Statistical Model Checking for Probabilistic Hyperproperties*

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Abstract. In this paper, we propose the temporal logic HyperPCTL\textsuperscript{*} that extends PCTL\textsuperscript{*} and HyperPCTL to reason about probabilistic hyperproperties. It allows expressing probabilistic hyperproperties with nested temporal and probability operators. We show that HyperPCTL can express important probabilistic information-flow security policies. Furthermore, for the first time, we investigate statistical model checking (SMC) algorithms for HyperPCTL\textsuperscript{*} specifications in discrete-time Markov chains (DTMC). To this end, we first study SMC for HyperPCTL\textsuperscript{*} specifications with non-nested probability operators for a desired confidence or significance level. Unlike existing SMC algorithms which are based on sequential probability ratio tests (SPRT), we use the Clopper-Pearson confidence interval to avoid the need of a priori knowledge on the indifference margin. Then, we extend the proposed SMC algorithms to HyperPCTL\textsuperscript{*} specifications with multiple probability operators that are nested in different ways. Finally, we evaluate the proposed algorithms on two examples, dining cryptographers and probabilistic causation.

1 Introduction

Hyper temporal logics extends trace-based temporal logics by allowing the argument of temporal specifications that involve multiple executions of the system of interest. They can describe important information-flow security policies, such as non-interference and differential privacy as well as consistency models in concurrent computing \cite{4,7,5}, which are not definable by trace-based languages. In hyper temporal logics, atomic propositions are associated with different paths. These labeled atomic propositions in combination with other logic operators specify which path should satisfy what properties for every time instance.

In this paper, we introduce the temporal logic HyperPCTL\textsuperscript{*} that extends PCTL\textsuperscript{*} \cite{3} by allowing quantification over paths and HyperPCTL \cite{1} by allowing nested probability and temporal operators. Thus, HyperPCTL\textsuperscript{*} reduces to PCTL\textsuperscript{*}, when no more than one path variable is used. The path fragment of HyperPCTL\textsuperscript{*}, namely, the path formulas without the probability operator, is HyperLTL proposed in \cite{6}.

A major challenge in verifying or monitoring hyperproperties, specified in a hyper temporal logic, is that the computation complexity for symbolic verification grows at

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least exponentially in the number of quantifier alternations of the input formula \[2,6,5\]. To alleviate this issue, in this paper, to our knowledge, we introduce the first statistical approach to verify HyperPCTL* with probabilistic guarantees. Compared to the symbolic methods, the main advantage of statistical model checking (SMC) is improved scalability \[12,11,13,17,20\]. The general idea of SMC is to treat the problem of checking a temporal logic formula on a probabilistic system as multiple hypothesis testing problems \[19,11\]. By drawing samples from the underlying probabilistic system, the satisfaction of the formula can be inferred with high confidence levels.

The SMC algorithm proposed in this work is based on constructing confidence intervals for the temporal formulas, unlike most proposed methods that are based on the use of sequential probability ratio tests (SPRT) \[20,14,11,18\]. Thus, we do not require a priori knowledge on the indifference margin, as for the existing methods. To achieve this, we employ the Clopper-Pearson confidence interval, which is the optimal conservative confidence interval for given confidence or significance levels. On the other hand, SMC algorithms using other tighter but approximate confidence intervals, such as the Agresti-Coull interval \[10\], can be similarly derived.

For HyperPCTL* formulas with non-nested probability operators, we propose SMC algorithms for both fixed-size samples and for a desired confidence level in a sequential setup. For fixed-size samples, we provide both the exact and approximate confidence level for asserting a non-nested HyperPCTL* formula. Based on this, we introduce a sequential SMC algorithm for given HyperPCTL* formulas that achieves exactly or approximately the desired confidence or significance levels.

Furthermore, we extend the SMC algorithms from non-nested formulas to PCTL* formulas with nested probability operators, using a compositional approach. Due to the involvement of multiple executions in hyperproperties, the nesting law of HyperPCTL* is different for PCTL*, as studied in \[14,16,15\]. A nested HyperPCTL* formulas is constructed iteratively in two ways: (i) replacing a simple sub-formula with a non-nested formula, and (ii) quantifying free path variables in a non-nested formula. For both cases, we propose SMC algorithms for either fixed-size samples or for desired confidence or significance levels.

An advantage of the proposed SMC algorithms is the capability of verifying hyperproperties with arithmetic relations on probabilities. Thus, we implement them to verify probabilistic causation \[8\] on Markov chains and probabilistic non-interference for dining cryptographers \[9\]. The simulation results show that the proposed SMC algorithms provide the correct answer with the desired significance levels in all cases, while requiring very short analysis times.

The rest of the paper is organized as follows. We start with preliminaries in Section 2. HyperPCTL* is introduced in Section 3 before illustrating its applications in Section 4. SMC algorithms for non-nested HyperPCTL* are introduced in Section 5 before extending them to cover nested HyperPCTL* in Section 6. Finally, we show applicability of our verification methods in Section 7 before providing concluding remarks in Section 8.
2 Preliminaries

We denote the set of integers and real numbers by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Let \( \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\} \). For \( n \in \mathbb{N} \), let \([n] = \{1,\ldots,n\}\). The indicator function is denoted by \( \mathbb{1} \). We denote the (Borel) measure of a measurable set by \( \mu_B(\cdot) \). The cardinality of a set is denoted by \(|\cdot|\). For \( n \in \mathbb{N}_\infty \), we write \( \{s_1,\ldots,s_n\} \) as \( \{s\}_n \). For a sequence \( \sigma = \{\sigma(t)\}_{t \in \mathbb{N}} \), we denote by \( \sigma^{(i)} = \{\sigma(t+i)\}_{t \in \mathbb{N}} \) the \( i \)-index shift. For any set \( I \subseteq \mathbb{R}^n \), we denote its boundary, interior, and closure by \( \partial I, I^\circ \) and \( \bar{I} \). Given a map \( V : A \rightarrow B \) and \( A' \subseteq A \), let \( V(A') = \cup_{a \in A'} V(a) \). Finally, we make the convention that indexes start from 1.

2.1 Labeled Discrete-Time Markov Chains

Let \( \text{AP} \) be a finite set of atomic propositions. A labeled discrete-time Markov chain (DTMC) is a Markov chain with states labeled by the set of atomic propositions.

**Definition 1 (Labeled DTMC).** A labeled discrete-time Markov chain (DTMC) is a tuple \( \mathcal{M} = (S, T, \text{AP}, L) \) where:

- \( S \) is a finite set of states,
- \( T : S \times S \rightarrow [0,1] \) is the transition probability function with \( \sum_{s' \in S} T(s,s') = 1 \) for any state \( s \in S \),
- \( \text{AP} \) is a set of labels and \( L : S \rightarrow 2^\text{AP} \) is a labeling function.

A path of a DTMC \( \mathcal{M} = (S, T, \text{AP}, L) \) is defined as an infinite sequence \( s_0 s_1 s_2 \cdots \in S^\omega \) of states with \( T(s_i, s_{i+1}) > 0 \), for all \( i \geq 0 \). Let \( \text{Paths}(s) \) denote the set of all (infinite) paths starting in state \( s \) in \( \mathcal{M} \), and \( \text{Paths}_{\text{fin}}(s) \) denote the set of all finite prefixes of paths from \( \text{Paths}(s) \), which we sometimes refer to as finite paths.

2.2 Clopper-Pearson Confidence Interval

Let \( \text{Binom}(n, p) \) be the binomial distribution with the probability mass function \( f_{\text{Binom}}(k | n, p) = C_n^k p^k (1-p)^{n-k}, \ k \in 0,\ldots,n \), where \( C_n^k = n!/(k!(n-k)!)) \) is the binomial coefficient. Given a random variable \( X \sim \text{Binom}(n, p) \), we can construct a two-side Clopper-Pearson (CP) confidence interval of \( p \) as:

\[
\begin{align*}
    p &\in \begin{cases} 
        [0, 1 - \alpha \frac{1}{2}], & \text{if } X = 0 \text{ with confidence } 1 - \alpha, \\
        [\alpha \frac{1}{2}, 1], & \text{if } X = n \text{ with confidence } 1 - \alpha, \\
        [p_l, p_u], & \text{if } 0 < X < n \text{ with confidence } 1 - \alpha - \beta,
    \end{cases} \\
    p_l &= \left(1 + \frac{n - X + 1}{XF(\alpha | 2X, 2(n - X + 1))}\right)^{-1} \quad (1) \\
    p_u &= \left(1 + \frac{n - X}{(X + 1)F(1 - \beta | 2(X + 1), 2(n - X))}\right)^{-1}.
\end{align*}
\]

Here, \( F(q | a_1, a_2) \) is the \( q \) quantile from an F-distribution with \((a_1, a_2)\) degrees of freedom.
3 The Temporal Logic HyperPCTL$^*$

HyperPCTL$^*$ is an extension of PCTL$^*$ that enables handling hyperproperties. It is also a generalization of HyperPCTL [11] that allows for nested temporal and probability operators. For DTMCs, HyperPCTL$^*$ has the power to define properties involving multiple paths.

3.1 Syntax

Again, we use $\text{AP}$ to denote a set of atomic propositions, and $\Gamma$ and $\Sigma$ infinite sets of path and state variables, respectively. HyperPCTL$^*$ formulas are inductively defined by the following grammar:

$$\psi ::= \forall \sigma^\pi \cdot \psi \mid \exists \sigma^\pi \cdot \psi \mid \Phi$$
$$\Phi ::= \neg \Phi \mid \Phi \land \Phi \mid P \leadsto P$$
$$P ::= P^II(\varphi) \mid f(P, \ldots, P)$$
$$\varphi ::= a^\pi \mid \Phi^\pi \mid \neg \varphi \lor \varphi \lor \bigcirc \varphi \mid \varphi \land \Phi \mid p \leadsto p \mid p \leadsto P$$
$$p ::= P^II(\varphi) \mid P^II p \mid f(p, \ldots, p)$$

where $\pi \in \Gamma$ is a path variable, $\sigma \in \Sigma$ is a state variable, $P^\varphi \subseteq \Gamma$ is the set of all unquantified path variables in $\varphi$, $P^\varphi \subseteq P^\pi$, $\Lambda \subseteq \Gamma$, $a \in \text{AP}$ is an atomic proposition, $f : \mathbb{R}^n \to \mathbb{R}$ is an $n$-ary measurable function, and $\leadsto \in \{<, >, =, \leq, \geq\}$. In a quantified formula, $\sigma^\pi$ means that the path $\pi$ starts in state $\sigma$. Finally, a 0-ary function $f$ returns a constant. By abuse of notation if $\Pi$ has only one element, we omit the set notation; i.e., we write $P^\pi$ instead of $P(\pi)$.

We refer to $\Psi$ as quantified state formulas and similar to PCTL$^*$, we refer to $\varphi$ as a path formula and $\Phi$ as a state formula. Temporal operators $\bigcirc$ and $\land$ stand for ‘next’ and ‘until’, respectively. More logic operators are derived as follows: $\varphi \lor \varphi' \equiv \neg (\neg \varphi \land \neg \varphi')$, $\varphi \to \varphi' \equiv \neg \varphi \lor \varphi'$, $\bigcirc \varphi \equiv \text{true} \land \varphi$, and $\Box \varphi \equiv \neg \bigcirc \neg \varphi$.

Finally, note the difference that $p$ defined by (6) should have path variables that are not quantified by probability operators, while $P$ defined by (4) should not. Thus, without using probability operators, only $P$ can be used to construct HyperPCTL$^*$ state formulas, which can be further associated with another path variables, according to the second rule of (5). Hence, the syntax allows the formula $P^\{\pi_1, \pi_2\}(a^\pi_1 \land \varphi) \mid P^\{\pi_3\}(a^\pi_3 \land \varphi) \mid P^\{\pi_4\}(a^\pi_4 \land \varphi) \mid P^\{\pi_5\}(a^\pi_5 \land \varphi) \mid P^\{\pi_6\}(a^\pi_6 \land \varphi)$, where the path variable $\pi_4$ is not quantified by a probability operator before sub-formula $P^\{\pi_3\}(a^\pi_3 \land \varphi)$. This can cause the double indexing of $a_3$ with $\pi_2$ and $\pi_4$.

Remark 1. HyperPCTL$^*$ augments the standard PCTL$^*$ by associating atomic propositions with path variables to reason about multiple paths simultaneously. The labeled atomic propositions in combination with other logic operators specify which path should satisfy what properties for every time instance. Note that HyperPCTL$^*$ reduces to PCTL$^*$ [3] when no more than one path variable is used.

Remark 2. Compared to HyperPCTL [11], rule (5) is similar to the HyperPCTL state formulas, while rule (5) extends HyperPCTL path formulas by allowing nested temporal
operators in the same spirit as PCTL\(^\ast\). Furthermore, the arithmetic rules (4) and (6) generalize the HyperPCTL arithmetic rules to measurable \( n \)-ary functions.

**Remark 3.** In PCTL\(^\ast\), path formulas without the probability operator resemble LTL. Likewise, the path fragment of HyperPCTL\(^\ast\), namely, the path formulas without the probability operator, resemble HyperLTL \([2]\).

### 3.2 Semantics

We define the semantics of HyperPCTL\(^\ast\) on DTMCs via an assignment or instantiation of all path variables to the paths of the DTMCs. Formally, let \( \mathcal{M} = (S, T, AP, L) \) be a labeled DTMC. Let \( V : \Gamma \rightarrow \text{Paths} \) be an assignment of the path variables to the paths of \( \mathcal{M} \). For a set of path variables \( I \), let \( V[I] \) be the restriction of \( V \) on \( I \). In addition, we use \( \llbracket \cdot \rrbracket_V \) to denote the instantiation of \( V \) on a HyperPCTL\(^\ast\) formula and use \( V[s_I] \) to denote the assignment of state \( s \in S \) to the initial state of the paths in \( I \).

The satisfaction relation \( \models \) is defined for a HyperPCTL\(^\ast\) state formula \( \varphi \) for an assignment \( V \) as:

\[
V \models \forall s_I \varphi \quad \text{iff} \quad \text{for all } s \in S, V[s_I] \models \varphi
\]
\[
V \models \exists s_I \varphi \quad \text{iff} \quad \text{there exists } s \in S, V[s_I] \models \varphi
\]
\[
V \models \neg \varphi \quad \text{iff} \quad V \not\models \varphi
\]
\[
V \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad V \models \varphi_1 \text{ and } V \models \varphi_2
\]
\[
V \models \top \quad \text{iff} \quad V \models \top
\]
\[
V \models (\varphi \land \varphi_2) \quad \text{iff} \quad V \models \varphi_1 \text{ and } V \models \varphi_2
\]

and for path formulas as:

\[
V \models a \pi \quad \text{iff} \quad a \in L(\llbracket \pi \rrbracket_V(0))
\]
\[
V \models \pi \varphi \quad \text{iff} \quad V \models \llbracket \pi \varphi \rrbracket_V
\]
\[
V \models \neg \varphi \quad \text{iff} \quad V \not\models \varphi
\]
\[
V \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad V \models \varphi_1 \text{ and } V \models \varphi_2
\]
\[
V \models \top \quad \text{iff} \quad V \models \top
\]
\[
V \models (\varphi \land \varphi_2) \quad \text{iff} \quad V \models \varphi_1 \text{ and } V \models \varphi_2
\]
\[
V \models [f(p, \ldots, p)]_V \quad \text{iff} \quad V \models f(\llbracket p \rrbracket_V, \ldots, \llbracket p \rrbracket_V)
\]
\[
V \models [\mathbb{P}^H(\varphi)]_{V[s_I]} \quad \text{iff} \quad \mathbb{P}^H(\varphi)_{V[s_I]} \models \mathbb{P}^H(\varphi)_{V[s_I]}
\]
\[
V \models p \sim p \quad \text{iff} \quad \llbracket p \rrbracket_V \sim \llbracket p \rrbracket_V
\]

where \( V^{(i)} \) is the \( i \)-shift of path assignment \( V \), defined by \( V^{(i)}(\pi) = (V(\pi))^{(i)} \), and mapping \( \pi \) to the \( i \)-suffix of \( V(\pi) \), for all \( \pi \in \Gamma \).

**Example** To illustrate the semantics of HyperPCTL\(^\ast\), consider the DTMC \( \mathcal{M} \) in Figure 1 and the following HyperPCTL\(^\ast\) formula:

\[
\varphi = \exists \sigma_{1,2} \forall \sigma_{2,2}. (\mathbb{P}^{\{\pi_{1,2}\}}((a_{1,2}^{\pi_{1,2}} \land a_{2,2}^{\pi_{1,2}}) \land (a_{2,2}^{\pi_{1,2}} \land a_{2,2}^{\pi_{1,2}})) > 1/6)
\]
The formula asserts that there exists a state $\sigma_1$, such that for any state $\sigma_2$, the probability that two independent paths $\pi_1$ and $\pi_2$ starting from $\sigma_1$ and $\sigma_2$, respectively, that satisfy $a_1$ initially, and $a_2$ finally is greater than $1/6$. Clearly, if $\sigma_1 = s_2$, then this probability is 0. Therefore, we have that $M \not|= \varphi$.

### 3.3 Expressivity of HyperPCTL$^*$

**Theorem 1.** HyperPCTL$^*$ subsumes PCTL$^*$ with respect to DTMCs.

**Proof.** First, it is straightforward to see that any PCTL$^*$ formula $\varphi$ can be expressed in HyperPCTL$^*$ by adding one universal state quantifier and one path variable to $\varphi$; i.e., $\forall \sigma. \varphi'$, where $\varphi'$ is obtained by adding $\pi$ to each probability operator and atomic proposition in $\varphi$.

Now, we show that there exist formulas in HyperPCTL$^*$ that cannot be expressed in PCTL$^*$. Consider the DTMC shown in Figure 2 and the following HyperPCTL$^*$ formula:

$$\varphi = \forall \sigma. \left( \frac{P_{\pi}(\text{init} \rightarrow \Diamond (a_1 \wedge a_2))}{P_{\pi}(\text{init} \rightarrow \Diamond a_2^2)} = \frac{1}{2} \right).$$

Now, we prove that the $\varphi$ cannot be expressed in PCTL$^*$. By the syntax and semantics of PCTL$^*$ [3], it suffices to show that $\varphi$ cannot be expressed by a formula $P(\psi)$, where $\psi$ is a PCTL$^*$ path formula derived by concatenating a set of PCTL$^*$ state formulas $\Phi_1, \ldots, \Phi_n$ with $\wedge$, $\neg$, or temporal operators. These state formulas are either true or false on the states $s_0, s_1, s_2,$ and $s_3$. Thus, whether a path satisfies $\psi$, defines a subset of the paths $\text{Paths}(s_0) = \{s_0s_1^2, s_0s_2^2, s_0s_3^2\}$ in the DTMC. Since every path in $\text{Paths}(s_0)$ is taken with probability $1/3$, formula $P(\psi)$ can only evaluate to a value in $\{0, 1/3, 2/3, 1\}$. However, by the semantics of HyperPCTL$^*$, the fractional probability on the right side of the implication has value $1/2$, thus $\varphi$ evaluates to true and cannot be expressed by $P(\psi)$ in PCTL$^*$.

**Theorem 2.** HyperPCTL$^*$ subsumes HyperPCTL with respect to DTMCs.

**Proof.** Intuitively, the proof is based on the fact that PCTL$^*$ is strictly more expressive than PCTL. For instance, formula $\forall \sigma. (\pi_1. \pi_2). P(\pi_1. \pi_2) (a_1 U \Diamond a_2^2)) > c_1$ with nested temporal operators cannot be expressed in $\text{HyperPCTL}$. 

\[ \square \]
4 Applications of HyperPCTL∗

HyperPCTL∗ can be used to define security policies that stipulate probability relation of multiple executions. We illustrate this on three examples that cannot be expressed in HyperPCTL– generalized probabilistic causation, probabilistic noninterference, and quantitative information flow.

4.1 Conditional Probabilities and Generalized Causation

HyperPCTL∗ can handle conditional probabilities, which is not possible with PCTL∗. An example for the use of conditional probabilities is probabilistic causation [8]. Probabilistic causation asserts that the probability of occurring effect e if cause c happens is higher than the probability of occurring e when c does not happen. Now, let c and e be two formulas involving a set Π of paths. Then, the conditional probability of e given c can be specified by \( \frac{\mathbb{P}^Π(c^Π∧e^Π)}{\mathbb{P}^Π(c^Π)} \). With this, we can specify that c probabilistically causes e for the paths Π from a state σ as follows:

\[
\forall σ^Π. \left( \frac{\mathbb{P}^Π(c^Π∧e^Π)}{\mathbb{P}^Π(c^Π)} > \frac{\mathbb{P}^Π(¬c^Π∧e^Π)}{\mathbb{P}^Π(¬c^Π)} \right)
\]

(7)

The above formula can be quantified by more state variables. We note that HyperPCTL can express only very simple forms of probabilistic causation; namely, cases where the cause and the effect are simple propositional formulas. On the other hand, in HyperPCTL∗, the cause and the effect can be more complex formulas with temporal operators. For example, for two paths π1 and π2 that start from two different states, let c = \( \bigcirc(s^{π1} ∨ s^{π2}) \); i.e., at least one of the two executions is initially safe, and e = \( \bigboxdot(s^{π1} ∧ s^{π2}) \); i.e., eventually the two executions simultaneously stay safe forever. Then, one can rewrite (7), with Π = \( \{π_1, π_2\} \), and c and e are replaced by \( \bigcirc(s^{π1} ∨ s^{π2}) \) and \( \bigboxdot(s^{π1} ∧ s^{π2}) \). The resulting formula asserts that the probability of the two executions eventually stay safe if at least one of them is initially safe is higher than the case where both are initially unsafe.

4.2 Dinning Cryptographers

Three cryptographers sit round a table having dinner. Either one of the cryptographers or, alternatively, the National Security Agency (NSA) must pay for their meal. The cryptographers respect each other’s right to make an anonymous payment, but want to find out whether the NSA paid. So they decide to execute the following protocol:

- Every two cryptographers establish a shared one-bit secret by tossing an unbiased coin and only informs the cryptographer on the right of the outcome.
- Then, each cryptographer publicly states whether the two coins that it can see (the one it flipped and the one the left-hand neighbor flipped) agree, if he/she did not pay.
- However, if a cryptographer actually paid for dinner, then it instead states the opposite – disagree if the coins are the same and agree if the coins are different.
An even number of agrees indicates that the NSA paid, while an odd number indicates that a cryptographer paid.

The protocol can be modeled by a DTMC with the states labeled by the values of the Boolean variables mentioned below. In addition, the state labels \(C_{i}^{1,2,3}\) indicates that the cryptographer \(i\) paid, and \(C_{0}\) indicates that the NSA paid. The common shared secret between two cryptographers \(i\) and \(j\) are denoted by a Boolean variable \(l_{ij}\). The final result of the process is denoted by a Boolean variable \(r\), where \(r = 1\) if someone paid, and \(r = 0\) otherwise. We define an information-flow security condition that given that some cryptographer paid, the probability that either cryptographer \(i\) or \(j\) paid are equal irrespective of the common shared secret between them. This is specified by the following HyperPCTL* formula:

\[
\forall \sigma_{1}^{\pi_{1}}.\forall \sigma_{2}^{\pi_{2}}.\forall \sigma_{3}^{\pi_{3}}.\forall \sigma_{4}^{\pi_{4}}.\mathbb{P}^{\pi_{1}}\left(C_{i}^{\pi_{1}} \rightarrow \bigdiamond \left((l_{ij}^{\pi_{1}} = 0) \land \bigdiamond (r^{\pi_{1}} = 1)\right)\right) = \mathbb{P}^{\pi_{2}}\left(C_{i}^{\pi_{2}} \rightarrow \bigdiamond \left((l_{ij}^{\pi_{2}} = 1) \land \bigdiamond (r^{\pi_{2}} = 1)\right)\right) = \mathbb{P}^{\pi_{3}}\left(C_{j}^{\pi_{3}} \rightarrow \bigdiamond \left((l_{ij}^{\pi_{3}} = 0) \land \bigdiamond (r^{\pi_{3}} = 1)\right)\right) = \mathbb{P}^{\pi_{4}}\left(C_{j}^{\pi_{4}} \rightarrow \bigdiamond \left((l_{ij}^{\pi_{4}} = 1) \land \bigdiamond (r^{\pi_{4}} = 1)\right)\right),
\]

(8)

### 4.3 Quantitative Information Flow

*Quantitative information flow* (QIF) permits a restricted leakage of information from high security variables to low security variable. A general measure of the information of a discrete random variable and the information leakage from one discrete random variable to another is the Rényi entropy and Rényi divergence, defined as:

\[
H_{\alpha}(X) = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{n} p_{i}^{\alpha} \right), \quad H_{\alpha}(X \parallel Y) = \frac{1}{\alpha - 1} \log \left( \sum_{i=1}^{n} \frac{p_{i}^{\alpha}}{q_{i}^{\alpha-1}} \right).
\]

(9)

Here, \(X\) and \(Y\) are discrete random variables with \(n\) possible values, and \(p_{i}\) and \(q_{i}\) are the probability mass functions of \(X\) and \(Y\). The Rényi entropy reduces to max-entropy, Shannon entropy, Collision entropy, or min-entropy for \(\alpha = 0\), \(\alpha \to 1\), \(\alpha = 2\), or \(\alpha \to \infty\), respectively.

For an execution \(\pi\), let \(H_{\alpha}(\pi)\) be a high security input and \(L_{\pi}\) be a low security output. QIF with bound \(c\) for binary random variables can be written in PCTL* as:

\[
\forall \sigma^{\pi_{1},...,\pi_{5}}. \frac{1}{\alpha - 1} \log \left( \frac{P_{3}^{\alpha}}{(P_{1}P_{2})^{\alpha-1}} + \frac{P_{4}^{\alpha}}{(P_{1}(1 - P_{2}))^{\alpha-1}} \right) + \frac{(P_{2}(1 - P_{3}))^{\alpha}}{((1 - P_{3})P_{2})^{\alpha-1}} + \frac{(1 - P_{3} - P_{4} - P_{5})^{\alpha}}{((1 - P_{2})(1 - P_{3}))^{\alpha-1}} \leq c
\]

where
\[
P_1 = \mathbb{P}^{π_1}(L_{π_1} = 0) \quad P_2 = \mathbb{P}^{π_2}(\Diamond(H_{π_2} = 0))
\]
\[
P_3 = \mathbb{P}^{π_3}(L_{π_3} = 0 \land \Diamond(H_{π_3} = 0)) \quad P_4 = \mathbb{P}^{π_4}(L_{π_4} = 0 \land \Diamond(H_{π_4} = 1))
\]
\[
P_5 = \mathbb{P}^{π_5}(L_{π_5} = 1 \land \Diamond(H_{π_5} = 0)).
\]

5 Verifying Non-nested State Formulas

This section focuses on statistically verifying HyperPCTL* state formulas with non-nested probability operators on given states. Here, quantified state formulas can be verified on DTMCs by exploring on all the finite states. To simplify discussions, we introduce the following rule by replacing the arithmetic rules (4)(6) and the rules \(p \sim p, \ p \sim P, \ P \sim P\) in [3,5] by

\[
V \models (\llbracket P^{π_1}\varphi_1 \rrbracket_{V[I]} \ldots , \llbracket P^{π_n}\varphi_n \rrbracket_{V[I]} ) \in D \text { iff } (\Pr_{I_1}(\llbracket \varphi_1 \rrbracket_{V[I_1]} | P_1 \subseteq \text{Paths}(s_0)), \ldots , \Pr_{I_n}(\llbracket \varphi_n \rrbracket_{V[I_n]} | P_n \subseteq \text{Paths}(s_0)) ) \in D,
\]

where \(D \subseteq [0,1]^n\) is measurable. For example, the HyperPCTL* formula

\[
(f_1(P^{π_1}\varphi_1,f_2(P^{π_2}\varphi_2,P^{π_3}\varphi_3)) > c_1) \land (f_3(P^{π_3}\varphi_3) < c_2)
\]

can be equivalently written as \((P^{π_1}\varphi_1,P^{π_2}\varphi_2,P^{π_3}\varphi_3) \in I\), where

\[
D = \{(p_1,p_2,p_3) \in [0,1]^3 \mid f_1(p_1,f_2(p_2,p_3)) > c_1, f_3(p_2) < c_2\}.
\]

As a statistical approach is adopted, we assume that the exact probability of satisfying \(\varphi\) does not lie on the boundary of the test region \(D\), as stated in Assumption [1]. Furthermore, we assume that \(\varphi\) is verifiable for a finite time horizon \(H\) to simplify the termination condition for each sample.

**Assumption 1** In checking \(V \models (\llbracket P^{π_1}\varphi_1 \rrbracket_{V[I]} \ldots , \llbracket P^{π_n}\varphi_n \rrbracket_{V[I]} ) \in D\), we assume that (i) \(\varphi_i\) are verifiable for a finite time horizon \(H\) for \(i \in [n]\), (ii) the test region \(D\) is a simply connected domain with \(\mu_B(D) \neq 0\), and (iii)

\[
(\Pr_{I_1}(\llbracket \varphi_1 \rrbracket_{V[I_1]}), \ldots , \Pr_{I_n}(\llbracket \varphi_n \rrbracket_{V[I_n]})) \notin \partial D.
\]

**Remark 4.** Compared to previous studies on SMC using sequential probability ratio tests (SPRT) [2013], Assumption [1] is weaker as it requires no a priori knowledge on the indifference margin.

**Remark 5.** From Assumption [1] \(\llbracket P^{π_1}\varphi_1 \rrbracket_{V[I_1]} \ldots , \llbracket P^{π_n}\varphi_n \rrbracket_{V[I]} \) \(\in D_1\) and \(\llbracket P^{π_1}\varphi_1 \rrbracket_{V[I_1]} \ldots , \llbracket P^{π_n}\varphi_n \rrbracket_{V[I]} \) \(\in D_2\) are semantically equivalent if \(\overline{D_1} = \overline{D_2}\). In addition, \(P^{π} \in D\) and \(P^{π}(\neg \varphi) \in D^c\) are semantically equivalent with \(D^c = [0,1] \setminus D\).
5.1 Verifying $V \models \mathcal{V}_{\varphi} \in I_{V[s^I]}$

To verify $V \models \mathcal{V}_{\varphi} \in I_{V[s^I]}$ where $I \in [0, 1]$ and the initial state of the paths in $I$ is assigned to $s_0$, it suffices to solve the following hypothesis testing problem

$$
\begin{cases}
H_0 : & p_\varphi \in I, \\
H_1 : & p_\varphi \in [0, 1] \setminus I, \\
p_\varphi = \text{Pr}_I([\varphi]_{V[\pi] \in I} \mid I \in \text{Paths}(s_0)).
\end{cases}
$$

(10)

The SMC algorithms is designed by constructing the confidence intervals of $\varphi$. Given $\{\theta_1, \ldots, \theta_K\} \subseteq \text{Paths}(s_0)$ with $K = |I|$, we define

$$
\varphi([\theta]_K) = \begin{cases}
0, & \text{if } [\varphi]_{V[\pi_i \rightarrow \theta_i] \in [K]} \text{ is false}, \\
1, & \text{otherwise},
\end{cases}
$$

(11)

where $V[\pi_i \rightarrow \theta_i] \in [K]$ is the assignment derived from $V$ by assigning the sample path $\theta_i$ to the path variable $\pi_i$ for each $i \in [K]$.

Given $N$ independent sample paths $\{\theta\}_N \subseteq \text{Paths}(s_0)$ of the DTMC $M$ starting from the state $s$, each of length $H$, we consider the statistics

$$
T = \sum_{(j_1, \ldots, j_K) \in [N]^K} \varphi(\theta_{j_1}, \ldots, \theta_{j_K}).
$$

(12)

By the independence of sample paths $\{\theta\}_N$, the statistics $T$ is binomial in $T \sim \text{Binom}(N^K, p_\varphi)$. By Section 2.2, we can derive the confidence level for $p_\varphi \in I$. Let $I = \bigcup_{j=1}^M (a_j, b_j)$ with $a_j < b_j$. Without loss of generality, we assume that $b_j \neq a_{j+1}$; otherwise, $(a_j, b_j)$ and $(a_{j+1}, b_{j+1})$ can be merged into $(a_j, b_{j+1})$ by Assumption 1.

The confidence level for asserting $H_0$ is

$$
p_{H_0} = \sum_{j=1}^M F_{\text{CP}}(a_j, b_j \mid T, N^K)
$$

(13)

where $F_{\text{CP}}(\cdot \mid \alpha, \beta)$ is the cumulative probability of the beta distribution with parameters $(\alpha, \beta)$.

As computing the cumulative probabilities for the beta distribution can be computationally costly, we consider the lower bound of the confidence level $p_{H_0}$. If the maximum likelihood estimation of $p_\varphi$ satisfies $T/N^K \in (a_j, b_j)$ for some $J \in [M]$, then it holds that

$$
p_{H_0} \geq F_{\text{CP}}(a_j, b_j \mid T, N^K).
$$

(14)

By the law of large numbers, this bound is asymptotically tight for $N \rightarrow \infty$. The confidence level for asserting $p_{H_1}$ is derived similarly by swapping the two hypothesis $H_0$ and $H_1$. 
5.2 Verifying $V \models ([\mathbb{P}^{P_1} \varphi_1]_{V[s_1^n]}, \ldots, [\mathbb{P}^{P_n} \varphi_n]_{V[s_n^n]}) \in D$

The results derived in Section 5.1 extend to the non-nested formula involving multiple probability operators $V \models ([\mathbb{P}^{P_1} \varphi_1]_{V[s_1^n]}, \ldots, [\mathbb{P}^{P_n} \varphi_n]_{V[s_n^n]}) \in D$). Again, we consider the corresponding hypothesis testing problem

$$
\begin{cases}
H_0 : (p_{\varphi_1}, \ldots, p_{\varphi_n}) \in D, \\
H_1 : (p_{\varphi_1}, \ldots, p_{\varphi_n}) \in [0, 1]^n \setminus D,
\end{cases}
$$

(15)

where $p_{\varphi_i} = \Pr_{\Pi_i} ([\varphi]_{V[r_i \cap \Pi_i]} \mid \Pi_i \in \text{Paths}(s_i))$ for $i \in [n]$.

Given $N$ independent sample paths $\{\theta\}_N \subseteq \text{Paths}(s_0)$ of the DTMC $\mathcal{M}$ starting from the state $s$, each of length $H$, we consider for $i \in [n]$ the statistics,

$$
T_i = \sum_{(j_1, \ldots, j_{K_i}) \in [N]^K_i} \varphi_i(\theta_{j_1}, \ldots, \theta_{j_{K_i}}), \quad K_i = |\Pi_i|,
$$

(16)

where $T_i \sim \text{Binom}(N^{K_i}, p_{\varphi_i})$. For this multi-dimensional case, the exact confidence level $p_{H_0}$ of asserting $H_0$ is hard to derive. Therefore, we consider a lower bound of the confidence level $p_{H_0}$. If the maximum likelihood estimate of $(p_{\varphi_1}, \ldots, p_{\varphi_n})$ satisfies,

$$
\left(\frac{T_1}{N^{K_1}}, \ldots, \frac{T_n}{N^{K_n}}\right) \in \prod_{i \in [n]} [a_i, b_i] \subseteq D,
$$

(17)

for some hypercube $\prod_{i \in [n]} [a_i, b_i]$ contained in the test region $D$, then the confidence level for asserting $H_0$ is at least

$$
p_{H_0} \geq \prod_{i=1}^{n} F_{\text{CP}}(a_i, b_i \mid T, N^K),
$$

(18)

where $F_{\text{CP}}(a_j, b_j \mid T, N^K)$ is given by (13). And the same holds for $p_{H_1}$ by swapping the two hypothesis $H_0$ and $H_1$.

When implementing this verification approach, for each iteration, we can find a largest hypercube $\prod_{i \in [n]} [a_i, b_i]$ satisfying (17) to maximize the lower bound of the confidence level in (18). By Assumption 1 the Borel measure of the largest hypercube is non-zero. And by the law of large numbers, this bound is asymptotically tight for $N \to \infty$. Finally, note the difference between (13) and (18), where the test region is the union and product of intervals, respectively.

5.3 Sequential Verification

Following the previous discussions, we define the assertion function based on maximal likelihood estimation as

$$
\mathcal{A}(V \models ([\mathbb{P}^{P_1} \varphi_1]_{V[s_1^n]}, \ldots, [\mathbb{P}^{P_n} \varphi_n]_{V[s_n^n]}) \in D)) =
\begin{cases}
1, & \text{if } \left(\frac{T_1}{N^{K_1}}, \ldots, \frac{T_n}{N^{K_n}}\right) \in D \\
0, & \text{otherwise}
\end{cases}
$$

(19)

where $\mathcal{A} = 1$ stands for asserting $H_0$ and $\mathcal{A} = 0$ stands for asserting $H_1$. Based on the discussions in Sections 5.1 and 5.2 we derive the following theorem.
Algorithm 1 Sequential SMC of non-nested HyperPCTL$^*$

**Require:** DTMC $\mathcal{M}$, PCTL$^*$ formula $V = (([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)$, desired significance level $\alpha$, batch size $B$.

1. $N \leftarrow 0$, desired confidence level $p_A \leftarrow 0$
2. **for** $i \in [n]$ **do**
3. \hspace{1em} $K_i \leftarrow |\Pi_i|$, $T_i \leftarrow 0$
4. **end for**
5. **while** $p_A < 1 - \alpha$ **do**
6. \hspace{1em} $N \leftarrow N + B$
7. \hspace{2em} Draw $\theta_{N+1}, \ldots, \theta_{N+B} \in \text{Paths}(s_0)$ of length $H$.
8. \hspace{1em} **for** $i \in [n]$ **do**
9. \hspace{2em} $T_i \leftarrow T_i + \sum_{(j_1, \ldots, j_{K_i}) \in [N+B][K_i] \setminus \Pi_i} \varphi_i(\theta_{j_1}, \ldots, \theta_{j_{K_i}})$.
10. **end for**
11. Update $A(V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)$ by (19) and $p_A$
12. **end while**
13. **return** $A(V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)$ by Theorem 3

**Theorem 3.** Given $N$ independent sample paths $\{\theta\}_N \subseteq \text{Paths}(s_0)$ of the DTMC $\mathcal{M}$ starting from the state $s$, the confidence level $p_A$ for the assertion function $A(V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)$ given in (19), namely,

\[
A(V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)) = I(([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)),
\]

is at least (19), where $I$ is the indicator function.

**Proof.** From Section 2.2, the probability of $p_{\varphi} \in I$ is the summation of the probabilities of $p_{\varphi} \in (a_j, b_j)$ $j \in [M]$. The approximation (14) holds by the fact that $T/NK \rightarrow p_{\varphi}$ for $N \rightarrow \infty$ almost surely.

This allows us to propose an SMC algorithm for the non-nested HyperPCTL$^*$ formula $V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D)$ by monitoring the confidence level $p_A$ of $A(V = ([\mathbb{P}^H_1 \varphi_1]_{V[s_1^n]}), \ldots, [\mathbb{P}^H_n \varphi_n]_{V[s_n^n]} \in D))$ at each time. Given a desired significance level $\alpha$, the algorithm keeps on drawing sample paths until $p_A > 1 - \alpha$. To save computation costs, the samples may be drawn in batch size $b$. This is summarized by Algorithm 1 and Theorem 3.

**Theorem 4.** Algorithm 1 gives the correct answer with probability at least $1 - \alpha$.

**Proof.** Let $N$ be the step Algorithm 1 terminates, then $\Pr(\text{Answer is correct}) = \sum_{i \in \mathbb{N}} \Pr(\text{Answer is correct} | N = i) \Pr(N = i)$. By construction of the confidence intervals, $\Pr(\text{Answer is correct} | N = i) > 1 - \alpha$ for each $i \in \mathbb{N}$, and $\sum_{i \in \mathbb{N}} \Pr(N = i) = 1$, thus the theorem holds.
significance level $\alpha$

number of positive instantiation $\sum_i$ with unknown, we can estimate it statistically using the assertion. We note that $\Pi$ is simple. The main idea is as follows. For the instantiating of the path variables $\Pi_i \to \Theta_i$, where $i \in \mathbb{N}$ and $\Theta_i \subseteq \text{Paths}(s_0)$, let $T_i$ be the indicator of whether the sub-formula $\llbracket p_i \phi \rrbracket_{V[s_0]}$ is true under this instantiating. Though $T_i$ is unknown, we can estimate it statistically using the assertion $R_i$ of Algorithm 1 with significance level $\alpha_i$. Then, to verify the full formula $\psi$, we need to estimate the total number of positive instantiation $\sum_i T_i$. For any significance level $\alpha_0$, we can estimate it with $\sum R_i$ allowing some bounded error. Specifically, let $\{\sigma\}_N \subseteq \text{Paths}(s_0)$ be $N$

6 Verifying Nested State Formulas

Now we extend the SMC algorithms in Section 5 to nested HyperPCTL* formulas of finite time horizon $H$. In the sequel, we refer to HyperPCTL* (state) formulas without probability operator as simple formulas. A HyperPCTL* formula with nested probability operators is constructed iteratively in two ways: (i) replacing a simple sub-formula with an non-nested formula, and (ii) quantifying free path variables in a non-nested formula.

For (i), we consider the nested formula $\psi = \llbracket \llbracket p H \phi \rrbracket_{V[s_0]} \rrbracket_{J}$ with $\rho = (\llbracket p H_1 \phi_1 \rrbracket_{V[s_0]}, \ldots, \llbracket p H_n \phi_n \rrbracket_{V[s_0]}) \in I$, where $I, J \in [0, 1]$ are unions of intervals, and $p H \psi$ is a non-nested HyperPCTL* formula if $\rho$ is viewed as an atomic proposition. We note that $\Pi$ is disjoint with $\Pi_1, \ldots, \Pi_n$. To verify $\psi$, we following the compositional analysis similar to [16,14]. Specifically, we can statistically verify the sub-formula $\rho$ on each state $s$ of the DTMC $M$ with significance level $\alpha_s$, using Theorem 3 and Algorithm 1, and label the state with $\rho$ if the assertion by Theorem 4 is positive.

The overall significance level is $\alpha = \sum_{s \in S} \alpha_s + \alpha_0$.

To implement the sequential algorithm, given the overall significance level $\alpha$, we can split it into the summation $\sum_{s \in S} \alpha_s + \alpha_0$. The simplest way is $\alpha_s = \alpha_0 = \alpha / (|S| + 1)$ for $s \in S$, where $S$ is the number of states of the DTMC $M$. Then, we can employ Algorithm 1 to verify $\rho$ on each state $s$ with significance level $\alpha_s$ and then assert $\psi$ with significance level $\alpha_0$ using Algorithm 1 again. This is summarized by Algorithm 2.

For (ii), we consider verifying a HyperPCTL* formula $V \models \psi$ with $\psi = \llbracket p H_1, (p H_2 \phi \in I_2) \in I_1 \rrbracket_{V[s_0]}$, where $I_1, I_2 \in [0, 1]$ are unions of intervals, and $\phi$ is simple. The main idea is as follows. For the an instantiatiing of the path variables $\Pi_1 \to \Theta_i$, where $i \in \mathbb{N}$ and $\Theta_i \subseteq \text{Paths}(s_0)$, let $T_i$ be the indicator of whether the sub-formula $\llbracket p H_2 \phi \rrbracket_{V[s_0]}$ is true under this instantiating. Though $T_i$ is unknown, we can estimate it statistically using the assertion $R_i$ of Algorithm 1 with significance level $\alpha_i$. Then, to verify the full formula $\psi$, we need to estimate the total number of positive instantiation $\sum_i T_i$. For any significance level $\alpha_0$, we can estimate it with $\sum R_i$ allowing some bounded error. Specifically, let $\{\sigma\}_N \subseteq \text{Paths}(s_0)$ be $N$

Algorithm 2 Sequential SMC for nested HyperPCTL* (I)

Require: DTMC $M$, formula $V \models \llbracket p H \psi \rrbracket_{V[s_0]}$ with $\rho = (\llbracket p H_1 \phi_1 \rrbracket_{V[s_0]})$,

$\ldots, (\llbracket p H_n \phi_n \rrbracket_{V[s_0]}) \in I$, desired significance level $\alpha$.

1: Split $\alpha$ into $\sum_{s \in S} \alpha_s + \alpha_0$

2: for $s \in S$ do

3: Verify $V \models \llbracket \rho \rrbracket_{V[s_0]}$ by Algorithm 1 with significance level $\alpha_s$ and label $s$

4: end for

5: Verify $V \models \llbracket p H \psi \rrbracket_{V[s_0]}$ on the relabeled $M$ by Algorithm 1 with significance level $\alpha_0$. 
sample paths; then, we have statistical assertions
\[
R_i = A(V \mid \exists \sigma_2^i \in I_2 \forall \sigma_2^i, (j_1, \ldots, j_{K_1}) \in [N]^{K_1}, \Theta_i = \{\sigma_{j_1}, \ldots, \sigma_{j_{K_1}}\}
\]
\[
K_1 = |I_1|, \ i = \sum_{k=1}^{K_1} N_{k-1} j_k, \ (j_1, \ldots, j_{K_1}) \in [N]^{K_1},
\]
with significance level \( \alpha_i \) given by Theorem 4. The assertions \( R_i \) are used to estimate
\[
T = \sum_{i \in [N^K_1]} T_i
\]
with
\[
T_i = \mathbf{1}(\exists \sigma_2^i \in I_2 \forall \sigma_2^i, (j_1, \ldots, j_{K_1}) \in [N]^{K_1})
\]
Since \( R_i = T_i \) with significance \( \alpha_i \) for \( i \in [N^K_1] \), we have \( |T - R| > \Delta \) where
\[
\delta = 1 - F_{\text{Binom}}(\Delta \mid N^{K_1}, \max_{i \in [N^K_1]} \alpha_i),
\]
where \( F_{\text{Binom}} \) is the binomial cumulative distribution function. Therefore, the overall
significance level is
\[
\alpha = \delta + \arg\max_{x = R, \pm \Delta} \alpha_0(x)
\]
where \( \alpha_0(x) \) is the significance level given by Theorem 4 for \( x \) positive samples.

To implement the sequential algorithm, given the overall significance level \( \alpha \), we need to simultaneously decrease both \( \alpha_0 \) and \( \delta \), and thus, to increase both the number of samples \( N \) and the estimation margin \( \Delta \). As \( \delta \) is mainly determined by the expected value \( N^{K_1} \max_{i \in [N^K_1]} \alpha_i \), we start from \( \Delta = cN^{K_1} \max_{i \in [N^K_1]} \alpha_i \) and increase \( c \) by 1 when \( \delta > \alpha/2 \). This is summarized by Algorithm 3.

7 Evaluation

We numerically evaluate the proposed SMC algorithms on the Dining Cryptographers and Probabilistic Causation problems, described in Section 4. The simulations were performed on a laptop with Intel\textsuperscript{®} Core\textsuperscript{™} i7-7820HQ, 2.92GHz Processor and 32GB RAM. The accuracy of the proposed algorithms is estimated from 100 simulations for each case.

**Dining cryptographers** We statistically verified the protocol for three participants for wan approximate and simplified version of (3), namely,
\[
\varphi_{dc} = \forall \sigma_1 \in \exists \sigma_2^1 \cdot \exists \sigma_2^2 \cdot \exists \sigma_2^3 \cdot (C_1^{\sigma_2^1} \rightarrow \Diamond((t_{123}^{\sigma_2^1} = 0) \land \Diamond(r_{123} = 1)))
\]
\[
= \exists \sigma_2^1 \cdot (C_1^{\sigma_2^1} \rightarrow \Diamond((t_{23}^{\sigma_2^1} = 1) \land \Diamond(r_{23} = 1))) < \varepsilon
\]
and its negation, with the parameter \( \varepsilon \in \{0.05, 0.1, 0.2\} \) and significance level \( \alpha \in \{0.05, 0.1, 0.2\} \). We note that \( \varphi_{dc} \) is true and its negation is false for any \( \varepsilon > 0 \). The results for SMC of this specification and the negation are shown in Tables 1 and 2, respectively. The estimated accuracy agrees with the desired significance levels, showing that the proposed SMC algorithms provide the correct answer with the desired significance levels in all cases, while requiring very short analysis times. In addition, as expected, the analysis times increase with decrease in values for margin \( \varepsilon \) as well as significance \( \alpha \).
Algorithm 3 Sequential SMC for nested HyperPCTL$^\ast$ (II)

Require: DTMC $\mathcal{M}$, PCTL$^\ast$ formula $V \models [P^{H_1}(P^{H_2}\varphi) \in I_2]_{V[s_0]}$, desired significance level $\alpha$, parameter $c \leftarrow 2$.

1: $K_{1,2} \leftarrow |H_{1,2}|$, $T \leftarrow 0$, $N \leftarrow 0$, $T_i \leftarrow 0$ for $i \in [N]$.
2: Significance level $\tilde{\alpha} \leftarrow 1$, $\tilde{\alpha}_i \leftarrow 1$ for $i \in [N]$.
3: while $\tilde{\alpha} > \alpha$ do
4: $N \leftarrow N + 1$.
5: Draw $\sigma_{N+1} \in \text{Paths}(s_0)$ of length $H$.
6: for $i \in [N_{K_1}]$ do
7: Update $R_i$ by (20) and $\tilde{\alpha}_i$ by Theorem 3.
8: end for
9: $R \leftarrow \sum_{i \in N_{K_1}} R_i$, $\Delta \leftarrow cN_{K_1} \max_{i \in [N_{K_1}]} \tilde{\alpha}_i$.
10: Update $\delta$ by (21) and $\alpha$, $\bar{T}$ by (22).
11: if $\delta > \alpha/2$ then
12: $c \leftarrow c + 1$
13: end if
14: end while
15: return Assertion given by Theorem 3 with $\bar{T}$ positives among $N_{K_1}$ total samples for test region $I_1$.

| Margin $\varepsilon$ | Significance $\alpha$ | Accuracy | Sample Cost | Time(s) |
|----------------------|-----------------------|----------|-------------|---------|
| 0.2                  | 0.2                   | 0.98     | 163         | 0.039   |
| 0.1                  | 0.2                   | 0.95     | 603         | 0.143   |
| 0.05                 | 0.2                   | 0.85     | 1916        | 0.445   |
| 0.2                  | 0.1                   | 1.0      | 227         | 0.051   |
| 0.1                  | 0.1                   | 1.0      | 951         | 0.225   |
| 0.05                 | 0.1                   | 0.98     | 3433        | 0.801   |
| 0.2                  | 0.05                  | 1.0      | 301         | 0.073   |
| 0.1                  | 0.05                  | 1.0      | 1187        | 0.277   |
| 0.05                 | 0.05                  | 0.97     | 4577        | 1.081   |

Table 1. Statistically verifying $\varphi_{dc}$ for Dining Cryptographers of size three.

Probabilistic causation We statistically verified an approximate version of (7), namely, the hyperproperty:

$$\varphi_{pc} = \forall \sigma_{(\pi_1, \pi_2)} \frac{P(\pi_1, \pi_2)(c(\pi_1, \pi_2) \land e(\pi_1, \pi_2))}{P(\pi_1, \pi_2)(c(\pi_1, \pi_2))} - \frac{P(\pi_1, \pi_2)(\neg c(\pi_1, \pi_2) \land e(\pi_1, \pi_2))}{P(\pi_1, \pi_2)(\neg c(\pi_1, \pi_2))} > \varepsilon,$$

with $c = \bigcirc(s^{\pi_1} \lor s^{\pi_2})$ and $e = \bigdiamond s^{\pi_1} \land s^{\pi_2}$. We statistically verify $\varphi_{pc}$ with the parameter $\varepsilon \in \{0.05, 0.1, 0.2\}$ and significance level $\alpha \in \{0.05, 0.1, 0.2\}$ on a labeled Markov chain, where $\varphi_{pc}$ is true for $\varepsilon < 0.16$ by symbolic analysis. As shown in Table 3, the results of the proposed SMC algorithms agree with the symbolic analysis. The estimated accuracy agrees with the desired significance levels in most cases,
Table 2. Statistically verifying $\neg \varphi_{dc}$ for Dining Cryptographers of size three.

| Margin $\epsilon$ | Significance $\alpha$ | Accuracy | Sample Cost | Time(s) |
|------------------|---------------------|----------|-------------|---------|
| 0.2              | 0.2                 | 0.94     | 31          | 0.005   |
| 0.1              | 0.2                 | 0.99     | 25          | 0.004   |
| 0.05             | 0.2                 | 1.0      | 21          | 0.004   |
| 0.2              | 0.1                 | 0.97     | 49          | 0.008   |
| 0.1              | 0.1                 | 0.99     | 35          | 0.006   |
| 0.05             | 0.1                 | 1.0      | 29          | 0.005   |
| 0.2              | 0.05                | 0.95     | 74          | 0.012   |
| 0.1              | 0.05                | 1.0      | 42          | 0.007   |
| 0.05             | 0.05                | 1.0      | 35          | 0.005   |

Table 3. Statistically verify Probabilistic Causation $\varphi_{pc}$.

| Margin $\epsilon$ | Significance $\alpha$ | Assertion | Accuracy | Sample Cost | Time(s) |
|------------------|---------------------|-----------|----------|-------------|---------|
| 0.2              | 0.2                 | False     | 0.93     | 3485        | 1.782   |
| 0.1              | 0.2                 | True      | 0.82     | 3690        | 1.864   |
| 0.05             | 0.2                 | True      | 1.0      | 3020        | 1.554   |
| 0.2              | 0.1                 | False     | 0.99     | 6714        | 3.435   |
| 0.1              | 0.1                 | True      | 0.89     | 6605        | 3.386   |
| 0.05             | 0.1                 | True      | 1.0      | 6279        | 3.211   |
| 0.2              | 0.05                | False     | 1.0      | 10409       | 5.289   |
| 0.1              | 0.05                | True      | 0.93     | 10658       | 5.456   |
| 0.05             | 0.05                | True      | 1.0      | 10576       | 5.434   |

except for small deviations for the two boldface entries. We believe this was caused by randomness, exhibited since only 100 runs were used.

8 Conclusion

In this paper, we studied the problem of statistical model checking (SMC) of probabilistic hyperproperties on discrete-time Markov chains. First, to reason about probabilistic hyperproperties, we introduced the probabilistic temporal logic HyperPCTL$^*$ that extends PCTL$^*$ by allowing explicit and simultaneous quantification over paths. HyperPCTL$^*$ also generalizes HyperPCTL [1] by incorporating nested temporal and probability operators. In terms of SMC, we first considered SMC of HyperPCTL$^*$ specifications with non-nested probability operators using fixed-size samples or for desired confidence levels. Unlike existing SMC algorithms based on sequential probability ratio tests (SPRT), we used the Clopper-Pearson confidence interval to avoid the need of a priori knowledge on the indifference margin. Then, we extended the proposed SMC algorithms to be able to handle HyperPCTL$^*$ specifications with multiple probability operators that are nested in different ways. Finally, we evaluated these SMC algorithms on two classical security problems, dining cryptographers and probabilistic causation.


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