Non-polyhedral extensions of the Frank-and-Wolfe theorem
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Abstract
In 1956 Marguerite Frank and Paul Wolfe proved that a quadratic function which is bounded below on a polyhedron $P$ attains its infimum on $P$. In this work we search for larger classes of sets $F$ with this Frank-and-Wolfe property. We establish the existence of non-polyhedral Frank-and-Wolfe sets, obtain internal characterizations by way of asymptotic properties, and investigate stability of the Frank-and-Wolfe class under various operations.

Keywords
Quadratic optimization — asymptotes — Motzkin-sets — Frank-and-Wolfe theorem

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1. Introduction
In this paper we investigate extensions of the famous Frank and Wolfe theorem \cite{8, 5, 6, 7, 1}, which states that a quadratic function $f$ which is bounded below on a closed convex polyhedron $P$ attains its infimum on $P$. This has applications to linear complementarity problems, and a natural question is whether this property is shared by larger classes of non-polyhedral convex sets $F$.

The present work expands on \cite{14}, where the Frank-and-Wolfe property was successfully related to asymptotic properties of a set $F$. Following this line, we presently obtain a complete characterization of the Frank-and-Wolfe property within the class of Motzkin decomposable sets. In particular, the converse of a result of Kummer \cite{12} is obtained.

A second theme addresses versions of the Frank-and-Wolfe theorem where the class of quadratic functions is further restricted. One may for instance ask for sets $F$ on which convex or quasi-convex quadratics attain their finite infima. It turns out that this class has a complete characterization as those sets which have no flat asymptotes in the sense of Klee. As a consequence we obtain a version of the Frank-and-Wolfe theorem which extends a result of Rockafellar \cite{16, Sect. 27} and Belousov and Klatte \cite{4} on convex polynomials.

Invariance of Frank-and-Wolfe type sets under various operations such as finite intersections, unions, cross-products, sums, and under affine images and pre-images are also investigated.
The structure of the chapter is as follows. In section 2 we give the definition and collect basic information on FW-sets. In section 3 we consider quasi-Frank-and-Wolfe sets, where a version of the Frank and Wolfe theorem for quasi-convex quadratics is discussed. It turns out that the same class allows many more applications, as it basically suffices to have polynomial functions which have at least one convex sub-level set. In section 4 we consider sets with a generalized Motzkin decomposition of the form $F = K + D$ with $K$ compact and $D$ a closed convex cone. This class was used by Kummer [12], who proved a version of the Frank and Wolfe theorem in this class when $D$ is polyhedral. We give a new proof of this result and also establish its inverse, that is, if a Motzkin set satisfies the Frank and Wolfe theorem, then the cone $D$ must be polyhedral. Section 5 discusses invariance properties of the class of Motzkin sets with the Frank and Wolfe property.

Notations

We generally follow Rockafellar’s book [16]. The closure of a set $F$ is $\overline{F}$. The Euclidean norm in $\mathbb{R}^n$ is $\| \cdot \|$, and the Euclidean distance is $\text{dist}(x, y) = \|x - y\|$. For subsets $M, N$ of $\mathbb{R}^n$ we write $\text{dist}(M, N) = \inf \{\|x - y\| : x \in M, y \in N\}$. A direction $d$ with $x + td \in F$ for every $x \in F$ and every $t \geq 0$ is called a direction of recession of $F$, and the cone of all directions of recession is denoted as $0^+ F$.

A function $f(x) = \frac{1}{2} x^T Ax + b^T x + c$ with $A = A^T \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, c \in \mathbb{R}$ is called quadratic. The quadratic $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex on a convex set $F \subset \mathbb{R}^n$ if the sub-level sets of $f|_F : F \to \mathbb{R}$ are convex. Similarly, $f$ is convex on the set $F$ if $f|_F$ is convex.

2. Frank-and-Wolfe sets

The following definition is the basis for our investigation:

**Definition 1.** A set $F \subset \mathbb{R}^n$ is called a Frank-and-Wolfe set, for short a FW-set, if every quadratic function $f$ which is bounded below on $F$ attains its infimum on $F$.

In [14] this notion was introduced for convex sets $F$, but in the present note we extend it to arbitrary sets, as this property is not really related to convexity. The classical Frank-and-Wolfe theorem says that every closed convex polyhedron is a FW-set, cf. [8, 5, 6, 7]. Here we are interested in identifying and characterizing more general classes of sets with this property. We start by collecting some basic information about FW-sets.

**Proposition 1.** Affine images of FW-sets are again FW-sets.

**Proof.** Let $F$ be a FW-set in $\mathbb{R}^n$ and and $T : \mathbb{R}^n \to \mathbb{R}^m$ an affine mapping. We have to show that $T(F)$ is a FW-set. Let $f$ be a quadratic on $\mathbb{R}^m$ which is bounded below on $T(F)$, then $f \circ T$ is a quadratic on $\mathbb{R}^n$, which is bounded below on $F$, hence attains its infimum at some $x \in F$. Then $f$ attains its infimum at $Tx \in T(F)$. \qed

It is equally easy to see that every FW-set is closed, because if $x \in \overline{F}$, then the quadratic function $f = \| \cdot - x \|^2$ has infimum $0$ on $F$, and if this infimum is to be attained, then $x \in F$. As a consequence, a bounded set $F$ is FW iff it is closed, so there is nothing interesting to report on bounded FW-sets, and the property is clearly aimed at the analysis of unbounded sets.

One can go a little further than just proving closedness of FW-sets and get first information about their asymptotic behavior. We need the following:
Definition 2. An affine manifold \( M \) in \( \mathbb{R}^n \) is called an \( f \)-asymptote of the set \( F \subseteq \mathbb{R}^n \) if \( F \cap M = \emptyset \) and \( \text{dist}(F,M) = 0 \).

This expands on Klee [11], who introduced this notion for convex sets \( F \). The symbol \( f \) stands for flat asymptote. This allows us now to propose the following:

**Proposition 2.** Let \( F \) be a FW-set. Then \( F \) has no \( f \)-asymptotes.

**Proof.** Let \( M \) be an affine subspace such that \( \text{dist}(F,M) = 0 \). We have to show that \( M \cap F \neq \emptyset \). Let \( M = \{x \in \mathbb{R}^n : Ax - b = 0\} \) for a suitable matrix \( A \) and vector \( b \). Put \( f(x) = \|Ax - b\|^2 \), then \( f \) is quadratic, and \( \gamma = \inf \{f(x) : x \in F\} \geq 0 \). Now there exist \( x_n \in F \) and \( y_n \in M \) with \( \text{dist}(x_n, y_n) \to 0 \).

But \( Ay_n = b \), and \( \|A(x_n - y_n)\| \leq \|A\|\|x_n - y_n\| \to 0 \), hence \( Ax_n \to b \), which implies \( \gamma = 0 \). Now since \( F \) is a FW-set, this infimum is attained, hence there exists \( x \in F \) with \( f(x) = 0 \), which means \( Ax = b \), hence \( x \in M \). That shows \( F \cap M \neq \emptyset \), so \( M \) is not an \( f \)-asymptote of \( F \).

**Remark 1.** An immediate consequence of Propositions 1, 2 is that affine images of FW-sets, and in particular, projections of FW-sets, are always closed.

Yet another trivial fact is the following:

**Proposition 3.** Finite unions of FW-sets are FW.

We conclude this preparatory section by looking at invariance of the FW-class under affine pre-images. First we need the following:

**Proposition 4.** If \( F \subseteq \mathbb{R}^n \) is a FW-set and \( M \subseteq \mathbb{R}^m \) is an affine manifold, then \( F \times M \) is a FW-set in \( \mathbb{R}^n \times \mathbb{R}^m \).

**Proof.** Since translates of FW-sets are FW-sets, we may assume that \( M \) is a linear subspace, and then there is no loss of generality in assuming that \( M = \mathbb{R}^m \). Moreover, by an easy induction argument, we only need to consider the case when \( m = 1 \).

Let \( q \) be a quadratic function on \( \mathbb{R}^n \times \mathbb{R} \) bounded below on \( F \times \mathbb{R} \). We can write \( q(x,t) = \frac{1}{2}x^TAx + \frac{1}{2}bt^2 + tc^Tx + d^Tx + et + f \) for suitable \( A, b, c, d, \) and \( f \). Clearly, \( b \geq 0 \), as otherwise \( q \) could not be bounded below on \( F \times \mathbb{R} \). Now we have \( \inf_{(x,t) \in F \times \mathbb{R}} q(x,t) = \inf_{x \in F} \inf_{t \in \mathbb{R}} q(x,t) \).

First consider the case \( b > 0 \). Then the inner infimum in the preceding expression is attained at \( t = -\frac{ct + e}{b} \). Hence we have \( \inf_{(x,t) \in F \times \mathbb{R}} q(x,t) = \inf_{x \in F} q \left( x, -\frac{ct + e}{b} \right) \). Given that \( q \left( x, -\frac{ct + e}{b} \right) \) is a quadratic function of \( x \) and is obviously bounded below on \( F \), it attains its infimum over \( F \) at some \( \overline{x} \in F \). Therefore \( q \) attains its infimum over \( F \times \mathbb{R} \) at \( \left( \overline{x}, -\frac{ct + e}{b} \right) \).

Now consider the case \( b = 0, c \neq 0 \). Here \( F \) must be contained in the hyperplane \( c^Tx + e = 0 \). Substituting this, we get \( \inf_{(x,t) \in F \times \mathbb{R}} q(x,t) = \inf_{x \in F} \left\{ \frac{1}{2}x^TAx + d^Tx \right\} + f \). Hence, the quadratic function given by \( \frac{1}{2}x^TAx + d^Tx \) is bounded below on \( F \) and, for every minimizer \( \overline{x} \in F \) and every \( t \in \mathbb{R} \), the point \( (\overline{x}, t) \) is a minimizer of \( q \) over \( F \times \mathbb{R} \).

Finally, when \( b = 0, c = 0 \) it follows that we must also have \( e = 0 \), so \( q \) no longer depends on \( t \), and we argue as in the previous case.

**Remark 2.** As we shall see in the next section (example 1), the cross product \( F_1 \times F_2 \) of two FW-sets \( F_i \) is in general no longer a FW-set, so Proposition 4 exploits the very particular situation.
We have the following consequence:

**Corollary 1.** Let $F$ be a FW-set in $\mathbb{R}^n$ and $M$ an affine subspace of $\mathbb{R}^n$. Then $F + M$ is a FW-set.

**Proof.** $F \times M$ is a FW-set by Proposition 4, and its image under the mapping $(x,y) \to x + y$ is a FW-set by Proposition 1, and that set is $F + M$. □

Concerning pre-images, we have the following consequence of Proposition 4:

**Proposition 5.** Let $T$ be an affine operator and suppose the FW-set $F$ is contained in the range of $T$. Then $T^{-1}(F)$ is a FW-set.

**Proof.** Since the notion of a FW-set is invariant under translations and under coordinate changes, we can assume that $T$ is a surjective linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ and $F \subset \mathbb{R}^m$. Now $\tilde{F} = (T|_{\ker(T)}^{-1})(F)$ is an affine image of the FW-set $F$, hence by Proposition 1 is a FW-subset of $\mathbb{R}^n$. By Corollary 1 the set $\tilde{F} + \ker(T)$ is a FW-set, but this set is just $T^{-1}(F)$. □

**Remark 3.** It is not clear whether this result remains true when $F$ is not entirely contained in the range of $T$, i.e., when only $F \cap \text{range}(T) \neq \emptyset$. In contrast, see Corollary 3 and Proposition 9.

More sophisticated invariance properties of the class of FW-sets will be investigated later. For instance, one may ask whether or under which conditions finite intersections, cartesian products, or closed subsets of FW-sets are again FW.

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### 3. Frank-and-Wolfe theorems for restricted classes of quadratic functions

Following [14] it is also of interest to investigate versions of the Frank and Wolfe theorem, where the class of quadratic functions is further restricted. The following notion is from [14]:

**Definition 3.** A convex set $F \subset \mathbb{R}^n$ is called a quasi-Frank-and-Wolfe set, for short a qFW-set, if every quadratic function $f$, which is quasi-convex on $F$ and bounded below on $F$, attains its infimum on $F$.

Note that for the class of qFW-sets we have to maintain convexity as part of the definition, as otherwise absurd situations might occur, so the notion is precisely as introduced in [14].

**Remark 4.** Every convex FW-set is clearly a qFW-set. The converse is not true, i.e., qFW-sets need not be FW, as will be seen in Example 1. It is again clear that qFW-sets are closed, and that affine images of qFW-sets are qFW.

It turns out that $f$-asymptotes are the key to understanding the quasi-Frank-and-Wolfe property. We have the following:

**Theorem 1.** Let $F$ be a convex set in $\mathbb{R}^n$. Then the following statements are equivalent:

1. Every polynomial $f$ which has at least one nonempty convex sub-level set on $F$ and which is bounded below on $F$ attains its infimum on $F$. 

(2) \( F \) is a qFW-set.

(3) Every quadratic function \( q \) which is convex on \( F \) and bounded below on \( F \) attains its infimum on \( F \).

(4) \( F \) has no f-asymptotes.

(5) \( T(F) \) is closed for every affine mapping \( T \).

(6) \( P(F) \) is closed for every orthogonal projection \( P \).

\textit{Proof.} The implication (1) \( \implies \) (2) is clear, because for a quasi-convex function on \( F \) every sublevel set on \( F \) is convex. The implication (2) \( \implies \) (3) is also evident. Implication (3) \( \implies \) (4) follows immediately with the same proof as Proposition 2, because the quadratic \( f(x) = \|Ax - b\|^2 \) used there is convex.

Let us prove (4) \( \implies \) (5). We may without loss of generality assume that \( T \) is linear, as properties (4) and (5) are invariant under translations. Suppose \( T(F) \) is not closed and pick \( y \in \overline{T(F)} \setminus T(F) \). Put \( M = T^{-1}(y) \), then \( M \) is an affine manifold. Note that \( M \cap F = \emptyset \), because \( T(M) = \{ y \} \). Now pick \( y_n \in T(F) \) such that \( y_n \to y \), and choose \( x_n \in T^{-1}(y_n) \cap F \). Since \( T \) is affine there exist \( x'_n \in T^{-1}(y_n) \) such that \( x'_n \to x' \in T^{-1}(y) \). We have \( \|x_n - (x' - x'_n + x_n)\| \to 0 \), with \( x_n \in F \), and since \( x_n - x'_n \in \ker(T) \), we have \( x' - x'_n + x_n \in x' + \ker(T) = M \). That proves \( \dist(F, M) = 0 \), and so \( F \) has \( M \) as an f-asymptote, a contradiction.

The implication (5) \( \implies \) (6) is clear. Let us prove (6) \( \implies \) (1). We will prove this by induction on \( n \). For \( n = 1 \) the implication is clearly true, because any polynomial \( f : \mathbb{R} \to \mathbb{R} \) which is bounded below on a convex set \( F \subset \mathbb{R} \) satisfying (6) attains its infimum on \( F \), as (6) implies that \( F \) is closed. Suppose therefore that the result is true for dimension \( n - 1 \), and consider a polynomial \( f : \mathbb{R}^n \to \mathbb{R} \) which is bounded below on a set \( F \subset \mathbb{R}^n \) with property (6) such that \( S_\alpha := \{ x \in F : f(x) \leq \alpha \} \) is nonempty and convex for some \( \alpha \in \mathbb{R} \). We may without loss of generality assume that the dimension of \( F \) is \( n \), i.e., that \( F \) has nonempty interior, as otherwise \( F \) is contained in a hyperplane, and then the result follows directly from the induction hypothesis. If \( \alpha = \gamma := \inf \{ f(x) : x \in F \} \), then \( f \) clearly attains \( \alpha \), so we assume from now on that \( \alpha > \gamma \). If \( S_\alpha := \{ x \in F : f(x) \leq \alpha \} \) is bounded, then by the Weierstrass extreme value theorem the infimum of \( f \) over \( S_\alpha \) is attained, because by hypothesis (6) the set \( F \) is closed. But this infimum is also the infimum of \( f \) over \( F \), so in this case we are done. Assume therefore that \( S_\alpha \) is unbounded. Since \( S_\alpha \) is a closed convex set, it has a direction of recession \( d \), that is, \( x + td \in S_\alpha \) for every \( t \geq 0 \) and every \( x \in S_\alpha \). Fix \( x \in S_\alpha \). This means

\[
\gamma \leq f(x + td) \leq \alpha
\]

for every \( t \geq 0 \). Since \( t \mapsto f(x + td) \) is a polynomial on the real line, which is now bounded on \([0, \infty)\), it must be constant as a function of \( t \), so that \( f(x) = f(x + td) \) for all \( t \geq 0 \), and then clearly also \( f(x + td) = f(x) \) for every \( t \in \mathbb{R} \). But the argument is valid for every \( x \in S_\alpha \). By assumption \( F \) has dimension \( n \), so \( S_\alpha \) has nonempty interior. That shows \( f(x + td) = f(x) \) for all \( x \) in a nonempty open set contained in \( S_\alpha \) and all \( t \in \mathbb{R} \). Altogether, since \( f \) is a polynomial, we obtain

\[
f(x + td) = f(x) \text{ for every } x \in \mathbb{R}^n \text{ and every } t \in \mathbb{R}.
\]
Now let $P$ be the orthogonal projection onto the hyperplane $H = d^\perp$. Then $\tilde{f} := f_{|H}$ is a polynomial on the $(n - 1)$-dimensional space $H$ and takes the same values as $f$ due to (1). In particular, $\tilde{f} = f_{|H}$ is bounded below on the set $\tilde{F} = P(F)$.

We argue that the induction hypothesis applies to $\tilde{F}$. Indeed, $\tilde{F}$ being the image of $F$ under a projection, is closed by condition (6). Its dimension is $n - 1$, and moreover, every projection of $\tilde{F}$ is closed, because any such projection is also a projection of $F$.

It remains to prove that the restriction of $\tilde{f}$ to $\tilde{F}$ has a nonempty convex sub-level set. To this end it will suffice to prove that, for $\tilde{S}_\alpha := \{ x \in \tilde{F} : \tilde{f}(x) \leq \alpha \}$, one has $\tilde{S}_\alpha = P(S_\alpha)$. This will easily follow from the observation that $\tilde{f} \circ P = f$, which is an immediate consequence of (1). Let $x \in \tilde{S}_\alpha$. Since $x \in \tilde{F}$, we have $P(x') = x$ for some $x' \in F$, and hence $f(x') = (\tilde{f} \circ P)(x') = \tilde{f}(P(x')) = \tilde{f}(x) \leq \alpha$, which proves that $x' \in S_\alpha$. Therefore $x \in P(S_\alpha)$, which shows $\tilde{S}_\alpha \subseteq P(S_\alpha)$. To prove the opposite inclusion, let $x \in P(S_\alpha)$. We then have $x = P(x')$ for some $x' \in S_\alpha$. From the inclusion $S_\alpha \subset F$, it follows that $x \in P(F) = \tilde{F}$. On the other hand, $\tilde{f}(x) = f(x') \leq \alpha$. This shows $x \in \tilde{S}_\alpha$ and proves the inclusion $P(S_\alpha) \subseteq \tilde{S}_\alpha$ and hence our claim $\tilde{S}_\alpha = P(S_\alpha)$.

Altogether, $\tilde{f}$ now attains its infimum on $\tilde{F}$ by the induction hypothesis, and then $f$, having the same values, also attains its infimum on $F$. This proves the validity of (1).

\begin{remark}
The equivalence of (4) and (6) can already be found in [11].
\end{remark}

\begin{remark}
All that matters in condition (1) is the rigidity of polynomials. Any class $\mathcal{F}(L)$ of continuous functions defined on affine subspaces $L$ of $\mathbb{R}^n$ with the following properties would work as well: (i) $\mathcal{F}(L)$ is defined for every $L \subset \mathbb{R}^n$ and every $n$. (ii) If $f \in \mathcal{F}(\mathbb{R})$ is bounded below on a closed interval on $\mathbb{R}$, then $f$ attains its infimum. (iii) If $f \in \mathcal{F}(\mathbb{R}^n)$ and $H$ is a hyperplane in $\mathbb{R}^n$, then $f_{|H} \in \mathcal{F}(H)$. (iv) If $f \in \mathcal{F}(\mathbb{R}^n)$ is bounded (above and below) on some ray $x + \mathbb{R}^+d \subset \mathbb{R}^n$, then $f$ does not depend on $d$, i.e., $f(x) = f(x + td)$ for all $t \in \mathbb{R}$.

We had seen in section 2 that FW-sets have no $f$-asymptotes. Moreover, from the results of this section we see that if $F$ is convex and has no $f$-asymptotes, then it is already a $qFW$-set. This raises the question whether the absence of $f$-asymptotes also serves to characterize FW-sets, or if not, whether it does so at least for convex $F$. We indicate by way of two examples that this is not the case, i.e., the absence of $f$-asymptotes does not characterize Frank-and-Wolfe sets. Or put differently, there exist quasi-Frank-and-Wolfe sets which are not Frank-and-Wolfe.

\begin{example}
We construct a closed convex set $F$ without $f$-asymptotes, which is not Frank-and-Wolfe. We use Example 2 of [13], which we reproduce here for convenience. Consider the optimization problem

\begin{align*}
\text{minimize} & \quad q(x) = x_1^2 - 2x_1x_2 + x_3x_4 + 1 \\
\text{subject to} & \quad c_1(x) = x_2^2 - x_3 \leq 0 \\
& \quad c_2(x) = x_3^2 - x_4 \leq 0 \\
& \quad x \in \mathbb{R}^4
\end{align*}

then as Lou and Zhang [13] show the constraint set $F = \{ x \in \mathbb{R}^4 : c_1(x) \leq 0, c_2(x) \leq 0 \}$ is closed convex, and the quadratic function $q$ has infimum $\gamma = 0$ on $F$, but this infimum is not attained.

Let us show that $F$ has no $f$-asymptotes. Note that $F = F_1 \times F_2$, where $F_1 = \{ (x_1, x_3) \in \mathbb{R}^2 : x_1^2 - x_3 \leq 0 \}$, $F_2 = \{ (x_2, x_4) \in \mathbb{R}^2 : x_2^2 - x_4 \leq 0 \}$. Observe that $F_1 \cong F_2$, and that $F_1$ does not have
asymptotes, being a parabola. Therefore, $F$ does not have $f$-asymptotes either. This can be seen from the following:

**Proposition 6.** Any nonempty finite intersection of qFW-sets is again a qFW-set.

**Proof.** By Theorem 1 the result follows immediately from a theorem of Klee [11, Thm. 4], which says that finite intersections of sets without $f$-asymptotes have no $f$-asymptotes. □

**Corollary 2.** If $F_1, \ldots, F_m$ are qFW-sets, then the cartesian product $F_1 \times \cdots \times F_m$ is again a qFW-set.

**Proof.** Consider for the ease of notation the case of two sets $F_i \subset \mathbb{R}^{d_i}$, $i = 1, 2$. Then write

$$F_1 \times F_2 = \left( F_1 \times \mathbb{R}^{d_2} \right) \cap \left( \mathbb{R}^{d_1} \times F_2 \right).$$

Now $F_1 \times \mathbb{R}^{d_2}$ is also qFW, and so is $\mathbb{R}^{d_1} \times F_2$, and hence the result follows from Proposition 6. The fact that $F_1 \times \mathbb{R}^{d_1}$ is qFW is easily seen as follows: If $M$ is a $f$-asymptote of $F_1 \times \mathbb{R}^{d_1}$, then $L = \{ x : (x, y) \in M \text{ for some } y \}$ is a $f$-asymptote of $F_1$. □

**Remark 7.** Example 1 also tells us that the sum of FW-sets need not be a FW-set even when closed, as follows from the identity $F_1 \times F_2 = (F_1 \times \{0\}) + (\{0\} \times F_2)$. Note that even though $F_1 \times F_2$ fails to be FW, it remains qFW due to Corollary 2.

**Example 2.** Let $F$ be the epigraph of $f(x) = x^2 + \exp(-x^2)$ in $\mathbb{R}^2$. Then $q(x, y) = y - x^2$ is bounded below on $F$, but does not attain its infimum, so $F$ is not FW. However, $F$ has no $f$-asymptotes, so it is qFW. □

**Remark 8.** In [14] it is shown explicitly that the ice-cream cone is not qFW. Here is a simple synthetic argument. The ice cream cone $D \subset \mathbb{R}^3$ can be cut by a plane $L$ in such a way that $F = D \cap L$ has a hyperbola as boundary curve. Since $F$ has asymptotes, it is not qFW, hence neither is the cone $D$.

The method of proof in implication $(6) \implies (1)$ in Theorem 1 can be used to show that sub-level sets of convex polynomials are qFW-sets, see [3, Chap. II, §4, Thm. 13]. We obtain the following extension of [4, Thm. 3]:

**Corollary 3.** Let $F_0$ be a qFW-set and let $f_1, \ldots, f_m$ be convex polynomials on $F_0$ such that the set $F = \{ x \in F_0 : f_i(x) \leq 0, i = 1, \ldots, m \}$ is non-empty. Let $f$ be a polynomial which is bounded below on $F$ and has at least one nonempty convex sub-level set on $F$. Then $f$ attains its infimum on $F$. □

**Remark 9.** From Corollary 2 and Proposition 6 we learn that the class of qFW-sets is closed under finite intersections and cross products, while example 1 tells us that this is no longer true for FW-sets. Yet another invariance property of qFW-sets is the following:

**Corollary 4.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an affine operator, and let $F \subset \mathbb{R}^n$ be a qFW-set. If $T^{-1}(F)$ is nonempty, then it is a qFW-set, too.

**Proof.** We use property (4) of Theorem 1. Suppose $T^{-1}(F)$ had an $f$-asymptote $M$, then $T(M)$ would be an $f$-asymptote of $F$. □
Corollary 5. (See [4], [16, Cor. 27.3.1]). Let \( f \) be a polynomial which is convex and bounded below on a qFW-set \( F \). Then \( f \) attains its infimum on \( F \).

The following consequence of Theorem 1 is surprising.

Corollary 6. Let \( F \) be a convex cone. Then the following are equivalent:

1) \( F \) is a FW-set;
2) \( F \) is a qFW-set;
3) \( F \) is polyhedral.

Proof. (1) \( \implies \) (2) is clear, because \( F \) is convex. (2) \( \implies \) (3): Let \( F \subset \mathbb{R}^n \) be qFW, then by condition (iv) of Theorem 1 every orthogonal projection \( P(F) \) on any two-dimensional subspace of \( \mathbb{R}^n \) is closed. Therefore, by Mirkil’s theorem, which we give as Lemma 1 below, \( F \) is polyhedral. (3) \( \implies \) (1): By the classical Frank-and-Wolfe theorem every polyhedral convex cone is FW. \( \square \)

Lemma 1. (Mirkil’s theorem [15]). Let \( D \) be a convex cone in \( \mathbb{R}^n \) such that every orthogonal projection on any two-dimensional subspace is closed. Then \( D \) is polyhedral. \( \square \)

Remark 10. This result puts an end to hopes to get new results for the linear complementarity problem by investigating FW-cones.

We end this section with a nice consequence of Mirkil’s theorem. First we need the following characterization of \( f \)-asymptotes:

Proposition 7. For a closed convex set \( F \) and a linear subspace \( L \), the following statements are equivalent:

1) No translate of \( L \) is an \( f \)-asymptote of \( F \).
2) The orthogonal projection of \( F \) onto the orthogonal complement \( L^\perp \) is closed.
3) \( F + L \) is closed.

Proof. 1) \( \implies \) 2). Let \( x \in P_{L^\perp}(F) \). Since \( P_{L^\perp}^{-1}(x) = x + L \), we can easily prove that \( \text{dist}(F, x + L) = \text{dist}(P_{L^\perp}(F), x) = 0 \). Since \( x + L \) is not an \( f \)-asymptote of \( F \), we have \( F \cap (x + L) \neq \emptyset \), which amounts to saying that \( x \in P_{L^\perp}(F) \).

2) \( \implies \) 3). Let \( x_k \in F \) and \( y_k \in L \) (\( k = 1, 2, ... \)) be such that the sequence \( x_k + y_k \) converges to some point \( z \). Then \( P_{L^\perp}(z) = \lim P_{L^\perp}(x_k + y_k) = \lim P_{L^\perp}(x_k) \in P_{L^\perp}(F) \) due to closedness of \( P_{L^\perp}(F) \). But \( P_{L^\perp}(F) = (F + L) \cap L^\perp \subset F + L \), hence \( P_{L^\perp}(z) \in F + L \). Now \( z = P_{L^\perp}(z) + P_L(z) \in F + L + L = F + L \).

3) \( \implies \) 1). Let us assume that \( x + L \) is an \( f \)-asymptote of \( F \) for some \( x \). Then \( 0 \leq \text{dist}(x, F + L) \leq \text{dist}(x, (F + L) \cap L^\perp) = \text{dist}(x, P_{L^\perp}(F)) = \text{dist}(F, x + L) = 0 \), hence \( \text{dist}(x, F + L) = 0 \). Since \( F + L \) is closed, this implies \( x \in F + L \). This is equivalent to saying that \( F \cap (x + L) \neq \emptyset \), a contradiction to the assumption that \( x + L \) is an \( f \)-asymptote of \( F \). \( \square \)

The consequence of Mirkil’s Theorem we have in mind is the following:

Proposition 8. For a closed convex cone \( D \) in \( \mathbb{R}^n \) (with \( n > 2 \)), the following statements are equivalent:

1) \( D \) is polyhedral.
2) \( C + D \) is a convex polyhedron for every convex polyhedron \( C \).
3) \( L + D \) is closed for every \( (n - 2) \)-dimensional subspace \( L \).
4) \( D \) has no \( (n - 2) \)-dimensional \( f \)-asymptotes.
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\[ \text{Proof.} \] Implications 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3) are immediate. Implication 3) \(\Rightarrow\) 1) is a consequence of 3) \(\Rightarrow\) 2) of Proposition 7 combined with Mirkil’s Theorem. Implication 3) \(\Rightarrow\) 4) follows from 3) \(\Rightarrow\) 1) of Proposition 7. Finally, implication 4) \(\Rightarrow\) 3) can be easily derived from implication 1) \(\Rightarrow\) 3) of Proposition 7.

4. Motzkin type sets

Following [9, 10], a convex set \(F\) is called Motzkin decomposable, if it may be written as the Minkowski sum of a compact convex set \(C\) and a closed convex cone \(D\), that is, \(F = C + D\). Motzkin’s classical result states that every convex polyhedron has such a decomposition. We extend this definition as follows:

**Definition 4.** A closed set \(F \subset \mathbb{R}^n\) is called a Motzkin set, for short an \(M\)-set, if it can be written as \(F = K + D\), where \(K\) is a compact set, and \(D\) is a closed convex cone.

We shall continue to reserve the term Motzkin decomposable for the case where the set \(F\) is convex. A Motzkin set \(F\) which is convex is then clearly Motzkin decomposable.

**Remark 11.** Let \(F = K + D\) be a Motzkin set, then similarly to the convex case \(D\) is uniquely determined by \(F\). Indeed, taking convex hulls, we have \(\text{co}(F) = \text{co}(K) + \text{co}(D) = \text{co}(K) + D\), hence \(\text{co}(F)\) is a convex Motzkin set, i.e., a Motzkin decomposable set. Then from known results on Motzkin decomposable sets [9, 10], \(D = 0^+ \text{co}(F)\), the recession cone of \(\text{co}(F)\). Now if we define the recession cone of \(F\) in the same way as in the convex case, i.e., \(0^+ F = \{u \in \mathbb{R}^n : x + tu \in F\ \text{for all} \ x \in F \ \text{and all} \ t \geq 0\}\), then \(0^+ F \subset 0^+ \text{co}(F) = D \subset 0^+ F\), proving \(D = 0^+ F\). In particular, \(F\) and \(\text{co}(F)\) have the same recession cone.

**Theorem 2.** Let \(F\) be a Motzkin set in \(\mathbb{R}^n\), represented as \(F = K + D = K + 0^+ F\). Then the following are equivalent:

1. \(F\) is a FW-set.
2. The recession cone \(0^+ F\) of \(F\) is polyhedral.
3. \(F\) has no \(f\)-asymptotes.

**Proof.** We prove (1) \(\Rightarrow\) (2). Let \(P\) be an orthogonal projection of \(\mathbb{R}^n\) onto a subspace \(L\) of \(\mathbb{R}^n\). Since \(F = K + D\) is a FW-set, \(P(F)\) is closed. Since \(P(F) = P(K) + P(D)\) and \(P(F) = P(K) + P(D)\), this means \(P(K) + P(D) = P(K) + P(D)\). We have to show that this implies \(P(D) = P(D)\). This follows from the so-called order cancellation law, which we give as Lemma 2 below. It is applied to the convex sets \(A = \overline{P(D)}\), \(B = P(D)\), and for the compact set \(P(K)\). This shows indeed \(P(D) = P(D)\). This means every projection of \(D\) is closed, hence by Mirkil’s theorem (Lemma 1), the cone \(D\) is polyhedral.

**Lemma 2.** (Order cancellation law, see [10]). Let \(A, B \subset \mathbb{R}^n\) be convex sets, \(K \subset \mathbb{R}^n\) a compact set. If \(A + K \subset B + K\), then \(A \subset B\).

Let us now prove (2) \(\Rightarrow\) (1). Write \(F = K + D\) for \(K\) compact and \(D\) a polyhedral convex cone. Now consider a quadratic function \(q(x) = \frac{1}{2} x^T Ax + b^T x\) bounded below by \(\gamma\) on \(F\). Hence

\[
\inf_{x \in F} q(x) = \inf_{y \in K} \inf_{z \in D} q(y + z) = \inf_{y \in K} \left( q(y) + \inf_{z \in D} y^T Az + q(z) \right) \geq \gamma.
\]
Observe that for fixed $y \in K$ the function $q_y : z \mapsto y^T Az + q(z)$ is bounded below on $D$ by $\eta = \gamma - \max_{y' \in C} q(y')$. Indeed, for $z \in D$ we have

$$y^T Az + q(z) \geq \left( q(y) + \inf_{z' \in D} y^T A z' + q(z') \right) - q(y)$$

$$\geq \inf_{y \in K} \left( q(y) + \inf_{z' \in D} y^T A z' + q(z') \right) - \max_{y' \in K} q(y')$$

$$\geq \gamma - \max_{y' \in K} q(y') = \eta.$$

Since $q_y$ is a quadratic function bounded below on the polyhedral cone $D$, the inner infimum is attained at some $z = z(y)$. This is in fact the classical Frank and Wolfe theorem on a polyhedral cone. In consequence the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined as

$$f(y) = \inf_{z \in D} y^T Az + q(z),$$

satisfies $f(y) = y^T Az(y) + q(z(y)) > -\infty$ for every $y \in K$, so the compact set $K$ is contained in the domain of $f$. But now a stronger result holds, which one could call a parametric Frank and Wolfe theorem, and which we shall prove in Lemma 3 below. We show that $f$ is continuous relative to its domain. Once this is proved, the infimum (2) can then be written as

$$\inf_{x \in F} q(x) = \inf_{y \in K} q(y) + f(y),$$

and this is now attained by the Weierstrass extreme value theorem due to the continuity of $q + f$ on the compact $K$. Continuity of $f$ on $K$ is now a consequence of the following

**Lemma 3.** Let $D$ be a polyhedral convex cone and define

$$f(c) = \inf_{x \in D} c^T x + \frac{1}{2} x^T G x,$$

where $G = G^T$. Then $\text{dom}(f)$ is a polyhedral convex cone, and $f$ is continuous relative to $\text{dom}(f)$.

**Proof.** If $x^T G x < 0$ for some $x \in D$, then $\text{dom}(f) = \emptyset$, so we may assume for the remainder of the proof that $x^T G x \geq 0$ for every $x \in D$. The proof is now divided into three parts. In part 1) we establish a formula for the domain $\text{dom}(f)$. In part 2) we use this formula to show that $\text{dom}(f)$ is polyhedral, and in part 3) we show that the latter implies continuity of $f$ relative to $\text{dom}(f)$.

1) We start by proving that

$$\text{dom}(f) = \left\{ c : c^T x \geq 0 \text{ for every } x \in D \text{ such that } x^T G x = 0 \right\}.$$  (3)

The inclusion $\subseteq$ being obvious, we have to prove the following implication:

$$c^T x \geq 0 \text{ for every } x \in D \text{ such that } x^T G x = 0 \implies \inf_{x \in D} c^T x + \frac{1}{2} x^T G x > -\infty.$$

We establish this by induction on the number $l$ of generators of $D$. The case $l = 1$ being clear, let $l > 1$, and suppose the implication is correct for every polyhedral convex cone $D'$ with $l' < l$.
generators. Let \( c \) be such that \( c^T x \geq 0 \) for every \( x \in D \) having \( x^T Gx = 0 \). We have to show that \( c \in \text{dom}(f) \). Assume on the contrary that
\[
\inf_{x \in D} c^T x + \frac{1}{2} x^T Gx = -\infty,
\]
and choose a sequence \( x_k \in D \) with \( \|x_k\| \to \infty \) such that
\[
c^T x_k + \frac{1}{2} x_k^T Gx_k \longrightarrow -\infty.
\]
Passing to a subsequence, we can assume that the sequence \( y_k = x_k / \|x_k\| \) converges to some \( y \in D \). We must have \( y^T Gy = 0 \), as otherwise we would have \( c^T x_k + \frac{1}{2} x_k^T Gx_k = \|x_k\| \|c^T y_k + \frac{1}{2} \|x_k\|^2 y_k^T G y_k \to +\infty \), a contradiction. Hence, by our assumption, \( c^T y \geq 0 \). We cannot have \( c^T y > 0 \), as otherwise for large enough \( k \) we would have \( c^T x_k = \|x_k\| c^T y_k > 0 \) and thus \( c^T x_k + \frac{1}{2} x_k^T Gx_k > 0 \) due to \( x_k^T Gx_k \geq 0 \), which is impossible because of (5). Therefore \( c^T y = 0 \). This will be used later.

Collecting more facts about \( y \), note that as a consequence of our standing assumption \( x^T Gx \geq 0 \) for \( x \in D \), \( y \) is a minimizer of the quadratic form \( \frac{1}{2} x^T Gx \) over \( D \), which implies that \( Gy \) belongs to the positive polar cone of \( D \), that is, \( x^T Gy \geq 0 \) for every \( x \in D \). This property will also be used below.

Let \( E = \{e_1, \ldots, e_l\} \) be the set of generating rays of \( D \), and for \( i = 1, \ldots, l \) denote by \( D_i \) and \( \hat{D}_i \) the cones generated by \( E \setminus \{e_i\} \) and \( (E \setminus \{e_i\}) \cup \{y\} \), respectively. As the induction hypothesis applies to each \( D_i \), we have \( \inf_{x \in D_i} c^T x + \frac{1}{2} x^T Gx > -\infty \) for every \( i \), so the infimum \( m \) of \( c^T x + \frac{1}{2} x^T Gx \) over \( \bigcup_{i=1}^l D_i \) is finite.

Now observe that
\[
D = \bigcup_{i=1}^l \hat{D}_i. \tag{6}
\]
Indeed, the inclusion \( \supseteq \) being clear, take \( x \in D \) and write it as \( x = \sum_{i=1}^l \lambda_i e^i \) for certain \( \lambda_i \geq 0 \). Since \( y \in D \setminus \{0\} \), we have \( y = \sum_{i \in I} \mu_i e^i \) for some \( 0 \neq I \subset \{1, \ldots, l\} \) and \( \mu_i > 0 \). Put \( \nu = \min\{\lambda_i / \mu_i : i \in I\} =: \lambda_{i_0} / \mu_{i_0} \), then
\[
x = \sum_{i \in I} \lambda_i e^i + \sum_{j \notin I} \lambda_j e^j + \nu \left( y - \sum_{i \in I} \mu_i e^i \right) = \sum_{i \in I} (\lambda_i - \nu \mu_i) e^i + \sum_{j \notin I} \lambda_j e^j + \nu y.
\]
Since \( \lambda_i - \nu \mu_i \geq 0 \) for every \( i \in I \), and \( \lambda_{i_0} - \nu \mu_{i_0} = 0 \), we have shown \( x \in \hat{D}_{i_0} \). That proves (6).

Now, using (6), for every \( x \in D \) there exist \( i \in \{1, \ldots, l\} \), \( z \in D_i \), and \( \lambda \geq 0 \) such that \( x = z + \lambda y \). We then have \( c^T x + \frac{1}{2} x^T Gx = c^T z + \lambda c^T y + \frac{1}{2} z^T Gz + \lambda z^T Gy + \frac{1}{2} \lambda^2 y^T Gy = c^T z + \frac{1}{2} \lambda z^T Gz + \lambda z^T Gy \geq c^T z + \frac{1}{2} \lambda z^T Gz \geq m \), which gives \( \inf_{x \in D} c^T x + \frac{1}{2} x^T Gx = m \), contradicting (4). This shows that our claim (3) was correct.

2) Now by the Farkas-Minkowski-Weyl theorem (cf. [16, Thm. 19.1] or [17, Cor. 7.1a]) the polyhedral cone \( D \) is the linear image of the positive orthant of a space \( \mathbb{R}^p \) of appropriate dimension, i.e. \( D = \{ Zu : u \in \mathbb{R}^p, u \geq 0 \} \). Using (3), this implies
\[
\text{dom}(f) = \{ c : c^T Zu \geq 0 \text{ for every } u \geq 0 \text{ such that } u^T Z^T Gu = 0 \}.
\]
Now observe that if \( u \geq 0 \) satisfies \( u^T Z^T G u = 0 \), then it is a minimizer of the quadratic function \( u^T Z^T G u \) on the cone \( u \geq 0 \), hence \( Z^T G u \geq 0 \) by the Kuhn-Tucker conditions. Therefore we can write the set \( P = \{ u \in \mathbb{R}^p : u \geq 0, u^T Z^T G u = 0 \} \) as

\[
P = \bigcup_{I \subset \{1, \ldots, p\}} P_I,
\]

where the \( P_I \) are the polyhedral convex cones

\[
P_I = \{ u \geq 0 : Z^T G u \geq 0, u_i = 0 \text{ for all } i \in I, (Z^T G u)_j = 0 \text{ for all } j \notin I \}.
\]

For every \( I \subset \{1, \ldots, p\} \) choose \( m_I \) generators \( u_{i1}, \ldots, u_{im_I} \) of \( P_I \). Then,

\[
\text{dom}(f) = \left\{ c : c^T Z u \geq 0 \text{ for every } u \in P \right\} = \left\{ c : c^T Z u \geq 0 \text{ for every } u \in \bigcup_{I \subset \{1, \ldots, p\}} P_I \right\} = \bigcap_{I \subset \{1, \ldots, p\}} \left\{ c : c^T Z u \geq 0 \text{ for every } u \in P_I \right\} = \bigcap_{I \subset \{1, \ldots, p\}} \left\{ c : c^T Z u_{ij} \geq 0 \text{ for all } j = 1, \ldots, m_I \right\}.
\]

Since a finite intersection of polyhedral cones is polyhedral, this proves that \( \text{dom}(f) \) is a polyhedral convex cone.

3) To conclude, continuity of \( f \) relative to its domain now follows from polyhedrality of \( \text{dom}(f) \), and using [16, Thm. 10.2], since \( f \) is clearly concave and upper semicontinuous. This completes the proof of \( (2) \implies (1) \).

(1) \implies (3) was proved in Proposition 2. Let us prove \( (3) \implies (2) \). By Mirkil’s theorem (Lemma 1) it suffices to show that every orthogonal projection \( P(F) \) is closed. Suppose this is not the case, and let \( y \in P(F) \setminus P(F) \). Let \( L = y + \ker(P) \), then \( F \cap L = \emptyset \). Now choose \( y_n \in F \) such that \( P(y_n) \to P(y) = y \). Then \( y_n = P(y_n) + z_n \) with \( z_n \in \ker(P) \). Hence \( P(y) + z_n \in L \), but \( \| (P(y_n) + z_n) - (P(y) + z_n) \| \to 0 \), which shows \( \text{dist}(F, L) = 0 \). That means \( F \) has an \( f \)-asymptote, a contradiction. \( \square \)

Remark 12. The main implication \( (2) \implies (1) \) in Theorem 2 was first proved by Kummer [12]. Our proof of \( (2) \implies (1) \) is slightly stronger in so far as it gives additional information on the polyhedrality of the domain of \( f \) in Lemma 3.

Remark 13. We refer to Bank et al. [2, Thm. 5.5.1 (4)] for a result related to Lemma 3 in the case where \( G \succeq 0 \). For the indefinite case see also Tam [18].

Remark 14. The statement of Theorem 2 is no longer correct if one drops the hypothesis that \( F \) is a Motzkin set. We take the convex \( F = \{ (x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1 \} \), then \( F \), being limited by a hyperbola, has \( f \)-asymptotes, hence is not \( qFW \), but \( 0^+F \) is the positive orthant, which is polyhedral.

Corollary 7. A Motzkin decomposable set \( F \) without \( f \)-asymptotes is Frank-and-Wolfe.

Proof. Since \( F \) has no \( f \)-asymptotes and is convex, it is a \( qFW \)-set by Theorem 1. But then by Theorem 2, \( F \) is even a \( FW \)-set. \( \square \)
5. Invariance properties of Motzkin FW-sets

We have seen in example 1 that intersections of FW-sets need no longer be FW-sets, not even when convexity is assumed. In contrast, the class of qFW-sets turned out closed under finite intersections. This rises the question whether more amenable sub-classes of the class of FW-sets with better invariance properties may be identified. In response we show in this chapter that the class of Motzkin FW-sets, for short FWM-sets, is better behaved with regard to invariance properties.

Lemma 4. Consider a set of the form $K + D$, where $K$ is compact and $D$ is a polyhedral closed convex cone in $\mathbb{R}^n$. Let $L$ be a linear subspace of $\mathbb{R}^n$. Then there exists a compact set $K_0$ such that $(K + D) \cap L = K_0 + (D \cap L)$.

Proof. 1) We assume for the time being that the cone $D \cap L$ is pointed. For fixed $x \in K$ consider the polyhedron $P_x := (x + D) \cap L$. Define $M(P_x) = \{x' \in P_x : (x' - (D \cap L)) \cap P_x = \{x'\}\}$, and let $K(P_x)$ be the closed convex hull of $M(P_x)$. Then according to [9, Thm. 19] the set $K(P_x)$ is compact, and we have the minimal Motzkin decomposition $P_x = K(P_x) + (D \cap L)$. This uses the fact that $D \cap L$ is the recession cone of $P_x$. It follows that

$$(K + D) \cap L = \bigcup_{x \in D} (x + D) \cap L = \bigcup_{x \in K} K(P_x) + (D \cap L),$$

so all we have to do is show that the set $\bigcup_{x \in K} K(P_x)$ is bounded, as then its closure $K_0$ is the compact set announced in the statement of the Lemma. To prove boundedness of $\bigcup_{x \in K} K(P_x)$ it clearly suffices to show that $\bigcup_{x \in K} K(P_x)$ is bounded.

Let $\mathcal{F}$ be the finite set of faces of $D$, where we assume that $D$ itself is a face. Let $x' \in M(P_x)$, then $x'$ is in the relative interior of one of the faces $x + F$, $F \in \mathcal{F}$, of the shifted cone $x + D$.

We divide the faces $F \in \mathcal{F}$ of the cone $D$ into two types: $\mathcal{F}_1$ is the class of those faces $F \in \mathcal{F}$ for which there exists $d \in L$, $d \neq 0$, such that $d$ is a direction of recession of $F$, i.e., those where $F \cap L$ does not reduce to $\{0\}$. The class $\mathcal{F}_2$ gathers the remaining faces $D$ which are not in the class $\mathcal{F}_1$.

Now suppose the set $\bigcup_{x \in K} M(P_x)$ is unbounded. Then there exists a sequence $x_k \in K$ and $x'_k \in M(P_{x_k})$ with $\|x'_k\| \to \infty$. From the above we know that each $x'_k$ is in the relative interior of $x_k + F_k$ for some $F_k \in \mathcal{F}$. Since there are only finitely many faces, we can extract a subsequence, also denoted $x_k$ and satisfying $\|x'_k\| \to \infty$, such that the $x'_k$ are relative interior points of $x_k + F$ for the same fixed face $F \in \mathcal{F}$. Due to compactness of $K$ we may, in addition, assume that $x_k \to x \in K$. Using the definition of $M(P_{x_k})$ write $x'_k = x_k + t_k d_k \in L$ with $d_k \in F \subset D$, $\|d_k\| = 1$, $t_k > 0$, $t_k \to \infty$. Passing to yet another subsequence, assume that $d_k \to d$, where $\|d\| = 1$. It follows that $d \in L$, because in the expression $x'_k/t_k = x_k/t_k + d_k$ the middle term tends to $0$ due to compactness of $K$ and $t_k \to \infty$, while the left hand term is in $L$ because $x'_k$ belongs to $L$. Since $F$ is a cone, it also follows that $x + \mathbb{R} d \subset x + F$, hence $d \in F$. This shows that the face $F$ is in the class $\mathcal{F}_1$.

2) So far we have shown that $\bigcup_{F \in \mathcal{F}_2} \{x' \in M(P_x) : x \in K, x' \in \text{ri}(x + F)\}$ is a bounded set. It remains to prove that this set contains all points $x' \in M(P_x)$, $x \in K$, i.e., that $x' \in M(P_x)$ cannot be a relative interior point of any of the faces $x + F$ with $F \in \mathcal{F}_1$.

3) Contrary to what is claimed, consider $x \in K \setminus L$ such that $x' \in M(P_x)$ satisfies $x' \in \text{ri}(x + F)$ for some $F \in \mathcal{F}_1$. By definition of the class $\mathcal{F}_1$ there exists $d \in L \cap F$, $d \neq 0$. Since $x' \in L$ by the definition of $M(P_x)$, we have $x' + \mathbb{R} d \subset L$. But this line is also contained in $x + \text{span}(F)$, because we have $d \in \text{span}(F)$ and $x' = x + d'$ for some $d' \in F$, hence $x' + \mathbb{R} d \subset x + \text{span}(F)$. 


Since $x'$ is a relative interior point of $x + F$, there exists $\varepsilon > 0$ such that $N_{\varepsilon} = \{x' + sd : |s| < \varepsilon\}$ is contained in $x + F$. Since $d \in F \cap L \subseteq D \cap L$, we have arrived at a contradiction with the fact that $x' \in M(P_x)$. Namely, moving in $N_{\varepsilon}$ we can stay in $P_x$ while going from $x'$ slightly in the direction of $-d \in -(D \cap L)$. This contradiction shows that what was claimed in 2) is true. The Lemma is therefore proved for pointed $D \cap L$.

4) Suppose now $D$ is allowed to contain lines. With a change of coordinates we may arrange that $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^p$ and $D \subseteq \mathbb{R}^m \times \{0\}$, where the possibility $p = 0$ is not excluded and corresponds to the case where $D = D = \mathbb{R}^n$. Now consider the space $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ and define the cone $\tilde{D} \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ as $\tilde{D} = \{(x^+, x^-, 0) : x^+ \in \mathbb{R}^m, x^+ \geq 0, x^+ - x^- \in D\}$. Then $\tilde{D}$ is polyhedral and pointed. Let $T$ be the mapping $(x^+, x^-, y) \mapsto (x^+ - x^-, y)$, then $T(\tilde{D}) = D$. Since $T$ maps $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ onto $\mathbb{R}^m \times \mathbb{R}^p$, there exists a compact set $\tilde{K} \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ such that $T(\tilde{K}) = K$. Put $\tilde{L} = T^{-1}(L)$. Now since $\tilde{D}$ is pointed, the first part of the proof gives a compact $\tilde{K}_0 \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p$ such that $(\tilde{K} + \tilde{D}) \cap \tilde{L} = \tilde{K}_0 + (\tilde{D} \cap \tilde{L})$. Applying $T$ on both sides, and using the fact that $\tilde{L}$ is a pre-image, we deduce $(K + D) \cap L = T(\tilde{K}_0) + (D \cap L)$. On putting $K_0 = T(\tilde{K}_0)$ which is compact, we get the desired statement $(K + D) \cap L = K_0 + (D \cap L)$. That completes the proof of the Lemma. 

Corollary 8. Any finite intersection of sets of the form $K + D$ with $K$ compact and $D$ a polyhedral convex cone is again a set of this form.

Proof. It suffices to consider the case of two sets $F_i = K_i + D_i$ in $\mathbb{R}^n$, $i = 1, 2$, with compact $K_i$ and $D_i$ polyhedral convex cones. We build the set $F = F_1 \times F_2$ in $\mathbb{R}^n \times \mathbb{R}^p$, which is of the same form, because trivially $(K_1 + D_1) \times (K_2 + D_2) = (K_1 \times K_2) + (D_1 \times D_2)$, and since the product of two polyhedral cones is a polyhedral cone.

Now by Lemma 4 the intersection of $F_1 \times F_2$ with the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ is a set of the form $\mathcal{K} + \mathcal{D}$ with $\mathcal{K}$ compact and $\mathcal{D}$ a polyhedral convex cone, because the diagonal is a linear subspace. Finally, $F_1 \cap F_2$ is the image of $\mathcal{K} + \mathcal{D}$ under the projection $p : (x, y) \mapsto x$ onto the first coordinate, hence is of the form $p(\mathcal{K}) + p(\mathcal{D})$, and since $p(\mathcal{D})$ is a polyhedral convex cone, we are done.

We conclude with the following invariance property of the class $FWM$:

Proposition 9. If the pre-image of a $FWM$-set under an affine mapping is nonempty, then it is a $FWM$-set.

Proof. Let $T$ be an affine mapping and $F$ be a $FWM$-set such that $T^{-1}(F) \neq \emptyset$. Since translates of $FWM$-sets are $FWM$, there is no loss of generality in assuming that $T$ is linear. Then the restriction of $T$ to $\text{ker}(T) \perp$ is a bijection from $\text{ker}(T) \perp$ onto $R(T)$, and one has

$$T^{-1}(F) = \left(T \big|_{\text{ker}(T) \perp}\right)^{-1}(F \cap R(T)) + \text{ker}(T).$$

Since $R(T)$ is a subspace, hence a convex polyhedron, and $T^{-1}(F) \neq \emptyset$, the set $F \cap R(T)$ is $FWM$ by Corollary 8. Since $\left(T \big|_{\text{ker}(T) \perp}\right)^{-1}$ is an isomorphism from $R(T)$ onto $\text{ker}(T) \perp$, the set $\left(T \big|_{\text{ker}(T) \perp}\right)^{-1}(F \cap R(T))$ is $FWM$. Hence it suffices to observe that $\text{ker}(T)$, being a subspace, is $FWM$, and that the class of $FWM$-sets is closed under taking sums. 

\qed
Remark 15. It is worth mentioning that in general the affine pre-image of a Motzkin decomposable set need not be Motzkin decomposable. To wit, consider the ice cream cone $F$ in $\mathbb{R}^3$ and the mapping $T: (x_1, x_2, x_3) \mapsto (1, x_2, x_3)$, then the linear function $x_3 - x_2$ does not attain its infimum on $T^{-1}(F)$, which proves that $T^{-1}(F)$ is not Motzkin decomposable.

Remark 16. In Proposition 5 we had proved that the affine pre-image $T^{-1}(F)$ of a FW-set is FW if $F$ is contained in the range of $T$. A priori this additional range condition cannot be removed, because we have no result which guarantees that $F \cap \text{range}(T)$ is still a FW-set (if nonempty). As we just saw, this range condition can be removed for FWM-sets, and also for qFW-sets, so these two classes are invariant under affine pre-images without further range restriction.

Open question: Let $F$ be a FW-set and $L$ a linear subspace, is $F \cap L$ a FW-set?

Remark 17. Altogether we have found the class of FWM-sets to be closed under finite products, finite intersections, images and pre-images under affine maps. If we call a set FWMU if it is a finite union of FWM-sets, then sets in this class are still FW-sets. By De Morgan’s law the class FWMU remains closed under finite intersections. The class FWMU remains also closed under affine pre-images, because the pre-image of a union coincides with the union of the pre-images. Similarly the class FWMU remains closed under affine images.

6. Parabolic sets

As we have seen in Theorem 2, the search for new FW-sets does not lead very far beyond polyhedrality within the Motzkin class, because if a Motzkin set $F = K + D$ is to be FW, then its recession cone $D = 0^{+}F$ must already be polyhedral. The question is therefore whether one can find FW-sets which exhibit non-polyhedral asymptotic behavior, those then being necessarily outside the Motzkin class. The following result shows that such FW-sets do indeed exist.

Theorem 3. (Luo and Zhang [13]). Let $P$ be a closed convex polyhedron and define $F = \{x \in P : x^T Q x + q^T x + c \leq 0\}$, where $Q = Q^T \succeq 0$. Then $F$ is a FW-set. \hfill $\square$

The result generalizes the Frank and Wolfe theorem in the following sense: if we add just one convex quadratic constraint $x^T Q x + q^T x + c \leq 0$ to a linearly constrained quadratic program, then finite infima of quadratics are still attained. As example 1 shows, adding a second convex quadratic constraint already fails.

The question is now can the Luo-Zhang theorem, just like the Frank-and-Wolfe theorem, be extended from polyhedra $P$ to FWM-sets $F = K + D$? That means, if $F = K + D$ is a FWM-set, and if $Q = Q^T \succeq 0$, will the set $\mathcal{F} = \{x \in F : x^T Q x + q^T x + c \leq 0\}$ still be a FW-set? We show by way of a counterexample that the answer is in the negative.

Example 3. We consider the cylinder $F = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \leq 1\}$. Note that $F$ is a FWM-set, because it can be represented as $F = K + L$ for the compact convex set $K = \{(x_1,x_2,0,0) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \leq 1\}$ and the subspace $L = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$.

Now we add the convex quadratic constraint $x_3^2 \leq x_4$ to the constraint set $F$, which leads to the set

$\mathcal{F} = \{x \in F : x_3^2 \leq x_4\} = \{x \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \leq 1, x_3^2 \leq x_4\}$. 

We will show that $\mathcal{F}$ is no longer a FW-set. This means that the extension of Theorem 3 from polyhedra $P$ to FWM-sets $F$ fails.

Consider the quadratic function $q(x) = x_4x_1 - 2x_2x_3 + 2$. We claim that $q$ is bounded below on $\mathcal{F}$ by 0. Indeed, since $x_1 \geq 0$ on the feasible domain $\mathcal{F}$, we have $x_4x_1 \geq x_2^2x_1$ on the feasible domain, hence $q(x) \geq x_2^2x_1 - 2x_2x_3 + 2 = q(x_1, x_2, x_3, x_4^2)$, the expression on the right no longer depending on $x_4$. Let us compute the infimum of that expression on $\mathcal{F}$. This comes down to globally solving the program

$$\begin{align*}
(P) & \quad \text{minimize} & & x_2^2x_1 - 2x_2x_3 + 2 \\
& \quad \text{subject to} & & (x_1 - 1)^2 + x_2^2 \leq 1
\end{align*}$$

and it is not hard to see that $(P)$ has infimum 0, but that this infimum is not attained. (Solve for $x_3$ with fixed $x_1, x_2$ and show that the value at $(x_1, x_2, x_2/x_1)$ goes to 0 as $x_1 \to 0^+$, $(x_1 - 1)^2 + x_2^2 = 1$, but that 0 is not attained).

Now if $\bar{x} \in \mathcal{F}$ is a minimizing sequence for $q$, then $\bar{x}^k := (x_1^k, x_2^k, x_3^k, (x_4^k)^2) \in \mathcal{F}$ is also feasible and gives $q(x^k) \geq q(\bar{x}^k)$, so the sequence $\bar{x}^k$ is also minimizing, showing that the infimum of $q$ on $\mathcal{F}$ is the same as the infimum of $(P)$, which is zero. But then the infimum of $q$ on $\mathcal{F}$ could not be attained, as otherwise the infimum of $(P)$ would also be attained. Indeed, if the infimum of $q$ on $\mathcal{F}$ is attained at $\bar{x} \in \mathcal{F}$, then it must also be attained at $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4^2) \in \mathcal{F}$ because $q(\bar{x}) \geq q(\bar{x})$, and then the infimum of $(P)$ is attained at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, contrary to what was shown.

**Remark 18.** We can write the set $\mathcal{F}$ as $\mathcal{F} = K' \times F'$, where $K' = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 \leq 1\}$ is compact convex, and where $F'$ is the Luo-Zhang set $F' = \{(x_3, x_4) : x_3^2 \leq x_4\}$, which by Theorem 3 is a FW-set. This shows that the cross product of a convex FW-set (which is not FWM) and a compact convex set need no longer be a FW-set.

**Remark 19.** We can also write $\mathcal{F} = (K + L) \cap (F + M)$, where $L, M$ are linear subspaces of $\mathbb{R}^4$. Indeed, $K, L$ are as in Example 3, while $F = \{(0, 0, x_3, x_4) : x_3^2 \leq x_4\}$ and $M = \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}$. Here $K + L$ is FWM, while $F + M$ is a FW-set by Theorem 3.

**Remark 20.** Note that $\mathcal{F}$ is a qFW-set by Proposition 6, see also [13, Cor. 2].

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