Geometrization of Lie and Noether symmetries with applications in Cosmology

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Abstract. We derive the Lie and the Noether conditions for the equations of motion of a dynamical system in a \( n \)–dimensional Riemannian space. We solve these conditions in the sense that we express the symmetry generating vectors in terms of the special projective and the homothetic vectors of the space. Therefore the Lie and the Noether symmetries for these equations are geometric symmetries or, equivalently, the geometry of the space is modulating the motion of dynamical systems in that space. We give two theorems which contain all the necessary conditions which allow one to determine the Lie and the Noether symmetries of a specific dynamical system in a given Riemannian space. We apply the theorems to various interesting situations covering Newtonian 2d and 3d systems as well as dynamical systems in cosmology.

Keywords: Lie Symmetries, Noether symmetries, Cosmology
Pacs - numbers:98.80.-k,04.20.-q, 45.20.D-,02.20Sv

1. Introduction

In a Riemannian space the affinely parameterized geodesics are determined uniquely by the metric. Therefore one should expect a close relation between the geodesics as a set of homogeneous ordinary differential equations (ODE) linear in the highest order term and quadratically non-linear in first order terms, and the metric as a second order symmetric tensor. A system of such ODEs is characterized (not fully) by its Lie symmetries and correspondingly a metric is characterized (again not fully) by its collineations. Therefore it is reasonable to expect that the Lie symmetries of the system of geodesic equation of a metric will be closely related with the collineations of the metric. That such a relation exists it is easy to see by the following simple example. Consider on the Euclidian plane a family of straight lines parallel to the \( x \)–axis. These curves can be considered either as the integral curves of the ODE \( \frac{d^2y}{dx^2} = 0 \) or as the geodesics of the Euclidian metric \( dx^2 + dy^2 \). Subsequently consider a symmetry operation defined by a reshuffling of these lines without preserving necessarily their parametrization. According to the first interpretation this symmetry operation is a Lie symmetry of the ODE \( \frac{d^2y}{dx^2} = 0 \) and according to the second interpretation it is a (special) projective symmetry of the Euclidian two dimensional metric.

What has been said for a Riemannian space can be generalized to a space in which there is only a linear connection. In this case the geodesics are called autoparallels (or paths) and they comprise again a system of ODEs linear in the highest order term and quadratically non-linear.
in the first order terms. In this case one is looking for relations between the Lie symmetries of the autoparallels and the projective collineations of the connection.

A Lie point symmetry of an ordinary differential equation (ODE) is a point transformation in the space of variables which preserves the set of solutions of the ODE \([1–3]\). If we look at these solutions as curves in the space of variables, then we may equivalently consider a Lie point symmetry as a point transformation which preserves the set of the solution curves. Applying this observation to the geodesic curves in a Riemannian (affine) space, we infer that the Lie point symmetries of the geodesic equations in any Riemannian space are the automorphisms which preserve the set of these curves. However we know from Differential Geometry that the point transformations of a Riemannian (affine) space which preserve the set of geodesics are the projective transformations. Therefore it is reasonable to expect a correspondence between the Lie symmetries of the geodesic equations and the projective algebra of the space.

The equation of geodesics in an arbitrary coordinate frame is a second order ODE of the form

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + F(x^i, \dot{x}^j) = 0
\]  

(1)

where \(F(x^i, \dot{x}^j)\) is an arbitrary function of its argument and the functions \(\Gamma^i_{jk}\) are the connection coefficients of the space. However equation (1) is also the equation of motion of a dynamical system moving in a Riemannian (affine) space under the action of a velocity dependent force. According to the above argument we expect that the Lie symmetries of the ODE (1) for a given function \(F(x^i, \dot{x}^j)\) will be a subalgebra of the projective algebra of the space. This subalgebra is selected by means of certain constraint conditions which will involve geometric quantities of the space and the function \(F(x^i, \dot{x}^j)\) \([4–8]\).

The determination of the Lie point symmetries of a given system of ODEs consists of two steps (a) the determination of the conditions which the components of the Lie symmetry vector must satisfy and (b) the solution of the system of these conditions. Step (a) is formal and it is outlined in e.g. \([1–3]\). The second step is the key one and, for example, in higher dimensions where one has a large number of simultaneous equations the solution can be quite involved and perhaps prohibitive. However if one expresses the system of Lie symmetry conditions of (1) in terms of collineation (i.e. symmetry) conditions of the metric, then the determination of Lie symmetries is transferred to the geometric problem of determining the generators of a collineation group of the metric and, as it will be shown below, specifically the generators of the special projective group of the metric. In this field there is a vast amount of work from Differential Geometry waiting to be used. Indeed the projective symmetries are already known for many spaces or they can be determined by existing general theorems. For example the projective algebra and all its subalgebras are known for the important case of spaces of constant curvature \([10]\) and in particular for the flat spaces. This implies that, for example, the Lie symmetries of all Newtonian dynamical systems as well as those of Special Relativity can be determined by simple differentiation from the known projective algebra of these spaces!

What has been said for the Lie point symmetries of (1) applies also to Noether point symmetries (provided (1) follows from a Lagrangian). The Noether symmetries are Lie point symmetries which satisfy the additional constraint

\[
X^{[1]} L + L \frac{d\xi}{dt} = \frac{df}{dt}.
\]  

(2)

Noether symmetries form a closed subalgebra of the Lie point symmetries algebra. In accordance to the above this implies that the Noether symmetries will be related with a subalgebra of the projection algebra of the space where ‘motion’ occurs. As it will be shown this subalgebra is contained in the homothetic algebra of the space. As it is well known with each Noether point symmetry it is associated a conserved current (i.e. a Noether first integral). This leads us to the
important conclusion that the (standard) conserved quantities of a dynamical system depend on the space it moves and the type of force $F(x^i, \dot{x}^j)$ which modulates the motion. In particular in 'free fall', that is when $F(x^i, \dot{x}^j) = 0$, the orbits are the geodesics and the geometry of the space is the sole factor which determines the conserved quantities of motion. This conclusion is by no means trivial and means that the space where motion occurs is not a pathetic carrier of motion but it is the major modulator in the evolution of a dynamical system. In other words there is a strong and deep relation between Geometry of the space and Physics in that space!

The above matters have been discussed extensively in a series of interesting papers by Aminova [11–13] who has given a partial answer. Furthermore in a recent work [14] the authors have considered the KVs of the metric and their relation to the Lie symmetries of the system of affinely parameterized geodesics in maximally symmetric spaces of low dimension. In the same paper a conjecture is made, which essentially says that the maximally symmetric spaces of non-vanishing curvature do not admit further Lie symmetries.

The purpose of the present work is to present recent results concerning the above approach. More specifically we do the following:

a. We derive the general expression for the Lie symmetry conditions of (1) assuming a general (i.e. not necessarily symmetric) connection in terms of geometric quantities

b. For the case that the function $F$ depends only on the coordinates, i.e. $F(x^i)$ we give two theorems which establish the exact relation between the projective/homothetic algebra of the space and the Lie/Noether symmetry algebra of (1).

c. We consider applications in Newtonian Physics, General Relativity and in Cosmology.

The structure of the paper is as follows. In section 3 we determine the conditions for Lie symmetries in covariant form. We find that the major symmetry condition relates the Lie symmetries with the special projective symmetries of the connection. A similar result has been obtained by Prince and Crampin in [4] using the bundle formulation of second order ODEs. In section 4 we apply these conditions in the case of Riemannian spaces. We solve the symmetry conditions and in Theorem 1 we give the Lie symmetry vectors in terms of the collineations of the metric. In section 5 we apply Theorem 1 to the case of spaces of constant curvature of dimension $n$ and compute the complete set of Lie symmetries of the system of affinely parameterized geodesic equations in these spaces. We distinguish the case of the flat space and the spaces of non-zero curvature and prove the validity of the conjecture made in [14]. In the remaining sections we consider applications in General Relativity and in Cosmology.

2. Collineations of Riemannian spaces

A collineation in a Riemannian space is a vector field $X$ which satisfies an equation of the form

$$\mathcal{L}_X A = B$$

(3)

where $\mathcal{L}_X$ denotes Lie derivative, $A$ is a geometric object (not necessarily a tensor) defined in terms of the metric and its derivatives (e.g. connection, Ricci tensor, curvature tensor etc.) and $B$ is an arbitrary tensor with the same tensor indices as $A$. The collineations in a Riemannian space have been classified by Katzin et al. [15]. In the following we use only certain collineations.

A conformal Killing vector (CKV) is defined by the relation

$$\mathcal{L}_X g_{ij} = 2\psi \left( x^k \right) g_{ij}.$$  

(4)

If $\psi = 0$, $X$ is called a Killing vector (KV), if $\psi$ is a non-vanishing constant a homothetic vector (HV) and if $\psi_{ij} = 0$, a special conformal Killing vector (SCKV). A CKV is called proper if it is not a KV, HV or a SCKV.
Table 1. The projective algebra of a Riemannian space

| Collineation                              | A     | B     |
|-------------------------------------------|-------|-------|
| Killing vector (KV)                       | $g_{ij}$ | 0     |
| Homothetic vector (HV)                    | $g_{ij}$ | $\psi g_{ij}, \psi,_{;i} = 0$ |
| Conformal Killing vector (CKV)           | $g_{ij}$ | $\psi g_{ij}, \psi,_{;i} \neq 0$ |
| Projective collineation (PC)              | $\Gamma^i_{jk}$ | $2\phi_{(;j}\delta^i_{k)}$, $\phi,_{;i} \neq 0$ |
| Special Projective collineation (SPC)    | $\Gamma^i_{jk}$ | $2\phi_{(;j}\delta^i_{k)}$, $\phi,_{;i} \neq 0$ and $\phi,_{;jk} = 0$ |

Table 2. The projective algebra of the Euclidian space

| Collineation                              | Gradient | Non-gradient |
|-------------------------------------------|----------|--------------|
| Killing vectors (KV)                      | $S_I = \delta_I^j \partial_i$ | $X_{IJ} = \delta_I^j \delta^i_{[j} x_{i]} \partial_i$ |
| Homothetic vector (HV)                    | $H = x^j \partial_i$ |             |
| Affine Collineation (AC)                  | $A_{IJ} = x_I \delta^I_j \partial_i$ |             |
| Special Projective collineation (SPC)    | $P_I = S_I H$. |             |

Where the indices $I, J = 1, 2, \ldots, n$.

A projective collineation (PC) is defined by the equation

$$\mathcal{L}_X \Gamma^i_{jk} = 2\phi_{(;j}\delta^i_{k)}.$$  \hspace{1cm} (5)

If $\phi = 0$ the PC is called an affine collineation (AC) and if $\phi,_{;ij} = 0$ a special projective collineation (SPC). A proper PC is a PC which is not an AC, HV or KV or SPC. The PCs form a Lie algebra whose ACs, HV and KVs form subalgebras. It has been shown that if a metric admits a SCKV then also admits a SPC, a gradient HV and a gradient KV [16]. We summarize the above in the Table 1

In the following we shall need the symmetry algebra of spaces of constant curvature. In [10] it has been shown that the PCs of a space of constant non-vanishing curvature consist of proper PCs and KVs only and if the space is flat then the algebra of the PCs consists of KVs/HV/ACs and SPCs. For convenience we summarize these results in Table 2.

The Lie point symmetries of all Newtonian dynamical systems are amongst the vectors in the above table. Also the Noether point symmetries of all Newtonian dynamical systems (and more general all systems moving in a flat space) follow from the elements of the first two rows of Table 2 (apart form some differences in sign depending on the signature of the metric).

3. The Lie point symmetry conditions in an affine space

We consider the system of ODEs:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \sum_{m=0}^{n} P^i_{j_1 \ldots j_m} \dot{x}^{j_1} \ldots \dot{x}^{j_m} = 0$$  \hspace{1cm} (6)

where $\Gamma^i_{jk}$ are the connection coefficients of the space and $P^i_{j_1 \ldots j_m}(t, x^i)$ are smooth polynomials completely symmetric in the lower indices and derive the Lie point symmetry conditions in
geometric form using the standard approach. Equation (6) is quite general and covers most of the standard cases autonomous and non autonomous and in particular equation (1). Furthermore because the $\Gamma^i_{jk}$’s are not assumed to be symmetric, the results are valid in a space with torsion. Obviously they hold in a Riemannian space where the connection coefficients are given in terms of the Christoffel symbols.

The detailed calculation can be found in [8] and we shall not be repeat it here. In the following we summarize for convenience the results.

The terms $\dot{x}^j_1 \ldots \dot{x}^j_m$ for $m \leq 4$ give the equations:

\[ L_\eta P^i_j + 2\xi^i_{,t} P^i_j + \xi P^i_{,jt} + \eta^i_{,t} \eta^j_{,t} + \eta^j_{,t} P^i_j = 0 \]  
\[ L_\eta P^i_j + \xi_{,t} P^i_j + \xi P^i_{,jt} + (\xi_{,k} \delta^j_k + 2\xi_{,ij} \delta^j_k) P^k_j + 2\eta^j_{,t} \eta^j_{,t} \eta^i_{,t} + 2\eta^j_{,t} P^i_{jk} = 0 \]  
\[ L_\eta P^i_j k + L_\eta \Gamma^i_{jk} + \left( \xi_{,t} \delta^i_k + \xi_{,(k} \delta^i_{d)} P^i_{j,k} + \xi P^i_{jk,t} - 2\xi_{,t(j} \delta^i_k) + 3\eta^j_{,t} P^i_{dkj} = 0 \]  
\[ L_\eta P^i_{j,kd} - \xi_{,t} P^i_{j,kd} + \xi_{,e} \delta^i_{(k} P^e_{j,d)} + \xi P^i_{jkde} + 4\eta^j_{,t} P^i_{dkje} - \xi_{,(j|k} \delta^i_{d)} = 0 \]

and the conditions due to the terms $\dot{x}^j_1 \ldots \dot{x}^j_m$ for $m > 4$ are given by the following general formula:

\[ L_\eta P^i_{j_1 \ldots j_m} + P^i_{j_1 \ldots j_m} \delta^i + (2 - m) \xi_{,t} P^i_{j_1 \ldots j_m} + \xi_{,r} (2 - (m + 1)) P^i_{j_1 \ldots j_{m-1}} \delta^j_{j_m} + (m + 1) P^i_{j_1 \ldots j_{m+1}} \eta^j_{,t} + \xi_{,j} P^i_{j_1 \ldots j_{m-1}} \delta^i_{j_m} = 0. \]  

We note the appearance of the term $L_\eta \Gamma^i_{jk}$ in these expressions.

Eqn (1) is obtained for $m = 0$, $P^i = F^i$ in which case the Lie symmetry conditions read:

\[ L_\eta P^i_j + 2\xi^i_{,t} P^i_j + \xi P^i_{,jt} + \eta^i_{,t} \eta^j_{,t} = 0 \]  
\[ (\xi_{,k} \delta^j_k + 2\xi_{,ij} \delta^j_k) P^k_j + 2\eta^j_{,t} \eta^j_{,t} \eta^j_{,t} + \eta^j_{,t} P^i_{jk} = 0 \]  
\[ L_\eta \Gamma^i_{jk} - 2\xi_{,t(j} \delta^i_k) = 0 \]  
\[ \xi_{,(j|k} \delta^i_{d)} = 0. \]

If $F^i = 0$ we obtain the Lie symmetry conditions for the geodesic equations (see [8], [9]).

4. The autonomous dynamical system moving in a Riemannian space
We 'solve' the Lie symmetry conditions (12) - (15) for an autonomous dynamical system in the sense that we express them in terms of the collineations of the metric of the space where motion occurs.

Equation (15) means that $\xi_{,j}$ is a gradient KV of $g_{ij}$. This implies that the metric $g_{ij}$ is decomposable. Equation (14) means that $\eta^i$ is a projective collineation of the metric with projective function $\xi_{,r}$. The remaining two equations are constraint conditions, which relate the components $\xi, n^i$ of the Lie symmetry vector with the vector $F^i(x^j)$. Equation (12) gives

\[ (L_\eta g^{ij}) F_j + g^{ij} L_\eta F_j + 2\xi_{,t} g^{ij} F_j + \eta^j_{,tt} = 0. \]  

This equation is an additional restriction for $\eta^i$ because it relates it directly to the symmetries of the metric. Finally equation (13) gives

\[ -\delta^j_i \xi_{,tt} + (\xi_{,j} \delta^i_k + 2\delta^i_j \xi_{,k}) F^k + 2\eta^i_{,tt} + 2\Gamma^i_{jk} \eta^j_k = 0. \]

We conclude that the Lie symmetry equations are equations (16), (17) where $\xi(t, x)$ is a gradient KV of the metric and $\eta^i(t, x)$ is a special Projective collineation of the metric with projective function $\xi_{,t}$. We state this result in theorem 1 [8].
Theorem 1 The Lie point symmetries of the system of equations of motion of an autonomous system under the action of the force $F^j(x^i)$ in a general Riemannian space with metric $g_{ij}$, namely

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = F^i$$  \hspace{1cm} (18)

are given in terms of the generators $Y^i$ of the special projective algebra of the metric $g_{ij}$.

If the force $F^i$ is derivable from a potential $V(x^i)$, so that the equations of motion follow from the standard Lagrangian

$$L(x^j, \dot{x}^j) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^i)$$  \hspace{1cm} (19)

with Hamiltonian

$$E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x^i)$$  \hspace{1cm} (20)

it can be shown that the Noether conditions are

$$V_{,k} \eta^k + V \xi, t = -f, t \hspace{1cm} (21)$$
$$\eta^i, t g_{ij} - \xi, j V = f, j \hspace{1cm} (22)$$
$$L \eta g_{ij} = 2 \left( \frac{1}{2} \xi, t \right) g_{ij} \hspace{1cm} (23)$$
$$\xi, k = 0. \hspace{1cm} (24)$$

Last equation implies $\xi = \xi (t)$ and reduces the system as follows

$$L \eta g_{ij} = 2 \left( \frac{1}{2} \xi, t \right) g_{ij} \hspace{1cm} (25)$$
$$V_{,k} \eta^k + V \xi, t = -f, t \hspace{1cm} (26)$$
$$\eta, t = f, j. \hspace{1cm} (27)$$

Equation (25) implies that $\eta^i$ is a conformal Killing vector of the metric provided $\xi, t \neq 0$. Because $g_{ij}$ is independent of $t$ and $\xi = \xi (t)$ the $\eta^i$ must be is a HV of the metric. This means that $\eta^i (t, x) = T(t) Y^i (x)$ where $Y^i$ is a HV. If $\xi, t = 0$ then $\eta^i$ is a Killing vector of the metric. Equations (26), (27) are the constraint conditions, which the Noether symmetry and the potential must satisfy for the former to be admitted. The above lead to the following theorem [8].

Theorem 2 The Noether point symmetries of the Lagrangian (19) are generated from the homothetic algebra of the metric $g_{ij}$.

More specifically, concerning the Noether symmetries, we have the following.

All autonomous systems admit the Noether symmetry $\partial_t$ whose Noether integral is the Hamiltonian $E$ given in equation (20). For the rest of the Noether symmetries we have the following cases

Case I Noether point symmetries generated by the homothetic algebra.

The Noether symmetry vector and the Noether function $G(t, x^k)$ are

$$X = 2 \psi t + Y^i \partial_i, \hspace{0.5cm} G(t, x^k) = pt$$  \hspace{1cm} (28)
where $\psi_Y$ is the homothetic factor of $Y^i$ ($\psi_Y = 0$ for a KV and 1 for the HV) and $p$ is a constant, provided the potential satisfies the condition

$$\mathcal{L}_Y V + 2\psi_Y V + p = 0.$$  \tag{29}

**Case II** Noether point symmetries generated by the gradient homothetic Lie algebra i.e. both KVs and the HV must be gradient.

In this case the Noether symmetry vector and the Noether function are

$$\mathbf{X} = 2\psi_Y \int T(t) \, dt \partial_t + T(t) \, H^i \partial_i \quad \text{and} \quad G(t, x^k) = T^i \partial_i H + p \int T \, dt$$  \tag{30}

where $H^i$ is the gradient HV or a gradient KV, the function $T(t)$ is computed from the relation $T_{,tt} = mT$ where $m$ is a constant and the potential satisfies the condition

$$\mathcal{L}_H V + 2\psi_Y V + mH + p = 0.$$  \tag{31}

Concerning the Noether integrals we have the following result (not including the Hamiltonian)

**Corollary 1** The Noether integrals of Case I and Case II are respectively

$$I_{CI} = 2\psi_Y t E - g_{ij} Y^i \dot{x}^j + pt$$  \tag{32}

$$I_{CII} = 2\psi_Y \int T(t) \, dt \, E - g_{ij} H^i \dot{x}^j + T_{,t} H + p \int T \, dt.$$  \tag{33}

where $E$ is the Hamiltonian given in (20).

We remark that theorems 1 and 2 do not apply to generalized symmetries [17,18].

It would be of interest to examine if the above close relation of the Lie and the Noether symmetries of the second order ODEs of the form (6) with the collineations of the metric is possible to be carried over to some types of second order partial differential equations (PDEs). To this question it is not possible to give a global answer, due to the complexity of the study and the great variety of PDEs. However for the case of the generalized heat equation the problem has been solved (see [19]).

5. The Lie symmetries of geodesic equations in an Einstein space

Spaces of constant curvature are Einstein spaces whose curvature tensor is of the form $R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})$ where $R$ is a constant. In this section we generalize the results of the last section to proper Einstein spaces in which $R_{ab} = \frac{R}{2} g_{ab}$ but $^1 R_{abcd} \neq \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc})$.

Suppose $X^a$ is a projective collineation with projection function $\phi(x^a)$, so that $\mathcal{L}_X \Gamma^a_{bc} = \phi_{,b} \delta^a_c + \phi_{,c} \delta^a_b$. For a proper Einstein space:

$$\mathcal{L}_X g_{ab} = \frac{n(1 - n)}{R} \phi_{,ab}.$$

It follows that if $X^a$ generates either an affine or a special projective collineation then $\phi_{,ab} = 0$ hence $X^a$ reduces to a KV. This means that proper Einstein spaces do not admit homothetic vector, affine collineations or special projective collineations.

The above results and Theorem 1 lead to the following conclusion:

$^1$ Using the contracted Bianchi identity $(R^{ij} - \frac{1}{2} R g^{ij}){,}_{ij} = 0$ it follows that in an Einstein space of dimension $n > 2$ the curvature scalar $R =$constant.
Table 3. First family of 3d Newtonian dynamical systems admitting Lie symmetries

| Lie symmetry | \( F(x_\mu, x_\nu, x_\sigma) \) |
|-------------|---------------------------------|
| \( \frac{d}{dt} \partial_t + \partial_\mu \) | \( e^{-d x_\nu f_{\mu, \nu, \sigma}} (x_\nu, x_\sigma) \) |
| \( \frac{d}{dt} \partial_t + \partial_{\theta(\mu \nu)} \) | \( e^{-d \theta(\mu \nu)} f_{\mu, \nu, \sigma} (r_{(\mu \nu)}, x_\sigma) \) |
| \( \frac{d}{dt} \partial_t + R \partial_R \) | \( x_\mu^{1-d} f_{\mu, \nu, \sigma} \left( \frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu} \right) \) |
| \( \frac{d}{dt} \partial_t + x_\mu \partial_\mu \) | \( x_\mu^{1-d} f_{\mu, \nu, \sigma} (x_\nu, x_\sigma) \) |
| \( \frac{d}{dt} \partial_t + x_\nu \partial_\mu \) | \( e^{-d x_\nu x_\mu} \left[ \frac{x_\nu}{x_\mu} f_{\nu} (x_\nu, x_\sigma) + f_{\mu} (x_\nu, x_\sigma) \right] \partial_\mu + f_\nu \partial_\nu + f_\sigma \partial_\sigma \) |

**Theorem 3** The Lie point symmetries of the geodesic equations in a proper Einstein space of curvature scalar \( R \neq 0 \) are given by the vectors \( X = \xi(t, x) \partial_t + \eta^i(t, x) \partial_{x^i} \) where

\[
\begin{align*}
\xi(t, x) &= K t + L \\
\eta^i(t, x) &= D^i(x)
\end{align*}
\]

where \( D^i(x) \) is a KV of the metric and \( K, L \) are constants.

Theorem 3 extends and amends the conjecture of [14] to the more general case of Einstein spaces.

6. Applications in Newtonian Physics

One important problem to consider is to find all 2d and 3d dimensional autonomous Newtonian dynamical systems, that is, all dynamical systems moving in flat Euclidean space with equation of motion \( \ddot{x}^i = F^i(x^j) \) where \( i, j = 1, 2 \) or \( 1, 2, 3 \) which admit Lie and Noether symmetries and of course to determine the admitted symmetries. The answer to this question consists of finding all the forces \( F^i(x^j) \) which result in this property.

The importance of this problem is twofold. Indeed the knowledge of the Lie symmetries makes possible the computation of the invariants of the dynamical system; also the knowledge of the Noether symmetries gives the Noether integrals (conserved currents) of the system. Both these data can be used to simplify the equations of motion and even lead to their analytic solution.

The two dimensional case has been considered originally by [20] and the 3d case by [21, 22]. Both cases have been considered anew following the geometric approach of the previous paragraphs. More specifically the 2d case has been considered in [23] and the 3d case in [24]. In both cases it has been shown that the existing results were incomplete.

It is to be noted that from the applications point of view the 2d case is important because it applies to the mini super space which is particularly useful in cosmology. Indeed in the standard cosmological model and much of its extensions the resulting dynamical system reduces to a two dimensional dynamical system in a flat 2d Lorentz space (see [25]). For convenience we collect the results of the 3d case in Tables 3 and 4. Concerning the determination of all 3d Newtonian dynamical systems, which admit Noether point symmetries and subsequently the ones which are integrable via Noether integrals we have the results of the Table 5 and Table 6.

Noether symmetries

In order to show the use of these tables are used we consider two applications; one concerning the motion of a Newtonian system on a 2d space and the second motion in a 3d space. Before we
Table 4. Second family of 3d Newtonian dynamical systems admitting Lie symmetries

| Lie symmetry                  | $F_{\mu} (x_{\mu}, x_{\nu}, x_{\sigma})$ |
|-------------------------------|-------------------------------------------|
| $t \partial_{\mu}$           | $f_{\mu,\nu,\sigma} (x_{\nu}, x_{\sigma})$ |
| $t^2 \partial_{\mu} + t R \partial_R$ | $\frac{1}{x_{\mu}^2} f_{\mu,\nu,\sigma} \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ |
| $e^{\pm t \sqrt{m}} \partial_{\mu}$ | $-m x_{\mu} + f_{\mu,\nu,\sigma} (x_{\nu}, x_{\sigma})$ |
| $\frac{1}{\sqrt{m}} e^{\pm t \sqrt{m}} \partial_{\mu} + e^{\pm t \sqrt{m}} R \partial_R$ | $- \frac{m}{4} x_{\mu} + \frac{1}{x_{\mu}^2} f_{\mu,\nu,\sigma} \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ |

Table 5. First family of 3d Newtonian dynamical systems which admit Noether symmetries

| Noether          | $d = 0$ | $d \neq 2$ |
|------------------|---------|------------|
| $\frac{d}{dt} \partial_{\mu} + \partial_{\mu}$ | $c_1 x_{\mu} + f (x_{\nu}, x_{\sigma})$ | $e^{-dx_{\mu}} (x_{\nu}, x_{\sigma})$ |
| $\frac{d}{dt} \partial_{\mu} + \theta_{(\mu\nu)}$ | $c_1 \theta_{(\mu\nu)} + f (r_{(\mu\nu)}, x_{\sigma})$ | $e^{-d \theta_{(\mu\nu)}} f (r_{(\mu\nu)}, x_{\sigma})$ |
| $\frac{d}{dt} \partial_{\mu} + R \partial_R$ | $x_{\mu}^2 f \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ | $x_{\mu}^{2-d} f \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ |
| $\frac{d}{dt} \partial_{\mu} + x_{\mu} \partial_{\mu}$ | $c_1 x_{\mu}^2 + f (x_{\nu}, x_{\sigma})$ | $\varnothing$ |
| $\frac{d}{dt} \partial_{\mu} + x_{\nu} \partial_{\mu}$ | $c_1 x_{\mu} + c_2 \left( x_{\mu}^2 + x_{\nu}^2 \right) + f (x_{\sigma})$ | $\varnothing$ |

Table 6. Second family of 3d Newtonian dynamical systems which admit Noether symmetries

| Noether          | $V (x, y, z)$ |
|------------------|---------------|
| $t \partial_{\mu}$ | $c_1 x_{\mu} + f (x_{\nu}, x_{\sigma})$ |
| $t^2 \partial_{\mu} + t R \partial_R$ | $\frac{1}{x_{\mu}^2} f \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ |
| $e^{\pm t \sqrt{m}} \partial_{\mu}$ | $-m x_{\mu}^2 + c_1 x_{\mu} + f (x_{\nu}, x_{\sigma})$ |
| $\frac{1}{\sqrt{m}} e^{\pm t \sqrt{m}} \partial_{\mu} + e^{\pm t \sqrt{m}} R \partial_R$ | $- \frac{m}{8} R^2 + \frac{1}{x_{\mu}^2} f \left( \frac{x_{\nu}}{x_{\mu}}, \frac{x_{\sigma}}{x_{\mu}} \right)$ |

Proceed we recall that in order a 3d Newtonian dynamical system to be integrable via Noether point symmetries it must admit at least 3 Noether first integrals.

6.1. Newtonian motion on the two dimensional sphere

The motion of a system moving on the two dimensional sphere under the potential function $V (\theta, \phi)$ has Lagrangian

$$L (\phi, \theta, \dot{\phi}, \dot{\theta}) = \frac{1}{2} \left( \dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2 \right) - V (\theta, \phi)$$  \hspace{1cm} (36)
Table 7. The potentials $V(\theta, \phi)$ for which the dynamical system (36) is integrable via Noether point symmetries

| $V(\theta, \phi)$ | Noether Integral |
|-------------------|------------------|
| $F(\cos \theta \sin \phi)$ | $I_{CK}^{1,e,h}$ |
| $F(\sin \theta \sin \phi)$ | $I_{CK}^{2,e,h}$ |
| $F(\phi)$ | $I_{CK}^{3,e,h}$ |
| $F\left(\frac{1+\tan^2 \theta}{\sin^2 \phi (a-b \tan \theta)^2}\right)$ | $aI_{CK}^{1,e,h} + bI_{CK}^{2,e,h}$ |
| $F(a \cos \theta \sin \phi - K \cos \theta \cos \phi)$ | $aI_{CK}^{1,e,h} + bI_{CK}^{3,e,h}$ |
| $F(a \sin \theta \sin \phi - K \cos \phi)$ | $aI_{CK}^{2,e,h} + bI_{CK}^{3,e,h}$ |
| $F\left(\frac{a \cos \theta - b \sin \theta \sin \phi}{1+K \cos \phi}\right)$ | $aI_{CK}^{1,e,h} + bI_{CK}^{2,e,h} + cI_{CK}^{3,e,h}$ |

where

$\sin\phi = \begin{cases} \sin \phi & K = 1 \\ \sinh \phi & K = -1 \end{cases}$

$\cos\phi = \begin{cases} \cos \phi & K = 1 \\ \cosh \phi & K = -1 \end{cases}$

The equations of motion are:

\[
\ddot{\phi} - \sin\phi \cos\phi \dot{\theta}^2 + V_{,\phi} = 0 \quad (37)
\]

\[
\ddot{\theta} + 2 \frac{\cos\phi}{\sin\phi} \dot{\phi} \dot{\theta} + \frac{1}{\sin^2 \phi} V_{,\theta} = 0. \quad (38)
\]

The problem to find all potential functions for which the system is integrable via Noether point symmetries is to find all potentials for which the system admits at least one extra Noether symmetry, in addition to the trivial one which is the energy (Hamiltonian)

\[
E = \frac{1}{2} \left(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2\right) + V(\theta, \phi).
\]

In order to do that we consider the kinematic metric $g_{ij} = \frac{1}{2} \left(\dot{\phi}^2 + \sin^2 \phi \dot{\theta}^2\right)$ in the configuration space, which is defined by the kinematic part of the Lagrangian and apply theorem 2.

This has been done in [24] and the answer is given in Table 7. Where $I_{CK}^{1,2,3}$ are the Noether Integrals

\[
\begin{align*}
I_{CK}^{3} &= \dot{\phi} \sin^2 \phi, & I_{CK}^{1} &= \dot{\phi} \sin \theta + \dot{\theta} \cos \theta \sin \phi \cos \phi \\
I_{CK}^{2} &= \dot{\phi} \cos \theta - \dot{\theta} \sin \theta \sin \phi \cos \phi
\end{align*}
\]

Corollary 2 A dynamical system with Lagrangian (36) has one, two or four Noether point symmetries hence Noether integrals.

Proof: For the case of the free particle we have the maximum number of four Noether symmetries (the rotation group so(3) plus the $\partial_t$). In the case the potential is not constant the Noether symmetries are produced by the non-gradient KVs with Lie algebra $[X_A, X_B] = C_{AB}^C X_C$ where $C_{12}^{3} = C_{31}^{2} = C_{23}^{1} = 1$ for $\varepsilon = 1$ and $\bar{C}_{21}^{3} = \bar{C}_{23}^{1} = \bar{C}_{31}^{2} = 1$ for $\varepsilon = -1$. Because
the Noether point symmetries form a Lie algebra and the Lie algebra of the KVs is semisimple. The system will admit either none, one or three Noether symmetries generated from the KVs. The case of three is when \( V(\theta, \phi) = V_0 \) that is the case of geodesics, therefore the Noether point symmetries will be (including \( \partial_t \)) either one, two or four.

6.2. Newtonian motion in 3d space

We consider the dynamical system defined by the equations of motion

\[
\ddot{x}^\mu = -\frac{m}{4}x_\mu + \frac{1}{x_\mu^3} f_{\mu\nu\sigma} \left( \frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu} \right)
\]

This system is the well known Ermakov system which has a long history in the older and recent literature [26–30].

From Table 6 we find without any calculations that the system admits the Lie symmetry

\[
X = \frac{1}{\sqrt{m}} e^{t \sqrt{m}} \partial_t + e^{t \sqrt{m}} R \partial_R.
\]

7. Applications in cosmology

In this section we consider applications of Theorem 1 and Theorem 2 to dynamical systems occurring in Cosmology.

7.1. Lie and Noether symmetries of Bianchi class A homogeneous cosmologies with a scalar field

The Bianchi models in the ADM formalism are described by the metric

\[
\text{ds}^2 = -N^2(t) dt^2 + g_{\mu\nu} \omega^\mu \otimes \omega^\nu
\]

where \( N(t) \) is the lapse function and \( \{ \omega^\alpha \} \) is the canonical basis 1-forms which satisfy the Lie algebra \( \text{d}\omega^i = C^i_{jk} \omega^j \wedge \omega^k \) where \( C^i_{jk} \) are the structure constants of the algebra.

The spatial metric \( g_{\mu\nu} \) splits so that \( g_{\mu\nu} = \exp(2\lambda) \exp(-2\beta)_{\mu\nu} \) where \( \exp(2\lambda) \) is the scale factor of the universe and \( \beta_{\mu\nu} \) is a 3 \( \times \) 3 symmetric, traceless matrix, which can be written in a diagonal form with two independent quantities, known as the anisotropy parameters \( \beta_+, \beta_- \), as follows:

\[
\beta_{\mu\nu} = \text{diag} \left( \beta_+, -\frac{1}{2} \beta_+ + \frac{\sqrt{3}}{2} \beta_-, -\frac{1}{2} \beta_+ - \frac{\sqrt{3}}{2} \beta_- \right).
\]

It has been shown that the dynamics of the class A Bianchi models with a scalar field is described by the Lagrangian

\[
L = e^{3\lambda} \left[ R^* + 6\lambda - \frac{3}{2} (\beta_1^2 + \beta_2^2) - \dot{\phi}^2 + V(\phi) \right]
\]

where \( R^* \) is the Ricci scalar of the 3 dimensional spatial hypersurfaces given by the expression:

\[
R^* = -\frac{1}{2} e^{-2\lambda} \left[ N_1^2 e^{4\beta_1} + e^{-2\beta_1} \left( N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2} \right)^2 - 2 N_1 e^{\beta_1} \left( N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2} \right) \right] + \frac{1}{2} N_1 N_2 N_3 (1 + N_1 N_2 N_3).
\]

The constants \( N_1, N_2, \) and \( N_3 \) are the components of the classification vector \( n^\mu \) and \( \beta_1 = -\frac{1}{2} \beta_+ + \frac{\sqrt{3}}{2} \beta_- \), \( \beta_2 = -\frac{1}{2} \beta_+ - \frac{\sqrt{3}}{2} \beta_- \).
It is important to note that the curvature scalar $R^*$ does not depend on the derivatives of the anisotropy parameters $\beta_+, \beta_-$, equivalently of $\beta_1, \beta_2$.

The Euler Lagrange equations due to the Lagrangian (41) are:

\[
\ddot{\lambda} + \frac{3}{2} \dot{\lambda}^2 + \frac{3}{8} (\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{1}{4} \dot{\phi}^2 - \frac{1}{12} e^{-3\lambda} \frac{\partial}{\partial \lambda} \left( e^{3\lambda} R^* \right) - \frac{1}{2} V(\phi) = 0
\]

\[
\ddot{\beta}_1 + 3 \dot{\lambda} \dot{\beta}_1 + \frac{1}{3} \frac{\partial R^*}{\partial \beta_1} = 0
\]

\[
\ddot{\beta}_2 + 3 \dot{\lambda} \dot{\beta}_2 + \frac{1}{3} \frac{\partial R^*}{\partial \beta_2} = 0
\]

\[
\ddot{\phi} + 3 \dot{\phi} \dot{\lambda} + \frac{\partial V}{\partial \phi} = 0
\]

where a dot over a symbol indicates derivative with respect to $t$.

We apply Theorem 1 and Theorem 2 in order to compute the Lie and the Noether symmetries of class A Bianchi models with a scalar field. Similar (but incomplete) works on that topic can be found in [31–33].

We consider the four dimensional Riemannian space with coordinates $x^i = (\lambda, \beta_1, \beta_2, \phi)$ and metric

\[
ds^2 = e^{3\lambda} \left( 12d\lambda^2 - 3d\beta_1^2 - 3d\beta_2^2 - 2d\phi^2 \right). \tag{42}
\]

It is easy to show that this metric is the conformally flat FRW spacetime whose special projective algebra consists of the non gradient KVs

\[
Y^1 = \partial_{\beta_1}, \; Y^2 = \partial_{\beta_2}, \; Y^3 = \partial_\phi, \; Y^4 = \beta_2 \partial_{\beta_1} - \beta_1 \partial_{\beta_2} \]

\[
Y^5 = \phi \partial_{\beta_1} - \frac{3}{2} \beta_1 \partial_\phi, \; Y^6 = \phi \partial_{\beta_2} - \frac{3}{2} \beta_2 \partial_\phi
\]

and the gradient HV

\[
H^i = \frac{2}{3} \partial_\lambda, \; \psi = 1.
\]

The Lagrangian is written $L = T - U(x^i)$ where $T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$ is the geodesic Lagrangian, the potential function is

\[
U(x^i) = -e^{3\lambda} \left( V(\phi) + R^* \right) \tag{43}
\]

and we have used the fact that the curvature scalar does not depend on the derivatives of the coordinates $\beta_1, \beta_2$.

We apply Theorem 1 and Theorem 2 to determine the Lie and the Noether symmetries of the dynamical system with Lagrangian (41) in the following cases:

Case 1. Vacuum. In this case $\phi =$constant.

Case 2. Zero potential $V(\phi) = 0, \phi \neq 0$

Case 3. Constant Potential $V(\phi) =$constant, $\dot{\phi} \neq 0$

Case 4. Arbitrary Potential $V(\phi), \dot{\phi} \neq 0$.

The detailed calculations can be found in [8]. Below we give the results for Bianchi I in the Table 8, for Bianchi II in the Table 9, for Bianchi VI and VII in the Table 10 and for Bianchi VIII/IX spacetimes in the Table 11.

8. Conclusion

The Lie and the Noether symmetry vectors of the equations of motion in a general Riemannian space are not independent dynamical quantities but instead they are determined completely by the geometry of the space. Indeed Theorem 1 and Theorem 2 show that the Lie point
### Table 8. Bianchi I spacetime

| Bianchi I | Noether Sym. | Lie Sym. |
|-----------|--------------|----------|
| Vacuum    | $\partial_t, Y^1, Y^2, Y^4$ | $\partial_t, t\partial_t, Y^{1,2,4}, H^i$ |
|           | $2t\partial_t + H^i, t^2\partial_t + tH^i$ | $t^2\partial_t + tH^i$ |
| Zero Pot. | $\partial_t, Y^{1-6}, 2t\partial_t + H^i$ | $\partial_t, t\partial_t, Y^{1-6}$ |
|           | $t^2\partial_t + tH^i$ | $H^i, t^2\partial_t + tH^i$ |
| Constant Pot. | $\partial_t, Y^{1-6}$ | $\partial_t, Y^{1-6}, H^i$ |
|           | $\frac{1}{2}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$ | $\frac{1}{2}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$ |
| Arbitrary Pot. | $\partial_t, Y^{1,2,4}$ | $\partial_t, Y^{1,2,4}, H^i$ |
| Exponential Pot. | $\partial_t, Y^{1,2,4}$ | $\partial_t, Y^{1,2,4}, H^i$ |
|           | $2t\partial_t + H^i + \frac{4}{3}Y^3$ | $t\partial_t + \frac{2}{3}Y^3$ |

### Table 9. Bianchi II spacetime

| Bianchi II | Noether Sym. | Lie Sym. |
|------------|--------------|----------|
| Vacuum     | $\partial_t, Y^2$ | $\partial_t, Y^2$ |
|           | $6t\partial_t + 3H^i - 5Y^1$ | $\frac{1}{3}t\partial_t + H^i, t\partial_t - Y^1$ |
| Zero Pot.  | $\partial_t, Y^2, Y^3, Y^6$ | $\partial_t, Y^2, Y^3, Y^6$ |
|           | $6t\partial_t + 3H^i - 5Y^1$ | $\frac{1}{3}t\partial_t + H^i, t\partial_t - Y^1$ |
| Constant Pot. | $\partial_t, Y^2, Y^3, Y^6$ | $\partial_t, Y^2, Y^3, Y^6$ |
|           | $3H^i + Y^1$ | $3H^i + Y^1$ |
| Arbitrary Pot. | $\partial_t, Y^2$ | $\partial_t, Y^2, 3H^i + Y^1$ |
| Exponential Pot. | $\partial_t, Y^2$ | $\partial_t, Y^2, 3H^i + Y^1$ |
|           | $2t\partial_t + H^i - \frac{5}{3}Y^1 + \frac{4}{3}Y^3$ | $t\partial_t + \frac{2}{3}Y^3$ |

### Table 10. Bianchi VI/VII spacetimes

| Bianchi VI$_0$ / VII$_0$ | Noether Sym. | Lie Sym. |
|---------------------------|--------------|----------|
| Vacuum                    | $\partial_t, 6t\partial_t + 3H^i + \frac{1}{2}Y^1 - 2\sqrt{3}Y^2$ | $\partial_t, H^i + \frac{1}{2}Y^1 + \frac{\sqrt{3}}{2}Y^2$ |
|                           | $- 2Y^1 - 2\sqrt{3}Y^2$ | $2t\partial_t - Y^1 - \sqrt{3}Y^2$ |
| Zero Pot.                 | $\partial_t, Y^3, 6t\partial_t + 3H^i + \frac{1}{2}Y^1 - 2\sqrt{3}Y^2$ | $\partial_t, H^i + \frac{1}{2}Y^1 + \frac{\sqrt{3}}{2}Y^2$ |
|                           | $- 2Y^1 - 2\sqrt{3}Y^2$ | $Y^3, 2t\partial_t - Y^1 - \sqrt{3}Y^2$ |
| Constant Pot.             | $\partial_t, Y^3$ | $\partial_t, Y^3, H^i + \frac{1}{2}Y^1 + \frac{\sqrt{3}}{2}Y^2$ |
| Arbitrary Pot.            | $\partial_t$ | $\partial_t, H^i + \frac{1}{2}Y^1 + \frac{\sqrt{3}}{2}Y^2$ |
| Exponential Pot.          | $\partial_t, 6t\partial_t + 3H^i - 2Y^1 + \frac{1}{2}Y^1 + \frac{\sqrt{3}}{2}Y^2$ | $t\partial_t + \frac{1}{2}Y^3$ |
|                           | $- 2\sqrt{3}Y^2 + \frac{6}{7}Y^3$ | $t\partial_t + \frac{1}{2}Y^3$ |
Table 11. Bianchi VIII/IX spacetimes

| Bianchi VIII | Noether Sym. | Lie Sym. |
|--------------|--------------|----------|
| Vacuum       | $\partial_t$ | $\partial_t$, $\frac{2}{3}t\partial_t + H^i$ |
| Zero Pot.    | $\partial_t$, $Y^3$ | $\partial_t$, $Y^3$, $\frac{2}{3}t\partial_t + H^i$ |
| Constant Pot.| $\partial_t$, $Y^3$ | $\partial_t$, $Y^3$ |
| Arbitrary Pot.| $\partial_t$ | $\partial_t$ |

| Bianchi IX | Noether Sym. | Lie Sym. |
|-----------|--------------|----------|
| Vacuum    | $\partial_t$ | $\partial_t$ |
| Zero Pot. | $\partial_t$, $Y^3$ | $\partial_t$, $Y^3$ |
| Constant Pot.| $\partial_t$, $Y^3$ | $\partial_t$, $Y^3$ |
| Arbitrary Pot.| $\partial_t$ | $\partial_t$ |

symmetries are elements of the special projective algebra of the space and the Noether point symmetries elements of the homothetic algebra of the space. The selection of the particular vectors for a given ‘force’ modulating the motion of a particular system is done by means of certain compatibility conditions of differential nature which can easily be managed. Using the above theorems we have determined all 2d and 3d Newtonian dynamical systems which admit Lie and Noether point symmetries and have demonstrated how the results can be used to determine these symmetries in specific situations. Furthermore we have applied the theorems in the case of Cosmology and have determined the Lie and the Noether point symmetries of the Bianchi Class A homogeneous models. More applications of these theorems in cosmology can be found in [25, 34], where one is able to determine analytic solutions using the integration of field equations by means of Noether integrals.

Acknowledgments
This research was partially funded by the University of Athens Special Account of Research Grants no 10812.

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