Stochastic proximal splitting algorithm for stochastic composite minimization

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Abstract Supported by the recent contributions in multiple branches, the first-order splitting algorithms became central for structured nonsmooth optimization. In the large-scale or noisy contexts, when only stochastic information on the smooth part of the objective function is available, the extension of proximal gradient schemes to stochastic oracles is based on proximal tractability of the nonsmooth component and it has been deeply analyzed in the literature. However, there remained gaps illustrated by composite models where the nonsmooth term is not proximally tractable anymore. In this note we tackle composite optimization problems, where the access only to stochastic information on both smooth and nonsmooth components is assumed, using a stochastic proximal first-order scheme with stochastic proximal updates. We provide the iteration complexity (in expectation) under the strong convexity assumption on the objective function.

1 Introduction

In this paper we consider the following convex composite optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) + h(x),$$

where $f$ is the smooth component and $h$ is a proper, convex, lower-semicontinuous function. In literature, many applications from statistics [17] or signal processing, often motivated noisy contexts which allow access only to stochastic first order information of smooth function $f$, while $h$ is typically known to be a proximally-tractable convex function. In our terms, proximally-tractable means that the proximal map of a given function is possible to be computed in closed form or, at most, in linear time. Therefore, in these situations the following model

$$\min_{x \in \mathbb{R}^n} E[f(x; \xi)] + h(x)$$

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express better the real assumptions. However, the recent dimensionality inflation of machine learning \([6,21,29]\) and signal processing \([21,22]\) models gave birth to optimization problems with complicated regularizers or complicated many constraints. As practical examples we recall: parametric sparse representation \([22]\), group lasso \([6,21,29]\), CUR-like factorization \([27]\), graph trend filtering \([20,24]\), dictionary learning \([22,28]\). Motivated by all these models, in this note instead of assuming proximal-tractable \(h\), we consider that \(h\) is expressed as an expectation of stochastic proximally-tractable components \(h(\cdot;\zeta)\) (i.e. \(h(x) = \mathbb{E}[h(x;\zeta)]\)). Therefore our main results are focused towards the following model:

\[
\min_{x \in \mathbb{R}^n} F(x) := \mathbb{E}[f(x;\xi)] + \mathbb{E}[h(x;\zeta)],
\]

where \(\xi, \zeta\) are random variables associated with probability spaces \((\mathbb{P}, \Omega_1)\) and \((\mathbb{P}, \Omega_2)\). Functions \(f(\cdot;\xi) : \mathbb{R}^n \to \mathbb{R}\) are smooth with Lipschitz gradients and \(h(\cdot,\zeta) : \mathbb{R}^n \to (-\infty, +\infty]\) are proper convex and lower-semicontinuous, \(\mathbb{E}[\cdot]\) is the expectation over respective random variable. In general, many existing primal schemes encounter computational difficulties when a large (possibly infinite) number of constraints are present, since they require projections onto complicated feasible set.

**Contributions.** (i) We develop stochastic first-order splitting schemes relying on stochastic gradients and stochastic proximal updates, which naturally generalize the widely known SGD and SPP algorithms toward composite models with untractable regularizations; (ii) we provide \(O(1/k)\) iteration complexity estimates which were previously unknown for this type of schemes. In particular, for convex feasibility problems, the analysis yields naturally linear convergence rates.

We briefly recall further the milestone results from stochastic optimization literature with focus on the complexity of stochastic first-order methods.

### 1.1 Previous work

Great attention has been given in the last decade to the behaviour of stochastic first order schemes, with special focus in stochastic gradient descent (SGD), on varying models under different convexity properties, see \([8,9,11,12,14,17,19]\). Since the analysis of SGD naturally require various smoothness conditions, proper modifications are necessary to attack nonsmooth models. The stochastic proximal point (SPP) algorithm has been recently analyzed using various differentiability assumptions, see \([1,4,7,15,18,23,25]\) and has shown surprising analytical and empirical performances. In \([23]\) is considered the typical stochastic learning model involving the expectation of random particular components \(f(x;\xi)\) defined by the composition of a smooth function and a linear operator, i.e. \(f(x;\xi) = \ell(a_\xi^T x)\), where \(a_\xi \in \mathbb{R}^n\). Their complexity analysis requires smoothness and strong convexity to obtain in the quadratic mean and an \(O\left(\frac{1}{k}\right)\) convergence rate, using vanishing stepsize. The generalization of these convergence guarantees is undertaken in \([15]\), where no linear composition structure is required and an (in)finite number of constraints are included in the stochastic model, i.e. \(h(\cdot;\xi) = \mathbb{I}_{\mathcal{X}_\xi}(\cdot)\). However, the analysis of \([15]\) requires strong convexity and Lipschitz gradient continuity for each functional component.
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Note that our analysis surpasses these restrictions and provides a natural generalization of [13] to nonsmooth composite models. Further, in [4] a general asymptotic convergence analysis of slightly modified SPP scheme has been provided, under mild convexity assumptions on a finitely constrained stochastic problem. In [25] similar SPP algorithms are developed, which are also tailored for complicated feasible sets. The authors focus on mild convex models deriving optimal $O(1/\sqrt{\ell})$ rates. Recently, in [1], the authors analyze SPP schemes for shared minimizers stochastic optimization obtaining linear convergence results, for variable stepsize SPP. Also, without shared minimizers assumption, they obtain for SPP $O\left(\frac{1}{\sqrt{\ell}}\right)$ in convex Lipschitz continuous case and, furthermore, $O\left(\frac{1}{\ell}\right)$ in strongly convex case.

Splitting first-order schemes received significantly attention due to their natural insight and simplicity in contexts when a sum of two components are minimized (see [3,13]). However, when the information access is limited only to stochastic samples of the two components, extending the existing guarantees is not straightforward. Notice that overall, on one hand, SPP avoids any splitting in composite models by treating the constrained smooth problems as black box expectations (see [15,23]). On the other hand, until recently the composite nonsmooth models assumed proximally tractable $h$ in order to use extend proof arguments of the results from stochastic smooth optimization [17]. Therefore, only recently the full stochastic composite models with stochastic regularizers have been properly tackled [20], where almost sure asymptotic convergence is established for a stochastic splitting scheme, where each iteration represents a proximal gradient update using stochastic samples of $f$ and $h$.

In our paper we analyze the nonasymptotic behaviour of this scheme and point the relations with other algorithms from the literature.

The stochastic splitting schemes are also related to the model-based methods developed in [5]. Here, the authors developed a unified algorithmic framework, which generates stochastic schemes, for different models arising in learning applications. They assume their composite objective function to be the sum of a (weakly convex) stochastic component with bounded gradients and simple (proximally tractable) convex regularization. Although their framework is algorithmically more general, our analysis avoid these boundedness assumptions and allows objectives with a component having Lipschitz continuous gradient and do not require proximal tractability on regularizations.

**Notations.** We use notation $[m] = \{1, \cdots, m\}$. For $x, y \in \mathbb{R}^n$ denote the scalar product $\langle x, y \rangle = x^T y$ and Euclidean norm by $\|x\| = \sqrt{x^T x}$. The projection operator onto set $X$ is denoted by $\pi_X$ and the distance from $x$ to the set $X$ is denoted $\text{dist}_X(x) = \min_{z \in X} \|x - z\|$. The indicator function of a set $X$ is denoted: $I_X(x) = \begin{cases} 0, & \text{if } x \in X \\ \infty, & \text{otherwise} \end{cases}$. We use notations $\partial h(x; \xi)$ the subdifferential set and $g_h(x; \xi)$ for a subgradient of $h(\cdot; \xi)$ at $x$. In differentiable case we use the gradient notation $\nabla f(\cdot; \xi)$. 

\[ f(\cdot; \xi) \]
1.2 Preliminaries

We denote the set of optimal solutions with \( X^* \) and \( x^* \) any optimal point for \( \Pi \). For simplicity, in the sequel we consider that \( \Omega_1 = \Omega_2 \) and use only \( \xi \) as the sampling random variable.

**Assumption 1** The central problem \( \Pi \) has nonempty optimal set \( X^* \) and satisfies:

1. The function \( f(\cdot;\xi) \) has \( L_f \)-Lipschitz gradient, i.e. there exists \( L_f > 0 \) such that:
   \[
   \| \nabla f(x;\xi) - \nabla f(y;\xi) \| \leq L_f \| x - y \|, \quad \forall x, y \in \mathbb{R}^n, \xi \in \Omega.
   \]
   and \( f \) is \( \sigma_f \)-strongly convex, i.e. there exists \( \sigma_f \geq 0 \) satisfying:
   \[
   f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\sigma_f}{2} \| x - y \|^2 \quad \forall x, y \in \mathbb{R}^n. \tag{2}
   \]

2. There exists subgradient mapping \( g_h : \mathbb{R}^n \times \Omega \to \mathbb{R}^n \) such that \( g_h(x;\xi) \in \partial h(x) \) and \( \mathbb{E}[g_h(x;\xi)] \in \partial h(x) \).

3. \( h(\cdot;\xi) \) has bounded gradients on the optimal set: there exists \( S < \infty \) such that \( \mathbb{E} \| g_h(x^*;\xi) \|^2 \leq S \) for all \( x^* \in X^* \);

The first part of the above assumption is natural in the composite (stochastic) optimization \([3][13][15]\). The Assumption \( \Pi \) ensure the existence of a subgradient mapping for functions \( h(\cdot;\xi) \). Denote \( \partial F(x;\xi) = \nabla f(x;\xi) + \partial h(x;\xi) \). Moreover, since \( 0 \in \partial F(x^*) \) for any \( x^* \in X^* \), then we assume in the sequel that \( g_F(x^*) := \mathbb{E}[g_F(x^*;\xi)] = 0 \). Also the third part Assumption \( \Pi \) is standard in the literature related to stochastic algorithms.

Let closed convex sets \( \{X_\xi\}_{\xi \in \Omega} \) and \( X = \bigcap_{\xi \in \Omega} X_\xi \), then a favorable "conditioning" property for most projection methods is the linear regularity property: there exists \( \kappa > 0 \) such that
\[
\mathbb{E}[\text{dist}_{X_\xi}^2(x)] \geq \frac{\kappa}{2} \text{dist}_X^2(x) \quad \forall x \in \mathbb{R}^n. \tag{3}
\]

Given some smoothing parameter \( \mu > 0 \), define Moreau envelope of \( h(x;\xi) \) and the prox operator as follows:

\[
h_\mu(x;\xi) := \min_{z \in \mathbb{R}^n} h(z;\xi) + \frac{1}{2\mu} \| z - x \|^2
\]

\[\text{prox}_{h_\mu}(x;\xi) := \arg \min_{z \in \mathbb{R}^n} h(z;\xi) + \frac{1}{2\mu} \| z - x \|^2.\]

The approximate \( h_\mu(\cdot;\xi) \) inherits the same convexity properties of \( h(\cdot;\xi) \) and additionally has Lipschitz continuous gradient with constant \( \frac{1}{\mu} \), see \([16]\). In particular, when \( h(x;\xi) = \| X_\xi - x \| \) the prox operator becomes the projection operator \( \text{prox}_{h_\mu}(x;\xi) = \pi_{X_\xi}(x) \).
2 Stochastic Splitting Proximal Gradient Algorithm

In the following section we present the Stochastic Splitting Proximal Gradient (SSPG) algorithm and analyze its nonasymptotic convergence towards the optimal set of the original problem \((1)\). The asymptotic convergence of vanishing stepsize SSPG have been analyzed in [20].

Let \(x^0 \in \mathbb{R}^n\) be a starting point and \(\{\mu_k\}_{k \geq 0}\) be a nonincreasing positive sequence of stepsizes.

**Stochastic Splitting Proximal Gradient algorithm (SSPG):** For \(k \geq 0\) compute

1. Choose randomly \(\xi_k \in \Omega\) w.r.t. probability distribution \(P\).
2. Update:
   \[
   y^k = x^k - \mu_k \nabla f(x^k; \xi_k) \\
   x^{k+1} = y^k - \mu_k \nabla h_{\mu_k}(y^k; \xi_k) \quad (= \text{prox}_{h,\mu_k}(y^k; \xi_k)).
   \]
3. If the stopping criterion holds, then STOP, otherwise \(k = k + 1\).

The SSPG iteration \(x^{k+1} = \text{prox}_{h,\mu_k}(x^k - \mu_k \nabla f(x^k; \xi_k); \xi_k)\) is mainly a Stochastic Proximal Gradient iteration based on stochastic proximal maps [20]. Thus, the first step of algorithm SSPG consists of a varying-stepsize (vanilla) stochastic gradient update, while the second step rely on a stochastic proximal update, or equivalently a gradient step in the direction of the randomly sampled gradient of expected Moreau envelope \(h_{\mu}(\cdot)\). Further results will state that a diminishing stepsize is an appropriate choice to obtain convergence in expectation. By varying our central model, this general SSPG scheme recovers several well-known stochastic first order algorithms.

(i) In the smooth case \((h = 0)\), SSPG reduces to vanilla SGD [8]:
\[
x^{k+1} = x^k - \mu_k \nabla f(x^k; \xi_k).
\]

(ii) By considering proximal-tractable regularizers (i.e. \(h(\cdot; \xi) = h(\cdot)\)) or simple convex sets (i.e. \(h(\cdot; \xi) = 1_{\mathcal{X}}(\cdot)\), with \(\pi_{\mathcal{X}}(\cdot)\) computable in closed form), then we recover Proximal (or Projected, respectively) SGD [17]:
\[
x^{k+1} = \text{prox}_{h,\mu_k}(x^k - \mu_k \nabla f(x^k; \xi_k)) \quad \text{or} \quad x^{k+1} = \pi_{\mathcal{X}}(x^k - \mu_k \nabla f(x^k; \xi_k)).
\]

(iii) For nonsmooth objective functions, when \(f = 0\), SSPG is equivalent with SPP iteration [1,15,23]:
\[
x^{k+1} = \text{prox}_{h,\mu_k}(x^k; \xi_k).
\]

(iv) For CFPs (i.e. \(h(\cdot; \xi) = 1_{\mathcal{X}_{\xi}}(\cdot)\), the SSPG recovers Randomized Alternating Projections [2]:
\[
x^{k+1} = \pi_{\mathcal{X}_{\xi_k}}(x^k).
\]

Therefore, the below results will implicitly represent unifying convergence rates for these algorithms, under stated assumptions.
3 Iteration complexity in expectation

In this section we assume that the function $f$ is strongly convex and derive convergence rates for this particular case.

**Lemma 1** Let Assumption [2] hold and $\mu_k \leq \frac{1}{2L_f}$. Then the sequence $\{x^k\}_{k\geq 0}$ generated by SPSG satisfies:

$$
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq (1 - \sigma_f \mu_k) \mathbb{E}[\|x^k - x^*\|^2] + 2\mu_k \mathbb{E} \left[ F(x^*) - F(x^{k+1}; \xi_k) - \frac{1}{4\mu_k} \|x^{k+1} - x^k\|^2 \right].
$$

**Proof** First notice that from the optimality conditions of the subproblem $\min_z h(z; \xi) + \frac{1}{2\sigma_f} \|z - y\|$, the following relation holds:

$$
\nabla h(x^{k+1}; \xi_k) + \frac{1}{\mu_k} (x^{k+1} - y^{k+1}) = 0.
$$

Further we obtain the main recurrence:

$$
\|x^{k+1} - x^*\|^2 = \|x^k - x^*\|^2 + 2\mu_k \|x^{k+1} - x^k\|^2 + 2 \langle \mu_k \nabla f(x^k; \xi_k), x^* - x^{k+1} \rangle - \|x^{k+1} - x^k\|^2
$$

$$
= \|x^k - x^*\|^2 + 2\mu_k \|\nabla f(x^k; \xi_k)\|^2 + 2 \mu_k (\nabla f(x^k; \xi_k), x^* - x^{k+1}) - \|x^{k+1} - x^k\|^2
$$

$$
\leq \|x^k - x^*\|^2 + 2\mu_k (\nabla f(x^k; \xi_k), x^* - x^k) - \|x^{k+1} - x^k\|^2
$$

$$
= \|x^k - x^*\|^2 - 2\mu_k (\nabla f(x^k; \xi_k), x^k - x^*) + \frac{1}{4\mu_k} \|x^{k+1} - x^k\|^2 + h(x^{k+1}; \xi_k) + 2\mu_k (\nabla f(x^k; \xi_k), x^* - x^k) - \frac{1}{2} \|x^{k+1} - x^k\|^2 + 2\mu_k h(x^*; \xi_k).
$$

By taking expectation w.r.t. $\xi_k$ in both sides, we obtain:

$$
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq \|x^k - x^*\|^2 + 2\mu_k \mathbb{E}(f(x^k; \xi_k) - F(x^{k+1}; \xi_k))
$$

$$
+ 2\mu_k \|\nabla f(x^k)\|^2 + \frac{1}{2} \mathbb{E}[\|x^{k+1} - x^k\|^2] + 2\mu_k h(x^*)
$$

$$
\leq \|x^k - x^*\|^2 + 2\mu_k \mathbb{E}(f(x^k; \xi_k) - F(x^{k+1}; \xi_k))
$$

$$
+ 2\mu_k \left( F(x^*) - f(x^k) - \frac{\sigma_f}{2} \|x^k - x^*\|^2 \right) \frac{1}{2} \|x^{k+1} - x^k\|^2
$$

$$
= (1 - \sigma_f \mu_k) \|x^k - x^*\|^2 + 2\mu_k \mathbb{E} \left[ F(x^*) - F(x^{k+1}; \xi_k) - \frac{1}{4\mu_k} \|x^{k+1} - x^k\|^2 \right].
$$

Finally, by taking the expectation with the entire index history we obtain our result.

Further we present some lower bounds on the second term from the right hand side.
Lemma 2 Given $\mu > 0$, let Assumption 1 hold. Then $F(x) = f(x) + h(x)$ satisfies the following relations: given $k \geq 0$

(i) $E \left[ F(x^{k+1}; \xi_k) - F^* + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right] \geq -\mu_k E \left[ \left\| g_F(x^*; \xi) \right\|^2 \right]$.

(ii) Let $h(\cdot; \xi) = \|X_\xi(\cdot)\|$ and $\{X_\xi\}_{\xi \in \Omega}$ satisfying linear regularity with constant $\kappa$.

$$E \left[ F(x^{k+1}; \xi_k) - F^* + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$\geq -\mu_k E \left[ \left\| \nabla f(x^*; \xi) \right\|^2 \right] - \frac{2\mu_k}{\kappa} \left\| \nabla f(x^*) \right\|^2 + \frac{\kappa}{8\mu_k} E[\text{dist}_X^2(x^k)]$$

Proof In order to prove (i), let $z \in \mathbb{R}^n$. Then, given $x^* \in X^*$ and $g_F(x^*; \xi) \in \partial F(x^*; \xi)$, by convexity of $f(\cdot; \xi)$ we have:

$$E \left[ F(x^{k+1}; \xi_k) - F^* + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$\geq E \left[ \langle g_F(x^*; \xi_k), x^{k+1} - x^* \rangle + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$\geq E \left[ \langle g_F(x^*; \xi_k), x^k - x^* \rangle + \langle g_F(x^*; \xi_k), x^{k+1} - x^k \rangle + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$\geq E \left[ \langle g_F(x^*; \xi), x^k - x^* \rangle + \min_z \langle g_F(x^*; \xi), z - x^* \rangle + \frac{1}{4\mu_k} \left\| z - x^k \right\|^2 \right]$$

$$\geq \langle E[ g_F(x^*; \xi)], x^k - x^* \rangle - \mu_k E \left[ \left\| g_F(x^*; \xi) \right\|^2 \right] \quad \forall x^* \in X^*,$$

where we recall that we consider $E[ g_F(x^*; \xi)] = 0$. For the second part (ii) we have $h_\mu(x; \xi) := \frac{1}{2\mu} \text{dist}_X^2(x)$ and $h_\mu(x) := \frac{1}{2\mu} E \left[ \text{dist}_X^2(x) \right] \geq \frac{\kappa}{2} \text{dist}_X^2(x)$ for all $x$.

Then we derive that:

$$E \left[ F(x^{k+1}; \xi_k) - F^* + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$\geq E \left[ \left\| \nabla f(x^*; \xi_k), x^{k+1} - x^* \right\|^2 + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right] + E[h_\mu(x)]$$

$$\geq E \left[ \left\| \nabla f(x^*; \xi_k), x^{k+1} - x^* \right\|^2 + \frac{1}{4\mu_k} \left\| x^{k+1} - x^k \right\|^2 \right]$$

$$+ E \left[ \left\| \nabla f(x^*; \xi), x^k - x^* \right\|^2 + E[h_\mu(x)] \right]$$

$$\geq -\mu_k E \left[ \left\| \nabla f(x^*; \xi), x^k - x^* \right\|^2 \right] + E[h_\mu(x)]$$

$$= \mu_k E \left[ \left\| \nabla f(x^*; \xi), x^k - x^* \right\|^2 \right] + \frac{1}{4\mu_k} E[\text{dist}_X^2(x^k)]$$

$$\geq -\mu_k E \left[ \left\| \nabla f(x^*; \xi), x^k - x^* \right\|^2 \right] + \frac{\kappa}{4\mu_k} E[\text{dist}_X^2(x^k)]$$

C.S. $\geq -\mu_k E \left[ \left\| \nabla f(x^*; \xi) \right\|^2 \right] - \frac{\kappa}{4\mu_k} E[\text{dist}_X^2(x^k)]$
\[ \geq -\mu_k E\left[\|\nabla f(x^*; \xi)\|^2\right] - \|\nabla f(x^*)\| \sqrt{E[\text{dist}_X^2(x^k)] + \frac{\kappa}{4\mu_k} E[\text{dist}_X^2(x^k)]} \]

\[ \geq -\mu_k E\left[\|\nabla f(x^*; \xi)\|^2\right] - \frac{2\mu_k}{\kappa}\|\nabla f(x^*)\|^2 + \frac{\kappa}{8\mu_k} E[\text{dist}_X^2(x^k)]. \]

In the third and the last inequality we used \( \langle a, b \rangle + \frac{1}{2\kappa}\|a\|^2 \geq -\frac{\kappa}{2}\|a\|^2. \) Also in the fifth inequality we used the optimality conditions \( \langle \nabla f(x^*), z - x^* \rangle \geq 0, \forall z \in X. \) In the seventh inequality we used Jensen inequality: \( E[X] \leq \sqrt{E[X^2]}. \)

Next we present the main recurrences which will finally generate our nonasymptotic convergence rates.

**Theorem 2** The sequence \( \{x^k\}_{k \geq 0} \) generated by SPSG satisfies:

(i) Let Assumptions [7] hold and \( \mu_k = \mu_k \leq \frac{1}{2L_f} \), then:

\[ E[\|x^{k+1} - x^*\|^2] \leq (1 - \sigma_f \mu_k) E[\|x^k - x^*\|^2] + \mu_k^2 \Sigma, \]

where \( \Sigma = E\left[\|g_F(x^*; \xi)\|^2\right]. \)

(ii) In particular, additionally to Assumption [7] let \( h(\cdot; \xi) = \|X_\xi(\cdot) \) with linearly regular sets \( \{X_\xi\}_{\xi \in \Omega} \). Then the following recurrence holds:

\[ E[\|x^{k+1} - x^*\|^2] \leq (1 - \sigma_f \mu_k) E[\|x^k - x^*\|^2] + \mu_k^2 \Sigma - \frac{\mu_k^2 \kappa}{8} E[\text{dist}_X^2(x^k)], \]

where \( \Sigma = E\left[\|\nabla f(x^*; \xi)\|^2\right] + \frac{2}{\kappa} \|\nabla f(x^*)\|^2. \)

**Proof** The proof result straightforwardly from Lemmas 1 and 2.

**Remark 1** Consider deterministic setting \( F(\cdot; \xi) = F(\cdot) \) and \( \mu_k = \frac{1}{2L_f} \), then SPSG becomes the proximal gradient algorithm and Theorem 2(i) holds with \( g_F(x^*; \xi) = g_F(x^*) = 0 \), implying that \( \Sigma = 0 \). Thus the well-known iteration complexity estimate \( O\left( \frac{d}{\sigma_f} \log(1/\epsilon) \right) \) of proximal gradient algorithm is recovered up to a constant.

The above recurrences generates immediately the following convergence rates.

**Corollary 3** Under Assumption [7] the following convergence rates hold:

(i) Let \( \mu_k = \frac{1}{k}, \gamma \in (0, 1) \) then: \[ E[\|x^k - x^*\|^2] \leq O\left( \frac{1}{k^\gamma} \right) \]

(ii) Let \( \mu_k = \frac{1}{\kappa}, \gamma \in (0, 1) \) then: \[ E[\|x^k - x^*\|^2] \leq \begin{cases} O\left( \frac{1}{k} \right) & \text{if } \mu_0 \sigma_f > e - 1 \\ O\left( \frac{1}{\kappa} \right) & \text{if } \mu_0 \sigma_f > e - 1 \\ O\left( \frac{1}{\kappa} \right) & \text{if } \mu_0 \sigma_f < e - 1. \end{cases} \]

(iii) For constant stepsize \( \mu_k = \mu > 0 \), the recurrence from Theorem 2 implies:

\[ E[\|x^k - x^*\|^2] \leq (1 - \mu \sigma_f)^k \|x^0 - x^*\|^2 + \frac{\mu}{\sigma_f} \Sigma, \quad (6) \]

where \( \Sigma = E\left[\|g_F(x^*; \xi)\|^2\right]. \)
Moreover, consider convex feasibility problem where $f = 0$, $h(\cdot; \xi) = \mathbb{I}_{X_\xi}(\cdot)$ with $\kappa$-linearly regular sets $\{X_\xi\}_{\xi \in \Omega}$ and constant stepsize $\mu_k = \mu$. Then the SSPG sequence $\{x^k\}_{k \geq 0}$ converges linearly as follows:

$$\mathbb{E}[\text{dist}_X^2(x^k)] \leq \left(1 - \frac{\mu \kappa}{8}\right)^k \text{dist}_X^2(x^0).$$

**Proof** The proof for the first two results (i) and (ii) follows similar lines with [15, Corollary 15]. However, for completeness we present it in appendix. To prove (iii), notice that Theorem 2 straightforwardly implies:

$$\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq (1 - \sigma f \mu)\|x^k - x^*\|^2 + \mu^2 \Sigma$$

$$\leq (1 - \mu \sigma f )^k \|x^0 - x^*\|^2 + \mu^2 \Sigma \sum_{i=0}^{k-1} (1 - \mu \sigma f )^i$$

$$\leq (1 - \mu \sigma f )^k \|x^0 - x^*\|^2 + \frac{\mu \Sigma}{\sigma f}[1 - (1 - \mu \sigma f )^k]$$

$$\leq (1 - \mu \sigma f )^k \|x^0 - x^*\|^2 + \frac{\mu \Sigma}{\sigma f}.$$

Now under convex feasibility problem settings the second part of Theorem 2 states that:

$$\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq \|x^k - x^*\|^2 - \frac{\mu^2 \sigma h}{8} \text{dist}_X^2(x^k).$$

Since in this case, $X = \bigcap_{\xi \in \Omega} X_\xi = X^*$, then by choosing $x^* = \pi_X(x^k)$ and using $\|x^{k+1} - \pi_X(x^k)\| \geq \text{dist}_X(x^{k+1})$, then we obtain:

$$\mathbb{E}[\text{dist}_X^2(x^{k+1})] \leq \mathbb{E}[\|x^{k+1} - \pi_X(x^k)\|^2] \leq \left(1 - \frac{\mu \sigma h}{8}\right) \mathbb{E}[\text{dist}_X^2(x^k)],$$

which yields the linear convergence rate of SSPG.

**Remark 2** Although sublinear $O(1/k)$ convergence rates for strongly convex objectives are typically obtained in literature for many first-order stochastic schemes [8,15], these rates are new due to their generalization potential. Regarding second rate (ii), although it expresses a geometric decrease of the initial residual term, this rate states that, after $O\left(\frac{1}{\mu \sigma f } \log \left(\frac{1}{\delta}\right)\right)$ iterations, the sequence $\{x^k\}_{k \geq 0}$ will remain (in expectation) in a bounded neighborhood of the optimal point $\{x : \|x - x^*\|^2 \leq \frac{\mu \Sigma}{\sigma f}\}$. This fact suggests that only sufficiently small constant stepsizes guarantee the convergence of SSPG sequence.

In the convex feasibility setting, SSPG reduces to Randomized Alternating Projections algorithm for which the obtained linear rate obeys the rates from the literature up to a constant. However, we believe that using some refinements of proof arguments there might be obtained optimal rates w.r.t. the constants.
4 Conclusion

In this note we presented preliminary guarantees for stochastic gradient schemes with stochastic proximal updates, which unify some well-known schemes in the literature. For future work, would be interesting to analyze the empirical behaviour of our general scheme under various stepsize choices.

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5 Appendix

Proof (of Corollary 3) For simplicity denote $\theta_k = (1 - \mu_k \sigma_f)$, then Theorem 2 implies that:

$$E[\|x^{k+1} - x^*\|^2] \leq \left( \prod_{i=0}^{k} \theta_i \right) \|x^0 - x^*\|^2 + \sum_{i=0}^{k} \left( \prod_{j=i+1}^{k} \theta_j \right) \mu_i^2.$$ 

By using the Bernoulli inequality $1 - tx \leq \frac{1}{1 +tx} \leq (1 + x)^{-t}$ for $t \in [0, 1], x \geq 0$, then we have:

$$\prod_{i=l}^{u} \theta_i = \prod_{i=l}^{u} \left( 1 - \frac{\mu_0 \sigma_f}{\gamma f} \right) \leq \prod_{i=l}^{u} (1 + \mu_0 \sigma_f)^{-1/\gamma} = (1 + \mu_0 \sigma_f)^{-\sum_{i=1}^{u} \phi}.$$ 

(7)

On the other hand, if we use the lower bound

$$\sum_{i=0}^{u} \frac{1}{\gamma f} \geq \int_{l}^{u+1} \frac{1}{\tau \gamma} d\tau = \varphi_{1-\gamma}(u) - \varphi_{1-\gamma}(l).$$ 

(8)

then we can finally derive:

$$\sum_{i=0}^{k} \left( \prod_{j=i+1}^{k} \theta_j \right) \mu_i^2 = \sum_{i=0}^{m} \left( \prod_{j=i+1}^{k} \theta_j \right) \mu_i^2 + \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} \theta_j \right) \mu_i^2$$

$$\leq \sum_{i=0}^{m} (1 + \mu_0 \sigma_f)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \mu_i^2 + \mu_0 \sigma_f^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \sum_{i=0}^{m} \frac{1}{\gamma^2}$$

$$= (1 + \mu_0 \sigma_f)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \mu_0 \sum_{i=0}^{m} \frac{1}{i^2}$$

$$+ \mu_0 \sigma_f^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} (1 - \mu_j \sigma_f) \right) (1 - (1 - \sigma f \mu_i))$$

$$= \left( 1 + \mu_0 \sigma_f \right)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \frac{\mu_0}{\gamma^2} \sum_{i=0}^{m} \frac{1}{i^2}$$

$$+ \mu_0 \sigma_f \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} (1 - \mu_j \sigma_f) \right) (1 - (1 - \sigma f \mu_i))$$

$$\leq \left( 1 + \mu_0 \sigma_f \right)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \frac{\mu_0}{\gamma^2} \sum_{i=0}^{m} \frac{1}{i^2}$$

$$+ \mu_0 \sigma_f \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} (1 - \mu_j \sigma_f) \right) (1 - (1 - \sigma f \mu_i))$$

$$\leq \left( 1 + \mu_0 \sigma_f \right)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \frac{\mu_0}{\gamma^2} \sum_{i=0}^{m} \frac{1}{i^2} + \mu_0 \sigma_f \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} (1 - \mu_j \sigma_f) \right) (1 - (1 - \sigma f \mu_i))$$

$$\leq \left( 1 + \mu_0 \sigma_f \right)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \frac{\mu_0}{\gamma^2} \sum_{i=0}^{m} \frac{1}{i^2} + \mu_0 \sigma_f \sum_{i=m+1}^{k} \left( \prod_{j=i+1}^{k} (1 - \mu_j \sigma_f) \right) (1 - (1 - \sigma f \mu_i))$$

$$\leq \left( 1 + \mu_0 \sigma_f \right)^{\varphi_{1-\gamma}(m) - \varphi_{1-\gamma}(k)} \varphi_{1-\gamma}(m) + \frac{\mu_0 \sigma_f}{\gamma^2}.$$. 
By denoting the second constant $\tilde{\theta}_0 = \frac{1}{\gamma \mu_0 \sigma_f}$, then the last relation implies the following bound:

$$E \left[\|x^{k+1} - x^*\|^2 \right] \leq \tilde{\theta}_0^{\gamma - 1}(k) \|x^0 - x^*\|^2 + \tilde{\theta}_0^{\gamma - 1}(k) \varphi_{1-\gamma}(m) \varphi_{1-2\gamma}(m) \Sigma + \frac{\mu_{m+1}}{\sigma_f} \Sigma.$$

Denote $r_k^2 = E[\|x^k - x^*\|^2]$. To derive an explicit convergence rate order we analyze upper bounds on function $\phi$.

(i) First assume that $\gamma \in (0, \frac{1}{2})$. This implies that $1 - 2\gamma > 0$ and that:

$$\varphi_{1-2\gamma} \left( \frac{k}{2} \right) \leq \varphi_{1-2\gamma} \left( \frac{k}{2} \right) = \left( \frac{k}{2} \right)^{1-2\gamma} \left( 1 - \frac{1}{1 - 2\gamma} \right) \leq \left( \frac{k}{2} \right)^{1-2\gamma} \left( 1 - \frac{1}{1 - 2\gamma} \right). \quad (9)$$

On the other hand, by using the inequality $e^{-x} \leq \frac{1}{1+x}$ for all $x \geq 0$, we obtain:

$$\tilde{\theta}_0^{\varphi_{1-\gamma}(k) - \varphi_{1-\gamma}(\frac{k}{2})} \varphi_{1-2\gamma} \left( \frac{k}{2} \right) = e^{(\varphi_{1-\gamma}(k) - \varphi_{1-\gamma}(\frac{k}{2})) \ln \tilde{\theta}_0^{\varphi_{1-2\gamma}} \left( \frac{k}{2} \right)}$$

$$\leq \frac{\varphi_{1-2\gamma} \left( \frac{k}{2} \right)}{1 + [\varphi_{1-\gamma}(k) - \varphi_{1-\gamma}(\frac{k}{2})] \ln \frac{1}{\tilde{\theta}_0}} \leq \frac{\varphi_{1-2\gamma} \left( \frac{k}{2} \right)}{1 + [\varphi_{1-\gamma}(k) - \varphi_{1-\gamma}(\frac{k}{2})] \ln \frac{1}{\tilde{\theta}_0}} \leq \frac{2^{1-\gamma} \varphi_{1-2\gamma} \left( \frac{k}{2} \right)}{1 - (\frac{1}{2})^{1-\gamma} \ln \frac{1}{\tilde{\theta}_0}} = O \left( \frac{1}{k^{1/2}} \right).$$

Therefore, in this case, the overall rate will be given by:

$$r_k^2 \leq \tilde{\theta}_0^{O(k^{-1})} r_0^2 + O \left( \frac{1}{k^{1/2}} \right) \approx O \left( \frac{1}{k^{1/2}} \right).$$

If $\gamma = \frac{1}{2}$, then the definition of $\varphi_{1-2\gamma}(\frac{k}{2})$ provides that:

$$r_k^2 \leq \tilde{\theta}_0^{O(\sqrt{k})} r_0^2 + \tilde{\theta}_0^{O(\sqrt{k})} O(\ln k) + O \left( \frac{1}{\sqrt{k}} \right) \approx O \left( \frac{1}{\sqrt{k}} \right).$$

When $\gamma \in (\frac{1}{2}, 1)$, it is obvious that $\varphi_{1-2\gamma} \left( \frac{k}{2} \right) \leq \frac{1}{2^{1-\gamma}}$ and therefore the order of the convergence rate changes into:

$$r_k^2 \leq \tilde{\theta}_0^{O(k^{-1})} [r_0^2 + O(1)] + O \left( \frac{1}{k^{1/2}} \right) \approx O \left( \frac{1}{k^{1/2}} \right).$$

(ii) Lastly, if $\gamma = 1$, by using $\tilde{\theta}_0^{\ln (k+1)} \leq \left( \frac{1}{k} \right)^{\ln \frac{1}{\tilde{\theta}_0}}$ we obtain the second part of our result.