Independent random variables on Abelian
groups with independent the sum and difference

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Abstract

Let $X$ be a second countable locally compact Abelian group. Let $\xi_1$, $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$, $\mu_2$ such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Assuming that the connected component of zero of the group $X$ contains a finite number elements of order 2 we describe the possible distributions $\mu_k$.

Key words. Locally compact Abelian group, Kac–Bernstein theorem, Gaussian measure.

1. Introduction. The classical Kac–Bernstein theorem states:

Theorem A. Let $\xi_1$, $\xi_2$ be independent random variables. If the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, then the random variables $\xi_k$ are Gaussian.

Much research has been devoted to group analogues of this theorem (see [1] – [12]). In the present article we study the following question: Let $\xi_1$, $\xi_2$ be independent random variables with values in a locally compact Abelian group. Assume that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. What one can say about distributions of the random variables $\xi_k$. Before we pass to the proof of the main result recall some necessary notation and definitions.

Let $X$ be a second countable locally compact Abelian group. Denote by $Y = X^*$ the character group of the group $X$, and by $(x,y)$ the value of a character $y \in Y$ at a point $x \in X$. Denote by $T = \{z \in \mathbb{C} : |z| = 1\}$ the circle group (the one-dimensional torus). Denote by $M^1(X)$ the convolution semigroup of probability distributions on $X$. The characteristic function of a distribution $\mu \in M^1(X)$ we define by the formula

$$\hat{\mu}(y) = \int_X (x,y)d\mu(x).$$

A distribution $\gamma \in M^1(X)$ is called Gaussian ([13]), if its characteristic function is represented in the form

$$\hat{\gamma}(y) = (x,y)\exp\{-\varphi(y)\}, \quad y \in Y,$$

where $x \in X$ and $\varphi(y)$ is a continuous nonnegative function on $Y$ satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.$$  \hfill (2)

Denote by $\Gamma(X)$ the set of Gaussian distributions on the group $X$. A Gaussian distribution $\gamma$ is called symmetric if in (1) $x = 0$. Denote by $\Gamma^s(X)$ the set of symmetric Gaussian distributions on $X$. Denote by $I(X)$ the set of idempotent distributions on the group $X$, i.e. the set of shifts of Haar distributions $m_K$ of compact subgroups $K$ of the group $X$.

We will formulate now the following problem.

Problem 1. Let $X$ be a second countable locally compact Abelian group. Let $\xi_1$, $\xi_2$ be independent random variables with values in $X$ and distributions $\mu_1$, $\mu_2$. Assume that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Describe the possible distributions $\mu_k$.

A.L. Rukhin in [1] and [2] received some sufficient conditions for the group $X$ in order that distributions $\mu_1$ and $\mu_2$ were represented as convolutions of Gaussian and idempotent distributions. The complete descriptions of such groups has been obtained by the author.
Theorem B (4), see also (13 §7). Let $X$ be a second countable locally compact Abelian group. Assume that the connected component of zero of the group $X$ contains no elements of order 2. Let $\xi_1, \xi_2$ be independent random variables with values in $X$ and distributions $\mu_1, \mu_2$ such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Then $\mu_k \in \Gamma(X) \ast I(X)$, $k = 1, 2$, and $\mu_1 = \mu_2 \ast E_x$, $x \in X$.

If the connected component of zero of a group $X$ contains elements of order 2, then there exist independent random variables $\xi_1, \xi_2$ with values in $X$ and distributions $\lambda_1, \lambda_2$ such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent, but $\lambda_k \notin \Gamma(X) \ast I(X)$, $k = 1, 2$.

We note that if a distribution $\mu \in \Gamma(X) \ast I(X)$, then $\mu$ is invariant with respect to a compact subgroup $K$, and $\mu$ induces a Gaussian distribution on the factor-group $X/K$ under the natural homomorphism $X \rightarrow X/K$.

Problem 1 was solved in [3] for the group $X = \mathbb{T}$, and was solved in [11] for the group $X = \mathbb{R} \times \mathbb{T}$ and $a$-adic solenoids $\Sigma_a$. Taking into account Theorem B solution of Problem 1 is reduced to the description of distributions $\mu_1$ and $\mu_2$ for groups $X$ which contain elements of order 2. In the present article we solve Problem 1 for groups $X$ satisfying the following condition: the connected component of zero of the group $X$ contains a finite number of elements of order 2. Note that if this condition holds, under additional assumption that the random variables $\xi_1$ and $\xi_2$ are identically distributed Problem 1 was solved in [12].

We need some results about the structure of locally compact Abelian groups and the duality theory (see [13] Ch. 6). If $G$ is a closed subgroup of $X$, then denote by $A(Y,G) = \{y \in Y : (x,y) = 1 \text{ for all } x \in G\}$ its annihilator. The factor-group $Y/\overline{A(Y,G)}$ is topologically isomorphic to the character group of the group $G$. Put $X_{(n)} = \{x \in X : nx = 0\}$, $X^{(n)} = \{x \in X : x = n\bar{x}, \bar{x} \in X\}$. A group $X$ is said to be Corwin group if $X^{(2)} = X$. Denote by $\overline{Y^{(2)}}$ the closure of the subgroup $Y^{(2)}$. Consider the subgroup $X^{(2)}$. It is obvious that $A(Y,X^{(2)}) = Y^{(2)}$. Assume that $X^{(2)}$ is a finite subgroup, and let $n$ be the number of its elements $|X^{(2)}| = n$. Then $X^{(2)} \cong (X^{(2)})^* \cong Y/\overline{Y^{(2)}}$. Let $Y = \bigcup_{j=0}^{n-1} (y_j + \overline{Y^{(2)}})$ be a decomposition of the group $Y$ with respect to the subgroup $\overline{Y^{(2)}}$. If $G$ is a subgroup of $X^{(2)}$, then its annihilator $H = A(Y,G)$, as is easily seen, is of the form $H = \bigcup_{j=0}^{l-1} (y_j + \overline{Y^{(2)}})$. Denote by $c_X$ the connected component of zero of the group $X$. Denote by $\mathbb{Z}$ the group of integers, and by $\mathbb{Z}(m) = \{0, 1, \ldots, m-1\}$ the group of residue classes modulo $m$.

Let $f(y)$ be an arbitrary function on the group $Y$, and let $h \in Y$. Denote by $\Delta_h$ the finite difference operator

$$
\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.
$$

Let $K$ be a compact subgroup of the group $X$. Then the characteristic function of the Haar distribution $m_K$ is of the form

$$
m_K(y) = \begin{cases} 
1, & y \in A(Y,K), \\
0, & y \notin A(Y,K).
\end{cases}
$$

Denote by $E_x$ the degenerate distribution concentrated at a point $x \in X$. For $\mu \in M^1(X)$ denote by $\sigma(\mu)$ the support of $\mu$.

2. Solution of Problem 1. Let $\xi_1, \xi_2$ be independent random variables with values in the group $X$ and with distributions $\mu_1, \mu_2$. It is easily seen that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent if and only if the characteristic functions $\hat{\mu}_k(y) = \mathbb{E}[(\xi_k, y)]$ satisfy the equation

$$
\hat{\mu}_1(u + v)\hat{\mu}_2(u - v) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(-v), \quad u, v \in Y.
$$

In the sequel we need the following lemmas.
Lemma 1 ([4], see also ([14] §9)) Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in a group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \). If the sum \( \xi_1 + \xi_2 \) and the difference \( \xi_1 - \xi_2 \) are independent, then distributions \( \mu_k \) can be replaced by their shifts \( \mu'_k \) in such a manner that \( \sigma(\mu'_k) \subset M, \ k = 1, 2 \), where \( M \) is a subgroup of \( X \) such that \( M \) is topologically isomorphic to a group of the form \( \mathbb{R}^m \times K \), where \( m \geq 0 \) and \( K \) is a compact Corwin group.

Lemma 2 ([12]). Let a locally compact Abelian group \( X \) be of the form \( X = \mathbb{R}^m \times K \), where \( m \geq 0 \) and \( K \) is a compact Corwin group. Then \( Y^{(2)} = Y^{(2)} \) and \( X^{(2)} \subset cX \).

Lemma 3 ([12]). Let \( X \) be a finite Abelian group. Then for any function \( f(y) \) on the group \( Y \) there is a complex measure \( \delta \) on \( X \) such that \( \hat{\delta}(y) = f(y), \ y \in Y \). If all nonzero elements of \( X \) have order \( 2 \), and \( f(y) \) is a real-valued function, then \( \delta \) is a signed measure.

Lemma 4 ([14]). Let \( Y \) be a locally compact Abelian group, \( \psi(y) \) be a continuous function on \( Y \) satisfying the equation
\[
\Delta^2_h \Delta_{2k} \psi(y) = 0, \ h, k, y \in Y,
\]
and the conditions \( \psi(-y) = \psi(y), \ \psi(0) = 0 \). Let
\[
Y = \bigcup_{\alpha} (y_\alpha + \overline{Y}^{(2)})
\]
be the decomposition of the group \( Y \) with respect to the subgroup \( \overline{Y}^{(2)} \). Then the function \( \psi(y) \) can be represented in the form
\[
\psi(y) = \varphi(y) + r_\alpha, \ y \in y_\alpha + \overline{Y}^{(2)},
\]
where \( \varphi(y) \) is a continuous function on \( Y \) satisfying equation (2).

The main result of the article is the following theorem.

Theorem 1. Let \( X \) be a second countable locally compact Abelian group. Assume that the connected component of zero of the group \( X \) contains a finite number elements of order \( 2 \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in \( X \) and distributions \( \mu_1 \) and \( \mu_2 \) such that the sum \( \xi_1 + \xi_2 \) and the difference \( \xi_1 - \xi_2 \) are independent. Then the following statements hold.

1. There exists a subgroup \( G \subset X_{(2)} \) such that distributions \( p(\mu_k) \) (\( p \) is the natural homomorphism \( p : X \to X/G \)) are of the form:
\[
p(\mu_k) = \gamma * \pi_k * \mu_V * E_{x_k},
\]
where \( \gamma \in \Gamma^s(X/G) \), \( \pi_k \) are signed measures on \( (X/G)_{(4)} \), \( V \) is a compact Corwin subgroup of the factor-group \( X/G \), \( x_k \in X/G \).

2. If a group \( X \) is topologically isomorphic to a group of the form \( \mathbb{R}^m \times K \), where \( m \geq 0 \) and \( K \) is a compact Corwin group, then either \( \tilde{\mu}_1(y) \equiv 0 \) or \( \tilde{\mu}_2(y) \equiv 0 \) for each coset \( y_j + Y^{(2)} \) which disjoint with \( A(Y,G) \).

Proof. By Lemma 1 the distributions \( \mu_k \) can be replaced with their shifts \( \mu'_k \) in such a manner that \( \sigma(\mu'_k) \subset M, \ m \geq 0 \) and \( M \) is a subgroup of \( X \) such that \( M \) is topologically isomorphic to a group of the form \( \mathbb{R}^m \times K \), where \( m \geq 0 \) and \( K \) is a compact Corwin group. Thus, we may assume from the beginning that the group \( X \) is of the mentioned form. Then by Lemma 2 \( X_{(2)} \subset cX \), and hence \( X_{(2)} \) is a finite subgroup. We note that the character group \( Y \) is topologically isomorphic to a group of the form \( \mathbb{R}^m \times D \), where \( D \) is a discrete group without elements of order \( 2 \). It is obvious that \( Y^{(2)} = Y^{(2)} \).

Let \( |X_{(2)}| = n \), and
\[
Y = \bigcup_{j=0}^{n-1} (y_j + Y^{(2)}) \tag{4}
\]
be a decomposition of the group $Y$ with respect to the subgroup $Y^{(2)}$. Put $N_k = \{ y \in Y : \hat{\mu}_k(y) \neq 0 \}$, $k = 1, 2$, $N = N_1 \cap N_2$. It follows from equation (3) that $N$ is an open subgroup of $Y$ satisfying the following conditions: if $2y \in N$, then $y \in N$, and

$$N \cap Y^{(2)} = N^{(2)}. \quad (5)$$

Equation (3) also implies that

$$|\tilde{\mu}_1(u + v)||\tilde{\mu}_2(u - v)| = |\tilde{\mu}_1(u - v)||\tilde{\mu}_2(u + v)|$$

for all $u, v \in Y$. It follows from this that for arbitrary elements $a$ and $b$ from a given coset $y_j + Y^{(2)}$ we have the equality

$$|\tilde{\mu}_1(a)||\tilde{\mu}_2(b)| = |\tilde{\mu}_1(b)||\tilde{\mu}_2(a)|. \quad (6)$$

Denote by $H$ a union of cosets $y_j + Y^{(2)}$ such that $N \cap (y_j + Y^{(2)}) \neq \emptyset$. Since $N$ is a subgroup of $Y$, we conclude that $H$ is also a subgroup of $Y$. Changing if it is necessary the numeration, we can assume that

$$H = \bigcup_{j=0}^{l-1} (y_j + Y^{(2)}). \quad (7)$$

Moreover, we may assume that $y_j \in N$. Note also that (5) implies that

$$N = \bigcup_{j=0}^{l-1} (y_j + N^{(2)}). \quad (8)$$

Put $G = A(X, H)$. Then $H^* \cong X/G$ and $H = A(Y, G)$. It is clear that $G \subset X^{(2)}$. Assume that $N \cap (y_j + Y^{(2)}) = \emptyset$ for some $j$. If $a \in N_1 \cap (y_j + Y^{(2)})$, then $a \notin N_2 \cap (y_j + Y^{(2)})$, and the right-hand side of (5) vanishes. Hence, $\tilde{\mu}_2(b) = 0$ for any $b \in y_j + Y^{(2)}$. Thus, we proved statement 2 of Theorem 1. This reasoning also shows that if $N \cap (y_j + Y^{(2)}) \neq \emptyset$, then $N_1 \cap (y_j + Y^{(2)}) = N_2 \cap (y_j + Y^{(2)})$.

Consider the restriction of equation (3) to $N$. It is obvious that the functions $|\hat{\mu}_1(y)|$ and $|\hat{\mu}_2(y)|$ also satisfy equation (3). Put $f_k(y) = -\ln |\hat{\mu}_k(y)|$, $y \in N$, $k = 1, 2$. It follows from (3) that the functions $f_k(y)$ satisfy the equation

$$f_1(u + v) + f_2(u - v) = A(u) + A(v), \quad u, v \in N, \quad (9)$$

where $A(u) = f_1(u) + f_2(u)$. Apply the finite difference method to solve equation (9). Let $h$ be an arbitrary element of $N$. Substitute $u + h$ for $u$ and $v + h$ for $v$ in equation (9). Subtracting equation (9) from the resulting equation we obtain

$$\Delta_{2h} f_1(u + v) = \Delta_h A(u) + \Delta_h A(v), \quad u, v \in N. \quad (10)$$

Put in (10) $v = 0$ and subtract from (10) the obtained equation. We get

$$\Delta_v \Delta_{2h} f_1(u) = \Delta_h A(v) + \Delta_h A(0), \quad u, v \in N. \quad (11)$$

It follows from this that

$$\Delta^2_v \Delta_{2h} f_1(u) = 0, \quad u, v, h \in N. \quad (12)$$

The function $f_2(y)$ satisfies the same equation. Applying Lemma 4 we obtain the representations:

$$f_k(y) = \varphi_k(y) + p_{k,j}, \quad y \in y_j + N^{(2)}, \quad k = 1, 2.$$
Since equation (3) implies that $|\widehat{\varphi}(2y)| = |\widehat{\varphi}(2y)|$, $y \in Y$, the functions $\varphi_1(y)$ and $\varphi_2(y)$ coincide on $N(2)$ and hence, they also coincide on $N$, i.e.

$$\varphi_1(y) = \varphi_2(y) = \varphi(y), \quad y \in N.$$  

Taking this into account, we get from (3) that $p_{1,j} = -p_{2,j} = p_j$, $j = 0, 1, \ldots, l - 1$. It is obvious that $\varphi(y) \geq 0$.

Put $l_k(y) = \widehat{\mu}(y)/|\widehat{\mu}(y)|$, $y \in N$, $k = 1, 2$. The functions $l_1(y)$ and $l_2(y)$ satisfy the equation

$$l_1(u + v)l_2(u - v) = l_1(u)l_2(u)l_1(v)l_2(-v), \quad u, v \in N. \quad (11)$$

Moreover, $|l_k(u)| = 1$, $l_k(-u) = \overline{l_k(u)}$, $u \in N$, $l_k(0) = 1$, $k = 1, 2$. Putting in (11) $v = u$, and then $v = -u$, we find

$$l_k(2u) = l_k^2(u), \quad u \in N, \quad k = 1, 2. \quad (12)$$

Replacing in equation (11) $u$ by $v$ we get

$$l_1(u + v)l_2(v - u) = l_1(v)l_2(v)l_1(u)l_2(-u), \quad u, v \in N. \quad (13)$$

Multiplying (11) and (13) we find

$$l_k^2(u + v) = l_k^2(u)l_k^2(v), \quad u, v \in N. \quad (14)$$

It follows from (12) and (11) that the restriction of the function $l_1(y)$ to $N(2)$ is a character of the group $N(2)$. The same reasoning shows that the restriction of the function $l_2(y)$ to $N(2)$ is also a character of the group $N(2)$. Extend these characters from $N(2)$ to some characters of the group $H$. By the duality theorem there exist elements $x_k \in X/G$ such that $l_k(y) = (x_k, y)$, $y \in N(2)$. Put $l_k^\prime(y) = (-x_k, y)l_k(y)$, $y \in H$. Then

$$l_k^\prime(2y) = 1, \quad y \in N, \quad k = 1, 2, \quad (15)$$

and (12) implies that $l_k^\prime(y) = \pm 1$, $y \in N$, $k = 1, 2$. Hence,

$$l_k^\prime(y) = l_k^\prime(-y), \quad y \in N, \quad k = 1, 2, \quad (16)$$

and the functions $l_1^\prime(y)$ and $l_2^\prime(y)$ satisfy the equation

$$l_1^\prime(u + v)l_2^\prime(u - v) = l_1^\prime(u)l_2^\prime(u)l_1^\prime(v)l_2^\prime(v), \quad u, v \in N. \quad (17)$$

This implies that for any elements $a$ and $b$ from a given coset $y_j + N(2)$ the equality

$$l_1^\prime(a)l_2^\prime(b) = l_1^\prime(b)l_2^\prime(a) \quad (18)$$

holds. Fix an element $a \in y_j + N(2)$. It follows from (13) that for all $b \in y_j + N(2)$ either $l_1^\prime(b) \equiv l_2^\prime(b)$ or $l_1^\prime(b) \equiv -l_2^\prime(b)$. Consider the subgroup

$$L = N(2) \cup (y_j + N(2)),$$

where $y_j \in N$. Taking into account that $l_1^\prime(y) \equiv l_2^\prime(y)$ for $y \in N(2)$, and for $y \in y_j + N(2)$ either $l_1^\prime(y) \equiv l_2^\prime(y)$ or $l_1^\prime(y) \equiv -l_2^\prime(y)$, it easily follows from (17) that

$$l_k^\prime(u + v)l_k^\prime(u - v) = 1, \quad u, v \in L, \quad k = 1, 2. \quad (19)$$
We conclude now from (8) and (19) that
\begin{equation}
l'_k(u + 4v) = l'_k(u), \quad u, v \in N, \quad k = 1, 2,
\end{equation}
i.e. the functions \(l'_k(y)\) are invariant with respect to the subgroup \(N(4)\).

It follows from representation (4) that \(Y^{(2)} = \bigcup_{j=0}^{n-1} (2y_j + Y^{(4)})\). Taking into consideration (7) this implies that
\begin{equation}
H = \bigcup_{0 \leq i \leq n-1, 0 \leq j \leq l-1} (2y_i + y_j + Y^{(4)}).
\end{equation}
Similarly, (8) implies that
\begin{equation}
N = \bigcup_{0 \leq i \leq l-1} (2y_i + y_j + N^{(4)}).
\end{equation}
Taking into account (21), extend the functions \(l'_k(y)\) from \(N\) to some functions \(\tilde{l}'_k(y)\) on \(H\) by the formulas
\begin{align*}
\tilde{l}'_k(2y_i + y_j + u) &= l'_k(2y_i + y_j), \quad u \in Y^{(4)}, \quad 0 \leq i, j \leq l-1, \\
\tilde{l}'_k(2y_i + y_j + u) &= 1, \quad u \in Y^{(4)}, \quad l \leq i \leq n-1, \quad 0 \leq j \leq l-1, \quad k = 1, 2.
\end{align*}
It follows from (19) and (21) that the functions \(\tilde{l}'_k(y)\) also satisfy the condition
\begin{equation}
\tilde{l}'_k(y) = \tilde{l}'_k(-y), \quad y \in H, \quad k = 1, 2.
\end{equation}
By construction the functions \(\tilde{l}'_k(y)\) are invariant with respect to the subgroup \(Y^{(4)}\), in particular they satisfy the equation
\begin{equation}
\tilde{l}'_k(u + 4v) = \tilde{l}'_k(u), \quad u, v \in H, \quad k = 1, 2,
\end{equation}
and hence, they define some functions on the factor-group \(H/Y^{(4)}\). Set \(F = A(X/G, Y^{(4)}) \subset (X/G)^{y(4)}\) and note that \(F \cong (H/Y^{(4)})^\star\). Taking into account that \(H/Y^{(4)}\) is a finite group and applying Lemma 3 we conclude that there exist complex measures \(\delta_k\) on \(X/G\) supported in the group \(F\) such that \(\delta_k(y) = \tilde{l}'_k(y), \quad y \in H\). It is obvious that \(F \cong (\mathbb{Z}(4))^m\) for some \(m\). An arbitrary character of the group \((\mathbb{Z}(4))^m\) is of the form
\begin{equation}
((k_1, \ldots, k_m), (l_1, \ldots, l_m)) = \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^{m} k_j l_j \right\},
\end{equation}
\((k_1, \ldots, k_m) \in (\mathbb{Z}(4))^m, \quad (l_1, \ldots, l_m) \in ((\mathbb{Z}(4))^m)^\star\). The complex measures \(\delta_k\), considering as complex measures on \(F\), are defined by the formulas
\begin{equation}
\delta_k\{(k_1, \ldots, k_m)\} = \frac{1}{4^m} \sum_{(l_1, \ldots, l_m) \in ((\mathbb{Z}(4))^m)^\star} \tilde{l}'_k(l_1, \ldots, l_m) \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^{m} k_j l_j \right\}, \quad k = 1, 2.
\end{equation}
Note that (22) and (23) imply that
\begin{equation}
\tilde{l}'_k(3y) = \tilde{l}'_k(y), \quad y \in H, \quad k = 1, 2.
\end{equation}
We also note that \(\exp \left\{ \frac{i\pi}{2} \sum_{j=1}^{m} k_j l_j \right\} = \pm i\) if and only if when \(\sum_{j=1}^{m} k_j l_j\) is an odd number. Since (25) implies that
\begin{equation}
\tilde{l}'_k(l_1, \ldots, l_m) = \tilde{l}'_k(3l_1, \ldots, 3l_m),
\end{equation}
and the numbers \( \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^{m} k_j l_j \right\} \) and \( \exp \left\{ \frac{i\pi}{2} \sum_{j=1}^{m} 3k_j l_j \right\} \) are complex conjugate, it follows from (24) that all numbers \( \delta_k \{ (k_1, \ldots, k_m) \} \) are real, i.e. actually \( \delta_k \) are signed measures.

Return to the representations of the functions \( f_k(y) \) on \( N \). We have

\[
f_1(y) = \varphi(u) + p_j, \quad f_2(y) = \varphi(y) - p_j, \quad y \in y_j + N^{(2)}, \quad j = 0, 1, \ldots, l - 1.
\]

By Lemma 3, there exist signed measures \( \epsilon_1 \) and \( \epsilon_2 \) on \( (X/G)_{(2)} \) such that

\[
\widehat{\epsilon}_1(y) = e^{p_j}, \quad \widehat{\epsilon}_2(y) = e^{-p_j}, \quad y \in y_j + Y^{(2)}, \quad j = 0, 1, \ldots, l - 1. \tag{27}
\]

Put \( \pi_k = \delta_k \ast \epsilon_k, \quad k = 1, 2 \). Then \( \pi_k \) are signed measures on \( (X/G)_{(4)} \). Extend the function \( \varphi(y) \) from \( N \) to \( H \) retaining its properties ([17, §5.2]). Denote by \( \tilde{\varphi}(y) \) the extended function. Let \( \gamma \) be a symmetric Gaussian distribution on the factor-group \( X/G \) with the characteristic function \( \tilde{\gamma}(y) = \exp \{ -\tilde{\varphi}(y) \}, \quad y \in H \). Set \( V = A(X/G, N) \). It is easily seen that \( V \) is a compact Corwin subgroup. Thus, we obtained that the restriction to \( H \) of the characteristic functions of the distributions \( \mu_k \) can be represented in the form

\[
\tilde{\mu}_k(y) = \begin{cases} 
\exp\{ -\varphi(y) \} \pi_k(y)(x_k, y), & y \in N, \\
0, & y \in H \setminus N,
\end{cases}
\]

for \( k = 1, 2 \). The desired representation \( p(\mu_k) = \gamma \ast \pi_k \ast m_V \ast E_{x_k}, \quad k = 1, 2, \) results now from the following general remark: if \( \mu \in M^1(X) \), and \( H \) is a closed subgroup of \( Y \), then the restriction of the characteristic function \( \tilde{\mu}(y) \) to \( H \) is the characteristic function of the distribution \( p(\mu) \) on the factor-group \( X/G \), where \( G = A(X, H) \), and \( p \) is the natural homomorphism \( p : X \to X/G \). The theorem is proved completely.

3. Some remarks. Give some comments to Theorem 1.

Remark 1. Let \( X \) be a second countable locally compact Abelian group such that \( X \) is topologically isomorphic to a group of the form \( R^m \times K \), where \( m \geq 0 \) and \( K \) is a compact Corwin group. Assume also that \( X \) contains only one element of order 2, i.e. \( X_{(2)} \cong Z_{(2)} \). Then we can strengthen Theorem 1. We reason as in the proof of Theorem 1 and retain the same notation.

Since \( X_{(2)} \) is a finite group and \( Y^{(2)} = A(Y, X_{(2)}) \), we have \( X_{(2)} \cong (X_{(2)})^* \cong Y/Y^{(2)} \). Hence, a decomposition of the group \( Y \) with respect to the subgroup \( Y^{(2)} \) is of the form \( Y = Y^{(2)} \cup (y_1 + Y^{(2)}) \), and there are two possibilities for the subgroup \( H \): either \( H = Y \), and then \( G = A(X, Y) = \{0\} \), or \( H = Y^{(2)} \), and then \( G = A(X, Y^{(2)}) = X_{(2)} \).

1. Let \( H = Y \), then

\[
N = N^{(2)} \cup (y_1 + N^{(2)}). \tag{28}
\]

Since \( l_1'(y) = l_2'(y) \) for \( y \in N^{(2)} \) and \( l_1'(y) = \pm l_2'(y) = \pm 1 \) for \( y \in y_1 + N^{(2)} \), it follows from (28) that the functions \( l_k'(y) \) are characters of the subgroup \( N \). Extend these characters from \( N \) to some characters of the group \( Y \). By the duality theorem there exist elements \( t_k \in X \) such that \( l_k'(y) = (t_k, y), \quad y \in N \). Set \( z_k = x_k + t_k \). We obtain as a result the representation

\[
\mu_k = \gamma \ast \epsilon_k \ast m_V \ast E_{z_k}, \quad k = 1, 2,
\]

where \( \gamma \in \Gamma^s(X), \quad \epsilon_k \) are signed measures on \( X_{(2)} \), \( V \) is a compact Corwin subgroup of the group \( X, \quad z_k \in X \). Moreover, it follows from (27) that such that \( \epsilon_1 \ast \epsilon_2 = E_0 \).

2. Let \( H = Y^{(2)} \). Then \( N \subset Y^{(2)} \), and it follows from (27) that

\[
N = N^{(2)}. \tag{29}
\]

7
There are two possibilities for the subgroup $N$: either $N \neq \{0\}$ or $N = \{0\}$.

\textbf{a.} Assume that $N \neq \{0\}$. Put $W = N^*$. It follows from (24) that the group $W$ contains no elements of order 2. Consider the restriction of equation (23) to $N$ and apply Theorem B to the group $W$. We get that the restrictions of the characteristic functions $\hat{\mu}_k(y)$ to $N$ are the characteristic functions of some Gaussian distributions. It follows easily from this that

\begin{equation}
    p(\mu_k) = \gamma \ast m_N \ast E_{x_k}, \quad k = 1, 2,
\end{equation}

where $\gamma \in \Gamma^s(X/X(2))$, $V$ is a compact Corwin subgroup of the factor-group $X/X(2)$, $x_k \in X/X(2)$. Moreover, since either $\hat{\mu}_1(y) \equiv 0$ or $\hat{\mu}_2(y) \equiv 0$ for $y_1 + Y(2)$, it is not difficult to prove taking into account (30) that at least one of the distributions $\mu_k$ is represented in the form

\begin{equation}
    \mu_k = \lambda \ast m_N \ast E_{z_k},
\end{equation}

where $\lambda \in \Gamma^s(X)$, $U$ is a compact Corwin subgroup of $X$, $z_k \in X$.

\textbf{b.} Suppose that $N = \{0\}$. Obviously, it is possible only if $X$ is a compact group. We have $V = A(X/X(2), N) = X/X(2)$. Taking into account that the equality $N_1 \cap Y(2) = N_2 \cap Y(2) = N \setminus Y(2)$ always holds, we get the representation

\begin{equation}
    p(\mu_k) = m_{X/X(2)}, \quad k = 1, 2.
\end{equation}

Moreover, since either $\hat{\mu}_1(y) \equiv 0$ or $\hat{\mu}_2(y) \equiv 0$ for $y_1 + Y(2)$, it follows from (31) that $\mu_k = m_X$ at least for one of the distributions $\mu_k$. Let for example, $\mu_1 = m_X$. Returning to the random variables $\xi_k$, it means that $\xi_1$ and $2\xi_2$ are identically distributed random variables with distribution $m_X$.

As a result we obtain the following statement.

\textbf{Theorem 2.} Let $X$ be a second countable locally compact Abelian group such that $X$ is topologically isomorphic to a group of the form $R^m \times K$, where $m \geq 0$ and $K$ is a compact Corwin group. Assume also that the group $X$ contains only one element of order 2. Let $p$ be the natural homomorphism $p : X \rightarrow X/X(2)$, and $Y = Y(2) \cup (y_1 + Y(2))$ be a decomposition of the group $Y$ with respect to the subgroup $Y(2)$. Assume that $\xi_1, \xi_2$ are independent random variables with values in $X$ and distributions $\mu_1, \mu_2$ such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. Then either

\begin{equation}
    \mu_k = \gamma \ast \epsilon_k \ast m_V \ast E_{z_k},
\end{equation}

where $\gamma \in \Gamma^s(X)$, $\epsilon_k$ are signed measures on $X(2)$ such that $\epsilon_1 \ast \epsilon_2 = E_0$, $V$ is a compact Corwin subgroup of $X$, $z_k \in X$, $k = 1, 2$, or

\begin{equation}
    p(\mu_k) = \gamma \ast m_V \ast E_{x_k},
\end{equation}

where $\gamma \in \Gamma^s(X/X(2))$ $V$ is a compact Corwin subgroup of the factor-group $X/X(2)$, $x_k \in X/X(2)$, $k = 1, 2$, and at least one of the distributions $\mu_k$ is represented in the form

\begin{equation}
    \mu_k = \lambda \ast m_U \ast E_{z_k},
\end{equation}

where $\lambda \in \Gamma^s(X)$, $U$ is a compact Corwin subgroup of $X$, $z_k \in X$.

We illustrate Theorem 2 by the following example. Let $X = T$. Then $Y \cong Z$. In order not to complicate notation we assume that $Y = Z$. We have $T(2) \cong Z(2)$, and hence there are two possibilities: either $G = \{0\}$ or $G = T(2)$.

1. $G = \{0\}$. We observe that all compact Corwin subgroups $V$ of the group $T$ are of the form: either $V = T$, or $V$ is the subgroup of $m$th roots of 1, where $m$ is odd, i.e. $V \cong Z(m)$. If $V = T$, then by Theorem 2

\begin{equation}
    \mu_1 = \mu_2 = m_T.
\end{equation}
Let \( V \cong \mathbb{Z}(m) \). Then by Theorem 2

\[
\mu_k = \gamma \ast \epsilon_k \ast m_V \ast E_{z_k},
\]

where \( \gamma \in \Gamma^s(T) \), \( \epsilon_k \) are signed measures on \( T_{(2)} \) such that \( \epsilon_1 \ast \epsilon_2 = E_1 \), \( z_k \in T \), \( k = 1, 2 \). It follows from \( V \cong \mathbb{Z}(m) \) that \( N = \mathbb{Z}^{(m)} \), and hence (32) implies that the characteristic functions of distributions \( \mu_k \) are represented in the form

\[
\tilde{\mu}_1(n) = \begin{cases} 
\exp\{-\sigma n^2 + \text{int}_1\}, & n \in \mathbb{Z}^{(2m)}, \\
\exp\{-\sigma n^2 + \text{int}_1 + q\}, & n \in \mathbb{Z}^{(m)} \backslash \mathbb{Z}^{(2m)}, \\
0, & n \notin \mathbb{Z}^{(m)},
\end{cases}
\]

\[
\tilde{\mu}_2(n) = \begin{cases} 
\exp\{-\sigma n^2 + \text{int}_2\}, & n \in \mathbb{Z}^{(2m)}, \\
\exp\{-\sigma n^2 + \text{int}_2 - q\}, & n \in \mathbb{Z}^{(m)} \backslash \mathbb{Z}^{(2m)}, \\
0, & n \notin \mathbb{Z}^{(m)},
\end{cases}
\]

where \( \sigma \geq 0 \), \( t_k \), \( q \in \mathbb{R} \), \( k = 1, 2 \).

2. \( G = T_{(2)} \). It follows from (29) that \( N = \{0\} \). Hence, \( V = A(T/T_{(2)}) = T/T_{(2)} \). We get by Theorem 2 \( p(\mu_k) = m_{T/T_{(2)}} \), \( k = 1, 2 \), i.e. \( \tilde{\mu}_1(2n) = \tilde{\mu}_2(2n) = 0 \), \( n \in \mathbb{Z} \), \( n \neq 0 \). Moreover, either \( \tilde{\mu}_1(2n + 1) = 0 \) or \( \tilde{\mu}_2(2n + 1) = 0 \), \( n \in \mathbb{Z} \). This implies that at least one of the distributions \( \mu_k \), say \( \mu_1 = m_T \). As regards to the second distribution, we know only that \( \tilde{\mu}_2(2n) = 0 \), \( n \in \mathbb{Z} \), \( n \neq 0 \). Returning to the random variables \( \xi_k \) it means that \( \xi_1 \) and \( 2\xi_2 \) are identically distributed random variables with the distribution \( m_T \).

The obtained description of possible distributions \( \mu_k \) for the group \( X = T \) is the main content of article [3]. Similarly, it is possible to get from Theorem 2 the description of possible distributions \( \mu_k \) for the group \( \mathbb{R} \times T \) and for the \( a \)-adic solenoids \( \Sigma_a \) found in [11].

**Remark 2.** Let \( X = T^2 \). Then \( Y \cong \mathbb{Z}^2 \). Denote by \( x = (e^{it}, e^{is}) \) elements of the group \( X \), and by \( y = (m, n) \in \mathbb{Z}^2 \) elements of the group \( Y \). We will construct independent random variables \( \xi_1 \) and \( \xi_2 \) with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \) such that the sum \( \xi_1 + \xi_2 \) and the difference \( \xi_1 - \xi_2 \) are independent, and \( \mu_k = \gamma \ast \pi_k \), where \( \gamma \in \Gamma^s(X) \), and \( \pi_k \) are signed measures supported in \( X_{(4)} \), but not in \( X_{(2)} \). Thus, we will show that the statement in Theorem 1 that \( \pi_k \) are signed measures on \( (X/G)_{(4)} \), generally speaking can not be strengthen to the statement that \( \pi_k \) are signed measures on \( (X/G)_{(2)} \) (compare with Theorem 2).

Take \( \sigma > 0 \) such that

\[
\sum_{(m,n)\in\mathbb{Z}^2, (m,n)\neq(0,0)} e^{-\sigma(m^2+n^2)} < 1. \tag{33}
\]

Consider on the group \( Y \) the functions

\[
l_1(m, n) = \begin{cases} 
1, & (m, n) \in \{Y^2, (1, 0) + Y^4, (3, 0) + Y^4, (0, 1) + Y^4, \\
(0, 3) + Y^4, (1, 1) + Y^4, (3, 3) + Y^4\}, \\
-1, & (m, n) \in \{(1, 2) + Y^4, (3, 2) + Y^4, (2, 1) + Y^4, \\
(2, 3) + Y^4, (1, 3) + Y^4, (3, 1) + Y^4\}
\end{cases}
\]

and

\[
l_2(m, n) = \begin{cases} 
1, & (m, n) \in \{Y^2, (1, 0) + Y^4, (3, 0) + Y^4, (0, 1) + Y^4, \\
(0, 3) + Y^4, (1, 3) + Y^4, (3, 1) + Y^4\}, \\
-1, & (m, n) \in \{(1, 2) + Y^4, (3, 2) + Y^4, (2, 1) + Y^4, \\
(2, 3) + Y^4, (1, 1) + Y^4, (3, 3) + Y^4\}.
\end{cases}
\]
Consider on the group $X$ the functions

$$\rho_k(x) = \rho_k(e^{it}, e^{is}) = 1 + \sum_{(m,n) \in \mathbb{Z}^2, (m,n) \neq (0,0)} e^{-\sigma(m^2+n^2)}l_k(m,n)e^{-i(tm+sn)}, \quad k = 1, 2. \quad (34)$$

Inequality (33) implies that $\rho_k(x) > 0$ and it is obvious that

$$\int_X \rho_k(x)dm_X(x) = 1, \quad k = 1, 2.$$ 

Let $\mu_k$ be the distributions on the group $X$ with respect to the Haar distribution $m_X$, and let $\xi_k$ be independent random variables with values in $X$ and distributions $\mu_k$, $k = 1, 2$. It follows from (34) that

$$\hat{\mu}_k(m,n) = e^{-\sigma(m^2+n^2)}l_k(m,n), \quad (m,n) \in \mathbb{Z}^2, \quad k = 1, 2. \quad (35)$$

We can check directly that the functions $l_k(m,n)$ satisfy equation (3), and hence the characteristic functions $\hat{\mu}_k(m,n)$ also satisfy equation (3). Thus, the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent.

By the construction, the functions $\hat{l}_k(m,n)$ are invariant with respect to the subgroup $Y^{(4)}$, and hence, they define some functions $\hat{\pi}_k(m,n)$ on the factor-group $X/Y^{(4)} \cong (\mathbb{Z}(4))^2$. By Lemma 3 there exist complex measures $\pi_k$ on $X$ supported in $A(X,Y^{(4)}) = X^{(4)}$ such that

$$\hat{\pi}_k(m,n) = l_k(m,n), \quad (m,n) \in \mathbb{Z}^2, \quad k = 1, 2. \quad (36)$$

Since the functions $\hat{l}_k(m,n)$ satisfy the condition (26), reasoning as in the proof of Theorem 1 we get that the complex measures $\pi_k$ actually are signed measures. Since the characteristic functions $\hat{\pi}_k(m,n)$ are not invariant with respect to the subgroup $Y^{(2)}$, the supports of the signed measures $\pi_k$ do not contain in $X^{(2)}$. Let $\gamma$ be a symmetric Gaussian distribution on $X$ with the characteristic function

$$\hat{\gamma}(m,n) = e^{-\sigma(m^2+n^2)}, \quad (m,n) \in \mathbb{Z}^2. \quad (37)$$

It follows from (35)–(37) that $\mu_k = \gamma * \pi_k, \quad k = 1, 2$.

**Remark 3.** Assume that under conditions of Theorem 1 the independent random variables $\xi_1$ and $\xi_2$ are identically distributed, i.e. $\mu_1 = \mu_2 = \mu$. As has been proved in [12], this implies that

$$\mu = \gamma * \pi * mV \ast E_x,$$

where $\gamma \in \Gamma^*(X)$, $\pi$ is a signed measure on $X^{(2)}$ such that $\pi^{*2} = E_0$, $V$ is a compact Corwin subgroup of the group $X$, $x \in X$. Taking into account Theorem 1 and the example constructing in Remark 2, we see that even we assume that in Theorem 1 $G = \{0\}$, the distributions $\mu_k$ have generally speaking factorisation much more complicated than in the case of identically distributed random variables.

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