A Complete Unitary Similarity Invariant for Unicellular Matrices

Douglas Farenick

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

Tatiana G. Gerasimova

The Faculty of Mechanics and Mathematics, Kiev National Taras Shevchenko University, Volodymyrska St, 64, Kiev-33, 01033, Ukraine

Nadya Shvai

The Faculty of Mechanics and Mathematics, Kiev National Taras Shevchenko University, Volodymyrska St, 64, Kiev-33, 01033, Ukraine

Abstract

We present necessary and sufficient conditions for an $n \times n$ complex matrix $B$ to be unitarily similar to a fixed unicellular (i.e., indecomposable by similarity) $n \times n$ complex matrix $A$.

Keywords: unitary similarity problem, unicellular matrix, Toeplitz matrix, Volterra operator

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1. Introduction

A fundamental problem in matrix analysis is the unitary similarity problem [2, 3]. Under what necessary and sufficient conditions are two $n \times n$ complex matrices unitarily similar? A classical and purely algebraic solution to this problem due to Specht [7, 10]: two $n \times n$ complex matrices $A$ and $B$ are unitarily similar if and only if

$$\text{Trace } \omega(A, A^*) = \text{Trace } \omega(B, B^*),$$

for every word $\omega$ in two noncommuting variables $x$ and $y$. 
In many applications, the data one has about a particular matrix are not based on the trace of the matrix, but rather on some other analytical information: the spectrum or pseudospectrum, the numerical range or polynomial numerical hull, the singular values, a unitarily invariant norm, and so forth. Our concern in the present paper is with a solution to the unitary similarity problem that is based on a particular choice of unitarily invariant norm.

Let $M_n$ be the space of all $n \times n$ complex matrices; we denote the unitary group by $U_n$. Two matrices $A, B \in M_n$ are unitarily similar, which we express by $A \sim B$, if there is a $U \in U_n$ such that $B = U^* A U$. The norm under study is defined by

$$\| A \| = \sqrt{\text{spr} \ (A^* A)}, \tag{2}$$

where $\text{spr} \ X$ is the spectral radius of $X \in M_n$. The norm (2) has the property that $\| U^* A U \| = \| A \|$, for all $A \in M_n$ and $U \in U_n$, and it coincides with the largest singular value of $A$. Moreover, if $A \in M_n$ is considered as a linear transformation on the complex inner product space $\mathbb{C}^n$ with respect to the standard inner product $\langle \xi, \eta \rangle = \eta^* \xi$, for $\xi, \eta \in \mathbb{C}^n$, then

$$\| A \| = \max_{\langle \xi, \xi \rangle = \langle \eta, \eta \rangle = 1} | \langle A \xi, \eta \rangle |.$$

Let $\mathbb{C}[t]$ denote the ring of polynomials with complex coefficients. If $A \sim B$, then necessarily $\| f(A) \| = \| f(B) \|$ for all $f \in \mathbb{C}[t]$. Conversely, if $A, B \in M_n$ are such that $\| f(A) \| = \| f(B) \|$ for all $f \in \mathbb{C}[t]$, then $A$ and $B$ yield to the same matrix analysis:

(i) $A$ and $B$ have the same spectrum;
(ii) $A - zI$ and $B - zI$ have the same condition numbers, for all nonspectral $z$ in the complex plane;
(iii) $A$ and $B$ have the same polynomial numerical hulls and, in particular, the same numerical range;
(iv) $A$ and $B$ have the same spectral set;
(v) $A$ and $B$ have the same pseudospectrum.

Our first objective is to determine cases in which the condition $\| f(A) \| = \| f(B) \|$ for all $f \in \mathbb{C}[t]$ is also sufficient for $A \sim B$. In general it will not be so, for if one takes any two nonzero projections (selfadjoint idempotents) $P$ and $Q$, then one has $\| f(P) \| = \| f(Q) \|$ for all $f \in \mathbb{C}[t]$, independent of the ranks of $P$ and $Q$. Therefore, for questions concerning unitary similarity, the hypothesis $\| f(A) \| = \| f(B) \|$ for all $f \in \mathbb{C}[t]$ is relevant only for the analysis of nonnormal matrices.
Definition 1.1. A matrix $A \in M_n$ is said to be \textit{unicellular} if $A$ is not similar to a matrix $B \in M_n$ of the form $B = G \oplus H$, for some square matrices $G$ and $H$ of strictly smaller size than $B$.

Our use of the term unicellular matrix is motivated by the concept of unicellular operator or transformation in operator theory. If $A \in M_n$ is a unicellular matrix, then $A$ is unicellular in the sense of [5, §9], [6, §2.5] as a linear transformation on $\mathbb{C}^n$. Unicellular matrices are also said to be \textit{indecomposable by similarity}.

In this paper we present two main results. The first, Theorem 2.1, states that the unitary similarity class of any upper triangular unicellular Toeplitz matrix $R$ is determined by the values of $\|f(R)\|$ for various $f \in \mathbb{C}[t]$. If one drops the requirement that $R$ be Toeplitz, yet remain upper triangular and unicellular, then the values of $\|f(R)\|$, for $f \in \mathbb{C}[t]$, are insufficient to identify $R$ up to unitary similarity (Proposition 3.1). But with our second main result, Theorem 3.2, we augment the criterion slightly to obtain necessary and sufficient conditions that classify unicellular matrices up to unitary similarity (see, also, Proposition 5.1).

2. Upper Triangular Toeplitz Matrices

Definition 2.1. A matrix $R \in M_n$ is an \textit{upper triangular Toeplitz matrix} if

$$R = \begin{bmatrix}
  z_0 & z_1 & a_2 & \cdots & z_{n-1} \\
  0 & z_0 & z_1 & \ddots & \vdots \\
  0 & 0 & \ddots & \ddots & z_2 \\
  \vdots & \ddots & \ddots & \ddots & z_1 \\
  0 & \ldots & \ldots & 0 & z_0
\end{bmatrix}, \quad (3)$$

for some $z_0, z_1, \ldots, z_{n-1} \in \mathbb{C}$.

The set of all upper triangular Toeplitz matrices $R \in M_n$ is denoted by $\text{UpperToepl}_n$.

The main theorem of this section is:

Theorem 2.1. Let $R \in M_n$ be an upper triangular Toeplitz matrix \textit{(3)} with $z_1 \neq 0$. If $A \in M_n$ is any matrix for which $\|f(A)\| = \|f(R)\|$, for all $f \in \mathbb{C}[t]$, then $A \sim R$. 

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Before moving to the proof of Theorem 2.1, let us consider one of its consequences, namely Corollary 2.2 below, which is of interest in linear-algebraic analysis. For any $A \in M_n$, the unital algebra $\text{Alg} A$ generated by $A$ is

$$\text{Alg} A = \{ f(A) : f \in \mathbb{C}[t] \}.$$

In particular, $\text{UpperToepl}_n = \text{Alg} S$, where

$$S = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix}.$$

More generally, if $R \in \text{UpperToepl}_n$ is of the form (3) and satisfies $z_1 \neq 0$, then the range of $R - z_0 I$ is clearly $(n - 1)$-dimensional and so the kernel of $R - z_0 I$ is 1-dimensional. Thus, there is an invertible $X \in M_n$ for which $S = X(R - z_0 I)X^{-1}$, the Jordan canonical form of $R - z_0 I$. Hence, the abelian algebras $\text{UpperToepl}_n = \text{Alg} S$ and $\text{Alg} R$ are isomorphic. Because $\text{Alg} R$ is a subalgebra of $\text{UpperToepl}_n$, they can be isomorphic only if they are equal. Thus, if $R \in \text{UpperToepl}_n$ satisfies $z_1 \neq 0$, then $R$ is called a generator of $\text{UpperToepl}_n$. (Consideration of the Jordan form shows that this necessary condition on $z_1$ is also sufficient for $R \in \text{UpperToepl}_n$ to be a generator of $\text{UpperToepl}_n$, but we do not require this fact.)

**Corollary 2.2.** If $\varphi : \text{UpperToepl}_n \to M_n$ is a homomorphism such that $\|\varphi(X)\| = \|X\|$, for every $X \in \text{UpperToepl}_n$, then there is a $U \in U_n$ such that $\varphi$ is given by $\varphi(X) = U^*XU$.

**Proof.** Choose $R \in \text{UpperToepl}_n$ of the form (3) with $z_1 \neq 0$ and let $A = \varphi(R)$. Thus, $f(A) = \varphi(f(R))$, for all $f \in \mathbb{C}[t]$. By hypothesis, $\|f(A)\| = \|\varphi(f(R))\| = \|f(R)\|$, for all $f \in \mathbb{C}[t]$; therefore, Theorem 2.1 asserts that $A = U^*RU$ for some $U \in U_n$. Because $R$ generates $\text{UpperToepl}_n$, we conclude that $\varphi(X) = U^*XU$, for every $X \in \text{UpperToepl}_n$. □

We move now to the proof of Theorem 2.1.
2.1. Lemmas

Lemma 2.3. If

\[
Q = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & 0
\end{bmatrix}, \tag{4}
\]

then \[\sum_{k=1}^{\infty} (-1)^{k+1}Q^k = S.\]

Proof. Clearly \(Q = \sum_{k=1}^{\infty} S^k.\) Thus, \(I + Q = \sum_{j=0}^{\infty} S^j = (I - S)^{-1},\) whence \(I = (I - S)(I + Q).\) That is, \(S = I - (I + Q)^{-1} = \sum_{k=1}^{\infty} (-1)^{k+1}Q^k.\)

Lemma 2.4. Let \(Q \in M_n\) be given by (4). If

\[
A = \begin{bmatrix}
0 & 1 & a_{13} & \cdots & a_{1n} \\
0 & 1 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & a_{n-2,n} & \vdots \\
\vdots & \ddots & \ddots & 1 & 0
\end{bmatrix}
\]

has the property that \(\left\| \sum_{k=1}^{\infty} (-1)^{k+1}A^k \right\| \leq 1,\) then \(A = Q.\)

Proof. We proceed by induction on \(n.\) The base case is \(n = 3.\) In this case,

\[
\sum_{k=1}^{\infty} (-1)^{k+1}A^k = \begin{bmatrix}
0 & 1 & a_{13} - 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

The first row of the matrix above has Euclidean length at most 1, since \(A\) has norm at most 1. Thus, \(a_{13} = 1,\) implying that \(A = Q.\) This row condition extends unchanged to the induction step.
Assume now the statement holds in $n$-dimensional space and consider $A$, $Q$, and $S$ as acting on $\mathbb{C}^{n+1}$. Let $\tilde{A}$, $\tilde{Q}$, and $\tilde{S}$ denote the versions of $A$, $Q$, and $S$ that act on $\mathbb{C}^n$, and let $e_1, \ldots, e_n$ denote the canonical orthonormal basis vectors in $\mathbb{C}^n$. Hence, as a partitioned matrix, $A$ has the form

$$A = \begin{bmatrix} \tilde{A} & \eta \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where

$$\eta = e_n + \sum_{i=1}^{n-1} a_{i,n+1} e_i = [a_{1,n+1}, \cdots, a_{n-1,n+1}]^T \in \mathbb{C}^n.$$

Because

$$1 \geq \left\| \sum_{k=1}^{\infty} (-1)^{k+1} A^k \right\| = \left\| \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{A}^k \right\|,$$  \hspace{1cm} \text{(5)}

the induction hypothesis yields $\tilde{A} = \tilde{Q}$. Hence, using Lemma 2.3, we obtain

$$\sum_{k=1}^{\infty} (-1)^{k+1} A^k = \begin{bmatrix} \tilde{S} & \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{A}^{k-1} \eta \\ 0 & \cdots & 0 \end{bmatrix}.$$  \hspace{1cm} \text{(6)}

That is,

$$\sum_{k=1}^{\infty} (-1)^{k+1} A^k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \ast \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 & \ast \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix},$$  \hspace{1cm} \text{(6)}

Similar to the case $n = 3$, we have from $\text{(5)}$ that

$$1 \geq \left\| \sum_{k=1}^{\infty} (-1)^{k+1} A^k \right\|.$$

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for the matrix (6). But (7) holds for the matrix (6) only if the \(i\)-th entry in the final column of the matrix (6) is 0, for \(1 \leq i \leq n - 1\). Therefore, using \(\tilde{A} = \tilde{Q}\), we have \((\tilde{I} - \tilde{S})\eta = \lambda e_n\) for some complex number \(\lambda\). Hence,

\[
\eta = \lambda (\tilde{I} - \tilde{S})^{-1} e_n = \lambda (\tilde{I} + \tilde{S} + \tilde{S}^2 + \cdots + \tilde{S}^{n-2}) e_n = \lambda (e_n + e_{n-1} + \cdots + e_1).
\]

But on the other hand,

\[
\eta = e_n + \sum_{i=1}^{n-1} a_{i,n+1} e_i,
\]

which implies that \(\lambda = 1\) and \(a_{i,n} = 1\) for all \(1 \leq i \leq n - 1\). Therefore, \(A = Q\).

2.2. Proof of Theorem 2.1

Assume first that the matrix \(R\) in (3) has \(z_0 = 0\) and \(z_j = 1\) for \(1 \leq j \leq (n - 1)\); that is, assume that \(R = Q\), where \(Q\) has the form (4). Thus, the hypothesis is that \(A \in \mathbb{M}_n\) satisfies \(\|f(A)\| = \|f(Q)\|\), for all \(f \in \mathbb{C}[t]\).

By the Spectral Radius Formula,

\[
0 = \text{spr } Q = \lim_{k \to \infty} \|Q^k\|^{1/k} = \lim_{k \to \infty} \|A^k\|^{1/k} = \text{spr } A,
\]

which implies that \(A\) is nilpotent. Without loss of generality, \(A\) may be assumed to be in upper triangular form. Furthermore, using a diagonal unitary similarity transformation, the entries \(a_{i,i+1}\) may assumed to be nonnegative, for \(1 \leq i \leq n - 1\). Indeed, since \(1 = \|Q^{n-1}\| = \|A^{n-1}\| = |a_{12}a_{23}\cdots a_{n-1,n}|\), each \(a_{i,i+1}\) is nonzero; thus, we may assume that \(a_{i,i+1} > 0\) for all \(i\).

The numerical range, or field of values, \(W(X)\) of any \(X \in \mathbb{M}_n\) is given analytically by

\[
W(X) = \bigcap_{\alpha, \beta \in \mathbb{C}} \{ z \in \mathbb{C} : |\alpha z + \beta| \leq \|\alpha X + \beta 1\| \}.
\]

Hence, \(W(A) = W(Q)\). Let \(\Re(X) = \frac{1}{2}(X + X^*)\), for any \(X \in \mathbb{M}_n\), and observe that \(\frac{1}{2} + \Re(Q) = \frac{1}{2} \xi \otimes \xi\), where \(\xi = \sum_{i=1}^{n} e_i \in \mathbb{C}^n\) and \(\xi \otimes \xi\) denotes the outer product \(\xi \xi^* \in \mathbb{M}_n\) of \(\xi\) (a column vector) with its conjugate transpose \(\xi^*\). Thus, for every unit vector \(\gamma \in \mathbb{C}^n\), the real part of \(\langle Q\gamma, \gamma \rangle\) satisfies the inequality

\[
\Re(\langle Q\gamma, \gamma \rangle) \geq -\frac{1}{2}.
\]
Because $A$ and $Q$ have the same numerical range, $\Re(A)$ has the same property above. Now, if $P_i$ is the projection of $\mathbb{C}^n$ onto $\text{Span} \{e_i, e_{i+1}\}$, for each $1 \leq i \leq n - 1$, then $P_iAP_i$ as a linear transformation on the range of $P_i$ is given by

$$
\begin{bmatrix}
0 & a_{i,i+1} \\
0 & 0 
\end{bmatrix}.
$$

Therefore, the numerical range of $P_iAP_i$ is a disc of radius $\frac{1}{2}a_{i,i+1}$ centered at the origin. Because $W(P_iAP_i) \subseteq W(A) \subseteq \{z \in \mathbb{C} \mid \Re(z) \geq -1/2\}$, we conclude that each $a_{i,i+1} \leq 1$. However, under these conditions the equation $1 = \|A_{n-1}\| = a_{12}a_{23}\cdots a_{n-1,n}$ holds only if $a_{i,i+1} = 1$ for all $1 \leq i \leq n - 1$.

Hence, $A$ has the structure given in the hypothesis of Lemma 2.4. Moreover, by Lemma 2.3,

$$
1 = \|S\| = \left\| \sum_{k=1}^{\infty} (-1)^{k+1}Q^k \right\| = \left\| \sum_{k=1}^{\infty} (-1)^{k+1}A^k \right\|.
$$

Thus, $A$ satisfies all of the hypotheses of Lemma 2.4 yielding $Q = A$.

For the general case, we now suppose that $R \in \text{UpperToepl}_n$ satisfies $z_1 \neq 0$ and $A \in \mathbb{M}_n$ is such that $\|f(A)\| = \|f(R)\|$ for every $f \in \mathbb{C}[t]$. Therefore, the ideals $\mathfrak{J}_A$ and $\mathfrak{J}_R$ coincide, where for a given $X \in \mathbb{M}_n$

$$
\mathfrak{J}_X = \{ p \in \mathbb{C}[t] : p(X) = 0 \}.
$$

Because $R$ is a generator of $\text{UpperToepl}_n$, there is a $g \in \mathbb{C}[t]$ such that $Q = g(R)$. Let $B = g(A)$. Thus, $\|h(B)\| = \|h(Q)\|$, for every $h \in \mathbb{C}[t]$. By what we proved above, this yields $B = U^*QU$ for some $U \in \mathbb{U}_n$. As $Q$ generates $\text{UpperToepl}_n$, there is an $q \in \mathbb{C}[t]$ such that $R = q(Q)$. Hence,

$$
p(t) = t - q(g(t)) \in \mathfrak{J}_R = \mathfrak{J}_A.
$$

This implies that

$$
0 = p(A) = A - q(g(A)) = A - q(B) = A - U^*q(Q)U = A - U^*RU,
$$

which completes the proof.

3. Necessary and Sufficient Conditions for Unitary Similarity

If $A \in \mathbb{M}_n$ is unicellular — say with spectrum $\{\lambda\}$ — and if $B \in \mathbb{M}_n$ is any matrix for which $\|f(A)\| = \|f(B)\|$ for all $f \in \mathbb{C}[t]$, then $A$ and $B$ are
similar, as the condition implies that $\sigma(B) = \sigma(A)$ and that $(B - \lambda I)^{n-1} \neq 0$. But, unlike the case for generators of the upper triangular Toeplitz matrices, $A$ and $B$ need not be unitarily equivalent (Proposition 3.1 below). Therefore, one can have an invertible matrix $Z \in M_n$ with

$$\|f(A)\| = \|Zf(A)Z^{-1}\|, \text{ for all } f \in \mathbb{C}[t],$$

and yet $Z$ can fail to be unitary.

**Proposition 3.1.** If $0 < \alpha < \beta$, then the unicellular matrices

$$A = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & \beta & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

satisfy $\|f(A')\| = \|f(A)\|$ for all $f \in \mathbb{C}[t]$, but $A \not\sim A'$.

**Proof.** Note that $A' = W^*A^TW$, where $X \mapsto X^T$ denotes the transpose map and

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Because the norm is transpose invariant, $\|f(A')\| = \|f(A^T)\| = \|f(A)\| = \|f(A^T)\|$, for all $f \in \mathbb{C}[t]$. On the other hand, $A \not\sim A'$ by Littlewood’s algorithm [8] because $0 < \alpha < \beta$. (One also can verify directly that the equation $UA' = AU$ is impossible to satisfy with $U \in U_3$. Alternatively, the referee observed that the matrices $A$ and $A'$ fail to satisfy Specht’s tracial condition with the word $\omega(x, y) = xy^2x^2y$; hence, $A \not\sim A'$.)

**Notation 3.1.** If $1 \leq k \leq n$ and $X = [x_{ij}]_{i,j=1}^n \in M_n$, then $X_k = [x_{ij}]_{i,j=1}^k \in M_k$. That is, $X_k$ is the leading $k \times k$ principal submatrix of $X$.

The failure of $A$ and $A'$ in (8) to be unitarily similar is explained by the fact that the norms of $f(A_2)$ and $f(A'_2)$ do not always coincide, even though $\|f(A)\| = \|f(A')\|$ for all $f \in \mathbb{C}[t]$. This observation motivates our second main result of the present paper.

**Theorem 3.2.** Assume that $A \in M_n$ is an upper triangular matrix such that

(a) $a_{ii} = a_{kk}$ for all $1 \leq i, k \leq n$, and
Then the following statements are equivalent for an upper triangular matrix $A' \in M_n$:

1. $\|f(A_i)\| = \|f(A'_i)\|$, for all $f \in \mathbb{C}[t]$ and $1 \leq i \leq n$;
2. $A' = W^*AW$ for some diagonal unitary matrix $W \in U_n$.

Proof. We need only prove that first statement implies the second.

There is a diagonal unitary $W \in U_n$ such that the entries in the first superdiagonal of the upper triangular matrix $W^*AW$ are positive; therefore, without loss of generality we assume that $a_{i,i+1} > 0$ for $1 \leq i \leq (n-1)$. As we argued in the proof of Theorem 2.1, the condition $\|f(A_i)\| = \|f(A'_i)\|$, for all $f \in \mathbb{C}[t]$, implies that $A'$ has one point of spectrum, in this case $\lambda = a_{11}$. Therefore, by scalar translation $X \mapsto X - \lambda I$, we may assume without loss of generality that $\lambda = 0$. That is,

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & a_{n-1n} \\ 0 & & & & 0 \end{bmatrix}, \quad (9)$$

where $a_{\ell,\ell+1} > 0$, for $1 \leq \ell \leq n - 1$.

To complete the proof of theorem, it is sufficient to prove that the entries of $A$ in (9) are completely determined from the values of $\|f(A_i)\|$ for $1 \leq i \leq n$ and all $f \in \mathbb{C}[t]$.

We shall proceed by induction on $n \geq 3$.

Let $n = 3$. Thus,

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

The value of $a_{12}$ is determined via the fact that $\|A_2\| = a_{12}$, and so the value of $a_{23}$ is determined from the equation $a_{12}a_{23} = \|A^2\|$. Using $f(t) = t$, we have

$$\|A\|^2 = \frac{1}{2} \left( a_{12}^2 + a_{23}^2 + |a_{13}|^2 + \sqrt{(a_{12}^2 + a_{23}^2 + |a_{13}|^2)^2 - 4a_{12}^2a_{23}^2} \right).$$
which determines the value of $|a_{13}|$. Two similar calculations using the polynomials $f(t) = t - \frac{1}{a_{12}a_{23}}t^2$ and $g(t) = t - \frac{i}{a_{12}a_{23}}t^2$ determine the values of $|a_{13} - 1|$ and $|a_{13} - i|$. These last two quantities together with the value of $|a_{13}|$ determine the complex number $a_{13}$, thereby establishing the base case for the induction.

Assume now that the statement holds for all spaces of dimension up to and including $n - 1$; we will show the statement also holds for spaces of dimension $n$.

For convenience, we denote the entries of $A^k$ by $a_{ij}^{(k)}$. By the inductive hypothesis, the entries of the submatrix $A_{n-1}$ of $A$ are uniquely determined by the norms $\|f(A_j)\|$, for various $f \in \mathbb{C}[t]$ and $1 \leq j \leq (n - 1)$. Therefore, the only elements left to consider are those in the final column of $A$: $a_{in}$, $1 \leq i \leq (n - 1)$. We shall obtain these entries in an argument that requires $n - 1$ steps; each step uses the conclusion of the previous step.

**Step 1.** Recall $A^n = 0$ and $A^{n-1} \neq 0$. The elements of $A^{n-1}$ are zero except in the $(1, n)$ position, where we have

$$\|A^{n-1}\| = |a_{1n}^{(n-1)}| = a_{12}a_{23} \ldots a_{n-2,n-1}a_{n-1,n}.$$  

Hence, $a_{n-1,n}$ is uniquely determined by the norms $\|f(A_j)\|$ for various $f \in \mathbb{C}[t]$ and $1 \leq j \leq n$. This means, in addition, all of the entries of $A^{n-1}$ are now determined.

**Step i.** Assume that $3 \leq i \leq (n - 1)$ and that we have completed Steps 1 to $i-1$, giving us the values of $a_{j,n}$, for $j = n - i - 1, \ldots, n - 1$ and the entries of each $A^{n-j}$, for $j = 1, \ldots, i-1$. We aim to show that the value of $a_{n-i,n}$ is determined from the norms of various $f(A_j)$.

For each complex number $z \in \mathbb{C}$, let $g_z \in \mathbb{C}[t]$ be given by $g_z(t) = t^{n-i} + \frac{z}{a_{12}q}t^{n-1}$, where $q = a_{2,n}^{(n-2)}$ (as in Step 2). Thus,

$$g_z(A) = \begin{bmatrix} 0 & \cdots & 0 & a_{1,n-i+1}^{(n-i)} & a_{1,n-i+2}^{(n-i)} & \cdots & a_{1,n}^{(n-i)} + z \\ 0 & a_{2,n-i+2}^{(n-i)} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{m}^{(n-i)} \\ 0 & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$
Observe that $g_z(A)$ is a rank-1 perturbation (by a matrix unit) of $A^{n-i}$: namely,

$$g_z(A) = A^{n-i} + zE_{1,n}.$$ 

Suppose that there is a complex number $\tilde{a}^{(n-i)}$ such that

$$\| \tilde{A}^{n-i} + zE_{1,n} \| = \| A^{n-i} + zE_{1,n} \|, \text{ for all } z \in \mathbb{C},$$

where $\tilde{A}^{n-i}$ is the matrix obtained from $A^{n-i} + zE_{1,n}$ by replacing $a^{(n-i)}_{1,n}$ by $\tilde{a}^{(n-i)}_{1,n}$. We shall prove that $\tilde{a}^{(n-i)}_{1,n} = a^{(n-i)}_{1,n}$. Define a function $h : \mathbb{C} \to \mathbb{R}_+$ by $h(z) = \| A^{n-i} + zE_{1,n} \|$ and let $\gamma = \tilde{a}^{(n-i)}_{1,n} - a^{(n-i)}_{1,n}$. Thus, $h(z) = h(z + \gamma)$, for all $z \in \mathbb{C}$. In particular, $h(0) = h(k\gamma)$, for all positive integers $k$. However, as it is clear that $|h(z)| \to \infty$ as $|z| \to \infty$, the equations $h(0) = h(k\gamma)$, for all positive integers $k$, can hold only if $\gamma = 0$.

Thus, we have shown that the $(1, n)$-entry of $A^{n-i}$, namely $a^{(n-i)}_{1,n}$, is determined uniquely by the norms of various $f(A_j)$.

Because the first $n - i - 1$ entries in the first row of $A^{n-i-1}$ are zero and because the first $n - i - 1$ entries of the last column of $A$ are $a_{1,n}, \ldots, a_{1,n-i-1}$, we obtain from $A^{n-i} = A^{n-i-1}A$ that the $(1, n)$-entry of $A^{n-i}$ is given by

$$a^{(n-i)}_{1,n} = a^{(n-i-1)}_{1,n-i} + \sum_{k=1}^{i-1} a^{(n-i-1)}_{1,n-i+k} a_{1,n-i+k}.$$ 

(11)

Because the entries $a^{(n-i)}_{1,n}$, $a^{(n-i-1)}_{1,n-i}$, $a^{(n-i-1)}_{1,n-i+k}$, and $a_{1,n-i+k}$, for $1 \leq k \leq i - 1$, have already been determined from the norms of various $f(A_j)$ using the induction hypothesis and Steps 1 to $i - 1$, (11) implies that the value of $a^{(n-i)}_{1,n-i}$ is determined uniquely from the norms of various $f(A_j)$.

This completes the induction and, hence, the proof of the theorem. \qed

Note that an upper triangular unicellular matrix $A$ satisfies the hypothesis of Theorem 3.2 and so Theorem 3.2 solves the unitary similarity problem in the class of unicellular matrices.

4. Application

In the theory of integral equations, the classical Volterra operator $V$ of integration has some remarkably special properties \[3\]. The operator $V$ is defined as follows: for each $f \in L^2([0, 1])$, let $Vf \in L^2([0, 1])$ be given by

$$VF(t) = 2i \int_t^1 f(s) \, ds, \quad f \in L^2([0, 1]), \quad t \in [0, 1].$$
In the context of our work in this paper, the operator $V$ is unicellular, which in infinite dimensions is to say that its closed invariant subspaces are totally ordered by inclusion.

A question raised many years ago by Arveson [1, page 218] asks whether the norms $\|f(V)\|$, for $f \in \mathbb{C}[t]$, determine the unitary similarity class of $V$ in the set of irreducible compact operators on $L^2([0,1])$. Although this question remains open, we prove below that given any $\varepsilon > 0$ there is a unicellular piece $A$ of the Volterra operator whose norms $\|f(A)\|$ determine its unitarily similarity class and such that $A$ is within $\varepsilon$ of $V$ uniformly on $L^2([0,1])$.

**Proposition 4.1.** For every $\varepsilon > 0$ there is a finite-dimensional subspace $L \subset L^2([0,1])$ such that, if $P$ denotes the projection onto $L$, then

1. $PVP|_L$ is a unicellular operator whose unitary similarity orbit, as an operator on $L$, is completely determined by the norms $\|f(PVP|_L)\|$, for $f \in \mathbb{C}[t]$, and
2. $\|PVP - V\| < \varepsilon$.

**Proof.** We use an approximation scheme of Davies and Simon [3], which they employed to compute the norm of $V$. For each positive integer $m$, let $H_m$ be the Hilbert space spanned by the $m$ orthonormal functions $\sqrt{m} \chi_{E_j}$, $0 \leq j \leq n - 1$, where $E_j = \left[\frac{j}{m}, \frac{j+1}{m}\right)$. If $P_m$ is the projection with range $H_m$, then $P_mVP_m$ considered as an operator on $H_m$ has a matrix representation with respect to this orthonormal basis of $H_m$ that is given by

$$P_mVP_m|_{H_m} = \frac{i}{m}(1 + 2Q) ,$$

where $Q$ is the Toeplitz operator acting on $\mathbb{C}^m$ given by [4]. Therefore, by Theorem 2.1, the unitary similarity orbit of $P_mVP_m|_{H_m}$ is completely determined by the norms $\|f(P_mVP_m|_{H_m})\|$, for $f \in \mathbb{C}[t]$.

The sequence $\{P_m\}_m$ of finite-rank projections $P_m$ converges strongly to the identity operator. Hence, because $V$ is a compact operator, there is an $m$ such that $\|P_mVP_m - V\| < \varepsilon$. \qed

5. **Remark**

Theorem 3.2 is linked to higher-dimensional phenomena encoded by the **matricial spectrum** of $A \in M_n$ [3].
Because for every $A \in M_n$ the unital algebra $\text{Alg} A$ is abelian, there exist unital homomorphisms $\text{Alg} A \to M_k$, for all $1 \leq k \leq n - 1$. For a given $k$, let $\text{Hom} (A, M_k)$ denote the set of all unital homomorphisms $\text{Alg} A \to M_k$. If $\rho \in \text{Hom} (A, M_k)$, then there is a $k$-dimensional subspace $L \subseteq \mathbb{C}^n$ such that $\rho(A) \sim PAP_{|L}$, where $P \in M_n$ is the unique (selfadjoint) projection with range $L$. This subspace $L$ is necessarily semi-invariant under $A$; conversely, every $k$-dimensional semi-invariant subspace of $A$ determines an element $\rho \in \text{Hom} (A, M_k)$ [6, Theorem 3.3.1].

It is natural to consider the values of $\rho \in \text{Hom} (A, M_k)$ as higher order spectra. Specifically, consider the $k$-th matricial spectrum of $A$:

$$\sigma_k(A) = \{ \Lambda \in M_k : \Lambda = \rho(A) \text{ for some } \rho \in \text{Hom} (A, M_k) \}.$$

This set is closed under unitary similarity, and is itself a unitary similarity invariant of $A$. Theorem 3.2 is formulated in the context of leading principal submatrices of upper triangular matrices, which in a basis-free setting means that the formulation is in the context of invariant subspaces; if one strengthens that to semi-invariant subspaces, then a slightly weaker hypothesis on $B$ is afforded.

**Proposition 5.1.** Assume that $A, B \in M_n$ and that $A$ is unicellular. If for each $1 \leq k \leq n$ and each $\rho \in \text{Hom} (A, M_k)$ there is a $\varrho \in \text{Hom} (B, M_k)$ such that

$$\|\varrho(f(B))\| = \|\rho(f(A))\|, \text{ for all } f \in \mathbb{C}[t],$$

then $B \sim A$.

**Proof.** Without loss of generality we may assume that $A$ and $B$ are in upper triangular form with nonnegative entries along the superdiagonal above the main diagonal. With $A$, the entries $a_{i,i+1}$ are positive. Fix $i$ and consider $\Lambda = PAP_{|L}$ and $\Omega = PBP_{|L}$, where $L = \text{Span} \{e_i, e_{i+1}\}$. In this case, $\rho(X) = PXP_{|L}$, for $X \in \text{Alg} A \cup \text{Alg} B$, defines an element of $\text{Hom} (A, M_2)$ and $\text{Hom} (B, M_2)$ such that

$$\Lambda \sim \begin{bmatrix} 0 & a_{i,i+1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Omega \sim \begin{bmatrix} 0 & b_{i,i+1} \\ 0 & 0 \end{bmatrix}.$$  

Thus, $0 \neq \|\Lambda\| = \|\Omega\| = b_{i,i+1}$ Thus, $B$ satisfies the hypothesis of Theorem 3.2 which yields our conclusion. \qed
The power of working in higher dimensions is strikingly illustrated by an important theorem of Arveson \[2\]: if \(A, B \in M_n\) are irreducible, then \(A \sim B\) if and only if \(\|A \otimes C + I \otimes D\| = \|B \otimes C + I \otimes D\|\), for all \(C, D \in M_n\). This is to say that the norms of polynomials (of degree at most 1) in \(A\), over the ring \(M_n\), determine \(A\) up to unitary similarity. In comparison, Theorem 3.2 and Proposition 5.1 represent a hybrid of the matricial and scalar environments.

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