A NON-ZERO-SUM REINSURANCE-INVESTMENT GAME WITH DELAY AND ASYMMETRIC INFORMATION

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Abstract. In this paper, we investigate a non-zero-sum stochastic differential reinsurance-investment game problem between two insurers. Both insurers can purchase proportional reinsurance and invest in a financial market that contains a risk-free asset and a risky asset. We consider the insurers’ wealth processes with delay to characterize the bounded memory feature. For considering the effect of asymmetric information, we assume the insurers have access to different levels of information in the financial market. Each insurer’s objective is to maximize the expected utility of its performance relative to its competitor. We derive the Hamilton-Jacobi-Bellman (HJB) equations and the general Nash equilibrium strategies associated with the control problem by applying the dynamic programming principle. For constant absolute risk aversion (CARA) insurers, the explicit Nash equilibrium strategies and the value functions are obtained. Finally, we present some numerical studies to draw economic interpretations and find the following interesting results: (1) the insurer with less information completely ignores its own risk aversion factor, but imitates the investment strategy of its competitor who has more information on the financial market, which is a manifestation of the herd effect in economics; (2) the difference between the effects of different delay weights on the strategies is related to the length of the delay time in the framework of the non-zero-sum stochastic differential game, which illustrates that insurers should rationally estimate the correlation between historical performance and future performance based on their own risk tolerance, especially when decision makers consider historical performance over a long period of time.

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1. **Introduction.** In recent years, cooperation and competition gradually become popular phenomena in society. The game between companies has been one of the hottest issues in modern financial and economic research. A “win-win” or “multi-win” situation, which belongs to non-zero-sum game, is a phenomenon that players are most willing to see. It states that the sum of the gains/losses of the players is not zero. In most instances, the player pays attention to the wealth level of others due to the comparison psychology (see [5], [25]). For a company, the past performance and the information obtained from the financial market usually affect its current wealth level. Generally, a good past performance may bring the company more gain due to the delay feature (i.e. the memory feature or past-dependence feature), such as [15]. The wealth level of the company depends on its own investment behavior which is related to the level of information and what is more, different companies usually have access to different levels of information (i.e. asymmetric information, refer to [26]). In view of these phenomena, we consider a non-zero-sum stochastic differential reinsurance-investment game problem with delay and asymmetric information between two insurers.

The optimal reinsurance-investment problem is a classic problem of actuarial area. Insurance is a traditional and effective way to deal with risks, but with the increase in wealth, the underlying risks of insurance operation are increasing. Usually, an insurer signs a reinsurance contract to transfer some claim risk to the reinsurer. In this sense, reinsurance is an important risk management tool for insurers. Moreover, in order to increase surpluses and strengthen solvency, the insurer’s investment in the financial market is another essential tool for risk control. In recent years, many scholars have studied the optimal reinsurance and investment under a variety of objectives, for example, minimizing the probability of ruin ([8],[12],[27], etc.), maximizing the minimal expected utility of terminal wealth over a family of real-world probability measures ([40],[23], etc.), maximizing the utility of the terminal wealth ([4],[36],[28],[11],[41],[42],[24], etc.).

The majority of these literatures do not consider the impact of cooperation or competition on strategy. In the contexts of finance and insurance, there has been a rapid development in extending the optimal reinsurance-investment problem under the framework of stochastic differential game, such as [30], [39]. [17] considered the problem of optimal investment when agents take into account their relative performance by comparison to their peers in the form of a non-zero sum stochastic differential game. The so-called relative performance means that each agent takes into account a convex combination of his wealth and the difference between his wealth and the average wealth of the other investors. Many scholars have introduced this concept into the field of actuarial insurance. Under the objective of maximizing the utility of the relative performance, [6] studied a non-zero-sum stochastic differential investment and reinsurance game between two insurers whose surplus processes were modulated by continuous-time Markov chains; [37] considered a reinsurance and investment game between two insurers who have different opinions about some extra information. More studies on stochastic differential game problem may be found in [31], [33], [21], [14], [13], etc. In this paper, we consider a non-zero-sum stochastic differential reinsurance and investment game problem between two insurers. The objective of the insurer is to maximize the expected utility of its performance relative to its competitor by selecting appropriate reinsurance and investment strategies.
only considered the impact of asymmetric information in the insurance market, not that in the financial market. As mentioned above, the insurer’s investment in the financial market is an important tool for risk control. Therefore, it is necessary to study the impact of asymmetric information on the financial market. The partial information model is an important class of models of asymmetric information. [34] and [22] studied optimal consumption and investment problem with partial information. [9] researched optimal investment problem with over and under-reaction to information. [18] analyzed the Merton portfolio optimization problem when the growth rate is an unobserved Gaussian process whose level is estimated by filtering from observations of the stock price. In this paper, we introduce the partial information into the game problem and assume that the two insurers have access to different levels of information on the financial market. More precisely, we assume that insurer 1 have full information of the financial market while insurer 2 can only observe the stock price.

In addition, insurer’s strategy depends on the exogenous capital instantaneous inflow into or outflow from current wealth due to its bounded memory feature. [35] and [2] considered optimal reinsurance and investment problem with delay under the mean-variance criterion and exponential utility function, respectively. [38] researched an optimal proportional reinsurance problem for the compound Poisson risk model with delay and multiple dependent classes of insurance business under the criterion of maximizing the mean-variance utility of the terminal wealth. [3] studied a portfolio optimization problem with delay under Cox-Ingersoll-Ross model. However, there is only limited literature, which consider the effect of delay on strategy under the framework of non-zero-sum stochastic differential game so far. Motivated by these gaps, we consider the wealth processes of insurers with delay in the paper.

The main work of this article is summarized as follows. We build a model for the control problem of a non-zero-sum reinsurance and investment game with delay and asymmetric information. By applying the dynamic programming principle, we derive the general Nash equilibrium strategies under the objective of maximizing the expected utility of its performance relative to its competitor. For constant absolute risk aversion (CARA) insurers, the explicit Nash equilibrium strategies and the value functions are obtained. Through the expression of equilibrium investment strategies, we find that insurer 2 imitates the investment strategy of insurer 1 and its equilibrium investment strategy is wholly independent of its own risk aversion factor. An extreme phenomenon is that insurer 2 completely does not invest when no competition is involved. In addition, we present some numerical studies and sensitivity analysis to draw economic interpretations. The numerical results verify theoretical results and find that the effect of the delay weight on the strategies is related to the length of the delay time under the framework of a non-zero-sum stochastic differential game.

Our paper differs from previous research in at least three respects. (1) We consider the impact of financial market’s asymmetric information on equilibrium strategies, while [37] focused on the effect of extra information of the insurance market on reinsurance strategies. We find that the insurer with less information completely ignores its own risk aversion factor, but imitates the investment strategy of its competitor who has more information of the financial market. (2) We consider the effect of delay on strategies under the framework of a non-zero-sum stochastic differential game and find that the effect of the delay weight on the strategies is related to the
length of the delay time, which has rarely been studied in the past. (3) The price process of risky asset is described by the constant elasticity of variance (CEV) model which is closer to the actual situation than geometric Brownian motion (GBM) used by most studies about stochastic differential game in the field of insurance actuarial science.

The rest of the paper is organized as follows. Section 2 formulates a non-zero-sum stochastic differential reinsurance and investment game problem between two insurers with delay and asymmetric information. Section 3 formulates the non-zero sum stochastic differential game problem by means of the dynamic programming principle. Section 4 provides the explicit solutions for the equilibrium reinsurance/investment strategies of CARA insurers. Section 5 provides some numerical examples and sensitivity analysis to demonstrate the theoretical results. Section 6 concludes the paper. Appendix 7 presents the proof of the verification theorem.

2. Model. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space equipped with a filtration \(\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}\) satisfying the usual conditions, i.e., \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) is right continuous and \(P\) complete, where \(T > 0\) is a fixed time horizon.

2.1. Insurance market. We consider an insurance market containing two competing insurers and one reinsurer. The surplus process of insurer \(i \in \{1, 2\}\), denoted by \(\{X_i(t)\}_{t \geq 0}\), is depicted by the classic Cramér-Lundberg risk model

\[
X_i(t) = x_i^0 + c_i t - \sum_{n=1}^{N_i(t) + N(t)} \tilde{Y}_i^n,
\]

where \(x_i^0 > 0\) and \(c_i \geq 0\) are the initial surplus and the premium rate of insurer \(i\); \(N_i(t) + N(t)\) represents the number of claims up to time \(t\); \(\{\tilde{Y}_i^n \geq 0, n \geq 1\}\) is a list of independent identically distributed (i.i.d.) random variables with distribution function \(F_i(\tilde{y})\); \(\tilde{Y}_i^n\) represents the amount of the \(n\)-th claim of insurer \(i\). We assume that (a) \(\{\tilde{Y}_i^n \geq 0, n \geq 1\}\) has finite first moment \(\tilde{\mu}_i(0 < \tilde{\mu}_i < +\infty)\) and finite secondary moment \((\tilde{\sigma}_i)^2 < +\infty\); (b) \(\{\tilde{Y}_i^n \geq 0, n \geq 1\}\) is independent of \(N_i(t)\) and \(N(t)\); (c) \(N_1(t), N_2(t)\) and \(N(t)\) are three mutually independent Poisson processes with intensities \(\tilde{\lambda}_1 > 0, \tilde{\lambda}_2 > 0\) and \(\tilde{\lambda} > 0\), respectively. The surplus process \(\{X_i(t)\}_{t \geq 0}\) indicates that insurers 1 and 2 are subject to common impact that is represented by \(\{N(t)\}_{t \geq 0}\).

Refer to [20, 8, 19, 4, 11, 13] etc., the classic Cramér-Lundberg risk model can be approximated by the following diffusion process:

\[
dX_i(t) = c_i dt - \lambda_i \tilde{\mu}_i dt + \sqrt{\lambda_i (\tilde{\sigma}_i)^2} dW_i(t), \quad X_i(0) = x_i^0,
\]

where \(\lambda_i = \tilde{\lambda}_i + \tilde{\lambda}i\); \(\{W_i(t), t \geq 0\}\) is a standard \(P\)-Brownian motion. \(d\langle W_1(t), W_2(t)\rangle = \rho dt\), where \(\rho = \frac{\tilde{\lambda} E(Y_1^2) E(Y_2)}{\sqrt{\lambda_1 + \lambda_2} E(Y_1^2) (\lambda_1 + \lambda_2 + \lambda_2)} = \frac{\tilde{\lambda} \tilde{\mu}_1 \tilde{\mu}_2}{\sqrt{\lambda_1 \lambda_2 \sigma_1 \sigma_2}}\). According to expected value premium principle, we know that \(c_i = (1 + \tilde{\theta}_i) \lambda_i \tilde{\mu}_i\), where \(\tilde{\theta}_i > 0\) is the safety loading of insurer \(i\).

Suppose that both insurers can manage their claim risk through purchasing proportional reinsurance continuously. The reinsurance strategy of insurer \(i \in \{1, 2\}\) is characterized by \(\{q_i(t), t \geq 0\}\), \(q_i(t) \in [0, 1]\). Then, the reinsurer will cover \((1 - q_i(t))100\%\) of the claims while insurer \(i\) will cover the remaining at time \(t\). Therefore, \(q_i(t)\) can be understood as the risk reserve level of insurer \(i\) at time \(t\). The reinsurance premium rate at time \(t\) is \(\delta(q_i(t)) = (1 + \tilde{\theta}_i)(1 - q_i(t)) \lambda_i \tilde{\mu}_i\), where
\(\tilde{\theta}_i \geq \bar{\theta}_i\) is the relative safety loading of the reinsurer. The reinsurance is called cheap when \(\tilde{\theta}_i = \bar{\theta}_i\), while if \(\tilde{\theta}_i > \bar{\theta}_i\), it is called non-cheap. Then, the surplus process with proportional reinsurance strategy of insurer \(i\) is as following:

\[
dX_i(t) = \left[\theta_i + \mu_i q_i(t)\right] dt + q_i(t) \sigma_i dW_i(t),
\]

where \(\theta_i = (\tilde{\theta}_i - \bar{\theta}_i) \lambda_i \tilde{\mu}_i \leq 0\), \(\mu_i = \tilde{\theta}_i \lambda_i \tilde{\mu}_i\), \(\sigma_i = \sqrt{\lambda_i (\tilde{\sigma}_i)^2}\).

2.2. Financial market. Assuming that both the insurers can invest in a financial market which consists of a risk-free asset and a risky asset. The price process of the risk-free asset \(\{S_0(t)\}_{t \geq 0}\) follows:

\[
dS_0(t) = r_0 S_0(t) dt, \quad S_0(0) = 1,
\]

while the price process of risky asset \(\{S(t)\}_{t \geq 0}\) is described by the constant elasticity of variance (CEV) model

\[
dS(t) = S(t) \left[r dt + \sigma S^\beta(t) dW(t)\right], \quad S(0) = s_0,
\]

where \(r_0\) is the constant risk-free interest rate; \(r, \sigma S^\beta(t)\) (where \(\sigma > 0\) and \(\beta\) denote the instantaneous mean rates of return, the instantaneous volatility and constant elasticity parameter of the risky asset, respectively; \(\{W(t), t \geq 0\}\) is a standard \(P\)-Brownian motion and independent of \(\{W_i(t), t \geq 0\}, i = 1, 2\). As we all know, the CEV model reduces to a geometric Brownian motion when \(\beta = 0\). If \(\beta < 0\), the instantaneous volatility \(\sigma S^\beta(t)\) increases as the stock price decreases, and a distribution with a fatter left tail can be generated. If \(\beta > 0\), the instantaneous volatility \(\sigma S^\beta(t)\) increases as the stock price increases.

It is commonly accepted that asymmetric information plays an important role in the financial market as agents have access to different levels of information. We assume that insurer 1 have full information of this financial market while insurer 2 can only observe the partial information. In other words, insurer 1 knows exactly the parameter value of the stock price process, while insurer 2 only observes the stock price and the volatility, but does not know the instantaneous mean rates of return, because estimating expected returns is more difficult than estimating volatilities from an econometric point of view, see [32]. Refer to [34] and [22], we change from the original measure \(P\) to the risk neutral measure.

We define the market price of risk \(\varphi = (\varphi_t)_{t \in [0,T]}\) by \(\varphi_t = \frac{r - r_0}{\sigma S^\beta(t)}\) and the density process \(\phi = (\phi_t)_{t \in [0,T]}\) by

\[
\phi_t = \exp \left\{-\int_0^t \varphi_s dW(s) - \frac{1}{2} \int_0^t (\varphi_s)^2 ds\right\},
\]

and introduce the risk neutral probability measure \(Q\) by \(dQ = \phi_T dP\). Then, we have \(E^Q(X(t)) = E^P(\phi_t X(t))\) for arbitrary stochastic process \(X(t), t \in [0,T]\), where \(E^P\) and \(E^Q\) are expectation operators under measure \(P\) and \(Q\), respectively. Girsanov’s theorem guarantees that

\[
\hat{W}(t) = W(t) + \int_0^t \varphi_s ds, \quad t \in [0,T],
\]

is a \(Q\)-Brownian motion with respect to the filtration \(\mathcal{F}\). Then, the price process of a risky asset \(\{S(t)\}_{t \geq 0}\) under measure \(Q\) is given by

\[
dS(t) = S(t) \left[r_0 dt + \sigma S^\beta(t) d\hat{W}(t)\right], \quad S(0) = s_0.
\]
2.3. The game problem with delay and asymmetric information. By section 2.2, we know that all random variables and stochastic processes are defined on the filtration $\mathcal{F}_t$, but different measures are adopted by insurer 1 and insurer 2, i.e., insurer 1 adopt the original measure $P$, while insurer 2 adopt the risk neutral measure $Q$.

2.3.1. The wealth processes with delay and asymmetric information. Suppose that there are no transaction costs or taxes for investment and reinsurance, and short-selling of the risky asset is allowed. For insurer $i$ ($i \in \{1, 2\}$), let $\{l_i(t)\}_{t \geq 0}$ be a measurable process valued in $\mathbb{R}$ representing the amount that insurer $i$ invests in the risky asset at $t \geq 0$. Then, the remaining wealth $X_i(t) - l_i(t)$ is invested in the risk-free asset. Let $\{X_i^\pi(t)\}_{t \geq 0}$ be the surplus process of insurer $i$ after purchasing reinsurance protection $q_i(t)$ and making investment $l_i(t)$, where $\pi_i(t) = (q_i(t), l_i(t))$.

Then, the wealth process of insurer 1 with strategy $\pi_1(t)$ under the original measure $P$ is governed by the following stochastic differential equation (SDE):

$$
\begin{align*}
    dX_1^\pi(t) &= [\theta_1 + \mu q_1(t) + (r - r_0)l_1(t) + r_0X_1^\pi(t)] \, dt \\
    & \quad + q_1(t)\sigma_1 dW_1(t) + l_1(t)\sigma^3(t) d\bar{W}(t).
\end{align*}
$$

(6)

Correspondingly, under the risk neutral measure $Q$, the wealth process of insurer 2 with strategy $\pi_2(t)$ is as following:

$$
\begin{align*}
    dX_2^\pi(t) &= [\theta_2 + \mu_2 q_2(t) + r_0X_2^\pi(t)] \, dt + q_2(t)\sigma_2 dW_2(t) + l_2(t)\sigma^3(t) d\bar{W}(t).
\end{align*}
$$

(7)

In fact, since both insurers have bounded memory features, their strategies depend on the exogenous capital instantaneous inflow into or outflow from current wealth. Refer to [35], we consider insurer $i$’s wealth with delay. Denote by $\bar{Y}_i(t)$ and $Z_i(t)$ the average and pointwise delayed information of insurer $i$’s wealth in the past horizon $[t - h_i, t]$, respectively. That is

$$
\begin{align*}
    \bar{Y}_i(t) &= \frac{Y_i(t)}{\int_{t-h_i}^t e^{\alpha_i s}ds}, \\
    Z_i(t) &= X_i^\pi(t-h_i), \quad \forall t \in [0, T], \quad i = 1, 2,
\end{align*}
$$

(8)

where $Y_i(t) = \int_{t-h_i}^t e^{\alpha_i s}X_i^\pi(t+s)ds$ is the integrated delayed information of the insurer $i$’s wealth in the time horizon $[t - h_i, t]$; $\alpha_i > 0$ and $h_i > 0$ are the average parameter and delay parameter of insurer $i$, respectively.

Let $f_i(t, X_i^\pi(t) - \bar{Y}_i(t), X_i^\pi(t) - Z_i(t))$ represent the capital inflow/outflow amount of insurer $i$, where $X_i^\pi(t) - \bar{Y}_i(t)$ implies the average performance of the wealth in the time horizon $[t - h_i, t]$ and $X_i^\pi(t) - Z_i(t)$ accounts for the absolute performance of the wealth between $t$ and $t - h_i$. This means that the insurer’s wealth process has a memory feature. Specifically, insurer’s present wealth is related to its past performance. A good past performance indicates more potential gain and extra dividend to its shareholders. Contrarily, a poor past performance forces the insurer to seek further capital injection for covering the loss so that the final performance objective is still achievable. To make the problem solvable, we assume

$$
\begin{align*}
    f_i(t, X_i^\pi(t) - \bar{Y}_i(t), X_i^\pi(t) - Z_i(t)) &= B_i(X_i^\pi(t) - \bar{Y}_i(t)) + C_i(X_i^\pi(t) - Z_i(t)),
\end{align*}
$$

(9)

where $B_i$ and $C_i$ are nonnegative constants. In other words, the amount of the capital inflow/outflow is a linear weighted sum of the average performance and the absolute performance.
Then, considering the capital inflow/outflow function $f_i(t, X_i^{\pi_i}(t) - \tilde{Y}_i(t), X_i^{\pi_i}(t) - Z_i(t))$, the wealth processes of insurers under the measure $P'/Q$ are governed by the following stochastic differential delay equations (SDDEs), respectively:

$$dX_i^{\pi_i}(t) = \left[ \theta_1 + \mu_1q(t) + (r - r_0)l_1(t) + r_0X_i^{\pi_i}(t) - f_i(t, X_i^{\pi_i}(t) - \bar{Y}_1(t), X_i^{\pi_i}(t) - Z_1(t)) \right] dt + q_1(t)\sigma_1 dW_1(t) + l_1(t)\sigma S^\beta(t) dW(t)$$

$$= \left[ \theta_1 + \mu_1q(t) + (r - r_0)l_1(t) + A_1X_i^{\pi_i}(t) + B_1Y_1(t) + C_1Z_1(t) \right] dt + q_1(t)\sigma_1 dW_1(t) + l_1(t)\sigma S^\beta(t) dW(t),$$

(10)

$$dX_2^{\pi_2}(t) = \left[ \theta_2 + \mu_2q_2(t) + r_0X_2^{\pi_2}(t) - f_2(t, X_2^{\pi_2}(t) - \bar{Y}_2(t), X_2^{\pi_2}(t) - Z_2(t)) \right] dt + q_2(t)\sigma_2 dW_2(t) + l_2(t)\sigma S^\beta(t) dW(t)$$

$$= \left[ \theta_2 + \mu_2q_2(t) + A_2X_2^{\pi_2}(t) + B_2Y_2(t) + C_2Z_2(t) \right] dt + q_2(t)\sigma_2 dW_2(t) + l_2(t)\sigma S^\beta(t) dW(t),$$

(11)

where $A_i = r_0 - \bar{B}_i - C_i$, $B_i = \frac{1}{\gamma_i} B_i e^{-\alpha t}$, $i \in \{1, 2\}$. In addition, we assume that insurer $i$ is endowed with an initial wealth $x_i^0$ at time $-h_i$ and does not start the business (insurance/reinsurance/investment) until time 0, i.e., $X_i(t) = x_i^0 > 0, \forall t \in [-h_i, 0)$. Then the initial value of the integrated performance wealth is $Y_i(t) = \frac{e^\theta t}{\alpha_1} (1 - e^{-\alpha_1 h_i})$.

Denote $X_i^{\pi_i}(t) = x_i, Y_i(t) = y_i, Z_i(t) = z_i$ and $S(t) = s$, for any fixed $t \in [0, T]$. Then, we define the admissible strategy as follows.

**Definition 2.1.** (Admissible strategy) For any fixed $t \in [0, T]$, a strategy $\pi_i(t) = (q_i(t), l_i(t))$ is said to be admissible, if

(i) $\{\pi_i(t)\}_{t \in [0, T]}$ is $\mathcal{F}$-progressively measurable process;

(ii) $E^P \left[ \int_t^T (l_1(\ell))^2 (S(\ell))^\beta (q_1(\ell))^2 d\ell \right] < +\infty$ and

$E^Q \left[ \int_t^T (l_2(\ell))^2 (S(\ell))^\beta (q_2(\ell))^2 d\ell \right] < +\infty, \forall t \in [t, T]$;

(iii) $\forall (t, x_1, y_1, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\forall \ell \in [t, T], \forall \theta \in [1, +\infty)$, the equation (10) associated with $\pi_1(t)$ has a unique solution $X_i^{\pi_1}(t)$, which satisfies

$E^P_{t,x_1,y_1,s}[\sup \{|X_i^{\pi_1}(\ell)|\}] < +\infty$, where $E^P_{t,x_1,y_1,s}[\cdot]$ is the conditional expectation given $X_i^{\pi_1}(t) = x_1, Y_1(t) = y_1, S(t) = s$ under measure $P$;

(iv) $\forall (t, x_2, y_2, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\forall \ell \in [t, T], \forall \theta \in [1, +\infty)$, the equation (11) associated with $\pi_2(t)$ has a unique solution $X_i^{\pi_2}(t)$, which satisfies

$E^Q_{t,x_2,y_2,s}[\sup \{|X_i^{\pi_2}(\ell)|\}] < +\infty$, where $E^Q_{t,x_2,y_2,s}[\cdot]$ is the conditional expectation given $X_i^{\pi_2}(t) = x_2, Y_2(t) = y_2, S(t) = s$ under measure $Q$.

Let $\Pi_i$ denote the set of all admissible strategies of insurer $i$, and $\pi_i^*$ denote the optimal strategies of insurer $i$.

2.3.2. *The value functions of insurers.* Insurer $i, i \in \{1, 2\}$, is concerned with the average wealth over the period $[T - h_i, T]$ (i.e., $\bar{Y}_i(T)$). Then, $X_i^{\pi_i}(T) + \bar{\eta}_i Y_i(T)$ is considered, where $\bar{\eta}_i$ values the weight of $\bar{Y}_i(T)$, which implies the impact of average performance on the terminal wealth. In fact, $\bar{\eta}_i$ can also be understood as the correlation between past performance and future expectations of the insurer $i$.

Similar to [35], for the convenience of calculation and to make our modeling framework in accordance with that in literature (see [16], etc.) on stochastic control with delay, we consider the integrated performance $Y_i(T)$ rather than the average
one \( \hat{Y}_i(T) \) directly. Let \( \eta_i = \frac{\bar{\eta}_i}{\int_{h_i} \rho \pi \, ds}, i \in \{1, 2\} \), which can be considered as the weight between \( X_i^1(T) \) and \( Y_i(T) \). Then, we have \( \bar{\eta}_i \hat{Y}_i(T) = \eta_i Y_i(T) \). Thus, it is equivalent to considering \( \eta_i Y_i(T) \) in this paper.

In the competitive social environment, insurer \( i \) should consider not only the bounded memory feature of wealth, but also its performance relative to its competitor. Refer to [10], [6], [17], [37], the objective of the non-zero-sum differential game problem is to find a Nash equilibrium.

Let \( \pi_i \) be the value functions of the insurers, respectively.

Then, refer to [6], [37] and [14], for \( (t, x_1, x_2, y_1, y_2, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \), let

\[
V^1(t, x_1, x_2, y_1, y_2, s) = \sup_{\pi_1 \in \Pi_1} E^P \left[ U_1 \left( (X_1^1(t) + \eta_1 Y_1(T)) - k_1 (X_2^2(T) + \eta_2 Y_2(T)) \right) \right],
\]

\[
V^2(t, x_1, x_2, y_1, y_2, s) = \sup_{\pi_2 \in \Pi_2} E^Q \left[ U_2 \left( (X_2^2(t) + \eta_2 Y_2(T)) - k_2 (X_1^1(T) + \eta_1 Y_1(T)) \right) \right],
\]

be the value functions of the insurers, respectively.
For convenience, we denote $\dot{X}^{\pi_1}_i(t) = X^{\pi_1}_i(t) - k_iX^{\pi_j}_j(t)$ for $i \neq j \in \{1, 2\}$. Then, we have that: for insurer 1, under the original measure $P$,

$$d\dot{X}^{\pi_1}_i(t) = \left[ A_1X^{\pi_1}_i(t) - k_1A_2X^{\pi_2}_2(t) + B_1Y_1(t) - k_1B_2Y_2(t) + C_1Z_1(t) - k_1C_2Z_2(t) \
+ \theta_i - k_i\theta_2 + \mu_1q_1(t) - k_1\mu_2q_2(t) + (r - r_0)(l_1(t) - k_1l_2(t)) \right] dt \\
+ q_1(t)\sigma_1dW_1(t) - k_1q_2(t)\sigma_2dW_2(t) + (l_1(t) - k_1l_2(t))\sigma S^3(t)d\bar{W}(t),$$

(18)

with $\dot{X}^{\pi_1}_1(0) = X^{\pi_1}_1(0) - k_1X^{\pi_2}_2(0) = x_1^0 - k_1x_2^0$; for insurer 2, under the risk neutral measure $Q$,

$$d\dot{X}^{\pi_2}_2(t) = \left[ A_2X^{\pi_2}_2(t) - k_2A_1X^{\pi_1}_1(t) + B_2Y_2(t) - k_2B_1Y_1(t) + C_2Z_2(t) - k_2C_1Z_1(t) \
+ \theta_2 - k_2\theta_1 + \mu_2q_2(t) - k_2\mu_1q_1(t) \right] dt + q_2(t)\sigma_2dW_2(t) \\
- k_2q_1(t)\sigma_1dW_1(t) + (l_2(t) - k_2l_1(t))\sigma S^3(t)d\bar{W}(t),$$

(19)

with $\dot{X}^{\pi_2}_2(0) = X^{\pi_2}_2(0) - k_2X^{\pi_1}_1(0) = x_2^0 - k_2x_1^0$.

For any fixed $t \in [0, T]$, denote $X^{\pi}_i(t) = X^{\pi}_i(t) - k_iX^{\pi}_j(t) = x_i - k_ix_j = \hat{x}_i$. According to existing literatures, the optimal control problem with delay is infinite-dimensional in general. To make the problem solvable and finite-dimensional, we assume the following conditions: $C_i = \eta_i e^{-\alpha_i h}$, $B_i = (\alpha_i + A_i + \eta_i)\eta_i$.

3. Nash equilibrium.

3.1. Verification theorem. For convenience, we first provide some notations. Let $\mathcal{O} = [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. For $i \neq j \in \{1, 2\}$, denote by $C^{1,2,1,1,2}(\mathcal{O})$ the class of functions $g^i: \mathcal{O} \rightarrow \mathbb{R}$ such that

(i) for $\forall (\hat{x}_i, y_i, y_j, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $g^i(t, \hat{x}_i, y_i, y_j, s)$ is a once continuously differentiable function of $t$;

(ii) for $\forall (t, y_i, y_j, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $g^i(t, \hat{x}_i, y_i, y_j, s)$ is a twice continuously differentiable function of $\hat{x}_i$;

(iii) for $\forall (t, \hat{x}_i, y_i, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $g^i(t, \hat{x}_i, y_i, y_j, s)$ is a once continuously differentiable function of $y_i$;

(iv) for $\forall (t, \hat{x}_i, y_i, y_j) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $g^i(t, \hat{x}_i, y_i, y_j, s)$ is a twice continuously differentiable function of $s$.

For any $J^i(t, \hat{x}_i, y_i, y_j, s) \in C^{1,2,1,1,2}(\mathcal{O}), i \neq j \in \{1, 2\}$, we define variational operators for insurers:

$$A^{\pi_1}J^1(t, \hat{x}_1, y_1, y_2, s) \nonumber$$

$$= \left[ A_1x_1 - k_1A_2x_2 + B_1y_1 - k_1B_2y_2 + C_1z_1 - k_1C_2z_2 + \theta_1 - k_1\theta_2 \
+ \mu_1q_1(t) - k_1\mu_2q_2(t) + (r - r_0)(l_1(t) - k_1l_2(t)) \right] J^1_{x_1} + \frac{1}{2} \left( q_1(t)\sigma_1 \right)^2 \\
+ (k_1q_2(t)\sigma_2)^2 + (l_1(t) - k_1l_2(t))\sigma S^3(t)J^1_{y_1} + (x_1 - \alpha_1y_1 - e^{-\alpha_1h_1}z_1)J^1_{y_1} + (x_2 - \alpha_2y_2 - e^{-\alpha_2h_2}z_2)J^1_{y_2} + rsJ^1_s \\
+ (l_1(t) - k_1l_2(t))\sigma S^3(t)J^3_{s},$$

(20)

$$A^{\pi_2}J^2(t, \hat{x}_2, y_2, y_1, s) \nonumber$$

$$= \left[ A_2x_2 - k_2A_1x_1 + B_2y_2 - k_2B_1y_1 + C_2z_2 - k_2C_1z_1 + \theta_2 - k_2\theta_1 \
+ \mu_2q_2(t) - k_2\mu_1q_1(t) + (r - r_0)(l_2(t) - k_2l_1(t)) \right] J^2_{x_2} + \frac{1}{2} \left( q_2(t)\sigma_2 \right)^2 \\
+ (k_2q_1(t)\sigma_1)^2 + (l_2(t) - k_2l_1(t))\sigma S^3(t)J^2_{y_2} + (x_2 - \alpha_2y_2 - e^{-\alpha_2h_2}z_2)J^2_{y_2} + rsJ^2_s,$$
Theorem 3.1. \( \) Suppose \( J^i(t, \hat{x}_i, y_i, y_j, s), i \neq j \in \{1, 2\} \), satisfies a quadratic growth condition, i.e., there exists \( C > 0 \) such that
\[
|J^i(t, \hat{x}_i, y_i, y_j, s)| \leq C(1 + |\hat{x}_i|^2 + |y_i|^2 + |y_j|^2 + |s|^2),
\]
for all \( (t, \hat{x}_i, y_i, y_j, s) \in \mathcal{O}, i \neq j \in \{1, 2\} \).

(1) Suppose that
\[
-J^i_t(t, \hat{x}_i, y_i, y_j, s) - \sup_{\pi_i(t) \in \Pi_i} \{ A^{\pi_i} J^i(t, \hat{x}_i, y_i, y_j, s) \} \geq 0,
\]
and the SDE
\[
d\hat{X}^i(t) = dX^i(t) - k_i dX^{i, \gamma}(t),
\]
for \( i \neq j \in \{1, 2\} \), admits a unique solution, denote by \( \hat{X}^i \), given the initial condition \( \hat{X}^i(0) = \hat{x}_i \). Furthermore, the process \( \{\pi^*_i(t)\} \) for \( i \geq 0 \) lies in \( \Pi_i \). Then
\[
J^i = V^i, \quad \text{on } \mathcal{O},
\]
and \( \pi^*_i(t) \) is an optimal strategy.

Proof. The proof is similar to the proof of [6] and [37], see the appendix for details. \( \square \)
3.2. The nash equilibrium strategies. Under the assumptions of Theorem 3.1, we solve the insurers’ value functions and reinsurance-investment strategies by solving the HJB equation (29). The results are shown in Theorem 3.2.

**Theorem 3.2.** (The Nash equilibrium strategies) Suppose that \(k_1 k_2 < 1, J_1^{\hat{x}_i, \hat{y}_i} \neq 0\) for \(\forall (t, \hat{x}_i, \hat{y}_i, s) \in \hat{O}, i \neq j \in \{1, 2\}, S(t) = s \neq 0\) for \(\forall t \in [0, T]\). The Nash equilibrium strategy \(\pi_i^*(t) = (q_i^*(t), l_i^*(t))\) for insurer \(i\), where \(q_i^*(t)\) is the solution of the following coupled system of nonlinear equations,

\[
q_1^*(t) = \left( -\frac{\mu_1}{(\sigma_1)^2} J_1^{\hat{x}_1} + \frac{k_1 \sigma_2}{\sigma_1} \rho q_2^*(t) + 1 \right),
\]

\[
q_2^*(t) = \left( -\frac{\mu_2}{(\sigma_2)^2} J_2^{\hat{x}_2} + \frac{k_2 \sigma_1}{\sigma_2} \rho q_1^*(t) + 1 \right),
\]

\[
l_1^*(t) = \frac{-1}{1 - k_1 k_2} \left[ \frac{r - r_0}{(\sigma_1)^2} s J_1^{\hat{x}_1} + s J_2^{\hat{x}_2} + k_1 s J_2^{\hat{x}_1} \right],
\]

\[
l_2^*(t) = \frac{-1}{1 - k_1 k_2} \left[ \frac{k_2 (r - r_0)}{(\sigma_1)^2} s J_1^{\hat{x}_1} + s J_2^{\hat{x}_2} + k_2 s J_2^{\hat{x}_1} \right],
\]

where \(J_1, J_2\) are the solutions of the following system of coupled partial differential equations (PDEs)

\[
0 = J_1^t + \left[ A_1 J_1 - k_1 A_2 J_2 + B_1 y_1 - k_1 B_2 y_2 + C_1 z_1 - k_1 C_2 z_2 \right] J_1^{\hat{x}_1} + \frac{1}{2} (k_1 \sigma_2)^2 (1 - \rho^2) (q_2^*(t))^2 J_1^{\hat{x}_1, \hat{x}_1} + (x_1 - \alpha_1 y_1 - e^{-\alpha_1 h_1} z_1) J_1^{\hat{x}_1, \hat{y}_1} + (x_2 - \alpha_2 y_2 - e^{-\alpha_2 h_2} z_2) J_1^{\hat{x}_2, \hat{y}_2} + rs J_2^s - \frac{1}{2} \frac{r - r_0}{(\sigma_1)^2} s + \frac{(\mu_1)^2}{(\sigma_1)^2} J_1^{\hat{x}_1, \hat{x}_1} - \frac{1}{2} \sigma_2^2 s J_1^{\hat{x}_1, \hat{y}_1} - \frac{1}{2} \sigma_2^2 s J_1^{\hat{x}_1, \hat{y}_2} - \frac{1}{2} \sigma_2^2 s J_1^{\hat{x}_2, \hat{y}_1} - \frac{1}{2} \sigma_2^2 s J_1^{\hat{x}_2, \hat{y}_2},
\]

\[
0 = J_2^t + \left[ A_2 J_2 - k_2 A_1 J_1 + B_2 y_2 - k_2 B_1 y_1 + C_2 z_2 - k_2 C_1 z_1 + \theta_2 - k_2 \theta_1 \right] J_2^{\hat{x}_2} + \frac{1}{2} (k_2 \sigma_1)^2 (1 - \rho^2) (q_1^*(t))^2 J_2^{\hat{x}_2, \hat{x}_2} + (x_2 - \alpha_2 y_2 - e^{-\alpha_2 h_2} z_2) J_2^{\hat{x}_2, \hat{y}_2} + (x_1 - \alpha_1 y_1 - e^{-\alpha_1 h_1} z_1) J_2^{\hat{x}_1, \hat{y}_2} + rs J_2^s - \frac{1}{2} \frac{r - r_0}{(\sigma_2)^2} s + \frac{(\mu_2)^2}{(\sigma_2)^2} J_2^{\hat{x}_2, \hat{x}_2} - \frac{1}{2} \sigma_2^2 s J_2^{\hat{x}_2, \hat{y}_2} - \frac{1}{2} \sigma_2^2 s J_2^{\hat{x}_1, \hat{y}_2} - \frac{1}{2} \sigma_2^2 s J_2^{\hat{x}_1, \hat{y}_1} - \frac{1}{2} \sigma_2^2 s J_2^{\hat{x}_1, \hat{y}_2},
\]

with the terminal conditions

\[
J_1^T (\hat{x}_1, y_1, y_2, s) = U_1 (\hat{x}_1 + \eta_1 y_1 - k_1 \eta_2 y_2),
\]

\[
J_2^T (\hat{x}_2, y_2, y_1, s) = U_2 (\hat{x}_2 + \eta_2 y_2 - k_2 \eta_1 y_1).
\]

**Proof.** By (22)-(23) and \(q_i^*(t) \in [0, 1]\), we can easily get (32)-(33). By (24)-(25), we can get

\[
l_1^*(t) = -\frac{r - r_0}{\sigma_2} s J_1^{\hat{x}_1} - s J_2^{\hat{x}_1, \hat{y}_1} + k_1 l_1^*(t),
\]

\[
l_2^*(t) = -\frac{r - r_0}{\sigma_1} s J_2^{\hat{x}_2} - s J_2^{\hat{x}_2, \hat{y}_2} + k_2 l_2^*(t),
\]
\[ l_2(t) = -s \frac{J_2^2}{J_2^2} + k_2 l_1^s(t). \]

Solving the above equations simultaneously, we can get (34)-(35).

4. **Nash equilibrium of CARA insurers.** Assume that both insurers are constant absolute risk aversion (CARA) agents, i.e., each insurer has an exponential utility function:

\[ U_i(\hat{x}_i + \eta_i y_i - k_i \eta_j y_j) = -\frac{1}{\gamma_i} \exp(-\gamma_i(\hat{x}_i + \eta_i y_i - k_i \eta_j y_j)), \quad i \neq j \in \{1, 2\}, \tag{40} \]

where \( \gamma_i > 0 \) is the constant absolute risk aversion coefficient of insurer \( i \). Using standard dynamic programming techniques, we can obtain the following theorem.

For convenience, let \( K = 1 - k_1 k_2, \ K_\varrho = 1 - k_1 k_2 \rho^2, \ K_1 = 1 + k_1 k_2 (1 - 2 \rho^2), \ K_2 = 3 + (k_1 k_2)^2 - 4 k_1 k_2 \rho^2. \)

**Theorem 4.1.** Suppose that \( A_1 + \eta_1 = A_2 + \eta_2 \) and \( k_1 k_2 < 1 \). The equilibrium strategy \( \pi_i^*(t) = (q_i^*(t), l_i^*(t)), i = 1, 2 \), is given as follows. Define

\[
\begin{align*}
\hat{q}_1(t) &= \frac{1 - \gamma_1}{K \gamma_1 \psi_1(t)} \left[ \frac{1}{\psi_1(t)} \sigma_1 + \frac{1}{\psi_2(t)} \frac{k_1 \rho_2}{\sigma_2} + \frac{1}{\gamma_1 \psi_1(t)} \right], \\
\hat{q}_2(t) &= \frac{1}{K \gamma_2 \psi_2(t)} \left[ \frac{1}{\psi_2(t)} \sigma_2 + \frac{1}{\gamma_1 \psi_1(t)} \right],
\end{align*}
\]

where

\[
\phi^1(t) = \phi^2(t) = \exp \left\{ (A_1 + \eta_1)(T - t) \right\} = \exp \left\{ (A_2 + \eta_2)(T - t) \right\}. \tag{42}
\]

For \( q_i^*(t), i = 1, 2 \), we have the following cases:

(i) If \( \hat{q}_i(t) < 1 \) for \( i \in \{1, 2\} \), then \( (q_i^*(t), q_j^*(t)) = (\hat{q}_i(t), \hat{q}_j(t)) \);

(ii) If \( \hat{q}_i(t) \geq 1, \hat{q}_j(t) < 1 \), then \( (q_i^*(t), q_j^*(t)) = (\frac{\mu_1}{\sigma_1}, \frac{k_2 \rho_2}{\sigma_2}) \);

(iii) If \( \hat{q}_i(t) < 1, \hat{q}_j(t) \geq 1 \), then \( (q_i^*(t), q_j^*(t)) = (\frac{\mu_1}{\sigma_1}, 1) \);

(iv) If \( \hat{q}_i(t) \geq 1 \) for \( i \in \{1, 2\} \), then \( (q_i^*(t), q_j^*(t)) = (1, 1) \).

For \( l_i^*(t), i = 1, 2 \), we have

\[
\begin{align*}
l_1^*(t) &= \frac{r - r_0}{K \gamma_1 \phi^1(t)} \left[ \frac{r - r_0}{\sigma_2^2 s^2} \frac{1}{\sigma_3^2 s^2} \right], \tag{43} \\
l_2^*(t) &= k_2 l_1^*(t) \frac{r - r_0}{K \gamma_1 \phi^1(t)} \left[ \frac{r - r_0}{\sigma_2^2 s^2} \frac{1}{\sigma_3^2 s^2} \right]. \tag{44}
\end{align*}
\]

The value function of insurer \( i \) is

\[
J^i(t, \hat{x}_i, y_i, y_j, s) = -\frac{1}{\gamma_i} \exp \left\{ -\gamma_i \phi^i(t)(\hat{x}_i + \eta_i y_i - k_i \eta_j y_j) + G^i(t, s) \right\}, \quad i \neq j \in \{1, 2\}, \tag{45}
\]

where

\[
G^i(t, s) = g_1^i(t) s^{-2\beta} + g_2^i(t), \quad i = 1, 2. \tag{46}
\]

In the case of \( i = 1 \),

\[
g_1^1(t) = \frac{(r - r_0)^2}{4 \sigma_1^2 s^2} \left[ \exp\{-2r_0 \beta(T - t)\} - 1 \right], \tag{47}
\]

\[
g_2^1(t) = \frac{2 \beta + 1}{8 \beta} \left( r - r_0 \right)^2 \left[ 1 - \exp\{-2r_0 \beta(T - t)\} \right] - \frac{(2 \beta + 1)(r - r_0)^2}{4 r_0} (T - t) + \frac{k_1 \theta_2 - \theta_1}{A_1 + \eta_1} \phi^1(t) - 1 + D_1(t), \tag{48}
\]

\]
where

\[
D_1(t) = \begin{cases}
D_{11}(t), & (q_1^1(t), q_2^1(t)) = (\bar{q}_1(t), \bar{q}_2(t)); \\
D_{12}(t), & (q_1^2(t), q_2^2(t)) = \left(1, \frac{\mu_2}{\sigma_2} + \frac{K_2}{2(K_\rho)^2} \frac{\mu_1}{\sigma_1}\right); \\
D_{13}(t), & (q_1^3(t), q_2^3(t)) = \left(\frac{k_1}{k_1}, \frac{k_2}{k_1} \frac{\mu_2}{\sigma_2} \right); \\
D_{14}(t), & (q_1^4(t), q_2^4(t)) = (1, 1).
\end{cases}
\] (49)

\[
D_{11}(t) = \left\{ \frac{k_1}{K_\rho} \frac{\gamma_1}{\gamma_2} \left[ \frac{k_1}{2K_\rho} \frac{\gamma_1}{\gamma_2} (1 - \rho^2) - 1 \right] \left(\frac{\mu_2}{\sigma_2}\right)^2 + \frac{K_2}{2(K_\rho)^2} \left(\frac{\mu_1}{\sigma_1}\right)^2 \\
\quad + \frac{k_3}{K_\rho} \left[ \frac{k_1}{K_\rho} \frac{\gamma_1}{\gamma_2} - k_2 \right] \frac{\mu_1 \mu_2 \rho}{\sigma_1 \sigma_2} \right\} (T - t),
\]

\[
D_{12}(t) = -\frac{k_3 \gamma_1 \mu_2 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \left[ \frac{k_3 \gamma_1 \mu_2 (T - t) \rho}{(\sigma_2)^2 \gamma_2} + \frac{k_3 \gamma_1 \mu_2 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \right] \left[ (T - t) - 1 \right],
\]

\[
D_{13}(t) = \left(\frac{\mu_1}{\sigma_1}\right)^2 [\varphi^1(t) - 1] - \frac{k_3 \gamma_1 \mu_2 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \left[ \mu_2 \left[ \frac{\mu_1}{\sigma_1}\right] - \frac{k_3 \gamma_1 \mu_2 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \right] \left[ (T - t) - 1 \right],
\]

\[
D_{14}(t) = D_{13}(t).
\]

In the case of \(i = 2\),

\[
g_1^2(t) = 0,
\]

\[
g_2^2(t) = \frac{k_2 \theta_1 - \theta_2 \gamma_2}{A_2 + \eta_2} [\varphi^2(t) - 1] + D_2(t),
\] (51)

where

\[
D_2(t) = \begin{cases}
D_{21}(t), & (q_1^1(t), q_2^1(t)) = (\bar{q}_1(t), \bar{q}_2(t)); \\
D_{22}(t), & (q_1^2(t), q_2^2(t)) = \left(1, \frac{\mu_2}{\sigma_2} + \frac{K_2}{2(K_\rho)^2} \frac{\mu_1}{\sigma_1}\right); \\
D_{23}(t), & (q_1^3(t), q_2^3(t)) = \left(\frac{k_1}{k_1}, \frac{k_2}{k_1} \frac{\mu_2}{\sigma_2} \right); \\
D_{24}(t), & (q_1^4(t), q_2^4(t)) = (1, 1).
\end{cases}
\] (52)

\[
D_{21}(t) = \left\{ \frac{k_1}{K_\rho} \frac{\gamma_1}{\gamma_2} \left[ \frac{k_1}{2K_\rho} \frac{\gamma_1}{\gamma_2} (1 - \rho^2) - 1 \right] \left(\frac{\mu_2}{\sigma_2}\right)^2 + \frac{K_2}{2(K_\rho)^2} \left(\frac{\mu_1}{\sigma_1}\right)^2 \\
\quad + \frac{k_3}{K_\rho} \left[ \frac{k_1}{K_\rho} \frac{\gamma_1}{\gamma_2} - k_2 \right] \frac{\mu_1 \mu_2 \rho}{\sigma_1 \sigma_2} \right\} (T - t),
\]

\[
D_{22}(t) = \left(\frac{\mu_2}{\sigma_2}\right)^2 [\varphi^2(t) - 1] - \frac{k_3 \gamma_2 \mu_1 (T - t) \rho}{(\sigma_2)^2 \gamma_2} [\varphi^2(t) - 1] - \frac{k_3 \gamma_2 \mu_1 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \left[ (T - t) - 1 \right],
\]

\[
D_{23}(t) = \left(\frac{k_2 \gamma_2 \mu_1 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \right) \left[ \mu_1 \left[ \frac{\mu_1}{\sigma_2}\right] - \frac{k_3 \gamma_2 \mu_1 (T - t) \rho}{(\sigma_2)^2 \gamma_2} \right] \left[ (T - t) - 1 \right],
\]

\[
D_{24}(t) = D_{22}(t).
\]
Proof. To solve this problem, we conjecture that

\[ J^i(t, \tilde{x}, y_i, y_j, s) = -\frac{1}{\gamma_i} \exp \left\{ -\gamma_i \varphi^i(t)(\tilde{x}_i + \eta_i y_i - k_i \eta_j y_j) + G^i(t, s) \right\}, \quad i \neq j \in \{1, 2\}, \]

with \( \varphi^i(T) = 1 \) and \( G^i(T, s) = 0 \), where \( \varphi^i(t) \) and \( G^i(t, s) \) are functions to be determined later. Then we obtain:

\[ J^i_{\tilde{x}_i} = -\gamma_i \varphi^i(t) J^i, \quad J^i_{y_i} = k_i \gamma_i \varphi^i(t) J^i, \quad J^i_{y_j} = \gamma_i \varphi^i(t) J^i, \quad J^i_s = G^i_s J^i, \]

\[ J^i_{\tilde{x}_i, \tilde{x}_i} = (\gamma_i \varphi^i(t))^2 J^i, \quad J^i_{\tilde{x}_i, y_i} = -\gamma_i \varphi^i(t) G^i_{\tilde{x}_i y_i}, \quad J^i_{\tilde{x}_i, y_j} = \left( (G^i_{\tilde{x}_i y_j})^2 + G^i_{\tilde{x}_i s} \right) J^i. \]

Substituting the above derivatives into (32)-(35), we can derive that

\[ q^1_i(t) = \left( \frac{\mu_1}{\sigma^2} + \frac{k_1 \sigma_1^2}{\sigma^2} \rho \right) \right)^+ \]

\[ q^2_i(t) = \left( \frac{\mu_2}{\sigma^2} + \frac{k_2 \sigma_2^2}{\sigma^2} \rho \right) \right)^+ \]

\[ l^1_i(t) = \frac{1}{K^2} \left[ \frac{r - r_0}{\sigma^2} \right] \gamma_i \varphi^i(t) \right)^+ \]

\[ l^2_i(t) = \frac{1}{K^2} \left[ \frac{k_2 (r - r_0)}{\sigma^2} \right] \gamma_i \varphi^i(t) \right)^+ \]

Substituting (57)-(58) into (36)-(37), we can get the following HJB equations of insurers:

\[ 0 = -\gamma_i (\tilde{x}_i + \eta_i y_i - k_i \eta_j y_j) \varphi^i + G^i - \left[ A_1 x_1 - k_1 A_2 x_2 + B_1 y_1 - k_1 B_2 y_2 \right] \gamma_i \varphi^i(t) \]

\[ + C_1 z_1 - k_1 C_2 z_2 + (\theta_1 - k_1 \theta_2) + k_1 (\frac{\sigma_2 \mu_1}{\sigma_1} - \mu_2) q^2_i(t) \right)^+ \]

\[ + (x_2 - \alpha_2 y_2 - e^{-\alpha_2 h_2} z_2) k_1 \gamma_i \varphi^i(t) + r s G^i_s + \frac{1}{2} \sigma^2 s^2 \right)^+ \]

\[ - \frac{1}{K^2} \left[ \frac{(r - r_0)^2}{\sigma^2 s^2} \right] \gamma_i \varphi^i(t) \right)^+ \]

\[ 0 = -\gamma_i (\tilde{x}_2 + \eta_2 y_1 - k_2 \eta_1 y_1) \varphi^2 + G^2 - \left[ A_2 x_2 - k_2 A_1 x_1 + B_2 y_2 - k_2 B_1 y_1 \right] \gamma_2 \varphi^2(t) \]

\[ + C_2 z_2 - k_2 C_1 z_1 + (\theta_2 - k_2 \theta_1) + k_2 (\frac{\sigma_1 \mu_2}{\sigma_2} - \mu_1) q^1_i(t) \right)^+ \]

\[ + (x_1 - \alpha_1 y_1 - e^{-\alpha_1 h_1} z_1) k_2 \gamma_2 \varphi^2(t) + r s G^i_s + \frac{1}{2} \sigma^2 s^2 \right)^+ \]

\[ - \frac{1}{K^2} \left[ \frac{(r - r_0)^2}{\sigma^2 s^2} \right] \gamma_2 \varphi^2(t) \right)^+ \]

The observation shows that \( q^1_i(t) \) and \( q^2_i(t) \) do not depend on the state variables \( x_1, x_2, y_1, y_2 \) and \( s \). Then by the formulations of \( B_i \) and \( C_i \), HJB equations of the two insurers can be decomposed into the following equations:

\[ 0 = -(x_1 + \eta_1 y_1) \gamma_1 \varphi^1(t) + (A_1 + \eta_1) \varphi^1(t), \]

\[ 0 = -(x_2 + \eta_2 y_2) \gamma_2 \varphi^2(t) + (A_2 + \eta_2) \varphi^2(t). \]
reinsurance strategies in the following situations:

\[
0 = (x_2 + \eta_2 y_2)k_1 \gamma_1 [\varphi_1^t + (A_2 + \eta_2)\varphi^1(t)],
\]

\[
0 = G^1_t - \left( (\theta_1 - k_1 \theta_2) + k_1 (\frac{\sigma_2 \rho \mu_1}{\sigma_1} - \mu_2) q_2^t(t) \right) \gamma_1 \varphi^1(t)
\]

\[
+ \frac{1}{2} (k_1 \sigma_2)^2 (1 - \rho^2) (q_2^t(t))^2 (\gamma_1 \varphi^1(t))^2 + rs G^1_s + \frac{1}{2} \sigma^2 s^{2\beta+2} [(G^1_s)^2 + G^1_{ss}]
\]

\[
- \frac{1}{2} \left( r - r_0 \right)^2 + \left( \frac{\mu_1}{\sigma_1} \right)^2 - (r - r_0) s G^1_s - \frac{1}{2} \sigma^2 s^{2\beta+2} (G^1_s)^2,
\]

and

\[
0 = -(x_2 + \eta_2 y_2) \gamma_2 [\varphi_1^t + (A_2 + \eta_2)\varphi^2(t)],
\]

\[
0 = (x_1 + \eta_1 y_1)k_2 \gamma_2 [\varphi_1^t + (A_1 + \eta_1)\varphi^2(t)],
\]

\[
0 = G^2_t - \left( (\theta_2 - k_2 \theta_1) + k_2 (\frac{\sigma_1 \rho \mu_1}{\sigma_2} - \mu_1) q_1^t(t) \right) \gamma_2 \varphi^2(t)
\]

\[
+ \frac{1}{2} (k_2 \sigma_1)^2 (1 - \rho^2) (q_1^t(t))^2 (\gamma_2 \varphi^2(t))^2 + rs G^2_s + \frac{1}{2} \sigma^2 s^{2\beta+2} [(G^2_s)^2 + G^2_{ss}]
\]

\[
- \frac{1}{2} \left( \mu_2 \right)^2 - \frac{1}{2} \sigma^2 s^{2\beta+2} (G^2_s)^2.
\]

Conditionally on \( A_1 + \eta_1 = A_2 + \eta_2 \), and with the terminal conditions \( \varphi^i(T) = 1 \), the following formulation holds:

\[
\varphi^i(t) = \varphi^2(t) = \exp\{(A_1 + \eta_1)(T - t)\} = \exp\{(A_2 + \eta_2)(T - t)\}.
\]

From (55), (56) and (67), we know that \( (\frac{\mu_i}{\sigma_i})^{\gamma_i(t)} + \frac{k_i \sigma_i}{\sigma_i} \rho q_i^*(t) \land 1 > 0, \ i \neq j \in \{1, 2\} \). That is, \( q_i^*(t) > 0, i \in \{1, 2\} \). Then:

\[
q_1^*(t) = \frac{\mu_1}{(\sigma_1)^{2 \gamma_1(t)}} + \frac{k_1 \sigma_2}{\sigma_1} \rho q_2^*(t) \land 1,
\]

\[
q_2^*(t) = \frac{\mu_2}{(\sigma_2)^{2 \gamma_2(t)}} + \frac{k_2 \sigma_1}{\sigma_2} \rho q_1^*(t) \land 1.
\]

According to [6] and [14], in order to discuss the magnitude of \( q_i^*(t) (i \in \{1, 2\}) \) relative to 1, we first define \( \tilde{q}_i(t) (i \in \{1, 2\}) \) as (41), which is the solution of the following system of equations:

\[
\left\{ \begin{array}{l}
\tilde{q}_1(t) = \frac{\mu_1}{(\sigma_1)^{2 \gamma_1(t)}} + \frac{k_1 \sigma_2}{\sigma_1} \rho \tilde{q}_2(t), \\
\tilde{q}_2(t) = \frac{\mu_2}{(\sigma_2)^{2 \gamma_2(t)}} + \frac{k_2 \sigma_1}{\sigma_2} \rho \tilde{q}_1(t).
\end{array} \right.
\]

Obviously, \( \tilde{q}_i(t) > 0, t \in [0, T] \) for \( i \in \{1, 2\} \). Then, we can obtain the equilibrium reinsurance strategies in the following situations:

(a) If \( \tilde{q}_i(t) < 1 \) for \( i \in \{1, 2\} \), then \( (q_1^*(t), q_2^*(t)) = (\tilde{q}_1(t), \tilde{q}_2(t)) \). In this case, \( q_i^*(t) < 1, i \in \{1, 2\} \) is obviously established. (63) and (66) become

\[
0 = \frac{K^2}{2(K^2)^2} \left( \frac{\mu_1}{\sigma_1} \right)^2 + \frac{k_1 \gamma_1 \varphi^1(t)}{K^2 \gamma_2 \varphi^2(t)} \left( \frac{k_1}{K^2} \gamma_1 \varphi^1(t) \right) \left( 1 - \rho^2 \right) - 1 \left( \frac{\mu_2}{\sigma_2} \right)^2
\]

\[
+ \frac{1}{K^2} k_1 \frac{\mu_1 \mu_2}{\sigma_1 \sigma_2} + \frac{1}{K^2} \gamma_1 \varphi^1(t) \frac{G^1_t + r_0 s G^1_s + \frac{1}{2} \sigma^2 s^{2\beta+2} G^1_{ss}}{2(\sigma^2 s^{2\beta+2})} - \frac{(r - r_0)^2}{2(\sigma^2 s^{2\beta+2})} + (k_1 \theta_2 - \theta_1) \gamma_1 \varphi^1(t),
\]

(68)
If insurer 1 who has more comprehensive information. This shows that when insurer 2 has incomplete information about the financial market, it will take the interest rate of the risk-free asset as its expected return rate on the risky asset under the risk neutral measure \( Q \). Intuitively speaking, in the absence of sufficient financial market information, insurer 2 is more inclined to imitate the investment strategy of insurer 1 who has more comprehensive information. This

\[
G^i(t, \xi) = g^i_1(t)\xi + g^i_2(t), \quad i = 1, 2.
\]

Theorem 4.1, insurer 2 completely ignores its own risk aversion factor, but imitates the investment strategy of insurer 1 due to the effect of asymmetric information. From a mathematical perspective, the comparison between equation (4) and equation (5) shows that when insurer 2 has incomplete information about the financial market, it will take the interest rate of the risk-free asset as its expected return rate on the risky asset under the risk neutral measure \( Q \). Intuitively speaking, in the absence of sufficient financial market information, insurer 2 is more inclined to imitate the investment strategy of insurer 1 who has more comprehensive information. This

\[
(69)
\]

Equations (68) and (69) are nonlinear, let \( \xi = s^{-\beta} \), then \( G^i(t, s) = \tilde{G}^i(t, \xi) \), where

\[
\tilde{G}^i(t, \xi) = g^i_1(t)\xi + g^i_2(t), \quad i = 1, 2.
\]

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\[
0 = \frac{K_2}{2(K_\rho)^2} \left( \frac{\mu_2}{\sigma_2} \right)^2 + \frac{k_2}{K_\rho} \frac{\gamma_2}{\gamma_1} \varphi^2(t) \left( \frac{1}{2K_\rho} \frac{\gamma_1}{\gamma_2} (1 - \rho^2) - 1 \right) \left( \frac{\mu_1}{\sigma_1} \right)^2 + \left( k_2 \theta_1 - \theta_2 \right) \gamma_2 \varphi^2(t).
\]

Remark 1. According to the form of the equilibrium investment strategy in Theorem 4.1, insurer 2 completely ignores its own risk aversion factor, but imitates the investment strategy of insurer 1 due to the effect of asymmetric information. From a mathematical perspective, the comparison between equation (4) and equation (5) shows that when insurer 2 has incomplete information about the financial market, it will take the interest rate of the risk-free asset as its expected return rate on the risky asset under the risk neutral measure \( Q \). Intuitively speaking, in the absence of sufficient financial market information, insurer 2 is more inclined to imitate the investment strategy of insurer 1 who has more comprehensive information. This

\[
\tilde{G}^i(t, \xi) = g^i_1(t)\xi + g^i_2(t), \quad i = 1, 2.
\]

Remark 1. According to the form of the equilibrium investment strategy in Theorem 4.1, insurer 2 completely ignores its own risk aversion factor, but imitates the investment strategy of insurer 1 due to the effect of asymmetric information. From a mathematical perspective, the comparison between equation (4) and equation (5) shows that when insurer 2 has incomplete information about the financial market, it will take the interest rate of the risk-free asset as its expected return rate on the risky asset under the risk neutral measure \( Q \). Intuitively speaking, in the absence of sufficient financial market information, insurer 2 is more inclined to imitate the investment strategy of insurer 1 who has more comprehensive information. This

\[
\tilde{G}^i(t, \xi) = g^i_1(t)\xi + g^i_2(t), \quad i = 1, 2.
\]
is also consistent with the actual situation. In particular, when insurer 2 does not consider the wealth gap between itself and insurer 1, insurer 2 deposits all its wealth in a risk-free asset, and the amount invested in the risky asset is 0 (i.e., when \( k_2 = 0 \), we have \( l_2^*(t) = 0 \)). This situation is in line with the economic phenomenon, because when insurer 2 does not consider the wealth gap, it makes investment strategy only based on its observable financial market information. In this case, insurer 2 is more inclined to invest in the risk-free asset. Because the expected return rate of the risk-free asset is the same as that of the risky asset, and the volatility of the risk-free asset is zero while that of the risky asset is not.

**Remark 2.** From Theorem 4.1, we can find that the equilibrium reinsurance-investment strategy is independent of the current wealth. In addition, the investment strategy of insurer \( i, i \in \{1, 2\} \) is independent of its reinsurance strategy. These two conclusions are consistent with the conclusions of most existing literatures on reinsurance and investment, for example [4], [6], [37], [1] and [14]. Moreover, the asymmetric information does not affect reinsurance strategies. Because we suppose that two insurers have different levels of information in the securities market, and they have the same level of information in the insurance market.

**Remark 3.** From Theorem 4.1, the equilibrium reinsurance-investment strategy is related to delay parameters \( \bar{h}_1, C_i, h_i, \alpha_i, \bar{\eta}_i, i = 1, 2 \). Assume that there is no delay in the insurance market, i.e. \( \bar{h}_i = C_i = \bar{\eta}_i = 0 \), then \( A_i = r_0 \). Let

\[
\begin{align*}
\dot{q}_1(t) &= \frac{\mu_1}{(\sigma_1)^2} e^{-r_0(T-t)} + \frac{k_1 \sigma_2}{\sigma_1} \dot{q}_2(t), \\
\dot{q}_2(t) &= \frac{\mu_2}{(\sigma_2)^2} e^{-r_0(T-t)} + \frac{k_2 \sigma_1}{\sigma_2} \dot{q}_1(t).
\end{align*}
\]

That is

\[
\begin{align*}
\dot{q}_1(t) &= \frac{1}{K_1 \rho \sigma_1} e^{-r_0(T-t)} \left[ \frac{\mu_1}{\gamma_1 \sigma_1} + \frac{k_1 \rho \mu_2}{\gamma_2 \sigma_2} \right], \\
\dot{q}_2(t) &= \frac{1}{K_2 \rho \sigma_2} e^{-r_0(T-t)} \left[ \frac{\mu_2}{\gamma_2 \sigma_2} + \frac{k_2 \rho \mu_1}{\gamma_1 \sigma_1} \right] .
\end{align*}
\]

Then, the equilibrium reinsurance strategy without delay has the following cases:

(i) If \( \dot{q}_1(t) < 1 \) for \( i \in \{1, 2\} \), then \( (q_1^*(t), q_2^*(t)) = (\dot{q}_1(t), \dot{q}_2(t)) \);

(ii) If \( \dot{q}_1(t) \geq 1, \dot{q}_2(t) < 1 \), \( (q_1^*(t), q_2^*(t)) = \left( 1, \frac{\mu_2}{(\sigma_2)^2} e^{-r_0(T-t)} + \frac{k_2 \sigma_1 \rho}{\sigma_2} \right) \);

(iii) If \( \dot{q}_1(t) < 1, \dot{q}_2(t) \geq 1 \), \( (q_1^*(t), q_2^*(t)) = \left( \frac{\mu_1}{(\sigma_1)^2} e^{-r_0(T-t)} + \frac{k_1 \sigma_2 \rho}{\sigma_1}, 1 \right) \);

(iv) If \( \dot{q}_1(t) \geq 1 \) for \( i \in \{1, 2\} \), \( (q_1^*(t), q_2^*(t)) = (1, 1) \);

which is consistent with the results in the literature [6]. The equilibrium investment strategies \( l_1^*(t) \) and \( l_2^*(t) \) without delay are given by

\[
\begin{align*}
l_1^*(t) &= \frac{1}{K_1 \gamma_1 \sigma_1^2 s^2} e^{-r_0(T-t)} \left[ 1 + \frac{r - r_0}{2r_0} (1 - e^{-2r_0 \beta(T-t)}) \right], \\
l_2^*(t) &= k_2 l_1^*(t) = \frac{k_2}{K_1 \gamma_1 \sigma_1^2 s^2} e^{-r_0(T-t)} \left[ 1 + \frac{r - r_0}{2r_0} (1 - e^{-2r_0 \beta(T-t)}) \right].
\end{align*}
\]

**Remark 4.** Based on the assumption in Remark 3, we further assume that insurers do not consider competitive factors (i.e., \( k_i = 0, i \in \{1, 2\} \)), insurer 2’s wealth invested in the risky asset is 0 and insurer 1’s investment strategy degenerates to

\[
l_1^*(t) = \frac{r - r_0}{\gamma_1 \sigma_1^2 s} e^{-r_0(T-t)} \left[ 1 + \frac{r - r_0}{2r_0} (1 - e^{-2r_0 \beta(T-t)}) \right],
\]
which is consistent with Remark 3.3 in the literature [41].

Furthermore, if $\beta = 0$, the CEV model reduces to the GBM model. Then, the investment strategy of insurer 1 is given by

$$l_1^*(t) = \frac{r - r_0}{\gamma_1 \sigma^2} e^{-r_0(T-t)},$$

(73)

which is just the optimal investment strategy when there is only one risky asset in the literature [4] and is the optimal investment strategy derived by [7].

**Remark 5.** Based on the assumption in Remark 3, if insurer 2 has the same level information of the financial market as insurer 1 (i.e., two insurers have symmetric information about the financial market), the equilibrium investment strategies $l_1^*(t)$ and $l_2^*(t)$ are given by

$$l_1^*(t) = \frac{r - r_0}{K \sigma^2} e^{-r_0(T-t)} \left( 1 + \frac{k_1}{\gamma_1} \right) \left[ 1 + \frac{r - r_0}{2r_0} (1 - e^{-2r_0 \beta(T-t)}) \right],$$

$$l_2^*(t) = \frac{r - r_0}{K \sigma^2} e^{-r_0(T-t)} \left( 1 + \frac{k_2}{\gamma_2} \right) \left[ 1 + \frac{r - r_0}{2r_0} (1 - e^{-2r_0 \beta(T-t)}) \right].$$

Furthermore, if $\beta = 0$, the CEV model reduces to the GBM model. Then, $l_1^*(t)$ and $l_2^*(t)$ are given by

$$l_1^*(t) = \frac{r - r_0}{K \sigma^2} e^{-r_0(T-t)} \left( \frac{1}{\gamma_1} + \frac{k_1}{\gamma_2} \right),$$

$$l_2^*(t) = \frac{r - r_0}{K \sigma^2} e^{-r_0(T-t)} \left( \frac{1}{\gamma_2} + \frac{k_2}{\gamma_1} \right),$$

which is consistent with Theorem 5 in the literature [6].

**Remark 6.** Similar to [29] and [41], the equilibrium investment strategy $l_i^*(t)$ ($i = 1, 2$) can be simplified to

$$l_1^*(t) = \frac{r - r_0}{K \gamma_1 \sigma^2} e^{-(A_1+\gamma_1)(T-t)} + \frac{(r - r_0)^2}{2K \gamma_1 \sigma^2} e^{-(A_1+\gamma_1)(T-t)} \left[ 1 - e^{-2r_0 \beta(T-t)} \right],$$

$$l_2^*(t) = \frac{k_2(r - r_0)}{K \gamma_1 \sigma^2} e^{-(A_2+\gamma_2)(T-t)} + \frac{k_2(r - r_0)^2}{2K \gamma_1 \sigma^2} e^{-(A_2+\gamma_2)(T-t)} \left[ 1 - e^{-2r_0 \beta(T-t)} \right].$$

Thus, the equilibrium investment strategy $l_i^*(t)$ ($i = 1, 2$) contains two parts. The first part is akin to (73) in form except for the competitive factor, delay and the stochastic volatility $\sigma^2$, while the second part results from the fact that the insurer tries to hedge its investment against the additional volatility risk. $e^{-2r_0 \beta(T-t)} < 1$ when $\beta > 0$, it will cause positive deviation from the optimal investment strategy under the classical GBM model; similarly, $e^{-2r_0 \beta(T-t)} > 1$ when $\beta < 0$, it will cause negative deviation from the optimal investment strategy under the classical GBM model.

Then, we give some properties of equilibrium reinsurance and investment strategies for $\beta \geq 0$ in the following Corollary 1.

**Corollary 1.** Let $H_i = (1+\alpha_i h_i)e^{-\alpha_i h_i}$. If $\beta \geq 0$, some of the properties of $q_i^*(t)$ and $l_i^*(t)$ are given in Table 1 and Table 2, respectively.

**Proof.** The above conclusions can be obtained by taking partial derivatives of $q_i^*(t)$ and $l_i^*(t)$ with corresponding variables, respectively. \[\square\]
5. Numerical examples. In this section, we present some sensitivity analysis about the equilibrium reinsurance and investment strategies \( q^*_i(t) \), \( l^*_i(t) \) (\( i = 1, 2 \)) and only consider the situation of \( \hat{q}_i(t) < 1 \) (\( i \in \{1, 2\} \)) for universality. Through further investigation about numerical results, we shall have a clear knowledge about the effect of the model parameters on the equilibrium strategies. In the following numerical illustrations, unless otherwise stated, the basic model parameters are given in Table 3 and Table 4.

| Parameter   | Value | Parameter   | Value |
|-------------|-------|-------------|-------|
| \( \mu_1 \) | 5     | \( \mu_2 \) | 1     |
| \( \sigma_1 \) | 8     | \( \sigma_2 \) | 5     |
| \( h_1 \) | 2     | \( h_2 \) | 3     |
| \( \alpha_1 \) | 0.5   | \( \alpha_2 \) | 0.3   |
| \( \eta_1 \) | 0.05  | \( \eta_2 \) | /     |
| \( \gamma_1 \) | 0.3   | \( \gamma_2 \) | 0.1   |
| \( k_1 \) | 0.4   | \( k_2 \) | 0.3   |
| \( \rho \) | 0.5   | \( \rho \) | 0.5   |

Table 1. The properties of \( q^*_i(t) \).

| \( \frac{\partial q^*_i(t)}{\partial h_i} \) | \( \frac{\partial q^*_i(t)}{\partial \eta_i} \) | \( \frac{\partial q^*_i(t)}{\partial \eta_i} \) | \( \frac{\partial q^*_i(t)}{\partial A_i} \) | \( \frac{\partial q^*_i(t)}{\partial \gamma_i} \) |
|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| +                                       | +                                       | +                                       | 0                                       | +                                       |

Table 2. The properties of \( l^*_i(t) \).

| \( \frac{\partial l^*_i(t)}{\partial h_i} \) | \( \frac{\partial l^*_i(t)}{\partial \eta_i} \) | \( \frac{\partial l^*_i(t)}{\partial \eta_i} \) | \( \frac{\partial l^*_i(t)}{\partial A_i} \) | \( \frac{\partial l^*_i(t)}{\partial \gamma_i} \) |
|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| +                                       | +                                       | +                                       | 0                                       | +                                       |

From \( A_i = r_0 - \bar{B}_i - C_i, B_i = \frac{B_i}{\int_{-\infty}^{\infty} e^{a_i b_i} ds}, C_i = \eta_i e^{-\alpha_i h_i}, B_i = (\alpha_i + A_i + \eta_i) \eta_i, i \in \{1, 2\} \), we have \( A_i = \frac{\alpha_i r_0}{\alpha_i + \eta_i (1 - e^{-\alpha_i h_i})} - \eta_i \). We can get that \( \eta_2 = \frac{\alpha_2 \eta_1 (1 - e^{-\alpha_2 h_2})}{\alpha_1 (1 - e^{-\alpha_2 h_2})} \) due to the condition \( A_1 + \eta_1 = A_2 + \eta_2 \) in Theorem 4.1. Then, we can evaluate \( \eta_2 \) from the parameter values in Table 4. In addition, the data presented in Table 4 satisfies the condition \( k_1 k_2 < 1 \) mentioned in Theorem 4.1.

Figure 1 shows the change of insurers’ equilibrium reinsurance strategies over time under different correlation coefficients. When the correlation coefficient is determined, we can see that insurer \( i (i \in \{1, 2\}) \) should gradually increase its reserve level as time elapses. From a mathematical point of view, \( q^*_i(t) = \frac{1}{K_\mu \sigma_i} (\frac{\mu_i}{\gamma_i \sigma_i} + \)
k_i, \rho_j \mu_j / \sigma_j } / \varphi_1(t) = q_i^*(T) / \varphi_1(t) is an increasing function of \( t \). From an economic perspective, the closer to the terminal time \( T \), the less uncertain information about the claims in the insurance market, and the insurer is more reluctant to spend money to sign a reinsurance contract, so the reservation level of the insurer will increase. In addition, as can be seen from Figure 1, the larger the correlation coefficient is, the higher the reservation level is. This trend is consistent with \( \frac{\partial q_i^*(t)}{\partial \rho} > 0 \). This may be because the bigger the correlation coefficient is, the more insurance businesses in the common area between the two insurers may be undertaken. Due to the competitive factors and the psychology of competition in the insurance market, insurers will try to increase their relative performance by increasing their reserve level.

![Figure 1. Optimal reinsurance strategy.](image)

Figure 2 provides the trend of equilibrium investment strategies of insurers over time \( t \) under three kinds of elasticity parameter (i.e., \( \beta = 1 \), \( \beta = 0 \) and \( \beta = -1 \)). Both insurers’ investment strategies will increase over time when the elasticity parameter is fixed. From the perspective of economics, the closer to the terminal moment \( T \), the less uncertain information there is in the financial market, and the more inclined insurers are to adopt aggressive investment strategies to achieve better relative performance at the terminal moment. From Theorem 4.1, we already know that \( l_i^*(t) = k_2 l_i^*(t) \), i.e., insurer 2 imitates the investment strategy of insurer 1 which is obvious in Figure 2. In addition, we can see that \( l_i^*(t)|_{\beta=1} < l_i^*(t)|_{\beta=0} < l_i^*(t)|_{\beta=-1} \) (\( i \in \{1, 2\} \)). That is, compared to the case of \( \beta = 0 \), positive elasticity gives rise to a positive hedging demand; the hedging demand is negative for negative elasticity, which is consistent with Remark 6.

Figure 3 illustrates the effect of the constant absolute risk aversion coefficient and the competitive parameters. The equilibrium reinsurance strategies of insurers \( q_1^*(0) \) and \( q_2^*(0) \) have negative correlation with risk aversion coefficient \( \gamma_1 \) and \( \gamma_2 \), respectively, which is obvious in Figure 3. We also find that the equilibrium
Figure 2. Optimal investment strategy.

Figure 3. Effect of risk aversion parameters on strategies.
Investment strategy of insurer 1 (i.e., $l_1^*(0)$) is a decreasing function of $\gamma_1$ while the equilibrium investment strategy of insurer 2 (i.e., $l_2^*(0)$) is irrelevant with its own risk aversion factor $\gamma_2$ but negatively correlated with $\gamma_1$. As parameter $k_i$ indicates the extent to which insurer $i$ cares about the wealth gap between itself and its competitor ($i.e., insurer j, j \neq i \in \{1, 2\}$), a higher the value of $k_i$ results in insurer $i$ becoming more concerned with its performance compared with that of its competitor in the terminal period $T$. The equilibrium reinsurance strategy of insurer $i$ is an increasing function of $k_i$ because the greater the competition, the greater the risk that insurers will take on unknown claims rather than paying more for reinsurance contracts. Similarly, the larger the competitive parameter $k_i$ is, the more likely the insurer is to adopt aggressive investment strategy to achieve better relative performance at the terminal moment. It’s worth mentioning that the amount of money that insurer 2 invests in the risky asset is always 0 when competition is not considered. This phenomenon is consist with Remark 4. In other words, either the reinsurance strategy or the investment strategy of an insurer with the competition concern is riskier than that without the concern.

Figure 4 describes the effect of delay time (i.e., $h_i$) and delay weight (i.e., $\eta_i$) on the equilibrium reinsurance strategy (i.e., $q_i^*(0)$) and the equilibrium investment strategy (i.e., $l_i^*(0)$). As shown in Figure 4, both the equilibrium reinsurance strategy and the equilibrium investment strategy increase with the increase of the delay time, and increase with the increase of the delay weight. These curves are consistent with Corollary 1. For a given delay weight $\eta_i$, the longer the delay time $h_i$, the greater the integrated wealth $Y_i(T)$ in the objective function. Insurers are...
more able to bear more claim risks and risks in the securities market, so the reserve level and wealth invested in the risky asset will increase with the increase of \( h_i \). Delay weight \( \eta_i \) reflects the degree of insurer \( i \)'s concern about past integrated performance. For a given delay time \( h_i \), the bigger \( \eta_i \), the higher the degree of concern, the higher the reserve level of compensation, and the higher the amount of investment in the risky asset. In addition, the shorter the delay time, the smaller the difference of the influence of different delay weights on the equilibrium strategy. On the contrary, the longer the delay time, the greater the difference of the influence of different delay weights on the equilibrium strategy. For example, both gaps \( q^*_i(0)|_{\eta_1=0.1} - q^*_i(0)|_{\eta_1=0.05} \) and \( l^*_i(0)|_{\eta_1=0.1} - l^*_i(0)|_{\eta_1=0.05} \) will increase with the increase of \( h_i \). It also illustrates that the insurer managing risk according to the historical experience adopts a more rational management strategy. Therefore, in the framework of non-zero-sum stochastic differential game, the difference between the effect of different delay weights on the strategies is related to the length of the delay time.

6. Conclusion. In the study, we investigate a non-zero-sum stochastic differential reinsurance and investment game problem between two insurers while taking the delay and asymmetric information of both the financial market and insurance market into account. Each competitive insurer can purchase proportional reinsurance and invest in a financial market. We assume that the financial market consists of one risk-free asset and one risky asset whose price process is denoted by the CEV model. The two insurers have access to different levels of information. Insurer 1 has the full information of this financial market while insurer 2 observes only the stock price. Considering performance in terms of the capital inflow/outflow, we model the insurers’ wealth processes as SDDEs by adopting pointwise and average delayed terms. The objective of the non-zero-sum differential game problem is to maximize the expected utility of relative performance. By applying the dynamic programming principle, the HJB equations associated with the control problem are derived in Section 3. Explicit Nash reinsurance-investment equilibrium strategies and the value functions of the CARA insurers are obtained in Section 4. Moreover, some numerical examples and sensitivity analysis are presented in Section 5 to demonstrate the effect of the model parameters on the equilibrium strategies.

The main findings are as follows: (1) \( A_1 + \eta_1 = A_2 + \eta_2 \) is one of the sufficient conditions to ensure the game problem can be solved for CARA insurers in the framework of the non-zero-sum stochastic differential game; (2) insurer 2 imitates the investment strategy of insurer 1 due to the effect of asymmetric information; (3) the difference between the effect of different delay weights on the strategies is related to the length of the delay time.

Games between companies are a very common social phenomenon and an important current research issue in the economic and financial fields. In the future work, this study can be extended in the following directions: one is introducing multi-asset investment, which is closer to reality, the other is considering regime switching to better describe the stochastic market.

7. Appendix. The proof of the Theorem 3.1:

Proof. (i) As \( J^i(t, \hat{x}_i, y_i, y_j, s) \in C^{1,2,1,1,2}(\mathcal{O}), i \neq j \in \{1, 2\}, \) applying Itô’s lemma to \( J^i \) gives, for \( \forall(t, \hat{x}_i, y_i, y_j, s) \in \mathcal{O}, \pi^i(t) \in \Pi_i, t \in [0, T], t \leq \ell \leq T \)
and any stopping time \( \tau \in [t, +\infty) \), that under the measure \( P \)
\begin{align*}
J^1(\ell \wedge \tau, \hat{x}_1(\ell \wedge \tau), y_1(\ell \wedge \tau), y_2(\ell \wedge \tau), s(\ell \wedge \tau)) \\
= J^1(t, \hat{x}_1, y_1, y_2, s) + \int_t^{\ell \wedge \tau} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u)) \ + A^T J^1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))]du \\
+ \int_t^{\ell \wedge \tau} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))q_1(u)\sigma_1]dW_1(t) \\
- \int_t^{\ell \wedge \tau} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))k_1q_2(u)\sigma_2]dW_2(t) \\
+ \int_t^{\ell \wedge \tau} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))(l_1(u) - k_1l_2(u))\sigma(s(u))^{\beta}]dW(t),
\end{align*}
under the measure \( Q \)
\begin{align*}
J^2(\ell \wedge \tau, \hat{x}_2(\ell \wedge \tau), y_2(\ell \wedge \tau), y_1(\ell \wedge \tau), s(\ell \wedge \tau)) \\
= J^2(t, \hat{x}_2, y_2, y_1, s) + \int_t^{\ell \wedge \tau} [J^2_2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u)) \ + A^T J^2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u))]du \\
+ \int_t^{\ell \wedge \tau} [J^2_2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u))q_2(u)\sigma_2]dW_2(t) \\
- \int_t^{\ell \wedge \tau} [J^2_2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u))k_2q_1(u)\sigma_1]dW_1(t) \\
+ \int_t^{\ell \wedge \tau} [J^2_2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u))(l_2(u) - k_2l_1(u))\sigma(s(u))^{\beta}]dW(t).
\end{align*}
Define the non-negative function \( \phi(u) : [0, T] \rightarrow [0, +\infty) \) as 
\begin{align*}
\phi(u) =&(J^2_2(\hat{x}_2(u), y_2(u), y_1(u), s(u)))^2((q_2(u)\sigma_2)^2 + (k_2q_1(u)\sigma_1)^2 \\
- 2k_2q_1(u)\sigma_1q_2(u)\sigma_2) + (J^1_1(\hat{x}_1(u), y_1(u), y_2(u), s(u)))^2((q_1(u)\sigma_1)^2 \\
+ (k_1q_2(u)\sigma_2) - 2q_1(u)\sigma_1k_1q_2(u)\sigma_2) \\
+ (J^1_1(\hat{x}_1(u), y_1(u), y_2(u), s(u)))^2([l_1(u) - k_1l_2(u)]\sigma(s(u))^\beta)^2 \\
+ (J^2_2(\hat{x}_2(u), y_2(u), y_1(u), s(u)))^2([l_2(u) - k_2l_1(u)]\sigma(s(u))^\beta)^2.
\end{align*}
Choose the stopping times \( \tau_n \) for \( n = 1, 2, \cdots \), to satisfy that \( \tau_n = T \) if 
\[ \int_0^T \phi(u)du < n \] or else 
\[ \tau_n = \inf \{ \ell \in [t, T] : \int_t^\ell \phi(u)du \geq n \}. \]
Note that \( \tau_n \uparrow T \) as \( n \rightarrow +\infty \), and the stopped processes 
\begin{align*}
\int_t^{\ell \wedge \tau_n} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))q_1(u)\sigma_1]dW_1(t), \\
\int_t^{\ell \wedge \tau_n} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))k_1q_2(u)\sigma_2]dW_2(t), \\
\int_t^{\ell \wedge \tau_n} [J^1_1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u))(l_1(u) - k_1l_2(u))\sigma(s(u))^{\beta}]dW(t),
\end{align*}
are all true martingales under measure \( P \) for each \( n \). Furthermore,
\[
\begin{align*}
\int_t^{\ell \wedge \tau} & \left[ J^2_{x^2}(u, \tilde{x}_2(u), y_2(u), y_1(u), s(u))q_2(u)\sigma_2 \right] dW_2(t), \\
\int_t^{\ell \wedge \tau} & \left[ J^2_{x^2}(u, \tilde{x}_2(u), y_2(u), y_1(u), s(u))k_2q_1(u)\sigma_1 \right] dW_1(t), \\
\int_t^{\ell \wedge \tau} & \left[ J^2_{x^2}(u, \tilde{x}_2(u), y_2(u), y_1(u), s(u))(l_2(u) - k_2l_1(u))\sigma(s(u))^3 \right] d\tilde{W}(t),
\end{align*}
\]
are all true martingales under measure \( Q \) for each \( n \). Taking the expectation in (74) and (75), we obtain
\[
E_P[J^1(t, \tilde{x}_1(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{s}(t))]
\]
\[
= J^1(t, \tilde{x}_1(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{s}(t)) + \mathcal{A}^{\pi_1}J^1(t, \tilde{x}_1(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{s}(t))]
\]
As \( J^i \) satisfies the conditions (34) and (35), for \( \pi_i \in \Pi_i, i \neq j \in \{1, 2\} \), we have
\[
\int_t^{\ell \wedge \tau} [J^i(u, \tilde{x}_i(u), y_i(u), S(u)) + \mathcal{A}^{\pi_i}J_i(t, \tilde{x}_i(t), \tilde{y}_i(t), \tilde{y}_j(t), \tilde{s}(t))] du \leq 0.
\]
So, for all \( \pi_i \in \Pi_i \), we get that
\[
E_P[J^1(t, \tilde{x}_1(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{s}(t))] \leq J^1(t, \tilde{x}_1(t), \tilde{y}_1(t), \tilde{y}_2(t), \tilde{s}(t)),
\]
\[
E_Q[J^2(t, \tilde{x}_2(t), \tilde{y}_2(t), y_1(t), s(t))] \leq J^2(t, \tilde{x}_2(t), \tilde{y}_2(t), y_1(t), s(t)).
\]
Since \( J^i \) satisfies the quadratic growth condition, we have, for all \( \ell \in [t, T] \)
\[
\sup_{\ell \in [t, T]} |J^i(\ell) - J^i(\ell) - \varphi^i(\ell)|^2 + \sup_{\ell \in [t, T]} |\varphi^i(\ell)|^2 \leq C(1 + \sup_{\ell \in [t, T]} |\varphi^i(\ell)|^2 + \sup_{\ell \in [t, T]} |\varphi^i(\ell)|^2).
\]
Applying the dominated convergence theorem and sending \( n \) to infinity in (78) and (79). For all \( \pi_i \in \Pi_i \), we have
\[
E_P[J^1(t, \tilde{x}_1(t), y_1(t), y_2(t), \tilde{s}(t))] \leq J^1(t, \tilde{x}_1(t), y_1(t), y_2(t), \tilde{s}(t)),
\]
\[
E_Q[J^2(t, \tilde{x}_2(t), y_2(t), y_1(t), \tilde{s}(t))] \leq J^2(t, \tilde{x}_2(t), y_2(t), y_1(t), \tilde{s}(t)).
\]
As \( J^i \) is continuous with respect to \( t \) and \( \tilde{x}_i \), for all \( \pi_i \in \Pi_i \). By sending \( \ell \) to \( T \), we can again apply the dominated convergence theorem to obtain
\[
E_P[J^1(t, \tilde{x}_1(T), \tilde{y}_1(T), y_2(T), S(T))] \leq J^1(t, \tilde{x}_1(T), \tilde{y}_1(T), y_2(T), S(T)),
\]
\[
E_Q[J^2(t, \tilde{x}_2(T), y_2(T), \tilde{y}_1(T), S(T))] \leq J^2(t, \tilde{x}_2(T), y_2(T), \tilde{y}_1(T), S(T)).
\]
Because the choice of \( \pi_i \in \Pi_i \) is arbitrary, we can deduce that \( J^i \leq V^i \), for all \( (t, \tilde{x}_i, y_1(t), y_2(t), \tilde{s}(t)) \in \mathcal{O} \) and attain at \( \pi_i^* \).
(ii) Applying Itô’s Lemma to \( J_i^1(u, \hat{x}_i(u), y_i(u), y_j(u), s(u)) \) between \( t \in [0, T) \) and \( \ell \in [t, T) \), \( i \neq j \in \{1, 2\} \), we have
\[
E_P[J_1^1(\ell, \hat{x}_1(\ell), y_1(\ell), y_2(\ell), s(\ell))] = J_1^1(t, \hat{x}_1, y_1, y_2, s)
\]
\[
+ E_P\int_t^\ell J_1^1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u)) + A^\pi J_1^1(u, \hat{x}_1(u), y_1(u), y_2(u), s(u)) \, du,
\]
\[
E_Q[J_2^2(\ell, \hat{x}_2(\ell), y_2(\ell), y_1(\ell), s(\ell))] = J_2^2(t, \hat{x}_2, y_2, y_1, s)
\]
\[
+ E_Q\int_t^\ell J_2^2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u)) + A^\pi J_2^2(u, \hat{x}_2(u), y_2(u), y_1(u), s(u)) \, du.
\]

By the definition of \( \pi^*_i \in \Pi_i \), we have
\[
-J_i^1(u, \hat{x}_i(u), y_i(u), y_j(u), s(u)) - A^\pi J_i^1(u, \hat{x}_i(u), y_i(u), y_j(u), s(u)) = 0,
\]
and so
\[
E_P[J_1^1(\ell, \hat{x}_1(\ell), y_1(\ell), y_2(\ell), s(\ell))] = J_1^1(t, \hat{x}_1, y_1, y_2, s),
\]
\[
E_Q[J_2^2(\ell, \hat{x}_2(\ell), y_2(\ell), y_1(\ell), s(\ell))] = J_2^2(t, \hat{x}_2, y_2, y_1, s).
\]

Sending \( \ell \) to \( T \) and using the quadratic growth condition and the dominated convergence theorem, we have
\[
J_1^1(t, \hat{x}_1, y_1, y_2, s) = E_P[J_1^1(T, \hat{x}_1(T), y_1(T), y_2(T), s(T))],
\]
\[
J_2^2(t, \hat{x}_2, y_2, y_1, s) = E_Q[J_2^2(T, \hat{x}_2(T), y_2(T), y_1(T), s(T))].
\]

This implies that \( J_i^1(t, \hat{x}_i, y_i, y_j, s) \leq V_i^1(t, \hat{x}_i, y_i, y_j, s) \), \( i \neq j \in \{1, 2\} \). Together with the result in (i), it follows that \( J_i^1 = V_i^1 \), and \( \pi_i^* \) is an optimal control strategy.

\[ \square \]

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