ADOMIAN DECOMPOSITION METHOD APPLIED TO CONTINUOUS-TIME
BILINEAR STOCHASTIC PROCESSES WITH TIME-VARYING COEFFICIENTS

FATEH MERAHI*

Department of Mathematics, Faculty of Mathematics and Computer Science,
Batna 2 University, Batna, Algeria

Copyright © 2022 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In the present paper, we apply the Adomian decomposition method for bilinear stochastic processes with time-varying coefficients in both time and frequency domain. More precisely, we derived an analytical approximate solution and we prove its convergence to the exact solution in time domain, furthermore we give an analytical approximate solution in frequency domain, i.e, we derived analytical approximate transfer functions which converge to the exact transfer functions.

Keywords: Adomian decomposition method; continuous-time bilinear processes; approximate solution; spectral representation; transfer functions.

2010 AMS Subject Classification: Primary 40A05, 40A25, Secondary 45G05.

1. INTRODUCTION

The Adomian decomposition method (ADM) is an approach proposed by George Adomian (1923-1996) [3, 4, 5] which provides an analytic approximation to linear and/or nonlinear problems, boundary value problems, algebraic equations and differential equations [2, 1, 18, 20]. The general principle is based on the so-called 'reversion method' which consists of decomposing the solution of a nonlinear functional equation in a series of functions elegantly computed

*Corresponding author

E-mail address: f.merahi@univ-batna2.dz

Received March 02, 2022
The method can be applied directly for different type of (stochastic) differential equations providing a solution with fast speed of convergence \[7, 9, 10, 14, 16\]. So, the aim of this Note is to apply the ADM (as an alternative of Itô solution) for solving a continuous-time bilinear (COBL) equation defined as a stochastic differential equations (SDE) with time varying coefficients in time and frequency domain.

2. **BILINEAR STOCHASTIC DIFFERENTIAL EQUATION**

In this Note, we consider the first order continuous-time bilinear process (COBL(1,1)) driven by a stochastic differential equation of the form

\[
dX(t) = (\alpha(t)X(t) + \mu(t))dt + (\gamma(t)X(t) + \beta(t))dw(t), t \geq 0, X(0) = \eta,
\]

where \((w(t))_{t \geq 0}\) is a standard Brownian motion in \(\mathbb{R}\) defined on some basic probability space \((\Omega, \mathcal{F}, P)\) and \(\alpha(t), \mu(t)\) and \(\gamma(t)\) are complex measurable functions such that \(\forall t \geq 0, \alpha(t) \neq 0, \gamma(t) \neq 0\). To ensure the existence and uniqueness of the solution process \((X(t))_{t \geq 0}\) of equation (1), we assume in addition that, \(\forall T > 0, \int_0^T |\alpha(t)|dt < \infty, \int_0^T |\mu(t)|dt < \infty, \int_0^T |\gamma(t)|^2 dt < \infty\) and that the initial state \(\eta\) is a random variable defined on \((\Omega, \mathcal{F}, P)\), independent of \(w\) with \(E\{\eta\} = m(0)\) and \(Var\{\eta\} = K(0)\). The existence of the Itô solution process \((X(t))_{t \geq 0}\) of equation (1) in time domain is however ensured by general results on SDE (see e.g., [6])

\[
X(t) = \Phi(t) \left\{ X(0) + \int_0^t \Phi^{-1}(s)(\mu(s) - \gamma(s)\beta(s))ds + \int_0^t \Phi^{-1}(s)\beta(s)dW(s) \right\}
\]

where \(\Phi(t) = \exp\left\{ \int_0^t (\alpha(s) - \frac{1}{2}\gamma^2(s))ds + \int_0^t \gamma(s)dW(s) \right\}\) its mean function is \(\Psi(t) = \exp\left\{ \int_0^t \alpha(s)ds \right\}\), \(t \geq 0\). If the coefficients \(\alpha(t), \mu(t), \gamma(t)\) and \(\beta(t)\) are constant and satisfying the condition \(\gamma \neq 0, \alpha\beta \neq \mu\gamma, 2\alpha + \gamma^2 < 0\), then the process \((X(t))_{t \in \mathbb{R}}\) is a second-order stationary with

\[
E\{X(t)\} = -\frac{\mu}{\alpha}, K(0) = Var\{X(t)\} = \frac{(\alpha\beta - \mu\gamma)^2}{\alpha^2(2\alpha + \gamma^2)}\text{ and } Cov(X(t), X(t+h)) = K(0)e^{\alpha|h|}, h \in \mathbb{R}.
\]
3. **Decomposition Method Applied to COBL(1, 1) in Time-Domain**

Without loss of generality we assume that $\beta(t) = 0$, for all $t$, since this assumption can be fulfilled by the transformation $Y(t) = X(t) + \frac{\beta(t)}{\gamma(t)}$ with some additional assumptions on the differentiability of the functions $\gamma(t)$ and $\beta(t)$. The Adomian’s method suggest that the solution process $X(t)$ of SDE (1) is sought in the form

$$X(t) = \sum_{j=0}^{\infty} X_j(t)$$

Substituting (3) into (1) we obtain

$$\sum_{j=0}^{\infty} X_j(t) = \eta + L^{-1} \mu(t) + L^{-1} \left( \alpha(t) + \gamma(t)w^{(1)}(t) \right) \sum_{j=0}^{\infty} X_j(t),$$

in wherein $w^{(1)}(t) = \frac{dw(t)}{dt}$ and $L$ is the operator defined by $L^{-1}[.] = \int_{0}^{t}(...)ds$. So we have by identification

$$X_0(t) = \eta + \int_{0}^{t} \mu(s)ds$$

and $\forall n \geq 1$,

$$X_n(t) = \int_{0}^{t} \left( \alpha(s) + \gamma(s)w^{(1)}(s) \right) X_{n-1}(s)ds,$$

More precisely, we have

$$X_1(t) = \int_{0}^{t} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) X_0(s_1)ds_1$$

$$= \eta \int_{0}^{t} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1))ds_1 + \int_{0}^{t} \int_{0}^{s_1} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \mu(s_2)ds_2ds_1$$

$$X_2(t) = \int_{0}^{t} (\alpha(s_2) + \gamma(s_2)w^{(1)}(s_2)) X_1(s_2)ds_2$$

$$= \eta \int_{0}^{t} \int_{0}^{s_1} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) (\alpha(s_2) + \gamma(s_2)w^{(1)}(s_2)) ds_2ds_1$$

$$+ \int_{0}^{t} \int_{0}^{s_1} \int_{0}^{s_2} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) (\alpha(s_2) + \gamma(s_2)w^{(1)}(s_2)) \mu(s_3) ds_3ds_2ds_1$$

$$\vdots$$

$$X_n(t) = \eta \int_{0}^{t} \int_{0}^{s_1} \cdots \int_{0}^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \cdots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) ds_1 ds_2 \cdots ds_{n-1} ds_1$$

$$+ \int_{0}^{t} \int_{0}^{s_1} \cdots \int_{0}^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \cdots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) \mu(s_{n+1}) ds_{n+1} ds_n ds_{n-1} \cdots ds_2 ds_1$$
Finally, we approximate the solution by the truncated series

\[\forall N \geq 1, \Phi_N(t) = \sum_{n=0}^{N-1} X_n(t).\]

In order to show the convergence of the series (7) to the exact solution processes we introduce the concept of the so-called stochastic exponential associated with a centred Gaussian random variable \(Z\) and defined by \(e^Z = \exp\{Z - \frac{1}{2} E\{Z^2\}\}\). First, we have the following lemma

**Lemma 3.1.** If \(f\) is a real function defined on the interval \([t_0, T] \subset \mathbb{R}\). Then we have for all \(t \in [t_0, T]\) and \(\forall n \geq 2,

\[
\int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_n-1} f(s_1)\ldots f(s_{n-1}) f(s_n) ds_n ds_{n-1} \ldots ds_1 = \left[\int_{t_0}^{t} f(s) ds\right]^n / n! ,
\]

where \(t_0 \leq s_n \leq s_{n-1} \leq \ldots \leq s_2 \leq s_1 \leq t\).

**Proof.** By induction, we have for \(n = 2,\)

\[
\int_{t_0}^{t} \int_{t_0}^{s_1} f(s_1) f(s_2) ds_2 ds_1 = \int_{t_0}^{t} F(s_1) f(s_1) ds_1 = \frac{(F(t))^2}{2} ,
\]

where \(F(t) = \int_{t_0}^{t} f(s) ds\). Now assume that the equation (8) holds for \(n - 1\), it means

\[
\int_{t_0}^{s_1} \ldots \int_{t_0}^{s_n-1} f(s_2)\ldots f(s_{n-1}) f(s_n) ds_n ds_{n-1} \ldots ds_2 = \frac{(F(s_1))^{n-1}}{(n-1)!} ,
\]

which implies

\[
\int_{t_0}^{t} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_n-1} f(s_1)\ldots f(s_{n-1}) f(s_n) ds_n ds_{n-1} \ldots ds_1 = \int_{t_0}^{t} \frac{(F(s_1))^{n-1}}{(n-1)!} f(s_1) ds_1 = \frac{(F(t))^n}{n!} = \frac{\left[\int_{t_0}^{t} f(s) ds\right]^n}{n!} .
\]

\(\blacksquare\)

**Theorem 3.2** (Theorem 1, El-Kalla [13]). If \(q(t)\) is integrable and \(r(t)\) is derivable functions, then

\[
e^{-r(t)} \int e^{r(t)} q(t) dt = \sum_{k=0}^{\infty} (-1)^k \int \frac{dr(t)}{dt} \int k-fold \frac{dr(t)}{dt} \int k-fold q(t) dt dt \ldots dt .
\]

We are now in a position to prove the convergence of the approximate solution \(\Phi_N(t)\) to the exact solution process.
Theorem 3.3. Let $\Phi_N(t)$ is the approximate solution (7) given by Adomian decomposition method of the COBL(1,1) driven by the equation (1), then almost surely, we have

$$\lim_{N \to +\infty} \Phi_N(t) = X(t)$$

Proof. Applying Lemma 3.1 on the first term of $X_n(t)$, we obtain

$$\eta \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \ldots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) \, ds_n \ldots ds_1 = \eta \left( \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right)^n,$$

and Applying Theorem 3.2 on the infinite sum of second term of $X_n(t)$, we obtain

$$\sum_{n=0}^{\infty} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \ldots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) \mu(s_{n+1}) \, ds_n \ldots ds_1$$

$$= \mathcal{E} \left\{ \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right\} \int_0^t \mathcal{E} \left\{ - \int_0^s (\alpha(u) + \gamma(u)w^{(1)}(u)) \, du \right\} \mu(s) \, ds.$$

Then we can obtain

$$\lim_{N \to +\infty} \Phi_N(t)$$

$$= \eta \sum_{n=0}^{\infty} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \ldots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) \, ds_n \ldots ds_1$$

$$+ \sum_{n=0}^{\infty} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} (\alpha(s_1) + \gamma(s_1)w^{(1)}(s_1)) \ldots (\alpha(s_n) + \gamma(s_n)w^{(1)}(s_n)) \mu(s_{n+1}) \, ds_n \ldots ds_1$$

$$= \eta \sum_{n=0}^{\infty} \left( \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right)^n$$

$$+ \mathcal{E} \left\{ \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right\} \int_0^t \mathcal{E} \left\{ - \int_0^s (\alpha(u) + \gamma(u)w^{(1)}(u)) \, du \right\} \mu(s) \, ds$$

$$= \mathcal{E} \left\{ \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right\} \left( \eta + \int_0^t \mathcal{E} \left\{ - \int_0^s (\alpha(u) + \gamma(u)w^{(1)}(u)) \, du \right\} \mu(s) \, ds \right),$$

where $\mathcal{E} \left\{ \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right\}$ denotes the stochastic exponential of $\int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds$, so we get

$$\mathcal{E} \left\{ \int_0^t (\alpha(s) + \gamma(s)w^{(1)}(s)) \, ds \right\} = \exp \left\{ \int_0^t \left( \alpha(s) - \frac{\gamma^2(s)}{2} \right) \, ds + \int_0^t \gamma(s)w^{(1)}(s) \, ds \right\},$$
which implies
\[
\lim_{N \to +\infty} \Phi_N(t) = \exp \left\{ \int_0^t \left( \alpha(s) - \frac{\gamma^2(s)}{2} \right) ds + \int_0^t \gamma(s)w^{(1)}(s)ds \right\} \\
\times \left( \eta + \int_0^t \exp \left\{ -\int_0^s \left( \alpha(u) - \frac{\gamma^2(u)}{2} \right) du - \int_0^s \gamma(u)w^{(1)}(u)du \right\} \mu(s)ds \right),
\]
which is the exact Itô solution process (11) of SDE (1) and the proof is complete.

\[ \square \]

4. Decomposition Method in Frequency Domain

In frequency domain, the process \((X(t))_{t \in \mathbb{R}}\) admits the so-called Wiener-Itô orthogonal representation

\[
X(t) = f(t, 0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{i\lambda(r)} f(t, \lambda(r))dZ(\lambda(r)),
\]

where \(\lambda(r) = \sum_{i=1}^r \lambda_i\) and the integrals are multiple Wiener-Itô stochastic integrals with respect to the stochastic measure \(Z(d\lambda)\), \(f(t, 0) = E\{X(t)\}\), \(Z(d\lambda(r)) = \prod_{i=1}^r Z(d\lambda_i)\), (see Major [19] and Dobrushin [11]). and \(f(t, \lambda(r))\) are referred as the \(r\)th evolutionary transfer functions of \((X(t))_{t \in \mathbb{R}}\), uniquely determined and fulfill the condition \(\sum_{r \geq 0} \frac{1}{r!} \int_{\mathbb{R}^r} |f(t, \lambda(r))|^2 dF(\lambda(r)) < \infty\). As a property of the representation (10) is that for any \(f(t, \lambda(n))\) and \(f(s, \lambda(m))\), we have

\[
E \left\{ \int_{\mathbb{R}^n} f(t, \lambda(n))dZ(\lambda(n)) \int_{\mathbb{R}^m} f(t, \lambda(m))dZ(\lambda(m)) \right\} = \delta^m_n n! \int_{\mathbb{R}^n} \tilde{f}(t, \lambda(n)) \bar{f}(s, \lambda(n))dF(\lambda(n)),
\]

where \(\delta^m_n\) is the delta function and \(\tilde{f}(t, \lambda(r))\) is the symmetrized version of \(f\) by the vector \(\lambda(r)\) which is the average of those values of \(f\) taken by all possible permutations of entries of \(\lambda(r)\).

We need also the diagram formula given as follow

\[
\int_{\mathbb{R}} f(\lambda)Z(d\lambda) \int_{\mathbb{R}^n} g(\lambda(n)) Z(d\lambda(n)) = \int_{\mathbb{R}^{n+1}} g(\lambda(n)) f(\lambda_{n+1}) Z(d\lambda_{n+1})
\]

\[\left. + \sum_{k=1}^n \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} g(\lambda(n)) \tilde{f}(\lambda_k) dF(\lambda_k) Z(d\lambda_{n\setminus k}) \right.\]

where \(Z(d\lambda(n\setminus k)) = Z(d\lambda_1) ... Z(d\lambda_{k-1}) . Z(d\lambda_{k+1}) ... Z(d\lambda_n)\).
Theorem 4.1. Assume that the process \((X(t))_{t \geq 0}\) generated by the SDE (1) has a regular second-order solution. Then the evolutionary symmetrized transfer functions \(f(t, \lambda(r)), (t, r) \in \mathbb{R} \times \mathbb{N}\) of this solution are given by

\[
f^{(1)}(t, \lambda(r)) = \begin{cases} 
\alpha(t)f(t,0) + \mu(t), & \text{if } r = 0 \\
(\alpha(t) - i\lambda(r)) f(t, \lambda(r)) + r(\gamma(t)f(t, \lambda(r-1)) + \delta_{r=1}\beta(t)), & \text{if } r \geq 1 
\end{cases}
\]

Proof. See Bibi and Merahi [8].

Remark 4.2. The existence and uniqueness of the evolutionary symmetrized transfer functions \(f(t, \lambda(r)), (t, r) \in \mathbb{R} \times \mathbb{N}\) of this solution is ensured by general results on linear ordinary differential equations (see, e.g., [17], ch. 1) so,

\[
f(t, \lambda(r)) = \begin{cases} 
\phi_t(0) \left( f(0,0) + \int_0^t \varphi^{-1}_r(0) \mu(s)ds \right) & \text{if } r = 0 \\
\phi_t \left( \lambda(r) \right) \left( f(0, \lambda(r)) + \int_0^t \varphi^{-1}_r \left( \lambda(r) \right) \left( \gamma(s)f(s, \lambda(r-1)) + \delta_{r=1}\beta(s) \right) ds \right) & \text{if } r \geq 1 
\end{cases}
\]

where \(\phi_t \left( \lambda(r) \right) = \exp \left\{ \int_0^t \left( \alpha(s) - i\lambda(r) \right) ds \right\}\).

Applying ADM method to obtain approximate transfer functions \(F_N(t, \lambda(r))\) \(N \geq 1\), \((t, r) \in \mathbb{R} \times \mathbb{N}\) where

\[
F_N(t, \lambda(r)) = \sum_{n=0}^{N-1} f_n(t, \lambda(r)) \rightarrow f(t, \lambda(r)) \text{ as } N \rightarrow \infty.
\]

4.1. First Technique. We substitute the series \(f(t, \lambda(r)) = \sum_{n=0}^{\infty} f_n(t, \lambda(r))\) in the equations (12) we obtain

(1) For \(r = 0\), we have

\[Lf(t,0) = \alpha(t)f(t,0) + \mu(t),\]

which implies

\[
\sum_{n=0}^{\infty} f_n(t,0) = f(0,0) + L^{-1} \left( \alpha(t) \sum_{n=0}^{\infty} f_n(t,0) + \mu(t) \right),
\]

it follows, \(f_0(t,0) = f(0,0) + \int_0^t \mu(s)ds\) and \(\forall n \geq 1,\)

\[f_n(t,0) = \int_0^t \alpha(s)f_{n-1}(s,0)ds.\]
After computing the terms \((f_n(t, 0))_{n \geq 1}\) have the form

\[
f_1(t, 0) = f(0, 0) \int_0^t \alpha(s_1) ds_1 + \int_0^t \int_0^{s_1} \alpha(s_1) \mu(s_2) ds_2 ds_1,
\]
\[
f_2(t, 0) = f(0, 0) \int_0^t \int_0^{s_1} \alpha(s_1) \alpha(s_2) ds_2 ds_1 + \int_0^t \int_0^{s_1} \alpha(s_1) \alpha(s_2) \mu(s_3) ds_2 ds_3 ds_2 ds_1,
\]
\[\vdots\]
\[
f_n(t, 0) = f(0, 0) \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \alpha(s_1) \cdots \alpha(s_n) ds_n ds_{n-1} \cdots ds_2 ds_1 + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \alpha(s_1) \cdots \alpha(s_n) \mu(s_{n+1}) ds_{n+1} ds_n ds_{n-1} \cdots ds_2 ds_1.
\]

(2) For \(r = 1\), we have

\[
Lf(t, \lambda) = (\alpha(t) - i\lambda) f(t, \lambda) + (\gamma(t) f(t, 0) + \beta(t)),
\]

which implies

\[
\sum_{n=0}^{\infty} f_n(t, \lambda) = f(0, \lambda) + L^{-1} \left( (\alpha(t) - i\lambda) \sum_{n=0}^{\infty} f_n(t, \lambda) + \gamma(t) f(t, 0) + \beta(t) \right),
\]

it follows, \(f_0(t, \lambda) = f(0, \lambda) + \int_0^t (\gamma(s) f(s, 0) + \beta(s)) ds\) and \(\forall n \geq 1\),

\[
f_n(t, \lambda) = \int_0^t (\alpha(s) - i\lambda) f_{n-1}(s, \lambda) ds.
\]

After computing the terms \((f_n(t, \lambda))_{n \geq 1}\) have the form

\[
f_1(t, \lambda) = f(0, \lambda) \int_0^t (\alpha(s_1) - i\lambda) ds_1 + \int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda) (\gamma(s_2) f(s_2, 0) + \beta(s_2)) ds_2 ds_1,
\]
\[
f_2(t, \lambda) = f(0, \lambda) \int_0^t \int_0^{s_1} (\alpha(s_1) - i\lambda) (\alpha(s_2) - i\lambda) ds_2 ds_1
\]
\[\vdots\]
\[
f_n(t, \lambda) = f(0, \lambda) \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\alpha(s_1) - i\lambda) \cdots (\alpha(s_n) - i\lambda) ds_n ds_{n-1} \cdots ds_2 ds_1 + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\alpha(s_1) - i\lambda) \cdots (\alpha(s_n) - i\lambda) f(s_{n+1}, 0) + \beta(s_{n+1})) ds_{n+1} ds_n ds_{n-1} \cdots ds_2 ds_1.
\]

(3) For \(r \geq 2\), we have

\[
Lf(t, \lambda_{(r)}) = \left(\alpha(t) - i\lambda_{(r)}\right) f(t, \lambda_{(r)}) + r\gamma(t) f(t, \lambda_{(r-1)}),
\]
which implies

$$
\sum_{n=0}^{\infty} f_n(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) + L^{-1} \left( \left( \alpha(t) - i\lambda_{(r)} \right) \sum_{n=0}^{\infty} f_n(t, \lambda_{(r)}) + r\gamma(t) f(t, \lambda_{(r-1)}) \right),
$$

it follows, $f_0(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) + r \int_0^t \gamma(s) f(s, \lambda_{(r-1)}) ds$ and $\forall n \geq 1$,

$$
f_n(t, \lambda_{(r)}) = \int_0^t \left( \alpha(s) - i\lambda_{(r)} \right) f_{n-1}(s, \lambda_{(r)}) ds.
$$

After computing the terms $\left( f_n(t, \lambda_{(r)}) \right)_{n \geq 1}$ have the form

$$
f_1(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \left( \alpha(s) - i\lambda_{(r)} \right) ds + r \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \gamma(s_2) f(s_2, \lambda_{(r-1)}) ds_2 ds_1,
$$

$$
f_2(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \left( \alpha(s_2) - i\lambda_{(r)} \right) ds_2 ds_1
$$

$$+ r \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \left( \alpha(s_2) - i\lambda_{(r)} \right) \gamma(s_3) f(s_3, \lambda_{(r-1)}) ds_3 ds_2 ds_1,
$$

$$
\vdots
$$

$$
f_n(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} \left( \alpha(s_1) - i\lambda_{(r)} \right) \ldots \left( \alpha(s_n) - i\lambda_{(r)} \right) ds_n ds_{n-1} \ldots ds_2 ds_1
$$

$$+ r \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} \int_0^{s_n} \left( \alpha(s_1) - i\lambda_{(r)} \right) \ldots \left( \alpha(s_n) - i\lambda_{(r)} \right) \gamma(s_{n+1}) f(s_{n+1}, \lambda_{(r-1)}) ds_{n+1} ds_n ds_{n-1} \ldots ds_2 ds_1.
$$

**Theorem 4.3.** Assume that the process $(X(t))_{t \geq 0}$ generated by the SDE (1) has a regular second-order solution. We consider the process $(\mathcal{F}_N(t))_{t \geq 0}$ which has the following spectral representation

$$
\mathcal{F}_N(t) = \tilde{F}_N(t, 0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{i\lambda_{(r)}} \tilde{F}_N(t, \lambda_{(r)}) dZ(\lambda_{(r)}),
$$

where $\tilde{F}_N(t, \lambda_{(r)}) = \sum_{n=0}^{N-1} f_n(t, \lambda_{(r)}), \forall N \geq 1$. Then almost surely, we have

$$
\lim_{N \to +\infty} \mathcal{F}_N(t) = X(t).
$$

**Proof.** First applying the Lemma 3.1, Theorem 3.2, we obtain for all $r \geq 0$,

$$
\lim_{N \to +\infty} F_N(t, \lambda_{(r)}) = \lim_{N \to +\infty} \sum_{n=0}^{N-1} f_n(t, \lambda_{(r)}) = f(t, \lambda_{(r)}),
$$
by substitution in (14) we get
\[
\lim_{N \to +\infty} \mathcal{F}(t) = \tilde{f}(t, 0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}} e^{it\lambda(r)} \tilde{f}(t, \lambda(r)) d\lambda(r) = X(t),
\]
where the transfer functions \( f(t, \lambda(r)), (t, r) \in \mathbb{R} \times \mathbb{N} \) are given by (13) in theorem 4.1. \( \square \)

**Remark 4.4.** Theorem 4.3 means that, \( \forall n \geq 0 \) there exists a process \( (\chi_n(t))_{t \in \mathbb{R}} \) admits the spectral representation
\[
\chi_n(t) = f_n(t, 0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}} e^{it\lambda(r)} f_n(t, \lambda(r)) d\lambda(r), \forall n \geq 0,
\]
such that
\[
X(t) = \sum_{n=0}^{\infty} \chi_n(t).
\]

4.2. **Second Technique.** In this subsection, we present a different technique to decomposition the transfer functions \( f(t, \lambda(r)) \) by the substitution of the series \( f(t, \lambda(r)) = \sum_{n=0}^{\infty} g_n(t, \lambda(r)) \) and \( f(t, \lambda(r-1)) = \sum_{n=0}^{\infty} g_n(t, \lambda(r-1)) \), for all \( r \geq 1 \) in the differential equations (13), so

For \( r = 0 \), we have
\[
f(t, 0) = \sum_{n=0}^{\infty} g_n(t, 0),
\]
where \( g_0(t, 0) = f(0, 0) + \int_0^t \mu(s) ds = f_0(t, 0) \), and
\[
g_n(t, 0) = \int_0^t \alpha(s) g_{n-1}(s, 0) ds, \forall n \geq 1,
\]
this implies
\[
g_n(t, 0) = f_n(t, 0), \forall n \geq 0.
\]

Now for all \( r \geq 1 \), we substitute the series \( f(t, \lambda(r)) = \sum_{n=0}^{\infty} g_n(t, \lambda(r)) \) and \( f(t, \lambda(r-1)) = \sum_{n=0}^{\infty} g_n(t, \lambda(r-1)) \) in the differential equations
\[
f^{(1)}(t, \lambda(r)) = \left( \alpha(t) - i\lambda(r) \right) f(t, \lambda(r)) + r \left( \gamma(t) f(t, \lambda(r-1)) + \delta_{r=1} \beta(t) \right)
\]
we obtain

1. For \( r = 1 \), we have
\[
L f(t, \lambda) = (\alpha(t) - i\lambda) f(t, \lambda) + (\gamma(t) f(t, 0) + \beta(t)),
\]
which implies
\[
\sum_{n=0}^{\infty} g_n(t, \lambda) = f(0, \lambda) + \int_0^t \beta(s) ds + \int_0^t \sum_{n=0}^{\infty} \left[ (\alpha(s) - i \lambda) g_n(s, \lambda) + \gamma(s) g_n(s, 0) \right] ds,
\]

it follows, \( g_0(t, \lambda) = f(0, \lambda) + \int_0^t \beta(s) ds \) and
\[
g_n(t, \lambda) = \int_0^t (\alpha(s) - i \lambda) g_{n-1}(s, \lambda) ds + \int_0^t \gamma(s) g_{n-1}(s, 0) ds, \quad \forall n \geq 1.
\]

After computing the terms \( (g_n(t, \lambda))_{n \geq 1} \) have the form
\[
g_1(t, \lambda) = f(0, \lambda) \int_0^t (\alpha(s_1) - i \lambda) ds_1 + \int_0^t \int_0^{s_1} (\alpha(s_1) - i \lambda) \beta(s_2) ds_2 ds_1 + \int_0^t \gamma(s_1) g_0(s_1, 0) ds_1,
\]
\[
g_2(t, \lambda) = f(0, \lambda) \int_0^t \int_0^{s_1} (\alpha(s_1) - i \lambda) (\alpha(s_2) - i \lambda) \beta(s_3) ds_3 ds_2 ds_1 + \int_0^t \int_0^{s_1} \gamma(s_2) g_0(s_2, 0) ds_2 ds_1 + \int_0^t \gamma(s_1) g_1(s_1, 0) ds_1,
\]
\[
\vdots
\]
\[
g_n(t, \lambda) = f(0, \lambda) \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\alpha(s_1) - i \lambda) \cdots (\alpha(s_n) - i \lambda) ds_n ds_{n-1} \cdots ds_2 ds_1
\]
\[
+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\alpha(s_1) - i \lambda) \cdots (\alpha(s_n) - i \lambda) \beta(s_{n+1}) ds_{n+1} ds_n ds_{n-1} \cdots ds_2 ds_1
\]
\[
+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\alpha(s_1) - i \lambda) \cdots (\alpha(s_{n-1}) - i \lambda) \gamma(s_n) g_0(s_n, 0) ds_n ds_{n-1} \cdots ds_2 ds_1
\]
\[
+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} (\alpha(s_1) - i \lambda) \cdots (\alpha(s_{n-2}) - i \lambda) \gamma(s_{n-1}) g_1(s_{n-1}, 0) ds_{n-1} ds_{n-2} \cdots ds_2 ds_1 + \cdots
\]
\[
\vdots
\]
\[
+ \int_0^t \int_0^{s_1} \int_0^{s_2} (\alpha(s_1) - i \lambda) (\alpha(s_2) - i \lambda) \gamma(s_3) g_{n-3}(s_3, 0) ds_3 ds_2 ds_1
\]
\[
+ \int_0^t \int_0^{s_1} (\alpha(s_1) - i \lambda) \gamma(s_2) g_{n-2}(s_2, 0) ds_2 ds_1 + \int_0^t \gamma(s_1) g_{n-1}(s_1, 0) ds_1.
\]

(2) For \( r \geq 2 \), we have
\[
L f(t, \lambda_{(r)}) = \left( \alpha(t) - i \lambda_{(r)} \right) f(t, \lambda_{(r)}) + r \gamma(t) f(t, \lambda_{(r-1)}),
\]

which implies
\[
\sum_{n=0}^{\infty} g_n(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) + \sum_{n=0}^{\infty} \int_0^t \left[ (\alpha(s) - i \lambda_{(r)}) g_n(s, \lambda_{(r)}) + r \gamma(s) g_n(s, \lambda_{(r-1)}) \right] ds,
\]
it follows, \( g_0(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \) and \( \forall n \geq 1, \)

\[
g_n(t, \lambda_{(r)}) = \int_0^t \left( \alpha(s) - i\lambda_{(r)} \right) g_{n-1}(s, \lambda_{(r)}) ds + r \int_0^t \gamma(s) g_{n-1}(s, \lambda_{(r-1)}) ds.
\]

After computing the terms \( \left(g_n(t, \lambda_{(r)})\right)_{n \geq 1} \) have the form

\[
g_1(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \left( \alpha(s) - i\lambda_{(r)} \right) ds_1 + r \int_0^t \gamma(s_1) g_0(s_1, \lambda_{(r-1)}) ds_1,
\]

\[
g_2(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \left( \alpha(s_2) - i\lambda_{(r)} \right) ds_2 ds_1
\]

\[+ r \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \gamma(s_2) g_0(s_2, \lambda_{(r-1)}) ds_2 ds_1 + r \int_0^t \gamma(s_1) g_1(s_1, \lambda_{(r-1)}) ds_1,
\]

\[\vdots\]

\[
g_n(t, \lambda_{(r)}) = f(0, \lambda_{(r)}) \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} \left( \alpha(s_1) - i\lambda_{(r)} \right) \ldots \left( \alpha(s_n) - i\lambda_{(r)} \right) ds_1 ds_2 \ldots ds_{n-1} ds_n + r \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-2}} \left( \alpha(s_1) - i\lambda_{(r)} \right) \ldots \left( \alpha(s_{n-2}) - i\lambda_{(r)} \right) \gamma(s_{n-1}) g_0(s_{n-1}, \lambda_{(r-1)}) ds_{n-1} ds_{n-2} \ldots ds_1 + \ldots + r \int_0^t \int_0^{s_1} \ldots \int_0^{s_2} \left( \alpha(s_1) - i\lambda_{(r)} \right) \left( \alpha(s_2) - i\lambda_{(r)} \right) \gamma(s_3) g_{n-3}(s_3, 0) ds_3 ds_2 ds_1
\]

\[+ r \int_0^t \int_0^{s_1} \left( \alpha(s_1) - i\lambda_{(r)} \right) \gamma(s_2) g_{n-2}(s_2, \lambda_{(r-1)}) ds_2 ds_1 + r \int_0^t \gamma(s_1) g_{n-1}(s_1, \lambda_{(r-1)}) ds_1.
\]

**Theorem 4.5.** Under the condition of theorem 4.3, \( \forall n \geq 0 \) there exists a process \( \mathcal{X}_n(t) \in \mathbb{R}^n \) admits the spectral representation

\[
\mathcal{X}_n(t) = \mathcal{g}_n(t, 0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}} e^{it\lambda_{(r)}} \mathcal{g}_n(t, \lambda_{(r)}) dZ(\lambda_{(r)}), \forall n \geq 0,
\]

such that

\[
X(t) = \sum_{n=0}^{\infty} \mathcal{X}_n(t).
\]

**Proof.** The result follows immediately, because

\[
\sum_{n=0}^{\infty} g_n(t, \lambda_{(r)}) = \sum_{n=0}^{\infty} f_n(t, \lambda_{(r)}) = f(t, \lambda_{(r)}),
\]

then
\[
\sum_{n=0}^{\infty} \ddot{X}_n(t) = \sum_{n=0}^{\infty} \ddot{g}_n(t,0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{i\tilde{\lambda}(r)} \sum_{n=0}^{\infty} \ddot{g}_n(t,\tilde{\lambda}(r))d\tilde{Z}(\lambda(r))
\]

\[
= \sum_{n=0}^{\infty} \ddot{f}_n(t,0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{i\tilde{\lambda}(r)} \sum_{n=0}^{\infty} \ddot{f}_n(t,\tilde{\lambda}(r))d\tilde{Z}(\lambda(r))
\]

\[
= \ddot{f}(t,0) + \sum_{r \geq 1} \frac{1}{r!} \int_{\mathbb{R}^r} e^{i\tilde{\lambda}(r)} \ddot{f}(t,\tilde{\lambda}(r))d\tilde{Z}(\lambda(r))
\]

\[
= X(t).
\]

\section{Conclusion}

In this paper we propose the ADM for solving bilinear SDE driven by Brownian motion which behaves as a nonlinear stochastic differential equation. This method is applied in time and frequency domain where we have proved the convergence of the approximate processes to the exact solution.

\section{Conflict of Interests}

The author declares that there is no conflict of interests.

\section{References}

[1] K. Abbaoui, Y. Cherruault, Convergence of Adomian’s method applied to nonlinear equations, Math. Comput. Model. 20(9) (1994), 60-73.
[2] K. Abbaoui, Y. Cherruault, Convergence of Adomian’s method applied to differential equations, Computers Math. Appl. 28(5) (1994), 103-109.
[3] G. Adomian, Stochastic systems, Academic Press, New York, NY, (1983).
[4] G. Adomian, Nonlinear stochastic operator equations, Academic Press, London, (1986).
[5] G. Adomian, Solving frontier problems of physics: The decomposition method, Kluwer Academic Publishers, Dordrecht, (1994).
[6] L. Arnold, Stochastic differential equations, theory and applications, J. Wiley, New York, (1974).
[7] E. Babolian, J. Biazar, On the order of convergence of Adomian method, Appl. Math. Comput. 130 (2002), 383-387.
[8] A. Bibi, F. Merahi, A note on $L_2$-structure of continuous-time bilinear processes with time-varying coefficients, Int. J. Stat. Probab. 4 (2015), 150-160.

[9] A. Boumenir, M. Gordon, The rate of convergence for the decomposition method, Numer. Funct. Anal. Optim. 25 (2004), 15-25.

[10] Y. Cherruault, Convergence of Adomian’s method, Kybernetes, 18 (1989), 31-38.

[11] R.L. Dobrushin, Gaussian and their subordinated self-similar random generalized fields, Ann. Probab. 7 (1979), 1-28.

[12] I. El-Kalla, Error analysis of Adomian series solution to a class of nonlinear differential equations, Appl. Math. E-Notes, 7 (2007), 214-221.

[13] I. El-Kalla, New results on the analytic summation of Adomian series for some classes of differential and integral equations, Appl. Math. Comput. 217 (2010), 3756-3763.

[14] N. Himoun, K. Abbaoui, Y. Cherruault, New results of convergence of Adomian’s method, Kybernetes, 28 (1999), 423-429.

[15] N. Himoun, K. Abbaoui, Y. Cherruault, Short new results on Adomian method, Kybernetes, 32 (2003), 523-539.

[16] M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method, Appl. Math. Comput. 182 (2006), 536-543.

[17] W.G. Kelley, A. Peterson, The theory of differential equations. Springer, (2010).

[18] D. Lesnic, The decomposition method for initial value problems, Appl. Math. Comput. 181 (2006), 206-213.

[19] P. Major, Multiple Wiener-Itô integrals, Lecture Notes in Mathematics 849, Springer-Verlag, New York, (1981).

[20] R. Rach, On the Adomian decomposition method and comparisons with Picard’s method, J. Math. Anal. Appl. 128 (1987), 480-483.