An integrable spin chain for the SL(2,R)/U(1) black hole sigma model

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(Dated: May 14, 2012)

We introduce a spin chain based on finite-dimensional spin-1/2 SU(2) representations but with a non-hermitian ‘Hamiltonian’ and show, using mostly analytical techniques, that it is described at low energies by the SL(2,R)/U(1) Euclidian black hole Conformal Field Theory (CFT). This identification goes beyond the appearance of a non-compact spectrum: we are also able to determine the density of states, and show that it agrees with the formulas in [J. Math. Phys. 42, 2961 (2001)] and [JHEP 04, 014 (2002)], hence providing a direct ‘physical measurement’ of the associated reflection amplitude.

Introduction. The profound relation between quantum spin chains and quantum field theories (QFTs) is central to modern theoretical physics. Its simplest aspect is that the low-energy excitations of a chain are described by a QFT in the continuum limit. Conversely, some difficult, strongly interacting QFTs can be tackled using appropriate (usually antiferromagnetic) spin chains, for which a large variety of methods—including numerical—are available. The numerous success stories using this approach include understanding the θ-term in the O(3) sigma model [1], and developing bosonization techniques from the concept of Luttinger liquid [2].

The above examples all involve spin chains built with finite-dimensional representations, and their continuum limits are QFTs with a compact target. But many current problems of physics are concerned with strongly curved, non-compact targets. For instance, the CFT describing the transition between plateaux in the two-dimensional (2D) Integer Quantum Hall Effect (IQHE) is expected to be the low-energy limit of the non-compact 2D super sigma model on $U(1,1)[U(1)|U(1)|]$ at $\theta = \pi$. Also, the dual of $N = 4$ SUSY gauge theory in 4D is closely related with a 2D sigma model on $PSU(2,2|4)/[SO(4,1) \times SO(5)]$ [4]. While it seems extremely hard to solve these sigma models directly, one might hope that spin chain regularizations provide access to some of their properties. A priori, these chains should involve infinite-dimensional representations. In the IQHE such a chain indeed arises in the very anisotropic limit of the Chalker-Coddington network model, and involves alternating highest and lowest weight representations of $psl(2|2)$ [4].

Unfortunately, the technical difficulties encountered in the analysis of these infinite-dimensional spin chains are considerable. While numerical methods based on Hilbert space truncations are possible [2], analytical approaches have stalled. Despite much work on different aspects of the Bethe Ansatz (BA) in this case [2, 3, 7], it is not even known whether the antiferromagnetic non-compact XXX spin chains are gapless!—nor to what extent analyses based on coherent state representations and analogies with the compact case [8] make sense.

In this Letter we show how to construct a solvable, finite-dimensional, antiferromagnetic spin chain whose low-energy physics is described nevertheless by a non-compact CFT. This is obviously important progress, since the usual BA techniques can then be used, without insurmountable difficulties, to understand non-compact CFTs.

We illustrate this discovery with the SL(2,R)/U(1) sigma model, a gauged Wess-Zumino-Witten (WZW) model originally introduced in the context of black holes in string theory [9] and later intensively studied for its CFT features as well [10–12]. While it is tempting to assume that it (or the SL(2,R) WZW model) is the continuum limit of a spin chain based on infinite-dimensional representations of SL(2,R) or SL(2,C), this connection remains presently entirely speculative. In contrast, we here show how all the known features of the SL(2,R)/U(1) sigma model, including the non-compact spectrum and the highly non-trivial density of states, are obtainable starting from a rather modest-looking spin chain, one of whose aspects however is non-hermiticity.

The spin chain. The starting point is a $\mathbb{Z}_2$ staggered model, which was introduced in relation with the antiferromagnetic Potts model [16–18]. It is a variant of the six-vertex model but with alternating spectral parameters $(u, u + \frac{\pi}{2}, \ldots, u, u + \frac{\pi}{2})$ and $(0, \frac{\pi}{2}, \ldots, 0, \frac{\pi}{2})$ on the horizontal and vertical lines of the square lattice, respectively. Here, we use the Boltzmann weights: $a = \sin(\gamma - u), b = \sin u, c = \sin \gamma$, in Baxter’s notations [12], encoded in the matrix $R_{ij}(u)$, and we restrict to the regime $0 < \gamma < \frac{\pi}{2}$. The one-row transfer matrix with periodic boundary conditions, for a system of width $2L$, is

$$t(u) := \text{Tr}_0 R_{0,2L}(\bar{u}) R_{0,2L-1}(u) \ldots R_{02}(\bar{u}) R_{01}(u),$$
where \( \bar{u} := u - \pi/2 \). For simplicity, we suppose \( L \) even. In the very anisotropic limit \( u \to 0 \), the equivalent quantum Hamiltonian is

\[
H := \frac{1}{2} \sin 2\gamma \left[ t^{-1}(0)t'(0) + t^{-1}(\frac{\pi}{2}) t'\left(\frac{\pi}{2}\right) \right],
\]

where the prime denotes differentiation with respect to \( u \). In terms of Pauli matrices,

\[
H = \sum_{j=1}^{2L} \left[ -\frac{1}{2} \sigma_j \cdot \sigma_{j+2} + \sin^2 \gamma \sigma_j^z \sigma_{j+1}^z - \frac{1}{2} \cos 2\gamma I \right] + i \sin \gamma \left( \sigma_{j-1}^z - \sigma_{j+2}^z \right) (\sigma_j^z \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^-). \tag{2}
\]

The parameter \( \gamma \) defines the quantum algebra \( U_q(S\ell_2) \) for the \( R \)-matrix, through the relation \( q = e^{i\gamma} \). Like in the open XXZ chain, this makes \( H \) non-hermitian. However, the low-lying states we study here all have real energies. Among the conserved quantities of \( H \), one has

- the \( \mathbb{Z}_2 \) charge \( C := \prod_{j=1}^{L} c_{2j-1,2j} \),
- the total magnetization \( M := \frac{1}{2} \sum_{j=1}^{2L} \sigma_j^z \),
- the ‘quasi-momentum’ \( S := \frac{\pi - 2\gamma}{4\pi\gamma} \log \left[ t(0)t^{-1}(\frac{\pi}{2}) \right] \),

which reads

\[
S = \frac{\pi - 2\gamma}{4\pi\gamma} \log \left( \prod_{j=1}^{L} c_{2j,2j+1} \times \prod_{j=1}^{L} c_{2j-1,2j} \right). \tag{3}
\]

We have defined \( c_{ij} = P_j R_{ij}(-\pi/2)/\cos \gamma \), and \( P_j \) permutes the spins \( i \) and \( j \). The three above operators commute with \( H \) by the Yang-Baxter equations and the six-vertex ‘ice rule’. Below we derive the low-energy spectrum of (2), and establish a dictionary between the above conserved quantities and those of the \( SL(2,\mathbb{R})/U(1) \) sigma model.

**Low-energy spectrum from the Bethe Ansatz.** The model (2) is solvable by the Bethe Ansatz, and the low-energy states correspond to two sets of real roots \( \{\lambda_j\}_{j=1...r_0} \) and \( \{\mu_j\}_{j=1...r_1} \). The BA equations read [20]

\[
\left[ \frac{\cosh(\lambda_j - i\gamma)}{\cosh(\lambda_j + i\gamma)} \right]^{r_0} = -\prod_{\ell=1}^{r_0} \sinh \frac{1}{2} (\lambda_j - \lambda_{j+1} - 2i\gamma) \times \prod_{\ell=1}^{r_1} \cosh \frac{1}{2} (\lambda_j - \mu_{j+1} - 2i\gamma),
\]

\[
\left[ \frac{\cosh(\mu_j - i\gamma)}{\cosh(\mu_j + i\gamma)} \right]^{r_1} = -\prod_{\ell=1}^{r_0} \cosh \frac{1}{2} (\mu_j - \lambda_{j+1} + 2i\gamma) \times \prod_{\ell=1}^{r_1} \sinh \frac{1}{2} (\mu_j - \mu_{j+1} + 2i\gamma). \tag{4}
\]

and subject to two coupled linear integral equations:

\[
2\pi\eta_a(\lambda) + \sum_{b=0,1} \int_{-\Lambda_b}^{+\Lambda_b} \frac{d\mu}{2\sinh \gamma} \eta_b(\mu) K_{a-b}(\lambda - \mu) = \phi(\lambda), \tag{6}
\]

where \( a \in \{0,1\} \). It is convenient to define the kernels \( K_{a-b} \) and the source term \( \phi \) through their Fourier transform, with the convention \( \hat{f}(\omega) := \int d\lambda \, f(\lambda) e^{i\omega\lambda} \). One has:

\[
\hat{K}_0, \hat{K}_{\pm 1}, \hat{\phi} = -\frac{2\pi}{\sinh \gamma} \frac{2\pi}{\sinh \gamma} \frac{2\pi}{\sinh \gamma} \frac{2\pi}{\sinh \gamma}.
\]

Like in the XXZ model, the ground state (gs) corresponds to the limit \( \Lambda_a = \Lambda'_a \to \infty \) in (6), and the solution is simply obtained by Fourier transform \( \eta_0(\lambda) = \eta_1(\lambda) = \eta_{gs}(\lambda) := 1/(2\gamma' \cosh \frac{\Lambda}{2}) \). The central charge obtained from the scaling of the ground-state energy is \( c = 2 \).

The elementary excitations over the ground state (spinons) are holes in the root distributions \( \eta_0, \eta_1 \). Using standard kernel methods [21, 22], we get the dressed magnetic charge \( Z = \pi/(4\gamma) \) and the energy and momentum...
of a spinon of rapidity \( \lambda \):

\[
\epsilon_{\text{sp}}(\lambda) = -\frac{\pi \sin \gamma'}{\gamma' \cosh \frac{\pi \lambda}{2\gamma'}};
\]

\[
p_{\text{sp}}(\lambda) = 2\arctan \left( \tanh \frac{\pi \lambda}{2\gamma'} \right).
\]

The low-energy spinons \((\lambda \to \infty)\) thus have a linear dispersion, with Fermi velocity \( v_f = \frac{\pi \sin \gamma'}{\gamma' \cosh \frac{\pi \lambda}{2\gamma'}} \). Similarly, the quasi-momentum associated to a spinon is

\[
s_{\text{sp}}(\lambda) = \pm \frac{\pi - 2\gamma}{4\pi\gamma} \log \left[ \cosh \frac{\pi \lambda}{2\gamma'} \right],
\]

where the sign depends on which distribution \((\eta_0 \text{ or } \eta_1)\) the spinon lives in. Since \( s \propto (r_1 - r_0) \) for large \( L \) (see (14)), this quantity measures the difference between the two total root densities.

The main object of this Letter is the study of the conformal spectrum \( \{ \langle h, \tilde{h} \rangle \} \) through the finite-size behavior of the energy gap and the momentum [23]:

\[
\Delta E = E - E_{gs} \approx \frac{2\pi v_f}{L} (h + \tilde{h}), \quad p \approx \frac{2\pi}{L} (h - \tilde{h}).
\]

In analogy with the XXZ case [24], we assume that the Bethe states which converge to primaries in the continuum limit are combinations of a magnetic excitation (removal of \( m_0 \) roots \( \{ \lambda_j \} \)) and \( m_1 \) roots \( \{ \mu_j \} \) of the energy gap \((\omega_j)\) this quantity measures the difference between the two total root densities.

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\]
In Eqs. [14][15] (derived here for $e = 0$), $L_0$ is a cut-off depending only on $\gamma$, and $B$ is a correction term which we discuss below. More generally, for $e \neq 0$, a similar derivation yields

$$\Delta E \approx \frac{2\pi vF}{L} \left( \frac{\gamma m^2}{2\pi} + \frac{\pi e^2}{2\gamma} + \frac{2\gamma s^2}{\pi - 2\gamma} \right), \quad p = \frac{2\pi em}{L},$$

and hence

$$\hbar = \frac{(m + ek)^2}{4k} + \frac{s^2}{k - 2}, \quad \tilde{\hbar} = \frac{(m - ek)^2}{4k} + \frac{s^2}{k - 2}, \quad (16)$$

where $(m, e) \in \mathbb{Z}^2$, $s \in \mathbb{R}$, and we have set $k := \pi/\gamma$. Since $s$ is real, this is a non-compact spectrum.

**Relation to the sigma model.** Recall now that, while the SL(2, $\mathbb{R}$)/U(1) black hole (BH) model has central charge $c_{\text{BH}} = 2(k + 1)/(k - 2)$, the identity field in that theory is associated with a non-normalizable state. In fact, normalizable states arise mostly from continuous representations, and have conformal weights as in (16), but with the second term $s^2/(k - 2)$ replaced by a WZW-type term $-\gamma(j + 1)/(k - 2)$, with $j = -\frac{1}{2} + is$. The ‘bottom’ of the spectrum thus occurs at $\hbar_0 := 1/4(k - 2)$, leading to an effective central charge $c = c_{\text{BH}} - 2\hbar_0 = 2$ in our lattice model. The spectrum (16) is thus formally identical with the SL(2, $\mathbb{R}$)/U(1) one [11][15].

Since, in the large-$L$ limit, $s$ becomes a real parameter, the spectrum (16) is a collection of continua over the conformal weights of a compact boson. In the SL(2, $\mathbb{R}$)/U(1) theory, this boson describes excitations along the compact direction of the cigar (angular momentum of rotations around the tip), whereas $s$ is the angular momentum along the axis of the cigar. We have expressed in (15) the lattice operator $S$ measuring this angular momentum. In finite size, since $s \simeq \pi \bar{n}/(4 \log L)$, the $s^2$ terms in (16) correspond to the magnetic charge of a boson with effective compactification radius $R \propto \log L$.

As in ordinary quantum mechanics, there is actually little dynamical information in the spectrum (15) alone: what is really needed is the density of states. This can also be extracted from our finite-size calculation. Denoting $q = \exp(2i\pi \tau)$ the modular parameter, the partition function of our model on a torus reads, in the scaling limit,

$$Z = \frac{(qq)^{-2/24}}{[\eta(\tau)]^4} \sum_{m + \bar{n} \in \mathbb{Z}} q^h \bar{q}^\bar{h} = \frac{(qq)^{-2/24}}{[\eta(\tau)]^4} \sum_{e, m \in \mathbb{Z}} \int_{-\infty}^{+\infty} ds \, \rho(s) \, q^h \bar{q}^\bar{h},$$

where $\eta$ is the Dedekind eta function, and the density of states is

$$\rho(s) = \frac{2}{\pi} \left[ \log \frac{L}{L_0} + \partial_s (sB) \right], \quad (17)$$

where $B$ was introduced in (14). The logarithmic divergence with the IR cutoff is familiar in the sigma model [15], whereas the finite part of $\rho(s)$ is determined by requiring $\bar{m} \in \mathbb{Z}$ in (14). Consider the purely magnetic state $\Psi_{m_0, m_1, 0}$. Our WH technique only gives access to the function $B$ in the regime of large $s$ and $m$, where we get

$$B(\gamma, m, e = 0, s) \sim \left\{ \begin{array}{cl} -\log s & \text{for } s \gg m, \\ -\log m & \text{for } s \ll m. \end{array} \right. \quad (18)$$

We believe it will eventually be possible to obtain more complete results on $B$ by a deeper analysis of the BAE. For now, in order to interpolate between the above limiting behaviors, we compute $B$ numerically, by solving (4–5) at finite $L$: see results on Figs. [1][2]. The only adjustable parameter in these computations is $L_0$, which can be fixed, e.g., by imposing the value of $B$ in the ground state $m = e = s = 0$. Slow convergence with the system size is to be expected, because higher-order corrections to Eq. (14) are of order $1/\log L$.

The finite part of the density of states $\rho(s)$ in the SL(2, $\mathbb{R}$)/U(1) sigma model was calculated in [10][11] (see also [27]), and reads, in our parametrization,

$$B_{\text{BH}}(s) = \frac{1}{2s} \text{Im} \log \left[ \Gamma\left(\frac{1-m+ek}{2} - is\right)\Gamma\left(\frac{1+m-ek}{2} - is\right) \right]. \quad (19)$$

This function obeys the asymptotic behavior (18), and numerical agreement with the finite part in our model is good, as shown in Figs. [1][2]. Moreover, we have computed the values of $(B - B_{\text{BH}})$, where $B_{\text{BH}}$ stands for the ground-state value of $B$, in the limit $s \to 0$, to check that $L_0$ depends only on $\gamma$: see Fig. [3].

![FIG. 1.](image)

(Colour online) Finite part $B(s)$ of the density of states (17) for the continuum over the ground state of the $\mathbb{Z}_2$ model at $\gamma = \pi/5$, compared to $B_{\text{BH}}$.

To conclude, we have identified the continuum limit of our spin chain as the SL(2, $\mathbb{R}$)/U(1) black hole sigma model...
Spin chains have also appeared from a different viewpoint in the AdS/CFT conjecture [32]. It was discovered that many physical quantities on the gauge theory side can be related with the spectra of quantum spin chains [33]. These spectra in turn can be studied by techniques addressing directly the low-energy excitations [34], or via the BA. Recently, a powerful machinery has been developed along those lines to obtain results for the gauge theory at any coupling [35]. It is tempting to conjecture that spin chains such as ours might appear in this context as the ‘gauge theory’ side of some new interesting CFTs.

Acknowledgments. H.S. thanks G. Korchemsky, V. Schomerus and J. Teschner for many discussions about non-compact spin chains and CFTs. The authors wish to thank C. Candu for a careful reading of the manuscript and helpful remarks. The work of J.L.J. and H.S. is supported by the Agence Nationale de la Recherche (grant ANR-10-BLAN-0414). The work of Y.I. is supported by the European Research Council (grant CONFRA 228046).

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The behavior of the BA kernels changes at $\gamma = \frac{\pi}{2}$, and the continuum limit of the spin chain is different: see [18].

The WH method of [21] is a finite-field method, strictly valid only for finite ‘particle densities’ $m/L$ and $s/L$, and thus $m, s \propto L$. However, the energy is a polynomial in these densities, and hence the result [18] is reliable for all values of $m, s$.

A. B. Zamolodchikov, unpublished notes.

The careful reader might notice that (17) is twice the density in [11]. Our model indeed enjoys an additional $\mathbb{Z}_2$ symmetry (the exchange of $\{\lambda_j\}$ and $\{\mu_j\}$), and the target is actually two independent cigars.

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