Path-Integral Measures in Higher-Derivative Gravities

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A simple method of obtaining path-integral measures in higher-derivative gravities is presented. The measures are nothing but the generalized Lee-Yang terms.

§1. Introduction

Path-integral measures in higher-derivative gravities have been calculated by Buchbinder and Lyahovich. To obtain the results, they investigated the structure of the constraints of the theories, performing canonical quantization straightforwardly.

In the present paper, we propose a simple method of deriving path-integral measures. The measures are obtained by calculating the generalized Lee-Yang terms.

In §2 a generic higher-derivative system is considered. We review canonical formalism and path-integral quantization of the system, showing that the generalized Lee-Yang term gives the path-integral measure. The Lee-Yang terms are explicitly calculated for Einstein gravity in §3, and for four-derivative gravity in §4. The measures obtained there agree with the ones of Buchbinder-Lyahovich. The method is also applicable to the case of more-than-four-derivative gravities, which is studied in §5. Section 6 gives summary and discussion.

§2. Generalized Lee-Yang term

We consider a generic system with coordinates $x_a(t)$ ($a = 1, \ldots, N$) of Grassmann parities $\varepsilon_a$. The Lagrangian of the system is assumed to contain up to $n_a$-th derivative of $x_a(t)$

$$L = L(x_a, \dot{x}_a, \ddot{x}_a, \ldots, x_a^{(n_a)}),$$

where

$$x_a^{(r_a)} = (d/dt)^{r_a} x_a. \quad (r_a = 1, \ldots, n_a)$$

Canonical formalism of Ostrogradski$^\circ$ regards $x_a^{(s_a)}$ ($s_a = 1, \ldots, n_a - 1$) as independent coordinates $q^{s_a + 1}$:

$$x_a^{(s_a)} \rightarrow q^{s_a + 1},$$

$$L(x_a, \dot{x}_a, \ldots, x_a^{(n_a)}) \rightarrow L(q_a^1, \ldots, q_a^{n_a}, \dot{q}_a^{n_a}).$$

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Momenta conjugate to $q^{na}_a$ are defined as usual:

$$p^{a}_{na} \equiv \partial L_q/\partial \dot{q}^{na}_a. \quad (2.5)$$

The Hessian matrix of $L_q$ is

$$A^{ab} \equiv \partial (\partial/\partial \dot{q}^{na}_a) \partial L_q/\partial \dot{q}^{nb}_b. \quad (2.6)$$

If the system is nonsingular $\det A^{ab} \neq 0$, which is the only case considered hereafter, then the relation (2.5) can be inverted to give $\dot{q}^{na}_a$ as functions of $q^{ra}_a$ ($r_a = 1, \ldots, n_a$) and $p^{a}_{na}$:

$$\dot{q}^{na}_a = \dot{q}^{na}_a(q^r, p_n). \quad (2.7)$$

The Hamiltonian is defined by

$$H \equiv p^{a}_{na} q^{n+1}_a + p^{a}_{na} \dot{q}^{na}_a(q^r, p_n) - L_q(q^r, q^n(q^r, p_n)). \quad (2.8)$$

Canonical equations of motion $\dot{q}^r = (\partial/\partial p_r) H$ and $\dot{p}_r = -\partial H/\partial q^r$ determine the time development of the system. Path integral is

$$Z = \int \mathcal{D}q^{na}_a \mathcal{D}p^{a}_{na} \exp \left\{ i \int dt \left[ p^{a}_{na} \dot{q}^{na}_a - H(q^r, p_r) \right] \right\}. \quad (2.9)$$

After integration with respect to $p^{a}_{na}$ and $q^{n+1}_a$, this reduces to

$$Z = \int \mathcal{D}q^{1(a)}_a \mathcal{D}p^{a}_{na} \exp \left\{ i \int dt \left[ p^{a}_{na} \dot{q}^{1(a)}_a - \hat{H}(q^1, q^{1(s)}, p_n) \right] \right\}, \quad (2.10)$$

where

$$\hat{H}(q^1, q^{1(s)}, p_n) \equiv p^{a}_{na} \dot{q}^{1(a)}_a(q^1, q^{1(s)}, p_n) - L_q(q^1, q^{1(s)}, \dot{q}^{1(a)}_a(q^1, q^{1(s)}, p_n)), \quad (2.11)$$

$$q^{1(a)}_a \equiv (d/dt)^{s_a} q^1_a. \quad (2.12)$$

To proceed further, take the following special case:

$$L_q = \frac{1}{2} q^{na}_a A^{ab} q^{nb}_b + B^a q^{na}_a + C, \quad \det A^{ab} \neq 0, \quad (2.13)$$

where $A^{ab}, B^a$ and $C$ are arbitrary functions of $q^{na}_a$ with the properties

$$A^{ba} = (-)^s a + e_b + e_a \epsilon^b A^{ab}, \quad (A^{ab})^s = A^{ba},$$

$$(B^a)^* = (-)^s \epsilon^a B^a, \quad C^* = C. \quad (2.14)$$

In this case, the conjugate momenta are

$$p^{a}_{na} = (-)^s a A^{ab} \dot{q}^{nb}_b + B^a. \quad (2.15)$$

\(^*) In order to distinguish between left and right derivatives, we use the following notations:

$$\partial (\partial/\partial \theta) A$$ for left derivative,
$$\partial A/\partial \theta$$ for right derivative.
The Hamiltonian $\hat{H}$ in (2.11) becomes

$$\hat{H} = \frac{1}{2} (p^a_{na} - B^a) A_{ab} (-)^c (p^b_{na} - B^b) - C,$$

(2.16)

where $A_{ab}$ represents the inverse matrix of $A^{ab}$. Integration with respect to $p^a_{na}$ can be carried out in (2.10). The result is

$$Z = \int Dq^a_{na} \Delta \exp \left( i \int dt L \right).$$

(2.17)

In this expression, $L$ is the higher-derivative Lagrangian in the original configuration space

$$L \equiv \frac{1}{2} q^a_{na} A^{ab} q^b_{na} + B^a q^1_{na} + C.$$  

(2.18)

The measure $\Delta$ is given as the generalized Lee-Yang term

$$\Delta \equiv (\det A_{ab})^{-1/2} = (\det A^{ab})^{1/2}.$$  

(2.19)

§3. Measure in Einstein gravity

The Lagrangian is given by

$$\mathcal{L}^{(1)} = \mathcal{L}_E + \mathcal{L}_{\text{BRS}},$$

(3.1)

$$\mathcal{L}_E = - \frac{1}{\kappa^2} \sqrt{-g} R.$$  

(3.2)

For the BRS Lagrangian we take

$$\mathcal{L}_{\text{BRS}}^{(1)} = - i\delta \left[ \bar{c}_\mu \left( \frac{1}{\kappa} \partial_\nu \tilde{g}^{\mu\nu} - \frac{\alpha}{2} \eta^{\mu\nu} b_\nu \right) \right].$$

(3.3)

$$\mathcal{L}_{\text{GF}}^{(1)} = \frac{1}{\kappa} b_\mu \partial_\nu \tilde{g}^{\mu\nu} - \frac{\alpha}{2} \eta^{\mu\nu} b_\mu b_\nu,$$

(3.4)

$$\mathcal{L}_{\text{FP}}^{(1)} = - \frac{i}{2} (\partial_\mu \bar{c}_\nu + \partial_\nu \bar{c}_\mu) D^{\mu\nu} c^\rho.$$  

(3.5)

Here $\kappa$ is the gravitational constant; $\alpha$ is a gauge parameter; $\eta^{\mu\nu}$ is the Minkowski metric $(-+++)$; $c^\mu, \bar{c}_\mu$ are the Faddeev-Popov (FP) ghosts; $b_\mu$ are the Nakanishi-Lautrap (NL) fields; $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$; and

$$D^{\mu\nu} \equiv \tilde{g}^{\mu\sigma} \partial_\sigma + \tilde{g}^{\nu\sigma} \partial_\sigma - \tilde{g}^{\mu\nu} \partial_\sigma - (\partial_\mu \tilde{g}^{\nu\sigma}).$$

(3.6)

The Lagrangian $\mathcal{L}^{(1)}$ is invariant under the following BRS transformation:

$$\begin{align*}
\delta \tilde{g}^{\mu\nu} &= \kappa D^{\mu\nu} c^\rho, \\
\delta c^\mu &= - \kappa c^\lambda \partial_\lambda c^\mu, \\
\delta \bar{c}_\mu &= i b_\mu, \\
\delta b_\mu &= 0.
\end{align*}$$

(3.7)
Because the BRS Lagrangian $L^{(1)}_{	ext{BRS}}$ is introduced from the beginning, the system is made nonsingular. In order to calculate the Hessian matrix it is convenient to rewrite $L^{(1)}_{\text{GF}}$ by performing path-integration with respect to $b_\mu$ in advance:

$$L^{(1)}_{\text{GF}} \rightarrow L^{(1)'}_{\text{GF}} = \frac{1}{2\alpha k^2} \eta_{\mu\nu} \partial_\mu \tilde{g}^{\nu\rho} \partial_\sigma \tilde{g}_{\rho\sigma}. \quad (3.8)$$

Let the ADM coordinates be $(\lambda, \lambda_i, e_{ij})$:

$$g_{\mu\nu} = \begin{pmatrix} \lambda^k \lambda_k - \lambda^2 & \lambda_j e_{ij} \\ \lambda_i & \lambda^i \equiv \epsilon^{ij} \lambda_j \end{pmatrix}, \quad \lambda^i \equiv \epsilon^{ij} \lambda_j. \quad (3.9)$$

Expressing the Lagrangians (3.2), (3.8) and (3.5) in terms of the ADM coordinates, we have

$$L_E = \frac{1}{\kappa^2} \frac{1}{4\lambda} \sqrt{e} \left( -P^{(2)ijkl} + 2P^{(0)ijkl} \right) \dot{e}_{ij} \dot{e}_{kl} + \ldots, \quad (3.10)$$

$$L^{(1)'}_{\text{GF}} = \frac{1}{2\alpha k^2} \sqrt{e} \left\{ -\frac{1}{\lambda^4} \left( 1 - \eta_{mn} \lambda^m \lambda^n \right) \dot{\lambda}^2 - \frac{2}{\lambda^2} \eta_{mn} \lambda^m \epsilon^{ni} \dot{\lambda}_i 
+ \frac{1}{\lambda^3} \left[ (1 - \eta_{mn} \lambda^m \lambda^n) \epsilon^{ij} + \eta_{mn} \lambda^m \left( \epsilon^{ni} \dot{\lambda}_j + \epsilon^{nj} \dot{\lambda}_i \right) \right] \dot{\lambda}_{ij} 
+ \frac{1}{\lambda^2} \eta_{mn} \epsilon^{ni} \dot{\lambda}_i \dot{\lambda}_j + \frac{1}{\lambda^2} \eta_{mn} \epsilon^{ni} \lambda^m \left( \lambda^n \epsilon_{kl} - \epsilon^{nk} \lambda^l - \epsilon^{nl} \lambda^k \right) \dot{\lambda}_i \dot{e}_{kl} 
+ \frac{1}{4\lambda^2} \left[ -\epsilon^{ij} \epsilon_{kl} \right. 
+ \eta_{mn} \left( \lambda^m \epsilon^{ij} - \epsilon^{ni} \lambda^j - \epsilon^{mj} \lambda^i \right) \left( \lambda^n \epsilon_{kl} - \epsilon^{nk} \lambda^l - \epsilon^{nl} \lambda^k \right) \dot{e}_{ij} \dot{e}_{kl} \left\} 
+ \ldots, \quad (3.11)$$

$$L^{(1)}_{\text{FP}} = i \sqrt{e} \left\{ \frac{1}{\lambda} \dot{\epsilon}_{0c}^0 + \frac{1}{\lambda^2} \dot{\epsilon}_{0c}^0 \dot{\lambda} - \frac{1}{2\lambda} \dot{\epsilon}_{0c}^0 \epsilon^{ij} \dot{e}_{ij} + \frac{1}{\lambda} \dot{\epsilon}_{ij} \dot{\lambda}_j 
+ \frac{1}{\lambda} \dot{\epsilon}_{ij} \dot{\lambda}_j + \frac{1}{\lambda} \left[ \epsilon^{ij} \epsilon_{kl} - \epsilon^{ik} \epsilon_{jl} - \epsilon^{jl} \epsilon_{ik} \right] \lambda_j \dot{e}_{kl} \right\} 
+ \ldots, \quad (3.12)$$

where the projection operators

$$P^{(0)ijkl} \equiv \frac{1}{3} \epsilon^{ij} \epsilon_{kl}, \quad P^{(2)ijkl} \equiv \frac{1}{2} \left( \epsilon^{ik} \epsilon_{jl} + \epsilon^{il} \epsilon_{jk} \right) - \frac{1}{3} \epsilon^{ij} \epsilon_{kl} \quad (3.13)$$

have been introduced. In (3.10)-(3.12), only the highest-derivative terms have been written down explicitly.

These terms are sufficient to read the Hessian matrix

$$M^{(1)} = e_{kl} \begin{pmatrix} e_{ij} & \lambda_b & e^0 & \bar{c}_0 & \bar{c}^0 & \bar{c}_j \\ \lambda^i & \lambda^0 & \epsilon^{ij} & \epsilon_{ij} & \epsilon_{ij} \end{pmatrix} \begin{pmatrix} A^{(1)} & F^{(1)} \\ F^{(1)t} & B^{(1)} \end{pmatrix}, \quad (3.14)$$
The submatrix $A^{(1)}$

$$A^{(1)} = \begin{pmatrix} A^{(1)ij,kl} & A^{(1)ij,0} & A^{(1)ij,b} \\ A^{(1)0,kl} & A^{(1)00} & A^{(1)0b} \\ A^{(1)a,kl} & A^{(1)a0} & A^{(1)ab} \end{pmatrix}$$  \hspace{1cm} (3.15)$$

has the elements

$$A^{(1)ij,kl} = \frac{1}{\kappa^2} \sqrt{e} \left[ -P^{(2)ij,kl} + 2P^{(0)ij,kl} \right] + \frac{1}{2\alpha\kappa^2} \frac{e}{2\lambda^2} \left[ -e^{ij}e^{kl} + \eta_{mn} \left( \lambda^n e^{ij} - e^{mi} \chi^j - e^{mj} \chi^i \right) \left( \lambda^l e^{kl} - e^{nk} \lambda^l - e^{nl} \lambda^k \right) \right],$$  \hspace{1cm} (3.16)$$

$$A^{(1)ij,0} = \frac{1}{2\alpha\kappa^2} \left( \frac{e}{\lambda^3} \right) (1 - \eta_{mn} \lambda^m \lambda^n),$$  \hspace{1cm} (3.17)$$

$$A^{(1)ij,b} = \frac{1}{2\alpha\kappa^2} \eta_{mn} e^{mi} \left( \lambda^n e^{ij} - e^{mi} \chi^j - e^{mj} \chi^i \right),$$  \hspace{1cm} (3.18)$$

$$A^{(1)00} = \frac{1}{2\alpha\kappa^2} \left( \frac{e}{\lambda^3} \right) \eta_{mn} \lambda^m e^{nb},$$  \hspace{1cm} (3.19)$$

$$A^{(1)ab} = \frac{1}{2\alpha\kappa^2} \frac{2e}{\lambda^2} \eta_{mn} e^{ma} e^{nb}. $$  \hspace{1cm} (3.20)$$

The submatrix $B^{(1)}$ has the form

$$B^{(1)} = \begin{pmatrix} J^{(1)} \\ J^{(1)} \\ J^{(1)} \end{pmatrix}, \quad J^{(1)} = \begin{pmatrix} 1 & \sqrt{e} \lambda & 1 \\ 1 & -1 & 0 \end{pmatrix}. $$  \hspace{1cm} (3.22)$$

It is easy to calculate the determinants of these submatrices:

$$\det A^{(1)} \propto \lambda^{-16} e, $$  \hspace{1cm} (3.23)$$

$$\det B^{(1)} \propto \lambda^{-8} e^4. $$  \hspace{1cm} (3.24)$$

Using the fact $\Gamma^{(1)} B^{(1)} - 1 \Gamma^{(1)} t = 0$, we have

$$\det M^{(1)} = \frac{\det (A^{(1)} - \Gamma^{(1)} B^{(1)} - 1 \Gamma^{(1)} t)}{\det B^{(1)}} = \frac{\det A^{(1)}}{\det B^{(1)}} \propto \lambda^{-8} e^{-3}. $$  \hspace{1cm} (3.25)$$

Changing field variables from the ADM coordinates $(\lambda, \lambda_i, e_{ij})$ to the original ones $g_{\mu\nu}$ gives the factor

$$(g^{(0)})^m (-g)^n \mathcal{D}g_{\mu\nu} = \lambda^{-2m+2n+1} e^n \mathcal{D} \lambda \mathcal{D} \lambda_i \mathcal{D} e_{ij}$$

$$= (\det M^{(1)})^{1/2} \mathcal{D} \lambda \mathcal{D} \lambda_i \mathcal{D} e_{ij}. $$  \hspace{1cm} (3.26)$$

In the present case Eq.(3.25) shows $m = 1$ and $n = -3/2$. Thus the path-integral measure $\Delta^{(1)}$ is

$$\Delta^{(1)} = g^{00} (-g)^{-3/2}. $$  \hspace{1cm} (3.27)$$
Bringing back the $b_{\mu}$-integration returns the gauge-fixing Lagrangian $L^{(1)'}_{\text{GF}}$ to the original form $L^{(1)}_{\text{GF}}$. We finally obtain

$$Z^{(1)} = \int Dg_{\mu\nu} Db_{\mu} Dc^\mu D\bar{c}_\mu \Delta^{(1)} \exp \left( i \int L^{(1)} d^4 x \right). \quad (3.28)$$

§4. Measure in four-derivative gravity

The Lagrangian of four-derivative gravity is given by

$$L^{(2)} = L^{(2)}_E + L^{(2)}_{\text{HD}} + L^{(2)}_{\text{BRS}}, \quad (4.1)$$

$$L^{(2)}_{\text{HD}} = \sqrt{-g} \left( \beta_2 R_{\mu\nu} R^{\mu\nu} + \beta_0 R^2 \right), \quad (4.2)$$

$$L^{(2)}_{\text{BRS}} = L^{(1)}_{\text{BRS}} \quad (4.3)$$

with the parameters of the theory $\beta_2, \beta_0$. For the BRS Lagrangian we have adopted the same one as in the previous section. This is sufficient for making the theory nonsingular.

By the use of the ADM coordinates the Lagrangians are expressed as

$$L_E + L^{(2)}_{\text{HD}} = \frac{1}{4\lambda^3} \sqrt{\varepsilon} A^{(2)ij,kl}$$

$$\times \left[ \dot{e}_{ij} - \frac{\dot{\lambda}}{\lambda} (\dot{e}_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i) - \left( \nabla_i \dot{\lambda}_j + \nabla_j \dot{\lambda}_i \right) \right]$$

$$\times \left[ \dot{e}_{kl} - \frac{\dot{\lambda}}{\lambda} (\dot{e}_{kl} - \nabla_k \lambda_l - \nabla_l \lambda_k) - \left( \nabla_k \dot{\lambda}_l + \nabla_l \dot{\lambda}_k \right) \right]$$

$$+ \ldots, \quad (4.4)$$

$$L^{(2)'}_{\text{GF}} = L^{(1)'}_{\text{GF}}$$

$$= \frac{1}{2\alpha^2} \varepsilon \left[ - \frac{1}{\lambda^4} (1 - \eta_{mn} \lambda^m \lambda^n) \dot{\lambda}^2 \right.$$

$$- \frac{2}{\lambda^2} \eta_{mn} \lambda^m \varepsilon^{ni} \dot{\lambda}_i + \frac{1}{\lambda^2} \eta_{mn} \varepsilon^{ni} \varepsilon^{nj} \dot{\lambda}_i \dot{\lambda}_j \right]$$

$$+ \ldots, \quad (4.5)$$

$$L^{(2)}_{\text{FP}} = L^{(1)}_{\text{FP}}$$

$$= i \sqrt{\varepsilon} \left[ \frac{1}{\lambda} \dot{c}_0 c^0 + \frac{1}{\lambda^2} \dot{c}_0 c^0 \lambda + \frac{1}{\lambda} \dot{c}_i c^i + \frac{1}{\lambda^2} \dot{c}_i c^i \left( - \frac{1}{\lambda} \lambda^i \dot{\lambda} + \varepsilon^{ij} \dot{\lambda}_j \right) \right]$$

$$+ \ldots, \quad (4.6)$$

where $A^{(2)ij,kl}$ represents

$$A^{(2)ij,kl} \equiv \frac{d}{4} \beta_2 P^{(2)ij,kl} + 4 (\beta_2 + 3 \beta_0) P^{(0)ij,kl} \quad (4.7)$$

and $\nabla_i$ denotes the covariant derivative associated with $e_{ij}$. We assume $\beta_2 \neq 0$ and $\beta_2 + 3 \beta_0 \neq 0$ from now on. In this case the method of obtaining path-integral measures explained in §2 is applicable without any complexity.
The Hessian matrix

\[
M^{(2)} = \begin{pmatrix}
\epsilon_{kl} & \lambda & \epsilon^0 & \tilde{c}_0 & \epsilon^1 & \tilde{c}_j \\
\lambda & \epsilon_{0} & \epsilon^0 & \tilde{c}_0 & \epsilon^1 & \tilde{c}_j \\
\epsilon^0 & \epsilon^0 & \epsilon_{ij} & \tilde{c}_0 & \epsilon^1 & \tilde{c}_j \\
\epsilon^1 & \epsilon^1 & \epsilon_{ij} & \tilde{c}_0 & \epsilon^1 & \tilde{c}_j \\
\end{pmatrix}
\]

(4.8)

can be read from (4.4)-(4.6) as follows:

\[
A^{(2)} = \begin{pmatrix}
A^{(2)ij,kl} & A^{(2)ij,0} & A^{(2)ij,b} \\
A^{(2)0,kl} & A^{(2)00} & A^{(2)0b} \\
A^{(2)a,kl} & A^{(2)a0} & A^{(2)ab} \\
\end{pmatrix}
\]

(4.9)

has the elements

\[
A^{(2)ij,kl} = \frac{\sqrt{e}}{2\lambda^3} A^{(2)ij,kl},
\]

(4.10)

\[
A^{(2)ij,0} = -\frac{\sqrt{e}}{2\lambda^4} A^{(2)ij,kl} \left( \dot{e}_{kl} - \nabla_k \lambda_l - \nabla_l \lambda_k \right),
\]

(4.11)

\[
A^{(2)ij,b} = -\frac{\sqrt{e}}{2\lambda^3} A^{(2)ij,kl} \left( \nabla_k \delta^b_l + \nabla_l \delta^b_k \right),
\]

(4.12)

\[
A^{(2)00} = \frac{1}{2\alpha\kappa^2} \left( -\frac{2e}{\lambda^3} \right) (1 - \eta_{mn} \lambda^m \lambda^n)
\]

\[
+ \frac{\sqrt{e}}{2\lambda^5} A^{(2)ij,kl} \left( \dot{e}_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i \right) \left( \dot{e}_{kl} - \nabla_k \lambda_l - \nabla_l \lambda_k \right),
\]

(4.13)

\[
A^{(2)0b} = \frac{1}{2\alpha\kappa^2} \left( -\frac{2e}{\lambda^3} \right) \eta_{mn} \lambda^m e^{nb}
\]

\[
+ \frac{\sqrt{e}}{2\lambda^4} A^{(2)ij,kl} \left( \dot{e}_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i \right) \left( \nabla_k \delta^b_l + \nabla_l \delta^b_k \right),
\]

(4.14)

\[
A^{(2)ab} = \frac{1}{2\alpha\kappa^2} \frac{2e}{\lambda^2} e^{ma} e^{nb}
\]

\[
+ \left( \dot{\nabla}_i \delta^a_j + \dot{\nabla}_j \delta^a_i \right) \frac{\sqrt{e}}{2\lambda^3} A^{(2)ij,kl} \left( \nabla_k \delta^b_l + \nabla_l \delta^b_k \right),
\]

(4.15)

and \(B^{(2)}\) is

\[
B^{(2)} = B^{(1)}.\]

(4.16)

The determinant of \(A^{(2)}\) can be calculated in this case as

\[
\det A^{(2)} \propto \lambda^{-28} e.
\]

(4.17)

Therefore we have

\[
\det M^{(2)} \propto \lambda^{-20} e^{-3},
\]

(4.18)
where we have taken into account Eq. (3.24) and used the fact $\Gamma^{(2)} B^{(2)-1} \Gamma^{(2)t} = 0$. The path-integral expression obtained is thus

$$Z^{(2)} = \int Dg_{\mu\nu} Db_\mu Dc^\mu D\bar{c}_\mu \Delta^{(2)} \exp \left( i \int \mathcal{L}^{(2)} d^4x \right)$$  \hspace{1cm} (4.19)

with the measure

$$\Delta^{(2)} = (g^{00})^4 (-g)^{-3/2}.$$  \hspace{1cm} (4.20)

This is in agreement with the result of Buchbinder-Lyahovich. 

§5. Measures in more-than-four-derivative gravities

We consider the system described by the Lagrangian

$$\mathcal{L}^{(n+2)} = \mathcal{L}_E + \mathcal{L}_{\text{HD}}^{(n+2)} + \mathcal{L}_{\text{BRS}}^{(n+2)},$$  \hspace{1cm} (5.1)

$$\mathcal{L}_{\text{HD}}^{(n+2)} = \sqrt{-g} \left[ R^{\mu\nu} h_2 \left( \frac{D^2}{A^2} \right) R_{\mu\nu} + R h_0 \left( \frac{D^2}{A^2} \right) R \right].$$  \hspace{1cm} (5.2)

For the BRS Lagrangian we take the following form:

$$\mathcal{L}_{\text{BRS}}^{(n+2)} = -i \delta \left[ \bar{c}_\mu \omega \left( \frac{\Box}{A^2} \right) \left( \frac{1}{\kappa} \partial_\nu \bar{g}^{\mu\nu} - \frac{\alpha}{2} \eta^{\mu\nu} b_\nu \right) \right]$$

$$= \mathcal{L}_{\text{GF}}^{(n+2)} + \mathcal{L}_{\text{FP}}^{(n+2)},$$  \hspace{1cm} (5.3)

$$\mathcal{L}_{\text{GF}}^{(n+2)} = \frac{1}{\kappa} b_\mu \omega \left( \frac{\Box}{A^2} \right) \partial_\nu \bar{g}^{\mu\nu} - \frac{\alpha}{2} \eta^{\mu\nu} b_\nu \omega \left( \frac{\Box}{A^2} \right) b_\nu,$$  \hspace{1cm} (5.4)

$$\mathcal{L}_{\text{FP}}^{(n+2)} = -\frac{i}{2} (\partial_\mu \bar{c}_\nu + \partial_\nu \bar{c}_\mu) \omega \left( \frac{\Box}{A^2} \right) D^\mu b^\nu.$$  \hspace{1cm} (5.5)

Here $h_2, h_0, \omega$ are polynomials of degree $n > 0$

$$h_2(x) = \sum_{k=0}^{n} \beta_{2k} x^k,$$  \hspace{1cm} (5.6)

$$h_0(x) = \sum_{k=0}^{n} \beta_{0k} x^k,$$  \hspace{1cm} (5.7)

$$\omega(x) = \sum_{k=0}^{n} \gamma_k x^k.$$  \hspace{1cm} (5.8)

$D^2 \overset{d}{=} D^\mu D_\mu$ is the covariant D’Alembertian associated with $g_{\mu\nu}$; $\Box \overset{d}{=} \partial^\mu \partial_\mu$ the ordinary D’Alembertian; and $\Lambda$ a dimensionless constant. In order to make the theory nonsingular we have introduced a polynomial $\omega(x)$ into the BRS Lagrangian. The NL fields $b_\mu$ are treated as independent canonical coordinates here. This is a point different from in the previous two sections, where $b_\mu$ were regarded as dependent fields to be integrated out in the course of path-integral calculation.
In the ADM coordinates the Lagrangians are written as follows:

\[
\mathcal{L}_E + \mathcal{L}_{\text{HD}}^{(2)} + \mathcal{L}_{\text{HD}}^{(n+2)} = \left( \frac{1}{2\lambda} \right)^{2n} \frac{1}{4\lambda^7} \sqrt{e} A^{(n+2)ij,kl} \\
\times \left[ e_{ij}^{(n+2)} - \frac{\lambda^{(n+1)}}{\lambda} (\dot{e}_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i) - \left( \nabla_i \lambda^{(n+1)}_j + \nabla_j \lambda^{(n+1)}_i \right) \right] \\
\times \left[ e_{kl}^{(n+2)} - \frac{\lambda^{(n+1)}}{\lambda} (\dot{e}_{kl} - \nabla_k \lambda_l - \nabla_l \lambda_k) - \left( \nabla_k \lambda^{(n+1)}_l + \nabla_l \lambda^{(n+1)}_k \right) \right] \\
+ \ldots
\]

\[
\mathcal{L}_{\text{GF}}^{(n+2)} = \gamma_n \left( \frac{1}{2\lambda} \right)^{2n} \left[ \frac{1}{\kappa} \sqrt{e} \left( \frac{1}{\lambda^2} \lambda^{(n+1)} b_0^{(n)} - \frac{1}{\lambda^2} \lambda b^{(n+1)}_i + \frac{1}{\lambda} \epsilon^{ij} \lambda^{(n+1)} b_j^{(n)} \right) \\
- \frac{\alpha}{2} \eta^{\mu\nu} b_\mu^{(n)} b_\nu^{(n)} \right]
\]

\[
\mathcal{L}_{\text{FP}}^{(n+2)} = \gamma_n \left( \frac{1}{2\lambda} \right)^{2n} i \sqrt{e} \left[ \frac{1}{\lambda^2} \epsilon^{(n+1)} c^0(n+1) + \frac{1}{\lambda^2} \epsilon^{(n+1)} c^0(n+1) + \frac{1}{\lambda^2} \epsilon^{(n+1)} c^i(n+1) \\
+ \frac{1}{\lambda^2} \epsilon^{(n+1)} c^0 \left( - \frac{1}{\lambda} \lambda \lambda^{(n+1)} + \epsilon^{ij} \lambda^{(n+1)} \right) \right]
\]

where we have used the notation

\[
A^{(n+2)ij,kl} \equiv \beta_2 n \eta^{(2)ij,kl} + 4 \left( \beta_2 n + 3 \beta_0 n \right) P^{(0)ij,kl}.
\]

We assume \( \beta_2 n \neq 0 \) and \( \beta_2 n + 3 \beta_0 n \neq 0 \) for simplicity.

For the Hessian matrix

\[
M^{(n+2)} = \begin{pmatrix}
\epsilon_{ij} & \lambda & b_0 & b_\alpha & c^0 & \bar{c}_0 & \bar{c}^i & \bar{c}_j \\
\lambda & A^{(n+2)} & \Gamma^{(n+2)} & \\
b_0 & \bar{b}_0 & B^{(n+2)} \\
b_\alpha & \bar{b}_\alpha & \\
c^0 & \bar{c}^i & \\
\bar{c}_0 & \bar{c}^i & \\
\bar{c}_i & \end{pmatrix}
\]

we have the following form. The submatrix \( A^{(n+2)} \)

\[
A^{(n+2)} = \begin{pmatrix}
A^{(n+2)ij,kl} & A^{(n+2)ij,0} & A^{(n+2)ij,b} & A^{(n+2)ij,\bar{0}} & A^{(n+2)ij,\bar{b}} \\
A^{(n+2)0,kl} & A^{(n+2)0,0} & A^{(n+2)0,b} & A^{(n+2)0,\bar{0}} & A^{(n+2)0,\bar{b}} \\
A^{(n+2)a,kl} & A^{(n+2)a,0} & A^{(n+2)a,b} & A^{(n+2)a,\bar{0}} & A^{(n+2)a,\bar{b}} \\
A^{(n+2)\bar{0},kl} & A^{(n+2)\bar{0},0} & A^{(n+2)\bar{0},b} & A^{(n+2)\bar{0},\bar{0}} & A^{(n+2)\bar{0},\bar{b}} \\
A^{(n+2)a,\bar{kl}} & A^{(n+2)a,\bar{0}} & A^{(n+2)a,\bar{b}} & A^{(n+2)a,\bar{\bar{0}}} & A^{(n+2)a,\bar{\bar{b}}} \\
\end{pmatrix}
\]
has the elements

\[ A^{(n+2)ij,kl} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \frac{\sqrt{e}}{2\Lambda^{3}} A^{(n+2)ii,kl}, \tag{5.15} \]

\[ A^{(n+2)ij,0} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \left( -\frac{\sqrt{e}}{2\lambda^{3}} \right) A^{(n+2)ii,kl} \left( \dot{e}_{kl} - \nabla_{k}\lambda_{l} - \nabla_{l}\lambda_{k} \right), \tag{5.16} \]

\[ A^{(n+2)ij,b} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \left( -\frac{\sqrt{e}}{2\lambda^{3}} \right) A^{(n+2)ii,kl} \left( \nabla_{k}\delta_{l}^{b} + \nabla_{l}\delta_{k}^{b} \right), \tag{5.17} \]

\[ A^{(n+2)ij,0} = 0, \tag{5.18} \]

\[ A^{(n+2)ij,b} = 0, \tag{5.19} \]

\[ A^{(n+2)00} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \frac{\sqrt{e}}{2\lambda^{3}} A^{(n+2)ii,kl} \times \left( \dot{e}_{ij} - \nabla_{i}\lambda_{j} - \nabla_{j}\lambda_{i} \right) \left( \dot{e}_{kl} - \nabla_{k}\lambda_{l} - \nabla_{l}\lambda_{k} \right), \tag{5.20} \]

\[ A^{(n+2)0b} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \frac{\sqrt{e}}{2\lambda^{3}} A^{(n+2)ii,kl} \times \left( \dot{e}_{ij} - \nabla_{i}\lambda_{j} - \nabla_{j}\lambda_{i} \right) \left( \nabla_{k}\delta_{l}^{b} + \nabla_{l}\delta_{k}^{b} \right), \tag{5.21} \]

\[ A^{(n+2)00} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} \left( -\frac{1}{\kappa} \right) \frac{\sqrt{e}}{\lambda^{2}} \delta_{i}^{b}, \tag{5.22} \]

\[ A^{(n+2)0b} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} \left( -\frac{1}{\kappa} \right) \frac{\sqrt{e}}{\lambda^{2}} \delta_{a}^{b} \times \left( \dot{e}_{ij} - \nabla_{i}\lambda_{j} - \nabla_{j}\lambda_{i} \right), \tag{5.23} \]

\[ A^{(n+2)ab} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} \left( -\frac{1}{\kappa} \right) \frac{\sqrt{e}}{\lambda^{2}} \delta_{a}^{b} \left( \nabla_{k}\delta_{l}^{b} + \nabla_{l}\delta_{k}^{b} \right), \tag{5.24} \]

\[ A^{(n+2)00} = 0, \tag{5.25} \]

\[ A^{(n+2)0b} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} \left( -\alpha \right) \eta_{a}^{b}. \tag{5.26} \]

\[ A^{(n+2)00} = 0, \tag{5.27} \]

\[ A^{(n+2)0b} = 0, \tag{5.28} \]

\[ A^{(n+2)0b} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} \left( -\alpha \right) \eta_{a}^{b}. \tag{5.29} \]

The submatrix \( B^{(n+2)} \) is

\[ B^{(n+2)} = \left( \frac{1}{\Lambda \lambda} \right)^{2n} \gamma_{n} B^{(1)}. \tag{5.30} \]

It turns out that the determinants are

\[ \det A^{(n+2)} \propto \lambda^{-28n-28e}, \tag{5.31} \]

\[ \det B^{(n+2)} \propto \lambda^{-16n-8e}. \tag{5.32} \]

Taking into account \( \Gamma^{(n+2)} B^{(n+2)-1} \Gamma^{(n+2)t} = 0 \), we have

\[ \det M^{(n+2)} \propto \lambda^{-12n-20e}. \tag{5.33} \]
Therefore the measure in the path-integral
\[
Z^{(n+2)} = \int Dg_{\mu\nu}Db_{\mu}Dc^{\mu}D\bar{c}_{\mu}\Delta^{(n+2)} \exp \left( i \int \mathcal{L}^{(n+2)} d^4x \right)
\]
(5.34)
is given as
\[
\Delta^{(n+2)} = (g^{00})^{3n+4}(-g)^{-3/2}.
\]
(5.35)

§6. Summary and discussion

We have given a simple method of calculating path-integral measures in higher-derivative gravities. The measures are given as the generalized Lee-Yang terms. The results obtained in §§3 and 4 agree with the ones of Buchbinder and Lyahovich, while the result in §5 has been first reported in the present paper.

Our method proposed here is much simpler than Buchbinder and Lyahovich’s, which was to study the complex structure of the constraints of the theories and to perform canonical quantization straightforwardly.

For the four-derivative gravity their investigation covers more general cases than ours. The constraints \(\beta_2 \neq 0\) and \(\beta_2 + 3\beta_0 \neq 0\) have been imposed for the parameters in §4. In Ref[4] the cases of \((\beta_2 = 0, \beta_2 + 3\beta_0 \neq 0)\) and \((\beta_2 \neq 0, \beta_2 + 3\beta_0 = 0)\) were also investigated. It is not difficult, however, to extend our method to treat those exceptional cases. This will be discussed in a subsequent paper[1].

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*) A preliminary consideration on this subject has been given in Ref[4].