We consider the open XXZ quantum spin chain with nondiagonal boundary terms. For bulk anisotropy value $\eta = \frac{in}{p+1}$, $p = 1, 2, \ldots$, we propose an exact $(p+1)$-order functional relation for the transfer matrix, which implies Bethe-Ansatz-like equations for the corresponding eigenvalues. The key observation is that the fused spin-$\frac{p+1}{2}$ transfer matrix can be expressed in terms of a lower-spin transfer matrix, resulting in the truncation of the fusion hierarchy.
1 Introduction

A long outstanding problem has been to solve the open XXZ quantum spin chain with nondiagonal boundary terms, defined by the Hamiltonian

\[ H = \frac{1}{2} \left\{ \sum_{n=1}^{N-1} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh \eta \, \sigma_n^z \sigma_{n+1}^z \right) + \sinh \eta \left( \coth \xi_+ \sigma_1^x - \sigma_1^x - \coth \xi_+ \sigma_N^x - \frac{2\kappa_+}{\sinh \xi_+} \sigma_N^z \right) \right\}, \tag{1.1} \]

where \( \sigma_x, \sigma_y, \sigma_z \) are the standard Pauli matrices, \( \eta \) is the bulk anisotropy parameter, \( \xi_+, \kappa_+ \) are arbitrary boundary parameters, and \( N \) is the number of spins. Solving this problem (e.g., determining the Bethe Ansatz equations) is a crucial step in formulating the thermodynamics of the spin chain and of the boundary sine-Gordon model. Moreover, this problem has important applications in condensed matter physics and statistical mechanics.

A fundamental difficulty is that, in contrast to the special case of diagonal boundary terms (i.e., \( \kappa_\pm = 0 \)) considered in \([4, 1]\), a simple pseudovacuum state does not exist. Hence, most of the techniques which have been developed to solve integrable models cannot be applied. Moreover, it is not yet clear how to implement the few techniques (such as Baxter’s \( T - Q \) approach \([5]\) or the generalized algebraic Bethe Ansatz \([5, 6]\)) which do not rely on a pseudovacuum state.

We report here some progress on this problem. Namely, for bulk anisotropy value

\[ \eta = \frac{i\pi}{p+1}, \quad p = 1, 2, \ldots, \tag{1.2} \]

(and hence \( q \equiv e^{i\eta} \) is a root of unity, satisfying \( q^{p+1} = -1 \)), we propose an exact \( (p+1) \)-order functional relation for the fundamental transfer matrix, which implies Bethe-Ansatz-like equations for the corresponding eigenvalues. The key observation is that the fused transfer matrices \( t^{(j)}(u) \), which are constructed with a spin-\( j \) auxiliary space, satisfy the identity

\[ t^{(\frac{p+1}{2})}(u) = \alpha(u) \left[ t^{(\frac{p-1}{2})}(u + \eta) + \beta(u) \mathbb{I} \right], \tag{1.3} \]

where \( t^{(0)} = \mathbb{I} \) (identity matrix), and \( \alpha(u), \beta(u) \) are scalar functions. That is, the spin-\( \frac{p+1}{2} \) transfer matrix can be expressed in terms of a lower-spin transfer matrix, resulting in the truncation of the fusion hierarchy. We have verified this result explicitly for \( p = 1, 2, 3 \), and we conjecture that it is true for all positive integer values of \( p \). The simplest case

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\(^1\)This is distinct from the observation due to Belavin et al. \([7, 8]\) that, for the special case of quantum-group symmetry (i.e., \( \kappa_\pm = 0, \xi_\pm \to \infty \)), the fused transfer matrix \( t^{(\frac{p}{2})}(u) \) vanishes after quantum group reduction.
$p = 1$, which corresponds to the XX chain, has recently been analyzed in [9]. Similar higher-order functional relations have been obtained for the closed (periodic boundary conditions) 8-vertex model by Baxter [10] using a different method.

The outline of this article is as follows. In Section 2 we review the construction of the fundamental ($j = \frac{1}{2}$) transfer matrix which contains the Hamiltonian (1.1). In Section 3 we briefly review the so-called fusion procedure [11]-[17] and the construction of the higher-spin transfer matrices, which obey an infinite fusion hierarchy. In Section 4, we obtain the relation (1.3), to which we refer as the “truncation identity,” since it serves to truncate the fusion hierarchy. In Section 5 we present the exact functional relations which are obeyed by the fundamental transfer matrix. The corresponding sets of Bethe Ansatz equations are given in Section 6. We conclude with a brief discussion of our results in Section 7.

2 Fundamental transfer matrix

We recall [1] that the transfer matrix for an open chain is made from two basic building blocks, called $R$ (bulk) and $K$ (boundary) matrices.

An $R$ matrix is a solution of the Yang-Baxter equation

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).$$

(2.1)

(See, e.g., [13, 18, 19].) For the XXZ spin chain, the $R$ matrix is the 4 × 4 matrix

$$R(u) = \begin{pmatrix}
\sinh(u + \eta) & 0 & 0 & 0 \\
0 & \sinh u & \sinh \eta & 0 \\
0 & \sinh \eta & \sinh u & 0 \\
0 & 0 & 0 & \sinh(u + \eta)
\end{pmatrix},$$

(2.2)

where $\eta$ is the anisotropy parameter. This $R$ matrix has the symmetry properties

$$R_{12}(u) = P_{12} R_{12}(u) P_{12} = R_{12}(u)^{t_{12}},$$

(2.3)

where $P_{12}$ is the permutation matrix and $t$ denotes transpose. Moreover, it satisfies the unitarity relation

$$R_{12}(u) R_{12}(-u) = \zeta(u) I, \quad \zeta(u) = -\sinh(u + \eta) \sinh(u - \eta),$$

(2.4)

and the crossing relation

$$R_{12}(u) = V_1 R_{12}(-u - \eta)^{t_2} V_1, \quad V = -i\sigma_y.$$

(2.5)
Finally, it has the periodicity property

\[ R_{12}(u + i\pi) = -\sigma_z^1 R_{12}(u) \sigma_z^2 = -\sigma_z^1 R_{12}(u) \sigma_z^2. \]  

(2.6)

The matrix \( K^-(u) \) is a solution of the boundary Yang-Baxter equation \[20]\]

\[ R_{12}(u - v) \ K_1^-(u) \ R_{21}(u + v) \ K_2^-(v) = K_2^-(v) \ R_{12}(u + v) \ K_1^-(u) \ R_{21}(u - v). \]  

(2.7)

We consider here the following \( 2 \times 2 \) matrix \[2, 3]\]

\[ K^-(u) = \begin{pmatrix} \sinh(\xi_- + u) & \kappa_- \sinh 2u \\ \kappa_- \sinh 2u & \sinh(\xi_- - u) \end{pmatrix}, \]  

(2.8)

which evidently depends on two boundary parameters \( \xi_-, \kappa_- \). We set the matrix \( K^+(u) \) to be \( K^-(u - \eta) \) with \((\xi_-, \kappa_-)\) replaced by \((\xi_+, \kappa_+)\); i.e.,

\[ K^+(u) = \begin{pmatrix} -\sinh(u + \eta - \xi_+) & -\kappa_+ \sinh(2u + 2\eta) \\ -\kappa_+ \sinh(2u + 2\eta) & \sinh(u + \eta + \xi_+) \end{pmatrix}. \]  

(2.9)

The \( K \) matrices have the periodicity property

\[ K^\pm(u + i\pi) = -\sigma_z^1 K^\pm(u) \sigma_z^2. \]  

(2.10)

The fundamental transfer matrix \( t(u) \) for an open chain of \( N \) spins is given by \[11\]

\[ t(u) = \text{tr}_0 K_0^+(u) T_0(u) K_0^-(u) \hat{T}_0(u), \]  

(2.11)

where \( \text{tr}_0 \) denotes trace over the “auxiliary space” 0, and \( T_0(u) \), \( \hat{T}_0(\lambda) \) are so-called monodromy matrices \[11\]

\[ T_0(u) = R_0N(u) \cdots R_01(u), \quad \hat{T}_0(u) = R_{10}(u) \cdots R_{0N}(u). \]  

(2.12)

Indeed, Sklyanin has shown that \( t(u) \) constitutes a one-parameter commutative family of matrices

\[ [t(u), t(v)] = 0. \]  

(2.13)

The Hamiltonian \((1.1)\) is related to the first derivative of the transfer matrix

\[ H = \frac{t'(0)}{4 \sinh \xi_- \sinh \xi_+ \sinh^{2N-1} \eta \cosh \eta} - \frac{\sinh^2 \eta + N \cosh^2 \eta}{2 \cosh \eta} I. \]  

(2.14)

\[^{2}\text{As is customary, we usually suppress the “quantum-space” subscripts } 1, \ldots, N.\]
The corresponding energy eigenvalues $E$ are therefore given by
\[
E = \frac{\Lambda'(0)}{4 \sinh \xi_- \sinh \xi_+ \sinh^{2N-1} \eta \cosh \eta} - \frac{\sinh^2 \eta + N \cosh^2 \eta}{2 \cosh \eta},
\]
where $\Lambda(u)$ are eigenvalues of the transfer matrix.

The transfer matrix has the periodicity property
\[
t(u + i \pi) = t(u),
\]
as follows from (2.6), (2.10). Moreover, the transfer matrix has crossing symmetry
\[
t(-u - \eta) = t(u),
\]
which can be proved using a generalization of the methods developed in [21] for the special case of quantum-group symmetry. Finally, we note that the transfer matrix has the asymptotic behavior (for $\kappa_+ \neq 0$)
\[
t(u) \sim -\kappa_- \kappa_+ \frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}} \mathbb{I} + \ldots \quad \text{for} \quad u \to \infty.
\]

3 Fusion

Our main tool is the so-called fusion technique, by which higher-dimensional representations can be obtained from lower-dimensional ones. The fusion technique was first developed in [11, 12, 13] for $R$ matrices, and then later generalized in [14] - [17] for $K$ matrices. Following [12], we introduce the (undeformed) projectors
\[
P_{1\ldots m}^\pm = \frac{1}{m!} \sum_\sigma (\pm 1)^\sigma \mathcal{P}_\sigma,
\]
where the sum is over all permutations $\sigma = (\sigma_1, \ldots, \sigma_m)$ of $(1, \ldots, m)$, and $\mathcal{P}_\sigma$ is the permutation operator in the space $\bigotimes_{k=1}^m \mathbb{C}^2$. For instance,
\[
P_{12}^+ = \frac{1}{2} (\mathbb{I} + \mathcal{P}_{12}),
\]
\[
P_{123}^+ = \frac{1}{6} (\mathbb{I} + \mathcal{P}_{23} \mathcal{P}_{12} + \mathcal{P}_{12} \mathcal{P}_{23} + \mathcal{P}_{12} + \mathcal{P}_{23} + \mathcal{P}_{13}).
\]
The fused spin-($j, \frac{1}{2}$) $R$ matrix ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) is given by [12, 13]
\[
R_{(1\ldots 2j)2j+1}(u) = P_{1\ldots 2j}^+ R_{1,2j+1}(u) R_{2,2j+1}(u + \eta) \ldots R_{2j,2j+1}(u + (2j - 1)\eta) P_{1\ldots 2j}^+.
\]
The fused spin-$j$ $K^-$ matrix is given by 

$$K^-_{(1...2j)}(u) = P^+_{1...2j} K^-_{2j-1}(u + \eta) K^-_{2j-2}(2u + 2\eta) K^-_{2j-3}(2u + 3\eta) \ldots K^-_1(u + 2\eta)$$

The fused spin-$j$ $K^+$ matrix is given by $K^+_{(1...2j)}(-u - 2j\eta)$ with $(\xi_-, \kappa_-)$ replaced by $(\xi_+, \kappa_+)$,

$$K^+_{(1...2j)}(u) = K^-_{(1...2j)}(-u - 2j\eta) \bigg|_{(\xi_-, \kappa_-) \to (\xi_+, \kappa_+)}.$$  

The fused (boundary) matrices satisfy generalized (boundary) Yang-Baxter equations.

The fused transfer matrix $t^{(j)}(u)$ constructed with a spin-$j$ auxiliary space is given by

$$t^{(j)}(u) = \text{tr}_{1...2j} K^+_{(1...2j)}(u) T_{(1...2j)}(u) K^-_{(1...2j)}(u) \hat{T}_{(1...2j)}(u + (2j - 1)\eta),$$

where

$$T_{(1...2j)}(u) = R_{(1...2j)_N}(u) \ldots R_{(1...2j)_1}(u),$$

$$\hat{T}_{(1...2j)}(u + (2j - 1)\eta) = R_{(1...2j)_1}(u) \ldots R_{(1...2j)_N}(u).$$

The transfer matrix (2.11) corresponds to the fundamental case $j = \frac{1}{2}$; that is, $t^{(\frac{1}{2})}(u) = t(u)$.

The fused transfer matrices constitute commutative families

$$[t^{(j)}(u), t^{(k)}(v)] = 0.$$  

These transfer matrices also satisfy a so-called fusion hierarchy

$$t^{(j)}(u) = \zeta_{2j-1}(2u + (2j - 1)\eta) \left[ t^{(j - \frac{1}{2})}(u) t^{(\frac{1}{2})}(u + (2j - 1)\eta) - \frac{\Delta(u + (2j - 2)\eta) \zeta_{2j-2}(2u + (2j - 2)\eta)}{\zeta(2u + 2(2j - 1)\eta)} t^{(j-1)}(u) \right],$$

with $t^{(0)} = I$, and $j = 1, \frac{3}{2}, \ldots$. The quantity $\Delta(u)$, the so-called quantum determinant, is given by

$$\Delta(u) = \Delta \{ K^+(u) \} \Delta \{ K^-(u) \} \delta \{ T(u) \} \delta \left\{ \hat{T}(u) \right\},$$

where

$$R_{(1...2j)_{j-1}}(u) = P^+_{1...2j} R_{2j+1,2j}(u - (2j - 1)\eta) \ldots R_{2j+1,1}(u - \eta) P^+_{1...2j}$$

$$= R_{(1...2j)_{j-1}}(u - (2j - 1)\eta).$$  

(3.4)
where
\[
\delta \{ T(u) \} = \text{tr}_{12} \left\{ P_{12}^{-} T_1(u) T_2(u + \eta) \right\} = \zeta (u + \eta)^N,
\]
\[
\delta \{ \hat{T}(u) \} = \text{tr}_{12} \left\{ P_{12}^{-} \hat{T}_2(u) \hat{T}_1(u + \eta) \right\} = \zeta (u + \eta)^N,
\]
\[
\Delta \{ K^-(u) \} = \text{tr}_{12} \left\{ P_{12}^{-} K_1^{-}(u) R_{12}(2u + \eta) K_2^{-}(u + \eta) \right\}
= - \sinh 2u \left[ \sinh(u + \eta + \xi) \sinh(u + \eta - \xi) + \kappa^2 \sinh^2(2u + 2\eta) \right],
\]
\[
\Delta \{ K^+(u) \} = \text{tr}_{12} \left\{ P_{12}^{-} K_2^+(u + \eta) R_{12}(-2u - 3\eta) K_1^+(u) \right\}
= \Delta \{ K^-(u - 2\eta) \} \bigg|_{(\xi_-, \kappa_-) \to (\xi_+, \kappa_+)}.
\]

Moreover,
\[
\tilde{\zeta}_j(u) = \prod_{k=1}^j \zeta (u + k\eta), \quad \tilde{\zeta}_0(u) = 1.
\] (3.13)

4 Truncation identity

We now proceed to formulate the important identity (1.3), which serves to truncate the fusion hierarchy (3.10). To this end, we first derive separate “truncation” identities for the \( R \) and \( K \) matrices.

4.1 \( R \) matrix truncation

We recall that, in addition to the fusion approach described above, there is an alternative construction \[23\] of higher-spin \( R \) matrices based on quantum groups. Following the notation of \[8\], the spin-\((\frac{1}{2}, j)\) \( R \) matrix is given by
\[
R_{(\frac{1}{2}, j)}^{qq}(u) = \begin{pmatrix}
\sinh \left( u + \left( \frac{1}{2} + \hat{H} \right)\eta \right) & \sinh \eta \hat{F} \\
\sinh \eta \hat{E} & \sinh \left( u + \left( \frac{1}{2} - \hat{H} \right)\eta \right)
\end{pmatrix},
\] (4.1)
where the matrices \( \hat{H}, \hat{E} \) and \( \hat{F} \) have matrix elements
\[
(\hat{H})_{mn} = (j + 1 - n)\delta_{m,n}, \quad m, n = 1, 2, \ldots, 2j + 1,
\]
\[
(\hat{E})_{mn} = \omega_m \delta_{m,n-1}, \quad (\hat{F})_{mn} = \omega_n \delta_{m-1,n}, \quad \omega_n = \sqrt{[n]_q [2j + 1 - n]_q},
\] (4.2)
and
\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q = e^{\eta}.
\] (4.3)
These matrices form a \((2j + 1)\)-dimensional representation of the \(U_q(su(2))\) algebra

\[
[H, \dot{E}] = \dot{E}, \quad [\dot{H}, \vec{F}] = \vec{F}, \quad [\dot{E}, \vec{F}] = [2\dot{H}]_q .
\]

The corresponding spin-\((j, 1/2)\) \(R\) matrix \(R_{(j, 1/2)}^{qq}\) is then given by

\[
\alpha\beta \left( R_{(j, 1/2)}^{qq}(u) \right)_{\alpha,\beta'} = \beta\alpha \left( R_{(1/2,j)}^{qq}(u) \right)_{\beta',\alpha'} .
\]

We refer to these \(R\) matrices as “quantum group” \((qg)\) \(R\) matrices in order to distinguish them from the fused \(R\) matrices constructed previously \((3.3)\). The two sets of \(R\) matrices are related as follows \(\footnote{It is understood that the “null” rows and columns (i.e., those with only zero matrix elements) are to be pruned from the LHS. We have verified this relation explicitly for \(j = 1, \frac{3}{2}, 2\), and we conjecture that it is true for all \(j\).}

\[
B_{1\ldots 2j} A_{1\ldots 2j} R_{(1\ldots 2j)2j+1}(u) A_{1\ldots 2j}^{-1} B_{1\ldots 2j}^{-1} = \left[ 2j-1 \right] \prod_{k=1}^{2j-1} \sinh(u + k\eta) \right] R_{(1/2,j)}^{qq}(u + (2j - 1)\frac{\eta}{2}) ,
\]

where \(A_{1\ldots 2j}\) is the matrix of (unnormalized) Clebsch-Gordon coefficients in the decomposition of the tensor product of \(2j\) spin-\(\frac{1}{2}\) representations into a direct sum of \(su(2)\) irreducible representations. Moreover, \(B_{1\ldots 2j}\) is a \(u\)-independent diagonal matrix which renders symmetric the matrix on the LHS of \((4.6)\). The \(A\) and \(B\) matrices for the cases \(j = 1, \frac{3}{2}, 2\) are given in Appendix \[\footnote{It is understood that the “null” rows and columns (i.e., those with only zero matrix elements) are to be pruned from the LHS. We have verified this relation explicitly for \(j = 1, \frac{3}{2}, 2\), and we conjecture that it is true for all \(j\).}

\[
B_{1\ldots p+1} A_{1\ldots p+1} R_{(1\ldots p+1)p+2}(u) A_{1\ldots p+1}^{-1} B_{1\ldots p+1}^{-1} = \mu(u) \left( \begin{array}{ccc} \nu(u)\sigma^z & 0 & 0 \\ 0 & B_{1\ldots p-1} A_{1\ldots p-1} R_{(1\ldots p-1)p}(u + \eta) & A_{1\ldots p-1}^{-1} B_{1\ldots p-1}^{-1} \\ 0 & 0 & -\nu(u)\sigma^z \end{array} \right) .
\]

In view of the relation \((4.6)\), we see that the corresponding fused spin-\((\frac{p+1}{2}, \frac{1}{2})\) \(R\) matrix satisfies

\[
B_{1\ldots p+1} A_{1\ldots p+1} R_{(1\ldots p+1)p+2}(u) A_{1\ldots p+1}^{-1} B_{1\ldots p+1}^{-1} = \mu(u) \left( \begin{array}{ccc} \nu(u)\sigma^z & 0 & 0 \\ 0 & B_{1\ldots p-1} A_{1\ldots p-1} R_{(1\ldots p-1)p}(u + \eta) & A_{1\ldots p-1}^{-1} B_{1\ldots p-1}^{-1} \\ 0 & 0 & -\nu(u)\sigma^z \end{array} \right) .
\]
\[ \nu(u) = -\frac{1}{\mu(u)} \prod_{k=0}^{p} \sinh(u + k\eta) = -\frac{1}{\mu(u)} \left( \frac{i}{2} \right)^{p} \sinh((p + 1)u). \] (4.10)

In obtaining this result, we have used the identity (see 1.392 in [24])
\[ \prod_{k=0}^{p} \sinh(u + k\eta) \bigg|_{\eta = \frac{\pi}{p+1}} = \left( \frac{i}{2} \right)^{p} \sinh((p + 1)u). \] (4.11)

As will be explained in Section 4.2 below, it is actually more useful to consider the similarity transformation with a matrix \( C \) (instead of \( B \)), which results in a triangular (instead of a symmetric, block-diagonal) matrix. For later convenience, we present this result here:

\[
C_{1...p+1} A_{1...p+1} R_{(1...p+1)p+2}(u) A_{1...p+1}^{-1} C_{1...p+1}^{-1} \\
= \mu(u) \begin{pmatrix}
\nu(u)\sigma^z & 0 & 0 \\
0 & B_{1...p-1} A_{1...p-1} R_{(1...p-1)p}(u + \eta) A_{1...p-1}^{-1} B_{1...p-1}^{-1} & * \\
0 & 0 & -\nu(u)\sigma^z
\end{pmatrix}
\] (4.12)

The monodromy matrices therefore obey analogous relations

\[
C_{1...p+1} A_{1...p+1} T_{(1...p+1)}(u) A_{1...p+1}^{-1} C_{1...p+1}^{-1} \\
= \mu(u)^{N} \begin{pmatrix}
\nu(u)^{N} F & 0 & 0 \\
0 & B_{1...p-1} A_{1...p-1} T_{(1...p-1)(u + \eta)} A_{1...p-1}^{-1} B_{1...p-1}^{-1} & * \\
0 & 0 & (-\nu(u))^{N} F
\end{pmatrix},
\]

\[
C_{1...p+1} A_{1...p+1} \hat{T}_{(1...p+1)}(u + p\eta) A_{1...p+1}^{-1} C_{1...p+1}^{-1} \\
= \mu(u)^{N} \begin{pmatrix}
\nu(u)^{N} F & 0 & 0 \\
0 & B_{1...p-1} A_{1...p-1} \hat{T}_{(1...p-1)}(u + (p - 1)\eta) A_{1...p-1}^{-1} B_{1...p-1}^{-1} & * \\
0 & 0 & (-\nu(u))^{N} F
\end{pmatrix},
\] (4.13)

where \( F = \prod_{k=1}^{N} \sigma^z_k \). The \( C \) matrices for the cases \( j = 1, \frac{3}{2}, 2 \) are also given in Appendix A.

### 4.2 \( K \) matrix truncation

An explicit construction of nondiagonal higher-spin \( K \) matrices analogous to (4.1) is unfortunately not yet known. Nevertheless, considerable insight can be gained by first considering
the diagonal case (i.e., $\kappa_{\pm} = 0$). The spin-$j$ diagonal $K^-$ matrix is given explicitly by (see [14] for the case $j = 1$)

$$K_{(j)}^{-qg}(u) = \text{diag} \left( k_{(j)}^{(1)}(u), k_{(j)}^{(2)}(u), \ldots, k_{(j)}^{(2j+1)}(u) \right), \quad (4.14)$$

where

$$k_{(j)}^{(m)}(u) = \prod_{l=0}^{2j-m} \sinh(\xi_+ + u + l\eta) \prod_{l=0}^{m-2} \sinh(\xi_- - u - l\eta). \quad (4.15)$$

This $K$ matrix is related to the fused $K$ matrix (3.5) by

$$A_{1\ldots2j} \, K_{(1\ldots2j)}^{-qg}(u) \, A_{1\ldots2j}^{-1} = \left[ \prod_{l=1}^{2j-1} \prod_{k=1}^{l} \sinh(2u + (l+k)\eta) \right] K_{(j)}^{-qg}(u). \quad (4.16)$$

We find that the diagonal $K_{(j)}^{-qg}(u)$ with $j = \frac{p+1}{2}$ has the following “truncation” property for $\eta = \frac{i\pi}{p+1}$,

$$K_{\left(\frac{p+1}{2}\right)}^{-qg}(u) = \sinh(\xi_+ + u) \sinh(\xi_- - u) \times \begin{pmatrix} \left(\frac{i}{2}\right)^p \frac{\sinh((p+1)(\xi_+ + u))}{\sinh(\xi_+ + u) \sinh(\xi_- - u)} & 0 & 0 \\ 0 & 0 & K_{\left(\frac{p+1}{2}\right)}^{-qg}(u + \eta) \\ 0 & 0 & \left(-\frac{i}{2}\right)^p \frac{\sinh((p+1)(\xi_- - u))}{\sinh(\xi_- - u) \sin(\xi_+ + u)} \end{pmatrix}. \quad (4.17)$$

It follows that for $\eta = \frac{i\pi}{p+1}$, the diagonal fused spin-$\frac{p+1}{2}$ $K^-$ matrix satisfies the truncation identity

$$A_{1\ldots p+1} \, K_{(1\ldots p+1)}^{-}(u) \, A_{1\ldots p+1}^{-1} \mu_-(u) = \begin{pmatrix} \mu_-(u) & 0 & 0 \\ 0 & A_{1\ldots p-1} \, K_{(1\ldots p-1)}^{-}(u + \eta) \, A_{1\ldots p-1}^{-1} & 0 \\ 0 & 0 & \nu'_-(u) \end{pmatrix}, \quad (4.18)$$

where

$$\mu_-(u) = \frac{\Delta\{K^-(u - \eta)\}}{\sinh(2u - 2\eta)} \prod_{k=2}^{2p} \sinh(2u + k\eta) = -\frac{\Delta\{K^-(u - \eta)\} \sinh^2(2(p+1)u)}{2^{2p} \sinh(2u) \sinh(2u + \eta) \sinh(2u - \eta) \sinh(2u - 2\eta)}. \quad (4.19)$$

Footnote 3 applies here too.
the quantum determinant $\Delta\{K(u)\}$ is given in Eq. (3.12),

$$
\nu_-(u) = \frac{1}{\mu_-(u)} \left( \frac{i}{2} \right)^p \left[ \prod_{l=1}^{p} \prod_{k=1}^{l} \sinh(2u + (l+k)\eta) \right] \sinh((p+1)(\xi_+ + u))
= \frac{1}{\mu_-(u)} e^{\frac{1}{2}i\pi(p+2)} \cosh\left[\frac{p}{2}\right](p+1)u \sinh\left[\frac{p+1}{2}\right](p+1)u
\times \sinh((p+1)(\xi_+ + u)) \, ,
$$

(4.20)

and

$$
\nu'_-(u) = \pm \nu_-(u) \quad \text{with} \quad \xi_+ \rightarrow -\xi_+ .
$$

(4.21)

The standard notation $[x]$ denotes integer part of $x$.

We turn now to the more general nondiagonal case ($\kappa_\pm \neq 0$). The boundary matrices

$$
B_{1,2j} A_{1,2j} K_{(1,2j)}^+(u) A_{1,2j}^{-1} B_{1,2j}^{-1} ,
$$

like their bulk counterpart (4.6), are symmetric. However, for $\eta = \frac{i\pi}{p+1}$ and $j = \frac{p+1}{2}$, these matrices are not block-diagonal. In order to obtain a truncation identity for the full transfer matrix, it is desirable to have at least block-triangular matrices.

To this end, we consider a new similarity transformation, replacing the matrix $B$ by a new matrix $C$ which is also diagonal and $u$-independent. (See Appendix A for explicit expressions of the C matrices for $j = 1, \frac{3}{2}, 2$.) Indeed, we propose the following generalization of (4.18) for the nondiagonal case:

$$
C_{1,p+1} A_{1,p+1} K_{(1,p+1)}^+(u) A_{1,p+1}^{-1} C_{1,p+1}^{-1} .
$$

$$
\mu_{\mp}(u) \left( \begin{array}{ccc}
\nu_{\mp}(u) & 0 & \sigma_{\mp}(u) \\
0 & B_{1,p-1} A_{1,p-1} K_{(1,p-1)}^+(u + \eta) A_{1,p-1}^{-1} B_{1,p-1}^{-1} & * \\
\rho_{\mp}(u) & 0 & \nu'_-(u) \\
\end{array} \right) ,
$$

(4.22)

where $\mu_-(u)$ and $\nu'_-(u)$ are given by the same expressions (4.19), (4.21); but the expression (4.20) for $\nu_-(u)$ is now replaced by

$$
\nu_-(u) = \frac{1}{\mu_-(u)} e^{\frac{1}{2}i\pi(p+2)} \cosh\left[\frac{p}{2}\right](p+1)u \sinh\left[\frac{p+1}{2}\right](p+1)u n(u; \xi_-, \kappa_-) ,
$$

(4.23)

\text{This equation is formal, since $\sigma_{\mp} \rightarrow \infty$ (and $\rho_{\mp} \rightarrow 0$) in the limit $\eta \rightarrow \frac{i\pi}{p+1}$, as can be seen from Eqs. (4.26) and (4.28). However, substituting this result into the expression for the fused transfer matrix gives a final result (4.30) which is finite in the $\eta \rightarrow \frac{i\pi}{p+1}$ limit.}
where the function \( n(u; \xi, \kappa) \) is defined by

\[
n(u; \xi, \kappa) = \sinh \left( (p + 1)(\xi + u) \right) + \sum_{l=1}^{\left\lfloor \frac{p+1}{2} \right\rfloor} c_{p,l} \kappa^{2l} \sinh \left( (p + 1)u + (p + 1 - 2l)\xi \right).
\]

We have explicitly computed the coefficients \( c_{p,l} \) which appear in this function for values of \( p \) up to \( p = 5 \), and we find that they are consistent with the following formulas:

\[
c_{p,1} = p + 1,
\]

\[
c_{p,2} = \frac{1}{2}p(p - 1) - 1,
\]

and also \( c_{5,3} = 2 \). It remains a challenge to determine the coefficients \( c_{p,l} \) for all values of \( p \) and \( l \). Moreover,

\[
\sigma_-(u) = \frac{a \omega_-(u)}{\mu_-(u)}, \quad \rho_-(u) = \frac{\omega_-(u)}{a \mu_-(u)},
\]

where

\[
\omega_-(u) = \kappa_+^{p+1} \left[ \prod_{l=1}^{p} \prod_{k=1}^{l} \sinh(2u + (l + k - 1)\eta) \right] \prod_{k=0}^{p} \sinh(2u + 2k\eta)
\]

\[
= \frac{e^{\frac{1}{2}i\pi p(p+2)}}{2^{\frac{1}{2}p(p+1)}} \kappa_+^{p+1} \cosh^{\frac{p+2}{2}}((p + 1)u) \sinh^{\frac{p+3}{2}}((p + 1)u),
\]

and

\[
a = ([p + 1]q)^{-\frac{1}{2}}.
\]

Note that \( a \to \infty \) for \( \eta \to \frac{i\pi}{p+1} \).

The \( K^+ \) matrices are given, in view of Eq. (3.6) with \( \eta = \frac{i\pi}{p+1} \) and \( j = \frac{p+1}{2} \), by

\[
K^+_{(1 \ldots p+1)}(u) = K^+_{(1 \ldots p+1)}(-u - i\pi) \bigg|_{(\xi-, \kappa-) \to (\xi+, \kappa+)}.
\]

Hence, the “plus” quantities \( \mu_+, \nu_+ \), etc. can be readily obtained from the corresponding “minus” quantities \( \mu_-, \nu_- \), etc. by making the replacements \( u \to -u - i\pi \) and \( (\xi-, \kappa-) \to (\xi+, \kappa+) \).

### 4.3 Transfer matrix truncation

We are finally in position to formulate the truncation identity for the fused transfer matrix \( t^{(j)}(u) \) defined in (3.7). Recalling the results (4.13) for the monodromy matrices and (4.22) for the \( K \) matrices, we obtain (for \( \eta = \frac{i\pi}{p+1} \) and \( j = \frac{p+1}{2} \))

\[
t^{\left(\frac{p+1}{2}\right)}(u) = \alpha(u) \left[ t^{\left(\frac{p-1}{2}\right)}(u + \eta) + \beta(u)I \right],
\]

(4.30)
where

\[
\begin{align*}
\alpha(u) & = \mu(u)^{2N} \mu_-(u) \mu_+(u), \\
\beta(u) & = \nu(u)^{2N} \left[ \nu_-(u) \nu_+(u) + \nu'_-(u) \nu'_+(u) \right. \\
& \quad + \left. (-1)^N \left( \sigma_+(u) \rho_-(u) + \sigma_-(u) \rho_+(u) \right) \right].
\end{align*}
\]

(4.31)

Note that the factors of \(a\) from \(\sigma_\mp(u)\) and \(\rho_\mp(u)\) cancel in the expression for \(\beta(u)\); and hence, the result is finite for \(\eta \to \frac{i\pi}{p+1}\).

5 Functional relations

Combining the fusion hierarchy (3.10) and the truncation identity (4.30), it is straightforward to obtain – for any positive integer value of \(p\) – a \((p + 1)\)-order functional relation for the fundamental transfer matrix. We propose the following general form of the functional relations:

\[
\begin{align*}
f_0(u) & t(u) t(u + \eta) \ldots t(u + pn) \\
- f_1(u) t(u + \eta) & t(u + 2\eta) \ldots t(u + (p - 1)\eta) \\
- f_1(u + \eta) & t(u + 2\eta) t(u + 3\eta) \ldots t(u + pn) \\
- f_1(u + 2\eta) & t(u) t(u + 3\eta) \ldots t(u + 4\eta) \ldots t(u + pn) \\
- f_1(u + 3\eta) & t(u) t(u + \eta) t(u + 4\eta) \ldots t(u + pn) - \ldots \\
- f_1(u + pn) & t(u) t(u + \eta) \ldots t(u + (p - 2)\eta) \\
+ \frac{f_1(u) f_1(u + 2\eta)}{f_0(u)} & t(u + 3\eta) t(u + 4\eta) \ldots t(u + (p - 1)\eta) \\
+ \frac{f_1(u) f_1(u + 3\eta)}{f_0(u)} & t(u + \eta) t(u + 4\eta) t(u + 5\eta) \ldots t(u + (p - 1)\eta) + \ldots \\
+ \frac{f_1(u) f_1(u + (p - 1)\eta)}{f_0(u)} & t(u + \eta) t(u + 2\eta) \ldots t(u + (p - 3)\eta) \\
+ \frac{f_1(u + \eta) f_1(u + 3\eta)}{f_0(u)} & t(u + 4\eta) t(u + 5\eta) \ldots t(u + pn) \\
+ \frac{f_1(u + \eta) f_1(u + 4\eta)}{f_0(u)} & t(u + 2\eta) t(u + 5\eta) t(u + 6\eta) \ldots t(u + pn) + \ldots \\
+ \frac{f_1(u + \eta) f_1(u + pn)}{f_0(u)} & t(u + 2\eta) t(u + 3\eta) \ldots t(u + (p - 2)\eta)
\end{align*}
\]
\[ + \frac{f_1(u + 2\eta)}{f_0(u)} f_1(u + 4\eta) t(u)t(u + 5\eta)t(u + 6\eta)\ldots t(u + p\eta) \]
\[ + \frac{f_1(u + 2\eta)}{f_0(u)} f_1(u + 5\eta) t(u)t(u + 3\eta)t(u + 6\eta)t(u + 7\eta)\ldots t(u + p\eta) + \ldots \]
\[ + \frac{f_1(u + 2\eta)}{f_0(u)} f_1(u + p\eta) t(u)t(u + 3\eta)t(u + 4\eta)\ldots t(u + (p - 2)\eta) \]
\[ + \ldots = f_3(u). \]  

(5.1)

In particular, the first three functional relations are given by

\[ p = 1: \quad f_0(u)t(u)t(u + \eta) - f_1(u) - f_1(u + \eta) = f_3(u), \]
\[ p = 2: \quad f_0(u)t(u)t(u + \eta)t(u + 2\eta) - f_1(u)t(u + \eta) - f_1(u + \eta)t(u + 2\eta) - f_1(u + 2\eta)t(u) = f_3(u), \]
\[ p = 3: \quad f_0(u)t(u)t(u + \eta)t(u + 2\eta)t(u + 3\eta) - f_1(u)t(u + \eta)t(u + 2\eta) - f_1(u + \eta)t(u + 3\eta) - f_1(u + 2\eta)t(u + 3\eta) - f_1(u + 3\eta)t(u + \eta) = f_3(u). \]

(5.2)

Remarkably, only three distinct functions \( f_0, f_1, f_3 \) appear in the functional relations. The function \( f_0(u) \) is given by

\[
f_0(u) = \prod_{l=1}^{p} \prod_{k=1}^{l} \zeta(2u + (l + k)\eta) \]
\[ = \frac{e^{i\pi(p+1)^2}}{2^{p(p-1)}} \cosh^p((p+1)u) \sinh^{p-1}((p+1)u) \cosh((p+1)(u - \frac{i\pi}{2})). \]

(5.3)

This function has the periodicity property

\[ f_0(u) = f_0(u + \eta) \]

(5.4)

for \( \eta = \frac{i\pi}{p+1} \). The function \( f_1(u) \) is given by

\[ f_1(u) = \frac{f_0(u)\Delta(u - \eta)}{\zeta(2u)}, \]

(5.5)

where \( \zeta(u) \) is given by (2.4), and the quantum determinant \( \Delta(u) \) is given by (3.11). Finally, the function \( f_3(u) \) is given by

\[ f_3(u) = \alpha(u)\beta(u) \]
\[
\frac{(-1)^N p^{2N+1} \cosh \left( \frac{1}{2} p \right)((p+1)u) \sinh^{2N+2} \left( \frac{1}{2} p \right)((p+1)u)}{2p(p+1+2N)}
\times \left\{ n(u; \xi_-, \kappa_-) n(u; -\xi_+, \kappa_+) + n(u; -\xi_-, \kappa_+) n(u; \xi_+, \kappa_+)
+ 2(-1)^N (-\kappa_- \kappa_+)^{p+1} \sinh^2(2(p+1)u) \right\},
\]

where the function \( n(u; \xi, \kappa) \) is defined in Eq. (4.24). As already mentioned, we have computed the coefficients \( c_p, l \) which appear in this function only up to \( p = 5 \).

We remark that, in obtaining the result (5.1), we have made use of the relation
\[
f_1(u) = \alpha(u) \prod_{l=1}^{p-2} \prod_{k=1}^{l} \zeta(2u + (l + k + 2)\eta)
\]
which is satisfied by the function \( f_1(u) \) defined in (5.5).

6 Eigenvalues and Bethe Ansatz equations

The functional relations which we have obtained can be used to determine the eigenvalues of the transfer matrix. The commutativity relation (2.13) implies that the transfer matrix has eigenstates \(|\Lambda\rangle\) which are independent of \(u\),
\[
t(u)|\Lambda\rangle = \Lambda(u)|\Lambda\rangle,
\]
where \(\Lambda(u)\) are the corresponding eigenvalues. Acting on \(|\Lambda\rangle\) with the functional relation (5.1), we obtain the corresponding relation for the eigenvalues
\[
f_0(u)\Lambda(u)\Lambda(u+\eta)\ldots\Lambda(u+p\eta) - f_1(u)\Lambda(u+\eta)\Lambda(u+2\eta)\ldots\Lambda(u+(p-1)\eta)
+ \ldots = f_3(u).
\]

Similarly, it follows from (2.16) and (2.17) that the eigenvalues have the periodicity and crossing properties
\[
\Lambda(u + i\pi) = \Lambda(u), \quad \Lambda(-u - \eta) = \Lambda(u).
\]

Finally, (2.18) implies the asymptotic behavior (for \(\kappa_\pm \neq 0\))
\[
\Lambda(u) \sim -\kappa_- \kappa_+ \frac{e^{u(2N+4)+\eta(N+2)}}{2^{2N+1}} + \ldots \text{ for } u \to \infty.
\]

We shall assume that the eigenvalues have the form
\[
\Lambda(u) = \rho \prod_{j=-1}^{N} \sinh(u - u_j) \sinh(u + \eta + u_j),
\]
where $u_j$ and $\rho$ are ($u$-independent) parameters which are to be determined. Indeed, this expression satisfies the periodicity and crossing properties (6.3), and it has the correct asymptotic behavior (6.4) provided that we set

$$\rho = -8\kappa_- \kappa_+.$$  \hfill (6.6)

Evaluating the functional relation (6.2) at the root $u = u_j$, we obtain a set of Bethe-Ansatz-like equations

\begin{align*}
-f_1(u_j)\Lambda(u_j + \eta)\Lambda(u_j + 2\eta) \cdots \Lambda(u_j + (p-1)\eta) \\
+ \frac{f_1(u_j)f_1(u_j + 2\eta)}{f_0(u_j)}\Lambda(u_j + 3\eta)\Lambda(u_j + 4\eta) \cdots \Lambda(u_j + (p-1)\eta) \\
+ \frac{f_1(u_j)f_1(u_j + 3\eta)}{f_0(u_j)}\Lambda(u_j + \eta)\Lambda(u_j + 4\eta)\Lambda(u_j + 5\eta) \cdots \Lambda(u_j + (p-1)\eta) + \cdots \\
+ \frac{f_1(u_j)f_1(u_j + (p-1)\eta)}{f_0(u_j)}\Lambda(u_j + \eta)\Lambda(u_j + 2\eta) \cdots \Lambda(u_j + (p-3)\eta) + \cdots \\
+ [u_j \rightarrow u_j + \eta] = f_3(u_j), \quad j = -1, 0, \ldots, N, \tag{6.7}
\end{align*}

where $\Lambda(u)$ is given by (6.5). In particular, the first three cases are given by

\begin{align*}
p = 1 &: \quad f_1(u_j) + f_1(u_j + \eta) + f_3(u_j) = 0, \\
p = 2 &: \quad f_1(u_j)\Lambda(u_j + \eta) + f_1(u_j + \eta)\Lambda(u_j + 2\eta) + f_3(u_j) = 0, \\
p = 3 &: \quad f_1(u_j)\Lambda(u_j + \eta)\Lambda(u_j + 2\eta) + f_1(u_j + \eta)\Lambda(u_j + 2\eta)\Lambda(u_j + 3\eta) \\
&\quad + f_3(u_j) = 0. \tag{6.8}
\end{align*}

The simplest case $p = 1$ has recently been analyzed in \cite{9}.

The Bethe Ansatz equations may be written in a more explicit form by substituting the Ansatz (6.5) into (6.7), and then using the identity (4.11) to simplify the products. In this way, we obtain

\begin{equation*}
(-1)^{N+3} \left( \frac{i}{2} \right)^{2p(N+2)} \rho^{p-1} \times \left\{ f_1(u_j) \prod_{k=-1}^{N} \frac{\sinh((p+1)(u_j - u_k))}{\sinh(u_j - u_k)\sinh(u_j - u_k - \eta)\sinh(u_j + u_k + \eta)\sinh(u_j + u_k)} \right\}
\end{equation*}

\footnote{The functions $g_1(u)$ and $g_3(u)$ in \cite{1} are related to $f_1(u)$ and $f_3(u)$ as follows: \[ f_1(u) = -\sinh^2 2u \cosh^4 u \ g_1(u), \quad f_3(u) = -\sinh^2 2u \sinh^2 u \cosh^2 u \ g_3(u). \]}

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\[ + f_1(u_j + \eta) \prod_{k=1}^{N} \frac{\sinh((p+1)(u_j - u_k + \eta)) \sinh((p+1)(u_j + u_k + \eta))}{\sinh(u_j - u_k + \eta) \sinh(u_j - u_k) \sinh(u_j + u_k + 2\eta) \sinh(u_j + u_k + \eta)} \}
\]
\[ + \ldots = f_3(u_j), \quad j = -1, 0, \ldots, N. \quad (6.9) \]

The poles which occur when \( j = k \) are harmless, as they are canceled by corresponding zeros.

We conclude this section with a brief discussion of the diagonal case, \( \kappa_{\pm} = 0 \). In this case, the transfer matrix commutes with the matrix \( M \),

\[ [t(u), M] = 0, \quad (6.10) \]

where

\[ M = \frac{N}{2} - S^z, \quad S^z = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^z. \quad (6.11) \]

The two matrices can therefore be simultaneously diagonalized,

\[ t(u)|\Lambda^{(m)}\rangle = \Lambda^{(m)}(u)|\Lambda^{(m)}\rangle, \]
\[ M|\Lambda^{(m)}\rangle = m|\Lambda^{(m)}\rangle. \quad (6.12) \]

The asymptotic behavior of the eigenvalues is now given by

\[ \Lambda^{(m)}(u) \sim \frac{i}{2^{2N+2}}e^{u(2N+2)} \left( e^{2(N-m)\eta+\xi_- - \xi_+} + e^{2m\eta - \xi_- + \xi_+} \right) + \ldots \quad \text{for} \quad u \to \infty. \quad (6.13) \]

The eigenvalues are therefore given by

\[ \Lambda^{(m)}(u) = \rho^{(m)} \prod_{j=0}^{N} \sinh(u - u_j) \sinh(u + \eta + u_j), \quad (6.14) \]

where

\[ \rho^{(m)} = 2ie^{-\eta} \cosh(\xi_- - \xi_+ + (N - 2m)\eta), \quad (6.15) \]

and the roots \( u_j \) satisfy essentially the same Bethe Ansatz equations (6.7).

### 7 Discussion

We have demonstrated an approach for solving the open XXZ quantum spin chain with nondiagonal boundary terms for \( \eta = \frac{i\pi}{p+1} \). In particular, we have proposed exact \((p+1)\)-order functional relations (5.1) for the transfer matrix, and Bethe Ansatz equations (6.7).
(3.9) for the corresponding eigenvalues. We emphasize that the function $f_3(u)$ appearing in these equations involves the coefficients $c_{p,l}$ which we have computed only up to $p = 5$. (See Eqs. (5.6), (4.24), (4.25).) The complete determination of these coefficients must presumably await a more explicit construction of nondiagonal $K$ matrices for arbitrary spin.

A similar approach can certainly also be applied to the elliptic case (i.e., the open XYZ chain with nondiagonal boundary terms), since both a fusion hierarchy and a truncation identity can also be obtained for this case. Moreover, we expect that a similar approach can be applied to spin chains constructed with $R$ and $K$ matrices associated with any affine Lie algebra.

An important feature of our Bethe Ansatz equations is that, as in the case of the closed (periodic boundary conditions) XYZ chain, there is a fixed number of roots for all the eigenvalues. Because of this fact, it is not easy to compare our results in the diagonal limit ($\kappa_{\pm} \to 0$) with those of [4, 1]. A similar difficulty arises [25] when comparing the Bethe Ansatz equations of the closed XYZ chain in the trigonometric limit with the “conventional” Bethe Ansatz equations of the closed XXZ chain.

It should be interesting to use our results to determine, for low values of $p$, physical properties (thermodynamics, etc.) of the spin chains and of the associated quantum field theories. It would also be interesting to try to further simplify our system of Bethe Ansatz equations, and perhaps to generalize to the case of generic values of the bulk anisotropy.

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A Similarity transformations

We discuss in Section 4 certain similarity transformations on the fused $R$ and $K$ matrices. Here we give explicit expressions for the matrices $A, B, C$ corresponding to these similarity transformations for the first three cases, namely $j = 1, \frac{3}{2}, 2$. 
For $j = 1$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}, \quad (A.1)$$

$$B = \text{diag}(a, 1, a, 1), \quad C = \text{diag}(a, 1, 1, 1), \quad a = ([2]_q)^{-\frac{1}{2}}. \quad (A.2)$$

For $j = \frac{3}{2}$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}, \quad (A.3)$$

$$B = \text{diag}(a, 1, 1, a, 1, 1, 1, 1), \quad C = \text{diag}(a, \ldots, 1), \quad a = ([3]_q)^{-\frac{1}{2}}. \quad (A.4)$$

For $j = 2$,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & -\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A.5)$$
\[ B = \text{diag}(a_1, b_1, a_1, \ldots, 1), \quad C = \text{diag}(a_1, b_1, \ldots, 1), \]
\[ a = \left(\frac{[4]_q}{[2]_q}\right)^{-\frac{1}{2}}, \quad b = \left(\frac{[2]_q}{[3]_q}\right)^{-\frac{1}{2}}. \tag{A.6} \]

**Note added:**

Boundary quantum group symmetry \cite{26, 27} can be used to determine \cite{28} $K$ matrices for arbitrary spin. In particular, the coefficients $c_{p,l}$ appearing in Eq. (4.24) are given by

\[ c_{p,l} = \frac{(p+1)}{l!} \prod_{k=0}^{l-2} (p-l-k). \]

Thus, for $p \to \infty$, $c_{p,l} \sim \frac{p^l}{l!}$.

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