The Hilbert modular group and orthogonal groups

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Abstract

We derive an explicit isomorphism between the Hilbert modular group and certain congruence subgroups on the one hand and particular subgroups of the special orthogonal group $SO(2, 2)$ on the other hand. The proof is based on an application of linear algebra adapted to number theoretical needs.

Keywords: Hilbert modular group, Congruence subgroup, Orthogonal group, Discriminant kernel

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1 Introduction

A generalization of the classical elliptic modular group $SL_2(\mathbb{Z})$ was introduced by Blumenthal [2, 3] more than 100 years ago. $SL_2(\mathcal{O}_K)$ is the ordinary Hilbert modular group over a totally real number field $K$. Meanwhile the theory of Hilbert modular forms has been developed into various directions (cf. [6, 7, 18]). On the other hand Borcherds [4] established a product expansion for modular forms on $SO(2, n)$. If $n = 2$ and $K$ is a real-quadratic field these groups are basically isomorphic. More precisely $PSL_2(\mathcal{O}_K)$ is isomorphic to discriminant kernel of the orthogonal group $SO(2, 2)$ (cf. Theorem 2). Borcherds theory has led to important examples of Hilbert modular forms (cf. [5]), which in some cases allow to describe the graded ring of Hilbert modular forms (cf. [17, 19]). As Borcherds products are usually related with discriminant kernels, it is interesting to describe the associated congruence subgroups of $PSL_2(\mathbb{R})$ precisely.

In this paper we derive an explicit isomorphism based on linear algebra. Our approach can be extended to congruence subgroups. The results below can serve as an explicit dictionary, when passing between Borcherds theory and classical Hilbert modular forms.

Given a non-degenerate symmetric even matrix $T \in \mathbb{Z}^{m \times m}$ let

$$SO(T; \mathbb{R}) := \{ U \in SL_m(\mathbb{R}); \ U^\tau T U = T \}$$

denote the attached special orthogonal group. Let $SO_0(T; \mathbb{R})$ stand for the connected component of the identity matrix $I$ and $SO_0(T; \mathbb{Z})$ for the subgroup of integral matrices. The discriminant kernel

$$D(T; \mathbb{Z}) := \{ U \in SO_0(T; \mathbb{Z}); \ U \in I + \mathbb{Z}^{m \times m}T \}$$

(1)
is clearly a normal subgroup of $SO_0(T; \mathbb{Z})$. Given $N \in \mathbb{N}$ we set

$$T_N := \begin{pmatrix} 0 & 0 & N \\ 0 & T & 0 \\ N & 0 & 0 \end{pmatrix}$$

for the orthogonal sum with the rescaled hyperbolic plane.

2 The normalizer of the Hilbert modular group

Throughout this paper let $\mathbb{K} = \mathbb{Q}(\sqrt{m})$, $m \in \mathbb{N}$, $m > 1$ squarefree, be a real-quadratic number field with ring of integers and discriminant

$$\mathcal{O}_\mathbb{K} = \mathbb{Z} + \mathbb{Z} \omega_\mathbb{K}, \quad \omega_\mathbb{K} = \begin{cases} (m + \sqrt{m})/2, & m \equiv 1 \text{ mod } 4, \\ m + \sqrt{m}, & 4m, \text{ else.} \end{cases}$$

The non-trivial automorphism of $\mathbb{K}$ is given by

$$\mathbb{K} \to \mathbb{K}, \quad a = a + \beta \sqrt{m} \mapsto a' = a - \beta \sqrt{m}. \quad (4)$$

If $\ell \in \mathbb{N}$, $\sqrt{\ell} \notin \mathbb{K}$ we extend (4) to $\mathbb{K}(\sqrt{\ell})$ via $\sqrt{\ell} = \sqrt{\ell'}$. Moreover $a \gg 0$ means that $a \in \mathbb{K}$ is totally positive, i.e., $a > 0$ and $a' > 0$.

The (ordinary) Hilbert modular group (cf. [2,3,6]) is given by

$$\Gamma_\mathbb{K} := \text{SL}_2(\mathcal{O}_\mathbb{K}).$$

At first we are going to describe the normalizer

$$\mathcal{N}_\mathbb{K} := \{ M \in \text{SL}_2(\mathbb{R}); \ M^{-1} \Gamma_\mathbb{K} M = \Gamma_\mathbb{K} \}$$

of $\Gamma_\mathbb{K}$ in $\text{SL}_2(\mathbb{R})$. Therefore let $\mathcal{I}(L)$ denote the fractional ideal generated by the entries of a matrix $0 \neq L \in \mathbb{K}^{2 \times 2}$.

**Theorem 1** If $\mathbb{K}$ is a real-quadratic number field, the normalizer $\mathcal{N}_\mathbb{K}$ is equal to

$$\left\{ \frac{1}{\sqrt{\det L}} L; \ L \in \mathcal{O}_\mathbb{K}^{2 \times 2}, \ \det L > 0, \ \mathcal{I}(L)^2 = \mathcal{O}_\mathbb{K} \det L \right\}. \quad (5)$$

**Proof** Clearly the matrices in (5) belong to $\mathcal{N}_\mathbb{K}$. As $\Gamma_\mathbb{K}$ contains a $\mathbb{Z}$-basis of $\mathcal{O}_\mathbb{K}^{2 \times 2}$, we conclude that any $M \in \mathcal{N}_\mathbb{K}$ satisfies

$$M^{-1} \mathcal{O}_\mathbb{K}^{2 \times 2} M = \mathcal{O}_\mathbb{K}^{2 \times 2}.$$ 

Inserting the classical basis elements, we see that the product of any two entries of $M$ belongs to $\mathcal{O}_\mathbb{K}$. If $\alpha$ is any non-zero entry of $M$, we get

$$M = \frac{1}{\sqrt{\alpha^2}} L, \ L \in \mathcal{O}_\mathbb{K}^{2 \times 2}, \ \det L = \alpha^2, \ \mathcal{I}(L)^2 = \mathcal{O}_\mathbb{K} \alpha^2.$$ 

Thus $M$ belongs to (5). \qed

Now we define the group

$$\Sigma_\mathbb{K} = \left\{ \frac{1}{\sqrt{\ell}} L; \ \ell \in \mathbb{N}, \ L \in \mathcal{O}_\mathbb{K}^{2 \times 2} \right\} \cap \text{SL}_2(\mathbb{R}) \supseteq \Gamma_\mathbb{K}.$$ 

Note that $M_1, M_2 \in \Sigma_\mathbb{K}$ satisfy

$$(M_1 M_2)' = \pm M_1' M_2'.$$
due to the above extension of (4).

Let \( H \) denote the upper half-plane in \( \mathbb{C} \). Then a matrix \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Sigma_K \) acts on \( H^2 \) via

\[
\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \mapsto \begin{pmatrix} M(\tau_1) \\ M'(\tau_2) \end{pmatrix}, \quad M(\tau) = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}.
\]

(6)

It is well-known that a subgroup \( \Gamma' \subseteq \Sigma_K \) acts discontinuously if and only if the embedded group

\[
\{ \pm (M, M'); \ M \in \Gamma \} \subseteq \left( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \right) / \{ \pm (I, I) \}
\]

is discrete (cf. [6], I.2.1). The maximal discontinuous extension \( \Gamma' \) of \( \Gamma \) is called the Hurwitz–Maaß extension. It is described in [1,9,16] as

\[
\Gamma' = \left\{ \frac{1}{\sqrt{\det L}} L; \ L \in \mathcal{O}_{K}^{2 \times 2}, \ det L \gg 0, \ \mathcal{I}(L)^2 = \mathcal{O}_{K} \det L \right\}
\]

\[
= \left\{ \frac{1}{\sqrt{\ell}} L; \ \ell \in \mathbb{N}, \ L \in \mathcal{O}_{K}^{2 \times 2}, \ det L = \ell, \ \mathcal{I}(L)^2 = \mathcal{O}_{K} \ell \right\}
\]

(8)

where we define the generalized Atkin-Lehner matrices by

\[
V_{\ell} = \frac{1}{\sqrt{\ell}} \begin{pmatrix} \nu \ell & \mu(m + \sqrt{m}) \\ m - \sqrt{m} & \ell \end{pmatrix}, \quad \nu, \mu \in \mathbb{Z}, \quad \nu \ell - \mu m(m - 1)/\ell = 1.
\]

(9)

If \( A_{\ell} = \mathbb{Z} \ell + \mathcal{O}_{K} \) denotes the integral ideal in \( \mathcal{O}_{K} \) of reduced norm \( \ell \) for a squarefree divisor \( \ell \) of \( d_{K} \), we have

\[
V_{\ell} \Gamma_K \subseteq \mathcal{I}_{K} V_{\ell} = \frac{1}{\sqrt{\ell}} A_{\ell}^{2 \times 2} \cap SL_2(\mathbb{R}.
\]

(10)

Note that for squarefree divisors \( k, \ell \) of \( d_{K} \), we have

\[
\frac{1}{\sqrt{k}} A_{k} = \mathcal{O}_{K} \iff k = 1 \ or \ k = m,
\]

\[
\frac{1}{\sqrt{k}} A_{k} \cdot \frac{1}{\sqrt{\ell}} A_{\ell} = \frac{1}{\sqrt{f}} A_{f}, \quad f = \frac{k \ell}{\gcd(k, \ell)}.
\]

Note that due to (8) and Theorem 1, in some cases there exists a matrix \( M_0 \in \mathcal{N}_{K}, \ M_0 \notin \Gamma'_{K} \) of the fundamental unit \( \epsilon_0 \) satisfies \( \epsilon_0 \epsilon_0' = -1 \), we may choose

\[
M_0 = \frac{1}{\sqrt{\epsilon_0}} \begin{pmatrix} \epsilon_0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(11)

If \( m = \alpha^2 + \beta^2 \) for some \( \alpha, \beta \in \mathbb{N}, \ \alpha \ odd \), we choose \( \nu, \mu \in \mathbb{Z} \) satisfying \( \nu \alpha - 2 \mu \beta = 1, \ u := \beta + \sqrt{m} \in \mathcal{O}_{K}, \ uu' = -\alpha^2, \)

\[
L = \begin{pmatrix} \mu \alpha + \nu u & \mu u \\ u & \alpha \end{pmatrix}, \quad det L = u > 0,
\]

\[
\mathcal{I}(L) = \mathcal{O}_{K} \alpha + \mathcal{O}_{K} u, \quad \mathcal{I}(L)^2 = \mathcal{O}_{K} u, \quad u' < 0.
\]

Thus we may choose

\[
M_0 = \frac{1}{\sqrt{\ell}} L
\]

(12)

in this case. If we apply the theory of ambiguous ideals, then [15], 7.6 and 7.8, imply
Corollary 1 Let $\mathbb{K}$ be a real-quadratic number field with fundamental unit $\varepsilon_0$. If $\varepsilon_0 \varepsilon_0' = 1$ and $p \mid d_\mathbb{K}$ for some prime $p \equiv 3 \mod 4$, we have

$$N_\mathbb{K} = \Gamma^*_\mathbb{K}.$$  

In any other case
$$N_\mathbb{K} = \Gamma^*_\mathbb{K} \cup M_0 \Gamma^*_\mathbb{K}$$

holds with $M_0$ from (11) and (12).

We can quote Maaß [16] or apply the results from (8) and (10) in order to obtain

$$[\Gamma^*_\mathbb{K} : \Gamma_\mathbb{K}] = 2^{\nu - 1}, \quad \nu = \sharp \{p \text{ prime}; p \mid d_\mathbb{K}\}.$$

Remark 1 a) The field $\hat{\mathbb{K}}$ generated by the entries of the matrices in $\Gamma^*_\mathbb{K}$ is given by

$$\hat{\mathbb{K}} = \mathbb{Q}(\sqrt{p}; p \text{ prime, } p \mid d_\mathbb{K})$$

according to (8) and (9). Hence an analog of Theorem 4 in [13] also holds in this case:

$$\hat{\mathbb{K}} \supset \mathbb{Q} \text{ is unramified outside } 2d_\mathbb{K}.$$ 

b) If $\varepsilon \gg 0$ is a unit in $O_\mathbb{K}$, then (8) yields

$$\frac{1}{\sqrt{2 + \varepsilon + \varepsilon'}} \begin{pmatrix} \varepsilon + 1 & 0 \\ 0 & \varepsilon' + 1 \end{pmatrix} \in \Gamma^*_\mathbb{K}.$$ 

Thus the squarefree kernel $q$ of $2 + \varepsilon + \varepsilon'$ always divides $d_\mathbb{K}$.

3 The Hilbert modular group as an orthogonal group

If $T \in \mathbb{Z}^{2 \times 2}$ is an even symmetric matrix with $\det T < 0$ the associated half-space $H_T$ is defined to be

$$H_T := \{ z = x + iy \in \mathbb{C}^2; y^r Ty > 0, y_1 > 0 \}.$$ 

Given $\tilde{M} \in SO_0(T_N; \mathbb{R})$ (cf. (2)) we will always assume the form

$$\tilde{M} = \begin{pmatrix} \bar{\alpha} & a^r T \bar{\beta} \\ b & K \bar{\gamma} \\ \gamma d^r T \bar{\delta} \end{pmatrix}, \quad \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \in \mathbb{R}. \quad (13)$$

It is well-known (cf. [8,12]) that $\tilde{M}$ acts on $H_T$ via

$$z \mapsto \tilde{M}(z) := (\tilde{M}[z])^{-1} (-\frac{1}{2} z^r Tz \cdot b + Kz + c), \quad \tilde{M}(z) := -\frac{1}{2} \bar{\gamma} z^r Tz + d^r Tz + \bar{\delta}. \quad (14)$$

Note that $H^2$, where $H$ denotes the upper half-plane in $\mathbb{C}$, is the orthogonal half-space $H_P$, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Considering [12], Sect. 5, we obtain a surjective homomorphism of the groups

$$\Omega : SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \rightarrow SO_0(P_1; \mathbb{R}), \quad (U, V) \mapsto \begin{pmatrix} \alpha F V F \beta \gamma \\ \gamma F V F \delta \end{pmatrix}, \quad U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
with kernel \{±(l, I)\} satisfying
\[
\Omega(U, V) \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) = \left( \begin{array}{c} U(r_1) \\ V(r_2) \end{array} \right).
\]

Thus we have for \(M \in \Sigma_K\)
\[
\Omega(M, M') = \begin{pmatrix}
\alpha\alpha' & -\alpha'\beta & -\alpha\beta' \\
-\alpha'y' & \alpha\delta' & \beta'\gamma \\
-\alpha\gamma' & \beta\gamma' & \alpha'\delta
\end{pmatrix} \in SO_0(P_1; \mathbb{R}).
\]

Note that \(\Omega(M, M') \neq -I\) for all these \(M\). Now choose a basis \(B = (u, v)\) of \(\mathbb{K}\) over \(\mathbb{Q}\) and consider the base change
\[
\left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) = Gz, \quad G = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}, \quad \widehat{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & G & 0 \end{pmatrix}.
\]

We set
\[
\tilde{M} = \widehat{G}^{-1}\Omega(M, M')\widehat{G} \in SO_0(S_1; \mathbb{R}), \quad S = G^\text{tr}PG = \begin{pmatrix} 2uu' & uv' + u'v \\ uv' + u'v & 2vv' \end{pmatrix},
\]
\[
\tilde{M}(z) = (y', r_1 + \delta')(y' r_2 + \delta').
\]

If we define
\[
\varphi_B : \mathbb{K} \to \mathbb{Q}^2, \quad \alpha u + \beta v \mapsto \left( \begin{array}{c} \alpha \\ \beta \end{array} \right).
\]

a straightforward calculation yields that the description of \(\tilde{M}\) in (13) is given by
\[
\begin{align*}
\tilde{\alpha} &= \alpha\alpha', \quad \tilde{\beta} = -\beta\beta', \quad \tilde{\gamma} = -\gamma\gamma', \quad \tilde{\delta} = \delta\delta', \\
a &= \varphi_B(-\alpha\beta'), \quad b = \varphi_B(-\alpha\gamma'), \quad c = \varphi_B(\beta\delta'), \quad d = \varphi_B(\gamma\delta), \\
K &\text{ is the matrix that represents the endomorphism} \\
f_M : \mathbb{K} \to \mathbb{K}, \quad w \mapsto \alpha\delta w + \beta\gamma w', \quad \text{with respect to the basis} \ B.
\end{align*}
\]

Recalling the definition of the discriminant kernel from (1), we obtain

**Theorem 2.** Let \(\mathbb{K}\) be a real-quadratic field and let \(B = (u, v)\) be a \(\mathbb{Z}\)-basis of \(\sigma_\mathbb{K}\) with Gram matrix \(S\) from (17). Then the mappings
\[
\tilde{\varphi}_B : \Sigma_\mathbb{K}/\{±I\} \to SO_0(S_1; \mathbb{Q})/\{±I\}, \quad ±M \mapsto ±\tilde{M},
\]
\[
\tilde{\varphi} : \Gamma_\mathbb{K}^*/\{±I\} \to SO_0(S_1; \mathbb{Z})/\{±I\}, \quad ±M \mapsto ±\tilde{M},
\]

where \(\tilde{M}\) is defined by (18), are isomorphisms of the groups. Moreover the mapping
\[
\varphi_B : \Gamma_\mathbb{K}/\{±I\} \to D(S_1; \mathbb{Z}), \quad ±M \mapsto \tilde{M},
\]
is an isomorphism of the groups.

**Proof.** We get \(\tilde{\varphi}_B(\Sigma_\mathbb{K}/\{±I\}) \subseteq SO_0(S_1; \mathbb{Q})/\{±I\}\) and \(\tilde{\varphi}_B(\Gamma_\mathbb{K}^*/\{±I\}) \subseteq SO_0(S_1; \mathbb{Z})/\{±I\}\) directly from (18). As \(\tilde{\varphi}_B\) maps the group actions in (14) and (15) onto each other, \(\tilde{\varphi}_B\) is an injective homomorphism of the groups. In the notation of [12] we easily see that \(SO_0(S_1; \mathbb{Q})\) is generated by the matrices
\[
T_{\lambda}, \ 1, \ \lambda, \ \lambda \in \mathbb{Q}^2, \ \text{diag} (\ell, 1, 1/\ell, \ 1) \in \mathbb{N},
\]
\( R_K, K = \begin{pmatrix} a & bm \\ b & a \end{pmatrix}, \ w = a + b\sqrt{m} \in K, \ w > 0, \ ww' = 1. \)

They appear as images of the matrices
\[
\pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}, \lambda \in K, \pm \frac{1}{\sqrt{2 + 2a}} \begin{pmatrix} w + 1 & 0 \\ 0 & w + 1 \end{pmatrix}
\]
in \( \Sigma_K \). Hence \( \tilde{\phi}_B \) is an isomorphism for the rationals. As \( \tilde{\phi}_B^{-1}(SO_0(S_1; \mathbb{Z})/\{\pm I\}) \) is a discontinuous subgroup of \( PSL_2(\mathbb{R}) \) containing \( \Gamma^*_K/\{\pm I\} \), we obtain equality from the result of Maaß [16].

In view of \( -I \notin \phi_B(\Gamma_K/\{\pm I\}) \) we conclude that \( \phi_B \) is an injective homomorphism of the groups with \( \phi_B(\Gamma_K/\{\pm I\}) \subseteq SO_0(S_1; \mathbb{Z}) \). We get
\[
(K - I) \in \mathbb{Z}^{2 \times 2}
\]
if and only if \( FM = fM - id \) satisfies
\[
(\phi_B(F_M(u)), \phi_B(F_M(v))) \in \mathbb{Z}^{2 \times 2}.
\]
If \( X \in \mathbb{Z}^{2 \times 2} \) satisfies
\[
(\sqrt{d_K}u, \sqrt{d_K}v) = (u, v)X
\]
the latter condition becomes equivalent to
\[
(\phi_B(F_M(\sqrt{d_K}u)), \phi_B(F_M(\sqrt{d_K}v))) = (\phi_B(F_M(u)), \phi_B(F_M(v)))X \subseteq \mathbb{Z}^{2 \times 2}X = d_K\mathbb{Z}^{2 \times 2}
\]
as
\[
SX = G^{tr}PGX = G^{tr}\sqrt{d_K}G = \pm d_KJ, \ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Thus (19) holds if and only if
\[
F_M(\sqrt{d_K}\sigma_K) \subseteq d_K\sigma_K.
\]
This is easily demonstrated for \( M \in \Gamma_K \) using \( \alpha\delta - \beta\gamma = 1 \). If \( \ell \) is a squarefree divisor of \( d_K \), \( \ell \neq 1, m \) we verify
\[
F_{\ell_1}(\sqrt{d_K}) \notin d_K\sigma_K.
\]
Thus \( \phi_B \) becomes an isomorphism, too. \( \square \)

Now we apply the results to congruence subgroups.

**Corollary 2** Let \( N \in \mathbb{N} \) and \( B = (u, v) \) be a \( \mathbb{Z} \)-basis of \( \sigma_K \) with Gram matrix \( S \) from (17).

(a) \( \phi_B \) maps
\[
\begin{align*}
\{ M &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K; \ \alpha\alpha' \equiv \delta\delta' \equiv 1 \mod N, \ \gamma \in N\sigma_K \} \bigg/ \{ \pm I \} \\
\end{align*}
\]
onto
\[
F_ND(S_N; \mathbb{Z})F_N^{-1}, \quad F_N = \text{diag}(1, 1, 1, N).
\]

(b) \( \phi_B \) maps the principal congruence subgroup
\[
\{ M \in \Gamma_K; M \equiv \varepsilon I \mod N\sigma_K, \varepsilon \in \mathbb{Z}, \varepsilon^2 \equiv 1 \mod N \} \bigg/ \{ \pm I \}
\]
onto
\[
D(NS_1; \mathbb{Z}).
\]
Proof  (a) Apply Theorem 2 and (18).

(b) Given $M$ in the principal congruence subgroup, then $\phi_B(\pm M) \in D(NS_1; \mathbb{Z})$ is a consequence of (18) and Theorem 2. On the other hand $\pm M \in \phi_B^{-1}(D(NS_1; \mathbb{Z}))$ implies $\beta, \gamma \in NO_K$ due to (18). Considering the representing matrix of $w \mapsto \alpha \delta' w$ and using $\alpha \delta \equiv 1 \mod NO_K$ we obtain

$$M \equiv \varepsilon I \mod NO_K, \ v \in \mathbb{Z}, \ v^2 \equiv 1 \mod N.$$  

$\square$

We add a Remark

Remark 2 $\sigma_K$ with the symmetric bilinear form $(\alpha, \beta) \mapsto \alpha \beta' + \alpha' \beta$ is a maximal even lattice. Thus Theorem 2 corresponds to the results on $SO(2, n), n \geq 3$ in [14].

4 The general Hilbert modular group

Given an integral ideal $0 \neq \mathcal{I} \subseteq \sigma_K$ the (general) Hilbert modular group with respect to $\mathcal{I}$ is defined by

$$\Gamma_K(\mathcal{I}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{K}); \ \alpha, \delta \in \sigma_K, \beta \in \mathcal{I}, \gamma \in \mathcal{I}^{-1} \right\}.$$  

Given a basis $B$ of $\mathbb{K}$ then Theorem 2 and (18) immediately imply

$$\phi_B(\Gamma_K(n\mathcal{I})/\{\pm I\}) = H_n \phi_B((\Gamma_K(\mathcal{I})/\{\pm I\})H_n^{-1},$$  

whenever $n \in \mathbb{N}$ and $H_n = \phi_B \left( \pm \frac{1}{\sqrt{n}} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \right) = \text{diag}(n, 1, 1/n)$. Hence we may assume that $\mathcal{I}$ is a primitive ideal, i.e.

$$\frac{1}{n} \mathcal{I} \subseteq \sigma_K, \ n \in \mathbb{N} \implies n = 1.$$  

In this case $\mathcal{I}$ contains a $\mathbb{Z}$-basis

$$N, t + \omega_K, N \in \mathbb{N}, \ t \in \mathbb{Z}, \ N | (t + \omega_K)(t + \omega_K'),$$  

where $N$ is the reduced norm of $\mathcal{I}$ (cf. [15], Proposition 4.2). It follows from [9], [10] or [16] that the so-called Maaß-Hurwitz extension $\Gamma_{K*}(\mathcal{I})$ is a maximal discontinuous subgroup of $SL_2(\mathbb{R})$ containing $\Gamma_K(\mathcal{I})$ with index

$$2^{v-1}, \ v = \sharp \{ p \text{ prime}; p | d_K \}.$$  

It is given by

$$\Gamma_{K*}(\mathcal{I}) = \left\{ \frac{1}{\sqrt{\det L}} L; \ L \in \begin{pmatrix} \sigma_K & \mathcal{I} \\ \mathcal{I}^{-1} & \sigma_K \end{pmatrix}, \ \det L \gg 0, \ \mathcal{I}(L)^2 = \sigma_K \det L \right\}$$  

$$= \bigcup_{\ell \in \mathbb{N} \text{ squarefree}, \ \ell | d_K} V_\ell \Gamma_K(\mathcal{I}).$$  

One may choose

$$V_\ell = \frac{1}{\sqrt{\ell}} \begin{pmatrix} v \ell & \mu \ell N \\ w' & \ell \end{pmatrix}, \ u = \frac{\ell}{\gcd(\ell, N)} + m + \sqrt{m}, \ v, \mu \in \mathbb{Z}, \ v\ell - \mu N w' / \ell = 1.$$  

Just as in Sect. 2 we have

$$V_\ell \Gamma_K(\mathcal{I}) = \Gamma_K(\mathcal{I}) V_\ell = \frac{1}{\sqrt{\ell}} \begin{pmatrix} A_\ell & A_\ell \mathcal{I} \\ A_\ell \mathcal{I}^{-1} & A_\ell \end{pmatrix} \cap SL_2(\mathbb{R}).$$
as well as
\[ V_\ell \Gamma_\mathcal{K}(\mathcal{I}) = \Gamma_\mathcal{K}(\mathcal{I}) \iff \ell = 1 \text{ or } \ell = m. \]

**Theorem 3** Let \( \mathcal{K} \) be a real-quadratic field and let \( \mathcal{I} \subseteq \mathcal{O}_\mathcal{K} \) be a primitive ideal with reduced norm \( N \). Assume that \( B = (u, v) \) is a \( \mathbb{Z} \)-basis of \( \mathcal{I} \) with Gram-matrix \( S \) from (17) and let \( T = \frac{1}{N} S \). Then the mappings
\[
\psi_B : \Gamma_\mathcal{K}(\mathcal{I})/\{\pm I\} \to \mathcal{D}(T_1; \mathbb{Z}), \quad \pm M \mapsto H_N^{-1} M H_N,
\]
\[
\tilde{\psi}_B : \Gamma_\mathcal{K}^*(\mathcal{I})/\{\pm I\} \to SO_0(T_1; \mathbb{Z})/\{\pm I\}, \quad \pm M \mapsto \pm H_N^{-1} \tilde{M} H_N,
\]
where \( H_N = \text{diag} (N, 1, 1, 1) \) and \( \tilde{M} \) is given by (18), are isomorphisms of the groups.

**Proof** Proceed exactly as in the proof of Theorem 2. Observe that \( \mathcal{I}^{-1} = \frac{1}{N} \mathcal{I}' \) and verify that
\[
(K - I) \in \mathbb{Z}^{2 \times 2} T \iff F_M(\sqrt{d_\mathcal{K} \mathcal{I}}) \subseteq d_\mathcal{K} \mathcal{I}.
\]
Use the basis (20) of \( \mathcal{I} \) in order to verify that
\[
F_{V_\ell}(\sqrt{d_\mathcal{K} \mathcal{I}}) \subseteq d_\mathcal{K} \mathcal{I} \iff \ell = 1 \text{ or } \ell = m,
\]
whenever \( \ell \) is a squarefree divisor of \( d_\mathcal{K} \).

We obtain an immediate application to congruence subgroups of \( \Gamma_\mathcal{K} \).

**Corollary 3** Let \( N \) be a squarefree divisor of \( d_\mathcal{K} \). Then the principal congruence subgroup
\[
\left\{ M \in \Gamma_\mathcal{K}; \ M \equiv \varepsilon I \mod A_N, \ \varepsilon \in \mathbb{Z}, \ \varepsilon^2 \equiv 1 \mod N \right\}/\{\pm I\}
\]
is isomorphic to
\[
G_N^{-1} \mathcal{D}(T_N; \mathbb{Z}) G_N, \quad G_N = \text{diag} (1, N, N, 1), \quad T = \begin{pmatrix} 2N & \omega_\mathcal{K} + \omega_\mathcal{K}' \\ \omega_\mathcal{K} + \omega_\mathcal{K}' & 2N \omega_\mathcal{K} \omega_\mathcal{K}' / N \end{pmatrix}
\]
via \( \phi_B \), where \( B = (N, \omega_\mathcal{K}) \).

**Proof** Apply Theorem 3 to \( \mathcal{I} = A_N = \mathbb{Z} N + \mathbb{Z} \omega_\mathcal{K} \) and follow the proof of Corollary 2 a).

\( \square \)

We add a final Remark.

**Remark 3** The congruence conditions with respect to \( N \) as opposed to more general \( \mathcal{I} \) in Corollaries 2 and 3 are equivalent to the restriction to ideals \( \mathcal{I} \) satisfying \( \mathcal{I} = \mathcal{I}' \). It is not only motivated by technical reasons, but also by the fact that symmetric Hilbert modular forms, i.e. \( f(\tau_2, \tau_1) = f(\tau_1, \tau_2) \), can be defined for these subgroups.

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