An analytical method for the inverse Cauchy problem of Lame equation in a rectangle

Yu Grigor’ev
Academy of Sciences of Republic Sakha (Yakutia), Yakutsk, Russia
North-Eastern Federal University, Yakutsk, Russia
E-mail: grigyum@yandex.ru

Abstract. In this paper, we present an analytical computational method for the inverse Cauchy problem of Lame equation in the elasticity theory. A rectangular domain is frequently used in engineering structures and we only consider the analytical solution in a two-dimensional rectangle, wherein a missing boundary condition is recovered from the full measurement of stresses and displacements on an accessible boundary. The essence of the method consists in solving three independent Cauchy problems for the Laplace and Poisson equations. For each of them, the Fourier series is used to formulate a first-kind Fredholm integral equation for the unknown function of data. Then, we use a Lavrentiev regularization method, and the termwise separable property of kernel function allows us to obtain a closed-form regularized solution. As a result, for the displacement components, we obtain solutions in the form of a sum of series with three regularization parameters. The uniform convergence and error estimation of the regularized solutions are proved.

1. Introduction
Let us have an element of construction with boundary $S = S_1 \cup S_2$ consisting of two parts $S_1$ and $S_2$. The part $S_1$ is accessible for all measurements and the part $S_2$ is not accessible. In engineering practice it is often necessary to reconstruct the stress-strain state of such structural element by means of measured data on $S_1$. In this case within an elastic model, we have the Cauchy problem for the Lame equation of the theory of elasticity. The Cauchy problem for elliptic equations is a so-called ill-posed problem. The main difficulty for solving such problems is numerical instability. For the Cauchy problem in linear elasticity, there exist numerous ways for numerical solving. Many of these methods can be classified as Tikhonov-type regularization methods, the finite element method, the boundary element method, etc. (sf. [1, 2]). With regard to analytical solutions, only the usage of Carleman’s method is known in the elasticity theory. In this paper, we present an analytical method for the Cauchy problem solution of the Lame equation in a rectangle. This method generalizes the Liu method [3] for an analytical solution of the Cauchy problem of the Laplace equation in a rectangle.

2. Formulation of the problem
In numerous publications under the name of Cauchy problem for the Laplace equation, there exist two types of problems. In them, it is necessary to find a harmonic function in $\Omega \in \mathbb{R}^3$. At first one has a cylindrical domain $\Omega = S \times (0, T)$ with lower base $S \in \mathbb{R}^2$ and lateral surface $S_1$. On $S$, the Cauchy conditions are given: $u = f$, $\partial u / \partial n = g$, and some boundary conditions are
posed on the lateral surface $S_1$, for example, $u = 0$, and on the upper base, there are no given boundary conditions. Such problems can be named as initial boundary-value problems for the Laplace equation (see Ch. 9 in [1]). For another type, we have a bounded domain $\Omega \in \mathbb{R}^3$ with a smooth boundary $S = S_1 \cup S_2$ and the Cauchy conditions are given on the part $S_1$ of the boundary $S$. We deal with the first type of the Cauchy problem, and it is analogous in the elasticity theory. It is necessary to note that the first type of Cauchy problem is not considered for analytical solutions in the elasticity theory. All analytical results in the elasticity theory concern the second-type problem and use Carleman’s method.

Let $x, y, z$ be the Cartesian coordinates. For convenience, we also simultaneously use the index notation: $x_i$ ($i = 1, 2, 3$). An equilibrium equation in the theory of elasticity is the Lame equation

$$Lu \equiv \mu \nabla^2 u + (\lambda + \mu) \nabla (\nabla \cdot u) = 0.$$  \hspace{1cm} (1)

Hooke’s law expresses connections between the components $\sigma_{ij}$ of stress tensor and the components $\varepsilon_{ij}$ of deformation tensor and for the linear elasticity has the form

$$\sigma_{ij} = (\lambda + 2\mu)(\nabla \cdot u)\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (i, j = 1, 2, 3).$$  \hspace{1cm} (2)

Here and below, the comma and an index (or a variable) denote the corresponding partial derivative. For simplicity we consider the plane deformation when the components of displacement $u$ have the form $u_x = u(x, y), \quad u_y = v(x, y), \quad u_z = 0$. In such a case, an equilibrium equation in stresses components has the form

$$\sigma_{xx,x} + \sigma_{xy,y} = 0,$$

$$\sigma_{xy,x} + \sigma_{yy,y} = 0.$$  \hspace{1cm} (3)

Let $\Omega \in \mathbb{R}^2$ be the rectangle $r \in (0, a) \times (0, b)$. The problem to be solved is

$$Lu(r) = 0, \quad r \in \Omega;$$

$$\sigma_{yy} = u_x = u_y = 0, \quad \sigma_{xy} = \sigma(x), \quad u_y = 0 \quad \text{on} \ y = b;$$

$$\sigma_{xx} = u_y = 0 \quad \text{on} \ x = 0 \text{ and } x = a.$$  \hspace{1cm} (4)

Thus, we have the Cauchy problem for the Lame equation with the Cauchy data on the upper base, with zero boundary conditions on the lateral sides of the rectangle, and with no boundary conditions on the lower base of the rectangle. On the upper base, only one boundary function $\sigma(x)$ of the four conditions is taken not to be equal to zero only for simplicity.

3. Solution method

3.1. Cauchy problem for the auxiliary harmonic function

Let us introduce an auxiliary function $f_0 = (\lambda + 2\mu) \nabla \cdot u$. It is well known that this function is harmonic. From Hook’s law (2), we have

$$\sigma_{xx} = \lambda \nabla \cdot u + 2\mu \varepsilon_{xx} = \lambda \nabla \cdot u + 2\mu u_{x,x} = \lambda \nabla \cdot u + 2\mu (\nabla \cdot u - u_{y,y}),$$

$$\sigma_{yy} = \lambda \nabla \cdot u + 2\mu \varepsilon_{yy} = \lambda \nabla \cdot u + 2\mu u_{y,y} = \lambda \nabla \cdot u + 2\mu (\nabla \cdot u - u_{x,x}),$$  \hspace{1cm} (5)

and for our auxiliary function, we have the expressions

$$f_0 = \sigma_{xx} + 2\mu u_{y,y},$$

$$f_0 = \sigma_{yy} + 2\mu u_{x,x}. $$  \hspace{1cm} (6)
Then, after differentiating these expressions and using equilibrium equations (3) and (1), we obtain

\[
\begin{align*}
f_{0,x} &= \sigma_{xx,x} + 2\mu u_{y,yy} = -\sigma_{xy,y} + 2\mu(\nabla \cdot u - u_{x,x},x) = -\sigma_{xy,y} + 2\mu \nabla \cdot u_x - 2\mu u_{x,xx} \\
&= -\sigma_{xy,y} + 2\mu \nabla \cdot u_x - 2[-\mu u_{x,yy} - (\lambda + \mu) \nabla \cdot u_x] = -\sigma_{xy,y} + 2\mu u_{x,yy} + 2f_{0,x},
\end{align*}
\]

which implies

\[
f_{0,x} = \sigma_{xy,y} - 2\mu u_{x,yy}.
\]

(7)

Analogically, we can obtain a similar formula,\n
\[
f_{0,y} = \sigma_{xy,x} - 2\mu u_{y,xx}.
\]

(8)

Let the boundary conditions of the Cauchy problem (4) provide the solution smoothness, \(u \in C^2(\Omega)\). In this case, all formulas (6)–(8) hold till the rectangle boundary. Then from formulas (6)–(8) and boundary conditions in (4), we obtain boundary conditions for the function \(f_0\) and thus have the following Cauchy problem for the harmonic function \(f_0\):

\[
\begin{align*}
\Delta f_0 &= 0, \quad \mathbf{r} \in \Omega, \\
f_0 &= 0, \quad f_{0,y} = -h(x) \quad \text{on } y = b, \\
f_0 &= 0 \quad \text{on } x = 0 \text{ and } x = a,
\end{align*}
\]

(9)

where the function \(h(x)\) is expressed in terms of the initial boundary functions from (4)

\[
h(x) = \sigma'(x).
\]

(10)

The regularized solution \(f_0^\alpha\) of this problem is obtained by using the Liu method [3] in the form

\[
f_0^\alpha(x, y) = \sum_{k=1}^{\infty} a_k^\alpha \frac{\sinh[k\pi(b - y)/a]}{\sinh(k\pi b/a)} \sin \frac{k\pi x}{a},
\]

\[
a_k^\alpha = \frac{2 \sinh(k\pi b/a)}{k\pi + \alpha \sinh(k\pi b/a)} \int_0^a h(\xi) \sin \frac{k\pi \xi}{a} d\xi,
\]

(11)

where \(\alpha\) is a parameter of regularization. In this method, the integral equation of the first kind is solved by Lavrent’ev’s regularization method using the Fourier method. The kernel in this integral equation has a termwise separability property. This property is the main reason for a solution to be obtained in closed form.

For solution (11), the following results presented in [3] are valid including the error estimation.

**Theorem 1.** If the Neumann datum \(h(x)\) is bounded in the interval \(x \in [0, a]\), then for any \(\alpha > 0\) and \(y_0 > 0\) for all \(x \in [0, a]\) and \(y \in [y_0, b]\), the regularized solution (11) converges uniformly, and this series can be termwise differentiated any number of times.

**Theorem 2.** Assume that the Neumann datum \(h(x) \in L^2(0, a)\). Then a sufficient and necessary condition for the Cauchy problem (9) to have a solution is

\[
\sum_{k=1}^{\infty} \frac{\sinh^2(k\pi b/a)}{k^2 \pi^2} \left( \int_0^a h(\xi) \sin \frac{k\pi \xi}{a} d\xi \right)^2 < \infty.
\]

(12)

**Theorem 3.** If the Neumann datum \(h(x)\) satisfies condition (12) and there exists an \(\varepsilon \in (0, 1)\) such that

\[
4 \sum_{k=1}^{\infty} \frac{a^2 \sinh^2(k\pi b/a)}{k^2 \pi^2} \left[ \int_0^a h(\xi) \sin \frac{k\pi \xi}{a} d\xi \right]^2 := M^2(\varepsilon) < \infty,
\]

(13)

then for any \(\alpha > 0\), the regularized solution \(f_0^\alpha(x, y)\) satisfies the following error estimation:

\[
\|f_0^\alpha(x, y) - f(x, y)\|_{L^2(0,a)} \leq \alpha^2 M(\varepsilon).
\]

(14)
3.2. Cauchy problems for the displacement components

After determining \( f_0^\alpha(x, y) \), we have Poisson equations for the components of a displacement from the equilibrium equations (1). It can be shown that, for all components \( u_x(x, y) \) and \( u_y(x, y) \), we can find Cauchy boundary conditions. And for the \( u_x(x, y) \), we have the following Cauchy problem for the Poisson equations:

\[
\Delta u_x = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)} f_{0,x}, \quad r \in \Omega,
\]

\[
u_x = 0, \quad u_{x,y} = \frac{1}{\mu} \sigma(x) \quad \text{on } y = b,
\]

\[
u_{x,x} = 0 \quad \text{on } x = 0 \text{ and } x = a.
\]

For the component \( u_y(x, y) \), we have another Cauchy problem

\[
\Delta u_y = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)} f_{0,y}, \quad r \in \Omega,
\]

\[
u_y = u_{y,y} = 0 \quad \text{on } y = b,
\]

\[
u_y = 0 \quad \text{on } x = 0 \text{ and } x = a.
\]

These problems are solved by the method similar to the Liu method. Thus, for the solution \( u_x \) and \( u_y \) of the Cauchy problem (4), we have regularized expressions similar to (11) but with additional regularization parameters \( \beta \) and \( \gamma \). Now we give the final formulas for these solutions:

\[
u_x^{\alpha,\beta}(x, y) = u_x^\alpha(x, y) + v_x^{\alpha,\beta}(x, y),
\]

\[
u_y^{\alpha,\gamma}(x, y) = u_y^\alpha(x, y) + v_y^{\alpha,\gamma}(x, y);
\]

\[
u_x^\alpha(x, y) = \frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \sum_{k=1}^{\infty} u_{xk}(y) \cos \frac{k\pi x}{a},
\]

\[
u_x(x, y) = \frac{a_k^2}{\sinh(k\pi b/a)} \left[ y \cos \frac{k\pi (b - y)}{a} - b \frac{\sinh(k\pi y/a)}{\sinh(k\pi b/a)} \right];
\]

\[
u_y^{\beta}(x, y) = b_0(b - y) + \sum_{k=1}^{\infty} b_k \frac{\sinh(k\pi (b - y)/a)}{\sinh(k\pi b/a)} \cos \frac{k\pi x}{a},
\]

\[
\begin{align*}
b_k^\beta &= \frac{k\pi \sinh(k\pi b/a)}{k\pi + \beta \sinh(k\pi b/a)} \frac{a}{\mu} \left[ \frac{\lambda + \mu}{2(\lambda + 2\mu) \sinh(k\pi b/a)} \left( 1 - \frac{k\pi b}{a} \coth \frac{k\pi b}{a} \right) - \sigma_k \right];
\end{align*}
\]

\[
u_y^{\alpha}(x, y) = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)} \sum_{k=1}^{\infty} a_k \frac{\sinh(k\pi (b - y)/a)}{\sinh(k\pi b/a)} \sin \frac{k\pi x}{a};
\]

\[
u_y^{\gamma}(x, y) = \sum_{k=1}^{\infty} c_k^{\gamma} \frac{\sinh(k\pi (b - y)/a)}{\sinh(k\pi b/a)} \sin \frac{k\pi x}{a},
\]

\[
c_k^{\gamma} = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)} \cdot \frac{k\pi ba_k^\alpha}{k\pi + \gamma \sinh(k\pi b/a)},
\]

\[
\sigma_k = \int_0^\alpha \sigma(\xi) \cos \frac{k\pi \xi}{a} d\xi, \quad k \geq 1; \quad b_0 = -\frac{1}{a\mu} \int_0^\alpha \sigma(\xi)d\xi.
\]

If the boundary functions in (4) are bounded in closed intervals, then all of these regularized solutions converge uniformly and can be differentiated termwise any number of times. When the parameters \( \alpha, \beta, \gamma \to 0 \), then the regularized solutions \( u_x^{\alpha,\beta}(x, y) \), \( u_y^{\alpha,\gamma}(x, y) \) tend to the exact solutions. Error estimations for these solutions can be obtained analogically to Theorem 3. Effective numerical calculations can be realized by means of the formulas presented above.
Conclusions
The Liu method for solving the Cauchy problem for the Laplace equation is generalized to the Cauchy problem of the linear elasticity. This allows us to obtain a regularized solution of the problem in closed form by using the Fourier and Lavrent’ev methods. The obtained regularized solution converges uniformly and the error estimation of the regularized solution is proved. For simplicity, we only consider a two-dimensional rectangular domain. The method can be used for other boundary conditions, it can be generalized to the three-dimensional case. It is also interesting to note that the above-used auxiliary function $f_0 = (\lambda + 2\mu) \nabla \cdot \mathbf{u}$ is a scalar part of a regular quaternion function connected with the solution $\mathbf{u}$ of Lame equation (1) [4–6]. Thus it may be possible to use the above-presented method in the problem of regular extension to a domain of the quaternion function defined on a part of its boundary.

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