Einstein-Maxwell-Chern-Simons Black Holes

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Abstract. Black holes in 5-dimensional Einstein-Maxwell-Chern-Simons (EMCS) theory and their intriguing properties are discussed. For the special case of the CS coupling constant $\lambda = \lambda_{SG}$, as obtained from supergravity, a closed form solution is known for the rotating black holes. Beyond this supergravity value, the EMCS black hole solutions can e.g. exhibit nonuniqueness and form sequences of radially excited solutions. In the presence of a negative cosmological constant the black holes can possess an extra-parameter corresponding to a magnetic flux in addition to the mass, electric charge and angular momenta. This latter family of black holes possesses also a solitonic limit. Finally, a new class of squashed EMCS black hole solutions is discussed.

1. Introduction
The properties of black holes in higher dimensions can differ in many respects from those known in four dimensions. Most prominently, asymptotically flat vacuum black holes may possess a non-spherical horizon topology in more than four dimensions. But also for black holes with a spherical horizon surprises arise, as soon as a U(1) gauge field is coupled, i.e., Einstein-Maxwell (EM) black holes are considered. In odd dimensions the presence of a Chern-Simons (CS) term allows for further intriguing properties of the resulting Einstein-Maxwell-Chern-Simons (EMCS) black holes. In the following first asymptotically flat black hole solutions of EMCS theory will be discussed. Then a negative cosmological constant will be included, making contact with the celebrated AdS/CFT correspondence, a central topic of the conference.

2. Asymptotically Flat EMCS Black Holes
In odd dimensions $D = 2n + 1$ the Einstein-Maxwell action may be supplemented by an $\mathcal{A}F$ Chern-Simons term. In 5 dimensions the EMCS action reads

$$S = \int \frac{1}{16\pi G_5} \left\{ \sqrt{-g} \left( R - F_{mn} F^{mn} \right) - \frac{2\lambda}{3\sqrt{3}} \varepsilon^{mnpqr} A_m F_{np} F_{qr} \right\} d^5x \quad (1)$$

with Newton’s constant $G_5$, curvature scalar $R$, gauge potential $A_m$, field strength tensor $F_{mn}$, and CS coupling constant $\lambda$. In EM theory $\lambda = 0$, while $\lambda = 1$ in the bosonic sector of minimal
$D = 5$ supergravity. Variation of the action leads to the Einstein equations (8\pi G = 1)

$$G_{mn} = 2 \left( \mathcal{F}_{mr} \mathcal{F}_{n}^{r} - \frac{1}{4} \mathcal{F}_{rs} \mathcal{F}^{rs} \right),$$

which are unchanged w.r.t. the pure Maxwell case, and to the Maxwell-CS equations

$$\nabla_n \mathcal{F}^{mn} + \frac{\lambda}{2\sqrt{3}} \epsilon^{mnpr} \mathcal{F}_{np} \mathcal{F}_{qr} = 0.$$  \hspace{1cm} (3)

Clearly, the CS term breaks the charge symmetry $Q \rightarrow -Q$ present in Maxwell theory.

### 2.1. D = 5 Einstein-Maxwell Theory

The $D$-dimensional generalizations of the Kerr black holes are given by the Myers-Perry (MP) black holes, which possess $N = \lfloor \frac{D-1}{2} \rfloor$ independent angular momenta $J_i, i = 1, \ldots, N$ [1]. The inclusion of a Maxwell field, however, most of the time prevents the construction of black hole solutions in closed form, even if in odd dimensions the angular momenta are chosen to be of equal magnitude, i.e., when only cohomogeneity-1 solutions are sought. In five dimensions an appropriate ansatz for the metric and the gauge potential is given by

$$ds^2 = -F_0(r)dt^2 + F_1(r)dr^2 + \frac{1}{4} F_2(r)(\sigma_1^2 + \sigma_2^2) + \frac{1}{4} F_3(r)(\sigma_3 - 2\omega(r)dt)^2,$$

$$\mathcal{A} = a_0(r)dt + a_\varphi(r)\frac{1}{2}\sigma_3$$ \hspace{1cm} (4)

with $\sigma_1^2 + \sigma_2^2 = \bar{\theta}^2 + \sin^2 \bar{\theta}d\psi^2, \sigma_3 = d\phi + \cos \bar{\theta}d\psi$, and functions $F_0, \ldots, F_3, \omega, a_0$ and $a_\varphi$.

In this case (i) perturbative calculations in the charge or in the rotation parameter can be performed, (ii) the solutions can be obtained numerically, or (iii) near-horizon solutions can be found in the extremal limit. To obtain the latter, an adequate ansatz for the metric and the gauge potential in five dimensions is given by [2]

$$ds^2 = v_1 \left( \frac{dr^2}{r^2} - r^2 dt^2 \right) + v_2 \left[ \sigma_1^2 + \sigma_2^2 + v_3(\sigma_3 - kr dt)^2 \right],$$

$$\mathcal{A} = q_1 r dt + q_2 (\sigma_3 - kr dt)$$ \hspace{1cm} (5)

with constants $v_1, v_2, v_3, q_1, q_2$ and $k$.

An analogous ansatz holds in more than five dimensions. Solving the resulting algebraic system of equations for the constants leads to two distinct solutions. The first solution starts at the extremal MP black hole and satisfies $J = \sqrt{2(D-3)}A_H$, thus its angular momentum is proportional to its horizon area. The second solution starts at the extremal Reissner-Nordström (RN) black hole and satisfies $J = (D-1)J_H$, thus its angular momentum is proportional to its horizon angular momentum. As seen in Figs. 1, 2 both branches intersect at a critical solution (black cross) and continue beyond. Interestingly, only the name-giving parts of the MP and RN solutions up to the intersection point are realized globally [3, 4]. The domain of existence of these EM black holes is seen in Fig. 3 and corresponds to the area enclosed by the $\lambda = 0$ curve.

### 2.2. D = 5 minimal supergravity

For the case $\lambda = 1$ representing $D = 5$ minimal supergravity the black hole solutions are known in closed form [5, 6, 7]. The domain of existence is also seen in Fig. 3 and delimited by the $\lambda = 1$ curve. The charge symmetry $Q \rightarrow -Q$ is clearly broken here. The vertical line represents the extremal BMPV black holes [5]. For these black holes, as the charge $Q$ is kept fixed and the angular momentum $J$ increases, the mass $M$ remains constant.
Figure 1. Odd-$D$ EM near-horizon solutions: area vs charge (scaled with appropriate powers of the angular momentum).

Figure 2. Same as Fig. 1 for the horizon angular momentum.

Figure 3. Domain of existence of $D = 5$ EMCS black holes: angular momentum vs charge (scaled with appropriate powers of the mass) for CS coupling $\lambda = 1$ and $\lambda = 0$.

Figure 4. Same as Fig. 3 including CS coupling $\lambda = 2$. The shaded area represents counterrotating black holes. $\Omega = 0$ black holes are also indicated.

When considering the first law $dM = TdS + 2\Omega dJ + \Phi dQ$ for these BMPV black holes, one realizes that this implies that their horizon angular velocity $\Omega$ must be zero, while the global angular momentum is finite. Thus angular momentum is stored in the Maxwell field, where a negative fraction of the angular momentum resides behind the horizon. While the effect of rotation is to deform the horizon into a squashed 3-sphere [8]. The set of BMPV solutions ends at a critical solution with vanishing area.

2.3. $D = 5$ EMCS theory: $\lambda \neq 1$

Let us now consider the CS coupling as a free parameter and increase it above the supergravity value [9]. Then it becomes clear that the supergravity value represents the borderline between stability and instability, since at $\lambda = 1$ a zero mode is present, leading to a rotational instability for larger values of $\lambda = 1$ [8, 9].

Then solutions with vanishing horizon angular velocity no longer form a part of the boundary
of the domain of existence (except for the critical solution with vanishing area at the cusp), but reside well within the domain of existence. Now this part of the boundary is formed by a new type of black hole solutions. These are counterrotating in the sense, that the horizon angular velocity and the global angular momentum carry opposite signs [9]. In Fig. 4 all counterrotating black holes are represented by the shaded area.

![Figure 5](image-url) **Figure 5.** \( D = 5 \) EMCS near-horizon and global solutions: area vs angular momentum (\( \lambda = 5 \)). The asterisk marks the extremal static RN black hole.

For CS coupling \( \lambda > 2 \) further surprises appear. First of all, uniqueness of black holes with spherical horizon topology no longer holds [9]. Here distinct black holes with the same global charges are present. Second, there arises a strong mismatch between the near-horizon solutions and the global solutions [10, 11]. In fact, there are near-horizon solutions that correspond to (i) no global solution, (ii) one global solution, and (iii) many global solutions (possibly even infinitely many).

The area versus the angular momentum of the near-horizon solutions is illustrated in Fig. 5 for CS coupling \( \lambda = 5 \) for positive and negative charge. For comparison also the global solutions are included in the figure. Furthermore cusps and bifurcation points are noted. In particular, while well connected with the branches of near-horizon solutions, the RN solution is isolated from the \( Q < 0 \) global solutions. Moreover, the \( Q < 0 \) near-horizon solutions exhibit a non-static \( J = 0 \) solution, which corresponds to a set of distinct non-static global \( J = 0 \) solutions, that can be labeled by an integer \( n \). Thus there are black holes with a rotating horizon but vanishing global angular momentum. Also, the \( Q < 0 \) solutions always contain a degenerate \( (J \to -J) \) zero area solution.

The mass versus the angular momentum of the corresponding global solutions is exhibited in Fig. 6. Here the degeneracy present in the area versus angular momentum plot is lifted. Instead an intricate branch structure of the global solutions is revealed. In fact, in a regular manner, new branches of solutions arise, which form cusps and then bifurcate with previous branches. The cusps \( C \) and bifurcation points \( B \) are counted and labeled by integers in the figure. When the branches cross, a degenerate pair of \( J = 0 \) solutions appears, also labeled by a respective integer \( n \). As \( n \) increases, the mass of the rotating \( J = 0 \) solutions tends to the mass of the extremal RN black hole.

This intriguing branch structure for \( Q < 0 \) is also seen in Fig. 7, where the domain of existence of the EMCS solutions for \( \lambda = 5 \) is shown. While forming the boundary of the domain of existence, extremal EMCS solutions reside also deep within this domain along with the extremal static RN solution (black cross). Only the extremal \( n = 1 \) rotating \( J = 0 \) solutions
are part of the boundary. The solutions with vanishing area represent the boundary solutions at the cusps at maximal $Q/M$.

When analyzing the branch structure one realizes, that the integer number $n$ labeling the branches corresponds to the number of nodes of the gauge field function $a_\varphi$ as illustrated in Fig. 8 for $n = 1, \ldots, 7$. While solutions with more than thirty nodes have been constructed in a systematic way, one is lead to conjecture, that there is in fact an infinity sequence of solutions, $n = 1, \ldots, \infty$. Note, that the drag function of the metric increases its nodes in an analogous manner. This sequence of radially excited rotating black holes is reminiscent of other physical systems with radial excitations, such as the hydrogen atom.

3. EMCS Solutions with AdS Asymptotics

Let us now include a negative cosmological constant, $\Lambda = -6/L^2$, with AdS length scale $L$. Then the solutions are no longer asymptotically flat but asymptotically AdS. The dS/AdS generalizations of the MP black holes are known in closed form [12, 13]. But the corresponding EMCS black holes could be obtained in closed form so far only for the case of gauged supergravity ($\lambda = 1$), where in a particular limit they are known to preserve some amount of supersymmetry [14, 6, 7].

3.1. Charged solutions

We first address the question of how the presence of the negative cosmological constant affects the properties of the rotating black holes discussed above. The interesting new features present for the asymptotically flat solutions are basically retained in the presence of the cosmological term. In particular, when the CS coupling is sufficiently large, there appear counterrotating black holes. The black holes are no longer uniquely specified by their global charges. Instead an analogous branch structure arises, where the solution branches can be labeled by the node number of the magnetic gauge potential function.

For CS coupling in the vicinity of the supergravity value, we demonstrate in Fig. 9 as an example the $\lambda$-dependence of the area vs the charge and temperature. The extremal solutions form the lower boundary ($T_H = 0$). The charge symmetry breaking by the CS term is again clearly visible, with the zero area solutions only present for negative $Q$. 

![Figure 7](image1.png) **Figure 7.** Domain of existence of $D = 5$ EMCS black holes: angular momentum vs charge (scaled with appropriate powers of the mass) for CS coupling $\lambda = 5$.

![Figure 8](image2.png) **Figure 8.** $D = 5$ EMCS black holes: Node structure of the gauge field function $a_\varphi$ ($\lambda = 5$, $n = 1, \ldots, 7$). The numbering includes the node at infinity.
A new feature arises for small CS coupling $\lambda$. Here a near-horizon analysis shows, that the two types of branches (RN and MP for $\Lambda = 0$) no longer cross. This leads to a gap in the set of regular extremal black hole solutions, where the set of non-extremal solutions is limited by the so-called gap set, which seems to consist of singular solutions (as based on an extrapolation of the properties of the non-extremal solutions).

3.2. Magnetized solutions

Let us now include a new physical feature, namely a new parameter $c_m$ of the AdS solutions associated with the (finite) magnitude of the magnetic potential at infinity, i.e., for $r \to \infty$: $a_\phi \to c_m$. This parameter therefore determines the magnetic flux through the base space $S^2$ of the $S^3$ fibration of the $S^3$ (see Eqs. 4-5)

$$\Phi_m = \frac{1}{4\pi} \int_{S^2_{\infty}} F = -\frac{1}{2} c_m.$$  \hfill (8)

For simplicity we first address the case of static EM solutions, since, acting like a box, the AdS background also allows for static solitonic EM solutions [15]. In particular, by considering magnetic fields only, one obtains regular EM solitons in $D = 5$ (and higher odd dimension), which are characterized by the magnetic flux parameter $c_m$ [16]. Interestingly, there is a maximal value of $c_m$, where such solitons exist.

By imposing a regular horizon, static EM black hole solutions arise, which can be classified into two types. Figs. 10 and 11 show their horizon area and mass versus their temperature. In type I black holes the horizon area and the mass increase with the temperature. In contrast, in type II black holes both decrease as the temperature increases. Type II black holes can be deformed continuously into solitons, when the horizon size approaches zero. Thus they exist only below the maximal value of $c_m$ for solitons. Type I black holes on the other hand exist also for large values of $c_m$, but they become singular in the limit of vanishing horizon size.

Clearly the above black holes represent new families of static EM AdS black holes, whose properties differ from those of the known RN AdS black holes. This suggests that similar magnetized EMCS AdS solutions could also exist. However, in the EMCS case solutions with a magnetic field must necessarily rotate. Indeed rotating EMCS generalizations of the above soliton and black hole solutions exist [17]. These solutions then also carry an electric charge,
where we should now distinguish between the ordinary charge, the Page charge, and the \( R \) charge
\[
Q^{(R)} = -\frac{1}{2} \int_{S^3_\infty} \left( \tilde{F} + \frac{2\lambda}{3\sqrt{3}} A \wedge \tilde{F} \right),
\]
(9)
since the second term now contributes, and its prefactor differs for these three types of charges.

### 3.3. Magnetized squashed solutions

Let us now consider a last twist w.r.t. the plethora of EMCS solutions by considering solutions which are asymptotically only locally AdS. Thus we impose that the magnetized rotating EMCS solutions should asymptotically approach not a round \( S^3 \) sphere but instead a squashed sphere. As discussed in [18] the boundary metric can then be expressed as
\[
\begin{align*}
  ds_B^2 &= L^2 d\Omega^2_{(v)} - dt^2, \\
  d\Omega^2_{(v)} &= \frac{1}{4} \left( d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + v \cos \theta d\phi)^2 \right),
\end{align*}
\]
(10)
where \( v \) is a control parameter. Clearly, for \( v = 1 \) the \( S^3 \) sphere becomes round, whereas for \( v = 0 \) the solutions tend to AdS black strings and vortices, their boundary corresponding to \( S^2 \times S^1 \).

Then for a given \( \lambda \) a family of squashed magnetized soliton solutions arises smoothly from the AdS vacuum, which are labeled by the parameter \( v \) and satisfy the relation \( J = \Phi_m Q^{(R)} \). In the case of gauged supergravity (\( \lambda = 1 \)) these solitons retain some amount of supersymmetry [18]. In fact, the properties of these squashed susy solitons can be expressed simply in terms of the squashing parameter \( v \).

As one might expect, the solitons are related to a new class of squashed magnetized black holes. In the general case, these black holes are characterized by their mass, charge, angular momentum and magnetic flux parameter. In the limit of vanishing horizon radius solitons can be approached for an appropriate parameter choice. Families of black hole solutions for the case of gauged supergravity are exhibited in Figs. 12 and 13.

Generically the black holes possess an extremal limit with a finite horizon area. Among these extremal black holes a particular family stands out: the supersymmetric black holes. For these solutions the trace of the boundary stress tensors vanishes, yielding the condition \( c_m = \pm \frac{L}{\sqrt{3}} (1 - v^2) \). Like the susy solitons these susy black holes can be characterized by the squashing parameter \( v \).
This observation then leads to the following new picture for the classes of supersymmetric EMCS AdS black holes. Besides the previously known family of Gutowski-Reall black holes there exists a new family of squashed magnetized supersymmetric black holes. In fact, the new susy squared magnetized black holes intersect the Gutowski-Reall black holes at a critical configuration, where the squashing and magnetization vanish, i.e., $v = 1$ and $c_m = 0$.

Obviously, there are still many unanswered questions, and many avenues are open awaiting further investigations. This holds, in particular, w.r.t. the relevance of the above configurations in an AdS/CFT context.

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