Solar System Constraints on Gauss-Bonnet Dark Energy

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Abstract. Quadratic curvature Gauss-Bonnet gravity may be the solution to the dark energy problem, but a large coupling strength is required. This can lead to conflict with laboratory and planetary tests of Newton’s law, as well as light bending. The corresponding constraints are derived. If applied directly to cosmological scales, the resulting bound on the density fraction is $|\Omega_{GB}| \lesssim 3.6 \times 10^{-32}$.

Keywords: dark energy, experimental gravity tests, string cosmology

PACS: 04.25.Nx, 04.50.+h, 04.80.Cc, 95.36.+x

GAUSS-BONNET GRAVITY AND THE SOLAR SYSTEM

Corrections to Einstein gravity, such as the string-motivated Gauss-Bonnet term $\mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ could explain the current accelerated expansion of our universe. On its own, in four dimensions, the Gauss-Bonnet term does not contribute to the gravitational field equations. Coupling it to a scalar field will produce a non-trivial effect, which could act as effective dark energy. Including the corresponding higher order scalar kinetic terms, we obtain the ghost-free, quadratic curvature, gravitational Lagrangian

$$\mathcal{L} = \sqrt{-g} \left\{ R - (\nabla \phi)^2 + \xi_1 \mathcal{L}_{GB} + \xi_2 G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \xi_3 (\nabla \phi)^2 \nabla^2 \phi + \xi_4 (\nabla \phi)^4 \right\}. \quad (1)$$

The gravity modifications will not only be felt at cosmological scales, but also within the solar system where high precision gravitational measurements have been performed. The fields there are relatively weak and slowly varying, allowing us to use the approximate post-Newtonian metric [1]

$$ds^2 = -(1 + 2\Phi/c^2)(c\, dt)^2 + (1 - 2\Psi/c^2)\, dx^i\, dx_i + \mathcal{O}(\epsilon^{3/2}), \quad (2)$$

with $\Phi, \Psi \sim c^2\epsilon$, and $\partial_i \sim \epsilon^{1/2}$. We take $\phi = \phi_0 + \mathcal{O}(\epsilon)$, with $\phi_0$ a constant. For standard Einstein gravity $\Phi = \Psi = -Gm/r$. We find the expansion parameter satisfies $\epsilon \lesssim 10^{-5}$.

A perturbative analysis of the gravitational field equations can now be performed. However, it should be noted that if the field-dependent couplings $\xi_i(\phi)$ are to produce cosmological acceleration, they must be large. With this in mind we make no assumptions on the relative magnitude of $\xi_i(\phi_0)$ and $\epsilon$. For simplicity we will assume the $\xi_i(\phi)$, and all of their derivatives, are of the same order. This is the case, for example, when $\phi$ arises from a toroidal compactification of a higher dimensional space [2].

To leading order in $\epsilon$, the scalar field equation is

$$c^4 \Delta \phi = -4\xi_1 \mathcal{D}(\Phi, \Psi) + \mathcal{O}(\epsilon^2, \xi \epsilon^3 / r^2), \quad (3)$$
where \( \xi' = \partial \xi / \partial \phi \), evaluated at \( \phi = \phi_0 \). The Einstein equations take the form
\[
\Delta \Phi = 4 \pi G \rho_m - 2 \xi' \phi (\Phi + \Psi, \phi) + \partial (\xi^2, \xi^3 / r^2),
\]
(4)
\[
\Delta \Psi = 4 \pi G \rho_m - 2 \xi' \phi (\Psi, \phi) + \partial (\xi^2, \xi^3 / r^2),
\]
(5)
with \( \rho_m \) the matter energy density in the solar system. We have defined the operators
\[
\Delta X = \sum_i X_{ii}, \quad \mathcal{D}(X, Y) = \sum_{i,j} X_{ij} Y_{ij} - \Delta X \Delta Y.
\]
(6)

To agree with observation, the solution to the above equations must be close to the usual Einstein gravity results. We can therefore assume \( \Phi = -Gm/r + \mathcal{O}(\xi^3) \), etc., from which we obtain, to leading order [3]
\[
\phi = \phi_0 - 2 \xi' \left( \frac{Gm}{c^4 r^4} \right), \quad \Phi = -\frac{Gm}{r} - \frac{64 \xi' \xi^2 (Gm)^3}{7 c^4 r^7}, \quad \Psi = -\frac{Gm}{r} - \frac{32 \xi' \xi^2 (Gm)^3}{7 c^4 r^7}.
\]
(7)

We see there are mass-dependent, \( 1/r^7 \) corrections, which are not covered by the usual parametrised post-Newtonian formalism [1]. This is in agreement with [4], but not [5] (which does not allow for the possibility that the couplings \( \xi \), could be large).

**Planetary motion**

Planets in our solar system experience a gravitational acceleration \( g_{\text{acc}} = -Gm/r^2 \), resulting in elliptical orbits with period \( 2\pi \sqrt{a^3/(Gm)} \), where \( a \) is the semi-major axis of the planet and \( m \) is the sun’s mass. Corrections to the Newtonian potential alter the effective mass felt by the planets [6, 7]. From (7) we obtain [3]
\[
g_{\text{acc}}(r) = -\frac{d\Phi}{dr} = -\frac{Gm}{r^2} \left[ 1 - \frac{64 \xi' \xi^2 \xi^2 r_g^7}{7 c^4 r^7} \right] \equiv -\frac{G(m + \delta m)}{r^2}
\]
(8)

where \( r_g \equiv Gm/c^2 \approx 1.5 \) km is gravitational radius of the sun. To agree with observation, the correction must be smaller than the uncertainty in \( a \), so \( \delta m / m < 3 \delta a / a \).

The strongest bound comes from Mercury (with \( a \approx 5.8 \times 10^7 \) km and \( \delta a \approx 0.11 \) m [8])
\[
|\xi'|| \lesssim \frac{\sqrt{3a^5 \delta a}}{8r_g} \approx 3.8 \times 10^{16} \text{km}^2.
\]
(9)

Applying this directly to Gauss-Bonnet density fraction [2] in cosmology, we find
\[
|\Omega_{GB}| = \left| 4 \xi' d\phi / dt \right| \lesssim 8.8 \times 10^{-30}
\]
(10)

if \( d\phi / dt \sim H \), and if \( \xi' \) has comparable values on local and cosmological scales. This value is far short the 0.7 required to solve the dark energy problem.
For a cosmological constant $\Phi = -Gm/r - r^2 c^2 \Lambda/6 + \cdots$. The corresponding bound comes from Mars [7] ($a \approx 2.3 \times 10^8$ km, $\delta a \approx 0.66$ m [8]) and is

$$|\Lambda| \lesssim \frac{9 r_g \delta a}{a^4} \lesssim 1.2 \times 10^{-34} \text{ km}^{-2}.$$  

(11)

This implies $\Omega_\Lambda = \Lambda/(3H^2) \lesssim 7.3 \times 10^{11}$, which is vastly weaker than the corresponding cosmological constraint ($\Omega_\Lambda \lesssim 1$).

Cassini spacecraft

An even stronger constraint is obtained from signals between the earth (at $r_\oplus \approx 1.5 \times 10^8$ km) and Cassini spacecraft (at $r_e \approx 1.3 \times 10^9$ km) as it travelled to Saturn. For a round trip, the sun’s gravitational field produces a time delay in the signals of [3]

$$c\Delta t = 2 \int_{r_{\text{ray}}} \left( \sqrt{\frac{g_{xx}}{g_{tt}}} - 1 \right) dx \approx -2 \int_{r_{\text{ray}}} (\Phi + \Psi) dx \approx 4r_g \ln \frac{r_\oplus r_e}{4b^2} + \frac{1024 \xi' r_g^3}{b^6},$$  

(12)

where the impact parameter $b$, is the smallest value of $r$ on the signal’s path. In 2002 it fell to its lowest value, $b \approx 1.1 \times 10^6$ km.

Rather than directly measure $\Delta t$, the Cassini experiment actually found the frequency shift in the signal [9]

$$y_{gr} = \frac{d\Delta t}{dt} = \frac{d\Delta t}{db} \frac{db}{dt} = -10^{-5} \frac{s}{b} \left[ 2 + (2.1 \pm 2.3) \times 10^{-5} \right].$$  

(13)

Requiring the Gauss-Bonnet correction (12) to be within the measured range (13) implies the bounds

$$|\xi'| \lesssim 1.6 \times 10^{14} \text{ km}^2; \quad |\Omega_{GB}| \lesssim 3.6 \times 10^{-32}.$$  

(14)

A table-top laboratory test of Newton’s law

Laboratory tests will also constrain modified gravity, as we will illustrate with the experiment described in [10]. It consists of a 60 cm copper bar, suspended at its midpoint by a tungsten wire. Two 7.3 kg masses 105 cm from the bar produce a torque $N_{105}$ on the bar, and an $m \approx 43$ g mass 5 cm to the side of bar produces a comparable torque $-N_5$. By changing the positions of the masses, the ratio $R = N_{105}/N_5$ was determined and compared to theory

$$\delta_R = \frac{R_{\text{expt}}}{R_{\text{Newton}}} - 1 = (1.2 \pm 7) \times 10^{-4}.$$  

(15)

The Gauss-Bonnet term affects all the masses, and gives cross-terms due its non-linearity. However we can ignore these complications, and just use (7) for the mass $m,$
since it gives the dominant correction. A mass at $\vec{X} = (X, Y, Z)$ produces a torque

$$N = \int_{\text{bar}} d^3x (\vec{x} \wedge \vec{F}) \, \propto \int_{\text{bar}} d^3x \left[ \frac{yX - xY}{r} \frac{d\Phi}{dr} \right]_{r = |\vec{X} - \vec{x}|}. \tag{16}$$

We find $\delta N_5/N_5 \approx -0.003 (G m \xi_1')^2 c^{-4}$ cm$^{-6}$. Requiring $\delta N_5/N_5 < \delta R$, gives the bound $|\xi_1'| \lesssim 1.3 \times 10^{16}$ km$^2$, which is comparable to the planetary constraint (9).

**DISCUSSION**

Extrapolating solar system constraints to cosmological scales suggests that the density fraction $\Omega_{GB}$ is far too small to solve the dark energy problem. However our analysis features many assumptions, which while credible, could be violated and thus offer a way round the constraints. Clearly at least one of them must be broken if Einstein-Gauss-Bonnet gravity is to explain the acceleration of our universe.

In particular, we applied solar system results directly to cosmological scales. This assumes no significant spatial or temporal evolution of the field $\phi$. Significant variation in the couplings $\xi_i$ seems to offer the best way to save Gauss-Bonnet dark energy. Another possibility is that $\phi$ couples differently to dark matter and baryons, which will also break the relation between the two scales.

Instead, it may be that our assumptions on the form of the theory should be changed. The scalar field could be coupled directly to the Einstein-Hilbert term, as in Brans-Dicke gravity. Additionally, the couplings $\xi_i$ and their derivatives could be of different orders. Both these changes open up the possibility of the corrections to Einstein gravity cancelling within the solar system. Alternatively $\phi$ could be given a large mass, which would suppress the quadratic curvature effects, as they operate via the scalar field. However this is also likely to inhibit acceleration.

**ACKNOWLEDGMENTS**

I am grateful to the Netherlands Organisation for Scientific Research (NWO) for financial support, to my collaborators C. Charmousis and L. Amendola, and of course to the conference organisers themselves.

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