A Multi-Frequency Sampling Method for the Inverse Source Problems with Sparse Measurements

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Abstract. We consider the inverse source problems with multi-frequency sparse near field measurements. In contrast to the existing near field operator based on the integral over the space variable, a multi-frequency near field operator is introduced based on the integral over the frequency variable. A factorization of this multi-frequency near field operator is further given and analyzed. Based on such a factorization, we introduce a single-receiver multi-frequency sampling method to reconstruct a shell support of the source. Its theoretical foundation is derived from the properties of the factorized operators and a properly chosen point spread function. Numerical examples are provided to illustrate the multi-frequency sampling method with sparse near field measurements. Finally we briefly discuss how to extend the near field case to the far field case.

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1 Introduction

Inverse source problems merit numerous applications in medical imaging, geophysics, non-destructive testing, and many others. A survey on the state of the art of the mathematical theory and numerical approaches can be found in the monograph [22] and the review paper [6]. In the full aperture case when the multi-frequency measurements are available all around the unknown source, the uniqueness of the source with compact support is a direct consequence of the Fourier theory, see for instance [7, 18]. Several numerical methods were proposed for reconstructing the source, see e.g., an iterative method [9] and Fourier method [38, 39]. We also refer to [11, 40] on direct sampling methods for imaging point sources. However, in applications where measurements are only

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available at a limited number of sparse receivers, we are led to the inverse source problem with sparse multi-frequency measurements. Generally, neither the exact source nor its exact support is uniquely determined by the sparse data. Some lower bounds for the support of an extended source was established in [37] in terms of multi-frequency sparse far field measurements. For extended sources we refer to [20] and [1] for a factorization method and a direct sampling method, respectively, using multi-frequency sparse far field measurements; in these work they show that a strip support of the source can be uniquely determined by multi-frequency measurements at a single observation direction and provide sampling methods to image such a strip. For the multi-frequency sparse near field measurements, the corresponding uniqueness and direct sampling method can be found in a recent work [27]. For point sources, the uniqueness on the locations as well as the source strengths was established in [24, 26, 27] using multi-frequency sparse far field or near field measurements. In particular, the smallest number of sensors needed was given in terms of the number of point sources. We also refer to [34] regarding a study on multi-frequency sampling methods in waveguides.

Sampling methods have attached a lot of attentions in the last 30 years. Classical sampling methods such as the linear sampling method [14] and the factorization method [28] are independent of certain a priori information on the geometry and physical properties of the unknown scatterers. Based on a factorization of the far field or the near field operator, one may derive a criteria to reconstruct the unknown object and design an indicator function which is large inside the underlying object and relatively small outside. We refer to the monographs [12, 15, 29] for a comprehensive introduction. There have been recent efforts on other types of sampling methods such as orthogonality sampling method [19, 21, 35], direct sampling method [4, 5, 23, 25, 31], reverse time migration [13], and other direct reconstruction methods [2, 3]. These sampling methods inherit many advantages of the classical ones. The main feature of these sampling methods is that only inner product of the measurements with some suitably chosen functions is involved in the imaging function and thus these sampling methods seem very robust to noises.

In two recent papers [1, 27], under certain assumption, the shell support of the source can be uniquely determined by multi-frequency measurement at a given receiver. However, the theoretical basis is far less developed. In particular, the indicator function given in [27] is merely based on an observation that the scattered field due to an extend source can be considered as the one from a superposition of monopoles. This work contributes to a novel multi-frequency sampling method. Different from [1, 27], in this work we introduce a multi-frequency near field operator \( N_x \) based on the integral over the frequency range \( K \) at a single receiver \( x \) and study its factorization

\[
N_x = P_x T_x P_x^*.
\]

Based on the coercivity of the middle operator, we show that for any sampling point \( z \),

\[
C_1 \| P_x g_{xz} \|_{L^2(D)}^2 \leq \langle (N_x g_{xz}, g_{xz})_{L^2(K)} \rangle \leq C_2 \| P_x g_{xz} \|_{L^2(D)}^2.
\]
for some positive constants $C_1$ and $C_2$, where $g_{xz}$ is a properly chosen function that depends on the sampling point $z$ so that $P^*_x g_{xz}$ is a certain point spread function and the function $\|P^*_x g_{xz}\|_{L^2(D)}^2$ is expected to image a shell support of the source (which is supported in $D$). The factorization of the multi-frequency near field operator and the choice of $g_{xz}$ play important roles in our sampling method. Note that such a sampling method is in the spirit of the recent work [33] on a modified sampling method with near field measurements at a fixed frequency. Based on these observations, $|\langle \mathcal{N}_x g_{xz}, g_{xz} \rangle_{L^2(K)}|$ is applied as the indicator function for imaging the shell support of the source. By adding the indicators functions with respect to a few receivers, we illustrate numerically how to image a support of the extended source.

The paper is further organized as follows. We introduce the model of the forward problem and inverse source problem with multi-frequency sparse near field measurements in Section 2. We further introduce the multi-frequency near field operator $\mathcal{N}_x$, provide its factorization and investigate its properties in Section 3. In Section 4, we design an imaging function with properly chosen $g_{xz}$. Furthermore we provide its theoretical foundation based on the factorization and a point spread function. Section 5 presents numerical examples to illustrate our multi-frequency sampling method with sparse near field measurements. Finally, in Appendix A we briefly discuss how to extend the near field case to the far field case.

2 Mathematical model and the inverse problem

We consider the acoustic wave propagation due to a source $f$ in a homogeneous isotropic medium in $\mathbb{R}^3$ with speed of sound $c > 0$. Denote by $D$ the support of the unknown source, which is a bounded Lipschitz domain in $\mathbb{R}^3$ with connected complement $\mathbb{R}^3 \setminus \overline{D}$. The mathematical model of the scattering of time-harmonic wave leads to the nonhomogeneous Helmholtz equation

$$\Delta u^s + k^2 u^s = -f \quad \text{in} \quad \mathbb{R}^3,$$

(2.1)

where the wave number is $k = \omega / c$ with frequency $\omega > 0$. The scattered field $u^s$ is required to satisfy the Sommerfeld radiation condition

$$\lim_{r = |x| \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

(2.2)

uniformly for all directions $x/|x|$. Throughout the paper we write $u^s(\cdot,k)$ to emphasize the dependence on the wave number $k$. If not otherwise stated, we consider multiple wave numbers in a bounded interval, i.e.,

$$k \in K := (0,k_{\max}]$$

(2.3)

with an upper bound $k_{\max} > 0$. 
The unique radiating solution \( u^s \) to the scattering problem (2.1)-(2.2) takes the form
\[
 u^s(x,k) = \int_D \Phi_k(x,y) f(y) ds(y), \quad x \in \mathbb{R}^3, \quad k \in K
\]  (2.4)
with
\[
 \Phi_k(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y
\]  (2.5)
being the fundamental solution to the Helmholtz equation.

A practical inverse problem is to recover the source support \( D \) from the multi-frequency sparse measurements where the scattered fields are available at finitely many points in the set
\[
 \Gamma_L := \{x_1, x_2, \cdots, x_L\} \subset \mathbb{R}^3 \setminus \overline{D}.
\]
Accordingly, we obtain the following multi-frequency sparse near field measurements:
\[
 \mathcal{M}_N := \{u^s(x,k) \mid x \in \Gamma_L, k \in K\}. \quad (2.6)
\]

Generally speaking, uniqueness of the source support \( D \) does not hold for the near field measurements \( \mathcal{M}_N \). An approximate support containing \( D \) can be uniquely determined. We refer to [1, 20, 27, 37] for more details on the uniqueness.

3 The multi-frequency near field operator and its factorization

In this section, we introduce a multi-frequency near field operator given by the measurements \( \mathcal{M}_N \) (2.6) and study its factorization which plays a key role in the analysis and design of our imaging algorithm. We assume that the source function \( f \in L^2(\mathbb{R}^3) \) is real-valued and satisfies that
\[
 C_f \geq f(x) \geq c_f \quad \text{or} \quad -C_f \leq f(x) \leq -c_f, \quad x \in D
\]  (3.1)
for two constants \( C_f > c_f > 0 \). To begin with, we define the scattered fields with negative frequencies by
\[
 u^s(x,-k) := \overline{u^s(x,k)}, \quad x \in \Gamma_L, \quad k \in K. \quad (3.2)
\]
Note that \( f \) is real-valued, we then have the following integral representation for the scattered fields:
\[
 u^s(x,k) = \int_D \Phi_k(x,y) f(y) ds(y), \quad x \in \Gamma_L, \quad k \in [-k_{\max}, k_{\max}] \setminus \{0\}. \quad (3.3)
\]

**Remark 3.1.** Note that we have used the assumption (3.1) to derive the integral representation (3.3) via (3.2). The assumption on the source \( f \) in [20] in the far-field case is weaker due to the particular structure of the far field operator with two (opposite) observation
directions and by knowing the phase of $f$. Similarly we can relax the assumption (3.1) on $f$ to
\[ C_f \geq e^{i\theta} f(x) \geq c_f \quad \text{or} \quad -C_f \leq e^{i\theta} f(x) \leq -c_f, \quad x \in D \] (3.4)
for some fixed $\theta \in [0, 2\pi)$. In this way, we modify the definition of $u^s(x,-k)$ by
\[ u^s(x,-k) := \overline{u^s(x,k)} e^{-2i\theta}, \quad x \in \Gamma_L, \quad k \in K \] (3.5)
to derive Eq. (3.3). The main result can then be verified to hold with this modification.

In the near field case, we have not found a more relaxed assumption, however we remark that a more relaxed assumption on $f$ can be derived in the far field case, see the Appendix A.

For each $x \in \Gamma_L$, we define the integral operator $N_x: L^2(K) \rightarrow L^2(K)$ by
\[ (N_x g)(t) := \int_K u^s(x,t-s)g(s)ds, \quad t \in K, \] (3.6)
which we will call the multi-frequency near field operator. Different from the usual near field operator which uses full-aperture measurements with a single frequency, this operator seeks to explore the relations among different frequencies. The factorization of the usual near field operator plays an important role in the modified sampling method [33] so that no asymptotic assumptions on the distance between the measurement surface and the scatterers were made there. In the same spirit, we seek to given a characterization of the unknown source support $D$ by the multi-frequency near field operator $N_x$ together with its factorization.

To factorize $N_x$, we first define $P_x: L^2(D) \rightarrow L^2(K)$ by
\[ (P_x \psi)(t) := \int_D e^{it|x-y|} \psi(y)dy, \quad t \in K. \] (3.7)
We can directly derive that its adjoint $P_x^*: L^2(K) \rightarrow L^2(D)$ is given by
\[ (P_x^* \phi)(y) := \int_K e^{-is|x-y|} \phi(s)ds, \quad y \in D. \] (3.8)
Finally we define the operator $T_x: L^2(D) \rightarrow L^2(D)$ by
\[ (T_x h)(y) := \frac{f(y)}{4\pi|x-y|} h(y), \quad y \in D. \] (3.9)
It directly follows that $T_x$ is a self-adjoint operator since $f$ is real-valued. Now we are ready to give the following factorization of the multi-frequency near field operator $N_x$.

**Theorem 3.1.** For each $x \in \Gamma_L$, the following factorization holds:
\[ N_x = P_x T_x P_x^*. \] (3.10)
Proof. Inserting the integral representation (3.3) into the definition (3.6) of the multi-frequency near field operator $N_x$, we have that

\[
(N_x g)(t) = \int_K \int_D e^{i(t-s)|x-y|} f(y) dy g(s) ds
\]

\[
= \int_D e^{i|x-y|} \frac{f(y)}{4\pi|x-y|} \int_K e^{-i|x-y|} g(s) ds dy
\]

\[
= (P_x T_x P_x^* g)(t), \quad t \in K.
\]

This completes the proof. \(\square\)

Theorem 3.1 provides a symmetric factorization which allows us to study a factorization method, which is expected to be similar to the factorization method using multi-frequency far field measurements [20]. Alternatively, our goal here is to utilize this factorization to introduce a different multi-frequency sampling method. We remark that this type of sampling method may still work even in the case of non-symmetric factorization (see for instance [33]).

### 4 The multi-frequency sampling method

This section is devoted to a multi-frequency sampling method using sparse near field measurements $M_N$ (2.4). To begin with, we provide the following property of the middle operator $T_x$.

**Lemma 4.1.** Assume that $f \in L^2(\mathbb{R}^3)$ satisfies (3.1). For each $x \in \Gamma_L$, the operator $T_x : L^2(D) \to L^2(D)$ is self-adjoint and coercive, i.e.

\[
\frac{c_f}{4\pi r_2} \| h \|_{L^2(D)}^2 \leq \left| \langle T_x h, h \rangle_{L^2(D)} \right| \leq \frac{C_f}{4\pi r_1} \| h \|_{L^2(D)}^2, \quad \forall h \in L^2(D),
\]

where

\[
r_1 := \inf \{|x-y| : y \in D\}, \quad r_2 := \sup \{|x-y| : y \in D\}.
\]

**Proof.** Recall the definition of $T_x$ (3.9), we have that

\[
(T_x h, h)_{L^2(D)} = \int_D \frac{f(y)}{4\pi|x-y|} |h(y)|^2 dy.
\]

Note from the assumption (3.1) that

\[
C_f \geq f(x) \geq c_f \quad \text{or} \quad -C_f \leq f(x) \leq -c_f, \quad x \in D,
\]

then the inequality (4.1) follows by the definition of $r_1$ and $r_2$. This proves the lemma. \(\square\)
We now prove the following theorem which plays an important role in the analysis and design of our imaging algorithm.

**Theorem 4.1.** Assume that $f \in L^2(\mathbb{R}^3)$ satisfies (3.1). For each $x \in \Gamma_L$, the following inequality holds:

$$
\frac{c_f}{4\pi r^2} \| \mathcal{P}_x^*g \|_{L^2(D)}^2 \leq \| \mathcal{N}_x g \|_{L^2(K)} \leq \frac{c_f}{4\pi r_1} \| \mathcal{P}_x^*g \|_{L^2(D)}^2, \quad \forall g \in L^2(K).
$$

**Proof.** With the help of the factorization (3.10) in Theorem 3.1, we have that

$$(\mathcal{N}_x g, g)_{L^2(K)} = (\mathcal{P}_x T_x \mathcal{P}_x^* g, g)_{L^2(K)} = (T_x \mathcal{P}_x^* g, \mathcal{P}_x^* g)_{L^2(D)}.$$ 

Then the theorem follows from Lemma 4.1. This proves the theorem. \qed

The following theorem gives a properly chosen $g_{xz}$ such that $(\mathcal{P}_x^* g_{xz})$ is a point spread function.

**Theorem 4.2.** For each $x \in \Gamma_L$ and $z \in \mathbb{R}^3$, we define $g_{xz} \in L^2(K)$ by

$$
g_{xz}(s) := e^{is|x-z|}, \quad s \in K. \quad (4.4)
$$

It holds that

$$(\mathcal{P}_x^* g_{xz})(y) = \begin{cases} k_{\text{max}}, & \text{if } t = 0, \\
\frac{e^{ik_{\text{max}}t} - 1}{it}, & \text{if } t \neq 0,
\end{cases} \quad y \in D, \quad (4.5)
$$

where $t := |x-z| - |x-y|$. In particular, $(\mathcal{P}_x^* g_{xz})(y) = (\mathcal{P}_x^* g_{xz})(y^*)$ if $|y-x| = |y^*-x|$.

Furthermore, $|(\mathcal{P}_x^* g_{xz})(y)| \leq k_{\text{max}}, \quad y \in D$, where “$=$” holds if and only if $t = 0$. The following asymptotic holds:

$$
|(\mathcal{P}_x^* g_{xz})(y)| = O \left( \frac{1}{t} \right), \quad t \to \infty.
$$

**Proof.** From the definition of $\mathcal{P}_x^*$ (3.8), we have that

$$(\mathcal{P}_x^* g_{xz})(y) = \int_0^{k_{\text{max}}} e^{is(|x-z|-|x-y|)} ds, \quad y \in D.
$$

Then Eq. (4.5) follows from a direct computation. It then follows that $(\mathcal{P}_x^* g_{xz})(y) = (\mathcal{P}_x^* g_{xz})(y^*)$ if $|y-x| = |y^*-x|$.

Furthermore we derive that

$$
|(\mathcal{P}_x^* g_{xz})(y)| = \int_0^{k_{\text{max}}} e^{is\pi} ds \leq \int_0^{k_{\text{max}}} |e^{is\pi}| ds \leq k_{\text{max}}, \quad y \in D,
$$

where the equality holds if and only if $t = 0$.

Finally, the asymptotic of $|(\mathcal{P}_x^* g_{xz})(y)|$ for large $t$ follows from (4.5). This proves the theorem. \qed
To illustrate the asymptotic property of \(|(P_s^* g_{xz})(y)|\) as a function of \(t\), we refer to Fig. 1 where we plot \(|(P_s^* g_{xz})(y)|\) with respect to different \(k_{\text{max}}\). In particular, Fig. 1 implies that the resolution is better with larger wave number \(k_{\text{max}}\).

Now for given \(y \in D\) and sampling point \(z\), \(|(P_s^* g_{xz})(y)|\) peaks at \(t = 0\), i.e. at sampling point \(z\) such that \(|x - z| = |x - y|\). In another word, \(|(P_s^* g_{xz})(y)|\) as a function of sampling point \(z\) peaks at a sphere centered at \(x\) with radius \(|x - y|\). Furthermore, turning to \(\|P_s^* g_{xz}\|^2_{L^2(D)}\) as a function of sampling point \(z\), we expect the function \(\|P_s^* g_{xz}\|^2_{L^2(D)}\) is capable of imaging the smallest shell support \(A_D(x) := B_{r_2}(x) \setminus B_{r_1}(x)\) centered at the measurement point \(x\) with \(r_1\) and \(r_2\) given by (4.2). See Fig. 2 for an illustration.

It is not feasible to compute \(\|P_s^* g_{xz}\|^2_{L^2(D)}\) directly since \(D\) is what we aim to reconstruct. However, we are able to relate \(\|P_s^* g_{xz}\|^2_{L^2(D)}\) to an indicator function given by measurements. Precisely, letting \(g = g_{xz}\) in Theorem 4.1 yields that

\[
\frac{c_f}{4 \pi r_2} \|P_s^* g_{xz}\|^2_{L^2(D)} \leq |(N_s g_{xz}, g_{xz})|_{L^2(K)} \leq \frac{c_f}{4 \pi r_1} \|P_s^* g_{xz}\|^2_{L^2(D)}.
\]

Figure 1: Plot of \(|(P_s^* g_{xz})(y)|\) as a function of \(t\) in \([-5, 5]\). Left: \(k_{\text{max}} = 3\). Middle: \(k_{\text{max}} = 5\). Right: \(k_{\text{max}} = 10\).

Figure 2: The smallest shell support \(A_D(x) := B_{r_2}(x) \setminus B_{r_1}(x)\) centered at the measurement point \(x\) with \(r_1 = \inf\{|x - y|: y \in D\}\) and \(r_2 = \sup\{|x - y|: y \in D\}\).
It is then expected that \( \| P^*_x g_x \|_{L^2(D)} \) is qualitatively the same as \( |(N_x g_x \cdot g_x)|_{L^2(K)} | \) for a single measurement point \( x \). We remark that the values of \( c_f \) and \( C_f \) may introduce an impact on the qualitative behavior of \( |(N_x g_x \cdot g_x)|_{L^2(K)} | \); in particular if \( C_f/c_f \) (which represents the contrast of the source) is too large which poses difficulty in imaging methods (without knowing a priori that the source is high contrast) as far as we know, the value \( k_{\text{max}} \) will be large enough (which implies a higher resolution of \( |P^*_x g_z| \), see Fig. 1) in order to yield a satisfactory image.

From above, it is expected that the measurement driven function \( |(N_x g_x \cdot g_x)|_{L^2(K)} | \) is expected to image the shell support \( A_D(x) \) without the knowledge of \( D \). Consequently for all \( x \in \Gamma_L \), we introduce the indicator function

\[
I(z) := \sum_{x \in \Gamma_L} |(N_x g_x \cdot g_x)|_{L^2(K)} |, \quad z \in \mathbb{R}^3.
\] (4.6)

To summarize our qualitative imaging method, the indicator function \( I(z) \) is expected to image the shell support \( A_D(x) \) for a single measurement point \( x \). With the increase of measurement points, an approximate source support \( \bigcap_{x \in \Gamma_L} A_D(x) \) is expected to be reconstructed by plotting \( I(z) \) over a sampling region.

## 5 Numerical examples

In this section, we present some numerical examples to illustrate the performance of the multi-frequency sampling method.

We generate the synthetic data \( u^s \) using the finite element computational software Netgen/NGSolve [36]. To be more precise, the computational domain is \( \{ x : |x| < 4 \} \) and the measurements are on the sphere \( \{ x : |x| = 3 \} \). We apply a radial Perfectly Matched Layer (PML) in the domain \( \{ x : 3.5 < |x| < 4 \} \) and choose PML absorbing coefficient \( 5i \). See Fig. 3 for an illustration. In all of the numerical examples, we apply the second-order finite element to solve for the wave field, where the source is constant 1, the mesh size is chosen as 0.5 outside the source and a finer mesh inside the source, the set of wave numbers is \( \{ k : k = 1, 2, 3, \ldots, 11 \} \). We further add 5% Gaussian noise to the synthetic data \( u^s \) to implement the indicator function given by (4.6) in Matlab, for the visualization, we plot \( I(z) \) over the sampling region \( \{ x = (x_1, x_2, x_3) : |x_j| < 3, j = 1, 2, 3 \} \) and we always normalize it such that its maximum value is 1.

We first illustrate the performance of the multi-frequency sampling method with one measurement point. In this example, the measurement point is \( (3,0,0) \). The source has support given by a ball

\[
\{ x : |x_1|^2 + |x_2|^2 + |x_3|^2 < 1 \}.
\] (5.1)

We plot the exact ball, the three-dimensional view of the reconstruction, and its iso-surface view with iso-value \( 7 \times 10^{-1} \) in Fig. 4. As expected, the reconstruction is a shell support of the exact ball.
Figure 3: Mesh view of the computational domain.

Figure 4: Reconstructions of a ball with one measurement point \((3,0,0)\). Left: 3D view of the exact ball. Middle: 3D view of the reconstruction. Right: iso-surface view of the reconstruction with iso-value \(8 \times 10^{-1}\).

What may we reconstruct with a little bit more measurements points? Now we still consider the reconstruction of the same ball \((5.1)\) but with 3 measurement points given by Table 1, where these 3 measurement points are on the upper half sphere. We plot both the three-dimensional reconstruction and its cross-section views in Fig. 5. It is observed that the location of ball is clearly indicated with only 3 measurement points.

To further shed light on the performance of our imaging algorithm, we consider the case when multiple (but still sparse) measurement points are available all around the source. In the following examples, we consider 14 measurement points which are roughly equally distributed on the measurement sphere. We remark that only \(14 \times 11 = 154\) measurements (in three-dimensional space and frequency) are used in this case. The polar coordinates of the 14 measurement points are given in Table 2.

Table 1: 3 measurement points: \((3\sin \theta \cos \varphi, 3\sin \theta \cos \varphi, 3\cos \theta)\). Angles are in degrees.

| \(\varphi\)          | \(\theta\)          |
|----------------------|----------------------|
| -180.000000000000    | 45.00000000000000    |
| -90.000000000000     | 45.00000000000000    |
| 0.00000000000000     | 45.00000000000000    |
Table 2: 14 measurement points: \((r_m \sin \theta \cos \phi, r_m \sin \theta \cos \phi, r_m \cos \theta)\). Angles are in degrees.

| \(\phi\)          | \(\theta\)          |
|-------------------|---------------------|
| 0.0000000000000000 | 90.000000000000000 |
| 180.00000000000000 | 90.000000000000000 |
| 90.00000000000000  | 90.000000000000000 |
| -90.00000000000000 | 90.000000000000000 |
| 90.00000000000000  | 0.0000000000000000  |
| 90.00000000000000  | 180.000000000000000 |
| 45.00000000000000  | 54.735610317245346 |
| -45.00000000000000 | 54.735610317245346 |
| 135.00000000000000 | 125.2648962754654  |
| -135.00000000000000| 125.2648962754654  |

Fig. 6 shows the reconstruction of the ball (5.1) using multiple but sparse measurements. Compared with the reconstruction using 3 measurement points, it is observed that both the location and shape of the ball are better reconstructed.

We continue to consider a variety of different geometries of the support: a cube (reconstruction in Fig. 7) given by

\[
\{ x = (x_1, x_2, x_3) : |x_1| < 1, |x_2| < 1, |x_3| < 1 \},
\]

a rounded cylinder (reconstruction in Fig. 8) given by

\[
x = (x_1, x_2, x_3) : \sqrt{x_1^2 + x_2^2} < 1 \quad \text{for} \quad |x_3| < 1,
\]

\[
\sqrt{x_1^2 + x_2^2 + (x_3 - 1)^2} < 1 \quad \text{for} \quad 1 < x_3 < 2,
\]

\[
\sqrt{x_1^2 + x_2^2 + (x_3 + 1)^2} < 1 \quad \text{for} \quad -2 < x_3 < -1
\}

a peanut-shape support (reconstruction in Fig. 9) given by

\[
x = (x_1, x_2, x_3) : \sqrt{(x_1 - 0.5)^2 + x_2^2 + x_3^2} < 1 \quad \text{or} \quad \sqrt{(x_1 + 0.5)^2 + x_2^2 + x_3^2} < 1
\}

an L-shape support (reconstruction in Fig. 10) given by

\[
\{ x : -0.5 < x_1 < 0, -1.5 < x_2 < 0.5, |x_3| < 0.75 \} \cup \{ x : 0 < x_1 < 1.5, 0 < x_2 < 0.5, |x_3| < 0.75 \}
\]

and two balls (reconstruction in Fig. 10) given by

\[
\{ x : \sqrt{|x_1 + 1|^2 + |x_2|^2 + |x_3|^2} < 0.7 \} \cup \{ x : \sqrt{|x_1 - 1|^2 + |x_2|^2 + |x_3|^2} < 0.7 \}.
\]
Figure 5: Reconstructions of the ball with 3 measurement points located on the upper sphere. Top left: exact ball. Top right: iso-surface view of the reconstruction with iso-value $8 \times 10^{-1}$. Bottom left: $x_1 x_2$ cross section view of the reconstruction. Bottom middle: $x_2 x_3$ cross section view of the reconstruction. Bottom right: $x_1 x_3$ cross section view of the reconstruction.

Figure 6: Reconstructions of the ball with 14 measurement points. Top left: exact ball. Top middle and top right: iso-surface view of the reconstruction with iso-values $7 \times 10^{-1}$ and $8 \times 10^{-1}$ respectively. Bottom left: $x_1 x_2$ cross section view of the reconstruction. Bottom middle: $x_2 x_3$ cross section view of the reconstruction. Bottom right: $x_1 x_3$ cross section view of the reconstruction.
Figure 7: Reconstructions of the cube with 14 measurement points. Top left: exact cube. Top middle and top right: iso-surface view of the reconstruction with iso-values $7 \times 10^{-1}$ and $8 \times 10^{-1}$ respectively. Bottom left: $x_1x_2$ cross section view of the reconstruction. Bottom middle: $x_2x_3$ cross section view of the reconstruction. Bottom right: $x_1x_3$ cross section view of the reconstruction.

Figure 8: Reconstructions of the rounded cylinder with 14 measurement points. Top left: exact rounded cylinder. Top middle and top right: iso-surface view of the reconstruction with iso-values $7 \times 10^{-1}$ and $8 \times 10^{-1}$ respectively. Bottom left: $x_1x_2$ cross section view of the reconstruction. Bottom middle: $x_2x_3$ cross section view of the reconstruction. Bottom right: $x_1x_3$ cross section view of the reconstruction.
Figure 9: Reconstructions of the peanut with 14 measurement points. Top left: exact peanut. Top middle and top right: iso-surface view of the reconstruction with iso-values $7 \times 10^{-1}$ and $8 \times 10^{-1}$ respectively. Bottom left: $x_1 x_2$ cross section view of the reconstruction. Bottom middle: $x_2 x_3$ cross section view of the reconstruction. Bottom right: $x_1 x_3$ cross section view of the reconstruction.

Figure 10: Reconstructions of the L-shape (top) and two balls (bottom) with 14 measurement points. Left: exact support. Middle: iso-surface view of the reconstruction with iso-values $8 \times 10^{-1}$. Right: $x_1 x_2$ cross section view of the reconstruction.
Figure 11: Reconstructions of the L-shape with 14 measurement points with different noise levels. Top: iso-surface (with iso-value $8 \times 10^{-1}$) view of the reconstruction. Bottom: $x_1 x_2$ cross section view of the reconstruction. Noise levels from left to right: 10%, 20%, 50%.

Here for these two examples we set mesh size as 0.4 due to the increasing demand in forward problem simulation.

Finally, we give an example to illustrate the performance with respect to different noise levels in Fig. 11. The reconstruction is observed to be robust with respect to the noises being added.

We observe from the numerical examples that 1) the shell support of the source can be reconstructed from only one measurement point; 2) the location of a single source can be reconstructed from a few (sparse) measurement points (where we use 3 measurement points with 11 different frequencies); 3) both the shape and location of a single source can be well reconstructed from multiple (but sparse) measurement points (where we use 14 measurement points with 11 different frequencies) that are roughly equally distributed all around the source. It is also observed that the concave part of the support (see Fig. 9 for the peanut and top of Fig. 10 for the L-shape) may be reconstructed; 4) the reconstruction is observed to be robust with respect to the noises being added; 5) the imaging algorithm also works when there are multiple sources (see bottom of Fig. 10).

Appendix A. Extension to far field measurements

In this appendix, we discuss briefly the indicator function defined by the multi-frequency sparse far field measurements. From the asymptotic behavior of Hankel functions, we
deduce that the corresponding far field pattern of $u^\infty$ (2.4) has the form
\[ u^\infty(\hat{x},k) = \int_D e^{-ik\hat{x} \cdot y} f(y) dy, \quad \hat{x} \in S^2, \quad k \in K, \tag{A.1} \]
where $S^2$ denotes the unit sphere. We consider the following multi-frequency sparse far field measurements:
\[ M_F := \{ u^\infty(\hat{x},k) | \hat{x} \in \Theta_L, k \in K \} \tag{A.2} \]
with $\Theta_L := \{ \pm \hat{x}_1, \pm \hat{x}_2, \ldots, \pm \hat{x}_L \} \subset S^2$. In view of (A.1), we have
\[ u^\infty(\hat{x},-k) = u^\infty(-\hat{x},k), \quad \hat{x} \in \Theta_L, \quad k \in K. \]
Therefore, the multi-frequency sparse far field measurements $M_F$ (A.2) gives the following data:
\[ u^\infty(\hat{x},k), \quad \hat{x} \in \Theta_L, \quad k \in [-k_{\max}, k_{\max}] \setminus \{0\}. \]
We define the multi-frequency far field operator $F_L: L^2(K) \to L^2(K)$ by
\[ (F_L \phi)(t) := \int_K u^\infty(\hat{x},t-s) \phi(s) ds, \quad t \in K. \tag{A.3} \]
The analogous results of Theorems 3.1 and 4.1 are formulated in the following theorem.

**Theorem A.1.** The far field operator $F_L: L^2(K) \to L^2(K)$ has a factorization of the form
\[ F_L = Q_L T_L Q_L^*. \tag{A.4} \]
Here, $Q_L: L^2(D) \to L^2(K)$ is given by
\[ (Q_L \psi)(t) := \int_D e^{-ik\hat{x} \cdot y} \psi(y) dy, \quad t \in K, \]
and its adjoint $Q_L^*: L^2(K) \to L^2(D)$ is given by
\[ (Q_L^* \phi)(y) := \int_K e^{ik\hat{x} \cdot y} \phi(s) ds, \quad y \in D. \]
The operator $T_L: L^2(D) \to L^2(D)$ is a multiplication operator given by $T_L \phi = f \phi$, where $f \in L^2(\mathbb{R}^3)$ is the source with support $D$. Moreover, if $|f| < C_f$ and $\Re(\epsilon^{i\theta} f(\hat{x})) > c_f$, a.e., for two positive constant $c_f$ and $C_f$ and a fixed $\theta \in (0, 2\pi)$, then it holds that
\[ c_f \| Q_L^* \phi \|^2_{L^2(D)} \leq \| (F_L \phi_t)_{L^2(K)} \| \leq C_f \| Q_L^* \phi \|^2_{L^2(D)}, \quad \forall \phi \in L^2(K), \quad \hat{x} \in \Theta_L. \]
Motivated by the Theorem A.1, we introduce the following indicator function:
\[ I(z) := \sum_{\hat{x} \in \Theta_L} \left| (F_L \phi_{\hat{x}z}, \phi_{\hat{x}z})_{L^2(K)} \right|, \quad z \in \mathbb{R}^3, \tag{A.5} \]
where $\phi_{xz} \in L^2(K)$ is given by

$$\phi_{xz}(t) = e^{it\cdot z}, \quad t \in K.$$  \hfill (A.6)

Consequently, for $y \in D$ we have

$$\left| (Q^* \phi_{xz})(y) \right| = \left| \int_K e^{it\cdot \tilde{\xi}} ds \right|, \quad \tilde{\xi} = \tilde{x} \cdot (z - y),$$  \hfill (A.7)

where $\phi_{xz}$ is given by (A.6). Similarly, $| (Q^* \phi_{xz})(y) |$ is a point spread function of $\tilde{\xi}$.

We remark that the factorization (A.4) has been derived in [20] where a factorization method is investigated. As can be seen, we have allowed a complex-valued $f$ in the case of far field measurements. We finally remark that the results in this appendix can be directly extended to the two dimensional case.

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