Quines are the fittest programs
Nesting algorithmic probability converges to constructors

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Abstract—In this article we explore the limiting behavior of the universal prior distribution applied over multiple meta-level hierarchy of program and output data of a Turing machine. We were motivated to reduce the effect of Solomonoff’s assumption that all computable functions/hypothesis of the same length are equally likely, by weighing each program in turn by the algorithmic probability of their description number encoding. In the limiting case we converge the set of all possible program strings of a fixed-length to a distribution of self-replicating quines and quine-relays - having the structure of a constructor. We discuss how experimental algorithmic information theory provides insights towards understanding the fundamental metrics proposed in this work and reflect on the significance of these result in the constructor theory of life.

I. INTRODUCTION

A dichotomy exist in algorithmic information theory, where, the length of the data string is considered a false indication of its complexity, while it uses the length of programs to calculate the likelihood of the occurrence of the data string. While short programs are more likely to produce a data, it is not considered how likely is the short program to be generated by another higher level program.

We consider a fixed length model, where the data, the program, and all higher level programs have the same length, and are input programs or outputs of the same universal Turing machine. We explore the properties of the final distribution of the data string provided we consider multiple levels of meta-programs.

In Section II we provide the background of Solomonoff’s algorithmic probability and the coding theorem method used to estimate it. Thereafter, in Section III we present the idea and derive the distribution occuring by nesting algorithmic probability over multiple levels of a universal Turing machine, and the idea of an universal constructor in Section IV. In Section V the limiting behavior of this distribution is analysed in depth. We extensively consider a specific case in Section VI as an experiment. We conclude this article in Section VII with discussion on our results and possibilities of future research.

II. ALGORITHMIC PROBABILITY

The universal Solomonoff algorithmic probability \( \xi \) of a program \( p \) on a universal Turing machine (UTM) \( U \) for an output \( x \) is:

\[
\xi(x) = \sum_{p: U(p) = x} 2^{-|p|}
\]

This naturally formalizes Occam’s razor (law of parsimony) and Epicurus’s principle of multiple explanations by assigning larger prior credence to theories that require a shorter algorithmic description.

Consider a Turing machine with \( n \) symbols and \( m \) states, which can be enumerated by running a finite number of programs. The algorithmic probability of a string \( x \) can be approximated as:

\[
\xi(x) \approx D_{n,m}(x) = \frac{|U \text{ outputs } x \text{ and halts}|}{|U \text{ halts}|}
\]

i.e. counting how many programs produce the given output divided by the total number of programs that halt. This technique is called the Coding Theorem Method (CTM) \([2]\).

Approximating \( \xi(x) \) using CTM, although theoretically computable, is extremely expensive in terms of computation time. The space of possible Turing machines may span thousands of billions of instances. Moreover, being uncomputable, every Turing machine in-principle needs to be run forever to know if it will halt. However, for small Turing machines, it is possible to judiciously stop the computation and declare the enumeration as non-halting if it hasn’t halted within a maximum time step (called the Busy Beaver runtime). However, for 5 states the value is still unknown as there are over 26 trillion cases to enumerate and from partial enumerations so far, we know the value is \( \geq 47176870 \). It is intractable to run so many machines for so long iterations, thus, often the CTM is estimated from a time limited by running the TM for \( t \) cycles. This is based on \([3]\) and hypothesizes an exponential decay \([4]\) in the number of halting TM with run duration.

We do not consider a special halt state thus allowing us to explore the complete state space of program \([5]\). This would also includes programs with a halt state by encoding them as states that loop on themselves, moves the tape head arbitrarily and writes back the read character. This is further motivated by the naturally occurring computing hardware like DNA or the brain, having a fixed resource for storing the program, e.g. the number of base-pairs or the number of neurons. We are interested in the distribution of the computing output generated by a set of fixed size programs. Thus the related concept of Kolmogorov complexity doesn’t have much significance, considering the length is fixed.

Let the description number of \( n \) symbols and \( m \) states be encoded as binary strings of length \( l \). Thus, all \( 2^l \) possible programs have length \( |p| = l \), and when run for \( t \) time steps
Now we can pose the question: why should all programs be considered equiprobably? Considering a higher hierarchy of meta-programming, there is some physical process that generates that program on the program part of the tape of an UTM. Normally, for the standard definition of algorithmic probability, it is considered an unbiased coin flip, or the infinite programming monkey theorem. Since the universal distribution is not known apriori, there is no other preference than an uniform distribution to start with.

However, we can feedback the universal distribution on the programs and understand what effect it has on the data. Thus introducing a hierarchy of UTM levels, where the output data of one UTM is the input program for the lower levels. Weighting the contribution of each program based on the probability that they themselves physically occur on the program part of the UTM tape, we get:

$$\xi^0_{U_0 U_1}(x_i^0) = \sum_{x^2:U(x^2)\rightarrow x^1} 2^{-|x^2|} \xi^2_{U_2}(x^2)$$

Assuming, $U_1 = U_2 = U$, in our model motivated from the invariant theorem

$$\xi^0_{x_i^0} = \sum_{x^2:U(x^2)\rightarrow x^1} 2^{-|x^2|} \sum_{x^2:U(x^2)\rightarrow x^1} 2^{-|x^2|} \xi^0_{x_2^0}$$

Now consider these 3 programs would print the output and reach an attractor state early. Matters to us is the frequency of programs (as in the CTM), not at what stage in the computation of our t steps it reached a stable attractor state. So even within same length programs we expect an non-uniform distribution as we do for the data.

**(Fig. 2)** 2 level Nesting of algorithmic probability

Considering the constant length $|x_k^2| = |x_k^3| = |x_i^0| = l$, the equation for $\xi^0_{x_i^0}$ equals:

$$2^{-2l} |x_1^2:U(x_1^2)\rightarrow x_i^0| |x_2^2:U(x_2^3)\rightarrow (x_j^1:U(x_j^2)\rightarrow x_i^0)|$$

Two levels of hierarchy is shown in Fig. 2. Indeed it is possible to extend this argument to arbitrary many levels, with no particular reason to choose $\xi^0_{x_i^0}$ over $\xi^0_{x_i^0}$ for the expected distribution of the data occurring physically. We are interested in two specifics of $w$ approaching a large number. Let,

$$A^w = |x^w : U(x_k^w) \rightarrow (x_j^{w-1} : U(x_j^{w-1}) \rightarrow x_i^{w-2})|$$

and,

$$B^w = |x^{w-1} : U(x_j^{w-1}) \rightarrow x_i^{w-2})|$$

Since, $A \leq B$, for all $w$, what are the properties of the strings that ‘survive’ over these generations? This is explored in Section \[V\]
IV. Universal Constructors

Universal constructor is a self-replicating machine foundational in automata theory, complex systems and artificial life. John von Neumann was motivated to study abstract machines which is complex enough such that it could grow/evolve like biological organisms. The simplest such machine when executed should at least replicate itself.

The design of a self-replicating machine consists of:
- a program or description of itself
- a universal constructor mechanism that can read any description and construct the machine description encoded in that description
- a universal copy machine that can make copies of any description (if this allows mutating the description it is possible to evolving to a higher complexity)

Additionally, it might have a overall operating system (which can be part of the world rule or compiler) and additional functions as payloads. This is shown in Fig. 3. The payload can be very complex like a learning agent like AIXI or an instance of an evolving neural network.

![Fig. 3. John von Neumann’s system of self-replicating automata](image)

It is generally run in 2 steps: first the universal constructor is used to construct a new machine encoded in the description (thereby interpreting the description as program), then, the copy machine is used to create a copy of that description in the new machine (thereby interpreting the description as data). This is similar to the DNA translation and DNA replication process respectively. The cell’s dynamics is the operating system which also performs the metabolism as the extra functions when it is not reproducing.

Von Neumann designed an universal constructor in a 2-dimensional cellular automata with 29 states. If we consider the Turing machine automata, this translates to printing out the description on the tape which can be executed as a program by another Turing machine. Note that the rest of the Turing machine mechanism like the tape head and movement are akin to the underlying cellular automata rules that automatically apply to the new cells where the replicate machine manifests.

The very design of the 3 parts of a constructor suggests that it cannot be algorithmic random as it should be possible to compress parts of its description. We are interested in the complexity and probability of constructors, which forms a subset of all possible configurations a automata can possess.

V. Fitness of Quines

Quines are programs which takes no input and produces a copy of its own source code as its output. It may not have other useful output. In computability theory, such self-replicating (self-reproducing or self-copying) programs are fixed points of an execution environment, as a function transforming programs into their outputs. The quine concept can be extended to multiple levels of recursion, called ouroboros programs or quine-relays. Quines are also a limiting case of algorithmic randomness as their length is same as their output.

In our model we consider the entire set of $2^l$ strings. Each string is represented by a node in Fig. 4 with the arrows representing the mapping by running the string as a program on the UTM. Thus note that, while many-to-one arrows are possible, one-to-many is not. We partition the set into 2 types: attractors and repellers. The entire space might have multiple connected components. Each connected component consists of an attractor basin, made of quines or quine-relays, as shown in the right side of the red line in Fig. 4. Each node in the attractor basin might have a trail of repeller nodes, which, over cycles of algorithmic probability converge to the node on the basin.

![Fig. 4. Attractor and repeller strings](image)

Over multiple cycles the final set will only include the nodes on the right, with one-to-one mapping, thus conserving the number of strings in subsequent cycles. We denote this specific number of cycles with $M$, the meta-level at which a uniform distribution results in only attractors after $M$ cycles. The number of attractors is given by

$$Q = |x^M: \xi^0M(x_i^M) \neq 0|$$

At this stage, each string has a one-to-one mapping. The algorithmic probability for these constructor strings depends on the number of paths leading to these attractor basis over these cycles. $M$ is dependent on the number of considered state and symbols, the specification of the Turing machine, the length of the programs and the time approximation for estimating the algorithmic probability at each level.

$M$ being at least as (semi-)uncomputable as $\xi$, we can only hope to study these fundamental metrics under reasonable approximations via Experimental Algorithmic Information Theory (EAIT) [8, 9].
VI. Experiments

Here, we consider a particular case to illustrate the developed formalism of nesting algorithmic probability. We take the 2 state 2 symbol Turing machine as it is both non-trivial as well as within the bounds of exhaustive enumeration. The details of this machine can be found in [5]. The program (description number) is encoded as:

\[
[QMW]^Q_1R_1[QMW]^Q_1R_0[QMW]^Q_0R_1[QMW]^Q_0R_0
\]

This gives the values \(q_\delta = 12\), as the length of the description number required to store a program for this machine. Thus, the space of this encoding allows \(P = 2^{12} = 4096\) possible programs. The tape is also of length \(c = z = 12\) and consists of all zeros with the tape head on the left most character: 000000000000. The machine is run for \(z = q_\delta = 12\) iteration. A Python script that emulates this restricted model of the Turing machine for all 4096 cases is available at [10].

At level 3, we start with a uniform distribution of all programs, to produce the standard universal distribution. The mapping is shown in Fig. 5 (the high resolution svg is available at [10]).

![Fig. 5. Level 3 (the high resolution svg is available at [10)](image)

We observe that, at this level itself, the number of possible programs for the next generation gets reduced from 4096 to 21, with the following frequency of occurrence.

- P0 : 1886
- P4095 : 640
- P1365 : 64
- P2730 : 64
- P3840 : 17
- P3968 : 11
- P3584 : 10
- P1792 : 2
- P2560 : 2
- P192 : 1
- P2944 : 1

Also, at this level itself we find that the machines P0 and P4095 are self-replicating, and can already predict the limiting behaviour.

At level 2, these 21 programs gets further mapped to just 3 programs, with the following frequency.

- P0 : 16
- P4095 : 3
- P2048 : 2

This is shown in Fig. 6.

![Fig. 6. Level 2](image)

At level 1, the 3 programs finally converge to the quines P0 and P4095 as shown in Fig. 7.

![Fig. 7. Level 1](image)

The frequency is as follows:

- P0 : 2
- P4095 : 1

At level 0, we reach the attractor states with a uniform one-to-one mapping.

![Fig. 8. Level 0](image)

So, to calculated the overall algorithmic probability of these two fixed points, we calculate the cumulative frequency over these 4 cycles. Thus, at level 2, the total frequency of P4095 consists of adding up the frequency of Programs P4095, P2047, P4032 from the previous step, totaling to 640 + 64 + 10 = 714. For P2048, we add up the frequency of
Programs $P_{1365}, P_{1344}$, totaling to $64 + 10 = 74$. The rest of the 16 programs total to 3308 cases that reach $P_0$. At level 1, $P_{2048}$ also reach the $P_0$ attractor, giving the final algorithmic probability of the quines:

$$P_0 : 3382 \quad P_{4095} : 714$$

A similar experimentation on this space of program for the 2-2-1 Turing machine is conducted as part of the Wolfram Physics Project exploring the rulial space of Turing machines \[11\].

The 2-2-1 UTM with 4096 programs does not show much diversity and the quines are simple. It remains to be seen what the encodings of quines in larger spaces reveal about the structure and complexity of constructors. The 2 symbol 4 state UTM has $2^{32}$ programs making an exhaustive enumeration for each over at least $2^{32}$ time steps (so that the write head’s causal cone covers the full tape) difficult. We need to look at alternative approaches (e.g. using quantum computation) to tame this overwhelmingly large state space.

**VII. DISCUSSIONS**

In this research we extend the idea of algorithmic probability to meta-programming under the assumption of fixed length. We present a mathematical formulation and derive the properties of this distribution analytically. The dynamics of the space of strings show some interesting properties: starting from an uniform distribution of programs, converges to an distribution of constructors. The number of cycles for this convergence, the space of attractor strings, and their behavior with respect to larger states and length are fundamental metrics in the space of programs \[12\], reminiscence of Chaitin’s Omega number and Kleene’s recursion theorems.

The idea of using the fixed-point, called the Y-combinator in lambda calculus $\lambda f. (\lambda x. f(xx))(\lambda x. f(xx))$ to describe the genetic code \[13\] is pioneered by Gregory Chaitin \[14\] as the field meta-biology. This is in line with the constructor theory \[15\] approach for understanding physical transformations in biology. In the field of transcendental/recreational programming the DNA structure can be used to code the Gödel number (similar to the description number) of any Ruby script \[16\]. In our future research, we intend to explore the signification of these results for artifical life applications \[17\], \[18\] in synthetic biology. The space and probability of constructors \[19\] can inform the subset of DNA encoding for in vitro experimentation.

**REFERENCES**

[1] Ray J Solomonoff. A formal theory of inductive inference. part i. Information and control, 7(1):1–22, 1964.
[2] Leonid Anatolevich Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. Problemy Peredači Informacii, 10(3):30–35, 1974.
[3] Cristian S Calude and Michael A Stay. Most programs stop quickly or never halt. arXiv preprint cs/0610153, 2006.
[4] Fernando Soler-Toscano, Hector Zenil, Jean-Paul Delahaye, and Nicolas Gauvrit. Calculating kolmogorov complexity from the output frequency distributions of small turing machines. PloS one, 9(5), 2014.
[5] Aritra Sarkar, Zaid Al-Ars, and Koen Bertels. Quantum accelerated estimation of algorithmic information. arXiv preprint arXiv:2006.00987, 2020.
[6] Marcus Hutter. Universal artificial intelligence: Sequential decisions based on algorithmic probability. Springer Science & Business Media, 2004.
[7] Kenneth O Stanley, Jeff Clune, Joel Lehman, and Risto Miikkulainen. Designing neural networks through neuroevolution. Nature Machine Intelligence, 1(1):24–35, 2019.
[8] Jean-Paul Delahaye and Hector Zenil. On the kolmogorov-chaitin complexity for short sequences. ArXiv, abs/0704.1043, 2007.
[9] Hector Zenil. Experimental algorithmic information theory — complexity and randomness. http://www.mathrix.org/experimentalAIT/ 2020.
[10] Aritra Sarkar. Advanced-research-centre/QuBio. https://github.com/Advanced-Research-Centre/QuBio/tree/master/Project_01/classical/ 2020.
[11] Stephen Wolfram. Exploring rulial space: The case of turing machines. https://writings.stephenwolfram.com/2020/06/explo... 2020.
[12] Stephen Wolfram. A new kind of science, volume 5. Wolfram media Champaign, IL, 2002.
[13] Gregory J Chaitin. Meta math! the quest for omega. arXiv preprint math/0404335, 2004.
[14] Gregory Chaitin. Proving Darwin: making biology mathematical. Vintage, 2012.
[15] Chiara Marletto. Constructor theory of life. Journal of The Royal Society Interface, 12(104):20141226, 2015.
[16] Yusuke Endoh. mame/doublehelix. https://github.com/mame/doubleheli... 2020.
[17] Vincent Noeaux, Yusuke T Maeda, and Albert Libchaber. Development of an artificial cell, from self-organization to computation and self-reproduction. Proceedings of the National Academy of Sciences, 108(9):3473–3480, 2011.
[18] Anne Condon, Hélène Kirchner, Damien Lariiviére, Wallace Marshall, Vincent Noireaux, Tsvi Tlusty, and Eric Fourmentin. Will biologists become computer scientists? EMBO reports, 19(9):e46628, 2018.
[19] Moshe Sipper and James A Reggia. Go forth and replicate. Scientific American, 285(2):34–43, 2001.