DIMENSION AND MEASURE OF BAKER-LIKE SKEW-PRODUCTS OF $\beta$-TRANSFORMATIONS

DAVID FÄRM AND TOMAS PERSSON

Abstract. We consider a generalisation of the baker’s transformation, consisting of a skew-product of contractions and a $\beta$-transformation. The Hausdorff dimension and Lebesgue measure of the attractor is calculated for a set of parameters with positive measure. The proofs use a new transversality lemma similar to Solomyak’s [11]. This transversality, which is applicable to the considered class of maps holds for a larger set of parameters than Solomyak’s transversality.

1. Introduction

In [1], Alexander and Yorke considered fat baker’s transformations. These are maps on the square $[0,1) \times [0,1)$, defined by

$$(x, y) \mapsto \begin{cases} 
(\lambda x, 2y) & \text{if } y < 1/2 \\
(\lambda x + 1 - \lambda, 2y - 1) & \text{if } y \geq 1/2
\end{cases},$$

where $1/2 < \lambda < 1$ is a parameter, see Figure 1. They showed that the srb-measure of this map is the product of Lebesgue-measure and (a rescaled version of) the distribution of the corresponding Bernoulli convolution

$$\sum_{k=1}^{\infty} \pm \lambda^k.$$  

Together with Erdős’ result [3], this implies that if $\lambda$ is the inverse of a Pisot-number, then the srb-measure is singular with respect to the Lebesgue measure on $[0,1) \times [0,1)$.

2010 Mathematics Subject Classification. Primary 37D50, 37C40, 37C45.

Both authors were supported by EC FP6 Marie Curie ToK programme CODY. Part of the paper was written when the authors were visiting Institut Mittag-Leffler in Djursholm. The authors are grateful for the hospitality of the institute. The authors would like to thank Lingmin Liao for pointing out the articles [2] and [4].

Figure 1. The fat baker’s transformation for $\lambda = 0.6$. 1
In [11], Solomyak proved that for almost all $\lambda \in (\frac{1}{2}, 1)$, the distribution of the corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm \lambda^k$ is absolutely continuous with respect to Lebesgue measure. Hence this implies that the SRB-measure of the fat baker’s transformation is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$. Solomyak’s proof used a transversality property of power series of the form $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$, where $a_k \in \{-1, 0, 1\}$. More precisely, Solomyak proved that there exists a $\delta > 0$ such that if $x \in (0, 0.64)$ then
\begin{equation}
|g(x)| < \delta \implies g'(x) < -\delta.
\end{equation}
This property ensures that if the graph of $g(x)$ intersects the $x$-axis it does so at an angle which is bounded away from 0, thereby the name transversality. The constant 0.64 is an approximation of a root to a power series and cannot be improved to something larger than this root. A simplified version of Solomyak’s proof appeared in the paper [6], by Peres and Solomyak. We will make use of the method from this simpler version.

In this paper we consider maps of the form
\begin{equation}
(x, y) \mapsto \begin{cases} 
(\lambda x, \beta y) & \text{if } y < 1/\beta \\
(\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta 
\end{cases},
\end{equation}
where $0 < \lambda < 1$ and $1 < \beta < 2$, see Figure 2. Using the above mentioned transversality property (1) of Solomyak one can prove that for almost all $\lambda \in (0, 0.64)$ and $\beta \in (1, 2)$ the SRB-measure is absolutely continuous with respect to Lebesgue measure provided $\lambda \beta > 1$, and the Hausdorff dimension of the SRB-measure is $1 + \frac{\log \beta}{\log 1/\lambda}$ provided $\lambda \beta < 1$.

A problem with this approach is that the condition $\lambda < 0.64$ is very restrictive when $\beta$ is close to 1. Then the above method yields no $\lambda$ for which the SRB-measure is absolutely continuous, and it does not give the dimension of the SRB-measure for any $\lambda \in (0.64, 1/\beta)$.

We prove that these results about absolute continuity and dimension of the SRB-measure hold for sets of $(\beta, \lambda)$ of positive Lebesgue measure, even when $\lambda > 0.64$. This is done by extending the interval on which the transversality property (1) holds. This can be done in our setting, since in our class of maps, not every sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in \{-1, 0, 1\}$ occurs in the power series $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$ that we need to consider in the proof. To control which sequences that occur, we will use some results of Brown and Yin [2] and Kwon [4] on natural extensions of $\beta$-shifts.

The paper is organised as follows. In Section 2 we recall some facts about $\beta$-transformations and $\beta$-shifts. We then present the results of Brown and
Yin, and Kwon in Section 3. In Section 4 we state our results, and give the proofs in Section 5. The transversality property is stated and proved in Section 6.

2. \(\beta\)-shifts

Let \(\beta > 1\) and define \(f_\beta : [0, 1] \to [0, 1]\) by \(f_\beta(x) = \beta x \mod 1\). For \(x \in [0, 1]\) we associate a sequence \(d(x, \beta) = (d_k(x, \beta))_{k=1}^{\infty}\) defined by \(d_k(x, \beta) = \lfloor \beta f_\beta^{k-1}(x) \rfloor\) where \(\lfloor x \rfloor\) denotes the integer part of \(x\). If \(x \in [0, 1]\), then \(x = \phi_\beta(d(x, \beta))\), where

\[
\phi_\beta(i_1, i_2, \ldots) = \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}
\]

This representation, among others, of real numbers was studied by Rényi [8]. He proved that there is a unique probability measure \(\mu_\beta\) on \([0, 1]\) invariant under \(f_\beta\) and equivalent to Lebesgue measure. We will use this measure in Section 6.

We let \(S_\beta^+\) denote the closure in the product topology of the set \(\{d(x, \beta) : x \in [0, 1]\}\). The compact symbolic space \(S_\beta^+\) together with the left shift \(\sigma\) is called a \(\beta\)-shift. If we define \(d_-(1, \beta)\) to be the limit in the product topology of \(d(x, \beta)\) as \(x\) approaches 1 from the left, we have the equality

\[
S_\beta^+ = \{(a_1, a_2, \ldots) \in [0, 1, \ldots, [\beta]]^{\mathbb{N}} : \sigma^k(a_1, a_2, \ldots) \leq d_-(1, \beta) \forall k \geq 0\},
\]

where \(\sigma\) is the left-shift. This was proved by Parry in [5], where he studied the \(\beta\)-shifts and their invariant measures. Note that \(d_-(1, \beta) = d(1, \beta)\) if and only if \(d(1, \beta)\) contains infinitely many non-zero digits. A particularly useful property of the \(\beta\)-shift is that \(\beta < \beta'\) implies \(S_\beta^+ \subset S_{\beta'}^+\). The map \(\phi_\beta : S_\beta^+ \to [0, 1]\) is not necessarily injective, but we have \(d(\cdot, \beta) \circ f_\beta = \sigma \circ d(\cdot, \beta)\).

3. Symmetric \(\beta\)-shifts

Let \(\beta > 1\) and consider \(S_\beta^+\). The natural extension of \((S_\beta^+, \sigma)\) can be realised as \((S_\beta, \sigma)\), with

\[
S_\beta = \{(\ldots, a_{-1}, a_0, a_1, \ldots) : (a_n, a_{n+1}, \ldots) \in S_\beta^+ \forall n \in \mathbb{Z}\},
\]

where \(\sigma\) is the left shift on bi-infinite sequences. We will use the concept of cylinder sets only in \(S_\beta^+\). A cylinder set is a subset of \(S_\beta^+\) of the form \([a_{-n}, a_{-n+1}, \ldots, a_0] = \{(\ldots, b_{-1}, b_0, b_1, \ldots) \in S_\beta : a_k = b_k \forall k = -n, \ldots, 0\}\).

We define \(S_\beta^-\) to be the set

\[
S_\beta^- = \{(b_1, b_2, \ldots) : \exists (a_1, a_2, \ldots) \in S_\beta^+ \text{ s.t. } (\ldots, b_2, b_1, a_1, a_2, \ldots) \in S_\beta \}
\]

\[= \{(b_1, b_2, \ldots) : (\ldots, b_2, b_1, 0, 0, \ldots) \in S_\beta\}.\]

We will be interested in the set \(S\) of \(\beta\) for which \(S_\beta^+ = S_\beta^-\). This set was considered by Brown and Yin in [2]. We now describe the properties of \(S\) that we will use later on.
Consider a sequence of the digits \(a\) and \(b\). Any such sequence can be written in the form
\[
(a^{n_1}, b, a^{n_2}, b, \ldots),
\]
where each \(n_k\) is a non-negative integer or \(\infty\). We say that such a sequence is allowable if \(a \in \mathbb{N}\), \(b = a - 1\), and \(n_1 \geq 1\). If the sequence \((n_1, n_2, \ldots)\) is also allowable, we say that \((a^{n_1}, b, a^{n_2}, b, \ldots)\) is derivable, and we call \((n_1, n_2, \ldots)\) the derived sequence of \((a^{n_1}, b, a^{n_2}, b, \ldots)\). For some sequences, this operation can be carried out over and over again, generating derived sequences out of derived sequences. We have the following theorem.

**Theorem 1** (Brown–Yin [2], Kwon [4]). \(\beta \in S\) if and only if \(d(1, \beta)\) is derivable infinitely many times.

The “only if”-part was proved by Brown and Yin in [2] and the “if”-part was proved by Kwon in [4]. Using this characterisation of \(S\), Brown and Yin proved that \(S\) has the cardinality of the continuum, but its Hausdorff dimension is zero.

There is a connection between numbers in \(S\) and Sturmian sequences. We will not make any use of the connection in this paper, but refer the interested reader to Kwon’s paper [4] for details.

For our main results in the next section, it is nice to know whether \(S\) contains numbers arbitrarily close to 1. The following proposition is easily proved using Theorem 1.

**Proposition 1.** \(\inf S = 1\).

**Proof.** We prove this statement by explicitly choosing sequences \(d(1, \beta)\) corresponding to numbers \(\beta \in S\) arbitrarily close to 1. We do this by first finding some sequences that are infinitely derivable, and then we find the corresponding \(\beta\) by solving the equation \(1 = \phi_\beta(d(1, \beta))\). Let us first remark that the sequence \((1, 0, 0, \ldots)\) is its own derived sequence.

The sequence \(d(1, \beta) = (1, 1, 0, (1, 0)^\infty)\) is clearly derivable infinitely many times. It’s derived sequence is \((2, 1, 1, \ldots)\), and the derived sequence of this sequence is \((1, 0, 0, \ldots)\). One finds numerically that the corresponding \(\beta\) is given by \(\beta = 1.801938\ldots\) and that \(1/\beta = 0.554958\ldots\)

There are however smaller numbers in the set \(S\). Consider the sequence \(d(1, \beta) = (1, 0, (1, 0, 0)^\infty)\). It’s derived sequence is \((1, 1, 0, (1, 0)^\infty)\), which derives to \((2, 1, 1, \ldots)\), and so on. Solving for \(\beta\) we find that \(\beta = 1.558980\ldots\) and \(1/\beta = 0.641445\ldots\) Now, for all natural \(n\), let \(\beta_n\) be such that
\[
d(1, \beta_n) = (1, 0^n, (1, 0^{n+1})^\infty).
\]
Then, for \(n \geq 2\), the derived sequence of \(d(1, \beta_n)\) is the sequence \(d(1, \beta_{n-1})\). Hence all sequences \(d(1, \beta_n)\) are infinitely derivable, and so \(\beta_n \in S\). Moreover it is clear that \(\beta_n \to 1\) as \(n \to \infty\). See Table 1.

\[
4. \text{Results}
\]

Let \(0 < \lambda < 1\) and \(1 < \beta < 2\). Put \(Q = [0, 1) \times [0, 1)\) and define \(T_{\beta, \lambda} : Q \to Q\) by
\[
T_{\beta, \lambda}(x, y) = \begin{cases} 
(\lambda x, \beta y) & \text{if } y < 1/\beta \\
(\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta
\end{cases}
\]

Table 1. Some numerical values of $\beta_n$.

| $n$ | $\beta_n$     | $1/\beta_n$     |
|-----|---------------|------------------|
| 1   | 1.558980...   | 0.641445...      |
| 2   | 1.438417...   | 0.695209...      |
| 3   | 1.365039...   | 0.732580...      |
| 4   | 1.315114...   | 0.760390...      |
| 5   | 1.278665...   | 0.782066...      |

Figure 3. The set $\Lambda$ for $\beta = 1.2$ and $\lambda = 0.8$ (left) and $\beta = 1.8$ and $\lambda = 0.4$ (right).

Denote by $\nu$ the 2-dimensional Lebesgue measure on $Q$. For any $n \in \mathbb{N}$ we define the measure

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ T_{\beta,\lambda}^{-n}.$$

The $\text{SRB}$-measure (it is unique as noted below) of $T_{\beta,\lambda}$ is the weak limit of $\nu_n$ as $n \to \infty$.

The $\text{SRB}$-measures are characterised by the property that their conditional measures along unstable manifolds are equivalent to Lebesgue measure. The existence of such measures was established for invertible maps by Pesin [7] and extended to non-invertible maps by Schmeling and Troubetzkoy [10]. We denote the $\text{SRB}$-measure of $T_{\beta,\lambda}$ by $\mu_{\text{SRB}}$. Using the Hopf-argument used by Sataev in [9] one proves that the $\text{SRB}$-measure is unique. (Sataev’s paper is about a somewhat different map, but the argument goes through without changes.)

The support of $\mu_{\text{SRB}}$ is the set

$$\Lambda = \text{closure} \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q)$$

of which we have examples in Figure 3. One can estimate the dimension from above by covering the set $\Lambda$ with the natural covers, consisting of the pieces of $T_{\beta,\lambda}^n(Q)$. This gives us the upper bound, that the Hausdorff dimension of $\Lambda$ is at most $1 + \frac{\log \beta}{\log 1/\lambda}$. If $\lambda/\beta > 1$ this is a trivial estimate, since then $1 + \frac{\log \beta}{\log 1/\lambda} > 2$. 
The following theorem states that in the case when \( \lambda \beta < 1 \), there is a set of parameters of positive Lebesgue measure for which the estimate above is optimal.

**Theorem 2.** Let \( 1 < \beta < 2 \) and \( \gamma = \inf \{ \beta' \in S : \beta' \geq \beta \} \). Then for Lebesgue almost every \( \lambda \in (0, 1/\gamma) \) the Hausdorff dimension of the SRE-measure of \( T_{\beta, \lambda} \) is \( 1 + \frac{\log \beta}{\log 1/\lambda} \).

Recall from Proposition 1 that \( \inf S = 1 \). This implies that when \( \beta \) gets close to 1, Theorem 2 gives the dimension of the SRE-measure for a large set of \( \lambda > 0 \), which is not obtainable using Solomyak’s transversality from [11], described in the introduction.

In the area-expanding case, when \( \lambda \beta > 1 \), we have the following theorem.

**Theorem 3.** For any \( \gamma \in S \), there is an \( \varepsilon > 0 \) such that for all \( \beta \) with \( 1/\beta \in [1/\gamma, 1/\gamma + \varepsilon] \), and Lebesgue almost every \( \lambda \in (1/\beta, 1/\gamma + \varepsilon) \) the SRE-measure of \( T_{\beta, \lambda} \) is absolutely continuous with respect to Lebesgue measure.

Since \( \inf S = 1 \) by Proposition 1, there are \( \beta \) arbitrarily close to 1 for which we have a set of \( \lambda \) of positive Lebesgue measure, where the SRE-measure is absolutely continuous. In particular, this means that for these parameters, the set \( \Lambda \) has positive 2-dimensional Lebesgue measure.

Let us comment on the relation between Theorem 3 and the results of Brown and Yin in [2]. Brown and Yin considers any \( \beta > 1 \). In the case \( 1 < \beta < 2 \) their result is the following. They consider the map

\[
(x, y) \mapsto \begin{cases} 
(\frac{1}{\beta}x, \beta y) & \text{if } y < \frac{1}{\beta}, \\
(\frac{1}{\beta}x + \frac{1}{\beta}, \beta y - 1) & \text{if } y \geq \frac{1}{\beta},
\end{cases}
\]

Hence their map is similar to ours when \( \lambda = \frac{1}{\beta} \). They proved that the Lebesgue measure restricted to the set \( \Lambda \) is invariant if \( \beta \in S \).

5. Transversality

The main results of this paper, Theorem 2 and Theorem 3, only deal with \( 1 < \beta < 2 \). However, the arguments in this section work just as well for larger \( \beta \), so for the rest of this section we will be working with a fixed \( \beta > 1 \).

Consider the set of power series of the form

\[
g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k)x^k,
\]

where \( (a_1, a_2, \ldots) \) and \( (b_1, b_2, \ldots) \) are sequences in \( S_{\beta}^+ \).

**Lemma 1.** There exist \( \varepsilon > 0 \) and \( \delta > 0 \) such that for any power series \( g \) of the form (4), \( x \in [0, 1/\beta + \varepsilon] \) and \( |g(x)| < \delta \) implies that \( g'(x) < -\delta \).

**Proof.** Let

\[
0 < \varepsilon < \min \left\{ \frac{1 - 1/\beta}{2}, \frac{1}{\beta} \right\}
\]

and assume that no such \( \delta \) exists. We will show that if \( \varepsilon \) is too small, then we get a contradiction.
By assumption, there is a sequence \( g_n \) of power series of the form (4) and a sequence of numbers \( x_n \in [0, 1/\beta + \varepsilon] \), such that \( \lim_{n \to \infty} g_n(x_n) = 0 \) and \( \liminf_{n \to \infty} g_n(x_n) \geq 0 \). We can take a subsequence such that \( g_n \) converges term-wise to a series

\[
g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k)x^k
\]

with \((a_1, a_2, \ldots), (b_1, b_2, \ldots) \in S^+_\beta\), and such that \( x_n \) converges to some number \( x_0 \in [0, 1/\beta + \varepsilon] \). Clearly, \( g(x_0) = 0 \) and \( g'(x_0) \geq 0 \), so looking at (4) we note that \( x_0 \neq 0 \).

Assume first that \( x_0 \in (0, 1/\beta] \). Let \( \beta_0 = 1/x_0 \geq \beta \). Then \( g(x_0) = 0 \) and \((a_1, a_2, \ldots), (b_1, b_2, \ldots) \in S^+_\beta_0\) implies that

\[
(6) \quad \phi_{\beta_0}(a_1, a_2, \ldots) - \phi_{\beta_0}(b_1, b_2, \ldots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1.
\]

Both of the sums in (6) are in \([0, 1]\), since they equal \( \phi_{\beta_0}(a_1, a_2, \ldots) \) and \( \phi_{\beta_0}(b_1, b_2, \ldots) \) respectively. We conclude that

\[
\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.
\]

We must therefore have \((a_1, a_2, \ldots) = (0, 0, \ldots)\), and \( b_k \) must be nonzero for at least some \( k \). From (4) we then get \( g'(x) = -\sum_{k=1}^{\infty} kb_k x^{k-1} < 0 \) for all \( x \in (0, 1/\beta] \), contradicting the fact that \( g'(x_0) \geq 0 \).

Assume instead that \( x_0 \in (1/\beta, 1/\beta + \varepsilon] \). We write

\[
g(x) = 1 + h_1(x) - h_2(x),
\]

where

\[
(8) \quad h_1(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad h_2(x) = \sum_{k=1}^{\infty} b_k x^k.
\]

Since \((b_1, b_2, \ldots) \in S^+_\beta\), we have \( h_2(1/\beta) \leq 1 \). Moreover, for \( x \geq 0 \) we have

\[
0 \leq h_2'(x) \leq \sum_{k=1}^{\infty} |\beta| k x^{k-1} = \frac{|\beta|}{(1-x)^2}.\]

Therefore we have

\[
(9) \quad h_2(x_0) \leq 1 + \int_{1/\beta}^{1/\beta + \varepsilon} \frac{|\beta|}{(1-x)^2} \, dx = 1 + \frac{|\beta| \varepsilon}{(1-1/\beta - \varepsilon)(1-1/\beta)}.
\]

Since \( g(x_0) = 0 \) we see from (7) and (9) that

\[
h_1(x_0) \leq \frac{|\beta| \varepsilon}{(1-1/\beta - \varepsilon)(1-1/\beta)}.
\]

If we have \( \frac{|\beta| \varepsilon}{(1-1/\beta - \varepsilon)(1-1/\beta)} \geq x_0 \), then let \( k = 0 \). Otherwise, let \( k \) be the largest integer such that \( x_0^k > \frac{|\beta| \varepsilon}{(1-1/\beta - \varepsilon)(1-1/\beta)} \). Since \( h_1(x) \) is of the form (8) and all its terms are non-negative we must have \( a_i = 0 \) for \( i \leq k \). This implies that

\[
(10) \quad h_1'(x) \leq \sum_{i=k+1}^{\infty} |\beta| i x^{i-1} \leq |\beta| \frac{(k+1)x^k + k x^{k+1}}{(1-x)^2} = x^k [\beta] \frac{k+1 + k x}{x(1-x)^2}.
\]
By the maximality of $k$, we have $x_0^{k+1} \leq \frac{[\beta]^{e}}{(1-1/\beta-e)(1-1/\beta)}$, so (10) and (5) implies

\[(11) \quad h'_1(x_0) \leq \frac{[\beta]^2 \varepsilon}{(1-1/\beta-e)(1-1/\beta)} \frac{k+1+kx_0}{x_0(1-x_0)^2} \leq \frac{[\beta]^2 \varepsilon(2k+1)}{(1-1/\beta-e)^4x_0}.
\]

To estimate $h'_2(x_0)$ from below, we note that since $h_2(x)$ is of the form \[8\], we must have $h'_2(x) \geq 0$ for all $x$. We also have $h_2(x_0) \geq 1$ since $0 = g(x_0) = h_1(x_0) - h_2(x_0)$. Since $h_2(0) = 0$, this implies

\[(12) \quad h'_2(x_0) \geq \frac{h_2(x_0)}{x_0} \geq \frac{1}{x_0}.
\]

Now, if we can choose $\varepsilon$ so small that $g'(x_0) = h'_1(x_0) - h'_2(x_0) < 0$, we get a contradiction to the fact that $g'(x_0) \geq 0$. By (11) and (12) we see that it is enough to choose $\varepsilon$ so small that

\[\frac{[\beta]^2 \varepsilon(2k+1)}{(1-1/\beta-e)^4x_0} - \frac{1}{x_0} < 0 \iff \varepsilon < \frac{(1-1/\beta-e)^4}{[\beta]^2(2k+1)}.
\]

So, by (10) it is sufficient to choose

\[\varepsilon < \frac{(1-1/\beta)^4}{2^4[\beta]^2(2k+1)}.
\]

To get a bound on $k$ recall that by definition, either $k = 0$ or it satisfies

\[x_0^k > \frac{[\beta]^{e}}{(1-1/\beta-e)(1-1/\beta)}.
\]

By (5) we get

\[k < \frac{\log([\beta]^{e}) - \log(1-1/\beta-e) - \log(1-1/\beta)}{\log(x_0)} < \frac{\log([\beta]^{e})}{\log(1+1/\beta)}.
\]

Inserting this estimate into (13), we get the sufficient condition

\[\varepsilon < \frac{(1-1/\beta)^4}{2^4[\beta]^2[\log([\beta]^{e})]^{2} + 2^4[\beta]^2} \iff \frac{2^5[\beta]^2}{\log(1+1/\beta)} \varepsilon \log([\beta]^{e}) + 2^4[\beta]^2 \varepsilon < \left(1-1/\beta\right)^4.
\]

But $\varepsilon \log\varepsilon \to 0$ as $\varepsilon$ shrinks to 0, so it is clear that we can find an $\varepsilon > 0$ satisfying (14).

\[\square
\]

**Remark 1.** Let us give an explicit formula for which $\varepsilon$ we can choose in the case $1 < \beta < 2$. For such $\beta$ we have $[\beta] = 1$. By (5) we have $\varepsilon \leq \frac{1-1/\beta}{2}$, so it follows that $\varepsilon \leq \frac{-\log \varepsilon}{\log(1-1/\beta)}$. This implies that (14) is satisfied if

\[\varepsilon \log \varepsilon \left(\frac{2^5}{2\log(1+1/\beta)} + \frac{2^4}{2\log(1-1/\beta)}\right) < \left(1-1/\beta\right)^4.
\]
Finally we use that \( -\varepsilon \log \varepsilon < \frac{3}{4} \sqrt{\varepsilon} \) and conclude that it is sufficient to pick any
\[
\varepsilon \leq \frac{16}{9} \left( \frac{(1 - 1/\beta)^8}{(2^{3/4} + 2^{5/4})^2} \right).
\]

6. Proofs

Before we give the proofs of Theorems 2 and 3, we make some preparations that will be used in both proofs.

For fixed \( 1 < \beta < 2 \) and \( 0 < \lambda < 1 \), the set \( \Lambda \) satisfies
\[
\Lambda = \{ (x, y) : \exists a \in S_{\beta} \text{ such that } x = \pi_1(a, \lambda), \ y = \pi_2(a, \beta) \},
\]
where
\[
\pi_1(a, \lambda) = (1 - \lambda) \sum_{k=0}^{\infty} a_{-k} \lambda^k,
\]
\[
\pi_2(a, \beta) = \sum_{k=1}^{\infty} a_k \beta^{-k}.
\]

To see this one can argue as follows. Recall that \( \Lambda \) is the closure of the set \( \bigcap_{n=0}^{\infty} T^n_{\beta, \lambda}(Q) \). For each \( (x, y) \in \bigcap_{n=0}^{\infty} T^n_{\beta, \lambda}(Q) \), we have that \( (x, y) = T^n_{\beta, \lambda}(x_n, y_n) \) for some sequence \( (x_n, y_n) \in Q \) with \( T_{\beta, \lambda}(x_{n+1}, y_{n+1}) = (x_n, y_n) \). This means that there is a sequence \( a \in S_{\beta} \) such that
\[
(x, y) = T^n_{\beta, \lambda}(x_n, y_n) = \left( \lambda^nx_n + (1 - \lambda) \sum_{k=0}^{n-1} a_{-k} \lambda^k, \ y \right),
\]
and
\[
T^n_{\beta, \lambda}(x, y) = (x_n, y_n) = \left( x_n, \beta^ny_n - \sum_{k=1}^{n} \beta^{-k} a_k \right).
\]

Hence
\[
x = \lambda^nx_n + (1 - \lambda) \sum_{k=0}^{n-1} a_{-k} \lambda^k,
\]
\[
y = \beta^{-n}y_n + \sum_{k=1}^{n} \beta^{-k} a_k.
\]

Letting \( n \to \infty \) we get that all points \( (x, y) \in \bigcap_{n=0}^{\infty} T^n_{\beta, \lambda}(Q) \) are of the form \( (\pi_1(a, \lambda), \pi_2(a, \beta)) \).

For any point \( (x, y) \in \Lambda \), there is sequence \( (x^{(k)}, y^{(k)}) \) of points from \( \bigcap_{n=0}^{\infty} T^n_{\beta, \lambda}(Q) \) that converges to \( (x, y) \). But each of the points \( (x^{(k)}, y^{(k)}) \) is of the form \( (\pi_1(a^{(k)}, \lambda), \pi_2(a^{(k)}, \beta)) \) for some \( a^{(k)} \in S_{\beta} \). Since the space \( S_{\beta} \) is closed we conclude that \( (x, y) \in \Lambda \) is also of this form.

On the other hand, \( T_{\beta, \lambda}(\pi_1(a, \lambda), \pi_2(a, \beta)) = (\pi_1(\sigma a, \lambda), \pi_2(\sigma a, \beta)) \), so the set of points of the form \( (\pi_1(a, \lambda), \pi_2(a, \beta)) \) is contained in \( \Lambda \). This proves (15).
We are now going to describe the unstable manifolds using the symbolic representation. Let
\begin{equation}
\pi(a, \beta, \lambda) = (\pi_1(a, \lambda), \pi_2(a, \beta)).
\end{equation}
Consider a sequence $a \in S_\beta$ and the corresponding point $p = \pi(a, \beta, \lambda)$. In the symbolic space, $T_{\beta, \lambda}$ acts as the left-shift, so the local unstable manifold of $p$ corresponds to the set of sequences $b$ such that $a_k = b_k$ for $k \leq 0$.

For $\lambda \leq 1/2$, $\pi$ is injective on $S_\beta$ so the local unstable manifold of $p$ is unique. If $\lambda > 1/2$, then $\pi$ need not be injective on $S_\beta$, so the local unstable manifold of $p$ need not be unique. Indeed, when $\pi$ is not injective there are $a \neq b$ such that $p = \pi(a, \beta, \lambda) = \pi(b, \beta, \lambda)$, giving rise to different unstable manifolds.

Because of the description \((13)\) we have that $\pi(b, \beta, \lambda)$ is in the unstable manifold of $\pi(a, \beta, \lambda)$ if $(b_1, b_2, \ldots) \leq (a_1, a_2, \ldots)$. Hence for the unstable manifold of $\pi(a, \beta, \lambda)$, there is a maximal $c$, with $c_k = a_k$ for all $k \leq 0$, such that $\pi(c, \beta, \lambda)$ is contained in the unstable manifold. For this $c$ we have that the unstable manifold is the set
\[ \{(x, y) : x = \pi_1(a, \lambda), y \leq \pi_2(c, \beta)\}, \]
i.e. a vertical line. So, if $a$ is such that $(a_1, a_2, \ldots)$ does not end with a sequence of zeros, then the unstable manifold has positive length. Since $\Lambda$ is a union of unstable manifolds, we conclude that $\Lambda$ is the union of line-segments of the form $\{(x, y) : x \text{ fixed}, 0 \leq y \leq c\}$.

We will be using the symbolic representation of $\Lambda$ given by \((15)\), so we transfer the measure $\mu_{\text{srb}}$ to a measure $\eta$ on $S_\beta$ by $\eta = \mu_{\text{srb}} \circ \pi(-, \beta, \lambda)$. We take a closer look at this measure $\eta$ before we start the proofs. Recall, from Section \([2]\) the probability measure $\mu_\beta$ on $[0, 1]$ that is invariant under $f_\beta$ and equivalent to Lebesgue measure. We get a shift-invariant measure on $S_\beta^+$ by taking $\mu_\beta \circ \phi_\beta$ and it can be extended in the natural way to a shift-invariant measure $\eta_\beta$ on $S_\beta$.

Since $\mu_{\text{srb}}$ and $\mu_\beta$ are the unique srb-measures for $T_{\beta, \lambda}$ and $f_\beta$ respectively, we conclude that $\mu_\beta$ is the projection of $\mu_{\text{srb}}$ to the second coordinate. Thus $\eta$ and $\eta_\beta$ coincide on sets of the form $\{(a : a_k = b_k, k = 1, \ldots, n)\}$. By invariance $\eta$ and $\eta_\beta$ will coincide. Since $\eta_\beta$ does not depend on $\lambda$ by construction, $\eta$ does not depend on $\lambda$. We now get the following estimates using the relation between $\eta$ and $\mu_\beta$.

\begin{align}
\eta([a_{-n} \ldots a_0]) &= \mu_\beta\left(\phi_\beta\left(\{(x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0\}\right)\right) \\
&\leq K \text{diameter}\left(\phi_\beta\left(\{(x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0\}\right)\right) \\
&\leq K \beta^{-(n+1)},
\end{align}
where $K < \infty$ is a constant. It follows from \((17)\) that for $\eta$ almost all $a \in S_\beta$, the sequence $(a_1, a_2, \ldots)$ does not end with a sequence of zeros. As already noted, this means that the unstable manifold is a vertical line segment of positive length. Hence for $\eta$ almost all $a$ the corresponding unstable manifold is of positive length. We will use this fact in the proofs that follow.
Proof of Theorem 2. Let $\beta > 1$ and pick any $\beta' \geq \beta$ such that $\beta' \in S$. For $\eta$ almost every sequence $a$, the local unstable manifold of $\pi(a, \beta, \lambda)$ corresponding to $a$, contains a vertical line segment of positive length. Note that this length does not depend on $\lambda$. Let $\omega_\delta$ be the set of sequences $a$, such that the corresponding local unstable manifold of $\pi(a, \beta, \lambda)$ has a length of at least $\delta > 0$. Take $\delta > 0$ so that $\omega_\delta$ has positive $\eta$-measure. Then the set $\Omega_\delta = \pi(\omega_\delta, \beta, \lambda)$ has the same positive $\mu_{SRB}$-measure. Consider the restriction of $\mu_{SRB}$ to $\Omega_\delta$ and project this measure to $[0, 1) \times \{0\}$. Let $\mu_{SRB}^a$ denote this projection.

Take an interval $I = (c, d)$ with $0 < c < d < 1/\beta'$. Let $t$ be a number in $(0, 1)$. We estimate the quantity

$$J(t) = \int_I \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{1}{|x_1 - x_2|^t} \, d\mu_{SRB}(x_1) d\mu_{SRB}(x_2) \, d\lambda.$$

If this integral converges, then for Lebesgue almost every $\lambda \in I$, the dimension of $\mu_{SRB}^a$ is at least $t$, and so the dimension of $\mu_{SRB}$ is at least $1 + t$. Writing $J(t)$ as an integral over the symbolic space we have that

$$J(t) = \int_I \int_{\omega_\delta} \int_{\omega_\delta} \frac{1}{|\pi_1(a, \lambda) - \pi_1(b, \lambda)|^t} \, d\eta(a) d\eta(b) \, d\lambda.$$

Since $\eta$ does not depend on $\lambda$ we can change order of integration and write

$$J(t) = \int_I \int_{\omega_\delta} \int_I \frac{1}{|\pi_1(a, \lambda) - \pi_1(b, \lambda)|^t} \, d\lambda d\eta(a) d\eta(b).$$

Now, $a, b \in S_\beta \subset S_{\beta'}$, so for $a$ and $b$ with $a_j = b_j$ for $j = -k + 1, \ldots, 0$ and $a_{-k} \neq b_{-k}$, we have

$$|\pi_1(a, \lambda) - \pi_1(b, \lambda)|^t = \lambda^k |\pi_1(\sigma^{-k} a, \lambda) - \pi_1(\sigma^{-k} b, \lambda)|^t = \lambda^{kt} |g(\lambda)|^t,$$

where $g$ is of the form (14). Since $I = [c, d] \subset [0, 1/\beta']$, we can use the transversality from Lemma 4 to conclude that

$$\int_I \frac{d\lambda}{|\pi_1(a, \lambda) - \pi_1(b, \lambda)|^t} \leq c^{-kt} \int_I \frac{d\lambda}{|g(\lambda)|^t} \leq C c^{-kt}$$

for some constant $C$. We can write $S_\beta \times S_\beta = A \cup B$, where

$$A = \bigcup_{k=1}^{\infty} \bigcup \left[0, a_{-k+1}, \ldots, a_0\right] \times \left[1, a_{-k+1}, \ldots, a_0\right]$$

$$\cup \bigcup_{k=1}^{\infty} \bigcup \left[1, a_{-k+1}, \ldots, a_0\right] \times \left[0, a_{-k+1}, \ldots, a_0\right],$$

and

$$B = \bigcup_{a \in S_\beta} \{a\} \times \{a\}.$$
Since \( \eta(a) = 0 \) for all \( a \in S_\beta \), we can replace \( \omega_\delta \times \omega_\delta \) by \( A \) in the estimates, so after using (18) we get

\[
J(t) \leq \sum_{k=1}^{\infty} \sum_{[a_{k+1},a_0]} 2C e^{-kt} \int_{[0,a_{k+1},a_0]} \int_{[1,a_{k+1},a_0]} d\eta d\eta
\]

\[
\leq \sum_{k=1}^{\infty} \sum_{[a_{k+1},a_0]} 2CK e^{-kt} \beta^{-k} \int_{[1,a_{k+1},a_0]} d\eta
\]

\[
\leq 2CK \sum_{k=0}^{\infty} e^{-kt} \beta^{-k},
\]

by (17) and the fact that \( \eta \) is a probability measure. This series converges provided that \( t < \frac{\log \beta}{\log 1/\epsilon} \).

We have now proved that for a.e. \( \lambda \in I = (c,d) \), the dimension of the srb-measure is at least \( 1 + \frac{\log \beta}{\log 1/\epsilon} \). To get the result of the theorem, we let \( \epsilon > 0 \) and write \( I = (0, 1/\beta') \) as a union of intervals \( I_n = (c_n, d_n) \) such that \( \frac{\log \beta}{\log 1/cn} > \frac{\log \beta}{\log 1/c} - \epsilon \). Then the dimension is at least \( 1 + \frac{\log \beta}{\log 1/c} \geq 1 + \frac{\log \beta}{\log 1/\epsilon} - \epsilon \) for a.e. \( \lambda \in I \). Since \( \epsilon \) and \( \beta' \) was arbitrary this proves the theorem. \( \square \)

**Proof of Theorem 3** In [6], Peres and Solomyak gave a simplified proof of Solomyak’s result from [11], about the absolute continuity of the Bernoulli convolution \( \sum_{k=1}^{\infty} \pm \lambda^k \). The proof that follows uses the method from [6] and we refer to that paper for omitted details.

Let \( \gamma \in S \), pick \( \epsilon \) according to Lemma 1 and let \( \beta \) be such that \( 1/\beta \in [1/\gamma, 1/\gamma + \epsilon) \). Let \( \mu_{srb}^s \) be the projection of \( \mu_{srb} \) to \([0,1] \times \{0\}\). We form

\[
D(\mu_{srb}^s, x) = \liminf_{r \downarrow 0} \frac{\mu_{srb}^s(B_r(x))}{2r},
\]

where \( B_r(x) = (x-r, x+r) \), and note that \( \mu_{srb}^s \) is absolutely continuous with respect to Lebesgue measure if \( D(\mu_{srb}^s, x) < \infty \) for \( \mu_{srb}^s \) almost all \( x \). Since we already have absolute continuity in the vertical direction, it would then follow that \( \mu_{srb} \) is absolutely continuous with respect to the two-dimensional Lebesgue measure. If

\[
S = \int_I \int_{[0,1]} D(\mu_{srb}^s, x) d\mu_{srb}^s(x) d\lambda < \infty,
\]

for an interval \( I \), then \( \mu_{srb}^s \) is absolutely continuous for almost all \( \lambda \in I \). So if we prove that \( S \) is bounded for \( I = [c, 1/\gamma + \epsilon] \), where \( c > 1/\beta \) is arbitrary, then we are done.

Let \( I = [c, 1/\gamma + \epsilon] \) for some fixed \( c > 1/\beta \). By Fatou’s Lemma we get

\[
S \leq \liminf_{r \downarrow 0} (2r)^{-1} \int_I \int_{[0,1]} \mu_{srb}^s(B_r(x)) d\mu_{srb}^s(x) d\lambda
\]

\[
= \liminf_{r \downarrow 0} (2r)^{-1} \int_I \int_{\gamma} \eta(B_r(a, \lambda)) d\eta(a) d\lambda.
\]

where \( B_r(a, \lambda) = \{ b : |\pi_1(a, \lambda) - \pi_1(b, \lambda)| < r \} \). We have

\[
\eta(B_r(a, \lambda)) = \int_{\gamma} \chi_{\{ b \in S_\gamma : |\pi_1(a, \lambda) - \pi_1(b, \lambda)| \leq r \}}(a) d\eta(b),
\]

and
where $\chi$ is the characteristic function. Since $\eta$ is independent of $\lambda$, we can change the order of integration and we get

$$S \leq \liminf_{r \to 0} (2r)^{-1} \int_{S_\gamma} \int_{S_\gamma} \mu_{\text{Leb}} \{ \lambda \in I : |\pi_1(a, \lambda) - \pi_1(b, \lambda)| \leq r \} \, d\eta(a) \, d\eta(b),$$

where $\mu_{\text{Leb}}$ is the one-dimensional Lebesgue measure. Now, $a, b \in S_\gamma$, so for $a$ and $b$ with $a_j = b_j$ for $j = -k + 1, \ldots, 0$ and $a_{-k} \neq b_{-k}$, we have

$$|\pi_1(a, \lambda) - \pi_1(b, \lambda)| = \lambda^k |\pi_1(\sigma^{-k}a, \lambda) - \pi_1(\sigma^{-k}b, \lambda)| = \lambda^k |g(\lambda)|,$$

where $g$ is of the form (4). Since $I = [c, 1/\gamma + \varepsilon]$ we can use the transversality from Lemma 1 and we get

$$\mu_{\text{Leb}} \{ \lambda \in I : |\pi_1(a, \lambda) - \pi_1(b, \lambda)| \leq r \} \leq \mu_{\text{Leb}} \{ \lambda \in I : |g(\lambda)| \leq rc^{-k} \} \leq \tilde{K} rc^{-k},$$

for some constant $\tilde{K} < \infty$. As in the proof of Theorem 2, we can disregard the set

$$B = \bigcup_{a \in S_\beta} \{a\} \times \{a\}.$$

and after using (17) we get

$$S \leq \liminf_{r \to 0} (2r)^{-1} \sum_{k=1}^{\infty} \sum_{[a_{-k+1}, \ldots, a_0]} 2\tilde{K} c^{-k} r c^{-k} \int_{[a_{-k+1}, \ldots, a_0]} \int_{[1, a_{-k+1}, \ldots, a_0]} d\eta \, d\eta$$

$$\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1}, \ldots, a_0]} \tilde{K} K c^{-k} \beta^{-k} \int_{[1, a_{-k+1}, \ldots, a_0]} d\eta$$

$$\leq \tilde{K} K \sum_{k=0}^{\infty} (c\beta)^{-k},$$

which converges since $c\beta > 1$. Since $c > 1/\beta$ was arbitrary, we are done. □

References

[1] J. C. Alexander, J. A. Yorke, Fat baker’s transformations, Ergodic Theory & Dynamical Systems 4 (1984), 1–23.
[2] G. Brown, Q. Yin, $\beta$-transformation, natural extension and invariant measure, Ergodic Theory and Dynamical Systems, 20 (2000), 1271–1285.
[3] P. Erdős, On a family of symmetric Bernoulli convolutions, American Journal of Mathematics 61 (1939), 974–976.
[4] D. Kwon, The natural extensions of $\beta$-transformations which generalize baker’s transformations, Nonlinearity, 22 (2009), 301–310.
[5] W. Parry, On the $\beta$-expansion of real numbers, Acta Mathematica Academiae Scientiarum Hungaricae 11 (1960), 401–416.
[6] Y. Peres, B. Solomyak, Absolute continuity of Bernoulli convolutions, a simple proof, Mathematical Research Letters 3 (1996), no. 2, 231–239.
[7] Ya. Pesin, Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties, Ergodic Theory and Dynamical Systems 12 (1992), no. 1, 123–151.
[8] A. Rényi, Representations for real numbers and their ergodic properties, Acta Mathematica Academiae Scientiarum Hungaricae 8 (1957), 477–493.
[9] E. Sataev, Ergodic properties of the Belykh map, Journal of Mathematical Sciences, 95 (1999), 2564–2575.
[10] J. Schmeling, S. Troubetzkoy, *Dimension and invertibility of hyperbolic endomorphisms with singularities*, Ergodic Theory and Dynamical Systems 18 (1998), no. 5, 1257–1282.

[11] B. Solomyak, *On the random series $\sum \pm \lambda^n$ (an Erdős problem)*, Annals of Mathematics 142:3 (1995), 611–625.

David Färn, Institute of Mathematics, Polish Academy of Sciences ulica Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland

Current address: Centre for Mathematical Sciences, Box 118, 22 100 Lund, Sweden

E-mail address: david@maths.lth.se

Tomas Persson, Institute of Mathematics, Polish Academy of Sciences ulica Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland

Current address: Centre for Mathematical Sciences, Box 118, 22 100 Lund, Sweden

E-mail address: tomasp@maths.lth.se