Action-angle Variables for Generic 1D Mechanical Systems

(Draft)

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Abstract

We consider a 1D mechanical system

\[ \tilde{H}(P, Q) = P^2 + G(Q) \]

in action-angle variable \((P, Q)\) where \(G\) is a 2\(\pi\)-periodic analytic function with non degenerate critical points. Then, we consider a small analytic perturbation of \(\tilde{H}\) of the form

\[ H^*(P, Q; \hat{P}) = P^2 + G(Q) + \eta F(P, Q; \hat{P}) =: P^2 + G^*(P, Q; \hat{P}), \quad \eta \ll 1, \]

where the perturbed potential \(G^*\) may depend on the action \(P\) and also on parameters \(\hat{P}\) (“the adiabatic actions”); indeed, this is the form of a finite dimensional mechanical system close to an exact simple resonance after averaging over fast angles and disregarding the exponentially small remainder, see [5].

Up to a finite number of separatrices and elliptic/hyperbolic points the phase space of \(H^*\) is divided into a finite number of open connected components foliated by invariant circles. On every connected component we perform a (Arnold–Liouville) symplectic action-angle transformation which integrates the system.
We give a complete and quantitative description of the analyticity properties of
such integrating transformations, estimating, in particular, how such transfor-
mations differ from the integrating transformation for $\bar{H}$; compare Theorem 6.1
below.

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1 Set up and notations

(i) **Norms on finite dimensional vector spaces**
\[ |\cdot| \text{ denotes the standard Euclidean norm on } \mathbb{C}^n \text{ and its subspaces.} \]
For linear maps and matrices (which are identified), \(|\cdot|\) denotes the “operator norm”
\[ |A| = \sup_{u \neq 0} \frac{|Au|}{|u|}. \]
\(|k|_1\) denotes the 1-norm \(\sum |k_j|\).
\(|M|_\infty\), with \(M\) matrix (or vector), denotes the maximum norm \(\max_{ij} |M_{ij}|\) (or \(\max_i |M_i|\)).

(ii) **Open covers**
Given a set \(D \subseteq \mathbb{R}^m\), \(r > 0\) we denote by \(D_r \subseteq \mathbb{C}^m\) the complex open neighborhood of \(D\) formed by points \(z \in \mathbb{C}^m\) such that \(|z - y| < r\), for some \(y \in D\).
Given \(s > 0\), we denote by \(T^n_s\) the open complex neighborhood of \(^1\) \(\mathbb{T}^n\) given by
\[ T^n_s := \{ x \in \mathbb{C}^n : \max_{1 \leq j \leq n} |\text{Im} x_j| < s \}/2\pi\mathbb{Z}^n. \]

(iii) **Norms of analytic functions**
Given a real–analytic function \(^2\) \(f : T^n_s \to \mathbb{C}, f(x) = \sum_{k \in \mathbb{Z}^n} f_k e^{ikx}\), we define the “sup–Fourier norm”
\[ \|f\|_s := \sup_{k \in \mathbb{Z}^n} |f_k| e^{|k|_1 s} < \infty. \] (1)
Analogously, if \(f : D_r \times T^n_s \to \mathbb{C}, f(y, x) = \sum_{k \in \mathbb{Z}^n} f_k(y) e^{ikx}\), we let
\[ \|f\|_{r,s} := \sup_{k \in \mathbb{Z}^n} (\sup_{y \in D_r} |f_k(y)| e^{|k|_1 s}). \] (2)
If the (real) domain needs to be specified, we let:
\[ \|f\|_{D, r,s} := \|f\|_{r,s}. \] (3)
Given a bounded holomorphic function \(f : T^n_s \to \mathbb{C}\), we set
\[ |f|_s := \sup_{T^n_s} |f|. \] (4)

---

\(^1\)\(\mathbb{T}^n\) denotes the standard flat \(n\)-dimensional torus \(\mathbb{R}^n/(2\pi\mathbb{Z}^n)\).

\(^2\)\(f_k\) denotes Fourier coefficients.
Given a bounded holomorphic function \( f : D_r \times \mathbb{T}_n^s \to \mathbb{C}^m \), or \( f : D_r \to \mathbb{C}^m \) with \( D \subseteq \mathbb{R}^n \) we set
\[
|f|_{D,r,s} = |f|_{r,s} := \sup_{D_r \times \mathbb{T}_n^s} |f|, \quad \text{or, respectively,} \quad |f|_{D,r} = |f|_r := \sup_{D_r} |f|. \tag{5}
\]
Notice that the following relations between the two norms \( \| \cdot \| \) and \( | \cdot | \) hold: for \( \sigma > 0 \), we have
\[
\|f\|_{r,s} \leq |f|_{r,s} \leq \coth^n(\sigma/2) \|f\|_{r,s+\sigma} \leq (1 + 2/\sigma)^n \|f\|_{r,s+\sigma}. \tag{6}
\]

(iv) Scale of Banach spaces of real–analytic periodic functions
For \( s \geq 0 \), we denote by \( \mathbb{B}_s^n \) the following Banach space of real–analytic functions on \( \mathbb{T}^n \) with vanishing average:
\[
\mathbb{B}_s^n := \{ f : \mathbb{T}^n \to \mathbb{C} \text{ s.t. } \|f\|_s < \infty \text{ and } f_0 = 0 , \ f_k = f_{-k} \}.
\]

2 Standard form of parametrized 1D mechanical systems
Let \( r_0, R_0, s_0 > 0 \), \( \hat{D} \subset \mathbb{R}^{n-1} \) and consider the Hamiltonian
\[
H^* := P_n^2 + \mathcal{G}^*(P,Q_n), \tag{7}
\]
real–analytic for
\[
(P,Q_n) \in D_{r_0} \times \mathbb{T}_{s_0}, \quad \text{where} \quad D := \hat{D} \times (-R_0, R_0). \tag{8}
\]
This Hamiltonian represents a 1D system in the symplectic variables \( (P_n, Q_n) \in \mathbb{R} \times \mathbb{T} \) depending on the parameter \( \hat{P} = (P_1, ..., P_{n-1}) \in \mathbb{R}^{n-1} \).

Now, we want to reduce (7) to a parameterized 1D mechanical system (i.e., with a potential independent of \( P_n \)). We do this via the following “normalization lemma”, where \( H^* \) is considered as a function of \( 2n \) variables \( (P,Q) \).

---

3 Indeed:
\[
\sum_{k \in \mathbb{Z}^n} e^{-|k|_1 \sigma} = \left( \sum_{k \in \mathbb{Z}^n} e^{-|k|_1 \sigma} \right)^n = \left( 1 + 2 \sum_{j \geq 1} e^{-j \sigma} \right)^n = \left( \frac{e^\sigma + 1}{e^\sigma - 1} \right)^n = \coth^n(\sigma/2).
\]

4 The bar denotes complex conjugate.
Lemma 2.1 (Standard form of parametrized 1D mechanical systems)
Let \( D = \hat{D} \times (-R_0, R_0), \hat{D} \subset \mathbb{R}^{n-1}, s_0, r_0, R_0 > 0, 0 < \hat{s} \leq s_0/2, \hat{G} \in \mathcal{B}_{s_0}^1 \) and let
\[
H^*(P, Q) := P_n^2 + G^*(P, Q_n), \quad \text{with} \quad |G^* - \hat{G}|_{D,r_0,s_0} \leq \eta_0
\]
and
\[
\eta_0 \leq \frac{r_0^2}{64} \min \left\{ \frac{s_0}{\pi}, 1 \right\}. \tag{9}
\]
Then, there exists a symplectic transformation
\[
\Phi_{\text{mech}} : (p, q) \in \hat{D}_{r_0} \times (-R_0, R_0)_{r_0/2} \times T_{\hat{s}} \times T_{s_0} \mapsto (P, Q) \in D_{r_0} \times T_{\hat{s} + 16\pi\eta_0/r_0^2} \times T_{s_0}, \tag{10}
\]
of the form
\[
(P, Q) = \Phi_{\text{mech}}(p, q) : \left\{ \begin{array}{l}
\dot{P} = \hat{P} \\
P_n = p_n + a_*(\hat{p}, q_n) = P_n - P_n^*(\hat{p}) + P(\hat{p}, q_n) \\
Q_n = q_n
\end{array} \right. \right. \tag{11}
\]
such that \( H^* \circ \Phi_{\text{mech}}(p, q) =: H_{\text{mech}}(p, q_n) \) has the “standard form”
\[
\left\{ \begin{array}{l}
H_{\text{mech}}(p, q_n) = (1 + b(p, q_n))(p_n - P_n^*(\hat{p}))^2 + G(\hat{p}, q_n)
\end{array} \right. \tag{12}
\]
where \( G := G^*(\hat{p}, P(\hat{p}, q_n), q_n) + (P(\hat{p}, q_n))^2 \).

Furthermore, the following estimates hold:
\[
|P_n^*|_{\hat{D},r_0} \leq 2\eta_0/r_0 \leq r_0/8, \quad |a_*|_{\hat{D},r_0,s_0} \leq 4\eta_0/r_0, \quad |b_*|_{\hat{D},r_0/2,s_0} \leq 16\pi\eta_0/r_0^2, \tag{13}
\]
and
\[
|G - \hat{G}|_{\hat{D},r_0,s_0} \leq 2\eta_0, \quad |b_*| \leq \frac{32}{r_0^2} \eta_0, \quad |\partial_{p_n} b_*| \leq \frac{64}{r_0^2} \eta_0, \quad |p_n \cdot b(p, q_n)| \leq \frac{10}{r_0} \eta_0, \quad |p_n \cdot \partial_{p_n} b(p, q_n)| \leq \frac{100}{r_0^2} \eta_0, \tag{14}
\]
where by
\[
|\cdot|_* := \sup_{\hat{D}_{r_0} \times (-R_0, R_0)_{r_0/2} \times T_{s_0}} |\cdot|. \tag{15}
\]
Proof Recalling (A1), (9) and noting that \( \bar{G} \) does not depend on \( p_n \), by Cauchy estimates we have

\[
\sup_{p_n \in (-R_0, R_0)_{3r_0/4}} |\partial_{p_n} G^*(\cdot, p_n, \cdot)|_{\hat{D}, r_0, s_0} \leq \frac{4}{r_0} \eta_0 ,
\]

\[
\sup_{p_n \in (-R_0, R_0)_{3r_0/4}} |\partial^2_{p_n} G^*(\cdot, p_n, \cdot)|_{\hat{D}, r_0, s_0} \leq \frac{32}{r_0^2} \eta_0 ,
\]

\[
\sup_{p_n \in (-R_0, R_0)_{3r_0/4}} |\partial^3_{p_n} G^*(\cdot, p_n, \cdot)|_{\hat{D}, r_0, s_0} \leq \frac{384}{r_0^3} \eta_0 .
\]

As one easily checks by (9), the fixed point equation

\[
P(\hat{p}, q_n) = -\frac{1}{2} \partial_{p_n} G^*(\hat{p}, P(\hat{p}, q_n), q_n)
\]

has a unique solution \( P = P(\hat{p}, q_n) \) with

\[
|P|_{\hat{D}, r_0, s_0} \leq 2 \eta_0 / r_0 \leq r_0 / 8 .
\]

Set

\[
P^*_n(\hat{p}) := \langle P(\hat{p}, q_n) \rangle , \quad a_*(\hat{p}, q_n) := P(\hat{p}, q_n) - \langle P(\hat{p}, q_n) \rangle ,
\]

where \( \langle \cdot \rangle \) denotes the average with respect to \( q_n \). This proves the first two estimates in (13). Let \( \phi = \phi(\hat{p}, q_n) \) the unique function satisfying \( a_* = \partial_{q_n} \phi \) with \( \langle \phi \rangle = 0 \). By the second estimate in (13) we get \( |\phi|_{\hat{D}, r_0, s_0} \leq 8 \pi \eta_0 / r_0 \). Set \( b_* := -\partial_{\hat{p}} \phi \); then the third estimate in (13) follows by Cauchy estimates. Now, let \( \Phi_{\text{mech}} \) be the symplectic transformation in (11) obtained by the generating function \( \hat{p} \cdot \hat{Q} + p_n Q_n + \phi(\hat{p}, Q_n) \). By the estimates on \( a_* \) and \( b_* \) in (13) and (9) it turns out that the canonical transformation \( \Phi_{\text{mech}} \) is well defined with respect to the domains in (10). Then \( \Phi_{\text{mech}} \) casts \( H^* \) into

\[
\left( p_n - P^*_n(\hat{p}) + P(\hat{p}, q_n) \right)^2 + G^*(\hat{p}, p_n - P^*_n(\hat{p}) + P(\hat{p}, q_n), q_n)
\]

\[
= (1 + b(p, q_n))(p_n - P^*_n(\hat{p}))^2 + G(\hat{p}, q_n) ,
\]

where \( G \) was defined in (12) and\(^5\) omitting, for brevity, the dependence on \( \hat{p}, q_n \),

\[
b = \frac{G^*(P + p_n - P^*_n) - G^*(P) - \partial_{p_n} G^*(P)(p_n - P^*_n)}{(p_n - P^*_n)^2} = \int_0^1 (1 - t) \partial_{p_n}^2 G^*(P + t(p_n - P^*_n)) dt .
\]

\(^5\)Using (17).
Then the first estimate in (14) follow by (12), (A1), (9), (18). Note that by (18) and (19) if \( p_n \in (-R_0, R_0)_{R_0/2} \) then \( P + p_n - P_n^* \in (-R_0, R_0)_{3r_0/4} \). By (20) and (16) we get the second estimate in (14). Using the first equality in (20) and the first estimate in (16) we get

\[
|\langle p_n - P_n^* \rangle \cdot b(p, q_n) | \leq \frac{8}{r_0} \eta_0
\]

and, therefore, by (13), the second inequality in (14) and (9) we get the fourth estimate in (14). By (20), and the last estimate in (16) we get

Finally, by the first equality in (20), we have

\[
\partial_{p_n} b = \frac{\partial_p G^*(P + p_n - P_n^*) - \partial_{p_n} G^*(P)}{(p_n - P_n^*)^2} - \frac{2b}{p_n - P_n^*},
\]

and, by (16) and the second estimate in (14),

\[
|\langle p_n - P_n^* \rangle \cdot \partial_{p_n} b(p, q_n) | \leq \left| \frac{\partial_p G^*(P + p_n - P_n^*) - \partial_{p_n} G^*(P)}{p_n - P_n^*} \right| + 2|b| \leq \frac{96}{r_0^2} \eta_0 .
\]

Then by (13), the third inequality in (14) and (9) we get the last estimate in (14).

Remark 2.1 In the oscillatory regime, where \( q_n \) is not an angle, we could perform a canonical transformation (of the form in (21) below) which translates \( p'_n = p_n - P_n^*(\hat{p}) \), so that \( H_{\text{mech}} \) simplifies a little bit, but such translation is impossible in the rotational regime, where \( q_n = q'_n \) is an angle, since the transformation of \( \hat{q} = \hat{q}' - P_n^*(\hat{p})q'_n \) is not 2\( \pi \)-periodic in \( q_n \).

Remark 2.2 The symplectic transformation in (11) belongs to the group \( \mathfrak{G} \) of symplectic transformation of the special form:

\[
\Phi : (p, q) \mapsto (P, Q) : \left\{ \begin{array}{ll}
\hat{P} = \hat{p} \\
P_n = P_n^*(p, q_n)
\end{array} \right. \quad \left\{ \begin{array}{ll}
\hat{Q} = \hat{q} + \hat{q}^*(p, q_n) \\
Q_n = Q_n^*(p, q_n)
\end{array} \right. \quad (21)
\]

where, in general, \( Q, q \) may belong either to \( \mathbb{T}^n \) or to \( \mathbb{R}^n \). Notice that restricting the relation \( dP \wedge dQ = dp \wedge dq \) onto the planes \( \hat{p} = \hat{P} = \text{const} \), one has that \( dP_n \wedge dQ_n = dp_n \wedge dq_n \), i.e.:

For every fixed \( \hat{p} \), the map \( (p_n, q_n) \mapsto (P_n^*(p, q_n), Q_n^*(p, q_n)) \) is also symplectic.

Note that \( \int_0^1 (1 - t) dt = 1/6 \).
Notation 1 For a transformation $\Phi \in \mathcal{G}$ we let $\tilde{\Phi}$ denote the $(n + 1)$-dimensional map
\[ \tilde{\Phi} : (p, q_n) \mapsto (P, Q_n) := (\hat{p}, P_n(p, q_n), q_n(p, q_n)) . \] (22)

Remark 2.3 If $\Phi_i \in \mathcal{G}$, then
\[ (\Phi_1 \circ \Phi_2)_{\tilde{\Phi}} = \tilde{\Phi}_1 \circ \tilde{\Phi}_2 , \] (23)
and, furthermore,
\[ \tilde{\Phi}(E) \times \mathbb{T}^{n-1} = \Phi(E \times \mathbb{T}^n) , \quad \forall \ \Phi \in \mathcal{G} , \quad \forall \ E \subseteq \mathbb{R}^n \times \mathbb{T} . \] (24)

Relation (24) implies, in particular, that, for any map $\Phi \in \mathcal{G}$, the map $\tilde{\Phi}$ is volume-preserving.

3 Morse non-degenerate potentials

Definition 3.1 Let $s_0, M, \beta > 0$ and let $\mathcal{G}$ be a $2\pi$-periodic holomorphic function. We say that $\mathcal{G}$ is $(M, \beta, s_0)$-Morse–non–degenerate if
\[ |\mathcal{G}|_{s_0} \leq M , \] (25)
\[ \min_{\theta \in \mathbb{R}} (|\mathcal{G}'(\theta)| + |\mathcal{G}''(\theta)|) \geq \beta , \quad \min_{1 \leq i < j \leq 2N} |\bar{E}_i - \bar{E}_j| \geq \beta , \] (26)
where
\[ \bar{E}_i := \mathcal{G}(\bar{\theta}_i) , \quad 1 \leq i \leq 2N , \] (27)
are the $2N$ distinct critical values (“critical energies”) and $\bar{\theta}_i$ the corresponding non-degenerate critical points of $\mathcal{G}$.

Note that by (25) and (26) we have\(^7\)
\[ \frac{M}{\beta} \geq \frac{1}{2} , \quad \frac{M}{\beta s_0} \geq \frac{1}{3} . \] (28)

We may assume that, up to translation, the unique absolute maximum of $\mathcal{G}$ is attained at
\[ \bar{\theta}_0 := \bar{\theta}_{2N} - 2\pi = -\pi \]

---

\(^7\)The first estimate is obvious since $|\bar{E}_i| \leq M$; then the second estimate directly follows if $s_0 \leq 1$, otherwise it follows by the first inequality in (26) and Cauchy estimates, which imply $\beta \leq \frac{M}{s_0} + \frac{2M}{s_0^2}$. 

Then, the relative strict non-degenerate minimum and maximum points follow in alternating order:
\[ \bar{\theta}_0 := -\pi < \bar{\theta}_1 < \bar{\theta}_2 < \ldots < \bar{\theta}_{2N-1} < \bar{\theta}_{2N} := \pi, \]
\[ \left\{ \begin{array}{l}
\bar{\theta}_{2j} \quad \text{maximum points} \\
\bar{\theta}_{2j-1} \quad \text{minimum points}
\end{array} \right. \]
(29)

and:

the odd energies \( \bar{E}_1, \ldots, \bar{E}_{2N-1} \) are the \( N \) (local) minimal energies
the even energies \( \bar{E}_2, \ldots, \bar{E}_{2N} \) are the \( N \) (local) maximal energies

and
\[ \bar{E}_0 := \bar{E}_{2N} \]
(30)
is the unique global maximum.

By Cauchy estimates we get
\[ \max_{i} |\partial_i^k \bar{G}| \leq k! M/s_0^k, \quad |\partial_i^k \bar{G}|_{\sigma} \leq k! M/(s_0 - \sigma)^k. \]

Note that by (26), (25) and Cauchy estimates it follows that
\[ \beta \leq 2M, \quad \beta s_0 \leq 2M, \quad \beta s_0^2 \leq 4M. \]
(31)

Obviously the second assumption in (26) directly implies (31). In the particular important case in which \( \bar{G} \) is minus cosine we can explicitly evaluate
\[ \bar{G}(\theta) = -\cos \theta \quad \Rightarrow \quad \left\{ \begin{array}{l}
M = \cosh s_0, \quad N = 1, \quad \bar{\theta}_1 = 0, \quad \bar{\theta}_2 = \pi \\
\bar{E}_1 = -1, \quad \bar{E}_2 = 1, \quad \beta = 1
\end{array} \right. \]
(32)

Lemma 3.1 We have
\[ 2\theta_* \leq \bar{\theta}_i - \bar{\theta}_{i-1} \leq 2\pi, \quad N \leq \frac{\pi}{2\theta_*}, \]
(33)
where
\[ \theta_* := \sqrt{\frac{\beta s_0^3}{3M}}. \]
(34)

---

\(^8\)The first estimates directly follows by (25) and the second inequality in (26). The other two estimates follow by contradiction: otherwise: i) in a point \( \bar{\theta}_0 \) with \( \bar{G}''(\bar{\theta}_0) = 0 \), one has, choosing \( \theta = \pm s_0 \) according to the sign of \( \bar{G}'(\theta_0)/\bar{G}(\theta_0) \), that \( |\bar{G}(\theta_0 + \theta)| \geq |\bar{G}(\theta_0)| + |\beta| - M \theta^2 / s_0^2 \geq \beta s_0 - M > M \), contradicting (25) and, so, proving the second estimate in (31); ii) in a point \( \theta_1 \) with \( \bar{G}'(\theta_1) = 0 \), one has \( |\bar{G}(\theta_1 + \theta)| \geq |\bar{G}(\theta_1)| + \bar{G}''(\theta_1)\theta^2 / 2 - M \theta^3 / s_0^3 \), then, by (26), \( \sup_{|\theta| < s_0} \bar{G}(\theta_1 + \theta) \geq \beta s_0^2 / 2 - M > M \), contradicting (25) and, so, proving the third estimate in (31).
Proof Since \( \bar{G} \) is convex in \( \bar{\theta}_{2j-1} \) and concave in \( \bar{\theta}_{2j} \), there exists \( \bar{\theta}_{2j-1} < \bar{\theta} < \bar{\theta}_{2j} \) such that \( \bar{G}''(\bar{\theta}) = 0 \). By (26) and Cauchy estimates we get
\[
\bar{G}'(\bar{\theta} + \theta) \geq \beta - 3Ms_0^3\theta^2.
\]
Then \( \bar{G}'(\bar{\theta} + \theta) > 0 \), when \( |\theta| < \theta_* \), which implies \( \bar{\theta}_{2j-1} \leq \bar{\theta} - \theta_* < \bar{\theta} + \theta_* \leq \bar{\theta}_{2j} \), proving the first estimate in (33) (from which the estimate on \( N \) directly follows).

Lemma 3.2 For every \( 1 \leq i \leq 2N \)
\[
|\bar{G}'(\bar{\theta}_i + \theta)| \geq \frac{\beta}{2}|\theta|, \quad \forall \theta \in \mathbb{R}, \ |\theta| \leq \theta_* := \frac{\beta s_0^3}{6M}, \quad \text{and} \quad \beta s_0^3 \leq 1, \quad (35)
\]
\[
\min |\bar{G}'| \geq \frac{\beta^2 s_0^3}{32M}. \quad (36)
\]

Proof We will consider only the case\(^9 \) \( i = 2j \). For\(^10 \) \( 0 \leq \theta \leq \theta_* \) we get by Cauchy estimates
\[
\bar{G}'(\bar{\theta}_i + \theta) \geq \bar{G}''(\bar{\theta}_i)\theta - \frac{3M}{s_0^3}\theta^2 \geq \beta\theta - \frac{3M}{s_0^3}\theta^2 \geq \frac{\beta}{2}\theta.
\]
Noting that, as in (33), we have \( 2\theta_* \leq \bar{\theta}_i - \bar{\theta}_{i-1} \leq 2\pi \), we get (36).
Regarding the minimum of \( \bar{G}' \) in the interval\(^11 \) \([\bar{\theta}_{i-1} + \theta_*/2, \bar{\theta}_i - \theta_*/2]\) if it is achieved at the endpoints then the inequality holds, otherwise if it is achieved in an inner point \( \theta_* \) then \( \bar{G}''(\theta_*) = 0 \) and, by (26), \( \bar{G}'(\theta_*) \geq \beta \geq \frac{\beta^2 s_0^3}{32M} \) by (36). \( \blacksquare \)

Fix \( 1 \leq j \leq N \) and consider a minimum point \( \bar{\theta}_{2j-1} \), thanks to (26) the function \( \bar{G} \) is strictly increasing, resp. strictly decreasing, in the interval \([\bar{\theta}_{2j-1}, \bar{\theta}_{2j}]\), resp. \([\bar{\theta}_{2j-2}, \bar{\theta}_{2j-1}]\), then we can invert \( \bar{G} \) on the above intervals obtaining two functions
\[
\bar{\Theta}_{2j} : [E_{2j-1}, E_{2j}] \to [\bar{\theta}_{2j-1}, \bar{\theta}_{2j}] \quad \text{and} \quad \bar{\Theta}_{2j-1} : [E_{2j-1}, E_{2j-2}] \to [\bar{\theta}_{2j-2}, \bar{\theta}_{2j-1}] \quad (38)
\]
such that
\[
\bar{G}(\bar{\Theta}_i(E)) = E, \quad \bar{\Theta}_i(\bar{G}(\theta)) = \theta, \quad \forall 1 \leq i \leq 2N.
\]
Note that \( \bar{\Theta}_i \) is increasing, resp. decreasing, if \( i \) is even, resp. odd. The functions \( \bar{\Theta}_i \) have a holomorphic extension as it is shown below.

\(^9\)The case \( i = 2j - 1 \) is analogous.
\(^10\)The case \( -\theta_* \leq \theta \leq 0 \) is analogous.
\(^11\)Note that on such interval \( \bar{G}' > 0 \).
4 The perturbed potential and the analytic properties of its inverse

4.1 Perturbed potential

Recalling (12) and (14), we now consider a perturbation \( \mathcal{G}(\theta, \hat{I}) \) of the function \( \bar{\mathcal{G}}(\theta) \) depending also on a parameter \( \hat{I} \in \hat{D} \) satisfying

\[ |\mathcal{G}(\theta, \hat{I}) - \bar{\mathcal{G}}(\theta)|_{\hat{D}, r_0, s_0} \leq \eta, \tag{39} \]

where \( \eta \) is a small parameter that we assume to satisfy the condition

\[ \eta \leq \eta_0 = \eta_0(M, \beta, s_0, r_0) := \frac{\beta^9 s_0^{15}}{2^{120} M^9} \min \left\{ r_0^2, \frac{r_0^3}{\sqrt{M}}, \frac{\beta^{45} s_0^{75}}{2^{321} M^{44}} \right\}. \tag{40} \]

Note that by (28) and (31)

\[ s_* := \min\{s_0, 1\} \geq \max \left\{ \frac{s_0 \beta}{4M}, \frac{s_0^2 \beta}{4M} \right\}. \tag{41} \]

In particular, by (25)

\[ |\mathcal{G}|_{\hat{D}, r_0, s_0} \leq M + \eta. \tag{42} \]

Moreover, by (39) and Cauchy estimates

\[ |\partial_\mathcal{I} \mathcal{G}(\theta, \hat{I})|_{\hat{D}, r_0/2, s_0} \leq 2\eta/r_0. \tag{43} \]

By (26), for \( \eta \leq \eta_0 \) small enough, we can continue the critical points \( \bar{\theta}_i \) (defined in (29)), resp. critical energies \( \bar{E}_i \), of \( \bar{\mathcal{G}} \) obtaining critical points \( \theta_i(\hat{I}) \), resp. critical energies \( E_i(\hat{I}) \), of \( \mathcal{G}(\cdot, \hat{I}) \), solving the implicit function equation

\[ \partial_{\bar{\theta}} \mathcal{G}(\theta_i(\hat{I}), \hat{I}) = 0 \tag{44} \]

and then evaluating

\[ \mathcal{G}(\theta_i(\hat{I}), \hat{I}) =: E_i(\hat{I}), \tag{45} \]

respectively. Note that, by definition (recall (30))

\[ E_0 = E_{2N}. \tag{46} \]

Note also that \( \theta_i(\hat{I}) \), and \( E_i(\hat{I}) \) are analytic functions of \( \hat{I} \in \hat{D}_{r_0} \). More precisely we have the following
Lemma 4.1 Assume that
\[ \eta \leq \eta_0 \leq \frac{\beta s_0^2}{16} \left( \frac{12M}{\beta s_0^2} + 1 \right)^{-1} \] (47)
(which is implied by (40)). Then, for every \( 1 \leq i \leq 2N \), there exists a holomorphic function \( \theta_i(\hat{I}) \) with
\[ |\theta_i - \bar{\theta}_i|_{D,r_0} \leq 2\eta/\beta s_0 \leq s_0/8 \] solving equation (44). Moreover
\[ |\partial_{\hat{I}}\theta_i|_{D,r_0/2} \leq 4\eta/\beta s_0 r_0 \] (49)

Proof We know that \( \partial_{\bar{\theta}}\mathcal{G}(\bar{\theta}_i) = 0 \). Then, considering \( \hat{I} \in \hat{D}_{r_0} \) as a parameter, we want to find \( \chi_i = \chi_i(\hat{I}) \), with
\[ |\chi_i| \leq \rho := 2\eta/\beta s_0 \leq s_0/2 \] solving the equation
\[ \partial_{\bar{\theta}}\mathcal{G}(\hat{I}, \bar{\theta}_i + \chi_i) = 0 \] (50)
Introducing the parameter \( \epsilon \) we want to solve the equation
\[ \mathcal{F}(y, \epsilon) = \mathcal{F}(y, \epsilon; \hat{I}) = 0 \] where \( \mathcal{F}(y, \epsilon; \hat{I}) := \partial_{\bar{\theta}}\mathcal{G}(\bar{\theta}_i + y) + \epsilon \partial_y \mathcal{G}(\hat{I}, \bar{\theta}_i + y) \),
with \( \mathcal{G} := \mathcal{G} - \bar{\mathcal{G}} \), finding \( y = y(\epsilon; \hat{I}) \) for every \( |\epsilon| \leq 1 \). Then \( \chi_i = \chi_i(\hat{I}) := \chi_i(1; \hat{I}) \) solves (50) (recalling (39)). We are going to solve \( \mathcal{F} = 0 \) by the Implicit Function Theorem. First note that
\[ \mathcal{F}(0, 0) = \partial_{\bar{\theta}}\mathcal{G}(\bar{\theta}_i) = 0 \] Denote by \( \gamma := 1/\partial_{\bar{\theta}}\mathcal{F}(0, 0) = 1/\partial_{\bar{\theta}}\mathcal{G}(\bar{\theta}_i) \) and note that by (26)
\[ |\gamma| \leq 1/\beta \]
In order to apply a quantitative version of the Implicit Function Theorem, we have to verify the following conditions:
\[ \sup_{|\epsilon| \leq 1} |\mathcal{F}(0, \epsilon)| \leq \frac{\rho}{2|\gamma|} \]
and
\[ \sup_{|\epsilon| \leq 1, |y| \leq \rho} |1 - \gamma \partial_y \mathcal{F}(y, \epsilon)| \leq \frac{1}{2} \].
Since, by Cauchy estimates,
\[ \sup_{|\epsilon| \leq 1} |F(0, \epsilon)| = |\partial_\theta G(\hat{I}, \bar{\theta})| \leq \frac{\eta}{s_0} \leq \frac{\rho}{2|\gamma|}, \]
the first condition follows by (26). Regarding the second condition we note that
\[ \partial_y F(y, \epsilon) = \partial_\theta \theta \bar{G}(\bar{\theta} + y) + \epsilon \partial_\theta \theta G(\hat{I}, \bar{\theta} + y). \]
Then, by Cauchy estimates and noting that \( |y| \leq \rho \leq s_0 / 2 \), we get
\[ \sup_{|\epsilon| \leq 1, |y| \leq \rho} |1 - \gamma \partial_y F(y, \epsilon)| \leq |\gamma| \frac{6M\rho}{(s_0/2)^3} + |\gamma| \frac{2\eta|\gamma|}{s_0^2} \left( \frac{12M}{\beta s_0^2} + 1 \right) \leq \frac{1}{2} \]
by (47).
(49) follows (48) and Cauchy estimates. □

Note that by (40) and (31) we get
\[ \eta_0 \leq \frac{M}{16}. \] (51)

Then by (51) and (42)
\[ |G|_{\hat{D}, r_0, s_0} \leq 2M. \] (52)

Recalling (45), by (39), (48) and Cauchy estimates we get, for \( \eta \leq \eta_0 \),
\[ |E_i - \bar{E}_i|_{\hat{D}, r_0} \leq \left( \frac{4M}{\beta s_0^2} + 1 \right) \eta \leq 2\eta. \] (53)

Then by Cauchy estimates
\[ |\partial_I E_i|_{\hat{D}, r_0/2} \leq 4\eta/r_0. \] (54)

By (33),(34), (48), (40), (53), we note that \( \theta_i(\hat{I}) \) and \( E_i(\hat{I}) \) maintain the same order (w.r.t. \( i \)) of \( \theta_i \) and \( \bar{E}_i \); moreover (recalling also (31))
\[ \inf_{\hat{I} \in \hat{D}, \theta \in \mathbb{R}} \min_{x \in \hat{I}} \left( |\partial_\theta G(\theta, \hat{I})| + |\partial_\theta \theta G(\theta, \hat{I})| \right) \geq \frac{\beta}{2}, \]
\[ \inf_{\hat{I} \in \hat{D}, i \neq j} \min_{\hat{I}} |E_i(\hat{I}) - E_j(\hat{I})| \geq \frac{\beta}{2}. \] (55)

Finally by (48), (33), (40), (31) and (34) we get
\[ \inf_{\hat{I} \in \hat{D}, \eta} |\theta_i(\hat{I}) - \theta_{i-1}(\hat{I})| \geq \theta_* = \sqrt{\frac{\beta s_0^2}{3M}}. \] (56)
Lemma 4.2 There exists a function $\tilde{G}(\theta, \hat{I})$ holomorphic in $\{ |\theta| < \theta_\circ \} \times \hat{D}_{r_0}$ where

$$\theta_\circ := \frac{\beta s_0^3}{2^7 M} \leq \frac{s_0}{2^7},$$

such that

$$\partial_\theta G(\theta_i(\hat{I}) + \theta, \hat{I}) = \theta \tilde{G}(\theta, \hat{I}).$$

Moreover $\tilde{G}(0, \hat{I}) = \partial_{\theta\theta} G(\theta_i(\hat{I}), \hat{I})$ and

$$\sup_{\{ |\theta| < \theta_\circ \} \times \hat{D}_{r_0}} \frac{1}{|\tilde{G}|} \leq \frac{4}{\beta}.$$  

Proof For brevity we skip to write the immaterial dependence on $\hat{I}$. By Taylor expansion we get

$$\tilde{G}(\theta) = \partial_{\theta\theta} G(\theta_i) + \theta \int_0^1 (1 - t) \partial_{\theta\theta\theta} G(\theta_i + t\theta) dt.$$  

Note that the above expression is well posed since, by (48), $|\text{Im} \theta_i| \leq s_0/8$ and $|\theta| < \theta_\circ \leq s_0/2^7$. Then, by Cauchy estimates and (52), we also get

$$\sup_{\{ |\theta| < \theta_\circ \} \times \hat{D}_{r_0}} |\partial_{\theta\theta\theta} G(\theta_i + t\theta)| \leq \frac{2^7 M}{s_0^3}.$$  

Then by (55) we have that, uniformly on $\{ |\theta| < \theta_\circ \} \times \hat{D}_{r_0}$,

$$|\tilde{G}| \geq \frac{\beta}{2} - \theta_\circ \frac{2^7 M}{s_0^3} = \frac{\beta}{4}.$$  

Lemma 4.3 For every real $\hat{I} \in \hat{D}$, there are no more critical points of $\theta \to \tilde{G}(\theta, \hat{I})$ than $\theta_1(\hat{I}), \ldots, \theta_{2N}(\hat{I})$; namely if $\theta^* \in (\theta_{2N}(\hat{I}) - 2\pi, \theta_{2N}(\hat{I})]$ satisfies $\partial_\theta \tilde{G}(\theta^*, \hat{I}) = 0$, then $\theta^* = \theta_i(\hat{I})$ for some $i = 1, \ldots, 2N$.

Proof We assume by contradiction that there exists $\theta^*$ satisfying $\partial_\theta \tilde{G}(\theta^*, \hat{I}) = 0$, with $\theta_{i-1}(\hat{I}) < \theta^* < \theta_i(\hat{I})$ for some $i = 1, \ldots, 2N$. By Lemma 4.2 we have that, for every $j = 1, \ldots, 2N$,

$$|\theta^* - \theta_j(\hat{I})| \geq \theta_\circ.$$  

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Since, by (39) and Cauchy estimates we have
\[ |\bar{G}'(\theta^*)| \leq \eta/s_0 \leq \beta/4, \]
we get \(|\bar{G}''(\theta^*)| \geq \beta/2\). Then there exist \(\bar{\theta}^*\) with \(|\bar{\theta}^* - \theta^2| \leq 2\eta/s_0\beta\) and \(\bar{G}'(\bar{\theta}^*) = 0\). This means that \(\bar{\theta}^* = \hat{\theta}_j\) for some \(j = 1, \ldots, 2N\). Then by (48)
\[ |\theta^2 - \theta_j(\hat{I})| \leq |\theta^2 - \bar{\theta}^*| + |\bar{\theta}^* - \theta_j(\hat{I})| \leq 4\eta/s_0\beta, \]
which by (40) (recall (57)) contradicts (58).

4.2 Rescaled potentials

Now let us introduce the affine functions (considering \(\hat{I}\) as a parameter)
\[
\bar{\gamma}_i(\bar{\theta}) := \frac{\bar{\theta}_i - \bar{\theta}_{i-1}}{2} \bar{\theta} + \frac{\bar{\theta}_i + \bar{\theta}_{i-1}}{2}, \quad \gamma_i(\bar{\theta}, \hat{I}) := \frac{\theta_i(\hat{I}) - \theta_{i-1}(\hat{I})}{2} \bar{\theta} + \frac{\theta_i(\hat{I}) + \theta_{i-1}(\hat{I})}{2},
\]
\[
\bar{\lambda}_i(E) := (-1)^i \frac{2E - E_i - E_{i-1}}{E_i - E_{i-1}}, \quad \lambda_i(E, \hat{I}) := (-1)^i \frac{2E - E_i(\hat{I}) - E_{i-1}(\hat{I})}{E_i(\hat{I}) - E_{i-1}(\hat{I})}. \quad (59)
\]

Lemma 4.4 For every \(0 \leq i \leq 2N\)
\[ |\partial_\theta \bar{\gamma}_i| \leq \pi, \quad |\partial_\theta \gamma_i|_{D,r_0} \leq 2\pi, \quad |\partial_E \bar{\lambda}_i| \leq \frac{2}{\beta}, \quad |\partial_E \lambda_i|_{D,r_0} \leq \frac{4}{\beta}. \quad (60)\]
Moreover, for \(\sigma > 0\), we have
\[
\sup_{[-1,1]_\sigma \times D_{r_0}} |\gamma_i - \bar{\gamma}_i| \leq \frac{2\eta}{\beta s_0}(2 + \sigma), \quad \sup_{\{|E| \leq 2M\} \times D_{r_0}} |\lambda_i - \bar{\lambda}_i| \leq \frac{48M\eta}{\beta^2} \quad (61)\]
and
\[
\sup_{[-1,1]_\sigma} |\text{Im} \, \bar{\gamma}_i|, \quad \sup_{[-1,1]_\sigma \times D_{r_0}} |\text{Im} \, \gamma_i| \leq \frac{2\eta}{\beta s_0}(2 + \sigma) + \pi\sigma. \quad (62)\]

Proof The first estimate follows by (29); then the second one follows by (48), (40) and (36). The third and fourth estimates follow by (26) and (55), respectively. The first estimate in (61) follows by (48) and noting that \(|\theta| \leq 1 + \sigma\) for \(\theta \in [-1, 1]_\sigma\). Regarding the second estimate in (61) we first note that by (26),(55) and (53)
\[
\left| \frac{1}{E_i(\hat{I}) - E_{i-1}(\hat{I})} - \frac{1}{E_i - E_{i-1}} \right| \leq \frac{4\eta}{\beta^2}.
\]
Then, by (51), (53), (26) and the first estimate in (31), we get

\[ |\lambda_i - \tilde{\lambda}_i| \leq 2(|E| + |E_i| + |E_{i-1}|) \left| \frac{1}{E_i(\hat{I}) - E_{i-1}(\hat{I})} - \frac{1}{\bar{E}_i - \bar{E}_{i-1}} \right| + \frac{|E_i - \bar{E}_i| + |E_{i-1} - \bar{E}_{i-1}|}{|E_i - \bar{E}_{i-1}|} \leq \frac{40M\eta}{\beta^2} + \frac{4\eta}{\beta} \leq \frac{48M\eta}{\beta^2}. \]

Finally, since \( \bar{\gamma}_i(\text{Re} \theta) \in \mathbb{R} \), we have \( \text{Im} \left( \gamma_i(\theta, \hat{I}) \right) = \text{Im} \left( \gamma_i(\theta, \hat{I}) - \bar{\gamma}_i(\text{Re} \theta) \right) \) and then, for \( (\theta, \hat{I}) \in [-1, 1]_\sigma \times \hat{D}_{r_0} \), we get

\[ \left| \text{Im} \left( \gamma_i(\theta, \hat{I}) \right) \right| \leq \left| \gamma_i(\theta, \hat{I}) - \bar{\gamma}_i(\text{Re} \theta) \right| \leq \left| \gamma_i(\theta, \hat{I}) - \gamma_i(\theta) \right| + \left| \gamma_i(\theta) - \bar{\gamma}_i(\text{Re} \theta) \right|. \]

Then (62) follows by (60) and (61).

We also have

\[ \bar{\lambda}_i^{-1}(\bar{E}) = \frac{1}{2} \left( \frac{1}{E_i(\hat{I}) - E_{i-1}(\hat{I})} \bar{E} + \bar{E_i} + \bar{E}_{i-1} \right), \] \( \lambda_i^{-1}(\bar{E}, \hat{I}) = \frac{1}{2} \left( \frac{1}{E_i(\hat{I}) - E_{i-1}(\hat{I})} \bar{E} + \bar{E_i} + \bar{E}_{i-1} \right), \] \( \lambda_i^{-1}(\lambda_i(E), \hat{I}) = \frac{1}{2} \left( \frac{E_i(\hat{I}) - E_{i-1}(\hat{I})}{E_i - E_{i-1}} \left( \frac{2E - \bar{E}_i - \bar{E}_{i-1}}{E_i - E_{i-1}} + E_i(\hat{I}) + E_{i-1}(\hat{I}) \right) \right) = \frac{E_i(\hat{I}) - E_{i-1}(\hat{I})}{E_i - E_{i-1}} E + \frac{E_{i-1}(\hat{I})E_i - E_i(\hat{I})E_{i-1}}{E_i - E_{i-1}}. \]

We have

\[ |\lambda_i^{-1}(\bar{\lambda}_i(E), \hat{I}) - E| \leq \frac{4\eta}{\beta} |E| + \frac{4\eta M}{\beta}, \quad \forall E \in \mathbb{C}, \ \hat{I} \in \hat{D}_{r_0}, \] \( \text{(64)} \)

by (53),(25) and (26).

Let us introduced the ”rescaled” functions \( \tilde{\mathcal{G}}_i(\theta) \) and \( \check{\mathcal{G}}_i(\theta, \hat{I}) \) by

\[ \tilde{\mathcal{G}}_i := \tilde{\lambda}_i \circ \tilde{\mathcal{G}} \circ \gamma_i, \quad \check{\mathcal{G}}_i := \lambda_i \circ \mathcal{G} \circ \gamma_i \] \( \text{(65)} \)

(recall (59)). Recalling (44) and (45) For every real \( \hat{I} \), these functions are bijective from \([-1, 1] \) to \([-1, 1] \); in particular

\[ \tilde{\mathcal{G}}_i(\pm 1) = \check{\mathcal{G}}_i(\pm 1, \hat{I}) = \pm (-1)^i, \quad \partial_\theta \tilde{\mathcal{G}}_i(\pm 1) = \partial_\theta \check{\mathcal{G}}_i(\pm 1, \hat{I}) = 0. \] \( \text{(66)} \)
We also set
\[ \tilde{G}_i^*(\theta, \hat{I}) = \tilde{G}_i(\theta, \hat{I}) - g_i(\theta). \] (67)

Note that by (66) we have
\[ \tilde{G}_i^*(\pm 1, \hat{I}) = \partial_\theta \tilde{G}_i^*(\pm 1, \hat{I}) = 0. \] (68)

**Lemma 4.5** Let \( 0 \leq i \leq 2N \) and \( \eta \leq \eta_0 \). Then \( \tilde{g}_i \), resp. \( \tilde{G}_i \), has analytic extension on \([-1,1]_s \times \tilde{D}_{r_0} \), where
\[ s := s_0/2\pi \] (69)

with
\[ \sup_{[-1,1]_s} |\tilde{g}_i|, \quad \sup_{[-1,1]_s \times \tilde{D}_{r_0}} |\tilde{G}_i| \leq \frac{8M}{\beta} =: \tilde{M}. \] (70)

Moreover
\[ \sup_{[-1,1]_s \times \tilde{D}_{r_0}} |\tilde{G}_i^*| = \sup_{[-1,1]_s \times \tilde{D}_{r_0}} |\tilde{G}_i - \tilde{g}_i| \leq \frac{70M}{\beta^2 s_0^2} \eta =: \tilde{\eta} \] (71)

and
\[ \min_{\theta \in [-1-s,1+s]} \left( |\partial_\theta \tilde{G}_i(\theta)| + |\partial_\theta \tilde{g}_i(\theta)| \right) \geq \frac{\beta^2 s_0^3}{3\pi M^2} =: \tilde{\beta}, \] (72)

\[ \inf_{\hat{I} \in \tilde{D}_{r_0}} \min_{\theta \in [-1-s,1+s]} \left( |\partial_\theta \tilde{G}_i(\theta, \hat{I})| + |\partial_\theta \tilde{g}_i(\theta, \hat{I})| \right) \geq \frac{\tilde{\beta}}{4}. \] (73)

**Proof** By (62) we get
\[ \sup_{[-1,1]_s} \left| \text{Im} \left( \tilde{\gamma}_i(\theta) \right) \right|, \quad \sup_{[-1,1]_s \times \tilde{D}_{r_0}} \left| \text{Im} \left( \gamma_i(\theta, \hat{I}) \right) \right| \leq \frac{2\eta}{\beta s_0} (2 + s) + \pi s \leq \frac{s_0}{2}, \] (74)

then (recall (39)) \( \tilde{G}_i \) in (65) is well defined and holomorphic in \([-1,1]_s \times \tilde{D}_{r_0} \). The case of \( g_i \) is similar.

(70) follows by the definition of \( \tilde{\lambda}_i, \lambda_i \) in (59), by (26), (55) and since here \(|E|, |E_i|, |E_i(\hat{I})| \leq M + \eta \) (recall (25) and (42)).
Moreover by (61), (60), (39)

\[ |\tilde{G}_i - \tilde{G}_i| \leq |(\lambda_i - \bar{\lambda}_i) \circ G \circ \gamma_i| + |\bar{\lambda}_i \circ G \circ \gamma_i - \bar{\lambda}_i \circ \bar{G} \circ \bar{\gamma}_i| \]

\[ \leq \frac{48M\eta}{\beta^2} + \frac{2}{\beta} |G \circ \gamma_i - \bar{G} \circ \bar{\gamma}_i| \]

\[ \leq \frac{48M\eta}{\beta^2} + \frac{2}{\beta} \left( |(G - \bar{G}) \circ \gamma_i| + |G \circ \gamma_i - \bar{G} \circ \bar{\gamma}_i| \right) \]

\[ \leq \frac{48M\eta}{\beta^2} + \frac{2\eta}{\beta} \left( \frac{\beta}{s_0 / 2} \sup_{[-1, 1] \times B_{r_0}} |\gamma_i - \bar{\gamma}_i| \right) \]

\[ \leq \frac{48M\eta}{\beta^2} + \frac{2\eta}{\beta} + \frac{8M\eta}{\beta^2 s_0^2} (2 + \frac{s_0}{2\pi}) \]

\[ \leq \frac{70M\eta}{\beta^2 s_0^2}, \]

recalling the definition of \( s_* \) in (41).

Noting that

\[ \partial_\theta \tilde{G}_i = (-1)^i \frac{\theta_i - \tilde{\theta}_{i-1}}{E_i - E_{i-1}} \partial_\theta G \circ \bar{\gamma}_i, \quad \partial_\theta \tilde{G}_i = (-1)^i \frac{(\tilde{\theta}_i - \tilde{\theta}_{i-1})^2}{2(E_i - E_{i-1})} \partial_\theta G \circ \bar{\gamma}_i, \]

by (26) and (33)

\[ |\partial_\theta \tilde{G}_i| + |\partial_\theta \tilde{G}_i| \geq \frac{\theta_*}{M} |\partial_\theta G \circ \bar{\gamma}_i| + \frac{\theta_*^2}{M} |\partial_\theta G \circ \bar{\gamma}_i| \geq \frac{\theta_*^2}{\beta M} \left( |\partial_\theta G \circ \bar{\gamma}_i| + |\partial_\theta G \circ \bar{\gamma}_i| \right) \]

\[ \geq \frac{\theta_*^2 \beta}{3\pi M} \cdot \frac{\beta^2 s_0^3}{\pi M} = \frac{\beta^2 s_0^3}{3\pi M^2}. \]

Similarly by

\[ \partial_\theta \tilde{G}_i = (-1)^i \frac{\theta_i - \theta_{i-1}}{E_i - E_{i-1}} \partial_\theta G \circ \gamma_i, \quad \partial_\theta \tilde{G}_i = (-1)^i \frac{(\theta_i - \theta_{i-1})^2}{2(E_i - E_{i-1})} \partial_\theta G \circ \gamma_i, \]

and by (55), (56) we get

\[ |\partial_\theta \tilde{G}_i| + |\partial_\theta \tilde{G}_i| \geq \frac{\theta_*}{2M} |\partial_\theta G \circ \gamma_i| + \frac{\theta_*^2}{4M} |\partial_\theta G \circ \gamma_i| \]

\[ \geq \frac{\theta_*^2}{2\pi M} \left( |\partial_\theta G \circ \gamma_i| + |\partial_\theta G \circ \gamma_i| \right) \]

\[ \geq \frac{\theta_*^2 \beta}{4\pi M} = \frac{\beta^2 s_0^3}{12\pi M^2}. \]
Note that by (31) and (36) we get
\[ \tilde{M} \geq 4, \quad \tilde{\beta} \leq \frac{8}{3\pi} < 1 \]  
and
\[ \frac{12\pi\tilde{\beta}}{M} \leq 1, \quad \frac{3\pi^2\tilde{\beta} \tilde{s}}{M} \leq 1, \quad \frac{3\pi^3\tilde{\beta}^2 \tilde{s}^2}{M} \leq 1, \quad \frac{3\pi^4\tilde{\beta}^3 \tilde{s}^3}{M} \leq 1. \]  

**Lemma 4.6** For every \( 1 \leq i \leq 2N \),
\[ |\partial_{\theta} \tilde{G}_i(\pm 1 + \theta)| \geq \frac{\tilde{\beta}}{8} |\theta|, \quad \forall |\theta| \leq \tilde{\theta}_z := \frac{\tilde{s}^3}{2^9 M} < \min\left\{ \frac{\tilde{s}}{8}, \frac{1}{216} \right\}, \]  
\[ \min_{[-1+\theta_0,1-\theta_0]} |\partial_{\theta} \tilde{G}_i| \geq \frac{\tilde{\beta} \bar{\theta}_0}{8}, \quad \forall 0 \leq \bar{\theta}_0 \leq \tilde{\theta}_z \]  
\[ \inf_{[-1+\theta_0,1-\theta_0]} |\partial_{\theta} \tilde{G}_i| \geq \frac{\tilde{\beta} \bar{\theta}_0}{16}, \quad \bar{\theta}_1 := \frac{\tilde{s}^2 \bar{\theta}_0}{2^7 M} < \frac{1}{211} \tilde{\theta}_0. \]  

In particular
\[ \frac{1}{|\partial_{\theta} \tilde{G}_i(\tilde{\theta})|} \leq \frac{8}{\tilde{\beta}} \left( \frac{2}{\tilde{\theta}_z} + \frac{1}{|1 - \tilde{\theta}|} + \frac{1}{|1 + \tilde{\theta}|} \right), \quad \forall \tilde{\theta} \in [-1,1], \quad \tilde{\theta}_* := \frac{\tilde{s}^2 \tilde{\theta}_0}{2^7 M} < \frac{1}{211} \tilde{\theta}_z. \]  

Finally
\[ |\partial_{\theta} \tilde{G}_i^*(\theta, \hat{I})| = |\partial_{\theta} \tilde{G}_i(\theta, \hat{I}) - \partial_{\theta} \tilde{G}_i(\theta)| \leq \frac{2^{11} \tilde{M}}{\tilde{\beta}^2 \tilde{s}^4} \tilde{\eta} |\partial_{\theta} \tilde{G}_i(\theta)|, \quad \forall \theta \in [-1,1], \quad \hat{I} \in \hat{D}_{r_0}. \]  

**Proof** We will consider only the case\(^{12}\) \( i = 2j \). For\(^{13}\) \( |\theta| \leq \tilde{\theta}_z < \tilde{s}/2 \) we get by Cauchy estimates
\[ |\partial_{\theta} \tilde{G}(\pm 1 + \theta)| \geq |\partial_{\theta} \tilde{G}(\pm 1)||\theta| - \frac{48\tilde{M}}{s^3} |\theta|^2 \geq \frac{\tilde{\beta}^2}{4} |\theta| - \frac{48\tilde{M}}{s^3} |\theta|^2 \geq \frac{\tilde{\beta}}{8} |\theta|. \]

Regarding the minimum of \( \partial_{\theta} \tilde{G} \) in the interval\(^{14}\) \([-1 + \tilde{\theta}_0, 1 - \tilde{\theta}_0]\) if it is achieved in an inner point \( \tilde{\theta}_* \), then \( \partial_{\theta_0} \tilde{G}(\tilde{\theta}_*) = 0 \) and, by (72), \( \partial_{\theta_0} \tilde{G}(\tilde{\theta}_*) \geq \tilde{\beta} \); otherwise the minimum is

\(^{12}\) The case \( i = 2j - 1 \) is analogous.
\(^{13}\) The case \(-\tilde{\theta}_z \leq \theta \leq 0\) is analogous.
\(^{14}\) Note that on such interval \( \partial_{\theta} \tilde{G} > 0 \).
achieved at the endpoints and (78) follows from (77).

By (78), (70) and Cauchy estimates we have that for every \( \theta \in [-1 + \tilde{\theta}_0, 1 - \tilde{\theta}_0] \) and \( |\tilde{\theta}| \leq \tilde{\theta}_1 \)

\[
|\partial_{\theta} \tilde{G}_i(\theta + \tilde{\theta})| \geq \frac{\tilde{\beta} \tilde{\theta}_0}{8} - \frac{8 \tilde{M}}{s^2} \tilde{\theta}_1 \geq \frac{\tilde{\beta} \tilde{\theta}_0}{16},
\]

showing (79).

(80) directly follows from (77) and (79) taking \( \tilde{\theta}_0 = \tilde{\theta}^\sharp \), namely defining

\[\tilde{\theta}^\star := \frac{\tilde{\beta} \tilde{s}^2 \tilde{\theta}^\sharp}{2^7 \tilde{M}} = \frac{\tilde{\beta}^2 \tilde{s}^5}{2^{16} \tilde{M}^2} < \frac{1}{2^{11} \tilde{\theta}^\sharp}.\]

Let us finally prove (81). By (79) it follows that

\[
\inf_{[-1 + \tilde{\theta}_z/2, 1 - \tilde{\theta}_z/2]_{\tilde{\theta}_z}} |\partial_{\theta} \tilde{G}_i| \geq \frac{2 \tilde{\beta} \tilde{s}^3}{2^{10} \tilde{M}}. \tag{82}
\]

Recalling (68) by Cauchy estimates and (71), (77) we get

\[
|\partial_{\theta} \tilde{G}_i^*(\pm 1 + \theta, \hat{I})| \leq \frac{8 \tilde{\eta}}{s^2} \tilde{\theta} \leq \frac{64 \tilde{\eta}}{\tilde{\beta} \tilde{s}^2} |\partial_{\theta} \tilde{G}_i^*(\pm 1 + \theta)|, \quad \forall |\theta| \leq \tilde{\theta}_z, \quad \hat{I} \in \hat{D}_{r_0}.
\]

For every \( \theta \in [-1 + \tilde{\theta}_z/2, 1 - \tilde{\theta}_z/2]_{\tilde{\theta}_z} \) and \( \hat{I} \in \hat{D}_{r_0} \) by Cauchy estimates and (71) we get

\[
|\partial_{\theta} \tilde{G}_i^*(\theta, \hat{I})| \leq \frac{2 \tilde{\eta}}{s} \leq \frac{2^{11} \tilde{M}}{\tilde{\beta}^2 \tilde{s}^4} \tilde{\eta} |\partial_{\theta} \tilde{G}_i^*(\theta)|
\]

proving (81). \( \square \)

### 4.3 Inverting the rescaled unperturbed potential

Define, for \( \rho > 0 \), the complex sets\(^{15}\)

\[
\mathbb{C}_* := \{ z \in \mathbb{C} \mid \text{Im} \, z = 0 \implies \text{Re} \, z > 0 \}, \quad \Omega_\rho := [-1, 1]_\rho \cap (\mathbb{C}_* - 1) \cap (1 - \mathbb{C}_*). \tag{83}
\]

We define the square root and the (natural) logarithm on \( \mathbb{C}_* \) in order to have \( \sqrt{\hat{I}} = 1 \) and \( \ln 1 = 0 \).

We want to invert the function \( \tilde{g}_i \) solving

\[
\tilde{g}_i(\theta) = \tilde{E}. \tag{84}
\]

\(^{15}\) In particular \( (\mathbb{C}_* - 1) \cap (1 - \mathbb{C}_*) = \{ z \in \mathbb{C} \mid \text{Im} \, z = 0 \implies -1 < \text{Re} \, z < 1 \}.\)
Lemma 4.7  Let
\[ \tilde{r}_0 := \frac{\beta^4 s^6}{233 M^3} = \frac{s_0^{18}}{24834\pi^{10}} \left( \frac{\beta}{M} \right)^{11} < \frac{1}{244} . \] (85)

There exists a holomorphic function \( \tilde{\Theta}_i \) defined on \( \Omega_{\tilde{r}_0} \) such that \( x = \tilde{\Theta}_i(\tilde{E}) \) solves (84), namely
\[ \tilde{g}_i(\tilde{\Theta}_i(\tilde{E})) = \tilde{E} \] (86)
and
\[ \sup_{\Omega_{\tilde{r}_0}} |\tilde{\Theta}_i| \leq 2 . \] (87)

Moreover there exist two holomorphic functions \( \tilde{\Theta}_{i,+}, \tilde{\Theta}_{i,-} \), defined on \( B_{\sqrt{\tilde{r}_0}}(0) \) with
\[ \sup_{B_{\sqrt{\tilde{r}_0}}(0)} |\tilde{\Theta}_{i,\pm}| \leq \frac{2}{\sqrt{\tilde{r}_0}} \] (72)

and
\[ \sup_{B_{\sqrt{\tilde{r}_0}}(0)} \inf_{\tilde{r}_{1/4}} |\tilde{\Theta}_{i,\pm}| \geq \frac{\tilde{s}}{2\sqrt{M}} \] (88)

such that
\[ \tilde{\Theta}_i(\tilde{E}) = \begin{cases} \pm 1 \mp \sqrt{1 \mp \tilde{E}} \tilde{\Theta}_{i,\pm}(\sqrt{1 \mp \tilde{E}}) \end{cases} , \quad \text{when} \quad \mp 1 \mp \tilde{E} \in \mathbb{C} \cap B_{\tilde{r}_0}(0) . \] (90)

Moreover, on the real, \( \tilde{\Theta}_i \) is bijective from \([-1, 1]\) to itself and is strictly increasing, resp. decreasing, if \( i \) is even, resp. odd; in particular \( \tilde{\Theta}_{2j}(\mp 1) = \tilde{\Theta}_{2j-1}(\pm 1) = \mp 1 \).

Finally
\[ |\tilde{\Theta}_i(\tilde{E}) \pm (-1)^i| \geq \frac{\tilde{s}}{64} \sqrt{\frac{\tilde{r}_0}{M}} \sqrt{|\tilde{E} \pm 1|} , \quad \forall \tilde{E} \in \Omega_{\tilde{r}_1} , \quad \tilde{r}_1 := \frac{\tilde{s}}{2^{10}} \sqrt{\frac{\tilde{r}_0^3}{M}} \] (91)

and, for \( \rho \leq \tilde{r}_0/2 \),
\[ \tilde{\Theta}_i(\Omega_{\rho}) \subseteq [-1, 1]_{s(\rho)} , \quad \text{where} \quad s(\rho) := \max \left\{ \frac{4\rho^{1/3}}{\beta^{1/2}}, \frac{4\rho}{\tilde{r}_0} \right\} . \] (92)

\[ ^{16}\text{Actually they are defined on the larger ball of radius } r_*/2, \text{ with } r_* \text{ defined in (104).} \]
Proof We consider only the case $i = 2j$, the case $i = 2j - 1$ being analogous. We start inverting equation (84) for $x$ and $\tilde{E}$ close to $-1$. Set

$$\tilde{c}_{2j} := \partial_{\psi_n\psi_n} \tilde{g}_{2j}(-1)/2 > 0.$$ 

Note that by Cauchy estimates and (72)

$$\frac{\tilde{\beta}}{2} \leq \tilde{c}_{2j} \leq \frac{\tilde{M}}{\tilde{s}^2}. \quad (93)$$

We have

$$\tilde{g}_{2j}(-1 + \theta) - \tilde{g}_{2j}(-1) \overset{(66)}{=} \tilde{g}_{2j}(-1 + \theta) + 1 =: \theta^2 \hat{g}_{2j}(\theta), \quad (94)$$

for a suitable function $\hat{g}_{2j}$ analytic for $|\theta| < \tilde{s}$. By definition

$$\hat{g}_{2j}(\theta) = \frac{\tilde{g}_{2j}(-1 + \theta) + 1}{\theta^2}$$

and, by Cauchy estimates,

$$|\hat{g}_{2j}(\theta) - \tilde{c}_{2j}| \leq \frac{8\tilde{M}}{\tilde{s}^3} |\theta|, \quad \forall |\theta| \leq \tilde{s}/2, \quad (95)$$

then, recalling (93),

$$\sup_{|\theta| \leq \tilde{s}/2} |\hat{g}_{2j}(\theta)| \leq \frac{5\tilde{M}}{\tilde{s}^2}. \quad (96)$$

Moreover

$$\hat{g}_{2j}(0) = \tilde{c}_{2j}. \quad (97)$$

By (94), (84) is equivalent, in the variable $\theta = x + 1$, to

$$\theta^2 \hat{g}_{2j}(\theta) = \tilde{g}_{2j}(-1 + \theta) + 1 = \tilde{E} + 1. \quad (98)$$

We define the square root $\sqrt{\cdot}$ on $\mathbb{C}_*$ defined in (83) such that it coincide with the positive square root on the positive reals, namely if $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$, then $\sqrt{z} := \sqrt{r}e^{i\theta/2}$, so that $\text{Re} \sqrt{z} > 0$. Thus for $\tilde{E} + 1 \in \mathbb{C}_*$ we can define $\sqrt{\tilde{E} + 1}$. Set

$$\tilde{\rho} := \frac{\tilde{\beta}\tilde{s}^3}{2\tilde{M}}. \quad (99)$$
and note that, by (76),
\[ \tilde{\rho} \leq \min \left\{ \frac{\tilde{s}}{3\pi^{3/2}}, \frac{1}{3\pi^{4/2}} \right\}. \] (100)

Then
\[ |\theta| \leq \tilde{\rho} \quad \Rightarrow \quad |\hat{G}_{2j}(\theta) - \hat{c}_{2j}| \leq \frac{8M}{\tilde{s}^3} |\theta| \leq \frac{\tilde{\beta}}{24} \leq \frac{1}{8} \hat{c}_{2j} \]
\[ \Rightarrow \quad \text{Re} \hat{G}_{2j}(\theta) \geq \frac{1}{2} \hat{c}_{2j} \quad \Rightarrow \quad \hat{G}_{2j}(\theta) \in \mathbb{C}_{*}. \] (101)

Thus for every $|\theta| \leq \tilde{\rho}$ we can define the holomorphic function $\sqrt{\hat{G}_{2j}(\theta)}$. Let us consider now the equation
\[ f(\theta) := \theta \sqrt{\hat{G}_{2j}(\theta)} = w. \] (102)
Setting
\[ T := 1/\partial_\theta f(0) = 1/\sqrt{\hat{G}_{2j}(0)} = 1/\sqrt{\hat{c}_{2j}}. \]
recalling (97). If the smallness condition
\[ \sup_{|\theta| \leq \tilde{\rho}} |1 - T \partial_\theta f(\theta)| = \sup_{|\theta| \leq \tilde{\rho}} \left| 1 - \frac{\theta \partial_\theta \hat{G}_{2j}(\theta)}{2 \sqrt{\hat{G}_{2j}(\theta)}} \right| \leq \frac{1}{2} \] (103)
is satisfied, then by a quantitative Inverse Function Theorem there exists an analytic function $g(w)$ defined for
\[ |w| \leq r_* := \frac{\tilde{\rho}}{2|T|} = \frac{1}{2} \tilde{\rho} \sqrt{\hat{c}_{2j}}. \] (104)
such that $g(0) = 0$ and $\theta = g(w)$ satisfies equation (102). In order to prove (103) we first note that by (93) and (101) we have
\[ |\hat{G}_{2j}(\theta)| \geq \frac{\tilde{\beta}}{4} \quad \text{and} \quad |\sqrt{\hat{G}_{2j}(\theta)} - \sqrt{\hat{G}_{2j}(0)}| \leq \frac{1}{\sqrt{\tilde{\beta}}} |\hat{G}_{2j}(\theta) - \hat{G}_{2j}(0)| \leq \frac{\sqrt{\tilde{\beta}}}{24}, \quad \forall |\theta| \leq \tilde{\rho}. \] (105)
Thus we get, recalling (93),
\[ \left| 1 - \frac{\sqrt{\hat{G}_{2j}(\theta)}}{\sqrt{\hat{G}_{2j}(0)}} \right| = \left| \frac{\sqrt{\hat{G}_{2j}(\theta)} - \sqrt{\hat{G}_{2j}(0)}}{\sqrt{\hat{G}_{2j}(0)}} \right| \leq \frac{1}{2^{3/2}} \]
and by Cauchy estimates (96), (100), (101) and (99)

\[ \left| \frac{\theta \partial_\theta \hat{G}_{2j}(\theta)}{2\sqrt{\hat{c}_{2j}}/\sqrt{\hat{\bar{c}}_{2j}(\theta)}} \right| \leq \frac{20\sqrt{2}\tilde{\rho}M}{\tilde{\beta}3^3} = \frac{5\sqrt{2}}{2^5}, \]

thus condition (103) is satisfied. Then we find an analytic function \( g(w) \) defined for \( |w| \leq r_* \) with

\[ \sup_{|w| \leq r_*} |g(w)| \leq \tilde{\rho}, \quad (106) \]

such that \( \theta = g(w) \) solves (102). Then we can define

\[ \theta(E) := g(\sqrt{\tilde{E} + 1}), \quad \forall |\tilde{E} + 1| \leq r_*^2, \quad \tilde{E} + 1 \in \mathbb{C}_*, \quad (107) \]

solving

\[ \theta(E)\sqrt{\hat{G}_{2j}(\theta(E))} = \sqrt{\tilde{E} + 1} \]

and, squaring,

\[ \theta^2(E)\hat{G}_{2j}(\theta(E)) = \tilde{E} + 1 \]

We note that we can write

\[ g(w) = w\tilde{g}(w) \]

for a suitable analytic function \( \tilde{g} \) defined for \( |w| \leq r_*/2 \) with

\[ \sup_{|w| \leq r_*/2} |\tilde{g}(w)| \leq \frac{\tilde{\rho}}{2r_*}. \quad (108) \]

Indeed, by (106) and Cauchy estimate,

\[ \sup_{|w| \leq r_*/2} |g'(w)| \leq \frac{\tilde{\rho}}{2r_*} \]

and, recalling that \( g(0) = 0 \), we get

\[ |g(w)| \leq \frac{\tilde{\rho}}{2r_*}|w|, \quad \forall |w| \leq r_*/2, \]

proving (108).

Moreover we claim that

\[ g([0,r_*/8]) \supset [0,\tilde{\rho}/24]. \quad (109) \]

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Indeed we first prove that
\[ \tilde{g}(r_*/8) \geq \frac{2}{3\sqrt{\tilde{c}_{2j}}} \],
(110)
from which we get
\[ g(r_*/8) \geq \frac{r_*}{12\sqrt{\tilde{c}_{2j}}} = \frac{\tilde{\rho}}{24} \]
and (109) follows. Let us prove (110). By (108) and Cauchy estimates we get
\[ \sup_{|w| \leq r_*/8} |\tilde{g}'(w)| \leq \frac{4\tilde{\rho}}{3r_*^2}, \]
then, noting that
\[ \tilde{g}(0) = g'(0) = 1/\sqrt{\tilde{c}_{2j}(0)} = 1/\sqrt{\tilde{c}_{2j}}, \]
we get
\[ \tilde{g}(r_*/8) \geq \frac{1}{\sqrt{\tilde{c}_{2j}}} - \sup_{|w| \leq r_*/8} |\tilde{g}'(w)| \frac{r_*}{8} = \frac{1}{\sqrt{\tilde{c}_{2j}}} - \frac{\tilde{\rho}}{6r_*} = \frac{2}{3\sqrt{\tilde{c}_{2j}}}, \]
proving (110) (and (109)).

Choosing \( \tilde{\Theta}_{2j,-} := \tilde{g} \) and recalling (104) and (93) we get the first estimate in (88) in the case \( E \) close to -1, the case close to +1 is analogous. The second estimate in (88) follows from (89), (87), (85), (75), (76) and Cauchy estimates.

Now we consider the case of \( E \) far away from \( \pm 1 \). Thanks to (109) (and the analogous estimate in the case we are close to +1), it remains to invert \( 17 \tilde{g}(\theta) = \tilde{c}_{2j}(\theta) \) for\(^{17} \)
\[ x \in [-1 + \tilde{\rho}/24, 1 - \tilde{\rho}/24]. \]

First of all we claim that
\[ \partial_\theta \tilde{G}(\theta) \geq m := \frac{1}{2\tilde{\beta}} \tilde{\beta} \tilde{\rho}, \quad \forall \theta \in [-1 + \tilde{\rho}/24, 1 - \tilde{\rho}/24]. \]
(111)

In order to prove (111) we note that, by (100), \( m \leq \tilde{\beta}/8 \). Then if, by contradiction, there exists \( \tilde{\theta} \in [-1 + \tilde{\rho}/24, 1 - \tilde{\rho}/24] \) with \( \partial_\theta \tilde{G}(\tilde{\theta}) < m \), then by (72) we have \( |\partial_{\theta\theta} \tilde{G}(\tilde{\theta})| \geq \tilde{\beta}/2 \). To fix ideas we consider the case \( \partial_{\theta\theta} \tilde{G}(\tilde{\theta}) \leq -\tilde{\beta}/2 \). Then, recalling that \( \tilde{G} \) is strictly increasing, we get for \( x \geq \tilde{\theta} \),
\[ 0 < \partial_\theta \tilde{G}(\theta) < m - \frac{\tilde{\beta}}{2}(\theta - \tilde{\theta}) + \frac{\tilde{\beta}}{6} \frac{\tilde{\beta}}{2} \]

---

\(^{17}\)Skipping for brevity the \( 2j \) subscript from now on.

\(^{18}\)Recall (100).
and, choosing \( x := \bar{\theta} + \bar{\rho}/2^5 \), we have

\[
0 < m - \frac{\bar{\beta} \bar{\rho}}{2^6} + \frac{3 \bar{M} \bar{\rho}^2}{2^{10} s^3} \quad (99) \leq \left( -\frac{1}{2^7} + \frac{3}{2^{17}} \right) \bar{\beta} \bar{\rho} < 0 ,
\]

which is a contradiction, proving (111).

Then, fixing \( \bar{x} \in [-1 + \bar{\rho}/24, 1 - \bar{\rho}/24] \), we want to apply the Inverse Function Theorem in the ball \( |x - \bar{x}| \leq \rho_1 \) where

\[
\rho_1 := \frac{\bar{s} \bar{r} \rho}{2^{11} M} \quad (99) \leq \frac{\bar{s} \bar{r} \rho}{2^{18} M^2} \leq \min \left\{ \frac{\bar{\rho}}{2^{11}} \cdot \frac{s}{3^3}, \frac{s}{9 \pi^6 2^{18} \bar{M}}, \frac{1}{9 \pi^7 2^{18}} \right\} , \quad (112)
\]

recalling (76). We have, by Cauchy estimates

\[
\sup_{|x - \bar{x}| \leq \rho_1} \left| 1 - \frac{\partial \bar{G}(\theta)}{\partial \bar{G}(\bar{x})} \right| \leq 8 \bar{M} \rho_1 \leq \frac{\bar{s} \bar{r} \rho}{2^{8} m} \leq \frac{1}{2} \quad (111)
\]

We incidentally note that, by (111) and Cauchy estimates, we get

\[
|\partial \bar{G}(\theta)| \geq \frac{m}{2} = \frac{\bar{s} \bar{r} \rho}{2^8} , \quad \forall \ x \in [-1 + \bar{\rho}/24, 1 - \bar{\rho}/24]_{\rho_1} . \quad (113)
\]

Set\(^{19}\)

\[
\bar{r}_0 = \frac{m^2 \bar{s}^2}{2^8 \bar{M}} = \frac{\bar{\rho}^2 \bar{s}^2}{2^{19} \bar{M}} \leq \frac{\bar{s}^4 \bar{r}^6}{2^{33} \bar{M}^3} , \quad \bar{E}_* := \bar{G}(\bar{x}) .
\]

Then by the Inverse Function Theorem there exists a holomorphic function

\[
\bar{\Theta} : \{|\bar{E} - \bar{E}_*| < \bar{r}_0 \} \to \{|x - \bar{x}| \leq \rho_1 \}
\]

inverting \( \bar{G} \). We have by (107), (106) (and recalling that \( \rho_1 \leq \bar{\rho} \))

\[
\sup_{\Omega_{\bar{r}_0}} |\bar{\Theta}| \leq 1 + \bar{\rho} \leq 2 ,
\]

proving (87). Anyway we note that \( \bar{\Theta}_+ , \bar{\Theta}_- \) are actually defined on the larger domain

\[
B_{\bar{r}_*/2}(0) \supset B_{\sqrt{\bar{r}_0}}(0) .
\]

\(^{19}\)Actually we could choose a larger \( \bar{r}_0 \), namely \( \bar{r}_0 := \frac{\rho_1}{2^{11} M} \), where \( T = 1/\partial \bar{G}(\bar{x}) \). Then recall (111).
We now prove (91), showing only the + case, the − case being analogous. We first note that by (90), the second estimate in (88) (and (85)), we have that (91) holds when
\[ \frac{\tilde{r}_0}{2} \leq \tilde{E} \leq \frac{1}{2} \] and \( \tilde{E} \in \mathbb{C}_\ast \cap B_{\bar{r}_1}(0) \). Consider now a point \( \tilde{E} \in \Omega_{\tilde{r}_0/32} \) with \( |\tilde{E} + 1| \geq \tilde{r}_0/16 \); then there exists a real point \( \tilde{E}_0 \in [-1 + \tilde{r}_0/16, 1 - \tilde{r}_0/16] \) such that \( |\tilde{E} - \tilde{E}_0| \geq \tilde{r}_0/24 \). Since the function \( \tilde{\Theta}_{2j} \) is increasing on the real we have that
\[ \tilde{\Theta}_{2j}(\tilde{E}_0) + 1 \geq \tilde{\Theta}_{2j}(-1 + \tilde{r}_0/16) + 1 \geq \sqrt{\frac{\tilde{r}_0}{16} \frac{s}{2\sqrt{M}}} \]
by (90) and the second estimate in (88). Therefore, by (87) and Cauchy estimates, we get
\[ |\tilde{\Theta}_{2j}(\tilde{E}) + 1| \geq \frac{s}{8} \sqrt{\frac{\tilde{r}_0}{M}} - \frac{64\tilde{r}_1}{\tilde{r}_0} \geq \frac{s}{16} \sqrt{\frac{\tilde{r}_0}{M}} \] (114)
Then (91) follows noting that \( \sqrt{|\tilde{E} + 1|} \leq 4 \).

We finally prove (92). By (88), (90) and noting that \( \rho < 1 \) by (85), we have that
\[ 1 \pm \tilde{E} \in \mathbb{C}_\ast \cap B_{\rho^{2/3}}(0) \supset \mathbb{C}_\ast \cap B_{\rho}(0) \implies 1 \pm \tilde{\Theta}_{2j}(\tilde{E}) \in B_{\rho^{2/3} \sqrt{2\bar{r}}}(0) \] (114)
Take now \( \tilde{E} \in \Omega_{\rho} \) but \( |\tilde{E} + 1| \geq \rho^{2/3} \); then
\[ d := \text{dist}(\tilde{E}, \partial \Omega_{\rho}) \geq \min\{\rho^{2/3}, \tilde{r}_0/2\} \]
and by (87) and Cauchy estimates
\[ |\tilde{\Theta}_{2j}(\tilde{E}) - \tilde{\Theta}_{2j}(\text{Re} \tilde{E})| \leq \frac{2}{d} |\text{Im} \tilde{E}| \leq \frac{2\rho}{\bar{r}} \leq s(\rho) , \]
since \( \tilde{\bar{r}} < 1 \) by (75). The above estimate and (114) conclude the proof of (92).

4.4 Inverting the rescaled perturbed potential
We now find \( \tilde{\Theta}_i \), the inverse of \( \tilde{G}_i \).

---

20Indeed the following result holds: if, for \( 0 < r \leq 1/2 \), \( z \in \Omega_r \) but \( |z + 1| \geq r \) then there exists a real \( z_0 \in [-1 + r, 1 - r] \) such \( |z - z_0| \leq \sqrt{2 - \sqrt{3}} r \leq 2r/3 \).
Lemma 4.8 Assume that \( \tilde{\eta} \) in (71) satisfies
\[
\tilde{\eta} \leq \frac{\beta^2 s^2 \rho_1^2}{211 M} = \frac{\beta^6 s^{12}}{247 M^5}.
\]
Then, for every \( 1 \leq i \leq 2N \), there exists an analytic function \( \chi_i(\theta, \hat{I}) \), with\(^{21}\)
\[
\sup_{[-1, +1] \times \partial \mathcal{D}_r} |\chi_i| \leq \frac{16}{\sqrt{\beta} \rho_1} \cdot \sup_{[-1, +1] \times \partial \mathcal{D}_r} |\hat{G}_i^*| \leq \frac{16\tilde{\eta}}{\sqrt{\beta} \rho_1} \quad \text{and} \quad \chi_i(\pm 1, \hat{I}) = 0,
\]
such that the analytic function\(^{22}\)
\[\tilde{\Theta}_i(\tilde{E}, \hat{I}) := \Theta_i(\tilde{E}) + \chi_i(\tilde{E}, \hat{I}), \quad \tilde{E} \in \Omega_{r_0}, \quad \hat{I} \in \mathcal{D}_{r_0},\]
solves
\[\tilde{G}_i(\tilde{E}, \hat{I}) = \tilde{E}.
\]
Moreover\(^{23}\)
\[\tilde{\Theta}_i(\tilde{E}, \hat{I}) = (-1)^i \left( \pm 1 \mp \sqrt{1 \mp \tilde{E} \Theta_i(\pm 1 \mp \tilde{E}, \hat{I})} \right), \quad \text{when} \quad \pm \tilde{E} \in \mathbb{C} \cap B_{r_0}(0),
\]
where
\[\tilde{\Theta}_i(\mp 1, \hat{I}) := \tilde{\Theta}_i(\mp y \Theta_i(\pm 1, \hat{I}))\]
with
\[\tilde{x}_i(\mp z, \hat{I}) := \mp \frac{(\mp 1)^i}{z} \chi_i(\mp 1) \mp \frac{(\mp 1)^i}{z} + \mp z, \hat{I} \).
\]
In particular \( \tilde{\Theta}_{2j}(\mp 1, \hat{I}) = \tilde{\Theta}_{2j-1}(\mp 1, \hat{I}) = \mp 1 \). Finally the following estimates hold
\[
\sup_{|z| < \rho_1, \hat{I} \in \partial \mathcal{D}_{r_0}} |\hat{x}_i(\pm 1, \hat{I})| \leq \frac{32\tilde{\eta}}{\sqrt{\beta} \rho_1^2} \leq \frac{1}{6},
\]
and
\[
\sup_{|y| < \sqrt{\eta} \rho_1, \hat{I} \in \partial \mathcal{D}_{r_0}} |\hat{\Theta}_i(\pm 1, \hat{I})| \leq \frac{2}{\sqrt{\beta}}.
\]
\(^{21}\)\( \rho_1 \) and \( \tilde{G}_i^* \) where defined in (112) and (67), respectively. Recall also (71).
\(^{22}\)\( \tilde{\Theta}_i \) was defined in Lemma (4.7). \( \Omega_{r_0} \) was defined in (83) and (85).
\(^{23}\)Recalling the definition of \( \tilde{r}_0 \) in (85).
\(^{24}\)Note that \( \tilde{\chi}_i(\pm 1, \hat{I}) = 0 \).
Proof Let us introduce some notations. For $s > 0$, let $\tilde{B}^s$ the Banach space of analytic function $\chi(\theta, \tilde{I})$ on the complex neighborhood $\chi : [-1, +1]_s \times \tilde{D}_{r_0} \rightarrow \mathbb{C}$, with bounded sup-norm

$$|\chi|_s := \sup_{[-1, +1]_s \times \tilde{D}_{r_0}} |\chi|$$

and such that $\chi(\pm 1, \tilde{I}) = 0$. Let $\bar{B}^s$ be the (closed) subspace of $\tilde{B}^s$ with $\partial \chi(\pm 1, \tilde{I}) = 0$. For $\rho > 0$, let $\bar{B}^s_{\rho}$, resp. $\hat{B}^s_{\rho}$ be the closed ball of $\tilde{B}^s$, resp. $\hat{B}^s$, with center 0 and radius $\rho$. Recalling the definition of $\rho_1$ in (112), let us consider the two parameters $\rho_2, r_2 \geq 0$ with

$$\rho_2 \leq \frac{\tilde{s}^2}{28\tilde{M}} \leq \frac{1}{8}\rho_1 \leq \frac{1}{215} \rho^2 \overset{(100)}{\leq} \frac{1}{237} \tilde{s}^2, \quad r_2 := \frac{\tilde{\beta} \rho_1}{16} \rho_2 \leq \frac{\tilde{s}^2}{211\hat{M}}. \quad (125)$$

Let us define the function

$$\mathcal{F}_i : \tilde{B}^s_{\rho_1} \times \hat{B}^s_{r_2} \rightarrow \hat{B}^s_{\rho_1},$$

$$(\mathcal{F}_i(\chi, \tilde{G})) (\theta, \tilde{I}) := \tilde{g}_i(\theta + \chi(\theta, \tilde{I})) - \tilde{G}_i(\theta) + \tilde{G}(\theta + \chi(\theta, \tilde{I}), \tilde{I}), \quad (126)$$

where $\tilde{g}_i$ was defined in (65). In the following we will often drop the index $i$ for brevity, writing $\mathcal{F}, \tilde{g}_i$, instead of $\mathcal{F}_i, \tilde{g}_i$. Note that $\mathcal{F}$ is well defined since

$$\sup_{[-1, +1]_{\rho_1} \times \tilde{D}_{r_0}} |\text{Im} (\theta + \chi(\theta, \tilde{I}))| \leq \rho_1 + \rho_2 \leq \frac{\tilde{s}}{4} \quad (127)$$

by (112), (100) and (125). We want to find $\chi = \chi(\tilde{G})$ that solves the implicit function $\mathcal{F}(\chi, \tilde{G}) = 0$, for $\chi$ and $\tilde{G}$ small since $\mathcal{F}(0, 0) = 0$. Consider the functional

$$\chi \rightarrow \tilde{\chi} := \partial_{\chi} \mathcal{F}(0, 0)[\chi]$$

defined as $\tilde{\chi}(\theta) = \partial_{\tilde{G}} \tilde{g}(\theta) \chi(\theta)$. Then for a given $\tilde{\chi} \in \tilde{B}^\rho_{\rho_1}$ we have that

$$\chi = (\partial_{\chi} \mathcal{F}(0, 0))^{-1}[\tilde{\chi}] =: T\tilde{\chi} \quad (128)$$

given by

$$\chi(\theta) := \frac{\tilde{\chi}(\theta)}{\partial_{\tilde{G}} \tilde{g}(\theta)} \quad (129)$$

We have to show that the above expression is well defined for $\tilde{\chi} \in \hat{B}^\rho_{\rho_1}$. We first claim that

$$\sup_{[-1, +1]_{\rho_1} \cap \{x+1<\rho\}} |\chi(\theta)| \leq \frac{8|\tilde{\chi}|_{\rho_1}}{\tilde{\beta} \rho_1} \quad (130)$$
We will prove (130) only in the case \( \check{G} = \check{G}_{2j} \), the case \( \check{G} = \check{G}_{2j-1} \) being analogous. Recalling the definition of \( \hat{G} = \hat{G}_{2j} \) in (94) and setting \( \theta := x + 1 \), we have that

\[
\partial_{\theta} \check{G}(\theta) = \partial_{\theta} \check{G}(\theta - 1) = 2\theta \hat{G}(\theta) + \theta^2 \partial_{\theta} \hat{G}(\theta) .
\]

By (105), (100), Cauchy estimates and (99), we get for \( |\theta| < \check{\rho} \)

\[
|2\hat{G}(\theta) + \theta \partial_{\theta} \hat{G}(\theta)| \geq 2|\hat{G}(\theta)| - |\theta||\partial_{\theta} \hat{G}(\theta)| \geq \frac{\check{\beta}}{2} - \rho \frac{20\check{M}}{\check{s}^3} \geq \frac{\check{\beta}}{4} .
\]

Then for every \( |x + 1| = |\theta| < \check{\rho} \),

\[
|\partial_{\theta} \check{G}(\theta)| = |\partial_{\theta} \check{G}(\theta - 1)| \geq |\theta| \frac{\check{\beta}}{4} . \tag{131}
\]

Therefore, recalling (129), we get

\[
\sup_{[-1,1]_{\rho_1} \cap \{ \frac{\rho_1}{2} \leq |x + 1| < \check{\rho} \}} |\chi(\theta)| \leq \frac{|\check{\chi}|_{\rho_1}}{(\rho_1/2)(\check{\beta}/4)} = \frac{8|\check{\chi}|_{\rho_1}}{\check{\beta} \rho_1} . \tag{132}
\]

Recalling that \( \check{\chi}(-1) = 0 \), by Cauchy estimates we get

\[
|\check{\chi}(\theta - 1)| \leq |\theta| \sup_{|\theta| \leq \rho_1/2} |\check{\chi}(\theta - 1)| \leq |\theta| \frac{2|\check{\chi}|_{\rho_1}}{\rho_1} , \quad \forall |\theta| \leq \frac{\rho_1}{2} .
\]

Then, by (131) we get

\[
\sup_{[-1,1]_{\rho_1} \cap \{ |x - 1| \leq \frac{\rho_1}{2} \}} |\chi(\theta)| \leq \frac{8|\check{\chi}|_{\rho_1}}{\check{\beta} \rho_1} . \tag{133}
\]

By (132) and (133) we finally get (130). Moreover, since \( \check{\chi}(-1) = \partial_{\theta} \check{\chi}(-1) = 0 \) by (131) (recall (129)) we have that \( \chi(-1) = 0 \). Analogously we can show that

\[
\sup_{[-1,1]_{\rho_1} \cap \{ |x - 1| < \check{\rho} \}} |\chi(\theta)| \leq \frac{8|\check{\chi}|_{\rho_1}}{\check{\beta} \rho_1} \tag{134}
\]

and \( \chi(1) = 0 \). Moreover, by (113)

\[
\sup_{[-1+\check{\rho}/24,1-\check{\rho}/24]_{\rho_1}} |\chi| \leq \frac{2^8}{\check{\beta} \check{\rho}} |\check{\chi}|_{\rho_1} . \tag{135}
\]
Noting that by\(^{25}\) (112)
\[-1, 1]_{\rho_1} \subset [-1 + \rho/24, 1 - \rho/24]_{\rho_1} \cup \{|x + 1| < \rho\} \cup \{|x - 1| < \rho\},
we have that, by (135), (130), (134) (and (112)), the expression in (129) is well defined, 
\(\chi \in \tilde{B}^{\rho_1}\) and, moreover, we have
\[
\sup_{[-1, 1]_{\rho_1}} |\chi| = |\chi|_{\rho_1} \leq \frac{8|\tilde{\chi}|_{\rho_1}}{\beta_{\rho_1}}, \quad \text{i.e.}
\]
\[
|\left(\partial_{\chi} F(0, 0)\right)^{-1}|_{L(\tilde{B}^{\rho_1}, \tilde{B}^{\rho_1})} = |T|_{L(\tilde{B}^{\rho_1}, \tilde{B}^{\rho_1})} \leq \frac{8}{\beta_{\rho_1}}. \tag{136}
\]
We now apply the Implicit Function Theorem in Banach spaces. First we note that, since \(\rho_1 \leq \tilde{s}/4\), we have
\[
\sup_{|\tilde{G}| \leq \tilde{r}_2} |\tilde{G}|_{\tilde{s}} \leq |F(0, \tilde{G})|_{\rho_1} \leq r_2 \tag{125}, \tag{136}
\]
for \(\chi \in \tilde{B}^{\rho_1}\) set
\[
R := \partial_{\theta} \tilde{G}(\theta + \chi) - \partial_{\theta} \tilde{G}(\theta)
\]
and note that, by Cauchy estimates, (127) and (125), we get
\[
|R|_{\rho_1} \leq \frac{8\tilde{M}}{\tilde{s}^2} |\chi|_{\rho_1} \leq \frac{8\tilde{M} \rho_2}{\tilde{s}^2} \leq \frac{\tilde{\beta}_{\rho_1}}{2^5}
\]
and, for \(\tilde{G} \in \tilde{B}^{\tilde{s}}_{\tilde{r}_2}\), also
\[
|\partial_{\theta} \tilde{G}(\theta + \chi)|_{\rho_1} \leq \frac{2r_2}{\tilde{s}} = \frac{\tilde{s} \rho_1}{8\tilde{s}} \leq \frac{\tilde{\beta}_{\rho_1}}{2^{40}}.
\]
Then, by (136), we obtain
\[
\sup_{\tilde{B}^{\rho_1}_{\tilde{r}_2} \times \tilde{B}^{\tilde{s}}_{\tilde{r}_2}} |Id - T\partial_{\chi} F(\chi, \tilde{G})|_{L(\tilde{B}^{\rho_1}, \tilde{B}^{\rho_1})}
= \sup_{\chi \in \tilde{B}^{\rho_1}} |\chi' - T\left(\partial_{\theta} \tilde{G}(\theta + \chi) + \partial_{\theta} \tilde{G}(\theta + \chi)\right)\chi'|_{\rho_1}
\leq |T||R|_{\rho_1} + |\partial_{\theta} \tilde{G}(\theta + \chi)|_{\rho_1}
\leq \frac{8}{\tilde{s} \rho_1} \left(\frac{\tilde{s} \rho_1}{2^5} + \frac{\tilde{s} \rho_1}{2^{40}}\right) \leq \frac{1}{2}. \tag{138}
\]
\(^{25}\)In particular \(|\rho/24 + i\rho_1| < \rho\).
Since estimates (137) and (138) are satisfied we can apply the Implicit Function The-orem finding, for every $\tilde{G} \in \tilde{B}_{r_2}^\delta$ a function $\chi = \chi_i \in \tilde{B}_{\rho_1}^\rho$ solving (126), namely $\mathcal{F}_i(\chi_i, \tilde{G}) = 0$. In particular the theorem can be applied with $\tilde{G} := \tilde{G}_i^\ast$ defined in (67). Indeed, by (68), $\tilde{G}_i^\ast \in \tilde{B}_2^\delta$. Moreover choosing $r_2$ in (125) as

$$r_2 := \sup_{[-1,1] \times \hat{D}_r} |\tilde{G}_i^\ast|,$$

we have that the conditions in (125) are satisfied by assumption (115) (recall (71)). Then the (first) estimate in (116) directly follows by (125).

Then we have $F_i(\chi_i, \tilde{G}_i^\ast) = 0$, which is equivalent to

$$\tilde{G}_i(\theta + \chi_i(\theta, \hat{I}), \hat{I}) = \tilde{G}_i(\theta).$$

Evaluating at $x = \tilde{\Theta}_i(\hat{E})$ we get (recall (86)) (118).

We now prove (123). Recalling (122), we distinguish the case $\rho_1/2 \leq |z| < \rho_1$, where we directly obtain (123), and the case $|z| < \rho_1/2$ where we have, uniformly for $\hat{I} \in \hat{D}_r$,

$$|\hat{\chi}_{i,\pm}(z, \hat{I})| \leq \sup_{[-1,1] \times \hat{D}_r} |\partial_{\theta}\chi_i| \leq \frac{16\tilde{\eta}}{\beta \rho_1} \frac{1}{\rho_1/2},$$

recalling that $\hat{\chi}_{i,\pm}(\pm 1, \hat{I}) = 0$ and by Cauchy estimates. The first inequality in (123) is proved. The second one follows noting that

$$\frac{32\tilde{\eta}}{\beta \rho_1^2} \leq \frac{\beta \delta^2}{26M} \leq 2\tilde{\delta} \leq \frac{1}{6^4}.$$

Finally (124) follows by (120), (123) and (88).

In the following we will assume the condition\footnote{Recall (76).}

$$\tilde{\eta} \leq \frac{\beta^9 s^18}{2^{73}M^8}.$$

(139)

**Remark 4.1** Note that, recalling (69)-(72), condition (139) is implied by\footnote{The two conditions are equivalent w.r.t. the dependence on the parameter, we choose the smaller constant $2^{-197}$ in (140) for brevity.}

$$\eta \leq \frac{\beta^{28} s_{10}^4 s_{2}^2}{2^{197}M^{27}},$$

(140)

which follows by (40).
Lemma 4.9 Assume (139). Set\(^{28}\)
\[
\tilde{r} := \frac{\tilde{\beta}^{10}s^{19}}{2^{88}M^{9}} = \frac{s_{0}^{9} \beta^{29}}{2^{134}3^{10} \pi^{20} M^{29}}, \quad \rho_{\ast} := \frac{\tilde{\beta}^{6}s^{13}}{2^{49}M^{6}} = \frac{s_{0}^{31} \beta^{18}}{2^{76}3^{6} \pi^{19} M^{18}}. \tag{141}
\]
Then
\[
\tilde{E} \in \Omega_{\tilde{r}} \implies \tilde{\Theta}_{i}(\tilde{E}), \tilde{\Theta}_{i}(\tilde{E}, \hat{I}) \in [-1, 1]_{\rho_{\ast}/2}, \quad \forall \hat{I} \in \hat{D}_{\rho_{0}}. \tag{142}
\]

**Proof** First we note that, recalling (85),
\[
\frac{4\tilde{r}}{\tilde{r}_{0}} = \frac{\rho_{\ast}}{2}\tag{143}
\]
(which implies \(\tilde{r} \leq \tilde{r}_{0}/2\)) and
\[
\frac{4\tilde{r}^{1/3}}{\sqrt{\beta}} \geq \frac{4\tilde{r}}{\tilde{r}_{0}}
\]
by (75). Then by (92) we prove that, actually,
\[
\tilde{\Theta}_{i}(\tilde{E}) \in [-1, 1]_{\rho_{\ast}/2}. \tag{144}
\]
Then by (115),(117) and (116) we get \(\tilde{\Theta}_{i}(\tilde{E}, \hat{I}) \in [-1, 1]_{\delta_{s}/2}\) since
\[
\frac{2^{22}M^{2}}{\beta^{3} \tilde{s}^{5}} \tilde{\eta} \leq \frac{\rho_{\ast}}{4},
\]
by (139) and (76). \(\blacksquare\)

Lemma 4.10 Assume (139). Then
\[
|\tilde{\partial}_{\theta}\tilde{G}_{i}(\theta + \chi_{i}(\theta)) - \tilde{\partial}_{\theta}\tilde{G}_{i}(\theta)| \leq \frac{2^{71}M^{8}}{\beta^{3} \tilde{s}^{18}} \tilde{\eta} |\tilde{\partial}_{\theta}\tilde{G}_{i}(\theta)|, \quad \forall \theta \in [-1, +1]_{\rho_{\ast}}. \tag{145}
\]

**Proof** We start considering a neighborhood of \(\pm 1\). We apply Lemma 7.5 with \(f \sim \tilde{\partial}_{\theta}\tilde{G}_{i}, \chi \sim \chi_{i}, 0 \sim \pm 1, r \sim \rho_{1}\) (defined in (112)), \(M_{2} \sim \frac{2M}{\tilde{s}}\) (recall (70)), \(\eta \sim \frac{16\tilde{M}}{\beta \rho_{1}}\) (recall (116)), \(\rho \sim \rho_{3}\), where, recalling (72),
\[
\rho_{3} := \frac{\rho_{2}^{2} \tilde{s}}{32M} = \frac{\tilde{\beta}^{5} \tilde{s}^{11}}{2^{41}M^{5}}. \tag{146}
\]
\(^{28}\)Recall (69)-(72)
By assumption\(^{29}\) (139) we can apply Lemma 7.5 obtaining
\[
|\partial_b \tilde{G}_i(\theta + \chi_i(\theta)) - \partial_b \tilde{G}_i(\theta)| \leq \frac{32}{\tilde{\beta}\rho_1\rho_3} \tilde{\eta}|\partial_b \tilde{G}_i(\theta)|, \quad \forall |\theta \pm 1| < \rho_3. \quad (145)
\]
Applying (78) with\(^{30}\) \(\tilde{\theta}_0 \rightsquigarrow \rho_3/2\)
\[
\inf_{[-1+\rho_3/2,1-\rho_3/2]_\rho_3} |\partial_b \tilde{G}_i| \geq \frac{\tilde{\beta}\tilde{\theta}_0}{16}.
\]
Then by (70), Cauchy estimates (116)
\[
|\partial_b \tilde{G}_i(\theta + \chi_i(\theta)) - \partial_b \tilde{G}_i(\theta)| \leq \frac{2^7 \tilde{M}}{\tilde{\beta}^2 \rho_1} \tilde{\eta} \leq \frac{2^{12} \tilde{M}}{\tilde{\beta}^2 \rho_1 \rho_3} \tilde{\eta} |\partial_b \tilde{G}_i(\theta)|
\]

Note that by (76), (116) and (139)
\[
\rho_* \leq \tilde{\theta}_*/2^{57} \quad \text{and} \quad \sup_{[-1+1,\tilde{\theta}_*/16]_\rho_3} |\chi_i| \leq \frac{\tilde{\theta}_*}{32},
\]
then (recall (81))
\[
\theta \in [-1,1]_{\rho_*} \implies \theta + \chi_i(\theta) \in [-1,1]_{\tilde{\theta}_*/16}. \quad (146)
\]
We claim that
\[
|\partial_b \tilde{G}_1(\theta + \chi_i(\theta), \hat{I}) - \partial_b \tilde{G}_i(\theta)| \leq \frac{2^{72} \tilde{M}^8}{\tilde{\beta}^6 \tilde{g}^{18}} \tilde{\eta} |\partial_b \tilde{G}_i(\theta)|, \quad \forall \theta \in [-1,1]_{\rho_*}, \quad \hat{I} \in \hat{D}_{\rho_0}. \quad (147)
\]

\(^{29}\)Which implies condition \(\eta \leq r/8\) of Lemma 7.5.
\(^{30}\)Note that the condition \(\tilde{\theta}_0 \leq \tilde{\theta}_2\) is satisfied by (76). Note also that
\[
\rho_* := \frac{3\tilde{\beta}^2 \rho_3}{2^8 M} < \frac{1}{2^{12} \rho_3}.
\]
Indeed by (146), (81) and (143) we have
\[
|\partial_\theta \tilde{G}_i(\theta + \chi_i(\theta), \hat{I}) - \partial_\theta \tilde{G}_i(\theta)| \\
\leq |\partial_\theta \tilde{G}_i(\theta + \chi_i(\theta), \hat{I}) - \partial_\theta \tilde{G}_i(\theta + \chi_i(\theta))| + |\partial_\theta \tilde{G}_i(\theta + \chi_i(\theta)) - \partial_\theta \tilde{G}_i(\theta)| \\
\leq \frac{2^{11} \tilde{M}}{\beta^2 \hat{g}^4} |\partial_\theta \tilde{G}_i(\theta + \chi_i(\theta))| + \frac{2^{17} \tilde{M}^8}{\beta^9 \hat{g}^{18}} |\partial_\theta \tilde{G}_i(\theta)| \\
\leq \left(1 + \frac{\tilde{g}^{14}}{2^{60} \hat{M}^7} + \frac{2^{11} \tilde{M}}{\beta^2 \hat{g}^4} \right) \frac{2^{17} \tilde{M}^8}{\beta^9 \hat{g}^{18}} |\partial_\theta \tilde{G}_i(\theta)| \\
\leq \frac{2^{72} \tilde{M}^8}{\beta^9 \hat{g}^{18}} |\partial_\theta \tilde{G}_i(\theta)|.
\]

By (139) and (147) we also have
\[
\frac{1}{2} |\partial_\theta \tilde{G}_i(\theta)| \leq |\partial_\theta \tilde{G}_i(\theta + \chi_i(\theta), \hat{I})| \leq 2 |\partial_\theta \tilde{G}_i(\theta)|, \quad \forall \theta \in [-1, 1], \hat{I} \in \hat{D}_r.
\]

### 4.5 Inverting the original potentials

Recalling (65) we set
\[
\Theta_i := \bar{\gamma}_i \circ \tilde{\Theta}_i \circ \bar{\lambda}_i, \quad \Theta_i := \gamma_i \circ \tilde{\Theta}_i \circ \lambda_i,
\]

respectively. Note that for \( \hat{I} \in \hat{D} \) (namely \( \hat{I} \) real)
\[
\Theta_{2j}(\cdot, \hat{I}) : [E_{2j-1}(\hat{I}), E_{2j}(\hat{I})] \to [\theta_{2j-1}(\hat{I}), \theta_{2j}(\hat{I})], \\
\Theta_{2j-1}(\cdot, \hat{I}) : [E_{2j-1}(\hat{I}), E_{2j-2}(\hat{I})] \to [\theta_{2j-2}(\hat{I}), \theta_{2j-1}(\hat{I})],
\]

moreover \( \Theta_i \) is increasing, resp. decreasing (as a function of \( \theta \)), if \( i \) is even, resp. odd. Note also that
\[
\partial_\theta \Theta_i(E, \hat{I}) = 1/\partial_\theta \tilde{G}(\Theta_i(E, \hat{I}), \hat{I})
\]

and
\[
\Theta_{2j-1}(E_{2j-2}(\hat{I}), \hat{I}) = \theta_{2j-2}(\hat{I}), \quad \Theta_{2j}(E_{2j-1}(\hat{I}), \hat{I}) = \theta_{2j-1}(\hat{I}), \\
\Theta_{2j}(E_{2j-1}(\hat{I}), \hat{I}) = \theta_{2j-1}(\hat{I}), \quad \Theta_{2j}(E_{2j}(\hat{I}), \hat{I}) = \theta_{2j}(\hat{I}).
\]
Regarding the derivatives we have by (44) and (45)

\[ \partial_j \theta_i (\hat{I}) = - \frac{\partial_{gi} G(\theta_i (\hat{I}), \hat{I})}{\partial_{gin} G(\theta_i (\hat{I}), \hat{I})} \]  

(154)

and

\[ \partial_i E_i (\hat{I}) = \partial_i G(\theta_i (\hat{I}), \hat{I}) - \frac{\partial_{gi} G(\theta_i (\hat{I}), \hat{I})}{\partial_{gin} G(\theta_i (\hat{I}), \hat{I})} \partial_{gi} G(\theta_i (\hat{I}), \hat{I}) . \]  

(155)

Now, recalling that \( \tilde{\Theta}_{i,\pm} \) (defined in (120)) are holomorphic in \( B_{\sqrt{\rho}_0}(0) \times \hat{D}_{\rho_0} \) (see (124)) and noting that

\[ |y| < \frac{1}{2} \sqrt{\rho_0} \beta \leq \sqrt{\frac{\rho_0}{2}} \sqrt{E_{2j}(\hat{I}) - E_{2j-1}(\hat{I})}, \]

we can define, for

\[ |y| < \frac{1}{2} \sqrt{\rho_0} \beta = \frac{\rho_0}{2} \frac{s_0^9 \beta^6}{2^{25} \xi^5 M^{11/2}} =: r_0 , \quad \hat{I} \in \hat{D}_{\rho_0} , \]

the holomorphic functions

\[ \Theta_{i,\pm}(y, \hat{I}) := \frac{\theta_i (\hat{I}) - \theta_{i-1}(\hat{I})}{\sqrt{2} \sqrt{(-1)^i (E_i(\hat{I}) - E_{i-1}(\hat{I}))}} \tilde{\Theta}_{i,\pm} \left( \frac{\sqrt{2}}{\sqrt{(-1)^i (E_i(\hat{I}) - E_{i-1}(\hat{I}))}} y, \hat{I} \right) \]  

(157)

By (124) and (55) we get

\[ \sup_{|y| < r_0, \hat{I} \in \hat{D}_{\rho_0}} |\Theta_{i,\pm}(y, \hat{I})| \leq \frac{4\pi}{\sqrt{\beta \rho}} = \frac{4\sqrt{3}\pi M}{s_0^{3/2} \beta^{3/2}}. \]  

(158)

**Lemma 4.11** The following equalities hold\(^{31}\)

\[ \Theta_{2j-1}(E, \hat{I}) = \theta_{2j-1}(\hat{I}) - \sqrt{E - E_{2j-1}(\hat{I})} \Theta_{2j-1, -} \left( \sqrt{E - E_{2j-1}(\hat{I})}, \hat{I} \right) , \]

\[ \Theta_{2j-1}(E, \hat{I}) = \theta_{2j-2}(\hat{I}) + \sqrt{E_{2j-2}(\hat{I}) - E} \Theta_{2j-1, +} \left( \sqrt{E_{2j-2}(\hat{I}) - E}, \hat{I} \right) , \]

\[ \Theta_{2j}(E, \hat{I}) = \theta_{2j-1}(\hat{I}) + \sqrt{E - E_{2j-1}(\hat{I})} \Theta_{2j, -} \left( \sqrt{E - E_{2j-1}(\hat{I})}, \hat{I} \right) , \]

\[ \Theta_{2j}(E, \hat{I}) = \theta_{2j}(\hat{I}) - \sqrt{E_{2j}(\hat{I}) - E} \Theta_{2j, +} \left( \sqrt{E_{2j}(\hat{I}) - E}, \hat{I} \right) . \]  

(159)

\(^{31}\)Where they are meaningful, namely, e.g., in the third equality \( E - E_{2j-1}(\hat{I}) \in \mathbb{C}_+, |E - E_{2j-1}(\hat{I})| < r_0^2 \) and \( \hat{I} \in \hat{D}_{\rho_0} \).
Moreover
\[
\Theta_{2j,-}(0, \hat{I}) = \sqrt{\frac{2}{\partial_{\theta\theta} G(\theta_{2j-1}(\hat{I}), \hat{I})}} = \Theta_{2j-1,-}(0, \hat{I}),
\]
\[
\Theta_{2j,+}(0, \hat{I}) = \sqrt{-\frac{2}{\partial_{\theta\theta} G(\theta_{2j}(\hat{I}), I)}} = \Theta_{2j-1,+}(0, \hat{I})
\]
and
\[
\Theta_{2j,-}(y, \hat{I}) = \Theta_{2j-1,-}(-y, \hat{I}), \quad \Theta_{2j,+}(y, \hat{I}) = \Theta_{2j-1,+}(-y, \hat{I}).
\]

Finally, for every \( \hat{I} \in \hat{D}_{\nu} \)
\[
|\Theta_{i,\pm}(0, \hat{I})| \geq \frac{s_0}{2\sqrt{M}}
\]
and
\[
\sup_{|y| \leq r_{\nu}, \hat{I} \in \hat{D}_{\nu}} \frac{1}{|\Theta_{i,\pm}(y, \hat{I})|} \leq \frac{4\sqrt{M}}{s_0},
\]
where
\[
r_{\dagger} := \frac{s_0^{3/2} \beta^{15/2}}{247M^{1/2}} \overset{(31)}{=} \frac{r_{\circ}}{2^{30}}.
\]

**Proof** The equalities in (159) follows by (149), (119) and (157).
(160) follows by (159) and (150). For example, in order to prove the first\(^{32}\) equality in (160), we insert the third expression in (159) into the second equality in (150) obtaining\(^{33}\)
\[
0 = \left( G(\theta_{2j-1}(\hat{I}) + y \Theta_{2j,-}(y, \hat{I})) - E \right) \frac{1}{y^2} = -1 + \frac{1}{2} \partial_{\theta\theta} G(\theta_{2j-1}(\hat{I}), \hat{I})(\Theta_{2j,-}(y, \hat{I}))^2 + O(y)
\]
where \( y \) is short for \( \sqrt{E - E_{2j-1}(\hat{I})} \). Then we conclude taking \( y \to 0 \).

We finally prove the first equality in (161). Let us fix \( \hat{I} \) and, for brevity, let us omit to write it. Again let \( y \) be short for \( \sqrt{E - E_{2j-1}} \). Inserting the first and the third equalities in (159) into (150) we get
\[
G(\theta_{2j-1} - y \Theta_{2j,-}(y)) = y^2 + E_{2j-1} = G(\theta_{2j-1} + y \Theta_{2j,-}(y)),
\]
\(^{32}\)The other being analogous.
\(^{33}\)Developing in Taylor expansion w.r.t. \( y \).
In particular we have that \( \mathcal{G}(\theta_{2j-1} - y \Theta_{2j-1,-}(y)) \) is an even function of \( y \). Then
\[
\mathcal{G}(\theta_{2j-1} + y \Theta_{2j-1,-}(y)) = y^2 + E_{2j-1} = \mathcal{G}(\theta_{2j-1} + y \Theta_{2j-1,-}(y)) .
\]
(165)

Now we claim that for \( y \) close to 0 there exists a unique function \( g(y) \) with Re \( g(0) > 0 \) such that
\[
\mathcal{G}(\theta_{2j-1} + yg(y)) = y^2 + E_{2j-1} .
\]
(166)

In order to prove the claim we define the analytic function
\[
\tilde{\mathcal{G}}(\theta) := (\mathcal{G}(\theta_{2j-1} + \theta) - E_{2j-1})/\theta^2 ,
\]
and note that
\[
\text{Re } \tilde{\mathcal{G}}(\theta) = \text{Re } \frac{1}{2} \partial_{\theta\theta} f(\theta_{2j-1}) > 0 .
\]
(167)

Then, for \( y \) close to 0, it is well defined the function \( \sqrt{\tilde{\mathcal{G}}(yg(y))} \). We can rewrite (166) as
\[
g^2(y) \tilde{\mathcal{G}}(yg(y)) = y^2
\]
and the above equation has a unique solution \( g(y) \) with Re \( g(0) > 0 \). Indeed, taking the square root, it is equivalent to
\[
g(y) \sqrt{\tilde{\mathcal{G}}(yg(y))} = y ,
\]
which, by the implicit function theorem and since \( \sqrt{\tilde{\mathcal{G}}(0)} \neq 0 \) (by (167)), has a unique solution \( g(y) \) with Re \( g(0) > 0 \). So we have proved that there exists a unique function \( g(y) \) with Re \( g(0) > 0 \) solving (166). Then by (165) we conclude that \( \Theta_{2j,-}(y) = \Theta_{2j-1,-}(-y) \) noting that by (160) (and our definition of the square root)
\[
\text{Re } \Theta_{2j,-}(0) = \text{Re } \Theta_{2j-1,-}(0) > 0 .
\]

Finally recalling that by (48) \( |\theta_i(\hat{I}) - \tilde{\theta}_i| \leq s_0/8 \) and that \( \tilde{\theta}_i \in \mathbb{R} \), by Cauchy estimates and (52) we have
\[
|\partial_{\theta\theta} \mathcal{G}(\theta_i(\hat{I}), \hat{I})| \leq \frac{4M}{(7s_0/8)^2} \leq \frac{8M}{s_0^2} .
\]

Then by (160) we get (162). Finally by (158), Cauchy estimates, (162) and (164) we get (163).

**Remark 4.2** Note that \( M, \beta, \eta, E, \mathcal{G}, \) etc., respectively \( r_0, y, \hat{I} \), have the same “physical” dimension of an energy, respectively of an action (the square root of an energy). The parameters with “~” are adimensional.
5 The action variables

Remark 5.1 As well known, in the proof of Arnold–Liouville’s theorem, one first introduces the actions through line integrals

\[ I_n^{(i)}(E, \hat{I}) := \frac{1}{2\pi} \oint_{H_{\text{mech}}^{-1}(E; \hat{I})} p_n dq_n \]  

(168)

as functions of energy \( E \) and then defines the integrated Hamiltonian \( E^{(i)} \) inverting such functions; in particular, one has

\[ I_n^{(i)} \left( E^{(i)}(\hat{I}, I_n), \hat{I} \right) = I_n \]  

(169)

Indeed, all the fine analytic properties of \( E^{(i)} \) will be described through the functions \( E \rightarrow I_n^{(i)}(E, \hat{I}) \).

5.1 Actions for the unperturbed Hamiltonian

Recall (27). For \( 1 \leq j \leq N \) set

\[ j_0 := j - 1 \quad \text{if} \quad \bar{E}_{2j - 2} < \bar{E}_{2j} \quad \text{and} \quad j_\circ := j \quad \text{otherwise}. \]  

(170)

Note that

\[ \bar{E}_{2j_0} = \min\{\bar{E}_{2j - 2}, \bar{E}_{2j}\}. \]

For \( 1 \leq j < N \) set

\[ j_- := \max\{i < j \; \text{s.t.} \; \bar{E}_{2i} > \bar{E}_{2j}\}, \quad j_+ := \min\{i > j \; \text{s.t.} \; \bar{E}_{2i} > \bar{E}_{2j}\} \]

\[ j_* = j_- \quad \text{if} \quad \bar{E}_{2j_-} < \bar{E}_{2j_*} \quad \text{and} \quad j_* = j_+ \quad \text{otherwise}. \]  

(171)

Note that

\[ \bar{E}_{2j} < \bar{E}_{2j_*} = \min\{\bar{E}_{2j_-}, \bar{E}_{2j_*}\}. \]  

(172)

Let us set

\[ \bar{E}_i^{(i)} := \bar{E}_i, \quad E^{(i)}_i(\hat{I}) := E_i(\hat{I}), \quad \text{for} \quad 1 \leq i \leq 2N, \]

\[ \bar{E}^{(2j-1)}_\pm := \bar{E}_{2j}, \quad E^{(2j-1)}_\pm(\hat{I}) := E_{2j}(\hat{I}), \quad \text{for} \quad 1 \leq j \leq N, \]

\[ E^{(2j)}_\pm := E_{2j}, \quad E^{(2j)}_\pm(\hat{I}) := E_{2j}(\hat{I}), \quad \text{for} \quad 1 \leq j < N. \]  

(173)
Let us define the functions

\[ I_{n}^{(2j-1)}(E) := \frac{1}{\pi} \int_{\bar{\Theta}_{2j-1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad \bar{E}_{-}^{(2j-1)} \leq E \leq \bar{E}_{+}^{(2j-1)}, \quad 1 \leq j \leq N, \]

\[ I_{n}^{(2j)}(E) := \frac{1}{\pi} \int_{\Theta_{2j+1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad \bar{E}_{-}^{(2j)} \leq E \leq \bar{E}_{+}^{(2j)}, \quad 1 \leq j \leq N - 1, \]

\[ I_{n}^{(2N)}(E) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{E - G(\theta)} \, d\theta, \quad E \geq \bar{E}_{-}^{(2N)} \]

\[ I_{n}^{(0)}(E) := -I_{n}^{(2N)}(E) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{E - G(\theta)} \, d\theta, \quad E \geq \bar{E}_{-}^{(2N)}. \quad (174) \]

**Remark 5.2** We want to show that the functions in (174) have an analytic extension for complex \( E \). Moreover, while it is immediate to evaluate the derivative of the last two, namely

\[ \partial_{E} \bar{I}_{n}^{(2N)}(E) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{E - G(\theta)}} \, d\theta, \]

\[ \partial_{E} \bar{I}_{n}^{(0)}(E) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{E - G(\theta)}} \, d\theta, \quad (175) \]

it is not obvious to justify the formal derivation\(^{34}\)

\[ \partial_{E} \bar{I}_{n}^{(2j-1)}(E) = \frac{1}{2\pi} \int_{\Theta_{2j-1}(E)} \frac{1}{\sqrt{E - G(\theta)}} \, d\theta, \]

\[ \partial_{E} \bar{I}_{n}^{(2j)}(E) = \frac{1}{2\pi} \int_{\Theta_{2j+1}(E)} \frac{1}{\sqrt{E - G(\theta)}} \, d\theta. \quad (176) \]

Set

\[ \bar{I}_{n}^{(2j-1),-}(E) := \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad \bar{E}_{2j-1} \leq E \leq \bar{E}_{2j-2}, \quad 1 \leq j \leq N, \]

\[ \bar{I}_{n}^{(2j-1),+}(E) := \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad \bar{E}_{2j-1} \leq E \leq \bar{E}_{2j}, \quad 1 \leq j \leq N, \]

\[ \bar{I}_{n}^{(2j),-}(E) := \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad E \geq \bar{E}_{2j}, \quad 1 \leq j \leq N, \]

\[ \bar{I}_{n}^{(2j),+}(E) := \frac{1}{\pi} \int_{\Theta_{2j+1}(E)} \sqrt{E - G(\theta)} \, d\theta, \quad E \geq \bar{E}_{2j}, \quad 0 \leq j < N. \quad (177) \]

\(^{34}\) See Remark 5.4 below.
Remark 5.3 Note that in \( \bar{I}_n^{(2j)-}(E) \) the interval of integration does depend on \( E \), while this does not happen in \( \bar{I}_n^{(2j)+}(E) \). This fact makes this last case simpler. In particular \( \bar{I}_n^{(2j)+}(E) \) are defined for every \( E \in \mathbb{C} + \bar{E}_{2j} \).

In view of (174) and (177) we get
\[
\bar{I}_n^{(2j)-}(E) = \bar{I}_n^{(2j-1)-}(E) + \bar{I}_n^{(2j-1)+}(E), \quad \text{for } 1 \leq j \leq N \tag{178}
\]
\[
\bar{I}_n^{(2j)+}(E) = \bar{I}_n^{(2j+1)-}(E) + \bar{I}_n^{(2j+1)+}(E) + \sum_{j-i-j_i} \bar{I}_n^{(2i)-}(E) + \bar{I}_n^{(2i)+}(E),
\]
for \( 1 \leq j \leq N \),
\[
2\bar{I}_n^{(2N)}(E) = \bar{I}_n^{(0)+}(E) + \bar{I}_n^{(2N)-}(E) + \sum_{1 \leq i \leq N-1} \bar{I}_n^{(2i)-}(E) + \bar{I}_n^{(2i)+}(E).
\]

We want to express the functions \( \bar{I}_n^{(i)}(E) \) in terms of the rescaled potentials \( \tilde{G}_i \) (defined in (65)). Given \( G \) as in (25) we set, for \( 1 \leq i \leq 2N \),
\[
\tilde{G}_i := G_{[\bar{\theta}_{i-1}, \bar{\theta}_i]}.
\]
Recalling (65) we get
\[
\tilde{\tilde{G}}_i = \hat{\lambda}_i \circ \tilde{G}_i \circ \tilde{\tau}_i.
\]
Recalling (59), (65), (149) we set
\[
\begin{align*}
\tilde{I}_n^{(2j-1)-}(\tilde{E}) & := \frac{1}{\pi} \int_{\tilde{\theta}_{2j-1}(\tilde{E})}^{1} \sqrt{\tilde{E} - \tilde{G}_{2j-1}(\tilde{\theta})} \ d\tilde{\theta}, \\
\tilde{I}_n^{(2j-1)+}(\tilde{E}) & := \frac{1}{\pi} \int_{\tilde{\theta}_{2j}(\tilde{E})}^{-1} \sqrt{\tilde{E} - \tilde{G}_{2j}(\tilde{\theta})} \ d\tilde{\theta}, \\
\tilde{I}_n^{(2j)-}(\tilde{E}) & := \frac{1}{\pi} \int_{-1}^{-1} \sqrt{\tilde{E} - \tilde{G}_{2j}(\tilde{\theta})} \ d\tilde{\theta}, \\
\tilde{I}_n^{(2j)+}(\tilde{E}) & := \frac{1}{\pi} \int_{-1}^{1} \sqrt{\tilde{E} - \tilde{G}_{2j+1}(\tilde{\theta})} \ d\tilde{\theta}.
\end{align*}
\]
Then

\[
I_n^{(2j-1),-}(E) = \frac{\theta_{2j-1} - \theta_{2j-2}}{2} \sqrt{E_{2j-2} - E_{2j-1}} \frac{1}{2} \tilde{I}_n^{(2j-1),-}(\tilde{\lambda}_{2j-1}(E)),
\]

\[
I_n^{(2j),+}(E) = \frac{\theta_{2j} - \theta_{2j-1}}{2} \sqrt{E_{2j} - E_{2j-1}} \frac{1}{2} \tilde{I}_n^{(2j),+}(\tilde{\lambda}_{2j}(E)),
\]

\[
I_n^{(2j),-}(E) = \frac{\theta_{2j} - \theta_{2j-1}}{2} \sqrt{E_{2j} - E_{2j-1}} \frac{1}{2} \tilde{I}_n^{(2j),-}(\tilde{\lambda}_{2j}(E)),
\]

\[
I_n^{(2j),+}(E) = \frac{\theta_{2j+1} - \theta_{2j}}{2} \sqrt{E_{2j+1} - E_{2j}} \frac{1}{2} \tilde{I}_n^{(2j),+}(\tilde{\lambda}_{2j+1}(E)).
\]

**Lemma 5.1** The functions \( \tilde{I}_n^{(2j-1),\pm}(\tilde{E}) \) in\(^{35}\) (181) has holomorphic extension on \( \Omega_{\tilde{\rho}} \).

Moreover, setting

\[
\tilde{E}(t) := \tilde{E} - (\tilde{E} + 1)t = -t + (1 - t)\tilde{E}, \quad \text{for } 0 \leq t \leq 1,
\]

the following formulas hold:

\[
\tilde{I}_n^{(2j-1),-}(\tilde{E}) = \frac{1}{\pi} \left( \tilde{E} + 1 \right)^{3/2} \sqrt{t} \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j-1} \left( \tilde{\Theta}_{2j-1}(\tilde{E}(t)) \right)} dt,
\]

\[
\tilde{I}_n^{(2j-1),+}(\tilde{E}) = \frac{1}{\pi} \left( \tilde{E} + 1 \right)^{3/2} \sqrt{t} \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j} \left( \tilde{\Theta}_{2j}(\tilde{E}(t)) \right)} dt
\]

(184)

and, for the derivatives,

\[
\partial_{\tilde{E}} \tilde{I}_n^{(2j-1),-}(\tilde{E}) = -\frac{\sqrt{\tilde{E} + 1}}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j-1} \left( \tilde{\Theta}_{2j-1}(\tilde{E}(t)) \right)} dt,
\]

\[
\partial_{\tilde{E}} \tilde{I}_n^{(2j-1),+}(\tilde{E}) = \frac{\sqrt{\tilde{E} + 1}}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j} \left( \tilde{\Theta}_{2j}(\tilde{E}(t)) \right)} dt
\]

(185)

**Proof** Note that \( \tilde{E}(t) \) defined in (183) describes the segment joining \( \tilde{E} \) and \(-1\). Moreover

\[
\tilde{E} \in \Omega_{\rho} \implies \tilde{E}(t) \in \Omega_{\rho}, \quad \forall \rho > 0, \quad \forall 0 \leq t \leq 1.
\]

\(^{35}\tilde{\rho} \) was defined in (141) and recall (83).
Then, for \( \dot{E} \in \Omega_x \) we have that \( \dot{E}(t) \in \Omega_x \) and, recalling Lemma 4.9, we also get that the following changes of variables are well defined:

\[
\begin{align*}
\bar{\theta} &= \bar{\Theta}_{2j-1}(\dot{E}(t)) \in [-1, 1]_{\rho \ast /2}, \\ 0 \leq t \leq 1, \\
\bar{\theta} &= \bar{\Theta}_{2j}(\dot{E}(t)) \in [-1, 1]_{\rho \ast /2}, \\ 0 \leq t \leq 1,
\end{align*}
\] (187)

with \( \rho \ast \) defined in (141). By (86) we get

\[
t = \frac{\ddot{E} - \ddot{\bar{g}}_{2j-1}(\bar{\theta})}{\dot{E} + 1}, \\
\tilde{t} = \frac{\ddot{E} - \ddot{g}_{2j}(\bar{\theta})}{\dot{E} + 1},
\]

and

\[
d\bar{\theta} = -\frac{\ddot{E} + 1}{\partial_\theta \ddot{g}_{2j-1}(\bar{\Theta}_{2j-1}(\dot{E}(t)))} dt \quad \text{or} \quad d\tilde{\theta} = -\frac{\ddot{E} + 1}{\partial_\theta \ddot{g}_{2j}(\bar{\Theta}_{2j}(\dot{E}(t)))} dt,
\]

(deriving (86)). Making the changes of variables (187) in the first and second integral in (181) respectively, we get (184).

We now show that the functions in (184) are holomorphic in \( \Omega_x \) and that the expressions in (185) hold. In order to prove the claim we first note that, for \( t \sim 1 \) (namely \( \dot{E}(t) \sim -1 \))

\[
\bar{\Theta}_{2j}(\dot{E}(t)) = -1 + \sqrt{(1 - t)(1 + \dot{E})} \bar{\Theta}_{2j-1}(\sqrt{(1 - t)(1 + \dot{E})})
\] (188)

by (90). This implies that the function

\[
\dot{E} \mapsto \int_0^1 \frac{\sqrt{t}}{\partial_\theta \ddot{g}_{2j}(\bar{\Theta}_{2j}(\dot{E}(t)))} dt
\] (189)

is holomorphic in \( \Omega_x \), since, for every \( \dot{E}_0 \in \Omega_x \) and every closed ball \( B \) centered in \( \dot{E}_0 \) and contained in \( \Omega_x \), the function

\[
\frac{d}{d\dot{E}} \frac{\sqrt{t}}{\partial_\theta \ddot{g}_{2j}(\bar{\Theta}_{2j}(\dot{E}(t)))} = \frac{\sqrt{t}(t - 1)\partial_\theta \ddot{g}_{2j}(\bar{\Theta}_{2j}(\dot{E}(t)))}{\left(\partial_\theta \ddot{g}_{2j}(\bar{\Theta}_{2j}(\dot{E}(t)))\right)^3}
\]

is dominated\(^{36}\) by an \( L^1 \)-function uniformly on \( B \). Then, by the Lebesgue’s theorem, the function in (189) is holomorphic in \( B \) and we can exchange the derivative with the

\(^{36}\)This follows by (142) and (80). In particular note that the denominator in the r.h.s. vanishes only for \( t = 1 \) where it behaves as \( (1 - t)^{3/2} \) (recall (188)); then the whole function behaves as \( (1 - t)^{-1/2} \), which is in \( L^1 \).
integral obtaining that
\[
\frac{d}{d\tilde{E}} \int_0^1 \frac{\sqrt{t}}{\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))} \, dt = \int_0^1 \frac{\sqrt{t}(t-1)\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))}{\left(\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))\right)^3} \, dt.
\]

As a direct consequence we have that \( \tilde{I}_n^{(2j-1)+} \) in (184) is holomorphic in \( \Omega_\varepsilon \) and
\[
\partial_\tilde{E} \tilde{I}_n^{(2j-1)+}(E) = \frac{1}{2\pi} \int_0^1 \left( \frac{3\sqrt{t}\sqrt{\tilde{E}+1}}{\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))} + \frac{2\sqrt{t}(t-1)(\tilde{E}+1)^{3/2}\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))}{\left(\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))\right)^3} \right) \, dt
\]
\[
+ \frac{\sqrt{\tilde{E}+1}}{\pi} \int_0^1 \frac{1}{\sqrt{t}} \frac{1}{\partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))} \, dt,
\]
where the last integral vanishes by the fundamental theorem of calculus\(^{37}\) proving (the second\(^{38}\) expression in) (185). \(\blacksquare\)

**Definition 5.1** Given \( A \subset \mathbb{C} \) and \( r > 0 \) we define\(^{39}\)
\[
A_{(r)} : \{ z \in \mathbb{C} : z = z_0 + it, \ z_0 \in A, \ |t| < r \}
\]
(190)

**Lemma 5.2** \( \tilde{I}_n^{(2j-1)+}(E) \), respectively \( \tilde{I}_n^{(2j-1)-}(E) \), in (177) has holomorphic extension on \( \tilde{\lambda}_{2j-1}(\Omega_\varepsilon) \), respectively \( \tilde{\lambda}_{2j-1}(\Omega_\varepsilon) \). Moreover
\[
\tilde{\lambda}_{2j-1}(\Omega_{\varepsilon/2}) \supset (E_{2j-1}, E_{2j})_{(r_1)};
\]
\[
\tilde{\lambda}_{2j-1}(\Omega_{\varepsilon/2}) \supset (\tilde{E}_{2j-1}, \tilde{E}_{2j-2})_{(r_1)};
\]
(191)

with
\[
r_1 := \frac{\beta \tilde{x}}{4}.
\]
(192)

\(^{37}\) By the above considerations \( \partial_\tilde{\theta} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t))) \sim \sqrt{1-t} \) for \( t \sim 1 \).

\(^{38}\) The proof of the first one is analogous.

\(^{39}\) i = \(\sqrt{-1}\).
Proof The first part is a direct consequence of (182) and of Lemma 5.1. The inclusions in (191) follow by (59) and (60).

Remark 5.4 Note that making the inverses of the change of variables (187) in the expression (185) we get

\[
\begin{align*}
\partial_E \tilde{I}_{n-1,-}^{(2j-1)}(\tilde{E}) &= \frac{1}{2\pi} \int_{\hat{\theta}_{2j-1}^{-1}(\tilde{E})}^{1} \frac{1}{\sqrt{\tilde{E} - \tilde{G}_{2j-1}(\tilde{\theta})}} d\tilde{\theta}, \\
\partial_E \tilde{I}_{n-1,+}^{(2j-1)}(\tilde{E}) &= \frac{1}{2\pi} \int_{-1}^{\hat{\theta}_{2j}^{-1}(\tilde{E})} \frac{1}{\sqrt{\tilde{E} - \tilde{G}_{2j}(\tilde{\theta})}} d\tilde{\theta}.
\end{align*}
\] (193)

showing also that the formal derivation in (176) is actually correct. Indeed, recalling (149),(180),(59), and making the change of variables \(\hat{\theta} = \gamma_{2j-1}^{-1}(\theta)\) and \(\tilde{\theta} = \gamma_{2j}^{-1}(\theta)\) in the first and in the second integral in (193), respectively, and summing the results, we get (176).

Lemma 5.3 The functions \(I_n^{(2j),\pm}(E)\) are holomorphic for \(E \in \mathbb{C}_* + \tilde{E}_{2j}\). Their derivatives are

\[
\begin{align*}
\partial_E I_n^{(2j),-}(E) &= \frac{1}{2\pi} \int_{\tilde{\theta}_{2j}}^{\hat{\theta}_{2j}} \frac{1}{\sqrt{\tilde{E} - \tilde{G}(\tilde{\theta})}} d\tilde{\theta}, \\
\partial_E I_n^{(2j),+}(E) &= \frac{1}{2\pi} \int_{\hat{\theta}_{2j}}^{\tilde{\theta}_{2j+1}} \frac{1}{\sqrt{\tilde{E} - \tilde{G}(\tilde{\theta})}} d\tilde{\theta}.
\end{align*}
\] (194)

Analogously the functions \(\tilde{I}_n^{(2j),\pm}(\tilde{E})\) are holomorphic for \(\tilde{E} \in \mathbb{C}_* + 1\). Their derivatives are

\[
\begin{align*}
\partial_E \tilde{I}_n^{(2j),-}(\tilde{E}) &= \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{\sqrt{\tilde{E} - \tilde{G}_{2j}(\tilde{\theta})}} d\tilde{\theta}, \\
\partial_E \tilde{I}_n^{(2j),+}(\tilde{E}) &= \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{\sqrt{\tilde{E} - \tilde{G}_{2j+1}(\tilde{\theta})}} d\tilde{\theta}.
\end{align*}
\] (195)

Proof We simply note that for \(E \in \mathbb{C}_* + \tilde{E}_{2j}\) we have that

\[E - \tilde{G}(\theta) \in \mathbb{C}_*, \quad \forall \theta_{2j-1} \leq \theta \leq \theta_{2j} \quad \text{and} \quad \forall \theta_{2j+1} \leq \theta \leq \theta_{2j} \]
in particular \( \min_{\tilde{\theta}_{2j-1} \leq \theta \leq \tilde{\theta}_{2j}} |E - \tilde{G}(\theta)| > 0 \) and \( \min_{\tilde{\theta}_{2j+1} \leq \theta \leq \tilde{\theta}_{2j}} |E - \tilde{G}(\theta)| > 0 \). So we can derive inside the integral obtaining (194). The case of \( \tilde{f}^{(2j)\pm}_n(\tilde{E}) \) is analogous.

**Lemma 5.4** For \( \tilde{E} \in C_* + 1 \) the following formulas hold:

\[
\tilde{f}^{(2j)\pm}_n(\tilde{E}) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{\tilde{E} - y}}{\partial_y \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y))} \, dy,
\]

\[
\tilde{f}^{(2j)\pm}_n(\tilde{E}) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{\tilde{E} - y}}{\partial_y \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y))} \, dy.
\]

(196)

and, for the derivatives,

\[
\partial_{\tilde{E}} \tilde{f}^{(2j)\pm}_n(\tilde{E}) = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{\sqrt{\tilde{E} - y} \, \partial_y \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y))} \, dy,
\]

\[
\partial_{\tilde{E}} \tilde{f}^{(2j)\pm}_n(\tilde{E}) = -\frac{1}{2\pi} \int_{-1}^{1} \frac{1}{\sqrt{\tilde{E} - y} \, \partial_y \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y))} \, dy.
\]

(197)

**Proof** Recalling Lemma 4.9, for \( y \in \Omega_{\tilde{E}} \) the following changes of variables are well defined:

\[
\tilde{\theta} = \tilde{\Theta}_{2j}(y) \in [-1, 1]_{\rho^\prime/2},
\]

\[
\tilde{\theta} = \tilde{\Theta}_{2j+1}(y) \in [-1, 1]_{\rho^\prime/2},
\]

(198)

with \( \rho^\prime \) defined in (141). Note that \( \tilde{\Theta}_{2j}(\pm 1) = \tilde{\Theta}_{2j+1}(\mp 1) = \pm 1 \) (recall Lemma 4.7). Recalling (86) we get

\[
d\tilde{\theta} = \frac{1}{\partial_y \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y))} \, dy \quad \text{or} \quad d\tilde{\theta} = \frac{1}{\partial_y \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y))} \, dt,
\]

(deriving (86)). Making the changes of variables (198) in the third and fourth integral in (181) respectively, we get\(^{40}\) (196). Note that if \( \tilde{E} \in C_* + 1 \) and \( -1 \leq y \leq 1 \) then \( \tilde{E} - y \in C_* \), so that \( \sqrt{\tilde{E} - y} \) is well defined. Deriving (196) we get (197). \( \blacksquare \)

\(^{40}\)One can easily control that the integrals in (196) absolutely converge.
Set
\[ c_\theta := \min_{1 \leq i \leq 2N-1} \inf_{E \in (E_{-i}^{(i)}, E_{+i}^{(i)})} \partial_E I_n^{(i)}(E). \] (199)

**Lemma 5.5** We have that
\[ \partial_E I_n^{(2j-1),+}(E) \geq \frac{s_0}{2\pi \sqrt{M}}, \quad \forall \bar{E}_{2j-1} < E < \bar{E}_{2j-1+1} \] (200)

and
\[ \partial_E I_n^{(2j),+}(E) \geq \frac{1}{2\pi \sqrt{E - E_{2j-1}}} \sqrt{\frac{\beta s_0^3}{3M}}, \quad \forall E > \bar{E}_{2j}. \] (201)

As a consequence
\[ c_\theta \geq \frac{\sqrt{\beta s_0^{3/2}}}{32M}. \] (202)

Moreover
\[ \partial_E I_n^{(2N)}(E) \geq \frac{1}{2\sqrt{E + M}}, \quad \forall E > \bar{E}_{2N}. \] (203)

**Proof** First we note that, since
\[ \bar{G}(\theta_{2j-1} + \theta) - \bar{G}(\theta_{2j-1}) = \bar{G}(\theta_{2j-1} + \theta) - \bar{E}_{2j-1} \leq \frac{M}{s_0^2} \theta^2 \]
for every \( \theta \), then
\[ \bar{\Theta}_{2j}(E) - \bar{\theta}_{2j-1}, \bar{\theta}_{2j-1} - \bar{\Theta}_{2j-1}(E) \geq \frac{s_0}{\sqrt{M}} \sqrt{E - \bar{E}_{2j-1}}. \] (204)

Therefore\(^{41}\)
\[ \partial_E I_n^{(2j-1),+}(E) = \frac{1}{2\pi} \int_{\bar{\theta}_{2j-1}}^{\bar{\Theta}_{2j}(E)} \frac{1}{\sqrt{E - \bar{G}(\theta)}} d\theta \geq \frac{1}{2\pi} \int_{\bar{\theta}_{2j-1}}^{\bar{\Theta}_{2j}(E)} \frac{1}{\sqrt{E - E_{2j-1}}} d\theta \geq \frac{s_0}{2\pi \sqrt{M}} \]
for \( \bar{E}_{2j-1} < E < \bar{E}_{2j} \). The estimates for \( \partial_E I_n^{(2j-1),-} \) is analogous.

For \( \bar{\theta}_{2j-1} \leq \theta \leq \bar{\theta}_{2j} \) we have
\[ E - \bar{G}(\theta) \leq E - \bar{E}_{2j-1}, \]

\(^{41}\)Recall (176) and (177).
then we get
\[ \partial_E \bar{I}_n^{(2j)}(E) = \frac{1}{2\pi} \int_{\theta_{2,j-1}}^{\theta_{2,j}} \frac{1}{\sqrt{E - G(\theta)}} \, d\theta \geq \frac{1}{2\pi} \int_{\theta_{2,j-1}}^{\theta_{2,j}} \frac{1}{\sqrt{E - \bar{E}_{2,j-1}}} \, d\theta \]
and (201) follows\(^{42}\) by (34).
(202) follows by (200), (201), (31) and (178).
Finally (203) directly follows by (175) and (25). \( \blacksquare \)

Recalling (173),(199) and (202) we have that, for \( 1 \leq i < 2N \), the function
\[ \bar{I}_n^{(i)} : E \in [\bar{E}_-^{(i)}, \bar{E}_+^{(i)}] \mapsto \bar{I}_n^{(i)}(E) \]
is strictly monotone increasing and, therefore, invertible. Its inverse
\[ \bar{E}^{(i)} : I_n \in [\bar{a}_-^{(i)}, \bar{a}_+^{(i)}] \mapsto \bar{E}^{(i)}(I_n) \in [\bar{E}_-^{(i)}, \bar{E}_+^{(i)}], \quad (205) \]
where
\[ \bar{a}_-^{(i)} := \bar{I}_n^{(i)}(\bar{E}_-^{(i)}), \quad \text{for } 1 \leq i < 2N, \quad \bar{a}_-^{(2N)} := \bar{I}_n^{(2N)}(\bar{E}_-^{(2N)}). \quad (206) \]
Note that actually
\[ \bar{a}_-^{(2j-1)} := 0, \quad \forall 1 \leq j \leq N. \quad (207) \]
Moreover, by (203), also the function
\[ \bar{I}_n^{(2N)} = -\bar{I}_n^{(0)} : E \in [\bar{E}_-^{(2N)}, \infty) \mapsto [\bar{a}_-^{(2N)}, \infty) \]
is strictly monotone increasing and, therefore, invertible. We have the corresponding inverse functions
\[ \bar{E}^{(2N)} : I_n \in [\bar{a}_-^{(2N)}, \infty) \mapsto [\bar{E}_-^{(2N)}, \infty), \quad \bar{E}^{(0)} : I_n \in (-\infty, -\bar{a}_-^{(2N)}) \mapsto [\bar{E}_-^{(2N)}, \infty). \quad (208) \]

5.2 The domains of definitions of the action functions for the perturbed Hamiltonian
Let us consider now a real analytic Hamiltonian
\[ H_{\text{mech}}(p_n, q_n) = \left(1 + b(p_n, q_n)\right) \left(p_n - p_n^*(\hat{p})\right)^2 + G(\hat{p}, q_n) \quad (209) \]
\(^{42}\)The estimates for \( \partial_E \bar{I}_n^{(2j)} \) is analogous.
with holomorphic extension on
\[(p, q_n) = (\hat{p}, P_n, q_n) \in \hat{D}_{r_0} \times (-R_0, R_0)_{r_0/2} \times \mathbb{T}_{s_0}.
\]
Assume also that for some \(\eta \geq 0\),
\[|G - \bar{G}|_{\hat{D}, r_0, s_0} \leq \eta, \quad |P^*_n|_{\hat{D}, r_0} \leq \eta / r_0 \quad (210)
\]
with \(\bar{G}\) as in (25)-(27) and
\[|b|_* \leq \frac{\eta}{r_0^2}, \quad |\partial p_n b|_* \leq \frac{\eta}{r_0^2}, \quad |p_n \cdot b(p, q_n)|_* \leq \frac{\eta}{r_0}, \quad |p_n \cdot \partial p_n b(p, q_n)|_* \leq \frac{\eta}{r_0^2}, \quad (211)
\]
with \(|\cdot|_* := \sup_{\hat{D}_{r_0} \times (-R_0, R_0)_{r_0/2} \times \mathbb{T}_{s_0}} |\cdot|, \)
\[\text{(212)}\]

**Remark 5.5** Note that the Hamiltonian \(H_{\text{mech}}\) in (12) is of the form (209)-(211) with \(\eta = 100\eta_0\) by (18) and (14).

Assume \(R_0 \geq 2\sqrt{M}\), \(\text{(213)}\)

with \(M\) defined in (25). Note that
\[\eta \leq \frac{r_0^2}{32}, \quad (214)\]
as it is implied by (40).

Note that \(\hat{p}\) is a parameter with no dynamical meaning since its conjugated variable \(\hat{q}\) does not appear in the Hamiltonian. Note that
\[\nabla_{(p_n, q_n)} H_{\text{mech}}(p, q_n) = 0 \quad \Rightarrow \quad p_n = P^*_n(\hat{p}), \quad \partial_{q_n} G(p, q_n) = 0 .
\]
Let us consider the equation
\[P^*_n(\hat{p}) + \frac{z}{\sqrt{1 + b(p, q_n)}} - p_n = 0 \quad (215)\]
Since \(|b|_* \leq \frac{\eta}{r_0^2} \leq \frac{1}{2}\) by (211) and \(214\), we have
\[\text{Re} (1 + b(p, q_n)) \geq \frac{1}{2}, \quad \forall \hat{p} \in \hat{D}_{r_0}, \quad p_n \in (-R_0, R_0)_{r_0/2}, \quad q_n \in \mathbb{T}_{s_0} \quad (216)\]
and, therefore, \(\sqrt{1 + b(p, q_n)}\) is well defined on \(\hat{D}_{r_0} \times (-R_0, R_0)_{r_0/2} \times \mathbb{T}_{s_0}\).

\[\text{Recall (15).}\]
\[\text{See (223) below.}\]
Lemma 5.6 Assume (214). Then there exists a (unique) real analytic function \( \tilde{P} : (-R_0, R_0)_{\frac{r_0}{4}} \times T_{s_0} \times \hat{D}_{r_0} \to \mathbb{C} \) with

\[
|\tilde{P}| := \sup_{z \in (-R_0, R_0)_{\frac{r_0}{4}}} |\tilde{P}(z, \cdot, \cdot)|_{\hat{D}, r_0, s_0} \leq \frac{4}{r_0} \eta \leq \frac{r_0}{8}, \tag{217}
\]

such that

\[
p_n = P(z, q_n, \tilde{p}) := z + \tilde{P}(z, q_n, \tilde{p}) \tag{218}
\]
solves (215). Moreover

\[
P : (-R_0, R_0)_{\frac{r_0}{4}} \times T_{s_0} \times \hat{D}_{r_0} \to (-R_0, R_0)_{\frac{r_0}{2}} \tag{219}
\]

Proof We first note that if \( \tilde{P} \) satisfies the first inequality in (217), then, by (214), it also satisfies the second one. Therefore, if \( z \in (-R_0, R_0)_{\frac{r_0}{4}} \), then \( z + \tilde{P} \in (-R_0, R_0)_{\frac{r_0}{2}} \) and (219) holds. Let us define \( \tilde{P} = \tilde{P}(z, q_n, \tilde{p}) \) as the solution of the fixed point equation

\[
\tilde{P} = \Phi(\tilde{P}) := P_n^*(\tilde{p}) + \left( \frac{1}{\sqrt{1 + b(\tilde{p}, z + \tilde{P}, q_n)}} - 1 \right) z \tag{220}
\]
in the closed ball \( B \) of \( \tilde{P} \) satisfying (217). We immediately see that \( \Phi(B) \subseteq B \), since, by\(^{45}\) (211) and (216),

\[
|\Phi(\tilde{P})| = \frac{\eta}{r_0} + 2|bz|_* \leq \frac{3\eta}{r_0}.
\]

Moreover \( \Phi \) is a contraction since by the fifth and the third equation in\(^{46}\) (14)

\[
|\Phi(\tilde{P}) - \Phi(\tilde{P}')| \leq 2 \left( b(\tilde{p}, z + \tilde{P}, q_n) - b(\tilde{p}, z + \tilde{P}', q_n) \right) z \leq \frac{4\eta}{r_0^2} |\tilde{P} - \tilde{P}'| \leq \frac{1}{2} |\tilde{P} - \tilde{P}'|.
\]

Then equation (220) is solved by the Contraction Theorem. \( \blacksquare \)

\(^{45}\)Recall the notation in (212).

\(^{46}\)Omitting for brevity to write \( \hat{P}, q_n \) we have, setting \( \theta = \theta(t) = (1 - t)\tilde{P}' + t\tilde{P} \)

\[
(b(z + \tilde{P}) - b(z + \tilde{P}'))z = (\tilde{P} - \tilde{P}') \int_0^1 \partial_{p_n} b(z + \theta) \cdot z \, dt
\]

\[
= (\tilde{P} - \tilde{P}') \left[ \int_0^1 \partial_{p_n} b(z + \theta) \cdot (z + \theta) \, dt - \int_0^1 \partial_{p_n} b(z + \theta) \cdot \theta \, dt \right].
\]

Finally note that, for every \( 0 \leq t \leq 1 \) we have \( |\theta(t)|_* \leq 8\eta_0/r_0 \leq r_0/4 \) by (217) and (214).
Obviously\(^{47}\)

\[ p_n = \mathcal{P}\left( \pm \sqrt{E - G(\hat{p}, q_n)}, q_n, \hat{p} \right) \]

solves (w.r.t. \( p_n \))

\[ H_{\text{mech}}(\hat{p}, p_n, q_n) = E, \]  \hspace{1cm} (221)

according to \( \pm (p_n - P^*_n(\hat{p})) \geq 0 \). Note that, for real \( q_n, \hat{p}, z \mapsto \mathcal{P}(z, q_n, \hat{p}) \) is an increasing function of (real) \( z \in (-R_0, R_0) \), since

\[ \partial_z \mathcal{P} \geq 1 - \frac{16\eta}{r_0^2} \geq \frac{1}{2} \]  \hspace{1cm} (222)

by (214)-(218) and Cauchy estimates. Note also that

\[ \mathcal{P}(0, q_n, \hat{p}) = P^*_n(\hat{p}). \]

**Remark 5.6** We will assume that\(^{48}\), for real \( E \),

\[ E + M < R_0^2, \]  \hspace{1cm} (223)

so that \( \mathcal{P}\left( \pm \sqrt{E - G(\hat{p}, q_n)}, q_n, \hat{p} \right) \) is well defined.

Outside a zero measure set contained in the set of critical energies \( \{ H_{\text{mech}} = E_i \} \), \( 1 \leq i \leq 2N \), the phase space \( \hat{D} \times (-R_0, R_0) \times \mathbb{T}^n \) is decomposed in \( 2N+1 \) open connected components \( C^i \), \( 0 \leq i \leq 2N \), on which we will define the action-angle transformation. The \( C^i \) are normal sets with respect to the variable \( p_n \) and are defined as follows.

For \( i = 2j - 1 \) odd, \( 1 \leq j \leq N \),

\[ C^{2j-1} := \left\{ (p, q) \in \hat{D} \times (-R_0, R_0) \times \mathbb{T}^n \ : \right. \]

s.t.

\[ \mathcal{P}\left( - \sqrt{E_{2j} (\hat{p})} - G(\hat{p}, q_n), q_n, \hat{p} \right) < p_n < \mathcal{P}\left( \sqrt{E_{2j} (\hat{p})} - G(\hat{p}, q_n), q_n, \hat{p} \right), \]

\[ \Theta_{2j-1}(E_{2j} (\hat{p}), \hat{p}) < q_n < \Theta_{2j}(E_{2j} (\hat{p}), \hat{p}) \}

\( \setminus \left\{ p_n = P^*_n(\hat{p}), q_n = \Theta_{2j-1}(\hat{p}) \right\} \)  \hspace{1cm} (224)

\(^{47}\)For real values of \( \hat{p}, q_n, E \) such that \( E \geq G(\hat{p}, q_n) \).

\(^{48}\)Recall (213).
where \( j_0 \) was defined in (170).
For \( i = 2j \) even, \( 1 \leq j \leq N - 1 \),
\[
\mathcal{C}^{2j} := \left\{ (p, q) \in \hat{D} \times (-R_0, R_0) \times \mathbb{T}^m \ \text{s.t.} \right. \\
\mathcal{P}\left(-\sqrt{E_{2j}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right) < p_n < \mathcal{P}\left(\sqrt{E_{2j}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right), \\
\Theta_{2j_0+1}(E_{2j}, (\hat{p})) < q_n < \Theta_{2j_0}(E_{2j}, (\hat{p})), \\
\left\{ \mathcal{P}\left( -\sqrt{E_{2j}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right) \leq p_n \leq \mathcal{P}\left(\sqrt{E_{2j}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right), \\
\Theta_{2j_0+1}(E_{2j}, (\hat{p})) \leq q_n \leq \Theta_{2j_0}(E_{2j}, \hat{p}) \right\}, \\
\right. \\
\] (225)

where \( j_-, j_+, j_0 \) were defined in (171).
Finally
\[
\mathcal{C}^{2N} := \left\{ (p, q) \in \hat{D} \times (-R_0, R_0) \times \mathbb{T}^m \ \text{s.t.} \right. \\
R_0 > p_n > \mathcal{P}\left(\sqrt{E_{2N}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right) \} \\
\] (226)
\[
\mathcal{C}^{0} := \left\{ (p, q) \in \hat{D} \times (-R_0, R_0) \times \mathbb{T}^m \ \text{s.t.} \right. \\
-R_0 < p_n < \mathcal{P}\left(-\sqrt{E_{2N}}(\hat{p}) - G(\hat{p}, q_n), q_n, \hat{p}\right) \} \\
\] (227)

Note that actually in \( \mathcal{C}^i \) with \( 1 \leq i < 2N \), \( q_n \) is not an angle!
Remark 5.7 The phase space regions $C^i$ are closely related to the phase portrait of the Morse potential $\bar{G}$, which is the limiting potential, as $\eta$ goes to zero, of the potential $G$ of $H_{\text{mech}}$. Roughly speaking, $\bigcup_{i=0}^{2N} C^i$ is a bounded region around $p_n = 0$ of the phase space of $H_{\text{mech}}$ minus the separatrices (stable/unstable manifolds) issuing from critical hyperbolic lower dimensional manifolds (in the 1D projection: critical hyperbolic points). The suffix $i$, for $1 \leq i \leq 2N - 1$, labels the internal regions trapped by separatrices, where the actual motion of the 1D projection $(p_n, q_n)$-system is oscillatory, while $i = 0, 2N$ labels the two external regions, where the $q_n$-coordinate rotates (positive velocity for $i = 2N$ and negative velocity for $i = 0$).

5.3 Action variables and their adimensional version

On the above connected components $C^i$, $0 \leq i \leq 2N$, we want to define action angle variables integrating $H_{\text{mech}}$. We first define the action variables as a function of the energy $E$ and of the dummy variable $\hat{p}$.
Define the analytic function $b_{\sharp} = b_{\sharp}(z, \theta, \hat{p})$ as follows:

$$b_{\sharp}(z, \theta, \hat{p}) := \frac{1}{2\sqrt{1 + b(\hat{p}, P(z, \theta, \hat{p}), \theta)}} + \frac{1}{2\sqrt{1 + b(\hat{p}, P(-z, \theta, \hat{p}), \theta)}} - 1 \quad (228)$$

($b$ defined in (209)). Note that $b_{\sharp}$ is even w.r.t. $z$ and that

$$\sup_{z \in (-R_0, R_0) \cap \mathbb{R}} |b_{\sharp}(z, \hat{p}, \theta)|_{\partial \mathcal{D}, r_0, s_0} \leq |b|_{\ast} \leq \frac{\eta}{r_0^2}, \quad (229)$$

by (211).

**Remark 5.8** In the following, for brevity, we will often omit to write the immaterial dependence on $\hat{p}$ and/or on $\theta$.

Moreover we have

$$\partial_z b_{\sharp}(z) = \frac{-\partial_{\hat{p}} b(P(z)) \partial_z P(z)}{4(1 + b(P(z)))^{3/2}} + \frac{\partial_{\hat{p}} b(P(-z)) \partial_z P(-z)}{4(1 + b(P(-z)))^{3/2}}$$

and, since $z \partial_{\hat{p}} b(P(z)) = P(z) \partial_{\hat{p}} b(P(z)) - \bar{P}(z) \partial_{\hat{p}} b(P(z))$, we get

$$\sup_{z \in (-R_0, R_0) \cap \mathbb{R}} |z \partial_{\hat{p}} b(z)|_{\partial \mathcal{D}, r_0, s_0} \leq \frac{\eta}{r_0^2} + \frac{r_0 \eta}{8 r_0^3} \leq \frac{2\eta}{r_0^2}.$$

by (211) and (217). Therefore, since

$$\sup_{z \in (-R_0, R_0) \cap \mathbb{R}} |\partial_z P|_{\partial \mathcal{D}, r_0, s_0} \leq 2$$

by Cauchy estimates and (217),(218), we finally obtain that

$$\sup_{z \in (-R_0, R_0) \cap \mathbb{R}} |z \partial_z b_{\sharp}(z)|_{\partial \mathcal{D}, r_0, s_0} \leq \frac{8\eta}{r_0^2}. \quad (230)$$

**Lemma 5.7** Assume that $g(z)$ is holomorphic on $[0, R]$. If $g$ is even\(^{50}\) then one can define $G(v)$ holomorphic on $[0, R^2]$, such that $G(z^2) = g(z)$.

---

\(^{49}\)Note that, if $\text{Re}(1 + b) \geq 1/2$ (recall (216)), then $|(1 + b)^{-1/2} - 1| \leq |b|$.

\(^{50}\)Namely $g(z) = g(-z)$ for $z$ close to zero.
Moreover, since \( v_0 \) is even, it is actually holomorphic on \([-R, R]\). Denoting by \( D_r(0) := \{|z| < r\} \), we have that, since \( g \) is holomorphic and even on \( D_r(0) \), \( g(z) = \sum_{j \geq 0} a_{2j} z^{2j} \), where the power series has a radius of convergence \( \geq r \). Then \( G(v) := \sum_{j \geq 0} a_{2j} v^{2j} \) has radius of convergence \( \geq r^2 \). It remains to define \( G \) in the set \( \Omega := [0, R] \setminus D_{r^2}(0) \). Note that \( \Omega \subset \mathbb{C}_s \); then we set \( G(v) := g(\sqrt{v}) \) for \( v \in \Omega \), noting that \( z := \sqrt{v} \in [0, R] \). This follows noting that if \( v \in D_{r^2}(v_0^2) \), with \( v_0 \in \mathbb{R} \), \( v_0 > r \), then \( \sqrt{v} \in D_r(v_0) \). This last fact is equivalent to proving that \( D_{r^2}(v_0^2) \subseteq s(D_r(v_0)) \), where \( s(v) := v^2 \).

\[ b_1(z^2, \hat{p}, \theta) := b_1(z, \hat{p}, \theta) \]

with (recall (229))

\[ \sup_{v \in (0, R^2)^2 \cap 2^j} |b_1(v, \hat{p}, \theta)| \leq \frac{\eta}{r^2} \}

Moreover, since \( v \partial_v b_1(v) = \frac{1}{2} \sqrt{v} \partial_\theta b_2(\sqrt{v}) \),

\[ \sup_{v \in (0, R^2)^2 \cap 2^j} |v \partial_v b_1(v)| \leq \frac{4\eta}{r^2} \]

by (230).

For \( i = 2j - 1 \) odd, \( 1 \leq j \leq N \), and \( E_{2j - 1} < E < E_{2j} \), we set

\[ I_{n(2j-1)}^n(E) = I_{n(2j-1)}^n(E, \hat{p}) \]

\[ := \frac{1}{2\pi} \int_{\Theta_{2j-1}(E)} \left[ P\left( \sqrt{E - \mathcal{G}(\theta)}, \theta \right) - P\left( -\sqrt{E - \mathcal{G}(\theta)}, \theta \right) \right] d\theta \]

\[ (215),(228) \Rightarrow \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - \mathcal{G}(\theta)} \left( 1 + b_2(\sqrt{E - \mathcal{G}(\theta)}, \theta) \right) d\theta \]

\[ (231) \Rightarrow \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - \mathcal{G}(\theta)} \left( 1 + b_1(E - \mathcal{G}(\theta), \theta) \right) d\theta \]

\[ 51 \]This inclusion follows noting that, for every angle \( \theta \), we have \(|(v_0 + re^{i\theta})^2 - v_0^2| \geq r^2 \). The last inequality follows noting that it is equivalent to \(|re^{2i\theta} + 2v_0 e^{i\theta}| = |re^{i\theta} + 2v_0| \geq r \), that follows from \( v_0 > r \).
Analogously, for $i = 2j$ even, $1 \leq j \leq N - 1$ and $E_{2j} < E < E_{2j+1}$, we set (recall (171))

$$I^{(2j)}_n(E) = I^{(2j)}_n(E, \hat{p}) := \frac{1}{2\pi} \int_{\Theta_{2j+1}(E)} \left[ \mathcal{P}\left( \sqrt{E - \mathcal{G}(\theta)}, \theta \right) - \mathcal{P}\left( -\sqrt{E - \mathcal{G}(\theta)}, \theta \right) \right] d\theta$$

$$= \frac{1}{\pi} \int_{\Theta_{2j+1}(E)} \sqrt{E - \mathcal{G}(\theta)} \left( 1 + b_\uparrow(\sqrt{E - \mathcal{G}(\theta)}, \theta) \right) d\theta$$

Finally for $E > E_{2N}$ we set

$$I^{(2N)}_n(E) = I^{(2N)}_n(E, \hat{p}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}\left( \sqrt{E - \mathcal{G}(\theta)}, \theta \right) d\theta , \quad (236)$$

$$I^{(0)}_n(E) = I^{(0)}_n(E, \hat{p}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{P}\left( -\sqrt{E - \mathcal{G}(\theta)}, \theta \right) d\theta . \quad (237)$$

Recalling the definition of $b_\uparrow$ in (231) we set

$$\tilde{b}(v) = \tilde{b}(v, \hat{p}, \theta) := b_\uparrow(v) + 2v^2 b_\uparrow(v) , \quad (238)$$

with

$$\sup_{v \in \left(0, R_0^2\right), \theta / 64} |\tilde{b}(v)| \leq \frac{9\eta}{r_0^2} \leq \frac{9}{32} , \quad (239)$$

by (232), (233) and (214).

Recalling Remark 5.2 we have the following

**Remark 5.9** We want to show that the functions in (234)-(237) have an analytic extension for complex $E$. Moreover, while it is immediate to evaluate the derivative of
the last two\textsuperscript{52}, namely

\[
\partial_E I_n^{(2N)}(E) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{E - G(\theta)}} \partial_z \mathcal{P} \left( \sqrt{E - G(\theta)}, \theta \right) d\theta,
\]

\[
\partial_E I_n^{(0)}(E) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{E - G(\theta)}} \partial_z \mathcal{P} \left( -\sqrt{E - G(\theta)}, \theta \right) d\theta.
\]  

(241)

it is not obvious to justify the formal derivation\textsuperscript{53}

\[
\partial_E I_n^{(2j-1)}(E) = \frac{1}{2\pi} \int_{\Theta_{2j-1}(E)} \frac{1}{\sqrt{E - G(\theta)}} \left( 1 + \tilde{b}(E - G(\theta), \theta) \right) d\theta,
\]

\[
\partial_E I_n^{(2j)}(E) = \frac{1}{2\pi} \int_{\Theta_{2j}(E)} \frac{1}{\sqrt{E - G(\theta)}} \left( 1 + \tilde{b}(E - G(\theta), \theta) \right) d\theta.
\]  

(242)

Recalling (234) and (178) we split the integral \( \int_{\Theta_{2j}(E)} = \int_{\Theta_{2j-1}(E)} + \int_{\Theta_{2j-1}(E)} \) obtaining

\[
I_n^{(2j-1)}(E) = I_n^{(2j-1),+}(E) + I_n^{(2j-1),-}(E),
\]  

(243)

where, for 1 \( \leq j \leq N \),

\[
I_n^{(2j-1),-}(E) = I_n^{(2j-1),-}(E, \hat{t}) := \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - G(\theta)} \left( 1 + b_1(E - G(\theta), \theta) \right) d\theta,
\]

\[
I_n^{(2j-1),+}(E) = I_n^{(2j-1),+}(E, \hat{t}) := \frac{1}{\pi} \int_{\Theta_{2j-1}(E)} \sqrt{E - G(\theta)} \left( 1 + b_1(E - G(\theta), \theta) \right) d\theta.
\]

(244)

Analogously, recalling (235) and (178) we split the integral

\[
\int_{\Theta_{2j+1}(E)} = \int_{\Theta_{2j+1}(E)} + \sum_{j_- < i < j_+} \left( \int_{\Theta_{2i-1}} + \int_{\Theta_{2i+1}} \right) + \int_{\Theta_{2j+1}(E)},
\]

(245)

\[
I_n^{(2j)}(E) = I_n^{(2j-1),+}(E) + \sum_{j_- < i < j_+} \left( I_n^{(2i),-}(E) + I_n^{(2i),+}(E) \right) + I_n^{(2j-1),+}(E),
\]

\textsuperscript{52}And also the second derivatives:

\[
\partial_{EE} I_n^{(2N)}(E) = -\frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{\sqrt{E - G(\theta)} \partial_{zz} \mathcal{P} \left( \sqrt{E - G(\theta)}, \theta \right) - \partial_z \mathcal{P} \left( \sqrt{E - G(\theta)}, \theta \right)}{(E - G(\theta))^{3/2}} d\theta,
\]

\[
\partial_{EE} I_n^{(0)}(E) = \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{\sqrt{E - G(\theta)} \partial_{zz} \mathcal{P} \left( -\sqrt{E - G(\theta)}, \theta \right) + \partial_z \mathcal{P} \left( -\sqrt{E - G(\theta)}, \theta \right)}{(E - G(\theta))^{3/2}} d\theta.
\]

\textsuperscript{53}See Remark 5.10 below.

57
for $1 \leq j < N$, where
\[
I_{n}^{(2j),-}(E) = \frac{1}{\pi} \int_{\theta_{2j-1}}^{\theta_{2j}} \sqrt{E - \tilde{G}(\theta)} \left(1 + b_{t,j}(E - \tilde{G}(\theta))\right) d\theta, \quad 1 \leq j \leq N, \\
I_{n}^{(2j),+}(E) = \frac{1}{\pi} \int_{\theta_{2j-1}}^{\theta_{2j+1}} \sqrt{E - \tilde{G}(\theta)} \left(1 + b_{t,j}(E - \tilde{G}(\theta))\right) d\theta, \quad 0 \leq j < N.
\]

Finally
\[
2I_{n}^{(2N)}(E) = I_{n}^{(0),+}(E) + I_{n}^{(2N),-}(E) + \sum_{1 \leq i \leq N-1} I_{n}^{(2i),-}(E) + I_{n}^{(2i),+}(E). \tag{246}
\]

### The adimensional action

Recalling the definition of $\tilde{G}$, given in (65) we set
\[
\tilde{I}_{n}^{(2j),-}(\tilde{E}) = \tilde{I}_{n}^{(2j),-}(\tilde{E}, \tilde{I}) = \frac{1}{\pi} \int_{\tilde{\Theta}_{2j-1}}^{\tilde{\Theta}_{2j}} \sqrt{\tilde{E} - \tilde{G}_{2j-1}(\tilde{\theta})} \left(1 + \tilde{b}_{t,2j-1}(\tilde{E} - \tilde{G}_{2j-1}(\tilde{\theta}), \tilde{\theta})\right) d\tilde{\theta},
\]
\[
\tilde{I}_{n}^{(2j),+}(\tilde{E}) = \tilde{I}_{n}^{(2j),+}(\tilde{E}, \tilde{I}) = \frac{1}{\pi} \int_{\tilde{\Theta}_{2j}}^{\tilde{\Theta}_{2j+1}} \sqrt{\tilde{E} - \tilde{G}_{2j}(\tilde{\theta})} \left(1 + \tilde{b}_{t,2j}(\tilde{E} - \tilde{G}_{2j}(\tilde{\theta}), \tilde{\theta})\right) d\tilde{\theta},
\]
\[
\tilde{I}_{n}^{(2j),-}(\tilde{E}) = \tilde{I}_{n}^{(2j),-}(\tilde{E}, \tilde{I}) = \frac{1}{\pi} \int_{\tilde{\Theta}_{2j-1}}^{\tilde{\Theta}_{2j+1}} \sqrt{\tilde{E} - \tilde{G}_{2j+1}(\tilde{\theta})} \left(1 + \tilde{b}_{t,2j+1}(\tilde{E} - \tilde{G}_{2j+1}(\tilde{\theta}), \tilde{\theta})\right) d\tilde{\theta}, \tag{248}
\]

where
\[
\tilde{b}_{t,i}(\tilde{v}, \tilde{\theta}) = \tilde{b}_{t,i}(\tilde{v}, \tilde{\theta}) := b_{t,i} \left(-1\right)^{i} \frac{E_{i} - E_{i-1}}{2} \frac{\tilde{v}}{r_{0}} \frac{1}{\gamma_{i}(\tilde{\theta})}, \tag{249}
\]
($b_{t}$ defined in (231)). Note that, by (232) we have (recalling (53))
\[
\sup_{\tilde{v} \in (0, R_{0}^{2}/\Delta_{i})_{\alpha, 2\theta_{M}}} |\tilde{b}_{t,i}(\tilde{v}, \tilde{\theta})|_{D, r_{0}, s_{0}} \leq \frac{\eta}{r_{0}^{7}}, \quad \text{where} \quad \Delta_{i} := (-1)^{i}(E_{i} - E_{i-1})/2. \tag{250}
\]
Recalling (59), (65), (149), (244) we have\(^54\)

\[
\begin{align*}
I_n^{(2j-1),-}(E) &= \frac{\theta_{2j-1} - \theta_{2j-2}}{2} \sqrt{\frac{E_{2j-2} - E_{2j-1}}{2}} I_n^{(2j-1),-}(\lambda_{2j-1}(E)), \\
I_n^{(2j-1),+}(E) &= \frac{\theta_{2j} - \theta_{2j-1}}{2} \sqrt{\frac{E_{2j} - E_{2j-1}}{2}} I_n^{(2j-1),+}(\lambda_{2j}(E)), \\
I_n^{(2j),-}(E) &= \frac{\theta_{2j} - \theta_{2j-1}}{2} \sqrt{\frac{E_{2j} - E_{2j-1}}{2}} I_n^{(2j),-}(\lambda_{2j}(E)), \\
I_n^{(2j),+}(E) &= \frac{\theta_{2j+1} - \theta_{2j}}{2} \sqrt{\frac{E_{2j+1} - E_{2j}}{2}} I_n^{(2j),+}(\lambda_{2j+1}(E)).
\end{align*}
\]  

(251)

Let set\(^55\)

\[
\tilde{b}_t(\hat{v}, \hat{\theta}) = \tilde{b}_t(\hat{v}, \hat{p}, \hat{\theta}) = \tilde{b}_t(\hat{v}, \hat{\theta}) + 2\hat{v} \hat{p} \tilde{b}_t,\hat{i}(\hat{v}, \hat{\theta}) = \tilde{b}\left((-1)^i \frac{E_i - E_{i-1}}{2}, \hat{v}, \hat{\gamma}_i(\hat{\theta}) \right),
\]

\[
\sup_{\hat{\theta} \in (0,R_0^2/\Delta_0)_{\partial\Omega_M}} |\tilde{b}_t(\hat{v})|_{\hat{D}, r_0, s_0} \leq \frac{9\eta}{r_0^2},
\]

(252)

recalling (239) and (250).

**Lemma 5.8** The functions \( \tilde{I}_n^{(2j-1),+}(\hat{E}, \hat{I}) \) in\(^56\) (248) are holomorphic in \( \Omega \times \hat{D}_{r_0} \). Moreover the following formulas hold:

\[
\begin{align*}
\tilde{I}_n^{(2j-1),-}(\hat{E}) &= -\frac{(\hat{E} + 1)^{3/2}}{\pi} \int_0^1 \sqrt{t} \frac{1 + \tilde{b}_{\hat{t},2j-1}(t(1 + \hat{E}), \hat{\Theta}_{2j-1}(\hat{E}(t)))}{\partial_\hat{\theta} \hat{\Theta}_{2j-1}(\hat{E}(t))} dt, \\
\tilde{I}_n^{(2j-1),+}(\hat{E}) &= \frac{(\hat{E} + 1)^{3/2}}{\pi} \int_0^1 \sqrt{t} \frac{1 + \tilde{b}_{\hat{t},2j}(t(1 + \hat{E}), \hat{\Theta}_{2j}(\hat{E}(t)))}{\partial_\hat{\theta} \hat{\Theta}_{2j}(\hat{E}(t))} dt.
\end{align*}
\]

(253)

\(^{54}\)Note that

\[
E - \mathcal{G}(\theta) = (-1)^i \frac{E_i - E_{i-1}}{2} \left( \lambda_i(E) - \hat{\mathcal{G}}_i(\gamma_i^{-1}(\theta)) \right).
\]

\(^{55}\)Recall (249).

\(^{56}\)\( \hat{x} \) was defined in (141) and recall (83).
and, for the derivatives,

\[
\partial_{\bar{E}} \tilde{I}_n^{(2j-1),-}(\bar{E}) := -\frac{\sqrt{\bar{E} + 1}}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \frac{1 + \bar{b}_{2j-1}(t(1 + \bar{E}), \hat{\Theta}_{2j-1}(\bar{E}(t)))}{\partial_{\hat{\vartheta}} \hat{\Theta}_{2j-1}(\hat{\Theta}(\bar{E}(t)))} \, dt,
\]

\[
\partial_{\bar{E}} \tilde{I}_n^{(2j-1),+}(\bar{E}) := \frac{\sqrt{\bar{E} + 1}}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \frac{1 + \bar{b}_{2j}(t(1 + \bar{E}), \hat{\Theta}_{2j}(\bar{E}(t)))}{\partial_{\hat{\vartheta}} \hat{\Theta}_{2j}(\hat{\Theta}(\bar{E}(t)))} \, dt. \tag{254}
\]

**Proof** We proceed as in the proof of Lemma 5.1. Recalling (186) we have that if \( \bar{E} \in \Omega_\varepsilon \) then \( \bar{E}(t) \in \Omega_\varepsilon \) (with \( \bar{E}(t) \) defined in (183)) and, recalling Lemma 4.9, we also get that the following changes of variables are well defined:

\[
\theta = \hat{\Theta}_{2j-1}(\bar{E}(t), \hat{I}) \in [-1, 1]_{\rho_{x,2}}, \quad 0 \leq t \leq 1, \quad \hat{I} \in \hat{D}_{r_0},
\]

\[
\tilde{\theta} = \hat{\Theta}_j(\bar{E}(t), \hat{I}) \in [-1, 1]_{\rho_{x,2}}, \quad 0 \leq t \leq 1, \quad \hat{I} \in \hat{D}_{r_0}, \tag{255}
\]

with \( \rho_x \) defined in (141). By (118) we get

\[
t = \frac{\bar{E} - \bar{\vartheta}_{2j-1}(\hat{\vartheta})}{\bar{E} + 1}, \quad t = \frac{\bar{E} - \bar{\vartheta}_{2j}(\hat{\vartheta})}{\bar{E} + 1}
\]

and

\[
d\tilde{\vartheta} = -\frac{\bar{E} + 1}{\partial_{\hat{\vartheta}} \hat{\Theta}_{2j-1}(\hat{\Theta}(\bar{E}(t)))} \, dt \quad \text{or} \quad d\tilde{\vartheta} = -\frac{\bar{E} + 1}{\partial_{\hat{\vartheta}} \hat{\Theta}_{2j}(\hat{\Theta}(\bar{E}(t)))} \, dt.
\]

Making the change of variables (255) in the first and second integral in (248), respectively, we get (253). The proof of (254) is analogous to the one of (185). \( \square \)

**Definition 5.2** For \( z, w \in \mathbb{C} \) we define the segment

\[
(z, w) := \{(1 - t)z + tw, \quad 0 < t < 1\}. \tag{256}
\]

**Lemma 5.9** \( \tilde{I}_n^{(2j-1),+}(E, \hat{I}) \), respectively \( \tilde{I}_n^{(2j-1),-}(E, \hat{I}) \), in (244) is holomorphic on \( (\lambda_{2j}^{-1}, id)\)(\( \Omega_\varepsilon \times \hat{D}_{r_0} \)), respectively \( (\lambda_{2j-1}^{-1}, id)\)(\( \Omega_\varepsilon \times \hat{D}_{r_0} \)). In particular

\[
\partial_{E} \tilde{I}_n^{(2j-1),-}(E) = \frac{\theta_{2j-1} - \theta_{2j-2}}{2} \sqrt{\frac{2}{E_{2j-2} - E_{2j-1}}} \partial_{E} \tilde{I}_n^{(2j-1),-}(\lambda_{2j-1}(E)),
\]

\[
\partial_{E} \tilde{I}_n^{(2j-1),+}(E) = \frac{\theta_{2j} - \theta_{2j-1}}{2} \sqrt{\frac{2}{E_{2j} - E_{2j-1}}} \partial_{E} \tilde{I}_n^{(2j-1),+}(\lambda_{2j}(E)). \tag{257}
\]

\[57\]Where, obviously, \( (\lambda_{2j}^{-1}, id)(\Omega_\varepsilon \times \hat{D}_{r_0}) := \{(E, \hat{I}) = (\lambda_{2j}^{-1}(\bar{E}, \hat{I}), \hat{I}) \mid (\bar{E}, \hat{I}) \in \Omega_\varepsilon \times \hat{D}_{r_0}\} \).
Moreover for every fixed \( \hat{I} \in \hat{D}_{r_0} \) the function \( E \mapsto I_{n}^{(2j-1),+}(E, \hat{I}) \), respectively \( E \mapsto I_{n}^{(2j-1),-}(E, \hat{I}) \), is holomorphic on

\[
(E_{2j-1}(\hat{I}), E_{2j}(\hat{I}))_{r_1}, \text{ respectively } (E_{2j-1}(\hat{I}), E_{2j-2}(\hat{I}))_{r_1},
\]

with \( r_1 = \beta \bar{r}/4 \) define in (192).

**Proof** The first part is a direct consequence of (251) and of Lemma 5.8. (257) follows by (251) and (59). (258) follows by (59) and (60).

Analogously to Remark 5.4 we have

**Remark 5.10** Note that making the inverses of the change of variables (255) in the expression (254) we get

\[
\begin{align*}
\partial_{\hat{E}} I_{n}^{(2j-1),-}(\hat{E}) &:= \frac{1}{2\pi} \int_{\Theta_{2j-1}(\hat{E})}^{1} \frac{1 + \tilde{b}_{2j-1}(\hat{E} - \hat{G}_{2j-1}(\hat{\theta}), \hat{\theta})}{\sqrt{\hat{E} - \hat{G}_{2j-1}(\hat{\theta})}} d\hat{\theta}, \\
\partial_{\hat{E}} I_{n}^{(2j-1),+}(\hat{E}) &:= \frac{1}{2\pi} \int_{-1}^{\Theta_{2j}(\hat{E})} \frac{1 + \tilde{b}_{2j}(\hat{E} - \hat{G}_{2j}(\hat{\theta}), \hat{\theta})}{\sqrt{\hat{E} - \hat{G}_{2j}(\hat{\theta})}} d\hat{\theta},
\end{align*}
\]

(259)

showing also that the formal derivation in (242) is actually correct. Indeed, recalling (149), (180), (59), and making the change of variables \( \hat{\theta} = \gamma_{2j-1}^{-1}(\theta) \) and \( \hat{\theta} = \gamma_{2j}^{-1}(\theta) \) in the first and in the second integral in (259), respectively, and summing the results, we get (242).

Let us define

\[
E(t) := E - (E - E_{2j-1})t = tE_{2j-1} + (1 - t)E.
\]

Recalling (59), (65), (149), (252) we have that

\[
\begin{align*}
\tilde{E} = \lambda_i(E) & \implies \\
\tilde{E}(t) &= \lambda_i(E(t)), \\
(-1)^i \frac{E_i - E_{i-1}}{2}(1 + \tilde{E}) &= E - E_{2j-1}, \quad \text{for } i = 2j, 2j - 1, \\
\Theta_i(E(t)) &= \gamma_i \circ \tilde{\Theta}_i(\tilde{E}(t)), \\
\tilde{b}(t(E - E_{2j-1}), \Theta_i(E(t))) &= \tilde{b}_i(t(1 + \tilde{E}), \tilde{\Theta}_i(\tilde{E}(t))), \\
\partial_{\tilde{b}} \mathcal{G}_i \left( \Theta_i(E(t)) \right) &= (-1)^i \frac{E_i - E_{i-1}}{\theta_i - \tilde{\theta}_{i-1}} \partial_{\tilde{b}} \tilde{\mathcal{G}}_i \left( \tilde{\Theta}_i(\tilde{E}(t)) \right),
\end{align*}
\]

(261)
where \( \tilde{E}(t) \) and \( E(t) \) were defined in (183) and (260) respectively. By (261) and recalling (254), (257) we get the following

**Lemma 5.10** We have

\[
\partial_E I_n^{(2j-1)}(E) = \frac{\sqrt{E - E_{2j-1}}}{2\pi} \cdot \int_0^1 \frac{dt}{\sqrt{t}} \left( 1 + \tilde{b}(t(E - E_{2j-1}), \Theta_{2j}(E(t))) - 1 + \tilde{b}(t(E - E_{2j-1}), \Theta_{2j-1}(E(t))) \right) \partial_\theta G(\Theta_{2j-1}(E(t))) \partial_\theta G(\Theta_{2j}(E(t))) dt.
\]

**Proof** We also give a more direct proof. It is convenient to split the first integral in (242) as

\[
\int_{\Theta_{2j}(E)}^{\Theta_{2j-1}(E)} = \int_{\Theta_{2j-1}(E)}^{\Theta_{2j}(E)} + \int_{\Theta_{2j-1}(E)}^{\Theta_{2j}(E)}
\]

and make the changes of variables

\[
\theta = \Theta_{2j-1}(E(t)), \quad 0 \leq t \leq 1,
\]

\[
\theta = \Theta_{2j}(E(t)), \quad 0 \leq t \leq 1,
\]

respectively in the first and second integral. Note that in both cases, by (150), \( t = \frac{E - G(\theta)}{E - E_{2j-1}} \), while

\[
d\theta = -\frac{E - E_{2j-1}}{\partial_\theta G(\Theta_{2j-1}(E(t)))} dt, \quad d\theta = -\frac{E - E_{2j-1}}{\partial_\theta G(\Theta_{2j}(E(t)))} dt,
\]

respectively (recalling (152)). Recalling (153) we get (262). \( \square \)

**Lemma 5.11** The functions \( \tilde{I}_n^{(2j)}(\tilde{E}) \) in (248) are holomorphic in\(^{58}\)

\[
\Omega_{r_0^2/2^8M}^{(2j),-} := \left( C_* + 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j}} - 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j+1}} - 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j+1}} - 1 \right) = \Omega_{r_0^2/2^8M}^{(2j),+}
\]

\[
\Omega_{r_0^2/2^{8}M}^{(2j),+} := \left( C_* + 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j+1}} - 1 \right)
\]

\[
\Omega_{r_0^2/2^{8}M}^{(2j),-} := \left( C_* + 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j+1}} - 1 \right)
\]

\[
\Omega_{r_0^2/2^{8}M}^{(2j),+} := \left( C_* + 1 \right) \cap \left( 1, \frac{R_0^2}{\Delta_{2j+1}} - 1 \right)
\]

\[^{58}\]The intervals in (263) are non empty by (213).
Moreover the following formulas hold:

\[
\mathcal{I}_n^{(2j),-}(\tilde{E}) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{\tilde{E} - y} \left( 1 + \tilde{b}_{1,2j}(\tilde{E} - y, \tilde{\Theta}_{2j}(y)) \right)}{\partial_{\tilde{\theta}} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y))} \, dy ,
\]

\[
\mathcal{I}_n^{(2j),+}(\tilde{E}) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{\tilde{E} - y} \left( 1 + \tilde{b}_{1,2j+1}(\tilde{E} - y, \tilde{\Theta}_{2j+1}(y)) \right)}{\partial_{\tilde{\theta}} \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y))} \, dy ,
\]  

(264)

(where \(\tilde{b}_{i,\pm}\) were defined in (249)) and, for the derivatives,

\[
\partial_{\tilde{E}} \mathcal{I}_n^{(2j),-}(\tilde{E}) = \frac{1}{2\pi} \int_{-1}^{1} \left( 1 + \tilde{b}_{2j}(\tilde{E} - y, \tilde{\Theta}_{2j}(y)) \right) \frac{1}{\sqrt{\tilde{E} - y}} \partial_{\tilde{\theta}} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y)) \, dy ,
\]

\[
\partial_{\tilde{E}} \mathcal{I}_n^{(2j),+}(\tilde{E}) = -\frac{1}{2\pi} \int_{-1}^{1} \left( 1 + \tilde{b}_{2j+1}(\tilde{E} - y, \tilde{\Theta}_{2j+1}(y)) \right) \frac{1}{\sqrt{\tilde{E} - y}} \partial_{\tilde{\theta}} \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y)) \, dy ,
\]  

(265)

(where \(\tilde{b}_{i}\) were defined in (252)).

**Proof** Recalling Lemma 4.9, for \(y \in \Omega_{\tilde{r}}\) the following changes of variables are well defined:

\[
\tilde{\theta} = \tilde{\Theta}_{2j}(y) \in [-1,1]_{\rho_{*}/2} ,
\]

\[
\tilde{\theta} = \tilde{\Theta}_{2j+1}(y) \in [-1,1]_{\rho_{*}/2} ,
\]  

(266)

with \(\rho_{*}\) defined in (141). Note that \(\tilde{\Theta}_{2j}(\pm 1) = \tilde{\Theta}_{2j+1}(\mp 1) = \pm 1\) (recall Lemma 4.8).

Recalling (118) we get

\[d\tilde{\theta} = \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j}(\tilde{\Theta}_{2j}(y))} \, dy \quad \text{or} \quad d\tilde{\theta} = \frac{1}{\partial_{\tilde{\theta}} \tilde{G}_{2j+1}(\tilde{\Theta}_{2j+1}(y))} \, dt ,\]

(deriving (118)). Making the changes of variables (266) in the third and fourth integral in (248) respectively, we get\(^{59}\). Note that if \(\tilde{E} \in \mathbb{C}_{\rho_{*}} + 1\) and \(-1 \leq y \leq 1\) then \(\tilde{E} - y \in \mathbb{C}_{\rho_{*}}\), so that \(\sqrt{\tilde{E} - y}\) is well defined. Moreover

\[
\tilde{E} \in (1, R_{0}^{2}/\Delta_{i} - 1, r_{0}^{2}/2^{s} M) \quad \Rightarrow \quad \tilde{E} - y \in (0, R_{0}^{2}/\Delta_{i} r_{0}^{2}/2^{s} M) , \quad \forall -1 \leq y \leq 1 .
\]  

(267)

\(^{59}\)One can easily control that the integrals in absolutely converge.
Deriving (and recalling (252)) we get (265).

5.4 Closeness of the rescaled unperturbed and perturbed actions

By (147) and (148) we get, for every $-1 < y < 1$ and $\hat{I} \in \hat{D}_{r_0}$,

$$\left| \frac{1}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} - \frac{1}{\partial_y \bar{G}_i(\bar{\Theta}_i(y))} \right| \leq \left| \frac{\partial_y \bar{G}_i(\bar{\Theta}_i(y))}{\partial_y \bar{G}_i(\bar{\Theta}_i(y))} - \frac{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} \right| \leq \frac{2^{73}\beta^8}{\beta^9 s_18} \left| \frac{\eta}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} \right| \leq \frac{1}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))}, \quad (268)$$

where the second inequality holds by (69)-(72) and the last inequality follows from (40). Moreover for every $\bar{E} \in \Omega_{r_0^2/2^8\beta}$, $-1 < y < 1$ and $\hat{I} \in \hat{D}_{r_0}$, we have (recalling (267))

$$\left| \frac{1 + \bar{b}_i(\bar{E} - y, \tilde{\Theta}_i(y))}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} - \frac{1}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} \right| \leq \frac{\bar{b}_i(\bar{E} - y, \tilde{\Theta}_i(y))}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} + \left| \frac{1}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))} - \frac{1}{\partial_y \bar{G}_i(\bar{\Theta}_i(y))} \right| \leq \left( \frac{18 r_0^2}{\beta^{28} s_0^{45} s_2} + \frac{2^{193}M^{27}}{\beta^{28} s_0^{45} s_2} \right) \eta \left| \partial_y \tilde{G}_i(\tilde{\Theta}_i(y)) \right| \leq \frac{1}{\partial_y \tilde{G}_i(\tilde{\Theta}_i(y))}, \quad (270)$$

where the second inequality follows by (239), (252), (273) and the last inequality follows from (40).

By (142) and (186) we have that

$$\bar{E} \in \Omega_x \implies \tilde{\Theta}_i(\bar{E}), \tilde{\Theta}_i(\bar{E}, \hat{I}), \tilde{\Theta}_i(\bar{E}(t)), \tilde{\Theta}_i(\bar{E}(t), \hat{I}) \in [-1, 1]_{\rho_*/2}, \quad \forall t \in [0, 1], \forall \hat{I} \in \hat{D}_{r_0},$$

64
where \( \mathbf{\bar{r}} \) and \( \rho_* \) were defined in (141). Then, for every \( \mathbf{\bar{E}} \in \Omega_\mathbf{\bar{r}}, \mathbf{\bar{I}} \in \mathcal{D}_{r_0}, 0 \leq t \leq 1, \)

\[
\left| \frac{1}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} - \frac{1}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} \right| \leq \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_*^2} \left| \frac{\eta}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} \right| \tag{273}
\]

\[
\leq \frac{1}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))}, \tag{274}
\]

where the second inequality holds by (69)-(72) and the last inequality follows from (40) and

\[
\left| 1 + \frac{\mathbf{\ddot{b}}_i(t(1 + \mathbf{\bar{E}}), \mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} - \frac{1}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} \right|
\leq \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_*^2} \right) \left| \frac{\eta}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))} \right| \tag{275}
\]

\[
\leq \frac{1}{\partial_\theta \mathcal{G}_i(\mathbf{\bar{\Theta}}_i(\mathbf{\bar{E}}(t)))}, \tag{276}
\]

The odd case

**Lemma 5.12** We have that \( \mathbf{\bar{I}}_n^{(2j-1), \pm}(\mathbf{\bar{E}}) \) and \( \mathbf{\bar{I}}_n^{(2j-1), \pm}(\mathbf{\bar{E}}, \mathbf{\bar{I}}) \) are holomorphic functions in the sets\(^{60} \Omega_\mathbf{\bar{r}} \) and \( \Omega_\mathbf{\bar{r}} \times \mathcal{D}_{r_0} \) respectively. Moreover, for \( \mathbf{\bar{I}} \in \mathcal{D}_{r_0} \) and \( \mathbf{\bar{E}} \in \Omega_\mathbf{\bar{r}} \) with \( \text{Re} \mathbf{\bar{E}} < 1 \), the following estimate holds

\[
|\partial_\mathbf{\bar{E}} \mathbf{\bar{I}}_n^{(2j-1), \pm}(\mathbf{\bar{E}}, \mathbf{\bar{I}}) - \partial_\mathbf{\bar{E}} \mathbf{\bar{I}}_n^{(2j-1), \pm}(\mathbf{\bar{E}})| \leq \eta \left( \frac{36}{r_0^2} + \frac{2^{194} M^{27}}{\beta^{28} s_0^{45} s_*^2} \right) \frac{2^{55} M^{7}}{\beta^{7} s_0^{13}} \left( 1 + \ln \frac{1}{1 - \text{Re} \mathbf{\bar{E}}} \right) \tag{277}
\]

**Proof** The “-” term in (277) is bounded by

\[
\sqrt{\mathbf{\bar{E}} + 1} \int_0^1 \frac{1}{\sqrt{t}} \left| 1 + \frac{\mathbf{\ddot{b}}_{2j-1}(t(1 + \mathbf{\bar{E}}), \mathbf{\bar{\Theta}}_{2j-1}(\mathbf{\bar{E}}(t)))}{\partial_\theta \mathcal{G}_{2j-1}(\mathbf{\bar{\Theta}}_{2j-1}(\mathbf{\bar{E}}(t)))} - \frac{1}{\partial_\theta \mathcal{G}_{2j-1}(\mathbf{\bar{\Theta}}_{2j-1}(\mathbf{\bar{E}}(t)))} \right| dt \tag{278}
\]

\(^{60}\text{Recall (83) and (141).}\)
By (275) it is bounded by

\[
\eta \left( \frac{18}{r_0^2} + \frac{2^{103} M^{27}}{\beta^{28} s_0^{45} s^2} \right) \sqrt{|\tilde{E} + 1|} \frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \left| \partial_{\theta} \tilde{g}_{2j-1} \left( \tilde{\Theta}_{2j-1}(\tilde{E}(t)) \right) \right| dt.
\]

The “+” term is estimated analogously\(^{61}\). Then we get

\[
|\partial_{E} \tilde{I}_{n}^{(2j-1),\pm}(\tilde{E}, \tilde{I}) - \partial_{E} \tilde{I}_{n}^{(2j-1),\pm}(\tilde{E})| \\
\leq \eta \left( \frac{36}{r_0^2} + \frac{2^{104} M^{27}}{\beta^{28} s_0^{45} s^2} \right) \sqrt{|\tilde{E} + 1|} \frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{t}} \left| \partial_{\theta} \tilde{g}_i \left( \tilde{\Theta}_i(\tilde{E}(t)) \right) \right| dt,
\]

(279)

where \( i = 2j \) in the + case and \( i = 2j - 1 \) in the − case.

We estimate the integral in (279) in the following

**Lemma 5.13** For \( \tilde{E} \in \Omega_\tilde{r} \) with \( \text{Re} \tilde{E} < 1 \) we have

\[
\int_0^1 \frac{1}{\sqrt{t}} \left| \partial_{\theta} \tilde{g}_i \left( \tilde{\Theta}_i(\tilde{E}(t)) \right) \right| dt \leq \frac{2^{31} \tilde{M}}{\tilde{\beta}^3 \tilde{s}^4} \left( \frac{1}{\sqrt{|1 + \tilde{E}|}} + \ln \frac{1}{1 - \text{Re} \tilde{E}} \right).
\]

\[(280)\]

**Proof** Recalling (272), (80) and (77) we get

\[
\left| \partial_{\theta} \tilde{g}_i \left( \tilde{\Theta}_i(\tilde{E}(t)) \right) \right| \leq \frac{8}{\beta} \left( \frac{2^{10} \tilde{M}}{\tilde{\beta}^3 \tilde{s}^3} + \frac{1}{|1 - \tilde{\Theta}_i(\tilde{E}(t))|} + \frac{1}{|1 + \tilde{\Theta}_i(\tilde{E}(t))|} \right),
\]

Recalling (91), (141), (85) and noting that \( \tilde{r} \leq \tilde{r}_0 \), we get

\[
\left| \partial_{\theta} \tilde{g}_i \left( \tilde{\Theta}_i(\tilde{E}(t)) \right) \right| \leq \frac{2^{26} \tilde{M}}{\tilde{\beta}^3 \tilde{s}^4} \left( 1 + \frac{1}{\sqrt{|\tilde{E}(t) + 1|}} + \frac{1}{\sqrt{|\tilde{E}(t) - 1|}} \right).
\]

We claim that the integral in (280) is bounded by

\[
\frac{2^{26} \tilde{M}}{\tilde{\beta}^3 \tilde{s}^4} \left( 2 + \frac{\pi}{\sqrt{|1 + \tilde{E}|}} + 4 + 2 \ln \frac{1}{1 - \text{Re} \tilde{E}} \right)
\]

\(^{61}\)With \( 2j \) instead of \( 2j - 1 \).
and (280) follows. We now prove the above claim. Note that $1 + \dot{E}(t) = (1 - t)(1 + \ddot{E})$ and $|\dot{E}(t) - 1| \geq 1 - \text{Re} \dot{E}(t) = (1 - \text{Re} \ddot{E}) + (1 + \text{Re} \ddot{E})t$, for every $t \in [0, 1]$. Then $\int_0^1 dt/\sqrt{t|\ddot{E}(t) + 1|} = \pi/\sqrt{1 + 1}$. Moreover in estimating $J := \int_0^1 dt/\sqrt{|\ddot{E}(t) - 1|}$ we have two cases: if $\text{Re} \ddot{E} \leq 0$ then $|\ddot{E}(t) - 1| \geq 1$, $\forall t \in [0, 1]$ and $J \leq 2$; otherwise, when $\text{Re} \ddot{E} \geq 0$, we have $|\ddot{E}(t) - 1| \geq 1 - \text{Re} \ddot{E} + t$, $\forall t \in [0, 1]$, then, setting $\xi := 1 - \text{Re} \ddot{E}$ (note that $0 < \xi \leq 1$), we get

$$J \leq \int_0^1 \frac{dt}{\sqrt{1 + \text{Re} \ddot{E} + t}} = 2 \int_0^{1/\sqrt{\xi}} \frac{ds}{\sqrt{1 + s^2}} \leq 4 \int_0^{1/\sqrt{\xi}} \frac{ds}{1 + s} = 4 \ln(1 + \frac{1}{\sqrt{\xi}}) \leq 4 + 2 \ln \frac{1}{\xi},$$

proving the claim. □

Inserting the estimate (280) in (279) we get

$$|\partial E \dot{E}^{(2j-1), \pm}(\ddot{E}, \hat{I}) - \partial E \dot{E}^{(2j-1), \pm}(\ddot{E})| \leq \eta \left( \frac{36}{r_0^2} + \frac{1341 M^2}{\beta^5 s_0^4 s^2} \right) \frac{2^{31} \tilde{M}}{\beta^3 s^4} \left( 1 + \ln \frac{1}{1 - \text{Re} \ddot{E}} \right).$$

By (69)-(72) we get (277). □

• The even case

**Lemma 5.14** For $\text{Re} \ddot{E} > 1$ we have

$$\int_{-1}^1 \frac{1}{\sqrt{\ddot{E} - y \partial_y \tilde{G}_i(\hat{\Theta}_i(y))}} dy \leq \frac{2^{31} \tilde{M}}{\beta^3 s^4} \frac{1}{\sqrt{\text{Re} \ddot{E}}} \ln \left( 4 + \frac{1}{\text{Re} \ddot{E} - 1} \right). \quad (281)$$

**Proof** Recalling (272), (80) and (77) we get, for every $-1 < y < 1$,

$$\frac{1}{\partial_y \tilde{G}_i(\hat{\Theta}_i(y))} \leq \frac{8}{\beta} \left( \frac{2^{10} \tilde{M}}{\beta^3 s^3} + \frac{1}{|1 - \hat{\Theta}_i(y)|} + \frac{1}{|1 + \hat{\Theta}_i(y)|} \right),$$

Recalling (91), (141), (85) we get

$$\frac{1}{\partial_y \tilde{G}_i(\hat{\Theta}_i(y))} \leq \frac{2^{26} \tilde{M}}{\beta^3 s^4} \left( 1 + \frac{1}{\sqrt{|y + 1|}} + \frac{1}{\sqrt{|y - 1|}} \right) \leq \frac{2^{26} \tilde{M}}{\beta^3 s^4} \left( 2 + \frac{1}{\sqrt{1 - y}} \right).$$

Note that for $\ddot{E} \in \Omega$ we have $1/\sqrt{|1 + \ddot{E}|} \geq 1/2$. 62
\[
\frac{2^{26}\tilde{M}}{\tilde{\beta}^3 s^4} \int_{-1}^{1} \left( 2 + \frac{1}{\sqrt{1 - y}} \right) \frac{1}{\sqrt{\Re \tilde{E} - y}} \frac{d y}{\sqrt{\Re \tilde{E} - y}} \leq \frac{2^{31}\tilde{M}}{\tilde{\beta}^3 s^4} \frac{1}{\sqrt{\Re \tilde{E}}} \ln \left( 4 + \frac{1}{\Re \tilde{E} - 1} \right)
\]

and (281) follows. \[\square\]

**Lemma 5.15** For every \( \hat{I} \in D_{r_0} \) and \( \tilde{E} \in \Omega_{r_0/2 sM}^{(2j) \pm} \) with \( \Re \tilde{E} > 1 \), we have

\[
|\partial_{\tilde{E}} I_n^{(2j) \pm}(\tilde{E}, \hat{I}) - \partial_{\tilde{E}} \tilde{I}_n^{(2j) \pm}(\tilde{E})| \leq \eta \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_*^2} \right) \frac{2^{55} M^7}{\beta^{28} s_0^{13} s_*^2} \frac{1}{\sqrt{\Re \tilde{E}}} \ln \left( 4 + \frac{1}{\Re \tilde{E} - 1} \right). \tag{282}
\]

**Proof** Let us consider the “-” case, the “+” one is analogous. Then the quantity on the l.h.s. of (282) is bounded by

\[
\frac{1}{2\pi} \int_{-1}^{1} \frac{1}{|\sqrt{\tilde{E} - y}|} \left| \frac{1 + \tilde{b}_{2j} (\tilde{E} - y, \tilde{\Theta}_{2j}(y))}{\partial_{\tilde{\theta}} \tilde{\Theta}_{2j}(\tilde{\Theta}_{2j}(y))} - \frac{1}{\partial_{\tilde{\theta}} \tilde{\Theta}_{2j}(\tilde{\Theta}_{2j}(y))} \right| \frac{d y}{\sqrt{\Re \tilde{E} - y}} \leq \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{|\sqrt{\tilde{E} - y}|} \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_*^2} \right) \eta \left| \partial_{\tilde{\theta}} \tilde{\Theta}_{i}(\tilde{\Theta}_{i}(\tilde{E})) \right| \frac{d y}{\sqrt{\Re \tilde{E} - y}} \leq \eta \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_*^2} \right) \frac{2^{31}\tilde{M}}{\tilde{\beta}^3 s^4} \frac{1}{\sqrt{\Re \tilde{E}}} \ln \left( 4 + \frac{1}{\Re \tilde{E} - 1} \right).
\]

By (69)-(72) we get (282). \[\square\]

### 5.5 Estimates on \( \partial_{E} I_n^{(2j-1)} \)

**Remark 5.11** We will assume, to fix ideas, that \( \bar{E}_{2j-2} < \tilde{E}_j \) so that \( j_0 = j - 1 \) in (170), (173) and \( \bar{E}_{2j_0} = \bar{E}_{2j-2} = \bar{E}_{2j-2}^{(2j-1)} \); analogously \( E_{2j_0} = E_{2j-2} = E_{2j-2}^{(2j-1)} \).

**Lemma 5.16** \( \partial_{E} I_n^{(2j-1)} \) is a holomorphic function of the complex variable \( \zeta = E - \bar{E}_{2j-1} \) for

\[
|\zeta| < r_* := \min \left\{ \frac{s_0^{18}}{277 M^{11}}, \frac{r_0^2}{28} \right\}. \tag{283}
\]
In particular
\[
\partial E I^{(2j-1)}_n(E, \hat{I}) = \frac{1}{2\pi} \int_0^1 \frac{G(E - E_{2j-1}, (1-t)(E - E_{2j-1}), \hat{I})}{\sqrt{t} \sqrt{1-t}} \, dt,
\]
for a suitable holomorphic function \(G(v, y, \hat{I})\) satisfying
\[
\sup_{v \in (0, R_0^2), |y| < r_s} |G|_{\hat{D}, r_0} \leq \frac{32\sqrt{M}}{s_0 \beta}.
\]

Then
\[
\sup_{|E - E_{2j-1}| < r_s} |\partial E I^{(2j-1)}_n(E, \hat{I})|_{\hat{D}, r_0} \leq \frac{16\sqrt{M}}{s_0 \beta}.
\]

**Proof** Set
\[
w = w(t) := \sqrt{E - E_{2j-1}} \sqrt{1-t} = \sqrt{E(t) - E_{2j-1}},
\]
then by \((159)\)
\[
\Theta_{2j-1}(E(t)) = \theta_{2j-1} - w(t)\Theta_{2j-1, -}(w(t)), \quad \Theta_{2j}(E(t)) = \theta_{2j-1} + w(t)\Theta_{2j, -}(w(t)),
\]
for \(|w(t)| < r_s\), \((288)\)

\((r_s \text{ defined in } (156))\), which is implied by
\[
|E - E_{2j-1}| < r_s \leq \min\{r_{s, 2}^2, r_{s, 8}^2\} = \min\left\{\frac{s_0^{18}}{250^4 \pi^{10} M^{11}}, \frac{r_0^2}{28^8}\right\}, \quad E - E_{2j-1} \in C_s.
\]

Since \(\partial_{\theta} G(\theta_{2j-1}) = 0\) we have that the function
\[
\hat{G}(\theta) := \frac{\partial_{\theta} G(\theta_{2j-1} + \theta)}{\theta} = \int_0^1 \partial_{\theta \theta} G(\theta_{2j-1} + \theta y) dy.
\]
is holomorphic. By Taylor expansion, Cauchy estimates, \((52)\) and \((55)\)
\[
|\hat{G}(\theta)| = \left| \frac{\partial_{\theta} G(\theta_{2j-1} + \theta)}{\theta} \right| \geq |\partial_{\theta \theta} G(\theta_{2j-1})| - \frac{8M}{s_0^3} |\theta| \geq \frac{\beta}{2} - \frac{\beta}{4} = \frac{\beta}{4}
\]
for
\[
|\theta| \leq \frac{\beta s_0^3}{32M} =: \theta_s.
\]
By (288) we have that
\[ \partial_\theta G(\Theta_{2j}(E(t))) = w(t)\Theta_{2j-1}(w(t))\partial_\theta \Theta_{2j-1}(w(t)) \]
\[ \partial_\theta G(\Theta_{2j-1}(E(t))) = -w(t)\Theta_{2j-1}(w(t))\partial_\theta \Theta_{2j-1}(w(t)) \]  \hspace{1cm} (291)

for \( |w(t)| < r_\circ \).

Let us define the holomorphic function
\[ g(v, w, \hat{I}) := \frac{1 + \hat{b}(v - w^2, \theta_{2j-1} + w\Theta_{2j-1}(w))}{\Theta_{2j-1}(w)} \partial_\theta \Theta_{2j-1}(w) + \frac{1 + \hat{b}(v - w^2, \theta_{2j-1} - w\Theta_{2j-1}(w))}{\Theta_{2j-1}(w)} \partial_\theta \Theta_{2j-1}(w). \]  \hspace{1cm} (292)

By (161) it follows that \( g \) is an even function w.r.t. \( w \). Then
\[ g(v, w, \hat{I}) =: G(v, w^2, \hat{I}) \]  \hspace{1cm} (293)

for a suitable function \( G \). Recalling (288) and noting that, for \( |w| < r_\circ \),
\[ |w\Theta_{2j,\pm}(w)| \leq \frac{4\sqrt{3}\pi M}{s_0^{3/2}\beta^{3/2}} = \frac{s_0^9}{2^{259}\pi^5 M^{11/2}} \frac{\beta^6}{s_0^{3/2}\beta^{3/2}} = \frac{s_0^{15/2}}{2^{233}\sqrt{3}\pi^4 M^{9/2}} \]
\[ \leq \min \left\{ \theta_\circ, \frac{s_0}{8} \right\}, \]  \hspace{1cm} (294)

we have that
\[ \sup_{v \in (0, R_0^2), \|w\| < r_\circ} \sup_{|w| < \sqrt{r_*}} |g(v, w, \hat{I})|_{D, r_0} = \sup_{v \in (0, R_0^2), \|w\| < r_\circ} \sup_{|w| < \sqrt{r_*}} |G(v, w^2, \hat{I})|_{D, r_0} \leq \frac{32\sqrt{M}}{s_0\beta} \]  \hspace{1cm} (295)

by (239), (294), (290) and (162).

By (291) and (293) we get
\[ \frac{1 + \hat{b}(t(E - E_{2j-1}), \Theta_{2j}(E(t)))}{\partial_\theta G(\Theta_{2j}(E(t)))} = \frac{1 + \hat{b}(t(E - E_{2j-1}), \Theta_{2j-1}(E(t)))}{\partial_\theta G(\Theta_{2j-1}(E(t)))} = \frac{g(E - E_{2j-1}, w(t), \hat{I})}{w(t)} \frac{G(E - E_{2j-1}, w^2(t), \hat{I})}{w(t)}. \]  \hspace{1cm} (296)

By (262)
\[ \partial E\frac{I_n^{(2j-1)}}{2\pi} = \frac{1}{2\pi} \int_0^1 \frac{g(E - E_{2j-1}, w(t), \hat{I})}{\sqrt{t}\sqrt{1 - t}} dt = \frac{1}{2\pi} \int_0^1 \frac{G(E - E_{2j-1}, w^2(t), \hat{I})}{\sqrt{t}\sqrt{1 - t}} dt \]  \hspace{1cm} (297)
and (284) follows. Then (286) follows from (285) and since \( \int_0^1 \frac{1}{\sqrt{1-t}} \frac{1}{t} dt = \pi \).

**Remark 5.12** By Lemma 5.16 the function \( \partial_E I_n^{(2j-1)}(E, \hat{I}) \) is a holomorphic function of the complex variable \( \zeta = E - E_{2j-1} \). This is not the case of the functions \( \partial_E I_n^{(2j-1),\pm}(E, \hat{I}) \) and \( \partial_E I_n^{(2j-1),\pm}(E) \) that are holomorphic functions of \( \sqrt{\zeta} \) only. The holomorphicity of \( \partial_E I_n^{(2j-1)} = \partial_E I_n^{(2j-1),+} + \partial_E I_n^{(2j-1),-} \) is due to parity cancellations.

**Lemma 5.17** Set

\[
\mathfrak{r}_2 := \min \left\{ \frac{s_0^{49} \beta^{30}}{2214 M^{30}}, \frac{r_0^2}{210 M} \right\} < \frac{\beta r_1^2}{210 M^2}. \tag{298}
\]

The function \( \partial_E I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - z M, \hat{I}) \), initially defined for \( 0 < z < \mathfrak{r}_2 \) and \( \hat{I} \in \hat{D} \), has holomorphic extension to the complex set \( \{ z \in \mathbb{C}_* \ s.t. \ |z| < \mathfrak{r}_2 \} \times \hat{D}_0 \). In particular

\[
\partial_E I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - z M, \hat{I}) = \varphi^{(2j-1)}(z, \hat{I}) + \psi^{(2j-1)}(z, \hat{I}) \ln z, \tag{299}
\]

where \( \varphi^{(2j-1)}(z, \hat{I}) \) and \( \psi^{(2j-1)}(z, \hat{I}) \) are holomorphic function in the set \( \{|z| < \mathfrak{r}_2\} \times \hat{D}_0 \) with

\[
sup_{\{|z| < \mathfrak{r}_2\} \times \hat{D}_0} |\varphi^{(2j-1)}(z, \hat{I})| \leq \frac{8^{44} M^8}{s_0^{15} \beta^{17/2}}, \quad \text{sup} \sup_{\{|z| < \mathfrak{r}_2\} \times \hat{D}_0} |\psi^{(2j-1)}(z, \hat{I})| \leq \frac{27 \sqrt{M}}{\beta s_0}, \tag{300}
\]

and

\[
\inf_{\hat{I} \in \hat{D}_0} |\psi^{(2j-1)}(0, \hat{I})| \geq \frac{s_0}{32 \sqrt{M}}. \tag{301}
\]

Moreover the functions \( \partial_E I_n^{(2j-1),-}(E_{2j-2}(\hat{I}) - z M, \hat{I}) \) and \( \partial_E I_n^{(2j-1),+}(E_{2j-2}(\hat{I}) - z M, \hat{I}) \) have holomorphic extension to the complex sets \( \{ z \in \mathbb{C}_* \ s.t. \ |z| < \mathfrak{r}_2 \} \times \hat{D}_0 \) and \( \{|z| < \mathfrak{r}_2\} \times \hat{D}_0 \), respectively, with

\[
\partial_E I_n^{(2j-1),-}(E_{2j-2}(\hat{I}) - z M, \hat{I}) = \varphi^{(2j-1),-}(z, \hat{I}) + \psi^{(2j-1)}(z, \hat{I}) \ln z, \tag{302}
\]

and

\[
\text{sup} \sup_{\{|z| < \mathfrak{r}_2\} \times \hat{D}_0} |\varphi^{(2j-1),\pm}(z, \hat{I})| \leq \frac{2^{84} M^8}{s_0^{15} \beta^{17/2}}. \tag{303}
\]

\( \mathfrak{r}_1 \) was defined in (164). Use (31) to prove the inequality.
Finally

\[
\sup_{\{z|<r_2\} \times \hat{D}_{r_0/2}} |\partial f^{(2j-1)}(z, \hat{I})| \leq M_\varphi \eta, \quad \sup_{\{z|<r_2\} \times \hat{D}_{r_0/2}} |\partial f^{(2j-1)}(z, \hat{I})| \leq M_\psi \eta, \quad (304)
\]

where \(M_\varphi, M_\psi\) are suitable large constants.

**Remark 5.13** The constants \(M_\varphi, M_\psi\) can be explicitly evaluated but we do not do this here!

**Proof** First set

\[
\zeta := zM. \quad (\text{305})
\]

Note that the function

\[
\sqrt{E - E_{2j-1}} = \sqrt{\Delta E - \zeta} = \sqrt{u(\zeta)},
\]

with \(\Delta E := E_{2j-2} - E_{2j-1}\), \(u = u(\zeta) := \Delta E - \zeta\)

is holomorphic for \(|\zeta| < \beta/2\). Noting that by (25) we have \(\beta \leq E_{2j-2} - E_{2j-1} \leq 2M\), recalling (53), (213) (and (40)) we have

\[
u(\zeta) \in (\beta, 2M)_{r_0^2/2^7} \subset (0, R_0^2/2)_{r_0^2/2^7} \quad \text{and} \quad |u(\zeta)| < 5M \quad (306)
\]

for \(|\zeta| < r_2 M\). Indeed by (26) we have \(\text{Re} (\bar{E}_{2j-2} - \bar{E}_{2j-1}) = \bar{E}_{2j-2} - \bar{E}_{2j-1} \geq \beta\) and

\[
\text{Re} \Delta E = \text{Re} (E_{2j-2} - E_{2j-1}) \geq \beta - 4\eta \geq \beta - \frac{\beta^2}{64M} \geq \frac{\beta}{2}. \quad (307)
\]

In particular note that

\[
|\zeta| \leq \beta/4 \quad \implies \quad |\Delta E - \zeta| \geq \beta/4. \quad (307)
\]

Recalling that

\[
E = E_{2j-2}(\hat{I}) - zM,
\]
we set

\[ I_1 := \frac{\sqrt{u}}{2\pi \sqrt{t}} \frac{1 + \hat{b}(t(E - E_{2j-1}), \Theta_{2j}(E(t)))}{\partial \theta G(\Theta_{2j}(E(t)))} \]

\[ (261) \]

\[ \frac{\theta_{2j} - \theta_{2j-1}}{E_{2j} - E_{2j-1}} \frac{\sqrt{u}}{2\pi \sqrt{t}} \frac{1 + \hat{b}_2(t(1 + \hat{E}), \Theta_{2j}(\hat{E}(t)))}{\partial \hat{\theta} \hat{G}_{2j}(\hat{\Theta}_{2j}(\hat{E}(t)))} \]

for \( \hat{E} = \lambda_{2j}(E_{2j-2}(\hat{I}) - zM) \),

\[ I_2 := -\frac{\sqrt{u}}{2\pi \sqrt{t}} \frac{1 + \hat{b}(t(E - E_{2j-1}), \Theta_{2j-1}(E(t)))}{\partial \theta G(\Theta_{2j-1}(E(t)))} \]

\[ (261) \]

\[ \frac{\theta_{2j-1} - \theta_{2j-2}}{E_{2j-1} - E_{2j-2}} \frac{\sqrt{u}}{2\pi \sqrt{t}} \frac{1 + \hat{b}_{2j-1}(t(1 + \hat{E}), \Theta_{2j-1}(\hat{E}(t)))}{\partial \hat{\theta} \hat{G}_{2j-1}(\hat{\Theta}_{2j-1}(\hat{E}(t)))} \]

for \( \hat{E} = \lambda_{2j-1}(E_{2j-2}(\hat{I}) - zM) \),

(recalling that \( \hat{E}(t) = -t + (1 - t)\hat{E} \) was defined in (183)). Then we split

\[ \partial_E I_n^{(2j-1)} = I_1 + I_2 + I_3 + I_4, \]

where

\[ I_1 := \int_{1-t_1}^1 (I_1 + I_2) dt, \]

\[ I_2 := \int_{0}^{t_1} I_2 dt, \]

\[ I_3 := \int_{0}^{1-t_1} I_1 dt, \]

\[ I_4 := \int_{t_1}^{1-t_1} I_2 dt, \]

(recalling (164)).

In the beginning we consider the real case, namely \( 0 < \zeta < r_2 M \) (and, therefore, \( 0 < z < r_2 \)) and \( \hat{I} \in D \). Note that in this case we have \( u \geq \beta/4 > 0 \). Then we will
rewrite the functions $I_i$ in a different way such that it clearly appears that $I_1, I_3, I_4$ actually have a holomorphic extension for $|\zeta| < r_2 M$ (and $\hat{I} \in \mathcal{D}_r$), while $I_2$ has a holomorphic extension for $|\zeta| < r_2 M, \zeta \in \mathbb{C}^*$, (and $\hat{I} \in \mathcal{D}_r$) due to the presence of a logarithmic term. Note that we will omit to explicitly write the dependence on the dummy variable $\hat{I}$, with respect to which all the estimates are uniform.

- **Study of $I_1$.**

Recalling the definition of $w(t)$ in (287), we have that

$$1 - t_1 \leq t \leq 1 \implies |w(t)| \leq 2\sqrt{M}\sqrt{t}^{(310)} < r_\circ \implies (288) \text{ holds}.$$  

Then, recalling the definition of $G$ in (293) and (296), we get

$$I_1 = \frac{1}{2\pi} \int_{1-t_1}^1 \frac{G(E - E_{2j-1}, (1-t)(E - E_{2j-1}), \hat{I})}{\sqrt{t}\sqrt{1-t}} \, dt,$$

which by (285) and (213) gives that $I_1$ is actually a holomorphic function of $z = (E - E_{2j-1})/M$ in the ball $\{|z| < r_2\}$ with estimate

$$\sup_{\{|z| < r_2\} \times \mathcal{D}_r_0} |I_1| \leq \frac{32\sqrt{M}}{s_0\beta_0}. \quad (311)$$

- **Study of $I_2$.**

Set

$$v = v(t) := \sqrt{E_{2j-2} - E(t)} = \sqrt{t\Delta E + (1-t)\zeta} = \sqrt{\zeta + ut}, \text{ where } u = u(\zeta) := \Delta E - \zeta.$$

By (159)

$$\Theta_{2j-1}(E(t)) = \theta_{2j-2} + v(\Theta_{2j-1}, +) \text{,} \quad (312)$$

when, recalling (156), $|v| < r_\circ$. Since (recall (164))

$$r_2 \leq r_1^2/8M$$

by (298) and (31) we have that

$$0 \leq t \leq t_1 = \frac{r_1^2}{26M} \implies |v(t)| \leq \frac{r_1}{2} \leq \frac{r_\circ}{231}. \quad (313)$$

Then, by (158)

$$|v(t)\Theta_{2j-1,+}(v(t))| \leq \frac{16r_1 M}{s_0^{3/2}\beta_3^{3/2}} \leq \frac{s_0^{10}}{243 M^6} \leq \frac{\theta_2}{230}, \quad \forall 0 \leq t \leq t_1.$$

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In conclusion, by Lemma 4.2 we have that, for every $0 \leq t \leq t_1$,
\[
\partial_\theta G(\Theta_{2j-1}(E(t))) = v(t)\Theta_{2j-1,+}(v(t)) \tilde{G}(v(t)\Theta_{2j-1,+}(v(t))) = v(t)G_o(v(t)),
\]
where
\[
G_o(v) := \Theta_{2j-1,+}(v) \tilde{G}(v\Theta_{2j-1,+}(v)), \quad \text{with} \quad \tilde{G}(0) = \partial_{\theta\theta}G(\theta_{2j-2})
\]
is a holomorphic function in the ball $|v| < r_o$ and, again by Lemma 4.2 and (163), we get
\[
\sup_{\{|v|<r_1\} \times \hat{D}_{r_0}} \frac{1}{|G_o|} \leq \frac{16\sqrt{M}}{\beta s_0}.
\]

We have
\[
I_2 = \frac{\sqrt{u}}{2\pi} \int_0^{t_1} \frac{1 + \tilde{b}(t(\Delta E - \zeta), \Theta_{2j-1}(E(t)))}{\sqrt{t} \partial_\theta G(\Theta_{2j-1}(E(t)))} dt
\]
\[
= \frac{\sqrt{u}}{2\pi} \int_0^{t_1} \frac{1 + \tilde{b}(t(\Delta E - \zeta), \theta_{2j-2} + v(t)\Theta_{2j-1,+}(v(t)))}{\sqrt{t} v(t)G_o(v(t))} dt.
\]

Let us consider the holomorphic function
\[
b(v) = b(y, v) = b(y, v, \hat{I}) := \frac{1}{G_o(v)} \left(1 + \tilde{b}(y, \theta_{2j-2} + v\Theta_{2j-1,+}(v))\right)
\]
with
\[
\sup_{(0,R_o^2)_{\hat{I}/64} \times \{|v|<r_1\} \times \hat{D}_{r_0}} |b(v, v, \hat{I})| \leq \frac{64\sqrt{M}}{\beta s_0}.
\]

Split it in its even and odd part w.r.t. $v$, namely $b(v) = b_e(v) + b_o(v)$, where $b_e(v) := (b(v) + b(-v))/2$ and $b_o(v) := (b(v) - b(-v))/2$, for which the same estimate as (318) holds. Since $b_e(v)$ is an even function there exists a holomorphic function $b_e(w)$ such that $b_e(v) = b_e(v^2)$ with estimate
\[
\sup_{(0,R_o^2)_{\hat{I}/64} \times \{|v|<r_1^2\} \times \hat{D}_{r_0}} |b_e(v, w, \hat{I})| \leq \frac{64\sqrt{M}}{\beta s_0}.
\]

---

64 Recall (316) and (239).
65 Omitting for brevity the dependence on $v$ and $\hat{I}$.
On the other hand, since \( b_0(v) \) is odd, the function \( b_0(v)/v \) is also holomorphic with estimate
\[
\sup_{(0,R_0^2)_{3/64}\times\{|v|<r_1/2\}\times\tilde{D}_{r_0}} |b_0(v, \hat{v})/v| \leq \frac{2^7\sqrt{M}}{r_1\beta s_0}.
\] (320)
by Cauchy estimates. Since \( b_0(v)/v \) is an even function there exists a holomorphic function \( b_0(w) \) such that \( b_0(v)/v = b_0(v^2) \) with estimate
\[
\sup_{(0,R_0^2)_{3/64}\times\{|w|<r_1^2/4\}\times\tilde{D}_{r_0}} |b_0(v, w, \hat{v})| \leq \frac{2^7\sqrt{M}}{r_1\beta s_0}.
\] (321)
Recollecting \( b(v, v) = \nu b_0(v, v^2) + b_e(v, v^2) \).
Noting that \( v^2(t) = \zeta + ut \) and setting
\[
I_{2,e}(\zeta) := \frac{\sqrt{u}}{4\pi} \int_0^{t_1} \frac{b_e(ut, \zeta + ut)}{\sqrt{t}} \frac{\sqrt{\zeta + ut}}{\sqrt{\zeta}} \, dt, \\
I_{2,o}(\zeta) := \frac{\sqrt{u}}{4\pi} \int_0^{t_1} \frac{b_o(ut, \zeta + ut)}{\sqrt{t}} \, dt = \frac{\sqrt{u}}{2\pi} \int_0^{T_1} \frac{b_o(us^2, \zeta + us^2)}{\sqrt{s}} \, ds,
\]
we get
\[ I_2 = I_{2,e} + I_{2,o}. \]
Recalling that \( |u| < 5M \) and \( r_2 \leq r_1^2/8M \) we get, for every \( 0 \leq s \leq \sqrt{t_1} \),
\[
|us^2| \leq |ut_1| < r_1^2/8, \quad |\zeta + us^2| < r_1^2/4, \quad (322)
\]
for \( |\zeta| \leq r_2 M \). It is obvious that \( I_{2,o} \) is holomorphic, moreover, by (321)
\[
\sup_{|\zeta| \leq r_2 M} |I_{2,o}(\zeta)| \leq \frac{2^7 M}{r_1 \beta s_0} \frac{\sqrt{t_1}}{\beta s_0} = \frac{2^4 \sqrt{M}}{\beta s_0}. \] (323)
Regarding \( I_{2,e} \) we split it as
\[
I_{2,e} = I_{2,1} + I_{2,2} := \frac{\sqrt{u}}{4\pi} \int_0^{16\zeta/u} + \frac{\sqrt{u}}{4\pi} \int_{16\zeta/u}^{t_1}.
\]
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We claim that \( I_{2,1} \) is a holomorphic function of \( \zeta \) in the ball \( \{ |\zeta| < r_2 M \} \). Indeed changing variable \( t = y^2 \zeta/u \) we get

\[
I_{2,1} = \frac{1}{2\pi} \int_0^4 \frac{b_e(\zeta y^2, \zeta(1 + y^2))}{\sqrt{1 + y^2}} \, dy,
\]

which is obviously holomorphic on \( \{ |\zeta| < r_2 M \} \) with estimate

\[
\sup_{|\zeta| < r_2 M} |I_{2,1}(\zeta)| \leq \frac{2^{10} \sqrt{M}}{\beta s_0} \tag{324}
\]

by (319). On the other hand

\[
I_{2,2} = \frac{\sqrt{u}}{4\pi} \int_{16\zeta/u}^{t_1} \frac{b_e(ut, \zeta + ut)}{\sqrt{u} \sqrt{\zeta + ut}} \, dt,
\]

substituting \( w = ut \) becomes

\[
I_{2,2} = \frac{1}{4\pi} \int_{16\zeta}^{u t_1} \frac{b_e(w, \zeta + w)}{w \sqrt{1 + \frac{\zeta}{w}}} \, dw = \frac{1}{4\pi} \int_{16\zeta}^{u t_1} \frac{b_e(w, \zeta + w)}{w \sqrt{1 + \frac{\zeta}{w}}} \, dw. \tag{325}
\]

By (319) we write, for \( |\zeta|, |w| < r_\natural^2/2 \)

\[
b_e(w, \zeta + w) = \sum_{h \geq 0} b_h(\zeta) w^h, \tag{326}
\]

for suitable holomorphic functions \( b_h(\zeta) \) satisfying

\[
\sup_{|\zeta| < r_\natural} |b_h(\zeta)| \leq M_\natural r_\natural^{-h}, \quad \text{with} \quad M_\natural := \frac{64 \sqrt{M}}{\beta s_0}, \quad r_\natural := r_\natural^2/2. \tag{327}
\]

Let us develop, for \( |y| < 1 \),

\[
\frac{1}{\sqrt{1 + y}} = \sum_{k \geq 0} c_k y^k, \quad c_k := \left( \frac{-1/2}{k} \right) \tag{328}
\]

and note that \( |c_k| < 1 \). For \( |\zeta| < |w| < r_\natural \), we have

\[
\frac{b_e(w, \zeta + w)}{\sqrt{1 + \frac{\zeta}{w}}} = \sum_{n \in \mathbb{Z}} d_n(\zeta) w^n, \quad \text{where} \quad d_n(\zeta) := \sum_{k \geq \max\{0, -n\}} c_k b_{k+n}(\zeta) \zeta^k, \tag{329}
\]
in particular, for \( n \geq 0 \)
\[
d_n(\zeta) = \sum_{k \geq 0} c_k b_{k+n}(\zeta) \zeta^k, \quad d_{-n}(\zeta) = \sum_{k \geq n} c_k b_{k-n}(\zeta) \zeta^k = \zeta^n \sum_{k \geq 0} c_{k+n} b_k(\zeta) \zeta^k. \tag{330}
\]
By (327) we have, for \( n \geq 0 \),
\[
\sup_{|\zeta| < \frac{r}{4}} |d_n(\zeta)| \leq 2M_\sharp r_{-n}^2, \quad \sup_{|\zeta| < \frac{r}{4}} |d_{-n}(\zeta)| \leq 2M_\sharp |\zeta|^n. \tag{331}
\]
Recalling (322) and we note that, in the real case, for every \( 0 < \zeta < r_2 M \) (and \( \hat{I} \in D \)) we have \( 16 \zeta \leq w \leq \eta_1 \) and then \( \zeta/w \leq 1/16 \). Therefore, for every \( 0 < \zeta < r_2 M \), the first series in (329) totally converges in the interval \( 16 \zeta \leq w \leq \eta_1 \) and we get
\[
I_{2,2} = \frac{1}{4\pi} \int_{16\zeta}^{\eta_1} \frac{b_k(w, \zeta + w)}{w} dw = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} d_n(\zeta) \int_{16\zeta}^{\eta_1} w^{n-1} dw = \psi(z) \ln z + I_{2,3}, \tag{332}
\]
where
\[
\psi(z) := -\frac{d_0(zM)}{4\pi} \tag{333}
\]
and
\[
I_{2,3} := \frac{1}{4\pi} d_0(\zeta) \ln(u_1/16M) + \frac{1}{4\pi} \sum_{n \neq 0} d_n(\zeta) \left((u_1)^n - (16\zeta)^n\right). \tag{334}
\]
Note that, except for the first one, all the other addenda in the last line are holomorphic functions of \( \zeta \) in the ball \( \{|\zeta| < r_2 M\} \); for example by (330)
\[
\sum_{n > 0} \frac{d_{-n}(\zeta)}{n(16\zeta)^n} = \sum_{n > 0} \frac{1}{n16^n} \sum_{k \geq 0} c_{k+n} b_k(\zeta) \zeta^k.
\]
Recalling (331) and (307)
\[
\sup_{|\zeta| < \frac{r}{4\pi}} |\psi(z)| \leq 2M_\sharp, \tag{335}
\]
which, recalling (327), implies (300). By (326),(328),(329), (317),(315), (160) we have
\[
d(0) = c_0 b_0(0) = b_0(0, 0) = b(0, 0) = 2 \frac{1 + \tilde{b}(0, \theta_{2j-2})}{\mathcal{G}_0(0)} = 2 \frac{1 + \tilde{b}(0, \theta_{2j-2})}{\Theta_{2j-1, +}(0) \mathcal{G}(0)}
\]
\[
= 2 \frac{1 + \tilde{b}(0, \theta_{2j-2})}{\sqrt{-2/\partial_\theta \mathcal{G}(\theta_{2j-2}) \partial_\theta \mathcal{G}(\theta_{2j-2})}} = -\sqrt{2} \frac{1 + \tilde{b}(0, \theta_{2j-2})}{\sqrt{-\partial_\theta \mathcal{G}(\theta_{2j-2})}} \tag{336}
\]
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so that
\[ \psi(0) = \frac{1 + \tilde{b}(0, \theta_{2j-2})}{\sqrt{8\pi} \sqrt{-\partial_{\theta\theta} G(\theta_{2j-2})}}. \] (337)

Then, by (52) and Cauchy estimates we get (301).

From (334), (325), (332), (331), (322), (307), (310), (298) we get

\[ \sup_{|z| < r_2} |I_{2,3}(zM)| \leq 2M_* \ln(27M/r_2^2) + 2M_* \sum_{n > 0} \left( \frac{1}{4^n} + \frac{1}{2^n} + \left(\frac{4r_2}{\beta t_1}\right)^n + \frac{1}{16^n}\right) \]
\[ \leq 2M_* \ln(27M/r_2^2) + 16M_* \leq 2M_* \ln \left(\frac{210M^{15}}{s_0^3 \beta^{15}}\right) + 16M_* \]
\[ \leq 2^{14} M_* \sqrt{\frac{M}{s_0 \beta}} \leq 2^{20} \frac{M}{s_0^{3/2} \beta^{3/2}}, \] (338)

by (28) and also\(^{66}\)

\[ \ln \frac{2^{101} M^{15}}{s_0^{23} \beta^{15}} \leq \ln \frac{2^{109} M^{23}}{s_0^{23} \beta^{23}} \leq 109 + 23 \ln \frac{M}{s_0 \beta} \leq 109 + 23 \sqrt{\frac{M}{s_0 \beta}} \leq 2^{12} \sqrt{\frac{M}{s_0 \beta}}. \] (339)

Recollecting we have
\[ I_2(z) = I_{2,0}(zM) + I_{2,1}(zM) + I_{2,3}(zM) + \psi(z) \ln z. \] (340)

\[ \textbf{• Study of } I_3. \]

We claim that
\[ |z| < r_2 \implies \bar{E} = \lambda_{2j}(E_{2j-2} - zM) \in \Omega_{\xi}. \] (341)

In order to prove (341) we note that, since \(\tilde{\lambda}_{2j}\) (defined in (59)) is an increasing function\(^{67}\),
\[ -1 = \tilde{\lambda}_{2j}(\bar{E}_{2j-1}) < \tilde{\lambda}_{2j}(\bar{E}_{2j-2}) < \tilde{\lambda}_{2j}(\bar{E}_{2j}) = 1, \] (342)

recalling that
\[ \bar{E}_{2j-1} < \bar{E}_{2j-2} < \bar{E}_{2j} \] (343)

by Remark 5.11. By (60) we have
\[ |\lambda_{2j}(E_{2j-2} - zM) - \lambda_{2j}(\bar{E}_{2j-2})| \leq \frac{4}{\beta} \left(|E_{2j-2} - \bar{E}_{2j-2}| + Mr_2\right) \]
\[ \leq \frac{4}{\beta} (2\eta + Mr_2) \]

\[ \text{Noting that } \ln x \leq \sqrt{x}. \]

\[ \text{Note that } \lambda_i(\bar{E}_{i'}) \text{ is real.} \]

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\[
|\lambda_2(\bar{E}_{2j-2}) - \bar{\lambda}_2(\bar{E}_{2j-2})| \leq \frac{48M\eta}{\beta^2} \leq \frac{96\eta}{\beta}
\]
by (61). Then
\[
|\lambda_2(E_{2j-2} - zM) - \bar{\lambda}_2(E_{2j-2})| \leq \frac{2^7\eta}{\beta} + \frac{4Mr_2}{\beta} \leq \frac{2^7\eta}{\beta} + \frac{s_0^4\beta^{29}}{2^{12}M^{29}} < \frac{\tilde{r}}{2}
\]
(\tilde{r} defined in (141)).
Moreover, since
\[
1 - \bar{\lambda}_2(\bar{E}_{2j-2}) = 2\bar{\lambda}_2(\bar{E}_{2j-2}) - 2\bar{\lambda}_2(E_{2j-2}) + \bar{\lambda}_2(E_{2j-2}) = \lambda_2(E_{2j-2}) - 2zM
\]
by (345) we get
\[
1 - \text{Re } \tilde{E} \geq \frac{\beta}{2M}.
\]
Analogously we have
\[
|1 + \tilde{E}| \geq 1 + \text{Re } \tilde{E} \geq \frac{\beta}{2M}.
\]
Finally (342), (344), (345), (346) imply (341).
By (308) and recalling (309) we get
\[
I_3 = \frac{\theta_{2j} - \theta_{2j-1}}{E_{2j} - E_{2j-1}} \sqrt{u} \tilde{I}_3,
\]
with \(\tilde{I}_3 := \int_0^1 t \frac{1}{\sqrt{t}} \frac{1 + \tilde{b}_2(t(1 + \tilde{E}), \tilde{\Theta}_{2j}(\tilde{E}(t)))}{\sqrt{t} \partial_\theta \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))} dt\),
\[
\text{where } \tilde{E} = \lambda_2(E_{2j-2} - zM). \text{ Since } \tilde{E} \in \Omega_{\tilde{r}} \text{ by (341), we can apply estimate (276) obtaining}
\]
\[
|\tilde{I}_3| \leq \int_0^1 \frac{2}{\sqrt{t} \partial_\theta \tilde{G}_{2j}(\tilde{\Theta}_{2j}(\tilde{E}(t)))} dt.
\]
Since by (345) we also have \(\text{Re } \tilde{E} < 1\), by (280), (345), (346), (28) we get
\[
|\tilde{I}_3| \leq \frac{2^{32}M}{\beta^3 s^4} \left(\frac{1}{\sqrt{|1 + \tilde{E}|}} + \ln \frac{1}{1 - \text{Re } \tilde{E}}\right) \leq \frac{2^{38}M \sqrt{M}}{\beta^3 s^4 \sqrt{\beta}} \leq \frac{2^{68}M^{15/2}}{s_0^{13} \beta^{15/2}} ,
\]
\(^{68}\)Noting that \(\ln x \leq \sqrt{x}\).
by (69)-(72). Then by (347), (306), (55) we get

$$|I_3| \leq \frac{271M^8}{s_0 ^{13} \beta^{17/2}}. \quad (348)$$

- **Study of $I_4$.**

It is similar to the case $I_3$. By (308) and recalling (309) we get

$$I_4 = \frac{\theta_{2j-1} - \theta_{2j-2}}{E_{2j-1} - E_{2j-2}} \sqrt{\bar{u}} I_4, \quad \text{with} \quad \tilde{I}_4 := \int_{t_1}^{1-t_1} \frac{1 + \bar{b}_{2j-1}(t(1 + \bar{E}), \bar{\Theta}_{2j-1}(\bar{E}(t)))}{\sqrt{t} \partial_b \bar{G}_{2j-1}(\bar{\Theta}_{2j-1}(\bar{E}(t)))} \, dt,$$

where

$$\bar{E} = \lambda_{2j-1}(E_{2j-2} - zM).$$

In the integral in (349) we make the change of variable

$$t = (1 - t_1)\bar{t} + t_1$$

such that

$$\bar{E}(t) = \bar{E}(\bar{t}) := \bar{E} - (\bar{E} + 1)\bar{t}, \quad \text{where} \quad \bar{E} := \bar{E} - (\bar{E} + 1)t_1 \quad (350)$$

and

$$\tilde{I}_4 = \sqrt{1-t_1} \int_0^{1-t_2} \frac{1 + \bar{b}_{2j-1}
(1 + \bar{E})(\bar{t} + t_2), \bar{\Theta}_{2j-1}(\bar{E}(\bar{t}))}{\sqrt{t} + t_2 \partial_b \bar{G}_{2j-1}(\bar{\Theta}_{2j-1}(\bar{E}(\bar{t}))} \, d\bar{t}, \quad (351)$$

where

$$t_2 := \frac{t_1}{1-t_1}.$$ 

Since $\bar{\lambda}_{2j-1}$ (defined in (59)) is an increasing function$^{69}$,

$$-1 = \bar{\lambda}_{2j-1}(E_{2j-1}) < \bar{\lambda}_{2j-1}(E_{2j-2}) = 1, \quad (352)$$

recalling (343). By (60) we have

$$|\lambda_{2j-1}(E_{2j-2} - zM) - \lambda_{2j-1}(E_{2j-2})| \leq \frac{4}{\beta} \left(|E_{2j-2} - \bar{E}_{2j-2}| + M \bar{r}_2 \right) \leq \frac{4}{\beta}(2\eta + M \bar{r}_2) \quad (53)$$

$^{69}$Note that $\bar{\lambda}_i(E_i)$ is real.
and
\[ |\lambda_{2j-1}(\bar{E}_{2j-2}) - \bar{\lambda}_{2j-1}(\bar{E}_{2j-2})| \leq \frac{48M\eta}{\beta^2} \leq \frac{96\eta}{\beta} \]
by (61). Then
\[ |\bar{\vec{E}} - 1| \leq \frac{2\eta}{\beta} + \frac{4M\bar{r}}{\beta} \leq \frac{2\eta}{\beta} + \frac{s_0^{20}\beta^{29}}{2^{212}M^{29}} < \frac{\bar{r}}{2} \]  
(353)
(\bar{r} defined in (141)). By (350) and (310) we have
\[ |\bar{E} - \bar{\vec{E}}| \leq 3t_1 \]
and, therefore, by (353)
\[ |\bar{E} - 1| \leq \bar{r} + 3t_1. \]  
(354)
Set\(^{70}\)
\[ \bar{E} - 1 =: x_1 + ix_2, \quad |x_i| \leq \frac{\bar{r}}{2}, \]  
(355)
by (353). We note that
\[ -\frac{1}{2} < \text{Re} \bar{E} - 1 = x_1 - (1 + x_1)t_1 \leq -\frac{t_1}{2} < 0, \]  
(356)
indeed, the first inequality is immediate by (354); regarding the second one we note that, if \(x_1 \leq 0\) it is obvious, otherwise, when \(x_1 > 0\), we have \(x_1 - (1 + x_1)t_1 < x_1 - t_1 \leq \frac{\bar{r}}{2} - t_1 \leq -\frac{t_1}{2} < 0\) since
\[ \bar{r} < t_1 \]
(recalling (141), (310), (31)). Moreover
\[ |\text{Im} \bar{E}| = |\text{Im} \bar{E}|(1 - t_1) < |\text{Im} \bar{E}| \leq \frac{\bar{r}}{2}, \]  
(357)
by (355). Recollecting by (356) and (357) we get
\[ \bar{E} \in \Omega_{\bar{r}}, \quad \text{Re} \bar{E} < 1. \]  
(358)
By (239), (252), we have that for every \(0 \leq t \leq 1 - t_1\)
\[ \left| \bar{b}_{2j-1}\left( (1 + \bar{E})(\bar{i} + t_2), \bar{\Theta}_{2j-1}(\bar{E}(\bar{i})) \right) \right| \leq \frac{9\eta}{\bar{r}_0^2} \leq 1 \]
\(^{70}\)With \(x_i \in \mathbb{R}.\)
by (40). Then by (351) we get

\[ |\tilde{I}_4| \leq \int_0^1 \sqrt{t} \left| \partial_\theta \Theta_{2j-1}(\tilde{\Theta}_{2j-1}(\tilde{E}(\tilde{t})) \right| d\tilde{t}, \]

Since \( \tilde{E} \in \Omega_\tilde{\theta} \) by (273) we get

\[ |\tilde{I}_4| \leq \int_0^1 \sqrt{\tilde{t}} \left| \partial_\theta \tilde{\Theta}_{2j-1}(\tilde{\tilde{\Theta}}_{2j-1}(\tilde{E}(\tilde{t}))) \right| d\tilde{t}. \]

Since (358) holds, we can apply estimate (280) obtaining

\[ |\tilde{I}_4| \leq \frac{2^{33} \check{M}}{\beta^{3/4}} \left( \frac{1}{\sqrt{|1 + \tilde{E}|}} + \ln \frac{1}{1 - \text{Re} \tilde{E}} \right). \]

Then by (354), (356) we get

\[ |\tilde{I}_4| \leq \frac{2^{35} \check{M}}{\beta^{3/4}} \ln \frac{1}{t_1} \leq \frac{2^{68} M}{\beta^7} \frac{100 M^{15}}{s_{0}^{23} \beta^{15}} \leq \frac{2^{80} M^{15/2}}{\beta^{15/2} s_{0}^{27/2}} \]

by (69)-(72), (310) and (339). Then by (349), (306), (55) we get

\[ |I_4| \leq \frac{2^{83} M^{8}}{\beta^{17/2} s_{0}^{27/2}}. \]  \hfill (359)

\[ \bullet \text{ Proof of (300)} \]

Recalling (309) and (340) we set

\[ \varphi(z) = I_1(zM) + I_{2,0}(zM) + I_{2,1}(zM) + I_{2,3}(zM) + I_3(zM) + I_4(zM). \]  \hfill (360)

Then by (311), (323), (324), (338), (348), (359) we get

\[ \sup_{|z| < r_2} |\varphi(z, \tilde{I})| \leq \frac{32 \sqrt{M}}{s_{0}^{3/2}} + \frac{2^4 \sqrt{M}}{\beta s_{0}} + \frac{2^{10} \sqrt{M}}{s_{0}^{3/2} \beta^{3/2}} \frac{M}{s_{0}^{3/2} \beta^{3/2}} + \frac{2^{71} M^{8}}{s_{0}^{13} \beta^{17/2}} + \frac{2^{83} M^{8}}{\beta^{17/2} s_{0}^{27/2}}. \]

Then by (28) and (31) also the first estimate in (300) follows.

---

\( ^{71} \)With \( \tilde{E} \) instead of \( \tilde{E} \).

\( ^{72} \)Again with \( \tilde{E} \) instead of \( \tilde{E} \).
Concerning the functions \( \partial_E I_n^{(2j-1),\pm} \)

Recalling (308) and (309) we have that
\[
\partial_E I_n^{(2j-1),+} = I_3 + I_5, \quad \partial_E I_n^{(2j-1),-} = I_2 + I_4 + I_6,
\]
with
\[
I_5 := \int_{1-t_1}^{1} \mathcal{I}_1 dt, \quad I_6 := \int_{1-t_1}^{1} \mathcal{I}_2 dt.
\]

We have to consider only the term
\[
I_5 = \frac{\sqrt{u}}{2\pi} \int_{1-t_1}^{1} \frac{1 + \hat{b}(t(E - E_{2j-1}), \Theta_{2j}(E(t)))}{\sqrt{1 - t} \partial_\theta \mathcal{G}(\Theta_{2j}(E(t)))} dt,
\]
the term \( I_6 \) being analogous. Noting that \( r_2 M < r_* \) we can argue as in Lemma 5.16, obtaining
\[
|I_5| = \left| \frac{1}{2\pi} \int_{1-t_1}^{1} \frac{1 + \hat{b}(t(E - E_{2j-1}) - w^2(t), \theta_{2j-1} + w(t)\Theta_{2j-1}(w(t)))}{\sqrt{1 - t} \Theta_{2j-1}(w(t)) \mathcal{G}(w(t)\Theta_{2j-1}(w(t)))} dt \right| \leq \frac{32\sqrt{M}}{s_0 \beta}
\]
(assuming as in (295) and recall the definition of \( w(t) \) in (287)). This proves (303). □

Lemma 5.18 If
\[
|z| \leq \frac{s_0^4 \beta^2 r_2^2}{2^{52} M^2}, \quad z \in \mathbb{C}_*,
\]
then
\[
\left| \frac{\partial_{EE} I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I})}{(\partial_{EE} I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I}))^3} \right| \geq \frac{s_0^4 \beta^3}{2^{30} M^3} \frac{1}{\sqrt{|z|}},
\]

Proof

Deriving (299) w.r.t. \( z \) we get, for \( |z| \leq r_2/2, z \in \mathbb{C}_* \)
\[
M|\partial_{EE} I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I})|
\geq \frac{1}{|z|} |\psi^{(2j-1)}(0, \hat{I})| - \frac{2}{r_2} \sup_{|z| < r_2} |\psi^{(2j-1)}(z, \hat{I})|(1 + |\ln z|) - \frac{2}{r_2} \sup_{|z| < r_2} |\phi^{(2j-1)}(z, \hat{I})|
\geq \frac{1}{|z|} \frac{s_0}{32 \sqrt{M}} - \frac{28 \sqrt{M}}{\beta s_0 r_2} (1 + |\ln z|) - \frac{285 M^8}{s_0 s_1^{13} \beta^{17/2} r_2}
\]

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by Cauchy estimates, (300) and (301). Then, using that for \(|z| \leq 1/e^2, z \in \mathbb{C}_*, |\ln z| \leq 1/\sqrt{|z|}\), we get, for \(z\) as in (362), that the following estimate holds:

\[
|\partial E I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I})| \\
\geq \frac{1}{|z|} \frac{2^9}{s_0} - \frac{2^{85}M^7}{s_0 s_1^{13} \beta^{17/2} s_2} \geq \frac{1}{|z|} \frac{s_0}{2^{6}M^{5/2}}.
\]

Since \(|\ln z| \leq 1/|z|^{1/6}\) for \(|z| \leq e^36, z \in \mathbb{C}_*,\) and recalling (299) and (300) we get for \(z\) as in (362)

\[
|\partial E I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I})| \leq \frac{2^{84}M^8}{s_0 s_1^{13} \beta^{17/2}} + \frac{2^{7} \sqrt{M}}{\beta s_0 |z|^{1/6}} \leq \frac{2^{8} \sqrt{M}}{\beta s_0 |z|^{1/6}}.
\]

(363) follows.

Lemma 5.19 The functions \(\partial E I_n^{(2j-1)}(E, \hat{I})\) have holomorphic extension to\(^{73}\)

\[
E \in (E_+^{(2j-1)}, E_-^{(2j-1)}) - 3r_2M/2, r_2M \cap \{ \text{Re } E > E^{(2j-1)}_+ + r_2M/2 \} \quad \hat{I} \in \hat{D}_r_0
\]

with uniform estimate

\[
|\partial E I_n^{(2j-1)}(E, \hat{I})| \leq \frac{2^{79}M^{15/2}}{s_0^{13} \beta^{8} \sqrt{s_2}}.
\]

Proof We proceed in a way similar to Lemma 5.17. First we define \(I_1, I_2\) as in (308) but with

\[
\hat{E} = \lambda_{2j}(E), \quad \text{respectively } \quad \hat{E} = \lambda_{2j-1}(E).
\]

Then we define \(I_1\) and \(I_3\) as in (309), while

\[
I_7 := \int_0^{1-t_1} \mathcal{I}_2 \, dt,
\]

so that

\[
\partial E I_n^{(2j-1)} = I_1 + I_3 + I_7.
\]

The estimate of the term \(I_1\) and \(I_3\) are as in Lemma 5.17. The estimates of the term in \(I_7\) is similar to the one of \(I_4\) in Lemma 5.17. More precisely, since \(E = E_1 + E_2\) with

\(^{73}\)r_2 \text{ was defined in } (298).
\( E_1 \in (\bar{E}_{2j-1} + r_2M/2^6, \bar{E}_{2j-1} - 3rM/2) \) and \( |E_2| < r_2M, \ \text{Re} \ E_2 \geq 0 \), we have, recalling (59) and the definition of \( \bar{r} \) in (141),

\[
|\lambda_{2j-1}(E) - \bar{\lambda}_{2j-1}(E)| \leq \frac{48M\eta}{\beta^2} \leq \frac{r_2}{2^6} \leq \frac{\bar{r}}{4},
\]

\[
|\bar{\lambda}_{2j-1}(E) - \bar{\lambda}_{2j-1}(E_1 + \text{Re} \ E_2)| = \frac{2|\text{Im} \ E_2|}{E_{2j-2} - E_{2j-1}} \leq |\bar{\lambda}_{2j-1}(E) - \bar{\lambda}_{2j-1}(E_1)|
\]

\[
= \frac{2r_2M}{E_{2j-2} - E_{2j-1}} \leq \frac{2r_2M}{\beta} \leq \frac{\bar{r}}{4},
\]

\[
-1 + \frac{r_2}{2^9} \leq -1 + \frac{r_2M}{2^8(E_{2j-2} - E_{2j-1})} = \bar{\lambda}_{2j-1}(\bar{E}_{2j-1} + r_2M/2^9) < \bar{\lambda}_{2j-1}(E_1)
\]

\[
= \text{Re} \left( \bar{\lambda}_{2j-1}(E) \right) \leq \bar{\lambda}_{2j-1}(E_1 + \text{Re} \ E_2) < \bar{\lambda}_{2j-1}(\bar{E}_{2j-2} - 3rM/2 + r_2M)
\]

\[
= 1 - \frac{r_2M}{E_{2j-2} - E_{2j-1}} \leq 1 - \frac{r_2}{2}, \quad (367)
\]

(recalling (25)) and noting that

\[
\text{Re} \left( \bar{\lambda}_{2j-1}(E) \right) = \bar{\lambda}_{2j-1}(E_1 + \text{Re} \ E_2), \quad \text{Im} \left( \bar{\lambda}_{2j-1}(E) \right) = \frac{2\text{Im} \ E_2}{E_{2j-2} - E_{2j-1}}.
\]

Recalling (367) we get

\[
\ddot{E} = \lambda_{2j-1}(E) \in \Omega_{\bar{r}}, \quad -1 + \frac{r_2}{210} \leq \text{Re} \ \ddot{E} \leq 1 - \frac{r_2}{4}. \quad (368)
\]

By (308),(365) and recalling (309) we get\(^{74}\)

\[
I_7 = \frac{\theta_{2j-1} - \theta_{2j-2}}{E_{2j-1} - E_{2j-2}} \sqrt{E - E_{2j-1}} \tilde{I}_7, \quad \text{with}
\]

\[
\tilde{I}_7 := \int_0^{1-t_1} \frac{1 + \dot{\bar{b}}_{2j-1}(t(1 + \dot{E}), \dot{\bar{\Theta}}_{2j-1}(\dot{E}(t)))}{\sqrt{t} \ \partial_\theta \tilde{\Gamma}_{2j-1}(\dot{\bar{\Theta}}_{2j-1}(\dot{E}(t)))} \ dt. \quad (369)
\]

By (368), we can apply estimate (276) obtaining

\[
|\tilde{I}_7| \leq \int_0^1 \frac{2}{\sqrt{t} \ \partial_\theta \tilde{\Gamma}_{2j-1}(\dot{\bar{\Theta}}_{2j-1}(\dot{E}(t)))} \ dt.
\]

---

\(^{74}\)Recalling that \( u = E - E_{2j-1} \) by (305).
Since by (368) we also have $\text{Re } \bar{E} < 1$, by (280) we get
\[
|\tilde{I}_7| \leq \frac{2^{32} \tilde{M}}{\beta^4 s^4} \left( \frac{1}{\sqrt{|1 + \tilde{E}|}} + \ln \frac{1}{1 - \text{Re } \tilde{E}} \right) \leq \frac{2^{68} M^7}{s_0^{13} \beta^7} \left( \frac{2^5}{\sqrt{r_2}} + \ln \frac{4}{r_2} \right) \leq \frac{2^{74} M^7}{s_0^{13} \beta^7 \sqrt{r_2}},
\]
by (69)-(72) and (368). Then by (369), (368), (55) we get
\[
|I_7| \leq \frac{2^{78} M^{15/2}}{s_0^{13} \beta^8 \sqrt{r_2}}.
\]
Then (364) follows recalling (311) and (348).

For every $r, \bar{\eta} \geq 0$ set\(^{75}\)
\[
\mathcal{E}_2^{j-1}(r, \bar{\eta}) := (\bar{E}_-^{(2j-1)}, \bar{E}_+^{(2j-1)})_r \cap \{ \text{Re } E < \bar{E}_+^{(2j-1)} - \bar{\eta} \},
\]
\[
\mathcal{E}_*^{j-1}(r, \bar{\eta}) := \mathcal{E}_2^{j-1}(r, \bar{\eta}) \cap \{ \text{Re } E > \bar{E}_-^{(2j-1)} + r_2 M/2^9 \},
\]
\[
\mathcal{E}_{*\ast}^{j-1}(r, \bar{\eta}) := \mathcal{E}_2^{j-1}(r, \bar{\eta}) \cap \{ \text{Re } E > \bar{E}_+^{(2j-1)} + r_2 M/2^8 \}.
\]
Note that the above families of sets satisfy $\mathcal{E}_2^{j-1}(r, \bar{\eta}) \supset \mathcal{E}_*^{j-1}(r, \bar{\eta}) \supset \mathcal{E}_{*\ast}^{j-1}(r, \bar{\eta})$ and are increasing with $r$ and decreasing with $\bar{\eta}$.

Note that
\[
r_2 \leq \min \left\{ \beta \bar{r}, \frac{r_*}{8 M}, \frac{r_3}{M} \right\},
\]
where $\bar{r}, r_*, r_2$ and $r_3$ were defined in (141),(283),(298) and (387), respectively. Set
\[
r_4 := \frac{r_2 M}{2^5} = \min \left\{ \frac{s_0^{49} \beta^{30}}{2^{219} M^{29}}, \frac{r_0^2}{2^{15}} \right\}.
\]
The following lemma is justified in view of Remark 5.12.

**Lemma 5.20**

i) The functions $\partial_E I_n^{(2j-1),\pm}(E, \hat{I})$ and $\partial_E \bar{I}_n^{(2j-1),\pm}(E)$ are holomorphic on $\mathcal{E}_2^{j-1}(2r_4, 2\bar{\eta}) \times \hat{D}_{r_0}$ and $\mathcal{E}_*^{j-1}(2r_4, 2\bar{\eta})$, respectively.

ii) Moreover, for $2\eta \leq \bar{\eta} \leq r_4$ we have
\[
\sup_{\mathcal{E}_2^{j-1}(2r_4, 2\bar{\eta})} \left| \partial_E I_n^{(2j-1),\pm}(E, \hat{I}) \right|, \quad \sup_{\mathcal{E}_*^{j-1}(2r_4, 2\bar{\eta})} \left| \partial_E \bar{I}_n^{(2j-1),\pm}(E) \right| \leq \frac{2^{80} M^{15/2}}{s_0^{13} \beta^8 \sqrt{r_2}} + \frac{2^8 \sqrt{M}}{\beta s_0} \ln \frac{M}{\bar{\eta}}.
\]

\(^{75}\)Recall the definition of $E_{\pm}^{(i)}$ in (173).
and

\[
\sup_{\varepsilon_{n}^{2j-1}(3\varepsilon_{2}/2\bar{\eta}) \times \hat{D}_{\eta}} \left| \partial_{EE} I_{n}^{(2j-1), \pm}(E, \hat{I}) \right|, \quad \sup_{\varepsilon_{n}^{2j-1}(3\varepsilon_{2}/2\bar{\eta})} \left| \partial_{EE} \tilde{I}_{n}^{(2j-1), \pm}(E) \right|
\]

\[
\leq \frac{9^{9} M^{13/2}}{s_{0}^{13} \beta^{2} \varepsilon_{2}^{3/2}} + \frac{2^{9} \sqrt{M}}{\beta s_{0} \bar{\eta}}. \tag{374}
\]

**Proof** Here we prove only the minus case, namely the case of \( \partial_{EE} I_{n}^{(2j-1), -} \) and \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -} \); the case of \( \partial_{EE} I_{n}^{(2j-1), +} \) and \( \partial_{EE} \tilde{I}_{n}^{(2j-1), +} \) is completely analogous.

i) By Lemma 5.19 we have that \( \partial_{EE} I_{n}^{(2j-1), -}(E, \hat{I}) \) is holomorphic on\(^{76}\)

\[
E \in (\tilde{E}_{2j-1}, \tilde{E}_{2j-2} - 3\varepsilon_{2} / 2\bar{\eta}) \cap \{ \text{Re} \ E > \tilde{E}_{2j-1} + \varepsilon_{2} / 2\bar{\eta} \} \quad \hat{I} \in \hat{D}_{\eta} \quad (375)
\]

with uniform estimate

\[
|\partial_{EE} I_{n}^{(2j-1), -}(E, \hat{I})| \leq \frac{2^{79} M^{15/2}}{s_{0}^{13} \beta^{2} \varepsilon_{2}^{3/2}} \tag{376}
\]

and the same also holds for\(^{77}\) \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -}(E) \). Moreover by Lemma 5.17 \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -}(E, \hat{I}) \) is holomorphic for \( \hat{I} \in \hat{D}_{\eta} \) and \( \hat{I} \in \hat{D}_{\eta} \) and \( E \) belonging to the set

\[
B_{\varepsilon_{2}}(\tilde{E}_{2j-2}(\hat{I})) \cap (\tilde{E}_{2j-2}(\hat{I}) - C_{s}) \tag{377}
\]

and, as above (recall footnote 77), the same also holds for \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -}(E) \) on the set

\[
B_{\varepsilon_{2}}(\tilde{E}_{2j-2}) \cap (\tilde{E}_{2j-2} - C_{s}).
\]

By (53) we have that

\[
B_{\varepsilon_{2}}(\tilde{E}_{2j-2}(\hat{I})) \cap (\tilde{E}_{2j-2}(\hat{I}) - C_{s}) \supset B_{\varepsilon_{2}}(\tilde{E}_{2j-2}) \cap \{ \text{Re} \ E < \tilde{E}_{2j-2} - 2\bar{\eta} \}
\]

\[
\supset B_{7\varepsilon_{2}/8}(\tilde{E}_{2j-2}) \cap \{ \text{Re} \ E < \tilde{E}_{2j-2} - 2\eta \}, \tag{378}
\]

since \( \eta \leq \varepsilon_{2} / 16 \) by (40). Moreover we have that

\[
(\tilde{E}_{2j-1}, \tilde{E}_{2j-2} - 3\varepsilon_{2} / 2\bar{\eta}) \cup B_{7\varepsilon_{2}/8}(\tilde{E}_{2j-2}) \supset (\tilde{E}_{2j-1} - \varepsilon_{2} / 2\bar{\eta}, \tilde{E}_{2j-2})_{\varepsilon_{2}}. \tag{379}
\]

Then by (375)-(379) we get that \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -}(E) \) and \( \partial_{EE} \tilde{I}_{n}^{(2j-1), -}(E, \hat{I}) \) are holomorphic for

\[
E \in \varepsilon_{2j-1}(2\varepsilon_{2}, 2\bar{\eta}) \cap \{ \text{Re} \ E > \tilde{E}_{2j-1} + \varepsilon_{2} / 2\bar{\eta} \} = \varepsilon_{2j-1}(2\varepsilon_{2}, 2\bar{\eta})
\]

\(^{76}\)Recall (173) and Remark 5.11.

\(^{77}\) Indeed this is a particular case of Lemma 5.19 with \( \eta = 0 \) in (39).
and \( \hat{I} \in \hat{D}_{r_0} \). This proves the part \( i \) of the lemma.

\( ii \) Assume now \( 2\eta \leq \bar{\eta} \leq r_4 \). By (299) and (300) we have that for \( E \in B_{\tau_2M}(\hat{E}_{2j-2}(\hat{I})) \cap (E_{2j-2}(\hat{I}) - \mathbb{C}_s) \), \( \hat{I} \in \hat{D}_{r_0} \) it results

\[
|\partial E I_n^{(2j-1),-}(E, \hat{I})| \leq \frac{2^{84}M^8}{s_3s_0^{13}\beta^{17/2}} + \frac{2^7\sqrt{M}}{\beta s_0} \left| \ln \frac{E_{2j-2}(\hat{I}) - E}{M} \right| .
\]

(380)

Take

\[ E \in B_{7\tau_2M/8}(\hat{E}_{2j-2}) \cap \{ \text{Re } E < \hat{E}_{2j-2} - 2\bar{\eta} \}. \]

Recalling (378) and (380) we get

\[
|\partial E I_n^{(2j-1),-}(E, \hat{I})| \leq \frac{2^{84}M^8}{s_3s_0^{13}\beta^{17/2}} + \frac{2^8\sqrt{M}}{\beta s_0} \ln \frac{M}{\eta},
\]

(381)

since

\[
|E_{2j-2}(\hat{I}) - E| \geq |\hat{E}_{2j-2} - E| - |E_{2j-2}(\hat{I}) - \hat{E}_{2j-2}| \geq 2\bar{\eta} - 2\eta \geq \bar{\eta}
\]

and \( M/\bar{\eta} > M/r_4 \geq 2^{10} \). By (379), (376) and (381) we get, recalling the definition of \( r_2 \) in (298) and (41),

\[
\sup_{\varepsilon^{2j-1}(2r_4,2\bar{\eta}) \times \hat{D}_{r_0}} \left| \partial E I_n^{(2j-1),-}(E, \hat{I}) \right| \leq \frac{2^{79}M^{15/2}}{s_0^{13}\beta^8 \sqrt{F_2}} + \frac{2^{84}M^8}{s_3s_0^{13}\beta^{17/2}} + \frac{2^8\sqrt{M}}{\beta s_0} \ln \frac{M}{\eta}
\]

\[
\leq \frac{2^{80}M^{15/2}}{s_0^{13}\beta^8 \sqrt{F_2}} + \frac{2^8\sqrt{M}}{\beta s_0} \ln \frac{M}{\eta}
\]

proving (373) (the estimate on \( \partial E I_n^{(2j-1),-}(E) \) is analogous).

Let us now prove (374). We use (373) with \( \bar{\eta} = r_4 \), namely (recall (372), (298), (31))

\[
\sup_{\varepsilon^{2j-1}(2r_4,2r_4) \times \hat{D}_{r_0}} \left| \partial E I_n^{(2j-1),-}(E, \hat{I}) \right| \leq \frac{2^{80}M^{15/2}}{s_0^{13}\beta^8 \sqrt{F_2}} + \frac{2^8\sqrt{M}}{\beta s_0} \ln \frac{25}{r_2}
\]

\[
\leq \frac{2^{81}M^{15/2}}{s_0^{13}\beta^8 \sqrt{F_2}}.
\]

Then by Cauchy estimates

\[
\sup_{\varepsilon^{2j-1}(3r_4/2,2r_4) \times \hat{D}_{r_0}} \left| \partial EE I_n^{(2j-1),-}(E, \hat{I}) \right| \leq \frac{2^{90}M^{13/2}}{s_0^{13}\beta^8 \sqrt{F_2}}.
\]

(383)
By (299) we have, for $|z| < r_2$,

$$-M\partial_{EE}I_n^{(2j-1)}(E_{2j-2}(\hat{I}) - zM, \hat{I}) = \partial_z \varphi(z, \hat{I}) + \partial_z \psi(z, \hat{I}) \ln z + \frac{1}{z} \psi(z, \hat{I}),$$

with, by (300) and Cauchy estimates,

$$\sup_{\{|z| < r_2/2\} \times \hat{D}_{r_0}} |\partial_z \varphi(z, \hat{I})| \leq \frac{2^{85} M^8}{s \cdot s_0^{13} \beta^{17/2} r_2}, \quad \sup_{\{|z| < r_2/2\} \times \hat{D}_{r_0}} |\partial_z \psi(z, \hat{I})| \leq \frac{2^6 \sqrt{M}}{\beta s_0 r_2};$$

then, using again (300) and (382), we get, for

$$E \in B_{r_2 M/2}(E_{2j-2}(\hat{I})) \cap (E_{2j-2}(\hat{I}) - \mathbb{C}_*), \quad \hat{I} \in \hat{D}_{r_0},$$

that

$$|\partial_{EE}I_n^{(2j-1)}(E, \hat{I})| \leq \frac{2^{85} M^7}{s \cdot s_0^{13} \beta^{17/2} r_2} + \frac{2^8}{s_0 \beta \sqrt{M} r_2} \left| \ln \frac{E_{2j-2}(\hat{I}) - E}{M} \right| + \frac{2^7 \sqrt{M}}{\beta s_0 |E_{2j-2}(\hat{I}) - E|} \leq \frac{2^{85} M^7}{s \cdot s_0^{13} \beta^{17/2} r_2} + \frac{2^9}{s_0 \beta \sqrt{M} r_2} \ln \frac{1}{r_2} + \frac{2^9 \sqrt{M}}{\beta s_0 |E_{2j-2}(\hat{I}) - E|}.$$

By (382) we have that, for

$$E \in B_{r_2 M/2}(E_{2j-2}(\hat{I})) \cap \{\Re E \leq E_{2j-2} - 2\eta\}, \quad \hat{I} \in \hat{D}_{r_0},$$

$$|\partial_{EE}I_n^{(2j-1)}(E, \hat{I})| \leq \frac{2^{85} M^7}{s \cdot s_0^{13} \beta^{17/2} r_2} + \frac{2^9}{s_0 \beta \sqrt{M} r_2} \ln \frac{1}{r_2} + \frac{2^9 \sqrt{M}}{\beta s_0 \eta}. \quad (384)$$

Recalling (372), by (383) and (384) (and (31) and (41)) the estimate (374) follows (the estimate on $\partial_{EE}I_n^{(2j-1)}(E)$ is analogous). \[\blacksquare\]

By (243) and lemmata 5.16,5.17,5.20 we get the following

**Corollary 5.1** The function $I_n^{(2j-1)}(E, \hat{I})$ is holomorphic for $\hat{I} \in \hat{D}_{r_0}$ and

$$E \in (E_{2j-1}^{-}, E_{2j-1}^{+})_{2r_4} \cap \{E_{2j-1}^{(2j-1)}(\hat{I}) - \mathbb{C}_*\}.$$

\[78\] Noting that, for $z = \frac{E_{2j-2}(\hat{I}) - E}{M} = re^{i\theta}, r > 0, -\pi < \theta < \pi$, with $|z| = r \leq r_2/2 < 1/16$, we have

$$|\ln |z| \leq |\ln r| + \pi \leq 2 \ln \frac{1}{r} \leq 2 \ln \frac{1}{r_2} + \frac{r_2}{r}.$$
5.6 Estimates on $\partial_E I_n^{(2j)}$

Deriving (246) w.r.t. $E$ we get

$$
\partial_E I_n^{(2j),-}(E) = \frac{1}{2\pi} \int_{\theta_2 j - 1}^{\theta_2 j} \frac{1 + \tilde{b}(E - G(\theta), \theta)}{\sqrt{E - G(\theta)}} d\theta, \\
\partial_E I_n^{(2j),+}(E) = \frac{1}{2\pi} \int_{\theta_2 j}^{\theta_2 j + 1} \frac{1 + \tilde{b}(E - G(\theta), \theta)}{\sqrt{E - G(\theta)}} d\theta,
$$

(385)

with $\tilde{b}$ defined in (238). Note that by (239) the function $\tilde{b}(E - G(\theta), \theta)$ is well defined for

$$
E \in (E_{2j}(\hat{I}), P_0^2 - 2M) r_0^2 / 2, \quad \hat{I} \in \hat{D}_r, \quad \theta \in T_{s_0}^1.
$$

(386)

Lemma 5.21 Set

$$
r_3 := \min\left\{ \frac{s_0^6 \beta^3}{2^{25} M^3}, \frac{r_0^2}{2^{8} M} \right\}.
$$

(387)

The function $\partial_E I_n^{(2j),-}(E_{2j}(\hat{I}) + zM, \hat{I}) + \partial_E I_n^{(2j),+}(E_{2j}(\hat{I}) + zM, \hat{I})$, initially defined for $0 < z < r_3$ and $\hat{I} \in \hat{D}_r$, has holomorphic extension to the complex set $\{ z \in \mathbb{C} \text{ s.t. } |z| < r_3 \} \times \hat{D}_r$. In particular

$$
\partial_E I_n^{(2j),-}(E_{2j}(\hat{I}) + zM, \hat{I}) + \partial_E I_n^{(2j),+}(E_{2j}(\hat{I}) + zM, \hat{I}) = \varphi(z, \hat{I}) + \psi(z, \hat{I}) \ln z,
$$

(388)

where $\varphi(z, \hat{I})$ and $\psi(z, \hat{I})$ are holomorphic function in the set $\{ |z| < r_3 \} \times \hat{D}_r$ with

$$
\sup_{\{ |z| < r_3 \} \times \hat{D}_r} |\varphi(z, \hat{I})| \leq \frac{2^{20} M}{\beta^{3/2} s_0^2 s_*}, \quad \sup_{\{ |z| < r_3 \} \times \hat{D}_r} |\psi(z, \hat{I})| \leq \frac{16}{\sqrt{\beta}},
$$

(389)

and

$$
\inf_{\hat{I} \in \hat{D}_r} |\psi(0, \hat{I})| \geq \frac{s_0}{4\sqrt{M}}.
$$

(390)

Finally

$$
\sup_{\{ |z| < r_3 \} \times \hat{D}_r/2} |\partial_\hat{I} \varphi(z, \hat{I})| \leq M_\varphi \eta, \quad \sup_{\{ |z| < r_3 \} \times \hat{D}_r/2} |\partial_\hat{I} \psi(z, \hat{I})| \leq M_\psi \eta.
$$

(391)

$M_\varphi, M_\psi$ definite in (304)
Proof We first note that, recalling (386),
\[
\tilde{b}(E_{2j}(\hat{I}) + zM - \mathcal{G}(\theta), \theta)
\]
is well defined for
\[
|z| \leq \frac{r_0^2}{2^{8}M}, \quad \hat{I} \in \hat{D}_{r_0}, \quad \theta \in \mathbb{T}_{s_0/2}.
\]
Setting
\[
\theta_\bullet := \frac{\beta_{s_0}^3}{2^{10}M} \leq s_0/4,
\]
we split
\[
2\pi \left( \partial E I^{(2j)}_{n^2} - (E) + \partial E I^{(2j)}_{n^2} + (E) \right) = I_1 + I_2 + I_3
\]
\[
:= \int_{\theta_{2j-1}}^{\theta_{2j}} + \int_{\theta_{2j}}^{\theta_{2j+1}} \frac{1 + \tilde{b}(E - \mathcal{G}(\theta), \theta)}{\sqrt{E - \mathcal{G}(\theta)}} d\theta.
\]
We first consider the more relevant integral which is \( \int_{\theta_{2j-1}}^{\theta_{2j+1}} \cdot \). Since the interval of integration is symmetric w.r.t. \( \theta_{2j} \) we can consider the "even part" of the integrand, namely, changing variable \( \theta = \theta_{2j} + \vartheta \)
\[
I_2 = \int_{\theta_{2j-1}}^{\theta_{2j+1}} \frac{1 + \tilde{b}(E - \mathcal{G}(\theta), \theta)}{\sqrt{E - \mathcal{G}(\theta)}} d\theta
\]
\[
= \int_{0}^{\theta_{2j}} \left( \frac{1 + \tilde{b}(E - \mathcal{G}(\theta_{2j} + \vartheta), \theta_{2j} + \vartheta)}{\sqrt{E - \mathcal{G}(\theta_{2j} + \vartheta)}} + \frac{1 + \tilde{b}(E - \mathcal{G}(\theta_{2j} - \vartheta), \theta_{2j} - \vartheta)}{\sqrt{E - \mathcal{G}(\theta_{2j} - \vartheta)}} \right) d\vartheta.
\]
Since \( \mathcal{G} \) has a maximum at \( \theta_{2j} \) we have that
\[
\mathcal{G}(\theta_{2j} + \vartheta) = E_{2j} - \beta_0 \vartheta^2 - \vartheta^3 \mathcal{G}_*(\vartheta),
\]
where \( \beta_0 = -\partial_{\theta^0} \mathcal{G}(\theta_{2j}(\hat{I}), \hat{I})/2 \) with
\[
\beta/4 \leq |\beta_0| \leq M/s_0^2
\]
by (55), (52) and
\[
\sup_{|\vartheta| \leq 2\theta_\bullet} |\mathcal{G}_*(\vartheta)| \leq 8M/s_0^3.
\]
We set
\[
\zeta = w^2 M := zM, \quad E = E_{2j}(\hat{I}) + \zeta = E_{2j}(\hat{I}) + w^2 M = E_{2j}(\hat{I}) + zM.
\]
For the moment being, we consider only real \( \hat{I} \in \hat{D} \), so that \( \mathcal{G} \) is real on real. In particular we think \( \zeta > 0, w > 0 \). We also have \( \beta_0 > 0 \).

Then we have

\[
I_2 = \int_0^{\theta^*} \left( \frac{1 + \tilde{b}(\zeta + \beta_0 \bar{\vartheta}^2 + \partial^3 \mathcal{G}_*(\vartheta), \theta_2 + \bar{\vartheta})}{\sqrt{\zeta + \beta_0 \bar{\vartheta}^2 + \partial^3 \mathcal{G}_*(\vartheta)}} + \frac{1 + \tilde{b}(\zeta + \beta_0 \vartheta^2 - \partial^3 \mathcal{G}_*(-\vartheta), \theta_2 - \vartheta)}{\sqrt{\zeta + \beta_0 \vartheta^2 - \partial^3 \mathcal{G}_*(-\vartheta)}} \right) d\vartheta.
\]

Then we split

\[
I_2 = I_4 + I_5 := \int_0^{4w\sqrt{M/\beta_0}} + \int_0^{\theta^*}.
\]

Changing variable \( \vartheta = wy \) we get

\[
I_4 = \int_0^{4\sqrt{M/\beta_0}} \left( \frac{1 + \tilde{b}(w^2M + \beta_0 w^2 y^2 + w^3 y^3 \mathcal{G}_*(wy), \theta_2 + wy)}{\sqrt{M + \beta_0 y^2 + w y^3 \mathcal{G}_*(wy)}} + \frac{1 + \tilde{b}(w^2M + \beta_0 w^2 y^2 - w^3 y^3 \mathcal{G}_*(-wy), \theta_2 - wy)}{\sqrt{M + \beta_0 y^2 - w y^3 \mathcal{G}_*(-wy)}} \right) dy.
\]

We note that for \( |w| \leq \sqrt{r} \) and \( 0 \leq y \leq 4\sqrt{M/\beta_0} \) we have

\[
|wy^3 \mathcal{G}_*(wy)| \leq 2^9 \sqrt{r} M^{5/2}/\beta_0^{3/2} s_0^3 \leq M/8
\]

by (387). Regarding the term \( \beta_0 y^2 \) it is positive when \( \hat{I} \in \hat{D} \); if \( \hat{I} \in \hat{D}_{r_0} \) then \( \hat{I} = \hat{I}_1 + \hat{I}_2 \)
with \( \hat{I}_1 \in \hat{D} \) and \( |\hat{I}_2| < r_0 \); then \( \beta_0 = \beta_1 + \beta_2 \) with

\[
\beta_1 := -\partial_{\theta \theta} \mathcal{G}(\theta_2(\hat{I}_1), \hat{I}_1)/2 \geq \beta/4 > 0
\]

and 

\[
\beta_2 := -\frac{1}{2}(\partial_{\theta \theta} \mathcal{G}(\theta_2(\hat{I}_2), \hat{I}_2) - \partial_{\theta \theta} \mathcal{G}(\theta_2(\hat{I}_1), \hat{I}_1)), \text{ with (recall (39),(48),(25))}
\]

\[
|\beta_2| \leq |\partial_{\theta \theta} \mathcal{G}(\theta_2(\hat{I}_2)) - \partial_{\theta \theta} \mathcal{G}(\theta_2(\hat{I}_1))| + 8\eta/s_0^2 \leq 16\eta M/\beta s_0^3 + 8\eta/s_0^2 \leq \frac{\beta}{2^{16}} \leq \frac{|\beta_0|}{2^{12}}.
\]

Recollecting we have, also in the complex case,

\[
\text{Re } (M + \beta_0 y^2 \pm wy^3 \mathcal{G}_*(\pm wy)) \geq M/2.
\]

Then the modulus of the integrand function in \( I_4 \) is, for every \( 0 \leq y \leq 4\sqrt{M/\beta_0} \), bounded (recall also (239)) by \( 8/\sqrt{M} \). Then \( I_4 \) defines a even\(^79\) holomorphic function of \( w \) in \( |w| \leq \sqrt{r} \) with

\[
|I_4| \leq 16/\sqrt{M}
\]

\(^79\)Since the integrand is even w.r.t. \( w \).
uniformly; equivalently $I_4$ is a holomorphic function of $z$ in $|z| \leq r_3$ with the same bound.

Let us consider now the term $I_5$. We rewrite it as

$$I_5 = \int_{4w \sqrt{M/\beta_0}}^{\theta^*} \frac{1}{\vartheta} G(w^2/\vartheta^2, \vartheta) d\vartheta,$$

where

$$G(\xi, \vartheta) := \frac{1 + b(M\xi \vartheta^2 + \beta_0 \vartheta^2 + \vartheta^3 G_*(\vartheta), \theta_2 + \vartheta)}{\sqrt{\beta_0 + M\xi + \vartheta G_*(\vartheta)}} \cdot \frac{1 + b(M\xi \vartheta^2 + \beta_0 \vartheta^2 - \vartheta^3 G_*(-\vartheta), \theta_2 - \vartheta)}{\sqrt{\beta_0 + M\xi - \vartheta G_*(-\vartheta)}}.$$

By (395) and (392) we obtain that

$$\sup_{|\vartheta| < 2\theta^*} |\vartheta G_*(\vartheta)| \leq 16M\theta^*/s_0^3 \leq |\beta_0|/8.$$

Then $G$ is holomorphic and bounded by

$$\frac{8}{\sqrt{|\beta_0|}} \leq \frac{16}{\sqrt{\beta}}$$

on the set

$$\{||\xi| \leq |\beta_0|/2M\} \times \{|\vartheta| \leq 2\theta^*\},$$

and even w.r.t. $\vartheta$. In particular

$$G(\xi, \vartheta) = \sum_{h,k \geq 0} G_{hk} \xi^h \vartheta^{2k}$$

for suitable coefficients $G_{hk}$ satisfying

$$|G_{hk}| \leq \frac{16}{\sqrt{\beta}} (2M/|\beta_0|)^h (1/4\theta^*_2)^k.$$

Then, using that the series totally converges on the above set, we get

$$I_5 = \sum_{h,k \geq 0} G_{hk} w^{2h} \int_{4w \sqrt{M/\beta_0}}^{\theta^*} \vartheta^{2(k-h)-1} d\vartheta$$

$$= \sum_{h \neq k} G_{hk} w^{2k} \left( \frac{\theta^*_2(2-k-h)}{2(k-h)} - \frac{(4w)^{2(k-h)}(M/\beta_0)^{k-h}}{2(k-h)} \right) + \sum_{h \geq 0} G_{hh} w^{2h} (\ln \frac{\theta^* \sqrt{\beta_0}}{4\sqrt{M}} - \ln w)$$

$$= \psi(z) \ln z + \varphi_1(z),$$

(398)
where
\[ \psi(z) := -\sum_{h \geq 0} \frac{1}{2} G_{hh} z^h \] (399)

and
\[ \varphi_1(z) := -2 \ln \frac{\theta \sqrt{\beta_0}}{4 \sqrt{M}} \psi(z) + \sum_{h \geq 0} \varphi_{1,h} z^h, \quad \varphi_{1,h} := \sum_{k \geq 0, k \neq h} \frac{G_{hk} \theta_\star 2^{(k-h)} + G_{kh} (16M/\beta_0)^{h-k}}{2(k-h)} \] (400)

Now we note that the above representation formula for \( I_5 \) also holds for complex value of \( \hat{I} \) and that the functions \( \psi(z) \) and \( \varphi_1(z) \) are well defined (since their series totally converge) and holomorphic for \( |z| < r_3 \).

Indeed by (397),(394) and (387) we get
\[ \sup_{|z| \leq r_3} |\psi(z)| \leq \frac{16}{\sqrt{\beta}} \] (namely the second estimate in (389) holds) and
\[ |\varphi_{1,h}| \leq \frac{32}{\sqrt{\beta}} \left( \frac{4M}{\theta_\star} \right)^h \]

Then, recalling (400) and (394), we also get\(^80\)
\[ \sup_{|z| \leq r_3} |\varphi_1(z)| \leq \frac{2^9}{\sqrt{\beta}} \left( 1 + \ln \frac{M}{\beta \theta^2_\star} \right) = \frac{2^9}{\sqrt{\beta}} \left( 1 + \ln \frac{2^{20}M^3}{\beta^3 s_0^2} \right) \leq \frac{2^{16}M}{\beta^{3/2} s_0^2} \] (401)

Note that by (399)
\[ \psi(0) = -\frac{1}{2} G_{00} = -\frac{1}{2} G(0,0) = -\frac{1 + \tilde{b}(0, \theta_{2j})}{\sqrt{\beta_0}}, \]
then by (394) and (239) we get (390).

We finally consider the terms \( I_1 \) and \( I_3 \) (recall (393)), which are analogous. First we note that
\[ \bar{E}_{2j} - \bar{G}(\bar{\theta}) \geq \frac{\beta \theta^2_\star}{4}, \quad \forall \bar{\theta}_{2j-1} \leq \bar{\theta} \leq \bar{\theta}_{2j} - \theta_\star, \quad \forall \bar{\theta}_{2j} + \theta_\star \leq \bar{\theta} \leq \bar{\theta}_{2j+1}. \] (402)

\(^80\)In the last inequality we use that \( \ln x < x \) and (31).
Indeed, considering the case $\bar{\theta}_{2j} + \theta_* \leq \tilde{\theta} \leq \bar{\theta}_{2j+1}$ (the other case being analogous), we have, since in such interval $\bar{G}$ is decreasing,

$$E_{2j} - \bar{G}(\tilde{\theta}) \geq \bar{E}_{2j} - \bar{G}(\bar{\theta}_{2j} + \theta_*) \geq -\frac{1}{2} \partial_{\theta_0} \bar{G}(\bar{\theta}_{2j}) \theta_*^2 - \frac{M}{s_0} \theta_*^3 \geq \frac{\beta \theta_*^2}{2} - \frac{M}{s_0} \theta_*^3 \geq \frac{\beta \theta_*^2}{4}$$

by (25), Cauchy estimates, (26) and (392). We now consider a point $\theta \in (\theta_{2j} + \theta_*, \theta_{2j+1})$ (recall the definition in 5.2). This means that there exists $0 \leq t \leq 1$ such that

$$\theta = t(\theta_{2j} + \theta_*) + (1 - t) \theta_{2j+1}.$$ 

Then, set

$$\bar{\theta} = t(\bar{\theta}_{2j} + \theta_*) + (1 - t) \bar{\theta}_{2j+1},$$

with $\bar{\theta}_{2j} + \theta_* \leq \bar{\theta} \leq \bar{\theta}_{2j+1}$. By (48) we get

$$|\theta - \bar{\theta}| \leq 4\eta / \beta s_0 . \tag{403}$$

For

$$E = E_{2j}(\hat{I}) + zM, \quad |z| < r_3,$$

we have

$$|(E - G(\theta)) - (E_{2j} - G(\bar{\theta}))| \leq |E_{2j} - E_{2j}| + r_3 M + 4\eta |G(\theta) - G(\bar{\theta})| \leq 3\eta + r_3 M + \frac{4\eta M}{\beta s_0^2} \leq \frac{\beta \theta_*^2}{8}$$

by (53),(25),(403), (387) and (40). Then by (402) we get

$$|E - G(\theta)| \geq \frac{\beta \theta_*^2}{8}, \quad \text{for} \quad \theta \in (\theta_{2j} + \theta_*, \theta_{2j+1})$$

(analogously for $\theta \in (\theta_{2j-1}, \theta_{2j} - \theta_*)$). Hence, recalling (393) (and (239))

$$|I_1|, \, |I_3| \leq \frac{2^5}{\sqrt{3} \beta \theta_*} = \frac{215 M}{\beta^{3/2} s_0^3} .$$

Recalling (396),(398), (401) (and (31)) we get (389).

We omit the proof of (391).  

For every $r, \tilde{\eta} \geq 0$ set $^81$

$$\mathcal{E}^{2j}(r, \tilde{\eta}) := (E^{(2j)}_-, R_0^2 - 2M)r \cap \{ \Re E > E^{(2j)}_- + \tilde{\eta} \}. \tag{404}$$

Note that this family of sets is increasing with $r$ and decreasing with $\tilde{\eta}$.  

$^81$Recall the definition of $E^{(i)}_\pm$ in (173).
Lemma 5.22 The functions $\partial_E I_n^{(2j),\pm}(E, \hat{I})$ are holomorphic for
$$E \in \mathcal{E}^{2j}(r_0^2/2^7, \tilde{\eta}), \quad \forall \tilde{\eta} \geq \frac{4M}{\beta s_0^3} \eta$$
and $\hat{I} \in \hat{D}_{r_0}$ with uniform estimate
$$\sup_{\mathcal{E}^{2j}(r_0^2/2^7, \tilde{\eta}) \times \hat{D}_{r_0}} |\partial_E I_n^{(2j),\pm}(E, \hat{I})| \leq \frac{2^9}{\sqrt{\beta}} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\eta M^2} \right).$$

Proof We consider only $\partial_E I_n^{(2j),+}(E)$ since the argument for $\partial_E I_n^{(2j),-}(E)$ is analogous. Recalling (177) and splitting the integral we have that
$$2\pi \partial_E I_n^{(2j),+}(E) = \int_{\theta_{2j}}^{\theta_{2j+1}} + \int_{\tilde{\theta}_{2j+1}}^{\tilde{\theta}_{2j+1}} \frac{1 + \tilde{b}(E - \mathcal{G}(\theta), \theta)}{\sqrt{E - \mathcal{G}(\theta)}} d\theta.$$  

For $|\text{Im} \theta| < s_0$ by (39) we have
$$|E - \mathcal{G}(\theta)| \geq |E - \mathcal{G}(\theta)| - \eta \geq |\text{Re}(E - \mathcal{G}(\theta))| - \eta$$
and, by (35) and recalling that $\mathcal{G}(\theta)$ is decreasing for $\theta \in [\tilde{\theta}_{2j}, \tilde{\theta}_{2j+1}]$ we have, for $E$ as in (405),
$$|E - \mathcal{G}(\theta)| \geq \tilde{\eta} + \tilde{E}_{2j} - \mathcal{G}(\theta) - \eta \geq \left\{ \begin{array}{ll} \frac{\tilde{\eta}}{2} + \frac{\beta}{2} (\theta - \tilde{\theta}_{2j})^2 & \text{if } \tilde{\theta}_{2j} \leq \theta \leq \tilde{\theta}_{2j} + \theta_z \\
\frac{\beta}{2} \theta_z^2 & \text{if } \tilde{\theta}_{2j} + \theta_z \leq \theta \leq \tilde{\theta}_{2j+1} \end{array} \right.$$  

where $\theta_z = \frac{\beta s_0^3}{6M}$ was defined in (35). Moreover by (239) we get
$$|1 + \tilde{b}(E - \mathcal{G}(\theta), \theta)| \leq 2.$$  

Then by (409), (410) we get$^{82}$
$$\left| \int_{\tilde{\theta}_{2j}}^{\tilde{\theta}_{2j+1}} \frac{1 + \tilde{b}(E - \mathcal{G}(\theta), \theta)}{\sqrt{E - \mathcal{G}(\theta)}} d\theta \right| \leq \int_{\tilde{\theta}_{2j}}^{\tilde{\theta}_{2j+1}} \frac{2}{\sqrt{|E - \mathcal{G}(\theta)|}} d\theta \leq \frac{2^8}{\sqrt{\beta}} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\eta M^2} \right).$$  

$^{82}$Using that, for $x > 9$ we have $\int_0^x (1 + x^2)^{-1/2} dx = \ln(\sqrt{1 + x^2} + x) \leq 2 \ln x$.  

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For $|\theta - \bar{\theta}_j| \leq 2\eta/\beta s_0$ and $E$ as in (405) we have by (408), (48) and Cauchy estimates

$$|E - \mathcal{G}(\theta)| \geq |E - \bar{\mathcal{G}}(\theta)| - \eta = |E - E_{2j} + \bar{\mathcal{G}}(\bar{\theta}_j) - \bar{\mathcal{G}}(\theta)| - \eta \geq \bar{\eta} - \frac{2\eta M}{\beta s_0^2} - \eta \geq \frac{\bar{\eta}}{4},$$

by (31); then, by (410) and (48), we get

$$\left| \int_{\theta_{2j}}^{\bar{\theta}_{2j}} 1 + \frac{\tilde{b}(E - \mathcal{G}(\theta), \theta)}{\sqrt{E - \bar{\mathcal{G}}(\theta)}} \, d\theta \right| \leq \frac{8\eta}{\sqrt{\bar{\eta}} \beta s_0} \leq \frac{1}{\sqrt{M}},$$  \hspace{1cm} (412)

where the last inequality follows by (40) and (405). The estimate for the last integral in (407) is analogous (even better). Then by (411), (412) (and (31)) we get (406). \qed

By (245) and lemmata 5.21, 5.22 we get the following

**Corollary 5.2** The function $I_n^{(2j)}(E, \hat{I})$ is holomorphic for $\hat{I} \in \hat{D}_{r_0}$ and

$$E \in (E_{-}^{(2j)}, E_{+}^{(2j)})_{2r_4} \cap \{E_{-}^{(2j)}(\hat{I}) + \mathbb{C}_{+} \} \cap \{E_{+}^{(2j)}(\hat{I}) - \mathbb{C}_{+} \}.$$  \hspace{1cm} (415)

**5.7 Closeness of the unperturbed and perturbed actions**

**Lemma 5.23** Let $1 \leq j \leq N$. For

$$\frac{2^5 M \eta}{\beta} \leq \bar{\eta} \leq r_4 \overset{(372)}{=} \frac{r_2 M}{2^5},$$  \hspace{1cm} (413)

we have

$$\left| \partial_{E} I_{n}^{(2j-1), \pm}(E, \hat{I}) - \partial_{E} I_{n}^{(2j-1), \pm}(E) \right| \leq \left( \frac{2^{78} M^7}{s_0^{12} \beta^7 r_2^{3/2}} + \frac{s_0^{12} M}{\bar{\eta}} \right) \frac{2^{17} M^{1/2}}{\beta^2 s_0 \eta}$$ \hspace{1cm} (414)

for$^{83}$

$$E \in \mathcal{E}_{2j-1}^{2j-1}(r_4, \bar{\eta}) \cap \{ \text{Re } E \geq E_{-}^{(2j-1)} + r_2 M/2^7 \}, \quad \hat{I} \in \hat{D}_{r_0},$$

$^{83}$ $\mathcal{E}_{2j-1}^{2j-1}(r_4, \bar{\eta})$ was defined in (370).
Proof By (182), (257) and (277) we have

\[
\left| \frac{\sqrt{2} \sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \partial_{E} I_{n}^{(2j-1),-}(\lambda_{2j-1}(\tilde{E})) - \frac{\sqrt{2} \sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \partial_{E} I_{n}^{(2j-1),-}(\tilde{\lambda}_{2j-1}(\tilde{E})) \right|
\]

\[
= |\partial_{E} I_{n}^{(2j-1),-}(\tilde{E}, \tilde{I}) - \partial_{E} I_{n}^{(2j-1),-}(\tilde{E})| \leq \eta \left( \frac{36}{r_{0}^{2}} + \frac{2^{194} M^{27}}{\beta^{28} s_{0}^{45} s_{*}^{2}} \right) \frac{2^{555} M^{7}}{\beta^{7} s_{0}^{13}} \left( 1 + \ln \frac{1}{1 - \text{Re} \tilde{E}} \right),
\]

(416)

for \( \tilde{I} \in \hat{D}_{r_{0}} \) and \( \tilde{E} \in \Omega_{\tilde{r}} \) with \( \text{Re} \tilde{E} < 1 \). Making the substitution \( \tilde{E} = \tilde{\lambda}_{2j-1}(E) \), (416) becomes

\[
\left| \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \partial_{E} I_{n}^{(2j-1),-}(\lambda_{2j-1}(\tilde{E})) - \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \partial_{E} I_{n}^{(2j-1),-}(E) \right|
\]

\[
\leq \eta \left( \frac{36}{r_{0}^{2}} + \frac{2^{194} M^{27}}{\beta^{28} s_{0}^{45} s_{*}^{2}} \right) \frac{2^{555} M^{7}}{\beta^{7} s_{0}^{13}} \left( 1 + \ln \frac{E_{2j-2} - E_{2j-1}}{2(E_{2j-2} - \text{Re} E)} \right),
\]

(417)

for \( \tilde{I} \in \hat{D}_{r_{0}} \) and

\[
E \in \tilde{\lambda}_{2j-1}^{-1}\left( \Omega_{\tilde{r}} \cap \{ \text{Re} \tilde{E} < 1 \} \right).
\]

Recalling (63) and (26) we note that

\[
\tilde{\lambda}_{2j-1}^{-1}\left( \Omega_{\tilde{r}} \cap \{ \text{Re} \tilde{E} < 1 \} \right)
\]

\[
\supseteq (E_{2j-2}, E_{2j-1})_{\beta \tilde{r} / 2} \cap (C_{*} + E_{2j-1}) \cap \{ \text{Re} E < E_{2j-2} \}
\]

\[
\supseteq C^{2j-1}(2r_{4}, 0) \cap (C_{*} + E_{2j-1}),
\]

(418)

where the last inclusion holds since \( 4r_{4} \leq \beta \tilde{r} / 2 \) by (371) (recall also (173)). By (418) we have that

(417) holds for \( E \in C^{2j-1}(2r_{4}, 0) \cap (C_{*} + E_{2j-1}) \)

(419)

and \( \tilde{I} \in \hat{D}_{r_{0}} \). By (64) we have, for \( |E| \leq 2M \) and \( \tilde{I} \in \hat{D}_{r_{0}} \),

\[
|\lambda_{i}^{-1}(\tilde{\lambda}_{i}(E), \tilde{I}) - E| \leq \frac{12 \eta M}{\beta} \leq \min \left\{ \frac{\tilde{\eta}}{4}, \frac{r_{4}}{25} \right\} \leq \frac{r_{2M}}{2^{10}},
\]

(420)

where the last inequality follows by (40), (413) and (31). By (420) and (31)

\[
\frac{\tilde{\eta}}{4} \geq \frac{4 \eta M}{\beta} \geq 2 \eta.
\]

(421)
Then
\[ E \in \mathcal{E}^{2j-1}(r_4, \tilde{\eta}) \cap \{ \text{Re} \ E \geq \tilde{E}^{(2j-1)}_\pm + r_2 M/2^7 \} \]
\[ \implies \lambda_{2j-1}^{-1}(\bar{\lambda}_{2j-1}(E), \hat{I}) \in \mathcal{E}^{2j-1}_*(3r_4/2, \tilde{\eta}/4) \subset \mathcal{E}^{2j-1}_*(2r_4, 2\tilde{\eta}) \]  \quad (422)
for every \( \hat{I} \in \hat{D}_{r_0} \). For
\[ E \in \mathcal{E}^{2j-1}(r_4, \tilde{\eta}) \cap \{ \text{Re} \ E \geq \tilde{E}^{(2j-1)}_\pm + r_2 M/2^7 \} \subset \mathcal{E}^{2j-1}_*(3r_4/2, \tilde{\eta}/4) \]
by (422) and (420) we have that
\[ \left| \partial_E I_n^{(2j-1)} - \lambda_{2j-1}^{-1}(\lambda_{2j-1}(E)) \right| \leq \sup_{\mathcal{E}^{2j-1}_*(3r_4/2, \tilde{\eta}/4) \times \hat{D}_{r_0}} \left| \partial_{EE} I_n^{(2j-1)} \right| \frac{12\eta M}{\beta} \]
\[ \leq \left( \frac{2^{94}M^{15/2} + 2^{16}M^{3/2}}{s_0^{13}\beta^9 \tilde{\eta}^{3/2}} \right) \eta \]  \quad (423)
by (374) (used with \( \tilde{\eta} \sim \tilde{\eta}/8 \)).

Let us estimate
\[ \left| \sqrt{E_{2j-2} - E_{2j-1}} - \sqrt{\tilde{E}_{2j-2} - \tilde{E}_{2j-1}} \right| \]
\[ = \left| \sqrt{E_{2j-2} - \tilde{E}_{2j-1}} \right| \left| 1 + \frac{E_{2j-2} - \tilde{E}_{2j-2} + E_{2j-1} - E_{2j-1}}{E_{2j-2} - \tilde{E}_{2j-1}} - 1 \right| \]
\[ \leq 8\frac{\sqrt{M}}{\beta} \eta, \]  \quad (424)

\[ ^{84} \text{Note that } \mathcal{E}^{2j-1}_*(3r_4/2, \tilde{\eta}/4) \text{ is a convex set.} \]
by (25), (53). Then we have

\[ \left| \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\bar{\theta}_{2j-1} - \bar{\theta}_{2j-2}} - \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \right| \leq \frac{\sqrt{3M}}{\bar{s}_0^3} \left( \frac{8\sqrt{M}}{\beta} \eta + \frac{8M\eta}{s_0^{5/2}\beta^{3/2}} \right) \]

\[ \leq \frac{2^6 M^{3/2}}{s_0^{4/3}\beta^{2}} \eta, \quad (425) \]

by (424), (48), (33), (56) and (31). Note that by (26)

\[ \left| \frac{\bar{\theta}_{2j-1} - \bar{\theta}_{2j-2}}{\sqrt{E_{2j-2} - E_{2j-1}}} \right| \leq \frac{2\pi}{\sqrt{\beta}}. \quad (426) \]

For \( E \) as in (415) we have by (422), (425) and (373) that

\[ \left| \left( \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} - \sqrt{E_{2j-2} - E_{2j-1}} \right) \partial_{E I_n^{(2j-1)} - \lambda_{2j-1}^{-1}(\bar{\lambda}_{2j-1}(E))} \right| \]

\[ \leq \frac{2^6 M^{3/2}}{s_0^{4/3}\beta^{2}} \eta \left( \frac{2^{80} M^{15/2}}{s_0^{13} \beta^8 \sqrt{\bar{\xi}_2}} + \frac{2^8 \sqrt{M}}{\beta s_0^{5/2} \ln \eta} \right). \quad (427) \]
Then we have for $E$ as in (415)

\[
\left| \partial_E I_n^{(2j-1),-} \left( \lambda_{2j-1}^{-1} \left( \tilde{\lambda}_{2j-1}(E) \right) \right) - \partial_E \bar{I}_n^{(2j-1),-}(E) \right|
\]

\[
\leq \frac{2\pi}{\sqrt{\beta}} \left| \frac{\sqrt{E_{2j-2} - E_{2j-1}}}{\theta_{2j-1} - \theta_{2j-2}} \left( \partial_E I_n^{(2j-1),-} \left( \lambda_{2j-1}^{-1} \left( \tilde{\lambda}_{2j-1}(E) \right) \right) - \partial_E \bar{I}_n^{(2j-1),-}(E) \right) \right|
\]

(426), (419)

\[
\leq \frac{2^9 M^{3/2}}{s_0^{4/3} \beta^{5/2}} \eta \left( \frac{2^{80} M^{15/2}}{s_0^{13/3} \beta^8 \sqrt{s_2}} + \frac{2^8 \sqrt{M}}{\beta s_0} \ln \frac{M}{\bar{\eta}} \right)
\]

\[
+ \eta \left( \frac{36}{r_0^2} + \frac{2^{194} M^{27}}{\beta^{28} s_0^{15/2} s_2^2} \right) \frac{2^{58} M^7}{\beta^{15/2} s_0^{13/2}} \left( 1 + \ln \frac{E_{2j-2} - E_{2j-1}}{2\bar{\eta}} \right)
\]

\[
\leq \eta \left( \frac{2^{80} M^9}{s_0^{17/2} \beta^{21/2} \sqrt{s_2}} \right) + \eta \left( \frac{1}{r_0^2} + \frac{2^{194} M^{27}}{\beta^{28} s_0^{45/2} s_2^2} \right) \frac{2^{66} M^7}{\beta^{15/2} s_0^{13}} \ln \frac{M}{\bar{\eta}}
\]

(427), (419)

\[
\leq \eta \left( \frac{1}{r_0^2} + \frac{2^{194} M^{27}}{\beta^{28} s_0^{45/2} s_2^2} \right) \frac{2^{66} M^7}{\beta^{15/2} s_0^{13}} \ln \frac{M}{\bar{\eta}}
\]

(428)

using ((31) and) that by (413) and (372)

\[
\frac{M}{\bar{\eta}} \geq \frac{M}{r_4} = \frac{2^5}{r_2}
\]

and, in the last inequality, (298).
Recollecting for $E$ as in (415) we have that (414) follows by

\[
\left| \partial_E f_n^{(2j-1),-}(E) - \partial_E \bar{f}_n^{(2j-1),-}(E) \right|
\]

\[
\leq \left| \partial_E f_n^{(2j-1),-}(\lambda_{2j-1}^{-1}(\bar{\lambda}_{2j-1}(E))) - \partial_E \bar{f}_n^{(2j-1),-}(E) \right|
\]

\[
+ \left( \frac{2^{94}M^{15/2}}{s_0^{13} \beta^{9} \tau_2^{3/2}} + \frac{2^{16}M^{3/2}}{\beta^2 s_0 \eta} \right) \eta
\]

\[
\leq \eta \left( \frac{1}{r_0^2} + \frac{2^{194}M^{27}}{\beta^{28} s_0^{15} s_2^{12}} \right) \frac{2^{66}M^7}{\beta^{11/2} s_0^{12} \ln \frac{M}{\eta}} + \left( \frac{2^{14}M^{15/2}}{s_0^{13} \beta^{9} \tau_2^{3/2}} + \frac{2^{18}M^{3/2}}{\beta^2 s_0 \eta} \right) \eta
\]

\[
= \left[ \frac{2^{10}M}{r_0^2} + \frac{2^{204}M^{28}}{\beta^{28} s_0^{45} s_2^{12}} \right] \frac{2^{40}M^{11/2}}{\beta^{11/2} s_0^{12} \ln \frac{M}{\eta}} + \left( \frac{2^{78}M^7}{s_0^{12} \beta^7 \tau_2^{3/2}} + \frac{M}{\eta} \right) \frac{2^{16}M^{1/2}}{\beta^{2} s_0 \eta}
\]

\[
\leq \left( \frac{2^{78}M^7}{s_0^{12} \beta^7 \tau_2^{3/2}} + \frac{2^{17}M^{1/2}}{\beta^2 s_0 \eta} \right)
\]

where in the last inequality we have used that

\[
\frac{2^{41}M^{11/2} s_2^{12}}{\beta^{11/2} s_0^{12} \tau_2} \ln \frac{M}{\eta} \leq \left( \frac{2^{78}M^7 s_2^{12}}{s_0^{12} \beta^7 \tau_2^{3/2}} + \frac{M}{\eta} \right).
\]

(429)

In order to prove (429) we first note that, for $a, b > 0$, 

\[
\min_{x > 0} (x + b - a \ln x) = a + b - a \ln a.
\]

Using the above formula with

\[
a = \frac{2^{41}M^{11/2} s_2^{12}}{\beta^{11/2} s_0^{12} \tau_2}, \quad b = \frac{2^{78}M^7 s_2^{12}}{s_0^{12} \beta^7 \tau_2^{3/2}}
\]

(31)

This last estimate follows by (298) and (31).
Corollary 5.3 Let $1 \leq j \leq N$. If (413) holds we have that the functions $\partial_E I_n^{(2j-1)}(E, \hat{I})$, $\partial_E \bar{I}_n^{(2j-1)}(E)$ are holomorphic on $E^{2j-1}(r_4/4, \bar{\eta})$ with

$$\sup_{E^{2j-1}(r_4/4, \bar{\eta})} \left| \partial_E I_n^{(2j-1)}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j-1)}(E) \right| \leq \left( \frac{2^{78}M^7}{s_0^{12} \beta^2 r_2^{3/2}} + \frac{s_1^{12} M}{\bar{\eta}} \right) \frac{2^{22}M^{1/2}}{\beta^2 s_0} \eta. \tag{431}$$

Then

$$\sup_{E^{2j-1}(r_4/4, \bar{\eta}) \times \hat{D}_{r_0}} \left| \partial_E I_n^{(2j-1)}(E, \hat{I}) \right| \leq \left( \frac{2^{78}M^7}{s_0^{12} \beta^2 r_2^{3/2}} + \frac{s_1^{12} M}{\bar{\eta}} \right) \frac{2^{23}M^{1/2}}{\beta^2 s_0 r_0} \eta. \tag{432}$$

Proof By (414), (415), (178) and (243) we have

$$\left| \partial_E I_n^{(2j-1)}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j-1)}(E) \right| \leq \left( \frac{2^{78}M^7}{s_0^{12} \beta^2 r_2^{3/2}} + \frac{s_1^{12} M}{\bar{\eta}} \right) \frac{2^{18}M^{1/2}}{\beta^2 s_0} \eta \tag{433}$$

for $E \in E^{2j-1}(r_4, \bar{\eta}) \cap \{ \text{Re} E \geq \bar{E}_{(2j-1)}^{(2j-1)} + r_4/4 \}$, $\hat{I} \in \hat{D}_{r_0}$.

By Lemma 5.16, (283) and (372) the function

$$\partial_E I_n^{(2j-1)}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j-1)}(E)$$

is holomorphic on the closed ball

$$B_{3r_4/4}(\bar{E}_{(2j-1)}^{(2j-1)} + r_4/4)$$

and (433) holds on one of its diameter, namely on

$$B_{3r_4/4}(\bar{E}_{(2j-1)}^{(2j-1)} + r_4/4) \cap \{ \text{Re} E = \bar{E}_{(2j-1)}^{(2j-1)} + r_4/4 \}.$$

\textsuperscript{85}By Cauchy estimates.
Then, by the Borel-Caratheodory Theorem\textsuperscript{86}, we get, for every $\hat{I} \in \hat{D}_r,$

$$
\sup_{|E - E^{(2j-1)} - \mathbf{r}_4/4| \leq \mathbf{r}_4/2} \left| \partial_E I_n^{(2j-1)}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j-1)}(E) \right| \\
\leq \left( \frac{278 M^7}{s_0^{12} \beta^7 r_{\mathbf{r}_2}^{3/2}} + \frac{s_0^{12} M}{\eta} \right) \frac{2^{22} M^{1/2}}{\beta^2 s_0} \eta.
$$

Combining this estimate with (433) we get (431).

\textbf{Lemma 5.24} Let $0 \leq j \leq N.$ For $\hat{I} \in \hat{D}_r$ and\textsuperscript{87}

$$
E \in \mathcal{E}^{(2j)}(r_0^2 \beta/2^{10} M, \eta), \quad \forall \eta \leq \frac{2^5 M}{\beta s_0^2} \eta
$$
we have

$$
\left| \partial_E I_n^{(2j),\pm}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j),\pm}(E) \right| \leq \left[ \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_0^2} \right) \frac{2^{55} M^7}{\beta^{15/2} s_0^{13}} \ln \left( 4 + \frac{\beta}{2 \eta} \right) + \frac{2^{21} M^2}{\beta^{5/2} s_0^3} \right] \eta.
$$

\textbf{Proof} For $\hat{I} \in \hat{D}_r$ and $\bar{E} \in \Omega^{(2j),-}_{r_0^2/2^9 M}$ (defined in (263)) with $\text{Re} \bar{E} > 1,$ namely

$$
\bar{E} \in \Omega_{\mathbf{r}_2} := \left( 1, \frac{2 R_0^2}{E_{2j} - E_{2j-1}} - 1 \right)_{r_0^2/2^9 M} \cap \{ \text{Re} \bar{E} > 1 \}.
$$

we have, deriving (182) (recall (59)) and (282), that

$$
\left| \sqrt{2 \sqrt{E_{2j} - E_{2j-1}} - \theta_{2j} - \theta_{2j-1}} \partial_E I_n^{(2j),-}(\lambda_{2j}^{-1}(\bar{E})) \right| = \frac{\sqrt{2 \sqrt{E_{2j} - E_{2j-1}} - \theta_{2j} - \theta_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \partial_E \bar{I}_n^{(2j),-}(\lambda_{2j}^{-1}(\bar{E}))
$$

$$
\leq \left\{ \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{45} s_0^2} \right\} \frac{2^{55} M^7}{\beta^{15/2} s_0^{13}} \ln \left( 4 + \frac{1}{\text{Re} \bar{E} - 1} \right).
$$

\textsuperscript{86} The Borel-Caratheodory Theorem says that if $f$ is a holomorphic function on the closed disc of radius $R$ centered in $z_0$ and $|f(z)|$ is bounded by $M > 0$ on a diameter of the above disc then, for every $0 < r < R,$

$$
\sup_{|z - z_0| \leq r} |f(z)| \leq \frac{2r}{R - r} M + \frac{R + r}{R - r} |f(z_0)| \leq \frac{R + 3r}{R - r} M.
$$

We use the theorem with $z := E, z_0 := \bar{E}_n^{(2j-1)} + r_4/4, f := \partial_E I_n^{(2j-1)} - \partial_E \bar{I}_n^{(2j-1)}, R := 3r_4/4, r := r_4/2, M :=$ the right hand side of (433).

\textsuperscript{87} Recalling (404) and (173) $\mathcal{E}^{(2j)}(r, \eta) := (E_{2j}, r_0^2 - 2M) \cap \{ \text{Re} E > E_{2j} + \eta \}. $
Making the substitution \( \dot{E} = \dot{\lambda}_{2j}(E) \), (437) becomes (recalling (59))

\[
\left| \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \partial_E I_n^{(2j-1)}(\lambda_{2j}^{-1}(\dot{\lambda}_{2j}(E))) \right| \leq \eta \left( \frac{18}{r_0^2} + \frac{2^{193}M^{27}}{\beta^{28}s_0^{45}s_*^{12}} \right) \frac{2^{55}M^7}{\beta^7s_0^{13}} \ln \left( 4 + \frac{1}{\Re \lambda_{2j}(E) - 1} \right),
\]

(438)

for \( \dot{I} \in \dot{D}_{r_0} \) and

\[
E \in \lambda_{2j}^{-1}(\Omega_4) \supset \left( \tilde{E}_{2j} - \frac{\tilde{E}_{2j}}{2}, \frac{\tilde{E}_{2j}}{2} + \frac{3\tilde{E}_{2j-1}}{2} \right) \cap \{ \Re E > \tilde{E}_{2j} \} \supset \mathcal{E}^{2j}(r_0^2\beta/2^{10}M, 0),
\]

(439)

recalling (63) and (26).

For \( E \) as in (434), by (59) and (26) we get

\[
\Re \lambda_{2j}(E) - 1 = \frac{2\tilde{\eta}}{E_{2j} - E_{2j-1}} \geq \frac{2\tilde{\eta}}{\beta},
\]

then, by (437), we get

\[
\left| \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \partial_E I_n^{(2j-1)}(\lambda_{2j}^{-1}(\dot{\lambda}_{2j}(E))) \right| \leq \eta \left( \frac{18}{r_0^2} + \frac{2^{193}M^{27}}{\beta^{28}s_0^{45}s_*^{12}} \right) \frac{2^{55}M^7}{\beta^7s_0^{13}} \ln \left( 4 + \frac{\beta}{2\tilde{\eta}} \right),
\]

(440)

Note that by (420), (434), (40), (31)

\[
E \in \mathcal{E}^{2j}(r_0^2\beta/2^{10}M, \tilde{\eta}) \implies \lambda_{2j}^{-1}(\dot{\lambda}_{2j}(E)) \in \mathcal{E}^{2j}(r_0^2/2^8, \tilde{\eta}/2).
\]

(441)

By (441), (420), (406) and Cauchy estimates we have that

\[
\left| \partial_E I_n^{(2j)}(\lambda_{2j}^{-1}(\dot{\lambda}_{2j}(E))) - \partial_E I_n^{(2j)}(E) \right| \leq \frac{12\eta M}{\beta} \sup_{\mathcal{E}^{2j}(r_0^2/2^8, \tilde{\eta}/2) \times \dot{D}_{r_0}} |\partial_E I_n^{(2j)}| \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta s_0^3}{\tilde{\eta}M^2} \right) \eta.
\]

(442)

\[\text{Indeed we have by (406) used with } \tilde{\eta} \sim \tilde{\eta}/4 \text{ and Cauchy estimates}\]

\[
\sup_{\mathcal{E}^{2j}(r_0^2/2^8, \tilde{\eta}/2) \times \dot{D}_{r_0}} |\partial_E I_n^{(2j)}(E, \dot{I})| \leq \frac{2^{17}}{\sqrt{\beta} \max\{r_0^2, \tilde{\eta}\}} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta s_0^3}{\tilde{\eta}M^2} \right).
\]

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Arguing as in (424) we get

\[ \left| \sqrt{E_{2j} - E_{2j-1}} - \sqrt{\bar{E}_{2j} - \bar{E}_{2j-1}} \right| \leq 8 \frac{\sqrt{M}}{\beta} \eta. \]  

(443)

Arguing as in (425) we get

\[ \left| \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} - \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \right| \leq \frac{2^6 M^{3/2}}{s_0^4 \beta^2} \eta. \]  

(444)

By (26)

\[ \left| \frac{\bar{\theta}_{2j} - \bar{\theta}_{2j-1}}{\sqrt{E_{2j} - E_{2j-1}}} \right| \leq \frac{2\pi}{\sqrt{\beta}}. \]  

(445)

For \( E \) as in (434) we have by (441), (444) and (406) (used with \( \bar{\eta} \sim \bar{\eta}/2 \)) that

\[ \left| \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} - \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \right| \leq \frac{2^{17} M^{3/2}}{s_0^4 \beta^2} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\bar{\eta} M^2} \right) \eta. \]  

(446)

Then we have for \( E \) as in (434)

\[ \left| \partial_E I_n^{(2j),-}(\lambda_{2j}^{-1}(\bar{\lambda}_{2j}(E))) - \partial_E I_n^{(2j),-}(E) \right| \]

\[ \leq \frac{2\pi}{\sqrt{\beta}} \left( \frac{\sqrt{E_{2j} - E_{2j-1}}}{\theta_{2j} - \theta_{2j-1}} \right) \left( \partial_E I_n^{(2j),-}(\lambda_{2j}^{-1}(\bar{\lambda}_{2j}(E))) - \partial_E I_n^{(2j),-}(E) \right) \]

\[ \leq \frac{2^{17} M^{3/2}}{s_0^4 \beta} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\bar{\eta} M^2} \right) \eta + \eta \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta \beta_{28} s_0^4 s_0^2} \right) \frac{2^{17} M^{7}}{\beta^{15/2} s_0^3} \ln \left( 4 + \frac{\beta}{2 \bar{\eta}} \right) \]

\[ \leq \eta \left( 18 \frac{r_0^2}{\beta \beta_{28} s_0^4 s_0^2} + \frac{2^{193} M^{27}}{\beta^{15/2} s_0^3} \right) \frac{2^{17} M^{7}}{\beta^2} \ln \left( 4 + \frac{\beta}{2 \bar{\eta}} \right). \]  

(447)
Recollecting for \( E \) as in (443) we have

\[
\left| \partial_E I_{n}^{(2j)}(\cdot - E) - \partial_E \bar{I}_{n}^{(2j)}(\cdot - E) \right| \\
\leq \left| \partial_E I_{n}^{(2j)}(\cdot - E) - \left( \lambda_{2j}^{-1}(\lambda_{2j}(E)) \right) - \partial_E \bar{I}_{n}^{(2j)}(\cdot - E) \right| \\
+ \frac{2^{21} M}{\beta^{3/2} \min \{ r_0^2, \tilde{\eta} \} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\tilde{\eta} M^2} \right)} \eta \\
\leq \eta \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{15} s_2^2} \right) \frac{2^{56} M^7}{\beta^{15/2} s_0^{13}} \ln \left( 4 + \beta \frac{2}{\tilde{\eta}} \right) + \frac{2^{21} M}{\beta^{3/2} \min \{ r_0^2, \tilde{\eta} \} \left( \frac{M}{\beta s_0^3} + \ln \frac{\beta^3 s_0^6}{\tilde{\eta} M^2} \right)} \eta \\
\leq \left[ \left( \frac{18}{r_0^2} + \frac{2^{193} M^{27}}{\beta^{28} s_0^{15} s_2^2} \right) \frac{2^{57} M^7}{\beta^{15/2} s_0^{13}} \ln \left( 4 + \beta \frac{2}{\tilde{\eta}} \right) + \frac{2^{21} M^2}{\beta^{5/2} s_0^{3} \tilde{\eta}} \right] \eta
\]

proving (435). 

Let us define the rectangle

\[
\mathcal{R}^{(2j)}(r, \tilde{\eta}) := \{ E_{-}^{(2j)} + \tilde{\eta} < \text{Re} E < \bar{E}_{+}^{(2j)} - \tilde{\eta}, \ |\text{Im} E| < r \}. \tag{448}
\]

**Corollary 5.4** Let \( 1 \leq j < N \). Let

\[
\frac{2^5 M}{\beta s_0^2} \eta \leq \tilde{\eta} \leq \frac{r_2 M}{2^5}. \tag{449}
\]

Then \( \partial_E I_{n}^{(2j)}(E, \hat{I}) \) and \( \partial_E \bar{I}_{n}^{(2j)}(E) \) are holomorphic for \( E \in \mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \) and \( \hat{I} \in \hat{D}_{r_0} \) with estimates

\[
\sup_{\mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \times \hat{D}_{r_0}} \left| \partial_E I_{n}^{(2j)}(E, \hat{I}) - \partial_E \bar{I}_{n}^{(2j)}(E) \right| \leq \left( \frac{2^{75} M^6}{s_0^{21/2} \beta^6 r_2^{3/2}} + \beta \frac{2^{25} M^{5/2}}{\tilde{\eta} \beta^4 s_0^{9/2} r_2} \right) \eta. \tag{450}
\]

Then\(^{89}\)

\[
\sup_{\mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \times \hat{D}_{r_0}} \left| \partial_E I_{n}^{(2j)}(E, \hat{I}) \right| \leq \left( \frac{2^{75} M^6}{s_0^{21/2} \beta^6 r_2^{3/2}} + \beta \frac{2^{26} M^{5/2}}{\tilde{\eta} \beta^4 s_0^{9/2} r_2 r_0} \right) \eta. \tag{451}
\]

**Proof** First note that (449) implies both (413) and (434). Moreover, recalling (415), (448), (26), (298) and (31), we have

\[
\mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \subset \mathcal{E}^{2j-1}(r_4, \tilde{\eta}) \cap \{ \text{Re} E \geq E_{-}^{(2j-1)} + r_2 M/2^7 \}
\]

\(^{89}\)By Cauchy estimates.

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and, by (372) and (370),
\[ \mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \subset \mathcal{E}^{2j}(r_0^2 \beta/2^{10} M, \tilde{\eta}). \]

Therefore we can apply estimates (414) and (435), recalling (178), (245) and the fact that by (33) \( N \leq 4\sqrt{M/\beta s_0^2} \), obtaining
\[
\sup_{\mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \times \hat{D}_{r_0}} \left| \partial_E I_n^{(2j)}(E, \hat{I}) - \partial_E \bar{I}_n^{(2j)}(E) \right| \leq \left( \frac{2^{78} M^7}{s_0^{12} \beta^7 r_2^{3/2}} + \frac{s_0^{12} M^7}{\beta^2 s_0 \tilde{\eta}} \right) \frac{2^{22} M^{1/2}}{\beta^2 s_0 \eta} 
+ \frac{8 M^{1/2}}{\beta \beta^2 s_0^{3/2}} \left[ \left( \frac{18}{\beta^2 s_0^{10} r_2^{2}} + \frac{2^{193} M^{27}}{\beta^2 s_0^{4} s_0^2 \tilde{\eta}} \right) \frac{2^{57} M^7}{\beta^3 s_0^{13}} \ln \left( 4 + \frac{\beta}{2 \beta} \right) + \frac{2^{21} M^2}{\beta^3 s_0^{3/2} \tilde{\eta}} \right] \eta 
\leq \left[ \frac{2^{75} M^5}{s_0^{17/2} \beta^5 r_2^{3/2}} + \frac{2^{16} M^4}{\beta^4 s_0^{10} r_2^{2}} \ln \frac{\beta}{\tilde{\eta}} + \frac{\beta}{\tilde{\eta}} \right] \frac{2^{25} M^{5/2}}{\beta^4 s_0^{9/2} r_2^{2}} \eta,
\]
where in the last inequality we have used (31) and that
\[
\frac{2^{16} M^4}{\beta^4 s_0^{10} r_2^{2}} \ln \frac{\beta}{\tilde{\eta}} \leq \frac{2^{75} M^6}{s_0^{21/2} \beta^6 r_2^{3/2}} + \frac{\beta}{\tilde{\eta}}.
\]
The last estimates can be proved as (429) substituting \( a \) and \( b \) in (430) with
\[
a := \frac{2^{16} M^4}{\beta^4 s_0^{10} r_2^{2}}, \quad b := \frac{2^{75} M^6}{s_0^{21/2} \beta^6 r_2^{3/2}}.
\]

**Corollary 5.5** For \( i = 0,2,N, \) and \( \hat{I} \in \hat{D}_{r_0} \) and\(^{90}\)
\[
E \in \mathcal{E}^{i}(r_0^2 \beta/2^{10} M, \tilde{\eta}), \quad \forall \tilde{\eta} \geq \frac{2^{5} M}{\beta s_0^2 \eta},
\]
we have
\[
\left| \partial_E I_n^{(i)}(E, \hat{I}) - \partial_E \bar{I}_n^{(i)}(E) \right| \leq \left( \frac{2^{56} M^{13/2}}{r_2 \beta^3 s_0^{29/2}} \ln \left( 4 + \frac{\beta}{2 \beta} \right) + \frac{2^{24} M^{5/2}}{\beta^3 s_0^{9/2} \tilde{\eta}} \right) \eta.
\]

\(^{90}\)Recalling (404), (173), (30) and (46) we have \( \mathcal{E}^{2N}(x, \tilde{\eta}) = \mathcal{E}^{0}(x, \tilde{\eta}) = (\bar{E}_{2N}, r_0^2 - 2 M) \cap \{ \text{Re } E > \bar{E}_{2N} + \tilde{\eta} \} \).
Then

\[
|\partial_E I_n^{(i)}(E, \hat{I})| \leq \left( \frac{256M^{13/2}}{r_2\beta s_0^{29/2}} \ln \left( 4 + \frac{\beta}{2\eta} \right) + \frac{224M^{5/2}}{\beta^3 s_0^{9/2} \eta} \right) \frac{2\eta}{r_0}.
\]  

(454)

**Proof** By estimate (435), recalling (178), (247) and the fact that by (33) \(N \leq 4\sqrt{M/\beta s_0^3} \), we get

\[
|\partial_E I_n^{(i)}(E, \hat{I}) - \partial_E \bar{I}_n^{(i)}(E)| \leq \frac{8M^{1/2}}{\beta s_0^{3/2}} \left[ \left( \frac{18}{r_0^2} + \frac{2^{193}M^{27}}{\beta s_0^{45} \beta^{15/2} s_0^2} \right) \frac{2^{57}M^7}{\beta^{15/2} s_0^3} \ln \left( 4 + \frac{\beta}{2\eta} \right) + \frac{2^{21}M^2}{\beta^{5/2} s_0^3 \eta} \right] \eta.
\]

(298) 

(41)

\[
(456)
\]

\[
\leq \frac{8M^{1/2}}{\beta s_0^{3/2}} \left[ \frac{1}{r_2} \frac{2^{53}M^{6}}{\beta^{15/2} s_0^{13}} \ln \left( 4 + \frac{\beta}{2\eta} \right) + \frac{2^{21}M^2}{\beta^{5/2} s_0^3 \eta} \right] \eta.
\]

\[
\min_{1 \leq i \leq 2N-1} \inf_{\hat{I} \in \hat{D}} \inf_{E \in (E_n^{(i)}(\hat{I}), E_n^{(i)}_{+}(\hat{I}))} \partial_E I_n^{(i)}(E, \hat{I}) \geq \frac{\sqrt{\beta s_0^{3/2}}}{64M},
\]  

(457)

As a consequence

\[
\partial_E I_n^{(2N)}(E, \hat{I}), -\partial_E I_n^{(0)}(E, \hat{I}) \geq \frac{1}{4\sqrt{E + 3M/2}}, \forall E < R_0^2 - 2M, \hat{I} \in \hat{D}.
\]  

(458)

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91By Cauchy estimates.

5.8 The energy as a function of the actions

Lemma 5.5 also holds (with slightly modified constants) for the perturbed action:

**Lemma 5.25** We have that

\[
\partial_E I^{(2j-1)}(E, \hat{I}) \geq \frac{s_0}{8\sqrt{M}}, \quad \forall E^{(2j-1)}(\hat{I}) < E < E^{(2j-1)}(\hat{I}) \uparrow \hat{I} \in \hat{D}
\]  

(455)

and

\[
\partial_E I^{(2j)}(E, \hat{I}) \geq \frac{1}{16\sqrt{E - E^{(2j-1)}}} \frac{\sqrt{\beta s_0^3}}{M}, \quad \forall E^{(2j-1)}(\hat{I}) < E < E^{(2j-1)}(\hat{I}) \uparrow \hat{I} \in \hat{D}.
\]  

(456)

As a consequence

\[
\min_{1 \leq i \leq 2N-1} \inf_{\hat{I} \in \hat{D}} \inf_{E \in (E_n^{(i)}(\hat{I}), E_n^{(i)}_{+}(\hat{I}))} \partial_E I_n^{(i)}(E, \hat{I}) \geq \frac{\sqrt{\beta s_0^{3/2}}}{64M}.
\]

Moreover

\[
\partial_E I_n^{(2N)}(E, \hat{I}), -\partial_E I_n^{(0)}(E, \hat{I}) \geq \frac{1}{4\sqrt{E + 3M/2}}, \forall E_{2N}(\hat{I}) < E < R_0^2 - 2M, \hat{I} \in \hat{D}.
\]  

(458)
**Proof** First we note that, since
\[
G(\theta_{2j-1} + \theta) - G(\theta_{2j-1}) = G(\theta_{2j-1} + \theta) - E_{2j-1} \leq \frac{2M}{s_0^2} \theta^2
\]
for every \( \theta \), then
\[
\Theta_{2j}(E) - \theta_{2j-1}, \ \theta_{2j-1} - \Theta_{2j-1}(E) \geq \frac{s_0}{\sqrt{2M}} \sqrt{E - E_{2j-1}}.
\]
Therefore, by (459)
\[
\partial E I_n^{(2j-1)+}(E) = \frac{1}{2\pi} \int_{\theta_{2j-1}}^{\Theta_{2j}(E)} \frac{1}{\sqrt{E - G(\theta)}} \left( 1 + \hat{b}(E - G(\theta), \theta) \right) d\theta,
\]
and (239) follows by (34) and (48).
(457) follows by (455), (456), (31) and (245). Finally (458) directly follows by (241), (219), (222), (42).

By Lemma 5.25, (243), (245) and (247) we have that, for \( \hat{I} \in \hat{D} \) and \( 1 \leq i \leq 2N \), the functions \( I_n^{(i)}(E, \hat{I}) \), \(-I_n^{(0)}(E, \hat{I})\) are strictly increasing increasing w.r.t. \( E \) and, therefore, invertible with inverse functions
\[
E^{(i)}(\hat{I}, \cdot) : I_n \in [a_i^{(i)}(\hat{I}), a_i^{(i)}(\hat{I})] \mapsto E^{(i)}(\hat{I}, I_n) \in [E^{(i)}_-(\hat{I}), E^{(i)}_+(\hat{I})], \quad 1 \leq i < 2N,
\]
\[
E^{(2N)}(\hat{I}, \cdot) : I_n \in [a_{2N}^{(2N)}(\hat{I}), a_{2N}^{(2N)}(\hat{I})] \mapsto [E^{(2N)}_-(\hat{I}), R_0^2 - 2M],
\]
\[
E^{(0)}(\hat{I}, \cdot) : I_n \in [a_{2N}^{(0)}(\hat{I}), a_{2N}^{(0)}(\hat{I})] \mapsto [E^{(0)}_-(\hat{I}), R_0^2 - 2M],
\]
\[\]
where
\[
\tilde{a}^{(i)}_{\pm} (\hat{I}) := I^{(i)}_n (E^{(i)}_{\pm} (\hat{I}), \hat{I}) , \quad \text{except for}
\]
(463)
\[
\tilde{a}^{(2N)}_{\pm} (\hat{I}) := I^{(2N)}_n (R^2_0 - 2M, \hat{I}) , \quad \tilde{a}^{(0)}_{\pm} (\hat{I}) := I^{(0)}_n (R^2_0 - 2M, \hat{I}) .
\]
(464)

Note that actually
\[
\tilde{a}^{(2j-1)}_{\pm} := 0 , \quad \forall 1 \leq j \leq N .
\]
(465)

Set
\[
\mathcal{B}^i := \left\{ I = (\hat{I}, I_n) \mid \hat{I} \in \hat{D} , \quad \tilde{a}^{(i)} (\hat{I}) < I_n < \tilde{a}^{(i)}_{\pm} (\hat{I}) \right\} \subseteq \hat{D} \times \mathbb{R} \subseteq \mathbb{R}^n .
\]
(466)

By construction
\[
I^{(i)}_n (E^{(i)} (I), \hat{I}) = I_n \quad \text{on} \quad \mathcal{B}^i .
\]
(467)

For \( \lambda > 0 \) we define
\[
\begin{align*}
\tilde{a}^{(2j-1)} (\hat{I}, \lambda) &:= 0 , \\
\tilde{a}^{(2j-1)}_+ (\hat{I}, \lambda) &:= I^{(2j-1)}_n (E^{(2j-1)}_+ (\hat{I}) - \lambda, \hat{I}) , \quad (1 \leq j \leq N) , \\
\tilde{a}^{(2j)}_+ (\hat{I}, \lambda) &:= I^{(2j)}_n (E^{(2j)}_+ (\hat{I}) + \lambda, \hat{I}) , \quad (1 \leq j < N) , \\
\tilde{a}^{(2N)}_+ (\hat{I}, \lambda) &:= I^{(2N)}_n (E^{(2N)}_+ (\hat{I}) + \lambda, \hat{I}) , \\
\tilde{a}^{(0)}_+ (\hat{I}, \lambda) &:= I^{(0)}_n (R^2_0 - 2M, \hat{I}) .
\end{align*}
\]
and
\[
\mathcal{B}^i (\lambda) := \left\{ I = (\hat{I}, I_n) \mid \hat{I} \in \hat{D} , \quad \tilde{a}^{(i)} (\hat{I}, \lambda) < I_n < \tilde{a}^{(i)}_{\pm} (\hat{I}, \lambda) \right\} \subseteq \hat{D} \times \mathbb{R} \subseteq \mathbb{R}^n .
\]
(468)

Note that
\[
\tilde{a}^{(i)}_+ (\hat{I}, 0) = \tilde{a}^{(i)} (\hat{I}) , \quad \mathcal{B}^i (0) = \mathcal{B}^i .
\]

Set
\[
\begin{align*}
\mathbb{E}^{2j-1} (\lambda, \hat{I}) &:= (E^{(2j-1)}_- (\hat{I}), E^{(2j-1)}_+ (\hat{I}) - \lambda) , \quad 1 \leq j \leq N , \\
\mathbb{E}^{2j} (\lambda, \hat{I}) &:= (E^{(2j)}_- (\hat{I}) + \lambda, E^{(2j)}_+ (\hat{I}) - \lambda) , \quad 1 \leq j < N , \\
\mathbb{E}^i (\lambda, \hat{I}) &:= (E^{(i)}_- (\hat{I}) + \lambda, R^2_0 - 2M) , \quad i = 0, 2N .
\end{align*}
\]
(469)
Note that by construction
\[ \forall I^* = (\hat{I}^*, I_n^*) \in \mathcal{B}^i(\lambda) \quad \exists! \ E^* \in \mathcal{E}^i(\lambda, \hat{I}^*) \text{ s.t. } I_n^{(i)}(E^*, \hat{I}^*) = I_n^*. \quad (470) \]

Define the following domains of \( \mathbb{C}^n \):
\[ \mathcal{D}^i(\lambda, r_0) := \{(E, \hat{I}) \mid E \in \mathcal{E}^i(\lambda, \hat{I}^*), |\hat{I} - \hat{I}^*| < r_0 \} \text{ with } \hat{I}^* \in \hat{D}. \quad (471) \]

**Lemma 5.26** We have, for \( 0 < \lambda \leq r_4/2 \)
\[
\sup_{\mathcal{D}^i(\lambda, r_0)} |\partial_{\hat{E}} I_n^{(i)}(E, \hat{I})| \leq c_{1,0} \ln \frac{M}{\lambda},
\]
\[
\sup_{\mathcal{D}^i(\lambda, r_0)} |\partial_{\hat{E}} I_n^{(i)}(E, \hat{I})| \leq c_{1,1} \eta \ln \frac{M}{\lambda},
\]
\[
\sup_{\mathcal{D}^i(\lambda, r_0/2)} |\partial_{\hat{I}} I_n^{(i)}(E, \hat{I})| \leq c_{0,1} \eta,
\]
\[
\sup_{\mathcal{D}^i(\lambda, r_0/2)} |\partial_{\hat{I}} I_n^{(i)}(E, \hat{I})| \leq c_{0,2} \eta,
\]
for suitable constants \( c_{i,j} \).

**Proof** We omit the details. \( \blacksquare \)

**Lemma 5.27** Let \( 0 < \lambda \leq r_4/2 \) and set
\[
\rho = \rho(\lambda) := \begin{cases} 
\min \left\{ \frac{\beta s_0^3 \lambda}{2^{16} c_{2,0} (c_{0,1} + 1) M^2} \frac{r_0}{2} \right\} & \text{if } \lambda_0 \leq \lambda \leq \frac{M}{2}, \\
\min \left\{ \sqrt{\frac{3^5}{3^2} \lambda} \frac{r_0}{\ln \frac{M}{\lambda}}, \frac{r_0}{2} \right\} & \text{if } 0 < \lambda \leq \lambda_0,
\end{cases}
\]

where \( \lambda_0 := M \exp \left( \frac{-2^{97} M^{19/2}}{s_0^{15} \beta^{19/2}} \right) \).

For every \( 0 \leq i \leq 2N \), the function \( \mathcal{E}^{(i)}(I) \) has analytic extension on the complex \( \rho \)-neighborhood of \( \mathcal{B}^i(\lambda) \), namely \( \mathcal{B}_\rho^i(\lambda) \). Finally
\[
\Xi(\mathcal{B}_\rho^i(\lambda)) \subset \mathcal{D}^i(\lambda, \rho), \quad \text{where} \quad \Xi(I) := (\mathcal{E}^{(i)}(I), \hat{I}). \quad (474)
\]

\(^{94}r_4 \) defined in (372).
Proof We consider first the case $1 \leq i < 2N$. Set

$$\mathcal{F}(E, I) := I_n^{(i)}(E, \hat{I}) - I_n.$$ 

Fix $I^* \in B^i(\lambda)$. By (470) there exist unique $E^* = E^{(i)}(I^*) \in \mathbb{E}^i(\lambda, \hat{I}^*)$ such that $\mathcal{F}(E^*, I^*) = 0$. We want to apply the Implicit Function Theorem to $\mathcal{F}$ in order to find a holomorphic function

$$E^{(i)} : \{|I - I^*| \leq \rho\} \to \{|E - E^*| \leq c\lambda\}, \quad c \leq 1/2, \quad (475)$$

(c to be chosen below see (479)) such that

$$\mathcal{F}(E^{(i)}(I), I) = 0.$$ 

This is possible since

$$\partial_E \mathcal{F}(E^*, I^*) = \partial_E I_n^{(i)}(E^*, \hat{I}^*) \neq 0$$

by Lemma 5.25. By a quantitative version of the Implicit Function Theorem we have to check that

$$\sup_{|I - I^*| \leq \rho} |I_n^{(i)}(E^*, \hat{I}) - I_n| \leq \frac{c\lambda}{2} \delta_*, \quad (476)$$

$$\sup_{|E - E^*| \leq c\lambda, |I - I^*| \leq \rho} |\partial_E I_n^{(i)}(E, \hat{I}) - \partial_E I_n^{(i)}(E^*, \hat{I}^*)| \leq \frac{\delta_*}{2}, \quad (477)$$

where

$$\delta_* := |\partial_E I_n^{(i)}(E^*, \hat{I}^*)|.$$ 

Recalling (472) we have that

$$|I_n^{(i)}(E^*, \hat{I}) - I_n| \leq |I_n^{(i)}(E^*, \hat{I}) - I_n^{(i)}(E^*, \hat{I}^*)| + |I_n^* - I_n|$$

$$\leq (c_{0,1} + 1)\rho \leq \frac{c\lambda}{2} \delta_*$$

by (473). Then, taking

$$\rho \leq \frac{c\lambda\delta_*}{2(c_{0,1} + 1)} \quad (478)$$

we satisfy (476). Moreover by (472) and since $c \leq 1/2$ we have

$$|\partial_E I_n^{(i)}(E, \hat{I}) - \partial_E I_n^{(i)}(E^*, \hat{I}^*)| \leq \frac{2c_{2,0}}{\lambda} c\lambda + c_{1,1}\eta(\ln \frac{2M}{\lambda})\rho$$

$$\leq 2c_{2,0} + c_{1,1}\eta(\ln \frac{2M}{\lambda}) \frac{c\lambda\delta_*}{2(c_{0,1} + 1)} \leq \frac{\delta_*}{2}$$

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if
\[ c \leq \frac{\delta_*}{8c_{2,0}} \quad \text{and} \quad c \leq \frac{c_{0,1} + 1}{4c_{1,1}\eta\lambda\ln(2M/\lambda)}. \]
The second restriction on \( c \) is always satisfied for \( \eta \) small enough by (40). The first restriction is satisfied if we take\(^{95}\)
\[ c \leq \min \left\{ \frac{1}{2}, \frac{\delta_*}{8c_{2,0}} \right\}, \]
which is implied choosing\(^{95}\)
\[ c := \frac{\sqrt{\beta s_0^{3/2}}}{2^9c_{2,0}M}, \quad \text{(479)} \]

since
\[ \delta_* \geq \frac{\sqrt{\beta s_0^{3/2}}}{64M} \quad \text{(480)} \]
by (457).

On the other hand for \( \lambda \) small enough we could always take \( c := 1/2 \) (but for simplicity we take \( c \) as in (479) in any case). Indeed, recalling lemmata 5.17 and 5.21,
\[ \lambda \leq M \exp \left( -\frac{2^{97}M^{19/2}}{s_0^{15}\beta_{319/2}} \right), \]
we have the better (w.r.t. (480)) estimates from below\(^{96}\)
\[ \delta_* \geq \frac{s_0}{2^6\sqrt{M}} \ln \frac{M}{\lambda} \geq \frac{2^{91}M^9}{s_0^4\beta_{319/2}}. \]

Then, taking \( \rho \) as in (473) and recalling (480) we have that (478) is satisfied. Finally (474) follows by (475).

We omit the details of the case \( i = 0, 2N. \)

### 5.9 Derivatives of the energy

We have
\[ \partial_{I_n}E^{(i)} = \frac{1}{\partial E I_n^{(i)}}, \quad \partial I E^{(i)} = -\frac{\partial E I_n^{(i)}}{\partial E I_n^{(i)}}, \quad \partial_{I_n I_n}E^{(i)} = -\frac{\partial E E I_n^{(i)}}{\partial E I_n^{(i)}} \]
\[ \partial_{I_i I_i}E^{(i)} = \frac{\partial E E I_n^{(i)}\partial E I_n^{(i)}}{(\partial E I_n^{(i)})^3} - \frac{\partial E I_n^{(i)}}{(\partial E I_n^{(i)})^2} , \]
\[ \partial_{I_i I_i}E^{(i)} = -\frac{\partial E I_n^{(i)}}{\partial E I_n^{(i)}} + \frac{2\partial E E I_n^{(i)}\partial E I_n^{(i)}}{(\partial E I_n^{(i)})^2} - \frac{\partial E E I_n^{(i)}\partial E I_n^{(i)}}{(\partial E I_n^{(i)})^3}, \quad \text{(481)} \]

\(^{95}\)Even if we did not evaluate \( c_{2,0} \), it is very large!

\(^{96}\)Recall in particular (299), (300), (301), (41).
where $E^{(i)}$ and $I_n^{(i)}$ are evaluated in $(I_n^{(i)}(E, \hat{I}), \hat{I})$ and $(E, \hat{I})$, respectively.

### 5.10 The cosine case

Let us introduce the parameter $\eta > 0$. Let us consider the case in which the unperturbed potential is $(-\eta)$-cosine, namely

$$\tilde{g}(\theta) = -\eta \cos \theta.$$ 

Given $s_0 > 0$, we obviously have that this function is $(M, \beta, s_0)$-Morse-non-degenerate according to Definition 3.1 with, recalling (32),

$$M = \eta \cosh s_0, \quad \beta = \eta. \quad (482)$$

We have only two critical points, namely $N = 1$: the minimum $\bar{\theta}_1 = 0$ and the maximum $\bar{\theta}_2 = \pi$, with corresponding critical energies $\bar{E}_1 = -\eta$ and $\bar{E}_2 = \eta$, respectively. The functions $\bar{\Theta}_i$, $i = 1, 2$ (defined in (38)) are $\bar{\Theta}_1(E) = -\arccos(-E/\eta)$ and $\bar{\Theta}_2(E) = \arccos(-E/\eta)$.

We will put the apex $\ast$ (instead of the bar $\bar{\cdot}$) to mean that we are consider exactly the $(-\eta)$-cosine case.

Set

$$E := E/\eta.$$ 

Then recalling (174) we have that the action variable corresponding to the $-\eta$ cos potential is

$$I_n^{(1),\ast}(E) = \frac{2\sqrt{\eta}}{\pi} \int_0^{\arccos(-E)} \sqrt{E + \cos \theta} d\theta, \quad (483)$$

with inverse function (recall (205))

$$E^{(1),\ast}(0, 4\sqrt{2\eta}/\pi) \to (-\eta, \eta).$$

The derivative of $I_n^{(1),\ast}$ is (recall also (176))

$$\partial_E I_n^{(1),\ast}(E) = \frac{1}{\pi \sqrt{\eta}} \int_0^{\arccos(-E)} \frac{d\theta}{\sqrt{E + \cos \theta}} = \frac{2}{\pi \sqrt{\eta}} \int_0^1 \frac{dy}{\sqrt{1 - E + 2Ey^2 - (E + 1)y^4}}$$

$$= \frac{2}{\pi \sqrt{\eta}} \int_0^1 \frac{dy}{\sqrt{1 - y^4 - E(1 - y^2)^2}} > 0.$$ 

\footnote{Or, which is equivalent, in $I$ and $(E^{(i)}(I), \hat{I})$, respectively.}
making the change of variables $\theta = \arccos \left( (E + 1)y^2 - E \right)$. Moreover

$$
\partial_{EE} I_n^{(1),*}(E) = \frac{1}{\pi \eta^{9/2}} \int_0^1 \frac{(1 - y^2)^2}{(1 - y^4 - E(1 - y^2)^2)^{3/2}} dy > 0.
$$

In particular $\partial_E I_n^{(1),*}(E)$ is an increasing function. Analogously we have that

$$
\partial_{EEE} I_n^{(1),*}(E) = \frac{3}{2 \pi \eta^{5/2}} \int_0^1 \frac{(1 - y^2)^4}{(1 - y^4 - E(1 - y^2)^2)^{5/2}} dy > 0. \quad (484)
$$

Similarly all the derivatives of $I_n^{(1),*}$ are positive functions.

Note that (by the Lebesgue’s theorem)

$$
\lim_{E \to -1^+} \partial_E I_n^{(1),*}(E) = \frac{1}{\sqrt{2 \eta}}, \quad \lim_{E \to -1^+} \partial_{EE} I_n^{(1),*}(E) = \frac{1}{8 \sqrt{2 \eta^{3/2}}}. \quad (485)
$$

The fact that $c_{-\cos}$ (defined in (199)) is equal to the first limit in (485), namely

$$
\inf_{-1 < E < 1} \partial_E I_n^{(1),*}(E) = \frac{1}{\sqrt{2 \eta}}, \quad (486)
$$

follows since $\partial_E I_n^{(1),*}(E)$ is increasing. By direct calculation (or general arguments, recall(299)),

$$
\lim_{E \to -1^-} \partial_{EE} I_n^{(1),*}(E) = +\infty.
$$

Since $\partial_{EE} I_n^{(1),*} > 0$, by (484) and (485) we have

$$
\inf_{-1 < E < 1} \partial_{EE} I_n^{(1),*}(E) \geq \frac{1}{8 \sqrt{2 \eta^{3/2}}} > 0. \quad (487)
$$

We also have, recalling (481),

$$
\partial_{I_n} E^{(1),*}(I_n) = \frac{1}{\partial_{EE} I_n^{(1),*}(E(I_n),*)(I_n))} > 0, \quad \partial_{I_n I_n} E^{(1),*}(I_n) = -\frac{\partial_{EE} I_n^{(1),*}(E(I_n),*)(I_n))}{\partial_{EE} I_n^{(1),*}(E(I_n),*)(I_n))} < 0 \quad (488)
$$

and, by a direct calculation$^{98}$ and (485),

$$
\inf_{0 < I_n < 4 \sqrt{\eta/\pi}} -\partial_{I_n I_n} E^{(1),*}(I_n) = \lim_{E \to -1^+} \frac{\partial_{EE} I_n^{(1),*}(E)}{(\partial_{EE} I_n^{(1),*}(E))^{3/2}} = \frac{1}{4}. \quad (489)
$$

$^{98}$The function is increasing.
Recalling (174) we get

\[ I_n^{(2),*}(E) = \frac{\sqrt{\eta}}{\pi} \int_0^\pi \sqrt{E + \cos \theta} d\theta = -I_n^{(0),*}(E), \tag{490} \]

with inverse functions

\[ E^{(2),*} : (2\sqrt{2\eta}/\pi, +\infty) \to (\eta, +\infty), \quad E^{(0),*} : (-\infty, -2\sqrt{2\eta}/\pi) \to (\eta, +\infty). \]

The derivatives of \( I_n^{(2),*} \) are

\[ \partial_{E} I_n^{(2),*}(E) = \frac{1}{2\sqrt{\eta}} \int_0^\pi \frac{d\theta}{\sqrt{E + \cos \theta}} > 0, \quad \partial_{EE} I_n^{(2),*}(E) = -\frac{1}{4\pi \eta^{3/2}} \int_0^\pi \frac{d\theta}{(E + \cos \theta)^{3/2}} < 0. \tag{491} \]

Note also that \( \partial_{EE} E^{(2),*} > 0 \). We have

\[ E \geq 2\eta \implies \partial_{E} I_n^{(2),*}(E) \leq \frac{1}{\sqrt{2E}}, \quad -\partial_{EE} I_n^{(2),*}(E) \geq \frac{1}{8\sqrt{2E^{3/2}}}. \tag{492} \]

We get

\[ \partial_{I_n^{(2),*}} E(I_n) = \frac{1}{\partial_{E} I_n^{(2),*}(E^{(2),*}(I_n)} > 0, \quad \partial_{I_n^{(2),*}} I_n(I_n) = -\frac{\partial_{EE} I_n^{(2),*}(E^{(2),*}(I_n))}{\left(\partial_{E} I_n^{(2),*}(E^{(2),*}(I_n))\right)^3} > 0 \tag{493} \]

and

\[ \inf_{I_n > 2\sqrt{2\eta}/\pi} \partial_{I_n^{(2),*}} E(I_n) > 0, \]

since, by direct calculation,

\[ \lim_{E \to 1^+} \frac{-\partial_{EE} I_n^{(2),*}(E)}{\left(\partial_{E} I_n^{(2),*}(E)\right)^3} = +\infty \]

and by (491)

\[ \lim_{E \to +\infty} \frac{-\partial_{EE} I_n^{(2),*}(E)}{\left(\partial_{E} I_n^{(2),*}(E)\right)^3} = 2. \]

By the previous limits, (493) and (490) we get

\[ \inf_{I_n > 4\sqrt{2}/\pi} \left| \partial_{I_n^{(2),*}} E(I_n) \right| = \inf_{I_n < -4\sqrt{2\eta}/\pi} \left| \partial_{I_n^{(2),*}} I_n(I_n) \right| \geq c_{**} > 0, \tag{494} \]

for a suitable \( c_{**} > 0 \) (that can be explicitly evaluated!).
5.11 The cosine-like case

An important class of Morse non–degenerate functions, as we will shortly show, is the following.

Lemma 5.28 Let $s_0, \eta > 0$. If $G$ satisfies

$$|G(\theta) + \eta \cos \theta|_{s_0} \leq c\eta, \quad \text{for some} \quad 0 < c \leq \frac{1}{4} \min\{1, s_0^2\},$$

then it is $(M, \beta, s_0)$–Morse–non–degenerate\(^{99}\) with

$$\beta = \frac{\eta}{4}, \quad M = \eta(c + \cosh s_0) \leq \eta\left(\frac{1}{4} + \cosh s_0\right).$$

Moreover $G$ has only two non–degenerate critical points (a maximum and a minimum).

Proof We have, by Cauchy estimates,

$$\frac{1}{\eta}(|G'(\theta)| + |G''(\theta)|) \geq |\sin \theta| + |\cos \theta| - \frac{c}{s_0} - 2\frac{c}{s_0^2} \geq 1 - \frac{c}{s_0} - 2\frac{c}{s_0^2} \geq \frac{1}{4}.$$

We can choose $M$ as above since $|\cos \theta|_{s_0} = \cosh s_0$. Regarding the last sentence we note that for $\theta \in (-\pi, \pi]$ we have only two critical points, a minimum in $(-\pi/6, \pi/6)$ and a maximum in $(-\pi, -5\pi/6) \cup (5\pi/6, \pi]$. Indeed we have that, setting $g(\theta) := \eta^{-1}G(\theta) + \cos \theta$, $\eta^{-1}G'(\theta) = \sin \theta + g'(\theta)$, so that

$$\eta^{-1}G'(\theta) = \sin \theta + g'(\theta) \geq \sin \theta - c/s_0 \geq \sin \theta - 1/4. \quad (495)$$

This implies that $G''(\pi/6) \geq \eta/4$, $G''(-\pi/6) \leq -\eta/4$. Then, by continuity, there exists a critical point of $G$ in $(-\pi/6, \pi/6)$. Moreover such point is a minimum and there are no other critical points in $(-\pi/6, \pi/6)$ since there $G$ is strictly convex:

$$\eta^{-1}G''(\theta) = \cos \theta + g''(\theta) \geq \sqrt{3}/2 - 2c/s_0^2 \geq \sqrt{3}/2 - 1/2 > 0.$$ 

Similarly in $(-\pi, -5\pi/6) \cup (5\pi/6, \pi]$ there is only one critical point, which is a maximum. Finally, by (495), $G'(\theta) \geq \eta/4$ for $\theta \in [\pi/6, 5\pi/6]$ and, analogously, $G'(\theta) \leq -\eta/4$ for $\theta \in [-5\pi/6, -\pi/6]$, so that there are no other critical points. \hfill \qed

---

\(^{99}\)According to Definition 3.1.

\(^{100}\)This is exactly the value of $\rho_2$ in (298) in the case of a $(2\eta e^s_0, \eta/4, s_0)$–Morse-non-degenerate function.
\[ r_2^* = r_2^*(\eta, s_0, r_0) := \min \left\{ \frac{s_0^{49}}{2^{304} e^{30s_0}}, \frac{r_0^2}{2^{111} \eta e^{s_0}} \right\} \]  

and

\[ \eta_\circ^* = \eta_\circ^*(\eta, s_0, r_0) := \frac{s_0^{27/2} (r_2^*)^4 \eta}{2^{135} s_0^{12} e^{6s_0}}. \]  

Note that

\[ \eta_\circ^*(\eta, s_0, r_0) \leq \frac{1}{16} \eta_\circ(\eta e^{s_0}, \eta, s_0, r_0) = \frac{s_0^{15}}{2^{124} e^{9s_0}} \min \left\{ r_0^2, \frac{r_0^3}{\sqrt{\eta e^{s_0/2}}}, \frac{\eta s_0^{75}}{2^{321} e^{44s_0}} \right\}. \]  

**Definition 5.3** Given \( r_0, s_0, \eta > 0 \), we say that a holomorphic function \( G : \hat{D}_{r_0} \times \mathbb{T}_{s_0} \to \mathbb{C} \) is \((-\eta)\)-cosine-like if

\[ |G(\theta, \hat{I}) + \eta \cos \theta|_{\hat{D}_{r_0}, s_0} \leq \frac{1}{16} \eta_\circ(2\eta e^{s_0}, \eta/4, s_0, r_0). \]  

**Proposition 5.1** Assume that \( G \) is \((-\eta)\)-cosine-like. Then for every \( \hat{I} \in \hat{D}_{r_0} \) the function \( G(\theta, \hat{I}) \) is \((2\eta e^{s_0}, \eta/4, s_0)\)-Morse-non-degenerate and, therefore, by (457)

\[ \inf_{E_1(\hat{I}) < E < E_2(\hat{I})} \partial E I_n^{(1)}(E, \hat{I}) \geq \frac{s_0^{3/2}}{2^8 e^{s_0} \sqrt{\eta}}. \]  

Moreover

\[ \inf_{E_1(\hat{I}) < E < E_2(\hat{I})} \partial E I_n^{(1)}(E, \hat{I}) \geq \frac{1}{16}, \quad \inf_{E_1(\hat{I}) < E < E_2(\hat{I})} \partial E I_n^{(2)}(E, \hat{I}) \geq 2, \]  

with \( c_{\bullet \bullet} \) defined in (494).

**Remark 5.14** Imposing a stronger condition in (499) and using Lemma 5.17 (in particular (299) and (301)) we can prove that (500) holds with \( 1/4 \sqrt{\eta} \) on the right hand side.

---

101 Note that \( \cosh s_0 \leq e^{s_0} \).
102 According to Definition 5.3.
103 According to Definition 3.1.
Proof Set
\[ \eta^* := |\mathcal{G} + \eta \cos \theta^*|_{D,r_0,s_0} \leq \eta_0^*(\eta, s_0, r_0) \leq \frac{1}{16} \eta_0(2\eta e^{s_0}, \eta/4, s_0, r_0). \] (502)

Then the Morse-non-degeneracy of \( \mathcal{G} \) follows by Lemma 5.28 (with \( G \rightsquigarrow \mathcal{G} \)).

Let us now prove the first estimate in (500).

We note that by Lemma 5.18 for \( E^2(\hat{I}^0) - s_0^2 \eta^2 \leq E < E^2(\hat{I}^0) \)

we have
\[ \frac{\partial_{E E I_n^{(1)}(E, \hat{I})}}{\partial_{E I_n^{(1)}(E, \hat{I})}} \geq \frac{s_0^2}{2^{20} e^{2 s_0} r^*} \geq 1. \] (503)

Let us now consider the case
\[ E_1(\hat{I}) < E < E_2(\hat{I}) - \frac{s_0^4 \eta^2}{2^{20} e^{2 s_0} r^*} \leq E_2(\hat{I}) - 2^{15} e^{s_0} \eta^* \] (504)

By (502) and (53) we have that \( |E_1(\hat{I}) + \eta|, |E_2(\hat{I}) - \eta| \leq 2\eta^* \leq 2\eta^* \).

Then we have that, if \( E \) satisfies (504) then it also satisfies
\[ -\eta - 4\eta^* < E < \eta - 2^{14} e^{s_0} \eta^*. \] (505)

Recalling (372) we set
\[ r^* := \frac{r^* \eta e^{s_0}}{2^4}, \quad \tilde{\eta}^* := \frac{s_0^4 \eta^2}{2^{38} e^{s_0}} \geq 2^{14} e^{s_0} \eta^*. \]

Since
\[ 2^8 e^{s_0} \eta^* \leq \tilde{\eta}^* \leq r^*/4, \]

we have
104 Note that in this case \( \bar{E}_1 = -\eta, \bar{E}_2 = \eta \).
105 This is exactly condition (413) for the present case.
we can apply Corollary 5.3 obtaining

\[
\sup_{E^{2j-1}(r_1^*/4, \hat{\eta}^*/2) \times \hat{D}_{r_0}} \left| \partial E I_n^{(2j-1)}(E, \hat{I}) - \partial E I_n^{(2j-1),*}(E) \right| \leq \frac{2^{99} e_{s_0}}{s_0^{12} \eta^3} + \frac{2^{25} e_{s_0}}{s_0 \eta^3} \eta^*.
\]

By Cauchy estimate we get (noting that \(r_1^*/8 \geq \hat{\eta}^*/2\))

\[
\sup_{E^{2j-1}(r_1^*/8, \hat{\eta}^*/2) \times \hat{D}_{r_0}} \left| \partial E I_n^{(2j-1)}(E, \hat{I}) - \partial E I_n^{(2j-1),*}(E) \right| \leq \frac{2^{66} s_0^{12} e_{2s_0}}{s_0^5 (r_1^*)^2 \eta^3 / \eta^*} \eta^*.
\]

Now we note that if \(E\) satisfies (504), then it also satisfies (505) and, therefore,

\[
E \in E^{2j-1}(r_1^*/8, \hat{\eta}^*/2),
\]

then, for \(\hat{I} \in \hat{D}\), by (486), (487), (500), (506), (507) we have\(^{107}\)

\[
\frac{\partial E I_n^{(1)}}{(\partial E I_n^{(1)})^3} \geq \frac{1}{4} \left( 1 - \frac{2^{72} s_0^{12} e_{2s_0}}{s_0^5 (r_1^*)^2 \eta^*} \eta^* \right) - \frac{2^{90} s_0^{12} e_{5s_0}}{s_0^{19/2} (r_1^*)^2 \eta^*} \eta^* \geq \frac{1}{16},
\]

where the last inequality follows by (502) and (497). Recalling (503) this conclude the proof of the first estimate in (501).

We omit the proof of the second inequality in (501).
6 The theorem on the universal analytic behavior of actions

Let \( r_0, s_0, M, \beta, \eta > 0 \) and \( \hat{D} \) a domain of \( \mathbb{R}^{n-1} \). Consider a real analytic function on \( T_{s_0} \times \hat{D}_{r_0} \). We will make the following assumptions on \( \mathcal{G} \).

There exists a real–analytic function \( \theta \rightarrow \bar{\mathcal{G}}(\theta) \in \mathcal{B}_{s_0}^1 \) such that the following assumptions hold\(^{108}\)

\[
|\mathcal{G} - \bar{\mathcal{G}}|_{\hat{D}_{r_0,s_0}} \leq \eta_c = \eta_c(M, \beta, s_0, r_0) \quad (A1)
\]

\[
\bar{\mathcal{G}} \in \mathcal{B}_{s_0}^1 \text{ is } (M, \beta)-\text{Morse–non–degenerate} \quad (A2)
\]

In alternative to (A2) we will later consider the following stronger\(^{109}\) assumption: \( \mathcal{G} \) is \((-\eta)\)-cosine–like according to Definition 5.3, namely

\[
|\mathcal{G}(\theta, \hat{I}) + \eta \cos \theta|_{\hat{D}_{r_0,s_0}} \leq \eta \cdot \| \hat{I} \|_{(\mathcal{G}, s_0, r_0)} . \quad (A2')
\]

Theorem 6.1 (Universal analytic behavior of actions)

Part I. Assume that \( \mathcal{G} \) satisfies (A1) and (A2).

(i) (Universal behavior at critical energies)

There exist real-analytic functions \( \phi_{\pm}^{(i)}(z, \hat{I}), \psi_{\pm}^{(i)}(z, \hat{I}) \), with holomorphic extension on \(^{110}\)

\( |z| < r_2, \hat{I} \in \hat{D}_{r_0} \), with estimates

\[
\sup_{|z| < r_2, \hat{I} \in \hat{D}_{r_0}} |\phi_{\pm}^{(i)}| \leq \frac{2^{84}M^8}{s_0 \beta^{17/2}}, \quad \sup_{|z| < r_2, \hat{I} \in \hat{D}_{r_0}} |\psi_{\pm}^{(i)}| < \frac{2^{9} \sqrt{M}}{\beta^{3/2}},
\]

\[
\sup_{|z| < r_2, \hat{I} \in \hat{D}_{r_0/2}} |\partial \phi_{\pm}^{(i)}| \leq M_\phi, \quad \sup_{|z| < r_2, \hat{I} \in \hat{D}_{r_0/2}} |\partial \psi_{\pm}^{(i)}| < M_\psi \eta, \quad (508)
\]

where \( M_\phi, M_\psi \) where defined in (304). Moreover\(^{111}\)

\[
J_n^{(i)} \left( E_{\pm}^{(i)}(\hat{I}) \equiv \text{M}z, \hat{I} \right) = \phi_{\pm}^{(i)}(z, \hat{I}) + \psi_{\pm}^{(i)}(z, \hat{I}) z \log z , \quad \text{for} \quad 0 < z < r_2, \hat{I} \in \hat{D} . \quad (509)
\]

\(^{108}\)\( \eta_c \) defined in (40), while \( (M, \beta)\)-Morse-non-degeneracy in Definition 3.1.

\(^{109}\)Recall (482) and (499).

\(^{110}\)\( r_2 \) was defined in (298).

\(^{111}\)For \( 0 \leq i \leq 2N \) except \( i = 0, 2N \) and the + sign.
(ii) (Analyticity at minimal energies)

In the case of relative minimal critical energies (i.e., \( i = 2j - 1 \))

\[
\psi^{(2j-1)}_\pm = 0,
\]

while in the other cases\(^\text{112}\)

\[
|\psi_{\pm}^{(i)}(0, \hat{t})| \geq \frac{s_0}{32\sqrt{M}}.
\]

(iii) (Perturbative behavior away from critical energies)

Let

\[
\frac{2^5M}{\beta s^2} \eta \leq \tilde{\eta} \leq \frac{r_4 M}{2^5}.
\]

For \( 1 \leq j \leq N \), we have that the functions \( \partial_E I_n^{(2j-1)}(E, \hat{t}), \partial_E \hat{I}_n^{(2j-1)}(E) \) are holomorphic on\(^\text{113}\) \( \mathcal{E}^{2j-1}(r_4/4, \tilde{\eta}) \times \hat{D}_r \) with

\[
\sup_{\mathcal{E}^{2j-1}(r_4/4, \tilde{\eta}) \times \hat{D}_r} \left| \partial_E I_n^{(2j-1)}(E, \hat{t}) - \partial_E \hat{I}_n^{(2j-1)}(E) \right| \leq \left( \frac{2^{78}M^7}{s_0^{12} \beta^7 r_2^{3/2} \eta} + \frac{s_1^{12}M}{\tilde{\eta}} \right) \frac{2^{22}M^{1/2}}{\beta^2 s_0 \eta}.\]

For \( 1 \leq j < N \), the functions \( \partial_E I_n^{(2j)}(E, \hat{t}) \) and \( \partial_E \hat{I}_n^{(2j)}(E) \) are holomorphic on\(^\text{114}\) \( \mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \times \hat{D}_r \) with estimates

\[
\sup_{\mathcal{R}^{(2j)}(r_4/4, \tilde{\eta}) \times \hat{D}_r} \left| \partial_E I_n^{(2j)}(E, \hat{t}) - \partial_E \hat{I}_n^{(2j)}(E) \right| \leq \left( \frac{2^{75}M^6}{r_0^{21/2} \beta^6 r_2^{3/2} \eta} + \frac{\beta}{\tilde{\eta}} \right) \frac{2^{25}M^{5/2}}{\beta^2 s_0^{5/2} r_2}.\]

Finally for \( i = 0, 2N \), the functions \( \partial_E I_n^{(i)}(E, \hat{t}) \) and \( \partial_E \hat{I}_n^{(i)}(E) \) are holomorphic on\(^\text{115}\) \( \mathcal{E}^i(r_0^2 \beta/2^{10} M, \tilde{\eta}) \times \hat{D}_r \) with estimates

\[
\left| \partial_E I_n^{(i)}(E, \hat{t}) - \partial_E \hat{I}_n^{(i)}(E) \right| \leq \left( \frac{2^{56}M^{13/2}}{r_2 \beta^8 s_0^{29/2} \eta} \ln \left( 4 + \frac{\beta}{2\tilde{\eta}} \right) + \frac{2^{24}M^{5/2}}{\beta^3 s_0^{9/2} \eta} \right) \eta.
\]

(iv) (Estimates on the derivatives of the actions)

We have

\[
\min_{1 \leq i \leq 2N-1} \inf_{1 \in D} \inf_{E \in (E_-(i), E_+(i))} \partial_E I_n^{(i)}(E, \hat{t}) \geq \frac{\sqrt{3} s_0^{3/2}}{64M} =: \frac{1}{c}.
\]

\(^{112}\) Namely \( \psi^{(2j)}_\pm \) and \( \psi^{(2j-1)}_\pm \).

\(^{113}\) Recall (370).

\(^{114}\) Recall (448).

\(^{115}\) Recall (404).
\[ \partial E_n^{(2N)}(E, \hat{I}), -\partial E_n^{(0)}(E, \hat{I}) \geq \frac{1}{4\sqrt{E + 3M/2}}, \ \forall E \in (\hat{E}_n^{(0)}(\hat{I}) < E < R_0^2 - 2M, \ \hat{I} \in \hat{D}. \]  

(517)

(v) (Estimates on the derivatives of the energies)

Take \( 0 < \lambda \leq \frac{r_4}{2} \). Recalling the definition of \( B_\lambda \) in \((468)\) and of \( \rho_\lambda \) in \((473)\) we have that the following estimates hold uniformly in \( I \in B_\lambda \rho_\lambda(\hat{I}) \) and for every \( 1 \leq i < 2N \),

\[ \left| \partial I_n E^{(i)} \right| \leq \bar{c}^3 c_2,0 \frac{1}{\lambda}, \]

\[ \left| \partial I_n I E^{(i)} \right| \leq \left( \bar{c}^3 c_2,0 c_0,1 \frac{1}{\lambda} + c^2 c_1,1 \ln \frac{M}{\lambda} \right) \eta, \]

\[ \left| \partial I I E^{(i)} \right| \leq \left( \bar{c}^3 c_0,2 + 2c^2 c_0,1 c_1,1 \eta \ln \frac{M}{\lambda} + \bar{c}^3 c_2,0 c_0,1 \eta \right) \eta, \]  

(518)

where the above constants are defined in \((472)\) and \((516)\). The same estimates hold for the case \( i = 0, 2N, \) with \( 8R_0 \) instead of \( \bar{c} \).

Part II (cosine-like case)

If \( G \) satisfies \((A2')\) then,

\[ \inf_{\hat{I} \in \hat{D}} \inf_{E_1(\hat{I}) < E < E_2(\hat{I})} \partial E_n^{(1)}(E, \hat{I}) \geq \frac{S_0^{3/2}}{2^8 e^{390} \sqrt{\eta}} \]  

(519)

and

\[ \inf_{E_1(\hat{I}) < E < E_2(\hat{I}), \hat{I} \in \hat{D}} -\partial^2 I_n E^{(1)} \geq \frac{1}{16}, \ \inf_{E_2(\hat{I}) < E < R_0^2 - 2M, \hat{I} \in \hat{D}} \partial^2 I_n E^{(2)} \geq 2. \]  

(520)

6.1 Proof of Theorem 6.1

(i) follows by Lemmata 5.17 and (5.21), noting that \( r_2 < r_3 \) (defined in (298) and (387) respectively) and using (245) (and (33), (31)).

(ii) (510) follows by Lemma 5.16. (511) follows by (301) and (390).

(iii) follows by Corollaries 5.3, 5.4 and 5.5.

(iv) (516) is (457). (517) is (458).

(iv) (518) follows by (481), (472), (474) and (516) (517 in the case \( i = 0, 2N \)).

Part II

(519) is (500). (520) is (501) (recall (481)).
7 Appendix

7.1 Technical lemmata on holomorphic functions

Lemma 7.1 Let \( f(\zeta) \) be a holomorphic function on the domain \( C_r \cap \{|z| < r\} \) for some \( r > 0 \). Assume that \( f'(\zeta) = a(\zeta) + b(\zeta) \ln \zeta \), for some functions \( a(\zeta), b(\zeta) \) holomorphic in \( \{ |z| < r \} \). Then \( f(\zeta) = \phi(\zeta) + \chi(\zeta) \zeta \ln \zeta \) where \( \phi(\zeta), \chi(\zeta) \) are holomorphic functions in \( \{ |z| < r \} \) with

\[
\chi(\zeta) = \frac{1}{\zeta} \int_0^\zeta b(z)dz, \quad \phi(\zeta) = c + \int_0^\zeta (a(\zeta) - \chi(\zeta))dz, \tag{521}
\]

for some constant \( c \).

Proof We first note\(^{116}\) that \( \chi(\zeta) \) is holomorphic in \( \{ |z| < r \} \). It is immediate to see that \( \chi,\phi \) must satisfy the equations \( \zeta \chi' + \chi = b \) and \( \phi' + \chi = a \). Since every solution of the homogeneous equation \( \zeta g' + g = 0 \) has the form \( g(\zeta) = \text{const} / \zeta \), we have that the only solution of the inhomogeneous equation which is continuous at \( \zeta = 0 \) is the one defined in (521). The formula for \( \phi \) is obvious. \( \square \)

Lemma 7.2 Let \( f(\zeta) \) be holomorphic on \( C_r \cap \{|z| < r\} \) and continuous on \( \{ |z| < r \} \), for some \( r > 0 \). Then \( f(\zeta) \) is holomorphic on \( \{ |z| < r \} \).

Proof It directly follows by the following well known result in complex analysis\(^{117}\): Suppose \( \Omega \) is a region, \( L \) is a straight line or a circular are, \( \Omega \setminus L \) is the union of two regions \( \Omega_1 \) and \( \Omega_2 \), \( f \) is continuous in \( \Omega \), and \( f \) is holomorphic in \( \Omega_1 \) and in \( \Omega_2 \). Then \( f \) is holomorphic in \( \Omega \). \( \square \)

Lemma 7.3 Assume that \( \chi(\theta) \) is a holomorphic function on the complex ball \( B_r = B_r(0) \) for some radius \( r > 0 \), with \( \chi(0) = 0 \) and \( \sup_{B_r} |\chi| \leq \eta \). Then the function \( \phi(\theta) := \chi(\theta)/\theta \) for \( \theta \neq 0 \) and \( \phi(0) := \chi'(0) \), is holomorphic in \( B_r \) with \( \sup_{B_r/2} |\phi| \leq 2\eta/r \).

Proof By Cauchy estimates \( \sup_{B_r/2} |\chi'| \leq 2\eta/r \), then, since \( \chi(0) = 0 \), \( |\chi(\theta)| \leq \frac{2\eta}{r} |\theta| \) for every \( \theta \in B_{r/2} \). \( \square \)

\(^{116}\) Writing \( b(\zeta) = \sum_{n \geq 0} b_n \zeta^n \), we get \( \chi(\zeta) = \sum_{n \geq 0} \frac{b_n}{n+1} \zeta^n \).

\(^{117}\) See, e.g., Theorem 16.8 of Rudin’s book, Real and Complex Analysis.
Lemma 7.4 Let \( g, \phi \) be holomorphic on \( B_r \) with \( \sup_{B_r} |g| \leq M_1, \sup_{B_r} |\phi| \leq \epsilon \leq 1/4 \) and \( \inf_{B_{r/2}} |g| =: g_0 > 0 \). Then

\[
|g(\theta + \theta \phi(\theta)) - g(\theta)| \leq \frac{2M_1}{g_0} \epsilon |g(\theta)|, \quad \forall |\theta| < \frac{r}{2}.
\]

Proof If \( |\theta| < r/2 \) then \( |\theta + \theta \phi(\theta)| < 3r/4 \) and, by Cauchy estimates,

\[
|g(\theta + \theta \phi(\theta)) - g(\theta)| \leq \sup_{B_{3r/4}} |g'| \frac{r}{2} \leq 2M_1 \epsilon \leq \frac{2M_1 \epsilon}{g_0} |g(\theta)|.
\]

Lemma 7.5 Let \( f, \chi \) be holomorphic on \( B_r \), satisfying \( f(0) = \chi(0) = 0 \), \( f'(0) \neq 0 \), \( \sup_{B_r} |f| \leq M_2 \), \( \sup_{B_r} |\chi| \leq \eta \leq r/8 \). Then for every \( \rho \leq r/2 \)

\[
|f(\theta + \chi(\theta)) - f(\theta)| \leq \frac{2}{\rho} \eta |f(\theta)|, \quad \forall \theta \in B_{\rho}. \tag{522}
\]

Proof First we note that \( \rho \leq r/4 \) since by Cauchy estimates \( |f'(0)| \leq M_2/r \). Set \( g(\theta) := f'(\theta) \) for \( \theta \neq 0 \) and \( g(0) := f'(0) \) and set also \( \phi(\theta) := \chi(\theta)/\theta \) for \( \theta \neq 0 \) and \( \phi(0) := \chi'(0) \). \( g \) and \( \phi \) are holomorphic on \( B_r \). By Lemma 7.3

\[
\sup_{B_{r/2}} |\phi| \leq \frac{2\eta}{r} \leq \frac{1}{4}, \quad \sup_{B_{r/2}} |g| \leq \frac{2M_2}{r}.
\]

We have (recalling that \( \rho \leq r/4 \))

\[
\inf_{B_r} |g| \geq |g(0)| - \sup_{B_\rho} |g'| \rho \geq |f'(0)| - \frac{8M_2}{r^2} \rho \geq \frac{1}{2} |f'(0)|.
\]

By Lemma 7.4 (with \( r \sim 2\rho \)) we have

\[
|g(\theta + \theta \phi(\theta)) - g(\theta)| \leq \frac{8M_2}{r} \frac{2\eta}{r} |g(\theta)| = \frac{1}{\rho} \eta |g(\theta)|, \quad \forall \theta \in B_{\rho}.
\]

Moreover, by the last estimate, we get

\[
|g(\theta + \theta \phi(\theta)) \phi(\theta)| \leq |g(\theta + \theta \phi(\theta))| \frac{2\eta}{r} \leq \frac{2\eta}{r} (1 + \frac{\eta}{\rho} |g(\theta)|) \leq \frac{1}{\rho} \eta |g(\theta)|, \quad \forall \theta \in B_{\rho},
\]

This shows that

\[
|g(\theta + \theta \phi(\theta))(1 + \phi(\theta)) - g(\theta)| \leq \frac{2}{\rho} \eta |g(\theta)|, \quad \forall \theta \in B_{\rho},
\]

which is equivalent to (522) (dividing by \( |\theta| \)).
References

[1] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. Mathematical aspects of classical and celestial mechanics, volume 3 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.

[2] D. Bambusi, A. Fusè, M. Sansottera. Exponential stability in the perturbed central force problem, Regular and Chaotic Dynamics volume 23, pages 821–841(2018), doi https://doi.org/10.1134/S156035471807002X

[3] L. Biasco, and L. Chierchia. KAM Theory for secondary tori, arXiv:1702.06480v1 [math.DS] (21 Feb 2017)

[4] L. Biasco, and L. Chierchia. On the measure of Lagrangian invariant tori in nearly–integrable mechanical systems. Rend. Lincei Mat. Appl. 26 (2015), 1–10

[5] L. Biasco, and L. Chierchia. On the topology of nearly–integrable Hamiltonians at simple resonances. To appear in Nonlinearity (2020)

[6] A.G. Medvedev, A.I. Neishtadt, D.V. Treschev, Lagrangian tori near resonances of near–integrable Hamiltonian systems, Nonlinearity, 28:7 (2015), 2105–2130