ASYMPTOTIC BEHAVIOR FOR THE RADIAL EIGENVALUES OF $p$-LAPLACIAN IN CERTAIN ANNULAR DOMAINS

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Abstract. In this paper we prove an asymptotic behavior for the radial eigenvalues to the Dirichlet $p$-Laplacian problem $-\Delta_p u = \lambda |u|^{p-2}u$ in $\Omega$, $u = 0$ on $\partial \Omega$, where $\Omega$ is an annular domain $\Omega = \Omega_{R, \overline{R}}$ in $\mathbb{R}^N$.

1. Introduction

This paper investigates an asymptotic behavior for the radial eigenvalues $\lambda_k = \lambda_k(R, \overline{R})$ (when $0 < \overline{R} = R + 1$ and $R \to +\infty$) to the following eigenvalue problem

$$\begin{align*}
-\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in} \quad \Omega_{R, \overline{R}}, \\
u &= 0 \quad \text{on} \quad \partial \Omega_{R, \overline{R}},
\end{align*}$$

where $\Omega_{R, \overline{R}} = \{x \in \mathbb{R}^N : R < |x| < \overline{R}\}$, with $0 < R < \overline{R}$ constants in $\mathbb{R}$, is the annular domain, and we suppose that

$$1 < p \leq N.$$

In particular, when $p = 2$, we obtain the Dirichlet Laplacian problem

$$\begin{align*}
-\Delta u &= \lambda u \quad \text{in} \quad \Omega_{R, \overline{R}}, \\
u &= 0 \quad \text{on} \quad \partial \Omega_{R, \overline{R}}.
\end{align*}$$

Since we are interested only in the radial eigenvalues of (1.1), we can rewrite (1.1) as the following 1-dimensional eigenvalue problem

$$\begin{align*}
(r^{N-1} |u'(r)|^{p-2}u'(r))' + \lambda r^{N-1} |u(r)|^{p-2}u(r) &= 0 \quad \text{in} \quad (R, \overline{R}), \\
u(R) &= u(\overline{R}) = 0.
\end{align*}$$

We remark that for every $1 \leq k \in \mathbb{N}$, if we denote by $\lambda_k$ the $k$-th eigenvalue of (1.3) and by $\lambda_k^{\text{rad}}$ the $k$-th radial eigenvalue of (1.1),

$$\lambda_k = \lambda_k^{\text{rad}}.$$

In order to study the solution of (1.1), one can make a standard change of variables, see for example [1, 5].

If $N > p$, let $t = -\frac{A}{r^{N-p}(r^{N-p})'} + B$ and $v(t) = u(r)$, where

$$A = \frac{(R\overline{R})^{\frac{N-p}{p-1}}}{R^{\frac{N-p}{p-1}} - \overline{R}^\frac{N-p}{p-1}} \quad \text{and} \quad B = \frac{\overline{R}^\frac{N-p}{p-1}}{R^{\frac{N-p}{p-1}} - \overline{R}^\frac{N-p}{p-1}},$$

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then the problem (1.3) (hence (1.1)) transforms into the boundary value problem for the nonautonomous ODE

$$\begin{cases}
|v'(t)|^{p-2}v'(t)'' + \lambda q(t)|v(t)|^{p-2}v(t) = 0 & \text{in } (0,1), \\
v(0) = v(1) = 0.
\end{cases}$$

where

$$q(t) := q_{R,R}(t) = \left( \frac{p-1}{N-p} \right)^p \frac{A^{(p-1)/p}}{(B-t)^{N-1}}.$$ 

In the case $p = N$, one sets $r = R\left( \frac{R}{R} \right)^{t}$ and $v(t) = u(r)$, obtaining again the problem (1.4), now with

$$q(t) := q_{R,R}(t) = \left[ R \left( \frac{R}{R} \right)^{t} \ln \left( \frac{R}{R} \right) \right]^p.$$ 

Let $\lambda_k$ be the $k$-th eigenvalue of (1.4) and let $\varphi_k$ be an eigenfunction corresponding to $\lambda_k$. Since $q$ satisfies

$$q \in C^1([0,1]), \ q(t) > 0 \text{ for } 0 \leq t \leq 1.$$ 

It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \lim_{k \to \infty} \lambda_k = \infty,$$

and that $\varphi_k$ has exactly $k - 1$ zeros in $(0,1)$, see [2, 3].

Motivated by the work of Zhang [7], whose purpose was to compute an estimate for eigenvalues of the Dirichlet $p$-Laplacian (1.4), $p > 1$, we propose to prove an asymptotic behavior for the $\lambda_k(R,R)$ in the form:

$$\lim_{R \to +\infty} \lambda_k(R,R + 1).$$

The following estimate is known, see Zhang [7, Remark 2.1]. We suppose that

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)},$$

$$\bar{q}-(R,R) = \int_0^1 \min\{1,q(t)\}dt$$

and

$$\bar{q}+(R,R) = \int_0^1 \max\{1,q(t)\}dt.$$ 

In [7] (with $T = 1$), the author proved the inequality

$$\left( \frac{k\pi_p}{\bar{q}+(R,R)} \right)^p \leq \lambda_k(R,R) \leq \left( \frac{k\pi_p}{\bar{q}-(R,R)} \right)^p.$$ 

In S.S. Lin [4, Lemma A.1], the author proves an asymptotic behavior for all eigenvalues (that is, radial and non-radial eigenvalues), of Dirichlet problem (1.2), which is the following result.

**Lemma 1.1** ([4]). Let $\lambda_{k,j}(R,R + 1)$ be the $j$-th eigenvalue of

$$\phi'' + \frac{N-1}{r} \phi' - \frac{\alpha_k}{r^2} \phi = -\lambda_{k,j}(R,R + 1) \phi, \text{ in } (R,R + 1),$$

$$\phi(R) = \phi(R + 1) = 0,$$
where $\alpha_k = k(k + N - 2)$, and let $\lambda_j = j^2\pi^2$ be the $j$-th eigenvalue of
\[
\phi'' = -\lambda_\phi, \text{ in } (0, 1),
\]
$\phi(0) = \phi(1) = 0$,
where $k = 0, 1, 2 \cdots$ and $j = 1, 2, 3, \cdots$. Then
\[
\lim_{R \to +\infty} \lambda_{k,j}(R, R + 1) = \lambda_j.
\]

In this paper, we will prove a generalization of the results of [4], for the radial eigenvalues of $p$-Laplacian, in the cases when $p = r + 1$ and $N = 2r + 1$, with $r \in \mathbb{N}$; $p = N$ and $p = 2 < 3 \leq N$. The last case is another proof of Lin’s result for the radial eigenvalues. It is noteworthy that it is not yet known a characterization for all eigenvalues of (1.1). The results of [4] are consequence of some results of Bessel functions, while in our paper we use a totally different approach following the work of Zhang [7].

We state the main result.

**Theorem 1.2.** (i) Suppose that $N = p$. Then
\[
\lim_{R \to +\infty} \bar{q}_+(R, R + 1) = \lim_{R \to +\infty} \bar{q}_-(R, R + 1) = 1.
\]
In particular, by (1.8)
\[
\lim_{R \to +\infty} \lambda_k(R, R + 1) = (k\pi)^p,
\]
that is, the eigenvalues $\lambda_k(R, R + 1)$ converge asymptotically to the eigenvalues of the problem
\[
\begin{cases}
(|v'(t)|^{p-2}v'(t))' + \lambda |v(t)|^{p-2}v(t) = 0 \text{ in } (0, 1), \\
v(0) = v(1) = 0.
\end{cases}
\]

(ii) Suppose that $p = 2$ and $N \geq 3$. Then,
\[
\lim_{R \to +\infty} \bar{q}_+(R, R + 1) = \lim_{R \to +\infty} \bar{q}_-(R, R + 1) = 1.
\]
In particular, by (1.8)
\[
\lim_{R \to +\infty} \lambda_k(R, R + 1) = (k\pi)^2,
\]
that is, the eigenvalues $\lambda_k(R, R + 1)$ converge asymptotically to the eigenvalues of the problem
\[
\begin{cases}
v''(t) + \lambda v(t) = 0 \text{ in } (0, 1), \\
v(0) = v(1) = 0.
\end{cases}
\]

2. Proofs

**Proof of Theorem 1.2** (i): By (1.6), we have
\[
q(t) = R \left( \frac{\bar{R}}{R} \right)^t \ln \left( \frac{\bar{R}}{R} \right)^p.
\]
Therefore,
\[
q'(t) = p \left[ R \left( \frac{\bar{R}}{R} \right)^t \ln \left( \frac{\bar{R}}{R} \right)^p \right]
\]
\[
\left( \ln \left( \frac{\bar{R}}{R} \right) \right)^2 > 0,
\]
hence, the function $q$ is increasing at $t$ and

$$0 < q(0) \leq q(t) \leq q(1), \quad \forall \, t \in [0, 1],$$

that is,

$$\left[R \ln \left( \frac{R}{R} \right) \right]^p \leq q(t) \leq \left[R \ln \left( \frac{R}{R} \right) \right]^p, \quad \forall \, t \in [0, 1].$$

If $R > 0$ and $\overline{R} = R + 1$, we have

$$q(0) = \left[R \ln \left( \frac{R + 1}{R} \right) \right]^p = \left[\ln \left( 1 + \frac{1}{R} \right) \right]^p$$

and

$$q(1) = \left[(R + 1) \ln \left( \frac{R + 1}{R} \right) \right]^p = \left[\ln \left( 1 + \frac{1}{R} \right) + \ln \left( 1 + \frac{1}{R} \right) \right]^p.$$

Therefore,

(2.2) \quad \lim_{R \to +\infty} q(0) = 1

and

(2.3) \quad \lim_{R \to +\infty} q(1) = 1,

where we used the Euler Number

$$e = \lim_{R \to +\infty} \left(1 + \frac{1}{R}\right)^R.$$

By (2.2) and (2.3), we obtain that

$$\lim_{R \to +\infty} q(t) = 1 \text{ uniformly in } t \in [0, 1].$$

As

$$\bar{q}_-(R, R + 1) = \int_0^1 \min\{1, q(t)\} \, dt = \int_0^1 \frac{1 + q(t) - |1 - q(t)|}{2} \, dt,$$

we conclude that

(2.4) \quad \lim_{R \to +\infty} \bar{q}_-(R, R + 1) = 1.

As

$$\bar{q}_+(R, R + 1) = \int_0^1 \max\{1, q(t)\} \, dt = \int_0^1 \frac{1 + q(t) + |1 - q(t)|}{2} \, dt,$$

we conclude that

(2.5) \quad \lim_{R \to +\infty} \bar{q}_+(R, R + 1) = 1.

It follows from (1.8) that

(2.6) \quad \left(\frac{k \pi_p}{\bar{q}_+(R, R + 1)}\right)^p \leq \lambda_k(R, R + 1) \leq \left(\frac{k \pi_p}{\bar{q}_-(R, R + 1)}\right)^p.

By (2.4), (2.5) and by limits in (2.6), we obtain

$$\lim_{R \to +\infty} \lambda_k(R, R + 1) = (\pi_p k)^p.$$

The proof of $(\pi_p k)^p$ is a solution of (1.11), according to Zhang [7] (see also del Pino and Manasevich [6]).
Proof of (ii): By (1.5), if \( N > p \), we have
\[
q(t) = \left( \frac{p - 1}{N - p} \right)^p A^{(\frac{p-1}{p})} (B - t)^{\frac{p(N-1)}{N-p}}.
\]
Therefore,
\[
q'(t) = \left( \frac{p - 1}{N - p} \right)^p \frac{p(N-1)}{N-p} (B - t)^{\frac{p(N-1)}{N-p} - 1} > 0,
\]
and the function \( q \) is increasing at \( t \). Hence,
\[
0 < q(0) \leq q(t), \quad \forall t \in [0, 1].
\]
Since \( p = 2 \) and \( N \geq 3 \), by (2.7), we have
\[
q(t) = (N-2)^{-2} \left( \frac{R - R^{N-2}}{R^{N-2}} - t \right)^{\frac{N-2}{N-1}}.
\]
In particular,
\[
q(0) = \frac{1}{(N-2)^2} R^2 \left( \frac{R - R^{N-2}}{R^{N-2}} \right)^2
\]
and
\[
q(1) = \frac{1}{(N-2)^2} R^2 \left( \frac{R - R^{N-2}}{R^{N-2}} \right)^2.
\]
If \( R > 0 \) and \( \overline{R} = R + 1 \), we have
\[
q(0) = \frac{1}{(N-2)^2} R^2 \left( \frac{(R+1)^{N-2} - R^{N-2}}{(R+1)^{N-2}} \right)^2
\]
\[
= \frac{1}{(N-2)^2} R^2 (R + 1 - R)^2 \left( \frac{(R+1)^{N-3} + (R+1)^{N-4} R^{N-4} + \cdots + (R+1) R^{N-4} + R^{N-3}}{(R+1)^{N-2}} \right)^2
\]
\[
= \frac{1}{(N-2)^2} R^2 \left( \sum_{j=0}^{N-3} \frac{(R + 1)^{N-(3+j)} R^j}{(R+1)^{N-2}} \right)^2
\]
and we conclude that
\[
\lim_{R \to +\infty} q(0) = 1,
\]
where we used that
\[
R^2 \left( \sum_{j=0}^{N-3} \frac{(R + 1)^{N-(3+j)} R^j}{(R+1)^{N-2}} \right)^2 = \left( 1 - \frac{1}{R + 1} \right)^2 \left( \sum_{j=0}^{N-3} \left( 1 - \frac{1}{R + 1} \right)^j \right)^2 \to (N-2)^2,
\]
as \( R \to +\infty \).

Similarly, \( q(1) \) as above.
and we conclude that
\[(2.9) \lim_{R \to +\infty} q(1) = 1.\]

Therefore, by (2.8) and (2.9), we obtain that
\[\lim_{R \to +\infty} q(t) = 1 \text{ uniformly in } t \in [0,1].\]

As
\[\bar{q}-(R, R+1) = \int^1_0 \min \{1, q(t)\} dt = \int^1_0 \frac{1 + q(t) - |1 - q(t)|}{2} dt,\]
we conclude that
\[(2.10) \lim_{R \to +\infty} \bar{q}-(R, R+1) = 1.\]

As
\[\bar{q}+(R, R+1) = \int^1_0 \max \{1, q(t)\} dt = \int^1_0 \frac{1 + q(t) + |1 - q(t)|}{2} dt,\]
we conclude that
\[(2.11) \lim_{R \to +\infty} \bar{q}+(R, R+1) = 1.\]

Since by (1.7) we have \(\pi_2 = \pi\), it follows from (1.8) that
\[(2.12) \left(\frac{k\pi}{\bar{q}+(R, R+1)}\right)^2 \leq \lambda_k(R, R+1) \leq \left(\frac{k\pi}{\bar{q}-(R, R+1)}\right)^2.\]

By (2.10), (2.11) and by limits in (2.12), we obtain
\[\lim_{R \to +\infty} \lambda_k(R, R+1) = \pi^2 k^2.\]

This proves the item \((ii)\).  \(\square\)

3. Additional results

Similar to Theorem 1.2 we can get the following result. Let \(r \in \mathbb{N}\). Suppose that \(p = r + 1\) and \(N = 2r + 1\). Then,
\[(3.1) \lim_{R \to +\infty} \bar{q}+(R, R+1) = \lim_{R \to +\infty} \bar{q}-(R, R+1) = 1.\]

In particular, by (1.8)
\[(3.2) \lim_{R \to +\infty} \lambda_k(R, R+1) = (k\pi p)^p = (\pi r + 1 k)^r.\]

Indeed, since \(p = r + 1\) and \(N = 2r + 1\), then
\[\left(\frac{p - 1}{N - p}\right)^p = 1, \quad \frac{N - p}{p - 1} = 1, \quad \frac{(p - 1)p}{N - p} = r + 1, \quad \frac{p(N - 1)}{N - p} = 2(r + 1)\]
and by (2.7), we have
\[q(t) = \left(\frac{R^2}{R - t}\right)^{r+1}.\]

If \(R > 0\) and \(\overline{R} = R + 1\), we have
\[q(t) = \frac{(R^2 + R)^{r+1}}{(R + 1 - t)^{2(r+1)}}.\]
In particular,
\[ q(0) = \left( \frac{R}{R + 1} \right)^{r+1} < 1 \]
and
\[ q(1) = \left( \frac{R + 1}{R} \right)^{r+1} > 1. \]

Therefore,
\[ \lim_{R \to +\infty} q(0) = 1 \]  
and
\[ \lim_{R \to +\infty} q(1) = 1. \]

By (3.3) and (3.4), we obtain that
\[ \lim_{R \to +\infty} q(t) = 1 \text{ uniformly in } t \in [0, 1]. \]

As
\[ \bar{q}_-(R, R + 1) = \int_0^1 \min\{1, q(t)\} \, dt = \int_0^1 \frac{1 + q(t) - |1 - q(t)|}{2} \, dt, \]
we conclude that
\[ \lim_{R \to +\infty} \bar{q}_-(R, R + 1) = 1. \]

As
\[ \bar{q}_+(R, R + 1) = \int_0^1 \max\{1, q(t)\} \, dt = \int_0^1 \frac{1 + q(t) + |1 - q(t)|}{2} \, dt, \]
we conclude that
\[ \lim_{R \to +\infty} \bar{q}_+(R, R + 1) = 1. \]

It follows from (3.7) that
\[ \left( \frac{k \pi_p}{q_+(R, R + 1)} \right)^p \leq \lambda_k(R, R + 1) \leq \left( \frac{k \pi_p}{q_-(R, R + 1)} \right)^p. \]

By (3.5), (3.6) and by limits in (3.7) we obtain
\[ \lim_{R \to +\infty} \lambda_k(R, R + 1) = (\pi_p k)^p = (\pi_{r+1} k)^{r+1}. \]

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