3D spinfoam quantum gravity: matter as a phase of the group field theory

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Abstract
An effective field theory for matter coupled to three-dimensional quantum gravity was recently derived in the context of spinfoam models (Freidel and Livine 2006 Phys. Rev. Lett. 96 221301 (Preprint hep-th/0512113)). In this paper, we show how this relates to group field theories and generalized matrix models. In the first part, we realize that the effective field theory can be recast as a matrix model where couplings between matrices of different sizes can occur. In a second part, we provide a family of classical solutions to the three-dimensional group field theory. By studying perturbations around these solutions, we generate the dynamics of the effective field theory. We identify a particular case which leads to the action of Freidel and Livine (2006 Phys. Rev. Lett. 96 221301 (Preprint hep-th/0512113)) for a massive field living in a flat non-commutative spacetime. The most general solutions lead to field theories with nonlinear redefinitions of the momentum which we propose to interpret as living on curved spacetimes. We conclude by discussing the possible extension to four-dimensional spinfoam models.

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1. Introduction: 2D and 3D group field theories

A large class of constrained or deformed topological BF field theories can be covariantly quantized via the spinfoam technology [2]. In particular, spinfoam models provide viable candidates for a covariant theory of quantum gravity in arbitrary spacetime dimensions [3, 4]. They can be regarded [5, 6] as implementing the physical scalar product in loop quantum gravity by defining spacetime histories interpolating between the spin network quantum states of the gravitational field.

Technically, they appear as two-complexes coloured by statistical-like weights assigned to the faces, edges and vertices of the dual two-skeleton of a chosen triangulation of spacetime.
If the theory is topological, the discretized theory is equivalent to the continuum theory since no local degrees of freedom are present. If the theory under consideration does admit local degrees of freedom, one needs to suppress the dependence on the triangulation. This procedure can be implemented by computing a sum over triangulations in order to recover the infinite number of degrees of freedom of the theories of interest. This is achieved by considering dual field theory formulations named group field theories (GFT).

A \(d\)-dimensional GFT [7, 8] is an abstract, nonlocal field theory of simplicial geometries living on \((d)\) copies of a group manifold. The theory is built such that its Feynman evaluations reproduce the associated dual spinfoam amplitudes on the two-complexes given by the Feynman diagrams of the GFT, consequently generalizing the matrix model (MM) technology to higher dimensions. In fact, there is a crystal clear relationship between GFTs in two dimensions and MMs. Let \(G\) denote an arbitrary semi-simple, compact Lie group.

The 2D GFT is a theory of complex, nonlocal fields \(\varphi : G^2 \to \mathbb{C}\) satisfying a reality condition and a right invariance property under the diagonal \(G\) action:

\[
\varphi(g_1, g_2) = \overline{\varphi(g_2, g_1)} \quad \text{and} \quad \varphi(g_1 h, g_2 h) = \varphi(g_1, g_2), \quad \forall h \in G. \tag{1}
\]

The dynamics of the theory are encoded in the following action:

\[
S_{2D}[\varphi] = \frac{1}{2} \int_{G^{\times 2}} dg_1 dg_2 \varphi(g_1, g_2)\varphi(g_2, g_1)
- \sum_n \frac{\lambda_n}{n!} \int_{G^{\times n}} \prod_{i=1}^n dg_i \varphi(g_1, g_2)\varphi(g_2, g_3) \cdots \varphi(g_n, g_1), \tag{2}
\]

where \(\lambda_n\)'s are coupling constants and \(dg_i, i = 1, \ldots, n\), denotes the normalized Haar measure on the \(i\)th copy of the compact group \(G\). Because of the right invariance symmetry of the theory, we are free to fix the gauge and reformulate the theory in terms of a field \(\phi\) on a single copy of \(G\) defined as \(\varphi(g_1, g_2) = \phi(g_1 g_2^{-1})\). The 2D GFT action then reads

\[
S_{2D}[\phi] = \frac{1}{2} \int_{G^{\times 2}} dg \phi(g)\phi(g^{-1})
- \sum_n \frac{\lambda_n}{n!} \int_{G^{\times n}} \prod_{i=1}^n dg_i \delta(g_1 g_2 \cdots g_n)\phi(g_1)\phi(g_2) \cdots \phi(g_n). \tag{3}
\]

We can decompose the field \(\phi\) along the unitary, irreducible representations \((\rho, \mathbb{V}_\rho)\) of \(G\) using the Peter–Weyl theorem: \(L^2(G) \simeq \bigoplus_\rho \mathbb{V}_\rho \otimes \mathbb{V}_\rho^*\). We obtain the following expansion:

\[
\phi(g) = \sum_\rho d_\rho \phi_{\rho ab}(g)^{ab}. \tag{4}
\]

Here, \(d_\rho \in \mathbb{N}\) is the dimension of the representation \(\rho : G \to \text{Aut}(\mathbb{V}_\rho)\), the indices \(a, b = 1, \ldots, d_\rho\) are matrix indices associated with the matrix \(\rho(g)\) representing the group element \(g\), and \(\phi_{\rho} \in \mathbb{V}_\rho \otimes \mathbb{V}_\rho^* \simeq \text{End}(\mathbb{V}_\rho)\) is the matrix Fourier coefficient of the function \(\phi\).

In other words, each \(\phi_{\rho}\) is a rank \(d_\rho = N\) matrix. More precisely, the symmetry requirement of equation (1) implies that the matrices are Hermitian \(\phi_{\rho} = (\phi_{\rho})^\dagger\), i.e., \(\phi_{\rho} \in \text{u}(N)\). The 2D GFT action is then re-expressed in terms of Hermitian matrix fields:
\[ S_{3D}[^\phi, \rho] = \sum_\rho d_\rho \left[ \frac{1}{2} \text{tr}(\phi_\rho^2) - \sum_n \frac{\lambda_n}{n!} \text{tr}(\phi_\rho^n) \right]. \] (5)

This is the action of a tower of decoupled matrix models\(^1\) of all possible sizes \(N = d_\rho \in \mathbb{N}\). In the case of interest here, where \(G = \text{SU}(2)\), the unitary, irreducible representations are labelled by spins \(I \in \mathbb{N}/2\) and the sum is over half-integers.

The Feynman evaluation \(I_{GFT}(T^*)\) associated with a given diagram \(T^*\) equates the topological invariant partition function \(Z_{BF}(T)\) of two-dimensional BF theory discretized on the triangulation \(T\) dual to the two-complex \(T^*\) defined by the GFT diagram:

\[ I_{GFT}(T^*) = Z_{BF}(T) \equiv \sum_\rho d_\rho^{\chi(T)}, \] (6)

where \(\chi(T)\) denotes the Euler characteristic of the surface triangulated by \(T\), or equivalently, of the diagram \(T^*\). It is explicitly given in terms of the number of vertices \(V\), edges \(E\) and faces \(F\) of the diagram \(T^*\) by \(\chi(T^*) = F - E + V\). This explains the presence of this Euler characteristic in the Feynman evaluation. Indeed, each face corresponds to a closed loop, i.e. a tracing leading to a \(d_\rho\) factor. Similarly, each vertex carries a \(d_\rho\) contribution since the dimension factorizes the whole action. Finally, each edge is associated with a propagator which is trivially given by inverting the kinetic term of the action and yields accordingly a \(1/d_\rho\) contribution.

The sum over representations collapses to a single term if one considers BF theory on a surface with boundaries supporting the canonical (spin network) states. These states select a particular value of \(\rho\) [9] which, from the MM perspective regarded as zero-dimensional (scalar) QCD, corresponds to the representation in which the quarks transform, i.e., to the number of colours involved in the theory.

The same type of rationale remains true in higher dimensions. We now describe the three-dimensional case.

The 3D GFT defines a manifold independent, covariant formulation of 3D quantum gravity. Three-dimensional general relativity is a topological field theory and can accordingly be quantized through the spinfoam procedure. The resulting discretized path integral is called the Ponzano–Regge (PR) model [10] and was actually the first quantum gravity model ever written. The dual GFT, considered by Boulatov [11], is defined on the Cartesian cube \(G^3 \times \eta, \phi : G^3 \times \eta \rightarrow \mathbb{C}\), where \(G = \text{Spin}(\eta)\), with \(\eta\) being the diagonal form of a three-dimensional metric. For simplicity, we will consider Riemannian quantum gravity in what follows, that is, we will work in Euclidean signatures where \(\text{SO}(\eta) = \text{SO}(3)\) and \(\text{Spin}(\eta) = \text{SU}(2)\). To ensure that the theory is torsion free, one requires a global right invariance of the field under diagonal \(G\) rotations:

\[ \phi(g_1 h, g_2 h, g_3 h) = \phi(g_1, g_2, g_3), \quad \forall h \in G. \] (7)

Also, one requires the following reality condition:

\[ \phi(g_1, g_2, g_3) = \overline{\phi}(g_3, g_2, g_1). \] (8)

The most general framework also requires some specific transformation properties of \(\phi\) under the permutation group in order to generate all possible topologically inequivalent triangulations each with different weights (see, e.g., [12] and references therein). We will neglect this aspect in the present work since it would unnecessarily complicate the derivation and eventually lead to the same results up to some numerical factors.

\(^1\) To avoid the trivial matrix model of size \(1 \times 1\) corresponding to the trivial \(\rho_0\) mode, we can require the field \(\phi\) to have a vanishing integral over \(G\): \(\rho_0 = \int \phi = 0\).
The dynamics are that of a nonlocal $\varphi^4$ theory:

$$S_{3D}[\varphi] = \frac{1}{2} \int_{G^{\times 3}} dg_1 dg_2 dg_3 \varphi(g_1, g_2, g_3) \varphi(g_3, g_2, g_1) \tag{10}$$

$$- \frac{\lambda}{4!} \int_{G^{\times 6}} \prod_{i=1}^{6} dg_i \varphi(g_1, g_2, g_3) \varphi(g_3, g_5, g_4) \varphi(g_4, g_2, g_6) \varphi(g_6, g_5, g_1). \tag{9}$$

The interaction term depicted above is a tetrahedron interaction chosen such that the GFT Feynman diagrams generate (oriented) triangulations. More generally, we should, in principle, include all closed spin network evaluations, which can then be interpreted as dual to arbitrary 3-cells. For example, in [13], Freidel and Louapre include a ‘pillow’ interaction term,

$$S_{3D}[\varphi] = \int_{G^{\times 6}} \prod_{i=1}^{6} dg_i \varphi(g_1, g_2, g_3) \varphi(g_2, g_3, g_4) \varphi(g_4, g_5, g_6) \varphi(g_5, g_6, g_1), \tag{10}$$

which is the only other nontrivial quartic interaction term. Moreover, they show that including it renders the partition function Borel-summable. From this perspective, including all possible consistent interaction terms is natural in an effective QFT approach when studying the renormalization of the group field theory.

As for the two-dimensional case, we can expand the field $\varphi$ in terms of the unitary, irreducible representations $D^I$ of $G = SU(2)$. For all triples of unitary, irreducible representations $I, J, K$, let $\Psi^K_I : \mathbb{V}_I \otimes \mathbb{V}_J \to \mathbb{V}_K$ and $\Phi^{IJ}_K : \mathbb{V}_K \to \mathbb{V}_I \otimes \mathbb{V}_J$ denote the Clebsch–Gordan intertwining operators. Using the right invariance property of the field, one obtains the following decomposition:

$$\varphi(g_1, g_2, g_3) = \varphi_{I_1 I_2 I_3} \psi_{I_1 I_2} \left( \bigotimes_{i=1}^{3} \sqrt{d_I^I(g_i)} \right), \tag{11}$$

Here, the dots ‘.’ stand for tensor index contraction. The normalized, three-valent intertwining operator $\psi_{I_1 I_2 I_3} \in \text{Hom}_G(\mathbb{V}^*_{I_1} \otimes \mathbb{V}^*_{I_2} \otimes \mathbb{V}^*_{I_3}, \mathbb{C})$ is related to the Clebsch–Gordan map $\psi_{I_1 I_2}$ by the following evaluation called a $3j$-symbol:

$$\psi_{I_1 I_2 I_3} (e_{a_1} \otimes e_{a_2} \otimes e_{a_3}) = \left( \begin{array}{ccc} I_1 & I_2 & I_3 \\ a_1 & a_2 & a_3 \end{array} \right) \equiv \epsilon^{(I_1 - I_2 - I_3)} \epsilon_{a_3} \psi_{I_1 I_2} (e_{a_2} \otimes e_{a_3}) \in \mathbb{R}, \tag{12}$$

where $\epsilon^{(I)}$ is the scalar product associated with the bijective intertwiner $\epsilon_I : \mathbb{V}_I \to \mathbb{V}^*_I$. The symbol $\epsilon^{(I)}_{a_1 a_2 a_3}$ denotes the associated adjoint operator. The ‘tensor’ fields $\varphi_{I_1 I_2 I_3}$ are given in terms of the Fourier modes $\tilde{\varphi}_{I_1 I_2 I_3}$ of the Peter–Weyl decomposition of the GFT field by $\varphi_{I_1 I_2 I_3} = \tilde{\varphi}_{I_1 I_2 I_3} \prod_{i=1}^{6} \sqrt{d_I^I}$. The Boulouik theory can then be understood as a generalization of the matrix model based on 3-tensors instead of matrices.

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2 We will denote

$$\mathbb{V}_I = \mathbb{C}[e_a | a = -I, \ldots, I] = \mathbb{C}[(I, a)]_a \cong \mathbb{C}^{2I+1},$$

for the complex vector space associated with the spin $I$ unitary, irreducible representation of $SU(2)$.

3 The Clebsch–Gordan coefficients, well known from the quantum mechanics of angular momentum, are defined by the following evaluations:

$$\psi_{I J K}(e_a \otimes e_b) = \sum_{c}^{(I \ J \ K)} \begin{pmatrix} a & b & c \\ I & J & K \end{pmatrix}, \quad \text{and} \quad \psi_{I J K}(e_a \otimes e_b) = \sum_{c}^{(I \ J \ K)} \begin{pmatrix} a & b & c \\ I & J \end{pmatrix}.$$
tensor field corresponds to an elementary two-simplex whose propagation builds simplicial three-geometries. The action reads

$$S_{3D} = \sum_{\{I\}} \left| \frac{1}{2} \psi_{I_1, I_2, I_3} \right|^2 - \frac{\lambda}{4!} \psi_{I_1, I_2, I_3} \psi_{I_1, I_4, I_5} \psi_{I_2, I_6, I_7} \left\{ I_1, I_2, I_3 \right\} \right|, \quad (13)$$

where the summation symbol is a sum over all spins $I$ (and implicitly all magnetic numbers $a$) appearing on the right-hand side of the sum and

$$\left\{ I_1, I_2, I_3 \right\} = \left\{ a_1, a_2, a_3 \right\} \left\{ I_1, I_2, I_3 \right\} \left\{ I_4, I_5, I_6 \right\} \left\{ I_7 \right\}, \quad (14)$$

is the $6j$-symbol in a particular orientation configuration.

Once again, the GFT is built such that its Feynman diagram evaluations generate the path integral of three-dimensional gravity discretized on the triangulations dual to the two-complexes defined by the diagrams. Indeed, if $T^*$ denotes a Feynman diagram of Boulatov’s GFT, we have the following equality:

$$I_{\text{GFT}}(T^*) = Z_{\text{PR}}(T) \equiv \sum_{\{I_e\}} \prod_e dI_e \prod_t \left\{ I_{1t}, I_{2t}, I_{3t} \right\} \left\{ I_{4t}, I_{5t}, I_{6t} \right\}, \quad (15)$$

where the products are, respectively, on the edges $e$ and tetrahedra $t$ of the triangulation $T$, dual to the cellular complex $T^*$, on which the Ponzano–Regge model is defined.

To summarize, GFTs appear to be extremely powerful tools for generating simplicial manifolds of arbitrary topologies and dimensions. Their use is further highlighted in the study of non-topological theories where one needs to recover the infinite number of degrees of freedom from lattice discretized models. That said, a clear physical interpretation of the GFT is still lacking despite some progress [7]. What is the physical content of the fields? What is the meaning of the coupling constant? What are the classical symmetries? What are the non-perturbative properties of these theories? These issues remain open. In the following, we reconsider 2D GFTs as non-abstract theories and introduce a physical field theory sharing many GFT properties but whose physical interpretation is perfectly controlled.

1.1. 2D spinfoams embedded into 3D spinfoams: effective field theory

The coupling of point particles to the PR model [14] has unravelled an intriguing relationship between the Feynman diagrams of quantum field theory (QFT) and spinfoam models: Feynman diagrams have been shown to yield natural quantum gravity observables [15, 16]. Moreover, the expectation values of such observables, attached to graphs embedded in the spinfoam, can be re-expressed as deformed Feynman amplitudes issued from a QFT on a flat, non-commutative spacetime. This leads to the notion of an effective field theory (EFT) describing the dynamics of matter once the quantum gravity fluctuations have been integrated out [1].

The non-commutativity in spacetime coordinates is encoded in a well-defined $\star$-product (which is not of Moyal type) whose detailed construction can be found in the original work [1, 17]. The key point of that analysis is that although spacetime remains flat, the momentum space becomes a curved manifold $G$ isomorphic, as a manifold, to Spin($\eta$). In the limit where the Newton constant $G_N$ is sent to zero, the star-product becomes commutative, momentum space becomes flat and one recovers ordinary QFT. The crucial point that we will develop in the what follows is the fact that such Feynman diagrams are, in fact, two-dimensional spinfoams. Accordingly, we are led to the study of spinfoam models embedded into higher dimensional quantum geometrical backgrounds. From now on, we will concentrate on the Riemannian $G = \text{SU}(2)$ case. We expect that the core of our results will translate to the non-compact case.
Consider the generalized two-dimensional GFT defined by the (momentum space) action
\[
S[\phi] = \frac{1}{2} \int_G dg (g) \mathcal{K}(g) \phi(g^{-1}) - \sum_n \frac{\lambda_n}{n!} \int_G \prod_{i=1}^n dg_i \delta(g_1g_2 \cdots g_n) \prod_i \phi(g_i). \tag{16}
\]
Here the group variables \(g\) are now considered as deformed momenta of (scalar) particles propagating in 3D quantum gravity, in contrast with the non-embedded, abstract 2D GFT. We will assume that the usual symmetry requirements of the GFT on the scalar field \(\phi\) have been relaxed. The group function \(\mathcal{K}\) denotes the kinetic kernel in momentum space, which is assumed to obey an \(\text{Ad}(G)\)-invariance, that is, to be central. Finally, \(\lambda_n\)'s are the coupling constants for the \(n\)-vertex interactions of the scalar field and \(\delta(g_1 \cdots g_n)\) describes the momentum conservation at each vertex.

Let \(\Gamma \subset M\) denote a Feynman diagram of the theory. It can be embedded into a triangulated two-surface \((\Sigma, \Delta)\), which, in turn, can be embedded into a 3D triangulation \(T\) of the spacetime manifold \(M\). If the triangulation \(T\) is homeomorphic to the three-sphere, the Feynman amplitude \(I(\Gamma)\) associated with the diagram \(\Gamma\) can be written as a three-dimensional state sum model \(\mathcal{Z}(\Gamma, T)\) via a topological duality transformation \([1, 17]\) and integrated over the group (momentum) variables:
\[
I(\Gamma) = \mathcal{Z}(\Gamma, T) \equiv \sum_{\{I_e\}} \prod_{e \in \Gamma} dI_e \prod_{e \in \Gamma} \mathcal{P}(I_e) \prod_t \left\{ \begin{array}{c} I_{t_1} \quad I_{t_2} \quad I_{t_3} \\ I_{t_4} \quad I_{t_5} \quad I_{t_6} \end{array} \right\}, \tag{17}
\]
where \(\mathcal{P}(I)\) is the Fourier mode appearing in the Peter–Weyl decomposition of the propagator \(\mathcal{P} \equiv i\mathcal{K}^{-1} = \sum \mathcal{P}(I) \chi^I\) in terms of characters \(\chi^I(g) = \text{tr} D^I(g)\). It is a function of the length quantum numbers of the three-dimensional quantum geometry theory. This is due to the fact that the two-dimensional spinfoam is embedded into a (triangulated) three-dimensional spacetime. More precisely, the discretization procedure of 3D gravity leading to the Fonzano–Regge model assigns a physical length vector \(t^a_e = \int e^a \in \mathbb{R}^3, a = 1, 2, 3\), to each one-simplex \(e\) of the chosen triangulation \(T\), where \(e^a = e^a_{\mu} dx^\mu\) is the dual co-frame expressed into a particular local basis \(\{dx^\mu\}_\mu\) of the co-tangent space. In the quantum framework, the lengths are quantized and take only discrete values (because we are working in the Riemannian framework thus with a compact gauge group) encoded in the representation labels \(I_e\) assigned to the one-simplices \(e\) of the triangulation. Accordingly, the mode \(\mathcal{P}(I_e)\) can be understood as the propagator assigned to the edge \(e\) of the diagram in (quantum) position space; it depends only on the length separating its boundary vertices \(t(e), s(e)\). It is the signature of the presence of a particle, i.e., of the obstruction to the flatness of the connection along the corresponding edge.

Let us now make the description more concrete and consider specific choices of propagators.

- \(\mathcal{P}(g) = 1\). This leads to the ordinary 2D GFT action (provided the fields obey the appropriate symmetry conditions) regarded from a three-dimensional perspective. The corresponding spinfoam amplitude is that of the PR model with insertion of an observable fixing the representations label to zero along the edges of the Feynman diagrams: \(\mathcal{P}(I) = \delta_{I_0}\).

Since the spins are interpreted as length quantum numbers, we can readily see that the 2D GFT forces the lengths of the one-simplices supporting the edges of the Feynman diagram to vanish. In other words, the embedding of the diagrams of the MM into the (simplicial) spacetime manifold is degenerate (non-injective). It defines an immersion where all the points of the surface are mapped onto a point. This is due to the fact that the GFT propagator along an edge \(e\) is given by a delta function on position space:
$G(x_{\ell(e)} - x_{\ell'(e)}) = \delta(x_{\ell(e)} - x_{\ell'(e)})$ in the Abelian no-gravity limit. In the $G_N \neq 0$ case, the absence of resolution beyond the Planck length in non-commutative spacetime implies the replacement of the delta function by its deformed analogue which is also concentrated on zero lengths but has non-zero width (it is the first Bessel function [17]). It is interesting to remark that the obtained spinfoam amplitude is, in fact, a gauged fixed Ponzano–Regge partition function [14]. The gauge fixing procedure, i.e., the killing of the sum over representations, occurs along the 2D GFT diagram $\Gamma$. Note, however, that the GFT can generate the maximal trees [14] usually used to gauge-fix the gauge symmetries in the Ponzano–Regge model, only at the classical, or tree level.

- $\mathcal{P}(g) = \delta(g)$. This choice leads to a theory free of matter: there are no particles travelling in the diagram, i.e., there are no topological defects in spacetime. As remarked in [19], modifying the 2D GFT so as to include such a propagation term leads to the exact Ponzano–Regge amplitude.
- $\mathcal{P}(g) = i(P^2(g) - M^2 - i\epsilon)^{-1}$, where $\epsilon \to 0^+$ is a regulator. Here, the momentum $P : G \to g$ is defined through the projection of the group element $g$ on the basis of Pauli matrices $\{\sigma^a\}$, spanning the tangent space $su(2)$:

$$P^a(g) := \frac{\kappa}{2i} \text{tr}(g \sigma^a),$$

(18)

with $a = 1, 2, 3$ and $\kappa$ is the Planck mass related to the Newton constant by $\kappa = (4\pi G_N)^{-1}$. The renormalized mass $M = \kappa \sin \theta$, where $\theta = \frac{\pi}{2}$ is expressed in terms of the deficit angle $\theta$ of the conical singularity created by the particle of bare mass $m$. It takes into account the gravitational feed-back. The expectation value of the associated PR observable

$$\mathcal{P}(I) \equiv P_\theta(I) = \frac{i}{2\kappa^2} \frac{e^{-ikI(\theta - i\epsilon)}}{\cos \theta},$$

(19)

corresponds to the PR amplitude with the insertion of Feynman propagators along the particles worldlines.

It is interesting to see that the 2D GFT considered as a physical field theory (not just as a computational tool) whose Feynman diagrams are embedded into spacetime can generate three-dimensional Ponzano–Regge geometries. It suggests a tight link between two-dimensional and three-dimensional spinfoam models. This paper is devoted to the detailed study of this relationship. The plan of the paper is the following. Having investigated the GFT structure of the EFT in the introduction, we show that it can be reformulated as a MM in section 2. The question of the effect of the nontrivial propagation term involving a squared momentum is analysed. As we will see, it will result in a richer structure where couplings between matrices of different sizes can occur. We will describe in detail the Feynman rules of this new theory and give an interpretation of the variables involved in terms of three-dimensional quantum geometry.

In section 3, we discuss the derivation of the EFT from Boulatov’s GFT. This issue has been tackled in [18, 19], where the authors generalize the Boulatov GFT to include particles. Here, we will give a simpler answer, showing that the effective QFT is a particular phase of the 3D GFT. We will first define a particular dimensional reduction yielding the two-dimensional GFT from its 3D counterpart, by selecting a particular 2D phase of 3D quantum gravity. We will then exhibit a one-parameter family of solutions to Boulatov’s field equations corresponding to nontrivial geometrical backgrounds. The effective QFT will then be understood as describing (surface-like) perturbations around these particular instantons of the Boulatov model. This fact highlights the interpretation of matter as excited states of geometry in three spacetime dimensions. In fact, in the most general case, we will see that
the obtained perturbed action is that of the EFT corrected by a nonlinear redefinition of the momentum in terms of higher degree momenta. We will interpret these corrections as the signature of a nontrivial background corresponding to a spacetime metric which is not flat.

Section 4 is a four-dimensional outlook where we sketch how to apply the same techniques for the 4D GFT. We conjecture that we will be led to a field theory of string-like excitations.

Finally, the appendices contain an analysis of the classical solutions to the EFT and an explicit calculation of the general three-dimensional perturbations of Boulatov’s GFT which may be needed in generalizing our work to the fermionic case.

2. 3D non-commutative field theory as a matrix model

The fact that the effective field theory (EFT) in momentum space defined by the action (16) with kinetic kernel \( \mathcal{K}(g) = P^2(g) - M^2 \) is defined in terms of fields living on a group manifold turns it into a two-dimensional GFT. We thus expect a reformulation as an MM. We now explore in detail this interesting analogy.

The expression of the EFT as an MM goes through the recasting of the action in terms of representation labels. We first use the group structure of the momentum space manifold to develop the field \( \phi \) in Fourier modes. We then consider the matrices \( \phi_{ab} \) as the dynamical fields of the EFT. Note that, \textit{a priori}, these matrices are not Hermitian, unless we impose the reality condition \( \phi(g^{-1}) = \phi^*(g) \). The action of the EFT then reads

\[
S_{\text{eff}}[\phi_I] = \frac{1}{2} \sum_{I,J} \phi_I K_{IJ} \phi_J - \sum_{n,J} \lambda_n n! \text{tr}(\phi^n_J),
\]

(20)

where we have used the notation \( \phi_I K_{IJ} \phi_J := \phi_{Iab} K_{IJK} \phi_{Jcd} \) and the kinetic kernel \( K_{IJ} \in \text{Aut}(\mathbb{V}_I \otimes \mathbb{V}_J) \) is given by

\[
K_{IJ} = d_I d_J \int_G \text{d}g \mathcal{K}(g) D^I(g) \otimes D^J(g^{-1}),
\]

(21)

with \( \mathcal{K}(g) = P^2(g) - M^2 \). To display the propagator \( \mathcal{P} \), we next invert the kinetic part of the action. Inserting matrix indices, we obtain the following propagation term,

\[
K^{-1}_{abcd} = \int_G \text{d}g \mathcal{K}(g)^{-1} D^I(g) \text{tr}(D^J(g^{-1}))_{dc},
\]

(22)

such that \( K^{-1}_{ab} K_{cd} = \delta_{bc} \delta_{de} \).

We can now express the group function \( K^{-1} \) in the basis of the unitary, irreducible representations of \( G \). To this aim, we use the fact that \( P^2(g) \equiv P^2(\alpha) = \kappa^2 \sin^2(\alpha) \) is \( \text{Ad}(G) \)-invariant and also has a central inverse\(^4\). The (regulated) inverse kernel \( K^{-1}(\alpha) = (\kappa^2 (\sin^2 \alpha - \sin^2 \theta - i \epsilon))^{-1} \) can thus be expanded in terms of characters

\[
K^{-1}(g) = (P^2(g) - M^2 - i \epsilon)^{-1} = -i \sum_I \mathcal{P}_I(\chi_I(g),
\]

(23)

with the kernel \( \mathcal{P}_I(\chi) \) defined in the previous section.

Using the integration formula of the tensor product of three representations

\[
\int_G \text{d}g D^I(g) \otimes D^J(g) \otimes D^K(g^{-1}) = \frac{1}{d_K} \Phi^{IJK}_{IJ},
\]

(24)

\(4\) Here, we are using the diffeomorphism mapping \( SU(2) \) onto the unit three-sphere \( S^3 \) in quaternion space \( \mathbb{H} \), to parametrize the group manifold through an angle \( \alpha \in [0, 2\pi] \) and a unit vector \( n \in S^2 \). Accordingly, any group element \( g \) is parametrized by \( g(\alpha, n) = \cos \alpha \mathbb{I} + i \sin \alpha n_a \sigma^a \), and the squared momentum yields \( P^2(g) = \kappa^2 \sin^2 \alpha \).
we obtain the propagator in representation space

\[ P_{IJ} = iK^{-1} \sum_{K} \frac{\mathcal{P}_0(K)}{d_J} \phi_{IJ}^{K/J} \psi_{IJ}^{K/J}. \]  

(25)

The Feynmanology of the EFT cast as a generalized MM follows immediately and is depicted in figure 1. One can verify that setting \( \mathcal{P}_0(K) = \delta_{K,0} \) in (25) leads to the correct trivial propagation of the GFT encountered previously, that is, \( P_{IJ} = \delta_{IJ}/d_J \).

We close this section with a set of remarks. The first remark is that the diagrams generated by the EFT have a fat structure, namely the propagators are built out of two lines, each carrying a matrix index. This property is shared by any theory whose dynamical fields have a matrix structure, such as, for instance, non-Abelian gauge theories where the double index structure of the gauge field (carrying e.g. colour indices) can be geometrically interpreted as the composite nature of the gauge particles [23]. Indeed, if the theory contains fermionic fields, they will carry only one colour index, as they live in colour space, which implies that the gauge field can be geometrically, and only in that sense, regarded as composed of a fermion and an anti-fermion. As a result, the Feynmanology of the EFT generates two-dimensional piecewise-linear manifolds where the two lines of the propagators correspond to the gluing of two elementary two-cells each defined by a Feynman loop. In general, the MMs or 2D GFTs generate two-complexes dual to triangulations of surfaces. The same is true here. However, if we restrict ourselves to the diagrams containing loops of order three (going through three vertices), the theory generates two-dimensional triangulations which, when embedded in a three-dimensional triangulation, are related via a duality transformation to the insertion of particles along the one-simplices (propagators) of the 3D triangulation in the Ponzano–Regge model.

Second, let us stress an important deviation from standard MMs which is the nontriviality of the propagator (see figure 1), taking into account the matter propagation. This aspect is also present in string theory where the presence of matter fields (the embedding of the worldsheet in target space) modifies the propagation properties of the dual MM formulation. The immediate consequence is the dynamical coupling of different rank matrices. More precisely, the Feynmanology teaches us that the matrices evolve dynamically into matrices of different sizes. This nontrivial fact has no gauge field theory formulation counterpart. However, to parallel the geometrical composite interpretation discussed above, we see that the matrix field of the EFT can be regarded as composed of interacting ‘fermions’ belonging...
to a theory where the number of colours changes dynamically through interactions, like the number of spin states in ordinary QFT.

Finally, we can give a geometrical interpretation of the ‘colour’ labels appearing in the EFT, that is, relate the surface theory generated by the EFT to three-dimensional quantum gravity. Here, the size of the matrices is related to the length quantum numbers of the three-dimensional quantum geometry theory because the EFT carries information about its embedding into a (triangulated) three-dimensional spacetime. We have seen that the physical quantum length numbers are given by the representation labels assigned to the one-simplices $e$ of the triangulation. At the classical level, these physical length vectors are measured with respect to a particular frame. This frame is usually chosen to be associated with a given trivialization of the spin bundle over the spacetime spin manifold by a global section, piecewise constant in each tetrahedron of the triangulation. Let us consider a such tetrahedron and focus on one of its boundary triangles assumed to support a (three-valent) Feynman loop of the EFT. The loop is built out of three propagators each carrying a spin on its end points. By virtue of the Feynman rules (figure 1), the representation labels before and after a vertex are constrained to match. Accordingly, the loop carries three spin, say $I_1$, $I_2$, and $I_3$ each sitting on one of the summits 123 of the considered triangle, taken in cyclic order. Regarding the geometrical interpretation discussed above, it appears clearly that the three spins encode the quantum distance to the origin of the chosen frame. Accordingly, the representation labels $I_{ij} = I_i - I_j$ assigned to the edges $ij$, $i$, $j = 1, 2, 3$, measure the quantum lengths of the three boundary edges. If we choose the reference frame appropriately, namely such that its origin coincides with one of the summits, say the vertex 1, and its three axis are along the three segments emerging from 1, the corresponding length accordingly reads $I_1 = 0$ and the quantum number $I_2 \equiv I$ measures the length of the segment 12. Concentrating on the associated propagator, we see that the fat structure collapses:

$$P_{ab00}^{10} = \sum_{K,e} P_0(K) \delta_{K1} \epsilon_{Ibe} \epsilon_{Iea}$$

$$= P_0(I) \delta_{ab},$$

where $\epsilon_{Iab}$ denote the matrix elements of the dual pairing intertwiner $\epsilon_I$ and the delta function is on the vector space $V_I$. Switching back to the spinfoam perspective, we can readily see that we have recovered the propagator appearing in the PR model as a sole function of the quantum length number $I$ associated with the corresponding one-simplex. This is the three-dimensional geometrical picture of the surface theory generated by the EFT; the size of the matrix fields $\phi_i$ and $\phi_j$ living at the endpoints of the propagator $ij$ encode the physical length of the one-simplex supporting the propagator. The nontriviality of the EFT propagation does not constrain the rank of the two matrices to match which would lead to one-simplices of zero physical length, as discussed in the introduction. In this sense, the effective action recast in terms of representation labels (20) can be understood as a QFT living on a discrete, quantum geometry background. The representation indices $I$ are the counterparts of the position vectors $x$ in ordinary QFT.

We have just discussed the fact that the Feynman diagrams of the EFT are two-dimensional spinfoams whose evaluations, once embedded into spacetime, yield three-dimensional spinfoam amplitudes of particles propagating in the 3D PR geometry. Now, we know that the PR partition function can be related to the Feynman integrals of a 3D GFT. This leads to the following natural question: is it possible to relate the EFT to a three-dimensional GFT? As we explain below, the EFT, and thus matter excitations, can be understood as a 2D phase of Boulatov’s 3D GFT.
3. Matter as perturbations around classical solutions to the 3D GFT

We now show how to obtain the EFT dynamics as perturbations around a nontrivial geometrical background defined by a classical solution to the 3D GFT. As we are about to see, these perturbations are of a special type, namely they are surface-like excitations. We call these types of perturbations shape two-dimensional phases of Boulatov’s theory.

We start by showing how to reduce the Ponzano–Regge model to its two-dimensional sub-sector. We define the 2D phase of the Boulatov GFT by restricting the set of fields to those depending only on two of their arguments which in turn are functions of a unique group element by virtue of gauge invariance:

\[ \varphi(g_1, g_2, g_3) := \psi(g_1, g_3) = \psi(g_1 g_3^{-1}). \]  

(28)

Then it is easy to check that

\[ S_{3D}[\varphi = \psi] = S_{2D}[\psi], \]

because we are considering normalized Haar measures yielding a unit volume for the compact group \( G \). If we only consider the tetrahedron interaction term in the 3D GFT, we will obtain \( \psi^3 \) and \( \psi^4 \) interaction terms for the 2D phase. If we want to obtain interaction terms with higher powers of the field \( \psi \), we can include higher order terms in the 3D GFT.

From the point of view of the Fourier transform \( \varphi_{I_1 I_2 I_3} \), the restriction \( \varphi = \psi \) implies setting \( I_2 = 0 \) and thus reducing the 3-tensors to (square) matrices: the triangles of the 3D GFT collapse to double lines since we have forced one of the boundary segments to have zero length. Accordingly, we only generate 2D cellular decomposition and not 3D triangulations anymore; the ansatz (28) defines the two-dimensional phase of the Boulatov group field theory.

Next, we introduce the ‘instantons’ 5 around which we are willing to study classical perturbations. Consider the one parameter family of fields

\[ \varphi_f(g_1, g_2, g_3) = \sqrt{\frac{3!}{\lambda}} \int_G dh \delta(g_1 h) f(g_2 h) \delta(g_3 h), \]  

(29)

which is parametrized by the function \( f : SO(3) \to \mathbb{R} \), an arbitrary function of \( L^2(SO(3)) \). The reality of \( f \) is imposed by the reality condition constraining the fields of the 3D GFT. The choice of \( G = SO(3) \) is to impose a purely even Fourier decomposition to avoid non-analyticity issues. We also require \( f \) to satisfy the following normalization condition:

\[ \int_G \text{d}g \ f^2 = 1. \]  

(30)

An example of such a function is provided by the character \( \chi^I, I \in \mathbb{N} \). This field provides a whole family of solutions to the field equations of the Boulatov group field theory:

\[ \varphi(g_3, g_2, g_1) = \frac{\lambda}{3!} \int dg_4 dg_5 \varphi(g_3, g_5, g_4) \varphi(g_4, g_2, g_6) \varphi(g_6, g_5, g_1). \]  

(31)

Accordingly, \( \varphi_f \) defines a nontrivial, three-dimensional background geometry. Even if the link between GR and the GFT at the classical level is obscure, we make a first step toward clarifying this issue in this paper and parallel the procedure of studying perturbations of Einstein’s theory around a given classical solution. Thus, we now question the classical GFT about its behaviour in the neighbourhood of the solution \( \varphi_f \), i.e., study the first-order perturbations around this ‘instantonic’ solution:

\[ S^{(f)}_{3D}[\varphi] \equiv \delta S_{3D}[\varphi] = S_{3D}[\varphi_f + \varphi] - S_{3D}[\varphi_f]. \]  

(32)

5 Let us nevertheless point out that the evaluation of the action on these solutions diverges because of a factor \( \delta(\mathbb{1}) \).
We obtain the usual quadratic term in $\phi^2$ corrected with $\phi^2\phi^2\phi$ terms and the usual quartic interaction in $\phi^4$ plus a new $\phi^3\phi f$ cubic term [20] (see appendix B for a detailed computation). For the present analysis, we are particularly interested in 2D perturbations. To this aim, we expand

$$ \psi(g_1, g_2, g_3) = \phi_f(g_1, g_2, g_3) + \psi(g_1, g_3). $$

This leads us to the following key result:

$$ S_{3D}^{(f)}[\psi] = \frac{1}{\kappa^2} \int_{G} dg \psi(g) K_f(g) \psi(g^{-1}) - \frac{\mu}{32} \int_{G^{\ast}} \prod_{i=1}^{3} dg_i \delta(g_1 \cdots g_3) \prod_{i=1}^{3} \psi(g_i) $$

$$ - \frac{\lambda}{4!} \int_{G^{\ast}} \prod_{i=1}^{4} dg_i \delta(g_1 \cdots g_4) \prod_{i=1}^{4} \psi(g_i). $$

Here, the kinetic term is given by

$$ \forall g \in G, \quad K_f(g) = \frac{\kappa^2}{2} \left[ 1 - 2 \left( \int_{G} dh f(h)^2 \right) - 1 \left( \int_{G} dh f(h)f(hg) \right) \right]. $$

where the quadratic contribution in $(\int f)^2$ comes from quartic interactions of the type $\int \phi_f \phi_f \psi \psi$ while the convolution term in $f \circ f$ comes from $\int \phi_f \phi_f \psi$ contributions. From here on, we require that $f$ is central[6], i.e $Ad(G)$-invariant. This is to ensure that the kinetic term $K(g)$ be constant on the conjugacy classes and thus depend only on the norm of the momentum (and not its direction).

The interaction term involves a cubic and a quartic coupling. The cubic term is issued from quartic interactions of the form $\int \phi_f \phi_f \psi \psi$. The strength of the coupling is governed by the coupling constant $\mu = \sqrt{6\lambda(\int f)}$ which depends on the quartic coupling $\lambda$.

This perturbed action is the action of an EFT with a nonlinear redefinition of the momentum with cubic and quartic interactions. Indeed, decomposing the central function $f$ on the basis of characters, $f = \sum_I f_I \chi_I$ with the normalization constraint $\sum_I f_I^2 = 1$, we can separate the kinetic part into a generalized momentum term, which vanishes on zero momenta (i.e. on $g = 1$), and a constant term yielding a massive contribution

$$ \forall g \in G, \quad K_f(g) = Q_f^2(g) - M_0^2, $$

with the generalized momentum $Q_f$ given by

$$ Q_f^2(g) = \frac{\kappa^2}{2} \sum_{I \in \mathbb{N}} f_I^2 \left( 1 - \frac{\chi_I(g)}{d_I} \right), $$

and the mass term by

$$ M_0^2 = \kappa^2 f_0^2 \leq \kappa^2. $$

Furthermore, since the (absolute value of the) character associated with the representation $I$ is bounded by the associated dimension, $\forall g \in G, \forall I \in \mathbb{N}, |L_I(g)| \leq 1$, the squared momentum

6 Note, however, that there are no technical obstructions in relaxing the conjugation invariance requirement. The theory would be non-isotropic, but still mathematically well-defined. $f$ admits a generic Peter–Weyl decomposition $f(g) = \text{tr} f_I D^I(g)$, where $f_I$ is a $d_I \times d_I$ matrix. The normalization condition reads $\int f^2 = \sum_I \frac{1}{d_I} \text{tr} f_I^2 = 1$, with $\epsilon_I$ the isomorphism between the representation $V^I$ and its complex conjugate as introduced previously. Then the kinetic term in the action would be expressed in terms of the convolution:

$$ f \circ f(g) = \sum_I \frac{1}{d_I} \text{tr} f_I^2 f_I D^I(g). $$

Thus the kinetic term will depend explicitly on the momentum $P$ and not only on its norm $P^2$. 

$Q_f^2$ is an absolutely convergent series bounded from below by zero and from above by the squared Planck mass,
\[
\forall g \in G, \quad 0 \leq Q_f^2(g) \leq \kappa^2.
\]
Accordingly, the kinetic term of the perturbed action is always positive which implies that the theory is free from instabilities. Actually, the $\kappa^2$ bound is loose, the true maximal bound depends on the function $f$. The important point is that the momentum $Q_f$ is still bounded, reflecting the compactness of the gauge group $SU(2)$. Finally, we can express the cubic coupling constant in terms of the mass $M_Q$. Since $M_Q = \kappa \int f = \kappa f_0$, it is straightforward to obtain the relation\(^7\)
\[
\lambda = \frac{\mu^2 \kappa^2}{3M_Q^3},
\]
between the cubic and quartic coupling constants. Assuming that $\lambda$ is held fixed, if $\int f = 0$, then both $M_Q$ and $\mu$ vanishes: the theory is massless and without cubic interaction. On the other hand, assuming $\mu$ fixed, then the quartic coupling $\lambda$ is determined by its ratio with the (dimensionless) mass $M_Q/\kappa$: the cubic term will prevail for large $f_0$ while $\lambda$ would blow up as $f_0$ goes to 0.

The generalized momentum $Q_f$ is, in fact, a nonlinear redefinition of the momentum $P$ of the EFT. As we have seen above, $Q_f$ is given by an infinite sum over representation labels $I$. Each term of order $I$ generates powers of the momentum of degree $2I$. Indeed, the characters can be expressed in terms of the Chebyshev polynomials of the second kind:
\[
\chi^I(\alpha) = U_{2I}(\cos \alpha) = \sum_{r=0}^{I} (-1)^r \binom{2I-r}{r} (2\cos \alpha)^{2I-2r}.
\]
This only involves powers of the squared cosine\(^8\) which can in turn be expressed as powers of $\sin^2 \alpha = P^2(\alpha)/\kappa^2$. Accordingly, the characters are polynomials in $P^2$:
\[
\chi^I = d_I - \sum_{n=1}^{I} d_n c_n^{(I)} (\frac{2}{\kappa} P)^n,
\]
and we obtain
\[
Q_f^2 = k_1[I] P^2 + k_2[I] \frac{P^4}{\kappa^2} + \cdots,
\]
where $k_n[I] = \frac{1}{2} \sum_I c_n^{(I)} f_I^n \in \mathbb{R}$. The explicit value of these coefficients depends on the coefficients $c_n \in \mathbb{R}$ computable order by order through the definition of the Chebyshev polynomials. For instance, the first-order coefficient\(^9\) is given by $k_1[I] = \frac{1}{2} \int f \Delta f$ with $C(I) = I(I+1)$ and $\Delta$ respectively denoting the Casimir and Laplace operator on $SU(2)$.

\(^7\) This reminds of the relation between 't Hooft's coupling constant $\lambda$ and the coupling constant of the Yang–Mills interaction $g_N^2 = \frac{g_N^2}{g_N^2 N}$, where $N$ is the number of colours of the Yang–Mills theory.

\(^8\) This is the reason why we required that $f$ be a function on $SO(3)$ allowing only even modes in the Peter–Weyl decomposition, otherwise we would get terms in $\sqrt{1 - (P/\kappa)^2}$. Actually, to work with a well-defined EFT defined on $SU(2)$ and not only $SO(3)$ requires a four-dimensional point of view \([22]\). The four-momentum is defined as $\pi_t = P$, and $\pi_s = \kappa \sqrt{1 - (P/\kappa)^2}$, with the mass-shell condition $\pi_s^2 = \kappa^2$.

\(^9\) We can compute these coefficients by matching the Taylor expansion of the character in $\alpha$,
\[
\chi^I(\alpha) = \chi^I(e^{2i\alpha t_j}) = \text{tr}_I(1) + i 2\alpha \text{tr}_I J_z - 2^3 \frac{\alpha^2}{3!} \text{tr}_I J_z^2 - i 2^3 \frac{\alpha^3}{3!} \text{tr}_I J_z^3 + o(\alpha^4),
\]
with its expansion in $\sin^2 \alpha$:
\[
\chi^I(\alpha) = U_{2I}(\cos \alpha) = d_I - \sum_{n=1}^{I} d_n c_n^{(I)} \sin^2 \alpha = d_I - d_I c_I^{(I)} \alpha^2 + o(\alpha^4),
\]
Thus $d_I c_1 = 2 \text{tr}_I J_z^2 = \frac{2}{3} \text{tr}_I J_z^2 = \frac{2}{3} d_I I(I+1)$. Note that $\text{tr}_I J_z^2 = \text{tr}_I J_z^3 = 0$. 

\[Q^2\] is an absolutely convergent series bounded from below by zero and from above by the squared Planck mass,
\[
\forall g \in G, \quad 0 \leq Q_f^2(g) \leq \kappa^2.
\]
Note that $\sum f^2 I C(I)$ does not necessarily converge for arbitrary $f \in L^2$. Actually the higher order coefficients $k_n[f]$ will have similar expressions involving higher powers of the Casimir $C(I)$. The simplest assumption in order to get a meaningful perturbative expansion is that the mode decomposition of $f$ is finite, i.e., involves a finite number of representations (this can be naturally achieved by choosing $f$ appropriately or by replacing SU(2) with the quantum group $U_q(\mathfrak{su}(2))$ at root of unity).

Hence, we have shown that the generalized momentum $Q_f$ can be defined perturbatively in inverse powers of the Planck mass. Let us keep in mind that the Planck mass in 3D is simply the inverse Newton constant $G_N$ and does not contain any factor in $\hbar$. Thus the perturbation of the momentum $Q_f$ in $1/\kappa$ is purely classical and does not require a quantum gravity interpretation.

The order zero, corresponding to the solution $f = 0$, is simply a group field theory where the matter degrees of freedom are frozen (trivial propagator). The first order gives the EFT dynamics while the higher orders provide corrections to the EFT in $1/\kappa$. The mass of the EFT and the corrections are dictated by the function $f$, that is, by the classical solution (29) to Boulatov’s theory. These high order modes correspond to higher derivatives in the matter action. They can be interpreted as further gravitational corrections to the scalar field dynamics producing new (unphysical) resonances. The question is now to understand how to interpret these new corrections, keeping in mind that some specific solutions do not yield any corrective terms; they only appear in the most general case. Our explanation of this fact is the following.

First, note that the presence of high order derivatives is a common fact when working with effective field theories where the high energy modes have been integrated out. For instance, the first-order term in the expression of the generalized momentum, when $Q_f^2 = P^2$, already involves arbitrary high derivatives by virtue of the $\star$-product associated with the group Fourier transform mapping the momentum group manifold onto non-commutative spacetime [17]. However, here, we are generating corrections to a theory which is already deformed, not simply to flat QFT. We interpret these corrections to the EFT’s dynamics as a signature of the non-flatness of the metric corresponding to the background geometry generated by the instanton around which we are perturbing. Indeed, in the seminal work [1, 17], the action for the EFT is derived from a theory of shape point particles creating local conical defects in spacetime. As a result, the EFT is formulated on a flat (non-commutative) spacetime and describes the matter field dynamics once the gravitational fluctuations around a flat metric have been integrated out. However, the three-dimensional Einstein equations in the presence of a scalar shape field allow, as classical solutions, more complicated metrics than a simple locally flat, spinning cone metric generated by point particles. Accordingly, it should possible to write the effective field theory of the scalar field coupled to gravity on a curved geometry by integrating out the gravitational fluctuations around the chosen background geometry. We interpret our instantonic solution as generating such a background geometry solution to Einstein’s equations in the presence of a scalar field and not simply in the presence of a (finite) collection of point particles. From this perspective, we interpret the deformed momentum $Q_f$ as the Fourier transform of a covariant derivative for a non-flat metric, mapping the momentum group manifold to a curved spacetime manifold. This momentum can then be re-expressed as a nonlinear function in the ‘flat’ momentum $P$. This interpretation would be confirmed by studying the field+gravity fluctuations around a nontrivial classical metric. We postpone these investigations to future work.

We close this discussion with a remark concerning the generalized momentum $Q_f$. It is important to note that the mass term $M_Q$ is the mass with respect to the generalized momentum $Q_f$. Namely, it is a singularity of the propagator $(Q_f^2 - M_Q^2)^{-1}$. However, it is
not the ‘physical’ mass $M_P$ with respect to the flat momentum $P$ which would be defined as
the singularity for the propagator $(Q_f[P]^2 - M_P^2)^{-1}$ expressed in terms of $P$.

To conclude this section, we tune the classical solution (29) as a means to first obtain
exactly the EFT and then to compute the first-order corrections. Indeed, we can choose the
function $f$ such that its mode decomposition involves solely terms of order lower than 2, i.e.,
$f_I = 0, \forall I \geq 2$. In this case, $f$ is a linear combination of the characters of the trivial and
adjoint representations,

$$f = f_0 + f_1 \chi^1,$$

with the constraint $f_0^2 + f_1^2 = 1$. This choice leads to the kinetic term

$$K_f(g) = k_1[f]P^2(g) - \kappa^2 f_0^2,$$

that is, the action of a massive scalar field of mass $M_P^2 = \frac{M_P^2}{k_1[f]} = \frac{\kappa^2}{2} f_0^2$. Note that $M_P$ is
different from $M_Q$.

We can also generate massless actions simply by choosing an instanton associated with a
function $f$ such that its mode decomposition involves a single spin $J$, i.e,

$$f = \pm \chi^J.$$

Accordingly, the kinetic term of the perturbed action yields

$$K_f(g) = P^2(g) \left[ \frac{1}{3} C(J) + \frac{1}{2} \sum_{n=1}^{J} c_n^{(J)} \left( \frac{P(g)}{\kappa} \right)^{2(n-1)} \right].$$

We can readily see that, regarding the momentum $P$, the zero mass mode remains a solution
but we have generated new resonances as soon as $J$ is greater than 1. These resonances will
always be unphysical, that is, either complex or heavier than the Planck mass because of
the positivity of $Q_f$. For instance, we can compute the first correction by considering the
background geometry defined by the function $f = \pm \chi^2$. This choice leads to a massless
scalar field action with quartic momenta $K_f = 2P^2 - \frac{2}{\kappa^2} P^4$. Consequently, we obtain a new
(physical) resonance $M_{(2)} = \kappa \sqrt{3}/2 > \kappa$ in addition to the massless mode.

4. Four-dimensional outlook

This last section is dedicated to a discussion of the four-dimensional extension of the above
work. The four-dimensional GFT was first written by Ooguri in [24]. The Feynmanology
of the theory generates spinfoam amplitudes of 4D BF theory with semi-simple, compact
symmetry group $G$ discretized on two-complexes dual to four-dimensional triangulations.
The action is a functional on the space of complex fields on $G \times 4$ given by

$$S_{4D}[\psi] = \frac{1}{2} \int_{G \times 4} \psi(g_1, g_2, g_3, g_4) \psi(g_4, g_3, g_2, g_1) - \frac{\lambda}{5!} \int_{G \times 10} \prod_{i=1}^{10} dg_i \psi(g_1, g_2, g_3, g_4)$$

$$\times \psi(g_4, g_5, g_6, g_7) \psi(g_7, g_8, g_9, g_6) \psi(g_9, g_6, g_2, g_1) \psi(g_10, g_8, g_5, g_1),$$

where we require the field $\psi$ to satisfy the same reality and symmetry requirements than in
two and three dimensions:

$$\psi(g_1, g_2, g_3, g_4) = \overline{\psi}(g_4, g_3, g_2, g_1) \quad \text{and} \quad \psi(g_1h, g_2h, g_3h, g_4h) = \psi(g_1, g_2, g_3, g_4),$$

$\forall h \in G.$
The classical field equations are given by
\[
\varphi(g_4, g_3, g_2, g_1) = \frac{\lambda}{4!} \int_{G^{10}} \prod_{i=5}^{10} dg_i \varphi(g_4, g_5, g_6, g_7) \varphi(g_7, g_8, g_9) \varphi(g_9, g_6, g_2, g_10) 
\times \varphi(g_{10}, g_8, g_5, g_1).
\]

An immediate generalization of the techniques developed above leads to the identification of the following two-parameter family of classical solutions,
\[
\varphi_{f_1, f_2}(g_1, g_2, g_3, g_4) = \sqrt{\frac{4!}{2}} \int_G dh \delta(g_1 h) f_1(g_2 h) f_2(g_3 h) \delta(g_4 h),
\]
labelled by the couple of functions \((f_1, f_2) \in L^2(G)^{x^2}\) satisfying the normalization constraint \(\int dg f_1(g) f_2(g) = 1\). An example of such a couple of functions is given by the characters, \(f_1 = f_2 = \delta\). This ansatz can easily be generalized to the group field theories for the Barrett–Crane model [3].

Following the proposal developed in this paper, we can try perturbing the 4D GFT around these classical solutions by a lower dimensional phase and analyse the resulting dynamics. It is straightforward to check that two-dimensional perturbations do not acquire any nontrivial propagator. Then we would need to move to 3D perturbations. We expect that the obtained action will lead to a field theory whose fundamental excitations are no longer particles but one-dimensional manifolds, i.e., a string field theory.

In such a framework, the Feynman diagrams of the theory are networks \(\Sigma\) of elementary surfaces; the propagators are assigned to surfaces along which two three-cells (the Feynman loops of the theory) are glued together. Accordingly, discretizing the surface \(\Sigma\) and choosing a triangulation \(T\) adapted to such diagrams \((\Sigma \subset T)\), it is possible to perform a (four-dimensional) duality transformation on the associated amplitudes by replacing the 'momenta' associated with the triangles \(\Delta \subset T\) by dual variables assigned to the tetrahedra sharing the given triangle, or equivalently to the dual edges \(e^\nu \subset T^*\) of the dual triangulation. We expect that these transformed amplitudes correspond to the spinfoam model [26] of the canonical Baez–Perez proposal [27]:

\[
Z(T, \Sigma) = \sum_{\{\rho_\Lambda\}_{\Lambda \in \Sigma}} \prod_{\Lambda \in \Sigma} d_{\rho_\Lambda} \prod_{\rho_\Lambda} \mathcal{P}_T(\rho_\Lambda) \prod_{s} \{15j\}_s,
\]

where the symbol \(s\) labels the four-simplices of the triangulation, \(\{15j\}_s\), denotes a 15j-symbol constructed from the ten representations and the five intertwining operators associated with the ten triangles and five tetrahedra building a given four-simplex \(s\), \(\mathcal{P}_T(\rho_\Lambda)\) is the

\footnote{Following the original paper [12], the Barrett–Crane spinfoam amplitude can be generated by the same GFT action (40) as the topological BF theory, but the field \(\varphi\) must be constrained. The symmetry group is the Lorentz group \(G = SO(n)\), where \(n = (\sigma^2, +, +, +)\) with \(\sigma = 1\) in the Riemannian case \(\sigma = i\) in the Lorentzian case. We introduce two projectors,
\[
\mathcal{P}_\varphi(g_i) = \int_{SO(n)} dg_i \varphi(g_i, g), \quad \mathcal{Q}_\varphi(g_i) = \int_{SO(n)^4} [dh_i h^*_i] \varphi(g, h_i),
\]
respectively projecting onto gauge invariant fields and \(SO(n)\)-invariant fields, where \(\varphi\) is a three-dimensional flat metric whose isometry group leaves a given fixed internal vector invariant. Accordingly, its signature fixes \(SO(n)\) to be \(SO(3)\) in the Riemannian case and \(SO(3)\) or \(SO(1, 2)\) in the Lorentzian case. The Barrett–Crane model is defined by the restriction \(\varphi \in \text{Im} \mathcal{P}_\varphi\). Since \(\mathcal{P}\) and \(\mathcal{Q}\) do not commute, the operator \(\mathcal{P}_\varphi \mathcal{Q}\) is not a projector. This creates normalization ambiguities for the field \(\varphi\) which lead to ambiguities in the precise GFT interaction for the constrained model [25]. We can generalize the classical solutions found for the full GFT to this constrained GFT by applying the operator \(\mathcal{P}_\varphi \mathcal{Q}\) to the field \(\delta(g_1 h_1) f_1(g_2 h_1) f_2(g_3 h_1) \delta(g_4)\). In particular, the \(\mathcal{Q}\) projection implies that the fields will decompose only onto the simple representations of \(SO(n)\) and that we will work with the spherical kernels on \(SO(n)\) instead of the characters.}
string 'propagation' term depending on the unitary, irreducible representation label \( \rho_\Delta \) of \( G \) associated with the triangle \( \Delta \) and on the string tension \( T \). The details of this four-dimensional construction are currently under study [28].

5. Conclusion

The study of the EFT of matter coupled to quantum gravity achieved in this paper has revealed two new aspects. The first is that this field theory can be re-expressed as a generalized matrix model. The nontrivial dynamics imply that the matrices can change size during propagation. The novelty is the interpretation of the dimensions of the matrices from the three-dimensional quantum gravity perspective. It appears clearly that the size of the matrix fields encodes the physical length quantum numbers of the Ponzano–Regge model. The second major aspect unravelled by the present work is the role of classical solutions to the GFT. We have identified a one-parameter family of solutions to Boulatov’s field equations such that perturbations around these nontrivial geometrical backgrounds generate the dynamics of the EFT. In fact, the most general solutions lead to higher order derivative corrections to the EFT. We have interpreted these corrections as the signature of the non-flatness of the background geometry around which we have integrated the gravitational fluctuations. This geometry is generated by the instantonic solutions to the GFT, and can be understood as the geometry associated with a solution to the Einstein equations in the presence of a scalar field. As a result, the associated EFT would be defined on a curved spacetime. This is in contrast with the original setting of the (flat) EFT describing the dynamics of matter once the gravitational fluctuations around a flat background geometry, punctured with local topological defects, have been integrated out. The proof of these interpretations will be studied elsewhere. To make progress in this direction, it is now urgent to study in great detail the precise relationship between the group field theory and general relativity as classical theories.

The lessons of this paper are two-fold. First, the crucial role played by the non-perturbative aspects of the GFT. It has been suspected for a long time that these effects should play a significant role in the formulation of quantum gravity theories [7, 20]. Here, for the first time, we have explicitly shown that nontrivial solutions to the classical field equations could generate nontrivial geometrical properties of spacetime such as, for instance, matter propagation on a quantum gravity background. This leads to the second point emphasized by our work: matter is defined by excited geometry states. Indeed, we have generated the dynamics of matter purely from field solutions to the (vacuum) GFT. We have not introduced matter degrees of freedom by hand, as in the pure spinfoam context; matter appears as a particular phase of the field theory of simplicial geometry, i.e., the GFT. In this sense, the work presented here is in striking contrast with the GFT models containing extra data to model the particle content of the theory [18, 19]. In fact, our work shows that no extra data are needed: matter is a particular phase of the geometry and is already somehow contained in our quantum gravity models. It would be interesting to extend our results to the non-scalar case, that is, generate the effective field theory of spinning fields from the GFT following the procedure developed here.

It seems therefore tempting to apply the same type of rationale to the four-dimensional case. We have discussed the prototype ideas in the last section of this paper. It now appears clearly that the extension of the three-dimensional spinfoam quantum gravity formalism to 4D will require the introduction of string-like excitations of the geometry. We have derived solutions to Ooguri’s field equations and the next logical step is to study the perturbations around the background defined by such a solution. Of course, four-dimensional BF theory is non-geometrical until the simplicity constraints are implemented. Accordingly, the physical
situation will require some extra incomes, such as, for instance, starting from the Barrett–Crane GFT [12].

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Appendix A. Field solutions to the effective non-commutative QFT

In this appendix, we show how to compute some classical solutions to the equations of motion of the effective non-commutative QFT defined by the action $S_{\text{eff}}$. On the one hand, it is interesting to compare them to the classical solutions of the 2D group field theory and see the effect of the new kinetic term with $P^2(g)$. On the other hand, it provides us with new classical solutions to the Boulatov–Ooguri 3D group field theory. Determining the classical solutions of the GFT will naturally provide informations on the semi-classical regime of spinfoam models [7, 21].

We work with a cubic interaction term, but we hope that these considerations will be generalizable to higher order interactions. The equation of motion of the 2D GFT is [20]

$$\phi(g) = \frac{\lambda}{2} \int_G \phi(h) \phi(h^{-1} g) = \frac{\lambda}{2} \phi \circ \phi(g),$$  \hspace{1cm} (A.1)

where $\circ$ stands for the convolution product on SU(2). The classical field $\phi$ is thus a projector (up to the factor $\lambda/2$ for $\circ$). Looking for solutions invariant under conjugation, i.e., which only depends on the angle of rotation $\theta$ of the group element $g$, the only solutions are actually the characters $\chi^j$: $\phi(g) = d_j \chi^j(g) \lambda/2$ for each spin $j \in \mathbb{N}/2$ provides an infinite number of solutions of the equation of motion.

Now let us look at the equation of motion for the EFT for a massless scalar field. The only difference is the factor $P^2(g)$:

$$P^2(g) \phi(g) = \frac{\lambda}{2} \phi \circ \phi(g).$$  \hspace{1cm} (A.2)

Since $P^2(g)$ contains $\chi^1(g)$, these equations are going to couple the different representations and the solutions will not be as simple as for the 2D GFT. In order to solve this equation, it is more convenient to expand the field $\phi$ in representations. Once again, we will only look for fields invariant under conjugation. We will also assume that $\phi$ is even (it is a field over SO(3)) and only decomposes onto integer spins. Then we decompose $\phi$ onto the characters: $\phi(g) = \sum_{j \in \mathbb{N}} \phi_j \chi_j$. Taking into account that $\chi^1 \chi^j = (\chi^{j-1} + \chi^j + \chi^{j+1})$ for $j \geq 1$, the equation of motion becomes

$$3 \phi_0 + \phi_1 = 3 \phi_0 \chi^1 - \frac{1}{4} \phi_0 \chi^1 - \frac{1}{4} \sum_{j \geq 1} \phi_j (\chi^{j-1} + \chi^j + \chi^{j+1}) = \frac{\lambda}{2} \sum_j \phi_j^2 d_j \chi_j.$$

This translates to a set of recursion relations on the $\phi_j$'s:

$$3 \phi_0 = \phi_1 = \lambda \phi_0^2,$n  \phi_j = \frac{1}{2} (\phi_{j-1} + \phi_{j+1}) = \frac{\lambda}{d_j} \phi_j^2.$$

The first equation gives $\phi_1$ in terms of the initial value $\phi_0$. The other equation determines $\phi_{j+1}$ in terms of $\phi_j$ and $\phi_{j-1}$ as soon as $j \geq 1$. We see that we have fewer solutions as above
and that the structure of the solutions are actually very different due to the coupling between representations induced by the factor $P^2(g)$. We have not been able to obtain a closed form for the classical field $\phi(g)$ in terms of the normalization $\phi_0$. We can nevertheless discuss the asymptotical behaviour of $\phi_j$ when $j$ goes to $\infty$. Indeed recognizing the left-hand term in (A.3) as a discretized second derivative\(^{11}\), we see that $\phi_j$ should go asymptotically as $1/j$. More precisely, the asymptotical solution is $-2/\lambda j$.

This can actually be compared to the classical equation in the continuum:

$$\Delta \phi = -\lambda \phi^2.$$  

Assuming $\phi$ to be invariant under rotation and thus to only depend on the radial coordinate $r$, this equation becomes

$$\partial_r^2 (r\phi) = -\lambda (r\phi)^2.$$  

This equation looks like the continuum limit of the recursion relation (A.3). Its obvious solution is $(r\phi(r)) = -2/\lambda r$. To make the correspondence more explicit, we would like to identify $r\phi(r)$ to $\phi_j$ and the discrete difference $(\phi_{j-1} + \phi_{j+1})/2 - \phi_j$ to the second derivative $\partial_r^2 (r\phi)$. This is actually realized through the Duflo map introduced in [22], where the authors prove that this correspondence is made true in the context of the non-commutative geometry and $*$-product underlying the effective theory $S_{\text{eff}}$ for matter coupled to 3D quantum gravity.

### Appendix B. Full perturbations of the 3D GFT

We now describe the full perturbed action around the considered classical solutions to the 3D group field theory without restricting ourselves to the 2D sector, that is, we explicitly compute the full action $S'_{3D}[\phi] = S_{3D}[\phi_j + \phi] - S_{3D}[\phi_j]$ defined in equation (32). It is straightforward to compute:

$$S'_{3D}[\phi] = S_{3D}[\phi] - \frac{1}{2} \int_{G^{*3}} \prod_{i=1}^{3} dg_i \psi(g_1, g_2, g_3) \psi(g_3, g_2, g_1) \int_G df(g) f(gg_1g_3^{-1})$$

$$- \int_{G^{*4}} \prod_{i=1}^{4} dg_i f(g_2g_1^{-1}) \psi(g_1, g_2, g_3) f(g_4g_1^{-1}) \psi(g_3, g_4, g_1)$$

$$- \sqrt{\lambda/3!} \int_{G^{*5}} \prod_{i=1}^{5} dg_i f(g_4g_1^{-1}) \psi(g_1, g_2, g_3) \psi(g_3, g_4, g_5) \psi(g_5, g_2, g_1).$$  

(B.1)

We obtain two new quadratic terms which produce a nontrivial propagator for the GFT and a new cubic interaction term with coupling constant $\sqrt{\lambda}$. Here, we insist on the fact that we are simply perturbing the GFT action around a nontrivial field configuration. Therefore the non-perturbative partition function does not change at all although the structure of its perturbative expansion might get modified.

The field $\phi(g_1, g_2, g_3)$ is gauge-invariant. Thus it is actually a function of two ‘loop variables’, $g_1g_3^{-1}$ and $g_2g_3^{-1}$. Following the logic of the paper, we introduce the following generic multi-component ansatz for the field:

$$\psi(g_1, g_2, g_3) = \sum_{\alpha} \psi_\alpha (g_1g_3^{-1}) A_{\alpha} (g_2g_3^{-1}).$$  

(B.2)

\(^{11}\lambda = 0\) is a special case. The interaction term $\phi^3$ disappears from the equations of motion and we are dealing with a free scalar field. It is then straightforward to check that we have a two-parameter family of solutions of the type $\phi_j = a_j + b_j$, $j \geq 1$, which have a vanishing discretized second derivative. As soon as $\lambda$ is turned on, the physical content of the theory changes completely.
where $\alpha$ is an abstract index. We consider $\psi_\alpha(g)$ as field variables while we hold $A_\alpha(g)$ fixed. However, this is only a point of view and both $\psi_\alpha$ and $A_\alpha$ could be considered as variables. Assuming that the field $\varphi$ is still real, the kinetic term of the action $S_{3D}$ with a nontrivial background field now reads

$$\frac{1}{2} \int_G d\psi_\alpha(g) \bar{\psi}_\beta(g) \left[ \left( 1 - \int_G dhf(h)f(hg) \right) \int_k dA_\alpha(k) \bar{A}_\beta(k) \\
- \left( \int_G dhf(h)A_\alpha(hg) \right) \left( \int_G dk f(k)\bar{A}_\beta(kg) \right) \right].$$

This allows a coupling between the abstract (internal) indices $\alpha, \beta$ labelling the field components and the momentum $g$. This might allow us to derive actions for matter fields with spin.

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