BRANCHING AND BIFURCATION

PATRICK M. FITZPATRICK

Department of Mathematics
University of Maryland, 4176 Campus Dr
College Park, MD 20742, USA

JACOBO PEJSACHOWICZ *

INDAM, Dipartimento di Scienze Matematiche
Politecnico di Torino, Duca degli Abruzzi 24
10129 Torino, Italy

Dedicated to Norman Dancer

Abstract. By relating the set of branch points $\mathcal{B}(f)$ of a Fredholm mapping $f$ to linearized bifurcation, we show, among other things, that under mild local assumptions at a single point, the set $\mathcal{B}(f)$ is sufficiently large to separate the domain of the mapping. In the variational case, we will also provide estimates from below for the number of connected components of the complement of $\mathcal{B}(f)$.

Introduction. Let $X$ and $Y$ be real Banach spaces and $f: U \to Y$ be a $C^k$-Fredholm mapping of index 0 defined on an open subset $U$ of $X$. A point $x_* \in U$ is called a branch point for $f$ provided that there is no neighborhood of $x_*$ on which $f$ is one-to-one, that is to say, there are sequences $\{x_n\}$ and $\{z_n\}$ in $U$ that converge to $x_*$ and for all $n$, $f(x_n) = f(z_n)$ while $x_n \neq z_n$. We denote by $\mathcal{B}(f)$ the set of all branch points of $f$. Fredholm maps are locally proper, and hence locally closed. Therefore $x_*$ is a branch point for $f$ if and only if $f$ fails to be a local homeomorphism at $x_*$.

According to the inverse function theorem, $\mathcal{B}(f)$ is contained in the set of critical points of the map, $\mathcal{C}(f) \equiv \{x \in U \mid Df(x) \text{ is noninvertible}\}$. Simple examples show that in general it is only a proper closed subset of $\mathcal{C}(f)$. Indeed, it is shown in [10, Theorem 2] that, when $f$ is proper and $\dim X \geq 3$, no isolated critical point of the map can be a branch point. However, by the Caccioppoli global inverse function theorem [12], branch points always exist if $f$ is proper and $U$ is connected but not simply connected. The structure of the set of branch points both in finite and infinite dimensions has been intensively studied (see [3, 13, 14] and their references).

Our first observation here is that there is a simple relation between branch points of maps between Banach spaces and bifurcation from a trivial branch of the solutions of a parametrized family of equations. We will use this relation, together with two topological invariants of paths of linear Fredholm operators, namely the parity and the spectral flow, in order to show that, under mild local assumptions at a single
point, the set $B(f)$ is big enough to separate the domain of $f$. Moreover, in the variational case, that is, when $f$ is the gradient of a $C^2$ functional defined on an open subset of a Hilbert space, we will also provide estimates from below for the number of connected components of the complement of $B(f)$ in $U$.

While clearly distinct in general, in many concrete cases the sets $B(f)$ and $C(f)$ coincide. Because of this, the difference between these two concepts is often blurred in the mathematical literature. For example, under the assumptions of the Ambrosetti-Prodi theorem [3] we necessarily have $C(f) = B(f)$. In this paper we will establish a weaker form of this well-known theorem under assumptions which do not require the above two sets to be equal.

In the same vein, we will show that the zero-set of a proper Fredholm map of degree $\pm 1$ is connected, provided that $B(f)$ has a dense and connected complement, improving in this way an old result by Smith and Stuart for $k$-set contractive perturbations of the identity [30], where an analogous assumption was made about the complement of $C(f)$.

The methods of this paper heavily depend on the results obtained in [19, 20, 17]. In order to make this paper self contained we will explicitly describe the concepts and results in these papers that we will need here, together with their background.

The first section is devoted to branch points of general Fredholm maps. We state our main results and prove them, after first reviewing the necessary material regarding the relationship of parity to bifurcation. We also show here how the topology of the set of branch points influences the topological properties of the zero-set of the map.

The second section deals with branch points in the case when the mapping $f$ is the gradient of a $C^3$ functional on a Hilbert space $H$. After reviewing the concept of spectral flow and its relationship to bifurcation of critical points, it is proved that a local condition which involves the signature of a quadratic form infinitesimally defined at a single point of the domain implies that $B(f)$ disconnects the domain $U$ of $f$. We also obtain two estimates from below for the number of connected components of $U \setminus B(f)$.

1. Branching and linearized bifurcation for Fredholm maps. Throughout this section, $X$ and $Y$ are real Banach spaces and $B(x, r)$ denotes the open ball of radius $r$ centered at $x$. Let $U$ be an open neighborhood of $0$ in $X$ and let $\Lambda$ be a $C^k$ Banach manifold. Given a $C^k$ map $h: \Lambda \times U \to Y$ we will consider the associated $C^k$-family of maps $\{h_\lambda: U \to Y\}_{\lambda \in \Lambda}$, parametrized by $\Lambda$, with $h_\lambda$ defined by $h_\lambda(x) = h(\lambda, x)$. Assuming that $h(\lambda, 0) = 0$ for all $\lambda \in \Lambda$, solutions of the equation $h(\lambda, x) = 0$ of the form $(\lambda, 0)$ are called trivial and the set $\Lambda \times \{0\} \subseteq \{(\lambda, x) | h(\lambda, x) = 0\}$ is called the trivial branch. In what follows we will identify this set with $\Lambda$.

A bifurcation point of solutions of the equation $h(\lambda, x) = 0$ from the trivial branch is a point $(\lambda_*, 0)$ such that every neighborhood of $(\lambda_*, 0) \in \Lambda \times U$ contains a nontrivial solution of this equation.

There is a very simple relation between branch points of $C^k$ maps and bifurcation points for solutions of parametrized families of equations.

Proposition 1.1. Given a $C^k$ map $f: U \to Y; k \geq 0$, defined on an open subset $U$ of $X$, the point $u_* \in U$ is a branch point of $f$ if and only if $(u_*, 0)$ is a bifurcation point from the trivial branch for solutions of the family of maps
Given a Corollary 1.2, the following corollary will suffice.

Proof. Clearly we have \( h_u(0) = 0 \) for all \( u \in B(u_*, \delta) \). If \( f(u_n) = f(v_n) \) with \( u_n \neq v_n \) and with \( \{ u_n \} \to u_* \) and \( \{ v_n \} \to u_* \), then \( x_n = v_n - u_n \neq 0 \) belong to \( B(0, \delta) \) for big enough \( n \) and hence \( (u_n, x_n) \) are nontrivial solutions of \( h(u, x) = 0 \) with \( \{ x_n \} \to 0 \) and \( \{ u_n \} \to u_* \). Therefore \( (u_*, 0) \) is a bifurcation point of \( h \). The converse is clear.

Proposition 1.1 allows us to study the set of branch points of a map by restricting the family \( h \) to finite dimensional planes of \( X \) through \( u_* \) and then using known results of several parameter bifurcation theory.

In the present paper we will use the one-parameter version only, and therefore the following corollary will suffice.

Corollary 1.2. Given a \( C^k \) map \( f : U \to Y \) as above and a \( C^k \) path \( \gamma : (-\delta, \delta) \to U \) with \( \gamma(0) = u_* \), \( u_* \) is a branch point for \( f \) provided that \( 0 \) is a bifurcation point from the trivial branch for the family \( h_\gamma \) defined near \( (0, u_*) \in \mathbb{R} \times U \) by

\[
h_\gamma(\lambda, x) = f(x + \gamma(\lambda)) - f(\gamma(\lambda)).
\]

Proof. If \( 0 \) is a bifurcation point for \( h_\gamma \), then \( u_* = \gamma(0) \) is a bifurcation point for the family \( h \) defined in (1) and therefore a branch point.

A linear Fredholm operator \( T : X \to Y \) is a bounded operator such that \( \text{im} T \) is closed and both \( \ker T \) and \( \text{coker} T \) are finite dimensional. The index of a Fredholm operator \( T \) is defined by \( \text{ind} T = \dim \ker T - \dim \text{coker} T \). We will denote by \( \Phi_i(X, Y) \) the collection of Fredholm operators of index \( i \). The sets \( \Phi_i(X, Y) \) are the connected components of the set of Fredholm operators \( \Phi(X, Y) \). The latter is an open subset of \( \mathcal{L}(X, Y) \), the space of all bounded operators from \( X \) to \( Y \), equipped with the norm topology. \( \Phi_0(X, Y) \) contains the set \( GL(X, Y) \) of invertible operators as an open subset. Moreover, the Riesz characterization of \( \Phi_0(X, Y) \) asserts that \( T \in \Phi_0(X, Y) \) if and only if there exists compact operator \( K \) such that \( T - K \in GL(X, Y) \).

Given a \( k \geq 1 \), a \( C^k \) map \( f : U \to Y \) is said to be a Fredholm map if \( Df(x) \) is a Fredholm operator for every \( x \in U \). The index of \( f \) is nothing but the index of \( Df(x) \) provided the later is independent of \( x \). In what follows, except when otherwise stated, “Fredholm” means “Fredholm of index 0”.

Let us briefly describe “linearized bifurcation” for families of Fredholm maps of index 0 and its relation to branching.

Let \( h : \Lambda \times U \to Y \) be such that \( h_\lambda \) is a Fredholm map of index 0 for any \( \lambda \in \Lambda \). The family of linearizations of \( h \) along the trivial branch, defined by \( L_\lambda \equiv D_x h(\lambda, 0) \), is a family of linear Fredholm operators \( L : \Lambda \to \Phi_0(X, Y) \).\(^1\) It follows from the implicit function theorem that if \( \lambda_* \) is a bifurcation point, then \( L_{\lambda_*} = D_x h(\lambda_*, 0) \) cannot be invertible, which, since \( L_{\lambda_*} \) is Fredholm of index 0, is equivalent to the assertion that \( \ker L_{\lambda_*} \neq \{0\} \). In other words, the set \( \text{Bi} f(h) \) of all bifurcation points is a closed subset of the singular set

\[
\Sigma(L) \equiv \{ \lambda \in \Lambda \mid \ker L(\lambda) \neq \{0\} \}
\]

of the family \( L \) of linearizations of \( h \) along the trivial branch.

\(^1\)We write the parameter \( \lambda \) as a subscript in order distinguishes it from the variables of the problem.
The simplest result of one-parameter bifurcation theory which detects points of \( \Sigma(L) \) that belong to \( Bif(h) \) is the following well-known Crandall-Rabinowitz bifurcation theorem.

**Theorem 1.3.** Let \( h: (a, b) \times U \rightarrow Y \) be a \( C^2 \) mapping, where \( U \) is a neighborhood of 0. For all \( \lambda \), assume \( h(\lambda, 0) = 0 \) and let \( L_\lambda \equiv D_x h(\lambda, 0) \) be the family of linearizations of \( h \) along the the trivial branch. Let the parameter \( \lambda_* \in (a, b) \) have the property that

1. \( L_{\lambda_*} \) is Fredholm of index 0 and \( \ker L_{\lambda_*} \) is generated by a vector \( v \neq 0 \);
2. \( \hat{L}_{\lambda_*}(v) \notin \text{im } L_{\lambda_*} \).

Then \( \lambda_* \) is a bifurcation point for nontrivial solution of \( h(\lambda, x) = 0 \), and there is a \( C^{k-1} \) curve of nontrivial solutions branching from \( (\lambda_*, 0) \).

The Crandall-Rabinowitz theorem combined with Corollary 1.2 leads to the following proposition which characterizes branch points of a Fredholm map \( f \) among those belonging to \( C(f) \).

Let \( x_* \in C(f) \) be a critical point such that \( \dim \ker Df(x_*) = 1 \) and let \( v \neq 0 \) be a vector which spans the kernel. If we choose \( u \neq 0 \) and apply Corollary 1.2 along the path \( \gamma(\lambda) = \lambda u + x_* \), we deduce that \( x_* \) is a branch point for \( f \) whenever \( \lambda = 0 \) is a bifurcation point from the trivial branch of the parametrized family

\[
h(\lambda, x) = f(x + \lambda u + x_*) - f(\lambda u + x_*),
\]

defined near \( (0, x_*) \in \mathbb{R} \times U \). On the other hand,

\[
L_\lambda = Df(\lambda u + x_*) \quad \text{and} \quad \hat{L}_0 v = D^2 f(x_*)(u, v).
\]

Therefore, applying the Crandall-Rabinowitz theorem to the family (4) we obtain

**Proposition 1.4.** Let the mapping \( f: U \rightarrow Y \) be \( C^2 \)-Fredholm. Let \( x_* \in C(f) \) be a critical point such that \( \dim \ker Df(x_*) = 1 \) and let \( v \neq 0 \) be a vector which spans the kernel. Then \( x_* \in U \) is a branch point of \( f \) provided that, for some \( u \), \( D^2 f(x_*)(u, v) \) is not in the image of \( Df(x_*) \).

1.1. **The separation property.** Our main goal is to obtain from the local assumption of Proposition 1.4 at the point \( x_* \) a global separation property of \( B(f) \).

The following theorem is an example of what we have in mind. We postpone the proof until after we introduce the necessary tools.

**Theorem 1.5.** Let \( U \) be an open simply connected subset of \( X \) and the mapping \( f: U \rightarrow Y \) be both \( C^2 \)-Fredholm and proper on closed bounded sets. Let \( x_* \in C(f) \) be such that

1. \( \ker Df(x_*) \) is generated by a single vector \( v \neq 0 \);
2. for some \( u \neq 0 \), \( D^2 f(x_*)(v, u) \notin \text{im } Df(x_*) \).

Then \( x_* \in B(f) \) and \( B(f) \) disconnects \( U \).

**Remark.** Observe that irrespective of \( U \) being simply connected we may deduce from the above theorem that \( B(f) \) always disconnects a small enough neighborhood \( W \) of \( x_* \). This plainly follows from the fact that, under assumptions i), ii) of Theorem 1.5, there is an open neighborhood \( W \) of \( x_* \) such that \( B(f) \cap W \) is a one codimensional submanifold of \( W \). For a proof of this latter assertion see either the remark after the proof of the forthcoming Lemma 1.10 or Lemma 2.1 in [3].

\[ ^2 \text{dot stands for derivative.} \]
**Example 1.** Consider the problem

\[
\begin{cases}
-Lu(x) = \phi(x, u(x)), & x \in \Omega \\
u(x) = 0 & x \in \partial \Omega.
\end{cases}
\]  

(5)

Here \( L = L(x, D) \) is a self-adjoint uniformly elliptic operator of second order with smooth coefficients, defined on a bounded domain \( \Omega \) in \( \mathbb{R}^n \) with a smooth boundary, and \( \phi(x, u) \) is a smooth function. The mapping \( f: C_0^{2,\alpha} (\Omega) \to C^{0,\alpha}(\Omega) \), defined by

\[
f(u)(x) = Lu(x) + \phi(x, u(x)),
\]

is a smooth Fredholm map of index 0 and proper on closed bounded sets. At \( u \equiv 0 \) the Fréchet derivative of \( f \) is given by \( Df(0)h = Lh + \phi_u(x,0)h \) and its Hessian given by \( D^2f(0)(v,w) = \phi_{uu}(x,0)vw \).

Assume that \( \ker Df(0) \) is generated by a single eigenfunction \( v \), and that the function \( g(x) \equiv \phi_{uu}(x,0) \) does not vanish identically. Let \( \mu \in C_0^{2,\alpha} \) be such that \( \mu(x) > 0 \) on \( \Omega \). If we take in Theorem 1.6 \( u \equiv \mu \cdot g \), then \( D^2f(0)(u,v) = \mu^2g^2v \). Since \( \int \mu^2g^2(x) dx > 0 \), and since \( w \in \text{im} \, Df(0) \) if and only if \( \int w(x)v(x)dx = 0 \), it follows that \( D^2f(0)(u,v) \notin \text{im} \, Df(0) \). Therefore \( B(f) \) disconnects \( C_0^{2,\alpha}(\Omega) \).

If, in Theorem 1.5, we also assume that \( f \) is proper, \( u = v \) and no points of \( U \setminus B(f) \) are mapped by \( f \) into \( f(B(f)) \), more can be said (compare with [3]).

**Theorem 1.6.** Let \( U \) be an open simply connected subset of \( X \) and the mapping \( f: U \to Y \) be \( C^2 \)-Fredholm and proper. Assume that

i) \( \ker Df(x_*) \) is generated by \( v \neq 0 \);

ii) \( D^2f(x_*)(v,v) \notin \text{im} \, Df(x_*) \).

Furthermore, assume that \( f^{-1}[f(B(f))] = B(f) \). Then \( x_* \) is a branch point for \( f \) and not only does \( B(f) \) disconnect \( X \), but \( f(B(f)) \) disconnects \( Y \). Moreover there are two components \( C^+ \) and \( C^- \) of \( Y \setminus f(B(f)) \) such that for \( y \in C^- \) the equation \( f(x) = y \) has no solutions while for \( y \in C^+ \) the same equation has exactly two solutions.

**Remark.** In the terminology of [3], a point \( x \in C(f) \) for which \( \ker Df(x) \) is generated by \( v \neq 0 \) and \( D^2f(x)(v,v) \notin \text{im} \, Df(x) \) is called an ordinary singular point. In [3, Theorem 2.6] it is assumed that \( C(f) \) is connected and every point in \( C(f) \) is an ordinary singular point. Under this stronger assumption, \( B(f) = C(f) \) is a codimension one submanifold of \( U \) and the same is true for the set of singular values \( f(B(f)) \) of the mapping \( f \). But then necessarily \( Y \setminus f(B(f)) = C^+ \cup C^- \) which is a much more precise result. In the case considered in [3] the mapping \( f \) is globally equivalent to a fold \( f(x,t) = (x,t^2) \). Assuming that \( ii) \) holds at only one point covers more general classes of mappings.

**Remark.** It follows from Theorem 1.6 that if \( x_* \) is an ordinary singular point of a proper \( C^2 \)-Fredholm map verifying \( f^{-1}[f(B(f))] = B(f) \), then the base point degree (see section 1.3) \( \deg_{v}(f,V,f(x_*)) \), with respect to any open bounded subset \( V \) of \( U \) on which is defined, must vanish.

Using a generalization of the Crandall-Rabinowitz theorem proved in [17, Theorem 6.18], there is the following improvement of Theorem 1.5.

**Theorem 1.7.** Let \( U \) be an open simply connected subset of \( X \) and the mapping \( f: U \to Y \) be \( C^2 \)-Fredholm. Let \( x_* \in U \) be such that

i) \( Df(x_*) \) has an odd dimensional kernel \( N \);

ii) there exists a direction \( u \neq 0 \) such that

\[
D^2f(x_*)(u,N) \oplus \text{im} \, Df(x_*) = Y.
\]
Then \( x_* \in \mathcal{B}(f) \) and \( \mathcal{B}(f) \) disconnects \( U \).

**Remark.** Applying multi-parameter bifurcation results from [2, 16] to families associated to restrictions of the map \( f \) to \( k \)-dimensional planes through the point \( x_* \) with \( k > 1 \), appropriate modifications of \((i)\) and \((ii)\) of Theorem 1.7 provide more general sufficient conditions for branching. However, the precise statements are cumbersome and we omit them here.

1.2. **The parity of a path of Fredholm operators.** For compact perturbations of the identity, Theorem 1.5 may be proved using the the homotopy invariance property of the Leray-Schauder degree. To establish this theorem in the Fredholm setting it is necessary to make an appeal to a homotopy invariant of paths of linear Fredholm operators called parity. We will take \( X = Y \) is this and the following two subsections. There is no loss of generality in doing so because, if there is a Fredholm operator of index zero from \( X \) to \( Y \), then \( X \) and \( Y \) are isomorphic. Our constructions are invariant under the action of isomorphisms and everything we prove for the case \( X = Y \) holds in general.

Let \( \Sigma \equiv \Phi_0(X) \setminus \text{GL}(X) \) be the set all singular, that is, noninvertible, Fredholm operators. The singular set \( \Sigma \) is stratified by the following countable disjoint collection of submanifolds:

\[
\Sigma = \bigcup_{k \geq 1} \Sigma_k, \quad \text{where each } \Sigma_k \equiv \{ T \in \Phi_0(X) \mid \dim \ker T = k \}.
\]

Each stratum \( \Sigma_k \) is a submanifold of \( \Phi_0(X) \) of codimension \( k^2 \).

As we observed before, the implicit function theorem tells us that bifurcation arises only at points \((\lambda_*, 0)\) of the trivial branch at which \( L_{\lambda_*} \) belongs to \( \Sigma \).

Roughly speaking, the parity of a path \( L: I \to \Phi_0(X) \) of Fredholm operators is an intersection index, which counts (mod-2) the number of intersection of the path with the singular set \( \Sigma \). More precisely, using approximations by smooth paths and elementary transversality, any continuous path \( L: [a, b] \to \Phi_0(X) \) with invertible end-points can be approximated in \( C^0(I; L(X)) \) by a smooth path \( \hat{L} \) that is transversal to \( \Sigma_k \), for every \( k \geq 1 \). Since, for \( k > 1 \) the codimension of \( \Sigma_k \) is greater than one, the intersection of a transversal path with \( \Sigma_k \) is empty in this case. Moreover, by the Local Representation of Transversality [1, Theorem 17.1], transversal intersections are isolated. Therefore \( \hat{L} \) can have only a finite number of intersections points with \( \Sigma \) and they are all transversal intersections with the top stratum \( \Sigma_1 \).

We call a path \( L: [0, 1] \to \Phi_0(X) \) admissible if it has invertible end-points. The **parity** of an admissible path \( L: [0, 1] \to \Phi_0(X) \), denoted either by \( \sigma(L) \) or \( \sigma(L, [a, b]) \), is the element of the multiplicative group \( \mathbb{Z}_2 = \{\pm 1\} \) defined by

\[
\sigma(L) \equiv (-1)^k,
\]

where \( k \) is the number of transversal intersections with \( \Sigma \) of a path \( \hat{L} \) that is close enough to \( L \) and intersects \( \Sigma \) transversally (see [17, section 3]).

Parity has the typical properties of an index of intersection, despite the fact that the manifold \( \Sigma_1 \) is not a closed submanifold. Indeed, \( \sigma(L) \) is multiplicative under concatenation of paths, composition and direct sum of operators. What is most important, it is invariant under homotopies keeping the end points of the path invertible, and even by free homotopies in the case of closed paths. Moreover, \( \sigma(L) = 1 \) if and only if the path \( L \) can be deformed into a path of invertible operators keeping the end-points invertible. The proof of all these properties can be found in [17].
1.3. The base-point degree. The parity was used in [19] in order to define a notion of orientation and, by way of orientation, an oriented degree for $C^2$-Fredholm maps $f: U \to X$ defined on a simply connected open subset $U$ of $X$ that are proper on closed bounded subsets of $U$. It was called the base-point degree in [19] because of its dependence on the choice of a base-point $b \in U$. In [26], the above degree theory was extended to $C^1$-Fredholm maps, using the same approach.

Given a $C^2$-Fredholm mapping $f: U \to X$, we first assume that $f$ has regular points and we choose one of them, call it $b$, as a base-point. Then, the choice of an orientation $\varepsilon(b) = \pm 1$ at the point $b$ induces an orientation $\varepsilon(x) = \pm 1$ at any other regular point $x$ of $f$, which is defined by the rule

$$
\varepsilon(x) = \varepsilon(b) \cdot \sigma(Df \circ \gamma),
$$

where $\gamma$ is any path in $U$ joining $x$ to the base-point $b$. Indeed, since $U$ is simply connected, by the homotopy and concatenation properties of the parity, $\varepsilon(x)$ is independent of the choice of path joining the regular point $x$ to $b$, and hence the orientation $\varepsilon$ is properly defined.

If the map $f$ is proper on the closure of an open bounded set $\Omega$ whose closure is contained in $U$, and $y \not\in f(\partial \Omega)$ is a regular value of the restriction of $f$ to $\Omega$, the degree is defined as usual, by summing the orientations (relative to $b$) of points in $f^{-1}(y) \cap \Omega$. Namely,

$$
\deg_b(f, \Omega, y) \equiv \sum_{x \in f^{-1}(y) \cap \Omega} \varepsilon(x).
$$

If $f^{-1}(y) = \emptyset$ or if all points of the domain are critical points of $f$, then, by definition, $\deg_b(f, \Omega, y) \equiv 0$, for any choice of base-point $b \in U$.

Using the Sard-Smale theorem regarding approximation by regular values, this definition extends to a degree, $\deg_b(f, \Omega, y)$, defined on triples $(f, \Omega, y)$, where $f: U \to X$ is $C^2$-Fredholm and proper on the closure of an open bounded set $\Omega$ whose closure is contained in $U$, and $y \not\in f(\partial \Omega)$.

The base-point degree has all the usual properties of a degree, except for the homotopy invariance. Indeed, a theorem of Kuiper asserts that if $X$ is an infinite dimensional separable Hilbert space, then $GL(X)$ is contractible. Using this it is easy to see that there cannot be a degree theory for Fredholm mappings that both extends the Leray-Schauder degree and has the homotopy invariance property.

A natural substitute for the homotopy property is the following homotopy variation property (see [19]): Let $h: [a, b] \times U \to X$ be a Fredholm homotopy that is proper on $\Omega$ and such that $y \not\in h([0, 1] \times \partial \Omega)$. Assume that $b$ is a base-point for both $h(0, \cdot)$ and $h(1, \cdot)$ Then, if $L$ is the path defined by $L_{\lambda} \equiv D_{x} h(\lambda, b)$,

$$
\deg_b(h_{0}, \Omega, y) = \sigma(L) \deg_b(h_{1}, \Omega, y).
$$

Remark. There are many other approaches to the construction of an oriented (that is, taking values in $\mathbb{Z}$) degree theory for general Fredholm mappings (see [15, 11, 9]). However, the relation between parity and the behavior of the degree under general homotopies seems not to have been observed prior to [19].

1.4. The Krasnoselskij bifurcation principle and its consequences. The homotopy variation property suffices to establish the following extension of the classical Krasnoselskij bifurcation principle for compact perturbations of the identity to $C^2$-Fredholm maps of index 0 (see [26] for the $C^1$ case).
Theorem 1.8. Let $U$ be a neighborhood of $0$ and $h: [a, b] \times U \to X$ be a family of $C^2$ maps. Assume that for all $\lambda \in [a, b]$, $h(\lambda, 0) = 0$ and $L_\lambda \equiv D_x h(\lambda, 0)$ is Fredholm. If $L_a$ and $L_b$ are invertible and $\sigma(L) = -1$, then the interval $[a, b]$ contains some bifurcation point for nontrivial solutions of $h(\lambda, x) = 0$.

Proof. Assume there are no bifurcation points. Then, if $r > 0$ is sufficiently small, the only solutions of $h(\lambda, x) = 0$ in the closed cylinder $[a, b] \times \overline{B}(0, r)$ are those that are trivial. Since $\Phi_0(X)$ is open, for small enough $r$, the restriction of $h$ to $[a, b] \times \overline{B}(0, r)$ is a Fredholm family. Furthermore, $h$ is proper on $\overline{B}(0, r)$ because, as a consequence of the forthcoming Proposition 1.12, Fredholm maps are locally proper. Choose $b = 0$ as the base-point for the degree. Then, by the very definition of the base-point degree,

$$\deg_0(h_a, B(0, r), 0) = \deg_0(h_1, B(0, r), 0) = 1.$$ 

But there are no solutions of $h(\lambda, x) = 0$ on $[0, 1] \times \partial B(0, r)$. We appeal to the homotopy invariance relation (8), in the case $\Omega = B(0, r)$, to deduce that $\sigma(L) = 1$. This contradicts our assumption. \hfill $\square$

Let $U$ be an open subset of $X$ and the mapping $f: U \to X$ be $C^1$-Fredholm. A continuous path $\gamma: [a, b] \to U$ is said to be admissible with respect to $Df$ provided that the path $Df \circ \gamma$ has invertible ends.

Corollary 1.9. Let $U$ be an open subset of $X$ and the mapping $f: U \to X$ be both $C^2$-Fredholm. Let the path $\gamma: [a, b] \to U$ be admissible with respect to $Df$ with $\sigma(Df \circ \gamma) = -1$. Then there is a $\lambda_*$ in $(a, b)$ for which $\gamma(\lambda_*)$ is a branch point for $f$.

Proof. For $r > 0$ sufficiently small consider the family of $C^2$-Fredholm mappings $h: [a, b] \times \overline{B}(0, r) \to X$ defined by $h(\lambda, x) \equiv f(x + \gamma(\lambda)) - f(\gamma(\lambda))$. For each $\lambda \in [a, b]$, $h(\lambda, 0) = 0$ and $L_\lambda \equiv D_x h(\lambda, 0) = Df(\gamma(\lambda))$. Since $\sigma(L) = -1$, according to Theorem 1.8 there is a bifurcation point $\lambda_*$ for nontrivial solutions of $h(\lambda, x) = 0$. But then, according to Corollary 1.2, $\gamma(\lambda_*)$ is a branch point for $f$. \hfill $\square$

1.5. Proof of Theorem 1.5. We already have shown, in the discussion preceding the statement of this theorem, that $x_*$ is a branch point for $f$. It remains to show that $B(f)$ disconnects $U$. We begin with the following lemma which explains why assumption ii) of the Crandall-Rabinowitz theorem is a transversality condition.

Lemma 1.10. A $C^1$-path $L: (-\delta, \delta) \to \Phi_0(X, Y)$ for which $L_0 \in \Sigma_1$ is transverse to $\Sigma_1$ at $L_0$ if and only if for every $v \neq 0$ in $\ker L_0$, $L_0 v \notin \im L_0$.

Proof. We compute the tangent space $T_S(\Sigma_1)$ to the submanifold $\Sigma_1$ of $\Phi_0(X, Y)$ at an operator $S \in \Sigma_1$, and so establish a criterion for determining when a $C^1$ path crosses $\Sigma_1$ at $S$ transversally.

Each tangent vector $R \in T_S(\Sigma_1)$ is of the form $R = M_0$, where $M: (-\delta, \delta) \to \Sigma_1$ is a smooth path such that $M_0 = S$. Since the dimension of the kernels of the path $M$ is constantly one, the kernels form a vector bundle and taking a local smooth section of this bundle we get a smooth path $e: (-\delta, \delta) \to X$ with the property that $M_\lambda(e(\lambda)) = 0$ for all $\lambda$. Differentiating at $\lambda = 0$ we get

$$M_0(e(0)) + M_0(\dot{e}(0)) = 0.$$ 

It follows from this that if $R \in T_S(\Sigma_1)$, then for every $v \in \ker S$, $Rv \in \im S$. 


On the other hand, the subspace of all elements $R \in \mathcal{L}(X,Y)$ verifying the above condition is clearly of codimension one in $\mathcal{L}(X,Y)$. Thus, comparing dimensions, we deduce that $R$ belongs to $T_S(\Sigma_1)$ if and only if $R$ maps $\ker S$ into $\text{im} S$. The proof is complete. □

The above lemma extends to several parameter families with exactly the same proof. We will only use the following conclusion of this lemma.

**Lemma 1.11.** A $C^2$-Fredholm map $f : U \to Y$ verifies assumptions i) and ii) of Theorem 1.5 at $x_*$ if and only if the family $Df : U \to \Phi_0(X,Y)$ is transversal to the stratum $\Sigma_1$ at $x_*$. 

As was already observed in the remark following the statement of Theorem 1.5, from the Local Representation of Transversality [1, Theorem 17.1] it follows that assumptions i) and ii) of Theorem 1.5 imply that there exists an open neighborhood $W$ of $x_*$ such that the set of branch points $B(f) \cap W$, which coincides with $C(f) \cap W$ by Theorem 1.5, is a one-codimensional submanifold of $W$.

We turn to the details of the proof of Theorem 1.5. Firstly, choose $\delta > 0$ sufficiently small so that the segment between $x_- = x_* - \delta u$ and $x_+ = x_* + \delta u$ is contained in $U$, and consider the path $\eta : [-\delta, \delta] \to U$ defined by $\eta(t) = x_* + tu$. From the hypothesis of Theorem 1.5 and Lemma 1.10 it follows that the path $L = Df \circ \eta$ crosses $\Sigma_1$ at 0 transversally. Since transversal intersections are isolated, by possibly choosing a smaller $\delta$ we can assume that the only intersection point of $L$ with $\Sigma_1$ is $L_0$. But then, $L_\lambda$ is invertible for all $\lambda$ with $0 < |\lambda| < \delta$, and by the very definition of parity we have that $\sigma(L, [-\delta, \delta]) = -1$.

Consider now any path $\gamma : [0, 1] \to U$ joining $x_-$ to $x_+$. Since $U$ is simply connected it follows easily from the homotopy and concatenation properties of the parity that the parity of $Df$ along any admissible path depends only on its end-points. Therefore, since $\gamma$ and $\eta$ have the same end points, $\sigma(Df \circ \gamma) = \sigma(Df \circ \eta) = -1$.

We appeal now to Proposition 1.9 in order to conclude that there is a branch point for $f$ on the image of the path $\gamma$. Thus, no paths in the open set $U \setminus B(f)$ can join $x_-$ to $x_+$, and therefore the points $x_-$ and $x_+$ belong to different connected components of his set. This proves Theorem 1.5.

Theorem 1.7 follows from [17, Theorem 6.18] using the same argument.

**Example 2.** We want to show that, in Theorem 1.5, the assumption that $U$ is simply connected is essential. To do so, let $H$ be an infinite dimensional separable Hilbert space and let $X \equiv H \times \mathbb{R}$. Also, for $\lambda \in [-\delta, \delta]$, define $L_\lambda(h, r) \equiv (h, \lambda r)$. Then $\Sigma(L) = \{0\}$ and $\sigma(L, [-\delta, \delta]) = -1$, because $\ker L_0 = \{(0, r)\}$, $\text{im} L_0 = H \times 0$ and therefore $L_0(0, 1) \notin \text{im} L_0$. According to Kuiper's theorem, there is a path of isomorphism $\tilde{L}$ in $GL(X)$ joining $L_\delta$ with $L_{-\delta}$. Let $S^1 \subseteq \mathbb{C}$ be the unit circle. We can identify the interval $[-\delta, \delta]$ with a closed neighborhood of 1 \in $S^1$ via the exponential function and thereby, $\tilde{L}$ provides an extension of $L : [-\delta, \delta] \to \Phi_0(X)$ to a continuous family $L : S^1 \to \Phi_0(X)$. By the concatenation property of parity, $\sigma(L, S^1) = -1$. Using standard approximation by smooth paths, we can assume that $L$ is smooth.

Consider now the Hilbert manifold $M = S^1 \times X$ and define $f : M \to M$ by $f(\lambda, h, r) = (\lambda, L_\lambda x)$. Clearly $f$ is a smooth Fredholm mapping of index 0, and if $m = (\lambda, h, r) \in M$, then $Tf(m) : T_m M \to T_m M$ is given by the operator matrix

$$
\begin{pmatrix}
L_\lambda & \frac{\partial}{\partial \lambda} L_\lambda \\
0 & I
\end{pmatrix}
$$
Since \( L_\lambda \) is transversal to \( \Sigma \) at \( \lambda = 0 \) and \( \Sigma(L) = \{0\} \), \( B(f) = C(f) = X \times \{1\} \). In local coordinates at \( \lambda = 1 \) the mapping \( f \) is given by \( f(\lambda, h, r) = (\lambda, h, \lambda r) \). Thus, arguing as above, if \( m \in M \) is of the form \( m = (1, h, r) \) and \( u = (0, 0, 1) \), then the transversality condition \( D^2f(m)(u, u) \notin im Df(m) \) holds at \( m \). However, clearly \( B(f) \) has a connected complement in \( M \). On the other hand, every Hilbert manifold is diffeomorphic to an open subset of a Hilbert space [22]. Hence, up to a diffeomorphism, we have exhibited a mapping \( f: U \subseteq X \rightarrow Y \) verifying the second hypothesis of Theorem 1.5 at each critical point while \( U \setminus B(f) \) is connected.

**Remark.** Concrete examples of closed paths in \( \Phi_0(X) \) with parity \(-1\) that are induced by linear ordinary differential operators can be easily found.

### 1.6. Proof of Theorem 1.6

The proof is based on one of the many versions of the Lyapunov-Schmidt reduction regarding the reformulation of a problem in an infinite dimensional space as one in a finite dimensional space. We illustrate the general scheme in the next proposition. To simplify notation, we take \( x = 0, f(0) = 0 \).

**Proposition 1.12.** Let \( U \subseteq X \) be a neighborhood of \( 0 \) in \( X \) and \( f: U \rightarrow Y \) be a \( C^k \)-Fredholm map with \( f(0) = 0 \). Let \( Z \) be any complementary subspace of \( im Df(0) \) in \( Y \). There is a \( C^k \)-local change of coordinates \( \psi: V \subseteq Y \rightarrow W \subseteq X \) at \( 0 \) such that \( f(\psi(y)) = y - k(y) \), with \( k(V) \subseteq Z \).

**Proof.** Let \( N = ker Df(0) \). Let \( h: U \subseteq X \rightarrow Z \) be any \( C^k \) map, defined on a neighborhood of \( 0 \), such that the restriction of \( Dh(0) \) to \( N \) is an isomorphism between \( N \) and \( Z \) and let \( \phi(x) \equiv f(x) + h(x) \). Clearly, the derivative \( D\phi(0) = Df(0) + Dh(0) \) is injective, and therefore is an isomorphism since \( D\phi(0) \), being a finite dimensional perturbation of a Fredholm operator, is itself Fredholm. We appeal to the inverse function theorem to find two neighborhoods \( W \) and \( V \) of \( 0 \) such that \( \phi|_W : W \rightarrow V \) is a local diffeomorphism at \( 0 \). Taking \( \psi = \phi^{-1} \), we get \( f \circ \psi(y) = y - k(y) \) with \( k(V) \subseteq Z \). The proof is complete.

There are two consequences of this proposition that deserve mention. Firstly, we observe that for \( z \in V \cap Z \), the equation \( f(x) = z \) holds in \( W \) if and only if \( x = \psi(y) \) and \( y - k(y) = z \) belongs to \( V \). But then necessarily \( y \) belongs to \( Z \cap V \). Therefore \( \psi \) induces a diffeomorphism between the set of solutions of the equation \( f(x) = z \) in \( W \) and the set of solutions of the equation \( g(y) = z \), where \( g \) is the restriction of the map \( y - k(y) \) to \( Z \cap V \). Since \( Z \cong \mathbb{R}^n \) we obtain in this way a bijection between solutions in \( W \) of the original equation with solutions of a finite system of nonlinear equations \( g(y) = z \) in the same number of unknowns. The system \( g \) is clearly related to what is classically known as the “bifurcation equation” of the Lyapunov-Schmidt method. Secondly, since \( f \circ \psi \) is a compact perturbation of the identity, the map \( f \) is locally proper in the sense that each point has neighborhood on whose closure it is proper. Therefore \( f \) also is locally closed. It follows in particular that \( B(f) \) is the set of points where \( f \) fails to be a local homeomorphism, a property asserted in the introduction.

We turn to the details of the proof of Theorem 1.6. That \( B(f) \) separates \( U \) follows from Theorem 1.5. In order to prove the remaining part, let \( \beta \in Y^* \) be a nontrivial functional which vanishes on \( im Df(0) \) and let \( v \) be a generator of \( N = ker Df(0) \) with \( \beta D^2f(0)(v, v) = 1 \). Since, by hypothesis \( w = D^2f(0)(v, v) \notin im Df(0) \) we can apply Proposition 1.12, taking as \( Z \) the subspace generated by the vector \( w \).

Recall that there is an open neighborhood \( U' \) of \( 0 \) such that \( B(f) \cap U' \) is a one-codimensional manifold, which, by [3, Lemma 2.1], is locally defined as the zero-set
of a function $\eta: U' \rightarrow \mathbb{R}$ for which
\[ d\eta(0)v = \beta D^2 f(0)(v, v) = 1. \] (9)

Let us define $h: U' \rightarrow Z$ by $h(x) = \eta(x) w$, and let $\phi(x) = f(x) + h(x)$. Clearly, $\phi$ restricted to $B \cap U'$ coincides with $f$ and $D\phi(0)v = w$. Moreover, it follows from (9) that $D\phi(0)$ is invertible (see the proof of [3, Lemma 2.3]) and therefore $\phi$ restricts to a diffeomorphism between two neighborhoods $W$ and $V$ of $0$ in $U$ and $Y$ respectively.

Now consider the real valued function $g$ defined for sufficiently small enough $t$ by
\[ g(t) \equiv \beta f\psi(tw) = \beta(tw - \eta(\psi(tw)))w = t - \eta(\psi(tw)), \] (10)
where $\psi = \phi^{-1}: V \rightarrow W$. Up to the isomorphism $Z \cong \mathbb{R}$, $g$ coincides with the map $g$ of Proposition 1.12. We have $g(0) = 0$. Moreover, since $D\psi(0)w = v$, from the right-hand side of (10) and (9) we get
\[ g'(0) = 1 - d\eta(0)(D\psi(0)w) = 0. \]

On the other hand, using the left-hand of (10), we obtain
\[ g''(0) = \beta D^2 f(0)[D\psi(0)w, D\psi(0)w] + \beta Df(0) \circ D^2 \psi(0)[w, w] = \beta D^2 f(0)[v, v] = 1. \]

It follows that $g(t) = t^2 + o(t^2)$, so that $0$ is a strict local minimum of $g$. Therefore the equation $g(t) = s$ has no solutions for $s < 0$ and exactly two solutions for $0 < s < \delta$, if $\delta$ is small enough.

Let $\mathcal{M} = \mathcal{B}(f) \cap W$. Since the restriction of $\psi$ to $\mathcal{M}$ coincides with the restriction of $f$ and since $f^{-1}(\mathcal{B}(f)) = \mathcal{B}(f))$, it follows that the restriction of $f$ to $\mathcal{M}$ is a diffeomorphism of $\mathcal{M}$ with $\mathcal{N} = f(\mathcal{M})$. Therefore $\mathcal{N}$ is also a submanifold of codimension one in $V$, $0$ belongs to $\mathcal{N}$ and, since the restriction of $f$ to $\mathcal{M}$ is a diffeomorphism, $f^{-1}(0) \cap \mathcal{M} = \{0\}$. Notice that, for $s \neq 0$ but close to zero in $\mathbb{R}$, $sw \notin \mathcal{N}$, for, if not, $w$ would be tangent to $\mathcal{N}$ at 0, which is impossible by (9).

On the other hand, the map $f$, being proper, sends closed sets into closed ones. Therefore $V' = V \setminus f(U \setminus W)$ is an open neighborhood of $0$ such that $f^{-1}(V') \subseteq W$.

Let us take $s > 0$ so small that both $sw$ and $-sw$ belong to $V'$. Then, if $x$ solves $f(x) = \pm sw$, we deduce from (10) that $x = \psi(tw)$ and $g(t) = s$. Thus $f^{-1}(\pm sw)$, depending on the sign, is either empty or consists in exactly two points. On the other hand, since $f$ is proper and $f^{-1}(\mathcal{B}(f)) = \mathcal{B}(f))$, $f: U \setminus \mathcal{B}(f) \rightarrow Y \setminus f(\mathcal{B}(f))$ is a proper local homeomorphism.

A simple argument, analogous to the one used above in finding the subset $V'$ (see either [24, Page 8] or [3, Theorem 1.2]), shows that for such a map the cardinality of $f^{-1}(y)$ is a locally constant function on $Y \setminus f(\mathcal{B}(f))$, and therefore it must be constant on the components of this open set. In particular $\pm sw$ belong to two different components $C^\pm$ of $Y \setminus f(\mathcal{B}(f))$. This completes the proof of Theorem 1.6.

1.7. **Branch points and connected zero-sets.** In [30], using the degree for $k$-set contractive perturbations of the identity, Smith and Stuart proved an interesting theorem for $C^1$ maps the form $f = \text{Id} - F$ where $F$ is a $k$-set contraction. Assuming that $F: \hat{\Omega} \rightarrow X$ is a continuous $k$-set-contraction, with $k < 1$, defined on the closure of a bounded connected open subset $D$ of a real Banach space $X$ that is continuously differentiable on $D$, they proved that, if $F(x) \neq x$ on $\partial\Omega$, $\deg(f|\hat{\Omega}, 0) = \pm 1$, and the set $\mathcal{R}(f) = \Omega \setminus \mathcal{C}(f)$ of regular points of $f|\Omega$ is connected and dense, then the set of fixed points of $F$, that is, the set of zeroes of $f$, is connected.

Differentiable mappings of the above form are Fredholm of index 0. Moreover they are proper on closed bounded sets. If $U'$ has the base-point degree described above ([26]) we will prove the result of [30] for all $C^1$-Fredholm maps as above.
Moreover we will improve it, by substituting \( \mathcal{R}(f) \) with \( \Omega \setminus \mathcal{B}(f) \), a set which is strictly larger than \( \mathcal{R}(f) \) in many cases. For example, if \( f(x) = x^3 \) and \( \Omega = (-1, 1) \) then \( \mathcal{R}(f) \) is disconnected but \( \Omega \setminus \mathcal{B}(f) \) is connected.

**Theorem 1.13.** Let \( f : X \rightarrow X \) be \( C^1 \)-Fredholm map that is proper on closed bounded sets. Let \( \Omega \) be an open bounded connected subset of \( X \) for which \( 0 \notin f(\partial \Omega) \) and \( \deg_b(f, \Omega, 0) = \pm 1 \) for some \( b \in X \). Assume that \( \Omega \setminus \mathcal{B}(f_{|\Omega}) \) is connected and dense in \( \Omega \). Then the set \( S = \{ x \in \Omega | f(x) = 0 \} \) is connected. In particular, if \( f(x) = 0 \) for some regular point \( x \), then \( x \) is the only zero of \( f \) in \( \Omega \).

**Proof.** We argue by contradiction. Assume that there is a non-trivial separation \( S_1, S_2 \) of \( S \). Since both sets are compact, they are at a positive distance one from the other. Choose two open bounded disjoint subsets of \( \Omega \), \( \Omega_1 \) and of \( \Omega_2 \), with \( S_1 \subseteq \Omega_1 \) and \( S_2 \subseteq \Omega_2 \).

By the additivity property of the base-point degree
\[
\pm 1 = \deg_b(f, \Omega, 0) = \deg_b(f, \Omega_1, 0) + \deg_b(f, \Omega_2, 0)
\]
Thus at least one of the numbers on the right hand side does not vanish. Assume that
\[
\deg_b(f, \Omega_1, 0) \neq 0.
\]
Let us choose a point \( x_0 \in S_2 \). It follows from the homotopy property of the base-point degree that \( \deg_b(f, \Omega, y) \) depends only on the connected component of \( y \) in \( Y \setminus f(\partial \Omega) \). Using the density hypothesis, we can approximate \( x_0 \) with a point \( x' \in \Omega \) not belonging to \( \mathcal{B}(f_{|\Omega}) \) and such that
\[
\pm 1 = \deg_b(f, \Omega, 0) = \deg_b(f, \Omega, f(x')).
\]
Then, by the existence property of the base-point degree, \( f \) is a local homeomorphism at \( x' \). Choose two connected open neighborhoods \( U' \subseteq \Omega_2 \) and \( V' \subseteq Y \setminus f(\partial \Omega) \) of \( x' \) and \( f(x') \) respectively such that \( f_{|U'} \) is a homeomorphism of \( U' \) with \( V' \). It follows from [32, Theorem 2] that \( U' \) contains regular points of \( f \). Since the change of a base-point induces only a change in sign \( \pm 1 \) for the degree, without loss of generality, we can take as base-point any regular point \( b \) of \( f \) belonging to \( U' \subseteq \Omega \). Let \( y \in V' \) be any regular value of \( f_{|\Omega} \). By the definition of the base-point degree,
\[
\pm 1 = \deg_b(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \varepsilon(b) \sigma(Df \circ \gamma_x),
\]
where each \( \gamma_x \) is any path joining \( b \) to \( x \). Since \( \Omega \setminus \mathcal{B}(f) \) is connected we can chose all the joining paths contained in this set. But then, by Theorem 1.8, \( \sigma(Df \circ \gamma_x) = 1 \) for all \( x \in f^{-1}(y) \). Substituting in (14) we get
\[
\pm 1 = \deg_b(f, \Omega, y) = n \varepsilon(b),
\]
where \( n \) is the cardinality of \( f^{-1}(y) \). It follows then from (15) that \( n = 1 \) and hence \( U' \) must contain the only solution of the equation \( f(x) = y \). Therefore \( \deg_b(f, \Omega_1, 0) = 0 \), which contradicts (12). \( \square \)

**2. Branching in the variational case.** Throughout this section, \( H \) is a real separable Hilbert space. We consider \( C^1 \)-Fredholm maps \( f : U \rightarrow H \), where \( U \subseteq H \) is open, that are of variational type, in the sense that there is \( C^2 \) functional \( \psi : U \rightarrow \mathbb{R} \) for which \( f = \nabla \psi \). In this case the zeroes of the map \( f \) are the critical points of the potential \( \psi \). For all \( x \in U \), the operator \( Df(x) \) is self-adjoint. Of course, the derivative \( Df(x) \) is the Hessian of \( \psi \) at \( x \), and it will be also denoted by \( \mathcal{H}_\psi(x) \).
The Morse index, $m(\psi, x)$, of a critical point $x$ is the dimension (finite or infinite) of the negative spectral space of the self-adjoint operator $Df(x) = H\psi(x)$. A critical point $x$ of $\psi$ is said to be non-degenerate provided $Df(x)$ is invertible. If moreover $Df(x)$ is essentially positive, that is, if its negative spectral space is finite dimensional, then the Morse index is locally constant with respect to the $C^2$ topology in the space of potentials.

As in the previous section, we want to use bifurcation theory for critical points of one parameter families of functionals, that is, variational bifurcation, in order to study the branch set $\mathcal{B}(f)$.

A well-known principle of variational bifurcation for compact perturbations of the identity states that if along the trivial branch there are two points at which the Hessians of the potential function are non-degenerate and have different Morse indices, then there is bifurcation of critical points from the trivial branch somewhere between these two parameter values. Among the many versions of this principle, whose origins go back to Krasnoselskij and his school, we quote the following theorem, proved in [23].

**Theorem 2.1.** Let the functional $\psi : [a, b] \times U \to \mathbb{R}$ be $C^2$ and assume that, for all $\lambda \in [a, b]$, $\nabla \psi_\lambda = \text{Id} - C_\lambda$, with $C_\lambda$ a compact mapping. Assume that for all $\lambda \in [a, b]$, 0 is a critical point of $\psi(\lambda, \cdot)$, and it is non-degenerate at $\lambda = a$ and $\lambda = b$. If

$$ \Delta m \equiv m(\psi_a, 0) - m(\psi_b, 0) \neq 0, $$

then there is bifurcation of nontrivial critical points of $\psi$ from the trivial branch $[a, b] \times \{0\}$.

In the above theorem, $\Delta m$ is properly defined because both Morse indices are finite. Indeed, the derivative of a compact mapping is compact, and a linear symmetric compact perturbation of the identity is always essentially positive.

In this section we will mainly focus on the **strongly indefinite** case, that is, the case in which the Hessian at the critical point 0 of both $\psi_\lambda$ and $-\psi_\lambda$ has an infinite dimensional negative spectral space. The strongly indefinite case occurs in variational elliptic problems of odd order, such as Hamiltonian systems. In order to deal with this kind of functionals we will substitute $\Delta m$ with the spectral flow.

### 2.1. Spectral flow of a path of self-adjoint Fredholm operators

The spectral flow of a path $L : [a, b] \to \Phi_0(H)$ of self-adjoint Fredholm operators that has invertible end-points is, loosely speaking, the number of negative eigenvalues of $L_a$ that become positive as $\lambda$ goes from $a$ to $b$ minus the number of positive eigenvalues that become negative. This notion was introduced in the case of elliptic self-adjoint operators in [6] and then extended in several forms for general bounded and unbounded operators [29, 28, 21, 20, 27]. Here we describe a variant of the construction of spectral flow in [20, Theorem 5.3], emphasizing the parallelism with the construction of parity described in the previous section.

Let us denote by $\Phi_S(H)$ the set of self-adjoint Fredholm operators on $H$. Similarly to the general case, the set $\Sigma$ of singular operators in $\Phi_S(H)$ is stratified, the strata $\Sigma_k \equiv \{ T \in \Phi_S(H) | \dim \ker T = k; k \geq 1 \}$ being submanifolds, in this case of codimension $k(k+1)/2$. In particular, the top stratum also is of codimension 1. However, in sharp contrast of the properties of the top stratum in the non self-adjoint case, here the top stratum $\Sigma_1$ is co-oriented in $\Phi_S(H)$. More precisely, its normal bundle posses a natural orientation inherited from the evolution of the first eigenvalue.
Indeed, if \( L: (-a, a) \rightarrow \Phi_S(X) \) is a smooth path crossing \( \Sigma_1 \) transversely at \( \lambda = 0 \), since 0 is a simple eigenvalue of \( L_0 \), for \( \lambda \) small enough \( L_\lambda \) possesses a smooth branch \( \mu(\lambda) \) of simple eigenvalues of \( L_\lambda \) with \( \mu(0) = 0 \). The corresponding normalized eigenvectors \( e(\lambda) \) also are smooth functions of \( \lambda \). Transversal crossing means that if \( v \) generates \( \ker L_0 \), then \( L_0 v \not\in \im L_0 \), which, because each \( L_\lambda \) is symmetric, is equivalent to \( \langle L_0 e(\lambda), v \rangle \neq 0 \). Differentiating the identity \( L_\lambda e(\lambda) = \mu(\lambda)e(\lambda) \) at \( \lambda = 0 \), we see that \( \langle L_0 e(0), e(0) \rangle \neq 0 \) if and only if \( \mu'(0) \neq 0 \). Therefore, to each transversal crossing of a path with \( \Sigma_1 \), a multiplicity \( \pm 1 \) can be assigned according to the sign of \( \mu'(0) \).

Any continuous path \( L: [a, b] \rightarrow \Phi_S(H) \) with invertible end-points can be arbitrarily closely approximated by a smooth path \( \tilde{L}: [a, b] \rightarrow \Phi_S(H) \) that is transversal to \( \Sigma = \cup_{k \geq 1} \Sigma_k \). Again, since the codimension of the higher strata are greater than one, \( \tilde{L} \) crosses only \( \Sigma_1 \) transversely. By definition, the spectral flow \( sf(L) \) of a continuous path \( L: [a, b] \rightarrow \Phi_S(H) \) with invertible end-points is defined by

\[
\text{sf}(L) = \sum_{\lambda \in L^{-1}(\Sigma_1)} \text{sgn} \mu'(\lambda),
\]

where \( \tilde{L} \) is any \( C^1 \) path that is transversal to \( \Sigma \) and close enough to \( L \) (see [20, Theorem 3.12]) and the path \( \mu \) is as in the preceding paragraph.

The main properties of spectral flow can be summarized as follows: It is invariant under homotopies of paths in \( \Phi_S(H) \) with invertible end-points (and free homotopies of closed paths). It is additive under concatenation. Moreover \( sf(L) = 0 \) if and only if \( L \) is homotopic to a path of invertible self-adjoint operators.

### 2.2. Spectral flow and bifurcation

Spectral flow is related to linearized bifurcation in the variational case, as it is easy to guess from its construction, which parallels that of parity. The precise relation between the two invariants is

\[
\sigma(L) = (-1)^{sf(L)},
\]

whenever both are defined. Moreover,

\[
\text{sf}(L) = \Delta m \equiv m(L_b) - m(L_a),
\]

if \( L \) is a path of essentially positive positive Fredholm operators. Taking into account both of these relations, it follows that the next theorem is, at the same time, a refinement, in the variational case, of Theorem 1.8 and an extension to the Fredholm setting of Theorem 2.1.

**Theorem 2.2** ([19, 27]). Let \( U \subseteq H \) be a neighborhood of 0 and \( \psi: [a, b] \times U \rightarrow \mathbb{R} \) be a family of \( C^2 \) functionals. Assume that, for all \( \lambda \in [a, b] \), \( \nabla \psi_\lambda(0) = 0 \) and \( L_\lambda \), the Hessian of \( \psi(\lambda, \cdot) \) at 0, is Fredholm. Also assume that \( L_a \) and \( L_b \) are invertible. Then:

i) If \( sf(L) \neq 0 \), the interval \( (a, b) \) contains at least one point of bifurcation of critical points of \( \psi_\lambda \) from the trivial branch.

ii) If \( \Sigma(L) = \{ \lambda \in [a, b] \mid \dim \ker L_\lambda \geq 1 \} \) is a finite subset of \( (a, b) \), then \( \psi \) possesses at least \( |sf(L)|/d(L) \) bifurcation points in \( (a, b) \), where

\[
\text{d}(L) \equiv \max \{ \dim \ker L_\lambda \mid \lambda \in [a, b] \}
\]

is the order of degeneracy of the path \( L \) in \( [a, b] \).
A regular crossing point for a differentiable path $L$ is a point $\lambda \in \Sigma(L)$ at which the crossing form $Q(\lambda)$, defined as the restriction of the quadratic form $(\langle L_\lambda h, h \rangle)$ to ker $L_\lambda$, is non-degenerate. In [19, Theorem 4.1] it is proved that regular crossing points are isolated, and moreover, for paths with only regular crossing points,

$$sf(L, I) = \sum_{\lambda \in \Sigma(L)} \sigma(Q(\lambda)),$$

where $\sigma$ stands for the signature. Recall that the signature of a quadratic form $Q$ is the number of positive eigenvalues of a representing symmetric matrix minus the number of the negatives ones.

**Remark.** The relations between the parity, spectral flow and signature sharpen our understanding of the comparison between two classical bifurcation results, frequently attributed to Krasnosel’skij. Consider a family of the quite special form $f(\lambda, x) = x - \lambda C(x)$, where the mapping $C$ is both $C^2$ and compact, and $C(0) = 0$. We have $L_\lambda = -\lambda K$, where $K \equiv DC(0)$. In the variational case, the first theorem asserts that when $C = \nabla \psi$, every characteristic value $\lambda$, of $K$ is a bifurcation point from the trivial branch of solutions of $x - \lambda C(x) = 0$. In the absence of variational structure, the second bifurcation theorem asserts that the same can be proved only for characteristic values of odd multiplicity. Indeed, it is easy to see that, for small enough $\delta$, if $I = [\lambda_\ast - \delta, \lambda_\ast + \delta]$ we have

$$sf(L, I) = \sigma(Q(\lambda_\ast)) = m \quad \text{and} \quad \sigma(L, I) = (-1)^{\sigma(Q(\lambda_\ast))} = (-1)^m,$$

where $m = \dim \ker(\text{Id} - \lambda_\ast K)$.

2.3. A criterion for branching in the presence of a potential function.

**Proposition 2.3.** Let $U \subseteq H$ be open and $\psi: U \to \mathbb{R}$ be a $C^3$ functional, with Fredholm gradient $f = \nabla \psi$. Assume a $C^2$ path $\gamma: [a, b] \to U$ is admissible with respect to $DF$ and that $sf(DF \circ \gamma) \neq 0$. Then there is a $\lambda_\ast$ in $(a, b)$ for which $\gamma(\lambda_\ast)$ is a branch point for $f$.

**Proof.** For $r > 0$ sufficiently small consider the family of $C^2$-Fredholm mappings $h: [a, b] \times B(0, r) \to Y$ defined by

$$h(\lambda, x) = f(x + \gamma(\lambda)) - f(\gamma(\lambda)).$$

Each $h_\lambda$ is the gradient of

$$\psi_\lambda(x) = \psi(x + \gamma(\lambda)) - \langle f(\gamma(\lambda)), x \rangle.$$

For each $\lambda \in [a, b]$, $h(\lambda, 0) = 0$ and $L_\lambda \equiv D_x h(\lambda, 0) = Df(\gamma(\lambda))$. Since $sf(L) \neq 0$, according to Theorem 2.2 there is a bifurcation point $\lambda_\ast$ for nontrivial solutions of $h(\lambda, x) = 0$. Clearly $\gamma(\lambda_\ast)$ is a branch point for $f$. $\square$

Given a $C^3$ functional $\psi: U \to \mathbb{R}$, with Fredholm gradient $f$, a point $x \in U$ and a direction $u \neq 0$ in $H$, our sufficient conditions for branching will be given in terms of a quadratic form $Q(x, u): \ker Df(x) \to \mathbb{R}$, defined, at $v \in \ker Df(x)$, by

$$Q(x, u)[v] = \langle D^2 f(x)(u, v), v \rangle.$$  \hspace{1cm} (17)

**Theorem 2.4.** Let $U \subseteq H$ be open and simply connected and let $\psi: U \to \mathbb{R}$ be a $C^3$ functional with Fredholm gradient $f = \nabla \psi$. Let $x_\ast \in U$ be a point such that, for some $u \in H$, the quadratic form $Q(x_\ast, u)$ defined by (17) is non-degenerate and $\text{sgn} \ Q(x_\ast, u) \neq 0$. Then $x_\ast$ is a branch point for $f$ and $B(f)$ disconnects $U$. 
Proof. For $a > 0$ sufficiently small, consider the path $L: [-a, a] \to \Phi_S(H)$ defined by $L_{\lambda} = Df(x_\ast + \lambda u)$. The crossing form of the path $L$ at $\lambda = 0$ is easily seen to be $Q(x_\ast, u)$. It follows from our hypothesis that $\lambda = 0$ is a regular crossing point of $L$ whose crossing form has a non-vanishing signature. Taking a small enough $a > 0$, it follows from (16) that 0 is the only singular point of $L$ in the interval $[-a, a]$ and $\text{sf}(L, [-a, a]) = \text{sgn} Q(x_\ast, u) \neq 0$.

It is convenient to write $L = Df \circ \eta$ where $\eta: [-a, a] \to U$ given by $\eta(t) = x_\ast + tu$. According to the preceding proposition, there is a branch point for $f$ on the image of $\eta$. Since the only singular point of $L$ is $\lambda = 0$, this point must be $x_\ast$. Consider now any path $\gamma([-a, a] \to U$ joining $x^- = x_\ast - au$ with $x^+ = x_\ast + au$. As in the case of parity, since $U$ is simply connected, the spectral flow of a path depends only on its end points, and therefore $\text{sf}(Df \circ \gamma) = \text{sf} L \neq 0$. According to Proposition 2.3, there must be a point $\lambda_\ast \in [-a, a]$ such that $\gamma(\lambda_\ast)$ belongs to $B(f)$. This shows that $x^+$ and $x^-$ belong to different components of the open set $U \setminus B(f)$ and concludes the proof. \hfill \Box

Example 3. Problem (5), described in the example 1, is of variational type. Under standard growth assumptions on $\phi$, weak solutions of (5) are critical points of the functional $\psi$ defined on $H = H_0^1(\Omega)$ by

$$\psi(u) = q(u) + \int_{\Omega} \rho(x, u(x)) \, dx,$$

where $q$ is the bounded quadratic form on $H_0^1(\Omega)$ associated to the unbounded self adjoint operator $L$ and $\rho(x, t) = \int_0^t \phi(x, s) \, ds$.

It is well-known that $f = \nabla \psi$ is Fredholm. Moreover, the Hessian $Df(u)$ has the property that $Df(u)h = 0$ if and only if $h \in H^2(\Omega) \cap H_0^1(\Omega)$ and $Lh + \phi_u(x, u(x))h = 0$. Assume that $\dim \ker Df(0) > 0$ and that $g(x) = \phi_{uu}(x, 0)$ does not vanishes identically. Taking $u = \mu g$, where $\mu \in H_0^1(\Omega)$ and strictly positive on $\Omega$, one can easily check that the quadratic form $Q(0, u)$ of Theorem 2.4 is defined, for $h \in \ker Df(0)$ by $Q(0, g)h = \int_{\Omega} \mu(x) g^2(x) h^2(x) \, dx$. This form is positive definite on $\ker Df(0)$ and hence it is non-degenerate and $\text{sig} Q(0, g) = \dim \ker Df(0) > 0$. By Theorem 2.4, $B(f)$ separates $H_0^1(\Omega)$ irrespective of the dimension of the kernel.

Example 4. The discussion in Example 3 can be easily extended to strongly indefinite systems

$$-\Delta u = H_v(x, u, v) \quad \text{in } \Omega,$$

$$-\Delta v = H_u(x, u, v) \quad \text{in } \Omega,$$

$$u = v = 0 \quad \text{on } \partial \Omega,$$

where $\Omega$ is a open bounded upset of $\mathbb{R}^n$ with smooth boundary. Such problems were studied in [4, 5, 31, 34], among others.

Under appropriate growth assumptions (see [34]) on the Hamiltonian $H$, weak solutions of (19) are critical points of the functional $\psi$ defined on $X = H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$\psi(u, v) = \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx - \int_{\Omega} H(x, u, v) \, dx.$$
The Hessian \( L_z = \mathcal{H}_\psi(z) \) of \( \psi \) at \( z = (u, v) \in X \) is given by
\[
< L_z(k, h)(\tilde{k}, \tilde{h}) >_X = \int_{\Omega} \langle \nabla k, \nabla \tilde{h} \rangle \, dx + \int_{\Omega} \langle \nabla \tilde{k}, \nabla h \rangle \, dx - \int_{\Omega} a(x, z(x)) \tilde{k} \tilde{h} + b(x, z(x))(\tilde{k} h + k \tilde{h}) + c(x, z(x))h \tilde{h} \, dx,
\]
(21)
where \( a(x, u, v) = H_{uu}(x, u, v), \ b(x, u, v) = H_{uv}(x, u, v), \ c(x, u, v) = H_{vv}(x, u, v) \).

The main difference with the previous example is that the spectrum of the Hessian \( L_z \) is neither essentially positive nor essentially negative. Hence the functional \( \psi \) is of strongly indefinite type. Branching can be detected by means of the spectral flow but not via the Morse index. Assume \( \text{Ker} L_0 \neq \{0\} \). It is easy to see that, given a direction \( z = (u, v) \in X \) the quadratic form form \( Q(0, z) \) is defined on \( (h, k) \in \text{ker} L_0 \) by
\[
Q(0, z)(h, k) = \int_{\Omega} [u(x)c_u(x, 0) + v(x)c_v(x, 0)]k^2 \, dx - \int_{\Omega} [u(x)a_u(x, 0) + v(x)a_v(x, 0)]h^2 + 2[u(x)b_u(x, 0) + v(x)b_v(x, 0)]hk \, dx.
\]
Assuming, for example, that the matrix
\[
\begin{pmatrix}
a_u(x, 0) & b_u(x, 0) \\
b_u(x, 0) & c_u(x, 0)
\end{pmatrix}
\]
is point-wise positive definite in \( \Omega \), taking \( z = (u, 0) \) with \( u \) strictly positive in \( \Omega \) we get that
\[
Q(0, z)(h, k) = -\int_{\Omega} u a_u(x, 0) h^2 + 2u b_u(x, 0) h k + u c_u(x, 0) k^2 \, dx
\]
is negative definite on \( \text{Ker} L_0 \) and hence \( \text{sig} Q(0, z) = -\dim \text{Ker} L_0 \neq 0 \). Therefore \( \mathcal{B}(\nabla \psi) \) separates \( X \).

2.4. The number of components of \( U \setminus \mathcal{B}(f) \).

**Theorem 2.5.** Let \( U \) be an open simply connected subset of \( H \) and \( \psi: U \to \mathbb{R} \) a \( C^3 \) functional with Fredholm gradient \( f = \nabla \psi \). Assume there is a sequence of admissible paths \( \{\gamma_n: [a, b] \to U\} \) with the property that
\[
\lim_{n \to \infty} |\text{sf}(Df \circ \gamma_n)| = \infty.
\]
Then \( U \setminus \mathcal{B}(f) \) has infinitely many components.

**Proof.** Define \( \mathcal{R}(f) \) to be the set of regular points of \( f \) in \( U \). Fix a point \( x_0 \) in \( \mathcal{R}(f) \). For each component \( C \) of \( U \setminus \mathcal{B}(f) \) intersecting \( \mathcal{R}(f) \), define \( \text{i}(C) = \text{sf}(Df \circ \gamma) \), where \( \gamma: [a, b] \to U \) is a path joining \( x_0 \) to a regular point of \( f \) in \( C \). We claim that this is properly defined, that is, it is independent of the choice of joining path. Indeed, we can connect the end-points \( \gamma_1(b), \gamma_2(b) \) of two such paths by a path \( \gamma \) entirely contained in \( C \). Since \( \gamma(t) \in U \setminus \mathcal{B}(f) \) for all \( t \), by Proposition 2.3 the spectral flow of \( Df \circ \gamma \) must vanish. Since \( U \) is simply connected, it follows from the concatenation property that the paths \( Df \circ \gamma_1 \) and \( Df \circ \gamma_2 \) have the same spectral flow.
For any admissible path $\gamma$ joining two points in $R(f)$ if $C_{\gamma(a)}$ and $C_{\gamma(b)}$ are the connected components containing the end-points, then again by the concatenation property of spectral flow, we have the relation

$$\text{sf}(Df \circ \gamma) = i(C_{\gamma(b)}) - i(C_{\gamma(a)}).$$

(22)

Therefore, if the number of components of $U \setminus B(f)$ were finite, the number of those intersecting $R(f)$ would also be finite, and the function $\rightarrow \text{sf}(L \circ \gamma)$ would take only a finite number of values, contradicting the assumption regarding $\{\text{sf}(L \circ \gamma_n)\}$. □

In the absence of a sequence as in the above theorem one can still estimate the number of path components of $U \setminus B(f)$ provided that the map $Df: U \to \Phi_S(H)$ is transversal to $\Sigma$. In view of the preceding remarks on the stratification of $\Sigma$, in this case $C(f)$ is a union of manifolds $C_i = (Df)^{-1}(\Sigma_i)$. Moreover if a path $\gamma: [0, 1] \to U$ is transversal to the manifold $C_i$ of critical points of $f$, then $Df \circ \gamma$ is transversal to $\Sigma_i$ and hence $Df \circ \gamma$ intersects only the top stratum $\Sigma_1$ transversally a finite number of times. By elementary transversality theory [1], any path $\gamma$ in $U$ may be arbitrarily closely approximated by a path $\tilde{\gamma}$ such that $Df \circ \tilde{\gamma}$ has the same spectral flow as $Df \circ \gamma$. We will use this fact in the proof of the following theorem.

**Theorem 2.6.** Let $U$ be an open simply connected subset of $H$ and $\psi: U \to \mathbb{R}$ a $C^3$ functional with Fredholm gradient $f = \nabla \psi$. Assume that $Df: U \to \Phi_S(H)$ is transversal to $\Sigma = \cup_{i \geq 1} \Sigma_i$. If

$$s = \sup \{|\text{sf}(Df \circ \gamma)| \ ; \gamma: [0, 1] \to U \text{ admissible with respect to } Df\} < \infty,$$

then $U \setminus B(f)$ has at least $s + 1$ connected components.

**Proof.** Chose a path $\gamma: [0, 1] \to U \setminus B(f)$ that attains the maximum $s$ and has the property that the composition $Df \circ \gamma$ crosses transversally $\Sigma_1$ (necessarily a finite number of times). The preceding discussion regarding transversality tells us that such a choice can be made.

Let $0 < \mu_1 < \mu_2 < \cdots < \mu_k < 1$ be the singular points of $Df \circ \gamma$ and set $\mu_0 = 0$, $\mu_{k+1} = 1$. We enumerate the components of $U \setminus B(f)$ traversed by $\gamma$ in an increasing order according to the ordering of $\mu_i$. More specifically, $\gamma([0, \mu_1)) \subseteq C_0$, $\gamma([\mu_1, \mu_2)) \subseteq C_1$, etc. We obtain in this way associated to $\gamma$ a list $(C_0, C_1, \ldots, C_k)$ of components of $U \setminus B(f)$, with $\gamma(0) \in C_0$ and $\gamma(1) \in C_k$.

If there is a pair of indices $i < j$ such that $C_i = C_j = C$, we can use the density of transversal paths again in order to obtain a shorter path $\tilde{\gamma}$ with the same spectral flow, but with a smaller list of components. To do so, we connect $\gamma(\mu_i)$ with $\gamma(\mu_{j+1})$ by means of a path $\eta$: $[\mu_i, \mu_{j+1}] \to C$ that is transversal to $\cup_{r \geq 1} C_r$. This won’t change the spectral flow. The new path $\tilde{\gamma}$, defined on $[0, \mu_i]$ as the restriction of $\gamma$ followed by $\eta$ on $[\mu_i, \mu_{j+1}]$ and by $\gamma$ again on $[\mu_{j+1}, 1]$, will still have only transversal intersections with $C_1$, but the component $C$ will appear in its list one time less.

Iterating this procedure we obtain a path $\tilde{\gamma}$ such that $L = Df \circ \tilde{\gamma}$ has spectral flow $s$, has a finite singular set $\Sigma(L) = \{t_1, \ldots, t_p\}$ for which each $\dim \ker L \circ \tilde{\gamma}(t_i) = 1$, but which never traverses twice the same component of $U \setminus B(f)$.

Let $S = \{t_{i_1}, \ldots, t_{i_m}, 1 \leq j \leq m\} \subseteq \Sigma(L)$ be the set of all bifurcation points of the family $h_\lambda(x) \equiv f(x + \tilde{\gamma}(\lambda)) - f(\tilde{\gamma}(\lambda))$. Then $\tilde{\gamma}(t_{i_j})$ are the only intersection points of $\tilde{\gamma}([0, 1])$ with $B(f)$. For $1 \leq i \leq r$, there is a an neighborhood $V_i$ of $x_i = \tilde{\gamma}(t_i)$ such that $C(f) \cap V_i = B(f) \cap V_i$ is a one-codimensional submanifold of $V_i$ whose complement in $V_i$ has exactly two components (see [3]). If $\tilde{\gamma}(t_{j_p}) = \tilde{\gamma}(t_{k_q})$, for some
If \( p \neq q \) then \( \tilde{\gamma} \) must traverse the same component of \( U \setminus B(f) \) twice, contradicting our construction. Hence, the restriction of \( \gamma \) to \( S \) is an injective function from \( S \) into \( B(f) \). Since \( s = |s f(Df \circ \gamma)| = |s f(Df \circ \tilde{\gamma})| \), from ii) of Theorem 2.2 it follows that \( m \geq s \) and therefore \( \tilde{\gamma} \) must traverse at least \( s + 1 \) components of \( U \setminus B(f) \). \( \square \)

2.5. Further comments regarding applications. Here we briefly consider a couple of examples to which Theorems 2.5 and 2.6 might be possibly applied.

Firstly, we sketch a possible application of Theorem 2.5. According to that theorem, in order to show that the complement of \( B(f) \) has an infinite number of connected components, it suffices to find a sequence paths \( \{ \gamma_n \} \), each admissible with respect to \( Df \), with \( \{|Df \circ \gamma_n|\} \rightarrow \infty \). We consider how it may be possible to verify this assumption. The paper [7] deals with the ordinary Morse index for the functional \( \psi \) defined on \( H^1_0(\Omega) \) (see Example 1), whose critical points are weak solutions of the problem

\[
\begin{align*}
-\Delta u &= g(x,u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(23)

It is shown that for super-linear problems with sub-critical growth, bounds for the Morse index of a critical point \( u \) are equivalent to bounds in the \( L^\infty \) norm of \( u \). Since the minimax methods provide solutions with arbitrary large norm ([32, 8]), it follows that \( \psi \) has critical points \( u_n \) of arbitrary large Morse index. If among them there would be an infinite number of non-degenerate critical points, then keeping one of them fixed and joining it by a path to the others we will get the needed sequence \( \{ \gamma_n \} \) with \( \text{sf}(\gamma_n) \rightarrow \infty \). We now will show that, in order to verify this, it is enough to know that an infinite number of solutions \( u_n \) belong to \( H^1_0 \setminus B(f) \), where \( f = \nabla \psi \). By the variational characterization of eigenvalues, the dimension of the negative eigenspace of a linear compact self-adjoint perturbation of a positive definite operator is a lower semi-continuous function of the operator. Denoting by \( E^-_v(\psi) \) the negative eigenspace of the Hessian of \( \psi \) at (not necessarily a critical point of \( \psi \)) \( v \) and with \( m_v(\psi) \) its dimension, for any \( u \in H^1_0 \) we can find a neighborhood \( U_u \) such that \( m_v \geq m_u \), for all \( v \in U \). If \( u \in H^1_0 \setminus B(f) \), then, by taking a smaller \( U \) if necessary we can assume that the restriction of \( f \) to \( U \) is injective. But then, by [33, Theorem 2], \( U \) must contain some regular point \( v \) of \( f \) such that \( m_v \geq m_u \). Hence if an infinite number of (possibly degenerate) solutions \( u_n \) of (23) belongs to \( H^1_0 \setminus B(f) \), we can, by this procedure, obtain near to them a sequence \( \{ v_n \} \) of regular points of \( f \) keeping one of them fixed and joining it by a path to the others we will get the needed sequence \( \{ \gamma_n \} \) with \( \text{sf}(\gamma_n) \rightarrow \infty \), and thereby show that \( H^1_0 \setminus B(f) \) has an infinite number of connected components.

Papers [4, 5] deal with the problem of Example 4. They extend the results of [7] to renormalized Morse indices of the associated strongly indefinite functional. In [5], using a variant of Floer homology it is shown that, for even Hamiltonians of a special form and for \( n > 0 \), there is a sequence of critical points \( \{ z_k \} \) of the functional such that the renormalized Morse indices of \( z_k \) diverge to infinity as \( k \rightarrow \infty \). Their renormalized Morse indices are defined via the spectral flow of \( Df \) along particular affine families, but since they do not depend on the path their results satisfy the assumptions of Theorem 2.5. However, in order to accommodate Floer’s theory, the authors work on some special interpolation spaces. Therefore, in order to apply Theorem 2.5 to this case one must overcome a few technical problems.

Finally, let us observe that Theorem 2.6 can also be applied to the functional (20), using the comparison theorem for the spectral flow and some calculations of
the latter obtained in [4, 34]. Here what is needed is an explicit description of the transversality of $Df$ to $\Sigma$ in terms of the data of the problem.

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E-mail address: pmf@math.umd.edu
E-mail address: jacobo.pejsachowicz@polito.it