Dyonic Instantons in Five-Dimensional Gauge Theories

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Abstract

We show that there exist finite energy, non-singular instanton solutions for five-dimensional theories with broken gauge symmetry. The soliton is supported against collapse by a non-zero electric charge. The low-energy dynamics of these solutions is described by motion on the ADHM moduli space with potential.
Introduction

Instantons appear as BPS particle-like configurations in five-dimensional supersymmetric Yang-Mills theories. If the gauge symmetry is broken however, the instanton shrinks to zero size resulting in a singular solution. In this letter, we show that when the gauge group is broken to the maximal torus this collapse may be stabilised by a non-zero electric charge.

We consider $\mathcal{N} = 1$ supersymmetric Yang-Mills in five dimensions with arbitrary gauge group $G$ of rank $r$. The field content consists of a vector field with field strength $F$, a single real adjoint scalar $\phi$, and two pseudo-Majorana spinors. In five dimensions the gauge coupling constant $g$ has dimension $-1/2$ and the theory is weakly coupled in the infra-red. It is also non-renormalisable and is to be considered as an effective theory for some, undetermined, short-distance physics.

The classical moduli space of the theory is $\mathbb{R}^r/W$, where $W$ is the Weyl group, and is parametrised by the vacuum expectation value (VEV) of $\phi$ which is taken to lie in the Cartan sub-algebra: $\langle \phi \rangle = v \cdot H$. We choose $v$ such that $G$ is broken to $U(1)^r$. As well as these surviving, local, Abelian symmetries, the theory contains a $SU(2)_R \times U(1)_I$ global symmetry. The former is the usual R-symmetry for theories with eight supercharges, while the latter has conserved current $J = \ast \text{Tr}(F \wedge F)$ and conserved charge equal to the instanton winding number, $k$. The supersymmetry algebra includes a single, real, scalar central charge that includes contributions from each local and global Abelian symmetry [1],

$$Z = \frac{8\pi^2 k}{g^2} + v \cdot q ,$$

where the $r$-vector $q$ denotes electric charge under $U(1)^r$. It was further shown in [1] that the central charge is altered by the presence of Chern-Simons terms which may be generated at one-loop. In this paper we shall restrict ourselves to actions without such terms.

In the remainder of the paper, we shall demonstrate the existence of classical soliton solutions charged under both $U(1)_I$ and local gauge symmetries. These configurations therefore carry both topological and electrical charge and we christen them dyonic instantons in analogy with four-dimensional monopoles. In the simplest case of a single instanton in $SU(2)$ gauge group, the equations may be solved explicitly. For higher instanton charge in $SU(N)$ gauge group, we derive an expression for the electric charge in terms of the ADHM construction [2]. We further describe the low-energy dynamics of instantons in spontaneously broken gauge groups by a 0+1 dimensional sigma-model on the ADHM moduli space with a potential given by

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The $\mathcal{N} = 1$ theory has eight supercharges. The arguments also apply to the $\mathcal{N} = 2$ theories with sixteen supercharges providing four of the scalar fields are set to zero. In these theories, the dyonic instantons are 1/4-BPS states.
the length of a tri-holomorphic Killing vector. The dyonic instantons appear as BPS solutions to the equations of motion of this massive sigma-model.

The Bogomol’nyi Equations

The existence of dyonic instantons may be demonstrated by a simple Bogomol’nyi type argument, entirely analogous to that used for 1/4-BPS states of the four-dimensional \( N = 4 \) Yang-Mills theory \([3]\). One starts with the energy density for the bosonic fields,

\[
H = \frac{1}{2} \mathrm{Tr} \int d^4 x \left\{ E^2 + \frac{1}{2} F^2 + (D_0 \phi)^2 + (F_{\mu \nu})^2 \right\},
\]

where \( \mu = 1, 2, 3, 4 \) is a spatial index, and \( E_\mu = \partial_0 A_\mu - D_\mu A_0 \). Completing the square in the usual fashion, we find,

\[
H = \frac{1}{2} \mathrm{Tr} \int d^4 x \left\{ (E_\mu - D_\mu \phi)^2 + \frac{1}{4} (F_{\mu \nu} - \ast F_{\mu \nu})^2 + (D_0 \phi)^2 + \frac{1}{2} F_{\mu \nu} \ast F_{\mu \nu} + 2 E_\mu D_\mu \phi \right\}. 
\]

The final two terms are both total derivatives, an observation that requires use of Gauss’ law,

\[
D_\mu E_\mu = ig[\phi, D_0 \phi].
\]

If the electric field has the usual five-dimensional leading order behaviour for a point-like, electrically charged, configuration,

\[
E_\mu = \frac{q}{2 \pi^2 x^4} x_\mu + \ldots,
\]

where \ldots denotes terms suppressed by \( 1/x \), then the resulting energy bound coincides with that derived using the supersymmetry algebra, \( H \geq |Z| \). The inequality is saturated by time-independent solutions \( (\partial_0 = 0) \) with the time-component of the gauge field given by \( A_0 = -\phi \) and the remaining gauge fields determined by the usual self-duality condition,

\[
F_{\mu \nu} = \ast F_{\mu \nu}. \tag{2}
\]

Finally, Gauss’ law requires the adjoint scalar to obey the covariant Laplace equation in the background of the instanton,

\[
D^2 \phi = 0. \tag{3}
\]

Given a self-dual field strength, there exists a unique solution to (3) for each VEV of \( \phi \), a result which can be seen through study of the instanton zero-modes \([3]\). In the
simplest case of a single instanton in $SU(2)$ gauge group, solutions to both (2) and (3) are well known. In singular gauge they are given by,

$$A_\mu = \frac{2}{g} \frac{\rho^2}{x^2(x^2 + \rho^2)} \eta^a_{\mu\nu} \frac{\sigma^a}{2} x^\nu ; \quad \phi = v \frac{x^2}{x^2 + \rho^2} \frac{\sigma^3}{2} ,$$

where $\eta^a$ are the self-dual 't Hooft matrices, $\sigma^a$ the Pauli matrices in the role of $SU(2)$ gauge generators, and $\rho$ a collective coordinate determining the scale size of the instanton. The five-dimensional field strength is therefore,

$$E_\mu = D_\mu \phi = \frac{2v \rho^2 x^\nu}{(x^2 + \rho^2)^2} \eta^3_{\nu\lambda} \eta^a_{\mu\lambda} \frac{\sigma^a}{2} .$$

Notice that in this gauge, the leading order term does not lie fully in the Cartan sub-algebra. The electric charge is given by the gauge invariant integral over the spatial boundary,

$$q = \int d^4S_\mu \frac{1}{v} \text{Tr}(\phi E_\mu) = 4\pi^2 \rho^2 v ,$$

and a dyonic instanton of given electric charge $q$ is therefore seen to stabilise at size $\rho \sim (q/v)^{-1/2}$.

In order to demonstrate the BPS nature of these solutions, we consider the supersymmetry transformations of a five-dimensional $\mathcal{N} = 1$ gauge theory. These are obtained from the dimensional reduction of the six-dimensional, chiral, $(1,0)$ supersymmetric gauge theory. In this way we find that, for a bosonic configuration, supersymmetry is preserved if and only if

$$\frac{1}{4} F_{\mu\nu} \Gamma_{\mu\nu} \epsilon - \frac{1}{2} \Gamma_\mu (E_\mu \Gamma_0 - D_\mu \phi \Gamma_5) \epsilon = 0 ,$$

where $\Gamma_0, \Gamma_1, \Gamma_2, ..., \Gamma_5$ are an eight dimensional representation of the Clifford algebra in six dimensions. The vanishing of these two terms therefore requires $\Gamma_{1234} \epsilon = \epsilon$ and $\Gamma_{05} \epsilon = -\epsilon$. Since the six-dimensional theory is chiral (i.e. $\Gamma_{012345} \epsilon = -\epsilon$) the second condition follows from the first and one half of the $\mathcal{N} = 1$ supersymmetries are thus preserved. Note that if we instead considered the maximal $\mathcal{N} = 2$ theory, obtained by dimensional reduction of the non-chiral, six-dimensional $(1,1)$ gauge theory, then these two conditions are independent and the solution preserves one quarter of the supersymmetry. This mirrors the situation for “string-web” type states in four-dimensions; they too preserve four supercharges, regardless of the number of supersymmetries of their parent theory.

The ADHM Construction

For higher instanton winding number, explicit solutions are not known. For classical gauge groups however, a formal expression for solutions to (4) may be derived using
the ADHM construction \[2\]. In recent years, this technique has been employed extensively by Dorey, Khoze, Mattis and collaborators in order to calculate instanton contributions to correlation functions in four-dimensional supersymmetric gauge theories. See for example \[4, 5, 6\]. Moreover, these authors have succeeded in casting solutions to equation (3) in the language of the ADHM variables. We now briefly review these solutions for instantons of winding number \(k\) in \(SU(N)\) gauge groups and determine an expression for the electric charge of the dyonic instanton in terms of its collective coordinates.

A detailed discussion of the ADHM construction for \(SU(N)\) gauge group, as well as its application to solving further equations, may be found in \[6\]. The construction starts with a complex \(N + 2k \times 2k\) matrix \(\Delta\). This matrix is linear in the spatial coordinates \(x^\mu\) and, in a certain basis, has components,

\[
\Delta = \begin{pmatrix}
\omega_{ia} \\
\alpha^j_{ia} + x^\beta_{j\alpha} \delta^j_i
\end{pmatrix},
\]

where the ranges of the indices are \(a = 1, \ldots, N\), \(i, j = 1, \ldots, k\), and \(\alpha, \beta = 1, 2\). The \(2 \times 2\) matrix \(x\) encodes the spatial coordinates in the usual quaternionic form, \(x = x^\mu \sigma^\mu\), with \(\sigma^\mu = (1, \sigma^a)\). The \(N \times 2k\) matrix \(\omega\) and the \(2k\)-dimensional square matrix \(a\) are both independent of \(x^\mu\).

The next step is to define a matrix of orthonormal null vectors of \(\Delta^\dagger\). This \(N + 2k \times N\) dimensional complex matrix \(U\) satisfies \(\Delta^\dagger U = U^\dagger \Delta = 0\). The ADHM ansatz for the gauge potential is then \(A_\mu = U^\dagger \partial_\mu U\) and is seen to yield a self-dual field strength provided the ADHM constraints \(\Delta^\dagger \Delta = f^{-1} \otimes \sigma^0\) are satisfied, with \(f\) a real, arbitrary, invertible \(k \times k\) matrix. This construction can be shown to realise the full \(4kN\) dimensional hyperKähler moduli space of instantons.

In order to exhibit solutions to (3) in this language, we first introduce a basis for the Cartan sub-algebra in which \(\langle \phi \rangle = \text{diag} (v^1, \ldots, v^N)\) with \(\sum_a v^a = 0\). Then solutions to the adjoint Higgs equation of motion involves the \(N + 2k \times N + 2k\) real, constant matrix \(A\) and is given by \[4, 6\],

\[
\phi = U^\dagger A U \quad \text{with} \quad A = \begin{pmatrix}
\langle \phi \rangle^a \\
0
\end{pmatrix},
\]

where \(A'\) is a real, constant \(k \times k\) matrix required to satisfy the linear, algebraic equation

\[
\frac{1}{2} \text{tr}_2 \{ A', \omega^\dagger \omega \} - \frac{1}{2} \text{tr}_2 \left( [a^\dagger, A] a - a^\dagger [a, A'] \right) = \text{tr}_2 \omega^\dagger \langle \phi \rangle \omega,
\]

where \(\text{tr}_2\) denotes the trace over the \(\alpha, \beta = 1, 2\) indices of \(\omega\) and \(a\). The expression for the electric field is now easily determined \[4\],

\[
E_\mu = D_\mu \phi = -U^\dagger (\partial_\mu \Delta) f \Delta^\dagger A U - U^\dagger \Delta f (\partial_\mu \Delta^\dagger) U.
\]
Once again employing the notation $q = (q^1, \ldots, q^N)$, with $\sum_a q^a = 0$, the electric charge may be expressed as

$$q^a = 8\pi^2 \sum_{\alpha=1}^k \sum_{i,j=1}^k \left[ \omega^{i\alpha}_a \omega^a_{\alpha j} (v^a \delta^j_i - A'^j_{\delta^i_j}) \right].$$

(7)

where there is no summation over the $a$ index on the right-hand side.

**Low-Energy Dynamics**

The expressions (5) and (7) that we have derived for the electric charge are not quantised. This of course is to be expected for Noether charges in a classical theory. In order to see how its quantisation arises, we examine the low-energy dynamics of the soliton system. Again, the situation here is entirely analogous to the corresponding problem for 1/4-BPS states [4].

We first consider the low-energy dynamics of instantons in the case of vanishing VEV, for which the full ADHM moduli space of solitons exists. Let this space be parametrised by coordinates $X^p$, (for example, $p = 1, \ldots, 4kN$ for gauge group $SU(N)$), with the corresponding instanton zero mode denoted as $\delta_p A_\mu$. These zero modes are subject to Gauss’ law and must satisfy $D_\mu \delta_p A_\mu = 0$. The small-velocity interactions of instantons are described in the usual Manton approximation by a 0+1 dimensional sigma-model on this space with metric,

$$g_{pq} = \Tr \int d^4 x \left\{ \delta_p A_\mu \delta_q A_\mu \right\} .$$

(8)

This metric is singular at the points of vanishing instanton size. Turning now to the situation with non-zero VEV, it is natural to describe the low-energy dynamics in terms of the same sigma-model, extended to include a potential term [7],

$$V = \frac{1}{2} \Tr \int d^4 x \left( \mathcal{D}_\mu \phi \right)^2 = \frac{1}{2} v \cdot q .$$

(9)

This potential has a nice geometrical interpretation in terms of the ADHM moduli space metric: it is equal to half the norm squared of a tri-holomorphic Killing vector generated by the gauge transformation that asymptotes to $v \cdot H$ [8]. Moreover, denoting as $\epsilon \cdot K^p$ the components of the Killing vector on the ADHM moduli space that is generated by the gauge transformation that asymptotes to $\epsilon \cdot H$, it was further shown in [8] that the electric charge is given by,

$$q = g_{pq} (v \cdot K^p) K^q .$$

(10)

To see how this works in practice, we may return to the simplest example of the single instanton in $SU(2)$. In this case it is known that the instanton moduli space is
$C^2 \times C^2 / \mathbb{Z}_2$, where the first factor parametrises the centre of mass of the instanton, the radial coordinate of the second factor parametrises the scale size of the instanton, and the angular coordinates parametrise the three gauge orientation modes. We choose to write the flat metric on $C^2 / \mathbb{Z}_2$ as the singular limit of the Eguchi-Hanson metric,

$$ds^2 = H(r)dr \cdot dr + H^{-1}(d\psi + \omega \cdot dr)^2,$$

where $H = 2/r$ and $\nabla \times \omega = \nabla H$. Furthermore, $\psi$ has range $0 \leq \psi < 4\pi$ and $r = \pi^2 \rho^2$, where the coefficient $\pi^2$ has been determined in order to derive the correct numerical value for the potential below. The singularity of the metric at $r = 0$ reflects the singularity of the corresponding instanton solution (4) at $\rho = 0$. An advantage of the above description of $C^2 / \mathbb{Z}_2$ is that the tri-holomorphic Killing vector that arises from $U(1)$ gauge transformations (with period $2\pi$) is manifest and is given by $2\partial \psi$. The potential (9) becomes,

$$V = 4v^2 g_{\psi \psi} = 4v^2 H^{-1} = 2\pi^2 v^2 \rho^2,$$  

(11)

whose functional form is in agreement with (4) and (5). Notice that, unlike the similar potential on Taub-NUT space that appears in the low-energy effective dynamics of 1/4-BPS states [8, 7], the instanton potential (11) is unbounded.

So far, we have only described the low-energy dynamics of instantons embedded in broken gauge groups. We have yet to recover a description of the dyonic instantons. In fact, these appear as solutions to the equations of motion of the massive sigma model [7, 9]. Indeed, one may easily verify that solutions to the first order equation,

$$\frac{dX^p}{dt} = v \cdot K^p,$$  

(12)

also satisfy the equations of motion. Compactness of the gauge group ensures that the orbits of these solutions are compact and they therefore have the interpretation of excited instanton states rather than scattering states. They have mass given by $H = v \cdot q$ and are identified with the dyonic instantons.

Let us now consider the fermions in our discussion. Their properties follow from supersymmetry and the BPS nature (6) of the dyonic instantons. The bosonic collective coordinates are accompanied by $4kN$ fermionic collective coordinates (this number is increased to $8kN$ for the $\mathcal{N} = 2$ theory) and the massive sigma model described above is supersymmetrised accordingly. Indeed, a potential given by the length of a tri-holomorphic Killing vector, as in equations (4) and (10), is of the form that allows a supersymmetric completion of up to 8 supercharges [10] and it is easily seen that solutions to (12) saturate the BPS-bound of the sigma-model supersymmetry algebra, reflecting their BPS nature in the underlying five-dimensional theory. In terms of the ADHM language, expressions for the (non-kinetic) fermion terms of the massive sigma model, as well as the supersymmetry transformations, have been constructed in [4, 5, 6].
For the $\mathcal{N} = 1$ theory, one may consider the addition of hypermultiplets in various representations of the gauge group. There exists a single real, bare mass parameter, $m$, which may be assigned to each hypermultiplet. Restricting ourselves to $N_f$ hypermultiplets in the fundamental representation of an $SU(N)$ gauge group, and denoting the bare masses in $N_f$-vector form $\vec{m}$, these parameters generically break the global flavour group to its Cartan sub-algebra and induce a further term in the central charge of the form $\vec{m} \cdot \vec{S}$ where $\vec{S}$ denotes the charge of a state under the surviving Abelian flavour symmetries. An examination of the instanton mass [5, 6] reveals that the instanton is indeed charged under these flavour symmetries, courtesy of fundamental fermion zero modes.

We may now turn to the question of the BPS spectrum in the quantum theory. In the case of vanishing VEV, naive semi-classical quantisation of the scaling collective coordinate would appear to lead to a continuous spectrum of particles. The interpretation of this is not well understood. With non-vanishing VEV, the problem is different. Uncharged instantons become singular and their existence in the theory can only be determined by appealing to new ultra-violet degrees of freedom, generically arising from string theory. This reflects the non-renormalisability of five-dimensional Yang-Mills theory. Dyonic instantons however suffer from neither of these problems and it is sensible to attempt to determine their spectrum without recourse to string theory. Quantisation of sigma-models with potential has been considered recently in [7] in the context of 1/4-BPS states. These authors find that the states of the theory are described by suitable, normalisable differential forms satisfying,

$$i(d - i_{v \cdot K})\Omega = \pm \ast (d - i_{v \cdot K}) \ast \Omega,$$

where $i_{v \cdot K}$ denotes contraction with the Killing vector field $v \cdot K$ and $\ast$ is the Hodge dual. Moreover, the quantisation of electric charge now becomes apparent arising, as for the more familiar dyons in four-dimensions, through the quantisation of a periodic variable parametrising the orbits of the Killing vector fields.

The BPS spectrum of five-dimensional gauge theories has been derived in the context of fivebrane webs (see for example [11]) where the minimal instanton states were indeed found to be dyonic. It would be interesting to reproduce these results from field theory.

Finally, we turn to six-dimensional theories with sixteen supercharges. Consider first the $(1, 1)$ theory that propagates on the IIB fivebrane. The methods in this paper may be employed to construct an instanton string in this theory. The identification of this soliton with a bound state of D-strings and fundamental strings will be explored in a forthcoming publication [12]. The relationship to the $(2, 0)$ theory that exists on coincident IIA fivebranes is more subtle: the 0+1-dimensional ADHM sigma model with eight supercharges has been proposed as a matrix model description of this theory [13]. The massive sigma model described above corresponds to the spontaneously
broken theory on separated fivebranes. It is natural to interpret the states described by (12) as “W-boson” strings of the 2-form gauge potential.

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