The quantum measurement process: an exactly solvable model

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An exactly solvable model for a quantum measurement is discussed, that integrates quantum measurements with classical measurements.

The \( z \)-component of a spin-\( \frac{1}{2} \) test spin is measured with an apparatus, that itself consists of magnet of \( N \) spin-\( \frac{1}{2} \) particles, coupled to a bath. The initial state of the magnet is a metastable paramagnet, while the bath starts in a thermal, gibbsian state. Conditions are such that the act of measurement drives the magnet in the up or down ferromagnetic state, according to the sign of \( s_z \) of the test spin.

The quantum measurement goes in two steps. On a timescale \( 1/\sqrt{N} \) the collapse takes place due to a unitary evolution of test spin and apparatus spins; on a larger but still short timescale this collapse is made definite by the bath. Then the system is in a ‘classical’ state, having a diagonal density matrix. The registration of that state is basically a classical process, that can already be understood from classical statistical mechanics.

Quantum mechanics has filled last century with respect and debate. Respect for its accurate description of e.g. the solid state, quantum chemistry, high energy physics and the early universe. Debate has remained about its foundations: what exactly is this theory standing for?

One of the most fundamental questions is still: how should we describe and understand a quantum measurement? The first answer this question came from Bohr, who stated that the apparatus should be classical. This separation between classical world versus quantum world (apparatus versus test system) was felt to be unnatural. von Neumann considered an apparatus as macroscopic and postulated that it induces a collapse of the wavefunction (reduction of the wavepacket), so his complete quantum mechanics consists of Schrödinger dynamics plus the collapse postulate. Here a similar separation has been introduced, namely macroscopic versus microscopic.

To investigate the matter, several models have been proposed, which did not converge to a unique picture. Here we discuss here an exactly solvable model, from which the general structure, found before in a more complicated bosonic model, can be read off. As foreseen on general grounds, the measurement takes place in two steps: on timescale \( \tau_{\text{collapse}} \ll \hbar/T \), the quantum timescale, a collapse of the wavefunction (reduction of the wavepacket) takes place, while on a timescale \( \tau_{\text{reg}} \gg \hbar/T \) the registration of the measurement occurs. We shall discuss that the registration just coincides with a measurement of a ‘classical’ Ising spin \( s_z = \pm 1 \) with the same apparatus.

Our solution for the measurement problem is compatible with the statistical interpretation of quantum mechanics and rules out several competing interpretations.

Classical measurement

Since our approach aims to give a unified view of quantum and classical measurements, let us first recall how to measure a classical Ising spin (classical two-state system), which is in a definite state \( s_z = \pm 1 \). It is known that some classical systems may indeed be approximately described as a two-state object, e.g. a classical brownian particle in a double-well potential with well-separated minima and a steep potential barrier in between.

Our apparatus (A) consists of a magnet (M) coupled to a bath (B). The magnet contains \( N \) Ising spins \( \sigma_z^{(n)} = \pm 1 \) having a mean-field interaction between all quartets

\[
H_M = -\frac{J}{4N^2} \sum_{ijkl=1}^N \sigma_z^{(i)} \sigma_z^{(j)} \sigma_z^{(k)} \sigma_z^{(l)} - \frac{1}{4}NJm^4, \tag{1}
\]

with \( m = (1/N) \sum_n \sigma_z^{(n)} \) the fluctuating magnetization. In the standard Curie-Weiss model all pairs would be coupled, and a second order phase transition occurs. The quartic interaction has been chosen in order to have a first order transition, as it happens in a bubble chamber, where an oversaturated gas goes into droplets of its stable phase, the liquid, when triggered by a particle.

The interaction between the test system S and the apparatus

\[
H_{SA} = -gs_z \sum_n \sigma_z^{(n)} = -gs_zN\bar{m}_z \tag{2}
\]

is turned on at \( t = 0 \), the beginning of the measurement, and turned off at the final time \( t_f \).

Initially the magnet starts in the paramagnetic state; each spin has chance \( \frac{1}{2} \) to be up or down, implying a zero average magnetization, \( m \equiv \langle \bar{m} \rangle = 0 \).

At a critical temperature \( T_c \) the magnet undergoes a phase transition to a state with magnetization \( \pm m_c \). Due

\textsuperscript{*}The author who presented this contribution.
F = \frac{Jm^4}{4} - gs z m - T(\frac{1 + m}{2} \ln \frac{2}{1 + m} + \frac{1 - m}{2} \ln \frac{2}{1 - m}).

At time \( t = 0^+ \) the coupling \( g \) between the test-spin and the apparatus is turned on, which puts the magnet in an external field \( gs z = \pm g \), see Eq. (2). If \( g \) is large enough and \( s_z = +1 \), the interaction suppresses the barrier near \( m = 0.7 \), see Figure 2, and for \( s_z = -1 \) it will suppress the one near \( m = -0.7 \). This is the magnetic analog of a bubble in a bubble chamber, where a supercritical gas is triggered to bubbles of its liquid state by a test particle.

Let us denote by \( \uparrow \) and \( \downarrow \) the \( s_z = \pm 1 \) cases. With the field turned on, the magnetization will move from \( m = 0 \) to the minimum of \( F \). This is possible due to a weak coupling to the bath, which allows to dump the excess energy in the bath. The dynamics of \( m \) was coined on the basis of detailed balance alone, but that is not enough to fix it. We consider the model where all three spin components of all \( N \) apparatus spins are weakly coupled to independent Ohmic bosonic baths (sets of harmonic oscillators). The proper dynamics appears to be:

\[
\dot{m} = \gamma h(1 - \frac{m}{\tanh \beta h}), \quad \hbar = gms z + Jm^3,
\]

where \( \gamma \ll \frac{1}{\hbar} \) is a small parameter charactering the weak coupling to the bath. For \( s_z = +1 \), \( m \) will go to the right in Fig. (2). After the \( m \) has approached the minimum \( m_1 \), the apparatus is decoupled, \((g \to 0)\). The magnetization then move from \( m_1 \) to the \( g = 0 \) minimum \( m_1 \approx m_0 - 0.004 \). It will stay there up to a hopping time \( \exp(N) \); for large \( N \) this means “for ever”. Whether or when the apparatus is read off (“observation”) is immaterial.

The measurement has been performed: if \( s_z \) was \( +1 \), the apparatus has ended up with magnetization \( +m_1 \), and for \( s_z = -1 \) the magnetization went to \( m_1 = -m_1 \).

Repeating the measurement will then allow to determine an ensemble with spins having \( s_z = 1 \) with probability \( p_1 \) and \( s_z = -1 \) with probability \( p_1 = 1 - p_1 \).

The statistical interpretation of quantum mechanics.

We can describe the quantum measurement by adopting the statistical interpretation put forward by Einstein, see e.g. Ref. [2]. The most important aspects are: 1) A quantum state is described by a density matrix. 2) A single system does not have “its own” density matrix or wavefunction; 3) Each quantum state describes an ensemble of identically prepared systems; this also holds for a pure state \(|\psi\rangle\langle\psi|\).

In this approach a quantum measurement must describe an ensemble of measurements on an ensemble of systems. The task is to show that all possible outcomes occur with Born probabilities and proper correlations between test system and apparatus.
In our model the above setup carries over immediately to the quantum situation. First, the Ising spin $s_z$ should be replaced by the $2 \times 2$ Pauli matrix $\hat{s}_z$, and the apparatus spins $\sigma_z(n)$ by $\hat{\sigma}_z(n)$. The magnetization operator $\hat{m} = (1/N) \sum_n \hat{\sigma}_z(n)$ will enter the Hamiltonians (1) and (3). In the Hamiltonian of the bath and the interaction Hamiltonian between the magnet spins and the bath, there will occur creation and annihilation operators for the bosons, while the interaction term has the Pauli operators for the spins.

The test spin may start in an unknown quantum state, that is to say, its spin-averages $\langle \hat{s}_x \rangle$, $\langle \hat{s}_y \rangle$ and $\langle \hat{s}_z \rangle$ are unknown and arbitrary; our measurement will determine the latter. On the basis where $\hat{s}_z$ is diagonal the initial density matrix $\rho(0)$ has the elements $r_{\uparrow\uparrow}(0) = 1 - r_{\downarrow\downarrow}(0) = \frac{1}{2}(1 + \langle \hat{s}_z \rangle)$, $r_{\uparrow\downarrow}(0) = r_{\downarrow\uparrow}(0) = \frac{1}{2}(\langle \hat{s}_z \rangle - i \langle \hat{s}_y \rangle)$.

The full density matrix of the system $\mathcal{D}$ reads initially: $\mathcal{D}(0) = \rho(0) \otimes R_M(0) \otimes R_B(0)$, where $R_M(0) = 2^{-N} \Pi_n \hat{\sigma}_z(0)$ describes the paramagnet, so each spin is described by the identity matrix $\hat{\sigma}_z(n)_{ij} = \delta_{ij}$, while $R_B(0)$ the equilibrium (Gibbs) state of the bath. In particular, there is no initial correlation between test system and apparatus, in order to avoid any bias in the measurement.

Selection of the collapse basis

The dynamics is set by the von Neumann equation $i\hbar \frac{d}{dt} \mathcal{D} = [H, \mathcal{D}]$, where $H$ is the full Hamiltonian operator, including also the bath and the coupling between magnet and bath. The state of the test spin is $\rho(t) = tr_{\mathcal{M},B} \mathcal{D}(t)$. For its evolution only the quantum version of the interaction Hamiltonian (3) remains,

$$\frac{d}{dt} r_{ij} = -gN(s_i - s_j) tr_{\mathcal{M},B}[\hat{m}, \hat{D}_{ij}], \quad (5)$$

where $i, j = \uparrow, \downarrow$ and $s_i = +1$, $s_i = -1$ are the eigenvalues of $\hat{s}_z$. It follows that the diagonal elements are conserved in time: $r_{\uparrow\uparrow}(t) = p_\uparrow$, $r_{\uparrow\downarrow}(t) = p_\downarrow$. This happens because the spin has no dynamics of its own. This conservation is a sine-qua-non for a reliable measurement. The off-diagonal elements are endangered, and actually will collapse.

We learn from this that the selection of the collapse basis is a direct consequence of forces exerted by the apparatus on the test system: The choice of the interaction Hamiltonian sets the basis on which its system-sector diagonalizes. Zurek has claimed that the selection would be imposed by the coupling to the environment, even though the difficulties to control these couplings make it an undesired candidate for such an important issue. The above argument does in no way invoke the environment and thus rules out Zurek’s picture.

It will take place on a short timescale, where both the spin-spin interactions and the spin-bath interactions are still inactive. The problem is then simple: the evolution of $N$ independent apparatus spins, coupled to the test spin. For $\mathcal{D}_{\uparrow\downarrow}$ this means that each $\hat{\sigma}_z^{(n)} = \text{diag}(1,1)$ evolves as $\text{diag}(e^{2gt/\hbar}, e^{-2igt/\hbar})$, implying

$$r_{\uparrow\downarrow}(t) \equiv tr_{\mathcal{M},B} \mathcal{D}_{\uparrow\downarrow}(t) (t) = r_{\uparrow\downarrow}(0) [\cos \frac{2gt}{\hbar}]^N. \quad (6)$$

For short times this exposes a Gaussian decay, $r_{\uparrow\downarrow}(0) \exp(-t^2/\tau_{\text{collapse}}^2)$, with collapse time

$$\tau_{\text{collapse}} = \frac{\hbar}{g\sqrt{2N}} \ll \frac{\hbar}{T}. \quad (7)$$

In the estimate we used that $g \sim J \sim T$ and $N \gg 1$.

The recurrent peaks of the cosines, at $t_k = k\pi T/2g$, are suppressed by the bath, which brings a factor $\sim \exp(-\gamma T N)$, that vanishes when $N$ is large enough. A small dispersion in the $g$’s is quite realistic, and would bring an additional reduction by $\exp(-k^2\pi^2(g^2 - (g_i)^2)/2N)$. In conclusion, the collapse is a quantum coherent process on a short timescale $1/\sqrt{N}$, agreeing with von Neumann’s postulate if $N$ is macroscopic. The recurrence of the peaks is suppressed on a later timescale, due to a small coupling of the macroscopic magnet to the bath (decoherence). This is an effect of the ‘environment’, but it is not the main cause of the collapse.

Registation of the quantum measurement.

The off-diagonal sectors of the density matrix having decayed, it is time to study the diagonal ones. Now the bath and the coupling to its have to be specified in detail. For simplicity each spin of the magnet is assumed to have its own subbath. These subbaths are all identical but independent, consisting of harmonic oscillators in x,y and z-direction, which are coupled bilinearly to the components of the spins, and start out in their Gibbs state. Working out this quantum problem we observe complete analogy to the above description termed “classical measurement”. In particular, the evolution of $m(t)$, Eq. (4) announced above, is derived form first principles. The characteristic timescale is much larger than the quantum time,

$$\tau_{\text{reg}} = \frac{1}{\gamma g} \sim \frac{1}{\gamma J} \gg \frac{\hbar}{T}. \quad (8)$$

Result of the measurement.

After the measurement, at $t_\uparrow \gg \tau_{\text{reg}}$, the common state of test spin and apparatus is
$D(t_{f}) = p_{1} \left| \uparrow \right\rangle \langle \uparrow \left| \otimes \rho^{(1)}_{\uparrow\uparrow}(t_{f}) \otimes \cdots \otimes \rho^{(N)}_{\uparrow\uparrow}(t_{f}) \right.
\left. + p_{j} \left| \downarrow \right\rangle \langle \downarrow \left| \otimes \rho^{(1)}_{\downarrow\downarrow}(t_{f}) \otimes \cdots \otimes \rho^{(N)}_{\downarrow\downarrow}(t_{f}) \right. \right) \tag{9}$

where $\rho^{(n)}_{\uparrow\uparrow}(t_{f}) = \frac{1}{2} \text{diag}(1 + m_{\uparrow}, 1 - m_{\uparrow})$ is the Gibbs density matrix of spin $n$ of the magnet.

In words: with probability $p_{1}$ one finds the spin in the up-state, and the magnet with magnetization up, and likewise in the down-sector. The off-diagonal sectors, called “Schrödinger cats”, are eliminated by the collapse.

We can now turn off the apparatus (by setting $g = 0$). The above state will change slightly, because $m$ now goes to the $g = 0$ minimum of the free energy, and the system will remain there forever. Eq. (9) will hold in this modified form.

Let us stress the meaning of Eq. (9) according to the statistical interpretation. In doing a series of experiments, there are two possible outcomes, connected with the magnetization of the apparatus being up or down, which occurs with probabilities $p_{1}$ and $p_{j}$, respectively. In each such event, the $z$-component of the test spin is equal to $+1$ or $-1$, correspondingly. The ensemble of spins having $+1$ is described by the pure state density matrix $|\uparrow\rangle \langle \uparrow|$, or simply by the wavefunction $|\uparrow\rangle$. A similar statement holds for the down spins.

Notice that within the Copenhagen interpretation one assumes that once a single event has happened, a single spin does have its own collapsed wavefunction.

Another issue, namely whether single events can be accounted for by quantum mechanics, has to be answered negatively. Within the statistical interpretation, quantum mechanics is a theory about the statistics of outcomes of many events.

**Conclusion**

The initial paramagnetic state of the apparatus consists of many microstates, so a statistical description is called for, and we have retained that for the quantum measurement. This is possible within the statistical interpretation of quantum mechanics, which states that any quantum state describes an ensemble of systems. A theory of quantum measurements must therefore describe an ensemble of measurements on an ensemble of identically prepared systems.

It was found that the collapse occurs quite fast after the start of the measurement. It goes in two steps: the collapse occurs due to interaction of the test system with the macroscopic apparatus, and later is made definite by bath induced decoherence.

The registration of the measurement occurs in a “classical” state, a state that has collapsed already. Here a naive classical approach and a detailed quantum approach yield exactly the same outcome. The pointer variable ends up in a stable thermodynamic state. Whether the outcome is observed or not is immaterial.

For a macroscopic apparatus the collapse will almost be instantaneous, yielding the basis for the postulate of von Neumann. Our theory can be tested by mapping out the $N$-dependence.

**Other interpretations of quantum mechanics.**

Our approach makes some of the interpretations, that caused much dispute in the past, obsolete. A multi-universe picture is incompatible, since a collapse does occur. Mind-body problems do not show up, because the act of observation is no more than gathering information about the classical final state of the apparatus. Not environment induced selection but the coupling to the apparatus is causing the collapse. Gravitation plays no role. Extensions of quantum mechanics, like spontaneous or stochastic localization and spontaneous collapse models, are not needed.

We find no support for interpretations that attribute a special role to pure states of test system, e.g. the modal interpretation and Bohmian mechanics. The assumption of an underlying pure state for the whole system is unnecessary and would anyhow be problematic for describing the statistical nature of the apparatus in realistic setups.

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