Design a Fractional Chaotic Logistic Dynamical System

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Abstract. In the present article, a new procedure to design a continuous fractional chaotic dynamical system with delay time from a logistic map by using sampled data representation of continuous time Caputo fractional models. An exact expression for the solution of the constructed method is found. In addition, the latest chaotic attractor that results is seen. Finally, using the MATLAB software, all theoretical results were numerically confirmed in this study.

1. Introduction

Continuous chaotic dynamical systems with delay time have been extensively studied in recent years, for example, Wang et al. (2000) [1] investigated anti-control of chaos in continuous time systems using feedback of time delay. Wang et al. (2007) [2] also investigated a Lotka-volterra predator-prey device with implosive ratio harvesting prey amplitude perturbation skills. From the standpoint of equilibrium and chaos control, Ghosh (2008) [3] implemented a multiple delayed Rossler method. Suneel (2006) [4], Garca-Martinez (2013)[5], Campos-Canton (2009) [6] have modeled analog-electronics realizations for chaotic models in the discrete case. Acho (2017)[7] proposed a new procedure for transforming a chaotic Logistic map into a chaotic structure with a continuous time delay. A new multidimensional discrete chaotic structure was proposed by Li et al. (2018) [8].

Bhalekar(2010), Bhalekar(2017), Bhalekar(2016), Bhalekar(2012) [9–13] investigated chaos in fractional dynamical systems with delay time. Almatroud et al. (2020) [14] proposed a new fractional order discrete time chaotic system with two quadratic nonlinearities concepts that had no equilibria. Patil and Bhalekar (2020) [15] investigated a new chaotic dynamical structure in three dimensions.

This work’s key contribution is that it provides an alternative method for generating fractional chaotic systems from discrete time only once. Furthermore, the proposed model’s chaotic proof behavior is easily demonstrated. This work can be thought of as a modern technique for converting a chaotic system from a discrete time domain to a continuous fractional chaotic system. In addition, the Mittag-leffler function and certain fractional calculus properties are used to find the solution to the proposed model. Results are described numerically and graphically, and they are found to be in excellent alignment with the solution form.

The following is how the rest of the paper is organized, Preliminaries and notations are introduced in Section 2. The system is introduced in Section 3. Sections 4 and 5 provide empirical simulations and conclusions, respectively.

2. Preliminaries and notations

In this section, we give some definitions and properties of the fractional calculus, Metag-Lever function, chaotic function and differential transformation method [16–25].

2.1 Fractional Calculus

Definition 1 [15]. The Caputo fractional derivative \( D_C^\gamma \) of order \( \gamma \geq 0 \) for a real valued function \( g \in C(a,b) \), is defined as
\[
D_\gamma^T g(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \frac{g^{(n)}(r)}{(t-r)^{n-\gamma}} dr, \quad n-1 < \gamma < n \in N \tag{1}
\]

where \( \Gamma \) is a gamma function.

**Definition 2 [15].** The Caputo fractional integral operator \((I_\gamma^T)\) of order \( \gamma \geq 0 \) for a real valued function \( g \in C(a,b) \), is defined as

\[
I_\gamma^T g(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{g(r)}{(r-t)^{1-\gamma}} dr, \quad 0 < \gamma < 1, \quad \gamma = 0 \tag{2}
\]

**Proposition 1 [22].** Assume that the function \( g: [0, \infty) \) is such that the Laplace transform \( \mathcal{L} \) exists on \([a, \infty)\) with some \( a \in \mathbb{R} \). Let \( \gamma > 0 \) and \( n = \lceil \gamma \rceil \). Then, for \( s > \max(0,a) \) we have

\[
\mathcal{L}D_0^\gamma g(s) = \frac{1}{s^\gamma} \mathcal{L}g(s)
\]

And

\[
\mathcal{L}D_0^\gamma g(s) = s^\gamma \mathcal{L}g(s) - \sum_{k=1}^{n} s^{\gamma-k} g^{(k-1)}(0)
\]

**Definition 3 [22].** The Mittag-Leffler function \( E_\gamma(z) \) for any \( \gamma > 0 \) is defined as follows

\[
E_\gamma(z) = \sum_{m=1}^{\infty} \frac{z^m}{\Gamma(\gamma m + 1)}
\]

**Proposition 2 [22].** The following relations are hold for any \( \gamma, t, \delta > 0 \) and \( \lambda \in \mathbb{C} \)

i. If \( y(t) = E_\gamma(-\delta t^\gamma) \) then \( \mathcal{L}y(t) = \lambda^{-\gamma} (\lambda^\gamma - \delta)^{-1} \)

ii. \[
\frac{d}{dt} E_\gamma(-\delta t^\gamma) = \sum_{m=1}^{\infty} \frac{(\delta m)^{m-1}}{\Gamma(\gamma m)} \left( \frac{\delta m^m}{\gamma} \right)
\]

### 2.2 Logistic Chaotic Map

The Logistic map is a discrete time map that despite its formal simplicity, exhibits an unexpected degree of complexity. Historically it has been one of the most important and paradigmatic systems during the early days of research on deterministic chaos. The logistic map can be defined as [18],

\[
\omega(n+1) = \lambda \omega(n)(1 - \omega(n)), \quad n = 0, 1, 2, ...
\]

\[
\omega(0) = \omega_0 \tag{6}
\]

where, the initial value \( \omega_0 \in [0, 1] \) and \( \lambda \in (0, 4) \). Recently, the author Acho in (2015) [26] designed a new chaotic logistic map as follows

\[
\omega(n+1) = -1.3 \text{sgn} \left( \omega(n) \right) (1 - \omega(n)), \quad n = 0, 1, 2, ...
\]

\[
\omega(0) = 0.1 \tag{7}
\]

In the mathematical modeling above system can be considered simple, it is easy to our work. Hence, this system will be invoked later in this paper.

### 2.3 Dynamical representation of sampled data system

The representation for any sampled data of a fractional linear control system in continuous time can be written in the following form [16],

\[
D_\gamma^T x(t) = Ay(t) + Bu(t)
\]

\[
u(t) = E y(t_k) \tag{8}
\]
where, \( y \in \mathbb{R}^d \) and \( u \in \mathbb{R}^p \) are the state and the control input respectively. The matrix \( E \) is the stabilizing gain state feedback control. The sequence \( \{t_k\} \) is a sampling instants and assumed strictly increasing on time. \( D^\gamma_t \) is the Caputo fractional derivative of order \( 0 < \gamma < 1 \). The alternative representation of the system (8) is given in the following form

\[
\begin{align*}
D^\gamma_t y(t) &= Ay(t) + Bu(t - \varphi(t)) \\
y(0) &\in \mathbb{R}^d \\
u(t) &= E y(t)
\end{align*}
\]

where \( \varphi(t) \) is a delay function which it satisfies \( \frac{d\varphi(t)}{dt} = 1 \) almost everywhere with discontinuities at resetting to zero at time \( t_i, i = 1, 2, \ldots \). The delay function is given as \( t - \varphi(t) = t_k, t \in [t_k, t_{k+1}] \) when the periodic sampling is assumed, then \( t_{k+1} = t_k + T, T > 0 \) is named the sampling period.

### 3. System Description

Consider a continuous chaotic dynamical systems with delay time as follows

\[
\begin{align*}
D^\gamma_t y(t) &= \delta y(t) - 1.3\delta \text{sgn} \left( y(t - \varphi(t)) \right) \left( 1 - y(t - \varphi(t)) \right), t \in [t_k, t_{k+1}] \\
y(0) &= 0.1 \\
y(t) &= 0, t \in [-T, 0]
\end{align*}
\]

Where, the parameter \( \delta > 0 \), and \( y(t) \) is a state function and \( \varphi(t) \) is defined in equation (9).

The approach is based on the Laplace transform of the Caputo fractional method on. Firstly, applying the Laplace transform of the Caputo fractional integral of both sides of equation (10), we obtain:

\[
\lambda^\gamma \mathcal{L} y(\lambda) - \lambda^{\gamma-y} y(t_k) = \delta \mathcal{L} y(\lambda) + \delta \mathcal{L} \left( g(t_k) \right) \left( \lambda \right)
\]

where,

\[
g(t_k) = -1.3\text{sgn} \left( y(t - \varphi(t)) \right) \left( 1 - y(t - \varphi(t)) \right)
\]

Therefore,

\[
(\lambda^\gamma - \delta) \mathcal{L} y(\lambda) = \lambda^{\gamma-y} y(t_k) + \delta \mathcal{L} \left( g(t_k) \right) \left( \lambda \right)
\]

Multiplying above equation by \( (\lambda^\gamma - \delta) \), then under the condition \( \lambda^\gamma \in \rho(\delta) \), we get

\[
\mathcal{L} y(\lambda) = \lambda^{\gamma-y}(\lambda^\gamma - \delta)^{-1} y(t_k) + (\lambda^\gamma - \delta)^{-1}\delta \mathcal{L} \left( g(t_k) \right) \left( \lambda \right)
\]

Now, we need to find the inverse Laplace transform of the functions on the right hand side of above equation. Using Proposition 2., we have

\[
y(t) = \mathcal{E} \left( -\delta (t - t_k)^\gamma \right) y(t_k) + \delta g(t_k) \int_{t_k}^{t} \sum_{m=1}^{\infty} \frac{(\delta)^m(r - t_k)^{ym-1}}{\Gamma(my)} dr
\]

This gives

\[
y(t) = \mathcal{E} \left( -\delta (t - t_k)^\gamma \right) y(t_k) + 2 \delta g(t_k) \int_{t_k}^{t} (r - t_k)^{ym-1} dr
\]

Therefore,

\[
y(t) = \mathcal{E} \left( -\delta (t - t_k)^\gamma \right) y(t_k) + g(t_k) - g(t_k) \int_{t_k}^{t} (r - t_k)^{ym} dr
\]

Using definition 3. we get
\[ y(t) = E_y(-\delta (t - t_k))y_k + g(t_k)(1 - E_y(-\delta(t - t_k))) \quad t \in [t_k, t_{k+1}] \]  

(18)

4. Main Results

**Definition 4.** A state \( y(t) \in \mathbb{C}([0, B]; R) \) is a solution of the fractional dynamical system in (1) if, for each periodic delay function \( \varphi(t) \), it satisfies the integral equation (18).

**Theorem 1.** For a sufficiently large value of \( \delta \) and a periodic delay function \( \varphi(t) \) the fractional dynamical system in (10) is chaotic.

**Proof:** The equation (18) is an equivalent to original system (10), therefore, if we replace \( t \) by \( t_{k+1} \) in equation (18) then, we obtain

\[ y(t_{k+1}) = E_y(-\delta (t_{k+1} - t_k))y_k + g(t_k)(1 - E_y(-\delta(t_{k+1} - t_k))) \]  

(19)

From the assumption of this theorem then \( \delta \) is a sufficiently large to get \( E_y(-\delta (t_{k+1} - t_k)) \rightarrow 0 \). Consequently, \( y(t_{k+1}) = g(t_k) \) which is precisely system (7) with \( \omega(n) = y(n) \). It is observed that the result in this theorem may be considered as strong tool to design other systems. Actually, from the above result in this theorem the fractional dynamical system in (10) is chaotic if the corresponding system \( \omega(n + 1) = g(\omega(n)) \) is chaotic too, which seems obvious since it is a filter version of a chaotic discrete signal.

5. Numerical Experiments

The fractional dynamical system in (10) was implemented by using MATLAB R2015a with the following parameters

\[ \gamma = 0.5, 0.8, 1, \delta = 10, 15 \text{ and } \varphi(t) = t - i, t \in [i, i + 1], i = 1, 2, \ldots \]

The numerical results of this work are shown in Fig. (1-8). The expected result on chaos realization of these experiments results is validated, specially, the new chaotic attractor displayed in Fig. we can see that the effectiveness of the fractional parameter \( \gamma \) on the chaotic behavior of the proposed model is found although it is simple. Finally the chaotic behavior of the system (10) depends on the parameter \( \delta \). Precisely, the values of the parameter \( \delta \) play important role in determining the chaotic behavior of system (10) where it increases by its increasing.
Fig. 1. Numerical result: $\varphi(t)$ versus $t$, using a sampling rate of one second.

Fig. 2. Numerical result: $y(t)$ versus $t$, using the values of parameters $\gamma = 0.5$ and $\delta = 10$.

Fig. 3. Numerical result: $y(t)$ versus $t$, using the values of parameters $\gamma = 0.5$ and $\delta = 20$.
Fig. 4. Numerical result: $y(t)$ versus $t$, using the values of parameters $\gamma = 0.8$ and $\delta = 15$.

Fig. 5. Numerical result: $y(t)$ versus $t$, using the values of parameters $\gamma = 1$ and $\delta = 15$. 
Fig. 6. Numerical result: $y(t - \varphi(t))$ versus $y(t)$ using the values of parameters $\gamma = 0.5$ and $\delta = 15$.

Fig. 7. Numerical result: $y(t - \varphi(t))$ versus $y(t)$ using the values of parameters $\gamma = 0.8$ and $\delta = 15$. 
6. Conclusions

Our work offers a new mathematical method for designing a fractional, chaotic, disorderly system. To prove the chaotic behavior of chaotic systems, we use methods such as bifurcation diagrams, Lyapunov exponents, and chaotic attractors. The method used to prove chaos in this paper could be considered simple. The presence of chaos for such a proposed fractional structure has been explored through numerical experiments, as has the form of chaotic behavior and its range. It has been confirmed that the chaotic behavior and its range are dependent on the values of fractional order. The proposed system may also be applied to actual models and used in different systems such as the industry, medicine, biomedicine, engineering and environment for data analysis.

References

[1] Wang F., Chen G., Yu X., (2000). Anticontrol of chaos in continuous-time systems via time-delay feedback, Chaos. An Interdisciplinary Journal of Nonlinear Science, 10(4), 771-779.
[2] Wang F., Zen G., (2007). Chaos in a Lotka-Volterra predator-prey system with periodically impulsive ratio-harvesting the prey and time delays Chaos. Solitons & Fractals, 32(4), 1499-1512.
[3] Ghosh D., Chowdhury A., Saha P., (2008). Multiple delay R’ossler system: Bifurcation and chaos control, Chaos. Solitons & Fractals, 35(3), 472-485.
[4] Suneel M., (2006). Electronic circuit realization of the logistic map. Sadhana, 31(1), 69-78.
[5] García-Martínez M., Campos-Cantón on I., CamposCantón on E., ‘Celikovsky’y S., (2013). Difference map and its electronic circuit realization. Nonlinear Dynamics, 74(3), 819-830.
[6] Campos-Cantón on I., Campos-Cantón on E., Murguía S., Rosu C., (2009). A Simple Electronic Circuit Realization of the Tent Map, Chaos. Solitons & Fractals, 42, 12-16.
[7] Acho L., (2017). A Continuous-time Delay Chaotic System Obtained from a Chaotic Logistic Map. International Conference Innsbruck, Austria Modelling, Identification and Control.
[8] Li W., Wang C., Feng K., (2018). A Multidimensional discrete Digital Chaotic Encryption System. International Journal of Distributed Sensor Networks, 25.
[9] Bhalekar S. Gejji,V., (2010). Fractional ordered Liu system with time delay. Commun. Nonlinear Sci. 15/178–2191.
[10] Bhalekar S., (2017). Dynamics of fractional order complex Ucar system, In Fractional Order Control and Synchronization of Chaotic Systems. Springer, Cham, 747–771.
[11] Bhalekar S., (2016). Stability and bifurcation analysis of a generalized scalar delay differential equation, Chaos. An Interdisciplinary Journal of Nonlinear Science, 26 (8), 084306.

[12] Bhalekar S., (2013). Stability analysis of a class of fractional delay differential equations. Pramana, 81(2), 215–224.

[13] Bhalekar S., (2012). Dynamical analysis of fractional order U/ar prototype delayed system. Signal, Image and Video Processing, 6 (3), 513–519.

[14] Almatroud A., Khennaoui A., Ouannas A., Grassi G., Alsawalha M., Garsi A., (2020). Dynamical Analysis of a New Chaotic Fractional Discrete Time System and Its Control. Entropy, 22, 1344.

[15] Patil M., Bhalekar S., (2020). A new Fractional Order Chaotic Dynamical System and Its Synchronization Using Optimal Control. arXiv: 2007.03168v1.

[16] Podlubny I., (2009). Fractional Differential Equations. Academic Press, New York, 1999.

[17] Zhu H., Zhou S., He Z., (2009). Chaos synchronization of the fractional-order Chen's system. Chaos Solitons Fractals 41, 2733–2740.

[18] Sprott J., (2003). Chaos and Time-Series Analysis. Oxford University Press, Oxford.

[19] Butaïneh A., Alomari, A., Noorani M., Hashim I., Nazar R., (2009). Series solutions of systems of nonlinear fractional differential equations. Acta Appl. Math. 105, 189–198.

[20] Li C., Peng G., (2004). Chaos in Chen’s system with a fractional order. Chaos Solitons Fractals 22, 443–450.

[21] Wang J., Xiong X., Zhang Y., (2006). Extending synchronization scheme to chaotic fractional-order Chen systems. Physica A 370, 279–285.

[22] Diethelm K., Ford N., (2002). Analysis of fractional differential equations. J. Math. Anal. Appl. 265, 229–248.

[23] Diethelm K., Ford N., Freed A., (2002). A predictor–corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29, 3–22.

[24] Zhou T., Li C., (2005). Synchronization in fractional-order differential systems. Physica D 212, 111–125.

[25] He J., (2009). An elementary introduction to the homotopy perturbation method. Comput. Math. Appl. 57, 410–412.

[26] Acho L., (2015). A discrete-time chaotic oscillator based on the logistic map: A secure communication scheme and a simple experiment using Arduino. Journal of the Franklin Institute, 352(8), 3113-3121.

[27] Chandrashekhar Meshram, Rabha W. Ibrahim, Ahmed J. Obaid, Sarita Gajbhiye Meshram, Akshay Kumar Meshram, Alaa Mohamed Abd El-Latif, Fractional chaotic maps based short signature scheme under human-centered IoT environments, Journal of Advanced Research, 2020, ISSN 2090-1232, https://doi.org/10.1016/j.jare.2020.08.015.