1. Introduction

An anti self dual (ASD) connection minimizes the classical Yang-Mills functional, whose minimal value is expressed in terms of the characteristic classes and hence is a topological invariant, if an ASD connection exists. In non commutative geometry, notion of ASD does not make sense, since space does not appear in general. Connes reformulated the Yang-Mills functional over a compact four manifold by using the Dixmier trace, which reinterprets the functional from a non commutative setting. Connes-Yang-Mills action functional $\text{CYM}(A)$ is given by taking infimum value of the pre Connes-Yang-Mills action functional $\text{Tr}_\omega(\nabla^2)$ in the equivalent class of non commutative connections $\nabla$ associated to $A$, where $\text{Tr}_\omega$ is the Dixmier trace. It is not easy to formulate a non commutative Yang-Mills action functional without space, even though the pre Connes-Yang-Mills action functional exists. Actually $\text{CYM}(A)$ is defined by using the associated connections over the underlying four manifold, and its formulation is not straightforward from non commutative viewpoint.

Let $A$ be a $\ast$-algebra, and $E$ be a finitely generated projective $\mathcal{A}$ module. Our basic question is what kind of non commutative connections over $E$ would possess ‘nice’ properties. Our idea is to use the Nahm transform between connections, which transforms an ASD connection to another ASD connection in the case of the 4-dimensional flat torus. Classical Nahm transform provides a kind of dual connection over the Picard torus of covering group $\Gamma = \mathbb{Z}^4$ of the manifold $X$ (for example, $\Gamma = \pi_1(X)$ the fundamental group), and hence it concerns commutative property of the group. In this paper we generalize it in a non commutative way, by constructing a higher version of the Connes-Yang-Mills action functional by using the Dixmier $\Gamma$-trace. As an output, we obtain a finitely generated projective $C^*_r(\Gamma)$ module with a non commutative connection on it.

Corresponding to the output in section 3, we quantize the group $C^*$-algebra using spectral triple and we need its smooth subalgebra to introduce differential on the algebra. On the other hand we do not need to quantize the algebra on the input side in section 5. We actually need a Hilbert module with a connection on it, and does not require smooth subalgebra, since four manifold as a space is assigned in the side.
Let $\mathcal{E}$ be a Hilbert $C^*_r(\Gamma)$ module, where $\Gamma$ is a discrete group. We introduce a Dixmier $\Gamma$-trace:

$$\text{Tr}^\Gamma_{\omega} : \mathcal{L}^{(1,\infty)}(\mathcal{E}) \to \mathbb{C}$$

and use it to formulate a $\Gamma$-Connes-Yang-Mills action functional $CYM^\Gamma$. Our motivation to construct it arose from a question how to formulate a higher Nahm transform. Let us explain it below in detail.

Let $(X,g)$ be a Riemannian 4-manifold, and $E \to X$ be a unitary bundle with a unitary connection $A$. Consider $\tilde{X}$, a $\Gamma$-principal bundle of $X$ ($\tilde{X}/\Gamma = X$). A typical example is when $\tilde{X}$ is the universal cover of $X$ with fundamental group $\Gamma$.

Let $C^\infty(\Gamma)$ be a dense holomorphic closed subalgebra of $C^*_r(\Gamma)$ containing $C^\Gamma$. Then consider the set of smooth sections $\mathcal{E}_0 = C^\infty(E \otimes \tilde{X} \times_\Gamma C^\infty(\Gamma))$ which is a right $C^\infty(\Gamma)$ module. The connection $A$ naturally extends to a connection $\hat{A}$ on $\mathcal{E}_0$. In our case, the Hodge star $\ast$ also acts on $\Omega^2(\mathcal{E}_0)$, and hence the notion of anti-self duality on a connection makes sense. We induce an equivalence between a minimizer for $\hat{A}$ of the $\Gamma$-Connes-Yang-Mills action functional and the ASD condition. This follows from the next coincidence combined with theorem 14 in [7]:

**Theorem 1.1.** Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Let $\Gamma = \mathbb{Z}^4$ be a covering group of $X$. Then the Connes-Yang-Mills functional and its higher analogue coincide:

$$CYM(A) = CYM^\Gamma(\hat{A}).$$

Nahm transform is roughly described as below. Suppose $X$ is spin and $D_A$ is the Dirac operator on $S \otimes E := SE$, where $S$ is the spinor bundle on $X$. $\text{ind } D_A$, the index of the twisted Dirac operator $D_A$ over $SE \otimes \tilde{X} \times_\Gamma C^*_r(\Gamma)$ consists of a formal difference of finitely generated projective $C^*_r(\Gamma)$ modules, and is called the higher index in noncommutative geometry.

For $\Gamma = \mathbb{Z}^m$, the twisted Dirac operator $D_A$ is regarded as a family of Dirac operators over the Picard torus as in the classical case. Suppose $\ker D_A = 0$ at each parameter value, and hence $\text{ind } D_A = -[\text{coker } D_A]$ holds. In the case $\text{coker } D_A \subset \mathcal{H} := L^2(SE \otimes \tilde{X} \times_\Gamma C^*_r(\Gamma))$ forms a vector bundle over the Picard torus. Take the connection $\hat{d}_A$ given by the trivial connection composed with the orthogonal projection onto $\text{coker } D_A$. The assignment:

$$\prod_{\rho \in \text{Pic}} (E \otimes L_\rho, A) \to (\text{coker } D_A, \hat{d}_A)$$

is called the Nahm transform. So it assigns the higher index with the induced connection. The following is known:

**Lemma 1.2.** Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Then the Nahm transform assigns an ASD connection to another ASD connection.

The higher index exists for a general discrete group, where we will have no reasonable parameter space, while $C^*$-algebra module exists. In the absence of the underlying space, we have to use a noncommutative connection to formulate higher Nahm transform. With a spectral triple, there is a connection on a Hilbert $C^*_r(\Gamma)$ module, which is induced from the trivial one.
By using the orthogonal projection onto coker $D_A$, we obtain a non commutative connection $\hat{d}_A$ as the induced connection on coker $D_A$. So we have given a higher Nahm transform from a bundle with a connection $(E, A)$ to a finitely generated projective $C^r_\ast(\Gamma)$ module coker $D_A$ with a non commutative connection $\hat{d}_A$:

$$(E, A) \to (E \otimes \tilde{X} \times_\Gamma C^r_\ast(\Gamma), A) \to (\text{coker } D_A, \hat{d}_A).$$

Lemma 1.2 can be reinterpreted as below in terms our formulation:

**Corollary 1.3.** Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Let $\Gamma = \mathbb{Z}^4$ be a covering group of $X$.

The higher Nahm transform sends the minimizer of the higher Connes-Yang-Mills functional to the minimiser of the Connes-Yang-Mills functional.

There arise two questions:

**Question:**

1. Let $X$ be a compact four manifold with $\pi_1(X) = \Gamma$, and $E \to X$ be a unitary bundle. Consider a number:

$$\tau_{T^\Gamma}(X) := \inf_{A: \text{ASD}} \text{Tr}_\omega(\nabla^2_{\hat{d}_A}) \in \mathbb{R}.$$  

When is $\tau_{T^\Gamma}(X)$ a topological number?

2. Compare it with another number:

$$\tau_{T^\Gamma}'(X) := \inf_{A} \text{Tr}_\omega(\nabla^2_{\hat{d}_A}) \in \mathbb{R}.$$  

A pri-ori the inequality holds:

$$\tau_{T^\Gamma}'(X) \leq \tau_{T^\Gamma}(X)$$

if an ASD connection exists. When the equality holds?

In the case of the 4-dimensional flat torus, we know both the answers affirmatively.

## 2. Nahm Transform

In this section we quickly review some basic things on the Nahm transform.

### 2.1. Complex surface

Let $X$ be a 4-manifold with Riemannian metric $g$. Suppose that $X$ has a complex structure $J$ compatible with $g$ and define a $(1, 1)$-form $\omega$ by

$$\omega(a, b) = g(a, Jb).$$

We write $\Lambda^i_X, \Lambda^{i,j}_X$ for $\Lambda^i T^*X, \Lambda^{i,j}(T^*X \otimes \mathbb{C})$. Then we have the decompositions

$$\Lambda^2_X \otimes \mathbb{C} = \Lambda^{2,0}_X \oplus \Lambda^{1,1}_X \oplus \Lambda^{1,0}_X,$$

and

$$\Lambda^{1,1}_X = (\Lambda^{0,0}_X \otimes \mathbb{C}) \cdot \omega \oplus \Lambda^{1,1}_{X,0}.$$
Here $\Lambda^{1,1}_{X,0}$ is the orthogonal complement of $\omega$ in $\Lambda^{1,1}_X$. We also have

$$
\Lambda^+_X \otimes \mathbb{C} = \Lambda^{2,0}_X \oplus \Lambda^{0,0}_X \cdot \omega \oplus \Lambda^{0,2}_X,
$$
(2.1)

Here $\Lambda^+_X$ is the self-dual part of $\Lambda^{2,0}_X$ and $\Lambda^-_X$ is the anti-self dual of $\Lambda^{2,0}_X$.

**Proposition 2.1.** Let $A$ be a unitary connection on a Hermitian complex vector bundle $E$ on $X$. Then $A$ is ASD if and only if $F_A$ is $(1,1)$-type and $(F_A, \omega) = 0$ at each point.

Note that $F_A$ is $(1,1)$-type if and only if $\bar{\partial}_A$ defines a holomorphic structure on $E$. So we have the following:

**Proposition 2.2.** Let $A$ be a unitary connection on a Hermitian complex vector bundle $E$ and suppose that $\bar{\partial}_A$ defines a holomorphic structure on $E$. Then $A$ is ASD if and only if $(F_A, \omega) = 0$ at each point of $X$.

We will consider the ASD equation on $\mathbb{R}^4$ with the standard Riemannian metric. The complex structures $J$ on $\mathbb{R}^4$ compatible with the Riemannian structure are parametrized by $SO(4)/U(2)(\cong S^2)$. It is easy to see that

$$
\Lambda^- \otimes \mathbb{C} = \bigcap_{J \in S^2} \Lambda_{J,1}^{1,1}.
$$

Hence we have

**Proposition 2.3.** Let $E$ be a Hermitian complex vector bundle on $\mathbb{R}^4$. A unitary connection $A$ on $E$ is ASD if and only if $F_A$ is $(1,1)$ type for all complex structures $J$ compatible with the Riemannian metric.

### 2.2. Nahm transform.

Let $X$ be a closed, Riemannian 4-manifold. Take a spin structure $s$ on $X$. Then we have the spinor bundles $S^\pm$ on $X$. Choose a complex vector bundle $E$ with a Hermitian metric. For each unitary connection $A$ on $E$ we have the twisted Dirac operator $D_A : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$.

Note that if

$$
\ker(D_A : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)) = 0,
$$
(2.2)

then $D_A^*$ is surjective.

Suppose that we have a continuous family $\{A_y\}_{y \in Y}$ of unitary connections on $E$. We assume that $A_y$ satisfies (2.2) for all $y \in Y$. Since $D^*_{A_y}$ is surjective, the dimension of $\ker D^*_{A_y}$ is constant and

$$
\tilde{E} := \prod_{y \in Y} \ker D^*_{A_y}
$$

defines a subbundle of the trivial Hilbert bundle

$$
\tilde{H} := Y \times L^2(S^- \otimes E)
$$

over $Y$ with fiber $H = L^2(S^- \otimes E)$. We have the covariant derivative corresponding to the trivial connection:

$$
\nabla : \Gamma(\tilde{H}) \to \Gamma(\tilde{H} \otimes T^*X).
$$
Let \( i : \hat{E} \hookrightarrow H \) be the inclusion and \( p : H \to \hat{E} \) be the \( L^2 \)-projection. Then we get a connection \( \hat{A} \) on \( \hat{E} \) with covariant derivative

\[
\nabla_{\hat{A}} = p \nabla i.
\]

We call \( \hat{A} \) the Nahm transform of \( \{A_y\}_{y \in Y} \).

2.3. Nahm transform of ASD connections on a 4-manifold with \( b_1 = 4 \). We will follow the discussion in Section 3.2 of [9].

Let \( X \) be a closed, oriented, spin, smooth 4-manifold with \( b_1 = 4 \). We denote by \( \hat{X} \) the Picard torus:

\[
\hat{X} : = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) \cong T^4.
\]

We can think of \( \hat{X} \) as the moduli space of \( U(1) \)-flat connections on \( X \).

Let \( A \) be a connection on a Hermitian vector bundle \( E \) over \( X \). We have the family \( \{A_\rho\}_{\rho \in \hat{X}} \) of connections parametrized by \( \hat{X} \). Here \( A_\rho = A \otimes \rho \).

Let \( S^\pm \) be the spinor bundles over \( X \). Then we have the twisted Dirac operators

\[
D_{A_\rho} : \Gamma(E \otimes L_\rho \otimes S^+ ) \to \Gamma(E \otimes L_\rho \otimes S^- ),
\]

\[
D^*_{A_\rho} : \Gamma(E \otimes L_\rho \otimes S^- ) \to \Gamma(E \otimes L_\rho \otimes S^+ ).
\]

Here \( L_\rho \) is the flat line bundle corresponding to \( \rho \).

From now on, we assume that \( A \) is ASD (and hence \( A_\rho \) is also ASD) and that the following condition holds:

\[
\ker D^*_{A_\rho} = 0 \quad (\forall \rho \in \hat{X}).
\]

Applying the Nahm transform in Section 2.2 to \( \{A_\rho\}_{\rho \in \hat{X}} \), we obtain a connection \( \hat{A} \) on the bundle \( \hat{E} \to \hat{T} \). More precisely, the bundle

\[
\coprod_{\rho \in \hat{X}} L^2(S^- \otimes E \otimes L_\rho ) \to \hat{X}
\]

does not have a natural trivialization. This means that we can not apply the Nahm transform to \( \{A_\rho\}_{\rho \in \hat{X}} \) directly. To avoid this issue, we consider the family of ASD connections \( \{A_{\tilde{\rho}}\}_{\tilde{\rho} \in H^1(X; \mathbb{R})} \) parameterized by the universal cover \( H^1(X; \mathbb{R}) \) of \( \hat{X} \). Then the flat line bundle \( L_{\tilde{\rho}} \) can be considered to be the pair \((\mathbb{C}, \tilde{\rho})\) of the trivial bundle complex line bundle \( \mathbb{C} \) over \( X \) and the flat connection \( \tilde{\rho} \). Since \( L_{\tilde{\rho}} \) is trivial as a topological complex line bundle, we can think that the operators \( D_{\tilde{\rho}} \) act on the the same space \( \Gamma(S^- \otimes E) \).

Therefore we can apply the Nahm transform to this family and we get a connection \( \tilde{A} \) on

\[
\tilde{E} = \coprod_{\tilde{\rho} \in H^1(X; \mathbb{R})} \ker D^*_{A_{\tilde{\rho}}} \to H^1(X; \mathbb{R}).
\]

Recall that the corresponding covariant derivative \( \nabla_{\tilde{A}} \) is given by the formula:

\[
\nabla_{\tilde{A}} = p \nabla i.
\]
Here $i$ is the family of inclusions $i_{\tilde{\rho}} : \ker D_{A_{\tilde{\rho}}} \hookrightarrow L^2(S^- \otimes E)$, $p$ is the family of projections $p_{\tilde{\rho}} : L^2(S^- \otimes E) \to \ker D_{A_{\tilde{\rho}}}$ and $\nabla$ is the trivial connection. If $\tilde{\rho} \in H^1(X; \mathbb{Z})$, we have\[ D^*_{A_{\tilde{\rho}}} = u^{-1}D^*_{A_0}u, \]where $u : X \to U(1)$ is the harmonic gauge transform corresponding to $\tilde{\rho}$. Hence:

(2.5) \[ i_{\tilde{\rho}} = i_{0u}, \quad p_{\tilde{\rho}} = u^{-1}p_0. \]

By (2.4) and (2.5), we can see that the connection $\tilde{A}$ naturally descends to a connection $\hat{A}$ on the bundle:

$$\hat{E} = \coprod_{\rho \in \hat{X}} \ker D^*_\rho$$

because

$$\hat{E} = \coprod_{\tilde{\rho} \in H^1(X; \mathbb{R})} \ker D^*_\tilde{\rho} / H^1(X; \mathbb{Z}),$$

where the action of $H^1(X; \mathbb{Z})$ is the adjoint of the harmonic gauge transforms $u : X \to U(1)$.

**Theorem 2.4.** The connection $\hat{A}$ on $\hat{E}$ is ASD.

**Proof.** We will show outline of the proof.

Put $V := H^1(X; \mathbb{R})(\cong \mathbb{R}^4)$ and let $\pi : V \to \hat{X}$ be the projection. Then $\hat{A}$ is ASD if and only if $\pi^*\hat{A} (= \hat{A})$ is ASD. The Riemannian metric on $X$ naturally induces a metric on $V$. Take any complex structure $\tilde{J}$ on $V$ compatible with the metric. As explained in Section 3.1.3 of [9], $\hat{E}$ has a natural holomorphic structure, and an infinite dimensional version of Lemma (3.1.20) of [9] shows that the connection $\hat{A}$ is compatible with the holomorphic structure. Therefore

$$F_{\hat{A}} \in \Lambda^{1,1}_j \otimes \mathfrak{g}_E$$

at each point. By Proposition 2.3, $\hat{A}$ (and hence $\hat{A}$) is ASD.

q.e.d.

### 3. Higher Nahm Transform

Let $(X, g)$ be a compact Riemannian four manifold, and $E \to X$ be a unitary bundle equipped with a connection $A$. Denote $\tilde{X}$ a $\Gamma$-cover of $X$ with covering group $\Gamma$. For example, $\Gamma = \pi_1(X)$ is the fundamental group and $\tilde{X}$ is the universal cover. Consider the associated $C^*_r(\Gamma)$ bundle:

$$\gamma := \tilde{X} \times_\Gamma C^*_r(\Gamma).$$

Assume that $X$ is spin, and let $S$ be the spinor bundle. Denote $S \otimes E := S_E$. Let us introduce an $L^2$-inner product on the smooth sections of $S_E \otimes \gamma$:

$$(f, g) := \int_X <f(x), g(x)> \ vol \in C^*_r(\Gamma)$$

where $< \ >$ is the Hilbert module inner product, and denote its completion by $L^2(S_E \otimes \gamma)$. 
A connection $A$ on $E$ induces the twisted connection on $E \otimes \gamma$ by:
$$\nabla_A(f \otimes \sigma) = \nabla_A(f) \otimes \sigma + f \otimes d\sigma.$$ 
Hence $A$ induces a Dirac operator $D_A$ on $S_E \otimes \gamma$:
$$D_A^\pm: L^2(X; S_E^\pm \otimes \gamma) \to L^2(X; S_E^\mp \otimes \gamma).$$
Both ker $D_A$ and coker $D_A$ consists of finitely generated projective $C_r^*(\Gamma)$ modules after compact perturbation. The higher index is defined as their formal difference:
$$\text{ind} \ D_A := [\ker D_A] - [\coker D_A] \in K_0(C_r^*(\Gamma)).$$
Suppose ker $D_A^+$ = 0, and consider the finitely generated projective $C_r^*(\Gamma)$ module:
$$E_A := \coker D_A^- = \ker D_A^+ \subset L^2(X; S_E^- \otimes \gamma).$$

**Lemma 3.1.** Suppose ker $D_A^+$ = 0. Then there is the $C_r^*(\Gamma)$ module projection:
$$P: L^2(X; S_E^- \otimes \gamma) \to E_A$$

**Proof.** Notice that ker $(D_A^-)^* = \ker D_A^+ = 0$ by the assumption. Then we consider the bounded operator:
$$P := \text{id} - (D_A^-)^*(D_A^-(D_A^-)^*)^{-1}D_A^-$$
on $L^2(X; S_E^- \otimes \gamma)$. It is easy to check that this satisfies the required properties. q.e.d.

**Remark 3.2.** Notice that in general the projection does not exists for a Hilbert $C_r^*(\Gamma)$ module embedding $\mathcal{E} \hookrightarrow \mathcal{H}$.

Actually $\mathcal{E} \oplus \mathcal{E}^\perp$ do not coincide with $\mathcal{H}$ in general.

### 3.1. Quatized calculus.

We recall a notion of connection on a finitely generated projective module associated to a spectral triple, in non commutative geometry.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, where $\mathcal{A}$ is a unital $*$-algebra represented in a Hilbert space $\mathcal{H}$ as $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$. $D$ is an unbounded operator on $\mathcal{H}$ such that $[D, a] \in \mathcal{L}(\mathcal{H})$ is bounded for any $a \in \mathcal{A}$, and $(1 + D^2)^{-1} \in K(\mathcal{H})$ is compact. Suppose Ker $D$ = 0 and define:
$$\hat{d}(a) := i[F, a] \in \mathcal{L}(H)$$
where $F = \frac{D}{\|D\|}$. Let us consider the linear space:
$$\hat{\Omega}_F := \text{span} \{ a^0 \hat{a} a^1 \ldots \hat{a} a^q \}$$
spanned by vectors of the form $a^0 \hat{a} a^1 \ldots \hat{a} a^q$ for $q \geq 0$, where $a^i \in \mathcal{A}$. It admits a structure of graded algebra $\hat{\Omega}_F = \oplus^q \hat{\Omega}_F^q$. We call an element in this space as a non commutative differential form.

**Lemma 3.3.** It satisfies the following two properties:
\begin{enumerate}
\item $\hat{d}^2 = 0$,
\item $\hat{d}(a_1 a_2) = (\hat{d} a_1) a_2 + a_1 \hat{d} a_2$.
\end{enumerate}
Let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$. A
connection $A$ on $\mathcal{E}$ is a linear map with a derivation property:
\[
\nabla_A : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \hat{\Omega}^1_F, \\
\nabla_A(ea) = (\nabla Ae)a + e\hat{d}a \\
e \in \mathcal{E}, a \in \mathcal{A}.
\]

**Remark 3.4.** If $\mathcal{E} = \Gamma(S)$ denotes the space of smooth sections of the spinor bundle $S$ over a closed spin manifold $X$, then $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A} = \mathcal{C}^\infty(X)$. In this case a canonical choice of spectral triple is the Dirac spectral triple $(\mathcal{C}^\infty(X), L^2(X, S), D)$ where $D$ is the Dirac operator on the spinor bundle. It is known that for $a \in \mathcal{C}^\infty(X)$, the equality holds:
\[
\left[ D, a \right] = c(da)
\]
and hence, one recovers the classical 1-form $da \in \mathcal{C}^\infty(X, T^*X)$ via the Clifford multiplication:
\[
c : TX \to \text{End } S.
\]

Assume that $\mathcal{C}^*_r(\Gamma)$ admits a dense subalgebra $\mathcal{C}^\infty(\Gamma)$ containing $\mathcal{C}\Gamma$ closed under holomorphic functional calculus. Let $\mathcal{A} = \mathcal{C}^\infty(\Gamma)$, and $H \otimes \mathcal{C}^\infty(\Gamma)$ be the trivial $\mathcal{C}^\infty(\Gamma)$ module equiped with the trivial connection $\hat{d}$ on $H$ given by:
\[
\hat{d}(h \otimes u) := h \otimes \hat{d}(u) \in H \otimes \hat{\Omega}^1 = (H \otimes \mathcal{C}^\infty(\Gamma)) \otimes_{\mathcal{C}^\infty(\Gamma)} \hat{\Omega}^1_F.
\]

Here $H \otimes \mathcal{C}^\infty(\Gamma)$ is viewed as a dense subspace of its $\mathcal{C}^*$-module completion $H \otimes \mathcal{C}^*_r(\Gamma)$.

It is well known as Kasparov’s stabilization that there exists a Hilbert $\mathcal{C}^*_r(\Gamma)$ module isomorphism:
\[
L^2(X; S_E \otimes \gamma) \oplus (H \otimes \mathcal{C}^*_r(\Gamma)) \cong H \otimes \mathcal{C}^*_r(\Gamma).
\]

Let us fix this isomorphism, and let:
\[
P' : H \otimes \mathcal{C}^*_r(\Gamma) \to L^2(X; S_E \otimes \gamma)
\]
be the projection.

Recall lemma 3.1. Composed with another projection $P = P_A : L^2(X; S_E \otimes \gamma) \to \mathcal{E}_A$, one obtains the projection:
\[
Q_A := P_A \circ P' : H \otimes \mathcal{C}^*_r(\Gamma) \to \mathcal{E}_A.
\]

Denote by $\mathcal{E}_A^\infty$ the dense subspace of $\mathcal{E}_A$ obtained by intersecting $\mathcal{E}_A$ with smooth sections $\mathcal{C}^\infty(X, S_E \otimes \Gamma \mathcal{C}^\infty(\Gamma))$ in $L^2(X, S_E \otimes \gamma)$. The trivial connection on $H \otimes \mathcal{C}^\infty(\Gamma)$ induces the connection on $\mathcal{E}_A^\infty$ by composition:
\[
\hat{d}_A := Q_A \circ \hat{d} : \mathcal{E}_A^\infty \to \mathcal{E}_A^\infty \otimes_\mathcal{A} \hat{\Omega}^1_F.
\]

**Definition 3.5.** A higher Nahm transform is given by:
\[(E, A) \to (\mathcal{E}_A^\infty, \hat{d}_A).
\]

Notice that the right hand side is a non commutative object and does not involve space.
3.2. Dixmier trace and Connes-Yang-Mills functional. In [7], Connes made use of Dixmier trace to study the Yang-Mills functional. We recall the definitions and constructions.

For every positive element $A \in \mathcal{K}(\mathcal{H})$, put:

$$\mu_n = \inf_{T \in R_n} \| A - T \|$$

where $R_n = \{ Q \in \mathcal{L}(\mathcal{H}) : \text{rank}(Q) \leq n \}$, and

$$\delta_N(A) = \sup \{ \text{Tr}(AP) ; \text{rk}(P) \leq N \}$$

where the supremum is over all projections $P$ whose ranks are no more than $N$. Equivalently, if $\lambda_i \geq 0$ is a decreasing sequence of eigenvalues of $A$, then

$$\delta_N(A) = \sum_{i=1}^{N} \lambda_i \quad \mu_n(A) = \lambda_n.$$  

For $T \in \mathcal{K}(\mathcal{H})$, let us define a norm:

$$\| T \|_{(1,\infty)} := \sup_N \frac{1}{\log(1 + N)} \sum_{n=1}^{N} \mu_n(T).$$

This gives rise to the Banach ideal of $\mathcal{K}(\mathcal{H})$:

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{ T \in \mathcal{K}(\mathcal{H}) : \| T \|_{(1,\infty)} < \infty \}.$$  

Denote its positive cone by:

$$\mathcal{L}^{(1,\infty)}_+(\mathcal{H}) = \{ T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) : T \geq 0 \}.$$  

This is the domain of the Dixmier trace.

For every linear form $\omega$ on $l^{\infty}$, denote

$$\omega - \lim(\{ a_n \}) := \omega(\{ a_n \}).$$

Let $\omega$ be a state (positive linear functional with norm 1) on $l^{\infty}$ satisfying the following properties:

- $\omega - \lim(\{ a_n \}) \geq 0$ if $a_n \geq 0$;
- $\omega - \lim(\{ a_n \}) = \lim a_n$ if $\{ a_n \}$ is convergent;
- $\omega$ is invariant under dilation $D_2$, i.e.,

$$(3.1) \quad \omega - \lim(\{ a_n \}) = \omega - \lim(a_1, a_1, a_2, a_2, \ldots) \quad \forall \{ a_n \} \in l^{\infty}.$$  

**Definition 3.6.** The Dixmier trace $\text{Tr}_\omega : \mathcal{L}^{(1,\infty)}_+(\mathcal{H}) \to [0,\infty)$ is defined by:

$$\text{Tr}_\omega(A) = \omega - \lim_{N \to \infty} \frac{1}{\log(N + 1)} \delta_N(A)$$

$$= \omega - \lim_{N \to \infty} \frac{1}{\log(N + 1)} \sum_{n=1}^{N} \mu_n(A).$$

We extend it to $\text{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H}) \to \mathbb{C}.$

Consider a complex vector bundle $E$ on a closed manifold $X$ of dimension $n$ and the Hilbert space $\mathcal{H} = L^2(X, E)$.
For $m \in \mathbb{Z}$, denote by $\Psi^m(X, E)$ the space of pseudo differential operators over $X$ of order $m$. For $P \in \Psi^{-n}(X, E)$, denote by $\sigma_P \in C^\infty(S^*X, \text{End} E)$ its principal symbol. Define the Wodzicki residue of $P$ by:

$$\text{Res}(P) = \frac{1}{(2\pi)^n} \int_{S^*X} \text{tr}_E \sigma_{-n}(P)dv.$$ 

Here, $dv$ is the volume element defined by 

$$(-1)^{\frac{n(n+1)}{2}} \alpha \wedge (d\alpha)^{(n-1)},$$

where $\alpha = \sum_i \xi_i dx_i$ in local coordinate, is the canonical 1-form on $T^*X$, and $S^*X$ is the unit sphere bundle.

Below is the Connes’ trace formula:

**Theorem 3.7.** \cite{7} Let $E \to X$ and $\mathcal{H}$ be as above. Let $P \in \Psi(X, E)$ be a pseudo differential operator of order $-n$. Then $P \in L^{1,\infty}(\mathcal{H})$ and the equality holds:

$$\text{Tr}_\omega(P) = \frac{1}{n} \text{Res}(P).$$

Let $X$ be a closed oriented spin 4-manifold. Let $E$ be a finite dimensional complex vector bundle over $X$. Consider now the Hilbert space $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{E} \otimes_\mathcal{A} \mathcal{H}_0 = L^2(X, E \otimes S) = L^2(X, S_E)$$

where $\mathcal{E} = C^\infty(E)$, $\mathcal{A} = C^\infty(X)$ and $\mathcal{H}_0 = L^2(X, S)$. Here $\mathcal{H}$ is again a module over $\mathcal{A}$ because $\mathcal{A}$ is commutative.

Let $A_c : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} C^\infty(X, T^*X)$ be a connection on $\mathcal{E}$. These data give the Dirac spectral triple $(\mathcal{A}, \mathcal{H}_0, D)$, where $D = D_{A_c}$ acting on sections of $S_E$. Then one obtains a quasized connection replacing $da$ for $a \in C^\infty(X)$ by $[D[D]^{-1}, a] \in \mathcal{K}(\mathcal{H}_0)$, when $\ker D = 0$. It is a pseudo differential operator of order $-1$.

Denote $F = D[D]^{-1}$ and $\Omega_F^1$ be the span of quasized 1 forms $a[F, b] \in \mathcal{L}(\mathcal{H}_0)$ for $a, b \in \mathcal{A}$.

$A : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega_F^1$ induces the quasized covariant derivative:

$$A : \mathcal{E} \otimes_\mathcal{A} \Omega_F^1 \to \mathcal{E} \otimes_\mathcal{A} \Omega_F^2$$

given by:

$$A(e \otimes a[F, b]) := A(e) \otimes a[F, b] + e \otimes [F, a][F, b].$$

The quasized curvature is defined by the composition:

$$\theta := A^2 : \mathcal{E} \otimes_\mathcal{A} \Omega_F^2.$$

It follows from the formula:

$$\theta(ea) = \theta(e)a$$

that $\theta$ can be regarded as an element in $\mathcal{L}(\mathcal{H})$, induced by the composition:

$$\theta : \mathcal{E} \otimes_\mathcal{A} \mathcal{H}_0 \to \mathcal{E} \otimes_\mathcal{A} \mathcal{L}(\mathcal{H}_0) \otimes_\mathcal{A} \mathcal{H}_0 \to \mathcal{E} \otimes_\mathcal{A} \mathcal{H}_0.$$

There is a well defined surjective map $c : \Omega_F^1 \to \Omega_A^1$ such that $c([F, a]) = da$. Denote by $\theta_c$ and $\theta$ the curvature of $A_c$ and the quasized connection $A$ respectively.
Theorem 3.8 (Connes [7]). Suppose $X$ is of four dimension. The square of the quantized curvature $\Theta^2 \in L^{1,\infty}(\mathcal{H})$ gives rise to a positive functional:

$$A \to \inf_{c(A)=A} \text{Tr}_\omega(\Theta^2)$$

independent of $\omega$. The function coincide with the classical Yang-Mills functional up to a constant.

3.3. Pre Yang-Mills functional. We propose a model for pre Yang-Mills functional using Dixmier trace for a finitely generated projective module. If the algebra involved is non commutative, then it is in general difficult to formulate an analogy to the Connes-Yang-Mills functional in the commutative case.

Let $(A, \mathcal{H}_0, D)$ be a spectral triple. Assume the spectral triple is n-summable, i.e., $|D|^{-n} \in L^{1,\infty}(\mathcal{H}_0)$. In our setting, $A = C^\infty(\Gamma)$ for $\Gamma$, a covering group of a closed 4-manifold $X$ and we will assume $n = 4$. Let $F = D|D|^{-1} \in L(\mathcal{H}_0)$ and $\Omega_F(A) = \{a[F,b]|a,b \in A\} \subset L(\mathcal{H}_0)$ be as before.

Let $E$ be a finitely generated right projective $A$-module, with a connection $\nabla_E : E \to E \otimes_A \Omega_F(A)$. By definition $\nabla_E$ satisfies the equality:

$$\nabla_E(ea) = (\nabla_E e)a + e[F,a] \quad e \in E, a \in A.$$ 

We define its composition $\circ (\nabla_E)^2 : E \otimes_A \mathcal{H}_0 \to E \otimes_A \mathcal{H}_0$ with the contraction $\cdot$ by:

$$\circ (\nabla_E)^2(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta)$$

when $\nabla_E \xi = \sum \xi_\alpha \otimes \omega_\alpha$ and $\xi_\alpha, \omega_\alpha \in \Omega_F(A)$.

It can be checked that $[F,b]^n \in L^{1,\infty}(\mathcal{H})$ for $b \in A$. Assume $n = 4$. We therefore obtain that the quantized curvature

$$\Theta = (\nabla_E)^2 \in L(\mathcal{H})$$

admits a Dixmier trace:

$$\text{Tr}_\omega(\Theta^2) < \infty.$$ 

See page 680 of [7] and [9]. We propose that a pre Connes-Yang-Mills functional is given by a map:

$$\nabla_E \to \text{Tr}_\omega(\Theta^2)$$

for every connection $\nabla_E$ on $\mathcal{E}$, where $\Theta = (\nabla_E)^2$ is the quantized curvature.

Remark 3.9. Unlike the commutative case, we have no space, and hence do not have the surjective map $c : \Omega_F^1 \to \Omega_A^1$ mapping $i[F,a]$ to $da$ as in [7]. So we can not take minimum of the right hand side. Hence the name.

4. Dixmier $\Gamma$ trace

Let $\mathcal{E}$ be a Hilbert $A$ module, and $L(\mathcal{E})$ be the set of bounded $A$-linear endomorphisms. Let $\mathcal{K}(\mathcal{E}) \subset L(\mathcal{E})$ be the set of compact endomorphisms given by the norm closure of finite rank projections. $\mathcal{E}$ is reduced to a Hilbert space $\mathcal{H}$ when $A = \mathbb{C}$. 
4.1. Dixmier $\Gamma$-trace. Dixmier $\Gamma$-trace is an instance of type II non commutative geometry. In [2], Dixmier trace was introduced on von Neumann algebras. Our specific context here is to reformulate this. Let $A = C^*_r(\Gamma)$ be the reduced group $C^*$-algebra. Denote by the canonical von Neumann trace:

\begin{equation}
C^*_r(\Gamma) \to \mathbb{C}
\end{equation}

determined by:

$$\sum_{\gamma \in \Gamma} a_{\gamma} \gamma \mapsto a_e.$$

Let $\mathcal{E}$ be a Hilbert $C^*_r(\Gamma)$-module. Denote by $K(\mathcal{E})$ the algebra of $A$-linear compact endomorphisms on $\mathcal{E}$, and by $K(\mathcal{E})_+$ the subset of positive elements.

Kasparov’s stabilization implies that there is a Hilbert module isomorphism $\mathcal{E} \oplus \mathcal{H} \otimes C^*_r(\Gamma) \cong \mathcal{E}$, where $\mathcal{H}$ is a separable Hilbert space. Passing through this, a compact endomorphism on a Hilbert module can be represented as an infinite matrix with entries in $C^*_r(\Gamma)$. Let:

$$\text{tr}_{\Gamma} = \text{tr} \circ \text{Tr} : K(\mathcal{E})_+ \to [0, \infty]$$

be the trace induced by (4.1), and extend it to $\text{tr} : K(\mathcal{E}) \to \mathbb{C}$, where $\text{Tr}$ is the matrix trace. When $\text{tr}_{\Gamma}(A) < \infty$, then $A$ is known as a $\Gamma$-trace class operator. Note that when $\Gamma$ is trivial, $A$ is a compact operator in the usual sense and $\text{tr}_{\Gamma}(A)$ is the operator trace of $A$.

For $t \geq 0$ and an operator $A \in K(\mathcal{E})_+$, recall that the generalised singular value function:

$$\mu(A) : [0, \infty) \to [0, \infty)$$

is given by:

$$\mu_t(A) = \inf\{s \geq 0 : \text{tr}(\chi_{(s, \infty)}(A)) \leq t\}$$

where $\chi$ is the characteristic function (see [10, 11, 13]).

When $\Gamma$ is trivial, a compact positive operator $A$ on a Hilbert space has point spectrum only. In the case $\mu_n(A)$ is the $(n + 1)$-th largest eigenvalue of $A$, taken into account of multiplicity for $n \in \{0\} \cup \mathbb{N}$. More precisely we have:

**Lemma 4.1.** Let $\{\lambda_i\}_{i=0}^{\infty}$ be the set of eigenvalues of a compact positive operator $A$ on a Hilbert space, counted with multiplicity, decreasing to 0 as $i \to \infty$. Then:

$$\mu_t(A) = \lambda_i \quad \text{where} \quad \sum_{j=0}^{i-1} \dim E_{\lambda_j} \leq t < \sum_{j=0}^{i} \dim E_{\lambda_j}.$$ 

Here $E_{\lambda_j}$ stands for the eigenspace of $\lambda_j$.

**Proof.** By assumption, $A = \oplus_j \lambda_i E_i$. Then:

$$\chi_{(s, \infty)}(A) = \oplus_{\lambda_i > s} P_{E_{\lambda_i}} \quad \text{and} \quad \text{Tr}(\chi_{(s, \infty)}(A)) = \sum_{\lambda_i > s} \dim E_{\lambda_i}.$$

By definition, we have:

$$\mu_t(A) = \inf\{s \geq 0 : \sum_{\lambda_i > s} \dim E_{\lambda_i} \leq t\}$$

which gives rise to the statement. q.e.d.
Suppose $P$ is a projection in $\mathcal{L}(\mathcal{E})$ of $\Gamma$-trace class. Then the $\Gamma$-rank or $\Gamma$-dimension is given by:

$$\text{rk}_\Gamma P := \text{tr}_\Gamma(P) \in \mathbb{R}.$$  

The rank does not have to be an integer. For every $r > 0$ and a positive element $A \in \mathcal{K}(\mathcal{E})^+$, let us put:

$$\delta^\Gamma_r(A) := \sup \{ \text{tr}_\Gamma(AP) : \text{rk}_\Gamma(P) \leq r \}$$

where the supremum is taken over all projections $P$ with $\Gamma$-rank at most $r$ in $\mathcal{K}(\mathcal{E})$. Observe the equality:

$$\delta^\Gamma_r(A) = \int_0^r \mu_t(A)dt.$$  

See Lemma A.2 in [2] for more details.

For $A \in \mathcal{K}(\mathcal{E})^+$, let us introduce a norm:

$$\|A\|_{(1,\infty)} := \sup_{t>0} \frac{1}{\log(1+t)} \delta^\Gamma_t(A)$$  

$$= \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_t(A)ds.$$  

We then have a subset of $\mathcal{K}(\mathcal{E})$:

$$\mathcal{L}^{(1,\infty)}_+(\mathcal{E}) = \{ T \in \mathcal{K}(\mathcal{E})^+ : \|T\|_{(1,\infty)} < \infty \}$$

and an ideal of $\mathcal{K}(\mathcal{E})$:

$$\mathcal{L}^{(1,\infty)}(\mathcal{E}) = \{ T \in \mathcal{K}(\mathcal{E}) : \text{Re}(T)_\pm, \text{Im}(T)_\pm \in \mathcal{L}^{(1,\infty)}_+(\mathcal{E}) \}.$$  

A state $\omega$ defining the Dixmier $\Gamma$ trace exists by the following lemma. See for example [2, 19].

**Lemma 4.2.** There exists a state $\omega$ on $L^\infty(0, \infty)$ satisfying the following conditions:

- $\omega(C_0(0, \infty)) = 0$;
- If $f$ is real-values in $L^\infty(0, \infty)$, then
  
  $$\text{ess lim inf}_{t \to \infty} f(t) \leq \omega(f) \leq \text{ess lim sup}_{t \to \infty} f(t);$$

- $\omega(f) = 0$ vanishes, if the essential support of $f$ is compact;
- For any $a > 0$, the equality:
  
  $$\omega = \omega \circ D_a$$

holds, where $D_a$ is the dilation $D_a(f)(x) = f(x/a)$ for $f \in L^\infty(0, \infty)$.

Again, denote the evaluation of $\omega$ at $f \in L^\infty(0, \infty)$ by:

$$\omega - \lim_{r \to \infty} f(r) := \omega(f).$$

Let $A$ be a positive operator in $\mathcal{K}(\mathcal{E})$. The absolute Dixmier $\Gamma$ trace on $\mathcal{L}^{(1,\infty)}(\mathcal{E})$ is given by:

$$\text{Tr}^\Gamma_0(A) := \omega - \lim_{r \to \infty} \frac{1}{\log(r+1)} \delta^\Gamma_r(A).$$
Definition 4.3. Let $A$ be a self adjoint operator. The Dixmier $\Gamma$ trace:

$$\text{Tr}_\Gamma^\omega : \mathcal{L}^{(1,\infty)}(\mathcal{E}) \to \mathbb{C}$$

is defined by:

$$\text{Tr}_\Gamma^\omega (A) = \text{Tr}_\Gamma^\omega (A_+) - \text{Tr}_\Gamma^\omega (A_-).$$

For arbitrary $A$, it is given by:

$$\text{Tr}_\Gamma^\omega (A) = \text{Tr}_\Gamma^\omega (\text{Re} \ A) + i \text{Tr}_\Gamma^\omega (\text{im} \ A).$$

5. Higher Yang-Mills instantons

Let $X$ be an oriented closed smooth 4-manifold with a spin structure. Let $\tilde{X}$ be a $\Gamma$-cover of $X$ with covering group $\Gamma$. For example, $\Gamma = \pi_1(X)$ is the fundamental group and $\tilde{X}$ is the universal cover. Let $\gamma = \tilde{X} \times_\Gamma \mathbb{C}^\ast r(\Gamma) \to X$ be the Mischenko-Fomenko line bundle with flat connection $\nabla^{\gamma}$. Let $S \to X$ be the spinor bundle and denote its lift by $\tilde{S}$ on the universal covering space $\tilde{X}$. Let $E \to X$ be a unitary bundle equipped with a unitary connection $A = \nabla^{E}$.

Consider:

$$E_0 = C^\infty (X, \tilde{E} \times_\Gamma C_r^*(\Gamma)) = C^\infty (X, E \otimes \gamma),$$

$$A = C^\infty (X),$$

$$\mathcal{H} = L^2 (X, S) = L^2 (X, S^+) \oplus L^2 (X, S^-)$$

where $\mathcal{E}_0$ is a finitely generated $(A, C_r^*(\Gamma))$ bi-module in the sense that it is a left module over $C^\infty (X)$ and is a right module over $C_r^*(\Gamma)$. $\mathcal{H}$ is a $A$ bi-module.

Therefore, we form the Hilbert $C^*_r(\Gamma)$-module:

$$\mathcal{E} = \mathcal{E}_0 \otimes_A \mathcal{H} = C^\infty (X, E \otimes \gamma) \otimes_{C^\infty (X)} L^2 (X, S)$$

$$= L^2 (X, E \otimes S \otimes \gamma).$$

Let $\Omega^1_A$ be the space of smooth 1-forms of $A$:

$$\Omega^1_A = \{ \sum adb| a, b \in A \} = C^\infty (X, T^* X).$$

Let $D : \mathcal{H} \to \mathcal{H}$ be the Dirac operator on $S$. Locally $D$ has the form $\sum_i r(e_i) \nabla_{e_i}^H$, where $\{ e_i \}_i$ is a local orthonormal frame and $r(e_i)$ is the Clifford multiplication by $e_i$. $\nabla^H$ is the spin connection so that:

$$[\nabla_{e_i}^H, r(\alpha)] = r(\nabla_{e_i} \alpha) \quad \alpha \in \Omega^1_A$$

holds. Then it can be checked the equality:

$$[D, f] = r(df) \quad f \in A.$$

Observe that a one form in $\Omega^1_A$ acts on $\mathcal{H}$ as a bounded operator via the inclusion:

$$r : \Omega^1_A \hookrightarrow \mathcal{L}(\mathcal{H}), \quad r(adb) = a[D, b].$$

The space of connections $A = \nabla^{E_0}$ on $\mathcal{E}_0$ forms an affine space over:

$$C^\infty (X, T^* X \otimes E \otimes_{A^\ast r(\Gamma)} \text{End} \gamma)$$

where $\Gamma$ acts on $\text{End} \gamma$ by conjugation.
Assume that \( \ker D^+ = 0 \) holds, and let:

\[
F = D^+|D^+|^{-1}
\]

be the unitary part of \( D \). Introduce the quantized differential by:

\[
\hat{da} := i[F, a] \quad a \in \mathcal{A}
\]

and denote by \( \Omega^1_F \) the space of quantized 1-forms:

\[
\Omega^1_F = \{ \sum a \hat{db} \in \mathcal{L}(\mathcal{H}) | a, b \in \mathcal{A} \}.
\]

Similarly, one has the quantized differential forms of degree \( k \):

\[
\Omega^k_F = \{ \sum a^0 \hat{da}^1 \cdots \hat{da}^k \in \mathcal{L}(\mathcal{H}) | a^j \in \mathcal{A} \}
\]

and their differentials:

\[
\alpha \in \Omega^k_F \rightarrow \hat{d}\alpha := i(F\alpha - (-1)^k \alpha F) \in \Omega^{k+1}_F.
\]

**Lemma 5.1.** ([7])

(1) There exists a unique bimodule linear map which satisfies the property:

\[
c : \Omega^1_F \rightarrow \Omega^1_A \quad \hat{da} \mapsto da.
\]

This map is surjective and the image of self adjoint elements of \( \Omega^1_F \) are the real forms.

(2) For \( \alpha \in \Omega^1_F \), we have

\[
A c(\hat{da}) = d[c(\alpha)]
\]

where \( A \) stands for the projection onto the anti-symmetric tensor.

**Remark 5.2.** Recall ([7]) the meaning of \( c \) on \( \Omega^2_F \) in lemma 5.1 and its relation to the principal symbol. The principal symbol of the pseudo differential operator \( da(a \in \mathcal{A}) \) of order \(-1\) is given by:

\[
\sigma_{-1}(\hat{da}) = r(da - \langle \xi, da \rangle \xi) \in C^\infty(S^*X, Cl(T^*X)).
\]

Notice that the assignment:

\[
da \mapsto da - \langle \xi, da \rangle \xi
\]

induces a linear injection:

\[
C^\infty(X, T^*X) \hookrightarrow C^\infty(S^*X, Cl(T^*X)).
\]

In particular \( \sigma_{-1}(\hat{da}) \) and hence \( \hat{da} \) uniquely determines \( da \). This gives a well defined linear map \( c \) on \( \hat{da} \in \Omega^1_F \) with \( c(\hat{da}) = da \), which coincides with \( \sigma_{-1}(\hat{da}) \) up to this identification.

Then Connes defined \( c : \Omega^2_F \rightarrow C^\infty(X, (T^*X) \otimes (T^*X)) \) by the image of \( \sigma_{-2} \) in \( C^\infty(S^*X, Cl(T^*X)) \) using the homomorphism:

\[
\pi : C^\infty(X, (T^*X) \otimes (T^*X)) \rightarrow C^\infty(S^*X, Cl(T^*X))
\]

determined by \( \eta_1 \otimes \eta_2 \mapsto r(\eta_1 - \langle \xi, \eta_1 \rangle r(\eta_2 - \langle \xi, \eta_2 \rangle) \) for \( \eta_1, \eta_2 \in T^*_x X \) and \( \xi \in S^*_x X \).
A q-connection $\nabla^{E_0}$ on $E_0$ is given by a linear map:

$$\nabla^{E_0}: E_0 \rightarrow E_0 \otimes_A \Omega^1_F$$

such that $\nabla^{E_0}(\xi \cdot x) = (\nabla^{E_0}\xi) \cdot x + \xi \otimes \hat{dx}$ holds for $\xi \in E_0$ and $x \in A$. Its curvature:

$$\Theta := (\nabla^{E_0})^2: E_0 \rightarrow E_0 \otimes_A \Omega^2_F$$

is an $A$ module. Hence it induces an endomorphism:

$$\Theta : E = E_0 \otimes_A H \rightarrow E$$

by contraction.

Every q-connection $\nabla^{E_0}: E_0 \rightarrow E_0 \otimes_A \Omega^1_F$ determines uniquely a classical connection $\nabla^{E_0}_c: E_0 \rightarrow E_0 \otimes_A \Omega^1_A$ by composition with the bimodule map:

$$\nabla^{E_0}_c := (1 \otimes c) \circ \nabla^{E_0}.$$  

Denote the von Neumann trace by:

$$\tau(T) = \langle T\delta e, \delta e \rangle$$

for an element $T$ of $\text{End}_{C^*_r(\Gamma)}(\gamma)$.

**Lemma 5.3.** The curvature $\Theta_c$ associated to the connection:

$$\nabla^{E_0}_c: C^\infty(X, E \otimes \gamma) \rightarrow C^\infty(X, T^*X \otimes E \otimes \gamma)$$

is the antisymmetric part of the curvature $\Theta$ associated to the q-connection $\nabla^{E_0}$:

$$\Theta_c := (\nabla^{E_0}_c)^2 = Ac(\Theta).$$

Here $A$ acts on $T^*X \otimes T^*X$ and does nothing to $\text{End}_{C^*_r(\Gamma)}(\gamma)$.

**Proof.** The proof essentially follows from Lemma 5.1. We only need to verify the equality for a component of $\nabla^{E}$ of the form $\omega = adb \in \Omega^1_F$.

The quantized curvature $\Theta$ can be represented by:

$$d\omega + \omega \otimes \omega = \hat{da}\hat{db} + a^2 \hat{db}\hat{db}.$$  

The antisymmetric part of $c(\Theta)$ will have the form:

$$Ac(da\otimes db + a^2\hat{db}\hat{db}) = A(da \otimes db) + A(a^2\hat{db} \otimes db) = da \wedge db.$$

On the other hand, the component of $\Theta_c$ is:

$$d[c(\omega)] = d(adb) = da \wedge db.$$

The lemma is then proved. q.e.d.

Let $A_c$ be a unitary connection on $E$. It induces a connection $A_c$ on $E \otimes \gamma$:

$$A_c: C^\infty(X, E \otimes \gamma) \rightarrow C^\infty(X, T^*X \otimes E \otimes \gamma)$$

Replacing 1-forms appeared in $A_c$, $A_c$ respectively by $[F, \alpha]$, we obtain two quantized connections:

$$A \text{ on } C^\infty(X, E), \quad A_c \text{ on } C^\infty(X, E \otimes \gamma) = E_0.$$  

Both the curvatures $\theta = A^2$ and $\Theta = A_c^2$ are pseudo differential operators of order $-2$.

$$\Theta : L^2(X, E \otimes S \otimes \gamma) \rightarrow L^2(X, E \otimes S \otimes \gamma)$$

is a pseudodifferential operator of order $-2$. Its principal symbol $\sigma(\Theta)$ is an element of

$$C^\infty(S^*X, \text{End}(\pi^*(E \otimes S \otimes \gamma))) = C^\infty(S^*X, \text{Cl}(T^*X) \otimes \text{End} \pi^*(E \otimes \gamma)).$$
Thus, this gives rise to
\[ c(\Theta) \in C^\infty(X, T^*X \otimes T^*X \otimes \text{End} E \otimes \text{End}_{C^*}(\gamma)). \]
Let \( \Theta_{ij} \) be the decomposition of \( \Theta \) according to \( \text{End} E \). So
\[ c(\Theta_{ij}) \in C^\infty(X, T^*X \otimes T^*X \otimes \text{End}_{C^*}(\gamma)). \]
Denote by
\[ \|c(\Theta_{ij})\| \in C^\infty(X, \text{End}_{C^*}(\gamma)) \]
the \( C^\infty(X, \text{End}_{C^*}(\gamma)) \)-valued norm.

**Lemma 5.4.** Let \( X \) be a closed, oriented, spin, smooth 4-manifold with \( b_1 = 4 \). Let \( \Gamma = \mathbb{Z}^4 \) be a covering group of \( X \). (Existence of a \( \mathbb{Z}^4 \)-cover is ensured by \( b_1 = 4 \).) Then the formula:
\[ \text{Tr}_\omega^\Gamma(\Theta^2) = \int_{S^*X} \tau \circ \text{Tr}_E \sigma_{2}(\Theta)(x, \xi) d\xi \]
holds, where \( \Theta_{ij} \) is local representation by the \( \text{End}_{C^*}(\gamma) \) valued matrix.

**Proof.** The quantized curvature \( \Theta \) is a pseudo differential operator of order \(-2\) and is self-adjoint, i.e., \( \Theta_{ij} = \Theta_{ji} \) (similar argument as in [7]). Denote \( P = \Theta^2 = \Theta \Theta^* \). Then
\[ \sigma_{-4}(P) = \sigma_{-2}(\Theta)\sigma_{-2}(\Theta)^* \]
with
\[ \text{Tr}_E(\sigma_{-2}(\Theta)\sigma_{-2}(\Theta)^*) = \sum_{i,j} \sigma_{-2}(\Theta_{ij})\sigma_{-2}(\Theta_{ij})^* \]
which is identified with \( \sum_{i,j} c(\Theta)c(\Theta)^* \) under \( \pi \) in (5.1). See Remark 5.2. Thus, from the trace formula (6.3), we have
\[ \text{Tr}_\omega^\Gamma(\Theta^2) = \int_{S^*X} \tau \circ \text{Tr}_E \sigma_{-4}(\Theta^2)(x, \xi) d\xi \]
\[ = \sum_{i,j} \int_X \tau \circ \text{Tr}_S(c(\Theta_{ij})c(\Theta_{ij})^*) dx \]
\[ = \sum_{i,j} \int_X \tau(\|c(\Theta_{ij})\|^2) dx. \]
Note that \( \text{rank}[H_1(X; \mathbb{Z})] = 4 \) is used here to ensure \( X \) a \( \Gamma = \mathbb{Z}^4 \) cover. So we obtain the equality:
\[ \text{Tr}_\omega^\Gamma(\Theta^2) = \sum_{i,j} \int_X \tau(\|c(\Theta_{ij})\|^2). \]
q.e.d.

For a 4-manifold \( X \) with a \( \Gamma = \mathbb{Z}^4 \) cover, we obtain a \( \Gamma \)-trace formula for the higher Connes-Yang-Mills functional:

**Theorem 5.5.** Let \( X \) be a closed, oriented, spin, smooth 4-manifold with \( b_1 = 4 \). Let \( \Gamma = \mathbb{Z}^4 \) be a covering group of \( X \). (Existence of a \( \mathbb{Z}^4 \)-cover is ensured by \( b_1 = 4 \).) Let \( \Theta_c = \mathcal{A}_c^2 \) be the curvature of the classical connection.
associated to a quantized connection $A$, and put the quantized curvature by $\Theta = A$. Then the formula holds:

$$\inf_{c(A) = A_c} \text{Tr}_\omega(\Theta^2) = \sum_{i,j} \int_X \tau(\|c(\Theta_{ij})\|^2).$$

**Proof.** In view of Lemmas 5.4 and 5.3, the proof follows essentially from the proof Theorem 14 in [7]. Namely, observe that locally $T^*_X \mathbb{X} \otimes T^*_X \mathbb{X} \cong \Lambda^2 \mathbb{R}^4 \oplus S^2 \mathbb{R}^4$ is orthogonally decomposed into the symmetric part and the antisymmetric part. By the triangle inequality with Lemma 5.3, we obtain the estimate:

$$\tau(\|c(\Theta_{ij})\|^2) \geq \tau(\|A(\Theta_{ij})\|^2) = \tau(\|\Theta_{ij}\|^2).$$

Note that all numbers above are nonnegative because $\Theta^2$ is a positive operator. The proof then is implied by the previous lemma. q.e.d.

**Theorem 5.6.** Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Let $\Gamma = \mathbb{Z}^4$ be a covering group of $X$. We have $\theta^2 \in L^{1,\infty}(\mathcal{H})$ and $\Theta^2 \in L^{1,\infty}(\mathcal{E})$ and also the equality holds:

$$\inf_{c(A) = A_c} \text{Tr}_\omega(\Theta^2) = \inf_{c(A) = A_c} \text{Tr}_\omega(\Theta^2).$$

Moreover the positive functional is independent of $\omega$.

**Proof.** Note that $\theta^2$ is a pseudo differential operator of order $-4$, and $\Theta^2$ is a pseudo differential operator of order $-4$ with coefficient in $C^\infty_r(\Gamma)$. Then $\theta^2 \in L^{1,\infty}(\mathcal{H})$ follows from Theorem 1 of [7] and $\Theta^2 \in L^{1,\infty}(\mathcal{E})$ follows from Proposition 6.3 below.

To show the identity, by Theorem 5.5 and Theorem 14 in [7], we only need to verify the equality:

$$\int_X \text{Tr}_E \theta^2_c = \int_X \tau \circ \text{Tr}_E (\Theta^2_c).$$

Recall that the connection $A_c$ on $E \to X$ gives rise to a family of connections $A_\rho$ on the family of complex vector bundles:

$$E = \{ \tilde{E} \otimes \Gamma C_\rho \}_{\rho \in \hat{T}^4} \to X \times \hat{T}^4.$$ Denote by $\theta_{A_\rho, c}$ the curvature of $A_\rho$, which actually coincides with $\theta_c = \theta_{A_c}$. The twisted connection $A_c$ on $\hat{E} \otimes \Gamma C^*_c(\Gamma) \to X$ is identified with the above family of connections. By Fourier transform, evaluation at the identity in $C^\infty_r(\Gamma)$ corresponds to averaging of elements of $C(\hat{T}^4)$ over $\hat{T}^4$. So $\tau \circ \text{Tr}_E (\Theta^2_c)$ is to average the family involving curvatures $\int_{\hat{T}^4} \text{Tr}_E \theta^2_{A_c, \rho}$ on constant 1 function on $\hat{T}^4$, which corresponds to $\delta_c$ in $C^\infty_r(\Gamma)$. Now the equality:

$$\theta_{A_c, \rho}(1) = \theta_{A_c} + (d\rho)(1) = \theta_{A_c},$$

follows since $\rho$ is a flat connection. Set the volume of $\hat{T}^4$ to be 1. We have:

$$\tau \circ \text{Tr}_E (\Theta^2_c) = \int_{\hat{T}^4} \text{Tr}_E \theta^2_{A_c, \rho} = \text{Tr}_E \theta^2_{A_c} = \text{Tr}_E \theta^2,$$

and the proof is complete. q.e.d.
Definition 5.7. A higher Connes-Yang-Mills instanton is a connection of the form $A_c$ which attains local minimum of the higher Connes-Yang-Mills action functional given by:

$$CYM^\Gamma(A_c) := 16\pi^2 \inf_{c(A)=A_c} I(A).$$

The corollary below follows immediately from Theorem 14 in [7] and Theorem 5.6.

Corollary 5.8. Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Let $\Gamma = \mathbb{Z}^3$ be a covering group of $X$. Then the Connes-Yang-Mills functional and its higher analogue coincide:

$$CYM(A_c) = CYM^\Gamma(A_c).$$

In particular a Connes-Yang-Mills instanton is equivalent to its higher version.

Lemma 1.2 gives the following:

Corollary 5.9. Let $X$ be a closed, oriented, spin, smooth 4-manifold with $b_1 = 4$. Let $\Gamma = \mathbb{Z}^3$ be a covering group of $X$. The higher Nahm transform sends the minimizer of the higher Connes-Yang-Mills functional to the minimiser of the Connes-Yang-Mills functional.

6. Some properties of Dixmier $\Gamma$-trace

By definition, Dixmier $\Gamma$-trace is positive and vanishes on the ideal of $\Gamma$-trace class operators. We also have the following property for the Dixmier $\Gamma$-trace. Let $\mathcal{E}$ be a Hilbert $C^*_r(\Gamma)$ module.

Lemma 6.1. For every $A \in \mathcal{L}^{(1,\infty)}(\mathcal{E})$ and $Y \in \mathcal{L}(\mathcal{E})$ bounded, we have:

$$\text{Tr}_\omega^\Gamma(AY) = \text{Tr}_\omega^\Gamma(YA).$$

Proof. Because every bounded linear operator on $\mathcal{E}$ can be written as a linear combination of unitary operators (see, for example, Lemma on page 209 VI 1.6 of [18] adapted to the case of Hilbert module), we only need to verify the equality:

$$\delta^\Gamma_r(A) = \delta^\Gamma_r(U^*AU)$$

for a unitary operator $U$. Now

$$\delta^\Gamma_r(U^*AU) = \sup_P \{\text{tr}(U^*AUP)\} = \sup_P \{\text{tr}(AUPU^*)\} = \delta^\Gamma_r(A)$$

because $U^*PU$ is a projection of rank $r$ if $P$ does. q.e.d.

In the following, we will introduce some examples of pseudo differential operators arising in the domain of Dixmier $\Gamma$ trace together with an analogue of Connes’ trace theorem [7] (see also [11]) when $\Gamma$ is free abelian. We recall some pseudo differential calculus on a closed manifold with coefficient in a $C^*_r(\Gamma)$-bundles of finite type in [3].

Let $\mathcal{V} \rightarrow X$ be a flat $C^*_r(\Gamma)$-bundle over a closed manifold $X$, whose fiber is isomorphic to $(\mathcal{C}^*_r(\Gamma))^N$ for some $N \in \mathbb{N}$. For example, the tensor product of a complex vector bundle $E$ over $X$ with the Mischenko-Fomenko bundle $\gamma = X \times \Gamma C^*_r(\Gamma)$ forms such a bundle $\mathcal{V} = E \otimes \gamma$. 
A pseudo differential operator acts on the set of smooth sections:

\[ E^\infty := C^\infty(X, \mathcal{V}) \]

and one can take the closure to obtain a Sobolev space \( H^l(X, \mathcal{V}) \). Denote the Hilbert module over \( C^*_r(\Gamma) \):

\[ E := L^2(X, \mathcal{V}) = H^0(X, \mathcal{V}). \]

As in the classical case, a linear operator:

\[ P : E^\infty \to E^\infty \]

is a pseudo differential operator of order \( m \) if it can be expressed as:

\[ P = \sum_j P_j + R \]

where \( R \) is a smoothing operator and \( P_j \) are pseudo differential operators with support in the domain of \( \psi_i \) for an atlas \( \{ \psi_i \} \) of \( \mathcal{V} \to X \). We denote by \( \Psi^m_\Gamma(X, \mathcal{V}) \) the space of pseudo differential operators on \( X \) with coefficient in \( \mathcal{V} \) of order \( m \).

A pseudo differential operator on \( X \) with coefficient in the \( C^*_r(\Gamma) \)-bundle \( \mathcal{V} \) can be also constructed by gluing. Let \( X = \bigcup_j \Omega_j \) be a locally finite covering of \( X \) by coordinate neighbourhoods, and \( P_j \) be pseudo differential operators of order \( m \) on \( \Omega_j \). Let \( \sum_j \psi_j = 1 \) be partition of unity subordinate to the given covering and let \( \phi_j \in C^\infty_0(\Omega_j) \) with \( \phi_j|_{\text{supp } \psi_j} = 1 \).

Then

\[ P = \sum_j \phi_j P_j \psi_j + R \]

is a pseudo differential operator of order \(-m\) where \( R \in \Psi^{-\infty}_\Gamma(X, \mathcal{V}) \) is a smoothing operator.

One can check that any pseudo differential operator \( P \in \Psi^m_\Gamma(X, \mathcal{V}) \) of order \( m \) extends to:

\[ P : H^m(X, \mathcal{V}) \to H^0(X, \mathcal{V}). \]

In particular, any \( P \in \Psi^m_\Gamma(X, \mathcal{V}) \) with \( m \leq 0 \) extends to a bounded linear operator

\[ P : E \to E. \]

Moreover \( P \in \mathcal{K}(E) \) holds if \( m \leq -\dim X \). In particular, if \( m < -\dim X \), \( P \in \mathcal{K}(E) \) is a \( \Gamma \)-trace class operator.

**Example 6.2.** Let \( X = \mathbb{T}^n \) be the flat torus and \( \Delta \) be the Laplacian determined by the square of the Dirac operator on \( \mathbb{T}^n \). Let

\[ \Delta_\gamma : C^\infty(\mathbb{T}^n, S \otimes \gamma) \to C^\infty(\mathbb{T}^n, S \otimes \gamma) \]

be the twist of \( \Delta \) by the Mischenko-Fomenko line bundle \( \gamma \), induced from \( \Delta \). Then:

\[ (1 + \Delta_\gamma)^{-\frac{n}{2}} \in \mathcal{L}^{(1,\infty)}(E) \]

where \( E = L^2(\mathbb{T}^n, S \otimes \gamma) \). We will verify this by calculating its Dixmier \( \Gamma \) trace explicitly in the next subsection. See Proposition 6.5.
Proposition 6.3. Let $\Gamma = \mathbb{Z}^n$. Let $X$ be a closed manifold of dimension $n$ with a $\Gamma$-cover. Let $V \rightarrow X$ be a flat $C^*_\Gamma(\Gamma)$-bundle whose fiber is isomorphic to $(C^*_\Gamma(\Gamma))^N$ for some $N \in \mathbb{N}$. Let $E = L^2(X, V)$ be the Hilbert module of $L^2$-sections of $V$. Then every pseudo differential operator $P \in \Psi^{-n}_\Gamma(X, V)$ of order $-n$ has finite Dixmier $\Gamma$ trace, i.e.,

$$P \in \mathcal{L}^{(1, \infty)}(E) \quad \text{and} \quad \text{Tr}_\omega^\Gamma(P) < \infty.$$  

Proof. We will first prove the special case when $X = T^n$ and $V = \gamma$ as in the previous example. A pseudo differential operator $P$ of order $-n$ has the form

$$P = B(1 + \Delta_\gamma)^{-\frac{n}{2}}$$

where $B \in \Psi^0_\Gamma(T^n, V)$ is a bounded linear operator on $E$ and $(1 + \Delta_\gamma)^{-\frac{n}{2}} \in \Psi^{-n}_\Gamma(T^n, V)$ is a compact operator on $E$. From Proposition 6.5 we have (6.1). Then noting that $\mathcal{L}^{(1, \infty)}(E)$ is an ideal of $\mathcal{L}(E)$, we have

$$P \in \mathcal{L}^{(1, \infty)}(E) \quad E = L^2(T^n, \gamma).$$

Next, in general let $X$, $V$ and $P$ be as assumed in the proposition. Then

$$P = \sum_j \phi_j P\psi_j + R$$

where $R$ is a smoothing operator. Let us fix a covering $\{\Omega_j\}_j$ of $X$ such that $V|_{\Omega_j}$ is trivial. To see $\text{Tr}_\omega^\Gamma(P) < \infty$ in view of (6.2) we only need to verify finiteness:

$$\text{Tr}_\omega^\Gamma(\phi_j P\psi_j) < \infty \quad \forall j.$$  

Embed $\Omega_j$ in $T^n$ and define the operator to be $0$ on other coordinate patch of $T^n$. Then $\phi_j P\psi_j$ can be viewed as a pseudo differential operator on $T^n$ with coefficient in $V$. But we have already verified:

$$\phi_j P\psi_j \in \mathcal{L}^{(1, \infty)}(E)$$

and hence $P \in \mathcal{L}^{(1, \infty)}(E)$. q.e.d.

The principal symbol $\sigma_P$ of $P \in \Psi^m_\Gamma(X, V)$ can be identified as:

$$\sigma_P \in C^\infty(S^*X, \text{End } V).$$

Define the $\Gamma$-residue of $P$ by:

$$\text{Res}_\Gamma(P) := \int_{S^*X} \text{Tr}_V(\sigma_P(x, \xi))d\nu.$$  

Here $\text{Tr}_V$ is the composition of matrix trace $\text{Tr}$ with the von Neumann trace $\text{tr}$, regarding an element of $\text{End}(V)$ as a matrix with entries in $C^*_\Gamma(\Gamma)$ locally.

Proposition 6.4. Under the condition in Proposition 6.3, the equality holds:

$$\text{Tr}_\omega^\Gamma(P) = \frac{1}{n} \text{Res}_\Gamma(P).$$  

Proof. Note that $\Psi^{-n-1}(X, V)$ consists of $\Gamma$-trace class operators on which $\text{Tr}_\omega^\Gamma$ vanishes. Therefore $\text{Tr}_\omega^\Gamma$ is a well-defined linear functional on the quotient space $\Psi^{-n}(X, V)/\Psi^{-n-1}(X, V)$, which is identified with the space of
order $-n$ principal symbols $\sigma_P$ in $C^\infty(S^*X, \text{End} \mathcal{V})$. Therefore, $\text{Tr}_\omega(P)$ depends only on trace of the principal symbol of $P$,

$$(x, \xi) \mapsto \text{Tr}_\mathcal{V}[\sigma_P(x, \xi)]$$

which is a continuous function on $S^*X$. We denote this function by $f_P$. As explained before,

$$T : f_P \to \text{Tr}_\omega(P)$$

gives rise to a distribution on $C(S^*X)$. Note that this distribution is positive. By the Riesz-Markov-Kakutani representation theorem on linear functionals on continuous functions, the distribution $T$ is given by a positive measure $\mu$ on $S^*X$, i.e., $T(f_P) = \int f_P d\mu$ and that is:

$$(6.3) \quad \text{Tr}_\omega(P) = \int_{S^*X} f_P(x, \xi) d\mu(x, \xi) = \int_{S^*X} \text{Tr}_\mathcal{V}(\sigma_P(x, \xi)) d\mu(x, \xi).$$

Because $\text{Tr}_\omega$ is invariant under unitary transformation, the measure $\mu$ is invariant under isometry. Because both $\text{Tr}_\omega$ and $\text{Res}_\mathcal{V}$ can be reduced locally to an atlas trivialising $\mathcal{V}$, it is sufficient to show the equality for any closed manifold with trivial $C^*_r(\Gamma)$-line bundle. Let $X = S^n$. As explained in [1], the group of isometries $SO(n+1)$ on $\mathbb{R}^{n+1}$ induces a action on $S^*S^n$ as a homogeneous space and the volume form on $S^*S^n$ of the induced Riemannian metric is invariant under the action of $SO(n+1)$. Then uniqueness of invariant measure on $S^*S^n$ shows that the measure $\mu$ is proportional to the volume form. Note that uniqueness of invariant measure on homogeneous spaces can be found in [17]. Now the proposition is proved because the scaling constant is completely determined by the example of $T^n$. q.e.d.

6.1. $\mathbb{Z}^n$-Dixmier trace for flat torus. Let $X = (\mathbb{R}/\mathbb{Z})^n$ be the $n$-torus and $D$ be the Dirac operator on $X$. $D$ has point spectrum only. In particular, associated to each element $x$ in the integer lattice $\mathbb{Z}^n$ is an eigenvector of the Laplacian $\Delta = D^2$ with eigenvalue $\|x\|^2$. We call $\mathbb{Z}^n$ the spectral lattice of $D^2$. The pseudo differential operator $(D^2 + 1)^{-\frac{n}{2}}$ has order $-n$ and is a compact operator on $L^2(M, S)$ where $S$ is the spinor bundle.

Let $D$ be the Dirac operator on $\mathbb{T}^n$ and $D_\gamma$ be its twist by the Mischenko-Fomenko line bundle over $\mathbb{T}^n$. We verify the following:

**Proposition 6.5.** The equality holds:

$$\text{Tr}_\omega((1 + D^2)^{-\frac{n}{2}}) = \text{Tr}_\omega((1 + D_\gamma^2)^{-\frac{n}{2}}).$$

The statement remains true when replacing the operators by any order $-n$ pseudo differential operator obtained from functional calculus of $D, D_\gamma$.

**Proof.** Denote $A = (D^2 + 1)^{-\frac{n}{2}}$. In [7], the Dixmier trace was computed by:

$$\text{Tr}_\omega(A) = \lim_{V_r \to \infty} \sum_{\lambda \in \mathbb{Z}^n, \|\lambda\|^2 \leq r} \frac{(1 + \|\lambda\|^2)^{-\frac{n}{2}}}{\log(V_r + 1)} = \frac{\Omega_n}{n} = \frac{1}{n(2\pi)^n} \int_{S^{n-1}} \sigma_{-n}(A)$$

which is known as Connes’ trace formula, where $V_r = \frac{\Omega_n}{n} r^n$ is the volume of ball of radius $r$ in $\mathbb{R}^n$ and $\Omega_n$ is the area of the unit sphere $S^{n-1}$.
We recall the contents in Section 4.1. Consider a sequence of lattice points $x_0, \ldots, x_m, \ldots \in \mathbb{Z}^n$ with $\|x_i\| > \|x_j\|$ for all $i > j$, whose union satisfies the coincidence below as sets:

$$\{\|x_i\|: i = 0, 1, \ldots\} = \{\|x\|: x \in \mathbb{Z}^n\}.$$  

The operator $A$ has the set of eigenvalues:

$$\{(\|x\|^2 + 1)^{-\frac{n}{2}} : x \in \mathbb{Z}^n\}.$$  

Let $\lambda_1 > \lambda_2 > \cdots$ be the ordered set of eigenvalues. Then:

$$\lambda_i = (\|x_i\|^2 + 1)^{-\frac{n}{2}}.$$  

The dimension of the eigenspaces admits the formula:

$$\sum_{j=0}^{i-1} \dim E_{\lambda_j} = \#\{x \in \mathbb{Z}^n : \|x\| \leq \|x_i\|\}.$$  

Let us denote the right hand side by $N_0, \|x_i\|$, the number of lattice points in the ball centered at the origin with radius $\|x_i\|$. Denote the number of lattice points on the sphere centered at the origin with radius $\|x_i\|$ by:

$$S_0, \|x_i\| := N_0, \|x_i\| - N_0, \|x_i-1\|.$$  

Using lemma 4.1 we have the equality:

$$\mu_t(A) = ((\|x_i\|^2 + 1)^{-\frac{n}{2}}$$

where $N_{0, \|x_{i-1}\|} \leq t < N_{\|x_i\|}$ and $\mu_n$ is the $(n+1)$-th largest eigenvalue counted with multiplicity.

Therefore, the Dixmier trace of $A$ is calculated by:

$$\text{Tr}_w(A) = \lim_{t \to \infty} \frac{1}{\log(t+1)} \int_0^t \mu_s(A) ds$$

$$= \lim_{N \to \infty} \frac{1}{\log(N+1)} \sum_{n=0}^{N} \mu_n(A)$$

$$= \lim_{t \to \infty} \frac{1}{\log(N_{0, \|x_i\|}+1)} \sum_{j=0}^{i} (\|x_j\|^2 + 1)^{-\frac{n}{2}} S_{0, \|x_j\|}$$

$$= \lim_{i \to \infty} \frac{1}{\log N_{0, \|x_i\|}} \sum_{j=0}^{i} \|x_j\|^{-n} S_{0, \|x_j\|}$$

$$= \frac{\Omega_n}{n}$$

and hence it coincides with $\frac{\Omega_n}{n} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Denote by $\gamma$ the Mischenko-Fomenko line bundle and $\tilde{S} \otimes_{\mathbb{Z}^n} C^*_r(\mathbb{Z}^n)$ the $C^*$-algebra bundle obtained by twisting $S \to \mathbb{T}^n$ by $\gamma$ over $\mathbb{T}^n$. The Dirac operator $D$ induces the action $D_\gamma$ on the $L^2$-sections:

$$\mathcal{E} = L^2(\mathbb{T}^n, \tilde{S} \otimes_{\mathbb{Z}^n} C^*_r(\mathbb{Z}^n))$$

which is a Hilbert module over $C^*_r(\mathbb{Z}^n)$. Under the identification:

$$C^*_r(\mathbb{Z}^n) \cong C(\hat{\mathbb{Z}}^n) = \int_{\alpha \in \mathbb{Z}^n} \mathbb{C}_\alpha,$$
\[ \mathcal{E} \text{ admits a direct integral decomposition:} \]
\[ \mathcal{E} = \int_{\alpha \in \hat{\mathbb{Z}}^n} L^2(T^n, \hat{S} \otimes \mathbb{C}_\alpha). \]
Under this decomposition, \( D_\gamma \) is identified with a family of Dirac operators \( D_\alpha \) on \( H_\alpha = L^2(T^n, \hat{S} \otimes \mathbb{C}_\alpha) \) parametrised by \( \alpha \in \hat{\mathbb{Z}}^n \):
\[ D_\gamma = \{ D_\alpha \}_{\alpha \in \hat{\mathbb{Z}}^n}. \]

\( D_\alpha \) has point spectrum. We claim that \( D_\alpha \) has spectral lattice which differs by shift of \( i\alpha \) from that of \( D = D_0 \), where \( 0 \) is the trivial representation. To see this, recall that \( \alpha \in \hat{\mathbb{Z}}^n \) gives rise to a representation \( \pi_\alpha : \mathbb{Z}^n \to \mathbb{C}^* \) given by:
\[ \pi_\alpha(m) = e^{i\langle \alpha, m \rangle} \quad m \in \mathbb{Z}^n. \]
A smooth section of \( \mathbb{R}^n \times \mathbb{Z}^n \mathbb{C}_\alpha \) is identified as a function on \( \mathbb{R}^n \) with values in \( \mathbb{R}^n \times \mathbb{C}_\alpha \), i.e., \( f \in \Gamma(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{C}_\alpha)^{\mathbb{Z}^n} \) which is \( \mathbb{Z}^n \)-invariant in the sense:
\[ f(x) = f(x + m) e^{i\langle \alpha, m \rangle} \quad x \in \mathbb{R}^n. \]
Then \( g(x) := e^{i\langle \alpha, x \rangle} f(x) \) is \( \mathbb{Z}^n \)-invariant:
\[ g(x) = g(x + m) \quad x \in \mathbb{R}^n, m \in \mathbb{Z}^n. \]
So \( g \) is a section of \( \mathbb{R}^n \times \mathbb{Z}^n \mathbb{C}_0 \) and hence \( e^{i\langle \alpha, x \rangle} D_\alpha f = Dg = De^{i\langle \alpha, x \rangle} f \). This follows from the fact that \( D_0 \) and \( D_\alpha \) are viewed as the same operator after identifying these spaces they act on by the unitary operator:
\[ e^{i\langle \alpha, x \rangle} : L^2(T^n, \mathbb{R}^n \times \mathbb{Z}^n \mathbb{C}_\alpha) \to L^2(T^n, \mathbb{R}^n \times \mathbb{Z}^n \mathbb{C}_0). \]
Thus,
\[ D_\alpha = e^{-i\langle \alpha, x \rangle} D e^{i\langle \alpha, x \rangle} = D + i\alpha. \]
The claim is proved.

Let \( A = (1 + D_0^2)^{-\frac{n}{2}} \in \mathcal{K}(\mathcal{E}) \) be the positive compact endomorphism on the Hilbert \( C_\alpha^* (\mathbb{Z}) \)-module. It is represented by:
\[ h = \{ P_\alpha \}_{\alpha \in \hat{\mathbb{Z}}^n} \]
where \( P_\alpha = (D_\alpha^2 + 1)^{-\frac{n}{2}} \) is a family of compact operators. Denote by \( E_n \) the 1-dimensional eigenspace of \( D \) associated to \( n \in \mathbb{Z}^n \). Then \( P_\alpha \) admits spectral decomposition:
\[ P_\alpha = \oplus_{n \in \mathbb{Z}^n} (||n + i\alpha||^2 + 1)^{-\frac{n}{2}} E_n. \]
For \( s \geq 0 \), we have:
\[ \chi(s, \infty)(A) = \int_{\alpha \in \hat{\mathbb{Z}}^n} \chi(s, \infty)(P_\alpha) d\alpha. \]
Note that the integrant \( \chi(s, \infty)(P_\alpha) \) is a finite rank projection, whose operator trace is equal to:
\[ \text{Tr}(\chi(s, \infty)(P_\alpha)) = \# \{ x \in \mathbb{Z}^n : (||x + i\alpha||^2 + 1)^{-\frac{n}{2}} > s \} \]
The von Neumann trace \( \text{tr} \) on the dual under Fourier transform is to integrate over \( \alpha \in \hat{\mathbb{Z}}^n : \)
\[ \text{tr}(\chi(s, \infty)(A)) = \int_{\alpha \in \hat{\mathbb{Z}}^n} \text{Tr}(\chi(s, \infty)(P_\alpha)) d\alpha. \]
By definition:
\[
\mu_t(A) = \inf \left\{ s \geq 0 : \int_{\alpha \in \hat{Z}^n} \# \left\{ m \in Z^n : (|m + i\alpha|^2 + 1)^{-\frac{1}{2}} > s \right\} d\alpha \leq t \right\}
\]
\[
= \inf \left\{ (|x|^2 + 1)^{-\frac{1}{2}} : \int_{\alpha \in \hat{Z}^n} \# \left\{ m \in Z^n : |m + i\alpha| \leq |x| \right\} d\alpha \leq t \right\}.
\]
Denote the integrant \( \# \{ m \in Z^n : |m + i\alpha| \leq |x| \} \) by \( N_{\alpha, |x|} \). Observe that:
\[
\int_{\alpha \in \hat{Z}^n} N_{\alpha, |x|} d\alpha \sim \text{Vol } B_{|x|} \text{ as } |x| \to \infty,
\]
where \( B_{|x|} \) is volume of the ball of radius \( |x| \) in \( \mathbb{R}^n \).
Thus we obtain:
\[
\mu_t(A) = \inf \left\{ (|x|^2 + 1)^{-\frac{1}{2}} : \text{Vol } B_{|x|} \leq t \right\}.
\]
Because \( \text{Vol } B_{|x|} = \frac{\Omega_n}{n} |x|^n \), and \( (|x|^2 + 1)^{-\frac{1}{2}} \sim |x|^{-n} \) when \( x \) is large, we have
\[
\mu_t(A) \sim \frac{1}{t} \frac{\Omega_n}{n}
\]
when \( t \) is large. The Dixmier \( \Gamma \)-trace is then equal to:
\[
\text{Tr}_{\Gamma}^t(A) = \lim_{t \to \infty} \frac{1}{\log(t+1)} \int_1^t \mu_s(A) = \frac{\Omega_n}{n} \lim_{t \to \infty} \frac{\log(t)}{\log(t+1)} = \frac{\Omega_n}{n}
\]
q.e.d.

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