Some ways of combining optimum interval upper limits

S. Yellin

Department of Physics, Stanford University, Stanford, CA 94305, USA

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When backgrounds are not well enough controlled to measure the value of some physical constant, one may still obtain an upper limit on the constant. A single experiment may have several detectors, each of which can alone be used to derive an upper limit from the set of detected events, and the experiment can be run during multiple periods. There can also be more than one experiment which produces an upper limit. Six methods are discussed for producing a combined upper limit. These methods all assume use of either the optimum interval method or the maximum gap method for finding an upper limit in the face of poorly known background.

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I. INTRODUCTION

In a measurement based on detection of events there can be “signal” events from a physical process of interest characterized by the size of some physical constant, and there can be “background” events which individually cannot be distinguished from signal events, but which come from other physical processes. Although the Feldman-Cousins method can be used to set limits when the uncertainty on the background contribution is much smaller than the contribution from the signal, that condition is not necessarily met. Instead the optimum interval method (or the simpler, but weaker maximum gap method) can be used to produce an upper limit. The event distribution in some measured quantity is used to obtain an upper limit on the size of the physical constant that could produce the signal. If the event distribution differs from what would be expected from the signal alone, the maximum gap and optimum interval methods can get stronger limits than would be obtained by simply counting events. They use an interval in the range of the measured quantity that gets the strongest (lowest) upper limit, while taking the proper statistical penalty for the freedom to choose an interval which is especially free of background. The methods involve first transforming the measured quantity of each event into the cumulative probability of its value – the probability of a random event from the assumed signal having a lower value of the measured quantity than the observed event has. In the absence of background, the probability distribution of this transformed quantity is uniform between zero and unity. The optimum interval and maximum gap methods take advantage of any non-uniformity in the background to find an upper limit on the signal by using, in a statistically proper manner, an interval with especially few events within the range of this transformed quantity. An example considered in various parts of this paper is the search for weakly interacting massive particles (WIMPs) which pass through the Earth while orbiting within the Milky Way. The signal size corresponds to the value of the WIMP cross section on nucleons. The measured quantity for each event is the energy each WIMP candidate deposits in a detector. The shape of the expected WIMP energy spectrum can be computed for each assumed WIMP mass, this shape can be integrated to get the cumulative probability distribution expected for WIMPs of each mass, and the optimum interval method can then be used to set a cross section upper limit as a function of WIMP mass.

This paper discusses various ways of using the optimum interval method when combining measurements to produce a combined upper limit. Most of the discussion will be for exactly two measurements, although all methods can in principle, and sometimes in practice, be generalized to an arbitrary number. The method chosen should be selected based on the circumstances of the measurements, but as much as feasible be selected “blindly”, without testing which method gets the strongest limit – it would bias the result low to try several methods and choose the one that produces the strongest limit. We require each method to be statistically valid – for example a 90% confidence level upper limit must be obtained by a method whose probability of mistakenly excluding the truth is no more than 10%. Monte Carlo simulations with software implementing all discussed methods have confirmed this property.

We would like the methods to have two virtues: A)
get a stronger limit than either measurement alone if the data justify it, and B) be robust against one of the two having worse backgrounds than the other. To get an idea of the extent to which each method has these virtues, we will test their performance under two scenarios. For both scenarios assume that two measurements are made by two experiments with the same measured quantity range and sensitivity, and that there is no signal. Assume there is no background in experiment 2. Test “A” checks for virtue A by seeing how strong a limit results if experiment 1 also has no background, and test “B” tests for virtue B by seeing how much experiment 2’s limit is hurt if it’s combined with an experiment 1 suffering from an extremely large background over its entire range. The first three of the methods discussed perform well on test A, but very badly on test B. The fourth performs well on test A, and better than the first three on B. The last two are designed for use with high backgrounds in both experiments, but their performance can also be investigated for the two low background tests. They perform better than the methods appropriate for low backgrounds on test B, but not test A.

Another test, called here “test C”, uses CDMS low background data for each of the six methods. The CDMS experiment used very low temperature germanium crystals in an attempt to detect both the sound and the ionization from WIMPs scattering on nuclei. We imagine the CDMS exposure split into two equal halves of identical effective net exposure and efficiencies, with the WIMP candidate events arbitrarily allocated between the two halves. Each method for combining experiments is used to combine the two halves for all \(2^4\) possible ways of assigning events to the two halves.

Since A, B, and C don’t properly test performance in the face of high backgrounds, the last two methods are also challenged with data from a CDMS high background experiment.

It is up to the user to choose a method based on what is known in advance about the backgrounds.

II. METHODS FOR LOW BACKGROUNDs

A. Simple Merging

We begin with a method that is especially easy to understand and implement. If two or more measurements are almost background free then each of their limits could benefit from a larger exposure. In that case if they’re sufficiently alike a good way to combine them is the “simple merging” method: treat them as one bigger experiment. Combine the events from the measurements into one set of events, giving a single event distribution as a function of the measured quantity used, and compute the expected spectrum of the combined set of events. Then use the optimum interval method to compute a joint upper limit. Simple merging has been used by the CDMS collaboration to combine WIMP search results from its germanium detectors, and has also been used for the same purpose by the EDELWEISS and CRESST collaborations.

Although simple merging can be used to combine data from different types of detectors, extra care must be taken to do it well. Consider, for example, the following scenario: One wishes to get an upper limit for WIMPs of mass 20 GeV/c\(^2\) from combining an experiment with Ge detectors and an experiment which uses Xe, and one of the Xe experiments has recoil energy of 40 keV. Take the maximum velocity of WIMPs bound within the Galaxy to be \(\sim 544\) km/s, from which it follows that a WIMP of mass 20 GeV/c\(^2\) cannot produce an elastic scatter in Xe with more recoil energy than about 33 keV. The 40 keV event must therefore be from background. The optimum interval method applied to Xe alone would correctly ignore this event because its transformed quantity couldn’t be in the optimum interval; there is probability unity of a WIMP event depositing less energy than 40 keV, so the event would be at the very end of the transformed range. But such a WIMP could produce an event with recoil energy as high as 48 keV in Ge, so if recoil energies are merged, the 40 keV background event would not be at the very end of the transformed range of the merged experiments, and could make the upper limit unnecessarily conservative. Such a problem can be prevented by a variation of simple merging, “cumulative probability merging”. Instead of merging recoil energies, those energies are first transformed into cumulative probabilities for each experiment, and then the transformed quantities are merged. Cumulative probability merging would put the 40 keV Xe event at the end of the merged range, where it would have no effect on the merged upper limit.

A nice feature of cumulative probability merging is that the same upper limit would result if one or both experiments used some quantity other than recoil energy. If, to pick a strange example, the Ge experimenters expressed their events in terms of recoil momentum, not energy, while the Xe experimenters described their events by recording in place of recoil energy the minimum WIMP mass that could produce an elastic scatter with the observed recoil energy, the
transformed values for each experiment would be the same probabilities as if both expressed their data in terms of recoil energy. In fact, experiments may be merged which differ so greatly that they don’t even measure the same physical characteristic of the event. All other methods described in this paper for combining measurements automatically have the same ability.

For the conditions of test A, simple merging results in one combined measurement with twice the exposure, but still no events, so the upper limit is made stronger by a factor of 1/2. For the conditions of test B, however, experiment 1 badly pollutes the otherwise clean experiment 2 data, resulting in a very much weaker upper limit. For test C, both simple merging and cumulative probability merging of the two half CDMS experiments are equivalent to the method used for the published CDMS result [4] from simple merging of data from all well functioning detectors.

B. Combining with Confidence Levels

For most methods discussed in this paper, measurements are combined based on upper limit confidence levels from them, rather than from using their events directly as in simple merging. For a given value of the physical constant being sought, call “µ” the expectation value of the number of events from the physical process. Take “p” to be the confidence level by which the measurement excludes a particular value of the physical constant. If this confidence level is found by the maximum gap or optimum interval method, p itself has a certain probability distribution under the assumption of no background, and with correct assumptions about the physical process whose strength is being measured. In order to see what the probability distribution is, imagine a huge number of independent similar experiments, all having no background and all having the same true value of the physical constant corresponding to µ expected number of events. These could be generated by a Monte Carlo simulation. For experiment i, the p confidence level upper limit µi(p) can be computed. By the definition of “confidence level”, p = 0.9 means that for 90% of the experiments µ < µi(p), while there is only a 10% probability that the upper limit µi(p) is mistakenly found to be higher than µ. In general, p is the probability of µ < µi(p). In a region of p where the probability distribution of µ is continuous and non-zero, the probability that µ < µi(p + dp) is p + dp and the probability that µ < µi(p) is p, so there is probability dp that µ is between µi(p) and µi(p + dp). There is probability dp of the true value of the constant being between the p and p + dp confidence level values, so the probability distribution of p has d(Probability)/dp = 1 in regions where the probability density of p is non-zero and continuous. The probability density is zero at values of p for which none of the experiments produces that confidence level, as when p < 0 or p is greater than its maximum possible value. There is probability e^{-µ} of there being zero events in the entire range of measurement, in which case the optimum interval is the entire range of the measurement, and the corresponding p is at its maximum possible value, 1 − e^{-µ}. Thus the probability density of p is zero for p > 1 − e^{-µ}, and the density has a delta function at p = 1 − e^{-µ} whose integral is e^{-µ}. To summarize, if p is correctly computed for the optimum interval method (or for the maximum gap method), then in the absence of background it has probability zero of being below zero or above 1 − e^{-µ}, has probability e^{-µ} of being exactly 1 − e^{-µ}, and is otherwise uniformly distributed with unit density for 0 < p < 1 − e^{-µ}.

Maximum gap or optimum interval results can be combined to form some function q(p1, p2, µ1, µ2) in such a way that if the observed q is too large for an assumed value of the physical constant being investigated, the value will be excluded. A value for a physical constant is conservatively excluded to the 90% confidence level when it is so large that a random pair of measurements similar to measurements 1 and 2, and with no unknown background adding to the signal, would have a 90% probability of q being smaller than was observed. The probability of q being smaller than was observed can be computed from the probability distributions of p1 and p2. Different methods correspond to different choices for q as a function of results from the two measurements. Choose measurement 1 to be the one with the larger expected number of events: µ1 ≥ µ2.

C. Minimum Probability

The “minimum probability” method takes q to be the smaller of p1 and p2. The probability of q being smaller than z is

\[
P_{\text{MinP}}(z; \mu_1, \mu_2) = \begin{cases} 
0 & z \leq 0 \\
2z - z^2 & 0 < z < 1 - e^{-\mu_2} \\
1 & z \geq 1 - e^{-\mu_2}
\end{cases}
\] (1)

P_{\text{MinP}} jumps by e^{-2\mu_2} at z = 1 - e^{-\mu_2}.

For test A the minimum probability upper limit of the combined experiment improves the single experiment limit by a factor of 1/2. But this method
completely fails test B; the combined experiment increases the upper limit by a large factor.

The result of test C for the minimum probability upper limit is shown in Fig. 1. The figure shows the ratio between the minimum probability upper limit curve and the simple merging upper limit curve for combining two equal halves of the CDMS experiment [4]. When the ratio is below the horizontal dotted line at 1.0, the minimum probability upper limit is stronger than the already published simple merging one.

![Figure 1](image1.png)

**FIG. 1:** Ratio of the minimum probability upper limit to the simple merging limit from combining two equal halves of CDMS [4]. The solid curve is the ratio averaged over all ways of allocating events between the two halves, and the dashed curves are the maximum and minimum values of the ratio as the way of dividing up the events between the two halves is varied.

**D. Probability Product**

The “probability product” method takes \( q = p_1 p_2 \). The probability that \( q \) is smaller than \( z \) is

\[
P_{\text{ProdP}}(z, \mu_1, \mu_2) =
\begin{align*}
0 & \quad z \leq 0 \\
\frac{z \log \left(1 - e^{-\mu_1} \right) \log \left(1 - e^{-\mu_2} \right)}{z(1 - e^{-\mu_1 - \mu_2})} & \quad 0 \leq z < z_{\text{max}} \\
1 & \quad z \geq z_{\text{max}}
\end{align*}
\]

where \( z_{\text{max}} = (1 - e^{-\mu_1})(1 - e^{-\mu_2}) \). At \( z = z_{\text{max}} \), \( P_{\text{ProdP}} \) jumps by \( e^{-(\mu_1 + \mu_2)} \).

For test A the probability product upper limit of the combined experiments improves the 90% confidence level upper limit by a factor of 1/2. This method, like the other methods discussed so far, completely fails test B.

The result of test C for the probability product upper limit is shown in Fig. 2.

![Figure 2](image2.png)

**FIG. 2:** Ratio of the probability product upper limit to the simple merging limit from combining two equal halves of CDMS [4]. The solid curve is the ratio averaged over all ways of allocating events between the two halves, and the dashed curves are the maximum and minimum values of the ratio as the way of dividing up the events between the two halves is varied.

**E. Summed Maximum Gap**

We next discuss a method which should be effective when the backgrounds of separate measurements are low, but which unlike the other methods discussed so far is not too badly hurt if the background of one is much worse than that of another. Suppose two measurements have their upper limits determined using the maximum gap method. A “gap” between neighboring events has its size characterized by “\( s \)”, the expectation value of the number of events from the signal. The maximum gap, “\( x \)”, is the maximum over all gaps of \( s \). A “summed maximum gap” for two
measurements takes \( q = x_1 + x_2 \), where \( x_1 \) and \( x_2 \) are the maximum gaps of the two measurements. If the two measurements have their upper limits determined by the optimum interval methods, \( x_1 \) and \( x_2 \) can instead be taken as the maximum gaps that would give the same confidence levels as the optimum interval method gave. In terms of the \( C_0 \) function defined in reference \([2]\), the “summed maximum gap” method takes \( q = x_1 + x_2 \), where \( x_1 \) and \( x_2 \) are \( X_0(p_1, \mu_1) \) and \( X_0(p_2, \mu_2) \), and \( X_0(p, \mu) \) is an inverse function to \( C_0(x, \mu) \); it’s the value for which \( C_0(X_0(p, \mu), \mu) = p \).

The probability that a random pair of background-free measurements will give \( q \) smaller than some value, \( z \), given \( \mu_1 \) and \( \mu_2 \) is \( P_{\text{SumG}} = \)

\[
0 \leq z \leq \mu_2 \\
\int_{-\mu_2}^{\mu_1} dx \frac{\partial C_0(x, \mu_1)}{\partial x} C_0(z - x, \mu_2) \\
\int_{\mu_2}^{\mu_1} dx \frac{\partial C_0(x, \mu_2)}{\partial x} C_0(z - x, \mu_1) \\
+ C_0(0, \mu_1) C_0(z, \mu_2) \\
e^{-\mu_1} C_0(z - \mu_1, \mu_2) \quad \mu_1 \leq z \leq \mu_1 + \mu_2 \\
e^{-\mu_1} C_0(z - \mu_2, \mu_1) \quad z > \mu_1 + \mu_2. \tag{3}
\]

\( P_{\text{SumG}} \) has a discontinuity at \( z = \mu_1 + \mu_2 \); it jumps by \( e^{-(\mu_1 + \mu_2)} \). Although the integrals can be evaluated in closed form, those forms haven’t yet been made simple enough to use. But \( P_{\text{SumG}}(z, \mu_1, \mu_2) \) can be evaluated numerically, and possibly tabulated. The 90\% confidence level upper limit cross section is the cross section for which \( 0.9 = P_{\text{SumG}}(x_1 + x_2, \mu_1, \mu_2). \)

Test A for the summed effective maximum gap method gives a factor of 1/2 improvement in the upper limit compared to the limit for each experiment alone. For test B, the 90\% confidence level upper limit for \( \mu \) is at \( \mu = 6.679 \), the solution of

\[
0.9 = \int_0^\mu dx \frac{\partial C_0(x, \mu)}{\partial x} C_0(\mu - x, \mu). \tag{4}
\]

The 90\% confidence level upper limit is therefore increased by a factor of 6.679/2.303 = 2.90 over its value for experiment 1 alone.

The result of test C for the summed effective maximum gap upper limit is shown in Fig.\( 3.\)can be much worse than the background of the other one. It may be that because the different measurements have different backgrounds, one gets a stronger limit for some sets of physical assumptions and another gets a stronger limit for other sets of physical assumptions. This situation is likely if the different measurements are results from different parts of the detection apparatus. For example, the CDMS collaboration normally combines results from many detectors using the simple merging method \([4]\). But it has also lowered its analysis threshold and modified its cuts to provide improved sensitivity to lower WIMP masses despite the high and uncertain backgrounds at low energies \([8][9]\). With a low threshold, detectors with especially low background could give a stronger limit alone than they could when merged with detectors with poorer background rejection. And because the backgrounds were differently distributed in the different detectors, which detector gave the strongest limit could depend on the WIMP mass being tested. The limit would be biased low by simply choosing the lowest one for each WIMP mass, because the lowest one could benefit from a statistical fluctuation, rather than from superior background rejection. There must be a statistical penalty paid for allowing the choice.

FIG. 3: Ratio of the summed effective maximum gap upper limit to the simple merging limit from combining two equal halves of CDMS \([4]\). The solid curve is the ratio averaged over all ways of allocating events between the two halves, and the dashed curves are the maximum and minimum values of the ratio as the way of dividing up the events between the two halves is varied.

III. METHODS FOR HIGH BACKGROUNDS

We next discuss two methods intended for the case of large backgrounds. For a particular set of physical assumptions, the background of one measurement...
If the background is high, the statistical penalty will be a small fraction of the upper limit. High backgrounds may require the high statistics version \[10\] of the optimum interval method. If the differences between backgrounds in different measurements are large enough, the advantage from making a choice will be greater than the statistical penalty that must be paid.

### A. Serialization

A simple way to automatically choose the measurement with best background rejection is “serialization”: concatenate the measured quantity ranges and apply the optimum interval method to the resulting total range. The concatenation is illustrated in Fig. 4.

The optimum interval will include the part of the measurement range which gives the strongest limit, and will therefore tend to use the measurement with the strongest limit. Because the optimum interval method automatically takes the correct statistical penalty for its freedom to choose the interval, serialization automatically takes the correct statistical penalty for its freedom to choose as its optimum interval one mainly or entirely in the part of the concatenated range for the measurement which seems to be most free of background. The result of this method may depend on the order in which the measurements are concatenated, but no additional statistical penalty need be taken for the freedom to choose the order provided the choice is made “blindly”, without knowledge of which order gives the strongest limit. None of the other methods discussed in this paper depends on the way measurements are ordered.

For both tests A and B, \(\mu_1 = \mu_2 \equiv \mu\), and the optimum interval 90\% confidence level upper limit for experiment 2 alone corresponds to a signal size for which \(\mu = -\log(0.1) = 2.30\). Serialization of the two data sets under the conditions of test A results in one combined measurement with twice the exposure, but no events, resulting in an upper limit improved by a factor of 1/2. For test B the 90\% confidence limit on \(\mu\) becomes the \(\mu\) for which \(\bar{C}_{Max}(0.9, 2\mu) = C_0(\mu, 2\mu)\) after serialization (notation of Ref. \[2\]), resulting in a factor of 4.78/2.30 = 2.1 increase (weakening) of the upper limit from experiment 2 alone. While the background of experiment 1 hurts, it doesn’t hurt as much as for simple merging. But if experiment 1 is the first in the series and scenario A is modified to have many background events at the bottom of the energy range of experiment 2, then instead of a factor of 1/2 strengthening of the upper limit, serialization will suffer approximately the same factor of 2.1 weakening of the limit in the modified test A as in test B. Even modifying scenario A by adding just one background event at the bottom of the range of experiment 2 will result in the combination producing its upper limit where \(1 - e^{-2\mu}(1 + 2\mu) = \bar{C}_{Max}(0.9, 2\mu)\), or \(\mu = 2.167\), a factor of 0.94, instead of 0.50, of the experiment 2 limit alone. None of the other methods discussed in this paper are significantly weakened by background events very near threshold.

The result of test C for the serialization upper limit is shown in Fig. \[5\]. The previously discussed methods all give on the average about the same upper limit as simple merging, depending on WIMP mass, but the serialization upper limit for test C is on the average weaker than the simple merging one, and for some ways of distributing the events is much weaker.

Tests A, B, or C are inappropriate for the recommended use of the serialization method. They all have at least one experiment or detector with low background, while the serialization method is intended for use with high background. To remedy this deficiency we demonstrate the performance of the serialization method with high background data, a CDMS low threshold analysis \[8\]. Figure \[6\] allows comparison between simple merging and serialization for a 90\% confidence level upper limit reanalysis of CDMS data taken at a shallow site. Although the
FIG. 5: Ratio of the serialization upper limit to the simple merging limit from combining two equal halves of CDMS [4]. The solid curve is the ratio averaged over all ways of allocating events between the two halves, and the dashed curves are the maximum and minimum values of the ratio as the way of dividing up the events between the two halves is varied.

published results [8] combined germanium and silicon detectors, the silicon detectors had too much background and too little mass for their inclusion to improve the combined limit. Only germanium detectors were used for Fig. 6, which compares merging with concatenation of six measurements. This figure shows that at the low WIMP masses for which the reanalysis was performed and for which the high background is important, the serialization method gives a stronger limit than does simple merging. At high masses, the optimum interval occurs where backgrounds are low, and simple merging is a more sensitive way of combining detectors.

B. Minimum Limit

The statistical penalty required by the serialization method must also include a penalty for the possibility that the optimum interval may cross the boundary between the ranges for two measurements. The reason for using intervals, rather than some other non-contiguous point set, is because backgrounds tend to be larger in some parts of the experimental range than others. For example, the background may be especially large at the low end of the experimental range, where detector signals are especially small. Concatenating experimental ranges may well put a small background region adjacent to a large background one, in which case the optimum interval method would not choose an interval which extends much past the point at which the two ranges join, even though it must take a penalty for the possibility of doing so. Under such circumstances, a somewhat stronger limit can be obtained by instead directly choosing the measurement which gives the strongest limit at a given confidence level, while taking the correct statistical penalty for the freedom to make such a choice: the “minimum limit” method.

Assume the confidence level by which an assumed physical constant is excluded as too high decreases, or at least doesn’t increase, when the assumed value decreases. This monotonic decrease of confidence level with value of the constant has small exceptions for the optimum interval method, but let’s neglect such effects. When measurements with higher upper limits than the strongest one have the upper limit value of the physical constant lowered until it’s the same as the strongest limit, the confidence levels by which they’re rejected will be lowered. Thus finding the minimum limit for the same confidence level is equivalent to finding the maximum confidence level for the

FIG. 6: Comparison [11] of high background methods applied to CDMS low threshold Ge data [8]. The dashed curve shows the 90% CL upper limit from simple merging, the dark solid curve shows the result from the serialization method, the dotted curve is for the minimum limit, and the light solid curve shows what the minimum limit would be if there had been no penalty for the freedom to choose the detector with the minimum limit.
given upper limit. “Minimum limit” is equivalent to “maximum probability”. For two measurements, $q$ is the larger of $p_1$ and $p_2$. More generally, $n$ results are combined by taking $q$ to be the largest of all the probabilities, $p_i$. If the highest $p_i$ for a given cross section is smaller than $z$, then all the $p_i$ are smaller than $z$. The probability of all $p_i$ being smaller than $z$ is equal to the product of the probabilities of each having $p_i < z$. I.e., it is 0 if $z < 0$, 1 if $z \geq 1$, and is otherwise $z^k$, where $k$ is the number of measurements with $1 - e^{-\mu_i} > z$. The resulting function has discontinuities wherever $k$ changes. Here’s an intuitive justification for lowering $k$ when it results in a measurement with $\mu_i < -\log(1 - z)$, corresponding to $1 - e^{-\mu_i} > z$. Suppose, for example, the maximum probability for a particular cross section is 96%, and for that cross section measurement $i$ has $\mu_i = 3$. Then measurement $i$ cannot get a 96% upper limit no matter what events are in it, because the exclusion cross section from measurement $i$ cannot be greater than what one would get from zero events, for which the confidence level would be $1 - e^{-3} = 95% < 96%$. In that case, measurement $i$ shouldn’t be included in the penalty that must be paid for the multiple chances of accidentally getting a maximum probability as high as 96%.

The algorithm for finding the, say, 90% confidence level upper limit starts with finding the upper limit for each measurement to the $z$ confidence level, where $z^k = 0.9$, and initially $k = n$. If for that value of $z$ the number of measurements with $1 - e^{-\mu_i} > z$ is $k$, conclude the algorithm by taking the minimum upper limit as the overall 90% one. Otherwise repeat with $k \leftarrow k' = k - 1$ until the number is $\geq k'$. If the number is $> k'$ then there is no $z$ for which the confidence level for rejection is exactly 90%. Conservatively choose the lowest confidence level above 90% for which there is a $z$ which rejects it. I.e., find the lowest $z$ for which there are $k'+1$ measurements with $1 - e^{-\mu_i} \geq z$.

For both tests A and B the 90% CL upper limit of $\mu = 2.30$ from experiment 2 alone becomes $\mu = 2.97$ from the combined measurement.

The result of test C for the minimum limit upper limit is shown in Fig. 7. The decision on whether a penalty is needed for the existence of a second experiment can be ambiguous if the two halves are exactly equal in exposure, so the two halves of the CDMS exposure were made very slightly unequal. As for the serialization method, the minimum limit test C result is on the average weaker than the simple merging one. Although the minimum limit result can be stronger than the one from simple merging, unlike the other methods discussed in this paper it cannot be stronger than the strongest of the limits from the individual measurements.

Like the serialization method, the minimum limit method has been tested on CDMS low threshold data taken at a shallow site. Figure 6 shows that for this experiment the minimum limit is somewhat stronger than the serialization one, probably because it need not take a penalty for the possibility of the optimum interval overlapping two detectors. As for the serialization method, the minimum limit gives the strongest limit at low masses, where backgrounds are high. The lighter solid curve in the figure shows by its difference with the dotted curve how much of a penalty had to be taken for the freedom to choose the detector which gave the minimum limit. If CDMS had been able to tell in a blind manner, without seeing the events actually used for setting a limit, which detector to use for each WIMP mass, the light solid curve would have been the upper limit.
IV. CONCLUSIONS

Six methods have been discussed for obtaining a combined upper limit for multiple experiments, or parts of experiments, which may individually be analyzed using the optimum interval method. The combination methods also apply if the individual measurements are analyzed using the maximum gap method. None of the methods require that the different experiments use similar detectors or procedures, so long as they both can set an upper limit on the same physical constant. Some combination methods are especially useful if the individual measurements have low enough background to be limited in sensitivity by the amount of running time each has had. Others are most useful if the individual measurements have high and different backgrounds, and one wishes to use only the most sensitive one while paying the appropriate statistical penalty for the possibility that the apparently best one may seem more sensitive simply because of a statistical fluctuation. The various combination methods were briefly examined for their ability to achieve sensitivity when two experiments have very low background, and they were examined for their sensitivity when only one experiment has low background, while the other has very high background. Qualitative results are shown in Table I. This table applies mainly to methods useful for combining measurements which have low backgrounds. The methods appropriate for high backgrounds were tested on high background experimental data.

Software is publicly available for applying the methods discussed in this paper.

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TABLE I: Sensitivity of the combination methods for two sample experiments, each with either low or high backgrounds.

| Method               | Both Low | One Low, One High |
|----------------------|----------|-------------------|
| Simple Merging       | Good     | Bad               |
| Minimum Probability  | Good     | Bad               |
| Probability Product  | Good     | Bad               |
| Summed Maximum Gap   | Good     | Fair              |
| Serialization        | Fair     | Fair              |
| Minimum Limit        | Fair     | Fair              |