Abstract. We construct a quasi-inverse of the cochain map on the negative cyclic complexes of the second kind induced from the quasi-Yoneda embedding on a curved dg algebra. This gives an explicit formula for the Chern character of a perfect module.

1. Introduction

1.1. Overview. Let \( k \) be a field and let \( G \) be either \( \mathbb{Z} \) or \( \mathbb{Z}/2 \). Curved differential \( G \)-graded \( k \)-algebras (for short, cdg algebras) and their modules appear in relation with deformations of dg algebras [5], Koszul duals of nonhomogeneous quadratic algebras [8], mirror duals of Fano varieties in mirror symmetry [9], and LG-models in LG/CY correspondence [13], etc.

Given a cdg algebra \( A := (A, d, h) \), its ordinary negative cyclic homology \( HN_* (A) \) (and its Hochschild homology \( HH_* (A) \)) vanishes if its curvature \( h \) is nonzero (see [2, 7]). Thus, its negative cyclic homology of the second kind \( HN^{II}_* (A) \) is used as a pseudo-equivalence (called “quasi-Morita”) invariant of cdg algebras. Moreover, if \( A \) is a cofibrant dg algebra \( (A, d, h = 0) \), then the natural comparison map \( HN_* (A) \to HN^{II}_* (A) \) is an isomorphism [7].

On the other hand, there is another “quasi-Morita” invariant of \( A \): the Grothendieck group \( K_0 (A) \) of the idempotent completion \( \text{Perf}(A) \) for the homotopy category \( \text{Perf}(A) \) of the dg category \( \text{Perf}(A) \) of perfect right \( A \)-modules.

Two invariants are naturally related by the Chern Character map with values in the negative cyclic homology of the second kind

\[
\text{Ch}^{II}_{HN} : K_0 (A) \to HN^{II}_0 (A).
\]

In this paper, we obtain a formula for \( \text{Ch}^{II}_{HN} (P) \in HN^{II}_0 (A) \) which manifests the various known results [1, 9, 11] as special cases and is extended to a global geometric version for smooth curved algebras via Chern-Weil theory in [3].

1.2. Main results. Let \( A = (A, d, h) \) be a cdg algebra. A right \( A \)-module \( P = (P, \delta_P) \) is perfect if and only if \( P \) is a direct summand of a twist \( N_\alpha := (N, d_F + \alpha) \) of shifted free right \( A \)-module \( N = \oplus_{i=1}^l A[n_i] \) with...
\[ \alpha \in \text{End}_{A}^{1}(N), \quad d_{F}(\alpha) + \alpha^{2} = \lambda_{-h}. \] Here \( \lambda_{-h} \) is the left multiplication by \( -h \).

Then, there are a non-unital dg algebra homomorphism \( F := i \circ (\ ) \circ j : \text{End}_{A}(P) \to \text{End}_{A}(N_{\alpha}) \), a cdg algebra isomorphism \((\text{id}_{P}, \alpha) : \text{End}_{A}(N_{\alpha}) \to \text{End}_{A}(N_{0})\) inducing a cochain map \((\text{id}_{P}, \alpha)_{\ast} \circ F_{\ast}(u) : \)

\[ \prod_{\mathcal{N}}(\text{End}_{A}(P)) \to \prod_{\mathcal{N}}(\text{End}_{A}(N_{\alpha})) \to \prod_{\mathcal{N}}(\text{End}_{A}(N_{0})) \]

and the Chern character of \( P \) is given by

\[ \text{Ch}_{\mathcal{N}}^{II}(P) = \text{Tr}((\text{id}_{P}, \alpha)_{\ast}(F_{\ast}(u)([1_{P}]))) \in \prod_{\mathcal{N}}^{II}(A) \]

where \( \text{Tr} : \prod_{\mathcal{N}}^{II}(\text{End}_{A}(N_{0})) \to \prod_{\mathcal{N}}^{II}(A) \) is the generalized trace map; see § 2.4.1 for the definition.

More concretely, in § 2.4 we show that for a perfect \( A \)-module \( P \) given by an idempotent \( \pi \) on \( N_{\alpha} \), \( \text{Ch}_{\mathcal{N}}^{II}(P) \) is represented by a cocycle in the normalized negative cyclic complex of the second kind \((\mathcal{C}^{II}(A)[[u]], \bar{b} + u\mathcal{B}):\)

\[ (1.1) \]

\[ \sum_{j=0}^{\infty}(-1)^{j}\text{Tr}(\pi[\alpha^{j}]) + \sum_{n \geq 1} \sum_{(j_{0}, \ldots, j_{2n}) \in \mathbb{Z}^{2n+1}_{\geq 0}} c_{n,J} \text{Tr}((2\pi - 1_{N})(\alpha^{j_{0}}|\pi|\alpha^{j_{1}}|\pi| \cdots |\pi|\alpha^{j_{2n}}))u^{n} \]

where \( J = \sum_{k=0}^{2n} j_{k} \), \( c_{n,J} := (-1)^{n+J} \frac{(2n)!}{2n!} \), \( \alpha^{r} = \underbrace{\alpha|\cdots|\alpha}_{r} \) for \( r = j, j_{0}, \ldots, j_{2n} \).

Moreover, if \( h = 0 \) and \( \mathbb{G} = \mathbb{Z} \), then the Chern character \( \text{Ch}_{\mathcal{N}}^{II}(P) \) of the first kind is represented by the same formula above but with a finite sum. In § 2.4.3, we briefly mention a Chern character formula for a homotopy direct summand of \( N_{\alpha} \).

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1.4. Conventions and notation. This paper is a sequel of [3]. Unless otherwise stated, we freely use the conventions from [3] and the definitions and notation from Section 2 of [3], in particular for the category of mixed complexes, the (normalized) Hochschild chain complexes (of the second kind), the (normalized) Hochschild homology (of the second kind), the (normalized) mixed Hochschild chain complexes (of the second kind), the (normalized) negative cyclic chain complexes (of the second kind), the (normalized) negative cyclic homology (of the second kind) etc. We refer to [10] for similar definitions and properties for \( A_{\infty} \)-categories.

2. Categorical Chern characters

2.1. Definitions of Chern characters.
2.1.1. Given a small cdg category $\mathcal{A}$, let $q\text{Perf}(\mathcal{A})$ be the closure of $\mathcal{A}$ in the cdg category $q\text{Mod}(\mathcal{A})$ of right quasi-modules over $\mathcal{A}$ under shift, direct sum, twist, and passage to a direct summand. The objects of $q\text{Perf}(\mathcal{A})$ are called perfect right quasi-modules over $\mathcal{A}$. Let $\text{Perf}(\mathcal{A})$ be the full subcategory of $q\text{Perf}(\mathcal{A})$ consisting of all right $\mathcal{A}$-modules, called perfect right $\mathcal{A}$-modules.

2.1.2. The homotopy category $[\text{Perf}(\mathcal{A})]$ of $\text{Perf}(\mathcal{A})$ is naturally triangulated. Let $K_0([\text{Perf}(\mathcal{A})])$ be the Grothendieck group of the triangulated category $[\text{Perf}(\mathcal{A})]$. By the tautological assignments $P \mapsto [1_P]$, we obtain a homomorphism $\text{taut}_{II} : K_0([\text{Perf}(\mathcal{A})]) \to \mathcal{H}N^I_0(\text{Perf}(\mathcal{A}))$. Together with induced maps from natural embeddings, $\text{taut}_{II}$ makes the following diagram define Chern character maps (see [I]):

$$
\begin{array}{ccc}
K_0([\text{Perf}(\mathcal{A})]) & \xrightarrow{\text{taut}_{II}} & \mathcal{H}N^I_0(\text{Perf}(\mathcal{A})) \\
\downarrow & & \downarrow \\
\mathcal{H}N_0(\text{Perf}(\mathcal{A})) & \xrightarrow{\cong} & \mathcal{H}N^I_0(\text{Perf}(\mathcal{A})) \\
\downarrow & & \downarrow \\
\text{Ch}_{\mathcal{H}N}^{II} & \longrightarrow & \text{Ch}_{\mathcal{H}N}^{II}
\end{array}
$$

where the dotted arrows make sense when $\mathcal{A}$ is a dg category.

2.2. Hochschild morphism for a semifunctor. For an arbitrary non-unital functor $F : \mathcal{A} \to \mathcal{B}$ between dg categories, there is a functorial cochain map $F_* : (C(\mathcal{A}), b) \to (C(\mathcal{B}), b)$. We will construct its extension

$$
F_*(u) : (C(\mathcal{A})[[u]], b + uB) \to (C(\mathcal{B})[[u]], b + uB)
$$

by letting $F_*(u) := p(u) \circ F_*^e \circ \iota(u)$; see §2.2.1 for the definition of $F_*^e$ and §2.2.2 for the definitions of $\iota(u), p(u)$.

2.2.1. In §2.2.1 we allow that $\mathcal{A}$ is a dg category possibly without identities. Denote the dg category obtained from $\mathcal{A}$ by adjoining identities by $\mathcal{A}^+$; its objects are the same objects as $\mathcal{A}$ and its morphisms are defined by

$$
\mathcal{A}^+(x, y) = \mathcal{A}(x, y) \text{ if } x \neq y, \quad \mathcal{A}^+(x, x) = \mathcal{A}(x, x) \oplus k \cdot e_x, |e_x| = 0.
$$

The composition on $\mathcal{A}^+$ is defined in the obvious way. Consider the subcomplex $D$ of $(C(\mathcal{A}^+), b^+)$ generated by $e_x$ and $a_0[a_1] \cdots [a_n]$ with $a_i = e_x$ for some $i > 0$ and any $x \in \mathcal{A}$. We denote $(C(\mathcal{A}^+)/D, b^e)$ by $(C^e(\mathcal{A}), b^e)$ and call it the non-unital Hochschild complex of $\mathcal{A}$. Let $\text{cod}(a_i)$ denote the codomain of $a_i$ and define Connes’ differential $B^e : C^e(\mathcal{A}) \to C^e(\mathcal{A})$ by

$$
B^e(a_0[a_1] \cdots [a_n]) := e_{\text{cod}(a_0)}[a_0[a_1] \cdots [a_n]]
$$

so that $MC^e(\mathcal{A}) := (C^e(\mathcal{A}), b^e, B^e)$ is a mixed complex and called the the non-unital Hochschild mixed complex of $\mathcal{A}$.
Remark 2.1. The following ones hold; see [10] § 3.5.

(i) The non-unital Hochschild complex of $\mathcal{A}$ is, by definition, the reduced Hochschild complex of $\mathcal{A}^+$; see [6] § 1.4.2.

(ii) For a dg category $\mathcal{A}$ with identities, the composition of the natural maps

$$j : (C(\mathcal{A}), b) \hookrightarrow (C(\mathcal{A}^+), b^+) \rightarrow (C^e(\mathcal{A}), b^e)$$

is a quasi-isomorphism.

(iii) Any semifunctor $F : \mathcal{A} \rightarrow \mathcal{B}$ induces a canonical dg functor

$$F^+ : \mathcal{A}^+ \rightarrow \mathcal{B}^+$$

and a morphism of mixed complexes

$$F^e : (C^e(\mathcal{A}), b^e, B^e) \rightarrow (C^e(\mathcal{B}), b^e, B^e)$$

compatible with the quasi-isomorphisms in item (ii).

(iv) The construction of non-unital Hochschild complex is functorial; for semifunctors $F : \mathcal{A} \rightarrow \mathcal{B}, G : \mathcal{B} \rightarrow \mathcal{C}$,

$$(G \circ F)^e = (G^e \circ F^e).$$

2.2.2. Let $\mathcal{A}$ be a dg category with identities as usual. The map $j : (C(\mathcal{A}), b) \rightarrow (C^e(\mathcal{A}), b^e)$ in (2.1) is not compatible with Connes’ operators, but we can still construct homotopy inverses between $(C(\mathcal{A})[[u]], b+uB)$ and $(C^e(\mathcal{A})[[u]], b^e+uB^e)$ as $k[[u]]$-modules which extends $j$. This construction was done by Shklyarov (see [12]) as follows:

As graded $k$-modules

$$C^e(\mathcal{A}) = C(\mathcal{A}) \oplus C^+(\mathcal{A})$$

where $C^+(\mathcal{A}) = \{ a' \in C^e(\mathcal{A}) | a' = e_{\text{cod}(a_1)}[a_1] \cdots [a_n], n \geq 1 \}$ and $j(a) = (a, 0)$. Consider the following $k$-linear maps:

$$\iota(u) : (C(\mathcal{A})[[u]], b+uB) \rightarrow (C^e(\mathcal{A})[[u]], b^e+uB^e)$$

$$a \mapsto a + us^esN(a)$$

and

$$p(u) : (C^e(\mathcal{A})[[u]], b^e+uB^e) \rightarrow (C(\mathcal{A})[[u]], b+uB)$$

$$(a, a') \mapsto a + (1 - t^{-1})\mu(a'),$$

where $\mu : C^+(\mathcal{A}) \rightarrow C(\mathcal{A})$; $e_{\text{cod}(a_1)}[a_1] \cdots [a_n] \mapsto 1_{\text{cod}(a_1)}[a_1] \cdots [a_n]$.

For a dg algebra $A$, $\iota(u), p(u)$ are shown to be homotopy inverses over $k((u))$ in [12] § 3.2 and the same proof for a dg category $\mathcal{A}$ works over $k[[u]]$ as follows:

Remark 2.2.

(i) $\iota(u), p(u)$ are $k[[u]]$-linear morphisms.
(ii) \( \iota(u), p(u) \) are homotopy inverses as \( k[[u]] \)-morphisms: \( p(u) \circ \iota(u) \) is homotopic to \( \text{id}_{C(A)[[u]]} \) with a homotopy \( H(u) := u(1 - t^{-1}) \) and \( \iota(u) \circ p(u) \) is homotopic to \( \text{id}_{C^e(A)[[u]]} \) with a homotopy \( H^e(u) : (a, a') \mapsto (0, s^e \mu(a')) \).

2.3. A lemma. Suppose that \( P \) is a direct summand of \( N \) in a dg category \( \mathcal{A} \). In other words, there are degree 0 closed homomorphisms \( i : P \to N \) and \( j : N \to P \) such that \( 1_P = j \circ i \) and denote \( \pi := i \circ j \). Then, we have a non-unital functor \( F : \{P, N\} \to \{N\} \) of full dg subcategories of \( \mathcal{A} \) defined by

\[
\begin{align*}
\text{for } f & \in \text{End}P, f' & \in \text{Hom}(N, P), f'' & \in \text{Hom}(P, N), g & \in \text{End}(N), \\
& \gamma_P := 1_P + \sum_{i=1}^{\infty} (-1)^i \frac{(2i)!}{2(4i)!} 2 \cdot 1_P \left[ \cdots \left[ 1_P \right] \cdots \right] \gamma^i; \text{ and } \\
& \eta_\pi := \pi + \sum_{i=1}^{\infty} (-1)^i \frac{(2i)!}{2(4i)!} (2 \pi - 1_N) \left[ \cdots \left[ \pi \right] \cdots \right] \gamma^i \\
& = \pi - (2 \pi - 1_N) \left[ \pi \right] + 6 (2 \pi - 1_N) \left[ \pi \right] u^2 + \cdots.
\end{align*}
\]

**Lemma 2.3.** There is a cochain map \( F_* : C\{P, N\}[[u]] \to C\{N\}[[u]] \) between the negative cyclic complexes such that:

1. The composition \( F_* \circ \text{inc}_* \) is the identity in the homology level \( H N_* \{N\} \to H N_* \{N\} \).
2. When \( u = 0 \), it is \( F_* \), i.e., \( F_*|_{u=0} = F_* \).
3. Let \( \text{quot} : C\{N\}[[u]] \to C\{N\}[[u]] \) denote the quotient map. Then, \( \text{quot} \circ F_* (\gamma_P) = \eta_\pi. \)

In particular \( \left| 1_P \right| = \left| \eta_\pi \right| \) in \( H N_* \{P, N\} \).

**Proof.** Since the semifunctor \( F \) above does not preserve the unit \( 1_P \), we need a new idea. There are so-called non-unital mixed Hochschild complex \( (C^e(A), b^e, B^e) \) and cochain maps

\[
\begin{align*}
\iota(u) &: (C(A)[[u]], b + uB) \to (C^e(A)[[u]], b^e + uB^e); \text{ and } \\
p(u) &: (C^e(A)[[u]], b^e + uB^e) \to (C(A)[[u]], b + uB),
\end{align*}
\]

which are homotopy inverses to each other; see Remark 2.2. The semifunctor \( F \) induces a cochain map

\[
F_*^e : (C^e\{P, N\}[[u]], b^e + uB^e) \to (C^e\{N\}[[u]], b^e + uB^e)
\]

as explained in Remark 2.1(iii) such that \( F_*^e|_{C^e\{N\}[[u]]} = \text{id}_{C^e\{N\}[[u]]} \). Let \( F_*^e(u) := p(u) \circ F_*^e \circ \iota(u) \). Then (1) and (2) are clear. Item (3) is a straightforward computation from the definitions of \( \iota(u) \) and \( p(u) \). \( \square \)
Corollary 2.4. Two cycles $1_P$ and $\eta_\pi$ are homologous in $(\overline{C}\{P,N\}[[u]], b + uB)$.

We note that Corollary 2.4 is proved in [3] by a very different method.

2.4. A Chern character formula. Let $\mathcal{A} = (A, d, h)$ be a cdg algebra and let $\mathcal{P} = (P, \delta_P) \in \text{Perf}(\mathcal{A})$. Recall that there is a canonical isomorphism

\begin{equation}
HN^*_H(\text{Perf}(\mathcal{A})) \cong HN^*_H(A)
\end{equation}

induced by the embeddings $\mathcal{A} \to q\text{Perf}(\mathcal{A})$ and $\text{Perf}(\mathcal{A}) \to q\text{Perf}(\mathcal{A})$. In this subsection we apply Corollary 2.4 to get a formula for $\text{Ch}_{H^*}^{H}(P)$, which works as well as for $\text{Ch}_{H^*}(P)$ if $h = 0$ and $\mathbb{G} = \mathbb{Z}$.

Let $\mathcal{N}_0 := (N, d_F)$ denote a finitely generated free cdg module over $\mathcal{A} = (A, d, h)$, i.e., a finite sum of $A$’s up to degree shifts with the induced differential $d_F$ from the differential $d$ of $A$. We call $\mathcal{N}_0$ a \textit{finitely generated free quasi-module} over $\mathcal{A}$.

Let $P$ be a direct summand of twisted $A$-module $N_\alpha := (N, d_F + \alpha)$ of $\mathcal{N}_0$ with $\alpha \in \text{End}^1_A(N)$. Note that $d_F(\alpha) + \alpha^2 = \lambda - h$ where $\lambda - h$ is the left multiplication by $-h$. The module $\mathcal{N}_0$ is called a \textit{finitely generated semi-free module} over $\mathcal{A}$ and note that any perfect $\mathcal{A}$-module $P$ is represented as a direct summand of a finitely generated semi-$\mathcal{A}$-module. Consider the cdg isomorphism

\((\text{id}_N, \alpha) : \{N_\alpha\} \to \{N_0\}\)

between full cdg subcategories of $q\text{Perf}(\mathcal{A})$ consisting of the indicated object only. For example, $\{N_\alpha\}$ is the dg algebra $\text{End}_A(N_\alpha) := (\text{End}_A(N), d_{N_\alpha} = [d_F + \alpha, \cdot])$. Then we get a Chern character formula for $P \in \text{Perf}(\mathcal{A})$ as in the following proposition.

Proposition 2.5. Let $P \in \text{Perf}(A, d, h)$ be a direct summand of $N_\alpha = (N, d_F + \alpha) \text{ given by a closed idempotent } \pi : N \to N$.

1. Under \(2.3\), $\text{Ch}_{H^*}^{H}(\mathcal{P})$ is representable by a cocycle

\[\text{Tr}((\text{id}_N, \alpha)_*(\eta_\pi)) \in \overline{C}^I(A, d, h)[[u]].\]

Here Tr is the $k[[u]]$-linear extension of the generalized trace map

\begin{equation}
\overline{C}^I(N_0) \to \overline{C}^I(A, d, h)
\end{equation}

defined in §2.4.

2. Suppose that $\mathbb{G} = \mathbb{Z}$ and $h = 0$. We may write $N = \oplus_{i=1}^l A[n_i]$ for some integers $n_i$, $l \geq 0$ for which $\alpha : N \to N$ is a strictly upper triangular $l \times l$-matrix of entries $\alpha_{ij} \in \text{Hom}^1_A(A[n_j], A[n_i])$.

Then under the canonical isomorphism $HN_*(\text{Perf}(A, d)) \cong HN_*(A, d)$, $\text{Ch}_{H^*}(\mathcal{P})$ is representable by a cocycle

\[\text{Tr}((\text{id}_N, \alpha)_*(\eta_\pi)) \in \overline{C}(A, d)[[u]].\]
Proof. In this proof we write simply $\mathcal{A}$ for the right quasi-module $(A, d)$ over $(A, d, h)$. (1) The cdg functor $(\text{id}_N, \alpha)$ can be extendable to a cdg functor $(\text{id}_N, \alpha) : \{N_\alpha, N_0, \mathcal{A}\} \to \{N_0, \mathcal{A}\}$ which is a left inverse of the inclusion $\{N_0, \mathcal{A}\} \to \{N_\alpha, N_0, \mathcal{A}\}$. Hence for every $x \in C^II\{N_0\}[[u]]$ we note that $x$ and $(\text{id}_N, \alpha)_*(x)$ are homologous in $\overline{C}^II\{N_\alpha, N_0, \mathcal{A}\}[[u]]$. The map $\text{Tr}$ in (2.4) can be extendable to a morphism $\widetilde{\text{Tr}} : \overline{\text{MC}}^II\{N_0, \mathcal{A}\} \to \overline{\text{MC}}^II(\mathcal{A})$ which is a left inverse of the inclusion morphism of mixed complexes; see §2.4.1 for the extension. Again note that $z$ and $\text{Tr}(z)$ are homologous in $\overline{C}^II\{N_0, \mathcal{A}\}[[u]]$ for any $z \in \overline{C}^II\{N_0\}[[u]]$. Now by Corollary 2.4 and letting $x = \eta_\pi$ and $z = (\text{id}_N, \alpha)_*(\eta_\pi)$ we conclude the proof.

(2) We have shown that $\widetilde{\text{Tr}} \circ (\text{id}_N, \alpha)_*$ is a left inverse of the inclusion $\overline{\text{MC}}^II(\mathcal{A}) \to \overline{\text{MC}}^II\{N_\alpha, N_0, \mathcal{A}\}$. Since $\alpha$ is strictly upper triangular, we can apply (2.5). Therefore $\widetilde{\text{Tr}} \circ (\text{id}_N, \alpha)_*$ restricted to $\overline{\text{MC}}\{N_\alpha, N_0, \mathcal{A}\}$ lands in $\overline{\text{MC}}(\mathcal{A})$, which establishes the proof. □

2.4.1. Let $L$ be a graded $k$-module. For example $L$ is a finite sum of shifted $A$’s. For positive integers $m_1, m_2, n$, let $\text{Mat}_{m_1 \times m_2}(L^\otimes n)$ denote the $k$-module of all the $m_1 \times m_2$ matrices with entries in $L^\otimes n$. There is the matrix multiplication

$\bullet : \text{Mat}_{m_1 \times m_2}(L^\otimes k^{n_1}) \otimes_k \text{Mat}_{m_2 \times m_3}(L^\otimes k^{n_2}) \to \text{Mat}_{m_1 \times m_3}(L^\otimes k^{n_1+n_2})$

$$(a_{ij}) \otimes (b_{ij}) \mapsto (\sum_j a_{ij} \otimes b_{jk})$$

by the tensor algebra of $L$. If $\text{tr} : \text{Mat}_{m \times m}(L^\otimes k^n) \to L^\otimes k^n$ denotes the supertrace map, then for $\phi_i \in \text{Mat}_{m_i \times m_{i+1}}(L)$ $i = 0, \ldots, n$, $m_{n+1} = m_0$, we define the generalized trace map $\text{Tr}$ by letting

$$\text{Tr}(\phi_0 \otimes \phi_1 \otimes \ldots \otimes \phi_n) := \text{tr}(\phi_0 \bullet \phi_1 \bullet \ldots \bullet \phi_n).$$

This is the graded version of the generalized trace map of [6 § 1.2.1]. Note that for $m \geq m_i$

$$(2.5) \quad \text{Tr}(\phi_0 \otimes \ldots \otimes \phi_{i-1} \otimes \phi^\otimes m \otimes \phi_i \otimes \ldots \otimes \phi_n) = 0$$

if $\phi$ is a strictly upper triangular square matrix in $\text{Mat}_{m_i \times m_i}(L)$.

2.4.2. Let $\mathcal{A} = (A, d, h)$ be a cdg algebra. Let $qF(\mathcal{A})$ denote the full subcategory of $q\text{Perf}\mathcal{A}$ consisting of finitely generated free quasi-modules and let $sF(\mathcal{A})$ denote the full subcategory of $\text{Perf}\mathcal{A}$ consisting of finitely generated semi-free modules. The proofs of Lemma 2.3 and Proposition 2.5 show there are quasi-isomorphisms.
All inclusions are induced from the embeddings. The cochain maps \( \{(id,\alpha)\_s\} \) and \( \{(id,0)\_s, (id,\alpha)\_s\} \) are induced from cdg functors. The right bottom map \( F\_s(u) \) is induced from a semifunctor and non-unital mixed Hochschild complexes. All maps except the right bottom map are induced from morphisms between mixed complexes. The cochain maps \( Tr \) and \( \{(id,0)\_s, (id,\alpha)\_s\} \) are left inverses of the corresponding inclusions, respectively. Diagram (2.6) commutes in the homology level. Therefore the composition \( Tr \circ \{(id,0)\_s\} \circ quot \circ F\_s(u) \) fits in a commutative diagram

(2.7) \( HN^II(A) \xrightarrow{\text{canon}} HN^II(q\text{Perf}A) \xrightarrow{\text{canon}} HN^II(\text{Perf}A). \)

Thus for \( \mathcal{P} \) in Proposition 2.5, \( Ch^II_HN(\mathcal{P}) \) is represented by a cocycle in \( C^II(A)[[u]] \):

(2.8) \[
\sum_{j=0}^{\infty} (-1)^j Tr(\pi_\alpha^j) + \sum_{n \geq 1} \sum_{(j_0,...,j_{2n}) \in \mathbb{Z}^{2n+1}} c_{n,J} Tr((2\pi - 1)_N(\pi^j|\pi^j|\pi^j|\cdots|\pi^j|2^{2n}))u^n
\]

where \( J = \sum_{k=0}^{2n} j_k, \ c_{n,J} := (-1)^{n+J} \frac{2^n}{(2n)!} \) and \( \alpha^r = \alpha_{|...|\alpha} \) for \( r = j, j_0, ..., j_{2n} \).

This proves formula (5.24) of [1]. When \( u \) is specialized to 0, this recovers [9, Theorem 2.14].

When \( G = \mathbb{Z} \) and \( h = 0 \), the composition \( Tr \circ \{(id,\alpha)_s\} \circ quot \circ F\_s(u) \) restricted to \( C(\text{Perf}A)[[u]] \) is a quasi-inverse of the cochain map \( C(A)[[u]] \to C(\text{Perf}A)[[u]] \) induced from the Yoneda embedding \( A \to \text{Perf}A \). Therefore diagram (2.7) without the superscripts \( II \) makes sense and commutes. Hence for \( \mathcal{P} \) in Proposition 2.5 (2), \( Ch^II_HN(\mathcal{P}) \) is represented by a cocycle in \( C(A)[[u]] \):

(2.6) \[
\xymatrix{ & C^II(qF(A),sF(A))[u] \ar[rr]|{\{(id,0)\_s, (id,\alpha)\_s\}} \ar[dd]|{\{id,\alpha\}_s} & & C^II(q\text{Perf}A)[u] \\
C^II(qF(A))[u] \ar[rr]|{\{id,\alpha\}_s} & & C^II(sF(A))[u] \ar[rr]|{quot} & & C^II(\text{Perf}A)[u] \\
C^II(A)[u] \ar[rr]|{quot} & & C^II(sF(A))[u] \ar[rr]|{F\_s(u)} & & C^II(\text{Perf}A)[u].}
\]
\[
\sum_{j=0}^{l-1} (-1)^j \text{Tr}(\pi[\alpha^j]) + \sum_{n \geq 1, 0 \leq j_0, \ldots, j_{2n} \leq l-1} c_{n,j} \text{Tr}((2\pi - 1)N)[\alpha^{j_0}\pi[\alpha^{j_1}]\pi[\alpha^{j_2}]\cdots\pi[\alpha^{j_{2n}}])u^n.
\]

When \( u \) is specialized to 0, this recovers [11, Theorem 1.1].

2.4.3. Let \( Q = (Q, \delta_Q) \in \text{Perf}(\mathcal{A}) \) be represented by a homotopy direct summand of a finitely generated semi-free \( \mathcal{A} \)-module \( N_\alpha = (N, d_N + \alpha) \). Then, there are closed \( \mathcal{A} \)-module homomorphisms \( i : Q \to N, j : N \to Q \) such that \( j \circ i = \text{id}_Q + [\delta_Q, H_1], H_1 \in \text{End}^{-1}_{\mathcal{A}}(Q) \), \( i \circ j = \pi + [d_N + \alpha, H_2], H_2 \in \text{End}^{-1}_{\mathcal{A}}(N) \).

One can check that the following family \( F := \{F_n\}_{n \geq 1} : \{N_\alpha, Q\} \to \{N_\alpha\} \) defines a non-unital \( A_\infty \)-functor. For \( f_i \in \text{End}_\mathcal{A}(Q) \) with \( \text{cod}(f_1) = \text{dom}(f_n) = Q \),

\[
F_n(f_1 \otimes \cdots \otimes f_n) := i \circ f_1 \circ H_{i_1} \circ f_2 \circ H_{i_2} \circ \cdots \circ H_{i_{n-1}} \circ f_n \circ j
\]

where \( H_{i_j} \) is \( H_1 \) or \( H_2 \) depending on \( f_{i_j} \). For others, it is defined in the obvious way via (2.2). This extends to a unital \( A_\infty \)-functor \( F^e = \{F^e_n\}_{n \geq 1} : \{N_\alpha, Q\} \to \{N_\alpha\} \) and with this \( F \), Lemma [2.3] is true (see [10] for the definitions of \( A_\infty \)-category and \( A_\infty \)-functor, and for similar above facts about \( A_\infty \)-category). Thus,

\[
\text{Ch}^{II}_{HN}(Q) = \text{Tr}((\text{id}, \alpha)_*(F^e(u)(\gamma_Q)))
\]

where \( F^e(u) := p(u) \circ F^e \circ i(u) \).

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