THEOREMS OF BORSUK-ULAM TYPE FOR FLATS AND COMMON TRANSVERSALS

R.N. KARASEV

Abstract. In this paper some results on the topology of the space of $k$-flats in $\mathbb{R}^n$ are proved, similar to the Borsuk-Ulam theorem on coverings of sphere. Some corollaries on common transversals for families of compact sets in $\mathbb{R}^n$, and on measure partitions by hyperplanes, are deduced.

1. Introduction

Let us remind some classical results, which are generalized in this paper. One of the most important results is the Borsuk-Ulam theorem on coverings of the sphere [4].

**Theorem** (The Borsuk-Ulam theorem). If the sphere $S^n$ is covered by a family of $n + 1$ closed (or open) sets $X_1, \ldots, X_{n+1}$, then at least one of $X_i$ contains a pair of antipodal points of $S^n$.

Note that the sphere $S^n$ is considered to be a unit sphere in $\mathbb{R}^{n+1}$, and the points $x$ and $-x$ are called antipodal. This theorem can be reformulated for coverings of a ball.

**Theorem** (The Borsuk-Ulam theorem for coverings of the ball). Let a ball $B^n \subset \mathbb{R}^n$ be covered by closed (or open) sets $X_1, \ldots, X_{n+1}$, and for any $i = 1, \ldots, n+1$ the intersection $X_i \cap \partial B^n$ does not contain a pair of antipodal points. Then the intersection $\bigcap_{i=1}^{n+1} X_i$ is non-empty.

In this paper we are going to consider the configuration spaces of $k$-flats ($k$-dimensional affine subspaces) in $\mathbb{R}^n$, and prove the generalizations of the Borsuk-Ulam theorem for coverings of such spaces.

Let us state another famous theorem about intersections of convex sets in $\mathbb{R}^n$, the Helly theorem [10].

**Theorem** (Helly’s theorem). Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^n$. The family $\mathcal{F}$ has a common points, iff any subfamily $\mathcal{G} \subseteq \mathcal{F}$ with size $|\mathcal{G}| \leq n + 1$ has a common point.

We are going to use the generalization of the Helly theorem from [1].

**Theorem** (The colored Helly theorem). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{n+1}$ be families of convex compact sets in $\mathbb{R}^n$. Suppose that for any system of representatives $\{X_i \in \mathcal{F}_i\}_{i=1}^{n+1}$ the intersection $\bigcap_{i=1}^{n+1} X_i$ is non-empty. Then for some $i$ the intersection $\bigcap \mathcal{F}_i$ is non-empty.

2000 Mathematics Subject Classification. 52A35,55M30,55M35.

Key words and phrases. Borsuk-Ulam theorem, common transversals, Helly-type theorems.

This research was partially supported by the Dynasty Foundation.
The Helly theorem gives a condition for existence of a common point for the whole family in terms of existence of a common point for subfamilies of given size. It also makes sense to find a $k$-flat that intersects every set in the family, and try to find some sufficient conditions on its existence. Let us make a definition.

**Definition 1.** If a $k$-flat $L$ intersects every set of the family $\mathcal{F}$, then $L$ is called *(common)* $k$-transversal for the family $\mathcal{F}$.

In fact, the existence of $k$-transversal for arbitrary finite family of convex sets cannot be deduced from any Helly-type theorem. Thus it makes sense to modify the conditions in Helly-type theorems to provide a common transversal, see [7, 6] for example.

Let us state a result on common transversals from [13, 17], that generalizes the Helly theorem.

**Theorem (The Horn-Klee theorem).** Let $1 \leq k \leq d$ be integers, let $\mathcal{F}$ be a family of convex compact sets in $\mathbb{R}^d$. Then the following conditions are equivalent:

1) Every $k$ or less sets of $\mathcal{F}$ have a common point;
2) Every flat of codimension $k - 1$ in $\mathbb{R}^d$ can be translated to intersect every member of $\mathcal{F}$;
3) Every flat of codimension $k$ in $\mathbb{R}^d$ is contained in a flat of codimension $k - 1$, that is a transversal for $\mathcal{F}$.

In Section 8 we prove some results on common transversals, close to the colored Helly theorem and the Horn-Klee theorem. In Section 7 we prove some Borsuk-Ulam-type theorems, that give sufficient conditions for existence of a common $k$-transversal for $n + 1$-element families of (possibly non-convex) sets in $\mathbb{R}^n$.

Let us state the result on measure partitions from [21, 22], that can be deduced from the Borsuk-Ulam theorem.

**Theorem (The “ham sandwich” theorem).** Suppose that $d$ absolutely continuous probabilistic measures $\mu_1, \ldots, \mu_d$ are given in $\mathbb{R}^d$. Then there is a half-space $H \subset \mathbb{R}^d$ such that for any $i = 1, \ldots, d$

$$
\mu_i(H) = 1/2.
$$

It is natural to ask, whether we can partition the measures into parts of arbitrary measure. More precisely, suppose we are given numbers $(\alpha_1, \ldots, \alpha_d) \in [0, 1]^d$, and try to find a half-space $H \subset \mathbb{R}^d$ such that for any $i = 1, \ldots, d$ the measure is $\mu_i(H) = \alpha_i$. This cannot be done in general, it is sufficient to consider several uniform measures on concentric balls.

Some additional conditions on the measures are required, in the papers [3, 6, 13, 2, 5] it was shown that it is sufficient to require the supports of the measures to be separated, i.e. for any system of representatives $x_i \in \text{conv supp } \mu_i$, the points $(x_1, \ldots, x_d)$ should be affine independent. In Section 6 we study the measure partitions by hyperplanes and prove some generalization of this result.
2. THEOREMS ON THE CANONICAL BUNDLE OVER THE GRASSMANNIAN

In this section we study the configuration space of $k$-flats in $\mathbb{R}^n$ and obtain its properties, which are required in the study of common transversals and measure partitions.

Denote the index set $[n] = \{1, 2, \ldots, n\}$.

**Definition 2.** The subsets $X$ and $Y$ of a linear space $L$ are called separated, if there exists a linear function (a polynomial of degree 1) $l : L \rightarrow \mathbb{R}$ such that $l(X) < 0$ and $l(Y) > 0$.

**Definition 3.** The families $F$ and $G$ of subsets of a linear space $L$ are called separated, if there exists a linear function $l : L \rightarrow \mathbb{R}$ such that $l(X) < 0$ for any $X \in F$, and $l(Y) > 0$ for any $Y \in G$.

**Definition 4.** Two families of segments $A$ and $B$ in the real line are called equalized, if one of the following alternatives holds:

1) All right ends of $A$ coincide in $a$, all left ends of $B$ coincide in $b$, either $a$ is to the left of $b$, or every segment of $A \cup B$ contains the segment $[ba]$;
2) All right ends of $B$ coincide in $b$, all left ends of $A$ coincide in $a$, either $b$ is to the left of $a$, or every segment of $B \cup A$ contains the segment $[ab]$.

Denote $\gamma_n^k$ the canonical vector bundle over the Grassmannian $G_n^k$ of linear $k$-subspaces in $\mathbb{R}^n$.

In the sequel the continuous dependence of a convex compact set on some parameter is considered in the Hausdorff metric.

**Theorem 1.** Consider $n + 1$ closed subsets $V_1, V_2, \ldots, V_{n+1}$ in $\gamma_n^1$, such that the intersection of $V_i$ with any fiber $L$ is a non-empty segment (possibly one point), depending continuously on $L$. Then one of the alternatives holds:

1) There exists a fiber $L$ and $i \in [n + 1]$ such that $V_i \cap L$ is contained in every $V_j \cap L$, for $j \neq i$;
2) For any partition of the family $\{V_i\}$ into non-empty subfamilies $F_1$ and $F_2$ there is a fiber $L$ such that the families of segments

$$F_1(L) = \{U \cap L : U \in F_1\} \quad \text{and} \quad F_2(L) = \{U \cap L : U \in F_2\}$$

are equalized in $L$.

It is clear that a pair of equalized families of segments is either separated, or has a common point. Thus Theorem 1 implies the following.

**Corollary 2.** Consider $n + 1$ closed subsets $V_1, V_2, \ldots, V_{n+1}$ in $\gamma_n^1$, such that the intersection of $V_i$ with any fiber $L$ is a non-empty segment (possibly one point), depending continuously on $L$. Then either all the sets $V_i$ have a common point; or for any partition of the family $\{V_i\}$ into non-empty subfamilies $F_1$ and $F_2$ there is a fiber $L$ such that the families of segments

$$F_1(L) = \{U \cap L : U \in F_1\} \quad \text{and} \quad F_2(L) = \{U \cap L : U \in F_2\}$$

are separated in $L$. 
Now we are going to state some results for arbitrary $k$ and the canonical bundle $\gamma^k_n \to G^k_n$.

Let us make some definitions.

**Definition 5.** Let $K \subset \mathbb{R}^n$ be a convex compact set. A pair of points on $\partial K$ is called **antipodal** w.r.t. $K$, if they can be enclosed into a pair of support hyperplanes of $K$ with opposite outer normals.

In other words, the points $x$ and $y$ are antipodal w.r.t. $K$, iff the segment $[xy]$ is an affine diameter of $K$.

**Definition 6.** A family of compact sets $F$ in $\mathbb{R}^n$ is called **non-antipodal**, if none of the sets $V \in F$ contains a pair of points, antipodal w.r.t. $\text{conv} \bigcup F$.

In the sequel we assume that for any vector bundle we have some norm on the fibers, that has a smooth unit ball, and depends continuously on the fiber. In particular, we can consider the standard Euclidean norm on $\gamma^k_n$, or some other norm.

**Theorem 3.** Consider $n+1$ compact sets $V_1, V_2, \ldots, V_{n+1}$ in $\gamma^k_n$ such that for any $i = 1, \ldots, n+1$ the intersection with fiber $V_i \cap L$ is nonempty and depends continuously on $L$ in the Hausdorff metric. Suppose also that for any fiber $L \in G^k_n$ the family $\{V_i \cap L\}_{i=1}^{n+1}$ is non-antipodal in $L$. Then there exists a point $x$ in some fiber $L$ such that the distances from $x$ to all $V_i \cap L$ ($i = 1, \ldots, n+1$) are equal.

**Theorem 4.** Consider $n+1$ compact sets $V_1, V_2, \ldots, V_{n+1}$ in $\gamma^k_n$ such that for any $i = 1, \ldots, n+1$ the intersection with fiber $V_i \cap L$ is nonempty and depends continuously on $L$ in the Hausdorff metric. Suppose also that for any fiber $L \in G^k_n$ the family $\{V_i \cap L\}_{i=1}^{n+1}$ is non-antipodal in $L$, and the union $\bigcup_{i=1}^{n+1} V_i \cap L$ is convex. Then the sets $V_i$ have a common point.

The following theorem gives a partial solution to the conjecture on the fields of polytopes in the vector bundle $\gamma^k_n$ (see [19], Conjecture 1 and Theorem 12).

**Theorem 5.** Consider $m$ continuous sections $s_1, \ldots, s_m$ of the bundle $\gamma^k_n$, such that for any fiber $L \in G^k_n$ the polytope $P(L) = \text{conv}\{s_1(L), \ldots, s_m(L)\}$ has non-empty interior. Then there exists a fiber $L \in G^k_n$ and a pair of disjoint support half-spaces of $P(L)$, say $H_1, H_2 \subset L$, such that the union $H_1 \cup H_2$ contains at least $n+1$ points of $\{s_1(L), \ldots, s_m(L)\}$.

In Sections 6, 7, 8 we deduce some corollaries from the above theorems, and prove some results using the similar technique.

3. SOME TOPOLOGICAL ASSERTIONS

Let us state some definitions of equivariant topology, see the book [14] for more detailed discussions.

**Definition 7.** Let $G$ be a compact Lie group or a finite group. A space $X$ with continuous action of $G$ is called a **$G$-space**. A continuous map of $G$-spaces, commuting with the action of $G$ is called a **$G$-map** or an **equivariant map**. A $G$-space is called **free** if the action of $G$ is free.
There exists the universal free $G$-space $EG$ such that any other $G$-space maps uniquely (up to $G$-homotopy) to $EG$. The space $EG$ is homotopy trivial, the quotient space is denoted $BG = EG/G$. For any $G$-space $X$ and an Abelian group $A$ the equivariant cohomology $H^*_G(X, A) = H^*(X \times_G EG, A)$ is defined, and for free $G$-spaces the equality $H^*_G(X, A) = H^*(X/G, A)$ holds.

In this paper we consider the action of $G = Z_2$ only. Note that

$$H^*_G(pt, Z_2) = H^*(\mathbb{R}P^\infty, Z_2) = Z_2[w] = \Lambda,$$

where the dimension of the generator is $\dim w = 1$. Since any $G$-space $X$ can be mapped to the point $\pi_X : X \to pt$, we have a natural map $\pi_X^* : \Lambda \to H^*_G(X, Z_2)$, the image $w$ under this map will be denoted $w$, if it does not make a confusion. The generator element of $Z_2$ will be denoted $\sigma$.

It is important to use the Čech or Alexander-Spanier cohomology to consider arbitrary closed subsets and their cohomology, because of their continuity property w.r.t. intersections.

**Definition 8.** The cohomology index of a $Z_2$-space $X$ is the maximal $n$ such that the power $w^n \neq 0$ in $H^*_G(X, Z_2)$. If there is no maximum, we consider the index equal to $\infty$. Denote the index of $X$ by hind $X$.

It is quite clear that the index of a non-free $Z_2$-space equals $\infty$. From the explicit description of the cohomology of $\mathbb{R}P^n$ it follows that the index of the $n$-dimensional sphere with the antipodal action of $Z_2$ ($x \mapsto -x$) equals $n$.

Let us state the following well-known lemma.

**Lemma 1** (The generalized Borsuk-Ulam theorem for odd maps). *If there exists an equivariant map $f : X \to Y$, then hind $X \leq$ hind $Y$.*

The lemma follows from the definition of index and the naturality of maps $\pi_X$ and $\pi_Y$. It is also called the monotonicity property of index.

Let us introduce some geometrical construction for a $Z_2$-space.

**Definition 9.** Let $X$ be a free $Z_2$-space. Take the product $X \times I$, where $I = [0, 1]$ is the segment, and define the action of $Z_2$ by $\sigma(x, t) = (\sigma(x), 1 - t)$. The space $X \times I$ is free, so we put $B(X) = (X \times I)/Z_2$. Informally, $B(x)$ is obtained from $X$ by gluing a segment to any pair $\{x, \sigma(x)\}$ and introducing the respective topology on the set of segments. The natural map $B(X) \to X/Z_2$ is a fiber bundle with fiber $I$, and a homotopy equivalence.

**Definition 10.** Define the map $i_X : X \to B(X)$ by the formula $i_X(x) \mapsto (x, 0)$. It identifies $X$ with sphere bundle of the line bundle $B(X) \to X/Z_2$. In the sequel we always identify $X$ with a subset of $B(X)$ by the map $i_X$.

Note that the space $B(X)$ has a natural $Z_2$-action, given by $(x, t) \mapsto (\sigma(x), t)$, with fixed-point set $(X \times \{1/2\})/Z_2$.

The following Theorem is a slight generalization of Lemma 5.5 from [23].

**Theorem 6.** Suppose that the $Z_2$-spaces $X$ and $Y$ have hind $X = hind Y = n$. Then for any equivariant map $f : X \to Y$ the map $f^* : H^n(Y, Z_2) \to H^n(X, Z_2)$ is nontrivial.
Moreover, there does not exist a (continuous, non-equivariant) map $h : B(X) \to Y$ such that $f = h \circ i_X$.

Proof. The spaces $X$ and $Y$ are obviously free. Consider the Thom exact sequences

$$
\ldots \xrightarrow{\delta_X} H^k(X, Z_2) \xrightarrow{i_X^*} H^k(B(X), Z_2) \xrightarrow{\pi_X^*} H^k(B(X), X, Z_2) \xrightarrow{\delta_X} H^{k-1}(X, Z_2) \ldots,
$$

$$
\ldots \xrightarrow{\delta_Y} H^k(Y, Z_2) \xrightarrow{i_Y^*} H^k(B(Y), Z_2) \xrightarrow{\pi_Y^*} H^k(B(Y), Y, Z_2) \xrightarrow{\delta_Y} H^{k-1}(Y, Z_2) \ldots.
$$

Note that $f$ gives a natural continuous map $B(X) \to B(Y)$, hence the above exact sequences are mapped into each other by $f^*$, which commutes with the exact sequence maps. Let the Thom classes in $H^1(B(X), X, Z_2)$ and $H^1(B(Y), Y, Z_2)$ be $u_X$ and $u_Y$; let the images of $w \in H^n_G(pt, Z_2)$ in $H^*(B(X), Z_2)$ and $H^*(B(Y), Z_2)$ be $w_X$ and $w_Y$. It is clear that $f^*(u_Y) = u_X, f^*(w_Y) = w_X$.

By the definition of index we have $\pi_Y^*(u_X w_X^n) = w_X^{n+1} = 0$ and $\pi_Y^*(u_Y w_Y^n) = w_Y^{n+1} = 0$, from the exactness there exists a class $v \in H^n(Y, Z_2)$ such that $\delta_Y(v) = u_Y w_Y^n$. We have $\delta_X(f^*(v)) = f^*(\delta_Y(v)) = u_X w_X^n \neq 0$ and therefore $f^*(v) \neq 0$, the first claim of the theorem is proved.

To prove the second claim, note that the existence of $h$ would imply $f^*(v) \in \text{Im} \ i_X^*$, and from the exactness of the Thom sequence $\delta_X(f^*(v)) = 0$, which contradicts the formulas in the previous paragraph. □

Now we are going to deduce a corollary from Theorem 6 on coverings of $Z_2$-spaces. We need a definition first.

**Definition 11.** The points $x$ and $\sigma(x)$ of some $Z_2$-space $X$ are called antipodal.

The term “antipodal” was already defined for the points on the boundary of a convex compact set. But actually it does not lead to a confusion in the sequel.

**Theorem 7.** Let a compact metric $Z_2$-space $X$ with hind $X = n$ be covered by a family of closed sets $\mathcal{F} = \{U_1, U_2, \ldots, U_{n+2}\}$, so that none of $U_i$ contains a pair of antipodal points. Then for any partition of the family $\{U_i\}$ into non-empty subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ there exists a point $x \in X$ such that

$$
x \in \bigcap \mathcal{F}_1 \quad \text{and} \quad \sigma(x) \in \bigcap \mathcal{F}_2.
$$

Moreover, if the covering $\mathcal{F}$ is induced by some closed covering $\mathcal{G} = \{V_1, V_2, \ldots, V_{n+2}\}$ of $B(X)$ then the family $\mathcal{G}$ has a common point.

Proof. Suppose $W$ is a subset of some metric space $M$ and put

$$
W(\varepsilon) = \{x \in M : \text{dist}(x, W) \leq \varepsilon\}.
$$

From the compactness consideration we can assume that

$$
\exists \varepsilon > 0 : \forall i = 1, \ldots, n+2 \quad \text{dist}(U_i, \sigma(U_i)) > 2\varepsilon.
$$

Consider a partition of unity $\{f_i\}_{i=1}^{n+2}$, subordinated to the covering by $U_i(\varepsilon)$, such that $f_i > 0$ on $U_i$. 
Now consider the functions \( g_i(x) = f_i(x) - f_i(\sigma(x)) \). They together give an equivariant map \( g : X \to \mathbb{R}^{n+2} \), the image of \( g \) is contained in the hyperplane \( H \), given by the equation \( y_1 + \ldots + y_{n+2} = 0 \), and does not contain the origin. Thus the map \( g \) induces the map \( h : X \to S^n \) to the unit sphere of \( H \), given by the formula \( h(x) = g(x)/|g(x)| \).

The map \( h : X \to S^n \) is equivariant and by Theorem \( \text{[1]} \) the map \( h^* : H^n(S^n, Z_2) \to H^n(X, Z_2) \) is nontrivial, hence \( h \) must be surjective.

Take a partition \( [n+2] = I_1 \cup I_2 \) and the corresponding partition \( \{U_i(\varepsilon)\}_{i=1}^{n+2} = \mathcal{F}_1(\varepsilon) \cup \mathcal{F}_2(\varepsilon) \).

Take the number \( c = \sqrt{|I_1||I_2|(n+2)} \) and consider a point \( y \in S^n \) with coordinates

\[
y_i = \frac{|I_2|}{c} \quad i \in I_1 \\
y_i = -\frac{|I_1|}{c} \quad i \in I_2.
\]

There exists a point \( x \in X \) such that \( y = h(x) \), hence \( x \in \bigcap \mathcal{F}_1(\varepsilon) \), \( \sigma(x) \in \bigcap \mathcal{F}_2(\varepsilon) \). Going to the limit with \( \varepsilon \to 0 \) and applying the compactness consideration we obtain the first claim of the theorem.

Now suppose that the covering is induced by a covering of \( B(X) \), the functions \( f_i \) give therefore a partition of unity on \( B(X) \). They give the map \( f \), that maps \( X \) to the hyperplane \( H_1 \), given by the equation \( y_1 + \ldots + y_{n+2} = 1 \). If the image of \( f \) contains the point \((\frac{1}{n+2}, \ldots, \frac{1}{n+2})\) then, similar to the above reasoning the sets \( \{V_i\} \) have a common point. Otherwise, \( f \) gives a map \( h_1 : B(X) \to S^n \) to the unit sphere of the hyperplane \( H_1 \). Let us show that the maps \( h \) and \( h_1 \mid X \) are homotopy equivalent. Put for any \( t \in [0, 1] \)

\[
g_t(x) = f(x) - (1-t)f(\sigma(x)) \\
h_t(x) = \frac{g_t(x) - t(\frac{1}{n+2}, \ldots, \frac{1}{n+2})}{|g_t(x) - t(\frac{1}{n+2}, \ldots, \frac{1}{n+2})|}
\]

then \( h_t \) is the required homotopy. By the second claim of Theorem \( \text{[1]} \) \( h \) (and its homotopy equivalent \( h_1 \)) cannot be extended from \( X \) to \( B(X) \) that is a contradiction. \( \square \)

Let us prove another theorem on coverings.

**Theorem 8.** Let a compact metric \( Z_2 \)-space \( X \) with \( \text{hind} X = n \) be covered by a family of closed sets \( \mathcal{F} = \{U_1, U_2, \ldots, U_N\} \). Suppose that none of \( U_i \) contains a pair of antipodal points. Then there exists a point \( x \in X \) such that the number of sets \( U_i \), that contain either \( x \) or \( \sigma(x) \) is at least \( n+2 \).

**Proof.** As in the previous theorem, consider the corresponding partition of unity \( f_i : X \to \mathbb{R}^+ \). Consider the map \( g(x) = f(x) - f(\sigma(x)) \) and assume the contrary, i.e. for any \( x \in X \) at most \( n+1 \) of the coordinates of \( g(x) \) are nonzero.

Since the sum of the positive coordinates \( g_i(x) \) is 1, the sum of negative coordinates is \(-1\), the image of \( g \) is contained in some \( n-1 \)-dimensional simplicial complex. This complex has free \( Z_2 \)-action, and its index is at most \( n-1 \) from the dimension considerations, thus we have a contradiction with Lemma \( \text{[1]} \). \( \square \)

Let us state a theorem, that strengthens the Lyusternik-Schnirelmann theorem on the category of \( \mathbb{R}P^n \).
Definition 12. An invariant subset $U \subseteq X$ of a $Z_2$-space $X$ is called inessential, if $\text{hind} \ U = 0$.

Note that the index of $U$ is zero, iff there exists an equivariant map of $U$ to the zero-dimensional sphere (a pair of points), here the continuity of cohomology is used.

Theorem 9. Let a compact metric $Z_2$-space $X$ with $\text{hind} \ X = n$ be covered by a family of closed inessential invariant subsets $\mathcal{F} = \{U_1, U_2, \ldots, U_N\}$. Then $N \geq n + 1$ and some $n + 1$ sets of the family $\mathcal{F}$ have a common point.

Proof. For any $U_i$ we have a $Z_2$-equivariant map $f_i : U_i \to \{-1, +1\}$. Extend $f_i$ to an equivariant map $f_i : X \to [-1, +1]$ so that $f_i$ is zero outside some $\epsilon$-neighborhood of $U_i$. Finally, the functions $f_i$ give an equivariant map $f : X \to \mathbb{R}^N \setminus \{0\}$, hence an equivariant map $h : X \to S^{N-1}$ can be defined by $h(x) = f(x) / |f(x)|$. By Lemma 1 $n \leq N - 1$ and the first claim is proved.

To prove the second claim assume the contrary: for sufficiently small $\epsilon$ it would mean that at most $n$ of the coordinates of $h(x)$ can be nonzero. Thus the image if $h$ is contained in some $n - 1$-dimensional subset of $S^{N-1}$, that contradicts with Lemma 1. \hfill \Box

Another analogue of Theorem 7 can be proved for a product of $Z_2$-spaces (see Theorem 2 from [16]).

Theorem 10. Let compact metric $Z_2$-spaces $X$ and $Y$ have indexes $\text{hind} \ X = n$, $\text{hind} \ Y = m$, $m \geq n$.

Let the set $B(X) \times B(Y)$ be covered by a family of closed subsets $\mathcal{F} = \{V_{ij}\}_{i \in [n+2], j \in [m+2]}$. Denote the projections $\pi_X : B(X) \times B(Y) \to B(X)$, $\pi_Y B(X) \times B(Y) \to B(Y)$.

Suppose that for any $i \in [n+2]$ the set $\pi_X(\bigcup_{j \in [m+2]} V_{ij}) \cap X$ does not contain a pair of antipodal points, and for any $j \in [m+2]$ the set $\pi_Y(\bigcup_{i \in [n+2]} V_{ij}) \cap Y$ does not contain a pair of antipodal points either.

Let $\{a_i\}_{i \in [n+2]}$ be positive integers with sum equal to $m + 2$. Then there exists a map $\tau : [m+2] \to [n+2]$ such that

$$\forall i \in [n+2] \ |\tau^{-1}(i)\| = a_i$$

and

$$\bigcap_{j \in [m+2]} U_{\tau(j), j} \neq \emptyset.$$
**Proof of Theorem 10** Similar to the previous proofs, let us pass to the partition of unity $\phi_{ij} : B(X) \times B(Y) \to \mathbb{R}^+$. Consider the functions $f_i = \sum_{j \in [m+2]} \phi_{ij}$ and $y_j = \sum_{i \in [n+2]} \phi_{ij}$. Similar to the proof of Theorem 7, the maps $f$ and $g$ can be considered as maps of $B(X) \times B(Y)$ to $n+1$ and $m+1$-dimensional linear subspaces respectively. Denote the inclusions

$$i_X : B(X) \to B(X) \times B(Y) \quad i_X(x) = x \times y_0$$

and

$$i_Y : B(Y) \to B(X) \times B(Y) \quad i_Y(y) = x_0 \times y.$$  

From the proof of Theorem 6 it follows that the maps

$$i_X^* \circ f^* : H^{n+1}(\Delta^{n+1}, \partial \Delta^{n+1}, Z_2) \to H^{n+1}(B(X), X, Z_2))$$

and

$$i_Y^* \circ g^* : H^{m+1}(\Delta^{m+1}, \partial \Delta^{m+1}, Z_2) \to H^{m+1}(B(Y), Y, Z_2))$$

are nontrivial, and by the Künneth formula the following map

$$(f \times g)^* : H^{n+m+2}(\Delta^{n+1} \times \Delta^{m+1}, \partial (\Delta^{n+1} \times \Delta^{m+1}), Z_2) \to H^{n+m+2}(B(X) \times B(Y), X \times B(Y) \cup B(X) \times Y, Z_2)$$

is nontrivial, and therefore $f \times g$ is surjective.

Consider the preimage of

$$\left(\frac{a_1}{m+2}, \ldots, \frac{a_{n+2}}{m+2}\right) \times \left(\frac{1}{m+2}, \ldots, \frac{1}{m+2}\right) \in \Delta^{n+1} \times \Delta^{m+1},$$

denote it $p$. The matrix $\{\phi_{ij}(p)\}$ (considered as the bipartite graph incidence matrix) satisfies the conditions of Lemma 2 that gives the required map $\tau$, compare [15].

---

4. **Geometry and topology of the space of $k$-flats**

Let us describe the space of all $k$-flats in $\mathbb{R}^n$. For any $k$-flat $\alpha$ there is a unique $(n-k)$-dimensional linear subspace of $\mathbb{R}^n$, orthogonal to $\alpha$, denote it $g(\alpha)$, and a unique intersection point $\alpha \cap g(\alpha)$. Thus the space of $k$-flats is parameterized by the total space $\gamma^{n-k}_n$ of the canonical vector bundle $\gamma_n \to G^{n-k}_n$. In the sequel we identify the space of $k$-flats with $\gamma^{n-k}_n$, thus introducing the topology on the space of $k$-flats.

In any vector bundle the group $Z_2$ acts by the fiber-wise map $x \mapsto -x$. Let us calculate the index of the space of spheres $S(\gamma^{n-k}_n)$ under this action.

**Theorem 11.** $\text{hind} S(\gamma^{n-k}_n) = n - 1$.

**Proof.** The space $S(\gamma^{n-k}_n)$ can be viewed as the space of pairs $(n, L)$, where $n$ is a unit vector, and $L$ is a $k$-dimensional linear subspace of $\mathbb{R}^n$, orthogonal to $n$. Hence there is a natural equivariant map from $S(\gamma^{n-k}_n)$ to the $(n-1)$-dimensional sphere $(n, L) \mapsto n$, and by Lemma 1 $\text{hind} S(\gamma^{n-k}_n) \leq n - 1$.

Let us calculate the cohomology $H^*_G(S(\gamma^{n-k}_n), Z_2)$ as the cohomology of the quotient space $G^{1,k}_n = S(\gamma^{n-k}_n)/Z_2$. This space is identified with the set of pairs $(l, L)$, where $l$ is one-dimensional subspace $\mathbb{R}^n$, $L$ is $k$-dimensional subspace, and $l \perp L$. 


The map \( \pi : (l, L) \mapsto l \) sends \( G_{n}^{1,k} \) to \( \mathbb{R}P^{n-1} \). It is easy to check that the one-dimensional generator \( w \) of \( H^{*} (\mathbb{R}P^{n-1}, Z_{2}) \) is mapped under \( \pi^{*} \) to the element \( w \in H^{*} (G_{n}^{1,k}, Z_{2}) \), let us prove that \( w^{n-1} \neq 0 \). Thus the space \( G_{n}^{1,k} \) can be viewed as the space of \( k \)-dimensional subspaces in the fibers of the complementary canonical bundle \( \eta \mapsto \mathbb{R}P^{n-1} \).

Note that the flag bundle \( F(\eta) \) of the vector bundle \( \eta \) is identified with the flag manifold \( F(\mathbb{R}^{n}) \), and the natural projection \( \pi_{F} : F(\eta) \to \mathbb{R}P^{n-1} \) maps a flag

\[
0 \subset L_{1} \subset L_{2} \subset \cdots \subset L_{n} = \mathbb{R}^{n}
\]
to the line \( L_{1} \), considered as an element of \( \mathbb{R}P^{n-1} \).

Let the map \( \rho : F(\eta) \to G_{n}^{1,k} \) map a flag \( L_{1} \subset L_{2} \subset \cdots \subset L_{n} \) to the pair, consisting of \( L_{1} \) and the orthogonal complement to \( L_{1} \) in \( L_{k+1} \). Evidently \( \pi_{F} = \pi \circ \rho \). The map \( \pi_{F}^{*} \) is known to give the injective map on the cohomology mod 2. Hence the map \( \pi^{*} \) is injective, and \( w^{n-1} \neq 0 \) in the cohomology of \( G_{n}^{1,k} \).

Under the notation of the previous section we formulate the lemma.

**Lemma 3.** There exists a map from \( B(S(\gamma_{n}^{n-k})) \) to the space of balls \( B(\gamma_{n}^{n-k}) \), identical on \( S(\gamma_{n}^{n-k}) \).

**Proof.** Let a pair \( (s, t) \) represent some element \( B(S(\gamma_{n}^{n-k})) \), let us map it to the combination \( (1 - t)s - ts \in B(\gamma_{n}^{n-k}) \). The pair \( (-s, 1 - t) \) is mapped to the same point, thus the map \( B(S(\gamma_{n}^{n-k})) \to B(\gamma_{n}^{n-k}) \) is defined. \( \square \)

Now we can deduce a theorem.

**Theorem 12.** Let \( S(\gamma_{n}^{n-k}) \) be covered by a family of closed sets \( \mathcal{F} = \{ U_{1}, U_{2}, \ldots, U_{n+1} \} \).

Suppose that none of \( U_{i} \) contains a pair of antipodal points. Then for any partition of the family \( \{ U_{i} \} \) into non-empty subfamilies \( \mathcal{F}_{1} \) and \( \mathcal{F}_{2} \) there exists a point \( c \in S(\gamma_{n}^{n-k}) \) such that \( c \in \bigcap \mathcal{F}_{1} \) and \( \sigma(c) \in \bigcap \mathcal{F}_{2} \).

Moreover, if the covering \( \mathcal{F} \) is induced by some closed covering \( \mathcal{G} = \{ V_{1}, V_{2}, \ldots, V_{n+1} \} \) of \( B(\gamma_{n}^{n-k}) \), then \( \mathcal{G} \) has a common point.

**Proof.** The first claim follows from Theorems 7 and 11 to prove the second claim we use Lemma 3 to obtain a covering of \( B(\gamma_{n}^{n-k}) \) from the covering of \( B(S(\gamma_{n}^{n-k})) \). \( \square \)

Let us consider the oriented Grassmannian \( G_{n}^{k+} \). It has a natural action of \( Z_{2} \) by the change of the orientation. Note that \( G_{n}^{k+} \sim C_{n}^{n-k+} \), hence it is sufficient to consider the case \( 2k \leq n \) to calculate the index of this action. The following theorem summarizes the data on the index of the oriented Grassmannian from the papers 11, 12.

**Theorem 13.** Let \( 2k \leq n \), and let \( 2^{*} \) be the minimal power of two, satisfying \( 2^{*} \geq n \).

1) If \( k = 1 \), then \( \text{hind} \ G_{n}^{1+} = \text{hind} \ S_{n-1} = n - 1 \);
2) If \( k = 2 \), then \( \text{hind} \ G_{n}^{2+} = 2^{*} - 2 \);
3) If \( k > 2 \), then in the case \( n = 2k = 2^{*} \) we have \( 2^{*+1} \leq \text{hind} \ G_{n}^{k+} \leq 2^{*} - 1 \) and \( 2^{*} - 2 \leq \text{hind} \ G_{n}^{k+} \leq 2^{*} - 1 \) in other cases.

In all cases \( \text{hind} \ G_{n}^{k+} \geq n - k \), the equality holds for \( k = 1 \), \( k = 2 \) and \( n = 2^{*} \).
5. Proofs for the theorems on the canonical bundle

Proof of Theorem 11. Consider the space of balls $B(\gamma^1_n)$, we can choose the balls large enough so that $B(\gamma^1_n)$ contains all the sets $V_i$.

Let us define the subsets $U_i$ of the space $S(\gamma^1_n)$ as follows. Take some $s \in S(\gamma^1_n)$, lying in the fiber $L$. Choose the farthest from $s$ point on each of the segments $V_i \cap L$, denote it $f_i(s)$. These points depend on $s$ continuously. Now denote the nearest to $s$ point of these point by $f(s)$, it also depends continuously on $s$.

Now put
\[ U_i = \{s \in S(\gamma^1_n) : f(s) = f_i(s)\}. \]

These sets are closed. If some $U_i$ contains an antipodal pair $s, s' \in S(\gamma^1_n)$, then the segment $V_i \cap L$ satisfies the first alternative.

Otherwise we apply Theorem 12. For any partition of the index set $[n + 1] = I_1 \cup I_2$ there is a pair of antipodal points $s, s' \in S(\gamma^1_n)$ such that
\[ s \in \bigcap_{i \in I_1} U_i, \quad s' \in \bigcap_{i \in I_2} U_i. \]

Consider the families of segments $A = \{V_i \cap L\}_{i \in I_1}$ and $B = \{V_i \cap L\}_{i \in I_2}$, without loss of generality assume that $s$ is to the left of $s'$. In this case all right ends of segments of $A$ coincide in $a$, all left ends of segments of $B$ coincide in $b$, and either $a$ is to the left of $b$, or all the segments of $A \cup B$ contain $[ba]$, i.e. the families of segments are equalized. \hfill $\square$

Proof of Theorem 13. Denote
\[ U_i = \{x \in \gamma^k_n : x \text{ is in the fiber } L, \text{ dist}(x, V_i \cap L) = \min_{j=1,...,n+1} \text{dist}(x, V_j \cap L)\}. \]

The set $V_i \cap L$ depends continuously on $L$, hence the sets $U_i$ are closed. Denote the bundle of spheres of size $R$ in $\gamma^k_n$ by $S(\gamma^k_n)$. Let us show that for large enough $R$ the sets $U_i \cap S(\gamma^k_n)$ do not contain antipodal pairs.

Assume the contrary. Let the set $U_i \cap S(\gamma^k_n)$ contain a pair of antipodal points $R_m x_m$ and $-R_m x_m$ for some sequence of radii $R_m \to +\infty$. It means that the closest to $R_m x_m$ and $-R_m x_m$ points of $\bigcup V_i$ belong to the same set $V_i$, denote these points by $y_m$ and $z_m$. From the compactness considerations we assume that the points $x_m, y_m, z_m$ tend to some points $x, y, z$ in $L$.

It is easy to see that $\text{conv} \bigcup_{i=1}^{n+1} \{V_i \cap L\}$ contains two points $y, z$, that belong to the same $V_i$. Besides, the spheres with centers $R_m x_m, -R_m x_m$ and radii $|R_m x_m - y_m|, |R_m x_m - z_m|$, tend to some disjoint support half-spaces for $\bigcup_{i=1}^{n+1} \{V_i \cap L\}$, containing $y$ and $z$ respectively. This is a contradiction with the non-antipodality of $\{V_i \cap L\}$.

Hence, for large enough $R$ the sets $U_i \cap S(\gamma^k_n)$ do not contain antipodal pairs.

Applying Theorem 12 we find a common point for the family $\{U_i\}$, that is exactly what we need. \hfill $\square$

Proof of Theorem 14. Let us apply Theorem 13 and find $x \in L$, equidistant from $V_i \cap L$.

Suppose that the distance is positive. Let the set of closest to $x$ points of $\bigcup_{i=1}^{n+1} V_i \cap L$ be $K$. Obviously, $K$ intersects all of $V_i$, since $K$ and $\bigcup_{i=1}^{n+1} V_i \cap L$ are convex, then $K$ is contained in some support half-space $H$ for $\bigcup_{i=1}^{n+1} V_i \cap L$. But the opposite to $H$ support
half-space cannot intersect \( V_i \) from the non-antipodality condition on \( \{ V_i \cap L \} \), so it cannot be a support half-space for \( \bigcup_{i=1}^{j+1} V_i \cap L \). That is a contradiction.

\[ \square \]

**Proof of Theorem 5.** Consider the space \( S(\gamma^k_n) \) and define its closed subspaces
\[
U_i = \{(n,L) : L \in G^k_n, \; n \in S(L), \; (n, s_i(L)) = \max_{j \in [m]} (n, s_j(L))\}.
\]
Since the interiors of \( P(L) \) are non-empty, the sets \( U_i \) does not contain antipodal pairs. Now Theorems 11 and 8 imply this theorem directly.

\[ \square \]

## 6. Partitioning measures by hyperplanes

Let us formulate the generalization of the theorem from [3, 6, 18], that claims that any \( n + 1 \) convex compact sets in \( \mathbb{R}^n \) can either be intersected by a hyperplane; or any two non-empty disjoint subfamilies of this family can be separated by a hyperplane. We need some definition about measures, see the book [20] for the general treatment of measures.

**Definition 13.** A measure \( \mu \) on \( \mathbb{R}^n \) is called probabilistic if \( \mu(\mathbb{R}^n) = 1 \).

We are going to consider such measures that the measure of a half-space depends continuously on the half-space. We call such measures continuous. For a measure to be continuous in this sense it is sufficient that the measure is absolutely continuous in the common sense.

**Definition 14.** A pair of a continuous probabilistic measure with compact support \( \mu \) and a number \( \varepsilon \in [0,1/2) \) is called a measure with deviation. If we consider several measures \( \mu_i \) with deviation, the deviation of each measure is denoted \( \varepsilon(\mu_i) \).

**Definition 15.** Let \( \mu \) be a measure with deviation in \( \mathbb{R}^n \). A hyperplane \( h \) (reliably) intersects the measure \( \mu \), if \( h \) partitions \( \mathbb{R}^n \) into half-spaces \( H_1 \) and \( H_2 \), and
\[
\mu(H_1), \mu(H_2) \geq \varepsilon(\mu).
\]

**Definition 16.** Let \( \mu \) be a measure with deviation in \( \mathbb{R}^n \). A half-space \( H \) (almost) contains the measure \( \mu \), if
\[
\mu(H) > 1 - \varepsilon(\mu).
\]

**Definition 17.** Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two families of measures with deviations in \( \mathbb{R}^n \). A hyperplane \( h \) (almost) separates the families \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), if \( h \) partitions \( \mathbb{R}^n \) into half-spaces \( H_1 \) and \( H_2 \), and any \( \mu \in \mathcal{M}_1 \) is almost contained in \( H_1 \), and any \( \mu \in \mathcal{M}_2 \) is almost contained in \( H_2 \).

Corollary 2 implies the following claim.

**Corollary 14.** Let \( \mathcal{M} \) be a family of \( n + 1 \) measures with deviation in \( \mathbb{R}^n \). Then either there exists a hyperplane that reliably intersects all the measures of \( \mathcal{M} \); or for any partition of \( \mathcal{M} \) into non-empty \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) there exists a hyperplane, that almost separates \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \).
Proof. Put $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_{n+1}\}$, and denote $V_i$ the set of hyperplanes that reliably intersect $\mu_i$. Now applying Corollary [2] we obtain the required alternative. \qed

We are going to generalize some results of [2, 5].

**Definition 18.** Consider a family of $n$ measures with deviation $\{\mu_i\}_{i=1}^n$ in $\mathbb{R}^n$. Denote by $X \subseteq \gamma_n^i$ the set of hyperplanes, that reliably intersect all these measures, consider the natural projection $p : X \to G_n^1$. The family of measures $\{\mu_i\}_{i=1}^n$ is flat, if the projection $p$ can be lifted to the universal covering $\pi : S^{n-1} \to G_n^1$, i.e. $p = \pi \circ \tilde{p}$ for some continuous $\tilde{p}$.

**Theorem 15.** Consider a flat family of measures with deviation $\{\mu_i\}_{i=1}^n$ in $\mathbb{R}^n$ and the numbers $\{\alpha_i\}$ such that for any $i \in [n]$ either $\alpha_i = \varepsilon(\mu_i)$, or $\alpha_i = 1 - \varepsilon(\mu_i)$. Then there exists a half-space $H \subset \mathbb{R}^n$ such that for all $i = 1, \ldots, n$

$$\mu_i(H) = \alpha_i.$$

The flatness condition on $p$ here generalizes the condition on separated supports from Theorem 1 in [2]. If the supports are separated, then the family of measures is flat, independent of deviations. In [5] a similar result is proved, the separated supports condition is replaced by the following condition: there exist $n$ separated compacts $C_1, \ldots, C_n$, such that any hyperplane, that reliably intersects $\mu_i$, intersects $C_i$. This condition implies the flatness condition of Theorem [5] too.

In Theorem [13] the deviations are not equal to $1/2$, but going to the limit, we can prove it when some of the deviations are $1/2$. If all the deviations are $1/2$, we obtain the “ham sandwich” theorem.

**Proof.** Denote by $V_i$ the set of hyperplanes that reliably intersect $\mu_i$. The following is similar to the proof of Theorem [11].

Consider the ball bundle $B(\gamma_n^i)$, take the balls large enough so that it contains all the sets $V_i$ and define the maps $f_i : S(\gamma_n^i) \to B(\gamma_n^i)$ ($i = 1, \ldots, n$) and $f : S(\gamma_n^i) \to B(\gamma_n^i)$ as in the proof of Theorem [11]. Now define close subsets $U_0, U_1, \ldots, U_n \subseteq S(\gamma_n^i)$.

Take the projection of $X$ to $G_n^1$, denote its image by $Y$. Put $Z = p^{-1}(Y) \cap S(\gamma_n^i)$. The set $Z$ is a two-fold cover of $Y$ and by the flatness condition the covering $Z \to Y$ is trivial, i.e. $Z = Z_1 \cup Z_2$, where $p : Z_1 \to Y$ and $p : Z_2 \to Y$ are bijections.

Now put $U_0 = Z_1$ and

$$U_i = \{s \in S(\gamma_n^i) \setminus \text{int } U_0 : f(s) = f_i(s)\}.$$

The set $U_0$ does not contain antipodal pairs by definition. Suppose that some $U_i$ contains an antipodal pair $s, s' \in S(\gamma_n^i)$ in the fiber $L$. In this case the segment $V_i \cap L$ is contained in all other segments $V_j \cap L$. The length of $V_i \cap L$ is positive, since $\varepsilon(\mu_i) < 1/2$. Then $p(s) = p(s') \in \text{int } Y$, i.e. one of the points $s, s'$ is in $\text{int } U_0$, that is a contradiction with the definition of $U_i$.

Now put $I = \{0, 1, \ldots, n\}$,

$$I_1 = \{i = 1, \ldots, n : \alpha_i = 1 - \varepsilon(\mu_i)\}$$

and $I_2 = I \setminus I_1$. Applying Theorem [12] we find an antipodal pair $s, s' \in S(\gamma_n^i)$ such that

$$\forall i \in I_1 \ f(s) = f_i(s), \quad \forall i \in I_2 \setminus \{0\} \ f(s') = f_i(s'), \quad s' \in \text{bd } U_0.$$
It follows that the segments $V_i \cap L$ intersect in the unique point, which is the right end for $V_i \cap L (i \in I_1)$, and the left end for $V_i \cap L (i \in I_2 \setminus \{0\})$. It is easy to see that this point designates the required hyperplane. □

7. BORSUK-ULAM TYPE THEOREMS FOR FLATS

Let us make a definition and state a corollary of Theorems 3 and 4.

**Definition 19.** A set $X \subseteq \mathbb{R}^n$ is called $l$-convex, if its projection to any $l$-dimensional subspace of $\mathbb{R}^n$ is convex.

**Corollary 16.** Suppose $\mathcal{F}$ is a non-antipodal family of $n+1$ compact sets in $\mathbb{R}^n$, then there exists a $k$-flat equidistant from all the sets of $\mathcal{F}$. If, in addition, the union $\bigcup \mathcal{F}$ is $(n-k)$-convex, then $\mathcal{F}$ has a common $k$-transversal.

**Proof.** Let $\mathcal{F} = \{K_i\}_{i=1}^{n+1}$. Denote $V_i$ the set of $k$-flats, intersecting $K_i$. In this case the sets $V_i \cap L$ are projections of $K_i$ to $L$, hence they form a non-antipodal family. Applying Theorems 3 or 4 to $V_i$ we obtain the required result. □

**Definition 20.** Consider two subsets $X, Y \subseteq \mathbb{R}^n$. The deviation of $X$ from $Y$ is the following number

$$\delta(X, Y) = \sup_{x \in X} \text{dist}(x, Y).$$

**Corollary 17.** Suppose $\mathcal{F}$ is a non-antipodal family of $n+1$ compact sets in $\mathbb{R}^n$, then there exists a $k$-flat $M$ such that the deviations of all the sets of $\mathcal{F}$ from $M$ are equal.

**Proof.** Let $\mathcal{F} = \{K_i\}_{i=1}^{n+1}$. Denote

$$V_i = \{M \in \gamma_{n-k} : \delta(\bigcup \mathcal{F}, M) = \delta(K_i, M)\}.$$

Similar to the proof of Theorem 3 we note that for large enough radius of balls in the ball bundle $B(\gamma_{n-k})$, none of the sets $V_i$ contain antipodal points in $S(\gamma_{n-k})$. Hence there exists non-empty intersection $\bigcap_{i=1}^{n+1} V_i$. □

Note again, that in Corollaries 16 and 17 the distance can be taken in any norm with smooth unit ball.

The fact hind $S(\gamma_n^k) = n-1$ leads to another generalization of the Borsuk-Ulam theorem. Let us give some definitions.

**Definition 21.** Let $S^{n-1} \subseteq \mathbb{R}^n$ be the unit sphere. A $k$-subsphere is an intersection of a $k$-dimensional linear subspace $L \subseteq \mathbb{R}^n$ with $S^{n-1}$.

**Definition 22.** Let $S^{n-1} \subseteq \mathbb{R}^n$ be the unit sphere. A $k$-half-sphere is a half of some $k$-subsphere.

**Theorem 18.** Let $V_1, \ldots, V_n$ be open subsets of $S^{n-1}$ such that each of $V_i$ intersects any $k$-subsphere. Then there exists a $k$-half-sphere that intersects every $V_i$. 
Proof. The space of \( k \)-subspheres is parameterized by the Grassmannian \( G^k_n \), the space of all \( k \)-half-spheres is parameterized by the half-spaces in the fibers of \( \gamma^k_n \), with boundaries containing the origin, i.e. parameterized by \( S(\gamma^k_n) \).

Denote \( U_i \) the set of \( k \)-half-spheres disjoint with \( V_i \). This set is compact and does not contain antipodal pairs. By Theorem 11 hind \( S(\gamma^k_n) = n - 1 \), and by the generalized Borsuk-Ulam theorem the sets \( U_i \) cannot cover \( S(\gamma^k_n) \), that is what we need. \( \square \)

Theorem 19. Let \( V_1, \ldots, V_{n+1} \) be open subsets of \( S^{n-1} \) such that each of \( V_i \) intersects any \( k \)-subsphere. Then either there exist a \( k \)-half-sphere that intersects every \( V_i \); or for any partition \([n+1] = I_1 \cup I_2\) into non-empty sets there exists a pair of \( k \)-half-spheres \( H_1 \) and \( H_2 \), being the complementary halves of one \( k \)-subsphere, such that \( V_i \cap H_1 = \emptyset \) for any \( i \in I_1 \), and \( V_i \cap H_2 = \emptyset \) for any \( i \in I_2 \).

Proof. As in the previous theorem, denote \( U_i \) the set of \( k \)-half-spheres disjoint with \( V_i \). Now the claim follows from Theorems 7 and 11. \( \square \)

8. Helly-type theorems for common transversals

Here we state several theorems, close to the Horn-Klee theorem and its generalizations from [8]. V.L. Dolnikov has some similar results (private communication) which are not published yet.

Theorem 20. Suppose \( n + 1 \) families of 1-convex compact sets \( \{ F_i \}_{i \in [n+1]} \) are given in \( \mathbb{R}^n \). Let any two sets of the same family have non-empty intersection. Then one of the alternatives holds.

1) The family \( \bigcup_{i \in [n+1]} F_i \) has \( n - 1 \)-transversal (a hyperplane);

2) For any partition of the index set \([n+1]\) into non-empty \( I_1 \) and \( I_2 \) there exists a hyperplane \( h \) and a set of representatives \( C_i \in F_i \) \((i \in [n+1])\) so that the sets \( \{ C_i \}_{i \in I_1} \) are on one side of \( h \), while the sets \( \{ C_i \}_{i \in I_2} \) are on the other side.

Proof. For any line \( l \in G^1_n \) denote \( \pi_l \) the orthogonal projection onto this line, and put

\[ V_i(l) = \bigcap_{C \in F_i} \pi_l(C). \]

These sets are nonempty, since for any \( i \) any two of the segments in \( \{ \pi_l(C) \}_{C \in F_i} \) have an intersection. It is clear that \( V_i(l) \) depend continuously on \( l \). Put \( V_i = \bigcup_{l \in G^1_n} V_i(l) \).

Apply Corollary 2 to the family \( \{ V_i \} \). The first alternative of Corollary 2 obviously corresponds to the first alternative of this theorem.

In the other case, for any partition \([n+1] = I_1 \cup I_2\) there exists a hyperplane \( h \), that separates \( \{ V_i \}_{i \in I_1} \) and \( \{ V_i \}_{i \in I_2} \). Consider the projection onto the line \( l \perp h \), take some directions as “left” and “right” on this line. Without loss of generality we can assume that \( \{ V_i \}_{i \in I_1} \) are to the left of \( \pi_l(h) \), and \( \{ V_i \}_{i \in I_2} \) are to the right of \( \pi_l(h) \). The right end of the segment \( V_i \) \((i \in I_1)\) is a right end of some \( \pi_l(C_i) \) \((C_i \in F_i)\), the left end of the segment \( V_i \) \((i \in I_2)\) is a left end of some \( \pi_l(C_i) \) \((C_i \in F_i)\). Hence \( \{ C_i \}_{i \in [n+1]} \) are the required system of representatives for the partition \([n+1] = I_1 \cup I_2\). \( \square \)
Theorem 21. Let $0 < k < n$ and suppose that $n - k + 1$ families $\{F_i\}_{i \in [n-k+1]}$ of convex compact sets are given in $\mathbb{R}^n$. Then one of the following alternatives holds.

1) There exists a system of representatives $K_i \in F_i$ such that $\bigcap_{i \in [n-k+1]} K_i = \emptyset$;
2) There exists $i \in [n-k+1]$ such that in $F_i$ any $k + 1$ or less sets have a $k-1$-transversal;
3) There exists a family of parallel $k$-flats $\{\alpha_i\}_{i \in [n-k+1]}$ such that for any $i \in [n-k+1]$ the flat $\alpha_i$ is a $k$-transversal for $F_i$.

If $2k \leq n$, then the third alternative is only possible in the case $k = 1$ or $k = 2$ and $n = 2^l$.

Proof. Suppose the first alternative does not hold. Take any $L \in G^{n-k}_n$ and consider the projection of all the families to $L$. By the colored Helly theorem for some $i$ the family $\pi_L(F_i)$ has a common point, this point corresponds to some $k$-transversal to $F_i$, orthogonal to $L$. Denote

$$U_i = \{L \in G^{k+}_n : \bigcap \pi_L(F_i) \neq \emptyset\}.$$

Suppose that the alternative (2) fails. Consider the subfamily (its cardinality should be exactly $k + 1$) $K_1, K_2, \ldots, K_{k+1} \in F_i$ that does not have a $k-1$-transversal. For any $L \in U_i$ take the corresponding $k$-transversal $\alpha$ for the family $F_i$, and compare the orientation on $\alpha$, given by any system of representatives $x_i \in K_i \cap \alpha$ ($i \in [k + 1]$) with the orientation of $\alpha$, corresponding to $L$. All the possible systems $(x_1, \ldots, x_{k+1})$ give the same orientation, since they are never contained in a single $k-1$-flat. If the orientations coincide, assign the sign ‘+’ to $L$ otherwise assign ‘−’ to it.

Thus the sets $U_i$ are mapped $Z_2$-equivariantly to $\{+1, -1\}$, and by Theorem 13 and Theorem 3 the sets $U_i$ should have a common point, that is equivalent to the alternative (3). Theorem 13 tells, that it is only possible when $k = 1$, or $k = 2$ and $n = 2^l$. □

In the case, when the number of families is small compared to $n$, Theorem 21 can be strengthened.

Theorem 22. Let $n = 2k + 1 \geq 3$ and suppose that 2 families $F_1, F_2$ of convex compact sets are given in $\mathbb{R}^n$. Then one of the following alternatives holds.

1) There exist two representatives $K_1 \in F_1, K_2 \in F_2$ such that $K_1 \cap K_2 = \emptyset$;
2) In some of the families $F_i$ any $k + 2$ or less sets have a $k$-transversal.

Theorem 23. Let $k > 2, 2k < n + 2$ and suppose that $k$ families $F_1, \ldots, F_k$ of convex compact sets are given in $\mathbb{R}^n$. Suppose that for some $m \leq n - k + 1$ the inequality

$$2^{\lceil \log_2 n \rceil} \geq k 2^{\lceil \log_2 (n-m) \rceil} + 2$$

holds. Then one of the following alternatives holds.

1) There is a system of representatives $K_1 \in F_1, \ldots, K_k \in F_k$ such that $\bigcap_{i=1}^k K_i = \emptyset$;
2) In some of the families $F_i$ any $m + 1$ or less sets have a $m-1$-transversal.

The inequality in the statement of Theorem 23 looks quite complicated, but it is true, for example, in the case $n \geq k(2n - 2m - 1) + 2$.

We are going to use the following lemma (Lemma 5.4 from [23]).
Lemma 4. For compact $Z_2$-invariant subsets $X$ and $Y$ of some free $Z_2$-space $Z$ the following inequality holds

$$\text{hind} (X \cup Y) \leq \text{hind} X + \text{hind} Y + 1.$$  

The following lemma generalizes the reasoning in the proof of Theorem 21.

Lemma 5. Let $k + 1 \leq m \leq n - 1$ and suppose that the family $\mathcal{F} = \{K_1, K_2, \ldots, K_{k+1}\}$ of convex compact sets in $\mathbb{R}^n$ has no $k - 1$-transversal. Then the set of oriented $m$-transversals for $\mathcal{F}$ can be $Z_2$-mapped to $G_{n-k}^{m-k+}$. 

Proof. Define a vector bundle $\eta \to K_1 \times \cdots \times K_{k+1}$ as follows. For any system of representatives $(x_1, \ldots, x_{k+1}) \in K_1 \times \cdots \times K_{k+1}$ the affine hull $L(x_1, \ldots, x_{k+1})$ has the dimension $k$, otherwise $\mathcal{F}$ would have a $k - 1$-transversal. The quotient space $M(x_1, \ldots, x_{k+1}) = \mathbb{R}^n/L(x_1, \ldots, x_{k+1})$ is an $n - k$-dimensional vector space, and together these vector spaces form the bundle $\eta$.

The space $K_1 \times \cdots \times K_{k+1}$ is contractible, hence any vector bundle over it is trivial. Fix some isomorphism of vector bundles

$$\phi : \eta \to \mathbb{R}^{n-k} \times K_1 \times \cdots \times K_{k+1},$$

and consider its composition with the projection to the first factor

$$\psi : \eta \to \mathbb{R}^{n-k}.$$

Let the set of oriented $m$-transversals for $\mathcal{F}$ be $T \subseteq \gamma_{n-m+}$. For any $m$-flat $\tau \in T$ we can choose the $k + 1$ points

$$x_1(\tau) \in \tau \cap K_1, x_2(\tau) \in \tau \cap K_2, \ldots, x_{k+1}(\tau) \in \tau \cap K_{k+1},$$

since the sets $K_i$ are strictly convex, the points $x_i$ may be chosen to depend continuously on $\tau$.

The image of $\tau$ under the natural map $\mathbb{R}^n \to M(x_1(t), \ldots, x_{k+1}(t))$ is an oriented $m - k$-dimensional subspace of $M(x_1(t), \ldots, x_{k+1}(t))$, and after the map $\psi : M \to \mathbb{R}^{n-k}$ it becomes an $m - k$-dimensional subspace in $\mathbb{R}^{n-k}$. Thus the required map of $T$ to $G_{n-k}^{m-k+}$ is defined. 

Proof of Theorem 22. It is sufficient to prove the theorem for strictly convex compact sets, from the compactness considerations.

Denote the set of oriented hyperplane transversals for $\mathcal{F}_i$ by $Y_i \subseteq \gamma_{n+1}^i$. Denote the natural projection of this set to $G_{k+}^{1+} = S^{n-1}$ by $X_i = \pi_\gamma (Y_i)$.

If the first alternative fails, then $X_1 \cup X_2 = S^{n-1}$ and by Lemma 4 for one of $X_i$ we have

$$\text{hind} X_i \geq k.$$ 

Now assume the contrary: there exist $k+2$ sets $K_1, \ldots, K_{k+2}$ in $\mathcal{F}_i$ without $k$-transversal.

By Lemma 5 the set $Y_i$ can be mapped equivariantly to $G_{k+}^{1+} = S^{k-1}$ by some map $f_i$. The natural projection $\pi_\gamma : Y_i \to X_i$ has segments as fibers, thus this map is an equivariant homotopy equivalence. Hence $\text{hind} X_i \leq k - 1$ and we obtain a contradiction with Lemma 1.

\[\square\]
Proof of Theorem 23. Again, consider the sets to be strictly convex.

Denote by $Y_i \subseteq \gamma^{k-1+}$ the set of $n-k+1$-transversals for $\mathcal{F}_i$, $X_i = \pi_\gamma(Y_i) \subseteq G^{n-1+}$ the corresponding set of directions. The projection $\pi_\gamma : Y_i \rightarrow X_i$ has a convex set as a fiber, and has a section $\tau_i : X_i \rightarrow Y_i$.

If the first alternative fails, by the colored Helly theorem for the projections of these families to $k-1$-dimensional subspaces, we obtain $\bigcup_{i=1}^k X_i = G^{n-k+1+}$.

If the second alternative fails, then by Lemma 5 each of $Y_i$ (and therefore $X_i$) can be equivariantly mapped to $G^{n-k-m+1+}$. By Lemma 1 and Theorem 13 we obtain

$$\text{hind} X_i \leq 2^{\lceil \log_2(n-m) \rceil} - 1.$$ 

Then by Lemma 2

$$\text{hind} G^{n-k+1+} \leq k2^{\lceil \log_2(n-m) \rceil} - 1.$$ 

But Theorem 13 gives an estimate $\text{hind} G^{n-k+1+} \geq 2^{\lceil \log_2 n \rceil} - 2$, that leads to the contradiction. □

9. Acknowledgments

The author thanks V.L. Dolnikov for the discussion and useful remarks on the presentation of these results.

References

[1] I. Bárány. A generalization of Carathéodory’s theorem. // Discrete Math., 40, 1982, 141–152.
[2] I. Bárány, A. Hubard, J. Jerónimo. Slicing convex sets and measures by a hyperplane. // Discrete and Computational Geometry, 39, 2008, 67–75.
[3] T. Bisztriczky. On separated families of convex bodies. // Arch. Math., 54, 1990, 193-199.
[4] K. Borsuk. Drei Sätze über die $n$-dimensionale euklidische Sphäre. // Fund. Math., 20, 1933, 177–190.
[5] F. Breuer. Uneven splitting of ham sandwiches. // Discrete and Computational Geometry, DOI 10.1007/s00454-009-9161-7.
[6] S.E. Cappell, J.E. Goodman, J. Pach, M. Sharir, R. Wenger. Common tangents and common transversals. // Adv. in Math., 106, 1994, 198-215.
[7] V.L. Dol’nikov. Transversals of families of sets in $\mathbb{R}^n$ and a connection between the Helly and Borsuk theorems (In Russian), // Sb., Math. 79(1), 1994, 93–107; translation from Mat. Sb., 184(5), 1993, 111–132.
[8] V.L. Dolnikov. Helly-type theorems for transversals and their applications. Doctor of Mathematics thesis. Yaroslavl’ State University, 2001.
[9] F. Harary. Graph Theory. Reading, MA, Addison-Wesley, 1994.
[10] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. // Jber Deutsch. Math. Verein., 32, 1923, 175–176.
[11] H.L. Hiller. On the cohomology of real grassmanian. // Trans. Amer. Math. Soc., 257(2), 1980, 521–533.
[12] H.L. Hiller. On the height of the first Stiefel-Whitney class. // Proc. Amer. Math. Soc., 79(3), 1980, 495–498.
[13] A. Horn. Some generalization of Helly’s theorem on convex sets. // Bull. Amer. Math. Soc., 55, 1949, 923–929.
[14] Wu Yi Hsiang. Cohomology theory of topological transformation groups. Berlin-Heidelberg-New-York, Springer Verlag, 1975.
R.N. Karasev. Colored version of the Knaster-Kuratowski-Mazurkiewicz lemma (In Russian). // Modelirovaniye i analiz informatsionnyh sistem, 13(2), 2006, 66–70.
16 R.N. Karasev. Colored versions of the Sperner theorem and the KKM theorem. // Third Russian-German Geometry Meeting dedicated to 95th birthday of A.D. Alexandrov, Abstracts, St.-Petersburg, Russia, June 18–23, 2007, 17–18.
17 V. Klee. On certain intersection properties of convex sets. // Canad. J. Math., 3, 1951, 272–275.
18 V. Klee, T. Lewis, B. Von Hohenbalken. Appollonius revisited: supporting spheres for sundered systems. // Discrete and Computational Geometry, 18, 1997, 385–395.
19 V.V. Makeev. Some extremal problems for vector bundles (In Russian). // St. Petersbg. Math. J., 19(2), 2008, 261–277; translation from Algebra Anal., 19(2), 2007, 131–155.
20 G.E. Shilov. Integral, Measure, and Derivative: A Unified Approach. Dover Publications, 1977.
21 A.H. Stone, J.W. Tukey. Generalized ‘Sandwich’ Theorems. // Duke Math. J., 9, 1942, 356–359.
22 H. Steinhaus. Sur la division des ensembles de l’espaces par les plans et des ensembles plans par les cercles. // Fund. Math., 33, 1945, 245–263.
23 A.Yu. Volovikov, E.V. Shchepin. Antipodes and embeddings (In Russian). // Sb. Math., 196(1), 2005, 1–28; translations from Mat. Sb., 196(1), 2005, 3–32.

E-mail address: r_n_karasev@mail.ru

Roman Karasev, Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700