Sandpiles on fractal bases: 
pile shape and phase segregation

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Abstract

Sandpiles have become paradigmatic systems for granular flow studies in statistical physics. New directions of investigations are discussed here. Rather than varying the nature of the pile (sand, salt, rice,...) we have investigated changes in the boundary conditions. We have investigated experimentally and numerically sandpile structures on bases having a fractal perimeter. This type of perimeter induces the formation of a quite complex set of ridges and valleys. A screening effect of the valleys is observed and depends on the angle of repose. Binary granular systems have also been investigated on such bases: a spectacular demixing is obtained along the valleys. This type of phase segregation is discussed with respect to numerical studies.
1. Introduction

Despite their everyday familiarity, sandpiles and granular flows have become paradigmatic systems of new complexity problems in physics [1, 2]. Granular matter shows behaviors that are intermediate between those of solids and liquids: powders pack like solids but flow like liquids. However, granular flows are non-Newtonian [3]. Thus, dry and wet granular frictions are a practical and theoretical challenge [4].

The most basic property of sandpiles is certainly the angle of repose [5], i.e. the angle \( \theta \) made between the horizontal and the apparent surface of the pile. This angle can take values between \( \theta_r \) (the angle below which the pile is stationary) and \( \theta_d \) (the angle above which avalanches flow down the surface). In between \( \theta_r \) and \( \theta_d \), the sandpile manifests some bistability: it can be either stationary or in a state of avalanches.

The symmetry of the base on which the sandpile is constructed is a relevant parameter which can sometimes lead to exotic pile shapes. When the base is a disk (Figure 1a), the pile has a conic shape; every point of the surface being characterized by the angle of repose \( \theta \). When the base is a square (see Figure 1b), a pyramidal pile is usually obtained. Again, every point of the surface is characterized by a local angle \( \theta \). However, ridges emerge within a four-fold structure (see the grey lines in Figure 1b). It should be noticed that the ridges are characterized by an angle less than \( \theta \). For more complex bases having a constant convexity, it has been proven that the pile is a so-called tectohedron geometrical object [6] for which all facets are inclined with similar \( \theta \) angles. The facets meet on a network of ridges which can describe...
the pile shape. When the base has a low symmetry, various distinct states exist for the pile shape (see the grey lines in Figure 1c). Up to now, the case of non-convex bases has not received much attention.

Intuitively, one can imagine that for non-convex bases, the pile exhibits numerous valleys in addition to the network of ridges. The situation is then more complex.

In order to import other basic symmetry conditions, it seems of interest to consider fractal-like systems. This should lead to a wide variety of length scales as well as simple power laws for characterizing physical properties [7].

In section 2, we focus our experimental investigations on the effects of such bases with a fractal perimeter as the one illustrated in Figure 1d. This will underline thereby the interest for studying such sandpiles and the effect of the boundary conditions on the pile structure. We also present some experiments with binary mixtures of sand. In section 3, some simulation work is presented allowing some qualitative interpretation of our findings.

2. Experiments

Figure 2 presents a picture of a sandpile built on the base illustrated in Figure 1d. The perimeter of such a base results from 3 iterations of a generator and has a fractal dimension close to $D_f = \frac{\ln 5}{\ln 3}$. The arrows point to a large valley and inner sub-valleys. The fractal perimeter implies that a hierarchy of valleys emerges on the sides of the pile.

One could first ask if well placed holes in the inner part of the base instead of a fractal perimeter can produce the same phenomenon. The answer is
that the formation of valleys can be created using holes in the base but the valleys and ridges are not especially hierarchically distributed. A fractal perimeter allows for the study of “natural” structures. These structures look like natural erosion patterns which have been also recognized as being fractal objects [10].

Other sandpiles on various bases with different fractal dimensions $D_f$ have been numerically and experimentally investigated. The pictures are not shown here because in all cases, a hierarchical structure of valleys and ridges is formed. However, depending on the repose angle $\theta_r$ of the pile, different shapes are observed on the same fractal base. It can be understood that the pile extension is small and avalanches are mainly dissipated on the valleys close to the center of the base if the sand has a high repose angle. Thus, only a limited part of the fractal perimeter participates to the dissipation dynamics or ejection of grains when $\theta > \theta_r$. However, with a low repose angle, the pile is large and the whole fractal perimeter participates to the dissipation of avalanches.

Consider further the case of a binary sand mixture. It is known that the difference in the repose angles of two kinds of particles can produce phase segregation inside the sandpile [11]. Demixing is often observed but depending on various parameters like (i) the input flow, (ii) the grain size and (iii) the morphology of each species. A spectacular self-stratification can be sometimes produced. This has also been proven experimentally [11], numerically [12] and theoretically [13]. However, this self-stratification observed in a vertical Hele Shaw cell cannot be visualized from the exterior of a tri-dimensional sandpile.
However, on bases with a fractal perimeter, the situation is quite different. As an example, a mixture of dark and white sand grains has been used: (i) white grains: with diameter between 0.2 – 0.3 mm and (ii) brown grains: with diameter between 0.07 – 0.1 mm. The two species have different angles of repose. The angles $\theta_r$ of brown and white grains have been estimated to be $38^\circ$ and $32^\circ$ respectively. We have checked that this mixture leads to an internal self-stratification as observed in a vertical Hele Shaw cell. When this mixture is poured on a classical regular base (a disk), no specific structure is observed on the conic surface. However, on the base of Figure 1d, a phase segregation is clearly observed along the valleys (see Figure 3). Indeed, valleys remains in dark and ridges are in white. Also in the subvalleys, the phase segregation is observed. Due to the fractality of the perimeter, the phase segregation results in alternating vertical strates. Thus, the phase segregation can be visualized from the pile exterior itself for sandpile having a fractal-like perimeter. This effect is relevant in e.g. industry where granular piles do not have especially a conic shape.

3. Simulations

In order to find out some information on the geometrical structure made of valleys and ridges, we have performed numerical simulations of pile shapes. These simulations are based on the common cellular automaton which has been extensively discussed in the relevant literature [14, 15]. In this model, the sandpile is built on a two-dimensional lattice, where the grains occupy only a single lattice site. Here, $h_{i,j}$ denotes the height of the sandpile at co-
ordinate \(i, j\). At each time step, an arbitrary number \(N\) of grains is dropped at the top of the central column of the lattice. Then, the pile is relaxed as follows. The dynamics of each grain at position \(i, j\) on the sandpile surface is governed by the four local angles of repose: \(\tan^{-1} |h_{i,j} - h_{i+1,j}|\), \(\tan^{-1} |h_{i,j} - h_{i-1,j}|\), \(\tan^{-1} |h_{i,j} - h_{i,j+1}|\), and \(\tan^{-1} |h_{i,j} - h_{i,j-1}|\). When one or several of these angles is larger than the repose angle \(\tan^{-1}(z_c)\), some grains are assumed to roll on the surface until they reach a stable configuration where the local slope is less than the repose angle. The “rolling grain” follows downward or/and sideways random paths on the pile surface. Figure 4 presents a tri-dimensional sketch of the top of the pile where a single grain is deposited and rolls down until it reaches a stable local configuration. The simulation stops when the sandpile reaches one site on the border of the base and when the pile shape is then nearly invariant.

The model is thus based on the classical sandpile models encountered in the scientific literature since the introduction of the Bak-Tang-Wiesenfeld model [14]. The new ingredient we introduce in view of the observation in section 2 is the shape of the base which is herein considered to have a fractal perimeter. In addition to the perimeter geometry, only one parameter controls the pile shape: \(z_c\). For convenience, only integer values of \(z_c\) have been considered in this work. Lattice sizes up to 200 \(\times\) 200 have been used for the larger bases.

A typical example of the network of ridges that is numerically obtained is illustrated in Figure 5. The base is the one of Figure 1d. The structure is markedly branched and reaches a high level of complexity. The main features of the simulated structures are recognized to be those of the experimental
sandpiles: both valleys and ridges exist and are hierarchically distributed.

One should note that the network of ridges of Figure 5 is calculated with a low $\theta_r$ value ($z_c = 2$) in order to reach all parts of the perimeter. The partial screening effect has deep analogies with the diffusion of entities through fractal interfaces studied by Sapoval and coworkers [16].

The amazing phase segregation described in section 2 can be also simulated. The above numerical model can be generalized to the case of two distinct species. Following the Head and Rodgers model [17], four parameters should then be considered: $z_{c}^{\alpha\beta}$ corresponds to the maximum slope on which a particle of type $\alpha = 1, 2$ can remain on the top of a particle $\beta = 1, 2$ without starting to roll down. A typical set of parameter values giving self-stratification is: $z_{c}^{11} = 5$, $z_{c}^{12} = 4$, $z_{c}^{21} = 3$ and $z_{c}^{22} = 3$. Figure 6 presents the top view of a pile obtained with this set of parameter values. The binary sandpile was built on a base with a fractal perimeter. In Figure 6, a phase segregation is clearly observed at proximity of the holes, i.e. near the valleys. This is consistent with experimental results.

However, it is not yet clear how to choose the relevant parameters $z_{c}^{\alpha\beta}$. Indeed, the diagonal elements $z_{\alpha\alpha}$ of the $z_c$-matrix are obviously the angles of repose of each pure species $\alpha$. These parameters can be measured in the macroscopic world. However, the non-diagonal elements of $z_c$ do not have a macroscopic counterpart. It should be noted that some sets of parameter values lead to “exotic” patterns which are not encountered in the experiments.

In the light of our numerical results, the phase segregation is interpreted as follows. The species having the largest angle $\theta_r$, i.e. the brown species, cannot reach the whole perimeter areas, before the other type, whence avalanches
are dissipated in the primary valleys. Also, the complete network of ridges of Figure 4 cannot form. The white species has however a low angle of repose and reaches the borders of the base. Thus, primary valleys are controlled by the largest angle of repose and secondary valleys and ridges are controlled by the second angle of repose leading approximately to the picture of dark valleys and white ridges as in Figure 3.

4. Conclusion

In summary, we have investigated experimentally sandpiles on bases with a fractal perimeter. The shapes of the sandpiles exhibit a hierarchical (fractal) structure of valleys and ridges. Phase segregation has been found to be visualized. The repose angle $\theta_r$ seems to be the fundamental parameter since $\theta_r$ allows or not the pile to take the whole fractal shape or not. Moreover, we have shown that lattice models provide useful numerical tools for describing qualitatively the phenomenology that we discovered herein. More precise experimental studies should be made in the future.

Our work also suggests new directions of investigations. Instead of varying the nature of the pile (sand, rice,...) [18], we have suggested hereabove to change the conditions on the boundaries where avalanches are dissipated. Theoretical investigations of these are fascinating goals.

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Figure Captions

**Figure 1** — Illustration of bases on which sandpiles are built: (a) a circular base; (b) the four-fold network of ridges of the pile built on a square base; (c) irregular convex polygon and its network of ridges; (d) a base having a fractal perimeter \(D_f = \frac{\ln 5}{\ln 3}\). The perimeter of the bases is drawn in black while the network of ridges is denoted in grey.

**Figure 2** — Sandpile made on the fractal base as in Figure 1d. The diameter of the sand grains is in the range 0.2 – 0.3 mm. Valleys and subvalleys are indicated.

**Figure 3** — Sandpile on the fractal base as in Figure 1c. A mixture of two kinds of sand grains has been used: (i) diameter of white grains: 0.2 – 0.3 mm and (ii) diameter of brown grains: 0.07 – 0.1 mm. Phase segregation (demixing) is clearly observed along the valleys.

**Figure 4** — Tri-dimensionnal sketch of the rule for the present cellular automaton model. Each grain is deposited at the top of the pile and rolls down following the relaxation rule. The parameter is herein \(z_c = 3\).

**Figure 5** — Top view of a numerical simulation of the network of ridges obtained for a granular pile on the base as in Figure 1d.
Figure 6 — Top view of a typical configuration of the binary pile using the parameters $z_{c11}^c = 5$, $z_{c12}^c = 4$, $z_{c21}^c = 3$ and $z_{c22}^c = 3$. 
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