EULER CHARACTERISTICS OF DIFFERENTIAL EQUATIONS AND SPECTRAL CURVES

KAZUKI HIROE

Abstract. We show a coincidence of Euler characteristics of a differential equation with irregular singularities on a compact Riemann surface and the associated spectral curve which is recently called irregular spectral curve. Also we present a comparison of local invariants, so called Milnor formula which links the Komatsu-Malgrange irregularity of the differential equation and the Milnor number of the spectral curve.

Introduction

This paper presents a numerical comparison between cohomology groups of a differential equation with irregular singularities on a Riemann surface and those of associated spectral curve which is sometimes called an irregular spectral curve.

A main result in this paper is the following. We consider a differential equation

\[ dw = Aw \]

where \( A \) is a square matrix of size \( n \) whose entries are meromorphic 1-forms on a compact Riemann surface \( X \) of genus \( g \). In particular this differential equation is allowed to have several regular/irregular singular points \( a_1, a_2, \ldots, a_k \) on \( X \). Then we can define a divisor on the cotangent bundle \( T^*X \) as the zero locus of the characteristic polynomial

\[ \det(yI_n - A) \]

and this is called spectral curve \( C_A \). Singular points appear as poles of \( A \) and the zero locus of the characteristic polynomial will pass through the line at infinity \( y = \infty \). Thus it is natural to consider the spectral curve as a divisor on a compactified cotangent bundle \( \overline{T^*X} \). Let us assume one of the following conditions is satisfied at each singular points \( a_1, a_2, \ldots, a_k \).

1. The Hukuhara-Turrittin-Levelt normal form of the germ \( A_{a_i} \) of \( A \) at \( a_i \) is multiplicity free (see Definition 3.2).
2. The germ \( A_{a_i} \) is regular semisimple over \( \mathbb{C}[z_{a_i}] \) (see Definition 3.9).

Then we can show the following coincidence of the index of rigidity of the differential equation and the Euler characteristic of the spectral curve.

1991 Mathematics Subject Classification. 34M35, 14J17, 14H70.
Key words and phrases. Index of rigidity, Euler characteristic, Milnor number, Komatsu-Malgrange irregularity, Irregular spectral curve.

The author is supported by JSPS Grant-in-Aid for Young Scientists (B) Grant Number 17K14222.
Theorem 0.1 (Theorem 4.5, Corollary 4.6). Let \( \nabla_A \) be the algebraic connection defined by the differential equation \( dw = Aw \). Suppose that \( C_A \) is integral and irreducible. Moreover suppose that \( C_A \) is smooth on \( T^*X \). Then the index of rigidity \( \text{rig}(\nabla_A) \) of \( \nabla_A \) and the Euler characteristic \( \chi(\tilde{C}_A) \) of the normalization \( \tilde{C}_A \) of \( C_A \) coincide with each other, i.e.,

\[
\text{rig}(\nabla_A) = \chi(\tilde{C}_A).
\]

Moreover assume that \( \nabla_A \) is irreducible. Then we have the numerical coincidences of cohomology groups,

\[
h^i_{\text{dR}}(X, j_*(\mathcal{E}nd\nabla_A)) = h^i(\tilde{C}_A, \mathbb{C}), \quad i = 0, 1, 2.
\]

Here \( h^i(*) := \dim_{\mathbb{C}} H^i(*) \).

This fact has been known by Kamimoto [13] and Oshima [25] for Fuchsian differential equations on \( \mathbb{P}^1 \).

Let us look at the equation

\[
h^1_{\text{dR}}(X, j_*(\mathcal{E}nd\nabla_A)) = h^1(\tilde{C}_A, \mathbb{C})
\]

in our main theorem. This can be seen as an analogy of the well-known fact on the infinitesimal deformations of a holomorphic Higgs bundle: the genus of the corresponding spectral curve is equal to half of the dimension of the space of the infinitesimal deformations, see [11] and [24]. That is to say, cohomology group \( H^1_{\text{dR}}(X, j_*(\mathcal{E}nd\nabla_A)) \) is known to be identified with the space of isotipical infinitesimal deformations of \( \nabla_A \) by THEOREM 4.10 in [5] and also see Lemma 4.7 in [1]. Here the isotipical deformation means the deformation of \( \nabla_A \) under the condition that the HTL-normal forms at \( a_i, \ i = 1, 2, \ldots, k \) are kept fixed. Thus we may say that the main theorem gives an analogy of the fact for holomorphic Higgs bundles to irregular meromorphic connections following the philosophy of the nonabelian Hodge correspondence [27], [4].

On the other hand, it has been known a similar comparison of Euler characteristics of differential equations and another geometric counterparts, \( \ell \)-adic sheaves by Katz in [15], [16]. Furthermore, he pointed out a similarity between local properties, namely, the singularities of differential equations and the ramifications of local Galois actions on \( \ell \)-adic sheaves. He gave a table of analogies in [14]. Even in our case, we have obtained a following local comparison theorem in the process of showing our main theorem. As we saw, the spectral curve \( C_A \) had intersections with the line at infinity \( X_\infty = \overline{T^*X \setminus T^*X} \) at \( \infty_{a_i} = (\infty, a_i) \), \( i = 1, 2, \ldots, k \) and these intersection points may have singularities resulting from the irregular singularities of the corresponding differential equations. These singularities of irregular spectral curves are discussed also in [19] and [28]. Then we can show that the Milnor number of \( C_A \) at \( \infty_{a_i} \) can be computed from the Komatsu-Malgrange irregularity of the corresponding local differential module \( M_{A_{a_i}} \) as follows.

Theorem 0.2 (Theorem 4.3). The Milnor number of \( C_A \) at \( \infty_{a_i} \) for each \( i = 1, 2, \ldots, k \) is

\[
\mu(C_A)_{\infty_{a_i}} = -\delta(\text{End}_{\mathbb{C}[z_{a_i}]}(M_{A_{a_i}})) - r_{C_{A_{a_i}}} + 2(n - 1)(C_A, X_\infty)_{\infty_{a_i}} + 1.
\]
Here \((C, C')_a\) is the intersection number of divisors \(C, C'\) at \(\infty_a\), \(r_{C_{A}}\) is the number of branches of the germ \(C_{A}\), and

\[
\delta(\text{End}_{\mathbb{C}[[z]]} (M_A)) = \text{rank } \text{End}_{\mathbb{C}[[z]]} (M_A) + \text{Irr(End}_{\mathbb{C}[[z]]} (M_A)) - \dim \text{Hom}_{\mathbb{C}[[z]]} (\text{End}_{\mathbb{C}[[z]]} (M_A), \mathbb{C}[[z]])^{\text{hor}}.
\]

By using the \(\delta\)-invariant of a singularity of a plane curve germ, this formula can be written in a simpler form.

**Corollary 0.3** (Remark 4.4).

\[
2\delta(C_A) - 2(n - 1)(C_A, X_{\infty})_a = -\delta(\text{End}_{\mathbb{C}[[z]]} (M_A)).
\]

In Katz’ table of analogies [14], the irregularity of a differential equation corresponds to the swan conductor of the local Galois action on an \(\ell\)-adic sheaf. On the arithmetic geometry side, the comparison formula of the swan conductor and the Milnor number has been studied, which is called Deligne’s Milnor formula, see [6]. Our formula might be seen as a variant of this Milnor formula if we follow Katz’ table.

**Acknowledgements.** The author would like to express his gratitude to Professors Shingo Kamimoto and Toshio Oshima who gave him many inspirations through fruitful discussions. The essential part of the idea to prove the main theorem is based on their pioneering works in the case of Fuchsian differential equations on \(\mathbb{P}^1\). The most part of this work had been done when the author was a member of the Department of Mathematics in Josai University and the work would never been completed without the support from Josai University. Finally the author would like to thank Professor Akane Nakamura. Many discussions with her were very much inspiring and encouraging.

### 1. Spectral curves of differential equations

#### 1.1. Compactified cotangent bundle on a Riemann surface.

Let \(X\) be a compact Riemann surface of genus \(g\) and consider a compactification of \(T^*X\) defined by

\[
\overline{T^*X} := \mathbb{P}(\mathcal{O}_X \oplus T^*X)
\]

which is the projective bundle of the vector bundle \(\mathcal{O}_X \oplus T^*X\). The complement of \(T^*X\) is denoted by \(X_{\infty} := \overline{T^*X}\backslash T^*X\). The natural projection

\[
\pi : \overline{T^*X} \to X
\]

enables us to regard this surface as a ruled surface. Thus the Neron-Severi group \(\text{NS} \overline{T^*X} := \text{Pic} \overline{T^*X} / \text{Pic}^0 \overline{T^*X}\) is generated by \(X_0\), the zero section of \(T^*X\) and a fiber \(f\), namely,

\[
\text{NS} \overline{T^*X} \cong \mathbb{Z}X_0 \oplus \mathbb{Z}f.
\]

This lattice has the \(\mathbb{Z}\)-bilinear form determined by the intersection numbers of generators,

\[
(X_0, f) = 1, \quad (f, f) = 0, \quad (X_0, X_0) = -\deg T^*X = 2g - 2.
\]
1.2. Spectral curve of differential equation. Let

\[ D := \{a_1, a_2, \ldots, a_p\} \subset X \]

be a finite set. We consider a differential equation with poles on \( D \),

\[ dw = Aw \]

where \( A \in M(n, \Omega_X(*D)(X)) \). We may assume that the set of all poles of \( A \) is exactly \( D \). To this differential equation, we shall associate a divisor on \( T^*X \) in an explicit way as follows.

First let us take a complex atlas \( \{(U_i, z_i)\}_{i=1,2,\ldots,k} \) of \( X \) with an open covering \( X = \bigcup_{i=1}^{k} U_i \) and local coordinates \( z_i : U_i \to \mathbb{C} \). On this local coordinate system the canonical 1-form \( \theta \in \Omega_{T^*X}(T^*X) \) can be expressed as

\[ \theta = \eta_i dz_i \]

with the fiber coordinate \( \eta_i \) of \( T^*X \) on \( U_i \) for \( i = 1, 2, \ldots, k \).

Then the canonical 1-form \( \theta \) extends to the meromorphic 1-form \( \bar{\theta} \) on \( T^*X \) of the form

\[ \bar{\theta} = \eta_i \zeta_i dz_i \]

where \( \zeta_i : \eta_i \in \mathbb{P}^1 \) is the fiber coordinate of \( T^*X = \mathbb{P}(\mathcal{O}_X \oplus T^*X) \) on \( U_i \).

Let us denote the trivialization of \( A \) on \( U_i \) by \( A_i = A_i(z_i) dz_i \), \( A_i(z_i) \in M(n, \mathbb{C}(z_i)) \). Then the pullback \( \pi^*A \) by the projection \( \pi : T^*X \to X \) can be written in the same form

\[ \pi^*A([\zeta_i : \eta_i], z_i) = A_i(z_i) dz_i \]

on \( \pi^{-1}(U_i) \cong \mathbb{P}^1 \times U_i \).

Then \( \det \left( \frac{\eta_i}{\zeta_i} I_n - A_i(z_i) \right) \) gives a meromorphic function of \( \pi^{-1}(U_i) \) for each \( i = 1, 2, \ldots, k \). Since the compatibility follows immediately from the definition, the collection

\[ \left\{ \left( \pi^{-1}(U_i), \det \left( \frac{\eta_i}{\zeta_i} I_n - A_i(z_i) \right) \right) \right\}_{i=1,2,\ldots,k} \]

defines a Cartier divisor on \( T^*X \). The corresponding Weil divisor is the spectral curve of the differential equation \( dw = Aw \) and denoted by

\[ C_A \subset T^*X. \]

Remark 1.1. Spectral curves are usually defined as characteristic polynomials of Higgs bundles, though we have constructed them from differential equations as above. These two constructions of spectral curves can be linked through the notion of lambda connections.

For \( \lambda \in \mathbb{C}, \nabla_\lambda := \lambda d - A \) defines a lambda connection on the trivial bundle \( \mathcal{O}_X^{\oplus n} \), i.e., a \( \mathbb{C} \)-linear map

\[ \nabla_\lambda : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n} \otimes \Omega_X(*D) \]

satisfying

\[ \nabla_\lambda(f v) = \lambda df \otimes v + f \otimes \nabla_\lambda v \]

for \( f \in \mathcal{O}_X, v \in \mathcal{O}_X^{\oplus n} \). Then the spectral curve \( C_A \) defined above is nothing but that of the Higgs bundle \( (\mathcal{O}_X^{\oplus n}, \nabla_0) \). This is so called irregular Higgs bundle which has been studied by several researchers, \[4, 8, 19, 23\] and so on.
1.3. **Arithmetic genus of spectral curve.** We denote the divisor class of the spectral curve $C_A$ by the same notation. The arithmetic genus $g_a(C_A)$ of $C_A$ can be obtained by the genus formula

$$g_a(C_A) = \frac{1}{2}(C_A, C_A + K_{T^*X}) + 1.$$  

Our complex surface $T^*X$ is a ruled surface of which the Neron-Severi group is well-understood. Thus standard argument enables us to examine the explicit value of $g_a(C_A)$, see V.2 in [9] and [8] for example.

Let us first determine the coefficients $a$, $b$ in the expression $C_A = aX_0 + bf \in \text{NS} T^*X$. Since the projection $\pi|_{C_A}: C_A \to X$ is of degree $n$, we have $(C_A, f) = n$. Thus

$$n = (C_A, f) = (aX_0 + bf, f) = a.$$  

Next we note that $(X_\infty, X_0) = 0$ and $(X_\infty, f) = 1$. This shows that $b = (nX_0 + bf)X_\infty = (C_A, X_\infty)$ and we have

$$C_A = nX_0 + (C_A, X_\infty)f \in \text{NS} T^*X.$$  

Also note that

$$K_{T^*X} = -2X_0 + (2g - 2 + \deg T^*X)f = -2X_0 + (4g - 4)f.$$  

Finally, the genus formula leads us to

(1) $$g_a(C_A) = \frac{1}{2}(nX_0 + (C_A, X_\infty)f, (n - 2)X_0 + ((C_A, X_\infty) + 4g - 4)f) + 1 = \frac{1}{2}(n^2(2g - 2) + (2n - 2)(C_A, X_\infty)) + 1.$$  

2. **Local formal theory on differential equations.**

Here we recall the Hukuhara-Turrittin-Levelt theory on local structure of differential equations and the notion of irregularity introduced by Komatsu [18] and Malgrange [21].

2.1. **Differential modules over differential fields.** First let us fix notation. Let $\mathbb{C}[z]$ and $\mathbb{C}((z))$ denote the ring of formal power series and the field of formal Laurent series respectively. Similarly $\mathbb{C}\{z\}$ and $\mathbb{C}\{z\}$ denote the ring of convergent power series and the field of convergent Laurent series. Let $\mathcal{P} := \bigcup_{s\in\mathbb{Z}_{>0}} \mathbb{C}(\langle z^\frac{1}{s}\rangle)$ be the field of Puiseux series. Also $\mathcal{P}^{\text{conv}}$ denote the field of convergent Puiseux series. Set $\mathcal{P}^+ := \bigcup_{s\in\mathbb{Z}_{>0}} \mathbb{C}[\langle z^\frac{1}{s}\rangle]$ $\mathcal{P}^- := \bigcup_{s\in\mathbb{Z}_{>0}} z^{-\frac{1}{s}}\mathbb{C}[z^{-\frac{1}{s}}]$. Then we can decompose

$$\mathcal{P} = \mathcal{P}^- \oplus \mathcal{P}^+.$$  

The order of $f(z) = \sum_{r\in\mathbb{Q}} a_r z^r$ is the number

$$\text{ord} f(z) := \min \{ r \in \mathbb{Q} \mid a_r \neq 0 \}.$$
Similarly, the order of Puiseux series $G(z) = \sum_{r \in \mathbb{Q}} G_r z^r \in \mathcal{P} \otimes \mathbb{C} M(n, \mathbb{C})$ with matrix coefficients is defined by $\text{ord} G(z) := \min \{ r \in \mathbb{Q} \mid G_r \neq 0 \}$ where $0_n$ is the zero matrix of size $n$.

Let $K$ be one of the following fields: $\mathbb{C}(z)$, $\mathbb{C}(\{z\})$, $\mathbb{C}(\{z^{1/2}\})$, $\mathbb{C}(\{z^{1/3}\})$, $\mathcal{P}$ and $\mathcal{P}^{\text{conv}}$. A differential module $M$ over $K$ is a $K$-module with the derivation $\nabla_M \in \text{End}_K(M)$ satisfying the Leibniz rule $\nabla_M(km) = \frac{d}{dz} k \cdot m + k \cdot \nabla_M m$ for $k \in K$ and $m \in M$. Suppose that $M$ is finite of rank $n$ over $K$ and choose a basis $e = \{e_1, e_2, \ldots, e_n\}$. Then the matrix $G = (g_{i,j})_{1 \leq i, j \leq n}$ defined by

$$\nabla_M e_i = \sum_{j=1}^n g_{i,j} e_j$$

gives the matrix form of $\nabla_M \in \text{End}_K(M)$, that is

$$\frac{d}{dz} - G \in \text{End}_K(K^{\otimes n}).$$

We call $G$ the matrix of $\nabla_M$ with respect to $e$. Conversely, $G \in M(n, K)$ defines a differential module $M_G := K^{\otimes n}$ with the derivation $\nabla_{M_G} := \frac{d}{dz} - G$.

For two matrices $G, G'$ of $M$, there exists a base change matrix $X \in \text{GL}(n, K)$ and we have

$$G' = X G X^{-1} + \left( \frac{d}{dz} X \right) X^{-1}.$$ 

Let us recall some operations on finite differential modules. For differential modules $M$ and $M'$, the direct product $M \oplus M'$ is naturally defined as $K$-modules equipped with the derivation

$$\nabla_{M \oplus M'}(m + m') := \nabla_M m + \nabla_{M'} m' \quad (m \in M, m' \in M').$$

Also we can define the tensor product $M \otimes_K M'$ with the derivation

$$\nabla_{M \otimes_K M'}(m \otimes m') := \nabla_M m \otimes m' + m \otimes \nabla_{M'} m' \quad (m \in M, m' \in M').$$

The dual module of $M$ is $M^* := \text{Hom}_K(M, K)$ with the derivation $\nabla_{M^*}$ satisfying the following. If $G$ is the matrix of $\nabla_M$ with respect to a basis $e$, then $-\xi G$ is the matrix of $\nabla_{M^*}$ with respect to the dual basis $f$ of $e$.

The identification

$$\text{Hom}_K(M, M') \cong M^* \otimes_K M$$

induces the differential module structure on $\text{Hom}_K(M, M')$.

### 2.2. Hukuhara-Turrittin-Levelt normal forms.

We shall review the Hukuhara-Turrittin-Levelt theory which gives a formal classification of local differential equations. We use the notation

$$\text{diag}(A_1, A_2, \ldots, A_k)$$

which stands for a block diagonal matrix with the diagonal entries $A_i \in M(n_i, K)$. Recall that the substitution $\xi: f(z) \to f(e^{2\pi i z})$ for $f(z) \in \mathbb{C}(\{z^{1/2}\})$ generates the Galois group

$$\text{Gal}(\mathbb{C}(\{z^{1/2}\})/\mathbb{C}(\{z\})) \cong \mu_s$$

where $\mu_s$ is the cyclic group which consists of $s$th roots of 1 in $\mathbb{C}$. 

\[6\]
Definition 2.1 (HTL cell). Take \( q(z) \in \mathcal{P}^- \) and set \( r = \min \{ s \in \mathbb{Z}_{>0} \mid q(z) \in z^{-s-\frac{1}{2}} \mathbb{C}[z^{-1}] \} \). Then the elementary Hukuhara-Turrittin-Levelt cell \( E_{q(z),R} \) for the above \( q(z) \) and \( R \in M(n, \mathbb{C}) \) is

\[
E_{q(z),R} := \text{diag} \left( (q(z)I_n + R, \xi(q(z))(z)I_n + R, \ldots, \xi^{r-1}(q(z))I_n + R) z^{-1} \right) \in M(rn, \mathbb{C}(\{ z^{-\frac{1}{2}} \}))
\]

Here we call the integers \( n \) and \( r \) multiplicity and ramification index of \( E_{q(z),R} \) respectively.

Definition 2.2 (HTL normal form). A Hukuhara-Turrittin-Levelt normal form is a matrix

\[
\text{diag}(E_{q_1(z),R_1}, \ldots, E_{q_m(z),R_m})
\]

with elementary HTL cells \( E_{q_i(z),R} \) for \( i = 1, 2, \ldots, m \) such that

\[
\text{Gal}(\mathcal{P}/\mathbb{C}(\{ z \})) \cdot q_i(z) \cap \text{Gal}(\mathcal{P}/\mathbb{C}(\{ z \})) \cdot q_j(z) = \emptyset, \quad \text{if } i \neq j.
\]

Theorem 2.3 (Hukuhara-Turrittin-Levelt, [12], [29], [20]). Let \( M \) be a differential module over \( \mathbb{C}(\{ z \}) \) of rank \( n \) and \( \tilde{M} := \mathbb{C}(\{ z \}) \otimes \mathbb{C}(\{ z \}) M \) the formalization of \( M \). Then there exists an HTL normal form

\[
\text{diag}(E_{q_1(z),R_1}, \ldots, E_{q_m(z),R_m})
\]

as a matrix of \( \tilde{M} := \mathcal{P} \otimes \mathbb{C}(\{ z \}) \tilde{M} \) with respect to a suitable basis. Furthermore, if two differential modules \( M \) and \( M' \) over \( \mathbb{C}(\{ z \}) \) share a same HTL normal form, then \( \tilde{M} \cong \tilde{M}' \).

The HTL normal form induces the following decomposition of \( M \).

Theorem 2.4 (see (7.15) in [2] and COROLLARY 3.3 in [20]). We use the same notation as in Theorem 2.3. There exists a differential module \( M_{E_{q_i(z),R_i}} \) over \( \mathbb{C}(\{ z \}) \) whose HTL normal form is \( E_{q_i(z),R_i} \) for each \( i = 1, 2, \ldots, m \) and we have a decomposition

\[
\tilde{M} \cong \bigoplus_{i=1}^{m} M_{E_{q_i(z),R_i}}
\]

as differential modules over \( \mathbb{C}(\{ z \}) \).

2.3. Komatsu-Malgrange irregularity. Let us recall that the index of a \( \mathbb{C} \)-linear endomorphism \( \Phi \) is

\[
\chi(\Phi) := \dim_{\mathbb{C}} \text{Ker} \Phi - \dim_{\mathbb{C}} \text{Coker} \Phi.
\]

The Komatsu-Malgrange irregularity is an analytic invariant of local differential equations defined as follows.

Definition 2.5 (Komatsu-Malgrange irregularity). Let \( M \) be a finite differential module over \( \mathbb{C}(\{ z \}) \) and \( \tilde{M} := \mathbb{C}(\{ z \}) \otimes \mathbb{C}(\{ z \}) M \) its formalization.

Then the Komatsu-Malgrange irregularity of \( M \) is

\[
\text{Irr}(M) := \chi(\nabla_{\tilde{M}}) - \chi(\nabla_M).
\]
If $M$ has the HTL-normal form
\[
\text{diag}(E_{q_1(z), R_1}, \ldots, E_{q_m(z), R_m}),
\]
then it is known that the Komatsu-Malgrange irregularity is
\[
\text{Irr}(M) = -\sum_{i=1}^{m} r_i - 1 \sum_{j=0}^{r_i - 1} \text{ord} \xi^j(q_i)(z) = -\sum_{i=1}^{m} r_i \text{ord} q_i(z).
\]
Here $r_i$ are ramification indices of $E_{q_i(z), R_i}$ for $i = 1, 2, \ldots, m$.

3. LOCAL COMPARISON: MILNOR FORMULA

In this section we deal with a local differential module and define its characteristic polynomial with respect to a fixed basis. The zero locus of this characteristic polynomial may have a singularity at infinity which corresponds to the irregular singularity of the differential module. We shall compare these singularities and obtain a comparison formula between the irregularity of differential module and the Milnor number of the characteristic polynomial.

3.1. HUKUHARA-TURRITTI-LEVELT normal form and decomposition of characteristic polynomial.

**Definition 3.1 (characteristic polynomial).** Let us consider a finite differential module $M$ over $\mathbb{C}(\{z\})$ of rank $n$. Fix a matrix $G \in M(n, \mathbb{C}(\{z\}))$ of $M$ with respect to a basis $e$. Then the characteristic polynomial of $M$ with respect to $e$ is
\[
\det(yI - G) \in \mathbb{C}(\{z\})[y].
\]
The characteristic polynomial may have a singularity at $(y, z) = (\infty, 0)$. We shall see that the HTL normal form of $M$ have some information on the singularity.

**Definition 3.2 (multiplicity free HTL normal form).** An HTL normal form
\[
\text{diag}(E_{q_1(z), R_1}, \ldots, E_{q_m(z), R_m})
\]
is said to be multiplicity free when all HTL cells $E_{q_i(z), R_i}$, $i = 1, 2, \ldots, m$, are multiplicity one, namely, $R_i \in M(1, \mathbb{C})$ for all $i = 1, 2, \ldots, m$.

**Proposition 3.3.** Let $M$ be a differential module over $\mathbb{C}(\{z\})$ of rank $n$ and fix a matrix $G \in M(n, \mathbb{C}(\{z\}))$ of $M$.

Suppose that the $M$ has the multiplicity free HTL normal form
\[
\text{diag}(E_{q_1(z), R_1}, \ldots, E_{q_m(z), R_m}).
\]
Then the characteristic polynomial $\det (yI - G) \in \mathbb{C}(\{z\})[y]$ decomposes as follows,
\[
\det(yI - G) = \prod_{i=1}^{m} \prod_{j=0}^{r_i - 1} \left( y - \frac{\tilde{q}_{i,j}(z)}{z} \right).
\]
Here $\tilde{q}_{i,j}(z) \in \mathcal{P}^{\text{conv}}$ satisfies that
\[
\text{pr}^{-1}(\tilde{q}_{i,j}(z)) = \xi^j(q_i)(z)
\]
for each $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, r_i - 1$, and $\text{pr}^{-1}: \mathcal{P} \to \mathcal{P}^{-}$ is the projection along the decomposition $\mathcal{P} = \mathcal{P}^{-} \oplus \mathcal{P}^{+}$.
Proof. For $q_i(z)$, $i = 1, 2, \ldots, m$, define

$$E_{q_i(z)}^o := \text{diag} \left( q_i(z), \xi(q_i(z)), \ldots, \xi^{r_i-1}(q_i(z)) \right) z^{-1} \in M(r_i, \mathbb{C}(\langle z^{-k} \rangle)).$$

Then the multiplicity free condition leads to

$$\text{diag}(E_{q_1(z)}, R_1, \ldots, E_{q_m(z)}, R_m) \equiv \text{diag}(E_{q_1(z)}^o, \ldots, E_{q_m(z)}^o) \pmod{z^{-1} \mathbb{C}[z^{\frac{1}{n}}]}.$$ 

Here $s := \text{lcm}\{r_1, r_2, \ldots, r_m\}$. Thus there exists $X \in \text{GL}(n, \mathbb{C}(\langle z^{\frac{1}{s}} \rangle))$ such that

$$XGX^{-1} + \left( \frac{d}{dz} X \right) X^{-1} \equiv \text{diag}(E_{q_1(z)}^o, \ldots, E_{q_m(z)}^o) \pmod{z^{-1} \mathbb{C}[z^{\frac{1}{s}}]}.$$ 

Since ord $\left( \frac{d}{dz} X \right) X^{-1} \geq -1$, we have

$$XGX^{-1} \equiv \text{diag}(E_{q_1(z)}^o, \ldots, E_{q_m(z)}^o) \pmod{z^{-1} \mathbb{C}[z^{\frac{1}{s}}]}.$$ 

Note that all the entries $\xi^i(q_j(z))$ in $\text{diag}(E_{q_1(z)}^o, \ldots, E_{q_m(z)}^o)$ are mutually different. Thus applying the Lemma 3.4 below repeatedly, we can find $X' \in \text{GL}(n, \mathbb{C}(\langle z^{\frac{1}{s}} \rangle))$ so that $X'XGX(X'X)^{-1}$ is a diagonal matrix and

$$X'XG(X'X)^{-1} \equiv \text{diag}(E_{q_1(z)}^o, \ldots, E_{q_m(z)}^o) \pmod{z^{-1} \mathbb{C}[z^{\frac{1}{s}}]}.$$ 

This leads us to the decomposition

$$\det (yI_n - G) = \prod_{i=1}^m \prod_{j=0}^{r_i-1} \left( y - \tilde{q}_{[i,j]}(z) \right)$$

with $\tilde{q}_{[i,j]}(z) \in \mathbb{C}(\langle z^{\frac{1}{s}} \rangle)$ satisfying

$$\tilde{q}_{[i,j]}(z) \equiv \xi^i(q_i(z)) \pmod{\mathbb{C}[z^{\frac{1}{s}}]}$$

for each $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, r_i - 1$. Since the field $\mathcal{P}^{\text{conv}}$ is algebraically closed, the equation (2) coincides with the decomposition in $\mathcal{P}^{\text{conv}}[y]$. Thus the formal Puiseux series $\tilde{q}_{[i,j]}(z)$ should be convergent power series. \hfill $\square$

The following lemma is just a slight modification of the standard and well-known argument in the local formal theory of differential equations, so called the splitting lemma, see Lemma 3 in the section 3.2 in [3] for example.

**Lemma 3.4.** Let us consider $A(t) = t^r \sum_{i=0}^{\infty} A_i t^i \in M(n, \mathbb{C}(\langle t \rangle))$ and suppose that $A_0 = \text{diag}(A_{01}^{11}, A_{02}^{22}) \in M(n, \mathbb{C})$ with $A_{0j}^{jj} \in M(n_j, \mathbb{C})$, $j = 1, 2$, and the sets of eigenvalues of $A_{01}^{11}$ and $A_{02}^{22}$ respectively are disjoint. Then there exists

$$T(z) = \begin{pmatrix} I_{n_1} & T_{12}(t) \\ T_{21}(z) & I_{n_2} \end{pmatrix}, \quad T_{jk}(t) = \sum_{i=1}^{\infty} T_{ij}^{jk} t^i, \; j, k = 1, 2$$

such that

$$T(t)A(t)T^{-1}(t) = \begin{pmatrix} B_{11}(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix}$$

where $B_{jj}(t) = t^r \sum_{i=0}^{\infty} B_{ij}^{jj} t^i \in M(n_j, \mathbb{C}(\langle t \rangle))$ with $B_{0j}^{jj} = A_{0j}^{jj}$, $j = 1, 2$. 


Proof. The proof is almost the same as that of the splitting lemma in the local theory of differential equations.

Let us write

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix},$$

where $A_{jk}(t) = t^j \sum_{i=0}^{\infty} A_i^{jk}t^i \in M(n_j \times n_k, \mathbb{C}(t))$, $j, k = 1, 2$. Then the equation

$$T(t)A(t) = \begin{pmatrix} B_{11}(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix} T(t)$$

is equivalent to

$$B_{jj}(t) = A_{jj}(t) + T_{jk}(t)A_{kj}(t)$$

$$A_{jk}(t) + T_{jk}(t)A_{kk}(t) = B_{jj}(t)T_{jk}(t)$$

for $1 \leq j \neq k \leq 2$. Comparing the coefficients of the powers of $t$ on both sides, we have

$$T_n^{jk}A_{0k} - A_0^{jk}T_n^k = \sum_{\mu=1}^{n-1} (A_{\mu-\mu}^{jk}T_\mu - T_\mu^{jk}A_{\mu-\mu})$$

$$- \sum_{\nu=1}^{n-2} T_{\nu}^{jk} \sum_{\mu=1}^{n-\nu-1} A_{\mu-\nu-\mu}^{jk}T_\mu + \sum_{\mu=1}^{n-1} A_{\mu-\mu}^{jk}$$

for $n \geq 1$. Recall that the equation

$$TA_0^{22} - A_0^{11}T = C$$

for a given $C \in M(n_1 \times n_2, \mathbb{C})$ has the unique solution $T \in M(n_1 \times n_2, \mathbb{C})$ since the sets of eigenvalues of $A_0^{11}$ and $A_0^{22}$ respectively are disjoint, see Lemma 24 of the section A.1 in [3] for example. Thus the above equations determine $T_n^{jk}$, $n = 1, 2, \ldots, \text{inductively}$. \hfill \Box

Since the HTL normal form is multiplicity free, the decomposition in Theorem 22 is

$$\tilde{M} \cong \bigoplus_{i=1}^{m} M_{\tilde{\mathcal{E}}(\tilde{\mathcal{F}}_{\tilde{E}, R_i})}$$

is the irreducible decomposition. Correspondingly, the following proposition shows that the decomposition in Proposition 5.3 is the irreducible decomposition with the irreducible components

$$\prod_{j=0}^{r_i-1} \left( y - \frac{\tilde{q}_{i,j}(z)}{z} \right) \in \mathbb{C}[[z]][y], \quad i = 1, 2, \ldots, m.$$

Proposition 3.5. We use the same notation as in Proposition 5.3. The Galois orbit of $\tilde{q}_{i,j}(z) \in \mathcal{P}^{\text{conv}}$ is

$$\text{Gal}(\mathcal{P}/\mathbb{C}(z)) \cdot \tilde{q}_{i,j}(z) = \{ q_{i,0}(z), q_{i,1}(z), \ldots, q_{i}(z_{r_i-1}) \}$$

$$= \{ \tilde{q}_{i,0}(z), \xi_{i,0}(z), \ldots, \xi_{i,0}^{r_i-1}(z) \}$$

for each $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, r_i-1$. In particular $\tilde{q}_{i,j}(z) \in \mathbb{C}[[z^{1/n_i}]]$. \hfill \Box
Proof. The decomposition
\[
\det(yI_n - G) = \prod_{i=1}^{m} \prod_{j=0}^{r_i-1} \left( y - \frac{\tilde{q}_{i,j}(z)}{z} \right) \in \mathbb{C}[[z]][y]
\]
tells us that
\[
\text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot \tilde{q}_{i,j}(z) \subset \{ \tilde{q}_{k,l}(z) \mid k = 1, 2, \ldots, m, l = 0, 1, \ldots, r_k \}.
\]
Let \( pr^- : \mathcal{P} \to \mathcal{P}^- \) be the projection along the decomposition \( \mathcal{P} = \mathcal{P}^- \oplus \mathcal{P}^+ \). Since \( pr^- \) is compatible with the Galois action, we have
\[
pr^- \left( \text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot \tilde{q}_{i,j}(z) \right) = \text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot \xi^i(q_i(z)) = \text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot q_i(z).
\]
Thus we have
\[
\text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot \tilde{q}_{i,j}(z) = (pr^-)^{-1} \left( \text{Gal}(\mathcal{P}/\mathcal{C}([[z]])) \cdot q_i(z) \right)
\]
\[
= \{ q_{i,0}(z), q_{i,1}(z), \ldots, q_{i,r_i-1}(z) \}
\]
\[
= \{ \tilde{q}_{i,0}(z), \tilde{q}_{i,1}(z), \ldots, \xi^{i-1}(\tilde{q}_{i,0})(z) \}.
\]

3.2. Milnor formula. By Proposition 3.3, the decomposition in Proposition 3.3 can be rewritten as follows,
\[
\det(yI_n - G) = \prod_{i=1}^{m} \prod_{j=0}^{r_i-1} \left( y - \frac{\xi^i(\tilde{q}_i)(z)}{z} \right),
\]
where \( \tilde{q}_i(z) \in \mathbb{C}[[\{ z^{\frac{1}{r_i}} \}] \) and
\[
pr^- (\tilde{q}_i(z)) = (q_i)(z)
\]
for \( i = 1, 2, \ldots, m \). Moreover this is the irreducible decomposition with the irreducible components
\[
\prod_{j=0}^{r_i-1} \left( y - \frac{\xi^i(\tilde{q}_i)(z)}{z} \right) \in \mathbb{C}[[z]][y].
\]

We now investigate the singularity of the zero locus of each irreducible components at \((y, z) = (\infty, 0)\). To be more precise, let us put \( y = \frac{2}{z} \) and consider the homogenized polynomial
\[
\prod_{j=0}^{r_i-1} \left( \eta - \frac{\xi^i(\tilde{q}_i)(z)}{z} \right).
\]
The restriction to \( \eta = 1 \) gives
\[
\prod_{j=0}^{r_i-1} \left( 1 - \frac{\xi^i(\tilde{q}_i)(z)}{z} \right) = \prod_{j=0}^{r_i-1} \left( -\frac{\xi^i(\tilde{q}_i)(z)}{z} \right) \cdot \prod_{j=0}^{r_i-1} \left( \zeta - \frac{z}{\xi^i(\tilde{q}_i)(z)} \right).
\]
Proposition 3.6. Let us fix $F_0$ and set $\mathfrak{F}_0$. Then the zero locus of $p_i r_j := \text{ord } q_i(z)$, $i = 1, 2, \ldots, m$.

\textbf{Proposition 3.7. Let us fix an $i \neq j \in \{1, 2, \ldots, m\}$ such that $\text{ord } \frac{\partial F_i}{\partial \zeta} < 0$ and $\text{ord } \frac{\partial F_j}{\partial \zeta} < 0$. Then the intersection number of $C_{\tilde{q}_i}$ and $C_{\tilde{q}_j}$ is}

$$(C_{\tilde{q}_i}, C_{\tilde{q}_j}) = p_ir_j + p_ir_i + r_ir_j - \text{Irr}(\text{Hom}_{C(z)}(M_{E_{q_i}, R_i}, M_{E_{q_j}, R_j})).$$

\textbf{Proof. Since the plane curve germ $C_{\tilde{q}_i}$ is parametrized by $z(t) = t^{r_i}$, $\zeta(t) = \frac{r_i}{\tilde{q}_i(t^{r_i})}$ and the germ $C_{\tilde{q}_j}$ is defined by $\prod_{k=1}^r \left( \zeta - \frac{z}{\tilde{q}_j(z)} \right) = 0$. Then the intersection number of them is computed as follows, see 1.2 in [30] for example.}

$$(C_{\tilde{q}_i}, C_{\tilde{q}_j}) = \text{ord } t \prod_{k=1}^r \left( \zeta(t) - \frac{t^{r_i}}{\xi^k(\zeta)(t^{r_i})} \right)$$

$$= r_i \text{ord } z \prod_{k=1}^r \left( \frac{z}{\tilde{q}_i(z)} - \frac{z}{\xi^k(\tilde{q}_j)(z)} \right)$$

$$= r_i \text{ord } z \prod_{k=1}^r \left( \frac{1}{\tilde{q}_i(z)} - \frac{1}{\xi^k(\tilde{q}_j)(z)} \right) + r_ir_j$$

$$= r_i \text{ord } z \prod_{k=1}^r \left( \frac{\xi^k(\tilde{q}_j)(z) - \tilde{q}_i(z)}{\tilde{q}_i(z)\xi^k(\tilde{q}_j)(z)} \right) + r_ir_j$$

$$= r_i \text{ord } z \prod_{k=1}^r \left( \xi^k(\tilde{q}_j)(z) - \tilde{q}_i(z) \right) + p_ir_j + p_ir_i + r_ir_j$$

$$= p_ir_j + p_ir_i + r_ir_j - \text{Irr}(\text{Hom}_{C(z)}(M_{E_{q_i}, R_i}, M_{E_{q_j}, R_j})).$$

\[ \square \]

\textbf{Proposition 3.7. Let us fix an $i \in \{1, 2, \ldots, m\}$ and suppose that $\text{ord } \frac{\partial }{\partial \zeta} < 0$. Then the Milnor number of the germ $C_{\tilde{q}_i}$ is}

$$\mu(C_{\tilde{q}_i}) = (2p_i + r_i - 1)(r_i - 1) - \text{Irr} \left( \text{End}_{C(z)}(M_{E_{q_i}, R_i}) \right).$$

\textbf{Proof. Set } $F_i(\zeta, z) := \prod_{k=1}^r \left( \zeta - \frac{z}{\xi^k(\tilde{q}_j)(z)} \right)$. Then the germ $C_{\tilde{q}_i}$ is defined by $F_i = 0$. Then the Milnor number can be obtained by

$$\mu(C_{\tilde{q}_i}) = (F_i, \frac{\partial}{\partial \zeta} F_j) + 1 - (F_j, z).$$
We refer to COROLLARY 7.16 and THEOREM 7.18 in [10] for this fact.

If we note that \( \frac{\partial}{\partial \zeta} F_i = \sum_{k=1}^{r_i} \prod_{1 \leq l < r_i} \left( \zeta - \frac{z}{\xi^l(\tilde{q}_l)}(z) \right) \), then

\[
(F_i, \frac{\partial}{\partial \zeta} F_j) = \text{ord}_z \prod_{1 \leq l, k \leq r_i, l \neq k} \left( \frac{z}{\xi^k(\tilde{q}_l)}(z) - \frac{z}{\xi^l(\tilde{q}_k)}(z) \right)
\]

\[
= \text{ord}_z \prod_{1 \leq l, k \leq r_i, l \neq k} \left( \xi^l(\tilde{q}_l)(z) - \xi^k(\tilde{q}_k)(z) \right) + r_i(r_i - 1) + 2p_i(r_i - 1)
\]

\[
= -\text{Irr} \left( \text{End}_{\mathbb{C}[z]}(M_{E_{q_i} R_i}) \right).
\]

Also we have \((F_j, z) = r_i\). Thus combining these equations, we have

\[
\mu(C_{\tilde{q}_i}) = (2p_i + r_i)(r_i - 1) - \text{Irr} \left( \text{End}_{\mathbb{C}[z]}(M_{E_{q_i} R_i}) \right) + 1 - r_i
\]

\[
= (2p_i + r_i - 1)(r_i - 1) - \text{Irr} \left( \text{End}_{\mathbb{C}[z]}(M_{E_{q_i} R_i}) \right).
\]

\(\square\)

Now we compute the Milnor number of the zero locus of the characteristic polynomial \(\det(y I_n - G)\) at \((y, z) = (\infty, 0)\) as follows. Let us suppose that \(M\) has a singularity at \(z = 0\). Then \(G\) has a pole at \(z = 0\) and the zero locus of the homogenization of the characteristic polynomial

\[
\zeta^n \det \left( \frac{\eta}{\zeta} I_n - G \right)
\]

pass through the point \(([\zeta : \eta], z) = ([0 : 1], 0)\). Let us denote the zero locus by \(C_G\) and the Milnor number of \(C_G\) at \(([\zeta : \eta], z) = ([0 : 1], 0)\) by \(\mu(C_G)_\infty\).

**Theorem 3.8** (Milnor formula). Let us take a differential module \(M\) over \(\mathbb{C}[z]\) of rank \(n\) and a matrix \(G\) of \(M\) as in Proposition 3.3. Suppose that \(M\) has a singularity at \(z = 0\). Then the Milnor number of \(C_G\) at \(([\zeta : \eta], z) = ([0 : 1], 0)\) is

\[
\mu(C_G)_\infty = -n^2 - \text{Irr} \left( \text{End}_{\mathbb{C}[z]}(M) \right) + 2(n - 1)(C_G, \zeta)_\infty + (m - r_{C_G}) + 1.
\]

Here \(r_{C_G}\) is the number of branches of the germ of \(C_G\) at \(([\zeta : \eta], z) = ([0 : 1], 0)\) and \((C_G, \zeta)_\infty\) is the intersection number of \(C_G\) and \(\zeta\) at \(([\zeta : \eta], z) = ([0 : 1], 0)\).

**Proof.** First we assume that \(\frac{\partial \tilde{q}_i(z)}{\partial z} < 0\) for all \(i = 1, 2, \ldots, m\). Then the decomposition in Proposition 3.3 shows that the homogenized characteristic polynomial

\[
\zeta^n \det \left( \frac{\eta}{\zeta} I_n - G \right) = \prod_{i=1}^{m} \left( \prod_{j=1}^{r_i-1} \left( \frac{-\xi^j(\tilde{q}_i)(z)}{z} \right) \right) \cdot \prod_{j=0}^{r_i-1} \left( \zeta - \frac{z}{\xi^j(\tilde{q}_i)(z) \eta} \right)
\]

defines a reduced plane curve germ \(C_G\) at \(([\zeta : \eta], z) = ([0 : 1], 0)\) with branches \(C_{\tilde{q}_i}, i = 1, 2, \ldots, m\). Then by Propositions 3.6 and 3.7, the Milnor
number of $C_G$ is
\[
\mu(C_G)_\infty = \sum_{i=1}^m \mu(C_{q_i}) + 2 \sum_{1 \leq j < k \leq m} (C_{q_j}, C_{q_k}) - r_{C_G} + 1
\]
\[= - \sum_{1 \leq i, j \leq m} \text{Irr}(\text{Hom}_{C\{\{z\}\}}(M_{E_{q_i}(z), r_i}, M_{E_{q_j}(z), r_j}))
\]
\[+ \sum_{i=1}^m (2p_i + r_i - 1)(r_i - 1) + 2 \sum_{1 \leq j < k \leq m} (p_j r_k + p_k r_j + r_j r_k)
\]
\[- m + 1
\]
\[= -\text{Irr}(\text{End}_{C\{\{z\}\}}(M)) + \sum_{k} (2p_i r_i + r_i r_i - 2(p_i + r_i)) + m
\]
\[+ 2 \sum_{1 \leq j < k \leq m} (p_j r_k + p_k r_j + r_j r_k) - m + 1
\]
\[= -\text{Irr}(\text{End}_{C\{\{z\}\}}(M)) + 2 \sum_{i=1}^m p_i \sum_{i=1}^m r_i + \sum_{i=1}^m r_i \sum_{i=1}^m r_i
\]
\[- 2 \sum_{i=1}^m (p_i + r_i) + 1.
\]
Now let us note that $n = \sum_{i=1}^m r_i$ and
\[(C_G, \zeta) = \sum_{i=1}^m (F_i, \zeta) = \sum_{i=1}^m \text{ord}_i \frac{f_i}{q_i(t^{r_i})}
\]
\[= \sum_{i=1}^m (p_i + r_i).
\]
Then we have
\[
\mu(C_G)_\infty = -\text{Irr}(\text{End}_{C\{\{z\}\}}(M)) + 2 \sum_{i=1}^m (p_i + r_i) n + n^2
\]
\[= -2 \sum_{i=1}^m (p_i + r_i) + 1
\]
\[= -\text{Irr}(\text{End}_{C\{\{z\}\}}(M)) + 2(n - 1) \sum_{i=1}^m (p_i + r_i) - n^2 + 1
\]
\[= -n^2 - \text{Irr}(\text{End}_{C\{\{z\}\}}(M)) + 2(n - 1)(C_G, \zeta) + 1.
\]
On the other hand, let us assume that there exists $i \in \{1, 2, \ldots, m\}$ such that $\text{ord}_{\hat{q}_i(z)} \geq 0$. Then $\text{pr}^{-1}(\hat{q}_i(z)) = q_i(z)$ must be 0 and $\text{ord}_{\hat{q}_j(z)} < 0$ for the other $j \in \{1, 2, \ldots, m\}\{i\}$ because $q_j(z) \neq 0$ by the definition of HTL normal forms. We may put $i = m$ by permuting the indices if necessary.

Let us note that $m \geq 2$ in this case. If $m = 1$, then $\frac{\hat{q}_i(z)}{z} = G$. Hence $G$ has no pole at $z = 0$ and $M$ has no singularity at $z = 0$. 

14
In a way similar to the above argument, we can show that

\[
\mu(C_G) = \sum_{i=1}^{m-1} \mu(C_{q_i}) + 2 \sum_{1 \leq j < k \leq m-1} (C_{q_j}, C_{q_k}) - r C_G + 1
\]

\[
= - \sum_{1 \leq i, j \leq m-1} \text{Irr}(\text{Hom}_{C\{z\}}(M_{E_{q_i}(z), R_i}, M_{E_{q_j}(z), R_j}))
\]

\[
+ 2 \sum_{i=1}^{m-1} p_i \sum_{i=1}^{m-1} r_i + \sum_{i=1}^{m-1} r_i \sum_{i=1}^{m-1} r_i - 2 \sum_{i=1}^{m-1} (p_i + r_i) + 1.
\]

Now let us notice that

\[
\text{Irr}(\text{End}_{C\{z\}}(M)) = \sum_{1 \leq i, j \leq m-1} \text{Irr}(\text{Hom}_{C\{z\}}(M_{E_{q_i}(z), R_i}, M_{E_{q_j}(z), R_j}))
\]

\[
+ 2 \sum_{i=1}^{m-1} \text{Irr}(\text{Hom}_{C\{z\}}(M_{E_{q_{n-1}}(z), R_{n-1}}, M_{E_{q_i}(z), R_i}))
\]

\[
+ \text{Irr}(\text{End}_{C\{z\}}(M_{E_{q_{n-1}}(z), R_{n-1}}))
\]

\[
= \sum_{1 \leq i, j \leq m-1} \text{Irr}(\text{Hom}_{C\{z\}}(M_{E_{q_i}(z), R_i}, M_{E_{q_j}(z), R_j}))
\]

\[
+ 2 \sum_{i=1}^{m-1} p_i,
\]

\[
\sum_{i=1}^{m} r_i \sum_{i=1}^{m} r_i = \sum_{i=1}^{m-1} r_i \sum_{i=1}^{m-1} r_i + 2 \sum_{i=1}^{m-1} r_i + 1
\]

\[
= \sum_{i=1}^{m-1} r_i \sum_{i=1}^{m-1} r_i + 2(n - 1) + 1,
\]

\[
\sum_{i=1}^{m} (p_i + r_i) = \sum_{i=1}^{m-1} (p_i + r_i) + 1,
\]

\[
(C_G, \zeta) = \sum_{i=1}^{m} (p_i + r_i) - 1,
\]

where we use the fact \(p_m = 0\) and \(r_m = 1\). Then it follows that

\[
\mu(C_G) = -\text{Irr}(\text{End}_{C\{z\}}(M)) + 2 \sum_{i=1}^{m} p_i \sum_{i=1}^{m} r_i + \sum_{i=1}^{m} r_i \sum_{i=1}^{m} r_i
\]

\[
- 2 \sum_{i=1}^{m} (p_i + r_i) - 2(n - 1) + 2
\]

\[
= -n^2 - \text{Irr}(\text{End}_{C\{z\}}(M)) + 2(n - 1)(C_G, \zeta) + 2
\]

\[
= -n^2 - \text{Irr}(\text{End}_{C\{z\}}(M)) + 2(n - 1)(C_G, \zeta) + (m - r C_G) + 1.
\]

\[\Box\]
3.3. **Regular semisimple over \( \mathbb{C}[z] \).** In order to decompose the characteristic polynomial in accordance with the HTL normal form of the corresponding differential module, we have assumed the multiplicity free condition in Proposition 3.3. We now discuss another condition which we call regular semisimplicity over \( \mathbb{C}[z] \) and see that the previous argument is also valid under this condition.

**Definition 3.9.** Let us consider a differential module \( M \) over \( \mathbb{C}[[z]] \) of rank \( n \) with a matrix \( G \). If there exists \( X \in \text{GL}(n, \mathbb{C}[z]) \) such that

\[
XGX^{-1} + \left( \frac{d}{dz} X \right) X^{-1} = \text{diag}(q_1(z), q_2(z), \ldots, q_n(z))z^{-1}
\]

with mutually different polynomials \( q_i(z) \in \mathbb{C}[z^{-1}] \) of \( z^{-1} \), then we say that \( G \) is **regular semisimple over \( \mathbb{C}[z] \)** with the HTL normal form \( \text{diag}(q_1(z), q_2(z), \ldots, q_n(z))z^{-1} \).

**Proposition 3.10.** Let \( M \) be a differential module over \( \mathbb{C}[[z]] \) of rank \( n \) and fix a matrix \( G \in M(n, \mathbb{C}[[z]]) \) of \( M \).

Suppose that \( G \) is regular semisimple over \( \mathbb{C}[z] \) with the HTL normal form

\[
\text{diag}(q_1(z), q_2(z), \ldots, q_n(z))z^{-1}.
\]

Then the characteristic polynomial \( \det(yI_n - G) \in \mathbb{C}[[z]] \) decomposes as follows,

\[
\det(yI_n - G) = \prod_{i=1}^{n} \left( y - \frac{\tilde{q}_i(z)}{z} \right).
\]

Here \( \tilde{q}_i(z) \in \mathbb{C}[[z]] \) satisfies that

\[
\text{pr}^{\leq 0}(\tilde{q}_i(z)) = q_i(z)
\]

for each \( i = 1, 2, \ldots, n \), and \( \text{pr}^{\leq 0} : \mathbb{C}[[z]] \to \mathbb{C}[z^{-1}] \) is the projection along the decomposition \( \mathbb{C}[[z]] = \mathbb{C}[z^{-1}] \oplus z\mathbb{C}[z] \).

**Proof.** By the assumption, there exists \( X \in \text{GL}(n, \mathbb{C}[z]) \) such that

\[
XGX^{-1} + \left( \frac{d}{dz} X \right) X^{-1} = \text{diag}(q_1(z), q_2(z), \ldots, q_n(z))z^{-1}.
\]

Since \( \text{ord} \left( \frac{d}{dz} X \right) X^{-1} \geq 0 \), it follows that

\[
XGX^{-1} \equiv \text{diag}(q_1(z), q_2(z), \ldots, q_n(z))z^{-1} \pmod{\mathbb{C}[z]}.
\]

The regular semisimplicity assures that all \( q_i(z) \) are mutually different polynomials of \( z^{-1} \). Thus the result follows from same argument in Proposition 3.8. \( \square \)

The argument in Propositions 3.6, 3.7 and Theorem 3.8 is valid without any change even for this case. Thus the Milnor formula as we saw in Theorem 3.8 holds for this regular semisimple case.

**Theorem 3.11.** Let us take a differential module \( M \) over \( \mathbb{C}[[z]] \) of rank \( n \) and a matrix \( G \) of \( M \) as in Proposition 3.10. Suppose that \( M \) has a singularity at \( z = 0 \). Then the Milnor number of \( C_G \) at \( ([\zeta : \eta], z) = ([0 : 1], 0) \) is

\[
\mu(C_G)_{\infty} = -n^2 - \text{Irr} (\text{End}_{\mathbb{C}[[z]]}(M)) + 2(n - 1)(C_G, \zeta)_{\infty} + (n - r_{CG}) + 1.
\]
4. Global comparison: Euler characteristics

In the previous section, we have obtained a comparison formula of local singularities of a differential module and its characteristic polynomial. This local comparison deduces the following coincidence of the global invariants, namely we shall show the matching of the index of rigidity of a differential equation and the Euler characteristics of the corresponding spectral curve.

We now come back to the differential equation
\[ dw = Aw, \quad A \in M(n, O_X(*D)(X)), \]
on the Riemann surface \( X \). We use the same notation as in Section 11. Recall that \( D = \{ a_1, a_2, \ldots, a_p \} \) is the set of poles of \( A \). This equation defines an algebraic connection \( \nabla_A := d - A \) on the trivial algebraic vector bundle \( \mathcal{O}^\text{alg}_{U} \) where \( U := X \setminus D \). Here \( \mathcal{O}_{U, \text{alg}} \) is the sheaf of regular functions on the Zariski open subset \( U \subset X \).

Let \( \nabla_A^* \) be the dual connection of \( \nabla_A \), i.e., the dual bundle \( (\mathcal{O}^\text{alg}_{U})^* \) with the connection
\[ \nabla_A^*(\phi)(s) = -\phi(\nabla_A s) \]
where \( f \) and \( s \) are sections of \( (\mathcal{O}^\text{alg}_{U})^* \) and \( \mathcal{O}^\text{alg}_{U} \) respectively.

Further define the endomorphism connection as the tensor product,
\[ \mathcal{E}\text{nd}(\nabla_A) := \nabla_A^* \otimes \nabla_A. \]

**Definition 4.1** (index of rigidity, Katz [17]). The index of rigidity of \( \nabla_A \) is the Euler characteristic
\[ \text{rig}(\nabla_A) = \sum_{i=0}^{2} (-1)^{i} \dim_{\mathbb{C}} H^i_{\text{dR}}(X, j_* (\mathcal{E}\text{nd}(\nabla_A))). \]

Here \( j_* (\mathcal{E}\text{nd}(\nabla_A)) \) is the middle extension by the embedding \( j: U \hookrightarrow X \) of \( \mathcal{E}\text{nd}(\nabla_A) \) and \( H^i_{\text{dR}}(X, j_* (\mathcal{E}\text{nd}(\nabla_A))) \) are hypercohomology groups of the algebraic de Rham complex of \( j_* (\mathcal{E}\text{nd}(\nabla_A)) \), see II.6 in [17] and 2.9 in [15] for more detailed treatment.

The Euler-Poincare formula by Deligne, Gabber and Katz, see THEOREM 2.9.9 in [15], gives a decomposition of \( \text{rig}(\nabla_A) \) into a sum of local invariants as follows,
\[ \text{rig}(\nabla_A) = (2 - 2g) \text{rank}(\mathcal{E}\text{nd}(\nabla_A)) - \sum_{a \in D} \delta(\text{End}_{\mathbb{C}[[z]]}(M_{A_a})), \]
where
\[ \delta(\text{End}_{\mathbb{C}[[z]]}(M_{A_a})) := \text{rank} \text{End}_{\mathbb{C}[[z]]}(M_{A_a}) + \text{Irr}(\text{End}_{\mathbb{C}[[z]]}(M_{A_a})) - \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}[[z]]}((\text{End}_{\mathbb{C}[[z]]}(M_{A_a})), \mathbb{C}[[z]])^{\text{hor}}, \]
and \( M^{\text{hor}} := \{ m \in M \mid \nabla_M m = 0 \} \) for a differential module \( M \).

Take \( a \in D \) and choose a local coordinate \( (U_i, z_i) \) containing \( a \in U_i \). Then we can write \( A = A_i dz_i, A_i \in M(n, \mathbb{C}(z_i)) \) on \( U_i \). Let us put \( z_a := z_i - a \). Then power series expansion of \( A_i \), defines \( A_a \in M(n, \mathbb{C}[[z_a]]) \) and a differential module \( M_{A_a} := \mathbb{C}[[z_a]]^{\oplus n} \) with the derivation \( \nabla_{M_{A_a}} := \frac{d}{dz_a} - A_a. \)

Let us denote the point \((\xi_i : \eta_i, z_i) = ([0 : 1], a) \) by \( \infty_a \).

The following assumption enables us to apply the results in Section 3.1 to the connection \( \nabla_A \).
Assumption 4.2. For each \( a \in D \), the HTL normal form of \( M_{Aa} \) is multiplicity free or \( A_a \) is regular semisimple over \( \mathbb{C}[z_a] \).

Theorem 4.3. Under Assumption 4.2 we have the following. For each \( a \in D \) the Milnor number of \( C_A \) at \( \infty_a \) is
\[
\mu(C_A)_{\infty_a} = -\delta(\mathrm{End}_{\mathbb{C}[[z_a]]}(M_{Aa})) - r_{C_A} + 2(n-1)(C_A, X_\infty)_{\infty_a} + 1.
\]

Here \( (C, C')_{\infty_a} \) is the intersection number of divisors \( C, C' \) at \( \infty_a \).

Proof. We need to compute \( \dim \mathbb{C} \mathrm{Hom}_{\mathbb{C}[[z_a]]}(\mathrm{End}_{\mathbb{C}[[z_a]]}(\widetilde{M_{A_a}}, \mathbb{C}[[z_a]])^{\mathrm{hor}} \). Let
\[
diag(E_{q_1}^{a}(z_a), R_1^a), \ldots, E_{q_m}^{a}(z_a), R_m^a)
\]
be the multiplicity free HTL normal form of \( M_{A_a} \). Then we have the irreducible decomposition
\[
\widetilde{M_{A_a}} \cong \bigoplus_{i=1}^{m_a} M_{E_{q_i}^{a}(z_a), R_i^a}.
\]

This decomposition shows that
\[
\mathbb{C} \mathrm{Hom}_{\mathbb{C}[[z_a]]}(\mathrm{End}_{\mathbb{C}[[z_a]]}(\widetilde{M_{A_a}}, \mathbb{C}[[z_a]])^{\mathrm{hor}} \cong \mathbb{C} \mathrm{Hom}_{\mathbb{C}[[z_a]]}(\widetilde{M_{A_a}}, \widetilde{M_{A_a}})^{\mathrm{hor}}.
\]

\[
\cong \bigoplus_{i=1}^{m_a} \mathbb{C} \mathrm{Hom}_{\mathbb{C}[[z_a]]}(E_{q_i}^{a}(z_a), R_i^a)
\]

by Schur’s lemma since \( M_{E_{q_i}^{a}(z_a), R_i^a} \) are irreducible for \( i = 1, 2, \ldots, m_a \).

Then the desired equation directly comes from Theorem 3.8. \( \square \)

Remark 4.4. Let us introduce the \( \delta \)-invariant of a singularity which is defined by using the Milnor number as follows,
\[
\delta(C_{A_a}) := \frac{1}{2} \left( \mu(C_{A_a}) + r_{C_A} - 1 \right).
\]

See [22] for a geometric meaning of this invariant. Here we note that the germ \( C_{A_a} \) is reduced by the multiplicity free condition. Then the above formula can be rewritten in a natural form,
\[
2\delta(C_{A_a}) - 2(n-1)(C_A, X_\infty)_{\infty_a} = -\delta(\mathrm{End}_{\mathbb{C}[[z]]}(M_{A_a})).
\]

Theorem 4.5. We use the same notation as above. Suppose that \( \nabla_A \) satisfies Assumption 4.2 and \( C_A \) is integral and irreducible. Moreover suppose that \( C_A \) is smooth on \( T^*X \). Then the Euler characteristic \( \chi(\widetilde{C_A}) \) of the normalization \( \widetilde{C_A} \) of \( C_A \) coincides with the index of rigidity of \( \nabla_A \), i.e.,
\[
\chi(\widetilde{C_A}) = \text{rig}(\nabla_A).
\]

18
Proof. Since $C_A$ is smooth on $T^*X$, possible singularities are only on $C_A \cap X_\infty = \{ \infty_a \mid a \in D \}$. Hence the Euler characteristic $\chi(C_A)$ can be computed by the formula
\[ \chi(C_A) = (2 - 2g_a(C_A)) + 2 \sum_{a \in D} \delta(C_{Aa}), \]
see EXERCISES IV 1.8 in [9] for example. We have already computed the arithmetic genus $g_a(C_A)$ in the equation (1). Thus
\[ \chi(C_A) = n^2(2 - 2g) + (2 - 2n)(C_A, X_\infty) + 2 \sum_{a \in D} \delta(C_{Aa}). \]
Finally the formula in Remark 4.4 shows that
\[ \chi(C_A) = n^2(2 - 2g) + (2 - 2n)(C_A, X_\infty) \]
\[ = (2 - 2g)\text{rank}(\text{End}(\nabla_A)) - \sum_{a \in D} \delta(\text{End}_{\mathbb{C}(\{z_a\})(M_{Aa}))} = \text{rig}(\nabla_A). \]

Corollary 4.6. Let $\nabla_A$ be as in Theorem 4.5 and moreover assume that $\nabla_A$ is irreducible. Then we have the following numerical coincidences of the cohomology groups,
\[ h_i^{\text{dR}}(X, j_* (\text{End}\nabla_A)) = h_i(C_A, \mathbb{C}), \quad i = 0, 1, 2. \]
Here $h^i(*) := \dim \mathbb{C}H^i(*)$.

Proof. By the irreducibility and duality, we have
\[ h_0^{\text{dR}}(X, j_* (\text{End}\nabla_A)) = h_2^{\text{dR}}(X, j_* (\text{End}\nabla_A)) = 1. \]
which shows that
\[ h_i^{\text{dR}}(X, j_* (\text{End}\nabla_A)) = 1 = h_i(C_A, \mathbb{C}) \]
for $i = 0, 2$. Thus Theorem 4.5 implies that
\[ h_1^{\text{dR}}(X, j_* (\text{End}\nabla_A)) = 2 - \text{rig} \nabla_A = 2 - \chi(C_A) = h_1(C_A, \mathbb{C}). \]

References
[1] D. Arinkin, Fourier transform and middle convolution for irregular D-modules. Preprint arXiv:0808.0699 2008, 29 pp.
[2] D. Babbitt, V. Varadarajan, Formal reduction theory of meromorphic differential equations: a group theoretic view. Pacific J. Math. 109 (1983), no. 1, 1–80.
[3] W. Balser, Formal power series and linear systems of meromorphic ordinary differential equations. Universitext. Springer-Verlag, New York, 2000. xviii+299 pp.
[4] O. Biquard, P. Boalch, Wild non-abelian Hodge theory on curves. Compos. Math. 140 (2004), no. 1, 179–204.
[5] S. Bloch, H. Esnault, Local Fourier transforms and rigidity for $\mathcal{D}$-modules. Asian J. Math. 8 (2004), no. 4, 587–605.
[6] P. Deligne, La formule de Milnor, Groupes de Monodromie en Géométrie Algébrique, SGA 7II, Springer Lecture Note in Mathematics 340 (1973), 197–211.
[7] P. Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics 163 (1970).
[8] O. Dumitrescu, M. Mulase, Quantization of spectral curves for meromorphic Higgs bundles through topological recursion. *Topological recursion and its influence in analysis, geometry, and topology*, 179–229, Proc. Sympos. Pure Math., 100, Amer. Math. Soc., Providence, RI, 2018.

[9] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, xvi+496 pp.

[10] A. Hefez, Irreducible plane curve singularities. Real and complex singularities, 1–120. *Lecture Notes in Pure and Appl. Math.*, 232, Dekker, New York, 2003.

[11] N. Hitchin, The self-duality equations on a Riemann surface. *Proc. London Math. Soc.* (3) 55 (1987), no. 1, 59–126.

[12] M. Hukuhara, Sur les points singuliers des équation différentielles linéaires. III., *Mem. Fac. Sci. Kyushu Imp. Univ. A.*, 2 (1942), 125–137.

[13] S. Kamimoto, A relation between structures of turning points and singular points, personal note,(2013).

[14] N. Katz, On the calculation of some differential Galois groups, *Invent. Math.* 87 (1987), 13–61.

[15] N. Katz, Exponential sums and differential equations. *Annals of Mathematics Studies*, 124. Princeton University Press, Princeton, NJ, 1990. xii+430 pp.

[16] N. Katz, Exponential sums over finite fields and differential equations over the complex numbers: some interactions, *Bull. AMS* 23 No.2 (1990), 269–309.

[17] N. Katz, Rigid local systems, *Annals of Mathematics Studies*, vol. 139, Princeton University Press, 1996.

[18] H. Komatsu, On the index of ordinary differential operators, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 18 (1971), 379–398.

[19] M. Kontsevich, Y. Soibelman, Wall-crossing structures in Donaldson-Thomas invariants, integrable systems and Mirror Symmetry, Preprint, arXiv:1303.3253, 2013, 111 pp.

[20] A. Levelt, Jordan decomposition for a class of singular differential operators, *Ark. Mat.* 13 (1975), 1–27.

[21] B. Malgrange, Remarques sur les équations différentielles à points siguliers irréguliers, *Lecture Notes in Mathematics* 712, Springer-Verlag 1979, 77–86.

[22] J. Milnor, Singular points of complex hypersurfaces. *Annals of Mathematics Studies*, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968 iii+122 pp.

[23] T. Mochizuki, Wild harmonic bundles and wild pure twistor D-modules, *Astérisque* 340, Société Mathématique de France, 2011.

[24] N. Nitsure, Moduli space of semistable pairs on a curve. *Proc. London Math. Soc.* (3) 62 (1991), no. 2, 275–300.

[25] T. Oshima, Complex linear ordinary differential equations on the Riemann sphere and multivariable hypergeometric functions (in Japanese), *Proceedings of 14th Oka Symposium*, 2016.

[26] C. Sabbah, An explicit stationary phase formula for the local formal Fourier-Laplace transform. Singularities I, 309–330, *Contemp. Math.*, 474, Amer. Math. Soc., Providence, RI, 2008.

[27] C. Simpson, Higgs bundles and local systems, *Inst. Hautes Etudes Sci. Publ. Math.* (1992), no. 75, 5–95.

[28] S. Szabó, The birational geometry of unramified irregular Higgs bundles on curves. (English summary) *Internat. J. Math.* 28 (2017), no. 6, 1750045, 32 pp.

[29] H. Turritin, Convergent solutions of linear homogeneous differential equations in the neighborhood of an irregular singular point, *Acta. Math.* 93, (1955), 27–66.

[30] C. Wall, Singular points of plane curves. *London Mathematical Society Student Texts*, 63. Cambridge University Press, Cambridge, 2004. xii+370 pp.

E-mail address: kazuki@math.s.chiba-u.ac.jp

DEPARTMENT OF MATHEMATICS AND INFORMATICS, CHIBA UNIVERSITY, 1-33, YAGOCHO, INAGE-KU, CHIBA-SHI, CHIBA, 263-8522 JAPAN