Pseudogap and superconductivity emerging from quantum magnetic fluctuations: a Monte Carlo study
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The origin of the pseudogap behavior, found in many high-$T_c$ superconductors, remains one of the greatest puzzles in condensed matter physics. One possible mechanism is fermionic incoherence, which near a quantum critical point allows pair formation but suppresses superconductivity. Employing quantum Monte Carlo simulations of a model of itinerant fermions coupled to ferromagnetic spin fluctuations, represented by a quantum rotor, we report numerical evidence of pseudogap behavior, emerging from pairing fluctuations in a quantum-critical non-Fermi liquid. Specifically, we observe enhanced pairing fluctuations and a partial gap opening in the fermionic spectrum. However, the system remains non-superconducting until reaching a much lower temperature. In the pseudogap regime the system displays a “gap-filling” rather than “gap-closing” behavior, consistent with experimental observations. Our results provide the first unambiguous lattice model realization of a pseudogap state in a strongly correlated system, driven by superconducting fluctuations.

Introduction
Even though unconventional and high-$T_c$ superconductivity arises in a diverse set of materials, many of them share similar features in their phase diagram. One prominent feature is a superconducting (SC) dome, which emerges near the termination point of a non-SC phase with either spin or charge order. The second feature is anomalous transport and non Fermi-liquid (nFL) behavior around the putative quantum critical points (QCP). These features have led to the proposal that soft quantum-critical fluctuations of the order parameter serve as the source for the universal behavior and mediate singular interaction that gives rise to superconductivity with nontrivial pairing symmetry, strange metal behavior, and intertwined orders.

In many unconventional superconductors, most notably the cuprates, there is a third feature: the “pseudogap”, a gap-like feature in the fermionic spectrum above the superconducting phase. Despite decades of investigation, the origin (or origins) of the pseudogap remain intensely debated. One class of proposals names exotic, possibly topological order in the particle-hole channel as the origin [1–3], while another points to pairing fluctuations in the strong coupling regime [4–7].

It is clear, that understanding of the coupling between fermionic excitations near the Fermi surface (FS) and bosonic quantum critical fluctuations [9–13] is needed to describe these three features. The development of quantum Monte Carlo (QMC) algorithms for a class of models of this type has created a feasible way to study this physics in an unbiased manner [14–16]. In QMC models, FS fermions couple to bosonic fluctuations, representing certain collective modes of low-energy fermions [17–23]. The bosonic part is bestowed with independent (non-fermionic) dynamics and can be tuned to criticality to mimic the situation in real materials. Crucially, these models are free of the sign-problem plaguing most fermionic QMC, allowing for a realistic test of theory and comparison with experiment.

In this work, we consider a variant of a quantum critical model, in which the bosons represent critical fer-
FIG. 1. Model and Phase diagram. (a) Sketch of the model in Eq. (1). Deep yellow dots and grids of the top and bottom layers represent fermion degrees of freedom with nearest hopping strength $t_1$ and next-nearest hopping $t_2$. Blue arrows and grid in the middle layers denote bosonic parts with an unit vector representing $\theta \in [0, 2\pi)$ of the rotor on each site. The on-site coupling $K$ between fermions and bosons is shown by the vertical dashed lines. The system size is $L \times L$. (b) $U - T$ phase diagram of the model obtained from QMC simulation. The inset zooms in to the vicinity of the pseudogap (PG, yellow), ferromagnetic (FM, blue) and superconducting (SC, orange) regions. The blue points on the FM phase boundary are determined by finite size scaling with fixed $T$ or $U$. Notably, for $U = 5.9$, as temperature gets lower, the system first enters into the FM phase at $T \approx 0.13$, then exits it at $T \approx 0.08$. The yellow points of the PG boundary are determined from the onset of a pseudogap in the single-particle spectrum, as shown in Fig. 2. The red points denoting an onset of s-wave superconductivity are determined from the onset of a full gap in the spectrum as well as Kosterlitz-Thouless scaling of the pairing susceptibility. The maximum of superconducting phase transition temperature $T_c$ is approximately 0.05. The green dot and dashed line, are the phase boundary of the (uncoupled) quantum rotor model [8].

romagnetic spin fluctuations (a “spin-fermion” model). Compared to earlier works on ferromagnetism, the coupling strength of our model is stronger in two aspects. First, the spin system is an $XY$ quantum rotor model that is inherently more strongly fluctuating than an Ising model, analyzed earlier [24, 25]. Second, the coupling constant $K$ between the fermionic and bosonic sectors is set to larger values than in previous works. As we see below, these modifications allow us to reveal new behavior, including a pseudogap.

In the normal state, at low enough temperatures we found in the bosonic sector near the QCP an overdamped dynamics with linear frequency response ($z = 2$ scaling). This is different from the $z = 3$ behavior, found in Ising systems, and is a result of a non-conservation of the order parameter in our model.

In the temperature range, where the bosonic susceptibility is linear in frequency, we observed several remarkable features. The uniform susceptibility deviates from Curie-Weiss behavior and actually becomes weaker at smaller $T$. In the fermionic sector, we found a gap-like feature in the density of states (DOS). Unlike in a BCS superconductor, the size of the gap remains roughly independent on temperature, while the DOS becomes progressively depleted (filled) upon lowering (raising) temperature. Importantly, the scaling behavior of the pairing susceptibility clearly shows that the system is not in a superconducting state. We thus identify the spectral gap in such a state as a pseudogap.

We note that the “gap-filling” behavior observed in our numerical results has also been observed in tunneling and photoemission experiments on the cuprates [26], and has been obtained in a class of $\gamma-$models of quantum-critical pairing [6]. Our results, obtained from unbiased large-scale QMC simulations, confirm the existence of a pseudogap behavior from pairing fluctuations in a quantum-critical system with itinerant fermions.

The quantum-critical spin dynamics and normal state fermionic properties that we found are consistent with recent theoretical predictions for nFLs at finite temperature, obtained within the modified Eliash-
sites with hopping amplitudes hop between nearest-neighbor (next-nearest-neighbor) top and the bottom layers. Fermions in each layer can \( \lambda \) lattice, with layer index \( \lambda \) and \( \lambda = 4 \), extracted from QMC, are consistent with theoretical predictions (see Ref. 6 and Methods). Our results therefore provide the first unambiguous numerical realization of the transition from nFL to pseudogap and eventually to superconductivity, lending support to the scenario of pairing fluctuations driven pseudogap phenomena.

**Model**

We consider a model of itinerant fermions coupled to \( SO(2) \) quantum rotors, as shown in Fig. 1(a) (rotors are in the middle layer). The model is described by

\[
\hat{H} = \hat{H}_{qr} + \hat{H}_t + \hat{H}_{qr-f},
\]

where

\[
\hat{H}_{qr} = \frac{U}{2} \sum_i \hat{I}_i^2 - t_b \sum_{\langle i,j \rangle} \cos (\hat{\theta}_i - \hat{\theta}_j),
\]

\[
\hat{H}_t = -t_1 \sum_{\langle i,j \rangle \sigma \lambda} \hat{c}_i^{+ \sigma \lambda} \hat{c}_j^{\sigma \lambda} - t_2 \sum_{\langle i,j \rangle \sigma \lambda} \hat{c}_i^{+ \sigma \lambda} \hat{c}_j^{\sigma \lambda} - \mu \sum_{i \sigma \lambda} \hat{n}_i^{\sigma \lambda},
\]

\[
\hat{H}_{qr-f} = -\frac{K}{2} \sum_{i \lambda} \left( \hat{\theta}_i^\sigma \hat{\theta}_i^\lambda \cdot \cos \hat{\theta}_i + \hat{\theta}_i^\sigma \hat{\theta}_i^\lambda \cdot \sin \hat{\theta}_i \right).
\]

The first term \( \hat{H}_{qr} \) describes a quantum rotor model on a square lattice. Here \( \hat{I}_i \) is the angular momentum of 2D rotor \( \hat{\theta}_i \) at site \( i \). The second term \( \hat{H}_t \) describes two identical copies of spin-1/2 fermions on a square lattice, with layer index \( \lambda = 1 \) and 2 representing the top and the bottom layers. Fermions in each layer can hop between nearest-neighbor (next-nearest-neighbor) sites with hopping amplitudes \( t_1 \) \( (t_2) \), and the chemical potential \( \mu \) controls the fermion density. The last term \( \hat{H}_{qr-f} \) couples quantum rotors and fermions via an on-site ferromagnetic interaction that tends to align \( XY \) component of a fermion spin with the direction of a rotor on each site.

The absence of fermion-rotor coupling, rotors develop quasi-long-range ferromagnetic order via a Kosterlitz-Thouless(KT) transition [8, 29]. At zero temperature, ferromagnetic order becomes long range. The KT transition line in \( (T, U) \) plane terminates at a QCP at \( (U/t_b)_{c} = 4.25(2) \) [8, 30, 31]. As we turn on the fermion-rotor coupling, fermion contributions shift the KT phase boundary towards larger \( U \) and \( T \). More importantly, the phase transition now involves fermion spins, which at \( T = 0 \) also order ferromagnetically below \( U_c \). This allows us to study quantum phenomena near a ferromagnetic QCP in a metal. Due to the antiunitary symmetry and the presence of two copies of fermions, this model can be simulated via QMC techniques without the sign problem (see SI for details).

This setup then allows us to analyze the universal behavior near a QCP with high numerical accuracy and large system sizes.

**Results**

We express all quantities in units of \( t_b \). In the simulations we set \( K = 4, t_1 = 1, t_2 = 0.2 \) and \( \mu = 0 \). We varied \( U \) and the temperature \( T \) and constructed the phase diagram of the model, Fig. 1 (b), which features a paramagnetic-ferromagnetic transition and several other transitions/phases. The magnetic transition at a finite temperature is of KT type. As \( U \) increases, the transition temperature decreases and terminates at a QCP at \( U_c \). The \( T = 0 \) transition upon varying \( U \) belongs to \( XY \) universality class as the coupling to rotors creates an easy plane for fermion spins. Fermion spins order ferromagnetically in the \( XY \) plane, breaking a spin-rotational symmetry. We observe a superconducting dome around the QCP. Above the dome, we find convincing evidence of pseudogap behavior in the range of \( T \), comparable to \( T_c \).

First, by measuring correlation functions of Cooper pairs in various pairing channels, we find that the dominant pairing channel is spin-triplet and odd under the interchange between the top and the bottom layers (layer-singlet), i.e., \( \Delta(r) = \frac{1}{\sqrt{2}} (\hat{c}_{r1}^{\uparrow} \hat{c}_{r2}^{\downarrow} - \hat{c}_{r2}^{\uparrow} \hat{c}_{r1}^{\downarrow}) = \frac{1}{\sqrt{2}} (\hat{c}_{r1}^{\uparrow} \hat{c}_{r2}^{\downarrow} + \hat{c}_{r1}^{\downarrow} \hat{c}_{r2}^{\uparrow}) \), where 1 and 2 label layers. In the classification of 2D irreducible representations,
there are two distinct types of spin-triplet pairing – one the spin-triplet channel. In the geometry of our model, introduce an effective interaction that is attractive in phase transition, soft dynamical bosonic fluctuations long being known that near a ferromagnetic quantum fluctuations, associated with the QCP. Indeed, it has observation is a direct evidence that superconductivity origina- tion with the corresponding intermediate temperature scale in the dynamic bosonic susceptibility $\chi$ in Fig. 4 (b). The gap becomes 'deeper' as temperature goes lower indicating the presence of a pseudogap. At $T = 0.05$, $N(\omega = 0) \approx 0$ and the full gap indicates the onset of superconductivity. (b) Data collapse of the pairing susceptibility $P_s$ versus temperature at $U = 6$ for system sizes $L = 12, 14, 16, 18$, consistent with a KT transition. The best fit coefficients are $A = 0.75$, $T_c = 0.048$, which is consistent with the temperature of the fully-gapped spectrum in (a).

FIG. 2. Pseudogap and superconductivity. (a) Local DOS $N(\omega)$ for various temperatures at $U = 6$ with $L = 12$. For $T = 1/4$, far above the pseudogap, the system exhibits a Fermi liquid spectrum. At $T_{PG} = 0.1$, the superconducting fluctuations begin to play important role and a noticeable gap forms at $\omega = 0$. This gap-forming temperature is consistent with the corresponding intermediate temperature scale in the dynamic bosonic susceptibility $\chi$ in Fig. 4 (b). The gap becomes 'deeper' as temperature goes lower indicating the presence of a pseudogap. At $T = 0.05$, $N(\omega = 0) \approx 0$ and the full gap indicates the onset of superconductivity. (b) Data collapse of the pairing susceptibility $P_s$ versus temperature at $U = 6$ for system sizes $L = 12, 14, 16, 18$, consistent with a KT transition. The best fit coefficients are $A = 0.75$, $T_c = 0.048$, which is consistent with the temperature of the fully-gapped spectrum in (a).

this is an $s$-wave gap, as $\Delta(0)$ is finite. This observation is a direct evidence that superconductivity originates from the interaction mediated by soft bosonic fluctuations, associated with the QCP. Indeed, it has long been known that near a ferromagnetic quantum phase transition, soft dynamical bosonic fluctuations introduce an effective interaction that is attractive in the spin-triplet channel. In the geometry of our model, there are two distinct types of spin-triplet pairing – one is odd under momentum inversion in a layer and even under layer interchange (e.g., $p$-wave layer-triplet), the other is even within each layer and odd under layer interchange ($s$-wave layer-singlet). By analogy with previous studies of the pairing mediated by small $q$ fluctuations [24], one expects the leading instability to be towards the $s$-wave layer-singlet, spin-triplet order. The numerical finding of the largest pairing correlations in this channel thus affirms the crucial role of soft ferromagnetic bosonic fluctuations in the formation of a SC dome.

Second, we obtained the fermionic spectral function and the local DOS $N(\omega)$. For this, we first computed, within QMC the imaginary-time fermion Green’s function and then converted it to real frequency via stochas-tic analytic continuation method (See Methods and SI for details). We show the results for $N(\omega)$ in Fig. 2 (a). At low $T$, inside the SC dome, there is clear evidence for an $s$-wave gap. The data show that, that as $T$ increases, the magnitude of the gap slightly increases, rather than shrinks, as would be the case in a BCS superconductor. Simultaneously, $N(\omega)$ for $\omega$ smaller than the gap increases and gradually fills in the states within the gap, ultimately restoring its normal-state value. This phenomenon has been termed gap-filling. It is consistent with experimental observations in many strongly-correlated unconventional superconductors at $T \geq T_c$ [5, 32–34]. At smaller $T \leq T_c$, the DOS displays gap-closing behavior, like in a conventional BCS superconductor. Guided by the experimental evidence [32, 34] that gap-filling behavior holds at $T \geq T_c$, we defined the pseudogap region as the one where the DOS gets filled in upon increasing $T$. We set the lower boundary of this region to where the DOS at the Fermi energy significantly deviates from thermally activated behavior of $e^{-\Delta/\kappa_B T}$. The upper boundary of the pseudogap region is set at $T_{PG}$, at which the dip of $N(\omega)$ at the Fermi energy becomes invisible. The pseudogap region, obtained this way, is plotted in yellow in Fig. 1(b).

Third, to determine the actual SC transition temperature, $T_c$, we performed scaling analysis of the pairing susceptibility $P_s = \frac{1}{L^2} \int_0^\beta \sum_i (\Delta^+(\tau_i, -\tau) \Delta^-(\tau_i, -\tau))$, using KT scaling for the pairing susceptibility $P_s = L^{2-\eta_s} \int_0^\beta \exp(-T_{KT}^\alpha \tau)$ for $T > T_c$, with $\eta_{KT} = 1/4$ [22, 35, 36]. We show the results in Fig. 2 (b). The data for $P_s$ for various system sizes and temperatures collapse onto a single curve. We fitted the curve by the formula above and extracted $T_c = 0.048$. This agrees with the lower boundary of the pseudogap region. The upper boundary, $T_{PG}$, is about twice larger in our sim-
argue that our unbiased numerical QMC simulations diagram in Fig. 1 (b). Based on this comparison, we been associated with the actual $T \leq T_c$ closing behavior at small $T$. Further, Eliashberg calculations below $T_c$ compared with $T = 5.9$, $T = 0.05$ in (c). The spectral weights are normalized with the same scale. The system size is $L = 12$ and the twisted boundary condition in the fermion hopping is applied such that the momentum resolution is 4 times larger in both $k_x$ and $k_y$ directions.

We have also computed within QMC the superfluid density, $\rho_S(T)$, which has been widely used in QMC simulations to determine $T_c$. At $T_c$, $\rho_S(T_c) = \alpha T_c$, where $\alpha$ is a dimensionless constant [36] and usually set to $2/\pi$, based on the analysis of the $XY$ model [37]. This criterion, although qualitatively correct, typically overestimates $T_c$ [35]. In our case, $T_c$ determined this way is somewhat higher than $T_c$, obtained from a more rigorous scaling analysis.

We analyzed the QMC data within the quantum critical theory of itinerant ferromagnets [38, 39], extended to finite $T$ [28] and modified to include two layers of itinerant fermions and superconductivity. We computed fermionic and bosonic self-energies near $U_c$ and found good agreement with the simulations in the normal state (see SI). We extracted the effective fermion-boson coupling from this comparison, and used it to compute the onset temperature for the pairing within the Eliashberg theory for quantum-critical pairing [6]. This theory does not differentiate between pair formation and superconductivity, hence the result has to be compared with $T_{PG}$, extracted from simulations. We obtained theoretical $T_{PG} \sim 0.08$, quite consistent with $T_{PG} \sim 0.1$, extracted from QMC data, see Fig. 1 (b). Further, Eliashberg calculations below $T_{PG}$ show gap-closing behavior at small $T$ and gap-filling behavior at $T \leq T_{PG}$. The boundary between two regimes has been associated with the actual $T_c$, based on the analysis of phase fluctuations [6]. This is shown in the phase diagram in Fig. 1 (b). Based on this comparison, we argue that our unbiased numerical QMC simulations are consistent with the theory and provide strong evidence for pseudogap behavior, originating from pre-formed pairs above $T_c$, near a ferromagnetic QCP in a metal.

**Magnetic dynamics and re-entrance effect**

The pairing behavior also has an impact on the magnetic phase transition and the quantum dynamics of the rotors. As shown in Fig. 1(b), the phase boundary of the paramagnetic-ferromagnetic transition exhibits a reentrance behavior at $U \sim 5.9$, close to the QCP. For example, at $U = 5.9$, upon reducing the temperature, the system first enters the ferromagnetic state and then returns to the paramagnetic one. This can also be seen from the Fermi surface behavior. In Fig. 3, we plot the Fermi surface, $G(k, \tau = \beta/2) \sim N(k, \omega = 0)$, evolution with temperature. At intermediate temperature $T = 0.1$, the Fermi surface splits due to the ferromagnetic order. However, the split vanishes both either increasing or lowering the temperature. This re-entrance phenomenon is the direct consequence of the pseudogap and superconducting fluctuations, which suppress the fermion DOS and hence the electron-hole contribution to magnetic order. Similar behavior has been seen previously in an antiferromagnetic model [17], but no pseudogap was reported there. We emphasize that the paramagnetic-ferromagnetic phase boundary starts to bend to the left roughly at $T_{PG}$, which is well above the SC dome, indicating that SC fluctuations without phase coherence in the pseudogap region are responsible for the magnetic dynamics.

In addition, we measured the inverse dynamical bosonic susceptibility of the rotors across different regions of the phase diagram. Our results are summarized in Fig. 4, showing data for three representative $U$ at various temperatures. To study the dynamics, we subtract the static part of the inverse susceptibility and focus on the spin polarization $\chi^{-1}(q, \omega) - \chi^{-1}(q = 0, \omega = 0)$ (for details of what follows see SI). Deep in the ferromagnetic phase, Fig. 4 (a), we find an $\omega^2$ dependence (dynamical exponent $z = 1$). This is similar to that of the bare rotor model, and indicates that the fermionic contribution to the dynamics is negligible because of the spin gap. Similarly, deep in the Fermi liquid phase, Fig. 4 (c), we find an $\omega^2$ dependence, except at the lowest frequencies, which furthermore extrapolates to a nonzero value. The saturation is readily understood as resulting from the non-analyticity of the Lindhard function which implies $\chi^{-1}(q = 0, \omega \to 0) \neq \chi^{-1}(q \to 0, \omega = 0)$ at weak coupling.

[FIG. 3. Re-entrance. Evolution of the Fermi surface (FS) from non-interacting system with $H_f$ in (a), to the nFL FS subjected to strong ferromagnetic correlation at $U = 5.9$, $T = 0.1$ in (b), and eventually to the pseudogap FS at $U = 5.9$, $T = 0.05$ in (c). The spectral weights are normalized with the same scale. The system size is $L = 12$ and the twisted boundary condition in the fermion hopping is applied such that the momentum resolution is 4 times larger in both $k_x$ and $k_y$ directions.]


FIG. 4. Magnetic dynamics. Inverse bosonic dynamic susceptibility $\chi$ versus $\omega_n$ in three different regions, at $U = 3, 6, 8$, corresponding (a) in the ferromagnetic phase, (b) in the pseudogap and SC phases, and (c) disorder phase. (a) log-log plot for various system size $L = 6, 8, 10, 12, 14$, each of which includes various $\beta = 12, 16, 20, 24$. Red line is a quadratic line of $\chi^{-1} \sim \omega^2$ for low frequency part $\omega_n < 1$. (b) log-log plot for various $\beta = 10, 16, 20, 24, 30$ with $L = 12$. At temperature $T = 0.1 (\beta = 10)$, the fermions are in the quantum critical regime, the bosonic susceptibility is the linear function of $\omega$, as indicated by the orange line. When temperature gets lower, the fermion goes into pseudogap phase, prompting the bosonic scaling behavior to deriviate from linear function. And upon entering the SC phase, the $\chi^{-1} \sim \omega^2$ (the blue line, a guide to the eye) as in the ferromagnetic phase. (c) Bosonic susceptibility in the disordered phase at $U = 8$. Plot for system size $L = 6, 8, 10, 12$ with various $\beta = 12, 16, 20, 24$. At high frequency all data points successfully merge together, as indicated by the red line quadratic in $\omega$ for $\omega > 1$. At low frequency, the $\chi^{-1} \sim \omega$ as indicated by the blue line, which is a guide to the eye, due to the non-conserved rotor order parameter.

Quantitatively different behavior indicating strong fermionic correlations.

First, at higher frequencies we find a linear frequency dependence ($z = 2$), which does not saturate to a finite value. This is surprising at first glance, since Landau damping for a ferromagnet has an $\omega/q$ form ($z = 3$) rather than linear $\omega$. We note that, in purely electronic models $\chi^{-1}(q, \omega)$ is required to be non-analytic at any coupling strength due to spin conservation, and non-analytic behavior was seen previously in simulations of Ising-ferromagnets. However, in our simulations of the $XY$ model, the order parameter is not conserved, leading to linear frequency dependence, in direct contrast to Ising model studied in Refs. [24, 25].

Second, at lower Matsubara frequencies accessible at lower temperatures, the $\omega^2$ behavior is again restored even in the quantum-critical region. As discussed above, this is again a direct result of the formation of a gap - this time the pseudogap, which depletes the low-energy fermion density of states and reduces the fermionic feedback on the bosons.

From the analysis above, we see that the spin dynamics is consistent with the quantum-critical behavior and pseudogap physics.

Summary and discussion

In this work, we performed a large-scale quantum Monte Carlo simulation of a ferromagnetic spin-fermion model. we reported direct spectral and thermodynamic evidence of the formation of a pseudogap prior to the superconducting transition. Within such a pseudogap phase, the temperature evolution of the fermion spectral gap exhibits a gap-filling behavior, in sharp contrast with that of a conventional superconductor. Moreover, we found that the dynamics of the spin fluctuations display a different behavior than the well-known Landau damping behavior with $z = 3$.

Remarkably, we were able to reconcile all these features with theoretical predictions of Eliashberg theory and its generalization to the $\gamma$-model. Experimentally, pseudogap phases have been observed in various unconventional superconductors, most notably the underdoped cuprates. Our results imply that a pseudogap arising from strong dynamical fluctuations should be ubiquitous in quantum-critical metals, and we expect this to be an extremely fruitful direction for future research.

Methods

QMC simulations and data analysis – We employ the determinant quantum Monte Carlo (DQMC) method [14, 24] to simulate the Hamiltonian in Eq. (1). The quantum rotor model plays the role of the auxiliary field in the conventional DQMC and the quantum rotor model can be efficiently simulated with non-local update scheme developed in our previous work [8]. For each realization of the rotor in spacetime, the fermion determinant is evaluated with the kinetic part and the coupling part of the Hamiltonian in-
cluded as the configurational weight and the Markov chain of the Monte Carlo process is carried out according the weight. Detailed measurements of the physical observables are given in the SI.

In order to obtain real-frequency spectral functions, the stochastic analytic continuation (SAC) scheme is employed to obtain the spectral function \( N(\omega) \) from the imaginary-time correlation function \( G(\tau) \).

\[
G(\tau) = \int_{-\infty}^{\infty} d\omega \frac{e^{-\omega(\tau-\beta/2)}}{2 \cosh(\beta \omega/2)} N(\omega)
\]  

(3)

It is known that the problem of inverting the Laplace transform is equivalent to finding the most probable spectral function \( N(\omega) \) out of its exponentially many suggestions to match the QMC correlation function \( G(\tau) \) with respect to its stochastic errors, and such transformation has been converted to a Monte Carlo sampling process [40–42]. This QMC-SAC approach has been successfully applied to quantum magnets ranging from the simple square lattice Heisenberg antiferromagnet [43] to deconfined quantum critical point and quantum spin liquids with their fractionalized excitations [44–46].

Theoretical analysis –

We analyzed the QMC data for fermionic and bosonic response using the modified Eliashberg theory, which is a low energy effective dynamical theory for itinerant fermions near a QCP at finite temperatures. The theory accepts as parameters the static properties of a coupled fermion-boson system near a QCP, e.g. fermion band structure, bosonic susceptibility, etc., and computes the dynamical response of the system in terms of the fermionic self energy \( \Sigma(k, \omega_n) \) and bosonic self energy (polarization) \( \Pi(q, \Omega_n) \), taking into account the low energy excitations near the FS. It accounts for deviations from the canonical \( T \rightarrow 0 \) quantum critical behavior, e.g. deviations from the \( \Sigma \sim \omega_n^{2/3} \) NFL self energy, and from the Landau damping \( \Pi \sim \Omega_n/(v_F |q|) \) as discussed in the main text. For details on the method see Refs. [25, 28].

We applied the theory to our QMC data, both to verify our assumptions on the normal state of the system and to extract the effective fermion-boson coupling. In the bare theory, the coupling \( \tilde{g} \sim K^2 \), but it is renormalized by fermions with energies of order of the bandwidth, so it should be extracted by fitting from the QMC data. We present results for \( U = 6 \) which is almost above the QCP in Fig. S8 in SI, showing good agreement between theory and data. For details of the fitting procedure and a discussion of the quality of the fits are presented in SI. We found \( \tilde{g} = 6.3 \pm 0.2 \), representing a 20% renormalization of the bare vertex \( K \), which is consistent with earlier works [28].

Finally, we used the obtained \( \tilde{g} \) to predict \( T_{PG} \) within Eliashberg theory (the \( \gamma \)–model). Our model corresponds to \( \gamma = 1/3 \). The analytical prediction for \( T_{PG} \) can be found in Ref. [6], and details of the conversion from our \( \tilde{g} \) to the \( \gamma \)–model parameters are in the SI. We found \( T_{PG} \approx 0.08 \), in good agreement with the QMC \( T_{PG} \approx 0.1 \).

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Supplementary Information for
Pseudogap and superconductivity emerging from quantum magnetic fluctuations: a Monte Carlo study

In this supplementary material, we present the details of quantum Monte Carlo (QMC) implementation and more results of the phase diagram in different phases, as well as the theoretical analysis upon the QMC data.

I. DETAILS OF QMC SIMULATION

A. Quantum Rotor Model

The Monte Carlo simulation on the quantum rotor model (QRM) starts from employing proper basis for the Hamiltonian. As shown in Eq. 2 in main text, the boson part has global U(1) symmetry under \( \theta \) representation. We adopt the representation of the angle variable \( \theta \) for each site, ranging between \([0, 2\pi)\), which is the eigenstates of the potential part. With canonical commutation relation \([\hat{\theta}_i, \hat{n}_j] = i\delta_{i,j}\), the QRM Hamiltonian can be expressed as,

\[
H_{qr} = \hat{T} + \hat{U} = \frac{U}{2} \sum_i \left( -i \frac{\partial}{\partial \theta_i} \right)^2 - t_b \sum_{\langle i,j \rangle} \cos(\hat{\theta}_i - \hat{\theta}_j) \tag{4}
\]

and the partition function writes,

\[
Z = \text{Tr} \left\{ e^{-\beta \left[ -\frac{U}{2} \sum_i \left( -i \frac{\partial}{\partial \theta_i} \right)^2 - t_b \sum_{\langle i,j \rangle} \cos(\hat{\theta}_i - \hat{\theta}_j) \right]} \right\} \tag{5}
\]

Using Trotter decomposition, we can divided \( \beta \) into \( M \) slices with step \( \Delta \tau = \beta/M \), and insert the complete sets of \( \{\theta_i\} \) on each time slice. We have,

\[
Z = \int D\theta \prod_{l=0}^{M-1} \langle \{\theta(l + 1)\} | e^{-\Delta \tau \hat{T}} | \{\theta(l)\} \rangle \tag{6}
\]

The states follow the periodic boundary condition \( \{\theta(M)\} = \{\theta(0)\} \). For the potential part, \( \theta_i(l) \)-s are the eigenstates of \( V \) and can be directly calculated. For the kinetic part, if one inserts a complete basis of \( J_i(l) \) as the integer-valued angular momentum at site \( i \) and time slice \( l \), the left kinetic part writes,

\[
T(l) = \sum_{\{J\}} \prod_i e^{-\frac{\Delta \tau U}{2} |J_i(l)|^2} \langle \theta_i(l + 1) | J_i(l) \rangle \langle J_i(l) | \theta_i(l) \rangle \tag{7}
\]

The term \( \langle \theta_i(l) | J_i(l) \rangle \) equals to a complex value \( e^{i J_i(l) \theta_i(l)} \). Next, we transfer the square term of \( J_i(l) \) into linear term with the help of the Poisson summation formula,

\[
T(l) = \prod_i \sum_J e^{-\frac{\Delta \tau U}{2} J_i^2} e^{i J_i (\theta_i(l) - \theta_i(l+1))} \tag{8}
\]

\[
= \prod_i \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dJ e^{2\pi i J m} e^{-\frac{\Delta \tau U}{2} J^2} e^{i J (\theta_i(l) - \theta_i(l+1))} \tag{9}
\]

\[
= \prod_i \sum_{m=-\infty}^{\infty} \sqrt{\frac{2\pi}{\Delta \tau U}} e^{-\frac{1}{2\Delta \tau U^{1/2}} (\theta_i(l) - \theta_i(l+1) - 2\pi m)^2}. \tag{10}
\]

Then we remedy this by Villian approximation as

\[
T(l) \approx \prod_i e^{\frac{\Delta \tau U}{2} \cos(\theta_i(l) - \theta_i(l+1))} \tag{11}
\]
where the kinetic part of QRM can be regarded as the effective interaction along imaginary time axis. We finally map the QRM to 3D anisotropic $XY$ model [8], and the space-time configuration of the rotors, as shown in the Fig. 1(a) of the main text, plays the role of the usual auxiliary field for the determinant QMC simulations, which we will discuss next.

### B. Determinantal QMC implementations

The determinant quantum Monte Carlo (DQMC) is designed to deal with the interacting fermion lattice model with quartic interactions [47] and to decouple the quartic interactions into auxiliary bosonic fields coupled with fermion bilinears. In recent years, there are new developments that one can bestow the bosonic auxiliary field with quartic interactions [47] and to decouple the quartic interactions into auxiliary bosonic fields coupled with various Fermi surface geometries [14], and this is the path we are implementing our model in this work.

In DQMC, one transfers the non-eigenstates to a series of classical configurations, such as the space-time rotor configurations in previous section, and then sample in the configuration space in the form matrix expressions (or the determinant of matrices). To start with, one writes down the path integral of partition function,

$$Z = \text{Tr}\{e^{-\beta \hat{H}}\} = \text{Tr}\{\prod_{m=1}^{M} e^{-\Delta \tau \hat{H}}\}$$

(11)

Here, $M = \beta/d\tau$, denoting the number of the imaginary time slices. $\hat{H}$ is the total Hamiltonian contains bosonic, fermionic parts and their interaction. Trace operation can be divided into trace for fermion $\text{Tr}_f$ and boson $\int d\theta$, which we express the bosonic degrees of freedom as $\theta$. Next, we insert a series of unit operators with periodic boundary condition $\{\theta(M)\} = \{\theta(0)\}$ and make $\Delta \tau \to 0$,

$$Z = \text{Tr}_f \left\{ \int D\theta (\{\theta\}) \prod_{l=0}^{M-1} e^{-\Delta \tau \hat{H}_f} |\{\theta\}\rangle \langle \{\theta\}| \right\}$$

(12)

$$= \text{Tr}_f \left\{ \int D\theta \prod_{l=0}^{M-1} |\{\theta(l+1)\}\rangle e^{-\Delta \tau \hat{H}_f} e^{-\Delta \tau \hat{H}_q-f} |\{\theta(l)\}\rangle \right\}$$

(13)

Next, utilizing the Eq. (10), we write the partition function as

$$Z = \int D\theta \left( \prod_{l=0}^{M-1} \prod_{i} e^{\frac{\Delta \tau}{\Delta \tau} \cos(\theta(l) - \theta_i(l+1))} \right) \left( \prod_{l=0}^{M-1} \prod_{i} e^{\Delta \tau t_q \sum_{(j,i)} \cos(\theta_i(l) - \theta_j(l))} \right) \text{Tr}_f \left\{ \prod_{l=0}^{M-1} e^{-\Delta \tau \hat{H}_f e^{-\Delta \tau \hat{H}_q-f}} \right\}$$

(14)

The bosonic part can be taken away from the trace of fermion. Furthermore, the kinetic part of free fermion is independent of configuration and can be calculated at the beginning of the simulation, while the interaction part of boson and fermion depends on the configurations of $\theta$. The calculation process on $\text{Tr}_f$ is always displayed as the determinant. Finally, the total weight of configuration is,

$$Z = \int D\theta \ W_b(\{\theta\}) \det(1 + \prod_{l=0}^{M-1} e^{-\Delta \tau \hat{H}_f e^{-\Delta \tau \hat{H}_q-f(\theta(l))}})$$

$$= \int D\theta \ W_b(\{\theta\}) \det(1 + B(\beta,0)(\theta))$$

$$= \int D\theta \ W_b(\{\theta\}) W_f(\{\theta\})$$

(15)

where $W_b(\{\theta\})$ is the weight of bosonic part, and $H_f, H_{q-f}$ is the matrix in fermionic layer, spin, coordinate representation. So far, we map the model to series classical configuration and obtain its weight. Using Markov chain, we sample the configuration of $\{\theta\}$ and implement local update scheme and global update - Wolff update scheme to avoid critical slowing down, see algorithm analysis and details of QRM in [8].
C. QMC sign problem

We find an antiunitray transformation \( K = i \sigma_y K \) under which the model is invariant, where \( \sigma_y \) is Pauli matrix on layer basis, \( K \) is the complex conjugation operator. Thus the model is free of sign problem, and

\[
W_f = \det(1 + B'(\beta, 0)_{\{y\}})^2
\]  

(16)

\( B' \) is \( 2N \times 2N \) dimension matrix for single layer fermions, where \( N = L \times L \) is the number of sites. Since the model is symmetric for two layers, the Green’s function is the same for both layers with same site and spin index, i.e. \( G_{11}^{\sigma\sigma'} = G_{22}^{\sigma\sigma'} \), where \( G_{\lambda\lambda'}^{\sigma\sigma'} = \langle \hat{T} \hat{c}_{i\sigma\lambda} \hat{c}_{j\sigma\lambda}' \rangle \).

D. Controlling finite-size effects

The simulation of DQMC is restricted to the finite system size, as the computational complexity scales as \( \sim O(\beta N^3) \) \cite{14}. Therefore, it is necessary to reduce the finite size effect in simulations to save computing time and resource. To increase the momentum resolution on finite size simulations, we introduce one magnetic field perpendicular to the lattice plane called \( z \)-direction flux. The magnetic field changes the dispersion relation of the free system and mimic the DOS to the infinite system \cite{48}. Since the lattice site is finite, the flux must be quantized. The magnetic field is introduced via the Peirls phase factors on the bonds,

\[
\hat{c}_{i\sigma\lambda}^+ \hat{c}_{j\sigma\lambda} \rightarrow e^{i \int_{i\lambda}^{j\lambda} A_{\sigma\lambda}(r) dr} \hat{c}_{i\sigma\lambda}^+ \hat{c}_{j\sigma\lambda} = e^{i A_{ij} / i\sigma\lambda} \hat{c}_{j\sigma\lambda}
\]  

(17)

with \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \Phi_0 \) the flux quanta. We take Landau gauge \( \mathbf{A}(\mathbf{r}) = -B(y, 0, 0) \), which is independent of spin and layer index. To satisfy the periodic boundary condition, the boundary hopping terms must have different form compared with that of the inner bonds. Furthermore, we hope the flux on each area of lattice plane is equivalent. Since the model has the next-nearest hopping term, the square area encircled by four adjacent sites can be divided into four triangular parts. We design the magnetic field to satisfy this condition and for the nearest-neighbor hopping the phases \( A_{ij} \) read,

\[
A_{ij} = \begin{cases} 
\frac{2\pi}{\Phi_0} B \cdot i_y, \leftarrow \text{hopping} \\
\frac{-2\pi}{\Phi_0} B \cdot i_y, \rightarrow \text{hopping} \\
0, \uparrow, \downarrow \text{ hopping} \\
\frac{2\pi}{\Phi_0} B \cdot L \cdot i_x, \uparrow \text{ hopping} \\
\frac{-2\pi}{\Phi_0} B \cdot L \cdot i_x, \downarrow \text{ hopping}
\end{cases}
\]  

(18)
For the next-nearest-neighbor hopping,

\[
A_{ij} = \begin{cases}
+ \frac{2\pi}{\phi_0} B \cdot i_y, \uparrow \text{ hopping} \\
- \frac{2\pi}{\phi_0} B \cdot i_y, \downarrow \text{ hopping} \\
+ \frac{2\pi}{\phi_0} B \cdot i_y, \swarrow \text{ hopping} \\
- \frac{2\pi}{\phi_0} B \cdot i_y, \nearrow \text{ hopping} \\
+ \frac{2\pi}{\phi_0} B \cdot (L i_x - i_y), \uparrow \text{ hopping(boundary crossing)} \\
- \frac{2\pi}{\phi_0} B \cdot (L i_x - i_y), \downarrow \text{ hopping(boundary crossing)} \\
+ \frac{2\pi}{\phi_0} B \cdot (L i_x + i_y), \swarrow \text{ hopping(boundary crossing)} \\
- \frac{2\pi}{\phi_0} B \cdot (L i_x + i_y), \nearrow \text{ hopping(boundary crossing)}
\end{cases}
\]

where \( B = \frac{\Phi_0}{L^2} \) is the unit magnetic flux, and \( i_x, i_y \) are the indices of site range between 1 and \( L \) in the \( x \) and \( y \) lattice directions. Various arrows represent the direction of hopping terms from site \( i \) to \( j \). Note that when \( L \to \infty \), the magnetic field approaches to 0, and the Hamiltonian goes back to the original one. Method of adding z-flux significantly reduces finite size effect. However, the magnetic field breaks the translation symmetry, i.e., the momentum \( k \) is not valid for fermion. In DQMC simulation, we add the z-flux when measuring bosonic observables, e.g., bosonic susceptibility. While for fermionic observables e.g. spectral functions, superfluid density, we drop it.

II. PHYSICAL OBSERVABLES

To be able to obtain the phase diagram of our model in the main text, we measure different physical observables in DQMC simulations, and analyze their behavior. Besides the main results presented in the main text, here we give a detailed description of the rest of them.

A. Pairing susceptibility and superfluid density

Superconductivity is expected to enhance near the QCP [49–51], but the detailed competition of the pseudogap, nFL and superconductivity in our system still needs to be revealed with different physical observables.

As for the pairing, considering the interaction is on-site and layer and spin symmetric, we compute a number of different on-site channels and find the strongest one occurs in the \( s \)-wave channel with orbital-singlet and spin-triplet, with the order parameter

\[
\Delta(r) = \frac{1}{\sqrt{2}} (\hat{c}_{r1\uparrow} \hat{c}_{r2\downarrow} + \hat{c}_{r1\downarrow} \hat{c}_{r2\uparrow})
\]

where 1,2 are layer indices. We construct the pairing susceptibility defined as,

\[
P_s = \frac{1}{L^2} \int_0^\beta \sum_i (\Delta^\dagger(r_i, \tau) \Delta(0, 0)).
\]

\( P_s \) captures the dynamic pair-pair correlation, which increases as temperature goes lower. Since the SC pair is quasi-long-range order below \( T_c(T_{KT}) \), \( P_s \) will exhibit scaling behavior with system size as \( P_s = L^{2-\eta} f(L, \exp(-\frac{A}{T-T_c})^\eta) \) as \( T \) approaches \( T_c \) from above, and thus at \( P_s \propto L^{2-\eta} \) with \( \eta_{KT} = 1/4 \) at the transition at the thermodynamic limit [22, 35, 36].

Another supporting evidence for the establishment of quasi-long-range order of \( s \)-wave pairing is the superfluid density \( \rho_s \). \( \rho_s \) describes tendency towards pairing, and is regarded as an amount to measure the ratio of superconductive electron density over the entire itinerant electrons [52, 53]. The method to calculate
\( \rho_s \) is derived from linear response of external magnetic field. Adding a vector potential \( A_x(\mathbf{r}, t) \) with harmonic frequency \( \omega \) to the bond of the free fermion system and expand to the second order, one can deduce that the total induced current density \( J_x(\mathbf{q}, \omega) \) is indicated as,

\[
\langle J_x(\mathbf{q}, \omega) \rangle = -[(\langle -k_x \rangle - \Lambda_{xx}(\mathbf{q}, \omega)] A_x(\mathbf{q}, \omega) \quad (22)
\]

where \( k_x \) is the kinetic energy density. \( \Lambda_{xx}(\mathbf{q}, \omega) \) is the current-current correlation function, which is associated with the paramagnetic current density \( j_x^p(\mathbf{r}, \tau) \),

\[
\Lambda_{xx}(\mathbf{q}) = \frac{1}{4} \sum_{i\sigma} \int_0^\beta d\tau e^{-i\mathbf{q}\cdot\mathbf{r}} \langle j_x^p(\mathbf{r}, \tau) j_x^p(0, 0) \rangle \quad (23)
\]

And \( j_x^p(\mathbf{r}, \tau) \) is defined as,

\[
j_x^p(\mathbf{r}, \tau) = i t \sum_{\lambda} (\hat{c}_{i,\lambda}^\dagger(\tau) \hat{c}_{i+\mathbf{e}_x,\lambda}(\tau) - \hat{c}_{i+\mathbf{e}_x,\lambda}^\dagger(\tau) \hat{c}_{i,\lambda}(\tau)) \quad (24)
\]

The criteria of superconductivity comes from the Meissner effect that if the current density response of a superconductor in a static, \( \omega = 0 \), long wavelength \( q_y = 0 \), the London equation is given by,

\[
J_x(q_y) = -\rho_s A_x(q_y) \quad (26)
\]

where, \( \rho_s \) is the superfluid density to be calculated. The response is always transverse \( \mathbf{q} \cdot \mathbf{A} \) that one can take the different order of the long wavelength transverse and longitudinal limit and obtain,

\[
\rho_s = \langle -k_x \rangle - \Lambda_{xx}(q_x = 0, q_y = 0, i\omega = 0) = 0 = \langle -k_x \rangle - \Lambda_{xx}(q_x = 0, q_y = 0, i\omega = 0) \quad (27)
\]

which means \( \rho_s \) can be ultimately calculated by current-current correlation function. In the thermodynamic limit, one expects the value of \( \rho_s \) has an universal jump at transition point according to the renormalization theory of KT phase transition [54], as \( \rho_s = \frac{(\pi T_{KT})}{2} \). We thus obtain \( T_{KT} \) by plotting \( \rho_s(T) \), and looking for the crossing of \( \rho_s \) with \( \frac{\pi T_{KT}}{2} \). It is also noted that in the correlated electron systems, such crossing temperature actually turns to overestimate the \( T_c \) compared with that obtained from the pairing susceptibility [35, 36], we have also confirmed such behavior in our simulation and would therefore pay more attention to the \( T_c \) from the finite size scaling of \( P_s \).

### B. Bosonic susceptibilities

In the DQMC, we compute the dynamic bosonic susceptibility

\[
\chi(h, T, \mathbf{q}, \omega_n) = \frac{1}{L^2} \int d\tau \sum_{ij} e^{i\omega_n \tau - i\mathbf{q}\cdot\mathbf{r}_{ij}} \langle \theta_i(\tau) \theta_j(0) \rangle . \quad (28)
\]

For the bare rotor model, the behavior of dynamic susceptibility has the recognized form,

\[
\chi_0(\mathbf{q}, \omega_n) = \frac{1}{\omega_n^2 - q^2 - \xi_c^2} , \quad (29)
\]

where \( \xi_c \) is the correlation length of bosonic field, which diverges at the critical point (in the lattice simulation such divergence is parameterized by the inverse distance towards the QCP in terms of the control parameter of the transition such as \( U - U_c \)). When taking \( q = 0 \), \( \omega_n = 0 \), the dynamic susceptibility goes back to the uniform static susceptibility \( \chi \), which only depends on temperature and tuning parameter \( U \). Our results of the uniform static susceptibility of the coupled system can not be captured by generic Curie-Weiss form in Eq. (29). Non-monotonous behaviour versus temperature is observed with fixing \( U \) and reducing \( T \) in a wide parameter range in the phase diagram (for example, as will be discussed in Fig. 6). This is the interesting fact that the coupling has altered the nature of the scaling in QCP and in our case such behavior serves as a signature of substantial superconducting fluctuations.

As shown in the Fig. 4 in the main text, the \( \omega_n \)-dependence dynamic susceptibility follows non-conserved bosonic order rule [27], and display continuous behavior at \( \omega_n = 0 \), from which the novel quantum critical scaling behavior of our system is fully revealed. In the DQMC simulations, we also explore \( q \)-dependence and show the results in the following section.

### III. PSEUDOGAP, SUPERCONDUCTIVITY AND THE REPRESENTATIVE SCANS IN THE PHASE DIAGRAM

#### A. Scan at the maximum at \( T_c \)

We start with the scan at \( U = 6 \) as a function of reducing temperature. In Fig. 2 in the main text, the \( N(\omega) \) is presented at various temperatures. It is clear that there emerges a pseudogap at the temperature
FIG. 5. Superfluid density \( \rho_s \) versus temperature at \( U = 6 \) for system sizes \( L = 12, 14, 16, 18 \). The onset temperature of SC fluctuation \( T_{KT} \) is approximated by the crossover temperature for curve of \( \rho_s(L \to \infty) \) and linear function with slope \( \frac{2}{\pi} \). For \( L = 12, 14 \), such temperature is at the scale of \( T \sim 0.1 \), consistent with the onset of pseudogap in Fig. 2 in main text.

FIG. 6. Bosonic static susceptibility \( \chi(q = 0, \omega = 0) \) versus temperature at \( U = 6 \) for system sizes \( L = 6, 8, 10, 12, 14 \). Considering finite size effect, the nonmonotonous behavior of \( \chi(T) \) depicts a crossover of two phase, whose transition temperature is approximately at \( T = 0.09 \).

\[ T = 0.1 \ (\beta = 10), \] and as the temperature is further reduced, the pseudogap is steadily widened, and at the temperature of \( T = 0.05 \ (\beta = 20) \), the full gap opens in the single-particle spectrum.

These two temperature/energy scales of the pseudogap region, are consistent with that in other physical observables. Fig. 5 shows the \( \rho_s(T) \) along the same scan, and one sees that the crossing temperature for larger system sizes are at the onset of the pseudogap temperature \( T = 0.1 \). Fig. 6 shows the uniform static susceptibility, and one observe a clear suppression of \( \chi \), i.e. the deviation from the Curie-Weiss behavior due to the onset of superconductivity fluctuations, at the same temperature scale of \( T = 0.1 \). As the pseudogap spectra gradually evolve into a full gap when temperature is decreasing, the superconducting fluctuation becomes stronger, and eventually renders the system into the quasi-long-range order of the s-wave pairing state. This can be seen in Fig. 2(b) in the main text, where the data collapse of the pairing susceptibility \( P_s \) using KT phase transition critical characters for different system sizes are presented. One sees that at the temperature scale \( T \approx 0.05 \), a power-law divergence of \( P_s \) is established. These criteria, together with the results from DOS and \( \omega_n \)-dependence bosonic dynamic susceptibility, reveals two distinct temperature/energy scales of the fermionic SC properties.

In addition, at \( U = 6 \), the momentum dependence \( \chi(|q|, \omega = 0) \) falls nicely with the power-law of \(|q|^{-2} \), as shown in Fig. 7, which is similar to the bare rotor model in Eq. (29).

B. Scans at re-entry regime

In this section, we focus at \( U = 5, 9 \), where the re-entry phenomenon is detected from the boundary of KT phase transition. We analyze the data in the KT phase at smaller \( U \) versus \( T \) near the QCP. We first use the scaling of the uniform bosonic susceptibility to determine the \( U_{KT} \) as

\[ \chi(U) = L^{2 - \eta} f(L \cdot \frac{U - U_{KT}}{U_{KT}}). \]
FIG. 8. Rescaled bosonic static susceptibility at $U = 5.9$ versus temperature for system sizes $L = 12, 14, 16, 18$. Quasi-long-range order is expected to exist at the temperature where the rescaled susceptibility increases with system size, otherwise, in the disorder phase.

FIG. 9. Bosonic static susceptibility $\chi(q = 0, \omega = 0)$ versus temperature at $U = 5.9$ for system sizes $L = 6, 8, 10, 12, 14$. Considering finite size effect, the nonmonotonous behavior of $\chi(T)$ depicts a crossover of two phase, whose transition temperature is approximately at $T = 0.09$.

\[
\exp\left(-\frac{A}{(U-U_{KT})^{1/2}}\right) \text{ at fixed } T \text{ with } \eta = 1/4. \]

It is expected that if one scale the $\chi L^{-7/4}$, the curves of different system sizes will cross at the $U_{KT}$ and this is indeed what we saw in Fig. 8. Here we fix $U = 5.9$ and show the uniform susceptibility with different temperature. The KT scaling of the uniform susceptibility manifests, signifying the establish of the quasi-long-range order of the ferromagnetic rotor degrees of freedom. At $T \gtrsim 0.1$ and $T \lesssim 0.08$, the system is obviously located in the disorder phase. At intermediate temperatures at $\beta = 11, 12$ as calculated, there is clear evidence forming quasi-long-range order. Because the rescaled susceptibility of bigger size is above the small size at fixed temperature, illustrating the fitting power of uniform susceptibility greater than critical exponent 1.75. Therefore, as a function of temperature, the system undergoes two KT phase transitions, i.e., shows the re-entry phenomenon.

Furthermore, the uniform susceptibility also manifests the bending behavior along the temperature axis at $\beta = 12$ shown in Fig. 9.

C. Scans at ferromagnetic phase

When $U$ is goes from QCP to small value, the ferromagnetic properties of bosonic part gradually increases. The pseudogap phase is strongly suppressed, and the crossover temperature drops to the temperature lower than we could explore. Similar to previous method, this can be seen from the bosonic static susceptibility. Fig. 10 manifests that at $U = 5.5$, the crossover temperature is lower than $T = 0.05$. Thus we think the pseudogap phase disappear quickly at small $U$ in the ferromagnetic regime, as shown in the phase diagram in the main text.

D. Scans in the disorder phase

The pseudogap phase found in the vicinity of QCP actually extends to the disorder phase at larger $U$. Here

FIG. 10. Bosonic static susceptibility $\chi(q = 0, \omega = 0)$ versus temperature at $U = 5.5$ for system sizes $L = 6, 8, 10, 12, 14$. The nonmonotonous behavior of $\chi(T)$ is not observed at the lowest temperature $T = 0.05$ calculated, indicating the pseudogap crossover temperature is lower than that.
we present a temperature scan at $U = 8$ with $N(\omega)$ at different temperatures. The behavior in Fig. 11 is similar with that in Fig. 2 in the main text, only that the onset of pseudogap now happens at slightly lower temperature of $T \sim 0.08$. However, the superconducting phase is clearly happening at a much lower temperature compared with that in QCP. For here even with $\beta = 24$ the full gap is still not opened, in sharp contrast with the corresponding curve in phase diagram in the main text. Thus we think the SC phase domed at QCP does not extend as much as the pseudogap phase at large $U$, as shown in the phase diagram in the main text.

IV. THEORETICAL ANALYSIS

A. Modified Eliashberg theory

We analyzed the QMC data for fermionic and bosonic response using the modified Eliashberg theory (mET), which is a low energy effective dynamical theory for itinerant fermions near a QCP at finite temperatures. Within this theory, one obtains and solves the set of self-consistent equations for fermionic self-energy $\Sigma(k, \omega_n) \approx \Sigma(\omega_n)$ and bosonic propagator $\Pi(q_\perp, \Omega_n)$, related to fermionic Green’s function $G(k, \omega_n)$ and bosonic susceptibility $\chi(q, \Omega_n)$ as

$$G^{-1}(k, \omega_n) = i\omega_n - \varepsilon_k + i\Sigma(k, \omega_n), \quad \chi^{-1}(q_\perp, \Omega_n) = \chi_0^{-1} \left[ (r(T) + q^2 + 2\Omega_n^2 + \Pi(q_\perp, \Omega_n)) \right].$$

The input parameters for mET are

$$k_F, v_F, c, \chi(T) = \frac{\chi_0}{r(T)} \bar{g}.$$  (32)

These are, respectively, the fermionic Fermi vector, Fermi velocity, bosonic velocity, static bosonic susceptibility, and the effective static boson-fermion vertex. At a QCP, $r(T) \to 0$. At the bare level, $\bar{g} = K^2 \chi_0$, see Eq. (1) in the main text, but it get substantially renormalized by fermions with energies of order of the bandwidth, and we treat $\bar{g}$ as a fitting parameter. The details of mET approach have been discussed in Refs. [25, 28], and we refer an interested reader to those works for details.

The fermionic self energy consists of a thermal contribution $\Sigma_T(\omega_n)$ and quantum contribution $\Sigma_Q(\omega_n)$,

$$\Sigma(\omega_n) = \Sigma_T(\omega_n) + \Sigma_Q(\omega_n).$$  (33)

The thermal contribution is a solution of a self-consistent equation,

$$\Sigma_T(\omega_n) = \frac{\bar{g}T}{\pi} \frac{S(A_n)}{[\omega_n + \Sigma_T(\omega_n)]}$$  (34)

where (for $n \geq 0$), $A_n = \frac{v_F}{\sqrt{r(T)[\omega_n + \Sigma_T(\omega_n)]}}$ and $S(x) = \frac{\cosh^{-1}(1/x)}{\sqrt{1-x^2}}$. The quantum contribution is

$$\Sigma_Q(\omega_n) = \frac{\bar{g}T}{\pi} \sum_{m \neq n} \frac{T(A_m, B_{mn})}{[\omega_m + \Sigma_T(\omega_m)]}.$$  (35)

where $B_{mn} = \frac{\bar{g}k_F v_F [\omega_m - \omega_n]/\pi]^{1/3}}{[\omega_m + \Sigma_T(\omega_m)]}$ and

$$T(x, y) = \int_0^\infty \frac{z^2 dz}{\sqrt{z^2 + 1}(z^3 + za^2 + y^2)}.$$  (36)

The expressions are cumbersome, but allows one to straightforwardly compute $\Sigma_T$ and $\Sigma_Q$ numerically. The outcome of the computations is the following. At $T \to 0$, $\Sigma_T$ vanishes and $\Sigma_Q \propto \omega_n^{2/3}$, leading to the well-known nFL fermionic behavior and $z = 3$ dynamical scaling. At a finite $T$, there exists a wide range of temperatures, where the variations of $\Sigma_T$ and $\Sigma_Q$ with $\omega_n$ nearly compensate one another, leading to a fairly flat total self energy $\Sigma(\omega_n)$. Roughly, this happens because $\Sigma_T$ ($\Sigma_Q$) are decreasing (increasing) functions of $\omega_n$. 

![FIG. 11. Local DOS $N(\omega)$ for various temperature at $U = 8.0$ with $L = 12$. The onset temperature of pseudogap phase is approxiametly at $T = 0.08$. While at the lowest temperature at $T = 0.04$, $N(\omega = 0)$ is still far from 0, indicating the SC phase boundary is far less than $T = 0.04$.](image-url)
The bosonic self-energy has the form,
\[
\Pi(q,\Omega_n) = \frac{2 g T k_F}{v_F} \times 
\sum_{m=-n}^{1} \frac{1}{(\Omega_n + |\Sigma'(\omega_n + m)| + |\Sigma'(\omega_m)|)^2 + \omega_n^2 v_F^2 q^2}.
\]
(37)

For \(v_F q \gg \omega_n\), \(\Sigma\), and at low \(T\), \(\Pi \propto \Omega_n/(v_F q)\) has the form of a canonical Landau damping of the bosons. This gives rise to \(z = 3\) scaling. However, at \(q = 0\) and at a small but finite \(\Omega_n\),
\[
\Pi \propto \frac{\Omega_n}{\Sigma'(\omega_n)},
\]
(38)
and the scaling changes to \(z = 2\).

Eq. (37) is justified for small/moderate frequencies. At larger \(\Omega_n\), vertex corrections play a role [27, 55, 56]. In the cases when a boson represents a conserved quantity, vertex corrections remove the dependence of \(\Pi(q,\Omega)\) on the self-energy, as required by the Ward identity. However, in almost all QMC simulations to date, the boson is not a conserved quantity, and therefore the effect of vertex corrections must be computed in case-by-case basis. The results are that for an Ising spin, ladder vertex corrections are strong, and the dressed \(\Pi(q,\Omega) \propto \Omega_n/\sqrt{\Omega_n^2 + v_F^2 q^2}\), obeys an “effective” Ward identity. Any violation of this identity must arise from additional diagrams, e.g. Aslamasov-Larkin diagrams, which are expected to be weak. For an SU(2) spin, vertex corrections are also strong, but do not cancel out the dependence of \(\Pi\) on \(\Sigma\). In this situation, Eq. (37) is good only for order of magnitude estimates for \(\Omega_n \lesssim v_F q\). The case of an SO(2) spin, which we have in our simulations, is much better in this regard because ladder vertex corrections actually vanish. In this situation, corrections to Eq. (37) only come from non-ladder diagrams. These are normally quite small diagrams, so we expect Eq. (37) to be a fairly decent approximation.

B. Data Analysis

Since MET is valid in the vicinity of a QCP, we picked data for \(U = 6\) to perform the analysis. This is because it is near the critical \(U = 5.9\), but the reentrance effect, shown in Fig. 1 in the main text, is weaker there. Nevertheless, the system develops both a PG and magnetic order for low \(T\). This poses several constraints. First, we are limited to \(T > 0.1\) to avoid a significant pseudogap. This means that there are very few data points that are valid for a low-energy theory. We picked as a cutoff in frequency \(\omega_F = k_F v_F/2\), and as a cutoff in momentum \(q_F = v_F^{-1}\), which leaves about 5 Matsubara frequencies within our window at \(T = 0.1\). Second, the presence of even a small Zeeman gap distorts the self energy such that the self-energy obeys
\[
\Sigma_{FM} = \Sigma(\omega_n) + \frac{\Delta_{FM}^2}{\omega_n}
\]
(39)
where \(\Sigma(\omega_n)\) appears in Eq. (33). In terms of the inputs to our theory, Eq. (32), \(k_F, v_F, \chi_0, r(T)\) were obtained from the band structure and from the QMC data for the bosonic propagator. Unfortunately, we were not able to extract \(c\) reliably from the data, and so in our calculations we set \(c = 0\) for simplicity (we checked that varying \(c\) doesn’t qualitatively change our results). We then fit the fermionic self energy at the FS to Eq. (39), using \(\Delta_{FM}\) and \(\bar{g}\) as fitting parameters. We present the data for \(\Sigma_{FM}\) along the BZ diagonal in Fig. 12(a), showing excellent agreement with the data. The fit parameters were
\[
\Delta_{FM} = 0.34 \pm 0.01, \bar{g} = 6.3 \pm 0.2.
\]
(40)
\(\Delta_{FM}\) is on order of an inverse lattice vector \(\pi/L, L = 12\), consistent with the splitting seen in e.g. Fig. 3(b) in the main text. \(\bar{g}\) is a bit higher than the bare \(\bar{g}_0 = 4.2\) obtained from the model parameters, and represents about a 20% increase in the interaction vertex \(K\), consistent with previous QMC at strong coupling. We expect the coupling in Eq. (40) to be somewhat overestimated because we neglected \(c > 0\) effects. We checked our fits by comparing the theoretical and QMC bosonic self-energy. We present the comparison in Fig. 12(b), which shows a fairly good agreement. The quantitative discrepancies are not surprising, both because of the Zeeman gap and due to the issues discussed above and in the background section. For our purposes, it is enough that the theory correctly predicts the deviation from \(z = 3\) scaling, and that the slope of the theoretical and QMC data are comparable, confirming that our estimate of \(\bar{g}\) is reasonable.

To confirm that the onset of the pseudogap in our simulations is consistent with theoretical predictions, we computed \(T_{PG}\) within the \(\gamma\)-model (see Ref. [6] and references within). To facilitate comparison with previous works, we supply here the conversion between the coupling \(\bar{g}\) in our model, and the effective coupling \(\bar{g}_\gamma\) in the \(\gamma\)-model. It is,
\[
\bar{g}_\gamma = \frac{1}{2\pi^2} \frac{\bar{g}^2}{k_F v_F} \left(\frac{2}{3\sqrt{3}}\right)^3.
\]
(41)
Our model corresponds to a $\gamma = 1/3$ model, for which $T_{PG} = 4.4\bar{g}_\gamma$. Using the extracted $\bar{g}$ from Eq. (40) we find $T_{PG} = 0.08$, which is in good agreement with the measurement of $T_{PG} \sim 0.1$.

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