More on axial anomalies of Lifshitz fermions

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Abstract

We show that the gauge and metric field contribution to the axial anomaly of a four-dimensional massless Lifshitz fermion theory with anisotropy scaling exponent $z$ is identical to the relativistic case, hereby extending the results found in arXiv:1103.5693 to arbitrary values of $z$. This is in accordance with the fact that the axial anomaly is an infra-red phenomenon in disguise. We also provide some new models that realize baryon and lepton number violation in non-relativistic theories of gravity. In all cases, the volume of space exhibits a lower bound that is fixed by the gravitational coupling parameters.
1 Introduction

It has been known for a long time that a massless Dirac fermion in four space-time dimensions exhibits a quantum anomaly in the axial current conservation law when coupled to external gauge and/or metric fields. The anomalous term in the divergence of the axial current \( J_5^\mu = \bar{\Psi} \gamma^\mu \gamma_5 \Psi \) is proportional to the characteristic classes \( \text{Tr}(F \wedge F) \) and \( \text{Tr}(R \wedge R) \) written in terms of the curvature two-forms \( F \) and \( R^a_{\ b} \) of the gauge and metric fields, respectively, with corresponding coefficients \( 1/4 \pi^2 \) and \( 1/96 \pi^2 \) (in appropriate units that will be used throughout this paper) [1, 2]. The coefficient of anomaly is half of that when computed for a Weyl fermion. This result describes the local form of the axial anomaly, whereas its integral over space-time has been used to study possible violations of chiral symmetry via the Atiyah-Singer index theorem (for an overview of the subject from a physicist’s perspective see, for instance, [3] and references therein). The axial charge \( Q_5 \) is not conserved when the four-dimensional Dirac operator in a given background exhibits an asymmetry in the number of positive and negative chirality zero modes, leading to violations of lepton and baryon number in the theory [4]. Generalizations to higher dimensions have also been worked out systematically [5] and the results found many important applications in physics.

In this paper we examine the occurrence of axial anomalies in the non-relativistic fermion field theories of Lifshitz type. For this we consider higher order generalizations of the Dirac operator of the form \( Q = i \gamma^0 \partial_0 + i \gamma^i \partial_i (\partial^k - M^2)^\alpha \) acting on spinors \( \Psi \), where \( M \) is an arbitrary mass scale and \( \alpha \) is a free parameter. The resulting theory of Lifshitz fermions, which is described by the Lagrangian density \( \bar{\Psi} Q \Psi \) exhibits anisotropic scaling in space and time by assigning dimensions \([L] = -1, [T] = -z \) and \([\Psi] = z/2 \) with parameter \( z = 2\alpha + 1 \). This model has an axial current which is conserved at the classical level. Its time component \( J_5^0 \) is the same as in the relativistic case (the same also applies to the axial charge \( Q_5 \)), but the spatial components \( J_5^i \) are more complicated as they involve derivative terms that depend upon \( \alpha \); their precise form will not be needed for the purposes of the present work. Then, a natural question is whether there are anomalies in the axial current conservation law that arise upon quantization of the fermions in the background of gauge and/or metric fields, and, if so, what will be their form compared to the relativistic case, \( \alpha = 0 \). The result turns out to be independent of the parameters \( M \) and \( \alpha \), and, in this sense, the structure of the axial anomalies appears to be universal.

The present work generalizes earlier results on the subject that were obtained for Lifshitz fermions with anisotropy scaling parameter \( z = 3 \) coupled to background gauge fields [6] and gravity [7] in four space-time dimensions. Here, \( z = 2\alpha + 1 \) is left arbitrary and it can even assume fractional values. For practical reasons we restrict attention to four space-time dimensions, although we expect the same universality to govern the form of the axial anomalies in higher dimensional Lifshitz fermion models as in relativistic theories [5]. An intuitive explanation for all this is provided by the fact that the axial anomaly is an infra-red phenomenon in disguise, and, hence, any
higher order corrections to the Dirac operator that become relevant in the ultra-violet regime ought to leave its form unaltered (the way it works in relativistic theories is nicely explained in [5], but see also [8, 9, 10] for earlier work on the subject). The introduction of the mass scale $M$ into the definition of $Q$ serves precisely the purpose of showing by explicit computation that the axial anomaly is indeed independent of it and the order of the higher derivative corrections to the fermion operator. The result should be contrasted to the form of the Weyl anomaly in Lifshitz theories that exhibit anisotropic local scale invariance. In that case one does not expect the result to be the same as for relativistic fields because the Weyl anomaly is an ultra-violet phenomenon that is sensitive to higher order corrections at short distances. Some partial results in this direction can be found in [11] for scalar field models, whereas generalizations to other field theories of Lifshitz type have not been addressed so far in the literature.

Thus, the computation of axial anomalies in non-relativistic massless fermion models is a much simpler problem compared to other type of anomalies and at the same time it has many interesting applications related to chiral symmetry breaking at a Lifshitz point. A closely related problem is the parity anomaly of three-dimensional fermion theories coupled to external gauge and/or metric fields. The methods we use in this paper suffice to demonstrate the universality of the parity violating piece of the effective action induced by massless fermions regardless the order of the three-dimensional fermion operator. This result is a manifestation of the close relation that exists between different type of anomalies in odd and even number of space-time dimensions generalizing the framework that was established long time ago for relativistic theories (see, for instance, [12] and references therein). We will say more about this later focusing, in particular, to the gravitational case and the universal character of the induced three-dimensional Chern-Simons action.

In section 2, we employ the path integral method to extract the local form of the axial anomaly of Lifshitz fermion models. The computation is performed rather easily for gauge field backgrounds and the result is identical to the relativistic case, as expected on general grounds. Confirming the universality of the axial anomaly for metric field backgrounds is much more involved computationally by path integral methods. For this reason we employ an indirect method based on the integrated form of the anomaly and the associated index theorem for the Dirac-Lifshitz operator. Then, using this framework, the coefficient of the axial anomaly follows from the computation of the $\eta$-invariant of the associated three-dimensional Lifshitz operator on certain geometries. In section 3, we briefly review the notion of $\eta$-invariant and its relation to anomalies in relativistic fermion theories in three and four dimensions. In section 4, we compute the $\eta$-invariant of higher order fermion operators on homogeneous geometries and show that the result is independent of the parameters $M$ and $\alpha$. This suffices to prove our claim about the gravitational contribution to the axial anomaly of Lifshitz fermion. In section 5, we present some applications to gravitational theories of Lifshitz type and examine the possibility of chiral symmetry breaking by instantons, thus generalizing the results found in [7]. Finally, section 6 contains our conclusions.


2 Path integral derivation of axial anomaly

Coupling the Lifshitz fermion theory to external fields amounts to replacing $\partial_\mu$ by $D_\mu$, as usual. Then, the interacting Dirac-Lifshitz operator takes the form

$$Q = i\gamma^\mu D_\mu = i\gamma^0 D_0 + \frac{1}{2}i\gamma^i[D_i(-D_k D^k + M^2)^{\alpha} + (-D_k D^k + M^2)^{\alpha} D_i]$$

up to a factor ordering ambiguity that turns out to be irrelevant for the computation of the anomaly. To avoid unnecessary complications or ambiguities in the definition of the operator (2.1), and in view of the applications that will be discussed later, coupling to geometry is taken with respect to metrics $-(dx^0)^2 + g_{ij}(x^0, x)dx^i dx^j$. Such backgrounds arise naturally in the canonical decomposition of the metric field in space-times of the form $I \times \Sigma_3$ choosing the lapse and shift functions as $N = 1$ and $N_i = 0$.

The anomalous divergence of the axial current can be found by applying Noether’s procedure to the fermionic path integral of the interacting theory, which is most conveniently described in the Euclidean domain after Wick rotation of time $x^0 \rightarrow i t$. The measure $(D \bar{\Psi})(D \Psi)$ is not invariant under chiral rotations $\delta \Psi = i \epsilon \gamma_5 \Psi$, giving rise to a non-trivial contribution to the divergence of the axial current (see, for instance, [13] and references therein to the original papers). Using the eigen-states $\phi_n(t, x)$ of the interacting fermion operator $i\gamma^\mu D_\mu$, the result takes the intermediate form

$$\nabla_\mu J_5^\mu(t, x) = 2 \lim_{\Lambda \rightarrow \infty} \sum_n \phi^\dagger_n(t, x) \gamma_5 e^{-\frac{(i\gamma^\mu D_\mu)^2}{\Lambda^2 z}} \phi_n(t, x)$$

after introducing a cut-off $\Lambda$ to regulate the infinite sum that otherwise is ill-defined. Note that $\Lambda$ is raised to the power $2z$ to match the scaling properties of the square of the Dirac-Lifshitz operator for general values of $z$.

The gauge field contribution to the axial anomaly can be computed relatively easy for all values of the anisotropy scaling parameter $z$. First, using the algebra of the Dirac gamma-matrices, it is convenient to rewrite the square of the Dirac-Lifshitz operator appearing in the regulator as

$$(i\gamma^\mu D_\mu)^2 = -D_\mu D^\mu - \frac{1}{4} [\gamma^\mu, \gamma^\nu] [D_\mu, D_\nu] ,$$

where $D_\mu = \partial_\mu - i A_\mu$ provides the minimal coupling of the theory to an external gauge field $A_\mu$, which can be Abelian or non-Abelian. Next, using the plane wave basis of solutions of the free Dirac-Lifshitz operator, $\exp(ik_\mu x^\mu)$, we can extract the explicit gauge field dependence of the anomalous term that appears on the right-hand side of equation (2.2). Since the action of the interacting Dirac-Lifshitz operator on plane waves amounts to replacing $D_\mu$ by $D_\mu + ik_\mu$ everywhere, it follows that the
anomalous divergence of the axial current in a gauge field background takes the form,

\[
\nabla_\mu J^\mu_5 = 2 \lim_{\Lambda \to \infty} \Lambda^{z+3} \text{Tr} \int \frac{d^4k}{(2\pi)^4} e^{-k^2} k^{3z} \gamma_5 \exp \left\{ -\frac{i k^{2a}}{2\Lambda^{z+1}} \left[ \gamma^0, \gamma^i \right] \left( F_{0i} + 2a F_{0j} \hat{k}_j \hat{k}^j \right) \right\} 
\]

\[
- \frac{i k^{4a}}{4\Lambda^2} \left[ \gamma^j, \gamma^k \right] \left( F_{jk} + 4a F_{jl} \hat{k}_j \hat{k}^l \right) + \cdots \right\} .
\]

(2.4)

In writing this equation we have also rescaled \( k_0 \) to \( \Lambda^z k_0 \) and \( k_i \) to \( \Lambda k_i \) conforming with the anisotropic scaling of the components of the momentum vector \( k_\mu = (k_0, k_i) \) that is implied by the scaling of the space-time coordinates \( x_\mu = (x_0, x_i) \) and introduced the unit three-momentum vector with components \( \hat{k}_i \), setting \( k_i = k \hat{k}_i \), where \( k^2 = k_i k^i \). Here, we have depicted the most relevant terms that originate from the exponential of \( (i \gamma^\mu D_\mu)^2 \) after replacing \( D_\mu \) by \( D_\mu + i k_\mu \). The factor \( \exp(-k_0^2 - k^2z) \) originates from \( D_\mu D^\mu \) in the decomposition (2.3), which is taken in the Euclidean domain, whereas the two terms shown explicitly in curly brackets originate from the electric and magnetic components \( \left[ \gamma^0, \gamma^i \right] [D_0, D_i] \) and \( \left[ \gamma^j, \gamma^k \right] [D_j, D_k] \), respectively, appearing on the right-hand side of (2.3). The terms that are omitted carry explicit dependence on the mass scale \( M \), but they do not contribute to the final result. Such terms yield contributions that either vanish as \( \Lambda \to \infty \) or their trace is zero because they do not contain sufficient number of gamma-matrices multiplied with \( \gamma_5 \).

The calculation proceed by expanding the exponential in curly brackets and noting that the only relevant terms arise to second order. We also perform the Gaussian integration over \( k_0 \), picking up a factor of \( \sqrt{\pi} \), and use the identity \( \text{Tr}(\gamma_5 [\gamma^0, \gamma^i] [\gamma^j, \gamma^k]) = -16 \epsilon^{0ijk} \). Then, after taking the limit \( \Lambda \to \infty \), we obtain

\[
\nabla_\mu J^\mu_5 = \frac{1}{4\pi^{7/2}} \epsilon^{0ijk} \int d^3k e^{-k^2} k^{3z-3} \text{Tr} \left( F_{0i} F_{jk} + 2a F_{0j} F_{ik} \hat{k}_j \hat{k}^j + 4a F_{0l} F_{jl} \hat{k}_j \hat{k}^j \right) ,
\]

(2.5)

setting \( z = 2a + 1 \). The calculation is completed by integrating out the three-momenta. Using polar coordinates in momentum space with \( \hat{k} = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta) \) and substituting \( x = k^2 \), it follows easily that

\[
\int d^3k e^{-k^2} k^{3z-3} = 4\pi \int_0^\infty dk \ k^{3z-1} e^{-k^2} = \frac{1}{z} \pi^{3/2}
\]

(2.6)

and

\[
\int d^3k e^{-k^2} k^{3z-3} \hat{k}_i \hat{k}_j = \frac{4\pi}{3} \delta_{ij} \int_0^\infty dk \ k^{3z-1} e^{-k^2} = \frac{1}{3z} \pi^{3/2} \delta_{ij} .
\]

(2.7)

The final result is independent of the scaling parameter \( z \),

\[
\nabla_\mu J^\mu_5 = \frac{1}{4\pi^2} \epsilon^{0ijk} \text{Tr}(F_{0i} F_{jk}) = \frac{1}{4\pi^2} \text{Tr} (F \wedge F) ,
\]

(2.8)

and coincides with the anomaly of the axial current conservation law of a Dirac fermion.
Similarly, we may compute the gravitational contribution to the axial anomaly under the minimal substitution of $\partial_\mu$ by $D_\mu = \partial_\mu + (1/8) [\gamma_{ab}, \gamma_b] \omega^{ab}_\mu$ written in terms of the spin connection $\omega$. The regulated sum shown in (2.2) is much more difficult to evaluate in this case as one has to introduce point splitting and use the curved space analogue of plane waves given in terms of the geodesic interval in space-time. Keeping track of the relevant terms is also quite tricky, in general, apart from the $z = 1$ case that corresponds to Dirac fermions. For higher values of $z$ one encounters multiple derivatives of the geodesic interval which in the coincidence limit correspond to Synge-DeWitt tensors of rank bigger than 2. There are many sources of such terms that contribute to the final result and one has to sum them up carefully. For $z = 3$ the intermediate calculations are already quite tedious (see [7] for details) and one expects that this method will be practically impossible to carry out to the end for general values of $z$. A closely related (but practically simpler) calculational method could be developed by applying path integral methods to suitably chosen models of supersymmetric quantum mechanics with Hamiltonian equal to the square of the Dirac-Lifshitz fermion operator. This method was successfully employed in the past to compute the form of anomalies in relativistic theories [5], particularly in higher dimensions where the conventional approach becomes intractable, but it is not currently known how to generalize it to encompass non-relativistic fermion systems. Thus, we should resort to other means for the efficient calculation of the gravitational contribution to the axial anomaly for general values of the anisotropy scaling parameter $z$.

The axial anomaly is necessarily a total derivative term, as it obstructs the divergence of a current. It is also gauge invariant, meaning that the anomalous term is solely expressed via the field strength $F$ and/or $R$. In four dimensions there is only one such term given by the topological density $\text{Tr}(F \wedge F)$ when fermions couple to gauge fields. Thus, the whole purpose of the path integral computation is to determine the coefficient of this term, which, as we saw before, turns out to be universal. By the same token, when fermions couple to metric fields, there are two possible terms given by the topological densities $\text{Tr}(R \wedge R) = R^a_b \wedge R^b_a$ and $\text{Tr}(R \wedge \ast R) = (-1/2) \epsilon_{abcd} R^a_b \wedge R^c_d$ in four dimensions. The second term is easily ruled out by the trace identities of Dirac gamma-matrices; close inspection of the anomalous term (2.2) reveals that the trace cannot yield more than one epsilon symbol, and, hence only $\text{Tr}(R \wedge R)$ and not $\text{Tr}(R \wedge \ast R)$ can possibly contribute to the axial anomaly. Then, the only remaining task is to evaluate its coefficient and verify that in all cases the result is the same, namely

$$\nabla_\mu J^\mu_5 = -\frac{1}{96\pi^2} \epsilon^{0ijk} R^{ab}_{0i} R_{ab}^{jk} = \frac{1}{96\pi^2} \text{Tr}(R \wedge R) \, .$$

The method that we will follow relies on the integrated form of the anomaly and its relation to the index theorem of the Dirac-Lifshitz operator. An important ingredient in the computation is the $\eta$-invariant of the fermion operator restricted on the three-dimensional leaves of space-time foliation and its intimate relation to the gravitational Chern-Simons action via the anomaly, which turns out to be universal.
3 The $\eta$-invariant and axial anomalies

In this section we make a small detour to summarize some basic facts about the index theorem of the Dirac operator and the associated notion of $\eta$-invariant, which play important role in relativistic quantum field theory. Later we will extend the results to the more general class of Dirac-Lifshitz fermion operators and use them to determine the coefficient of the axial anomaly by spectral methods.

Recall that the $\eta$-invariant of a self-adjoint matrix $A$ with real eigen-values $\lambda_k$ is the signature of $A$, which is defined to be

$$\eta_A = \text{sign}(A) = \sum_{\lambda_k>0} \text{sign} \lambda_k - \sum_{\lambda_k<0} 1,$$

assuming that there are no zero eigen-values in the spectrum, which otherwise are excluded. As such, $\eta_A$ is invariant under rescaling of the matrix by any positive number and it counts the spectral asymmetry between positive and negative eigen-values of $A$. If $A$ depends upon a parameter $\delta$, the spectrum will flow as $\delta$ varies continuously and level crossing may occur in the system. The $\eta$-invariant of $A(\delta)$ changes by $\pm 2$ units every time a negative (respectively positive) eigen-value at $\delta_0$ crosses to positive (respectively negative) values at $\delta_1$ as $\delta$ varies from $\delta_0$ to $\delta_1$.

Computing the $\eta$-invariant is a notoriously difficult task when the matrix $A$ is infinite dimensional, since the spectrum of $A$ cannot be determined in general and the definition of $\eta_A$ becomes ill-defined as it stands. The standard prescription in those cases is to consider first the $\eta$-function of $A$ obtained by zeta-function regularization, as

$$\eta_A(s) = \sum_{\lambda_k} \text{sign} \lambda_k |\lambda_k|^{-s},$$

and then define the corresponding $\eta$-invariant as

$$\eta_A = \eta_A(0).$$

Typically, $A$ is a self-adjoint operator on a compact manifold so that its spectrum is discrete and $\eta_A(s)$ can be analytically extended to a meromorphic function on the entire complex $s$-plane which is regular at $s = 0$. Yet, direct computation of the $\eta$-invariant by spectral methods can only be done in a few cases. Such examples will be given later.

The Atiyah-Patodi-Singer index theorem is the main mathematical framework that necessitated the introduction of the $\eta$-invariant of various operators [14] (but see also the textbooks [15, 16] for more details). Restricting attention to the Dirac operator $i\gamma^\mu D_\mu$ on a four-dimensional Riemannian manifold $M_4$, the difference of its positive and negative chirality zero modes, known as the index of $i\gamma^\mu D_\mu$ (we will use $D$ in
short), is given by

\[
\text{Ind}(D) = -\frac{1}{192\pi^2} \int_{M^4} \text{Tr}(R \wedge R) + \frac{1}{192\pi^2} \int_{\partial M^4} \text{Tr}(\theta \wedge R) - \frac{1}{2} \eta_D(\partial M^4) .
\] (3.4)

The first term provides the bulk contribution to the index, which is obtained by integration over \( M^4 \) of (a half times) the local form of the axial anomaly of a Dirac fermion in the background of a metric field. It involves the (appropriately normalized) characteristic class \( \text{Tr}(R \wedge R) \) that is written in terms of the curvature two-form \( R = d\omega + \omega \wedge \omega \) associated to the spin-connection \( \omega \) of the metric \( g \) on \( M^4 \). The other two terms arise when \( M^4 \) has a boundary \( \partial M^4 \). The local boundary term is the integral over \( \partial M^4 \) of the (appropriately normalized) secondary characteristic class \( \text{Tr}(\theta \wedge R) \), [17], which is written in terms of the second fundamental form \( \theta = \omega - \omega^0 \) and accounts for the possible deviation of the metric \( g \) from a cross-product form \( g_0 \) at the boundary. Finally, the last term is a non-local boundary contribution to the index provided by the \( \eta \)-invariant of the tangential part of the Dirac operator restricted to \( \partial M^4 \). Here, it is implicitly assumed that the four-component spinors \( \Psi \) have a specific fall-off rate close to \( \partial M^4 \) (known as APS boundary conditions) and so what we are really computing is the \( L^2 \)-index of normalizable zero modes of \( i\gamma^\mu D_\mu \).

An immediate consequence of the Atiyah-Patodi-Singer index theorem is that the \( \eta \)-invariant of the three-dimensional Dirac operator on a compact manifold \( \Sigma^3 \) without boundary, which is endowed with a Riemannian metric, is related to the gravitational Chern-Simons action [17]

\[
W_{CS}(\Sigma^3) = \int_{\Sigma^3} \text{Tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) \] (3.5)

as follows,

\[
\frac{1}{2} \eta_D(\Sigma^3) + \frac{1}{192\pi^2} W_{CS}(\Sigma^3) = \text{constant} .
\] (3.6)

This is easily seen by considering a four-dimensional spin manifold \( M^4 \) bounded by \( \Sigma^3 \) and choosing a metric on it that has cross-product form near the boundary \( \partial M^4 = \Sigma^3 \). Thus, although both the \( \eta \)-invariant and the Chern-Simons action depend explicitly on the conformal class of the metric on \( \Sigma^3 \), their (appropriately weighted) sum is in fact independent of it. The coefficient of \( W_{CS} \) is uniquely fixed by the anomaly. We also note for completeness that both \( \eta_D \) and \( W_{CS} \) are odd under parity, thus flipping sign under orientation reversing diffeomorphisms on the manifold \( \Sigma^3 \).

Equation (3.6) provides an indirect way to compute the \( \eta \) invariant of the three-dimensional Dirac operator, up to a constant, based on axial anomalies in four space-time dimensions. Conversely, one may use this formula to extract the coefficient of the anomalous term \( \text{Tr}(R \wedge R) \) in the divergence of the axial current using spectral methods, provided that the \( \eta \)-invariant can be computed explicitly for a certain class of three-metrics. This is the strategy that will also be applied later to Lifshitz models.
Alternatively, on a four-dimensional manifold of the form \( M_4 \simeq I \times \Sigma_3 \), where \( I \) is an interval (it can also be \( I \simeq \mathbb{R} \) in the physical applications) endowed with the metric

\[
ds^2 = dt^2 + g_{ij}(t, x)dx^i dx^j ,
\]

the Atiyah-Patodi-Singer index theorem for the Dirac operator implies the following useful relation

\[
\frac{1}{2} \Delta \eta_D(\Sigma_3) + \frac{1}{192\pi^2} \Delta W_{CS}(\Sigma_3) = \text{integer} ,
\]

where \( \Delta \) denotes the difference of any given quantity at the two end-points of the Euclidean time interval \( I \). One may view (3.6) as special case of the more general relation (3.8) by just shrinking one of the two ends of the cylinder \( I \times \Sigma_3 \) to a point.

For example, for \( \Sigma_3 \simeq S^3 \), let us assume without loss of generality that one of the two boundaries is endowed with the constant curvature (round) metric. At that boundary the \( \eta \)-invariant vanishes, as there can be no asymmetry in the spectrum of the Dirac operator, whereas \( W_{CS} \) assumes the value \( 16\pi^2 \). Shrinking that boundary to a point by conformal rescaling does not affect the values of \( \eta_D \) and \( W_{CS} \) and (3.8) reduces to (3.6) with constant term equal to \( 16\pi^2 / 192\pi^2 = 1/12 \) modulo integers.

The integer appearing on the right-hand side of equation (3.8) is the index of the Dirac operator on \( I \times \Sigma_3 \). If the space-time metric (3.7) has a cross-product form everywhere, i.e., \( g_{ij} \) is independent of \( t \), the three-dimensional metric at the two ends of the cylinder (boundaries of space-time) will be the same, and, hence, \( \Delta \eta_D = 0 = \Delta W_{CS} \) giving zero index. Otherwise, if the metric is warped by depending explicitly on \( t \), the \( \eta \)-invariant and the Chern-Simons action will be different at the two end-points in general. Furthermore, it can be shown in this case that the index of the four-dimensional Dirac operator is provided by the spectral flow,

\[
\text{Ind}(D) = \Delta S(\Sigma_3) ,
\]

where \( \Delta S(\Sigma_3) \) is the net number of level crossings that may occur in the spectrum of the three-dimensional Dirac operator on \( \Sigma_3 \) as the metric \( g_{ij}(t, x) \) deforms from one end-point of the interval \( I \) to the other end-point (recall that the \( \eta \)-invariant jumps by \( \pm 2 \) units when an eigen-value crosses from negative to positive values or conversely, and, therefore, each level crossing contributes \( \pm 1 \) units to the index). This formula and its generalization to Lifshitz models is very useful for the applications that will be discussed later.

There is another occurrence of the \( \eta \)-invariant of the three-dimensional Dirac operator in quantum field theory. This time, one is interested in extracting the parity-violating piece of the effective action \( \Gamma_{\text{eff}} \),

\[
\exp (-\Gamma_{\text{eff}}) = \int (D\bar{\Psi})(D\Psi) \left( - \int d^3 x \bar{\Psi}(x)i\gamma^i D_i \Psi(x) \right) ,
\]

where
which is obtained by integrating out the fermions. Here, the Dirac operator is minimally coupled to external fields, such as gauge and/or metric fields, and, therefore, the fermion effective action depends explicitly on them. There is a natural relation between the parity anomaly in odd dimensions and the gauge and/or gravitational contribution to axial anomalies in even dimensions [12]. In our case, the imaginary part of the effective action that provides the parity violating piece is

\[ \text{Im} \Gamma_{\text{eff}} = \frac{\pi}{2} \eta_D, \]  

(3.11)

which in turn is related to the parity violating Chern-Simons action. The derivation is based on the formal relation (choosing the branch \( \log(-1) = i\pi \))

\[ \frac{1}{2} \log \frac{\det(-A)}{\det(A)} = \frac{i\pi}{2} \eta_A \]  

(3.12)

and it can be made rigorous by zeta-function regularization of the determinant of the Dirac operator [18]. Thus, the Chern-Simons term of topologically massive gauge and/or gravitational theories in three dimensions is induced radiatively [19, 20, 21] (but see also [22] for earlier work on the subject). A closely related subject is the structure of the induced anomalous vacuum current in odd dimensions and the \( \eta \)-function regularization of the vacuum charge of a fermion field (see, for instance, [23] for a general and rigorous discussion of this subject).

4 Universality of \( \eta \)-invariant in Lifshitz models

The index of the Dirac-Lifshitz operator \( i\gamma^\mu D_\mu \) defined by (2.1) (we will use \( D \) in short) follows from the integrated form of the axial anomaly. Thus, on a space-time \( M_4 \) with boundaries we have

\[ \text{Ind}(D) = -\frac{1}{192\pi^2} \int_{M_4} \text{Tr}(R \wedge R) + \frac{1}{192\pi^2} \int_{\partial M_4} \text{Tr}(\theta \wedge R) - \frac{1}{2} \eta_D(\partial M_4), \]  

(4.1)

in exact analogy with the index of the Dirac operator (3.4), provided that the coefficient of the anomalous term in (2.9) is the same as for the Dirac operator in a metric background. Here, the \( \eta \)-invariant refers to the operator of order \( z \)

\[ i\gamma^i D_i = \frac{1}{2} i\gamma^i [D_i(-D_k D^k + M^2)^a + (-D_k D^k + M^2)^a D_i] \]  

(4.2)

acting on two-component spinors on the three-dimensional boundary \( \partial M_4 \).

As before, the Atiyah-Patodi-Singer theorem (4.1) implies the following identity between \( \eta_D \) and the gravitational Chern-Simons action on a compact three-manifold
\[\frac{1}{2} \eta_D(\Sigma_3) + \frac{1}{192\pi^2} W_{CS}(\Sigma_3) = \text{constant}. \quad (4.3)\]

Then, comparison with equation (3.6) for the Dirac operator implies that \(\eta_D(\Sigma_3)\) ought to be equal to \(\eta_D(S^3)\) up to a constant. For \(\Sigma_3 \simeq S^3\), in particular, this constant has to be zero because \(S^3\) admits a constant curvature metric for which there can be no spectral asymmetry for \(D\) nor \(\bar{D}\). Conversely, proving that the \(\eta\)-invariant is universal, i.e.,

\[\eta_D(S^3) = \eta_D(S^3) \quad (4.4)\]

for any given metric on \(S^3\), suffices to show that the coefficient of the anomaly in the axial current conservation law (2.9) is indeed \(-1/192\pi^2\). This is precisely what we will show next by restricting attention to a particular class of metrics that depend continuously on a particular modulus (anisotropy parameter). Hence, we will provide an alternative derivation of the anomaly by spectral methods, as advertised before.

The class of metrics that will be used in the following allow for exact computation of the \(\eta\)-invariant. Then, as consequence of the Atiyah-Patodi-Singer theorem, equation (4.4) should hold true for all metrics on \(S^3\). This way we will also be able to provide model geometries for which the index of the four-dimensional fermion operator can be computed explicitly by spectral flow. Finally, we will derive geometric conditions for having non-zero index on certain backgrounds. The presentation we follow here has many common elements to our earlier work [7], but it is now applicable to Lifshitz operators of arbitrary order \(z\).

First, to set up the stage, we consider some general aspects of the eigenvalue problem of the Dirac operator \(i\gamma^iD_i\) on a three-dimensional manifold \(\Sigma_3\) with a Riemannian metric \(g\) acting on two-component spinors,

\[i\gamma^iD_i \Psi(x) = \zeta \Psi(x); \quad \Psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (4.5)\]

where

\[i\gamma^iD_i = i\gamma^i E_i \left( \partial_i + \frac{1}{8}[\gamma_I, \gamma_K] \omega^{IK} \right). \quad (4.6)\]

Here, \(E_i^I\) are the components of the inverse dreibeins associated to \(g_{ij}\) with tangent space indices \(I\) and \(\omega^{IK}\) are the components of the corresponding spin connection. Also, the gamma matrices \(\gamma^I\) are chosen in terms of the Pauli matrices, as usual,

\[\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.7)\]
and satisfy the anti-commutation relations $[\gamma^I, \gamma^J]_+ = 2\delta^{IJ}$. A useful relationship is provided by Lichnerowicz’s formula for the square of the Dirac operator \cite{24},

$$(i\gamma^i D_i)^2 = -D^2 + \frac{1}{4} R,$$  \hspace{1cm} (4.8)

which is written in terms of the Bochner Laplacian operator $D^2 = D_i D^i$ acting on spinors (rather than scalars) and the Ricci scalar curvature $R$ associated to the three-dimensional metric $g$.

The eigen-value problem for the more general class of operators $i\gamma^i D_i$ of order $z = 2\alpha + 1$,

$$i\gamma^i D_i \Psi(x) = Z \Psi(x),$$  \hspace{1cm} (4.9)

is definitely a more complex problem. Even if the spectrum $\zeta$ of the Dirac operator can be determined for a given metric $g$, the eigen-values $Z$ will not be easy to obtain in general. Note at this point, based on Lichnerowicz’s formula (4.8), that we may express the operators $i\gamma^i D_i$ in the form

$$i\gamma^i D_i = \frac{1}{2} i\gamma^i \left[D_i \left((i\gamma^k D_k)^2 - \frac{1}{4} R + M^2 \right)^\alpha + \left((i\gamma^k D_k)^2 - \frac{1}{4} R + M^2 \right)^\alpha D_i \right]$$  \hspace{1cm} (4.10)

from which it follows that $i\gamma^i D_i$ and $i\gamma^i D_i$ do not commute in general. Their commutator differs from zero by derivatives of the curvature, which provide a geometric obstruction to find a common system of eigen-spinors $\Psi$. However, the situation simplifies drastically when the geometry is homogeneous, since $R$ is the same at all points of space and all its derivatives vanish by definition. Only in this case the two operators exhibit a common system of eigen-spinors and the corresponding eigen-values are simply related by

$$Z = \zeta \left(\zeta^2 - \frac{1}{4} R + M^2 \right)^\alpha.$$  \hspace{1cm} (4.11)

From now on we restrict attention to homogeneous geometries on $\Sigma_3$ and to be more specific we consider the class of Bianchi IX metrics on $\Sigma_3 \simeq S^3$ of the following type

$$ds^2 = \gamma \left[(\sigma^1)^2 + (\sigma^2)^2 + \delta^2 (\sigma^3)^2 \right],$$  \hspace{1cm} (4.12)

using the left-invariant one-forms $\sigma^I$ of the group $SU(2)$ that satisfy the relations

$$d\sigma^I + \frac{1}{2} \epsilon^I_{JK} \sigma^J \wedge \sigma^K = 0.$$  \hspace{1cm} (4.13)

Here, we are actually assuming that the geometry has enlarged isometry $SU(2) \times U(1)$ by taking two of the metric coefficients equal to each other. Then, the parameters of the metric are the conformal factor $\gamma$ and the anisotropy parameter $\delta$, which can range from 0 to $\infty$. These metrics are often referred in the literature as Berger spheres.
The spectrum of the Dirac operator can be completely determined on such geometries and the same thing also applies to the spectrum of the more general operators $i \gamma^i D_i$. If we were considering the more general class of Bianchi IX metrics, without imposing axial symmetry, the eigen-value problem of the Dirac operator would have not been tractable. For later use we also write the Ricci scalar curvature for this particular class of three-dimensional metrics \((4.12)\),
\[
R = \frac{1}{2\gamma} (4 - \delta^2) .
\] (4.14)

There is a dual basis of vector fields $f_i$ associated to $\sigma^I$ with $< f_i, \sigma^J > = \delta^I_J$ that satisfy the commutation relations $[f_i, f_j] = -\epsilon_{ij}^K f_K$ and represent the $SU(2)$ Killing vector fields of this particular class of geometries. Both $\sigma^I$ and $f_i$ can be written in terms of three Euler angles, but the explicit expressions will not be needed here. Then, the Dirac operator for the class of axially symmetric Bianchi IX geometries \((4.12)\) takes the following form (see also the Appendix)
\[
 i \gamma^i D_i = \begin{pmatrix}
 if_3 / \delta & if_1 - f_2 \\
 if_1 + f_2 & -if_3 / \delta
\end{pmatrix} + \frac{1}{4\delta} (\delta^2 + 2) ,
\]
(4.15)
where we have taken $\gamma = 1$ without loss of generality. The conformal factor $\gamma$ can be easily reinstated by noting that $i \gamma^i D_i$ scales uniformly as $1/\sqrt{\gamma}$, but this will not affect the value of the $\eta$-invariant on Berger spheres, since it only depends on $\delta$. For later use we note that the same thing applies to the operators $i \gamma^i D_i$ that scale uniformly by $1/(\sqrt{\gamma})^{2\alpha+1}$ and their $\eta$-invariant is also be inert to $\gamma$ (actually, in this case, we also have to rescale appropriately the mass parameter $M$, but it does not really matter since the $\eta$-invariant will turn out to be independent of $M$ as well).

The spectrum of the Dirac operator \((4.15)\) on Berger spheres is given by the following discrete set of eigen-values \([25]\) (but see also \([26]\))
\[
\zeta_\pm = \frac{\delta}{4} \pm \frac{1}{2\delta} \sqrt{4\delta^2 pq + (p - q)^2} ; \quad (p, q) \in \mathbb{N}^2
\]
(4.16)
with multiplicities $p + q$ for each pair $(p, q)$. The positive integers $p$ and $q$ are not ordered, which means that the conjugate pairs $(q, p)$ also yield the same eigen-values $\zeta_\pm$ with the same multiplicities. There are additional eigen–values arising formally for $q = 0$,
\[
\zeta_0 = \frac{\delta}{4} + \frac{p}{2\delta} ; \quad p \in \mathbb{N} ,
\]
(4.17)
but their multiplicity is $2p$ rather than $p$. Thus, the complete spectrum of the Dirac operator is $\zeta_\pm$ and $\zeta_0$ forming a two–dimensional state lattice which is labeled by the quantum numbers $(p, q)$. The eigen–values $\zeta_+$ and $\zeta_0$ are positive definite for all $\delta$,
whereas $\zeta_-$ can assume positive as well as negative values by tuning their parameters. Likewise, the spectrum of the more general operator $i\gamma^D_i$ takes the form

$$Z = \zeta \left( \zeta^2 + \frac{1}{8}(\delta^2 - 4) + M^2 \right)^{\alpha}$$

(4.18)

following from (4.11) and (4.14) with $\gamma = 1$. The corresponding eigen-values $Z_{\pm}$ and $Z_0$ are obtained by setting $\zeta = \zeta_{\pm}$ and $\zeta_0$ respectively.

We also note that the sign of the eigenvalues $\zeta$ and $Z$ is the same irrespective of $\alpha$ and $M$, since

$$\zeta^2 + \frac{1}{8}(\delta^2 - 4) > 0, \quad \forall \delta \in [0, \infty).$$

(4.19)

This can be verified directly for $\zeta = \zeta_{\pm}$ or $\zeta_0$ and it is important for computing the $\eta$-invariant of the corresponding operators by separating the contribution of their positive and negative modes. Although $\zeta_{\pm}$ and $\zeta_0$ are positive definite for all values of $\delta$, $\zeta_-$ can be either negative or positive depending upon $\delta$. The eigen-values $\zeta_-$ can also become zero at special values of $\delta$ for which there exist positive integer solutions $(p, q)$ to the equation

$$\delta^2 = 2\sqrt{4pq\delta^2 + (p - q)^2}.$$  

(4.20)

The corresponding zero modes $\Psi$ are often referred as harmonic spinors in the mathematics literature [24, 25], and, clearly, they can only exist for $\delta \geq 4$. The presence of zero modes should be properly accounted in the computation of the $\eta$-invariant.

There are some important points that need to be emphasized before proceeding further. The anisotropy parameter $\delta$ accounts for the distortion of the three-sphere. The value $\delta = 1$ corresponds to the round metric on $S^3$. When $\delta$ decreases the sphere is squashed until it becomes fully flattened in the extreme case $\delta = 0$; in the latter case $S^3$ collapses to $S^2$ but the geometry is completely regular there. When $\delta$ increases the sphere is stretched until it becomes singular as $\delta \to \infty$. Note that the Ricci scalar curvature (4.14) is positive definite for $\delta < 2$ and it turns negative for $\delta > 2$. Then, by Lichnerowicz’s formula (4.13), the Dirac operator cannot exhibit any zero modes for $\delta < 2$. Zero modes are allowed to exist only for $\delta > 2$. Note at this point that when the curvature of space is positive definite, all eigen-values $\zeta_-$ are negative and clearly the same thing applies to all eigen-values $Z_-$. For negative curvature, however, level crossing becomes possible provided that $\delta > 4$. The critical value $\delta = 4$ is the threshold for the existence of solutions to equation (4.20). This value is far beyond Lichnerowicz’s bound $\delta > 2$, and, thus, in this context, it corresponds to Berger spheres with sufficiently negative curvature. At $\delta = 4$ the eigen-value $\zeta_-$ with $p = q = 1$ becomes zero and then it turns positive for all $\delta > 4$. As $\delta$ increases further, there are more and more eigen-values $\zeta_-$ that become positive, until they all undergo level crossing in the extreme limit $\delta \to \infty$.

The computation of the $\eta$-invariant $\eta_D(S^3)$ relies on the separation of positive and negative modes. First, we assume that $\delta \leq 4$ so that the eigen-values $Z_-$ are all neg-
We have to evaluate the following sums when $\delta$ with integer $n$.

The terms that are omitted vanish at $\mathcal{Z}$.

Thus, according to definition,

$$\eta_D = \lim_{s \to 0} \eta_D(s) = \lim_{s \to 0} \sum_{\text{eigenvalues}} (\text{sign} \mathcal{Z}) |\mathcal{Z}|^{-s}, \tag{4.21}$$

we have to evaluate the following sums when $\delta < 4$,

$$\eta_D(s) = \sum_{p,q > 0} (p + q) \left( \left( \frac{\delta^2}{2} + X \right) \left[ \left( \frac{\delta^2}{2} + X \right)^2 + \frac{\delta^2}{2} (\delta^2 - 4) + 4\delta^2 M^2 \right]^\alpha \right)^{-s} - \sum_{p,q > 0} (p + q) \left( \left( -\frac{\delta^2}{2} + X \right) \left[ \left( -\frac{\delta^2}{2} + X \right)^2 + \frac{\delta^2}{2} (\delta^2 - 4) + 4\delta^2 M^2 \right]^\alpha \right)^{-s} + \sum_{p > 0} 2p \left( \left( \frac{\delta^2}{2} + p \right) \left[ \left( \frac{\delta^2}{2} + p \right)^2 + \frac{\delta^2}{2} (\delta^2 - 4) + 4\delta^2 M^2 \right]^\alpha \right)^{-s}, \tag{4.22}$$

setting $X = \sqrt{4\delta^2 pq + (p - q)^2}$ for notational convenience. The first line refers to the contribution of $\mathcal{Z}_+$, the second to $\mathcal{Z}_-$ and the third to $\mathcal{Z}_0$. Note that in writing down (4.22) all eigen-values $\zeta$ in (4.18) have been rescaled by a factor of $2\delta$, which is irrelevant, however, for the final result that is obtained at $s = 0$.

Next, we compute the individual sums by expanding all fractions in powers of $\delta$ (up to the appropriate order) and then set $s = 0$. The contribution of the modes $\mathcal{Z}_0(\delta)$ is the easiest to evaluate. Using the power series expansion

$$\left( \left( \frac{\delta^2}{2} + p \right) \left[ \left( \frac{\delta^2}{2} + p \right)^2 + \frac{\delta^2}{2} (\delta^2 - 4) + 4\delta^2 M^2 \right]^\alpha \right)^{-s} = \frac{1}{p^{(2\alpha + 1)s}} \left[ 1 - s\delta^2 \left( \frac{2\alpha + 1}{2p} + \frac{2\alpha (2M^2 - 1)}{p^2} \right) + \frac{s\delta^4}{8p^2} \left( (s + 1)(2\alpha + 1)^2 - 2\alpha(2\alpha + 3) \right) \right. + \ldots \right] \tag{4.23}$$

we find that the last group of terms in (4.22) assumes the following expansion written in terms of Riemann zeta–functions,

$$I_0(s) = 2\zeta((2\alpha + 1)s - 1) - s\delta^2 \left[ (2\alpha + 1)\zeta((2\alpha + 1)s) + 4\alpha(2M^2 - 1)\zeta((2\alpha + 1)s + 1) \right] + \frac{s\delta^4}{4} \left[ (s + 1)(2\alpha + 1)^2 - 2\alpha(2\alpha + 3) \right] \zeta((2\alpha + 1)s + 1) + \ldots. \tag{4.24}$$

The terms that are omitted vanish at $s = 0$ as they contain the factor $s\zeta((2\alpha + 1)s + n)$ with integer $n \geq 2$ and $\zeta(s)$ is absolutely convergent for $\text{Re} s > 1$; they include a
term of the form $\delta^4/p^3$ in (4.23) as well as all terms of order $\delta^6$ and higher which come multiplied with $1/p^{n+1}$ with $n \geq 2$. Then, taking into account that $\zeta(-1) = -1/12$, $\zeta(0) = -1/2$ and that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1 (and, thus, $(2\alpha + 1)s\zeta((2\alpha + 1)s + 1)$ equals 1 at $s = 0$), we obtain

$$I_0(0) = -\frac{1}{6} - \frac{4\alpha}{2\alpha + 1}(2M^2 - 1)\delta^2 - \frac{2\alpha - 1}{4(2\alpha + 1)}\delta^4. \quad (4.25)$$

The contribution of the modes $Z_\pm(\delta)$ to the $\eta$–invariant is more difficult to extract because of the double sums that are involved. Again, we expand the fractions in power series of $\delta$

$$\left(\frac{-\delta^2}{2} + X\right)^{\alpha} = \frac{1}{X^{(2\alpha + 1)s}} \left[1 - \frac{s\delta^2}{2X} \left(\frac{2\alpha + 1}{2X} + \frac{2\alpha(2M^2 - 1)}{X^2}\right) + \cdots\right]$$

omitting all terms of the form $1/X^{n+2}$ with $n \geq 2$ that do not contribute to the final result when $s = 0$, as will be seen shortly. Then, the first two sums in the general expression (4.22) for $\eta_D(s)$, which we denote respectively by $I_\pm(s)$, are combined together as follows,

$$I_+(s) - I_-(s) = -(2\alpha + 1)s\delta^2f \left(\frac{(2\alpha + 1)s + 1}{2}\right) - 2sf \left(\frac{(2\alpha + 1)s + 3}{2}\right) \left\{\alpha[2\alpha - 1 - (s + 1)(2\alpha + 1)](2M^2 - 1)\delta^4 + \frac{1}{48}(2\alpha(2\alpha - 1)(2\alpha + 7) + (s + 1)(2\alpha + 1)[(s + 2)(2\alpha + 1)^2 - 6\alpha(2\alpha + 3)]\right\} + \cdots, \quad (4.27)$$

setting for convenience

$$f(s) = \sum_{p,q > 0} \frac{p + q}{X^{2s}} = \sum_{p,q > 0} \frac{p + q}{4\delta^2pq + (p - q)^2}s.$$

(4.28)
The function $f(s)$ is absolutely convergent on the complex $s$–plane for $\text{Re } s > 3/2$, which justifies the suppression of the higher order terms in the power series expansion above (all such terms are multiplied with $s$ and vanish at $s = 0$). Furthermore, the residues of the function $f(s)$ at the two special points $s = 1/2$ and $3/2$, which are relevant for the calculation, are (see, for instance, [27])

$$
\text{res } f(s)|_{s=1/2} = \frac{\delta^2 - 1}{6}, \quad \text{res } f(s)|_{s=3/2} = \frac{1}{2\delta^2},
$$

and, therefore, the resulting contribution of these terms to the $\eta$-invariant is

$$
I_+(0) - I_-(0) = \frac{\delta^2}{3} \left( 1 + \frac{12\alpha}{2\alpha + 1}(2M^2 - 1) \right) - \frac{\delta^4}{3} \left( 1 - \frac{10\alpha - 1}{4(2\alpha + 1)} \right).
$$

Putting all together, it turns out that the $\eta$-invariant is independent of the exponent $\alpha$ and the scale parameter $M$. The final result reads

$$
\eta_D(0) = I_+ - I_0 = -\frac{1}{6} (1 - \delta^2)^2.
$$

As such, it coincides with the $\eta$-invariant of the Dirac operator on Berger spheres with anisotropy parameter $\delta$ and it is related to the gravitational Chern-Simons action as follows (see also the Appendix),

$$
\frac{1}{192\pi^2} W_{\text{CS}}(\text{Berger}) = \frac{1}{12} \left( 2 - 2\delta^2 + \delta^4 \right) = \frac{1}{2} \eta_D(0) + \frac{1}{12},
$$

in agreement with formula (3.6) for the Dirac operator. This proves our claim (4.4) for the $\eta$-invariant on Berger spheres with $\delta < 4$ and suffices to show that the coefficient of the anomaly is indeed universal.

Next, we examine the case $\delta > 4$ that involves level crossing. Every time an eigenvalue $Z_-$ crosses from negative to positive values, $\eta_D$ jumps by 2 to $\eta_D + 2$, since the spectral asymmetry, as calculated for $\delta < 4$, changes by 2. Let us denote by

$$
\mathcal{C}(\delta_c) = \{(p, q) \in \mathbb{N}^2; \delta_c^2 = 2\sqrt{4pq\delta_c^2 + (p - q)^2}\}
$$

the set of positive integers $(p, q)$ that account for harmonic spinors at each one of the special values $\delta_c < \delta$ arising for fixed $\delta > 4$. Since the zero modes at $\delta_c$ have multiplicity $p + q$, the quantity

$$
S(\delta) = \sum_{\delta_c < \delta} \left[ \sum_{(p,q) \in \mathcal{C}(\delta_c)} (p + q) \right]
$$

provides the total number of modes that crossed from negative to positive values as $\delta$ was varying from $\delta < 4$ to any given value $\delta > 4$. Thus, for $\delta > 4$, the $\eta$-invariant of
the operator $\mathcal{D}$ on Berger spheres is shifted by twice the number of these modes and equals

$$\eta_{\mathcal{D}} = -\frac{1}{6}(\delta^2 - 1)^2 + 2S(\delta).$$

(4.35)

$S(\delta)$ is also inert to the parameters $\alpha$ and $M$ which shows that level crossing is the same as for the Dirac operator. It can be easily seen that $S(\delta)$ is always an even integer, but there is no closed expression for it for general values of $\delta$.

The results we have presented here generalize our earlier work on the subject \cite{7} to all values of the anisotropy scaling parameter $z$ and show that the axial anomaly is an infra-red phenomenon, as in relativistic quantum field theory. The universality of the $\eta$-invariant also shows that the parity breaking piece of the induced fermion action in three-dimensions is provided by the Chern-Simons term with the same coefficient as in (3.11) irrespective of the order of the fermion operator $i\gamma^i \mathcal{D}_i$. Finally, the model geometries that were used to perform the calculation provide simple examples for which the index of the four-dimensional fermion operator can be computed explicitly by spectral flow as $\text{Ind}(\mathcal{D}) = \pm S(\delta)$ (the sign depends on the choice of orientation).

In the next section, we consider four-dimensional backgrounds that can give rise to violations of the chiral charge conservation law in gravitational theories, and, thus, lead to baryon and lepton number violation when the index differs from zero (note at this end that the axial anomaly will not obstruct the conservation law $\Delta Q_5 = 0$ if the index is zero).

### 5 Applications to non-relativistic theories of gravity

So far the geometry has been treated as background for fermion propagation without reference to any metric field equations. Space-times of the form $I \times \Sigma_3$ with Euclidean line element (3.7) will lead to violations of the chiral charge conservation if the gravitational field equations for the metric $g_{ij}(t, x)$ induce level crossing for the fermion operator on $\Sigma_3$ as $t$ varies in $I$. In Einstein gravity this cannot happen for various reasons (see, for instance, \cite{28}), but in non-relativistic gravitational theories of Hořava-Lifshitz type \cite{29} there are regular solutions that can support such exotic effects.

For definiteness, we concentrate on gravitational instanton solutions in $3 + 1$ dimensions with $SU(2) \times U(1)$ isometry for which $\Sigma_3 \simeq S^3$ is a Berger sphere with metric (4.12). In Einstein gravity these are the familiar Taub-NUT and Eguchi-Hanson instantons for which $I$ is the semi-infinite line and the anisotropy parameter $\delta$ changes between 0 and 1 as $t$ varies between the two end-points of $I$; in these cases one side of $I$ terminates at a removable singularity, which is a nut or a bolt respectively. The field equations do not allow $\delta$ to exceed 1 without ruining the regularity and completeness of the four-dimensional metrics. Thus, in this case, level crossing cannot occur and the index of the four-dimensional Dirac operator on such metric backgrounds is zero. More generally, for non-compact four-metrics with non-negative Ricci scalar curva-
ture, there can be no bound state solutions of the Dirac equation; if such states existed, they would have been covariantly constant, and, hence, non-normalizable \[24\] leading to contradiction.

On the other hand, in Hořava-Lifshitz theory with detailed balance (meaning that the potential term of the action is derivable from a local superpotential functional \(W[g]\)) the instantons are eternal solution of the gradient flow equation for the three-metric \(g\) derived from \(W\) \[30\]. In this case, \(I \simeq R\) and the instanton interpolates between two distinct vacua of the three-dimensional action \(W\) (viewed as fixed points of the gradient flow equation) as \(t\) ranges from \(-\infty\) to \(+\infty\). Typically, on \(S^3\), \(W[g]\) admits a maximally symmetric vacuum for which \(\delta = 1\), but there should be other less symmetric vacua that coexist and provide the other end-point of the instantons. Then, within our Bianchi IX mini super-space model, we pose the following problem: under what conditions \(W[g]\) admits another less symmetric vacuum with \(\delta > 4\), in order to have violation of chiral charge conservation induced by instantons, and what is the phase of the non-relativistic gravitational theory that admits such instantons as bona fide Euclidean solutions? The results we describe below show that \(Q_5\) cannot be always conserved in Hořava-Lifshitz gravity with anisotropy scaling exponent \(z \geq 3\). For lower values of \(z\) there seems to be no window in the space of couplings for having non-conservation of chiral charge. Curiously, Hořava-Lifshitz gravity becomes power counting renormalizable only for \(z \geq 3\), which was the reason to view it as substitute of general relativity in the deep ultra-violet regime in the first place \[29\]. Thus, it is interesting to study the problem of chiral symmetry breaking in these models.

Recall that the superpotential functional \(W[g]\) is taken to be the action of three-dimensional gravity on \(\Sigma_3\) augmented with higher derivative terms. A simple choice is provided by topologically massive gravity \[22\] containing the gravitational Chern-Simons term \[3.5\] and a three-dimensional cosmological constant \(\Lambda_w\),

\[
W_{\text{TMG}}[g] = \frac{2}{\kappa_w^2} \int_{\Sigma_3} d^3 x \sqrt{\text{det} g} \left( R - 2 \Lambda_w \right) + \frac{1}{2\omega} W_{\text{CS}}[g] .
\]

(5.1)

The corresponding theory of Hořava-Lifshitz gravity has anisotropy scaling parameter \(z = 3\) unless the Chern-Simons term is absent in which case \(z = 2\). The vacua of topologically massive gravity satisfy the following field equations, which are split on purpose into traceless and trace parts,

\[
R_{ij} - \frac{1}{3} R g_{ij} - \kappa_w^2 \frac{\omega}{\omega} C_{ij} = 0 \quad \text{and} \quad R = 6 \Lambda_w ,
\]

(5.2)

where \(C_{ij}\) is the Cotton tensor of the three-metric \(g\).

For the class of Bianchi IX metrics \(4.12\), the traceless part of the field equations reduce to a single algebraic relation among the anisotropy parameter \(\delta\) and the con-
formal factor $\gamma$,
\[
\frac{1 - \delta^2}{3} + \frac{\kappa^2_w \delta (1 - \delta^2)}{2 \omega \sqrt{\gamma}} = 0 .
\] (5.3)

Thus, apart from the maximally symmetric solution $\delta = 1$, there is one more solution for
\[
\frac{1}{3} + \frac{\kappa^2_w \delta}{2 \omega \sqrt{\gamma}} = 0
\] (5.4)

provided that the Chern-Simons coupling $\omega$ is negative (for a given choice of orientation on $S^3$). The two metrics can coexist as vacua of topologically massive gravity only when $\Lambda_w$ is non-negative. This follows from the trace part of the field equations $R = 6 \Lambda_w$ using (4.14). Then, the corresponding instanton of Hořava-Lifshitz gravity interpolates between two Berger spheres with the same non-negative curvature, and, thus, there can be no net level crossing in the spectrum of the three-dimensional fermion operator. The only way to have level crossing is to be in a unimodular phase of the theory in which the trace part of the field equations decouples from the dynamics and the two end-points are allowed to have different curvature (positive for $\delta = 1$ and sufficiently negative for the other solution (5.4)).

This possibility is naturally realized when the generalized DeWitt metric in super-space arising in the canonical formulation of the theory [29]
\[
G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}
\] (5.5)

has $\lambda = 1/3$ and projects any tensor to its traceless part ($G^{ijkl}$ has similar form and properties). Then, the volume of space is preserved by the dynamics in $3 + 1$ dimensions. Since the volume of the Berger sphere (4.12) is fixed $\text{Vol}(S^3) = 16\pi^2 \delta \gamma^{3/2}$, we can rewrite the defining equation of the anisotropic fixed point (5.4) as follows by eliminating $\gamma$,
\[
\text{Vol}(S^3) = 54\pi^2 \delta^4 \left(-\frac{\kappa^2_w}{\omega}\right)^3 .
\] (5.6)

Thus, in this case, the index of the Dirac-Lifshitz operator will be non-zero on the corresponding gravitational instanton background provided that $\delta > 4$, which in turn imposes a lower bound on the volume of space written in terms of the ratio of the couplings $\kappa^2_w$ to $\omega$ as follows,
\[
\text{Vol}(S^3) > 13824\pi^2 \left(-\frac{\kappa^2_w}{\omega}\right)^3 ,
\] (5.7)

reproducing the results reported in [7]. Note that the second fixed point (5.4) is absent when there is no Chern-Simons term in $W[g]$, and, hence, there is no instanton to induce violation of chiral charge conservation when the anisotropy scaling exponent is lowered to $z = 2$. 

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Next, we consider the case of $z = 4$ Hořava-Lifshitz gravity in $3 + 1$ dimensions (see, for instance, [31]) choosing as $W[g]$ the action of new massive gravity on $\Sigma_3$, [32].

\[
W_{\text{NMG}}[g] = \frac{2}{\kappa_w^2} \int_\Sigma d^3x \sqrt{\det g} \left( R - 2\Lambda_w \right) + \frac{1}{m^2} \int_\Sigma d^3x \sqrt{\det g} \left( R_{ij} R^{ij} - \frac{3}{8} R^2 \right). \tag{5.8}
\]

We want to examine if the results described above persist in this case, leaving again a window for violation of chiral charge conservation by gravitational instanton effects. The choice (5.8) is not the most general one that can be made. One may augment it with a gravitational Chern-Simons term, as will be discussed briefly later, and also change the relative coefficient of the quadratic curvature terms in (5.8), which, however, will not be discussed here for simplicity (such a generalization spoils some nice properties of new massive gravity as higher curvature extension of three dimensional Einstein gravity, but it is legitimate in the context of $z = 4$ Hořava-Lifshitz gravity).

For Berger spheres, the traceless part of the classical equations of motion of new massive gravity reduce consistently to the following algebraic relation (see [33] for details),

\[
\frac{1 - \delta^2}{3} + \frac{\kappa_w^2 (1 - \delta^2)(4 - 21\delta^2)}{48m^2\gamma} = 0. \tag{5.9}
\]

Thus, apart from the maximally symmetric solution $\delta = 1$ we also have another solution for

\[
1 + \frac{\kappa_w^2 (4 - 21\delta^2)}{16m^2\gamma} = 0. \tag{5.10}
\]

Since we are ultimately interested in the coexistence of vacua with $\delta > 4$, we impose the restriction $m^2 > 0$ on the coupling of the quadratic curvature terms (otherwise, if $\delta^2 < 4/21$, the existence of a second solution will require $m^2 < 0$). The trace part of the field equations yields

\[
\frac{4 - \delta^2}{2\gamma} = 6\Lambda_w + \frac{\kappa_w^2}{64m^2\gamma^2} (3\delta^2 - 4)(7\delta^2 - 4). \tag{5.11}
\]

Setting $\delta = 1$, we find that $\Lambda_w$ is non-negative for $m^2 > 0$. The other solution (5.10) does not satisfy the trace relation for $\delta > 4$, since the left-hand side will be negative and the right-hand side positive. Thus, the only way that both solutions can coexist when $\delta > 4$ is to decouple the trace part of the field equations from the dynamics.

As before, this only happens in the unimodular phase of the theory corresponding to the choice $\lambda = 1/3$ in the DeWitt metric. Then, the volume of space remains fixed and can be expressed in terms of $\delta$ by eliminating $\gamma$ from the fixed point (5.10). We have, in particular, the following relation

\[
\text{Vol}(S^3) = \frac{\pi^2 \kappa_w^3}{4m^3} \delta(21\delta^2 - 4)^{3/2} > 8\pi^2 (83)^{3/2} \frac{\kappa_w^3}{m^3}, \tag{5.12}
\]
using $\delta > 4$ to establish the lower bound in the volume of space in terms of the couplings. This inequality provides the window for having violation of chiral charge conservation in the corresponding $3 + 1$ theory of Hořava-Lifshitz gravity.

Further generalizations arise by adding the gravitational Chern-Simons term to $W_{NMG[g]}$ that yield the so called three-dimensional generalized new massive gravity [32]. This theory has a maximally symmetric solution with $\delta = 1$. Other Berger sphere solutions satisfy the traceless part of the field equations, which boil down to

$$\frac{1}{3} + \frac{\kappa_w^2 \delta}{2\omega \sqrt{\gamma}} + \frac{\kappa_w^2 (4 - 21 \delta^2)}{48 m^2 \gamma} = 0. \quad (5.13)$$

For simplicity, we assume $m^2 > 0$ so that there is only one admissible solution of equation (5.13), whereas $\omega$ can take either positive or negative values. The trace part of the field equations is the same as before, (5.11), because the Cotton tensor is traceless. Thus, for $m^2 > 0$, (5.11) cannot be possibly satisfied for both $\delta = 1$ and $\delta > 4$ and we are forced again to consider the unimodular variant of the theory that corresponds to $\lambda = 1/3$. Then, the volume of space is preserved throughout the evolution and it is given by

$$\text{Vol}(S^3) = \frac{1}{4} \pi^2 \delta^4 \left(-\frac{3\kappa_w^2}{\omega} + \sqrt{\left(\frac{3\kappa_w^2}{\omega}\right)^2 + \frac{\kappa_w^2}{m^2} \left(21 - \frac{4}{\delta^2}\right)}\right)^3 \geq 64\pi^2 \left(-\frac{3\kappa_w^2}{\omega} + \sqrt{\left(\frac{3\kappa_w^2}{\omega}\right)^2 + \frac{83\kappa_w^2}{4m^2}}\right)^3. \quad (5.14)$$

The lower bound is obtained by demanding $\delta > 4$ so that chiral charge conservation can be violated in the corresponding Hořava-Lifshitz gravity and it encompasses the two simpler cases that were discussed before. The analysis is more intricate when $m^2 < 0$, as there can be more than one vacua with $\delta \neq 1$, and, hence, more instantons that interpolate between them; the details are left to the interested reader.

Similar considerations apply to theories based on Born-Infeld generalizations of three-dimensional gravity [34] that include higher and higher derivative terms. The anisotropy scaling exponent $z$ of the corresponding Hořava-Lifshitz theory increases accordingly and there can be several Berger sphere solutions that coexist and support the end-points of instantons. We expect that the general features of our results will remain qualitatively the same in these cases too, leaving windows in the space of gravitational coupling parameters for having chiral symmetry breaking by gravitational instanton effects. This novelty differentiates the non-relativistic gravitational theories of Lifshitz type from Einstein gravity and will have phenomenological consequences in bouncing models of the early universe if Hořava-Lifshitz gravity plays a role. We also expect these results to be valid more generally beyond the mini-superspace model.
geometries, which were only used here for illustrative purposes.

In all examples that were considered above, it was found that non-conservation of chiral charge can only occur in the phase of Hořava-Lifshitz gravity with $\lambda = 1/3$, where the volume of space decouples from the dynamics. This phase of the theory is not inflicted with many of the pathologies that otherwise haunt non-relativistic models of gravitation. It may very well be that only this phase becomes relevant in the deep ultra-violet regime, as substitute for ordinary gravity, and not its variants with $\lambda \neq 1/3$. Flowing from the ultra-violet to the infra-red is a difficult problem that could be properly investigated within the asymptotic safety scenario for quantum gravity (for a review see, for instance, [35] and references therein) treating $\lambda$ as a coupling that undergoes renormalization. Proper treatment of the problem may also require turning on other operators that are lying outside the class of Hořava-Lifshitz models to ensure well-behaved transition to ordinary large scale physics. All these issues are currently under consideration.

6 Conclusions

We have investigated the occurrence of axial anomalies in non-relativistic fermion theories of Lifshitz type which are minimally coupled to gauge and/or gravitational fields in $3 + 1$ space-time dimensions. Our main conclusion is that the results are identical to the relativistic case, hereby generalizing earlier work on subject [6, 7] to theories with arbitrary anisotropy scaling exponents. This is not surprising in retrospect conforming with the general idea that the axial anomaly (unlike others) is an infra-red phenomenon. Also, the universal form of the $\eta$-invariant can be intuitively understood (prior to zeta-function regularization) by noting that the operator $(-D^2 + M^2)^{\alpha}$ is positive definite; hence, the number of positive and negative modes is the same for both operators $i\gamma^i D_i$ and $i\gamma^i \partial_i$. It was confirmed that the same conclusion holds for the regulated $\eta$-invariants.

Although the result has its own value, we have also considered some applications in the context of non-relativistic gravitational theories of Lifshitz type and found that (unlike ordinary gravity) violation of chiral charge conservation can be induced by instantons - and possibly other configurations - in certain regions of their parameter space.

There are a few technical questions that remain open and should be addressed carefully elsewhere. It is not yet clear whether there are any restrictions on the allowed values of the anisotropy scaling parameter $z$ that insure self-adjointness of the Dirac-Lifshitz operator in the strict mathematical sense. Also, it is interesting to generalize our results to higher dimensions and compare with the anomalies of the corresponding relativistic quantum field theories. Finally, it remains to explore the role of anomalies in condensed matter physics systems that exhibit quantum criticality.
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A Chern-Simons action for Berger spheres

In this appendix we provide the components of the connection one-form of the Berger sphere with metric \( ds^2 = \gamma ((\sigma_1)^2 + (\sigma_2)^2 + \delta^2(\sigma_3)^2) \). These expressions are needed for writing down the Dirac operator on such backgrounds as well as for evaluating the gravitational Chern-Simons action \( W_{\text{CS}}(\text{Berger}) \) used in the main text. We have, in particular,

\[
\omega_{12} = \frac{\delta^2 - 2}{2} \sigma_3, \quad \omega_{13} = \frac{\delta}{2} \sigma_2, \quad \omega_{23} = -\frac{\delta}{2} \sigma_1, \quad (A.1)
\]

noting that they are all independent of the scale factor \( \gamma \).

The gravitational Chern-Simons action can be easily evaluated in steps as follows,

\[
W_{\text{CS}} = \int_{S^3} \text{Tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right)
\]

\[
= 2 \int_{S^3} \left( \omega_{12} \wedge d\omega_{12} + \omega_{13} \wedge d\omega_{13} + \omega_{23} \wedge d\omega_{23} + 2 \omega_{12} \wedge \omega_{23} \wedge \omega_{31} \right)
\]

\[
= (\delta^4 - 2\delta^2 + 2) \int_{S^3} \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \quad (A.2)
\]

using along the way \( d\sigma_1 + \sigma_2 \wedge \sigma_3 = 0 \) and cyclic permutations. Since the integral over \( S^3 \) of the volume form \( \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \) equals \( 16\pi^2 \), we obtain the final result

\[
W_{\text{CS}}(\text{Berger}) = 16\pi^2 (\delta^4 - 2\delta^2 + 2). \quad (A.3)
\]

The extrema of this action, as function of \( \delta \in [0, \infty) \), occur at \( \delta = 1 \) and \( \delta = 0 \) in accordance with the fact that the round sphere and the fully squashed Berger sphere are the only metrics in this class that are conformally flat. The minimum corresponds to \( \delta = 1 \), whereas \( \delta = 0 \) is a local maximum.
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