat{H} = -\mu \hbar \vec{B}(t) \vec{\sigma}, \quad \vec{B}(t) = B(\sin \theta \cos \varphi(t), \sin \theta \sin \varphi(t), \cos \theta)
\]

(18)

with \( \varphi(t) = \omega t \) and constant \( \omega, B \) and \( \theta \).

To diagonalize the second quantized \( H_{\text{eff}} \), we define the constant \( \alpha \) by \[13\]

\[
\tan \alpha = \frac{\hbar \omega \sin \theta}{2 \mu \hbar B + \hbar \omega \cos \theta}
\]

(19)

or equivalently \( 2 \mu B \sin \alpha = \hbar \omega \sin(\theta - \alpha) \), and the basis vectors

\[
w_+(t) = \begin{pmatrix}
\cos \frac{1}{2}(\theta - \alpha) e^{-i\varphi(t)} \\
\sin \frac{1}{2}(\theta - \alpha)
\end{pmatrix}, \quad w_-(t) = \begin{pmatrix}
\sin \frac{1}{2}(\theta - \alpha) e^{-i\varphi(t)} \\
-\cos \frac{1}{2}(\theta - \alpha)
\end{pmatrix}
\]

which satisfy \( w_\pm(0) = w_\pm(T) \) with \( T = \frac{2\pi}{\omega} \), and

\[
w_\pm(t) \hat{H} w_\pm(t) = \mp \mu \hbar B \cos \alpha, \quad w_\pm(t) i\hbar \partial_t w_\pm(t) = \frac{\hbar \omega}{2} (1 \pm \cos(\theta - \alpha))
\]

The effective Hamiltonian \( H_{\text{eff}} \) is now diagonal, and the exact solution of the Schrödinger equation \( i\hbar \partial_t \psi(t) = \hat{H} \psi(t) \) is given by \[13\]

\[
\psi_\pm(t) = w_\pm(t) \exp \{-\frac{i}{\hbar} \int_0^t dt' [w_\pm(t') \hat{H} w_\pm(t') - w_\pm(t') i\hbar \partial_{t'} w_\pm(t')]}.
\]

(20)

This exact solution may be regarded either as an exact version of the adiabatic phase or as the cyclic nonadiabatic phase in our formulation.

We examine some limiting cases of this exact solution:

(i) For adiabatic limit \( \hbar \omega / (\hbar \mu B) \ll 1 \), we have from (19)

\[
\alpha \simeq [\hbar \omega / 2\hbar \mu B] \sin \theta,
\]

and if one sets \( \alpha = 0 \) in (20), one recovers the Berry’s phase \[2\]

\[
\psi_\pm(T) \simeq \exp \{i\pi (1 \pm \cos \theta)\} \exp \{\pm \frac{i}{\hbar} \int_0^T dt \mu B \} w_\pm(T)
\]

(21)

with

\[
w_+(t) = \begin{pmatrix}
\cos \frac{1}{2} \theta e^{-i\varphi(t)} \\
\sin \frac{1}{2} \theta
\end{pmatrix}, \quad w_-(t) = \begin{pmatrix}
\sin \frac{1}{2} \theta e^{-i\varphi(t)} \\
-\cos \frac{1}{2} \theta
\end{pmatrix}.
\]
For non-adiabatic limit $\hbar \mu B/(\hbar \omega) \ll 1$, we have from (19)
\[ \theta - \alpha \simeq [2\hbar \mu B/\hbar \omega] \sin \theta \]
and if one sets $\alpha = \theta$ in (20), one obtains the trivial geometric phase
\[ \psi_{\pm}(T) \simeq w_{\pm}(T) \exp \{ \pm i \int_0^T dt [\mu \hbar B \cos \theta] \} \]
with
\[ w_{+}(t) = \begin{pmatrix} e^{-i\varphi(t)} & 0 \\ 0 & 1 \end{pmatrix}, \quad w_{-}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]
This shows that the “monopole-like” phase in (21) is smoothly connected to a trivial phase in the exact solution, and thus the geometric phase is topologically trivial.\[ \Box \]

5 Chiral anomaly

It is known that all the anomalies in gauge field theory \[14, 15\] are understood in the path integral as arising from the non-trivial Jacobians under symmetry transformations \[16, 17\]. For example, in the fermionic path integral
\[ \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \{ i \int d^4x [\bar{\psi} i \gamma^\mu (\partial_\mu - igA_\mu) \psi] \} \]
and for infinitesimal chiral transformation
\[ \psi(x) \rightarrow e^{i\omega(x)\gamma^5} \psi(x), \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\omega(x)\gamma^5}, \]
we have
\[ \mathcal{D}\bar{\psi} \mathcal{D}\psi \rightarrow \exp \{ -i \int d^4x \omega(x) g^2 \frac{\epsilon^{\mu\nu\alpha\beta}}{16\pi^2} F^\mu_{\nu\alpha} F^\alpha_{\nu\beta} \} \mathcal{D}\bar{\psi} \mathcal{D}\psi. \]
The anomaly is integrated for a finite transformation, and it gives rise to the so-called Wess-Zumino term \[18\].

Based on this observation, one recognizes the following differences between the geometric phases and chiral anomaly: \[19\]
1. The Wess-Zumino term is added to the classical action in path integral, whereas the geometric term appears inside the classical action sandwiched by field variables as in (3). Geometric phases are thus state-dependent.
2. The topology of chiral anomaly, which is provided by gauge fields, is exact, whereas the topology of the adiabatic geometric phase, which is valid only in the adiabatic limit, is trivial.
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