CHARACTERIZATIONS OF NORM–PARALLELISM IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. In this paper, we consider the characterization of norm–parallelism problem in some classical Banach spaces. In particular, for two continuous functions \( f, g \) on a compact Hausdorff space \( K \), we show that \( f \) is norm–parallel to \( g \) if and only if there exists a probability measure (i.e. positive and of full measure equal to 1) \( \mu \) with its support contained in the norm attaining set \( \{ x \in K : |f(x)| = \|f\| \} \) such that \( \left| \int_{K} f(x)g(x)d\mu(x) \right| = \|f\| \|g\| \).

1. Introduction and preliminaries

Let \((X, \| \cdot \|)\) be a normed space and denote, as usual, by \( B_X \) and \( S_X \) its closed unit ball and unit sphere, respectively, and denote the topological dual of \( X \) by \( X^* \). Let \( C_b(\Omega) \) and \( C(K) \) denote the Banach spaces of all bounded continuous functions on a locally compact Hausdorff space \( \Omega \), with the usual norm \( \|f\| = \sup_{x \in \Omega} |f(x)| \) (\( f \in C_b(\Omega) \)) and all continuous functions on a compact Hausdorff space \( K \), with the usual norm \( \|f\| = \max_{x \in K} |f(x)| \) (\( f \in C(K) \)), respectively. By \( C_u(B_X, X) \) we denote the space of all uniformly continuous \( X \)-valued functions on \( B_X \) endowed with the supremum norm. Given a bounded function \( f : S_X \rightarrow X \), its numerical radius is

\[
v(f) := \sup\{ |x^*(f(x))| : \|x^*\| = x^*(x) = 1 \}.
\]

Let us comment that for a bounded function \( f : B_X \rightarrow X \), the above definitions apply by just considering \( v(f) := v(f|_{S_X}) \).

Recall that an element \( x \in X \) is said to be norm–parallel to another element \( y \in X \) (see [12, 17]), in short \( x \parallel y \), if

\[
\| x + \lambda y \| = \|x\| + \|y\| \quad \text{for some } \lambda \in \mathbb{T}.
\]

Here, as usual, \( \mathbb{T} \) is the unit circle of the complex plane. In the framework of inner product spaces, the norm–parallel relation is exactly the usual vectorial parallel relation, that is, \( x \parallel y \) if and only if \( x \) and \( y \) are linearly dependent. In the setting of normed linear spaces, two linearly dependent vectors are norm–parallel, but the converse is false in general. Notice that the norm–parallelism is symmetric and \( \mathbb{R} \)-homogenous, but not transitive (i.e., \( x \parallel y \) and \( y \parallel z \) \( \Rightarrow \) \( x \parallel z \); see [18, Example 2.7], unless \( X \) is smooth at \( y \); see [15, Theorem 3.1]).
In the context of continuous functions, the well-known Daugavet equation
\[ \|T + Id\| = \|T\| + 1 \]
is a particular case of parallelism. Here \(Id\) denotes, as usual, the identity function. Applications of this equation arise in solving a variety of problems in approximation theory; see [14] and the references therein.

Some characterizations of the norm–parallelism for operators on various Banach spaces and elements of an arbitrary Hilbert \(C^*-\)module were given in [2, 3, 7, 9, 13, 15, 16, 17, 18].

In particular, for bounded linear operators \(T, S\) on a Hilbert space \((H, [\cdot, \cdot])\), it was proved in [17, Theorem 3.3] that \(T \parallel S\) if and only if there exists a sequence of unit vectors \(\{\xi_n\}\) in \(H\) such that
\[ \lim_{n \to \infty} |[T\xi_n, S\xi_n]| = \|T\| \|S\|. \]
Further, for compact operators \(T, S\) it was obtained in [16, Theorem 2.10] that \(T \parallel S\) if and only if there exists a unit vector \(\xi \in H\) such that
\[ |[T\xi, S\xi]| = \|T\| \|S\|. \]

In [18], the authors also considered the characterization of norm parallelism problem for operators when the operator norm is replaced by the Schatten \(p\)-norm \((1 < p < \infty)\). More precisely, it was proved in [18, Proposition 2.19] that \(T \parallel S\) in the Schatten \(p\)-norm if and only if
\[ \left| \text{tr}([T^{p-1}U^*S]) \right| = \|T\|_{p}^{p-1}\|S\|_{p}, \]
where \(T = U|T|\) is the polar decomposition of \(T\).

Some other related topics can be found in [1, 5, 6, 7, 10, 16], and the references therein.

It is our aim in the next section to give characterizations of the norm–parallelism in \(C_b(\Omega)\) and \(C(K)\). More precisely, for \(f, g \in C_b(\Omega)\) we prove that \(f \parallel g\) if and only if there exists a sequence of probability measures \(\mu_n\) concentrated at the set \(\{x \in \Omega : |f(x)| \geq \|f\| - \varepsilon\}\) \((\varepsilon > 0)\) such that
\[ \lim_{n \to \infty} \int_{\Omega} \overline{f(x)} g(x) d\mu_n(x) = \|f\|^{-1}\|g\| \lim_{n \to \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x). \]
Moreover, for \(f, g \in C(K)\) we show that \(f \parallel g\) if and only if there exists a probability measure \(\mu\) with support contained in the norm attaining set \(\{x \in K : |f(x)| = \|f\|\}\) such that
\[ \int_{K} \overline{f(x)} g(x) d\mu(x) = \|f\| \|g\|. \]

Finally, in the next section, we state a characterization of the norm-parallelism for uniformly continuous \(X\) valued functions on \(B_X\) to the identity function. Actually, we show that if \(X\) is a Banach space and \(f \in C_u(B_X, X)\), then \(f \parallel Id\) if and only if \(\|f\| = v(f)\).
2. Main results

We begin with the following results, which will be useful in other contexts as well.

**Lemma 2.1.** [17, Theorem 4.1] Let \( X \) be a normed space. For \( x, y \in X \) the following statements are equivalent:

(i) \( x \parallel y \).

(ii) There exists a norm one linear functional \( \varphi \) over \( X \) such that \( \varphi(x) = \|x\| \) and \( |\varphi(y)| = \|y\| \).

**Lemma 2.2.** [8, Theorem 3.1] Let \( \Omega \) be a locally compact Hausdorff space and let \( f, g \in C^b(\Omega) \). Then

\[
\lim_{t \to 0^+} \frac{\|f + tg\| - \|f\|}{t} = \inf_{\epsilon > 0} \sup_{x \in M^\epsilon_f} Re(e^{-i \arg(f(x))} g(x)),
\]

where \( M^\epsilon_f = \{ x \in \Omega : |f(x)| \geq \|f\| - \epsilon \} \).

We use some techniques of [8] to prove the following theorem. Recall that a probability measure is a positive measure of total mass 1.

**Theorem 2.3.** Let \( \Omega \) be a locally compact Hausdorff space and let \( f, g \in C^b(\Omega) \). Then the following statements are equivalent:

(i) For every \( \epsilon > 0 \), there exists a sequence of probability measures \( \mu_n \) concentrated at \( M^\epsilon_f \) such that

\[
\|f\| \lim_{n \to \infty} \int_{\Omega} |f(x)g(x)| \mu_n(x) = \|g\| \lim_{n \to \infty} \int_{\Omega} |f(x)|^2 \mu_n(x).
\]

(ii) \( f \parallel g \).

Proof. (i)\( \Rightarrow \) (ii) Suppose (i) holds. Let \( \epsilon > 0 \). So, there exist a sequence of probability measures \( \mu_n \) concentrated at \( M^\epsilon_f \) and \( \lambda \in \mathbb{T} \) such that

\[
\|g\| \lim_{n \to \infty} \int_{\Omega} |f(x)|^2 \mu_n(x) = \lambda \|f\| \lim_{n \to \infty} \int_{\Omega} f(x)g(x) \mu_n(x),
\]

and hence

\[
\lim_{n \to \infty} \int_{\Omega} \overline{f(x)} \left( \|g\| f(x) - \lambda \|f\| g(x) \right) \mu_n(x) = 0. \quad (2.1)
\]

Let us now define a linear functional \( \varphi : \text{span}\{f, (\|g\|f - \lambda \|f\|g)\} \longrightarrow \mathbb{C} \) by setting

\[
\varphi(\alpha f + \beta (\|g\|f - \lambda \|f\|g)) = \alpha \|f\| \quad (\alpha, \beta \in \mathbb{C}).
\]
Thus

\[
\left\| \alpha f + \beta \left( \|g\| f - \lambda \|f\| g \right) \right\|^2 = \sup_{x \in \Omega} \left( |\alpha|^2 |f(x)|^2 + 2\text{Re} \left[ \overline{\alpha} \beta f(x) (\|g\| f(x) - \lambda \|f\| g(x)) \right] \right.
\]

\[
\left. + |\beta|^2 \|g\| f(x) - \lambda \|f\| g(x) \|^2 \right) \right) - \epsilon)^2 + 2\text{Re} \left[ \overline{\alpha} \beta \int_{\Omega} \overline{f(x)} (\|g\| f(x) - \lambda \|f\| g(x)) d\mu_n(x) \right].
\]

(since $\int_{\Omega} \beta^2 \|g\| f(x) - \lambda \|f\| g(x) \|^2 d\mu_n(x) \geq 0$)

\[\geq |\alpha|^2 (\|f\| - \varepsilon)^2 + 2\text{Re} \left[ \overline{\alpha} \beta \int_{\Omega} \overline{f(x)} (\|g\| f(x) - \lambda \|f\| g(x)) d\mu_n(x) \right]. \quad (2.2)\]

Letting $\varepsilon \to 0^+$ and $n \to \infty$ in (2.2), then by (2.1) we obtain

\[\left\| \alpha f + \beta \left( \|g\| f - \lambda \|f\| g \right) \right\|^2 \geq |\alpha| \|f\| \]

Hence $|\varphi \left( \alpha f + \beta \left( \|g\| f - \lambda \|f\| g \right) \right) | \leq \left\| \alpha f + \beta \left( \|g\| f - \lambda \|f\| g \right) \right\| \text{ and } \varphi(f) = \|f\|$. Thus $\|\varphi\| = 1$. Therefore, the Hahn-Banach theorem extends $\varphi$ to a linear functional $\tilde{\varphi}$ on $C_b(\Omega)$, with $\|\tilde{\varphi}\| = 1$. Since $\tilde{\varphi}(f) = \|f\|$ and $\tilde{\varphi}(\|g\| f - \lambda \|f\| g) = 0$, we get $\tilde{\varphi}(\lambda g) = \|g\|$, hence $|\tilde{\varphi}(g)| = \|g\|$. Now, by Lemma 2.1, we conclude that $f \parallel g$.

(ii)$\Rightarrow$(i) Let $f \parallel g$. By Lemma 2.1, there exists a norm one linear functional $\varphi$ over $C_b(\Omega)$ such that $\varphi(f) = \|f\|$ and $|\varphi(g)| = \|g\|$. So, there exists $\lambda \in \mathbb{T}$ such
that \( \varphi(\lambda g) = \|g\| \). From \( \|\varphi\| = 1, \varphi(f) = \|f\| \) and \( \varphi(\lambda g) = \|g\| \) it follows that

\[
\frac{\|f + re^{i\theta}(\|g\|f - \lambda\|f\|g)\| - \|f\|}{r} \geq \frac{\varphi(f + re^{i\theta}(\|g\|f - \lambda\|f\|g)) - \|f\|}{r}
\]

\[
= \frac{\varphi(f) + re^{i\theta}\|g\|\varphi(f) - re^{i\theta}\|f\|\varphi(\lambda g) - \|f\|}{r}
\]

\[
= \frac{\|f\| + re^{i\theta}\|g\|\|f\| - re^{i\theta}\|f\||\|g\| - \|f\|}{r} = 0
\]

for all \( r > 0 \) and all \( \theta \in [0, 2\pi) \). Hence, by Lemma 2.2, we get

\[
\inf_{\varepsilon > 0} \sup_{x \in M_f} \Re\left(e^{i\theta}e^{-i\arg(f(x))}(\|g\|f(x) - \lambda\|f\|g(x))\right) \geq 0
\]

for all \( \theta \in [0, 2\pi) \). Thus for all \( \varepsilon > 0 \) the set

\[
K_\varepsilon := \left\{e^{-i\arg(f(x))}(\|g\|f(x) - \lambda\|f\|g(x)) : x \in M_f\right\}
\]

contains at least one element with nonnegative real part under all rotations around the origin. Hence the values of the function \( \|g\|f - \lambda\|f\|g \) on \( M_f^\varepsilon \) are not contained in an open half plane with boundary that contains the origin. So, for all \( \varepsilon > 0 \) the closed convex hull of the set \( K_\varepsilon \) contains the origin. The convex hull of \( K_\varepsilon \) consists of points of the form \( \int_{\Omega} e^{-i\arg(f(x))}(\|g\|f(x) - \lambda\|f\|g(x))d\mu_\varepsilon(x) \), where \( \mu_\varepsilon \) is a probability measure supported on a finite subset of \( M_f^\varepsilon \) (see [11], chap. 3). Let \( n_0 \in \mathbb{N} \) and \( \frac{1}{n_0} < \varepsilon \). Thus for every \( n \geq n_0 \), there is a probability measure \( \mu_n \) concentrated at \( M_f^\varepsilon \) such that

\[
\left|\int_{\Omega} e^{-i\arg(f(x))}(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)\right| < \frac{1}{n}. \tag{2.3}
\]

We have

\[
\left|\int_{\Omega} \overline{f(x)}(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)\right|
\]

\[
= \left|\int_{\Omega} \overline{f(x)} - \|f\|e^{-i\arg(f(x))}\right|(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)
\]

\[
+ \int_{\Omega} \|f\|e^{-i\arg(f(x))}\right|(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)
\]

\[
\leq \left|\int_{\Omega} \overline{f(x)} - \|f\|e^{-i\arg(f(x))}\right|(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)\right| \tag{2.4}
\]

\[
+ \|f\|\int_{\Omega} e^{-i\arg(f(x))}(\|g\|f(x) - \lambda\|f\|g(x))d\mu_n(x)\right|.
\]
Furthermore, since \( \|f\| - \frac{1}{n} \leq |f(x)| \leq \|f\| \) for all \( x \in M_f \) we have
\[
\left| \int_{\Omega} \left( \overline{f(x)} - \|f\|e^{-i\arg(f(x))} \right) \left( \|g\|f(x) - \lambda\|f\|g(x) \right) d\mu_n(x) \right|
\leq \frac{1}{n} \int_{\Omega} \|g\|f(x) - \lambda\|f\|g(x) d\mu_n(x)
\leq \frac{1}{n} \left( \|g\| \int_{\Omega} |f(x)| d\mu_n(x) + \|f\| \int_{\Omega} |g(x)| d\mu_n(x) \right)
\leq \frac{2}{n} \|f\| \|g\|. \tag{2.5}
\]
By (2.3), (2.4) and (2.5) we get
\[
\left| \int_{\Omega} \overline{f(x)} \left( \|g\|f(x) - \lambda\|f\|g(x) \right) d\mu_n(x) \right| \leq \frac{2}{n} \|f\| \|g\| + \|f\| \frac{1}{n}. \tag{2.6}
\]
Taking \( \lim_{n \to \infty} \) in (2.6), we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \overline{f(x)} \left( \|g\|f(x) - \lambda\|f\|g(x) \right) d\mu_n(x) = 0,
\]
and hence
\[
\|f\| \lim_{n \to \infty} \int_{\Omega} \overline{f(x)}g(x) d\mu_n(x) = \|g\| \lim_{n \to \infty} \int_{\Omega} |f(x)|^2 d\mu_n(x).
\]

Next, we present a characterization of the norm–parallelism for continuous functions on a compact Hausdorff space \( K \). We will need the following lemma.

**Lemma 2.4.** [8, Theorem 3.1] Let \( K \) be a compact Hausdorff space and let \( f, g \in C(K) \). Then
\[
\lim_{t \to 0^+} \frac{\|f + tg\| - \|f\|}{t} = \max_{x \in M_f} \text{Re}\left( e^{-i\arg(f(x))} g(x) \right),
\]
where \( M_f = \{ x \in K : |f(x)| = \|f\| \} \).

**Theorem 2.5.** Let \( K \) be a compact Hausdorff space and let \( f, g \in C(K) \). Then the following statements are equivalent:

(i) There exists a probability measure \( \mu \) with support contained in the norm attaining set \( M_f \) such that
\[
\left| \int_{K} \overline{f(x)}g(x) d\mu(x) \right| = \|f\| \|g\|.
\]

(ii) \( f \parallel g \).

**Proof.** We proceed as in the proof of Theorem 2.3.

(i)\(\Rightarrow\) (i) Suppose (i) holds. So, there exists \( \lambda \in \mathbb{T} \) such that
\[
\int_{K} \overline{f(x)} \left( \|g\|f(x) - \lambda\|f\|g(x) \right) d\mu(x) = 0.
\]
Thus
\[
\left\| \alpha f + \beta (\|g\| f - \lambda \|f\| g) \right\|^2 \\
\geq |\alpha|^2 \|f\|^2 + |\beta|^2 \int_{M_f} \left( \|g\| f(x) - \lambda \|f\| g(x) \right)^2 d\mu(x) \geq |\alpha|^2 \|f\|^2,
\]
for any \( \alpha, \beta \in \mathbb{C} \). Let \( \varphi : \text{span}\{f, (\|g\| f - \lambda \|f\| g)\} \to \mathbb{C} \) be the linear functional defined as
\[
\varphi(\alpha f + \beta (\|g\| f - \lambda \|f\| g)) = \alpha \|f\| \quad (\alpha, \beta \in \mathbb{C}).
\]
Hence \( \varphi(f) = \|f\|, \varphi(\lambda g) = \|g\| \) and \( \|\varphi\| = 1 \). By the Hahn-Banach theorem, \( \varphi \) extends to a linear functional \( \tilde{\varphi} \) on \( \mathcal{C}(K) \), of the same norm. Since \( \tilde{\varphi}(f) = \|f\| \) and \( |\tilde{\varphi}(g)| = \|g\| \), Lemma 2.1 yields \( f \parallel g \).

(ii) \( \Rightarrow \) (i) Let \( f \parallel g \). By Lemma 2.1, there exist \( \lambda \in \mathbb{T} \) and a norm one linear functional \( \varphi \) over \( \mathcal{C}(K) \) such that \( \varphi(f) = \|f\| \) and \( \varphi(\lambda g) = \|g\| \). Hence by Lemma 2.4, we get
\[
\max_{x \in M_f} \text{Re} \left( e^{-i \arg(f(x))} \left( \|g\| f(x) - \lambda \|f\| g(x) \right) \right) \geq 0.
\]
So, the convex hull of the set \( \{ \overline{f(x)} \left( \|g\| f(x) - \lambda \|f\| g(x) \right) : x \in M_f \} \) consists of points of the form \( \int_K \overline{f(x)} \left( \|g\| f(x) - \lambda \|f\| g(x) \right) d\mu(x) \), where \( \mu \) is a probability measure supported on a finite subset of \( M_f \). Then there is a sequence \( \mu_n \) of probability measures such that
\[
\lim_{n \to \infty} \int_K \overline{f(x)} \left( \|g\| f(x) - \lambda \|f\| g(x) \right) d\mu_n(x) = 0.
\]
By the Banach-Alaoglu compactness theorem in dual space, there is a probability measure \( \mu \) such that \( \lim_{i \to \infty} \mu_n = \mu \). Thus the support of \( \mu \) is contained in \( M_f \) and we obtain
\[
\left| \int_K \overline{f(x)} g(x) d\mu(x) \right| = \|f\| \|g\|.
\]
\( \square \)

As a consequence of Theorem 2.5 we have the following result.

**Corollary 2.6.** Let \( K \) be a compact Hausdorff space and let \( f, g \in \mathcal{C}(K) \). If \( M_f = \{x_0\} \), then the following statements are equivalent:

(i) \( f \parallel g \).

(ii) \( \{x_0\} \subseteq M_g \).

We closed this paper with the following equivalence theorem. More precisely, we state a characterization of the norm–parallelism for uniformly continuous \( X \) valued functions on \( \mathbb{B}_X \) to the identity function. Note that since \( \mathbb{B}_X \) is convex and bounded, then every function in \( \mathcal{C}_u(\mathbb{B}_X, X) \) is also bounded. Before stating our result, let us quote a result from [4].
Lemma 2.7. [4, Corollary 2.4] Let $X$ be a Banach space. Let $0 < \theta < 2$ and suppose $y \in \mathbb{B}_X$ and $y^* \in \mathbb{B}_{X^*}$ satisfy $\text{Rey}^*(y) > 1 - \theta$. Then, there are $z \in S_X$ and $z^* \in S_{X^*}$ such that
\[
z^*(z) = 1, \quad \|y - z\| < \sqrt{2\theta} \quad \text{and} \quad \|y^* - z^*\| < \sqrt{2\theta}.
\]

Theorem 2.8. Let $X$ be a Banach space and let $f \in C_u(\mathbb{B}_X, X)$. Then the following statements are equivalent:
(i) $v(f) = \|f\|.$
(ii) $f \parallel Id,$
where $Id$ stands for the identity function.

Proof. (i)⇒(ii) Let $v(f) = \|f\|.$ For every $\varepsilon > 0$, we may find $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $|x^*(f(x))| > \|f\| - \varepsilon$. Let $x^*(f(x)) = \frac{\lambda}{\|\lambda\|} |x^*(f(x))|$ with $\lambda \in \mathbb{T}$. We have
\[
1 + \|f\| \geq \|Id + \lambda f\| \geq \|x + \lambda f(x)\|
\]
\[
\geq |x^*(x + \lambda f(x))| = |x^*(x) + \lambda x^*(f(x))|
\]
\[
= \left|1 + \lambda \frac{\lambda}{\|\lambda\|} |x^*(f(x))|\right| = 1 + |x^*(f(x))| > 1 + \|f\| - \varepsilon.
\]
Thus
\[
1 + \|f\| \geq \|Id + \lambda f\| > 1 + \|f\| - \varepsilon.
\]
Letting $\varepsilon \to 0^+$, we obtain $\|Id + \lambda f\| = 1 + \|f\|$, or equivalently, $f \parallel Id$.

(ii)⇒(i) Let $f \parallel Id$. So, there exists $\lambda \in \mathbb{T}$ such that
\[
\|Id + \lambda f\| = 1 + \|f\|.
\]
Fix $0 < \varepsilon < 1$. Since $f \in C_u(\mathbb{B}_X, X)$, there exists $0 < \delta < \varepsilon$ such that
\[
\|y - z\| < \delta \Rightarrow \|f(y) - f(z)\| < \varepsilon \quad (y, z \in \mathbb{B}_X).
\]
Since $1 + \|f\| = \sup_{y \in \mathbb{B}_X} \|y + \lambda f(y)\|$, there exists $y \in \mathbb{B}_X$ such that
\[
\|y + \lambda f(y)\| > 1 + \|f\| - \frac{\delta^2}{2}.
\]
Then we may find $y^* \in S_{X^*}$ such that
\[
\text{Rey}^*(y + \text{Rey}^*(\lambda f(y))) > 1 + \|f\| - \frac{\delta^2}{2},
\]
which yields
\[
\text{Rey}^*(y) > 1 - \frac{\delta^2}{2} \quad (2.8)
\]
and
\[
\text{Rey}^*(\lambda f(y)) > \|f\| - \frac{\delta^2}{2} \quad (2.9)
\]
By (2.8) and Lemma 2.7, there are $z \in S_X$ and $z^* \in S_{X^*}$ such that
\[
z^*(z) = 1, \quad \|y - z\| < \delta \quad \text{and} \quad \|y^* - z^*\| < \delta.
\]

(2.10)
So, by (2.10) and (2.7) we get
\[ \|f(y) - f(z)\| < \varepsilon. \] (2.11)

By (2.10) and (2.11) it follows that
\[ \left| \text{Re}z^*(\lambda f(z)) - \text{Re}y^*(\lambda f(y)) \right| \leq \left| \text{Re}z^*(\lambda f(z) - \lambda f(y)) \right| + \left| \text{Re}(z^* - y^*)(\lambda f(y)) \right| \]
\[ \leq \|\lambda f(z) - \lambda f(y)\| + \|z^* - y^*\| \leq \varepsilon + \delta, \]
whence
\[ \left| \text{Re}z^*(\lambda f(z)) - \text{Re}y^*(\lambda f(y)) \right| < \varepsilon + \delta. \] (2.12)

So, by (2.9) and (2.12) we obtain
\[ \text{Re}z^*(\lambda f(z)) > \|f\| - \frac{\delta^2}{2} - (\varepsilon + \delta) > \|f\| - 3\varepsilon. \]

This implies
\[ \|f\| \geq \nu(f) \geq \text{Re}z^*(\lambda f(z)) > \|f\| - 3\varepsilon. \]

Letting \( \varepsilon \to 0^+ \), we conclude \( \nu(f) = \|f\| \). \( \square \)

As an immediate consequence of Theorem 2.8 we have the following result.

Corollary 2.9. Let \( X \) be a Banach space and let \( f \in \mathcal{C}_u(\mathbb{B}_X, X) \). If \( f \parallel Id \), then
\[ \|f\| = \sup_{x \in S_X} \|f(x)\|. \]

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