Reduced critical processes for small populations

Minzhi Liu 1 2
Vladimir Vatutin 3

Abstract

Let \{Z(n), n \geq 1\} be a critical Galton-Watson branching process with finite variance for the offspring size of particles. Assuming that 0 < Z(n) \leq \varphi(n), where either \varphi(n) = an for some a > 0 or \varphi(n) = o(n) as n \to \infty, we study the structure of the process \{Z(m,n), 0 \leq m \leq n\}, where Z(m,n) is the number of particles in the process at moment m \leq n having a positive number of descendants at moment n.

Keywords: critical branching process, reduced processes, conditional limit theorem;
Mathematics Subject Classification: Primary 60J80; secondary 60G50.

1 Introduction and main results

Let \{Z(n), n \geq 0\} be a Galton-Watson branching process with Z(0) = 1 in which particles produce children in accordance with probability generating function

\[ f(s) = \mathbb{E}s^ζ = \sum_{k=0}^{\infty} f_k s^k \]

and let Z(m,n) be the number of particles in the process at moment m \leq n having a positive number of descendants at moment n. The process \{Z(m,n), 0 \leq m \leq n\} is called a reduced process.

Reduced processes for ordinary Galton–Watson branching processes were introduced by Fleischmann and Prehn 5, who discussed the subcritical case. The distance to the most recent common ancestor (MRCA) for the supercritical Galton–Watson processes and for the critical processes with possibly infinite variance of the offspring size has been investigated by Zubkov 18. Fleischmann and Siegmund-Schultze 6 proved a functional conditional limit theorem establishing, under the condition \{Z(n) > 0\} convergence of the reduced critical Galton–Watson branching process to the Yule process. Different questions related to the problem of the distribution of the MRCA for the k particles selected at random among the Z(n) ≥ k particles existing in the population at moment n were considered, for instance, in [1], [2], [3], [7]–[13].

However, all these papers do not consider the situation when the size of the population at moment n is bounded from above. In the present paper, we study the structure of a critical reduced process and investigate the asymptotic behavior of the number of its particles under the condition that the size of the population is bounded and positive at the moment of observation. Note that the critical Galton-Watson process given its extinction moment is fixed was investigated in [14] for the single-type case and in [17] for the multitype setting.

It is known (see, for instance, [3], Chapter I, Section 9 or [15], Chapter II, Section 5) that if \[ \mathbb{E}ξ = 1, \quad 2B := \text{Var}ξ \in (0, \infty), \] \[ \text{(1.1)} \]
then
\[ Q(n) := P(Z(n) > 0) \sim \frac{1}{Bn} \quad \text{as} \quad n \to \infty \]  
(1.2)
and, for any \( y \geq 0 \)
\[ \lim_{n \to \infty} P \left( \frac{Z(n)}{Bn} \leq y | Z(n) > 0 \right) = 1 - e^{-y}. \]  
(1.3)
In addition (see [6]), for any fixed \( t \in [0, 1) \) and all \( s \in [0, 1] \)
\[ \lim_{n \to \infty} E \left[ s^{Z(n)} | Z(n) > 0 \right] = s^{1-t}. \]  
(1.4)

In this note we study the asymptotic properties of the reduced process when the condition \( \{ Z(n) > 0 \} \) is replaced either by the assumption that \( \{ 0 < Z(n) \leq B \varphi(n) \} \) for a function \( \varphi(n) = o(n) \) as \( n \to \infty \) or by the assumption that \( \{ 0 < Z(n) \leq aBn \} \) for some \( a > 0 \). Our main results are contained in two theorems which we formulate below.

**Theorem 1.1** If g.c.d.\( \{ k : f_k > 0 \} = 1 \), condition (1.1) is valid, and \( \varphi(n) \to \infty \) in such a way that \( \varphi(n) = o(n) \), then for any \( x \in (0, \infty) \)
\[ \lim_{n \to \infty} E \left[ s^{Z(n-x\varphi(n), n)} | 0 < Z(n) \leq B \varphi(n) \right] = \frac{s^{1-t} - e^{-(1-s)/x}}{1-s}. \]  

Let
\[ \beta(n) := \max \{ 0 \leq m < n : Z(m, n) = 1 \} \]
be the birth moment of the MRCA of all particles existing in the population at moment \( n \) and let \( d(n) := n - \beta(n) \) be the distance from the point of observation \( n \) to the birth moment of the MRCA.

**Corollary 1.2** Under the conditions of Theorem 1.1
\[ \lim_{n \to \infty} P \left( d(n) \leq x\varphi(n)| 0 < Z(n) \leq B \varphi(n) \right) = x \left( 1 - e^{-1/x} \right). \]

To give a complete description of possible situations we present the following statement.

**Theorem 1.3** If g.c.d.\( \{ k : f_k > 0 \} = 1 \) and condition (1.1) is valid, then, for any fixed \( t \in [0, 1) \) and any \( a > 0 \)
\[ \lim_{n \to \infty} E \left[ s^{Z(n)-a} | 0 < Z(n) \leq aBn \right] = s^{1-t} - \frac{1-e^{-(1-ts)a/(1-t)}}{1-e^{-a}}. \]

**Corollary 1.4** Under the conditions of Theorem 1.3
\[ \lim_{n \to \infty} P \left( d(n) \leq tn| 0 < Z(n) \leq aBn \right) = \frac{1 - e^{-a/t}}{1 - e^{-a}}. \]
2 Proof of Theorem 1.1

For convenience of references we recall Faà di Bruno’s formula for the derivatives of composite functions:

If \( i_r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, r = 1, 2, ..., k, I_k := i_1 + \cdots + i_k \) and

\[
\mathcal{D}(k) := \{(i_1, ..., i_k) : 1 \cdot i_1 + 2 \cdot i_2 + \cdots + k i_k = k\},
\]

then

\[
\frac{d^k}{dz^k} [F(G(z))] = \sum_{\mathcal{D}(k)} \frac{k!}{i_1! \cdots i_k!} F(I_k)(G(z)) \prod_{r=1}^{k} \left( \frac{G^{(r)}(z)}{r!} \right)^{i_r}. 
\]

We split the proof of Theorem 1.1 into several lemmas.

Let

\[
f_0(s) := s \quad \text{and} \quad f_{n+1}(s) := f(f_n(s)), n \geq 0.
\]

Below for arbitrary \( x > 0 \) we agree consider \( f_{xn}(s) \) as \( f_{\lfloor xn \rfloor}(s) \). Besides, the symbol \( \sim \) will be usually used (if no otherwise is stated) for \( n \to \infty \).

**Lemma 1** If condition (1.1) is valid then, for any fixed \( k \in \mathbb{N} := \{1, 2, ...\} \) and any fixed \( x \in (0, \infty) \)

\[
f_{n}^{(k)}(f_{xn}(0)) \sim \frac{k! x^2 (Bxn)^{k-1}}{(x + 1)^{k+1}} \quad \text{as} \quad n \to \infty. \tag{2.1}
\]

**Proof.** In view of (1.2) we may rewrite (1.3) in terms of probability functions and Laplace transforms as follows: for any \( \lambda > 0 \)

\[
\lim_{n \to \infty} B_{n} \left( f_{n}(f_{\lambda xn}(0)) - f_{n}(0) \right) = \frac{1}{1 + \lambda}. \tag{2.2}
\]

Since

\[
\log f_{xn}(0) \sim -(1 - f_{xn}(0)) \sim -1/Bxn \quad \text{as} \quad n \to \infty, \tag{2.3}
\]

we conclude that

\[
\lim_{n \to \infty} B_{n} \left( f_{n}(f_{\lambda xn}(0)) - f_{n}(0) \right) = \frac{x}{x + \lambda}.
\]

Clearly, the prelimiting and limiting functions in the previous relations are analytical in the complex semi-plane \( \text{Re} \lambda > 0 \). Therefore, the derivatives of any order of the prelimiting functions converge to the respective derivatives of the limiting function for each \( \lambda \) with \( \text{Re} \lambda > 0 \). Thus,

\[
\lim_{n \to \infty} B_{n} \frac{d^k f_{n}(f_{\lambda xn}(0))}{d\lambda^k} = (-1)^k \frac{k! x}{(x + \lambda)^{k+1}}. \tag{2.4}
\]

In particular,

\[
B_{n} \frac{df_{n}(f_{\lambda xn}(0))}{d\lambda} = B_{n} f_{n}'(f_{\lambda xn}(0)) f_{\lambda xn}(0) \log f_{xn}(0) \sim (1) \frac{1! x}{(x + \lambda)^2}.
\]
Hence, setting $\lambda = 1$ and taking into account (2.3) we conclude that

$$f'_n(f_{xn}(0)) \sim \frac{1}{(x+1)^2},$$

proving the lemma for $k = 1$. Assume that (2.1) is proved for all $k < j$. Since

$$\frac{d^r}{d^r \lambda} f^\lambda_{xn}(0) = f^\lambda_{xn}(0) \log f^\lambda_{xn}(0), r = 1, 2, \ldots,$$

using Faà di Bruno's formula and induction hypothesis we get

$$B_n \frac{d^j f_n(f^\lambda_{xn}(0))}{d^j \lambda} = B_n \sum_{D(j)} \frac{j!}{i_1! \cdots i_j!} f^{(i_j)}_{n}(f^\lambda_{xn}(0)) \prod_{r=1}^{j} \left( \frac{1}{r!} \frac{d^r}{d^r \lambda} f^\lambda_{xn}(0) \right)^{i_r}$$

$$= B_n \log f_{xn}(0) \sum_{D(j)} \frac{j!}{i_1! \cdots i_j!} f^{(i_j)}_{n}(f^\lambda_{xn}(0)) \prod_{r=1}^{j} \left( \frac{1}{r!} f^\lambda_{xn}(0) \right)^{i_r}$$

$$\sim (-1)^j B_n \frac{d^j f_n(f^\lambda_{xn}(0))}{(B_{xn})^j} \sim (-1)^j \frac{j! x}{(x+\lambda)^{j+1}}.$$

Hence, setting $\lambda = 1$ we obtain

$$f^{(j)}_{n}(f_{xn}(0)) \sim \frac{j! x^2 (B_{xn})^{j-1}}{(x+1)^{j+1}}$$

justifying the induction step.

Lemma 1 is proved.

Lemma 2 If condition (1.1) is valid, $m = n - x \varphi(n)$, where $x \in (0, \infty)$ and $\varphi(n) = o(n)$ as $n \to \infty$, then, for any fixed $j \in \mathbb{N} := \{1, 2, \ldots\}$

$$f^{(j)}_{m}(f_{x\varphi(n)}(0)) \sim \frac{j! (B x \varphi(n))^{j+1}}{B^2 n^2} \text{ as } n \to \infty.$$

Proof. It is known (see, for instance [3], Chapter 1, Section 9, Corollary 1) that under condition (1.1)

$$\lim_{n \to \infty} n^2 [f_{n+1}(0) - f_n(0)] = \frac{1}{B}.$$

We consider for $\lambda > 0$ the function

$$f_m(f^\lambda_{x\varphi(n)}(0)) = f_m(e^{\lambda \log f_{x\varphi(n)}(0)})$$

and find $r$ such that

$$1 - f_{r+1}(0) < 1 - f^\lambda_{x\varphi(n)}(0) \leq 1 - f_r(0).$$

In view of (1.2) we know that

$$1 - f^\lambda_{x\varphi(n)}(0) \sim \lambda (1 - f_{x\varphi(n)}(0)) \sim \frac{\lambda}{B x \varphi(n)}.$$
Hence we get
\[ r \sim \frac{x \varphi(n)}{\lambda} = o(n) \text{ as } n \to \infty. \]

Then for \( n - m = x \varphi(n) \)
\[
\lim_{n \to \infty} \frac{n^2}{x \varphi(n)} \left[ f_m(f_r(0)) - f_m(0) \right]
= \lim_{n \to \infty} \frac{1}{x \varphi(n)} \sum_{k=0}^{r-1} n^2 \left[ f_m(f_{k+1}(0)) - f_m(f_k(0)) \right]
= \lim_{n \to \infty} \frac{1}{x \varphi(n)} \sum_{k=0}^{r-1} \frac{n^2}{(m+k)^2} (m+k)^2 \left[ f_{m+k+1}(0) - f_{m+k}(0) \right]
= \frac{1}{B} \lim_{n \to \infty} \frac{1}{x \varphi(n)} \sum_{k=0}^{r-1} 1 \frac{1}{m} = \frac{1}{B \lambda}.
\]

Thus,
\[
\lim_{n \to \infty} \frac{n^2}{x \varphi(n)} \left[ f_m(e^\lambda \log f_{x \varphi(n)}(0)) - f_m(0) \right] = \frac{1}{B \lambda}, \quad \lambda > 0. \tag{2.4}
\]

According to the similar reason in the proof of Lemma 1, we have that for each \( k \geq 1 \)
\[
\lim_{n \to \infty} \frac{Bn^2}{x \varphi(n)} \frac{d^k}{d \lambda^k} F_m(e^\lambda \log f_{x \varphi(n)}(0)) = (-1)^k \frac{k!}{\lambda^{k+1}}. \tag{2.5}
\]

By Faà di Bruno’s formula we have
\[
\frac{d^k}{d \lambda^k} \left[ F_m(e^\lambda \log f_{x \varphi(n)}(0)) \right]
= \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} f_m^{(i_k)}(e^\lambda \log f_{x \varphi(n)}(0)) \prod_{r=1}^{k} \left( \frac{\log f_{x \varphi(n)}(0)}{r!} \right)^{i_r}
= \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} f_m^{(i_k)}(e^\lambda \log f_{x \varphi(n)}(0)) e^{\lambda i_k \log f_{x \varphi(n)}(0)} \prod_{r=1}^{k} \left( \log f_{x \varphi(n)}(0) \right)^{i_r}
= (\log f_{x \varphi(n)}(0))^k \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} f_m^{(i_k)}(e^\lambda \log f_{x \varphi(n)}(0)) e^{\lambda i_k \log f_{x \varphi(n)}(0)} \prod_{r=1}^{k} \left( \frac{1}{r!} \right)^{i_r}.
\]

Recalling (2.3) we get
\[
\frac{Bn^2}{x \varphi(n)} \frac{d^k}{d \lambda^k} \left[ F_m(e^\lambda \log f_{x \varphi(n)}(0)) \right] \bigg|_{\lambda=1}
\sim (-1)^k \sum_{D(k)} \frac{k!}{i_1! \cdots i_k!} B^2 n^2 \frac{f_m^{(i_k)}(f_{x \varphi(n)}(0))}{B x \varphi(n)^{k+1}} \prod_{r=1}^{k} \left( \frac{1}{r!} \right)^{i_r} \sim (-1)^k k!.
\]

In particular,
\[
\frac{Bn^2}{x \varphi(n)} \frac{d}{d \lambda} \left[ F_m(e^\lambda \log f_{x \varphi(n)}(0)) \right] \bigg|_{\lambda=1}
\sim -\frac{n^2}{(x \varphi(n))^2} f_m'(f_{x \varphi(n)}(0)) \sim (-1) 1!
\]
giving
\[ f'(f_{x\varphi(n)}(0)) \sim \frac{(x\varphi(n))^2}{n^2}. \]

Now, by induction we prove that, for any \( k \geq 1 \), as \( n \to \infty \),
\[ \frac{B^2n^2}{(xB\varphi(n))^{k+1}} f_m^{(k)}(f_{x\varphi(n)}(0)) \sim k!. \]

This is true for \( k = 1 \) and if this is true for \( k < j \) then, in view of (2.5) and the induction hypothesis
\[ B^2n^2 \left( \frac{d^j}{dx^j} \frac{1}{x\varphi(n)} \right) \left( f_m^{(j)}(f_{x\varphi(n)}(0)) \right) \prod_{r=1}^{j} \frac{1}{(rt)^r} \]
\[ \sim (1) \sum_{i_1!i_2! \cdots i_j!} \frac{B^2n^2}{(xB\varphi(n))^{j+1}} f_m^{(j)}(f_{x\varphi(n)}(0)) \]
\[ = (1) \frac{B^2n^2}{(xB\varphi(n))^{j+1}} f_m^{(j)}(f_{x\varphi(n)}(0)) \sim (-1)^j j!. \]

Hence the lemma follows.

Let
\[ \mathcal{H}(n) := \{ 0 < Z(n) \leq B\varphi(n) \}. \]

Lemma 3 If the conditions of Theorem 1.1 are valid and \( \varphi(n) = o(n) \) as \( n \to \infty \), then
\[ \mathbb{P}(\mathcal{H}(n)|Z(0) = 1) \sim \frac{\varphi(n)}{n^2B}. \]

Proof. It is known [12] that if the conditions of Theorem 1.1 are valid and \( k, n \to \infty \) in such a way that the ratio \( k/n \) remains bounded then
\[ \lim_{n \to \infty} n^2B^2 \left( 1 + \frac{1}{Bn} \right)^{k+1} \mathbb{P}(Z(n) = k|Z(0) = 1) = 1. \quad (2.6) \]

Therefore,
\[ \mathbb{P}(\mathcal{H}(n)|Z(0) = 1) = \sum_{1 \leq k \leq B\varphi(n)} \mathbb{P}(Z(n) = k|Z(0) = 1) \]
\[ \sim \frac{1}{n^2B^2} \sum_{1 \leq k \leq B\varphi(n)} 1 \sim \frac{\varphi(n)}{n^2B} \]
as desired.

The next lemma is crucial for the proof of Theorem 1.1.
Lemma 4 Under the conditions of Theorem 1.1 for any \( x \in (0, \infty) \) and any \( j \geq 1 \)

\[
\lim_{n \to \infty} \mathbb{P}(Z(n - x\varphi(n), n) = j|\mathcal{H}(n)) = \frac{x}{(j - 1)!} \int_0^{1/x} z^{j-1}e^{-z}dz.
\]

Proof. Clearly, for any \( j \geq 1 \)

\[
\mathbb{P}(Z(m, n) = j) = \sum_{k=j}^{\infty} \mathbb{P}(Z(m) = k; Z(m, n) = j)
\]

\[
= \sum_{k=j}^{\infty} \mathbb{P}(Z(m) = k) C_k^j f_{k-m}(0) (1 - f_{n-m}(0))^j
\]

\[
= \frac{(1 - f_{n-m}(0))^j}{j!} f_m^j (f_{n-m}(0)). \quad (2.7)
\]

This representation, (1.2) and Lemma 2 give

\[
\mathbb{P}(Z(n - x\varphi(n), n) = j) = \frac{(1 - f_{x\varphi(n)}(0))^j}{j!} f_{n-x\varphi(n)}(f_{x\varphi(n)}(0))
\]

\[
\sim \frac{1}{j!} \frac{1}{x(\int xB\varphi(n))^j} \frac{Bn^2}{B}\left(\frac{x\varphi(n)}{Bn^2}\right)^j \sim \frac{x\varphi(n)}{Bn^2}. \quad (2.8)
\]

Let now \( Z_1^*(m), \ldots, Z_j^*(m) \) be i.i.d. random variables distributed as \( \{Z(m)|Z(m) > 0\} \), and let \( \eta_1, \ldots, \eta_j \) be i.i.d. random variables having exponential distribution with parameter 1. It is not difficult to understand, using (1.3) that

\[
\lim_{n \to \infty} \mathbb{P}(\mathcal{H}(n)|Z(n - x\varphi(n), n) = j)
\]

\[
= \lim_{n \to \infty} \mathbb{P}(Z_1^*(x\varphi(n)) + \cdots + Z_j^*(x\varphi(n)) \leq B\varphi(n))
\]

\[
= \lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1^*(x\varphi(n))}{Bx\varphi(n)} + \cdots + \frac{Z_j^*(x\varphi(n))}{Bx\varphi(n)} \leq \frac{1}{x}\right)
\]

\[
= \mathbb{P}\left(\eta_1 + \cdots + \eta_j \leq \frac{1}{x}\right) = \frac{1}{(j - 1)!} \int_0^{1/x} z^{j-1}e^{-z}dz. \quad (2.9)
\]

Combining this result with Lemma 3 and (2.8) we see that

\[
\mathbb{P}(Z(n - x\varphi(n), n) = j|\mathcal{H}(n))
\]

\[
= \frac{\mathbb{P}(Z(n - x\varphi(n), n) = j)|\mathcal{H}(n)}{\mathbb{P}(\mathcal{H}(n))}
\]

\[
\sim \frac{x\varphi(n)}{Bn^2} \frac{n^2B}{\varphi(n)} \frac{1}{(j - 1)!} \int_0^{1/x} z^{j-1}e^{-z}dz = \frac{x}{(j - 1)!} \int_0^{1/x} z^{j-1}e^{-z}dz.
\]
Proof of Theorem 1.1. By the dominated convergence theorem we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ s^{Z(n-x\varphi(n), n) | \mathcal{H}(n)} \right] = \sum_{j=1}^{\infty} \lim_{n \to \infty} \mathbb{P} (Z(n-x\varphi(n), n) = j | \mathcal{H}(n)) s^j
\]
\[
= \sum_{j=1}^{\infty} \frac{x}{(j-1)!} \int_0^{1/x} s^j z^{j-1} e^{-z} dz
\]
\[
= x s \int_0^{1/x} e^{(s-1)z} dz = \frac{xs}{1-s} \left( 1 - e^{-(1-s)/x} \right).
\]

Theorem 1.1 is proved.

Proof of Corollary 1.2. Since
\[
\mathbb{P} (d(n) \leq x\varphi(n) | \mathcal{H}(n)) = \mathbb{P} (Z(n-x\varphi(n), n) = 1 | \mathcal{H}(n)),
\]
the desired statement follows from Lemma 4 with \( j = 1 \).

3 Proof of Theorem 1.3
Similarly to (2.9) we have
\[
\lim_{n \to \infty} \mathbb{P} (0 < Z(n) \leq aBn | Z(nt, n) = j)
\]
\[
= \lim_{n \to \infty} \mathbb{P} \left( \frac{Z^*(n(1-t))}{Bn(1-t)} + \cdots + \frac{Z^*(n(1-t))}{Bn(1-t)} \leq \frac{a}{1-t} \right)
\]
\[
= \mathbb{P} \left( \eta_1 + \cdots + \eta_j \leq \frac{a}{1-t} \right) = \frac{1}{(j-1)!} \int_0^{a/(1-t)} z^{j-1} e^{-z} dz. \tag{3.1}
\]

Besides,
\[
\lim_{n \to \infty} \mathbb{P} (Z(n) \leq aBn | Z(n) > 0) = 1 - e^{-a}
\]
and, by (2.7), (1.2) and Lemma 1
\[
\mathbb{P} (Z(nt, n) = j) = \frac{(1-f_n(1-t)(0))^j}{j!} \frac{j!}{f_n(j)} \frac{(1-t)^j}{j!} \frac{Bn(1-t)^j}{(1-t+1)^{j+1}}
\]
\[
\sim \frac{1-t}{Bn} \frac{1}{t^{j-1}}.
\]
Therefore,
\[
\lim_{n \to \infty} \mathbb{P} (Z(nt, n) = j | 0 < Z(n) \leq aBn)
\]
\[
= \lim_{n \to \infty} \mathbb{P} (0 < Z(n) \leq aBn | Z(nt, n) = j) \mathbb{P} (Z(nt, n) = j)
\]
\[
= \frac{1-t}{1-e^{-a}} \frac{1}{(j-1)!} \int_0^{a/(1-t)} (t z)^{j-1} e^{-z} dz. \tag{3.2}
\]
As a result we get
\[
\lim_{n \to \infty} E \left[ s^{Z(nt,n)} | 0 < Z(n) \leq aBn \right] = \frac{(1-t)s}{1-e^{-a}} \sum_{j=1}^{\infty} \frac{\int_0^{a/(1-t)} e^{(st-z)j-1} dz}{(j-1)!}
\]
\[
= \frac{(1-t)s}{1-e^{-a}} \int_0^{a/(1-t)} e^{(st-z)j-1} dz
\]
\[
= \frac{(1-t)s}{1-e^{-a}(1-ts)} \left( 1 - e^{-(1-t)s/(1-t)} \right).
\]

Theorem 1.3 is proved.

**Proof of Corollary 1.4** Since
\[
P(d(n) \leq tn | 0 < Z(n) \leq aBn) = P(Z(n(1-t), n) = 1 | 0 < Z(n) \leq aBn),
\]
the desired statement follows from (3.2) with \(j = 1\) and \(1 - t\) for \(t\).

References

[1] Athreya, K. B. (2012) Coalescence in the recent past in rapidly growing populations. *Stochastic Processes and their Applications*, 122, 3757–3766.

[2] Athreya, K. B. (2012) Coalescence in critical and subcritical Galton-Watson branching processes. *Journal of Applied Probability*, 49, 627–638.

[3] Athreya, K. B. and Ney, P. E. (1972) *Branching processes*. Springer-Verlag, Berlin-Heidelberg-New York.

[4] Durrett, R. (1978) The genealogy of critical branching processes. *Stochastic Processes and their Applications*, 8, 101–116.

[5] Fleischmann, K., Prehn, U. (1974) Ein Grenzübergang fü r subkritische Verzweigungsprozesse mit eindlich vielen Typen von Teilchen. *Math. Nachr.*, 64, 233–241.

[6] Fleischmann, K., Siegmund-Schultze, R. (1977) The structure of reduced critical Galton-Watson processes. *Math. Nachr.*, 79, 233–241.

[7] Harris, S. C., Johnston, S. G. G., and Roberts, M. I. (2017) The coalescent structure of continuous-time Galton-Watson trees. https://arxiv.org/pdf/1703.00299.pdf

[8] Johnston, S. G. G. (2017) Coalescence in supercritical and subcritical continuous-time Galton-Watson trees. https://arxiv.org/pdf/1709.008500v1.pdf

[9] Lambert, A. (2003) Coalescence times for the branching process. *Advances in Applied Probability*, 35, 1071–1089.

[10] Lambert, A. (2016) Probabilistic models for the subtrees of life. https://arxiv.org/abs/1603.03705
[11] Le, V. (2014) Coalescence times for the Bienaymé-Galton-Watson process. *Journal of Applied Probability*, **51**, 209–218.

[12] Nagaev, S. V. and Vakhtel, V. I. (2006) On the local limit theorem for a critical Galton–Watson process. *Theory Probab. Appl.*, **50**, 400–419.

[13] O’Connell, N. (1995) The genealogy of branching processes and the age of our most recent common ancestor. *Advances in Applied Probability*, **27**, 418–442.

[14] Seneta, E. (1967) The Galton-Watson process with mean one. *Journal of Applied Probability*, **4**, 489–495.

[15] Sewast’yanov, B. A. (1974) Verzweigungsprozesse. Mathematische Lehrbucher und Monographien. II. Abteilung: Mathematische Monographien, Band 34. Akademie-Verlag, Berlin, xi+326.

[16] Vatutin, V. A., Dyakonova, E. E. (2008) Limit theorems for reduced processes in random environment. *Theory Probab. Appl.*, **52**, 277–302.

[17] Vatutin, V. A. and D’yakonova, E. E. (2015) Decomposable branching processes with a fixed extinction moment. *Proc. Steklov Inst. Math.*, **290**, 103–124.

[18] Zubkov, A. M. (1975) Limit distributions of the distance to the nearest common ancestor. *Theory Probab. Appl.*, **20**, 602–612.