Blow-up prevention by nonlinear diffusion in a 2D Keller-Segel-Navier-Stokes system with rotational flux

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Abstract

This paper investigates the following Keller-Segel-Navier-Stokes system with nonlinear diffusion and rotational flux

\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n S(x, n, c) \nabla c), \quad x \in \Omega, t > 0, \\
n_t + u \cdot \nabla c &= \Delta c - c + n, \quad x \in \Omega, t > 0, \\
u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\
\nabla \cdot u &= 0, \quad x \in \Omega, t > 0, \\
\end{align*}

(KSNF)

where $\kappa \in \mathbb{R}, \phi \in W^{2,\infty}(\Omega)$ and $S$ is a given function with values in $\mathbb{R}^{2 \times 2}$ which fulfills

$$|S(x, n, c)| \leq C_S$$

with some $C_S > 0$. Systems of this type describe chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production

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through cells. If \( m > 1 \) and \( \Omega \subset \mathbb{R}^2 \) is a **bounded** domain with smooth boundary, then for all reasonably regular initial data, a corresponding initial-boundary value problem for \((KS\text{N}F)\) possesses a global and bounded (weak) solution, which significantly improves previous results of several authors. Moreover, the **optimal condition** on the parameter \( m \) for global existence is obtained. Our approach underlying the derivation of main result is based on an entropy-like estimate involving the functional

\[
\int_{\Omega} (n_{\varepsilon} + \varepsilon)^m + \int_{\Omega} |\nabla c_{\varepsilon}|^2,
\]

where \( n_{\varepsilon} \) and \( c_{\varepsilon} \) are components of the solutions to \((2.1)\) below.

**Key words:** Navier-Stokes system; Keller-Segel model; Global existence; Nonlinear diffusion

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1 Introduction

Chemotaxis, the biased movement of cells in response to chemical gradients, plays an important role in coordinating cell migration in many biological phenomena (see Hillen and Painter [6]). For example, the fruit fly Drosophila melanogaster navigates up gradients of attractive odours during food location, and male moths follow pheromone gradients released by the female during mate location. In 1970 Keller and Segel [8] proposed a mathematical model describing chemotactic aggregation of cellular slime molds. But in their model, they did not take into account the relationship between cells and their environment. So the model can be used to describe that bacterial chemotaxis was viewed as locomotion in an otherwise quiescent fluid. Yet suspensions of aerobic bacteria often develop locomotion in an otherwise quiescent fluid. Yet suspensions of aerobic bacteria often develop flows from the interplay of chemotaxis and buoyancy. Tuval and his cooperator [17] described the above biological phenomena and proposed the mathematical model consisting of oxygen diffusion and consumption, chemotaxis, and fluid dynamics

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n\chi(c)\nabla c), \quad x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, t > 0, \\
    u_t + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, \quad x \in \Omega, t > 0
\end{align*}
\]

in a domain \( \Omega \subset \mathbb{R}^N (N \geq 1) \), where \( n, c, u, \) and \( P \) denote, respectively, the density of cells, chemical concentration, velocity field and pressure of the fluid. The coefficient \( \kappa \) is related to the strength of nonlinear fluid convection, \( \phi \) stands for the potential of the gravitational field within which the cells are driven through buoyant forces, the function \( \chi(c) \) measures the chemotactic sensitivity, and \( f(c) \) represents the oxygen consumption rate. Some modeling approaches suggested that an adequate description of bacterial motion near surfaces of their surrounding fluid should involve rotational components in the cross-diffusive flux (see [23, 24]), so the natural generalizations of chemotaxis-fluid systems should model
the evolution of the cell density, as the following form

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (nS(x, n, c)\nabla c), \quad x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, t > 0, \\
  u_t + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, \quad x \in \Omega, t > 0
\end{align*}
\]

where \( S \) stands for the chemotactic sensitivity. Moreover, since the diffusion of bacteria (or, more generally, of cells) in a viscous fluid is more like movement in a porous medium, the authors in [2] extended the above model to one with a porous medium-type diffusion

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (nS(x, n, c)\nabla c), \quad x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - n f(c), \quad x \in \Omega, t > 0, \\
  u_t + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, \quad x \in \Omega, t > 0
\end{align*}
\]

where \( m > 1 \). Concerning the framework where the chemical is produced by the cells instead of consumed, then the corresponding chemotaxis-fluid model is then the quasilinear Keller-Segel-Navier-Stokes system of the form (see [1, 6])

\[
\begin{align*}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (nS(x, n, c)\nabla c), \quad x \in \Omega, t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - c + n, \quad x \in \Omega, t > 0, \\
  u_t + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
  \nabla \cdot u &= 0, \quad x \in \Omega, t > 0
\end{align*}
\]

Due to the presence of the tensor-valued sensitivity as well as the strongly nonlinear term \((u \cdot \nabla)u\) and lower regularity for \( n \), the mathematical analysis of (1.1) regarding global and bounded solutions is far from trivial. Some simplified cases of the system (1.1) have been studied. When \( \kappa = 0 \), which is corresponding to the chemotaxis-Stokes system, the results focused on the global existence and boundedness of the solutions, for example, Wang and Xiang ([19]) dealt with the case \( m = 1 \) in 2-dimensional space; while for \( m \neq 1 \), Li, Wang and Xiang ([9]), Peng and Xiang ([12]) considered the problem with the spatial dimension \( N = 2 \) and \( N = 3 \), respectively. When \( \kappa \neq 0 \), \( m = 1 \) and \( |S(x, n, c)| \leq C_S(1 + n)^{-\alpha} \) for some \( C_S \geq 0 \) and \( \alpha > 0 \), Wang, Winkler and Xiang ([18]) and Ke and Zheng ([7]) considered the
global existence of the solution for the case $N = 2$ and $N = 3$, respectively. But till now, as far as we know, it is still not clearly that in the case that $\kappa \neq 0$ and $\alpha = 0$, whether the solution of the system (1.1) is bounded or not. At the same time, we also noticed that when dealing with the problem of $\kappa = 0$ and $\alpha = 0$, or $\kappa \neq 0$ and $\alpha > 0$, Li, Wang and Xiang [9] and Wang, Winkler and Xiang [18] both added the assumption that the domain is convex. Whether the convexity of the domain is necessary also arouses our interest. By considering the key energy functional

$$\int_{\Omega} n^m + \int_{\Omega} |\nabla c|^2,$$

we can obtain the global existence and boundedness of the solution for the system (1.1), which corresponding to the case that $\kappa \neq 0$ and $\alpha = 0$, in a more general non-convex domain.

In this paper, we shall subsequently consider the chemotaxis-Navier-Stokes system (1.1) along with the initial data

$$n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.2)$$

and under the boundary conditions

$$(nS(x, n, c)\nabla c) \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, \quad x \in \partial \Omega, \quad t > 0, \quad (1.3)$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where we assume that the chemotactic sensitivity tensor $S(x, n, c)$ be satisfied

$$S \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{2 \times 2}) \quad (1.4)$$

and

$$|S(x, n, c)| \leq C_S \quad \text{for all} \quad (x, n, c) \in \Omega \times [0, \infty)^2 \quad (1.5)$$

with some $C_S > 0$. Throughout this paper, we assume that

$$\phi \in W^{2,\infty}(\Omega) \quad (1.6)$$
and the initial data \((n_0, c_0, u_0)\) fulfills

\[
\begin{cases}
  n_0 \in C^\kappa(\Omega) \text{ for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\
  c_0 \in W^{2,\infty}(\Omega) \text{ with } c_0, w_0 \geq 0 \text{ in } \Omega, \\
  u_0 \in D(A),
\end{cases}
\]

where \(A\) denotes the Stokes operator with domain \(D(A) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)\), and \(L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) | \nabla \cdot \varphi = 0\}\). (see [14]).

Within the above frameworks, our main result concerning global existence and boundedness of solutions to (1.1)-(1.3) is as follows.

**Theorem 1.1.** Let \(m > 1\), \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth boundary, and assume (1.4)-(1.7) hold. Then the problem (1.1)-(1.3) admits a global-in-time weak solution \((n, c, u, P)\), which is uniformly bounded in the sense that

\[
\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t > 0
\]

with some positive constant \(C\).

**Remark 1.1.** (i) If \(u \equiv 0\), Theorem 1.1 is (partly) coincides with Theorem 4.1 of [20], which is optimal according to the fact that the 2D fluid-free system admits a global bounded classical solution for \(m > 1\) as mentioned by [15] (see also [20]).

(ii) Theorem 1.1 extends the results of Li, Wang and Xiang [9], who proved the possibility of boundedness in the case that \(\Omega\) is a bounded convex domain \(\Omega \subset \mathbb{R}^2\) with smooth boundary, \(\kappa = 0\) and \(S\) satisfies (1.4) as well as (1.5) with some \(m > 1\).

This paper is organized as follows. In Section 2, we do some preliminary works and propose a approximate problem. In Section 3, we use some iteration technique to establish the necessary a priori estimates. Finally, in Section 4, we obtain the global existence and boundedness of the solutions for the system (1.1)-(1.3) in a bounded domain.
2 Preliminaries

In order to construct a weak solutions by an approximation procedure, we construct the approximate problems as follows

\[
\begin{aligned}
n_{\varepsilon t} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} &= \Delta (n_{\varepsilon} + \varepsilon)^m - \nabla \cdot (n_{\varepsilon} S_{\varepsilon}(x,n_{\varepsilon},c_{\varepsilon}) \nabla c_{\varepsilon}), & x \in \Omega, \ t > 0, \\
c_{\varepsilon t} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} &= \Delta c_{\varepsilon} - c_{\varepsilon} + n_{\varepsilon}, & x \in \Omega, \ t > 0, \\
u_{\varepsilon t} + \nabla P_{\varepsilon} &= \Delta u_{\varepsilon} - \kappa (Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + n_{\varepsilon} \nabla \phi, & x \in \Omega, \ t > 0, \\
\nabla \cdot u_{\varepsilon} &= 0, & x \in \Omega, \ t > 0, \\
\nabla n_{\varepsilon} \cdot \nu = \nabla c_{\varepsilon} \cdot \nu = 0, & u_{\varepsilon} = 0, & x \in \partial \Omega, \ t > 0, \\
n_{\varepsilon}(x,0) = n_0(x), c_{\varepsilon}(x,0) = c_0(x), & u_{\varepsilon}(x,0) = u_0(x), & x \in \Omega,
\end{aligned}
\]

(2.1)

where

\[
S_{\varepsilon}(x,n,c) := \rho_{\varepsilon}(x) \chi_{\varepsilon}(u) S(x,n,c), \quad x \in \bar{\Omega}, \ n \geq 0, \ c \geq 0,
\]

\[
\rho_{\varepsilon} \in C^\infty_0(\Omega) \text{ such that } 0 \leq \rho_{\varepsilon} \leq 1 \text{ in } \Omega \text{ and } \rho_{\varepsilon} \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0,
\]

\[
\chi_{\varepsilon} \in C^\infty_0([0,\infty)) \text{ such that } 0 \leq \chi_{\varepsilon} \leq 1 \text{ in } [0,\infty) \text{ and } \chi_{\varepsilon} \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon \searrow 0,
\]

and

\[
Y_{\varepsilon} w := (1 + \varepsilon A)^{-1} w \text{ for all } w \in L^2_\sigma(\Omega)
\]

is a standard Yosida approximation.

By the well-established fixed-point arguments (see Lemma 2.1 of [22], [21] and Lemma 2.1 of [11]), we could show the local solvability of system (2.1).

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary, and assume (1.4)-(1.7) hold. For any \( \varepsilon \in (0,1) \), there exist \( T_{\max,\varepsilon} \in (0,\infty) \) and a classical solution \((n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}, P_{\varepsilon})\) of system (2.1) in \( \Omega \times [0,T_{\max,\varepsilon}) \). Here

\[
\begin{aligned}
n_{\varepsilon} &\in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})), \\
c_{\varepsilon} &\in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})) \cap \bigcap_{p > 1} L^\infty([0,T_{\max,\varepsilon}); W^{1,p}(\Omega)), \\
u_{\varepsilon} &\in C^0(\bar{\Omega} \times [0,T_{\max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max,\varepsilon})) \cap \bigcap_{\gamma \in (0,1)} C^0([0,T_{\max,\varepsilon}); L^2(A^\gamma)), \\
P_{\varepsilon} &\in C^{1,0}(\bar{\Omega} \times (0,T_{\max,\varepsilon})).
\end{aligned}
\]

(2.2)
Moreover, \( n_\varepsilon \) and \( c_\varepsilon \) are nonnegative in \( \Omega \times (0, T_{\max,\varepsilon}) \), and if \( T_{\max,\varepsilon} < +\infty \), then
\[
\lim_{t \uparrow T_{\max,\varepsilon}} \sup \left[ \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \right] = \infty
\]
for all \( p > 2 \) and \( \gamma \in (\frac{1}{2}, 1) \).

Lemma 2.2. ([10]) Let \( T \in (0, \infty] \), \( \sigma \in (0, T) \), \( A > 0 \) and \( B > 0 \), and suppose that \( y : [0, T) \to [0, \infty) \) is absolutely continuous and such that
\[
y'(t) + Ay(t) \leq h(t) \quad \text{for a.e.} \quad t \in (0, T)
\]
with some nonnegative function \( h \in L^1_{\text{loc}}([0, T)) \) satisfying
\[
\int_t^{t+\sigma} h(s) \, ds \leq B \quad \text{for all} \quad t \in (0, T - \sigma).
\]
Then
\[
y(t) \leq \max\{y_0 + B, \frac{B}{A^\gamma} + 2B\} \quad \text{for all} \quad t \in (0, T).
\]

3 Some basic priori estimates

In order to establish the global solvability of system (2.1), in this section, we plan to derive some estimates for the approximate system (2.1), which plays a significant role in obtaining the main result. Let us first state two basic estimates on \( n_\varepsilon \) and \( c_\varepsilon \).

Lemma 3.1. ([7]) The solution of (2.1) satisfies
\[
\int_\Omega n_\varepsilon = \int_\Omega n_0 \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{3.1}
\]
as well as
\[
\int_\Omega c_\varepsilon \leq \max\{\int_\Omega n_0, \int_\Omega c_0\} \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}).
\]

According to Lemma 3.1, we can obtain the following energy-type equality, which was also used in Lemma 3.3 in [7] (see also [26, 18]).

Lemma 3.2. Let \( m > 1 \). Then there exists \( C > 0 \) independent of \( \varepsilon \) such that the solution of (2.1) satisfies
\[
\int_\Omega n_\varepsilon + \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \leq C \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{3.2}
\]
Moreover, for all \( t \in (0, T_{\max, \varepsilon} - \tau) \), it holds that one can find a constant \( C > 0 \) independent of \( \varepsilon \) such that
\[
\int_{t}^{t+\tau} \int_{\Omega} \left[ (n_{\varepsilon} + \varepsilon)^{2m-4} |\nabla n_{\varepsilon}|^2 + |\nabla c_{\varepsilon}|^2 + |\nabla u_{\varepsilon}|^2 \right] \leq C, \tag{3.3}
\]
where \( \tau = \min\{1, \frac{T}{6} T_{\max, \varepsilon}\} \).

In order to obtain the boundedness of \( n_{\varepsilon} \), we need to give higher norm estimates on \( c_{\varepsilon} \).

**Lemma 3.3.** Let \( (n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}) \) be the solution of \((2.2)\) and \( \tau = \min\{1, \frac{T}{6} T_{\max, \varepsilon}\} \). Then for any \( q > 2 \), there exists \( C := C(q, K) \) independent of \( \varepsilon \) such that
\[
\|c_{\varepsilon}(\cdot, t)\|_{L^q(\Omega)} \leq C \text{ for all } t \in (0, T_{\max, \varepsilon}). \tag{3.4}
\]

**Proof.** Let \( p > 3 + 4(m-1) \). Multiplying the second equation in \((2.1)\) by \( c_{\varepsilon}^{p-1} \), using the fact \( \nabla \cdot u_{\varepsilon} = 0 \), and applying the integration by parts, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} c_{\varepsilon}^p + (p-1) \int_{\Omega} c_{\varepsilon}^{p-2} |\nabla c_{\varepsilon}|^2 + \int_{\Omega} c_{\varepsilon}^p
\]
\[
= \int_{\Omega} c_{\varepsilon}^{p-1} n_{\varepsilon}
\]
\[
\leq \int_{\Omega} c_{\varepsilon}^{p-1} (n_{\varepsilon} + \varepsilon)
\]
\[
\leq \|n_{\varepsilon} + \varepsilon\|_{L^{p-2(m-1)}(\Omega)} \left( \int_{\Omega} c_{\varepsilon} \left( \frac{p-1) p-2(m-1)}{m-1} \right) \right) \frac{m-1}{p-2(m-1)} \text{ for all } t \in (0, T_{\max, \varepsilon})
\]
by the Hölder inequality. Now, due to the Gagliardo–Nirenberg inequality and \((3.1)\), for some positive constants \( \kappa_0 \) and \( \kappa_1 \), we derive
\[
\left( \int_{\Omega} c_{\varepsilon} \left( \frac{p-1) p-2(m-1)}{m-1} \right) \right) \frac{m-1}{p-2(m-1)}
\]
\[
= \|c_{\varepsilon}^{p} \|_{L^{p-1}(p-1) p-2(m-1)}^{(p-1) p-2(m-1)} \|L^{2(m-1)}(\Omega)}
\]
\[
\leq \kappa_0 (\|\nabla c_{\varepsilon}^{2(m-1)} \|_{L^{2(m-1)}} \|L^{2(m-1)}(\Omega)} + \|c_{\varepsilon}^{p} \|_{L^{p-1}(\Omega)}^{2(m-1)} \|L^{2(m-1)}(\Omega)}^{2(m-1)}
\]
\[
\leq \kappa_1 (\|\nabla c_{\varepsilon}^{p} \|_{L^{2(m-1)}} + 1).
\]
So that, in light of \((3.5)\) and the Young inequality, we derive that for all \( t \in (0, T_{\max, \varepsilon}) \),
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} c_{\varepsilon}^p + (p-1) \int_{\Omega} c_{\varepsilon}^{p-2} |\nabla c_{\varepsilon}|^2 + \int_{\Omega} c_{\varepsilon}^p \leq \kappa_1 \|n_{\varepsilon} + \varepsilon\|_{L^{p-2(m-1)}(\Omega)} \left( \|\nabla c_{\varepsilon}^{2(m-1)} \|_{L^{2(m-1)}} + 1 \right)
\]
\[
\leq \frac{(p-1)}{2} \left( \int_{\Omega} c_{\varepsilon}^{p-2} |\nabla c_{\varepsilon}|^2 + C_1(p) \kappa_1 (p-2(m-1)) \|n_{\varepsilon} + \varepsilon\|_{L^{p-2(m-1)}(\Omega)} \left( \|\nabla c_{\varepsilon}^{2(m-1)} \|_{L^{2(m-1)}} + 1 \right) + \kappa_1 \|n_{\varepsilon} + \varepsilon\|_{L^{p-2(m-1)}(\Omega)} \right),
\]
where we have used the fact that \( \frac{p-2(m-1) - 1}{p-2(m-1)} + \frac{1}{p-2(m-1)} = 1 \). In view of \( p > 3 + 4(m-1) \), again, from the Young inequality, there exist positive constants \( C_3 \) and \( C_4 \) such that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} c_\varepsilon^p + \frac{(p-1)}{2} \int_{\Omega} c_\varepsilon^{p-2} |\nabla c_\varepsilon|^2 + \int_{\Omega} c_\varepsilon^p \leq C_2 \| n_\varepsilon + \varepsilon \|_{p-2(m-1) L^{p-2(m-1)}(\Omega)}^{p-2(m-1)} + C_3 \text{ for all } t \in (0, T_{max,\varepsilon}).
\] (3.6)

In the following, we will estimate the integrals on the right-hand side of (3.6). In view of the Gagliardo-Nirenberg inequality, for some \( C_4, C_5, \) and \( C_6 > 0 \) which are independent of \( \varepsilon \), we may derive from (3.3) that

\[
\int_{t}^{t+\tau} \left( \| n_\varepsilon + \varepsilon \|_{p-2(m-1) L^{p-4(m-1)}(\Omega)}^{p-2(m-1)} + C_3 \right) ds \\
= \int_{t}^{t+\tau} \left( \| (n_\varepsilon + \varepsilon)^{m-1} \|_{p-2(m-1) L^{p-4(m-1)}(\Omega)}^{p-2(m-1)} + C_3 \right) ds \\
\leq C_4 \int_{t}^{t+\tau} \left( \| \nabla (n_\varepsilon + \varepsilon)^{m-1} \|_{L^2(\Omega)}^2 (n_\varepsilon + \varepsilon)^{m-1} \|_{L^{m-1}(\Omega)}^{p-1} + \| (n_\varepsilon + \varepsilon)^{m-1} \|_{L^{m-1}(\Omega)}^{p-2(m-1)} + C_3 \right) ds \\
\leq C_5 \int_{t}^{t+\tau} \left( \| \nabla (n_\varepsilon + \varepsilon)^{m-1} \|_{L^2(\Omega)}^2 \right) + C_3 \\
\leq C_6,
\]

where \( \tau = \min\{1, \frac{1}{6} T_{max,\varepsilon} \} \). Therefore, (3.4) holds by applying Lemma 2.2 and the Hölder inequality.

Based on Lemma 3.2 and Lemma 3.3, we can get a series of important estimates of \( n_\varepsilon \) and \( c_\varepsilon \).

**Lemma 3.4.** Let \( m > 1 \). Then the solution of (2.1) satisfies

\[
\int_{\Omega} (n_\varepsilon + \varepsilon)^m + \int_{\Omega} |\nabla c_\varepsilon|^2 \leq C \text{ for all } t \in (0, T_{max,\varepsilon}) \text{ and any } \varepsilon > 0
\] (3.7)

and

\[
\int_{t}^{t+\tau} \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m} \leq C \text{ for all } t \in (0, T_{max,\varepsilon} - \tau) \text{ and any } \varepsilon > 0,
\] (3.8)

where \( \tau = \min\{1, \frac{1}{6} T_{max,\varepsilon} \} \).

**Proof.** Multiplying the first equation of (2.1) by \((n_\varepsilon + \varepsilon)^{m-1}\), integrating the product in \( \Omega \),
and noticing $\nabla \cdot u_\varepsilon = 0$, one obtains
\[
\frac{1}{m} \frac{d}{dt} \| n_\varepsilon + \varepsilon \|_{L^m(\Omega)}^m + (m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
= - \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon) \\
= (m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
\leq C_S (m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} |\nabla n_\varepsilon| |\nabla c_\varepsilon| \quad \text{for all} \quad t \in (0, T_{\text{max,} \varepsilon})
\]
by using (3.10). Then, by using the Young inequality, we have
\[
\frac{1}{m} \frac{d}{dt} \| n_\varepsilon + \varepsilon \|_{L^m(\Omega)}^m + (m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
\leq \frac{m - 1}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 + \frac{(m - 1)C_S^2}{2} \int_\Omega (n_\varepsilon + \varepsilon)|\nabla c_\varepsilon|^2 \quad \text{for all} \quad t \in (0, T_{\text{max,} \varepsilon}).
\]
(3.9)

On the other hand, in view of Lemma 3.2 and invoking the Gagliardo–Nirenberg inequality, we infer with some $\gamma_0 > 0$ and $\gamma_1 > 0$ that
\[
\int_\Omega (n_\varepsilon + \varepsilon)^{2m} \\
= \| (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} \|_{L^{\frac{4m}{2m-1}}(\Omega)}^{\frac{4m}{2m-1}} \\
\leq \gamma_0 \| (\nabla (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} |\nabla (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} |L^{\frac{4m}{2m-1}}(\Omega) \\
+ \| (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} |L^{\frac{4m}{2m-1}}(\Omega) \\
\leq \gamma_1 \| (\nabla (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} |L^{\frac{4m}{2m-1}}(\Omega) + \gamma_1.
\]

We then achieve, with the help of the above inequality, that
\[
m(m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
= \frac{4m(m - 1)}{(2m - 1)^2} \| \nabla (n_\varepsilon + \varepsilon)^{\frac{2m-1}{2}} \|_{L^2(\Omega)}^2 \\
\geq \frac{1}{\gamma_1 (2m - 1)^2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m - 1}.
\]
(3.10)

Here, the Young inequality allows to be written as
\[
\frac{(m - 1)C_S^2}{2} \int_\Omega (n_\varepsilon + \varepsilon)|\nabla c_\varepsilon|^2 \\
\leq \varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + C_1(\varepsilon_1) \int_\Omega |\nabla c_\varepsilon|^{\frac{4m}{2m-1}},
\]
where
\[
\varepsilon_1 = \frac{1}{\gamma_1 (2m - 1)^2}
\]
(3.11)
and

\[ C_1(\varepsilon_1) = \frac{2m - 1}{2m} (\varepsilon_1 2m)^{-\frac{1}{2m-1}} \left( (m - 1)C_3^2 \right)^{\frac{2m}{2m-1}}. \]

In light of (3.4), there exist positive constants \( l_0 > \frac{1}{m-1} \) and \( C_2 \), such that

\[ \|c_\varepsilon(\cdot, t)\|_{L^0((\Omega))} \leq C_2 \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \] (3.12)

Next, with the help of the Gagliardo–Nirenberg inequality and (3.12), we derive that

\[ C_1(\varepsilon_1) \int_\Omega |\nabla c_\varepsilon|^{\frac{4m}{2m-1}} \leq C_3 \|\Delta c_\varepsilon\|_{L^4(\Omega)}^{\frac{4m}{2m-1}} \|c_\varepsilon\|_{L^0((\Omega))}^{\frac{1-a}{2m}} + C_3 \|c_\varepsilon\|_{L^0((\Omega))}^{\frac{4m}{2m-1}} \leq C_4 \|\Delta c_\varepsilon\|_{L^2(\Omega)} + C_4 \]

with some positive constants \( C_3 \) and \( C_4 \), where

\[ a = \frac{1}{2} + \frac{l_0 - \frac{2m-1}{4m}}{\frac{1}{2} + \frac{l_0}{4m}} \in (0, 1). \]

This, together with the Young inequality and \( a \frac{4m}{2m-1} < 2 \) (due to \( l_0 > \frac{1}{m-1} \)), yields

\[ C_1(\varepsilon_1) \int_\Omega |\nabla c_\varepsilon|^{\frac{4m}{2m-1}} \leq \frac{1}{4} \|\Delta c_\varepsilon\|_{L^2(\Omega)}^2 + C_5. \] (3.13)

Taking \( -\Delta c_\varepsilon \) as the test function for the second equation of (2.1), and using the Young inequality, it yields that for all \( t \in (0, T_{\text{max},\varepsilon}) \)

\[ \frac{1}{2} \frac{d}{dt} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \]

\[ = - \int_\Omega n_\varepsilon \Delta c_\varepsilon + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon) \Delta c_\varepsilon \]

\[ = - \int_\Omega n_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \nabla (u_\varepsilon \cdot \nabla c_\varepsilon) \]

\[ = - \int_\Omega n_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \nabla (\nabla u_\varepsilon \cdot \nabla c_\varepsilon), \] (3.14)

where we have used the fact that

\[ \int_\Omega \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) = \frac{1}{2} \int_\Omega u_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 = 0 \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \]

Meanwhile, we can further use Gagliardo-Nirenberg inequality and the elliptic regularity (4) to conclude that for some \( C_6 > 0 \),

\[ \|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \leq C_6 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \text{ for all } t \in (0, T_{\text{max},\varepsilon}). \]
This, together with the Cauchy-Schwarz inequality and the Young inequality, yields
\begin{align}
- \int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon, \nabla c_\varepsilon) \\
\leq \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 \\
\leq C_6 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \\
\leq C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4} \|\Delta c_\varepsilon\|^2_{L^2(\Omega)} \tag{3.15} \text{ for all } t \in (0, T_{max, \varepsilon}).
\end{align}

Applying the Cauchy-Schwarz inequality, one obtain
\begin{align}
- \int_\Omega n_\varepsilon \Delta c_\varepsilon \leq \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{3.16}
\end{align}

From (3.14) and (3.15), we thus infer that
\begin{align}
\frac{d}{dt} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \leq 2 \int_\Omega n_\varepsilon^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}, \tag{3.17}
\end{align}

Collecting (3.9), (3.13)–(3.17), we derive that for all $t \in (0, T_{max, \varepsilon}),$
\begin{align}
\frac{d}{dt} (\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + m(m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
+ \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq m\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + 2 \int_\Omega n_\varepsilon^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_7, \\
\leq m\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + 2 \int_\Omega (n_\varepsilon + \varepsilon)^2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_7.
\end{align}

Moreover, it follows from the Young inequality and $m > 1,$ that
\begin{align}
\frac{d}{dt} (\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + m(m - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \\
+ \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq 2m\varepsilon_1 \int_\Omega (n_\varepsilon + \varepsilon)^{2m} + 2 + 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_8 \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{3.18}
\end{align}

By substituting (3.10) into (3.18) and using (3.11), we find that
\begin{align}
\frac{d}{dt} (\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + \left(\frac{1}{\gamma_1} \frac{4m(m - 1)}{(2m - 1)^2} - 2m\varepsilon_1\right) \int_\Omega (n_\varepsilon + \varepsilon)^{2m} \\
+ \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \\
= \frac{d}{dt} (\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + \left(\frac{1}{\gamma_1} \frac{2m(m - 1)}{(2m - 1)^2}\right) \int_\Omega (n_\varepsilon + \varepsilon)^{2m} \\
+ \frac{1}{2} \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \\
\leq 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_9 \text{ for all } t \in (0, T_{max, \varepsilon}).
\end{align}
Therefore, we derive from the Young inequality that

\[ \frac{d}{dt}(\|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon\|^2_{L^2(\Omega)}) + 2\int_\Omega n_\varepsilon^m + 2\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{\gamma_1 (2m - 1)^2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m} \leq 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_{10} \]

\[ \leq 2C_6^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} (\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \|n_\varepsilon + \varepsilon\|^m_{L^m(\Omega)}) + C_{10} \text{ for all } t \in (0, T_{max,\varepsilon}), \]

(3.19)

where we have used the fact that $2\int\Omega n_\varepsilon^m \leq \frac{1}{\gamma_1 (2m - 1)^2} \int\Omega (n_\varepsilon + \varepsilon)^{2m} + C_{10}$, $m > 1$ and the Young inequality. Now, again, from the Gagliardo–Nirenberg inequality, (3.3), and Lemma 3.2 there exist constants $\gamma_3 > 0$ and $\gamma_4 > 0$, such that

\[ \int_t^{t+\tau} \int_\Omega (n_\varepsilon + \varepsilon)^m \]

\[ = \int_t^{t+\tau} ||(n_\varepsilon + \varepsilon)^{m-1}|| \frac{m-1}{m-1}_{L\frac{m-1}{m-1}(\Omega)} \]

\[ \leq \gamma_3 \int_t^{t+\tau} \|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|_{L^\frac{1}{m-1}(\Omega)} \|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|_{L^\frac{1}{m-1}(\Omega)} \]

\[ \leq \gamma_4 \int_t^{t+\tau} \|\nabla (n_\varepsilon + \varepsilon)^{m-1}\|_{L^2(\Omega)}^2 + \gamma_4 \text{ for all } t \in (0, T_{max,\varepsilon} - \tau), \]

(3.20)

where $\tau = \min\{1, \frac{1}{6} T_{max,\varepsilon}\}$. Therefore, by (3.20), we conclude that

\[ \int_t^{t+\tau} \int_\Omega (n_\varepsilon + \varepsilon)^m \leq \gamma_5 \text{ for all } t \in (0, T_{max,\varepsilon} - \tau). \]

(3.21)

Thus, for $t \in (0, T_{max,\varepsilon})$, if we write

\[ y(t) := \|n_\varepsilon(\cdot, t) + \varepsilon\|^m_{L^m(\Omega)} + \|\nabla c_\varepsilon(\cdot, t)\|^2_{L^2(\Omega)} \]

and

\[ \rho(t) = 2C_6^2 \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2; \]

(3.19) implies that

\[ y'(t) + h(t) \leq \rho(t)y(t) + C_{11} \text{ for all } t \in (0, T_{max,\varepsilon}), \]

(3.22)

where

\[ h(t) = \frac{1}{\gamma_1 (2m - 1)^2} \int_\Omega (n_\varepsilon + \varepsilon)^{2m}(\cdot, t) \geq 0. \]

Next, by using estimates (3.21) and (3.3), one obtains

\[ \int_t^{t+\tau} \rho(s)ds \leq C_{12} \]

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and
\[
\int_t^{t+\tau} y(s) \, ds \leq C_{13},
\]
for all \( t \in (0, T_{\text{max},\varepsilon} - \tau) \). For given \( t \in (0, T_{\text{max},\varepsilon}) \), using estimates (3.21) and (3.3) again, one can choose \( t_0 \geq 0 \) such that \( t_0 \in [t - \tau, t) \) and
\[
y(\cdot, t_0) \leq C_{14}.
\]
This, together with (3.22) and the Gronwall lemma, yields
\[
y(t) \leq y(t_0) e^{\int_{t_0}^t \rho(s) \, ds} + \int_{t_0}^t e^{\int_s^t \rho(\tau) \, d\tau} C_{11} \, ds
\leq C_{14} e^{C_{12}} + \int_{t_0}^t e^{C_{12}} C_{11} \, ds
\leq C_{14} e^{C_{12}} + e^{C_{12}} C_{11} \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]
(3.23)

Finally, collecting (3.22) and (3.23), it yields (3.7) and (3.8).

Lemma 3.5. Let \( m > 1 \). There exists a positive constant \( C \) independent of \( \varepsilon \), such that
\[
\int_{\Omega} |\nabla u_\varepsilon(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]
(3.24)

Proof. Firstly, applying the Helmholtz projection to both sides of the first equation in (2.1), then multiplying the result identified by \( Au_\varepsilon \), integrating by parts, and using the Young inequality, we find that
\[
\frac{1}{2} \frac{d}{dt} \|A_{\varepsilon}^T u_\varepsilon\|^2_{L^2(\Omega)} + \int_{\Omega} |Au_\varepsilon|^2 \leq \int_{\Omega} Au_\varepsilon \mathcal{P}(-\kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon) + \int_{\Omega} \mathcal{P}(n_\varepsilon \nabla \phi) Au_\varepsilon
\leq \frac{1}{2} \int_{\Omega} |Au_\varepsilon|^2 + \kappa^2 \int_{\Omega} |(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon|^2 + \|\nabla \phi\|^2_{L^\infty(\Omega)} \int_{\Omega} n_\varepsilon^2 \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]
(3.25)

Noticing that \( \|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \leq \|u_\varepsilon\|_{L^2(\Omega)} \), it follows from the Gagliardo-Nirenberg inequality and the Cauchy-Schwarz inequality that with some \( C_1 > 0 \) and \( C_2 > 0 \)
\[
\kappa^2 \int_{\Omega} |(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon|^2 \leq \kappa^2 \|Y_\varepsilon u_\varepsilon\|^2_{L^4(\Omega)} \|\nabla u_\varepsilon\|^2_{L^4(\Omega)}
\leq \kappa^2 C_1 [\|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} [\|Au_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}]
\leq \kappa^2 C_1 C_2 [\|Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} [\|Au_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}] \quad \text{for all } t \in (0, T_{\text{max},\varepsilon}).
\]
(3.26)
Now, from the fact that $D(A^{\frac{1}{2}}) := W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2(\Omega)$ and (3.2), it follows that
\[
\| \nabla Y_\varepsilon u_\varepsilon \|_{L^2(\Omega)} = \| A^{\frac{1}{2}} Y_\varepsilon u_\varepsilon \|_{L^2(\Omega)} = \| Y_\varepsilon A^{\frac{1}{2}} u_\varepsilon \|_{L^2(\Omega)} \leq \| A^{\frac{1}{2}} u_\varepsilon \|_{L^2(\Omega)} \leq \| \nabla u_\varepsilon \|_{L^2(\Omega)},
\] (3.27)

Due to Theorem 2.1.1 in [14], $\| A(\cdot) \|_{L^2(\Omega)}$ defines a norm equivalent to $\| \cdot \|_{W^{2,2}(\Omega)}$ on $D(A)$. This, together with the Young inequality and estimates (3.27) and (3.26), yields
\[
\kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon|^2 \leq C_3 \| A u_\varepsilon \|_{L^2(\Omega)} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 \leq \frac{1}{4} \| A u_\varepsilon \|_{L^2(\Omega)}^2 + \kappa^4 C_1^2 C_2^2 \| \nabla u_\varepsilon \|_{L^2(\Omega)}^4
\]
for all $t \in (0, T_{max,\varepsilon})$.

By the fact that $\| A^{\frac{1}{2}} u_\varepsilon \|_{L^2(\Omega)}^2 = \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2$, we conclude that
\[
z'(t) \leq \rho(t) z(t) + h(t) \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}),
\] (3.28)

where
\[
z(t) := \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2,
\]
as well as
\[
\rho(t) = 2\kappa^4 C_1^2 C_2^2 \int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2
\]
and
\[
h(t) = 2 \| \nabla \phi \|_{L^\infty(\Omega)}^2 \int_\Omega n_\varepsilon(\cdot, t).
\]

However, (3.3) along with (3.8) warrants that for some positive constant $\alpha_0$,
\[
\int_t^{t+\tau} \int_\Omega |\nabla u_\varepsilon|^2 \leq \alpha_0 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon} - \tau)
\] (3.29)
and
\[
\int_t^{t+\tau} \int_\Omega n_\varepsilon^2 \leq \alpha_0 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon} - \tau)
\] (3.30)
with $\tau = \min\{1, \frac{1}{6} T_{max,\varepsilon}\}$. Now, (3.29) and (3.30) ensure that for all $t \in (0, T_{max,\varepsilon} - \tau)$
\[
\int_t^{t+\tau} \rho(s) ds \leq 2 C^2_3 \alpha_0
\]
and
\[\int_t^{t+\tau} h(s) ds \leq 4\|\nabla \phi\|_{L^\infty(\Omega)}^2 \alpha_0.\]

For given \(t \in (0, T_{max, \varepsilon})\), applying (3.29) again, we can choose \(t_0 \geq 0\) such that \(t_0 \in [t - \tau, t)\) and
\[\int_{\Omega} |\nabla u_\varepsilon(\cdot, t_0)|^2 \leq C_4,\]
which combined with (3.28) implies that
\[z(t) \leq z(t_0) e^{\int_{t_0}^t \rho(s) ds} + \int_{t_0}^t e^{\int_{t_0}^s \rho(\tau) d\tau} h(s) ds\]
\[\leq C_4 e^{2C_2^2 \alpha_0} + \int_{t_0}^t e^{2C_2^2 \alpha_0} h(s) ds\]
\[\leq C_4 e^{2C_2^2 \alpha_0} + e^{2C_2^2 \alpha_0} \|\nabla \phi\|_{L^\infty(\Omega)}^2 \alpha_0 \text{ for all } t \in (0, T_{max, \varepsilon})\]
by integration. The claimed inequality (3.24) thus results from (3.31).

**Lemma 3.6.** Let \(m > 1\). Then there exists a positive constant \(C\) independent of \(\varepsilon\) such that the solution of (2.7) satisfies
\[\|\nabla c_\varepsilon(\cdot, t)\|_{L^{2m}(\Omega)} \leq C \text{ for all } t \in (0, T_{max}).\]  

**Proof.** Considering the fact that \(\nabla c_\varepsilon \cdot \nabla \Delta c_\varepsilon = \frac{1}{2} \Delta |\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2\), by a straightforward computation using the second equation in (2.1) and several integrations by parts, we find that
\[
\frac{1}{2m} \frac{d}{dt} \|\nabla c_\varepsilon\|_{L^{2m}(\Omega)}^{2m} \]
\[= \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} \nabla c_\varepsilon \cdot \nabla (\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)\]
\[= \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |\nabla c_\varepsilon|^2 - \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 - \int_{\Omega} |\nabla c_\varepsilon|^{2m}
- \int_{\partial\Omega} n_\varepsilon \nabla \cdot (|\nabla c_\varepsilon|^{2m-2} \nabla c_\varepsilon) + \int_{\Omega} (u_\varepsilon \cdot \nabla c_\varepsilon) \nabla \cdot (|\nabla c_\varepsilon|^{2m-2} \nabla c_\varepsilon)\]
\[= -\beta \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |\nabla |\nabla c_\varepsilon|^2|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla c_\varepsilon|^{2m-2} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} - \int_{\Omega} |\nabla c_\varepsilon|^{2m}
- \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 - \int_{\Omega} n_\varepsilon |\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon - \int_{\Omega} n_\varepsilon \nabla c_\varepsilon \cdot \nabla (|\nabla c_\varepsilon|^{2m-2})
+ \int_{\Omega} (u_\varepsilon \cdot \nabla c_\varepsilon) |\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon + \int_{\Omega} (u_\varepsilon \cdot \nabla c_\varepsilon) \nabla c_\varepsilon \cdot \nabla (|\nabla c_\varepsilon|^{2m-2})\]
\[- \frac{2(m - 1)}{m^2} \int_{\Omega} |\nabla |\nabla c_\varepsilon|^m|^2 + \frac{1}{2} \int_{\partial\Omega} |\nabla c_\varepsilon|^{2m-2} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} - \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2
- \int_{\Omega} n_\varepsilon |\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon - \int_{\Omega} n_\varepsilon \nabla c_\varepsilon \cdot \nabla (|\nabla c_\varepsilon|^{2m-2}) - \int_{\Omega} |\nabla c_\varepsilon|^{2m}
+ \int_{\Omega} (u_\varepsilon \cdot \nabla c_\varepsilon) |\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon + \int_{\Omega} (u_\varepsilon \cdot \nabla c_\varepsilon) \nabla c_\varepsilon \cdot \nabla (|\nabla c_\varepsilon|^{2m-2}).
\]
for all $t \in (0, T_{max})$. Here, since $|\Delta c_\varepsilon| \leq \sqrt{2}|D^2 c_\varepsilon|$, by utilizing the Young inequality, we can estimate

$$
\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon \\
\leq \sqrt{2} \int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon| \\
\leq \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + 2 \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2m-2} \\
\leq \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + 2 \int_\Omega (n_\varepsilon + \varepsilon)^2 |\nabla c_\varepsilon|^{2m-2}
$$

(3.34)

and, similarly,

$$
\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)|\nabla c_\varepsilon|^{2m-2} \Delta c_\varepsilon \\
\leq \sqrt{2} \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon||\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon| \\
\leq \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + 2 \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2} \\
\leq \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + 2 \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m}
$$

(3.35)

for all $t \in (0, T_{max})$. Again, from the Young inequality, we have

$$
- \int_\Omega n_\varepsilon \nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2m-2}) \\
= -(m - 1) \int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2(m-2)} \nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
\leq \frac{m - 1}{8} \int_\Omega |\nabla c_\varepsilon|^{2m-4} |\nabla|\nabla c_\varepsilon|^2|^2 + 2(m - 1) \int_\Omega |n_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2}
$$

(3.36)

and

$$
\int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)|\nabla c_\varepsilon|^{2m-2} \nabla c_\varepsilon \cdot \nabla(|\nabla c_\varepsilon|^{2m-2}) \\
= (m - 1) \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)|\nabla c_\varepsilon|^{2(m-2)} \nabla c_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2 \\
\leq \frac{m - 1}{8} \int_\Omega |\nabla c_\varepsilon|^{2m-4} |\nabla|\nabla c_\varepsilon|^2|^2 \\
+ 2(m - 1) \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^2 |\nabla c_\varepsilon|^{2m-2} \\
\leq \frac{m - 1}{2m^2} \int_\Omega |\nabla|\nabla c_\varepsilon|^m|^2 + 2(m - 1) \int_\Omega |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m}.
$$

(3.37)
Observe that
\[
\int_{\partial \Omega} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} |\nabla c_\varepsilon|^{2m-2} \leq C \int_{\partial \Omega} |\nabla c_\varepsilon|^{2m} = C ||\nabla c_\varepsilon||_{L^2(\partial \Omega)}^2.
\] (3.38)

Let us take \( r \in (0, \frac{1}{2}) \). Due to Proposition 4.22 (ii) of [5], we have that \( W^{r+\frac{1}{2}, 2}(\Omega) \hookrightarrow L^2(\partial \Omega) \) is compact, so that,
\[
||\nabla c_\varepsilon|^m||_{L^2(\partial \Omega)}^2 \leq C \int_{\partial \Omega} |\nabla c_\varepsilon|^{2m} = C ||\nabla c_\varepsilon||_{W^{r+\frac{1}{2}, 2}(\Omega)}^2.
\] (3.39)

Now, let us pick \( a = \frac{2m+2r-1}{2m} \). By \( r \in (0, \frac{1}{2}) \) and \( \beta > 1 \), it implies that \( r + \frac{1}{2} \leq a < 1 \). Therefore, from the fractional Gagliardo–Nirenberg inequality and Lemma 3.4, for some positive constants \( \delta_0, \delta_1 \) and \( C_1 \), we conclude
\[
||\nabla c_\varepsilon|^m||_{W^{r+\frac{1}{2}, 2}(\Omega)}^2 \leq \delta_0 ||\nabla c_\varepsilon|^m||_{L^2(\Omega)}^{\alpha} ||\nabla c_\varepsilon|^{1-a}||_{L^{\frac{2m}{m-a}}(\Omega)} + \delta_1 ||\nabla c_\varepsilon|^\beta||_{L^{\frac{2m}{m-\beta}}(\Omega)} \leq C_1 ||\nabla c_\varepsilon|^m||_{L^2(\Omega)}^2 + C_1.
\] (3.40)

Combining (3.38)–(3.40), using the Young inequality and the fact that \( a \in (0, 1) \), it yields
\[
\int_{\partial \Omega} \frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} |\nabla c_\varepsilon|^{2m-2} \leq C_2 ||\nabla c_\varepsilon|^m||_{L^2(\Omega)}^2 + C_2 \leq \frac{(m-1)}{2m^2} \int_{\Omega} |\nabla |\nabla c_\varepsilon|^m||_{L^2(\Omega)}^2 + C_3.
\] (3.41)

Now, together with (3.33)–(3.37) and (3.41), we can derive that, for some positive constant \( C_4 \),
\[
\frac{1}{2m} \frac{d}{dt} ||\nabla c_\varepsilon||_{L^2(\Omega)}^{2m} + m - \frac{1}{2m^2} \int_{\Omega} |\nabla |\nabla c_\varepsilon|^m||_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2m-2} |D^2 c_\varepsilon|^2 + \int_{\Omega} |\nabla c_\varepsilon|^{2m} \leq 2m \int_{\Omega} n_\varepsilon^2 |\nabla c_\varepsilon|^{2m-2} + 2m \int_{\Omega} u_\varepsilon^2 |\nabla c_\varepsilon|^{2m} + C_4 \text{ for all } t \in (0, T_{\max}).
\] (3.42)

We proceed to estimate the first term on the right-hand side of (3.42). By using the Young inequality, we conclude that
\[
2m \int_{\Omega} n_\varepsilon^2 |\nabla c_\varepsilon|^{2m-2} \leq 2m \int_{\Omega} (n_\varepsilon + \varepsilon)^2 |\nabla c_\varepsilon|^{2m-2} \leq \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2m} + C_5 \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m} \text{ for all } t \in (0, T_{\max})
\] (3.43)
\[ 2m \int_{\Omega} |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m} \leq \int_{\Omega} |\nabla c_\varepsilon|^{2m+1} + C_6 \int_{\Omega} u_\varepsilon^{4m+2} \text{ for all } t \in (0, T_{\max}), \tag{3.44} \]

where \( C_5 = \frac{m}{m-1} \left( \frac{1}{2} m \right)^{\frac{m-1}{m}} (2m)^m \) and \( C_6 = (2m)^{2m+1} \). On the other hand, due to (3.7), we derive from the Gagliardo–Nirenberg inequality that for some positive constants \( C_7 \) and \( C_8 \)

\[
\int_{\Omega} |\nabla c_\varepsilon|^{2m+1} \leq (2m)^{2m+1} \int_{\Omega} u_\varepsilon^{4m+2} + C_9 \quad \text{for all } t \in (0, T_{\max}). \tag{3.45}
\]

Inserting (3.45) into (3.44), we derive that

\[
\int_{\Omega} |u_\varepsilon|^2 |\nabla c_\varepsilon|^{2m} \leq \frac{m-1}{2m^2} \int_{\Omega} |\nabla c_\varepsilon|^{2m} + C_9 \text{ for all } t \in (0, T_{\max}). \tag{3.46}
\]

Substituting (3.43) and (3.46) into (3.42), we have

\[
\frac{1}{2m} \frac{d}{dt} \|\nabla c_\varepsilon\|_{L^2(\Omega)}^{2m} + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2m} \leq C_5 \int_{\Omega} (n_\varepsilon + \varepsilon)^{2m} + C_6 \int_{\Omega} u_\varepsilon^{4m+2} + C_{10} \text{ for all } t \in (0, T_{\max}).
\]

Next, since \( W^{1,2}(\Omega) \hookrightarrow L^p(\Omega) \) for any \( p > 1 \), the boundedness of \( \|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \) (see Lemma \ref{lem:regularity}) implies that there exists a positive constant \( C_{11} \) such that

\[ \|u_\varepsilon(\cdot, t)\|_{L^{4m+2}(\Omega)} \leq C_{11} \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \]

which together with (3.8) yields to (3.32) by using Lemma \ref{lem:regularity}. This completes the proof of Lemma \ref{lem:estimates}. \qed

**Lemma 3.7.** Let \( m > 1 \). Then for all \( p > 1 \), there exists a positive constant \( C \) independent of \( \varepsilon \), such that the solution of (2.1) from Lemma \ref{lem:existence} satisfies

\[ \|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{3.47} \]
Proof. Let $p > \max\{1, m-1\}$. Taking $(n_\varepsilon + \varepsilon)^{p-1}$ as the test function for the first equation of (2.1), combining with the second equation, and using (1.5), the Young inequality and the fact $\nabla \cdot u_\varepsilon = 0$, we obtain, for all $t \in (0, T_{\max,\varepsilon})$,

$$\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \int_\Omega (n_\varepsilon + \varepsilon)^{m-p+3} \|\nabla n_\varepsilon\|^2$$

\[
\leq (p-1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} n_\varepsilon |\nabla n_\varepsilon| |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla c_\varepsilon|,
\]

\[
\leq (p-1) C_S \int_\Omega (n_\varepsilon + \varepsilon)^{p-1} |\nabla n_\varepsilon| |\nabla c_\varepsilon|,
\]

\[
\leq \frac{m(p-1)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{m-p+3} |\nabla n_\varepsilon|^2 + \frac{(p-1) C_S^2}{2m} \int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m} |\nabla c_\varepsilon|^2,
\]

which implies that

$$\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{m(p-1)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{m-p+3} |\nabla n_\varepsilon|^2$$

\[
\leq \frac{(p-1) C_S^2}{2m} \int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m} |\nabla c_\varepsilon|^2 \tag{3.48}
\]

for all $t \in (0, T_{\max,\varepsilon})$. In the following, we will estimate the right-hand side of (3.48). In fact, due to $m > 1$, we conclude from (3.32) that

$$\int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m} |\nabla c_\varepsilon|^2$$

\[
\leq \left( \int_\Omega (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right) \frac{m-1}{m} \left( \int_\Omega |\nabla c_\varepsilon|^2 \right)^{\frac{1}{m}},
\]

\[
\leq C_1 \left( \int_\Omega (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right) \frac{m-1}{m} \text{ for all } t \in (0, T_{\max})
\]

by using the Hölder inequality. These together with (3.2) and $m > 1$ implies that

$$C_1 \left( \int_\Omega (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right) \frac{m-1}{m}$$

\[
= C_1 \|n_\varepsilon + \varepsilon\|_{L^{m(p+1-m)/m-1}((\Omega))} \left\| (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right\|_{L^{m(p+1-m)/m-1}((\Omega))} \left\| (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right\|_{L^{m(p+1-m)/m-1}((\Omega))} \leq C_2(\|\nabla(n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1}\|_{L^2(\Omega)} \left\| (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right\|_{L^{m(p+1-m)/m-1}((\Omega))} + 1),
\]

\[
\leq C_2(\|\nabla(n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1}\|_{L^2(\Omega)} \left\| (n_\varepsilon + \varepsilon)^{m(p+1-m)/m-1} \right\|_{L^{m(p+1-m)/m-1}((\Omega))} + 1) \] 

\[
\leq \frac{m(p-1)}{4} \int_\Omega (n_\varepsilon + \varepsilon)^{m-p+3} |\nabla n_\varepsilon|^2 + C_4 \text{ for all } t \in (0, T_{\max})
\]

by using the Gagliardo–Nirenberg inequality as well as the Young inequality and the fact that

$$\frac{2(mp - m^2 + 1)}{m(p+m-1)} < 2.$$
Inserting (3.49) into (3.48), we have
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|^p_{L^p(\Omega)} + \frac{m(p - 1)}{4} \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 \leq C_5.
\]
Therefore, (3.47) holds by using Lemma 2.2 and some basic calculation. This completes the proof of Lemma 3.7.

Lemma 3.8. Let \( m > 1 \) and \( \gamma \in (\frac{1}{2}, 1) \). Then one can find a positive constant \( C \) independent of \( \varepsilon \), such that
\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\max, \varepsilon}),
\]
\[
\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, T_{\max, \varepsilon})
\]
as well as
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\max, \varepsilon})
\]
and
\[
\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T_{\max, \varepsilon}).
\]

Proof. Firstly, applying the variation-of-constants formula to the projected version of the third equation in (2.1), we derive that
\[
u_\varepsilon(\cdot, t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} P[n_\varepsilon(\cdot, \tau) \nabla \phi - \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon]d\tau \text{ for all } t \in (0, T_{\max, \varepsilon}).
\]
Now, picking \( h_\varepsilon = P[n_\varepsilon(\cdot, \tau) \nabla \phi - \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon] \), then, in view of the standard smoothing properties of the Stokes semigroup, we derive that for all \( t \in (0, T_{\max, \varepsilon}) \) and \( \gamma \in (\frac{1}{2}, 1) \), there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[
\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_1 \int_0^t (t-\tau)^{-\gamma - \frac{2(p_0 - 1)}{p_0}} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot, \tau)\|_{L^{p_0}(\Omega)}d\tau 
\]
by using (1.7), where \( p_0 \in (1, 2) \) satisfies that
\[
p_0 > \frac{2}{3 - 2\gamma}.
\]
In light of (3.47), for some positive constant $C_3$, it has

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_3 \text{ for all } t \in (0, T_{max}).$$

Employing the Hölder inequality and the continuity of $\mathcal{P}$ in $L^p(\Omega; \mathbb{R}^2)$ (see [3]), there exist positive constants $C_4, C_5, C_6$ and $C_7$ such that

$$\|h_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_4 \|(Y_{\varepsilon} u_\varepsilon \cdot \nabla) u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} + C_4 \|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)}$$

$$\leq C_5 \|Y_{\varepsilon} u_\varepsilon\|_{L^{2p_0/p_0}(\Omega)} \|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} + C_5$$

$$\leq C_6 \|\nabla Y_{\varepsilon} u_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} + C_5$$

$$\leq C_7 \text{ for all } t \in (0, T_{max,\varepsilon}),$$

where we have used the fact that $W^{1,2}(\Omega) \hookrightarrow L^{2p_0/p_0}(\Omega)$ and the boundedness of $\|\nabla u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}$.

Collecting (3.50), (3.51) and (3.52), we conclude that

$$\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}$$

$$\leq C_8 \int_0^t (t - \tau)^{-\gamma - \frac{2}{p_0} - \frac{1}{2}} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot, \tau)\|_{L^p(\Omega)} d\tau$$

$$\leq C_9 \int_0^t (t - \tau)^{-\gamma - \frac{2}{p_0} - \frac{1}{2}} e^{-\lambda(t-\tau)} \|h_\varepsilon(\cdot, \tau)\|_{L^p(\Omega)} d\tau \text{ for all } t \in (0, T_{max,\varepsilon}),$$

which together with the fact that $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$ by $\gamma > \frac{1}{2}$ yields

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{10} \text{ for all } t \in (0, T_{max,\varepsilon}).$$

(3.53)

In view of (3.53) and (3.52), we may use (1.7), the fact that $m > 1$, and the smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ to see that there exists $C_{11} > 0$ such that

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{11} \text{ for all } t \in (0, T_{max}).$$

(3.54)

Then, the boundedness of $n_\varepsilon$ can be obtained by the well-known Moser-Alikakos iteration procedure (see e.g. Lemma A.1 of [15]). Indeed, by using (3.53) and (3.54), we see that the hypotheses of Lemma A.1 of [15] are valid provided that we take the parameter $p$ in Lemma 3.7 appropriately large. Thus, we obtain

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{12} \text{ for all } t \in (0, T_{max}).$$
The proof of Lemma 3.8 is completed.

With all above regularization properties of each component \( n_\varepsilon, c_\varepsilon, u_\varepsilon \) at hand, we can show the existence of global bounded solutions to the regularized system (2.1).

Lemma 3.9. Let \( m > 1 \) and \( \gamma \in \left(\frac{1}{2}, 1\right) \). Let \( (n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)_{\varepsilon \in (0, 1)} \) be classical solutions of (2.7) constructed in Lemma 2.1 on \([0, T_{\text{max}}]\). Then the solution is global on \([0, \infty)\). Moreover, one can find \( C > 0 \) independent of \( \varepsilon \in (0, 1) \) such that

\[
\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, \infty)
\]

and

\[
\|c_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty)
\]

as well as

\[
\|u_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty).
\]

In addition, we also have

\[
\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, \infty).
\]

Then, with the help of Lemma 3.9, we can straightforwardly deduce the uniform H"older properties of \( c_\varepsilon, \nabla c_\varepsilon \) and \( u_\varepsilon \) by the standard parabolic regularity theory as the proof of Lemmas 3.18–3.19 in [21] (see also [25]).

Lemma 3.10. Let \( m > 1 \). Then one can find \( \mu \in (0, 1) \) such that for some \( C > 0 \)

\[
\|c_\varepsilon(\cdot, t)\|_{C^{0, \frac{\mu}{2}}(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (0, \infty)
\]

as well as

\[
\|u_\varepsilon(\cdot, t)\|_{C^{0, \frac{\mu}{2}}(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (0, \infty),
\]

and for any \( \tau > 0 \) there exists \( C(\tau) > 0 \) fulfilling

\[
\|\nabla c_\varepsilon(\cdot, t)\|_{C^{0, \frac{\mu}{2}}(\Omega \times [t, t+1])} \leq C \text{ for all } t \in (\tau, \infty).
\]
4 Prove of the main result

In this section, we will give the prove of the main result. Based on the above lemmas, we will construct a weak solution as the limit of classical solutions to approximating systems (2.1). Applying the idea of [25] (see also [21] and [10]), we first state the definition of the solution as follows.

**Definition 4.1.** Let $T > 0$ and $(n_0, c_0, u_0)$ fulfills (1.7). Then a triple of functions $(n, c, u)$ is called a weak solution of (1.1)-(1.3) if the following conditions are satisfied

$$\left\{ \begin{array}{l}
n \in L^{1}_{loc}(\bar{\Omega} \times [0, T)), \\
c \in L^{1}_{loc}([0, T]; W^{1,1}(\Omega)), \\
u \in L^{1}_{loc}([0, T]; W^{1,1}(\Omega)),
\end{array} \right.$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$, moreover,

$$n^m \text{ belong to } L^{1}_{loc}(\bar{\Omega} \times [0, \infty)),$$

$$cu, \ nu \text{ and } n\nabla c \text{ belong to } L^{1}_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^2)$$

and

$$-\int_{0}^{T} \int_{\Omega} n\varphi_t - \int_{\Omega} n_0\varphi(\cdot, 0) = \int_{0}^{T} \int_{\Omega} n^m \Delta \varphi + \int_{0}^{T} \int_{\Omega} n\nabla c \cdot \nabla \varphi + \int_{0}^{T} \int_{\Omega} nu \cdot \nabla \varphi$$

for any $\varphi \in C^{\infty}_{0}(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial \Omega \times (0, T)$, as well as

$$-\int_{0}^{T} \int_{\Omega} c\varphi_t - \int_{\Omega} c_0\varphi(\cdot, 0) = -\int_{0}^{T} \int_{\Omega} \nabla c \cdot \nabla \varphi - \int_{0}^{T} \int_{\Omega} c\varphi + \int_{0}^{T} \int_{\Omega} n\varphi + \int_{0}^{T} \int_{\Omega} cu \cdot \nabla \varphi$$

for any $\varphi \in C^{\infty}_{0}(\bar{\Omega} \times [0, T))$ and

$$-\int_{0}^{T} \int_{\Omega} u\varphi_t - \int_{\Omega} u_0\varphi(\cdot, 0) = \kappa \int_{0}^{T} \int_{\Omega} u \otimes u \cdot \nabla \varphi - \int_{0}^{T} \int_{\Omega} \nabla u \cdot \nabla \varphi - \int_{0}^{T} \int_{\Omega} n\nabla \varphi \cdot \varphi$$

for any $\varphi \in C^{\infty}_{0}(\bar{\Omega} \times [0, T); \mathbb{R}^2)$ fulfilling $\nabla \varphi \equiv 0$ in $\Omega \times (0, T)$. If for each $T > 0$, $(n, c, u) : \Omega \times (0, \infty) \rightarrow \mathbb{R}^4$ is a weak solution of (1.1)-(1.3) in $\Omega \times (0, T)$, then we call $(n, c, u)$ a global weak solution of (1.1)-(1.3).
In order to use the Aubin-Lions Lemma (see e.g. [13]), we will need the regularity of the time derivative of bounded solutions. Employing almost exactly the same arguments as that in the proof of Lemmas 3.22–3.23 in [21] (the minor necessary changes are left as an easy exercise to the reader), and taking advantage of Lemma 3.9, we conclude the following Lemma.

**Lemma 4.1.** Let $m > 1$ and $\varsigma > \max\{m, 2(m-1)\}$. Then for all $\varepsilon \in (0, 1)$, there exists a positive constant $C$ independent of $\varepsilon$ such that

$$\|\partial_t n_\varepsilon(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} \leq C \text{ for all } t \in (0, \infty).$$

Moreover, let $\varsigma > \max\{m, 2(m-1)\}$. Then for all $T > 0$ and $\varepsilon \in (0, 1)$, one can find $C(T)$ independent of $\varepsilon$ such that

$$\int_0^T \|\partial_t (n_\varepsilon + \varepsilon)^\varsigma(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} dt \leq C(T) \text{ for all } t \in (0, T)$$

and

$$\int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\varsigma|^2 \leq C(T) \text{ for all } t \in (0, T).$$

Finally, we can prove the main result.

**Proof of Theorem 1.1** In conjunction with Lemma 3.9 and the Aubin-Lions compactness lemma (see e.g. Simon [13]), we thus infer the existence of a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ along which

$$n_\varepsilon \rightharpoonup n \text{ weakly star in } L^\infty(\Omega \times (0, \infty)), \quad (4.1)$$

$$n_\varepsilon \to n \text{ in } C^0_{loc}([0, \infty); (W_0^{2,2}(\Omega))^*), \quad (4.2)$$

$$c_\varepsilon \to c \text{ in } C^0_{loc}(\bar{\Omega} \times [0, \infty)), \quad (4.3)$$

$$\nabla c_\varepsilon \to \nabla c \text{ in } C^0_{loc}(\bar{\Omega} \times (0, \infty)), \quad (4.4)$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c \text{ weakly star in } L^\infty(\Omega \times (0, \infty)) \quad (4.5)$$

as well as

$$u_\varepsilon \to u \text{ in } C^0_{loc}(\bar{\Omega} \times [0, \infty)) \quad (4.6)$$
and
\[ Du_\varepsilon \rightharpoonup D u \quad \text{weakly star in} \quad L^\infty(\Omega \times (0, \infty)) \] (4.7)
holds for some limit \((n, c, u) \in (L^\infty(\Omega \times (0, \infty)))^4\) with nonnegative \(n\) and \(c\). On the other hand, Lemma 4.1 implies that for each \(T > 0\), \((n^\varepsilon)_{\varepsilon \in (0,1)}\) is bounded in \(L^2((0, T); W^{1,2}(\Omega))\), so that, using Aubin-Lions lemma again, one may obtain \(n^\varepsilon \rightharpoonup z^\varepsilon\) for some nonnegative measurable \(z : \Omega \times (0, \Omega) \rightarrow \mathbb{R}\). Thus, (4.1) and the Egorov theorem yields to \(z = n\) necessarily, and thereby
\[ n_\varepsilon \rightarrow n \quad \text{a.e. in} \quad \Omega \times (0, \infty) \] (4.8)
holds.

Due to these convergence properties (see (4.1)–(4.8)), applying standard arguments we may take \(\varepsilon = \varepsilon_j \searrow 0\) in each term of the natural weak formulation of (2.1) separately to verify that in fact \((n, c, u)\) can be complemented by some pressure function \(P\) in such a way that \((n, c, u, P)\) is a weak solution of (1.1)-(1.3). In the end, we can infer from the boundedness of \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\) and the Banach-Alaoglu theorem that \((n, c, u)\) is bounded. \(\square\)

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