Type II Unprojection

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Abstract

Answering a question of M. Reid, we define and prove the Gorensteiness of the type II unprojection.

1 Introduction

Unprojection is a philosophy, due to M. Reid, which is based on adjunction, and aims to construct and analyse Gorenstein rings in terms of simpler ones. Geometrically, it is an inverse of projection and can also be considered as a modern version of Castelnuovo blow-down.

The case that has been well understood by now is the unprojection of type Kustin–Miller (or type I), which was studied in [PR] and [P], and is originally due, in a different formulation using complexes, to A. Kustin and M. Miller [KM]. Examples by Reid, A. Corti, S. Altınok, G. Brown and others (cf. [A], [CPR], [K], [R]) suggest that there are also other types of unprojections that appear naturally and are useful in algebraic geometry.

The purpose of the present work is to provide the foundations for the unprojection of type II. The initial data for the unprojection is due to Reid, who also gave some indications about how it should look like (cf. [K] Section 9). The main difficulties compared to the unprojection of type Kustin–Miller are the existence of nonlinear relations and the increase of codimension by more than one. We give for the first time a precise definition (Definition 2.2) of the generic type II unprojection and in Theorem 2.15 we prove that it is Gorenstein.

As a demonstration of our methods, we define using base change, and prove using generic perfection, the Gorensteiness of the complete intersection type II unprojection (Section 3). In Section 4 we discuss calculational aspects and applications to algebraic geometry. Finally, in Section 5 we state some remarks and open questions.
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2 Generic type II unprojection

2.1 Definition of the unprojection ring

Fix positive integers \( k, n \) with \( k \geq 1 \) and \( n \geq 2 \). Our ambient ring will be a polynomial ring \( \mathcal{O}_{\text{amb}} \) over the integers \( \mathbb{Z}[a_{ij}, z, w_p] \) where \( 1 \leq i \leq k+1, 1 \leq j \leq n \) and \( w_p \) is a finite number of indeterminants \( w_1, w_2, \ldots \) that we use only for extra flexibility in the applications. We assign \( a_{ij} \) to have weight equal to \( i \) for \( 1 \leq i \leq k+1 \), \( z \) to have weight \( k+2 \). We also give each \( w_p \) a positive number as weight.

Following Reid (cf. [Ki] Section 9.5), we define \( I_D \) to be the ideal of \( \mathcal{O}_{\text{amb}} \) generated by the \( 2 \times 2 \) minors of the \( 2 \times n(k+1) \) matrix \( M \), where

\[
M = \begin{pmatrix}
    a_{21} & \cdots & a_{2n} & \cdots & a_{k+1,1} & \cdots & a_{k+1,n} & za_{11} & \cdots & za_{1n} \\
    a_{11} & \cdots & a_{1n} & \cdots & a_{k1} & \cdots & a_{kn} & a_{k+1,1} & \cdots & a_{k+1,n}
\end{pmatrix}
\]

(2.1)

We notice that \( I_D \) is a homogeneous ideal of \( \mathcal{O}_{\text{amb}} \). We set

\[
D = V(I_D) \subset \text{Spec} \mathcal{O}_{\text{amb}}, \quad \mathcal{O}_D = \mathcal{O}_{\text{amb}}/I_D
\]

and

\[
\mathcal{O}_{\tilde{D}} = \mathbb{Z}[x_1, \ldots, x_n, t, w_p], \quad \tilde{D} = \text{Spec}(\mathcal{O}_{\tilde{D}}).
\]

Remark 2.1 The ring \( \mathcal{O}_D \) is not normal. Geometrically, the normalisation of \( D \) is the map

\[
q : \tilde{D} \to D
\]

with

\[
a_{ij} = x_j t^{i-1}, \quad z = t^{k+1}
\]

for \( 1 \leq i \leq k+1, 1 \leq j \leq n \). Since \( q \) is an isomorphism in codimension one, we have

\[
\omega_D = q_* \omega_{\tilde{D}} = q_* \mathcal{O}_{\tilde{D}}.
\]

(2.2)

We also see that the ideal \( I_D \) is prime of codimension \( nk \) in \( \mathcal{O}_{\text{amb}} \).
Let $I_X$ be a homogeneous prime ideal of $\mathcal{O}_{\text{amb}}$ such that $I_X \subset I_D$, $I_X$ has codimension $nk - 1$ in $\mathcal{O}_{\text{amb}}$, and the integral domain $\mathcal{O}_X = \mathcal{O}_{\text{amb}}/I_X$ is normal and Gorenstein. We denote by $K(X)$ to be the field of fractions of $\mathcal{O}_X$, and we set

$$I_D^{-1} = \{ f \in K(X) : fI_D \subset \mathcal{O}_X \}.$$ 

We will consider $\mathcal{O}_D$ as a quotient ring of $\mathcal{O}_X$. Using (2.2), $\omega_D$ is a Cohen–Macaulay $\mathcal{O}_X$-module which needs $k+1$ generators $e_0, \ldots, e_k$, as $\mathcal{O}_D$-module. It is also clear (cf. [K] Section 9.5) that we can choose them so that the module of relations between them is generated by relations of the form

$$M^t \begin{pmatrix} e_i \\ e_{i+1} \end{pmatrix} = 0.$$ 

**Definition 2.2** The generic type II unprojection is the $\mathcal{O}_X$-subalgebra of $K(X)$ 

$$\mathcal{O}_X[I_D^{-1}] \subset K(X)$$

generated by $I_D^{-1}$.

The relation with adjunction theory is as follows:

Since, by assumption, $I_D$ has codimension one in the Gorenstein ring $\mathcal{O}_X$, we have as in [PR] p. 563 a short exact sequence

$$0 \rightarrow \omega_X \rightarrow \text{Hom}_{\mathcal{O}_X}(I_D, \omega_X) \xrightarrow{\text{res}_D} \omega_D \rightarrow 0$$

where res$_D$ is the Poincaré residue map. Using the Gorensteiness of $\mathcal{O}_X$ we get the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \xrightarrow{\text{res}_D} \omega_D \rightarrow 0 \quad (2.3)$$

$I_D$ contains a regular element of $\mathcal{O}_X$, therefore $\text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is canonically isomorphic to $I_D^{-1}$, and we will consider it as a submodule of $K(X)$.

After giving a presentation of $\mathcal{O}_X[I_D^{-1}]$ (Proposition 2.6), we prove in Theorem 2.15 that it is Gorenstein.

**Remark 2.3** Assume the ideal $I_D \subset \mathcal{O}_X$ is principal, say $I_D = (a)$. Then it clear that the ring $\mathcal{O}_X[I_D^{-1}]$ is a localization of $\mathcal{O}_X$ hence Gorenstein. In the following, we will assume that $I_D$ is not a principal ideal of $\mathcal{O}_X$. 
2.2 Presentation of the unprojection ring

By the assumptions on $I_D$ and $\mathcal{O}_X$, we have a natural valuation map

$$\text{val}_D : K(X)^* \to \mathbb{Z}$$

It is convenient to extend it by assigning $\text{val}_D 0 = +\infty$.

**Remark 2.4** For an element $s \in I_D^{-1}$ we have $\text{val}_D(s) \geq -1$. We claim that every $s \in \mathcal{O}_X[I_D^{-1}]$ with $\text{val}_D(s) \geq 0$ is in $\mathcal{O}_X$. Indeed, by the normality of $\mathcal{O}_X$ it is enough to prove that for every codimension one prime ideal $I_E \subset \mathcal{O}_X$ we have $\text{val}_E(s) \geq 0$. Let $I_E \neq I_D$, then there exists a point $p \in E$ and $f \in I_D$ with $f(p) \neq 0$. For sufficiently large $k$ we have $sf^k \in \mathcal{O}_X$, as a consequence $s$ is defined at the point $p$.

We fix $s_0, s_1, \ldots, s_k \in \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \subset K(X)$ such that $\text{res}_D(s_i) = e_i$ for $0 \leq i \leq k$. By the adjunction sequence (2.3), there exist elements $c_{i,j,p}, d_{j,p} \in \mathcal{O}_X$ (for $1 \leq i \leq k$, $1 \leq j \leq n$ and $0 \leq p \leq k - 1$), such that

$$a_{i+1,j} s_p + a_{ij} s_{p+1} - c_{i,j,p} = 0 \in K(X),$$
$$za_{1,j} s_p + a_{k+1,j} s_{p+1} - d_{j,p} = 0 \in K(X).$$

Moreover, by the same sequence it follows that any (inhomogeneous) linear relation between $s_0, \ldots, s_k$ is generated by these relations.

Let $\mathcal{O}_X[T_0, \ldots, T_k]$ be the polynomial ring over $\mathcal{O}_X$ in $k + 1$ indeterminants, and denote by

$$\phi : \mathcal{O}_X[T_0, \ldots, T_k] \to \mathcal{O}_X[I_D^{-1}]$$

the $\mathcal{O}_X$-algebra homomorphism with $\phi(T_i) = s_i$. We set

$$f_{i,j,p}^a = a_{i+1,j} T_p + a_{ij} T_{p+1} - c_{i,j,p},$$
$$f_{j,p}^b = za_{1,j} T_p + a_{k+1,j} T_{p+1} - d_{j,p}.$$ 

These elements are in the kernel of $\phi$ and moreover generate the submodule of linear relations. Our aim now is to describe the quadratic relations and prove that together with the linear they generate the kernel of $\phi$. We need the following lemma.

**Lemma 2.5** (Quadratic relations) The following are true.

a) Let $0 \leq j, m \leq k$, $1 \leq i, l \leq k + 1$ and $1 \leq t \leq n$. Assume $l + m = i + j$. Then

$$\text{val}_D(a_{il}s_j - (-1)^{j-m}a_{lt}s_m) \geq 0.$$
b) Let $0 \leq i, j, l, m \leq k$. Assume $l + m = i + j$. Then
\[
\text{val}_D(s_is_j - s_is_m) \geq -1.
\]

c) Let $0 \leq i, j, l, m \leq k$. Assume $l + m = i + j - (k + 1)$. Then
\[
\text{val}_D(s_is_j - (-1)^{k+1}zis_m) \geq -1.
\]

**Proof** For a) we notice that assuming $i < l$, we have
\[
a_{ij} - (-1)^{j-m}a_{lt} = v_l - v_{i+1} + v_{i+2} - \cdots + (-1)^{i-l-1}v_{l-1},
\]
where $v_l = a_{qls} + a_{q+1,ls}$. Taking into account the linear relations the result follows.

For b) it is enough to show that
\[
\text{val}_D(s_is_j - s_is_{j+1}) \geq -1.
\]

We have
\[
a_{11}(s_is_j - s_is_{j+1}) = s_j(a_{11}s_i + a_{21}s_{i-1}) - s_{i-1}(a_{11}s_{j+1} + a_{21}s_{j}).
\]
Taking into account the linear relations the result follows.

For c), using b) it is enough to show that for $i \geq 1$ we have
\[
\text{val}_D(s_is_i - (-1)^{k+1}zs_{i-1}s_{0}) \geq -1.
\]

We have
\[
a_{11}(s_is_i - (-1)^{k+1}zs_{i-1}s_{0}) = s_i(a_{11}s_k - (-1)^{k+1}a_{k+1}s_{0}) - (-1)^{k+1}s_0(a_{11}zs_{i-1} + a_{k+1}s_{i}),
\]
so the result follows from a). QED

Using Proposition 2.5 for $i, j$ with $i + j \leq k$ there exists polynomial $g^{a}_{ij}(T_0, \ldots, T_k) \in \ker(\phi)$, of the form
\[
g^{a}_{ij} = T_iT_j - T_0T_{i+j} + \text{linear terms}.
\]
In addition, for $i, j$ with $i+j \geq k+1$, there exist polynomial $g^{b}_{ij}(T_0, \ldots, T_k) \in \ker(\phi)$, of the form
\[
g^{b}_{ij} = T_iT_j - (-1)^{k+1}zT_0T_{i+j-k-1} + \text{linear terms}.
\]
In both cases by linear terms we mean terms of total degree in $T_0, \ldots T_n$ at most one.
Proposition 2.6 We have

\[ \ker(\phi) = (f^a_{i,j,p}, f^b_{j,p}, g^a_{ij}, g^b_{ij}), \]

with indices as above.

Proof Taking into account the relations \( g^a, g^b \), it is enough to restrict to polynomials \( h(T_0, \ldots, T_k) \) which have total degree with respect to the variables \( (T_1, \ldots, T_k) \) at most one.

We use induction on the total degree (with respect to all variables) \( p \) of the polynomial \( h \).

We already noticed that for \( p = 1 \) it is true.

Assume it is true for all polynomials of total degree at most \( p - 1 \) in the kernel of \( \phi \) which have total degree with respect to the variables \( (T_1, \ldots, T_k) \) at most one. Let \( h(T_0, T_1, \ldots, T_k) \) be a polynomial of total degree exactly \( p \) in the kernel of \( \phi \), which has \( (T_1, \ldots, T_k) \)-degree at most one. Assume

\[ h(T_0, \ldots, t_k) = e_0 T_0^p + T_0^{p-1} u_1(T_1, \ldots, T_k) + L(T_0, \ldots, T_k) \]

where \( e_0 \in \mathcal{O}_X \), \( u_1 \) is (homogenously) linear in \( (T_1, \ldots, T_k) \), and \( L(T_0, \ldots, T_k) \) has total degree at most \( p - 1 \). Write

\[ h(T_0, \ldots, T_k) = T_0^{p-1}(e_0 T_0 + u_1) + L. \]

We have

\[ \text{val}_D(e_0 s_0 + u_1(s_1, \ldots, s_k)) \geq 0, \]

otherwise \( \text{val}_D h(s_0, \ldots, s_k) \leq -k \), which contradicts \( h(s_0, \ldots, s_k) = 0 \).

Therefore, by Remark 2.4

\[ e_0 s_0 + u_1(s_1, \ldots, s_k) \in \mathcal{O}_X. \]

By the case \( p = 1 \), there exists \( c \in \mathcal{O}_X \) such that

\[ e_0 T_0 + u_1(T_1, \ldots, T_k) - c \in (f^a, f^b, g^a, g^b). \]

We finish the proof by using the inductive hypothesis on the polynomial

\[ cT_0^{p-1} + L(T_0, \ldots, T_k). \]

QED
2.3 The theorem

Remark 2.7 As in [PR] Lemma 1.1, we can assume that $s_0 \in \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ is injective, write
$$s_0: I_D \to I_N \subset \mathcal{O}_X$$
for the image of $s_0$.

We will use the following rather general result

Proposition 2.8 Let $\mathcal{O}_X$ be a Cohen–Macaulay ring, $I$ an ideal of codimension one, $\phi: I \to \mathcal{O}_X$ an injective $\mathcal{O}_X$-homomorphism with image the ideal $J \subset \mathcal{O}_X$. Assume that if for $a \in \mathcal{O}_X$ we have $I \subset (a)$, then $a$ is a unit of $\mathcal{O}_X$. Then $J$ has codimension one in $\mathcal{O}_X$.

Proof By the assumption, there exists an $\mathcal{O}_X$-regular element $w$ in $I$. $\phi(w)$ is not a unit, otherwise for all $b \in I$ we have
$$b = \phi(w)(\phi(w))^{-1}b = (\phi(w))^{-1}\phi(b),$$
contradicting our assumptions. Since $\phi$ is injective, $\phi(w)$ is a regular element of $J$, so $J$ has codimension at least one.

Assume now that $J$ has codimension at least two. Since $\mathcal{O}_X$ is Cohen–Macaulay, there exists an $\mathcal{O}_X$-regular sequence of length two contained in $J$. That immediately implies that the natural map from $\mathcal{O}_X$ to $\text{Hom}_{\mathcal{O}_X}(J, \mathcal{O}_X)$ is an isomorphism. As a result, $\phi^{-1}: J \to I$ is given by multiplication by an element $a \in \mathcal{O}_X$. If $a$ is a unit, we get $J = I$, a contradiction since $I$ has codimension one. If $a$ is not a unit, then $I \subset (a)$, contradicting our assumptions. So $J$ can only have codimension one. QED

Corollary 2.9 We have $\text{codim}_{\mathcal{O}_X} I_N = 1$.

Proof Suppose that for an element $a \in \mathcal{O}_X$ we have
$$I_D \subset (a) \subset \mathcal{O}_X.$$
Recall that we assume (Remark 2.3) that the ideal $I_D$ of $\mathcal{O}_X$ is not principal, therefore $a \notin I_D$. Let $w_1 \in I_D$ be a nonzero element. We have that $a$ divides $w_1$, and since $I_D$ is prime, there exists $w_2 \in I_D$ with $w_1 = aw_2$. Continuing this way, we get an increasing sequence of ideals of $\mathcal{O}_X$
$$(w_1) \subset (w_2) \subset (w_3) \subset \ldots$$
such that \( w_i = aw_{i+1} \). Since \( \mathcal{O}_X \) is Noetherian, this sequence becomes stationary, so there exists an index \( i \) and \( b \in \mathcal{O}_X \) with \( w_{i+1} = bw_i \). Therefore \( (ab - 1)w_i = 0 \), and since \( \mathcal{O}_X \) is an integral domain we get \( a \) a unit. Then, the Corollary follows from Proposition 2.8. QED

For simplicity, we set

\[ R_1 = \mathcal{O}_X[T_0, \ldots, T_k]/(f^a, f^b, g^a, g^b). \]

By Proposition 2.6, \( R_1 \) is isomorphic under \( \phi \) to \( \mathcal{O}_X[I_D^{-1}] \subset K(X) \), so it is an integral domain and therefore \( T_0 \) is an \( R_1 \)-regular element. Let \( (T_0) \) be the ideal of \( R_1 \) generated by \( T_0 \).

**Proposition 2.10** We have the following isomorphisms of \( \mathcal{O}_X \)-modules

\[ R_1/(T_0) \cong \text{Hom}_{\mathcal{O}_X} (I_D, \mathcal{O}_X)/(s_0) \cong \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X)/(i_N) \cong \omega_N, \]

where \( i_N \in \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X) \) is the natural inclusion \( i_N : I_N \to \mathcal{O}_X \). Notice that in the second and third quotient we divide by the \( \mathcal{O}_X \)-submodules generated by the respective elements, while in the first with the ideal (that is, the \( R_1 \)-submodule) generated by \( T_0 \).

**Proof** By Corollary 2.9 we have that \( N \) has codimension one in \( X \). Therefore, the adjunction exact sequence

\[ 0 \to \mathcal{O}_X \to \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X) \to \omega_N \to 0 \]

gives the isomorphism

\[ \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X)/(i_N) \cong \omega_N. \]

The isomorphism \( s_0 : I_D \to I_N \) induces an isomorphism

\[ s_0^* : \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X) \to \text{Hom}_{\mathcal{O}_X} (I_D, \mathcal{O}_X) \]

with \( (s_0^*)(i_N) = s_0 \). The isomorphism

\[ \text{Hom}_{\mathcal{O}_X} (I_D, \mathcal{O}_X)/(s_0) \cong \text{Hom}_{\mathcal{O}_X} (I_N, \mathcal{O}_X)/(i_N) \]

follows.

Now we will prove that

\[ R_1/(T_0) \cong \text{Hom}_{\mathcal{O}_X} (I_D, \mathcal{O}_X)/(s_0). \]
Using the isomorphism $R_1 \cong \mathcal{O}_X[I_D^{-1}]$, is it enough to prove that

$$\mathcal{O}_X[I_D^{-1}]/(s_0) \cong \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)/(s_0)$$

as $\mathcal{O}_X$-modules. Again, the first quotient is with respect to the ideal generated by $s_0$, while the second is with respect to the submodule.

Denote by $\psi$ the composition

$$\text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \to \mathcal{O}_X[I_D^{-1}] \to \mathcal{O}_X[I_D^{-1}]/(s_0),$$

(the first map is the natural inclusion).

Clearly $s_0$ is in the kernel of $\psi$. Using the relations $g^a, g^b$ we get that the map $\psi$ is surjective.

**CLAIM** $\ker(\psi) = (s_0)$.

Proof of Claim: Assume $u(s_1, \ldots, s_k) \in \ker(\psi)$, where $u(T_1, \ldots, T_k) \in \mathcal{O}_X[T_1, \ldots, T_k]$ has total degree at most one. There exists $h(T_0, \ldots, T_K) \in \mathcal{O}_X[T_0, \ldots, T_k]$ with

$$u(s_1, \ldots, s_k) = s_0 h(s_0, \ldots, s_k).$$

(Equality inside $\mathcal{O}_X[I_D^{-1}]$.) As a consequence, $\text{val}_D(h(s_0, \ldots, s_k)) \geq 0$. By Remark 2.11, $h(s_0, \ldots, s_k) \in \mathcal{O}_X$. Therefore, $u(s_1, \ldots, s_k)$ is inside the $\mathcal{O}_X$-submodule of $\text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X)$ generated by $s_0$. QED

**Remark 2.11** To prove that $R_1$ Gorenstein it is enough to prove that $R_1/(T_0)$ is Gorenstein. This is because $I_D$ and $\mathcal{O}_X$ are graded so $\mathcal{O}_X[I_D^{-1}]$ has a natural grading and with that, $T_0$ is a homogeneous element of positive degree (cf. [BH] Exerc. 3.6.20(c), p. 144).

**Remark 2.12** To prove that $R_1/(T_0)$ is Gorenstein, it is enough, due to Proposition 2.10, to prove

$$\text{grade}_{\mathcal{O}_X}(m, \omega_N) = \dim \mathcal{O}_X - 1,$$

where $m = (a_{ij}, z, w_p)$ is the unique maximal homogeneous ideal of $\mathcal{O}_X$, and

$$\text{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \cong \omega_N,$$

the isomorphism as $\mathcal{O}_X$-modules.

**Proposition 2.13** We have

$$\text{grade}_{\mathcal{O}_X}(m, \omega_N) = \dim \mathcal{O}_X - 1.$$
Proof By Remark 2.1

$$\text{grade}_{\mathcal{O}_X}(m, \omega_D) = \dim \mathcal{O}_X - 1.$$ 

Using the two adjunction sequences for $\omega_D$ and $\omega_N$ the result follows using the arguments of step 2 of the proof of [PR] Theorem 1.5. QED

Remark 2.14 It is clear that the map

$$h_3 : \mathcal{O}_D \to \omega_D$$

with

$$h_3(a) = a \res(s_0)$$

is injective. Write $T$ for the cokernel of $h_3$. From Remark 2.11 it follows that as an $\mathcal{O}_X$-module $T$ is isomorphic to a direct sum of $k$ copies of $\mathcal{O}_X/(a_{11}, \ldots, a_{k+1,n})$.

Therefore, the support of $T$ in $\mathcal{O}_X$ has codimension in $\text{Spec}(\mathcal{O}_X)$ equal to $(k + 1)n - (nk - 1) = n + 1 \geq 3$. Using [BH] Corollary 3.5.11, we get

$$\text{Ext}^1_{\mathcal{O}_X}(T, \mathcal{O}_X) = \text{Ext}^2_{\mathcal{O}_X}(T, \mathcal{O}_X) = 0.$$

We are now ready to prove our main theorem.

Theorem 2.15 The ring $\mathcal{O}_X[I_D^{-1}]$ is Gorenstein

Proof We follow the pattern of the proof of [PR] Theorem 1.5.

Using Remarks 2.11, 2.12, and Proposition 2.13 it is enough to prove that

$$\text{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \cong \omega_N$$

as $\mathcal{O}_X$-modules.

Start with the two columns consisting of canonical maps

$$
\begin{array}{c}
I & \mathcal{O}_X \\
\cap & \cap \\
\mathcal{O}_X & \text{Hom}_{\mathcal{O}_X}(I_D, \omega_X) \\
\downarrow & \downarrow \\
\mathcal{O}_D & \omega_D
\end{array}
$$
We define three maps
\[ h_1 : I_D \to \mathcal{O}_X, \quad h_1(a) = s_0(a) \in \mathcal{O}_X, \]
\[ h_2 : \mathcal{O}_X \to \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X), \quad (h_2(a))(b) = s_0(ab) \in \mathcal{O}_X, \]
\[ h_3 : \mathcal{O}_D \to \omega_D, \quad h_3(a) = a \text{res}(s_0) \in \omega_D. \]

It is clear that all three maps \( h_1, h_2, h_3 \) are injective (compare Remark 2.14). In addition,
\[ \text{coker } h_1 = \mathcal{O}_N, \quad \text{coker } h_2 \cong \omega_N, \quad \text{coker } h_3 = T, \]
where \( T \) was calculated in Remark 2.14. We have a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \to & I & \xrightarrow{h_1} & \mathcal{O}_X & \to & \mathcal{O}_N & \to & 0 \\
\cap & & \cap & & \cap & & \cap & & \\
0 & \to & \mathcal{O}_X & \xrightarrow{h_2} & \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) & \to & \omega_N & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_D & \xrightarrow{h_3} & \omega_D & \to & T & \to & 0
\end{array}
\]
which induces a map \( d : \mathcal{O}_N \to \omega_N \). We claim that this map is injective. Indeed, let \( x \in \mathcal{O}_X \) such that \( \text{res}_N \circ s^*(m_x) = 0 \in \omega_N \), where \( m_x \in \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X) \) is the multiplication by \( x \). Then there exists \( w \in \mathcal{O}_X \) such that
\[ m_x = h_2(w) \]
hence,
\[ xa = ws_0(a) \in \mathcal{O}_X \]
for all \( a \in I_D \), which implies that
\[ x - ws_0 = 0 \in \text{Hom}_{\mathcal{O}_X}(I_D, \mathcal{O}_X). \]
Using the valuation \( \text{val}_D \) we get \( w \in I_D \). Therefore \( x = h_1(w) \).

Using the Snake Lemma, we get an exact sequence
\[ 0 \to \mathcal{O}_N \to \omega_N \to T \to 0 \]
Taking the long exact associated to \( \text{Hom}_{\mathcal{O}_X}(-, \omega_X) \), and using the vanishings from Remark 2.14, we get
\[ \text{Ext}^1_{\mathcal{O}_X}(\omega_N, \mathcal{O}_X) \cong \text{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_N, \mathcal{O}_X) \cong \omega_N, \]
which finishes the proof of the theorem. QED

The next proposition is a corollary of Proposition 2.6 and Theorem 2.15.

**Proposition 2.16** The map \( \phi \) induces an isomorphism

\[
O_X[I_D^{-1}] \cong \frac{O_{amb}[T_0, \ldots, T_k]}{I_X + (f^a, f^b, g^a, g^b)}.
\]

Moreover, if we consider \( O_X[I_D^{-1}] \) as an \( O_{amb}[T_0, \ldots, T_k] \)-module via \( \phi \), it is perfect of grade \((nk - 1) + (k + 1) = nk + k\).

**Proof** The isomorphism is immediate from Proposition 2.6. Using [BV] p. 210 remarks after Proposition 16.19, the perfectness follows from Theorem 2.15. We have

\[
\text{grade } O_X[I_D^{-1}] = \dim O_{amb}[T_0, \ldots, T_k] - \dim O_X[I_D^{-1}].
\]

Hence, it is enough to prove that \( \dim O_X = \dim O_X[I_D^{-1}] \). Set

\[ J = (f^a, f^b, g^a, g^b) \cap O_X[T_0]. \]

Using the relations \( g^a, g^b \) we see that \( O_X[I_D^{-1}] \) is a finitely generated \( O_X[T_0]/J \)-module. As a result, the extension

\[
O_X[T_0]/J \subset O_X[I_D^{-1}]
\]

is integral, so it is enough to prove \( \dim O_X = \dim O_X[T_0]/J \). This follows from the facts that \( T_0 \) is a regular element of \( O_X[T_0]/J \) and the quotient by the ideal generated by \( T_0 \) is isomorphic to the ring \( O_X/I_N \) (cf. proof of Theorem 2.15). QED

3 Complete intersection type II unprojection

3.1 Generic complete intersection type II

The generic complete intersection type II is the specialization of the generic type II to the case where \( I_X \) is a complete intersection. More precisely, the initial data is the triple

\[
I_X^g \subset I_D^g \subset O_{amb}^g
\]

(g for generic) defined as follows.
Fix as in Subsection 2.1 positive integers $k, n$ with $k \geq 1$ and $n \geq 2$. $I_D^g$ is the ideal generated by the $2 \times 2$ minors $u_1, \ldots, u_l$ of the matrix $M$ defined in \(\text{(2.1)}.\ We define \[ I_X^g = (f^1, \ldots, f^{nk-1}), \] where for $1 \leq p \leq nk - 1$ we have \[ f^p = \sum_{j=1}^l w_{pj} u_j. \]

Finally $\mathcal{O}_{amb}^g$ is the polynomial ring over the integers in all the indeterminants appearing above, \[ \mathcal{O}^g_{amb} = \mathbb{Z}[a_{ij}, z, w_{pj}]. \]

We put weights for $a_{ij}$ and $z$ as in Subsection 2.1, while for $w_{pj}$ we make the unique minimal choice of positive weights such that all $f^p$ become homogeneous.

Our configuration satisfies the conditions of Subsection 2.1 so setting, for simplicity, \[ R_1^g = \mathcal{O}^g_X[(I_D^g)^{-1}], \] we have that $R_1^g$ is a Gorenstein ring (Theorem 2.15). In addition it has the presentation described in Proposition 2.16 and by the same Proposition is a perfect $\mathcal{O}_{amb}^g[T_0, \ldots, T_k]$-module of grade $nk + k$.

### 3.2 Complete intersection type II

Assume we have a triple \[ I_X^s \subset I_D^s \subset \mathcal{O}_{amb}^s \] (s for specific) as follows.

$I_D^s$ is the ideal generated by the $2 \times 2$ minors $\hat{u}_1, \ldots, \hat{u}_l$ of a matrix $\hat{M}$, of the form defined in \(\text{(2.1)},\ but with elements $\hat{a}_{ij}$ instead of $a_{ij}$, and $\hat{z}$ instead of $z$. We assume that $I_D^s$ has codimension in $\mathcal{O}_{amb}^s$ equal to $nk$, which is the maximal possible.

$\mathcal{O}_{amb}^s$ is an equidimensional Gorenstein ring, not necessarily local or graded, containing all elements with hats.

$I_X^s$ has codimension $nk - 1$ in $\mathcal{O}_{amb}^s$ and is generated by an $\mathcal{O}_{amb}^s$-regular sequence $\hat{f}^1, \ldots, \hat{f}^{nk-1}$, so there are $\hat{w}_{pj} \in \mathcal{O}_{amb}^s$ such that \[ \hat{f}^p = \sum_{j=1}^l \hat{w}_{pj} \hat{u}_j. \]
for $1 \leq p \leq nk - 1$.

We consider the ring homomorphism

$$\hat{\cdot}: \mathcal{O}_{amb}^g[T_0, \ldots, T_k] \to \mathcal{O}_{amb}^s[T_0, \ldots, T_k]$$

which sends elements without hats to the corresponding elements with hats and $T_i$ to itself.

**Definition 3.1** The type II unprojection for our initial data (3.1) is the $\mathcal{O}_X^s$-algebra

$$R_1^s = R_1^g \otimes \mathcal{O}_{amb}^s[T_0, \ldots, T_k],$$

the tensor product over the ring $\mathcal{O}_{amb}^g[T_0, \ldots, T_k]$.

We have the following theorem.

**Theorem 3.2** The ring $R_1^s$ is Gorenstein.

**Proof** By [BV] Theorem 3.5, it is enough to prove that the ring $R_1^s$ has grade at least equal to $nk + k$ as $\mathcal{O}_{amb}^s[T_0, \ldots, T_k]$-module. For that we argue as in Proposition 2.16 taking into account that $I_D$ contains an $\mathcal{O}_X^s$-regular element. QED

**Remark 3.3** The reason we defined the unprojection using base change is that we expect that the ring $\mathcal{O}_X^s/(I_D)^{-1}$ is not always Gorenstein. An example in the unprojection of type Kustin–Miller is the following. Let $R$ be the homogeneous coordinate ring of the plane cuspidal cubic $x_0^2x_1 - x_2^3$ and $I = (x_1, x_2)$ the reduced ideal of the cusp. Then the ring $R[I^{-1}]$ is isomorphic to the homogeneous coordinate ring of the twisted cubic, hence it is not Gorenstein.

4 Calculations and Applications

4.1 Computations

The simplest case, due to Reid, is the complete intersection type II unprojection for parameters $k = 1$ and $n = 2$, in which we pass from codimension 1 to a codimension 3 Pfaffian. We refer to [Ki] Section 9.5 for the explicit calculations.
In [P2], which is work under progress, we explicitly calculate the linear relations for the type II unprojection for parameters $k = 1$ and any $n \geq 2$. The main idea is to identify the projective resolution of $\omega_D$ as a twisted sum of two Koszul complexes.

The calculation of the quadratic relations turns out to be more difficult. We expect that, assuming 2 is invertible, there are symmetric formulas, but so far, we have been able to write them down only for the complete intersection type II with $k = 1$ and $n \leq 3$.

4.2 Applications

One of the first appearances of the type II unprojection in algebraic geometry is in the elliptic involutions between Fano threefold hypersurfaces in [CPR]. According to [CPR] Section 7.3, it is expected that the various unprojection types will eventually serve as the basis for a classification for all (possibly singular) Fano threefolds.

Another application of the type II unprojection, is in the construction of new K3 surfaces, Fano threefolds, and Calabi–Yau threefolds. In [Al], Altınok constructs a large number of codimension three K3 surfaces using type II unprojection, and conjectures the existence of more K3 surfaces and Fano threefolds, including a codimension four Fano with $H^0(-K_X) = 0$ (cf. [Ki] Example 9.14).

In the Magma K3 database (cf. [ABR] Section 6) due to Brown, there is a large number of additional K3 surfaces which are likely to exist as type II unprojections, including cases in codimensions 6, 7 and 8. We believe that the ideas of the present work together with explicit singularity calculations can establish the existence of many of them.

Finally, there is work in progress by A. Buckley and B. Szendrői, which is expected to lead to the construction of new Calabi–Yau threefolds using type II unprojection (cf. [BS]).

5 Final remarks and questions

Remark 5.1 An interesting problem is to find conditions which will guarantee that $R^s_1 = O^s_X[(I^s_D)^{-1}]$, compare Remark 3.3. More generally, for given unprojection data, how can we intrinsically distinguish the relations of $O^s_X[(I^s_D)^{-1}]$ that are also present in the generic case from those that occur because $O^s_X$ is not sufficiently generic?
Remark 5.2 How can we construct a projective resolution of $R_1$ as $\mathcal{O}_{amb}[T_0, \ldots, T_k]$-module using projective resolutions of $\mathcal{O}_D$ and $\mathcal{O}_X$? In the case of unprojection of type Kustin–Miller it was done in [KM].

Remark 5.3 What other types of type II unprojection format exist, in addition to the complete intersection one?

Remark 5.4 In [CR], Corti and Reid pose the problem of interpreting the Gorenstein formats arising from unprojection as solutions to universal problems. What can be said about the type II case?

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