About the stability of the tangent bundle of $\mathbb{P}^n$ restricted to a surface

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Abstract Let $X$ be a smooth projective surface over $\mathbb{C}$ and let $L$ be a line bundle on $X$ generated by its global sections. Let $\phi_L : X \to \mathbb{P}^r$ be the morphism associated to $L$; we investigate the $\mu$-stability of $\phi_L^* T_{\mathbb{P}^r}$ with respect to $L$ when $X$ is either a regular surface with $p_g = 0$, a K3 surface or an abelian surface. In particular, we show that $\phi_L^* T_{\mathbb{P}^r}$ is $\mu$-stable when $X$ is K3 and $L$ is ample and when $X$ is abelian and $L^2 \geq 14$.

Keywords Tangent bundle · Stability · Exceptional bundle · Surface

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1 Introduction

Given a line bundle $L$ generated by its global sections on a smooth projective variety $X$, one can consider the kernel of the evaluation map

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0 \quad (1)$$

and its dual $E_L = M_L^*$. The $\mu$-stability of this bundle is equivalent to that of $\phi_L^* T_{\mathbb{P}^r}$, where $\phi_L : X \to \mathbb{P}^r$ is the morphism associated to $L$. It has been studied in the case of a curve by Paranjape [9] with Ramanan and in his Ph.D. thesis [8]; in particular, the latter contains the statements on which rely all our results contained in a former paper [3] and in this one. Later Ein and Lazarsfeld [4] showed that $M_L$ is stable if $\deg L > 2g$ and Beauville [2] investigated the case of degree $2g$.

The aim of this paper is to study this problem in the case of projective surfaces. Here we consider the $\mu$-stability of a sheaf with respect to a chosen linear series $H$, which
generalises the definition given in the case of curves: a vector bundle $E$ is said to be $\mu-$stable with respect to $H$ if for each proper torsion-free quotient sheaf $F$ we have $\mu(F) > \mu(E)$, where $\mu(F) = \frac{c_1(F) \cdot H}{rk F}$ is the slope of $F$ (see [5]).

After studying these vector bundles in Sect. 2, we gather some results which hold on curves in Sect. 3 and then in Sect. 4 we obtain some results about regular surfaces, including the following

**Theorem 1** Let $X$ be a smooth projective K3 surface over $\mathbb{C}$ and let $L$ be an ample line bundle generated by its global sections on $X$; then the vector bundle $E_L$ is $\mu-$stable with respect to $L$.

Finally, in Sect. 5 we study the case of abelian surfaces, showing the following

**Theorem 2** Let $X$ be a smooth projective abelian surface over $\mathbb{C}$ and let $L$ be a line bundle on $X$ generated by its global sections such that $L^2 \geq 14$. Then the vector bundle $E_L$ is $\mu-$stable with respect to $L$.

## 2 Simplicity and rigidity of $E_L$

Let us briefly recall the geometric interpretation of $E_L$: since $L$ is generated by its global sections, the morphism $\phi_L : X \to \mathbb{P}(H^0(L)) \cong \mathbb{P}^r$ is well-defined and we have $L = \phi_L^* O_{\mathbb{P}^r}(1)$; thus, from the dual sequence of (1) and from the well-known Euler exact sequence it follows that $E_L = \phi_L^* T_{\mathbb{P}^r} \otimes L^*$ and the $\mu-$stability of $E_L$ is equivalent to the $\mu-$stability of $\phi_L^* T_{\mathbb{P}^r}$.

In the next sections we will deal with the problem of whether or not these bundles are $\mu-$stable, but let us first of all underline that they satisfy in almost any case a less strong property, the simplicity.

**Proposition 1** Let $X$ be a smooth projective complex variety and $L$ be a big line bundle generated by its global sections on $X$; if $\dim X \geq 2$ then $E_L$ is simple.

**Proof** If we tensor with $E_L$ the short exact sequence (1) in cohomology we get an exact sequence

$$
0 \rightarrow H^0(M_L \otimes E_L) \rightarrow H^0(L) \otimes H^0(E_L) \xrightarrow{\alpha} H^0(L \otimes E_L) \rightarrow \cdots
$$

Since $H^0(L^*) \cong H^1(L^*) \cong 0$ by Ramanujam-Kodaira vanishing theorem (see [7]), we also have $H^0(L)^* \cong H^0(E_L)$. Now, by tensoring the dual sequence of (1) with $L$ we obtain in cohomology the following exact sequence

$$
0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(L) \otimes H^0(L)^* \xrightarrow{\alpha} H^0(L \otimes E_L) \rightarrow H^1(\mathcal{O}_X) \rightarrow \cdots
$$

where the morphism $\alpha$ is the same morphism as in (2). Hence $H^0(M_L \otimes E_L) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}$, i.e. $E_L$ is simple. $\square$

In the case of regular surfaces, under mild assumptions, which hold for example if $X$ is a K3 surface, these bundles are also rigid.
Proposition 2 Let $X$ be a smooth projective regular surface and $L$ as above; if the multiplication map $H^0(K_X) \otimes H^0(L) \to H^0(K_X \otimes L)$ is surjective, then $E_L$ is rigid.

Proof The morphism $\alpha$ in sequence (3) is surjective because $X$ is regular. Let us show that $H^1(E_L) \cong 0$: indeed, by tensoring (1) with $K_X$ in cohomology we get an exact sequence

$$0 \to H^0(M_L \otimes K_X) \to H^0(L \otimes H^0(K_X) \to H^0(L \otimes K_X) \to H^1(M_L \otimes K_X) \to H^0(L) \otimes H^1(K_X) = 0$$

Since we assumed $\varphi$ surjective, we have $H^1(E_L) \cong H^1(M_L \otimes K_X)^* \cong 0$ by the Serre’s duality theorem. Then from the exact sequence (2) it follows that $\text{Ext}^1(E_L, E_L) \cong H^1(M_L \otimes E_L) = 0$, i.e. $E_L$ is rigid. \qed

Remark 1 The cohomology sequence associated to (1) shows that $H^0(M_L) = 0$, hence for all subsheaves $N$ of $M_L$ we have $H^0(N) = 0$; in particular $O_X$ cannot be a subbundle of $M_L$.

3 Some results on vector bundles on curves

Let us briefly recall some facts about vector bundles on curves.

Given $C$ a smooth projective curve of genus $g \geq 2$, the Clifford index of a line bundle $L$ on $C$ is defined as $c(L) = \deg L - 2(h^0(L) - 1)$ and the Clifford index of the curve $C$ is $c(C) = \min\{c(L) \mid h^0(L) > 1, h^1(L) > 1\}$.

In a former paper [3, p. 424] we showed the following

Theorem 3 Let $C$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field $k$ and let $L$ be a line bundle on $C$ generated by its global sections such that $\deg L \geq 2g - c(C)$. Then:

1. $E_L$ is semi-stable;
2. $E_L$ is stable except when $\deg L = 2g$ and either $C$ is hyperelliptic or $L \cong K(p + q)$ with $p, q \in C$.

In the case $L = K_C$ more was already known: in [9, Corollary 3.5 p. 509] Paranjape and Ramanan showed the following

Theorem 4 Let $C$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$; $E_{K_C}$ is always semistable and it is also stable if $C$ is not hyperelliptic.

The proof of Theorem 3 was essentially based on the following lemma, shown by Paranjape [8, pp. 1–17].

Lemma 1 Let $F$ be a vector bundle on $C$ generated by its global sections and such that $H^0(C, F^*) = 0$; then $\deg F \geq \text{rk } F + g - h^1(C, \det F)$. Moreover, if $h^1(C, \det F) \geq 2$ then $\deg F \geq 2\text{rk } F + c(\det F) \geq 2\text{rk } F + c(C)$.

4 About regular surfaces

Before restricting to the case of regular surfaces, let us see a few statements which hold for every surface.
Lemma 2  Let $F$ be a vector bundle of rank 2 generated by its global sections on a smooth projective surface $X$ and assume moreover that $h^0(\det F) = 2$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \det F \longrightarrow 0 \quad (4)$$

Proof  We cannot have $F = \mathcal{O}_X^2$ because $h^0(\det F) = 2$; then, since $F$ is of rank 2 generated by its global sections, we have $h^0(F) \geq 3$. There is a section $s \in H^0(X, F)$ which is zero only in a finite number of points and we have the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{I}_Z \det F \longrightarrow 0 \quad (5)$$

where $Z$ is the zero locus of $s$. In cohomology we obtain

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, F) \longrightarrow H^0(X, \mathcal{I}_Z \det F) \longrightarrow \cdots$$

Since $h^0(F) \geq 3$, we get $h^0(\mathcal{I}_Z \det F) \geq 2$, but $h^0(\mathcal{I}_Z \det F) \leq h^0(\det F) = 2$. Since $\det F$ is generated by its global sections, from $h^0(\mathcal{I}_Z \det F) = h^0(\det F) = 2$ it follows that $\mathcal{I}_Z \det F = \det F$ and $Z = \emptyset$. Therefore the sequence (5) becomes (4). $\square$

Remark 2  Under the assumptions of Lemma 2, we have $F \cong \pi^* \mathcal{O}_{P^1} (1)$ where $\pi : X \to P^1$ is the fibration defined by the global sections of $\det F$.

Proposition 3  Let $X$ be a smooth projective surface over $\mathbb{C}$ and let $L$ be a line bundle on $X$ generated by its global sections. Let $C$ be a smooth irreducible curve on $X$ such that $H^1(L(-C)) = 0$. Then $(E_L)_C = E_{(L|_C)} \oplus \mathcal{O}_C^r$, with $r = h^0(L(-C))$.

Proof  Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with $L$, in cohomology we get

$$0 \longrightarrow H^0(X, L(-C)) \longrightarrow H^0(X, L) \longrightarrow H^0(X, L|_C) \longrightarrow 0$$

So we have the following diagram

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
L^*_{|C} & H^0(X, L|_C)^* \otimes \mathcal{O}_C & E_{(L|_C)} \\
\downarrow & \downarrow & \downarrow \\
0 & H^0(X, L)^* \otimes \mathcal{O}_C & (E_L|_C) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

The proof is now straightforward. $\square$
Corollary 1 Let $X$ be a smooth projective regular surface over $C$ such that $p_g = 0$ and let $C$ be a smooth irreducible curve on $X$ of genus $g \geq 2$ such that $L = \mathcal{O}_X(K_X + C)$ and $\mathcal{O}_X(C)$ are generated by their global sections; then $E_L$ is $\mu$–semistable with respect to $c$ and it is also $\mu$–stable if $c(C) > 0$.

Proof By Proposition 3 $(E_L)|_C \cong E_{(L|_C)}$, since $r = p_g = 0$; on the other hand, $L|_C = K_C$, so the statement follows from Theorem 4.

When $r \neq 0$, the restriction to the curve is no longer semistable, but in the case of K3 surfaces Proposition 3 is enough to prove the $\mu$–stability.

Proof of Theorem 1 Let $C \in |L|$ be a smooth irreducible curve of genus $g \geq 2$. We will consider three cases: $c(C) \geq 2$, $c(C) = 1$ and $c(C) = 0$.

By Proposition 3 we have $(E_L)|_C = E_{K_C} \oplus \mathcal{O}_C$, since $L|_C \cong K_C$; moreover $\mu(E_L) = \frac{2g - 2}{g} < 2$. Let us suppose that $g \geq 3$: if $g = 2$ then $C$ is hyperelliptic and we will deal with the case $c(C) = 0$ later. Let $F$ be a torsion-free quotient sheaf of $E_L$ of rank $0 < \text{rk } F < g$; then $F|_C$ is a quotient of $(E_L)|_C$ and we can suppose that it is a vector bundle on $C$. There is a diagram of the form

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & (E_L)|_C & \longrightarrow & E_{K_C} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W & \longrightarrow & F|_C & \longrightarrow & G \oplus \tau & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

where $G$ is a vector bundle generated by its global sections, $W$ is either $\mathcal{O}_C$ or $0$ and $\tau$ is a torsion sheaf on $C$, hence $\deg W = 0$ and length $\tau \geq 0$. So we get $\mu(F) = \frac{\deg G + \text{length } \tau}{\text{rk } F}$.

Let us recall once more that $E_L$ and all its quotients are globally generated.

1. If $\text{rk } G = 0$, then $\text{rk } (F) = 1$ and we always have $\mu(F) \geq 2$. Indeed, we cannot have $F = \mathcal{O}_X$, since this would imply the existence of a non-zero section $\mathcal{O}_X \hookrightarrow M_L$, in contradiction with $H^0(X, M_L) = 0.$ Hence $F = \mathcal{O}_X(D)$ with $D > 0$ an effective base-point free divisor such that $D.C \geq 1$ because $C$ is ample; since $\mathcal{O}_C(D)$ is globally generated we have then $D.C \geq 2$.

2. If $\text{rk } G > 0$, then $G$ is generated by its global sections such that $H^0(C, G^*) = 0$, because $G^*$ is a subbundle of $M_{K_C}$ and $H^0(C, M_{K_C}) = 0$; the hypotheses of Lemma 1 then hold and, since $\mu(F) \geq \frac{\deg G}{\text{rk } F}$, we have:

(a) if $h^1(\text{det } G) < 2$, since $g \geq 3$, then by Lemma 1

$$\mu(F) \geq 1 + \frac{g - 2}{\text{rk } G + 1} > 1 + \frac{g - 2}{g} = \mu(E_L).$$

(b) If $h^1(\text{det } G) \geq 2$, then by Lemma 1

$$\mu(F) \geq 2 + \frac{\text{c(det } G) + \deg \tau - 2}{\text{rk } G + 1} \geq 2 > \mu(E_L)$$

if $c(\text{det } G) \geq 2$, in particular if $c(C) \geq 2$, but also if $c(\text{det } G) = 1$ and $\deg \tau > 0$.

This shows that $\mu(F) > \mu(E_L)$ in the case $c(C) \geq 2$. 

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We now deal with the case \( c(C) = 1 \). We can repeat the above proof by applying Lemma 1 and it does not work only if \( h^1(\det G) \geq 2 \), \( \deg \tau = 0 \) and \( c(\det G) = 1 \). If \( g = 3 \) then \( \mu(E_L) = \frac{4}{3} \) and we always have \( \mu(F) > \frac{4}{3} \).

From now on we assume \( g \geq 4 \); then either the curve is trigonal or a smooth plane quintic of genus \( g = 6 \) (see [6]).

1. If there is a \( g_3^1 \) on \( C \), the only line bundles which compute the Clifford index are \( O_C(g_3^1) \) and \( O_C(K_C - g_3^1) \).

   (a) If \( \det G = O_C(g_3^1) \), since \( h^1(\det G) \geq 2 \), by Lemma 1 we have \( \deg G \geq 2 \rk G + 1 \), hence in this case \( \rk G = 1 \). Then \( \rk F = 2 \) and \( \det F_C = O_C(g_3^1) \); it follows that \( \det F = O_X(D) \) with \( D.C = 3 \). By the Hodge index theorem then, since \( g \geq 4 \), we have \( D^2 \leq \frac{9}{2g-2} < 2 \), so \( D^2 = 0 \) and \( D = kE \) with \( k \geq 1 \) and \( E \) an elliptic curve; since \( D.C = 3 \) and \( C.E \geq 2 \), this implies \( k = 1 \) and \( h^0(O_X(D)) = 2 \); by Lemma 2, it follows from \( h^1(\det F^*) = 0 = \text{Ext}^1(O_X, \det F) \) that \( F = O_X \oplus \det F \), hence \( h^0(F^*) > 0 \), which is impossible.

   (b) If \( \det G = O_C(K_C - g_3^1) \) we have \( \deg G = 2g - 5 \) and \( \rk G \leq g - 3 \) by Lemma 1, hence

   \[
   \mu(F) \geq \frac{2g - 5}{\rk G + 1} \geq \frac{2g - 5}{g - 2} = 2 - \frac{1}{g - 2} > \mu(E_L)
   \]

   if \( g > 4 \). If \( g = 4 \) we have \( \deg G = 3 \) and we fall in the former case.

2. If there is a \( g_3^2 \) on \( C \), the genus is \( g = 6 \) and the only line bundle which computes the Clifford index is \( O_C(g_3^2) \cong O_C(K_C - g_3^2) \).

   If \( \det G = O_C(g_3^2) \), since \( h^1(\det G) \geq 2 \), by Lemma 1 \( \deg G \geq 2 \rk G + 1 \), hence \( \rk G \leq 2 \) and \( \rk F \leq 3 \). Therefore we get

   \[
   \mu(F) = \frac{5}{\rk G + 1} \geq \frac{5}{3} = \mu(E_L)
   \]

   Let us investigate whether equality can hold or not; suppose that \( \rk F = 3 \). Since \( F \) is of rank \( > 2 \) generated by its global sections, there is a short exact sequence

   \[
   0 \longrightarrow O_X \longrightarrow F \longrightarrow V \longrightarrow 0
   \]

   with \( V \) of rank \( 2 \) generated by its global sections such that \( \det V = \det F = O_X(D) \) with \( D.C = 5 \). By the Hodge index theorem then \( D^2 \leq 2 \); however the case \( D^2 = 2 \) cannot occur, since otherwise \( (C - 2D)^2 = -2 \) and by Riemann-Roch theorem at least one between \( C - 2D \) and \( 2D - C \) would be effective, contradicting \( (C - 2D).C = 0 \) and the ampleness of \( C \). If \( D^2 = 0 \), then \( D = kE \) with \( k \geq 1 \) and \( E \) an elliptic curve; since \( D.C = 5 \) and \( C.E \geq 2 \), this implies \( k = 1 \) and \( h^0(O_X(D)) = 2 \), so by Lemma 2 there is a short exact sequence

   \[
   0 \longrightarrow O_X \longrightarrow V \longrightarrow \det V \longrightarrow 0
   \]

   and in cohomology we obtain \( h^1(V^*) = h^1(V) = 0 \). As a consequence we have \( \text{Ext}^1(O_X, V) = 0 \) and \( F = O_X \oplus V \), impossible since it would imply \( h^0(F^*) > 0 \).

   Then \( \mu(F) > \mu(E_L) \) also if \( c(C) = 1 \).

   Suppose now that \( C \) is a hyperelliptic curve; then the morphism \( \phi_L : X \longrightarrow P^8 \) induces a double covering \( \pi : X \longrightarrow F \) where \( F \subset P^8 \) is a rational surface of degree \( g - 1 \) which is either smooth or a cone over a rational normal curve (see [1, p. 129]). let \( i : F \hookrightarrow P^8 \) be
the embedding and \( H = i^*O_{\mathbb{P}^d}(1) \) the ample hyperplane section of \( F \) such that \( \pi^*H = L \); whenever \( E_H \) is \( \mu \)-stable with respect to \( H \), this yields the \( \mu \)-stability of \( E_L \) with respect to \( L \), because \( \pi \) is a finite covering (see [5], Lemma 3.2.2).

If \( g = 2 \) then \( F = \mathbb{P}^2 \) (see [1, p. 129]) and it is well-known that its tangent bundle is \( \mu \)-stable with respect to \( O_{\mathbb{P}^2}(1) \) (see [5, Section 1.4]).

If \( g \geq 3 \), we have \( H^2 = g - 1 \). On the surface \( F \) we have the short exact sequence

\[
0 \longrightarrow H^* \longrightarrow H^0(F, H)^* \otimes \mathcal{O}_F \longrightarrow E_H \longrightarrow 0
\]

We know that the curve \( H \) is rational, so \( p_a(H) = 0 \); we consider a smooth curve \( \Gamma \in |2H| \).

By the adjunction formula we have \( 0 = p_a(H) = 1 + \frac{1}{2}(H^2 + H.K_F) \), so we get \( H.K_F = -H^2 - 2 = -g - 1 \); using the adjunction formula once more we then obtain

\[
p_a(\Gamma) = 1 + \frac{1}{2}(\Gamma^2 + \Gamma.K_F) = 1 + 2H^2 + H.K_F = g - 2.
\]

Since \( g \geq 3 \) we have \( p_a(\Gamma) \geq 1 \). Since \( H \) is ample, we deduce \( H^0(F, \mathcal{O}_F(-H)) = H^1(F, \mathcal{O}_F(-H)) = 0 \) (see [7]). Then from the short exact sequence

\[
0 \longrightarrow \mathcal{O}_F(H - \Gamma) \longrightarrow \mathcal{O}_F(H) \longrightarrow \mathcal{O}_\Gamma(H) \longrightarrow 0
\]

and from the associated cohomology sequence it follows that \( H^0(F, \mathcal{O}_F(H)) \cong H^0(F, \mathcal{O}_\Gamma(H)) \), hence \( (E_H)|_{\Gamma} = E_{\mathcal{O}_\Gamma(H)} \).

Moreover, \( \deg \mathcal{O}_\Gamma(H) = H.\Gamma = 2g - 2 > 2p_a(\Gamma) = 2g - 4 \). Since \( \mathcal{O}_\Gamma(H) \) is a line bundle on a smooth projective curve \( \Gamma \) of genus \( \geq 1 \) of degree \( > 2p_a(\Gamma) \), \( (E_H)|_{\Gamma} \) is stable (see [4]).

Since \( E_H \) is \( \mu \)-stable with respect to \( 2H \), it is also \( \mu \)-stable with respect to \( H \) and this ends the proof. \( \square \)

Remark 3 Throughout the proof the ampleness of \( L \) is needed only when \( C \) is a smooth plane quintic of genus \( g = 6 \) to show that we cannot have equality between slopes. Indeed, if we only assume that \( L \) is generated by its global sections and \( L^2 \geq 2 \) then \( E_L \) is still \( \mu \)-semistable with respect to \( L \) and also \( \mu \)-stable unless \( C \) is a smooth plane quintic of genus \( g = 6 \).

5 About abelian surfaces

In this section we study the same problem when \( X \) is an abelian surface over \( \mathbb{C} \) and we give the proof of Theorem 2.

Proposition 4 Let \( X \) be an abelian surface over \( \mathbb{C} \); then there is no irreducible hyperelliptic curve of genus \( g \geq 6 \) and no irreducible trigonal curve of genus \( g \geq 8 \) on \( X \).

Proof Take \( d = 2 \) or 3 and suppose that there is a \( d \)-gonal irreducible curve \( C \) of genus \( g \geq 2d + 2 \) on \( X \). Then there is an exact sequence of sheaves on \( X \)

\[
0 \longrightarrow F^* \longrightarrow H^0(g_d^1) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_C(g_d^1) \longrightarrow 0
\]

where \( F \) is a vector bundle of rank 2 such that \( c_1(F) = C \) and \( c_2(F) = d \). Dualising the above exact sequence we get

\[
0 \longrightarrow \mathcal{O}_X^2 \longrightarrow F \longrightarrow \mathcal{O}_C(K_C - g_d^1) \longrightarrow 0
\]
It follows from the assumption on the genus that $c_1(F)^2 - 4c_2(F) = 2g - 2 - 4d > 0$, so $F$ is Bogomolov unstable (see [10]). Therefore, there exists a line bundle $\mathcal{O}_X(A)$ on $X$ such that $\mu(\mathcal{O}_X(A)) > \mu(F)$, i.e. $2A.C > C^2$, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow F \longrightarrow \mathcal{I}_2 \otimes \mathcal{O}_X(B) \longrightarrow 0$$

with $A + B = C$, $A.B + \deg \mathcal{I}_2 = d$ and $(A - B)^2 > 0$ (see [10]). Hence we can construct the following diagram

Since $i$ is an isomorphism outside $C$, $\mathcal{O}_C(K_C - g_d^1) > 0$ and $B$ is effective. By the Hodge index theorem $A^2B^2 \leq (A.B)^2 \leq d^2$. Since $K_X = 0$, $A^2$ and $B^2$ are even numbers and $A^2 > B^2$ because $2A.C > C^2$, hence we must have $B^2 \leq 2$.

If $B^2 = 2$, then $d = 3$ and $A^2 = 4$ and we would have $6 - 2A.B > 0$, so $A.B \leq 2$ in contradiction with $A^2B^2 = 8$. Therefore $B^2 = 0$, which means that $B = kE$ where $E$ is an elliptic curve and $k \geq 1$; on the other hand we know that $0 \leq A.B \leq d$. In fact $A.B > 0$, otherwise by the Hodge index theorem it would follow $B = 0$ against the fact that $\mathcal{O}_C(K_C - g_d^1) > 0$; hence $1 \leq kA.E \leq d$. Since $A.E = 1$ would imply that $A$ itself is elliptic, the only possibility is $k = 1$ and $A.B > 1$. In this case we have $\mathcal{O}_C(K_C - g_d^1) = 1$, hence by the snake lemma we have the following diagram
where $\tau$ and $\tau'$ are two torsion sheaves with support respectively on the zero-locus of $s$ and $\sigma$. Hence the exactness of the third line implies that $C$ is reducible, against our assumptions.

$\square$

**Proof of Theorem 2** Since $L$ is generated by its global sections such that $L^2 \geq 14$, the general member of $|L|$ is a smooth irreducible curve of genus $g \geq 8$. Given a nontrivial $\alpha \in \text{Pic}^0(X)$, we can find $C \in |L \otimes \alpha^{-1}|$ smooth irreducible of genus $g \geq 8$. The $\mu-$stability of $E_L$ with respect to $L$ is equivalent to the $\mu-$stability of $E_L$ with respect to $C$. Since we have $H^0(\alpha) = H^1(\alpha) = 0$, it follows from Proposition 3 that $(E_L)|_C \cong E_L(C)$. Moreover, $L(C) \cong K_C \otimes \alpha_C$, so by Theorem 3 $E_L$ is $\mu-$stable with respect to $C$ if $c(C) \geq 2$. By the hypothesis on the genus of $C$ and by Proposition 4 the cases $c(C) = 0, 1$ cannot occur, so there is nothing more to prove.

$\square$

**Remark 4** In the case $g(C) \leq 7$ the same proof shows the $\mu-$stability of $E_L$ if $c(C) \geq 2$. Moreover, it is possible to show that $E_L$ is $\mu-$stable with respect to $L$ also if either $C$ is a smooth plane quintic of genus $g = 6$ or if $C$ is a trigonal curve of genus $g = 4$.

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