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Chiral Anomaly, Dirac Sea and Berry monopole in Wigner Function Approach

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Abstract
Within Wigner function formalism, the chiral anomaly arises naturally from the Dirac sea contribution in un-normal-ordered Wigner function. For massless fermions, the Dirac sea contribution behaves like a 4-dimensional or 3-dimensional Berry monopole in Euclidian momentum space, while for massive fermions, although Dirac sea still leads to the chiral anomaly but there is no Berry monopole at infrared momentum region. We discuss these points explicitly in a simple and concrete example.

Keywords: Wigner function, chiral anomaly, Berry monopole.

1. Introduction
Under the background of the electromagnetic field, the axial vector current of a chiral fermion system is not conserved at the quantum level, which is called Adler-Bell-Jackiw anomaly or chiral anomaly. There are many methods to study this anomaly, such as Feynmann diagram, functional integral and so on. In recent years, there have been a considerable amount of work on the chiral kinetic theory which are devoted incorporating the chiral anomaly into the kinetic theory in a consistent way \cite{1,2,3,4,5,6,7,8,9}. Among these publications, most works connect the chiral anomaly in the chiral kinetic theory with Berry monopole in momentum space.

It has been shown in \cite{10} that the Dirac sea or vacuum contribution from the anti-commutation relations between antiparticle field operators in un-normal-ordered Wigner function plays a central role to generate chiral anomaly in quantum kinetic theory both for massive and for massless fermion systems. In this work, we will take a simple and concrete example to illustrate these points. In particular, we find that for massless fermion system, the chiral anomaly is associated with the singular 4-dimensional divergence \[ \partial_{\mu}\left[ p_{\mu}\delta(p^2) \right] \], which behaves like a 4-dimensional Berry monopole in Euclidian momentum space after Wick rotation, or

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like a 3-dimensional Berry monopole after integrating over the zero component of momentum. For massive fermion system, the chiral anomaly is associated with the 4-dimensional divergence $\partial_{\mu} \delta^p(p^2 - m^2)$. However it does not give rise to the singularity like 4-dimensional or 3-dimensional Berry monopole at infrared momentum region.

2. The divergence of axial vector current from Wigner function approach

We will take the collisionless fermion system near equilibrium under static and homogenous electromagnetic fields as a simple and concrete example. We consider the massless fermion system first and then generalize the main results to massive case. Our starting point is the following covariant and gauge-invariant Wigner function for spin-1/2 fermion \[ W_{\alpha\beta}(x, p) = \frac{1}{(2\pi)^3} \int d^4y e^{-ip \cdot y} \langle \bar{\Psi}_\beta(x + y/2) U(x + y/2, x - y/2) \Psi_\alpha(x - y/2) \rangle, \] where $\langle \cdot \rangle$ represents the ensemble average, $\Psi(x)$ is the Dirac filed operator, $\alpha, \beta$ are Dirac spinor indices, and $U(x + y/2, x - y/2)$ is a gauge link along a straight line from $x - y/2$ to $x + y/2$. The dynamical equation for $W(x, p)$ is given by \[ i\gamma^\mu p_\mu + \frac{i}{2} \nabla \cdot \vec{p} W(x, p) = 0, \]

where $\nabla_\mu = \partial_\mu - Q F_{\mu\nu} \partial^\nu$. It should be noted that there is no normal ordering in the Wigner matrix above. This plays a central role to give rise to the chiral anomaly in the following. Since $W(x, p)$ is a $4 \times 4$ matrix, we can decompose it by the 16 independent $\Gamma$-matrices,

\[ W = \frac{1}{4} \left( \mathcal{F} + i \gamma^5 \mathcal{P} + \gamma^\mu V_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{A}_{\mu\nu} \right). \]

The vector current $J^\mu(x)$ and axial vector current $A^\mu(x)$ can be expressed as the 4-momentum integration of $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$. We can expand $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ order by order in $\hbar$ as

\[ \mathcal{V}^\mu = \mathcal{V}^\mu_{(0)} + \hbar \mathcal{V}^\mu_{(1)} + \hbar^2 \mathcal{V}^\mu_{(2)} + \cdots, \]

\[ \mathcal{A}^\mu = \mathcal{A}^\mu_{(0)} + \hbar \mathcal{A}^\mu_{(1)} + \hbar^2 \mathcal{A}^\mu_{(2)} + \cdots. \]

As mentioned above, we will consider the specific fermion system near equilibrium. Hence we choose the zeroth order solutions $\mathcal{V}^\mu_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ as equilibrium distribution in free field theory \[ v_{(0)} = \frac{p^\mu \delta(p^2)}{(2\pi)^3} \sum_s \left( Z_n^s + Z^s \right), \]

\[ a_{(0)} = \frac{p^\mu \delta(p^2)}{(2\pi)^3} \sum_s sZ_n^s, \]

where $Z_n^s, Z^s$ are given by

\[ Z_n^s = \frac{2}{(2\pi)^3} \left[ \theta(u \cdot p) n_F(u \cdot p - \mu_s) + \theta(-u \cdot p) n_F(-u \cdot p + \mu_s) \right], \]

\[ Z^s = -\frac{2}{(2\pi)^3} \theta(-u \cdot p). \]

Here $n_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi-Dirac distribution, $\beta = 1/T$ is the inverse temperature, $\nu^\mu$ is the fluid velocity, and $\mu_s = \mu + s\mu_s$ with $s = \pm 1$ is the chemical potential for right-hand/left-hand fermions respectively. For simplicity, we will assume that there is no vorticity in the system, i.e. the fluid velocity $\nu^\mu$ is uniform. Note that $Z^s$ is the vacuum term or Dirac sea term which comes from the anticommutation of creation
and annihilation operators of antifermions as described in [10, 16, 17]. This term is universal and does not
depend on the specific distribution \( n_F(x) \) at all. From the dynamical equation of Wigner function at order \( \hbar \),
one can obtain \( \mathcal{A}_t \) near equilibrium

\[
\mathcal{A}_t \equiv Q F^{\mu \nu} p_{\nu} \delta'(p^2) \sum_{s} (\mathcal{Z}_n^{s} + \mathcal{Z}_v^{s}),
\]

where \( F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma} \). The equation satisfied by \( \mathcal{A}_t \) can be obtained from Eq. (2),

\[
\partial_\mu \mathcal{A}_t - Q F_{\mu \nu} \partial_\nu p_{\nu} = 0.
\]

The 4-momentum integration of Eq. (11) gives

\[
\partial_\mu J^\mu_A = - \frac{Q^2}{8\pi^2} F_{\mu \nu} \tilde{F}_{\mu \nu} \times (C_n + C_v),
\]

where \( C_n, C_v \) are defined as

\[
C_n = -2\pi^2 \int d^4 p \partial_\mu p_{\mu} \left[ p_{\nu} \delta'(p^2) \sum_s (\mathcal{Z}_n^s) \right],
\]

\[
C_v = -4\pi^2 \int d^4 p \partial_\mu p_{\mu} \left[ p_{\nu} \delta'(p^2) \mathcal{Z}_v^s \right].
\]

All discussions above is for massless fermion system. For massive fermion case, it turns out that we can just
set \( \mu_s = \mu, \mu_5 = 0 \) and change the on-shell condition in Eq. (13) as \( \delta'(p^2) \to \delta'(p^2 - m^2) \),
and Eq. (12) becomes

\[
\partial_\mu J^\mu_A = - \frac{Q^2}{8\pi^2} F_{\mu \nu} \tilde{F}_{\mu \nu} \times (C_n + C_v),
\]

where \( P \) is the pseudoscalar component of Wigner function in Eq. (3).

### 3. Chiral anomaly for massless fermion system

For massless fermion case, since \( \mathcal{Z}_v^s \) vanish rapidly at infinity in the phase space, \( C_n \) must be zero.
However the Dirac sea term \( C_v \) can keeps nonzero at \( p_0 = -\infty \) and could contribute non-zero value

\[
C_n = 0,
\]

\[
C_v = \frac{1}{2\pi} \int d^4 p \partial_\mu [p_{\mu} \theta(-p^0) \delta'(p^2)] = \frac{1}{2\pi} \int d^4 p \partial_\mu [p_{\mu} \delta'(p^2)].
\]

We can calculate this integral by two methods. First we can use the regularization

\[
\delta'(x) = \frac{1}{\pi} \Im \frac{1}{(x + i\epsilon)^2},
\]

followed by Wick rotation and obtain

\[
C_v = \frac{1}{2\pi} \Im \int d^4 p \partial_\mu \left[ \frac{p_{\mu}}{(p^2 + i\epsilon)^2} \right] = \frac{1}{2\pi} \int d^4 p E \partial_{pE} \left( \frac{p_E^\mu}{p_E^2} \right) = 1,
\]

where we have used the Gauss theorem in 4-dimensional momentum space or the identity \( \partial_{pE}(p_E^\mu/p_E^2) = 2\pi^2 \delta^4(p_E) \). It is obvious that \( p_\rho \delta'(p^2) \) plays the role of the Berry curvature of a 4-dimensional monopole in Euclidean momentum space, which was pointed out in Ref. [4].
We can also calculate the integral by brute force. After integrating over the zero component of momentum and keeping the non-vanishing term, we obtain
\[
C_v = \int \frac{d^3p}{2\pi} \hat{p} \cdot \left( \frac{\hat{p}}{2\hat{p}^2} \right) = 1, \tag{20}
\]
where we have used the Gauss theorem in 3-dimensional momentum space or the identity \( \partial_\mu \theta(-p^0) \delta'(p^3 - m^2) = 2\pi \delta(p^3) \). Here \( \hat{p}/2\hat{p}^2 \) is just the usual Berry curvature in 3-dimensional momentum space. For massless fermion system, we note that only vacuum or Dirac sea term contributes to the chiral anomaly in form of 4-dimensional or 3-dimensional Berry monopole.

4. Chiral anomaly for massive fermion system

For massive fermion system, it turns out that the coefficients \( C_n, C_C \) can be obtained by replacing the on-shell condition \( \delta'(p^3) \) with \( \delta'(p^3 - m^2) \). Similar to the massless fermion system, \( C_n \) always vanishes for normal distribution, while \( C_C \) is given by
\[
C_C = \frac{1}{\pi} \int d^3p \partial_\mu \left[ \frac{p_\mu}{(p^2 - m^2 + i\epsilon)^2} \right] = \frac{1}{2\pi^2} \int d^4p E \partial_\mu \left( \frac{p_\mu}{(p_E^2 + m^2)^2} \right) = 1, \tag{21}
\]
Again we can calculate this integral by Wick rotation
\[
C_C = \frac{1}{2\pi^2} \text{Im} \int d^3p \partial_\mu \left( \frac{p_\mu}{(p^2 - m^2 + i\epsilon)^2} \right) = \frac{1}{2\pi^2} \int d^4p E \partial_\mu \left( \frac{p_\mu}{(p_E^2 + m^2)^2} \right) = 1, \tag{22}
\]
or directly integrate over \( p_0 \)
\[
C_C = \int \frac{d^3p}{2\pi} \hat{p} \cdot \left( \frac{\hat{p}}{2(p^2 + m^2)} \right) = 1, \tag{23}
\]
where we have used Gauss theorem in 4-dimensional and 3-dimensional momentum space, respectively.

The chiral anomaly derived from Dirac sea contribution for massless or massive case is universal and independent of the phase space normal distribution function at zero momentum.

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