The Pauli Exclusion Principle and the problems of its theoretical foundation and experimental verification

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Abstract. How Wolfgang Pauli came to the formulation of his exclusion principle before the creation of modern quantum mechanics and the following from it the discovery of the conception of spin, based on the recollections of main participants of this story, is presented in Introduction. Then the modern state of the Pauli Exclusion Principle (PEP) is discussed.

PEP can be considered from two viewpoints. On the one hand, it asserts that particles with half-integer spin (fermions) are described by antisymmetric wave functions, and particles with integer spin (bosons) are described by symmetric wave functions. This is the so-called spin-statistics connection (SSC). As we will discuss, the physical reasons why SSC exists are still unknown. On the other hand, according to PEP, the permutation symmetry of the total wave functions can be only of two types: symmetric or antisymmetric, both belong to the one-dimensional representations of the permutation group, all other types of permutation symmetry are forbidden. While the solution of the Schrödinger equation may have any permutation symmetry.

We discuss this aspect of PEP and demonstrate that the proof of PEP in some textbooks on quantum mechanics is incorrect. The indistinguishability principle is insensitive to the permutation symmetry of the wave function and cannot be used as a criterion for the verification of PEP. However, as follows from the analysis of possible scenarios, the permission of states with permutation symmetry more general than symmetric and antisymmetric (degenerate permutation states) leads to contradictions with the concepts of particle identity and their independence. Thus, the existence in our Nature particles only in nondegenerate permutation states (symmetric and antisymmetric) is not accidental, as was accepted before, and so-called symmetrization postulate may not be considered as a postulate, since all symmetry options for the total wave function, except the antisymmetric and symmetric, cannot be realized. From this an important conclusion follows: we may not expect that in future some unknown elementary particles that are not fermions or bosons can be discovered.

In the last part of the paper, experimental verifications of PEP are discussed. It is shown that if really a small part of electrons exists that does not obey PEP, these non-Pauli electrons could not be detected in spectroscopic experiments, in which “forbidden” X-ray transitions are measured.

Keywords. Pauli Exclusion Principle; Spin-Statistics connection; Indistinguishability principle; Permutation symmetry.

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1. Historical survey and still unsolved problems

This review devoted to one of the fundamental principle of quantum mechanics – the Pauli Exclusion Principle (PEP). Wolfgang Pauli published his principle more than 90 years ago, basing on experimental data. In subsections 1.1 and 1.2 I present a short historical survey of the discovery of PEP and following from it the discovery of the conception of spin, which plays a fundamental role in quantum mechanics. Pauli formulated his principle for electrons, in subsections 1.3 and 1.4 I discuss the general formulations of PEP valid for all identical particles and still unsolved problems.

1.1 Discovery of PEP

Wolfgang Pauli formulated his principle before the creation of the contemporary quantum mechanics (1925-1927). He arrived at the formulation of his principle trying to explain regularities in the anomalous Zeeman effect in strong magnetic fields. But in his Princeton address [1], Pauli recalled that the history of the discovery goes back to his student days in Munich. At that time the periodic system of the elements was well known and the series of whole numbers 2, 8, 18, 32... giving the lengths of the periods in this table was zealously discussed in Munich. A great influence on Pauli had his participation in the Niels Bohr guest lectures at Göttingen in 1922, when he met Bohr for the first time. In these lectures Bohr reported on his theoretical investigations of the periodic system. Bohr emphasized that the question of why all electrons in an atom are not bound in the innermost shell, is the fundamental problem in these studies. However, no convincing explanation for this phenomenon could be given on the basis of classical mechanics.

In his first studies of the Zeeman effect, Pauli was interested in the explanation of the simplest case, the doublet structure of the alkali spectra. In December of 1924 Pauli submitted a paper on the Zeeman effect [2], in which he showed that Bohr’s theory of doublet structure, which was based on the non-vanishing angular moment of a closed shell, such as K-shell of the alkali atoms, is incorrect and closed shell has no angular and magnetic moments (it is instructive to stress: the young scientist corrected the most authoritative physicist and published his paper without any problems in a leading physical journal). Pauli came to the conclusion that instead of the angular momentum of the closed
shells of the atomic core, a new quantum property of the electron had to be introduced. In that paper he wrote remarkable for that time, prophetic words. Namely:

“According to this point of view, the doublet structure of alkali spectra ... is due to a particular **two-valuedness** of the quantum theoretic properties of the electron, which cannot be described from the classical point of view.”

This non-classical two-valued nature of electron is now called *spin*. In anticipating the quantum nature of the magnetic moment of electron before the creation of modern quantum mechanics, Pauli exhibited a striking intuition⁴.

Based on his results on the classification of spectral terms in a strong magnetic field, Pauli came to the conclusion that a single electron must occupy an entirely nondegenerate energy level. In the paper submitted for publication on January 16, 1925 Pauli formulated his principle as follows [3]:

“In an atom there cannot be two or more equivalent electrons, for which in strong fields the values of all four quantum numbers coincide. If an electron exists in an atom for which all of these numbers have definite values, then this state is ‘occupied’.”

In this paper Pauli used four quantum numbers for characterization of the state of the electron in atom: \( n, l, j = l \pm 1/2, \) and \( m_j \) (in the modern notations); by \( n \) and \( l \) he denoted the well-known at that time the principal and angular momentum quantum numbers, by \( j \) and \( m_j \) - the total angular momentum and its projection, respectively. Thus, Pauli characterized the electron by some additional quantum number \( j \), which in the case of \( l = 0 \) was equal to \( \pm 1/2 \). For this new fourth quantum number of the electron, Pauli did not give any physical interpretations, since he was sure, as we cited above, that it cannot be described in the terms of classical physics.

Introducing two additional possibilities for electron states, Pauli obtained \( 2(2l + 1) \) possibilities for the set \( (n, l, j, m_j) \). That led to the correct numbers 2, 8, 18, 32 for the lengths of the periods in the Periodic Table.

As Pauli noted in his Nobel Prize lecture [4]:

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¹ Another well-known example of the Pauli intuition is the Pauli prediction of neutrino in 1930. Pauli made this prediction without any experimental and theoretical indications that this chargeless and, as thought at that time, massless particle can exist. He tried to save the energy conservation law in the \( \beta \)-decay, because he did not agree with Niels Bohr who at that time was sure that the energy conservation law does not valid for microparticles. It turns out that Pauli was right.
“...physicists found it difficult to understand the exclusion principle, since no meaning in terms of a model was given to the fourth degree of freedom of the electron.”

Although not all physicists! Young scientists first Ralph Kronig and then George Uhlenbeck and Samuel Goudsmit did not take into account the Pauli words that the electron fourth degree of freedom cannot be described by classical physics and suggested the classical model of the spinning electron. Below I will describe in some details the discovery of spin using the reminiscences of main participants of this dramatic story.

1.2 Discovery of spin

Kronig recalled [5] that on January 7, 1925, at the age of 20, he, as a traveling fellow of the Columbia University, arrived at the small German university town of Tübingen to see Landé and Gerlach. At the Institute of Physics Kronig was received by Landé with the remark that it was a very opportune moment, since he was expecting Pauli the following day and he just received a long and very interesting letter from Pauli. In that letter Pauli described his exclusion principle.

Pauli’s letter made a great impression on Kronig and it immediately occurred to him that the additional to the orbital angular momentum $I$ the momentum $s = 1/2$ can be considered as an intrinsic angular momentum of the electron. The same day Kronig performed calculations of the doublet splitting. The results encouraged him, although the obtained splitting was too large, by a factor of 2. He reported his results to Landé. Landé recommended telling these results to Pauli. Next day Pauli arrived to Tübingen and Kronig had an opportunity to discuss with him his idea. According to Kronig [5], Pauli did not believe that his idea had any connection with reality.

Later Kronig discussed his model in Copenhagen with Heisenberg, Kramers and others and they also did not approve it. Under the impression of the negative reaction of most authoritative physicists and the serious problems in his calculations Kronig did not publish his ideas about a spinning electron. In the letter to van der Waerden [6] Kronig wrote about the difficulties he met in his studies of the spinning electron:

"First, the factor 2 already mentioned. Next, the difficulty to understand how a rotation of the electron about its axis would yield a magnetic moment of just one magneton. Next, the necessity to assume, for the rotating charge of an electron of classical size, velocities surpassing the velocity of light. Finally, the smallness of the magnetic
moments of atomic nuclei, which were supposed, at that time, to consist of proton and electrons”

Independently of Kronig, the Dutch physicists Uhlenbeck and Goudsmit after reading the Pauli paper on his exclusion principle also arrived at the idea of the spinning electron. In his address, delivered at Leiden on the occasion of his Lorentz Professorship, Uhlenbeck [7] told in detail the story of their discovery and its publication.

According to Uhlenbeck, he and Goudsmit were greatly affected by the Pauli exclusion principle, in particular by the fourth quantum number of the electron. For them it was a mystery, why Pauli did not suggest any concrete picture for it. Due to their conviction that every quantum number corresponds to a degree of freedom, they decided that the point model for the electron was not appropriate and the electron should be assumed as a small sphere that could rotate. However, very soon they recognized that the rotational velocity at the surface of the electron had to be many times larger the velocity of light. As Uhlenbeck recalled further,

“...we had not the slightest intention of publishing anything. It seems so speculative and bold, that something ought to be wrong with it, especially since Bohr, Heisenberg and Pauli, our great authorities, had never proposed anything of this kind. But of course we told Ehrenfest. He was impressed at once, mainly, I feel, because of the visual character of our hypothesis, which was very much in his line ... and finally said that it was either highly important or nonsense, and that we should write a short note for Naturwissenschaften and give it to him....Goudsmit and myself felt that it might be better for present not to publish anything; but when we said this to Ehrenfest, he answered: ‘Ich habe Ihren Brief schon längst abgesandt; Sie sind beide jung genug um sich eine Dummheit leisten zu können’.

Thus, the short letter of Uhlenbeck and Goudsmit was transmitted by Ehrenfest to the editor of Naturwissenschaften and soon published [8]. Then in February 1926 they published a paper in Nature [9].

Bohr, who was initially rather skeptic about the hypothesis of the spinning electron and did not approve the Kronig idea, gradually changed his mind. The meeting with Einstein became crucial. Einstein said Bohr that the problems arising in the model of spinning electron are naturally resolved in the theory of relativity. Under the influence of

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2 English translation of an essential part of Uhlenbeck address represented in Ref. [6].
3 I have already sent your letter some time ago. You are both young enough and can afford yourself a foolishness.
Bohr’s opinion on the idea of spinning electron, Heisenberg at last also removed his objections.

However, Pauli did not! His deep intuition did not allow him to admit the hypothesis of the spin as an intrinsic angular momentum of the rotating electron. Pauli’s objections resulted from the wrong factor 2 in the doublet splitting, but mainly from the classical nature of the spin hypothesis. After the Lorentz festival (December 1925) Pauli met Bohr in Berlin and in strong words expressed his dissatisfaction that Bohr changed his position. Pauli was convinced that a new "Irrlehre 4" has arisen in atomic physics, as van der Waerden wrote in his recollections [6].

Meanwhile in April 1926 a young English physicist Llewellyn Thomas, who had spent half a year in Copenhagen with Bohr, published a letter in Nature [10] where he presented a relativistic calculation of the doublet splitting. Thomas demonstrated that the wrong factor 2 disappears and the relativistic doublet splitting does not involve any discrepancy. In the end Thomas noted, “…as Dr. Pauli and Dr. Heisenberg have kindly communicated in letters to Prof. Bohr, it seems possible to treat the doublet separation as well as the anomalous Zeeman effect rigorously on the basis of the new quantum mechanics.” Thus, at this time Pauli was certain that the problem can be treated rigorously by the quantum mechanical approach. The relativistic calculations by Thomas finally deleted all his doubts.

In his Nobel Prize lecture Pauli recalled [4]:

“Although at first I strongly doubted the correctness of this idea because of its classical mechanical character, I was finally converted to it by Thomas [10] calculations on the magnitude of doublet splitting. On the other hand, my earlier doubts as well as the cautious expression 'classically non-describable two-valuedness' experienced a certain verification during later developments, as Bohr was able to show on the basis of wave mechanics that the electron spin cannot be measured by classically describable experiments... and must therefore be considered as an essentially quantum mechanical property of the electron.”

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4 Heresy.
Thus, after the creation of quantum mechanics it became clear that Pauli was right in not agreeing with the classical interpretation of the fourth degree of freedom. The spin in principle cannot be described by classical physics.

Let us mention how young the main participants of this story were. They were between 20 and 25. In 1925 even the creators of quantum mechanics: Werner Heisenberg (1901-1976), Paul Dirac (1902-1984), Wolfgang Pauli (1900-1958), Enrico Fermi (1901-1954), and some others were of the same age. Namely: Heisenberg – 24, Dirac – 23, Pauli - 25, Fermi – 24. At that time Erwin Schrödinger had 38 and their great authority Niels Bohr was 40 years old.

1.3 Creation of quantum mechanics and a general formulation of PEP

The conceptions of quantum mechanics were formulated in 1925 by Heisenberg, Born, and Jordan [12, 13] in the matrix formalism. It is interesting to mention that Heisenberg even did not know the word “matrix” and created the mathematical formalism for his study in Ref. [12]. Born, who used matrices in his lectures, in the paper with Jordan [13] noted that Heisenberg introduced in quantum mechanics the well-known in mathematics matrix calculus. In 1926 Schrödinger basing on the wave-particle dualism, suggested by de Broglie [13], introduced the wave function $\psi$ describing micro-particles and formulated his famous wave equation [14, 15].

The first studies devoted to application of the newborn quantum mechanics to many-particle systems were performed independently by Heisenberg [16] and Dirac [17] in the same 1926. In these studies, the Pauli principle, formulated as the prohibition for two electrons to occupy the same quantum state, was obtained as a consequence of the antisymmetry of the Schrödinger wave function. In both papers [16, 17] the antisymmetric many-electron wave functions were constructed and it was concluded that these functions cannot have two particles in the same state. Dirac represents an N-electron antisymmetric function as a determinant\(^5\) constructed with one-electron wave functions $\psi_{n_i}$:

\(^5\) It is important to note that the determinantal representation of the electronic wave function, which at present is widely used in atomic and molecular calculations, was first introduced not by Slater, but by Dirac [17] in 1926. In 1929 Slater [18] inserted spin functions into the determinant and used the determinantal
\[ \Psi_{n_1n_2...n_r}(1,2,...,r) = \begin{pmatrix} \psi_{n_1}(1) & \psi_{n_1}(2) & \cdots & \psi_{n_1}(r) \\ \psi_{n_2}(1) & \psi_{n_2}(2) & \cdots & \psi_{n_2}(r) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n_r}(1) & \psi_{n_r}(2) & \cdots & \psi_{n_r}(r) \end{pmatrix} \]  

where number of electrons \( N = r \). After presenting the many-electron wave function in the determinantal form Dirac [17] wrote:

"An antisymmetric eigenfunction vanishes identically when two of the electrons are in the same orbit. This means that in the solution of the problem with antisymmetric eigenfunctions there can be no stationary states with two or more electrons in the same orbit, which is just Pauli's exclusion principle."

Thus, with the creation of quantum mechanics, the prohibition on the occupation numbers of electron system states was supplemented by the prohibition of all types of permutation symmetry of electron wave functions except for the antisymmetric ones.

The first quantum-mechanical calculation of the doublet splitting and the anomalous Zeeman effect for atoms with one valence electron was performed by Heisenberg and Jordan [19]. They used the Heisenberg matrix approach and introduced the spin vector \( \mathbf{s} \) with components \( s_x, s_y, s_z \) with commutation relations the same as for the components of the orbital angular moment \( \mathbf{l} \). Application of the perturbation theory led to results, which were in a full accordance with experiment.

In 1927 Pauli [20] studied the spin problem using the Schrödinger wave function approach. Pauli introduced the spin operators \( s_x, s_y, s_z \) acting on the wave functions, which depend on the three spatial coordinates, \( q \), and a spin coordinate. Pauli took \( s_z \) as a spin coordinate. The latter is discrete with only two values. Therefore the wave function \( \psi(q, s_z) \) can be presented as a two-component function with components \( \psi_{\alpha}(q) \) and \( \psi_{\beta}(q) \) corresponding to \( s_z = \frac{1}{2} \) and \( s_z = -1/2 \), respectively. The operator, acting on the two-component functions, can be presented as a matrix of the second order. Pauli obtained an explicit form of the spin operators, representing them as \( s_x = \frac{1}{2}\sigma_x, s_y = \frac{1}{2}\sigma_y, \) and \( s_z = \frac{1}{2}\sigma_z \), where \( \sigma \) are the famous Pauli matrices. Applying this formalism to the problem of the doublet splitting and the anomalous Zeeman effect,

representation of the electronic wave function (so-called Slater’s determinants) for calculations of the atomic multiplets.
Pauli obtained, as it can be expected, the same results as Heisenberg and Jordan [19] obtained by the matrix approach.

The Pauli matrices were used by Dirac in his derivation of the Schrödinger equation for the electron [21]. In this study Dirac created the rigorous relativistic quantum theory of the electron, which includes naturally the conception of spin. We will not go inside the Dirac relativistic theory, but will discuss some consequences from it first analyzed by Schrödinger in his remarkable paper [22]. In that paper Schrödinger [22] revealed that from the Dirac equation for the electron follows the rapid oscillatory motion of the massless charge with the velocity $c$ around a center of mass, which he named \textit{Zitterbewegung}. This original picture developed by Schrödinger induced a broad discussion of the origin of spin. Below we critically analyze this discussion, which at present is still going on.

If one expresses Dirac’s dynamic variables via the spin variables, spin appears as an orbital angular momentum of the \textit{Zitterbewegung} [23, 24], see also [25-27] and recent publication by Hestenes [28]. These studies demonstrate how the conception of spin follows from the relativistic quantum mechanics.

However, some authors, see for instance Ref. [29-32], basing on so-called \textit{stochastic electrodynamics} [33, 34], claimed that from it follows the classical origin of the electron spin. In these publications the model of spin was considered as not following from quantum mechanics. So Muradlier [31] even in the title of his paper stressed that spin has a classical origin. The point is that the authors of stochastic electrodynamics, Marshall [33, 34] and Boyer who discussed it in series of papers [35-38], inserted in classical electrodynamics the zero-point radiation, or the zero-point field (ZPF), depending on the Planck constant $\hbar$ and connected with discussed above \textit{Zitterbewegung} [22].

The creators of stochastic electrodynamics have stressed that ZPF has a classical nature. Thus Boyer in all his numerous publications names ZPF as classical, in spite that he obtained, using this “classical” ZPF, the exact quantum expressions for the dispersion forces [35]. As was demonstrated in papers [39,40], the stochastic electrodynamics allows to obtain the Lamb shift that is a pure quantum electrodynamics effect.
In his publication in 2018, Boyer [38] tried to prove that the quantum Planck constant ħ inserted in classical physics plays role only as a scaling factor. He noted that if one put ħ → 0 in quantum theory it loses quantum properties, while classical physics remains classical. This viewpoint may not be considered as correct; it is a fallacy. If some quantum conceptions can be used in classical physics, they do not become classical. In contrary, the inclusion of the zero-point radiation in classical electrodynamics provides it by the quantum properties. The zero-point radiation is a quantum phenomenon, its energy equal to $\frac{1}{2} \hbar \omega_0$. In the classical limit when $\hbar \to 0$, it does not exist.

The same is true in respect to the electron spin $s = 1/2 \hbar$. It is evident that in classical physics $s = 0$. Pauli was completely right when he stressed that the spin is a quantum property of electron that cannot be defined in classical physics,

After this discussion of origin of the spin conception let us return to PEP.

In 1932 Chadwick [41] discovered neutron. In the same year Heisenberg [42] considered consequences of the model, in which the nuclei are built from protons and neutrons, but not from electrons and protons, as was accepted before. Heisenberg assumed that the forces between all pairs of particles are equal and in this sense the proton and neutron can be considered as different states of one particle. He introduced a variable $\tau$, the value $\tau = -1$ was assigned to the proton state, the value $\tau = 1$ to the neutron state. Wigner [43] called $\tau$ as isotopic spin (at present named also as isobaric spin). The isotopic spin has only two values and as in the fermion case can be represented as $\tau = \frac{1}{2}$. Taking into account that for protons and neutrons their nuclear spin $s = \frac{1}{2}$ too, Wigner studied the nuclear charge-spin supermultiplets for Hamiltonian not depending on the isotope and nuclear spins.

After discovery of various types of elementary particles, it was revealed that all discovered particles are described by wave functions of only two types of symmetry: symmetric and completely antisymmetric. As a result, the exclusion principle formulated Pauli for electrons was formulated for all elementary particles in the following general form:

The only possible states of a system of identical particles possessing spin $s$ are those for which the total wave function transforms upon interchange of any two particles as

$$P_i^j \Psi(1,\ldots,i,\ldots,j,\ldots,N) = (-1)^{2s} \Psi(1,\ldots,i,\ldots,j,\ldots,N)$$

(2)
That is, it is symmetric for integer values of $s$ and antisymmetric for half-integer values of $s$.

This general formulation holds also for composite particles. It was studied by Ehrenfest and Oppenheimer [44]. The authors considered some clusters of electrons and protons; it can be atoms, molecules or nuclei (at that time the neutron had not been discovered). They formulated a rule, according to which the statistics of a cluster depends upon the number of particles from which they are built up. In the case of odd number of particles, it is the Fermi-Dirac statistics, while in the case of even number it is the Bose-Einstein statistics. It was stressed that this rule is valid, if the interaction between composite particles does not change their internal states; that is, the composite particle is stable enough to preserve its identity.

![Fig. 1](image)

**Fig. 1** The statistics of composite particles

A good example of such stable composite particle is the atomic nucleus. It consists of nucleons: protons and neutrons, which are fermions because they have spin $s=1/2$. Depending on the value of the total nuclear spin, one can speak of boson nuclei or fermion nuclei, see Fig. 1. The nuclei with an even number of nucleons have an integer value of the total spin $S$ and are bosons; the nuclei with an odd number of nucleons have a half-integer value of the total spin $S$ and are fermions. A well-known system, in which the validity of PEP for composite particles was precisely checked in experiment, was the $^{16}\text{O}_2$ molecule, see a detailed discussion in Ref. [45].

The generalized formulation of PEP can be considered from two aspects. On the one hand, it asserts that particles with half-integer spin (fermions) are described by antisymmetric wave functions, and particles with integer spin (bosons) are described by symmetric wave functions. This is the so-called *spin-statistics connection* (SSC). On the other hand, PEP is not reduced only to SSC. It can be considered from another aspect - the restrictions on the allowed symmetry types of many-particle wave functions. Namely,
only two types of permutation symmetry are allowed: symmetric and antisymmetric. Both belong to the one-dimensional representations of the permutation group; while all other types of permutation symmetry are forbidden.

Below we will discuss the first aspect of PEP. The second aspect will be considered in Section 2.

1.4. Spin-statistics connection

As follows from the previous text, the initial formulation of PEP by Pauli for electrons and its generalization for all elementary particles is based on analysis of experimental data. Pauli himself was never satisfied by that. In his Nobel Prize lecture Pauli said [4]:

“Already in my initial paper, I especially emphasized the fact that I could not find a logical substantiation for the exclusion principle nor derive it from more general assumptions. I always had a feeling, which remains until this day, that this is the fault of some flaw in the theory.”

Let us stress that this was said in 1946, or after the Pauli well-known theorem [46] of the relation between spin and statistics. The point is that in this theorem, Pauli did not give a direct proof. He showed that due to some physical contradictions, the second quantization operators for particles with integral spins cannot obey the fermion commutation relations; while for particles with half-integral spins their second quantization operators cannot obey the boson commutation relations. Pauli was not satisfied by such kind of negative proof. Very soon it became clear that he was right.

The Pauli theorem [46], is implicitly assumed that particles can obey only two types of commutation relations: boson or fermion relations. However, this fact was not rigorously proved and stemmed from known at that time experimental data. In 1953 Green [47] and then independently Volkov [48] showed that more general, paraboson and parafermion trilinear commutation relations, satisfying all physical requirements and containing the boson and fermion commutation relations as particular cases, can be introduced. A corresponding parastatistics is classified by its rank $p$. For the parafermi statistics $p$ is the maximum occupation number. For $p = 1$ the parafermi statistics becomes identical to the Fermi-Dirac statistics (for more details see book by Ohnuki and Kamefuchi [49]).

Up to date the elementary particles obeying the parastatistics are not detected. In 1976, Kaplan [50] revealed that the parafermi statistics is realized for quasiparticles in a crystal lattice, e.g. for the Frenkel excitons or magnons, but due to a periodical crystal
field, the Green trilinear commutation relations are modified by the quasi-impulse conservation law. Later on, it was shown that introduced by Kaplan the modified parafermi statistics [50] is valid for different types of quasiparticles in a periodical lattice: polaritons [51,52], defectons [53], delocalized holes in crystals [54], and some others [55, 56].

After 1940 numerous proofs of SSC have been published, but none of them were rigorous; see, for instance, the Pauli criticism [57] on the proofs of such high-level physicists as Feynman [58] and Schwinger [59]. In the comprehensive book by Duck and Sudarshan [60] practically all proofs of the spin-statistics connection published at that time were criticized, see also Refs. [61, 62].

In his famous lectures Feynman [63] even apologized in the front of audience:

"Why is it that particles with half-integral spin are Fermi particles whose amplitudes add with the minus sign, whereas particles with integral spin are Bose particles whose amplitudes add with the positive sign? We apologize for the fact that we cannot give you an elementary explanation... It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics".

After this Feynman comment, it appeared many publications, in which authors claimed that they fulfilled the Feynman requirement and proposed a simple explanation of SSC. However, these proofs cannot be based in the frame of traditional quantum mechanics.

I would like to note that in 1997 Berry and Robbins [64] presented the original proof of SSC. However, in next paper [65], see also Ref. [66], they came to conclusion that their proof [64] is incorrect, since they found some alternative constructions to introduced in Ref. [64] transported spin basis, which lead to the wrong exchange sign. To the best of my knowledge, Berry and Robbins have been unique authors that criticized their proof of SSC.

It should be mentioned that publications of simple, according to authors, proofs of SSC still continues, see recent papers [67-71]. But all these proofs are outside of quantum mechanics. For instance, Jabs [67] for proving SSC postulated a special procedure for the exchange of identical particle that includes an additional rotation and differs from the simple definition of exchange in quantum mechanics. The same drawback has the
relativistic proof by Bennet [68] based on the proof [67]. Santamato and De Martini [69-71] proved the spin-statistics theorem in the frame of specially developed *conformal quantum geometrodynamics* where wave functions are not applied, although some “wave function” is used, but it is the same for fermions and bosons, since it does not change upon permutations. Their proof is essentially based on introduced by the authors a special “intrinsic helicity” of elementary particles [69], which has not been known in physics of elementary particles. For authors [69], it was not important that this property has not been detected in experiments. The neglect of experimental evidence is typical for such mathematical approaches to physics.

Thus, to the best of my knowledge, we still have no simple answer, what are the physical reasons that identical particles with half-integer spin are described by antisymmetric functions and identical particles with integer spin are described by symmetric functions. As Berry and Robbins [66] emphasized in 2000, the relation between spin and statistics “cries out for understanding”. Unfortunately, at present it still “cries”.

2. Theoretical foundations of PEP

2.1 Indistinguishability of identical particles and the symmetry postulate.

In this Section we will discuss the second aspect of PEP. As follows from our discussion of SSC in subsection 1.4, the first aspect is still waiting its solution. We noted above that according to the general formulation of PEP on p.10 (*in proofs to put a real page number*) only two types of permutation symmetry are allowed: symmetric and antisymmetric (both are the one-dimensional representations of the permutation group). However, the Schrödinger equation is invariant under any permutation of identical particles. The Hamiltonian of an identical particle system commutes with the permutation operators,

\[ [P, H] = 0, \]  \hspace{1cm} (3)

From this follows that the solutions of the Schrödinger equation may belong to any representation of the permutation group, including multi-dimensional representations.
The question might be asked: whether the Pauli principle limitation on the solutions of the Schrödinger equation follows from the fundamental principles of quantum mechanics or it is an independent principle?

Depending on the answer on this question, physicists studying the foundations of quantum mechanics can be divided on two groups.

Some physicists, including the founder of quantum mechanics Dirac [72] (see also books by Schiff [73] and Messiah [74]), had assumed that there are no laws in Nature that forbid the existence of particles described by wave functions with more complicated permutation symmetry than those of bosons and fermions, and that the existing limitations are only due to the properties of the known elementary particles. Messiah [74, 75] has even introduced the term symmetrization postulate to emphasize the primary nature of the constraint on the allowed types of the wave function permutation symmetry.

The independence of his exclusion principle from other fundamental principles of quantum mechanics was formulated quite general by Pauli in his Princeton address [1]:

“The exclusion principle could not be deduced from the new quantum mechanics but remains an independent principle which excludes a class of mathematically possible solutions of the wave equation. This excess of mathematical possibilities of the present-day theory, as compared with reality, is indications that in the region where it touches on relativity, quantum theory has not yet found its final form.”

There is another viewpoint on this problem; according to it, the symmetrization postulate is not an independent principle and can be derived from fundamental principles of quantum mechanics; in particular, from the principle of indistinguishability of identical particles. Proofs of this idea have been represented not only in articles, see critical comments on some publications in Refs. [75, 76], but also in textbooks [77-79], including the famous textbook by Landau and Lifshitz [78] translated into many languages. The incorrectness of the proof in the book by Corson [77] was noted by Girardeau [76], the proofs represented in Refs. [77-79] were critically analyzed in my first paper on the PEP [80] (a more detailed criticism was given in Refs. [81, 45]). Nevertheless, incorrect proofs of the symmetrization postulate have been still appeared in current literature.

In review by Canright and Girvin [82] devoted to the fractional statistics, the authors presented the same erroneous proof as it is in books [77-79]. It should be mention that the creators of the fractional statistics Leinaas and Mirheim in their paper on the fractional
statistics [83] accepted wrong ideas of Mirman [84] and pointed out that for identical particles the word “exchange” has no physical meaning and therefore the indistinguishability principle used in quantum mechanics also has no physical sense. Although the studies by authors [83] in 2D space were really pioneer and correct, the part in their paper devoted to 3D space was wrong. If really the exchange, that is, a permutation of two particles, has no physical sense, then the permutation group could not be applied in physics and fermions and bosons could not be defined. Nevertheless, this wrong idea was widely accepted in consequent studies in the fractional statistics field, see for instance book by Khare [85]. In Ref. [86], Section 5.4, I discuss it in detail.

Even in the recently published, very good in many fundamental aspects of quantum mechanics and quantum chemistry, book by Piela [87], the represented proof of PEP has the same errors, as in the cited above textbooks. Thus, it is worth-while to analyze the proof of the symmetrization postulate once more.

The typical argumentation (it is the same in Refs. [77-79, 82, 87]) is the following. From the requirement that the states of a system obtained by permutations of identical particles must all be physically equivalent, one concludes that the transposition of any two identical particles should multiply the wave function only on an insignificant phase factor,

$$P_{12} \Psi(x_1, x_2) = \Psi(x_2, x_1) = e^{i\alpha} \Psi(x_1, x_2) \quad (4)$$

where $\alpha$ is a real constant and $x$ is the set of spatial and spin variables. One more application of the permutation operator $P_{12}$ gives

$$\Psi(x_1, x_2) = e^{i2\alpha} \Psi(x_1, x_2) \quad (5)$$

or

$$e^{2i\alpha} = 1 \quad \text{and} \quad e^{i\alpha} = \pm 1. \quad (6)$$

Since all particles are assumed to be identical, the wave function should be changed in exactly the same way under transposition of any pair of particles, i.e. it should be either totally symmetric or totally antisymmetric.

This simple proof at first glance looks quite convincing. It seems that the simplicity of this proof hypnotizes readers to accept it without any criticism. However, the proof presented by Eqs. (4) – (6) contains two essential incorrectness’s at once. The first incorrectness is simply follows from the group theory formalism. Namely: Eq. (4) is valid...
only for one-dimensional representations. The application of a group operation to one of basis functions, belonging to some multi-dimensional representation with dimension \( f \), transforms it as a linear combination of basis functions. Namely,

\[
P_{12} \Psi_i = \sum_{k=1}^{f} \Gamma_{ki}(P_{12}) \Psi_k
\]

(7)

where coefficients \( \Gamma_{ki}(P_{12}) \) form a square matrix of the order \( f \) where \( f \) is the dimension of this multi-dimensional representation.

The application of the permutation operator \( P_{12} \) to both sides of Eq. (7) leads to the identity and we cannot arrive at any information about the symmetry, in contrary with Eq. (5). By requiring that under permutations the wave function must change by no more than a phase factor, one actually postulates that the representation of the permutation group, to which the wave function belongs, is one-dimensional. Thus, this proof is based on the initial statement, which is proved then as a final result.

The second incorrectness in the proof discussed follows from physical considerations laying in the base of quantum mechanics. This proof is directly related to the behavior of the wave function. However, since the wave function is not an observable, the indistinguishability principle is related to it only indirectly via the expressions of measurable quantities. Since in quantum mechanics, the physical quantities are expressed as bilinear forms of wave functions, the indistinguishability principle requires the invariance of these bilinear forms and can be formulated as:

\[
\langle P \Psi | \hat{L} | P \Psi \rangle = \langle \Psi | \hat{L} | \Psi \rangle
\]

(8)

where \( \hat{L} \) is an arbitrary operator. Often, one limits oneself to the requirement that the probability of a given configuration of a system of identical particles must be invariant under permutations [76,88],

\[
P | \Psi(x_1, ..., x_N)|^2 = | \Psi(x_1, ..., x_N)|^2.
\]

(9)

For a function satisfying Eq. (9), it is sufficient that under permutations it would change as

\[
P \Psi(x_1, ..., x_N) = e^{i\alpha_p(x_1, ..., x_N)} \Psi(x_1, ..., x_N),
\]

(10)

i.e. unlike the requirement of condition (4), in the general case the phase is a function of coordinates and the permutation, and Eq. (5) evidently does not hold.
As was discussed above, most proofs of the symmetry postulate contain unjustified constraints. Proofs of the symmetry postulate without imposing additional constraints have been given by Girardeau [76, 88], who based it on Eq. (9), and in my paper [80] where it was based on Eq. (8). As was noted later by the author [89, 81, 45] these proofs, basing on the indistinguishability principle in the forms (8) and (9), are incorrect, because Eqs. (8) and (9) are valid only for non-degenerate states. In a degenerate state, the system can be described with the equal probability by any one of the basis vectors of the degenerate state. As a result, we can no longer select a pure state (the one that is described by the wave function) and should regard a degenerate state as a mixed one, where each basis vector enters with the same probability. The possibility of expressing the density matrix through only one of the functions implies that the degeneracy with respect to permutations has been eliminated. However, the latter cannot be achieved without violating the identity of the particles.

Thus, we must sum both sides of Eqs. (8) and (9) over all wave functions that belong to the degenerate state. For instance, the expression of the probability density, which described via the diagonal element of the density matrix, in the case of a degenerate state has the following form

\[
D_t^{[\lambda]}(x_1, \ldots, x_N; x_1, \ldots, x_N) = \frac{1}{f_{\lambda}} \sum_{r=1}^{f_{\lambda}} \Psi_{rr}^{[\lambda]}(x_1, \ldots, x_N)^* \Psi_{rr}^{[\lambda]}(x_1, \ldots, x_N),
\]

where the expression (11) is written for the case of the \(f_{\lambda}\)-dimensional representation \(\Gamma^{[\lambda]}\) of the permutation group \(\pi_N\) and the wave functions \(\Psi_{rr}^{[\lambda]}\) are constructed by the Young operators \(\omega_{rr}^{[\lambda]}\), see Appendix, Eq. (21).

In 1937 V. Fock presented a proof of PEP in his unpublished lectures on quantum mechanics [90]. In his proof Fock substituted the correct expression (10) in equation for an arbitrary operator (8) and applied the variational theorem. However, as we discussed above, Eq. (8) is valid only in the case of non-degenerate states. The Fock proof is failed if one applies it to the expression for degenerate states.

It can be proved that for every representation \(\Gamma^{[\lambda]}\) of the permutation group \(\pi_N\), the probability density, Eq. (11), is a group invariant, that is, it is invariant upon action of an arbitrary permutation. For an arbitrary finite group, it was proved in Ref. [91]. Thus, for every permutation of the group \(\pi_N\)

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6 I am grateful to S. Zagoulaev for this information.
The Eq. (12) means that for all irreducible representations \( \Gamma^{[\lambda]} \) of the permutation group \( \pi_N \), the diagonal element of the full density matrix (and all reduced densities matrices as well) transforms according to the totally symmetric one-dimensional representation of \( \pi_N \) and in this respect one cannot distinguish between degenerate and nondegenerate permutation states. The probability density (11) obeys the indistinguishability principle even in the case of multi-dimensional representations of the permutation group. Thus, the indistinguishability principle is insensitive to the symmetry of wave function and cannot be used as a criterion for selecting the correct symmetry.

Although the symmetrization postulate cannot be rigorously derived from other quantum-mechanical postulates, there are physical arguments indicating that the description of an identical particle system by the multi-dimensional representations of the permutation group leads to contradictions with the concept of the particle identity and their independency. In next subsection we will discuss these arguments in detail.

2.2 Arbitrary permutation symmetry and arising contradictions with the concept of particle identity and their independence

Let us consider a quantum system of \( N \) identical particles without the restrictions imposed by PEP. Thus, we assume that all possible permutation symmetries of this system can be realized. The permutation symmetry of \( N \) particle system is characterized by the irreducible representations of the permutation group \( \pi_N \), which are labeled by the symbol \([\lambda]\) of the corresponding Young diagram with \( N \) boxes and denoted by \( \Gamma^{[\lambda]} \) or simply by \([\lambda]\), see Appendix.

The basis functions of an arbitrary irreducible representation \( \Gamma^{[\lambda]} \) can be constructed by applying the Young operators \( \omega_{r_s}^{[\lambda]} \), see Eq. (21) in Appendix, to the non-symmetrized product of one-electron orthonormal functions (spin-orbitals) \( \phi_n(i) \)

\[
\Phi_0 = \phi_1(1)\phi_2(2)\cdots\phi_N(N),
\]

numbers \( i \) in the argument of the function denote the set of particle spin and space coordinates. For simplicity, we consider the case when all one-particle functions in Eq. (13) are different. There will be no qualitative changes in the results, if some of them coincide. Thus,
The symmetry postulate demands that only one-dimensional irreducible representations, either \([N]\) or \([1^N]\), are realized in Nature; all other irreducible representations are forbidden. In this subsection we examine the situation that arises when no symmetry constraints are imposed.

One of the consequences of the different permutation symmetry of wave functions for bosons and fermions is the dependence of the energy on the particle statistics. For the same law of dynamic interaction, the so-called exchange terms, which are appeared in the one-particle approximation (Hartree-Fock approach), enter the expression for the energy of fermion and boson system with opposite signs. The expression for the energy in the state with symmetry, described by an arbitrary Young diagram \([\lambda]\) with N boxes, was obtained in Ref. [92] in a general case of non-orthogonal one-particle functions. In the case when all functions in Eq. (13) are different and orthogonal one gets

\[
E_{\lambda}^{[\lambda]} = \sum_{a} \langle \phi_{a} | h | \phi_{a} \rangle + \sum_{a < b} \left[ \langle \phi_{a} \phi_{b} | g | \phi_{b} \phi_{a} \rangle + \Gamma_{tt}^{[\lambda]}(P_{ab}) \langle \phi_{a} \phi_{b} | g | \phi_{b} \phi_{a} \rangle \right]
\]

where \(\Gamma_{tt}^{[\lambda]}\) is the diagonal matrix element of the transposition \(P_{ab}\) of functions \(\phi_{a}\) and \(\phi_{b}\) in Eq.(13); \(h\) and \(g\) are one- and two-particle interaction operators, respectively. Only exchange terms in Eq.(15) depend upon the permutation symmetry \([\lambda]\) of the state. For one-dimensional representations, \(\Gamma_{tt}^{[\lambda]}(P_{ab})\) does not depend on the number of particles and the permutation \(P_{ab}: \Gamma^{[N]}(P_{ab}) = 1\) and \(\Gamma^{[1^N]}(P_{ab}) = -1\) for all \(P_{ab}\) and \(N\). For multi-dimensional representations, the matrix elements \(\Gamma_{tt}^{[\lambda]}(P_{ab})\) depend on \([\lambda]\) and \(P_{ab}\); in general, they are different for different pairs of identical particles.

Thus,

1) transitions between states with different symmetry \([\lambda\lambda]\) are strictly forbidden;

2) each state of \(N\) particle system with different \([\lambda\lambda]\) has a different analytical formula for its energy.

---

8 The matrices of transpositions for all irreducible representations of groups \(\pi_{1} \rightarrow \pi_{6}\) are presented in book [93], Appendix 5.
From this follows that each type of symmetry $[\lambda_N]$ corresponds to a *definite kind of particles with statistics determined by this permutation symmetry*.

On the other hand, the classification of state with respect to the Young diagrams $[\lambda_N]$ is connected exclusively with identity of particles. Therefore, it must be some additional *inherent particle characteristics*, which establishes for the $N$ particle system to be in a state with definite permutation symmetry, like integer and half-integer values of particle spin for bosons and fermions; and this inherent characteristic has to be different for different $[\lambda_N]$. So, the particles belonging to the different types of permutation symmetry $[\lambda_N]$ *are not identical*, as it is in the particular cases of bosons, $[N]$, or fermions, $[1^N]$, where they do not depend on $N$.

Let us trace down the genealogy of the Young diagrams for systems with different number of particles. In Fig. 2 the genealogy for all permitted $[\lambda_N]$ with $N = 2$ to $4$ is presented.

![Fig. 2. The Young diagrams for $N = 2$-4 and their genealogy](image)

We called the hypothetical particles characterized by the multi-dimensional representations of the permutation group as *intermedions* implying that they obey some intermediate between fermion and boson statistics.

According to Fig. 2, for bosons and fermions there are two non-intersecting chains of irreducible representations: $[N] \rightarrow [N+1]$ and $[1^N] \rightarrow [1^{N+1}]$, respectively; and the energy expressions for bosons and fermions have the analytical form, which does not depend on the number of particles in a system. The situation drastically changes, if we put into
consideration the Young diagrams describing the multi-dimensional representations. In this case, as we showed above, different $\lambda_N$ describe particles with different statistics. The number of different statistics depends upon the number of particles in a system and rapidly increases with $N$. For the multi-dimensional representations, we cannot select any non-intersecting chains, as in the fermion and boson cases.

As follows from Fig. 2, the intermedion particles with a definite $\lambda_N$ in the $N$th generation can originate from particles with different kinds $\lambda_{N-1}$ in the $(N-1)$th generation, even from fermions or bosons. Thus, the $N$-particle state $\lambda_N$ stems from the particles in the $(N-1)$th generation with wave function, which must be in general described by a linear combination of wave functions with different permutations symmetry $\lambda_{N-1}$. However, this linear combination does not describe identical particles; see Eqs. (16) and (17) below and their discussion.

The physical picture, in which adding one particle changes properties of all particles, cannot correspond to a system of independent particles, although, it cannot be excluded for quasiparticles (collective excitations) systems, in which quasiparticles are not independent. It is the case of excitons and magnons [50] or other quasiparticle systems [54], see also Chapter 5 in book [86].

For ideal gas, it is evident that adding a particle identical to a system of $N$ identical particles cannot change the properties of a new $(N+1)$-particle system. On the other hand, the interaction of identical particles does not change the permutation symmetry of non-interacting particle system; it can be rigorously proved, see Ref. [80].

Thus, the scenario, in which all symmetry types $\lambda_N$ are allowed and each of them corresponds to a definite particles statistics, contradicts to the concept of particle identity and their independency from each other.

Let us consider the possibility that for some type of intermedions with fixed $N$, a multi-dimensional representation $\lambda_N$ exists that stems only from one $\lambda_{N-1}$, as $[2^2]$ in the case $N = 4$, see Fig. 2. But in the process of reducing the number of particles we cannot avoid the case $N = 3$ where only one multi-dimensional representation exists with $\lambda_3 = [21]$ and this representation proceeds from both two-particle representations: $\lambda_2 = [2]$ corresponding to bosons and $\lambda_2 = [1^2]$ corresponding to fermions. Thus, if from the three-particle state $[21]$ delete on particles, the wave function describing the obtained two-particle state should be a linear combinations of symmetric and antisymmetric wave functions. However, on the contrary to the statements in Refs. [94, 95] about hypothetical
paraparticles, which were more emotional than physical, the wave function of two identical particles may not be described by some superposition

$$\Psi_n(x_1, x_2) = c_1 \Psi^{[2]}(x_1, x_2) + c_2 \Psi^{[2]}(x_1, x_2).$$

This superposition corresponds to non-identical particles, since it does not satisfy the indistinguishability principle. In fact,

$$P_{12}[\Psi_n(x_1, x_2)]^2 = |c_1 \Psi^{[2]}(x_1, x_2) - c_2 \Psi^{[2]}(x_1, x_2)|^2 \neq |\Psi_n(x_1, x_2)|^2. \quad (17)$$

Let us stress that the permutation group can be applied only to identical particles and these particles are transformed according to the irreducible representations $\Gamma^{[2]}$ of the permutation group, but not according to their linear combinations.

For two identical particles it is evident that they can be only in the pure fermion or boson states. However, if the multi-dimensional representations of the permutation group are permitted, the addition of the third particle identical to the two others changes the fermion (or boson) statistics on the intermedion statistics with $[\lambda_3] = [21]$, see, Fig. 2. As we discussed above, the interaction of identical particles does not change the permutation symmetry of non-interacting particle system, so the change of statistics takes place even in an ideal gas of identical particles. Again we obtained a contradiction with the concept of particle identity and their independence.

Nevertheless, it is worth-while to mentioned that the multi-dimensional representations of the permutation group can be used in quantum mechanics of identical particles, although not for the total wave function, but for its factorized parts [93].

It is also important to stress that the existence of so-called fractional statistics does not contradict PEP. According to fractional statistics, see subsection 5.4 in book [86], in the 2D-space a continuum of intermedium cases between boson and fermion cases can exist. First this was shown by topological approach by Leinaas and Myrheim [83] and then by Wilczek [96], who introduced anyons that obey any statistics. However, anyons are not particles, they are quasiparticles (elementary excitations) in 2D-space. Particles can exist only in 3D-space and for them, according to PEP, only boson and fermion symmetries are allowed.

Thus, as it was demonstrated above, the permission of multi-dimensional representations of the permutation group for the total wave function leads to contradictions with the concepts of particle identity and their independence. All
contradictions in discussed scenarios are resolved, if only the one-dimensional irreducible representations of the permutation group (symmetric and antisymmetric) are permitted. Although the so-called symmetrization postulate cannot be derived from other fundamental principles of quantum mechanics, it may not be considered as a postulate, since all symmetry options for the total wave function, except the one-dimensional irreducible representations, corresponding fermions and bosons, cannot be realized. These arguments can be considered as an answer on the one aspect of the PEP: why in Nature only completely symmetric or antisymmetric multi-particle states are realized. However, the problem of connection between the value of spin and the permutation symmetry of wave function (SSC) is still unsolved.

3. Experimental verifications

All experimental data known to date agree with PEP. The theoretical ideas for experimental searches of possible violations of PEP where discussed in reviews [97-99]. The published experimental tests of PEP were classified in the review report by Gillaspy [100], see also Ignatiev [99]. Below I will discuss only the spectroscopic verifications.

Okun [97] stressed that except the violations of PEP it can be violations of the electric charge conservation law, due to a possible decay of the electron which may not be excluded. This idea was first discussed by Okun and Zeldovich [101] and Ignatiev et al. [102]. The probable decay of one electron on the 1s² atomic shell will induce the allowed in quantum mechanics radiative transition 2p⁶ → 1s¹, which can be measured, while the transition to the filled 1s² K-shell can be only if PEP is violated. However, as can be concluded from the discussion in Dirac’s book [72] and analyzed by Amado and Primakoff [103], in the framework of quantum mechanics with the commutation-invariant Hamiltonian the transitions to a filled shell are forbidden, even if PEP is violated.

Amado and Primakoff [103] discussing the experiments by Reines-Sobel [104] for electrons and by Logan-Ljubicic [105] for nuclei have showed that due to the identity of particles these experiments cannot test PEP, since the transitions from the normal antisymmetric fermionic states are possible only to the states with the same antisymmetric character (well-known spectroscopic selection rules). The non-Pauli electrons, if they are existing, must have another symmetry and differ from “normal” electrons. The paper [103] was cited in most experimental publications devoted to the verification of PEP, although it was misinterpreted. In consequent experimental publications it was accepted that there are no prohibitions on the transition of non-Pauli electrons to the filled K-shell,
only it can be expected that these transitions were \textit{already} occurred (the last statement was mentioned also in the Amado-Primakoff paper [103]). Thus, it was concluded that the K-shell radiation can be measured if experiment is performed with “new” ejected electrons, as was suggested first in the Ramberg and Show experiment [106]. That publication induced a great number of experimental verifications made according to the scheme represented on Fig. 3.

![Fig. 3](image)

**Fig. 3** The schematic representation of the formation of the Pauli-forbidden atomic inner-shell populations in the experimental search of the non-Pauli electrons.

Last years the systematic spectroscopic study of the validity of PEP using Ramberg-Show approach has been carried out by the VIP collaboration [107-109]. In their experiments they performed a search of X-rays produced by the Pauli-forbidden atomic transition from the $2p$ shell to the closed $1s^2$ shell of Cu atoms, forming the non-Pauli $1s^3$ shell, see Fig. 3. The obtained probability that PEP is violated (it is denoted as $\frac{1}{2} \beta^2$), according to their last measurements [108, 109], was

\[
\frac{1}{2} \beta^2 < 6 \times 10^{-29}. \tag{18}
\]

In the experiments performed in the Los Alamos laboratory by Elliott et al. [110], Pb instead of Cu was used. They reported a much stronger limit on the violation of PEP for electrons. Namely:

\[
\frac{1}{2} \beta^2 < 2.6 \times 10^{-39}. \tag{19}
\]
This limit was obtained by a modified procedure of the experimental data processing. The authors [110] did not divide electrons on “new” and “old”, they took into account all free electrons. The application of this approach to the VIP data also changes their limit on ten orders.

However, these experiments in principle cannot lead to the verification of PEP and not only due to the noted by Amado and Primakoff [103] superselection rule. Below we present a list of problems arising in applications of experimental scheme on Fig. 3.

1. Usually experimenters have considered the violation of PEP as a small admixture of the symmetric wave functions to the antisymmetric ones. They started from the Ignatiev-Kuzmin [111] and Greenberg-Mohapatra [112] theoretical suggestions; last years Greenberg’s quon model [113] has been applied. These approaches are based on the second quantization formalism, in which only the symmetric and antisymmetric states are defined. In general, this limitation on the permutation symmetry of the possible states is not valid, because the solutions of the Schrödinger equation may belong to any representation of the permutation group, see Eq. (4) and discussion in the end of Subsection 1.3. The violation of PEP means that there are some electrons described by wave functions with arbitrary permutation symmetry, not necessarily the symmetric one.

2. The electrons not satisfying PEP are not described by the antisymmetric wave functions, therefore they may not be mixed with the “normal” electrons that are the fermions, since the transitions between states with different permutation symmetry are strictly forbidden, just this was stressed in Ref. [103]. Thus, the transitions may take place only inside the group of probable non-Pauli electrons and these electrons are not identical to the “normal” Pauli electrons; in other case they must be characterized by the antisymmetric wave functions.

3 The last but not the least. Since the Pauli and probable non-Pauli electrons may not possess the same permutation symmetry, it is quite doubtful that the non-Pauli electrons can be located on the same shell as the “normal” Pauli electrons (fermions). We must take into account that the energy of identical particle system depends upon its permutation symmetry, see Eq. (15). Even if one assumes that in transition $2p^7 \rightarrow 1s^3$ (Fig.3) the only non-Pauli electron can be involved, then the important question about the magnitude of transition energy must be arise. In this
connection it should be mentioned that in the discussed experiments the energy of
the forbidden transition was calculated for the Pauli forbidden electron shells, but
all electrons were considered as fermions, see Refs. [107, 110]. However, in the
case of non-Pauli electrons the transition energy would be in another energy region
than the estimated energy for the fermion system. Thus, even if really a small part
of electrons exists that does not obey PEP, it is very improbable that the X-ray
transitions of these non-Pauli electrons could be detected in the experiments
described above.

From these comments follows that the numerous spectroscopic experiments that have
been performed for the verification of PEP, including experiments [104-110], could not
be appropriate for this aim.

Concluding remarks

As we showed in subsection 2.1, the indistinguishability principle is insensitive to the
permutation symmetry of wave function and is satisfied by wave functions with arbitrary
symmetry; they can belong to the multi-dimensional representations of the permutation
group characterized by the Young diagrams $[\lambda_N]$ of general type. So, the
indistinguishability principle cannot be used for the verification of PEP and the proofs
based on it, including proofs in textbooks [77-79, 87], are incorrect.

However, as it was demonstrated in subsection 2.2, different scenarios, in which an
arbitrary permutation symmetry (multi-dimensional representations) was permitted, lead
to contradictions with the concept of particle identity and their independence. Thus, the
symmetrization postulate may not be considered as a postulate, since particles describing
by wave functions with symmetry, corresponding to multi-dimensional representations
of the permutation group, may not exist. These arguments explain why in Nature only
completely symmetric or antisymmetric multi-particle states are realized and can be
considered as an theoretical substantiation of PEP.

The realization in Nature only one-dimensional permutation symmetry (symmetric
and antisymmetric) is by no means accidental, as was accepted [72]. From this an
important conclusion follows: we may not expect that in future some unknown
elementary particles can be discovered that are not fermions or bosons.

In Section 3 we presented arguments that if a small part of electrons exists that do not
obey PEP, these non-Pauli electrons could not be detected in spectroscopic experiments,
in which “forbidden” X-rays have been measured. Nevertheless, it must not be any
doubts in PEP. All experimental data completely confirm PEP. At present we do not know any phenomena described by quantum mechanics where PEP was not satisfied.

This is confirmed also by very precise calculations of H₂ molecule, in which, certainly, PEP was taken into account. The quantum mechanical calculations of its dissociation energy and the first ionization potential [114, 115] are in a complete agreement with very precise experimental values, see Table 1.1 in [116]. From this follows not only an additional confirmation of PEP, but also a rather general conclusion that molecules obey the same quantum-mechanical laws that obey traditionally physical objects: atoms and solids; at nanoscale we should not distinguish between chemical and physical systems.

Appendix. Necessary minimum knowledge on the permutation group.

The permutation symmetry is classified according to the irreducible representations of the permutation group \( \pi_N \). The latter are labeled by the Young diagrams

\[
[\lambda] = [\lambda_1 \lambda_2 \ldots \lambda_k],
\]

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k, \quad \sum_{i=1}^k \lambda_i = N
\]

where \( \lambda_i \) is represented by a row of \( \lambda_i \) cells. The presence of several rows of identical length \( \lambda_i \) is convenient to indicate by a power of \( \lambda_i \). For example,

\[
\begin{align*}
[\lambda] &= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \\
[\lambda] &= \begin{bmatrix} 2 \\ 1^2 \end{bmatrix}
\end{align*}
\]

It is obvious that one can form from two cells only two Young diagrams:

\[
\begin{align*}
\begin{bmatrix} 2 \\ 1^2 \end{bmatrix}
\end{align*}
\]

\[\text{For a more detailed treatise see books [93] and [117].}\]
For the permutation group of three elements, $\pi_3$, one can form from three cells three Young diagrams:

- [3]
- [2 1]
- [1^3]

The group $\pi_4$ has five Young diagrams:

- [4]
- [3 1]
- [2^2]
- [2 1^2]
- [1^4]

Each Young diagram $[\lambda]$ uniquely corresponds to a specific irreducible representation $\Gamma^{[\lambda]}$ of the group $\pi_N$. The assignment of a Young diagram determines the permutation symmetry of the basis functions for an irreducible representation, i.e. determines the behavior of the basis functions under permutations of their arguments. A diagram with only one row corresponds to a function symmetric in all its arguments. A Young diagram with one column corresponds to a completely antisymmetric function. All other types of diagrams correspond to intermediate types of symmetry. There are certain rules that enable one to find the matrices of irreducible representations of the permutations group from the form of the corresponding Young diagram. Such rules are especially simple in the case of the so-called standard orthogonal representation (this is the Young-Yamanouchi representation; see Ref. [93]).

The basis functions for an irreducible representation $\Gamma^{[\lambda]}$ can be constructed by means of the so-called normalized Young operators [93]^9,

$$\omega^{[\lambda]}_{rt} = \frac{f_\lambda}{\sqrt{N!}} \sum_p \Gamma^{[\lambda]}_{rt}(P)P$$  \hspace{1cm} (21)$$

---

^9 Operator (21) should not be mixed up with the operator that symmetrizes the rows and antisymmetrizes the columns in Young diagram, which Hamermesh [118] is named as Young operator.
where the summation over $P$ runs over all the $N!$ permutations in the group $\pi_N$, $\Gamma_{rt}^{[\lambda]} (P)$ are the matrix elements and $f_\lambda$ is the dimension of the irreducible representation $\Gamma^{[\lambda]}$. The application of operator (21) to a nonsymmetrized product of orthonormal one-particle functions $\varphi_\alpha$

$$\Phi_0 = \varphi_1(1)\varphi_2(2) \ldots \varphi_N(N)$$

(22)

produces a normalized function

$$\Phi_{rt}^{[\lambda]} = \omega_{rt}^{[\lambda]} \Phi_0 = \sqrt{\frac{f_\lambda}{N!}} \sum_P \Gamma_{rt}^{[\lambda]} (P) P \Phi_0$$

(23)

transforming in accordance with the representation $\Gamma^{[\lambda]}$.

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