STABILITY IN MEASURE FOR UNCERTAIN HEAT EQUATIONS

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Abstract. Uncertain heat equation is a type of uncertain partial differential equations driven by Liu processes. As an important part in uncertain heat equation, stability analysis has not been researched as yet. This paper first introduces a concept of stability in measure for uncertain heat equation, and proves a stability theorem under strong Lipschitz condition that provides a sufficient for an uncertain heat equation being stable in measure. Moreover, some examples are given.

1. Introduction. Since its inception ten years ago, uncertainty theory has become an axiomatic mathematical system to model human belief degrees. As an important component part in uncertainty theory, uncertain differential equation (UDE) was driven by Liu process, which is a Lipschitz continuous uncertain process with stationary and independent normal increments. The pioneering work of UDE was done by Liu [4] in 2008. Following that, UDE theory was researched by many scholars. The existence and uniqueness of an UDE was proved by Chen and Liu [1]. And, they obtained the analytic solution for linear UDE. In addition, Liu [9] and Yao [24] also got the analysis solutions for two special types of nonlinear UDE. In 2013, Yao and Chen [21] did a particularly important work to reduce an UDE to a family of ordinary differential equations. This results made it possible to analyze UDE, and lots of significative works (Yao and Chen [21], Yao [22]) have been obtained such as the inverse uncertainty distribution of solution, expected value of solution, extreme value of solution, first hitting time of solution and time integral of solution. Beyond that, some numerical methods were designed to solve general UDE by Yao and Chen [21], Yang and Shen [14], Yang and Ralescu [15], Gao [2], and Wang et al. [13].

The concept of stability in measure for UDE was first proposed by Liu [5], and the corresponding stability theorem was proved by Yao et al. [23]. Soon afterwards, different types of stability for UDE were discussed, for example, stability in mean (Yao et al. [25]), stability in moment (Sheng and Wang [10]), almost sure stability (Liu et al. [8]), exponential stability (Sheng and Gao [11]) and stability in inverse distribution (Yang et al. [17]). Furthermore, Tao and Zhu [12] researched the attractivity analysis of UDE. For more detailed exposition of uncertain differential equation, the readers may consult Yao’s recent book [26].

As an extension of UDE, uncertain partial differential equation (UPDE) was first proposed by Yang and Yao [16]. And they derived uncertain heat equation

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(UHE) whose heat source is often affected by the uncertain interference. In addition, they also obtained the solution and inverse uncertainty distribution of solution for a special class of UHE. Later then, under linear growth condition and Lipschitz condition, Yang and Ni [18] proved an existence and uniqueness theorem of solution for general UHE. Yang [19] defined a concept of $\alpha$-path for UHE, and showed that the solution of an UHE can be represented by a spectrum of solutions for ordinary heat equations. Besides, he also got the inverse uncertainty distribution of solution and expected value of solution for an UHE via $\alpha$-path. Moreover, Yang and Ni [20] considered the extreme values of solution for an UHE.

However, stability analysis of UHE has not been researched as yet. This paper first introduces a concept of stability for UHE, and aims at proving a stability theorem. The rest of this paper is set out as follows. Section 2 introduces some works of UHE. Section 4 obtains a stability theorem under strong Lipschitz condition that provide a sufficient for an UHE being stable in measure. At last, Section 5 addresses a brief summary.

2. Uncertain heat equation. This section gives a brief introduction for UHE. A heat equation describes the variation of temperature in a given region over time. However, heat source often suffers the interference of noise in practice. In order to described the noise, two processes are used, one is a Wiener process that is based on probability theory, another is a Liu process that is based on uncertainty theory (see Appendix A). If we consider noise as Wiener process, then heat equation turns into stochastic heat equation. Nevertheless, Yang and Yao [16] pointed that it is unreasonable to model the heat conduction process via stochastic heat equation. Therefore, Yang and Yao [16] proposed a one-dimensional UHE whose the noise of heat source is described by Liu process as follows,

\[
\begin{aligned}
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} &= f(t,x) + \sigma(t,x)\dot{C}_t \\
U_{0,x} &= \varphi(x), \quad t > 0, x \in \mathbb{R}
\end{aligned}
\]

where $a^2$ is a constant called thermal diffusivity, $\dot{C}_t = dC_t/dt$ denotes the time white noise, $C_t$ is a Liu process, $f(t,x)$ is a heat source, $\sigma(t,x)$ is a diffusion term of heat source, and $\varphi(x)$ is a given initial temperature at time $t = 0$. They proved that the solution of UHE (1) is

\[
U_{t,x} = \int_{-\infty}^{+\infty} K(t, x-y)\varphi(y)dy + \int_{0}^{t} \int_{-\infty}^{+\infty} K(t-s, x-y)f(s,y)dyds \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} K(t-s, x-y)\sigma(s,y)dydC_s
\]

where

\[
K(t,x) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2t}\right).
\]

For a general UHE,

\[
\begin{aligned}
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} &= f(t,x,U_{t,x}) + \sigma(t,x,U_{t,x})\dot{C}_t \\
U_{0,x} &= \varphi(x), \quad t > 0, x \in \mathbb{R}
\end{aligned}
\]
where \( \varphi(x) \) is a bounded real-valued function. Yang and Ni \([18]\) proved an existence and uniqueness theorem of solution for UHE (3) under linear growth condition
\[
|f(t, x, u)| + |\sigma(t, x, u)| \leq Q(1 + |u|), \quad \forall x \in \mathbb{R}, t \geq 0
\]
and Lipschitz condition
\[
|f(t, x, u) - f(t, x, v)| + |\sigma(t, x, u) - \sigma(t, x, v)| \leq Q|u - v|, \quad \forall x \in \mathbb{R}, t \geq 0
\]
for some constant \( Q \). As a corollary, the solution (2) is unique for UHE (1) if \( f(t, x), \sigma(t, x) \) and \( \varphi(x) \) are bounded functions.

3. **Stability in measure.** This section defines a concept of stability in measure for an UHE, and gives an example to explain the definition.

**Definition 3.1.** An UHE
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t, x, U_{t,x}) + \sigma(t, x, U_{t,x})\dot{C}_t
\]
is said to be stable in measure if for any two solutions \( U_{t,x} \) and \( V_{t,x} \) with different initial values \( U_{0,x} \) and \( V_{0,x} \), respectively, we have
\[
\lim_{\sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \to 0} \mathcal{M}\{|U_{t,x} - V_{t,x}| \leq \epsilon, \text{ for all } t \geq 0 \text{ and } x\} = 1
\]
for any given number \( \epsilon > 0 \).

**Example 3.1.** Consider the UHE
\[
\frac{\partial U_{t,x}}{\partial t} - \frac{\partial^2 U_{t,x}}{\partial x^2} = p + q\dot{C}_t
\]
where \( p \) and \( q \) are two real numbers. Since its solution with different initial values \( U_{0,x} \) and \( V_{0,x} \) are
\[
U_{t,x} = \int_{-\infty}^{+\infty} K(t, x - y)U_{0,y}dy + pt + qC_t,
\]
\[
V_{t,x} = \int_{-\infty}^{+\infty} K(t, x - y)V_{0,y}dy + pt + qC_t
\]
respectively, where
\[
K(t, x) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right).
\]
We have
\[
\lim_{\sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \to 0} \mathcal{M}\{|U_{t,x} - V_{t,x}| \leq \epsilon, \text{ for all } t \geq 0 \text{ and } x\}
\]
\[
= \lim_{\sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \to 0} \mathcal{M}\left\{\int_{-\infty}^{+\infty} K(t, x - y)|U_{0,y} - V_{0,y}|dy \leq \epsilon, \text{ for all } t \geq 0 \text{ and } x\right\}
\]
\[
= 1
\]
for any given number \( \epsilon > 0 \). Thus, this UHE is stable in measure.
4. **A sufficient condition.** This section proves a sufficient condition for an UHE being stable in measure. As a corollary, we also give a sufficient condition for other types of UHE being stable in measure.

**Theorem 4.1.** Assume the UHE
\[
\frac{\partial U_{t,x}}{\partial t} - a^2 \frac{\partial^2 U_{t,x}}{\partial x^2} = f(t,x,U_{t,x}) + \sigma(t,x,U_{t,x}) \dot{C}_t
\]  

has a unique solution for each given initial value. Then it is stable in measure if the coefficients \( f(t,x,u) \) and \( \sigma(t,x,u) \) satisfy strong Lipschitz condition
\[
|f(t,x,u) - f(t,x,v)| + |\sigma(t,x,u) - \sigma(t,x,v)| \leq Q(t,x)|u - v|, \quad \forall x,u,v \in \mathbb{R}, t \geq 0
\]
where \( Q(t,x) \) is some positive function satisfying
\[
\int_0^\infty \int_{-\infty}^{+\infty} K(t-s,x-y)Q(s,y)dyds < +\infty.
\]

**Proof.** Let \( U_{t,x} \) and \( V_{t,x} \) be the solutions for the UHE (4) with different initial values \( U_{0,x} \) and \( V_{0,x} \), respectively. Then for a Lipschitz continuous sample path \( C_t(\gamma) \), we get
\[
U_{t,x}(\gamma) = \int_{-\infty}^{+\infty} K(t,x-y)U_{0,y}dy + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)f(s,y,U_{s,y}(\gamma))dyds
\]
\[
+ \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)\sigma(s,y,U_{s,y}(\gamma))dydC_s(\gamma)
\]
and
\[
V_{t,x}(\gamma) = \int_{-\infty}^{+\infty} K(t,x-y)V_{0,y}dy + \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)f(s,y,V_{s,y}(\gamma))dyds
\]
\[
+ \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)\sigma(s,y,V_{s,y}(\gamma))dydC_s(\gamma).
\]

By the strong Lipschitz condition, we obtain
\[
|U_{t,x}(\gamma) - V_{t,x}(\gamma)|
\]
\[
\leq \int_{-\infty}^{+\infty} K(t,x-y)|U_{0,y} - V_{0,y}|dy
\]
\[
+ \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)|f(s,y,U_{s,y}(\gamma)) - f(s,y,V_{s,y}(\gamma))|dyds
\]
\[
+ \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)|\sigma(s,y,U_{s,y}(\gamma)) - \sigma(s,y,V_{s,y}(\gamma))|dy|dC_s(\gamma)|
\]
\[
\leq \int_{-\infty}^{+\infty} K(t,x-y)|U_{0,y} - V_{0,y}|dy
\]
\[
+ \int_0^t \int_{-\infty}^{+\infty} K(t-s,x-y)Q(s,y)|U_{s,y}(\gamma) - V_{s,y}(\gamma)|dydC_s(\gamma)
\]

where \( Q(t,x) \) is some positive function satisfying
\[
\int_0^\infty \int_{-\infty}^{+\infty} K(t-s,x-y)Q(s,y)dyds < +\infty.
\]
\[ + L(\gamma) \int_0^t \int_{-\infty}^{+\infty} K(t - s, x - y)Q(s, y)|U_{s,y}(\gamma) - V_{s,y}(\gamma)| \, dy \, ds \]

\[ = \int_{-\infty}^{+\infty} K(t, x - y)|U_{0,y} - V_{0,y}| \, dy \]

\[ + (1 + L(\gamma)) \int_0^t \int_{-\infty}^{+\infty} K(t - s, x - y)Q(s, y)|U_{s,y}(\gamma) - V_{s,y}(\gamma)| \, dy \, ds \]

where \( L(\gamma) \) is the Lipschitz constant of \( C(t) \). It follows from the Grönwall’s inequality that

\[ |U_{t,x}(\gamma) - V_{t,x}(\gamma)| \leq \int_{-\infty}^{+\infty} K(t, x - y)|U_{0,y} - V_{0,y}| \, dy \cdot \exp \left\{ (1 + L(\gamma)) \int_0^t \int_{-\infty}^{+\infty} K(t - s, x - y)Q(s, y) \, dy \, ds \right\} \]

\[ \leq \int_{-\infty}^{+\infty} K(t, x - y)|U_{0,y} - V_{0,y}| \, dy \cdot \exp \left\{ (1 + L(\gamma)) \int_0^t \int_{-\infty}^{+\infty} K(t - s, x - y)Q(s, y) \, dy \, ds \right\} \]

for any \( t \geq 0 \) and \( x \). From Lemma A.4, there exists a real number \( N \) such that

\[ M\{\gamma|L(\gamma) \leq N\} \geq 1 - \epsilon \]

for any given \( \epsilon > 0 \). Taking

\[ \delta = \epsilon \sup_{t \geq 0, x \in \mathbb{R}} \exp \left\{ - (1 + N) \int_0^\infty \int_{-\infty}^{+\infty} K(t - s, x - y)Q(s, y) \, dy \, ds \right\}, \]

when \( \sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \leq \delta \), we get

\[ \int_{-\infty}^{+\infty} K(t, x - y)|U_{0,y} - V_{0,y}| \, dy \leq \delta \int_{-\infty}^{+\infty} K(t, x - y) \, dy = \delta \]

for any \( t > 0 \). Then \( |U_{t,x}(\gamma) - V_{t,x}(\gamma)| \leq \epsilon \) for any \( t > 0 \) and \( x \) provided that \( \sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \leq \delta \) and \( L(\gamma) \leq N \). It means

\[ M\{\{U_{t,x} - V_{t,x} \leq \epsilon \text{ for all } t > 0 \text{ and } x\}\} \geq 1 - \epsilon \]

as long as \( \sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \leq \delta \). That is,

\[ \lim_{\sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \to 0} M\{\{U_{t,x} - V_{t,x} \leq \epsilon \text{ for all } t > 0 \text{ and } x\}\} = 1. \]

Obviously, we have

\[ \lim_{\sup_{x \in \mathbb{R}} |U_{0,x} - V_{0,x}| \to 0} M\{\{U_{0,x} - V_{0,x} \leq \epsilon \text{ for all } x\}\} = 1. \]

Thus the theorem is proved. \( \square \)

**Remark 1.** If an UHE satisfies linear growth condition and strong Lipschitz condition on \( t \times x = \mathbb{R}^+ \times \mathbb{R} \), then it is stable in measure.
Corollary 1. If \( f(t,x) \) and \( \sigma(t,x) \) are both bounded functions on \( t \times x = \mathbb{R}^+ \times \mathbb{R} \), then the UHE
\[
\frac{\partial U(t,x)}{\partial t} - \frac{\partial^2 U(t,x)}{\partial x^2} = f(t,x) + \sigma(t,x)\dot{C}_t
\]
is stable in measure.

Proof. From the work of Yang [18], the UHE has a unique solution for a given bounded initial value \( \varphi(x) \). And, it is obvious that \( f(t,x) \) and \( \sigma(t,x) \) satisfy strong Lipschitz condition. Based on Theorem 4.1, the corollary is thus proved. \( \square \)

Example 4.1. Consider the UHE
\[
\frac{\partial U(t,x)}{\partial t} - \frac{\partial^2 U(t,x)}{\partial x^2} = \sin(x)e^{-t} + \dot{C}_t.
\]
Since \( f(t,x,u) = \sin(x)e^{-t} \) and \( \sigma(t,x,u) = 1 \) are both bounded functions on \( t \times x = \mathbb{R}^+ \times \mathbb{R} \), this UHE is stable in measure.

Example 4.2. Consider the UHE
\[
\frac{\partial U(t,x)}{\partial t} - \frac{\partial^2 U(t,x)}{\partial x^2} = \cos(x) + \frac{1}{t+1}\dot{C}_t.
\]
Since \( f(t,x,u) = \cos(x) \) and \( \sigma(t,x,u) = 1/(t+1) \) are both bounded functions on \( t \times x = \mathbb{R}^+ \times \mathbb{R} \), this UHE is stable in measure.

5. Conclusion. The main contribution of this paper was first to propose the concept of stability for UHE. A stability theorem was proved under strong Lipschitz condition to provide a sufficient condition for an UHE being stable. For future work, one may consider the other concepts of stability for uncertain heat equation, such as stability in mean, stability in distribution and almost sure stability, and can discuss the relationships of those concepts of stability.

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Appendix A. Uncertainty theory. This section introduces a few fundamental concepts and theorems in uncertainty theory including uncertain measure, uncertain process and uncertain field.

Definition A.1. (Liu [3]) Let \( \mathcal{L} \) be a \( \sigma \)-algebra on a nonempty set \( \Gamma \). A set function \( \mathcal{M}: \mathcal{L} \to [0,1] \) is called an uncertain measure if it satisfies the following axioms,
(i) Normality Axiom. \( \mathcal{M}\{\Gamma\} = 1 \) for the universal set \( \Gamma \);
(ii) Duality Axiom. \( \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1 \) for any event \( \Lambda \);
(iii) Subadditivity Axiom. For every countable sequence of events \( \Lambda_1, \Lambda_2, \cdots \), we have
\[
\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.
\]

Besides, Liu [5] defined the product uncertain measure on the product \( \sigma \)-algebre \( \mathcal{L} \) as follows.
(iv) Product Axiom. Let $\prod_{k} L_{k}, M_{k}$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $M$ is an uncertain measure satisfying

$$M \left\{ \prod_{k=1}^{\infty} \Lambda_{k} \right\} = \bigwedge_{k=1}^{\infty} M_{k} \{ \Lambda_{k} \}$$

where $\Lambda_{k}$ are arbitrarily chosen events from $L_{k}$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, L, M)$ to the set of real numbers $\mathbb{R}$. The uncertain variables $\xi_{1}, \xi_{2}, \cdots, \xi_{m}$ are said to be independent if

$$M \{ \bigcap_{i=1}^{m} \{ \xi_{i} \in B_{i} \} \} = \bigwedge_{i=1}^{m} M \{ \xi_{i} \in B_{i} \}$$

for any Borel sets $B_{1}, B_{2}, \cdots, B_{m}$ of real numbers.

**Definition A.2.** (Liu [4]) Let $T$ be a totally ordered set (e.g. time) and let $(\Gamma, L, M)$ be an uncertainty space. An uncertain process is a function $X_{t}(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{ X_{t} \in B \}$ is an event for any Borel set $B$ of real numbers at each time $t$.

**Definition A.3.** (Liu [5]) An uncertain process $C_{t}$ is said to be a Liu process if

1. $C_{0} = 0$ and almost all sample paths are Lipschitz continuous;
2. $C_{t}$ has stationary and independent increments;
3. every increment $C_{s+t} - C_{s}$ is a normal uncertain variable with an uncertainty distribution

$$\Phi(x) = \left( 1 + \exp \left( -\frac{\pi x}{\sqrt{3t}} \right) \right)^{-1}, \ x \in \mathbb{R}.$$ 

**Lemma A.4.** (Yao et al. [23]) Let $C_{t}$ be a Liu process. Then there exists an uncertain variable $Q$ such that for each $\gamma$, $Q(\gamma)$ is a Lipschitz constant of the sample path $C_{t}(\gamma)$, and

$$\lim_{x \to +\infty} M\{ \gamma \in \Gamma | Q(\gamma) \leq x \} = 1.$$ 

**Definition A.5.** (Liu [7]) Let $T$ be a partially ordered set (e.g. time×space) and let $(\Gamma, L, M)$ be an uncertainty space. An uncertain field is a function $X_{t}(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{ X_{t} \in B \}$ is an event for any Borel set $B$ of real numbers at each $t$. 

Let $h(t, c)$ be a continuously differentiable function. Then $Z_{t} = h(t, C_{t})$ has an uncertain differential

$$dZ_{t} = \frac{\partial h}{\partial t}(t, C_{t}) dt + \frac{\partial h}{\partial c}(t, C_{t}) dC_{t}.$$ 

**Definition A.6.** (Liu [7]) Let $T$ be a partially ordered set (e.g. time×space) and let $(\Gamma, L, M)$ be an uncertainty space. An uncertain field is a function $X_{t}(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{ X_{t} \in B \}$ is an event for any Borel set $B$ of real numbers at each $t$. 

Theorem 5.1. If $X_{t}$ is an uncertain field, then $X_{t}$ is a Liu process if and only if $X_{t}$ is a Liu stochastic process.
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