Spectral determinant on Euclidean isosceles triangle envelopes of fixed area as a function of angles: absolute minimum and small-angle asymptotics

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Abstract

We study extremal properties of the determinant of Friederichs self-adjoint Laplacian on the Euclidean isosceles triangle envelopes of fixed area as a function of angles. Small-angle asymptotics show that the determinant grows without any bound as an angle of triangle envelope goes to zero. We prove that the equilateral triangle envelope (the most symmetrical geometry) always gives rise to a critical point of the determinant and find the critical value. Moreover, if the area of envelopes is not too large, then the determinant achieves its absolute minimum only on the equilateral triangle envelope and there are no other critical points, whereas for sufficiently large area the equilateral triangle envelope corresponds to a local maximum of the determinant.

1 Introduction

For surfaces with smooth varying metrics extremal properties of determinants of Laplacians and compactness of families of isospectral metrics were studied by Osgood, Phillips, and Sarnak in a series of papers \[29, 30, 31, 32\], see also \[33\] for a review of the results.

The case of a surface without boundary is the most simple: for all (smooth) metrics in a given conformal class and of given (fixed) area the determinant attains its unique absolute maximum on a unique constant curvature metric, called uniform, and goes to zero as the metric degenerates \[8, 40\].

The case of a surface of type \((p, n)\), obtained by removing \(n > 0\) distinct open discs from a closed surface of genus \(p\), is more involved: for all flat (curvature zero) metrics of given (fixed) total boundary length in a given conformal class the determinant of Dirichlet Laplacian attains its unique absolute maximum on a unique flat metric, also called uniform. It is the one for which the boundary has constant geodesic curvature. In the case \(p = 0\) the determinant goes to zero as the uniform metric degenerates \[31\], but this is no longer true for higher genus surfaces \[16\], where for some degenerations the determinant remains bounded by a positive constant from below, or even increases without any bound. In particular, this is why the argument in \[30, 31, 32\] allows to prove compactness of families of isospectral metrics in the natural \(C^\infty\)-topology for \(np = 0\) but not for \(np > 0\).
The isospectral compactness of flat metrics on \((p, n)\)-type surfaces with \(n > 0\) and negative Euler characteristic was demonstrated in [17], where the total area (instead of total boundary length) of flat metrics is fixed. We also refer to [18] for results on extremal properties of the determinant in the Bergman metric on the moduli space of genus two surfaces.

As discussed in [32], in the case \(p = 0, n \geq 3\) the space of uniform metrics can be identified with a subset of the space \(\mathcal{C}_n\) of conical metrics \(m = c^2 \prod_{j=1}^n |z - \tau_j|^{2\beta_j}|dz|^2\) on the Riemann sphere \(\mathbb{T} = \mathbb{C} \cup \{\infty\}\). Here \(c\) is a scaling factor, the Gauss-Bonnet theorem requires that the orders \(\beta_j > -1\) of conical singularities satisfy \(\sum_{j=1}^n \beta_j = -2\), by using a suitable Möbius transformation we always normalize the (complex) coordinates \(\tau_1, \ldots, \tau_n\) of conical singularities so that \(\tau_1 = -1, \tau_2 = 0,\) and \(\tau_3 = 1\).

As is well-known, the spectrum of the Friederichs self adjoint extension of the Laplace-Beltrami operator \(\Delta_m\) on \((\mathbb{T}, m)\) is discrete and the corresponding zeta regularized spectral determinant \(\det \Delta_m\) can be introduced in the usual way, e.g. [16]. In the same paper [16] it was shown that the determinant \(\det \Delta_m\), considered as a function on the subspace \(\mathcal{C}_n^*\) of \(\mathcal{C}_n\) consisting of the metrics with fixed orders \(\beta_j\) of conical singularities, is real analytic. Extremal properties of the determinant \(\det \Delta_m\) on the metrics of unit area with four conical singularities of order \(-1/2\) were studied in [23, Sec 3.5]. We also note that some variational formulas for \(\det \Delta_m\) as a function on \(\mathcal{C}_n^*\) can be obtained as an immediate consequence of results in [20, Proposition 1], [12].

An explicit expression for \(\det \Delta_m\) in terms of the coordinates \(c, \tau_j,\) and \(\beta_j\) on \(\mathcal{C}_n\) was found in [4] and rigorously proved in [13, Sec. 3.2]; see also [1] for the most recent progress towards a rigorous mathematical proof of a similar explicit formula for the determinant of Dirichlet Laplacian on polygons in [3].

In this paper we study extremal properties of the determinant of the Friederichs selfadjoint extension of the Laplacian on the Euclidean isosceles triangle envelopes of fixed area as a function of angles. The envelopes are glued from two congruent Euclidean isosceles triangles \(ABC\) and \(CBA'\) in accordance with the scheme on Fig. 1: The polygon \(ABA'C\) is first folded along the line \(BC\), then the side \(AC\) is glued to \(CA'\) and \(AB\) is glued to \(A'B\). The internal angles of the triangles are \(\pi(\beta+1), \pi(-3\beta-1),\) and \(\pi(\beta+1).\) We use the parameter \(\beta \in (-1,-1/2)\) as a measure of the angles.

The Euclidean isosceles triangle envelopes can equivalently be viewed as the Riemann sphere \(\mathbb{T}\) equipped with family of flat metrics \(m_\beta = c_\beta^2 |z|^2 - 1|z|^{-4\beta} dz|dz|^2\) having three conical singularities. This family forms a subspace in \(\mathcal{C}_3\) consisting of metrics with singularities of orders \(\beta_1 = \beta_3 = \beta\) and \(\beta_2 = -2 - 2\beta\) and (fixed) area
\[
S = c_\beta^2 \int_\mathbb{C} |z|^2 - 1|z|^{-4\beta} dz \wedge \frac{dz \wedge d\bar{z}}{-2i},
\]
the scaling factor \(c_\beta > 0\) is uniquely determined by the area \(S\) of envelope and the value of angle parameter \(\beta\). The Laplace-Beltrami operator \(\Delta^S_\beta\) on the sphere \(\mathbb{T}\) equipped with the metric \(m_\beta\) is nothing but the Euclidean Laplacian on the envelope of total area \(S\) with angles prescribed by the values of \(\beta\). We pick the Friederichs selfadjoint extension of \(\Delta^S_\beta\), which we still denote by \(\Delta^S_\beta\).

We show that for the envelopes of fixed area \(S \neq S(\beta)\) the function
\[
(-1, -1/2) \ni \beta \mapsto \log \det \Delta^S_\beta
\]
is real analytic and grows without any bound as $\beta \to -1^+$ or $\beta \to -1/2^-$ (i.e. as the envelopes degenerate, or equivalently, as an internal angle of the triangle $ABC$ on Fig. 1 goes to zero). We also prove that $\beta = -2/3$, which corresponds to the equilateral triangle envelope (the most symmetrical geometry), is a critical point of the function (1.1) and the critical value is given by

$$\log \det \Delta_{-2/3}^S = \frac{2}{3} \log \pi + \frac{1}{3} \log \left(\frac{2}{3}\right) - 2 \log \Gamma \left(\frac{2}{3}\right) + \frac{1}{3} \log S.$$}

Moreover, it turns out that for all not too large values of the area $S$ (in particular, for $S \leq 1$) the function (1.1) achieves its \underline{absolute minimum} only at $\beta = -2/3$ (i.e. only on the equilateral triangle envelope) and there are no other critical points, whereas for sufficiently large values of $S$ (e.g. for $S \geq 2$) the critical point $\beta = -2/3$ corresponds only to a \underline{local maximum} of the function (1.1).

As an important step of the proof we explicitly express $\log \det \Delta_{\beta}^S$ as a function of $S$ and $\beta$, which may be of independent interest. This is done by using an original approach based on the Meyer-Vietoris type formula for determinants of Laplacians [5], formulas for the determinants of Friederichs Dirichlet Laplacians on cones [34], integral representations for Barnes double zeta function [35], and the Polyakov-Alvarez anomaly formula [2, 29, 32]. The Meyer-Vietoris type formula (aka Burghelea-Friedlander-Kappeler or BFK formula) for determinants of Laplacians works very much like a substitution for the insertion lemma in [32, 33, 15, 17].

We also illustrate our purely analytical results by graphs based on explicit evaluations of the determinant for rational values of $\beta$ and uniform approximations for the
function (1.1), cf. Fig. 2 and Fig. 5.

To the best of our knowledge no other analytical results on extremal properties of determinants under variation of angles (or, equivalently, orders of conical singularities) are available yet, except for the one in an earlier work of the author [13, Section 3.1], where the determinant on the Riemann sphere equipped with family of singular spherical metrics is studied as an illustrating example.

This paper is organized as follows. In the next section (Section 2) we state the problem in the most straightforward and naive way and formulate our main results for the Euclidean isosceles envelopes of unit area (Theorem 2.1). In Section 3 we present our main technical tool: Proposition 3.1, this is where we explicitly express the determinant as a function of angles. In Section 4 we reformulate the problem in terms of flat singular metrics on the Riemann sphere. We then prove Proposition 3.1 in Section 5. Section 6 is devoted to the proof of Theorem 2.1. In Section 7 we demonstrate how the results of Theorem 2.1 change when considering the Euclidean isosceles triangle envelopes of non-unit area. Finally, Appendix A contains auxiliary results on derivatives of Barnes double zeta function that are mainly used in the proof of Theorem 2.1, Section 6; derivatives of Barnes double zeta functions first appear in Proposition 3.1.

2 Statement of the problem and main results

Consider an isosceles triangle envelope of unit area glued from two congruent isosceles triangles $ABC$ and $CBA'$ in accordance with the scheme on Fig. 1. As already discussed in Introduction 1, the triangles are first glued and folded along the side $BC$, then the side $AC$ is glued to $CA'$ and $AB$ is glued to $A'B$. The internal angles of the triangles are $\pi(\beta+1)$, $\pi(-2\beta-1)$, and $\pi(\beta+1)$. We shall use the parameter $\beta \in (-1, -1/2)$ as a measure of the angles. The pairwise glued vertices of triangles $ABC$ and $CBA'$ form the vertices $A$, $B$, and $C$ of the triangle envelope.

Let $(x, y)$ be a system of Cartesian coordinates in the Euclidean plane of the polygon $ABA'C$ on Fig. 1. Clearly, this induces a system of coordinates $(x, y)$ and a Euclidean metric on the envelope. Let $L^2$ stand for the space of functions $f$ on the envelope with finite norms

$$\|f\| = \left( \iint_{ABA'C} |f(x, y)|^2 \, dx \, dy \right)^{1/2};$$

in particular, $\|1\| = \sqrt{S} = 1$.

Consider the Euclidean Laplacian $\Delta_\beta = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2}$ as an unbounded operator in the Hilbert space $L^2$ initially defined on the functions $u$ that are smooth on the envelope and supported outside of its vertices $A$, $B$, and $C$. The functions $u$ can also be considered as functions in the polygon $ABA'C$. They are the smooth functions supported outside of the vertices $A, B, A', C$, and such that the value $u(x, y)$ and the values of all derivatives $\frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} u(x, y)$ at any point $P = (x, y)$ of the side $AC$ (resp. $AB$) coincide with those at the point $P' = (x', y')$ of $CA'$ (resp. $BA'$) satisfying $|PA| = |P'A'|$; here $|XY|$ stands for the Euclidean distance between $X$ and $Y$. The points $P$ and $P'$ are getting identified after gluing $AC$ to $CA'$ and $AB$ to $BA'$.
Next we introduce the spectral zeta regularized determinant of $\Delta_\beta$. We only outline the standard well-known steps and refer to [16] for further details (see also Section 4).

The operator $\Delta_\beta$ is densely defined, but not essentially selfadjoint. We consider its Friederichs selfadjoint extension, which we still denote by $\Delta_\beta$ and call the Friederichs Laplacian or Laplacian for short. (The Laplacian $\Delta_\beta$ can also be viewed as an operator of a certain boundary value problem in the polygon $ABA'C$, but we do not discuss this here.) The spectrum of $\Delta_\beta$ consists of isolated eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$ of finite multiplicity and the determinant $\det \Delta_\beta$ is formally introduced as their product with excluded eigenvalue $\lambda_0 = 0$. The standard zeta regularization gives the meaning to the infinite product: First the corresponding spectral zeta function is defined by the equality

$$\zeta_\beta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}, \quad \Re s > 1;$$

asymptotics of the spectral counting function of $\Delta_\beta$ guarantees convergence of the series and analyticity of $s \mapsto \zeta_\beta(s)$ in the half plane $\Re s > 1$. Then short time heat trace asymptotics are used to demonstrate that $s \mapsto \zeta_\beta(s)$ has an analytic continuation to a neighbourhood of $s = 0$. Finally, the determinant of Friederichs Laplacian $\Delta_\beta$ is rigorously defined in terms of $\zeta_\beta$ as follows:

$$\det \Delta_\beta := \exp(-\zeta_\beta'(0)), \quad -1 < \beta < -1/2.$$

We are now in position to formulate the main results of this paper.

**Theorem 2.1** (Determinant on Euclidean isosceles triangle envelopes of unit area).

1. The function $(-1, -1/2) \ni \beta \mapsto \log \det \Delta_\beta$ is real analytic.

2. As the isosceles triangle envelope of unit area degenerates (i.e. as an internal angle of triangle $ABC$ goes to zero, or, equivalently, as $\beta \to -1^+$ or $\beta \to -1/2^-$), the determinant $\det \Delta_\beta$ grows without any bound in accordance with the following small-angle asymptotics

$$\log \det \Delta_\beta = -\frac{\log(\beta + 1)}{6(\beta + 1)} + \left(\frac{1}{6} \log(8\pi) - 4 \log A\right) \frac{1}{\beta + 1}$$

$$- \log(\beta + 1) - \log 2 + O(\beta + 1) \quad \text{as} \quad \beta \to -1^+, \tag{2.1}$$

$$\log \det \Delta_\beta = \frac{\log(-2\beta - 1)}{12(2\beta + 1)} - \left(\frac{1}{6} + \frac{\log \pi}{12} - 2 \log A\right) \frac{1}{2\beta + 1}$$

$$- \frac{3}{4} \log(-2\beta - 1) - \frac{1}{2} \log 2 - \frac{1}{4} \log \pi + O(2\beta + 1) \quad \text{as} \quad \beta \to -1/2^-, \tag{2.2}$$

where $A$ is the Glaisher-Kinkelin constant.

3. The determinant $\det \Delta_\beta$ reaches its absolute minimum only on the equilateral triangle envelope. More precisely, for $-1 < \beta < -1/2$ we have

$$\log \det \Delta_\beta \geq \log \det \Delta_{-2/3} = \frac{2}{3} \log \pi + \frac{1}{3} \log \frac{2}{3} - 2 \log \Gamma\left(\frac{2}{3}\right)$$

with equality iff $\beta = -2/3$. The function $(-1, -1/2) \ni \beta \mapsto \log \det \Delta_\beta$ is concave up, its graph is depicted on Fig. 2.
As it was already discussed in Introduction 1, for the Euclidean isosceles triangle envelopes of fixed non-unit area \( S \neq S(\beta) \) the results remain essentially the same provided that the area \( S \) is not too large: The equilateral triangle envelope (i.e. the most symmetrical geometry) always gives rise to a critical point of the determinant, but the critical point turns from the absolute minimum to a local maximum of the determinant as the area \( S \) of the envelopes increases. We postpone a more detailed discussion to Section 7.

Figure 2: Graph of the function \((-1, -1/2) \ni \beta \mapsto \log \det \Delta_{\beta}\) for the Friederichs Laplacian \(\Delta_{\beta}\) on isosceles Euclidean triangle envelopes of unit area glued from two copies of triangle with interior angles \(\pi(\beta + 1), \pi(-1 - 2\beta),\) and \(\pi(\beta + 1),\) cf. Fig 1. For the graph we use the representation of \(\log \det \Delta_{\beta}\) as a function of \(\beta\) found in Proposition 3.1 and evaluate \(\log \det \Delta_{\beta}\) for some rational values of \(\beta\) by using Remark A.3 (the points marked with Diamond symbol on the graph), we also uniformly approximate \(\log \det \Delta_{\beta}\) by using Proposition 3.1 together with estimate (A.2) in Lemma A.1, where we take \(N = 4\) (solid line).

3 Determinant as a function of angles

Consider the Friederichs Laplacians \(\Delta_{\beta}\) on the Euclidean isosceles triangle envelopes of unit area. In Proposition 3.1 below we explicitly express \(\log \det \Delta_{\beta}\) as a function of \(\beta \in (-1, -1/2)\).

**Proposition 3.1** (Determinant as a function of angles). *For the determinant of the Friederichs Laplacian \(\Delta_{\beta}, -1 < \beta < -1/2,\) on the Euclidean isosceles triangle envelopes*
of unit area we have
\[
\log \det \Delta_\beta = \frac{1}{6} \left( 2\beta + \frac{4}{\beta + 1} - \frac{1}{2\beta + 1} \right) \log 2 - \frac{1}{6} \left( \frac{2}{\beta + 1} - \frac{1}{2\beta + 1} - 1 \right) \log c_\beta
\]
\[-4\zeta_B'(0; \beta + 1, 1, 1) - 2\zeta_B'(0; -2\beta - 1, 1, 1) + 2\zeta_B''(-1) - \log(\beta + 1) - \frac{1}{2} \log(-2\beta - 1) - \frac{5}{2} \log 2 - \log \pi. \tag{3.1}
\]

Here
\[
c_\beta = \frac{2}{\Gamma(-\beta - 1/2) \Gamma(\beta + 1)} \left( \frac{\pi}{\sin \pi(-2\beta - 1)} \right)^{1/2} > 0 \tag{3.2}
\]
is the scaling factor that comes from uniformization of the envelopes in Section 4, \( \zeta_B \) stands for the Barnes double zeta function initially defined by the double series
\[
\zeta_B(s; a, b, x) = \sum_{m,n=0}^{\infty} (am + bn + x)^{-s}, \quad \Re s > 2, a > 0, b > 0, x \in \mathbb{R}, \tag{3.3}
\]
and then extended by analyticity to the disk \(|s| < 1\) \cite{27, 35}, \( \zeta_R(s) \) is the Riemann zeta function, and the prime in \( \zeta'_B \) and \( \zeta'_R \) denotes the derivative with respect to \( s \).

Even though the assertion of Proposition 3.1 can be obtained as a consequence of much more general results \cite[Theorem 1.1 and Proposition 3.3]{13} (uniformization in Section 4 below is still needed), we decided to present a complete independent proof. First, because the proof of Proposition 3.1 is much more visual, transparent, and simple (due to particularly simple geometry of Euclidean isosceles triangle envelopes: spherical topology, Euclidean metrics, explicit formulas for local holomorphic and geodesic polar coordinates, only one angle parameter \( \beta \), readily available Meyer-Vietoris type formulas, etc.). Second, it makes this paper self-contained and can be of its own interest. We only note that the celebrated partially heuristic Aurell-Salomonson formula for determinants of Laplacians on polyhedra with spherical topology \cite[(50)]{4} returns a result equivalent to the one in Proposition 3.1; for details we refer to \cite[Section 3.2]{13}. We also refer to \cite{1} for the most recent progress towards a rigorous mathematical proof of a similar Aurell-Salomonson formula in \cite{3}, though the results and methods in \cite{1} seem not to be directly related to ours (in the correction notice the authors refer to a pre-print which is not yet available at the time of writing this paper).

We prove Proposition 3.1 in Section 5, the proof is preceded by Section 4, Lemma 5.1, and Lemma 5.2.

### 4 Uniformization

In this section we introduce a singular conformal metric \( m_\beta \) on the Riemann sphere \( \overline{\mathbb{C}} \) so that for each \( \beta \) the resulting metric sphere \( (\overline{\mathbb{C}}, m_\beta) \) is isometric to the corresponding Euclidean isosceles triangle envelope glued in accordance with the scheme on Fig. 1. Then we reintroduce the Laplacian \( \Delta_\beta \) as an unbounded selfadjoint operator in the \( L^2 \)-space of functions on \( \overline{\mathbb{C}} \). In Section 2 this was done in an equivalent naive way, which
is good for the statement of the problem and formulation of our main results, but not suitable for the methods we use in the proof.

The Schwarz-Christoffel transformation

\[ w(z) = c_\beta \int_0^z (\hat{z}^2 - 1)^{\beta/2} d\hat{z} \]

maps the upper complex half plane \( \mathbb{H} \) onto an isosceles triangle \( ABC \) with interior angles \( \pi(\beta + 1), \pi(-1 - 2\beta), \) and \( \pi(\beta + 1) \), where \(-1 < \beta < -1/2\). Later on the scaling factor \( c_\beta > 0 \) will be chosen so that the triangle \( ABC \) has an area of 1/2. The pullback of the Euclidean metric by this conformal transformation induces the metric

\[ m_\beta = c_\beta^2 |z^2 - 1|^{2\beta}|z|^{-4-4\beta} |dz|^2, \quad -1 < \beta < -1/2, \]  

(4.1)
on \( \mathbb{H} \). Similarly, the lower half plane equipped with the same metric (4.1) is isometric to a Euclidean triangle congruent to the triangle \( ABC \). Thus we obtain the following uniformization of the Euclidean isosceles triangle envelopes of unit area: the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) equipped with the metric (4.1) is isometric to a triangle envelope glued from two congruent Euclidean isosceles triangles of area 1/2 in accordance with the scheme on Fig. 1. The vertex \( A \) of the envelope corresponds to the point \( z = -1 \), the vertex \( B \) corresponds to \( z = 0 \), and \( C \) to \( z = 1 \). The points 0 and \( \pm 1 \) of \( \overline{\mathbb{C}} \) will also be called vertices.

It is natural to pick the local complex coordinates \( z \in \mathbb{C} \) and \( 1/z \) near \( \infty \) on \( \overline{\mathbb{C}} \). In the coordinate \( z \) the Laplacian \( \Delta_\beta \) takes the form

\[ \Delta_\beta = -4c_\beta^{-2}|z^2 - 1|^{-2\beta}|z|^{4+4\beta} \partial_z \partial_{\bar{z}}, \]

where \( \partial_z = \frac{\partial}{\partial z} \) and \( \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} \) are the complex derivatives. The change of coordinates \( z \mapsto 1/z \) leads to the corresponding representations of the metric \( m_\beta \) and the Laplacian \( \Delta_\beta \) in the local coordinate near \( \infty \).

In a vicinity of the vertex \( z = \pm 1 \) there is a local complex isothermal coordinate \( w_\pm \) that brings the metric (4.1) into the form \( m_\beta = |w_\pm|^{2\beta}|dw_\pm|^2 \) for \( |w_\pm| < \delta \) with some \( \delta > 0 \), see e.g. [36],[37, Lemma 3.4] or [32, Section 1]. For the holomorphic change of coordinate \( w_\pm = w_\pm(z) \) we have

\[ w_\pm(z) = \left( (\beta + 1)c_\beta \int_{\pm 1}^z (\hat{z}^2 - 1)^{\beta/2} d\hat{z} \right)^{1/2+1}, \]  

(4.2)

where \( z \) is the same as in (4.1), the sign + (resp. -) is taken for the vertex \( z = 1 \) (resp. \( z = -1 \)). Let us also notice that in the geodesic polar coordinates

\[ (r, \theta) = ((\beta + 1)^{-1}|w_\pm|^\beta + 1, \arg w_\pm) \]

near the vertex \( z = \pm 1 \) the metric takes the form

\[ m_\beta = dr^2 + (1 + \beta)^2 r^2 d\theta^2. \]
The latter one is the metric of a standard cone (surface of revolution) created by rotating an angle of size \( \alpha \in (0, \pi/6) \), \( \sin \alpha = \beta + 1 \), around one of its rays. Similarly, in the local isothermal coordinate

\[
w_0 = w_0(z) = \left( -(2\beta + 1)c_\beta \int_{0}^{z} (\hat{z}^2 - 1)^{\beta - 2 - 2\beta} d\hat{z} \right)^{-\frac{1}{2}}
\]

(4.3)

in a vicinity of the vertex \( z = 0 \) we have \( m_\beta = |w_0|^{-4 - 4\beta} |dw_0|^2 \) for \( |w_0| < \delta \) with some \( \delta > 0 \). In the geodesic polar coordinates

\[
(r, \theta) = \left( (-2\beta - 1)^{-1}|w_0|^{-2\beta - 1}, \arg w_0 \right)
\]

near \( z = 0 \) the metric \( m_\beta \) takes the form of a standard cone of vertex angle \( \alpha \in (0, \pi/2) \) such that \( \sin \alpha = -1 - 2\beta \).

The space \( L^2 \), considered as a space of functions on \( \hat{C} \), has the norm

\[
\|f\| = c_\beta \left( \int_{C} |z^2 - 1|^{2\beta} |z|^{-4 - 4\beta} |f(z, \bar{z})|^2 dz d\bar{z} \right)^{1/2}.
\]

(4.4)

The operator \( \Delta_\beta \) in the Hilbert space \( L^2 \) is initially defined on the smooth functions supported outside of the vertices \( z = 0 \) and \( z = \pm 1 \). This operator is densely defined but not essentially selfadjoint (due to the singularities at \( z = 0 \) and \( z = \pm 1 \)). We consider the Friederichs selfadjoint extension of \( \Delta_\beta \), which we still denote by \( \Delta_\beta \). Recall that it is the only selfadjoint extension with domain in \( H^1 \) [14]; here \( H^1 \) stands for the Sobolev space of all functions \( u \in L^2 \) with finite Dirichlet integral \( \|\nabla_\beta u\|^2 \), where \( \nabla_\beta u \) is the gradient of a function \( u \) with respect to the metric \( m_\beta \) and \( \|\nabla_\beta u\|^2 \) is the integral in (4.4) with \( f = \nabla_\beta u \). The well-known definition of the zeta regularized spectral determinant \( \det \Delta_\beta \) was already discussed in Section 2; for more details, we refer to [16].

Finally, in order to find the announced in Proposition 3.1 value of the scaling factor \( c_\beta \), we first note that the length of the side \( AB \) is the metric distance between the vertices \( z = -1 \) and \( z = 0 \) (or, what is the same, the length of \( BC \) is the metric distance between \( z = 0 \) and \( z = 1 \)). The length of \( AB \) is given by

\[
|AB| = |BC| = c_\beta \int_{0}^{1} |z^2 - 1|^{\beta} |z|^{-2 - 2\beta} |dz| = c_\beta \frac{\Gamma \left( -\beta - \frac{1}{2} \right) \Gamma(\beta + 1)}{2\sqrt{\pi}}.
\]

For the total area of the envelope we thus have

\[
1 = 2|AB| \cdot |BC| \cdot \sin \frac{\pi}{2}(-1 - 2\beta) \cdot \cos \frac{\pi}{2}(-1 - 2\beta)
= \frac{1}{4\pi} c_\beta^2 \left( \Gamma \left( -\beta - 1/2 \right) \Gamma(\beta + 1) \right)^2 \cdot \sin \pi(-2\beta - 1),
\]

this implies (3.2).

5 Proof of Proposition 3.1

We need some preliminaries before we can formulate the first lemma preceding the proof of Proposition 3.1.
Let $\mathbb{C}_\epsilon$ (respectively $\mathbb{C}_\epsilon$) stand for the Riemann sphere $\overline{\mathbb{C}}$ (respectively the complex plane $\mathbb{C}$) with three small disks $|w_-| < \epsilon$, $|w_0| < \epsilon$, and $|w_+| < \epsilon$ removed. Here $w_-$ and $w_0$ are the local complex isothermal coordinates (4.2) and (4.3) in vicinities of the vertices. Note that the construction of $\mathbb{C}_\epsilon$ is nearly identical to the one for $\Sigma_3$ in [32, Section 1].

As it was discussed in Section 4, any Euclidean isosceles triangle envelope of unit area can equivalently be viewed as the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (Riemann sphere) endowed with the metric $m_\beta$ in (4.1). Notice that the surface $(\mathbb{C}_\epsilon, m_\beta)$ with boundary $\partial \mathbb{C}_\epsilon$ (a flat pair of pants) is isometric to the Euclidean isosceles triangle envelope of unit area with $\epsilon$-cones around the vertices removed (i.e. the surface can be glued from two congruent Euclidean isosceles triangles with excision of circular sectors of radius $\frac{1}{\beta+1}\epsilon^{\beta+1}$ at the vertices $A$, $A'$ and $C$, and of radius $-\frac{1}{2\beta+1}\epsilon^{-2\beta-1}$ at the vertex $B$, see Fig. 1 and Fig. 3).

By $\Delta_{\beta}[\mathbb{C}_\epsilon]$ we denote the selfadjoint Dirichlet Laplacian on $(\mathbb{C}_\epsilon, m_\beta)$ (since there are no singularities of $m_\beta$ on $\mathbb{C}_\epsilon$, the Dirichlet Laplacian $\Delta_{\beta}[\mathbb{C}_\epsilon]$ initially defined on the smooth functions is essentially selfadjoint). The operator $\Delta_{\beta}[\mathbb{C}_\epsilon]$ is positive and its spectrum consists of discrete eigenvalues of finite multiplicity. Let $\det \Delta_{\beta}[\mathbb{C}_\epsilon]$ stand for the zeta regularized spectral determinant of $\Delta_{\beta}[\mathbb{C}_\epsilon]$.

We will also be using the standard spherical metric

$$m = 4(1 + |z|^2)^{-2}|dz|^2$$

(5.1)
as a reference metric on $\mathbb{C}$. The surface $(\mathbb{C}, m)$ is isometric to a unit sphere in $\mathbb{R}^3$ (and $(\mathbb{C}, m)$ is isometric to a unit sphere in $\mathbb{R}^3$ with three holes). Similarly to the notation for the determinant generated by the metric $m_\beta$ on $\mathbb{C}_\epsilon$ introduced above, we denote by $\det \Delta|_{\mathbb{C}}$ the determinant of the selfadjoint Dirichlet Laplacian $\Delta|_{\mathbb{C}}$ on $(\mathbb{C}, m)$.

**Lemma 5.1.** For $-1 < \beta < -1/2$ the determinants of the Dirichlet Laplacians $\Delta_\beta|_{\mathbb{C}}$ and $\Delta|_{\mathbb{C}}$ satisfy

$$\log \frac{\det(\Delta_\beta|_{\mathbb{C}})}{\det(\Delta|_{\mathbb{C}})} = \frac{3\beta^2 + 4\beta}{3} \log \epsilon + \frac{\beta^2 + 3\beta + 1}{3(\beta + 1)} \log 2$$

$$+ \frac{1}{6} \left( \frac{2}{\beta + 1} - \frac{1}{2\beta + 1} + 1 \right) \log c_\beta - \frac{4}{3} + o(1) \quad \text{as} \quad \epsilon \to 0^+,$$

where $c_\beta$ is the scaling factor found in (3.2).

**Proof of Lemma 5.1.** Introduce the metric potentials

$$\chi_\beta(z) = \log c_\beta + \beta \log |z + 1| + \beta \log |z - 1| - 2(1 + \beta) \log |z|, \quad (5.2)$$

$$\psi(z) = \log 2 - \log(1 + |z|^2), \quad (5.3)$$

and $\varphi_\beta = \chi_\beta - \psi$. In terms of these potentials we have $m_\beta = e^{2\chi_\beta} |dz|^2$ for the metric (4.1), $m = e^{2\psi} |dz|^2$ for the metric (5.1), and $m_\beta = e^{2\varphi_\beta} m$. The potential $\psi$ is smooth on $\mathbb{C}$, while $\chi_\beta$ and $\varphi_\beta$ are smooth on $\mathbb{C}_\epsilon$ (but not on $\mathbb{C}$).

The well-known Polyakov-Alvarez formula [2, 29, 32] implies

$$\log \frac{\det(\Delta_\beta|_{\mathbb{C}})}{\det(\Delta|_{\mathbb{C}})} = -\frac{1}{12\pi} \int_{\mathbb{C}_\epsilon} (|\nabla \varphi_\beta|^2 + 2K\varphi_\beta) dA$$

$$- \frac{1}{6\pi} \int_{\partial \mathbb{C}_\epsilon} k\varphi_\beta ds - \frac{1}{4\pi} \int_{\partial \mathbb{C}_\epsilon} \partial_n \varphi_\beta ds. \quad (5.4)$$

Here $\nabla$ is the gradient, $K = 1$ is the Gaussian curvature, $dA$ is the area, $k$ is the geodesic curvature of the boundary $\partial \mathbb{C}_\epsilon$, $\partial_n$ is the outward normal derivative to $\partial \mathbb{C}_\epsilon$, and $ds$ is the arc length — all with respect to the spherical metric $m$.

The Liouville’s equation for the curvature $K_\beta = 0$ of the metric $m_\beta$ on $\mathbb{C}_\epsilon$ reads $e^{-2\chi_\beta} (-4\partial_z \partial_{\bar{z}} \chi_\beta) = K_\beta$. Hence $\chi_\beta$ is a harmonic function on $\mathbb{C}_\epsilon$ and

$$- \Delta \varphi_\beta = -e^{-2\psi} (-4\partial_z \partial_{\bar{z}} \varphi_\beta) = e^{-2\psi} (-4\partial_z \partial_{\bar{z}} \psi) = 1 \quad (5.5)$$

is the curvature of $m$ on $\mathbb{C}_\epsilon$. Now we use Green’s formula together with (5.5) and write (5.4) in the form

$$\log \frac{\det(\Delta_\beta|_{\mathbb{C}})}{\det(\Delta|_{\mathbb{C}})} = -\frac{1}{12\pi} \left( \int_{\mathbb{C}_\epsilon} \varphi_\beta dA + \int_{\partial \mathbb{C}_\epsilon} \varphi_\beta(\partial_n \varphi_\beta) ds \right)$$

$$- \frac{1}{6\pi} \int_{\partial \mathbb{C}_\epsilon} k\varphi_\beta ds - \frac{1}{4\pi} \int_{\partial \mathbb{C}_\epsilon} \partial_n \varphi_\beta ds. \quad (5.6)$$

For the area integral in (5.6) we have

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{C}_\epsilon} \varphi_\beta dA = \int_{\mathbb{C}} \chi_\beta dA - \int_{\mathbb{C}} \psi dA = 4\pi \left( (\beta - 1) \log 2 + \log c_\beta + 1 \right).$$
Indeed, based on (5.2) and (5.3) both integrals over \( \mathbb{C} \) can be easily evaluated as follows:

\[
\int_{\mathbb{C}} \chi_{\beta} \, dA = \int_{\mathbb{C}} \chi_{\beta} \left( -4 \partial_{\bar{z}} \partial_z \psi \right) \frac{dz \wedge d\bar{z}}{-2i}
\]

\[
= \lim_{\delta \to 0^+} \left( \oint_{|z| = \delta} + \oint_{|z| = 1 - \delta} + \oint_{|z| = 1/\delta} \right) \left( \chi_{\beta} \partial_n \psi - \psi \partial_n \chi_{\beta} \right) |dz| \]

\[
= -2\pi \beta \psi(-1) - 2\pi \psi(0) - 2\pi \psi(1) - 2\pi (2 \log 2 - 2 \log c_{\beta})
\]

\[
= 4\pi (\beta \log 2 + \log c_{\beta}),
\]

\[
\int_{\mathbb{C}} \psi \, dA = \int_{\mathbb{C}} \left( \log 2 - \log(1 + |z|^2) \right) \frac{4}{(1 + |z|^2)^2} \frac{dz \wedge d\bar{z}}{-2i} = 4\pi (\log 2 - 1).
\]

Next we find the asymptotics of all other integrals in (5.6) as \( \epsilon \to 0^+ \). Since the equations of all three components of the boundary \( \partial \mathcal{C}_\epsilon \) are written in different local coordinates, it is convenient to treat the components in the corresponding local coordinates separately. The computations for all three components are very similar, we present detailed evaluations only for the components \( |w_\pm| = \epsilon \) encircling the vertices \( z = \pm 1 \).

Clearly, the geodesic curvature of the circle \( |w_\pm| = \epsilon \) with respect to the usual Euclidean metric \( |dw_\pm|^2 \) is \( \epsilon^{-1} \). In order to find the geodesic curvature \( k \) with respect to the spherical metric \( m = e^{2\psi_\pm} |dw_\pm|^2 \) we use the equality

\[
k = -e^{-\psi_\pm} (\epsilon^{-1} + \partial_{|w_\pm|} \psi_\pm),
\]

where \( \psi_\pm \) is the (smooth) potential of the spherical metric \( m \) written in the local holomorphic coordinate \( w_\pm \). For the arc length we have \( ds = e^{\psi_\pm} |dw_\pm| \). We also recall that \( m_\beta = |w_\pm|^{2\beta} |dw_\pm|^2 \) and \( \partial_n = e^{-\psi_\pm} \partial_{|w_\pm|} \). Thus for the components of \( \partial \mathcal{C}_\epsilon \) defined by the equations \( |w_\pm| = \epsilon \) we obtain

\[
\oint_{|w_\pm| = \epsilon} \varphi_\beta \partial_n \varphi_\beta \, ds = \oint_{|w_\pm| = \epsilon} \left( \beta \log |w_\pm| - \psi_\pm(w_\pm) \right) \partial_{|w_\pm|} \left( \beta \log |w_\pm| - \psi_\pm(w_\pm) \right) |dw_\pm|
\]

\[
= -\int_0^{2\pi} \left( \beta \log \epsilon - \psi_\pm(\epsilon e^{i\theta}) \right) (\beta/\epsilon + O(1)) \epsilon \, d\theta = -2\pi \beta^2 \log \epsilon + 2\pi \beta \psi_\pm(0) + o(1),
\]

\[
\oint_{|w_\pm| = \epsilon} k \varphi_\beta \, ds = -\int_{|w_\pm| = \epsilon} \left( \epsilon^{-1} + \partial_{|w_\pm|} \psi_\pm(w_\pm) \right) (\beta \log \epsilon - \psi_\pm(w_\pm)) |dw_\pm|
\]

\[
= -2\pi \beta \log \epsilon + 2\pi \psi_\pm(0) + o(1),
\]

\[
\oint_{|w_\pm| = \epsilon} \partial_n \varphi_\beta \, ds = \oint_{|w_\pm| = \epsilon} \partial_{|w_\pm|} \left( \beta \log |w_\pm| - \psi_\pm(w_\pm) \right) |dw_\pm| = -2\pi \beta + o(1).
\]

These estimates would be useless without knowing the value of \( \psi_\pm \) at zero. Fortunately, it is not hard to show that

\[
\psi_\pm(0) = \psi(\pm 1) - \log |w_\pm'(\pm 1)| = -\frac{\beta}{\beta + 1} \log 2 - \frac{1}{\beta + 1} \log c_{\beta}
\]

with the metric potential \( \psi \) given in (5.3) and the function \( w_\pm(z) \) given in (4.2).
In exactly the same way we also find that
\[ \oint |w_0| = \epsilon \partial_n \varphi_0 \, ds = -8\pi(\beta + 1) \log \epsilon - 4\pi(\beta + 1) \psi_0(0) + o(1), \]
\[ \oint |w_0| = \epsilon k\varphi_0 \, ds = 8\pi(\beta + 1) \log \epsilon + 2\pi \psi_0(0) + o(1), \]
\[ \oint |w_0| = \epsilon \partial_n \varphi_0 \, ds = -4\pi(\beta + 1) \log \epsilon - 2\pi \psi_0(0) + o(1), \]
where for the potential \( \psi_0(w_0) \) of the spherical metric \( m \) written in the local holomorphic coordinate \( w_0 = w_0(z) \) in (4.3) we have
\[ \psi_0(0) = \psi(0) - \log |w_0'(0)| = \log 2 + \frac{1}{2\beta + 1} \log \epsilon. \]  
(5.8)

We have evaluated all integrals in the right hand side of (5.6). Combining the results we obtain
\[ \log \frac{\det(\Delta \beta |_{w_\pm| < \epsilon})}{\det(\Delta |_{w_\pm| < \epsilon})} = -\frac{1}{3}((\beta - 1) \log 2 + \log c_\beta + 1) \]
\[ + \left( \frac{\beta(\beta + 2)}{3} \log \epsilon - \frac{\beta + 2}{3} \psi_\pm(0) + \beta \right) \]
\[ + \left( \frac{2\beta(\beta + 1)}{3} \log \epsilon + \frac{\beta}{3} \psi_0(0) - (\beta + 1) \right) + o(1) \quad \text{as} \quad \epsilon \to 0^+ \]

with \( \psi_\pm(0) \) and \( \psi_0(0) \) given in (5.7) and (5.8). This completes the proof of Lemma 5.1.

Lemma 5.2. Let \( -1 < \beta < -1/2 \). Then

1. The determinants of Dirichlet Laplacians \( \Delta \beta |_{w_\pm| \leq \epsilon} \) and \( \Delta |_{w_\pm| \leq \epsilon} \) satisfy
\[ \log \frac{\det(\Delta \beta |_{w_\pm| \leq \epsilon})}{\det(\Delta |_{w_\pm| \leq \epsilon})} = -\frac{\beta^2 + 2\beta}{6} \log \epsilon + \frac{\beta^2 - 2\beta}{6} \log 2 = \frac{1}{3(\beta + 1)} \log c_\beta \]
\[ - \frac{5}{12} \beta - \frac{1}{2} \log(\beta + 1) - 2\zeta_B'(0; \beta + 1, 1, 1) + 2\zeta_R'(-1) + o(1) \quad \text{as} \quad \epsilon \to 0^+. \]
2. The determinants of Dirichlet Laplacians $\Delta_\beta|_{|w_\pm| \leq \epsilon}$ and $\Delta|_{|w_\pm| \leq \epsilon}$ satisfy

\[
\log \frac{\det(\Delta_\beta|_{|w_\pm| \leq \epsilon})}{\det(\Delta|_{|w_\pm| \leq \epsilon})} = -\frac{2\beta^2 + 2\beta}{3} \log \epsilon - \frac{2\beta^2 + 2\beta + 1}{3(2\beta + 1)} \log 2 + \frac{1}{3(2\beta + 1)} \log c_\beta
\]
\[
+ \frac{5}{6}(\beta + 1) - \frac{1}{2} \log(-2\beta - 1) - 2\zeta'_{B}(0; -2\beta - 1, 1, 1) + 2\zeta'_{R}(-1) + o(1) \quad \text{as } \epsilon \to 0^+.
\]

Proof of Lemma 5.2. The Polyakov-Alvarez formula [2, 29, 32] for Dirichlet Laplacians in two metrics on the disk $|w_\pm| \leq \epsilon$ reads:

\[
\log \frac{\det(\Delta|_{|w_\pm| \leq \epsilon})}{\det(\Delta_\beta|_{|w_\pm| \leq \epsilon})} = -\frac{1}{6\pi} \left( \frac{1}{2} \int_{|x| \leq \epsilon} |\nabla_{\flat} \psi_\pm|^2 \frac{d\nu}{-2i} + \oint_{|w_\pm| = \epsilon} k_\flat \psi_\pm |d\nu_\pm| \right) - \frac{1}{4\pi} \int_{|w_\pm| = \epsilon} (\partial_n \psi_\pm) |d\nu_\pm|.
\]

(5.9)

Here $\psi_\pm$ stands for the potential of the spherical metric $m = e^{2\psi_\pm}|d\nu_\pm|^2$ written in the local coordinate $w_\pm = w_\pm(z)$ in (4.2) and the symbol $\flat$ refers to the flat metric $|d\nu_\pm|^2$ in the disk $|w_\pm| \leq \epsilon$. Thus $\nabla_{\flat}$ is the gradient, $k_\flat = 1/\epsilon$ is the geodesic curvature of the circle $|w_\pm| = \epsilon$, $\partial_n$ is the outer normal derivative, and $\Delta_\beta|_{|w_\pm| \leq \epsilon}$ is the selfadjoint Dirichlet Laplacian. As is well-known,

\[
\log \det(\Delta_\beta|_{|w_\pm| \leq \epsilon}) = -\frac{1}{3} \log \epsilon + \frac{1}{3} \log 2 - \frac{1}{2} \log(2\pi) - \frac{5}{12} - 2\zeta'_{R}(-1); \quad \text{see e.g. [39, first equality in (28)] or [34, Corollary 1].}
\]

We note that in the Polyakov-Alvarez formula (5.9) only the integral involving the geodesic curvature $k_\flat$ gives a nonzero input of $-\frac{4}{3}\psi_\pm(0) + o(1)$ as $\epsilon \to 0^+$, while all other integrals go to zero. Taking into account (5.7) we thus obtain

\[
\log \det(\Delta|_{|w_\pm| \leq \epsilon}) = -\frac{1}{3} \log \epsilon + \frac{2\beta + 1}{3(\beta + 1)} \log 2 + \frac{1}{3(\beta + 1)} \log c_\beta
\]
\[
- 2\zeta'_{R}(-1) - \frac{5}{12} - \frac{1}{2} \log(2\pi) + o(1) \quad \text{as } \epsilon \to 0^+.
\]

(5.10)

In order to complete the proof of the first assertion, we also need to know the behaviour of the determinant of the Dirichlet Laplacian $\Delta_\beta|_{|w_\pm| \leq \epsilon}$ as $\epsilon \to 0^+$. We rely on the explicit formula found in [34]. It gives

\[
\log \det(\Delta_\beta|_{|w_\pm| \leq \epsilon}) = -\frac{1}{6} \left( \frac{\nu + 1}{\nu} \right) \log \ell - \nu + 1 + \frac{1}{6} \nu \log 2 - (7 - 2 \log 2) \frac{1}{12\nu}
\]
\[
- 2i \int_0^{\infty} \frac{\Gamma(\nu(1 + iy))}{\Gamma(\nu(1 - iy))} \frac{dy}{e^{2\pi y} - 1} - \log \Gamma(\nu + 1) + \frac{2}{\nu} \zeta'_{H}(-1, \nu + 1),
\]

(5.11)

where $\nu = (\beta + 1)^{-1}$, $\ell = \sqrt{2(\beta + 1)^{-1/2}\epsilon^{\beta + 1}}$, and $\zeta_H$ stands for the Hurwitz zeta function; see the last formula in [34]. As it was noticed in [19, Appendix B], the integral representation

\[
\zeta'_{B}(0; a, b, x) = \left(-\frac{1}{2}\zeta_H\left(0, \frac{x}{a}\right) + \frac{a}{b} \zeta_H\left(-1, \frac{x}{a}\right) - \frac{1}{12} \frac{b}{a}\right) \log a + \frac{1}{2} \log \Gamma\left(\frac{x}{a}\right)
\]
\[
- \frac{1}{4} \log(2\pi) - \frac{a}{b} \zeta'_H\left(-1, \frac{x}{a}\right) - \frac{a}{b} \zeta'_{H}\left(-1, \frac{x}{a}\right) + i \int_0^{\infty} \frac{\Gamma\left(\frac{x + iby}{a}\right)}{\Gamma\left(\frac{x - iby}{a}\right)} \frac{dy}{e^{2\pi y} - 1}
\]
from [35, Proposition 5.1] allows to express the right hand side of (5.11) in terms of the Barnes double zeta function (3.3). Thus we arrive at the equality
\[
\log \det(\Delta_{\beta}|_{\omega+|\leq \varepsilon}) = -\frac{\beta^2 + 2\beta + 2}{6} \log \varepsilon + \frac{\beta^2 + 2\beta + 2}{6(\beta + 1)} \log 2
\]
\[\quad - 2\zeta'_{B}(0; \beta + 1, 1, 1) - \frac{5}{12} (\beta + 1) - \frac{1}{2} \log(\beta + 1) - \frac{1}{2} \log(2\pi).
\]
This together with (5.10) completes the proof of the first assertion.

For the proof of the second assertion we use exactly the same methods and obtain
\[
\log \det(\Delta_{\beta}|_{\omega+|\leq \varepsilon}) = -\frac{1}{3} \log \varepsilon - \frac{1}{3} \frac{1}{2\beta + 1} \log c_{\beta}
\]
\[\quad - 2\zeta'_{B}(-1) - \frac{5}{12} - \frac{1}{2} \log(2\pi) + o(1) \quad \text{as} \quad \varepsilon \to 0^+,
\]
\[
\log \det(\Delta_{\beta}|_{\omega+|\leq \varepsilon}) = -\frac{1}{6} \left(4\beta^2 + 4\beta + 2\right) \log \varepsilon - \frac{1}{6} \left(2\beta + 1 + \frac{1}{2\beta + 1}\right) \log 2
\]
\[\quad - 2\zeta'_{B}(0; -2\beta - 1, 1, 1) + \frac{5}{12} \left(2\beta + 1\right) - \frac{1}{2} \log(-2\beta - 1) - \frac{1}{2} \log(2\pi).
\]
This implies the second assertion and completes the proof of Lemma 5.2. □

Proof of Proposition 3.1. Here we glue the results of Lemma 5.1 and Lemma 5.2 together by using the Meyer-Vietoris type formula for determinants of Laplacians [5], widely known as the BFK (Burghelea-Friedlander-Kappeler) formula, see also [24, 25]. For the selfadjoint Laplacian \(\Delta\) on the unit sphere \((\mathbb{S}, m)\) the formula reads
\[
\det \Delta = 4\pi \det(\Delta|_{\mathbb{S}}) \cdot \det(\Delta|_{\omega+|\leq \varepsilon}) \cdot \det(\Delta|_{\omega|\leq \varepsilon}) \cdot \det(\Delta|_{\omega-|\leq \varepsilon}) \cdot \frac{\det N|_{\partial \mathcal{C}_\varepsilon}}{\ell(\partial \mathcal{C}_\varepsilon, m)}, \quad (5.12)
\]
where \(4\pi\) is the total area of the unit sphere, \(\ell(\partial \mathcal{C}_\varepsilon, m)\) stands for the length of the boundary \(\partial \mathcal{C}_\varepsilon\) in the spherical metric \(m\), and \(N|_{\partial \mathcal{C}_\varepsilon}\) is the Neumann jump operator on \(\partial \mathcal{C}_\varepsilon\) (a first order classical pseudodifferential operator). All other determinants in (5.12) are exactly the same as in Lemma 5.1 and Lemma 5.2.

As is well-known,
\[
\det \Delta = \exp(1/2 - 4\zeta'_{B}(-1)),
\]
see e.g. [29], and the quotient \(\det N|_{\partial \mathcal{C}_\varepsilon}/\ell(\partial \mathcal{C}_\varepsilon, m)\) in (5.12) is conformally invariant (as it was noticed in [38], this can be most easily seen from (5.12) together with Polyakov and Polyakov-Alvarez formulas for the determinants of Laplacians in it; see also [7, 9]).

It is also well-known that the BFK formula remains valid for the flat singular metrics if one picks the Friederichs selfadjoint extensions of the corresponding Laplacians (and there are no singularities on the boundaries): the formula and its deduction hold true without any changes thanks to the same structure of short time heat trace asymptotics. A non-exhaustive list of references where the BFK formula occurs in the context of singular metrics is [10, 11, 13, 20, 21, 22, 26]. For the Friederichs selfadjoint extension \(\Delta_{\beta}\) the formula reads
\[
\det \Delta_{\beta} = \det(\Delta_{\beta}|_{\mathbb{S}}) \cdot \det(\Delta_{\beta}|_{\omega+|\leq \varepsilon}) \cdot \det(\Delta_{\beta}|_{\omega|\leq \varepsilon}) \cdot \det(\Delta_{\beta}|_{\omega-|\leq \varepsilon}) \cdot \frac{\det N_{\beta}|_{\partial \mathcal{C}_\varepsilon}}{\ell(\partial \mathcal{C}_\varepsilon, m_{\beta})},
\]
where the area of envelope does not appear because it was normalized to 1 and the value of the quotient \( \det N_{\beta}|_{\partial C_\epsilon}/\ell(\partial C_\epsilon, m) \) is the same as the value of the one in (5.12). (Let us also note that the generalizations of BFK formula to the case of non-Friederichs selfadjoint extensions or non-flat singular metrics are way more involved, e.g. [13, 26].) Summing up we come to the equality

\[
\det \Delta_\beta = \frac{1}{4\pi} \exp\left(\frac{1}{2} - 4\zeta_R'(-1)\right) \times \lim_{\epsilon \to 0^+} \left( \frac{\det(\Delta_\beta|_{w_+|\leq \epsilon})}{\det(\Delta|_{w_+|\leq \epsilon})} \cdot \frac{\det(\Delta_\beta|_{w_0|\leq \epsilon})}{\det(\Delta|_{w_0|\leq \epsilon})} \cdot \frac{\det(\Delta_\beta|_{|w_0|\leq \epsilon})}{\det(\Delta|_{|w_0|\leq \epsilon})} \right).
\]

In the limit this together with Lemma 5.1 and Lemma 5.2 implies the stated formula (3.1) for \( \log \det \Delta_\beta \) and completes the proof of Proposition 3.1.

\[\text{□}\]

6 Absolute minimum and small-angle asymptotics

**Proof of Theorem 2.1.** 1. The first assertion is a direct consequence of the explicit formula for \( \log \Delta_\beta \) deduced in Proposition 3.1, where all terms in the right hand side are real analytic on the open interval \((-1, -1/2)\); for the properties of Barnes double zeta function see e.g. [27, 35].

2. The small-angle asymptotic expansions (2.1) and (2.2) readily follow from the explicit formula for \( \log \det \Delta_\beta \) found in Proposition 3.1, the well-known asymptotics (A.1) for the derivative of Barnes double zeta function, and the asymptotics

\[
\log c_\beta = \frac{1}{2} \log(\beta + 1) + \frac{1}{2} \log 2 - \frac{1}{2} \log \pi - 2(\beta + 1) \log 2 + O\left((\beta + 1)^2\right) \quad \text{as} \ \beta \to -1^+,
\]

\[
\log c_\beta = \frac{1}{2} \log(-2\beta - 1) - \frac{1}{2} \log \pi + (2\beta + 1) \log 2 + O\left((2\beta + 1)^2\right) \quad \text{as} \ \beta \to -1/2^-
\]

for the scaling factor (3.2).

3. Let us first show that \( \beta = -2/3 \) is a critical point of the right hand side in (3.1). From Lemma A.2 that we prove in Appendix A it follows that

\[
\frac{d}{d\beta}\{-4\zeta'_B(0; \beta + 1, 1, 1) - 2\zeta'_B(0; -2\beta - 1, 1, 1)\} = 0 \quad \text{for} \quad \beta = -2/3.
\]

This together with the formulas (3.1) for \( \log \det \Delta_\beta \) and (3.2) for the scaling factor \( c_\beta \) gives

\[
\frac{d}{d\beta} \log \det \Delta_\beta \bigg|_{\beta = -2/3} = \frac{1}{6} \left(-16 \log 2 - 8 \left(-\Psi\left(\frac{1}{3}\right) + \Psi\left(\frac{1}{6}\right) + \pi \cot \frac{\pi}{3}\right)\right),
\]

where \( \Psi \) is the digamma function and

\[
-\Psi\left(\frac{1}{3}\right) + \Psi\left(\frac{1}{6}\right) + \pi \cot \frac{\pi}{3} = -2 \log 2
\]

by the Gauss’s digamma theorem. This demonstrates that

\[
\frac{d}{d\beta} \log \det \Delta_\beta \bigg|_{\beta = -2/3} = 0,
\]

\[\text{□}\]
so \( \beta = -2/3 \) is a critical point of the function \( \beta \mapsto \log \det \Delta_{\beta} \).

For the rational values of \( a \) the derivative \( \zeta'_{B}(0; a, 1, 1) \) of the Barnes double zeta can be expressed in terms of \( \zeta'_{R}(-1) \) and gamma functions, see e.g. [6] and Remark A.3 in Appendix A. In particular, this allows to find the critical value, cf. (A.13).

We have demonstrated that the equilateral triangle envelope (the most symmetrical geometry) gives rise to a critical point of the determinant on the isosceles triangle envelopes of unit area. In the remaining part of the proof we show that it provides the determinant with the absolute minimum and there are no other critical points.

It suffices to show that the second derivative of the function

\[
(-1, -1/2) \ni \beta \mapsto \log \det \Delta_{\beta}
\]

is strictly positive. With this aim in mind we intend to approximate the second derivative of the right hand side in (3.1) by an elementary function.

We start with the term involving the scaling factor \( c_{\beta} \). Due to (3.2) we have

\[
-2 \log c_{\beta} = 2 \log \Gamma(\beta + 1) + 2 \log \Gamma(-\beta - 1/2) + \log \sin(\pi(-2\beta - 1)) - \log(4\pi).
\]

Expanding \( \log \Gamma(\beta + 2) = \log \Gamma(\beta + 1) + \log(\beta + 1) \) into the Taylor series we obtain

\[
\log \Gamma(\beta + 1) = -\log(\beta + 1) - \gamma(\beta + 1) + \sum_{k=2}^{\infty} \frac{\zeta_{R}(k)}{k}(-\beta - 1)^{k}, \quad |\beta + 1| < 1,
\]

where we can replace \( \beta + 1 \) by \( -\beta - 1/2 \) to get a similar representation for \( \log \Gamma(-\beta - 1/2) \).

Thus for the second derivative of the term involving \( c_{\beta} \) we obtain

\[
- \frac{1}{6} \frac{d^{2}}{d\beta^{2}} \left( \left( \frac{2}{\beta + 1} - \frac{1}{2\beta + 1} - 1 \right) \log c_{\beta} \right) > \frac{d^{2}}{d\beta^{2}} \left[ \frac{1}{12} \left( \frac{2}{\beta + 1} - \frac{1}{2\beta + 1} - 1 \right) \right]
\]

\[
\times \left( \log \frac{\sin(\pi(-2\beta - 1))}{\pi(\beta + 1)^{2}(2\beta + 1)^{2}} - \gamma + 2 \sum_{k=2}^{3} \frac{\zeta_{R}(k)}{k}((-\beta - 1)^{k} + (\beta + 1/2)^{k}) \right)
\]

where the Leibniz’s test was implemented to estimate the sums of infinite series on the interval \(-1 \leq \beta \leq -1/2\).

Next we consider the terms involving the Barnes double zeta functions. We find suitable approximations for Barnes double zeta function derivatives in Lemma A.1, see Appendix A. The inequality (A.3) from Lemma A.1, where we take \( N = 2 \), implies

\[
\frac{d^{2}}{d\beta^{2}} \left( -4 \zeta'_{B}(0; \beta + 1, 1, 1) - 2 \zeta'_{B}(0; -2\beta - 1, 1, 1) \right) + \left( \frac{1}{12} - \zeta'_{R}(-1) \right) \left( \frac{8}{(\beta + 1)^{3}} - \frac{2}{(\beta + 1/2)^{3}} \right) \geq -\frac{\zeta_{R}(3)(-3\beta - 1)}{15} > -\frac{1}{5}.
\]

17
In total we get

\[
\frac{d^2}{d\beta^2} \log \det \Delta_\beta > \frac{d^2}{d\beta^2} \left[ \frac{1}{12} \left( \frac{2}{\beta + 1} - \frac{1}{2\beta + 1} - 1 \right) \right]
\times \left( \log \frac{\sin(\pi(-2\beta - 1))}{\pi(\beta + 1)^2(2\beta + 1)^2} - \gamma + 2 \sum_{k=2}^{3} \frac{\zeta_R(k)}{k} \left( (-\beta - 1)^k + (\beta + 1/2)^k \right) \right)
\times \left( \frac{1}{6} \left( \frac{4}{\beta + 1} - \frac{1}{2\beta + 1} \right) \log 2 + \frac{1}{2} \log \frac{1}{(\beta + 1)^2(-2\beta - 1)} \right)
\times \left( -\frac{1}{5} - \left( \frac{1}{12} - \zeta^\prime_R(-1) \right) \left( \frac{8}{(\beta + 1)^3} - \frac{2}{(\beta + 1/2)^3} \right) \right).
\] (6.2)

One can check that the elementary function in the right hand side of the latter inequality is strictly positive on the interval $-1 < \beta < -1/2$. This is straightforward and we omit the details; see Fig. 4 for a graph of the elementary function.

Figure 4: Graph of the elementary function in the right hand side of (6.2).

We have demonstrated that the function $\beta \mapsto \log \det \Delta_\beta$ is concave up. This completes the proof of Theorem 2.1.

7 Determinant on envelopes of non-unit area

The Laplacian $\Delta^S_\beta$ on the Euclidean isosceles triangle envelopes of area $S \neq S(\beta)$ is generated by the metric $m^S_\beta = S \cdot m_\beta$ on $\mathbb{C}$, where $m_\beta$ is the same as in (4.1). Since for the Friederichs Laplacians we have $\Delta^S_\beta = \frac{1}{S} \Delta_\beta$, differentiating the spectral zeta function $\zeta^S_\beta(s) = \sum_{j>0}(\lambda_j/S)^{-s}$ of $\Delta^S_\beta$ with respect to $s$ we arrive at the standard rescaling property

\[
\log \det \Delta^S_\beta = \log \det \Delta_\beta - \zeta_\beta(0) \log S.
\] (7.1)
A more serious task is to find that
\[ \zeta_\beta(0) = -\frac{13}{12} + \frac{1}{6(\beta + 1)} - \frac{1}{12(2\beta + 1)}. \] \hfill (7.2)

This can be done either by repeating the steps in the proof of Proposition 3.1 (now for
the metric \( m_\beta^S \)) or by relying on a more general result [13, Section 1.2].

**Remark 7.1.** 1. Theorem 2.1.1 together with (7.1) and (7.2) implies that
\[ (-1, -1/2) \ni \beta \mapsto \log \det \Delta_\beta^S \] \hfill (7.3)
is a real analytic function (for the isosceles triangle envelopes of any fixed area \( S \)).

2. With the help of (7.1), (7.2) the small-angle asymptotics for \( \log \det \Delta_\beta^S \) can be
easily obtained from those in Theorem 2.1.2. In particular, it turns out that
the determinant \( \det \Delta_\beta^S \) grows without any bound as the isosceles triangle envelope of
area \( S \) degenerates (i.e. as an internal angle of triangle ABC of area \( S/2 \) goes to
zero, or, equivalently, as \( \beta \to -1^+ \) or \( \beta \to -1/2^- \)).

3. For \( \zeta_\beta(0) \) in (7.2) we have \( \frac{d}{d\beta} \zeta_\beta(0) \big|_{\beta = -2/3} = 0 \) and \( \zeta_{-2/3}(0) = -\frac{1}{3} \). Therefore as an
immediate consequence of Theorem 2.1.3 we conclude that \( \beta = -2/3 \) is a critical
point of the function \( \beta \mapsto \log \det \Delta_\beta^S \) for any \( S > 0 \). Thanks to (7.1) for
the critical value we have
\[ \log \det \Delta_{-2/3}^S = \frac{2}{3} \log \pi + \frac{1}{3} \log \frac{2}{3} - 2 \log \Gamma \left( \frac{2}{3} \right) + \frac{1}{3} \log S. \]

Recall that \( \beta = -2/3 \) corresponds to the equilateral triangle envelope, i.e. to the
most symmetrical geometry.

4. Since \( \frac{d^2}{d\beta^2} \zeta_\beta(0) > 0 \), it is also true that for any \( S \leq 1 \) the function (7.3) is concave
up, has exactly one critical point, and
\[ \log \det \Delta_\beta^S \geq \log \det \Delta_{-2/3}^S \quad \text{for} \quad -1 < \beta < -1/2 \]
with equality iff \( \beta = -2/3 \); cf. Theorem 2.1.3. In other words, the determinant
of Friederichs Laplacian on the isosceles triangle envelopes of fixed area \( S \leq 1 \)
reaches its absolute minimum on the equilateral triangle envelope.

5. In fact, the estimate (6.2) together with (7.1) and (7.2) is also capable of showing
that \( \beta = -2/3 \) still corresponds to the absolute minimum of \( \log \det \Delta_\beta^S \) for the
areas \( S \) slightly greater than one. However, with further increase of \( S \) the value of
\[ \frac{d^2}{d\beta^2} \zeta_\beta(0) \big|_{\beta = -2/3} \log S = 27 \log S \]
becomes greater than
\[ \frac{d^2}{d\beta^2} \log \det \Delta_\beta \big|_{\beta = -2/3} \approx 17.614 \]
(approximately for \( S > 1.92 \)). As a result, the second derivative
\[
\frac{d^2}{d\beta^2} \log \det \Delta^S_{\beta} \bigg|_{\beta = -2/3}
\]
becomes negative, cf. (7.1), and the critical point \( \beta = -2/3 \) turns into a local maximum of \( \log \det \Delta^S_{\beta} \), see Fig. 5.

![Graph of the function \((-1, -1/2) \ni \beta \mapsto \log \det \Delta^S_{\beta}\) for the Euclidean isosceles triangle envelopes of total area \( S = 1.92 \) (on the left) and \( S = 3 \) (on the right). The dashed line corresponds to the critical point \( \beta = -2/3 \) or, equivalently, to the equilateral triangle envelope (the most symmetrical geometry). For the graphs we use the formula for \( \log \det \Delta_{\beta} \) found in Proposition 3.1 together with rescaling formulas (7.1), (7.2). We find the exact values of \( \log \det \Delta^S_{\beta} \) for some rational values of \( \beta \) by using Remark A.3 (the points marked with Diamond symbol). We also uniformly approximate \( \log \det \Delta^S_{\beta} \) by using the estimate (A.2) in Lemma A.1, where we take \( N = 4 \) (solid line).

A similar effect also appears on a sphere with positive constant curvature (spherical) metrics of normalized area and two antipodal singularities, see [13, Section 3.1].

A Derivatives of Barnes double zeta function

The asymptotics
\[
\zeta'_B(0; a, 1, 1) = \left( \frac{1}{12} - \zeta'_R(-1) \right) \frac{1}{a} - \frac{1}{4} \log(2\pi) + \frac{\gamma a}{12}
+ \sum_{k=2}^{\infty} \frac{B_{2k} \zeta_R(2k - 1)}{2k(2k - 1)} a^{2k-1}, \quad a \to 0^+, \tag{A.1}
\]
for the derivative of the Barnes double zeta function (3.3) with respect to its first argument is well known, see e.g. [27, 35]. In this paper we use an improvement of this result, which we formulate and prove in Lemma A.1 below.
Lemma A.1 (Approximations for derivatives of Barnes double zeta function). For \( a > 0 \) and any \( N \geq 2 \) we have

\[
\left| \zeta_B'(0; a, 1, 1) - \left( \frac{1}{12} - \zeta_R'(-1) \right) \frac{1}{a} + \frac{1}{4} \log(2\pi) - \frac{\gamma a}{12} - \sum_{k=2}^{N-1} \frac{B_{2k} \zeta_R(2k-1)}{2k(2k-1)} a^{2k-1} \right| \leq \frac{|B_{2N} \zeta_R(2N-1)|}{2N(2N-1)} a^{2N-1},
\]

(A.2)

\[
\left| \frac{\partial^2}{\partial a^2} \zeta_B'(0; a, 1, 1) - 2 \left( \frac{1}{12} - \zeta_R'(-1) \right) \frac{1}{a^3} - \sum_{k=2}^{N-1} \left( 1 - \frac{1}{k} \right) B_{2k} \zeta_R(2k-1) a^{2k-3} \right| \leq \left( 1 - \frac{1}{N} \right) |B_{2N} \zeta_R(2N-1)| a^{2N-3};
\]

(A.3)

for \( N = 2 \) the sums with respect to \( k \) do not appear. Here \( \gamma = -\Gamma'(1) \) is the Euler’s constant, \( B_n \) is the \( n \)-th Bernoulli number, \( \zeta_R \) is the Riemann zeta function, and the prime denotes the derivative of \( \zeta_B(s; a, b, x) \) with respect to \( s \).

Proof. We rely on the integral representation\(^1\)

\[
\zeta_B'(0; a, 1, 1) = \frac{1}{12} \left( a + \frac{1}{a} \right) (\gamma - \log a) - \frac{1}{4} \log a + \frac{5}{24} a - \frac{1}{4} \log(2\pi) + J(a),
\]

(A.4)

where \( \gamma = -\Gamma'(1) \) and

\[
J(a) = \int_0^\infty \frac{1}{e^t - 1} \left[ \frac{1}{2t} \coth \frac{t}{2a} - \frac{a}{4} \text{csch}^2 \frac{t}{2} - \frac{1}{12} \left( a + \frac{1}{a} \right) \right] dt.
\]

We have

\[
J(a) = \int_0^\infty \left( \frac{t}{e^t - 1} e^{t/a} - 1 \right) + \frac{1}{e^t - 1} \left[ \frac{1}{2t} - \frac{a}{4} \text{csch}^2 \frac{t}{2} - \frac{1}{12} \left( a + \frac{1}{a} \right) \right] dt.
\]

(A.5)

Expanding the factor \( \frac{t}{e^t - 1} \) into the Taylor series and applying the Leibniz’s test we obtain

\[
\left| \frac{t}{e^t - 1} - 1 + \frac{1}{2} t - \frac{1}{12} t^2 - \sum_{k=2}^{N-1} \frac{B_{2k} t^{2k}}{(2k)!} \right| \leq \frac{|B_{2N}|}{(2N)!} t^{2N}, \quad t \geq 0.
\]

(A.6)

\(^1\)The representation of \( \zeta_B'(0; a, 1, 1) \) in terms of \( J(a) \) was first found in [13]. Other known integral representations of \( \zeta_B'(0; a, 1, 1) \), see e.g. [27, 35, 28] and references therein, are not good for our purposes as it is hard to differentiate them with respect to the parameter \( a \). To verify the validity of a representation for \( \zeta_B'(0; a, 1, 1) \) it suffices to check it only for the rational values of \( a \) (due to the analytic regularity of \( \zeta_B'(0; a, 1, 1) \) in the half plane \( \Re a > 0 \)). It is known that for rational values of \( a \) the value of \( \zeta_B'(0; a, 1, 1) \) can be expressed in terms of Riemann zeta and gamma functions, see e.g. [6] and Remark A.12. The above representation returns exactly the same result indeed; a detailed evaluation of \( J(a) \) for rational values of \( a \) can be found in [3].
It is not hard to check that
\[
\int_0^\infty \left( \frac{1}{t^2 - 2t + 1} \right) \frac{1}{e^{t/a} - 1} = \frac{1}{2} \left( a + 3 \right) \log a + \left( \frac{1}{12} - \zeta'_R(-1) \right) \frac{1}{a} - \frac{5}{24a}. \tag{A.7}
\]

We also note that the well known identity \( \zeta(n)(n-1)! = \int_0^\infty \frac{t^{n-1} dt}{e^t - 1} \) gives
\[
\int_0^\infty \frac{B_{2k} t^{2k-2}}{(2k)!} \frac{1}{e^{t/a} - 1} dt = \frac{B_{2k}}{(2k)!} \frac{t^{2k-2}}{e^t - 1} \int_0^\infty \frac{t^{2k-2}}{e^{t/a} - 1} dt = \frac{B_{2k} \zeta_R(2k-1)}{2k(2k-1)} a^{2k-1}. \tag{A.8}
\]
Now (A.5)–(A.8) lead to the estimate
\[
\left| J(a) - \frac{1}{12} \left( a + 3 \right) \log a - \left( \frac{1}{12} - \zeta'_R(-1) \right) \frac{1}{a} + \frac{5}{24a} \right| - \sum_{k=2}^{N-1} \frac{B_{2k} \zeta_R(2k-1)}{2k(2k-1)} a^{2k-1} \leq \frac{|B_{2N} \zeta_R(2N-1)|}{N(N-1)} a^{2N-1}.
\]
This together with (A.4) completes the proof of (A.2).

In order to prove (A.3) we first write
\[
\zeta'_B(0; a, 1, 1) - \left( \frac{1}{12} - \zeta'_R(-1) \right) \frac{1}{a} + \frac{1}{4} \log(2\pi) - \frac{\gamma a}{12}
= \int_0^\infty \left( \frac{t}{e^t - 1} - 1 + \frac{1}{2} t - \frac{1}{12} t^2 \right) \frac{1}{t^2(e^{t/a} - 1)} dt \tag{A.9}
= \int_0^\infty \frac{F(at)}{t(e^t - 1)} dt,
\]
where we introduced the notation
\[
F(r) = \frac{1}{e^t - 1} - \frac{1}{r} + \frac{1}{2} - \frac{1}{12} r \left( = \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} r^{2k-1} \right). \tag{A.10}
\]
Since \(|F'(r)/r| < 1\) and \(|F''(r)| < 1\) we have
\[
\int_0^\infty \left| \frac{\partial}{\partial a} F(at) \right| \frac{1}{t(e^t - 1)} dt = \int_0^\infty \left| F'(at) \right| \frac{1}{e^t - 1} dt < \int_0^\infty \frac{at}{e^t - 1} dt = \frac{\pi^2}{6} a
\]
\[
\int_0^\infty \left| \frac{\partial^2}{\partial a^2} F(at) \right| \frac{1}{t(e^t - 1)} dt = \int_0^\infty \left| F''(at) \right| \frac{t}{e^t - 1} dt < \int_0^\infty \frac{t}{e^t - 1} dt = \frac{\pi^2}{6}.
\]
This demonstrates that
\[
\int_0^\infty \frac{\partial^2}{\partial a^2} F(at) \frac{1}{t(e^t - 1)} dt = \int_0^\infty \frac{\partial^2}{\partial a^2} F(at) \frac{1}{t(e^t - 1)} dt.
\]
Now as a consequence of (A.9) we conclude that
\[
\frac{\partial^2}{\partial a^2} \zeta_B'(0; a, 1, 1) - 2 \left( \frac{1}{12} - \zeta_R'(-1) \right) \frac{1}{a^3} = \int_0^\infty \frac{\partial^2}{\partial a^2} F(at) \frac{1}{t(e^t - 1)} \, dt. \tag{A.11}
\]
For \(at \geq 0\) the second partial derivative \(\frac{\partial^2}{\partial a^2} F(at) = t^2 F''(at)\) can be written as the following convergent alternating series:
\[
\frac{\partial^2}{\partial a^2} F(at) = t^2 \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k-3)!} (at)^{2k-3}.
\]
The Leibniz’s test gives
\[
\left| \frac{\partial^2}{\partial a^2} F(at) - \sum_{k=2}^{N-1} \frac{B_{2k}}{2k(2k-3)!} t^{2k-3} \right| \leq \frac{|B_{2N}|}{2N(2N-3)!} t^{2N-3}.
\]
This together with (A.11) leads to (A.3) (in the same way as in the proof of (A.2)).

**Lemma A.2.** For the Barnes double zeta function \(\zeta_B(s; a, b, x)\) we have
\[
\frac{\partial}{\partial a} \{2\zeta_B'(0; a, 1, 1) + \zeta_B'(0; 1 - 2a, 1, 1)\} = 0 \quad \text{for} \quad a = 1/3.
\]
As before the prime stands for the derivative with respect to \(s\).

**Proof.** As in the proof of Lemma A.1 we conclude that
\[
\frac{\partial}{\partial a} \zeta_B'(0; a, 1, 1) = - \left( \frac{1}{12} - \zeta_R'(-1) \right) \frac{1}{a^2} + \frac{\gamma}{12} + \int_0^\infty \frac{F'(at)}{e^t - 1} \, dt, \quad a > 0,
\]
where \(F(r)\) is the same as in (A.10); cf. (A.9). This immediately implies the assertion.

**Remark A.3** (Particular values of Barnes double zeta derivative). If \(a\) is a rational number, then the value of the derivative \(\zeta_B'(0; a, 1, 1)\) can be expressed in the following way:
\[
\zeta_B(0; p/q, 1, 1) = \frac{1}{pq} \left( \zeta_R(-1) - \frac{\log q}{12} \right) + \left( S(q, p) + \frac{1}{4} \right) \log \frac{q}{p} + \sum_{k=1}^{p-1} \left( \frac{1}{2} - \frac{k}{p} \right) \log \Gamma \left( \left( \frac{kp}{p} \right) + \frac{1}{2} \right) + \sum_{j=1}^{q-1} \left( \frac{1}{2} - \frac{j}{q} \right) \log \Gamma \left( \left( \frac{jp}{q} \right) + \frac{1}{2} \right). \tag{A.12}
\]
Here \(p\) and \(q\) are coprime natural numbers, \(S(q, p) = \sum_{j=1}^{p} \left( \frac{j}{p} \right) \left( \frac{jp}{p} \right)\) is the Dedekind sum, and the symbol \(\left( \cdot \right)\) is defined so that \(\left( x \right) = x - \lfloor x \rfloor - 1/2\) for \(x\) not an integer and \(\left( x \right) = 0\) for \(x\) an integer; for details we refer to [13, Appendix A]. In particular, \(\zeta_B(0; 1, 1, 1) = \zeta_R(-1)\) and for the reciprocals of the natural numbers the general formula (A.12) simplifies to
\[
\zeta_B'(0; 1/q, 1, 1) = \frac{1}{q} \zeta_R'(-1) - \frac{1}{12q} \log q - \sum_{j=1}^{q-1} \frac{j}{q} \log \Gamma \left( \frac{j}{q} \right) + \frac{q-1}{4} \log 2\pi.
\]
In the context of this paper Remark A.3 becomes extremely helpful if $\beta$ is a rational number, i.e. the angles of triangles $ABC$ and $CBA'$ are rational multiples of $\pi$, cf. Fig. 1. Thus by using (3.1) together with (A.12) for the envelope of unit area glued from two right isosceles triangles we obtain

$$\log \det \Delta_{-3/4} = \frac{1}{4} \log \pi - \log \Gamma \left( \frac{3}{4} \right) \approx 0.0829.$$ 

Similarly, for the equilateral triangle envelope of unit area we get

$$\log \det \Delta_{-2/3} = \frac{2}{3} \log \pi + \frac{1}{3} \log \frac{2}{3} - 2 \log \Gamma \left( \frac{2}{3} \right) \approx 0.0217;$$

(A.13)

for the envelope of unit area glued from two congruent isosceles triangles with interior angles $\pi/6$, $2\pi/3$ and $\pi/6$ we have

$$\log \det \Delta_{-5/6} = \frac{1}{6} \log \pi + \frac{37}{72} \log 2 + \frac{1}{48} \log 3 + \zeta'_R(-1) - \frac{1}{4} \log \Gamma \left( \frac{2}{3} \right) \approx 0.3287.$$ 

On Fig. 2 and Fig. 5 all points marked by the diamond symbol were found in exactly the same way. Let us also note that for some exceptional values of $\beta$ it is possible to find all eigenvalues of the Friederichs Laplacian $\Delta^S_\beta$ and then the corresponding zeta regularized spectral determinant, see [4, Section B].

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