A POTENTIAL REDUCTION METHOD FOR TENSOR COMPLEMENTARITY PROBLEMS

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Abstract. As an extension of linear complementary problem, tensor complementary problem has been effectively applied in n-person noncooperative game. And a multitude of researchers have focused on its properties and theories, while the valid algorithms for tensor complementary problem is still deficient. In this paper, stimulated by the potential reduction method for linear complementarity problem, we present a new algorithm for the tensor complementarity problem, which combines the idea of damped Newton method and the interior point method. Utilizing the new algorithm, we settle the tensor complementarity problem with the underlying tensor being diagonalizable and positive definite. Furthermore, the global convergence of the iterative scheme is theoretically guaranteed and the given preliminary numerical experiments indicate the efficiency of the method.

1. Introduction. As a natural extension for the matrix case, high order tensors have attracted much attention of researchers in recent decade [4, 7, 8, 24, 25, 26, 36, 37, 38, 39, 42]. Particularly, it finds wide applications in higher-order statistics [2], signal and image processing [27], blind source separation [43], hypergraph theory [5, 6], and so on.

Let $\mathcal{A} = (A_{i_1\ldots i_m})$ be an $m$-th order $n$-dimensional tensor. Denote the set of $m$-th order $n$-dimensional tensors by $T_{m,n}$. If the entries $A_{i_1i_2\ldots i_m}$ are invariant under any permutation of their indices, $\mathcal{A}$ is called symmetric. We denote the set of all symmetric tensors by $S_{m,n}$.

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Recently, tensor complementarity problem (TCP) was introduced by Song and Qi [30], where the involved function is defined by some homogeneous polynomial of degree $m - 1$ with $m > 2$. It is known that the tensor complementarity problem is a generalization of the linear complementarity problem [9], and also a subclass of nonlinear complementarity problems [11], which has been studied by many scholars [3, 19, 30, 31].

Suppose $A \in T_{m,n}$ and $q \in \mathbb{R}^n$, the tensor complementarity problem (TCP($A, q$)) is defined as finding a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Ax^{m-1} + q \geq 0, \quad x^\top (Ax^{m-1} + q) = 0,$$

where $Ax^{m-1} \in \mathbb{R}^n$ is defined by

$$(Ax^{m-1})_i = \sum_{i_2, \cdots, i_m = 1}^n A_{i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \cdots, n.$$

Quite recently, an explicit relationship between the solutions of an $n$-person non-cooperative game and the tensor complementarity problem was established, which builds a bridge between these two classes of problems [14]. In addition, it is proved that finding a Nash equilibrium point of the multilinear game is equivalent to finding a solution of the resulted tensor complementarity problem [14]. Furthermore, several more interesting problems for the TCP also attracted plenty of attention [1, 3, 10, 13, 15, 30, 31, 40] such as the existence of solutions, the structure of the solution set and the error bound problem. To the best of our knowledge, the TCP problems with general tensors are not easy to solve numerically. However, it may be feasible for the TCP problems with tensors having special structures.

Song and Qi [29] proved the solution existence for the tensor complementarity problem under conditions that all diagonal entries of the underlying tensor are positive, as well as they showed the equivalence property between the positive definiteness of a real tensor and the solution existence of the corresponding TCP. Song and Yu [32] studied the boundedness of the solution set for the TCP with underlying $R_0$-tensors. Che, Qi and Wei [3] established the nonempty and compactness of the solution set if the underlying tensor is positive definite or strictly copositive. Bai, Huang and Wang [1] investigated the global uniqueness and solvability of the TCP. Besides, the authors obtained the same conclusion about nonempty and compactness solution set with $P$-tensor tensors. Ding, Luo and Qi [10] proposed a modified version of $P$-tensors including odd-order tensors and proved corresponding tensor complementarity problems possessing nonempty and compact solution sets. Wang, Huang and Bai [40] established ER-tensors which contains many important classes of tensors as subclasses, and also investigated the nonempty and compact solution set of this class TCP. Gowda, Luo and Qi [13] described various equivalent conditions for a $Z$-tensor to have the global solvability property with complementarity problems.

For the sparse TCP, Luo, Qi and Xiu [19] considered the solution existence of the tensor complementarity problem with the underlying tensor $A$ being a $Z$-tensor. Furthermore, Xie, Li and Xu [41] established an iterative solution method for solving the problem which may be the first algorithm for the tensor complementarity problem. Note that the primal-dual potential method is an efficient solution method for solving the LCP [33, 34], as well as it can be extended to the generalized linear complementarity problem [20]. Now a question arises naturally: Can we establish
A potential efficient method for tensor complementarity problem? This is the main motivation of this manuscript.

In this paper, by combining the damped Newton method and the interior point method, we propose a potential reduction method for the tensor complementarity problem. Moreover, the global convergence is established under milder conditions. The remainder of this paper is organized as follows. In Section 2, some preliminaries of tensors are presented, apart from basic notions of some structured tensors. In Section 3, we give an equivalent reforestation for the tensor complementarity problem, and also some additional properties for the equivalent reforestation are established. In Section 4, an implementation iterative scheme is proposed for the tensor complementarity problem, as well as the convergence of this algorithm is demonstrated. In Section 5, the numerical experiments are provided to show the efficiency of the proposed method. Finally, some remarks are drawn in the last section.

Before move on, we make some comments on notations that will be employed in the sequel. Tensors considered in this paper are \( m \)-th order and \( n \)-dimensional, in which \( m, n \geq 2 \) are positive integers and \( m \) is even. Throughout this paper, we use bold and calligraphic letters like \( \mathcal{A}, \mathcal{D} \) for higher order tensors and uppercase letters like \( A, X \) for matrices. Then let \( A_{i_1i_2\cdots i_m} \in \mathbb{R} \) be the \( i_1i_2\cdots i_m \)-th entry of tensor \( A \), as well as \( A_{ij} \in \mathbb{R} \) be the \( ij \)-th entry of matrix \( A \). Denote the diagonal matrix as \( \mathcal{D}_x = \text{diag}(x) \) with diagonal entries \( x_i \), \( i = 1, 2, \cdots, n \). Let letters \( x, y, q \) be vectors which are usually \( n \)-dimensional column vectors with \( x_i, y_i, q_i \) for their \( i \)-th entry. The inner product of vectors \( x, y \in \mathbb{R}^n \) is denoted by \( x^\top y \). \( e \) means the all one vector and \( 0 \) means the all zero vector. Using lower case letters and Greek alphabet, say \( s, \theta \), to refer to scalars. Let \( [n] = \{1, 2, \cdots, n\} \), as well as \( \mathbb{R}_+^n \), \( \mathbb{R}_+^{n \times n} \) denote the nonnegative and positive orthant of \( \mathbb{R}^n \), respectively. Denote \( \| \cdot \| \) and \( \| \cdot \|_\infty \) as the Euclidean norm and the infinity norm of a vector or a matrix, correspondingly. The Jacobian matrix of mapping \( F: \mathbb{R}^n \rightarrow \mathbb{R}^m \) at \( x \in \mathbb{R}^n \) is denoted by \( \nabla F(x) \).

For tensor \( \mathcal{A} \) and matrix \( X \in \mathbb{R}^{n \times n} \), their product on mode-\( k \) [18] is defined as

\[
(\mathcal{A} \times_k X)_{i_1i_2\cdots j_k\cdots i_m} = \sum_{i_k = 1}^n A_{i_1i_2\cdots i_k\cdots i_m} X_{i_kj_k},
\]

and we define

\[
\mathcal{A} X^{m-1} = \mathcal{A} \times_2 X \times_3 X \cdots \times_m X.
\]

2. Preliminaries. In this section, we recall some basic definitions, and also some useful existing conclusions are collected.

**Definition 2.1.** [29] A tensor \( \mathcal{A} \in \mathcal{T}_{m,n} \) is called

(i) an \textbf{R-tensor} iff the following system is inconsistent

\[
\begin{cases}
0 \neq x \geq 0, \ t \geq 0, \\
(\mathcal{A} x^{m-1})_i + t = 0, \ 	ext{if} \ x_i > 0, \\
(\mathcal{A} x^{m-1})_j + t \geq 0, \ 	ext{if} \ x_j = 0;
\end{cases}
\]

(ii) an \textbf{R_0-tensor} iff the above system is inconsistent for \( t = 0 \).

Obviously, this definition is a natural extension of the definition of Karamardian’s class for regular matrices. Moreover, some equivalent definitions are introduced in [29].
Theorem 2.2. Let $A \in T_{m,n}$, then

(i) $A$ is an $R_0$-tensor if and only if the TCP($A, 0$) has a unique solution $0$;
(ii) $A$ is an $R$-tensor if and only if it is an $R_0$-tensor and the TCP($A, e$) has a unique solution $0$, where $e = (1, 1, \cdots, 1)^T$.

Definition 2.3. [23] Tensor $A \in S_{m,n}$ is said to be positive definite if $A x^m > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$. The set of all positive definite tensors is denoted by $PD_{m,n}$.

Definition 2.4. [29] Tensor $A \in T_{m,n}$ is said to be

(i) semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in [n]$ such that $x_k > 0$ and $(Ax^{m-1})_k \geq 0$;
(ii) strictly semi-positive iff for each $x \geq 0$ and $x \neq 0$, there exists an index $k \in [n]$ such that $x_k > 0$ and $(Ax^{m-1})_k > 0$.

It is not difficult to see that each strictly semi-positive tensor is certainly both $R$-tensor and $R_0$-tensor. In addition, a positive definite tensor must be strictly semi-positive under the condition that the underlying vector $x$ is nonnegative. Furthermore, the positive definite tensor is an $R_0$-tensor.

The following conclusion is based on the positive definiteness of tensors [3].

Theorem 2.5. If $A$ is positive definite, then the TCP($A, q$) has a nonempty, compact solution set.

Similar to the diagonalizable matrices [12], diagonalizable tensors were defined by Che et al. [3].

Definition 2.6. Suppose that $A \in S_{m,n}$, $A$ is called diagonalizable if $A$ can be represented as

$$A = D \times_1 X \times_2 X \cdots \times_m X,$$

where $X \in \mathbb{R}^{n \times n}$ is nonsingular and $D$ is a diagonal tensor. Denote all diagonalizable tensors by $SD_{m,n}$.

For a positive definite tensor $A \in SD_{m,n}$, the following conclusion shows the property of the Jacobian matrix $\nabla F(x)$, where $F(x) = Ax^{m-1} + q$ and $x \in \mathbb{R}^n$.

Theorem 2.7. [3] Let $A \in SD_{m,n}$ be positive definite, then the Jacobian matrix $\nabla F(x) = (m - 1)Ax^{m-2}$ is positive definite with $x \neq 0$.

Theorem 2.8. [3] If $A$ is diagonalizable and positive definite, then the TCP($A, q$) has an unique solution.

3. New reformulation of the TCP. In this section, we firstly establish equivalent reforestation of the TCP($A, q$), and then some interesting conclusions are discussed.

Certainly, the TCP($A, q$) can be equivalently reformulated as the following system:

$$H(x, y) = \begin{bmatrix} x \circ y \\ y - Ax^{m-1} - q \end{bmatrix} = 0,$$  \hspace{1cm} (2)

$$x \geq 0, \quad y \geq 0.$$
where \( x \circ y = (x_1 y_1, x_2 y_2, \cdots, x_n y_n)^T \). To move on, the following notion is needed.

**Definition 3.1.** For the deformation of TCP(\(A, q\)), we denote the feasible set of (2) by

\[
S_+ = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y = Ax^{m-1} + q, \ x \geq 0, \ y \geq 0\},
\]

and its strictly feasible set by

\[
S_{++} = \{(x, y) \in S_+ \mid x > 0, \ y > 0\}.
\]

Let the set

\[
C = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x_i y_i = 0, \ i = 1, 2, \cdots, n\},
\]

and we denote the solution set of TCP(\(A, q\)) by \( S^* \), then \( S^* = S_+ \cap C \).

For interior point methods to solve nonlinear complementarity problems, it is difficult to ensure that the iterative sequence satisfies the feasibility since the involved function is nonlinear. So, a multitude of researchers explore the so-called infeasible interior point method [21, 22, 28]. Motivated by this method, we define a new set \( \Omega_{++} \) for the TCP(\(A, q\)) such that

\[
\Omega_{++} = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \|y - Ax^{m-1} - q\|_{\infty} < \epsilon\},
\]

where \( \epsilon > 0 \) is a small tolerance.

In last section, we have given the conclusion that if \( A \) is positive definite, then the TCP(\(A, q\)) has a nonempty, compact solution set. Furthermore, if \( A \) is an even order diagonalizable and positive definite tensor, then the TCP(\(A, q\)) has an unique solution, and also the corresponding matrix \((m - 1)A^{m-2}\) is positive definite. In this section, we will adopt the potential reduction method for solving TCP(\(A, q\)) under an even order positive definite diagonal tensor, as well as obtain its approximate solution.

Let \( x \in \mathbb{R}^n_+ \), \( y \in \mathbb{R}^n_+ \), \( h \in \mathbb{R}^n \), \( D_x = \text{diag}(x) \) and \( D_y = \text{diag}(y) \), we consider the following system of equations

\[
\begin{bmatrix}
 D_y & D_x \\
 -(m - 1)A^{m-2} & I
\end{bmatrix}
\begin{bmatrix}
 \Delta x \\
 \Delta y
\end{bmatrix}
= \begin{bmatrix}
 h \\
 -y + Ax^{m-1} + q
\end{bmatrix}.
\]

(3)

The system has the following property which can be obtained directly form Lemma 4.1 in [16].

**Theorem 3.2.** Suppose that \( A \in SD_{m,n} \cap PD_{m,n} \), then for all \((x, y) \in \Omega_{++}\), the matrix

\[
H(x, y)' = \begin{bmatrix}
 D_y & D_x \\
 -(m - 1)A^{m-2} & I
\end{bmatrix}
\]

is nonsingular.

From Theorem 3.2, we known that nonsingular matrix \( H(x, y)' \) is exactly the Jacobian matrix of \( H(x, y) \) at the point \((x, y)\). Therefore, we can construct the following Newton equation for \( H(x, y)' + H(x, y)'(\Delta x, \Delta y)^T = 0 \) at a given point \((x, y) \in \Omega_{++}\),

\[
\begin{bmatrix}
 D_y & D_x \\
 -(m - 1)A^{m-2} & I
\end{bmatrix}
\begin{bmatrix}
 \Delta x \\
 \Delta y
\end{bmatrix}
= \begin{bmatrix}
 -D_x y \\
 -y + Ax^{m-1} + q
\end{bmatrix}.
\]

(4)

According to the fact that \( H(x, y)' \) is nonsingular, it follows that (4) exists an unique solution \((\Delta x^*, \Delta y^*)\) which is called affine-scaling direction. Correspondingly, by the approximate Newton equation

\[
\begin{bmatrix}
 D_y & D_x \\
 -(m - 1)A^{m-2} & I
\end{bmatrix}
\begin{bmatrix}
 \Delta x \\
 \Delta y
\end{bmatrix}
= \begin{bmatrix}
 -D_x y + \frac{x^T y e}{\lambda} \\
 -y + Ax^{m-1} + q
\end{bmatrix},
\]

(5)
we construct the centering direction which is the unique solution \((Δx^c, Δy^c)\) of (5).
The convex combination of the above two directions is
\[
(Δx, Δy) = β(Δx^c, Δy^c) + (1 - β)(Δx^a, Δy^a), \quad β ∈ [0, 1].
\]
Consequently, \((Δx, Δy)\) is the unique solution of the following equations:
\[
\begin{bmatrix}
  D_y \\
  -(m - 1)Ax^{m-2}
\end{bmatrix}
\begin{bmatrix}
  Δx \\
  Δy
\end{bmatrix}
= \begin{bmatrix}
  -D_xy + β\frac{xe}{n} \\
  -y + Ax^{m-1} + q
\end{bmatrix}.
\]

4. Algorithm and its convergence. In this section, we extend the potential reduction methods to tensor complementary problems, and also present the corresponding algorithm. Furthermore, the convergence of this algorithm is established.

The potential reduction methods for the LP and the LCP are based on a special potential function, as well as the concerned problem is solved by an iterative procedure which makes the potential function value tend to \(-∞\) along with the progress of iterative procedure.

To extend the potential reduction method to the TCP\((\mathcal{A}, q)\), we adopt the potential function proposed in [17, 20, 33, 34, 35]:
\[
ϕ(z) = (n + s) \ln(x^T y + ||y - Ax^{m-1} - q||^2) - \sum_{i=1}^n \ln x_i y_i,
\]
where \(z = (x, y)\) and \(s > 0\) is a parameter.

In order to explain the relationship between the potential function and the solution of TCP\((\mathcal{A}, q)\), we provide the following conclusion.

**Theorem 4.1.** If \(\lim_{k→+∞} ϕ(z^k) = -∞\), where \(\{z^k\} ⊂ Ω_+\) and \(z^k = (x^k, y^k)\), then
\[
\lim_{k→+∞} (x^k)^T y^k = 0 \quad \text{and} \quad \lim_{k→+∞} \frac{y^k - A(x^k)^{m-1} - q}{||y - Ax^{m-1} - q||^2} = 0.
\]

**Proof.** Since \(z = (x, y) ∈ Ω_+\), then
\[
ϕ(z) = s \ln(x^T y + ||y - Ax^{m-1} - q||^2) + n \ln(x^T y + ||y - Ax^{m-1} - q||^2)
- \sum_{i=1}^n \ln x_i y_i
≥ s \ln(x^T y + ||y - Ax^{m-1} - q||^2) + n \ln x^T y - \sum_{i=1}^n \ln x_i y_i
= s \ln(x^T y + ||y - Ax^{m-1} - q||^2) + n \ln \frac{x^T y}{(\prod_{i=1}^n x_i y_i)^\frac{1}{n}} + n \ln n
≥ s \ln(x^T y + ||y - Ax^{m-1} - q||^2) + n \ln n.
\]
It follows that if \(ϕ(z^k) → -∞\), the \(\lim_{k→+∞} (x^k)^T y^k = 0\) and \(\lim_{k→+∞} \frac{y^k - A(x^k)^{m-1} - q}{||y - Ax^{m-1} - q||^2} = 0\).

Based on this, we will establish a line search method to generate a sequence \(\{z^k\} ⊂ Ω_+\), such that sequence \(\{ϕ(z^k)\}\) is strictly decreasing and diverges to \(-∞\) as \(k → +∞\).

First, we discuss the choice of the searching direction. For the current point \(z ∈ Ω_+\), using the following nonlinear system of equations
\[
\begin{bmatrix}
  D_y \\
  -(m - 1)Ax^{m-2}
\end{bmatrix}
\begin{bmatrix}
  Δx \\
  Δy
\end{bmatrix}
= \begin{bmatrix}
  -D_xy + β\frac{xe}{n} \\
  -y + Ax^{m-1} + q
\end{bmatrix}.
\]
to generate a searching direction, where parameter $\beta \in [0, 1]$. However, in our
given numerical experiments, taking $\beta \in (0, 1]$ to bias the searching direction for
the interior of the nonnegative orthant $(x, y) \geq 0$, then we can move further along
the searching direction before violating the nonnegative constraints.

Consequently, we denote the solution of equations (7) by $d = (\Delta x, \Delta y)^T$, and
also it is a descent direction of potential function $\varphi(z)$ at $z$ as showed in the following
conclusion.

**Theorem 4.2.** Let $d = (\Delta x, \Delta y)^T$ be a solution of (7), then
$$\nabla \varphi(z)^T d \leq -(1 - \beta)s < 0,$$
and there exists a scalar $\bar{\theta} > 0$, such that for all $\theta \in (0, \bar{\theta})$
$$z + \theta d \in \Omega_{++}, \quad \varphi(z + \theta d) - \varphi(z) < -\alpha \theta (1 - \beta)s < 0.$$

*Proof.* Obviously, function $\varphi(z)$ is continuously differentiable on the domain $\Omega_{++}$,
and its gradient is
$$\nabla \varphi(z) = \frac{(n+s)}{x^T y + ||y - Ax^{m-1} - q||^2} \left( y - 2||y - Ax^{m-1} - q||(m-1)Ax^{m-2} \right) /
+ 2||y - Ax^{m-1} - q||e
+ \begin{pmatrix} D_x^{-1} e \\ D_y^{-1} e \end{pmatrix}.$$

From (7), it follows that
$$D_x \Delta x + D_y \Delta y = -D_x y + \beta \frac{x^T y}{n} e,$$
and
$$-[(m-1)Ax^{m-2}]^T \Delta x + \Delta y = -y + Ax^{m-1} + q.$$

Therefore,
$$\nabla \varphi(z)^T d = \frac{(n+s)}{x^T y + ||y - Ax^{m-1} - q||^2} \left[ e^T (D_y \Delta x + D_x \Delta y) - 2||y - Ax^{m-1}
- q||(m-1)Ax^{m-2} \right) /
+ 2||y - Ax^{m-1} - q||e
+ \begin{pmatrix} D_x^{-1} e \\ D_y^{-1} e \end{pmatrix}.$$

According to the fact that the function $H$ is continuously differentiable on the domain $\Omega_{++}$ which is an open set, we conclude that there exists a scalar $\bar{\theta} > 0$ such
that for all $\theta \in (0, \bar{\theta})$, we have $z + \theta d \in \Omega_{++}$ and
$$\varphi(z + \theta d) - \varphi(z) \leq \theta (\nabla \varphi(z)^T d + (1 - \alpha)(1 - \beta)s) \leq -\alpha \theta (1 - \beta)s < 0.$$

From the above theorem, it follows that the potential function $\varphi(z)$ is monoton-
ously decreasing. Consequently, we can obtain the following conclusion about the
solution of TCP($A$, $q$).

**Theorem 4.3.** Let $\{x^k\} \subset \Omega_{++}$. If $\lim_{k \to +\infty} x^k = \bar{z}$ for some $\bar{z} = (\bar{x}, \bar{y}) \in \Omega_{++}$, and
$\bar{x}_j \bar{y}_j = 0$ for some $j \in [n]$, then $\bar{z}$ is a solution of the TCP($A$, $q$).
Proof. The sequence \( \{z^k\} \) is bounded owing to \( \lim_{k \to +\infty} z^k = \bar{z} \). Then there exists \( \xi \in \mathbb{R} \) such that
\[
\ln x_i^k y_i^k \leq \xi, \quad i \in [n], \ k = 1, 2, \ldots.
\]
Therefore, for every \( k = 1, 2, \ldots \)
\[
(n + s) \ln (x^k)^\top y^k + ||y^k - A(x^k)^{m-1} - q||^2 = \varphi(z^k) + \sum_{i=1}^n \ln x_i^k y_i^k 
\leq \varphi(z^0) + (n - 1)\xi + \ln x_j^k y_j^k.
\]
Letting \( k \to +\infty \) yields
\[
\lim_{k \to +\infty} (n + s) \ln (x^k)^\top y^k + ||y^k - A(x^k)^{m-1} - q||^2 = -\infty,
\]
or equivalently
\[
\lim_{k \to +\infty} (x^k)^\top y^k = 0 \quad \text{and} \quad \lim_{k \to +\infty} ||y^k - A(x^k)^{m-1} - q||^2 = 0.
\]
which imply that \( \bar{z} \) is a solution of the TCP\((A, q)\).

Based on the above analysis, we arrive at the position to give the description of our potential reduction algorithm for solving the TCP\((A, q)\).

**Algorithm 1.**

**Step 1 (Initialization)** Let \( s > 0, \varepsilon > 0, \alpha \in (0, 1), \beta \in [0, 1), \sigma > 0, \) and \( \rho \in (0, 1) \) be given. Choose any \( z^0 \in \Omega_++, \beta_0 \in [0, \beta) \) and set \( k = 0 \).

**Step 2 (Direction generation)** Solve the system of equations (7) at \( z = z^k \) to obtain the search direction \( d^k \).

**Step 3 (Stepsize determination)** Let \( l_k \) be the smallest nonnegative integer \( l \) such that the following two conditions hold:
\[
z^k + \sigma \rho^l d^k \in \Omega_++, \quad \varphi(z^k + \sigma \rho^l d^k) - \varphi(z^k) \leq -\alpha \sigma \rho^l (1 - \beta_k) s.
\]
Set \( z^{k+1} = z^k + \sigma \rho^l d^k \).

**Step 4 (Termination verification)** If \( z^{k+1} \) satisfies \( f(z^{k+1}) = (z^{k+1})^\top y^{k+1} + ||y^{k+1} - A(z^{k+1})^{m-1} - q||^2 < \varepsilon, \) then \( z^{k+1} \) is the approximate solution of TCP\((A, q)\).

else if \( z^{k+1} - z^k < \varepsilon \) and \( \ln x_j^{k+1} y_j^{k+1} = 0 \) for some \( j \in [n] \), then \( z^{k+1} \) is the approximate solution; otherwise pick any \( \beta_{k+1} \in [0, \beta] \) and return to Step 2 with \( k = k + 1 \).

To prove the convergence of the algorithm, we make a few remarks. Firstly, the direction \( d^k \) determined in Step 2 is well defined in view of Theorem 3.2. Secondly, Theorem 4.2 justifies that the integer \( l^k \) in Step 3 is also well defined. In fact, \( l^k \) can be determined in a finite number of trials with starting \( l = 0 \) and increasing \( l \) by one each time until either the first or the second equation in Step 3 fails to hold. Next, the sequence of function values \( \varphi(z^k) \) is strictly decreasing owing to Step 3 of the algorithm. Finally, for the stopping rule required in the algorithm, according to the Theorem 4.1 and Theorem 4.3, our choice is to give a tolerance \( \varepsilon \), and also to check if the quantity \( f(z^{k+1}) \) is less than \( \varepsilon \) or \( ||z^{k+1} - z^k|| < \varepsilon \) with \( \ln x_j^{k+1} y_j^{k+1} = 0 \) for some \( j \in [n] \). If so, \( z^{k+1} \) is taken to be an approximate solution of TCP\((A, q)\); otherwise we return to Step 2 to generate the next iteration.

In the remainder of this section, we establish a limiting property of the sequence \( \{z^k\} \) generated by the algorithm. We need the following technical lemma [35].
Lemma 4.4. Let
\[ \phi(u, v) = (n + s) \ln(||u||^2 + e^T v) - \sum_{i=1}^{n} \ln v_i, \quad (u, v) \in \mathbb{R}^n \times \mathbb{R}^n, \]
where \( s > 0 \) is a constant. Then for all \( \mu > 0 \) and \( \nu \in \mathbb{R} \),
\[ \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : ||u||^2 + e^T v \geq \mu, \phi(u, v) \leq \nu \} \]
is a compact set.

Before stating the main convergence result for the potential reduction algorithm, we present an important conclusion as follow.

Theorem 4.5. Let \( A \in SD_{m,n} \) be positive definite, if the potential reduction algorithm is initialized at \( x^0 = (x^0, y^0) \in \Omega_{++} \), then it generates an iterated sequence \( \{x^k\} = \{(x^k, y^k)\} \subset \Omega_{++} \), satisfying that both the sequences \( \{(x^k)^T y^k\} \) and \( \{||y^k - A(x^k)^{m-1} - q||^2\} \) are bounded.

Proof. From Theorem 4.2, it follows the fact that \( \{\varphi(x^k)\} \) is decreasing, and according to the proof of Theorem 4.1, we gain that
\[ \varphi(x^k) \geq s \ln((x^k)^T y^k + ||y^k - A(x^k)^{m-1} - q||^2) + n \ln n, \]
then
\[ 0 \leq (x^k)^T y^k + ||y^k - A(x^k)^{m-1} - q||^2 \leq \exp\left(\frac{\varphi(x^k) - n \ln n}{s}\right), \quad \forall k \geq 0, \]
which implies the sequence \( \{f(x^k)\} \) is bounded, where \( f(x^k) = (x^k)^T y^k + ||y^k - A(x^k)^{m-1} - q||^2 \). Since \( x^k \in \Omega_{++} \), \( (x^k)^T y^k > 0 \) and \( ||y^k - A(x^k)^{m-1} - q||^2 > 0 \), then the sequence \( \{(x^k)^T y^k\} \) and \( \{||y^k - A(x^k)^{m-1} - q||^2\} \) are all bounded. \( \Box \)

According to this above result, it follows the main convergence result for the potential reduction algorithm.

Theorem 4.6. Suppose that \( A \in SD_{m,n} \cap PD_{m,n} \), then the iterated sequence \( \{x^k\} \) generated by the potential reduction algorithm satisfies the following properties:
(i) the iterated sequence \( \{x^k\} \) is bounded;
(ii) every accumulation point of \( \{x^k\} \) is a solution of TCP\((A, q)\).

Proof. We assume for contradiction that \( \{x^k\} \) is unbounded, then there exists a subsequence \( \{(x^{i_k}, y^{i_k})\} \), \( i_k \in [k] \) such that
\[ \lim_{i_k \rightarrow +\infty} ||(x^{i_k}, y^{i_k})|| = +\infty, \quad \lim_{i_k \rightarrow +\infty} \frac{(x^{i_k}, y^{i_k})}{||x^{i_k}, y^{i_k}||} = (x^*, y^*) \neq 0. \]

Consider the two sequences \( \{r^{i_k}\} \) and \( \{s^{i_k}\} \) defined for every \( k \geq 0 \) by
\[ r^{i_k} = (x^{i_k})^T y^{i_k}, \quad s^{i_k} = ||y^{i_k} - A(x^{i_k})^{m-1} - q||^2. \]
From Theorem 4.5, it follows that the two sequences are bounded. Dividing \( \{r^{i_k}\} \) and \( \{s^{i_k}\} \) by \( ||(x^{i_k}, y^{i_k})||^2 \), both them are tend to 0 with \( i_k \in [k] \) increase to +\infty, which implies that we get a nonzero solution \( (x^*, y^*) \) of TCP\((A, 0)\). In view of the positive definiteness of tensor \( A \) and the vector \( x^k \in \mathbb{R}^n_A \), \( A \) is an \( R_0 \)-tensor, and also the TCP\((A, 0)\) has the unique solution 0, which leads to a contradiction. Therefore, the sequence \( \{x^k\} \) must be bounded.
For the second part of the theorem, let $f(z) = x^Ty + ||y - Ax^{m-1} - q||^2$ and $z^*$ be the limit of a subsequence of $\{z^k\}$. It remains to show that $f(z^*) = 0$. In fact, if the assertion would not hold, then we obtain that $f(z^k) \geq \mu$ for some $\mu > 0$, with all $i_k \in [k]$. On account of the boundedness of $\{\varphi(z^k)\}$ (by $\varphi(z^0)$), it follows from Lemma 4.4 that $\{H(z^k) | i_k \in [k]\}$ is contained in a compact subset of $\mathbb{R}_{++}^{n_d}$, thus $(x^*)^Tv^* > 0$. In addition, owing to a nonempty and open set $\Omega_{++}$, it follows that $z^* \in \Omega_{++}$. According to the Theorem 3.2, the Jacobian matrix $H'(z^*)$ is nonsingular, and also the sequence $\{H'(z^k)^{-1} | i_k \in [k]\}$ converges to $H'(z^*)^{-1}$. Furthermore, due to $\beta_{ik} \leq \beta < 1$, we may assume that $\{\beta_{ik} | i_k \in [k]\}$ converges to a scalar $\beta_\ast \in (0, 1)$. In view of the fact that for every $i_k \geq 0$,

$$d^k = H'(z^k)^{-1}\left(-H(z^k) + \left[\beta_{ik} \frac{(x^k)^\top y^k}{n} e\right]\right),$$

it follows that $\{d^k | i_k \in [k]\}$ converges to the vector $d^\ast$ satisfying the equation

$$H'(z^*)d^\ast = -H(z^*) + \left[\beta_\ast \frac{(x^\ast)^\top y^\ast}{n} e\right].$$

Consequently, from Theorem 4.2 with $\alpha \in (0, 1)$, we conclude

$$\nabla \varphi(z^\ast)^\top d^\ast \leq -(1 - \beta_\ast) s < -\alpha(1 - \beta_\ast) s.$$

On the other hand, now that $\lim_{i_k(\in [k]) \to +\infty} z^k = z^\ast \in \Omega_{++}$, it follows from the continuity of $\varphi$ that $\lim_{i_k(\in [k]) \to +\infty} \varphi(z^k) = \varphi(z^\ast)$. According to the fact that the sequence $\{\varphi(z^k)\}$ is strictly decreasing, we obtain that the whole sequence $\{\varphi(z^k)\}$ converges to $\varphi(z^\ast)$, and also $\lim_{k \to +\infty} \varphi(z^{k+1}) - \varphi(z^k) = 0$. In addition, due to $\beta_{ik} \in [0, \beta]$, it follows from the step 3 of the algorithm that $\lim_{k \to +\infty} \rho_{ik} = 0$, or equivalently, $l_k \to +\infty$. From the facts that $\Omega_{++}$ is an open set containing $z^\ast$, it follows that the sequence $\{d^k | i_k \in [k]\}$ is bounded, and also $l_k \to +\infty$. Moreover, $z^k + \sigma_{\rho^k} d^k \in \Omega_{++}$ is satisfied for $l = l_k - 1$. Therefore the inequality $\varphi(z^k + \sigma_{\rho^k} d^k) - \varphi(z^k) \leq -\alpha \sigma_{\rho^k}(1 - \beta_k)s$ must be violated at $l = l_k - 1$ for all sufficiently large $i_k \in [k]$, that is

$$\varphi(z^k + \sigma_{\rho^k-l_k} d^k) - \varphi(z^k) > -\alpha \sigma_{\rho^k}(1 - \beta_k)s.$$

Dividing this inequality by $\sigma_{\rho^k-l_k}$ and letting $k \to +\infty$, we obtain in view of $H$ is continuously differentiable on $\Omega_{++}$ that

$$\nabla \varphi(z^\ast)^\top d^\ast \geq -\alpha(1 - \beta_\ast) s.$$

This leads to a contradiction, which shows that $f(z^*) = 0$.

5. Numerical experiments. In this section, we test the efficiency of the proposed algorithm for the TCP($\mathcal{A}, q$) under conditions that the tensor $\mathcal{A}$ is diagonalizable positive definite. Numerical experiments will be performed via Matlab R2009a on a 3.4 GHz Intel Pentium computers (Intel Corporation, Santa Clara, CA, USA) with a 4-core i5-5200 processor. The parameters used in the experiments are chosen as follows:

$$s = 1, \ \alpha = 0.6, \ \beta = 0.8, \ \sigma = 0.8, \ \rho = 0.4.$$

Notation Iter denotes the number of iterations. For each numerical experiment, we obtain numerical results under different parameters or different initial points.
Example 1. Consider the TCP($\mathcal{A}, q$) where $q = (0.01, 0.01)^\top$, $\mathcal{A}$ is a 4-th order 2-dimensional diagonalizable and positive definite tensor with entries

$$
\mathcal{A}(\cdot; 1, 1) = \begin{bmatrix} 2403 & 10 \\ 10 & 50 \end{bmatrix}, \quad \mathcal{A}(\cdot; 1, 2) = \begin{bmatrix} 10 & 50 \\ 50 & 250 \end{bmatrix},
$$

$$
\mathcal{A}(\cdot; 2, 1) = \begin{bmatrix} 10 & 50 \\ 50 & 250 \end{bmatrix}, \quad \mathcal{A}(\cdot; 2, 2) = \begin{bmatrix} 50 & 250 \\ 250 & 1250 \end{bmatrix}.
$$

This numerical experiment is performed under the condition that the parameter $\beta_0$ is fixed as 0.6 and the initial points are different, then the numerical results are shown as follows:

| $\varepsilon$ | Iter | Time (s) |
|---------------|------|----------|
| $10^{-3}$     | 16   | 0.128738 |
| $10^{-3}$     | 20   | 0.181242 |
| $10^{-3}$     | 23   | 0.168388 |
| $10^{-3}$     | 23   | 0.186210 |
| $10^{-5}$     | 25   | 0.182582 |
| $10^{-8}$     | 35   | 0.194419 |
| $10^{-8}$     | 38   | 0.189600 |
| $10^{-8}$     | 43   | 0.184916 |
| $10^{-8}$     | 50   | 0.185770 |
| $10^{-8}$     | 55   | 0.221645 |
| $10^{-8}$     | 58   | 0.186433 |

Regardless of which the initial points be chosen, the solution of the problem under this algorithm is uniquely determined, and the approximate optimal solution is $z^* = 10^{-3} \times (1, 2, 0, 0)^\top$.

Example 2. Consider the TCP($\mathcal{A}, q$) where $q = (0, 0)^\top$, $\mathcal{A}$ is a 4-th order 2-dimensional diagonalizable and positive definite tensor with entries

$$
\mathcal{A}(\cdot; 1, 1) = \begin{bmatrix} 256 & 0.0060 \\ 0.0060 & 0.1800 \end{bmatrix}, \quad \mathcal{A}(\cdot; 1, 2) = \begin{bmatrix} 0.0060 & 0.1800 \\ 0.1800 & 5.4000 \end{bmatrix},
$$

$$
\mathcal{A}(\cdot; 2, 1) = \begin{bmatrix} 0.0060 & 0.1800 \\ 0.1800 & 5.4000 \end{bmatrix}, \quad \mathcal{A}(\cdot; 2, 2) = \begin{bmatrix} 0.1800 & 5.4000 \\ 5.4000 & 162.0000 \end{bmatrix}.
$$

This numerical experiment is performed under the condition that the parameter $\beta_0$ can be choose as different numbers, as well as the initial points are randomly generated. The numerical result of this problem is presented in the following table.
Example 3. Consider the TCP($\mathcal{A}, q$) where $q = (0, 0, 0, 0, 0, 0)^T$, $\mathcal{A}$ is a 6-th order 6-dimensional diagonalizable and positive definite tensor whose entries are randomly generated.

The numerical result of this problem with random initial points and fixed parameters $\varepsilon = 10^{-6}$, $\beta_0 = 0.1$ is shown in the following table.

### Table 5.3. Numerical Results for Example 3.

| $x^*$ | Iter | Time(s) |
|-------|------|---------|
| $(0.0120, 0.0045, 0.0168, 0.0091, 0.0070, 0.0062)^\top$ | 23 | 27.810409 |
| $(0.0166, 0.0052, 0.0137, 0.0137, 0.0069, 0.0103)^\top$ | 23 | 30.307993 |
| $(0.0005, 0.0007, 0.0004, 0.0004, 0.0023, 0.0015)^\top$ | 34 | 31.502331 |
| $(0.0012, 0.0007, 0.0017, 0.0004, 0.0015, 0.0004)^\top$ | 35 | 25.909628 |
| $(0.0016, 0.0026, 0.0038, 0.0017, 0.0038, 0.0045)^\top$ | 29 | 36.274344 |
| $1.0e-003 \times (0.4151, 0.1557, 0.3255, 0.0922, 0.0142, 0.3467)^\top$ | 44 | 32.017439 |
| $1.0e-003 \times (0.0399, 0.2636, 0.3479, 0.2752, 0.4337, 0.4146)^\top$ | 42 | 29.446150 |

The solution of this problem with an random tensor under this algorithm is uniquely determined and the approximate optimal solution is $z^* = (x^*, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)^\top$.

Example 4. Consider the TCP($\mathcal{A}, q$) where $\mathcal{A}$ is an $m$-th order $n$-dimensional diagonalizable and positive definite tensor and $q$ is a $n$-dimensional zeros vector.

Using the Matlab software, firstly, we could randomly generate a diagonal tensor $D \in T_{m,n}$, as well as a nonsingular matrix $X \in T_{2,n}$. According to calculating the following equation $\mathcal{A} = D \times_1 X \times_2 X \times_{m} X$, it follows that $\mathcal{A}$ is a diagonalizable tensor. In addition, from the algorithm which proposed in [42], we can identify whether $\mathcal{A}$ is positive definite. If so, we find a specific example, otherwise repeat
the above steps. In general, we choose the first tensor which satisfies the above conditions as our numerical example.

This numerical experiment is performed under the condition that the parameters $\varepsilon, \beta_0$ are fixed as $10^{-6}, 0.1$ respectively, and also the initial points are randomly generated. The numerical result of these problems are presented in the following table.

Table 5.4. Numerical Results for Example 4.

| $m$ | $n$ | Iter | Time(s) |
|-----|-----|------|---------|
| 4   | 10  | 45   | 0.195758|
| 4   | 20  | 46   | 1.025905|
| 4   | 40  | 49   | 14.254993|
| 4   | 50  | 50   | 34.858052|
| 4   | 60  | 50   | 90.753663|
| 4   | 80  | 51   | 465.798026|
| 4   | 100 | 51   | 2702.279664|
| 6   | 10  | 101  | 332.915881|
| 6   | 20  | 123  | 3420.345758|

From these numerical results, we can conclude that Algorithm 4.1 is efficient and stable.

6. Conclusion. In this paper, we extend the efficient potential reduction algorithm to the TCP($A, q$), as well as what was considered is positive definite tensor which have an important position both in theory and application. According to the fact that the tensor complimentary problem does not possess the linear properties which could guarantee the feasibility of the iterative sequence, we propose an infeasible interior point method based on potential function. In addition, the convergence analysis of the proposed algorithm is established. Furthermore, some numerical experiments are given to illustrate the effectiveness of this algorithm. As has been said before, the proposed iterative algorithm belongs to an interior method which has widely served in mathematical optimization. Finally, we leave the discussion of the computational complexity of the proposed algorithm for future research.

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