BAUER-FURUTA INVARIANTS UNDER $\mathbb{Z}_2$-ACTIONS

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Abstract. S. Bauer and M. Furuta defined a stable cohomotopy refinement of the Seiberg-Witten invariants. In this paper, we prove a vanishing theorem of Bauer-Furuta invariants for 4-manifolds with smooth $\mathbb{Z}_2$-actions. As an application, we give a constraint on smooth $\mathbb{Z}_2$-actions on homotopy $K3\#K3$, and construct a nonsmoothable locally linear $\mathbb{Z}_2$-action on $K3\#K3$. We also construct a nonsmoothable locally linear $\mathbb{Z}_2$-action on $K3$.

1. Introduction

F. Fang proved a mod $p$ vanishing theorem of Seiberg-Witten invariants under a cyclic group action of prime order $p$ [11]. This theorem, later extended by the author [25], is used when X. Liu and the author constructed nonsmoothable locally linear group actions on elliptic surfaces in [19, 21].

On the other hand, S. Bauer and M. Furuta defined a stable cohomotopy refinement of the Seiberg-Witten invariants [5, 3]. In general, Bauer-Furuta invariants may be non-trivial even if ordinary Seiberg-Witten invariants are trivial [2, 3, 13].

In this paper, we investigate Bauer-Furuta invariants under involutions. In fact, we prove a vanishing theorem of Bauer-Furuta invariants for 4-manifolds with smooth $\mathbb{Z}_2$-actions. As an application, we give a constraint on smooth $\mathbb{Z}_2$-actions on homotopy $K3\#K3$, and construct an example of locally linear action on $K3\#K3$ which can not be smoothed.

To state our results, we need some preliminaries. Let $X$ be an oriented smooth 4-manifold with an orientation-preserving smooth $\mathbb{Z}_2$-action. Fixing a $\mathbb{Z}_2$-invariant metric on $X$, we have a $\mathbb{Z}_2$-action on the frame bundle. Suppose that the $\mathbb{Z}_2$-action on $X$ lifts to a Spin$^c$-structure $c$ over $X$. Fix a $\mathbb{Z}_2$-invariant connection $A_0$ on the determinant line bundle $L$ of $c$. Then the Dirac operator $D_{A_0}$ associated to $A_0$ is $\mathbb{Z}_2$-equivariant, and the $\mathbb{Z}_2$-index of $D_{A_0}$ can be written as $\text{ind}_{\mathbb{Z}_2} D_{A_0} = k_+ C_+ + k_- C_- \in R(\mathbb{Z}_2) \cong \mathbb{Z}[t]/(t^2 - 1)$, where $C_+$ (resp. $C_-$) is the complex 1-dimensional representation on which the generator of $\mathbb{Z}_2$ acts by multiplication of $+1$ (resp. $-1$), and $R(\mathbb{Z}_2)$ is the representation ring of $\mathbb{Z}_2$.

Let $b_i$ be the $i$-th Betti number of $X$ and $b_+$ the rank of a maximal positive definite subspace $H^+(X; \mathbb{R})$ of $H^2(X; \mathbb{R})$. For any group $G$ and any $G$-space $V$, let $V^G$ be the fixed point set of the $G$-action. Let $b_+^G = \dim H^+(X; \mathbb{R})^G$.

Our main theorem is:

**Theorem 1.1.** Suppose the following conditions are satisfied:

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(1) $b_1 = 0$, $b_+ \geq 2$ and $b_{Z_2}^+ \geq 1$,
(2) $b_+ - b_{Z_2}^+$ is odd,
(3) $d(c) = 2(k_+ + k_-) - (1 + b_+) = 1$,
(4) $2k_\pm < 1 + b_{Z_2}^+$.

Then the Bauer-Furuta invariant of $c$ is zero.

Remark 1.2. The number $d(c)$ is the virtual dimension of the Seiberg-Witten moduli space for the Spin$^c$-structure $c$. The Bauer-Furuta invariant is $\mathbb{Z}/2$-valued if $d(c) = 1$ and $k_+ + k_-$ is even, and is always 0 if $d(c) = 1$ and $k_+ + k_-$ is odd [5]. Note also that, when $d(c) = 1$ and $b_1 = 0$, the ordinary Seiberg-Witten invariant is trivial by definition.

The strategy of the proof of Theorem 1.1 is inspired by Bauer’s arguments in [4]. The monopole map of a spin structure is Pin(2)-equivariant, and to analyse the monopole map, Bauer made use of the equivariant obstruction theory on Bredon cohomology. In fact, he proved that images of the forgetting map from a Pin(2)-equivariant stable cohomotopy group to $S^1$-equivariant one are multiples of 2, which implies that the value of the Seiberg-Witten invariant is even in some situations.

When a $\mathbb{Z}_2$-action on the Spin$^c$-structure is given, we have a $\mathbb{Z}_2 \times S^1$-action on the monopole map, instead of the above Pin(2)-action. By calculating Bredon cohomology groups explicitly, we can show the required vanishing of the Bauer-Furuta invariant.

As an application of Theorem 1.1 we construct a nonsmoothable locally linear $\mathbb{Z}_2$-action on $K3\#K3$. (Cf. [19, 21].)

**Theorem 1.3.** There exists a locally linear $\mathbb{Z}_2$-action on $X = K3\#K3$ which can not be smooth with respect to any smooth structure on $X$.

To prove Theorem 1.3 we basically follow the strategy of [19, 21]; In the first step, we give a constraint on smooth actions. In the second step, we construct a locally linear action which would violate the constraint.

To obtain a constraint on smooth actions, we use Theorem 1.1 together with the non-vanishing result of the Bauer-Furuta invariant of $K3\#K3$ by Furuta, Kametani and Minami [13]. On the other hand, to construct a locally linear action, we invoke a realization theorem due to A. L. Edmonds and J. H. Ewing [10].

As a byproduct of our argument, we also have:

**Theorem 1.4.** There exists a locally linear $\mathbb{Z}_2$-action on $K3$ with isolated fixed points satisfying $b_{Z_2}^+ = 3$ which can not be smooth with respect to any smooth structure on $K3$.

Remark 1.5. J. Bryan proved every smooth $\mathbb{Z}_2$-action on $K3$ with isolated fixed points satisfies $b_{Z_2}^+ = 3$ [7]. Moreover, the nonsmoothable involution in Theorem 1.4 has the same fixed point data and action on the $K3$ lattice as a symplectic automorphism of order 2 on $K3$ called Nikulin involution [24, Section 5].

The proof of Theorem 1.4 is parallel to that of Theorem 1.3. The difference is that the $G$-spin theorem gives a sufficient constraint on smooth involutions in this case.

The paper is organized as follows. In Section 2, we give a brief review on Bauer-Furuta invariants, in particular, a description of the invariants as obstruction classes. In Section 3,
we review on the equivariant obstruction theory. In Section 4, we prove our main theorem (Theorem 1.1). In Section 5, we explain the applications (Theorem 1.3 and Theorem 1.4).

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2. Bauer-Furuta invariants

The purpose of this section is to give a brief review on Bauer-Furuta invariants. (See [5, 3, 26] for details.)

2(i). Equivariant Bauer-Furuta invariants. Suppose that a 4-manifold $X$ with a $\mathbb{Z}_2$-action satisfies conditions in Theorem 1.1. Let $S^+$ and $S^-$ be the positive and negative spinor bundle of the Spin$^c$-structure $c$, and $L$ the determinant line bundle.

The Seiberg-Witten equations are a system of equations for $U(1)$-connections $A$ on $L$ and positive spinors $\phi \in \Gamma(S^+)$,

\[
\begin{align*}
D_A \phi &= 0, \\
F_A^+ &= q(\phi),
\end{align*}
\]

where $D_A$ denotes the Dirac operator, $F_A^+$ denotes the self-dual part of the curvature $F_A$, and $q(\phi)$ is the trace free part of the endomorphism $\phi \otimes \phi^*$ of $S^+$ and this endomorphism is identified with an imaginary-valued self-dual 2-form via the Clifford multiplication.

The action of the gauge transformation group $G = \text{Map}(X; U(1))$ is given by $u(A, \phi) = (A - 2u^{-1}du, u\phi)$ for $u \in G$.

Fix $k > 4$. Let $C$ be the $L^2_k$-completion of $\Omega^1(X) \oplus \Gamma(S^+)$, and $U$ be the $L^2_{k-1}$-completion of $\Gamma(S^-) \oplus i\Omega^+(X) \oplus \Omega^0(X)/\mathbb{R}$, where $\Omega^k(X)$ is the space of $k$-forms, $\Omega^+(X)$ is the space of self-dual 2-forms, and $\mathbb{R}$ is the space of constant functions on $X$. Fix a $\mathbb{Z}_2$-invariant connection $A_0$ on $L$, and let us define the monopole map $\mu: C \to U$ by

\[
\mu(a, \phi) = (D_{A_0+ia}\phi, F_{A_0+ia}^+ - q(\phi), d^*a).
\]

When the lift of the $\mathbb{Z}_2$-action to the Spin$^c$-structure $c$ is given, $\mu$ is $\mathbb{Z}_2 \times S^1$-equivariant.

Now we will review on finite dimensional approximations. Decompose the monopole map $\mu$ into the sum of the linear part $D$ and the quadratic part $Q$, i.e., $\mu = D + Q$, where $D: C \to U$ is given by

\[
D(a, \phi) = (D_{A_0}\phi, d^+a, d^*a),
\]

and $Q$ is the rest.

Theorem 2.2 ([3]). There exists a finite dimensional linear subspace $W_f \subset U$ which has the following properties:

1. $W_f$ and images of $D$ spans $U$, i.e., $U = W_f + \text{im} D$. 

(2) For each finite dimensional linear subspace \( W \subset U \) which contains \( W_f \), let \( V = D^{-1}(W) \). Then a \( \mathbb{Z}_2 \times S^1 \)-equivariant map \( f_W : S^V \to S^W \) between one-point compactifications of \( V \) and \( W \) is defined from \( \mu \). (The base points of \( S^V \) and \( S^W \) at infinity are denoted by \( \ast \).

(3) If \( W' = U \oplus W \), where \( U \) and \( W \) are finite dimensional linear subspaces, and \( W \supset W_f \), then \( f_{W'} \) and \( \text{id}_U \wedge f_W \) are \( \mathbb{Z}_2 \times S^1 \)-equivariant homotopic as pointed maps:

\[
S^V \cong S^{U \oplus V} \to S^{W'} \cong S^{U \oplus W}.
\]

Remark 2.3. More concretely, the map \( f_W \) is defined as follows. It is known that the monopole map \( \mu \) is proper, and extends to the map \( \mu^+ \) between \( S^c \) and \( S^U \). For each finite dimensional linear subspace \( W \) of \( U \), let \( S(W^\perp) \) be the unit sphere in the orthogonal complement \( W^\perp \) of \( W \). In [5], a retraction \( \rho_W : S(U) \setminus S(W^\perp) \to S^W \) is given, and it is proved that \( \text{im } \mu^+ \cap S(W^\perp) = \emptyset \) when \( W \) contains \( W_f \). Then \( f_W = (\rho_W \circ \mu^+|_{S^V}) : S^V \to S^W \).

Theorem 2.2 enables us to specify a well-defined class in the following equivariant stable cohomotopy group:

\[
[f_W] \in \{\text{ind}_{\mathbb{Z}_2} D, H^+\}^{\mathbb{Z}_2 \times S^1} = \text{colim}_{U \subset U} [S^U \wedge S^{V-W}, S^U \wedge S^0/\mathbb{Z}_2 \times S^1],
\]

where \( S^{V-W} \) is the desuspension of \( S^V \) by \( S^W \) which can be identified with the space of maps from \( S^W \) to \( S^V \). Note that the space \( U \) is the \( \mathbb{Z}_2 \times S^1 \)-universe which consists of the following representations: \( C_+, C_-, R_+, R_- \), where \( S^1(\subset C) \) acts on \( C_+, C_- \) by multiplication, and on \( R_+, R_- \) trivially, and \( \mathbb{Z}_2 \) acts on \( C_+, R_+ \) trivially, and on \( C_-, R_- \) by multiplication of \( \pm 1 \). The class \( [f_W] \) is the \( \mathbb{Z}_2 \)-equivariant Bauer-Furuta invariant of \((X,c)\), and we denote it by BF\[^{\mathbb{Z}_2}(c)\).

In the case without any extra group action on the Spin\(^c\)-structure, a similar construction determines an element \([f_W]\) in \{ind \( D, H^+ \)\}\(^{S^1}\), and this is the original Bauer-Furuta invariant, denoted by BF\((c)\). Note that the original Bauer-Furuta invariant BF\((c)\) is obtained from BF\[^{\mathbb{Z}_2}(c)\) via the map forgetting the \( \mathbb{Z}_2 \)-action \( \phi : \{\text{ind}_{\mathbb{Z}_2} D, H^+\}^{\mathbb{Z}_2 \times S^1} \to \{\text{ind } D, H^+\}^{S^1} \), by BF\((c)\) = \( \phi(BF^{\mathbb{Z}_2}(c)) \).

In general, it is difficult to calculate stable cohomotopy groups. However, in several cases of \( S^1 \), stable cohomotopy groups \{ind \( D, H^+ \)\}\(^{S^1}\) are calculated in [5]. In fact, when the virtual dimension \( d(c) \) of the moduli of \( c \) is 1, we know

\[
(2.4) \quad \{\text{ind } D, H^+\}^{S^1} \cong \begin{cases} 
\mathbb{Z}_2, & \text{when ind } D \text{ is even,} \\
\{0\}, & \text{otherwise.}
\end{cases}
\]

In the \( \mathbb{Z}_2 \times S^1 \)-case, \{ind \( D, H^+\)\}\(^{\mathbb{Z}_2 \times S^1}\) is attained by suspension by finite copies of \( C_+, C_- \) and \( R_+ \):

Proposition 2.5. Suppose \( d(c) = 1 \) and \( b_+ \geq 2 \). Let \( V_0 = k_+ C_+ \oplus k_- C_- \) and \( W_0 = H^+ \). (If one of \( k_\pm \) is negative, say \( k_+ \), then add \(-k_+ C_+ \) to both of \( V_0 \) and \( W_0 \) to obtain actual representations.) There is a representation \( V' \subset U \) of the form

\[
V' = a_+ C_+ \oplus a_- C_- \oplus b R_+,
\]
such that
\[ [S^V \wedge S^{V_0}, S^{V'} \wedge S^{W_0}]_{\mathbb{Z}_2 \times S^1} \cong \{ \text{ind}_{\mathbb{Z}_2} D, H^+ \}_{\mathbb{Z}_2 \times S^1}. \]

For the proof, we invoke the equivariant Freudenthal suspension theorem.

**Theorem 2.6** ([23], Chapter IX, Theorem 1.4). Let \( G \) be a compact Lie group, \( U \) a representation, \( Y \) a \( G \)-space, and \( X \) a \( G \)-CW complex (see [3]). For each subgroup \( H \subset G \), let \( c^H(Y) \) be the connectivity of \( Y^H \). If the following hold,

1. \( \dim X^H \leq 2c^H(Y) \) for all subgroups \( H \) with \( U^H \neq 0 \),
2. \( \dim X^H \leq c^K(Y) - 1 \) for all pairs of subgroups \( K \subset H \) with \( U^H \neq U^K \),

then the suspension map
\[ S^U : [X, Y]^G \to [S^U \wedge X, S^U \wedge Y]^G, \]
is bijective.

**Proof of Proposition 2.5.** The proof is divided into two steps.

1. For each subgroup \( H \subset G = \mathbb{Z}_2 \times S^1 \) such that there is a representation \( V \subset U \) with \( V^H \neq 0 \), by adding copies of \( V \) to \( V' \) if necessary, the following is satisfied:
\[ \dim V^H + \dim V'^H \leq 2(\dim V^H + \dim W^H_0 - 1). \]

2. For each pair of subgroups \( K \subset H \subset G \) such that there is a representation \( V \subset U \) with \( V^H \neq V^K \), by adding copies of \( V \) to \( V' \) if necessary, the following is satisfied:
\[ \dim V^H + \dim V'^H \leq (\dim V^K + \dim W^K_0 - 1) - 1. \]

If (1) and (2) are satisfied, then \([S^V \wedge S^{V_0}, S^{V'} \wedge S^{W_0}]_{\mathbb{Z}_2 \times S^1}\) is in stable range by Theorem 2.6. Therefore it suffices to prove that we can take a direct sum of finite copies of \( \mathbb{C}_\pm \) or \( \mathbb{R}_+ \) as \( V' \). First, note that \( U \) contains only four types of representation, and therefore the number of orbit types in them is finite. For (1), it suffices to add a direct sum of finite copies of \( \mathbb{R}_+ \) to \( V' \). For (2), we can prove that it is not necessary to add \( \mathbb{R}_- \) to \( V' \) under the assumption \( b_+ \geq 2 \), and it suffices to add a finite sum of \( \mathbb{C}_\pm \) or \( \mathbb{R}_+ \) to \( V' \).

Take a large \( V' \) as in Proposition 2.5 and put \( V = V' \oplus V_0 \) and \( W = V' \oplus W_0 \). We use the following notation. Let \( V_\mathbb{R}^\pm \) be the \( \mathbb{R}_\pm \)-component of \( V \), and \( V_\mathbb{C}^\pm \) the \( \mathbb{C}_\pm \)-component of \( V \), and similarly \( W_\mathbb{R}^\pm \) and \( W_\mathbb{C}^\pm \). Let \( V_\mathbb{C} = V_\mathbb{C}^+ \oplus V_\mathbb{C}^- \), and \( V_\mathbb{R}, W_\mathbb{C} \) and \( W_\mathbb{R} \) are also similarly defined.

2(ii). **Bauer-Furuta invariants as obstruction classes.** In this subsection, we describe (non-equivariant) Bauer-Furuta invariants in terms of ordinary obstruction theory. First, note the following:

**Proposition 2.7** ([5], Proposition 3.4). If \( \text{ind } D > 0 \), then \( \{ S^V, S^W \}^{S^1} \cong \{ S^V/S^1, S^W \} \).

Note that, if \( d(c) = 1 \), then \( \text{ind } D > 0 \). Thus, by the Freudenthal theorem, \( \{ S^V, S^W \}^{S^1} \) is isomorphic to non-equivariant (unstable) cohomotopy group \([ S^V/S^1, S^W ] \) for sufficient large \( V, W \). Then we can analyse \([ S^V/S^1, S^W ] \) by the standard obstruction theory.
Proposition 2.8. Suppose $d(c) = 1$ and put $n = \dim S^V/S^1$. (Note that $\dim S^W = n - 1$.) Then $H^r(S^V/S^1, *, \pi_r(S^W)) = 0$ when $r \neq n$, and $H^n(S^V/S^1, *, \pi_n(S^W)) \cong \mathbb{Z}/2$.

Proof. It is clear that $H^r(S^V/S^1, *, \pi_r(S^W)) = 0$ when $r < n - 1$ or $r > n$. For sufficient large $V$, the codimension of $S^{V_k}$ in $S^V/S^1$ is also large. Now note that $(S^V/S^1, *)$ has the same homotopy type as $(S^V \wedge S^1 \wedge \mathbb{C}P(V_C), *)$ (cf. [13]) and $S^W \cong S^{n-1}$. Then we see the rest case when $r = n - 1, n$. □

Then the standard obstruction theory implies the following:

Theorem 2.9 (See [13], Chapter VI). There exists a subgroup $J$ of $H^n(S^V/S^1, *, \pi_n(S^W))$ such that

\[ [S^V/S^1, S^W] \cong H^n(S^V/S^1, *, \pi_n(S^W))/J. \]

The isomorphism is given by correspondence that $f \in [S^V/S^1, S^W]$ is mapped to the difference obstruction class $d(f, \underline{0})$, where $\underline{0}$ is the map which collapse whole $S^V$ to the base point.

Remark 2.10. The subgroup $J$ in Theorem 2.9 is given as follows. Let $(K, L) = (S^V/S^1, *)$ and $Y = S^W$. Suppose a map $f : K \to Y$ is given. Let $K^r$ be the $r$-skeleton of $K$, and put $\bar{K}^r = K^r \cup L$. Let $\Theta$ be the set of maps $g : \bar{K}^{n-1} \to Y$ such that $g|L = f|L$, and put $\theta_0 = f|\bar{K}^{n-1}$. Let $R^n(K, L; f) := \pi_1(\Theta, \theta_0)$. Then each element in $R^n(K, L; f)$ can be represented by a homotopy $h_t : \bar{K}^{n-1} \to Y$ such that

\[ h_0 = h_1 = f|\bar{K}^{n-1}, \quad h_t|L = f|L, \quad 0 \leq t \leq 1. \]

Let us define the map $\xi : R^n(K, L; f) \to H^n(K, L; \pi_n(S^W))$ by $\xi([h_t]) = d(f, f; h_t)$, which is the obstruction for deforming $f$ to itself via the homotopy $h_t$. Then $J = \text{im} \xi$.

The meaning of Theorem 2.9 is as follows. If we take a map, say $\underline{0}$, as origin, then the cohomology $H^n(S^V/S^1, *, \pi_n(S^W))$ itself is identified with the set of maps modulo homotopies which are fixed on $(n - 2)$-skeleton. To obtain the required homotopy set given via homotopies which are not necessarily fixed on $(n - 2)$-skeleton, we need to know when two maps whose difference is represented by an element of the cohomology $H^n(S^V/S^1, *, \pi_n(S^W))$ are in fact homotopic. The subgroup $J$ measures this. Note that (2.3) can be proved by calculating $J$ directly.

Thus, when the map $f_W$ is a finite dimensional approximation of the monopole map, the Bauer-Furuta invariant $BF(c)$ can be written as the difference obstruction class $d(f_W, \underline{0})$.

3. Equivariant obstruction theory

In some special situations, equivariant Bauer-Furuta invariants can be written as equivariant obstruction classes. In this section, we give a brief review on the equivariant obstruction theory. For more details, see [23, 6, 22, 8].
3(i). Bredon cohomology. First, we review on Bredon cohomology which is the foundation of the equivariant obstruction theory. Let $G$ be a compact Lie group. We recall the notion of $G$-CW complex.

**Definition 3.1.** A $G$-CW complex $K$ is the union of sub $G$-spaces $K^n$ such that
- $K^0$ is a disjoint union of orbits $G/H$,
- $K^n$ is obtained from $K^{n-1}$ by attaching $n$-cells $\sigma \simeq G/H_\sigma \times D^n$ via attaching $G$-maps $a_\sigma: G/H_\sigma \times S^{n-1} \to K^{n-1}$,
- $K$ has the colimit topology with respect to $(K^n)$.

We denote the maximal degree of cells in $K$ by cell-dim $K$. In general, cell-dim $K = \dim K/G$. We need several notions to define Bredon cohomology.

**Definition 3.2.** The category of canonical orbits of $G$, denoted by $O_G$, is the category whose objects are cosets $G/H$, where $H$ are closed subgroups, and morphisms are $G$-homotopy classes of $G$-maps $G/H \to G/I$.

**Remark 3.3.** Let $N(H, I)$ be the subgroup of $a \in G$ such that $a^{-1}Ha \subset I$. Then we have an identification, $G$-map$(G/H, G/I) \simeq N(H, I)/I = (G/I)^H$.

Let $Abel$ be the category of Abelian groups.

**Definition 3.4.** We call a contravariant functor $O_G \to Abel$ a ($G$-)coefficient system.

**Remark 3.5.** For a closed subgroup $H$, let $WH$ be the Weyl group of $H$, and $W_0H$ its identity component. Then, for a coefficient system $M: O_G \to Abel$, $M(G/H)$ is a $\mathbb{Z}[WH/W_0H]$-module.

Let us consider the category of whole $G$-coefficient systems, denoted by $C_G$, whose morphisms are given by natural transformations. It is known that $C_G$ is an Abelian category ([20], p.258). Therefore we can develop homological algebra in it.

Let $K$ be a $G$-CW complex. We have coefficient systems,

$$C_n(K) = H_n(K^n, K^{n-1}; \mathbb{Z}),$$

which are defined by $C_n(K)(G/H) = H_n((K^n)^H, (K^{n-1})^H; \mathbb{Z})$. The connecting homomorphisms of homology of the triples $((K^n)^H, (K^{n-1})^H, (K^{n-2})^H)$ induce a homomorphism of $G$-coefficient systems,

$$\partial: C_n(K) \to C_{n-1}(K),$$

which satisfies $\partial \circ \partial = 0$.

Let $M$ be a $G$-coefficient system, and define the cochain complex $(C^n_G(K; M), \delta)$ by

$$C^n_G(K; M) = \text{Hom}_{C_G}(C_n(K), M), \quad \delta = (\partial)^*.$$

Its cohomology is the Bredon cohomology, denoted by $H^n_G(K; M)$.

The coefficient systems of chain, $C_n(K)$, can be identified as follows.

**Proposition 3.6.** We have an isomorphism,

$$C_n(K) \cong \bigoplus_{\sigma: n\text{-cell}} H_0(G/H_\sigma; \mathbb{Z}),$$
where $H_0(G/H; \mathbb{Z})$ is defined by $H_0(G/H; \mathbb{Z})(G/H) = H_0((G/H)_0; \mathbb{Z})$.

Proof. This follows from

$$H_n((K^n)_H, (K^{n-1})_H) \cong \bigoplus_{\sigma \in \text{n-cell}} H_n((G/H_{\sigma})_H \times (D^n, S^{n-1})) \cong \bigoplus_{\sigma \in \text{n-cell}} H_0((G/H_{\sigma})_H).$$

Remark 3.7. Note that

$$H_0((G/H_{\sigma})_H; \mathbb{Z}) \cong F\pi_0((G/H_{\sigma})_H) \cong F[G/H, G/H]_G,$$

where, for a set $S$, $FS$ denotes the free Abelian group generated by $S$. The coefficient system $H_0(G/H)$ has a special property in the next.

Proposition 3.8. Let $M$ be a coefficient system. There is an isomorphism,

$$\text{Hom}_{\mathcal{C}_G}(H_0(G/H), M) \cong M(G/H),$$

via $\phi \mapsto \phi(1_{G/H})$.

Proof. The inverse is given as follows. Suppose that $m_\phi \in M(G/H)$ to be $\phi(1_{G/H})$ is given. Recall that $H_0(G/H; \mathbb{Z})(G/I) \cong F[G/I, G/H]_G$. For a $G$-map $f : G/I \to G/H$, $M(f)m_\phi$ is in $M(G/I)$. Then, the correspondence $m_\phi \mapsto M(f)m_\phi$ determines a morphism $\phi$ in $\mathcal{C}_G$.

\[\square\]

Corollary 3.9. The cochain group $C^n_G(K; M)$ can be identified with the space of functions such that an n-cell $\sigma$ is mapped to an element of $M(G/H_{\sigma})$.

Remark 3.10. To define $H^n_G(K; M)$, the coefficient system $M$ need not to be defined on the whole $\mathcal{O}_G$, but on a subcategory $\mathcal{O}_G^K$ given below. Let $\text{Iso}(K)$ be the set of isotropy groups in $K$. Objects of $\mathcal{O}_G^K$ are cosets $G/H$ where $H \in \text{Iso}(K)$. Morphisms are $G$-homotopy classes of $G$-map between them.

For a $G$-CW pair $(K, L)$, the relative Bredon cohomology $H^n_G(K, L; M)$ is defined from

$$C_n(K, L) = H_n(K^n, K^{n-1}; M),$$

where $K^n = K \cup L$. Also there is an identification $C_n(K, L) = \bigoplus_{\sigma} H_0(G/H_{\sigma})$, where $\sigma$ runs over n-cells in $K \setminus L$.

This definition works well due to the following.

Proposition 3.11. For a $G$-CW pair $(K, L)$, $C_n(K)$, $C_n(L)$ and $C_n(K, L)$ are projective in $\mathcal{C}_G$.

This is proved from the fact that the coefficient system $H_0(G/H)$ is projective, which can be easily seen from Proposition 3.8.

For a $G$-CW pair $(K, L)$, we have an exact sequence,

$$0 \to C_n(L) \to C_n(K) \to C_n(K, L) \to 0.$$

When we apply the functor $\text{Hom}_{\mathcal{C}_G}(-, M)$ to this, the resulting sequence is also exact by Proposition 3.11. Thus we obtain long exact sequences of Bredon cohomology.
In general, it is not easy to calculate Bredon cohomology. However, the following simple case is rather easy.

**Proposition 3.12** (Cf. [8], Chapter II, 3). Let \( G \) be a compact Lie group, and \( G_0 \) its identity component. Let \( M \) be a \( G \)-coefficient system, and put \( M_0 = M(G/\{e\}) \). (Note that \( M_0 \) is a \( \mathbb{Z}[G/G_0] \)-module.) Suppose that \((K, L)\) is a \( G \)-CW pair such that \( G \) acts on \( K \setminus L \) freely. Let us consider the complex:

\[
\to \text{Hom}_{\mathbb{Z}[G/G_0]}(H_{n-1}(\bar{K}^{n-1}/G_0, \bar{K}^{n-2}/G_0), M_0) \xrightarrow{\delta_{n-1}} \text{Hom}_{\mathbb{Z}[G/G_0]}(H_n(\bar{K}^n/G_0, \bar{K}^{n-1}/G_0), M_0) \xrightarrow{\delta_n} \text{Hom}_{\mathbb{Z}[G/G_0]}(H_{n+1}(\bar{K}^{n+1}/G_0, \bar{K}^n/G_0), M_0) \to,
\]

where \( \delta_r \) are determined from connecting homomorphisms of homology of triples

\((\bar{K}^r/G_0, \bar{K}^{r-1}/G_0, \bar{K}^{r-2}/G_0)\).

Then the \( n \)-th Bredon cohomology \( H^*_G(K, L; M) \) is isomorphic to \( \ker \delta_n/\text{im} \delta_{n-1} \).

**Proof.** By the assumption, we have \( C_n(K, L) \cong \bigoplus_\sigma \text{Hom}(G/\{e\}) \). Then note that \( H_0(G) \cong H_0(G/G_0) \) and Proposition 3.8. \( \square \)

**Remark 3.13.** If \((K, L)\) is a \( G \)-CW complex, then \((K/G_0, L/G_0)\) has an ordinary CW complex structure with a cellular \( G/G_0 \)-action induced from the \( G \)-CW structure on \((K, L)\). In the case of Proposition 3.12, we can calculate the Bredon cohomology of \((K, L)\) from this CW complex structure of \((K/G_0, L/G_0)\) with the \( G/G_0 \)-action.

3(ii). **Equivariant obstruction theory.** We can develop equivariant obstruction theory on Bredon cohomology. Let \((K, L)\) be a \( G \)-CW pair. Let \( \text{Iso}(K, L) \) be the set of isotropy groups of \( K \setminus L \).

**Definition 3.14.** Fix \( n \geq 1 \). Let \( Y \) be a \( G \)-space such that, for each closed subgroup \( H \in \text{Iso}(K, L) \), \( Y^H \) is non-empty, path-connected and \( n \)-simple. Then the coefficient system \( \pi_n(Y) \) is defined by \( \pi_n(Y)(G/H) = \pi_n(Y^H) \).

Suppose that a \( G \)-map \( f : \bar{K}^n = K^n \cup L \to Y \) is given. We ask when \( f \) can be extended \( G \)-equivariantly over \( \bar{K}^{n+1} \). For each \((n+1)\)-cell \( \sigma \), the composite map of the attaching map \( a_\sigma : G/H_\sigma \times S^n \to \bar{K}^n \) with \( f \) defines an element of \( \pi_n(Y^{H_\sigma}) \) since \( f \circ a_\sigma \) maps \( eH_\sigma \times S^n \) into \( Y^{H_\sigma} \). This correspondence gives a well-defined cocycle,

\[
c_f \in C_{G}^{n+1}(K, L; \pi_n(Y)),
\]
and \( f \) extends to \( \bar{K}^{n+1} \) if and only if \( c_f = 0 \).

Similarly, for \( G \)-maps \( f_0, f_1 : \bar{K}^n \to Y \) and a \( G \)-homotopy \( h_t \) on \( \bar{K}^{n-1} \) rel \( L \) such that \( h_0 = f_0|\bar{K}^{n-1} \) and \( h_1 = f_1|\bar{K}^{n-1} \), we can define the difference cochain,

\[
d(f_0, f_1; h_t) \in C_{G}^{n}(K, L; \pi_n(Y)),
\]

such that \( \delta d(f_0, f_1; h_t) = c_f_1 - c_f_0 \). Then \( h_t \) extends over \( \bar{K}^n \) if and only if \( d(f_0, f_1; h_t) = 0 \). Furthermore, for given \( f_0 : \bar{K}^n \to Y \) and \( \beta \in C_{G}^{n}(K, L; \pi_n(Y)) \), there exists \( f_1 : \bar{K}^n \to Y \) which coincides with \( f_0 \) on \( \bar{K}^{n-1} \) and satisfies \( \beta = d(f_0, f_1) \), where the homotopy is assumed constant.

With these understood, a standard argument proves the following.
Theorem 3.15. Let \( n = \text{cell-dim}(K \setminus L) \), and denote the set of \( G \)-homotopy classes rel \( L \) of \( G \)-maps by \([K, Y]^G_{\text{rel} L}\). If \( H^k_G(K, L; \pi_k(Y)) = 0 \) for \( k \neq n \), then there exists a subgroup \( J' \subset H^0_G(K, L; \pi_n(Y)) \), and
\[
[K, Y]^G_{\text{rel} L} \cong H^0_G(K, L; \pi_n(Y))/J'.
\]

As in [2(i)], let \( f_W: S^V \to S^W \) be a finite dimensional approximation for sufficiently large \( V \) and \( W \). If \( (K, L) = (S^V, \ast) \) and \( Y = S^W \), and the conditions in Theorem [3.15] are satisfied, then the equivariant Bauer-Furuta invariant \( BF^{Z_2}(c) \) can be identified with the difference obstruction class \( d(f_W, \emptyset) \in H^0_{Z_2 \times S^1}(K, L; \pi_n(Y))/J' \).

4. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. The proof of Theorem 1.1 is divided into two steps. The first step proves Bredon cohomology groups vanish in all degrees except the top. Then, by Theorem 3.15, \( \{\text{ind}_{\mathbb{Z}_2} D, H^+\}^{Z_2 \times S^1} \) is written in terms of the top-degree Bredon cohomology. The second step compares the top-degree Bredon cohomology with the top-degree ordinary cohomology via the map forgetting the group action, and observes that the forgetting map is the 0-map. Then, it follows that the vanishing of the Bauer-Furuta invariant, since this is an image of the forgetting 0-map. Our argument is analogous to that in [8], Chapter II, 4 as well as [4].

4(i). Vanishing of Bredon cohomology in low degrees. Recall that we took sufficiently large \( V \) and \( W \) in [2(ii)]. We assume a \( \mathbb{Z}_2 \times S^1 \)-CW complex structure on \((S^V, \ast)\) fixed. (Later, we will give a \( \mathbb{Z}_2 \times S^1 \)-CW complex structure on \((S^V, \ast)\) concretely.)

Note that
\[
\text{Iso}(S^V, \ast) = \left\{ \{e\}, \mathbb{Z}_2, \tilde{\mathbb{Z}}_2, \{1\} \times S^1, \mathbb{Z}_2 \times S^1 \right\}.
\]
where, by assuming \( \mathbb{Z}_2 = \{\pm 1\} \) and \( S^1 \subset \mathbb{C} \), \( \mathbb{Z}_2 \) is the subgroup generated by \((-1, 1)\), and \( \tilde{\mathbb{Z}}_2 \) generated by \((-1, -1)\).

Lemma 4.1. For each \( H \in \text{Iso}(S^V, \ast) \), \((S^W)^H\) is a k-sphere, where \( k \geq 1 \).

Proof. This follows from the fact that \( W^H \) is a linear subspace of \( W \), \( W^H \supset W^{\mathbb{Z}_2 \times S^1} \) and \( \dim W^{\mathbb{Z}_2 \times S^1} \geq b^\mathbb{Z}_2^+ \geq 1 \). \( \square \)

Therefore, \( \pi_\ast(W) \) is well-defined.

Now, we prove the vanishing of Bredon cohomology in low degrees.

Lemma 4.2. Let \( n = \text{cell-dim} S^V \). If \( 2k_+ < 1 + b^\mathbb{Z}_2^+ \), then \( C^k_{\mathbb{Z}_2 \times S^1}(S^V, \ast; \pi_k(W)) = 0 \) for \( k \leq n - 2 \).

Proof. Let \( \sigma \) be a \( k \)-cell, and \( \varphi \) be a \( k \)-cochain. Put \( G = \mathbb{Z}_2 \times S^1 \). If \( H_\sigma = \{e\} \), then \( \varphi(\sigma) = 0 \) since \( \pi_k(W^G) = \pi_k(S^{n-1}) = 0 \).

If \( H_\sigma = \{1\} \times S^1 \), then \( k \leq \dim V^\mathbb{R} \). Since \( (S^W)^{S^1} = S^{W_\mathbb{R}} \), \( W_\mathbb{R} \cong V^\mathbb{R} \oplus H^+ \) and \( b^+ > 0 \), we have \( \pi_k(W^G) = \pi_k(S^{n-1}) = 0 \). Similarly, in the case when \( H_\sigma = \mathbb{Z}_2 \times S^1 \), we can show \( \pi_k(W^G) = 0 \) by \( b^\mathbb{Z}_2^+ > 0 \).
Now suppose that $H_\sigma = \mathbb{Z}_2$. Note that the condition $2k_+ < 1 + k_+^2$ is equivalent to the condition
\[
\dim(S^V)/\mathbb{Z}_2 < 1 + \dim(S^W)/\mathbb{Z}_2.
\]
On the other hand, $k \leq \text{cell-dim}(S^V)/\mathbb{Z}_2 = \dim(S^V)/\mathbb{Z}_2 - 1$. Therefore $\pi_k(S^W)(G/\mathbb{Z}_2) = 0$. The case when $H_\sigma = \mathbb{Z}_2$ is similar.\hfill \Box

4(ii). **Calculation in high degrees.** Let $(S^V)_s$ be the singular part of $S^V$, that is,
\[
(S^V)_s = \bigcup_{\{e\} \neq H \in \text{Iso}(S^V,*)} (S^V)_H.
\]

**Lemma 4.3.** If necessary, by adding several copies of $U = \mathbb{C}_+ \oplus \mathbb{C}_-$ to $V$ and $W$, the following holds.

\[
(4.4) \quad \dim(S^V)_s \leq \dim S^V - 4.
\]

**Proof.** This follows from the fact that $U^{\mathbb{Z}_2} = \mathbb{C}_+ \oplus \{0\}$, $U^{\mathbb{Z}_2} = \{0\} \oplus \mathbb{C}_-$ and these subspaces have complex codimension 1 in $U = U^{\{e\}}$. (Note that $U^{S^1} = U^{\mathbb{Z}_2 \times S^1} = \{0\}$.) \hfill \Box

Let $n = \text{cell-dim } S^V$. Then, Lemma 4.3 and the long exact sequence imply that
\[
H^k_{\mathbb{Z}_2 \times S^1}(S^V, *, M) \cong H^k_{\mathbb{Z}_2 \times S^1}(S^V, (S^V)_s, M),
\]
for $k = n, n - 1$. Therefore, by Proposition 3.12, $H^k_{\mathbb{Z}_2 \times S^1}(S^V, *, M)$ for $k = n, n - 1$ can be calculated from $S^V/S^1$.

Note that $S^V/S^1$ decomposes as
\[
(4.5) \quad S^V/S^1 = * \cup V_\mathbb{R} \times (\mathbb{R}_{>0} \times C_P(V_\mathbb{C}) \cup \{0\}).
\]

Recall that we may assume $V_\mathbb{R}$ contains no $\mathbb{R}_-$ by Proposition 2.5. Therefore, by giving a $\mathbb{Z}_2$-CW complex structure on $C_P(V_\mathbb{C})$, we can fix that on $S^V/S^1$ via the decomposition (4.5).

Now, we give a $\mathbb{Z}_2$-CW complex structure on $C_P(V_\mathbb{C})$. For $a \mathbb{C}_+ \oplus b \mathbb{C}_-$, we denote $C_P(a \mathbb{C}_+ \oplus b \mathbb{C}_-)$ by $P^{a,b}$. We have a filtration as $P^{a,b} \supset P^{a-1,b} \supset P^{a-2,b}$ if $a \geq 2$. Note that
\[
P^{a,b} \setminus P^{a-1,b} = \{[z_1, \ldots, z_a, w_1, \ldots, w_b] \mid z_a \neq 0\} \cong (a - 1)\mathbb{C}_+ \oplus b\mathbb{C}_-.
\]

Certainly, this is an open disk. However, $\mathbb{Z}_2$ still acts on it non-trivially. Hence, we need to take a subdivision. It suffices to subdivide $C_0^n$.

The space $C_0^b$ is identified with $\mathbb{R}^{2b} = \{(x_1, y_1, \ldots, x_b, y_b)\}$ by $w_i = x_i + \sqrt{-1}y_i$. Then, a $\mathbb{Z}_2$-equivariant subdivision is given as follows:
\[
\{x_1 > 0\} \cup \{x_1 < 0\} \cup \{x_1 = 0, y_1 > 0\} \cup \{x_1 = 0, y_1 < 0\} \cup \ldots.
\]

Let us denote the open disk with the above subdivision by $D^{a-1,b}$. We may assume $a, b \geq 2$, and have the decomposition:
\[
P^{a,b} = D^{a-1,b} \cup D^{a-2,b} \cup P^{a-2,b}.
\]
According to the above cell structure, we can write down its (ordinary) chain complex $\mathbb{Z}_2$-equivariantly.

$$
\begin{array}{cccccc}
0 & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & C_{n-2} & \xrightarrow{\partial_{n-2}} & C_{n-3} & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \| & & \\
\mathbb{Z}[\mathbb{Z}_2] & & \mathbb{Z}[\mathbb{Z}_2] & & \mathbb{Z}[\mathbb{Z}_2] \oplus \mathbb{Z}[\mathbb{Z}_2] & & \mathbb{Z}[\mathbb{Z}_2] \oplus \mathbb{Z}[\mathbb{Z}_2] & & \\
\end{array}
$$

where $\partial_n$ is given by the matrix over $\mathbb{Z}$ as follows,

\begin{equation}
\partial_n = \begin{pmatrix}
1 & -1 \\
-1 & 0
\end{pmatrix}.
\end{equation}

**Proposition 4.4.** If $b_+ - \bar{b}_+$ is odd, then $H^{n-1}_{\mathbb{Z}_2 \times S^1}(S^V, (S^V)_s; \mathbb{Z}_n-1(S^W)) = 0$.

**Proof.** When $b_+ - \bar{b}_+$ is odd, the $\mathbb{Z}_2$-action on $S^W$ reverses the orientation of $S^W$. Therefore the $\mathbb{Z}_2$-action on $\pi_{n-1}(S^W) \cong \mathbb{Z}$ is non-trivial. With this understood, we use Proposition **8.12**. In fact, by (4.6), we have $\ker (\partial_{n-1} + H^1_{\mathbb{Z}_2 \times S^1}(S^V, (S^V)_s; \mathbb{Z}_n-1(S^W))) = 0$ for the $(n-1)$-th cochain group. \qed

**Remark 4.8.** If $b_+ - \bar{b}_+$ is even, then $H^{n-1}_{\mathbb{Z}_2 \times S^1}(S^V, (S^V)_s; \mathbb{Z}_n-1(S^W)) \cong \mathbb{Z}_2$.

4(iii). **Completion of the proof.** So far, under the assumptions of Theorem 1.1, we have the following:

$$
\begin{align*}
H^k(S^V/S^1, \ast; \pi_k(S^W)) &= 0 \text{ for } k \neq n, \\
\{S^V, S^W\}_{S^1} &\cong H^n(S^V/S^1, \ast; \pi_n(S^W))/J, \\
H^k_{\mathbb{Z}_2 \times S^1}(S^V, \ast; \pi_k(S^W)) &= 0 \text{ for } k \neq n, \\
\{S^V, S^W\}_{\mathbb{Z}_2 \times S^1} &\cong H^n_{\mathbb{Z}_2 \times S^1}(S^V, \ast; \pi_n(S^W))/J'.
\end{align*}
$$

Let us compare $H^n(S^V/S^1, \ast; \pi_n(S^W))$ and $H^n_{\mathbb{Z}_2 \times S^1}(S^V, \ast; \pi_n(S^W))$. Note that the following commutative diagram:

\begin{align*}
0 & \longrightarrow H^n(S^V/S^1, \ast; \pi_n(S^W)) & & \longrightarrow & \text{Hom}_\mathbb{Z}(C_n, \mathbb{Z}_2) & & \longrightarrow & \text{Hom}_\mathbb{Z}(C_{n-1}, \mathbb{Z}_2) \\
\phi_* & & \phi & & & & \phi & \\
0 & \longrightarrow H^n_{\mathbb{Z}_2 \times S^1}(S^V, \ast; \pi_n(S^W)) & & \longrightarrow & \text{Hom}_{\mathbb{Z} [\mathbb{Z}_2]}(C_n, \mathbb{Z}_2) & & \longrightarrow & \text{Hom}_{\mathbb{Z} [\mathbb{Z}_2]}(C_{n-1}, \mathbb{Z}_2),
\end{align*}

where $\phi$ is the map forgetting the $\mathbb{Z}_2$-action. Then, a direct calculation shows,

\begin{equation}
\begin{align*}
\text{im } \phi_* &= 0.
\end{align*}
\end{equation}

We have another commutative diagram,

\begin{align*}
H^n(S^V/S^1, \ast; \pi_n(S^W)) & \longrightarrow H^n(S^V/S^1, \ast; \pi_n(S^W))/J \cong \{S^V, S^W\}_{S^1} \\
\phi_* & \longrightarrow \phi_* & & \phi_* & & \phi_*
\end{align*}

Recall that $BF(c) = \bar{\phi}(BF_{\mathbb{Z}_2}(c))$ and $\phi_*$ is the 0-map. Then, the diagram implies $BF(c) = 0$. Thus, Theorem 1.1 is established.
4(iv). **Remarks.** Before ending this section, we give several remarks.

**Remark 4.10.** We can give an alternative proof of the mod $p$ vanishing theorem of ordinary Seiberg-Witten invariants in $\text{11}, \text{25}$ in the case when $b_1 = 0$ by the same method as in this section.

**Remark 4.11.** In the case when $d(c) = 1$, suppose we are given a $Z_p$-action of odd prime $p$ instead of a $Z_2$-action. Then some part of arguments in this section also works, however some part does not. For example, under appropriate assumptions on $Z_p$-index of the Dirac operator, we can prove the vanishing of Bredon cohomology groups in low degrees. However, even if the $(n-1)$-th Bredon cohomology vanishes, $(4.9)$ does not hold. Therefore, we can not expect such a vanishing result of Bauer-Furuta invariants.

**Remark 4.12.** Also, in the cases when $d(c) \geq 2$, it would not be easy to prove such a vanishing result, since $(n-2)$-th ordinary cohomology does not vanish.

**Remark 4.13.** In Theorem $\text{11}$ we supposed that the $Z_2$-action on $X$ lifts to a $Z_2$-action on $c$. Now, let us suppose that the $Z_2$-action on $X$ only preserves the Spin$^c$-structure $c$, i.e., $\iota^* c \cong c$ for the involution $\iota$ generating the $Z_2$-action. Then, $\iota$ also preserves the determinant line bundle $L$. Therefore the $Z_2$-action lifts to $L$ by the theorem of Hattori and Yoshida $\text{14}$. Let $P_{SO}$ be the frame bundle of $X$ and $P_{U(1)}$ the $U(1)$-bundle for $L$. Recall that the Spin$^c$-structure $c$ is given by a double covering $P_{Spin^c(4)} \rightarrow P_{SO} \times_X P_{U(1)}$. Since the $Z_2$-action on $P_{SO} \times_X P_{U(1)}$ is given, two cases may occur on the lifting of the $Z_2$-action to $c$. The first case is that the $Z_2$-action on $X$ lifts to a $Z_2$-action on $c$ as above. We call such a case that the $Z_2$-action is even type with respect to $c$. The second case is that the $Z_2$-action on $X$ does not lift as a $Z_2$-action on $c$, however it is covered by a $Z_4$-action on $c$. We call such a case that the $Z_2$-action is odd type with respect to $c$.

In the odd case, the $Z_4$-index of the Dirac operator is written as $\text{ind}_{Z_4} D_{A_0} = k_1 C_1 + k_3 C_3$, where $C_j$ is the complex 1-dimensional representation of weight $j$. (This is because the $Z_4$-lift of the generator of $Z_2$ acts on spinors as multiplication by $\pm \sqrt{-1}$.)

In the odd case, we can also prove the following result similar to Theorem $\text{11}$. 

**Theorem 4.14.** Suppose the following conditions are satisfied:

1. $b_1 = 0$, $b_+ \geq 2$ and $b_+^2 \geq 1$,
2. $b_+ - b_+^2$ is odd,
3. $d(c) = 2(k_1 + k_3) - (1 + b_+) = 1$,
4. $2k_j < 1 + b_+^2$, for $j = 1, 3$.

Then the Bauer-Furuta invariant of $c$ is zero.

The proof of Theorem $\text{4.14}$ is analogous to that of Theorem $\text{11}$. In the odd case, the finite dimensional approximation $f_W : V \rightarrow W$ is $Z_4 \times S^1$-equivariant, and $V$ and $W$ are direct sums of finite copies of $C_1$, $C_3$, $R_+$. Note that every point in $V$, $W$ have an isotropy which contains a subgroup $Z_2 = \langle (g^2, -1) \rangle$, where $g$ is a generator of $Z_4$. Let $G = Z_4 \times S^1/(\langle g^2, -1 \rangle) \cong Z_2 \times S^1$. Then $f_W$ is $G$-equivariant. Now, an argument similar as in previous subsections proves Theorem $\text{4.14}$.
5. Applications

In this section, we prove Theorem 1.3 as an application of Theorem 1.1. Below, the Euler number of a manifold $Y$ is denoted by $\chi(Y)$, and the signature by $\text{Sign}(Y)$. We assume every manifold is oriented and every $\mathbb{Z}_2$-action is orientation-preserving, unless stated otherwise.

5(i). Constraint on smooth actions on $K_3\#K_3$. Rewriting the condition on the $\mathbb{Z}_2$-index of the Dirac operator in Theorem 1.1 by the $G$-index theorem, we can relate fixed point data to the value of the Bauer-Furuta invariant.

Let $X$ be a simply-connected smooth spin 4-manifold with a $\mathbb{Z}_2$-action. When $X$ is simply-connected, the spin structure on $X$ is unique up to equivalence. Therefore every involution $\iota: X \to X$ preserves the spin structure and also the Spin$^c$-structure $c_0$ which is determined by the spin structure. Then, as mentioned in Remark 4.13, $\iota$ lifts to $c_0$ as a $\mathbb{Z}_2$ (even type) or $\mathbb{Z}_4$ (odd type) action.

By [1], the $\mathbb{Z}_2$-action is even type if and only if the fixed point set is discrete. Now, for an even-type action, the $G$-spin theorem [1] claims that

$$\text{ind}_g D = k_+ - k_- = \frac{1}{4} \sum_{p \in X^{\mathbb{Z}_2}} \epsilon(p),$$

$$\text{ind} D = k_+ + k_- = -\frac{1}{8} \text{Sign}(X),$$

where $\epsilon: X^{\mathbb{Z}_2} \to \{\pm 1\}$ is the sign assignment determined by the lift of the $\mathbb{Z}_2$-action to $c_0$. By solving the above equations, we obtain

$$(5.1)\quad 2k_\pm = -\frac{1}{8} \text{Sign}(X) \pm \frac{1}{4} \sum_{p \in X^{\mathbb{Z}_2}} \epsilon(p).$$

Note that both $k_+$ and $k_-$ are even because of the quaternion structure of Dirac index. Note also that the sum $\sum \epsilon(p)$ is a multiple of 8, because $\frac{1}{8} \text{Sign}(X)$ is even.

If the Bauer-Furuta invariant of $c_0$ is non-trivial, then Theorem 1.1 implies a constraint on the sign assignment $\epsilon$. Furuta, Kametani and Minami proved that a spin manifold which has the same rational cohomology ring as $K_3\#K_3$ has non-trivial Bauer-Furuta invariant for every spin structure [13]. Therefore, we obtain the following.

**Proposition 5.2.** Let $X$ be a smooth closed simply-connected spin manifold which has the same intersection form as $K_3\#K_3$. Suppose an orientation-preserving smooth $\mathbb{Z}_2$-action on $X$ is given, and let $c_0$ be the Spin$^c$-structure which is determined by a spin structure. If $b_+^{\mathbb{Z}_2} = 5$ and fixed points are discrete, then the lift of the $\mathbb{Z}_2$-action to $c_0$ as above satisfies

$$(5.3)\quad \sum_{p} \epsilon(p) = \pm 8.$$

**Proof.** Let $m$ be the number of $X^{\mathbb{Z}_2}$. First, we prove $m = 10$ if $b_+^{\mathbb{Z}_2} = 5$. By the ordinary Lefschetz formula, we have $\chi(X/\mathbb{Z}_2) = \frac{1}{2} (\chi(X) + m)$. On the other hand, the $G$-signature theorem implies that $\text{Sign}(X/\mathbb{Z}_2) = \frac{1}{2} \text{Sign}(X)$. From the equation $1 + b_+^{\mathbb{Z}_2} = \frac{1}{2} (\chi(X/\mathbb{Z}_2) + \text{Sign}(X))$, we get
Sign($X/\mathbb{Z}_2$), we have $m = 10$. Since $\sum \varepsilon(p)$ is a multiple of 8, we have $\sum \varepsilon(p) = -8, 0$ or 8. If $\sum \varepsilon(p) = 0$, then $2k_\pm = 4 < 6 = 1 + b_1^C$. Therefore Theorem 5.1 implies BF($c_0$) = 0. However this contradicts with the non-vanishing result for $K3#K3$ in [13]. □

5(ii). Atiyah-Bott’s criterion for $\varepsilon(p)$. Before going further, let us recall Atiyah-Bott’s criterion for $\varepsilon(p)$. Let $X$ be a spin 4-manifold, and suppose that a smooth involution $\iota: X \to X$ with isolated fixed points is given. When an $\iota$-invariant metric is fixed, $\iota$ lifts to the frame bundle $F$ as $\iota_*: \tilde{F} \to F$. A spin structure on $X$ is given by a double cover $\varphi: \tilde{F} \to F$, where $\tilde{F}$ is a Spin(4) bundle. Suppose that $\iota$ lifts to $\tilde{F}$.

For distinct fixed points $P$ and $Q$, we shall compare $\varepsilon(P)$ and $\varepsilon(Q)$. For this purpose, we take a path $s$ in $F$ starting from a point $y \in F_P$ and ending at $y' \in F_Q$. Then the path $-\iota_*s$ has the same starting point and the end point as $s$, where “$-$” means the multiplication by $-1$ on each fiber. Thus, by connecting $s$ with $-\iota_*s$, we obtain a circle $C$ in $F$.

Then, Atiyah-Bott’s criterion is given as follows.

Proposition 5.4 ([1]). The preimage $\varphi^{-1}(C)$ has two components if and only if $\varepsilon(P) = \varepsilon(Q)$. In other words, the preimage $\varphi^{-1}(C)$ is connected if and only if $\varepsilon(P) = -\varepsilon(Q)$.

5(iii). Edmonds-Ewing’s construction of locally linear $\mathbb{Z}_2$-actions. To construct locally linear $\mathbb{Z}_2$-actions, we invoke the realization theorem by Edmonds and Ewing [10]. We summarize their result in a very special case for our purpose.

Theorem 5.5 ([10]). Suppose that we are given a $\mathbb{Z}_2$-invariant bilinear unimodular even form $\Psi: V \times V \to \mathbb{Z}$ which satisfies the following:

1. As a $\mathbb{Z}[\mathbb{Z}_2]$-module, $V \cong T \oplus F$, where $T$ is a trivial $\mathbb{Z}[\mathbb{Z}_2]$-module with rank $T = n$, and $F$ is a free $\mathbb{Z}[\mathbb{Z}_2]$-module.
2. For any $v \in V$, $\Psi(v, gv) \equiv 0 \mod 2$, where $g$ is the generator of $\mathbb{Z}_2$.
3. The $G$-signature theorem is satisfied, i.e., $\text{Sign}(g, (V, \Psi)) = 0$.

Then, there exists a locally linear $\mathbb{Z}_2$-action on a simply-connected 4-manifold $X$ such that its intersection form is $\Psi$, and the number of fixed points is $n + 2$.

Remark 5.6. Since the form $\Psi$ is assumed even in Theorem 5.5, the homeomorphism type of $X$ is unique by Freedman’s theorem.

For our application, we need to recall their construction precisely. Their construction is an equivariant handle construction.

Let $B_0$ be a unit ball in $\mathbb{C}^2$ on which $\mathbb{Z}_2$ acts by multiplication of $\pm 1$. Let us take a $\mathbb{Z}_2$-invariant knot $K$ in $S_0 = \partial B_0$. Then a framing of $K$ can be represented by an equivariant embedding $S^1 \times D^2 \to S_0$ for some $\mathbb{Z}_2$-action on $S^1 \times D^2$. In particular, 0-framing is represented by $f_0: S^1 \times D^2 \to S_0$ such that $f_0(S^1 \times \{0\}) = K$, and $f_0(S^1 \times \{1\})$ has linking number 0 with $K$, and the $\mathbb{Z}_2$-action on $S^1 \times D^2 \subset \mathbb{C}^2$ is given by $g(z, w) = (-z, -w)$. An arbitrary $r$-framing of $K$ can be represented by a map $f_r: S^1 \times D^2 \to S_0$ given by $f_r(z, w) = f_0(z, z^r w)$. Then $f_r$ is equivariant if the $\mathbb{Z}_2$-action on $S^1 \times D^2$ is given by $g(z, w) = (-z, (-1)^r w)$.

For a given $K$ and a framing $r$, we can construct a 4-manifold with a $\mathbb{Z}_2$-action as $W = B_0 \cup_{f_r} D^2 \times D^2$. 


Let $H_1, \ldots, H_n$ be copies of $D^2 \times D^2$ on which $\mathbb{Z}_2$ acts by $g(z, w) = (-z, -w)$. Note that, if the framing $r$ is even, then we can attach $H_i$ to $B_0$ equivariantly via $f_r$.

Edmonds-Ewing’s construction of locally linear actions is divided into three steps.

**Step 1.** Under the assumption of Theorem 5.5, by changing basis of $T$, we may assume $\Psi|T$ is represented by a matrix $(a_{ij})$ such that $a_{ii}$ is even and $a_{ij}$ is odd whenever $i \neq j$. (See [10], Section 6.) Then we can take a $n$-component link $L_T$ in $S_0 = \partial B_0$ representing the matrix $(a_{ij})$ such that each component of $L_T$ is $\mathbb{Z}_2$-invariant as follows. For every two $\mathbb{Z}_2$-invariant knots $K$ and $K'$ in $S_0$, the linking number $\text{lk}(K, K')$ is odd by [10], Lemma 5.1. Take an arbitrary $n$-component link $L_0$. By moving various components of $L_0$ across various components of $L_0$ equivariantly, we can alter the off-diagonal entries of the linking matrix of $L_0$ by arbitrary multiples of 2. Then we can arrange the link to represent the given matrix $(a_{ij})$. Thus $\Psi|T$ is represented by a framed link $L_T$ in $S_0$.

Now it is not difficult to realize the other part of $\Psi$ by a link, and therefore we obtain a framed link $L$ in $S_0$ which realizes the given $\mathbb{Z}_2$-invariant form $\Psi$.

**Step 2.** Since the diagonal entries of $(a_{ij})$ are assumed even, we can attach $H_1, \ldots, H_n$ and free 2-handles to $B_0$ along $L$ equivariantly. Thus we obtain a 4-manifold $X_0$ with a smooth $\mathbb{Z}_2$-action,

$$X_0 = B_0 \cup H_1 \cup \cdots \cup H_n \cup \text{(free handles)}.$$

**Step 3.** The boundary of $X_0$ is an integral homology 3-sphere $\Sigma$ with a free $\mathbb{Z}_2$-action. Under the assumptions of Theorem 5.5, Edmonds and Ewing proved that there exists a contractible 4-manifold $Z$ with a locally linear $\mathbb{Z}_2$-action such that its boundary is $\Sigma$ with the given free $\mathbb{Z}_2$-action, and it has exact one fixed point. Now $X = X_0 \cup Z$ is the required manifold with the required action.

Note that the action constructed above is smooth except near the final fixed point, that is, smooth on $X_0$. We shall determine the sign assignment $\varepsilon$ on $X_0$. Note that each of $B_0$ and $H_1, \ldots, H_n$ has exact one fixed point. We compare the value of $\varepsilon$ of the fixed point in each $H_i$ with that of $B_0$. It suffices to consider on $W = B_0 \cup H_i$ for each $i$.

Recall that $W$ is given as $W = B_0 \cup f_r, D^2 \times D^2$, for a knot $K$ in $S_0 = \partial B_0$ and a framing $r$. Let $P$ be the fixed point in $B_0$ and $Q$ the fixed point in $D^2 \times D^2$.

**Proposition 5.7.** Suppose $K$ is a trivial knot in $\partial B_0$ which bounds a $\mathbb{Z}_2$-invariant embedded disk $D_0$ in $B_0$ containing $P$. If $r \equiv 2 \text{ mod } 4$, then $\varepsilon(P) = \varepsilon(Q)$. If $r \equiv 0 \text{ mod } 4$, then $\varepsilon(P) = -\varepsilon(Q)$.

**Proof.** Let $\iota: W \to W$ be the generator of the $\mathbb{Z}_2$-action. In $H_i = D^2 \times D^2$, the disk $D_1 := D^2 \times \{0\}$ is $\iota$-invariant and bounded by the knot $K$. Let $N(D_0)$ be a tubular neighborhood of $D_0$. Then $N(D_0)$ can be identified with $D^2 \times D^2$ via a $\mathbb{Z}_2$-equivariant map $\tilde{f}_0: D^2 \times D^2 \to N(D_0)$. The restriction of $\tilde{f}_0$ to $N(D_0) \cap \partial B_0$ gives a 0-framing of $K$.

Let $S$ be the 2-sphere obtained by attaching $D_1$ to $D_0$ along $K$. Then a tubular neighborhood $N(S)$ of $S$ can be identified with the manifold $D_0^2 \times D^2 \cup f_r, D_1^2 \times D^2$, where the attaching map $\tilde{f}_r: \partial D_1 \times D^2 \to \partial D_0 \times D^2$ is given by $\tilde{f}_r(z, w) = (z, z^r w)$. 
The restriction of the frame bundle $F$ of $W$ to $S$ can be identified with the $SO(4)$-bundle obtained by attaching $D_0 \times SO(4)$ with $D_1 \times SO(4)$ via the map $f': K = D_0 \cap D_1 \to U(2) \subset SO(4)$ given by
\[
f'(z) = \begin{pmatrix} z^{-2} & 0 \\ 0 & z^{-r} \end{pmatrix},
\]
i.e., $F|_S = D_0 \times SO(4) \cup_{f'} D_1 \times SO(4)$. In the above matrix, $z^{-2}$ comes from the Euler class of the tangent bundle of $S$, and $z^{-r}$ comes from the $r$-framing. Note that the $\mathbb{Z}_2$-action on each $D_i \times SO(4)$ is given by $\iota_*(z, v) = (-z, -v)$.

Take a path $\bar{s}$ in $S$ starting from $P$ and ending at $Q$. Then $\bar{s}$ and $\iota \bar{s}$ form a circle $\bar{C}$ in $S$, which divides $S$ into two disks: $S = D^+ \cup D^-$. For $i = 0, 1$, let $s_i : \bar{s} \cap D_i \to F|_{D_i} = D_i \times SO(4)$ be the constant section given by $s_i(t) = (t, I)$, where $I$ is the identity element of $SO(4)$. Gluing $s_0$ and $s_1$, we obtain the lift $s$ of $\bar{s}$ to $F$. Let $C = s \cup -\iota^* s$.

A spin structure on $F|_{D^+}$ can be given by a trivialization of $F|_{D^+}$. We shall concretely give such a trivialization as follows. Let $D^+_i = D_i \cap D^+$ and $K^+ = K \cap D^+$. On $D^+_i$, assume the identity map as the trivialization $\phi_0 = \text{id} : D^+_0 \times SO(4) \rightarrow F|_{D^+_0} = D^+_0 \times SO(4)$. Then, the trivialization $\phi_0$ on $D^+_0$ determines a trivialization on $K^+$ as $\phi' : K^+ \times SO(4) \rightarrow F|_{K^+} = K^+ \times SO(4)$. If we view $\phi'$ in $D^+_1$, this is given by $\phi'(z, v) = (z, f'(z)v)$. Then extend this trivialization $\phi'$ over $D^+_1$ as $\phi_1 : D^+_1 \times SO(4) \rightarrow F|_{D^+_1} = D^+_1 \times SO(4)$. Gluing $\phi_0$ and $\phi_1$ along $K^+$, we obtain a trivialization $\phi : D^+ \times SO(4) \rightarrow F|_{D^+} = D^+_0 \times SO(4) \cup_{f'} D^+_1 \times SO(4)$.

Note that the trivialization $\phi$ above determines a double covering $\varphi : \tilde{F} \rightarrow F$ on $D^+$ which gives the spin structure. Now we will apply Proposition 5(iv). Let $p : D^+ \times SO(4) \rightarrow SO(4)$ be the projection, and $C'$ be the circle in $SO(4)$ given by $C' = p \circ \phi^{-1}(C)$. Then $\varphi^{-1}(C')$ is connected if and only if $C'$ represents the generator of $\pi_1(SO(4)) \cong \mathbb{Z}_2$. Furthermore, by the construction above, we can see that $C'$ represents the generator of $\pi_1(SO(4))$ if and only if $(r + 2)/2$ is odd. 

Thus, if each component of the link $L_T$ bounds an $\mathbb{Z}_2$-invariant embedded disk, the sign assignment $\varepsilon$ of $X_0$ is determined by diagonal entries $a_{ii}$ mod 4 in the attaching matrix $(a_{ij})$.

5(iv). **Nonsmoothable locally linear $\mathbb{Z}_2$-action on $K3\#K3$**. Now, we construct a nonsmoothable locally linear $\mathbb{Z}_2$-action on $X = K3\#K3$. Note that the intersection form of $X$ is isomorphic to $4E_8 \oplus 6H$, where $H$ is the hyperbolic form. By Theorem 5.3, if we give an appropriate $\mathbb{Z}_2$-action on the intersection form, then we have a locally linear $\mathbb{Z}_2$-action on $X$.

Let $\mathbb{Z}_2$ act on $\Psi = 4E_8 \oplus 6H$ as follows. On a $2H$ summand, let $\mathbb{Z}_2$ act by permutation of two $H$’s. Similarly, on $2E_8 \oplus 2E_8$ summand, let $\mathbb{Z}_2$ act by permutation of two $2E_8$’s, and on the rest $4H$ trivially. The trivial part is denoted by $T$. 

Let us consider the matrix, 

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
\end{pmatrix}.
\] 

(5.8)

Since the symmetric form represented by \( A \) is even, indefinite and unimodular, it is isomorphic to \( 4H \). Therefore, we may assume \( \Psi|_T \) is represented by the matrix \( A \). Furthermore, the matrix \( A \) can be realized by a link whose every component bounds a \( \mathbb{Z}_2 \)-invariant embedded disk as follows. Let \( p: S^3 \to S^2 \) be the Hopf fibration. Take distinct 8 points in \( S^2 \). Then the inverse image of 8 points by \( p \) forms a required link.

As in \( \S 5(iii) \) by equivariant handle construction, we can construct a smooth action on a manifold \( X_0 \) with a boundary, and Theorem 5.5 says that this action is extended to the whole \( X \) as a locally linear action.

If the smooth action on \( X_0 \) is extended smoothly on the whole \( X \), then we have \( \sum \varepsilon(P) = \pm 8 \) by Proposition 5.2. However, this is a contradiction because \( \sum \varepsilon(P) \) can not be \( \pm 8 \) by Proposition 5.7 for the action constructed on the matrix \( A \). Thus, the smooth action on \( X_0 \) can not be extended to the whole \( X \) smoothly.

Now, we claim more strongly that the locally linear action constructed above is nonsmoothable. First, we shall clarify what nonsmoothable means. Let \( X \) be an oriented topological manifold and \( G \) a finite group. If \( X \) admits a smooth structure and a smooth structure \( \sigma \) is specified, then we write the manifold with the smooth structure \( \sigma \) by \( X_\sigma \). Let \( LL(G, X) \) be the set of equivalence classes of orientation-preserving locally linear \( G \)-actions on \( X \). Here, two locally linear actions are said equivalent if there exists a homeomorphism \( f \) of \( X \) such that one action is conjugate to the other by \( f \). Similarly, let \( C^\infty(G, X_\sigma) \) be the set of equivalence classes of orientation-preserving smooth \( G \)-actions on \( X_\sigma \). Here, the equivalence of two smooth actions is given via a diffeomorphism of \( X_\sigma \). Then we have a forgetful map \( \Phi_\sigma: C^\infty(G, X_\sigma) \to LL(G, X) \) forgetting the smooth structure.

**Definition 5.9.** A locally linear action is called nonsmoothable with respect to the smooth structure \( \sigma \) if its class is not contained in the image of \( \Phi_\sigma \).

Let \( X_\sigma \) be a closed smooth 4-manifold and \( G = \mathbb{Z}_2 \). Since a spin structure on \( X_\sigma \) is defined from the tangent bundle, the definition implicitly uses the smooth structure on \( X_\sigma \). Hence, at present, the sign assignment \( \varepsilon \) seems to depend on classes of \( C^\infty(G, X_\sigma) \).

We claim that the sign assignment can be defined for locally linear actions, and it depends only on classes in \( LL(G, X) \). As in [9], topological spin structures on topological manifolds can be defined as follows. An oriented topological \( n \)-manifold \( M \) has the tangent micro bundle \( \tau M \). By Kister-Mazur’s theorem [17], \( \tau M \) can be identified with a \( \mathbb{R}^n \)-bundle
\(\tau'M\) whose structural group is the topological group \(\text{STop}(n)\) which consists of orientation-preserving homeomorphisms of \(\mathbb{R}^n\) preserving the origin. It is known that, when \(n \geq 2\), the inclusion \(\text{SO}(n) \to \text{STop}(n)\) induces isomorphisms of \(\pi_0, \pi_1\) and \(\pi_2\). (See [16] and [12].) Hence there is the unique double covering \(\varphi_0: \text{SpinTop}(n) \to \text{STop}(n)\). Let \(F\) be the principal \(\text{STop}(n)\)-bundle of “frames” of \(\tau'M\). Then, a topological spin structure is given by a double covering \(\varphi: \hat{F} \to F\) whose restriction to each fibre is \(\varphi_0\).

Suppose that we are given a locally linear \(\mathbb{Z}_2\)-action on a simply-connected topological 4-manifold \(X\) with isolated fixed points. Then the tangent micro bundle \(\tau X\) is \(\mathbb{Z}_2\)-locally linear in the sense of [18], and the corresponding \(\mathbb{R}^n\)-bundle \(\tau'X\) becomes a \(\mathbb{Z}_2\)-equivariant bundle. Let \(F\) be the principal \(\text{STop}(4)\)-bundle of frames of \(\tau'X\). Then, a \(\mathbb{Z}_2\)-action on \(F\) is induced from that on \(\tau'X\). By considering as in [1] and [9], we see that the \(\mathbb{Z}_2\)-action is even type if and only if fixed points are isolated.

Now, we can define the sign assignment by using Atiyah-Bott’s criterion itself on the topological spin structure \(\varphi: \hat{F} \to F\) as follows. Since the action is assumed locally linear, on the fiber of \(F\) over each fixed point \(P\), there is a point \(y_P\) which is mapped to \(-y_P\) by the \(\mathbb{Z}_2\)-action, where \(-1\) means the multiplication of \(-1\) which is considered as an element of \(\text{SO}(4) \subset \text{STop}(4)\). For distinct fixed points \(P\) and \(Q\), by taking a path \(s\) connecting such a \(y_P\) with such a \(y_Q\), we can define the sign assignment as in Proposition 5.4:

**Definition 5.10.** For each pair \((P, Q)\) of fixed points, let \(s\) be a path in \(F\) as above, and \(C\) the circle formed by \(s\) and \(-\iota_*s\). Define \(\varepsilon'(P, Q)\) by

\[
\varepsilon'(P, Q) = 1, \quad \text{if } \varphi^{-1}(C) \text{ has 2 components},
\]

\[
\varepsilon'(P, Q) = -1, \quad \text{if } \varphi^{-1}(C) \text{ is connected}.
\]

This definition does not depend on smooth structures, and is well-defined if \(X\) is simply-connected. Furthermore, if the action is realized by a smooth action,

\[
\varepsilon'(P, Q) = \varepsilon(P)\varepsilon(Q).
\]

Now, we complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We have a locally linear action constructed from the matrix \(A\) in (5.8). If this action is smoothable for a smooth structure, then \(\sum \varepsilon(p)\) must be \(\pm 8\) by Proposition 5.2. However, this is impossible by Proposition 5.7 and the relation (5.11). \(\square\)

5(v). **Nonsmoothable locally linear \(\mathbb{Z}_2\)-action on \(K3\).** The argument in previous subsections enables us to prove Theorem 1.4. The proof is parallel to that of Theorem 1.3.

Since both \(k_+\) and \(k_-\) are even and \(\text{Sign}(K3) = -16\), (5.1) implies the following constraint.

**Proposition 5.12.** Let \(X\) be a simply-connected spin manifold which has the same intersection form as \(K3\). Suppose that an orientation-preserving smooth \(\mathbb{Z}_2\)-action on \(X\) is given, and let \(c_0\) be the \(\text{Spin}^c\)-structure which is determined by a spin structure. If \(b_1^{\mathbb{Z}_2} = 3\) and fixed points are discrete, then the lift of \(\mathbb{Z}_2\)-action to \(c_0\) satisfies,

\[
\sum_p \varepsilon(p) = \pm 8.
\]
Now, we shall construct a nonsmoothable locally linear action on $K3$. The intersection form of $X = K3$ is isomorphic to $2E_8 \oplus 3H$. Let $Z_2$ act on the $2E_8$ summand by permutation, and on the $3H$ summand trivially. Let us consider the matrix,

$$B = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 2
\end{pmatrix}.$$

Since the symmetric form represented by $B$ is even, indefinite and unimodular, $B$ is isomorphic to $3H$. By using the Hopf fibration, we can find a link representing the matrix $B$ whose every component bounds a $Z_2$-invariant embedded disk. As in §5(iii) by equivariant handle construction, we obtain a locally linear $Z_2$-action associated to $B$. If this action is smoothable, the lift of the action to the spin structure should satisfy $\sum \varepsilon(p) = \pm 8$ by Proposition 5.12. However this is impossible by Proposition 5.7 and the relation (5.11). Thus, Theorem 1.4 is established.

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