QUASI FINITE LOOP SPACES ARE MANIFOLDS

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Abstract. It is an old conjecture, that finite $H$-spaces are homotopy equivalent to manifolds. Here we prove that this conjecture is true for loop spaces. Actually, we show that every quasi finite loop space is equivalent to a stably parallelizable manifold. The proof is conceptual and relies on the theory of $p$-compact groups. On the way we also give a complete classification of all simple $2$-compact groups of rank $2$.

1. Introduction

It is an old question in the theory of $H$-spaces, whether finite $H$-spaces are equivalent to differentiable manifolds. The first major result in this direction is due to Browder who showed in a series of papers that every simply connected finite $H$-space is homotopy equivalent to a closed topological manifold and, if the dimension is not congruent to $2$ mod $4$, then this manifold can be taken to be smooth and stably parallelizable.

The first examples of finite $H$-spaces which are not compact Lie groups, were constructed using Zabrodsky’s method of mixing homotopy types [11] [29]. Pedersen analyzed Zabrodsky’s method in detail and showed that, in particular, $H$-spaces in the genus of a compact Lie group are homotopy equivalent to stably parallelizable smooth manifolds [23] [24].

In [5], Capell and Weinberger got further results of this type for finite $H$-spaces. They put some extra conditions on the fundamental group; e.g that the fundamental group is a finite $p$-group ($p$ odd) or infinite with at most $2$-torsion. Under these assumptions they were able to show that such finite $H$-spaces are equivalent to topological manifolds.

In this paper we will concentrate on finite loop spaces in general, and show that all finite loop spaces are homotopy equivalent to stably parallelizable smooth manifolds.

A loop space is a triple $(L, BL, e)$ where $L$ and $BL$ are topological spaces, with $BL$ pointed, and where $e : \Omega BL \to L$ is a homotopy equivalence. By abuse of notation we denote this loop space also by $L$. Then $L$ is an $H$-space with classifying space $BL$. Properties of loop spaces are inherited from $L$; e.g $L$ is called finite if the space $L$ is homotopy equivalent to a finite $CW$-complex, quasi finite if $H^*(L; \mathbb{Z})$ vanishes in large degrees and is a finitely generated abelian group in each degree, and simply connected if the space $L$ is so. Since all components of an $H$-space are homotopy equivalent, we can restrict ourselves to connected loop spaces.

Theorem 1.1. For any connected quasi finite loop space $L$, there exists a stably parallelizable, smooth, finite dimensional closed manifold $M$ such that $L$ and $M$ are homotopy equivalent.

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This theorem also says that every quasi finite loop space is actually finite. This was already proved in [21] by methods similar to those also used in this paper.

The proof of the theorem is based on ideas and techniques developed by Pedersen in the above mentioned work [23] [24]. He constructed a special 1-torus for the $H$-spaces $X$ he considered. A special 1-torus is a fibration of the form $S^1 \to X \to Y$ with special extra properties (see the next section). In particular, the map $S^1 \to X$ factors through the inclusion $S^1 \subset S^3$. Using the fact that $Y$ is a quasi finite, stably reducible, nilpotent Poincaré complex he could prove that $X$ is homotopy equivalent to a stably parallelizable manifold.

The advantage of working with finite loop spaces comes from the fact that after $p$-adic completion we get $p$-compact groups. Hence we can treat $L^\wedge_p$ almost like a compact Lie group. This will allow us to construct particular subgroups of $L^\wedge_p$, which are used to construct special 1-tori for $L$.

The theory of $p$-compact groups will also provide enough information to show that $L/S^1$ is a quasi finite, stably reducible, nilpotent Poincaré complex, which is the other main ingredient to make Pedersen proof applicable. Our proof relies on work of Bauer [2] who used ideas of Klein [13] to construct an analogue of the one point compactification of the adjoint representation of a compact Lie group for $p$-compact groups (see Section 3).

Unfortunately, special 1-tori do not exist for all quasi finite loop spaces; e.g. they do not exist for products of $SO(3)$’s. The rational cohomology $H^*(L; \mathbb{Q})$ of a quasi finite loop space is an exterior algebra generated by odd dimensional classes. We say that $L$ is small if $H^*(L; \mathbb{Q})$ is generated by classes of degree less than or equal to 3. Otherwise, we call $L$ large. We have to treat small and large quasi finite loop spaces differently. For large loop spaces we follow the ideas of Pedersen and construct special 1-tori. For small quasi finite loop spaces we have a complete classification.

**Theorem 1.2.** Let $L$ be a small, connected, $\mathbb{Z}$-finite loop space. Then there exists a compact Lie group $G$ isomorphic to a product of $S^3$’s, a central elementary abelian subgroup $E \subset G$, and a torus $T$ such that $G/E \times T$ and $L$ are homotopy equivalent.

**Remark 1.3.** Since any finite loop space $L$ (actually any finite $H$-space) is equivalent to a product $L' \times T$ where $T$ is a torus and $L'$ is a finite loop space (respectively, a finite $H$-space) with finite fundamental group we may assume for both theorems that $\pi_1(L)$ is finite. And this we will do in all what follows. We call a quasi finite loop space semi simple if it is connected and if $\pi_1(L)$ is finite. Hence we only need to prove the above theorems for semi simple quasi-finite loop spaces.

The paper is organized as follows. In the next section we prove Theorem 1.2. Section 3 is devoted to a discussion of stable reducibility and completions. In Section 4 we discuss the notion of special 1-tori and reduce the proof of our main theorem for large finite loop spaces to the existence of local special 1-tori. In Section 5 we classify simple 2-compact groups of rank 2. This is needed to construct completed special 1-tori for $p$-compact groups. This is worked out in Section 6. In Section 7 we use an arithmetic square argument to construct special 1-tori for finite loop spaces.

We had developed our own argument for proving stable reducibility of $L/S^1$ but unfortunately we couldn’t apply it in our situation (see Section 8). We explain our argument in the last section, since we think that it is interesting in it’s own right and since it shows the power of Klein’s homotopy theoretic version of the adjoint representation of a compact Lie group [13]. The techniques are similar to those used...
in [2], but much simpler and very much motivated by a geometric argument for the stable reducibility of homogeneous spaces.

Several proofs of this paper rely on the theory of p-compact groups. For a general reference we refer the reader to the survey articles [15] and [19] and the references given there.

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2. The case of small finite loop spaces

In this section we assume that L is a small, semi simple, connected, quasi finite loop space. We notice that p-adic completion makes L into a p-compact group. That is \( \langle L_\wedge^p, BL_\wedge^p, e_\wedge^p \rangle \) is a connected p-compact group.

Theorem 1.2 is part of the following statement.

Theorem 2.1. Let L be a connected quasi finite loop space. Then the following conditions are equivalent:

(i) The exterior algebra \( H^*(L; \mathbb{Q}) \) is generated by 3-dimensional classes; i.e. L is small.

(ii) The polynomial algebra \( H^*(BL; \mathbb{Q}) \) is generated by 4-dimensional classes.

(iii) There exists a compact connected Lie group G isomorphic to a product of \( S^3 \)'s and a central elementary abelian subgroup \( E \subset G \) such that \( L \cong G/E \) (as spaces).

Proof. In the case of a p-compact group, Theorems of this type are proved in [9] (Theorems 0.5A and 0.5B). We will be using those results.

The equivalence of (i) and (ii) follows easily from an Eilenberg-Moore spectral sequence argument.

Let us assume that (ii) holds, and let \( L_\wedge^p \) denote the associated p-compact group. If \( p = 2 \) we can apply Theorem 0.5B of [9]. In our case this says that there exists a connected compact Lie group \( G \) isomorphic to a product of \( S^3 \)'s and a central elementary abelian subgroup \( E \subset G \) such that \( BL_2^\wedge \cong B(G/E)_2^\wedge \). Obviously, the number of \( S^3 \)'s is determined by the number of 4-dimensional generators of \( H^*(BL_\wedge^p; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q} \) or of \( H^*(BL; \mathbb{Q}) \).

Now we consider the case of an odd prime \( p \). The universal cover of \( \tilde{L}_\wedge^p \) of \( L_\wedge^p \) is a simply connected p-compact group [16] and splits into a product of simply connected simple p-compact groups [20] or [8], i.e. \( \tilde{L} \cong \prod_i X_i \). Since \( H^*(B\tilde{L}_\wedge^p; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q} \cong H^*(BL_\wedge^p; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q} \), the number of factors is determined by the number of generators of \( H^*(BL; \mathbb{Q}) \) and, for each factor \( X_i \), the algebra \( H^*(BX_i; \mathbb{Z}_2^\wedge) \otimes \mathbb{Q} \) is a monogenic polynomial algebra generated by a 4-dimensional class. Hence \( W_{X_i} \cong \mathbb{Z}/2 \) and \( W_{X_i}^\wedge \cong \prod_i W_{X_i} \) is an elementary abelian 2-group. Now Theorem 0.5A of [9] tells us that \( BL_\wedge^p \cong BH_\wedge^p \) where \( H \) is a product of \( S^3 \). Since \( H_\wedge^p \) is center free, \( BH_\wedge^p \cong B\tilde{L}_\wedge^p = BL_\wedge^p \).

The number of factors again equal the number of generators of \( H^*(BL; \mathbb{Q}) \).

By construction \( G \cong H \), and since \( B(G/E)_p^\wedge \cong BG_0^\wedge \) for odd primes, we have \( BL_\wedge^p \cong B(G/E)_0^\wedge \) for all primes. Since \( BL \) and \( BG/E \) are both rationally products of Eilenberg-MacLane spaces and since they have the same rational cohomology, we also know that the rationalizations \( BL_0 \) and \( B(G/E)_0 \) are equivalent. That is to say
that $BL$ is in the adic genus of $BG/E$ and hence $L$ in the adic genus of $G/E$. By [21] (Theorem 6.1), the adic genus of $G/E$ is rigid, that is every space in the adic genus of $G/E$ is actually equivalent to $G/E$. This implies condition (iii).

On the other hand if $L \simeq G/E$, then the exterior algebra $H^*(L; \mathbb{Q})$ is generated by 3-dimensional classes. This finishes the proof.

\[ \square \]

3. Stable reducibility and completions

For a quasi-finite space $K$ we define the homological dimension $\text{hd}_\mathbb{Z}(K)$ of $K$ by the degree of the largest non vanishing integral homology group. For a Poincaré complex this equal the formal dimension of $K$ which is given by the degree of the fundamental class. We call a quasi-finite Poincaré complex $K$ of homological dimension $\text{hd}_\mathbb{Z}(X) = n$ stably reducible, if, for some $r \in \mathbb{N}$ there exists a map $S^{n+r} \to S^r \wedge K = \Sigma^r K$ such that $H_{n+r}(S^{n+r}; \mathbb{Z}) \to H_{n+r}(\Sigma^r K; \mathbb{Z})$ is an isomorphism or, equivalently, if the Hurewicz map $h : \pi_{n+r}(\Sigma^r K) \to H_{n+r}(K; \mathbb{Z})$ is an epimorphism. These conditions are also equivalent to the fact that the Spivak normal bundle is stably trivial and to the fact that the top cell splits off stably.

Using the techniques of completions we will break the question of stable reducibility down to local ones. First we have to recall some notions.

Let $R$ be a commutative ring with unit. A space $K$ is called $R$-finite if $H_*(K; R)$ vanishes in large degrees and is a finitely generated $R$-module in each degree. In particular, quasi finiteness is nothing but $\mathbb{Z}$-finiteness. For such spaces the $R$-homological dimension of $K$, denoted by $\text{hd}_R(K)$, is given by the degree of the largest non vanishing homology group (with coefficients in the ring $R$). A $R$-finite space $K$ with $\text{hd}_R(K) = n$ is a Poincaré complex if $H^*(K; R)$ and $H_*(K; R)$ satisfy the usual Poincaré duality properties with respect to a fundamental class $[K]_R \in H_n(K; R)$.

We call a $R$-finite Poincaré complex $K$ with $\text{hd}_R(K) = n$ $R$-stably reducible if, for some $r \in \mathbb{N}$, there exists a map $S^{n+r} \to \Sigma^r K$ such that the induced map $H_{n+r}(S^{n+r}; R) \to H_{n+r}(\Sigma^r K; R)$ is an isomorphism or, equivalently, if the Hurewicz map $\pi_{n+r}(\Sigma^r K) \otimes R \to H_{n+r}(\Sigma^r K; R)$ is an epimorphism.

For a space $K$ we denote the $p$-adic completion by $K^\wedge_p$. If $K$ is nilpotent or $p$-good completion induces an isomorphism in mod-$p$ homology and cohomology. Hence, for such spaces $K$ and $K^\wedge_p$ have the same mod-$p$ properties.

**Lemma 3.1.** Let $K$ be a $\mathbb{Z}$-finite, nilpotent, Poincaré complex of homological dimension $n$. Then the following are equivalent:

(i) $K$ is stably reducible.
(ii) For all primes $p$, $K$ is mod-$p$ stably reducible.
(iii) For all primes $p$, the completion $K^\wedge_p$ is mod-$p$ stably reducible.

**Proof.** Since $K$ is a Poincaré complex of dimension $n$, we know that $H_{n+r}(\Sigma^r K; \mathbb{F}_p) \cong H_{n+r}(\Sigma^r K; \mathbb{Z}) \otimes \mathbb{F}_p \cong \mathbb{F}_p$. We consider the exact sequence

\[ \pi_{n+r}(\Sigma^r K) \to H_{n+r}(\Sigma^r K; \mathbb{Z}) \cong \mathbb{Z} \to Q \to 1 \]

where $Q$ is the cokernel of the Hurewicz map. We choose $r$ big enough so that we are in the stable range and $Q$ does not depend on $r$. Then $Q = 0$ if and only if $Q \otimes \mathbb{F}_p = 0$. 


for all primes. That is \( K \) is stably reducible if and only if \( K \) is mod-\( p \) stably reducible for all \( p \).

The equivalence between (ii) and (iii) follows from the fact that \( \pi_\ast(\Sigma^r K) \otimes \mathbb{F}_p \cong \pi_\ast(\Sigma^r K^p) \otimes \mathbb{F}_p \). □

We record another easy lemma.

**Lemma 3.2.** Let \( K \) be a \( \mathbb{Z}_{(p)} \)-finite, nilpotent, Poincaré complex of \( \mathbb{Z}_{(p)} \)-homological dimension \( n \). Then the following are equivalent:

(i) \( K \) is \( \mathbb{Z}_{(p)} \) stably reducible.

(ii) \( K \) is mod-\( p \) stably reducible.

(iii) \( K^p \) is mod-\( p \) stably reducible.

**Proof.** The equivalence of (ii) and (iii) is exactly as in the above lemma. Hence one only needs to establish the equivalence of (i) and (ii). This follows rather trivially since the map \( \pi_{n+r}(\Sigma^r K) \otimes \mathbb{Z}_{(p)} \rightarrow H_{n+r}(\Sigma^r K; \mathbb{Z}_{(p)}) \) is an epimorphism if and only if the map \( \pi_{n+r}(\Sigma^r K) \otimes \mathbb{F}_p \rightarrow H_{n+r}(\Sigma^r K; \mathbb{F}_p) \) is an epimorphism. □

In [2] (Theorem 1.3), Bauer showed that for a p-compact group \( X \) there exists a p-completed sphere spectrum \( S_X \) with a stable \( X \)-action whose dimension equals the \( \mathbb{F}_p \)-homological dimension \( \text{hd}_{\mathbb{F}_p}(X) \). Here we have to work in the category of simplicial spaces and have to replace the loop space \( X \) by the associated simplicial group. Moreover, if \( X \) is the completion of a compact Lie group, we can take for \( S_X \) the one point compactification of the Lie algebra of \( G \) with the adjoint action. In particular, if \( G \) is abelian, \( S_G \) has the trivial action. He also showed that, for a p-compact subgroup \( Y \subset X \) there exists a map of spectra \( S_X \rightarrow X \wedge Y S_Y \) which induces an isomorphism in \( H_n(\_ ; \mathbb{F}_p) \). Here, we take the mod-\( p \) cohomology of spectra. This result enables us to prove the following proposition.

**Proposition 3.3.** Let \( X \) be a p-compact group and let \( T \rightarrow X \) be a p-compact subtorus. Then, the \( \mathbb{F}_p \)-finite homogeneous space \( X/T \) is mod-\( p \) stably reducible.

**Proof.** Since \( T \) is the completion of an abelian compact Lie group, the sphere spectra \( S_T \) carries the trivial \( T \) action. Therefore, we get a map \( S_X \rightarrow X_+ \wedge T S_T \cong X/T_+ \wedge S_T \) which induces an isomorphism in mod-\( p \) homology in the right degree. In particular, this tells us that the top cell of \( X/T \) splits off stably and that \( X/T \) is mod-\( p \) stably reducible. □

4. **Special 1-tori and the proof of Theorem 1.1**

In [24] Pedersen introduced the concept of special 1-tori for spaces, which is his main concept to get control of the surgery obstructions (see [24] Proposition 2.1). We will recall his notion. Actually, we only need the p-local version. A fibration \( F \rightarrow E \rightarrow B \) is called orientable if \( \pi_1(B) \) acts trivially on the set \([F, F] \) of homotopy classes of self equivalences of \( F \).
**Definition 4.1.** A nilpotent space $K$ admits a $p$-local special 1-torus if, up to homotopy, there exists a diagram of orientable fibrations

\[
\begin{array}{c}
S_1^{(p)} \rightarrow S_3^{(p)} \rightarrow S_2^{(p)} \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
S_1^{(p)} \rightarrow K \rightarrow B \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
* \rightarrow A \rightarrow A \\
\end{array}
\]

such that

(i) $A$ is $\mathbb{Z}_{(p)}$-finite.

(ii) $B$ is $\mathbb{Z}_{(p)}$-finite and $\mathbb{Z}_{(p)}$-stably reducible.

(iii) Localized at 0, the diagram is homotopy equivalent to

\[
\begin{array}{c}
S_1^{0} \rightarrow S_3^{0} \rightarrow S_2^{0} \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
S_1^{0} \rightarrow A_0 \times S_3^{0} \rightarrow A_0 \times S_2^{0} \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
* \rightarrow A_0 \rightarrow A_0 \\
\end{array}
\]

where all vertical fibrations are trivial.

Using the notion of special 1-tori Pedersen could prove the following result (see [24] Theorem 1.4).

**Theorem 4.2.** (Pedersen) Let $X$ be a $\mathbb{Z}$-finite $H$-space. If for every prime $p$ the localization $X_{(p)}$ admits a special 1-torus, then $X$ is homotopy equivalent to a smooth stably parallelizable manifold.

Obviously, there also exists a notion of a global special 1-torus. For the proof of the above theorem, Pedersen first showed that, under the above assumption, $X$ has a global special 1-torus. Then he used this extra structure to prove that the finiteness obstruction for $X$ vanishes. Since $H$-spaces are stably reducible [4], their Spivak normal bundle is stably trivial. The existence of a special 1-torus then also implies the vanishing of the surgery obstruction for the existence of a homotopy equivalence to a stably parallelizable manifold (see [24] 2.1 and 4.2).

To prove Theorem 1.1, it is therefore only left to show that any large $\mathbb{Z}$-finite loop space admits $p$-locally a special 1-torus. And this is a consequence of the following proposition.

**Proposition 4.3.** Let $L$ be a large connected $\mathbb{Z}$-finite loop space. Then there exists a loop space $N$ and a fibration $A \rightarrow BS_3^{(p)} \rightarrow BN_{(p)}$ such that $A$ is simple, $\mathbb{Z}_{(p)}$-finite and such that $N$ and $L$ are homotopy equivalent spaces. Moreover, localized at 0, there exists a left inverse $s : BM_0 \rightarrow BS_3^{0}$ of $f$, i.e. $sf_0 = id_{BS_3^{0}}$.

The proof of this proposition will be given in Section 7.

**Corollary 4.4.** For a large $\mathbb{Z}$-finite loop space $L$ and a prime $p$, the localization $L_{(p)}$ admits a $p$-local special 1-torus.
Proof. Let $S^1 \subset S^3$ be the maximal torus of $S^3$. Since the loop space $N$ of the last proposition is equivalent to $L$ we only have to prove the claim for $N$ or equivalently, we may assume that there exist a fibration $BS_3^3(p) \to BL(p)$ with the desired properties.

Passing to classifying spaces and localizations, and taking homotopy fibers we get a commutative diagram of fibration sequences

![Diagram]

Here $B$ is the homotopy fiber of the composition $BS_3^1(p) \to BS_3^3(p) \to BL(p)$. As the homotopy fiber of maps between simply connected spaces, $A$ and $B$ are simple.

The three left columns of the above diagram will establish a p-local special 1-torus for $L(p)$. All rows of this $3 \times 3$-diagram are given by principal fibrations and therefore orientable. The same holds for the two left columns. For the right column we have a pull back diagram

![Diagram]

The bottom row is an orientable fibration. Hence, this also holds for the top row. This shows that the above $3 \times 3$-diagram consists of orientable fibrations.

Since $A$ is $\mathbb{Z}((p))$-finite, a Serre spectral sequence argument shows that the same holds for $B$.

Localized at 0, there exists a left inverse $s : BL_0 \to BS_0^3$. Since $s g_0 = s f i_0 = i_0$, this left inverse establishes rationally compatible left inverses for all vertical arrows between the second and third row of the above large diagram. In particular this shows that, localized at 0, the vertical fibrations of the $3 \times 3$-diagram are trivial and that this diagram satisfies the third condition of special 1-tori.

To complete the proof it remains to show that $B$ is $\mathbb{Z}_{(p)}$-stably reducible. We pass to completions. Then $L_0^\wedge$ becomes a p-compact group. We get a fibration $B_p^\wedge \to BS_0^1 \wedge \to BL_p^\wedge$. Since $B$ was $\mathbb{Z}_{(p)}$-finite and simple, $B$ and $B_p^\wedge$ have isomorphic mod-p homology. This shows that $B_p^\wedge$ is $\mathbb{F}_p$-finite, that $S_0^1 \wedge \to L_0^\wedge$ is a monomorphism of p-compact groups and that $B$ is equivalent to the homogeneous space $L_p^\wedge/S_p^1$. By Proposition 3.3, $B$ is $\mathbb{F}_p$-stably reducible and by Lemma 3.2, $B$ is $\mathbb{Z}_{(p)}$-stably reducible. This completes the proof and shows that $L(p)$ admits a p-local special 1-torus.

Proof of Theorem 1.1: We already discussed the case of small quasi finite loop spaces. Let $L$ be a large $\mathbb{Z}$-finite loop space. By Corollary 4.4 every localization $L(p)$ admits a p-local special 1-torus. By Theorem 4.2 this implies that $L$ is homotopy equivalent to a smooth stably parallelizable manifold.
Remark 4.5. Unfortunately, there exists no global version of Proposition 4.3; i.e. a large quasi finite loop space $N$ might not contain a $S^3$ or a $S^1$ as a subgroup. Hence, in general there exists no sequence of fibrations of the form

$$S^1 \to L \to L/S^1 \to BS^1 \to BN,$$

where $L$ and $N$ are homotopy equivalent. If such a sequence were to exist, the argument of Section 8 would give a proof of the stable reducibility of $L/S^1$.

5. 2-Compact Groups of Rank 2

In this section we will classify all simple 2-compact groups of rank 2. For the convenience of the reader and to fix notation we recall some material about $p$-compact groups.

A $p$-compact group $X$ is a loop space $X = (X, BX, e)$ such that $BX$ is $p$-complete and pointed and such that $X$ is $\mathbb{F}_p$-finite. Every $p$-compact group $X$ has a maximal torus $T_X$, a maximal torus normalizer $N_X$, and a Weyl group $W_X$ acting on $T_X$. These loop spaces fit into a diagram

$$\begin{array}{ccc}
BT_X & \to & BN_X \\
\downarrow & & \downarrow \\
& & BW_X \\
& & BX
\end{array}$$

Here, $BT_X \simeq K((\mathbb{Z}_p^n, 2)$ is homotopy equivalent to an Eilenberg-MacLane space of degree 2. The top row is a fibration and determines the action of $W_X$ on $T_X$, actually on $L_X := \pi_1(T_X) \cong (\mathbb{Z}_p^n)$. We call $L_X$ the associated $W_X$-lattice and $n$ the rank of $X$. This action can also be described by a representation $W_X \to Gl(L_X)$. If $X$ is connected, this representation is faithful and makes the finite group $W_X$ into a pseudo reflection group. And if in addition $p = 2$, then $W_X$ is a 2-adic reflection group. We call $X$ simple if $X$ is connected and if the associated representation $W_X \to Gl(L_X \otimes \mathbb{Q})$ is irreducible. For details and further notions we refer the reader to the survey articles [15] and [19] and the references mentioned there.

The following theorem might be known to the experts. But since we couldn’t find a reference for it, we will also include a proof.

**Theorem 5.1.** Any simple 2-compact group $X$ of rank 2 is isomorphic to the 2-adic completion of $SU(3)$, $Spin(5) = Sp(2)$, $SO(5)$ or $G_2$.

The rest of this section is devoted to the proof of this statement. For compact connected Lie groups we will abuse notation and denote by $G$ the associated 2-compact group obtained by 2-completion.

Let $U$ be a finite dimensional vector space over $\mathbb{Q}_2$ with an action of a finite group $W$ defined by a homomorphism $W \to Gl(U)$. A $W$-lattice $L$ of $U$ is a $\mathbb{Z}_2^n$-lattice $L \subset U$ of maximal rank fixed under the action of $W$; i.e. $L$ is a $\mathbb{Z}_2^n[W]$-module and $L \otimes \mathbb{Q} \cong U$.

We say that two $W$-lattices $L$ and $L'$ of $U$ are isomorphic if $L \cong L'$ as $\mathbb{Z}_2^n[W]$-modules. A $W_1$-lattice $L_1$ and a $W_2$-lattice $L_2$ are called weakly isomorphic if there exists an isomorphism $W_1 \cong W_2$ such that $L_1$ and $L_2$ are isomorphic as $W_1$ lattices.

We say that two $p$-compact groups $X$ and $Y$ have the same Weyl group data if the representations $W_X \to Gl(L_X)$ and $W_Y \to Gl(L_Y)$ are weakly isomorphic. Renaming
the elements of $W_Y$ we always can identify $W_Y$ with $W_X$ and assume that the two lattices are actually isomorphic.

From the Clark-Ewing list [6] we get a complete list of all irreducible reflection groups of rank 2 defined over $\mathbb{Q}^\wedge$. These are given by the dihedral groups $D_6$, $D_8$ and $D_{12}$ with their standard representation as reflection groups. In fact, these are the only dihedral groups which can be represented as reflection groups over $\mathbb{Q}^\wedge$. The first is the rational Weyl group representation of $SU(3)$, the second of $Spin(5)$ or $SO(5)$ and the last of the exceptional Lie group $G_2$. The classification of Clark and Ewing only works up to weak equivalence.

The Lie groups $Spin(5)$ and $Sp(2)$ are isomorphic. Hence, the Weyl groups $W_{Spin(5)}$ and $W_{Sp(2)}$ are also isomorphic and the associated lattices $L_{Spin(5)}$ and $L_{Sp(2)}$ weakly isomorphic. In the following, we will always use the one of these two which seems to be more appropriate.

The universal cover $\tilde{X}$ of a $p$-compact group $X$ is again a $p$-compact group and, if $\pi_1(X)$ is finite, $X$ and $\tilde{X}$ have the same rational Weyl group data and $\tilde{X} \cong X/\mathbb{Z}$ where $\mathbb{Z} \subset \tilde{X}$ is a central subgroup [16]. Simple $p$-compact groups have finite fundamental groups [16]. Therefore, Theorem 5.1 is a consequence of the following classification result for simply connected simple 2-compact groups.

**Theorem 5.2.** Let $G = SU(3)$, $Sp(2)$ or $G_2$. A simply connected 2-compact group $X$ has the same rational Weyl group data as $G$ if and only if $X$ and $G$ are isomorphic as 2-compact groups.

For the proof of this theorem we first have to classify all 2-adic lattices of the representation $W_G \longrightarrow GL(L_G \otimes \mathbb{Q})$.

**Lemma 5.3.** Let $U := (\mathbb{Q}^\wedge)^2$ and $W \longrightarrow GL(U)$ be a reflection group.

(i) If $W = D_6$, $D_{12}$ then, up to isomorphism, there exists exactly one $W$-lattice of the representation $W \longrightarrow GL(U)$.

(ii) If $W = D_8$ each $W$-lattice of $U$ is isomorphic either to $L_{SO(5)}$ or to $L_{Spin(5)}$. And both lattices are weakly isomorphic.

**Proof.** For $D_6$ and $D_{12}$ this follows from [1] (Proposition 4.3 and Theorem 6.2).

Now let $W = D_8$. In this case we have two non isomorphic lattices $L_{SO(5)}$ and $L_{Spin(5)}$. Let $L$ be another $W$-lattice of $U$. For a large $r$, the lattice $2^r L$, the submodule of all elements divisible by $2^r$, is a submodule of $L_{SO(5)} \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$. We choose $r$ minimal with this property, i.e. $2^r L \subset L_{SO(5)}$ but $2^r L \not\subset 2L_{SO(5)}$. Since $L \otimes \mathbb{Q} \cong L_{SO(5)} \otimes \mathbb{Q}$, we get a short exact sequence of $\mathbb{Z}_2[W]$-modules

$$0 \longrightarrow 2^r L \longrightarrow L_{SO(5)} \stackrel{\rho}{\longrightarrow} Q \longrightarrow 0.$$ 

The minimality of $r$ implies that $Q$ is a finite cyclic group; i.e. $Q \cong \mathbb{Z}/2^s$ generated by either $\rho((1,0))$ or $\rho((0,1))$. The dihedral group $D_8$ is generated by the three elements $\sigma_1, \sigma_2, \tau$, where $\sigma_i$ multiplies the $i$-th coordinate by $-1$ and $\tau$ exchanges the two coordinates. Since the automorphism group of $Q$ is abelian, the action of $W$ on $Q$ factors through the abelianization of $W$, $ab(W)$. It follows that the element $\sigma_1 \sigma_2 = \sigma_1 \tau \sigma_1 \tau$ acts trivially on $Q$. Hence the elements $(1,0), (0,1) \in M$ are mapped onto elements of order 2 in $Q$. Thus, either $Q = 0$ or $Q = \mathbb{Z}/2$. In the first case, we have $L \cong M = L_{SO(5)}$. In the second case, $D_8$ acts trivially on $Q$ with $\rho((1,0)) = \rho((0,1)) \neq 0$ in $\mathbb{Z}/2$ and consequently $L \cong L_{Spin(5)}$. This proves the first part of (ii).
The second part follows from the facts that $L_{Sp(2)}$ and $L_{Spin(5)}$ are weakly isomorphic and that $L_{Sp(2)}$ and $L_{SO(5)}$ are isomorphic.

**Proof of Theorem 5.2:** If $X$ and $G$ have the same rational Weyl group data, the above lemma shows that they also have the same 2-adic Weyl group data. We can assume that $W := W_G = W_X$ and that $L := L_G = L_X$. We also can identify the maximal tori $T := T_G \cong T_X$.

For $G = SU(3)$ or $G_2$, this implies $X \cong G$. For $SU(3)$ this follows from [17] and for $G_2$ from [27].

For $Spin(5) = Sp(2)$ uniqueness result are only known in terms of the maximal torus normalizer [22] [26]. We have to show that $N_G \cong N_{Sp(2)}$ as loop spaces; i.e. $BN_G \cong BN_{Sp(2)}$.

Since $X$ and $Sp(2)$ have the same rational Weyl group data, they have isomorphic rational cohomology. Hence, $H^*(X; \mathbb{Z}_2) \otimes \mathbb{Q}$ is an exterior algebra with generators in degree 3 and 7. If $H^*(X; \mathbb{Z}_2)$ has 2-torsion, then $X$ and $G_2$ have isomorphic mod-2 cohomology [12]. The Bockstein spectral sequence then shows that $X$ does not have the right rational cohomology. Therefore, $X$ has no 2-torsion, $H^*(X; \mathbb{Z}_2)$ is an exterior algebra with generators in degree 3 and 7 and $H^*(BX; \mathbb{F}_2) \cong \mathbb{F}_2[x_4, x_8]$ is a polynomial algebra generated by a class of degree 4 and one of degree 8. Since $H^*(BX; \mathbb{F}_2)$ is a finitely generated module over $H^*(BT; \mathbb{F}_2)$, the composition

$$H^*(BX; \mathbb{F}_2) \cong H^*(BX; \mathbb{Z}_2) \otimes \mathbb{F}_2 \longrightarrow H^*(BT; \mathbb{Z}_2)^W \otimes \mathbb{F}_2 \cong H^*(BSp(2); \mathbb{F}_2) \longrightarrow H^*(BT; \mathbb{F}_2)$$

is a monomorphism. The isomorphism $H^*(BT; \mathbb{Z}_2)^W \otimes \mathbb{F}_2 \cong H^*(BSp(2); \mathbb{F}_2)$ follows from the fact that $X$ and $Sp(2)$ have the same 2-adic Weyl group data (Lemma 5.3). Since the first and third term are both polynomial algebras of the same type,

$$H^*(BX; \mathbb{F}_2) \longrightarrow H^*(BT; \mathbb{Z}_2)^W \otimes \mathbb{F}_2 \cong H^*(BSp(2); \mathbb{F}_2)$$

is an isomorphism.

Let $t \subset T$ denote the elements of order 2 and $H := S^3 \times S^3 \subset Sp(2)$ the obvious subgroup. We have a chain of inclusion $t \subset T \subset H \subset Sp(2)$ and $H = C_{Sp(2)}(t)$. The action of $D_8$ on $t$ factors through the $\mathbb{Z}/2$-action on $t$ given by switching the coordinates.

Now we use Lannes’ T-functor theory (e.g. see [25]). We get a map $f : Bt \longrightarrow BX$ which looks in mod-2 cohomology like the map $BT \longrightarrow BSp(2)$. This map is $\mathbb{Z}/2$-equivariant up to homotopy. The mod-2 cohomology of the classifying space $BC_X(t) := map(Bt, BX)_{/f}$ of the centralizer $C_X(t)$ can be calculated with the help of Lannes’ T-functor and $H^*(BC_X(t); \mathbb{F}_2) \cong H^*(BC_{Sp(2)}(t); \mathbb{F}_2) \cong H^*(BH; \mathbb{F}_2)$. Moreover, the Weyl group of $C_X(t)$ is given by the elements of $D_8$ acting trivially on $t$. Hence $W_{C_X(t)} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. By [9] (Theorem 0.5B), this implies that $BC_X(t) \simeq BH$. We will identify $C_X(t)$ with $H$. The $\mathbb{Z}/2$-action on $t$ induces a $\mathbb{Z}/2$-action on $H$. Since $Bt \longrightarrow BX$ was $\mathbb{Z}/2$-equivariant up to homotopy, the inclusion $BC_X(t) \longrightarrow BX$ extends to a map $BY := BH_{h\mathbb{Z}/2} \longrightarrow BX$. In this case, the homotopy orbit space $BY$ happens to be a 2-compact group and has the same Weyl group as $X$. That is $N_Y \cong N_X$. Moreover, the space $BY$ fits into a fibration

$$BH \longrightarrow BY \longrightarrow B\mathbb{Z}/2,$$

which is classified by obstructions in $H^*(B\mathbb{Z}/2; \pi_*(SHE(BH)))$. Here, $SHE(BH)$ is the space of self equivalences of $BH$ homotopic to the identity. Since $SHE(BH) \simeq (B\mathbb{Z}/2)^2$ [7] and since $\mathbb{Z}/2$ acts on $\pi_2(B^2(\mathbb{Z}/2)^2) \cong (\mathbb{Z}/2)^2$ by switching the coordinates,
all obstruction groups vanish and the above fibration splits. This shows that \( BY \simeq B(H \times \mathbb{Z}/2) := BH' \) and that \( BN_X = BN_Y \simeq BN_{H'} = BN_{Sp(2)}. \) That is \( X \) and \( Sp(2) = Spin(5) \) have isomorphic maximal torus normalizer and shows that \( X \cong Sp(2). \)

**Remark 5.4.** The only simply connected 2-compact group of rank 1 is \( S^3. \) Hence, we get the following complete list (up to isomorphism) of connected 2-compact groups of rank 2, namely \( S^1 \times S^1, \ S^1 \times S^3, \ U(2), \ S^1 \times SO(3), \ S^3 \times S^3, \ S^3 \times SO(3), \ SO(4), \ SU(3), \ Sp(2), \ SO(5) \) and \( G_2. \)

The following corollary is needed for later purpose. For a compact connected Lie group we denote by \( \overline{X} \) the associated center free quotient.

**Corollary 5.5.** For any simple connected 2-compact group \( X \) of rank 2, there exists a homomorphism \( S^3 \longrightarrow X \) such that the composition \( S^3 \longrightarrow X \longrightarrow \overline{X} \) is a monomorphism.

**Proof.** Because of Theorem 5.1 we only have to check this for the compact connected Lie groups \( SU(3), \ Sp(2), \ SO(5) \) and \( G_2. \) There exists a chain of monomorphisms \( S^3 = SU(2) \subset SU(3) \subset G_2. \) Both groups, \( SU(3) \) and \( G_2, \) are 2-adically center free. This proves the claim in these two cases. Let \( S^3 \subset Sp(2) \) denote the inclusion into the first coordinate. Since the intersection of \( S^3 \) and the center of \( Sp(2) \) is trivial, the composition \( S^3 \subset Sp(2) \longrightarrow SO(5) \) is also a monomorphism. This proves the claim in the other cases. \( \square \)

6. **Particular subgroups of \( p \)-compact groups.**

In this section we will construct particular subgroups of large \( p \)-compact subgroups. A \( p \)-compact group \( X \) is called large, if the exterior algebra \( H^*(X; \mathbb{Z}_p) \otimes \mathbb{Q} \) has a generator of degree \( \geq 5. \) We want to prove the following proposition.

**Proposition 6.1.** Let \( X \) be a large semi simple connected \( p \)-compact group. Let \( r := \dim_{\mathbb{Q}_p} H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \) be the dimension of the \( \mathbb{Q}_p \)-vector space \( H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}. \) Then there exists a compact Lie Group \( G \) and a map \( f : BG^\wedge_p \longrightarrow BX \) such that the following hold:

(i) \( G \cong S^3 \times H \) with \( H \) semi simple and it’s universal cover \( \tilde{H} \) isomorphic to \( (S^3)^{r-1}. \) If \( p \) is odd, we can choose \( G = (S^3)^r. \)

(ii) The induced map \( H^4(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \longrightarrow H^4(BG^\wedge_p; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \) is an isomorphism.

(iii) The homotopy fiber \( X/G^\wedge_p \) of \( f \) is simple and \( \mathbb{F}_p \)-finite.

**Proof.** Comparing the statement with Proposition 3.1 of [21] there is an extra assumption on the generators of \( H^*(X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \) and the corresponding additional output is that \( G \) contains a factor \( S^3. \) Actually, for odd primes, the statements of both propositions are the same. Therefore we only have to prove the statement for \( p = 2. \) Again, for a compact connected Lie group we denote by \( G \) the associated 2-compact group.

Let \( X \) be a semi simple 2-compact group, i.e. \( \pi_1(X) \) is a finite 2-group. If \( H^*(X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \) is not generated by classes of degree 3, i.e. the polynomial algebra \( H^*(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \) is not generated by classes of degree 4, then the Weyl group \( W_X \) is non abelian [9] (Theorem 0.5B), but a honest reflection group, since \( W_X \) is defined over \( \mathbb{Q}_p^\wedge. \) That is,
$W_X$ is generated by elements of order 2 fixing a hyperplane of codimension 1. The universal cover $\tilde{X}$ of $X$ splits into a direct product $\tilde{X} \cong \prod X_i$ of simple, simply connected pieces \[8\]. Since $X$ and $\tilde{X}$ have isomorphic Weyl groups, we can assume that $X_1$ has a non abelian Weyl group $W_1$. For this piece we will construct a monomorphism $BG_1 := BS^3 \rightarrow BX_1$, such that the composition $BS^3 \rightarrow BX_1 \rightarrow B\tilde{X}_1$ is also a monomorphism (see below). Here, $\overline{X}_1$ denotes the center free quotient of $X_1$. Moreover, the map $BG_1 \rightarrow BX_1$ will induce an isomorphism in $H^4(-; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$.

Having done this we can proceed similarly as in \[21\]. For all other pieces there exists monomorphisms $BG_i \rightarrow BX_i$ inducing an isomorphism on $H^4(-; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ such that $G_i$ is isomorphic to $S^3$ or to $SO(3)$ (see \[21\]). This produces a homomorphism $\prod G_i \rightarrow \prod X_i \cong \tilde{X} \rightarrow X$ of p-compact groups. The kernel $K$ of this homomorphism, which might be nontrivial, is a central subgroup of $G_1 \times \prod_{i>1} G_i$ \[18\]. Since the center free quotient $\overline{X}$ is isomorphic to $\prod_i \overline{X}_i$ we have a homomorphism $X \rightarrow \overline{X}_1$. By construction the composition $S^3 \rightarrow X_1 \rightarrow \overline{X}_1$ is a monomorphism. We get a commutative diagram

$$
\begin{array}{ccc}
K & \longrightarrow & S^3 \times \prod_{i>1} G_i \\
\downarrow & = & \downarrow \\
K & \longrightarrow & S^3 \\
& \longrightarrow & \overline{X}_1
\end{array}
$$

where the right arrow in the bottom row is a monomorphism. Since $\overline{X}_1$ is center free the composition $K \rightarrow S^3 \times \prod_{i>1} G_i \rightarrow S^3$ is trivial. Therefore, $K$ is a subgroup of $\prod_{i>1} G_i$ and the map $S^3 \times \prod_{i>1} G_i \rightarrow X$ factors through a monomorphism $G := S^3 \times ((\prod_{i>1} G_i)/K) \rightarrow X$ with all the desired properties.

It remains to show that, for a simple, simply connected 2-compact group $X$ with non abelian Weyl group, there exists a monomorphism $S^3 \rightarrow X$ inducing isomorphisms in $H^4(-; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ such that $S^3 \rightarrow X \rightarrow \overline{X}$ is also a monomorphism. Let $W' \subset W_X$ be a subgroup of the Weyl group of $X$ generated by two non commutating reflections of $W_X$. Let $T \subset T_X \subset T_X$ denote the connected component of the fixed-point set of the $W_X$-action on $T_X$, which has codimension 2. The centralizer $C := C_X(T)$ is a connected 2-compact group, whose Weyl group $W_C$ contains $W'$ \[16\]. There exists a finite covering of $C$ which splits into a product $Y \times T$ where $Y$ is a simply connected 2-compact group of rank 2 with Weyl group isomorphic to $W_C$. The action of $W'$ on the maximal torus $T_Y$ of $Y$ gives rise to an irreducible representation over $\mathbb{Q}_2^\wedge$. Otherwise, $W'$ would split into a product and the two chosen reflections would commute. Hence, the 2-compact group $Y$ is simple and of rank 2. By Corollary 5.5 there exists a monomorphism $S^3 \rightarrow Y$ such that $S^3 \rightarrow Y \rightarrow \overline{Y}$ is a monomorphism. Putting all these homomorphisms and groups into a diagram we get

$$
\begin{array}{ccc}
S^3 & \longrightarrow & Y \\
\downarrow & = & \downarrow \\
S^3 & \longrightarrow & Y/K \\
& \longrightarrow & \overline{X}
\end{array}
$$

Here, $K$ denotes the kernel of $Y \rightarrow \overline{X}$. In particular, $K$ is a central subgroup of $Y$. Since $S^3 \rightarrow Y \rightarrow Y/K \rightarrow \overline{Y}$ is monomorphism, this also holds for the composition of the first two arrows. Moreover, since $Y/K \rightarrow \overline{X}$ is a monomorphism, the same holds for the composition $S^3 \rightarrow Y/K \rightarrow \overline{X}$. This proves the above claim and finishes the proof of the proposition. $\Box$

7. Proof of Proposition 4.3

In this section, we want to prove Proposition 4.3. The proof is based on an arithmetic square argument. First we need a statement about the existence of a particular subloop space, a global version of Proposition 6.1.

**Proposition 7.1.** Let $L$ be a large semi simple $\mathbb{Z}$-finite loop space. Then there exists a semi simple compact Lie group $G$, loop spaces $M$ and $N$ and a fibration

$$A \to BM \to BN$$

such that the following hold:

(i) $A$ is simple and $\mathbb{Z}$-finite.

(ii) $G \cong S^3 \times H$ and the universal cover of $H$ is isomorphic to a product of $S^3$'s.

(iii) The spaces $G$ and $M$ as well as $L$ and $N$ are homotopy equivalent.

(iv) $H^4(BN; \mathbb{Q}) \to H^4(BM; \mathbb{Q})$ is an isomorphism.

(v) There exists a commutative diagram

$$
\begin{array}{ccc}
BM_p & \to & BN_p \\
\downarrow & & \downarrow \\
BG_p & \to & BL_p
\end{array}
$$

where the vertical maps are equivalences. The same holds for the rationalizations of the classifying spaces.

**Proof.** This statement is a refinement of Proposition 1.4 of [21]. The proof of that statement is an arithmetic square argument which uses its $p$-completed version, namely Proposition 3.1 of [21], as input. The proof carries over word for word. We only have to replace that proposition by a $p$-completed version of the above claim, namely by Proposition 6.1. In particular, the bottom row in the diagram of (v) is the map constructed in Proposition 6.1. Claim (ii), which is not part of Proposition 1.4 of [21], is a consequence of the same formula in Proposition 6.1. □

**Remark 7.2.** The above proposition establishes an oriented fibration $G \to L \to L/G$. And the existence of such an oriented fibration is already sufficient to show that the finiteness obstruction vanishes and that every quasi finite loop space is actually finite (see [21]). The existence of a special tori is needed for the vanishing of the appropriate surgery obstruction.

For the proof of Proposition 4.3 we need two more lemmas.

**Lemma 7.3.** Let $A \in \text{Gl}(n, \mathbb{Z}_p^n)$. Then, there exists a vector $v = (v_1, \ldots, v_n) \in (\mathbb{Z}_p^n)^n$ such that $v_i$ is a square of a $p$-adic unit for all $i$ and such that $Av$ is a vector whose components are given by elements of $\mathbb{Z}_p$.

**Proof.** Let $B := A^{-1}$. We have to solve the following problem: Find a vector $w \in \mathbb{Z}_p^n$ such that $Bw \neq 0$ has as components squares of $p$-adic units. The question whether a $p$-adic unit is a square, can be decided by reducing to $\mathbb{Z}/p$ for $p$ odd or to $\mathbb{Z}/8$ for $p = 2$. In both cases the reduction $\overline{B}$ of $B$ is an invertible matrix and induces therefore an epimorphism on $(\mathbb{Z}/p)^n =: V$. In particular, if $\overline{v} \in V$ is a vector with components given by squares mod $p$ such that all entries are units in $\mathbb{Z}/p$, there exists a vector $w \in (\mathbb{Z})^n$ such that $\overline{B}w = \overline{v}$. Hence, $Bw$ is a vector whose components are squares of nontrivial $p$-adic units. For $p = 2$, the same argument works, we only have to replace $\mathbb{Z}/p$ by $\mathbb{Z}/8$. □
Proof of Proposition 4.3. Let $M$ and $N$ denote the loop spaces and $G$ the Lie group constructed in Proposition 7.1 Since $BM_p^\wedge \simeq BG_p^\wedge$ and $BM_0 \simeq BG_0$ we have a pull back diagram

$$
\begin{array}{c}
BM_{(p)} \longrightarrow & BG_p^\wedge \\
\downarrow & \downarrow \\
BG_0 \longrightarrow & BG_p^\flat \frac{A}{\rightarrow} (BG_p^\wedge)_0
\end{array}
$$

Here $BG_p^\wedge$ is the formal $p$-adic completion of the rationalization $BG_0$ in the sense of Sullivan, and $(BG_p^\wedge)_0$ the localization at 0 of $BG_p^\wedge$. The map $A$ is an equivalence between the homotopy equivalent spaces $BG_p^\wedge$ and $(BG_p^\wedge)_0$, and induces a continuous map in homotopy. The homotopy groups $\pi_\ast((BG_p^\wedge)_0$ carry a natural topology, since $\pi_\ast(BG_p^\wedge) \simeq \pi_\ast(BG) \otimes \mathbb{Z}_p$ (details may be found in [28]). $(BG_p^\wedge)_0 \simeq K(\mathbb{Q}_p^\wedge, 4)$ is a rational Eilenberg-MacLane space. Since self maps of rational Eilenberg-MacLane spaces are determined by the induced maps in homotopy, and since $A$ induces a continuous map in homotopy, we can think of $A$ as a matrix in $GL(n, \mathbb{Q}_p^\wedge)$ inducing a continuous self equivalence of $(\mathbb{Q}_p^\wedge)^n$. Such matrices can be written as a product $BR$ where $B \in GL(n, \mathbb{Z}_p^\wedge)$ and $R \in GL(n; \mathbb{Q})$. For example, this follows from the fact that the adic genus of products of $S^1$'s is rigid. Since $R$ can be realized as a self equivalence of $BG_0$, replacing $A$ by $B$ does not change the homotopy type of the pull back. Hence we may assume that $A \in GL(n, \mathbb{Z}_p^\wedge)$.

We have an analogous pull back diagram as above for the classifying space $B\tilde{M}$ of the universal cover $\tilde{M}$ of $M$ with the same gluing map $A$, namely

$$
\begin{array}{c}
B\tilde{M}_{(p)} \longrightarrow & BS^3_p \times B\tilde{H}_p^\wedge \\
\downarrow & \downarrow \\
BS^3_0 \times B\tilde{H}_0 \longrightarrow & (BS^3 \times B\tilde{H})_p^\wedge \frac{A}{\rightarrow} ((BS^3 \times B\tilde{H})^\wedge)_0
\end{array}
$$

Here we used the fact that $G \simeq S^3 \times H$ with $H = \tilde{H}/K$, and $\tilde{H}$ a product of $S^3$'s. Since every equivalence $BS^3_p^\wedge \longrightarrow BS^3_p^\wedge$ induces in $\pi_4(BS^3_p^\wedge)$ multiplication by a non trivial square unit of $\mathbb{Z}_p^\wedge$, Lemma 7.3 shows that there exists a map $BS^3 \longrightarrow BS^3_p^\wedge \times B\tilde{H}_p^\wedge$ such that the composition

$$
BS^3_{(p)} \longrightarrow BS^3_p^\wedge \longrightarrow BS^3_p^\wedge \times B\tilde{H}_p^\wedge \longrightarrow ((BS^3 \times B\tilde{H})^\wedge)_0 \frac{A^{-1}}{\rightarrow} (BS^3 \times B\tilde{H})_p^\wedge
$$

lifts to a map $BS^3_{(p)} \longrightarrow BS^3_0 \times B\tilde{H}_0$. Moreover, localized at 0, composition with the projection on the first factor is an equivalence. This establishes a map $BS^3_{(p)} \longrightarrow B\tilde{M}_{(p)}$ such that the completion of the composite $BS^3_{(p)} \longrightarrow B\tilde{M}_{(p)} \longrightarrow BM_{(p)}$ is induced by the monomorphism $S^3_p^\wedge \longrightarrow S^3_p^\wedge \times \tilde{H}_p^\wedge \longrightarrow S^3_p^\wedge \times \tilde{H}/K^\wedge = G^\wedge_p$ of $p$-compact groups. This shows that the homotopy fiber of $BS^3_{(p)} \longrightarrow BM_{(p)}$ is simple and $\mathbb{Z}_p$-finite as is the homotopy fiber of the composite map $f : BS^3_{(p)} \longrightarrow BM_{(p)} \xrightarrow{g} BN_{(p)}$. Since $H^4(BN_0; \mathbb{Q}) \simeq H^4(BM_0; \mathbb{Q})$, there exists a left inverse $s : BN_0 \longrightarrow BM_0$ for $g_0$. Projection onto the first factor gives a left inverse of $BS^3_0 \longrightarrow BM_0 \simeq BS^3_0 \times B\tilde{H}_0$. This shows that, localized at 0, the map $f : BS^3_{(p)} \longrightarrow BN_{(p)}$ has a left inverse and finishes the proof of the proposition. \qed
8. Stable Reducibility of abelian quotients

Let \( L \) be a finite loop space. Let \( T \) be an abelian compact Lie group, and let \( B\varphi : BT \to BL \) be a monomorphism. We will show in this section that the \( \mathbb{Z} \)-finite fiber \( L/T \) is stably reducible. Unfortunately, here we use a stronger assumption than we are able to produce in our case. In general, the fibration sequence \( S^1 \to L \to L/S^1 \to BS^1 \) cannot be extended one further step to the right. This is only possible after completion.

We begin with the case of a Lie group \( L \) to develop our intuition.

**Lemma 8.1.** Let \( \varphi : T \to L \) be a monomorphism between compact Lie groups, where \( T \) is abelian, then \( L/T \) is stably reducible.

**Proof.** Let \( \mathcal{L} \) and \( \mathcal{T} \) be the Lie algebras of \( L \) and \( T \) respectively. It is easy to see that the tangent bundle of \( L/T \) is given by \( L \times_T (\mathcal{L}/\mathcal{T}) \), where the action of \( T \) on the Lie algebras is given by the adjoint representation. Since \( T \) is abelian, the bundle \( L \times_T \mathcal{T} \) is trivial. On adding it to the tangent bundle of \( L/T \), we get the bundle \( L \times_T \mathcal{L} \). Notice that the adjoint action of \( T \) on \( \mathcal{L} \) extends to \( L \) and hence \( L \times_T \mathcal{L} \) is trivial. This shows that \( L/T \) has a stably trivial tangent bundle, which is equivalent to being stably reducible. \( \square \)

One would like to extend this argument to the case of \( L \) being a finite loop space. The theory that best preserves the analogy with compact Lie groups has been developed by John Klein in [13]. For a topological group \( G \), Klein defines the dualizing \( G \)-spectrum, \( D_G = \text{Map}(EG_+, S[G]^G) \), where \( S[G] \) is the spectrum \( \Sigma^\infty G_+ \) with the left \( G \) action. The residual right \( G \) action on \( S[G] \) induces the \( G \) action on \( D_G \). The spectrum \( D_G \) is the appropriate notion of the Adjoint representation. This is justified by the fact that for a compact Lie group \( G \), there is an equivalence \( D_G = S^{Ad} \), where \( S^{Ad} \) denotes the one-point compactification of the adjoint representation of \( G \).

The magic of the spectrum \( D_G \) lies in the following two theorems of Klein [13]

**Theorem 8.2.** Assume that \( BG \) is a finitely dominated space. Then the following are equivalent:

(i) \( BG \) is a Poincaré duality space of formal dimension \( n \).

(ii) \( D_G \) has the (unequivariant) homotopy type of a sphere spectrum of dimension \( -n \).

Moreover, in the above two cases, the Thom spectrum \( EG_+ \wedge_G D_G \) is the Thom spectrum of the Spivak normal bundle of \( BG \).

**Theorem 8.3.** Assume \( 1 \to H \to G \to Q \to 1 \) is an extension of topological groups, then if \( BH \) is a finitely dominated Poincaré duality space, then there is a weak equivalence of spectra

\[ D_G \cong D_H \wedge D_Q \]

Moreover, one may replace \( D_H \) by an \( H \)-equivalent spectrum so as to make the above equivalence \( H \)-equivariant.

We will apply the above theorems to the extension of topological groups

\[ 1 \to \Omega(L/T) \to T \to L \to 1 \]

In order to make sense of this extension, we must work in the model-category of simplicial groups and replace the above groups by equivalent topological groups (c.f [10]).

Let us record a simple lemma about simplicial groups that will be useful in the sequel.
Lemma 8.4. Let $sH \rightarrow sG$ be an acyclic fibration of simplicial groups. Then on taking realizations one gets an extension of topological groups

$$1 \rightarrow K \rightarrow H \rightarrow G \rightarrow 1$$

with a contractible kernel $K$. Moreover, there is an $H$-equivalence of spectra $D_H \cong D_G$.

Proof. The realization of an acyclic fibration of simplicial groups is an acyclic Serre fibration. Hence we get an extension of topological groups with a contractible kernel $K$. We may consider the space $EH/K$ as a model for $EG$. Then the required equivalence is induced by the (2-sided) $H$-equivalence $S[H] \rightarrow S[G]$ and is given by

$$D_H = \text{Map}(EH_+, S[H])^H \rightarrow \text{Map}(EH_+, S[G])^H = \text{Map}(EG_+, S[G])^G = D_G$$

We are now ready to prove the main theorem of this section

Theorem 8.5. Let $L$ be a finite loop space of formal dimension $n$, and let $B\varphi : BT \rightarrow BL$ be a monomorphism, where $T$ is an abelian compact Lie group of rank $r$, then the $\mathbb{Z}$-finite space $L/T$ is stably reducible.

Proof. For a connected space $X$, let $s\Omega X$ denote the simplicial Kan loop group of $X$ (c.f [10]). The map $B\varphi$ induces a simplicial homomorphism

$$s\varphi : s\Omega BT \rightarrow s\Omega BL$$

The properties of model categories allow us to factor $s\varphi$ through an acyclic cofibration followed by a fibration. Consequently, we may assume that $s\varphi$ is a fibration. Unfortunately, $s\Omega BT$ is a highly non-abelian model for $T$. This is the price one has to pay for obtaining a fibration. Fortunately however, the simplicial group $s\Omega BT$ is related to an abelian simplicial model for $T$ via adjointness. We have an acyclic fibration

$$s\Omega BT \rightarrow sT^a$$

where $sT^a$ is a simplicial abelian model for $T$.

Now taking the realization of $s\varphi$, we obtain an extension

$$1 \rightarrow K \rightarrow T^f \rightarrow L^f \rightarrow 1$$

where $T^f$ and $L^f$ denote the free models of $T$ and $L$, we obtain by realizing the simplicial Kan loop groups $s\Omega BT$ and $s\Omega BL$ respectively. The group $K$ is clearly equivalent to $\Omega(L/T)$. Using Theorem 8.3, one obtains a $K$-equivariant equivalence

$$D_{T^f} \cong D_K \wedge D_{L^f}$$

Hence $K$ acts trivially on $D_{L^f}$. In fact, $D_{L^f}$ is a sphere spectrum of dimension $n$ with a trivial $K$ action. One sees this as follows: by Theorem 8.2, $D_K$ is a sphere of dimension $r - n$. Moreover, by Lemma 8.4 $D_{T^f} \cong D_{T^a} \cong D_T$ is also a sphere of dimension $r$, thus $D_{L^f}$ is a sphere of dimension $n$ with a trivial $K$ action. Hence $D_K \cong \Sigma^{-n}D_{T^f}$ as a $K$-spectrum. It now follows from Theorem 8.2 that the Thom spectrum of the Spivak Normal bundle of $L/T$ is given by $\Sigma^{-n}EK_+ \wedge_K D_{T^f}$. Thus to show that $L/T$ is stably reducible, it is sufficient to show that $D_{T^f}$ is equivalent to a sphere spectrum with a trivial $T^f$ action. This follows by applying Lemma 8.4 to notice that $D_{T^f}$ is equivalent to $D_{T^a}$ which clearly has a trivial $T^f$ action. \qed
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