Coherent states sometimes look like squeezed states and vice versa: the Paul trap†

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Abstract. Using the Paul trap as a model, we point out that the same wavefunctions can be variously coherent or squeezed states, depending upon the system to which they are applied.

1. Introduction

Elsewhere \([1]–[4]\) we have investigated time-dependent Schrödinger equations that are quadratic in position, \(q\), and momentum, \(p = -i\partial_q\):

\[
\{2i\partial_t - [1 + g_2(t)]p^2 + g_0(t)\frac{1}{2}(qp + pq) + g_1(t)p - 2h_2(t)q^2 - 2h_1(t)q - 2h_0(t)\}\Psi(q, t) = 0.
\]

(1)

We use dimensionless variables (\(\bar{\hbar} = \hbar = m = 1\)) and the \(g\)’s and \(h\)’s are time-dependent functions. This study yielded methods for obtaining the symmetries of these systems. It also provided explicit analytical relationships among various subclasses of these equations. From these we could obtain implicit and sometimes explicit analytical solutions for the number, coherent and squeezed states and also for the associated uncertainty relations.

In this paper, we will apply the techniques so obtained to the Paul trap system. We will thereby be able to provide insight into what the coherent and squeezed states of the Paul trap are and into what uncertainty relations are satisfied.

† Our discussion of the physical distinction between coherent and squeezed states is most appropriate as a vehicle for us to pay homage to the late Dan Walls. Since the time of their experimental discovery in the mid-1980s, his name has closely been associated with the physics of squeezed states.
In section 2, we will set up the formalism for the particular subclass of equation (1) that contains the Paul trap as a special case. That is, all \( h_i(t) = g_i(t) = 0 \) except \( h_2(t) \equiv f(t) \). We go on, in section 3, to describe the coherent states, squeezed states and uncertainty relations for this subclass. The physical Paul trap is described in section 4.

In section 5, we discuss coherent states, squeezed states and uncertainty relations for the Paul trap. This is the main thrust of this paper. Specifically, the reader will see how the same physical states can be viewed either as coherent states of the Paul trap or as squeezed states of the harmonic oscillator and vice versa.

2. Time-dependent quadratic systems

We now concentrate on the aforementioned particular subclass of equation (1), the time-dependent, harmonic-oscillator, Schrödinger equation. We have that

\[
[\partial_{qq} + 2i\partial_t - 2f(t)q^2]\Psi(q, t) = 0. \tag{2}
\]

Here the function \( f(t) \) is a continuously differentiable and integrable function of \( t \). From the general results [1]–[4] for equation (1), we have that equation (2) admits a symmetry algebra that is a product of a Heisenberg–Weyl algebra, \( \mathfrak{w}_1 \), with a \( su(1, 1) \) algebra. The subalgebras \( \mathfrak{w}_1 \) and \( su(1, 1) \) are the ones associated with coherent states and squeezing.

The time-dependent lowering and raising (ladder) operators in \( \mathfrak{w}_1 \) are given by

\[
A(t) = i[\xi(t)p - \dot{\xi}(t)q] \quad A^\dagger(t) = -i[\xi^*(t)p - \dot{\xi}^*(t)q]. \tag{3}
\]

Here \( q \) and \( p = -i\partial_q \) are the ordinary position and momentum operators

\[
q = \frac{a + a^\dagger}{\sqrt{2}} \quad p = \frac{a - a^\dagger}{i\sqrt{2}}. \tag{4}
\]

It is straightforward to demonstrate [1, 2] that the time-dependent functions \( \xi \) and \( \xi^* \) in equation (3) are two linearly independent, complex solutions of the second-order differential equation:

\[
\ddot{\gamma} + 2f(t)\gamma = 0. \tag{5}
\]

Therefore, the time-dependent functions \( \xi \) and \( \xi^* \) are determined by the particular Hamiltonian.

The Wronskian of these solutions is

\[
\xi(t)\dot{\xi}^*(t) - \dot{\xi}(t)\xi^*(t) = -i. \tag{6}
\]

On account of Wronskian (6), the ladder operators \( A(t) \) and \( A^\dagger(t) \) satisfy the commutation relation

\[
[A(t), A^\dagger(t)] = I \tag{7}
\]

where \( I = 1 \) is the identity operator.

† The Hamiltonian is \( H = -\frac{i}{2}\partial_{qq} + f(t)q^2 \).

‡ The ‘dot’ over a function of \( t \) indicates ordinary differentiation with respect to \( t \).
3. Coherent/squeezed states and uncertainty relations

3.1. Coherent states

First, consider $|\alpha; t\rangle$, the displacement-operator coherent states (DOCSs). They are defined by

$$|\alpha; t\rangle = D_A(\alpha)|0; t\rangle = \exp[\alpha A(t) - \alpha^* A(t)]|0; t\rangle.$$  (8)

In the above, $D_A(\alpha)$ is the displacement operator and $\alpha$ is a complex constant. Similarly, the equivalent ladder-operator coherent states (LOCSs) are defined as

$$A(t)|\alpha; t\rangle = \alpha|\alpha; t\rangle.$$  (9)

In equation (8), the extremal state $|0; t\rangle$ is a member of a set of number states, $\{|n; t\rangle, n = 0, 1, 2, \ldots\}$, which are eigenfunctions of the number operator $A\dagger(t)A(t)$ [1, 2, 6]:

$$A\dagger(t)A(t)|n; t\rangle = n|n; t\rangle.$$  (10)

An important conceptual point for these time-dependent systems is that the number states, $|n; t\rangle$, are in general not eigenstates of the Hamiltonian. That is,

$$H|n; t\rangle \neq i\hbar|n; t\rangle \neq (\text{real constant})|n; t\rangle.$$  (11)

In particular, the extremal state $|0; t\rangle$ is technically not the ground state.

3.2. Uncertainty relations and squeezed states

The operators, $A$ and $A\dagger$, are linear functionals of $q$ and $p$. With them, generalized position and momentum operators can be defined:

$$Q = \frac{A + A\dagger}{\sqrt{2}}, \quad P = \frac{A - A\dagger}{i\sqrt{2}}.$$  (12)

Their associated commutation relation is

$$[Q,P] = iO = i[A, A\dagger]$$  (13)

where $O$ is a Hermitian operator.$\dagger$

Now consider the uncertainty product

$$U(Q, P) = (\Delta Q)^2(\Delta P)^2 = [(Q^2 - \langle Q \rangle^2)[(P^2 - \langle P \rangle^2]]$$  (14)

$$= [((Q - \langle Q \rangle)^2][(P - \langle P \rangle)^2]].$$  (15)

Applying the Schwartz inequality to equation (15) we have

$$\frac{1}{4}\{\{\hat{Q}, \hat{P}\}\}^2 \geq \frac{1}{4}|\langle \hat{Q} \rangle| + |\langle \hat{P} \rangle|^2$$  (16)

$$\geq \frac{1}{4}\langle \hat{Q} \rangle^2 + \frac{1}{4}\langle \hat{P} \rangle^2$$  (17)

$$\geq \frac{1}{4}\langle \hat{O} \rangle^2$$  (18)

\{ , \} being the anticommutator and

$$\hat{Q} \equiv Q - \langle Q \rangle, \quad \hat{P} \equiv P - \langle P \rangle.$$  (19)

$\dagger$ Because of equation (7), for the systems studied in this paper $O = I$, but for now we continue with the general $O$.  

New Journal of Physics 2 (2000) 18.1–18.9 (http://www.njp.org/)
Equation (17) is the Schrödinger uncertainty relation [7]. Equality is satisfied by states $|B, C\rangle$ that are collinear in $\hat{Q}$ and $\hat{P}$:

$$\hat{Q}|B, C\rangle = -iB\hat{P}|B, C\rangle$$  \hspace{1cm} (20)

where $B$ is a complex constant. Now going to wavefunction notation, for what will be the minimum-uncertainty squeezed states (MUSSs), equation (20) can be rewritten as [8, 9]

$$(Q + iBP)\Psi_{ss}(t) = C\Psi_{ss}(t)$$  \hspace{1cm} (21)

$$C \equiv \langle Q \rangle + iB\langle P \rangle.$$

(22)

In general, $B$ and $C$ are both complex. The solutions to equation (21) are ‘squeezed states’ for the system.

$B$ can be understood to be the complex squeeze factor by (i) multiplying equation (20) on the left first by $\hat{Q}$ and then by $\hat{P}$ and taking the expectation values, (ii) then doing the same for the adjoint equation (multiplying on the right) and (iii) finally using equation (15). This yields

$$B = i\frac{(\Delta Q)^2}{\langle \hat{P}\hat{Q} \rangle} = -i\frac{\langle \hat{Q}\hat{P} \rangle}{\langle \hat{P}\hat{Q} \rangle}$$  \hspace{1cm} (23)

$$B^* = -i\frac{(\Delta Q)^2}{\langle \hat{P}\hat{Q} \rangle} = -i\frac{\langle \hat{Q}\hat{P} \rangle}{\langle \hat{P}\hat{Q} \rangle}$$  \hspace{1cm} (24)

$$|B|^2 = \frac{(\Delta Q)^2}{(\Delta P)^2}.$$

(25)

Thus, $|B|$ yields the relative uncertainties of $Q$ and $P$.

For the particular case $B = 1$, the MUSSs are the minimum-uncertainty coherent states (MUCSs). These satisfy equality of expression (18), the Heisenberg uncertainty relation. That these are coherent states is easily seen from equation (21), which then reduces to $\sqrt{2}$ times the LOCS equation (9).

To intuitively see this, consider equation (21) for the simple harmonic oscillator. The solutions are

$$\psi_{ss}(q) \propto \exp\left(-\frac{1}{2}\frac{(q - \langle q \rangle)^2}{B} - i\langle p \rangle q\right).$$  \hspace{1cm} (26)

$B$ is an extremely complicated functional [10, 11] of the complex parameter $\lambda$ of the standard $su(1, 1)$ squeeze operator†:

$$S_a(\lambda(t)) = \exp\left[\frac{1}{2}\lambda(t)a^\dagger a - \frac{1}{2}\lambda^*(t)aa\right] \quad \lambda(t) \equiv r(t)e^{i\theta(t)}.$$  \hspace{1cm} (27)

The exact relationship depends on whether one defines the displacement-operator squeezed states (DOSSs) as

$$|\alpha, \lambda; t\rangle = D_A(\alpha)S_a(\lambda)|0\rangle$$  \hspace{1cm} (28)

or as $S_a(\lambda)D_A(\alpha)|0\rangle$.

We will now describe this formalism and then apply it to a well-known and important system, the Paul trap. The aim is to obtain a deeper insight into the ‘coherent’ and ‘squeezed’ states of this system.

† Here, we can allow $\lambda = \lambda(t)$; but note that, if, in contrast to the present case, the operators defining the squeeze operator have time derivatives (e.g. the $su(1, 1)$ operators $M_-$ and $M_+$ of [3]), then this possibility leads to time-derivative complications and an inequivalent result.
4. The Paul trap

The Paul trap is a dynamically stable environment for charged particles [12]–[14]. It has been of great use in areas from quantum optics to particle physics.

Its main structure consists of two parts. The first is an annular ring–hyperboloid of revolution, whose symmetry is about the \( x-y \) plane at \( z = 0 \). The distance from the origin to the ring focus of the hyperboloid is \( r_0 \). The inner surface of this ring electrode is a time-dependent electrical equipotential surface. The second part of the structure consists of two end caps. These are hyperboloids of revolution about the \( z \) axis. The distance from the origin to the two foci is usually \( d_0 = r_0/\sqrt{2} \). The two end-cap surfaces are time-dependent equipotential surfaces with sign opposite to that of the ring. The electrical field within this trap is a quadrupole field. When oscillatory potentials are applied, a charged particle can be dynamically stable.

Paul gives a delightful mechanical analogy [14]. Think of a mechanical ball put at the centre of a saddle surface. With no motion of the surface, it will fall off the saddle. However, if the saddle surface is rotated with an appropriate frequency about the axis normal to the surface at the inflection point, the particle will be stably confined. The particle is oscillatory about the origin both in the \( x \) and in the \( y \) direction. However, its oscillation in the \( z \) direction is restricted to be bounded from below by some \( z_0 > 0 \).

The potential energy can be parametrized as [12]

\[
V(x, y, z, t) = V_x(x, t) + V_y(y, t) + V_z(z, t)
\]

where

\[
V_x(x, t) = +\frac{e}{2r_0^2} V(t)x^2 \equiv \frac{1}{2}\Omega_x(t)x^2
\]

\[
V_y(y, t) = +\frac{e}{2r_0^2} V(t)y^2 \equiv \frac{1}{2}\Omega_y(t)y^2
\]

\[
V_z(z, t) = -\frac{e}{r_0} V(t)z^2 \equiv \frac{1}{2}\Omega_z(t)z^2
\]

In the above,

\[
V(t) = V_{dc} - V_{ac}\cos[\omega(t-t_0)]
\]

is the ‘dc’ plus ‘ac time-dependent’ electrical potential that is applied between the ring and the end caps. These potentials can be used to solve the classical motion problem. The result is oscillatory Mathieu functions for the bound case [12]. The oscillatory motion goes both positive and negative in the \( x-y \) plane, but is constrained to be positive in the \( z \) direction.

Exact solutions for the quantum case were first investigated in detail by Combescure [15]. In general, work has concentrated on the \( z \) coordinate, but not entirely [16]. Elsewhere [17] we will look at the symmetries, separations of variables and the number and coherent state solutions of the three-dimensional Paul trap, both in Cartesian and cylindrical coordinates.

5. Coherent physics of the Paul trap

5.1. Coherent states and the classical motion

With the background established in the previous two sections, we want to discuss the coherent/squeezed states of the Paul trap. We focus on the interesting \( z \) coordinate and, in
particular, use as reference the lovely discussion in Schrade et al. [18]. Using equation (32), the Hamiltonian is
\[ H = i\dot{\alpha} = -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{1}{2} \Omega_z(t)z^2. \] (34)

The connection between the notation of section 2 and that of Schrade et al. [18] is
\[ q = z \quad 2f = \Omega_z \quad \xi(t) = \frac{1}{\sqrt{2}}\epsilon(t). \] (35)

The ladder operators, \(A(t)\) and \(A^\dagger(t)\), are
\[ A(t) = \frac{1}{\sqrt{2}[\epsilon(t)p - \dot{\epsilon}(t)z]} = \frac{1}{2}\{a[\epsilon(t) - i\dot{\epsilon}(t)] + a^\dagger[-\epsilon(t) - i\dot{\epsilon}(t)]\} \] (36)
\[ A^\dagger(t) = -\frac{i}{\sqrt{2}[\epsilon^*(t)p - \dot{\epsilon}^*(t)z]} = \frac{1}{2}\{a^\dagger[\epsilon^*(t) + i\dot{\epsilon}^*(t)] + a[-\epsilon^*(t) + i\dot{\epsilon}^*(t)]\}. \] (37)

Recall that, because of equation (35), \(\epsilon(t)\) is a complex solution to the differential equation (5) and its complex conjugate, \(\epsilon^*(t)\), is the other linearly independent solution. The Wronskian of \(\epsilon(t)\) and \(\epsilon^*(t)\) is a constant:
\[ W(\epsilon, \epsilon^*) = \epsilon(t)\dot{\epsilon}^*(t) - \dot{\epsilon}(t)\epsilon^*(t) = -2i. \] (38)

Note from equation (32), that Newton’s classical equation of motion is
\[ F = \ddot{z}_c(t) = -\frac{dV(z,t)}{dz} = \frac{2e}{r_0^2}Y(t)z \equiv -\Omega_z(t)z_c(t) \] (39)
the solutions being Mathieu functions. However, equation (35) shows that equation (39) is exactly the form of equation (5) for \(\epsilon(t)\) and \(\epsilon^*(t)\). This means that a combination of \(\epsilon(t)\) and \(\epsilon^*(t)\), up to normalization, follows the classical-motion Mathieu function solutions for \(z_c(t)\) and \(p_c(t) = \dot{z}_c(t)\). Therefore, we can conveniently and with foresight write these classical solutions in the forms most convenient for the quantum study:
\[ z_c(t) = \frac{i}{2}\{[\epsilon^*(t)\epsilon_0 - \epsilon(t)\epsilon_0^*]p_0 + [\epsilon(t)\dot{\epsilon}_0^* - \epsilon^*(t)\dot{\epsilon}_0]z_0\} \] (40)
\[ p_c(t) = \frac{i}{2}\{[\epsilon^*(t)\epsilon_0 - \dot{\epsilon}(t)\epsilon_0^*]p_0 + [\dot{\epsilon}(t)\dot{\epsilon}_0^* - \dot{\epsilon}^*(t)\dot{\epsilon}_0]x_0\} = \dot{z}_c(t) \] (41)
where \(z_0\) and \(p_0\) are initial position and momentum and \(\epsilon_0 = \epsilon(t_0)\).

Now, using equations (12), (36) and (37), with \(Q \rightarrow Z\),
\[ Z(t) = \frac{i}{2}\{[\epsilon(t) - \epsilon^*(t)]p - [\dot{\epsilon}(t) - \dot{\epsilon}^*(t)]z\} \] (42)
\[ = \frac{i}{\sqrt{2}}\{a[\text{Im}\dot{\epsilon}(t)] + \text{Im}\epsilon(t)] - a^\dagger[\text{Im}\dot{\epsilon}(t)] - \text{Im}\epsilon(t)]\} \] (43)
\[ P(t) = \frac{i}{2}\{[\epsilon(t) + \epsilon^*(t)]p - [\dot{\epsilon}(t) + \dot{\epsilon}^*(t)]z\} \] (44)
\[ = \frac{1}{i\sqrt{2}}\{a[\text{Re}\epsilon(t)] - \text{Re}\dot{\epsilon}(t)] - a^\dagger[\text{Re}\epsilon(t)] + \text{Re}\dot{\epsilon}(t)]\}. \] (45)

It follows that
\[ [Z(t), P(t)] = -\frac{i}{2}[\epsilon^*(t)\dot{\epsilon}(t) - \epsilon(t)\dot{\epsilon}^*(t)] = i \] (46)
where we have used the Wronskian (38).
Using equations (8), (36) and (37), one can calculate that the coherent-state wavefunctions for the Paul trap go as \[18\]

$$\Psi_{cs}(z,t) \sim \exp\left[-\frac{1}{2} \left(\frac{-i\dot{\epsilon}(t)}{\epsilon(t)}\right) (z - z_{cl}(t))^2 + izp_{cl}(t)\right].$$

(47)

The Gaussian form of \(\Psi\) can be readily verified [3, 4] by noting that (i)

$$\left(-i\dot{\epsilon}(t)/\epsilon(t)\right) = 1/\phi(t) \left(1 - \frac{i}{2} \dot{\phi}(t)\right)$$

and (ii) \(\phi(t) = \epsilon(t)\epsilon^*(t)\)

(48)

(49)

5.2. Uncertainty relations

However, with these wavefunctions, Schrade et al [15] found that the \(z-p\) Heisenberg uncertainty relation is not satisfied. Rather, the \(z-p\) Schrödinger uncertainty relation is satisfied [1, 2, 18, 19]:

$$(\Delta z)^2(\Delta p)^2 = \frac{1}{4} [1 + \frac{1}{4} \dot{\phi}^2(t)]$$

(50)

where \(\phi\) is given in equation (49)‡.

We observe that this is not only correct but also to be expected. These wavefunctions were generated by the \(Z-P\) variables and hence should satisfy the Heisenberg uncertainty relation for \(Z\) and \(P\). Contrariwise, note that equation (36) can be written as

$$A(t) = \frac{1}{2} \{a[\epsilon(t) - i\dot{\epsilon}(t)] + a^\dagger[-\epsilon(t) - i\dot{\epsilon}(t)]\} \equiv \mu a + \nu a^\dagger.$$ (51)

However, with equation (38) one has that

$$1 = |\mu|^2 - |\nu|^2.$$ (52)

That is, \(A\) and \(A^\dagger\) must be related to \(a\) and \(a^\dagger\) by a Holstein–Primakoff/Bogoliubov transformation [20, 21] of the form

$$S_a^{-1}(\lambda)aS_a(\lambda) = (\cosh r)a + (e^{i\theta} \sinh r)a^\dagger$$

(53)

$$S_a^{-1}(\lambda)zs_a(\lambda) = \frac{1}{\sqrt{2}} [(\cosh r + e^{-i\theta} \sinh r)a + (\cosh r + e^{i\theta} \sinh r)a^\dagger].$$

(54)

However, there remains one further complication. The coefficient of \(a\) on the right-hand side of equation (53) is real whereas that of equation (51) is complex. There is a phase offset. This is related to the fact that there are four parameters in equation (51) and only two in equation (53). That there are only two parameters in equation (53) follows from the fact that the entire squeezed-state transformation from \((z, p)\) to \((Z, P)\) also involves a displacement. (See equation (28).) The displacement supplies the other two parameters, as we now demonstrate.

† The function (48) also arises in number-operator states. The relevant equation is \(A(t)|0; t\rangle = 0\), when \(A(t)\) of equation (36) is used. This is the 0-eigenvalue case of equation (9).

‡ The right-hand side of equation (50) is equal to \(1/4\) for all \(t\) only if \(\dot{\phi} = 0\); that is, for a harmonic oscillator or a driven oscillator [19].
5.3. Squeezed states

Combine, in the $A$ representation, the DOCS definition of equation (8) with the LOCS definition of equation (9):

$$A(t)D_A(\alpha)|0; t\rangle = \alpha D_A(\alpha)|0; t\rangle. \quad (55)$$

Next, insert $I = S_a^{-1}(\lambda)S_a(\lambda)$ in front of all the operators and multiply on the left by $S_a(\lambda)$ of equation (27). On regrouping, we obtain the equation

$$(S_aAS_a^{-1})(S_aD_A(\alpha)S_a^{-1})S_a(\lambda)|0; t\rangle = \alpha(S_aD_A(\alpha)S_a^{-1})S_a(\lambda)|0; t\rangle. \quad (56)$$

Given equations (51) and (52) and using both the BCH relation [1, 2]

$$\exp \left( \frac{\lambda a^\dagger a - \lambda^* aa}{2} \right) = \exp \left( \gamma_+ \frac{a^\dagger a}{2} \right) \exp[\gamma_3(a^\dagger a + \frac{1}{2})] \exp \left( -\frac{aa}{2} \right) \quad (57)$$

$$\gamma_- = -e^{-i\theta} \tanh r \quad \gamma_+ = e^{i\theta} \tanh r \quad \gamma_3 = \ln \cosh r \quad (58)$$

and also the theorem [22]

$$\exp(X) \exp(Y) \exp(-X) = \exp(e^XY e^{-X}) \quad (59)$$

it follows that

$$S_a(\lambda)D_A(\alpha)S_a^{-1}(\lambda) = \exp(\beta a^\dagger - \beta^* a) \equiv D_a(\beta) \quad (60)$$

$$\beta = \alpha v^* - \alpha^* u \quad \beta^* = \alpha^* v - \alpha u^*. \quad (61)$$

In addition, we see that

$$S_a(\lambda)AS_a^{-1}(\lambda) = va + ua^\dagger \quad (62)$$

where the coefficients $u$ and $v$ are

$$u = \nu \cosh r - \mu e^{i\theta} \sinh r \quad v = \mu \cosh r - \nu e^{-i\theta} \sinh r. \quad (63)$$

Note that $u$ and $v$ are functions of $t$ because $\mu$ and $\nu$ are. Furthermore, we have

$$|\nu|^2 - |\mu|^2 = |\mu|^2 - |\nu|^2 = 1. \quad (64)$$

Therefore, equation (56) becomes†

$$\{va + ua^\dagger\}D_a(\beta)S_a(\lambda)|0; t\rangle = \{\alpha(\lambda, \beta)\}D_a(\beta)S_a(\lambda)|0; t\rangle = \{\alpha(\lambda, \beta)\}|\alpha(\lambda, \beta), t\rangle. \quad (65)$$

This satisfies both the displacement-operator definition of squeezed states (to the right of the brackets) and the ladder-operator definition of squeezed states (the curly brackets) [3]. This completes the transformation of a coherent state in the $(A, A^\dagger)$ representation into a squeezed state in the $(a, a^\dagger)$ representation. Since the squeezing operator is invertible, we can obviously reverse this process‡.

† In equation (65), $\alpha$ is indeed the quantity of equation (55). However, because of our construction, it can be obtained from equations (61) and (63) as a functional of $\lambda$ and $\beta$.

‡ Because of the isomorphism of the two Heisenberg–Weyl algebras, $\{a, a^\dagger, I\}$ and $\{A(t), A^\dagger(t), I\}$, the converse of this result follows. Starting with a coherent-state equation in the $(a, a^\dagger)$ representation, $aD_a(\eta)|0\rangle = \eta D_a(\eta)|0\rangle$, where $\eta$ is complex, we can map this equation into an analogous squeezed-state equation in the $(A, A^\dagger)$ representation with an appropriate squeezing operator in the $(A, A^\dagger)$ representation. This is done by going through a procedure analogous to that above with the aid of the equations $a = \rho A + \sigma A^\dagger$, $\rho = \frac{1}{2}(e^*(t) - i\epsilon^*(t))$ and $\sigma = \frac{1}{2}(\epsilon(t) - i\epsilon(t))$ and the condition $|\sigma|^2 - |\rho|^2 = 1$.
6. Conclusion

The coherent states of \((Z, P)\) or \((A, A^\dagger)\) are squeezed states of \((z, p)\) or \((a, a^\dagger)\) and vice versa. This makes sense. The fundamental potentials are of different widths, so their coherent-state Gaussians are also of different widths. Indeed, equation (47) shows this. For the \((z, p)\) uncertainty relation, \([-\dot{\epsilon}(t)/\epsilon(t)]\) is a squeeze factor.

Acknowledgments

MMN acknowledges the support of the United States Department of Energy and the Alexander von Humboldt Foundation. DRT acknowledges a grant from the Natural Sciences and Engineering Research Council of Canada.

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