Analytical approximation of the stress-energy tensor of a quantized scalar field in static spherically symmetric spacetimes

Arkady A. Popov
Department of Mathematics, Kazan State Pedagogical University, Mezhlauk 1 st., Kazan 420021, Russia

Analytical approximations for $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ of a quantized scalar field in static spherically symmetric spacetimes are obtained. The field is assumed to be both massive and massless, with an arbitrary coupling $\xi$ to the scalar curvature, and in a zero temperature vacuum state. The expressions for $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ are divided into low- and high-frequency parts. The contributions of the high-frequency modes to these quantities are calculated for an arbitrary quantum state. As an example, the low-frequency contributions to $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ are calculated in asymptotically flat spacetimes in a quantum state corresponding to the Minkowski vacuum (Boulware quantum state). The limits of the applicability of these approximations are discussed.

PACS number(s): 04.62.+v, 04.70.Dy

I. INTRODUCTION

The interest in the vacuum polarization effects in strong gravitational fields is connected mainly with investigations of the early Universe and the construction of a self-consistent model of black hole evaporation. The main objects to calculate from quantum field theory in curved spacetime are the quantities $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ where $\varphi$ is the quantum field and $T_\mu^\nu$ is the stress-energy tensor operator for $\varphi$. The latter quantity is of particular interest as a source term in the semiclassical Einstein field equations:

$$G_\nu^\mu = 8\pi G \langle T_\mu^\nu \rangle. \quad (1)$$

Except for very special spacetimes, on which quantum matter fields are propagated, and for boundary conditions with a high degree of symmetry (see, for example [1]), it is not possible to obtain exact expressions for these quantities. Numerical computations of these quantities are as a rule extremely intensive [2, 3, 4, 5]. Thus it is useful, when possible, to have analytical approximations to $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$. One of the most widely used techniques to obtain information about these quantities is the DeWitt-Schwinger expansion [6]. It may be used to give the expansions for $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ in terms of powers of $mL$ where $m$ is the mass of the quantized field and $L$ is the characteristic scale of change of the background gravitational field. For conformally coupled massless fields approximate calculations have also been made. For $\langle T_\mu^\nu \rangle$ in static Einstein spacetimes ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) these include the approximations of Page, Brown, and Ottewill [7, 8, 9]. These results have been generalized to arbitrary static spacetimes by Zannias [10]. A different approach to the derivation of approximate expressions for $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ for conformally coupled massless fields in static spacetimes has been proposed by Frolov and Zel’nikov [11]. Their calculations were based primarily on geometric arguments and the common properties of the stress-energy tensor rather than on a field theory. Using the methods of quantum field theory the expressions for $\langle \varphi^2 \rangle$ and $\langle T_\mu^\nu \rangle$ of a scalar field in static spherically symmetric asymptotically flat spacetimes have been obtained by Anderson, Hiscock, and Samuel [4]. They assumed that the field is massive or massless with an arbitrary coupling $\xi$ to the scalar curvature and in a zero temperature quantum state or a nonzero temperature thermal state. The result was presented as a sum of two parts, numerical and analytical:

$$\langle T_\mu^\nu \rangle_{\text{ren}} = \langle T_\mu^\nu \rangle_{\text{numeric}} + \langle T_\mu^\nu \rangle_{\text{analytic}}. \quad (2)$$

The analytical part of their expression is conserved. This has a trace equal to the trace anomaly for the conformally invariant field. For these reasons they proposed to use $\langle T_\mu^\nu \rangle_{\text{analytic}}$ directly as an approximation for $\langle T_\mu^\nu \rangle_{\text{ren}}$. An analogous result has been obtained by Groves, Anderson, and Carlson [12] in the case of a massless spin-2 field in a general static spherically symmetric spacetime.

There are two questions about all of these approximations. First, what are the limits of applicability of these approximations? And second, how can one describe quantum states not considered in the above mentioned works?

The DeWitt-Schwinger expansion is independent of the quantum state. The criterion for the validity of the DeWitt-Schwinger approximation is well known: $mL \gg 1$. The validity of the other approximations discussed above was investigated by the authors by means of comparison with earlier known results.

In this paper, approximate expressions for $\langle \varphi^2 \rangle_{\text{ren}}$ and $\langle T_\mu^\nu \rangle_{\text{ren}}$ of a quantized scalar field in static spherically symmetric spacetimes are derived. The field is assumed to be both massless or massive with an arbitrary coupling $\xi$ to the scalar curvature $R$, and in a zero temperature vacuum state. The expressions for $\langle \varphi^2 \rangle_{\text{ren}}$ and $\langle T_\mu^\nu \rangle_{\text{ren}}$ are divided
into low- and high-frequency parts. The Anderson-Hiscock-Samuel approach \[4\] is used for derivation of the high-
frequency contributions to these quantities. Being an ultraviolet effect, the high-frequency contribution is general in the
sense that its value does not depend on the quantum state in which \(\langle \varphi^2 \rangle_{\text{ren}}\) and \((T_\mu^\nu)_{\text{ren}}\) are taken. This contribution contains all the ultraviolet divergences and can be renormalized. The low-frequency contributions are determined by the quantum state and, as an example, such contributions are calculated in asymptotically flat spacetime in a quantum state corresponding to the Minkowski vacuum (in generally accepted terminology this corresponds to the choice of the Boulware quantum state). Both parts of \((T_\mu^\nu)_{\text{ren}}\) are separately conserved. For a conformally invariant field the trace of the high-frequency part of \((T_\mu^\nu)_{\text{ren}}\) is equal to the trace anomaly and that of the low-frequency part of \((T_\mu^\nu)_{\text{ren}}\) is equal to zero. The validity of the approximations is discussed.

In Sec. II the expressions for the Euclidian Green’s function and the unrenormalized stress-energy tensor of a scalar field with arbitrary mass and curvature coupling in a general static spherically symmetric spacetime are derived. In Sec. III the WKB approximations for \(\langle \varphi^2 \rangle\) and \((T_\mu^\nu)\) of a very massive field are discussed. Section IV describes the method of calculating the high-frequency contributions to these quantities and the renormalization procedure for \(\langle \varphi^2 \rangle\) and \((T_\mu^\nu)\). The low-frequency contributions in the case of asymptotically flat spacetimes are derived and the renormalized expressions for \(\langle \varphi^2 \rangle\) and \((T_\mu^\nu)\) are displayed in Sec. V. The results are summarized in Sec. VI. In the Appendixes some technical results are derived and long expressions are displayed.

The units \(\hbar = c = 1\) are used throughout the paper.

II. AN UNRENNORMALIZED EXPRESSION FOR \(\langle \varphi^2 \rangle\) AND \((T_\mu^\nu)\)

In this section the main points of the Anderson-Hiscock-Samuel approach \[4\] for obtaining unrenormalized expressions for \(\langle \varphi^2 \rangle\) and \((T_\mu^\nu)\) are outlined.

First of all the metric of the static spherically symmetric spacetime is analytically continued into Euclidean form

\[
ds^2 = fd\tau^2 + d\rho^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where \(f\) and \(r\) are functions of \(\rho\), and \(\tau\) is the Euclidean time \((\tau = it, \text{where} \ t\ \text{is the coordinate corresponding to the timelike Killing vector, which always exists in static spacetime})\).

The regularization of \(\langle \varphi^2 \rangle\) and \((T_\mu^\nu)\) is achieved by using the method of point splitting. When the points are separated one can show that

\[
\langle \varphi^2 \rangle_{\text{unren}} = G_E(x, \bar{x}),
\]

\[
\langle T_\mu^\nu \rangle_{\text{unren}} = (1/2 - \xi) (g^{\alpha\sigma} G_E;\alpha\sigma + \dot{g}_E^{\nu}) + (2\xi - 1/2) \delta_E^{\nu} g^{\sigma\alpha} G_E;\sigma\alpha - \xi (G_E;\mu_\nu + \dot{g}_E^{\nu} + g^{\sigma\alpha} G_E;\sigma\alpha \right)

+ \dot{g}_E^{\nu} + 2\xi \delta_E^{\nu} (m^2 + \xi R) G_E + \xi (R_E^{\nu_\nu} - \delta_E^{\nu} R/2) G_E - \delta_E^{\nu} m^2 G_E/2,
\]

where \(m\) is the mass of the scalar field, \(\xi\) is its coupling to the scalar curvature \(R\), and \(g^{\alpha\beta}\) is the bivector of parallel transport of a vector at \(\bar{x}\) to one at \(x\).

The integral representation for the Euclidean Green’s function \(G_E(x, \bar{x})\) of a scalar field in a static spherically symmetric spacetime used by Anderson et al. \[3\] is the following:

\[
G_E(x, \bar{x}) = \frac{1}{4\pi^2} \int_0^\infty d\omega \cos[\omega(\tau - \bar{\tau})] \sum_{l=0}^\infty (2l + 1) P_l(\cos \gamma) C_{\omega l} \left( p_{\omega l}(\rho_\omega) - q_{\omega l}(\rho_\omega) \right),
\]

where \(P_l\) is a Legendre polynomial, \(\cos \gamma = \cos \theta \cos \bar{\theta} + \sin \theta \sin \bar{\theta} \cos(\varphi - \bar{\varphi})\), \(C_{\omega l}\) is a normalization constant, \(\rho_\omega\) and \(\rho_\omega\) represent the lesser and greater of \(\rho\) and \(\bar{\rho}\), respectively, and the modes \(p_{\omega l}(\rho)\) and \(q_{\omega l}(\rho)\) satisfy the equation

\[
\left\{ \frac{d^2}{d\rho^2} + \left[ \frac{1}{2f} \frac{df}{d\rho} + \frac{1}{r^2} \frac{dr^2}{d\rho} \right] \frac{d}{d\rho} - \left[ \frac{\omega^2}{f} + \frac{l(l+1)}{r^2} + m^2 + \xi R \right] \right\} \left( p_{\omega l} \right) = 0
\]

and the Wronskian condition

\[
C_{\omega l} \left( p_{\omega l} \frac{dq_{\omega l}}{d\rho} - q_{\omega l} \frac{dp_{\omega l}}{d\rho} \right) = \frac{-1}{r^2 f^{1/2}}.
\]

Above, it is assumed that the field is in a zero temperature vacuum state defined with respect to the timelike Killing vector.
By the change of variables

\[ p_{\omega l} = \frac{1}{\sqrt{2\pi^2 W}} \exp \left\{ \int_{\rho} f^\rho W f^{-1/2} d\rho \right\}, \]

\[ q_{\omega l} = \frac{1}{\sqrt{2\pi^2 W}} \exp \left\{ - \int_{\rho} f^\rho W f^{-1/2} d\rho \right\}, \]

one sees that the Wronskian condition \( \mathcal{B} \) is satisfied by \( C_{\omega l} = 1 \) and the mode equation \( \mathcal{C} \) gives the following equation for \( W(\rho) \):

\[ W^2 = \omega^2 + \frac{f}{r^2} l(l+1) + \frac{f'}{4r^2} + V + \frac{f''}{8W^2} + \frac{f}{4W^2} - \frac{5f (W^2)'^2}{16W^4}, \]

where

\[ V = f m^2 + \left( 2\xi - \frac{1}{4} \right) \frac{f'}{r^2} + \frac{f'^2}{2r^2} - f \left( \frac{f''}{2r^2} - \frac{(r^2)'^2}{4r^2} \right) \]

\[ + \xi f \left( \frac{f''}{f} - 2 \frac{(r^2)''}{2f} + \frac{f'(r^2)'}{4f^2} - \frac{(r^2)''}{4f^2} \right). \]

The prime denotes a derivative with respect to \( \rho \).

Now one can rewrite expressions \( \mathcal{D}, \mathcal{E} \) using expressions \( \mathcal{F}, \mathcal{G} \) and then suppose \( \rho = \tilde{\rho}, \theta = \tilde{\theta}, \phi = \tilde{\phi} \). The superficial divergences in the sums over \( l \) that appear in this case can be removed as in Refs. \( \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P} \):

\[ \langle \varphi^2 \rangle_{\text{unren}} = B_1, \]

\[ \langle T^i_i \rangle_{\text{unren}} = \left[ \frac{1}{2} g^i \xi \xrightarrow{\xi} f(g^i)^2 - \xi (g^{\rho})^2 \right] f \frac{\partial^2}{r^2 \partial_\xi^2} B_1 + \left( 2\xi - \frac{1}{2} \right) g^{\rho \rho} B_2 + \frac{1}{2} \xi f (g^i)^2 \]

\[ + \left( 2\xi - \frac{1}{2} \right) B_3 + \xi - \frac{f'}{2f} (g^{\rho})^2 \] \[ + \left[ \xi f \left( \frac{f''}{f} - 2 \frac{(r^2)''}{2f} + \frac{f'(r^2)'}{4f^2} - \frac{(r^2)''}{4f^2} \right) \xi \left( g^{\rho \rho} \right)^2 \right] B_4 \]

\[ + \xi \left( 1 + (g^{\rho \rho})^2 \right) \left( \frac{1}{4r^2} + \frac{m^2}{\xi} \right) B_3 + \xi \left( 1 + (g^{\rho \rho})^2 \right) \left( \xi \left( g^{\rho \rho} \right)^2 \right) B_4 \]

\[ + \xi \left( 1 + (g^{\rho \rho})^2 \right) \left( \frac{1}{4r^2} - \frac{m^2}{\xi} \right) \left( \xi \left( g^{\rho \rho} \right)^2 \right) B_4 \]

\[ + \xi R^i \left( \xi \left( g^{\rho \rho} \right)^2 \right) B_5 + i \xi f \xi' g^{\rho \rho} g^{\rho \phi} \sqrt{\frac{f}{r^2 \partial_\xi}} B_1 + \xi \left[ 2g^{\rho \rho} + 2f g^{\rho \gamma} g^{\rho \phi} \right] \sqrt{\frac{f}{r^2 \partial_\xi}} B_4, \]

\[ \langle T^\rho \rangle_{\text{unren}} = \left( 2\xi - \frac{1}{2} \right) g^{\rho \rho} \frac{\partial^2}{r^2 \partial_\rho^2} B_2 + \left( 2\xi - \frac{1}{2} \right) g^{\rho \rho} B_2 + \frac{1}{2} \xi f (g^\rho)^2 \]

\[ + \xi \left( 1 + (g^{\rho \rho})^2 \right) \left( \frac{1}{4r^2} + \frac{m^2}{\xi} \right) B_3 + \xi \left( 1 + (g^{\rho \rho})^2 \right) \left( \xi \left( g^{\rho \rho} \right)^2 \right) B_4 \]

\[ + \xi R^\rho \left( \xi \left( g^{\rho \rho} \right)^2 \right) B_5 + i \xi f \xi' g^{\rho \rho} g^{\rho \phi} \sqrt{\frac{f}{r^2 \partial_\xi}} B_1 + i \xi \left[ 2g^{\rho \rho} + 2f g^{\rho \gamma} g^{\rho \phi} \right] \sqrt{\frac{f}{r^2 \partial_\xi}} B_4, \]

\[ \langle T^\phi \rangle_{\text{unren}} = \left( 2\xi - \frac{1}{2} \right) g^{\rho \phi} \frac{\partial^2}{r^2 \partial_\phi^2} B_2 + \left( 2\xi - \frac{1}{2} \right) g^{\rho \phi} B_2 + \frac{1}{2} \xi f (g^\phi)^2 \]

\[ + \xi \left( 1 + (g^{\rho \phi})^2 \right) \left( \frac{1}{4r^2} + \frac{m^2}{\xi} \right) B_3 + \xi \left( 1 + (g^{\rho \phi})^2 \right) \left( \xi \left( g^{\rho \phi} \right)^2 \right) B_4 \]

\[ + \xi R^\phi \left( \xi \left( g^{\rho \phi} \right)^2 \right) B_5 + i \xi f \xi' g^{\rho \phi} g^{\rho \phi} \sqrt{\frac{f}{r^2 \partial_\xi}} B_1 + i \xi \left[ 2g^{\rho \phi} + 2f g^{\rho \phi} g^{\rho \phi} \right] \sqrt{\frac{f}{r^2 \partial_\xi}} B_4, \]

where

\[ B_1 = \frac{1}{4\pi^2} \int_0^\infty du \cos(u\varepsilon) \sum_{l=0}^{\infty} \frac{1}{r^2} \left[ \sqrt{\frac{f}{r^2}} (l + 1/2) \right] - 1, \]
\[
B_2 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^2} \left[ -\left( l + \frac{1}{2} \right) \sqrt{\frac{r^2}{f}} W + \left( l + \frac{1}{2} \right) \frac{(r^2)^2}{4r^2} \sqrt{\frac{1}{r^2 W}} \right. \\
+ \left( l + \frac{1}{2} \right) \frac{(r^2)^{r'}}{4} \sqrt{\frac{f}{r^2 W}} + \left( l + \frac{1}{2} \right) \frac{r^2}{16} \sqrt{\frac{f (W^2)^{r'}}{W^5}} + \left( l + \frac{1}{2} \right)^2 + \frac{u^2}{2} \\
+ \frac{r^2}{2f} V - \frac{(r^2)^{r'}}{4r^2} + \frac{r^2 f'}{16f} \left( \frac{r^2}{f} \right) + \left( r^2 \right) \frac{(f)}{8} \left( \frac{f}{r^2} \right) + \frac{r^2}{8} \left( \frac{f}{r^2} \right)'' \left( \frac{r^2}{f} \right) \\
- \frac{7r^2}{32} \left( \frac{f}{r^2} \right)'^2 \left( \frac{r^2}{f} \right)^{-2}, \right. \]
\]
\]
\[
B_3 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^2} \left[ \sqrt{\frac{f (l+1/2)^2}{W}} - \left( l + \frac{1}{2} \right)^2 + \frac{u^2}{2} + \frac{r^4}{8f} \left( \frac{f}{r^2} \right)'' \right. \\
+ \left. \frac{r^4 f'}{16f^2} \left( \frac{f}{r^2} \right) - \frac{5r^4 f}{32f} \left( \frac{f}{r^2} \right)'^2 + \frac{r^2}{2f} V \right], \]
\]
\[
B_4 = \frac{1}{4\pi^2} \int_0^\infty du \cos(ue) \sum_{l=0}^\infty \frac{1}{r^2} \left[ \left( l + \frac{1}{2} \right) \frac{(r^2)^2}{4r^2} \sqrt{\frac{1}{f r^2 W}} - \left( l + \frac{1}{2} \right) \frac{|r|}{4} \sqrt{\frac{f (W^2)r'}{W^3}} \\
+ \frac{(r^2)^{r'}}{4|r|} + \frac{|r| f'}{4f} \right], \]
\]
\[
\varepsilon = \sqrt{\frac{f}{r^2}} (\tau - \tilde{\tau}), \quad u = w \sqrt{\frac{r^2}{f}}. \]

III. WKB APPROXIMATION FOR \(\langle \varphi^2 \rangle\) AND \(\langle T^2 \rangle\) OF A VERY MASSIVE FIELD

Obtaining the exact solution of Eq. (10) is a very complicated problem. However, it can be solved iteratively if there is a small parameter of the considered task.

Let us evaluate the order of terms in this equation. If the characteristic scale of variation of the gravitational field is designated as \(L\):

\[
L^{-1} = \max \left\{ \left| \frac{1}{r} \right|, \left| \ln(f r^2) \right|', \left| \ln(f r^2) \right|''^{1/2}, \left| \ln(f r^2) \right|'''^{1/3}, \ldots \right\},
\]

then in case \(r_c = 1/m \ll L\), one can consider the quantity

\[
\varepsilon_{\text{WKB}} = \frac{r_c}{L} \ll 1
\]
as a small parameter of the iterative procedure. In this case as the zeroth-order term of the iterative procedure one can choose

\[
(W^2)_{(0)} = \omega^2 + \frac{f}{r^2} l(l + 1) + fm^2.
\]

The term of the next order [the iteration procedure demands the using of \((W^2)_{(0)}\) instead of \(W^2\) in the right hand side of Eq. (10) for calculation of the next order term]

\[
(W^2)_{(2)} = \frac{f}{4r^2} + V + \frac{f'}{8} (W^2)_{(0)}' + \frac{f}{4} (W^2)_{(0)}'' + \frac{5f (W^2)_{(0)}'}{16 (W^2)_{(0)}^2}
\]

(24)
has the order \((\varepsilon_{\text{WKB}})^2(W^2)_0\), i.e., for all values of \(w\) and \(l\) this term is much less than the zeroth-order term. For derivation it is convenient to choose as the zeroth-order term of the iterative procedure the following:

\[
(W^2)_{(0)} = \omega^2 + \frac{f}{r^2} \left( l + \frac{1}{2} \right)^2 + fm^2, \tag{25}
\]

because the addition to \((W^2)_{(0)}\) some of the addends from \((W^2)_{(2)}\) does not change the conclusions:

\[
(W^2)_{(0)} \gg (W^2)_{(2)} \gg (W^2)_{(4)} \gg \ldots \tag{26}
\]

or

\[
(W^2)_{(2)} \sim \varepsilon_{\text{WKB}}^2(W^2)_{(0)}, \quad (W^2)_{(4)} \sim \varepsilon_{\text{WKB}}^4(W^2)_{(0)}, \ldots \tag{27}
\]

and

\[
W^2 = (W^2)_{(0)} + (W^2)_{(2)} + (W^2)_{(4)} + \ldots \tag{28}
\]

This approach gives (Ref. [4]) the DeWitt-Schwinger expansions of \(\langle \varphi^2 \rangle\) and \(\langle T_\mu^\nu \rangle\) in terms of powers of \(1/(\mu L)\). Note that there are the different methods of derivation of the approximations for \(\langle \varphi^2 \rangle\) and \(\langle T_\mu^\nu \rangle\) in the large mass limit [7].

**IV. HIGH-FREQUENCY CONTRIBUTION TO \(\langle \varphi^2 \rangle\) AND \(\langle T_\mu^\nu \rangle\)**

In the case \(r_c \gg L\) or for a massless field a small parameter does not exist. But we can evaluate the contribution to \(\langle T_\mu^\nu \rangle\) of the high-frequency modes. For that it is necessary to impose a lower-limit cutoff \(u_0\) on the integrals over \(w\) in expressions (16)–(19) \((u \geq u_0 = w_0 \sqrt{r_c/l})\). As a small parameter of the iterative procedure we can choose

\[
\varepsilon_{\text{WKB}} = \frac{\sqrt{T}}{u_0 L} = \frac{|r|}{u_0 L} \ll 1. \tag{29}
\]

This also means [see the definitions of \(L\), Eq. (21), and \(u\), Eq. (20)]

\[
u_0 \gg \frac{|r|}{L} \geq 1. \tag{30}
\]

The zeroth-order solution of Eq. (10) can be chosen as follows:

\[
(W^2)_{(0)} = \omega^2 + \frac{f}{r^2} \left( l + \frac{1}{2} \right)^2. \tag{31}
\]

Then the second and fourth orders are

\[
(W^2)_{(2)} = V + \frac{f'}{8} \frac{(W^2)'_{(0)}}{(W^2)_{(0)}} + \frac{f''}{4} \frac{(W^2)''_{(0)}}{(W^2)_{(0)}} - \frac{5f}{16} \frac{(W^2)_{(0)}}{(W^2)_{(0)}}^2, \tag{32}
\]

\[
(W^2)_{(4)} = \frac{f'}{8} \frac{(W^2)_{(2)}}{(W^2)_{(0)}} + \frac{f''}{4} \frac{(W^2)_{(2)}}{(W^2)_{(0)}} - \frac{5f}{16} \frac{(W^2)_{(2)}}{(W^2)_{(0)}}^2
\]

\[
= \frac{1}{(W^2)_{(0)}} \left[ \frac{fV'}{8} + \frac{fV''}{4} \right] + \frac{1}{(W^2)_{(0)}} \left[ \left( \frac{f^2f'''}{64} - \frac{5fV'}{8} + \frac{f'''}{32} - \frac{fV'}{8} \right) (W^2)_{(0)}' \right]
\]

\[
+ \left( \frac{f'}{8} + \frac{3f'^2}{64} - \frac{fV'}{4} \right) (W^2)_{(0)}'' + \frac{3f'f''}{16} (W^2)_{(0)}''' + \frac{f^2}{16} (W^2)_{(0)}'''
\]

\[
- \frac{9f'^2}{32} (W^2)_{(0)}'' - \frac{7f'f''}{16} (W^2)_{(0)}' (W^2)_{(0)}'' + \frac{1}{(W^2)_{(0)}} \left[ \frac{27f'f''}{32} (W^2)_{(0)} ' (W^2)_{(0)}''
\]

\[
+ \frac{27f'^2}{16} (W^2)_{(0)}' (W^2)_{(0)}'' - \frac{1}{(W^2)_{(0)}} \left[ \frac{135f'^2}{128} (W^2)_{(0)}' \right]. \tag{33}
\]
The high-frequency contributions to the quantities $B_1$, $B_2$, $B_3$, and $B_4$ are obtained by substituting the WKB expansion of $W^2$ into expressions (14-19) and imposing a lower-limit cutoff $u_0$ on the integrals over $u$:

\[
B_1^{\text{WKB}} = \frac{1}{4\pi^2} \left\{ \frac{1}{r^2} S_0^2(\varepsilon, u_0) - \frac{V}{2f} S_1^2(\varepsilon, u_0) - \frac{r^2}{16f^2} \left[ f' \left( \frac{f}{r^2} \right) ' + 2f \left( \frac{f}{r^2} \right) '' \right] S_2^2(\varepsilon, u_0) \right. \\
+ \frac{5r^4}{32f^2} \left( \frac{f}{r^2} \right) ''^2 S_2^2(\varepsilon, u_0) + \frac{r^2}{16f^2} \left[ 6V^2 - f'V' - 2fV'' \right] S_2^2(\varepsilon, u_0) \\
+ \frac{r^4}{128f^3} \left[ 20Vf' + 40fV' - f'f'' - 2ff''' \left( \frac{f}{r^2} \right) ' \right. \\
+ \left( 40fV - 3f^2 \right) \\
+ 8f'' \left( \frac{f}{r^2} \right) '' - 12ff' \left( \frac{f}{r^2} \right) '' - 4f \left( \frac{f}{r^2} \right) '' \right] S_3^1(\varepsilon, u_0) + \frac{r^6}{512f^4} \left[ 21f^2 \right. \\
+ 56f'' - 280fV \left( \frac{f}{r^2} \right) ^2 + 84f^2 \left( \frac{f}{r^2} \right) ^{112} + 112f^2 \left( \frac{f}{r^2} \right) ' \left( \frac{f}{r^2} \right) '' \right. \\
+ 252ff' \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right) ' \left( \frac{f}{r^2} \right) ' \right. \\
+ S_2^1(\varepsilon, u_0) + \frac{r^8}{512f^5} \left[ - 231ff' \left( \frac{f}{r^2} \right) ' \right. \\
- 462f^2 \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right) ' \right. \\
\left. \right. \\
+ O \left( \frac{\varepsilon^2_{\text{WKB}}}{L^2} \right), \right. (34)
\]

\[
B_2^{\text{WKB}} = \frac{1}{4\pi^2} \left\{ - \frac{1}{r^4} S_{-1}^2(\varepsilon, u_0) + \frac{1}{4rf^6} \left[ \left( \frac{f}{r^2} \right)' \frac{f}{r^2} - 2Vr^4 \right] S_0^2(\varepsilon, u_0) \right. \\
+ \frac{1}{16f^2r^2} \left[ 4f \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right)' - 2f \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right)' - 2r^2 \left( \frac{f}{r^2} \right) '' \right] S_1^2(\varepsilon, u_0) \right. \\
+ \frac{7r^2}{32f^2} \left( \frac{f}{r^2} \right) ' S_2^2(\varepsilon, u_0) + \frac{1}{16f^2r^4} \left[ 4fr^2 \left( \frac{f}{r^2} \right)' V' - 2f \left( \frac{f}{r^2} \right)' V \right. \\
+ 2r^4V^2 - 2Fr^4V'' \right] S_0^2(\varepsilon, u_0) + \frac{1}{128f^3r^2} \left[ \left( 56fr^4V' - 48fr^2V \right) \left( \frac{f}{r^2} \right)' \right. \\
- 2ff' \left( \frac{f}{r^2} \right)' - 2f^2 \left( \frac{f}{r^2} \right)' + 12fr^4Vf' - r^4f'f'' + 4f^2 \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right) ' \right. \\
+ \left( 24fr^4V - 8fr^4f' - 4f^2 \left( \frac{f}{r^2} \right)', 12fr^2f' \left( \frac{f}{r^2} \right)' - 3r^4f^2 \left( \frac{f}{r^2} \right) ' \right. \\
+ \left( 8f^2r^4 \left( \frac{f}{r^2} \right)' - 12fr^4f' \right. \left( \frac{f}{r^2} \right) ' \left( \frac{f}{r^2} \right) ' \right. \\
\left. \right. \\
+ \frac{1}{512f^4} \left[ \left( 19fr^2f' + 20f^2 \left( \frac{f}{r^2} \right)' - 80fr^2f' \left( \frac{f}{r^2} \right)' - 280fr^4V \right. \\
+ 64fr^4f' \right. \left( \frac{f}{r^2} \right) ' + \left( 268fr^4f' - 160f^2r^2 \left( \frac{f}{r^2} \right)' \right. \left( \frac{f}{r^2} \right) ' \right. \\
+ 76f^2r^4 \left( \frac{f}{r^2} \right)' + 128f^2r^4 \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right) ' \left( \frac{f}{r^2} \right)' \\
\left. \right. \\
+ \frac{r^4}{512f^4} \left[ \left( 140fr^2f' - 259fr^2f' \right. \left( \frac{f}{r^2} \right) + \left( 518fr^2 \left( \frac{f}{r^2} \right)' \left( \frac{f}{r^2} \right) ' \left( \frac{f}{r^2} \right) ' \right. \\
\left. \right. \\
+ 1365r^8 \frac{f}{r^2} \left( \frac{f}{r^2} \right) ' S_3^2(\varepsilon, u_0) + O \left( \frac{\varepsilon^2_{\text{WKB}}}{L^4} \right), \right. (35)
\]

$$B_3^{HFC} = \frac{1}{4\pi^2} \left\{ \frac{1}{2f^2} S_0^1(\varepsilon, u_0) - \frac{V}{2f} S_1^1(\varepsilon, u_0) - \frac{r^2}{16f^2} \left[ f'' \left( \frac{f}{r} \right)^2 + 2f \left( \frac{f}{r} \right)^4 \right] \right\} S_2^2(\varepsilon, u_0)$$
$$+ \frac{5r^4}{32f^4} \left( \frac{f}{r} \right)^4 S_3^3(\varepsilon, u_0) + \frac{r^2}{16f^2} \left[ 6V'' - 3f'f'' \right] S_2^4(\varepsilon, u_0)$$
$$+ \frac{r^4}{128f^4} \left[ (40fV' + 20fV - 2f'd'' - 4f'd') \left( \frac{f}{r} \right)'' + (40fV - 3f'^2 \right.$$}

$$- 8f''') \left( \frac{f}{r} \right)'' - 12f'' \left( \frac{f}{r} \right)'' - 4f' \left( \frac{f}{r} \right)'' \right\} S_3^5(\varepsilon, u_0) + \frac{r^6}{512f^4} \left[ 21f^2$$

$$+ 56f'' - 280fV \left( \frac{f}{r} \right)'' + 84f' \left( \frac{f}{r} \right)'' + 252f'' \left( \frac{f}{r} \right)'' \right\} S_4^4(\varepsilon, u_0) - \frac{231r^8}{512f^4} \left[ f' \left( \frac{f}{r} \right)^3$$

$$+ 2f \left( \frac{f}{r} \right)'' \left( \frac{f}{r} \right)'' + 1155r^{10} \left( \frac{f}{r} \right)'' \right\} S_5^5(\varepsilon, u_0) + O \left( \frac{r^2}{L^2} \right),$$

$$B_4^{HFC} = \frac{1}{4\pi^2} \left\{ - \frac{(r^2)'}{2r^2} S_0^0(\varepsilon, u_0) - \frac{1}{4f^2} \left( f' \right)^2 S_1^1(\varepsilon, u_0) + \frac{1}{4f^2} \left[ (r^2)'V - r^2V' \right] S_2^2(\varepsilon, u_0)$$

$$- \frac{1}{32f^4} \left[ (f'(r^2)' - r^2f'') - 12r^2V \left( \frac{f}{r} \right)'' + (2f(r^2)' - 3r^2f') \left( \frac{f}{r} \right)'' \right.$$}

$$- 2fr^2 \left( \frac{f}{r} \right)'' \right\} S_3^3(\varepsilon, u_0) + \frac{r^2}{64f^4} \left[ 10r^2f' - 5f(r^2)' \right. \left( \frac{f}{r} \right)''$$

$$+ 20fr^2 \left( \frac{f}{r} \right)'' \right\} S_4^4(\varepsilon, u_0) + \frac{35r^6}{128f^3} \left( \frac{f}{r} \right)^3 S_5^5(\varepsilon, u_0) + O \left( \frac{r^2}{L^3} \right).$$

The mode sums and the integrals in these expressions are of the form

$$S_n^k(\varepsilon, u_0) = \int_{u_0}^{\infty} du \cos(\varepsilon u) \sum_{l=0}^{\infty} \frac{(l + 1/2)^{2k+1}}{u^2 + (l + 1/2)^2} - \text{subtraction terms},$$

where \( k \) and \( n \) are integers, \( k \geq 0 \) and \( n \geq -1 \). The subtraction terms for the sum over \( l \) can be obtained by expanding the function being summed in inverse powers of \( l \) and truncating at \( O(r^0) \). This subtraction corresponds to the removing of the superficial divergences in the sums over \( l \) discussed above:

$$S_{n-1}^0(\varepsilon, u_0) = \int_{u_0}^{\infty} du \cos(\varepsilon u) \sum_{l=0}^{\infty} \frac{(l + 1/2)^{2n+1}}{u^2 + (l + 1/2)^2} - \left( l + \frac{1}{2} \right)^2 + \frac{(2n - 1)u^2}{2},$$

$$S_n^0(\varepsilon, u_0) = \int_{u_0}^{\infty} du \cos(\varepsilon u) \sum_{l=0}^{\infty} \frac{(l + 1/2)^{2n+1}}{u^2 + (l + 1/2)^2} - 1.$$
The substitution of these expressions into Eqs. (34)-(37) and then into Eqs. (13)-(15) gives

\[ S_1^1(\varepsilon, u_0) = \frac{4}{\varepsilon^4} - \frac{u_0^4}{6} + \frac{7}{960} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) - \frac{31}{32256 u_0^2} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^4 \ln |\varepsilon| \right), \]

\[ S_0^0(\varepsilon, u_0) = \frac{1}{\varepsilon^2} + \frac{u_0^2}{2} - \frac{1}{24} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) + \frac{7}{3840 u_0^2} + \varepsilon^2 \left[ \frac{u_0^4}{8} + \frac{u_0^2}{96} - \frac{7}{2560} \right]
+ \frac{7}{3840} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) - \frac{31}{86016 u_0^2} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right), \]

\[ S_1^1(\varepsilon, u_0) = \frac{2}{\varepsilon^2} + u_0^2 - \frac{7}{1920 u_0^2} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \ln |\varepsilon| \right), \]

\[ S_n^0(\varepsilon, u_0) = \frac{(2n)!!}{(2n-1)!!} \frac{1}{\varepsilon^2} + \frac{(2n)!!}{(2n-1)!!} \frac{u_0^2}{2} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \ln |\varepsilon| \right) \quad (n \geq 2), \]

\[ S_0^0(\varepsilon, u_0) = -\left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) + \frac{1}{48 u_0^2} + \varepsilon^2 \left[ \frac{u_0^2}{4} + \frac{1}{48} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) - \frac{1}{32} \right]
- \frac{7}{2560 u_0^2} \right] + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right), \]

\[ S_2^2(\varepsilon, u_0) = -\frac{2}{3} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) + \varepsilon^2 \left[ \frac{u_0^2}{6} + \frac{7}{3840 u_0^2} \right] + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right), \]

\[ S_{n+1}^n(\varepsilon, u_0) = \frac{(2n)!!}{(2n+1)!!} \left[ - \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) + \varepsilon^2 \frac{u_0^2}{2} \right] + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right) \quad (n \geq 2), \]

\[ S_2^0(\varepsilon, u_0) = \frac{1}{6 u_0^2} + \varepsilon^2 \left[ - \frac{1}{4} + \frac{1}{6} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) - \frac{1}{96 u_0^2} \right] + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right) \quad (n \geq 2), \]

\[ S_{n+2}^n(\varepsilon, u_0) = \frac{(2n)!!}{(2n+3)!!} \left\{ \frac{1}{2 u_0^2} + \varepsilon^2 \left[ - \frac{3}{4} + \frac{1}{2} \left( C + \frac{1}{2} \ln |\varepsilon^2 u_0^2| \right) \right] \right\} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right) \quad (n \geq 1), \]

\[ S_{n+3}^n(\varepsilon, u_0) = -\frac{(2n)!!}{(2n+5)!!} \frac{3 \varepsilon^2}{4 u_0^2} + O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right) \quad (n \geq 0), \]

\[ S_k^k(\varepsilon, u_0) = O \left( \frac{1}{u_0^4} \right) + O \left( \varepsilon^2 \right) + O \left( \varepsilon^4 \right) \quad (k \geq 0, \ n \geq k + 4). \]

The substitution of these expressions into Eqs. (34)-(37) and then into Eqs. (13)-(15) gives \( \langle \phi^2 \rangle_{\text{HFC}}^{\text{unren}} \) and \( \langle T^\mu T^\nu \rangle_{\text{HFC}}^{\text{unren}} \) - the high-frequency contributions to \( \langle \phi^2 \rangle_{\text{unren}} \) and \( \langle T^\mu T^\nu \rangle_{\text{unren}} \). Let us note that the expansions of \( \langle \phi^2 \rangle_{\text{HFC}}^{\text{unren}} \) and \( \langle T^\mu T^\nu \rangle_{\text{HFC}}^{\text{unren}} \) in terms of powers of \( u_0 \) correspond to the DeWitt-Schwinger expansions of \( \langle \phi^2 \rangle_{\text{unren}} \) and \( \langle T^\mu T^\nu \rangle_{\text{unren}} \) in terms of powers of \( mL \).

The renormalizations of \( \langle \phi^2 \rangle \) and \( \langle T^\mu \rangle \) are achieved by subtracting the renormalization counterterms from \( \langle \phi^2 \rangle_{\text{unren}} \) and \( \langle T^\mu \rangle_{\text{unren}} \) and then letting \( \bar{\tau} \rightarrow \bar{\tau} \):

\[ \langle \phi^2 \rangle_{\text{ren}} = \lim_{\bar{\tau} \rightarrow \bar{\tau}} \left[ \langle \phi^2 \rangle_{\text{unren}} - \langle \phi^2 \rangle_{\text{DS}} \right], \]
\[ \langle T_\nu^\mu \rangle_{\text{ren}} = \lim_{\tau \to \tau} [\langle T_\nu^\mu \rangle_{\text{unren}} - \langle T_\nu^\mu \rangle_{\text{DI}}], \tag{54} \]

where

\[ \langle \phi^2 \rangle_{\text{DS}} = \frac{1}{8 \pi^2} + \frac{1}{8 \pi^2} \left[ \frac{m^2}{2} + \left( \xi - \frac{1}{6} \right) R \right] \left[ C + \frac{1}{2} \ln \left( \frac{m_{\text{DS}}^2 |\sigma|}{2} \right) \right] - \frac{m^2}{16 \pi^2} + \frac{1}{96 \pi^2} R_{\alpha \beta} \sigma^\alpha \sigma^\beta, \tag{55} \]

\[ C \] is Euler’s constant, \( R_{\alpha \beta} \) is the Ricci tensor, \( R = R_{\alpha \beta} \), \( \sigma \) is one-half the square of the distance between the points \( x \) and \( \bar{x} \) along the shortest geodesic connecting them, \( \sigma^\mu \) is the covariant derivative of \( \sigma \), and the constant \( m_{\text{DS}} \) is equal to the mass \( m \) of the field for a massive scalar field. For a massless field \( m_{\text{DS}} \) is an arbitrary parameter due to the infrared cutoff in \( \langle \phi^2 \rangle_{\text{DS}} \). A particular choice of the value of \( m_{\text{DS}} \) corresponds to a finite renormalization of the coefficients of terms in the gravitational Lagrangian and must be fixed by experiment or observation. The expression for \( \langle T_\nu^\mu \rangle_{\text{DS}} \) is displayed in Ref. [17] and is too long to be displayed here.

For the metric \( g_{\mu \nu} \) the calculations of quantities \( \sigma^\mu \) and \( g^{\mu \nu} \) are similar to those in Ref. [4]:

\[
\begin{align*}
\sigma^t &= (t - \bar{t}) + \frac{f^2}{24 f} (t - \bar{t})^3 - \frac{1}{120} \left( \frac{f^4}{8 f^2} - \frac{3 f^2 f''}{8 f} \right) (t - \bar{t})^5 + O \left( (t - \bar{t})^7 \right), \\
\sigma^\rho &= -\frac{f'}{4} (t - \bar{t})^2 - \frac{f' f''}{96} (t - \bar{t})^4 + O \left( (t - \bar{t})^6 \right), \\
\sigma^\theta &= 0, \\
\sigma &= \frac{1}{2} g_{\mu \nu} \sigma^\mu \sigma^\nu, 
\end{align*}
\]

\[
\begin{align*}
g^{\nu \bar{\nu}} &= \frac{g^{\rho \rho}}{f^2} = -\frac{1}{f} - \frac{f^2}{8 f^2} (t - \bar{t})^2 + \left( \frac{f^4}{384 f^3} - \frac{f^2 f''}{96 f} \right) (t - \bar{t})^4 + O \left( (t - \bar{t})^6 \right), \\
g^{t \bar{t}} &= g^{\bar{t} t} = \frac{f}{2 f} (t - \bar{t}) - \left( \frac{f^3}{96 f^2} + \frac{f' f''}{48 f} \right) (t - \bar{t})^3 + O \left( (t - \bar{t})^5 \right). 
\end{align*}
\]

The high-frequency contributions to \( \langle \phi^2 \rangle_{\text{unren}} \) and \( \langle T_\nu^\mu \rangle_{\text{unren}} \) contain all the ultraviolet divergences and can also be renormalized. This procedure gives the second-order WKB approximation for \( \langle \phi^2 \rangle_{\text{ren}} \) and the fourth-order one for components of \( \langle T_\nu^\mu \rangle_{\text{ren}} \) that contain the contributions of high-frequency modes only \((w > w_0)\):

\[
\begin{align*}
\langle \phi^2 \rangle_{\text{WKB}} &= \lim_{\tau \to \tau} \left[ \langle \phi^2 \rangle_{\text{HFC}} - \langle \phi^2 \rangle_{\text{DS}} \right] = \langle \phi^2 \rangle_{(0)} + \langle \phi^2 \rangle_{(2)} + O \left( \frac{1}{u_0^2 L^2} \right), \\
\langle T_\nu^\mu \rangle_{\text{WKB}} &= \lim_{\tau \to \tau} \left[ \langle T_\nu^\mu \rangle_{\text{HFC}} - \langle T_\nu^\mu \rangle_{\text{DS}} \right] = \langle T_\nu^\mu \rangle_{(0)} + \langle T_\nu^\mu \rangle_{(2)} + \langle T_\nu^\mu \rangle_{(4)} + O \left( \frac{1}{u_0^2 L^4} \right),
\end{align*}
\]

where \( \langle \phi^2 \rangle_{(0)}, \langle \phi^2 \rangle_{(2)} \), and the nontrivial components of \( \langle T_\nu^\mu \rangle_{\text{WKB}} \) of zeroth and second WKB orders have the form

\[
\langle \phi^2 \rangle_{(0)} = \frac{u_0^2}{8 \pi^2 r^2}, \tag{60}
\]

\[
\langle \phi^2 \rangle_{(2)} = \frac{m^2}{16 \pi^2} + \frac{1}{16 \pi^2} \left[ \frac{m^2}{2} + \left( \xi - \frac{1}{6} \right) R \right] \ln \left| \frac{4 u_0^2}{m_{\text{DS}}^2 r^2} \right| - \frac{f'^2}{96 \pi^2 f^2} + \frac{f''}{96 \pi^2 f^2} + \frac{f'(r^2)'}{96 \pi^2 f r^2}, \tag{61}
\]

\[
\langle T_\nu^\mu \rangle_{(0)} = \frac{u_0^4}{16 \pi^2 r^4}, \tag{62}
\]

\[
\langle T_\nu^\mu \rangle_{(2)} = -\frac{u_0^4}{48 \pi^2 r^4}, \tag{63}
\]

\[ \]
\[(T^\varphi_\theta)_{(0)} = (T^\varphi_\varphi)_{(0)} = -\frac{u_0^4}{48\pi^2 r^4}, \] (64)

\[(T^\varphi_t)_{(2)} = \frac{u_0^2}{4\pi^2 r^2} \left[ (\xi - \frac{1}{6}) \left( \frac{1}{2r^2} + \frac{5f'^2}{4f^2} - \frac{f''(r^2)'}{4f^2} + \frac{(r^2)'^2}{8r^4} - \frac{f'' - \frac{(r^2)''}{2r^2}}{4} \right) + m^2 \right], \] (65)

\[(T^\theta_\nu)_{(2)} = \frac{u_0^2}{4\pi^2 r^2} \left[ (\xi - \frac{1}{6}) \left( \frac{1}{2r^2} - \frac{f'^2}{4f^2} - \frac{f''(r^2)'}{4f^2} + \frac{(r^2)'^2}{8r^4} \right) - \frac{m^2}{4} \right], \] (66)

\[(T^\varphi_\varphi)_{(2)} = (T^\varphi_t)_{(2)} = \frac{u_0^2}{4\pi^2 r^2} \left[ (\xi - \frac{1}{6}) \left( \frac{5f'^2}{8f^2} - \frac{f''(r^2)'}{8r^2} + \frac{(r^2)'^2}{8r^4} - \frac{f'' - \frac{(r^2)''}{4r^2}}{4} \right) - \frac{m^2}{4} \right], \] (67)

\[u_0 = u_0 \sqrt{r^2/f} \gg 1. \] (68)

The quantities of the second WKB order \((\varphi^2)_{(2)}\) and the fourth WKB order \((T^\mu_\nu)_{(4)}\) are equivalent to the analytical approximations \(\langle \varphi^2 \rangle_{\text{analytic}}\) and \(\langle T^\mu_\nu \rangle_{\text{analytic}}\) of Anderson, Hiscock, and Samuel [3] (the constants \(m_{\text{os}}\) and \(u_0\) are equal to \(\mu\) and \(\lambda\) in their expressions for \(\langle \varphi^2 \rangle_{\text{analytic}}\) and \(\langle T^\mu_\nu \rangle_{\text{analytic}}\), respectively). In the coordinates [4] the expressions for \(\langle T^\mu_\nu \rangle_{(4)}\) are given in Appendix B.

The quantity \(\langle T^\mu_\nu \rangle_{\text{WKB}}\) is conserved and has the trace

\[\langle T^\mu_\nu \rangle_{\text{WKB}} = \frac{u_0^2}{4\pi^2 r^2} \left[ \frac{m^2}{2} + (\xi - \frac{1}{6}) \left( \frac{9f'^2}{4f^2} - \frac{3f''(r^2)'}{2f^2} - \frac{3f'' - \frac{3}{2}(r^2)''}{2f} \right) \right] + \langle T^\mu_\nu \rangle_{(4)}. \] (69)

For a conformally invariant field this trace is equal to the trace anomaly.

V. LOW-FREQUENCY CONTRIBUTION TO \(\langle \varphi^2 \rangle\) AND \(\langle T^\mu_\nu \rangle\)

The behavior of low-frequency modes is determined by the boundary conditions and the topological structure of the spacetime. If the spacetime is asymptotically flat and the characteristic scale of the gravitational field inhomogeneity \(\lambda\) is much less than the parameter \(\sqrt{f}/\omega_0\),

\[\frac{\lambda}{\sqrt{f}/\omega_0} \ll 1, \] (70)

the low-frequency contributions to \(\langle \varphi^2 \rangle\) and \(\langle T^\mu_\nu \rangle\) can be expanded in terms of powers of this small parameter. In the following the zeroth term of this expansion will be used for approximation of the low-frequency contributions to \(\langle \varphi^2 \rangle\) and \(\langle T^\mu_\nu \rangle\). This means that we choose the long-wave modes approximately coincident with long-wave modes of the Minkowski vacuum (in generally accepted terminology this corresponds to the choice of the Boulware quantum state). For these modes \(ds^2 = dT^2 + dx^2 + dy^2 + dz^2 = dT^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)\),

\[G_E(x, \tilde{x}) = \left( \frac{1}{2\pi} \right)^4 \int d\Omega d^3p \frac{\exp(i\Omega \Delta T + ip_\alpha \Delta x^\alpha)}{(\Omega^2 + p_\alpha^2 + p_B^2 + p^2 + m^2)} \]

\[= \frac{1}{4\pi^3} \int d\Omega e^{i\Omega \Delta T} \int_0^\infty dp \frac{p \sin(p\Delta r)}{\Delta r (\Omega^2 + p^2 + m^2)} \]

\[= \frac{1}{8\pi^2} \int d\Omega e^{i\Omega \Delta T} \exp\left( -\frac{\Delta r}{\sqrt{\Omega^2 + m^2}} \right) \]

\[= \frac{1}{8\pi^2} \int d\Omega e^{i\Omega \Delta T} \left[ \frac{1}{\Delta r - \sqrt{\Omega^2 + m^2} + O(\Delta r)} \right]. \] (71)
The first addend in the subintegral function must be removed because it gives a superficial divergence similar to that discussed above. Then the low-frequency contribution to $\langle \varphi^2 \rangle$ is

$$
\langle \varphi^2 \rangle_{\text{LFC}} = \lim_{\Delta \tau \to 0} \left\{ -\frac{1}{8\pi^2} \int_{-\Omega_0}^{\Omega_0} d\Omega e^{i\Omega \Delta \tau} \sqrt{\Omega^2 + m^2} \right\}
= -\frac{1}{8\pi^2} \left( \Omega_0 \sqrt{\Omega_0^2 + m^2} + m^2 \ln \left| \Omega_0 + \sqrt{\Omega_0^2 + m^2} \right| \right).
$$

(72)

In the case of a massless field or if $\Omega_0 \gg m$ this expression can be rewritten as

$$
\langle \varphi^2 \rangle_{\text{LFC}} = \frac{\Omega_0^2}{8\pi^2} - \frac{m^2}{16\pi^2} - \frac{m^2}{16\pi^2} \ln \left| \frac{4\Omega_0^2}{m^2} \right| + O \left( \frac{m^4}{\Omega_0^2} \right).
$$

(73)

If we take into account $\Omega_0 = \omega_0 / \sqrt{T}$ then

$$
\langle \varphi^2 \rangle_{\text{ren}} = \lim_{\hat{\tau} \to \tau} \left[ \langle \varphi^2 \rangle_{\text{unren}} - \langle \varphi^2 \rangle_{\text{DS}} \right] = \lim_{\hat{\tau} \to \tau} \left[ \langle \varphi^2 \rangle_{\text{unren}} - \langle \varphi^2 \rangle_{\text{LFC}} + \langle \varphi^2 \rangle_{\text{WKB}} + \langle \varphi^2 \rangle_{\text{LFC}} \right]
= \frac{R}{16\pi^2} \left( \xi - \frac{1}{6} \right) \ln \left| \frac{4u_0^2}{m^2_{\text{ren}}^2} \right| - \frac{f''}{96\pi^2 f^2} + \frac{f''(r^2)'}{96\pi^2 f r^2} + O \left( \frac{1}{L^2 u_0^2} \right).
$$

(74)

The corresponding expressions for $\langle T^\mu_{\nu} \rangle_{\text{LFC}}$ are

$$
\langle T^\mu_{\nu} \rangle_{\text{LFC}} = -\frac{\Omega_0^4}{48\pi^2} - \frac{m^2 \Omega_0^2}{16\pi^2} + \frac{m^4}{128\pi^2} \ln \left| \frac{4\Omega_0^2}{m^2} \right| + O \left( \frac{m^6}{\Omega_0^2} \right).
$$

(75)

$$
\langle T^\nu_{\nu} \rangle_{\text{LFC}} = \langle T^\varphi_{\varphi} \rangle_{\text{LFC}} = \frac{\Omega_0^4}{48\pi^2} + \frac{m^2 \Omega_0^2}{16\pi^2} + \frac{3m^4}{128\pi^2} \ln \left| \frac{4\Omega_0^2}{m^2} \right| + O \left( \frac{m^6}{\Omega_0^2} \right).
$$

(76)

The analytical approximation for $\langle T^\mu_{\nu} \rangle_{\text{ren}}$ in the case of asymptotically flat spacetimes is

$$
\langle T^\mu_{\nu} \rangle_{\text{ren}} = \lim_{\hat{\tau} \to \tau} \left[ \langle T^\mu_{\nu} \rangle_{\text{unren}} - \langle T^\mu_{\nu} \rangle_{\text{DS}} \right] = \lim_{\hat{\tau} \to \tau} \left[ \langle T^\mu_{\nu} \rangle_{\text{unren}} - \langle T^\mu_{\nu} \rangle_{\text{LFC}} + \langle T^\mu_{\nu} \rangle_{\text{WKB}} + \langle T^\mu_{\nu} \rangle_{\text{LFC}} \right]
= \langle T^\mu_{\nu} \rangle_{\text{WKB}} + \langle T^\mu_{\nu} \rangle_{\text{LFC}},
$$

(77)

$$
\langle T^\mu_{\nu} \rangle_{\text{ren}} = \frac{u_0^2}{4\pi^2 r^2} \left( \xi - \frac{1}{6} \right) \left( \frac{1}{2r^2} + \frac{5f''}{4f^2} - \frac{f''(r^2)'}{4fr'} + \frac{(r^2)''}{8r^4} - \frac{f''}{f} - \frac{(r^2)''}{2r^2} \right)
- \frac{m^4}{128\pi^2} + \frac{m^4}{64\pi^2} \ln \left| \frac{4u_0^2}{m^2_{\text{ren}}^2} \right| + \langle T^\mu_{\nu} \rangle_{(4)}+ O \left( \frac{1}{u_0^2 L^4} \right) + O \left( \frac{\lambda u_0}{rL^4} \right).
$$

(78)

$$
\langle T^\nu_{\nu} \rangle_{\text{ren}} = \frac{u_0^2}{4\pi^2 r^2} \left( \xi - \frac{1}{6} \right) \left( -\frac{1}{2r^2} - \frac{f''}{4f^2} - \frac{f''(r^2)'}{4fr'} + \frac{(r^2)''}{8r^4} \right)
+ \frac{3m^4}{128\pi^2} + \frac{m^4}{64\pi^2} \ln \left| \frac{4u_0^2}{m^2_{\text{ren}}^2} \right| + \langle T^\nu_{\nu} \rangle_{(4)}+ O \left( \frac{1}{u_0^2 L^4} \right) + O \left( \frac{\lambda u_0}{rL^4} \right)
$$

(79)

$$
\langle T^\varphi_{\varphi} \rangle_{\text{ren}} = \langle T^\varphi_{\varphi} \rangle_{\text{ren}} = \frac{u_0^2}{4\pi^2 r^2} \left( \xi - \frac{1}{6} \right) \left( \frac{(r^2)''}{4r^2} - \frac{f''}{4f} + \frac{5f''}{8f^2} \frac{(r^2)''}{8r^4} - \frac{f''(r^2)'}{8fr'} \right)
+ \frac{3m^4}{128\pi^2} + \frac{m^4}{64\pi^2} \ln \left| \frac{4u_0^2}{m^2_{\text{ren}}^2} \right| + \langle T^\varphi_{\varphi} \rangle_{(4)}+ O \left( \frac{1}{u_0^2 L^4} \right) + O \left( \frac{\lambda u_0}{rL^4} \right).
$$

(80)

This tensor is conserved,

$$
\langle T^\mu_{\nu} \rangle_{\text{ren; } \mu} = O \left( \frac{1}{u_0^2 L^3} \right) + O \left( \frac{\lambda u_0}{rL^5} \right).
$$

(81)
and has the trace

\[ \langle T^\mu_{\mu} \rangle_{\text{ren}} = \frac{u_0^2}{4\pi r^2} \left( \xi - \frac{1}{6} \right) \left( \frac{9f^2}{4f^2} - \frac{3f'(r^2)' - 3f''}{2f^2} \right) - \frac{m^4}{16\pi^2} + \frac{m^4}{16\pi^2} \ln\left| \frac{4u_0^2}{m^2r^2} \right| \]

\[ + \left( T^\mu_{\mu} \right)_{(4)} + O\left( \frac{1}{u_0^2L^4} \right) + O\left( \frac{\lambda u_0}{rL^4} \right). \]

(82)

For a conformally invariant scalar field this trace is also equal to the trace anomaly.

VI. CONCLUSION

In this paper, analytical approximations for \( \langle \varphi^2 \rangle_{\text{ren}} \) and \( \langle T^\mu_{\mu} \rangle_{\text{ren}} \) of quantized scalar fields in static spherically symmetric asymptotically flat spacetimes have been obtained. The field is assumed to be in a zero temperature vacuum state, with mass \( m \lesssim 1/L[\varphi] \) is the characteristic scale of variation of the gravitational field at the considered point] and with an arbitrary coupling \( \xi \) to the scalar curvature.

The necessary conditions for the validity of the analytical approximations (74),(78)-(80)

\[ \lambda \ll \frac{\sqrt{f(\rho)}}{u_0} \ll L(\rho), \]

(83)

where \( \lambda \) is the characteristic scale of the gravitational field inhomogeneity in asymptotically flat spacetime and \( u_0 \) is the constant of the WKB expansion.

If we consider a spherical body with radius \( r_0 > r_g \) \( (r_g \) is the gravitational radius of this body, \( \lambda \sim r_0 \) then the metric spacetime outside this body is

\[ ds^2 = -\left(1 - \frac{r_g}{r(\rho)}\right) dt^2 + d\rho^2 + r(\rho)^2(d\theta^2 + \sin^2\theta d\varphi^2), \]

(84)

where \( r(\rho) \) is the inverse function to the function

\[ \rho(r) = \rho_0 + r_g \left[ \sqrt{\frac{r}{r_g} \left( \frac{r}{r_g} - 1 \right)} - \sqrt{\frac{r_0}{r_g} \left( \frac{r_0}{r_g} - 1 \right)} \right]

\[ + \frac{1}{2} \ln \left( \frac{\sqrt{r/r_g - 1} + \sqrt{r_g/r}}{\sqrt{r/r_g - 1} - \sqrt{r_g/r}} \right) \left( \frac{\sqrt{r_0/r_g - 1} - \sqrt{r_0/r_g}}{\sqrt{r_0/r_g - 1} + \sqrt{r_0/r_g}} \right). \]

(85)

When \( r(\rho) \gg r_0 \)

\[ f(\rho) \sim 1, \quad L(\rho) \sim r(\rho), \]

(86)

and the conditions \( \text{(83)} \) can be satisfied by the choice of \( u_0 \),

\[ r_0 \ll u_0 \ll r(\rho). \]

(87)

This means that, far from the body where \( r(\rho) \gg r_0 \), the approximations \( \text{(74),(78)-(80)} \) are valid.

The presence in expressions \( \text{(74),(78)-(80)} \) the arbitrary parameter \( u_0 = u_0 r/\sqrt{f} \) is a generic feature of local approximation schemes \( \text{(2),(12),(15),(18),(19)} \). For a conformally invariant field this parameter can be absorbed into the definition of the constant \( m_{\text{QED}} \).

When the massless conformally coupled scalar field is in a zero temperature vacuum state and propagating in static spherically symmetric asymptotically flat spacetimes the expressions \( \text{(74),(78)-(80)} \) are equivalent to the approximations of Page, Brown, and Ottewill \( \text{(12),(18),(19)} \), Zamias \( \text{(15)} \), and Frolov and Zel’nikov \( \text{(11)} \) (for the particular choice of the arbitrary parameters in their expressions), and the analytical approximation of Anderson, Hiscock, and Samuel \( \text{(4)} \). Let us note that in this case the low-frequency contributions to \( \langle \varphi^2 \rangle_{\text{ren}} \) and \( \langle T^\mu_{\mu} \rangle_{\text{ren}} \) are equivalent to the low-frequency contributions which are given by the procedure of \( \text{(4)} \) for modes with \( w < u_0 \). This means that using such calculations in \( \text{(20),(21),(22)} \) is correct in the case of a conformally invariant field. Let us note also that the more exact procedure \( \text{(23)} \) gives different result.
Acknowledgments

I would like to thank S. V. Sushkov and N. R. Khusnutdinov for helpful discussions. This work was supported by the Russian Foundation for Basic Research Grant No. 02-02-17177 and by the SRPED Foundation of Tatarstan Republic Grant No. 06-6.5-110.

Appendix A

To calculate the quantities $S^k_n(\varepsilon, u_0)$ it is necessary to compute the various sums over $l$. We start from the sum in expression (39):

$$S(u, n) = \sum_{l=0}^{\infty} \left( \frac{(l + \frac{1}{2})^{2n+1}}{(u^2 + (l + 1/2)^2)^{(2n-1)/2}} - \left( l + \frac{1}{2} \right)^2 + (2n-1)\frac{u^2}{2} \right), \quad n \geq 0. \quad (A1)$$

For calculation of this sum we can use the Abel-Plana method [19, 24, 25].

$$\sum_{l=0}^{\infty} F(l + 1/2) = \int_{q}^{\infty} F(x)dx + \int_{q-i\infty}^{q+i\infty} \frac{F(z)}{1 + e^{2\pi x}}dz - \int_{q}^{q+i\infty} \frac{F(z)}{1 + e^{-i2\pi x}}dz, \quad (A2)$$

where $-1/2 < q < 1/2$, $f(z)$ is a holomorphic function for $Rez \geq q$, $f(z)$ satisfies the condition

$$|F(x + iy)| < \epsilon(x)e^{a|y|}, \quad 0 < a < 2\pi,$$

and $\epsilon(x)$ is an arbitrary function with asymptotic behavior

$$\epsilon(x) \to 0 \quad \text{for} \quad x \to +\infty. \quad (A3)$$

Using this formula we can calculate the sum (A1):

$$S(u, n) = \lim_{q \to +0} \left\{ \int_{q}^{\infty} \left[ \frac{x^{2n+1}}{(u^2 + x^2)^{(2n-1)/2}} - x^2 + (2n-1)\frac{u^2}{2} \right] dx \right. \quad (A4)$$

$$+ \int_{q-i\infty}^{q+i\infty} \left[ \frac{z^{2n+1}}{(u^2 + z^2)^{(2n-1)/2}} - z^2 + (2n-1)\frac{u^2}{2} \right] \frac{dz}{(1 + e^{2\pi x})} - \int_{q}^{q+i\infty} \left[ \frac{z^{2n+1}}{(u^2 + z^2)^{(2n-1)/2}} - z^2 + (2n-1)\frac{u^2}{2} \right] \frac{dz}{(1 + e^{-2\pi x})} \left\} \quad (n \geq 0). \quad (A5)$$

The sum in expression (40) can be calculated by the differentiation of $S(u, n)$:

$$\sum_{l=0}^{\infty} \left\{ \frac{(l + \frac{1}{2})^{2n+1}}{(u^2 + (l + 1/2)^2)^{(2n+1)/2}} - 1 \right\} = \left( \frac{d}{udu} \right) S(u, n) \quad (n \geq 0). \quad (A6)$$

The other sums in the expression for $S^k_n(\varepsilon, u_0)$ [Eq. (38)] can also be calculated by the differentiation of $S(u, n)$:

$$\sum_{l=0}^{\infty} \frac{(l + \frac{1}{2})^{2n+1}}{(u^2 + (l + 1/2)^2)^{(2n+2k+3)/2}} = \left( \frac{d}{udu} \right)^{k+2} S(u, n)$$
\begin{align}
\int_0^1 y\sqrt{1-y^2}dy &= \int_0^\infty \frac{ydy}{1 + e^{2\pi uy}} \left[ 1 - \frac{y^2}{2} - \frac{1!!}{4!!}y^4 - \frac{3!!}{6!!}y^6 + O(y^8) \right] \\
&= \frac{1}{48u^2} - \frac{7}{3840u^4} - \frac{31}{129024u^6} + O\left(\frac{1}{u^{10}}\right). \quad (A8)
\end{align}

Substituting this expression into Eqs. (A3) - (A7) and integrating in Eq. (A8) we obtain the resulting expressions (11) - (12).

**Appendix B**

\begin{align}
\left(T_i^I\right)_{(4)} &= \frac{1}{46080\pi^2 s f^4} \left\{ 360^8 s f^4 m^4 + \left[ 360^8 s f^2 f'^2 - 480 r^8 f^3 f'' - 480 r^8 f^3 f''' \right] m^2 \\
&+ 32^4 f^4 - 32^2 (r')^6 f^4 - 12 f'^2 r^2 s f^2 + 56(r')^2 r^4 f^4 + 16 f''' r^8 f^3 + 7 f^4 r^8 \\
&- 32 f'' (r')^6 r^6 f^3 - 16 f'^2 (r')^2 r^6 f^2 - 16 f'' (r')^2 r^4 f^2 - 16 f''' (r')^2 r^4 f^2 \\
&+ 4 f'^2 (r')^2 r^6 f^2 - 12 f'^2 (r')^2 r^6 f^2 - 14 f'^2 (r')^2 r^6 f^2 - 48 f'' (r')^2 r^6 f^3 + 48 f'' (r')^2 r^6 f^4 \\
&- 112 (r')^2 (r')^2 r^2 f^4 + 32 r'' (r')^2 (r')^2 r^4 f^3 + 40 (r')^4 f^4 + 32 (r')^4 f''' r^2 f^3 \\
&+ \left[ 5760^4 f^4 - 11520 (r')^4 r^4 f^4 + 2880 (r')^2 r^4 f^4 + 8640 f'^2 r^2 f^2 - 11520 f'' r^2 f^3 \\
&+ 5760 (r')^2 r^4 f^4 + 4320 f'^2 r^2 f^2 + 7560 f'^2 r^2 f^2 + 360 (r')^4 f^4 + 11520 f'' (r')^2 r^6 f^3 \\
&+ 2880 f'' (r')^2 r^4 f^2 - 2880 f'' (r')^2 r^4 f^2 - 5760 f''' r^8 f^2 - 2880 f'' (r')^2 r^4 f^3 \\
&+ 17280 f'^2 (r')^2 r^6 f - 11520 f'' (r')^2 r^6 f + 720 f'^2 (r')^2 r^6 f^2 + 11520 f'' (r')^2 r^6 f^3 \\
&+ 2880 (r')^2 (r')^2 r^2 f^4 + 14400 f'^2 (r')^2 r^6 f^2 - 5760 f'' (r')^2 r^6 f^3 \right] (\xi - 1/6)^2 \\
&+ \left[ 2880^6 f^4 - 5760^6 f^3 f' - 5760^6 f^3 f'' + 4320 r^2 f^2 f^2 + 720 r^4 f^3 (r')^2 \\
&- 2880 r^6 f^4 (r')^2 r^4 f^2 - 960 (r')^2 r^4 f^2 + 720 f'^2 r^2 f^2 - 960 (r')^2 r^2 f^4 \\
&- 960 (r')^4 f^4 + 1920 (r')^2 f'' r^6 f^3 + 960 (r')^2 f'' r^6 f^3 - 2160 f'^2 r^2 f^2 + 1440 (r')^2 r^4 f^4 \\
&+ 960 f''' r^8 f^3 + 2880 f'' (r')^2 r^4 f^2 + 1200 f' (r')^2 r^6 f^3 - 2160 f'^2 (r')^2 r^6 f^4 \\
&- 2880 f'^3 r^3 f^2 + 2880 f'^2 (r')^2 r^6 f^3 + 7290 f'^2 (r')^2 r^6 f + 2640 f'^2 (r')^2 r^6 f \\
&+ 900 f' (r')^2 r^4 f^2 + 2880 f'^2 (r')^2 r^4 f^2 + 1440 f'^2 (r')^2 r^4 f^2 \\
&- 3780 f'^3 r^3 f^2 - 1440 (r')^2 (r')^2 r^4 f^4 - 2400 (r')^2 (r')^2 r^4 f^3 - 4320 f'' f'' (r')^2 r^6 f^2 \right] (\xi - 1/6) \\
&+ \left[ -720 s^4 f^4 - 16 r^4 f^4 - 20 (r')^4 f^4 + 56 (r')^2 r^2 f^4 + 8 f'' (r')^2 r^4 f^4 \\
&- 28 (r')^2 r^4 f^4 - 16 f''' r^8 f^3 + 52 f' f' (r')^2 r^6 f^2 + 16 (r')^2 r^6 f^4 + 36 f'^2 r^8 f^2 \\
&+ 49 f'^3 r^8 f^3 - 4 f' (r')^2 r^2 f^4 + 4 f' f' (r')^2 r^6 f^2 + 4 f'' (r')^2 r^4 f^3 \\
&- 116 f'^2 f'' r^8 f - 22 f'^3 (r')^2 r^4 f - 3 f'^2 (r')^2 r^4 f^2 + 8 f' (r')^2 r^6 f^3 - 32 (r')^2 f''' r^6 f^3 \\
&+ 32 (r')^2 f''' r^6 f^3 \right] .
\end{align}
\[
\begin{align*}
-24(r^2)'(r^2)''r^4f^4 - 8f''(r^2)''r^6f^3 + \left( -2880r^4 f^4 + 4320(r^2)'r^2 f^4 + 5580(r^2)''r^4 f^4 \\
+ 5760f''(r^2)''r^4 f^3 - 15840(r^2)'r^2 f^4 + 6480f'''r^6 f^2 + 5760(r^2)''r^4 f^4 \\
- 2880f''r^6 f^3 - 5760(r^2)'''r^6 f^3 - 2880f'(r^2)'r^3 f^3 + 5760f''r^6 f^2 \\
+ 15840f'f''(r^2)'r^6 f^2 + 8640f'f'''r^8 f^2 + 2880f''(r^2)'r^3 f^3 - 20880f''f''r^8 f \\
- 9360f''(r^2)'r^6 f - 2160f''(r^2)'r^4 f^2 - 2880f''(r^2)'r^6 f^3 - 5760f''r^6 f^3 \\
+ 8640(3r^3)'r''r^4 f^4 - 5760f''(r^2)'r^6 f^3 + 8820f'f'f''r^8 f) (\eta/1-6)^2 \\
+ (28806f^4(r^2)'' - 720r^4 f^2(r^2)'^2 - 28806f^4) (\eta/1-6) m^2 \ln \left| \frac{4n_0^2}{m_{\text{tot},r}^2} \right|, \\
\end{align*}
\] 
(B1)

\[
\left( T^0_\phi \right)_{(4)} = \frac{1}{46080\pi^2 r^8 f^4} \left\{ -1080m^2 r^8 f^4 - 120 f''^2 r^2 m^2 r^8 f^2 + f'f^2 r^4 + 32f''(r^2)'r^4 f^3 \\
- 4f'f''^2 r^2 f - 24f''(r^2)'r^4 f^3 - 16f'(r^2)''r^6 f^3 - 16f'(r^2)'r^3 f^3 - 8f'(r^2)''r^4 f^3 \\
- 24f''(r^2)'r^2 f + 12f''^2 r^2 f^2 + 8f'f'''r^6 f^2 - 4f'f''r^2 f^2 + 16f'(r^2)''r^6 f^3 \\
- 8f'(r^2)'r^6 f + 32f'f'(r^2)'r^6 f^2 + \left[ 5760f''(r^2)'r^2 f^3 - 1440f'(r^2)'r^3 f^3 \\
- 5760f'(r^2)'r^4 f^3 - 2880f''(r^2)'r^6 f^3 + 2880f''(r^2)'r^6 f^2 + 2160f''(r^2)'r^4 f^2 \\
- 720f''r^4 + 1440f''r^4 f^2 + 2880f''(r^2)'r^6 f^2 \right] (\xi/1-6)^2 \\
+ \left[ -240f'f''r^6 f^2 + \left( 720r^2 r^2 r^4 f^4 - 1440f'(r^2)'r^6 f^3 - 1440f''r^2 r^8 f^2 \\
- 28806f^4) m^2 - 1260f''(r^2)'r^4 f^2 + 540f'f^2 r^4 + 480(r^2)'f'''r^6 f^3 - 720f''f''r^8 f \\
+ 480f''f'''r^6 f^3 - 720f''(r^2)'r^2 f^3 - 1200f''r''(r^2)'r^4 f^2 \\
+ 1200f''(r^2)'r^2 f^2 + 480f'f''r^2 f^2 + 720f''f''r^2 f^2 + 720f''r^2 f^2 r^4 f^2 + 720f''r^2 f^2 r^2 f^4 \\
+ 720f''r^4 f^3 - 240f''r^2 f^2 + \left[ 720m^2 r^4 f^4 + 12f''(r^2)''r^4 f^3 \\
- 16^r f^4 - 4r^2 f^4 + 8f'f''r^6 f^2 + 12f''r''r^2 f^2 + 8(r^2)'f'''r^6 f^3 - 8r^2 f''r^6 r^8 f^2 \\
+ 8f'(r^2)''r^6 f^3 - 24f'f''(r^2)'r^6 f^2 + 12f''(r^2)'r^2 f^2 + 4f'(r^2)'r^2 f^2 r^2 f^3 \\
- 16f'(r^2)''r^2 f^3 + 8r^2 f''(r^2)''r^4 f^4 - 8r^2 f'(r^2)''r^4 f^4 + 10f''(r^2)'r^6 f \\
- 7f''r^4 f^3 - 8f''r''(r^2)''r^6 f^3 + 4f''r^2 r^4 f^2 + 4(r^2)'f''r^2 f^2 + (-2880r^4 f^4 \\
- 4320r^2 r^2 f^4 - 2880f''(r^2)'r^6 f^2 - 2340(r^2)'f^4 f^4 - 1440f'f'''r^6 f^2 \\
+ 2880f''(r^2)'r^2 f^2 + 3420f''f''(r^2)'r^4 f^2 + 1440f''(r^2)'r^2 f^2 + 1440f''(r^2)'r^2 f^2 + 4320f''(r^2)'r^2 f^2 r^4 f^2 \\
+ 720f''(r^2)'r^6 f^3 + 2880f''(r^2)''r^6 f^3 + 5760(r^2)'r^2 f^4 - 5760(r^2)'r^2 f^4 r^4 f^4 \\
+ 1440f''(r^2)'r^4 f^3 - 1260f''r^4 f^3 - 2880f''r^4 f^3 + 2160f''r^4 f^3 f - 3420f''r^2 f^4 \\
+ 720f''r^2 f^4 + 2880f''r^2 f^3 r^2 f^2 + 720(r^2)'r^2 f^4 \right) (\xi/1-6)^2 + (-28806f^4 \\
+ 720(r^2)'r^4 f^4 + 1440f'(r^2)'r^6 f^3) (\xi/1-6) m^2 \ln \left| \frac{4n_0^2}{m_{\text{tot},r}^2} \right|, \right. \\
\end{align*}
\] 
(B2)
\[ \begin{align*}
&+ \left[ -5760 r^6 f^2 f'' - 1440 r^4 f^3 f''(r^2) + 2880 r^2 f^4 f'' - 1440 r^2 f^3 f'(r^2)''(r^2) \\
&+ 11520 r^6 f^3 f'(r^2)''' + 2880 r^4 f^2 f''(r^2) - 7920 r^4 f^3 f''(r^2)'' - 7920 r^4 f^3 f'(r^2)'' \\
&- 3600 r^4 f^2 f'(r^2)'' + 8640 r^4 f^3 f'(r^2)' + 2880 r^2 f^2 f'(r^2)'' \\
&+ 5040 r^6 f^3 f'(r^2)' + 6480 r^4 f^4 + 5760 r^2 f^2 f''(r^2) \\
&+ 10080 r^6 f^2 f''(r^2)'' \right] (\xi - 1/6)^2 \\
&+ \left[ -240 r^6 f^2 f'' + 480 r^4 f^3 f' + \left( -720 r^4 f^4(r^2)' - 1440 r^4 f^3 + 720 r^8 f^2 \right) \\
&+ 1140 r^6 f^4(r^2)' - 720 r^4 f^3 f'(r^2)' \\
&- 1920 r^8 f^2 f''' + 5400 r^6 f^4 f'' - 2580 r^6 f^4 f'' + 720 r^6 f^3 f''(r^2)' + 480 r^8 f^3 f''' \\
&- 2280 r^6 f^2 f'''(r^2)' + 720 r^6 f^3(r^2)' - 720 r^4 f^3 f''(r^2) + 1440 r^6 f^3 f''(r^2)' \\
&+ 2220 r^6 f^4 f''(r^2)' + 600 r^4 f^2 f'^2(2r^2)' - 960 r^6 f^2 f'(r^2)'' \right] (\xi - 1/6) \\
&\left[ -720 r^8 f^4 f^4 + 16 r^4 f^4 - 20 r^4 f^2 f'' + 12 r^6 f^2 f''(r^2)' \\
&+ 12 r^6 f^3 f''(r^2)' + 8 r^6 f^3 f''(r^2)' + 6 r^6 f^3 f''(r^2)' - 2 r^6 f^2 f'^2(2r^2)' - 8 r^6 f^3 f'(r^2)'' \\
&+ 12 r^6 f^4(r^2)' - 20 r^8 f^2 f'' + 4 r^4 f^2 f'(r^2)' - 8 r^4 f^3 f'(r^2)' + 3 r^4 f^2 f'(r^2)' \\
&+ 16 r^4 f^4(r^2)' + 52 r^8 f^2 f'' + 8 r^8 f^3 f''' - 8 r^6 f^4(2r^2)' - 32 r^6 f^2(2r^2)' \\
&+ 9360 r^6 f^2 f''(r^2)' + 72 r^8 f^2 f''(r^2)' - 3240 r^6 f^2 f''(r^2)' + 5040 r^6 f^2 f''(r^2)' \\
&+ 720 r^4 f^4(r^2)' - 5400 r^6 f^3 f'(r^2)' - 20880 r^2 f^4(2r^2)' + 11520 r^4 f^3 f''(r^2)' \\
&- 16560 r^8 f^2 f'' + 5400 r^6 f^3 f''(r^2)' + 8640 r^4 f^4(2r^2)' + 11520 r^6 f^3 f''(r^2)' \\
&- 5760 r^6 f^3 f''(r^2)' - 720 r^6 f^3 f''(r^2)' - 4320 r^4 f^3 f''' - 2880 r^8 f^3 f'' \\
&+ 5040 r^6 f^2 f'' + 5040 r^6 f^3 f''(r^2)' - 5760 r^6 f^4(2r^2)' + 720 r^8 f^4 \right] (\xi - 1/6)^2 \\
&+ \left( -720 r^4 f^3(r^2)' + 720 r^6 f^3 f'(r^2) + 1440 r^6 f^4(r^2)' + 1440 r^6 f^3 f'' \right)
\right] \left( \frac{4 v_0^2}{m_0^2 r^2} \right).
\end{align*}\]
[16] J. Matyjasek, Phys. Rev. D **61**, 124019 (2000); J. Matyjasek, *ibid* **63**, 084004 (2001), H. Koyama, Y. Nambu, and A. Tomimatsu, Mod. Phys. Lett. A **15**, 815 (2000).

[17] S. M. Christensen, Phys. Rev. D **14**, 2490 (1976).

[18] A. A. Popov and S. V. Sushkov, Phys. Rev. D **63**, 044017 (2001).

[19] A. A. Popov, Phys. Rev. D **64**, 104005 (2001).

[20] R. Balbinot, A. Fabbri, P. Nicolini, V. Frolov, P. Sutton, and A. Zelnikov, Phys. Rev. D **63**, 084029 (2001).

[21] V. Frolov, P. Sutton and A. Zelnikov, Phys. Rev. D **66**, 024014 (2002).

[22] R. Balbinot, A. Fabbri, P. Nicolini and P. Sutton Phys. Rev. D **66**, 024014 (2002).

[23] S. V. Sushkov, Gravitation Cosmol. **6**, 45 (2000).

[24] M. A. Evgrafov, *Analytic Functions* (Nauka, Moskow, 1968) (in Russian).

[25] S. V. Sushkov, Phys. Rev. D **62**, 064007 (2000).