Magneto-transport in periodic and quasiperiodic arrays of mesoscopic rings

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We study theoretically the transmission properties of serially connected mesoscopic rings threaded by a magnetic flux. Within a tight-binding formalism we derive exact analytical results for the transmission through periodic and quasiperiodic Fibonacci arrays of rings of two different sizes. The role played by the number of scatterers in each arm of the ring is analyzed in some detail. The behavior of the transmission coefficient at a particular value of the energy of the incident electron is studied as a function of the magnetic flux (and vice versa) for both the periodic and quasiperiodic arrays of rings having different number of atoms in the arms. We find interesting resonance properties at specific values of the flux, as well as a power-law decay in the transmission coefficient as the number of rings increases, when the magnetic field is switched off. For the quasiperiodic Fibonacci sequence we discuss various features of the transmission characteristics as functions of energy and flux, including one special case where, at a special value of the energy and in the absence of any magnetic field, the transmittivity changes periodically as a function of the system size.

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I. INTRODUCTION

Quantum transport in mesoscopic systems has been an exciting field of research in the past several years. Among the important aspects that has attracted much attention, both experimentally and theoretically, is the fluctuation of the magneto-conductance due to quantum coherence in such samples. For mesoscopic systems at very low temperatures the phonon scattering is insignificant, and the phase coherence length of the electrons becomes large compared to the system size. In the presence of a magnetic field one observes a specific, reproducible fluctuation pattern of the conductance, as the magnetic field or the Fermi level varies. The sample becomes equivalent to an electron waveguide where the transport properties are determined by the impurity configuration and the geometry of the conductor. In experimental and theoretical works, the Aharonov-Bohm (AB) effects in solid state devices in the forms of rings and cylinders that enclose a magnetic flux have been investigated with much vigor. The oscillations in the magneto-resistance were predicted to be dominated by a half-integer flux period $\phi_0/2$, where $\phi_0 = hc/e$ is the fundamental flux quantum. This was found experimentally for long cylinders and arrays of metal rings. For single rings, it has been discussed that both periods can be present. Among the theoretical studies on single ring systems, D’Amato et al., have discussed a tight-binding model of a disordered ring coupled to two external leads, and have calculated the Landauer conductance as a function of $\phi$. For strongly disordered rings and for arbitrary disorder with weakly coupled branches, they have found a dominant period $\phi_0$. Aldea et al., investigated the dc magneto-resistance in a two-probe ring geometry within a tight-binding formalism. They have presented analytical results for ordered single- and double-ring structures and discussed the localization effects due to disorder in such systems. AB effects for bound states such as neutral excitonic particles have also recently been studied.

In comparison, the behavior of the magneto-conductance in systems having serially connected rings, has received little attention. In one of the early studies on serially connected rings Deo and Jayannavar discussed quantum transport in these systems. The ‘band formation’ in such geometries has been analyzed and some magnetic properties of loop structures in the presence of an AB flux have been discussed. Takai and Ohta have also addressed similar problems where both the magnetic flux and an electrostatic potential are present. These works have relied on the solution of the continuous version of the Schrödinger equation for the ring systems as well as other geometries. Transmission through a serial arrangement of rings can equivalently be handled within a tight-binding formalism, in the spirit of the single-ring studies of Ref. and Ref. However,
only recently an attempt has been made in this direction by Li et al. They used a tight-binding model and a scattering-matrix technique to obtain closed expressions for transmission across a periodic array of identical rings.

The tight-binding formalism naturally facilitates the application of real-space renormalization-group methods to determine the bands and the transmittivity. It is also easy to incorporate disorder in this scheme. Additionally, it has also been successfully used to compute the dc conductivity of quasi-one-dimensional polyaniline chains, which resemble the ring-like mesoscopic objects quite closely. In this context, we believe that there is still scope to look deeper into the transport properties of serially connected rings. For example, a detailed analysis of the variation of the transmission properties as the number of atoms (scatterers) in each arm of the ring changes, keeping the electron energy $E$ and the flux $\phi$ constant, has been somewhat less attended to so far. Also, the effect of having rings arranged in geometries other than periodic (quasiperiodic or random, for example) on the overall transmittivity is something that has not drawn any attention at all. This, to our mind, is worth investigating in detail. In this communication we address some of these problems. We focus our attention on the transmission coefficient across (i) perfectly ordered arrays of identical rings, and (ii) across a quasiperiodic arrangement of the rings of two different sizes.

Apart from the results that already exist and present an interesting scenario, an additional motivation in undertaking the present work comes from a recent experiment on the measurement of low-temperature magnetic response of serially connected GaAs/AlGaAs mesoscopic rings, which resemble the ring-like mesoscopic objects quite closely. In this context, we believe that there is still scope to look deeper into the transport properties of serially connected rings. For example, a detailed analysis of the variation of the transmission properties as the number of atoms (scatterers) in each arm of the ring changes, keeping the electron energy $E$ and the flux $\phi$ constant, has been somewhat less attended to so far. Also, the effect of having rings arranged in geometries other than periodic (quasiperiodic or random, for example) on the overall transmittivity is something that has not drawn any attention at all. This, to our mind, is worth investigating in detail. In this communication we address some of these problems. We focus our attention on the transmission coefficient across (i) perfectly ordered arrays of identical rings, and (ii) across a quasiperiodic arrangement of the rings of two different sizes.

In section II we present the basic method of our calculation for the transmission coefficient, and apply it to discuss some of the results for the isolated rings. The periodic arrangement of the rings is considered in section III. In section IV we focus on the quasiperiodic geometries. The transmittivity for a Fibonacci array of rings at an arbitrary generation of the series has been calculated by suitably modifying the trace-antitrace formulation to include the magnetic flux. In section V we summarize.

### II. FROM A RING TO A DIMER

In this section we explain how we reduce a single ring geometry to a dimer. Let us concentrate on the simplest model of a ring which consists of identical ‘atoms’ each characterized by the same on-site potential energy $\epsilon$. The atoms are assumed to be equally spaced on the ring. The ring is attached at two sites $L$ and $R$ to two semi-infinite ordered leads to facilitate the transmission measurement as shown in Fig. 1(a). The leads are characterized by a constant on-site potential $\epsilon_0$ and a uniform hopping integral $t_0$ between the nearest neighbour sites. The leads can be attached to any two atoms, which is equivalent to saying that we may have different number of atoms in the ‘upper’ and ‘lower’ arms of the rings. A magnetic field of flux density $\phi$ penetrates the ring. We describe such a ring by the standard tight-binding Hamiltonian with the Peierls’ substitution

$$H = \epsilon \sum_i \langle i | i \rangle + t \sum_{ij} \left[ e^{i\gamma} \langle i | j \rangle + e^{-i\gamma} | j \rangle \langle i | \right]$$

where $t$ is the amplitude of the hopping integral for nearest-neighbor couplings of identical strength, and $| i \rangle$, $i = 1, \ldots, N_0$, denotes the tight-binding orbitals. The flux $\phi$ is measured in units of $\phi_0$ and enters the Hamiltonian via $\gamma = 2\pi \phi / N_0 \phi_0$. $N_0$ is the number of bonds (and sites) in the ring. The amplitudes of the wave function at the $j$th site in any arm of the ring (excluding the sites $L$ and $R$ at the junctions with the leads) are related to the
as shown in Fig. 1(a). This can be done analytically. It is straightforward to relate the amplitudes \( \psi_j, \psi_{j-1} \) on any arm of the ring to the set \( \psi_{j-1}, \psi_{j-2} \) on the same arm through a \( 2 \times 2 \) transfer matrix

\[
M = \begin{pmatrix}
E - \epsilon & -e^{-i\gamma} \\
1 & -e^{-2i\gamma}
\end{pmatrix}.
\]

The matrix has a determinant equal to \( \exp(-2i\gamma) \). The product of \( n \) such matrices is

\[
M^n = e^{-i(n-1)\gamma} \left[ U_{n-1} \left( \frac{E - \epsilon}{2t} \right) M - e^{-i\gamma} U_{n-2} \left( \frac{E - \epsilon}{2t} \right) \right]
\]

where \( \mathbb{I} \) is the \( 2 \times 2 \) identity matrix and \( U_n \) is the \( n \)th order Chebyshev polynomial of the second kind. Using this result together with the set of difference equations (2) we obtain the expressions for the effective on-site potentials of the contact points \( L \) and \( R \). For the general case of \( n \) and \( m \) atoms between \( L \) and \( R \) in the upper and the lower arm (excluding the contact points \( L \) and \( R \)), respectively, we get

\[
\tilde{\epsilon} = \epsilon - te^{i\gamma} \frac{M_{12}^n}{M_{11}^n} - te^{-i\gamma} \frac{(M_{12}^m)^*}{(M_{11}^m)}
\]

where \( M_{ij}^n \) is the \((i,j)\)th element of \( M^n \). The effective hopping integral includes the effect of the broken time-reversal symmetry resulting from the application of the magnetic field, and is denoted as \( \tilde{t}_F \) and \( \tilde{t}_B \) corresponding to the ‘forward’ and the ‘backward’ hopping. The forward hopping integral is given by

\[
\tilde{t}_F = \frac{te^{i\gamma}}{M_{11}^n} + \frac{te^{-i\gamma}}{(M_{11}^m)}
\]

and \( \tilde{t}_B = \tilde{t}_F \). The ring embedded in the ordered lead now reduces to a dimer comprising of the (modified) sites \( L \) and \( R \). The transfer matrices for these two sites are

\[
M(L) = \begin{pmatrix}
\frac{E - \tilde{\epsilon}}{t_F} & -\frac{t_0}{t_F} \\
1 & 0
\end{pmatrix}
\]

and

\[
M(R) = \begin{pmatrix}
\frac{E - \tilde{\epsilon}}{t_B} & -\frac{t_0}{t_B} \\
1 & 0
\end{pmatrix}
\]

with \( t_0 \) the hopping integral in the lead. The next step is to calculate the product \( P \) of the two matrices corresponding to the two sites \( R \) and \( L \), i.e., \( P = M(R)M(L) \). Using a well known formula the transmission coefficient of this effective dimer \( L-R \) can then be obtained as
with parametrization $E = \epsilon_0 + 2t_0 \cos k$. The lattice spacing has been chosen to be unity throughout. The above scheme has been tested to reproduce the results of the single-ring cases accurately.

We emphasize that the choice of the lead parameters is of much importance. For a given set of values of $\epsilon_0$ and $t_0$ in the lead we will be able to scan only those energy eigenvalues of our system which fall within the allowed ‘band’ of the lead, i.e., from $\epsilon_0 - 2t_0$ to $\epsilon_0 + 2t_0$. Thus it is in principle possible to choose ring parameters $\epsilon$ and $t$ such that there will be energies allowed in the ring structures that will be masked by the leads. Nevertheless, in the experimental situation the leads (not to be mistaken with the contacts) are usually made of the same material as the ring system. Therefore, we restrict ourselves to uniform values $t_0 = t$ and $\epsilon_0 = \epsilon$ throughout the system and therefore to an accessible energy band $\epsilon - 2t$ to $\epsilon + 2t$ in the following.

\section{III. PERIODIC ARRAY OF RINGS}

In this section we consider the variation of the transmission coefficient as a function of the flux at some fixed value of the energy. For convenience we restrict ourselves to the energy $E = \epsilon$, for which the Chebyshev polynomials assume particularly simple forms, and precise analytical expressions are obtained for the effective on-site potential $\tilde{\epsilon}$ and the hopping term $\tilde{t}_F$ and its complex conjugate. The hopping integral connecting one ring to the next is assumed to be same as that between the atoms in the rings, i.e., equal to $t$.

\subsection{A. Even-even case:}

We take $n = m = 2p$ with $p$ an integer. The number of sites in the ring, including the contact points, is $4p + 2$. Using the values of the appropriate Chebyshev polynomials at $E = \epsilon$, it is not difficult to work out with the help of Eqs. (9) and (11) that $\tilde{\epsilon} = 0$ and $\tilde{t}_F = \tilde{t}_B = (-1)^p 2t \cos (\pi \phi / \phi_0)$. The time-reversal symmetry is preserved in this case. Let us check if $E = \epsilon$ belongs to the spectrum of an infinite array of such rings. At $E = \epsilon$ the product transfer matrix corresponding to the dimer, which constitutes a ‘unit cell’ of the infinite array, becomes

\begin{equation}
P_{\text{even--even}} = (-1)^{p+1} \begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & \frac{\sigma}{2} \end{pmatrix}
\end{equation}

(10)

where $\sigma = 2t \cos (\pi \phi / \phi_0)$. In order to have $E = \epsilon$ in the spectrum of an infinite ordered array of such rings, one must have $|\text{Tr} P| \leq 2$. However, from (10) it is evident that $|\text{Tr} P| = 2$ only for $\phi / \phi_0 = 1/3, 2/3, 4/3, 5/3, \ldots$. At all other flux values the trace is greater than 2 as shown in Fig. 2. Therefore, at $E = \epsilon$ and at arbitrary flux, not equal to a special value such as above, the transmission coefficient across an array of this type of rings will decay exponentially as the system increases in size. This can also be checked using the formula for the transmission coefficient. For the special values of the flux mentioned above the transmission coefficient is precisely unity. This implies that for $E = \epsilon$, we can achieve totally ballistic transport by tuning the flux to a specific value. This is an example for an extended eigenstate under the influence of a magnetic field [18, 22, 23]. The phenomenon of full transmission at these specific flux values can be understood if we look at the values of $\tilde{\epsilon}$ and $\tilde{t}_F = \tilde{t}_B$. We have presented results here for $\epsilon = \epsilon_0 = 0$. Now, $\tilde{\epsilon}$ is zero, and equal to the on-site terms at the leads. As $\tilde{t}_F$ also becomes real and equal to unity, the electron essentially ‘sees’ an ordered array of identical sites connected by identical hopping integrals. The corresponding eigenstate is naturally extended and $E = 0$ happens to be the band center. In Fig. 3(a) we display the variation of the transmission coefficient against flux for this case. The transmission coefficient is periodic in flux with a pe-
eight-ring series when \( \phi = \phi_0/4 \) appears to be slightly enhanced compared to \( \phi = 0 \). If we select \( \phi = \phi_0/2 \), an array of arbitrary size becomes completely opaque to an incoming electron with \( E = 0 \). This is also clear from Fig. 3 and consistent with the findings in the single ring case [18].

B. Odd-odd case:

We now consider an equal but odd number of atoms in each arm, i.e., \( n = m = 2p+1 \) such that the total number of atoms in the ring remains even. In this case \( M_{11}^n = 0 \) at \( E = \epsilon \). Both \( \tilde{c} \) and \( \tilde{t} \) diverge as \( E \to \epsilon \), leading to a divergence of the trace, except at some special values of the flux for which the trace becomes exactly equal to zero leading to perfect transmission. An analytical attempt can be made to see this in the following way. Using Eqs. (5) and (6) one can show that, in the limit \( E \to \epsilon \), \( \tilde{c} = \epsilon + \frac{2t^2}{(p+1)(E-\epsilon)} \) and \( \tilde{t}_F = (-1)^p \frac{t\sigma}{(p+1)(E-\epsilon)} \), where the leading terms in the expressions for \( \tilde{c} \) and \( \tilde{t}_F \) have been retained as \( E \to \epsilon \). Here also \( \tilde{t}_B = \tilde{t}_F \). The elements of the transfer matrix across the dimer turn out to be

\[
P_{11}^\text{odd-odd} = (-1)^p \left\{ \frac{(p+1)(E-\epsilon)^2 - 2t^2}{\sigma t} \right\} - 1 \tag{11a}
\]

\[
P_{12}^\text{odd-odd} = (-1)^{p+1} \frac{(p+1)(E-\epsilon)^2 - 2t^2}{t\sigma} \tag{11b}
\]

\[
P_{21}^\text{odd-odd} = -P_{12}^\text{odd-odd} \tag{11c}
\]

\[
P_{22}^\text{odd-odd} = (-1)^{p+1} \frac{(p+1)(E-\epsilon)}{\sigma} \tag{11d}
\]

It can be easily verified that \( |\text{Tr} P_{\text{odd-odd}}| = 0 \) for \( \phi = m\phi_0 \), with \( m = 0, 1, 2, \ldots \). For all other values the trace diverges as \( E \to \epsilon \). Thus the transmission coefficient across an arbitrarily long array of the above rings is unity at the specified values of the flux and zero otherwise. This can easily be worked out using Eq. 6. There is no dependence of the transmission coefficient on the size of the system. For the zero flux case, the \( T-E \) diagram exhibits resonance at \( E = 0 \), in contrast to the previous case.

C. Even-odd case:

We next take \( n = 2p \) and \( m = 2p+1 \). Proceeding in the same spirit as in the odd-odd case, the effective on-site potential and the hopping matrix elements of the dimer read \( \tilde{c} = \epsilon + \frac{2t^2}{(p+1)(E-\epsilon)} \) and \( \tilde{t}_F = (-1)^p \frac{t\sigma}{(p+1)(E-\epsilon)} \), where the leading terms in the expressions for \( \tilde{c} \) and \( \tilde{t}_F \) have been retained as \( E \to \epsilon \). Here, \( \beta = (p+1)(E-\epsilon) \exp[i(2p+1)\chi] + t \exp[-i(2p+2)\chi] \), and \( \chi = \frac{2m\sigma/\phi_0}{4p+3} \).

\[
FIG. 3: (a) Transmission coefficient \( T \) versus magnetic flux \( \phi \) for an array of identical rings with \( n = m = 4 \). (b) for \( n = 5 \) and \( m = 4 \). The dotted, dashed, and solid lines correspond to two-, four- and eight-ring systems, respectively. We display the variation of the transmission coefficient as a function of the energy \( E \) for \( \phi = \phi_0/4 \) rings, and \( \phi/\phi_0 = 0 \). The overall transmission for \( \phi = \phi_0/4 \) appears to be slightly enhanced compared to \( \phi = 0 \). If we select \( \phi = \phi_0/2 \), an array of arbitrary size becomes completely opaque to an incoming electron with \( E = 0 \). This is also clear from Fig. 3 and consistent with the findings in the single ring case [18].

\[
FIG. 4: (a) Transmission coefficient \( T \) versus energy \( E \) for the same system as in Fig. 3 with (a) \( \phi = 0 \). (b) \( T-E \) diagram for \( \phi = \phi_0/4 \). The dashed and the solid lines correspond to four- and eight-ring systems, respectively.
\]
The elements of $P_{\text{even-odd}}$ are

$$P^{11}_{\text{even-odd}} = (-1)^p \left\{ \frac{[(p+1)(E-\epsilon)^2 - t^2]}{t\beta} \right\} \beta - \beta^* \right\}$$

$$\times \frac{1}{(p+1)(E-\epsilon)}$$ \hspace{1cm} (12a)

$$P^{12}_{\text{even-odd}} = (-1)^{p+1} \frac{[(p+1)(E-\epsilon)^2 - t^2]}{t\beta}$$ \hspace{1cm} (12b)

$$P^{21}_{\text{even-odd}} = -P^{12}_{\text{even-odd}}$$ \hspace{1cm} (12c)

$$P^{22}_{\text{even-odd}} = (-1)^{p+1} \frac{[(p+1)(E-\epsilon)]}{\beta}$$ \hspace{1cm} (12d)

It is not difficult to work out analytically that, as $E \to \epsilon$, the maximum value of $|\text{Tr} P_{\text{even-odd}}|$ is 2. That is, the trace is always bounded by 2 from above. This implies that $E = \epsilon$ is in the spectrum of an infinite array of these rings for all values of the flux. In Fig. 2 we show the variation of $|\text{Tr} P_{\text{even-odd}}|$ against $\phi/\phi_0$.

To study the transmission coefficient, let us first consider the case $\phi = 0$ such that

$$P_{\text{even-odd}}(\phi = 0) = (-1)^p \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}. \hspace{1cm} (13)$$

A product of such matrices will look like

$$P^{N}_{\text{even-odd}}(\phi = 0) = \begin{pmatrix} N+1 & -N \\ N & 1-N \end{pmatrix}.$$ \hspace{1cm} (14)

when $N$ is even. A similar expression can be worked out for odd values of $N$. In either case, it can be checked from Eq. (13) that with $\epsilon_0 = \epsilon = 0$ and $t_0 = t = 1$, the transmission coefficient exhibits a power-law decay for large values of $N$, i.e., $T \sim 1/N^2$. At $E = 0$, the matrix $P_{\text{even-odd}}$ is exactly the same as the transfer matrix for a periodic chain having $N$ sites with $\epsilon = 0$ with the electron at energy $E = -2t$ (which defines the band edge of the infinite ordered chain). Hence the $(E = 0, \phi = 0)$-combination for a periodic array of even-odd rings is equivalent to the $E = -2t$ situation of a periodic chain of atoms. Just as we analyzed the even-even case, here it is straightforward to show that unit transmission can be achieved by tuning the magnetic field so that $\phi = (2m+1)\phi_0/4$, irrespective of the size of the system. We show the variation of the transmission coefficient against $\phi/\phi_0$ in Fig. 2(b). The transmission coefficient is periodic, with a period equal to $\phi_0/2$.

IV. QUASIPERIODIC ARRAY OF RINGS

A. Fibonacci array

We follow the usual method of building a Fibonacci sequence recursively. We model the system by placing two different rings that the system is immersed in a constant magnetic field so that the flux $\phi_A$ and $\phi_B$ through the rings is proportional to their respective areas $S_A$ and $S_B$, $\gamma$ as defined in section II. Now, we consider the case $\phi_A$ and $\gamma_B$ related via

$$\gamma_B = \frac{N_B}{N_A} \gamma_A$$ \hspace{1cm} (15)

with $N_A$ and $N_B$ denoting the total number of scatterers in $A$-type and $B$-type rings, respectively.

Following the procedure described in section II we reduce each ring to an effective dimer, so that we finally have a Fibonacci sequence of these dimer-like objects $A$ and $B$, characterized by $P_A$ and $P_B$, respectively, which can be easily computed from (7), (8). The product transfer matrix across the $n$th generation Fibonacci chain is, as usual, $M(m+1) = M(m-1)M(m)$ (16)

with $M(1) = P_A$, and $M(2) = P_B P_A$. Keeping in mind that these matrices have a phase factor similar to $\exp(-i\gamma)$ as in (3), we have generalized the trace map (3) relation to

$$x_{m+1} = x_m x_{m-1} - \Delta_{m-1} x_{m-2}$$ \hspace{1cm} (17)

where $x_m = \text{Tr} M(m)$, and not the ‘half-trace’ as usual (32), and $\Delta_m = \text{det} M(m)$. At the start of the sequence, we have $x_1 = \text{Tr} P_A$ and $x_2 = \text{Tr}(P_B P_A)$. The above formula enables us to deal with situations where the phase accumulated by the electron in traversing one arm of the ring is not canceled by the phase accumulated from a trip along the other arm.

In Fig. 5 we show the distribution of the allowed energy values which correspond to $|x_m| \leq 2$ (32) within an energy range $[-2t, 2t]$. The use of the above trace-formula is essential in this case. We have also studied the change in the character of the ‘band’ as a magnetic field is switched on. Fig. 5(b) shows the allowed energy values for $\phi_A = \phi_0/4$ ($\phi_B \equiv N_B^2 N_A \phi_A = 11^2/10\pi \phi_A$). The pattern shows a marked change with respect to the earlier case.

It is also interesting to note that for such a quasiperiodic Fibonacci array of rings the map (17) leads to an invariant in the conventional sense (32). The invariant in this case is a function of $E$, $\phi_A$ and $\phi_B$, and is given by

$$J = \frac{1}{4} \left( \frac{x^2_{m+1}}{\Delta_{m+1}} + \frac{x^2_m}{\Delta_m} + \frac{x^2_{m-1}}{\Delta_{m-1}} - \frac{x_{m+1} x_m x_{m-1}}{\Delta_m \Delta_{m-1}} - 1 \right).$$ \hspace{1cm} (18)

For an ordered array of rings it is equal to zero. However, even for an ordered array of rings we have resonance ($T = \ldots$
1) and anti-resonance ($T = 0$) as a result of interference. Hence, the zeros of the invariant $J$ should not necessarily correspond to the $T = 1$ cases in a Fibonacci array as well. In order to compute the transmission coefficient for rings in a Fibonacci sequence we generalize the trace-antitrace formulation discussed in the literature [28]. The transmission coefficient for an $m$th generation sequence is given by

$$T(m) = \frac{4 \sin^2 k}{|z_m \cos k - y_m|^2 + |x_m|^2 \sin^2 k}$$  \hspace{1cm} (19)$$

where the antitraces $y_m$ and $z_m$ are given by $y_m = M_{21}(m) - M_{12}(m)$ and $z_m = M_{11}(m) - M_{22}(m)$. They are obtained recursively as

$$y_{m+1} = x_m y_{m-1} + \frac{\Delta_m}{\Delta_{m-2}} y_{m-2},$$

$$z_{m+1} = x_m z_{m-1} + \frac{\Delta_m}{\Delta_{m-2}} z_{m-2}. \hspace{1cm} (20)$$

FIG. 6: Variation of the transmission coefficient $T$ against $E$ for a 9th generation Fibonacci array of rings (solid line) and the invariant $J$ (dashed line) for (a) $\phi_A = 0$ and (b) $\phi_A = \phi_0/4$. $n = m = 4$ for the $A$-type ring and $n = m = 9$ for the $B$-type ring. (c) Same as (a), but $n = m = 3$ for $A$ and $n = m = 15$ for $B$. The energy resolution in all cases is $\Delta E/t = 0.0001t$.

We emphasize that the use of Eq. (19) in obtaining $T(m)$ depends crucially on setting $t_0 = t$ which, of course, has been our choice here. In Fig. 6 we show the variation of transmission coefficient and the invariant for a 9th generation sequence with $\phi_A = \phi_B = 0$ and $\phi_A = \phi_0/4$, $\phi_B = 4\phi_A$. The $A$- and $B$-rings have 4 and 9 atoms in each arm, respectively. This implies that the circumference of the $B$-ring is double the size of the $A$-ring. The invariant is real in this case. The zero-flux case is characterized by the appearance of several subbands symmetrically distributed around $E = 0$, while with $\phi_A = \phi_0/4$, the subbands get closer. We have scanned the energy range $[-2t, 2t]$ using various values of the energy interval $\Delta E$. The results are presented for relatively large $\Delta E$ for more clarity in the figures. On reducing the energy interval further we find that the plot becomes more dense within a subcluster. However, no new subband structure emerges for the cases we present. With increasing number of scatterers the invariant exhibits many more zeroes and the plot of the transmission coefficient becomes highly fragmented, see Fig. 6(c). A similar effect is also observed for the periodic arrays as well. It may be noted that the invariant becomes very close to zero at certain points which correspond to high transmittivity, and the
ranges of the energy where the invariant diverges correspond to the gaps in the spectrum. However, as we have already mentioned, finite transmission may also be seen for energies for which the invariant exhibits a finite value. Last, the behavior of the transmission coefficient as shown above is by no means unique and depends on the choice of the number of scatterers in each ring. For example, if the A-ring has 3 atoms in each arm, and the B-ring has 7 atoms in each arm, the size of the B-ring is again double the size of the A-ring. However, in this case, for \( \phi_A = \phi_B = 0 \) we have a transmission maximum at \( E = 0 \), in contrast to the case shown in Fig. 6.

**B. Variation of \( T \) against flux**

We now consider the transmission for a fixed electron energy \( E \) as the flux through the rings is varied. As \( \phi_B/\phi_A = S_B/S_A = N_B^2/N_A^2 \), the periodicity of the transmission coefficient should be sensitive to \( S_B/S_A \) independent of the order in which the rings are arranged, i.e., periodic or quasiperiodic. Let us work out a specific example. We consider \( E = 0 \). Let the A-ring have 2\( p \) atoms in each arm and the B-ring have 2\( q \) atoms in each arm, \( p \) and \( q \) being integers. Here, \( S_B/S_A = (2q+1)/(2p+1)^2 \).

The dimer matrices \( P_A \) and \( P_B \) commute and for rings arranged in an \( m \)-th generation Fibonacci array the transmission coefficient is given by

\[
T(m) = \frac{\tau_{pq}^m(\phi_A)}{[\tau_{pq}^m(\phi_A) + 1]^2},
\]

with

\[
\tau_{pq}^m(\phi_A) = 2^{2F_m} \cos^{2F_m-1} \left( \frac{\phi_A}{\phi_0} \right) \times \cos^{2F_m-2} \left( \frac{(2q+1)^2\phi_A}{(2p+1)^2\phi_0} \right).
\]

Here, \( F_m = F_{m-1} + F_{m-2} \) is the Fibonacci number in the \( m \)-th generation, with \( F_0 = F_1 = 1 \). The above formula reproduces the result of the corresponding even-ordered case. Resonance (\( T = 1 \)) is achieved when \( \tau_{pq}^m(\phi_A) = 1 \). Only if \( S_B = lS_A \), with integer \( l \), the period of the transmission coefficient remains \( \phi_0 \). This implies \( (2q+1) = \sqrt{l}(2p+1) \), which indicates that \( \phi_0 \)-periodicity in \( T \) cannot be achieved for arbitrary combination of \( p \) and \( q \). If \( S_B \neq lS_A \), the transmission coefficient has a period equal to \( (2p+1)^2\phi_0 \). In Fig. 7(a) we plot \( T \) against \( \phi_A/\phi_0 \) for a sixth-generation Fibonacci chain. The A-ring has 4 atoms and the B-ring has 5 atoms in each arm, respectively. The diagram shows a periodicity equal to \( 25\phi_0 \). In Fig. 7(b) a \( \phi_0 \)-periodic behavior is shown with the A-ring having 4 atoms in each arm and the B-ring having 5 atoms in each arm. So, both \( \phi_0 \) and \( (2p+1)^2\phi_0 \) periodicity can be present.

**C. A special case**

We now present an interesting feature that reveals the sensitivity of the transmission coefficient on the specific choices of the number of scatterers. We select \( \phi_A = 0 \). The A-ring has just one atom in each arm, while in the B-ring we have two atoms in each arm. Then at \( E = \sqrt{3} \), we get \( J = 0 \), \( P_B = -\mathbb{I} \), and

\[
P_A = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix},
\]

so that \( P_A^6 = -\mathbb{I} \). As a result, every 12th generation of the Fibonacci sequence consists of identical strings of \( P_A \)'s starting with the second generation. The transmission coefficients will consequently repeat every 12th generation. We thus have a case where even a quasiperiodic array of rings has periodically repeating values of the transmittivity as the system grows in size. We believe that this feature is also likely to be present for non-zero flux values as well. This aspect in under further investigation.

**V. SUMMARY**

We have studied the transmission spectra of mesoscopic rings in the presence of a magnetic flux within a tight-binding formalism. We address both periodic and quasiperiodic arrays of rings. In the spirit of a renormalization-group decimation method, we convert each ring into an effective dimer, and attach a series of such dimers to semi-infinite perfect leads to study
their transmittance. In view of a possible experimental realization we set the values of the on-site potential and the nearest neighbor hopping integral in the leads to be the same as those in the bulk. We find that for both geometries the transmission coefficient at fixed values of energy may exhibit both types of periods, viz, $\phi_0$ and $\phi_0/2$. For the periodic array of rings we find an interesting result: If a ring contains an even number of atoms in one of its arms and an odd number in the other arm, then in the absence of any magnetic flux an array of such rings shows a power-law decay in the transmission coefficient as the system grows in size. This has been shown analytically for a special energy $E = 0$. Apart from this, the field-induced resonance in periodic arrays of rings has also been discussed. For the quasiperiodic Fibonacci array of rings of two different sizes we find a drastic change in the distribution of allowed energy values compared to the purely one-dimensional Fibonacci chain \[32\]. The magnetic field alters it further. The trace maps and the Fibonacci invariant have been derived including the magnetic field.

The variation of the transmission coefficient as functions of the external flux, as well as the energy of the incident electron has been studied. The transmission coefficient exhibits different periodicities depending on the relative areas of the rings. The formulation has also been tested for other aperiodic sequences such as the Thue-Morse sequence. The basic nature of the periodic variation of the transmittivity as a function of the magnetic flux remains the same.

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