The Baumkuchen Theorem

MASANORI ANDO
Nara Gakuen University
Nara, Japan
m-ando@naragakuen-u.jp

TADAYUKI HARAGUCHI
Nara Gakuen University
Nara, Japan
t-haraguchi@naragakuen-u.jp

This paper is dedicated to Professor Hiro-Fumi Yamada, on the occasion of his retirement.

First, we will briefly introduce the baumkuchen and the pizza theorems, presented here as Theorems 1 and 2. Let $n$ be an even number of 4 or more. A Baumkuchen (annulus) and a pizza are divided into $2n$ equiangular slices by means of $n$ straight, concurrent cuts, as in Figure 1. Then the total areas of the gray slices is equal to that of the white slices. These theorems can be proved with elementary geometry, without utilizing integrals or radial cuts.

Figure 1 The baumkuchen (left) and the pizza (right) theorems at $n = 4$.

Throughout this paper, we assume the following: Let $D$ and $\tilde{D}$ be two disks centered at the point $O$ such that $\tilde{D}$ is a subset of $D$. Then the absolute complement $B$ of the interior of $\tilde{D}$ in $D$ represents the shape of a Baumkuchen. Let $P$ be a point of $\tilde{D}$, and let $n$ be an even number greater than or equal to 4. Then divide the sets $D$, $\tilde{D}$, and $B$ into $2n$ slices by constructing $n$ straight lines through $P$ such that the lines meet to form $2n$ equal angles. In the clockwise direction, let

$$A_1, A_2, \ldots, A_{2n}$$
be the intersections of the straight lines passing through the point $P$ and the boundary of $D$. Similarly, let

$$\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{2n}$$

be the intersections on the boundary of $\tilde{D}$ (see Figure 2).

We denote the areas of the slice $A_k A_{k+1} P$ and sector $A_k A_{k+1} O$ by $\text{Sl}(A_k)_P$ and $\text{Sec}(A_k)_O$, respectively. In particular, $\text{Sl}(A_{2n})_P$ and $\text{Sec}(A_{2n})_O$ are the areas of the slice $A_{2n} A_1 P$ and sector $A_{2n} A_1 O$, respectively. Similarly, we use $\text{Sl}(\tilde{A}_k)_P$ and $\text{Sec}(\tilde{A}_k)_O$ on the disk $\tilde{D}$.

Let $\text{Piece}(A_k)$ denote the area of piece

$$A_k A_{k+1} \tilde{A}_{k+1} \tilde{A}_k$$

on the Baumkuchen $B$. To prove the main theorems, we will need Lemmas 1 and 2. Suppose the point $P$ is on the boundary of the disk $\tilde{D}$ (see Figure 3). Then there are intersections

$$P, \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n.$$
Figure 4 If Piece($A_k$) can be divided by a straight line $C \tilde{C}$ passing through the center point $O$, then the two straight lines $F \tilde{F}$ and $G \tilde{G}$ pass through the center.

between $n$ straight lines passing through $P$ and the boundary of the disk $\tilde{D}$, where we number the intersections sequentially clockwise from point $P$.

**Lemma 1.** Suppose the point $P$ is on the boundary of the disk $\tilde{D}$. Then we have the following:

1. The $n$-sided polygon $\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n$ is regular.
2. For each $1 \leq k \leq (n/2) - 1$, the sum of $\text{Sl}(\tilde{A}_k)_{P}$ and $\text{Sl}(\tilde{A}_{k+\frac{n}{2}})_{P}$ is $2/n$ times the area of the disk $\tilde{D}$.
3. The sum of $\text{Sl}(\tilde{A}_{\frac{n}{2}})_{P}$ and the area of the figure surrounded by the line segments $\tilde{A}_1 P$, $\tilde{A}_n P$ and arc $\tilde{A}_1 P \tilde{A}_n$ is $\frac{2}{n}$ times the area of the disk $\tilde{D}$.

**Proof.** According to the inscribed angle theorem, for each $k = 1, 2, \ldots, n - 1$, the angle $\tilde{A}_k \tilde{A}_{k+1} O$ is twice the angle $\tilde{A}_k \tilde{A}_{k+1} P$. Thus, the $n$-sided polygon $\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n$ is regular. Since all opposite sides of the regular $n$-gon are parallel and of equal length, the sum of the areas of the triangles

$$\tilde{A}_k \tilde{A}_{k+1} P \quad \text{and} \quad \tilde{A}_{k+\frac{n}{2}} \tilde{A}_{k+\frac{n}{2}+1} P$$

is equal to the sum of the areas of the triangles

$$\tilde{A}_k \tilde{A}_{k+1} O \quad \text{and} \quad \tilde{A}_{k+\frac{n}{2}} \tilde{A}_{k+\frac{n}{2}+1} O,$$

where $k = 1, 2, \ldots, (n/2) - 1$ (see Figure 3). Thus, the sum of $\text{Sl}(\tilde{A}_k)_{P}$ and $\text{Sl}(\tilde{A}_{k+\frac{n}{2}})_{P}$ is $2/n$ times the area of the disk $\tilde{D}$. Using condition (2), we can easily deduce condition (3). \hfill \blacksquare

**Lemma 2.** Suppose point $P$ is on the boundary of the disk $\tilde{D}$. Then for each $1 \leq k \leq n$, the sum of Piece($A_k$) and Piece($A_{k+n}$) is $\frac{1}{n}$ times the area of the Baumkuchen $B$.

**Proof.** Step 1. Let $C$ and $\tilde{C}$ be two points on the arc $A_k A_{k+1}$ and the arc $\tilde{A}_k \tilde{A}_{k+1}$, respectively, such that the straight line $C \tilde{C}$ passes through the center point $O$. By the symmetry, the lengths of line segments

$$A_k \tilde{A}_k = A_{k+n} P \quad \text{and} \quad A_{k+1} \tilde{A}_{k+1} = A_{k+n+1} P$$
Figure 5  (Left) There is no straight line that divides Piece($A_k$) and passes through the center point $O$. (Right) If the work of pasting the pieces inductively is repeated, it can be divided by a straight line passing through the center.

are equal. Thus we obtain a piece $FG\tilde{G}\tilde{F}$ (see Figure 4) constructed by pasting the pieces

$$A_kC\tilde{C}\tilde{A}_k \quad \text{and} \quad A_{k+1}C\tilde{C}\tilde{A}_{k+1}$$

along with the line segments $A_{k+n}P$ and $A_{k+n+1}P$, respectively. Then $F$ and $G$ on boundary of the disk $D$ are points where the point $C$ is split into two points. Similarly, $\tilde{F}$ and $\tilde{G}$ are two points on the boundary of the disk $\tilde{D}$. It is clear that the sum of Piece($A_k$) and Piece($A_{k+n}$) is equal to the area of the piece $FG\tilde{G}\tilde{F}$. Then the length of the arc $\tilde{F}\tilde{G}$ is $1/n$ times of the circumference of the disk $\tilde{D}$ by condition (1) of Lemma 1. Since the area of the piece $FG\tilde{G}\tilde{F}$ is $1/n$ times the area of the Baumkuchen, so is the sum of Piece($A_k$) and Piece($A_{k+n}$).

Step 2. Suppose we cannot choose two points $C$ and $\tilde{C}$ so that the straight line $C\tilde{C}$ passes through the center point $O$. Let $C_{k_0}$ and $\tilde{C}_{k_0}$ be two points on the arcs $A_k\tilde{A}_{k+1}$ and $\tilde{A}_kA_{k+1}$, respectively, such that we can divide Piece($A_k$) by a line segment $C_{k_0}\tilde{C}_{k_0}$. Repeating our method from Step 1, we get a piece $F_{k_0}G_{k_0}\tilde{G}_{k_0}\tilde{F}_{k_0}$ (see Figure 5 (left)).

Then the lengths of the line segments $F_{k_0}\tilde{F}_{k_0}$ and $G_{k_0}\tilde{G}_{k_0}$ are equal. Inductively, we construct the piece $F_{k_j}G_{k_j}\tilde{G}_{k_j}\tilde{F}_{k_j}$ as follows: Let $C_{k_j}$ and $\tilde{C}_{k_j}$ be two points on the arc $F_{k_{j-1}}G_{k_{j-1}}$ and the arc $\tilde{F}_{k_{j-1}}\tilde{G}_{k_{j-1}}$, respectively, such that we can divide the piece $F_{k_{j-1}}G_{k_{j-1}}\tilde{G}_{k_{j-1}}\tilde{F}_{k_{j-1}}$ by a line segment $C_{k_j}\tilde{C}_{k_j}$. Since the lengths of the line segments $F_{k_{j-1}}\tilde{F}_{k_{j-1}}$ and $G_{k_{j-1}}\tilde{G}_{k_{j-1}}$ are equal, we can paste the piece $C_{k_j}F_{k_{j-1}}\tilde{F}_{k_{j-1}}\tilde{C}_{k_j}$ along $C_{k_j}\tilde{C}_{k_j}$ to the piece $C_{k_j}G_{k_{j-1}}\tilde{G}_{k_{j-1}}\tilde{C}_{k_j}$. That is, we get the piece $F_{k_j}G_{k_j}\tilde{G}_{k_j}\tilde{F}_{k_j}$ (see Figure 5 (right)). Then there exists a positive integer $l$ such that we can find two points $C_{k_l}$ and $\tilde{C}_{k_l}$ such that the straight line $C_{k_l}\tilde{C}_{k_l}$ passes through the center point $O$. Since the area of the piece $F_{k_j}G_{k_j}\tilde{G}_{k_j}\tilde{F}_{k_j}$ is $1/n$ times the Baumkuchen by the condition (1) of Lemma 1, so is the sum of Piece($A_k$) and Piece($A_{k+n}$).

We now have the following main theorem.

Theorem 1 (Baumkuchen theorem). For each $1 \leq k \leq n$, the sum of Piece($A_k$) and Piece($A_{k+n}$) is $1/n$ times the area of the Baumkuchen $B$. 


Proof. There is a disk $D'$ whose radius is the line segment $OP$. Let $B_1$ and $B_2$ be two Baumkuchens, where $B_1 = D \setminus \text{Int} D'$ and $B_2 = \tilde{D} \setminus \text{Int} D'$. Applying Lemma 2 to $B_1$ and $B_2$, we deduce the Baumkuchen theorem, as claimed. ■

Finally, we present the pizza theorem. It was proposed by L. J. Upton in this Magazine almost fifty years ago [11]. Goldberg first proved this theorem [3], and the results are widely introduced throughout the literature [1,2,4–10]. We close by giving the following new proof. (Refer to Figures 1 (right), 2, and 3).

**Theorem 2 (pizza theorem).** We can distribute the slices to $\frac{n}{2}$ people so that the sum of the areas is equal.

**Proof.** Let $\tilde{D}$ be the disk whose radius is the line segment $OP$, and let $B$ be the Baumkuchen given by the set $D \setminus \text{Int} \tilde{D}$. Then, we deduce the pizza theorem by applying Lemmas 1 and 2. In particular, we give the $k$-th person the four slices of

$$Sl(A_k)_P, \quad Sl(A_k + \frac{n}{4})_P, \quad Sl(A_k + n)_P, \quad \text{and} \quad Sl(A_k + \frac{3n}{4})_P.$$

■

**REFERENCES**

[1] Berzsenyi, G. (1994). The pizza theorem-part I. *Quantum.* 4(3): 29.
[2] Berzsenyi, G. (1994). The pizza theorem-part II. *Quantum.* 4(4): 29.
[3] Goldberg. M. (1968). Divisors of a circle (solution to problem 660). *Math. Mag.* 41(1): 46. doi.org/10.2307/2687962.
[4] Härterich, J., Iwata, S. (1989). Solutions to problem 1325. *Crux Math.* 15(4): 120–122.
[5] Hess. R. I. (1991). Solution to problem 1535. *Crux Math.* 17(6): 179–181.
[6] Hirschhorn, J., Hirschhorn, M., Hirschhorn, J. K., Hirschhorn, A., Hirschhorn, P. (1999). The pizza theorem. *Austral. Math. Soc. Gaz.* 26(2): 120–121.
[7] Humenberger, H. (2020). Fair sharing of triangular pizzas. *Math. Mag.* 93(3): 164–174 doi.org/10.1080/0025570X.2020.1742553.
[8] Mabry, R., Deiermann, P. (2009). Of cheese and crust: a proof of the pizza conjecture and other tasty results. *Amer. Math. Monthly.* 116(5): 423–438. doi.org/10.1080/00029890.2009.11920956
[9] Nelsen, R. B. (2004). Proof without Words: Four squares with constant area. *Math. Mag.* 77(2): 135. doi.org/10.1080/0025570X.1994.11996228.
[10] Pearce, C. E. M. (2000). More on the pizza theorem. *Austral. Math. Soc. Gaz.* 27(1): 4–5.
[11] Upton. L. J. (1968). Problem 660. *Math. Mag.* 40(1): 163. doi:10.2307/2688484.

**Summary.** We present the Baumkuchen theorem, which is related to the sums of the areas of a divided Baumkuchen (annulus). The theorem can be proved using the basic properties of elementary geometry. We also provide an original proof of the well-known pizza theorem.

**MASANORI ANDO** is an associate professor of mathematics at Nara Gakuen University in Japan. He received a Ph.D. in mathematics (2014) from Okayama University. His main interests are in integer partitions and Young diagrams. In particular, he uses a lot of bijective proofs. In his spare time, he enjoys the board game Catan.

**TADAYUKI HARAGUCHI** is an associate professor of mathematics at Nara Gakuen University in Japan. He received a Ph.D. in mathematics (2013) from Okayama University. His mathematical interests include diffeological and algebraic topology. In particular, he has been studying smooth homotopy theory on the category of diffeological spaces. When not doing mathematics, he likes playing Japanese chess and soft tennis.