Covariant central extensions of gauge Lie algebras

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Abstract
Motivated by positive energy representations, we classify those continuous central extensions of the compactly supported gauge Lie algebra that are covariant under a 1-parameter group of transformations of the base manifold.

1 Introduction
Let \( \pi: K \to M \) be a locally trivial bundle of finite dimensional Lie groups, with corresponding Lie algebra bundle \( \mathfrak{R} \to M \). We assume that the fibres \( \mathfrak{R}_x \) are semisimple. The group \( G = \Gamma_c(K) \) of compactly supported sections, called the (compactly supported) gauge group, is a locally convex Lie group with Lie algebra \( \mathfrak{g} = \Gamma_c(\mathfrak{R}) \), the (compactly supported) gauge Lie algebra.

In representation theory, one often wishes to impose positive energy conditions derived from a distinguished 1-parameter group \( \gamma_M: \mathbb{R} \to \text{Diff}(M) \) of transformations of the base. A lift \( \gamma: \mathbb{R} \to \text{Aut}(K) \) of \( \gamma_M \) induces a 1-parameter family \( \alpha: \mathbb{R} \to \text{Aut}(G) \) of automorphisms of the gauge group. If \( D \in \text{der}(\mathfrak{g}) \) is the derivation \( D(\xi) := \frac{d}{dt} \big|_{t=0} \alpha_t(\xi) \) induced by \( \alpha \), then the semidirect product

\[
G \rtimes \alpha \mathbb{R}
\]

is a locally convex Lie group with Lie algebra

\[
\mathfrak{g} \rtimes D \mathbb{R}.
\]

Since \([0 \oplus 1, \xi \oplus 0] = D(\xi)\), we will identify \( 0 \oplus 1 \) with \( D \) and write \( \mathfrak{g} \rtimes D \mathbb{R} = \mathfrak{g} \rtimes \mathbb{R}D \) accordingly. In this note, we give a complete classification of the continuous 1-dimensional central extensions \( \widehat{g} \) of \( \mathfrak{g} \rtimes D \mathbb{R} \). In other words, we determine the continuous second Lie algebra cohomology \( H^2(\mathfrak{g} \rtimes D \mathbb{R}, \mathbb{R}) \).

In order to describe the answer, write \( v \in \mathcal{V}(K) \) for the vector field on \( K \) that generates the flow of \( \gamma \), and write \( \pi_*v \in \mathcal{V}(M) \) for its projection to \( M \),

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which generates the flow of $\gamma_M$. Identifying $\xi \in \Gamma_c(\mathfrak{g})$ with the corresponding vertical left invariant vector field $\Xi_\xi$ on $K$, the action of the derivation $D$ on $\mathfrak{g} = \Gamma_c(\mathfrak{g})$ is described by $D\xi = L_\xi \mathfrak{g}$. For each fibre $\mathfrak{g}_x$, the universal invariant bilinear form $\kappa$ takes values in the $K$-representation $V(\mathfrak{g}_x)$, and $V := V(\mathfrak{g})$ is a flat bundle over $M$. In the (important!) special case that $\mathfrak{g}_x$ is a compact simple Lie algebra, $\kappa$ is simply the Killing form with values in $V(\mathfrak{g}_x) = \mathbb{R}$, and $V$ is the trivial real line bundle over $M$. Given a Lie connection $\nabla$ on $\mathfrak{g}$ and a closed $\pi_*\mathcal{V}$-invariant current $\lambda \in \Omega^1_c(M, \mathcal{V})$, there is a unique 2-cocycle $\omega_{\lambda, \mathcal{V}}$ on $\mathfrak{g} \rtimes \mathbb{R}D$ with

$$\omega_{\lambda, \mathcal{V}}(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)), \quad \omega_{\lambda, \mathcal{V}}(D, \xi) = \lambda(\kappa(L_\xi \nabla, \xi)) \quad \text{for} \quad \xi, \eta \in \mathfrak{g}.$$ 

The class $[\omega_{\lambda, \mathcal{V}}] \in H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ is independent of the choice of $\nabla$. One of our main results (Theorem 5.3) asserts that the map $\lambda \mapsto [\omega_{\lambda, \mathcal{V}}]$ is a linear isomorphism from the space of closed, $\pi_*\mathcal{V}$-invariant, $\mathcal{V}$-valued currents on $M$ to the continuous Lie algebra cohomology $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$.

Our motivation for classifying these central extensions comes from the theory of projective positive energy representations. If $G$ is a Lie group with locally convex Lie algebra $\mathfrak{g}$, and $\alpha : \mathbb{R} \to \text{Aut}(G)$ is a homomorphism defining a smooth $\mathbb{R}$-action on $G$, then the semidirect product $G \rtimes_\alpha \mathbb{R}$ is again a Lie group, with Lie algebra $\mathfrak{g} \rtimes \mathbb{R}D$. For every smooth projective unitary representation $\pi : G \rtimes_\alpha \mathbb{R} \to \text{PU}(\mathcal{H})$ of $G \rtimes_\alpha \mathbb{R}$, there exists a central Lie group extension $\hat{G}$ of $G \rtimes_\alpha \mathbb{R}$ by the circle group $\mathbb{T}$ for which $\pi$ lifts to a smooth linear unitary representation $\rho : \hat{G} \to \text{U}(\mathcal{H})$ (see [JN15] for details). The Lie algebra $\hat{\mathfrak{g}}$ can then be written as

$$\hat{\mathfrak{g}} = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D),$$

where $\omega$ is a Lie algebra 2-cocycle of $\mathfrak{g} \rtimes \mathbb{R}D$. The Lie bracket is

$$[zC + x + tD, z'C + x' + t'D] = \omega(x + tD, x' + t'D)C + [x, x'] + tD(x') - t'D(x),$$

and $d\rho(C) = i1$ by construction. We say that $\pi$ is a positive energy representation if the selfadjoint operator $H := id\rho(D)$ has a spectrum which is bounded below.

In [JN16] we address the problem of classifying the projective positive energy representations of the gauge group $G = \Gamma_c(K)$, for the smooth action $\alpha : \mathbb{R} \to \text{Aut}(G)$ induced by a smooth 1-parameter group $\gamma : \mathbb{R} \to \text{Aut}(K)$ of bundle automorphisms. We break this problem into the following steps:

(PE1) Classify the 1-dimensional central Lie algebra extensions $\hat{\mathfrak{g}}$ of $\mathfrak{g} \rtimes_D \mathbb{R}$.

(PE2) Determine which central extensions $\hat{\mathfrak{g}}$ fulfill natural positivity conditions imposed by so-called Cauchy–Schwarz estimates required for cocycles coming from positive energy representations (cf. [JN16]).

(PE3) For those $\hat{\mathfrak{g}}$, classify the positive energy representations that integrate to a representation of a connected Lie group $\hat{G}_0$ with Lie algebra $\hat{\mathfrak{g}}$. 

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In the present note we completely solve (PE1) for semisimple structure algebras \( \mathfrak{r}_e \), thus completing the first step in the classification of projective positive energy representations.

To proceed with (PE2), we assume in [JN16] that the vector field \( \pi \ast v \) on \( M \) has no zeros and generates a periodic flow, hence defines an action of the circle group \( T \) on \( M \). Under this assumption we then show that for every projective positive energy representation \( \rho \) of \( \mathfrak{g} \oplus R \mathfrak{D} \), there exists a locally finite set \( \Lambda \subseteq M/T \) of orbits such that the \( \mathfrak{g} \)-part of \( d\rho \) factors through the restriction homomorphism

\[
\mathfrak{g} = \Gamma_c(\mathfrak{r}) \to \Gamma_c(\mathfrak{r}|_{\Lambda_M}) \cong \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\psi(\mathfrak{t}),
\]

where \( \Lambda_M \subseteq M \) is the union of the orbits in \( M \), and

\[
\mathcal{L}_\psi(\mathfrak{t}) = \{ \xi \in C^\infty(\mathbb{R}, \mathfrak{t}) : (\forall t \in \mathbb{R}) \xi(t+1) = \psi^{-1}(\xi(t)) \}
\]

is the loop algebra twisted by a finite order automorphism \( \psi \in \text{Aut}(\mathfrak{t}) \). As the positive energy representations of covariant loop algebras and their central extensions, the Kac–Moody algebras ([Ka85]), are well understood ([PS86]), this allows us to solve (PE3). This result contributes in particular to “non-commutative distribution” program whose goal is a classification of the irreducible unitary representations of gauge groups ([A-T93]).

The structure of this paper is as follows. After introducing gauge groups, their Lie algebras and one-parameter groups of automorphism in Section 2, we describe in Section 3 a procedure that provides a reduction from semisimple to simple structure Lie algebras, at the expense of replacing \( M \) by a finite covering manifold \( \hat{M} \). In Section 4 we introduce the flat bundle \( V \), which is used in a crucial way in Section 5 for the description of the natural 2-cocycles on the gauge algebra. The first step (PE1) is completely settled in Section 5 where Theorem 5.3 describes all 1-dimensional central extensions of the gauge algebra.

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2 Gauge groups and gauge algebras

Let $\mathcal{K} \to M$ be a smooth bundle of Lie groups, and let $\mathfrak{K} \to M$ be the associated Lie algebra bundle with fibres $\mathfrak{K}_x = \text{Lie}(\mathcal{K}_x)$. If $M$ is connected, then the fibres $\mathcal{K}_x$ of $\mathcal{K} \to M$ are all isomorphic to a fixed structure group $K$, and the fibres $\mathfrak{K}_x$ of $\mathfrak{K}$ are isomorphic to its Lie algebra $\mathfrak{k} = \text{Lie}(K)$.

**Definition 2.1.** (Gauge group) The gauge group is the group $\Gamma(\mathcal{K})$ of smooth sections of $\mathcal{K} \to M$, and the compactly supported gauge group is the group $\Gamma_c(\mathcal{K})$ of smooth compactly supported sections.

**Definition 2.2.** (Gauge algebra) The gauge algebra is the Fréchet-Lie algebra $\Gamma(\mathfrak{K})$ of smooth sections of $\mathfrak{K} \to M$, equipped with the pointwise Lie bracket. The compactly supported gauge algebra $\Gamma_c(\mathfrak{K})$ is the LF-Lie algebra of smooth compactly supported sections.

The compactly supported gauge group $\Gamma_c(\mathcal{K})$ is a locally convex Lie group, whose Lie algebra is the compactly supported gauge algebra $\Gamma_c(\mathfrak{K})$.

**Proposition 2.3.** There exists a unique smooth structure on $\Gamma_c(\mathcal{K})$ which makes it a locally exponential Lie group with Lie algebra $\Gamma_c(\mathfrak{K})$ and exponential map $\exp: \Gamma_c(\mathfrak{K}) \to \Gamma_c(\mathcal{K})$ defined by pointwise exponentiation.

*Proof.* It suffices to prove this in the case that $M$ is connected. Let $V_t, W_t \subseteq \mathfrak{k}$ be open, symmetric 0-neighbourhoods such that the exponential $\exp: \mathfrak{k} \to K$ restricts to a diffeomorphism of $W_t$ onto its image, $V_t$ is contained in $W_t$, and $\exp(V_t) \cdot \exp(V_t) \subseteq \exp(W_t)$.

Choose a locally finite cover $(U_i)_{i \in I}$ of $M$ by open trivialising neighbourhoods for $\mathcal{K} \to M$, which possesses a refinement $(C_i)_{i \in I}$ such that $C_i \subset U_i$ is compact for all $i \in I$. Fix local trivialisations $\varphi_i: K \times U_i \to \mathcal{K}|_{U_i}$ of $\mathcal{K}$, which gives rise to local trivialisations $d\varphi_i: \mathfrak{k} \times U_i \to \mathfrak{K}|_{U_i}$ for $\mathfrak{K}$. Define $W_i := d\varphi_i(U_i \times W_K)$, and set

$$W_{\Gamma_c(\mathfrak{K})} := \{ \xi \in \Gamma_c(\mathfrak{K}) : \xi(C_i) \subseteq W_i \ \forall \ i \in I \}.$$ 

Similarly, $V_{\Gamma_c(\mathfrak{K})}$ is defined in terms of preimages over $C_i$ of $V_i := d\varphi_i(U_i \times V_K)$, and both $V_{\Gamma_c(\mathfrak{K})}$ and $W_{\Gamma_c(\mathfrak{K})}$ are open in $\Gamma_c(\mathfrak{K})$. Since the pointwise exponential $\exp: \Gamma_c(\mathfrak{K}) \to \Gamma_c(\mathcal{K})$ is a bijection of $W_{\Gamma_c(\mathfrak{K})}$ onto its image $W_{\Gamma_c(\mathcal{K})} := \exp(W_{\Gamma_c(\mathfrak{K})})$, the latter inherits a smooth structure. The same goes for its subset $V_{\Gamma_c(\mathcal{K})} := \exp(V_{\Gamma_c(\mathfrak{K})})$.

Inversion $W_{\Gamma_c(\mathcal{K})} \to W_{\Gamma_c(\mathcal{K})}$ and multiplication $V_{\Gamma_c(\mathcal{K})} \times V_{\Gamma_c(\mathcal{K})} \to W_{\Gamma_c(\mathcal{K})}$ are smooth, and for every $\sigma \in \Gamma_c(\mathcal{K})$, there exists an open 0-neighbourhood $W_\sigma \subseteq W_{\Gamma_c(\mathfrak{K})}$ such that $\text{Ad}_\sigma: W_\sigma \to W_{\Gamma_c(\mathfrak{K})}$ is smooth. It therefore follows from [[1833] p.14] (which generalises to locally convex Lie groups, cf. [[Ne06] Thm. II.2.1]), that $\Gamma_c(\mathcal{K})$ possesses a unique Lie group structure such that for some open 0-neighbourhood $U_{\Gamma_c(\mathfrak{K})} \subseteq W_{\Gamma_c(\mathfrak{K})}$, the image $\exp(U_{\Gamma_c(\mathfrak{K})}) \subseteq \Gamma_c(\mathcal{K})$ is an open neighbourhood of the identity. $\square$
Example 2.4. If $\mathcal{K} \to M$ is a trivial bundle, then the gauge group is $\Gamma(\mathcal{K}) = C^\infty(M, K)$, and the gauge algebra is $\Gamma(\mathfrak{k}) = C^\infty(M, \mathfrak{k})$. Similarly, we have $\Gamma_c(\mathcal{K}) = C^\infty_c(M, K)$ and $\Gamma_c(\mathfrak{k}) = C^\infty_c(M, \mathfrak{k})$ for their compactly supported versions. One can thus think of gauge groups as ‘twisted versions’ of the group of smooth $K$-valued functions on $M$.

The motivating example of a gauge group is the group $\text{Gau}(P)$ of vertical automorphisms of a principal fibre bundle $\pi: P \to M$ with structure group $K$.

Example 2.5. (Gauge groups from principal bundles) A vertical automorphism of a principal fibre bundle $\pi: P \to M$ is a $K$-equivariant diffeomorphism $\alpha: P \to P$ such that $\pi \circ \alpha = \alpha$. The group $\text{Gau}(P)$ of vertical automorphisms is called the gauge group of $P$. It is isomorphic to the group

$$C^\infty(P, K)^K := \{ f \in C^\infty(P, K); (\forall p \in P, k \in K) f(pk) = k^{-1}f(p)k \},$$

with isomorphism $C^\infty(P, K)^K \isom \text{Gau}(P)$ given by $f \mapsto \alpha_f$ with $\alpha_f(p) = pf(p)$.

In order to interpret $\text{Gau}(P)$ as a gauge group in the sense of Definition 2.1, we construct the bundle of groups $\text{Conj}(P) \to M$ with typical fibre $K$. For an element $k \in K$, we write $c_k(g) = kgk^{-1}$ for the induced inner automorphism of $K$, and also $\text{Ad}_k \in \text{Aut}(\mathfrak{k})$ for the corresponding automorphism of its Lie algebra $\mathfrak{k}$. Define the bundle of groups $\text{Conj}(P) \to M$ by

$$\text{Conj}(P) := P \times K / \sim,$$

where $\sim$ is the relation $(pk, h) \sim (p, c_k(h))$ for $p \in P$ and $k, h \in K$. We then have isomorphisms

$$\text{Gau}(P) \isom C^\infty(P, K)^K \isom \Gamma(\text{Conj}(P)),$$

where $f \in C^\infty(P, K)^K$ corresponds to the section $\sigma_f \in \Gamma(\text{Conj}(P))$ defined by $\sigma_f(\pi(p)) = [p, f(p)]$ for all $p \in P$. The bundle of Lie algebras associated to $\text{Conj}(P)$ is the adjoint bundle $\text{Ad}(P) \to M$, defined as the quotient

$$\text{Ad}(P) := P \times_{\text{Ad}} \mathfrak{k}$$

of $P \times \mathfrak{k}$ modulo the relation $(pk, X) \sim (p, \text{Ad}_k(X))$ for $p \in P$, $X \in \mathfrak{k}$ and $k \in K$. The compactly supported gauge group $\text{Gau}_c(P) \subseteq \text{Gau}(P)$ is the group of vertical bundle automorphisms of $P$ that are trivial outside the preimage of some compact set in $M$. Since it is isomorphic to $\Gamma_c(\text{Conj}(P))$, it is a locally convex Lie group with Lie algebra $\mathfrak{gau}_c(P) = \Gamma_c(\text{Ad}(P))$.

Remark 2.6. Gauge groups arise in field theory, as groups of transformations of the space of principal connections on $P$ (the gauge fields). If the space-time manifold $M$ is not compact, then one imposes boundary conditions on the gauge fields at infinity. Depending on how one does this, the group $\text{Gau}(P)$ may be too big to preserve the set of admissible gauge fields. One then expects the group of remaining gauge transformations to at least contain $\text{Gau}_c(P)$, or perhaps even some larger Lie group of gauge transformations specified by a decay condition at infinity (cf. [Wa10, Go04]).
An automorphism of \( \pi: \mathcal{K} \rightarrow M \) is a pair \((\gamma, \gamma_M) \in \text{Diff}(\mathcal{K}) \times \text{Diff}(M)\) with \(\pi \circ \gamma = \gamma_M \circ \pi\), such that for each fibre \(K_x\), the map \(\gamma|_{K_x}: K_x \rightarrow K_{\gamma_M(x)}\) is a group homomorphism. Since \(\gamma_M\) is determined by \(\gamma\), we will omit it from the notation. We denote the group of automorphisms of \(K\) by \(\text{Aut}(K)\).

**Definition 2.7.** (Geometric \(\mathbb{R}\)-actions) In the context of gauge groups, we will be interested in \(\mathbb{R}\)-actions \(\alpha: \mathbb{R} \rightarrow \text{Aut}(\Gamma(K))\) that are of geometric type. These are derived from a 1-parameter group \(\gamma: \mathbb{R} \rightarrow \text{Aut}(K)\) by

\[
\alpha_t(\sigma) := \gamma_t \circ \sigma \circ \gamma_{-1}^M. \tag{4}
\]

**Remark 2.8.** If \(K\) is of the form \(\text{Ad}(P)\) for a principal fibre bundle \(P \rightarrow M\), then a 1-parameter group of automorphisms of \(P\) induces a 1-parameter group of automorphisms of \(K\). If we think of the induced diffeomorphisms \(\gamma_M(t) \in \text{Diff}(M)\) as time translations, then the automorphisms of \(P\) encode the time translation behaviour of the gauge fields.

The 1-parameter group \(\alpha: \mathbb{R} \rightarrow \text{Aut}(\Gamma(K))\) of group automorphisms differentiates to a 1-parameter group \(\beta: \mathbb{R} \rightarrow \text{Aut}(\Gamma(\mathfrak{g}))\) of Lie algebra automorphisms given by

\[
\beta_t(\xi) = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \gamma_t \circ e^{\varepsilon \xi} \circ \gamma_{-1}^M. \tag{5}
\]

The corresponding derivation \(D := \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \beta_t\) of \(\Gamma(\mathfrak{g})\) can be described in terms of the infinitesimal generator \(v \in \mathfrak{k}(\mathcal{K})\) of \(\gamma\), given by \(v := \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \gamma_t\). We identify \(\xi \in \Gamma(\mathfrak{g})\) with the vertical, left invariant vector field \(\Xi_\xi \in \mathfrak{k}(\mathcal{K})\) defined by \(\Xi_\xi(k_x) = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} k_x e^{-\varepsilon \xi(x)}\). Using the equality \([v, \Xi_\xi] = \Xi_{D(\xi)}\), we write

\[
D(\xi) = L_v \xi. \tag{6}
\]

For \(\mathfrak{g} = \Gamma_c(\mathfrak{g})\), the Lie algebra \(\mathfrak{g} \rtimes_D \mathbb{R}\) then has bracket

\[
[\xi + t, \xi' + t'] = \left( [\xi, \xi'] + (t L_v \xi' - t' L_v \xi) \right) \oplus 0. \tag{7}
\]

### 3 Reduction to simple Lie algebras

In this note, we will focus attention on the class of gauge algebras with a semisimple structure group, not only because they are more accessible, but also because they are relevant in applications. We now show that every gauge algebra with a semisimple structure group can be considered as a gauge algebra of a bundle with a simple structure group which need not be the same for all fibers. Accordingly, the base manifold \(M\) is replaced by a not necessarily connected finite cover.

#### 3.1 From semisimple to simple Lie algebras

Let \(\mathfrak{g} \rightarrow M\) be a smooth locally trivial bundle of Lie algebras with semisimple fibres. We construct a finite cover \(\hat{M} \rightarrow M\) and a locally trivial bundle of Lie algebras \(\hat{\mathfrak{g}} \rightarrow \hat{M}\) with simple fibres such that \(\Gamma(\hat{\mathfrak{g}}) \simeq \hat{\Gamma}(\hat{\mathfrak{g}})\) and \(\Gamma_c(\mathfrak{g}) \simeq \Gamma_c(\mathfrak{g})\).
Because one can go back and forth between principal fibre bundles and bundles of Lie algebras, this shows that every gauge algebra for a principal fibre bundle with semisimple structure group is isomorphic to one with a simple structure group. Indeed, every principal fibre bundle \( P \to M \) with semisimple structure group \( K \) gives rise to the bundle \( \text{Ad}(P) \to M \) of Lie algebras. Conversely, every Lie algebra bundle \( \mathfrak{g} \to M \) with semisimple structure algebra \( k \) over the connected component \( M_i \) of \( M \) gives rise to a principal fibre bundle \( P_{\mathfrak{g}} \to M \) with semisimple structure group \( \text{Aut}(k_i) \) over \( M_i \) defined, for \( x \in M_i \), by \( P_{\mathfrak{g},x} := \text{Iso}(k_i, \mathfrak{g}_x) \) for \( x \in M_i \).

**Theorem 3.1.** (Reduction from semisimple to simple structure algebras) If \( \mathfrak{g} \to M \) is a smooth locally trivial bundle of Lie algebras with semisimple fibres, then there exists a finite cover \( \tilde{M} \to M \) and a smooth locally trivial bundle of Lie algebras \( \tilde{\mathfrak{g}} \to \tilde{M} \) with simple fibres such that there exist isomorphisms \( \Gamma(\mathfrak{g}) \simeq \Gamma(\tilde{\mathfrak{g}}) \) and \( \Gamma_c(\mathfrak{g}) \simeq \Gamma_c(\tilde{\mathfrak{g}}) \) of locally convex Lie algebras.

The finite cover \( \tilde{M} \to M \) is not necessarily connected, and the isomorphism classes of the fibres of \( \tilde{\mathfrak{g}} \to \tilde{M} \) are not necessarily the same over different connected components of \( \tilde{M} \).

**Proof.** For a finite dimensional semisimple Lie algebra \( \mathfrak{k} \), we write \( \text{Spec}(\mathfrak{k}) \) for the finite set of maximal ideals of \( \mathfrak{k} \), equipped with the discrete topology. We now define the set

\[
\tilde{M} := \bigcup_{x \in M} \text{Spec}(\mathfrak{g}_x)
\]

with the natural projection \( \text{pr}_{\tilde{M}} : \tilde{M} \to M \). Local trivialisations \( \mathfrak{g}|_U \simeq U \times \mathfrak{k} \) of \( \mathfrak{g} \) over open connected subsets \( U \subseteq M \) induce compatible bijections between \( \text{pr}_{\tilde{M}}^{-1}(U) \) and the smooth manifold \( U \times \text{Spec}(\mathfrak{k}) \). This provides \( \tilde{M} \) with a manifold structure for which \( \text{pr}_{\tilde{M}} : \tilde{M} \to M \) is a finite covering \( \Box \) We define

\[
\tilde{\mathfrak{g}} := \bigcup_{I_x \in \tilde{M}} \mathfrak{g}_x/I_x
\]

with the natural projection \( \pi : \tilde{\mathfrak{g}} \to \tilde{M} \). Local trivialisations \( \mathfrak{g}|_U \simeq U \times \mathfrak{k} \) of \( \mathfrak{g} \) yield bijections between \( \mathfrak{g}|_U \) and the disjoint union

\[
\bigcup_{I \in \text{Spec}(\mathfrak{k})} U_I \times (\mathfrak{k}/I),
\]

where \( U_I \simeq U \) is the connected component of \( \text{pr}_{\tilde{M}}^{-1}(U) \) corresponding to the maximal ideal \( I \subseteq \mathfrak{k} \) in the particular trivialisation. Since different trivialisations differ by Lie algebra automorphisms of the fibres, which permute the ideals in

\footnote{Note that non-isomorphic maximal ideals of \( \mathfrak{g}_x \) are always in different connected components of \( \tilde{M} \), whereas isomorphic maximal ideals may or may not be in the same connected component, depending on the bundle structure.}
and alike, the projection $\pi: \hat{\mathcal{R}} \to \hat{M}$ becomes a smooth locally trivial bundle of Lie algebras over $\hat{M}$.

The morphism $\Phi: \Gamma(\hat{\mathcal{R}}) \to \Gamma(\hat{\mathcal{K}})$ of Fréchet Lie algebras defined by

$$\Phi(\sigma)(x) := \sigma(x) + I_x$$

is an isomorphism; because the fibres are semisimple, the injection $\mathcal{R}_x/I_x \hookrightarrow \mathcal{R}_x$ allows one to construct the inverse

$$\Phi^{-1}(\tau)(x) = \sum_{I_x \in \text{Spec}(\mathcal{R}_x)} \tau(I_x).$$

Since the projection $\text{pr}_\hat{M}: \hat{M} \to M$ is a finite cover, this induces an isomorphism $\Phi: \Gamma_c(\hat{\mathcal{R}}) \to \Gamma_c(\hat{\mathcal{K}})$ of LF-Lie algebras.

Clearly, a smooth 1-parameter family of automorphisms of $\hat{\mathcal{R}} \to M$ acts naturally on the maximal ideals, so we obtain a smooth action on $\hat{M} \to M$ and on $\hat{\mathcal{K}} \to \hat{M}$. The action on $\hat{M}$ is locally free or periodic if and only if the action on $M$ is, and then the period on $\hat{M}$ is a multiple of the period on $M$.

**Example 3.2.** If $\mathfrak{k}$ is a simple Lie algebra, then $\hat{\mathcal{M}} = \mathcal{M}$.

**Example 3.3.** If $P = M \times K$ is trivial, then $\hat{\mathcal{M}} = M \times \text{Spec}(\mathfrak{k})$ and all connected components of $\hat{M}$ are diffeomorphic to $M$.

**Example 3.4.** If $\mathfrak{k}$ is a semisimple Lie algebra with $r$ simple ideals that are mutually non-isomorphic, then $\hat{\mathcal{M}} = \bigsqcup_{i=1}^r M$ is a disjoint union of copies of $M$.

**Example 3.5.** (Frame bundles of 4-manifolds) Let $M$ be a 4-dimensional Riemannian manifold. Let $P := \text{OF}(M)$ be the principal $O(4, \mathbb{R})$-bundle of orthogonal frames. Then $\mathfrak{f} = \mathfrak{so}(4, \mathbb{R})$ is isomorphic to $\mathfrak{su}(2, \mathbb{C}) \oplus \mathfrak{su}(2, \mathbb{C})$. The group $\pi_0(K)$ is of order 2, the non-trivial element acting by conjugation with $T = \text{diag}(-1, 1, 1, 1)$. Since this permutes the two simple ideals, the manifold $\hat{M}$ is the orientable double cover of $M$. This is the disjoint union $\hat{M} = M_L \sqcup M_R$ of two copies of $M$ if $M$ is orientable, and a connected twofold cover $\hat{M} \to M$ if it is not.

### 3.2 Compact and noncompact simple Lie algebras

A semisimple Lie algebra $\mathfrak{k}$ is called compact if its Killing form is negative definite. Every semisimple Lie algebra $\mathfrak{k}$ is a direct sum $\mathfrak{k} = \mathfrak{k}_{\text{cpt}} \oplus \mathfrak{k}_{\text{nc}}$, where $\mathfrak{k}_{\text{cpt}}$ is the direct sum of all compact ideals of $\mathfrak{k}$ (or, alternatively, its maximal compact quotient), and $\mathfrak{k}_{\text{nc}}$ is the direct sum of the noncompact ideals. Since the decomposition $\mathfrak{k} = \mathfrak{k}_{\text{cpt}} \oplus \mathfrak{k}_{\text{nc}}$ is invariant under $\text{Aut}(\mathfrak{k})$, every Lie algebra bundle bundle $\mathcal{R} \to M$ can be written as a direct sum

$$\mathcal{R} = \mathcal{R}_{\text{cpt}} \oplus \mathcal{R}_{\text{nc}}$$

(8)
of Lie algebra bundles over $M$. Correspondingly, we have the decomposition
\[ \hat{M} = \hat{M}_{cpt} \sqcup \hat{M}_{nc} \] (9)
of $\hat{M}$ into disjoint submanifolds, $\hat{M}_{cpt}$ and $\hat{M}_{nc}$, containing the maximal ideals $I_x \subset \mathfrak{A}_x$ with $\mathfrak{A}_x/I_x$ compact and noncompact, respectively. Writing $\hat{\mathfrak{A}}_{cpt}$ for the restriction of $\hat{\mathfrak{A}}$ to $\hat{M}_{cpt}$ and $\hat{\mathfrak{A}}_{nc}$ for its restriction to $\hat{M}_{nc}$, we find Lie algebra bundles $\hat{\mathfrak{A}}_{cpt} \to \hat{M}_{cpt}$ and $\hat{\mathfrak{A}}_{nc} \to \hat{M}_{nc}$ with compact and noncompact simple fibres respectively, and Fréchet Lie algebra isomorphisms
\[ \Gamma(\hat{\mathfrak{A}}_{cpt}) \simeq \Gamma(\hat{\mathfrak{A}}_{cpt}) \quad \text{and} \quad \Gamma(\hat{\mathfrak{A}}_{nc}) \simeq \Gamma(\hat{\mathfrak{A}}_{nc}) . \] (10)

## 4 Universal invariant symmetric bilinear forms

In Section 5 we will undertake a detailed analysis of the 2-cocycles of $\mathfrak{g} \rtimes_D \mathfrak{R}$ for compactly supported gauge algebras $\mathfrak{g} := \Gamma_c(\mathfrak{K})$ with semisimple structure group $\mathfrak{K}$. In order to describe the relevant 2-cocycles, we need to introduce universal invariant symmetric bilinear forms on the Lie algebra $\mathfrak{k}$ of the structure group. In the case that $\mathfrak{k}$ is a compact simple Lie algebra, this is simply the Killing form.

### 4.1 Universal invariant symmetric bilinear forms

Let $\mathfrak{t}$ be a finite dimensional Lie algebra. Then its automorphism group $\text{Aut}(\mathfrak{t})$ is a closed subgroup of $\text{GL}(\mathfrak{g})$, hence a Lie group with Lie algebra $\text{der}(\mathfrak{t})$. Since $\text{der}(\mathfrak{t})$ acts trivially on the quotient $V(\mathfrak{k}) := S^2(\mathfrak{k})/\text{der}(\mathfrak{t}) \cdot S^2(\mathfrak{k})$ of the twofold symmetric tensor power $S^2(\mathfrak{k})$, the the $\text{Aut}(\mathfrak{t})$-representation on $V(\mathfrak{t})$ factors through $\pi_0(\text{Aut}(\mathfrak{t}))$. The universal $\text{der}(\mathfrak{t})$-invariant symmetric bilinear form is defined by
\[ \kappa: \mathfrak{t} \times \mathfrak{t} \to V(\mathfrak{t}), \quad \kappa(x, y) := [x \otimes y] = \frac{1}{2} [x \otimes y + y \otimes x] . \]

We associate to $\lambda \in V(\mathfrak{t})^*$ the $\mathbb{R}$-valued, $\text{der}(\mathfrak{t})$-invariant, symmetric, bilinear form $\kappa_\lambda := \lambda \circ \kappa$. This correspondence is a bijection between $V(\mathfrak{t})^*$ and the space of $\text{der}(\mathfrak{t})$-invariant symmetric bilinear forms on $\mathfrak{t}$.

For the convenience of the reader, we now list some properties of $V(\mathfrak{t})$ for (semi)simple Lie algebras $\mathfrak{t}$, in which case $\text{der}(\mathfrak{t}) = \mathfrak{t}$. These results will be used in the rest of the paper. We refer to [NW09, App. B] for proofs and a more detailed exposition.

For a simple real Lie algebra $\mathfrak{t}$, we have $V(\mathfrak{t}) \simeq \mathbb{K}$, with $\mathbb{K} = \mathbb{C}$ if $\mathfrak{t}$ admits a complex structure, and $\mathbb{K} = \mathbb{R}$ if it does not, i.e., if $\mathfrak{t}$ is absolutely simple. The universal invariant symmetric bilinear form can be identified with the Killing form of the real Lie algebra $\mathfrak{t}$ if $\mathbb{K} = \mathbb{R}$ and the Killing form of the underlying
complex Lie algebra if $K = \mathbb{C}$. In particular, in the important special case that $\mathfrak{k}$ is a compact simple Lie algebra, the universal invariant bilinear form $\kappa: \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is simply the negative definite Killing form $\kappa: \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$, $\kappa(x, y) = \text{tr}(\text{ad} x \text{ad} y)$.

For a semisimple real Lie algebra $\mathfrak{t} = \bigoplus_{i=1}^{r} \mathfrak{t}_{i}^{m_{i}}$, where the simple ideals $\mathfrak{t}_{i}$ are mutually non-isomorphic, one finds $V(\mathfrak{t}) \simeq \bigoplus_{i=1}^{r} V(\mathfrak{t}_{i})^{m_{i}}$ with $V(\mathfrak{t}_{i})$ isomorphic to $\mathbb{R}$ or $\mathbb{C}$. The action of $\pi_{0}(\text{Aut}(\mathfrak{t}))$ on $V(\mathfrak{t})$ leaves invariant the subspaces $V(\mathfrak{t}_{i})^{m_{i}}$ coming from the isotypical ideals $\mathfrak{t}_{i}^{m_{i}}$. If $V(\mathfrak{t}_{i}) \simeq \mathbb{R}$, then the action of $\pi_{0}(\text{Aut}(\mathfrak{t}))$ on $V(\mathfrak{t}_{i})^{m_{i}}$ factors through the homomorphism $\pi_{0}(\text{Aut}(\mathfrak{t})) \to S_{m_{i}}$ that maps $\alpha \in \text{Aut}(\mathfrak{t})$ to the permutation it induces on the set of ideals isomorphic to $\mathfrak{t}_{i}$. If $V(\mathfrak{t}_{i}) \simeq \mathbb{C}$, then the action on $\mathbb{C}^{m_{i}}$ factors through a homomorphism $\pi_{0}(\text{Aut}(\mathfrak{t})) \to (\mathbb{Z}/2\mathbb{Z})^{m_{i}} \times S_{m_{i}}$, where the symmetric group $S_{m_{i}}$ acts by permuting components and $(\mathbb{Z}/2\mathbb{Z})^{m_{i}}$ acts by complex conjugation in the components.

### 4.2 The flat bundle $\mathcal{V} = V(\mathcal{R})$

If $\mathcal{R} \to M$ is a bundle of Lie algebras, we denote by $\mathcal{V} \to M$ the vector bundle with fibres $\mathcal{V}_{x} = V(\mathcal{R}_{x})$. It carries a canonical flat connection $\mathcal{d}$, defined by $\mathcal{d}\kappa(\xi, \eta) := \kappa(\nabla \xi, \eta) + \kappa(\xi, \nabla \eta)$ for $\xi, \eta \in \Gamma(\mathcal{R})$, where $\nabla$ is a Lie connection on $\mathcal{R}$, meaning that $\nabla[\xi, \eta] = [\nabla \xi, \eta] + [\xi, \nabla \eta]$ for all $\xi, \eta \in \Gamma(\mathcal{R})$. As any two Lie connections differ by a der($\mathcal{R}$)-valued 1-form, this definition is independent of the choice of $\nabla$ (cf. [1W13]).

If $\mathcal{R}$ has semisimple typical fibre $\mathfrak{t}$, then the isotypical ideals $\mathfrak{t}_{i}^{m_{i}}$ in the decomposition $\mathfrak{t} = \bigoplus_{i=1}^{r} \mathfrak{t}_{i}^{m_{i}}$ are $\text{Aut}(\mathfrak{t})$-invariant, so that we obtain a direct sum decomposition

$$\mathcal{V} = \bigoplus_{i=1}^{r} \mathcal{V}_{i}$$

of flat bundles.

If the ideal $\mathfrak{t}_{i}$ is absolutely simple, which is always the case if $\mathfrak{t}$ is a compact Lie algebra, then the structure group of $\mathcal{V}_{i}$ reduces to $S_{m_{i}}$. In particular, if $\mathfrak{t}$ is compact simple, then $\mathcal{V}$ is simply the trivial line bundle $M \times \mathbb{R} \to M$.

If the ideal $\mathfrak{t}_{i}$ possesses a complex structure, then the structure group of $\mathcal{V}_{i}$ reduces to $(\mathbb{Z}/2\mathbb{Z})^{m_{i}} \times S_{m_{i}}$. In particular, for $\mathfrak{t}$ complex simple, the bundle $\mathcal{V} \to M$ is the vector bundle with fibre $\mathbb{C}$, and $\alpha \in \text{Aut}(\mathfrak{t})$ flips the complex structure on $\mathbb{C}$ if and only if it flips the complex structure on $\mathfrak{t}$. If $\mathcal{R} = \text{Ad}(P)$ for a principal fibre bundle $P \to M$ with complex simple structure group $K$, then $\mathcal{V}$ is the trivial bundle $M \times \mathbb{C} \to M$.

### 5 Central extensions of gauge algebras

Let $\mathfrak{g}$ be the compactly supported gauge algebra $\Gamma_{c}(\mathcal{R})$ for a Lie algebra bundle $\mathcal{R} \to M$ with semisimple fibres. In this section, we will classify all possible central extensions of $\mathfrak{g} \times_{D} \mathbb{R}$, in other words, we will calculate the continuous second
Lie algebra cohomology $H^2(g \rtimes_D \mathbb{R}, \mathbb{R})$ with trivial coefficients. In [JN16] we will examine which of these cocycles comes from a positive energy representation.

**Remark 5.1.** For a cocycle $\omega$ on $g \rtimes_D \mathbb{R}$, the relation

$$\omega(D, [\xi, \eta]) = \omega(D\xi, \eta) + \omega(\xi, D\eta)$$  \hspace{1cm} (11)

shows that $i_D\delta\omega$ measures the non-invariance of the restriction of $\omega$ to $g \times g$ under the derivation $D$. It also shows that, if the Lie algebra $g$ is perfect, then the linear functional $i_D\delta\omega: g \to \mathbb{R}$ is completely determined by (11).

### 5.1 Definition of the 2-cocycles

We define 2-cocycles $\omega_{\lambda, \nabla}$ on $g \rtimes_D \mathbb{R}$ such that their classes span the cohomology group $H^2(g \rtimes_D \mathbb{R}, \mathbb{R})$. They depend on a $\nabla$-valued 1-current $\lambda \in \Omega^1_c(M, \nabla)'$, and on a Lie connection $\nabla$ on $\mathcal{R}$. Recall from Section 4 that $\kappa: \mathfrak{t} \times \mathfrak{t} \to V(\mathfrak{t})$ is the universal invariant bilinear form of $\mathfrak{t}$, and $\nabla \to M$ is the flat bundle with fibres $\nabla_x = V(\mathfrak{r}_x)$. In the important special case that $\mathfrak{t}$ is compact simple, $V(\mathfrak{t}) = \mathbb{R}$, $\kappa$ is the Killing form, and $\nabla$ is the trivial real line bundle.

A 1-current $\lambda \in \Omega^1_c(M, \nabla)' \in \mathfrak{g}$ is said to be

(L1) closed if $\lambda(dC^\infty(M, \nabla)) = 0$,

(L2) $\pi_*\nabla$-invariant if $\lambda(L_{\pi_*\nabla}\Omega^1_c(M, \nabla)) = \{0\}.$

Given a closed $\pi_*\nabla$-invariant current $\lambda \in \Omega^1_c(M, \nabla)'$, we define the 2-cocycle $\omega_{\lambda, \nabla}$ on $g \rtimes_D \mathbb{R}$ by skew-symmetry and the equations

$$\omega_{\lambda, \nabla}(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)),$$  \hspace{1cm} (12)

$$\omega_{\lambda, \nabla}(D, \xi) = \lambda(\kappa(L_{\nabla}\nabla, \xi)),$$  \hspace{1cm} (13)

where we write $\xi$ for $(\xi, 0) \in g \rtimes_D \mathbb{R}$ and $D$ for $(0, 1) \in g \rtimes_D \mathbb{R}$ as in (1). We define the $\text{der}(\mathcal{R})$-valued 1-form $L_{\nabla}\nabla \in \Omega^1(M, \text{der}(\mathcal{R}))$ by

$$(L_{\nabla}\nabla)_w(\xi) = L_{\nabla}(\nabla\xi)_w - \nabla_w L_{\nabla}\xi = L_{\nabla}(\nabla_w\xi) - \nabla_w L_{\nabla}\xi - \nabla_{[\pi_*\nabla, w]}\xi$$  \hspace{1cm} (14)

for all $w \in \mathfrak{X}(M)$, $\xi \in \Gamma(\mathcal{R})$. Since the fibres of $\mathcal{R} \to M$ are semisimple, all derivations are inner, so we can identify $L_{\nabla}\nabla$ with an element of $\Omega^1(M, \mathcal{R})$. Using the formulae

$$d\kappa(\xi, \eta) = \kappa(\nabla\xi, \eta) + \kappa(\xi, \nabla\eta),$$  \hspace{1cm} (15)

$$L_{\pi_*\nabla}\kappa(\xi, \eta) = \kappa(L_{\nabla}\xi, \eta) + \kappa(\xi, L_{\nabla}\eta),$$  \hspace{1cm} (16)

$$L_{\nabla}(\nabla\xi) - \nabla L_{\nabla}\xi = [L_{\nabla}\nabla, \xi],$$  \hspace{1cm} (17)

it is not difficult to check that $\omega_{\lambda, \nabla}$ is a cocycle. Skew-symmetry follows from (15) and (L1). The vanishing of $\delta\omega_{\lambda, \nabla}$ on $g$ follows from (14), the derivation property of $\nabla$ and invariance of $\kappa$. Finally, $i_D\delta\omega_{\lambda, \nabla} = 0$ follows from skew-symmetry, (17), (10), (L2) and the invariance of $\kappa$.
Note that the class \([\omega_{\lambda,\nabla}]\) in \(H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})\) depends only on \(\lambda\), not on \(\nabla\). Indeed, two connection 1-forms \(\nabla\) and \(\nabla'\) differ by \(A \in \Omega^2(M, \text{der}(\mathfrak{h}))\). Using \(\text{der}(\mathfrak{h}) \cong \mathfrak{h}\), we find

\[
\omega_{\lambda,\nabla'} - \omega_{\lambda,\nabla} = \delta \chi_A \quad \text{with} \quad \chi_A(\xi) := \lambda(A(\xi) \xi).
\]

### 5.2 Classification of central extensions

We now show that every continuous Lie algebra 2-cocycle on \(\mathfrak{g} \rtimes_D \mathbb{R}\) is cohomologous to one of the type \(\omega_{\lambda,\nabla}\) as defined in (12) and (13). The proof relies on a description of \(H^2(\mathfrak{g}, \mathbb{R})\) provided by the following theorem ([JW13, Prop. 1.1]).

**Theorem 5.2.** (Central extensions of gauge algebras) Let \(\mathfrak{g}\) be the compactly supported gauge algebra \(\mathfrak{g} = \Gamma_c(\mathfrak{h})\) of a Lie algebra bundle \(\mathfrak{h} \to M\) with semisimple fibres. Then every continuous 2-cocycle is cohomologous to one of the form

\[
\psi_{\lambda,\nabla}(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)),
\]

where \(\lambda: \Omega^1_c(M, \mathbb{V}) \to \mathbb{R}\) is a continuous linear functional that vanishes on \(\text{der}(\mathfrak{k})\), and \(\nabla\) is a Lie connection on \(\mathfrak{h}\). Two such cocycles \(\psi_{\lambda,\nabla}\) and \(\psi_{\lambda',\nabla'}\) are equivalent if and only if \(\lambda = \lambda'\).

Using this, we classify the continuous central extensions of \(\mathfrak{g} \rtimes_D \mathbb{R}\).

**Theorem 5.3.** (Central extensions of extended gauge algebras) Let \(\mathcal{K} \to M\) be a bundle of Lie groups with semisimple fibres, equipped with a 1-parameter group of automorphisms with generator \(v \in \mathfrak{x}(\mathcal{K})\). Let \(\mathfrak{g} = \Gamma_c(\mathfrak{k})\) be the compactly supported gauge algebra, and let \(\mathfrak{g} \rtimes_D \mathbb{R}\) be the Lie algebra \(\mathfrak{g}\). Then the map \(\lambda \mapsto [\omega_{\lambda,\nabla}]\) induces an isomorphism

\[
\left(\Omega^1_c(M, \mathbb{V})/\left(\text{der} \Omega^1_c(M, \mathbb{V}) + L_{\pi, v} \Omega^1_c(M, \mathbb{V})\right)\right)' \tilde{\to} H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})
\]

between the space of closed \(\pi, v\)-invariant \(\mathbb{V}\)-valued currents and \(H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})\).

**Proof.** Let \(\omega\) be a continuous 2-cocycle on \(\mathfrak{g} \rtimes_D \mathbb{R}\). If \(i: \mathfrak{g} \hookrightarrow \mathfrak{g} \rtimes_D \mathbb{R}\) is the inclusion, then \(i^*\omega\) is a 2-cocycle on \(\mathfrak{g}\). By Theorem 5.2 there exists a Lie connection \(\nabla\) and a continuous linear functional \(\varphi \in \mathfrak{g}'\) such that

\[
i^*\omega(\xi, \eta) = \lambda(\kappa(\xi, \nabla\eta)) + \varphi([\xi, \eta]), \quad \text{where} \quad \lambda \in \Omega^1_c(M, \mathbb{V})'.
\]

Using the cocycle property (cf. Rk. 5.1), we find

\[
\omega(D, [\xi, \eta]) = i^*\omega(L_v \xi, \eta) + i^*\omega(\xi, L_v \eta)
\]

and hence, using (16) and (17),

\[
\omega(D, [\xi, \eta]) = \lambda(\kappa(L_v \xi, \nabla\eta) + \kappa(\xi, \nabla L_v \eta)) + \varphi(L_v [\xi, \eta]) = \lambda(L_{\pi, v}\kappa(\xi, \nabla\eta)) + \lambda(\kappa(L_v \nabla [\xi, \eta])) + \varphi(L_v [\xi, \eta])
\]
In particular, \([\xi, \eta] = 0\) implies \(\lambda(L_{\pi, v}\kappa(\xi, \nabla\eta)) = 0\).

Now fix a trivialisation \(\mathcal{K}|_U \cong U \times K\) over an open subset \(U \subseteq M\). It induces the corresponding trivialisation \(\mathcal{V}|_U \cong U \times V(\mathfrak{k})\) of flat bundles. For \(f, g \in C_c^\infty(U)\) and \(X \in \mathfrak{k}\), we consider \(\xi = fX\) and \(\eta = gX\) as commuting elements of \(\Gamma_c(\mathfrak{k})\). With the local connection 1-form \(A \in \Omega^1(U, \mathfrak{k})\), we then have

\[
\kappa(\xi, \nabla\eta) = \kappa(fX, dg \cdot X + g[A, X]) = f \cdot dg \cdot \kappa(X, X).
\]

Since \([\xi, \eta] = 0\), we find \(\lambda(L_{\pi, v}\beta\kappa(X, X)) = 0\) for all 1-forms \(\beta = f \cdot dg\). Applying (18) to \(\Delta\), we see that \(\Delta(\mathfrak{g}) = 0\) and hence that \(\Delta = 0\) because \(\mathfrak{g}\) is perfect by [JW13, Prop. 2.4].

This shows surjectivity of the map \(\lambda \mapsto [\omega_{\lambda, \nabla}]\). Injectivity follows because \(\omega_{\lambda, \nabla} = \delta\chi\) implies \(\omega_{\lambda, \nabla}|_{\mathfrak{g} \times \mathfrak{g}} = \delta(\chi|_{\mathfrak{g}})\), hence \(\lambda = 0\) by Theorem 5.2.

**Remark 5.4.** If the Lie connection \(\nabla\) on \(\mathfrak{k}\) can be chosen so as to make \(v \in \mathfrak{X}(K)\) horizontal, \(\nabla_{\pi, v}\xi = L_v\xi\) for all \(\xi \in \Gamma(\mathfrak{k})\), then equation (14) shows that \(L_v\nabla = i_{\pi, v}R\), where \(R\) is the curvature of \(\nabla\). For such connections, (13) is equivalent to

\[
\omega_{\lambda, \nabla}(D, \xi) = \lambda(\kappa(i_{\pi, v}R, \xi)).
\]

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