AN APPLICATION OF THE SECOND Riemann Continuation Theorem TO COHOMOLOGY OF THE LIE ALGEBRA OF VECTOR FIELDS ON THE COMPLEX LINE.

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August 21, 2005

Abstract. We study cohomology groups of the Lie algebra of vector fields on the complex line, $W_1$, with values in the tensor fields in several variables. From a generalization by Scheja of the second Riemann (Hartogs) continuation theorem, we deduce a cohomology exact sequence of the subalgebra of $W_1$ consisting of vectors having a zero at the origin. As applications, we compute the cohomology algebra of $W_1$ with values in the functions on $\mathbb{C}^n$ explicitly, and establish a certain vanishing theorem for the cohomology of $W_1$ with values in the quadratic differentials in several variables, which is closely related to the moduli space of Riemann surfaces.

Introduction.

The topological Lie algebra of complex analytic vector fields on an open Riemann surface $O$, $L(O)$, acts continuously on the space of complex analytic tensor fields on the product space $O^n$ by the diagonal Lie derivative action. By a complex analytic analogue [Ka1] of the Bott-Segal addition theorem [BS], the computation of the (continuous) cohomology group of $L(O)$ with values in the space of tensor fields on $O^n$ was reduced to that for the case where $O$ is the complex line $\mathbb{C}$ together with a topological study of certain sheaves on the space $O^n$. The purpose of the present paper is to provide a geometric tool for the computation for this case $O = \mathbb{C}$.

We fix our notations. Let $W_1 := L(\mathbb{C}) = H^0(\mathbb{C}; \mathcal{O}_\mathbb{C}(T\mathbb{C}))$ be the Lie algebra of complex analytic vector fields on the complex line $\mathbb{C}$ with the topology of uniform convergence on compact sets. The closed subalgebra $L_0 := \{X \in W_1; X(0) = 0\}$

1991 Mathematics Subject Classification. Primary 58H10. Secondary 14H15, 17B56, 17B68, 32G15, 57R32.
of $W_1$ often plays more important roles than $W_1$ itself. We recall two kinds of $L_0$ modules.

(1) Let $\nu \in \mathbb{Z}$. The Lie algebra $L_0$ acts on the 1 dimensional complex vector space $1_\nu = \mathbb{C}1_\nu$ with the preferred base $1_\nu$ by

$$(\xi(z)\frac{d}{dz}) \cdot 1_\nu = \nu\xi'(0)1_\nu \quad (\xi(z)\frac{d}{dz} \in L_0).$$

The $L_0$ module $1_\nu$ is naturally isomorphic to the $\nu$-cotangent space $(T_0^*\mathbb{C})^{\otimes \nu}$ at the origin.

(2) For $\nu$ and $\lambda \in \mathbb{Z}$ we denote by $T^\lambda_\nu$ the Fréchet space of the meromorphic $\nu$-covariant tensor fields on $\mathbb{C}$ with a pole only at the origin of order $\leq \lambda$, $T^\lambda_\nu = H^0(\mathbb{C};\mathcal{O}_\mathbb{C}((T^*\mathbb{C})^{\otimes \nu} \otimes [0]^{\otimes \lambda}))$. Here $[0]$ is the line bundle induced by the divisor $0 \in \mathbb{C}$. The algebra $L_0$ acts on $T^\lambda_\nu$ by the Lie derivative

$$(\xi(z)\frac{d}{dz}) \cdot (f(z)dz^{\nu}) = (\xi(z)f'(z) + \nu\xi'(z)f(z))dz^{\nu},$$

where $\xi(z)\frac{d}{dz} \in L_0$ and $f(z)dz^{\nu} \in T^\lambda_\nu$.

When $\lambda = 0$, $T^0_0$ is a $W_1$ module, which is denoted by $T_\nu$ for simplicity. The $W_1$ module of $\nu$-covariant tensor fields on $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is denoted by $T^\infty_\nu$, i.e., $T^\infty_\nu := H^0(\mathbb{C} - \{0\};\mathcal{O}_\mathbb{C}((T^*\mathbb{C})^{\otimes \nu}))$.

Our purpose is to calculate the cohomology groups

$$H^\ast(W_1; \bigotimes_{i=1}^n T_{\nu_i}) \quad \text{and} \quad H^\ast(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\lambda_{\nu_i}),$$

for arbitrary integers $\nu_0, \nu_1, \ldots, \nu_n$ and $\lambda_1, \ldots, \lambda_n$. Here and throughout this paper $\otimes$ means the completed tensor product over the complex numbers $\mathbb{C}$. The Shapiro Lemma (Remark 2.4) implies the isomorphism $H^\ast(W_1; \bigotimes_{i=1}^n T_{\nu_i}) \cong H^\ast(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\lambda_{\nu_i})$, so that the calculation of the former is reduced to that of the latter.

First of all the cohomology group $H^\ast(W_1; T_\nu) \cong H^\ast(L_0; 1_\nu)$ was determined by Goncharova [Go]. Studying a filtration of the space $1_{\nu_0} \otimes T^\lambda_{\nu_1}$ derived from the action of the vector $e_0 := z\frac{d}{dz} \in L_0$ in detail, Feigin and Fuks [FF] determined the cohomology group $H^\ast(L_0; 1_{\nu_0} \otimes T^\lambda_{\nu_1})$ completely. In the present paper we study the cohomology group $H^\ast(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\lambda_{\nu_i})$ for the case $n \geq 2$ in a geometric way. Using a generalization by Scheja [S] of the second Riemann (Hartogs) continuation theorem, we introduce a cohomology exact sequence (3.2)

$$\cdots \to H^{q-n}(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T^\nu_{\nu_i}/T^\lambda_{\nu_i})) \overset{d_q}{\to} H^q(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\lambda_{\nu_i}) \to H^q_{L_0}(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(T^\lambda_{\nu_1}; \ldots, \nu_n)) \to \cdots,$$

which we call the fundamental exact sequence for the $L_0$ module $1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\lambda_{\nu_i}$. The $e_0$-invariants $C^\ast(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T^\nu_{\nu_i}/T^\lambda_{\nu_i}))$ is of finite dimension, and so is the first term. The third term is the $L_0$ equivariant cohomology of the space $\mathbb{C}^n - \{0\}$ with values in the sheaf of $L_0$ modules $1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(T^\lambda_{\nu_1}; \ldots, \nu_n)$ (§2). This term is reduced to the cohomology of $L_0$ with values in the tensor fields in fewer variables.
by the main result of our previous paper [Ka1] Theorem 5.3. In §1 we give an exposition of the equivariant cohomology theory for a Lie algebra.

Three results are deduced from the sequence (3.2).

(1) The cohomology of $L_0^{\text{alg.}} := z\mathbb{C}[z]\frac{dz}{dz}$ with values in the space of algebraic tensor fields on the complex torus $(\mathbb{C} - \{0\})^n$ coincides with that of $L_0$ with values in the space of complex analytic tensor fields of the same type on the torus (Corollary 3.5).

(2) The cohomology of $W_1$ with values in the algebra of complex analytic functions on $\mathbb{C}^n$, or equivalently, that of $W_1^{\text{alg.}} = \mathbb{C}[z]\frac{dz}{dz}$ with values in the algebra of polynomials in $n$ variables, $\mathbb{C}[z_1, \ldots, z_n]$, is determined explicitly (Corollary 3.7).

(3) A vanishing theorem (Theorem 4.7) on the quadratic differentials in $p$ variables:

$$H^q(W_1; \bigwedge^p Q) = 0, \quad \text{if } p > q,$$

where $Q := T_2$ and $\bigwedge^p$ denotes the completed $p$-fold alternating tensor product on the complex numbers $\mathbb{C}$. This suggests that the Virasoro equivariant $(p, q)$ cohomology of the dressed moduli of compact Riemann surfaces would vanish for $p > q$ [Ka2].

The author would like to express his own gratitude to Yukio Matsumoto for constant encouragement and to Shigeyuki Morita, Kazushi Ahara and Masanori Kobayashi for helpful discussions.

This paper is a revised version of an unpublished preprint written in 1993 (University of Tokyo, UTMS 93-18).

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1. Equivariant cohomology.
We begin by an exposition on the equivariant cohomology for a Lie algebra. Let \( g \) be a complex topological Lie algebra. A \( g \)-module means a complex topological vector space on which \( g \) acts continuously. The standard continuous cochain complex of the topological Lie algebra \( g \) with values in a \( g \)-module \( N \) is denoted by
\[
C^\ast(g; N) = \bigoplus_{p \geq 0} C^p(g; N).
\]
Here \( C^p(g; N) \) is the linear space of continuous alternating multi-linear mappings
\[
c : g \times \cdots \times g \to N.
\]
They can be regarded as continuous linear maps of the completed alternating tensor product \( \bigwedge^p g \) into \( N \). Hence we may identify \( C^p(g; N) \) with \( \text{Hom}(\bigwedge^p g, N) \). The cohomology group of the complex \( C^\ast(g; N) \) is called the (continuous) cohomology group of \( g \) with values in \( N \) and denoted by \( H^\ast(g; N) \).

When \( N \) is the trivial \( g \)-module \( \mathbb{C} \), we abbreviate them to \( C^\ast(g) \) and \( H^\ast(g) \) respectively. For details, see, for example, [HS].

If \( F \) is a sheaf of \( g \)-modules over a topological space \( M \), the cochain complex of sheaves over \( M \)
\[
C^\ast(g; F) : U \subset M \mapsto C^\ast(g; F(U))
\]
is defined. We denote by \( H^\ast_g(M; F) \) the hypercohomology group of the cochain complex of sheaves over \( M \) with respect to the functor \( \Gamma(M; \cdot) \) (= the sections of the sheaf \( \cdot \) over \( M \)) ([G] ch.0, §11.4, pp.32-) and call it the \( g \) equivariant cohomology group of \( M \) with values in the sheaf \( F \). Equivalently we define
\[
H^\ast_g(M; F) := H^\ast(\text{Total}(\Gamma(M; C^\ast_*)))
\]
for an injective right Cartan-Eilenberg resolution \( C^* = (C^{i,j})_{i,j \geq 0} \) of the complex \( C^\ast(g; F) \) (cf. ibid.loc.cit.). Needless to say this cohomology is an analogue of the equivariant cohomology of a space \( M \) acted on by a transformation group \( G \), i.e., the ordinary cohomology of the homotopy quotient \( M_G := E_G \times_G M \).

There exist two spectral sequences converging to the equivariant cohomology \( H^\ast_g(M; F) \)
\[
\begin{align*}
E_2^{p,q} &= H^p(H^q(M; C^\ast(g; F))) \\
E_2^{p,q} &= H^p(M; H^q(g; F)),
\end{align*}
\]
where we denote by \( H^\ast(g; F) \) the sheaf over \( M \) given by the presheaf
\[
U \subset M \mapsto H^\ast(g; F(U)).
\]

To study the \( E_2 \) term, we introduce a condition (1.2) related to a locally convex space \( F \):

1. \( F \) is a Fréchet nuclear space,
2. there exists a projective system \( \{ F_i, u_i^{i+1} : F_{i+1} \to F_i \}_{i=1}^\infty \) of the strong duals \( F_i \) of Fréchet nuclear spaces and compact maps \( u_i^{i+1} : F_{i+1} \to F_i \) satisfying a topological isomorphism \( F = \varprojlim F_i \), and
3. the natural projection \( F \to F_i \) has a dense image in each \( F_i \),

(cf. [K]). Then we have ([Sm]).
Lemma 1.3. (Küneth formula) Let $F$ be a locally convex space satisfying the condition (1.2) and $A^*$ a Fréchet cochain complex with $\dim H^q(A^*) < \infty$ for each $q$. Then we have

$$H^*(\text{Hom}(F, A^*)) = \text{Hom}(F, H^*(A^*)),$$

where Hom means the continuous linear maps.

Now we confine ourselves to the case $M$ is a finite dimensional complex analytic manifold and $F = \mathcal{O}_M(E)$ is the sheaf of germs of sections of a complex analytic vector bundle $E$ over $M$. Let $F$ be a locally convex space satisfying the condition (1.2). We denote by $A^q(U; E)$ the Fréchet space of $C^\infty$ sections of $E \otimes \bigwedge^q T^* M$ on an open subset $U$ of $M$. Consider the sheaf $\text{Hom}(F, \mathcal{O}_M(E))$ (resp. $\text{Hom}(F, A^q(E))$) given by

$$U \subset M \mapsto \text{Hom}(F, \mathcal{O}_M(E)(U)),
\quad (\text{resp. } U \subset M \mapsto \text{Hom}(F, A^q(U; E))),
$$

where we endow $\mathcal{O}_M(E)(U)$ (resp. $A^q(U; E)$) with the Fréchet topology of uniformly convergence on compact subsets (resp. with the $C^\infty$ topology). If $U \subset M$ is a Stein open subset, we have

$$(1.4) \quad H^*(\text{Hom}(F, A^*(U; E))) = \text{Hom}(F, \mathcal{O}_M(E)(U)) \quad \text{(in dim. 0)},$$

by Lemma 1.3 applied to the acyclic augmented cochain complex $A^*(U; E)$ with the augmentation $\mathcal{O}_M(E)(U) \hookrightarrow A^0(U; E)$.

Passing to the inductive limit, we obtain a fine resolution of the sheaf $\text{Hom}(F, \mathcal{O}_M(E))$

$$0 \to \text{Hom}(F, \mathcal{O}_M(E)) \to \text{Hom}(F, A^0(E)) \to \text{Hom}(F, A^1(E)) \to \cdots.$$

From (1.4) any Stein covering of $M$ is acyclic for the sheaf $\text{Hom}(F, \mathcal{O}_M(E))$.

**Proposition 1.5.** If $\mathfrak{g}$ satisfies the condition (1.2), and $M$, $E$ and $H^q(M; \mathcal{O}_M(E))$ for $q \neq 0$ are all finite dimensional, then the natural map

$$i^! E_2^{p, q} = H^p(H^q(M; C^*(\mathfrak{g}; \mathcal{O}_M(E)))) \to H^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E)))$$

is an isomorphism.

**Proof.** Using Lemma 1.3 again, we have

$$H^*(M; \text{Hom}(F, \mathcal{O}_M(E))) = \text{Hom}(F, H^*(M; \mathcal{O}_M(E))).$$

Substituting $F = \bigwedge^q \mathfrak{g}$ to this isomorphism, we obtain

$$H^q(M; C^p(\mathfrak{g}; \mathcal{O}_M(E))) = C^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E))),$$

as was to be shown. □

**Example 1.6.** Let $M$ be a compact Kähler manifold and $E$ the $n$-cotangent bundle $\bigwedge^n T^* M$. Since $\mathfrak{g}$ acts on $H^*(M; \mathcal{O}_M(\bigwedge^n T^* M)) = H^{n,*}(M) \subset H^{n,0}(M)$ trivially, we have $i^! E_2^{p, q} = H^p(\mathfrak{g}; H^q(M; \mathcal{O}_M(E)))$. 
Example 1.7. Let \( O \) be an open Riemann surface and \( S \) a finite subset of \( O \). We denote by \( L(O, S) \) the Lie algebra of complex analytic vector fields on \( O \) which have zeroes at all points in \( S \). \( L(O, S) \) is a Fréchet space satisfying the condition \( (1.2) \) with respect to the topology of uniform convergence on compact sets. Let \( E \to M \) be a complex analytic vector bundle over a finite dimensional Stein manifold \( M \). If the Lie algebra \( L(O, S) \) acts on the sheaf of topological linear spaces \( \mathcal{O}_M(E) \) continuously, then we have

\[
H^*(L(O, S); \mathcal{O}_M(E(M)) = H^*(L(O, S); \mathcal{O}_M(E(M)).
\]

Hence we obtain a spectral sequence

\[
\tilde{E}_2^{p,q} = H^p(M; \mathcal{H}^q) \Rightarrow H^{p+q}(L(O, S); \mathcal{O}_M(E(M)),
\]

where \( \mathcal{H}^q \) is a sheaf over \( M \) whose stalk at \( x \in M \) is given by

\[
\mathcal{H}^q_x = H^q(L(O, S); \mathcal{O}_M(E(x),)
\]

which we call the Rešetnikov spectral sequence. See [R] and [Ka1] §9.

Let \( E \) be a complex analytic vector bundle of finite rank over \( \mathbb{C}^n \). Consider the alternating Čech complex \( C^*(\mathcal{U}) = C^*(\mathcal{U}; \mathcal{O}_{\mathbb{C}^n}(E)) \) with respect to the \((n-1)\) dimensional Stein covering \( \mathcal{U} = \{U_j\}_{j=1}^n \) of \( \mathbb{C}^n \setminus \{0\} \) given by

\[
U_j = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_j \neq 0\}.
\]

In view of a theorem of Scheja [S],

\[
H^q(\mathbb{C}^n \setminus \{0\}; \mathcal{O}_{\mathbb{C}^n}(E)) = H^q(\mathbb{C}^n; \mathcal{O}_{\mathbb{C}^n}(E))
\]

for \( q \leq n - 2 \). Applying Lemma 1.3 to the acyclic augmented complex \( C^*(\mathcal{U}) \) with \( C^{-1}(\mathcal{U}) = \mathcal{O}_M(E(C^n)) \) and \( C^n(\mathcal{U}) = H^{n-1}(\mathbb{C}^n \setminus \{0\}; \mathcal{O}_{\mathbb{C}^n}(E)) \), we have

\[
H^*(\mathbb{C}^n \setminus \{0\}; \text{Hom}(F, \mathcal{O}_{\mathbb{C}^n}(E))) = \text{Hom}(F, H^*(\mathbb{C}^n \setminus \{0\}; \mathcal{O}_{\mathbb{C}^n}(E))).
\]

Consequently we obtain

**Proposition 1.9.** If \( g \) satisfies the condition \( (1.2) \), then the \( \tilde{E}_2 \) term \( (1.1) \) converging to the equivariant cohomology \( H_\mathbb{C}^*(\mathbb{C}^n \setminus \{0\}; \mathcal{O}_{\mathbb{C}^n}(E)) \) is given by

\[
\tilde{E}_2^{p,q} = \begin{cases} 
H^p(g; \mathcal{O}_{\mathbb{C}^n}(E(C^n))), & \text{if } q = 0, \\
H^p(g; H^{n-1}(\mathbb{C}^n \setminus \{0\}; \mathcal{O}_{\mathbb{C}^n}(E))), & \text{if } q = n - 1, \\
0, & \text{otherwise.}
\end{cases}
\]
2. Preliminaries.

The topological Lie algebras $W_1$ and $L_0$, the $L_0$ modules $1_\nu$ and $T^\lambda_\nu$, and the $W_1$ modules $T_\nu$ and $T^\infty_\nu$ are defined in Introduction. For $\lambda$ and $\nu \in \mathbb{Z}$ we denote by $\tau^\lambda_\nu$ the line bundle $(T^*\mathbb{C})^{\otimes \nu} \otimes \{0\}^{\otimes \lambda}$ over the complex line $\mathbb{C}$, where $\{0\}$ is the line bundle induced by the divisor $0 \in \mathbb{C}$. If $\lambda = 0$, we denote $\tau^\lambda_\nu := \tau^0_\nu$. We have $T^\lambda_\nu = H^0(\mathbb{C}; \mathcal{O}_\mathbb{C}(\tau^\lambda_\nu)), T_\nu = H^0(\mathbb{C}; \mathcal{O}_\mathbb{C}(\tau_\nu))$ and $T^\infty_\nu = H^0(\mathbb{C} - \{0\}; \mathcal{O}_\mathbb{C}(\tau_\nu))$. We denote by $\tau^\lambda_1,\ldots,\tau^n_\nu$ the line bundle over $\mathbb{C}^n$ given by

\[(pr^*_{1} \tau_{\nu_1}^\lambda) \otimes \cdots \otimes (pr^n_\nu \tau_{\nu_n}^\lambda), \quad (\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_n \in \mathbb{Z}),\]

where $pr_i : \mathbb{C}^n \to \mathbb{C}$ is the $i$-th projection. By the diagonal Lie derivative action, $1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau^\lambda_1,\ldots,\tau^n_\nu)$ ($\nu_0 \in \mathbb{Z}$) is a sheaf of $L_0$ modules. From the nuclear theorem we have $L_0$ isomorphisms

\[1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\nu_{\nu_i} = H^0(\mathbb{C}^n; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau^\lambda_1,\ldots,\tau^n_\nu)).\]

If $\lambda_1 = \cdots = \lambda_n = \lambda$ and $\nu_1 = \cdots = \nu_n = \nu$, we abbreviate $(\tau^\lambda_\nu)^n := \tau^\lambda_{\nu_1,\ldots,\nu_n}$.

We recall a basic fact on an $L_0$ module. We have no essential difference between these variants.

**Lemma 2.1.** For the $L_0$ modules $N = 1_{\nu_0} \otimes \bigotimes_{i=1}^n T^\nu_{\nu_i}$ and $1_{\nu_0} \otimes \bigotimes_{i=1}^n (T^\infty_{\nu_i}/T^\nu_{\nu_i})$, the inclusion $C^*(L_0; N)^{e_0} \subset C^*(L_0; N)$ induces a cohomology equivalence.

**Proof.** (cf. [Ka1]§2). The action of the multiplicative group $\mathbb{C}^\times := \mathbb{C} - \{0\}$ on the complex line

\[T_t : \mathbb{C} \to \mathbb{C}, \quad z \mapsto tz, \quad (t \in \mathbb{C}^\times)\]

induces the actions $T_t$ on $N$ and $L_0$ itself such that

\[t \frac{d}{dt} T_t = T_t e_0 = e_0 T_t.\]

Using the averaging operator

\[C^*(L_0; N) \to C^*(L_0; N)^{e_0}, \quad c \mapsto \int_0^1 (T_{\exp 2\pi \sqrt{-1} \theta \cdot c}) d\theta,
\]

we obtain the desired cohomology equivalence. □

Now we introduce some variants of $L_0, W_1$ and $T^\nu_\nu$. We define $W^\text{alg.}_1$, $W^\text{conv.}_1$, $L^\text{alg.}_0$ and $L^\text{conv.}_0$ by

\[W^\text{alg.}_1 := \mathbb{C}[z] \frac{d}{dz}, \quad \text{(the polynomial vector fields)},\]

\[W^\text{conv.}_1 := \mathcal{O}_\mathbb{C}(T^*\mathbb{C})_0 = \mathbb{C}[z] \frac{d}{dz}, \quad \text{(the germs at } 0 \in \mathbb{C}),\]

\[L^\text{alg.}_0 := \{ X \in W^\text{alg.}_1 ; X(0) = 0 \}, \quad \text{and}\]

\[L^\text{conv.}_0 := \{ X \in W^\text{conv.}_1 ; X(0) = 0 \}.
\]

Clearly $W^\text{alg.}_1 \subset W^\text{conv.}_1$ and $L^\text{alg.}_0 \subset L^\text{conv.}_0$. $T^\text{alg.}_\nu$ (resp. $T^\text{conv.}_\nu$) defined by

\[T^\text{alg.}_\nu := \frac{1}{z^\nu} \mathbb{C}[z] dz, \quad \text{resp. } T^\text{conv.}_\nu := \mathcal{O}_\mathbb{C}(\tau_\nu)^0 = \frac{1}{z^\nu} \mathbb{C}[z] dz^\nu\]

is a $L^\text{alg.}_0$ (resp. $L^\text{conv.}_0$) module. If $\lambda = 0$, we denote $T^\text{alg.}_\nu := T^\text{conv.}_\nu$. We have no essential difference between these variants.
Lemma 2.3. We have natural isomorphisms

(1) \( H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\lambda_i}) \cong H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{conv}}) \)
\[ \cong H^*(L_0^{\text{conv}}; 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{conv}}) \cong H^*(L_0^{\text{alg}}; 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{alg}}). \]

(2) \( H^*(W_1; \bigotimes_{i=1}^{n} T_{\nu_i}) \cong H^*(W_1; \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{conv}}) \)
\[ \cong H^*(W_1^{\text{conv}}; \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{conv}}) \cong H^*(W_1^{\text{alg}}; \bigotimes_{i=1}^{n} T_{\nu_i}^{\text{alg}}). \]

Proof. The second isomorphisms of (1) and (2) follow from [Ka1] Theorem 5.3, and the third from ibid., Corollary A.3.

We prove the first of (1) (cf. ibid., §2). That of (2) is proved in a similar manner. We define the \( L_0 \) modules \( N_{\varepsilon} \) (\( \varepsilon > 0 \)), \( N \), and \( N^{\text{conv}} \) by

\[ N_{\varepsilon} := H^0(\{|z| < \varepsilon\}; 1_{\nu_0} \otimes \mathcal{O}_C^{\lambda_1}(t_{\nu_1}, \ldots, \lambda_n)), \]
\[ N := 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\lambda_i} = \lim_{\varepsilon \to 0} N_{\varepsilon}, \quad \text{and} \]
\[ N^{\text{conv}} := 1_{\nu_0} \otimes \bigotimes_{i=1}^{n} T_{\nu_i}^{\lambda_i^{\text{conv}}} = \lim_{\varepsilon \to 0} N_{\varepsilon}. \]

The multiplicative group \( \mathbb{C}^\times \) acts on \( N_{\varepsilon} \) by \( T_t : N_{\varepsilon} \to N_{\varepsilon} \) (\( t \in \mathbb{C}^\times \)) as in the proof of Lemma 2.1. Hence \( H^*(L_0; N_{\varepsilon}) \cong H^*(C^*(L_0; N_{\varepsilon}^{\text{e}})). \)

By (2.2), if \( 0 < \delta < 1 \), the restriction homomorphism \( C^*(L_0; N_{\varepsilon}^{\text{e}}) \to C^*(L_0; N_{\delta\varepsilon}^{\text{e}}) \) has its inverse \( T_\delta \), and so is an isomorphism. Hence we obtain a series of isomorphisms

\[ C^*(L_0; N^{\text{conv}})^{\text{e}_0} = \lim_{\varepsilon \to 0} C^*(L_0; N_{\varepsilon})^{\text{e}_0} \]
\[ = C^*(L_0; N_{\varepsilon})^{\text{e}_0} = \lim_{\varepsilon \to 0} C^*(L_0; N_{\varepsilon})^{\text{e}_0} = C^*(L_0; N)^{\text{e}_0}. \]

The first isomorphism follows from [K] Lemma 3, p.372 (see also [Ka1] Lemma 4.3). This completes the proof. \( \square \)

Remark 2.4. For any \( \nu_1 \in \mathbb{Z} \) the \( W_1^{\text{alg}} \) module \( T_{\nu_1}^{\text{formal}} := \mathbb{C}[[z]]dz^{\nu_1} \) of \( \nu_1 \)-covariant formal tensor fields is just the co-induced module of the \( L_0^{\text{alg}} \) module \( 1_{\nu_1} \).

This implies \( T_{\nu_1}^{\text{formal}} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}} \) is the co-induced module of \( 1_{\nu_1} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}} \).

From the Shapiro lemma (the Frobenius reciprocity)

\[ H^*(W_{1}^{\text{alg}}; T_{\nu_1}^{\text{formal}} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}}) \cong H^*(L_0^{\text{alg}}; 1_{\nu_1} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}}). \]

On the other hand, \( C^*(W_{1}^{\text{alg}}; T_{\nu_1}^{\text{formal}} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}})^{\text{e}_0} = C^*(W_{1}^{\text{alg}}; T_{\nu_1}^{\text{alg}} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}^{\text{alg}})^{\text{e}_0}. \) Hence, by Lemma 2.3, we have

\[ H^*(W_1; \bigotimes_{i=1}^{n} T_{\nu_i}) \cong H^*(L_0; 1_{\nu_1} \otimes \bigotimes_{i=2}^{n} T_{\nu_i}). \]

This isomorphism can be described in the following explicit way. We remark the evaluation map

\[ ev : T_{\nu_1} \to 1_{\nu_1} \otimes f(0) \text{ and } f(z)dz^{\nu_1}. \]
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is an $L_0$ homomorphism. So we can define the composite map

$$H^*(W_1; \bigotimes_{i=1}^n T_{\nu_i}) \to H^*(L_0; \bigotimes_{i=1}^n T_{\nu_i}) \xrightarrow{ev} H^*(L_0; 1_{\nu_1} \otimes \bigotimes_{i=2}^n T_{\nu_i}),$$

which coincides with the Shapiro isomorphism.

In the succeeding sections we study the equivariant cohomology $H^*_L(C^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{C^n}(\tau_{\lambda_1}, \ldots, \lambda_n))$. The stalk at $z = (z_1, \ldots, z_n) \in C^n$ of the sheaf $H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{C^n}(\tau_{\lambda_1}, \ldots, \lambda_n))$, which appears in the "$E_2$ term (1.1) converging to the equivariant cohomology" is given as follows. For simplicity we assume, for some $0 \leq r_0 < r_1 < \cdots < r_l = n$,

$$z_i = 0, \quad \text{if } i \leq r_0$$

$$z_i = z_{r_k}, \quad \text{if } r_{k-1} < i \leq r_k,$$

$$z_{r_k} \neq 0, \quad \text{if } k \geq 1, \quad \text{and}$$

$$z_{r_k} \neq z_{r_j}, \quad \text{if } k \neq j.$$

Let $u_k \in H^2(L_0; (T_0)^{\otimes n})$ be the cohomology class of a 2 cocycle defined by

$$u_k(\xi_1(z) \frac{d}{dz}, \xi_2(z) \frac{d}{dz}) := \sum_{r_{k-1} < i \leq r_k} \int_0^{z_i} \det \begin{pmatrix} \xi'_1(z) & \xi'_2(z) \\ \xi''_1(z) & \xi''_2(z) \end{pmatrix} dz$$

for $\xi_1(z) \frac{d}{dz}$ and $\xi_2(z) \frac{d}{dz} \in L_0$.

**Lemma 2.7.** ([Ka1] Theorem 5.3.) The stalk at $z = (z_1, \ldots, z_n) \in C^n$ is given by

$$H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{C^n}(\tau_{\lambda_1}, \ldots, \lambda_n))(z_1, \ldots, z_n)$$

$$= C[u_1, \ldots, u_l] \otimes H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^{r_0} T^\lambda_{\nu_i}) \otimes \bigotimes_{k=1}^l H^*(W_1; \bigotimes_{r_{k-1} < i \leq r_k} T_{\nu_i}).$$
3. Fundamental exact sequence.

Using the Stein covering \( \{ U_j \}_{j=1}^n \) of \( \mathbb{C}^n - \{ 0 \} \) given by (1.8), we have an \( L_0 \) isomorphism

\[
H^{n-1}(\mathbb{C}^n - \{ 0 \}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^\lambda)) = 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^\lambda).
\]

From Proposition 1.9 (a corollary of a theorem of Scheja [S]), the \( ^{\prime}E_2 \) term (1.1) converging to \( H^*_{L_0}(\mathbb{C}^n - \{ 0 \}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^\lambda)) \) is given by

\[
^{\prime}E_2^{p,q} = \begin{cases} 
H^p(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda), & \text{if } q = 0, \\
H^p(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^\lambda)), & \text{if } q = n - 1, \\
0, & \text{otherwise.}
\end{cases}
\]

This means a cohomology exact sequence

\[
\cdots \to H^{q-n}(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^\lambda)) \overset{d_q}{\to} H^q(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda) \\
\to H^q_{L_0}(\mathbb{C}^n - \{ 0 \}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^\lambda)) \to \cdots,
\]

which we call the fundamental exact sequence for the \( L_0 \) module \( 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda \).

**Corollary 3.3.** If \( \nu_0 + \sum_{i=1}^n (\nu_i - \lambda_i) \leq n - 1 \), then

\[
H^p(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda) \cong H^p_{L_0}(\mathbb{C}^n - \{ 0 \}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1,\ldots,\nu_n}^\lambda)).
\]

**Proof.** The \( L_0 \) module \( 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^\lambda) \) is (topologically) generated by negative eigenvectors of \( e_0 = z \frac{d}{dz} \) under the given assumption. Hence \( C^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n (T_{\nu_i}^\times / T_{\nu_i}^\lambda))^{e_0} = 0 \). The corollary follows from the sequence (3.2). \( \square \)

As an application, we have

**Theorem 3.4.** Let \( \nu_0, \nu_1, \ldots, \nu_n, \lambda_1, \ldots, \lambda_n \) be integers satisfying

\[
\nu_0 + \sum_{i \in I} (\nu_i - \lambda_i) \leq \# I - 1
\]

for any non-empty index subset \( I \subset \{ 1, \ldots, n \} \). Then the restriction homomorphism

\[
H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda) \to H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\times)
\]

is an isomorphism.

**Proof.** We prove the theorem by induction on \( n \). We have \( C^*(L_0; 1_{\nu_0} \otimes T_{\nu_1}^\lambda)^{e_0} = C^*(L_0; 1_{\nu_0} \otimes T_{\nu_1}^\times)^{e_0} \), which implies the theorem for \( n = 1 \).

If \( n \geq 2 \), then, by Corollary 3.3, we have

\[
H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^\lambda) \cong H^*_0(\mathbb{C}^n - \{ 0 \}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\lambda_1,\ldots,\lambda_n}^\times))
\]
Consider the inclusion map \( i : (\mathbb{C}^x)^n \to \mathbb{C}^n - \{0\} \), where \( \mathbb{C}^x = \mathbb{C} - \{0\} \). We calculate the stalk at the point \( z \in (\mathbb{C}^n - \{0\}) - (\mathbb{C}^x)^n \) of the sheaf \( H^*(L_0; i_*1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_n)) \). For simplicity we assume the point \( z \) satisfies the condition (2.6). Lemma 2.7 (or [Ka1] Theorem 5.3) implies

\[
H^*(L_0; i_*1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_n))(z_1, \ldots, z_n)
= \mathbb{C}[u_1, \ldots, u_i] \otimes \lim_{\varepsilon \downarrow 0} (H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_0))(0 < |z| < \varepsilon))^{r_0})
\]

\[
\otimes \bigotimes_{k=1}^l H^*(W_1; \bigotimes_{r_k-1 < i \leq r_k} T_{\nu_i}).
\]

The second term of the RHS is isomorphic to \( H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^{r_0} T_{\nu_i}^x) \) by a similar method to the proof of Lemma 2.3. We have \( r_0 < n \) since \( z \in \mathbb{C}^n - \{0\} \). By the inductive assumption the term is isomorphic to \( H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^{r_0} T_{\nu_i}) \). Hence we have an isomorphism of sheaves over \( \mathbb{C}^n - \{0\} \)

\[
H^*(L_0; i_*1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_n)) \cong H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1}, \ldots, \nu_n))(\mathbb{C}^n - \{0\}).
\]

Comparing the \( E_2 \) terms, we obtain

\[
H^*_L(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}(\tau_{\nu_1}, \ldots, \nu_n)) \cong H^*_L(\mathbb{C}^n - \{0\}; i_*1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_n)).
\]

We write simply \( F := 1_{\nu_0} \otimes \mathcal{O}_{(\mathbb{C}^x)^n}(\tau_{\nu_1}, \ldots, \nu_n) \). For any open subset \( U \) of \( \mathbb{C}^n - \{0\} \) we have

\[
C^*(L_0; i_*F)(U) = C^*(L_0; (i_*F)(U)) = C^*(L_0; F(U \cap (\mathbb{C}^x)^n)) = i_*C^*(L_0; F(U)).
\]

Hence \( C^*(L_0; i_*F) = i_*C^*(L_0; F) \). If \( U \) is Stein, then \( U \cap (\mathbb{C}^x)^n \) is also Stein, so that \( H^q(U \cap (\mathbb{C}^x)^n; C^*(L_0; F)) = 0 \) for \( q \neq 0 \) from Proposition 1.5. This implies \( R^qi_*C^*(L_0; F) = 0 \) for \( q \neq 0 \), so that \( H^*(\mathbb{C}^n - \{0\}; i_*C^*(L_0; F)) = H^*((\mathbb{C}^x)^n; C^*(L_0; F)) \). Comparing the \( E_2 \)-terms, we have

\[
H^*_{L_0}(\mathbb{C}^n - \{0\}; i_*F) = H^*_{L_0}((\mathbb{C}^x)^n; F) = H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^x).
\]

This completes the induction. \( \square \)

We introduce an algebraic variant of the \( L_0 \) module \( T_{\nu}^x \) by

\[
T_{\nu}^{x\text{alg.}} := \lim_{\lambda \uparrow \infty} T_{\nu}^{\text{alg.}} = \lim_{\lambda \uparrow \infty} \frac{1}{z^\lambda} \mathbb{C}[z]dz^\nu = \mathbb{C}[z^{-1}, z]dz^\nu.
\]

The \( L_0^{\text{alg.}} \) module \( 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^{x\text{alg.}} \) is the space of algebraic tensors on the complex torus \( (\mathbb{C} - \{0\})^n \). It follows from Theorem 3.4 and Lemma 2.3

**Corollary 3.5.**

\[
H^*(L_0; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^x) = H^*(L_0^{\text{alg.}}; 1_{\nu_0} \otimes \bigotimes_{i=1}^n T_{\nu_i}^{x\text{alg.}}).
\]

We denote by \( F(\Omega) \) the Fréchet space of complex analytic functions on a complex analytic manifold \( \Omega \) with the topology of uniform convergence on compact sets. For any open subset \( U \subset \mathbb{C}^n \) the Lie algebra \( W_1 \) acts on the Fréchet space \( F(\Omega) \) by the diagonal Lie derivative action. Clearly we have an \( L_0 \) isomorphism \( F(\mathbb{C}^n) \cong (T_0)^{\otimes n} \). Now we determine the cohomology algebra \( H^*(W_1; F(\mathbb{C}^n)) \) explicitly by making a comparison with that with values in the functions on the configuration space

\[
P_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j \ (i \neq j) \} \ (n \geq 1).
\]
Theorem 3.6. The restriction homomorphism

\[ \iota_n : H^*(W_1; F(C^n)) \to H^*(W_1; F(P_n)) \]

is isomorphic for any \( n \geq 1 \).

It follows from [Ka1] Proposition 9.8

Corollary 3.7.

\[ H^*(W_1; F(C^n)) = H^*(P_n) \otimes \mathbb{C}[v_2, \ldots, v_n] \otimes \bigwedge^* (\nabla_0^1, \ldots, \nabla_n^1). \]

Here the (holomorphic) de Rham cohomology algebra of \( P_n \), \( H^*(P_n) \), is mapped into the algebra \( H^*(W_1; F(C^n)) \) in an obvious way (ibid., Corollary 9.9). The 2 cocycle \( v_i \) is defined by

\[ v_i(\xi_1(z) \frac{d}{dz}, \xi_2(z) \frac{d}{dz}) := \int_{z_{i-1}}^{z_i} \det \begin{pmatrix} \xi_1'(z) & \xi_2'(z) \\ \xi_1''(z) & \xi_2''(z) \end{pmatrix} dz \]

and the 1 cocycle \( \nabla_0^j \) by

\[ \nabla_0^j(\xi(z) \frac{d}{dz}) = \xi'(z_j) \]

for \( \xi(z) \frac{d}{dz}, \xi_1(z) \frac{d}{dz} \) and \( \xi_2(z) \frac{d}{dz} \in L_0 \).

Proof of Theorem 3.6. We prove the theorem by induction on \( n \). It is already known for the case \( n \leq 2 \) [FF]. From [Ka1], Corollary 9.9, the homomorphism \( \iota_n \) is surjective for any \( n \geq 1 \).

Suppose \( n \geq 2 \). We denote \( P_n^\times := P_n \cap (C - \{0\})^n \). Then we have a commutative diagram

\[
\begin{array}{ccc}
H^*(W_1; F(C^{n+1})) & \xrightarrow{\iota_{n+1}} & H^*(W_1; F(P_{n+1})) \\
\downarrow & & \downarrow \\
H^*(L_0; F(C^n)) & \xrightarrow{\iota'} & H^*(L_0; F(P_n^\times)),
\end{array}
\]

where the vertical homomorphisms are the Shapiro homomorphisms given in Remark 2.4, and the lower horizontal \( \iota' \) the restriction homomorphism. Since the left vertical is isomorphic from Remark 2.4 and \( \iota_{n+1} \) is surjective, it suffices to show \( \iota' \) is isomorphic. From Corollary 3.3 follows \( H^*_L(\mathbb{C}^n - \{0\}; \mathcal{O}_{\mathbb{C}^n}) = H^*(L_0; F(C^n)) \).

Consider the inclusion map \( i : P_n^\times \to \mathbb{C}^n - \{0\} \). If \((z_1, \ldots, z_n) \in \mathbb{C}^n - \{0\}, all the multiplicities of the set \{0, z_1, \ldots, z_n\} are not greater than \( n \). Hence we deduce an isomorphism of sheaves over \( \mathbb{C}^n - \{0\} \)

\[ H^*(L_0; i_*\mathcal{O}_{P_n^\times}) \cong H^*(L_0; \mathcal{O}_{\mathbb{C}^n}|_{\mathbb{C}^n - \{0\}}) \]

from the inductive assumption together with a similar method to the proof of Theorem 3.4. Moreover we have

\[ H^*_L(\mathbb{C}^n - \{0\}; i_*\mathcal{O}_{P_n^\times}) \cong H^*_L(P_n^\times; \mathcal{O}_{P_n^\times}) = H^*(L_0; F(P_n^\times)) \]

because \( P_n^\times \) is Stein. Consequently the restriction homomorphism \( \iota' \) is an isomorphism, which completes the induction.
4. Action of the symmetric group.  

The $n$-th symmetric group $\mathfrak{S}_n$ acts on the $L_0$ module $1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}$ in two ways: If $\sigma \in \mathfrak{S}_n$ and $1_{\nu_0} \otimes f(z_1, \ldots, z_n) dz_1^{\nu} \cdots dz_n^{\nu} \in 1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}$, then $\sigma(1_{\nu_0} \otimes f(z_1, \ldots, z_n) dz_1^{\nu} \cdots dz_n^{\nu})$ is given by

\[
1_{\nu_0} \otimes f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) dz_1^{\nu} \cdots dz_n^{\nu} \in 1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}
\]

in one way, and by

\[
(\text{sign } \sigma) 1_{\nu_0} \otimes f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) dz_1^{\nu} \cdots dz_n^{\nu} \in 1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}
\]

in the other way. We call the former the *symmetric action* and the latter the *alternating action* of the group $\mathfrak{S}_n$. The $L_0$ module $1_{\nu_0} \otimes (T^\lambda_\nu / T^\lambda_\nu)^{\otimes n}$ has the symmetric and the alternating actions of the group $\mathfrak{S}_n$ in a similar manner.

**Proposition 4.1.** The fundamental exact sequence (3.2) for the $L_0$ module $1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}$ is $\mathfrak{S}_n$ equivariant, if the group $\mathfrak{S}_n$ acts on $H^*(L_0; 1_{\nu_0} \otimes (T^\lambda_\nu / T^\lambda_\nu)^{\otimes n})$ by the symmetric (resp. alternating) action and on the other two terms by the alternating (resp. symmetric) action.

**Proof.** Consider the alternating $\check{\text{C}}$ech complex $C^*(\check{U}) = C^*(\check{U}; 1_{\nu_0} \otimes O_{\mathbb{C}^n}((\tau^\lambda_\nu)^n))$ with respect to the Stein covering $\check{U} = \{U_j\}_{j=1}^n$ of $\mathbb{C}^n - \{0\}$ given by (1.8). The cohomology group of the total complex of the double complex

\[
C^*(L_0; C^*(\check{U}; 1_{\nu_0} \otimes O_{\mathbb{C}^n}((\tau^\lambda_\nu)^n)))
\]

is the equivariant cohomology group $H^*_L(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes O_{\mathbb{C}^n}((\tau^\lambda_\nu)^n))$.

The symmetric group $\mathfrak{S}_n$ acts on the $\check{\text{C}}$ech complex as follows. Suppose a $q$-cochain $c \in C^q(\check{U})$ is given by the tensors

\[
1_{\nu_0} \otimes c_{i_0\ldots i_q}(z_1, \ldots, z_n) dz_1^{\nu} \cdots dz_n^{\nu} \quad (1 \leq i_0, \ldots, i_q \leq n),
\]

where $c_{i_0\ldots i_q}(z_1, \ldots, z_n) dz_1^{\nu} \cdots dz_n^{\nu} \in O_{\mathbb{C}^n}((\tau^\lambda_\nu)^n) \cap U_{i_0}. \ldots, U_{i_q}$. If $\sigma \in \mathfrak{S}_n$, the cochain $\sigma c$ is given by the tensors

\[
1_{\nu_0} \otimes (\sigma c)_{i_0\ldots i_q}(z_1, \ldots, z_n) dz_1^{\nu} \cdots dz_n^{\nu},
\]

\[
(\sigma c)_{i_0\ldots i_q}(z_1, \ldots, z_n) := (\text{sign } \sigma)c_{\sigma^{-1}(i_0)\ldots \sigma^{-1}(i_q)}(z_{\sigma(1)}, \ldots, z_{\sigma(n)}).
\]

Then the group $\mathfrak{S}_n$ acts on $H^0(C^*(\check{U})) = 1_{\nu_0} \otimes (T^\lambda_\nu)^{\otimes n}$ by the alternating action and on $H^{n-1}(C^*(\check{U})) = 1_{\nu_0} \otimes (T^\lambda_\nu / T^\lambda_\nu)^{\otimes n}$ by the symmetric action, as was to be shown. The rest is proved in a similar way. $\square$

To put the proposition into practice, we review on sheaves on which a finite transformation group acts. Here we regard a sheaf as an étale covering space.

Let $M$ be a Hausdorff space on which a finite group $G$ acts continuously. The isotropy group at a point $x \in M$ is denoted by $G_x$, i.e., $G_x := \{ \gamma \in G; \gamma x = x \}$. Let $\pi : F \to M$ be a sheaf of complex vector spaces on which the group $G$ acts continuously and linearly. The subset $F^G$ of $F$ defined by $F^G := \{ a \in F; \forall \gamma \in G_{\pi(a)}, \gamma a = a \}$ is a linear subsheaf of $F$. The functor $F \mapsto F^G$ is exact. Furthermore we have a canonical isomorphism

\[
C^*(M, F^G) \cong C^*(M, F)^G
\]
where $\mathcal{C}^*(M;\cdot)$ denotes the canonical resolution. See, for example, [B]I§2. The isomorphism (4.2) follows from the fact

$$(4.3) \quad \mathcal{F}^G(U) = \{ s \in \mathcal{F}(U) ; \forall x \in U, \forall \gamma \in G_x, \gamma s(x) = s(x) \}$$

and the definition of the resolution $\mathcal{C}^*(M;\cdot)$.

If $\varpi : M \to M/G$ is the quotient map, we have

$$H^*(M;\mathcal{F})^G = H^*(M/G;\varpi_*\mathcal{F})^G = H^*(\mathcal{C}^*(M/G;\varpi_*\mathcal{F})(M/G)^G)$$

$$= H^*(\mathcal{C}^*(M/G;\varpi_*\mathcal{F})(M/G)) = H^*(\mathcal{C}^*(M/G;(\varpi_*\mathcal{F})^G)(M/G))$$

$$= H^*(M/G;(\varpi_*\mathcal{F})^G).$$

The third equality follows from (4.3) and the fourth from (4.2). The quotient space $\mathcal{F}^G/G$ is a sheaf over $M/G$ and satisfies

$$(\mathcal{F}^G/G)(U) = \mathcal{F}(\varpi^{-1}(U))^G = (\varpi_*\mathcal{F})^G(U)$$

for any open subset $U$ of $M/G$. This means $\mathcal{F}^G/G = (\varpi_*\mathcal{F})^G$. Consequently we obtain an isomorphism

$$(4.4) \quad H^*(M;\mathcal{F})^G \cong H^*(M/G;\mathcal{F}^G/G).$$

For a topological vector space $V$, the completed $n$ fold symmetric (resp. alternating) tensor product is denoted by $S^n(V)$ (resp. $\bigwedge^n V$). We denote by $(\cdot)^{S_n,\text{sym.}} \ (\cdot)^{S_n,\text{alt.}}$ the space of invariants of the space $\cdot$ under the symmetric (resp. alternating) action of the group $S_n$. We have $S^n(V) = (V^{\otimes n})^{S_n,\text{sym.}}$ and $\bigwedge^n V = (V^{\otimes n})^{S_n,\text{alt.}}$.

Proposition 4.1 implies a cohomology exact sequence

$$(4.5) \quad \cdots \to H^{q-n}(L_0; 1_{\nu_0} \otimes S^n(T_\nu^\lambda/T_\nu)) \to H^q(L_0; 1_{\nu_0} \otimes \bigwedge^n T_\nu^\lambda)$$

$$\to H^q_{L_0}(\mathbb{C}^n - \{0\}; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}((\tau_\nu^\lambda)^n))^{S_n,\text{alt.}} \to \cdots,$$

which we call the fundamental exact sequence for the $L_0$ module $1_{\nu_0} \otimes \bigwedge^n T_\nu^\lambda$. From (4.4) the “$E_2$ term (1.1) converging to the third term of (4.5) is given by

$$E_2^{p,q} = H^p(\mathbb{C}^n - \{0\}/\mathcal{S}_n; H^q(L_0; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}((\tau_\nu^\lambda)^n))^{S_n,\text{alt.}}/\mathcal{S}_n).$$

By Lemma 2.7 we have

**Lemma 4.6.** If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ satisfies the condition (2.6), the stalk at $z$ of the sheaf $H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}((\tau_\nu^\lambda)^n))^{S_n,\text{alt.}}$ is given by

$$H^*(L_0; 1_{\nu_0} \otimes \mathcal{O}_{\mathbb{C}^n}((\tau_\nu^\lambda)^n))^{S_n,\text{alt.}}_{(z_1, \ldots, z_n)}$$

$$= \mathbb{C}[u_1, \ldots, u_l] \otimes H^*(L_0; 1_{\nu_0} \otimes \bigwedge^{r_0} T_\nu^\lambda) \otimes \bigotimes_{k=1}^l H^*(W_1; \bigwedge^{r_{k-r_k-1}} T_\nu).$$

Similar results to (4.5) and Lemma 4.6 hold for the $L_0$ module $1_{\nu_0} \otimes S^n T_\nu^\lambda$.

Finally we prove a certain vanishing theorem for the cohomology of $W_1$ with values in the quadratic differentials on $\mathbb{C}^n$. For the rest we denote $Q = T_2$, $Q^1 = T_2^1$ and $Q^\times = T_2^\times$. The cohomology groups $H^q(W_1; \bigwedge^p Q)$ and $H^q(L_0; \bigwedge^p Q^1)$ are closely related to the $(p,q)$ cohomology groups of the parameter (moduli) space and the total space of the universal family of dressed Riemann surfaces respectively [ADKP], [Ko2].
Theorem 4.7.

\[ H^q(L_0; 1_2 \otimes \bigwedge^{n-1} Q) = 0, \quad \text{if } q < n, \]
\[ H^q(W_1; \bigwedge^n Q) = 0, \quad \text{if } q < n. \]

Proof. We remark the first part implies the second part for each \( n \). The first part is already proved in [FF] for \( n \leq 2 \).

Suppose \( n \geq 2 \). If \( q \leq n \), we have
\[ H^q(L_0; 1_2 \otimes \mathcal{O}_{\mathbb{C}^n}((z_2)^n)) \cong \mathcal{O}_{\mathbb{C}^n-\{0\}} = 0 \]
by the inductive assumption and Lemma 4.6. It follows from the fundamental exact sequence (4.5)
\[ H^q(L_0; 1_2 \otimes \bigwedge^n Q) = \begin{cases} H^0(L_0; 1_2 \otimes \bigwedge^n Q), & \text{if } q = n, \\ 0, & \text{if } q < n. \end{cases} \]

Hence the following lemma completes the induction. Here we denote by \( q_\nu \) the class \((z_\nu-2dz_2 \mod Q) \in \mathbb{Q}^\times/Q\). Clearly \( e_0q_\nu = \nu q_\nu \).

Lemma 4.8.

\[ H^0(L_1; S^*(Q^\times/Q)) = \mathbb{C}[q_1] \quad \text{(the polynomial algebra in } q_1), \]
where \( L_1 \) denotes the subalgebra of \( L_0 \) generated by \( e_k := z^{k+1} \frac{d}{dz} \) (\( k \geq 1 \)).

Proof of Lemma 4.8. Assume there exists some \( \alpha \in S^m(Q^\times/Q)^{L_1} - \mathbb{C}q_1^m, m \geq 1 \). Let \( q_p, p \geq 0 \), be the maximal among those which appear in \( \alpha \) except \( q_1 \). Then we have
\[ \alpha = \sum_{i+j \leq m} q_1^i q_p^j f_{ij}(q-p-1, \ldots, q-s, \ldots). \]

There exists \( i_0 := \max\{i; \exists j \geq 1, f_{ij} \neq 0\} \) because of the choice of \( q_p \). Now we have
\[ 0 = e_{p+1} \cdot \alpha = \sum_{i+j \leq m} j(p+2)q_1^{i+1} q_p^{j-1} f_{ij}(q_p-1, \ldots, q_s, \ldots) + \sum_{i+j \leq m} q_1^i q_p^j e_{p+1}(f_{ij}(q_p-1, \ldots, q_s, \ldots)). \]

Since \( e_{p+1}(f_{ij}) \) has no \( q_1 \)'s, we have
\[ \sum_{j \leq m-i_0} j q_1^{i_0+1} q_p^{j-1} f_{i_0j}(q_p-1, \ldots, q_s, \ldots) = 0, \]
or equivalently, \( f_{i_0j} = 0 \) for any \( j \geq 1 \). This contradicts the choice of \( i_0 \). Hence \( S^*(Q^\times/Q)^{L_1} = \mathbb{C}[q_1], \) as was to be shown. \( \square \)

Consequently the proof of Theorem 4.7 is completed. \( \square \)
Theorem 4.9.

\[ H^q(L_0; \bigwedge^n Q^1) = 0, \quad \text{if } q < n. \]

**Proof.** The theorem for \( n = 1 \) is shown in [FF].

Suppose \( n \geq 2 \). By the inductive assumption and Theorem 4.7 we have

\[ H^q(L_0; \bigwedge^n Q^1) = 0 \]

for \( q < n \). Hence, by the fundamental exact sequence (4.5), \( H^q(L_0; \bigwedge^n Q^1) = 0 \) for \( q < n \), as was to be shown. \( \Box \)

**Remark 4.10.**

(1) We have \( H^n(W_1; \bigwedge^n Q) \neq 0 \) and \( H^n(L_0; \bigwedge^n Q^1) \neq 0 \). In fact, if \( \nabla_2 \) is the 1-cocycle in \( C^1(W_1; Q) \) defined by

\[ \nabla_2(\xi(z)\frac{d}{dz}) = \xi^{(3)}(z)dz^2, \quad (\xi(z)\frac{d}{dz} \in W_1), \]

then the cohomology class of the \( n \)-th power \( (\nabla_2)^n \) does not vanish in \( H^n(W_1; \bigwedge^n Q) \) nor \( H^n(L_0; \bigwedge^n Q^1) \). Hence the estimates in Theorems 4.7 and 4.9 are the best possibles.

(2) In Theorem 4.7 it is essential to take the alternating tensor product. By a computation due to Kazushi Ahara we have \( \dim H^4(W_1; Q^{\otimes 5}) = 3 \).

(3) Theorems 4.7 and 4.9 suggest that the Virasoro equivariant \((p,q)\) cohomology of the parameter space and the total space of the universal family of dressed Riemann surfaces would vanish for \( p > q \) [Ka2].

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