Non Hyperbolic Free-by-Cyclic and One-Relator Groups

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Abstract

We show that the free-by-cyclic groups of the form $F_2 \rtimes \mathbb{Z}$ act properly cocompactly on CAT(0) square complexes. We also show using generalised Baumslag-Solitar groups that all known groups defined by a 2-generator 1-relator presentation are either SQ-universal or are cyclic or isomorphic to $BS(1,j)$. Finally we consider free-by-cyclic groups which are not relatively hyperbolic with respect to any collection of subgroups.

1 Introduction

The recent far reaching work of Agol [1] and Wise [47] proves that a word hyperbolic group $G$ acting properly and cocompactly on a CAT(0) cube complex must be virtually special, implying that $G$ has a finite index subgroup which embeds in a right angled Artin group (RAAG). A host of very strong conclusions then apply to the group $G$, of which the two that will concern us here are being linear (which we take to mean over $\mathbb{C}$ although it is even true over $\mathbb{Z}$) and (if $G$ is not elementary) being large, namely $G$ has a finite index subgroup surjecting to a non abelian free group.

However, if $G$ is a finitely presented non hyperbolic group acting properly and cocompactly on a CAT(0) cube complex, then the above consequences need no longer hold, indeed $G$ can even be simple [13]. Therefore suppose we have a class of finitely presented groups which is believed to be a well behaved class, but which contains both word hyperbolic and non word hyperbolic examples. We can ask: first, do all examples in this class have a nice geometric
action, namely a proper cocompact action on a CAT(0) cube complex, and second: do they all enjoy our strong group theoretic properties which are a consequence of being a virtually special group, namely being linear or being large. Note that for non word hyperbolic groups, satisfying this geometric condition will not necessarily imply these group theoretic properties.

In this paper the classes of groups we will be interested in are the following three: groups of the form $F_k \rtimes \alpha \mathbb{Z}$ for $F_k$ a free group of finite rank $k$ and $\alpha$ an automorphism of $F_k$ (we refer to these as “free-by-cyclic groups”); the more general class of ascending HNN extensions $F_k \ast_{\theta}$ of finite rank free groups, where rather than $\theta$ having to be an automorphism, as in the free-by-cyclic case, we allow $\theta$ to be any injective endomorphism of $F_k$; and finally the class of groups admitting a presentation with 2 generators and 1 relator, which we refer to as 2-generator 1-relator groups. This last class neither contains nor is contained in either of the other two classes but there is considerable overlap.

In the free-by-cyclic case, it was recently shown in [30] that such word hyperbolic groups do act properly and cocompactly on a CAT(0) cube complex, and therefore are virtually special groups. However Gersten in [26] gave an example of a free-by-cyclic group which cannot act properly and cocompactly on any CAT(0) space, so this result cannot hold in general in the non word hyperbolic case. Moreover [6],[11] shows that there are free-by-cyclic groups which are not automatic, whereas groups that act nicely on CAT(0) cube complexes are automatic [41]. In Gersten’s example the free group has rank 3 but in Section 2 we consider free-by-cyclic groups of the form $F_2 \rtimes \alpha \mathbb{Z}$, none of which are word hyperbolic. Therefore it is of interest to show directly that they act properly and cocompactly on CAT(0) cube complexes, which we do in Section 2. This work is based on unpublished work of Bridson and Lustig. Those authors give us the method of changing the natural topological model of the standard 2 complex (shown in Figure 1) to get rid of a pocket of positive curvature, and in this way they go on to show that these groups act on 2-dimensional CAT(0) complexes. We expand on this by showing that one can build these complexes from squares. This also strengthens a result of T. Brady [7] who showed that there is a 2-complex of non-positive curvature made from equilateral triangles with fundamental group $F_2 \rtimes \alpha \mathbb{Z}$.

In Section 3 we consider 2-generator 1-relator groups. It is conceivable, but very definitely open, that a word hyperbolic 2-generator 1-relator group always acts properly and cocompactly on a CAT(0) cube complex (for instance see [48] Conjecture 1.9). However on moving to the non word hyper-
bolic case we see a different picture emerging because a group acting properly and cocompactly on any CAT(0) space cannot contain a Baumslag-Solitar group $BS(m, n)$ where $|m| \neq |n|$. Thus the examples of such nasty Baumslag-Solitar groups as $BS(2, 3)$ mean that we need not always have largeness nor linearity, or even residual finiteness. In fact even restricting to residually finite 2-generator 1-relator groups will not imply linearity in general. This is because it was shown in [10] that the group $\langle s, a, b | sas^{-1} = a^m, sbs^{-1} = b^n \rangle$ is not linear over any field if $|m|, |n| > 1$, and it was pointed out in [22] that the group $\langle t, a, b | tat^{-1} = b, tbt^{-1} = a^m \rangle$ which is indeed a 2-generator 1-relator group $\langle t, a | t^2 at^{-2} = a^m \rangle$ that is known to be residually finite, contains this as an index 2 subgroup when $m = n$.

However, in looking for a “large” property that we hope is held by all 2-generator 1-relator groups except for the soluble groups $BS(1, m)$ and $\mathbb{Z}$, including the non residually finite groups, we are led to the concept of a group $G$ being SQ-universal: namely that every countable group embeds in a quotient of $G$. It was conjectured by P. M. Neumann in [40] back in 1973 that a non cyclic 1-relator group is either SQ-universal or isomorphic to $BS(1, m)$.

Now it was shown in [45] that a group having a 1-relator presentation with at least 3 generators is SQ-universal, leaving the 2-generator 1-relator case. Also [42] from 1995 showed that all non elementary word hyperbolic groups are SQ-universal and this was generalized to non elementary groups which are hyperbolic relative to any collection of proper subgroups in [2] from 2007.

Recently the concept of a group being acylindrically hyperbolic, which is more general than being hyperbolic with respect to a collection of proper subgroups and which implies SQ-universality, was introduced in [44] and studied in [39] where one application was to 2-generator 1-relator groups. The authors divided these groups into three classes with the first consisting of groups that they could show were acylindrically hyperbolic. We prove in Theorem 3.2 that all the groups in their second case, which they show are not acylindrically hyperbolic, are indeed SQ-universal unless equal to $BS(1, m)$. In fact every group here is formed by taking an HNN extension with base equal to a quotient of some free-by-cyclic group $F_k \rtimes_\alpha \mathbb{Z}$ for $\alpha$ finite order, along with infinite cyclic edge groups. The proof proceeds by also identifying them as generalized Baumslag-Solitar groups, whereupon we show more generally in Theorem 3.2 that any generalized Baumslag-Solitar group either is SQ-universal or is isomorphic to $BS(1, m)$ or $\mathbb{Z}$.
This leaves their third case, which is exactly the class of 2-generator 1-relator groups that are ascending HNN extensions of finite rank free groups. Here we are not quite able to establish SQ-universality of all of these groups not equal to $BS(1, m)$ or $\mathbb{Z}$, though it is known to hold for the free-by-cyclic case, but we do show in Corollary 3.4 that the only possible exception would be a 2-generator 1-relator group equal to a strictly ascending HNN extension of a finite rank free group which either fails to be word hyperbolic and contains no Baumslag-Solitar subgroup, or does contain a Baumslag-Solitar subgroup (but does not contain $\mathbb{Z} \times \mathbb{Z}$) and where all finite index subgroups have first Betti number equal to 1. In both cases it is conjectured that no such examples exist, so we have established P. M. Neumann’s conjecture for all the known 2-generator 1-relator groups. We also obtain some general unconditional statements, such as Corollary 3.5 which says that if the relator is in the commutator subgroup of $F_2$ then $G$ is SQ-universal or equal to $\mathbb{Z} \times \mathbb{Z}$.

In the final section we show that free-by-cyclic groups formed using an automorphism $\alpha$ of polynomial growth are not hyperbolic relative to any collection of proper subgroups (thus SQ-universality cannot be established for all free-by-cyclic groups using only the result of [2]), but are acylindrically hyperbolic unless $\alpha$ has finite order as an outer automorphism.

In our final class of ascending HNN extensions of finitely generated free groups, it was shown for the word hyperbolic case in [29] that $F_k \ast \theta$ acts properly and cocompactly on a CAT(0) cube complex if $\theta$ is an irreducible endomorphism, although that still leaves the case where $\theta$ is not irreducible but $F_k \ast \theta$ is word hyperbolic. As for a non word hyperbolic group of the form $F_k \ast \theta$, there are some results on largeness for these groups in [13] which are not quite exhaustive but make it likely that largeness holds throughout. However the example mentioned above of the injective endomorphism $\theta(a) = a^m, \theta(b) = b^n$ of the rank two free group $F(a, b)$ for $m$ and $n$ both having modulus greater than 1, shows that $F_k \ast \theta$ need not be linear over any field. Moreover the possible existence in the non word hyperbolic case of Baumslag-Solitar subgroups $BS(1, m)$ where $|m| \neq 1$ again means that such a group need not have a proper cocompact action on any CAT(0) space. We finish by showing in Theorem 4.5 that the work in [30] and the Agol - Wise machinery answers Problem 17.108 in the recent edition of the Kourovka notebook [32], namely that Sapir’s example of a strictly ascending HNN extension of $F_2$ is indeed linear.

We would like to thank Martin Bridson and Martin Lustig for allowing us to reproduce the results from [10] here.
2 Square complexes for free-by-cyclic groups in the rank 2 case

2.1 Consequences

As mentioned before there are many nice consequences of word hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes. However, no automorphism of $F_2$ is hyperbolic since they all fix the conjugacy class of $[a, b]^{\pm 1}$, therefore, Agol’s theorem does not apply and the complexes constructed are not known to be virtually special, though, due to the work of various authors the groups are virtually special.

However, there are still advantages to having a group act on a CAT(0) cube complex. For instance, abelian subgroups are quasi-isometrically embedded, and such groups are biautomatic $[27, 41]$ and have a deterministic solution to the word problem in quadratic time $[25]$.

Groups which act on CAT(0) square complexes have the further nice property that all of their finitely presented subgroups also act properly and cocompactly on CAT(0) square complexes. This is proved using a tower argument (see $[9]$, p. 217) and the fact that a sub complex of a non-positively curved square complex is itself a non-positively curved square complex (this may fail in higher dimensions). The construction also shows that for $F_2$-by-$\mathbb{Z}$ groups their geometric dimension is equal to their CAT(0) dimension, namely 2.

2.2 Preliminaries

We assume that the reader is familiar with the basics of CAT(0) geometry for which the standard reference is $[9]$.

**Definition 2.1.** We say a metric space is non-positively curved if for each point there is a neighbourhood which is CAT(0).

In the following we will study 2 dimensional piecewise euclidean (PE) complexes. These are complexes built from polygonal subsets of $\mathbb{R}^2$ by gluing along edges, whereupon we put the natural path metric on the resulting complexes. For full details see $[9]$.

Square complexes are special examples of PE complexes where all the cells are squares.
The following theorems of Gromov [28] allow one to check whether a complex is non-positively curved just by looking at the links of vertices.

**Theorem 2.2.** [8] A PE complex with finitely many isometry types of cells is non positively curved if and only if the link of each vertex is a CAT(1) space.

In the two dimensional case the link of any vertex is a graph and so this can be reduced to the following.

**Lemma 2.3.** [9] A graph is CAT(1) if it contains no circuits of length less than $2\pi$.

**Definition 2.4.** We say that an action is proper if for each compact set $K$ the set \( \{ g \in G : gK \cap K \neq \emptyset \} \) is finite.

As these groups will be the fundamental groups of non-positively curved spaces, they have an action on the universal cover. Since the spaces are compact the action will be proper and cocompact and it will also be a free action since these groups are torsion free.

The groups $G_\phi$ that we shall be concerned with are mapping tori of $F_2 = F(x, y)$ by a single automorphism $\phi \in \text{Aut}(F_2)$. These groups have presentations of the form

$$\langle x, y, t | txt^{-1} = \phi(x), tyt^{-1} = \phi(y) \rangle.$$

We start by considering the case of automorphisms which are of finite order.

**Proposition 2.5.** If $\phi \in \text{Aut}(F_n)$ has order $q$ in $\text{Out}(F_n)$ then $G_\phi$ is the fundamental group of a non-positively curved 2-complex. Furthermore, this is finitely covered by $\Gamma \times S^1$ where $\Gamma$ is a graph with fundamental group $F_n$.

**Proof.** Every finite order automorphism $\phi$ of $F_n$ can be realised as an isometry of a finite graph $\Gamma$; see for instance [20] Theorem 2.1. Let $X = \Gamma \times [0, 1]/(x, 0) \sim (\phi(x), 1)$.

$X$ is locally isometric to $\Lambda \times (-\epsilon, \epsilon)$ where $\Lambda$ is a contractible subset of a graph. This will be CAT(0) and so $X$ is non-positively curved.

If we take the cover corresponding to the obvious map to $\mathbb{Z}_q$ this will be $\Gamma \times S^1$. 

\[\square\]
We now want to look at automorphisms of infinite order. We require the following lemmas to ensure that we account for the general case with our construction.

**Lemma 2.6.** $G_\phi$ is defined up to isomorphism by $[\phi] \in \text{Out}(F_2)$.

*Proof.* If $\phi = \text{ad}_g \psi$, for $\text{ad}_g \in \text{Inn}(F_2)$ the inner automorphism conjugation by $g$, then

$$G_\phi \cong \langle x, y, t | t^x t^{-1} = \phi(x), tyt^{-1} = \phi(y) \rangle$$

$$\cong \langle x, y, t | t^x t^{-1} = g\psi(x)g^{-1}, tyt^{-1} = g\psi(y)g^{-1} \rangle$$

$$\cong \langle x, y, t, t' | t^x t'^{-1} = \psi(x), t'y t'^{-1} = \psi(y), t' = g^{-1}t \rangle$$

$$\cong \langle x, y, t, t' | t^x t'^{-1} = \psi(x), t'y t'^{-1} = \psi(y) \rangle \cong G_\psi.$$ 

\[ \square \]

**Lemma 2.7.** $G_\phi$ is defined up to isomorphism by the conjugacy class of $[\phi] \in \text{Out}(F_2)$.

*Proof.* Let $\psi = \xi^{-1} \phi \xi$ then the following map defines an isomorphism.

$$\Omega : G_\psi \to G_\phi$$

$$g \mapsto \xi(g)$$

$$t \mapsto t.$$ 

\[ \square \]

As such we will restrict to conjugacy in $\text{Out}(F_2) = \text{GL}_2(\mathbb{Z})$.

In what follows we will need the following matrices:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$ 

**Lemma 2.8.** Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in \text{GL}_2(\mathbb{Z})$ be a matrix. Then at least one of $g, -g, Fg$ or $-Fg$ is conjugate to a matrix with all non-negative coefficients.

*Proof.* We will split into 2 cases. First we will deal with the case of matrices with no entry equal to 0. Conjugating and multiplying by $F$, we may assume that $|a| \geq |b|, |c|, |d|$. We may now replace $g$ by $-g$ to make $a > 0$. 

7
We note that every matrix of this form in $GL_2(\mathbb{Z})$ cannot have one negative entry, as if this is the case then we see $|ad - bc| \neq 1$.

We note that conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ changes the signs of $c$ and $b$. Using this we can assume that $a, c > 0$, then we know that $b, d < 0$ or $b, d > 0$. If we are in the second case we are done, so we assume we are in the first case.

We now conjugate by $R^{-1}$:

$$R^{-1}gR = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}.$$

We know that $a, c, a+b$ are $> 0$ so we see that $c+d \geq 0$. Now

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a-c & a+b-(c+d) \\ c & c+d \end{pmatrix}$$

so once again we know that $c, a-c, c+d$ are non-negative so $a+b-(c+d) \geq 0$.

If $g$ had an entry equal to 0 then either $g$ or $Fg$ is triangular. Further conjugating by $F$ and multiplying by $-1$ we can assume that $c = 0$ and $a = 1$. We now have two cases; namely $d = \pm 1$. In the case where $d = 1$ we can conjugate by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to make $b \geq 0$. If $d = -1$ then this matrix has order 2 and we do not worry about this case.

The following lemmas are from [19].

**Lemma 2.9.** If $g$ is a non-diagonal matrix with all entries non-negative, then there is a subtraction of one row from another, which reduces the sum of the entries and produces a non-negative result.

**Lemma 2.10.** Let $g \in GL_2(\mathbb{Z})$ with non-negative entries. Then there is a unique sequence of row subtractions which keep $g$ non-negative and reduce it to $I$ or $F$.

We can think of these operations as multiplication by $R$ or $L$ which gives us the following result.

**Corollary 2.11.** The semigroup generated by $-I, F, L$ and $R$ contains a conjugate of every infinite order matrix in $GL_2(\mathbb{Z})$.

As such we will only need to realise the automorphisms corresponding to these in our groups.
We will now construct non-positively curved complexes with $G_\phi$ as their fundamental groups, when $\phi$ is in the semigroup generated by $\lambda$:

- $\lambda : a \mapsto ba$
- $b \mapsto b$
- $\iota : a \mapsto a^{-1}$
- $b \mapsto b^{-1}$
- $\rho : a \mapsto a$
- $b \mapsto ab$
- $\sigma : a \mapsto b$
- $b \mapsto a$. 

We see that from the above this gives us all $F_2$-by-$\mathbb{Z}$ groups.

We start with the obvious 2-complex shown in Figure 1 for the automorphism $\lambda$. This has a repeated corner which means it cannot support a metric of non-positive curvature.

To get rid of the repeated corner we cut our building blocks along the dotted line identifying the triangles with the repeated corner, resulting in our basic building blocks shown in Figure 2.

By the above lemmas we can without loss of generality assume that our automorphism has the form $\phi = \eta_0 \ldots \eta_{n-1}\theta$ where $\eta_i = \rho$ or $\lambda$ and $\theta$ is one of the following finite order automorphisms:

- $\psi_1 : (a, b) \mapsto (a, b)$
- $\psi_2 : (a, b) \mapsto (a^{-1}, b^{-1})$
- $\psi_3 : (a, b) \mapsto (b, a)$
- $\psi_4 : (a, b) \mapsto (b^{-1}, a^{-1})$.

We can assume that we only apply one of these and we do it at the end. This is because in $\text{Out}(F_2)$ the first and second give central elements whereas $\psi_4$ is equal to the composition $\psi_2\psi_3$, but $\rho\psi_3 = \psi_3\lambda$. 

Figure 1: The 2-complex associated to $\lambda$
Figure 2: The basic building blocks for the construction with angles labelled
We can now glue these together to get an automorphism which is in the class defined previously, so up to isomorphism we have all groups $G_\phi$. We do this in the following way: in Figure 2 let $i = 0$ so that we have the positive and the negative vertices $t_0^\pm$ at each end of $t_0$. On performing the given gluing we have that $t_0^\pm$ are not identified, but if we further stick these two vertices together by identifying $a_0, b_0$ with $a_1, b_1$ respectively then the resulting 2-complex has fundamental group $F_2 \rtimes_\lambda \mathbb{Z}$ or $F_2 \rtimes_\rho \mathbb{Z}$. Now suppose that our automorphism $\phi = \eta_0 \ldots \eta_{n-1} \theta$, where $\theta$ is one of the four special finite order automorphisms above. For each $i$ between 0 and $n - 1$ we have a copy $C_i$ of the 2-complex associated to either $\lambda$ or $\rho$ in Figure 2 which contains the edge $t_i$. We then glue $C_i$ to $C_{i+1}$ by the identity between the $a_{i+1}, b_{i+1}$, which means that the vertex $t_{i+1}^+$ is identified with $t_{i+1}^-$. Finally we glue $C_{n-1}$ back to $C_0$ by identifying the $a_n, b_n$ with $a_0, b_0$ so that $t_{n-1}^+$ becomes equal to $t_0^-$.  

We say a vertex is at time $i$ if it is the vertex where $t_{i-1}$ and $t_i$ meet.

We start with the case of vertices of time not equal to 0, thus these will be where our complexes $C_{i-1}$ and $C_i$ are glued together by the identity between the $a_i, b_i$.

The link of such a vertex is shown in Figure 3. We now want to assign angles such that there are no circuits of length less than $2\pi$.

If we assign angles as in Figure 2 then we get two types of link as shown in Figure 3. Figure 3 (i) corresponds to the automorphisms at the $i$-th stage both being $\rho$ or both being $\lambda$. Figure 3 (ii) corresponds to when one automorphism is $\lambda$ and one is $\rho$.

We can see in either case that the link has no circuits of length less than $2\pi$.

We now look at the case of the vertex at time 0. This will have to take into account the map $\theta$. We can consider the link as being split into 2 halves as shown in Figure 4. The finite order maps defined earlier give vertex identifications. There are 16 possible links we may get in this way, corresponding to which automorphisms meet and to one of the 4 finite order automorphisms. All of these give a link which is homeomorphic to the 1 skeleton of a tetrahedron with the set of angles depicted in Figure 3.

### 2.3.1 Square Complexes

With a more careful assignment of angles we can see that the complexes above can be made into square complexes.
Figure 3: The possible links of a vertex.

Figure 4: The 2 halves of a link.
Figure 5: The 2 possible cases of automorphisms meeting at a vertex.
We split the automorphism $\phi$ into one of three types depending on its decomposition in the semigroup described earlier:

1. $\phi = \rho^n$ or $\lambda^n$

2. $\phi = (\rho \lambda)^n (\rho \theta)^\epsilon$ or $(\lambda \rho)^n (\lambda \theta)^\epsilon$ where $\theta \in \{\psi_3, \psi_4\}$ and $\epsilon \in \{0, 1\}$

3. all other automorphisms.

We will assign angles to our building blocks based on which meet at time $i$. We have depicted the two cases of our building blocks meeting at a vertex.

In case 1 each time our building blocks meet, it will be of the type depicted in Figure 5 i). In this case we keep the angle assignment we had before and make the edges labelled $a_j$ or $b_j$ on the vertical sides of the two rectangles of length 2 and the other edges of length 1. We then make the edge between the 2 triangles of length $\sqrt{2}$ and subdivide the rectangles into squares of edge length 1, thus replacing the two triangles with a new building block which is the square formed by gluing them together. The link of the original vertices in this complex are depicted in Figure 6 i), where the edges corresponding to $a_i$ and $b_i$ have been suppressed as they have valence 2.

In case 2 each time our building blocks meet, it will be of the type depicted in Figure 5 ii). In this case we collapse the triangles to lines and then subdivide the resulting rectangles into 2 squares of side length 1. The link of the original vertices in this complex are depicted in Figure 6 ii), where the edges corresponding to $a_i$ and $b_i$ have been suppressed as they have valence 2.
In case 3 we will have a mix of both Figures 5 i) and ii) and we collapse all the triangles to lines as we did for case 2. This introduces degenerate squares from building blocks meeting as in Figure 5 i), which could affect the topology of the overall complex if there were a cylinder of such squares. However here this will not happen as the only time a cylinder of degenerate squares could occur is if our automorphism is in case 1.

The links of vertices in these complexes are depicted in Figure 7, where the edges corresponding to $a_i$ and $b_i$ have been suppressed as they have valence 2. The case of edges of 0 length are the degenerate squares where in fact the two edges become identified.

In all the cases we see that the resulting complex will be a non-positively curved square complex.

3 SQ universal groups

A countable group $G$ is said to be SQ-universal (standing for Subgroup Quotient) if every countable group can be embedded in a quotient of $G$. This immediately implies that $G$ contains a non abelian free group, and in turn is implied by $G$ being large (having a finite index subgroup surjecting to a non abelian free group). However an infinite simple group $S$ containing $F_2$ would not be SQ-universal, nor would a just infinite group such as $\text{PSL}_n(\mathbb{Z})$ for $n \geq 3$.

As for examples of groups which are SQ-universal, we have all non elementary word hyperbolic groups by [42]. This means that word hyperbolic groups with property (T) provide lots of further examples of groups which are
SQ-universal but not large. Moreover by [2] a finitely generated group which is hyperbolic relative to any collection of proper subgroups is SQ-universal (or virtually cyclic).

An important class of groups in this area is 1-relator groups. It was shown in [3] in 1978 that any group with a presentation of deficiency at least 2 (thus any group having an \( n \)-generator 1-relator presentation for \( n \geq 3 \)) is large, leaving 2-generator 1-relator groups. The question of when such a group \( G \) contains \( F_2 \) has been known for some time: yes, unless \( G \) is isomorphic to a Baumslag-Solitar group of the form \( BS(1, n) \) (where \( n \in \mathbb{Z} - \{0\} \)) or is cyclic. Largeness is a different matter; for instance [24] showed that the group \( BS(m, n) \) is large if and only if \( m \) and \( n \) are not coprime. In [16] we undertook extensive computation suggesting that non large groups with a 2-generator 1-relator presentation are few and far between, however there are more than just Baumslag-Solitar examples. Moreover we did not see a clean criterion that presented itself for conjecture but in the case of SQ-universality there is a statement that appeared in [40] in 1973: a non cyclic 1-relator group is SQ-universal unless it is isomorphic to \( BS(1, n) \), thus if true this would be equivalent to containing \( F_2 \).

There have been a few results in this area since then; for instance [45] showed in 1974 that a group with an \( n \)-generator 1-relator presentation for \( n \geq 3 \) is SQ-universal (which was then subsumed by the largeness result mentioned above). As for 2-generator 1-relator groups, Edjvet’s thesis [23] proves SQ-universality in some useful cases. We showed in [14] Corollary 7.5 that it is true if the group is LERF but that is a very strong condition to impose.

Recently the concept of a group being acylindrically hyperbolic was introduced in [43]. It holds if our group is non elementary and is relatively hyperbolic with respect to a collection of proper subgroups. We will not need the definition here, just the fact also in [44] that such a group is SQ-universal.

This was followed up in [39] where the theory was applied to various situations, including 1-relator groups to obtain the following. Here a subgroup \( H \) of a group \( G \) is \( s \)-normal in \( G \) if \( H \) is infinite and moreover \( H \cap gHg^{-1} \) is infinite for all \( g \in G \). The relevance of this is that an \( s \)-normal subgroup of an acylindrically hyperbolic group must also be acylindrically hyperbolic, so for instance \( H \cong \mathbb{Z} \) being \( s \)-normal in \( G \) implies that \( G \) is not acylindrically hyperbolic (though it could certainly be SQ-universal or even large).
Proposition 3.1. ([39] Proposition 4.20) Let $G$ be a group with two generators and one defining relator. Then at least one of the following holds:

(i) $G$ is acylindrically hyperbolic;

(ii) $G$ contains an infinite cyclic $s$-normal subgroup. More precisely, either $G$ is infinite cyclic or it is an HNN-extension of the form

$$G = \langle a, b, t \mid a^t = b, w = 1 \rangle$$

of a 2-generator 1-relator group $H = \langle a, b \mid w(a, b) \rangle$ with non-trivial center, so that $a^r = b^s$ in $H$ for some $r, s \in \mathbb{Z}\setminus\{0\}$. In the latter case $H$ is (finitely generated free)-by-cyclic and contains a finite index normal subgroup splitting as a direct product of a finitely generated free group with an infinite cyclic group.

(iii) $G$ is isomorphic to an ascending HNN extension of a finite rank free group.

Moreover, the possibilities (i) and (ii) are mutually exclusive.

Thus this establishes that groups in class (i) are SQ-universal. We will show the same for groups in (ii) then discuss results for (iii). When considering groups in case (ii), we will use the class of generalized Baumslag-Solitar, or GBS, groups. These can be defined as those finitely generated groups which act on a tree with infinite cyclic vertex and edge stabilisers. The two recent papers [34] and [35] cover a lot of ground in this area and we now mention the points we will be using, referring to them for more detail.

We can describe a GBS group using the graph of groups theory, where a finite graph (possibly with loops and/or multiple edges) has labels consisting of a non zero integer at each end of each edge. This label tells us the index of this edge group in the adjacent vertex group, which determines the subgroup uniquely. In general many different finite labelled graphs can give rise to isomorphic GBS groups. One operation that can be performed without change of the underlying group is an elementary collapse. This is when one end of an edge $e$ next to a vertex $v$ is labelled $\pm 1$ and the edge is not a self loop. We can then contract this edge and multiply all other labels next to $v$ by the label at the other end of $e$. By doing this repeatedly, we may assume that any edge with an end labelled by $\pm 1$ is a self loop.
Note that all GBS groups have deficiency 1, that is they admit a presentation with one more generator than relator. In particular there always exists a surjective homomorphism from any GBS group to $\mathbb{Z}$.

**Theorem 3.2.** If the group $G$ is as in case (ii) of the preceding Proposition then $G$ is a generalized Baumslag-Solitar group. Moreover any generalized Baumslag-Solitar group is either SQ-universal or it is isomorphic to the Baumslag-Solitar group $BS(1, j)$ for some $j \in \mathbb{Z} \setminus \{0\}$ or is infinite cyclic.

**Proof.** For the first part we can use Theorem C of [33]. This states that the non cyclic finitely generated groups of cohomological dimension 2 that have an infinite cyclic $s$-normal subgroup are exactly the generalized Baumslag-Solitar groups. Now a 1-relator group has cohomological dimension 2 if the relator is not a proper power by [37], but a proper power gives rise to a group with torsion, whereas the groups in case (ii) are all torsion free.

Now let $\Gamma$ be the underlying graph of our graph of groups that results in the GBS group $G$. As mentioned above we assume that the only edge labels equal to $\pm 1$ appear on self loops.

It is well known that if $\Gamma$ contains more than one cycle (here we include self loops as cycles) then $G$ surjects to $F_2$ and so is SQ-universal, because we introduce a stable letter for each cycle when forming $G$, and all vertex subgroups can be quotiented out to leave only these stable letters which have no relations between them.

It is also known that if $\Gamma$ is a tree then $G$ is virtually $F_k \times \mathbb{Z}$ for $k \geq 2$ which is large, hence so is $G$. This can be seen by quoting Proposition 4.1 of [34] which states that a group is a GBS group with non trivial centre if and only if it is of the form $F_k \times_\alpha \mathbb{Z}$ with $\alpha$ having finite order in $\text{Out}(F_k)$. Now here $G$ will certainly have a non trivial centre, namely an element which is a common power of all the generators of the vertex subgroups as these form a generating set for $G$. Moreover in the case of a tree the surjective homomorphism $\theta$ from $G$ to $\mathbb{Z}$ has the property that no non trivial element of a vertex (or edge) group lies in its kernel, as if so then the whole vertex group does, thus so do the neighbouring edge groups and so on across the whole tree.

We now assume that $\Gamma$ has exactly one cycle $C$. First assume this is not a self loop. We pick one edge $e$ lying in $C$ and remove the interior of $e$ to form a tree $T$ and a group $H$ coming from considering $T$ as the corresponding graph of groups. Thus we have our homomorphism $\theta : H \to \mathbb{Z}$ as above, with $G$ obtained from $H$ by taking generators $h_1, h_2$ of the vertex groups at
each end \( v_1, v_2 \) of \( e \) and then adding a stable letter \( t \) which results in the presentation

\[
G = \langle H, t | th_1^m t^{-1} = h_2^n \rangle
\]

where \( m, n \) are the labels at each end of \( e \), neither of which are 0 or \( \pm 1 \).

We now obtain a surjection from \( G \) to a Baumslag-Solitar group using the following folklore lemma:

**Lemma 3.3.** Let \( G \) be an HNN extension of the group \( H \) amalgamating the subgroups \( A, B \) via the isomorphism \( \phi : A \to B \). Suppose we have a homomorphism \( \theta \) from \( H \) onto a quotient \( Q \) such that \( \phi \) descends to an isomorphism \( \bar{\phi} \) from \( \theta(A) \) to \( \theta(B) \), meaning that \( \bar{\phi} \) is well defined and bijective with \( \bar{\phi} \theta = \theta \phi \). (This occurs if and only if \( \phi(K) = L \) for \( K, L \) the kernels of the restriction of \( \theta \) to \( A, B \) respectively.)

Then on forming the HNN extension \( R \) of \( Q \) with stable letter \( s \) amalgamating \( \theta(A) \) and \( \theta(B) \) via \( \bar{\phi} \), we have that the original HNN extension \( G \) has this new HNN extension \( R \) as a quotient.

**Proof.** We define a homomorphism from the free product \( H \ast \langle t \rangle \) onto \( R \) sending \( t \) to \( s \) and \( h \in H \) to \( \theta(H) \in Q \). We see that this factors through \( G \) because any relation in \( G \) of the form \( tat^{-1} = \phi(a) \) has the left hand side mapped by \( \theta \) to \( s\theta(a)s^{-1} \) and the right hand side to \( \bar{\phi}\theta(a) \), and these two things are equal in \( R \) by the HNN construction.

Consequently in our case we have \( G \) surjects to \( BS(k_1m, k_2n) \), where \( k_1 = \theta(h_1) \) which is not equal to zero as mentioned above because \( H \) is formed from a tree, and similarly for \( k_2 \). Now a Baumslag-Solitar group \( BS(i, j) = \langle t, a | ta^d t^{-1} = a^j \rangle \) is known to be SQ-universal if neither of \( i, j \) equal \( \pm 1 \), by Lemma 1.4.3 of [23] if \( i \) and \( j \) are coprime, and by the well known trick of setting \( a^d \) equal to the identity when \( d \) divides \( i \) and \( j \), to get a surjection to \( \mathbb{Z} \ast \mathbb{Z}_d \) which is virtually free otherwise. As \( |m| \) and \( |n| \) are both greater than 1, we have that \( G \) surjects to an SQ-universal group and so itself is SQ-universal.

We are now only left with the case where there is a single self loop \( L \) in our graph \( \Gamma \), so that now \( v_1 = v_2 \) and \( h_1 = h_2 \). The above proof also works here by removing \( L \) this time, unless \( |k_1| = |m| = 1 \) (or \( |k_2| = |n| = 1 \) in which case we replace \( BS(i, j) \) with the isomorphic group \( BS(j, i) \)). If \( L \) is all of \( \Gamma \) then we have that \( G \) is just \( BS(1, j) \) which is soluble, and so is a genuine exception to being SQ-universal. Otherwise we note that \( |k_1| = \pm 1 \) implies that \( h_1 \) is mapped by \( \theta \) to a generator of \( \mathbb{Z} \). On taking any edge \( e_i \)
not equal to $L$ with one endpoint $v_1$, let $a_i$ be the label of $e_i$ at this end and $b_i$ the label at the other end of $e_i$, next to the vertex group $\langle x_i \rangle$, say. We must have $b_i$ dividing $a_i$ because of the relation $\theta(x_i^{a_i}) = \theta(h_1^{a_i})$ and the fact that $\theta(h_1) = \pm 1$. We now choose a particular edge $e_i$ and remove the loop $L$ and the vertex $v_1$ from $\Gamma$, to form a possibly disconnected graph that is a union of trees $T_k$, with $T_1$ the tree containing the vertex group $\langle x_i \rangle$.

We then take a prime $p$ dividing $b_i$ and consider the quotient of $G$ formed by setting $h_1$ and $x_i^p$ equal to the identity. We have a surjective homomorphism with domain the group obtained as a graph of groups from the tree $T_1$ and image $\mathbb{Z}_p$, consisting of the homomorphism to $\mathbb{Z}$ in the case of the tree $T_1$ and then composing with the map from $\mathbb{Z}$ to $\mathbb{Z}_p$. We now extend this to a homomorphism from $G$ onto $\mathbb{Z} \ast \mathbb{Z}_p$ which is SQ-universal as follows: send the stable letter $t$ to $1 \in \mathbb{Z}$, so that the relation $th_1t^{-1} = h_1^{a_i}$ obtained from the loop $L$ now has both sides sent to the identity. Moreover this holds for the relation $x_i^{b_i} = h_1^{a_i}$ obtained from the edge $e_i$ as $p$ divides $b_i$. Finally we send all vertex groups not in the component $T_1$ to the identity, with the $x_j^{b_j} = h_1^{a_j}$ relations obtained from the other edges $\{e_j : j \neq i\}$ that have endpoint $v_1$ all automatically satisfied.

Note: this result can be compared to [34] Theorem 6.7 in which the large GBS groups are determined, but there are cases for which the graph $\Gamma$ is a single cycle where the group $G$ is SQ-universal but not large.

We now come to case (iii), that of $G$ being equal to the HNN extension $F_k \ast \theta$, where $\theta : F_k \to F_k$ is injective but need not be surjective (if not then we call this a strictly ascending HNN extension of $F_k$). Here we can quote results of the first author in [14]. Theorem 5.4 of that paper states that $G$ will be SQ-universal whenever $\theta$ is an automorphism (unless $G \cong \mathbb{Z}, \mathbb{Z} \ast \mathbb{Z}$ or the Klein bottle group when the rank $k$ is 0 or 1). This is proved by showing that $\mathbb{Z} \ast \mathbb{Z} \leq G$ implies that $G$ is large, and then invoking Ol’shanskiǐ’s theorem on the SQ-universality of word hyperbolic groups and the result in [12] that not containing $\mathbb{Z} \ast \mathbb{Z}$ and being word hyperbolic are equivalent in the class of groups $F_k \ast \theta$ when $\theta$ is an automorphism.

In the case where $G$ is a strictly ascending HNN extension of a finite rank free group, we have further results but they are not quite definitive. Again we have that if $\mathbb{Z} \ast \mathbb{Z} \leq G$ then $G$ is large (or is equal to $\mathbb{Z} \ast \mathbb{Z}$ or the Klein bottle group) by [14] Corollary 4.6. However in the strictly ascending case there are examples where $G$ does not contain $\mathbb{Z} \ast \mathbb{Z}$ but does contain a Baumslag-Solitar subgroup, which must be of the form $BS(1, m)$ for $|m| \neq 1$ so that $G$
fails to be word hyperbolic. In [31] it is conjectured that a strictly ascending HNN extension of a finite rank free group is word hyperbolic if it does not contain a Baumslag-Solitar subgroup and this conjecture seems to be widely believed, but a proof might well require the machinery of train track maps to be developed in full for injective endomorphisms of $F_k$. Moreover it is an open question whether a 1-relator group (or indeed a group with a finite classifying space) containing no Baumslag-Solitar subgroups is word hyperbolic, so we would be covered in our case if any of these (or their intersection) turned out to be true.

As for when $G$ contains $BS(1, m)$ for $|m| \neq 1$, [14] Theorem 4.7 states that either $G$ is large, or $G$ is itself a Baumslag-Solitar group of the form $BS(1, n)$, or $G \neq BS(1, n)$ but $G$ has virtual first Betti number equal to 1 and it is conjectured that the last case does not occur. Putting all this together, we have our result on the SQ-universality of 2-generator 1-relator groups:

**Corollary 3.4.** If $G$ is a group given by a 2-generator 1-relator presentation that is not $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group then either $G$ is an SQ-universal group, or $G$ is a strictly ascending HNN extension $F_k * \theta$ of a free group $F_k$ which is not word hyperbolic and such that:

(i) either $G$ contains no Baumslag-Solitar subgroup (conjecturally this does not occur) or

(ii) $G$ contains a Baumslag-Solitar group $BS(1, m)$ for $|m| \neq 1$ but does not contain $\mathbb{Z} \times \mathbb{Z}$ and the virtual first Betti number of $G$ is 1 (conjecturally this only occurs if $G \cong BS(1, n)$ for $|n| \neq 1$).

We finish this section with a couple of unconditional results.

**Corollary 3.5.** If $G = \langle a, b | w(a, b) \rangle$ and $w$ is in the commutator subgroup of $F(a, b)$ (and without loss of generality cyclically reduced) then $G$ is SQ-universal, except when $G \cong \mathbb{Z} \times \mathbb{Z}$ for $w$ a cyclic permutation of $aba^{-1}b^{-1}$ or its inverse.

**Proof.** This proceeds by using the BNS invariant $\Sigma \subseteq S^1$ of $G$ in [4] and the proof is very similar to Theorem D of that paper. The idea is that $\Sigma$ is an open subset of $S^1$ and if $\Sigma \cup -\Sigma$ is not all of $S^1$ then we have a homomorphism $\chi : G \to \mathbb{Z}$ that expresses $G$ as a non-ascending HNN extension. The Magnus decomposition of this extension is such that $G$ is either in case (i) or case (ii) of Proposition 3.1, so that $G$ is SQ-universal by that theorem for case (i) or by Theorem 3.2 for case (ii).
Otherwise we have $\Sigma \cap -\Sigma \neq \emptyset$ as $S^1$ is connected, which means that the kernel of $\chi$ is finitely generated and consequently the 1-relator group $G$ can be expressed as $F_k \rtimes_\alpha \mathbb{Z}$, which is SQ-universal if $k \geq 2$. If $k = 1$ then $\alpha$ is the identity as $G$ surjects to $\mathbb{Z} \times \mathbb{Z}$, so $G = \mathbb{Z} \times \mathbb{Z}$ and therefore admits only the above 2-generator 1-relator presentations by [38] Theorem 4.11.

Finally an SQ-universal group can be thought of as one with many infinite quotients whereas an infinite residually finite group can be thought of as having many finite quotients. We see that all 2-generator 1-relator groups therefore have many quotients of some kind:

**Corollary 3.6.** A group with a 2-generator 1-relator presentation is either SQ-universal or residually finite.

**Proof.** This follows from Corollary [3.4] because [5] proved that a strictly ascending HNN extension of a finite rank free group is residually finite (though as mentioned, not necessarily linear).

### 4 Acylindrically hyperbolic mapping tori of free groups

For the three cases in the last section, we had that the groups in case (i) were all acylindrically hyperbolic whereas none in case (ii) were. However when considering groups in case (iii) for SQ-universality, we did this independently of results on acylindrically hyperbolic groups. It can therefore be asked which mapping tori of finite rank free groups are acylindrically hyperbolic and indeed this is exactly Problem 8.2 in Section 8 of [39]. Moreover a solution just for the 1-relator groups in this class would then completely determine which 2-generator 1-relator groups are acylindrically hyperbolic, which is their Problem 8.1.

It is clear that an ascending HNN extension $F_k \rtimes_\alpha \mathbb{Z}$ of $F_k$ formed using an automorphism $\alpha$ of finite order in $\text{Out}(F_k)$ will not be acylindrically hyperbolic because of the existence of an infinite order element in the centre. Thus a possible answer to Problem 8.2 is that all other ascending HNN extensions of $F_k$ are acylindrically hyperbolic with the exception of $BS(1,m)$ when $k = 1$. This would imply two mutually exclusive cases for these groups: either they are acylindrically hyperbolic or they are generalized Baumslag-Solitar groups, and it would also imply in answer to Problem 8.1 that a
1-relator group is acylindrically hyperbolic if and only if it does not contain an infinite cyclic $s$-normal subgroup. As a partial answer to Problem 8.2 we have

**Proposition 4.1.** If a finitely generated group $G$ of cohomological dimension 2 has a finite index subgroup $H$ splitting over $\mathbb{Z}$ then either $G$ is acylindrically hyperbolic or it is a generalized Baumslag-Solitar group.

*Proof.* We can apply [33] Theorem C to $H$ as it also has cohomological dimension 2. This implies that if $H$ splits over $A \cong \mathbb{Z} \leq H$ and $A$ is $s$-normal then $H$ is a generalized Baumslag-Solitar group and so is $G$ by [33] Corollary 3 (ii) as it is torsion free. Otherwise we can apply Corollaries 2.2 and 2.3 of [39] which state that if $H$ is an amalgamated free product or HNN extension over an edge group which is not $s$-normal and not equal to a vertex group under any inclusion then $H$ is acylindrically hyperbolic and therefore $G$ is by [39] Lemma 3.8. However if $A \cong \mathbb{Z}$ is equal to a vertex group then we have $H = BS(1,m)$ which is also a generalised Baumslag-Solitar group.

**Corollary 4.2.** An ascending HNN extension $F_k \ast \theta$ for $k \geq 2$ that virtually splits over $\mathbb{Z}$ is either acylindrically hyperbolic or is virtually $F_k \times \mathbb{Z}$.

*Proof.* All ascending HNN extensions of $F_k$ have geometric and thus cohomological dimension 2, so by Proposition 4.1 we obtain acylindrical hyperbolicity unless we have a generalised Baumslag-Solitar group. They are all also residually finite, but by [35] Corollary 7.7 a generalised Baumslag-Solitar group is not residually finite unless it is virtually $F_k \times \mathbb{Z}$ or $BS(1,m)$ when $k = 1$.

We now specialise to the case where the mapping torus is formed using an automorphism, so we are back in the class of free-by-cyclic groups $G = F_k \rtimes_\alpha \mathbb{Z}$, where we can say which such groups are acylindrically hyperbolic. Of course this will be true if $G$ is word hyperbolic or hyperbolic with respect to a collection of proper subgroups. There are plenty of examples of word hyperbolic free-by-cyclic groups when $k \geq 3$. When $k = 2$ there are none, but most are relatively hyperbolic with respect to the peripheral $\mathbb{Z} \times \mathbb{Z}$ subgroup because they will be the fundamental group of a finite volume hyperbolic 1-punctured torus bundle. The exceptions are when the monodromy has finite order, giving the virtually $F_2 \times \mathbb{Z}$ case which cannot be acylindrically hyperbolic, and parabolic monodromy where all groups will be commensurable with $G = F(a,b) \rtimes_\lambda \mathbb{Z}$. 

23
Here we consider the case where $[\alpha] \in \text{Out}(F_k)$ is a polynomially growing automorphism, with recent results on this in [18], which itself utilises the train track technology of Bestvina, Feighn and Handel. The facts from [18] Section 5 that we need are:

- If $[\alpha]$ is polynomially growing then there is a positive power $[\alpha^k]$ in $\text{UPG}(F_k)$, which is the subgroup of polynomially growing outer automorphisms whose abelianised action on $\mathbb{Z}^k$ has unipotent image.

- If $k \geq 2$ then every element of $\text{UPG}(F_k)$ has in its class an automorphism $\alpha$ of $F_k$ such that either:
  (i) There exists a non trivial $\alpha$-invariant splitting $F_k = B_1 \ast B_2$, so that $\alpha$ restricts to an automorphism of $B_1$ and also of $B_2$.
  (ii) There exists a non trivial splitting $F_k = B_1 \ast \langle x \rangle$, where $B_1$ is $\alpha$-invariant and $\alpha(x) = xw$ for $w$ an element of $B_1$.

We also use the following two statements which follow immediately from Corollary 4.22 and Theorem 1.4 in [43]:

**Lemma 4.3.** If $G$ is a finitely presented group that is torsion free and hyperbolic relative to a collection of proper subgroups $\{H_1, \ldots, H_l\}$ (the peripheral subgroups) then

(i) Any Baumslag-Solitar subgroup of $G$ is conjugate into some $H_i$.
(ii) Any $H_i$ is malnormal, so that if there is $g \in G$ with $H_i \cap gH_ig^{-1}$ non trivial then $g \in H_i$. Moreover if there is $g \in G$ with $H_i \cap gH_ig^{-1}$ non trivial then $i = j$.

**Theorem 4.4.** If $[\alpha] \in \text{Out}(F_k)$ is a polynomially growing automorphism then $G = F_k \rtimes_\alpha \mathbb{Z}$ is not hyperbolic relative to any collection of proper subgroups, but it is acylindrically hyperbolic unless $[\alpha]$ has finite order in $\text{Out}(F_k)$.

**Proof.** By taking a power of $\alpha$, which corresponds to a finite index subgroup, we can assume that $[\alpha]$ is in $\text{UPG}(F_k)$. Thus let us repeatedly apply options (i) and (ii) until we have decomposed $F_k$ into a free product of cyclic groups. These provide splittings over $\mathbb{Z}$ of this finite index subgroup and so Corollary 4.2 immediately implies the second part of the statement.

We can picture this process of repeatedly splitting $G$ over $\mathbb{Z}$ as a finite rooted tree, as in [18] Lemma 5.10 which involves describing it as a hierarchy. We have $F_k$ at the root vertex, with every other vertex being labelled by a proper non trivial free factor of $F_k$. To explain this, we split $F_k \rtimes_\alpha \mathbb{Z}$ using
either Case (i) or Case (ii), which in general changes the automorphism \( \alpha \) but only to something equal in \( \text{Out}(F_k) \) which we can also call \( \alpha \) for now.

If Case (i) is used then we have an \( \alpha \)-invariant decomposition \( F_k = B_1 * B_2 \), with stable letter \( t \) inducing \( \alpha \) by conjugation, which means that \( F_k \rtimes_\alpha \mathbb{Z} = \langle t, F_k \rangle \) has been split as an amalgamated free product \( \langle t, B_1 \rangle * \langle t, B_2 \rangle \) over \( \mathbb{Z} = \langle t \rangle \). We then draw two vertices labelled \( B_1 \) and \( B_2 \) as immediate descendents of the root vertex \( F_k \).

If however case (ii) is used to split \( F_k = B * \langle x \rangle \) where \( B \) has rank \( k - 1 \) and \( txt^{-1} = xw \) for \( w \in B \), then this corresponds to the HNN extension \( \langle t, F_k \rangle = \langle t, B \rangle * \langle x \rangle \) where \( x^{-1}tx = wt \), so now \( x \) is the stable letter of this HNN extension conjugating the infinite cyclic groups \( \langle wt \rangle \) and \( \langle t \rangle \). In this case we have only one immediate descendent vertex \( B \) of the root.

Having done this once, as the new vertex labels \( B_1, B_2 \) or \( B \) will be \( \alpha \)-invariant, we can take the appropriate restriction of \( \alpha \) and replace \( \langle t, F_k \rangle \) by the subgroup \( \langle t, B_1 \rangle, \langle t, B_2 \rangle \) or \( \langle t, B \rangle \), which is also free by cyclic but where the free part has smaller rank. Moreover this restriction will also be a polynomially growing outer automorphism whose abelianised action has unipotent image, so it is still in UPG. Thus we can then (on changing the restriction of \( \alpha \) by an inner automorphism) apply Case (i) or Case (ii) again, creating the next level of descendents and labelling them accordingly. This continues until we reach the leaf vertices (those with no descendents), which are each labelled by an infinite cyclic free factor of \( F_k \), so no further splitting occurs.

However each time we form the descendents of a vertex by applying Case (i) or (ii) to the free factor of \( F_k \) labelling this vertex, we are liable to change the given automorphism within the outer automorphism group of this free factor. Now we also need to keep track of the actual automorphisms and how they are induced, so there are further labels on each vertex (apart from the leaf vertices) consisting of an automorphism of \( F_k \) equal in \( \text{Out}(F_k) \) to \( \alpha \), along with a “stable letter” which is an element of \( F_k \rtimes_\alpha \mathbb{Z} \) inducing this automorphism under conjugation. This is defined inductively on the levels of the tree as follows: on splitting our root vertex \( F_k \), we have an automorphism equal to \( \alpha \) in \( \text{Out}(F_k) \) respecting this splitting. Now taking the composition of \( \alpha \) with an inner automorphism does not change \( F_k \rtimes_\alpha \mathbb{Z} \), so we can assume that \( \alpha \) respects this splitting, whereupon we label the root vertex \( F_k \) with this \( \alpha \) and the overall stable letter \( t \) that induces \( \alpha \) by conjugation. This completes our labelling for the zeroth level.

Now suppose that \( F_k = G_0, G_1, \ldots, G_n \) are vertices on each successive
level leading to the vertex $G_n$, which is not a cyclic group as otherwise it is a leaf vertex and therefore requires no further labelling. Therefore $G_n$ has successor(s) $G_{n+1}$ in case (ii) (as well as $G'_{n+1}$ if case (i) is taken). In order to describe the labels for $G_{n+1}$ (and $G'_{n+1}$) if this too is a leaf vertex, we suppose that $G_n$ is already labelled by the automorphism

$$
\gamma_n = \ell_{w_n} \ell_{w_{n-1}} \cdots \ell_{w_1} \alpha \text{ of } F_k,
$$

where $w_i$ is an element of $G_i$ and $\ell_x$ stands for the inner automorphism of $F_k$ consisting of conjugation by $x \in F_k$. Moreover we assume that both $G_n$ and $G_{n+1}$ (and $G'_{n+1}$) are $\gamma_n$-invariant. We further assume that also $G_n$ is the element $w_n w_{n-1} \cdots w_1 t \in F_k \rtimes_{\alpha} \mathbb{Z}$ which we call the stable letter $s_n$. Of course if now $G_{n+1}$ (or $G'_{n+1}$) is cyclic then we do not need to label these any further, but otherwise we apply the splitting to $G_{n+1}$ under the automorphism obtained by restricting $\gamma_n$ to $G_{n+1}$, thus we obtain an automorphism of $G_{n+1}$ equivalent to $\gamma_n|_{G_{n+1}}$ in $\text{Out}(G_{n+1})$ which preserves $G_{n+2}$ (and $G'_{n+2}$) and which acts accordingly in case (ii). Thus there is an element $w_{n+1} \in G_{n+1}$ such that we can define the automorphism $\gamma_{n+1}$ of $F_k$ to be

$$
\ell_{w_{n+1}} \gamma_n = \ell_{w_{n+1}} \ell_{w_n} \ell_{w_{n-1}} \cdots \ell_{w_1} \alpha,
$$

thus $\gamma_{n+1}$ restricts to an automorphism of $G_{n+1}$ that preserves $G_{n+2}$ (and $G'_{n+2}$). We then further label the vertex $G_{n+1}$ by this automorphism $\gamma_{n+1}$ and the stable letter $s_{n+1} = w_{n+1} s_n$ that will induce this map by conjugation.

Having completed the labelling, let us reverse the whole process and build up the free-by-cyclic group $G$ from these splittings. Let us start at a leaf vertex $G_{m+1} = \langle y \rangle$ of maximum depth. As the automorphism $\gamma_m$ obtained from the vertex $G_m$ above preserves $G_{m+1}$ and $y$ is part of a free basis for $F_k$, we must have $\gamma_m(y) = y$ as $[\gamma_m] = [\alpha] \in UPG(F_k)$ and so all eigenvalues of the abelianised map equal 1. Thus $s_m y s_m^{-1} = y$ giving us a copy of $\mathbb{Z}^2 = \langle s_m, y \rangle$ in $G$, which must therefore lie in a subgroup $C$ which is a conjugate of one of the peripheral subgroups $H_i$ and so $C$ is also malnormal. If $G_m$ gave rise in case (i) to $G_{m+1} = \langle y \rangle$ and $G'_{m+1}$, we would also have $G'_{m+1} = \langle z \rangle$ by maximality of depth. Thus $G_m = \langle y \rangle \ast \langle z \rangle$ and the same stable letter $s_m$ commutes with $z$, so we conclude by malnormality that $s_m, y, z \in C$. If however we had case (ii) then again there is $z$ with $G_m = \langle y \rangle \ast \langle z \rangle$ but now we have $\gamma_m(y) = y$ and $\gamma_m(z) = zy'$. Thus $z^{-1} s_m z = y' s_m \in C$ so again $z \in C$ by malnormality. Hence in either case we obtained from a leaf vertex a copy of $\mathbb{Z}^2$ lying in a conjugate $C$ of a peripheral subgroup and shown that $\langle s_m, G_m \rangle$ is in $C$ too.

26
We now contract edges starting at the leaf vertices, one edge at a time if the lower level vertex of this edge has one immediate descendent or a pair of vertices otherwise. We suppose that at each leaf vertex $G_j$ of the current contraction of the tree, we have $\langle s_j, G_j \rangle$ lying in a conjugate $C$ of a peripheral subgroup (although $C$ could vary over the current leaf vertices).

First suppose that $G_j$ is the only current descendent of $G_{j-1}$, so that $G_j$ was obtained by applying case (ii) to $G_{j-1}$ and the labelling indicates that $G_{j-1} = G_j \ast \langle x \rangle$, with $\gamma_{j-1}$ sending $x$ to $xu$ for $u$ a word in $G_j$. Thus $s_{j-1}xs_{j-1}^{-1} = xu$, but $s_{j-1} = w_j^{-1}s_j$ for $w_j \in G_j$ so $s_{j-1}$ is already in $C$. Consequently $x^{-1}s_{j-1}x = us_{j-1} \in C$ implies that $x$ and therefore $\langle s_{j-1}, G_{j-1} \rangle$ is in $C$ as well.

Now suppose that there are other current descendents of $G_{j-1}$ in addition to $G_j$. This means that after further contraction if necessary there will be the two leaf vertices $G_j$ and $G_{j}'$ which are the descendents of $G_{j-1} = G_j \ast G_{j}'$, and we have by our assumption that $\langle s_j, G_j \rangle$ lies in $C$ and similarly $\langle s_j', G_{j}' \rangle$ also lies in some conjugate $C'$ of a peripheral subgroup. Now $s_{j-1} \in C$ as above, but the same argument also says that $s_{j-1} = (w_j')^{-1}s'_j$ for $w_j' \in G_{j}'$. Thus the same element $s_{j-1}$ is in both $C$ and $C'$, hence by malnormality they are equal and we have $\langle s_{j-1}, G_{j-1} \rangle \leq C$. We can now continue contracting until we are left with the root, concluding that all of $G = \langle t = s_0, G_0 = F_k \rangle$ lies in $C$, thus $G$ is not relatively hyperbolic with respect to proper subgroups. 

This adds to results in the literature that provide a description of free-by-cyclic groups according to the type of hyperbolicity: for $G = F_k \rtimes_{\alpha} \mathbb{Z}$ we have that:

- $G$ is word hyperbolic if and only if no positive power of $\alpha$ sends $w \in F_k \setminus \{id\}$ to a conjugate of itself.
- $G$ is relatively hyperbolic if and only if $[\alpha]$ is not of polynomial growth. (The if direction requires one to accept the results of the unpublished manuscript [21] where the peripheral subgroups are the mapping tori of the polynomially growing subgroups under $[\alpha]$ of $F_k$, whereas the only if direction comes from Theorem 4.4.)
- $G$ is acylindrically hyperbolic but not relatively hyperbolic if and only if $[\alpha]$ is polynomially growing and of infinite order in $Out(F_k)$. (The same comment as above applies here, with “if” and “only if” reversed.)
- $G$ is not acylindrically hyperbolic if and only if $[\alpha] \in Out(F_k)$ has finite order.

Note that all free-by-cyclic groups are known to be large by [17], following
[14], [15] and [29], but are not known to be linear unless $k = 2$.

This leaves us with strictly ascending HNN extensions of finite rank free groups, which we have seen can be less well behaved and indeed need not be linear. However we finish with an example to show that the recent work of Hagen and Wise in [29] on cubulation of ascending HNN extensions of finitely generated free groups solves a problem in the Kourovka notebook [32] on linearity of a particular group of this kind. Problem 17.108 asks whether the group below is linear.

**Theorem 4.5.** The group $\langle a, b, t | tat^{-1} = ab, tbt^{-1} = ba \rangle$ is linear over $\mathbb{Z}$.

**Proof.** The group is clearly a strictly ascending HNN extension of a finitely generated free group using the injective endomorphism $\theta(a) = ab, \theta(b) = ba$ of $F_2$. Corollary 6.20 of [29] states that this group is virtually special, and hence linear over $\mathbb{Z}$, if $\theta$ is irreducible and the HNN extension is word hyperbolic.

Here a reducible endomorphism $\theta$ of $F_k$ is one where there is a free product decomposition $F_k = F_{k_1} \ast \ldots \ast F_{k_r} \ast C$ for $1 \leq r \leq k$ and where each $F_{k_i}$ is non trivial (though $C$ might be in which case $2 \leq r$) such that $\theta(F_{k_i})$ is sent into a conjugate of $F_{k_{i+1}}$ with $i$ considered modulo $r$.

Theorem A in [31] considered the word hyperbolicity of strictly ascending HNN extensions of a finite rank free group $F(X)$ and came up with a result when the endomorphism $\theta$ is an immersion, which is defined to mean that for all $x, y \in X \cup X^{-1}$ with $xy \neq e$, the word $\theta(x)\theta(y)$ admits no cancellation (so called because the standard selfmap of the $|X|$-petalled rose is an immersion). This implies that if $w = x_1 \ldots x_n$ is written as a reduced word on $X$ for $x_i \in X \cup X^{-1}$ then no cancellation can occur on the right hand side of $\theta(w) = \theta(x_1) \ldots \theta(x_n)$. Thus the word length of $\theta(w)$ is the sum of that for the $\theta(x_i)$. This theorem states that if there is no periodic conjugacy class, meaning that there is no $w \in F(X) - \{e\}$ and $i, j > 0$ such that $\theta^i(w)$ is conjugate to $w^j$ in $F(X)$, then the strictly ascending HNN extension $F(X) \ast_\theta$ is word hyperbolic. We can apply this to our group as $\theta$ is indeed an immersion and the fact that each generator maps to a word of length two implies that the word length of $\theta^i(w)$ is $2^i$ times that of $w$. But we can assume by conjugating that $w$ is cyclically reduced, in which case $\theta^i(w)$ is also cyclically reduced, as will be $w^j$ whose word length is $j$ times that of $w$. Now if $\theta^i(w)$ and $w^j$ are both cyclically reduced and are conjugate in $F_2$ then they must have the same length and be cyclic permutations of each other, thus $j = 2^i$. 

28
Next suppose the word length of \( w \) is even and set \( w = uv \) for \( u, v \) both half the length of \( w \). We have \( \theta^i(u)\theta^i(v) \) is a cyclic permutation of \( w^{2i} \) and so is equal to \( (sr)^{2i} \) without cancellation in this expression, where \( w = rs \) (but \( r \) and \( s \) are not necessarily of equal length). Thus \( \theta^i(u) = (sr)^{2i-1} = \theta^i(v) \). As an immersion must be injective, we obtain \( u = v \) and \( \theta^i(u) \) is a cyclic permutation of \( w^{2i-1} = u^{2i} \). Thus we can continue to cut \( w \) in half at each stage until it has odd length, whilst still preserving the fact that \( \theta^i(w) \) is a cyclic conjugate of \( w^{2i} \). When we reach this point, we consider the number of appearances of \( a^{\pm 1} \) and \( b^{\pm 1} \) in \( w \), which cannot be equal, thus nor can it be equal for \( w^{2i} \) or for any cyclic conjugate thereof. But as \( \theta(a) \) and \( \theta(b) \) both have equal exponent sums, this is true for any element in the image of \( w \) which is a contradiction.

Now showing irreducibility of an endomorphism is straightforward in rank 2 because any proper non trivial free factor is just the cyclic group generated by a primitive element. Thus to show \( \theta \) is irreducible it is enough to rule out there being a primitive element \( x \) such that \( \theta(x) \) or \( \theta^2(x) \) is conjugate to \( x^j \) for some \( j \neq 0 \). But this has been eliminated for \( j > 0 \), and also for \( j < 0 \) by considering \( \theta^2(x) \) and \( \theta^4(x) \) respectively. \( \square \)

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