ON CLOSURES IN SEMITOPOLOGICAL INVERSE SEMIGROUPS WITH CONTINUOUS INVERSION

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ABSTRACT. We study the closures of subgroups, semilattices and different kinds of semigroup extensions in semitopological inverse semigroups with continuous inversion. In particularly we show that a topological group \( G \) is \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if \( G \) is compact, a Hausdorff linearly ordered topological semilattice \( E \) is \( H \)-closed in the class of semitopological semilattices if and only if \( E \) is \( H \)-closed in the class of topological semilattices, and a topological Brandt \( \lambda^0 \)-extension of \( S \) is (absolutely) \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is \( S \). Also, we construct an example of an \( H \)-closed non-absolutely \( H \)-closed semitopological semilattice in the class of semitopological semilattices.

1. INTRODUCTION AND PRELIMINARIES

We shall follow the terminology of \[2,8,12,27,30\].

A subset \( A \) of an infinite set \( X \) is called cofinite in \( X \) if \( X \setminus A \) is finite.

Given a semigroup \( S \), we shall denote the set of idempotents of \( S \) by \( E(S) \). A semilattice is a commutative semigroup of idempotents. For a semilattice \( E \) the semilattice operation on \( E \) determines the partial order \( \leq \) on \( E \):

\[ e \leq f \quad \text{if and only if} \quad ef = fe = e. \]

This order is called natural. An element \( e \) of a partially ordered set \( X \) is called minimal if \( f \leq e \) implies \( f = e \) for \( f \in X \). An idempotent \( e \) of a semigroup \( S \) without zero (with zero \( 0_S \)) is called primitive if \( e \) is a minimal element in \( E(S) \) (in \( (E(S)) \setminus \{0_S\} \)). A maximal chain of a semilattice \( E \) is a chain which is properly contained in no other chain of \( E \). The Axiom of Choice implies the existence of maximal chains in any partially ordered set.

A semigroup \( S \) with the adjoined unit [zero] will be denoted by \( S^1 \) \([S^0]\) (cf. \[8\]). Next, we shall denote the unit (identity) and the zero of a semigroup \( S \) by \( 1_S \) and \( 0_S \), respectively. Given a subset \( A \) of a semigroup \( S \), we shall denote by \( A^* = A \setminus \{0_S\} \) and \( |A| \) the cardinality of \( A \). A semigroup \( S \) is called inverse if for any \( x \in S \) there exists a unique \( y \in S \) such that \( xyx = x \) and \( yxy = y \). Such an element \( y \) is called inverse of \( x \) and it is denoted by \( x^{-1} \).

If \( h: S \to T \) is a homomorphism (or a map) from a semigroup \( S \) into a semigroup \( T \) and if \( s \in S \), then we denote the image of \( s \) under \( h \) by \( (s)h \). A semigroup homomorphism \( h: S \to T \) is called annihilating if \((s)h = (t)h \) for all \( s, t \in S \).

Let \( S \) be a semigroup with zero and \( \lambda \) a cardinal \( \geq 1 \). We define the semigroup operation on the set \( B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\} \) as follows:

\[ (\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases} \]

and \((\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0 \), for all \( \alpha, \beta, \gamma, \delta \in \lambda \) and \( a, b \in S \). If \( S = S^1 \) then the semigroup \( B_\lambda(S) \) is called the Brandt \( \lambda \)-extension of the semigroup \( S \) \[13\]. Obviously, if \( S \) has zero then

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\end{itemize}
$\mathcal{J} = \{0\} \cup \{(\alpha, 0_S, \beta) \mid 0_S \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ and the semigroup $B_\lambda^0(S)$ is called the Brandt $\lambda^0$-extension of the semigroup $S$ with zero [19].

Next, if $A \subseteq S$ then we shall denote $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if $A$ does not contain zero, and $A_{0,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\} \cup \{0\}\text{ if } 0 \notin A, \text{ for } \alpha, \beta \in \lambda$.

We shall denote the semigroup of $\lambda \times \lambda$-matrix units by $B_\lambda$ and the subsemigroup of $\lambda \times \lambda$-matrix units of the Brandt $\lambda^0$-extension of a monoid $S$ with zero by $B_\lambda^0(1)$. We always consider the Brandt $\lambda^0$-extension only of a monoid with zero. Obviously, for any monoid $S$ with zero we have $B_\lambda^0(S) = S$. Note that every Brandt $\lambda$-extension of a group $G$ is isomorphic to the Brandt $\lambda^0$-extension of the group $G^0$ with adjoined zero. The Brandt $\lambda^0$-extension of the group with adjoined zero is called a Brandt semigroup [8, 27]. A semigroup $S$ is a Brandt semigroup if and only if $S$ is a completely $0$-simple inverse semigroup [27, Theorem II.3.5] (cf. also [27, Theorem II.3.5]). We also observe that the semigroup $B_\lambda$ of $\lambda \times \lambda$-matrix units is isomorphic to the Brandt $\lambda^0$-extension of the two-element monoid with zero $S = \{1_S, 0_S\}$ and the trivial semigroup $S$ (i.e. $S$ is a singleton set) is isomorphic to the Brandt $\lambda^0$-extension of $S$ for every cardinal $\lambda \geq 1$.

Let $\{S_t : t \in \mathcal{I}\}$ be a disjoint family of semigroups with zero such that $0_t$ is zero in $S_t$ for any $t \in \mathcal{I}$. We put $S = \{0\} \cup \bigcup \{S_t^* : t \in \mathcal{I}\}$, where $0 \notin \bigcup \{S_t^* : t \in \mathcal{I}\}$, and define a semigroup operation “$\cdot$” on $S$ in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_t^* \text{ for some } t \in \mathcal{I}; \\ 0, & \text{otherwise}. \end{cases}$$

The semigroup $S$ with the operation “$\cdot$” is called an orthogonal sum of the semigroups $\{S_t : t \in \mathcal{I}\}$ and in this case we shall write $S = \sum_{t \in \mathcal{I}} S_t$.

A non-trivial inverse semigroup is called a primitive inverse semigroup if all its non-zero idempotents are primitive [27]. A semigroup $S$ is a primitive inverse semigroup if and only if $S$ is an orthogonal sum of Brandt semigroups [27, Theorem II.4.3].

In this paper all topological spaces are Hausdorff. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $cl_Y(A)$ we denote the topological closure of $A$ in $Y$.

A (semi)topological semigroup is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an inverse topological semigroup. A topological inverse semigroup is an inverse topological semigroup with continuous inversion. We observe that the inversion on a (semi)topological inverse semigroup is a homeomorphism (see [10, Proposition II.1]). A semitopological group is a Hausdorff topological space with a separately continuous group operation. A semitopological group with continuous inversion is a quasitopological group. A paratopological group is called a group with a continuous group operation. A paratopological group with continuous inversion is a topological group.

Let $\mathcal{STSG}_0$ be a class of semitopological semigroups. A semigroup $S \in \mathcal{STSG}_0$ is called $H$-closed in $\mathcal{STSG}_0$, if $S$ is a closed subsemigroup of any topological semigroup $T \in \mathcal{STSG}_0$ which contains $S$ both as a subsemigroup and as a topological space. The $H$-closed topological semigroups were introduced by Stepp in [31], and there they were called maximal semigroups. A semitopological semigroup $S \in \mathcal{STSG}_0$ is called absolutely $H$-closed in the class $\mathcal{STSG}_0$, if any continuous homomorphic image of $S$ into $T \in \mathcal{STSG}_0$ is $H$-closed in $\mathcal{STSG}_0$. An algebraic semigroup $S$ is called:

- algebraically complete in $\mathcal{STSG}_0$, if $S$ with any Hausdorff topology $\tau$ such that $(S, \tau) \in \mathcal{STSG}_0$ is $H$-closed in $\mathcal{STSG}_0$;
- algebraically $h$-complete in $\mathcal{STSG}_0$, if $S$ with discrete topology $\mathfrak{d}$ is absolutely $H$-closed in $\mathcal{STSG}_0$ and $(S, \mathfrak{d}) \in \mathcal{STSG}_0$.

Absolutely $H$-closed topological semigroups and algebraically $h$-complete semigroups were introduced by Stepp in [32], and there they were called absolutely maximal and algebraic maximal, respectively.

Recall [1], a topological group $G$ is called absolutely closed if $G$ is a closed subgroup of any topological group which contains $G$ as a subgroup. In our terminology such topological groups are called $H$-closed in the class of topological groups. In [28] Raikov proved that a topological group $G$ is absolutely closed if and only if it is Raikov complete, i.e. $G$ is complete with respect to the two-sided uniformity. A
topological group $G$ is called $h$-complete if for every continuous homomorphism $h: G \to H$ the subgroup $f(G)$ of $H$ is closed \[9\]. In our terminology such topological groups are called absolutely $H$-closed in the class of topological groups. The $h$-completeness is preserved under taking products and closed central subgroups \[9\]. $H$-closed paratopological and topological groups in the class of paratopological groups studied in \[20\].

In \[32\] Stepp studied $H$-closed topological semilattice in the class of topological semigroups. There he proved that an algebraic semilattice $E$ is algebraically $h$-complete in the class of topological semilattices if and only if every chain in $E$ is finite. In \[23\] Gutik and Repovš established the closure of a linearly ordered topological semilattice in a topological semilattice. They proved the criterium of $H$-closedness of a linearly ordered topological semilattice in the class of topological semilattices and showed that every $H$-closed topological semilattice is absolutely $H$-closed in the class of topological semilattices. Also, such semilattices studied in \[6, 14\]. In \[3\] the structure of closures of the discrete semilattices $(\mathbb{N}, \min)$ and $(\mathbb{N}, \max)$ is described. Here the authors constructed an example of an $H$-closed topological semilattice in the class of topological semilattices which is not absolutely $H$-closed in the class of topological semilattices. The constructed example gives a negative answer on Question 17 from \[32\].

**Definition 1.1** \([19]\). Let $\mathcal{STSG}_0$ be a class of semitopological semigroups. Let $\lambda \geq 1$ be a cardinal and $(S, \tau) \in \mathcal{STSG}_0$. Let $\tau_B$ be a topology on $B^0_\lambda(S)$ such that

a) $(B^0_\lambda(S), \tau_B) \in \mathcal{STSG}_0$;

b) the topological subspace $(S_{\alpha, \alpha}, \tau_{B_{S_{\alpha, \alpha}}})$ is naturally homeomorphic to $(S, \tau)$ for some $\alpha \in \lambda$.

Then $(B^0_\lambda(S), \tau_B)$ is called a topological Brandt $\lambda^0$-extension of $(S, \tau)$ in $\mathcal{STSG}_0$.

In the paper \[24\] Gutik and Repovš established homomorphisms of the Brandt $\lambda^0$-extensions of monoids with zeros. They also described a category whose objects are ingredients in the constructions of the Brandt $\lambda^0$-extensions of monoids with zeros. Here they introduced finite, compact topological Brandt $\lambda^0$-extensions of topological semigroups and countably compact topological Brandt $\lambda^0$-extensions of topological semigroups. Here they introduced the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt $\lambda^0$-extensions of topological semigroups with zero. There they also described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt $\lambda^0$-extensions of topological semigroups with zeros. These investigations were continued in \[20, 21, 22\], where established countably compact topological Brandt $\lambda^0$-extensions of topological semigroups with zeros and pseudocompact topological Brandt $\lambda^0$-extensions of semitopological semigroups with zeros their corresponding categories. In the papers \[11, 15, 16, 19, 26\] were studied $H$-closed and absolutely $H$-closed topological Brandt $\lambda^0$-extensions of topological semigroups in the class of topological semigroups.

In Section 2 we study the closure of a quasitopological group in a semitopological inverse semigroup with continuous inversion. In particularly we show that a topological group $G$ is $H$-closed in the class of semitopological inverse semigroups with continuous inversion if and only if $G$ is compact.

Section 3 is devoted to the closure of a semitopological semilattice in a semitopological inverse semigroup with continuous inversion. We show that a Hausdorff linearly ordered topological semilattice $E$ is $H$-closed in the class of semitopological semilattices if and only if $E$ is $H$-closed in the class of topological semilattices. Also, we construct an example of an $H$-closed semitopological semilattice in the class of semitopological semilattices which is not absolutely $H$-closed in the class of semitopological semilattices.

In Section 4 we show that a topological Brandt $\lambda^0$-extension of $S$ is (absolutely) $H$-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is $S$. Also, we study the preserving of (absolute) $H$-closedness in the class of semitopological inverse semigroups with continuous inversion by orthogonal sums.
2. ON THE CLOSURE OF A QUASITOPOLOGICAL GROUP IN A SEMITOPOLOGICAL INVERSE SEMIGROUP WITH CONTINUOUS INVERSION

**Proposition 2.1.** Every left topological inverse semigroup with continuous inversion is semitopological semigroup.

*Proof.* We write an arbitrary right translation \( \rho_a : S \to S : x \mapsto xa \) of a left topological inverse semigroup \( S \) with continuous inversion \( \text{inv} : S \to S \) on three steps in the following way:

\[
\rho_a(x) = xa = (a^{-1}x^{-1})^{-1} = (\text{inv} \circ \lambda_{a^{-1}} \circ \text{inv})(x).
\]

This implies the continuity of right translations in \( S \).

It is well known that the closure of an inverse subsemigroup of a topological inverse semigroup is again a topological inverse semigroup (see: [10, Proposition II.1]). The following proposition extends this result to semitopological inverse semigroups with continuous inversion.

**Proposition 2.2.** The closure of an inverse subsemigroup \( T \) in a semitopological inverse semigroup \( S \) with continuous inversion is an inverse semigroup.

*Proof.* By Proposition 1.8(ii) from [30, Chapter I, Proposition 1.8(ii)] the closure \( \text{cl}_S(T) \) of \( T \) in a semitopological semigroup \( S \) is a semitopological semigroup. Then the continuity of the inversion \( \text{inv} : S \to S \) and Theorem 1.4.1 from [11] imply that \( \text{inv}(\text{cl}_S(T)) \subseteq \text{cl}_S(\text{inv}(T)) = \text{cl}_S(T) \) and hence we get that \( \text{inv}(\text{cl}_S(T)) = \text{cl}_S(T) \). This implies that \( \text{cl}_S(T) \) is an inverse subsemigroup of \( S \).

We observe that the statement of Proposition 2.2 is not true in the case of inverse topological semigroup. It is complete to consider the set \( \mathbb{R}^+ = [0, +\infty) \) of non-negative real numbers with usual topology and usual multiplication of real numbers. This implies that in Proposition 2.2 the condition that \( S \) has continuous inversion is essential.

In a compact topological semigroup the closure of a subgroup is a topological subgroup (see: [5, Vol. 1, Theorems 1.11 and 1.13]). Also, since for a topological inverse semigroup \( S \) the map \( f : S \to S : x \mapsto xx^{-1} \) is continuous, the maximal subgroup of \( S \) is closed, and hence the closure of a subgroup of a topological inverse semigroup is a subgroup. The previous observation implies that this is not true in the general case of topological semigroups. Also, the following example shows that the closure of a subgroup in a semitopological inverse semigroup with continuous inversion is not a subgroup.

**Example 2.3.** Let \( \mathbb{Z} \) be the discrete additive group of integers. We put \( \mathcal{A}(\mathbb{Z}) \) is the one point Alexandroff compactification of the space \( \mathbb{Z} \) with the remainder \( \infty \). We extend the semigroup operation from \( \mathbb{Z} \) onto \( \mathcal{A}(\mathbb{Z}) \) in the following way:

\[
n + \infty = \infty + n = \infty + \infty = \infty, \quad \text{for every} \quad n \in \mathbb{Z}.
\]

It is well known that \( \mathcal{A}(\mathbb{Z}) \) with such defined operation is a semitopological inverse semigroup with continuous inversion and \( \mathbb{Z} \) is not a closed subgroup of \( \mathcal{A}(\mathbb{Z}) \) [30].

A quasitopological group \( G \) is called precompact if for every open neighbourhood \( U \) of the neutral element of \( G \) there exists a finite subset \( F \) of \( G \) such that \( UF = G \) [2].

The following proposition gives examples quasitopological groups which are non-closed subgroups of some semitopological inverse semigroups with continuous inversion.

**Proposition 2.4.** For every non-precompact regular quasitopological group \( (G, \tau) \) there exists a regular semitopological inverse semigroup with continuous inversion which contains \( (G, \tau) \) as a non-closed subgroup.

*Proof.* Since the quasitopological group \( (G, \tau) \) is non-precompact there exists an open neighbourhood \( U \) of the neutral element \( e \) of the group \( G \) such that \( FU \neq G \) and \( UF \neq G \) for every finite subset \( F \) in \( G \). Let \( \mathcal{B}_e \) be a base of the topology \( \tau \) at the neutral element \( e \) of \( (G, \tau) \). Since the inversion is
continuous in \((G, \tau)\), without loss of generality we may assume that all elements of the family \(\mathcal{B}_e\) are symmetric, i.e., \(V = V^{-1}\) for every \(V \in \mathcal{B}_e\). We put
\[
\mathcal{B}_U = \{V \in \mathcal{B}_e : \text{cl}_G(V) \subseteq U\}.
\]
Since the quasitopological group \((G, \tau)\) is not precompact we have that \(F \cdot V \neq G\) and \(V \cdot F \neq G\) for every \(V \in \mathcal{B}_U\) and for every finite subset \(F\) in \(G\).

By \(G^0\) we denote the group \(G\) with a joined zero 0. Now, we put
\[
\mathcal{P}_0 = \{W_{g,V} = \{0\} \cup G \setminus \text{cl}_G(gV) : V \in \mathcal{B}_U, g \in G\} \cup \{W_{V,g} = \{0\} \cup G \setminus \text{cl}_G(Vg) : V \in \mathcal{B}_U, g \in G\}
\]
and \(\tau \cup \mathcal{P}_0\) is a subbase of a topology \(\tau_0\) on \(G^0\).

Since \((G, \tau)\) a quasitopological group, it is sufficient to show that the semigroup operation on \((G^0, \tau_0)\) is separately continuous in the following two cases: \(h \cdot 0 = 0\) and \(0 \cdot h = 0\), for \(h \in G\). Then for arbitrary subbase neighbourhoods \(W_{g_1\cdot v_1}, \ldots, W_{g_n\cdot v_n}\) and \(W_{V_1\cdot g_1}, \ldots, W_{V_n\cdot g_n}\) we have that
\[
h \cdot (W_{g_1\cdot v_1} \cap \cdots \cap W_{g_n\cdot v_n}) \subseteq W_{h\cdot g_1\cdot v_1} \cap \cdots \cap W_{h\cdot g_n\cdot v_n}.
\]

Also, since translations in the quasitopological group \((G, \tau)\) are homeomorphisms, for every open subbase neighbourhood \(V \in \mathcal{B}_U\) of the neutral element of \(G\) and every \(g \in G\) we have that \((W_{g,V})^{-1} \subseteq W_{V^{-1}\cdot g}^{-1}\). Therefore \((G^0, \tau_0)\) is a quasitopological inverse semigroup with continuous inversion.

Now for every open subbase neighbourhoods \(V_1, V_2 \in \mathcal{B}_U\) of the neutral element of \(G\) such that \(\text{cl}_G(V_1) \subseteq V_2\) and every \(g \in G\) the following conditions holds:
\[
\text{cl}_G(W_{g\cdot V_2}) \subseteq W_{g\cdot V_1}\quad \text{and} \quad \text{cl}_G(W_{V_2\cdot g}) \subseteq W_{V_1\cdot g}.
\]
Hence we get that the topological space \((G^0, \tau_0)\) is regular.

**Theorem 2.5.** A topological group \(G\) is \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if \(G\) is compact.

**Proof.** The implication \((\Rightarrow)\) is trivial.

\((\Leftarrow)\) Let a topological group \(G\) be \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion. Suppose to the contrary: the space \(G\) is not compact. Then \(G\) is \(H\)-closed in the class of topological groups and hence it is Raikov complete. If \(G\) is precompact then by Theorem 3.7.15 of [2], \(G\) is compact. Hence the topological group \(G\) is not precompact. This contradicts Proposition 2.4.

The obtained contradiction implies the statement of our theorem. \(\square\)

**Theorem 2.5** implies the following two corollaries:

**Corollary 2.6.** A topological group \(G\) is absolutely \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if \(G\) is compact.

**Corollary 2.7.** A topological group \(G\) is \(H\)-closed in the class of semitopological semigroups if and only if \(G\) is compact.

The following example shows that there exists a non-compact quasitopological group with adjoined zero which \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion.

**Example 2.8.** Let \(\mathbb{R}\) be the additive group of real numbers with usual topology. We put \(G\) is the direct quare of \(\mathbb{R}\) with the product topology. It is well known that \(G\) is a topological group. Let \(G^0\) be the group \(G\) with the adjoined zero 0. We define the topology \(\tau\) on \(G^0\) in the following way. For every non-zero element \(x\) of \(G^0\) the base of the topology \(\tau\) at \(x\) coincides with base of the product topology at \(x\) in \(G\). For every \((x_0, y_0) \in \mathbb{R}^2\) and every \(\varepsilon > 0\) we denote by
\[
O_\varepsilon(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \varepsilon\}
\]
the usual closed \( \varepsilon \)-ball with the center at the point \((x_0, y_0)\). We denote
\[
A(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} \cup \{(x, y_0) \in \mathbb{R}^2 : x \in \mathbb{R}\}
\]
and
\[
U_\varepsilon(x_0, y_0) = G^0 \setminus \left( O_\varepsilon(x_0, y_0) \cup A(x_0, y_0) \right).
\]
Now we put \( \mathcal{P}(0) = \{U_\varepsilon(x, y) : (x, y) \in \mathbb{R}^2, \varepsilon > 0\} \) and \( \mathcal{P}(0) \cup \mathcal{B}_G \) is a subbase of the topology \( \tau \) on \( G^0 \), where \( \mathcal{B}_G \) is a base of the topology of the topological group \( G \). Simple verifications show that \( (G^0, \tau) \) is a Hausdorff semitopological inverse semigroup with continuous inversion and \( (G^0, \tau) \) is not a regular space.

Then for any finitely many points \((x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2 \) and finitely many \( \varepsilon_1, \ldots, \varepsilon_n > 0 \) the following conditions hold:

(a) \( O_{\varepsilon_1}(x_1, y_1) \cup \cdots \cup O_{\varepsilon_n}(x_n, y_n) \) is a compact subset of the space \( (G^0, \tau) \);

(b) \( \text{cl}_{G^0} \left( U_{\varepsilon_1}(x_1, y_1) \cap \cdots \cap U_{\varepsilon_n}(x_n, y_n) \right) \cup O_{\varepsilon_1}(x_1, y_1) \cup \cdots \cup O_{\varepsilon_n}(x_n, y_n) = G^0 \).

This implies that \( (G^0, \tau) \) is an \( H \)-closed topological space and hence the semigroup \( (G^0, \tau) \) is \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion.

3. ON THE CLOSURE OF A SEMILATTICE IN A SEMITOPLOGICAL INVERSE SEMIGROUP WITH CONTINUOUS INVERSION

It is well known that the subset of idempotent \( E(S) \) of a topological semigroup \( S \) is a closed subset of \( S \) (see: [5, Vol. 1, Theorem 1.5]). We observe that for semitopological semigroups this statement does not hold [30]. Amassing, but the subset of all idempotent \( E(S) \) of a semitopological inverse semigroup \( S \) with continuous inversion is a closed subset of \( S \).

**Proposition 3.1.** The subset of idempotents \( E(S) \) of a semitopological inverse semigroup \( S \) with continuous inversion is a closed subset of \( S \).

**Proof.** First we observe that for any topological space \( X \) and any continuous map \( f : X \to X \) the set \( \text{Fix}(f) \) of fixed point of \( f \) is closed subset of \( X \) (see: [5, Vol. 1, Theorem 1.4] or [11, Theorem 1.5.4]). Since \( e^{-1} = e \) for every idempotent \( e \in S \), the continuity of inversion implies that \( E(S) \subseteq \text{Fix(inv)} \). Let be \( x \in S \) such that \( x \in \text{Fix(inv)} \). Since \( S \) is an inverse semigroup we obtain that \( xx = xx^{-1} \in E(S) \) and hence \( \text{Fix(inv)} \subseteq E(S) \). This completes the proof of the proposition.

**Proposition 3.1** implies the following

**Corollary 3.2.** The closure of a subsemilattice in a semitopological inverse semigroup \( S \) with continuous inversion is a subsemilattice of \( S \).

Since the closure of a subsemilattice in a Hausdorff topological semigroup is again a topological semilattice, an (absolutely) \( H \)-closed topological semilattice in the class of topological semilattices is (absolutely) \( H \)-closed in the class of topological semigroups [10]. In [32] Stepp proved that an algebraic semilattice \( E \) is algebraically \( h \)-complete in the class of topological semilattices if and only if every chain in \( E \) is finite. The following example shows that for every infinite cardinal \( \lambda \) there exists an algebraically \( h \)-complete semilattice \( E(\lambda) \) in the class of topological semilattices of cardinality \( \lambda \) such that \( E(\lambda) \) with the discrete topology is not \( H \)-closed in the class of semitopological semigroups.

**Example 3.3.** Let \( \lambda \) be any infinite cardinal. We fix an arbitrary \( a_0 \in \lambda \) and define the semigroup operation on \( \lambda \) by the formula:
\[
xy = \begin{cases} 
x, & \text{if } x = y; 
a_0, & \text{if } x \neq y.
\end{cases}
\]

The cardinal \( \lambda \) with so defined semigroup operation we denote by \( E(\lambda) \). It is obvious that \( E(\lambda) \) is a semilattice such that \( a_0 \) is zero of \( E(\lambda) \) and any two distinct non-zero elements of \( E(\lambda) \) are incomparable.
with respect to the natural partial order on $E(\lambda)$. Let be $a \notin E(\lambda)$. We extend the semigroup operation from $E(\lambda)$ onto $S = E(\lambda) \cup \{a\}$ in the following way:

$$aa = ax = xa = a_0, \quad \text{for any } x \in E(\lambda).$$

It is obvious that $S$ with so defined operation is not a semilattice.

We define a topology $\tau$ on $S$ in the following way. Fix an arbitrary sequence of distinct points $\{x_n: n \in \mathbb{N}\}$ from $E(\lambda)$ and put $U_n(a) = \{a\} \cup \{x_i: i \geq n\}$. Put all elements of the set $E(\lambda)$ are isolated points of the space $(S, \tau)$ and the family $\mathcal{B}(a) = \{U_n(a): n \in \mathbb{N}\}$ is a base of the topology $\tau$ at the point $a \in S$. Simple verifications show that $(S, \tau)$ is a metrizable 0-dimensional semitopological semigroup and $E(\lambda)$ is a dense subsemilattice of $(S, \tau)$. Also, we observe that by Theorem 9 from [32] the semilattice $E(\lambda)$ is algebraically $h$-complete in the class of topological semilattices.

**Remark 3.4.** We observe that for every infinite cardinal $\lambda$ and every Hausdorff topology $\tau$ on $E(\lambda)$ such that $(E(\lambda), \tau)$ is a semitopological semilattice we have that all non-zero idempotents of $(E(\lambda), \tau)$ are isolated points and moreover $(E(\lambda), \tau)$ is a topological semilattice. Also, a simple modification of the proof in the Example 3.3 shows that a semitopological semilattice $(E(\lambda), \tau)$ is $H$-closed in the class of semitopological semigroups if and only if the space $(E(\lambda), \tau)$ is compact.

Suppose that $E$ is a Hausdorff semitopological semilattice. If $L$ is a maximal chain in $E$, then by Proposition IV-1.13 of [12] we have that $L = \bigcap_{e \in L} (\uparrow e \cup \downarrow e)$ is a closed subset of $E$ and hence we proved the following proposition:

**Proposition 3.5.** The closure of a linearly ordered subsemilattice of a Hausdorff semitopological semilattice $E$ is a linearly ordered subsemilattice of $E$.

It is well known that the natural partial order on a Hausdorff semitopological semilattice is semiclosed (see [12] Proposition IV-1.13). Also, by Lemma 3 of [33] a semiclosed linear order is closed, and hence every linearly ordered set with a closed order admits the structure of a Hausdorff topological semilattice. This implies the following proposition:

**Proposition 3.6.** Every linearly ordered Hausdorff semitopological semilattice is a topological semilattice.

Propositions 3.5 and 3.6 imply

**Theorem 3.7.** A Hausdorff linearly ordered topological semilattice $E$ is $H$-closed in the class of semitopological semilattices if and only if $E$ is $H$-closed in the class of topological semilattices.

Theorem 3.7 and results obtained in the paper [23] imply Corollaries 3.8—3.12.

A linearly ordered semilattice $E$ is called complete if every non-empty subset of $S$ has inf and sup.

**Corollary 3.8.** A linearly ordered semitopological semilattice $E$ is $H$-closed in the class of semitopological semilattices if and only if the following conditions hold:

(i) $E$ is complete;

(ii) $x = \sup A$ for $A = \downarrow A \setminus \{x\}$ implies $x \in \text{cl}_E A$, whenever $A \neq \emptyset$; and

(iii) $x = \inf B$ for $B = \uparrow B \setminus \{x\}$ implies $x \in \text{cl}_E B$, whenever $B \neq \emptyset$.

**Corollary 3.9.** Every linearly ordered $H$-closed semitopological semilattice in the class of semitopological semilattices is absolutely $H$-closed in the class of semitopological semilattices.

**Corollary 3.10.** Every linearly ordered $H$-closed semitopological semilattice in the class of semitopological semilattices contains maximal and minimal idempotents.

**Corollary 3.11.** Let $E$ be a linearly ordered $H$-closed semitopological semilattice in the class of semitopological semilattices and $e \in E$. Then $\uparrow e$ and $\downarrow e$ are (absolutely) $H$-closed topological semilattices in the class of semitopological semilattices.
Corollary 3.12. Every linearly ordered semitopological semilattice is a dense subsemilattice of an $H$-closed semitopological semilattice in the class of semitopological semilattices.

Remark 3.13. Theorem 3.7, Example 7 and Proposition 8 from [23] imply that there exists a countable linearly ordered $\sigma$-compact 0-dimensional scattered locally compact metrizable topological semilattice which does not embeds into any compact Hausdorff semitopological semilattice.

At the finish of this section we construct an $H$-closed semitopological semilattice in the class of semitopological semilattices which is not absolutely $H$-closed in the class of semitopological semilattices.

A filter $\mathcal{F}$ on a set $X$ is called free if $\bigcap \mathcal{F} = \emptyset$.

Example 3.14 ([3]). Let $\mathbb{N}$ denote the set of positive integers. For each free filter $\mathcal{F}$ on $\mathbb{N}$ consider the topological space $\mathbb{N}_{\mathcal{F}} = \mathbb{N} \cup \{\mathcal{F}\}$ in which all points $x \in \mathbb{N}$ are isolated while the sets $F \cup \{\mathcal{F}\}$, $F \in \mathcal{F}$, form a neighbourhood base at the unique non-isolated point $\mathcal{F}$.

The semilattice operation $\min$ of $\mathbb{N}$ extends to a continuous semilattice operation $\min$ on $\mathbb{N}_{\mathcal{F}}$ such that $\min\{n, \mathcal{F}\} = \min\{\mathcal{F}, n\} = n$ and $\min\{\mathcal{F}, \mathcal{F}\} = \mathcal{F}$ for all $n \in \mathbb{N}$. By $\mathbb{N}_{\mathcal{F}, \min}$ we shall denote the topological space $\mathbb{N}_{\mathcal{F}}$ with the semilattice operation $\min$. Simple verifications show that $\mathbb{N}_{\mathcal{F}, \min}$ is a topological semilattice. Then by Theorem 2(i) of [3] the topological semilattice $\mathbb{N}_{\mathcal{F}, \min}$ is $H$-closed in the class of topological semilattices and hence by Theorem 3.7 it is $H$-closed in the class of semitopological semilattices.

Later by $E_2 = \{0, 1\}$ we denote the discrete topological semilattice with the semilattice operation $\min$.

Theorem 3.15. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ and $F \in \mathcal{F}$ be a set with infinite complement $\mathbb{N} \setminus F$. Then the closed subsemilattice $E = (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \cup ((\mathbb{N} \setminus F) \times \{1\})$ of the direct product $\mathbb{N}_{\mathcal{F}, \min} \times E_2$ is $H$-closed not absolutely $H$-closed in the class of semitopological semilattices.

Proof. The definition of the topological semilattice $\mathbb{N}_{\mathcal{F}, \min} \times E_2$ implies that $E$ is a closed subsemilattice of $\mathbb{N}_{\mathcal{F}, \min} \times E_2$.

Suppose the contrary: the topological semilattice $E$ is not $H$-closed in the class of semitopological semilattices. Since the closure of a subsemilattice in a semitopological semilattice is a semilattice (see [30] Chapter I, Proposition 1.8(ii))] we conclude that there exists a semitopological semilattice $S$ which contains $E$ as a dense subsemilattice and $S \setminus E \neq \emptyset$. We fix an arbitrary $a \in S \setminus E$. Then for every open neighbourhood $U(a)$ of the point $a$ in $S$ we have that the set $U(a) \cap E$ is infinite. By Theorem 2(ii) of [3] and Theorem 3.7 the subspace $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ of $E$ with the induced semilattice operation from $E$ is an $H$-closed in the class of semitopological semilattices. Therefore there exists an open neighbourhood $U(a)$ of the point $a$ in $S$ such that $U(a) \cap E \subseteq (\mathbb{N} \setminus F) \times \{1\}$ and hence the set $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite.

Since the subset $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ is an ideal of $E$, the $H$-closedness of $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ in the class of semitopological semilattices implies that $\mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ is a closed ideal in $S$ and hence we have that $x \cdot a \in \mathbb{N}_{\mathcal{F}, \min} \times \{0\}$ for every $x \in \mathbb{N}_{\mathcal{F}, \min} \times \{0\}$. Since for every open neighbourhood $U(a)$ of the point $a$ in $S$ the set $U(a) \cap ((\mathbb{N} \setminus F) \times \{1\})$ is infinite the semilattice operation in $E$ implies that for every $x \in (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \setminus \{(\mathcal{F}, 0)\}$ the set $x \cdot U(a)$ is infinite and hence we have that $x \cdot a \notin \mathbb{N} \times \{0\} = (\mathbb{N}_{\mathcal{F}, \min} \times \{0\}) \setminus \{(\mathcal{F}, 0)\}$. Therefore we obtain that $x \cdot a = (\mathcal{F}, 0)$. Now, since in $\mathbb{N}_{\mathcal{F}, \min}$ the sets $F \cup \{\mathcal{F}\}$, $F \in \mathcal{F}$, form a neighbourhood base at the unique non-isolated point $\mathcal{F}$, we conclude that $x \cdot U(a) \notin (F \cup \{\mathcal{F}\}) \times \{0\}$, which contradicts the separate continuity of the semilattice operation on $S$. Hence we get that $S \setminus E = \emptyset$. This implies that the topological semilattice $E$ is $H$-closed in the class of semitopological semilattices.

Now, by Theorem 3 of [3] the topological semilattice $E$ is not absolutely $H$-closed in the class of topological semilattices, and hence $E$ is not absolutely $H$-closed in the class of semitopological semilattices. □
Remark 3.16. Corollary 3.2 implies that the topological semilattice \( E \) determined in Theorem 3.15 is an example of a topological inverse semigroup which is \( H \)-closed but is not absolutely \( H \)-closed in the class of semitopological semigroups with continuous inversion.

Remark 3.17. Proposition 3.6 and Theorem 3.7 imply that Theorem 2 of [3] describes all \( H \)-closed semilattices in the class of semitopological semilattices which contain the discrete semilattice \((\mathbb{N}, \text{min})\) or the discrete semilattice \((\mathbb{N}, \text{max})\) as a dense subsemilattice.

4. On the Closure of Topological Brandt \( \lambda \)-Extensions in a Semitopological Inverse Semigroup with Continuous Inversion

In this section we study the preserving of \( H \)-closedness and absolute \( H \)-closedness by topological Brandt \( \lambda^0 \)-extensions and orthogonal sums of semitopological semigroups.

Theorem 4.1. Let \( S \) be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:

(i) \( S \) is absolutely \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion;

(ii) there exists a cardinal \( \lambda \geq 2 \) such that every topological Brandt \( \lambda^0 \)-extension of \( S \) is absolutely \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion;

(iii) for each cardinal \( \lambda \geq 2 \) every topological Brandt \( \lambda^0 \)-extension of \( S \) is absolutely \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion.

Proof. (i) \( \Rightarrow \) (iii). Suppose that the semigroup \( S \) is absolutely \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion. We fix an arbitrary cardinal \( \lambda \geq 2 \). Let \( B^0_\lambda(S) \) be a topological Brandt \( \lambda^0 \)-extension of \( S \) in the class of semitopological inverse semigroups with continuous inversion, \( T \) be a semitopological inverse semigroup with continuous inversion and \( h: B^0_\lambda(S) \to T \) be a continuous homomorphism.

First we observe that by Proposition 2.3 of [21], either \( h \) is an annihilating homomorphism or the image \( (B^0_\lambda(S))h \) is isomorphic to the Brandt \( \lambda^0 \)-extension \( B^0_\lambda((S_{a,\alpha})h) \) of the semigroup \( (S_{a,\alpha})h \) for some \( \alpha \in \lambda \). If \( h \) is an annihilating homomorphism then \( (S_{a,\alpha})h \) is a singleton, and therefore we have that \( (S_{a,\alpha})h \) is a closed subset of \( T \). Hence, later we assume that \( h \) is a non-annihilating homomorphism.

Next we show that for any \( \gamma, \delta \in \lambda \) the set \( (S_{\gamma,\delta})h \) is closed in the space \( T \). By Definition 1.1 there exists \( \alpha \in \lambda \) such that \( (S_{a,\alpha})h \) is a closed subset of \( T \). We define the maps \( \phi_h, \psi_h : T \to T \) by the formulae \( (x)\phi_h = (\alpha, 1_S, \gamma)h \cdot (x)h \cdot (\delta, 1_S, \alpha)h \) and \( (x)\psi_h = (\gamma, 1_S, \alpha)h \cdot (x)h \cdot (\alpha, 1_S, \delta)h \). Then the maps \( \phi_h \) and \( \psi_h \) are continuous because left and right translations in \( T \) and homomorphism \( h : B^0_\lambda(S) \to T \) are continuous maps. Thus, the full preimage \( A = ((S_{a,\alpha})h)\phi_h^{-1} \) is a closed subset of \( T \). Then the restriction map \( (\phi_h \circ \psi_h)|_A : A \to (S_{\gamma,\delta})h \) is a retraction, and therefore the set \( (S_{\gamma,\delta})h \) is a retract of \( A \). This implies that \( (S_{\gamma,\delta})h \) is a closed subset of \( T \).

Suppose to the contrary that \( (B^0_\lambda(S))h \) is not a closed subsemigroup of \( T \). By Lemma II.1.10 of [22], \( (B^0_\lambda(S))h \) is an inverse subsemigroup of \( T \). Since by Proposition 2.2 the closure of an inverse subsemigroup \( (B^0_\lambda(S))h \) in a semitopological inverse semigroup \( T \) with continuous inversion is an inverse semigroup, without loss of generality we may assume that \( (B^0_\lambda(S))h \) is a dense proper inverse subsemigroup of \( T \).

We fix an arbitrary \( x \in cl_T((B^0_\lambda(S))h) \setminus (B^0_\lambda(S))h \). Then only one of the following cases holds:

a) \( x \) is an idempotent of the semigroup \( T \);

b) \( x \) is a non-idempotent element of \( T \).

Suppose that case a) holds. By the previous part of the proof we have that every open neighbourhood \( U(x) \) of the point \( x \) in the topological space \( T \) intersects infinitely sets of the form \( (S_{a,\beta})h \), \( \alpha, \beta \in \lambda \). By Proposition 2.3 of [21], \( (B^0_\lambda(S))h \) is isomorphic to the Brandt \( \lambda^0 \)-extension \( B^0_\lambda((S_{a,\alpha})h) \) of the semigroup \( (S_{a,\alpha})h \) for some \( \alpha \in \lambda \), and since \( (B^0_\lambda(S))h \) is a dense subsemigroup of semitopological semigroup \( T \), the zero \( 0 \) of the semigroup \( (B^0_\lambda(S))h \) is zero of \( T \) (see [20, Lemma 23]). Then the semigroup operation of \( B^0_\lambda((S_{a,\alpha})h) \) implies that either \( 0 \in (\alpha, e, \alpha)h \cdot U(x) \) or \( 0 \in U(x) \cdot (\alpha, e, \alpha)h \) for every non-zero idempotent.
(α, e, α) of $B^0_0(S)$, $e \in E(S)$, $\alpha \in \lambda$. Now by the Hausdorffness of the space $T$ and the separate continuity of the semigroup operation of $T$ we have that either $(α, e, α)h \cdot x = 0$ or $x \cdot (α, e, α)h = 0$ for every non-zero idempotent $(α, e, α)$ of $B^0_0(S)$, $e \in E(S)$, $\alpha \in \lambda$. Since in an inverse semigroup any two idempotents commute we conclude that $(α, e, α)h \cdot x = x \cdot (α, e, α)h = 0$ for every non-zero idempotent $(α, e, α)$ of the semigroup $B^0_0(S)$.

We fix an arbitrary non-zero element $(α, s, β)h$ of the semigroup $(B^0_0(S))h$, where $α, β \in \lambda$ and $s \in S^*$. Then by the previous part of the proof we obtain that

$x \cdot (α, s, β)h = x \cdot (α, ss^{-1}s, β)h = x \cdot ((α, s, (α, s, β))h)h = x \cdot (α, ss^{-1}, α)h \cdot (α, s, β)h = 0 \cdot (α, s, β)h = 0$

and

$(α, s, β)h \cdot x = (α, ss^{-1}s, β)h \cdot x = ((α, s, β)(β, s^{-1}s, β))h \cdot x = (α, s, β)h \cdot (β, s^{-1}β, β)h \cdot x = (α, s, β)h \cdot 0 = 0$.

This implies that for every open neighbourhood $U(x)$ of the point $x$ in the space $T$ we have that $0 \in x \cdot U(x)$ and $0 \in U(x) \cdot x$. Then by Hausdorffness of the space $T$ and the separate continuity of the semigroup operation in $T$ we get that $x \cdot x = 0$, and hence $x = 0$. This implies that $E(T) = E((B^0_0(S))h)$.

Suppose that case (ii) holds. If $xx^{-1} = 0$, then $x \cdot xx^{-1} = 0 \cdot x = 0$, and similarly if $x^{-1} = 0$, then $x = xx^{-1} = x \cdot 0 = 0$. This implies that $xx^{-1}, x^{-1}x \in E((B^0_0(S))h) \setminus \{0\}$.

Then by Lemma 1.7.10 of [27] there exist idempotents $(α, e, α)$, $(β, f, β) \in B^0_0(S)$ such that $xx^{-1} = (α, e, α)h$ and $x^{-1}x = (β, f, β)h$, where $e, f \in (E(S))^*$ and $α, β \in \lambda$. Then we have that $x \cdot (β, f, β)h = (α, e, α)h \cdot x = x$. Since $e \in c_{\tau}((B^0_0(S))h) \setminus (B^0_0(S))h$, every open neighbourhood $U(x)$ of the point $x$ in the space $T$ intersects infinitely many sets $(S_{γ, δ})h$, $γ, δ \in \lambda$, and hence we obtain that either $U(x) \cdot (β, f, β)h \ni 0$ or $(α, e, α)h \cdot U(x) \ni 0$. Then the Hausdorffness of the space $T$ and the separate continuity of the semigroup operation on $T$ imply that $x \cdot (β, f, β)h = 0$ or $(α, e, α)h \cdot x = 0$. If $x \cdot (β, f, β)h = 0$ then $x = x \cdot xx^{-1} = x \cdot (β, f, β)h = 0$ and if $(α, e, α)h \cdot x = 0$ then $x = xx^{-1}x = (α, e, α)h \cdot x = 0$. All these two cases imply that $x = 0$, and hence we get that $T = (B^0_0(S))h$, which completes the proof of our theorem.

The implication (iii) ⇒ (ii) is trivial.

(iii) ⇒ (i). Suppose to the contrary: there exists semigroup $S$ such that $S$ is not absolutely $H$-closed semigroup $S$ in the class of semitopological inverse semigroups with continuous inversion and condition (ii) holds for $S$. Then there exists a semitopological inverse semigroup $T$ with continuous inversion and continuous homomorphism $h: S \to T$ such that $(S)h$ is non-closed subset of $T$. Now, by Proposition 2.2, without loss of generality we may assume that $(S)h$ is a proper dense inverse subsemigroup of $T$.

Next, for the cardinal $\lambda$ we define topologies $τ^B_T$ and $τ^S_S$ on Brandt $λ$-extensions $B^0_0(T)$ and $B^0_0(S)$, respectively, in the following way. We put

$B^T_{(α, t, β)} = \{(U(t))_{α, β}: 0 \notin U(t) \in B_T(t)\}$ and $B^S_{(α, s, β)} = \{(U(s))_{α, β}: 0 \notin U(s) \in B_S(s)\}$

are bases of topologies $τ^B_T$ and $τ^B_S$ at non-zero elements $(α, t, β) \in B^0_0(T)$ and $(α, s, β) \in B^0_0(S)$, respectively, $α, β \in \lambda$, where $B_T(t)$ and $B_S(s)$ are bases of topologies of spaces $T$ and $S$ at non-zero elements $t \in T$ and $s \in S$, respectively. Also, if $B_T(0_T)$ and $B_S(0_S)$ are bases at zeros $0_T \in T$ and $0_S \in S$ then we define $B^T_0 = \{(0) \cup \bigcup_{α, β \in \lambda} (U(0_T))_{α, β}: U(0_T) \in B_T(0_T)\}$ and $B^S_0 = \{(0) \cup \bigcup_{α, β \in \lambda} (U(0_S))_{α, β}: U(0_S) \in B_S(0_S)\}$

to be the bases of topologies $τ^B_T$ and $τ^S_S$ at zeros $0 \in B^0_0(T)$ and $0 \in B^0_0(S)$, respectively. Simple verifications show that if $T$ and $S$ are semitopological inverse semigroups with continuous inversion, then so are $(B^0_0(T), τ^B_T)$ and $(B^0_0(S), τ^S_S)$. Also the continuity of homomorphism $h: S \to T$ implies that the map $h_B: B^0_0(S) \to B^0_0(T)$ defined by the formulae

$$(α, s, β)h_B = \begin{cases} (α, (s)h, β), & \text{if } (s)h \neq 0_T; \\ 0, & \text{otherwise}, \end{cases}$$

$s \in S^*, α, β \in \lambda$, and $(0)h_B = 0$ is continuous. Also, by Theorem 3.10 of [24] so defines map $h_B: B^0_0(S) \to B^0_0(T)$ is a homomorphism. The definition of the topology $τ^B_T$ on $B^0_0(T)$ implies that the
homomorphic image \((B^0_\lambda(S))h_B\) is a dense proper subsemigroup of the semitopological inverse semigroup \((B^0_\lambda(T),\tau^B_T)\) with continuous inversion, which contradicts to statement (ii). The obtained contradiction implies the requested implication. □

Now, if we put \(h\) is a topological isomorphic embedding of semitopological semigroups with continuous inversions in the proof of Theorem 4.1 then we get the proof of the following theorem:

**Theorem 4.2.** Let \(S\) be a Hausdorff semitopological inverse monoid with zero and continuous inversion. Then the following conditions are equivalent:

(i) \(S\) is \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion;

(ii) there exists a cardinal \(\lambda \geq 2\) such that every topological Brandt \(\lambda^0\)-extension of \(S\) is \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion;

(iii) for each cardinal \(\lambda \geq 2\) every topological Brandt \(\lambda^0\)-extension of \(S\) is \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion.

Theorem 4.1 implies Corollary 4.3 which generalizes Corollary 20 from [17].

**Corollary 4.3.** For any cardinal \(\lambda \geq 2\) the semigroup of \(\lambda \times \lambda\)-units \(B_\lambda\) is algebraically \(h\)-complete in the class of semitopological inverse semigroups with continuous inversion.

Also, Theorems 4.1 and 4.2 imply the following corollary:

**Corollary 4.4.** For an inverse monoid \(S\) with zero the following conditions are equivalent:

(i) \(S\) is algebraically complete (algebraically \(h\)-complete) in the class of semitopological inverse semigroups with continuous inversion;

(ii) there exists a cardinal \(\lambda \geq 2\) such that the Brandt \(\lambda^0\)-extension of \(S\) is algebraically complete (algebraically \(h\)-complete) in the class of semitopological inverse semigroups with continuous inversion;

(iii) for each cardinal \(\lambda \geq 2\) the Brandt \(\lambda^0\)-extension of \(S\) is algebraically complete (algebraically \(h\)-complete) in the class of semitopological inverse semigroups with continuous inversion.

Theorems 4.5, 4.6 and 4.7 give a method of the construction of absolutely \(H\)-closed and \(H\)-closed semigroups in the class of semitopological inverse semigroups with continuous inversion.

**Theorem 4.5.** Let \(S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha\) be a semitopological inverse semigroup with continuous inversion such that

(i) \(S_\alpha\) is an absolutely \(H\)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any \(\alpha \in \mathcal{A}\); and

(ii) there exists an ideal \(T\) of \(S\) which is absolutely \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion such that \(S_\alpha \cdot S_\beta \subseteq T\) for all \(\alpha \neq \beta, \alpha, \beta \in \mathcal{A}\).

Then \(S\) is an absolutely \(H\)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

**Proof.** Suppose to the contrary that there exists a semitopological inverse semigroup \(K\) with continuous inversion and continuous homomorphism \(h: S \to K\) such that the image \((S)h\) is not a closed subsemigroup of \(K\). By Lemma II.1.10 of [27], \((S)h\) is an inverse subsemigroup of \(K\). Since by Proposition 2.2 the closure \(\text{cl}_K((S)h)\) of an inverse subsemigroup \((S)h\) in a semitopological inverse semigroup \(K\) with continuous inversion is an inverse semigroup, without loss of generality we may assume that \((S)h\) is a dense proper inverse subsemigroup of \(K\).

We observe that the assumption of the theorem states that \(T\) is an ideal of \(S\). This implies that \((T)h\) is an ideal in \((S)h\). Then by Proposition I.1.8(iii) of [30] the closure of an ideal of a semitopological semigroup is again an ideal, and hence we get that \((T)h\) is a closed ideal of the semigroup \(K\).

We fix an arbitrary \(x \in K \setminus (S)h\). Then only one of the following cases holds:

a) \(x\) is an idempotent of the semigroup \(K\);

b) \(x\) is a non-idempotent element of \(K\).
First we show that \( x \cdot y, y \cdot x \in (T)h \) for every \( y \in (S)h \). We fix an arbitrary open neighbourhood \( U(x) \) of the point \( x \) in the space \( K \). Since \( U(x) \) intersects infinitely many subsemigroups of \( K \) from the family \( \{(S_\alpha)h: \alpha \in \mathcal{A}\} \) we conclude that \( U(x) \cdot y \cap (T)h \neq \emptyset \) and \( y \cdot U(x) \cap (T)h \neq \emptyset \) for every \( y \in (S)h \).

Then the separate continuity of the semigroup operation in \( K \) implies that any open neighbourhood \( W(x \cdot y) \) and \( W(y \cdot x) \) of the points \( x \cdot y \) and \( y \cdot x \) in \( K \), respectively, intersect the ideal \((T)h\). This implies that \( x \cdot y, y \cdot x \in \text{cl}_K((T)h) \). Since the ideal \((T)h\) is closed in \( K \) we conclude that \( x \cdot y, y \cdot x \in (T)h \).

Suppose that case a) holds. Then there exists an open neighbourhood \( U(x) \) of the point \( x \) in the space \( K \) such that \( U(x) \cap (T)h = \emptyset \) and the neighbourhood \( U(x) \) intersects infinitely many semigroups from the family \( \{(S_\alpha)h: \alpha \in \mathcal{A}\} \). By the separate continuity of the semigroup operation in \( K \) we have that for every open neighbourhood \( U(x) \) of the point \( x \) in \( K \) such that \( U(x) \cap (T)h = \emptyset \) there exists an open neighbourhood \( V(x) \) of \( x \) in \( K \) such that \( x \cdot V(x) \subseteq U(x) \) and \( V(x) \cdot x \subseteq U(x) \). Now, the previous part of proof implies that \( x \cdot V(x) \cap (T)h \neq \emptyset \) and \( V(x) \cdot x \cap (T)h \neq \emptyset \), which contradicts the assumption \( U(x) \cap (T)h = \emptyset \). The obtained contradiction implies that \( E((S)h) = E(K) \).

Suppose that case b) holds. Then there exist idempotents \( e \) and \( f \) in \((S)h\) such that \( xx^{-1} = e \) and \( x^{-1}x = f \). We observe that \( e, f \notin (T)h \). Indeed, if \( e \in (T)h \) or \( f \in (T)h \), then we have that \( x = xx^{-1}x = ex \in (T)h \) and \( x = xx^{-1}x = xf \in (T)h \), because \((T)h\) is an ideal of the semigroup \( K \). Since \( x \in \text{cl}_K((S)h) \), every open neighbourhood of the point \( x \) in \( K \) intersects infinitely many semigroups from the family \( \{(S_\alpha)h: \alpha \in \mathcal{A}\} \), and hence we get that
\[
(U(x) \cdot f) \cap (T)h \neq \emptyset \quad \text{and} \quad (e \cdot U(x)) \cap (T)h \neq \emptyset.
\]

Then the Hausdorffness of \( K \) and the separate continuity of the semigroup operation in \( K \) imply that \( x = xx^{-1}x = x \cdot f = e \cdot x \in (T)h \). This contradicts the assumption that \( x \notin (T)h \). The obtained contradiction implies the statement of our theorem.

The proof of Theorem 4.6 is similar to the proof of Theorem 4.5.

**Theorem 4.6.** Let \( S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha \) be a semitopological inverse semigroup with continuous inversion such that

(i) \( S_\alpha \) is an \( H \)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion for any \( \alpha \in \mathcal{A} \); and

(ii) there exists an ideal \( T \) of \( S \) which is \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion such that \( S_\alpha \cdot S_\beta \subseteq T \) for all \( \alpha \neq \beta, \alpha, \beta \in \mathcal{A} \).

Then \( S \) is an \( H \)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

**Theorem 4.7.** Let a semitopological semigroup \( S \) with continuous inversion be the orthogonal sum of a family \( \{S_\alpha: \alpha \in \mathcal{I}\} \) of semitopological inverse semigroups with zeros. Then \( S \) is an (absolutely) \( H \)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion if and only if so is any element of the family \( \{S_\alpha: \alpha \in \mathcal{I}\} \).

**Proof.** First we observe that if \( S \) is a semitopological semigroup with continuous inversion then so is every semigroup from the family \( \{S_\alpha: \alpha \in \mathcal{I}\} \).

The implication (\( \Leftarrow \)) follows from Theorems 4.5 and 4.6.

First we shall prove the implication (\( \Rightarrow \)) in the case of absolute \( H \)-closedness.

Suppose to the contrary that there exists an absolute \( H \)-closed semigroup \( S \) in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family \( \{S_\alpha: \alpha \in \mathcal{I}\} \) of semitopological inverse semigroups and there exists a semigroup \( S_{\alpha_0} \) in this family such that \( S_{\alpha_0} \) is not absolute \( H \)-closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup \( K \) with continuous inversion and continuous homomorphism \( h: S_{\alpha_0} \to K \) such that the image \((S_{\alpha_0})h\) is not a closed subsemigroup of \( K \). By Lemma II.1.10 of [27], \((S_{\alpha_0})h\) is an inverse subsemigroup of \( K \). Since by Proposition 2.2 the closure \( \text{cl}_K((S_{\alpha_0})h) \) of an inverse subsemigroup \((S_{\alpha_0})h\) in a semitopological inverse semigroup \( K \) with continuous inversion is
an inverse semigroup, without loss of generality we may assume that \((S_{0\alpha})\) is a dense proper inverse subsemigroup of \(K\). Also, the semigroup \(K\) has zero because \((S_{0\alpha})\) contains zero.

We define a map \(f : S \to K\) by the formula

\[
(x)f = \begin{cases} 
0_K, & \text{if } x \in S \setminus S_{0\alpha}^*, \\
(x)h, & \text{if } x \in S_{0\alpha}^*,
\end{cases}
\]

where \(0_K\) is zero of the semigroup \(K\). Simple verifications show that so defined map \(f\) is a continuous homomorphism, but the image \((S)f = (S_{0\alpha})h\) is a dense proper subsemigroup of \(K\). This contradicts the assumption that the semigroup \(S\) is absolutely \(H\)-closed semigroup in the class of semitopological inverse semigroups with continuous inversion.

Now, we suppose that there exists an \(H\)-closed semigroup \(S\) in the class of semitopological inverse semigroups with continuous inversion which is an orthogonal sum of a family \(\{S_\alpha : \alpha \in \mathcal{J}\}\) of semitopological inverse semigroups and there exists a semigroup \(S_{0\alpha}\) in this family such that \(S_{0\alpha}\) is not \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion. Then there exists a semitopological inverse semigroup \(K\) with continuous inversion such that \(S_{0\alpha}\) is not a closed subsemigroup of \(K\). Since by Proposition \[22\] the closure \(\text{cl}_K(S_{0\alpha})\) of an inverse subsemigroup \(S_{0\alpha}\) in a semitopological inverse semigroup \(K\) with continuous inversion is an inverse semigroup, without loss of generality we may assume that \(S_{0\alpha}\) is a dense proper inverse subsemigroup of \(K\).

Next, we put \(S'\) be the orthogonal sum of the family \(\{S_\alpha : \alpha \in \mathcal{J} \setminus \{0\}\}\) and the semigroup \(K\). We determine a topology \(\tau\) on \(S'\) in the following way.

First we observe if the orthogonal sum \(T = \sum_{\alpha \in \mathcal{J}} T_{\alpha}\) is an inverse Hausdorff semitopological semigroup, then for every non-zero element \(t \in T\) there exists an open neighbourhood \(U(t)\) of \(t\) in \(T\) such that \(U(t) \subseteq T_{\alpha}\). Indeed, for every open neighbourhood \(W(t) \neq 0\) of \(t\) in \(T\) there exists an open neighbourhood \(U(t)\) of \(t\) in \(T\) such that \(tt^{-1} \subseteq W(t)\). The neighbourhood \(U(t)\) is requested.

We put that the bases of topologies at any point \(s\) of \(S \setminus S_{0\alpha}\) and of \(S' \setminus K\) coincide in \(S\) and in \(S'\), respectively. Also the bases at any point \(s\) of subspace \(K^* \subseteq S'\) coincide with the base at the point \(s\) of \(K^*\). The following family determines the base of the topology \(\tau\) at zero of the semigroup \(S'\):

\[
\mathcal{B}_0 = \{U \subseteq S' : \text{there exist an element } V \text{ of the base at zero of the topology of } S \text{ and an element } W \text{ of the base at zero of the topology of } K \text{ such that } U \cap S' \setminus K = V \cap S \setminus S_{0\alpha}, U \cap K = W \text{ and } U \cap S_{0\alpha} = W \cap S_{0\alpha}\}.
\]

Simple verifications show that \((S', \tau)\) is a Hausdorff semitopological inverse semigroup with continuous inversion and moreover \(S\) is a dense proper inverse subsemigroup of \((S', \tau)\), which contradicts the assumption of our theorem. The obtained contradiction implies the statement of the theorem.

Theorem 4.7 implies the following corollary:

**Corollary 4.8.** A primitive Hausdorff semitopological inverse semigroup \(S\) is (absolutely) \(H\)-closed in the class of semitopological inverse semigroups with continuous inversion if and only if so is every its maximal subgroup \(G\) with adjoined zero with an induced topology from \(S\).

**Remark 4.9.** We observe that the statements of Theorems 4.5, 4.6 and 4.7 hold for \(H\)-closed and absolute \(H\)-closed semitopological semilattices in the class of semitopological semilattices.

**Theorem 4.10.** An infinite semitopological semigroup of \(\lambda \times \lambda\)-matrix units \(B_\lambda\) id \(H\)-closed in the class of semitopological semigroups if and only if the space \(B_\lambda\) is compact.

**Proof.** Implication \((\Leftarrow)\) is trivial.

\((\Rightarrow)\). Suppose to the contrary that there exists a Hausdorff non-compact topology \(\tau_B\) on the semigroup \(B_\lambda\) such that \((B_\lambda, \tau_B)\) is an \(H\)-closed semigroup in the class of semitopological semigroups. By Lemma 2 of [18] every non-zero element of \(B_\lambda\) is an isolated point in \((B_\lambda, \tau_B)\). Then there exists an infinite open- and-closed subset \(A \subseteq B_\lambda \setminus \{0\}\).

Then we have that at least one of the following cases holds:
1) there exist finitely many \( i_1, \ldots, i_n \in \lambda \) such that if \( (i, j) \in A \) then \( i \in \{i_1, \ldots, i_n\} \);  
2) there exist finitely many \( j_1, \ldots, j_n \in \lambda \) such that if \( (i, j) \in A \) then \( i \in \{j_1, \ldots, j_n\} \);  
3) cases 1) and 2) don’t hold.

Suppose case 1) holds. Then there exists an element \( i_0 \in \{i_1, \ldots, i_n\} \) such that the set \( \{(i_0, j) : j \in \lambda\} \cap A \) is infinite. We denote \( A_{i_0} = \{(i_0, j) \in B_\lambda : (i_0, j) \in A\} \). It is obvious that \( A_{i_0} \) is infinite subset of the semigroup \( B_\lambda \). By Lemma 2 of [19] for every non-zero element \( b \in B_\lambda \) is an isolated point in \((B_\lambda, \tau_B)\) and hence \( A_{i_0} \) is an open-and-closed subset in the topological space \((B_\lambda, \tau_B)\). Since the left shift \( l_{(i_0, i)} : B_\lambda \to B_\lambda : x \mapsto (i_0, i) \cdot x \) is a continuous map for any \( i \in \lambda \), \( A_i = \{(i, j) \in B_\lambda : (i, j) \in A\} \) is an infinite open-and-closed subset in \((B_\lambda, \tau_B)\) for every \( i \in \lambda \). This implies that the set \( B_\lambda \setminus \{A_{i_1} \cup \cdots \cup A_{i_k}\} \) is an open neighbourhood of the zero in \((B_\lambda, \tau_B)\) for every finite subset \( \{i_1, \ldots, i_k\} \subseteq \lambda \).

Now, for every \( i \in \lambda \) we put \( a_i \notin B_\lambda \). We extend the semigroup operation from \( B_\lambda \) onto the set \( S = B_\lambda \cup \{a_i : i \in \lambda\} \) in the following way:

\[
\begin{align*}
(i) \quad a_i \cdot a_j &= a_i \cdot 0 = 0 \quad \text{for all } i, j \in \lambda; \\
(ii) \quad (s, p) \cdot a_i &= \begin{cases}
 a_s, & \text{if } p = i; \\
 0, & \text{if } p \neq i
\end{cases} \quad \text{for all } (s, p) \in B_\lambda \setminus \{0\} \quad \text{and } i \in \lambda; \\
(iii) \quad a_i \cdot (s, p) &= 0 \quad \text{for all } (s, p) \in B_\lambda \setminus \{0\} \quad \text{and } i \in \lambda.
\end{align*}
\]

Simple verifications show that so defines binary operation on \( S \) is associative, and hence \( S \) is a semigroup.

Next, we define a topology \( \tau_S \) on the semigroup \( S \) in the following way. For every element \( x \in B_\lambda \) we put that bases of topologies \( \tau_B \) and \( \tau_S \) at the point \( x \) coincide. Also, for every \( i \in \lambda \) we put

\[
\mathcal{B}_S(a_i) = \{\{a_i\} \cup C_i : C_i \text{ is a cofinite subset of } A_i\}
\]

is a base of the topology \( \tau_S \) at the point \( a_i \in S \). It is obvious that \((S, \tau_S)\) is a Hausdorff topological space. The separate continuity of the semigroup operation in \((S, \tau_S)\) follows from the cofinality of the set \( C_i \) in \( A_i \) for each \( i \in \lambda \). Therefore we get that the semitopological semigroup \((B_\lambda, \tau_B)\) is a dense proper subsemigroup of \((S, \tau_S)\), which contradicts the assumption of the theorem.

In case 2) the proof is similar.

Suppose that cases 1) and 2) don’t hold. By induction we construct an infinite sequence \( \{(x_i, y_i)\}_{i \in \mathbb{N}} \) in \( B_\lambda \) in the following way. First we fix an arbitrary element \( (x, y) \in A \) and denote \( (x_1, y_1) = (x, y) \).

Suppose that for some positive integer \( n \) we construct the finite sequence \( \{(x_i, y_i)\}_{i=1, \ldots, n} \). Since the set \( A \) is infinite and cases 1) and 2) don’t hold, there exists \( (x, y) \in A \) such that \( x \notin \{y_1, \ldots, y_n\} \) and \( y \notin \{x_1, \ldots, x_n\} \). Then we put \( (x_{n+1}, y_{n+1}) = (x, y) \).

Let \( a \notin B_\lambda \). We put \( T = B_\lambda \cup \{a\} \) and extend the semigroup operation from \( B_\lambda \) onto \( T \) in the following way:

\[
a \cdot x = x \cdot a = a \cdot a = 0, \quad \text{for every } x \in B_\lambda.
\]

Next, we define a topology \( \tau_T \) on the semigroup \( T \) in the following way. For every element \( x \in B_\lambda \) we put that bases of topologies \( \tau_B \) and \( \tau_T \) at the point \( x \) coincide. Also, we put

\[
\mathcal{B}_T(a) = \{\{a\} \cup C : C \text{ is a cofinite subset of the set } \{(x_i, y_i) : i \in \mathbb{N}\}\}
\]

is a base of the topology \( \tau_T \) at the point \( a \in T \). It is obvious that \((T, \tau_T)\) is a Hausdorff topological space, the semigroup operation in \((T, \tau_T)\) is separately continuous, and \( B_\lambda \) is a dense subsemigroup of \((T, \tau_T)\). This contradicts the assumption of the theorem.

The obtained contradictions imply the statement of our theorem. \( \square \)

**Remark 4.11.** By Theorem 2 [13] for every infinite cardinal \( \lambda \) there exists a unique Hausdorff pseudocompact topology \( \tau_c \) on the semigroup \( B_\lambda \) such that \((B_\lambda, \tau_c)\) is a semitopological semigroup. This topology is compact and it is described in Example 1 of [13].

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