UNBOUNDED NORM CONTINUOUS OPERATORS
AND STRONG CONTINUOUS OPERATORS ON
BANACH LATTICES

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Abstract. In this paper, using the unbounded norm convergence in Banach lattices, we define new classes of operators, named unbounded norm continuous (for short, \(un\)-continuous) and strong continuous operators. We study the properties of those operators and the relationship with other operators.

1. Introduction

The notion of unbounded order convergence (uo-convergence, for short) was investigated in [3]. A net \((x_\alpha)\) in Riesz space \(E\) unbounded order converges to \(x\) in \(E\), if \(|x_\alpha - x| \wedge u \rightarrow 0\) for all \(u \in E_+\). After that, A. Bahram-nezhad et al. proposed the definition of unbounded order continuous operators in [7]. A closely related notion of unbounded norm convergence (un-convergence, for short) was introduced and systematically studied in [4]. A net \((x_\alpha)\) in Banach lattice \(E\) unbounded norm converges to \(x\) in \(E\), if \(|x_\alpha - x| \wedge u \rightarrow 0\) for all \(u \in E_+\).

In [5], M. Kandic et al. gave the definition of (sequentially) un-compact operators and obtained the relationships between weakly compact operators and sequentially un-compact operators. Recently, O. Zabeti in [6] proposed a new so-called unbounded absolute weak convergence (uaw-convergence). A net \((x_\alpha)\) in Banach lattice \(E\) unbounded weak converges to \(x\) in \(E\), if \(|x_\alpha - x| \wedge u \wrightarrow 0\) for all \(u \in E_+\).

Now, we define new classes of operators:

Definition 1.1. An operator \(T : E \rightarrow F\) between two Banach lattice is said to be:

1. unbounded norm continuous (or, \(un\)-continuous for short), if \(x_\alpha \overset{un}{\rightarrow} 0\) in \(E\) implies \(Tx_\alpha \overset{un}{\rightarrow} 0\) in \(F\).
(2) unbounded σ-norm continuous (or, un-σ-continuous for short), if \( x_n \xrightarrow{un} 0 \) in \( E \) implies \( Tx_n \xrightarrow{un} 0 \) in \( F \).

**Definition 1.2.** An operator \( T : E \to F \) between two Banach lattices is said to be:

1. boundedly unbounded norm continuous (or, bun-continuous for short), if \( Tx_\alpha \xrightarrow{un} 0 \) in \( E \) for any norm bounded un-null net \((x_\alpha)\) in \( E \).
2. boundedly unbounded σ-norm continuous (or, bun-σ-continuous for short), if \( Tx_n \xrightarrow{un} 0 \) in \( E \) for any norm bounded un-null sequence \((x_n)\) in \( E \).

**Definition 1.3.** An operator \( T : E \to F \) from Banach lattice \( E \) to Banach space \( F \) is said to be:

1. strong continuous, if \( x_\alpha \xrightarrow{un} 0 \) in \( E \) implies \( T x_\alpha \to 0 \) in \( F \).
2. strong σ-continuous, if \( x_n \xrightarrow{un} 0 \) in \( E \) implies \( T x_n \to 0 \) in \( F \).

The collection of all un-continuous operators form Banach lattice \( E \) to Banach lattice \( F \) will be denoted by \( L_{un}(E, F) \), similarly, \( L_{un \sigma}(E, F) \).

The collection of all boundedly unbounded \( \sigma \)-norm continuous operators form Banach lattice \( E \) to Banach lattice \( F \) will be denoted by \( L_{bun}(E, F) \), similarly, \( L_{bun \sigma}(E, F) \).

The collection of all strong continuous operators form Banach lattice \( E \) to Banach space \( F \) will be denoted by \( L_s(E, F) \), similarly, \( L_{s \sigma}(E, F) \).

Let \( L(E, F) \) denote the vector space of all continuous operators, we have

\[
L_s(E, F) \subset L_{s \sigma}(E, F) \subset L(E, F)
\]

\[
L_s(E, F) \subset L_{un}(E, F), L_{s \sigma}(E, F) \subset L_{un \sigma}(E, F)
\]

2. **Results**

**Lemma 2.1.** Let \( B \) be a projection band and \( P \) the corresponding band projection, then \( P \) is un-continuous.

**Proof.** According to \( |P_B x_\alpha - P_B x| \wedge u \leq |x_\alpha - x| \wedge u \), it is obviously. \( \square \)

We know that norm convergence is un-convergence, so strong continuous operator is continuous operator. But the opposite is false, for example the identited operator is continuous operator, but it is not strong continuous operator. Let \((e_n)\) denote the stand basis of \( l_p \), \((1 \leq p < \infty)\), by [xxx], we have \( e_n \xrightarrow{un} 0 \), but \( \|Ie_n\| = 1 \).

**Proposition 2.2.** Let \( E, F \) be Banach lattices, if \( E \) has strong order unit, then \( L_s(E, F) = L_{s \sigma}(E, F) = L(E, F) \)

**Proof.** According to [5, theorem 2.3], if \( E \) has strong order unit, then the un-topology of \( E \) agrees with norm topology, so we have that. \( \square \)
Strong continuous operator is un-continuous operator, the reverse is false.

**Corollary 2.3.** Let $E, F$ be Banach lattices, if $F$ has strong order unit, then $L_{un}(E, F) = L_s(E, F)$.

Strong continuous operators maps *un*-compact subset to relatively compact subset. Strong continuous operator is strong $\sigma$-continuous operator, but the reverse is false. We have the following result.

**Proposition 2.4.** Let $E$ be Banach lattice and $F$ Banach space, $L_s(E, F) = L_{s\sigma}(E, F)$, if the one of the following valids:

1. $E$ has quasi-interior point;
2. $E$ has order continuous norm.

**Proof.** By [5, theorem 3.2], we have un-topology is metrizable iff $E$ has quasi-interior. A subset $A$ of $E$ is un-compact is sequentially un-compact and the norm topology is metrizable. $\square$

According to [5, proposition 5.3], we have that the set of all un-continuous functionals in $E'$ is an ideal. Now, we will ask when it is band.

**Proposition 2.5.** If $E$ is a Banach lattice, then the order bounded un-continuous functionals forms a band of $E^\sim$.

**Proof.** Let $0 \leq f_\beta \uparrow f$ in $E^\sim$ be order bounded un-continuous functionals. And let $0 \leq x_\alpha \xrightarrow{un} 0$ in $E$. Then for each $\beta$ and each $v \in R^+$ we have

$$0 \leq f(x_\alpha) \wedge v \leq (f - f_\beta)(x_\alpha) \wedge v + f_\beta(x_\alpha) \wedge v$$

Since $f_\beta \uparrow f$ and order bounded, so the first term converges to zero. And since $f_\beta$ is un-continuous, so the second term converges to zero, we have it is a band. $\square$

**Corollary 2.6.** If $E$ is a Banach lattice, then the order bounded un-$\sigma$-continuous functionals forms a band of $E^\sim$.

Now, we study the order structure of un-continuous operator.

Let $T : E \to F$ be a positive operator between Riesz spaces, we say that an operator $S : E \to F$ is dominated by $T$ whenever $|Sx| \leq T|x|$ holds for each $x \in E$.

**Problem 2.7.** (1) When the un-continuous operator has modulus and the modulus is un-continuous.

(2) Whether the un-continuous operator has dominated property.
Theorem 2.8. Let $E, F$ be Banach lattices, and $T : E \to F$ is lattice homomorphism and un-continuous operator, then $|T|$ is un-continuous operator.

Proof. Obviously, the modulus of $T$ exists. We show that $|T|$ is un-continuous.

Let $x_\alpha \xrightarrow{un} 0$, since $T$ is un-continuous operator, then we have $Tx_\alpha \wedge v \to 0$ for all $v \in F_+$. Since $|T|x = \sup\{|Ty| : |y| \leq x\}$ and $T$ is lattice homomorphism, so $|Ty_\alpha| = T|y_\alpha| \leq Tx_\alpha$. So, we have $Ty_\alpha \xrightarrow{un} 0$, $|T|x_\alpha \xrightarrow{un} 0$, $|T|$ is un-continuous. □

Corollary 2.9. Let $E, F$ be Banach lattices, and $T : E \to F$ is lattice homomorphism and un-σ-continuous operator, then $|T|$ is un-σ-continuous operator.

Theorem 2.10. If a positive un-continuous operator $T : E \to F$ dominates $S$, then $S$ is un-continuous.

Proof. Let $x_\alpha \xrightarrow{un} 0$, then $|x_\alpha| \xrightarrow{un} 0$ by [xxx]. Since $T$ is un-continuous operator, so $T|x_\alpha| \xrightarrow{un} 0$. Since $|Sx| \wedge u \leq (T|x|) \wedge u$ for all $u \in F_+$, so $Sx_\alpha \xrightarrow{un} 0$, so $S$ is un-continuous operator. □

Corollary 2.11. If a positive un-σ-continuous operator $T : E \to F$ dominates $S$, then $S$ is un-σ-continuous.

We do not know the relationship between un-continuous operators and continuous operators.

Problem 2.12. If or not, un-continuous operator is continuous operator, and the reverse?

Theorem 2.13. Let $E, F$ be Banach lattices, then we have the following assertions:

(1): If $T \in L_{un}(E, F)$ and $F$ has strong order unit, then $T$ is continuous operator;

(2): If $T : E \to F$ is an onto lattice homomorphism and continuous operator, then $T \in L_{un}(E, F)$.

Proof. (1): Assume $x_\alpha \to 0$, then $x_\alpha \xrightarrow{un} 0$. Since $T$ is un-continuous operator, so we have $Tx_\alpha \xrightarrow{un} 0$. Since $F$ has strong order unit, by [5, theorem 2.3], we have $Tx_\alpha \to 0$, so $T$ is continuous operator.

(2): Assume $x_\alpha \xrightarrow{un} 0$. Since $T$ is onto homomorphism, then for each $u \in E_+$, we have $v \in F_+$ such that $Tu = v$. Thus,

$$T(|x_\alpha| \wedge u) = |Tx_\alpha| \wedge v \to 0$$

for all $u \in E_+$. Hence, $T$ is un-continuous. □
Corollary 2.14. Let $E, F$ be Banach lattices, then we have the following assertions:

(1): If $T \in L_{un\sigma}(E, F)$ and $F$ has strong order unit, then $T$ is continuous operator;

(2): If $T : E \to F$ is an onto lattice homomorphism and continuous operator, then $T \in L_{un\sigma}(E, F)$.

Example 2.15. Consider the Lozanovsky-like example $T : L_p[0, 1] \to c_0$ $(1 < p < \infty)$, given by

$$T(f) = (\int_0^1 f(x)\sin xdx, \int_0^1 f(x)\sin 2xdx, \ldots)$$

for $f \in L_p[0, 1]$. Let $(f_n)$ be a norm bounded sequence in $L_p[0, 1]$ for which $f_n \xrightarrow{un}\ 0$ holds, so $f_n \xrightarrow{uaw}\ 0$. By [6, theorem 7], we have $f_n \xrightarrow{w}\ 0$ and thus $T(f_n) \xrightarrow{w}\ 0$. Since $c_0$ atomic and order continuous, by [5, proposition 4.16], $T(f_n) \xrightarrow{un}\ 0$. Hence, $T$ is $bun\sigma$-continuous.

Now, we study the adjoint property.

Theorem 2.16. Let $E, F$ be Banach lattices, $E'$ is atomic and, $E$, $E'$ and $F$ are order continuous. If $T : E \to F$ is a continuous operator, then $T' : F' \to E'$ is $bun$-continuous.

Proof. Let $(x'_\alpha)$ be a norm bounded un-null net in $F'$. Since $F$ has order continuous norm, by [5, theorem 8.1], $x'_\alpha \xrightarrow{w^*} 0$ in $F'$. Since $T$ is continuous operator, so $T'$ is $w^*$ continuous operator, thus $T'x'_\alpha \xrightarrow{w^*} 0$. Now, by [5, theorem 8.4], we have $T'x'_\alpha \xrightarrow{un}\ 0$. Hence, $T'$ is $bun$-continuous.

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References

[1] C. D. Aliprantis and O. Burkinshaw, Positive Operators, Springer, 2006.
[2] P. Meyer-Nieberg, Banach lattices, Universitext, Springer-Verlag, Berlin, 1991.
[3] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, J. Math. Anal. Appl. 415(2), 2014, 931-947.
[4] Y. Deng, M. O’Brien and V. G. Troitsky, Unbounded norm convergence in Banach lattices, Positivity.21(3),2018,963-974.
[5] M. Kandić, M. A. A. Marabe and V. G. Troitsky, Unbounded norm topology in Banach lattices, J. Math. Anal. Appl. 451(1), 2017, 259-279.
[6] Zabeti, Omid, Unbounded absolute weak convergence in Banach lattices, Positivity 22.3 (2018): 837-843.
[7] Bahramnezhad, Akbar, and Kazem Haghejad Azar, Unbounded order continuous operators on Riesz spaces, Positivity 22.3 (2018): 837-843.
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