Generalised Kummer construction
and the cohomology rings of $G_2$-manifolds

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Abstract. Intersection theory is used to calculate the cohomology rings of $G_2$-manifolds arising from the generalised Kummer construction. For one example, generators of the rational cohomology ring are found and their multiplication table is described.

Bibliography: 19 titles.

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The first examples of closed Riemannian manifolds with holonomy $G_2$ and Spin(7) were found by Joyce using the generalised Kummer construction [1]–[3] (see also [4]). The classical Kummer construction produces $K3$-surfaces from a four-dimensional torus. Its generalised version consists in resolving the singularities of orbifolds $T^n/\Gamma$, where $\Gamma$ is a group acting discretely on the torus ($n = 4$ and $\Gamma = \mathbb{Z}_2$ for $K3$-surfaces; $n = 7$ for $G_2$-manifolds; $n = 8$ for Spin(7)-manifolds). In [5], this construction was further generalised to the case of orbifolds $M^8/\Gamma$ where $M^8$ is an eight-dimensional Calabi-Yau manifold. This led to a construction of new examples of Spin(7)-manifolds. The examples obtained in those works, together with the $G_2$-manifolds obtained as twisted connected sums [6]–[8], exhaust all known examples of closed $G_2$- and Spin(7)-manifolds.

The cohomology rings of many $G_2$-manifolds obtained as twisted connected sums were calculated in [8]. It was also noted there that the cohomology rings of the $G_2$-manifolds obtained from the generalised Kummer construction remain uncomputed.

The ring structure in the cohomology of a $K3$-surface was described by Milnor using a rather general algebraic argument [9]. An explicit basis for the two-dimensional cohomology in which the intersection takes the canonical form had remained unknown until recently. In [10] we found such a basis using intersection theory on manifolds.

In this paper, we show how intersection theory can be applied to the calculation of the cohomology rings of the $G_2$-manifolds constructed in [1] and [2]. These manifolds form a finite series, so we present the calculation for a typical example. The other cases are treated similarly. Furthermore, we only consider the rational
cohomology ring. The integer cohomology ring can also be described similarly to [10], but it will require lengthy calculations.

We present results in terms of the intersection ring $H_*(X;\mathbb{Q})$, which is dual to the cohomology ring $H^*(X;\mathbb{Q})$.

Given a closed oriented $n$-dimensional smooth manifold $X$, the intersection product

$$H_k(X;\mathbb{Z}) \times H_l(X;\mathbb{Z}) \xrightarrow{\cap} H_{k+l-n}(X;\mathbb{Z})$$

is defined by the formula

$$u \cap v = D^{-1}(Du \cup Dv),$$

where

$$D: H_i(X;\mathbb{Z}) \to H^{n-i}(X;\mathbb{Z}), \quad i = 0, \ldots, n,$$

is the Poincaré duality operator. If the cycles $u \in H_k(X;\mathbb{Z})$ and $v \in H_l(X;\mathbb{Z})$ are represented by transversely intersecting smooth submanifolds $Y$ and $Z$, then their intersection is a smooth $(k+l-n)$-dimensional submanifold $W$ that represents the cycle $w \in H_{k+l-n}(X;\mathbb{Z})$ satisfying

$$u \cap v = w.$$  

The orientation of the intersection $Y \cap Z$ is defined as follows. Let $x \in Y \cap Z$, let $(e_1, \ldots, e_{k+l-n})$ be a basis in the tangent space to $Y \cap Z$ at $x$, and let $(e_1, \ldots, e_{k+l-n}, e'_1, \ldots, e'_{n-1})$ and $(e_1, \ldots, e_{k+l-n}, e_1'', \ldots, e''_{n-k})$ be positively-oriented bases in the tangent spaces to $Y$ and $Z$ at $x$. Then the orientation of $Y \cap Z$ is defined by the condition that the basis $(e_1, \ldots, e_{k+l-n})$ in the tangent space to $Y \cap Z$ has the same orientation (positive or negative) as the basis $(e_1, \ldots, e_{k+l-n}, e'_1, \ldots, e'_{n-1}, e_1'', \ldots, e''_{n-k})$ in the tangent space to $X$. This implies that the intersection product satisfies the standard anti-commutativity relation

$$u \cap v = (-1)^{(n-k)(n-l)}v \cap u, \quad \dim u = k, \quad \dim v = l, \quad \dim M = n.$$  

In general, not all cycles are representable by smooth submanifolds. Therefore, in order to define the intersection product geometrically, one needs to introduce a special class of chains. This was done by Lefschetz, who generalised Poincaré’s intersection theory for cycles of complementary dimensions $k$ and $n - k$ to cycles of arbitrary dimension. A detailed treatment of this construction was given in [11]. The term ‘Lefschetz ring’ was used in [11] for the homology group $H_*(X)$ with the intersection product, and the topological invariance of the Lefschetz ring is established using the isomorphism $D$ with the cohomology ring $H^*(X)$.

We now turn to a concrete example of Joyce’s manifold.

Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ be a 7-dimensional torus obtained by taking the quotient of the space $\mathbb{R}^7$ by the translations $x \to x + y$ by integer vectors, where $x \in \mathbb{R}^7$, $y \in \mathbb{Z}^7 \subset \mathbb{R}^7$. We denote the coordinates on $T^7$ arising from the Euclidean coordinates on $\mathbb{R}^7$ and defined up to the addition of integers by $x_1, \ldots, x_7$. Consider the transformations

$$\alpha((x_1, \ldots, x_7)) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),$$

$$\beta((x_1, \ldots, x_7)) = (b_1 - x_1, b_2 - x_2, x_3, x_4, -x_5, -x_6, x_7),$$

$$\gamma((x_1, \ldots, x_7)) = (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, -x_7)$$

(1)
of the torus $T^7$, which depend on the parameters $b_1, b_2, c_1, c_3$ and $c_5$. These transformations are involutions, that is,

$$\alpha^2 = \beta^2 = \gamma^2 = 1,$$

and they commute pairwise,

$$\alpha \beta = \beta \alpha, \quad \alpha \gamma = \gamma \alpha, \quad \beta \gamma = \gamma \beta.$$ 

It follows that the transformations above generate an action of the group $\Gamma = \mathbb{Z}_3^2$ on $T^7$, for any values of the parameters $b_1, b_2, c_1, c_3$ and $c_5$.

The following values of the parameters give the simply-connected manifold $M^7$ described in Example 3 in [2]:

$$b_1 = c_5 = 0, \quad b_2 = c_1 = c_3 = \frac{1}{2}. \quad (2)$$

Following [1] and [2], we list the basic facts on the topology of the $G_2$-manifold $M^7$, which is constructed from the data in (1) and (2).

1. The group $\Gamma$ acts on $H^* (T^7)$ by involutions. There are no nontrivial invariant subspaces in $H^1 (T^7)$ and $H^2 (T^7)$, while $H^3 (T^7)$ has a 7-dimensional invariant subspace generated by the cohomology classes of the following forms with integer periods:

$$dx_2 \wedge dx_4 \wedge dx_6, \quad dx_3 \wedge dx_4 \wedge dx_7, \quad dx_5 \wedge dx_6 \wedge dx_7,$$

$$dx_1 \wedge dx_2 \wedge dx_7, \quad dx_1 \wedge dx_3 \wedge dx_6, \quad (3)$$

$$dx_1 \wedge dx_4 \wedge dx_5, \quad dx_2 \wedge dx_3 \wedge dx_5.$$ 

It follows that

$$b^1 (T^7 / \Gamma) = b^2 (T^7 / \Gamma) = 0 \quad \text{and} \quad b^3 (T^7 / \Gamma) = 7.$$ 

Here we denote the Betti numbers of a space $Y$ by

$$b^k (Y) = \dim_{\mathbb{Q}} H^k (Y; \mathbb{Q}), \quad k \geq 0.$$ 

The 7-form $dx_1 \wedge \cdots \wedge dx_7$ is invariant with respect to $\Gamma$, which implies that the Hodge operator $\ast : H^k (T^7) \to H^{7-k} (T^7)$ corresponding to the flat metric on $T^7 = \mathbb{R}^7 / \mathbb{Z}^7$ takes invariant forms to invariant ones. Therefore,

$$b^5 (T^7 / \Gamma) = b^5 (T^7 / \Gamma) = 0 \quad \text{and} \quad b^4 (T^7 / \Gamma) = 7.$$ 

2. The action of $\Gamma = \mathbb{Z}_3^2$ is not free. The fixed points of each of the involutions $\alpha, \beta$ and $\gamma$ form 16 fixed three-dimensional tori:

$$\vartheta_1^\alpha, \vartheta_2^\alpha, \vartheta_3^\alpha, \vartheta_4^\alpha, x_5, x_6, x_7), \quad \vartheta_1^\alpha, \ldots, \vartheta_4^\alpha \in \left\{ 0, \frac{1}{2} \right\} \quad \text{for } \alpha;$$

$$\vartheta_1^\beta, \vartheta_2^\beta, x_3, x_4, \vartheta_5^\beta, \vartheta_6^\beta, x_7), \quad \vartheta_1^\beta, \vartheta_5^\beta, \vartheta_6^\beta \in \left\{ 0, \frac{1}{2} \right\}, \quad \vartheta_2^\beta \in \left\{ \frac{1}{4}, \frac{3}{4} \right\} \quad \text{for } \beta;$$

$$\vartheta_1^\gamma, x_2, \vartheta_3^\gamma, x_4, \vartheta_4^\gamma, x_6, \vartheta_7^\gamma), \quad \vartheta_5^\gamma, \vartheta_7^\gamma \in \left\{ 0, \frac{1}{2} \right\}, \quad \vartheta_1^\gamma, \vartheta_3^\gamma \in \left\{ \frac{1}{4}, \frac{3}{4} \right\} \quad \text{for } \gamma.$$
3. The group $\Gamma/\mathbb{Z}_2$ acts by nontrivial permutations on the set of fixed tori for each of the involutions $\alpha$, $\beta$ and $\gamma$. The $\Gamma$-orbit of each fixed torus consists of 4 tori. Using the parametrisation of the tori by quadruples of parameters $\vartheta_i$, we list the $\Gamma$-orbits of the fixed tori for each of the involutions $\alpha$, $\beta$ and $\gamma$:

- the orbits of the fixed tori for the involution $\alpha$ are split into two pairs corresponding to different values of $\vartheta^\alpha_4 \in \{0, 1/2\}$:
  \[
  \left\{ (0, 0, 0, \vartheta^\alpha_4), \left(0, \frac{1}{2}, 0, \vartheta^\alpha_4\right), \left(\frac{1}{2}, 0, \frac{1}{2}, \vartheta^\alpha_4\right), \left(\frac{1}{2}, \frac{1}{2}, 0, \vartheta^\alpha_4\right) \right\},
  \]
  \[
  \left\{ \left(0, \frac{1}{2}, \vartheta^\alpha_4\right), \left(0, \frac{1}{2}, 0, \vartheta^\alpha_4\right), \left(\frac{1}{2}, 0, 0, \vartheta^\alpha_4\right), \left(\frac{1}{2}, \frac{1}{2}, 0, \vartheta^\alpha_4\right) \right\};
  \]

- the orbits of the fixed tori for the involution $\beta$ are given by the parameters $\vartheta^\beta_5, \vartheta^\beta_6 \in \{0, 1/2\}$;
- the orbits of the fixed tori for the involution $\gamma$ are given by the parameters $\vartheta^\gamma_5, \vartheta^\gamma_6 \in \{0, 1/2\}$.

4. The products of elementary involutions $\alpha \beta$, $\beta \gamma$, $\alpha \gamma$ and $\alpha \beta \gamma$ have no fixed points, as they act by translations on certain coordinates. For example, $\alpha \beta(x_2) = x_2 + 1/2$.

5. Each of the involutions $\alpha$, $\beta$, $\gamma$ generates a $\mathbb{Z}_2$-action on $T^7$ such that

\[T^7/\mathbb{Z}_2 = T^3 \times (T^4/\mathbb{Z}_2),\]

where $T^4/\mathbb{Z}_2$ is a Kummer surface, which becomes a $K3$-surface after resolving the 16 singular points (see, for example, [12] and [13]). After taking the quotient by an involution, all generators of the group $\pi_1(T^7)$ are projected onto contractible loops in $T^4/\mathbb{Z}_2$. Therefore, $\pi_1(T^7/\Gamma) = 0$.

6. The set of singular points of the orbifold $T^7/\Gamma$ consists of 12 three-dimensional tori, which correspond to the $\Gamma$-orbits of the fixed tori for the involutions $\alpha$, $\beta$, $\gamma$. Each singular torus in $T^7/\Gamma$ has a neighbourhood of the form

\[U = T^3 \times (D/\mathbb{Z}_2),\]  \hspace{1cm} (4)

where $D = \{|z| \leq \tau: z \in \mathbb{C}^2\}$ is a small neighbourhood of zero in $\mathbb{C}^2$ with the $\mathbb{Z}_2$-action given by $z \mapsto -z$.

7. Topologically, the manifold $M^7$ is obtained from $T^7/\Gamma$ by the fibrewise resolution of singularities (4) in $D/\mathbb{Z}_2$. For each singular torus, the neighbourhood $U$ is removed and the product

\[\tilde{U} = T^3 \times V\]  \hspace{1cm} (5)

is glued instead. Here $V$ is a small neighbourhood of the zero section of the bundle $\gamma^2 \to \mathbb{C}P^1$, and $\gamma \to \mathbb{C}P^1$ is the tautological bundle over $\mathbb{C}P^1$. The zero section of the bundle $U \to T^3$ is replaced by $\mathbb{C}P^1$. As in the case of the resolution of singularities for the Kummer surface $T^3/\mathbb{Z}_2$, the self-intersection index of the homology class $[\mathbb{C}P^1]$ in the four-dimensional fibre of the bundle is equal to $-2$. The homology of the manifold $T^3 \times V$ is given by

\[H_*(T^3 \times V) = H_*(T^3) \otimes H_*(\mathbb{C}P^1).\]
8. The rational homology groups of the manifold $M^7$ are as follows:

$b^2 = 12$ and the generators are the 12 cycles given by the submanifolds $\mathbb{CP}^1$ obtained by resolving the 12 singular tori;

$b^3 = 43$ and the generators are the 7 cycles realised by the tori corresponding to the invariant 3-forms on $T^7$ together with the 12 sets of products of $\mathbb{CP}^1$ by the generating 1-cycles on the singular tori.

The Betti numbers $b^4 = 43$ and $b^5 = 12$ are obtained using Poincaré duality.

We construct manifolds realising generators of the group $H_*(M^7; \mathbb{Q})$.

t-cycles. We start by constructing generators coming from the homology of the quotient $T^7/\Gamma$. We choose three-dimensional tori $T_\alpha$, $T_\beta$ and $T_\gamma$ that are homologous to the fixed tori of the involutions $\alpha$, $\beta$ and $\gamma$, respectively, and have $\Gamma$-orbits consisting of eight tori each. For example, for the involution $\alpha$ we consider the torus

$$T_{\alpha} = \left\{ \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, x_5, x_6, x_7 \right), x_5, x_6, x_7 \in \mathbb{R} \right\}.$$ 

The $\Gamma$-orbit of the torus fixed by $\alpha$ is formed by singular tori homologous to $T_{\alpha}$, and we denote these singular tori by $T_{\alpha 1}, \ldots, T_{\alpha 4}$. The four-dimensional torus dual to $T_{\alpha}$ will be denoted by $T'_{\alpha}$; its intersection number with $T_{\alpha}$ is 1, that is,

$$T_{\alpha} \cap T'_{\alpha} = \text{pt},$$

and the $\Gamma$-orbit of $T'_{\alpha}$ consists of eight tori. For example, we can take

$$T'_{\alpha} = \left\{ x_1, x_2, x_3, x_4, \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}.$$ 

The tori corresponding to the involutions $\beta$ and $\gamma$ are defined similarly. By construction, for each of these tori, the integral of one of the forms in the first row of (3) over this torus in 1, and the integrals of all other forms in (3) over the torus are 0. For example, $\int_{T_{\alpha}} dx_5 \wedge dx_6 \wedge dx_7 = 1$. For each of the remaining four forms in (3), we construct the three-dimensional tori $T_1, \ldots, T_4$ and the dual four-dimensional tori $T'_1, \ldots, T'_4$ in the same way.

By construction, the orbits of the dual tori $T_i$ and $T'_i$ intersect at $64 = 8^2$ points, and the projections of these tori onto $T^7/\Gamma$ intersect at eight points that do not lie on singular tori. The intersection theory for $T^7$ implies that the tori $T_k$ and $T'_l$ with $k \neq l$ intersect in a cycle homologous to zero, where $k, l \in \{\alpha, \beta, \gamma, 1, 2, 3, 4\}$.

Consider the projections of the tori $T_k$ and $T'_l$ onto $T^7/\Gamma$. Denote the cycles represented by these projections by $t_k = [\pi(T_k)]$ and $t'_l = [\pi(T'_l)]$, where $\pi: T^7 \rightarrow T^7/\Gamma$ is the quotient projection. We have

$$t_k \cap t'_l = \begin{cases} 8 & \text{for } k = l, \text{ where } k, l \in \{\alpha, \beta, \gamma, 1, 2, 3, 4\}, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

The projections of the tori representing the cycles $t_k$ and $t'_l$ with $k \in \{\alpha, \beta, \gamma, 1, \ldots, 4\}$ and $l = 1, \ldots, 4$ do not intersect the singular tori, and therefore these projections are not affected by the resolution of singularities. We denote their representing cycles in $M^7$ by the same symbols.
The tori $T'_\alpha$, $T'_\beta$ and $T'_\gamma$ intersect the fixed tori. After the resolution of singularities, the projections of $T'_\alpha$, $T'_\beta$ and $T'_\gamma$ onto $T^7/\Gamma$ become cycles represented by embedded $K3$-surfaces. For example, the involution $\alpha$ acts on $T^\alpha$ as a reflection with 16 fixed points, in which $T^\alpha$ intersects the fixed tori of $\alpha$. If $\mathbb{Z}_2 = \langle \alpha \rangle$ is the group generated by $\alpha$, then the projection $T^7 \to T^7/\langle \alpha \rangle$ takes $T'_\alpha$ to a Kummer surface, on which the group $\mathbb{Z}_2^2 = \Gamma/\langle \alpha \rangle$ acts by translations. The projection $T^7 \to T^7/\Gamma$ takes the torus $T'_\alpha$ to a Kummer surface, on which the group $\mathbb{Z}_2$ acts by translations. The projection $T^7 \to T^7/\Gamma$ takes the torus $T'_\alpha$ to a Kummer surface that intersects each singular torus in 4 points. After resolving singularities a Kummer surface becomes a $K3$-surface. The tori $T'_\beta$ and $T'_\gamma$ are transformed similarly. We denote by $t'_\alpha$, $t'_\beta$ and $t'_\gamma$ the cycles in $H_4(M^7)$ represented by the resulting submanifolds.

**C-cycles.** For each involution $\delta \in \{\alpha, \beta, \gamma\}$ we index the singular tori $T_{\delta i}$ by the numbers $1$ to $4$. Let

$$C_{\alpha 1}, \ldots, C_{\alpha 4}, C_{\beta 1}, \ldots, C_{\gamma 4} \quad (\text{dim} = 2)$$

be two-dimensional submanifolds in the fibres of the bundles of the form

$$\widetilde{\pi}: \widetilde{U} = T^3 \times V \to T^3,$$

such that each of these twelve submanifolds is diffeomorphic to $\mathbb{C}P^1$, and these submanifolds generate the nontrivial homology of the fibre $V$.

For each singular torus $T_{\delta i}$, where $\delta \in \{\alpha, \beta, \gamma\}$ and $i = 1, \ldots, 4$, we choose three nontrivial loops $\lambda_{\delta ij}$, $j = 1, 2, 3$, that lie on this torus, pass through a certain basepoint and are obtained by varying one of the coordinates $x_i$. For each of these loops we choose a dual two-dimensional torus $\tau_{\delta ij} \subset T_{\delta i}$ that has index of intersection with the loop $\lambda_{\delta ij}$.

Consider the following submanifolds:

$$\lambda_{\delta ij} \times C_{\delta i} \quad (\text{dim} = 3), \quad \tau_{\delta ij} \times C_{\delta i} \quad (\text{dim} = 4), \quad T_{\delta i} \times C_{\delta i} \quad (\text{dim} = 5).$$

Here the product of a torus $W \subset T_{\delta i}$ and the submanifold $C_{\delta i}$ is understood as the product of $W$ and the submanifold $C_{\delta i} \subset V$ arising from (5). We denote the homological cycles represented by these submanifolds as follows:

$$c_{\delta i} = [C_{\delta i}], \quad c_{\delta ij} = [\lambda_{\delta ij} \times C_{\delta i}], \quad c'_{\delta ij} = [\tau_{\delta ij} \times C_{\delta i}], \quad c'_{\delta i} = [T_{\delta i} \times C_{\delta i}].$$

Next we find the intersections of these cycles.

a) Clearly, the intersection of any two such cycles with distinct first indices is zero, and the intersection of these cycles with a cycle of the form $t_k$ is also zero.

b) If the sum of the dimensions of the cycles is less than the dimension of the manifold, then their representing submanifolds become disjoint after a small perturbation. Therefore, the product of such cycles is zero:

$$c_{\delta i} \cap c_{\delta i} = 0, \quad c_{\delta i} \cap c_{\delta ij} = 0, \quad c_{\delta i} \cap c'_{\delta ij} = 0, \quad c_{\delta ij} \cap c_{\delta ik} = 0.$$  

c) After a translation, the two-dimensional tori $\tau_{\delta ij}$ become parallel in a three-dimensional torus, which implies that

$$c'_{\delta ij} \cap c'_{\delta ij} = 0.$$
Also, if a loop \( \lambda_{\delta_{ij}} \) lies on a two-dimensional torus \( \tau_{\delta_{ijk}} \), \( i \neq k \), then it can be moved away from this torus, eliminating the intersection. This implies that

\[
c_{\delta_{ij}} \cap c'_{\delta_{ik}} = 0 \quad \text{for} \quad j \neq k.
\]

d) Low-dimensional tori (loops and two-dimensional tori) in the fixed tori are homologous to low-dimensional subtori of \( T_k \). The latter are homologous to zero over the rationals, because there exist no \( \Gamma \)-invariant 1- and 2-forms on \( T^7 \). Consider the intersections of two cycles of the form \([\tau_{\delta_{ij}} \times C_{\delta_i}]\) and \([T_{\delta_i} \times C_{\delta_i}]\). By perturbing submanifolds homologous to \( C_{\delta_i} \) in each fibre, we obtain the intersection set of the form \( \pm 2\tau_{\delta_{ij}} \), which is homologous to zero. Therefore,

\[
c'_{\delta_{ij}} \cap c'_{\delta_{i}} = 0.
\]

The identities

\[
c'_{\delta_{ij}} \cap c'_{\delta_{ik}} = 0 \quad \text{for} \quad j \neq k, \quad c_{\delta_{ij}} \cap c'_{\delta_{i}} = 0,
\]

are proved similarly, because in this case the intersection set is homologous to \( \pm 2\lambda \), where \( \lambda \) is a loop on the torus \( T_{\delta_i} \).

e) On the torus \( T_{\delta_i} \), the loop \( \lambda_{\delta_{ij}} \) intersects the two-dimensional torus \( \tau_{\delta_{ij}} \) at a single point \( P \), so their index of intersection is 1. By perturbing slightly the component \( C_{\delta_i} \) in the products \( \lambda_{\delta_{ij}} \times C_{\delta_i} \) and \( \tau_{\delta_{ij}} \times C_{\delta_i} \), we reduce the intersection of these submanifolds to the self-intersection of \( C_{P_{1}} \) in the fibre over the point \( P \). This self-intersection index is equal to \(-2\). Therefore,

\[
c_{\delta_{ij}} \cap c'_{\delta_{ij}} = -2.
\]

f) The self-intersection of the cycle \( c'_{\delta_{i}} \) reduces to the self-intersection of \( CP^{1} \) in the fibres over points in the torus \( T_{\delta_i} \), similarly to the previous case. Therefore, a small perturbation of the submanifolds \( T_{\delta_i} \times C_{\delta_i} \) gives a transverse intersection set homologous to \(-2T_{\delta_i} \). Hence,

\[
c'_{\delta_{i}} \cap c'_{\delta_{i}} = -2t_{\delta}.
\]

We have therefore identified the ring structure of \( H_*(M^7; \mathbb{Q}) \) completely.

**Theorem.** The rational homology \( H_*(M^7; \mathbb{Q}) \) of the \( G_2 \)-manifold obtained by the generalised Kummer construction from the data (1) and (2) has generators in the following dimensions \( \leq \text{dim } M^7 = 7 \):

\[
\begin{align*}
\text{dim } = 2: & \quad c_{\delta_i}, \\
\text{dim } = 3: & \quad c_{\delta_{ij}}, \; t_{\delta}, \; t_{i}, \\
\text{dim } = 4: & \quad c'_{\delta_{ij}}, \; t'_{\delta}, \; t'_{i}, \\
\text{dim } = 5: & \quad c'_{\delta_{i}},
\end{align*}
\]

where \( \delta \in \{\alpha, \beta, \gamma\} \), \( i = 1, \ldots, 4 \), \( j = 1, 2, 3 \). All these cycles are represented by embedded submanifolds, and therefore are integral, that is, belong to \( H_*(M^7; \mathbb{Z}) \subset H_*(M^7; \mathbb{Q}) \).

The only nontrivial intersection products are

\[
\begin{align*}
c_{\delta_{i}} \cap c'_{\delta_{i}} & = -2, \\
c_{\delta_{ij}} \cap c'_{\delta_{ij}} & = -2, \\
t_{\delta} \cap t'_{\delta} & = 8, \\
t_{i} \cap t'_{i} & = 8,
\end{align*}
\]

\( c'_{\delta_{i}} \cap c'_{\delta_{i}} = -2t_{\delta}, \) (7)
Remarks. 1) To describe the rational cohomology ring we need to replace each generator \( a \in H_\ast \), \( \dim a = k \), by the generator \( \bar{a} \in H_\ast \), \( \deg \bar{a} = n - k \), and replace the relations \( a \cap b = c \) by \( \bar{a} \cup \bar{b} = \bar{c} \).

2) There is a similar description of the rational cohomology rings for other manifolds with special holonomy obtained by the generalised Kummer construction.

3) The first row in (7) implies that there is a pairing (Poincaré duality) between \( H_k \) and \( H_{n-k} \), which is defined by \( a \rightarrow a' \) on the given generators. This pairing is therefore given by the diagonal matrix with \( \det = 2^{12} \) for \( k = 2 \) and \( \det = 2^{36} 8^7 = 2^{57} \) for \( k = 3 \). The cycles given in (7) do not define a basis of the group \( H_\ast (M^7; \mathbb{Z})/\text{Torsion} \), because the corresponding integral pairing must be given by a unimodular matrix. Nontrivial products arise from the second row of (7).

Integral generators of the group \( H_\ast (M^7; \mathbb{Z})/\text{Torsion} \) could also be obtained by an explicit geometric construction, as was done in [10] for \( K3 \)-surfaces. However, such a construction would probably be rather cumbersome.

4) Simply-connected closed Kähler manifolds are formal by a theorem of Deligne, Griffiths, Morgan and Sullivan [14], [15]. This result cannot be generalised to the case of symplectic manifolds, which was shown first for manifolds of dimension \( \geq 10 \) in [16] (see also [17]) and later for the remaining dimension 8 in [18] (simply-connected closed manifolds of dimension \( \leq 6 \) are always formal). Kähler manifolds are manifolds with special holonomy \( U(n) \), so a natural question arises of whether all simply-connected closed manifolds with special holonomy are formal.

Manifolds with holonomy \( SU(n) \) and \( Sp(n) \) are clearly formal, because they are Kähler. For the remaining cases of quaternionic-Kähler manifolds (with holonomy \( Sp(n) Sp(1) \)), manifolds with holonomy \( G_2 \) (in dimension 7) and \( Spin(7) \) (in dimension 8), there are no known counterexamples to the conjecture above.

The only obstructions to the formality of simply-connected 7-dimensional closed manifolds are nontrivial triple Massey products of the form \( \langle a, b, c \rangle \), where \( a, b, c \in H^2 \) and \( a \cup b = b \cup c = 0 \). (For example, if \( b_1 = 1 \), then there could be only one obstruction, the triple Massey product \( \langle x, x, x \rangle \), where \( x \) is a generator of \( H^2 \). However, this Massey product is zero for reasons of dimension, and such a manifold is formal [19].) Our theorem gives many sets \( a, b, c \) of two-dimensional cohomology classes satisfying the conditions above, but we could not find any nontrivial Massey products.

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