The small index property for countable superatomic boolean algebras

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Abstract
It is shown that all the countable superatomic boolean algebras of finite rank have the small index property.

Keywords Countable atomic boolean algebra · Superatomic · Small index property

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1 Introduction

In [3] it was shown that the full symmetric group on a countably infinite set has the ‘small index property’ SIP, meaning that any subgroup having index strictly less than $2^{\aleph_0}$ contains the pointwise stabilizer of a finite set. The corresponding result for the group of order-preserving permutations of the set of rational numbers was given in [9]. The small index property has received a great deal of attention in quite a wide variety of special cases. Its model-theoretic significance is that its truth tells us that the natural topological group associated with the structure (under the topology of pointwise convergence) can be recovered from the pure group, and from this one can deduce that the structure is interpretable in the (abstract) automorphism group [5]. Most of the structures for which SIP has been studied are $\aleph_0$-categorical, but this is not required in the definition, and some non-$\aleph_0$-categorical cases have been looked at, for instance 1-transitive linear orders in [1], trees and cycle-free partial orders in [10], and rational Urysohn space [8].

In this paper we look at a class of countable structures which are not $\aleph_0$-categorical, namely the countable superatomic boolean algebras of finite rank, and show that their automorphism groups have the small index property. The definitions here are that the boolean algebra $B$ is atomic if every element is the least upper bound of the atoms...
below it, and it is superatomic if every homomorphic image is atomic. This notion was explored for instance in \([2, 7]\), and various equivalent conditions were given for superatomicity. Here we just look at the countable case (since for the most part, that is the context for considering \(\text{SIP}\)), where one can give an explicit description.

It is easiest to work with the order topology on a countable ordinal \(\alpha\), which by definition has as a base for its topology all open intervals \((a, b)\) where \(a, b \in \alpha\cup\{\pm \infty\}\). This is a compact space if and only if \(\alpha\) is a successor ordinal, which we now assume is the case, and then the family of clopen subsets forms a countable superatomic boolean algebra. Since for any successor ordinals \(\alpha\) and \(\beta\), \(\alpha + \beta\) is homeomorphic to \(\beta + \alpha\) under the order topology, by writing in Cantor normal form and reducing, the least successor ordinal in any homeomorphism class has the form \(\omega^\alpha \cdot a + 1\) for some finite \(a\) (as is shown in \([6]\) for instance), and in the terminology of \([2]\), it has cardinal sequence \((\aleph_0, \aleph_0, \ldots, a)\).

To analyze countable superatomic boolean algebras \(\mathbb{B}\), we consider the increasing sequence of ideals \((I_\beta : \beta \in On)\) given inductively as follows: \(I_0 = \{0\}\); assuming \(I_\beta\) has been defined, let \(I_{\beta + 1} \supsetneq I_\beta\) be such that \(I_{\beta + 1}/I_\beta\) is the ideal of \(\mathbb{B}/I_\beta\) generated by its set of atoms; for limits \(\lambda\), \(I_\lambda = \bigcup_{\beta < \lambda} I_\beta\). There is a least \(\alpha = \alpha(\mathbb{B})\) such that \(I_\alpha = I_{\alpha + 1}\), and (by superatomicity), this can only happen when \(I_\alpha = \mathbb{B}\). One can check that \(\alpha(\mathbb{B})\) is a successor, and \(\alpha - 1\) is called the rank of \(\mathbb{B}\). Clearly \(\mathbb{B}/I_{\alpha - 1}\) is finite, and \(\mathbb{B}\) is determined uniquely up to isomorphism by its rank and the number \(a\) of atoms of \(\mathbb{B}/I_{\alpha - 1}\) (called its ‘degree’). From the point of view of verifying the small index property, we can assume that \(|\mathbb{B}/I_{\alpha - 1}| = 2\) (so there is just one atom), since the general case can easily be derived from this (see Corollary 3.2). The ideal of a boolean algebra \(\mathbb{B}\) generated by its atoms is called the Fréchet ideal, written \(Fr(\mathbb{B})\). Thus in the above inductive step, \(I_{\beta + 1}/I_\beta = Fr(\mathbb{B}/I_\beta)\).

We remark that there is a direct connection between the rank of this superatomic boolean algebra \(\mathbb{B}\) and the Cantor–Bendixson rank of members of a successor ordinal \(\alpha\) whose family of clopen sets is isomorphic to \(\mathbb{B}\), since the points of Cantor–Bendixson rank less than \(\beta\) are precisely those elements of \(\alpha\) not divisible by \(\omega^\beta\).

Let us start then with an ordinal of the form \(\omega^\alpha\), where this is the ordinal power, and we consider the successor ordinal \(\omega^\alpha + 1\) under the order topology. For technical reasons, we work with the interval \(X = [1, \omega^\alpha]\) (which is order-isomorphic to \(\omega^\alpha + 1\)). If we let \(\mathbb{B}\) be the boolean algebra of clopen subsets of \(X\), then \(\mathbb{B}\) is superatomic of rank \(\alpha\) and degree 1. In fact one can verify that \(I_\beta\) is the family of ordinals in this range which are not divisible by \(\omega^{\beta + 1}\). For the proof that any countable superatomic boolean algebra is isomorphic to the algebra of clopen subsets of \(\omega^\alpha \cdot a + 1\) for some countable ordinal \(\alpha\) and finite \(a\), see \([2]\). For simplicity we assume that \(a = 1\), and deduce the result for general \(a\) in the final section.

We remark that the main part of our argument is contained in Lemmas 2.3–2.7, and the derivation of the main result in Sect. 3 is straightforward. The proof of Lemma 2.1 uses ideas from [3].

**2 The induction step**

We are aiming to prove that the group of homeomorphisms to itself \(G = G_\delta\) of \([1, \delta]\) satisfies the small index property for all countable \(\delta\) of the form \(\omega^\alpha\). At present we have
only succeeded in doing this for \( \alpha \) finite. This goes by induction on \( \alpha \). Since however the induction step goes through for successors for arbitrary \( \alpha \), the assumption that \( \alpha \) is finite is only made in the final section. So we let \( \delta = \omega^{\alpha+1} \), where \( \alpha \) is a countable ordinal, and this is the ordinal power. Let \( X = X_\delta = [1, \delta] \), under the order topology, and let \( \mathcal{B} \) be the boolean algebra of its clopen subsets. Then the group \( G = G_\delta \) of homeomorphisms of \( X \) to itself is the same as the automorphism group of \( \mathcal{B} \).

Let us write \( A_n = [\omega^\alpha (n - 1) + 1, \omega^\alpha \cdot n] \), for \( n \in [1, \omega) \). Thus the sets \( A_n \) form a partition of \( X \setminus \{\delta\} \) into clopen subsets. Throughout this section we write \( K \) for the subgroup of \( G \) which fixes each \( A_n \) setwise. We notice that each \( A_n \) is homeomorphic to \([1, \omega^\alpha] \), so \( K \) is isomorphic to the unrestricted direct product of the automorphism groups \( G_n \) of \( \mathcal{B} \) restricted to \( A_n \), each of which is isomorphic to \( G_{\omega^\alpha} \). For each \( n \) we fix a bijection \( \varphi_n \) from \( A_n \) to \( A_n \), which may be explicitly given by \( \varphi_n(\beta) = \omega^\alpha (n - 1) + \beta \) for \( \beta \in [1, \omega^\alpha] \). Then \( \varphi_n \) is a homeomorphism, and so it induces an isomorphism from \( G_1 \) to \( G_n \). Also \( \varphi_{ij} = \varphi_j \varphi_i^{-1} \) is a homeomorphism from \( A_i \) to \( A_j \), and clearly \( \varphi_{ij}^{-1} = \varphi_{ji} \).

Lemma 2.1 If \( N \) is a normal subgroup of \( G \) of index \( < 2^{\aleph_0} \), then \( K \leq N \).

Proof We write \( G_{\omega^\alpha} = H \), and identify \( K \) with \( H^\omega \) when desired. Let \([1, \omega) \) be written as the disjoint union \( \bigcup_{n \in [1, \omega)} Z_n \) of infinite sets \( Z_n \), and let \( \theta_n \) be a bijection from \([1, \omega) \) to \( Z_n \). Thus \( \theta_{mn} = \theta_n \theta_m^{-1} \) is a bijection from \( Z_m \) to \( Z_n \) (and its inverse is \( \theta_{nm} \)). Let \( h \) be an arbitrary member of \( K \). Then we may ‘copy’ \( h \) to each \( \bigcup_{i \in Z_n} A_i \) by letting \( h_n = \bigcup_{i \in Z_n} \varphi_{i \theta_n^{-1} i}^{-1} h \varphi_{i \theta_n^{-1} i} \), and \( h_n \) fixes all other points. Unravelling this, if \( i \in Z_n \), then \( \varphi_{i \theta_n^{-1} i} \) maps \( A_i \) to \( \varphi_{i \theta_n^{-1} i} \), where \( h \) now acts, and then \( \varphi_{i \theta_n^{-1} i}^{-1} \) maps this back to \( A_i \). Note that the actions of all the \( h_n \) are disjoint, since the support of \( h_n \) is contained in \( \bigcup_{i \in Z_n} A_i \).

Let \( M \) be a family of \( 2^{\aleph_0} \) pairwise almost disjoint infinite subsets of \([1, \omega) \) such that the union of any two is cofinite. For \( M \in \mathcal{M} \), let \( h_M \) agree with \( h_n \) on \( \bigcup_{i \in Z_n} A_i \) for all \( n \in M \), and fix all other members of \( X \). Then \( \{h_M : M \in \mathcal{M}\} \) is a family of \( 2^{\aleph_0} \) members of \( G \), so two of them must lie in the same left coset of \( N \). Hence \( h_M^{-1} h_M \in N \) for some \( M_1 \neq M_2 \) in \( \mathcal{M} \). Now if \( n \in M_1 \cap M_2 \), then \( h_{M_1} \) and \( h_{M_2} \) both agree with \( h_n \) on \( \bigcup_{i \in Z_n} A_i \), and so \( h_{M_1}^{-1} h_{M_2} \) is the identity on \( \bigcup_{i \in Z_n, n \in M_1 \cap M_2} A_i \), and it agrees with \( h_{M_1}^{-1} \) on \( \bigcup_{i \in Z_n} A_i \) for \( n \in M_1 \setminus M_2 \) and with \( h_{M_2} \) on \( \bigcup_{i \in Z_n} A_i \) for \( n \in M_2 \setminus M_1 \). We may therefore alternatively write \( h_{M_1}^{-1} h_{M_2} \) as \( h_{I_1}^{-1} h_{I_2} \) where \( I_1 = M_1 \setminus M_2 \) and \( I_2 = M_2 \setminus M_1 \) are disjoint and infinite, and with \( I_1 \cup I_2 \) cofinite.

Now let \( I_1 \) and \( I_2 \) be any disjoint infinite subsets of \([1, \omega) \) with cofinite union. Then there is a permutation \( \psi \) of \([1, \omega) \) which takes \( I_1 \) to \( J_1 \) and \( I_2 \) to \( J_2 \). Let \( k \in G \) be given by \( k(x) = \varphi_{\psi(m) \theta_n^{-1} \psi(m)}(x) \) if \( x \in A_i, i \in Z_m \). Thus for each \( m, k \) maps \( \bigcup_{i \in Z_m} A_i \) bijectively to \( \bigcup_{j \in Z_{\psi(m)}} A_j \). We want to show that \( h_{J_1}^{-1} h_{J_2} = k h_{I_1}^{-1} h_{I_2} k^{-1} \). Take any \( x \in X \setminus \{\delta\} \). Let \( n \in \omega \) and \( j \in Z_n \) be such that \( x \in A_j \). Let \( n = \psi(m) \),
such that any ‘copy’ of a coinfinite set lies in each of these restrictions lies in $G$. Observe that the argument so far started with any $h$ having Cantor–Bendixson rank $\omega\alpha$ whose support is contained in a coinfinte set lies in $N$. We calculate that

$$kh_1^{-1}h_2^{-1}(x) = \varphi_{ij}h_1^{-1}h_2^{-1}(x) = \varphi_{ij}h_2^{-1}(y) = \varphi_{ij}h_m(y)$$

$$= \varphi_{ij}\varphi_{i \ominus_1}^{-1}h\varphi_{i \ominus_1}^{-1}(y) = \varphi_{ij}\varphi_{i \ominus_1}^{-1}h\varphi_{i \ominus_1}^{-1}(y)$$

$$= \varphi_{i \ominus_1}^{-1}j h\varphi_{i \ominus_1}^{-1}(y) = \varphi_{i \ominus_1}^{-1}j h\varphi_{i \ominus_1}^{-1}(y)$$

$$= \varphi_{i \ominus_1}^{-1}j h\varphi_{j \ominus_1}^{-1}(x) = \varphi_{i \ominus_1}^{-1}j h\varphi_{j \ominus_1}^{-1}(x) = h_n(x) = h_j^{-1}h_2(x).$$

If $m \in I_1$, essentially the same calculation applies, with all elements involving $h$ replaced by their inverses. If $m \notin I_1 \cup I_2$, then the calculation is as follows:

$$kh_1^{-1}h_2^{-1}(x) = \varphi_{ij}h_1^{-1}h_2^{-1}(x) = \varphi_{ij}h_1^{-1}h_2^{-1}(y) = \varphi_{ij}(y) = x = h_j^{-1}h_2(x).$$

Since $N$ is normal, it follows that $h_j^{-1}h_2$ also lies in $N$. Take any such $J_1$ and $J_2$, and choose $J'_1, J'_2$ such that $J'_2 \subseteq J_1, |J_1 \setminus J'_2| = 1$, and $J'_1 = J_2$. By the above argument, there are members of $N$ which are equal to $h_j^{-1}h_2$ and $h_j^{-1}h_2$, and multiplying these, we get a member of $N$ which is equal to $h_j^{-1}h_k$ for some $i \in \omega$. By applying further conjugacies as necessary, every $h(i)$ lies in $N$.

The argument so far started with any $h \in K$, and found an infinite coinfinite set $Z$ such that any ‘copy’ of $h$ with support contained in $Z$ lies in $N$. If instead we start with $h'$ having support contained in an infinite coinfinite set $Z_1$, then we can find a partition of $[1, \omega)$ into $Z_n$ for $n \in [1, \omega)$, and $h \in K$ such that $h_1 = h'$, which as we have just shown, lies in $N$. In other words, any member of $K$ whose support is contained in a coinfinite set lies in $N$. Now cut $\omega$ into two infinite pieces, and write $h = h'^{\omega}$ where these are the restrictions of $h$ to two pieces of the given kind. By the above argument, each of these restrictions lies in $N$, and hence so does $h$. \qed

There is a rather larger subgroup of $G$ that we now need to consider, written $K^*$. Observe that $G$ fixes setwise the set $\{\omega^\alpha \cdot n : n \in [1, \omega)\}$, since these are the points having Cantor–Bendixson rank $\alpha$, so there is a homomorphism $\psi$ from $G$ to the group of all permutations of this set given by restriction, and $K^*$ is defined to be its kernel, which is thus the pointwise stabilizer of $\{\omega^\alpha \cdot n : n \in [1, \omega)\}$. This clearly contains $K$, since $\omega^\alpha \cdot n$ is the unique point of $A_n$ having Cantor–Bendixson rank $\alpha$. Using the fact that all the $A_n$ are homeomorphic, we see that we can permute them arbitrarily by members of $G$, and this means that $\psi$ is surjective. In practice we use a related map $\pi$ where for $g \in G$, $\pi(g)$ is the permutation on $[1, \omega)$ corresponding to $\psi$, which is therefore defined by $\pi(g)(m) = n$ if $\psi(g)(\omega^\alpha \cdot m) = \omega^\alpha \cdot n$. Thus $\pi$ is a homomorphism from $G$ to Sym$[1, \omega)$, also surjective and having kernel $K^*$.

**Lemma 2.2** If $N$ is a normal subgroup of $G$ of index $< 2^{\aleph_0}$, then for any $\sigma \in$ Sym$[1, \omega)$ there is $g \in N$ such that $\pi(g) = \sigma$. \hfill $\Box$
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Proof Using the above notation we observe that $NK^*/K^*$ is a normal subgroup of $G/K^*$ of index $< 2^{\aleph_0}$, and since Sym $\omega$ has no proper normal subgroups of index $< 2^{\aleph_0}$, it follows that $NK^* = G$, and this establishes precisely what is wanted. For given $\sigma \in \text{Sym}[1, \omega]$, let $g \in G$ be such that $\pi(g) = \sigma$ (for instance, $g$ may be taken in $K$). Since $NK^* = G$, we may write $g = hk$ where $h \in N, k \in K^*$, and it follows that $\pi(h) = \sigma$.

Now any open subset of $A_m$ containing $\omega^\delta \cdot m$ contains some set of the form $(x, \omega^\delta \cdot m]$ where $x < \omega^\delta \cdot m$ lies in $A_m$, so it follows by continuity that whenever $g(\omega^\delta \cdot m) = \omega^\delta \cdot n$, there is some $x < \omega^\delta \cdot m$ in $A_m$ such that $(x, \omega^\delta \cdot m]$ is mapped into $A_n$. More precisely, $g^{-1}A_n$ is an open set containing $\omega^\delta \cdot m$ which therefore contains $(x, \omega^\delta \cdot m]$ for some $x < \omega^\delta \cdot m$ lying in $A_m$ and therefore $A_m \setminus g^{-1}A_n \subseteq [\omega^\delta(m - 1) + 1, x]$. Let us call a subset of $A_m$ bounded if it is contained in $[\omega^\delta(m - 1) + 1, x]$ for some $x < \omega^\delta \cdot n$ in $A_m$, and it is cofinal if it is the complement of such a set (noting that this is not the usual sense of ‘cofinal’). Therefore, $A_m \setminus g^{-1}A_n$ is bounded, and a similar argument shows that $A_n \setminus gA_m$ is a bounded subset of $A_n$. In summary, the action of $G$ on $X$ is approximated by the symmetric group on $\{A_n : n \in [1, \omega)\}$. A key part of our argument concerns how the bounded pieces (forming the ‘errors’) are permuted by members of $G$.

Let $P$ be the family of ordered pairs of bounded clopen subsets of $X_{\omega^\delta}$. If $P$ and $Q$ are topological spaces, we denote by $P \sqcup Q$ the space obtained by taking their disjoint union, with the topology under which each of $P$ and $Q$ is clopen, and the subspace topology on each of them is their original topology. If $(P_1, Q_1), (P_2, Q_2) \in P$, we write $(P_1 \sqcup P_2, Q_1 \sqcup Q_2)$ as $(P_1, Q_1) + (P_2, Q_2)$. There is an associated equivalence relation $\sim$ on $P$ given by $(P_1, Q_1) \sim (P_2, Q_2)$ if $P_1 \sqcup P_2$ is homeomorphic to $P_2 \sqcup Q_1$. Now for $g \in G$ we let $P_i = P_i(g) = A_{\pi(g)(i)} \setminus gA_i$, and $Q_i = Q_i(g) = A_i \setminus g^{-1}A_{\pi(g)(i)}$, which as remarked above are bounded, so that $(P_i, Q_i) \in P$ if we identify $X_{\omega^\delta} = [1, \omega^\delta]$ with $A_i$, since by definition of $\pi$, $g(\omega^\delta \cdot i) = \omega^\delta \cdot \pi(g)(i)$, and using continuity of $g$. In the first case one considers, which is $\alpha = 1$, $P_i, Q_i$ are finite, and up to $\sim$-equivalence we can just replace $(P_i(g), Q_i(g))$ by $|P_i(g)| - |Q_i(g)|$. In the general case, these sets are not necessarily finite, though they are bounded and clopen, and so in a sense they are encoded by a finite amount of information, being compact sets. If $(P, Q) \in P$, we let $-(P, Q) = (Q, P)$, and this definition allows us to subtract members of $P$ (as well as add them).

Lemma 2.3 If $g \in G$ and $B$ is a cofinal subset of $A_i$ such that $gB \subseteq A_{\pi(g)(i)}$, then $(P_i(g), Q_i(g)) \sim (A_{\pi(g)(i)} \setminus gB, A_i \setminus B)$.

Proof Let us write $j$ for $\pi(g)(i)$. Then

$$(P_i(g), Q_i(g)) = (A_j \setminus gA_i, A_i \setminus g^{-1}A_j) = (A_j \setminus (A_j \cap gA_i), A_i \setminus (A_i \cap g^{-1}A_j)) \sim ((A_j \setminus (A_j \cap gA_i)) \cup ((A_j \cap gA_i) \setminus gB), (A_i \setminus (A_i \cap g^{-1}A_j)) \cup ((A_i \cap g^{-1}A_j) \setminus gB))$$

since $B \subseteq A_i \cap g^{-1}A_j \subseteq A_i, gB \subseteq A_j \cap gA_i \subseteq A_j$, and $g$ is a homeomorphism from $(A_i \cap g^{-1}A_j) \setminus B$ to $(A_j \cap gA_i) \setminus gB$. 

\(\square\)
Let us write $\sigma$.

Since $(A_j \setminus gB, A_i \setminus B)$, as desired.

\[ (P_i(hg), Q_i(hg)) = (A_k \setminus hgB, A_i \setminus B) \]

$\sim ((A_k \setminus hgB) \cup (A_j \setminus gB), (A_j \setminus gB) \cup (A_i \setminus B))$

$= ((A_k \setminus hgB), (A_j \setminus gB)) + ((A_j \setminus gB), (A_i \setminus B))$

$= (P_i(g), Q_i(g)) + (P_j(h), Q_j(h)).$

\[ \square \]

**Lemma 2.4** If $g, h \in G$ then for each $i$, $(P_i(hg), Q_i(hg)) \sim (P_i(g), Q_i(g)) + (P_{\pi(g)\pi(i)}(h), Q_{\pi(g)\pi(i)}(h))$.

**Proof** Let us write $j = \pi(g)\pi(i)$ and $k = \pi(h)\pi(j)$ and by continuity of $g$ and $h$ pick a cofinal subset $B$ of $A_i$ such that $gB \subseteq A_j$ and $hgB \subseteq A_k$. Then by Lemma 2.3,

\[ (P_i(hg), Q_i(hg)) = (A_k \setminus hgB, A_i \setminus B) \]

$\sim ((A_k \setminus hgB) \cup (A_j \setminus gB), (A_j \setminus gB) \cup (A_i \setminus B))$

$= ((A_k \setminus hgB), (A_j \setminus gB)) + ((A_j \setminus gB), (A_i \setminus B))$

$= (P_i(g), Q_i(g)) + (P_j(h), Q_j(h)).$

\[ \square \]

**Lemma 2.5** For any $g$ such that $\sigma = \pi(g)$ maps $[1, \omega)$ in a single infinite cycle, and $f : [1, \omega) \to \mathcal{P}$, there is $h \in K^*$ such that for each $i$, $(P_i(h^{-1}g), Q_i(h^{-1}gh)) \sim f(i)$.

**Proof** We choose $(R_i, S_i) \in \mathcal{P}$ inductively thus:

\[ (R_1, S_1) = (\emptyset, \emptyset) \]

\[ (R_{\sigma j+1}, S_{\sigma j+1}) = (R_{\sigma j+1}, S_{\sigma j+1}) - (P_{\sigma j+1}(g), Q_{\sigma j+1}(g)) + f(\sigma j 1) \text{ if } j \geq 0 \]

\[ (R_{\sigma j-1}, S_{\sigma j-1}) = (R_{\sigma j-1}, S_{\sigma j-1}) + (P_{\sigma j-1}(g), Q_{\sigma j-1}(g)) - f(\sigma j^{-1} 1) \text{ if } j \leq 0. \]

Since $\sigma$ has a single infinite cycle, this defines $(R_i, S_i)$ for all $i \in [1, \omega)$, and we note that $(R_{\sigma i}, S_{\sigma i}) - (R_i, S_i) = -(P_i(g), Q_i(g)) + f(i)$ for all $i$. For instance, if $i = \sigma j 1$ where $j < 0$, $(R_{\sigma j}, S_{\sigma j}) = (R_{\sigma j+1}, S_{\sigma j+1}) + (P_{\sigma j+1}(g), Q_{\sigma j+1}(g)) - f(\sigma j^{-1} 1)$, which gives $(R_{\sigma i}, S_{\sigma i}) - (R_i, S_i) = -(P_i(g), Q_i(g)) + f(i)$.

Now for each $i$ we shall carefully choose bounded clopen subsets $B_i, C_i$ of $A_i$ such that $(B_i, C_i) \sim (R_i, S_i)$, and $h \in K^*$ which maps $A_i \setminus B_i$ onto $A_i \setminus C_i$. Whichever $B_i$ and $C_i$ we take, the fact that $A_i \setminus B_i$ can be taken to $A_i \setminus C_i$ is immediate, since they are even order-isomorphic, and being clopen, we can act on the complement $\bigcup_{i \in \omega} B_i$ as we please. The main point therefore is to arrange that $(B_i, C_i) \sim (R_i, S_i)$. If this has been done, we can see that $(P_i(h), Q_i(h)) \sim -(R_i, S_i)$, and similarly $(P_i(h^{-1}), Q_i(h^{-1})) \sim (R_i, S_i)$. For this note first that since $h \in K^*$, $\pi(h) = \text{id}$. Hence

\[ (P_i(h), Q_i(h)) = (A_i \setminus hA_i, A_i \setminus h^{-1}A_i) \]

$= (A_i \setminus (h(A_i \setminus B_i) \cup hB_i), A_i \setminus (h^{-1}(A_i \setminus C_i) \cup h^{-1}C_i))$

$= (C_i \setminus hB_i, B_i \setminus h^{-1}C_i)$

$\sim (C_i, B_i) = -(R_i, S_i).$
To justify the final step, we have to see that \((C_i \setminus hB_i) \cup B_i\) is homeomorphic to \((B_i \setminus h^{-1}C_i) \cup C_i\). Now, \((C_i \setminus hB_i) \cup B_i\) is homeomorphic to \((h^{-1}C_i \setminus B_i) \cup B_i = B_i \cup h^{-1}C_i = (B_i \setminus h^{-1}C_i) \cup h^{-1}C_i\), which is homeomorphic to \((B_i \setminus h^{-1}C_i) \cup C_i\) as desired.

Fix \(i\) and let \(\sigma i = j\). Then by Lemma 2.4 we find that \((P_i(h^{-1}gh), Q_i(h^{-1}gh)) \sim (P_i(h), Q_i(h)) + (P_j(h^{-1}), Q_j(h^{-1})) \sim -(R_i, S_j) + (P_i(g), Q_i(g)) + (R_{\sigma i}, S_{\sigma i}) \sim f(i)\), as desired.

To choose \(B_i\) and \(C_i\) we note that in the case \(\alpha = 1\), any non-empty bounded clopen sets will serve, and this is because they will be finite, and so \(\bigcup_{i \in [1, \omega]} B_i\) and \(\bigcup_{i \in [1, \omega]} C_i\) are (countably) infinite sets with the discrete topology, so any bijection from the first to the second works. In the general case we have to take the topology into account. For this, let \(B_i' = B_i \cup B_i \cup \cdots\) \(\bigcup B_i = C_i \cup C_i \cup \cdots\) \(\bigcup C_i\) and \(C_i' = B_i' \cup C_i' \cup \cdots\) \(\bigcup C_i\), where \(B_i, B_i',\) and \(C_i, C_i'\) are still homeomorphic to bounded clopen subsets of \(\omega\). Now, \(\omega\) is homeomorphic to \(X\), so we can now map the disjoint union of all the \(B_i'\)'s to that of the \(C_i\)'s by taking \(B_i'\) to \(B_i\) and \(C_{i+1}\) to \(C_i\) where each individual map is a homeomorphism, and one verifies that it takes \(\bigcup_{i \in [1, \omega]} B_i\) 1–1 onto \(\bigcup_{i \in [1, \omega]} C_i\), so is suitable a choice for our extension of \(h\).

\[\Box\]

**Lemma 2.6** If there is no proper normal subgroup of \(G_{\omega^\alpha}\) of index < \(2^{\aleph_0}\), then the same is true for \(G = G_{\omega^\alpha+1}\).

**Proof** Suppose that \(N < G\) and \(|G : N| < 2^{\aleph_0}\), and we shall show that \(N = G\). Let \(g \in G\) be arbitrary. If \(\sigma\) permutes \(\omega\) in a single cycle, then by Lemma 2.2 there is \(h \in N\) such that \(\pi(h) = (\pi(g))^{-1}\sigma\). Thus \(\pi(h) = \pi(g)\pi(h) = \sigma\) is a single infinite cycle, and applying Lemma 2.2 again, there is \(k \in N\) such that \(\pi(k) = \pi(gh)\). By Lemma 2.5 we may replace \(k\) by a conjugate and suppose that \((P_i(k), Q_i(k)) \sim (P_i(gh), Q_i(gh))\) for each \(i\) (noting that the conjugating element has trivial \(\pi\) value). By Lemma 2.4,

\[
(P_i(k^{-1}gh), Q_i(k^{-1}gh)) \sim (P_i(gh), Q_i(gh)) + (P_{\sigma i}(k^{-1}), Q_{\sigma i}(k^{-1})) \\
\sim (P_i(k), Q_i(k)) + (P_{\sigma i}(k^{-1}), Q_{\sigma i}(k^{-1})) \\
\sim (P_i(k^{-1} \cdot k), Q_i(k^{-1} \cdot k)) \sim (\emptyset, \emptyset).
\]

Thus \(k^{-1}gh\) is a member of \(K^*\) such that \((P_i(k^{-1}gh), Q_i(k^{-1}gh)) \sim (\emptyset, \emptyset)\) for all \(i\). Let \(B_i = \{a \in A_i : k^{-1}gha \notin A_i\}\) and \(C_i = \{a \in A_i : (k^{-1}gh)^{-1}a \notin A_i\}\). Thus \(B_i\) and \(C_i\) are homeomorphic bounded clopen subsets of \(A_i\). Furthermore, \(A_i \setminus C_i = \{a \in A_i : (k^{-1}gh)^{-1}a \in A_i\} = (k^{-1}gh)A_i \cap A_i = (k^{-1}gh)(A_i \setminus B_i)\). Let \(l \in G\) agree with \(k^{-1}gh\) on \(A_i \setminus B_i\) and map \(B_i\) to \(C_i\). Thus \(l^{-1}k^{-1}gh\) fixes \(\bigcup_{i \in \omega}(A_i \setminus B_i)\) pointwise, and permutes points of \(\bigcup_{i \in \omega} B_i\). Also by Lemma 2.1, \(l \in K \leq N\).

Since each \(B_i\) is a bounded subset of \(A_i\), \(\bigcup_{i \in \omega} B_i\) has order-type \(\leq \omega^\alpha\), so there are bounded clopen subsets \(D_i\) of \(A_i\) such that \(B_i \subseteq D_i\) and \(\bigcup_{i \in \omega} D_i \cong \omega^\alpha\). Now \(Y = \bigcup_{i \in \omega} D_i\) is open, and has open complement, so is clopen. Let \(L\) be the subgroup of \(G\) comprising its elements whose support is contained in \(Y\). Since \(Y\) is clopen, any homeomorphism of \(Y\) to itself extends to a homeomorphism of \(X\) by fixing \(X \setminus Y\).
pointwise, and therefore $L$ is a subgroup of $G$ which is isomorphic to $G_{\omega^\omega}$. Since $N \cap L$ is a normal subgroup of $L$ of index $< 2^{\aleph_0}$, by assumption it follows that $L \leq N$. Since $l^{-1}k^{-1}gh$ fixes $\bigcup_{i \in \omega} (A_i \setminus B_i)$ pointwise it also fixes $\bigcup_{i \in \omega} (A_i \setminus D_i)$ pointwise, so lies in $L$, and hence also in $N$. Hence $g = kl(l^{-1}k^{-1}gh)h^{-1} \in N$ as required. \qed

**Corollary 2.7** For finite $\alpha$, $G_{\omega^\omega}$ has no proper normal subgroup of index $< 2^{\aleph_0}$.

**Proof** This follows by induction from the lemma. In the basis case $\alpha = 1$, $G_{\omega^\omega} \cong \text{Sym}(\omega)$, whose non-trivial normal subgroups are explicitly known, and are the alternating group, and the group of elements of finite support, both of which have index $2^{\aleph_0}$. \qed

### 3 The main result

We now use the results of Sect. 2 to complete the main proof.

**Theorem 3.1** The boolean algebra $\mathbb{B}$ of clopen subsets of $[1, \omega^\omega]$ for finite $\alpha$ has the small index property. This is, any subgroup $H$ of its automorphism group $G$ of index $< 2^{\aleph_0}$ contains the pointwise stabilizer $G_A$ of some finite $A \subseteq \mathbb{B}$.

**Proof** We use induction. The basis case $\alpha = 1$ follows from the small index property for $\text{Sym}(\omega)$ [3]. So now assume the result for $\alpha$, and we prove it for $\alpha + 1$. So $G = G_{\omega^{\alpha+1}}$, and as usual we let $A_n = [\omega^\omega(n - 1) + 1, \omega^\omega \cdot n]$. Let $K$ be the subgroup of $G$ fixing $\{A_n : n \in \omega\}$ setwise (not the same $K$ as before). We follow the proof of Lemma 2.1. As there, let $\pi : G \to \text{Sym}(1, \omega)$ where $\pi(g)(m) = n$ if $g(\omega^\omega \cdot m) = \omega^\omega \cdot n$. Then $K^*$ given above is the kernel of $\pi$.

Since $|G : H| < 2^{\aleph_0}$, we deduce that $|K : K \cap H| < 2^{\aleph_0}$, and also $|\pi K : \pi(K \cap H)| < 2^{\aleph_0}$. Now $\pi K$ is naturally isomorphic to $\text{Sym}(1, \omega)$, which by [3] has the small index property. Hence there is a finite $I \subseteq [1, \omega)$ such that any member $\sigma$ of $\text{Sym}(1, \omega)$ fixing $I$ pointwise lies in $\pi(K \cap H)$, so has the form $\pi(g)$ for some $g \in K \cap H$.

Let $G_1$ and $G_2$ be the restrictions of $G$ to $\bigcup_{i \in I} A_i$ and $\bigcup_{i \notin I} A_i$ respectively. That is, $G_1, G_2$ consist of all members of $G$ whose support is contained in $\bigcup_{i \in I} A_i, \bigcup_{i \notin I} A_i$ respectively. For each $i$, let $K_i = \{g \in G : \text{supp}(g) \subseteq A_i\}$. Then $K_i \cong G_{\omega^\omega}$ and $|K_i : K_i \cap H| < 2^{\aleph_0}$, so by the induction hypothesis, there is a finite family $B_i$ of clopen subsets of $A_i$ such that $(K_i)B_i \leq H$. Hence $H \geq \prod_{i \in I} (K_i)B_i = (G_1)\bigcup_{i \in I} B_i \cup \{A_i : i \notin I\}$. We let $Y = \bigcup_{i \in I} B_i \cup \{A_i : i \in I\}$ and show that $H \geq G_Y$. Since $G_Y = (G_1)_Y \times G_2$, it remains to see that $G_2 \leq H$. This follows by standard arguments as in [3] which we briefly sketch.

First write $(1, \omega) \setminus I$ as the disjoint union of infinite sets $M_i$ for $i \in [1, \omega)$. By considering the projections to $L_i = \text{the restriction of } G \text{ to } \bigcup_{j \in M_i} A_j$ we find as in [3] that $H$ projects onto some $L_i$. It follows that $H \cap L_i$ is a normal subgroup of $L_i$ of index $< 2^{\aleph_0}$. Since $L_i \cong G$, by Corollary 2.7, $L_i \leq H$. By conjugating by elements of $H$ arbitrarily permuting the members of $\omega \setminus I$ we deduce that $H$ contains the restriction of $G$ to $\bigcup_{j \in J} A_j$ for an arbitrary infinite cofinite subset $J$ of $\omega \setminus I$. Since $G_2$ is generated by such restrictions, it follows that $H \geq G_2$. \qed
Corollary 3.2 Any superatomic boolean algebra $\mathcal{B}$ of finite rank has the small index property.

Proof Let $\mathcal{B}$ have rank $\alpha$ and degree $a$. In the first case, if $a = 1$, the result is read off from the theorem. For the second part, let $\mathcal{B}$ be isomorphic to the family of clopen subsets of $[1, \omega^\alpha \cdot a]$, and for $1 \leq i \leq a$ let $A_i = [\omega^\alpha (i - 1) + 1, \omega^\alpha \cdot i]$. Note that each $A_i$ is clopen, so lies in $\mathcal{B}$. Furthermore, $G_{\{A_1, \ldots, A_a\}}$ is the direct product of its restrictions $G_i$ to the individual $A_i$s, each of which is isomorphic to $G_{\omega^\alpha}$. Now $|G_i : G_i \cap H| < 2^{\aleph_0}$, so by the theorem, $G_i \cap H \geq (G_i)_{B_i}$ for some finite set $B_i$ of clopen subsets of $A_i$. Hence $H \geq G_{B_1 \cup \ldots \cup B_a \cup \{A_i : 1 \leq i \leq a\}}$, concluding the proof.  

4 Concluding remarks

The main problem left open in what we have done is to prove the small index property for the homeomorphism group of $[1, \omega^\alpha]$ for an arbitrary countable ordinal $\alpha$, which by the argument of Corollary 3.2 would give SIP for all countable superatomic boolean algebras. As we have seen it would suffice to be able to handle the limit step. The methods given here do not seem to apply to this situation.

We remark that there is a related piece of work in the final chapter of [4], in which Hilton actually classified all the normal subgroups of $G_{\omega^\alpha \cdot a}$ which are contained in $K^*$, again for finite $\alpha$, of which there are $2^{2^{\aleph_0}}$, provided that $\alpha \geq 2$.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There is no conflict of interest.

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