CLASSIFICATION OF RANK 2 CLUSTER VARIETIES

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Abstract. We classify rank 2 cluster varieties (those whose corresponding skew-form has rank 2) according to the deformation type of a generic fiber $U$ of their $\mathcal{X}$-spaces, as defined by Fock and Goncharov. Our approach is based on the work of Gross, Hacking, and Keel for cluster varieties and log Calabi-Yau surfaces. We find, for example, that $U$ is “positive” (i.e., nearly affine) and either finite-type or non-acyclic (in the cluster sense) if and only if the monodromy of the tropicalization of $U$ is one of Kodaira’s matrices for the monodromy of an elliptic fibration. In the positive cases, we also describe the action of the cluster modular group on the tropicalization of $U$.

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1. Introduction

[FG09] defines a class of schemes, called cluster varieties, whose rings of global regular functions are upper cluster algebras. [GHK13a] describes how to view cluster varieties as certain blowups of toric varieties. We review this description, as well as [GHK11]’s construction of the tropicalization of a log Calabi-Yau surface. We then use these ideas to give a classification of rank 2 cluster varieties (those for which the symplectic leaves of the $\mathcal{X}$-space are 2 dimensional) and to describe their cluster modular groups.

By a log Calabi-Yau surface or a Looijenga interior, we mean a surface $U$ which can be realized as $Y \setminus D$, where $Y$ is a smooth, projective, rational surface over an algebraically closed field $k$ of characteristic 0, and the boundary $D$ is a choice of nodal anti-canonical divisor in $Y$. $D = D_1 + \ldots + D_n$ is either a cycle of smooth irreducible rational curves $D_i$ with normal crossings, or if $n = 1$, $D$ is an irreducible curve with one node. By a compactification of $U$, we mean such a pair $(Y, D)$ ([GHK] calls these compactifications with “maximal boundary”). We call $(Y, D)$ a Looijenga pair, as in [GHK11]. Every such $U$ can be obtained by performing certain blowups on a toric surface, cf. Lemma 2.9.

1.1. Outline of the Paper.

Cluster Varieties: [F] reviews [FG09]’s definition of cluster varieties and summarizes [GHK13a]’s description of cluster varieties as certain blowups of toric varieties (up to codimension 2). In particular, we review §5 of [GHK13a], which shows that log Calabi-Yau surfaces are roughly the same as fibers
of rank 2 cluster $\mathcal{X}$-varieties. Our classification of cluster varieties will up to deformation classes of these associated log Calabi-Yau surfaces. In §2.6 and §2.7 we review [FG09]’s definitions of the cluster complex $\mathcal{C}$ and the cluster modular group $\Gamma$.

The Tropicalization of $U$: In §3 we review [GHK11]’s construction of the tropicalization $U^\text{trop}$ of a log Calabi-Yau surface. $U^\text{trop}$ is homeomorphic to $\mathbb{R}^2$, but it has a natural integral linear structure that captures the intersection data of the boundary divisors. The integer points $U^\text{trop}(\mathbb{Z}) \subset U^\text{trop}$ generalize the cocharacter lattice $N$ for toric varieties in that they correspond to multiples of boundary divisors for certain compactifications of $U$. $U^\text{trop}$ itself generalizes $N_{\mathbb{R}} := N \otimes \mathbb{R}$.

The integral linear structure is singular at a point $0 \in U^\text{trop}$, and in §3.5 we examine the monodromy around this point. In §3.6, we discuss properties of lines in $U^\text{trop}$. For example, the monodromy in $U^\text{trop}$ may make it possible for lines to wrap around the origin and self-intersect. §3.7 introduces some automorphisms of $U^\text{trop}$ that we will see in §5 are induced by the action of $\Gamma$. In §3.8 we review some lemmas from [Man14] which will be useful for the classification in §4.

§3.9 shows that, although $U^\text{trop}$ does not in general determine the deformation type of $U$, it does at least determine the charge of $U$, which is the number of “non-toric blowups” necessary to realize a compactification of $U$ as a blowup of a toric variety.

Classification: §4 offers several equivalent classifications of rank 2 cluster varieties, or rather, of the deformation types of the log Calabi-Yau surfaces $U$ that arise as the fibers of cluster $\mathcal{X}$-varieties. The characterizations are based on several different properties of these varieties, including (but not limited to):

- The properties of the quivers associated to the cluster variety—e.g., Dynkin (finite-type), acyclic, or non-acyclic.
- The space of global regular functions on $U$—e.g., all constant, or including some, all, or no cluster $\mathcal{X}$-monomials.
- The intersection data of the boundary $D$ for a compactification of $U$—e.g., whether $(D_i \cdot D_j)$ is negative (semi)definite or not. We call the cases which are not negative semidefinite positive, as in [GHK11].
- The geometry of $U^\text{trop}$, including the monodromy and properties of lines.
- The intersection form $Q$ on the lattice $D^\perp \subset A_1(Y, \mathbb{Z})$ of curve classes which do not intersect any component of $D$.
- The intersection of the cluster complex (a subset of $\mathcal{X}^\text{trop}$) with $U^\text{trop}$—e.g., some, all, or none of $U^\text{trop}$.

For example, we find that $U$ corresponds to an “acyclic” cluster variety if and only if some straight lines in $U^\text{trop}$ do not wrap all the way around the origin. The cases where no lines wrap correspond to “finite-type” cluster varieties. We show that the inverse monodromies of $U^\text{trop}$ in these finite-type cases are Kodaira’s monodromy matrices $I_n$, $II$, $III$, and $IV$, from his classification of singular fibers in elliptic surfaces in [Kod63]. Similarly, the non-acyclic positive cases correspond to Kodaira’s matrices $I^*_n$, $II^*$, $III^*$, and $IV^*$—furthermore, the intersection form $Q$ on $D^\perp$ here is of type $D_{n+4}$ ($n \geq 0$) or $E_n$, $n = 8, 7, 6$, respectively (cf. Table 1). The deformation types for these cases are uniquely determined by $U^\text{trop}$, and we describe how to construct each of these cases explicitly.

Cluster Modular Groups: [FG09] defines a certain group $\Gamma$ of automorphisms of cluster varieties, called the cluster modular group. In §5 we explicitly describe the action of $\Gamma$ on $U^\text{trop}$ in all the positive cases (cf. Table 3). This action is interesting because, in addition to capturing most of
the relevant data about $\Gamma$, it preserves the scattering diagram which $[\text{GHK}11]$ and $[\text{GHKK}]$ use to construct canonical theta functions on the mirror. Symmetries of the scattering diagram induced by mutations were previously observed in Theorem 7 of $[\text{GP}09]$, although they did not put this in the language of cluster varieties or describe the full groups of automorphisms induced in this way.

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2. Cluster Varieties as Blowups of Toric Varieties

In $[\text{FG}09]$, Fock and Goncharov construct spaces called cluster varieties by gluing together algebraic tori via certain birational transformations called mutations. $[\text{GHK}13]$ interprets these mutations from the viewpoint of birational geometry, and thereby relates the log Calabi-Yau surfaces of $[\text{GHK}11]$ to cluster varieties. This section will summarize some of the main ideas from $[\text{GHK}13]$. 

2.1. Defining Cluster Varieties. The following construction is due to Fock and Goncharov $[\text{FG}09]$.

Definition 2.1. A seed is a collection of data

$$S = (N, I, E := \{e_i\}_{i \in I}, F, \langle \cdot, \cdot \rangle, \{d_i\}_{i \in I}),$$

where $N$ is a finitely generated free Abelian group, $I$ is a finite set, $E$ is a basis for $N$ indexed by $I$, $F$ is a subset of $I$, $\langle \cdot, \cdot \rangle$ is a skew-symmetric $\mathbb{Q}$-valued bilinear form, and the $d_i$’s are positive rational numbers called multipliers. We call $e_i$ a frozen vector if $i \in F$. The rank of a seed or of a cluster variety will mean the rank of $\langle \cdot, \cdot \rangle$.

We define another bilinear form on $N$ by

$$(e_i, e_j) := e_{ij} := d_j(e_i, e_j),$$

and we require that $e_{ij} \in \mathbb{Z}$ for all $i, j \in I$. Let $M = N^*$. Define

$$p_i^* : N \to M, \quad v \mapsto (v, \cdot), \quad p_i^* : N \to M, \quad v \mapsto (\cdot, v).$$

Let $K_i := \ker(p_i^*), N_i := \im(p_i^*) \subseteq M, \overline{e_i} := p_i^*(e_i)$, and $v_i := p_i^*(\epsilon_i)$. For each $i \in I$, define a “modified multiplier” $d_i'$ by saying that $v_i$ is $d_i'$ times a primitive vector in $M$.

Remark 2.2. Given only the matrix $(e_i, e_j)$ and the set $F$, we can recover the rest of the data, up to a rescaling of $\langle \cdot, \cdot \rangle$ and a corresponding rescaling of the $d_i$’s. This rescaling does not affect the constructions below, and it is common take the scaling out of the picture by assuming that the $d_i$’s are relatively prime integers (although we do not make this assumption). Also, notice that $\langle \cdot, \cdot \rangle$ and $\{d_i'\}$ together determine $\{d_i\}$, so when describing a seed we may at times give $\{d_i'\}$ instead of $\{d_i\}$.

Observations 2.3.

- $K_i$ is also equal to $\ker(v \mapsto \langle v, \cdot \rangle)$, so $\langle \cdot, \cdot \rangle$ induces non-degenerate skew-symmetric form on $N_i$. This also means that we could have equivalently defined the rank to be that of $\langle \cdot, \cdot \rangle$.

1 The construction of cluster varieties does not depend on the values of $(e_i, e_j)$ or $e_{ij}$ for $i, j \in F$, and so it is common to not include these coefficients in the data. When they are included in the data, as in $[\text{FG}09]$ and $[\text{GHK}13]$, they are not typically required to be integers. However, as $[\text{GHK}13]$ points out, if these are not integers, then the image of $p_i^*$ is not contained in $M$. $[\text{GHK}13]$ takes a slightly different fix to this (in which the $e_{ij}$ with $i, j \in F$ are again irrelevant), but it is essentially equivalent to our fix if we dropped the requirement that $(e_i, e_j) = -(e_j, e_i)$ when $i, j \in F$.  

• Define another skew-symmetric bilinear form on $N$ by $[e_i, e_j] := d_i d_j (e_i, e_j)$. Then $K_2 = \ker (v \mapsto [\cdot, v])$, so $[e_i, e_j]$ induces a non-degenerated skew-symmetric form on $\overline{N}_2$. We can extend this to $\overline{N}^{\text{stat}}_2$ (the saturation in $M$ of $\overline{N}_2$), and after possibly rescaling $[\cdot, \cdot]$ (and adjusting the $d_i$’s accordingly) we can identify this with the standard skew-symmetric form on $\overline{N}^{\text{stat}}_2$ with the induced orientation. We will denote this form and the induced symplectic form on $\overline{N}_{2,R}$ by $(\cdot, \cdot)$. Here and in the future, $\mathbb{R}$ in the subscript means the lattice tensored with $\mathbb{R}$.

• We note that the seed obtained from $S$ by replacing $(\cdot, \cdot)$ with $-[\cdot, \cdot]$ and $d_i$ with $d_i^{-1}$ produces the Langland’s dual seed $S'$ described in [FG09]. Switching to $S'$ essentially has the effect of switching the roles of (and negating) $p_i^*$ and $\overline{p}_i$. We also note that $p_2^*$ is the dual map to $p_1$.

• Since $(\cdot, e_i) = -d_i (e_i, \cdot)$, we see that $\text{im}(p_2^*)$ and $\text{im}(v \mapsto \langle v, \cdot \rangle)$ span the same subspace of $M_\mathbb{R}$. Thus, there is a canonical isomorphism $\overline{N}^{\text{stat}}_{2,R} \cong \overline{N}_{1,R}$. We easily see that this is a symplectomorphism with respect to the symplectic forms induced by $[\cdot, \cdot]$ and $(\cdot, \cdot)$.

Given a seed $S$ as above and a choice of non-frozen vector $e_j \in E$, we can use a mutation to define a new seed $\mu_j(S) := (N, I, E' = \{e_i\}_{i \in I}, F, (\cdot, \cdot), \{d_i\})$, where the $(e_i')$’s are defined by

$$(1) \quad e'_i = \mu_j(e_i) := \begin{cases} 
  e_i + e_i e_j & \text{if } e_{ij} > 0 \\
  -e_i & \text{if } i = j \\
  e_i & \text{otherwise.}
\end{cases}$$

Mutation with respect to frozen vectors is not allowed. Note that although the bases change, the form $(\cdot, \cdot)$ does not, so $K_1$ and $\overline{N}^{\text{stat}}_1$ are invariant under mutation. The same is true for $K_2$ and $\overline{N}^{\text{stat}}_2$, as can similarly be seen using the Langland’s dual seed and $[\cdot, \cdot]$—one can check that the procedure for obtaining $S'$ from $S$ commutes with mutation.

Given a lattice $L$ and some $v \in L^*$, we will denote by $z^v$ the corresponding monomial on $T_L := L \otimes \mathbb{k}^* = \text{Spec } \mathbb{k}[L^*]$ (more precisely, max-Spec of $\mathbb{k}[L^*]$). Corresponding to a seed $S$, we can define a so-called seed $\mathcal{X}$-torus $X_S := T_M = \text{Spec } \mathbb{k}[N]$, and a seed $\mathcal{A}$-torus $A_S := T_N = \text{Spec } \mathbb{k}[M]$. We define cluster monomials $X_i := z^{e_i} \in \mathbb{k}[N]$ and $A_i := z^{e_i} \in \mathbb{k}[M]$, where $\{e_i\}_{i \in I}$ is the dual basis to $E$.

**Remark 2.4.** We are departing somewhat from a common convention. In place of $M$, other authors typically use the superlattice $(M)^\circ \subset M \otimes \mathbb{Q}$ spanned over $\mathbb{Z}$ by vectors $f_i := d_i e_i^\circ$. They then take $A_i := (z^{f_i}) \in \mathbb{k}[M^\circ]$. It seems to this author that this significantly complicates the exposition and the formulas that follow, with little benefit, and so we do not follow this convention.

For any $j \in I$, we have a birational morphism $\mu_j^\mathcal{X} : X_S \rightarrow X_{\mu_j(S)}$ (called a cluster $\mathcal{X}$-mutation) defined by

$$(\mu_j^\mathcal{X})^* X_i' = X_i \left(1 + X_j^{\text{sign}(-e_{ji})} \right)^{-e_{ij}} \quad \text{for } i \neq j; \quad (\mu_j^\mathcal{X})^* X_j' = X_j^{-1}.$$

Similarly, we can define a cluster $\mathcal{A}$-mutation $\mu_j^\mathcal{A} : A_S \rightarrow A_{\mu_j(S)}$,

$$A_j (\mu_j^\mathcal{A})^* A_j' = \prod_{i : e_{ji} > 0} A_j^{e_{ji}} + \prod_{i : e_{ji} < 0} A_j^{-e_{ji}}; \quad (\mu_j^\mathcal{A})^* A_i' = A_i \quad \text{for } i \neq j.$$

Now, the cluster $\mathcal{X}$-variety $\mathcal{X}$ is defined by using compositions of $\mathcal{X}$-mutations to glue $A_{\mu_j(S)}$ to $X_S$ for every seed $S'$ which is related to $S$ by some sequence of mutations. Similarly for the cluster $\mathcal{A}$-variety $\mathcal{A}$, with $\mathcal{A}$-tori and $\mathcal{A}$-mutations. The cluster algebra is the subalgebra of $\mathbb{k}[M]$ generated by the the cluster variables $A_i$ of each seed that we can get to by some sequence of mutations. In this context,
the well-known Laurent phenomenon simply says that all the cluster variables are regular functions on $\mathcal{A}$.<sup>[GHK13a]</sup> uses this observation to give a simple geometric proof of the Laurent phenomenon. The ring of all global regular functions on $\mathcal{A}$ is called the <i>upper cluster algebra</i>. On the other hand, the $X_i$'s do not always extend to global functions on $\mathcal{X}$. When a monomial on a seed torus (i.e., a monomial in the $X_i$'s for a fixed seed) does extend to a global function on $\mathcal{X}$, we call it a <i>global monomial</i>, as in <sup>[GHK13a]</sup>.

### 2.1.1. Quivers and Seeds

We now describe a standard way to represent the data of a seed with the data of a (decorated) quiver. Each seed vector $e_i$ corresponds to a vertex $V_i$ of the quiver. The number of arrows from $V_i$ to $V_j$ is equal to $\langle e_i, e_j \rangle$, with a negative sign meaning that the arrows actually go from $V_j$ to $V_i$. Each vertex $V_j$ is decorated with the number $d_i$. Furthermore, the vertices corresponding to frozen vectors are boxed. Observe that all the data of the seed can be recovered from the quiver.

Now, a seed is called <i>acyclic</i> if the corresponding quiver contains no directed paths that do not pass through any frozen (boxed) vertices. A cluster variety is called acyclic if any of the corresponding seeds are acyclic. It is easy to see that a seed $S$ is acyclic if and only if there is some closed half-plane in $\mathbb{R}^2$ which contains $v_i$ for every $i \in I \setminus F$.

### 2.2. The Geometric Interpretation

As in <sup>[GHK13a]</sup>, for a lattice $L$ with dual $L^*$ and with $u \in L$, $\psi \in L^*$, define

$$m_{u,\psi,L} : T_L \dashrightarrow T_L$$

$$m_{u,\psi,L}^*(z^\psi) = z^\psi (1 + z^\psi)^{-\psi(u)} \quad \text{for } \varphi \in L^*.$$

One can check that the mutations above satisfy

$$\mu_j^X = m_{(\cdot, e_j), e_j, M}^* : z^u \mapsto z^u (1 + z^\varphi)^{-\varphi(u, e_j)}$$

$$\mu_j^A = m_{e_j, (\cdot, e_j), N}^* : z^\gamma \mapsto z^\gamma (1 + z^{(\cdot, e_j)})^{-\gamma(e_j)}.$$

The following Lemma, compiled from §3 of <sup>[GHK13a]</sup>, is what leads to the nice geometric interpretations of mutations and cluster varieties.

**Lemma 2.5**<sup>[GHK13a]</sup>. Suppose that $u$ is primitive in a lattice $L$. Let $\Sigma$ be a fan in $L$ with rays corresponding to $u$ and $-u$. Recall that the toric variety $TV(\Sigma)$ admits a $\mathbb{P}^1$ fibration $\pi$ with $D_u$ and $D_{-u}$ as sections, corresponding to the projection $L \to L/\mathbb{Z}(u)$.

The mutation $\mu_{u,\psi,L}$ is the birational map on $T_L \subset TV(\Sigma)$ coming from blowing up the “hypertorus”

$$H^+ := \{1 + z^\psi = 0\} \cap D_u$$

and then contracting the proper transforms of the fibers $F$ of $\pi$ which intersect this hypertorus. Furthermore, $\mu_j^X$ (and under certain conditions, $\mu_j^A$) preserve the centers of the blowups corresponding to $\mu_i^X$ (and, respectively, $\mu_i^A$) for each $i \neq j$.

Thus, a cluster $X$-mutation ($\mu_j^X$) corresponds to blowing up $\{X_j = -1\} \cap D_{(\cdot, e_j)}$, followed by blowing down some fibers of a certain $\mathbb{P}^1$ fibration, and repeating for a total of $d_j^i$ times (since $(\cdot, e_j)$ is $d_j^i$ times a primitive vector, and $m_{(\cdot, e_j), e_j, M} = [m_{(\cdot, e_j), d_j^i e_j, M}]^{d_j^i}$). The new seed torus is only different from the old one in that it is missing the blown-down fibers of the initial $\mathbb{P}^1$ fibration, but has gained
Figure 2.1. A mutation involves blowing up a hypertorus $H^+$ in $D_u$ (left arrow) and then contracting the proper transform $\tilde{F}$ of the fibers $F$ which hit $H^+$ (right arrow), down to a hypertorus $H^-$ in $D_{-u}$. $\tilde{E}$ denotes the exceptional divisor, with $E$ being its image after the contraction of $\tilde{F}$. The locus $p = \tilde{E} \cap \tilde{F}$ has codimension $2$ and does not appear in the cluster variety.

the exceptional divisor from the final blowup (except for the lower-dimensional set of points where this exceptional divisor intersects a blown-down fiber, represented by $p$ in Figure 2.1).

Since the centers of the blowups corresponding to the other mutations have not changed, this shows that the cluster $X$-variety can be constructed (up to codimension $2$) as follows: For any seed $S$, take a fan in $M$ with rays generated by $\pm(\cdot, e_i)$ for each $i$, and consider the corresponding toric variety. For each $i \in I \setminus F$, blow up the hypertorus $\{X_i = -1\} \cap D_{(\cdot, e_i)}$ $d'_i$ times, and then remove the first $(d'_i - 1)$ exceptional divisors. The cluster $X$ variety is then the complement of the proper transform of the toric boundary.

Remark 2.6. In this construction of $X$, the centers for the hypertori we blow up may intersect if $(\cdot, e_i) = (\cdot, e_j)$ for some $i \neq j$, so some care must be taken regarding the ordering of the blowups. Fortunately, this issue only matters in codimension at least $2$ (cf. [GHK13a] for more details). However, when we consider fibers of $X$ below, it is possible that some special fibers will have discrepancies in codimension $1$. We will use the notation $X^{ft}$ to denote that we are restricting to the variety constructed as above for some fixed ordering of the blowups, and keep in mind that while $X \setminus X^{ft}$ is codimension $2$ in $X$, there may be special fibers of $X$ whose intersection with $X \setminus X^{ft}$ is codimension $1$ in the fiber. As we will see below, $A$ is a torsor over what is perhaps the “most special” fiber of $X$. The failure of mutations to preserve the centers of blowups for $A$ may be viewed as a consequence of such codimension $1$ discrepancies in the special fiber.

Remark 2.7. We have seen that codimension $2$ issues arise as a result of missing points like $p$ in Figure 2.1 and also as a result of reordering the blowups. There are also missing contractible complete subvarieties—the $(d'_j - 1)$ exceptional divisors we remove when applying $(\mu_X^X)^*$.

These issues are relatively unimportant, since they do not affect the sheaf of regular functions on $X$. When we are interested in $X$ or its fibers up to these issues, we will say “up to irrelevant loci.”
2.3. The Cluster Exact Sequence. Observe that for each seed $S$, there is a not necessarily exact sequence

$$0 \rightarrow K_2 \rightarrow N \overset{p_2}{\rightarrow} M \rightarrow K_1^+ \rightarrow 0.$$  

Here, $M \rightarrow K_1^+$ is the map dual to the inclusion $K_1 \hookrightarrow N$. Tensoring with $k^*$ yields an exact sequence, and one can check (cf. Lemma 2.10 of [FG09]) that this sequence commutes with mutation. Thus, one obtains the exact sequence

$$1 \rightarrow \mathcal{H}_A \rightarrow A \overset{p_3}{\rightarrow} \mathcal{X} \xrightarrow{\lambda} \mathcal{H}_X \rightarrow 1,$$

where $\mathcal{H}_A := T_{K_2}$, and $\mathcal{H}_X := T_{K_1^+}$. Let $\mathcal{U} := p_2(A) \subset \mathcal{X}$. The sequence $1 \rightarrow \mathcal{H}_A \rightarrow A \rightarrow \mathcal{U} \rightarrow 1$ should be viewed as a generalization of the construction of toric varieties as quotients, with $\mathcal{U}$ being the generalization of the toric variety. In fact, Section 4 of [GHK13a] shows that the ring of global sections of $\mathcal{A}$ is (under certain assumptions) the Cox ring of $\mathcal{U}$. In this paper, we are more interested in the fibers of $\lambda$.

2.4. Looijenga Interiors. §5 of [GHK13a] shows that Looijenga interiors (i.e., log Calabi-Yau surfaces), as defined in [11], are exactly the surfaces (up to irrelevant loci) which arise as fibers of $\lambda|_{\mathcal{X}^0}$ for rank 2 cluster varieties. We explain this now.

Definitions 2.8. For a Looijenga pair $(Y, D)$ as in [11] we define a toric blowup to be a Looijenga pair $(\tilde{Y}, \tilde{D})$ together with a birational map $\tilde{Y} \rightarrow Y$ which is a blowup at a nodal point of the boundary $D$, such that $\tilde{D}$ is the preimage of $D$. Note that taking a toric blowup does not change the interior $U = Y \setminus D = \tilde{Y} \setminus \tilde{D}$. We also use the term toric blowup to refer to finite sequences of such blowups.

By a non-toric blowup $(\tilde{Y}, \tilde{D}) \rightarrow (Y, D)$, we will always mean a blowup $\tilde{Y} \rightarrow Y$ at a non-nodal point of the boundary $D$ such that $\tilde{D}$ is the proper transform of $D$. Let $(\tilde{Y}, \tilde{D})$ be a Looijenga pair where $\tilde{Y}$ is a toric variety and $\tilde{D}$ is the toric boundary. We say that a birational map $Y \rightarrow \tilde{Y}$ is a toric model of $(Y, D)$ (or of $U$) if it is a finite sequence of non-toric blowups.

Lemma 2.9 ([GHK11], Prop. 1.19). Every Looijenga pair has a toric blowup which admits a toric model.

According to [GHK], all deformations of $U$ come from sliding the non-toric blowup points along the divisors $\tilde{D}_i \subset D$ without ever moving them to the nodes of $D$. We call $U$ positive if some deformation of $U$ is affine. This is equivalent to saying that $D$ supports an effective $D$-ample divisor, meaning a divisor whose intersection with each component of $D$ is positive. We will always take the term $D$-ample to imply effective. See [13] for equivalent characterizations of $U$ being positive.

To see that Looijenga interiors are the same as fibers of $\lambda|_{\mathcal{X}^0}$ for rank 2 cluster varieties, up to irrelevant loci, we will need the following lemma from [GHK13a].

Lemma 2.10 ([GHK13a], Lemma 5.1). The intersection of the zero set of $1 + \varepsilon_i$ with $D_{\phi}$ and $\lambda^{-1}(\phi)$ (some fiber of $\lambda$) consists of $[e_i]$ points, where $[e_i]$ is the index of $e_i := p_1^*(\phi) \in \mathbb{N}^1$ (i.e., $e_i$ is $[e_i]$ times a primitive vector in $\mathbb{N}^1$).

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2This sequence actually generalizes the construction for toric varieties without boundary (i.e., just algebraic tori). However, we expect to show in a future paper how to allow for boundary components by allowing partial compactifications of $\mathcal{A}$ and $\mathcal{U}$.

3If $k$ is not algebraically closed, Lemma 2.9 might not be true, but it at least holds for $e_i$ primitive in $\mathbb{N}^1$. 

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Now, in light of Lemmas 2.9 and 2.10 and the description of $X^{\text{ft}}$ in [GHK13a], it is clear that for $\langle \cdot, \cdot \rangle$ rank 2, every fiber of $\lambda|_{X^{\text{ft}}}$ is a Looijenga interior, up to irrelevant loci. For the converse, we use the following:

**Construction 2.11.** Following Construction 5.3 of [GHK13a], let $U$ be a Looijenga interior. Choose a compactification $(Y, D)$ admitting a toric model $\pi : (Y, D) \to (\overline{Y}, \overline{D})$. Let $N_{\overline{Y}}$ be the cocharacter lattice of $\overline{Y}$. Let $(\cdot, \cdot) : N_{\overline{Y}}^2 \to \mathbb{Z}$ denote the standard wedge form.

Suppose that $\pi$ consists of $d'_i$ non-toric blowups at a point $q_i \in \overline{D}_{u_i}$, $i = 1, \ldots, s$, where $\overline{D}_{u_i}$ is the divisor corresponding to the ray $\mathbb{R}_{\geq 0} u_i \subset N_{\overline{Y}}$, $u_i \in N_{\overline{Y}}$ primitive. We can assume that the $q_i$’s are distinct. We extend this to a set $\mathcal{E} := \{u_1, \ldots, u_s, u_{s+1}, \ldots, u_m\}$ of not necessarily distinct primitive vectors generating $N_{\overline{Y}}$, and we choose positive integers $d'_{s+1}, \ldots, d'_m$.

Now, let $S$ be the seed with $N$ freely generated by a set $E = \{e_1, \ldots, e_m\}$, $I = \{1, \ldots, m\}$, $F := \{s+1, \ldots, m\}$, $\{d'_1\}$ as above, and $\langle e_i, e_j \rangle := u_i \wedge u_j$. Note that we can identify $N_{\overline{Y}}^{\text{sat}}$ with $N_{\overline{Y}}$ via the identification $v_i = d'_i u_i$. Similarly, we can identify $N_1 \cong N/K_1$ with $N_{\overline{Y}}$ via the identification $\langle e_i, \cdot \rangle = u_i$. Thus, each $\mathfrak{e}_i$ is primitive in $\overline{N}_1$.

Using $S$ to construct $X$, the interpretation of $X$-mutations from [2,2] together with Lemma 2.10 reveals that $U$ is deformation equivalent to the generic fibers of $\lambda$, up to irrelevant loci. A bit more work shows that $U$ is in fact isomorphic to some such a fiber, up to irrelevant loci.

This construction shows that:

**Theorem 2.12 ([GHK13a]).** Up to irrelevant loci, every Looijenga interior can be identified with the generic fiber of some rank 2 cluster $X$-variety, and conversely, any generic fiber of a rank 2 cluster $X$-variety is a Looijenga pair.

**Example 2.13.** Consider the case where $Y$ is a cubic surface, obtained by blowing up 2 points on each boundary divisor of $(\overline{Y} \cong \mathbb{F}^2, \overline{D} = D_1 + D_2 + D_3)$. We can take

$$
\mathcal{E} = \{(1, 0), (0, 1), (1, 0), (1, 1), (0, 0), (0, -1), (0, 1), (0, -1)\},
$$

with each $d_i = d'_i = 1$ and $F$ empty. Then the fibers of the resulting $X$-variety $X_1$ correspond to the different possible choices of blowup points on the $D_i$’s. The fiber $\mathcal{U}$ is very special, having four $(-2)$-curves. If we instead take $\mathcal{E} = \{(1, 0), (0, 1), (0, -1)\}$ with $\langle \cdot, \cdot \rangle$ given by

$$
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix},
$$

and each $d_i = d'_i = 2$, then the fibers of the resulting $X$-variety $X_2$ include only the surfaces constructed by blowing up the same point twice on each $D_i$ and then removing the three resulting $(-2)$-curves. $\mathcal{U}$ is the fiber where the blowup points are colinear and so there is one remaining $(-2)$-curve.

The deformation type of the fibers of $X^{\text{ft}}$ has only changed by the removal of certain $(-2)$-curves, i.e., by some irrelevant loci. Note that $X_2^{\text{ft}} = X_2$, and that $X_2$ can be identified (after filling in the removed $(-2)$-curves) with a subfamily of $X_1^{\text{ft}}$ whose fibers do not agree with those of $X_1$ in codimension 1.

These examples are well-known: the former corresponds to the Teichmüller space of the four-punctured sphere, while the latter corresponds to the Teichmüller space of the once-punctured torus (cf. §2.7 of [FG09]).
Definition 2.14. Consider a seed $S$. Assume each $\pi_i$, $i \in I \setminus F$, is primitive in $\overline{N}_1$. If $i \neq j$ implies $v_i \neq v_j$, we call $S$ minimal (this means that $d'_i$ is the total number of non-toric blowups taken on the divisor corresponding to $v_i$). On the other hand, if each $d'_i = 1$, we will call $S$ maximal.

Two seeds $S_1$ and $S_2$ (along with the associated cluster varieties) will be called equivalent if the generic fibers of the corresponding $\mathcal{X}$-varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ are of the same deformation type, up to irrelevant loci (or equivalently, if the not necessarily generic fibers of $\mathcal{X}_1^{\text{ff}}$ and $\mathcal{X}_2^{\text{ff}}$ are of the same deformation type, up to irrelevant loci).

Example 2.13 above demonstrates that we can often change the number of vectors in a seed without changing the equivalence class of the fibers. For example, consider a seed $\{N = \mathbb{Z}(E), I, E = \{e_1, \ldots, e_m\}, F, \langle \cdot, \cdot \rangle, \{d_i\}\}$ with each $d_i = d'_i$ such that each $\pi_i$ is primitive in $\overline{N}_1$. Given a collection of partitions $d_i = d_{i,1} + \ldots + d_{i,b_i}$, $d_{i,j} \in \mathbb{Z}_{\geq 0}$, we can define a new seed $S'$ as follows: Let $E' := \{e_{i,j}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, b_i$, and $N' := \mathbb{Z}(E')$. Define $\langle e_{i_1,j_1}, e_{i_2,j_2}\rangle' := \langle e_{i_1,j_1}, e_{i_2,j_2}\rangle$. We say the pair $(i,j) \in F'$ if $i \in F$. Finally, $d_{i,j}$ is as in the partitions. The corresponding space $\mathcal{X}'$ is equivalent to the original $\mathcal{X}$. By this method, we can show that:

Proposition 2.15. Every rank 2 seed is equivalent to a minimal seed and to a maximal seed.

Example 2.16. The first seed for the cubic surface in Example 2.13 is maximal, while the second seed is minimal.

2.4.1. The Canonical Intersection Form. For $S$ a maximal rank 2 seed and $(Y, D)$ a corresponding Looijenga pair,\textsuperscript{4} describes a natural way to identify $K_2 := \ker(p_2^*)$ with $D^\perp := \{C \in A_1(Y, \mathbb{Z}) | C \cdot D_3 = 0 \setminus i\}$, thus inducing a canonical symmetric bilinear form $Q$ on $K_2$. This identification of $K_2$ with $D^\perp$ is as follows: an element $v := \sum a_i e_i$ of $K_2$ corresponds to a relation $\sum a_i v_i = 0$ in $\overline{N}^{\text{sat}}$, which we recall can be identified with $N_{\mathcal{Y}}$, where $Y \rightarrow \overline{Y}$ is a toric model corresponding to $S$. Standard toric geometry says that this determines a unique curve class $C_v$ in $\pi^*[A_1(\overline{Y})]$ such that $C_v \cdot D_i = \sum a_j$ for each $i$, where the sum is over all $j$ such that $D_{v_j} = D_i$. So we can define an isomorphism $\iota : K_2 \cong D^\perp$ by

$$v \mapsto C_v - \sum_i a_i E_i,$$

where $E_i$ is the exceptional divisor corresponding to mutating with respect to $e_i$.

Finally, for $u_1, u_2 \in K_2$, define $Q(u_1, u_2) = \iota(u_1) \cdot \iota(u_2)$. We will see in [4] that $D^\perp$ together with this intersection pairing tells us quite a bit about the deformation type of $U$. In particular,\textsuperscript{4} tells us that $U$ is positive if and only if $Q$ is negative definite.

Recall that varying the fiber of $\mathcal{X}$ corresponds to changing the choices of non-toric blowup points on $D$. For some choices of blowup points, certain classes $C$ in $D^\perp$ may be represented by effective curves. Let $D^\perp_{\text{eff}} \subseteq D^\perp$ be the sublattice generated by the curve classes which are represented by an effective curve on some fiber.

Example 2.17. For the seed from Example 2.13 $K_2$ is generated by $\{e_2 - e_1, e_4 - e_3, e_6 - e_5, e_1 + e_3, e_5\}$. The corresponding curves in $D^\perp$ are $\{E_1 - E_2, E_3 - E_4, E_5 - E_6, L - E_1 - E_3 - E_5\}$, where $E_i$ is the exceptional divisor of the blowup corresponding to $e_i$, and $L$ is a generic line in $\overline{\mathcal{Y}} \cong \mathbb{P}^2$.

\textsuperscript{4}Every rank 2 seed is equivalent to one with this primitivity condition because they all have Looijenga pairs as the fibers of their corresponding $\mathcal{X}^{\text{ff}}$. 
Using $E_i \cdot E_j = -\delta_{ij}, \ L \cdot L = 1, \text{ and } L \cdot E_i = 0$ for each $i$, one easily checks that this lattice has type $D_4$. On the special fiber $U$, these four curve classes are effective, so $D^\perp_{\text{Eff}} = D^\perp$.

2.5. Tropicalizations of Cluster Varieties. [FG09] describes tropicalizations $A^{trop}$ and $X^{trop}$ of the spaces $A$ and $X$, respectively. Given a seed $S$, $A^{trop}$ can be canonically identified as an integral piecewise-linear manifold with $N_{R,S}$, and the integral points $A^{trop}(\mathbb{Z})$ of the tropicalization are identified with $N_S$. For a different seed $\mu_j(S)$, the identification is related by the integral piecewise-linear function $\mu_j : N_R \to N_R$, where we use the overline to indicate that $e_j$ is mapped by the same piecewise-linear function as the other vectors, rather than getting a special treatment. Similarly for $X^{trop}$ and $X^{trop}(\mathbb{Z})$ using $M_{R,S}, M_S$, and the dual seed mutations. We will use the subscript $S$ to indicate that we are equipping the tropical space with the vector space structure corresponding to the seed $S$.

Our interest in this paper is primarily with the fibers $U$ of $\lambda$. $U^{trop}$ can be canonically identified with $N^2 \otimes \mathbb{R} = p_2^*(A^{trop}) \subset X^{trop}$. We will spend [3] analyzing $U^{trop}$ in the rank 2 cases. [GHK11] has shown that in these cases, $U^{trop}$ has a canonical integral linear structure which is closely related to the geometry of the compactifications $(Y,D)$.

2.6. The Cluster Modular Group. A seed isomorphism $h : S \to S'$ is an isomorphism of the underlying lattices which respects all the seed data in the obvious way. This induces a cluster isomorphism $h : A \to A'$ and $h : X \to X'$ given by $h^* X_{h(c_i)} = X_i$ and $h^* A_{h(c_i)} = A_i$, respectively, as well as an isomorphism from $\mathcal{U} := p_2(A) \subset \mathcal{X}$ to $\mathcal{U}' := p_2(A') \subset \mathcal{X}'$. A seed transformation is a composition of seed mutations and seed isomorphisms, and a cluster transformation is a composition of cluster mutations and cluster isomorphisms (i.e., the corresponding maps on the varieties). [FG09] defines the cluster modular group $\Gamma$ to be the group of cluster automorphisms of a base seed $S$, modulo trivial cluster automorphisms (those which are the identity on both $A$ and $X$, hence on $U$).

We also define an extended cluster modular group $\hat{\Gamma}$ by allowing seed isomorphisms to reverse the sign of the skew-symmetric form on $N$. For example, for toric varieties, $\Gamma$ can be thought of as the subgroup of $\text{SL}(N)$ which preserves the fan, whereas $\hat{\Gamma}$ can be thought of as the subgroup of $\text{GL}(N)$ preserving the fan.

As motivation, we note that $\Gamma$ and $\hat{\Gamma}$ induce automorphisms of $U^{trop}$ which preserve the canonical scattering diagram of [GHK11]. We will analyze this action on $U^{trop}$ in [3].

2.7. The Cluster Complex. A seed $S$ with seed vectors $e_1, \ldots, e_n$ determines a cone $C_S \subset X^{trop}_S := M_{R,S}$ given by $e_i \geq 0$ for all $i$. The collection of all such cones in $X$ for every seed mutation equivalent to $S$ is called the cluster complex $C$. [GHKK] shows that $C$ forms a fan in $X$. It is a particularly nice piece of the “scattering diagram” that they use for constructing canonical theta functions on the “mirror” to $X$.

Note that a wall $W_i \subset e_i^\perp$ in some $C_S \subset C$ has a naturally associated vector $e_i^\perp := (e_i, \cdot) \in e_i^\perp$. The following is essentially a restatement of [FG09]’s Lemma 2.15, although our cluster complex is really the cone over their cluster complex. Recall that $U^{trop}_S := \overline{N}_{2,R,S}$ has a natural symplectic structure induced by $[\cdot, \cdot]$.

Proposition 2.18. $\Gamma$ is the group of vector space isomorphisms $g$ between $X^{trop}_S$ and $X^{trop}_{S'}$, for some fixed $S$ and varying $S'$ mutation equivalent to $S$, which take $C_S$ to $C_{S'}$, and the vectors $e_i^\perp$ to $e_i^\perp$.\footnote{Another perspective which might be worth exploring in the future would be to identify the tropicalizations of different fibers of $\lambda$ with different fibers of $\lambda$, with only $\lambda_e$ corresponding to what we call $U^{trop}$ here.}
and restrict to a symplectomorphism from $U_{S}^{\text{trop}}$ to $U_{S'}^{\text{trop}}$. Similarly for $\hat{\Gamma}$, but with the symplectic form on $U^{\text{trop}}$ possibly being negated.

Proof. The correspondence is as follows: if $g(\overline{e_{i,S}}) = \overline{e_{i,S'}}$, then as an element of $\Gamma$ we say $g(e_{i,S}) = e_{i,S'}$, and vice versa. Note that $g \in \Gamma$ is a trivial cluster automorphism if and only if the cluster isomorphism between $S$ and $S'$ is the identity map on the underlying lattice $N$, in which case the corresponding action on the cluster complex is also trivial. $g|_{U_{S}}$ being a symplectomorphism exactly means that it preserves the skew-symmetric pairing $[\cdot,\cdot]$, and therefore also the pairing $\langle\cdot,\cdot\rangle$ which is part of the seed data. □

Note that the condition of $e_{i,S}$ mapping to $e_{i,S'}$ can be replaced with the condition that the indexing of the walls is preserved: knowing $C_{S}$ and the form $\langle\cdot,\cdot\rangle$ on $N$ is enough to determine the $e_{i,S'}$'s up to reordering. We could also use the $v_{i,S}$'s in place of the $e_{i,S}$'s.

Remark 2.19. In [GHKK], the walls $e_{i,\perp}$ together with the attached functions $1 + z^{(e_{i,\cdot})}$ form an “initial scattering diagram,” which they use to produce the “consistent scattering diagram” that is central to their mirror construction. They show that the consistent scattering diagram does not depend on the choice of initial seed (up to certain transformations which they describe). $\Gamma$ may therefore be viewed as the group of symmetries of the scattering diagram which preserve the cluster complex. In general, the consistent scattering diagram may contain multiple copies of the cluster complex, corresponding to different cluster structures on the same space. These are not related by elements of $\Gamma$, but Remark 1.14 in [GHK13b] predicts that these different cluster complexes are related by an action of a Weyl group $W$ for the lattice $K_{2}$ and correspond to different cluster structures on the underlying varieties.

As a corollary of this fact that the induced action of $\Gamma$ on $X^{\text{trop}}$ (and in fact, on $A^{\text{trop}}$ too) preserves the scattering diagram, one concludes that the canonical theta functions constructed in [GHKK] are indeed $\Gamma$-equivariant, as predicted in [FG09]. In fact, [GHKK] predicts that the theta functions depend only on the underlying variety and not on cluster structure. This would mean that any automorphism of the variety must act equivariantly on the theta functions, even if it changes the cluster structure.

In §5 we will describe the action of the cluster modular group on $U^{\text{trop}}$. In many (conjecturally all) cases, every automorphism of $U^{\text{trop}}$ (preserving its canonical oriented integral linear structure described below) is induced by an element of the cluster modular group.

3. $U^{\text{trop}}$ as an Integral Linear Manifold

Recall that $U$ denotes a log Calabi-Yau surface. This section examines $U^{\text{trop}}$ with its canonical integral linear structure defined in [GHK11].

3.1. Some Generalities on Integral Linear Structures. A manifold $B$ is said to be (oriented) integral linear if it admits charts to $\mathbb{R}^{n}$ which have transition maps in $\text{SL}_{n}(\mathbb{Z})$. We allow $B$ to have a set $O$ of singular points of codimension at least 2, meaning that these integral linear charts only cover $B' := B \setminus O$. $B'$ has a canonical set of integral points which come from using the charts to pull back $\mathbb{Z}^{n} \subset \mathbb{R}^{n}$. Our space of interest, $B = U^{\text{trop}}$, will be homeomorphic to $\mathbb{R}^{2}$ and will typically have a singular point at 0 (which we say is also an integral point).

$B'$ admits a flat affine connection, defined using the charts to pull back the standard flat connection on $\mathbb{R}^{n}$. Furthermore, pulling back along these charts give a local system $\Lambda$ of integral tangent vectors on $B'$. We will be interested in the monodromy of $\Lambda$ around $O$. 
3.1.1. **Integral Linear Functions.** By a linear map \( \varphi : B_1 \to B_2 \) of integral linear manifolds, we mean a continuous map such that for each pair of integral linear charts \( \psi_i : U_i \to \mathbb{R}^n \), \( U_i \subset B_i \) with \( \varphi(U_1) \subset U_2 \), we have that \( \psi_2 \circ \varphi \circ \psi_1^{-1} \) is linear in the usual sense. \( \varphi \) is integral linear if it also takes integral points to integral points. By an integral linear function, we will mean an integral linear map to \( \mathbb{R} \) with its tautological integral linear structure.

We note that to specify an integral linear structure on an integral piecewise linear manifold (i.e., a manifold where transition functions are integral piecewise linear), it suffices to identify which piecewise linear functions are actually linear. These functions can then be used to construct charts. It therefore also suffices (in dimension 2) to specify which piecewise-linear straight lines are straight, since (piecewise-)straight lines form the fibers of (piecewise-)linear functions.

3.2. **Constructing** \( U^{\text{trop}} \).

**Notation 3.1.** Given a toric model \((Y,D) \to (\mathbb{Y},\mathbb{D})\), let \( N \) be the cocharacter lattice corresponding to \((\mathbb{Y},\mathbb{D})\) (contrary to [2]'s notation), and let \( \Sigma \subset N_{\mathbb{R}} \) be the corresponding fan. \( \Sigma \) has cyclically ordered rays \( \rho_i \), \( i = 1, \ldots, n \), with primitive generators \( \psi_i \) corresponding to boundary divisors \( D_i \subset \mathbb{D} \) and \( D_i \subset D \). Assume \( N_{\mathbb{R}} \) is oriented so that \( \rho_{i+1} \) is counterclockwise of \( \rho_i \). Let \( \sigma_{u,v} \) denote the closed cone bounded by two vectors \( u, v \), with \( u \) being the clockwise-most boundary ray. In particular, if \( u \) and \( v \) lie on the same ray, we define \( \sigma_{u,v} \) to be just that ray. We may use variations of this notation, such as \( \sigma_{i,i+1} := \sigma_{\psi_i,\psi_{i+1}} \) and \( v_\rho \) for the primitive generator of some arbitrary ray \( \rho \) with rational slope, but these variations should be clear from context.

We now use \((Y,D)\) to define an integral linear manifold \( U^{\text{trop}} \). As an integral piecewise-linear manifold, \( U^{\text{trop}} \) is the same as \( N_{\mathbb{R}} \), with 0 being a singular point and \( U^{\text{trop}}(\mathbb{Z}) := N \) being the integral points. Note that an integral \( \Sigma \)-piecewise linear (i.e., bending only on rays of \( \Sigma \)) function \( \varphi \) on \( U^{\text{trop}} \) can be identified with a Weil divisor of \( Y \) via \( W_\varphi := a_1 D_1 + \ldots + a_n D_n \), where \( a_i = \varphi(\psi_i) \in \mathbb{Z} \). We define the integer linear structure of \( U^{\text{trop}} \) by saying that a function \( \varphi \) on the interior of \( \sigma_{1-i,i} \cup \sigma_{i,i+1} \) is linear if it is \( \Sigma \)-piecewise linear and \( W_\varphi \cdot D_i = 0 \). This last condition is (for \( n \geq 2 \)) equivalent to

\[
3.3. \quad a_{i-1} + D_i^2 a_i + a_{i+1} = 0.
\]

**Remark 3.2.** This construction of \( U^{\text{trop}} \) naturally generalizes to higher dimensions, but the two-dimensional case is special in that the linear structure on \( U^{\text{trop}} \) is canonically determined by \((Y,D)\) (it does not depend on the choice of toric model). This is evident from the following atlas for \( U^{\text{trop}} \) (from [GHK2]): the chart on \( \sigma_{1-i,i} \cup \sigma_{i,i+1} \) takes \( \psi_{i-1} \) to \((1,0)\), \( \psi_i \) to \((0,1)\), and \( \psi_{i+1} \) to \((-1,-D_i^2)\), and is linear in between.

Furthermore, toric blowups and blowdowns do not affect the integral linear structure, so as the notation suggests, \( U^{\text{trop}} \) and \( U^{\text{trop}}(\mathbb{Z}) \) depend only on the interior \( U \).

**Example 3.3.** If \((Y,D)\) is toric, then \( U^{\text{trop}} \) is just \( N_{\mathbb{R}} \) with its usual integral linear structure. This follows from the standard fact from toric geometry that \( \sum_i (C \cdot D_i) \psi_i = 0 \) for any curve class \( C \).

Taking non-toric blowups changes the intersection numbers, resulting in a singularity at the origin.\footnote{We assume here that there are more than 3 rays in \( \Sigma \), so that \( \sigma_{1-i,i} \cup \sigma_{i,i+1} \) is not all of \( N_{\mathbb{R}} \). This assumption can always be achieved by taking toric blowups of \((Y,D)\). Alternatively, it is easy to avoid this assumption, but the notation and exposition becomes more complicated. We will therefore continue to implicitly assume that there are enough rays for whatever we are trying to do.}
Remark 3.4. Recall from standard toric geometry that any primitive vector \( v \in N \) corresponds to a prime divisor \( D_v \) supported on the boundary of some toric blowup of \(( Y, D )\), and a general vector \( kv \) with \( k \in \mathbb{Z}_{\geq 0} \) and \( v \) primitive corresponds to the divisor \( kD_v \). Two divisors on different toric blowups are identified if there is some common toric blowup on which their proper transforms are the same (equivalently, if they correspond to the same valuation on the function field). Since taking proper transforms under the toric model gives a bijection between boundary components of \(( Y, D )\) and boundary components of \(( \overline{Y}, \overline{D} )\) (and similarly for the boundary components of toric blowups), we see that points of \( U^{\text{trop}}( \mathbb{Z} ) \) correspond to multiples of divisors on compactifications of \( U \).

3.3. Another Construction of \( U^{\text{trop}} \). We now give another construction of the canonical integral linear structure, this time more closely related to the cluster picture. Given a seed \( S \), consider the non-frozen seed vectors \( \{ e_i \}_{i \in I \setminus F} \). Recall that \( v_i := p_2^i(e_i) \in U^{\text{trop}} := p_2^i( X^{\text{trop}} ) \subset X^{\text{trop}} \) (cf. \[ 2.3 \]). The integral linear structure on \( U^{\text{trop}} \) agrees with that of the vector space \( U^{\text{trop}}_S \) (with the lattice \( \mathbb{Z}_{2,5} \) as the integral points) on the complement of the rays \( \rho_i := \mathbb{R}_{\geq 0}v_i \) for \( i \in I \setminus F \). By repeatedly mutating, this determines the integral linear structure everywhere.

For yet another perspective, consider a line \( L \) in \( U^{\text{trop}}_S \) which crosses a ray \( \rho_i \) as above. Viewsed as a piecewise-straight line in \( U^{\text{trop}} \) with its canonical integral linear structure, \( L \) will appear to be bending away from the origin when it crosses \( \rho_i \). Lines \( L \) which appear straight in \( U^{\text{trop}} \) will appear to bend towards the origin in \( U^{\text{trop}}_S \) as follows: if \( u \) is a tangent vector to \( L \) on one side of \( \rho_i \), then on the other side, \( u - |u \wedge v_i|v_i \) will be a tangent vector pointing away from \( \rho_i \). Another way to state this perspective is that the “broken lines” (as in \[ \text{[GHK11]} \] and \[ \text{[GHKK]} \] ) in \( U^{\text{trop}} \) which are actually straight with respect to the canonical integral linear structure are exactly those which bend towards the origin as much as possible.

3.4. The Developing Map. We now describe a tool from \[ \text{[GHK11]} \] that is useful for doing explicit computations on \( U^{\text{trop}} \). Consider the universal cover \( \tilde{\xi} : \tilde{U}^{\text{trop}}_0 \to U^{\text{trop}}_0 := U^{\text{trop}}_0 \setminus \{ 0 \} \). Note that \( \tilde{U}^{\text{trop}}_0 \) has a canonical integral linear structure pulled back from \( U^{\text{trop}}_0 \). The integral points are \( \tilde{U}^{\text{trop}}_0( \mathbb{Z} ) := \xi^{-1}[U^{\text{trop}}_0( \mathbb{Z} )] \). Furthermore, a ray \( \rho \in U^{\text{trop}}_0 \) pulls back to a family of rays \( \rho^j, j \in \mathbb{Z} \), projecting to \( \rho \) (we arbitrarily choose a ray in \( \tilde{U}^{\text{trop}}_0 \) to be \( \rho_0 \) and then assign the other indices so that they increase as we go counterclockwise).

Suppose that \( v \in \rho_0 \) and \( v' \in \rho'_0 \) are primitive vectors in \( \tilde{U}^{\text{trop}}_0 \) spanning the integral points of \( \sigma_{v,v'} \). Then there is a unique linear map \( \delta_{\rho,\rho'} : \tilde{U}^{\text{trop}}_0 \to \mathbb{R}^2 \setminus \{ 0 \} \) such that \( \delta_{\rho,\rho'}(v) = (1,0) \) and \( \delta_{\rho,\rho'}(v') = (0,1) \). We call this the developing map with respect to \( \rho \) and \( \rho' \). We will often leave off the subscripts if they are not relavent, or we will write \( \delta_{\rho} \) if only the image \( \rho \) of the first ray is relavent. \( \delta \) is an integral linear immersion, and \( \delta(\tilde{U}^{\text{trop}}_0( \mathbb{Z} )) \subseteq \mathbb{Z}^2 \setminus \{ (0,0) \} \). A superscript \( j \in \mathbb{Z} \) on \( \delta \) will indicate that we are considering the \( j \)th sheet of \( \delta \) (e.g., \( \delta^j(\rho) := \delta(\rho^j) \) for \( \rho \in U^{\text{trop}}_0 \)).

Example 3.5. Consider the cubic surface (as in Example \[ 2.4 \]) constructed by taking two non-toric blowups on each of the three boundary divisors \( D_1, D_2, \) and \( D_3 \) of \( \mathbb{P}^2 \). The intersection matrix \( H := (D_i \cdot D_j) \) is \( H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \) and Equation \[ 3 \] (or the construction from charts) implies that \( \delta^0_{\rho_{D_1},\rho_{D_2}}(v_3) = (-1,1) \), and \( \delta^0(\nu) = (-1)^j \delta^0(\nu) \). See Figure \[ 3.2 \] (a).

Example 3.6. Consider \( (\overline{M}_{0,5}, D = D_1 + \ldots + D_5) \) constructed from the toric surface \( (\mathbb{P}^2, \overline{D} = D_1 + D_2 + D_4) \) by making toric blowups at \( D_1 \cap D_4 \) and \( D_2 \cap D_4 \), as well as one non-toric blowup
on each of $\mathcal{D}_1$ and $\mathcal{D}_2$. We then have five boundary components, each with self-intersection $-1$. A developing map takes the rays of the fan to $(1,0), (0,1), (-1,1), (-1,0), \text{and} (0,-1), \text{respectively, and then restarts with} (1,-1) \text{and} (1,0). \text{See Figure 3.2 (b).}

3.5. Monodromy About the Origin. We now consider what happens when we parallel transport a tangent vector $v$ in $T_p U^\text{trop}$ counterclockwise around the origin. We use the embedding of a cone in the tangent spaces of its points (which are all identified via parallel transport in the cone), and we use the notation $\delta^i := \delta^i_{\rho_D} \rho_{D_1}, \rho_{D_2}$.

**Example 3.7.** Suppose $Y \to \overline{Y}$ consists of a single non-toric blowup on, say, $D_1$. Then $\delta^0(v_1) = \delta^1(v_1) = (1,0)$. However, $\delta^0(v_2) = (0,1)$ while $\delta^1(v_2) = (1,1)$. We can view parallel transporting counterclockwise around the origin as parallel transporting up one sheet on the developing map, and then the monodromy tells us how to write the transported vector in terms of $\delta^1(v_1)$ and $\delta^1(v_2)$. Thus, the monodromy is

$$\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$ 

Similarly, the monodromy is in general given by $\mu = (\delta^1(v_1) \delta^1(v_2))^{-1}$ with respect to the basis and developing map $\{\delta^0(v_1) = (1,0), \delta^0(v_2) = (0,1)\}$. We may view $\mu^{-k}$ as a map $\tilde{U}_0^\text{trop} \to \tilde{U}_0^\text{trop}$ which lifts points up $k$ sheets. Note that the monodromy determines $U^\text{trop}$ as an integral linear manifold: $U^\text{trop}$ is the quotient of $\tilde{U}_0^\text{trop}$ by this $\mathbb{Z}$-action.

$\mu$ and $\mu^{-1}$ can always be factored into a product of unipotent matrices as follows: choose a toric model in which $k$ non-toric blowups are taken on the divisor $D_v$, for $v_1, \ldots, v_s \in N$ cyclically ordered counterclockwise. Then we have the factorization

$$\mu^{-1} = \mu_{v_s}^{-k_s} \cdots \mu_{v_1}^{-k_1}.$$
where \( \mu_{v_i} \) is given in an oriented unimodular basis \((v_i, v'_i)\) by the matrix \(
abla \mu_{v_i} = \begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix} \). More generally, in a basis where \( v_i = (a, b) \), the corresponding contribution to \( \mu^{-1} \) is

\[
\mu^{-k_i}_{(a,b)} := \begin{pmatrix} 1 - k_iab & k_i a^2 \\ -k_i b^2 & 1 + k_i ab \end{pmatrix}.
\]

Now \( \mu \) can of course be expressed as \( \mu_{v_1} \cdots \mu_{v_n} \). Alternatively (following from the fact that \( A\mu_s A^{-1} = \mu_{As} \)), the monodromy matrix is given by the product \( \mu = (\mu'_{v_n})^{k_n} \cdots (\mu'_{v_1})^{k_1} \) of matrices of the form

\[
(\mu'_{v_i})^{k_i} := \mu_{(a_i, b_i)}^{k_i} = \begin{pmatrix} 1 + k_i a_i b_i & -k_i a_i^2 \\ k_i b_i^2 & 1 - k_i a_i b_i \end{pmatrix},
\]

where \( (a_1, b_1) := v_1 \), and for \( i > 1 \), \( (a_i, b_i) := (\mu'_{v_{i-1}})^{k_{i-1}} \cdots (\mu'_{v_1})^{k_1} v_i \). This can be interpreted by saying that before we can apply the monodromy contribution corresponding to \( v_i \), we have to let the modifications we have made so far act on \( v_i \).

**Example 3.8.** In Example \ref{ex:mono}, we have \( \delta^1(v_1) = (-1, 0) \) and \( \delta^1(v_2) = (0, -1) \), so we thus see that the monodromy for the cubic surface is \( -\text{Id} \).

**Example 3.9.** Similarly, for Example \ref{ex:mono} we have \( \delta^1(v_1) = (1, -1) \) and \( \delta^1(v_2) = (1, 0) \), so the monodromy is

\[
\mu = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}
\]

with respect to the basis \( \{\delta^0(v_1) = (1, 0), \delta^0(v_2) = (0, 1)\} \).

We have that \( \mathcal{U}^\text{trop} \) is uniquely determined (as an integral linear manifold, up to isomorphism) by its monodromy, and that a factorization of the monodromy into unipotent elements with cyclically ordered eigenrays as above corresponds to a toric model for a Looijenga pair (up to deformation), and hence to a seed as in \ref{thm:seed}. By eigenray, we mean an eigenline with a chosen direction.

### 3.5.1. Mutations and Monodromy

We now describe the monodromy of \( \mathcal{U}^\text{trop} \) directly in terms of seed data. Use \( \mu_{\ast, S} \) to indicate that we are mutating a seed \( S \) with respect to a vector \( e_i \). We consider the induced map on \( \overline{\mathcal{N}}_\mathbf{Z} \), identified with \( \mathcal{N}_{\overline{\mathbf{Z}}} \) as in \ref{thm:seed} which we denote by \( \overline{\mu}_{\ast, S} \). This is not hard to describe—it is given by Equation \ref{eq:seed} with each \( e_i \) replaced by \( v_i := p_\mathbf{Z}^2(e_i) \), and \( (\cdot, \cdot) \) replaced by the induced non-degenerate bilinear form \( (\cdot \wedge \cdot) \) on \( \mathcal{N}_{\overline{\mathbf{Z}}} \). Assume that the \( v_i \)’s are positively ordered with respect to the orientation induced by this form.

Now we observe that, in the notation of Equation \ref{eq:seed}, \( \overline{\mu}_{\ast, S}^2 = \mu_{v_i}^{-d_i} \). Thus, the inverse monodromy \( \mu^{-1} \) of \( \mathcal{U}^\text{trop} \) is \( \mu^{-1} = \prod \overline{\mu}_{\ast, S}^{2} \), where the product is taken over all \( i \), with the \( v_i \)’s being ordered counterclockwise as we move from right to left in the product. Note that the \( v_i \)’s in this formula are not affected by the previous mutations!

Alternatively, by Equation \ref{eq:product} we have \( \mu = \overline{\mu}_{n,S_n}^2 \circ \overline{\mu}_{n-1,S_{n-1}}^2 \circ \cdots \circ \overline{\mu}_1^2 \), where \( S^1 := S \), and \( S^k := \mu_{k-1,S_{k-1}}^{-2}(S^{k-1}) \). That is, we apply the inverse mutation twice with respect to one vector, then twice with respect to the next vector in the new seed, and so on.

This straightforward way to compute the monodromy is potentially useful because in \ref{thm:seed} we classify cluster varieties in terms of their monodromies (among other things).
3.6. Lines in $U^\trop$. For us, a line $L$ in $U^\trop$ will simply mean the image of a linear map $L: \mathbb{R} \to U^\trop_0$ (we abuse notation by letting $L$ denote the map and its image). A line together with such a choice of linear map will be called a parametrized line.

The signed lattice distance of a parametrized line $L$ from the origin is given by the skew-form $L(t) \wedge L'(t)$, where we use the canonical identification of the vector from 0 to $L(t)$ with a vector in $T_{L(t)}$. Note that the lattice distance does not depend on $t$. We will write $L^{>0}$ to denote that a line $L$ has positive lattice distance from the origin (i.e., goes counterclockwise about the origin), or $L^{<0}$ to denote that it has negative lattice distance from the origin.

We will say that a parametrized line $L$ goes to infinity parallel to $q$ if, for any open cone $\sigma \ni q$, there is some $t_\sigma \in \mathbb{R}$ such that $t > t_\sigma$ implies $L(t) \in \sigma$, $L'(t) = q$ under parallel transport in $\sigma$. Similarly for coming from infinity parallel to $q$, with $t > t_\sigma$ replaced by $t < t_\sigma$ and $L'(t) = q$ replaced with $-L'(t) = q$.

We let $L(\infty)$ and $L(-\infty)$ denote the directions in which $L$ goes to and comes from infinity. We use the subscript $q$ to indicate that a line $L$ goes to infinity parallel to $q$. For example, $L^\sigma_q$ denotes a line which goes to infinity parallel to $q$ with the origin on its left.

We say that an unparametrized line goes to infinity parallel to $q$ if it admits a parametrization which does. In general, a line need not go to infinity at all. In fact, one characterization of $U$ being positive is that every line both goes to and comes from infinity, cf. §4.3.

We note that the monodromy about the origin in $U^\trop$ allows lines to wrap around the origin and self-intersect. We say that a line $L$ wraps if it intersects every ray, except possibly one, at least once. It wraps $k$ times if it hits each ray at at least $k$ times, except possibly for one line which it may hit only $(k-1)$ times.

Example 3.10. If $(Y, D)$ is the cubic surface introduced in Example [85], then for any ray $\rho \subset U^\trop$, $U^\trop \setminus \rho$ is isomorphic as an integral linear manifold to an open half-plane. Both ends of any line will go to infinity in the same direction. If we now make a non-toric blowup on some $D_{\rho}$, then in the new integral linear manifold, any line will self-intersect unless both ends will go to infinity parallel to $q$.

3.7. Some Integral Linear Automorphisms of $U^\trop$. Assume that $U$ is positive, so lines to infinity on both ends. Given a point $q$ in $U^\trop$, define

$$(7) \quad \nu_+(q) := L^\sigma_q(-\infty), \quad \nu_-(q) := L^\sigma_q(-\infty).$$

Intuitively, both operations correspond to “negating” a vector in the integral linear manifold, but using different choices of charts. These clearly lift to maps $\tilde{\nu}^\pm$ and $\tilde{\nu}^- : \tilde{U}^\trop_0 \to \tilde{U}^\trop_0$, which may be viewed as rotation $180^\circ$ clockwise or counterclockwise, respectively.

Lemma 3.11. $\nu_+$ and $\nu_-$ are integral linear.

Proof. This follows from $\tilde{\nu}^\pm$ being integral linear, which is clear since $180^\circ$ rotations of $\mathbb{R}^2$ are integral linear. \qed

We will see in Proposition [52] that $\nu_\pm$ are induced by $\Gamma$.

3.8. Useful Facts from [Man14]. The following is a restatement of a Lemmas 3.7 and Corollary 3.8 from [Man14].

Lemma 3.12. Let $L \subset U^\trop$ be a line which does not wrap. Let $u$ and $v$ be the directions in which $L$ goes to infinity. Let $\sigma_L \subset U^\trop$ be the closed cone which is bounded by $u$ and $v$ and which does
Definition 3.15. non-toric blowups are all along divisors corresponding to rays in \( \sigma_L \). Furthermore, if one restricts to \( \sigma_L \setminus \rho_u \) or \( \sigma_L \setminus \rho_v \), then the choices of blowup points here is uniquely determined.

Lemma 3.16. A Looijenga pair \((Y, D)\) has charge \( c(Y, D) := \dim(Y) + \text{rank}(\text{Pic}(Y)) - n \).

Proof. First note that, for \( n > 1 \), toric blowups increase \( n \) by 1, decrease \( \text{Tr}(H) \) by 3, and keep the charge constant, so Equation (8) is unaffected by toric blowups and blowdowns. Similarly, non-toric blowups decrease \( \text{Tr}(H) \) by 1 and increase the charge by 1, so the validity of the equation is also unaffected by non-toric blowups. Since every Looijenga pair is related to a copy of the toric pair \((\mathbb{P}^2, D)\) by some sequence of toric blowups, toric blowdowns, and non-toric blowups, it now suffices to just check this case. We have \( c(\mathbb{P}^2, D) = 0 \), \( n = 3 \) and \( \text{Tr}(H) = 3 \), so the equation holds.

 GHK constructs a family \( \mathcal{V} \to \text{Spec } B \) mirror to \( U \) which admits a canonical \( B \)-module basis of theta functions \( \{ \vartheta_q \}_{q \in U^{\text{trop}}(Z)} \). \( \text{GHK} \) shows that if \( U \) is positive, then it can be realized as a fiber of \( \mathcal{V} \), thus giving theta functions on \( U \). Recall from \([2.1]\) that a global monomial is regular function on \( X \) whose restriction to some seed \( X \)-torus is a monomial. We also call the restriction to a fiber \( U \subset X \) of such a function a global monomial. §3.6 of \([\text{Man14}]\) observes the following (phrased differently):

**Lemma 3.13.** Take \( \sigma_L \) as in Lemma 3.12. For any \( q \in \sigma_L \), \( \vartheta_q \) is a global monomial.

Assume \( U \) is positive, and let \( V \) denote a generic fiber of the mirror \( \mathcal{V} \). For \( q \in U^{\text{trop}}(Z) \), \( v \in V^{\text{trop}}(Z) \), we can define \( \vartheta_q^{\text{trop}}(v) := \text{val}_{D_v}(\vartheta_q) \), where \( D_v \) is the boundary divisor corresponding to \( v \) in some compactification of \( V \). \([\text{Man14}]\) extends \( \vartheta_q^{\text{trop}} \) to all of \( V^{\text{trop}} \) and describes its fibers explicitly. In particular Corollary 4.11 of \([\text{Man14}]\) implies:

**Lemma 3.14.** Sets of the form \( \{ \vartheta_q^{\text{trop}} = d < 0 \} \subset V^{\text{trop}} \) for fixed \( d \) are given by \( Z(L) \) for some line \( L \). Thus, if every line wraps, then every \( \vartheta_q^{\text{trop}} \) is non-positive everywhere, and in fact, \( f^{\text{trop}} \) is non-positive everywhere for every regular function on \( V \).

Proof. The last statement uses that every regular function is a linear combination of theta functions, and valuations of linear combinations of theta functions are given by taking the minima of the valuations of each term (Remark 4.4 and the preceding paragraph of \([\text{Man14}]\) explain why no cancellations occur).

3.9. The Tropicalization Determines the Charge. One natural question to ask is to what extent \( U^{\text{trop}} \) determines \( U \). We will see in the next section that in many cases, \( U \) is uniquely determined up to deformation by \( U^{\text{trop}} \). This is not always the case though: for example, there are two degree 8 Del Pezzo’s with an irreducible choice of anti-canonical divisor which have the same \( U^{\text{trop}} \) but are not deformation equivalent. This subsection shows that \( U^{\text{trop}} \) does at least determine the number of non-toric blowups.

**Definition 3.15.** The charge \( c(Y) \) of a Looijenga pair \((Y, D)\) is the number of non-toric blowups in a toric model for some toric blowup of \((Y, D)\).

**Lemma 3.16.** A Looijenga pair \((Y, D = D_1 + \ldots + D_n)\) with \( n > 1 \) and intersection matrix \( H := (D_i \cdot D_j) \) has charge

\[
c(Y, D) = 12 - 3n - \text{Tr}(H)
\]

More generally, the charge of a log Calabi-Yau variety \((Y, D = D_1 + \ldots + D_n)\) is given by \( c(Y, D) := \dim(Y) + \text{rank}(\text{Pic}(Y)) - n \).
An similar formula appears in [GHK]: 
\[ c(Y, D) = 12 - (n + K_Y^2) \].

**Proposition 3.17.** Suppose that \((Y, D)\) and \((Y', D')\) are two Looijenga pairs with the same tropicalization \(U^{\text{trop}}\). Then \(c(Y, D) = c(Y', D')\).

**Proof.** Let \(\Sigma_Y\) and \(\Sigma_{Y'}\) be the corresponding fans in \(U^{\text{trop}}\). There exists some nonsingular common refinement \(\Sigma\) which is the fan for a toric blowup of both \((Y, D)\) and \((Y', D')\). The intersection matrices for these two toric blowups are the same, since each can be determined from \(\Sigma\), so the claim follows from Lemma 3.16. \(\square\)

4. **Classification**

Here we give several equivalent classifications for the possible deformation classes of Looijenga pairs. These classifications are based on the intersection matrix \(H\) of \(D\), the intersection form \(Q\) on \(D^\perp_{\text{Eff}} \cong K_2\) (see §2.4.1), the monodromy \(\mu\) of \(U^{\text{trop}}\), the properties of lines in \(U^{\text{trop}}\), the properties of the quiver for a corresponding cluster structure, and various other properties. This may be viewed as a classification of rank-2 cluster varieties up to the notion of equivalence given in Definition 2.14. The classification is not totally new—for example, the cases that we refer to as “no lines wrap” or “some lines wrap” are simply the finite-type or acyclic cases, respectively, in the cluster language. However, we do offer new characterizations of these cases.

Throughout this section, \(D\) will be called **minimal** if it has no \((-1)\)-components.

4.1. **The Negative Definite Case.** The following are equivalent, and have all appeared (along with some other equivalent statements) in some form in [GHK11], [GHK], or [GHK13a].

- The intersection matrix \(H = (D_i \cdot D_j)\) is negative definite.
- Any developing map \(\delta\) as in §3.3 embeds the universal cover \(\tilde{U}_0^{\text{trop}}\) of \(U_0^{\text{trop}}\) into a strictly convex cone in \(\mathbb{R}^2\).
- The monodromy satisfies \(\text{Tr}(\mu) > 2\).
- All lines in \(U^{\text{trop}}\) wrap infinitely many times around the origin, meaning that they hit each ray infinitely many times.
- The quadratic form \(Q\) is not negative semi-definite.
- \(U\) and its deformations admit no non-constant global regular **functions**.
- \(D\) can be blown down to get a surface \(\overline{Y}\) with a cusp singularity. If \(D\) is minimal, \(D_i^2 \leq -2\) for all \(i\), and \(D_i^2 \leq -3\) for some \(i\).

See Example 1.9 of [GHK11] for the relationship between \(\mu\) and the cusp singularity on \(\overline{Y}\). In fact, much of [GHK11] is devoted to deformations of cusp singularities.

4.2. **The Strictly Negative Semi-Definite Case.** Once again, the following statements are all equivalent and can be found in [GHK11] and [GHK] (or follow easily).

- The intersection matrix \(H\) is negative semi-definite but not negative definite.
- Any developing map \(\delta\) for \(U_0^{\text{trop}}\) identifies the universal cover of \(U_0^{\text{trop}}\) with a half-plane in \(\mathbb{R}^2\).
- The monodromy \(\mu\) is \(SL_2(\mathbb{Z})\)-conjugate to a matrix of the form \(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\), with \(a > 0\).
- Lines in \(U^{\text{trop}}\) can be circles, or they can wrap infinitely many times around the origin.
• If $D$ is minimal, then $D \in D^\perp$, meaning that either $D_i^2 = -2$ for all $i$, or $D$ is irreducible with $D^2 = 0$.
• The quadratic form $Q$ is negative semi-definite but not negative definite (since $Q(D) = 0$).
• $(Y, D)$ is deformation equivalent to a Looijenga pair $(Y', D')$ which admits an elliptic fibration having $D'$ as a fiber.

As stated above, if $D$ is minimal then it is either irreducible or consists of $n > 1$ $(-2)$-curves.

The largest possible $n$ here is 9. This follows from Lemma 3.16, which says that the charge is $c(Y, D) = 12 - 3n - \text{Tr}(H) = 12 - n$. The charge is by definition non-negative, giving us $n \leq 12$.
Furthermore, the classifications below then imply that some lines do not wrap if $c(Y, D) \leq 2$, so then $n \leq 9$. A case with $n = 9$ can be explicitly constructed.

4.3. The Positive Cases. As a converse to the above cases, we have that the following are equivalent:

• The intersection matrix $H$ is not negative semi-definite.
• The developing map for $U^\text{trop}_0$ is not injective.
• Lines in $U^\text{trop}$ wrap at most finitely many times, so both ends of each line go to infinity.
• The quadratic form $Q$ is negative definite.
• $U$ is deformation equivalent to an affine surface.
• $U$ is a minimal resolution of $\text{Spec}(\Gamma(U, O_U))$, which is an affine surface with at worst Du Val singularities.
• $D$ supports a $D$-ample divisor.

If any of these conditions hold, we say that $U$ is positive. We have several sub-cases:

4.3.1. All Lines Wrap/Positive Non-Acyclic Cases.

Theorem 4.1. The following are equivalent:

1. Lines in $U^\text{trop}$ all wrap, but only finitely many times.
2. Every sheet of the developing map is convex, but the developing map is not injective.
3. Non-zero global regular functions on $U$ are not generically 0 along any boundary divisor of any compactification $(Y, D)$ of $U$ (i.e., the corresponding valuations are non-positive). On the other hand, there are enough global regular functions that $\dim \text{Spec} \Gamma(U, O_U) = 2$.
4. The inverse monodromy matrix $\mu^{-1}$ is conjugate to a Kodaira matrix\footnote{In [Kod63], Kodaira listed the matrices which can appear as monodromies about singular fibers of elliptic fibrations of surfaces. See Tables 1 and 2 for a list of these matrices.} of type $I_k^*$, $II^*$, $III^*$, or $IV^*$.
5. If $D$ is minimal, then either $D = D_1 + D_2$ with $D_2^2 = 0$ and $-1 \neq D_2^2 \leq 0$ (up to re-labelling), or $D$ is irreducible with $1 \leq D^2 \leq 4$.
6. $U$ can be constructed from $(\mathbb{P}^2, D)$, with $D = D_1 + D_2 + D_3$ a triangle of lines, by blowing up $d_i$ times on $D_i$ for each $i$, with $(d_1, d_2, d_3)$ as in the final column of Table 7. Equivalently, $U$ comes from a seed with $E = (e_1, e_2, e_3)$, $F = \emptyset$, $\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and multipliers $(d_1, d_2, d_3)$ as in the final column of Table 7.
7. $D^\perp_{\text{Eff}} = D^\perp$, and the quadratic form $Q$ is of type $D_n$ ($n \geq 4$) or $E_n$ ($n = 6, 7, \text{ or } 8$).
Proof. (1)⇔(2) is clear from the definitions. (1)⇔(3) follows immediately from Lemma 5.13 (the ring of global regular functions being two-dimensional is equivalent to positivity).

For (1)⇒(5), using the construction of $U^\text{trop}$ from charts in Remark 3.2, we can easily see that having any $D^2_i > 0$ with $D$ not irreducible would allow a line to not wrap. On the other hand, having every $D^2_i \leq -2$ would mean we are in a negative semi-definite case. So if $D$ is minimal and not irreducible, then $D^2_i$ must be 0 for some $i$. $D$ having more than one additional component would allow a non-convex sheet of the developing map, so the claim follows, except for when $D$ is irreducible. If $D$ is irreducible and $D^2 > 4$, then the proper transform of $D$ after taking a toric blowup would have positive self-intersection, which we have already ruled out, and $D^2 < 1$ would mean we are in a negative semi-definite case.

For (5)⇒(2), observe that in the $D^2_1 = D^2_2 = 0$ case, every sheet of any developing map is convex (but not strictly convex). The other cases come from non-toric blowups and toric blow-downs of this, so the sheets of their developing maps will of course still be convex (non-toric blowups make these sheets “more convex”).

(5)⇔(4) is a straightforward check. Note that we now have the equivalence of (1) through (5).

(6)⇒(7) is also straightforward. For $U$ generic, $D^{\perp}$ is generated by classes of the form $E_{i,j} - E_{i,j}$ (where $E_{i,j}$ denotes the exceptional divisor from a non-toric blowup on $D_i$), together with a class of the form $L - E_{1,j_1} - E_{2,j_2} - E_{3,j_3}$, where $L$ is the class of a generic line in $\mathbb{P}^2$. If we choose all the blowup points on each $D_i$ to be infinitely near, and choose the blowup points on different $D_i$’s to be colinear, then $D^{\perp}$ is generated by effective divisors with the correct intersections.

(7)⇒(1) because $Q$ of type $D_n$ or $E_n$ implies that $Q$ is negative definite, so by the above characterizations, we are not in an $H$ negative semi-definite case. We also cannot be in a some lines wrap case because, as we see below, $Q|_{D^n_{k\ell}}$ in these cases is a direct sum of $A_n$’s.

It now suffices to show that (5)⇒(6) (since (4)⇔(5), this means we are showing that $U^\text{trop}$ really does determine the deformation type of $U$ in these cases). For the $I_5^n$ case, we have $\mu^{-1} = \text{Id}$. Such a $U^\text{trop}$ contains a reflexive polytope with 3 integral points on the boundary, and this implies that $U$ must be an affine cubic surface (cf. Example 5.21 in [Man14]), which we know can be obtained as in Example 5.35.

Now for the $I_6^n$ cases, we can choose a compactification $(Y, D)$ of $U$ with $D^1_1 = D^2_2 = -1$ and $D^2_3 = -1 - k$. The divisor $C := D_1 + D_2$ has $C \cdot D_1 = C \cdot D_2 = C^2 = 0$, and $C \cdot D_3 = 2$. By Riemann-Roch, $\dim |C| \geq 1$. If $C$ is the only singular element of some pencil $\mathbb{P}^1 \subset |C|$, then (for $U$ generic in its deformation class) $Y \setminus C$ is a $\mathbb{P}^1$-bundle over $\mathbb{A}^1$, hence has Euler characteristic 2. So then $Y$ has Euler characteristic 5. However, we know from 3.9 that $U^\text{trop}$ determines the charge $c$ of $(Y, D)$, which in this situation is $6 + k$. One checks that the Euler characteristic of a Looijenga pair with $n$ boundary components and charge $c$ is $n + c$, which in this case is $9 + k > 5$. So $|C|$ must contain other singular curves. These must contain irreducible rational components $E_1, E_2$ with $E_1 \cdot D_3 = 1$ and $E^2_i = -1$. Blowing down either of these is a non-toric blowdown and reduces us to the $I^n_{k-1}$ case, so the claim follows by induction.

For the $IV^{*}$ case, we have a compactification of $U$ with $D = D_1 + D_2 + D_3$, $D^2 = -1$, $D^2_1 = D^2_2 = -2$. Note that $D \cdot D_1 = 1$, while $D \cdot D_2 = D \cdot D_3 = 0$, so $\dim |D| \geq 1$. Thus, there is some point on $D_1$ which we can blow up to get a new pair $(\hat{Y}, \hat{D})$, with exceptional divisor $E$, such $\hat{Y}$ admits an elliptic fibration with $\hat{D}$ being a fiber and $E$ being a section. Such a surface can be obtained by blowing up 9 base-points for a pencil of cubics in $\mathbb{P}^2$, with $E$ being the exceptional divisor of the final blowup.
$\tilde{D}$ then is the proper transform of one of the cubics $D_i$ of $\overline{D}$. Thus, after blowing $E$ down, we see that $Y$ must contain disjoint $(-1)$-curves hitting each component of $D$. Blowing down a $(-1)$-curve hitting, say, $D_2$, reduces to the $I_1^*$ case we have already dealt with.

A similar argument works for the $III^*$ case using a compactification of $U$ with $D = D_1 + D_2$, $D_1^2 = -1$, $D_2^2 = -2$, and blowing up a point in $D_1$ to get a surface with an elliptic fibration. The $II^*$ case is also similar, using $D$ irreducible with self-intersection 1 and blowing up some point in $D$ to get a surface with an elliptic fibration.

Table 1 summarizes the different cases from the above theorem.

| Kodaira Matrix | Cartan Form $Q$ | Monodromy $\mu$ | $(d_1, d_2, d_3)$ |
|----------------|----------------|------------------|------------------|
| $I_k^* (k \geq 0)$ | $D_{n+4}$ | $\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$ | $(2,2,2+n)$ |
| $IV^*$ | $E_6$ | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | $(2,3,3)$ |
| $III^*$ | $E_7$ | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $(2,3,4)$ |
| $II^*$ | $E_8$ | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ | $(2,3,5)$ |

Table 1. Cases where all lines wrap.

4.3.2. Not All Lines Wrap/Acyclic Cases.

**Theorem 4.2.** The following are equivalent:

1. $U^\text{trop}$ contains a line which does not wrap.
2. Some compactification of $U$ admits a toric model $Y \rightarrow \overline{Y}$ for which all the non-toric blowups are on divisors corresponding to rays in one half of $N_\overline{Y}$. I.e, there is some seed $S$ for which all of the non-frozen vectors’ images in $p_2^*(N)$ lie in one half of the plane.
3. Cluster varieties corresponding to $U$ are acyclic.
4. The intersection of the cluster complex $C \subset X^\text{trop}$ with $U^\text{trop}$ is nonempty (assuming that there are no frozen variables).
5. There exists a global monomial on $U$.
6. The quadratic form $Q$ on $D^\perp$ is negative definite, and $Q|_{D^\perp_{\text{Eff}}}$ is a direct sum of $A_n$’s. In fact, it is $A_{d'_1-1} \oplus \cdots \oplus A_{d'_m-1}$, where the $(d'_i)$’s are the modified multipliers for a minimal seed corresponding to $U$ (equivalently, $d'_i$ is the number of non-toric blowups on $D_i$ in a toric model for a compactification $U$).

**Proof.** (1)$\Leftrightarrow$(2) is Lemma 3.12. (2)$\Leftrightarrow$(3) was observed in §2.1.1.

For (2)$\Leftrightarrow$(4), note that for some seed vector $e_i$ for a seed $S$, the set $\{ e_i \geq 0 \} \cap U^\text{trop}$ is the same as the set $(v_i \wedge \cdot) \geq 0$, where $\wedge$ is the symplectic form on $U^\text{trop}$ induced by $\{ \cdot, \cdot \}$. The intersection of these positive half-spaces for all non-frozen $e_i$’s is clearly nonempty if and only if $S$ is as in (2).
For (5)⇒(1), note that for a global monomial $q_i$, the tropicalization $\vartheta_q^{\text{trop}}$ is positive somewhere, and so Lemma 3.14 implies that the fibers $\vartheta_q^{\text{trop}} = d < 0$ are lines which do not wrap.

(6)⇒(1) because if every line does wrap (possibly infinitely many times), then we have seen that either $Q$ is not negative-definite or $Q|_{D_{\epsilon}^\text{Eff}}$ is of type $D_n$ or $E_n$.

For (2)⇒(6), first note that $Q$ is negative definite on $D^\perp$ by positivity of $U$. Now, let $(Y, D) \rightarrow (\mathbb{Y}, \mathbb{D})$ be the toric model corresponding to a seed with all non-toric blowups corresponding to rays in one half of the plane $N_\mathbb{Y}$. For any curve $C$ in $\mathbb{Y}$, $\sum (C \cdot D_i)v_i = 0$ where $v_i$ is the primitive vector in $N_\mathbb{Y}$ corresponding to $D_i$. If $C$ is the image of an irreducible effective curve $C \in D^\perp$, then $C \cdot D_i \geq 0$ for all $i$, and $C \cdot D_i$ can only be positive if there is a non-toric blowup point somewhere in $C \cap D_i$.

Thus, each $C \cdot D_i$ must actually be 0, so $C$ must have been supported on an exceptional divisor. Thus, $D_{\epsilon}^\text{Eff}$ is generated by classes obtained by taking the $d'_i$ blowups to be infinitely near, and then taking the $d'_i - 1$ exceptional divisors which do not intersect $D$.

Let $C_U$ denote the union of all cones $\sigma_L$ for lines $L$ which do not wrap, where $\sigma_L$ is defined as in Lemma 3.12. We note that the argument for (2)⇔(4) above can be modified to prove the following:

**Proposition 4.3.** $C_U$ is the intersection of the cluster complex $C$ with $U^{\text{trop}} \subset X^{\text{trop}}$.

This justifies Man14 calling $C_U$ the cluster complex.

**4.3.3. No Lines Wrap/Finite-Type Cases.**

**Theorem 4.4.** The following are equivalent:

1. No Lines in $U^{\text{trop}}$ wrap.
2. No sheet of the developing map is convex.
3. The Laurent phenomenon holds for the $X$-space, meaning that each $X_i$ is a global monomial. Furthermore, the global monomials form an additive basis for the global function on $U$.
4. The inverse monodromy matrix $\mu^{-1}$ is a Kodaira matrix of type I_k, II, III, or IV.
5. Cluster structures for $U$ are of finite type, meaning that they have only a finite number of distinct seeds.
6. For some maximal seed, the corresponding quiver (after removing frozen vectors) is of type $A_k^n$, $A_2$, $A_3$, or $D_4$.
7. The cluster complex $C \subseteq X^{\text{trop}}$ contains all of $U^{\text{trop}}$, and in fact is all of $X^{\text{trop}}$ (assuming that there are no frozen variables).

**Proof.** (1)⇔(2) is obvious. (1)⇔(3) follows from Lemma 3.12.

To see that (1) implies (5), we need Lemma 3.12 which says that for any line $L^{d<0}_q$ which does not wrap, there are only finitely many $(-1)$-curves hitting boundary divisors corresponding to rays in the cone $\sigma_L$ bounded by $L^{d<0}_q(\pm \infty)$. Since no lines wrap, we can cover $U^{\text{trop}}$ by finitely many cones of the form $\sigma_L$, and so there are only finitely many $(-1)$-curves in $Y$ hitting the boundary. Since seeds correspond to certain finite subsets of this collection of $(-1)$-curves, the claim follows.

(5)⇔(6) follows from a well-known result of FZ03, which says that a cluster algebra is of finite type if and only if the matrix $(-\epsilon_{ij} + 2\delta_{ij})_{i,j \in I \setminus F}$ is a finite type Cartan matrix. One easily checks that the only quivers of this type which produce rank 2 cluster varieties are those listed in the statement of theorem, along with types $B_2$, $B_3$, and $G_2$, which are equivalent to types $A_3$, $D_4$, and $D_4$, respectively, in the sense of Definition 2.14.
One can easily check \((6) \Rightarrow (4)\) by explicit computations: the \(A_k^1, A_2, A_3,\) and \(D_4\) quivers correspond to the \(I_k, II, III,\) and \(IV\) matrices, respectively. \((4) \Rightarrow (1)\) is now automatic.

For \((5) \Leftrightarrow (7)\), recall that seeds are in bijection with cones of the cluster complex. For any boundary wall \(W\) of any cone in \(C\), both sides of \(W\) will always be in \(C\), so if there are only finitely many cones, then \(C\) must fill up all of \(A^trop\). Conversely, if there are infinitely many cones, then they must “bunch up” near some ray \(\rho\) which is not in \(C\).

Remark 4.5. Without frozen vectors, the \(I_k\) cases, \(k \geq 0\), are actually of rank 0. Thus, although we tend to ignore frozen vectors, they are necessary for constructing these examples. They are also necessary for many other examples—this was reflected in Construction 2.11 when we required that the vectors \(u_1, \ldots, u_m\) generate \(N_Y\).

Remark 4.6. We suggest here that the appearance of Kodaira’s matrices may have a deeper geometric significance. The symplectic heuristic behind [GHK11]’s mirror construction (see their §0.6.1) assumes that \(U\) admits a special Lagrangian torus fibration over \(U^{trop}\), or at least over a deformation of \(U^{trop}\) in which the singularity is factored into several singular points. We expect that, at least in the all-lines-wrap and no-lines-wrap cases, some symplectic deformation or degeneration of \(U\) (perhaps \(U := p_2(A) \subset X\) or something closely related) will indeed admit a special Lagrangian fibration over \(U^{trop}\). This is known explicitly for the \(I_k\) cases (cf. [CU13]), and in cases representing moduli of local systems this can be realized (with the singularity factored) as the Hitchin fibration (as explained to me by Andy Neitzke). Furthermore, we hope that \(U\) (or at least some analytic open subset of \(U\)) in these cases admits a hyperkahler structure, and that for some rotation of the complex structure, the SYZ fibration over \(U^{trop}\) (or over some neighborhood of \(0 \in U^{trop}\)) will become an elliptic fibration (again a standard part of the Hitchin system picture). Doing this without factoring the singularity in \(U^{trop}\) would of course require that the monodromy is one of Kodaira’s monodromies.

| Quiver | Kodaira Matrix | Cartan Form | Monodromy \(\mu\) | \((d_1, d_2, d_3)\) |
|--------|---------------|-------------|-------------------|-----------------|
| \(A_k^1\) \((k \geq 0)\) | \(I_k\) | \(A_{k-1}\) | \(
\begin{pmatrix}
1 & -k \\
0 & 1
\end{pmatrix}
\) | \((k,0,0)\) |
| \(A_2\) | \(II\) | \(A_0\) | \(
\begin{pmatrix}
0 & -1 \\
1 & 1
\end{pmatrix}
\) | \((1,1,0)\) |
| \(A_3\) | \(III\) | \(A_1\) | \(
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\) | \((2,1,0)\) |
| \(D_4\) | \(IV\) | \(A_2\) | \(
\begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix}
\) | \((3,1,0)\) |

Table 2. Cases where no lines wrap.
4.3.4. Some Lines Wrap and Some Do Not.

Proposition 4.7. The following are equivalent:

(1) Some lines in $U^\text{trop}$ wrap, while others do not.
(2) Some (but not all) sheets of the developing map are convex.
(3) Cluster varieties corresponding to $U$ are acyclic but not of finite type.
(4) The monodromy satisfies $\text{Tr}(\mu) \leq -2$, and if there is equality, then $\mu$ is conjugate to

$$\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

for some $a < 0$.

Proof. (1)$\Leftrightarrow$(2) is easy, and (1)$\Leftrightarrow$(3) follows immediately from Theorems 4.2 and 4.4. The equivalence with (4) follows because all the other possibilities have been eliminated by the previous theorems. □

5. Cluster Modular Groups

In this section, we explicitly describe the action of the cluster modular group $\Gamma$ on $U^\text{trop}$ in every positive rank 2 case. However, keeping track of frozen variables will overly complicate matters and will obscure certain meaningful symmetries. We therefore define a new group $\Gamma'$ for which we drop the requirement that frozen vectors are permuted by $\Gamma$ (we allow frozen vectors to be mapped anywhere). This may introduce more automorphisms than one wishes to consider, so one could also require that elements of $\Gamma'$ do not act trivially on both $U^\text{trop}$ and on the set of non-frozen vectors. $\Gamma$ can be recovered by taking the subgroup of $\Gamma'$ which is the stabilizer of the set of frozen vectors (roughly meaning that the corresponding cluster transformations extend over certain partial compactifications).

5.1. The Action on $U^\text{trop}$. Let $\text{Aut}(U^\text{trop})$ be the group of orientation preserving integral linear automorphisms of $U^\text{trop}$. As in Proposition 4.18, we have a natural map $r : \Gamma' \to \text{Aut}(U^\text{trop})$. Let $\kappa$ denote the kernel. Recall that elements of $\Gamma'$ are represented by certain cluster transformations, i.e., compositions of mutations and seed isomorphisms. Elements of $\kappa$ must act trivially on $U^\text{trop}$, so they come from the cluster transformations whose only seed isomorphisms are ones such that if $e_i \mapsto e_j$, then $v_i = v_j$. What we plan to describe is the image $G := r(\Gamma') \subseteq \text{Aut}(U^\text{trop})$. Note that if all seeds related to $S$ are minimal, then $G = \Gamma'$.

Conjecture 5.1. $G = \text{Aut}(U^\text{trop})$ for all rank 2 cases.

Recall $\nu_{\pm} \in \text{Aut}(U^\text{trop})$ from 3.7. We will see that at least these elements are always in $G$. Furthermore, from our descriptions of $G$ below, one can explicitly check that the conjecture holds for the all-lines-wrap and no-lines-wrap cases. Of course, for these cases, we have shown that $U^\text{trop}$ determines the deformation class of $U$. A little more work shows that $U^\text{trop}$ is even enough to completely identify the intersection of the cluster complex with $U^\text{trop}$ in these cases (ignoring frozen vectors), and it follows directly from this that every automorphism of $U^\text{trop}$ in these cases is induced by an element of $\Gamma'$.

We now note that when considering $U^\text{trop}$ with its canonical integral linear structure, mutating with respect to a seed vector $e_i$ for some seed $S$ does not change the positions of any of the $v_j$'s in $U^\text{trop}$ except for $v_i$. This is because the centers of the blowups corresponding to the $e_j$'s, $j \neq i$, are preserved by mutation, and the divisor containing the center is the one corresponding to $v_j$. Thus,

\[9\text{Since we are mainly interested in the image of the map } \Gamma' \to \text{Aut}(U^\text{trop}), \text{this extra requirement is not important.}\]
we only have to worry about what happens to $v_i$. This vector is negated with respect to the vector space structure $U_S^{\text{trop}}$. We now interpret what this means in different cases.

As in [3.3], we use the notation $\mu_{i,S}$ to indicate that we are mutating a seed $S$ with respect to a vector $e_i$. We let $S_{i_1, \ldots, i_k}$ denote the seed obtained from $S$ by mutating with respect to the seed vectors with indices $i_1$, then $i_2$, and so on up through $i_k$.

5.2. When Lines Do Not Wrap. In the toric case we of course have $G = \Gamma' = \text{SL}_2(\mathbb{Z})$.

We saw in Lemma 3.12 that if a line $L$ does not wrap, then (ignoring frozen vectors) there is a unique seed $S$ for which each $v_i$ is contained in $\sigma_L \setminus \rho$, where $\sigma_L$ is the cone bounded by $L$ and $\rho$ is either boundary ray of this cone. Assume the $v_i$’s are arranged in counterclockwise order $v_1, \ldots, v_s$.

Note that any line in $U_S^{\text{trop}}$ which does not intersect any $\rho_v$, is also a straight line in $U_S^{\text{trop}}$. $L_{v_i}^{>0}$ and $L_{v_i}^{<0}$ are two such lines. Thus, $\mu_{v_i}^X$ has the effect of applying $\nu_+$ to $v_1$, while $\mu_{v_i}^{-X}$ has the effect of applying $\nu_-$ to $v_s$.

Now note that $v_2, \ldots, v_s, v'_1 := \nu_+(v_1)$ are all contained in $\sigma_{L_1} \setminus \rho_{v_1}$, so we can repeat the process, mutating $v_2$, then $v_3$, and so on. Alternatively, we could have done the reverse, mutating $v_s$ first, then $v_{s-1}$, and so on. Since $v_{\pm}$ are integral linear automorphisms of $U_S^{\text{trop}}$ by Lemma 3.11, we see that $m_- := \nu^- \circ \mu_{s,1,2,\ldots,s-1} \circ \cdots \circ \mu_{1,S}$ is an element of $\Gamma$, and similarly for the reverse, $m_+ := \nu_+ \circ \mu_{1,S,s-1,\ldots,2} \circ \cdots \circ \mu_{s,S}$. We note that $r(m_{\pm}) = \nu_{\pm}$.

Of course, it might not be necessary to apply all $s$ mutations above before getting a seed isomorphic to the original one. For example, in the type $A_2$ case of Theorem 3.4, preforming a single mutation produces a seed isomorphic to the original. We may thus obtain fractional powers of $\nu_\pm$. It is not hard to see that all elements of $r(\Gamma')$ must be of this form, except in the $I_k$ cases (as we see below).

Thus, if not all lines wrap and we are not in an $I_k$ case ($k \geq 0$), then $G$ is cyclic.

In terms of developing maps and the notation of [3.3] $\delta^0[\nu^{+}(v)] = -\delta^1(v)$, which we may think of as $-\mu^{-1}(v)$. Similarly, $\delta^0[\nu^{-}(v)] = -\delta^{-1}(v) = -\mu(v)$. From this one can see a relationship between powers of $-\mu$ and symmetries of the scattering diagram in $U_S^{\text{trop}}$.

For example, if we are in a case where some lines wrap and others do not (cf. Proposition 3.7), then the monodromy has two eigenlines $\ell_1$ and $\ell_2$ in $\mathbb{R}^2$, or one eigenline with algebraic multiplicity 2 in the $\text{Tr}(\mu) = -2$ cases. Assume for now that $\text{Tr}(\mu) < -2$. Then $-\mu^{-1}$ has eigenvalues $\lambda$ and $\lambda^{-1}$ for some $\lambda \in (0, 1)$, and we can say $\ell_1$ is the eigenspace corresponding to $\lambda$. We can identify $U_S^{\text{trop}}$ with a half-space bounded by $\ell_1$, with the two outgoing rays of $\ell_1$ identified. Let $C$ be the cone bounded by $\ell_1$ and $\ell_2$ with $\ell_1$ as the clockwise-most boundary ray. Then the interior of $C$ is in fact $C_U$, the intersection of the cluster complex with $U_S^{\text{trop}}$—indeed, we see that $-\mu^{-1}$ moves vectors in the interior of $C$ counterclockwise, as one expects $\nu_+$. To do in $C_U$. Let $\sigma_L \subset C_U$ be a cone corresponding to a line $L$ which does not wrap, and let $\rho$ be either boundary ray of $\sigma_L$. Then $\sigma_L \setminus \rho$ is a fundamental domain for the action of $\langle \nu_\pm \rangle$ on $C_U$. We see that there is a similar action giving a periodic structure to the complement of $C$ with $\nu_+$ moving rays clockwise.

The cases where $\mu$ is conjugate to $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$ are essentially the same except that $\lambda = 1$, $\ell_1 = \ell_2$, and the complement of $C_U$ is just this eigenspace (a single ray in $U_S^{\text{trop}}$). So in any case where some lines wrap and others do not, we get $G \cong \mathbb{Z}$, with $\nu_\pm$ generating a finite index subgroup.

For the $II$, $III$, and $IV$ cases, $-\mu^{-1}$ has finite order, and so $G$ will also have finite order. One can explicitly compute $G$ in these cases to get the groups listed in Table 3.
For the $I_k$ cases, $k \geq 1$, there are non-trivial cluster automorphisms which fix the non-frozen seed vectors. These form an infinite cyclic group $N$, generated by $\mu^{1/k}$, which is normal and has index 2 in $G$. $\nu^2_k = \mu^{-2k}$, so $\nu_+ \mu$ generates an index 2$k$ subgroup, while $\mu \circ \nu_+$ generates a subgroup $H \cong \mathbb{Z}/2\mathbb{Z}$. We thus have $G = N \rtimes H \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

We note that powers of the cluster transformations for the $II, III$, and $IV$ cases give the trivial cluster transformations described in [FG09], Proposition 1.8, for their cases $h = 3, 4$, and 6, respectively. The $I_2$ case with no frozen vectors (cf. Remark 4.5) corresponds to [FG09]’s $h = 2$ case.

5.3. When All Lines Wrap. Consider the $I_0^*$ case. Take a minimal seed as in the second part of Example 2.13. That is, take the seed $S$ with no frozen vectors and with $(\cdot, \cdot)$ given by
\[
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix},
\]
and each $d_i = d_i' = 2$. We can identify $v_1$, $v_2$, and $v_3$ with $(2, 0)$, $(0, 2)$, and $(-2, -2)$ in $\mathbb{Z}$, respectively. We have $v'_1 := \mu_3.S(v_1) = v_1$, $v'_2 := \mu_3.S(v_2) = (-4, -2)$, and $v'_3 := \mu_3.S(v_3) = (2, 2)$. Note that there is a vector space isomorphism $\alpha$ taking the ordered triplet $(v'_1, v'_3, v'_2)$ to the ordered triplet $(v_1, v_2, v_3)$. Thus, $\alpha \circ \mu_{3,S}$ gives an element of $\Gamma$ which induces an automorphism $\alpha \in G$.

Note that $\alpha$ takes the ordered triplet $(v_1, v_2, v_3)$ to the ordered triplet $(v_1, v_1 + v_2, v_2)$ (addition done in $\sigma_{v_1,v_2}$ as defined in Notation 3.1). It is not hard to see from this that $\Gamma'$ acts transitively on the scattering rays, so every scattering ray in $U^{trop}$ looks the same. Thus, any automorphism of $U^{trop}$ is induced by an element of $\Gamma'$, so $G = \text{Aut}(U^{trop}) = \text{SL}_2(\mathbb{Z})/\{\pm \text{Id}\} = \text{PSL}_2(\mathbb{Z})$. Since we had no frozen vectors and every seed in the above cluster structure is minimal, we have $\Gamma = \Gamma' = G = \text{PSL}_2(\mathbb{Z})$, agreeing with [FG09]'s computation of this $\Gamma$ in their Lemma 2.32.

For the other cases where all lines wrap, the elements of $\Gamma$ are obtained similarly: Take the initial seed as above with different $(d'_1, d'_2, d'_3)$. In the $I_k^*$ cases, take $d'_1 = 2 + k$, $d'_2 = d'_3 = 2$. Then we obtain an element of $G$ exactly as above. However, unlike before, we cannot cycle the roles of the three seed vectors—$v_1$ is special. What we find is that $\Gamma = \mathbb{Z}$—If we identify $U^{trop} \setminus \rho_{v_1}$ with the upper half

| Classification | $G$ |
|---------------|-----|
| $I_0$ (Toric) | $\text{SL}_2(\mathbb{Z})$ |
| $I_k$ ($k > 0$) | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ |
| $II$ | $\mathbb{Z}/5\mathbb{Z}$ |
| $III$ | $\mathbb{Z}/3\mathbb{Z}$ |
| $IV$ | $\mathbb{Z}/4\mathbb{Z}$ |
| Some lines wrap. | $\mathbb{Z}$ |
| $I_0^*$ ($k > 0$) | $\text{PSL}_2(\mathbb{Z})$ |
| $II^*$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $III$ | $\{\text{Id}\}$ |
| $IV$ | $\{\text{Id}\}$ |

Table 3. The isomorphism class of the image $G$ of $\Gamma' \to \text{Aut}(U^{trop})$ for the positive rank 2 cases. If all seeds are minimal, then $\Gamma' = G$, and if there are no frozen vectors, then $\Gamma = \Gamma'$.
plane, with \( v_2 = (0, 1) \) and \( v_3 = (-1, 1) \), then \( x \in \mathbb{Z} \) corresponds to the automorphism taking \( v_2 \) to \( (-x, 1) \) and \( v_3 \) to \( (-x - 1, 1) \). In particular, we have that \( \pm k \) corresponds to \( \nu_\pm \).

For the \( IV^*, III^* \), and \( II^* \) cases, we take \( d_2' = 3 \), \( d_3' = 2 \), and \( d_1' = 3, 4, \) or 5, respectively. In the \( I_\psi^* \) case, when we apply the mutation with respect to \( v_3 \), we can then compose with the seed isomorphism \( v_3' \mapsto v_3, v_1' \mapsto v_2, \) and \( v_2' \mapsto v_1 \). This is the only nontrivial element of \( G \) in this case, so we have \( G \cong \mathbb{Z}/2\mathbb{Z} \). On can check that this non-trivial element is in fact \( \nu_+ = \nu_- \). In the \( III^* \) and \( II^* \) cases, we do not even have this element, and \( G \) is trivial.

We note that there is an orientation reversing automorphism in each of these three cases which, after mutating with respect to \( v_3 \) takes \( v_i' \mapsto v_i \), for each \( i \). Thus, one can obtain extra, potentially interesting symmetries of the scattering diagram by considering \( \hat{\Gamma} \) (as in §2.6) in place of \( \Gamma \).

In the \( I_{\psi^*}, III^* \), and \( II^* \) cases, one can check that \( \nu_\pm \) are trivial. Thus, in conjunction with what we have seen in the other cases, we have found that:

**Proposition 5.2.** \( \nu_\pm \) are induced by the cluster modular group \( \Gamma' \) (which we do not require to preserve frozen vectors) in all the positive cases.

We have now described \( G \) in all the positive cases. We summarize these findings in Table 3.

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