Dynamics of modular matings
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Abstract

We develop a dynamical theory for the family of holomorphic correspondences $F_a$ proved by the current authors to be matings between the modular group and parabolic rational maps in the Milnor slice $\text{Per}_1(1)$ (BLT). Such a mating endows the complement of the limit set of $F_a$ with the geometry of the hyperbolic plane, equipped with the action of the modular group. We introduce bi-infinite coding sequences for geodesics in this complement, utilising continued fraction expressions of endpoints; we prove landing theorems for periodic and preperiodic geodesics, and we establish a stronger Yoccoz inequality for repelling fixed points of these correspondences than Yoccoz’s classical inequality for quadratic polynomials. We deduce that the connectedness locus of the family $F_a$ is contained in a particular lune in parameter space.

1 Introduction

An $(m : n)$ holomorphic correspondence on the Riemann sphere $\hat{\mathbb{C}}$ is an $n$-valued function $F : z \to w$ (with $m$-valued inverse $F^{-1} : w \to z$) defined implicitly by a polynomial equation $P(z, w) = 0$, where $P$ has degree $m$ in $z$ and $n$ in $w$. In this paper we study the dynamics under iteration of members of a particular one complex parameter family of $(2 : 2)$ holomorphic correspondences, namely the family $F_a : z \to w$, given by

$$\left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3,$$

introduced by the first author, together with Christopher Penrose, in [BP]. For this family, there exists an open set $K \subset \mathbb{C}$, called the Klein Combination Locus, such that for all $a \in K$ the Riemann sphere admits a partition into two subsets completely invariant under $F_a$, denoted $\Lambda_a$ and $\Omega_a$. The limit set $\Lambda_a$ is closed and is the union of two closed subsets, the forwards limit set $\Lambda_a^-$, restricted to which $F_a$ is $(2 : 1)$, and the backwards limit set $\Lambda_a^+$, on which $F_a$ is $(1 : 2)$; the intersection $\Lambda_a^- \cap \Lambda_a^+$ consists of a parabolic fixed point of $F_a$ of multiplier 1 (see Section 2). The complement $\Omega_a = \mathbb{C} \setminus \Lambda_a$ of the limit set is called the regular set. The connectedness locus $C_\Gamma$ for the family $F_a$ is the set of $a \in \mathbb{C}$
such that the limit set $\Lambda_a$ is connected (see Section 2). In [BL1], the current authors prove that when $a \in \Gamma$, the correspondence $F_a$ behaves like a parabolic quadratic map of the form

$$P_A(z) = z + 1/z + A$$

on a doubly pinched neighbourhood of $\Lambda_a$, using parabolic-like maps (see [L]), and like the modular group on its complement. To make the statement precise, let us recall that, for every $A \in \mathbb{C}$, the map $P_A$ has a parabolic fixed point at $z = \infty$ with multiplier 1, and basin of attraction $\mathcal{A}_A(\infty)$; the filled Julia set $K_A$ of $P_A$ is defined to be $K_A = \mathbb{C} \setminus \mathcal{A}_A(\infty)$. The Main Theorem in [BL1] states that for every $a \in \Gamma$, the correspondence $F_a$ is hybrid conjugate on a doubly pinched neighbourhood of $\Lambda_a$ to a rational map of the form $P_A$ acting on a doubly pinched neighbourhood of its filled Julia set $K_A$, and on its regular set $\Omega_a$ the correspondence $F_a$ is conformally conjugate to the pair of generators

$$\alpha : z \to z + 1 \quad \text{and} \quad \beta : z \to \frac{z}{z + 1}$$

of the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$ acting on the complex upper half-plane $\mathbb{H}$. In other words, for every $a \in \Gamma$, the correspondence $F_a$ is a mating between a rational map of the form $P_A : z \to z + 1/z + A$ and $\Gamma$. In view of this theorem, we call the correspondences in the family $F_a$, $a \in M_{\Gamma}$ modular matings. We define the modular Mandelbrot set $M_{\Gamma}$ to be $M_{\Gamma} := \Gamma \cap \overline{D}(4, 3)$. See Figure 3.

In the current paper we develop a complete dynamical theory for these modular matings. For a general holomorphic correspondence on the Riemann sphere, analysing dynamics under iteration is a very challenging problem, even in the $(2 : 2)$ case. It is akin to the problem of analysing the dynamics of the group generated by a general pair of fractional linear transformations, and it is well known that for such a group it is a non-trivial problem even to determine whether it is discrete. Isolated examples of correspondences (for example the arithmetic-geometric mean) are well understood. The family of correspondences $F_a$, $a \in M_{\Gamma}$, is the simplest 1-parameter family to be fully explored, and as such it should provide a paradigm for the study of other families of correspondences displaying “discreteness” in their action on the sphere for some subset of parameter space.

For $a \in \Gamma$, the Main Theorem in [BL1] guarantees the existence of a canonical conformal homeomorphism

$$\varphi_a : \mathbb{C} \setminus \Lambda(F_a) \to \mathbb{H},$$

conjugating the correspondence restricted to $\Omega(F_a)$ to the pair of generators $\alpha, \beta$ of the modular group defined above (see Figure 1 and Section 3). We call this map the Böttcher map for the correspondence $F_a$, $a \in \Gamma$, because it plays a role in our theory analogous to that of the classical Böttcher map

$$\varphi_c : \mathbb{C} \setminus K(Q_c) \to \mathbb{C} \setminus D$$
Figure 1: Above: tessellation of $\mathbb{H}$ by copies of a fundamental domain for the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$. Below: mating between a ‘Douady rabbit’ in $\text{Per}_1(1)$ and the modular group, realised by the correspondence $F_a$ with $a = 4.53926 + 0.439437i$. There are two copies of the rabbit, $\Lambda_{a,-}$ on the left and $\Lambda_{a,+}$ on the right. They intersect in a parabolic fixed point $P$ for the correspondence (in the centre of the computer plot).

which conjugates a quadratic polynomial $Q_c : z \to z^2 + c$ on the complement of its filled Julia set $K(Q_c)$ to the map $z \to z^2$ on the complement of the closed unit disc, whenever $c$ lies in the Mandelbrot set $\mathcal{M}$, the set of $c \in \mathbb{C}$ such that $K_c$ is connected. Let us recall that external rays are the lines $R_\theta = \{z | \arg(\varphi_c(z)) = \theta\}$, preimages under the Böttcher map $\varphi_c$ of lines of constant argument in $\mathbb{C}$, and are important tools to understand the dynamics of a polynomial. Douady and Hubbard proved that periodic rays land (that is, there exists a limit, necessarily in $\partial K(Q_c)$, for $\varphi_c^{-1}(re^{2\pi i \theta})$ as $r \to 1$), and that repelling periodic points are landing points of at least one and at most finitely many external rays. By considering properties of the linearisation of $Q_c$ in the neighbourhood of a repelling periodic cycle, Yoccoz proved, for $Q_c$ with $c \in \mathcal{M}$, an inequality constraining the (complex) logarithm of the multiplier of a repelling periodic cycle to lie in a certain disc in $\mathbb{C}$ (see page 2) an inequality which he applied to estimate the size of the limbs of the Mandelbrot set.

There are strong parallels between the dynamical theories for the family $F_a$ and the family $Q_c$, but also essential differences. The core similarity is that both theories are constructed using isometries of the hyperbolic metric on $\mathbb{H}$: the map $z \to z^2$ on the punctured disc $\mathbb{C} \setminus \mathbb{H}$ lifts to the isometry $z \to 2z$ on its universal cover $\mathbb{H}$, and external rays in $\mathbb{C} \setminus K(Q_c)$ are the preimages, under the lift of the Böttcher map, of geodesics in $\mathbb{H}$ from points of $\mathbb{R} \subset \partial \mathbb{H}$ to $\infty$. The core difference between the theories comes from fact that $z \to 2z$ is a hyperbolic
isometry (it has two fixed points, 0 and ∞ on ∂H), whereas \( z \rightarrow \alpha(z) \) and \( z \rightarrow \beta(z) \) are two parabolic isometries (with one fixed point each, ∞ and 0, respectively). The modular group endows \( \mathbb{H} \) with a richer structure than that which the angle-doubling map gives to \( \hat{\mathbb{C}} \setminus \mathbb{D} \), but conversely there is no single point in \( \mathbb{H} \) which has properties for \( \Gamma \) analogous to those enjoyed by \( \infty \in \hat{\mathbb{C}} \setminus \mathbb{D} \) for the angle-doubling map. For the family \( \mathcal{F}_a \) the role of external rays in the Douady-Hubbard theory for \( \mathbb{Q}_c \) is replaced by that of geodesics in \( \hat{\mathbb{C}} \setminus \Lambda_a \) with initial point in \( \Lambda_a, - \) and final point in \( \Lambda_a, + \). The Douady-Hubbard theory makes frequent use of a Green’s function \( G_c : \mathbb{C} \setminus K(Q_c) \to \mathbb{R}^+ \) which interacts with the dynamics via \( G_c(Q_c(z)) = 2G_c(z) \). The Green’s function \( G_a \) for \( \Lambda_a \) has no such simple relationship with the two branches of \( \mathcal{F}_a \), but for points close to \( \Lambda_a, - \) we prove that there exist lower and upper bounds for the multiplying factor \( G_a(\mathcal{F}_a(z))/G_a(z) \) (Lemma 2, Section 3.2). This weaker property suffices for our purposes.

In Proposition 1 (Section 3) we prove that the inverse \( \psi_a \) of the Böttcher map extends continuously to the end points of the imaginary axis \( I \) in \( \mathbb{H} \) (the geodesic running from 0 to ∞) sending both 0 and ∞ to the parabolic fixed point \( \Lambda_a, - \cap \Lambda_a, + \) (see Fig. 1), in other words that this geodesic lands, at both ends. In Proposition 2 (Section 4) we prove that periodic geodesics land. Here, by a periodic geodesic we mean one which is mapped to itself by some finite sequence of branches of \( \mathcal{F}_a \): under the Böttcher map \( \varphi_a \) it corresponds to a geodesic in \( \mathbb{H} \) fixed by some finite product of positive powers of \( \alpha \) and \( \beta \). (Note that while \( I \) is not itself a periodic geodesic, its end-points are fixed by \( \alpha \) and \( \beta \) respectively.) We then explain how one may code a geodesic in \( \mathbb{H} \) from a point in \( \mathbb{R}^- \) to a point in \( \mathbb{R}^+ \) by a bi-infinite sequence \( S \in \{\alpha, \beta\}^\mathbb{Z} \) combining the continued fraction expansions of these two points. The 2-sided shift operator on bi-infinite sequences imposes a dynamic on the space of geodesics from \( \mathbb{R}^- \) to \( \mathbb{R}^+ \) in which the periodic geodesics play an organising role. By adapting a technique due to Benini and Lyubich [BeLy], in Section 5 we prove the deeper converse to Proposition 2 that repelling periodic points are landing points of periodic geodesics. More precisely:

**Theorem 1.** For every \( a \in \mathbb{C}_\Gamma \),

(i) every repelling fixed point of the 2-to-1 restriction \( f_a \) of \( \mathcal{F}_a \) to a (doubly pinched) neighbourhood of \( \Lambda_{-}(\mathcal{F}_a) \) is the landing point of exactly one cycle of periodic geodesics;

(ii) the points of every repelling cycle of period \( m \geq 1 \) of \( f_a \) are landing points of at least one and at most two cycles of periodic geodesics.

As a consequence of Theorem 1(i), we may define the combinatorial rotation number of a repelling fixed point \( \hat{z} \) of \( f_a \) (where \( a \in \mathbb{C}_\Gamma \)) to be the rotation number of the unique periodic cycle of geodesics landing at \( \hat{z} \). In Section 6 we prove an inequality of Pommerenke-Levin-Yoccoz type for our correspondences at such a fixed point:
Figure 2: Discs in the log(ζ)-plane permitted by the Yoccoz inequality: on the left for matings between the modular group and quadratic polynomials, and on the right for the classical case of quadratic polynomials. In both diagrams the discs plotted correspond to all 0 < p/q ≤ 1/2 with q ≤ 8.

**Theorem 2.** If \( a \in \mathcal{C}_\Gamma \) and \( p_a \) is a repelling fixed point of \( f_a \) which has multiplier \( \zeta \) and combinatorial rotation number \( p/q \in \mathbb{Q}/\mathbb{Z} \), where \( 0 < p/q < 1 \), then \( \log \zeta \) lies in the disc in the right hand half of \( \mathbb{H} \) which has boundary circle tangent to the imaginary axis at \( 2\pi i \nu \) and radius \( r_{p/q} \), where

\[
r_{p/q} = \frac{2p \log([q/p] + 1)}{q^2} \quad \text{if} \quad 0 < p/q \leq 1/2,
\]

and

\[
r_{p/q} = \frac{2(q-p) \log([q/(q-p)] + 1)}{q^2} \quad \text{if} \quad 1/2 \leq p/q < 1.
\]

In the original Yoccoz inequality for quadratic polynomials, the corresponding radius \( r_{p/q} \) is \( 1/q \) \([3]\). Writing \( 2\pi i \nu \) for the imaginary part of \( \log \zeta \), we deduce that whereas in the classical Yoccoz case for \( c \) to be in \( \mathcal{M} \) the value of \( \log \zeta \) (where \( \zeta \) is the derivative of \( Q_c \) at a repelling fixed point) must lie in a strip whose width goes to zero linearly with \( \nu \), in our situation for \( a \) to be in \( \mathcal{M}_\Gamma \) the value of \( \log \zeta \) has to lie in a strip whose width goes to zero at a rate proportional to \( \nu^2 \log(1/\nu) \) (for \( \nu = 1/q \) this is \( (\log q)/q^2 \)): see Figure 2. For repelling orbits of period greater than 1 the bound on the modulus of the derivative is more complicated: see Remark 4 at the end of Section 6 for an example.

In Section 7 we apply Theorem 2 to prove that the modular Mandelbrot set \( \mathcal{M}_\Gamma \) is contained in a subset of \( \mathbb{C} \) of a particular form. By the *lune* \( \mathcal{L}_\theta \) in the parameter plane we shall mean the closed subset of the \( a \)-plane bounded by the two arcs of circles which pass through the points \( a = 1 \) and \( a = 7 \) and meet the real axis at angles \( \pm \theta \) at these points (so the lune \( \mathcal{L}_{\pi/2} \) is the disc \( \overline{B}(4,3) \)). The plot on the right in Figure 3 suggests the possibility that \( \mathcal{M}_\Gamma \subset \mathcal{L}_{\pi/3} \). We do not know if this is the case, but we prove:

**Theorem 3.** There exists an angle \( \theta \) in the half-open interval \( \pi/3 \leq \theta < \pi/2 \) such that \( \mathcal{M}_\Gamma \subset \mathcal{L}_\theta \) and \( \mathcal{M}_\Gamma \) only meets \( \partial \mathcal{L}_\theta \) at the vertex \( a = 7 \).
Figure 3: The modular Mandelbrot set $M_\Gamma$. On the left the white region is the round disc which has centre $a = 4$ and radius 3; on the right it is the lune $L_\pi/3$, of vertex angle $2\pi/3$. We prove that $M_\Gamma \subset L_\theta$ for some $\theta$ with $\pi/3 \leq \theta < \pi/2$ (Theorem 3).

Writing $L_{p/q}$ for the $p/q$-limb of $M_\Gamma$, the set of parameter values where the fixed point of $F_a$ is repelling or neutral of combinatorial rotation number $p/q$, and writing $a_{p/q}$ for the root point of $L_{p/q}$, the value of $a$ where the derivative of $F_a$ at its fixed point is $e^{2\pi ip/q}$, the computations in the proof of Theorem 3 specialise to:

**Corollary 1.** As $p/q > 0$ converges to zero, $a_{p/q}$ converges to $a = 7$ tangentially to a straight line at angle $2\pi/3$ to the positive real axis and $\text{diam}(L_{p/q})$ converges to zero.

Indeed the picture in the $a$-plane close to $a = 7$ is the image of the lefthand plot in Figure 2 under the transformation $\log \zeta \rightarrow 7 + 6(i \log \zeta)^{2/3}$.

For the convenience of the reader, Table 1 exhibits parallels between the Douady-Hubbard theory for quadratic polynomials $Q_c$ and the theory developed below for the quadratic correspondences $F_a$.

In a sequel [BL3] to the present paper we shall show that when $a$ lies in the parameter space lune $L_\theta$ established by Theorem 3 we may choose a dynamical space lune $L_a$, depending analytically on $a$, on which to perform a surgery to transform the correspondence $F_a$ into a rational map of the form $z \rightarrow z + 1/z + A$, that is to say a quadratic rational map whose conjugacy class lies in Milnor’s slice $\text{Per}_1(1)$. Then, using the theory of holomorphic motions we shall prove the following conjecture, in which $\mathcal{M}_1$ denotes the connectedness locus of the family $\text{Per}_1(1)$:
Quadratic polynomials $Q_c$ are related to quadratic correspondences $F_a$ through the mapping $\varphi_c: \hat{C} \setminus K(Q_c) \to \hat{C} \setminus B$ which relates these. The Minkowski $\gamma$-function relates these, as discussed in Sections 3 and 4.

Conjecture 1. There exists a homeomorphism $\chi: M_\Gamma \to M_1$ such that for each $a \in M_\Gamma$ the correspondence $F_a$ is a mating between the quadratic rational map $\chi(a) \in \text{Per}_1(1)$ and the modular group $\Gamma$.

In [PR] Carsten Petersen and Pascale Roesch prove that $M_1$ is homeomorphic to the classical Mandelbrot set $M$. Thus proving Conjecture 1 will yield a mathematical justification of the resemblance between $M_\Gamma$ and $M$ first noted experimentally almost three decades ago in [BP].

The anti-holomorphic case. Contemporaneously with our work, but independently of it, Lee, Lyubich, Makarov and Mukherjee [LLMM] investigated matings between quadratic anti-rational maps and a discrete group generated by hyperbolic reflections, as part of their innovative programme of study of the dynamics of Schwarz reflection maps associated to quadrature domains. Their notion of a mating between a map and a group differs from ours, in that theirs is a single-valued function, which on an invariant closed connected subset has the behaviour of an anti-rational map on its filled Julia set, and which on the complement is associated to a discrete group of isometries of $\mathbb{H}$ generated by a reflection and a rotation. Despite the differences, there is a close relationship between the two settings. The dynamics of an (anti-holomorphic) Schwarz reflection maps in the one parameter family considered in [LLMM] strongly parallels that of a $(2:1)$ restriction of a holomorphic correspondence in the one parameter family investigated in [BP], [BL], and the present paper. See Remark 1 at the end of Section 2, where the relationship between the two theories is elucidated.

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2 Preliminaries: limit sets for $\mathcal{F}_a$

In this Section we note some general definitions and properties of $(2 : 2)$ holomorphic correspondences, and recall terminology and results from [BL1], which we shall use throughout the paper.

A $(2 : 2)$ holomorphic correspondence on the Riemann sphere is a 2-valued function $F : z \to w$ (with 2-valued inverse $F^{-1} : w \to z$) defined implicitly by a polynomial equation $P(z, w) = 0$, where $P$ has the form

$$P(z, w) = (az^2 + bz + c)w^2 + (dz^2 + ez + f)w + (gz^2 + hz + j).$$

The graph of $F$, the Riemann surface $P(z, w) = 0$, supports two natural involutions: $I_-$ which interchanges the two values of $z$ which map to the same $w$, and $I_+$ which interchanges the two values of $w$ which are images of the same $z$. We say that $F$ is a map of triples if $(I_+ I_-)^3$ is the identity. Equivalently the immediate images $w_2, w_3$ of any point $z_1$, and the immediate pre-images of $w_2$ and $w_3$ fit together in a closed diagram:

```
\begin{tikzpicture}
  \node (z1) at (0,0) {$z_1$};
  \node (z2) at (1,1) {$z_2$};
  \node (z3) at (2,0) {$z_3$};
  \node (w1) at (1,2) {$w_1$};
  \node (w2) at (1,1) {$w_2$};
  \node (w3) at (2,2) {$w_3$};
  \draw (z1) -- (w1);
  \draw (z2) -- (w2);
  \draw (z3) -- (w3);
\end{tikzpicture}
```

which at isolated points may take one of the degenerate forms below:

```
\begin{tikzpicture}
  \node (z1) at (0,0) {$z_1$};
  \node (z2) at (1,1) {$z_2$};
  \node (w1) at (1,2) {$w_1$};
  \node (w2) at (2,1) {$w_2$};
  \draw (z1) -- (w1);
  \draw (z2) -- (w2);
\end{tikzpicture}
```

In the diagram on the left, $(z_2, w_1)$ is a fixed point of $I_-$ and $(z_1, w_2)$ is a fixed point of $I_+$. We remark that $dw/dz = 0$ for the forwards branch $z_2 \to w_1$, and $dz/dw = 0$ for the backwards branch $w_2 \to z_1$; thus the four points in the diagram are all forwards or backwards critical points or values of the correspondence. In the diagram on the right, $(z_1, w_1)$ is fixed by both $I_-$ and $I_+$; the point $(z_1, w_1)$ is a double point of the Riemann surface $P(z, w) = 0$.

It follows from the definition, or equivalently from the diagrams above, that every map of triples necessarily factorises into the deleted covering correspondence $\text{Cov}_C^0$ of a cubic $C : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ post-composed by a Möbius transformation $M$. Here $C$ is the map which identifies each triple of points $\{z_1, z_2, z_3\}$ of the diagram condition to a single point, and $M$ is the map which sends each $z_j$ in the diagram to the corresponding $w_j$. Up to pre- and post-multiplication
by Möbius transformations, $C$ can take one of three forms, depending on the numbers and types of critical points of the correspondence:

(i) $C(z) = z^3$;
(ii) $C(z) = z^3 - 3z$;
(iii) $C(z) = z^2(z + a)/(z + 1)$ ($a \neq 0, 1, 9$).

These forms correspond to the three possible topological types for the graph of a $(2 : 2)$ holomorphic correspondence, namely two (intersecting) spheres, a single (self-intersecting) sphere, or (the generic case) an elliptic curve. Our interest is in the second case, and the particular sub-case that the Möbius transformation $M$ is an involution (which we denote $J$). This restriction reduces us to the two (complex) parameter family of reversible maps of triples which have graph a single sphere. Our final restriction, reducing us to one parameter, is to require that one of the fixed points of $J$ be also a critical point of $C$. Hence, we can write $F_a = J_a \circ C_0$, where $C_0$ denotes the deleted covering correspondence of $C(z) = z^3 - 3z$, that is to say the 2-to-2 correspondence $Z \rightarrow W$ on $\hat{C}$ defined by

$$\frac{C(Z) - C(W)}{Z - W} = 0,$$

that is, $Z^2 + ZW + W^2 = 3$,

and $J_a$ denotes the conformal involution on $\hat{C}$ which has fixed points $Z = 1$ and $Z = a$, namely

$$J_a(Z) = \frac{(a + 1)Z - 2a}{2Z - (a + 1)}.$$

Thus the correspondence $F_a : Z \rightarrow W$ is given by

$$Z^2 + ZJ(W) + (J(W))^2 = 3,$$

which under the change of coordinates

$$Z = (az + 1)/(z + 1), \text{ and } W = (aw + 1)/(w + 1)$$
is the relation
\[
\left(\frac{az + 1}{z + 1}\right)^2 + \left(\frac{az + 1}{z + 1}\right)\left(\frac{aw - 1}{w - 1}\right) + \left(\frac{aw - 1}{w - 1}\right)^2 = 3.
\]

By construction, the involution $J_a$ conjugates $F_a$ to its inverse $F_a^{-1}$. In the $z$-coordinate $J$ is the involution $z \to -z$ (whatever the value of $a$). The forwards critical points of $F_a$ (the points where $dW/dZ$ vanishes for a branch $Z \to W$ of $F_a$, see [BP] Section 2) are $Z = -1$ and $+1$. The corresponding forwards critical values are $W = J(+2)$ and $J(-2)$ respectively. The point $Z = \infty$ is a double point of $F_a$ (see [BP] Section 2).

As $F_a = J_a \circ Cov_0^C$, the two images of $Z \in \mathbb{C}$ under $F_a$ are the images under $J_a$ of the two preimages of $C(Z) = Z^3 - 3Z$ in $\mathbb{C}$ different from $Z$. Using this decomposition, we will now construct fundamental domains for the action of $F_a$. As $Z = 1$ is a critical point of the cubic $C$, and $C(-2) = C(1)$, we deduce that $Cov_0^C(-2) = 1$ under both branches, and $Cov_0^C((-\infty, -2])$ is a smooth curve running through $Z = 1$, symmetric about the real axis, and asymptotic to the directions $\pm \pi/3$ as $Z$ tends to $\infty$. On the other hand, $J_a$ fixes the circle through $Z = 1$ and $Z = a$ having centre on the real axis, and sends points inside the disc bounded by this circle to the exterior of the disc, and vice versa. The standard fundamental domain $\Delta_{Cov}^s$ is the subset of the $Z$-plane to the right of the curve $Cov_0^C((-\infty, -2])$. The standard fundamental domain $\Delta_J^s$ is the subset of the $Z$-plane exterior to the circle through $Z = 1$ and $Z = a$ which has centre on the real axis (see Figure 4 on the left). Define $D := \{a : |a - 4| \leq 3\}$, and note that for $a = 1$ the correspondence is not defined. When $a \in D \setminus \{1\}$, we have
\[
\Delta_{Cov}^s \cup \Delta_J^s = \hat{\mathbb{C}} \setminus \{1\}.
\]

Moreover, when $|a - 4| < 3$ these satisfy the condition that their boundaries are transverse to the parabolic axis at $Z = 1$ (see Proposition 3.5 in [BL1] for a proof). When $a = 7$ there are three attracting-repelling directions at $Z = 1$ and the domain boundaries are transverse to all three.

More generally, the Klein Combination Locus, $K$, is the set of values of the parameter $a \in \hat{\mathbb{C}}$ such that there exist fundamental domains $\Delta_{Cov}$ for $Cov_0^C$ and $\Delta_J$ for $J$ which together cover all of $\hat{\mathbb{C}}$ except for the single point $Z = 1$ ($z = 0$) (in the present article, as in [BL1], fundamental domains are open, so do not include their boundaries). A pair of fundamental domains $(\Delta_{Cov}, \Delta_J)$ satisfying this condition is called a Klein combination pair. Since the standard fundamental domains are a Klein combination pair, we have that $D \setminus \{1\} \subset K$. When $a \in K$ we can always choose $\Delta_{Cov}$ and $\Delta_J$ such that the Jordan curves $\partial \Delta_{Cov}$ and $\partial \Delta_J$ are smooth at the parabolic fixed point $Z = 1$ ($z = 0$) of $F_a$, and transverse to the attracting-repelling axis there (see Proposition 3.8 in [BL1]).
Figure 5: Boundaries of the standard domains, plotted in the $z$-coordinate for the correspondence $F_a$ with $a = 4.53926 + 0.439437i$, together with their images under $F_a^{-1}$ and $F_a^{-2}$. The domain $\Delta_{Cov}$ is the part of $\hat{C}$ outside the outer closed blue curve, $\Delta_J$ is the part of $\hat{C}$ to the left of the slanting straight red line. The set shaded grey is the backwards limit set $\Lambda_{a,-} = \bigcap_{n=0}^{\infty} F_a^{-n}(\hat{C} \setminus \Delta_{Cov})$. By construction, when $a \in K$ and $(\Delta_{Cov}, \Delta_J)$ is a Klein combination pair, $F_a$ restricted to the domain $\Delta_{Cov}$ is a 1-to-2 correspondence (see Figure 4 on the right, and Proposition 3.4 in [BL1] for a proof):

$$F_a : \Delta_{Cov} \rightarrow J(\hat{C} \setminus \Delta_{Cov}) \subset \Delta_{Cov}.$$ 

Provided that $\Delta_{Cov}$ and $\Delta_J$ have smooth boundaries at the parabolic point and that these are transverse to the attracting-repelling axis there,

$$\Lambda_{a,+} := \bigcap_{n=0}^{\infty} F_a^n(\Delta_{Cov})$$

is independent of the choice of the Klein combination pair $(\Delta_{Cov}, \Delta_J)$. We call $\Lambda_{a,+}$ the forwards limit set of $F_a$. Moreover, $\hat{C} \setminus \Delta_{Cov} \subset J(\Delta_{Cov})$ and the restriction of $F_a$ is a 2-to-1 map from the first of these sets onto the second. We define the backwards limit set of $F_a$ (see Figure 5) to be

$$\Lambda_{a,-} := \bigcap_{n=0}^{\infty} F_a^{-n}(\hat{C} \setminus \Delta_{Cov}).$$

Also by construction, $\Lambda_{a,-} = J(\Lambda_{a,+})$ and $\Lambda_{a,-} \cap \Lambda_{a,+}$ consists of a single point, the parabolic fixed point ($Z = 1$) of $F_a$. We denote $\Lambda_{a,-} \cup \Lambda_{a,+}$ by $\Lambda_a$. Finally, $F_a$ acts properly discontinuously on $\Omega_a = \hat{C} \setminus \Lambda_a$, with fundamental domain $\Delta_{corr} := \Delta_{Cov} \cap \Delta_J$.

Further restricting the codomain of the 2-to-1 branch of $F_a$ defined above to $\overline{\Delta_J}$, yields a 2-to-1 map which we denote $f_a$. This inherits the property that $f_a^{-1}(\overline{\Delta_J}) \subset \overline{\Delta_J}$. Modulo boundary curves, $\overline{\Delta_J} \setminus f_a^{-1}(\overline{\Delta_J})$ is the disjoint union of $\Delta_{Cov} \cap \Delta_J$ and its two images under $Cor_r^{C}$: The map $f_a$ has unique critical point $Z = -1$ in $\Delta_J$ (the other forwards critical point $Z = +1$ of $F_a$ being a
fixed point on the boundary ∂Δ_1.

The connectedness locus C_Γ is the set of parameter values a ∈ K such that ∆_a− (or equivalently ∆_a+ or ∆_a) is connected. The modular Mandelbrot set, M_Γ, is C_Γ ∩ D. It is proved in [BL3] that M_Γ = C_Γ.

**Remark 1.** Replacing the involution J_a in the construction above by reflection ι_a in the circle which has centre Z = a and passes through Z = 1, yields a family of (2 : 2) anti-holomorphic correspondences

$$\mathcal{G}_a := \iota_a \circ \text{Cov}_0^C,$$

where, as in the definition of F_a, the function C is the cubic C(Z) = Z^3 − 3Z. Let ∆_a denote the disc in the Z-plane bounded by the circle, and (by analogy with the definition of Δ_J) let Δ_a denote its complement in \(\hat{\mathbb{C}}\). The Klein combination condition becomes the condition that there exists a transversal \(\Delta_{\text{Cov}}\) for C which contains \(\overline{\Delta_a} \setminus \{Z = 1\}\), or, equivalently, that C is injective on \(\overline{\Delta_a}\). The correspondences in the family \(\mathcal{G}_a\) which satisfy this condition are intimately related to the family of (anti-holomorphic) Schwarz reflection maps \(\sigma_a\) investigated in [LLMM]. This relationship may be deduced from the discussion in Section 10 of [LLMM], but to assist readers we shall describe it explicitly here.

Before we get into the details, recall that in [LLMM] the notion of a mating between an anti-rational map and a group is not a correspondence, but a map, which on an invariant simply-connected closed subset of the Riemann sphere is conjugate to an anti-rational map on its filled Julia set, and on the complement of this subset behaves like a certain map associated to a group of automorphisms of \(\mathbb{H}\). Now let Σ denote the quotient sphere \(\hat{\mathbb{C}}/C\), and suppose that a is such that C is injective on the closure \(\overline{\Delta_a}\) of \(\Delta_a\). Let \(g_a : \mathcal{G}_a^{-1}(\Delta_a) \to \Delta_a\) denote the pinched anti-quadratic-like 2-to-1 branch of \(\mathcal{G}_a\) restricted to \(\Delta_a\) as domain and codomain, in analogy to the branch \(f_a\) of \(F_a\) defined a few lines before this Remark, and note that \(g_a \circ \iota_a\) is the corresponding 2-to-1 restriction of \(\mathcal{G}_a^{-1}\) to \(\Delta_a = \iota_a(\Delta_a)\). As C is univalent on \(\Delta_a\), its image \(C(\Delta_a) \subset \Sigma\) is the quadrature domain \(\Omega_a\) defined in [LLMM], Section 3.2. When restricted to \(\Delta_a\), the function C conjugates \(g_a \circ \iota_a\) to a 2-to-1 map from a subset of \(\Omega_a\) onto the whole of \(\Omega_a\). This 2-to-1 map is precisely the Schwarz reflection \(\sigma_a\) ([LLMM], Section 3.2) associated to the quadrature domain \(\Omega_a\), as is apparent from the formula in Proposition 2.3 of [LLMM] for the Schwarz reflection map associated to a quadrature domain in general. We deduce that the restriction of \(C \circ \iota_a\) to \(\Delta_a\) conjugates the pinched anti-quadratic-like map \(g_a\) on \(\Delta_a\) to the pinched anti-quadratic-like map \(\sigma_a\) on \(\Omega_a\). Under this conjugacy the backwards limit set \(\Lambda_a− = \bigcap_{n=0}^{∞} g_a−^n(\Delta_a)\) of \(\mathcal{G}_a\) is carried to the non-escaping set \(K_a = \bigcap_{n=0}^{∞} \sigma_a−^n(\Omega_a)\) of \(\sigma_a\), and the tiling on \(\Delta_a \setminus \Lambda_a−\) is carried to that on \(\Omega_a \setminus K_a\). Thus for any value of a such that the correspondence \(\mathcal{G}_a\) is a mating in the sense of [BP], [BL1] and the present paper, between a quadratic anti-rational map and the group of automorphisms of \(\mathbb{H}\) obtained from the modular group by...
substituting $z \to 1/z$ for $z \to -1/z$, the Schwarz reflection map $\sigma_a$ is a mating between the same map and group in the sense of [LLMM], and vice versa.

3 The Böttcher map

By Theorem A of [BL1], for every $a \in C^\Gamma$ there exists a conformal homeomorphism $\varphi_a : \Omega_a \to \mathbb{H}$ which conjugates the two branches of $F_a|\Omega_a$ to the automorphisms $\alpha : z \to z + 1$ and $\beta : z \to z/(z + 1)$ of $\mathbb{H}$. This $\varphi_a$ is unique, since any automorphism $h$ of $\mathbb{H}$ which conjugates both $\alpha$ to itself and $\beta$ to itself is necessarily the identity (to see this, observe that $h$ must fix both $\infty$ and 0, and that therefore $h$ has the form $h(z) = \lambda z$; the fact that $h^{-1} \alpha h(z) = \alpha(z)$ for all $z \in \mathbb{H}$ now implies that $\lambda = 1$). We shall refer to $\varphi_a$ as the ‘Böttcher map’. It is analogous to the map:

$$\varphi_c : \mathbb{C} \setminus K(Q_c) \to \mathbb{C} \setminus \mathbf{D}$$

defined by Douady and Hubbard [DH], from the complement of the filled Julia set $K(Q_c)$ of a quadratic polynomial $Q_c(z) = z^2 + c$ (with $c \in \mathcal{M}$) to the complement of the closed unit disc, conjugating $Q_c$ to $z \to z^2$.

**Notation.** For $a \in C^\Gamma$, the two branches of the correspondence $F_a$ become (single-valued) homeomorphisms when we restrict $F_a$ to $\Omega = \Omega(F_a)$. We denote these homeomorphisms by $g : \Omega \to \Omega$ and $h : \Omega \to \Omega$, where $g$ corresponds under $\varphi_a$ to $\alpha$, and $h$ corresponds to $\beta$.

We recall that $PSL(2, \mathbb{Z})$ is the free product of the subgroups $C_2$ generated by $\sigma : z \to -1/z$ and $C_3$ generated by $\rho : z \to -1 - 1/z$. Under the Böttcher map $\varphi_a$ the branch $g$ of $F_a|\Omega_a$ corresponds to $\sigma \rho$ and $h$ corresponds to $\sigma \rho^{-1}$. Thus $g^{-1} h g^{-1}$ corresponds to $\sigma$ and $g^{-1} h$ corresponds to $\rho$. We deduce:

**Lemma 1.** For $a \in C^\Gamma$, the Böttcher map $\varphi_a$ is the Riemann mapping of the simply-connected open set $\Omega(F_a)$,

$$\varphi_a : \Omega(F_a) \to \mathbb{H},$$

normalised to send the fixed point of $g^{-1} h g^{-1}$ to $i \in \mathbb{H}$ and the fixed point of $g^{-1} h$ to $(-1 + i \sqrt{3})/2 \in \mathbb{H}$.

Pursuing the same route as that followed by Douady and Hubbard for polynomials, we now consider the question as to how far the inverse of $\varphi_a$ extends to a continuous map from the boundary of $\mathbb{H}$ to the boundary of $\Omega = \Omega(F_a)$.

**Proposition 1.** The inverse $\psi_a : \mathbb{H} \to \Omega$ of the Riemann mapping $\varphi_a$ extends continuously to 0 and $\infty \in \mathbb{R} = \partial \mathbb{H}$, sending both these points to the fixed point $Z = 1$ of $F_a$.

**Proof** Let $I$ denote the imaginary axis in $\mathbb{H}$, the geodesic which runs from $0 \in \partial \mathbb{H}$ to $\infty \in \partial \mathbb{H}$. The homeomorphism $\psi_a : \mathbb{H} \to \Omega$ sends $I$ and its images
under the cyclic group generated by $\alpha : \mathbb{H} \to \mathbb{H}$ to an arc $\psi(I)$ in $\Omega$ and its images under the cyclic group generated by $g : \Omega \to \Omega$. The points $it \in I$, $t \to \infty$, lie on horocycles of $\alpha$ in a family converging to the fixed point $z = \infty$ of $\alpha$, on the boundary of $\mathbb{H}$. So their images $\psi_a(it)$, $t \to \infty$, lie on horocycles of $g$ in a family which converges to the fixed point $Z = 1$ of $g$ (see Step 3 in the proof of Theorem A in [BL1]). Thus the boundary point $Z = 1$ is accessible from within $\Omega$ by the path $\psi_a(I)$, and setting $\psi_a(\infty) = 1$ therefore extends $\psi_a$ continuously to the end point $z = \infty$ of $I$. The proof for the other end of $I$ is similar, with $\beta$ in place of $\alpha$ and $h$ in place of $g$.

Corollary 2. The inverse $\psi_a$ of $\varphi_a$ extends continuously to every $p/q \in \mathbb{Q} \subset \mathbb{R}$.

Proof This follows immediately from Proposition 1, since the orbit of $0 \in \mathbb{R}$ under $\text{PSL}(2, \mathbb{Z})$ is $\mathbb{Q}$.

We remark that the rationals perform the same role here as that of the dyadic rationals (those with finite binary expressions) in the case of quadratic polynomials. In Subsection 4.1 the quadratic irrationals will come into the picture for $\text{PSL}(2, \mathbb{Z})$, playing a role analogous to that played by the non-dyadic rationals for quadratic polynomials.

3.1 Digression: the Douady-Hubbard map and a tessellation of a neighbourhood of $M_\Gamma$

Recall that for quadratic polynomials $Q_c : z \to z^2 + c$, Douady and Hubbard constructed a canonical bijection $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$ by means of the ingenious assignment $\Phi(c) = \varphi_c(c)$. This map $\Phi$ enabled them to investigate $M$ via parameter rays in $\mathbb{C} \setminus M$. It is natural to ask whether there is an analogous construction in the parameter space of our family of correspondences. There is, and we outline it here, but we postpone the proof to a separate article [BL4], as this will be easier with the aid of some of the methods and results of [BL3].

Böttcher’s conjugacy $\varphi_c$ exists between every quadratic polynomial $Q_c$ and the squaring map $z \to z^2$ on a neighbourhood of $\infty$, whether or not $c \in M$. Douady and Hubbard observed that this conjugacy can always be extended to an open set containing the critical value $c$ of $Q_c$, although it can only be extended to the whole of $\mathbb{C} \setminus K(Q_c)$ if $c \in M$. For a correspondence $F_a$, we have no canonical point analogous to $\infty$ from which to begin: instead we start from a Klein combination pair $(\Delta_{\text{corr}}, \Delta_J)$ for $F_a$. For every $a$ in the Klein combination locus $\mathcal{K}$ there exists a fundamental domain $\Delta_{\text{mod}}$ for $\text{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$ such that there is a conformal homeomorphism $\varphi_a$ from $\Delta_{\text{corr}} = \Delta_{\text{corr}} \cap \Delta_J$ to $\Delta_{\text{mod}}$ which (i) sends the vertices of the ‘croissant’ $\Delta_{\text{corr}}$ to $0, \infty, i$ and $(-1 + i\sqrt{3})/2$, and (ii) is equivariant with respect to the side-pairings induced by Cov and $J$ on the boundaries of $\Delta_{\text{corr}}$, and the side-pairings $\rho : z \to -1 - 1/z$ and $\sigma : z \to -1/z$ on the boundaries of $\Delta_{\text{mod}}$. The existence of such a $\Delta_{\text{mod}}$ and (unique) $\varphi_a$ follows from the fact that the quotient orbifolds $\Delta_{\text{corr}}/F_a$ and $\mathbb{H}/\text{PSL}(2, \mathbb{Z})$.
are (uniquely) conformally isomorphic, each being conformally a sphere with 
two cone points and a puncture. But note that the shape of the tile \( \Delta_{mod} \) will 

vary with the choice of Klein combination pair, and with \( a \). We may extend 
the homeomorphism \( \varphi_a \) equivariantly to a simply-connected union of ‘tiles’ in 
\( \Omega(F_a) \), including the one containing the critical value \( v_a \) of \( F_a \), and we can then 
define 

\[
\Phi(a) := \varphi_a(v_a).
\]

This map \( \Phi \) can be shown to be well-defined on any simply-connected subset 
\( U \setminus \mathcal{M}_\Gamma \) of \( K \setminus \mathcal{M}_\Gamma \) and to be a conformal homeomorphism from \( U \setminus \mathcal{M}_\Gamma \) onto 
a neighbourhood in \( \mathbb{H} \) of the real interval \((-\infty, 0)\) in \( \partial\mathbb{H} \) (see [BL4] for details). 

We remark that although a neighbourhood of \( \mathcal{M}_\Gamma \) appears to be tiled in Figure 
6, the tile boundaries here are not those of the pull-back via \( \Phi \) of a tessellation of 
\( \mathbb{H} \) invariant under \( PSL(2, \mathbb{Z}) \): the figure is drawn by plotting parameter points 
where \( F_a^n(v_a) \) lies on \( \partial\Delta_{Cov}^n \) or \( \partial\Delta_J^n \) for some \( n \geq 0 \), and these boundaries lift 
to different curves on \( \mathbb{H} \) for different values of \( a \). However the tile vertices in 
Figure 6 are the pull-backs of the vertices of a \( PSL(2, \mathbb{Z}) \)-tessellation of \( \mathbb{H} \).

### 3.2 A Potential Function

In the Douady-Hubbard theory for quadratic polynomials a central role is played 
by a Green’s function \( G : \mathbb{C} \setminus K(Q_c) \to \mathbb{R}^>0 \) for the filled Julia set, which inter-

acts with the dynamics via the formula \( G(Q_c(z)) = 2G(z) \). We choose a 
potential function for the limit set of \( F_a \) which will play an analogous role in 
our theory, although its interaction with the dynamics is more complicated (see 
Lemma 2 below).

For every real \( k > 0 \) the function \( z \to k \log |z| \), from \( \mathbb{C} \setminus \mathbb{D} \) to \( \mathbb{R}^\geq0 \), is harmonic 
(since \( \log |z| \) is the real part of \( \log z \)) and takes the value 0 precisely on the unit.
circle $S^1$, the boundary of $D$. The equipotentials are the circles $C_R$, with centre the origin and radius $R > 1$. We extend $z \to k \log |z|$ to a continuous function $\overline{\mathbb{C}} \setminus D \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ by sending $\infty \in \overline{\mathbb{C}}$ to $\infty \in \mathbb{R}$.

Let $M : \mathbb{H} \to \overline{\mathbb{C}} \setminus D$ denote the conformal homeomorphism $z \to \zeta = M(z)$, where

$$M(z) = \frac{z + i}{z - i}.$$ 

**Definition 1.** Let $\Psi : \overline{\mathbb{C}} \setminus D \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ be the function $\zeta \to (\log |\zeta|)/2$, let $\chi : \mathbb{H} \cup \{\infty\} \to \mathbb{R}^{> 0} \cup \{\infty\}$ be the composition $\chi = \Psi \circ M$, and for $a \in K$ define

$$G : \overline{\mathbb{C}} \setminus \Lambda \to \mathbb{R}^{> 0} \cup \{\infty\}$$

to be the composition $G = \chi \circ \varphi_a$, where $\varphi_a$ is the Böttcher map defined in Proposition 1.

Note that:

1. $G$ is harmonic on $\overline{\mathbb{C}} \setminus \Lambda$;
2. setting $G(z) = 0 \forall z \in \Lambda$ extends $G$ to a continuous function $\overline{\mathbb{C}} \to \mathbb{R}^{\geq 0} \cup \{\infty\}$.

We remark that while the circles $M^{-1}(C_R)$ in $\mathbb{H}$ are not the level sets of the ‘height’ function $h(z) = y$ (for $z = x + iy \in \mathbb{H}$), they are close to these level sets in the following sense. For points on the imaginary axis, we have

$$\chi(iy) = y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} \ldots$$

so for points $iy$ with $y$ small the value of $\chi$ is close to that of the height function (this is the reason for choosing $k$ to be $1/2$ in the definition of $\Psi$). More generally for points $z = x + iy \in \mathbb{H}$ which have $y$ small and $|x|$ in a bounded interval, $h(z)$ differs from $\chi(z)$ by a bounded factor.

We record the following relationship between $\chi(z)$, $\chi(az)$ and $\chi(\beta z)$ for points $z \in \mathbb{H}$ which are close to the negative half of the real axis:

**Lemma 2.** (i) There exist a constant $\lambda_+ > 1$ and neighbourhood $N \subset \mathbb{H}$ of the interval $[-\infty, 0] \subset \partial \mathbb{H}$ such that $\chi(az) < \lambda_+ \chi(z)$ and $\chi(\beta z) < \lambda_+ \chi(z)$ for all $z \in N$.

(ii) For every real $K > 1$, there exist a constant $1 < \lambda_- < \lambda_+$ and neighbourhoods $N_1 \subset \mathbb{H}$ of the interval $[-K, -1] \subset \partial \mathbb{H}$ and $N_2 \subset \mathbb{H}$ of the interval $[-1, -1/K] \subset \partial \mathbb{H}$, such that $\chi(az) > \lambda_- \chi(z) \forall z \in N_1$, and $\chi(\beta z) > \lambda_- \chi(z) \forall z \in N_2$.

**Proof**

Conjugation by $M : z \to \zeta = (z+i)/(z-i)$ sends the automorphism $\alpha : z \to z+1$ of $\mathbb{H}$ to the automorphism $\zeta \to M\alpha M^{-1}(\zeta)$ of $\overline{\mathbb{C}} \setminus D$. 

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Since \( \chi(z) = (\log|\zeta|)/2 \), where \( \zeta = M(z) \), to prove (i) we must show that there exists a bound \( \lambda_+ \) such that every point \( \zeta \) in the complement of the unit disc which is sufficiently close to \( M[-\infty,0] \) has

\[
\log(|M\alpha M^{-1}(\zeta)|) < \lambda_+ \log(|\zeta|) \quad \text{and} \quad \log(|M\beta M^{-1}(\zeta)|) < \lambda_+ \log(|\zeta|).
\]

However \( M[-\infty,0] \) consists of the lower half of the unit circle, traversed clockwise, and so the points of \( \hat{C} \setminus \mathbb{D} \) close to \( M[-\infty,0] \) are the points \( \zeta = \Re - i\theta \) which have \( R = 1 + r \) with \( r > 0 \) small, and also have \( 0 \leq \theta \leq \pi \). But, if we neglect \( r^2 \), every point \( \zeta \) with \( |\zeta| = R = 1 + r \) has \( \log(|\zeta|) \sim r = |\zeta| - 1 \), and as the images of \( \zeta \) under \( M\alpha M^{-1} \) and \( M\beta M^{-1} \) are also close to the unit circle we are reduced to proving that there exists a constant \( \lambda_+ > 0 \) such that for \( \zeta = Re^{-i\theta} \), with \( R = 1 + r \) and \( 0 \leq \theta \leq \pi \), we have

\[
(|M\alpha M^{-1}(\zeta)| - 1) < \lambda_+ r \quad \text{and} \quad (|M\beta M^{-1}(\zeta)| - 1) < \lambda_+ r.
\]

However

\[
M\alpha M^{-1}(Re^{-i\theta}) = \frac{-(1 + 2i)R^{-i\theta} + 1}{-Re^{-i\theta} + (1 - 2i)},
\]

so

\[
|M\alpha M^{-1}(Re^{-i\theta})|^2 = \frac{5R^2 - 2R(\cos \theta + 2 \sin \theta) + 1}{5 - 2R(\cos \theta + 2 \sin \theta) + R^2},
\]

which, setting \( R = r + 1 \) and assuming \( r^2 \) to be negligible, simplifies to

\[
\frac{6 - 2(\cos \theta + 2 \sin \theta) + r(10 - 2(\cos \theta + 2 \sin \theta))}{6 - 2(\cos \theta + 2 \sin \theta) + r(2 - 2(\cos \theta + 2 \sin \theta))} \sim 1 + \frac{4r}{3 - (\cos \theta + 2 \sin \theta)}.
\]

Thus (still assuming \( r^2 \) negligible)

\[
|M\alpha M^{-1}(Re^{-i\theta})| - 1 = \frac{2r}{3 - (\cos \theta + 2 \sin \theta)}.
\]

When \( \tan \theta = 2 \) this function of \( \theta \) attains its maximum value

\[
\frac{2r}{3 - \sqrt{5}}.
\]

It follows that \( M\alpha M^{-1} \) (and hence also \( \alpha \)) increases potential by a factor of at most

\[
\frac{2}{3 - \sqrt{5}} = (3 + \sqrt{5})/2.
\]

A similar computation gives the same upper bound for the multiplier of \( M\beta M^{-1} \) (and hence also \( \beta \)) on potential. In both computations we assume \( r^2 \) to be negligible, so they are only valid in the limit as we approach the unit circle. But we may obtain a bound \( \lambda_+ \) which is valid on a sufficiently small neighbourhood of the unit circle by setting

\[
\lambda_+ = ((3 + \sqrt{5})/2) + \varepsilon.
\]

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for any $\varepsilon > 0$.

To prove the statement in (ii) concerning $\alpha$ we must find a constant $\lambda_- > 1$ such that for every point $\zeta$ in the complement of the unit disc which is sufficiently close to $M[-K,-1]$ we have

$$(|M\alpha M^{-1}(\zeta)| - 1) > \lambda_- (|\zeta| - 1)$$

For this we need a lower bound $> 1$ for

$$2r / 3 - (\cos \theta + 2 \sin \theta).$$

However for $0 < \theta \leq \pi/2$ we have

$$\cos \theta + 2 \sin \theta > 1,$$

and so provided we bound $\theta$ away from 0, so that say $0 < \delta < \theta < \pi/2$ for some small constant $\delta$, we can find a constant $k$ such that

$$\cos \theta + 2 \sin \theta > k > 1$$

for all $\theta \in (\delta, \pi/2)$, and thus we can find a constant $\lambda_- > 1$ with the desired property. Finally, the part of (ii) concerning $\beta$ follows from a calculation for $M\beta M^{-1}$, which for $\pi/2 \leq \theta < \pi$ gives the same constant. \hfill $\square$

4 Geodesics

When $a \in \mathcal{C}_R$, so that $\Lambda(F_a)$ is connected, we can use the Böttcher isomorphism $\varphi_a$ from $\Omega(F_a) = \tilde{\mathbb{C}} \setminus \Lambda(F_a)$ to $\mathbb{H}$ to pull back the hyperbolic metric on $\mathbb{H}$ to the hyperbolic metric on $\Omega(F_a)$. The geodesics for this metric become the analogues in our setting of the external rays defined by Douady and Hubbard for quadratic polynomials with connected Julia sets. In 4.1 we associate periodic geodesics to (finite) words $W$, and we prove that these geodesics land at both ends (Proposition 2). Generalising, in 4.2 we associate geodesics to bi-infinite sequences $S \in \{\alpha, \beta\}^\mathbb{Z}$, and in 4.3 we explain how this association is a manifestation of Minkowski’s question mark map. In 4.4 we consider Sturmian sequences: thought of as binary representations of real numbers these have orbit under the doubling map arranged in the same order as a rigid rotation of the circle $\mathbb{R}/\mathbb{Z}$. We conclude the section by computing bounds on the multipliers of Sturmian words and sequences.

4.1 Geodesics associated to finite words

As before we write $\alpha$ for $z \to z + 1$ and $\beta$ for $z \to z/(z + 1)$, now both acting on the boundary $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ as well as on $\mathbb{H}$ itself. We recall that $\beta$ is conjugate to $\alpha^{-1}$ (via the inversion $z \to -1/z$) and that both $\alpha$ and $\beta$ are
parabolic, having unique fixed point $\infty$ and 0 respectively. A (finite) word $W$ in the letters $\alpha$ and $\beta$ acts on the left on points $z \in \mathbb{H}$ (for example $W = \alpha^2 \beta^2 \alpha$ acts by $z \to \alpha^2(\beta^2(\alpha z))$).

**Lemma 3.** (i) For every word $W$ consisting of a finite sequence of positive powers of both $\alpha$ and $\beta$, there is a unique $W$-invariant geodesic $\gamma(W)$ in $\mathbb{H}$.

(ii) The end points of $\gamma(W)$ are a repelling fixed point $x^-(W)$ in $\mathbb{R}^{<0}$ and an attracting fixed point $x^+(W)$ in $\mathbb{R}^{>0}$.

**Proof**

The equation $Wz = z$ has the form $(a \alpha^2 \beta^2 + b)/(c \alpha^2 \beta^2 + d) = z$, with $a, b, c, d$ non-negative integers, so it has two real solutions if $W$ is hyperbolic, one real solution if $W$ is parabolic and no real solution if $W$ is elliptic. We first note that the hypothesis that the word $W$ contains positive powers of both $\alpha$ and $\beta$ is necessary.

(i) The products $\alpha \beta$ and $\beta \alpha$ both have trace 3. Building up $W$ inductively from either $\alpha \beta$ or $\beta \alpha$ by multiplying on one side or the other by $\alpha$ or $\beta$ either leaves the trace unchanged or increases it, since the matrices being multiplied together have all of their entries non-negative. We deduce that $W$ is hyperbolic, and so it has two fixed points, both real. As the product $-b/c$ of these two real numbers is negative, they lie on either side of 0. Denote these fixed points by $x^-(W)$ and $x^+(W)$ and let $\gamma(W)$ denote the geodesic in $\mathbb{H}$ which joins them, the axis of the loxodromic M"obius transformation $W$.

(ii) It will suffice to show that the derivative of $W$ at its fixed point $x^+(W)$ has modulus less than one, i.e. that this point is the attractor for the action of $W$ on $\gamma(W)$. But this derivative is $1/(|cz^+(W) + d|^2$, which can also be written as $(|x^+(W)|/|ax^+(W) + b|)^2$ (since $Wx^+(W) = x^+(W)$) and this is clearly less than 1 as a and $b$ are positive integers. □

**Example 1.** $W = \alpha \alpha \beta$.

$$\alpha \alpha \beta(z) = \frac{z}{z+1} + 2 = \frac{3z + 2}{z+1}$$

so $x^-(W) = -\sqrt{3} - 1$ and $x^+(W) = \sqrt{3} + 1$.

The orbit of $x^-(W)$ is the cycle

$$x^-(W) \to \beta x^-(W) \to \alpha \beta x^-(W) \to \alpha \alpha \beta x^-(W) = x^-(W),$$

that is, $P_0 = -\sqrt{3} - 1 \to P_1 = -\sqrt{3} + 1 \to P_2 = -\sqrt{3} \to P_0$.

The orbit of $x^+(W)$ is the cycle

$$x^+(W) \to \beta x^+(W) \to \alpha \beta x^+(W) \to \alpha \alpha \beta x^+(W) = x^+(W),$$

that is, $Q_0 = \sqrt{3} + 1 \to Q_1 = \sqrt{3} - 1 \to Q_2 = \sqrt{3} \to Q_0$. 

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Figure 7: The geodesics in $\mathbb{H}$ in Example 1 (the red vertical lines are at integer values on the real axis, from $-3$ to $+3$).

See Figure 7 for an illustration of these geodesics in $\mathbb{H}$.

**Proposition 2.** For every $a \in \mathbb{C}_\Gamma$ and every word $W$ consisting of a finite sequence of positive powers of both $\alpha$ and $\beta$, the inverse of the Böttcher map $\psi_a : \mathbb{H} \to \Omega$ extends continuously to $x^-(W)$ and $x^+(W)$. Under this extension,

- $\psi_a(x^-(W)) \in \partial \Lambda_-$ is a repelling periodic point of the 2-to-1 restriction $f_a$ of $\mathcal{F}_a$ defined on a neighbourhood of $\Lambda_-$;

- $\psi_a(x^+(W)) \in \partial \Lambda_+$ is a repelling periodic point of the restriction $Jf_aJ$ of $\mathcal{F}_a^{-1}$ defined on a neighbourhood of $\Lambda_+$, and thus an attracting periodic point of $\mathcal{F}_a$.

**Proof**

We follow the method of proof of Theorem 18.10 in [M]. First we modify our earlier definition of the potential function $G$ (Definition 1, Section 3.2), by pre-composing the map $\chi$ in that definition by a Möbius transformation which sends the geodesic $\gamma(W)$ to the imaginary axis, with the initial end of $\gamma(W)$ going to 0 and the final end to $\infty$. Denote this new potential function by $G_W$.

Now divide the imaginary axis in $\mathbb{H}$ into segments of Poincaré length $\ln(\mu(W))$, where $\mu(W)$ is the multiplier of $W$ (the square of the eigenvalue which is $> 1$). Correspondingly, parametrise $\varphi_a^{-1}(\gamma(W))$ as a path $p : \mathbb{R} \to \mathbb{C} \setminus \Lambda$ with $\mathbb{R}$ divided into unit intervals $I_k$ ($k \in \mathbb{Z}$) each mapped isometrically to the next by $f_a^q$, where $q$ is the length of the word $W$. Since $G_W(p(s))$ tends to zero as $s \to -\infty$ any limit point $\hat{z}$ of $\{p(s) : s \leq 0\}$ must belong to $\partial \Lambda_-$. Following the same reasoning as in the proof of Theorem 18.10 in [M] we deduce that the geodesic $\varphi_a^{-1}(\gamma(W))$ has a limit point $\hat{z}$ at its initial end, that this limit point is a fixed point of $f_a^q$, that the geodesic lands at $\hat{z}$, and that $\hat{z}$ is necessarily repelling or parabolic.

Conjugating $\mathcal{F}_a$ by $J$ sends $\mathcal{F}_a$ to $\mathcal{F}_a^{-1}$ and interchanges $\Lambda_-$ with $\Lambda_+$. The result for $x^+(W)$ follows. □
Remark 2. As $J\alpha J = \beta^{-1}$ and $J\beta J = \alpha^{-1}$, the orbit of $\psi_a(x^+(W))$ on $\Lambda_+(F_a)$ is not the $J$-image of the orbit of $\psi_a(x^-(W))$ on $\Lambda_-(F_a)$ for most words $W$ (an exception being $W = \alpha\beta$). As we vary the parameter $a$, the points of a periodic orbit may collapse together. For example, consider $a = 4.53926 - 0.439437i$ corresponding to a mating between $\text{PSL}(2,\mathbb{Z})$ and the (superattracting) ‘co-rabbit’. Here the orbit $\{P_0, P_1, P_2\}$ of $x^-(\alpha\alpha\beta)$ (the ends of geodesics to the left of the origin in Figure 7) are identified under $\psi_a$ to a single point, but the points of the orbit of $x^+(\alpha\alpha\beta)$ are not: at this value of $a$ the three periodic geodesics emanating from the fixed point in $\Lambda_-(F_a)$ have their opposite ends at distinct points of $\Lambda_+(F_a)$ (the points of the ‘rabbit orbit’, not the ‘co-rabbit orbit’).

Corollary 3. For $a \in \mathcal{M}_\Gamma$ the inverse $\psi_a : \mathbb{H} \to \Omega$ of the Böttcher map $\varphi_a$ extends continuously to all quadratic irrationals in $\mathbb{R}$.

Proof
Recall that a quadratic irrational is a root of a quadratic equation which has integer coefficients and real but not rational solutions, and that the positive quadratic irrationals are precisely the positive real numbers which have periodic or pre-periodic continued fraction expansions. We already know that periodic rays land, so it just remains to consider the strictly preperiodic case. Given any positive quadratic irrational $x$, the periodic ‘tail’ of the continued fraction is an $x^+(W)$ for some $W$, and so $x$ can be written $W'x^+(W)$ for some finite word $W'$ in $\alpha$ and $\beta$. The map $\psi_a$ extends continuously to $x^+(W)$ by Proposition 2, and so it extends to every point on the orbit of $x^+(W)$ under the correspondence, in particular to $W'x^+(W)$. As $x^-(W) = Jx^+(W')J$ for some $W'$ (usually different from $W$) the statement for negative quadratic irrationals follows. □

4.2 Geodesics associated to bi-infinite sequences

Let $S \in \{\alpha, \beta\}^{\mathbb{Z}}$. We think of $S$ as a bi-infinite sequence

$$\cdots g_n \cdots g_1 g_0 \cdot g_{-1} \cdots g_{-n} \cdots$$

where each $g_i$ is $\alpha$ or $\beta$, and the dot between $g_0$ and $g_{-1}$ is a position marker. In this section we will associate to $S$ a geodesic $\gamma(S)$, having left hand end-point in $\mathbb{R}^{\leq 0}$ and right hand end-point in $\mathbb{R}^{\geq 0}$, and we shall see (Lemmas 4 and 5 below) that this process generalises our earlier definition for a finite word.

Collecting up contiguous occurrences of each letter, we can write $S$ in the form:

$$S = \cdots \beta^{m_3} \alpha^{m_2} \beta^{m_1} \alpha^{m_0} \cdot \alpha^{n_0} \beta^{n_1} \alpha^{n_2} \beta^{n_3} \cdots$$

where $m_0, n_0 \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ and $m_i, n_i \in \mathbb{N} \cup \{\infty\}$ for $i > 0$ (and if any $m_i$ or $n_i$ is $\infty$ then subsequent $m_j$ or $n_j$ are undefined).

We may regard $S$ as a back-to-back pair of symbol sequences defining a pair of points $x^-(S) \in \mathbb{R}^{\leq 0}$ and $x^+(S) \in \mathbb{R}^{\geq 0}$, and thereby associate to $S$ the directed
geodesic $\gamma(S)$, which has ends as follows:

- $x^-(S) = -\infty$ when $m_0 = \infty$, and $x^-(S) = 0$ when $m_0 = 0$ and $m_1 = \infty$
- $x^-(S) = -[m_0; m_1, m_2, m_3, \ldots] \in \mathbb{R}^{<0}$, in all the other cases;
- $x^+(S) = \infty$ when $n_0 = \infty$, and $x^+(S) = 0$ when $n_0 = 0$ and $n_1 = \infty$
- $x^+(S) = [n_0; n_1, n_2, n_3, \ldots] \in \mathbb{R}^{>0}$, in all the other cases.

Note that there are dynamical systems associated to $S$ which have these points as attractors. Starting from the observation that $\alpha^{-1}(\mathbb{R}^{<0}) = (-\infty, -1)$ and $\beta^{-1}(\mathbb{R}^{<0}) = (-1, 0)$ are disjoint intervals in $\mathbb{R}^{<0}$ whose closures cover $\mathbb{R}^{<0}$, we can find an infinite composition of applications of $\alpha^{-1}$ and $\beta^{-1}$ for which the images of $\mathbb{R}^{<0}$ nest down to any chosen point of $\mathbb{R}^{<0}$ (this is what we do when we write down a continued fraction expression for the chosen point). Similarly any chosen point of $\mathbb{R}^{>0}$ is the limit of the images of a sequence of compositions of applications of $\alpha$ and $\beta$ to $\mathbb{R}^{>0}$. Formally, for $n > 0$ let $G_n$ denote the composite Möbius transformation $g_0^{-1} g_1^{-1} \ldots g_n^{-1}$, and $G_{-n}$ denote $g_{-1} g_{-2} \ldots g_{-n}$. Then for any choice of $z \in \mathbb{H} \cup \mathbb{R}^{<0}$ we have:

$$\lim_{n \to \infty} G_n(z) = x^-(S) \in \mathbb{R}^{\leq 0} \cup \{-\infty\}$$

and for any choice of $z \in \mathbb{H} \cup \mathbb{R}^{>0}$ we have:

$$\lim_{n \to \infty} G_{-n}(z) = x^+(S) \in \mathbb{R}^{\geq 0} \cup \{\infty\}.$$  

**Lemma 4.** If $W$ is a finite word in $\alpha$ and $\beta$, and $S(W) = \overline{W} \cdot \overline{W}$ (the bi-infinite sequence consisting of repeats of $W$), then

$$\gamma(S(W)) = \gamma(W)$$

where $\gamma(W)$ is as in Lemma 3.

**Proof**

Immediate from definitions. □

**Lemma 5.** Let $S = (g_n)_{n \in \mathbb{Z}}$, and let $\sigma$ denote the operation of moving the position marker in $S$ one place to the left (so $(\sigma S)_n = g_{n-1}$). Then

$$\gamma(\sigma(S)) = g_0(\gamma(S)).$$

**Proof**

This follows at once from the algorithms for $x^+(S)$ and $x^-(S)$ above, together with the observation that $x \to -x$ conjugates $\alpha$ and $\beta$ on $\mathbb{R}^{\geq 0}$ to $\alpha^{-1}$ and $\beta^{-1}$ respectively on $\mathbb{R}^{\leq 0}$. □

**Example revisited.** The bi-infinite sequence corresponding to the finite word $W = \alpha \alpha \beta$ of Example 4 is:

$$S = \ldots \alpha^2 \beta \alpha^2 \beta \cdot \alpha^2 \beta \alpha^2 \beta \ldots$$
so the definition above gives:

\[ x^{-}(S) = -[0; 1, 2, 1, \ldots] = -(\sqrt{3} - 1); \quad x^{+}(S) = [2; 1, 2, 1, \ldots] = \sqrt{3} + 1 \]

which agrees with our calculation in Example \[1\]. Furthermore

\[ \sigma(S) = \ldots \alpha^{2} \beta \alpha^{2} \beta \alpha^{2} \beta \ldots \]

which gives

\[ x^{-}(\sigma(S)) = -[2; 1, 2, 1, 2, \ldots] = -(\sqrt{3} + 1); \quad x^{+}(\sigma(S)) = [0; 1, 2, 1, 2, \ldots] = \sqrt{3} - 1 \]

confirming that \( \gamma(\sigma(S)) \) is indeed equal to \( \beta \gamma(S) \) in this example.

### 4.3 Minkowski’s ‘question mark’ map

In the preceding subsection we associated continued fraction expressions representing real numbers \( x^{-}(S) \) and \( x^{+}(S) \) to each marked bi-infinite sequence \( S \) of ‘\( \alpha \)’s and ‘\( \beta \)’s. Equally, we may associate binary expressions representing real numbers in the interval \([0, 1]\) to the sequences \( n_0, n_1, n_2, \ldots \) and \( m_0, m_1, m_2, \ldots \) which code \( S \), now with the symbols ‘1’ and ‘0’ corresponding to ‘\( \alpha \)’ and ‘\( \beta \)’ respectively. Indeed the correspondence between real numbers expressed as continued fractions, and real numbers in the interval \([0, 1]\) expressed in binary, is at the heart of the existence of matings between \( PSL(2, \mathbb{Z}) \) and quadratic polynomials in [BP]. The key is Minkowski’s ‘question mark’ map (see [Mi] p171-172).

Minkowski’s map is a homeomorphism from the unit interval \((0, 1) \subset \mathbb{R}\) to itself. We consider the following slightly modified version (which we denote by ‘?’ as did Minkowski his map). Our map is the homeomorphism from \((0, \infty) = \mathbb{R}^\mathbb{R}\) to the unit interval:

\[ ?([a_0; a_1, a_2 \ldots]) = 0.1 \ldots 10 \ldots 01 \ldots 1 \ldots \]

where on the left hand side is a continued fraction expression and on the right-hand side is a binary expression. Given any point \( x^{-}(S) \in \mathbb{R}^\mathbb{R} \) the point \(?(-x^{-}(S)) \in [0, 1] \) is the real number which has the binary expression \( 0.t_0t_1t_2\ldots \) where each \( t_j, \, j \geq 0 \) is 0 or 1 according to whether \( g_j \) is \( \beta \) or \( \alpha \) in the bi-infinite word \( S \). The bijection

\[ 0.t_0t_1t_2\ldots \leftrightarrow 0.g_0g_1g_2\ldots \text{ given by } 1 \leftrightarrow \alpha, \, 0 \leftrightarrow \beta \]

gives us an explicit formula for the correspondence between periodic external rays for a quadratic polynomial \( Q_c \) and periodic geodesics for a mating between \( Q_c \) and \( PSL(2, \mathbb{Z}) \). For example, the external ray labelled \( .110110110\ldots \) and its orbit under the doubling map (which has rotation number \( 2/3 \)), correspond to the periodic geodesic \( \gamma(W) \), \( W = \alpha \alpha \beta \), of Example \[1\] and its orbit under the cycle ‘apply \( \beta \) then \( \alpha \) then \( \alpha \)’:

\[ 3/7 = .011011\ldots \leftrightarrow [0; 1, 2, 1, 2, \ldots] = \sqrt{3} - 1 = -x^{-}(\alpha \alpha \beta); \]
\[ \frac{6}{7} = .110110 \ldots \leftrightarrow [2; 1, 2, 1, 2, \ldots] = \sqrt{3} + 1 = -x^{-}(\beta\alpha\alpha); \]
\[ \frac{5}{7} = .101101 \ldots \leftrightarrow [1; 1, 2, 1, 2, \ldots] = \sqrt{3} = -x^{\pm}(\alpha\beta\alpha). \]

### 4.4 Sturmian sequences, rotation numbers, bounds on multipliers

There are many equivalent definitions of the term ‘Sturmian’ (which is due to Morse and Hedlund); it is usually applied to infinite sequences, but can also be applied to bi-infinite sequences or to (finite) words. For the purposes of the current article:

**Definition 2.** An infinite or bi-infinite sequence in the symbols 0 and 1 is Sturmian if for each \( n \in \mathbb{N} \) the numbers of 1’s in any two blocks of \( n \) consecutive symbols differ by at most 1. A word \( W \) in 0’s and 1’s is Sturmian if the infinite sequence \( \cdot W \) is Sturmian (or equivalently the bi-infinite sequence \( W \cdot \overline{W} \)) is Sturmian. Here, as usual, \( W \) denotes a repeated sequence of blocks \( W \).

Sturmian sequences are also known as ‘balanced sequences’. They occur as the ‘cutting sequences’ of straight lines of rational or irrational slope on an integer grid of squares (with appropriate conventions where lines pass through vertices: see the proof of the Proposition below). A useful way to characterise an infinite Sturmian sequence is as a sequence of 0’s and 1’s such that the real number in \([0, 1]\) it represents in binary has orbit under the doubling map a sequence of points arranged in the same order around the circle \( \mathbb{R}/\mathbb{Z} \) as for a rigid rotation (see [BS]). A bi-infinite word is Sturmian if and only if the infinite sequence to the right (equivalently left) of the marker has this property wherever the marker is placed. This allows us to define a rotation number for such words.

**Definition 3.** The rotation number of a Sturmian sequence is that of the corresponding rigid rotation of the circle: equivalently it is the limiting frequency with which the digit ‘1’ appears in subwords of length \( n \), as \( n \) tends to \( \infty \).

The following Proposition lists the properties of Sturmian sequences that we shall make use of in our proofs in subsequent Sections:

**Proposition 3.** (i) For each rational \( 0 < p/q < 1 \) there is exactly one Sturmian word \( T_{p/q} \) (up to cyclic equivalence) of length \( q \) and rotation number \( p/q \), and there are two bi-infinite non-periodic Sturmian sequences of rotation number \( p/q \). Both ends of each of the non-periodic Sturmian sequences consist of repeated copies of \( T_{p/q} \).

(ii) For each irrational \( 0 < \nu < 1 \), the set of Sturmian sequences of rotation number \( \nu \) forms a Cantor set \( C_{\nu} \) contained in the real interval \((0, 1)\), when these sequences are regarded as binary expressions for real numbers. For each bi-infinite Sturmian sequence \( S \) of rotation number \( \nu \), and each shift (left or right), the orbit of \( x^{-}(S) \) under the shift is dense in \( C_{\nu} \), as is that of \( x^{+}(S) \).
Proof
We omit details, but these properties follow from the characterisation of a Sturmian sequence of rotation number $0 \leq \nu < 1$ as a sequence obtained by the 'staircase algorithm' [BS], applied to a straight line $L$ of slope $\nu$ superimposed on an integer grid of lines. This algorithm codes the maximum integer staircase that fits below $L$ by writing '0' for a horizontal move, and '1' for a horizontal plus vertical move. If $L$ passes through one or more vertices the rule to obtain a staircase is to cut $L$ at a point where it does not meet a grid line and then parallel translate the two halves of $L$ infinitesimally in opposite directions so that they no longer pass through any vertices. Thus for each irrational $\nu$ there is a countable set of Sturmian sequences of rotation number $\nu$ which have two continuations to bi-infinite Sturmian sequences (these correspond to the lines $L$ of slope $\nu$ which have continuations to the left which pass through a vertex), whereas all other Sturmian sequences of rotation number $\nu$ have unique continuations to bi-infinite Sturmian sequences. The result concerning density of every orbit in $C_\nu$, under either shift, follows from the fact that the intersections between $L$ and vertical grid lines, when projected to the vertical axis $\mathbb{R}$ and then to the circle $\mathbb{R}/\mathbb{Z}$, become the points of an orbit of an irrational rigid rotation of the circle, and thus dense in the circle under either forward or backward iteration.

Example 2. (a) $T_{1/3} = 001$. The two non-periodic bi-infinite Sturmian sequences of rotation number $1/3$ are $(001)(01)(001)$ and $(001)(0001)(001)$.
(b) The limit of the Sturmian words $10$, $101$, $10110$, $10110101$, $101101010101$, ... (obtained from 10 by repeatedly applying the substitutions $1 \rightarrow 10$, $0 \rightarrow 1$) is a Sturmian sequence of rotation number the golden mean. This has two continuations to the left yielding bi-infinite Sturmian sequences, namely ... $10110101 \cdot 10110101 \ldots$ and ... $10110101 \cdot 10110101 \ldots$

Corollary 4. If $A \subset \mathbb{R}/\mathbb{Z}$ is a closed invariant Sturmian subset on which the doubling map acts injectively, then the rotation number $\nu$ of the doubling map restricted to $A$ is a rational $p/q$, and the points of $A$ are the real numbers whose binary expressions are the cyclic permutations of $T_{p/q}$. In particular $A$ is finite.

Proof
Let $x \in A$. Since the doubling map restricted to $A$ is a homeomorphism we can continue the binary expression for $x$ to a bi-infinite Sturmian sequence. First suppose that $\nu$ is rational. If the bi-infinite sequence for $x$ is either of the two non-periodic sequences listed in part (i) of the Proposition, the orbit of $x$ under the doubling map is not injective, since it contains a point outside the orbit of $(T_{p/q})^\infty$ mapping onto this orbit. Now suppose that $\nu$ is irrational. In this case by part (ii) of the Proposition, since $A$ is closed we deduce that $A$ is the Cantor set $C_\nu$. But the doubling map on $C_\nu$ is non-injective, since it sends the two ends of the longest gap in $C_\nu$ to a single point (the longest gap has length 1/2, see [BS]).
We now replace ‘0’ by ‘\(\beta\)’ and ‘1’ by ‘\(\alpha\)’, and consider the finite Sturmian block \(T_{p/q}\) as a composition of \(q\) matrices, \(p\) of which are copies of \(\alpha\) and \(q - p\) of which are copies of \(\beta\). We shall establish upper and lower bounds on the multiplier \(\mu_{p/q}\) of \(T_{p/q}\) at \(x^{-}(T_{p/q})\), as a consequence of the following result:

**Proposition 4.** Let \(r > 1\).

(i) If \(W = \alpha^{r-1}\beta\) then the multiplier \(\mu(W)\) of \(W\) satisfies the inequality:

\[ r^2 < \mu(W) < (r + 1)^2. \]

(ii) If \(W\) is a word made up of \(s > 1\) blocks, each of the form either \(\alpha^{r-1}\beta\) or \(\alpha^r\beta\), then the multiplier \(\mu(W)\) of \(W\) satisfies the inequality:

\[ r^{2s} < \mu(W) < (r + 2)^{2s}. \]

(iii) The inequalities above also hold when \(\alpha^{r-1}\beta\) and \(\alpha^r\beta\) are replaced by \(\beta^{r-1}\alpha\) and \(\beta^r\alpha\).

**Proof**

We shall estimate the positions of points on the orbit of \((x - W)\) on \(\mathbb{R} < 0\), and the value of the derivative of \(\alpha\) or \(\beta\) (as appropriate) at each point, then multiply these derivatives together to get an estimate of the multiplier of the orbit. As the derivative of \(\alpha : x \to x + 1\) is 1 everywhere on \(\mathbb{R}\), we only have to compute the derivatives at orbit points where the map being applied is \(\beta : x \to x/(x+1)\).

(i) The unique point of the orbit of \((x^{-}(W))\) on \(\mathbb{R} < 0\), and the value of the derivative of \(\alpha\) or \(\beta\) (as appropriate) at each point, then multiply these derivatives together to get an estimate of the multiplier of the orbit. As the derivative of \(\alpha : x \to x + 1\) is 1 everywhere on \(\mathbb{R}\), we only have to compute the derivatives at orbit points where the map being applied is \(\beta : x \to x/(x+1)\).

Thus there is a Möbius conjugacy from \(\alpha^{r-1}\beta\) to the map \(z \to \lambda z/\lambda^{-1} = \lambda^2 z\), and the derivative of \(\alpha^{r-1}\beta\) at its expanding fixed point \(x_r\) is therefore

\[ \mu(W) = \lambda^2 = \left(\frac{(r + 1) + \sqrt{r^2 + 2r - 3}}{2}\right)^2. \]

In particular

\[ r^2 < \mu(W) < (r + 1)^2. \]

(ii) The \(s\) points of the orbit of \((x^{-}(W))\) which lie in \((-1, 0)\) all lie between \(x_r\) and \(x_{r+1}\), since the Minkowski question mark map preserves order and we know
that a binary sequence made up of blocks of the form ‘1 followed by \( r - 1 \) copies of 0’, or ‘1 followed by \( r \) copies of 0’, represents a real number which lies between the numbers represented by two periodic sequences made up of copies of just one of these blocks. Finally, since the derivative of \( \beta \) at \( x \) is \( 1/(1 + x)^2 \), which is monotonic in \( x \), and the multiplier of the orbit is the product of the values of the derivative of \( \beta \) at the points of the orbit which are in \((-1, 0)\), the general result follows from our initial special case calculation.

(iii) follows at once from the facts that \( z \to -1/z \) conjugates \( \alpha \) to \( \beta^{-1} \) and \( \beta \) to \( \alpha^{-1} \) on \( \mathbb{H} \), and that for any invertible matrix \( M \) the eigenvalues of \( M^{-1} \) are the inverses of those of \( M \).

\[ \square \]

**Corollary 5.**

(i) For all \( 0 < p/q \leq 1/2 \), the multiplier \( \mu_{p/q} \) of \( T_{p/q} \) satisfies

\[ |q/p|^{2p} < \mu_{p/q} < (1 + \lceil q/p \rceil)^{2p}. \]

(ii) For all \( 1/2 \leq p/q < 1 \), the multiplier \( \mu_{p/q} \) satisfies

\[ |q/(q-p)|^{2(q-p)} < \mu_{p/q} < (1 + \lceil q/(q-p) \rceil)^{2(q-p)}. \]

**Proof** First suppose that \( 0 < p/q \leq 1/2 \). The word \( T_{1/q} \) is (up to cyclic equivalence) \( \alpha^{q-1} \beta \), and, when \( p > 1 \), \( T_{p/q} \) is made up of blocks \( \alpha^{r-1} \beta \) and \( \alpha^r \beta \), where \( r = \lfloor q/p \rfloor \) and \( r + 1 = \lceil q/p \rceil \).

Now (i) and (ii) follow from parts (ii) and (iii) of Proposition 4 respectively. \( \square \)

## 5 The proof of Theorem \[1\]

Motivated by results for polynomial maps and having proved in Proposition 2 (Section 4.1) that every periodic geodesic lands, we now set out to prove Theorem \[1\] that for \( f_a \) with \( a \in \mathcal{C}_\Gamma \), every repelling periodic point in \( \Lambda_\Gamma \) is the landing point of a periodic geodesic.

We start by establishing notation for linearisation around a repelling fixed point, and some preparatory results concerning its properties. Let \( \hat{z} \in \Lambda_\Gamma \) be a repelling fixed point of \( f(= f_a) \), and let \( \omega \) denote the derivative of \( f \) at \( \hat{z} \) (so \( |\omega| > 1 \)). Let \( \mathbb{D} \) be the open unit disc and \( \lambda \) be a Koenigs linearisation, that is a conformal homeomorphism from \( \mathbb{D} \) to an open topological disc containing \( \hat{z} \), conjugating the map \( \times \omega \) (multiplication by \( \omega \)) to \( f \). Let \( C_{-n} \) be the circle in \( \mathbb{D} \) which has centre 0 and radius \( |\omega|^{-n} \), and let \( A_{-n} \) be the closed annulus which has boundaries \( C_{-n} \) and \( C_{-(n+1)} \).

Let \( \hat{\Lambda} = \lambda^{-1}(\Lambda) \) (where \( \Lambda = \Lambda(\mathcal{F}_a) = \Lambda_\Gamma(\mathcal{F}_a) \cup \Lambda_+ (\mathcal{F}_a) \)). The map \( \times \omega \) sends \( \hat{\Lambda} \cap A_{-n} \) bijectively to \( \hat{\Lambda} \cap A_{-(n-1)} \). We can extend \( \hat{\Lambda} \) in the obvious way to become a subset \( \bigcup_0^\infty \omega^n(\hat{\Lambda}) \subset \mathbb{C} \) invariant under \( \times \omega \), which we also denote by \( \hat{\Lambda} \). We may think of \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) as a covering space of the torus obtained from
A−n by identifying its boundaries via ×ω.

Let U = U0 be any component of \( \mathbb{C}^* \setminus \Lambda \). For each \( i \in \mathbb{Z} \) let \( U_i \) denote the component \( \omega^i(U_0) \). Under the map ×ω we have a bi-infinite sequence of components:

\[
\ldots \to U_{-n} \to \ldots \to U_{-1} \to U_0 \to U_1 \to \ldots \to U_n \to \ldots
\]

Intersecting these components with \( \mathbb{D} \) and applying the linearisation map \( \lambda \) we have a corresponding bi-infinite sequence of maps:

\[
f_i : \lambda(U_i \cap \mathbb{D}) \to \lambda(U_{i+1} \cap \mathbb{D})
\]

where each \( f_i \) is the branch of the correspondence which fixes \( \hat{z} \), denoted \( f \) a few lines above. The reason for the subscript ‘i’ is that if we now conjugate our sequence of maps \((f_i)_{i \in \mathbb{Z}}\) by the Böttcher conformal bijection: \( \varphi = \varphi_a : \hat{\mathbb{C}} \setminus \Lambda \to \mathbb{H} \) we obtain a sequence of maps

\[
h_i : \varphi(\lambda(U_i \cap \mathbb{D})) \to \varphi(\lambda(U_{i+1} \cap \mathbb{D}))
\]

each of which is a restriction of either \( \alpha : \mathbb{H} \to \mathbb{H} \) or \( \beta : \mathbb{H} \to \mathbb{H} \), and it is helpful to have a notation which allows us to distinguish these.

**Proposition 5.** For each \( i \in \mathbb{Z} \) the linearising map \( \lambda : U_i \cap \mathbb{D} \to \hat{\mathbb{C}} \setminus \Lambda \) extends to a conformal bijection \( \lambda_i : U_i \to \hat{\mathbb{C}} \setminus \Lambda \). The bijections \( \varphi \circ \lambda_i : U_i \to \mathbb{H} \) send the (bi-infinite) sequence of maps ‘×ω’

\[
\ldots \to U_{-n} \to \ldots \to U_{-1} \to U_0 \to U_1 \to \ldots \to U_n \to \ldots
\]

to the corresponding sequence of maps \( h_i : \mathbb{H} \to \mathbb{H} \), in a commuting ladder.

**Proof**
First note that each \( f_i \) is not just the locally defined branch of the correspondence fixing \( \hat{z} \), but is a well-defined conformal bijection \( \hat{\mathbb{C}} \setminus \Lambda \to \hat{\mathbb{C}} \setminus \Lambda \), the conjugate, via \( \varphi^{-1} \), of \( h_i : \mathbb{H} \to \mathbb{H} \), that is to say of either \( \alpha \) or \( \beta \).

Now, given any \( x_0 \in U_i \), choose \( n \) sufficiently large that \( x_{-n} = \omega^{-n}x_0 \) lies in \( U_{i-n} \cap \mathbb{D} \). Define \( \lambda_i(x_0) \) to be:

\[
\lambda_i(x_0) = f_{i-1} \circ f_{i-2} \circ \ldots \circ f_{i-n} \circ \lambda(x_{-n}).
\]

Proving that \( \lambda_i(x_0) \) is well-defined, and that when \( x_0 \in U_i \cap \mathbb{D} \) the definition agrees with that of \( \lambda(x_0) \), is a straightforward exercise. To see that

\[
\lambda_i : U_i \to \hat{\mathbb{C}} \setminus \Lambda
\]

is injective, observe that if we are given any two distinct points in \( U_i \) then by applying \( \omega^{-n} \) with \( n \) sufficiently large we can pull them back to a pair of distinct points in \( U_{i-n} \cap \mathbb{D} \); this pair then maps forward under the bijection.
To prove that $\lambda_i$ is surjective, we consider the sequence of conformal bijections

$$f_{i-1}^{-1}, f_{i-2}^{-1} \circ f_{i-1}^{-1}, f_{i-3}^{-1} \circ f_{i-2}^{-1} \circ f_{i-1}^{-1}, \ldots$$

from $\hat{C} \setminus \Lambda$ to itself.

This sequence forms a normal family (since $\hat{C} \setminus \Lambda$ is a hyperbolic surface), so some subsequence converges locally uniformly to a holomorphic map $\hat{C} \setminus \Lambda \to \hat{C}$. But on any compact subset of $\lambda(U_i \cap \mathbb{D})$ the whole sequence converges uniformly to the map which sends the compact set to the fixed point $\hat{z}$. By uniqueness of analytic continuation it follows that the only holomorphic map to which any subsequence of our self-maps of $\hat{C} \setminus \Lambda$ can converge is the constant map

$$\hat{C} \setminus \Lambda \to \hat{z} \in \hat{C}.$$ 

So for every $z \in \hat{C} \setminus \Lambda$ the images

$$f_{i-1}^{-1}(z), f_{i-2}^{-1} \circ f_{i-1}^{-1}(z), f_{i-3}^{-1} \circ f_{i-2}^{-1} \circ f_{i-1}^{-1}(z), \ldots$$

converge to $\hat{z}$. Thus for some $n$ the image $f_{i-n}^{-1} \circ f_{i-n+1}^{-1} \circ \ldots \circ f_{i-1}^{-1}(z)$ of $z$ lies in $\lambda(U_i \cap \mathbb{D})$ for some $j$. But, since the images of a line segment joining $z$ to a point of $U_i \cap \mathbb{D}$ under the sequence of maps must also converge uniformly to the single point $\hat{z}$, it is easily seen that $j = i - n$. It follows from our definition of the extension $\lambda_i$ that $z \in \lambda_i(U_i)$. That the homeomorphisms $\lambda_i$ form a commuting ladder follows from their definition, since they form such a ladder when restricted to the $U_i \cap \mathbb{D}$.

For a repelling cycle $\{z_0, \ldots z_{m-1}\}$, $m > 1$, in place of a fixed point of $F_a$, an analogue of the analysis above goes through in the obvious way.

Our next observation is there are restrictions on the itineraries that can occur for Fatou components of the linearised map, either at a repelling fixed point or a repelling cycle.

**Lemma 6.** For each repelling cycle $\{z_0, \ldots z_{m-1}\}$, $m \geq 1$, of $F_a$, there is a bound on the length of a sequence of consecutive occurrences of $\alpha$, or consecutive occurrences of $\beta$, that can occur in the itinerary $S$ of a Fatou component $U$ of the linearisation in a neighbourhood of the cycle.

**Proof**

First suppose that $U$ is a Fatou component at the point $z_0$ of the cycle, and the itinerary $S$ of $U$ contains $n$ consecutive $\alpha$’s. Replacing $U$ by its appropriate forward or backward image, we may suppose these $n$ consecutive $\alpha$’s lie immediately to the left of the marker point. Since $\alpha$ is the map $z \to z + 1$ on $\mathbb{H}$ and its boundary, it follows from Proposition 5 that we can choose a point $x_n \in U$
such that $\lambda(x_n)$ is arbitrarily close to $z_0$, its image $\varphi \lambda(x_n)$ is arbitrarily close to the boundary of $\mathbb{H}$, and $\text{Re}(\varphi \lambda(x_n)) < -n$.

Now, if for each positive integer $n$ the itinerary $S$ contains $n$ consecutive $\alpha$’s, we may construct a sequence of points $x_n$ in the appropriate components of the linearised map with the properties that
(i) the sequence $(\lambda(x_n))_{n \geq 1} \subset \Omega(F_a)$ converges to $z_0 \in \mathbb{C}$;
(ii) its image under $\varphi$, the sequence $(\varphi \lambda(x_n))_{n \geq 1} \subset \mathbb{H}$, converges to $-\infty \in \partial \mathbb{H}$.
But (i) and (ii) are contradictory, since the inverse of the B"ottcher map extends continuously to $-\infty$, sending $-\infty$ to $P$, the parabolic fixed point, yet $z_0 \neq P$.

We obtain a similar contradiction when we take $\beta$ in place of $\alpha$, and $0$ in place of $-\infty$.

**Notation.** Recall that $\gamma(S)$ is our notation for the geodesic in $\mathbb{H}$ which has itinerary $S$. It is convenient for the remaining proofs in the current section to introduce a notation for the image of $\gamma(S)$ in $\Omega(F_a)$ under the inverse $\varphi_a^{-1}$ of the B"ottcher map. We define
$$g(S) := \varphi_a^{-1}(\gamma(S)) \subset \mathbb{C} \setminus \Lambda(F_a).$$

In a linearising neighbourhood of a repelling fixed point $\mathbb{C}$ we can further pull back $g(S)$ to $\lambda^{-1}(g(S)) \subset \mathbb{D}$.

If $\gamma(S)$ is a geodesic in $\mathbb{H}$ with initial end point in $\mathbb{R}^- \subset \partial \mathbb{H}$, we parametrise $\gamma(S)$ in a neighbourhood of this end point by the potential $\chi$ (which we recall is approximately ‘height’) and thereby also parametrise the corresponding part of $g(S)$. We shall write $g_S$ for the parametrising function.

**Lemma 7.** If $\gamma(S_n), n > 0$, is any sequence of geodesics which have initial and final end points converging to the initial and final end points of $\gamma(S)$, then for each sufficiently small pair of positive real numbers $t_* < t^*$ the geodesics $g(S_n)$ converge uniformly to $g(S)$ on the interval $[t_*, t^*]$.

**Proof**
Working in the disc model of hyperbolic space, the inverse of the B"ottcher map is uniformly continuous in every closed annulus centred at the centre of the disc. Since the geodesics $\gamma(S_n)$ converge uniformly to the geodesic $\gamma(S)$ on the compact set $[t_*, t^*]$, the result follows.

Next, in place of the ‘fundamental domain’ $I_t(g_s)$ on the ‘dynamical ray’ $g_s$, defined in [BeLy], we define a ‘basic interval’ on the geodesic $g(S)$:

**Definition 4.** The basic interval is the subset $I_t(g(S)) := g_S[t/\lambda_+, t]$ of $g(S)$.

Here $\lambda_+$ is an upper bound on the multiplier for $\alpha$ and $\beta$ on the potential of a point $z$ close to the negative real axis in $\mathbb{H}$, as computed in Lemma 2(i) (Section 3.2). As $t$ tends to 0 the hyperbolic length of $I_t(g_S)$ tends to $\log(\lambda_+)$. Recall that $x^-(S)$ denotes the landing point of the geodesic $\gamma(S) \subset \mathbb{H}$ on $\mathbb{R}^-$.  

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Proposition 6. For each $K > 1$, the Euclidean length of every basic interval $I_t(g(S))$ with $x^-(S) \in [-K, -1/K]$ tends to zero, independently of $S$, as $t$ tends to 0.

Proof
Approaching the boundary of any bounded simply-connected domain in the plane, the density of a hyperbolic metric on the domain tends to zero (for a proof, see for example Lemma 2.3 of [BeLy]). For small $t > 0$ the basic intervals $I_t(g(S))$ with $x^-(S) \in [-K, -1/K]$ have uniformly bounded hyperbolic lengths and are contained in the neighbourhood of $\partial\Omega(F_a)$ bounded by the equipotential $\{z \in \Omega : G(z) = t\}$ (where $G$ is the potential function we chose in Definition 1 in Section 3.2). Hence the Euclidean length of $I_t(g(S))$ tends to zero with $t$. Uniformity with respect to $S$ follows from the compactness of each equipotential.

We now have all the ingredients to prove that there is at least one geodesic landing at each repelling periodic point, inspired by the methods in [BeLy].

Proof of Theorem 1

The proof is divided into four steps. For clarity of exposition, in the first three steps we consider the case of a repelling fixed point $\hat{z}$ and afterwards in Step 4 we list the modifications to Steps 1, 2 and 3 needed to prove the result for a repelling cycle of period $m > 1$.

Step 1: there exists a landing geodesic

We follow the strategy of the proof of Theorem 2.5 of [BeLy], constructing a landing geodesic as a limit of longer and longer segments of a convergent sequence of geodesics, the major difference from [BeLy] being that we add an overlapping ‘basic interval’ of geodesic at each stage, rather than adding a ‘fundamental domain’ to the end of the segment of geodesic already constructed.

Let $U$ be a component of the Fatou set in a linearising neighbourhood around $\hat{z}$ and $S$ be its itinerary. If $S$ is periodic, then so is $g(S)$, and in this case $g(S)$ lands, by Proposition 2. Otherwise, write $g_i$ for the $-i$th shift of $g_0 = g(S)$ (so $g_i = f_{a}^{-i}(g_0)$), and write $\gamma_i$ for the geodesic $\varphi_a(g_i) \subset \mathbb{H}$. Near to the boundary of $\mathbb{H}$ the geodesic $\gamma_i$ is parametrised by the potential function $\chi$, and near to $\Lambda$ the geodesic $g_i$ is parametrised by $G = \chi \circ \varphi_a$. We note that by Lemma 6 there is a bound on the length of consecutive appearances of the same letter in $S$, and it follows that the initial points of the geodesics $\gamma_i$ fall within some closed interval $[-K, -1/K]$ contained within $(\infty, 0) \subset \partial\mathbb{H}$. So, by Lemma 2(ii) (Section 3.2), there exists a ‘minimum multiplier’ $\lambda_-$ for the effect of $\alpha$ and $\beta$ on potential.

Let $V'$ be a linearising neighbourhood around $\hat{z}$, and $V \subset V'$ be its pre-image
least one of these images of circles must meet γ. We observe that the Euclidean length of I(γi) tends to zero (uniformly in i) as t tends to 0 by Proposition 6. Thus there exists tε > 0 such that the Euclidean length of I(γi) is less than ε for all i ≥ 0 and t < tε.

We claim there exists some i and t0 < tε with z0 := g: (t0) ∈ V′. To see this, observe that C \ Λ is foliated by images under φa ◦ λ of circles in D, and at least one of these images of circles must meet γi. Now pulling back by f−1 a sends points in the sphere back by ω−1 (so we can pull back into V) and reduces potential in H by a factor of at least λ− (so we can ensure t0 < tε). Remumber this gi as g0 and renumberr the gn accordingly, so that gn = f−n a (g0).

Let zn = f−n a (z0) ∈ g0, let t0 = γ(φa(zn)), let J1 = g: [t0, t0] and, inductively for n > 1, let Jn = f−1 a (Jn−1) ∪ I: (g: (g: n)). Figure 8 illustrates the points zn and geodesic segments Jn together with their φa-images in H. For each n ≥ 1, φa(zn) = h−1 n (φa(zn−1)), where h: is α or β (as defined in Proposition 5). As h−1 (φa(Jn−1)) overlaps I: (γn) = γn[t0/λ+, t0] (by Lemma 2(ii), Section 3.2), we deduce that φa(Jn) ≥ γn[t0, t0].

The point γ1 = f−1 a (z0) ∈ V, so J1 ⊂ V′ (since the Euclidean length of I: (g(S1)) < ε). Inductively, by similar reasoning, each of the intervals Jn ⊂ g(Sn) lies in V′. Moreover the sequence (tn) of lower ends of these intervals has limit zero (since, inductively, t0/t0 > (λ−)n).

Within our sequence (gn) of geodesics in Ω(Fa) we may choose a subsequence (gn) such that the left hand ends of the corresponding γn in H form a convergent sequence, and their right hand ends also converge. The fact that the itinerary S has a bound on the length of consecutive repetitions of the same
symbol ensures that the left hand ends of the geodesics \( \gamma_n \) are bounded away from 0 and \( \infty \) and hence that the \( \gamma_{n_j} \) converge to a genuine geodesic, not to the ‘degenerate geodesic’ consisting of the single point 0 or the single point \( \infty \). Thus the sequence of geodesic segments \( (J_{n_j}) \) converges to a segment \( g(S')(0,t_0) \) of some limit geodesic \( g(S') \). For any \( t_0 \) such that \( 0 < t_0 < t_0 \) this convergence is uniform on the interval \([t_*, t_0] \subset (0, t_0)\) by Lemma 7. Using the fact that close to the fixed point \( \hat{z} \) the map \( f_a \) is approximated by \( z \to \hat{z} + \omega(z-\hat{z}) \), it can now be verified easily that \( g(S')(0,t_{n_j}) \subset B(\hat{z}, A/|\omega|) \) for some constant \( A \) and \( n_j \) sufficiently large, and thus that \( g(S') \) restricted to \( (0,t_0] \) can be extended continuously to \( t = 0 \) by setting \( g(S')(0) = \hat{z} \). Then \( g(S') \) is a geodesic landing at the fixed point.

**Step 2: every landing geodesic is periodic**

Given any sequence \((\gamma_n)\) of geodesics in \( \mathbb{H} \) with left hand ends \( x^- (\gamma_n) \) converging to a point \( x \in \mathbb{R}_{<0} \), and with the corresponding geodesics \( g_n = \varphi_a^{-1}(\gamma_n) \) in \( \Omega(F_a) \) all landing at \( \hat{z} \), the method of Step 1 can be applied to construct a geodesic \( \gamma \) in \( \mathbb{H} \) with \( x^- (\gamma) = x \) and with \( \varphi_a^{-1}(\gamma) \) also landing at \( \hat{z} \). Thus the set \( A \) of left-hand end-points of images under \( \varphi \) of geodesics in \( \Omega(F_a) \) which land at \( \hat{z} \) is a closed subset of \( \mathbb{R}_{<0} \). This set \( A \), regarded as a subset of the circle obtained by identifying the end points \( -\infty \) and 0 of \( \mathbb{R}_{<0} \), is invariant under the doubling map defined by \( \alpha \) on \((-\infty,-1)\) and \( \beta \) on \((-1,0)\), and its cyclic order is preserved by this doubling map. Thus \( A \) is a closed invariant Sturmian subset of \( \mathbb{R}/\mathbb{Z} \). Moreover the doubling map is injective on \( A \), since \( F_a \) is linearisable in a neighbourhood of \( \hat{z} \). By Corollary 4 (Section 4.4), it follows that \( A \) is the (unique) finite Sturmian orbit of rotation number \( p/q \) for some rational \( p/q \).

**Step 3: counting cycles of landing geodesics**

By Step 2 the itinerary of a landing geodesic is necessarily of the form \( W^\infty \), where \( W = W_{p/q} \) is the Sturmian word of rotation number \( p/q \). There is only one such orbit.

**Step 4: modifications to Steps 1 to 3 needed to prove the Theorem for a repelling cycle of period \( m > 1 \)**

Step 1 is unchanged except that \( \hat{z} \) is replaced by one of the points of the repelling cycle and \( f_a \) is replaced by the first return map \( f_a^m \).

In Step 2 in the case \( m > 1 \), the closure of the set of left-hand end-points of \( \varphi_a \)-images of geodesics landing at points of the cycle becomes a union of \( m \) disjoint closed sets, each of which is invariant under the appropriate word \( W \) of length \( m \) in the symbols \( \alpha \) and \( \beta \) (the first return map) and has its cyclic order preserved by \( W \), and each of which has the same well-defined rotation number \( \nu \) under the first return map. To conclude the proof of this step we must exclude the possibility that these sets are infinite. Rather than generalising Corollary 4.
to this non-Sturmian situation, the easiest way to proceed is to apply the even more general Lemma 2.6 of [BeLy], which states that if \( f \) is a locally expanding map of a compact metric space \( X \) to itself, and \( A \) is a closed invariant subset of \( X \) restricted to which \( f \) is invertible, then \( A \) is finite. (Of course this lemma also provides an alternative proof of Step 2 for a fixed point.)

For Step 3 in the case \( m > 1 \), we observe that the combinatorics of itineraries of geodesics landing on repelling cycles for correspondences in our family are identical to the combinatorics of rays landing on repelling cycles for quadratic polynomials (see Milnor [M1], or Schleicher [S], for the latter: each step in these analyses goes through in the same way for our correspondences \( \mathcal{F}_a \)). The fact that there are at most two orbits of rays which land on any particular repelling cycle for a quadratic polynomial follows from the ‘tuning’ theory of Douady and Hubbard [DH]. The two orbit case corresponds to primitive components of the Mandelbrot set \( \mathcal{M} \): at the root point \( c \) of such a component the quadratic map \( Q_c : z \to z^2 + c \) has a parabolic cycle which has exactly two orbits of landing rays, and this landing pattern persists when the parabolic orbit is deformed into a repelling orbit. The single orbit case corresponds to landing patterns born at the root points of satellite components of \( \mathcal{M} \): the landing rays in such patterns have binary sequences which are renormalisable.

Finally we note in the corollary below that for a repelling fixed point \( \hat{z} \) not only there is a unique cycle of periodic geodesics which land at \( \hat{z} \), but there is just one cycle of Fatou components of the linearisation there. Similarly it can be proved that in the case of a repelling cycle of period \( m > 1 \) the linearised map has either exactly one or exactly two cycles of Fatou components.

**Corollary 6.** The Fatou components of the linearised map at a repelling fixed point \( \hat{z} \) form a single cycle.

**Proof**
The periodic geodesic landing at \( \hat{z} \) has itinerary \( W^\infty \) where \( W = W_{p/q} \) for some \( p/q \). If \( U \) and \( V \) are Fatou components of the linearisation then both must also have itinerary \( W^\infty \). But now \( U \) and \( V \) must be the same component, for we can join any point of \( \lambda(U) \) to a point of \( \lambda(V) \) by a path in \( \Omega \), and now applying \( W^{-n} \) for sufficiently large \( n \) shrinks this to a path contained within an arbitrarily small neighbourhood of \( \hat{z} \).  

6 The proof of Theorem 2

We recall [DH] that a quadratic map \( z \to z^2 + c \) with \( c \in \mathcal{M} \) has two fixed points: the beta-fixed-point is the landing point of the external ray of argument zero, so it has combinatorial rotation number 0. The other fixed point, known as the alpha-fixed-point, is a repeller precisely for those \( c \in \mathcal{M} \) which lie outside the closure of the main cardioid. Generically a correspondence in the family \( \mathcal{F}_a \)
has 4 fixed points, the parabolic fixed point at \( \Lambda_\pm \cap \Lambda_+ \) \((Z = 1, \text{ or equivalently, in the } z \text{-coordinate, } z = 0)\), which is always a double fixed point, and two others. When \( a \in K \), the Klein combination locus, these two others are one each in \( \Lambda_- \) and \( \Lambda_+ \). We are concerned here with the fixed point in \( \Lambda_- \), call it \( p_a \), in the Douady-Hubbard terminology the “alpha-fixed-point” of the 2-to-1 branch \( f_a \) of \( F_a \) defined earlier. Our Yoccoz inequality establishes bounds on the derivative 
\[ f'_a(p_a) = \zeta_a, \]
when \( a \in M_\Gamma \) and \( p_a \) is repelling.

Let \( U_0 \) be a (periodic) Fatou component of the linearised map at the repelling fixed point \( p_a \in \Lambda_{a,-} \) (as in the preceding section), and let \( T \) denote the torus \( \mathbb{C}^*/(x \zeta) \), so \( T \) is the quotient of \( \mathbb{C} \) by the lattice generated by \( z \rightarrow z + 2\pi i \) and \( z \rightarrow z + \tau \), where \( \tau \) is the principal value of the complex logarithm of \( \zeta \).

The image of \( U_0 \) in \( T \) is an annulus \( A \), homotopic to a closed \((-p,q)\)-curve wrapping around \( T \). Since this annulus is embedded in \( T \), we have, for a suitable choice of \( \tau \) mod \( 2\pi i \),
\[ \text{mod}(A) \leq \frac{2\pi \text{Re}(\tau)}{|2\pi ip - \tau q|^2} \]
(see [H], Proposition 3.2, for a justification). However it follows from Proposition \( \Box \) (Section 3) that \( A = U_0/(x \zeta^q) \) maps bijectively to the annulus \( \mathbb{H}/T_{p/q} \), where \( T_{p/q} \) is the Sturmian word of length \( q \) in the letters \( \alpha \) and \( \beta \) which corresponds to rotation number \( p/q \). But \( T_{p/q} \) acts on \( \mathbb{H} \) by
\[ z \rightarrow \mu(T_{p/q})z \]
and so the annulus \( \mathbb{H}/T_{p/q} \) is that obtained from the region \( 1 \leq |z| \leq \mu(T_{p/q}) \) in \( \mathbb{H} \) by identifying the bounding semicircles. Mapping \( z \) to \( \log z \) sends this region to a rectangle of side lengths \( \pi \) and \( \log(\mu(T_{p/q})) \). By applying Corollary \( \Box \) (Section 4.4) we deduce that in the case \( p/q \leq 1/2 \):
\[ \text{mod}(A) > \frac{\pi}{2p \log(\lceil q/p \rceil + 1)}. \]
From the two inequalities above, we have (still in the case \( p/q \leq 1/2 \)):
\[ \frac{\text{Re}(\tau)}{|\tau - 2\pi ip/q|^2} \geq \frac{q^2}{4p \log(\lceil q/p \rceil + 1)}, \]
which is equivalent to the statement for \( p/q \leq 1/2 \) in the theorem. (Given any \( r > 0 \), the set of points \( z = x + iy \) which satisfy the inequality \( x \geq |z|^2/2r \) form a disc of radius \( r \) tangent to the imaginary axis at the origin.) The statement for \( p/q \geq 1/2 \) follows from Corollary \( \Box \) in the same way.

From Theorem \( \Box \) we obtain the following practical criterion:

**Corollary 7.** Let \( a \in C_\Gamma \). If the derivative \( \zeta \) of \( f_a \) at a repelling fixed point has its argument in the interval \((0, \pi]\), then the principal value of the complex logarithm of \( \zeta \)
\[ \tau = \log(|\zeta|) + i \text{Arg}(\zeta) \]

...
lies in the part of $\mathbb{H}$ defined by 
\[ \Re(\tau) < 4.3\nu^2 \log(\nu^{-1} + 1) \quad \text{where} \quad \nu = \Im(\tau)/2\pi. \]

Here the multiplying factor of $4.3$ is chosen to ensure that the curve lies outside the union of the discs permitted by our Yoccoz inequality, not only outside their horizontal diameters. Note that as the graph is concave it suffices to find a multiplying factor such that the tangent to the curve where it crosses the horizontal $y = \pi$ does not meet $D_{1/2}$, the disc corresponding to $p/q = 1/2$. However we remark that we can sharpen our estimates of disc radii for rotation numbers of the form $\nu = 1/q$, by computing the multiplier of $\alpha^{q-1}\beta$ exactly. The disc $D_{1/2}$ corresponds to the Sturmian word $\alpha\beta$, which has multiplier $((3 + \sqrt{5})/2)^2$ (as we saw in the proof of Proposition 4). Repeating the calculation in the proof of Theorem 2 for the case $p/q = 1/2$, but now replacing the estimate $([2/1]+1) = 3$ by the sharper value $(3 + \sqrt{5})/2$ we obtain the value $\log((3 + \sqrt{5})/2)$ for the diameter of $D_{1/2}$, the largest disc, and hence the following absolute bound on the modulus of the derivative:

**Corollary 8.** If $a \in C_{\Gamma}$ then the derivative $\zeta$ of $f_a$ at its $\alpha$-fixed-point satisfies the inequality 
\[ |\zeta| \leq \frac{3 + \sqrt{5}}{2}. \]

In fact this bound is sharp (see Remark 6 in Section 7 below).

In Figure 2 in the Introduction, we plot some of the discs in the log $\zeta$-plane permitted by the Yoccoz inequality, on the left for matings between quadratic polynomials and $PSL(2, \mathbb{Z})$, and on the right for quadratic polynomials (here the disc for $\nu = p/q$ had radius $(\log 2)/q$). In the left-hand picture the discs lie entirely to the left of the curve (also illustrated):

\[ \{ \tau = 4.3\nu^2 \log(\nu^{-1} + 1) + 2\pi\nu i : 0 \leq \nu \leq 1/2 \}. \]

**Remark 3.** $\Re(\tau)$ has faster convergence to 0 as $\Im(\tau) \to 0$ than is the case for the classical Yoccoz inequality for quadratic polynomials, where the corresponding region in the upper half-plane is bounded by a straight line (see Figure 3). The underlying reason for the linear bound in the classical case (the right-hand picture in Figure 2) is that for the map $z \to 2z$ on $\mathbb{H}/\mathbb{Z}$ the multiplier of every period $q$ cycle is $2^q$. However in our case we are dealing with cycles made up from the parabolic maps $\alpha$ and $\beta$ on $\mathbb{H}$, and the multiplier of a $q$-cycle varies with the word $W$ which defines the cycle. The curved bound in the left-hand picture arises from the bounds we computed for the multipliers of the cycles $\alpha^{q-1}\beta$ corresponding to rotation number $1/q$ (Proposition 7(i)): these multipliers grow quadratically in $q$, not exponentially.

**Remark 4.** An obvious question is the nature of the Yoccoz inequalities for repelling periodic orbits of period greater than 1. As an example we compute these for period 2 orbits in the case that the rotation number of the first return map
is of the form $1/q$. Tuning theory (renormalisation) tells us that the itinerary of such a geodesic landing on a period two cycle is (a cyclic permutation of) $(W_{1/q})^\infty$ where $W_{1/q} = BA^{q-1}$ with $A = \alpha\beta$ and $B = \beta\alpha$. We easily compute that

$$\text{tr}(W_{1/q}) = \text{Fib}_{2q+1} + \text{Fib}_{2q-3}$$

where $\text{Fib}_n$ denotes the $n$th Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ...). Since $\text{Fib}_n = ((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n)/\sqrt{5}$ we can compute an exact formula for $\text{tr}(W_{1/q})$, and as $\text{tr}(W_{1/q}) = \mu_{1/q} + \mu_{1/q}^{-1}$ where $\mu_{1/q}$ is the multiplier of $W_{1/q}$ at its fixed point, we may deduce that, for large $q$,

$$\mu_{1/q} \sim \frac{1}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^{2(2q+1)}.$$

Thus for a repelling period 2 orbit of a correspondence in the family $F_a$, having first return map of rotation number $1/q$, the analogue of Figure 3 gives a linear bound, as in the quadratic polynomials case.

### 7 Proof of Theorem 3 and Corollary 1

We first compute the derivative of $f_a$ at its alpha-fixed-point, in preparation for applying our Pommerenke-Levin-Yoccoz inequality to prove Theorem 3.

We use the $z$ coordinate system, where the correspondence $F_a$ has equation:

\begin{equation}
\left( \frac{az + 1}{z + 1} \right)^2 + \left( \frac{az + 1}{z + 1} \right) \left( \frac{aw - 1}{aw + 1} \right) + \left( \frac{aw - 1}{w - 1} \right)^2 = 3.
\end{equation}

To find the alpha-fixed-point $z_0$ we set $w = z = z_0$ in (*) and get

$$(3a^2 - 3)z_0^4 + (a^2 - 8a + 7)z_0^2 = 0$$

which, ignoring the parabolic fixed point $z_0 = 0$ (the beta-fixed-point), gives us

$$z_0 = -\sqrt{\frac{7 - a}{3(a + 1)}}$$

where the minus sign before the square root indicate that we are taking the branch which has a negative value when $a$ is real and between +1 and 7.

To find the derivative at $z_0$ we differentiate (*) with respect to $z$, and set $z = w = z_0$. We deduce that at $z = z_0$ the value of $dw/dz$ is

$$\zeta = \left( \frac{z_0 - 1}{z_0 + 1} \right)^2 \left( \frac{3(az_0^2 - 1) + (1 - a)z_0}{3(az_0^2 - 1) + (a - 1)z_0} \right).$$

After we substitute $z_0^2 = (7 - a)/[3(a + 1)]$,

$$\zeta = \frac{(a^2 - 2a - 11) + (a + 1)(7 - a)z_0}{(a^2 - 2a - 11) - (a + 1)(7 - a)z_0}.$$
Remark 5. Notice that the expression for $\zeta$ can be written in the form

$$\zeta = \frac{1 + E}{1 - E}$$

and that $|\zeta| = 1$ if and only if $E$ is pure imaginary; also that $\zeta$ is real if and only if $E$ is real. Values where $|\zeta| = 1$ coincide with the boundary of the main component of the interior of $\mathcal{M}_\Gamma$. For example the value $\zeta = -1$, where the boundary of the main component cuts the real axis, is given by the positive solution to $a^2 - 2a - 11 = 0$, that is $a = 1 + 2\sqrt{3} = 4.464$.

Denote the open disc $\{a : |a - a_0| < r\}$ by $D(a_0, r)$. To prove Theorem it will suffice to prove the following three statements:

(i) There exists $\delta > 0$ such that $\mathcal{M}_\Gamma$ does not meet $D_1 = D(1, \delta)$;

(ii) For every $\alpha$ such that $\pi/3 < \alpha \leq \pi/2$, there is a disc neighbourhood $D_7 = D(7, \epsilon)$ with centre $a = 7$ and radius $\epsilon > 0$ such that $\mathcal{M}_\Gamma \cap D_7 \subset L_\alpha \cap D_7$;

(iii) $\partial D(4, 3) \setminus (D_1 \cup D_7)$ has a neighbourhood which does not meet $\mathcal{M}_\Gamma$.

Proof of (i). This is immediate from our formula for $\zeta$, which gives $\zeta \to \infty$ as $a \to 1$, so our Yoccoz inequality is violated for $a$ in a disc $D(1, \delta)$.

For the proof of (ii) and (iii), we change the parameter. We first note that since the lower half of the lune boundary is complex conjugate to the upper half, it will suffice to prove (ii) and (iii) for points on the upper half. Now let

$$b = \frac{a - 7}{a - 1}.$$ 

The upper half of the lune boundary becomes the straight line

$$b = te^{i(\pi - \alpha)}, \quad t \in \mathbb{R}^>0,$$

the point $z_0$ becomes

$$z_0 = -\sqrt{\frac{b}{b - 4}},$$

and $\zeta = \zeta(a)$ becomes

$$\zeta(b) = \frac{2 + 2b - b^2 + b(b - 4)z_0}{2 + 2b - b^2 - b(b - 4)z_0}$$

which we may write as

$$\frac{1 + E}{1 - E}$$

where

$$E = \frac{b(4 - b)}{2 + 2b - b^2} \sqrt{\frac{b}{b - 4}}.$$
Proof of (ii). Substituting \( b = te^{i(\pi - \alpha)} \) in our expression for \( E \) gives

\[
E = \frac{-te^{-i\alpha}(4 + te^{-i\alpha})}{2 - 2te^{-i\alpha} + t^2e^{-2i\alpha}} \sqrt{\frac{te^{-i\alpha}}{te^{-i\alpha} + 4}}
\]

the leading term of which is

\[
-t^{3/2}e^{-3i\alpha/2},
\]

and so we deduce that as \( t \to 0 \)

\[
\log \zeta = \log \left( \frac{1 + E}{1 - E} \right) \sim 2E \sim -2t^{3/2}e^{-3i\alpha/2} = 2t^{3/2}e^{i(\pi - 3\alpha/2)}.
\]

Thus as \( t \to 0 \) the complex number \( \log \zeta \) approaches 0 tangentially to a straight line of argument \( \pi - 3\alpha/2 \). When \( \pi/3 \leq \alpha \leq \pi/2 \) this line is in the positive quadrant: when \( \alpha = \pi/3 \) it is the imaginary axis and when \( \alpha = \pi/2 \) it is the line of argument \( \pi/4 \). By Corollary 7 it follows that for every value of \( \alpha \) in the interval \( \pi/3 < \alpha \leq \pi/2 \) the set \( M_\Gamma \cap D(7, \epsilon) \) lies inside \( L_\alpha \cap D(7, \epsilon) \) for a sufficiently small value of \( \epsilon > 0 \).

Proof of (iii). It will suffice to show that the entire parameter arc \( b = it \), \( 0 < t < \infty \), lies outside the region permitted by the Yoccoz inequality, since by continuity this will imply that the intersection of the arc with the complement of discs centred at its two ends has a neighbourhood which misses \( M_\Gamma \). We consider \( 0 < t \leq 1 \) and \( 1 < t < \infty \) separately.

For \( b = it \) we have

\[
E = \frac{it(4 - it)}{2 + 2it + t^2} \sqrt{\frac{-it}{4 - it}} = \frac{t^2 + 4it}{(2 + t^2) + 2it} \sqrt{\frac{-it}{4 - it}}.
\]

Now

\[
\left| \frac{t(4i + t)}{(2 + t^2 + 2it)} \right|^2 = t^2 \left( \frac{16 + t^2}{4 + 8t^2 + t^4} \right)
\]

which is strictly increasing for \( t \in [0, 1] \) so has value < \( \sqrt{17/13} < 6/5 \) there, and for \( 0 \leq t \leq 1 \) we also have

\[
\left| \sqrt{\frac{-it}{4 - it}} \right| < 1/2
\]

and so for this range of \( t \) we have

\[
|E| < \frac{3}{5}.
\]

We next estimate the arguments of the two factors of \( E \).

\[
\frac{t^2 + 4it}{(2 + t^2) + 2it} = \frac{t(4i + t)((2 + t^2) - 2it)}{(2 + t^2)^2 + 4t^2} = \frac{t(t^3 + 8t^2 + 2t + (8 + 2t^2)i)}{(2 + t^2)^2 + 4t^2}
\]
which certainly has argument in \([0, \pi/2]\) for all \(t \in [0, 1]\).

Also
\[
\arg \left( \frac{-i}{4 - t} \right) = \arg(1 - 4i) = -\arctan 4
\]
and as \(\arctan 4 > 5\pi/12\) we deduce that
\[
\arg(E) < \frac{\pi}{2} - \frac{5\pi}{24} = \frac{7\pi}{24}.
\]

Our formula for the derivative \(\zeta\) at \(z_0\) is
\[
\log(\zeta) = \log \left( \frac{1 + E}{1 - E} \right) = 2E(1 + E^2/3 + E^4/5 + E^6/7 + \ldots)
\]
so for \(|E| < 3/5\) we have
\[
\arg \left( 1 - \frac{|E|^2i}{3(1 - |E|^2)} \right) < \arg(1 + E^2/3 + E^4/5 + \ldots) < \arg \left( 1 + \frac{|E|^2i}{3(1 - |E|^2)} \right)
\]
which gives us
\[
\arg \left( 1 - \frac{3i}{16} \right) < \arg(1 + E^2/3 + E^4/5 + \ldots) < \arg \left( 1 + \frac{3i}{16} \right)
\]
so certainly
\[
-\frac{\pi}{15} < \arg(1 + E^2/3 + E^4/5 + \ldots) < \frac{\pi}{15}.
\]
Thus
\[
-\frac{\pi}{15} < \arg(\log(\zeta(b))) < \frac{7\pi}{24} + \frac{\pi}{15} = \frac{43\pi}{120} < \frac{3\pi}{8}
\]
which puts \(\log(\zeta)\) outside the region permitted by the Yoccoz inequality.

It remains to consider \(b = it\) for \(1 < t < \infty\). First we note that at \(t = 1\) (this corresponds to \(a = 4 + 3i\)) we have \(|E| = 0.563171\) and \(\arg(E) = 0.0749062\), so
\[
|\zeta| = |(1 + E)/(1 - E)| = 3.54691.
\]

It will suffice to show that for \(t \in (1, \infty)\) the value of \(|E| \in [0.56, 1.0]\) and \(\arg(E) \in [-0.075, +0.075]\), since \(|\zeta| = |(1 + E)/(1 - E)|\) will then be greater than \((3 + \sqrt{5})/2 = 2.618\ldots\), the maximum permitted by our Yoccoz inequality.

Squaring \(E\) gives a formula for \(E^2\) as a rational function of \(t\):
\[
E^2 = \frac{b^2(b - 4)}{(b^2 - 2b - 2)^2} = \frac{-it^3(it - 4)}{(-t^2 - 2it - 2)^2} = \frac{(t^8 + 16t^6 + 36t^4) - i(8t^5 - 16t^3)}{t^8 + 16t^6 + 72t^4 + 64t^2 + 16}.
\]
It is now easily shown that \(|E|\) increases monotonically for \(t \in [1, \infty)\), tending to the value of 1 as \(t \to \infty\), and that \(\arg(E)|\) has its maximum value on \(t \in [1, \infty)\) at \(t = 1\) (in fact, as \(t\) increases from 1, \(\arg(E)\) decreases to a minimum value of about \(-0.0388\) and then increases, tending to 0 as \(t \to \infty\): it passes through 0 at \(t = \sqrt{2},\) which corresponds to \(a = 3 + 2\sqrt{2}i\)). This completes the proof of Theorem 3.
Remark 6. The bound $(3 + \sqrt{5})/2$ on the modulus of the derivative at the fixed point (established in Corollary 8 in Section 6) is sharp. At the parameter value $a = 4$ the correspondence $F_a$ is a mating of $z \to z^2 - 2$ with $\text{PSL}(2, \mathbb{Z})$ and so the limit set is connected, indeed it is an interval. When $a = 4$, the fixed point $z_0$ is $-1/\sqrt{5}$ and our formula for the derivative $\zeta$ of $F_4$ at $z_0$ gives $\zeta = -(3 + \sqrt{5})/2$. We remark that the classical Yoccoz inequality gives a bound of 2 for the modulus of the derivative of $z \to z^2 + c$ at the $\alpha$-fixed-point, and that this bound is achieved for $z \to z^2 - 2$ at its fixed point $z = -1$.

Proof of Corollary [1]. As in our proof of Theorem 3 above, we work in terms of the parameter $b = (a - 7)/(a - 1)$ in place of $a$. In the proof above we established a formula for the multiplier at the alpha-fixed-point, namely:

$$\zeta(b) = \frac{1 + E}{1 - E}$$

where

$$E = \frac{b(4 - b)}{2 + 2b - b^2} \sqrt{\frac{b}{b - 4}},$$

and so for $b$ close to 0, we have that $E$ is close to zero and

$$\zeta(b) \sim 1 + 2E \sim 1 + ib^{3/2},$$

where we have neglected all powers of $b$ other than the leading term. Thus for values of $b$ close to the origin, the transformation from the $b$-plane to the $(\log \zeta)$-plane is given by:

$$\log \zeta \sim ib^{3/2},$$

and in the opposite direction it is given by

$$b \sim (i \log \zeta)^{2/3},$$

that is,

$$a \sim 7 + 6 (i \log \zeta)^{2/3},$$

and the statement of the Corollary follows. □

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