EQUIVARIANT ONE-PARAMETER FORMAL DEFORMATIONS OF HOM-LEIBNIZ ALGEBRAS

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Abstract. We introduce an equivariant 1-parameter formal deformation theory of Hom-Leibniz algebras equipped with finite group actions on Hom-Leibniz algebras. We define a suitable equivariant deformation cohomology which controls deformations.

1. Introduction

Gerstenhaber in a series of papers [6–10] introduced the algebraic deformation theory for associative algebras. Later following Gerstenhaber, deformation theory of other algebraic structures are studied extensively in various work, see, [21], [5], [11], [14], [22]. For example, A. Nijenhuis and R. Richardson studied formal deformation theory of Lie algebras [21]. It is well-known that to study deformation theory of a type of algebra one needs to define a suitable cohomology, called, deformation cohomology which controls deformation in question. In the case of associative algebras, deformation cohomology is Hochschild cohomology and for Lie algebras, the associated deformation cohomology is Chevalley-Eilenberg cohomology.

Hartwig, Larsson, and Silvestrov introduced the notion of Hom-Lie algebras in [12]. Hom-Lie algebras appeared in examples of $q$-deformations of the Witt and Virasoro algebras. In [19], Makhlof and Zusmanovich described Hom-Lie algebra structures on affine Kac-Moody algebras. Hom-Lie algebras are a generalization of Lie algebras. In the definition of Hom-Lie algebra, one replaces the Jacobi identity by Hom-Jacobi identity, which is obtained by twisting Jacobi identity by a self linear map. Lie algebras are a special case of Hom-Lie algebras by considering self linear maps as identity.

In [13], J.-L. Loday introduced some new types of non anti-commutative version of Lie algebras along with their (co)homologies. In this new type of algebras, the bracket satisfies Leibniz identity instead of Jacobi identity and algebras are called Leibniz algebra. Similar to Hom-Lie algebras, Makhlof and Silvestrov introduced the notion of Hom-Leibniz algebras in [10]. Hom-Leibniz algebras are generalizations of both Leibniz and Hom-Lie algebras. In Hom-Leibniz algebras, Leibniz identity is twisted by a self linear map and it is called Hom-Leibniz identity. For various Hom-types of algebra one may refer, e.g., to [15], [16], [17], [18], [23].

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Makhlouf and Silvestrov studied one-parameter formal deformation for Hom-associative and Hom-Lie algebras together with associated deformation cohomology of the first and second order for Hom-associative and Hom-Lie algebras [17]. Cheng and Cai defined cohomology of all orders for Hom-Leibniz algebras in [3].

In this paper, we first discuss a one-parameter formal deformation for Hom-Leibniz algebras. The main aim of this paper is to study deformation theory of Hom-Leibniz algebras in the equivariant world. We define a concept of finite group actions on Hom-Leibniz algebras along the line of Bredon cohomology of a G-space, [2] and introduce an equivariant cohomology that controls equivariant formal deformation for Hom-Leibniz algebras equipped with an action of a finite group. Note that, an action of finite group $G$ on Hom-Leibniz algebra $L$ naturally extends to the formal power series $L[[t]]$. The Hom-Leibniz algebra $L$ is a submodule of $L[[t]]$ under the action of $G$, and we could make $L[[t]]$ an algebra by bilinearly extending the multiplication of $L$, and the induced $G$-action is preserved in the sense that the induced multiplication is equivariant.

The paper is organized as follows. In section 2 we recall the definition of Hom-Leibniz algebra and cohomology of Hom-Leibniz algebras. We also show that there is a Gerstenhaber bracket on cochains for Hom-Leibniz cohomology. In section 3 we introduce a one-parameter formal deformation of Hom-Leibniz algebras and discuss obstructions to extend a $n$-deformation to $(n+1)$-deformation. We also study rigidity conditions for formal deformations. In section 4 we define finite group actions on Hom-Leibniz algebras and introduce equivariant cohomology for Hom-Leibniz algebras equipped with the actions of a finite group. In section 5 we define equivariant formal deformations and showed that equivariant cohomology controls such deformations. In the final section 6 we discuss the rigidity of equivariant deformation for Hom-Leibniz algebras equipped with an action of a finite group.

2. Cohomology of Hom-Leibniz Algebras

In this section, we recall the definition and cohomology of Hom-Leibniz algebras. We show that there is a Gerstenhaber bracket on cochains for Hom-Leibniz cohomology.

**Definition 2.1.** A Hom-Leibniz algebra is a $K$-linear vector space $L$ together with a $K$-bilinear map $[., .] : L \times L \to L$ and a $K$-linear map $\alpha : L \to L$ satisfying Hom-Leibniz identity:

$$[[\alpha(x), [y, z]] - [[x, y], \alpha(z)] - [[x, z], \alpha(y)].$$

A Hom-Leibniz algebra $(L_1, [., .], \alpha)$ is called multiplicative if $K$-linear map $\alpha$ satisfies $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, that is, $\alpha$ is an algebra morphism.
A morphism between Hom-Leibniz algebra \((L_1, [,], \alpha_1)\) to \((L_2, [,], \alpha_2)\) is a \(K\)-linear map \(\phi : L_1 \to L_2\) which satisfies \(\phi([x, y]_1) = [\phi(x), \phi(y)]_2\) and \(\phi \circ \alpha_1 = \alpha_2 \circ \phi\).

**Example 2.2.** Any Hom-Lie algebra is a Hom-Leibniz algebra as in the presence of skew-symmetry Hom-Leibniz identity is same as Hom-Jacobi identity.

**Example 2.3.** Given a Leibniz algebra \((L, [,])\) and a Leibniz algebra morphism \(\alpha : L \to L\), one always get a Hom-Leibniz algebra \((L, [,], \alpha, \alpha)\), where \([,]_\alpha = \alpha \circ [,]\).

**Example 2.4.** Let \(L\) is a two-dimensional \(\mathbb{C}\)-vector space with basis \(\{e_1, e_2\}\). We define a bracket operation as \([e_2, e_2] = e_1\) and zero else where and the endomorphism is given by the matrix

\[
\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

\((L, [,], \alpha)\) is a Hom-Leibniz algebra which is not Hom-Lie.

**Definition 2.5.** A Hom-vector space is a \(K\)-vector space \(M\) together with a \(K\)-linear map \(\beta : M \to M\) such that vector space operations are compatible with \(\beta\). We write a Hom-vector space as \((M, \beta)\).

**Definition 2.6.** Let \((L, [,], \alpha)\) be a Hom-Leibniz algebra. A \(L\)-bimodule is a Hom-vector space \((M, \beta)\) together with \(K\)-linear maps (left and right multiplications), \(m_l : L \otimes M \to M\) and \(m_r : M \otimes L \to M\) such that \(m_l, m_r\) satisfies Hom-Leibniz identity with respect to \(\beta\) and the following diagram commutes:

\[
\begin{array}{ccc}
L \otimes L \otimes M & \xrightarrow{[\cdot,\cdot] \otimes \beta} & L \otimes M \\
\downarrow \alpha \otimes m_l & & \downarrow m_l \\
L \otimes M & \xrightarrow{m_l} & M \\
\end{array}
\quad \quad \begin{array}{ccc}
M \otimes L \otimes L & \xrightarrow{\beta \otimes [\cdot]} & M \otimes L \\
\downarrow m_r \otimes \alpha & & \downarrow m_r \\
M \otimes L & \xrightarrow{m_r} & M \\
\end{array}
\]

Note that, any Hom-Leibniz algebra \((L, [,], \alpha)\) can be considered as a bimodule over itself by taking \(m = [,]\) and \(\beta = \alpha\). We now recall cohomology of Hom-Leibniz algebra \((L, [,], \alpha)\) with coefficient over itself, \([3]\).

Let \((L, [,], \alpha)\) be a Hom-Leibniz algebra. For \(n \geq 1\), suppose \(\phi : L^n \to L\) is an \(n\)-linear map satisfying,

\[\alpha \circ \phi(x_0, \ldots, x_{n-1}) = \phi(\alpha(x_0), \ldots, \alpha(x_{n-1}))\text{ for all }x_0, \ldots, x_{n-1} \in L.\]

Let \(C^n_{HL}(L, L)\) denotes a \(K\)-vector space whose elements are \(n\)-linear maps \(\phi : L \to L\) satisfying the above condition.
Proposition 2.7. For $n \geq 1$, we define $\delta^n : C^n_{HL}(L, L) \to C^{n+1}_{HL}(L, L)$ as follows,
\[
\delta^n \phi(x_0, \ldots, x_n) := [\alpha^{n-1}(x_0), \phi(x_1, \ldots, x_n)] + \sum_{i=1}^n (-1)^{i+1} [\phi(x_0, \ldots, \hat{x}_i, \ldots, x_n), \alpha^{n-1}(x_i)] + \sum_{0 \leq i < j \leq n} (-1)^{i+j} \phi(\alpha(x_0), \ldots, \alpha(x_i), [x_i, x_j], \alpha(x_{i+1}), \ldots, \hat{x}_j, \ldots, \alpha(x_n)).
\]
For $n = 1$ and for all $x, y \in L$, from the definition of $\delta^n$, we have
\[
\delta^1 \phi(x, y) = [x, \phi(y)] + [\phi(x), y] - \phi([x, y]).
\]
For $n = 2$ and for all $x, y, z \in L$, from the definition of $\delta^n$, we have
\begin{align*}
(1) \quad \delta^2 \phi(x, y, z) &= [\alpha(x), \phi(y, z)] + [\phi(x, z), \alpha(y)] - [\phi(x, y), \alpha(z)] \\
(2) \quad - \phi([x, y], \alpha(z)) + [\phi([x, z], \alpha(y)] + \phi([y, z], \alpha(x)).
\end{align*}

Proposition 2.7. For all $n \geq 1$, $\delta^{n+1} \circ \delta^n = 0$.

Proof. We show the result for $n = 1$, that is, $\delta^2 \circ \delta^1 = 0$. Let $x, y, z \in L$.
\[
\delta^2(\delta^1 \phi)(x, y, z) = [\alpha(x), (\delta^1 \phi)(y, z)] + [(\delta^1 \phi)(x, z), \alpha(y)] - [(\delta^1 \phi)(x, y), \alpha(z)] \\
- (\delta^1 \phi)([x, y], \alpha(z)) + (\delta^1 \phi)([x, z], \alpha(y)) + (\delta^1 \phi)(\alpha(x), [y, z]) \\
= [\alpha(x), [y, \phi(z)]] + [\alpha(x), [\phi(y), z]] + [\alpha(x), \phi([y, z])] \\
+ [[x, \phi(z)], \alpha(y)] + [[\phi(x), z], \alpha(y)] - [\phi([x, z]), \alpha(y)] \\
- [[x, \phi(y), \alpha(z)] - [[\phi(x), y], \alpha(z)] + [\phi([x, y]), \alpha(z)] \\
- [[x, y], \phi(\alpha(z))] - [[\phi(x), y], \alpha(z)] + [\phi([x, y]), \alpha(z)] \\
+ [[x, z], \phi(\alpha(y))] + [\phi([x, z]), \alpha(y)] - \phi([x, z], \alpha(y)) \\
+ [\alpha(x), \phi([y, z])]] + [\phi(\alpha(x)), [y, z]] - \phi(\alpha(x), [y, z]).
\]

The general result follows from induction on $n$, see section (5) of [3]. □

The space of $n$-coboundaries is defined as
\[
B^n_{HL}(L, L) = \{ \phi \in C^n_{HL}(L, L) : \phi = \delta^{n-1} f, f \in C^{n-1}_{HL}(L, L) \}.
\]
The space of $n$-cocycles is defined as
\[
Z^n_{HL}(L, L) = \{ \phi \in C^n_{HL}(L, L) : \delta^n \phi = 0 \}.
\]
As $\delta^{n+1} \circ \delta^n = 0$, we have $B^n_{HL}(L, L) \subseteq Z^n_{HL}(L, L)$. We define a $n$ th cohomology group of $(L, [., .], \alpha)$ as
\[
H^n_{HL}(L, L) = \frac{Z^n_{HL}(L, L)}{B^n_{HL}(L, L)}.
\]
2.1. Gerstenhaber bracket on cochains for Hom-Leibniz Cohomology. We define a Gerstenhaber bracket on cochains for Hom-Leibniz algebras cohomology and show that this bracket induce a graded Lie algebra structure on cohomology of Hom-Leibniz algebras.

Suppose for $n \geq -1, CH^n(L, L)$ be the space of all $(n + 1)$-linear maps $\phi : L^{n+1} \to L$ satisfying $\alpha \circ \phi = \phi \circ \alpha$.

Let $\phi \in CH^p(L, L)$ and $\psi \in CH^q(L, L)$, where $p \geq 0, q \geq 0$, we define $\psi \circ \phi \in CH^{p+q}(L, L)$ as follows:

$$\psi \circ \phi(x_0, \ldots, x_{p+q}) = \sum_{k=0}^{q} (-1)^{pk} \sum_{\sigma \in S_h(p, q-k)} sgn(\sigma) \psi(\alpha^p(x_0), \ldots, \alpha^p(x_{k-1}),$$

$$\phi(x_k, x_{\sigma(k+1)}, \ldots, x_{\sigma(k+p)}), \alpha^p(x_{\sigma(k+p+1)}), \ldots, \alpha^p(x_{\sigma(p+q)})).$$

Suppose $CH^*(L, L) = \bigoplus_{n \geq -1} CH^n(L, L)$.

We define a bracket $[., .]$ on $CH^*(L, L)$ as $[\psi, \phi] = \psi \circ \phi + (-1)^{pq+1} \phi \circ \psi$.

**Remark 2.8.** For $p = q = 1$,

$$\psi \circ \phi(x_0, x_1, x_2)$$

$$= \psi(\alpha(x_0), \phi(x_1, x_2)) - \psi(\phi(x_0, x_1), \alpha(x_2)) + \psi(\phi(x_0, x_2), \alpha(x_1)) = \psi \circ \alpha \phi.$$

For $\phi = \psi$, this is nothing but $\alpha$ associator of $\phi$.

**Proposition 2.9.** Suppose $m_1 \in CH^1(L, L)$ then $(L, m_1, \alpha)$ is a Hom-Leibniz algebra if and only if $[m_1, m_1] = 0$.

**Proof.**

$$[m_1, m_1] = 2(m_1(\alpha(x_0), m_1(x_1, x_2))$$

$$- m_1(m_1(x_0, x_1), m_1(x_2)) + m_1(m_1(x_0, x_2), \alpha(x_1))).$$

Thus from this formula it is clear that $(L, m_1, \alpha)$ is a Hom-Leibniz algebra if and only if $[m_1, m_1] = 0$. □

**Lemma 2.10.** Let $(L, m_0, \alpha)$ be a Hom-Leibniz algebra and $\phi \in CH^p(L, L)$, then $\delta \phi = -[\phi, m_0]$, where $m_0 = [\ . \ ]$ is the Leibniz bracket on $L$.

**Proof.** Let $x_0, x_1, \ldots, x_{p+1} \in L$. As $\delta \phi \in CH^{p+1}(L, L)$, from the coboundary formula we have,

$$\delta \phi(x_0, \ldots, x_{p+1})$$

$$= [\alpha^p(x_0), \phi(x_1, \ldots, x_{p+1})]$$

$$+ \sum_{i=1}^{p+1} (-1)^{i+1} \phi(x_0, \ldots, \hat{x_i}, \ldots, x_{p+1}), \alpha^p(x_i)$$

$$+ \sum_{0 \leq i < j \leq p+1} (-1)^{i+j} \phi(\alpha(x_0), \ldots, \alpha(x_{i-1}), [x_i, x_j], \alpha(x_{i+1}), \ldots, \hat{x_j}, \ldots, \alpha(x_{p+1})).$$
Now, $[\phi, m_0] = \phi \circ m_0 + (-1)^{p+1} m_0 \circ \phi$. Note that, $m_0 \in CH^1(L, L)$. Thus

$$\phi \circ m_0(x_0, \ldots, x_{p+1}) = \sum_{k=0}^{p} (-1)^k \{ \sum_{\sigma \in Sh(1, p-k)} sgn(\sigma) \phi(\alpha(x_0), \ldots, \alpha(x_{k-1}), [x_{k, x_{\sigma(k+1)}}, \alpha(x_{\sigma(k+2)}), \ldots, \alpha(x_{\sigma(p+1)})].$$

$$= \sum_{0 \leq k < j \leq (p+1)} (-1)^{j+1} \phi(\alpha(x_0), \ldots, \alpha(x_{k-1}), [x_{k, x_j}], \alpha(x_{j-1}), \hat{x}_j, \alpha(x_{j+1}), \ldots, \alpha(x_{(p+1)})),$$

And

$$m_0 \circ \phi(x_0, \ldots, x_{p+1})$$

$$= \sum_{\sigma \in Sh(p, 1)} sgn(\sigma) m_0(\phi(x_0, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}), \alpha^p(\alpha_{\sigma(p+1)})$$

$$+ (-1)^p [\alpha^p(x_0), \phi(x_1, \ldots, x_{p+1})]$$

$$= \sum_{1 \leq j \leq p+1} (-1)^{p+1-j} [\phi(x_0, x_1, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+1}), \alpha^p(x_j)]$$

$$+ (-1)^p [\alpha^p(x_0), \phi(x_1, \ldots, x_{p+1})].$$

Therefore,

$$[\phi, m_0] = (\phi \circ m_0 + (-1)^{p+1} m_0 \circ \phi)(x_0, \ldots, x_{p+1})$$

$$= \sum_{0 \leq k < j \leq (p+1)} (-1)^{j+1} \phi(\alpha(x_0), \ldots, \alpha(x_{k-1}), [x_{k, x_j}], \alpha(x_{j-1}), \hat{x}_j, \alpha(x_{j+1}), \ldots, \alpha(x_{(p+1)})$$

$$\alpha(x_{j-1}), \hat{x}_j, \alpha(x_{j+1}), \ldots, \alpha(x_{(p+1)})$$

$$+ \sum_{1 \leq j \leq p+1} (-1)^{j+2} [\phi(x_0, x_1, \ldots, x_{j-1}, \hat{x}_j, x_{j+1}, \ldots, x_{p+1}), \alpha^p(x_j)]$$

$$- [\alpha^p(x_0), \phi(x_1, \ldots, x_{p+1})] = -\delta \phi.$$

Thus, we have $\delta \phi = -[\phi, m_0].$ \(\square\)

Now, the graded $\mathbb{K}$-module $CH^*(L, L) = \bigoplus_p CH^p(L, L)$ together with the bracket $[\psi, \phi] = \psi \circ \phi + (-1)^{p+1} \phi \circ \psi$ is a graded Lie algebra, \([1]\).

If $\phi \in CH^p(L, L)$, then we define $|\phi| = p + 1$. We define a linear map $d : CH^*(L, L) \to CH^*(L, L)$ as follows:

$$d(\phi) = (-)^{|\phi|} \delta(\phi).$$

Using the graded Lie algebra structure on $CH^*(L, L)$, we prove the following lemma.

**Lemma 2.11.** The differential $d$ satisfies the following graded derivation formula:

$$d[\psi, \phi] = [d\psi, \phi] + (-1)^{|\psi|}[\psi, d\phi].$$

Here, $\psi, \phi \in CH^*(L, L).$
Proof. Let $\phi \in CH^p(L, L)$ and $\psi \in CH^q(L, L)$. To prove this lemma, we use property of graded Lie algebra.

$$
d[\psi, \phi] = (-1)^{p+q+1}\delta[\psi, \phi]$$

$$= -(-1)^{p+q+1}[[\psi, \phi], m_0]$$

$$= -(-1)^{p(q+1)}[\phi, [m_0, \psi]] - (-1)^p[\psi, [\phi, m_0]]$$

$$= (-1)^{q+1}[[m_0, \psi], \phi] + (-1)^q[\psi, d\phi]$$

$$= [(-1)^q\delta\psi, \phi] + (-1)^q[\psi, d\phi]$$

$$= [d\psi, \phi] + (-1)^q[\psi, d\phi].$$

□

From the lemma (2.11), $(CH^*(L, L), [., .], d)$ is a differential graded Lie algebra. The cohomology group of $(CH^*(L, L), [., .], d)$ is denoted by $(H^*_{HL}(L, L))$. It is clear from the lemma (2.11) that the bracket induces a bracket $[., .]$ on the cohomology level and we have the following theorem.

**Theorem 2.12.** $(H^*_{HL}(L, L), [., .])$ is a graded Lie algebra.

### 3. One-parameter formal deformation of Hom-Leibniz algebras

In this section, We define a concept of one-parameter formal deformation for Hom-Leibniz algebras. Let $(L, [., .], \alpha)$ be a Hom-Leibniz algebra. Suppose $\mathbb{K}[[t]]$ be the formal power series ring in one variable $t$ with coefficient in $\mathbb{K}$ and $L[[t]]$ be the set of formal power series whose coefficients are the elements of $L$.

**Definition 3.1.** An one-parameter formal deformation of Hom-Leibniz algebra $(L, [., .], \alpha)$ is given by a $\mathbb{K}[[t]]$-bilinear map $m_t : L[[t]] \times L[[t]] \rightarrow L[[t]]$ and a $\mathbb{K}[[t]]$-linear map $\alpha_t : L[[t]] \rightarrow L[[t]]$ of the form

$$m_t = \sum_{i \geq 0} m_i t^i \quad \text{and} \quad \alpha_t = \sum_{i \geq 0} \alpha_i t^i,$$

such that,

1. For all $i \geq 0$, $m_i : L \times L \rightarrow L$ is a $\mathbb{K}$-bilinear map and $\alpha_i : L \rightarrow L$ is a $\mathbb{K}$-linear map.
2. $m_0(x, y) = [x, y]$ is the original bracket on $L$ and $\alpha_0 = \alpha$ is the original twisting $\mathbb{K}$-linear map.
3. $(L[[t]], m_t, \alpha_t)$ satisfies the Hom-Leibniz identity, that is,

$$m_t(\alpha_t(x), m_t(y, z)) = m_t(m_t(x, y), \alpha_t(z)) - m_t(m_t(x, z), \alpha_t(y)).$$
Condition (3) in the last definition is equivalent to

\[ \sum_{i+j+k=n} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)) = 0. \]

For a Hom-Leibniz algebra \((L, [\cdot, \cdot], \alpha)\), a \(\alpha_j\)-associator is a map,

\[ \text{Hom}(L^{\times 2}, L) \times \text{Hom}(L^{\times 2}, L) \to \text{Hom}(L^{\times 3}, L), \]

\[ (m_i, m_k) \mapsto m_i \circ_{\alpha_j} m_k, \]

defined as

\[ m_i \circ_{\alpha_j} m_k(x, y, z) = m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)). \]

By using \(\alpha_j\)-associator, the deformation equation may be written as

\[ \sum_{i,j,k \geq 0} (m_i \circ_{\alpha_j} m_k) t^{i+j+k} = 0, \]

\[ \sum_{n \geq 0} \left( \sum_{j=0}^{n-j} \sum_{i=0}^{n-j} m_i \circ_{\alpha_j} m_{n-j-i} \right) t^n = 0. \]

Thus, for \(n = 0, 1, 2, \cdots\) we have the following infinite equations:

\[ \sum_{j=0}^{n} \sum_{i=0}^{n-j} m_i \circ_{\alpha_j} m_{n-j-i} = 0. \]

For \(n = 0\),

\[ m_0 \circ_{\alpha_0} m_0 = 0, \]

\[ m_0(\alpha_0(x), m_0(y, z)) - m_0(m_0(x, y), \alpha_0(z)) + m_0(m_0(x, z), \alpha_0(y)) = 0, \]

\[ [\alpha(x), [y, z]] - [[x, y], \alpha(z)] + [[x, z], \alpha(y)] = 0. \]

This is the original Hom-Leibniz relation.

For \(n = 1\), from the equation (4) we have,

\[ m_0 \circ_{\alpha_0} m_1 + m_1 \circ_{\alpha_0} m_0 + m_0 \circ_{\alpha_1} m_0 = 0, \]

\[ [\alpha(x), m_1(y, z)] - [m_1(x, y), \alpha(z)] + [m_1(x, z), \alpha(y)] + m_1(\alpha(x), [y, z]) - m_1([x, y], \alpha(z)) + m_1([x, z], \alpha(y)) = 0. \] This is same as \(\delta^2 m_1(x, y, z) = 0\).

**Definition 3.2.** A 2-cochain \(m_1\) is called an infinitesimal of the deformation \(m_t\). Suppose more generally that, \(m_n\) is the first non-zero term of \(m_t\) after \(m_0\), such \(m_n\) is called a \(n\)-infinitesimal of the deformation.

**Proposition 3.3.** Let \((L, [\cdot, \cdot], \alpha)\) is a Hom-Leibniz algebra and \((L_t, m_t, \alpha_t)\) be its one-parameter deformation then the infinitesimal of the deformation is a 2-cocycle of Hom-Leibniz cohomology.
Proof. Suppose $m_n$ is a $n$-infinitesimal of the deformation $m_t$. So, we have,

$m_i = 0$ for $0 < i < n$. From the equation (4), we have-

$\alpha(x, m_n(x, y), \alpha(z)) - [m_n(x, y), \alpha(z)] = m_n(x, y, \alpha(z)) = 0,
\delta^2 m_n(x, y, z) = 0.$

Thus, an infinitesimal of the deformation is a 2-cocycle of Hom-Leibniz cohomology.

3.1. Obstructions of deformations.

Definition 3.4. A $n$-deformation of a Hom-Leibniz algebra is a formal deformation of the form-

$m_t = \sum_{i=0}^{n} m_i t^i.$

$m_t$ satisfies the Hom-Leibniz identity,

$m_t(\alpha_t(x), m_t(y, z)) = m_t(m_t(x, y), \alpha_t(z)) - m_t(m_t(x, z), \alpha_t(y)).$

We say a $n$-deformation $m_t$ of a Hom-Leibniz algebra is extendable to a $(n + 1)$-deformation if there is an element $m_{n+1} \in C_{HL}^{n+1}(L, L)$ such that

$\bar{m}_t = m_t + m_{n+1} t^{n+1}$

and $\bar{m}_t$ satisfies all the conditions of formal deformations.

Now, we can rewrite the equation (3) in the following form using Hom-Leibniz cohomology.

$(5) \quad \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)) = 0.$

This is same as the following equation-

$\delta^2 m_{n+1}(x, y, z) = - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z))
- m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y))
= - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i \circ \alpha_j m_k.$
We define $n$th obstruction to extend a deformation of Hom-Leibniz algebra of order $n$ to order $n+1$ as

\begin{equation}
\text{Obs}_n := \sum_{i+j+k=n+1, i,j,k \geq 0} m_i \circ \alpha_j m_k = \delta^2 m_{n+1}.
\end{equation}

Note that, $\text{Obs}_n \in C^0_{HL}(L,L)$ as well as $\text{Obs}_n$ is a coboundary.

**Theorem 3.5.** A deformation of order $n$ extends to a deformation of order $n+1$ if and only if cohomology class of $\text{Obs}_n$ vanishes.

**Proof.** Suppose a deformation $m_t$ of order $n$ extends to a deformation of order $n+1$. From the obstruction equation (6), we have,

$$\text{Obs}_n = \sum_{i+j+k=n+1, i,j,k \geq 0} m_i \circ \alpha_j m_k = \delta^2 m_{n+1}.$$ 

As $\delta \circ \delta = 0$, we get cohomology class of $\text{Obs}_n$ vanishes.

Conversely, suppose cohomology class of $\text{Obs}_n$ vanishes, that is,

$$\text{Obs}_n = \delta^2(m_{n+1}),$$

for some 2-cochains $m_{n+1}$. We define a map $m'_t$ extending the deformation $m_t$ of order $n$ as follows-

$$m'_t = m_t + m_{n+1}t^{n+1}.$$ 

The map $m'_t$ satisfies the following equation for all $x, y, z \in L$.

$$\sum_{i+j+k=n+1, i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)) = 0.$$ 

Thus, $m'_t$ is a deformation of order $n+1$ which extends the deformation $m_t$ of order $n$. \qed

**Corollary 3.6.** If $H^3_{HL}(L,L) = 0$ then any 2-cocycle gives an one-parameter formal deformation of $(L,[\,\cdot\,],[\,\cdot\,],\alpha)$.

### 3.2. Equivalent and trivial deformations

Suppose $L_t = (L, m_t, \alpha_t)$ and $L'_t = (L, m'_t, \alpha'_t)$ be two one-parameter Hom-Leibniz deformation of $(L,[\,\cdot\,],[\,\cdot\,],\alpha)$, where $m_t = \sum_{i \geq 0} m_i t^i$, $\alpha_t = \sum_{i \geq 0} \alpha_i t^i$ and $m'_t = \sum_{i \geq 0} m'_i t^i$, $\alpha'_t = \sum_{i \geq 0} \alpha'_i t^i$.

**Definition 3.7.** Two deformation $L_t$ and $L'_t$ are said to be equivalent if there exists a $\mathbb{K}[[t]]$-linear isomorphism $\Psi_t : L[[t]] \to L[[t]]$ of the form $\Psi_t = \sum_{i \geq 0} \psi_t^i t^i$, where $\psi_0 = Id$ and $\psi_t : L \to L$ are $\mathbb{K}$-linear map such that the following relations holds:

\begin{equation}
\Psi_t \circ m'_t = m_t \circ (\Psi_t \otimes \Psi_t) \quad \text{and} \quad \alpha_t \circ \Psi_t = \Psi_t \circ \alpha'_t.
\end{equation}

**Definition 3.8.** A deformation $L_t$ of a Hom-Leibniz algebra $L$ is called trivial if $L_t$ is equivalent to $L$. A Hom-Leibniz algebra $L$ is called rigid if it has only trivial deformation upto equivalence.
Condition (17) may be written as
\[ \Psi_t(m'_t(x, y)) = m_t(\Psi_t(x), \Psi_t(y)) \] and \[ \alpha_t(\Psi_t(x)) = \Psi_t(\alpha'_t(x)), \quad \forall x, y \in L \]

The above conditions is equivalent to the following equations:
\begin{align*}
(8) & \quad \sum_{i \geq 0} \psi_i \left( \sum_{j \geq 0} m'_j(x, y) t^j \right) t^i = \sum_{i \geq 0} m_i \left( \sum_{j \geq 0} \psi_j(x) t^j, \sum_{k \geq 0} \psi_k(y) t^k \right) t^i, \\
(9) & \quad \sum_{i \geq 0} \alpha_i \left( \sum_{j \geq 0} \psi_j(x) t^j \right) t^i = \sum_{i \geq 0} \psi_i \left( \sum_{j \geq 0} \alpha'_j(x) t^j \right) t^i.
\end{align*}

This is same as the following equations:
\begin{align*}
(10) & \quad \sum_{i,j \geq 0} \psi_i(m'_j(x, y)) t^{i+j} = \sum_{i,j,k \geq 0} m_i(\psi_j(x), \psi_k(y)) t^{i+j+k}, \\
(11) & \quad \sum_{i,j \geq 0} \alpha_i(\psi_j(x)) t^{i+j} = \sum_{i,j \geq 0} \psi_i(\alpha'_j(x)) t^{i+j}.
\end{align*}

Comparing constant terms on both sides of the above equations we have,
\begin{align*}
m'_0(x, y) & = m_0(x, y), \quad \text{as} \quad \psi_0 = Id, \\
\alpha_0(x) & = \alpha'_0(x).
\end{align*}

Now, comparing coefficients of \( t \) we have,
\begin{align*}
(12) & \quad m'_1(x, y) + \psi_1(m'_0(x, y)) = m_1(x, y) + m_0(\psi_1(x), y) + m_0(x, \psi(y)), \\
(13) & \quad \alpha_1(x) + \alpha_0(\psi_1(x)) = \alpha'_1(x) + \psi_1(\alpha'_0(x)).
\end{align*}

The equation (12) is same as
\[ m'_1(x, y) - m_1(x, y) = [\psi_1(x), y] + [x, \psi_1(y)] - \psi_1([x, y]) = \delta m_1(x, y). \]

**Proposition 3.9.** An infinitesimal of two equivalent deformation determines same cohomology class.

**Proof.** Suppose \( L_t = (L, m_t, \alpha_t) \) and \( L'_t = (L, m'_t, \alpha'_t) \) be two equivalent one-parameter Hom-Leibniz deformation of \( (L, [\cdot, \cdot], \alpha) \). Suppose \( m_n \) and \( m'_n \) be two infinitesimal of the deformation \( m_t \) and \( m'_t \) respectively. Using equation (10) we get,
\begin{align*}
m'_n(x, y) + \psi_n(m'_0(x, y)) & = m_n(x, y) + m_0(\psi_n(x), y) + m_0(x, \psi_n(y)), \\
m'_n(x, y) - m_n(x, y) & = m_0(\psi_n(x), y) + m_0(x, \psi_n(y)) - \psi_n(m'_0(x, y)), \\
m'_n(x, y) - m_n(x, y) & = [\psi_n(x), y] + [x, \psi_n(y)] - \psi_n([x, y]) = \delta m_n(x, y).
\end{align*}

Thus an infinitesimal of two deformation determines same cohomology class.

**Theorem 3.10.** A non-trivial deformation of a Hom-Leibniz algebra is equivalent to a deformation whose infinitesimal is not a coboundary.
Proof. Let \((L_t, m_t, \alpha_t)\) be a deformation of Hom-Leibniz algebra \(L\) and \(m_n\) be the \(n\)-infinitesimal of the deformation for some \(n \geq 1\). Then by proposition \((3.3)\) \(m_n\) is a 2-cocycle, that is, \(\delta^2 m_n = 0\). Suppose \(m_n = -\delta \phi_n\) for some \(\phi_n \in C^1_{HL}(L, L)\), that is, \(m_n\) is a coboundary. We define a formal isomorphism \(\Psi_t\) of \(L[[t]]\) as follows:

\[
\Psi_t(a) = a + \phi_n(a)t^n
\]

We set \(\bar{m}_t = \Psi_t^{-1} \circ m_t \circ (\Psi_t \otimes \Psi_t)\) and \(\bar{\alpha}_t = \Psi_t^{-1} \circ \alpha_t \circ \Psi_t\). Thus we have a new deformation \(\bar{L}_t\) which is isomorphic to \(L_t\). By expanding the above equation and comparing coefficients of \(t^n\), we get,

\[
\bar{m}_n - m_n = \delta(\phi_n).
\]

So, \(\bar{m}_n = 0\). By repeating this argument, we can kill off any infinitesimal which is a coboundary. Thus, the process must be stopped if the deformation is non-trivial. \(\square\)

Corollary 3.11. Let \((L, [.,.], \alpha)\) be a Hom-Leibniz algebra. If \(H^2_{HL}(L, L) = 0\) then \(L\) is rigid.

4. Group action and equivariant cohomology

The notion of a finite group actions on a Leibniz algebra was introduced by authors in [20]. In this section, we introduce a notion of a finite group action on Hom-Leibniz algebras. We also define an equivariant cohomology of Hom-Leibniz algebras equipped with an actions of a finite group.

Definition 4.1. Let \(G\) be a finite group and \((L, [.,.], \alpha)\) is a Hom-Leibniz algebra. We say group \(G\) acts on the Hom-Leibniz algebra \(L\) from left if there is a function \(\Phi : G \times L \to L\), satisfying-

1. For each \(g \in G\), the map \(\psi_g : L \to L, x \mapsto gx\) is a \(K\)-linear map.
2. \(ex = x\), where \(e\) denotes identity elements of the group \(G\).
3. For all \(g_1, g_2 \in G\) and \(x \in L\), \((g_1g_2)x = g_1(g_2x)\).
4. For all \(g \in G\) and \(x, y \in L\), \([gx, y] = [g x, y]\) and \(\alpha(gx) = g \alpha(x)\).

We write a Hom-Leibniz algebra \((L, [.,.], \alpha)\) equipped with a finite group action \(G\) as \((G, L, [.,.], \alpha)\). An alternative way to present the above definition is the following:

Proposition 4.2. Let \(G\) be a finite group and \((L, [.,.], \alpha)\) is a Hom-Leibniz algebra. \(G\) acts on \(L\) from left if and only if there is a group homomorphism \(\Psi : G \to Iso_{Hom-Leib}(L), g \mapsto \psi_g\), where \(Iso_{Hom-Leib}(L)\) denotes group of ismorphisms of Hom-Leibniz algebras from \(L\) to \(L\).
Let $M, M'$ be Hom-Leibniz algebras equipped with an action of a group $G$. We say a $\mathbb{K}$-linear map $f : M \to M'$ is equivariant if for all $g \in G$ and $x \in M$, $f(gx) = gf(x)$. We write a set of all equivariant maps from $M$ to $M'$ as $\text{Hom}_{\mathbb{K}}^G(M, M')$.

A $G$-Hom-vector space is a Hom-vector space $(M, \beta)$ such that there is a group action $G$ on $M$ and $\beta : M \to M$ is an equivariant map. We denote an equivariant Hom-vector space as triple $(G, M, \beta)$.

**Example 4.3.** Any $G$-Hom-vector space $(G, M, \beta)$ together with the trivial bracket (i.e. $[x, y] = 0$ for all $x, y \in M$) is a Hom-Leibniz algebra equipped with an action of $G$.

**Example 4.4.** Let $V$ be $\mathbb{K}$-module which is a representation space of a finite group $G$. On

$$\tilde{T}(V) = V \oplus V \otimes 2 \oplus \cdots \oplus V \otimes n \oplus \cdots$$

there is a unique bracket that makes it into a Hom-Leibniz algebra by taking $\alpha = \text{Id}$ and verifies

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = \cdots [v_1, v_2], v_3], \cdots, v_n]$$

for $v_i \in V$ and $i = 1, \cdots, n$.

This is the free Hom-Leibniz algebra over the $\mathbb{K}$-module $V$. The linear action of $G$ on $V$ extends naturally to an action on $\tilde{T}(V)$.

**Definition 4.5.** Let $(G, L, [\_\_], \alpha)$ be a Hom-Leibniz algebra equipped with an action of a finite group $G$. A $G$-bimodule over $L$ is a $G$-Hom-vector space $(G, M, \beta)$ together with an equivariant $\mathbb{K}$-linear maps (left and right multiplications), $m_l : L \otimes M \to M$ and $m_r : M \otimes L \to M$ such that $m_l, m_r$ satisfies Hom-Leibniz identity with respect to $\beta$ and the following diagram commutes:

$$\begin{array}{ccc}
L \otimes L \otimes M & \xrightarrow{[\_\_]} & L \otimes M \\
| & \alpha \otimes m_l & |
\downarrow m_l & \alpha \otimes m_l & \downarrow m_l
\end{array}$$

$$\begin{array}{ccc}
L \otimes M & \xrightarrow{m_l} & M \\
m_l & \downarrow m_r & \downarrow m_r
\end{array}$$

$$\begin{array}{ccc}
M \otimes L & \xrightarrow{m_r} & M \\
m_r & \downarrow m_r & \downarrow m_r
\end{array}$$

**Remark 4.6.** Any $(G, L, [\_\_], \alpha)$ be a Hom-Leibniz algebra equipped with an action of a finite group $G$ is a $G$-bimodule over itself. In this paper, we shall only consider $G$-bimodule over itself.

We now introduce an equivariant cohomology groups of Hom-Leibniz algebras $(G, L, [\_\_], \alpha)$ equipped with an action of a finite group $G$.

Set

$$C^n_G(L, L) := \{c \in C^n_H(L, L) : c(\psi_g(a_1), \ldots, \psi_g(a_n)) = gc(a_1, \ldots, a_n)\}$$

$$= \{c \in C^n_H(L, L) : c(ga_1, \ldots, ga_n) = gc(a_1, \ldots, a_n)\}.$$

Where $C^n_H(L, L)$ is $n$-cochain group of the Hom-Leibniz algebra $(L, [\_\_], \alpha)$ and $C^n_G(L, L)$ consists of all $n$-cochains which are equivariant. Clearly, $C^n_G(L, L)$ is a submodule of $C^n_H(L, L)$ and $c \in C^n_G(L, L)$ is called an invariant $n$-cochain.
Lemma 4.7. If a $n$-cochain $c$ is invariant then $\delta^n(c)$ is also an invariant $(n+1)$-cochain. In otherwords, $c \in C^n_G(L, L) \implies \delta^n(c) \in C^{n+1}_G(L, L)$.

Proof. As $c \in C^n_G(L, L)$, we have $c(gx_0, gx_1, \ldots, gx_{n-1}) = gc(x_0, x_1, \ldots, x_{n-1})$ for all $g \in G$ and $x_0, x_1, \ldots, x_n \in L$. Now

$\delta^n(c)(\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_n))$

$= \delta^n(c)(gx_0, gx_1, \ldots, gx_n)$

$= [\alpha^{-1}(gx_0), \phi(gx_1, \ldots, gx_n)] + \sum_{i=1}^n (-1)^i \phi(\alpha(gx_0), \ldots, \alpha(gx_{i-1}), [gx_i, gx_j], \alpha(gx_{i+1}), \ldots, gx_j, \ldots, \alpha(gx_n))$

$= [g\alpha^{-1}(x_0), g\phi(x_1, \ldots, x_n)] + \sum_{i=1}^n (-1)^i [g\phi(x_0, \ldots, \hat{x}_i, \ldots, x_n), g\alpha^{-1}(x_i)]$

$= g\delta^n(c)(x_0, x_1, \ldots, x_n)$. Thus, $\delta^n(c) \in C^{n+1}_G(L, L)$. □

Definition 4.8. The cochain complex $\{C^n_G(L, L), \delta\}$ is called an equivariant cochain complex of $(G, L, [\cdot, \cdot], \alpha)$.

Definition 4.9. We define an equivariant cohomology of $(G, L, [\cdot, \cdot], \alpha)$ with coefficient over itself by

$H_G^n(L, L) := H^n(C^n_G(L, L))$

5. Equivariant formal deformations of Hom-Leibniz algebras

In this section, we introduce an one-parameter formal deformation theory of Hom-Leibniz algebras equipped with an action of a finite group $G$. We assume that the field $\mathbb{K}$ is of characteristics 0. We show that equivariant cohomology controls such equivariant deformations.

Definition 5.1. An equivariant one-parameter formal deformation of $(G, L, [\cdot, \cdot], \alpha)$ is given by $\mathbb{K}[[t]]$-bilinear and a $\mathbb{K}[[t]]$-linear map $m_t : L[[t]] \times L[[t]] \to L[[t]]$ and $\alpha_t : L[[t]] \to L[[t]]$ respectively of the form

$m_t = \sum_{i \geq 0} m_it^i$ and $\alpha_t = \sum_{i \geq 0} \alpha_it^i$,

where each $m_i : L \times L \to L$ is a $\mathbb{K}$-bilinear map and each $\alpha_i : L \to L$ is a $\mathbb{K}$-linear map satisfying the followings:

1. $m_0(x, y) = [x, y]$ is the original Hom-Leibniz bracket on $L$ and $\alpha_0(x, y) = \alpha(x, y)$.
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(2) $m_t$ and $\alpha_t$ satisfies the following Hom-Leibniz algebra condition:

$$m_t(\alpha_t(x), m_t(y, z)) = m_t(m_t(x, y), \alpha_t(z)) - m_t(m_t(x, z), \alpha_t(y)).$$

(3) For all $g \in G$, $x, y \in L$ and $i \geq 0$,

$$m_i(gx, gy) = gm_i(x, y) \text{ and } \alpha_i(gx, gy) = g\alpha_i(x, y)$$

That is, $m_i \in \text{Hom}_K^G(L \otimes L, L)$ and $\alpha_i \in \text{Hom}_K^G(L, L)$.

For all $n \geq 0$, condition (2) in the definition (5) is equivalent to

(14)

$$\sum_{i+j+k=n} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)) = 0.$$

Definition 5.2. An equivariant 2-cochain $m_t$ is called an equivariant infinitesimal of the deformation $m_t$. Suppose more generally that $m_n$ is the first non-zero term of $m_t$ after $m_0$, such $m_n$ is called an equivariant $n$-infinitesimal of the equivariant deformation.

Proposition 5.3. Let $G$ be a finite group and $(G, L, [\ldots], \alpha)$ is a Hom-Leibniz algebra. Suppose $(G, L_t, m_t, \alpha_t)$ be its equivariant one-parameter deformation then an equivariant infinitesimal of an equivariant deformation is a 2-cocycle of an equivariant Hom-Leibniz cohomology.

Proof. Let $m_n$ be an equivariant $n$-infinitesimal of an equivariant deformation $m_t$. Thus, $m_i = 0$ for all $0 < i < n$ and $m_i(ga, gb) = gm_i(a, b)$. From the equation (14), we have,

$$[\alpha(x), m_n(y, z)] - [m_n(x, y), \alpha(z)] + [m_n(x, z), \alpha(y)] + m_n(\alpha(x), [y, z])$$

$$-m_n([x, y], \alpha(z)) + m_n([x, z], \alpha(y)) = 0.$$

This is same as $\delta^2 m_n = 0$. Thus, desired result follows. \qed

An equivariant $n$-deformation of a Hom-Leibniz algebra equipped with a finite group action is a formal deformation of the form-

$$m_t = \sum_{i=0}^n m_i t^i,$$

such that

(1) For each $0 \leq i \leq n$, $m_i \in \text{Hom}_K^G(L \otimes L, L)$ and $\alpha_i \in \text{Hom}_K^G(L, L)$, that is, each $m_i$ and $\alpha_i$ are equivariant $K$-linear maps.

(2) $m_t$ satisfies the Hom-Leibniz identity, that is,

$$m_t(\alpha_t(x), m_t(y, z)) = m_t(m_t(x, y), \alpha_t(z)) - m_t(m_t(x, z), \alpha_t(y)).$$
We say an equivariant $n$-deformation $m_t$ of a Hom-Leibniz algebra $(G, L, [\cdot, \cdot], \alpha)$ is extendable to an equivariant $(n + 1)$-deformation if there is an element $m_{n+1} \in C^{n+1}_G(L, L)$ such that
\[
m_t = m_t + m_{n+1}t^{n+1}
\]
and $m_t$ satisfies all the conditions of equivariant formal deformations.

Now, for $n \geq -1$ we can rewrite the equation (14) in the following form using Hom-Leibniz cohomology
\[
\delta^2 m_{n+1}(x, y, z) = - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)) = 0.
\]
This is same as the following equation-
\[
\delta^2 m_{n+1}(x, y, z) = - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)).
\]

We define an equivariant $n$th obstructions to extend an equivariant deformation of Hom-Leibniz algebra of order $n$ to order $n + 1$ as
\[
\text{Obs}_n^G(x, y, z) = - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y)).
\]

Lemma 5.4. Suppose $m_t$ is an equivariant $n$-deformations, then $\text{Obs}_n^G \in C^{n}_G(L, L)$ is a cocycle for all $n \geq 1$.

Proof. As for all $i \geq 0$, $m_i \in \text{Hom}_K^G(L \otimes L, L)$ and $\alpha_i \in \text{Hom}_K^G(L, L)$. So for all $x, y \in L$, $m_i(gx, gy) = gm_i(x, y)$ and $\alpha_i(gx) = g\alpha_i(x)$. Now,
\[
\text{Obs}_n^G(gx, gy, gz) = - \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(gx), m_k(gy, gz))
\]
\[- m_i(m_k(gx, gy), \alpha_j(gz)) + m_i(m_k(gx, gz), \alpha_j(gy))
\]
\[= - g \sum_{i+j+k=n+1 \atop i,j,k \geq 0} m_i(\alpha_j(x), m_k(y, z)) - m_i(m_k(x, y), \alpha_j(z)) + m_i(m_k(x, z), \alpha_j(y))
\]
\[= g \text{Obs}_n^G(x, y, z)
\]
Thus, $\text{Obs}_n^G \in C^{n}_G(L, L)$. As $\text{Obs}_n^G(x, y, z) = \delta^2 m_{n+1}(x, y, z)$, we have $\text{Obs}_n^G$ is a cocycle. \hfill \Box

We can prove the following theorem along the same line as of non-equivariant case.
Theorem 5.5. An equivariant \( n \)-deformation extends to an equivariant \((n+1)\)-deformation if and only if cohomology class of \( \text{Obs}^G_n \) vanishes.

Corollary 5.6. If \( H^3_G(L, L) = 0 \) then any equivariant 2-cocycle gives an equivariant one-parameter formal deformation of \((G, L, [\cdot, \cdot], \alpha)\).

6. Rigidity of equivariant deformations

In this final section, we study rigidity conditions for equivariant deformations. Observe that an action of a finite group \( G \) on Hom-Leibniz algebra \( L \) induces an action on \( L[[t]] \) by bilinearity.

Definition 6.1. Given two equivariant deformations \( L^G_t = (G, L, m_t, \alpha_t) \) and \( L'^G_t = (L, m'_t, \alpha'_t) \) of \((G, L, [\cdot, \cdot], \alpha)\), where \( m_t = \sum_{i \geq 0} m_i t^i \), \( \alpha_t = \sum_{i \geq 0} \alpha_i t^i \) and \( m'_t = \sum_{i \geq 0} m'_i t^i \), \( \alpha'_t = \sum_{i \geq 0} \alpha'_i t^i \). We say \( L^G_t \) and \( L'^G_t \) are equivalent if there is a formal equivariant isomorphism \( \Psi_t : L[[t]] \to L[[t]] \) of the following form:

\[
\Psi_t(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \cdots ,
\]

Such that

1. \( \psi_0 = \text{Id} \) and for \( i \geq 1 \), \( \psi_i : L \to L \) is an equivariant \( \mathbb{K} \)-linear map.
2. \( \Psi_t \circ m'_t = m_t \circ (\Psi_t \otimes \Psi_t) \) and \( \alpha_t \circ \Psi_t = \Psi_t \circ \alpha'_t \).

Remark 6.2. Suppose \( L^G_t \) and \( L'^G_t \) are equivalent deformation. For every subgroup \( H \leq G \), \( H \)-fixed point set \( L^H \) is a Hom-Leibniz sub algebra. A formal equivariant isomorphism \( \Psi_t \) induces formal isomorphism \( L^H[[t]] \to L'^H[[t]] \) for all subgroups \( H \) of \( G \).

From the second condition of the definition we have the following equations:

\[
\sum_{i,j \geq 0} \psi_i(m'_j(x, y))t^{i+j} = \sum_{i,j,k \geq 0} m_i(\psi_j(x), \psi_k(y))t^{i+j+k},
\]

\[
\sum_{i,j \geq 0} \alpha_i(\psi_j(x))t^{i+j} = \sum_{i,j \geq 0} \psi_i(\alpha'_j(x))t^{i+j}.
\]

Comparing coefficients of infinitesimal on both sides of the above equations we have the following proposition,

Proposition 6.3. An equivariant infinitesimal of two equivalent equivariant deformation determines the same cohomology class.

Similar to the non-equivariant case, we have the following rigidity theorem for equivariant deformations.

Theorem 6.4. Let \((G, L, [\cdot, \cdot], \alpha)\) be a Hom-Leibniz algebra equipped with an action of finite group \( G \). If \( H^2_G(L, L) = 0 \) then \( L \) is equivariantly rigid.
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EQUIVARIANT ONE-PARAMETER FORMAL DEFORMATIONS OF HOM-LEIBNIZ ALGEBRAS

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