Impulsive quantum \((p, q)\)-difference equations

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Abstract

In this paper we study quantum \((p, q)\)-difference equations with impulse and initial or boundary conditions. We consider first order impulsive \((p, q)\)-difference boundary value problems and second order impulsive \((p, q)\)-difference initial value problems. Existence and uniqueness results are proved via Banach's fixed point theorem.

MSC: 05A30; 39A13; 34A37

Keywords: Quantum \((p, q)\)-difference equations; Impulsive differential equations; Existence; Uniqueness

1 Introduction and preliminaries

Let \(p, q\) be quantum constants satisfying \(0 < q < p \leq 1\). The \((p, q)\)-number, \([n]_{p,q}\), is defined by

\[ [n]_{p,q} = \frac{p^n - q^n}{p - q}. \]

If \(n\) is a positive integer, then

\[ [n]_{p,q} = p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1} \quad \text{and} \quad \lim_{(p,q)\to(1,1)} [n]_{p,q} = n. \]

The \((p, q)\)-difference of a function \(f\) on \([0, \infty)\) is defined by

\[ D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p - q)t}, \quad t \neq 0, \]

and \(D_{p,q}f(0) = f'(0)\). If \(f(t) = t^\alpha, \alpha \geq 0\), then we have

\[ D_{p,q}t^\alpha = [\alpha]_{p,q}t^{\alpha-1}. \]

Note that if the function \(f\) is defined on \([0, T]\), then the function \(D_{p,q}f(t)\) is defined on \([0, T/p]\). For some details of the shifting property and nonlocal boundary value problems
for first-order \((p, q)\)-difference equations, we refer the reader to [1]. In addition, in [2], the authors defined the second-order \((p, q)\)-difference by

\[
D_{pq}^2 f(t) = \frac{qf(p^2 t) - (p + q)f(pqt) + pf(q^2 t)}{pq(p - q)^2 t^2}.
\]

Then we see that if \(f(t)\) is defined on \([0, T]\) then the function \(D_{pq}^2 f(t)\) is defined on \([0, T/p^2]\).

The \((p, q)\)-integral of a function \(f\) on \([0, \infty)\) is defined by

\[
\int_0^t f(s) \, ds_{pq} = (p - q) t \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} t\right).
\]

If \(f(t) = t^\alpha, \alpha > 0\), then we have the formula

\[
\int_0^t s^\alpha \, ds_{pq} = \frac{p - q}{p^{\alpha+1} - q^{\alpha+1}} t^{\alpha+1}.
\]

Now we observe that if the function \(f\) is defined on a finite interval \([0, T]\) then the function \(\int_0^t f(s) \, ds_{pq}\) is defined on \([0, pT]\). In [1], the authors gave the formula of the double \((p, q)\)-integral

\[
\int_0^t \int_0^s f(r) \, dr_{pq} \, ds_{pq} = \frac{1}{p} \int_0^t (t - qs) f\left(\frac{1}{p} s\right) \, ds_{pq}
\]

\[
= \frac{1}{p} (p - q) t^2 \sum_{n=0}^\infty \frac{q^n}{p^{n+1} - q^{n+1}} f\left(\frac{q^n}{p^{n+1}} t\right),
\]

which implies that if \(f\) is defined on \([0, T]\), then the function \(\int_0^t \int_0^s f(r) \, dr_{pq} \, ds_{pq}\) is defined on \([0, p^2 T]\).

The \((p, q)\)-calculus was introduced in [3]. For some recent results, see [4–10] and references cited therein. For \(p = 1\), the \((p, q)\)-calculus is reduced to the classical \(q\)-calculus initiated by Jackson [11, 12]. See also [13, 14].

In [15, 16], M. Tunç and E. Göv defined the quantum \((p, q)\)-difference of a function \(f\) on the finite interval \([a, b]\) by

\[
aD_{pq} f(t) = \frac{f(pt + (1 - p)a) - f(qt + (1 - q)a)}{(p - q)(t - a)}, \quad t \neq a,
\]

and \(aD_{pq} f(a) = f'(a)\). The \((p, q)\)-difference of a power function \(f(t) = (t - a)^\alpha, \alpha \geq 0\), is given by

\[
D_{pq} (t - a)^\alpha = [a]_{pq} (t - a)^{\alpha-1}.
\]

Furthermore, they defined the \((p, q)\)-integral of a function \(f\) on \([a, b]\) as

\[
\int_a^t f(s) \, ds_{pq} = (p - q)(t - a) \sum_{n=0}^\infty \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} t + \left(1 - \frac{q^n}{p^{n+1}}\right) a\right).
\]
As is customary, we put the following relation:
\[
\int_{p}^{t} (s - a)^{\alpha} a_{D_{p,q}} s = \frac{p - q}{(p^{a+1} - q^{a+1})} (t - a)^{\alpha + 1}, \quad \alpha \geq 0.
\]
(1.8)

It is obvious that if \( a = 0 \), then equations (1.5)–(1.8) are reduced to (1.1)–(1.4), respectively.

The domain-shift properties of the \((p,q)\)-difference and \((p,q)\)-integral operators for a function \( f(t), t \in [a,b] \) are respectively given by
\[
a_{D_{p,q}} f(t), \quad t \in \left[ a, \frac{1}{p}(b - a) + a \right] \quad \text{and} \quad \int_{a}^{t} f(s) a_{D_{p,q}} s, \quad t \in \left[ a, p(b - a) + a \right].
\]

Also we remark that if \( p = 1 \), then both domains are reduced to \([a,b]\). For the shifting of the second order \((p,q)\)-difference and integral domains, we consider the following result.

**Lemma 1.1** Let \( f \) be a function defined on an interval \([a,b]\) with \( a \geq 0 \). The domains of \( a_{D_{p,q}}^{2} \) and \( \int_{a}^{t} \int_{a}^{t} f(s) a_{D_{p,q}}^{2} a_{D_{p,q}} f \) are
\[
\left[ a, \frac{1}{p^{2}}(b - a) + a \right] \quad \text{and} \quad \left[ a, p^{2}(b - a) + a \right],
\]
respectively.

**Proof** We have
\[
a_{D_{p,q}}^{2} f(t) = a_{D_{p,q}}(a_{D_{p,q}} f)(t) = a_{D_{p,q}} \left( \frac{f(pt + (1 - p)a) - f(qt + (1 - q)a)}{(p - q)(t - a)} \right)
\]
\[
= \left\{ \frac{f(p(pt + (1 - p)a) + (1 - p)a) - f(q(pt + (1 - p)a) + (1 - q)a)}{(p - q)((pt + (1 - p)a) - a)} - \frac{f(p(qt + (1 - q)a) + (1 - p)a) - f(q(qt + (1 - q)a) + (1 - q)a)}{(p - q)((qt + (1 - q)a) - a)} \right\}
\]
\[
= \frac{qf(p^{2}t + (1 - p^{2})a) - (p + q)f(pt + (1 - pq)a) + pf(q^{2}t + (1 - q^{2})a)}{pq(p - q)^{2}(t - a)^{2}}.
\]

Setting \( p^{2}t + (1 - p^{2})a = b \), we have
\[
t = \frac{1}{p^{2}}(b - a) + a.
\]

Then \( a_{D_{p,q}}^{2} f \) is defined on \([a, (b - a)/p^{2} + a]\).

Next we write the double \((p,q)\)-integral in the form of an infinite sum of a function \( f \) defined on \([a,b]\). We have
\[
\int_{a}^{t} \int_{a}^{t} f(r) a_{D_{p,q}}^{2} a_{D_{p,q}} s = \int_{a}^{t} (p - q)(s - a) \sum_{n=0}^{\infty} \frac{q^{n}}{p^{n+1}} f \left( \frac{q^{n}}{p^{n+1}} s + \left\{ 1 - \frac{q^{n}}{p^{n+1}} \right\} a \right) a_{D_{p,q}} s
\]
\[
= (p - q) \sum_{n=0}^{\infty} \frac{q^{n}}{p^{n+1}} \left[ \int_{a}^{t} (s - a) f \left( \frac{q^{n}}{p^{n+1}} s + \left\{ 1 - \frac{q^{n}}{p^{n+1}} \right\} a \right) a_{D_{p,q}} s \right].
\]
Now we consider
\[
\int_a^t (s-a)f \left( \frac{q^n}{p^{n+1}} s + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a \right) d_{p,q} s
\]
\[
= (p-q)(t-a) \sum_{m=0}^\infty \frac{q^m}{p^{m+1}} \left( \frac{q^n}{p^{n+1}} t + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a - a \right)
\]
\[
\times f \left( \frac{q^n}{p^{n+1}} \left[ \frac{q^n}{p^{n+1}} t + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a \right] \right)
\]
\[
= (p-q)(t-a)^2 \sum_{m=0}^\infty \frac{q^{2m}}{p^{2m+2}} f \left( \frac{q^{m+n}}{p^{m+n+2}} t + \left\{ 1 - \frac{q^{m+n}}{p^{m+n+2}} \right\} a \right),
\]
which leads to the expression
\[
\int_a^t \int_a^s f(r)d_{p,q} r d_{p,q} s
\]
\[
= (p-q)^2(t-a)^2 \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{q^{2m+n}}{p^{2m+n+3}} f \left( \frac{q^{m+n}}{p^{m+n+2}} t + \left\{ 1 - \frac{q^{m+n}}{p^{m+n+2}} \right\} a \right). \tag{1.9}
\]

For \( m = n = 0 \) and setting
\[
\frac{1}{p^2} t + \left\{ 1 - \frac{1}{p^2} \right\} a = b,
\]
we obtain \( t = p^2(b - a) + a \), which implies that \( \int_a^t \int_a^s f(s)d_{p,q} s d_{p,q} r \) is valid on \([a, p^2(b - a) + a]\). The proof is completed. \( \square \)

Before going to the next result, we would like to recall the operator \( a \Phi_r \) defined by
\[
a \Phi_r(m) = rm + (1 - r)a,
\]
where \( m, a \in \mathbb{R} \) and \( r \in [0, 1] \). Some properties of this operator can be found in [17].

**Lemma 1.2** Let \( f \) be a function defined on \([a, b]\). Then the double \((p, q)\)-integral of \( f \) can be written as a single one by
\[
\int_a^t \int_a^s f(r)d_{p,q} r d_{p,q} s = \frac{1}{p} \int_a^t (t - a \Phi_q(s))f(a \Phi_q(s)) d_{p,q} s, \quad t \in [a, p^2(b - a) + a]. \tag{1.10}
\]

**Proof** The double summation in (1.9) can be formulated by a single summation as
\[
\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{q^{2m+n}}{p^{2m+n+3}} f \left( \frac{q^{m+n}}{p^{m+n+2}} t + \left\{ 1 - \frac{q^{m+n}}{p^{m+n+2}} \right\} a \right)
\]
\[
= \sum_{n=0}^\infty \frac{q^n}{p^{n+3}} f \left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right) + \frac{q^{n+2}}{p^{n+5}} f \left( \frac{q^{n+1}}{p^{n+3}} t + \left\{ 1 - \frac{q^{n+1}}{p^{n+3}} \right\} a \right)
\]
\[ + \frac{q^{n+4}}{p^{n+5}} f\left(\frac{q^{n+2}}{p^{n+4}} t + \left\{ 1 - \frac{q^{n+2}}{p^{n+4}} \right\} a \right) + \frac{q^{n+6}}{p^{n+8}} f\left(\frac{q^{n+3}}{p^{n+5}} t + \left\{ 1 - \frac{q^{n+3}}{p^{n+5}} \right\} a \right) + \cdots \]
\[
= \frac{1}{p^n} f\left( \frac{1}{p^2} t + \left\{ 1 - \frac{1}{p^2} \right\} a \right) + \frac{q}{p^n} \left( 1 + \frac{q}{p} \right) f\left( \frac{q}{p^3} t + \left\{ 1 - \frac{q}{p^3} \right\} a \right) \\
+ \frac{q^2}{p^n} \left( 1 + \frac{q}{p} + \frac{q^2}{p^2} \right) f\left( \frac{q}{p^4} t + \left\{ 1 - \frac{q}{p^4} \right\} a \right) + \cdots \\
= \sum_{n=0}^{\infty} \frac{q^n}{p^{n+3}} \left( \frac{p^{n+1} - q^{n+1}}{p^n(p - q)} \right) f\left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right)
\]
\[
= \frac{1}{p - q} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( 1 - \frac{q^{n+1}}{p^{n+2}} \right) f\left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right)
\]

Substituting into (1.9) yields
\[
\int_a^t \int_a^s f(r) s_{d_{p,q} r} d_{p,q} s
\]
\[
= \frac{1}{p} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( t - \left\{ \frac{q^n}{p^{n+1}} t + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a \right\} \right)
\]
\[
\times f\left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right)
\]
\[
= \frac{1}{p} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( t - \Phi_a \left( \left\{ \frac{q^n}{p^{n+1}} t + \left\{ 1 - \frac{q^n}{p^{n+1}} \right\} a \right\} \right) \right)
\]
\[
\times f\left( \Phi_a \left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right) \right)
\]
\[
= \frac{1}{p} \int_a^t \left( t - \Phi_a(s) \right) f\left( \Phi_a \left( \frac{q^n}{p^{n+2}} t + \left\{ 1 - \frac{q^n}{p^{n+2}} \right\} a \right) \right) d_{p,q} s,
\]

which is completed the proof. \( \square \)

**Remark 1.3** If \( a = 0 \), then (1.10) is reduced to a result of Theorem 3 in [1].

The following theorem has been proved in [16].

**Theorem 1.4** The fundamental relations of \((p, q)\)-calculus can be stated as
\begin{enumerate}
  \item \( s_{d_{p,q} s} f(s) d_{p,q} s = f(t) \);
  \item \( f(a) d_{p,q} s = f(t) - f(a) \).
\end{enumerate}

In this paper we study the impulsive \((p, q)\)-difference equations with initial and boundary conditions. We consider four types of problems, two impulsive \((p, q)\)-difference equations of type I and two impulsive \((p, q)\)-difference equations of type II (explained in the next section). Existence and uniqueness results are proved via Banach's contraction mapping principle. Examples illustrating the obtained results are also constructed.

2 **Impulsive \((p, q)\)-difference equations**

In this section, we consider the first and second order \((p, q)\)-difference equations with initial or boundary conditions and also prove the existence and uniqueness of solutions.
for impulsive problems. Firstly, let $t_k, k = 1, \ldots, m$, be the impulsive points such that $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, $J_0 = [0, t_1]$ be the intervals such that $\bigcup_{k=0}^{m} J_k = [0, T] := J$. The investigations are based on $(p, q)$-calculus introduced in the previous section by replacing a point $a$ by $t_k$, quantum numbers $p$ by $p_k$ and $q$ by $q_k$, $k = 0, 1, \ldots, m$, and also applying the $(p_k, q_k)$-difference and $(p_k, q_k)$-integral operators only on a finite subinterval of $J$. In addition, the consecutive subintervals can be related with jump conditions which provide a meaning of quantum difference equations with impulse effects. There are two types of impulsive problems which will be established in the next two subsections. The consecutive domains of impulsive $(p, q)$-difference equations of type I are overlapped, while the unknown functions of impulsive equations of type II are defined on disconnected consecutive domains.

2.1 Impulsive $(p, q)$-difference equations of type I
Consider the first-order impulsive $(p, q)$-difference impulsive boundary value problem of the form

$$\begin{align*}
\Delta x(t_k) &= \varphi_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\
\alpha x(0) + \beta x(T) &= \gamma,
\end{align*}$$

(2.1)

where $\alpha, \beta$, and $\gamma$ are real constants with $\alpha \neq -\beta$, the quantum numbers $p_k, q_k$ satisfy $0 < q_k < p_k \leq 1$, $k = 0, 1, \ldots, m$, $f : [0, ((T - t_m)/p_m) + t_m] \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, are given functions, and $t_k D_{p_k,q_k}$ is the quantum $(p_k, q_k)$-difference operator starting at a point $t_k, k = 0, 1, \ldots, m$.

We remark that there are some overlapped intervals of domains of the first equation in (2.1). For example, if the unknown function $x(t)$ is defined on $J = [0, 2]$ and if there is an impulse point $t_1 = 1$, that is, $x(1^+) \neq x(1^-)$, with $p_0 = 1/2, q_0 = 1/3, p_1 = 1/4$, and $q_1 = 1/5$. Then we have the $(p, q)$-difference equations

$$\begin{align*}
0 D_{1/2,1/3} x(t) &= f(t, x(t)), \quad t \in (0, 2] \\
1 D_{1/4,1/5} x(t) &= f(t, x(t)), \quad t \in (1, 2].
\end{align*}$$

However, by the shifting property of $(p, q)$-integration applied to the two above equations, we have

$$x(t) = x(0) + \int_0^t f(s, x(s)) \, d_{1/2,1/3} s, \quad t \in (0, 1],$$

and

$$x(t) = x(1^-) + \int_1^t f(s, x(s)) \, d_{1/4,1/5} s, \quad t \in (1, 2],$$

respectively.
Theorem 2.1  The nonlinear first-order \((p,q)\)-difference boundary value problem (2.1) can be transformed into an integral equation

\[
x(t) = \frac{\gamma}{(\alpha + \beta)} - \frac{\beta}{(\alpha + \beta)} \left( \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f(s,x(s)) t_i d_{p_i,q_i} s + \sum_{j=1}^{m} \psi_j(x(t_j)) \right)
\]

\[
+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(s,x(s)) t_i d_{p_i,q_i} s + \sum_{j=1}^{k} \psi_j(x(t_j)) + \int_{t_k}^{t} f(s,x(s)) t_i d_{p_i,q_i} s, \quad t \in J, \quad (2.2)
\]

with \(\sum_{a}^{b} f(\cdot) = 0\), if \(b < a\).

Proof  From \(t_0 D_{p_0,q_0} x(t) = f(t,x(t)), t \in (t_0, (1/p_0)(t_1-t_0) + t_0]\), by taking the \((p_0,q_0)\)-integral, we obtain

\[
x(t) = x(0) + \int_{t_0}^{t} f(s,x(s)) t_0 d_{p_0,q_0} s, \quad t \in (t_0, t_1],
\]

by using Theorem 1.4 and the shifting property. Next, for \(t_1 D_{p_1,q_1} x(t) = f(t,x(t)), t \in (t_1, (1/p_1)(t_2-t_1) + t_1], \) where \(t_1\) is the first impulsive point in \(J\), we also obtain by applying the \((p_1,q_1)\)-integration,

\[
x(t) = x(t_1^+) + \int_{t_1}^{t} f(s,x(s)) t_1 d_{p_1,q_1} s, \quad t \in (t_1, t_2].
\]

By the impulsive condition \(x(t_1^+) = x(t_1) + \varphi_1(x(t_1)), \) it follows, for \(t \in (t_1, t_2]\), that

\[
x(t) = x(0) + \int_{t_0}^{t} f(s,x(s)) t_0 d_{p_0,q_0} s + \varphi_1(x(t_1)) + \int_{t_1}^{t} f(s,x(s)) t_1 d_{p_1,q_1} s.
\]

For \(t_2 D_{p_2,q_2} x(t) = f(t,x(t)), t \in (t_2, (1/p_2)(t_3-t_2) + t_2], \) we get

\[
x(t) = x(t_2^+) + \int_{t_2}^{t} f(s,x(s)) t_2 d_{p_2,q_2} s, \quad t \in (t_2, t_3],
\]

by \((p_2,q_2)\)-integration and

\[
x(t) = x(0) + \int_{t_0}^{t_1} f(s,x(s)) t_0 d_{p_0,q_0} s + \int_{t_1}^{t_2} f(s,x(s)) t_1 d_{p_1,q_1} s
\]

\[
+ \varphi_1(x(t_1)) + \varphi_2(x(t_2)) + \int_{t_2}^{t} f(s,x(s)) t_2 d_{p_2,q_2} s, \quad t \in (t_2, t_3],
\]

due to the impulsive condition \(x(t_2^+) = x(t_2) + \varphi_2(x(t_2)).\)

Repeating this process, we obtain, for \(t \in J_k, k = 0, 1, \ldots, m, \) that

\[
x(t) = x(0) + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(s,x(s)) t_i d_{p_i,q_i} s + \sum_{j=1}^{k} \varphi_j(x(t_j)) + \int_{t_k}^{t} f(s,x(s)) t_k d_{p_k,q_k} s. \quad (2.3)
\]
After that from the boundary condition \( \alpha x(0) + \beta x(T) = \gamma \), we have

\[
x(0) = \frac{\gamma}{(\alpha + \beta)} - \frac{\beta}{(\alpha + \beta)} \left( \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f(s, x(s)) t_i d_{p_i,q_i} s + \sum_{j=1}^{m} \phi_j(x(t_j)) \right).
\]

Putting the value of \( x(0) \) into (2.3), shows that (2.2) is true and the proof is completed. \( \square \)

**Remark 2.2** If \( \alpha \neq 0 \) and \( \beta = 0 \), then the boundary value problem (2.1) can be reduced to the initial value problem with initial condition \( x(0) = \gamma / \alpha \).

Before going to the second-order impulsive problem, we define

\[
\tau_k = \frac{1}{p_k-1} (t_k - t_{k-1}) + t_{k-1}, \quad k = 1, 2, \ldots, m,
\]

which are impulsive shifting points of the \((p_k,q_k)\)-derivative of the unknown function in our system. In addition, we introduce a notation

\[
\langle t_{i+1} \rangle_k = \begin{cases} 
  t_{i+1}, & t_{i+1} \leq t_k, \\
  t, & t_{i+1} > t_k.
\end{cases}
\]

For example,

\[
\sum_{i=0}^{2} (t_{i+1} - t_i) K_i = (t_1)_2 - t_0)K_0 + (t_2)_2 - t_2)K_1 + (t_3)_2 - t_2)K_2
\]

\[
= (t_1 - t_0)K_0 + (t_2 - t_1)K_1 + (t - t_2)K_2,
\]

where \( K_i \in \mathbb{R}, i = 0, 1, 2. \)

Now, we consider the second-order impulsive \((p,q)\)-difference initial value problem of the form

\[
\begin{cases}
  t_k D_{p_k,q_k}^2 x(t) = f(t, x(t)), & t \in (t_k, \frac{1}{p_k} (t_{k+1} - t_k) + t_k), \quad k = 0, 1, \ldots, m, \\
  \Delta x(t_k) = \phi_k(x(t_k)), & k = 1, 2, \ldots, m, \\
  t_k D_{p_k,q_k} x(t) - q_{k-1} D_{p_{k-1},q_{k-1}} x(t_k) = \phi^*_k(x(t_k)), & k = 1, 2, \ldots, m, \\
  x(0) = \lambda_1, & t_0 D_{p_0,q_0} x(0) = \lambda_2,
\end{cases}
\]

where \( f : [0, (T - t_m)/p_m^2 + t_m] \times \mathbb{R} \to \mathbb{R}, \phi_k : \mathbb{R} \to \mathbb{R} \) and \( \phi^*_k : \mathbb{R} \to \mathbb{R} \), are given functions, \( \lambda_1, \lambda_2 \) are given constants. Observe that the distance between the impulsive points \( t_k \) and \( \tau_k \) in the third equation of (2.4) depends on the value of \( p_k \) for \( k = 1, 2, \ldots, m \). Indeed,

\[
\tau_k - t_k = \frac{1}{p_k - 1} (t_k - t_{k-1}) + t_{k-1} - t_k = \frac{1 - p_{k-1}}{p_k - 1} (t_k - t_{k-1}),
\]

which has appeared by the shifting property of \((p,q)\)-calculus as discussed in the previous section.
Theorem 2.3  The impulsive initial value problem of type I given by the \((p, q)\)-difference equation (2.4) can be expressed as an integral equation of the form

\[
x(t) = \lambda_1 + \sum_{i=0}^{k-1} ((t_{i+1} - t_i) t_i \left[ \lambda_2 + \sum_{j=0}^{i-1} \left[ \int_{t_j}^{t_{j+1}} \int f(s, x(s)) \ nu_i d_p q_i s + \varphi_{j+1}^*(x(t_{j+1})) \right] \right] \\
+ \sum_{i=0}^{k-1} \left[ \frac{1}{p_r} \int_{t_i}^{t_{i+1}} (t_{i+1} - t_i) \Phi_{q_i} (s) f(\nu_i, \Phi_{p_i} (s)) \nu_i d_p q_i s + \varphi_{i+1}^*(x(t_{i+1})) \right] \\
+ \frac{1}{p_k} \int_{t_k}^{t} (t - t_k) \Phi_{q_k} (s) f(\nu_k, \Phi_{p_k} (s)) \nu_k d_p q_k s, \quad t \in I_k, k = 0, 1, \ldots, m,
\]

where \(f(\nu_i, \Phi_{p_i} (s)) = f(\nu_i, \Phi_{q_i} (s), x(\nu_i, \Phi_{p_i} (s))), r = 0, 1, \ldots, k, \) and \(\sum_{i=0}^{k} \) is 0, when \(b < a\).

Proof  By computing the \((p_0, q_0)\)-integral of both sides of the first equation of (2.4), we get

\[
t_0 D_{p_0, q_0} x(t) = t_0 D_{p_0, q_0} x(0) + \int_{t_0}^{t} f(s, x(s)) d_p q_0 s, \quad t \in \left(0, \frac{1}{p_0} t_0 \right].
\]

Applying another \((p_0, q_0)\)-integration, we obtain, for \(t \in (0, t_1]\),

\[
x(t) = x(0) + t_0 D_{p_0, q_0} x(0) + \int_{t_0}^{t} \int_{t_0}^{r} f(s, x(s)) d_p q_0 s d_p q_0 r
\]

\[
= \lambda_1 + \lambda_2 t + \frac{1}{p_0} \int_{t_0}^{t} (t - t_0 \Phi_{q_0} (s)) f(\nu_0, \Phi_{p_0} (s)) \nu_0 d_p q_0 s.
\]

For \(t \in (t_1, ((t_2 - t_1)/p_1^2) + t_1\], applying the double \((p_1, q_1)\)-integration to both sides of the first equation of (2.4), we have

\[
x(t) = x(t_1^+) + (t - t_1) t_1 D_{p_1, q_1} x(t_1^+) + \frac{1}{p_1} \int_{t_1}^{t} (t - t_1 \Phi_{q_1} (s)) f(\nu_1, \Phi_{p_1} (s)) \nu_1 d_p q_1 s,
\]

where \(t \in (t_1, t_2].\) Due to the impulsive conditions

\[
x(t_1^+) = x(t_1) + \varphi_{1}^*(x(t_1))
\]

\[
= \lambda_1 + \lambda_2 t_1 + \frac{1}{p_0} \int_{t_0}^{t_1} (t_1 - t_0 \Phi_{q_0} (s)) f(\nu_0, \Phi_{p_0} (s)) \nu_0 d_p q_0 s + \varphi_{1}^*(x(t_1))
\]

and

\[
t_1 D_{p_1, q_1} x(t_1^+) = t_0 D_{p_0, q_0} x(\tau_1) + \varphi_{1}^*(x(t_1))
\]

\[
= \lambda_2 + \int_{t_0}^{t_1} f(s, x(s)) d_p q_0 s + \varphi_{1}^*(x(t_1)),
\]
we have

\[ x(t) = \lambda_1 + \lambda_2 t_1 + \frac{1}{p_0} \int_{t_0}^{t_1} \left( t_1 - t_0 \phi_{q_0}(s) \right) f_{x} \left( \phi_{p_0}(s) \right) \phi_{q_0} \, ds + \phi_1(x(t_1)) \]

\[ + (t - t_1) \left[ \lambda_2 + \frac{1}{p_1} \int_{t_1}^{t} \left( t - t_1 \phi_{r_1}(s) \right) f_{x} \left( \phi_{p_1}(s) \right) \phi_{r_1} \, ds + \phi_1(x(t_1)) \right] \]

\[ + \frac{1}{p_1} \int_{t_1}^{t} \left( t - t_1 \phi_{r_1}(s) \right) f_{x} \left( \phi_{p_1}(s) \right) \phi_{r_1} \, ds, \quad t \in (t_1, t_2]. \]

Similarly, we deduce the integral equation (2.5), as desired. \[ \square \]

Now, the existence and uniqueness results for problems (2.1) and (2.4) will be proved by using the Banach’s contraction mapping principle. Let us define the space \( PC(J, \mathbb{R}) = \{ x : J \to \mathbb{R} : x(t) \) is continuous everywhere except for some \( t_k \) in which \( x(t_k^+) \) and \( x(t_k^-) \) exist and \( x(t_k^+) = x(t_k^-) \}, k = 1, 2, \ldots, m \}. \) The set \( PC(J, \mathbb{R}) \) is a Banach space equipped with the norm \( \| x \| = \sup \{ |x(t)| : t \in J \} \). For convenience, we put

\[ \Omega_1 = \frac{\beta}{\alpha + \beta} \sum_{i=0}^{m} (t_{i+1} - t_i), \]

\[ \Omega_2 = m \left( \frac{\beta}{\alpha + \beta} \right), \]

\[ \Omega_3 = \sum_{i=0}^{m} \left( t_{i+1} - t_i \right) \sum_{j=0}^{i-1} (t_{j+1} - t_j) \]

\[ + \sum_{r=0}^{m} \frac{(t_{r+1} - t_r)^2}{p_r + q_r}, \]

\[ \Omega_4 = \sum_{i=0}^{m} (t_{i+1} - t_i) i. \]

**Theorem 2.4** Let \( f : [0, ((T - t_m)/p_m) + t_m] \times \mathbb{R} \to \mathbb{R} \) and \( \varphi_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, \ldots, m \) be given functions satisfying

(H1) There exist positive constants \( L_1 \) and \( L_2 \) such that

\[ |f(t, x) - f(t, y)| \leq L_1 |x - y| \quad \text{and} \quad |\varphi_k(x) - \varphi_k(y)| \leq L_2 |x - y|, \]

for all \( t \in [0, ((T - t_m)/p_m) + t_m], x, y \in \mathbb{R} \) and \( k = 1, 2, \ldots, m \).

If

\[ L_1 \Omega_1 + L_2 \Omega_2 < 1, \quad \text{(2.6)} \]

then the boundary value problem (2.1) has a unique solution on \( J \).

**Proof** In view of Theorem 2.1, we define the operator \( A : PC(J, \mathbb{R}) \to PC(J, \mathbb{R}) \) by

\[ Ax(t) = \frac{\gamma}{(\alpha + \beta)} - \frac{\beta}{(\alpha + \beta)} \left( \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f(s, x(s)) \, ds + \sum_{j=1}^{m} \phi_j(x(t_j)) \right) \]

\[ + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(s, x(s)) \, ds + \sum_{j=1}^{k} \phi_j(x(t_j)) + \int_{t_k}^{t} f(s, x(s)) \, ds, \quad t \in J. \]
Define the ball $B_{r_1} = \{ x \in PC(I, \mathbb{R}) : \| x \| \leq r_1 \}$ where the positive constant $r_1$ is defined by

$$r_1 = \frac{|\gamma|\|\alpha + \beta\| + M_1 \Omega_1 + M_2 \Omega_2}{1 - (L_1 \Omega_1 + L_2 \Omega_2)}.$$

The Banach contraction mapping principle is used to claim that there exists a unique fixed point of the operator equation $x = Ax$ in $B_{r_1}$. By setting $\text{sup}_{t \in I} |f(t,0)| = M_1$, and $\text{sup}_{i} |\phi_i(0)| = M_2$ and using the inequalities $|f(t,x)| \leq |f(t,x) - f(t,0)| + |f(t,0)| \leq L_1 r_1 + M_1$ and $|\phi_i(x)| \leq |\phi_i(x) - \phi_i(0)| + |\phi_i(0)| \leq L_1 r_1 + M_2$, $i = 1, 2, \ldots, m$, we have

$$|Ax(t)| \leq \frac{|\gamma|}{|\alpha + \beta|} + \frac{|\beta|}{|\alpha + \beta|} \left( \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s,x(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))| \right)$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s,x(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))| + \int_{t_k}^{t} |f(s,x(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))|$$

$$\leq \frac{|\gamma|}{|\alpha + \beta|} + \frac{|\beta|}{|\alpha + \beta|} \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (L_1 r_1 + M_1) \, ds, \sum_{j=1}^{m} |L_2 r_1 + M_2| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))| \right)$$

$$+ \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (L_1 r_1 + M_1) \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))| + \int_{t_m}^{t} \Omega_1 r_1 + M_1 \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i))|$$

$$= \frac{|\gamma|}{|\alpha + \beta|} + L_1 \Omega_1 r_1 + L_2 \Omega_2 r_1 + M_1 \Omega_1 + M_2 \Omega_2 < r_1,$$

which leads to $AB_{r_1} \subset B_{r_1}$. To prove that $A$ is a contraction, we let $x, y \in B_{r_1}$. Then we have

$$|Ax(t) - Ay(t)|$$

$$\leq \frac{|\beta|}{|\alpha + \beta|} \left( \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} |f(s,x(s)) - f(s,y(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i)) - \phi_j(y(t_i))| \right)$$

$$+ \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s,x(s)) - f(s,y(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i)) - \phi_j(y(t_i))| + \int_{t_k}^{t} |f(s,x(s)) - f(s,y(s))| \, ds, \sum_{j=1}^{m} |\phi_j(x(t_i)) - \phi_j(y(t_i))|$$

$$\leq \frac{|\beta|}{|\alpha + \beta|} \left( L_1 \|x - y\| \sum_{i=0}^{m} (t_{i+1} - t_i) + mL_2 \|x - y\| \right).$$
\begin{align*}
+ L_1 \|x - y\| & \sum_{i=0}^{m} (t_{i+1} - t_i) + mL_2 \|x - y\| \\
= (L_1 \Omega_1 + L_2 \Omega_2) \|x - y\|. \\
\end{align*}

Therefore, \( \|Ax - Ay\| \leq (L_1 \Omega_1 + L_2 \Omega_2) \|x - y\|. \) By means of the Banach contraction mapping principle, the operator \( A \) has a unique fixed point in \( B_n \) which is a unique solution of boundary value problem (2.1). The proof is completed. \( \square \)

**Theorem 2.5** Assume that the functions \( f : [0, ((T - t_m)/p_m^2) + t_m] \times \mathbb{R} \to \mathbb{R} \) and \( \varphi_k : \mathbb{R} \to \mathbb{R}, \ k = 1, 2, \ldots, m, \) satisfy \((H_1)\). In addition, we suppose that the functions \( \varphi^*_k : \mathbb{R} \to \mathbb{R}, \ k = 1, 2, \ldots, m, \) satisfy \((H_2)\).

\((H_2)\) There exists a positive constant \( L_3 \) such that

\[ |\varphi^*_k(x) - \varphi^*_k(y)| \leq L_3 |x - y|, \]

for all \( x, y \in \mathbb{R}. \)

If

\[ L_1 \Omega_1 + L_2 m + L_3 \Omega_4 < 1, \tag{2.7} \]

then the boundary value problem (2.4) has a unique solution on \([0, T]\).

**Proof** The proof is similar to that of Theorem 2.4 and is omitted. \( \square \)

**Example 2.6** Consider the following first-order impulsive quantum \((p, q)\)-difference equation of type \( I \) subject to the boundary condition of the form:

\[
\begin{align*}
&kD^{\frac{1}{k+1+2}} x(t) = \frac{1}{18 + t^2} \left( \frac{x^2 + 2|x|}{1 + |t|} \right) + \frac{3}{2}, \quad t \in (k, 2k + 2], k = 0, 1, 2, \\
&\Delta x(k) = \frac{1}{6k} \sin x(t_k), \quad k = 1, 2, \\
&\frac{1}{2} x(0) + \frac{1}{3} x(3) = \frac{1}{4} 
\end{align*}
\]

(2.8)

Here \( p_k = 1/(k + 2), \) \( q_k = 1/(k + 3), \) \( k = 0, 1, 2, \alpha = 1/2, \beta = 1/3, \gamma = 1/4, t_k = k, \ k = 1, 2, \)

\( T = 3, \) and \( m = 2. \) The given data leads to constants \( \Omega_1 = 21/5, \) \( \Omega_2 = 14/5. \) Setting

\[ f(t, x) = \frac{1}{18 + t^2} \left( \frac{x^2 + 2|x|}{1 + |t|} \right) + \frac{3}{2} \quad \text{and} \quad \varphi_k(x) = \frac{1}{6k} \sin x, \]

we have \( |f(t, x) - f(t, y)| \leq (1/9)|x - y| \) and \( |\varphi_k(x) - \varphi_k(y)| \leq (1/6)|x - y| \) which satisfy Condition \((H_1)\) in Theorem 2.4 with \( L_1 = 1/9 \) and \( L_2 = 1/6. \) Since \( L_1 \Omega_1 + L_2 \Omega_2 = 14/15 < 1, \) by

Theorem 2.4, the boundary value problem (2.8) has a unique solution \( x \) on \([0, 3].\)
Example 2.7 Consider the following second-order impulsive quantum (p,q)-difference equation of type I with the initial conditions of the form:

\[
\begin{align*}
\sum_{\tau} D^2 x(t) &= \frac{1}{5(t+5)} \tan^{-1}|x(t)| + \frac{1}{2}, & t \in (k, k^2 + 5k + 4], k = 0, 1, 2, \\
\Delta x(k) &= \frac{[x(q_k)]}{10k(1 + |q_k|)}, & k = 1, 2, \\
\sum_{\rho} D x(t) &= (k-1) \sum_{\rho} D x(t) = \frac{1}{15k} \sin |x(t_k)|, & k = 1, 2, \\
x(0) &= \frac{3}{5}, & 0D_{\frac{1}{2}, 1} x(0) = \frac{\sqrt{7}}{7}.
\end{align*}
\] (2.9)

Here the quantum constants \( p_k, q_k \) and impulsive points \( t_k \), are as in Example 2.6. In addition, \( t_k = 2k, k = 1, 2 \), and initial constants \( \lambda_1 = 3/5, \lambda_2 = 5/7 \). Next, we can compute that \( \Omega_3 = 12.1365 \) and \( \Omega_4 = 3 \). Set

\[
f(t, x) = \frac{1}{5(t+5)} \tan^{-1}|x| + \frac{1}{2}, \quad \varphi_k(x) = \frac{|x|}{10k(1 + |x|)}, \quad \text{and} \quad \varphi_k^*(x) = \frac{1}{15k^2} \sin |x|.
\]

It is easy to see that \( f, \varphi_k, \) and \( \varphi_k^* \) satisfy \( (H_1) \) and \( (H_2) \) with \( L_1 = 1/25, L_2 = 1/10, \) and \( L_3 = 1/15 \). Therefore, we have \( L_1\Omega_3 + L_2m + L_3\Omega_4 = 0.8855 < 1 \). Hence the boundary value problem (2.9) has a unique solution \( x \) on \([0, 3]\) by Theorem 2.5.

2.2 Impulsive (p,q)-difference equations of type II

Now we study the first-order impulsive (p,q)-difference boundary value problem of the form

\[
\begin{align*}
\sum_{\tau} D_{\tau, k} x(t) &= f(t, x(t)), & t \in (k, t_{k+1}], k = 0, 1, \ldots, m, \\
x(t_k^+) - x(t_k^-) &= \varphi_k(x(t_k)), & k = 1, 2, \ldots, m, \\
\alpha x(0) + \beta x(t_{m+1}) &= \gamma,
\end{align*}
\] (2.10)

where \( f: J \times \mathbb{R} \to \mathbb{R} \) and the functions \( \varphi_k, k = 1, 2, \ldots, m, \) and constants \( \alpha, \beta, \gamma \) are defined as in Sect. 2.1. The constant \( \rho_k \) is defined by

\[
\rho_k = p_{k-1}(t_k - t_{k-1}) + t_{k-1}, \quad k = 1, 2, \ldots, m, m + 1.
\]

Then the lagging distance is \( t_k - \rho_k = (1 - p_{k-1})(t_k - t_{k-1}) \) which depends on the value of \( p_{k-1} \in (0, 1) \).

To observe the special characteristic of this type, by the shifting property of the (p,q)-derivative, we see that the unknown function \( x(t) \) is defined on \([t_0, \rho_1] \cup (t_k, \rho_{k+1}], \) \( k = 1, 2, \ldots, m. \)

Example 2.8 Let \( J = [0, 2] \) and \( t_1 = 1 \) be an impulsive point. Then

\[
\sum_{\tau} D_{\frac{1}{2}, t} x(t) = f(t, x(t)), & t \in [0, 1],
\]

and

\[
\sum_{\rho} D_{\frac{1}{2}, t} x(t) = f(t, x(t)), & t \in (1, 2],
\]
can be presented as
\[ x(t) = x(0) + \int_0^t f(s, x(s)) \, d_{\frac{1}{2}, \frac{1}{2}} s, \quad t \in \left[ 0, \frac{1}{2} \right], \]
and
\[ x(t) = x(1^+) + \int_1^t f(s, x(s)) \, d_{\frac{1}{2}, \frac{1}{2}} s, \quad t \in \left( 1, \frac{5}{4} \right]. \]

**Theorem 2.9** The first-order type II \((p, q)\)-difference boundary value problem (2.10) can be expressed as an integral equation
\[
x(t) = \frac{\gamma}{(\alpha + \beta)} - \frac{\beta}{(\alpha + \beta)} \left( \sum_{i=0}^{\rho_1} \int_{t_i}^{\rho_{i+1}} f(s, x(s)) \, d_{p, q_1} s + \sum_{j=1}^{m} \varphi_j(x(\rho_j)) \right) \\
+ \sum_{i=0}^{k-1} \int_{t_i}^{\rho_{i+1}} f(s, x(s)) \, d_{p, q_1} s + \sum_{j=1}^{k} \varphi_j(x(\rho_j)) + \int_{t_k}^{t} f(s, x(s)) \, d_{p, q_1} s, 
\]
with \(\sum_{a}^{b} \cdot = 0\), if \(b < a\).

**Proof** Firstly, the \((p_0, q_0)\)-integration of the first equation in (2.10) yields
\[ x(t) = x(0) + \int_{t_0}^{t} f(s, x(s)) \, d_{p_0, q_0} s, \quad t \in (t_0, \rho_1]. \]
In particular, for \(t = \rho_1\), we have
\[ x(\rho_1) = x(0) + \int_{t_0}^{\rho_1} f(s, x(s)) \, d_{p_0, q_0} s. \]
For \(k = 1\), by \((p_1, q_1)\)-integration, we obtain
\[ x(t) = x(t_1^+) + \int_{t_1}^{t} f(s, x(s)) \, d_{p_1, q_1} s, \quad t \in (t_1, \rho_2], \]
which leads to
\[ x(t) = x(0) + \int_{t_0}^{\rho_1} f(s, x(s)) \, d_{p_0, q_0} s + \varphi_1(x(\rho_1)) + \int_{t_1}^{t} f(s, x(s)) \, d_{p_1, q_1} s, \]
by using the impulse condition \(x(t_1^+) = x(\rho_1) + \varphi_1(x(\rho_1)).\)
Repeating the process for any \(t \in (t_k, \rho_{k+1}]\), we get
\[ x(t) = x(0) + \sum_{i=0}^{k-1} \int_{t_i}^{\rho_{i+1}} f(s, x(s)) \, d_{p_i, q_i} s + \sum_{j=1}^{k} \varphi_j(x(\rho_j)) + \int_{t_k}^{t} f(s, x(s)) \, d_{p_k, q_k} s. \]
Since
\[ x(\rho_{m+1}) = x(0) + \sum_{i=0}^{m} \int_{t_i}^{\rho_{i+1}} f(s, x(s)) \, d_{p_i, q_i} s + \sum_{j=1}^{m} \varphi_j(x(\rho_j)), \]
by the boundary condition, we have

\[ x(0) = \frac{γ}{(α + β)} - \frac{β}{(α + β)} \left( \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds \right) + \sum_{j=1}^{m} \psi_j(x(\rho_j)), \]

which implies that (2.11) holds. This completes the proof. \( \square \)

Next we define the points \( \rho_k^* = \rho_{k-1}^2 (t_k - t_{k-1}) + t_{k-1}, \ k = 1, 2, \ldots, m, m + 1. \) Now we consider the second-order type II impulsive \((p, q)-\)difference initial value problem of the form

\[
\begin{align*}
&\left\{ t_k D_{p_kq_k}^2 x(t) = f(t, x(t)), \quad t \in (t_k, t_{k+1}], k = 0, 1, \ldots, m, \\
&x(t_k^+) - x(\rho_k^*) = \phi_k(x(\rho_k^*)), \quad k = 1, 2, \ldots, m, \\
&t_k D_{p_kq_k} x(t_k^+) = \sum_{j=1}^{m} f_k(t_k, x(t_k), \ldots, x(t_j), x(t_{j+1}), \ldots, x(t_{m+1}), x(t_{m+1})) \\
&x(0) = \lambda_1, \quad \lambda_2 D_{p_0q_0} x(0) = \lambda_2,
\end{align*}
\]

which we can stated as an integral equation of the form

\[ x(t) = \lambda_1 + \sum_{k=0}^{m} \left[ \lambda_2 + \sum_{j=0}^{m} \left[ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds \right] \right] + \sum_{r=0}^{m} \left[ \int_{t_r}^{t_{r+1}} g(t, x(t)) \, dt \right] + \sum_{j=1}^{m} \psi_j(x(\rho_j)), \quad t \in (t_k, t_{k+1}], k = 0, 1, \ldots, m. \] (2.13)

**Proof** The mathematical induction will be used to prove that (2.13) holds. To do this, by applying the double \((p_0, q_0)-\)integration to the first equation of (2.12), we obtain

\[
x(t) = x(t_k^+) + \left( t - t_k \right) x(t_k^+), \quad t \in (t_k, t_{k+1}], \]

which implies that (2.13) is true for \( k = 0. \) In the next step, we suppose that (2.13) holds for \( t \in (t_k, \rho_k^*]. \) By mathematical induction, we shall show that (2.13) holds on \( (t_{k+1}, \rho_{k+2}^*]. \) Now, the double \((p_0, q_0)-\)integration of the first equation of (2.12) yields on \( t \in (t_{k+1}, \rho_{k+2}^*] \) that

\[
x(t) = x(t_{k+1}^+) + \left( t - t_{k+1} \right) x(t_{k+1}^+) + \left( t - t_{k+1} \right) x(t_{k+1}^+) \]

(2.14)
We have

\[ x(t_{k+1}^*) = x(\rho_{k+1}^*) + \varphi_{k+1}(x(\rho_{k+1}^*)) \]

\[ = \lambda_1 + \sum_{i=0}^{k} (\rho_{i+1}^* - t_i) \left[ \lambda_2 + \sum_{j=0}^{i-1} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} s + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} \right] + \sum_{r=0}^{k-1} \left\{ \frac{1}{p_r} \int_{t_r}^{t_{r+1}} (\rho_{r+1}^* - t_r \varphi_{q_r}(s)) f_k(t_r, \Phi_{\frac{1}{p_r}}(s)) d_{p_r} q_r s + \varphi_{r+1}(x(\rho_{r+1}^*)) \right\} \]

and

\[ t_{k+1}^* D_{\rho_{k+1}^*} x(t_{k+1}^*) = t_k D_{\rho_{k}^*} x(\rho_{k+1}^*) + \varphi_{k+1}(x(\rho_{k+1}^*)) \]

\[ = \lambda_2 + \sum_{j=0}^{k-1} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} q_j + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} + \int_{t_k}^{t_{k+1}} f(s, x(s)) \, ds d_{p_k} q_k s + \varphi_{k+1}(x(\rho_{k+1}^*)) \]

\[ = \lambda_2 + \sum_{j=0}^{k} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} q_j + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} . \]

Substituting above two values into (2.14), we obtain

\[ x(t) = \lambda_1 + \sum_{i=0}^{k} (\rho_{i+1}^* - t_i) \left[ \lambda_2 + \sum_{j=0}^{i-1} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} q_j + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} \right] + \sum_{r=0}^{k-1} \left\{ \frac{1}{p_r} \int_{t_r}^{t_{r+1}} (\rho_{r+1}^* - t_r \varphi_{q_r}(s)) f_k(t_r, \Phi_{\frac{1}{p_r}}(s)) d_{p_r} q_r s + \varphi_{r+1}(x(\rho_{r+1}^*)) \right\} + (t - t_{k+1}) \left[ \lambda_2 + \sum_{j=0}^{k} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} q_j + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} \right] \]

\[ + \frac{1}{p_{k+1}} \int_{t_{k+1}}^{t} (t - t_{k+1} \Phi_{\frac{1}{p_{k+1}}} (s)) f_k(t_{k+1}, \Phi_{\frac{1}{p_{k+1}}}(s)) d_{p_{k+1}} q_{k+1} s \]

\[ = \lambda_1 + \sum_{i=0}^{k+1} (\rho_{i+1}^* - t_i) \left[ \lambda_2 + \sum_{j=0}^{i-1} \left\{ \int_{t_j}^{t_{j+1}} f(s, x(s)) \, ds d_{p_j} q_j + \varphi_{j+1}(x(\rho_{j+1}^*)) \right\} \right] + \sum_{r=0}^{k} \left\{ \frac{1}{p_r} \int_{t_r}^{t_{r+1}} (\rho_{r+1}^* - t_r \varphi_{q_r}(s)) f_k(t_r, \Phi_{\frac{1}{p_r}}(s)) d_{p_r} q_r s + \varphi_{r+1}(x(\rho_{r+1}^*)) \right\} \]

\[ + \frac{1}{p_{k+1}} \int_{t_{k+1}}^{t} (t - t_{k+1} \Phi_{\frac{1}{p_{k+1}}} (s)) f_k(t_{k+1}, \Phi_{\frac{1}{p_{k+1}}}(s)) d_{p_{k+1}} q_{k+1} s, \]

which holds for \((t_{k+1}, \rho_{k+2}^*)\]. This completes the proof. \(\square\)

To investigate the impulsive \((p, q)\)-difference equations of type II, we define intervals of solutions as \(A_1 = (\bigcup_{k=0}^{m}(t_k, \rho_{k+1}^*)) \cup [0]\) and \(A_2 = (\bigcup_{k=0}^{m}(t_k, \rho_{k+1}^*)) \cup [0]\), and also the spaces
In proving our next results, we use the constants:

\[
\begin{align*}
\Omega_5 &= \frac{|\beta| + |\alpha + \beta|}{|\alpha + \beta|} \sum_{i=0}^{m} (\rho_{i+1} - t_i), \\
\Omega_6 &= \sum_{i=0}^{m} \left( (\rho_{i+1}^* - t_i) \sum_{j=0}^{i-1} (\rho_{j+1} - t_j) \right) + \sum_{r=0}^{m} \frac{(\rho_{r+1}^* - t_r)^2}{p_r + q_r}, \\
\Omega_7 &= \sum_{i=0}^{m} (\rho_{i+1}^* - t_i)i.
\end{align*}
\]

Applying Theorem 2.9 to define the operator on \( PC_1(A_1, \mathbb{R}) \) and following the method of Theorem 2.4, we can easily prove the existence of a unique solution of problem (2.10).

**Theorem 2.11** Assume that the functions \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( \varphi_k : \mathbb{R} \to \mathbb{R} \), \( k = 1, 2, \ldots, m \), satisfy condition (H1). If

\[
L_1 \Omega_5 + L_2 \Omega_2 < 1,
\]

then the boundary value problem of type II (2.10) has a unique solution on \( A_1 \).

**Theorem 2.12** Assume that the functions \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \), \( \varphi_k : \mathbb{R} \to \mathbb{R} \), and \( \varphi_k^* : \mathbb{R} \to \mathbb{R} \), \( k = 1, 2, \ldots, m \), satisfy (H1)–(H2). If

\[
L_1 \Omega_6 + L_2 m + L_3 \Omega_7 < 1,
\]

then the problem of type II (2.12) has a unique solution on \( A_2 \).

**Proof** To show the technique of computation of constants \( \Omega_6 \) and \( \Omega_7 \), we give a short proof. Now we prove that the operator equation \( x = Bx \) has a unique fixed point, where the operator \( B : PC_2(A_2, \mathbb{R}) \to PC_2(A_2, \mathbb{R}) \) is defined, in view of Theorem 2.10, by

\[
Bx(t) = \lambda_1 + \sum_{i=0}^{k} \left( (\rho_{i+1}^* - t_i) - m_{i+1} \right) \left[ \lambda_2 \sum_{j=0}^{i-1} \int_{t_j}^{\rho_{i+1}^*} f(s, x(s)) \, ds + \varphi_k^* (x(\rho_{i+1}^*)) \right] + \sum_{r=0}^{k-1} \frac{1}{p_r} \int_{t_r}^{\rho_{r+1}^*} (\rho_{r+1}^* - s) \varphi_k^* (s) \varphi_k^* (s) \, ds + \varphi_{r+1}^* (x(\rho_{r+1}^*)) + \int_{t_k}^{t} \left( t - t_k \varphi_k^* (s) \varphi_k^* (s) \right) \, ds, \quad t \in (t_k, \rho_{k+1}^*), k = 0, 1, \ldots, m.
\]

By a similar method as in Theorem 2.4, we can show that the operator \( B \) maps a subset of \( PC_2(A_2, \mathbb{R}) \) into subset of \( PC_2(A_2, \mathbb{R}) \). Next, we will prove that \( B \) is a contraction. Let
\(x, y \in PC_2(A_2, \mathcal{R}).\) Then we have

\[
\|Bx(t) - By(t)\| \leq \sum_{i=0}^{k} \left( \left| \rho_{i+1}^* - t_i \right| \left| L_1 \|x - y\|_2 + \int_{t_i}^{\rho_{i+1}^*} (\rho_{i+1}^* - t_i) \|A_{p,q}y\| L_3 \|x - y\|_2 \right) \right]
\]

which implies that \(\|Bx - By\|_2 \leq (L_1 \Omega_6 + L_2 m + L_3 \Omega_7) \|x - y\|_2.\) Condition (2.16) and the Banach contraction mapping principle guarantee that the impulsive \((p,q)\)-difference initial value problem of type II (2.12) has a unique solution on \(A_2.\) The proof is completed. \(\square\)

**Example 2.13** Consider the following first-order impulsive \((p,q)\)-difference equation of type II subject to the boundary condition of the form:

\[
\begin{align*}
\frac{d}{[k+1]_q x(t)} &= \frac{5}{6} \frac{1}{[k+2]_q x(t)} + \frac{3}{2} t \in (k, k+1], k = 0, 1, 2, \\
x(k) - x(\frac{3}{2} k + 1) &= \frac{1}{\alpha} \tan^{-1}(\frac{3}{2} k + 1), \quad k = 1, 2, \\
\frac{1}{2} x(0) + \frac{1}{3} x(\frac{1}{2}) &= \frac{1}{4}.
\end{align*}
\]

Here the quantum numbers are \(p_k = (k + 1)/(k + 2), q_k = (k + 1)/(k + 3), k = 0, 1, 2, f = [0, 3], \) \(t_k = k, k = 1, 2, \alpha = 1/2, \beta = 1/3, \gamma = 1/4, \) and \(p_k = (k^2 + k - 1)/(k + 1).\) We can find that
\( \Omega_2 = 2.8000, \Omega_5 = 2.6833, \) and
\[
A_1 = \left[ 0, \frac{1}{2} \right] \cup \left( 1, \frac{5}{3} \right] \cup \left( 2, \frac{11}{4} \right].
\]
By setting
\[
f(t,x) = \frac{5}{6(3+t)^2} \left( \frac{x^2 + 2|x|}{1 + |x|} \right) + \frac{3}{4} \quad \text{and} \quad \varphi_k(x) = \frac{1}{6k} \tan^{-1}(x),
\]
we see that the functions \( f \) and \( \varphi_k \) satisfy \((H_1)\) with \( L_1 = 5/27 \) and \( L_2 = 1/6 \), respectively. Then we get \( L_1 \Omega_5 + L_2 \Omega_2 = 0.9543 < 1 \). Therefore, by Theorem 2.11, the boundary value problem (2.17) has a unique solution \( x \) on \( A_1 \).

**Example 2.14** Consider the following second-order impulsive \((p, q)\)-difference equation of type II with the initial conditions of the form:
\[
\begin{align*}
\frac{d}{dt} x(t) &= \frac{1}{10(t+6)} \sin |x(t)| + \frac{5}{6}, \quad t \in (k, k+1], k = 0, 1, 2, \\
\frac{d}{dt} x(k^+) - x(k^+) &= \frac{3}{5(k+1)^2} \tan^{-1}(x((k^+)^2 - k^+)^{-1}), \quad k = 1, 2, \\
\frac{d}{dt} x(k^+) - (k-1)D x(k^+) &= \frac{1}{5k^2} \sin x((k-1)^2 - k+1)^{-1}), \quad k = 1, 2, \\
x(0) = \frac{3}{2}, \quad qD x(0) = \frac{3}{2}.
\end{align*}
\]

The quantum numbers \( p_k, q_k \), impulsive points \( t_k, \rho_k \), and interval \( J \) are defined the same as in Example 2.13. We have the constants \( \lambda_1 = 3/5, \lambda_2 = 5/7, \) and points \( \rho_1 = (k^2 + 2k^2 - k - 1)/(k+1)^2. \) Next we can find that \( \Omega_6 = 18.4273, \Omega_7 = 1.5694, \) and
\[
A_2 = \left[ 0, \frac{1}{4} \right] \cup \left( 1, \frac{13}{9} \right] \cup \left( 2, \frac{41}{16} \right].
\]
By setting
\[
f(t,x) = \frac{1}{20(t+6)} \sin |x| + \frac{5}{6}, \quad \varphi_k(x) = \frac{3}{5(k+1)^2} \tan^{-1}(x), \quad \text{and} \quad \varphi_k(x) = \frac{1}{5k^2} |x|,
\]
we deduce that \((H_1)-(H_2)\) are fulfilled with \( L_1 = 1/60, L_2 = 3/20, \) and \( L_3 = 1/5 \). Hence, it follows that \( L_1 \Omega_6 + L_2 \Omega_6 + L_3 \Omega_7 = 0.9210 < 1 \). Therefore, by applying Theorem 2.12, the boundary value problem (2.18) has a unique solution \( x \) on \( A_2 \).

### 3 Conclusion
In this research, we initiated the study of the first and second order \((p, q)\)-difference equations with initial or boundary conditions. Firstly, we let \( t_k, k = 1, \ldots, m, \) be the impulsive points such that \( 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T \) and \( J_k = (t_k, t_{k+1}], k = 1, \ldots, m, \) \( J_0 = [0, t_1] \) be the intervals such that \( \bigcup_{k=0}^m J_k = [0, T] := J. \) The investigations were based on \((p, q)\)-calculus introduced in the first section of this paper, by replacing a point \( a \) by \( t_k \), quantum numbers \( p \) by \( p_k \) and \( q \) by \( q_k \), \( k = 0, 1, \ldots, m, \) and also applying the \((p_k, q_k)\)-difference and \((p_k, q_k)\)-integral operators only on a finite subinterval of \( J \). In addition, the consecutive subintervals could be related with jump conditions which led to a meaning of quantum difference equations with impulse effects. There are two types of impulsive
problems. The consecutive domains of impulsive \((p, q)\)-difference equations of type I are overlapped, while the unknown functions of impulsive equations of type II are defined on disjoint consecutive domains. Four types of problems were considered, two impulsive \((p, q)\)-difference equations of type I and two impulsive \((p, q)\)-difference equations of type II. Existence and uniqueness results were proved via Banach's contraction mapping principle. Examples illustrating the obtained results were also presented.

Acknowledgements
Not applicable.

Funding
This work was financially supported by Rajamangala University of Technology Rattanakosin, Wang Klai Kangwon Campus, Thailand (Grant No. A-51/2561).

Availability of data and materials
Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 November 2019 Accepted: 18 February 2020 Published online: 28 February 2020

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