LOCAL AUTOMORPHISMS OF SOME QUANTUM MECHANICAL STRUCTURES

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Abstract. Let $H$ be a separable infinite dimensional complex Hilbert space. We prove that every continuous 2-local automorphism of the poset (that is, partially ordered set) of all idempotents on $H$ is an automorphism. Similar results concerning the orthomodular poset of all projections and the Jordan ring of all selfadjoint operators on $H$ without the assumption on continuity are also presented.

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The orthomodular lattice or quantum logic of projections on a Hilbert space plays fundamental role in the mathematical foundations of quantum mechanics. The interest in the poset of all skew projections (that is, idempotents) has been also aroused (see [15] and the references therein) since it can be defined on an arbitrary Banach space (or, more generally, topological vector space).

In relation to any algebraic structure, the importance of the study of automorphisms needs no justification. In a series of papers (see [12, 14] and their references), motivated by a problem of Larson [10, Some concluding remarks (5), p. 298] (also see [11]) we investigated the surprising and sometimes probably unexpected phenomenon when the local automorphisms of a given Banach algebra are all automorphisms. If this is the case, then one can say that the local actions of the automorphism group determines that group completely. In our papers we showed several Banach algebras (in fact, they were mainly $C^*$-algebras) which possess this property. The aim of this present paper is to study a similar problem for posets of idempotents and for the Jordan ring of selfadjoint operators.

Let us fix the notation and terminology that we shall use throughout. Let $H$ be a Hilbert space. Let $B(H)$ stand for the set of all bounded linear operators on $H$. The set of all skew projections (that is, idempotents) in $B(H)$ is denoted by $SP(H)$, and $P(H)$ stands for the set of all projections. For any $P,Q \in SP(H)$ we write $P \leq Q$ if $PQ = QP = P$ and we say that $P,Q$ are orthogonal if $PQ = QP = 0$. In what follows $SP(H)$ is regarded as a poset with the relation $\leq$ and $P(H)$ is viewed as an orthomodular poset with the additional map $P \mapsto I - P$ of orthocomplementation [6, Chapter 2].

The transformation $\phi : A \to A$ of the algebraic structure $A$ is called a local automorphism if for every $x \in A$, there is an automorphism $\phi_x$ of $A$ for which $\phi(x) = \phi_x(x)$. The map $\phi$ is called a 2-local automorphism if for every $x,y \in A$, there is an automorphism $\phi_{x,y}$ of $A$ for which $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$. In our mentioned papers we considered linear maps on various algebras which were local automorphisms and showed that they are in fact automorphisms. The main result of [1] says that every linear map which is a local automorphism of the algebra $B(H)$ on a separable infinite dimensional Hilbert space $H$ is an automorphism (see [12] for a stronger result). It turned out in [16] that the assumption of linearity in the result of [1] can be dropped provided we pay the price that we consider 2-local automorphisms instead of (1-)local automorphisms. More precisely, the main result of [16] tells us that every 2-local automorphism of $B(H)$ (no linearity is assumed) is an automorphism. For some results concerning 2-local automorphisms of CSL algebras see [1].

It was proved in [15] that in the case of an infinite dimensional separable Hilbert space $H$, the automorphisms of $SP(H)$ with respect to the partial order $\leq$ are exactly the maps
\[ P \mapsto TPT^{-1}, \quad P \mapsto TP^*T^{-1} \]
where $T$ is an invertible bounded linear or conjugate-linear operator of $H$.

The main result of the paper is the following theorem.

**Theorem.** Let $H$ be an infinite dimensional separable complex Hilbert space. Every continuous 2-local automorphism of the poset $SP(H)$ is an automorphism.

It is easy to see that the corresponding assertion for (1-)local automorphisms fails to be true. In fact, the local automorphisms of $SP(H)$ can be almost arbitrary.

In the proof of our result we shall use the following easy lemma.

**Lemma.** Let $A, B \in B(H)$. Suppose that for every $x \in H$ we have either $\langle Ax, x \rangle = 0$ or $\langle Bx, x \rangle = 0$. Then either $A = 0$ or $B = 0$.

**Proof.** The sets $\{ x \in H : \langle Ax, x \rangle = 0 \}$ and $\{ x \in H : \langle Bx, x \rangle = 0 \}$ are closed and their union is $H$. By Baire’s category theorem we obtain that one of these sets has nonempty interior. So, we can suppose that there exist $x_0 \in H$ and $\epsilon_0 > 0$ such that

$$\langle A(x_0 + \epsilon x), x_0 + \epsilon x \rangle = 0$$

for every unit vector $x \in H$ and $0 \leq \epsilon < \epsilon_0$. This gives us that

$$\langle Ax_0, \epsilon x \rangle + \langle A(\epsilon x), x_0 \rangle + \langle A(\epsilon x), \epsilon x \rangle = 0,$$

that is,

$$\epsilon \langle Ax_0, x \rangle + \epsilon \langle Ax, x_0 \rangle + \epsilon^2 \langle Ax, x \rangle = 0.$$

By the arbitrariness of $\epsilon$ we obtain that $\langle Ax, x \rangle = 0$ for every unit vector $x \in H$. This results in $A = 0$. \qed

**Proof of Theorem.** Let $\phi : SP(H) \to SP(H)$ be a continuous 2-local automorphism. Clearly, $\phi$ preserves the partial order $\leq$ and the orthogonality between the elements of $SP(H)$. Furthermore, $\phi$ preserves the rank of the finite rank idempotents.

In what follows we describe the form of $\phi$ on the set of all finite rank elements of $SP(H)$. We first show that $\phi$ is finitely orthoadditive on this set in question. To see this, let $P, Q \in SP(H)$ be orthogonal and of finite rank. By the monotony of $\phi$ we have $\phi(P), \phi(Q) \leq \phi(P + Q)$. Since $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$, it follows that $\phi(P) + \phi(Q) \leq \phi(P + Q)$. By the rank preserving property of $\phi$ we obtain that

$$\text{rank}(\phi(P) + \phi(Q)) = \text{rank}(\phi(P)) + \text{rank}(\phi(Q)) = \text{rank}(P) + \text{rank}(Q) = \text{rank}(P + Q) = \text{rank}(\phi(P + Q))$$

which yields that

$$\phi(P) + \phi(Q) = \phi(P + Q). \quad (1)$$

We now extend $\phi$ from the set of all finite rank projections to a Jordan homomorphism of the ideal $F(H)$ of all finite rank operators in $B(H)$. 
The additive map \( J : \mathcal{A} \to \mathcal{B} \) between the rings \( \mathcal{A}, \mathcal{B} \) is called a Jordan homomorphism if

\[
J(xy + yx) = J(x)J(y) + J(y)J(x) \quad (x, y \in \mathcal{A}).
\]

In \( \mathcal{A}, \mathcal{B} \) are algebras, then besides additivity, the Jordan homomorphisms are also supposed to be linear. Clearly, every homomorphism and every antihomomorphism are Jordan homomorphisms.

Let \( H_d \) denote an arbitrary \( d \)-dimensional (\( d \) is finite) subspace of \( H \). Consider the natural embedding \( B(H_d) \hookrightarrow B(H) \) and for any \( h \in H \) let \( \phi_h \) be defined by \( \phi_h(P) = \langle \phi(P)h, h \rangle \) for every projection \( P \) on \( H_d \). Since \( \phi \) is continuous, thus \( \phi_h \) is a bounded orthoadditive function. If \( d \geq 3 \), then by Gleason’s theorem [3, Theorem 3.2.16] there exists an operator \( T_h \) on \( H_d \) such that

\[
\phi_h(P) = \text{tr}T_hP \quad (P \in P(H_d)).
\]

Let \( P_1, \ldots, P_n \in P(H) \) be finite rank projections (their pairwise orthogonality is not assumed) and let \( \lambda_1, \ldots, \lambda_n \) be complex numbers. Define

\[
\psi(\sum_k \lambda_k P_k) = \sum_k \lambda_k \phi(P_k).
\]

We have to check that \( \psi \) is well-defined. To see this, let \( P_1', \ldots, P_n' \in P(H) \) be finite rank projections and \( \mu_1, \ldots, \mu_n \in \mathbb{C} \) be such that

\[
\sum_k \mu_k P_k = \sum_k \mu_k P_k'.
\]

Let \( H_d \) be a finite dimensional subspace of \( H \) of dimension \( d \geq 3 \) such that for the orthogonal projection \( P_{H_d} \) onto \( H_d \) we have \( P_k, P_k' \leq P_{H_d} (k = 1, \ldots, n) \). Let \( T_h \) denote the linear operator on \( H_d \) corresponding to \( \phi_h \) (see [3]). We compute

\[
\text{tr}T_h(\sum_k \mu_k P_k') = \sum_k \mu_k \phi(P_k') = \langle \sum_k \mu_k \phi(P_k')h, h \rangle.
\]

Since this holds true for every \( h \in H \), we obtain that \( \psi \) is well-defined. As \( F(H) \) is the linear span of its projections, the definition [4] clearly implies that \( \psi \) is a linear transformation on \( F(H) \). Since \( \psi \) sends projections to idempotents, it is now a standard argument to verify that \( \psi \) is a Jordan homomorphism of \( F(H) \). See, for example, the proof of [2, Theorem 2].

As \( F(H) \) is a locally matrix ring, it follows from a classical result of Jacobson and Rickart [3, Theorem 8] that \( \psi \) can be written as \( \psi = \psi_1 + \psi_2 \), where \( \psi_1 \) is a homomorphism and \( \psi_2 \) is an antihomomorphism. Let \( P \in P(H) \) be rank-one. Because \( \psi(P) = \phi(P) \) is also rank-one, we obtain that one of the idempotents \( \psi_1(P), \psi_2(P) \) is zero. Since \( F(H) \) is a simple ring, it is easy to see that this implies that either \( \psi_1 \) or \( \psi_2 \) is identically zero, that
is, ψ is either a homomorphism or an antihomomorphism of $F(H)$. In what follows we can assume without loss of generality that ψ is a homomorphism.

We show that ψ preserves the rank. Let $A \in F(H)$ be a rank-$n$ operator. Then there is a rank-$n$ projection $P$ such that $PA = A$. The rank of $\phi(P)$ is also $n$. We have $\psi(A) = \psi(P)\psi(A) = \phi(P)\psi(A)$ which proves that $\psi(A)$ is of rank at most $n$. If $Q$ is any rank-$n$ projection, then there are finite rank operators $U,V$ such that $Q = UAV$. Since $\phi(Q) = \psi(Q) = \psi(U)\psi(A)\psi(V)$ and the rank of $\phi(Q)$ is $n$, it follows that the rank of $\psi(A)$ is at least $n$. Therefore, ψ is rank preserving. We now refer to Hou’s work \cite{8} on the form of linear rank preservers on operator algebras. It follows from the argument leading to \cite{8}, Theorem 1.2 that there are linear operators $T, S$ on $H$ such that $\psi$ is of the form

\begin{equation}
\psi(x \otimes y) = (Tx) \otimes (Sy) \quad (x, y \in H)
\end{equation}

(recall that we have assumed that $\psi$ is a homomorphism). Here, for any $u, v \in H$, $u \otimes v$ stands for the operator defined by $(u \otimes v)(z) = \langle z, v \rangle u$ $(z \in H)$. We claim that $T, S$ are bounded. This follows from \cite{8}, Lemma 1 which states that if $T, S$ are linear operators on $H$ with the property that the map $x \mapsto (Tx) \otimes (Sx)$ is continuous on the unit ball of $H$, then $T, S$ are bounded. We infer from (5) that $(Tx, Sx) = (x, x)$ for every unit vector $x \in H$ ($\phi$ sends rank-one projections to idempotents). By polarization this implies that $(Tx, Sx) = (x, y)$ $(x, y \in H)$. Consequently, we have $S^*T = I$. From (5) we deduce that $\phi(P) = TPS^*$ for every rank-one projection $P$. By the finite orthoadditivity of $\phi$ appearing in \cite{B}, it follows that $\phi(P) = TPS^*$ holds true for every finite-rank projection $P$ as well.

Since $S^*T = I$, it follows that $Q = TS^*$ is an idempotent. We can write $\phi(P) = Q\phi(P)Q + Q\phi(P)(I - Q) + (I - Q)\phi(P)Q + (I - Q)\phi(P)(I - Q)$ for every $P \in SP(H)$. We claim that the two middle terms on the right hand side of the equality above are in fact missing. Denote

$\phi_{11}(P) = Q\phi(P)Q, \quad \phi_{12}(P) = Q\phi(P)(I - Q),$

$\phi_{21}(P) = (I - Q)\phi(P)Q, \quad \phi_{22}(P) = (I - Q)\phi(P)(I - Q).$

Let $P \in P(H)$ be fixed and let $P'$ be an arbitrary finite rank projection with $P' \leq P$. We know that $\phi(P') \leq \phi(P)$. Since $\phi(P') = TP'S^*$, we obtain that $\phi(P')Q = Q\phi(P') = \phi(P')$. Hence,

$\phi(P')\phi_{11}(P) = (\phi(P')Q)\phi(P)Q = \phi(P')\phi(P)Q = \phi(P')Q = \phi(P').$

Therefore, $TP'S^*\phi_{11}(P) = TP'S^*$. Similarly, we have $\phi_{11}(P)TP'S^* = TP'S^*$. By the arbitrariness of $P'$ it follows that

$\phi_{11}(P)TP'S^* = TPS^*\phi_{11}(P) = TPS^*.$

This means that

\begin{equation}
TPS^* \leq \phi_{11}(P).
\end{equation}
The local property of \( \phi \) implies that \( \phi(P) + \phi(I-P) = I \). Therefore, \( \phi(I-P) = I - \phi(P) \). Writing \( I-P \) for \( P \) in (6), we have

\[
Q - TPS^* = T(I-P)S^* \leq \phi_{11}(I-P) = Q - \phi_{11}(P).
\]

We deduce that \( TPS^* = \phi_{11}(P) \). We next compute

\[
TP'S^*\phi_{12}(P) = \phi(P')Q\phi(P)(I-Q) = \phi(P')\phi(P)(I-Q) = \phi(P')(I-Q) = \phi(P')Q(I-Q) = 0.
\]

Since this holds for every finite rank projection \( P' \) for which \( P' \leq P \), we infer that \( TPS^*\phi_{12}(P) = 0 \). Since \( \phi(I-P) = I - \phi(P) \), we have \( \phi_{12}(I-P) = -\phi_{12}(P) \). Therefore,

\[
0 = T(I-P)S^*\phi_{12}(I-P) = TS^*(-\phi_{12}(P)) - TPS^*(-\phi_{12}(P)) = -Q\phi_{12}(P) + TPS^*\phi_{12}(P) = -\phi_{12}(P) + TPS^*\phi_{12}(P) = -\phi_{12}(P)
\]

and, hence, we obtain \( \phi_{12} = 0 \). Similarly, one can verify that \( \phi_{21} = 0 \). Consequently, our map \( \phi \) is of the form

\[
(7) \quad \phi(P) = TPS^* + \phi_{22}(P) \quad (P \in P(H)).
\]

We know that \( \phi_{22}(P) = 0 \) for every finite rank projection \( P \) (recall that \( \phi(P) = TPS^* \) for all such \( P \)). We claim that \( \phi_{22}(P) = 0 \) holds for every \( P \in SP(H) \) as well.

Assume for a moment that \( \phi_{22}(P) \neq 0 \) for every projection \( P \) of infinite rank and infinite corank. We can choose uncountably many projections \( P_\alpha \) of infinite rank and infinite corank such that \( P_\alpha P_\beta \) is a finite rank projection for every \( \alpha \neq \beta \) (see, for example, the proof of [12, Theorem 1]). Using the local form of \( \phi \) we see that the rank of the idempotent \( \phi(P_\alpha)\phi(P_\beta) \) is equal to the rank of \( P_\alpha P_\beta \). On the other hand, referring to the injectivity of \( T \) and to the surjectivity of \( S^* \) (these follow from \( S^*T = I \)), we find that the range of \( TP_\alpha S^*TP_\beta S^* = TP_\alpha P_\beta S^* \) is the same as that of \( P_\alpha P_\beta \). By (6) this gives us that the rank of \( \phi_{22}(P_\alpha)\phi_{22}(P_\beta) \) is 0, that is, we have

\[
\phi_{22}(P_\alpha)\phi_{22}(P_\beta) = 0
\]

for every \( \alpha \neq \beta \). This means that the range of \( \phi_{22} \) contains uncountably many nonzero pairwise orthogonal idempotents which plainly contradicts the separability of \( H \). Therefore, \( \phi_{22}(P) = 0 \) holds for a projection \( P \) of infinite rank and infinite corank. The projections \( P \) and \( I-P \) can be connected by a continuous curve inside the set of projections (this is an easy consequence of the arcwise connectedness of the unitary group of \( B(H) \)). If \( R, R' \in P(H) \) are lying on the same arc in \( P(H) \), then by the continuity of \( \phi \), the idempotents \( \phi_{22}(R), \phi_{22}(R') \) are close enough to each other if \( ||R - R'|| \) is sufficiently small. Taking into account that the norm of a nonzero idempotent is not less than 1, this, together with \( \phi_{22}(P) = 0 \), yields that \( \phi_{22}(I-P) = 0 \). Hence, we have \( I-Q = \phi_{22}(I) = \phi_{22}(P) + \phi_{22}(I-P) = 0 \). 

Therefore,
and thus $\phi_{22} = 0$. Since $TS^* = Q = I = S^*T$, we obtain $S^* = T^{-1}$. Therefore, $\phi$ is of the form

$$\phi(P) = TPT^{-1} \quad (P \in P(H))$$

where $T$ is an invertible bounded linear operator on $H$. We show that our map $\phi$ is of this form on the whole set $SP(H)$. To verify this, we can obviously suppose that $T = I$. So, assume that $\phi$ is the identity on the set of all projections.

Let $P$ be any idempotent. Pick an arbitrary unit vector $x \in H$ and consider the operator $\phi(x \otimes x)(\phi(P))(x \otimes x)$. Taking into account the local property of $\phi$, the form of the automorphisms of $SP(H)$ and that $\phi$ is the identity on the set of all projections, we have either

$$\langle \phi(P)x, x \rangle x \otimes x = \phi(x \otimes x)\phi(P)\phi(x \otimes x) =$$

$$A \cdot x \otimes x \cdot A^{-1}APA^{-1}A \cdot x \otimes x \cdot A^{-1} = A \cdot ((Px, x) x \otimes x) \cdot A^{-1} =$$

$$\langle Px, x \rangle \phi(x \otimes x) = \langle Px, x \rangle x \otimes x,$$

or, using a similar computation,

$$\langle \phi(P)x, x \rangle x \otimes x = \overline{\langle Px, x \rangle x \otimes x},$$

or

$$\langle \phi(P)x, x \rangle x \otimes x = \langle P^*x, x \rangle x \otimes x,$$

or

$$\langle \phi(P)x, x \rangle x \otimes x = \overline{\langle P^*x, x \rangle x \otimes x}.$$  

This gives us that for every $x \in H$ we have either $\langle \phi(P)x, x \rangle = \langle Px, x \rangle$ or $\langle \phi(P)x, x \rangle = \langle P^*x, x \rangle$. Our lemma implies that for any $P \in SP(H)$ we have either $\phi(P) = P$ or $\phi(P) = P^*$. We assert that this results in either $\phi(P) = P$ for all $P \in SP(H)$ or $\phi(P) = P^*$ for all $P \in SP(H)$. Indeed, let $P \in SP(H)$ be a non-selfadjoint finite rank idempotent for which $\phi(P) = P$. Consider any non-selfadjoint finite rank idempotent $P'$ which is orthogonal to $P$. If $\phi(P') = P'^*$, then by $\phi(P) + \phi(P') = \phi(P + P')$ we would arrive at a contradiction. So, $\phi(P') = P'$ for every finite rank idempotent which is orthogonal to $P$. This implies that $\phi(R) = R$ for every finite rank idempotent with $P \leq R$. Let $P'$ be any non-selfadjoint finite rank idempotent. Suppose that $\phi(P') = P'^*$. If $R$ is any non-selfadjoint finite rank idempotent for which $P' \leq R$, then we have similarly as before that $\phi(R) = R^*$. Now, if $P, P' \leq R$, then we obtain on the one hand that $\phi(R) = R$ and on the other hand that $\phi(R) = R^*$. But this is a contradiction. Therefore, we have $\phi(P') = P'$ for every finite rank idempotent $P'$. By the monotony of $\phi$ it now follows that $P \leq \phi(P)$ for every $P \in SP(H)$. Putting $I - P$ in the place of $P$ we finally obtain $\phi(P) = P$ ($P \in SP(H)$).

In case $\psi$ is an antihomomorphism, we can follow a similar argument. The proof is complete. \qed
Examining the proof of our theorem, we can reach a result of the same spirit concerning the orthomodular poset \( P(H) \). First we need the form of the automorphisms of \( P(H) \) as an orthomodular poset (as it was mentioned in [13, Remark 4.5.], an automorphism of \( P(H) \) as a set is not necessarily an automorphism of it as an orthomodular poset). So, let \( \phi : P(H) \rightarrow P(H) \) be a bijection which preserves the partial order \( \leq \) as well as the orthocomplementation \( P \mapsto I - P \) in both directions. We easily get that \( \phi \) preserves the orthogonality between the elements of \( P(H) \) and then that \( \phi \) is orthoadditive. Using, for example, the result of Bunce and Wright [3] which solves the Mackey-Gleason problem (also see [3, 5]), we see that \( \phi \) can be extended to a bounded linear transformation of \( B(H) \). Since this map sends projections to projections, one can verify that it is a Jordan *-homomorphism of \( B(H) \). Since its range contains a rank-one operator and an operator with dense range (in fact, the range contains every projection), it is a trivial consequence of [12, Theorem 1] that this transformation is a Jordan *-automorphism of \( B(H) \) onto itself. As \( B(H) \) is a prime algebra (that is, \( AB(H)B = \{0\} \) implies that either \( A = 0 \) or \( B = 0 \)), it follows from a classical theorem of Herstein [7] that our map is either a *-automorphism or a *-anti-automorphism. The structure of those maps is well-known. In fact, they are of the form

\[
A \mapsto UAU^*, \quad A \mapsto VA^*V^*
\]

where \( U \) is unitary and \( V \) is antiunitary on \( H \). So, \( \phi \) is either of the form

\[
\phi(P) = UPU^* \quad (P \in P(H))
\]

or of the form

\[
\phi(P) = VPV^* \quad (P \in P(H))
\]

where \( U \) is unitary and \( V \) is antiunitary on \( H \). Now, we can prove the following result.

**Proposition.** Every 2-local automorphism of the orthomodular poset \( P(H) \) is an automorphism.

**Proof.** One can just follow the proof of our theorem. The only thing which deserves checking is that here we do not need the continuity of \( \phi \). To see this, let us go through those parts of the previous proof where we have used continuity. The first such place was where we applied Gleason’s theorem. But as \( \phi \) sends projections to projections, it follows that \( \phi \) is bounded, so we do not need continuity here. We next used the continuity when showing that the operators \( T, S \) in [3] are continuous. Since in our present case \( \psi \) sends projections to projections, it follows that for every unit vector \( x \in H \), the operator \( Tx \otimes Sx \) is a projection. This implies that \( Tx = Sx \) and \( \|Tx\| = \|Sx\| = 1 \). So, \( T = S \) is an isometry. The third appearance of the continuity of \( \phi \) was where we proved that if \( \phi_{22}(P) = 0 \) for a projection \( P \) of infinite rank and infinite corank, then \( \phi_{22} = 0 \). In our present case the terms in the decomposition \( \phi(R) = \phi_{11}(R) + \phi_{22}(R) \ (R \in P(H)) \) are projections.
Let \( P' \in P(H) \) be such that \( \| P - P' \| < 1 \). By the 2-local property of \( \phi \) we see that
\[
\| \phi(P) - \phi(P') \| = \| P - P' \| < 1.
\]
Since
\[
\| \phi_{22}(P') \| = \| \phi_{22}(P) - \phi_{22}(P') \| \leq \| \phi(P) - \phi(P') \| < 1,
\]
we deduce that \( \phi_{22}(P') = 0 \). As we can go from \( P \) to \( I - P \) in finitely many steps
\( P = P_0, P_1, \ldots, P_n = I - P \) such that \( \| P_{k-1} - P_k \| < 1 \), we obtain that \( \phi_{22}(I - P) = 0 \). This gives us that \( \phi_{22}(I) = 0 \) which implies \( \phi_{22} = 0 \). The proof is complete. \( \square \)

Beside the various structures of projections and idempotents, the Jordan ring \( B(H)_h \) of all selfadjoint operators on \( H \) is also well-known to be of great importance in the mathematical description of quantum mechanics. Our final result shows that the 2-local automorphisms of \( B(H)_h \) are necessarily automorphisms. To prove this, we first describe the automorphisms in question. So, let \( \phi : B(H)_h \rightarrow B(H)_h \) be a Jordan automorphism (this means that \( \phi \) is an additive bijection satisfying (2)). We have
\[
\phi(ABA) = \phi(A)\phi(B)\phi(A) \quad (A, B \in B(H)_h)
\]
(see, for example, [7, Lemma 2]). One can readily verify that \( \phi \) preserves the partial order and the orthogonality between the projections. Therefore, we obtain that \( \phi \) sends rank-one projections to rank-one projections, if \( \{P_n\}_n \) is a maximal system of pairwise orthogonal rank-one projections then the same holds for \( \{\phi(P_n)\}_n \), and \( \phi(I) = I \). Let \( P \) be any rank-one projection. Since \( \phi(P) \) is also rank-one, if \( \lambda \in \mathbb{R} \), then from the equality
\[
\phi(\lambda P) = \phi(P(\lambda P)P) = \phi(P)\phi(\lambda P)\phi(P)
\]
it follows that \( \phi(\lambda P) = f(\lambda)\phi(P) \) for some real number \( f(\lambda) \). It is easy to verify that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a ring-homomorphism with \( f(1) = 1 \). It is well-known and, in fact, it requires only elementary analysis to prove that this implies that \( f(\lambda) = \lambda \ (\lambda \in \mathbb{R}) \). So, we have \( \phi(\lambda P) = \lambda \phi(P) \). Since
\[
2\lambda \phi(P) = \phi(2\lambda P) = \phi(\lambda I)\phi(P) + \phi(P)\phi(\lambda I)
\]
for every rank-one projection, choosing a maximal orthogonal family of such projections, we obtain that
\[
2\lambda I = \phi(\lambda I)I + I\phi(\lambda I).
\]
That is, we have
\[
\phi(\lambda I) = \lambda I \quad (\lambda \in \mathbb{R}).
\]
It is now apparent that \( \phi \) is real-linear. Let us define \( \tilde{\phi} : B(H) \rightarrow B(H) \) by
\[
\tilde{\phi}(A + iB) = \phi(A) + i\phi(B) \quad (A, B \in B(H)_h).
\]
It is easy to check that \( \tilde{\phi} \) is a (linear) Jordan \(*\)-automorphism of \( B(H) \). Hence, just as in the discussion right before the formulation of our proposition, we find that \( \phi \) is of the form
\[
\phi(A) = UAU^* \quad (A \in B(H)_h)
\]
where \( U \) is an either unitary or antiunitary operator on \( H \).
Now, we are in a position to prove our final result which follows.

**Corollary.** Every 2-local automorphism of the Jordan ring $B(H)_h$ is an automorphism.

**Proof.** Let $\phi : B(H)_h \to B(H)_h$ be a 2-local automorphism. Clearly, $\phi|_{P(H)}$ is a 2-local automorphism of $P(H)$. By Proposition we obtain that there exists a unitary or antiunitary operator $U$ on $H$ such that

$$\phi(P) = UPU^* \quad (P \in P(H)).$$

We can assume without loss of generality that $\phi(P) = P$ for every $P \in P(H)$. Now, similarly as in the proof of our theorem, picking any $A \in B(H)_h$ and unit vector $x \in H$, considering the operator $\phi(x \otimes x)\phi(A)\phi(x \otimes x)$, we find that

$$\langle \phi(A)x, x \rangle x \otimes x = \langle Ax, x \rangle x \otimes x$$

which implies that

$$\langle \phi(A)x, x \rangle = \langle Ax, x \rangle.$$

Therefore, we have $\phi(A) = A$ ($A \in B(H)_h$) and this completes the proof. \qed

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