COMPARING POWERS AND SYMBOLIC POWERS OF IDEALS

CRISTIANO BOCCI & BRIAN HARBOURNE

Abstract. We develop tools to study the problem of containment of symbolic powers \( I^{(m)} \) in powers \( I^r \) for a homogeneous ideal \( I \) in a polynomial ring \( k[\mathbb{P}^N] \) in \( N + 1 \) variables over an arbitrary algebraically closed field \( k \). We obtain results on the structure of the set of pairs \((r, m)\) such that \( I^{(m)} \subseteq I^r \). As corollaries, we show that \( I^2 \) contains \( I^{(3)} \) whenever \( S \) is a finite generic set of points in \( \mathbb{P}^2 \) (thereby giving a partial answer to a question of Huneke), and we show that the containment theorems of [ELS] and [HH1] are optimal for every fixed dimension and codimension.

1. Introduction

Consider a homogeneous ideal \( I \) in a polynomial ring \( k[\mathbb{P}^N] \). Taking powers of \( I \) is a natural algebraic construction, but it can be difficult to understand their structure geometrically (for example, knowing generators of \( I^r \) does not make it easy to know its primary decomposition). On the other hand, symbolic powers of \( I \) are more natural geometrically than algebraically. For example, if \( I \) is a radical ideal defining a finite set of points \( p_1, \ldots, p_s \in \mathbb{P}^N \), then its \( m \)th symbolic power \( I^{(m)} \) is generated by all forms vanishing to order at least \( m \) at each point \( p_i \), but it is not easy to write down specific generators for \( I^{(m)} \), even if one has generators for \( I \).

Thus it is of interest to compare the two constructions, and a good deal of work has been done recently comparing powers of ideals with symbolic powers in various ways. See for example, [Ho], [S], [K], [ELS], [HH1], [CHHT] and [LS]. Here we ask when a power of \( I \) contains a symbolic power, or vice versa. The second question has an easy answer: if \( I \) is nontrivial (i.e., not \((0)\) or \((1)\)), then \( I^r \subseteq I^{(m)} \) if and only if \( m \leq r \) [PSC, Lemma 8.1.4]. Thus here we focus on the first question, and for that question all that it is easy to say is that if \( I \) is nontrivial and \( I^{(m)} \subseteq I^r \), then \( m \geq r \) (Lemma 2.3.3(a)). The problem of precisely for which \( m \geq r \) we have \( I^{(m)} \subseteq I^r \) is largely open.

As a stepping stone, we introduce an asymptotic quantity which we refer to as the resurgence, namely \( \rho(I) = \sup \{m/r : I^{(m)} \not\subseteq I^r \} \). In particular, if \( m > \rho(I)r \), then one is guaranteed that \( I^{(m)} \not\subseteq I^r \). Until recently it would not have been clear that the sup always exists, but results of [S] imply, for radical ideals at least, that it does, and [HH1], generalizing the result of [ELS], shows in fact that \( \rho(I) \leq N \)

\begin{flushleft}
\textbf{Date:} June 19, 2009.
\textbf{1991 Mathematics Subject Classification.} Primary: 14C20; Secondary: 13F20, 14N05, 14H20, 41A05.
\textbf{Key words and phrases.} fat points, Seshadri constants, symbolic powers.
\end{flushleft}

This research was partially supported by GNSAGA of INdAM (Italy) and by the NSA. We thank L. Avramov, M. Chardin, L. Chiantini, L. Ein, C. Huneke, S. Iyengar, D. Katz, J. Migliore, T. Marley and Z. Teitler for helpful comments.
and hence for a nontrivial homogeneous ideal $I$ we have $1 \leq \rho(I) \leq N$ (see Lemma 2.3.2 b)).

There are still, however, very few cases for which the actual value of $\rho(I)$ is known, and they are almost all cases for which $\rho(I) = 1$. For example, by Macaulay’s unmixedness theorem it follows that $\rho(I) = 1$ when $I$ is a complete intersection (also see [H0] and [LS]). And if $I$ is a monomial ideal, it is sometimes possible to compute $\rho(I)$ directly; for example, if $I$ defines three noncollinear points in $\mathbf{P}^2$, then one can show $\rho(I) = 4/3$ (see [BH]).

In this paper we give the first results regarding the structure of the set of pairs $(r, m)$ for which $I^{(m)} \subseteq I^r$. These results are in terms of numerical invariants of $I$. In particular, let $\alpha(I)$ be the least degree of a generator in any set of homogeneous generators of $I$, let $\omega(I)$ be the least degree $t$ such that $I$ is generated by forms of degree $t$ and less, and let $\operatorname{reg}(I)$ be the regularity of $I$. We also define an invariant $\gamma(I)$, which is like a Seshadri constant. We then obtain the following structural results. If $m/r \leq \alpha(I)/\gamma(I)$, we prove that $I^{(mt)} \subsetneq I^r$ for all $t > 0$ (Lemma 2.3.2). If in addition $I$ defines a zero dimensional subscheme, then we show $m/r \geq \operatorname{reg}(I)/\gamma(I)$ implies that $I^{(m_t)} \subseteq I^r$ (Corollary 2.3.4), and we show that $m/r > \omega(I)/\gamma(I)$ implies that $I^{(mt)} \subseteq I^r$ for all $t > 0$ (Corollary 2.3.9). From these results it follows that $\alpha(I)/\gamma(I) \leq \rho(I)$, and, when $I$ defines a zero-dimensional subscheme of $\mathbf{P}^N$, that $\rho(I) \leq \operatorname{reg}(I)/\gamma(I)$ (see Theorem 1.2.1).

By applying these results we give the first determinations of $\rho(I)$ in cases for which $\rho(I) > 1$ and $I$ is not monomial (see Theorem 2.4.3 a) and Proposition 2.5.1 a)). As a corollary, it follows that the upper bounds on $\rho$ coming from [ELS] and [HH1] are sharp (see Corollary 1.1.1).

Our original motivation for this work was a question of Huneke’s which is still open: if $I = I(S)$ is the ideal defining any finite set $S$ of points in $\mathbf{P}^2$, is it true that $I^{(3)} \subseteq I^2$? This question was prompted by the results of [HH1] and [ELS], which guarantee that $I^{(4)} \subseteq I^2$. The question of the containment $I^{(3)} \subseteq I^2$ turns out to be quite delicate. Here we show that containment holds at least when $S$ is a set of generic points (Theorem 1.1.1).

1.1. Comparison Invariants. As mentioned above, given any homogeneous ideal $0 \neq I \subseteq \mathbf{R} = k[\mathbf{P}^N]$, we define the resurgence, $\rho(I)$, of $I$ to be the supremum of all ratios $m/r$ such that $I^r$ does not contain $I^{(m)}$, where by $I^{(m)}$ we mean, as in [HH1], the contraction of $I^mR_A$ to $\mathbf{R}$, where $R_A$ is the localization of $\mathbf{R}$ by the multiplicative system $A$, and $A$ is the complement of the union of the associated primes of $I$. We refer to the maximum height among the associated primes of $I$ as the codimension, $\operatorname{cod}(I)$, of $I$.

The saturation sat($I$) of a homogeneous ideal $I$ is the ideal generated by all forms $F$ such that $(x_0, \ldots, x_N)^tF \subseteq I$ for some $t$ sufficiently large. If $I = \operatorname{sat}(I)$, we say $I$ is saturated. In any case, there is always a $t$ such that $I_j = \operatorname{sat}(I)_{j}$ for all $j \geq t$. The least such $t$ is the saturation degree, $\operatorname{satdeg}(I)$, of $I$.

In case $I$ is saturated and thus we have $I = I(X)$ for a subscheme $X \subseteq \mathbf{P}^N$, we may write $\rho(X)$ to mean $\rho(I)$. A case of particular interest to us here is when $I(X)$ is an intersection $I = I(X) = \cap_i I(L_i)^{m_i}$ of powers of ideals of linear subspaces $L_i \subseteq \mathbf{P}^N$, none of which contains another, in which case we refer to $X$ as a fat flat subscheme. Taking symbolic powers of $I(X)$ is then straightforward; since $I(L_i)^{m_i}$ is primary, $I^{(m)} = \cap_i I(L_i)^{m_i}$. A special case of particular importance is when
each space $L_i$ is a single reduced point $p_i$. In this case $X$ is known as a fat point subscheme and $I^{(m)}$ is just the saturation of $I^m$.

Now let $\rho(N,d)$ denote the supremum of $\rho(I)$ over homogeneous ideals $0 \neq I \subseteq k[P^N]$ of codimension $d$. (Since $d = N$ is an important special case, will just write $\rho(N)$ for $\rho(N,N).)$ The main theorem of [HH1] implies that $\rho(N,d) \leq d$. Here we show that in fact $\rho(N,d) = d$.

Although, as far as we know, it has not previously been shown for any $N$ or $d > 1$ that $\rho(N,d) = d$, examples of Ein (see Section 2.4 and [HH2]) show that $\lim_{N \rightarrow \infty} \rho(N,d) = d$. We obtain:

**Corollary 1.1.1.** For each $N \geq 1$ and $1 \leq d \leq N$, we have $\rho(N,d) = d$.

Our proof of Corollary 1.1.1 involves finding, for each $N$ and $d$, a sequence of subschemes $S_N(d,i) \subseteq P^N$ such that $\lim_{i \rightarrow \infty} \rho(S_N(d,i)) = d$. These subschemes can be taken to be fat flat subschemes, and, in fact, reduced.

Our main technical tool involves developing bounds, as discussed above, on $\rho(Z)$ for subschemes $Z \subseteq P^N$, mostly in terms of postualational invariants of $I(Z)$; i.e., invariants that are determined by the Hilbert functions of $I(Z)_m$ for which the Hilbert functions of $I(Z)$ and its symbolic powers remain the same. This is useful since postualational data is reasonably accessible, either computationally or theoretically (for example, [CnH] and [GHM] classify all sets of up to 8 points in $P^2$ according to the postualational data of fat point subschemes supported at the points).

1.2. Postualational Bounds and Seshadri Constants. We now discuss in detail the postualational invariants we will use. Given a homogeneous ideal $0 \neq I \subseteq R = k[P^N]$, let $\alpha(I)$ be the least degree $t$ such that the homogeneous component $I_t$ in degree $t$ is not zero. Thus $\alpha$ is, so to speak, the degree in which the ideal begins. It is also the degree of a generator of least degree, and it is the $M$-adic order of $I$ (i.e., the largest $t$ such that $I \subseteq M^t$), where $M$ is the maximal homogeneous ideal. If $Z \subseteq P^{N-1} \subseteq P^N$ is a subscheme contained in a hyperplane, in cases which are not clear from context we will use $\alpha_{N-1}(I(Z))$ or $\alpha_N(I(Z))$ to distinguish whether we are considering $\alpha$ for the ideal defining $Z$ in $P^{N-1}$ or in $P^N$. Let $\tau(I)$ be the least degree such that the Hilbert function becomes equal to the Hilbert polynomial of $I$ and let $\sigma(I) = \tau(I) + 1$.

Given a minimal free resolution $0 \rightarrow F_N \rightarrow \cdots \rightarrow F_0 \rightarrow I \rightarrow 0$ of $I$ over $R$, where $F_i$ is a graded $R$-module is $\oplus R[-b_{ij}]$, the Castelnuovo-Mumford regularity $\text{reg}(I)$ of $I$ is the maximum over all $i$ and $j$ of $b_{ij} - i$. If $I$ defines a 0-dimensional subscheme of $P^N$ (i.e., $I$ has codimension $N$), then $\text{reg}(I)$ is the maximum of $\text{satdeg}(I)$ and $\sigma(\text{sat}(I))$, hence if $I$ is already saturated (and so is the ideal of a 0-dimensional subscheme), then $\text{reg}(I) = \sigma(I)$ (see [GGP]). (We will only be concerned with the regularity in case $I$ defines a 0-dimensional subscheme.)

Our results depend on our developing bounds on $\rho(I)$. Our bounds involve the quantity $\gamma(I) = \lim_{m \rightarrow \infty} \alpha(I^{(m)})/m$ for a homogeneous ideal $0 \neq I \subseteq k[P^N]$. Because of the subadditivity of $\alpha$, this limit exists (see Remark III.7 of [HR2] or Lemma 2.3.1). Moreover, $\gamma(I) > 0$ (see Lemma 2.3.2). Given a subscheme $Z \subseteq P^N$, we will write $\gamma(Z)$ for $\gamma(I(Z))$. Since $\alpha(I^{(m)})$ is linear in $m$, note that $\alpha(I)/\gamma(I) = \lim_{m \rightarrow \infty} \alpha(I^{(m)})/\alpha(I^{(m)})$. Thus $\alpha(I)/\gamma(I)$ gives an asymptotic measure of the growth of $I^{(m)}$ compared to $I^m$. 
Our next result thus shows that $\rho(I)$ measures additional growth, in comparison to $\alpha(I)/\gamma(I)$ (hence the term resurgence for $\rho$).

**Theorem 1.2.1.** Let $0 \neq I \subseteq k[\mathbb{P}^N]$ be a homogeneous ideal.

(a) Then $\alpha(I)/\gamma(I) \leq \rho(I)$.

(b) If in addition $I$ defines a 0-dimensional subscheme, then $\rho(I) \leq \text{reg}(I)/\gamma(I)$.

Thus, for example, given $I = I(Z)$ for a fat point subscheme $Z$ with $\alpha(I) = \sigma(I)$, this theorem shows that computing $\rho(Z)$ is equivalent to computing $\gamma(Z)$. The quantity $\gamma$ is in that case essentially a uniform version of a multi-point Seshadri constant. Indeed, if $Z$ is a reduced finite generic set of $n$ points in $\mathbb{P}^N$, then $\gamma(Z) = n(\varepsilon(N, Z))^N$ (see Lemma 2.3.1), where, following the exposition of [HR2, HR3], $\varepsilon(N, Z)$ is the codimension 1 multipoint Seshadri constant for $Z = \{p_1, \ldots, p_n\}$; i.e., the real number

$$\varepsilon(N, Z) = \sqrt[N-1]{\inf \left\{ \frac{\deg(H)}{\sum_{i=1}^{n} \text{mult}_{p_i} H} \right\}},$$

where the infimum is taken with respect to all hypersurfaces $H$, through at least one of the points (see [D] and [X]). We also define $\varepsilon(N, n)$ to be $\sup \{\varepsilon(N, Z)\}$, where the supremum is taken with respect to all choices $Z$ consisting of $n$ distinct points $p_i$ of $\mathbb{P}^N$. In case $N$ is clear from context, we will write $\varepsilon(Z)$ for $\varepsilon(N, Z)$.

While it is in any case obvious from the definitions that $\gamma(Z) \geq n(\varepsilon(N, Z))^N$, equality can fail since the latter takes notice of hypersurfaces whose multiplicities at the points $p_i$ need not all be the same. (For example, if $Z$ is the reduced scheme consisting of $n = 4$ points in $\mathbb{P}^2$, 3 of them on a line and one off, then $5/3 = \gamma(Z) > n\varepsilon(2, Z) = 4/3$.)

1.3. Application to generic points. As an interesting example, consider $\mathbb{P}^N$ and some $s$, and let $I$ be the ideal of $n = \binom{s+N}{N}$ generic points of $\mathbb{P}^N$; then in Theorem 1.2.1 we have $\alpha(I) = s + 1 = \sigma(I) = \text{reg}(I)$. Although, in the case of $N = 2$, $\varepsilon(2, n)$ (and hence $\gamma(I)$) is known for $n < 10$, a famous and still open conjecture of Nagata [N] is equivalent to asserting that $\varepsilon(2, n) = 1/\sqrt{n}$ for $n \geq 10$. For no nonsquare $n \geq 10$ is $\varepsilon(2, n)$ currently known. However, it is not hard to show that $\varepsilon(2, n) = 1/\sqrt{n}$ if $n$ is any square. Thus we have the following corollary.

**Corollary 1.3.1.** If $n = \binom{s+N}{N}$, then for the subscheme $Z \subset \mathbb{P}^N$ consisting of the union of $n$ distinct generic points we have $\rho(Z) = \frac{s+1}{n(\varepsilon(N, Z))^{s+N}}$. If in addition $N = 2$ and $n$ is a square, then

$$\rho(Z) = \frac{s+1}{\sqrt{n}} = \sqrt[4]{\frac{s+1}{s+2}}.$$

We remark that there are infinitely many integers $n$ which are at the same time a square and of the form $\binom{s+N}{N}$. (An easy argument shows that $n = \binom{s+2}{2}$ is a square if and only if either $s + 1 = 2x^2$ for some $y$ such that $y^2 - 2x^2 = 1$, or $s + 2 = 2x^2$ for some $y$ such that $y^2 - 2x^2 = -1$. The fact that there are infinitely many such $x$ follows from the theory of Pell’s equation. The first few $s$ that arise are 0, 7, 48, 287, 1680, 9799, etc.)
2. Preliminaries

In this section we establish our postulational criteria for containment. We use two basic but surprisingly powerful ideas.

2.1. The Containment Principles. The first idea, given homogeneous ideals $I$ and $J$ in $k[P^N]$, is that by examining the zero loci of $I_0$ and $J_0$ (called $t$ degree envelopes in $\mathbb{P}^N$, we get a necessary criterion for containment. In particular, if $I \subseteq J$, then the zero locus of $I_0$ must contain the zero locus of $J_0$ in every degree $t$.

This is useful when trying to show that containment fails.

The second idea uses the obvious fact that $I^{(m)} \subseteq I^{(r)}$ if $r \leq m$, and the fact (when $I$ defines a 0-dimensional subscheme) that $(I^{(r)})_t = (I^r)_t$ for $t$ large enough. Given $r$, if we pick $m \geq r$ large enough, then $\alpha(I^{(m)})$ will be large enough so that $(I^{(r)})_t = (I^r)_t$ for all $t \geq \alpha(I^{(m)})$, and hence $(I^{(m)})_t \subseteq (I^{(r)})_t = (I^r)_t$ for $t \geq \alpha(I^{(m)})$. Since $(I^{(m)})_t = (0) \subseteq (I^r)_t$ for $t < \alpha(I^{(m)})$, we obtain $I^{(m)} \subseteq I^r$.

Given a homogeneous ideal $J \subseteq k[P^N]$, let $h_J(t) = \dim J_t$ denote its Hilbert function. Let $P_J$ denote the Hilbert polynomial. Thus $\alpha(J)$, defined when $J \neq 0$, is the least $t \geq 0$ such that $h_J(t) > 0$, and $\tau(J)$ is the least $t$ such that $h_J(t) = P_J(t)$.

2.2. Some Notation for Fat Flats. We now recall a convenient notation for denoting fat flats. Let $I \subseteq k[P^N]$ be any ideal of the form $I = \cap_i I(L_i)^{m_i}$, where each $L_i \subseteq P^N$ is a proper linear subspace, with no $L_i$ containing $L_j$, $j \neq i$, and where each $m_i$ is a nonnegative integer. The fat flat subscheme $Z$ defined by $I$ depends only on the spaces $L_i$, the integers $m_i$ and the space $P^N$ containing $Z$. Since the latter is usually clear from context, it is convenient to denote the subscheme formally by $Z = m_1L_1 + \cdots + m_nL_n$, and write $I = I(Z)$ for the defining ideal. In particular, $I(mZ) = I(Z)^{(m)}$ for each positive integer $m$.

Given a fat flat subscheme $Z = m_1L_1 + \cdots + m_nL_n \subseteq P^N$, the set $\text{Supp}(Z) = \{L_i : m_i > 0\}$ is called the support of $Z$. In case $Z$ is a fat point subscheme, we denote the sum $\sum_i m_i + N - 1$ by $\deg(Z)$; as is well known, $P_{I(Z)}(t) = \binom{t+N}{N} - \deg(Z)$. It is easy to see that $\deg(rZ)$ is a strictly increasing function of $r$.

2.3. Preliminary Lemmas. We begin by considering $\gamma(I)$.

Lemma 2.3.1. For any homogeneous ideal $0 \neq I \subseteq k[P^N]$, the limit

$$\gamma(I) = \lim_{m \to \infty} \alpha(I^{(m)})/m$$

exists. Moreover, if $I = I(Z)$, where $Z$ is the reduced subscheme $Z \subseteq P^N$ consisting of a finite generic set of $n$ points, we have $\gamma(Z) = n(\varepsilon(N,Z))^{N-1}$.

Proof. This is proved in Remark III.7 of [HR2]. For the reader’s convenience we recall the proof here.

First we show $\gamma(I)$ is defined. Note that $\alpha$ is subadditive (i.e., $\alpha(I^{(m_1+m_2)}) \leq \alpha(I^{(m_1)}) + \alpha(I^{(m_2)})$, and hence $\alpha(I^{(n)})/n \leq (qm/n)\alpha(I^{(m)})/m + \alpha(I^r)/n \leq \alpha(I^{(m)})/m + \alpha(I^r)/n$ for any positive integers $n = mq + r$, and $\alpha(I^{(n)})/n \leq \alpha(I^{(m)})/m$ if $r = 0$. Thus $\alpha(I^{(m)})/n! \leq \alpha(I^{(m)})/m$ whenever $m$ divides $n!$. Thus $\alpha(I^{(n!)/n})!$ is a non-increasing sequence, and hence has some limit $c$. In addition, for all $d \geq n!$, using integer division to write $d = q(n!)+r$ with $0 \leq r < n!$, we have $\alpha(I^{(d)})/d \leq \alpha(I^{(n!)/n!})/d + \alpha(I^r)/d$. It follows that the limit exists and is equal to $c$. 

For the second statement, argue as in the proof of Corollary 5 of [R] to reduce
to the case that the multiplicities are all equal. This uses the fact that the points
are generic and thus one can, essentially, average over the points. Now we see that
\(n(\varepsilon(N, Z))^{N-1}\) is, by definition, the infimum of the sequence \(\alpha(I^{(n)})/n\) whose limit
defines \(\gamma(I)\), but it is obvious from our argument above that \(\gamma(I) \leq \alpha(I^{(n)})/n\), and
hence \(\gamma(I)\) is the infimum.

We now give a criterion for containment to fail, and thence a lower bound for
\(\rho(I)\):

**Lemma 2.3.2** (Postulational Criterion 1). Let \(0 \neq I \subsetneq k[\mathbf{P}^N]\) be a homogeneous
ideal. Then \(\gamma(I) \geq 1\) and we have:

(a) If \(r\alpha(I) > \alpha(I^{(m)})\), then \(I^r\) does not contain \(I^{(m)}\).

(b) If \(m/r < \alpha(I)/\gamma(I)\), then, for all \(t \gg 0\), \(I^{rt}\) does not contain \(I^{(mt)}\). In
particular, \(1 \leq \alpha(I)/\gamma(I) \leq \rho(I)\).

**Proof.** For \(\gamma(I) \geq 1\), see [PSC] Lemma 8.2.2.

(a) This is because \((I^r)_t = 0\) but \((I^{(m)})_t \neq 0\) for \(t = \alpha(I^{(m)})\), since \(\alpha(I^r) = r\alpha(I) > \alpha(I^{(m)})\).

(b) Suppose \(m/r < \alpha(I)/\gamma(I)\). Let \(0 < \delta\) be such that \(m/r < \alpha(I)/(\delta + \gamma(I))\). By
definition, \(\alpha(I^{(mt)})/(mt) \leq \gamma(I) + \delta\) for \(t \gg 0\), so \(\alpha(I^{(mt)}) \leq mt(\gamma(I) + \delta) < r\alpha(I)\)
for \(t \gg 0\), and hence \(I^{rt}\) does not contain \(I^{(mt)}\) for \(t \gg 0\), which now implies \(\alpha(I)/\gamma(I) \leq \rho(I)\). Finally, by subadditivity, as in the proof of Lemma 2.3.1, we have \(\gamma(I) \leq \alpha(I)\), hence \(1 \leq \alpha(I)/\gamma(I)\). □

It is possible to give refined versions of Lemma 2.3.2 in which both \((I^r)_t\) and
\((I^{(m)})_t\) may be nonzero, but in which the zero locus of the former is bigger than
that of the latter. These refined versions are useful in doing examples and will be
the topic of a subsequent paper, [BH].

We next develop our criteria for containment to hold. First we recall a few well
known facts.

**Lemma 2.3.3.** Let \(0 \neq I \subsetneq k[\mathbf{P}^N]\) be a homogeneous
ideal.

(a) If \(I^{(m)} \subsetneq I^r\), then \(r \leq m\).

(b) We have \(\alpha(I^{(m)}) \leq m\alpha(I)\) and \(\alpha(I) \leq \text{reg}(I)\).

(c) If \(I\) defines a \(0\)-dimensional subscheme and \(t \geq \text{reg}(I)\), then \((I^r)_t = (\text{sat}(I^r))_t\); in particular, if \(I\) is saturated and defines a \(0\)-dimensional sub-
scheme, then \(t \geq r\sigma(I)\) implies \((I^r)_t = (I^{(r)})_t\).

**Proof.**

(a) We have \(I^m \subsetneq I^{(m)} \subsetneq I^r\), hence \(m\alpha(I) = \alpha(I^{(m)}) \geq \alpha(I^r) = r\alpha(I)\) so
\(m \geq r\).

(b) The claim \(\alpha(I^{(m)}) \leq m\alpha(I)\) follows by the subadditivity of \(\alpha\). The second
claim is immediate from the definition of regularity, since \(\text{reg}(I)\) is at least as much
as the degree of the homogeneous generator of greatest degree in any minimal set
of homogeneous generators of \(I\), while \(\alpha(I)\) is the degree of the generator of least
degree.

(c) We argue as in the proof of Proposition 2.1 of [AV]. By Theorem 1.1 of [GGP],
\(r\text{reg}(I) \geq \text{reg}(I^r) \geq \text{satdeg}(I^r)\), hence \(t \geq r\text{reg}(I)\) implies \((I^r)_t = (\text{sat}(I^r))_t\). The
second statement is just an instance of the first. □

Here we give a criterion for containment to hold:
Lemma 2.3.4 (Postulational Criterion 2). Let \( I \subseteq k[\mathbb{P}^N] \) be a homogeneous ideal (not necessarily saturated) defining a 0-dimensional subscheme. If \( \text{reg}(I) \leq \alpha(I^{(m)}) \), then \( I^{(m)} \subseteq I^r \).

Proof. First, \( \text{reg}(I) \leq \alpha(I^{(m)}) \leq m \alpha(I) \leq m \text{reg}(I) \), so \( r \leq m \), hence \( (I^{(m)})_t \subseteq (I^r)_t \) for all \( t \geq 0 \). Moreover, if \( I \) is not saturated, then the maximal homogeneous ideal \( M \) is an associated prime, so \( I^{(m)} = I^m \) for all \( m \geq 1 \), hence \( (I^{(m)})_t = I^m \subseteq I^r \). Thus we may as well assume that \( I \) is saturated. But \( (I^r)_t = (I^r)_t \) by Lemma 2.3.3 for \( t \geq \text{reg}(I) \), while \( \text{reg}(I) \leq \alpha(I^{(m)}) \) implies \( (I^{(m)})_t = \emptyset \subseteq (I^r)_t \) for \( t < \text{reg}(I) \). \( \square \)

As an application of Postulational Criterion 2 we have:

Corollary 2.3.5. Let \( I \subseteq k[\mathbb{P}^N] \) be a homogeneous ideal (not necessarily saturated) defining a 0-dimensional subscheme. If \( c \) is a positive real number such that \( mc \leq \alpha(I^{(m)}) \) for all \( m \geq 1 \), then \( I^{(m)} \subseteq I^r \) if \( m/r \geq \text{reg}(I)/c \); in particular, \( \rho(I) \leq \text{reg}(I)/c \).

Proof. By Lemma 2.3.4, \( \text{reg}(I) \leq mc \), or equivalently \( \text{reg}(I)/c \leq m/r \), implies \( I^{(m)} \subseteq I^r \). \( \square \)

Remark 2.3.6. Let \( I \subseteq k[\mathbb{P}^N] \) be a homogeneous ideal defining a 0-dimensional subscheme. Since we can evaluate limits on subsequences and since by subadditivity the sequence \( \alpha(I^{(im)})/(i!m) \) is non-increasing, we see that \( \gamma(I) \leq \alpha(I^{(m)})/m \) for all \( m \geq 1 \). Thus the \( c \) in Corollary 2.3.5 can be taken to be \( \gamma(I) \). It is reasonable to ask: why not just take \( c = \gamma(I) \)? Unfortunately, the exact value of \( \gamma(I) \) is rarely known even if \( I = I(Z) \) for a fat point subscheme \( Z = m_1p_1 + \cdots + m_np_n \) in \( \mathbb{P}^2 \), so it is useful that the statement not be in terms of \( \gamma(I) \). On the other hand, good lower bounds are known for \( \gamma(Z) \) in certain cases (see for example [B], [H1], [HR1], [ST] and [Tu], among many others). Also, exact values are known in some cases, such as when \( \text{Supp}(Z) \) consists of any \( n \leq 8 \) points in \( \mathbb{P}^2 \). (Since the subsemigroup of classes of effective divisors for a blow up of \( \mathbb{P}^2 \) at \( n \leq 8 \) points is polyhedral and the postulation for any such \( Z \) is known, one can explicitly determine \( \gamma(Z) \) in this situation if one knows the effective subsemigroup. The effective subsemigroups for all subsets of \( n \leq 8 \) points of the plane are now known, as a consequence of the classification of the configuration types of \( n \leq 8 \) points of \( \mathbb{P}^2 \), given in [GiH] for \( n \leq 6 \) and [GHM] for \( 7 \leq n \leq 8 \).)

Corollary 2.3.7. Let \( I = I(Z) \) for a nontrivial fat point subscheme \( Z \subseteq \mathbb{P}^N \). If \( \alpha(I) = \sigma(I) \), then \( \rho(I) = \alpha(I)/\gamma(I) \).

Proof. This is immediate from Corollary 2.3.5 Remark 2.3.6 and Lemma 2.3.2 \( \square \)

Remark 2.3.8. One can sometimes do better using non-postulational data. The paper [EHU] gives various bounds on the regularity under various assumptions. For another example that we will refer to in Section 3 let \( I \subseteq k[\mathbb{P}^N] \) be a homogeneous ideal defining a 0-dimensional subscheme. Then \( \text{reg}(I^r) \leq r \omega(I) + 2(\text{reg}(I) - \omega(I)) \) for any \( r \geq 2 \) by Theorem 0.4 of [Ch1] (or see Section 6 of [Ch2]), where \( \omega(I) \) is the maximum degree of a generator in any minimal set of homogeneous generators of \( I \). Replacing \( \text{reg}(I) \) by \( r \omega(I) + 2(\text{reg}(I) - \omega(I)) \) in the argument of the proof of Lemma 2.3.4 and then arguing as in Lemma 2.3.6 keeping in mind Remark 2.3.6 gives \( I^{(m)} \subseteq I^r \) if \( m/r \geq (\omega(I) + 2(\text{reg}(I) - \omega(I))/r)/\gamma(I) \).
Lemma 2.4.1. Let $I$ defined by the $s$ consisting of the degree among hypersurfaces that vanish on $mS$. First consider the case that $m = \omega(I)/\gamma(I)$. Then $I^{(m)} \subseteq I^r$ for all but finitely many pairs $(m, r)$ with $m/r \geq c$. In particular, if $m/r > \omega(I)/\gamma(I)$, then $I^{(mt)} \subseteq I^t$ for all $t \gg 0$.

Proof. By Remark 2.3.8, we have $I^{(m)} \subseteq I^r$ if $(m, r)$ is on or above the line $m = (\omega(I)/\gamma(I))r + 2(\text{reg}(I) - \omega(I))/\gamma(I)$. But $c$ is greater than the slope $\omega(I)/\gamma(I)$ of this line, so there are only finitely many pairs $(m, r)$ with $m/r \geq c$ below this line. The second statement is now immediate.

2.4. Constructions showing Optimality. To prove Corollary 1.1.1 it suffices to find subschemes $Z \subseteq P^N$ for which $\rho(Z)$ is large. Lemma 2.3.2 suggests where to look. We want a scheme $Z$ such that $\alpha(I(Z))$ is as large as possible, which means that $I(Z)$ should behave generically, from a postulatory point of view. On the other hand, we want $\gamma(Z)$ to be small, so among all $I(Z)$ with generic Hilbert function we want to examine those for which the Hilbert function of $I(Z)^{(m)}$ is as large as possible (and hence $\alpha(I(Z)^{(m)})$ is as small as possible).

This problem was studied in [GMS] in characteristic $0$ in the case that $N = m = 2$ with $Z = p_1 + \cdots + p_n$, a reduced set of points $p_i$; i.e., double points in the plane. They prove that the the set of singular points of a union of $s$ general lines (i.e., the pair-wise intersections of $s$ general lines) is a configuration of points in the plane having generic Hilbert function but for which the Hilbert function of the symbolic square of the ideal is as large as possible. This suggests, more generally, to look at the set of $N$-wise intersections of $s \geq N + 1$ general hyperplanes in $P^N$. More generally yet, for $1 \leq e \leq N$ and $s \geq e$, let $S_N(e, s, d)$ denote the reduced scheme consisting of the $e$-wise intersections of $s$ general hypersurfaces $H_1, \ldots, H_s$ in $P^N$ of respective degrees $d_i$ where $d = (d_1, \ldots, d_s)$. If $d_i = d$ for all $i$, we will write $S_N(e, s, d)$ for $S_N(e, s, d)$. If $d = 1$, we will write simply $S_N(e, s)$. Thus $S_N(N, N + 1)$ can be taken to be the set of coordinate vertices of $P^N$, and $S_N(1, N + 1)$ to be the union of the coordinate hyperplanes. In this notation, the examples of $\Omega$ having large $\rho$ are the codimension $e$ skeleta $S_N(e, N + 1)$ of the coordinate simplex in $P^N$ (hence $d_i = 1$ for all $i$); i.e., the $e$-wise intersections of $s = N + 1$ general hyperplanes in $P^N$. The case $e = N$ (i.e., of the coordinate vertices in $P^N$) is treated by Aris and Vatne (see Theorem 4.5 of [AV]).

It is easy to see that a general hyperplane section $H \cap S_N(e, s, d)$ is $S_{N-1}(e, s, d)$, defined by the $e$-wise intersections of the hypersurfaces $H \cap H_i \subseteq H$. We will denote $\alpha(I(mS_N(e, s, d)))$ by $\alpha_N(m, e, s, d)$, where $mS_N(e, s, d) \subseteq P^N$ is the subscheme consisting of the $e$-wise intersections of the $s$ hypersurfaces $H_i$, where each $e$-wise intersection is taken with multiplicity $m$.

In order to apply our bounds to $S_N(e, s, d)$, we need to determine the least degree among hypersurfaces that vanish on $mS_N(e, s, d)$.

Lemma 2.4.1. Let $1 \leq e \leq N$, $s \geq e$, and let $d = (d_1, d_2, \ldots, d_s)$. Let $I = I(mS_N(e, s, d)) \subseteq k[P^N]$. If $m = re$ for some $r$, then $r(d_1 + \cdots + d_s) \geq \alpha(I)$. If $d_1 = \cdots = d_s = 1$, then for any $m \geq 1$ we have $ms/e \leq \alpha(I)$, and hence we have equality if $m = re$.

Proof. First consider the case that $m = re$ is a multiple of $e$. Then the divisor $r(H_1 + \cdots + H_s)$ has degree $r(d_1 + \cdots + d_s) = m(d_1 + \cdots + d_s)/e$ and vanishes on each component of $S_N(e, s, d)$ with multiplicity $m$ (since each component of $S_N(e, s, d)$ is contained in exactly $e$ of the hypersurfaces $H_i$). Thus $r(d_1 + \cdots + d_s) \geq \alpha_N(m, e, s, d)$. 


Now assume \( d_1 = \cdots = d_s = 1 \). To show \( \alpha_N(m, e, s, d) \geq ms/e \), it is enough to show \( \alpha_N(m, e, s, d) \geq ms/e \), since by taking general hyperplane sections we have:

\[
\alpha_N(m, e, s, d) \geq \alpha_N-1(m, e, s, d) \geq \cdots \geq \alpha_e(m, e, s, d).
\]

Suppose it were true that \( \alpha_e(m, e, s, d) < ms/e \) for some \( m \). Let \( F \) be a form of degree \( d = \alpha_e(m, e, s, d) \) vanishing with multiplicity at least \( m \) at each point of \( S_e(e, s, d) \). Then \( F \) restricts to give a form on \( H_1 \) with \( d < ms/e \leq m(s-1)/(e-1) \), but \( H_1 \cap S_e(e, s, d) = S_{e-1}(e-1, s-1, d') \), where \( d' = (d_2, \cdots, d_s) \). And, by induction on the dimension (where dimension 1 is easy), we have \( m(s-1)/(e-1) \leq \alpha_{e-1}(m, e-1, s-1, d') \). Hence \( F \) vanishes identically on \( H_1 \). By symmetry, \( F \) vanishes on all of the hyperplanes \( H_i \). Dividing out by the linear forms defining the hyperplanes gives a form \( F' \) of degree \( d-s \) vanishing with multiplicity \( m-e \) at each point of \( S_e(e, s, d) \), and hence \( \alpha_e(m-e, e, s, d) \leq d-s < ms/e-s = (m-e)s/e \), hence again \( F' \) vanishes on all \( H_i \). Continuing in this way, we eventually obtain a form of degree less than \( s \) that vanishes on the \( s \) hyperplanes \( H_i \), which is a contradiction unless \( F = 0 \).

We still need to know \( \alpha(I(S_N(e, s, d))) \).

**Lemma 2.4.4.** Let \( 1 \leq e \leq N \), \( e \leq s \) and \( d_1 \leq d_2 \leq \cdots \leq d_s \). For \( S = S_N(e, s, d) \) we have \( \alpha(I(S)) = d_1 + \cdots + d_{s-e+1} \). If \( e = N \) and \( d_i = 1 \) for all \( i \), we have \( \alpha(I(S)) = \sigma(I(S)) = s - N + 1 \).

**Proof.** Clearly, \( \alpha(I(S)) \leq d_1 + \cdots + d_{s-e+1} \), since every intersection of \( e \) of the hypersurfaces must involve one of the hypersurfaces \( H_1, \ldots, H_{s-e+1} \). For the rest, let us refer to the union of the \( e \)-wise intersections of the hypersurfaces \( H_i \) as the codimension \( e \) skeleton of the \( H_i \), or just the \( e \)-skeleton. We will now show that any hypersurface \( H \) of degree \( d < d_1 + \cdots + d_{s-e+1} \) which vanishes on the \( e \)-skeleton also vanishes on the \((e-1)\)-skeleton. Since \( d < d_1 + \cdots + d_{s-e+1} \leq d_1 + \cdots + d_{s-(e-1)+1} \), this means that \( H \) also vanishes on the \((e-1)\)-skeleton, and so on, and thus vanishes on the \( 1 \)-skeleton and indeed the \( 0 \)-skeleton (i.e., the whole space, since a form of degree \( d \) cannot contain hypersurfaces whose degrees sum to more than \( d \)). Thus \( H \equiv 0 \), and this shows \( \alpha(I(S)) \geq d_1 + \cdots + d_{s-e+1} \) which gives equality.

So suppose \( H \) has degree \( d < d_1 + \cdots + d_{s-e+1} \) and vanishes on the \( e \)-skeleton. Thus for any indices \( i_1 < \cdots < i_{e-1} \) and any \( j \) not one of these indices, \( H \) vanishes on \( H_{i_1} \cap \cdots \cap H_{i_{e-1}} \cap H_j \). By Bertini (Theorem II.8.18 of [HT], taking hyperplane sections after uple embeddings), intersections of general hypersurfaces are smooth and, in dimension 2 or more, irreducible. Thus \( H_{i_1} \cap \cdots \cap H_{i_{e-1}} \) is irreducible. If it were not already contained in \( H \), we can intersect with \( H \) and do a degree calculation: \( H \cap H_{i_1} \cap \cdots \cap H_{i_{e-1}} \) has degree \( dd_{i_1} \cdots d_{i_{e-1}} \) whereas the union of the intersections of \( H_{i_1} \cap \cdots \cap H_{i_{e-1}} \) with all possible \( H_j \) (i.e., for all \( j \) not among the indices \( i_1, \ldots, i_{e-1} \)), has degree \( d_{i_1} \cdots d_{i_{e-1}} \sum_j d_j \), where the sum is over all \( j \) not among \( i_1, \ldots, i_{e-1} \). Clearly \( d < d_1 + \cdots + d_{s-e+1} \leq \sum_j d_j \) since the \( d_i \) are assumed to be nondecreasing. Since the total degree of the intersection \( H \cap H_{i_1} \cap \cdots \cap H_{i_{e-1}} \) is less than the sum of the degrees of the divisors contained in the intersection, it follows that \( H_{i_1} \cap \cdots \cap H_{i_{e-1}} \subseteq H \) for each component of the \((e-1)\)-skeleton, as claimed.

Finally, suppose \( e = N \) and \( d_i = 1 \) for all \( i \). Then as we have just seen, \( \alpha(S) = s - e + 1 \). But there are \( \binom{s}{e} \) points and \( \binom{(s-e)+N}{e} = \binom{s}{e} \) forms of degree \( s - e \) in \( N + 1 \) variables. Thus the number of conditions imposed by the points equals the number of points, hence \( \tau(I) = s - e \) so \( \sigma(I) = \alpha(I) = s - N + 1 \).
We now can obtain some results on $\rho(S_N(e, s))$. As noted above, Theorem 2.4.3(b) in the case $s = N + 1$ is due to L. Ein; Theorem 4.5 of [AV] implies $2 - 1/N \leq \rho(S_N(N, N + 1))$, using as Ein did the fact that the ideal is monomial.

**Theorem 2.4.3.** Let $1 \leq e \leq N$ and $e \leq s$. Then:

(a) $\rho(S_N(N, s)) = N(s - N + 1)/s$; and 
(b) $e(s - e + 1)/s \leq \rho(S_N(N, s))$. 
(c) More generally, given $d = (d_1, \ldots, d_s)$ with $d_1 \leq \cdots \leq d_s$, we have 
$$e(d_1 + \cdots + d_{s-e+1})/(d_1 + \cdots + d_s) \leq \rho(S_N(e, s, d)).$$

**Proof.** By Lemma 2.3.2, $\alpha(I(S_N(e, s))) = s - e + 1$, $\sigma(I(S_N(N, s))) = s - N + 1$, and $\gamma(I(S_N(e, s, d))) = d_1 + \cdots + d_{s-e+1}$, while by Lemma 2.4.1 we see that 
$$\gamma(I(S_N(e, s))) = \lim_{m \to \infty} \frac{\alpha(I(meS_N(e, s)))}{(me)} = s/e$$
and similarly $\gamma(I(meS_N(e, s, d))) \leq (d_1 + \cdots + d_s)/e$.

(a) By Corollary 2.3.7, we thus have $\rho(S_N(N, s)) = N(s - N + 1)/s$.
(b) By Lemma 2.3.2 we have $e(s - e + 1)/s \leq \rho(S_N(e, s))$.
(c) By Lemma 2.3.2 we have $e(d_1 + \cdots + d_{s-e+1})/(d_1 + \cdots + d_s) \leq \rho(S_N(e, s, d))$.

\(\square\)

2.5. **General Facts about $\rho$.** Here we take note of some general behavior of $\rho(I)$. 
To state the results, let $R = k[\mathbf{P}^N]$, let $x$ be an indeterminate with respect to which we have $R \subseteq R[x] = k[\mathbf{P}^{N+1}]$, and let the quotient $q : R[x] \to R$ correspond to the inclusion $\mathbf{P}^N \subseteq \mathbf{P}^{N+1}$. If $I \subseteq R$ is a homogeneous ideal, let $I' = IR[x]$ be the extended ideal. In case $I = I(Z)$ for some subscheme $Z \subseteq \mathbf{P}^N$, we will denote by $C(Z)$ the subscheme defined by $I'$; we note that $C(Z)$ is just the projective cone over $Z$.

**Proposition 2.5.1.** In the notation of the preceding paragraph, we have:

(a) $\rho(I) = \rho(I')$, hence $\rho(Z) = \rho(C(Z))$ for any nontrivial subscheme $Z \subseteq \mathbf{P}^N$;
(b) $\rho(I) = \rho(q^{-1}(I))$, hence if $I = I(Z)$ for a nontrivial subscheme $Z \subseteq \mathbf{P}^N \subseteq \mathbf{P}^{N+1}$, then $\rho(Z)$ is well defined, whether we regard $Z$ as being in $\mathbf{P}^N$ or $\mathbf{P}^{N+1}$; and 
(c) $\rho(mZ) \leq \rho(Z)$ for any fat flat subscheme $Z$.

**Proof.** (a) Since $R \to R[x]$ is flat, primary decompositions of ideals in $R$ extend to primary decompositions in $R[x]$ (see [Ma], Theorem 13, or Exercise 7, [AM]). Since $I$ and $I'$ have the same generators, whenever $I$ and $J$ are ideals in $R$, we have $I \subseteq J$ if and only if $I' \subseteq J'$. Taken together, this means $I^{(m)} \subseteq I'$ if and only if $(I')^{(m)} \subseteq (I')^r$, and hence that $\rho(I) = \rho(I')$.
(b) Note that $q^{-1}(I) = I' + (x)$, and use the facts that $(q^{-1}(I))^r = \sum_i (x^i)(I')^{r-i}$ and $(q^{-1}(I))^{(m)} = \sum_j (x^j)(I')^{(m-j)}$. If $(q^{-1}(I))^{(m)} \subseteq (q^{-1}(I))^r$, setting $x = 0$ gives $I^{(m)} \subseteq I'$, and hence $\rho(q^{-1}(I)) \geq \rho(I)$. And if $m/r > \rho(I')$, then $(m-j)/(r-j) \geq m/r$ for $0 \leq j < r$, so $x^j(I')^{(m-j)} \subseteq x^j(I')^{r-j}$ hence $(q^{-1}(I))^{(m)} \subseteq (q^{-1}(I))^r$, so $\rho(I) \geq \rho(q^{-1}(I))$.
(c) By definition we can find a ratio $s/r < \rho(I^{(m)})$ arbitrarily close to $\rho(I^{(m)})$ such that $(I^{(m)})^r$ does not contain $I^{(sm)}$, hence $I'^{rm}$ does not contain $I^{(sm)}$, so $sm/(sr) < \rho(I)$. \(\square\)
Equality in Proposition 2.5.1(c) can fail. For example, if $Z$ is the reduced union of three general points in $\mathbb{P}^2$, then $\rho(mZ) = 1$ if $m$ is even, while $\rho(mZ) = (3m + 1)/(3m)$ if $m$ is odd [BH].

3. Proofs

Proof of Corollary 1.1.1. The result of [HH1] shows that $\rho(N, e) \leq e$, while taking the limit as $s \to \infty$ in Theorem 2.4.3(b) shows $e \leq \rho(N, e)$. Alternatively, using Theorem 2.4.3(a), $i$ applications of Proposition 2.5.1(a), and then taking the limit for $s \to \infty$, we conclude $\rho(N + i, N) = N$ for all $N$ and $i$. □

Proof of Theorem 1.2.1. The upper bound is an immediate consequence of Corollary 2.3.5 and Remark 2.3.6. The lower bound is Lemma 2.3.2. □

Proof of Corollary 1.3.1. Since the points are generic and the number of points is the binomial coefficient $n = \binom{s + N}{s}$, we know $\alpha(I) = \sigma(I) = s + 1$. Also, by Lemma 2.3.1 we know $\gamma(I) = \frac{n}{\varepsilon(N, Z)} - 1$. Thus the result follows immediately from Theorem 1.2.1. When $N = 2$ and $n$ is a square, we know in addition that $n\varepsilon(2, n) = \sqrt{n}$. The fact that we can also write $\frac{s + 1}{\sqrt{n}} = \sqrt{2}\sqrt{\frac{s + 1}{2}}$ follows from $n = \frac{(s + 1)(s + 2)}{2}$. □

4. Additional Examples, Comments and Questions

The inspiration for this paper was a question Huneke asked the second author: if $S$ is a finite set of points in $\mathbb{P}^2$ with $I = I(S)$, is it true that $I(3) \subseteq I^2$?

We cannot yet answer this question, but we can show Huneke’s question has an affirmative answer in many cases. (Theorems 3.3 and 4.3 of [TY] give additional cases in which Huneke’s question has an affirmative answer.)

Theorem 4.1. Let $I = I(S)$, where $S$ is a set of $n$ generic points of $\mathbb{P}^2$. Then $I^2$ contains $I(3)$ for every $n \geq 1$.

Proof. Since for $n = 1, 2, 4$, $S$ is a complete intersection and hence $I^3 = I(3)$, the theorem is true in those cases, so assume $n$ is not 1, 2 or 4.

If $2\sigma(I) \leq \alpha(I(3))$, then $I^2$ contains $I(3)$ by Lemma 2.3.4. Since the points are generic and of multiplicity 1, they impose independent conditions in degrees $\alpha(I)$ or more, so $\sigma(I)$ is the largest $t$ such that $(t_3) < n$. Also, the Hilbert functions of ideals of fat point subschemes supported at 9 or fewer generic points are known (see, e.g., [H2]) so we can compute $\alpha(I(3))$ exactly. Here’s what happens for $n \leq 9$:

| $n$ | $\sigma(I)$ | $\alpha(I(3))$ |
|-----|-------------|----------------|
| 3   | 2           | 5              |
| 5   | 3           | 6              |
| 6   | 3           | 8              |
| 7   | 4           | 8              |
| 8   | 4           | 9              |
| 9   | 4           | 9              |

We see that $2\sigma(I) \leq \alpha(I(3))$, hence $I^2$ contains $I(3)$.

Now assume $n \geq 10$. Let $t = \sigma(I)$; then $(t_3) < n$ so $t^2 - t < 2n$. (Also note that since $n \geq 10$, we must have $t \geq 4$.) It is known (see [HH] or [MI]) that $n \geq 10$ points of multiplicity 3 impose independent conditions on forms of degree at least $\alpha(I(3))$, ...
so \(\alpha(I^{(3)})\) is the least \(d\) such that \(\left(\frac{d+2}{2}\right) > 6n\). Thus to show \(2t \leq d\), it is enough to show that \(\left(\frac{2t+1}{2}\right) \leq 6n\), which we will do using the fact that \(t^2 - t < 2n\) and hence \(3t^2 - 3t < 6n\). In fact, all we need do now is verify that \(\left(\frac{2t+1}{2}\right) \leq 3t^2 - 3t\), which is easy, keeping in mind that \(t \geq 4\). \qed

We can also give an affirmative answer to a stronger version of Huneke’s question in the case of \(n\) generic points.

**Theorem 4.2.** Let \(I = I(S_n)\), where \(S_n\) is a set of \(n\) generic points of \(\mathbb{P}^2\). Then \(I^r\) contains \(I^{(m)}\) whenever \(m/r > 3/2\).

**Proof.** This amounts to showing that \(\rho(S_n) \leq 3/2\). Again we can ignore \(n = 1, 2\) and 4. For \(n = 3, 5, 6, 8\) and 9 we use the upper bounds \(\sigma(I)/(nc(2,n))\) for \(\rho(S_n)\) obtained from Corollary 2.3.5, with \(c = nc(2,n)\); the values of \(nc(2,n)\) can be obtained from Nagata’s list of abnormal curves [N], or see [H4]. By Corollary 2.3.7, \(\sigma(I)/(nc(2,n)) = 3/2\) when \(n = 3\) and \(n = 6\). (Using refined methods that we will present in a subsequent paper, [H7], we can show that equality holds also for \(n = 8\) and 9, and that \(\rho(S_8) = 6/5\) and \(\rho(S_9) = 8/7\).)

| \(n\) | \(nc(2,n)\) | \(\sigma(I)\) | \(\frac{\sigma(I)}{nc(2,n)}\) |
|------|-------------|-------------|------------------|
| 3    | 2           | 4           | \(\frac{4}{2} = 2\) |
| 5    | 2           | 3           | \(\frac{3}{2} = 1.5\) |
| 6    | 12          | 3           | \(\frac{3}{12} = 0.25\) |
| 7    | 21          | 4           | \(\frac{4}{21} = 0.19\) |
| 8    | 48          | 4           | \(\frac{4}{48} = 0.083\) |
| 9    | 3           | 4           | \(\frac{4}{3} = 1.33\) |

In order to handle \(n = 7\), we see we need a better bound, which we obtain using Remark 2.3.8. We claim \(\rho(S_7) \leq 6/5\). We must show that if \(m/r > 6/5\), then \(\alpha(I^{(m)}) \geq r\omega(I) + 2(\text{reg}(I) - \omega(I))\), where here \(\omega(I) = 3\) and \(\text{reg}(I) = 4\) (see [H3] for the graded Betti numbers for the resolution of the ideal \(I^{(m)}\) for any \(m > 1\). From the Seshadri constant in the table above, we know \(\alpha(I^{(m)}) \geq 21m/8\). (In fact, it turns out that \(\alpha(I^{(m)}) = 21m/8\). Clearly \(\alpha(I^{(m)}) \geq 21m/8\), and one checks the cases \(m \leq 8\) directly to see that equality holds. For \(m \geq 8\), write \(m = 8i + j\) with \(0 \leq j < 8\) and use \(\alpha(I^{(8i+j)}) \leq i\alpha(I^{(8)}) + \alpha(I^{(j)}) = [21m/8].\) Thus \([21m/8] \geq 3r + 2\) (or, equivalently, \(21m/8 > 3r + 1\) implies \(\alpha(I^{(m)}) \geq r\omega(I) + 2(\text{reg}(I) - \omega(I))\). But for \(r > 6\), \(m/r \geq 6/5\) implies \(21m/8 > 3r + 1\) and hence \(\alpha(I^{(m)}) \geq r\omega(I) + 2(\text{reg}(I) - \omega(I))\). We now check \(r = 6\) individually. If \(r = 1\), clearly for any \(m \geq 1\) we have \(I^{(m)} \subseteq I^r\). For \(r = 2\) and \(m/r \geq 1.2\), we have \(m \geq 3\), \(\alpha(I^{(m)}) \geq 8\), and so \(\alpha(I^{(m)}) \geq 8 = 3r + 2\). Similarly for \(r = 3, 4, 5\) and 6. Thus \(\rho(S_7) \leq 1.2\). (We cannot do better than \(\rho(S_7) \leq 1.2\) using this argument, since \(m = 6\) and \(r = 5\) give \(m/r = 1.2\), yet fail to satisfy \(\alpha(I^{(m)}) \geq 3r + 2\).)

Now consider \(n > 9\). It is known that \(\varepsilon(n) \geq \sqrt{n - 1}/n\). See [Xu] for characteristic 0. It also follows from [H7] in all characteristics, as follows. Let \(s = \lfloor \sqrt{n} \rfloor\), and define \(0 \leq t \leq s\) so that either \(n = s^2 + 2t\) or \(n = s^2 + 2t + 1\). Let \(d = s\)
and \( r = s^2 + t \). First consider the case that \( n = s^2 + 2t \). Since \( r/d \geq \sqrt{n} \) by [HH1], and a little arithmetic shows that \( d/r \geq \sqrt{n-1}/n \). Now let \( n = s^2 + 2t + 1 \). Since now \( r/d \leq \sqrt{n} \), then \( \varepsilon(n) \geq r/n \) by [HH1], and it is easy to see that \( r/(nd) \geq \sqrt{n-1}/n \). So for \( n > 9 \) it is enough to check that \( \sigma(I)/(\sqrt{n-1}) \leq 3/2 \).

Now, \( \sigma(I) = t + 1 \) for the least \( t \) such that \( (t^2 + 2) \geq n \). Since for \( t = (\sqrt{8n + 1} - 3)/2 \) we have \( (t^2 + 2) \geq n \), we see \( \sigma(I) \leq (\sqrt{8n + 1} - 3)/2 + 2 \). It is not hard to check that \( (\sqrt{8n + 1} - 3)/2 + 2 \geq 3/2 \) for all \( n \geq 52 \).

We have left to deal with \( 10 \leq n \leq 51 \). For these few cases we can use the best lower bounds for \( \varepsilon(n) \) given in [HH1] (or the exact value if \( n \) is a square) instead of \( \sqrt{n-1}/n \), and we can use the exact value of \( \sigma(I) \) instead of \( (\sqrt{8n + 1} - 3)/2 + 2 \). Doing so we find that \( \sigma(I)/(\varepsilon(n)) < 3/2 \) for \( n = 11 \) (in this case even taking the conjectural value \( \varepsilon(n) = 1/\sqrt{n} \) gives only that \( \sigma(I)/(\varepsilon(n)) \leq 1.507) \), or \( n = 17, 22 \) or \( 37 \), in which case we have \( \varepsilon(n) \) being at least \( 4/17 \), \( 7/33 \) and \( 6/37 \), hence at least we obtain \( \sigma(I)/(\varepsilon(n)) \leq 3/2 \), but this suffices for the statement of the theorem (however, see the remark that follows).

For \( n = 11 \), argue as for \( n = 7 \). For \( n = 11 \), we have \( \omega(I) = 4 \), and \( \text{reg}(I) = 5 \), so \( r\omega(I) + 2(\text{reg}(I) - \omega(I)) = 4r + 2 \). Now \( \rho(S_{11}) \leq c \) if we pick \( c \) such that \( m/r \geq c \) implies \( \alpha(I^{(m)}) \geq 4r + 2 \). But \( \varepsilon(n) \geq 1/\sqrt{10} \) so \( \alpha(I^{(m)}) \geq m\sqrt{10} \geq rc\sqrt{10} \), and we just need \( c \) such that \( rc \sqrt{10} > 4r + 1 \) for \( r \geq 2 \). We see we need \( c > 4/\sqrt{10} + 1/(2\sqrt{10}) = (2\text{reg}(I) - 1)/(2\sqrt{n-1}) \) so \( c = 1.43 \) suffices; i.e., \( \rho(S_{11}) \leq 1.43 \).

Remark 4.3. In a subsequent paper, [BH1], we will compute \( \rho(S) \) for sets of points on irreducible plane conics. Our result for the case of \( 5 \) points on a smooth conic is \( \rho(S_5) = 6/5 \). Also, arguing in the case of \( n = 17, 22 \) and \( 37 \) generic points as we did for \( n = 11 \), we find, resp., that \( \omega(I) \) and \( \text{reg}(I) \) are \( 5, 6 \) and \( 8, 6, 7 \) and \( 9, \) and hence that \( (2\text{reg}(I) - 1)/(2\sqrt{n-1}) \) is \( 1.375, 1.418, 1.416 \), resp., so \( \rho(I) \) is, for example, at most \( 1.38, 1.42 \) and \( 1.42 \), resp. Thus in fact we can state a slightly stronger version of the preceding theorem: for a generic set \( S_n \) of \( n \) points of \( P^2 \), \( I^r \) contains \( I^{(m)} \) whenever \( m/r \geq 3/2 \) (rather than just \( m/r > 3/2 \)). (Alternatively, assuming characteristic \( 0 \), we can handle the cases \( n = 17, 22 \) and \( 37 \) simply by using a better estimate for \( \varepsilon(n) \): in characteristic \( 0 \), [HH1] shows \( \varepsilon(n) \) is at least \( 8/33, 42/197 \) and \( 12/73 \), resp.)

In fact, it may be possible that \( \rho(S) \leq \sqrt{2} \) whenever \( S \) is a generic finite set of points in \( P^2 \). [While this paper was under review we found that \( \rho(S_8) = 17/12 > \sqrt{2} \) [BH1], but we know no other cases for which \( \rho(S) > \sqrt{2} \).] In addition to Theorem 13.31 the following result gives some evidence for this possibility.

Proposition 4.4. Let \( S \) be a set of \( n = (d+2)(d+1)/2 + i \geq 10 \) generic points of \( P^2 \) and \((d+4)/2 \leq i \leq d+2 \). Then \( m/r \geq \sqrt{2} \) implies \( I^{(m)} \subseteq I^r \).

Proof. By Lemma 2.3.4 (Postulational Criterion 2), \( I^{(m)} \subseteq I^r \) if \( r\sigma(I) \leq \alpha(I^{(m)}) \), and hence if \( m/r \geq (\sigma(I))/(\varepsilon(S)) \), where \( \varepsilon(S) \geq c \). But here \( \sigma(I) = d+2 \) (since by our choice of \( i \) we have \( d^2/2 < n \leq (d+2)^2/3 \)) and, as in the proof of Theorem 4.2, we can take \( c = \sqrt{n-1}/n \) since \( n \geq 10 \). A little arithmetic using \( (d+4)/2 \leq i \) now shows that \( \sqrt{2} \geq (\sigma(I))/(\varepsilon(S)) \).}

Example 4.5. By the main theorems of [ELS] and [HH1], \( I^{(4)} \subseteq I^2 \) for \( I = I(S) \) for any finite subset \( S \subseteq P^2 \). Thus, in addition to asking, as Huneke did, if \( I^{(3)} \subseteq I^2 \),
one might also ask if \( I(4) \subseteq I^3 \) or if \( I(6) \subseteq I^4 \). We close by showing that the answer for the latter two is no.

In particular, let \( I = I(S) \) where \( S = S_2(2, s) \) is the set of \( n = \binom{s}{2} \) points of pairwise intersection of \( s \) general lines in \( \mathbb{P}^2 \). It is easy to check that \( \alpha(I(3)) = 2s - 1 \), since any form in \( I(3) \) of degree \( 2s - 2 \) must, by Bezout, vanish on each of the \( s \) lines, giving a form of degree \( s - 2 \) in \( I \), but \( \alpha(I) = s - 1 \), either by Bezout again or by Lemma 2.41. (Similarly, it follows that \( \alpha(I(m)) = \left( \frac{(m + 1)}{2} \right) s - 1 \) whenever \( m \) is odd.) Now by Lemma 2.33, using Lemma 2.12 it follows that \( I(3) \subseteq I^2 \) for all \( s \), and by Lemma 2.32 using Lemma 2.41 it follows that \( I^3 \) does not contain \( I(4) \) for \( s > 3 \) and that \( I^4 \) does not contain \( I(6) \) for \( s > 4 \).

References

[AV] A. Arsie and J. E. Vatne. A Note on Symbolic and Ordinary Powers of Homogeneous Ideals, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. Vol. II, 19-30 (2003) [http://www.uib.no/People/nmajv/03.pdf].

[AM] M. Atiyah and I. G. Macdonald. Introduction to Commutative Algebra, Addison-Wesley, Reading, MA, 1969, ix+128 pp.

[B] P. Biran. Constructing new ample divisors out of old ones, Duke Math. J. 98 (1999), no. 1, 113–135.

[BH] C. Bocci and B. Harbourne. The resurgence of ideals of points and the containment problem, to appear, Proc. Amer. Math. Soc.

[Ch1] M. Chardin. Regularity of ideals and their powers, Prépublication 364, Institut de Mathématiques de Jussieu, 2004.

[Ch2] M. Chardin. On the behavior of Castelnuovo-Mumford regularity with respect to some functors, preprint, 2007.

[CHHT] S. D. Cutkosky, H. T. Ha, H. Srinivasan and E. Theodorescu. Asymptotic behaviour of the length of local cohomology, Canad. J. Math. 57 (2005), no. 6, 1178–1192.

[D] J.P. Demailly. Singular Hermitian metrics on positive line bundles, Complex Algebraic Varieties (Bayreuth 1990) (K. Hulek et al., eds.), LNM, vol. 1507, Springer, 1992, pp. 87–104.

[ELS] L. Ein, R. Lazarsfeld and K. Smith. Uniform bounds and symbolic powers on smooth varieties, Invent. Math. 144 (2001), p. 241-252.

[EHU] D. Eisenbud, C. Huneke and B. Ulrich. The regularity of Tor and graded Betti numbers, Amer. J. Math. 128 (3), 2006, 573–605.

[GGP] A. V. Geramita, A. Gimigliano and Y. Pitteloud. Graded Betti numbers of some embedded rational n-folds, Math. Annalen 301 (1995), 363-380.

[GHM] A. V. Geramita, B. Harbourne and J. Migliore. Classifying Hilbert functions of fat point subschemes in \( \mathbb{P}^2 \), in preparation.

[GMS] A. V. Geramita, J. Migliore and L. Sabourin. On the first infinitesimal neighborhood of a linear configuration of points in \( \mathbb{P}^2 \), J. Algebra 298 (2006), no. 2, 563–611.

[GuH] E. Guardo and B. Harbourne. Resolutions of ideals of any six fat points in \( \mathbb{P}^2 \), J. Alg. 318 (2), 619–640 (2007).

[H1] B. Harbourne. Seshadri constants and very ample divisors on algebraic surfaces, J. Reine Angew. Math. 559 (2003) 115–122.

[H2] B. Harbourne. Complete linear systems on rational surfaces, Trans. Amer. Math. Soc. 289, 213–226 (1985).

[H3] B. Harbourne. An Algorithm for Fat Points on \( \mathbb{P}^2 \), Can. J. Math. 52 (2000), 123–140.

[H4] B. Harbourne. On Nagata’s Conjecture, J. Algebra 236 (2001), 692–702.

[HR1] B. Harbourne and J. Roé. Discrete Behavior of Seshadri Constants on Surfaces, Journal of Pure and Applied Algebra, 212 (2008), 616–627.

[HR2] B. Harbourne and J. Roé. Extendible Estimates of multipoint Seshadri Constants, preprint, [math.AG/0309064], 2003.

[HR3] B. Harbourne and J. Roé. Computing multi-point Seshadri constants on \( \mathbb{P}^2 \), to appear, Bulletin of the Belgian Mathematical Society - Simon Stevin.

[Ht] R. Hartshorne. Algebraic Geometry, Springer-Verlag, New York, 1977, xvi + 496.
A. Hirschowitz. *La méthode d’Horace pour l’interpolation à plusieurs variables*, Manus. Math. 50 (1985), 337–388.

M. Hochster. *Criteria for equality of ordinary and symbolic powers of primes*, Math. Z. 1973, 133, 53–65.

M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), no. 2, 349–369.

C. Huneke and M. Hochster. *Fine behavior of symbolic powers of ideals*, preprint, 2006.

V. Kodiyalam. *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc., Vol. 128, P. 407-411, 2000.

A. Li and I. Swanson. *Symbolic powers of radical ideals*, Rocky Mountain J. of Math. 36 (2006), 997–1009.

T. Mignon. *Systèmes de courbes planes à singularités imposées: le cas des multiplicités inférieures ou égales à quatre*, J. Pure Appl. Algebra 151 (2000), no. 2, 173–195.

M. Nagata. *On rational surfaces, II*, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. 33 (1960), 271–293.

T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek, and T. Szemberg. *A primer on Seshadri constants*, to appear in the AMS Contemporary Mathematics series volume “Interactions of Classical and Numerical Algebraic Geometry,” Proceedings of a conference in honor of A.J. Sommese, held at Notre Dame, May 22–24 2008.

J. Roë. *A relation between one-point and multi-point Seshadri constants*, J. Algebra 274, 643-651 (2004).

I. Swanson. *Linear equivalence of topologies*, Math. Zeitschrift, 234 (2000), 755–775.

T. Szemberg and H. Tutaj-Gasińska. *General blow ups of the projective plane*, Proc. Amer. Math. Soc. 130 (2002), no. 9, 2515–2524.

S. Takagi and K. Yoshihda. *Generalized test ideals and symbolic powers*, preprint, 2007, [math.AC/0701929](http://arxiv.org/abs/math.AC/0701929)

Z. Teitler. *On the intersection of the curves through a set of points in P^2*, to appear, Journal of Pure and Applied Algebra.

H. Tutaj-Gasińska. *A bound for Seshadri constants on P^2*, Math. Nachr. 257 (2003), no. 1, 108–116.

G. Xu. *Ample line bundles on smooth surfaces*, J. Reine Ang. Math. 469 (1995), 199–209.

Cristiano Bocci, Dipartimento di Scienze Matematiche e Informatiche "R. Magari", Università degli Studi di Siena, Pian dei mantellini, 44, 53100 Siena, Italy

E-mail address: [bocci24@unisi.it](mailto:bocci24@unisi.it)

Brian Harbourne, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130 USA

E-mail address: [bharbour@math.unl.edu](mailto:bharbour@math.unl.edu)