Spherically symmetric wormholes can be linearly stable

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In this work we study the problem of linear stability of gravitational perturbations in stationary and spherically symmetric wormholes. For this purpose, we employ the Newman-Penrose formalism which is well-suited for treating gravitational radiation in General Relativity, as well as the geometrical aspect of this theory. With this method we obtain a “master equation” that describes the behavior of perturbations that are “vacuum-like” and of odd-parity in the Regge-Wheeler gauge. This equation is later applied to a specific class of Morris-Thorne wormholes and also to the metric of an asymptotically flat scalar field wormhole. The analysis of the equations that these space-times yield reveals that they are stable with respect to the type of perturbations here studied.

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1. INTRODUCTION

Ever since the concept of “wormhole” was first introduced in the literature by Misner and Wheeler in 1957 [1], there has been general interest in the fascinating physical and geometrical properties that the non-trivial topology of these objects could possess. Previously, Einstein and Rosen had already proposed an interpretation of the Schwarzschild space-time consisting of two identical “sheets” connected through a “bridge”, this is nowadays known as an Einstein-Rosen bridge [2]. While the Schwarzschild metric can indeed admit the interpretation of wormhole, Fuller and Wheeler would later prove that it is not a traversable one since, in order for a signal to cross it, causality must be violated [3]. Nevertheless, these ideas can be considered as the precursors of the modern version of a wormhole, an hypothetical compact object that would allow communication between distant regions of the same universe, or even between two different universes. Several years later, Morris and Thorne established the characteristics that a traversable wormhole must feature, along with a general metric that should describe them [4]. The metric was assumed to be stationary and spherically symmetric. In their work they arrived to the unfortunate conclusion that a traversable wormhole requires of “exotic” matter, this is, matter that violates the energy conditions, as its gravitational source. Until now there is no observational evidence on the existence of wormholes, and thus, they have remained within the speculative realm of the theory of General Relativity. Moreover, there are additional concerns regarding the realistic existence of such objects in the universe, one of them is that of their stability.

It can be argued that, in order for a stellar object to be of any astrophysical interest, it has to stable. Otherwise, a slight perturbation or deviation from its initial state would result in the collapse of the space-time itself. Remarkable papers that solve this problem, at least in a linear theory and using analytical methods, are well-known. The stability of the Schwarzschild metric was first studied by Regge and Wheeler by adding a perturbation term $h_{\mu\nu}$ to the background metric and keeping terms up to first order of the perturbation [5]. Based on their results, later works then confirmed that said space-time is indeed stable against small perturbations [6, 7]. With a less straightforward approach, Teukolsky managed to find a “master equation” that describes gravitational, electromagnetic, and neutrino field perturbations of a spinning black hole [8]. Analyzing then the mentioned equation with numerical techniques, the stability of the first modes of vibration of the Kerr black hole was concluded [9]. From a practical point of view, this result was of great relevance to the existence and possible observation of a realistic black hole. However, from a theoretical standpoint, maybe the most interesting aspect of this series of papers is that of the derivation of the master equation. For this purpose, Teukolsky exploited the full potential of the Newman-Penrose formalism [10] and the underlying geometric properties of the Kerr metric, in particular, the fact that it is of type D in the algebraic classification of space-times. As impressive as the master equation is, unfortunately, it is only valid for vacuum space-times of type D. This rules out the possibility of applying it to wormholes.

It is clear that the problem of stability for black holes has been thoroughly studied. Over the years, these developments have contributed to the physical relevance of this outstanding prediction of General Relativity. On the other hand, wormholes are still only theoretical entities. They are commonly, but not uniquely, proposed as stationary and spherically symmetric space-times supported by a phantom scalar field, i.e., a scalar field whose kinetic energy has a reversed sign (sometimes referred too as ghost scalar field). Maybe the most simple model of such a wormhole is that of Ellis [11]. In recent years, many works have been written about the question of the linear stability of these type of wormholes and a handful of them report that they are generally unstable. Thus adding another problematic
issue to their set of particular properties. In [12], scalar field wormholes consisting of one asymptotically flat end, while the other end being asymptotically AdS, were proven to be unstable against axial perturbations. Similar results regarding the instability of particular cases of wormholes with phantom fields were obtained in papers [12, 15] when perturbing the space-time metric. Said perturbations were assumed to be only radial and, of course, time-dependent. Particularly in [14], the case of the Ellis wormhole was studied exhaustively. On the contrary, there have been a few other works [16, 17] studying certain wormholes that are supposedly stable. The perturbation analysis of [16] can be considered to be somewhat more general in the sense that it includes angular-dependent terms.

In this paper we will first develop a framework for treating linear gravitational perturbations using the Newman-Penrose formalism. This will benefit us with some previously used tools and results that have been found over the years for the problem of gravitational radiation in General Relativity. We will focus on what we shall call “vacuum-like” perturbations, that is, perturbations such that the first order change of the Ricci tensor is $\delta R_{\mu \nu} = 0$. Additionally, in this first approach, we will specialize to the odd-parity perturbations, named so by Regge and Wheeler in their paper [3]. The results we obtain will later on be applied to wormhole metrics, some of them supported by phantom scalar fields. The work is organized as follows. In section II we will study the general problem of gravitational perturbations within the tetrad formalism. This treatment will be particularized to stationary and spherically symmetric space-times in section III, here, the master equation will also be presented. In section IV the meaning of physical regularity that must be imposed to the mentioned perturbations will be discussed. Examples of the application of our master equation will be given in sections V and VI, first on the Morris-Thorne wormholes, and then on a specific phantom scalar field wormhole. Finally, conclusions and final comments will be given. We also included two appendixes containing mostly laborious calculations that were carried out throughout the paper.

II. GRAVITATIONAL PERTURBATIONS IN THE TETRAD FORMALISM

We shall follow the notation utilized by Newman and Penrose in their seminal paper [10] and hereafter may refer to this work as the NP paper. In this formalism a null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ is introduced into every point of a four-dimensional pseudo-Riemannian manifold of signature $(+1, -1, -1, -1)$ and metric $g_{\mu \nu}$. The vectors $l^\mu$ and $n^\mu$ are real, while $m^\mu$ and $\bar{m}^\mu$ are complex. In this paper we will use a bar over any given quantity to denote its complex conjugate. The vectors of the tetrad must also satisfy the orthogonal property $l^\mu n_{\mu} = -m^\mu \bar{m}_{\mu} = 1$, with the rest of the vector combinations being zero. The space-time metric can then be expressed as

$$g_{\mu \nu} = l_{\mu} n_{\nu} + n_{\mu} l_{\nu} - m_{\mu} \bar{m}_{\nu} - \bar{m}_{\mu} m_{\nu}. \quad (1)$$

This relation can be rewritten in a more compact way as $g_{\mu \nu} = z_{m \mu} z_{n \nu} \gamma_{m n}$ if one conveniently defines\(^1\)

$$z_{m \mu} = (l_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu}),$$

$$z_{m} = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu),$$

$$\gamma_{m n} = \gamma_{m n} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad (2)$$

where $\gamma_{m n} \gamma_{p m} = \delta_{n}^{p}$. Using (2) we can also write the orthogonality properties simply as $z_{m \mu}^\dagger z_{n \mu} = \gamma_{m n}$. The metric $\gamma$ will be used to raise or lower tetrad indices. Newman and Penrose define 12 complex spin coefficients that depend on linear combinations of the quantities $Z_{m n p} = z_{m \nu} \gamma_{n \mu} z_{p \rho}$, which are anti-symmetrical in their last two indices\(^2\), i.e., $Z_{m n p} = -Z_{m p n}$. Additionally, four differential operators are introduced

$$D = l^\mu \nabla_{\mu}, \quad \Delta = n^\mu \nabla_{\mu}, \quad \delta = m^\mu \nabla_{\mu}, \quad \delta^* = \bar{m}^\mu \nabla_{\mu}, \quad (3)$$

\(^1\) We will use Greek indices $(\mu, \nu = 0, 1, 2, 3)$ to denote tensor indices and lower-case Latin indices $(a, b, m, n = 0, 1, 2, 3)$ to denote tetrad indices.

\(^2\) In the case of $\gamma$ and $Z_{m n p}$ our notation will slightly vary from that of the NP paper. They use $\eta$ instead of $\gamma$ for the tetrad metric, and $\gamma_{p m n}$ instead of $Z_{m n p}$. Note that the order of the indices for these last quantities is also different.
or more compactly $D_m = z_m^\mu \nabla_\mu$ with $D_m = (D, \Delta, \delta, \delta^*)$. Using the 12 spin coefficients, along with the operators $\hat{\Sigma}$, Newman and Penrose obtained a set of numerous equations that are the equivalent of the Bianchi identities and the components of the Ricci and Weyl tensors in tetrad form, this is now known as the Newman-Penrose formalism. Since the Einstein field equations make use of the curvature tensors yielded by a given space-time metric, one can discuss any problem in General Relativity (at least its geometrical aspects) within this formalism.

Here we will develop a general framework for perturbation theory using the Newman-Penrose formalism. The scheme we follow is the typical one for linear gravitational perturbations, that is, we add a perturbation term $\tilde{h}_{\mu \nu}$ to a certain background metric $g_{\mu \nu}$, and then compute the components of the Ricci tensor keeping terms up to first order of the perturbation. The perturbation term is assumed to be small compared to its background counterpart. In this formalism, the perturbation term of the metric will be represented by a perturbation in the null tetrad, for example, $l_\mu = \tilde{l}_\mu + l_\mu$, where we will establish the convention that a tilde denotes any given background quantity and the hat denotes the perturbation term of said quantity. To proceed, we expand the perturbation terms of the tetrad in the basis of the background tetrad, hence,

$$
\begin{align*}
\tilde{z}_m^\mu &= \tilde{z}_m^\mu + \hat{z}_m^\mu = \hat{z}_m^\mu + \hat{\Sigma}_m^\mu \tilde{z}_m^\mu, \\
\tilde{z}_{m\mu} &= \tilde{z}_{m\mu} + \hat{z}_{m\mu} = \hat{z}_{m\mu} + \hat{\Sigma}_{m\mu}. \\
\end{align*}
$$

(4)

To maintain the vectors $l^\mu$ and $n^\mu$ real, the $\hat{\Sigma}_m^\mu$ matrix has to satisfy $\hat{\Sigma}_m^\mu \in \mathbb{R}$ and $\hat{\Sigma}_m^2 = \hat{\Sigma}_m^3$ for $m, n = 0, 1$. Additionally, we require that $\hat{\Sigma}_3^3 = \hat{\Sigma}_3^2$, $\hat{\Sigma}_2^2 = \hat{\Sigma}_3^3$, and that $\hat{\Sigma}_m^3 = \hat{\Sigma}_m^m$ for $m = 0, 1$ in order to $m^\mu$ and $\tilde{m}^\mu$ remain as complex conjugates of each other. To simplify notation we drop the hat off the perturbation terms $\hat{\Sigma}$ as complex conjugates of each other. To simplify notation we drop the hat off the perturbation terms $\hat{\Sigma}$. Using this result one can next verify that $g^{\mu\nu}g_{\mu\nu} = \delta_\mu^\nu$, and so, the fundamental equations of the formalism are consistent.

With the tetrad given by equation (4) the quantities $Z_{abc}$ related to the spin coefficients may be computed. However, note that the connection $\Gamma$ associated to the operator $\nabla$ appearing in these quantities is compatible with the metric $g$, not with the background metric $\bar{g}$. Naturally, the components of the connection $\Gamma$ can be expressed as $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \bar{\Gamma}^\rho_{\mu\nu}$. We obtain, thus,

$$
Z_{abc} = \bar{Z}_{abc} - \bar{\Gamma}_{cab} + \bar{D}_a \Omega_{cb} + \bar{D}_{ab} \Omega_c^p + \bar{D}_{ap} \Sigma^p_b + \bar{D}_{pb} \Sigma^p_a,
$$

(6)

where we have defined $\bar{\Gamma}_{cab} = \bar{z}_{c\alpha} \hat{\Gamma}_{\mu\nu}^{\alpha \mu \nu} \bar{z}^\beta_b$. The components of the perturbed connection may be found by using the compatibility condition $\nabla_\alpha g_{\mu\nu} = 0$ and the torsion free symmetry $\bar{\Gamma}_{abc} = \bar{\Gamma}_{acb}$. A straightforward, but somewhat long, calculation yields

$$
\bar{\Gamma}_{abc} = \bar{D}_{(a} \Omega_{c)b} + \bar{D}_{[b} \Omega_{a]c} + \bar{D}_{[c} \Omega_{a]b} + \bar{Z}_{(bc)p} \Sigma^p_a + \bar{Z}_{[ba]p} \Sigma^p_c + \bar{Z}_{[ca]p} \Sigma^p_b,
$$

(7)

with $\Xi^a_m = \Omega^a_m - \Sigma^a_m$ and also $\Pi^a_m = \Omega^a_m + \Sigma^a_m$, which will be used in the next equation. Substituting (7) in (6), and after some algebraic simplifications, we get $Z_{abc}$ in terms only of background quantities and metric perturbations, namely,

$$
Z_{abc} = \bar{Z}_{abc} + \bar{D}_{[b} \Sigma^a_{c]} - \bar{D}_{[c} \Sigma^a_{b]} + \bar{D}_{[b} \Sigma^a_{c]} + \Xi^m \bar{Z}_{c]am} + \bar{Z}_{am[c} \Pi^m_b + \Xi^m \bar{Z}_{[cb]m} + \bar{Z}_{mbc} \Sigma^m_a.
$$

(8)

In equation (4) we have explicitly indicated there are second order terms of $\Omega^a_m$. From this point forward we will omit the second order dependency in every equation for compactness and, unless otherwise noted, every equal sign should be understood as such only to first order of $\Sigma$ or $\Omega$.

Round brackets will be used to denote symmetrization of the indices enclosed, while square brackets to denote anti-symmetrization.
This equation is manifestly anti-symmetric in its last two indices as the quantity \( Z_{abc} \) should be. Though lengthy, equation (3) describes how the spin coefficients, which are necessary for the Newman-Penrose formalism, change to first order for any given perturbation \( \Sigma^m_n \).

### Perturbed Tetrad Rotations

Consider a transformation of the perturbation terms \( \Omega^{mn} \rightarrow \Omega^{mn} + \Omega^{'mn} \). From (3) it can be seen that

\[
g_{\mu\nu} \rightarrow g_{\mu\nu} + \Omega^{mn}(\bar{z}_{mn} \bar{z}_{\nu} + \bar{z}_{m\nu} \bar{z}_{n}),
\]

Since the expression in parenthesis is symmetric in its tetrad indices, the metric will then be invariant under these type of transformations if we demand that \( \Omega^{mn} = -\Omega^{'mn} \). Not only the metric will be invariant, but naturally, also any other scalar or tensor derived from it, so long as the tensor does not possess tetrad indices. Therefore, there exists liberty in choosing the perturbation tetrad \( \bar{z}^\mu = \Sigma^m_n z^\mu \) since the \( \Omega^{mn} \) that corresponds to a certain perturbed metric is not unique (recall that \( \Omega^m_n = -\gamma_{mp}\Sigma^p_q\gamma^qn \)). This of course is related to the group of Lorentz transformations that leave invariant the orthogonality properties of the formalism (see reference [18]). However, for this case, the parameters of the Lorentz group should be taken as infinitesimal.

Under the transformation \( \Omega^{mn} \rightarrow \Omega^{mn} + \Omega^{'mn} \), the previously defined \( \Xi^m_n \) is invariant, while

\[
\Pi^m_n \rightarrow \Pi^m_n + 2\Omega^m_n,
\]

with \( \Omega^m_n = \gamma_{mp}\Omega^pn \). Note that \( \Omega^{mn} = -\Omega^{'mn} \) implies that \( \Omega^nmn = \Sigma^mn \). Using these relations, we have that the quantities \( Z_{abc} \) transform as

\[
Z_{abc} \rightarrow Z_{abc} + 2\bar{Z}_{am}[\xi_{bc}] + \bar{Z}_{mbc}\Omega^m_n.
\]

Perhaps the most important benefit that the perturbed tetrad rotations provide lies in the differential operators \( D_m \). They evidently change as \( D_m \rightarrow D_m + \Sigma^m_n D_n \), but because there is some freedom in choosing the perturbation tetrad vectors, we may then conveniently pick them so that, for instance, \( D_m = \bar{D}_m + \chi \bar{D}_n \) for some fixed \( m \neq n \), and a scalar field \( \chi \). In the following section we take advantage of this particular property, simplifying thus our calculations.

It is important to notice that, when performing any rotation through \( \Omega^{mn} \rightarrow \Omega^{mn} + \Omega^{'mn} \), one has to be careful that the rotated vectors \( l^\mu \) and \( n^\mu \) end up being real, and that \( m^\mu \) and \( \bar{m}^\mu \) remain as complex conjugates. This restricts the possible valid rotations that can be done. Taking into account these restrictions, one can be convinced that there is a total of six degrees of freedom, which is consistent with the fact that the group of Lorentz transformations is a six parameter group.

### III. GRAVITATIONAL PERTURBATIONS IN SPHERICALLY SYMMETRIC SPACE-TIMES

For the remainder of this work we focus on four-dimensional stationary and spherically symmetric space-times \((M, g_{\mu\nu})\) whose line element, without loss of generality, can be written in the form

\[
ds^2 = g_0(r)dt^2 - g_1(r)dr^2 - g_2(r)\Omega^2,
\]

where we have introduced a radial coordinate \( r \) and the metric elements \( g_{0,1,2}(r) \), which are arbitrary functions of said coordinate. Also, \( \Omega^2 \) is the standard metric on the two-sphere. An orthonormal frame for this metric is simply given by

\[
X_0 = \frac{1}{\sqrt{g_0}}\partial_t, \quad X_1 = \frac{1}{\sqrt{g_1}}\partial_r, \quad X_2 = \frac{1}{\sqrt{g_2}}\partial_\theta, \quad X_3 = \frac{1}{\sqrt{g_2}\sin \theta}\partial_\phi.
\]

From frame (10), a null tetrad can be constructed by taking appropriate linear combinations of the \( X \) vectors. In this paper we will take advantage of the symmetries of the space-time, namely the fact that \( \partial_t \) and \( \partial_\phi \) are Killing
vectors, and choose $\tilde{l}^\mu$ and $\tilde{n}^\mu$ so that they lie in the subspace spanned by said Killing vectors. This can also be extended to axially symmetric space-times. Hence, the vectors of the null tetrad will be

$$
\tilde{l}^\mu = \frac{1}{\sqrt{2}}(X_0^\mu + X_3^\mu), \quad \tilde{n}^\mu = \frac{1}{\sqrt{2}}(X_0^\mu - X_3^\mu), \quad \tilde{m}^\mu = \frac{1}{\sqrt{2}}(X_1^\mu + iX_2^\mu).
$$

(11)

A direct evaluation of the spin coefficients of the Newman-Penrose formalism with metric (9) and tetrad (11) yields that the only non-vanishing coefficients are $\tilde{\kappa}, \tilde{\nu}, \tilde{\tau}, \tilde{\pi}, \tilde{\alpha}$ and $\tilde{\beta}$. Additionally, the following properties hold

$$
\tilde{\kappa} + \tilde{\nu}^* = \tilde{\tau} + \tilde{\pi}^* = \tilde{\alpha} + \tilde{\beta} = 0, \quad \tilde{\kappa} + \tilde{\nu} = -\tilde{\tau} - \tilde{\pi}, \quad \tilde{\alpha} = \frac{1}{4}(\tilde{\nu} + \tilde{\nu}^* + \tilde{\tau} + \tilde{\tau}^*)
$$

(12)

with $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$. Notice that as a consequence of our choice of vectors $\tilde{l}^\mu$ and $\tilde{n}^\mu$ we will have that $D\tilde{\phi} = \tilde{\Delta}\tilde{\phi} = 0$ for any background scalar quantity $\tilde{\phi}$, including these spin coefficients.

We now add a perturbation term $h_{\mu \nu}$ to the background metric introduced in this section. Following the pioneering work of Regge and Wheeler [5] we consider a perturbation of the form

$$
h_{\mu \nu} = \begin{bmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix},
$$

(13)

with $h_{\mu \nu}$ expressed in the coordinate basis $\{t, r, \theta, \varphi\}$. Regge and Wheeler obtained this particular (and simple) expression for $h_{\mu \nu}$ through a gauge transformation of the most general perturbation whose angular part consists of products of scalar, vector, and tensor spherical harmonics $Y_{l,m}(\theta, \varphi)$ on the 2-sphere. This gauge is sometimes called the Regge-Wheeler gauge. Furthermore, (13) represents a perturbation of $(-1)^{l+1}$ parity, which Regge and Wheeler named as odd, due to its negative symmetry under reflections about the origin. The $\varphi$-dependence of the perturbation can be eliminated without significant loss of information by setting $m = 0$. This is possible since the background space-time is spherically symmetric. Therefore, we have that $h_{0,1} = h_{0,1}(t, r, \theta)$.

Using the one-forms of the background tetrad $\{\tilde{l}_\mu, \tilde{n}_\mu, \tilde{m}_\mu, \tilde{\bar{m}}_\mu\}$ as a basis, we can write

$$
h_{\mu \nu} = f_0(\tilde{n}_\mu \tilde{n}_\nu - \tilde{l}_\mu \tilde{l}_\nu) + 2f_1[\tilde{l}_\mu \tilde{m}_\nu + \tilde{l}_\nu \tilde{m}_\mu - \tilde{n}_\mu \tilde{m}_\nu - \tilde{n}_\nu \tilde{m}_\mu],
$$

where $f_0 = h_0/\sqrt{q_0 q_2} \sin \theta$ and $f_1 = h_1/2\sqrt{q_1 q_2} \sin \theta$. It can be verified that an acceptable tetrad for the perturbed metric $\tilde{g}_{\mu \nu} = \tilde{g}_{\mu \nu} + h_{\mu \nu}$ is given by

$$
l_\mu = \tilde{l}_\mu + \frac{1}{2}f_0\tilde{n}_\mu - f_1(\tilde{m}_\mu + \tilde{\bar{m}}_\mu), \quad n_\mu = \tilde{n}_\mu - \frac{1}{2}f_0\tilde{l}_\mu + f_1(\tilde{m}_\mu + \tilde{\bar{m}}_\mu), \quad m_\mu = \tilde{m}_\mu, \quad \bar{m}_\mu = \tilde{\bar{m}}_\mu,
$$

(14)

from which the elements of $\Omega^m_n$ can be easily read off

$$
\Omega^m_n = \begin{bmatrix} 0 & f_0/2 & -f_1 & -f_1 \\ -f_0/2 & 0 & f_1 & f_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

It will more helpful, though, to represent the perturbation in terms of $\Sigma^m_n = -\gamma_{mp} \Omega^p_q \gamma^{qn}$, obtaining thus

$$
\Sigma^m_n = \begin{bmatrix} 0 & -f_0/2 & 0 & 0 \\ f_0/2 & 0 & 0 & 0 \\ f_1 & -f_1 & 0 & 0 \\ f_1 & -f_1 & 0 & 0 \end{bmatrix}.
$$

(15)
As it was already stated in the previous section, any given metric perturbation does not uniquely define $\Omega_m^n$, and consequently, $\Sigma_{m}^n$. We will be interested in a perturbed tetrad such that

$$
D\delta = (\chi_1 D + \chi_2 \Delta)\delta = 0, \quad \Delta\delta = (\xi_1 D + \xi_2 \Delta)\delta = 0,
$$

(16)

where again, $\delta$ is any background scalar quantity and $\chi_{1,2}$, $\xi_{1,2}$ are elements of $\Sigma_{m}^n$. It turns out that precisely the matrix given by (15) describes the perturbation tetrad with this desired property. Nonetheless, it is important to mention that one can always find, through an adequate tetrad rotation, a perturbation tetrad such that (16) holds in the matrix given by (15) describes the perturbation tetrad with this desired property. Nonetheless, it is important to realize that, apart from $\Lambda$, the $\Phi_{AB}$ quantities depend manifestly on the tetrad choice. Even upon fixing the background tetrad, $\Phi_{AB}$ will vary with perturbed rotations such as the ones described in the previous section. Since the Ricci tensor itself is invariant to these type of transformations, we look for expressions of its components in the coordinate basis and constructed from the quantities $\Phi_{AB}$ and $\Lambda$. More precisely, we will look for the components of $R_{\mu\nu}$ in the orthonormal frame (10).

In terms of the background tetrad basis, the orthonormal basis can be written as $X_{\alpha}^\mu = \bar{\Gamma}_{\alpha}^m z_{m}^\mu$. From (11), it can be easily seen that

$$
\bar{\Gamma}_{\alpha}^m = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -i & i \\
1 & -1 & 0 & 0 \\
\end{bmatrix}.
$$

Similarly, in terms of the perturbed tetrad we have that $X_{\alpha}^\mu = \Gamma_{\alpha}^m z_{m}^\mu$. Using the fact that $z_{m}^\mu = (\delta_{m}^n + \Sigma_{m}^n)\bar{z}_{n}^\mu$, we obtain $\Gamma_{\alpha}^n = \bar{\Gamma}_{\alpha}^m (\delta_{m}^n - \Sigma_{m}^n)$ to first order in $\Sigma$, or explicitly,

$$
\Gamma_{\alpha}^n = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 - f_{0}/2 & 1 + f_{0}/2 & 0 & 0 \\
-2f_{1} & 2f_{1} & 1 & 1 \\
0 & 0 & -i & i \\
1 + f_{0}/2 & -1 + f_{0}/2 & 0 & 0 \\
\end{bmatrix}.
$$

(19)

We can then write,
\[
X_0^\mu = \frac{1}{\sqrt{2}} \left[ (1 - \frac{f_0}{2}) l^\mu + \left( 1 + \frac{f_0}{2} \right) n^\mu \right], \quad X_1^\mu = \frac{1}{\sqrt{2}} \left[ m^\mu + \delta^\mu + 2f_1(n^\mu - l^\mu) \right], \quad X_2^\mu = \frac{i}{\sqrt{2}} \left[ \delta \mu - m^\mu \right], \quad X_3^\mu = \frac{1}{\sqrt{2}} \left[ \left( 1 + \frac{f_0}{2} \right) l^\mu - \left( 1 - \frac{f_0}{2} \right) n^\mu \right].
\]

The \( X \) vectors of equation (20) can be shown to be invariant under transformations \( \Sigma_m^\alpha \rightarrow \Sigma_m^{\prime \alpha} + \Sigma_m^{\prime \mu} \Sigma_m^\mu \), and to first order in \( \Sigma \), by noting that \( \Gamma^\alpha_{\mu \nu} \rightarrow \Gamma^\alpha_{\mu \nu} - \Gamma^\alpha_{\mu \nu} \Sigma_m^{\prime \mu} \Sigma_m^\nu \), and \( z^\mu \rightarrow z^\mu + \Sigma_m^{\prime \mu} z^\nu \). Thus, we may find the desired invariant equations for the Ricci tensors by contracting these vectors with the Ricci tensor field. Unfortunately, this has to be done for the 10 independent components of said tensor, yielding the following relations

\[
\hat{\mathcal{R}}_{00} = -\Phi_{00} - \hat{\Phi}_{22} + 2(3\Lambda - \Phi_{11}), \quad \hat{\mathcal{R}}_{01} = -\Phi_{01} - \Phi_{10} - \Phi_{21}, \quad \hat{\mathcal{R}}_{02} = \hat{\Phi}_{02} + \hat{\Phi}_{20} - 2(3\Lambda + \Phi_{11}), \quad \hat{\mathcal{R}}_{03} = -\phi_{00} + \hat{\Phi}_{22} + 2(3\Lambda - \Phi_{11}) f_0, \quad \hat{\mathcal{R}}_{11} = -\Phi_{11} = \hat{\Phi}_{22} + 2(3\Lambda - \Phi_{11}) f_0, \quad \hat{\mathcal{R}}_{12} = i(\Phi_{02} - \Phi_{20}), \quad \hat{\mathcal{R}}_{13} = -\Phi_{10} - \Phi_{12} + 4(3\Lambda - \Phi_{11} + \Phi_{00}) f_1, \quad \hat{\mathcal{R}}_{23} = i(\Phi_{10} + \Phi_{21} - \Phi_{12}), \quad \hat{\mathcal{R}}_{33} = -\Phi_{00} - \Phi_{22} - 2(3\Lambda - \Phi_{11}),
\]

where we have defined \( \hat{\mathcal{R}}_{\alpha \beta} = \hat{R}_{\mu \nu} X_\alpha^\mu X_\beta^\nu \). In equations (21) we have written only the perturbation terms (denoted by a hat), that is, the terms of first order in \( f_{0,1} \). Naturally, the background terms that should appear on both sides of the equations, which are of order zero, cancel each other out. Hereafter, we drop the tilde off the background quantities and so, any quantity or operator without a hat should be understood to be of the background space-time, except for the perturbation functions \( f_0 \) and \( f_1 \) (same convention as Appendix A).

Taking the results (A.10) from Appendix A, one can realize that the only non-vanishing components of \( \hat{\mathcal{R}}_{\alpha \beta} \) are

\[
\hat{\mathcal{R}}_{03} = 2(\delta_+ - 6\alpha) D f_1 + \left[ (\delta_+ - 2\kappa_+)\delta_- - (\delta_+ + \kappa_- + 3\pi_-)\delta_+ + 4(\kappa_-^2 - \kappa_+^2) + 2(3\Lambda - \Phi_{11}) \right] f_0, \\
\hat{\mathcal{R}}_{13} = 2 [D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_) + 2(3\Lambda - \Phi_{11} + \Phi_{00})] f_1 - (\delta_+ - 3\kappa_- + \pi_-) D f_0, \\
\hat{\mathcal{R}}_{23} = i(\delta_- - 2\kappa_+) D f_0 - 2i(\delta_+ + \kappa_- + 3\pi_-)(\delta_- - 2\kappa_) f_1.
\]

With the help of the commutator \([\delta_- - 2\kappa_+, \delta_+] = (\kappa_- + \pi_-)(\delta_- + 2\kappa_+)\), the \( \hat{\mathcal{R}}_{23} \) component of the past equations can be rewritten as

\[
\hat{\mathcal{R}}_{23} = i(\delta_- - 2\kappa_+)[D f_0 - 2(\delta_+ + 2\pi_-) f_1].
\]

In this paper we are interested in what we will call "vacuum-like linear perturbations", this is, perturbations such that the first order term \( \hat{R}_{\mu \nu} \) vanishes, which implies that \( \hat{R}_{\mu \nu} = 0 \). Hence, we can attempt to solve the system of equations (22) by making said assumption. Consider the \( \hat{\mathcal{R}}_{23} \) component, we have already factored it in a way that can be easily solved. Notice that the expression in parentheses cannot vanish since \( \delta_- \) is a differential operator and \( \kappa_+ \) is a scalar quantity. Thus, \( \hat{\mathcal{R}}_{23} \) will vanish if the expression in square brackets also does, or by the application of the operator in parentheses to the quantity in square brackets. We will examine the first possibility, that is,

\[
D f_0 = 2(\delta_+ + 2\pi_-) f_1.
\]

By inserting (23) in the \( \hat{\mathcal{R}}_{13} \) component of (22), an equation for the perturbation function \( f_1 \) can finally be found,

\[
[D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+) - (\delta_+ - 3\kappa_- + \pi_-)(\delta_+ + 2\pi_-) + 2(3\Lambda - \Phi_{11} + \Phi_{00})] f_1 = 0.
\]

We are left, however, with the \( \hat{\mathcal{R}}_{03} = 0 \) equation yet to solve with the inconvenient that the perturbation functions \( f_0 \) and \( f_1 \) have already been used to solve the other two equations in system (22). By applying the operator \( D \) in the \( \hat{\mathcal{R}}_{03} \) component, then using (23) and (24), this equation can be shown to vanish only if (see Appendix B),
\[(\delta_+ - 4\alpha)(3\Lambda - \Phi_{11} + \Phi_{00}) = 0.\]  

Unfortunately at this point, we are forced to abandon the generality that has been conserved until now regarding the spherically symmetric space-times here considered, and restrict ourselves to those for which equation (25) holds. By considering the explicit form of the spin coefficients and operators shown in \[A.2\], this past condition can be rewritten as

\[
\frac{1}{g_2\sqrt{2g_1}} \frac{d}{dr} \left[ g_2(3\Lambda - \Phi_{11} + \Phi_{00}) \right] = 0.
\]

If this equation is true everywhere in space-time the implication is that

\[3\Lambda - \Phi_{11} + \Phi_{00} = c/2g_2,\]  

where \(c\) is an integration constant chosen so for future convenience. We will be able to associate certain physical significance to a particular value of this constant in Section V. Examples that fulfill condition (26), besides vacuum space-times, are the solutions of the Einstein-scalar field equations \[R_{\mu\nu} = \pm \nabla_\mu \phi \nabla_\nu \phi,\] for this particular case, \(c = 0\) (see Appendix B for details). Many wormhole space-times arise as solutions to this class of field equations, hence, our results can be applied to them.

For reasons explained in the next section, we shall opt to replace the perturbation function \(f_1\) with \(Q = 2\sqrt{g_0}\sin \theta f_1\).

Substituting \(f_1 = Q/2\sqrt{g_0}\sin \theta\) in (24), and employing these two equalities, we at last arrive to our master equation for odd-parity perturbations,

\[
\left[D^2 + (\delta_- + 2\kappa_+)(\delta_- - 4\kappa_+) - (\delta_+ - 2\kappa_-)(\delta_+ + \pi_- + \kappa_-) + 2(3\Lambda - \Phi_{11} + \Phi_{00}) \right] Q = 0.
\]

The notation introduced throughout the paper allows us to easily identify the terms appearing in the master equation. The \(D\) operator is associated to the time dependence of the perturbation, the second term is associated with the angular part due to it containing the \(\delta_-\) operators, and the third term is related to the radial part because of the \(\delta_+\) operators. In \[27\] there also appears a background matter term which is purely radial. It is natural then to propose a separable ansatz of the form \(Q = T(t)R(r)\Theta(\theta)\). With such a proposed solution, the angular part of the master equation will yield the following differential equation when inserting the explicit expressions for the spin coefficients and operators,

\[
\frac{d^2\Theta}{d\theta^2} - \frac{1}{\tan \theta} \frac{d\Theta}{d\theta} = -l(l+1)\Theta.
\]

Equation (28) has for solution \(\Theta = \sin \theta d{P_l}(\cos \theta)/d\theta\), where \(P_l(\cos \theta)\) are the well-known Legendre polynomials. This result was of course, expected, owing to the spherical symmetry of the line element \[9\] and to the decomposition in tensor spherical harmonics of the perturbation that Regge and Wheeler previously used. In fact, this part of the solution is, obviously, the same that appears in their paper. It also can be verified that the radial equation yielded by \[27\] reduces to that of Regge-Wheeler when inserting the corresponding spin coefficients for the Schwarzschild metric. During the rest of this work we will analyze the radial part of the master equation \[27\], along with its properties.

IV. PHYSICAL REGULARITY OF THE PERTURBATION

In order for the gravitational perturbation to be of any physical relevance, it has to display a "acceptable" behavior throughout space-time, or at least asymptotically. One might naturally impose the condition that the metric
perturbation functions of $h_{\mu\nu}$ do not grow without bound as $r \to \infty$ and deem that as physical regularity. Nevertheless, due to the gauge freedom that exists in General Relativity, this condition is not quite precise. Fortunately, the Newman-Penrose formalism can also be used to describe more accurately what this acceptable behavior is expected to be by means of the so-called “peeling theorem” \[14\].

Consider the following vectors tangent to ingoing and outgoing radial null geodesics of the background metric,

$$ k_+^\mu = X_0^\mu \pm X_1^\mu = \frac{1}{\sqrt{2}} \left( l^\mu \pm n^\mu \pm m^\mu \mp \tilde{m}^\mu \right). $$

The next null rotations of our initial tetrad yield a new one such that the unperturbed part of $l''^\mu$ and $n''^\mu$ is aligned to the $k_+^\mu$ and $k_-^\mu$ vectors, respectively,

$$ l''^\mu = l^\mu + a_1 \tilde{m}^\mu + a_1 m^\mu + n_{\bar{a}1}^2 n^\mu, \quad m''^\mu = m^\mu + a_1 n^\mu, \quad n''^\mu = n^\mu, $$

with $a_1 = 1$ and $a_2 = -1/2$. In equations (29) and (30), and only in those equations, we temporarily restore the convention of section II in which any given quantity $\xi$ of the space-time is written as the sum of a background term and a perturbation term, i.e., $\xi = \tilde{\xi} + \xi$. Under transformations (29), the Weyl scalars we need change as (see reference \[18\])

$$ \psi'_0 = \psi_0 + 4a_1 \psi_1 + 6a_1^2 \psi_2 + 4a_1^3 \psi_3 + a_1^4 \psi_4, \quad \psi'_1 = \psi_1 + 3a_1 \psi_2 + 3a_1^2 \psi_3 + a_1^3 \psi_4, \quad \psi'_2 = \psi_2 + 2a_1^3 \psi_3 + a_1^4 \psi_4, \quad \psi'_3 = \psi_3 + 2a_2 \psi_3 + a_2^2 \psi_4. $$

If $a_1 = 1$ and $a_2 = -1/2$, then $\psi''_0 = \psi_0/4 - \psi_2/2 + \psi_4/4$. After substituting the expressions found in (A.7), the perturbed part of this Weyl scalar reduces to

$$ \tilde{\psi}_2'' = \frac{1}{2} (\delta_{-} + 2\kappa_{+}) \left[ (\delta_{-} - 2\kappa_{-}) f_0 - 2D f_1 \right]. $$

The physical significance of $\tilde{\psi}_2''$ can be revealed by applying the operator $D$ to (31), and then reducing it accordingly with some of the relations of the formalism here derived along with the master equation, thus obtaining

$$ D\tilde{\psi}_2'' = \frac{1}{2\sqrt{g_0} \sin \theta} \left[ (\delta_{-} + 2\kappa_{+}) (\delta_{-} - 4\kappa_{+}) + 2(3\Lambda - \Phi_{11} + \Phi_{00}) \right] Q. $$

We have already solved the angular part of the master equation whose terms appear again in (32). By making use of said solution and some properties of the Legendre equation, the past expression can be rewritten as

$$ \frac{\partial \tilde{\psi}_2''}{\partial t} = - \frac{i(l+1)}{g_2^{3/2}} \left[ (l-1)(l+2) + c \right] T(t) R(r) P_l(\cos \theta), $$

where we have made use of restriction (20) too. In the case of space-times that solve the Einstein-scalar field equations (and vacuum space-times too) we have that $c = 0$, and the meaning of

$$ \frac{\partial \tilde{\psi}_2''}{\partial t} = - \frac{i(l+2)!}{g_2^{3/2} (l-2)!} T(t) R(r) P_l(\cos \theta) $$

becomes clearer, as well as the reason behind the use of the perturbation function $Q$. The peeling theorem establishes that the Weyl scalar $\psi_2$ asymptotically decays at null infinity as $1/\lambda^3$, where $\lambda$ is the affine parameter of a null geodesic that reaches said infinity. Since $r$ is an appropriate radial coordinate, this affine parameter can be chosen as $\lambda' = r$ for the case of the background radial null geodesics to which the unperturbed part of $l''^\mu$ and $n''^\mu$ are tangent to. Furthermore, the metric component appearing in (31) goes asymptotically as $g_2(r) \sim r^2$. The perturbation function
$Q = T(t) R(r) \Theta(\theta)$, hence, manifestly describes the peeling property that the $\psi_2$ scalar should display at null infinity. From this analysis we can state that a regular behavior of $Q$ is one that does not alter the $1/r^3$ decay of the Weyl scalar $\psi_2''$ when $r \to \infty$. Also in this case, and from the reduced form of $\partial_l \psi_2''$, it can be seen that the $l = 0$ and $l = 1$ solutions will not yield a physical perturbation due to the vanishing of this Weyl scalar, i.e., the lowest multipole of gravitational radiation is the quadrupole ($l = 2$) [19]. The relation shown in equation (34) was previously found in the case of perturbations of the Schwarzschild black hole in [20]. There, it was also shown that $\psi_2''$ is invariant under infinitesimal null tetrad rotations and under gauge transformations as well, making this quantity measurable by any observer. Such properties are also valid for the $\psi_2''$ of the gravitational perturbations discussed in this paper.

V. THE MORRIS-THORNE WORMHOLES

In this section we will apply the master equation found in section III to the wormhole space-times introduced in [4]. The general line element is the following,

$$ds^2 = e^{2\Phi(r)} dt^2 - \frac{dr^2}{1 - b(r)/r} - r^2 d\Omega^2,$$  \hspace{1cm} (35)

where $\Phi(r)$ is known as the redshift function and $b(r)$ as the shape function. Both of these metric components fulfill certain conditions in order for the geometry of the space-time to be that of a wormhole. In particular, there exists a minimum radius $r = b_0 > 0$ such that $b(b_0) = b_0$. This value defines the throat of the wormhole, and hence, the domain of the radial coordinate is $r \in (b_0, \infty)$. It should be clarified that this coordinate decreases from positive infinity to $b_0$ as the throat is approached from one of the two universes it connects, and then increases back to infinity when emerging on the other universe. An additional requirement of the shape function is $1 - b(r)/r \geq 0$, along with $\Phi(r)$ being everywhere finite. This last condition on the redshift function is related to the non-existence of event horizons in the space-time so that hypothetical travelers may move from one universe to the other in both directions. If the wormhole is to be asymptotically flat, then the limits $\Phi(r) \to 0$ and $b(r)/r \to 0$ as $r \to \infty$ must also be imposed.

After this brief presentation on the features of the Morris-Thorne wormholes our intention next is to apply the master equation (34) to them. Nevertheless, it should be reminded that this equation is not valid for the entire family of Morris-Thorne metrics, but only for those that satisfy the condition (26). In fact, this condition determines a constraint on the redshift and shape functions, namely,

$$\frac{rb'(r) + b(r)}{2r} + (b(r) - r) \Phi'(r) = c.$$ \hspace{1cm} (36)

We will now show that the class of Morris-Thorne metrics defined by (36), satisfies the conditions that a wormhole must possess. The most compelling way to accomplish this is to rearrange the defining constraint of the class so that the shape function, without its first derivative, is in terms only of the redshift function. This will allow us to pick a suitable $\Phi(r)$, specifically an everywhere finite function, and find the corresponding expression for $b(r)$. Using the basic theory of first-order differential equations one can show that the desired relation between these functions is

$$b(r) = r + \frac{2e^{-2\Phi(r)}}{r} \left[(c - 1)F(r) + c_1 \right],$$

where $c_1$ is an integration constant and

$$F(r) = \int r e^{2\Phi(r)}dr.$$

The integration constant can be chosen so that the condition $b(b_0) = b_0$ on the minimum radius $r = b_0$ is fulfilled. Obtaining thus,

$$b(r) = r + \frac{2(c - 1)e^{-2\Phi(r)}}{r} \int_{b_0}^r e^{2\Phi(r')} dr'.$$ \hspace{1cm} (37)
From \( \text{(37)} \) and the fact that the integrand there is strictly positive in the domain of integration, it can be seen that the condition \( 1 - b(r)/r \geq 0 \) is satisfied if \( c < 1 \). This also implies that the vector \( \partial / \partial r \) remains everywhere space-like. Furthermore, by examining the limit \( r \to \infty \) for which \( \Phi(r) \to 0 \), one can realize that \( b(r)/r \to c \). Then, in order for the wormhole to be asymptotically flat, the constant \( c \) has to be set as \( c = 0 \). Recall that, for a vanishing value of \( c \), we concluded from the analysis in the previous section of the Weyl scalar \( \psi_2 \) that the lowest radiative multipole is the quadrupole. This is in full agreement with the expected behavior of gravitational perturbations in an asymptotically flat space-time. It can be argued, therefore, that only those space-times with \( c = 0 \) are physically meaningful.

We have obtained an upper bound for the constant \( c \) (and a fixed value of physical relevance) for which, given an appropriate redshift function, the metrics studied here possess indeed the geometry of a wormhole. With this, we shall examine the perturbation equation yielded by the line element \( \text{(35)} \). The relevant background spin coefficients and operators are

\[
\kappa_+ = -\pi_+ = \frac{i \cot \theta}{2\sqrt{2}r}, \quad \pi_+ + \kappa_- = -2\alpha = \frac{1}{r} \sqrt{\frac{1}{2} \left( 1 - \frac{b(r)}{r} \right)} \frac{1}{r}, \quad \pi_- - \kappa_- = \Phi(r) \sqrt{\frac{1}{2} \left( 1 - \frac{b(r)}{r} \right)},
\]

\[
\delta_+ = \sqrt{\frac{1}{2} \left( 1 - \frac{b(r)}{r} \right)} \frac{\partial}{\partial r}, \quad \delta_- = \frac{i}{\sqrt{2r}} \frac{\partial}{\partial \theta}, \quad D = \frac{1}{\sqrt{2}} \left( e^{-\Phi(r)} \frac{\partial}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \varphi} \right),
\]

which are computed using a tetrad of the form \( \text{(11)} \). With these expressions we obtain the following second-order partial differential equation from \( \text{(27)} \),

\[
e^{-2\Phi} \frac{\partial^2 Q}{\partial t^2} - \frac{1}{r^2} \left( \frac{\partial^2 Q}{\partial \theta^2} - \frac{1}{\tan \theta} \frac{\partial Q}{\partial \theta} \right) = \sqrt{1 - \frac{b}{r}} \frac{\partial}{\partial r} \left( \sqrt{1 - \frac{b}{r}} \frac{\partial Q}{\partial r} \right) - \Phi' \left( 1 - \frac{b}{r} \right) \frac{\partial Q}{\partial r} + \left( \Phi'(b - r) + \frac{rb' - 5b}{2r} \right) \frac{Q}{r^2} + 4(3\Lambda - \Phi_{11} + \Phi_{00})Q = 0.
\]

The equation obtained for the perturbation can be further simplified by considering the previously introduced ansatz for \( Q \), whose angular part has already been solved in section III, and with the additional assumption of an harmonic dependence on time, i.e., \( Q = e^{i\omega t} R(r) \sin \theta \partial \cos \theta / \partial \theta \). Furthermore, through the following coordinate change for \( r \),

\[
\frac{d}{dr_*} = \pm e^{i\Phi} \sqrt{1 - \frac{b}{r}} \frac{d}{dr} r
\]

and substituting equations \( \text{(20)} \), \( \text{(28)} \), the master equation for this restricted class of Morris-Thorne wormholes can finally be rewritten in a very compact form as,

\[
\frac{d^2 R}{dr_*^2} - \left( V(r) - \omega^2 \right) R = 0,
\]

(38)

with

\[
V(r) = e^{2\Phi} \left[ l(l + 1) + 3 \left( c - \frac{b}{r} \right) \right].
\]

In the coordinate transformation performed, one can always choose an adequate integration constant so that the throat of the wormhole is located at \( r_* = 0 \). Moreover, the \( r_* \) coordinate takes the positive sign for one side of the throat, and the negative sign for the other side. When \( r \to \infty \), one has that \( r_* \to \pm r / \sqrt{1 - c} \), where both coordinates coincide (up to the sign) if the wormhole joins asymptotically flat sides. The coordinate \( r_* \) thus takes values on the whole real line, i.e., \( r_* \in (-\infty, \infty) \).

With the domain of this new coordinate established, equation \( \text{(38)} \) then defines an eigenvalue problem for \( \omega^2 \) and the operator \( \mathcal{H} = -d^2 / dr_*^2 + V(r) \) which is linear and self-adjoint in a \( L^2(\mathbb{R}, dr_*) \) space. An operator of this type is sometimes called a Schrödinger operator. The stability analysis lies now in determining if there exist eigenvalues of the equation \( \mathcal{H}R = \omega^2 R \) which represent perturbations that grow without bound as \( t \to \infty \), but are physically regular.
TABLE I: Metric components of a few examples from the stable class of Morris-Thorne wormholes ($1 > c > 1/3$, $c = 0$ for asymptotically flat space-times).

| $e^{2\Phi(r)}$ | $1 - b(r)/r$ |
|---------------|-------------|
| 1            | $(1 - c)(1 - b_0^2/r^2)$ |
| $1 + e^{-(r/b_0)^2}$ | $(1 - c)e^{-2\Phi(r)}[1 - b_0^2(e^{2\Phi(r)} - e^{-1})/r^2]$ |
| $1 + b_0^2/(r^2 + b_0^2)$ | $(1 - c)e^{-2\Phi(r)}[1 - b_0^2(\ln[1/2 + r^2/2b_0^2] - 1)/r^2]$ |
| $1/2 + \arctan(r/b_0 - 1)/\pi$ | $(1 - c)e^{-2\Phi(r)}e^{2\Phi(r)} - b_0/\pi r - b_0^2 (\pi/2 - 1 + \ln[1 + (1 - r/b_0^2)])/\pi r^2$ |

otherwise. By equation and the peeling theorem, any eigenfunction $R \in L^2(\mathbb{R}, dr_x)$ will describe physically regular perturbations due to it being square-integrable. Since the operator $\mathcal{H}$ is self-adjoint, the eigenvalues $\omega^2$ must be real. Hence, considering the time dependent part of the proposed ansatz, any instability will appear as a purely imaginary $\omega$, this is, as a negative eigenvalue.

The discussion of the eigenvalue spectrum of follows in a fairly simple manner based on the properties of the potential $V(r)$ of the Schrödinger operator $\mathcal{H}$. First, we shall focus on the wormholes with vanishing $c$ since they represent asymptotically flat space-times. For this case it is readily seen that $V(r) \geq 0$ for all $r \in [b_0, \infty]$, due to the $1 - b(r)/r \geq 0$ condition and to the fact that $l$ takes positive integer values starting from $l = 2$. For a strictly positive potential there cannot exist negative eigenvalues (energy bound states) and thus, all of the vibrational modes of this class of wormholes are linearly stable under vacuum-like perturbations of odd-parity.

For the sake of completeness, we may also consider the cases of space-times with $c \neq 0$. Unlike the asymptotically flat metrics, and looking at equation for the Weyl scalar $\psi_2$, it must be noticed that the $l = 1$ modes can actually yield a gravitational perturbation. In order for the potential $V(r)$ to be everywhere positive, and thus implying stability for these wormholes too, the following lower bound has to be imposed for the discussed constant $1 > c > 1/3$, where once again the condition $1 - b(r)/r \geq 0$ is required to draw this conclusion. On the other hand, if $c < 1/3$, then $V(r)$ becomes negative in the region near the throat of the wormhole, potentially leading to the instability of one of the $l = 1$ modes.

To finalize this section we provide some examples of this class of stable Morris-Thorne wormholes in table 1. They are easily obtained utilizing equation for the shape function. This process requires only of a well-behaved and bounded redshift function as input and so, can be used to yield as many space-times as functions that exist of this type. Note that the asymptotically flat with $\Phi(r) = 0$ case reduces to the well-known Ellis wormhole, which additionally is a solution of the Einstein-scalar field equations with a negative sign. Unfortunately, since all of these wormholes belong to the family of Morris-Thorne metrics, they violate the energy conditions at least near their throats.

Interestingly enough, and though the $l = 0$ modes do not generate a gravitational perturbation, the potential $V(r)$ we deduce here reduces to that studied in for the Ellis metric when inserting the $l = 0$ value. In those works the instability of that wormhole follows due to their corresponding potential being negative. This indicates that the angular dependance of the solution proposed here is crucial to deduce stability. Of course, the reason why we obtain a different result lies in the type of perturbation we have analyzed during this work.

VI. A STABLE PHANTOM SCALAR FIELD WORMHOLE

In section III we mentioned that the master equation derived there is valid for solutions of the Einstein-scalar field equations. In fact, one of the examples of stable Morris-Thorne wormholes shown in table 1 is indeed a solution of this type, namely the Ellis metric. In what follows we will present one last example of a stable wormhole supported by a phantom scalar field, i.e., a solution to $R_{\mu\nu} = -\nabla_\mu \phi \nabla_\nu \phi$. This space-time was found in and interpreted as a rotating scalar field wormhole. Here, we will focus on its static version since our master equation can only be applied to that reduced form of the metric. Its line element in Boyer-Lindquist coordinates is

5 To obtain its more familiar line element, the transformation from the radial coordinate $r$ to $r_* = \pm \sqrt{r^2 - b_0^2}$ is needed. In this case the $r_*$ coordinate is the proper radial distance.
\[ ds^2 = f dt^2 - \frac{1}{f} \left[ dr^2 + (r^2 - 2rr_1 + r_1^2) d\Omega^2 \right], \]

with \( f = e^{-\phi_0(\lambda - \pi/2)} \) and \( \lambda = \arctan \left( \frac{(r - r_1)/\sqrt{r_0^2 - r_1^2}} {r - r_0} \right) \). In this coordinate system we have for the Boyer-Lindquist radius that \(-\infty < r_0 < \infty\), covering this way both universes. The quantities \( r_0 \) and \( r_1 \) are constant parameters whose units are that of length, and for which \( r_0^2 > r_1^2 \). The scalar field is given by \( \phi = \sqrt{2 + \phi_0^2/2(\lambda - \pi/2)} \), being \( \phi_0 \) a constant without units. In this wormhole the throat joins two asymptotically flat sides, nevertheless, these sides are not symmetrical. This can be seen when taking the asymptotic limits of the \( f \) function,

\[ \lim_{r \to -\infty} f = 1, \quad \lim_{r \to \infty} f = e^{\phi_0 \pi}. \]

By rescaling the \( t \) and \( r \) coordinates to \( t_- = e^{\phi_0 \pi/2} t \) and \( r_- = e^{-\phi_0 \pi/2} r \), it can be realized that indeed the other side of the throat is asymptotically flat as well. The wormhole becomes symmetric only if \( \phi_0 = 0 \), in which case, the line element reduces to that of the Ellis metric. It results convenient to replace the coordinate \( r \) with \( x = (r - r_1)/L \), where \( L^2 = r_0^2 - r_1^2 \). Thus,

\[ ds^2 = f dt^2 - \frac{L^2}{f} \left[ dx^2 + (x^2 + 1) d\Omega^2 \right], \quad (39) \]

and \( \lambda = \arctan x \). In these coordinates the throat of the wormhole is located at \( x = 0 \), while the upper and lower universes are described by \( x > 0 \) and \( x < 0 \), respectively.

To obtain the equation that governs the gravitational perturbations of this space-time we proceed with the same scheme as in the previous section. The spin coefficients and operators are

\[
\begin{align*}
\kappa_+ &= -\pi_+ = \frac{i \cot \theta}{2L} \sqrt{\frac{f}{2(x^2 + 1)}}, \\
\pi_+ + \kappa_- &= \frac{2x + \Phi_0}{2L(x^2 + 1)} \sqrt{\frac{f}{2}}, \\
\pi_- - \kappa_- &= -\Phi_0 \frac{1}{2L(x^2 + 1)} \sqrt{\frac{f}{2}}, \\
\delta_+ &= \frac{1}{L} \sqrt{\frac{f}{2(x^2 + 1)}} \frac{\partial}{\partial x}, \\
\delta_- &= \frac{i}{L} \sqrt{\frac{f}{2(x^2 + 1)}} \frac{\partial}{\partial \theta}, \\
D &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{f}} \frac{\partial}{\partial t} + \frac{1}{L \sin \theta} \sqrt{\frac{f}{x^2 + 1}} \frac{\partial}{\partial \phi} \right). \quad (40)
\end{align*}
\]

They yield the following expression when inserted into the master equation (27), along with the assumption of a similar ansatz as the one used throughout this paper \( Q = e^{i \omega t} X(x) \sin \theta dP_l(\cos \theta)/d\theta \),

\[- \frac{\omega^2}{f} Q + \frac{l(l+1)f}{L^2(x^2 + 1)} Q - \sqrt{f} \frac{\partial}{\partial x} \left( \sqrt{f} \frac{\partial Q}{\partial x} \right) + \frac{\Phi_0 f}{2L^2(x^2 + 1)} \frac{\partial Q}{\partial x} + \frac{3f}{L^2(x^2 + 1)^2} \left( \Phi_0 x + \frac{\phi_0^2}{4} - 1 \right) Q = 0. \]

Once again, a change of radial coordinate is needed to arrive at an eigenvalue equation as \( \mathcal{B} \), this is,

\[ \frac{d}{dx_*} = f \frac{d}{L \, dx} \]

Since \( f \) is regular for all \( x \in \mathbb{R} \) and because of the asymptotic form of said function at both infinities, the new coordinate ranges over the values \(-\infty < x_* < \infty \). A suitable integration constant can also be picked so that the throat is described by \( x_* = 0 \). As a result of this transformation, we have that

\[ \frac{d^2 X}{dx_*^2} - (V(x) - \omega^2) X = 0, \quad (41) \]

where now

\[ V(x) = \frac{f^2}{L^2(x^2 + 1)} \left[ l(l + 1) + \frac{3}{x^2 + 1} \left( \phi_0 x + \frac{\phi_0^2}{4} - 1 \right) \right]. \]
In this case equation (11) defines an eigenvalue problem for the operator $\mathcal{H} = -d^2/dx_+^2 + V(x)$ in the $L^2(\mathbb{R}, dx_+)$ space of square-integrable functions $X(x)$. Its properties are the same as those of the previous case in section V. Additionally, it can be easily verified that the second term that appears inside brackets in the expression of $V(x)$ has a global minimum $u_{\text{min}} = -3$ at the coordinate value $x = -\phi_0/2$. Hence, appealing to the fact that the $l = 2$ vibrational modes are the lowest possible, the potential $V(x)$ is strictly positive for all $x \in \mathbb{R}$. By the same arguments as those mentioned for the former class of Morris-Thorne wormholes, we can conclude that this scalar field wormhole is stable when perturbed by vacuum-like gravitational perturbations of odd-parity.

As mentioned before, and just like in the class of Morris-Thorne metrics previously discussed, the Ellis space-time is again a particular case of this phantom scalar field wormhole when the parameter $\phi_0$ vanishes\(^6\). The wormhole presented here also violates the energy conditions as a result of it being a solution of the Einstein-scalar field equations with a negative sign.

**CONCLUSIONS AND SOME ADDITIONAL COMMENTS**

We have utilized the Newman-Penrose formalism to obtain a so-called master equation that describes the linear behavior of gravitational perturbations in stationary and spherically symmetric space-times. The perturbations were assumed to be vacuum-like and of odd-parity in the Regge-Wheeler gauge. This framework allowed us to write the derived master equation in a compact (and may we dare say elegant) manner through the use of the spin coefficients and operators that characterize the formalism. Our master equation is not applicable, though, to the whole generality of space-times with spherical symmetry; this is due to a constraint on certain components of the Ricci tensor that has to be obeyed. Despite this, we showed that it is well-suited to analyze some interesting examples of metrics that describe wormholes, for instance, the solutions of the Einstein-scalar field equations. Other space-times that were found to be within the range of validity of our master equation belong to the family of Morris-Thorne wormholes. In fact, they define a particular class of wormholes that were concluded to be stable after we applied to them the aforementioned master equation. The explicit metric components of some of this type of space-times were presented too. Finally, we gave one last example of a static scalar field wormhole that, according to the properties of its corresponding master equation, is stable against the perturbations here studied.

It should be borne in mind that, while our results indicate stability for some wormholes, it is only with respect to perturbations of odd-parity. Future developments of this work include the study of their even-parity counterparts within the Newman-Penrose formalism. However, the complexity of the calculations involved for this purpose increases compared to the odd case. Another interesting aspect to determine is the possibility to generalize the scheme presented here for gravitational perturbations in the context of the tetrad formalism to axially symmetric space-times. This in turn implies a generalization of the Regge-Wheeler gauge to this kind of metrics. Yet again, the whole process may require of lengthy calculations that hopefully are still manageable from an analytical approach.

**APPENDIX A**

Here we show all the relevant quantities of the Newman-Penrose formalism calculated for metric (11) with background tetrad (11) and perturbation matrix (14). We will reference the equations of the NP paper from which our results are derived. To simplify notation, the use of the tilde for background quantities will be dropped and the hat will be kept for the perturbation terms. Thus, any quantity or operator without a hat should be understood to be of the background space-time, except for the perturbations functions $f_0$ and $f_1$.

From (NP 4.1a) and equation (8) of our text, the perturbation term of the spin coefficients is given by

\[
\dot{k} = \dot{Z}_{131} = D f_1 - \frac{1}{2} \delta f_0,
\]

\[
\dot{\nu} = -\dot{Z}_{242} = \Delta f_1 - \frac{1}{2} \delta^* f_0,
\]

\[
\dot{\rho} = \dot{Z}_{431} = (-\delta_+ + \kappa_+ - \tau_+) f_1,
\]

\[
\dot{\tau} = -\dot{Z}_{142} = 0,
\]

\[
\dot{\tau} = \dot{Z}_{231} = 0,
\]

\[
\lambda = -\dot{Z}_{442} = 0,
\]

\(^6\) For this particular case the relation between the proper radial length $r_+$ and the coordinates $x = x_+$ is $L x = r_+$, with $L = b_0$. 

\[ \hat{\mu} = -\hat{Z}_{342} = (\delta_- - \kappa_+ + \pi_+)f_1, \quad \hat{\alpha} = \hat{Z}_{331} = 0, \]
\[ \hat{\alpha} = \frac{1}{2}(\hat{Z}_{421} - \hat{Z}_{443}) = \frac{1}{2}(\nu f_0 - D f_1), \quad \hat{\beta} = \frac{1}{2}(\hat{Z}_{321} - \hat{Z}_{343}) = -\frac{1}{2}(\kappa f_0 - D f_1), \]
\[ \hat{\epsilon} = \frac{1}{2}(\hat{Z}_{121} - \hat{Z}_{143}) = \frac{1}{2}((-\delta_- + \kappa_+ - \pi_+)f_1 - \Delta f_0), \]
\[ \hat{\gamma} = \frac{1}{2}(\hat{Z}_{221} - \hat{Z}_{243}) = \frac{1}{2}((\delta_- - \kappa_+ + \pi_+)f_1 - D f_0), \]
\[ (A.1) \]

with the definitions \( \delta_\pm = (\delta \pm \delta^*)/2, \kappa_\pm = (\kappa \pm \kappa^*)/2, \) and \( \pi_\pm = (\pi \pm \pi^*)/2. \) In terms of the metric components these newly defined coefficients and operators take the explicit form

\[ \kappa_+ = -\pi_+ = \frac{i \cot \theta}{2\sqrt{2}g_2}, \quad \pi_- + \kappa_- = -2\alpha = \frac{g'_2}{2g_2\sqrt{2}g_1}, \quad \pi_- - \kappa_- = \frac{g_0'}{2g_0\sqrt{2}g_1}, \]
\[ \delta_+ = \frac{1}{\sqrt{2g_1}} \frac{\partial}{\partial r}, \quad \delta_- = \frac{i}{\sqrt{2g_2}} \frac{\partial}{\partial \theta}, \]
\[ (A.2) \]

where a prime in this set of equations denotes derivation with respect to the radial coordinate \( r. \) Note that \( \delta^*_+ = \delta_+, \delta^*_- = -\delta_-, \) and that \( \kappa_-, \pi_- \in \mathbb{R} \) while \( \kappa_+, \pi_+ \) are purely imaginary. With this notation, identity [13] can be expressed as

\[ \delta_+ \kappa_+ = 2\alpha \kappa_+. \]
\[ (A.3) \]

The linearized perturbed components of the Ricci tensor in tetrad form, sometimes called the Ricci identities, can be computed by the Newman-Penrose equations (NP 4.2). Thereby, we obtain

\[ D\hat{\rho} - \delta^* \hat{\kappa} = -\kappa^* \hat{\tau} - \kappa \tau - \kappa(3\hat{\alpha} + \hat{\beta}^* - \hat{\pi}) - \hat{\kappa}(3\alpha + \beta^* - \pi) + \hat{\Phi}_{00}, \]
\[ (NP \ 4.2a) \]
\[ D\hat{\sigma} - \delta^* \hat{\kappa} = -(\tau - \pi^* + \alpha^* + 3\beta)\kappa - (\tau^* - \pi^* + \alpha^* + 3\beta)\kappa + \hat{\psi}_0, \]
\[ (NP \ 4.2b) \]
\[ D\hat{\alpha} - \delta^* \hat{\epsilon} = (\hat{\rho} + \hat{\epsilon}^* - 2\hat{\epsilon})\alpha + \beta \hat{\epsilon}^* - \beta^* \hat{\epsilon} - \kappa \lambda - \kappa^* \hat{\gamma} + (\hat{\epsilon} + \hat{\rho})\pi + \hat{\Phi}_{10}, \]
\[ (NP \ 4.2c) \]
\[ D\hat{\gamma} - \Delta \hat{\epsilon} = (\hat{\tau} + \hat{\pi}^*)\alpha + (\hat{\tau}^* + \hat{\pi})\beta + \tau \pi + \tau^* \pi - \nu \kappa - \nu \kappa^* + \hat{\psi}_2 - \hat{\Lambda} + \hat{\Phi}_{11}, \]
\[ (NP \ 4.2d) \]
\[ D\hat{\lambda} - \delta^* \hat{\nu} = 2\pi \hat{\pi} + (\alpha - \beta^*)\hat{\pi} + (\hat{\alpha} - \hat{\beta}^*)\pi - \nu \kappa^* - \nu \kappa + \hat{\psi}_2 + 2\Lambda, \]
\[ (NP \ 4.2e) \]
\[ D\hat{\mu} - \delta^* \hat{\nu} = 3(\alpha^* + \beta^* + \tau^*)\hat{\nu} + (3\hat{\alpha} + \hat{\beta}^* + \pi^* + \hat{\tau}^*)\nu + \hat{\Psi}_4, \]
\[ (NP \ 4.2f) \]
\[ \delta \hat{\alpha} - \delta^* \hat{\beta} = \alpha \hat{\alpha}^* + \hat{\alpha} \alpha^* + \beta \hat{\beta}^* + \beta^* \alpha - 2\alpha \beta - 2\alpha \hat{\beta} - 2\hat{\alpha} \beta - \hat{\psi}_2 + \hat{\Lambda} + \hat{\Phi}_{11}, \]
\[ (NP \ 4.2g) \]
\[ \delta \hat{\beta} - \delta^* \hat{\pi} = -\nu^* \hat{\pi} - \nu \hat{\pi}^* + (\tau - 3\beta - \alpha^*)\nu^* + (\hat{\tau} - 3\hat{\beta} - \hat{\alpha}^*)\nu + \hat{\Phi}_{22}, \]
\[ (NP \ 4.2h) \]
\[ \delta \hat{\gamma} - \Delta \hat{\beta} = (\tau - \beta - \alpha^*) \hat{\gamma} + \hat{\mu} \tau - \hat{\sigma} \nu - \hat{\epsilon} \nu^* - \beta (\hat{\gamma} - \hat{\gamma}^* - \hat{\mu}) + \alpha \hat{\lambda}^* + \hat{\Phi}_{12}, \]
\[ (NP \ 4.2i) \]
\[ \delta \hat{\tau} - \Delta \hat{\sigma} = (\tau + \beta - \alpha^*) \hat{\tau} + (\hat{\tau} + \hat{\beta} - \hat{\alpha}^*) \tau - \kappa \nu^* - \kappa \nu + \hat{\Phi}_{22}, \]
\[ (NP \ 4.2j) \]

where we have taken advantage of the property \( \hat{D}_m \phi = 0 \) that our particular choice of tetrad gives us for arbitrary background scalars \( \phi. \) We have also omitted the background terms that should appear on both sides of these equations since they cancel each other out.

The commutators (NP 4.4) of the background differential operators of the formalism are

\[ [\Delta, D] = 0, \quad [\delta, D] = -\pi^* D + \kappa \Delta, \quad [\delta, \Delta] = -\nu^* D + \tau \Delta, \quad [\delta^*, \delta] = 2\alpha (\delta - \delta^*), \]
\[ (A.4) \]

which can be utilized to derived the commutation relations for our previously introduced operators \( \delta_\pm, \)

\[ [\delta_+, D] = \mp \pi_+ D + \kappa_+ \Delta, \quad [\delta_+, \Delta] = \kappa_\mp D \mp \pi_\mp \Delta, \quad [\delta_-, \delta_+] = -2\alpha \delta_-, \]
\[ (A.5) \]

When applying these commutators to \( \varphi \)-independent scalar quantities \( \phi, \) as will always be the case in this work, there is a further simplification \( [\delta_-, D] \phi = [\delta_-, \Delta] \phi = 0, \) since \( D \phi = \Delta \phi \) and \( \kappa_+ + \pi_+ = 0. \)
After some considerable algebraic steps, reduced equations for the linearized Ricci identities can be obtained by inserting the perturbed spin coefficients (A.1) into the (NP 4.2) equations presented above, along with the further aid of the commutators in (A.3) and the spin coefficient properties (12,17). Doing so yields

\[
\hat{\Phi}_{00} = -\hat{\Phi}_{22} = \frac{1}{2} \left[ (\delta_+ + \kappa_+ + 3\pi_-)\delta_+ - (\delta_- + 2\kappa_+)\delta_- + 4(\kappa_+^2 - \kappa_-^2) \right] f_0 \\
\quad - (\delta_+ - 6\alpha) D f_1,
\]

\[
\hat{\Phi}_{12} = \hat{\Phi}_{21} = -\hat{\Phi}_{01} = -\hat{\Phi}_{10} = \frac{1}{2} \left[ D^2 + (\delta_+ + \delta_- + \kappa_- + 3\pi_- + 4\kappa_+)(\delta_- - 2\kappa_+) \right] f_1 \\
\quad - \frac{1}{4} (\delta_+ + \delta_- + \pi_- - 3\kappa_- - 2\kappa_+) D f_0,
\]

\[
\hat{\Phi}_{11} = \hat{\Phi}_{20} = \hat{\Phi}_{02} = \hat{\Lambda} = 0. \tag{A.6}
\]

For the Weyl scalars of interest we obtain

\[
\hat{\psi}_0 = -\hat{\psi}_4^* = \frac{1}{2} (\delta_+ + \delta_- - \kappa_- + \pi_- + 2\kappa_+) \left[ (\delta_+ + \delta_-) f_0 - 2 D f_1 - 2(\kappa_+ + \kappa_-) [D f_1 + (\kappa_+ + \kappa_-) f_0] \right],
\]

\[
\hat{\psi}_2 = \delta_- D f_1 + (\kappa_- \delta_- - \kappa_+ \delta_+) f_0. \tag{A.7}
\]

**APPENDIX B**

A more detailed proof of the consistency condition (23) of the linearized Einstein field equations under vacuum-like perturbations is presented in this appendix. We also show that a background space-time that obeys the Einstein-scalar field equations satisfies this condition.

When applying the operator \( D \) to it, and using the commutators (A.4), the component \( \hat{\mathcal{R}}_{03} \) of the system (22) reduces to

\[
D\hat{\mathcal{R}}_{03} = 2(\delta_+ - 4\alpha + 2\pi_-) D^2 f_1 \\
+ \left[ (\delta_- + 2\kappa_+) \delta_- - (\delta_+ + 4\pi_-)(\delta_- + \pi_-) - (\delta_- + 3\kappa_- + \pi_-) \right] f_1 + 2 f_1[\delta_- + 4\kappa_+]D f_0. \tag{B.1}
\]

Expression (23) for the perturbation function \( f_0 \) can now be substituted in (B.1). The resulting terms can be rearranged as

\[
D\hat{\mathcal{R}}_{03} = 2(\delta_+ - 4\alpha + 2\pi_-) \left[ (D^2 + (\delta_- + 4\kappa_+)(\delta_- - 2\kappa_+)(\delta_+ - 2\pi_-))(\delta_- + 2\pi_-) \right] f_1 + 2 f_1[\delta_- + 4\kappa_+]D f_0
\]

Careful attention must be paid on the order in which the operators are being applied. The previous equation can be simplified by using (24) and defining the quantities \( \mathcal{A} = 2(\delta_- + 4\kappa_+)(\kappa_- + 2(3\Lambda - \Phi_{11} + \Phi_{00})) \), as well as \( \mathcal{B} = (\delta_+ + \kappa_- + 3\pi_-)(\kappa_- + 2\kappa_+ - 3\Lambda + \Phi_{11}) \). Despite the appearance of differential operators in these quantities, \( \mathcal{A} \) and \( \mathcal{B} \) should not be understood as such. They are merely scalar quantities, the operators \( \delta_\pm \) in them are to be applied only to the spin coefficients \( \kappa_\pm \). Hence, we can write

\[
D\hat{\mathcal{R}}_{03} = 2 \left[ (\delta_+ - 4\alpha + 2\pi_-) \mathcal{A} - \mathcal{B}(\delta_+ + 2\pi_-) \right] f_1. \tag{B.2}
\]

Expanding the first term of (B.2) results in

\[
D\hat{\mathcal{R}}_{03} = 2 f_1 \left[ (\delta_+ - 4\alpha + 2\pi_-) \mathcal{A} - 2 \mathcal{B} \right] + 2(\mathcal{A} - \mathcal{B})\delta_+ f_1. \tag{B.3}
\]

Using the background Ricci identities of the Newman-Penrose formalism (NP 4.2a) and (NP 4.2b), it can be proven that \( \mathcal{A} = \mathcal{B} = 4\kappa_+^2 - (\psi_0 + \psi_0^*)/2 - \Phi_{00} - 2(3\Lambda - \Phi_{11}) \). Another helpful identity, consequence of (A.3) and (A.5), is \((\delta_+ - 4\alpha)(\delta_- + 4\kappa_+)\kappa_+ = 0\). Equation (B.3) thereby simplifies to
\[ D\bar{\mathcal{R}}_{03} = -4f_1(\delta_+ - 4\alpha)(3\Lambda - \Phi_{11} + \Phi_{00}). \]

From the fact that on the previous equation only \( f_1 \) is time-dependent, and since \( D \) contains a time derivative, it follows that in order for the \( \bar{\mathcal{R}}_{03} \) component to vanish, the next condition has to hold

\[ (\delta_+ - 4\alpha)(3\Lambda - \Phi_{11} + \Phi_{00}) = 0. \] (B.4)

This is the result that was anticipated in section III of the main text.

Next we demonstrate that if \( \bar{R}_{\mu\nu} = \pm \nabla_{\mu}\phi \nabla_{\nu}\phi \) for a given static and spherically symmetric (even axially symmetric) space-time, which is true for the Einstein field equations minimally coupled to a massless scalar field \( \phi \) without a self-interaction potential, then (B.4) is satisfied when the null tetrad is given by \( \{1^n, n^n, m^n, \bar{m}^n\} \) in which the vectors \( l^\mu \) and \( n^\mu \) are appropriate combination of the Killing vectors \( \partial/\partial x^0 \) and \( \partial/\partial x^3 \), we have that \( \bar{R}_{\mu\nu}l^\mu n^\nu = 2(3\Lambda - \Phi_{11}) = 0 \). In the first equality sign we have used the (NP 4.3b) equations, the second equality is due to the components \( R_{ij} \) being zero. Furthermore, \( \Phi_{00} = -\bar{R}_{\mu\nu}l^\mu n^\nu/2 = 0 \). Therefore we obtain the desired result, \( 3\Lambda - \Phi_{11} + \Phi_{00} = 0 \), and (B.4) holds.

One last comment may be in order regarding the perturbations discussed throughout this paper in the context of this kind of space-times. Their field equations are of the type

\[ R_{\mu\nu} = \pm \nabla_{\mu}\phi \nabla_{\nu}\phi, \]
\[ \nabla^\mu \nabla_\mu\phi = 0. \] (B.5)

In the case of spherically symmetric space-times, both of this equations can be shown to hold up to first order of the metric perturbation \( \{\mathcal{L}\} \), with the additional condition that one does not add a perturbation term to the scalar field \( \phi \), i.e., \( \phi = 0 \). We assume, of course, that the background metric satisfies its particular set of field equations. Hence, the equal sign holds for the zero order terms of the linearized field equations (B.5). Since we are considering the vacuum-like property \( \bar{R}_{\mu\nu} = 0 \) too, the linearized version of the first equation is evidently satisfied when \( \phi = 0 \). For the second field equation this is not so easily seen, but is equally trivial. Its first order term is given by

\[ [\nabla^\mu \nabla_\mu\phi]^{(1)} = h^{\mu\nu}((\partial_{\mu\nu}^2 - \bar{\Gamma}_\mu^\alpha \nabla_\alpha)\phi - \bar{g}^{\mu\nu} \hat{\Gamma}_\mu^\alpha \nabla_\alpha\phi], \] (B.6)

where again a tilde denotes background quantities and a hat their respective perturbations, also \( \phi = \bar{\phi} \). All of the terms in the past equation vanish when performing their corresponding index contractions, this is mainly due to the exclusive \( r \)-dependence of the scalar field \( \phi \), combined with the zero entries of the background and perturbation inverse metrics, along with the vanishing of the following connection coefficients

\[ \bar{\Gamma}_\mu^\nu = \bar{\Gamma}_{\tau\nu} = \bar{\Gamma}_{\nu\tau} = \bar{\Gamma}_{\nu\tau} = 0. \] (B.7)

Thus, \( [\nabla^\mu \nabla_\mu\phi]^{(1)} = 0 \) and the vacuum-like perturbations of odd-parity studied in this paper are compatible with the linearized field equations (B.5).

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