ON THE HOMOLOGY OF ELEMENTARY ABELIAN GROUPS AS MODULES OVER THE STEENROD ALGEBRA

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Abstract. We examine the dual of the so-called "hit problem", the latter being the problem of determining a minimal generating set for the cohomology of products of infinite projective spaces as module over the Steenrod Algebra \( A \) at the prime 2. The dual problem is to determine the set of \( A \)-annihilated elements in homology. The set of \( A \)-annihilateds has been shown by David Anick to be a free associative algebra. In this note we prove that, for each \( k \geq 0 \), the set of \( k \) partially \( A \)-annihilateds, the set of elements that are annihilated by \(Sq^i \) for each \( i \leq 2^k \), itself forms a free associative algebra.

1. Introduction and Notations

Let \( \Bbb{F}_2 \) be the field of 2 elements and \( \Gamma = \{ \Gamma_{s,*} \}_{s \geq 0} \) be the bigraded \( \Bbb{F}_2 \)-space defined by
\[
\Gamma_{s,*} = H_*(B(\Bbb{Z}/2)^s, \Bbb{F}_2), \quad \text{for each } s \geq 0.
\]
The bigrading \((s,d)\) is by number of direct product factors of \( B(\Bbb{Z}/2)^s \) and homological degree. We shall say that an element \( x \) of bidegree \((s,d)\) has length \( s \) and degree \( d \). This paper studies \( \Gamma \) with its canonical structure as a right module over the Steenrod algebra \( A \). We are interested in particular in the problem of determining the graded vector spaces \( \Gamma_{s,*}^A = \{ a \in \Gamma_{s,*} \mid (a)Sq^k = 0, \forall k > 0 \} \) consisting of all elements of \( \Gamma \) that are annihilated by the Steenrod operations of positive degree. This problem and its dual (finding a minimal generating set for the cohomology of \( B(\Bbb{Z}/2)^s \) as a left \( A \)-module) have been much studied in recent years. The reader can find comprehensive bibliographies covering work done through 2000 in [7] and [8], and will find recent work in [4], [5] and [6]. Much progress has been made, but the general problem remains unsolved.

We will work in terms of the reduced homology groups of the smash products:
\[
\tilde{\Gamma}_{s,*} = \tilde{H}_*(B(\Bbb{Z}/2)^s, \Bbb{F}_2), \quad \text{for each } s \geq 1.
\]
We adopt also the convention \( \tilde{\Gamma}_{0,*} = H_*(\Bbb{F}_2) \), the homology of a point. We assemble the spaces \( \tilde{\Gamma}_{s,*} \) into a bigraded vector space
\[
\tilde{\Gamma} = \{ \tilde{\Gamma}_{s,*} \}_{s \geq 0}
\]
and write the associated vector spaces of \( A \)-annihilated elements \( \tilde{\Gamma}_{s,*}^A \subseteq \tilde{\Gamma}_{s,*} \). The vector spaces \( \Gamma_{s,*}^A \) are easily expressed in terms of the spaces \( \tilde{\Gamma}_{p,*}^A \) for \( p \leq s \), so a study of the smash products is sufficient. The natural mappings:
\[
B(\Bbb{Z}/2)^p \times B(\Bbb{Z}/2)^q \longrightarrow B(\Bbb{Z}/2)^{p+q}
\]
induce pairings of vector spaces:
\[
\tilde{\Gamma}_{p,*} \otimes \tilde{\Gamma}_{q,*} \longrightarrow \tilde{\Gamma}_{p+q,*}
\]
for all \( p, q \geq 0 \), which make \( \tilde{\Gamma} \) into a connected bigraded algebra. By the Künneth theorem, \( \tilde{\Gamma} \) is a free associative \( \Bbb{F}_2 \)-algebra. For each \( k \geq 1 \), it is convenient to represent the canonical generators,
\[
\gamma_k \in \tilde{\Gamma}_{1,k} = \tilde{H}_k(B(\Bbb{Z}/2)).
\]
Then we have \( \tilde{\Gamma} = \Bbb{F}_2\langle \{ \gamma_1, \gamma_2, \gamma_3, \ldots \} \rangle \) (we use the notation of Cohn [2]; for a field \( k \) and set \( X \), \( k\langle X \rangle \) is the free associative \( k \)-algebra generated by \( X \)). The Cartan formula implies that the bigraded vector space
\( \Gamma^A = \{ \tilde{\Gamma}_{s,d}^A \}_{s \geq 0} \) is a subalgebra of \( \tilde{\Gamma} \). Anick proves in [1] that this subalgebra is itself free. Now for each \( k \geq 0 \), and \( s, d \geq 0 \), define:

\[
\Delta(k)_{s,d} = \bigcap_{i=0}^{k} \ker(Sq^{2i} : \tilde{\Gamma}_{s,d} \to \tilde{\Gamma}_{s,d-2^i}),
\]

and set \( \Delta(k) = \{ \Delta(k)_{s,d} \}_{s,d \geq 0} \), a bigraded space called the “\( k \) partially \( \mathcal{A} \)-annihilateds.” Using a variant of Anick’s argument, we will show that

**Theorem 1.** For each \( k \geq 0 \), \( \Delta(k) \) is a free subalgebra of \( \tilde{\Gamma} \).

Note that if \( k \) is chosen so that \( d < 2^{k+2} \), then we have for any \( s \geq 0 \):

\[
\tilde{\Gamma}_{s,d}^A = \Delta(k)_{s,d}.
\]

Thus, determining the sets \( S_k \) such that \( \Delta(k) = \mathbb{F}_2(S_k) \) would solve the “\( \mathcal{A} \)-annihilated problem”, and the solution would be in terms of explicitly-given algebra generators. Furthermore, partial progress is meaningful, as the determination of the set \( S_k \) would determine all \( \mathcal{A} \)-annihilateds of degree \( d < 2^{k+2} \).

In this note, \( \ker Sq^p \) and \( \text{im} Sq^p \) will be understood to involve the restricted maps \( Sq^p : \tilde{\Gamma} \to \tilde{\Gamma} \).

**2. Proof of the Main Theorem**

In this section we prove that \( \Delta(k) \) is a tensor algebra, using a remarkable lemma of Anick [1], here stated for the case of \( \mathbb{Z}^t \)-graded algebras. Let \( k \) be a field. For \( t \geq 1 \), a \( k \)-algebra \( A \) is \( \mathbb{Z}^t \)-graded if \( A = \{ A_I \}_{I \in \mathbb{Z}^t} \), and multiplication in \( A \) is a family of maps,

\[
A_I \otimes A_J \to A_{I+J}.
\]

If \( x \in A_I \), we say the degree of \( x \) is \( I = (i_1, i_2, \ldots, i_t) \). Introduce a lexicographic ordering of degree as follows: \( I < J \) if and only if there is an integer \( r \) with \( 1 \leq r \leq t \) such that \( i_p = j_p \), if \( p < r \), and \( i_r < j_r \). The algebra \( A \) is said to be connected if \( A_I = 0 \) whenever \( I \) contains a negative entry, and \( A_{(0, \ldots, 0)} \cong k \). The positively-graded elements of \( A \) are the elements of the set

\[
A^+ = \bigcup \{ A_I \mid \text{all entries of } I \text{ are non-negative and at least one entry is positive} \}.
\]

**Definition 2.** Let \( A \) be a connected \( \mathbb{Z}^t \)-graded algebra (\( t \geq 1 \)) over a field \( k \). \( A \) is said to satisfy Anick’s Condition if whenever a relation,

\[
\sum_{i=1}^{n} a_i b_i = 0,
\]

holds in \( A \), where each \( b_i \neq 0 \), then there is a \( j \) such that

\[
a_j \in \sum_{i \neq j} a_i A.\]

**Lemma 3** (Anick [1]). Let \( A \) be a connected \( \mathbb{Z}^t \)-graded algebra over a field \( k \). Then \( A \) is a tensor algebra, \( A = k(\mathcal{X}) \), for some set of positively-graded elements \( \mathcal{X} \subset A^+ \), if and only if \( A \) satisfies Anick’s Condition.

Anick’s proof makes use of the work of Cohn [2] on so-called free ideal rings (firs). For completeness, we shall provide a proof that avoids as much of this machinery as possible. Furthermore, our working in the graded case allows us to simplify some of Cohn’s arguments.

**Proof.** The backward direction is the easier of the two. The proof is Anick’s [1]. Suppose that \( A \) is not a tensor algebra. Choose a minimal set \( \mathcal{X} \) of generators for \( A \), and write \( A = k(\mathcal{X})/R \) where \( R \) is the non-zero ideal of relations. Choose a non-zero \( \alpha \in R \) of minimal degree. Then \( \alpha \) can be expanded uniquely in the form:

\[
\alpha = \sum_{i=1}^{m} x_i Y_i,
\]

where the \( x_i \) are distinct elements of the generating set \( \mathcal{X} \), and \( Y_i \neq 0 \) for each \( i \). For each \( x \in k(\mathcal{X}) \), write \( \overline{x} \) for the corresponding element of \( A \). Since \( \alpha \in R \), we have:

\[
\sum_{i=1}^{m} \overline{x_i} Y_i = 0.
\]
Since \( \alpha \) is of minimal degree in \( R \), we have \( Y_i \neq 0 \) in \( A \), for each value of \( i \). So Eqn. (3) is a relation of the form (1). But if there were a \( j \) with \( 1 \leq j \leq m \) such that
\[
(4) \quad \sum_{i \neq j} x_i A,
\]
then the generating set \( X \) would not be minimal, contradicting our assumption. Thus \( A \) cannot satisfy Anick’s Condition.

For the forward direction, assume that \( A \) is a connected graded tensor algebra \( k \langle X \rangle \) on a generating set \( X \subset A^+ \). Suppose now that there is a relation,
\[
(5) \quad \sum_{i=1}^{n} a_i b_i = 0,
\]
for \( a_i, b_i \in A \), and each \( b_i \neq 0 \), as in the premise of Anick’s Condition. We may assume the summands are ordered so that \( \text{deg}(b_1) \geq \text{deg}(b_2) \geq \cdots \geq \text{deg}(b_n) \). Let \( I = \text{deg}(b_n) \), and let \( c \mu = cx_1 \cdots x_s \) be a term of degree \( I \) occurring in \( b_n \) \( (c \in k, x_i \in X) \). For any element \( a \in A \), we may write \( a = a_0 + a^* \mu \) for some \( a_0, a^* \in A \) such that \( \mu \) does not right-divide any term of \( a_0 \). Moreover, both \( a_0 \) and \( a^* \) are uniquely-determined since \( A \) is free. Observe, the function \( a \mapsto a^* \) is \( k \)-linear of degree \( -\text{deg}(\mu) = -I \) (This mapping is known as left transduction for \( \mu \), and \( a^* \) is known as the left cofactor of \( \mu \) in \( a \). See [3]).

Suppose \( b \in A \) is any single term. Then either \( \mu \) does not right-divide \( b \), in which case \( b^* = 0 \), or \( \mu \) does, and \( b^\mu = b \). Thus, if \( \text{deg}(b) \geq I \), then for any \( a \in A \), \( (ab)^* = ab^* \). By linearity of transduction, we have:
\[
(ab)^* = ab^*, \quad \text{for any } a, b \in A \text{ such that } \text{deg}(b) \geq I.
\]
Applying transduction for \( \mu \) to Eqn. (3), we have, since \( b_i \) has degree at least \( I \),
\[
0 = \left( \sum_{i=1}^{n} a_i b_i \right)^* = \sum_{i=1}^{n} a_i b_i^*.
\]
Finally, since \( b_i^* \neq 0 \) has degree \( (0, \ldots, 0) \), and \( A \) is connected, we have in fact shown that \( b_i^* \in k^x \). We obtain a relation of the form:
\[
a_n = \left( -\sum_{i=1}^{n-1} a_i b_i^* \right) (b_n^*)^{-1} = \sum_{i=1}^{n-1} a_i (-b_n^* )^{-1} b_i^*
\]
Therefore, \( a_n \in \sum_{i \neq n} a_i A \), as desired. \( \square \)

The following is a useful application of Lemma 3

**Lemma 4.** Let \( A \) be a connected \( \mathbb{Z}^l \)-graded algebra over a field \( k \), and suppose that \( A \) is a tensor algebra, \( A = k \langle X \rangle \), on some set of positively-graded elements \( X \subset A^+ \). Let \( S \) be any set of positively graded elements that form a minimal generating set for \( A \). Then \( A \) is the tensor algebra on \( S \).

**Proof.** Consider the canonical algebra mapping \( k \langle S \rangle \to A \). We must show that the kernel is zero. Suppose to the contrary there is a non-zero element of the kernel. We choose one of least degree; say \( \sum_{i=1}^{n} s_i Y_i \), where the elements \( s_i \) are distinct members of the set \( S \), and each \( Y_i \) is a non-zero element of \( k \langle S \rangle \). Then we have in \( A \) the relation:
\[
(6) \quad \sum_{i=1}^{n} s_i Y_i = 0.
\]
Our assumption that (6) is a relation of least degree assures that each \( Y_i \) is a non-zero element of \( A \). But by Lemma 3 \( A \) satisfies Anick’s condition. So there must be an index \( j \) with \( 1 \leq j \leq n \) and elements \( c_i \in A \) for \( i \neq j \) such that:
\[
(7) \quad s_j = \sum_{i \neq j} s_i c_i.
\]
Now for each index \( i \neq j \), the element \( c_i \) must be expressible as a non-commutative polynomial in the elements of \( S \). Further, since \( A \) is graded-connected, and each \( s_j \) has positive degree, none of these polynomials can involve the element \( s_j \). Hence, equation (7) expresses \( s_j \) in terms of the other generators. This dependence
would contradict the assumed minimality of the generating set \( S \). Thus there can be no relation of the form (4), and the result is proved.

Now we come to our main result.

**Theorem 5.** For \( k \geq 0 \), \( \Delta(k) \) is free as associative \( \mathbb{F}_2 \)-algebra.

**Proof.** We will show that \( \Delta(k) \) satisfies Anick’s Condition. Suppose there is a relation in \( \Delta(k) \),

\[
\sum_{i=1}^{n} a_i b_i = 0,
\]

where each \( b_i \neq 0 \). We want to show that there is an index \( j \) such that

\[
a_j \in \sum_{i \neq j} a_i \Delta(k).
\]

This will surely be the case if the elements \( a_1, \ldots, a_n \) were not distinct, so we may assume that the \( a_i \) are distinct. Now Eqn. (8) can be read as a relation in the connected tensor algebra \( \tilde{\Gamma} \). The fact that one such relation among the elements of \( \{a_i\} \) exists means that we can find one for which \( n \) is minimal. In other words, let

\[
\sum_{i=1}^{p} a_i c_i = 0,
\]

be a relation with minimal number of summands in \( \tilde{\Gamma} \), involving elements from the set \( \{a_i\} \), with each \( c_i \neq 0 \). Since \( \tilde{\Gamma} \) satisfies Anick’s Condition (by Lemma 3), there is an index \( j \) such that

\[
a_j = \sum_{1 \leq i \leq p, i \neq j} a_i d_i,
\]

for some \( d_i \in \tilde{\Gamma} \). We shall show that every \( d_i \) is in fact a member of \( \Delta(k) \). Let \( \ell \) be an integer, \( 0 \leq \ell \leq k \). Apply \( Sq^{2\ell} \) to both sides of Eqn. (10). Note, \( a_i Sq^q = 0 \) for each \( q \) satisfying \( 0 < q \leq 2\ell \) and every \( i \), since \( a_i \in \Delta(k) \). Hence, by the Cartan formula,

\[
0 = \sum_{i \neq j} a_i \left( d_i Sq^{2\ell} \right).
\]

If there are any indices \( i \) such that \( d_i Sq^{2\ell} \neq 0 \), then Eqn. (11) would represent a non-trivial relation among the elements of \( \{a_i\} \), of strictly fewer number of terms than the supposed minimal one. Therefore, \( d_i Sq^{2\ell} = 0 \) for each \( i \). But this is true for any \( 0 \leq \ell \leq k \), so each \( d_i \in \Delta(k) \). This shows that \( a_j \in \sum_{i \neq j} a_i \Delta(k) \), and so \( \Delta(k) \) satisfies Anick’s Condition. Hence \( \Delta(k) \) is a tensor algebra on a positively-graded generating set. \( \square \)

### 3. Analysis of \( \Delta(0) \)

In an effort to de-clutter our formulas, we use the notation:

\[
[i_1, i_2, \ldots, i_s] = \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_s} \in \tilde{\Gamma}_{s,*}, s \geq 1
\]

\[
[ ] = 1 \in \tilde{\Gamma}_{0,0}
\]

For \( s \geq 1 \), and integers \( m_i \geq 0 \), we define special elements of \( \tilde{\Gamma} \):

\[
\sigma(m_1, m_2, \ldots, m_s) \overset{df}{=} [2m_1 + 2, 2m_2 + 2, \ldots, 2m_s + 2] Sq^1,
\]

and we let \( S_0 \) be the set:

\[
S_0 = \{ \sigma(m_1, m_2, \ldots, m_s) | s \geq 1, m_1 \geq 0, \ldots, m_s \geq 0 \}.
\]

Our goal in this section is to prove that \( \Delta(0) = \ker Sq^1 \) is the free algebra on the set \( S_0 \).

**Lemma 6.** For each \( s \geq 1 \) one has in \( \tilde{\Gamma}_{s,*} \):

\[
\ker Sq^1 = \im Sq^1.
\]
Proof. $Sq^1$ acts as a differential on $\tilde{\Gamma}_{s,s}$, and the isomorphism:

$$\tilde{\Gamma}_{s,s} = (\tilde{\Gamma}_{1,s})^{\otimes s}$$

is an isomorphism of chain complexes. Since $\tilde{\Gamma}_{1,s}$ is acyclic, our results follows from the Künneth theorem. □

Given a monomial $\mu = [i_1, i_2, \ldots, i_s]$ in $\tilde{\Gamma}_{s,s}$, we define its weight to be the number of the indices $i_1, i_2, \ldots i_s$ that are odd.

Lemma 7. Let $\mu \in \tilde{\Gamma}$ be any monomial. Then $(\mu)Sq^1$ lies in the algebra generated by $S_0$.

Proof. We will prove the lemma by induction on $t$, the weight of $\mu$. The case $t = 0$ is tautological. Now suppose that $t \geq 1$ and that the lemma has been proved for all monomials $\mu$ of weights less than $t$. Let $\mu = [i_1, i_2, \ldots , i_s]$ be a given monomial of weight $t$. Choose an index $i_k$ that is odd; say, $i_k = 2m - 1$. Then $[i_k] = [2m]Sq^1$ and $[i_k]Sq^1 = 0$. Let $\alpha = [i_1, \ldots, i_{k-1}]$ and $\beta = [i_{k+1}, \ldots, i_s]$ so that using the product in $\tilde{\Gamma}$ we may write: $\mu = \alpha \cdot [i_k] \cdot \beta$. Then,

$$(\mu)Sq^1 = (\alpha \cdot [i_k] \cdot \beta)Sq^1 = (\alpha)Sq^1 \cdot [i_k] \cdot \beta + \alpha \cdot [i_k] \cdot (\beta)Sq^1 = (\alpha)Sq^1 \cdot [i_k] \cdot \beta + (\alpha)Sq^1 \cdot [2m] \cdot (\beta)Sq^1 + (\alpha)Sq^1 \cdot [2m] \cdot (\beta)Sq^1 + \alpha \cdot [i_k] \cdot (\beta)Sq^1 = (\alpha)Sq^1 \cdot ([2m] \cdot (\beta)Sq^1 + (\alpha) \cdot [2m] \cdot Sq^1 \cdot (\beta)Sq^1$$

But the right hand side of this equation is a sum of products of elements of the form $(\gamma)Sq^1$, where in each case, $\gamma$ is a monomial of weight less than $t$. So the inductive hypothesis implies that $(\mu)Sq^1$ lies in the algebra generated by $S_0$, and our inductive proof is complete. □

Combining Lemmas 6 and 7 we find:

Lemma 8. $\Delta(0)$ is generated as an algebra by the set $S_0$.

The next lemma will be useful in proving that the set $S_0$ is algebraically independent. In what follows, when we write “$\alpha$ expanded in terms of $\tilde{\Gamma}$”, we mean to express $\alpha$ as a sum of monomials $[i_1, i_2, \ldots , i_s] \in \tilde{\Gamma}_{s,s}$. By abuse of notation, we say that $i_j$ is a factor of the term $[i_1, i_2, \ldots , i_s]$, and so we may speak of odd or even factors of such a term.

Lemma 9. $S_0$ is a linearly independent subset of $\tilde{\Gamma}$.

Proof. Since $\tilde{\Gamma}$ is graded on length, any potential linear dependence will be of the form

$$(12) \sum_j \sigma(m_{j,1}, m_{j,2}, \ldots, m_{j,s}) = 0,$$

where the sum is over a positive number of indices $j$ and the value of $s$ is the same for all terms. We wish to show that such a relation is impossible if the sequences $\{m_{j,1}, m_{j,2}, \ldots, m_{j,s}\}$ are all distinct. But when a term $\sigma(m_{j,1}, m_{j,2}, \ldots, m_{j,s})$ is expanded in terms of $\tilde{\Gamma}$, the monomial $[2m_{j,1} + 1, 2m_{j,2} + 2, \ldots, 2m_{j,s} + 2]$ will appear, and this monomial cannot appear in any of the other expansions. So a relation of the form (12) is impossible. □

Preparing the proof of the next result, we define $W_k \tilde{\Gamma}$ to be the subspace of $\tilde{\Gamma}$ that is spanned by all the monomials $[i_1, i_2, \ldots , i_s]$ of weight $k$. Then as a bigraded vector space, $\tilde{\Gamma}$ splits as a direct sum:

$$\tilde{\Gamma} = \bigoplus_{k \geq 0} W_k \tilde{\Gamma}.$$

The product on $\tilde{\Gamma}$ is compatible with this direct sum decomposition, in the sense that:

$$W_k \tilde{\Gamma} \cdot W_l \tilde{\Gamma} \subseteq W_{k+l} \tilde{\Gamma}.$$

Proposition 10. $\Delta(0) = \mathbb{F}_2\langle S_0 \rangle$. 

Proof. We have already shown that the set of all products of elements from \( S_0 \) generates \( \ker Sq^1 \) as a vector space. All that remains is to prove that \( S_0 \) is an algebraically independent set. According to Lemma 3 it will be enough to prove that \( S_0 \) is a minimal generating set for the algebra \( \ker Sq^1 \). Suppose the contrary. Then it would be possible to express one particular generator, say, \( \sigma(m_1, m_2, \ldots, m_s) \), as a (non-commutative) polynomial in the other generators. But each generator \( \sigma(n_1, n_2, \ldots, n_t) \) lies in \( W_1 \Gamma \), so equations (13) and (14) imply that our expression of \( \sigma(m_1, m_2, \ldots, m_s) \) in terms of the other generators would reduce to an expression of \( \sigma(m_1, m_2, \ldots, m_s) \) as a linear combination of the others. According to Lemma 9 this is not possible. So \( S_0 \) must be a minimal generating set for the algebra \( \ker Sq^1 \), and our proof is complete. \( \square \)

Using Prop. [10] we can determine a formula for the dimension of the homogeneous components of \( \Delta(0) \).

**Proposition 11.** Let \( c_{s,d} \) be the dimension of the component of \( \Delta(0) \) in bidgree \( (s, d) \). Let \( \eta_{s,d} \) be defined by

\[
\eta_{s,d} = \begin{cases} 
0, & \text{if } d \text{ is even} \\
\frac{d}{s-1}, & \text{if } d \text{ is odd}
\end{cases}
\]

There is a recurrence relation,

\[
c_{0,0} = 1 \\
c_{s,0} = 0, \quad \text{if } s > 0 \\
c_{0,d} = 0, \quad \text{if } d > 0 \\
c_{s,d} = \sum_{r=1}^{s} \sum_{a=1}^{d} \eta_{r,a} c_{s-r,d-a}, \quad \text{if } s, d \geq 1
\]

Proof. Since \( \Delta(0) \) is a tensor algebra, \( c_{s,d} \) counts the number of terms of length \( s \) and degree \( d \). Partition the terms of \( \Delta(0)_{s,d} \) by length and degree of the first factor in the term. In what follows, “\( \sigma \)” always represents an algebra generator, such as \( \sigma(m_1, m_2, \ldots, m_k) \), while “\( r \)” represents a (possibly empty) product of algebra generators.

\[
\Delta(0)_{s,d} = \text{span} \left( \prod_{r,a} \{ \sigma \cdot \tau \mid \ell(\sigma) = r, \deg(\sigma) = a, \ell(\tau) = s - r, \deg(\tau) = d - a \} \right)
\]

Now since \( \tau \) is an arbitrary term in \( \Delta(0)_{s-r,d-a} \), we obtain the formula,

\[
c_{s,d} = \sum_{r,a} (\text{number of algebra generators } \sigma \text{ in bidegree } (r,a)) \cdot c_{s-r,d-a}
\]

For a typical \( \sigma \) in bidegree \( (r,a) \), we have \( \sigma = \sigma(m_1, \ldots, m_{r}) \) such that \( 2(m_1 + \cdots + m_r) + 2r - 1 = a \). Thus \( m_1 + \cdots + m_r = (a+1)/2 - r \). So the number of algebra generators \( \sigma \) in bidegree \( (r,a) \) is found by counting the number of ordered partitions of \( (a+1)/2 - r \) into \( r \) parts. Elementary combinatorics tells us that the number of such partitions is exactly \( \eta_{r,a} \) as defined above.

The base cases for the recurrence are easily verified. \( \square \)

As examples, we find closed formulas for \( c_{s,d} \) for small \( s \):

- \( c_{1,d} = \begin{cases} 
0, & \text{if } d \text{ even} \\
1, & \text{if } d \text{ odd}
\end{cases} \)
- \( c_{2,d} = \begin{cases} 
0, & \text{if } d \text{ even} \\
\frac{d}{2}, & \text{if } d \text{ odd}
\end{cases} \)
- \( c_{3,d} = \begin{cases} 
\frac{d}{2}, & \text{if } d \text{ even} \\
\frac{(d-1)}{4}, & \text{if } d \text{ odd}
\end{cases} \)

It was pointed out by the referee that the exactness of \( Sq^1 \) on \( \tilde{\Gamma}_{s,*} \) implies a nice reduction formula:

\[
c_{s,d} + c_{s,d+1} = \dim \left( \tilde{\Gamma}_{s,d+1} \right) = \binom{d}{s-1}
\]
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