Large amplitude behavior of the bulk viscosity of dense matter

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We study the bulk viscosity of dense matter, taking into account non-linear effects which arise in the large amplitude “supra-thermal” region where the deviation $\mu_\Delta$ of the chemical potentials from chemical equilibrium fulfills $\mu_\Delta \gtrsim T$. This regime is relevant to unstable modes such as r-modes, which grow in amplitude until saturated by non-linear effects. We study the damping due to direct and modified Urca processes in hadronic matter, and due to nonleptonic weak interactions in strange quark matter. We give general results valid for an arbitrary equation of state of dense matter and find that the viscosity can be strongly enhanced by supra-thermal effects. Our study confirms previous results on quark matter and shows that the non-linear enhancement is even stronger in the case of hadronic matter. Our results can be applied to calculations of the r-mode-induced spin-down of fast-rotating neutron stars, where the spin-down time will depend on the saturation amplitude of the r-mode.

I. INTRODUCTION

Compact stars are the only known objects that contain equilibrated matter that is compressed beyond nuclear density, making them a valuable laboratory for the study of the structure of matter under extreme conditions. In addition to hadronic matter they may also contain new forms of matter that involve deconfined quarks [1–4]. In contrast to the static properties of compact stars which only depend on the equation of state of matter [5], dynamic properties also depend on the low energy degrees of freedom and thereby might be able to discriminate more efficiently between different forms of strongly interacting matter. One of the dynamic properties of dense matter is viscosity, which determines the damping of mechanical perturbations, and a particularly important application is to the damping of r-mode oscillations of compact stars [6–10], which, at sufficiently low viscosity and high rotation rate, are unstable and can cause rapid spin-down of the star [11] via gravitational radiation. Since the r-mode is unstable, its exponential growth must eventually be stopped by some non-linear mechanism. Finding the relevant mechanism is important because it determines the amplitude at which the r-mode saturates, and hence the rate at which it spins down the star. Previously suggested mechanisms include mode coupling and the transformation of the r-mode energy into differential rotation [12–15], and friction between different layers of the star such as “surface rubbing” at the crust [16]. Because of the complexity of the problem, these mechanisms have to rely on approximations that are not always well controlled.

In this paper we consider an alternative mechanism which does not involve additional physics, but is already present in a quasi-static hydrodynamic description. At low amplitudes the bulk viscosity is amplitude-independent, but since the r-mode is unstable its amplitude grows, and unless stopped by other mechanisms will quickly enter the “supra-thermal” regime where the bulk viscosity grows with amplitude, and may become large enough to stop the growth of the mode [17–18]. The supra-thermal regime is characterized by $\mu_\Delta \gtrsim T$, where $T$ is the temperature and $\mu_\Delta$ is the amplitude of the oscillations in the chemical potential of the quantity whose re-equilibration causes the viscous damping. We will study the microscopic part of the problem via a comprehensive analysis of the bulk viscosity of dense matter. We leave the astrophysical aspects for future work. Since the precise phase structure and the equation of state of matter at high density is still unknown, we keep the dependence on those parameters as explicit as possible, and as well as numeric results we provide analytic approximations which prove to be surprisingly accurate. This allows us to obtain general results for the bulk viscosity valid for many different phases of matter, and enables us to estimate the involved uncertainties. We study in detail the cases of equilibrated npe-matter and strange quark matter, and consider both modified and direct Urca processes in the hadronic case. Yet, our general expressions can be applied to other equations of state and entirely different forms of strongly interacting matter.

II. BULK VISCOSITY OF DENSE MATTER

The bulk viscosity of a given form of matter is a measure of the energy dissipated when it is subjected to an oscillating cycle of compression and rarefaction. Bulk viscosity is known to be the dominant source for the damping of r-mode oscillations at high temperatures and low amplitudes. Consequently, bulk viscosity has been computed for many forms of dense matter [17–19,37]. Nearly all these studies restricted themselves to the sub-thermal regime $\mu_\Delta \ll 2\pi T$. However, as noted above, the astrophysically interesting scenario is one where the r-mode is unstable, and so the supra-thermal bulk viscosity may well become relevant. The influence of the supra-thermal regime has been studied numerically in [17] for the case of strange quark matter. This analysis showed that for large amplitude oscillations the viscosity can increase by orders of magnitude. Yet, because of their qualitatively different low energy degrees of freedom and weak-interaction equilibration channels, other forms of matter could show different behavior.
In the following we will derive the non-linear equations that determine the bulk viscosity due to weak interactions that interconvert the fermionic species that are present. We will then solve it for arbitrary amplitudes. We focus on weak interactions because their equilibration rate is comparable to typical compact star oscillation frequencies: strong interactions make a negligible contribution at these frequencies because their equilibration rate is much too fast.

The bulk viscosity of a given form of matter is defined by the response of the system to an oscillating compression and rarefaction. This corresponds to an oscillation in the densities of all exactly conserved quantities. We will assume that there is at least one such quantity whose density we call \( n^*_s \). In compact stars it is typically the baryon number density. We will study the energy dissipated as a result of a small harmonic oscillation \( \delta n^*_s \) around its equilibrium value \( \bar{n}^*_s \).

\[
n^*_s(\vec{r}, t) = \bar{n}^*_s(\vec{r}) + \delta n^*_s(\vec{r}, t) = \bar{n}^*_s(\vec{r}) + \Delta n^*_s(\vec{r}) \sin \left( \frac{2\pi t}{\tau} \right),
\]

where the amplitude of the oscillation is \( \Delta n^*_s \), and we assume \( \Delta n^*_s \ll \bar{n}^*_s \). The dissipated energy per volume due to the oscillation is given by

\[
\frac{dc}{dt} = -\zeta \left( \nabla \cdot \vec{v} \right)^2
\]

where \( \vec{v} \) is the local velocity of the fluid of the conserved quantity and the continuity equation for its particle number \( n^*_s \) reads

\[
\frac{\partial n^*_s}{\partial t} + \nabla \cdot (n^*_s \vec{v}) = 0.
\]

In the case that density varies slowly enough so that density gradients can be neglected, and using \( \Delta n^*_s/\bar{n}^*_s \ll 1 \), averaging over a whole oscillation period \( \tau = 2\pi/\omega \) gives the bulk viscosity as

\[
\zeta \approx -\frac{2}{\omega^2} \left\langle \frac{dc}{dt} \right\rangle \frac{\bar{n}^*_s^2}{(\Delta n^*_s)^2}.
\]

Using the relationship between fluctuations in volume and fluctuations of a conserved quantity,

\[
\frac{dn^*_s}{n^*_s} = -\frac{dV}{V}
\]

and the mechanical work done by a volume change

\[
dc = -\frac{p}{V} dV
\]

we can express the dissipated energy per volume averaged over one time period in terms of the induced pressure oscillation

\[
\left\langle \frac{dc}{dt} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{p}{n^*_s} \frac{dn^*_s}{dt} dt.
\]

To calculate the bulk viscosity we must calculate \( p(t) \). We will assume that the bulk viscosity arises from beta-equilibration of fermionic species. We further assume that, in the absence of weak interactions, there would be \( s \) conserved species, and that there is a single channel of weak interactions that can perform interconversion of species, leaving \( s - 1 \) exactly conserved fermion-number charges. We defer discussion of the general situation of several coupled channels to future studies. Subtracting the chemical potentials of the final state particles in the relevant weak channel from those of the initial state particles, we obtain the difference

\[
\mu^*_\Delta \equiv \sum \mu_i - \sum \mu_f.
\]

which is the quantity that is driven out of equilibrium by the driving density fluctuation, and whose re-equilibration leads to bulk viscosity. The quasi-equilibrium state can generally be described in terms of the driving density \( n^*_s \) and the ratio \( x \equiv n^*_1/n^*_s \) where \( n^*_1 \) is the density of one of the particle species whose number is changed by the equilibration process. For small oscillation amplitudes \( \Delta n^*_s/\bar{n}^*_s \ll 1 \) the pressure can then be expanded around its equilibrium value \( \bar{p} = p(\bar{n}^*_s) \)

\[
p = \bar{p} + \frac{\partial p}{\partial n^*_s} \bigg|_x \delta n^*_s + \frac{\partial p}{\partial x} \bigg|_{n^*_s} \delta x,
\]

where \( \delta x \) is the deviation of \( x \) from its beta-equilibrium value. The \( t \)-independent part \( \bar{p} \) as well as the term proportional to the driving density fluctuation \( \delta n^*_s \) do not contribute to the viscosity integral. The remaining susceptibility can be rewritten

\[
\frac{\partial p}{\partial x} \bigg|_{n^*_s} = \bar{n}^*_s^2 \frac{\partial \mu^*_\Delta}{\partial n^*_s} \bigg|_x.
\]

Because of weak interactions, \( x \) depends on time,

\[
\delta x(t) = \int_0^t \frac{dx}{dt} dt'.
\]

From Eqs. \( 1, 7, 9, 10, 11 \),

\[
\zeta = -\frac{1}{\pi} \frac{\bar{n}^*_s^3}{\Delta n^*_s} \int_0^\tau \frac{\partial \mu^*_\Delta}{\partial n^*_s} \int_0^t \frac{dx}{dt} dt' \cos (\omega t) dt.
\]

1 We do not study bulk viscosity arising from the interconversion of bosons, such as the light mesons that occur in color-flavor-locked phases.

2 In this counting we exclude fermions like neutrinos, which escape from compact stars and so are effectively not conserved.
We want to point out already at this point that, in contrast to the harmonic driving density oscillation $\delta n_s$ with amplitude $\Delta n_s$, the induced chemical potential fluctuation $\delta \mu_\Delta$ around the vanishing equilibrium value can have a more complicated anharmonic form.

The fluctuations of the density ratio can be obtained from an analogous expansion of the chemical potential fluctuation

$$\delta \mu_\Delta = \frac{\partial \mu_\Delta}{\partial n_s} \bigg|_x \delta n_s + \frac{\partial \mu_\Delta}{\partial x} \bigg|_{n_s} \delta x$$

(13)

which yields a linear equation relating $\mu_\Delta$ and $\delta x$

$$\frac{d\mu_\Delta}{dt} = C \omega \frac{\Delta n_s}{n_s} \cos (\omega t) + B \tilde{n}_s \frac{dx}{dt},$$

(14)

with the susceptibilities

$$C \equiv \tilde{n}_s \frac{\partial \mu_\Delta}{\partial n_s} \bigg|_x, \quad B = \frac{1}{\tilde{n}_s} \frac{\partial \mu_\Delta}{\partial x} \bigg|_{n_s}.$$  

(15)

Using (14) we obtain the bulk viscosity in terms of the chemical potential fluctuation,

$$\zeta = - \frac{1}{\pi} \frac{\tilde{n}_s}{\Delta n_s} \frac{C}{B} \int_0^\pi \mu_\Delta(t) \cos(\omega t)dt.$$  

(16)

In terms of a Fourier expansion of the periodic chemical potential fluctuation

$$\mu_\Delta(t) = \sum_{n=1}^\infty (a_n \sin(n\omega t) + b_n \cos(n\omega t))$$  

(17)

we see that the only component of $\mu_\Delta(t)$ that contributes to the viscosity is the component of the fundamental Fourier mode that lags the driving volume oscillation by a phase of $\pi/2$. This suggests that a truncated Fourier ansatz may provide a reliable approximation for the viscosity; we will explore this idea in Sec. III.

To obtain the temperature and amplitude dependence of the bulk viscosity, we now discuss the general form of the beta equilibration rate. We define the net equilibration rate

$$\Gamma^{(\pm)} = \Gamma^{(-)} - \Gamma^{(+)} = \tilde{n}_s \frac{\partial x}{\partial t},$$  

(18)

where we use the convention that $\Gamma^{(-)}$ is the rate for the process where $n_1$ is decreased, and $\Gamma^{(+)}$ is the rate for the inverse process. We study equilibration processes where the net rate takes the general form

$$\Gamma^{(\pm)} = - \tilde{T} T^\kappa \mu_\Delta \left( 1 + \sum_{j=1}^N \chi_j \left( \frac{\mu_\Delta}{T^2} \right)^j \right).$$  

(19)

where $N$ is the highest power of $\mu_\Delta$ arising in the rate. In terms of dimensionless variables

$$\varphi \equiv \omega t, \quad A(\varphi) \equiv \frac{\mu_\Delta(t)}{T}$$  

(20)

the differential equation for the chemical fluctuation eq. (14) can be written as

$$\frac{dA}{d\varphi} = d \cos (\varphi) - fA \left( 1 + \sum_{j=1}^N \chi_j A^{2j} \right),$$  

(21)

with the prefactors of the driving and feedback term given by

$$d = \frac{C \Delta n_s}{T \tilde{n}_s}, \quad f = \frac{B \tilde{T} T^\kappa}{\omega}.$$  

(22)

Note that the feedback term involves both linear and non-linear parts which are controlled by a single parameter $f$ and that its particular form is determined by the constants $\chi_j$ which parametrize the particular weak rate. The viscosity is then finally given by

$$\zeta = \frac{TC}{\pi \omega B} \frac{\tilde{n}_s}{\Delta n_s} \int_0^{2\pi} A(\varphi, d, f) \cos (\varphi) d\varphi,$$  

(23)

where $A$ is the periodic solution to eq. (21).

Before we discuss the general solution of these equations in detail, let us consider its asymptotic limits.

**Sub-thermal limit**

In the limit $\mu_\Delta \ll T$ corresponding to $A \ll 1$ the non-linear terms can be neglected

$$\left( \frac{d}{d\varphi} - f \right) A = d \cos (\varphi).$$  

(24)

Since this equation is linear, the fluctuation $A$ must be harmonic and only the $n = 1$ term in the Fourier ansatz eq. (17) is present. Inserting this ansatz yields the solution for the required Fourier coefficient

$$b_1 = - \frac{df}{1 + f^2}.$$  

(25)

Inserted in eq. (23) this yields the general sub-thermal result, denoted by a superscript $\prec$, for the bulk viscosity of an arbitrary form of matter which shows the characteristic resonant form

$$\zeta^\prec = \frac{C^2 \tilde{T} T^\kappa}{\omega^2 + (B \tilde{T} T^\kappa)^2} \zeta^{\prec\max} = \frac{2 \omega B \tilde{T} T^\kappa}{\omega^2 + (B \tilde{T} T^\kappa)^2}.$$  

(26)

As long as the combination of susceptibilities $C^2 / B$ does not vary too quickly with temperature, the sub-thermal viscosity has a maximum

$$\zeta^{\prec\max} \geq \frac{C^2}{2 \omega B} \text{ at } T_{\max} = \left( \frac{\omega}{\tilde{G} B} \right)^{1/2}.$$  

(27)

**Supra-Thermal limit**

The opposite, supra-thermal limit, $\mu_\Delta \gg T$, corresponds to $A \gg 1$. Since the feedback term in the differential equation is restraining, this limit can only be reached in
the limit of large driving terms $d \gg 1$. In this case only
the largest power of $A$ is relevant and eq. (21) reduces to

$$0 = d \cos(\varphi) - \chi_N f A^{2N+1} \Rightarrow \mathcal{A} \sim \left( \frac{\Delta n_\ast}{n_\ast} \right)^{\frac{2N}{\pi^2}}. \quad (28)$$

The viscosity scales correspondingly in this limit as

$$\zeta \sim \left( \frac{\Delta n_\ast}{n_\ast} \right)^{-\frac{2N}{\pi^2}} \quad (29)$$

and decreases at very large amplitudes.

**General solution**

After these limiting cases we will discuss the qualitative aspects of the general solution eq. (23). Due to the non-linearity of the differential equation (21) this requires a numeric solution. Yet, for each weak channel, characterized by the constants $\chi_N$, such a solution as a function of the two independent variables $d$ and $f$ has to be performed only once and is then valid for any equation of state and includes the complete dependence on the underlying parameters in eq. (22). The qualitative behavior of the solution as a function of the two independent parameters $d$ and $f$ is shown for hadronic matter with modified Urca process in fig. 1. Turning up the feedback term at fixed driving term increases the phase shift of the waveform from 0 to $\pi/2$ and at the same time decreases the amplitude, but the waveform stays harmonic. In contrast, turning up the driving term at fixed feedback increases the amplitude towards the supra-thermal regime $A > 1$ and the waveform becomes increasingly anharmonic. Recall, however, that only the phase shifted harmonic component in the Fourier expansion contributes to the viscosity eq. (23).

Motivated by the above expression eq. (27) for the maximum in the sub-thermal regime the general result can be written in the form

$$\zeta = \zeta_{\text{max}} \mathcal{I}(d, f) = \frac{C^2}{2\omega B} \mathcal{I}(d, f) \quad (30)$$

where the dimensionless function $\mathcal{I}$ that includes the non-trivial parameter dependence is given by

$$\mathcal{I}(d, f) \equiv \frac{2}{\pi d} \int_0^{2\pi} A(\varphi; d, f) \cos(\varphi) d\varphi \quad (31)$$

The expression $\mathcal{I}$ can then be tabulated as a function of the independent parameters $d$ and $f$. We believe that presenting our results in this form will make them easier to apply to calculations of r-mode damping, where the complete parameter dependence is required. The computation of the damping time of the mode involves an integral over the star of an expression that involves the bulk viscosity (e.g., [11]) which varies throughout the star because of its dependence on the amplitude of the mode and the susceptibilities, both of which are position-dependent. The function $\mathcal{I}(d, f)$ encapsulates the dependence of the bulk viscosity on the position-dependent parameters, allowing straightforward evaluation of the damping time integral.

The function $\mathcal{I}(d, f)$ is shown in fig. 2 for two examples: a model of strange quark matter and a model of hadronic matter; details of the models are discussed below. We see that the function has the same qualitative form in both cases. It has a global maximum value of 1, reached in the sub-thermal limit and a line of local maxima along a parabola in the $d$-$f$-plane. Thus the maximum value (27) of the sub-thermal viscosity is also the maximum in the general case and depends only on the equation of state, the density and the frequency but is independent of the weak rate. The weak rate influences, however, at what temperatures and amplitudes the local maxima are reached. As seen from eq. (22), the parameter $d$ is directly proportional to the amplitude, so that at moderate feedback an amplitude increase does initially not affect the viscosity at all, corresponding to the amplitude-independent sub-thermal result. But once the amplitude becomes sufficiently large we enter the supra-thermal regime and the viscosity increases strongly by orders of magnitude until it reaches its maximum. The size of the amplitude $A$ is denoted in fig. 2 by the darkness of shading of the surface. This qualitative behavior has already been observed in [17] but we find that at even higher amplitudes the viscosity decreases again according to the limiting behavior eq. (29). In contrast, at large feedback the viscosity becomes basically amplitude independent over the relevant parameter range as described by the sub-thermal result.

Let us now discuss the dependence of the viscosity on the underlying parameters in eq. (22). An amplitude increase (keeping all other variables fixed) results in a linear increase in the variable $d$ as shown by the dashed (blue online) curves in fig. 2. An increase in temperature changes the viscosity along a line shown by the solid (red online) curves. In order to assess the frequency and amplitude dependence we must take into account the prefactor in eq. (30). This prefactor, given by the maximum viscosity in the subthermal regime, is shown in fig. 3 for the hadronic model of fig. 2(a). It exhibits a monotonic increase with density and inverse angular frequency. An increase in angular frequency therefore changes the viscosity via a change of $\mathcal{I}$ towards the negative $f$-direction and furthermore via the overall prefactor featuring an additional $1/\omega$ dependence. A density increase has an even more indirect impact since it depends on the detailed form of the susceptibilities $C(\bar{n}_\ast)$ and $B(\bar{n}_\ast)$ which likewise arise in the prefactors of the viscosity. These dependencies will be studied in more detail below.

**III. STRANGE QUARK MATTER**

**A. General features**

It has been suggested that the true ground state of matter at high densities may be strange quark matter
Figure 1: Waveform $A(\phi) = \mu \Delta (\omega t)/T$ for different values of the two independent parameters. We show only the positive half-wave, on a logarithmic scale. Left panel: Fixed driving term $d = 1$, with varying feedback term $f = 0.001, 0.01, \ldots, 1000$. As $f$ rises, the phase lag increases from zero towards $\pi/2$, but at the same time the amplitude decreases as $1/f$. Right panel: Fixed feedback term $f = 1$, with varying driving term $d = 0.001, 0.01, \ldots, 1000$. As $d$ rises, the phase lag rises from $\pi/4$ to $\pi/2$ and the waveform becomes increasingly anharmonic, approaching a square wave in the limit.

(a) Hadronic matter, modified Urca process

(b) Quark matter, non-leptonic process

Figure 2: The function $I$ arising in the general solution eq. (23) for two models of dense matter. Left panel: hadronic matter with modified Urca equilibration. Right panel: quark matter with the non-leptonic equilibration process eq. (32). The function has a global maximum of 1 reached asymptotically for $d \to 0, f = 1$ and a line of slowly decreasing local maxima along a parabola in the $d$-$f$ plane. The shading of the surface denotes the size of the amplitude $A$ so that dark shades of grey represent the supra-thermal regime. Eq. (22) relates $d$ and $f$ to underlying physical parameters such as temperature $T$ and amplitude. An amplitude increase (keeping all other variables fixed) results in a linear increase in the variable $d$ as shown by the dashed (blue online) curves. An increase in temperature changes the viscosity along a line shown by the solid (red online) curves.
Since in this case only cubic non-linearities arise it is possible to obtain an approximate analytic solution to the non-linear equation \( \frac{d}{dt} \). Taking into account the above observation that only the leading Fourier coefficient in the expansion of the chemical potential oscillation contributes to the bulk viscosity it is natural to seek such a solution via a Fourier ansatz up to a given order \( O \)

\[
\mathcal{A}(t) = \sum_{n=-O}^{O} \tilde{A}_n e^{in\omega t} \tag{35}
\]

where the complex form is used to simplify the computation. In principle, the amplitude of the leading Fourier mode will depend on the truncation order \( O \), but analytically solving eq. \( \frac{d}{dt} \) via a computer algebra system to order \( O = 2 \) we find that the coefficients \( \tilde{A}_{\pm 2} \) vanish identically. Correspondingly anharmonicities do not directly contribute to the viscosity and are even absent to next to leading order so that we can restrict our analysis to the leading order \( O = 1 \). Although such a parameterization neglects any anharmonicities it properly captures both the amplitude and the phase shift of the oscillation even in the large amplitude regime. Due to the reality of the solution there is only one independent complex Fourier exponent determined by a non-linear algebraic equation.

In the case of quark matter where \( \chi_i = 0 \) for \( i > 1 \) and only the leading non-linear term \( \chi (\mu K / T)^3 \) is present an analytic solution of this equation is possible. In this case we can decompose the amplitude into real and imaginary parts \( \mathcal{A}_i = A_R + i A_I \), obeying coupled equations

\[
f A_R (1 + 3 \chi (A_{R}^{2} + A_{I}^{2})) + A_I = -d/2 \tag{36}
\]

\[
f A_I (1 + 3 \chi (A_{R}^{2} + A_{I}^{2})) - A_R = 0 \tag{37}
\]

Note that an analytic solution is only possible because the quark matter equations are cubic; other forms of matter with higher order non-linearities in eq. \( \frac{d}{dt} \) require a numeric solution. The above system of algebraic equations has a lengthy analytic solution which we refrain from giving here because, as we will see below, it can be very accurately approximated by a much simpler expression \( \Gamma_{\chi} \) constructed from a combination of the solutions in the sub-thermal and supra-thermal regimes. Therefore we now concentrate on the supra-thermal case, denoted by the index \( > \), where the temperature-dependent term can be neglected,

\[
\mathcal{A}^2 (\phi) = 2 A_R \cos (\phi) - 2 A_I \sin (\phi)
\]

\[
= \frac{3d}{2} \left( \frac{(q(z)^2 - 1)^2}{\sqrt{3}z q(z)^2} \cos (\phi) + \frac{q(z)^2 - 1}{z q(z)} \sin (\phi) \right), \tag{38}
\]

where the dimensionless quantity \( z \) is defined by

\[
z = \frac{9\sqrt{3}}{8} \chi d^2 f = \frac{9\sqrt{3} \chi}{8} \tilde{C}^2 B T^{\kappa-2} \left( \frac{\Delta n_s}{n_s} \right)^2 \tag{39}
\]
and

\[ q(z) = \left( \sqrt{z^2 + 1} - z \right)^{\frac{1}{3}}. \]  

(40)

Eq. (23) then yields the approximate analytic result for the bulk viscosity in the supra-thermal regime

\[ \zeta^> \approx \frac{2}{3\sqrt{3}} \frac{C^2}{B\omega} h\left( \frac{9\sqrt{3}\chi \tilde{\Gamma}C^2B}{8} T^{\kappa-2} \left( \frac{\Delta n_\kappa}{\bar{n}_\kappa} \right)^2 \right) \]  

(41)

in terms of the dimensionless function

\[ h(z) = \frac{9}{4z} \left( \left( \sqrt{z^2+1} - z \right)^{\frac{2}{3}} + \left( \sqrt{z^2+1} + z \right)^{\frac{2}{3}} - 2 \right). \]  

(42)

This function has a maximum at \( z_{max} = 3\sqrt{3} \). Since \( h(z_{max}) = 3\sqrt{3}/4 \), the corresponding maximum value of the viscosity is

\[ \zeta^>_{max} = \frac{2}{3\sqrt{3}} \frac{C^2}{B\omega} h(z_{max}) = \frac{C^2}{2B\omega} \]  

(43)

which strikingly is the same expression as in the sub-thermal limit eq. (27). Correspondingly the bulk viscosity has a universal upper bound \( \zeta_{max} \) that is independent of the particular weak damping process. It is directly proportional to the oscillation period with a coefficient that only depends on the response of the strongly interacting matter. However, the corresponding temperature limits, allows us to give a simple parameterization of the full function for all temperatures and amplitudes. We construct a weighted sum of the analytic results in the sub-thermal eq. (26) and the supra-thermal regime eq. (41),

\[ \zeta_{par} \approx \zeta^< + \theta(T_{max} - T) \frac{\zeta_{max} - \zeta^< \zeta^>}{\zeta_{max}} \]  

(45)

Studies of the damping of compact star oscillations previously took into account only the first, sub-thermal term in the parameterization eq. (45). The simple analytic form allows one to conveniently extend these studies in order to include large amplitude effects encoded in the second term. The small deviations of the simplified parameterization eq. (45) from the exact value of the bulk viscosity are negligible compared to the considerable uncertainties inherent in such a damping analysis. Evaluation of this expression requires knowledge of the susceptibilities \( B \) and \( C \) that depend on the equation of state.

| Matter/Channel            | \( \Gamma \) [MeV\(^{(3-\kappa)}\)] | \( \kappa \) | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) |
|--------------------------|-----------------------------------|------------|------------|------------|------------|
| quark non-leptonic       | \( 6.59 \times 10^{-12} \left( \frac{\mu_q}{300 \text{ MeV}} \right)^5 \) | 2          | \( \frac{1}{4\pi^2} \) | 0          | 0          |
| hadronic direct Urca     | \( 5.24 \times 10^{-15} \left( \frac{x n}{n_0} \right)^{\frac{1}{3}} \) | 4          | \( \frac{10}{17\pi^2} \) | \( \frac{1}{17\pi^2} \) | 0          |
| hadronic modified Urca   | \( 4.68 \times 10^{-19} \left( \frac{x n}{n_0} \right)^{\frac{1}{3}} \) | 6          | \( \frac{189}{367\pi^2} \) | \( \frac{21}{367\pi^2} \) | \( \frac{3}{1835\pi^6} \) |

Table I: Weak interaction parameters describing the considered damping process. Here \( \mu_q \) is the quark chemical potential, \( n \) is the baryon density, \( n_0 \) nuclear saturation density and \( x \) the proton fraction.

\[ \left( \zeta_{n_\kappa}/\bar{n}_\kappa \right)_{max} = \sqrt{\frac{8\omega}{3\chi B T^{\kappa-2} C^2 B}} \]  

(44)

at which this maximum is reached both depend on the weak rate. Knowing the upper bound \( \zeta_{max} \) and the functional behavior in the extreme sub-thermal and supra-thermal limits, allows us to give a simple parameterization of the full function for all temperatures and amplitudes. We construct a weighted sum of the analytic results in the sub-thermal eq. (26) and the supra-thermal regime eq. (41),
C. Models of quark matter

We now apply the results derived above to some simple models of quark matter. We start with the simplest model, free quarks in a “confining bag”. We will call this a “quark gas” (QG). We consider a 3-flavor quark and electron gas, with massless electron, up and down quarks and strange quark of mass $m_s$ with pressure

$$p_{\text{QG}} = \frac{1}{4\pi^2} \left( \mu_d^4 + \mu_u^4 + \mu_s^4 \right) - \frac{3}{2} m_s^2 \mu_s \mu_F - \frac{3}{2} m_s^4 \log \left( \frac{\mu_s + \mu_F}{m_s} \right) - B + \frac{\mu_s^4}{12\pi^2}$$

(46)

where the strange quark Fermi momentum is given by $p_F^2 = \mu_s^2 - m_s^2$. Here $B$ is the phenomenological bag constant that is important for the equilibrium composition of a strange star, but does not affect transport properties like the bulk viscosity studied in this work. The equilibrium state is determined from eq. (46) by taking into account charge neutrality and weak equilibrium with respect to both the explicitly considered non-leptonic channel as well as the quark Urca channel.

In quark matter there are multiple channels for beta equilibration: as well as the nonleptonic channel there are Urca channels which convert $d$ or $s$ quarks in to $u$ quarks and electrons, and emit neutrinos. However, at temperatures and oscillation frequencies of interest for compact star physics the Urca rates are much slower, and their contribution to the bulk viscosity is heavily suppressed. This means that the fractions $x_u$ and $x_s$ remain constant during the oscillation. The required susceptibilities then are given by

$$C_q = \bar{n} \left( \frac{\partial \mu_u}{\partial \bar{n}} - \frac{\partial \mu_d}{\partial \bar{n}} \right)_{\bar{x}_a, \bar{x}_a, \bar{x}_u}, \quad (47)$$

$$B_q = \frac{1}{\bar{n}} \left( \frac{\partial \mu_u}{\partial \bar{x}_u} - \frac{\partial \mu_d}{\partial \bar{x}_d} \right)_{\bar{n}, \bar{x}_u, \bar{x}_u}. \quad (48)$$

Taking into account charge neutrality, the above equation of state yields to leading order in $m_s/\mu_q$ the susceptibilities given in table (for the case $c = 0$).

According to eq. (43) the maximum viscosity of a quark gas is given by

$$\zeta_{\text{max}} \approx \frac{m_s^4}{12\pi^2 \omega}$$

(49)

which depends on density only through possible density-dependence of the strange quark mass.

In Fig. 4 we give a comparison between the parameterization eq. (45) and the full numeric solution. We show the amplitude dependence of the viscosity of strange quark matter for a range of temperatures. These results are analogous to those given by Madsen in his initial analysis of supra-thermal effects [17]. The analytic solution features the qualitative form that has been observed for the general result in fig. 2 and shows a striking agreement with the full solution in the physically relevant region of amplitudes below the maximum. Note that for temperatures around $T_{\text{max}}$, the parametrization eq. (45) overestimates the viscosity for amplitudes above $\Delta n/\bar{n}_{\text{max}}$, as can be seen for the $T = 10^9$ K curve in fig. 4. However, if such amplitudes are reached then supra-thermal bulk viscosity is overwhelmed, and other physics will have to be invoked to stop the growth of the mode.

We now examine the sensitivity of our results to uncertainties in the quark matter equation of state. We use an extension of the phenomenological parameterization proposed in [40] that allows us to study the behavior of the equation of state around chemical equilibrium. Expanding the ideal gas pressure to quartic order in $m_s$, the $m_s$-independent quartic terms in the individual quark are modified

$$p_{\text{par}} = \frac{1 - c}{4\pi^2} \left( \mu_d^4 + \mu_u^4 + \mu_s^4 \right) - \frac{3\mu_s^2}{4\pi^2}$$

$$+ \frac{3}{32\pi^2} \left[ 3 \log(2m_s/m_u) - 5 \right] - B + \frac{\mu_s^4}{12\pi^2}$$

(50)
where \( c \) is a new parameter which incorporates some effects of strong interactions between the quarks and \( m_s \) can parametrize here, in addition to corrections arising from the strange quark mass, also other interaction effects, like the pairing gap in color superconducting matter [40].

The bulk viscosity is sensitive (via the susceptibilities) to the parameters \( c \) and \( m_s \), but not to the bag constant. We show in fig. 5 the effect on the bulk viscosity of varying \( c \) and \( m_s \) within their expected range of values, at twice nuclear saturation density and a temperature \( T = 10^8 \) K. We calculate the bulk viscosity for an angular frequency of the oscillation of \( \omega = 8.4 \) kHz (corresponding to the r-mode of a pulsar with a period of 1 ms). We find that the uncertainty amounts to more than an order of magnitude. In contrast to the equilibrium composition of strange stars which proved to be strongly dependent on the parameter \( c \) [40], in the present case the effective strange quark mass has a larger impact.

Finally we show in fig. 6 the dependence of the viscosity of a quark gas on the density of the matter and the frequency of the oscillation. The density dependence is most pronounced in the supra-thermal regime and becomes basically irrelevant in the supra-thermal regime, in accordance with the density-independence of the maximum of the viscosity eq. (49). Further, we see that the viscosity increases strongly with frequency, according to the \( 1/\omega \)-dependence of the maximum eq. (49) which arises as a prefactor in eq. (45). Therefore, the results for a millisecond-pulsar given here in all other figures present a lower limit for the viscosity, whereas the damping of slower rotating stars is much faster.

### IV. HADRONIC MATTER

#### A. General features

The bulk viscosity has been calculated for various phases of nuclear matter (unpaired, superfluid, kaon-condensed etc) with flavor equilibration via either direct or modified Urca processes [19,23]. The lepton contribution has recently been calculated [41], and hyperonic matter has also been studied [25–28]. We concentrate on the simplest case of non-superfluid hadronic npe matter. We note however, that the generic properties of our results also apply to more complicated forms of matter like hyperonic and/or superfluid nuclear matter. In the case of hadronic matter we assume that weak equilibration

| \( B \) | \( C \) |
| --- | --- |
| quark matter (\( c=0 \)) | \[ \frac{2\pi^2}{3(1-c)\mu_q^2}\left(1+\frac{m_s^2}{12(1-c)\mu_q^2}\right) - \frac{m_s^2}{3(1-c)\mu_q} \] |
| hadronic matter | \[ \frac{8S}{n} + \frac{\pi^2}{(4(1-2x)S)^2} \left(4(1-2x)\left(\frac{\partial S}{n} - \frac{S}{3}\right) \right) \] |
| free hadron gas | \[ \frac{4m_s^2}{3(3\pi^2)^{1/3} n^{2/3}} \left(\frac{3\pi^2 n^{2/3}}{6m_s}\right) \] |

Table II: Strong interaction parameters describing the response of various models of dense matter. In the case of hadronic matter with baryon density \( n \) a quadratic ansatz in the proton fraction \( x \) parameterized by the symmetry energy \( S \) eq. (57) is employed. The expressions for a free hadron gas are given to leading order in \( n/m_s^3 \), and for quark matter with quark chemical potential \( \mu_q \) using eq. (50) to next to leading order in \( m_s/\mu_q \). The parameter \( c \) takes into account interaction effects within the employed quark matter model and vanishes for an ideal quark gas.

Figure 5: The dependence of the viscosity on parameters of the equation of state of strange quark matter using the simple parameterization eq. (50). We show the amplitude dependence at \( T = 10^8 \) K for \( \omega = 8.4 \) kHz and \( \tilde{n} = 2n_0 \). Dashed curves are for \( c = 0 \), solid curves are for \( c = 0.3 \). We show \( m_s = 100 \) MeV (lowest two curves, magenta online), \( m_s = 150 \) MeV (middle two curves, blue online) and \( m_s = 200 \) MeV (highest two curves, cyan online).
hadronic matter in \([19, 42, 43]\) in the standard case that isospin.

where the notation reflects that the equilibrating quantity is isospin.

\[ \mu_I \equiv \mu_n - \mu_p - \mu_e \]  

(52)

where the notation reflects that the equilibrating quantity in this case is isospin.

Taking into account the effect of supra-thermal oscillation amplitudes requires the non-linear \(\mu_I\)-corrections to the corresponding rates. These have been given for hadronic matter in \([19, 42, 43]\) in the standard case that only modified Urca processes are allowed.

\[ \Gamma_{hm}^{(\pm)} \mu_I = -3.5 \times 10^{13} \text{ergs cm}^{-3} \left( \frac{x_n}{n_0} \right)^{\frac{3}{2}} \frac{T^8}{11513} \]  

(53)

\[ \left( 14680 \frac{\mu_I^2}{\pi^2 T^2} + 7560 \frac{\mu_I^4}{\pi^4 T^4} + 840 \frac{\mu_I^6}{\pi^6 T^6} + 24 \frac{\mu_I^8}{\pi^8 T^8} \right) \]

and in the enhanced case when direct Urca processes dominate \([42, 43]\)

\[ \Gamma_{hd}^{(\pm)} \mu_I = -4.3 \times 10^{21} \text{ergs cm}^{-3} \left( \frac{x_n}{n_0} \right)^{\frac{3}{2}} \frac{T^6}{457} \]  

(54)

\[ \left( 714 \frac{\mu_I^2}{\pi^2 T^2} + 420 \frac{\mu_I^4}{\pi^4 T^4} + 42 \frac{\mu_I^6}{\pi^6 T^6} \right) \]

where \(T_8\) is the temperature in units of \(10^8\) K. Here we use the expressions given in \([12]\), but we note that the hadronic rates depend on model assumptions for the behavior of the strong interaction at high density (see \([43, 44]\)) and thereby involve uncertainties. These expressions yield the parameter values given in table \(1\). There are major differences between these hadronic rates and the corresponding one for strange quark matter. In quark matter, non-leptonic processes are naturally allowed and only particles that have a Fermi surface (quarks in this case) are involved. In contrast in hadronic matter such processes are absent (unless hyperons are present) and equilibration must proceed via semi-leptonic processes involving particles with no Fermi surface (neutrinos in this case) giving a much stronger temperature dependence. As noted before the simple analytic approximation suitable for strange quark matter is not applicable here. Nevertheless we will see that many qualitative aspects of that solution obtain in the general case. We note that although the prefactors of the non-linear terms decrease strongly as the power of \(\mu_I/T\) rises, it is not sufficient to neglect them since they enter the non-linear differential equation \([21]\) where they dominate at sufficiently large amplitudes.

According to eq. \((15)\), the susceptibilities for hadronic matter are

\[ C_h = \bar{n} \left( \frac{\partial \mu_n}{\partial n} \right)_{x_n} - \left( \frac{\partial \mu_p}{\partial n} \right)_{x_n} - \left( \frac{\partial \mu_e}{\partial n} \right)_{x_n} \]  

(55)

\[ B_h = \bar{n} \left( \frac{\partial n}{\partial x_n} \right)_{x_n} - \left( \frac{\partial \mu_p}{\partial x_n} \right)_{x_n} - \left( \frac{\partial \mu_e}{\partial x_n} \right)_{x_n} \]  

(56)

Computing these quantities requires the equation of state of dense neutron matter. We will perform calculations using two model equations of state of nuclear matter. The first one is the “hadron gas”, consisting of an electrically neutral beta-equilibrated mixture of free neutrons, protons, and electrons. The second one is “APR hadron matter”, using the well-known model by Akmal, Pandharipande and Ravenhall \([15]\) which relies on a potential model that reproduces scattering data at nuclear densities. As a low density extension of the APR data we
use [16] [17]. In order to make it easy to apply our general results to other equations of state, we implement the APR equation of state using the simple parameterization employed in [18] to approximate the dependence of the energy per particle on the proton fraction $x$ by a quadratic form

$$E(n, x) = E_s(n) + S(n)(1 - 2x)^2$$  \hspace{1cm} (57)

where $E_s$ and $S$ are the corresponding energy for symmetric matter and the symmetry energy. We perform a global quartic fit to the APR prediction for symmetric and pure neutron matter $E_n$ which then yields the symmetry energy as

$$S(n) = E_n(n) - E_s(n)$$  \hspace{1cm} (58)

and the complete pressure including the electron contribution reads

$$p(n, x, \mu_e) = n^2 \left( \frac{dE_s(n)}{dn} + \frac{dS(n)}{dn}(1 - 2x)^2 \right) + \frac{\mu_e^4}{12\pi^2}$$  \hspace{1cm} (59)

In the absence of oscillations the $\beta$-equilibrium condition $\mu_I = 0$ yields the electron chemical potential as

$$\mu_e = 4(1 - 2x)S(n)$$  \hspace{1cm} (60)

and the requirement of charge neutrality $n_p = n_e$ allows us to determine the proton fraction $x(n)$ so that the pressure becomes a function of the baryon density alone. With these explicit expressions the susceptibilities in table II can be computed and the general results in section II can be employed. In the following subsections we will discuss the numerical results for the bulk viscosity of nuclear matter, comparing it with those for one particular model of quark matter, the one given by eq. (60) with $m_s = 150$ MeV and $c = 0.3$.

### B. Sub-thermal case

When $\mu_\Delta \ll T$ we obtain from the analytic expression eq. (26) the results shown in fig. 7 where the bulk viscosity of strange quark matter discussed in the previous section is also included for comparison. Here and in the following plots we study matter at twice nuclear saturation density, $\bar{n} = 2n_0$, and a compression cycle with a high angular frequency $\omega = 8.4$ kHz corresponding to an r-mode in a pulsar with a period of 1 ms. We see in fig. 7 that the maximum bulk viscosity of hadronic matter as a function of temperature (or equivalently as a function of angular frequency) is roughly an order of magnitude smaller than the maximum value for strange quark matter. This is unrelated to the beta-equilibration rate: the maximum viscosity depends according to eq. (27) on the relevant susceptibilities of the matter in question.

Other features of the plot do depend on the equilibration rate. As we expect from [26], quark matter achieves its maximum viscosity at the lowest temperature, and has less suppression at low temperatures. This is because the nonleptonic equilibration only involves two particles in the initial and final state, each of which has a large Fermi momentum $\sim \mu_q$ and hence large phase space factors. This leads to a low $\kappa = 2$ and a large value of $\Gamma$ (table I). Thus the suppression at low temperature is only $T^2$, and, according to eq. (27), $T_{\text{max}}$ is relatively low. The next fastest is the direct Urca process in nuclear matter, which involves more particles (including neutrinos which have no Fermi surface and thus very little phase space) and therefore has a higher $\kappa$ and lower $\Gamma$, giving it stronger $T^4$ suppression at low temperatures, and a higher $T_{\text{max}}$. The slowest is the modified Urca process in nuclear matter, which involves additional spectator nucleons, raising $\kappa$ to 6 and further lowering $\Gamma$, raising $T_{\text{max}}$, and increasing the low-$T$ suppression to $T^6$.

Note that the right-most solid and dashed curves in fig. 7 for hadronic matter with modified Urca equilibration, correspond roughly to the leftmost of the three solid (red online) curves in fig. 2(a) that run along the surface from front to back.

We draw two important conclusions from fig. 7. First, we have retained the full resonant structure of the viscosity compared to previous analyses [19] [43] where a low temperature approximation $\Gamma BT\kappa \ll \omega$ was used. This allows us to see that the viscosity decreases again at large temperatures and the maximum [27] occurs at millisecond-scale frequencies at potentially physically relevant temperatures of the order $10^{10}$ K for direct Urca and $10^{13}$ K for modified Urca. This means that the resonant structure may be important in some astrophysical applications and from eq. (27) it is clear that it becomes increasingly important at lower frequencies. Second, we see in fig. 7 that for nuclear matter there is a considerable difference between the solid curves which are based on an interacting equation of state [14] [47] and the dashed curves which are for a free gas$^3$ of nucleons and electrons. These models have different susceptibilities $B$ and $C$, and the main effect of this is a vertical shift of the whole curve. The shift in $T_{\text{max}}$ is smaller because of the square root in eq. (27). Hadronic matter with interactions has been considered (with a more simplified equation of state) in [20] [44] but many analyses [11] [49] [50] rely on the simple analytic result$^4$ given by Sawyer [19] which is based on the free gas expression. We see that these differ by roughly a factor of three for the given density of $\bar{n} = 2n_0$, but according to fig. 8 this difference

$^3$ Note that strictly speaking there are no modified Urca processes in an ideal hadron gas. Yet, for comparison with previous studies we use here the interacting matter expression for the rate but the ideal gas expressions for the strong susceptibilities.

$^4$ Note that the numerical prefactor given in [19] is too large by two orders of magnitude, see also [43].
C. The supra-thermal regime

Beyond the sub-thermal limit, a numeric evaluation of eqs. (21) and (23) is required. As discussed in sect. II, the temperature and amplitude dependence of the bulk viscosity for a given form of matter can be expressed in terms of the function $\mathcal{I}(d, f)$ which was plotted in a form that is independent of the equation of state of hadronic matter with modified Urca processes in fig. 2.

Using this result, we show in fig. 8 plots of the amplitude dependence of the bulk viscosity at two temperatures (left panel: $T = 10^6$ K; right panel: $T = 10^9$ K) for the various forms of hadronic and quark matter considered in this paper. Here solid lines again show the results for interacting matter whereas the dashed lines show the free hadron/quark gas results.

Note that the right-most curves in fig. 8 for hadronic matter with modified Urca equilibration, corresponds roughly to the foremost of the three dashed (blue online) curves in fig. 2(a) that run along the surface from left to right.

At the lower temperature the viscosity reaches the supra-thermal regime already for small amplitudes, whereas at the higher temperature the sub-thermal regime extends to large amplitudes, giving a flat amplitude-independent plateau at low amplitudes. The stronger non-linear feedback in the hadronic cases leads to a significantly steeper rise that correlates with the largest power in eq. (15). Interestingly, despite these differences the maximum value reached by varying the amplitude is still roughly the same as the maximum value in the sub-thermal limit eq. (27), as has been analytically found in the case of strange quark matter. This is important since it means that oscillations are approximately equally damped at all temperatures once the amplitude becomes sufficiently large. The maximum arises for amplitudes of the order 0.01, 0.1 and 1 for strange quark matter and hadronic matter with direct and modified Urca, respectively. The supra-thermal enhancement of the bulk viscosity is so strong, particularly for hadronic matter, that it could well provide the main saturation mechanism for unstable r-modes, stopping their growth at amplitudes that are below the threshold for other competing saturation mechanisms (e.g. non-linear hydrodynamics) but large enough to allow spin-down of a neutron star via gravitational radiation on astrophysical time scales.

V. CONCLUSIONS

We have studied the bulk viscosity of dense matter including its non-linear behavior at large amplitudes. In particular we give a general solution for the bulk viscosity of degenerate matter that is valid for arbitrary equations of state and retains its full parameter dependence in sec. II. This allows one to include these supra-thermal effects in a systematic r-mode analysis. In the supra-thermal regime we give a general analytic result for strange quark matter with non-leptonic processes in sec. III. We found that the free hadron gas model of nuclear matter, used for example in [11] [49] [50] to compute the susceptibilities that enter the viscosity, is not accurate even in the sub-thermal regime, and may significantly underestimate the viscosity. Moreover, we find that the standard low temperature (high frequency) approximation is not applicable for temperatures around $10^{10}$ K and the full resonant form of the bulk viscosity is required. We confirm previous results for the amplitude-dependence of the bulk viscosity of strange quark matter [17] and find that these supra-thermal effects are parametrically even more important in nuclear matter, because of higher-order non-linearities in the amplitude-dependence of the Urca rate.

The most obvious application of our results is to the damping of unstable r-mode oscillations in neutron stars. As the amplitude of the mode enters the supra-thermal regime the viscosity will increase steeply above the sub-
thermal result and can exceed it by many orders of magnitude, but eventually it reaches an upper bound that is completely independent of the particular weak damping process and depends only on susceptibilities of the dense matter in question. The viscosity then decreases at even larger amplitudes. We conclude that if r-mode growth is not stopped by the supra-thermal bulk viscosity before this maximum is reached then other non-linear dynamic effects \[12–15\] will be required to stop it. We have already performed initial exploratory calculations of r-mode damping times, and these suggest that over a significant region of parameter space supra-thermal bulk viscosity is sufficient to saturate r-mode growth at a finite amplitude parameter \[\alpha_{\text{max}} < 1\], as was previously assumed \[11\]. This topic will be discussed in more detail in a forthcoming publication. There are several other directions in which our research could be developed. Other equations of state for quark matter could be studied, for example the perturbative equation of state \[52\], and also other phases with different equilibration mechanisms. The same is true for the various equations of state and phases of hadronic matter. Our analysis was for the case of a single equilibration channel, and it would be interesting to extend it to multiple channels, which may be relevant to both hadronic and quark matter (see appendix A of \[30\] and Ref. \[35\]). In quark matter the non-Fermi liquid enhancement of the Urca rate \[53\] should further increase resonant effects. The application of our results to r-modes in neutron stars also raises interesting questions concerning the correct treatment of the crust \[54\], and possible modification of the radial profile of the r-mode due to strong radial dependence of the bulk viscosity in layered stars such as hybrid stars.

Finally we note that at low temperatures the suprathermal enhancement of bulk viscosity becomes large, and the amplitude threshold for entering the suprathermal regime becomes low. This may not be relevant to the damping of r-modes because they are also damped by shear viscosity which becomes large at low temperature. But for other modes of compact stars, such as monopole pulsations, shear viscosity will not play such a significant role, and suprathermal bulk viscosity might be the dominant source of damping if external perturbations make the amplitude large enough. This might be relevant to old, cold, accreting stars in binary systems.
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