EFFECTIVE EQUATION FOR A SYSTEM OF MECHANICAL OSCILLATORS IN AN ACOUSTIC FIELD

CLAUDIO CACCIAPUOTI, RODOLFO FIGARI, AND ANDREA POSILICANO

ABSTRACT. We consider a one dimensional evolution problem modeling the dynamics of an acoustic field coupled with a set of mechanical oscillators. We analyze solutions of the system of ordinary and partial differential equations with time-dependent boundary conditions describing the evolution in the limit of a continuous distribution of oscillators.

1. INTRODUCTION

The investigation of the dynamics of spherical bubbles oscillating in a surrounding compressible fluid has been the subject of many papers in Applied Acoustics. Starting from the pioneering works [3, 4, 5] of C.A. Bjerknes and V.F.K. Bjerknes, at the beginning of last century, the analysis of the forces (secondary Bjerknes forces) exchanged between bubbles via the self-generated acoustic field was the basis of various theoretical attempt to examine the evolution of “bubbly liquids”. Here, we limit ourselves to cite a few review papers [8, 9, 13, 19] and a more recent work [17] where a thorough list of references on various aspects of this topic can be found.

The case of spherical bubbles of small linear size in comparison with their average mutual distances was the first to be analyzed. Under this hypothesis, only the far field emitted by the bubbles is relevant to the interaction forces and the bubbles can be considered point-like. Nevertheless, at the best of our knowledge, no attempts has been done to investigate the equations governing the evolution of the acoustic field and of the bubble dynamics in the limit of zero bubble radii.

It is worth mentioning that, at the times when the Bjerknes’ started their investigations on the acoustic field driven interactions between bubbles, many physicists were approaching the apparently much harder problem of producing a theory of the electromagnetic field together with its point sources (see e.g. [1, 2, 14, 15, 16]).

In the following, we present a one dimensional model of a coupled system of acoustic field and mechanical oscillators. Based on the results we obtained in [6], for the case of a finite number of oscillators, we address the problem of deducing the asymptotic form of the velocity field evolution equation when the oscillators are continuously distributed in a finite interval.

In our opinion, the asymptotic problem has reasons of mathematical interest. In fact, the small length scale does not refer here to the typical range of some inhomogeneity of the medium inside which the field evolves, but to the mutual distances of the sources producing and interacting with the field itself. In this respect, the problem is in no way typical of an homogenization problem for an hyperbolic system of evolution equations (see e.g. [7, 10, 11, 13, 20]).

In fact, we investigate here a scattering process: the limit Cauchy problem describes the evolution of the acoustic field velocity from one side to the other of a region filled with mechanical oscillators reacting to the field and becoming sources of secondary acoustic waves. At the
best of our knowledge rigorous results on limit effective equations for problems of this kind are not available.

We describe first the physical setup our system of equations is meant to model. We consider the following arrangement: an infinite pipe filled with a non-viscous, compressible fluid and \( n \) mechanical oscillators realized with thin walls of mass \( M_j \) positioned in the pipe perpendicularly to its axis. A system of springs with elastic constants \( K_j \) confines the walls around their equilibrium positions located in the points of coordinates \( s_j \), with respect to a system of coordinates whose \( x \)-axis coincides with the axis of the pipe.

We suppose that there is no friction between the fluid and the pipe and we analyze only one dimensional flows. The acoustic field is then described by the pressure field \( p(x,t) \) and the velocity field \( v(x,t) \). The motion of the mechanical oscillators is described through the displacements \( y_j(t) \) of the thin walls from their equilibrium positions \( s_j \) and the velocities \( z_j(t) = \dot{y}_j(t), \ j = 1, \ldots, n \).

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**Figure 1.** The figure shows the cross section of the whole system (on the left) and the details of two oscillators (on the right).

The field \( p(x,t) \) specifies the deviation of the pressure in the point \( x \) at time \( t \) with respect to an equilibrium pressure \( P_0 \). We analyze the dynamics of the whole system (acoustic field and mechanical oscillators) in the linearized acoustic and elastic regimes. Moreover we consider walls of zero thickness and we suppose that the contact between the fluid and the walls is maintained throughout the evolution. Under these hypotheses the dynamics is described by the following system of ordinary and partial differential equations with time dependent boundary conditions

\[
\begin{align*}
\frac{\partial p}{\partial t} &= -a^2 \rho_0 \frac{\partial v}{\partial x} & x \in \mathbb{R} \setminus S \\
\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} & x \in \mathbb{R} \setminus S \\
M_j \frac{dz_j}{dt} &= -K_j y_j - S (p(s_j^+) - p(s_j^-)) & j = 1, \ldots, n \\
z_j &= \frac{dy_j}{dt} = v(s_j) & j = 1, \ldots, n
\end{align*}
\]

where \( \rho_0 \) is a positive constant representing the density of the fluid, \( a \) is a positive constant indicating the velocity of sound in the fluid, \( S \) is the area of the transversal section of the pipe and \( S := (s_1, \ldots, s_n) \) is the set of the equilibrium positions of the walls. Equations (3) and (4) express the Newton’s laws for the mechanical oscillators. The two forces acting on each wall are the spring elastic force and the force due to the difference of pressure on its opposite sides. Aim of this paper is to characterize the effective behavior of the system in the limit of a continuous distribution of oscillators in a bounded region.
The Hilbert space of finite energy is defined by

$$e$$

and

$$C$$

The square norm of a vector $$\Psi$$,

$$\| \Psi \|^2 = \langle \langle \Psi, \Psi \rangle \rangle$$

The linear operator

$$\hat{A} : D(\hat{A}) \subset \mathcal{H} \to \mathcal{H}$$

$$D(\hat{A}) := \{ (p, v, y, \bar{z}) \mid p \in L^2(\mathbb{R}) \cap H^1(\mathbb{R}\backslash S), v \in H^1(\mathbb{R}), y \in \mathbb{C}^n, \bar{z} \in \mathbb{C}^n, v(s_j) = z_j, j = 1, \ldots, n \}$$

$$\hat{A}(p, v, y, \bar{z}) := \left( -\frac{1}{\rho_0} \frac{dp}{dx}, \frac{1}{\rho_0} \frac{dv}{dx}, \sum_{j=1}^n \left( \frac{K_j}{M_j} y_j + \frac{S}{M_j} \sigma_j \right) \bar{z}_j \right)$$

is real and skew-adjoint. Here $$\sigma_j \in \mathbb{C}$$ is defined by

$$\sigma_j := p(s_j^+) - p(s_j^-)$$

Given the initial conditions

$$p(x, 0); v(x, 0); y_j(0); z_j(0) \quad j = 1, \ldots, n$$

we consider the Cauchy problem

As it was shown in [6] existence and unicity, together with detailed properties of the solutions, can be proved formulating the problem as a unitary evolution in the space of finite energy. In the following, we fix the notation and we recall the main result obtained in [6].

We indicate with a capital Greek letter the generic vector $$\Psi := (p, v, y, \bar{z}) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$$, where $$L^2(\mathbb{R})$$ is the space of square-integrable functions on the real line,

$$\bar{y} = y_1 \bar{e}_1 + \cdots + y_n \bar{e}_n, \quad \bar{z} = z_1 \bar{e}_1 + \cdots + z_n \bar{e}_n$$

and $$\bar{e}_1, \ldots, \bar{e}_n$$ is the canonical orthonormal base in $$\mathbb{C}^n$$.

The Hilbert space of finite energy is defined by

$$\mathcal{H} := L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \mathbb{C}^n \oplus \mathbb{C}^n$$

with the scalar product

$$\langle \langle \Psi_1, \Psi_2 \rangle \rangle := \frac{1}{a^2 \rho_0} \langle p_1, p_2 \rangle + \rho_0 \langle v_1, v_2 \rangle + \frac{1}{S} \sum_{j=1}^n \left[ K_j y_j \bar{y}_j + M_j \bar{z}_j z_j \right]$$

where $$\langle \cdot, \cdot \rangle$$ indicates the standard scalar product in $$L^2(\mathbb{R})$$ and $$-$$ denotes complex conjugation. The square norm of a vector $$\Psi$$, $$\| \Psi \|^2 = \langle \langle \Psi, \Psi \rangle \rangle$$, defines the total energy of the system in the state $$\Psi$$

$$E_{tot} = \frac{S}{2} \| \Psi \|^2 = E_{ac} + E_{osc}$$

with

$$E_{ac} = \frac{S}{2a^2 \rho_0} \int_{-\infty}^{\infty} |p(x)|^2 dx + \frac{S \rho_0}{2} \int_{-\infty}^{\infty} |v(x)|^2 dx$$

$$E_{osc} = \sum_{j=1}^n \left[ \frac{K_j}{2} |y_j|^2 + \frac{M_j}{2} |z_j|^2 \right]$$

$$E_{ac}$$ is the energy stored in the acoustic field while $$E_{osc}$$ is the energy of the mechanical oscillators. Let $$H^1(\mathbb{R})$$ be the homogeneous Sobolev space of locally square-integrable functions with square-integrable (distributional) derivative and $$H^1(\mathbb{R})$$ the usual Sobolev space $$H^1(\mathbb{R}) := \dot{H}^1(\mathbb{R}) \cap L^2(\mathbb{R})$$. Similarly we define, for any open set $$\mathcal{O} \subseteq \mathbb{R}$$, the Sobolev spaces $$H^1(\mathcal{O}) := \{ f \in L^2(\mathcal{O}) : \frac{df}{dx} \in L^2(\mathcal{O}) \}$$ and $$H^2(\mathcal{O}) := \{ f \in H^1(\mathcal{O}) : \frac{df}{dx} \in L^2(\mathcal{O}) \}$$. The following theorem holds

**Theorem 1.** The linear operator

$$\hat{A} : D(\hat{A}) \subset \mathcal{H} \to \mathcal{H}$$

is well-defined and skew-adjoint.
and \( p_{\text{reg}} \in \bar{H}^1(\mathbb{R}) \) is the regular part of \( p \), i.e.

\[
p_{\text{reg}}(x) := p(x) - \frac{1}{2} \sum_{j=1}^{n} \sigma_j \text{sgn}(x - s_j).
\]

**Remark 1.** Notice that the left and right limits entering in the definitions of the \( \sigma_j \)'s exist and are finite by

\[
|p(x_1) - p(x_2)| \leq (x_2 - x_1)^{1/2} \left\| \frac{dp}{dx} \right\|_{L^2(s_j,s_{j+1})}, \quad s_j < x_1 \leq x_2 < s_{j+1}.
\]

Theorem 1 implies that the Cauchy problem

\[
\begin{align*}
\frac{d}{dt} \Psi_t &= \hat{A} \Psi_t \\
\Psi_{t=0} &= \Psi_0 \
\end{align*}
\]

has a unique solution expressed via the strongly continuous unitary group of evolution generated by \( \hat{A} \), \( \Psi_t = e^{t\hat{A}}\Psi_0 \). Moreover the unitary evolution preserves reality in such a way that one can consider the flow restricted to the real Hilbert space

\[
\mathcal{H}_r := L^2_r(\mathbb{R}) \oplus L^2_r(\mathbb{R}) \oplus \mathbb{R}^n \oplus \mathbb{R}^n,
\]

where \( L^2_r(\mathbb{R}) \) is the space of real-valued functions in \( L^2(\mathbb{R}) \). The operator \( \hat{A} \) is skew-adjoint so that \( \|\Psi_t\| = \|\Psi_0\| \), and the total energy remains constant. It is easy to check that the unique solution of (6)-(7) solves in \( \mathcal{H} \) the Cauchy problem (1)-(4), (5).

In fact from Theorem 1 the equation (6) is equivalent to the system

\[
\begin{align*}
\frac{\partial p}{\partial t} &= -a^2 \rho_0 \frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0} \frac{\partial p_{\text{reg}}}{\partial x}, \\
dy_j/dt &= z_j, \quad j = 1, \ldots, n, \\
dz_j/dt &= -\left( \frac{K_j}{M_j} y_j + \frac{S}{M_j} \sigma_j \right), \quad j = 1, \ldots, n.
\end{align*}
\]

Notice that if \( \Psi_0 \in D(\hat{A}) \) the solution \( \Psi_t \) belongs to \( D(\hat{A}) \) so that condition (4) is automatically satisfied at any time.

In the following section we investigate the effective response to an incoming acoustical wave of a chain of mechanical oscillators in the limit of a continuous distribution of oscillator masses.

**2. Effective equation in the limit of a continuous distribution of oscillators**

We consider a sequence of Cauchy problems indexed by the increasing number \( n \) of mechanical oscillators

\[
\begin{align*}
\frac{d}{dt} \Psi_{t}^{(n)} &= \hat{A}^{(n)} \Psi_{t}^{(n)} \\
\Psi_{t=0}^{(n)} &= \Psi_0^{(n)} \
\end{align*}
\]

where we denoted, as above, with \( \Psi_t^{(n)} \) the quadruple \( \Psi_t^{(n)} := (p_t^{(n)},v_t^{(n)},y_t^{(n)},z_t^{(n)}) \). We investigate the limit, for increasing \( n \), of the evolution equation satisfied by the velocity field alone.

We will make the following assumptions on masses, elastic constants and positions of the walls:
\( \mathcal{A}_1 \) The set \( S^{(n)} := (s_1^{(n)}, \ldots, s_n^{(n)}) \) of the equilibrium positions is contained in a bounded interval around the origin, \( s_j^{(n)} \in [-L/2, L/2] \) \( \forall n \) and \( \forall j = 1 \ldots n; \)

\( \mathcal{A}_2 \) The two measures \( \mu_M^{(n)} := \sum_{j=1}^{n} M_j^{(n)} \delta_{s_j^{(n)}} \) and \( \mu_K^{(n)} := \sum_{j=1}^{n} K_j^{(n)} \delta_{s_j^{(n)}}, \) where \( \delta_{s_j^{(n)}} \) is the Dirac mass supported at point \( s_j^{(n)}, \) are uniformly equivalent in the sense that \( 0 < c_1 < \sup \left[ L^2 K_j^{(n)}/a^2 M_j^{(n)} \right] < c_2 < \infty, \) here \( c_1 \) and \( c_2 \) are two positive constants. The two measures have total mass which is bounded uniformly in \( n: \sum_{j=1}^{n} M_j^{(n)} < c \sum_{j=1}^{n} K_j^{(n)} < c. \) Furthermore \( \mu_M^{(n)} \) and \( \mu_K^{(n)} \) weakly converge, when \( n \) tends to infinity, to measures supported in \([-L/2, L/2]\) which are absolutely continuous with respect to Lebesgue measure (the corresponding Radon-Nikodim derivatives will be denoted with \( S\rho_M \) and \( S\rho_K \)).

The initial conditions will be chosen independent of \( n. \) In particular we consider the case
\[
(10) \quad p^{(n)}(x, 0) = p_0(x); \quad v^{(n)}(x, 0) = v_0(x); \quad y_j^{(n)}(0) = 0; \quad s_j^{(n)}(0) = 0 \quad j = 1, \ldots, n,
\]
where \( p_0 \) and \( v_0 \) are in \( H^2(\mathbb{R}) \) and \( \text{supp}(p_0) \cap [-L/2, L/2] = \emptyset, \text{supp}(v_0) \cap [-L/2, L/2] = \emptyset. \)

Remark 2. The request of equivalence of the two measures \( \mu_M^{(n)} \) and \( \mu_K^{(n)} \) could be released in various ways but it is well motivated, from a physical point of view, preventing the occurrence of zero or infinite proper oscillation frequencies.

Initial conditions more general than (10) might of course be considered. Nevertheless (10) seem to be the most natural initial conditions which do not depend on \( n. \) Notice, in particular, that at each step \( n, \) corresponding to the zero eigenvalue of \( A^{(n)}, \) there are static solutions of problem (1)-(4). As it was shown in \([6]\) the static solutions are of the form \( \Psi_{st} = (p_{st}^{(n)}, y_{st}^{(n)}, 0) \) with \( \text{supp}(p_{st}) \subseteq [-L/2, L/2]. \) Being initial conditions (10) orthogonal to all static solutions we are guaranteed that a genuine scattering problem is analyzed.

Under the assumptions stated above we prove the following

**Theorem 2.** Let the sequence of sets \( S^{(n)} := (s_1^{(n)}, \ldots, s_n^{(n)}) \) and the sets of positive real numbers \( M^{(n)} := (M_1^{(n)}, \ldots, M_n^{(n)}) \) and \( K^{(n)} := (K_1^{(n)}, \ldots, K_n^{(n)}) \) be such that the assumptions \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) above are satisfied. Let \( v^{(n)} \in C^2(\mathbb{R}; L^2(\mathbb{R})) \cap C^1(\mathbb{R}; H^1(\mathbb{R})) \cap C(\mathbb{R}; H^2(\mathbb{R}) \setminus S^{(n)}) \) be the velocity field in the solution of problem (8) - (9) with initial conditions satisfying assumption \( \mathcal{A}_3 \).

Then
\[
\frac{\partial v^{(n)}}{\partial t} \text{ converge to } v \text{ and } \frac{\partial v}{\partial t} \text{ uniformly over compact subsets of } \mathbb{R}^2
\]
and
\[
\frac{\partial v^{(n)}}{\partial x}(t, \cdot) \text{ converges to } \frac{\partial v}{\partial x}(t, \cdot) \text{ in } L^2_{\text{loc}}(\mathbb{R}), \text{ uniformly in } t \text{ over compact intervals,}
\]
where \( v \in C^1(\mathbb{R}; C(\mathbb{R})) \cap C(\mathbb{R}; H^1_{\text{loc}}(\mathbb{R})) \) is the unique strong solution\(^1\) of the hyperbolic equation
\[
(11) \quad \left( 1 + \frac{\rho_M}{\rho_0} \right) \frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} + \frac{\rho_K}{\rho_0} v = 0.
\]
with initial data
\[
(12) \quad v|_{t=0} = v_0; \quad \frac{\partial v}{\partial t}|_{t=0} = -\frac{1}{\rho_0} \frac{dp_0}{dx}.
\]
\(^1\)see equation (25) below for more precise regularity properties of \( v. \)
Proof. By the basic properties of one-parameter groups of operators and their generators (see e.g. [12]), the map \( t \mapsto e^{t \hat{A}^{(n)}} \Psi_0 \) belongs to \( C^1(\mathbb{R}; \mathcal{H}) \cap C(\mathbb{R}; D(\hat{A}^{(n)})) \) if and only if \( \Psi_0 \in D(\hat{A}^{(n)}) \). In this case
\[
\frac{d}{dt} e^{t \hat{A}^{(n)}} \Psi_0 = \hat{A}^{(n)} e^{t \hat{A}^{(n)}} \Psi_0 = e^{t \hat{A}^{(n)}} \hat{A}^{(n)} \Psi_0,
\]
and, since \( \hat{A}^{(n)} \) is skew-adjoint,
\[
\| \hat{A}^{(n)} \Psi_t^{(n)} \| = \| e^{t \hat{A}^{(n)}} \hat{A}^{(n)} \Psi_0 \| = \| \hat{A}^{(n)} \Psi_0 \|.
\]
Therefore, since any initial condition \( \Psi_0 = (p_0, v_0, 0, 0) \) chosen according to assumption \( \mathcal{A}_3 \) is in \( D([\hat{A}^{(n)}]^2) \) for any \( n \), the path \( t \mapsto \Psi_t^{(n)} := e^{t \hat{A}^{(n)}} \Psi_0 \) is in \( C^2(\mathbb{R}; \mathcal{H}) \cap C^1(\mathbb{R}; D(\hat{A}^{(n)})) \cap C(\mathbb{R}; D([\hat{A}^{(n)}]^2)) \),
\[
\frac{d^2 \Psi_t^{(n)}}{dt^2} = [\hat{A}^{(n)}]^2 \Psi_t^{(n)}
\]
and
\[
||[\hat{A}^{(n)}]^2 \Psi_t^{(n)}|| = ||e^{t \hat{A}^{(n)}}[\hat{A}^{(n)}]^2 \Psi_0|| = ||[\hat{A}^{(n)}]^2 \Psi_0||.
\]
From the explicit characterization of domain and action of \([A^{(n)}]^2\)
\[
D([\hat{A}^{(n)}]^2) = \left\{ (p^{(n)}, v^{(n)}, y^{(n)}, z^{(n)}) \in D(\hat{A}^{(n)}) \left| \frac{dv^{(n)}}{dx} \in H^1(\mathbb{R}\setminus \mathcal{S}^{(n)}), \right. \right. \\
\frac{dp^{(n)}_{\text{reg}}}{dx} \in H^1(\mathbb{R}), \frac{dp^{(n)}_{\text{reg}}}{dx} (s^{(n)}) = \rho_0 \left( \frac{K_j^{(n)}}{M_j^{(n)}} y_j^{(n)} + \frac{S}{M_j^{(n)}} \sigma_j^{(n)} \right), j = 1, \ldots, n \right\},
\]
\[
[\hat{A}^{(n)}]^2 (p^{(n)}, v^{(n)}, y^{(n)}, z^{(n)}) = \left( a^2 \frac{dp^{(n)}_{\text{reg}}}{dx}, a^2 \frac{dv^{(n)}_{\text{reg}}}{dx} \right), \right.
\]
\[
- \sum_{j=1}^n \left( \frac{K_j^{(n)}}{M_j^{(n)}} y_j^{(n)} + \frac{S}{M_j^{(n)}} \sigma_j^{(n)} \right) \zeta_j^{(n)}, - \sum_{j=1}^n \left( \frac{K_j^{(n)}}{M_j^{(n)}} \zeta_j^{(n)} - a^2 \rho_0 S \sigma_j^{(n)} \right) \xi_j^{(n)} \right)
\]
where
\[
\zeta_j^{(n)} := \frac{dv^{(n)}}{dx} (s_j^{(n)+}) - \frac{dv^{(n)}}{dx} (s_j^{(n)-}),
\]
\[
\sigma_j^{(n)} := p^{(n)} (s_j^{(n)+}) - p^{(n)} (s_j^{(n)-}),
\]
\[
\frac{dv^{(n)}_{\text{reg}}}{dx} (x) := \frac{dv^{(n)}_{\text{reg}}}{dx} (x) - \frac{1}{2} \sum_{j=1}^n c_j^{(n)} \text{sgn} (x - s_j^{(n)}).
\]

We conclude that if \( \Psi_t^{(n)} = (p^{(n)}(t), v^{(n)}(t), y^{(n)}(t), z^{(n)}(t)) \) is the solution of (14), then
\[
v^{(n)} \in C^2(\mathbb{R}; L^2(\mathbb{R})) \cap C^1(\mathbb{R}; H^1(\mathbb{R})) \cap C(\mathbb{R}; H^2(\mathbb{R}\setminus \mathcal{S}^{(n)})), \quad z \in C^2(\mathbb{R}),
\]
and \((v^{(n)}, z^{(n)})\) is the strong solution of the following Cauchy problem with time-dependent boundary conditions:

\[
\begin{align*}
\frac{\partial^2 v^{(n)}}{\partial t^2} &= a^2 \frac{\partial}{\partial x} \left( \frac{\partial v^{(n)}}{\partial x} \right)_{\text{reg}}, \\
\frac{d^2 z^{(n)}}{dt^2} + \frac{K_j^{(n)}}{M_j^{(n)}} z_j^{(n)} &= \frac{a^2 \rho_0 S}{M_j^{(n)}} \zeta_j^{(n)}, \quad j = 1, \ldots, n, \\
v^{(n)}(t, s_j^{(n)}) &= z_j^{(n)}(t), \quad \frac{\partial v^{(n)}}{\partial t}(t, s_j^{(n)}) = \frac{dz_j^{(n)}}{dt}(t), \quad j = 1, \ldots, n, \\
v^{(n)}|_{t=0} = v_0, \quad \frac{\partial v^{(n)}}{\partial t}\big|_{t=0} = -\frac{1}{\rho_0} \frac{\partial p_0}{\partial x}, \\
z^{(n)}|_{t=0} = 0, \quad \frac{dz^{(n)}}{dt}\big|_{t=0} = 0.
\end{align*}
\]

Furthermore, by energy conservation and by (13) and (15), there exists a positive constant \(c\) depending only on the norm of the initial datum \(\Psi_0 = (p_0, v_0, 0, 0)\), such that the following bounds hold for any \(t\) and \(n\):

\[
\begin{align*}
\|v^{(n)}(t, \cdot)\|_{H^1(\mathbb{R})} &\leq c, \quad \left\| \frac{\partial v^{(n)}}{\partial t}(t, \cdot) \right\|_{H^1(\mathbb{R})} \leq c, \quad \left\| \frac{\partial^2 v^{(n)}}{\partial t^2}(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq c, \\
\left\| \left( \frac{\partial v^{(n)}}{\partial x} \right)_{\text{reg}} (t, \cdot) \right\|_{H^1(\mathbb{R})} &\leq c,
\end{align*}
\]

\[
\sum_{j=1}^n M_j^{(n)} |z_j^{(n)}(t)|^2 \leq c, \quad \sum_{j=1}^n M_j^{(n)} \left| \frac{K_j^{(n)}}{M_j^{(n)}} z_j^{(n)}(t) - \frac{a^2 \rho_0 S}{M_j^{(n)}} \zeta_j^{(n)}(t) \right|^2 \leq c.
\]

By the continuous injection \(H^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})\) one has

\[
\sup_{(t,x) \in \mathbb{R}^2} |v^{(n)}(t, x)| \leq c_1 \sup_{t \in \mathbb{R}} \|v^{(n)}(t, \cdot)\|_{H^1(\mathbb{R})}
\]

and (here \(x_1 < x_2 \) and \(t_1 < t_2\))

\[
\begin{align*}
|v^{(n)}(t_1, x_1) - v^{(n)}(t_2, x_2)| &\leq \sqrt{x_2 - x_1} \sup_{t \in \mathbb{R}} \left\| \frac{\partial v^{(n)}}{\partial x}(t, \cdot) \right\|_{L^2(\mathbb{R})} + (t_2 - t_1) \sup_{(t,x) \in \mathbb{R}} \left\| \frac{\partial v^{(n)}}{\partial t}(t, x) \right\| \\
&\leq \sqrt{x_2 - x_1} \sup_{t \in \mathbb{R}} \|v^{(n)}(t, \cdot)\|_{H^1(\mathbb{R})} + c_1 (t_2 - t_1) \sup_{t \in \mathbb{R}} \left\| \frac{\partial v^{(n)}}{\partial t}(t, \cdot) \right\|_{H^1(\mathbb{R})}.
\end{align*}
\]

Hence by (22) the sequence \(\{v^{(n)}\}_{n=1}^{\infty}\) is equi-bounded and equi-continuous. Thus, by Ascoli-Arzelà theorem, there exist a subsequence \(\{v^{(n_k)}\}_{k=1}^{\infty}\) and \(v \in C(\mathbb{R}^2)\) such that \(v^{(n_k)}\) converges to \(v\) uniformly over compact subsets of \(\mathbb{R}^2\).

Let us now take a bounded rectangle \(R = I_1 \times I_2 \subset \mathbb{R}^2\). Since the injection \(H^1(I_2) \hookrightarrow C(I_2)\) is a compact operator and, by (22), \(\left\{ \frac{\partial v^{(n_k)}}{\partial x} \right\}_{1}^{\infty}\) is bounded in \(L^\infty(I_1; H^1(I_2))\) and \(\left\{ \frac{\partial^2 v^{(n_k)}}{\partial x^2} \right\}_{1}^{\infty}\) is bounded in \(L^\infty(I_1; L^2(I_2))\), by corollary 4 in [21], section 8, applied to the triple of Banach spaces \(X = H^1(I_2) \subset B = C(I_2) \subset Y = L^2(I_2)\), there exists a sub-subsequence \(\left\{ \frac{\partial v^{(n_k)}}{\partial x} \right\}_{1}^{\infty}\) converging to \(\dot{v}_R \in C(I_1; C(I_2))\).
Let $\mathcal{H}_\alpha(I_2)$, $\alpha \in (0,1)$, be defined, as usual, by
\[
\mathcal{H}_\alpha(I_2) := \left\{ f \in L^2(I_2) : \int_{I_2} \int_{I_2} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2\alpha}} \, dx \, dy < \infty \right\}.
\]
$\mathcal{H}_\alpha(I_2)$ is a Banach space with norm
\[
\|f\|_{\mathcal{H}_\alpha(I_2)} = \left( \int_{I_2} \int_{I_2} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2\alpha}} \, dx \, dy \right)^{1/2}.
\]

Notice that $\text{sgn} (\cdot - s_j^{(n)}) \in \mathcal{H}_\alpha(I_2)$ for any $\alpha < 1/2$ and for any bounded $I_2 \subset \mathbb{R}$ and its $\mathcal{H}_\alpha(I_2)$ norm is bounded uniformly in $n$. Hence, by (16), (23), (24) and $A_2$, $\left\{ \frac{\partial \psi(n_k)}{\partial x} \right\}_1^\infty$ is bounded in $L^\infty(I_1; \mathcal{H}_\alpha(I_2))$, $\alpha < 1/2$. Moreover, by (22), $\left\{ \frac{\partial \psi(n_k)}{\partial x} \right\}_1^\infty$ is bounded in $L^\infty(I_1; L^2(I_2))$. Thus, since the injection $\mathcal{H}_\alpha(I_2) \hookrightarrow L^2(I_2)$, $\alpha > 0$, is a compact operator, again by corollary 4 in [21], section 8, applied to the triple of Banach spaces $X = \mathcal{H}_\alpha(I_2) \subset B = Y = L^2(I_2)$, there exists a sub-subsequence $\left\{ \frac{\partial \psi(n_k)}{\partial x} \right\}_1^\infty$ converging to $\psi'_{R}$ in $C(\tilde{I}^1; L^2(I_2))$.

Considering then the sequence of rectangles $R_m = (-m, m) \times (-m, m)$, by a standard diagonal argument, one obtains a further subsequence, which for notational convenience we continue to denote by $\left\{ \psi(n_k) \right\}_1^\infty$, such that $\left\{ \frac{\partial \psi(n_k)}{\partial x} \right\}_1^\infty$ converges to $\psi \in C(\mathbb{R}^2)$ uniformly over compact sets, and such that $\left\{ \frac{\partial \psi(n_k)}{\partial x} \right\}_1^\infty$ converges to $\psi' \in C(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R}))$ uniformly over compact time intervals. Defining
\[
v_1(t,x) := v(0,x) + \text{sgn}(t) \int_0^t \psi(s,x) \, ds, \quad v_2(t,x) := v(t,0) + \text{sgn}(x) \int_0^x \psi'(t,y) \, dy,
\]
one obtains $v^{(n_k)} \to v_1$ and $v^{(n_k)} \to v_2$, and so $v_1 = v_2 = v$, $\dot{v} = \frac{\partial v}{\partial t}$, $\dot{v}' = \frac{\partial v'}{\partial t}$.

Let us now multiply both sides of equation (17) by a function $g \in C^1_c(\mathbb{R}^2)$. Integrating by part we get
\[
- \int_0^\infty \left( \left( \frac{\partial v^{(n_k)}}{\partial t}(t,\cdot), \frac{\partial g}{\partial t}(t,\cdot) \right) - \left( \frac{\partial v^{(n_k)}}{\partial x}(t,0,\cdot), \frac{\partial g}{\partial x}(t,0,\cdot) \right) \right) \, dt
+ \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} M_j^{(n_k)} \frac{d z_j^{(n_k)}}{dt}(t) \frac{\partial g}{\partial t}(t,s_j^{(n_k)}) \, dt
- \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} K_j^{(n_k)} \dot{z}_j^{(n_k)}(t) g(t,s_j^{(n_k)}) \, dt.
\]

Using assumption $A_2$ and $A_3$ we conclude that
\[
\lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} M_j^{(n_k)} \frac{d z_j^{(n_k)}}{dt}(t) \frac{\partial g}{\partial t}(t,s_j^{(n_k)}) \, dt
= \lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} M_j^{(n_k)} \left( \frac{\partial v^{(n_k)}}{\partial t}(t,s_j^{(n_k)}) - \frac{\partial v}{\partial t}(t,s_j^{(n_k)}) \right) \frac{\partial g}{\partial t}(t,s_j^{(n_k)}) \, dt
+ \lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} M_j^{(n_k)} \frac{\partial v}{\partial t}(t,s_j^{(n_k)}) \frac{\partial g}{\partial t}(t,s_j^{(n_k)}) \, dt
\]
\[
= \int_0^\infty \left( \frac{\partial M_j}{\partial t}(t,\cdot), \frac{\partial g}{\partial t}(t,\cdot) \right) \, dt.
\]
and
\[
\lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} K_j^{(n_k)} z_j^{(n_k)} (t) g(t, s_j^{(n_k)}) \, dt \\
= \lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^{n_k} (v^{(n_k)}(t, s_j^{(n_k)}) - v(t, s_j^{(n_k)})) g(t, s_j^{(n_k)}) \, dt \\
+ \lim_{k \to \infty} \frac{1}{\rho_0 S} \int_0^\infty \sum_{j=1}^n K_j^{(n_k)} v(t, s_j^{(n_k)}) g(t, s_j^{(n_k)}) \, dt \\
= \int_0^\infty \langle \frac{\rho K}{\rho_0} v(t, \cdot), g(t, \cdot) \rangle \, dt.
\]

The limit function \( v \) then satisfies the equation (notice that \( \frac{\partial v}{\partial t}(0, \cdot) \) and \( \rho_M \) have disjoint supports)
\[
- \int_0^\infty \left\langle \left( 1 + \frac{\rho M}{\rho_0} \right) \frac{\partial v}{\partial t}, \frac{\partial g}{\partial t} \right\rangle \, dt - \int_0^\infty \left\langle \left( 1 + \frac{\rho M}{\rho_0} \right) \frac{\partial v}{\partial t}, \frac{\partial g}{\partial t} \right\rangle \\
= -a^2 \int_0^\infty \left\langle \frac{\partial v}{\partial x}, \frac{\partial g}{\partial x} \right\rangle \, dt - \int_0^\infty \left\langle \frac{p K}{\rho_0} v(t, \cdot), g(t, \cdot) \right\rangle.
\]

which is a weak form of equation (11) when the initial data \( v_0 \) is specified.

Posing \( w := 1 + \frac{\rho M}{\rho_0} \) and \( q := \frac{p K}{\rho_0} \), let us define the weighted \( L^2 \) space
\[
L^2(\mathbb{R}; w) := \{ f : \mathbb{R} \to \mathbb{C} \text{ measurable : } f \sqrt{w} \in L^2(\mathbb{R}) \}
\]
and the linear operator
\[
L : D(L) \subseteq L^2(\mathbb{R}; w) \to L^2(\mathbb{R}; w), \quad Lf := \frac{1}{w} \left( -a^2 \frac{d^2 f}{dx^2} + q f \right),
\]
\[
D(L) := \left\{ f \in L^2(\mathbb{R}; w) : f \in C^1(\mathbb{R}), \frac{df}{dx} \in AC(\mathbb{R}), \quad Lf \in L^2(\mathbb{R}; w) \right\}.
\]

Here \( AC(\mathbb{R}) \) denotes the set of absolutely continuous functions on the real line. Notice that \( L \) is in the limit point case at both \( +\infty \) and \( -\infty \) (use e.g. Theorem 6.3 in [22]) and so, by the theory of Sturm-Liouville operators (see e.g. [22]), \( L \) is self-adjoint (and positive). Hence, by the theory of abstract second-order equations (see e.g. [12], Chapter 2, Section 7), the Cauchy problem
\[
u \frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} - q v \\
v|_{t=0} = v_1 \in D(L), \quad \left. \frac{\partial v}{\partial t} \right|_{t=0} = v_2 \in D(\sqrt{L})
\]
has an unique strong solution
\[
v \in C^2(\mathbb{R}; L^2(\mathbb{R}; w)) \cap C^1(\mathbb{R}; D(\sqrt{L})) \cap C(\mathbb{R}; D(L)).
\]

Since, by hypotheses \( \mathcal{A}_2 \) and \( \mathcal{A}_3 \), \( v_1 \in D(L) \) and \( \frac{\partial v}{\partial x} \in D(\sqrt{L}) \), the regularity of the limit function \( v \) can be precised further: it is the unique strong solution of Cauchy problem (11)-(12) and satisfies (25).
Suppose now that the whole sequence \( \{v^{(n)}\}_{1}^{\infty} \) does not converge to \( v \). Then there exists a subsequence \( \{v^{(n^{'k})}\}_{1}^{\infty} \) such that for all \( k \) and some bounded rectangle \( R' \subset \mathbb{R}^2 \),

\[
\sup_{(t,x) \in R'} |v^{(n^{'k})}(t,x) - v(t,x)| \geq \varepsilon > 0
\]

(analogous relations hold for \( \frac{\partial v^{(n^{'k})}}{\partial t} \) and \( \frac{\partial v^{(n^{'k})}}{\partial x} \)). Since \( S^{(n^{'k})} \), \( M^{(n^{'k})} \) and \( K^{(n^{'k})} \) satisfy the assumptions \( A_1 \) and \( A_2 \) above, we get a contradiction and so the whole sequence converges to \( v \). □

We want to conclude the paper with few short remarks on the effective equation (11) we obtain in the limit. It is worth stressing that (11) is not simply a wave equation with space-dependent propagation speed \( \frac{a\sqrt{\rho_0}}{\sqrt{\rho_0 + \rho M}} \) taking into account the masses of the oscillators.

In fact, the potential term \( \frac{\rho K}{a^2 \rho_0} \) describes a source, depending linearly on the velocity field, due to the interaction of the field with the oscillators. In this respect, the problem we addressed is not a classical one in homogenization of the wave equation in highly inhomogeneous or random media. On the other hand, non-linear coupling between field and sources can be easily modeled. The higher-dimensional analogue of the problem treated in this paper is under investigation. Using techniques of self-adjoint extensions of symmetric operators, as we did in [6], we intend to make available two and three dimensional models of finite or infinite point emitters interacting via their own-generated acoustic field.

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HAUSDORFF CENTER FOR MATHEMATICS, INSTITUTE FOR APPLIED MATHEMATICS, BONN UNIVERSITY, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

Current address: Dipartimento di Scienza e Alta Tecnologia, Università dell’Insubria, Via Valleggio 11, 22100 Como, Italy

E-mail address: claudio.cacciapuoti@uninsubria.it

DIPARTIMENTO DI SCIENZE FISICHE, UNIVERSITÀ DI NAPOLI FEDERICO II, NAPOLI, ITALY

ISTITUTO NAZIONALE DI FISICA NUCLEARE, SEZIONE DI NAPOLI, VIA CINTIA, 80126 NAPOLI, ITALY

E-mail address: figari@na.infn.it

DIPARTIMENTO DI SCIENZA E ALTA TECNOLOGIA, UNIVERSITÀ DELL’INSUBRIA, VIA VALLEGGIO 11, 22100 COMO, ITALY

E-mail address: posilicano@uninsubria.it