Ultraviolet asymptotics of scalar and pseudoscalar correlators in hot Yang-Mills theory

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Abstract

Inspired by recent lattice measurements, we determine the short-distance \((a \ll r \ll 1/\pi T)\) as well as large-frequency \((1/a \gg \omega \gg \pi T)\) asymptotics of scalar (trace anomaly) and pseudoscalar (topological charge density) correlators at 2-loop order in hot Yang-Mills theory. The results are expressed in the form of an Operator Product Expansion. We confirm and refine the determination of a number of Wilson coefficients; however some discrepancies with recent literature are detected as well, and employing the correct values might help, on the qualitative level, to understand some of the features observed in the lattice measurements. On the other hand, the Wilson coefficients show slow convergence and it appears uncertain whether this approach can lead to quantitative comparisons with lattice data. Nevertheless, as we outline, our general results might serve as theoretical starting points for a number of perhaps phenomenologically more successful lines of investigation.

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1. Introduction

Many of the most interesting physical properties of a finite-temperature system are of an “infrared” type, i.e. to be extracted from the long-distance or short-frequency limit of appropriate 2-point correlation functions. For instance, in a weakly coupled Yang-Mills theory with the gauge coupling $g$, spatial correlation lengths originate at length scales $r \sim 1/(gT)$ or $r \geq \pi/(g^2T)$, depending on the global quantum numbers of the operator considered [1]. At the same time, transport coefficients, which reflect the real-time response of the system to small perturbations, arise at frequencies $\omega \sim g^4T/\pi$ (cf., e.g., ref. [2]). Infrared observables may either be genuinely non-perturbative [3, 4], or they do possess a weak-coupling series up to some order, but it is slowly convergent at temperatures relevant for heavy ion collision experiments (cf., e.g., ref. [5]). Hence, perhaps with a few exceptions (cf., e.g., ref. [6]), infrared observables need eventually to be determined via non-perturbative lattice simulations.

Despite the stated general picture, circumstances exist as well under which “ultraviolet” observables, measured at short distances ($r \lesssim 1/\pi T$) or large frequencies ($\omega \gtrsim \pi T$), are of physical interest. As an example on the former case, we may mention that in heavy quarkonium physics, the system has an additional “external” scale, the heavy quark mass, $M$. Given that normally $\pi T \ll M$, the inverse Bohr radius $r_B^{-1} \sim \alpha_s M \ll M$ of a quarkonium state could well be of the same order as the temperature, $r_B^{-1} \sim \pi T$. Changes in quarkonium properties caused by a finite temperature could therefore be due to thermal modifications of the quark–antiquark potential at $r \sim r_B \sim 1/(\pi T)$, in which case the weak-coupling expansion may converge faster than at large distances $r \gtrsim 1/(gT)$ [7]. As an example on the latter case, we remark that all lattice estimates of a spectral function, $\rho(\omega)$, from whose intercept, $\lim_{\omega \to 0} \rho(\omega)/\omega$, transport coefficients are determined, rely on “inverting” the relation

$$G(\hat{\tau}) = \int_0^\infty \frac{d\omega}{\pi} \rho(\omega) \cosh \left( \frac{1}{2} \hat{\tau} \right) \beta \omega \sinh \left( \frac{\beta \omega}{2} \right),$$

(1.1)

where $\beta \equiv 1/T$ and $G(\hat{\tau})$, $0 < \hat{\tau} < 1$, is a measured Euclidean correlator along the compact time direction. Since the “kernel” multiplying $\rho(\omega)$ in eq. (1.1) only depends on $\omega$ through $\beta \omega$, it is clear that $\rho(\omega)$ in the regime $\omega \sim \pi T$ gives an important contribution to $G(\hat{\tau})$, and needs to be well understood before infrared sensitive contributions from the range $\omega \ll \pi T$ can be reliably extracted.

In the present note we study the ultraviolet regime of certain 2-point correlation functions, and even take it to its extreme limit: not only do we consider $r \lesssim 1/(\pi T)$ but in fact $r \ll 1/(\pi T)$; not only $\omega \gtrsim \pi T$ but in fact $\omega \gg \pi T$. Only the ultraviolet cutoff, e.g. the lattice spacing $a$, is assumed to be even farther in the ultraviolet than our physical scales. In this situation the correlation functions can be determined within a framework similar to the Operator Product Expansion [8], as has recently been discussed in the Euclidean domain [9] and also more generally [10]. Apart from theoretical considerations, we would in principle also like
to make contact with the Euclidean lattice simulations described in ref. [11]. It is our experience, both from Euclidean [7] and, after analytic continuation, Minkowskian [12] domains that, when pursued to a sufficient order, the (continuum) weak-coupling expansion could work relatively well in the ultraviolet regime; however, the issue needs to be re-investigated when a new correlation function is considered or when the computation is organized as an Operator Product Expansion, and these are some of the goals of the present study.

The plan of this note is the following. In sec. 2 we define the basic observables considered. Section 3 contains an outline of the method used; the general results are given in sec. 4. The case of short distances is discussed more specifically in sec. 5, and that of large frequencies in sec. 6. We also briefly comment on the relation of our work to recently discussed “sum rules” in sec. 7. Section 8 offers a summary and outlook, whereas the three appendices collect together some details needed in the main text: the definitions and asymptotic expansions of the “master” sum-integrals appearing in pure Yang-Mills theory are listed in appendix A; a number of thermodynamic potentials playing a role in our study are given in appendix B; and fermionic effects are briefly discussed in appendix C.

2. Setup

Employing the convention $D_\mu =\partial_\mu - ig_0 A_\mu^a T^a$, with $T^a$ hermitean generators of $SU(N_c)$ normalized as $\text{Tr} [T^a T^b] = \frac{1}{2} \delta^{ab}$, and defining $F^a_{\mu\nu} = (2i/g_0)\text{Tr} \{T^a [D_\mu, D_\nu] \} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_0 f^{abc} A_\mu^b A_\nu^c$, the dimensionally regularized Euclidean action relevant for pure Yang-Mills theory at a finite temperature $T = \frac{1}{\beta}$ reads

$$S_E = \int_0^\beta d\tau \int d^{3-2\epsilon} x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \right\}. \tag{2.1}$$

The trace of the corresponding energy-momentum tensor is $D_{\mu\nu} F^a_{\mu\nu} F^a_{\mu\nu}$, where $D \equiv 4 - 2\epsilon$ is the space-time dimensionality. In order to define the operators whose correlation functions we are interested in, we note that it is the “geometric” structure $g_0^2 F^a_{\mu\nu} F^a_{\rho\sigma} = -2\text{Tr} \{[D_\mu, D_\nu] [D_\rho, D_\sigma] \}$ which requires no renormalization at the order of our computation. So, we define the gauge invariant scalar and pseudoscalar operators

$$\theta \equiv c_\theta g_0^2 F^a_{\mu\nu} F^a_{\mu\nu}, \quad \chi \equiv c_\chi \epsilon_{\mu\nu\rho\sigma} g_0^2 F^a_{\mu\nu} F^a_{\rho\sigma}, \tag{2.2}$$

where the $D$-dimensional Euclidean $\epsilon_{\mu\nu\rho\sigma}$ is handled as in ref. [13] (as long as the procedure is self-consistent the precise regularization has no impact on the final results). In analogy with ref. [11] the coefficients are chosen as

$$c_\theta = \frac{D - 4}{4g_0^2 \mu^{-2\epsilon}} = -\frac{b_0}{2} - \frac{b_1 g_0^2}{4} + \ldots, \tag{2.3}$$

$$c_\chi \equiv \frac{1}{64\pi^2}, \tag{2.4}$$
but in most of what follows we do not need to specify their values. For eq. (2.3), we have written

\[ g_0^2 = g^2 \mu^2 \left[ 1 - \frac{b_0}{\epsilon} g^2 + \left( \frac{b_0^2}{2 \epsilon^2} - \frac{b_1}{2 \epsilon} \right) g^4 + \ldots \right], \tag{2.5} \]

where

\[ b_0 = \frac{11 N_c}{3(4\pi)^2}, \quad b_1 = \frac{34 N_c^2}{3(4\pi)^4}, \tag{2.6} \]

and \( g^2 \) is the dimensionless renormalized coupling constant, evaluated at the \( \overline{\text{MS}} \) scheme renormalization scale \( \bar{\mu} \left( \mu^2 = \bar{\mu}^2 e^\gamma_E / 4\pi \right) \).

With this notation, the Euclidean correlators considered are defined as

\[ G_\theta(x) \equiv \langle \theta(x) \theta(0) \rangle_c, \quad G_\chi(x) \equiv \langle \chi(x) \chi(0) \rangle, \tag{2.7} \]

where \( \langle \ldots \rangle_c \) denotes the connected part, and the expectation value is taken at a finite temperature \( T \). (The disconnected part of \( G_\theta \) is \( \langle \theta \rangle^2 \), where \( \langle \theta \rangle = e - 3p \) is trace of the energy-momentum tensor; in a general regularization scheme \( \langle \theta \rangle \) is ultraviolet divergent. In the following we need the finite thermal part thereof, denoted by \( (e - 3p)(T) \).) We also consider the corresponding Fourier transforms,

\[ \tilde{G}_\theta(P) \equiv \int_x e^{-iP \cdot x} G_\theta(x), \quad \tilde{G}_\chi(P) \equiv \int_x e^{-iP \cdot x} G_\chi(x), \tag{2.8} \]

where short-distance singularities are regulated dimensionally. Finally we denote

\[ \Delta \tilde{G}_\theta(P) \equiv \tilde{G}_\theta(P) - \tilde{G}_\theta^{T=0}(P), \quad \Delta \tilde{G}_\chi(P) \equiv \tilde{G}_\chi(P) - \tilde{G}_\chi^{T=0}(P), \tag{2.9} \]

subtracting the zero-temperature parts at a fixed \( P \). In Euclidean signature, the components of the four-momentum and spacetime coordinate are denoted by

\[ P = (p_n, p), \quad p \equiv |p|, \quad x = (\tau, \mathbf{x}), \quad r \equiv |\mathbf{x}|, \tag{2.10} \]

where \( p_n = 2\pi T n \) are bosonic Matsubara frequencies.

### 3. Method

Carrying out the Wick contractions in the 1-loop and 2-loop Feynman graphs contributing to eq. (2.7) (cf. ref. 13 or fig. 1) and using the notation of appendix A for the various “master”
Figure 1: The graphs contributing to the connected 2-point correlators defined in eq. (2.7), up to 2-loop order. The wiggly lines denote gluons; the small dots the operators θ or χ (cf. eq. (2.2)); and the grey blob the 1-loop gauge field self-energy. Graphs obtained with trivial “reflections” from those shown have been omitted from the figure.

sum-integrals appearing in the result, we find the bare expressions (\(d_A \equiv N_c^2 - 1\))

\[
\frac{\tilde{G}_\theta(P)}{4d_A c^2 g_b^4} = (D - 2) \left[ -J_a + \frac{1}{2} J_b \right] + g_b^2 N_c \left\{ 2(D - 2) \left[ (D - 1) I_a + (D - 4) I_b \right] + (D - 2)^2 \left[ I_c - I_d \right] \right. \\
\left. + \frac{22 - 7D}{3} I_f - \frac{(D - 4)^2}{2} I_g + (D - 2) \left[ -3I_e + 3I_h + 2I_i - I_j \right] \right\},
\]

(3.1)

\[
\frac{\tilde{G}_\chi(P)}{-16d_A c^2 g_b^4(D - 3)} = (D - 2) \left[ -J_a + \frac{1}{2} J_b \right] + g_b^2 N_c \left\{ 2(D - 2) \left[ I_a + (D - 4) I_b \right] + (D - 2)^2 \left[ I_c - I_d \right] \right. \\
\left. - \frac{2D^2 - 17D + 42}{3} I_f - 2(D - 4) I_g + (D - 2) \left[ -3I_e + 3I_h + 2I_i - I_j \right] \right\}.
\]

(3.2)

Here, for brevity, structures containing \(\oint_Q\), which vanishes exactly in dimensional regularization, have been omitted (\(\oint_Q\) denotes a sum-integral with bosonic Matsubara frequencies).

The goal now is to obtain asymptotic expansions for the functions in eqs. (3.1), (3.2), valid in the Euclidean domain \(P^2 = p_n^2 + p^2 \gg (\pi T)^2\). (Interestingly, as has recently been reviewed in ref. [15], similar expansions may play a role also in the computation of high-order vacuum graphs at finite temperature.)

In order to obtain the expansions, we first carry out all the Matsubara sums, which can be done exactly. This has the effect of setting one or two of the propagators “on-shell”; the corresponding line is weighed by the Bose distribution. The coefficient can be identified
as a zero-temperature amplitude or 1-loop integral, with special kinematics. For instance, introducing the notation

\[ \left\{ \ldots \right\} Q \equiv \frac{1}{2} \sum_{q_\alpha = \pm i q} \{ \ldots \} , \quad \left\{ \ldots \right\} Q, R \equiv \frac{1}{4} \sum_{q_\alpha = \pm i q} \sum_{r_\alpha = \pm i r} \{ \ldots \} , \quad (3.3) \]

it is straightforward to verify that the sum-integral \( I_h \) (cf. eq. (A.10)) can be re-expressed as

\[
I_h = \int_{Q, R} \frac{P^4}{Q^2 R^2 (Q - R)^2 (R - P)^2} \\
+ \int_q \frac{n_b(q)}{q} \\
\times \int_R \left[ \frac{2P^4}{R^2 (Q - R)^2 (R - P)^2} + \frac{P^4}{R^2 (Q - R)^2 (Q - P)^2} + \frac{P^4}{(Q - P)^2 (Q - R)^2 (R - P)^2} \right] Q \\
+ \int_{q, r} \frac{n_b(q) n_b(r)}{r} \\
\times \left[ \frac{2P^4}{(Q - R)^2 (R - P)^2} + \frac{P^4}{(Q - R)^2 (Q - R - P)^2} + \frac{2P^4}{(R - P)^2 (Q + R - P)^2} \right] Q, R , \quad (3.4)
\]

where \( \int_Q \equiv \int d^D Q / (2\pi)^D \) and \( \int_q \equiv \int d^{D-1} q / (2\pi)^{D-1} \). It is important to realize that within the square brackets we can set \( Q^2 = 0 \) for any \( D \), and that therefore anything proportional to \( D \)-dependent powers of \( Q^2 \) vanishes exactly (this is the case particularly when the \( R \)-integration factorizes from the \( P \)-dependence).

In order to handle the remaining structures, we note that an integration variable appearing inside the Bose distribution is always ultraviolet safe (cut off by the temperature). Therefore, we can expand propagators as

\[
\left[ \frac{1}{(Q - R)^2} \right] Q = \left[ \frac{1}{R^2} + \frac{2Q \cdot R}{R^4} + \frac{4(Q \cdot R)^2}{R^6} + \ldots \right] Q , \quad (3.5)
\]

where we also made use of the on-shell condition \( Q^2 = 0 \). Such an expansion leads to the following types of vacuum integrals:

\[
\int_R \frac{R_\mu R_\nu}{(R^2)^m (|R - P|)^n} = \delta_{\mu \nu} A_{m, n} + P_\mu P_\nu B_{m, n} , \quad (3.6)
\]

\[
\int_R \frac{R_\mu}{(R^2)^m (|R - P|)^n} = P_\mu C_{m, n} , \quad (3.7)
\]

\[
\int_R \frac{1}{(R^2)^m (|R - P|)^n} = I_{m, n} . \quad (3.8)
\]

In dimensional regularization these can all be related to \( I_{m, n} \), which in turn reads

\[
I_{m, n} = \frac{(P^2)^{D - m - n} \Gamma(m + n - D) \Gamma(D - m) \Gamma(D - n)}{(4\pi)^{D/2} \Gamma(D - m - n) \Gamma(m) \Gamma(n)} . \quad (3.9)
\]

\(^2\)Various recipes for this can be found in the literature, but we have established all relations from scratch.
The remaining $P$-dependence often appears in the form $[(P \cdot Q)^2]_Q = q^2(\frac{r^2}{3-2\epsilon} - r_n^2)$, where we made use of rotational symmetry.

Within the last part of eq. (3.4), a similar expansion can be carried out with respect to $R$ in $1/(R - P)^2$, etc, but of course not in $1/(Q - R)^2$, since both variables are of the same magnitude in this case. The challenge then is to deal with the angular integrals over the directions of $q, r$. Letting $z = q \cdot r/qr, q_n = \sigma i q, r_n = \rho ir, \sigma = \pm, \rho = \pm$, the most difficult structures are the ones containing

$$\frac{1}{(Q - R)^2} = \frac{1}{2qr(\sigma \rho - z)}.$$  \hfill (3.10)

The averaging denoted by $[\ldots]_{Q,R}$ leads to

$$\frac{1}{4} \sum_{\sigma = \pm} \sum_{\rho = \pm} \frac{1}{\sigma \rho - z} = \frac{z}{1 - z^2}, \hfill (3.11)$$

$$\frac{1}{4} \sum_{\sigma = \pm} \sum_{\rho = \pm} \frac{\sigma}{\sigma \rho - z} = \frac{1}{4} \sum_{\sigma = \pm} \sum_{\rho = \pm} \frac{\rho}{\sigma \rho - z} = 0, \hfill (3.12)$$

$$\frac{1}{4} \sum_{\sigma = \pm} \sum_{\rho = \pm} \frac{\sigma \rho}{\sigma \rho - z} = \frac{1}{1 - z^2}. \hfill (3.13)$$

Antisymmetry kills the structure linear in $z$, unless there is another angular variable appearing in the numerator. Fixing the directions of $p, r$ and integrating over those of $q$, the latter case produces

$$\left\langle \frac{\hat{q} \cdot \hat{r} \cdot \hat{q}}{1 - (q \cdot r)^2} \right\rangle = \hat{r} \left\langle \frac{z^2}{1 - z^2} \right\rangle = \frac{\hat{r}}{D - 4} \langle 1 \rangle, \hfill (3.14)$$

where we made use of rotational symmetry and the dimensionally regularized angular integration measure. A subsequent integral over the directions of $r$ might contain

$$\left\langle (p \cdot \hat{r})^2 \right\rangle = \frac{p^2}{D - 1} \langle 1 \rangle. \hfill (3.15)$$

The case without any angular variable in the numerator yields

$$\left\langle \frac{1}{1 - z^2} \right\rangle = \frac{D - 3}{D - 4} \langle 1 \rangle. \hfill (3.16)$$

Setting finally $D = 4 - 2\epsilon$, we obtain the expansions listed in appendix A (eqs. (A.15)–(A.32)).

4. **Euclidean momentum-space correlators**

Inserting into eqs. (3.1), (3.2) the asymptotic expansions of the master sum-integrals from appendix A as well as the bare gauge coupling from eq. (2.5); expanding in $\epsilon$; and inserting
can be expressed as tensor, \( \hat{T} \), we rewrite (\( \bar{\alpha} \))

Here we have identified the structures of eqs. (B.2), (B.3) in the result, realizing that the vacuum parts, i.e. the first rows of eqs. (4.1), (4.2), can be compared with ref. [13]. If we rewrite \((\bar{\mu}/P)^{\alpha} = (\bar{\mu}e/P)^{\alpha}[1 - \alpha + ...]\) and re-expand the square brackets, we can reproduce the results of ref. [13].

As far as the thermal parts go we note that, within the accuracy of our computation, they can be expressed as

\[
\Delta \tilde{G}_\rho(P) = \frac{3}{P^2} \left( \frac{p^2}{3} - p_n^2 \right) \left[ 1 + g^2 N_c \left( \frac{22}{3} \ln \left( \frac{\bar{\mu}^2}{P^2} + \frac{203}{18} \right) \right) (e + p)(T) \right.
\]

\[
- \frac{2}{g^2 b_0} \left[ 1 + g^2 b_0 \ln \left( \frac{\bar{\mu}^2}{\zeta g P^2} \right) \right] (e - 3p)(T) + \mathcal{O}\left( g^4, \frac{1}{P^2} \right),
\]

\[
\Delta \tilde{G}_\chi(P) = \frac{3}{P^2} \left( \frac{p^2}{3} - p_n^2 \right) \left[ 1 + g^2 N_c \left( \frac{22}{3} \ln \left( \frac{\bar{\mu}^2}{P^2} + \frac{347}{18} \right) \right) (e + p)(T) \right.
\]

\[
+ \frac{2}{g^2 b_0} \left[ 1 + g^2 b_0 \ln \left( \frac{\bar{\mu}^2}{\zeta \chi P^2} \right) \right] (e - 3p)(T) + \mathcal{O}\left( g^4, \frac{1}{P^2} \right).
\]

Here we have identified the structures of eqs. (B.2), (B.3) in the result, realizing that the expressions proportional to \((p^2/3 - p_n^2)/P^2\) couple to the traceless part of the energy-momentum tensor, \( \hat{T}_{\mu\nu} \), satisfying \( \langle \hat{T}_{\mu\nu} \rangle = \langle T_{00} \rangle (\delta_{0\mu} \delta_{0\nu} - \delta_{\mu\nu} \delta_{00}/3) \), whereas the other terms couple to the trace part. In addition we have used renormalization group invariance of \((e - 3p)(T)\) as well as the theoretical expectation that the Wilson coefficients should be independent of the
“soft scale”, $T$, to provisionally add to the results the logarithmic terms on the second rows; the coefficients $\zeta_\theta, \zeta_\chi$ next to $P^2$ remain undetermined, because fixing them would necessitate a 3-loop computation. The qualitative structure of eq. (4.3) agrees with that put forward in ref. [10]. (Note that the term proportional to $e + p$ does not appear in classic vacuum studies like ref. [16], because it breaks Lorentz symmetry. In fact, expressed in another way, $(e + p)(T) = Ts(T)$, where $s$ denotes the entropy density.)

5. Short distances

As a first concrete application, we consider equal-time correlators in configuration space, measured recently in ref. [11]. This means that we need to inverse Fourier transform the correlators in eq. (2.8) in order to get back to the correlators of eq. (2.7). Since we are applying the inverse transform not to the full result but to a asymptotic expansion valid in the regime $P^2 \gg (\pi T)^2$, the inverse transform can (and must) be taken at zero temperature (i.e. omitting terms like $n_B(p)$). The master formula for an inverse Fourier transform in dimensional regularization reads

$$\int_P e^{iP \cdot x} \frac{P^4}{(4\pi)^2} \left( \frac{\bar{\mu}}{P} \right)^{2\epsilon} = \frac{\Gamma(D/2 - \alpha)}{(\pi^2 x)^{D(x/2) - 2\alpha}\Gamma(\alpha)},$$

We recall from eq. (2.10) that $r$ denotes a spatial separation in radial coordinates, $r = |x|$, whereas the temporal separation is chosen to vanish as in ref. [11], $\tau = 0$.

Employing eq. (5.1), we obtain for the structures appearing in the vacuum parts (the first rows of eqs. (4.1), (4.2)) the expansions

$$\int_P e^{iP \cdot x} \left( \frac{\bar{\mu}}{P} \right)^{2\epsilon} \left( \frac{P^4}{(4\pi)^2} \right) = \frac{12\mu^{-2\epsilon}}{\pi^4 r^8} \left( r\bar{\mu} \right)^{4\epsilon} 
\left[ 1 + \epsilon \left( \frac{31}{6} + 4\gamma_E - 4 \ln 2 \right) + \mathcal{O}(\epsilon^2) \right],$$

$$\int_P e^{iP \cdot x} \left( \frac{\bar{\mu}}{P} \right)^{4\epsilon} \left( \frac{P^4}{(4\pi)^2} \right) = \frac{24\mu^{-2\epsilon}}{\pi^4 r^8} \left( r\bar{\mu} \right)^{6\epsilon} 
\left[ 1 + \epsilon \left( \frac{-17}{2} + 6\gamma_E - 6 \ln 2 \right) + \mathcal{O}(\epsilon^2) \right].$$

Note that these are proportional to $\epsilon$, which is why we did not need to show terms of $\mathcal{O}(1)$ in eqs. (4.1), (4.2). The structures multiplying $e + p$ in eqs. (4.3), (4.4) yield

$$\int_P e^{iP \cdot x} \frac{1}{P^2} \left( \frac{P^2}{3} - \frac{P_n^2}{3} \right) = -\frac{2}{3\pi^2 r^4},$$

$$\int_P e^{iP \cdot x} \frac{1}{P^2} \left( \frac{P^2}{3} - \frac{P_n^2}{3} \right) \ln \left( \frac{\bar{\mu}^2}{P^2} \right) = -\frac{2}{3\pi^2 r^4} \left( 2 \ln \frac{r\bar{\mu}^2}{2} - \frac{3}{2} \right).$$

(To arrive at these it is helpful to consider the covariant integral $\int_P e^{iP \cdot x} P_\mu P_\nu / [P^2]^{\alpha}$ first.) Finally, the structures multiplying $e - 3p$ in eqs. (4.3), (4.4) contain kind of an ambiguity because, taken literally, the terms independent of $P$ yield $\delta^{(D)}(x)$, whereas according to
eq. (5.1) we get nothing. This ambiguity is of little significance, however, since the contact terms are of no interest at $r \neq 0$. In contrast, the terms with $\ln(\bar{\mu}^2/P^2)$ as an integrand yield a physical behaviour in dimensional regularization:

$$\int_{\mathbb{R}^d} e^{iP \cdot x} \ln \frac{\bar{\mu}^2}{P^2} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{iP \cdot x} \frac{1}{\epsilon} \left[ \left( \frac{\bar{\mu}}{P} \right)^{2\epsilon} - 1 \right] = \frac{1}{\pi^2 r^4}. \tag{5.6}$$

Inserting eqs. (5.2)–(5.6) into the inverse Fourier transforms of eqs. (4.1), (4.2), we get

$$G_{\theta}(r) = \frac{12d_A}{4\pi^2 c_{\theta}} \gamma_{\theta;1}(r) - \frac{2(e + p)}{\pi^2 r^4} \gamma_{\theta;e+p}(r) - \frac{2(e - 3p)}{\pi^2 r^4} \gamma_{\theta;e-3p}(r) + O\left(\frac{T^6}{r^2}\right), \tag{5.7}$$

$$G_{\chi}(r) = \frac{12d_A}{4\pi^2 c_{\chi}} \gamma_{\chi;1}(r) - \frac{2(e + p)}{\pi^2 r^4} \gamma_{\chi;e+p}(r) + \frac{2(e - 3p)}{\pi^2 r^4} \gamma_{\chi;e-3p}(r) + O\left(\frac{T^6}{r^2}\right), \tag{5.8}$$

where

$$\gamma_{\theta;1}(r) = g^4 + \frac{g^6 N_c}{(4\pi)^2} \left( \frac{44}{3} \ln \frac{\tau \bar{\mu} e^{\gamma_E}}{2} - \frac{1}{9} \right) + O(g^8), \tag{5.9}$$

$$\gamma_{\theta;e+p}(r) = g^4 + \frac{g^6 N_c}{(4\pi)^2} \left( \frac{44}{3} \ln \frac{\tau \bar{\mu} e^{\gamma_E}}{2} + \frac{5}{18} \right) + O(g^8), \tag{5.10}$$

$$\gamma_{\theta;e-3p}(r) = g^4 + O(g^6), \tag{5.11}$$

$$\gamma_{\chi;1}(r) = g^4 + \frac{g^6 N_c}{(4\pi)^2} \left( \frac{44}{3} \ln \frac{\tau \bar{\mu} e^{\gamma_E}}{2} + \frac{71}{9} \right) + O(g^8), \tag{5.12}$$

$$\gamma_{\chi;e+p}(r) = g^4 + \frac{g^6 N_c}{(4\pi)^2} \left( \frac{44}{3} \ln \frac{\tau \bar{\mu} e^{\gamma_E}}{2} + \frac{149}{18} \right) + O(g^8), \tag{5.13}$$

$$\gamma_{\chi;e-3p}(r) = g^4 + O(g^6). \tag{5.14}$$

The results of eqs. (5.7), (5.8) can be compared with those given in refs. [9, 11]. Inserting $c_{\theta}$ from eq. (2.3), our leading order results for the vacuum and trace anomaly parts of $G_{\theta}$ agree with ref. [4]. In contrast, we find a coefficient of the term proportional to $e + p$ to be larger by a factor 2 than in ref. [9]. We remark that this coefficient appears both at $O(g^4)$ and $O(g^6)$, and we get contributions at both orders which combine to produce the correct renormalization group invariant structure. In addition, the leading order result can even be worked out exactly: omitting terms that vanish in dimensional regularization, we get

$$\frac{G_{\theta}(r)}{4\pi^2 c_{\theta}} = \frac{G_{\chi}(r)}{-16c_{\chi}} = 4d_A g^4 \{ \partial_{\mu} \partial_{\nu} \Delta(x) \partial_{\mu} \partial_{\nu} \Delta(x) \}_{r=0} + O(g^6), \tag{5.15}$$

\(^3\)However the correct coefficient (in momentum space) can be found in the more recent ref. [17].
Figure 2: Numerical estimates of the Wilson coefficients $\gamma_{\theta;1}$, $\gamma_{\theta;e+p}$, $\gamma_{\chi;1}$, $\gamma_{\chi;e+p}$, cf. eqs. (5.9), (5.10), (5.12), (5.13), respectively. Thick lines correspond to choosing $\bar{\mu}$ such that the next-to-leading order correction vanishes (we refer to this value as $\bar{\mu}_{\text{opt}}$); thin lines correspond to $\bar{\mu} = 0.5\bar{\mu}_{\text{opt}}$ or $\bar{\mu} = 2.0\bar{\mu}_{\text{opt}}$ (the dependence on $\bar{\mu}$ is non-monotonic). The gauge coupling is solved from the 2-loop renormalization group equation, and $\Lambda_{\overline{\text{MS}}} = \lim_{\bar{\mu} \to \infty} \bar{\mu} \left[ b_0 g^2 - b_1 / 2 b_2 g \right]$.

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particularly in the $\chi$-channel reasonable apparent convergence is observed only at extremely short distances.

where $\Delta(x)$ is the scalar propagator in coordinate space,

$$\Delta(x) \equiv \sum_p \frac{e^{ip \cdot x}}{p^2} = \frac{T}{4\pi r} \text{Re} \left[ \coth(\pi T(r+i\tau)) \right] + O(\epsilon) \, .$$

(5.16)

Carrying out the derivatives in radial coordinates $[\partial_r \partial_r \Delta \partial_\mu \partial_\nu \Delta = \partial_r^2 \Delta \partial_\rho \Delta + 2 \partial_\rho \partial_r \Delta \partial_r \Delta + (2/r^2) \partial_\rho \partial_r \Delta \partial_\rho \Delta + \partial_r^2 \Delta \partial_\rho \Delta]$ and setting $\tau = 0$ in the end, we obtain an elementary if complicated expression, whose expansion in a small $rT$ yields

$$\frac{G_\theta(r)}{4c_\theta^2} = \frac{G_\chi(r)}{-16c_\chi^2} = 4d_A g^4 \left\{ \frac{3}{\pi^4 r^8} - \frac{2T^4}{45r^4} + \frac{8\pi^2 T^6}{315r^2} + O(1) \right\} + O(g^6) \, .$$

(5.17)

Identifying $e+p$ from eqs. (B.2), (B.5) we reproduce the two leading terms of eq. (5.7). (We can also read from here that the dimensionless expansion parameter is $\sim (r\pi T)^2.$)

Concerning $G_\chi$, it is stated in ref. [11] that it has the same Operator Production Expansion as $G_\theta$, whereas we find a different sign for the term proportional to the trace anomaly,
Our finding is consistent with ref. [16] (cf. eq. (3.14) there): if the vacuum terms are normalized to be equal in the two cases, as has been done in eqs. (5.7), (5.8), then the trace anomaly terms come with opposite signs, and cancel in the sum. We note, furthermore, that the numerical results of ref. [11] show a very different short-distance behaviour of the thermal parts of the two correlators (cf. fig. 5), which could also be a reflection of the fact that \(-G_\chi\) gets a positive contribution from the trace anomaly at short distances whereas \(G_\theta\) gets a negative contribution. (It must be stressed, though, that if the Wilson coefficients of the structures proportional to \(e + p\) and \(e - 3p\) were the same, as is the case at leading order, then the positive contribution from \(e - 3p\) could not overcome the negative contribution from \(e + p\) in \(-G_\chi\), given that \(-(e + p) + (e - 3p) = -4p < 0.\)

We conclude by noting that the Wilson coefficients in eqs. (5.9), (5.10), (5.12), (5.13) go beyond the accuracy of the analysis in refs. [9, 11]. With next-to-leading order corrections available, we should be in a position to estimate the coefficients numerically; this has been attempted in fig. [2]. Any sort of apparent convergence is only observed at very short distances, particularly in the \(\chi\)-channel; this unfortunate fact may not be totally unexpected [18]. In any case, it can be seen that the various Wilson coefficients could be numerically quite different even though they agree at leading order. In principle it would be nice to also determine the Wilson coefficients related to the trace anomaly terms (eqs. (5.11), (5.14)) at next-to-leading order but this would require a 3- or 4-loop computation. (Since the operator yielding \(e - 3p\) is Lorentz invariant and the Wilson coefficients should be \(T\)-independent, it might be possible to extract these coefficients from purely vacuum computations, however we have not unearthed literature where this would have been achieved.)

### 6. Large frequencies

We next turn to an “opposite” limit, that of large frequencies but vanishing spatial momenta: \(P = (p_n, 0)\). The results of eqs. (4.1)-(4.4) remain valid as a starting point. Furthermore, it has been argued in ref. [10] that the asymptotic expansions can be analytically continued to Minkowski signature, \(p_n \to -i[\omega + i0^+]\), extracting thereby the spectral functions, \(\rho(\omega) = \text{Im} \tilde{G}(-i[\omega + i0^+])\), even though this requires crossing the light-cone, \(P^2 = 0\), on which the asymptotic expansions are certainly not valid. In any case, within perturbation theory, the procedure should surely be justified, since we could imagine carrying out the analytic continuation before the asymptotic expansion in each individual master sum-integral, and expanding only subsequently in a large \(\omega/\pi T\).

Let us stress that we assume the analytic continuation to be carried out in the presence of an ultraviolet regulator for spatial momenta. The regulator is removed (i.e. the limit \(\epsilon \to 0\) is taken) only after the analytic continuation.
With these qualifications the structures in eqs. (6.1)–(6.4) yield (at \( p = 0 \) and \( \omega > 0 \))

\[
\frac{P^4}{(4\pi)^2} \left( \frac{\bar{\mu}}{F} \right)^{2\epsilon} \rightarrow \frac{\omega^4}{(4\pi)^2} e^\epsilon \left[ 1 + \epsilon \ln \frac{\bar{\mu}^2}{\omega^2} \right],
\]

\[
\frac{P^4}{(4\pi)^2} \left( \frac{\bar{\mu}}{F} \right)^{4\epsilon} \rightarrow \frac{\omega^4}{(4\pi)^2} 2e^\epsilon \left[ 1 + 2\epsilon \ln \frac{\bar{\mu}^2}{\omega^2} \right],
\]

\[
\frac{1}{P^2} \left( \frac{p^2}{3} - p_n^2 \right) \ln \frac{\bar{\mu}^2}{P^2} \rightarrow -\pi,
\]

\[
\ln \frac{\bar{\mu}^2}{P^2} \rightarrow \pi.
\]

Thereby we get the spectral functions

\[
\frac{\rho_0(\omega)}{4\epsilon^2_\theta\pi} = \frac{d\Lambda\omega^4}{(4\pi)^2} \tilde{\gamma}_{\theta;1}(\omega) - 2(e + p) \tilde{\gamma}_{\theta;e + p}(\omega) - 2(e - 3p) \tilde{\gamma}_{\theta;e - 3p}(\omega) + O\left(\frac{T^6}{\omega^2}\right),
\]

\[
\frac{\rho_\chi(\omega)}{-16\epsilon^2_\chi\pi} = \frac{d\Lambda\omega^4}{(4\pi)^2} \tilde{\gamma}_{\chi;1}(\omega) - 2(e + p) \tilde{\gamma}_{\chi;e + p}(\omega) + 2(e - 3p) \tilde{\gamma}_{\chi;e - 3p}(\omega) + O\left(\frac{T^6}{\omega^2}\right),
\]

where

\[
\tilde{\gamma}_{\theta;1}(\omega) = g^4 + \frac{g^6N_c}{(4\pi)^2} \left( \frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{73}{3} \right) + O(g^8),
\]

\[
\tilde{\gamma}_{\theta;e + p}(\omega) = \frac{11g^6N_c}{(4\pi)^2} + O(g^8),
\]

\[
\tilde{\gamma}_{\theta;e - 3p}(\omega) = g^4 + O(g^6),
\]

\[
\tilde{\gamma}_{\chi;1}(\omega) = g^4 + \frac{g^6N_c}{(4\pi)^2} \left( \frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{97}{3} \right) + O(g^8),
\]

\[
\tilde{\gamma}_{\chi;e + p}(\omega) = \frac{11g^6N_c}{(4\pi)^2} + O(g^8),
\]

\[
\tilde{\gamma}_{\chi;e - 3p}(\omega) = g^4 + O(g^6).
\]

After inserting \( c_\theta \) from eq. (2.3) the leading-order term of \( \tilde{\gamma}_{\theta;1} \) agrees with ref. [19], but we can now add to that result the first correction. Similarly, the leading-order term of \( \tilde{\gamma}_{\chi;1} \) agrees with a result given in ref. [11], but we can add a correction. The coefficients \( \tilde{\gamma}_{\theta;e + p}, \tilde{\gamma}_{\theta;e - 3p} \) agree with ref. [10], if the coefficient \( C \) introduced there is set to \( C = 1 \). The results for \( \tilde{\gamma}_{\theta;e + p}, \tilde{\gamma}_{\theta;e - 3p} \) have more recently been reproduced in ref. [17]. We are not aware of analogous results in the literature for \( \tilde{\gamma}_{\chi;e + p}, \tilde{\gamma}_{\chi;e - 3p} \).

The coefficients \( \tilde{\gamma}_{\theta;1}, \tilde{\gamma}_{\chi;1} \), for which next-to-leading order values are given in eqs. (6.7), (6.10), are estimated numerically in fig. 3. Like in sec. 5, reasonable apparent convergence is observed only extremely deep into the ultraviolet regime.
Figure 3: Numerical estimates of the Wilson coefficients $\tilde{\gamma}_{\theta;1}$, $\tilde{\gamma}_{\chi;1}$, cf. eqs. (6.7), (6.10), respectively. Unspecified conventions are as in fig. 2. Reasonable apparent convergence is observed only at extremely large frequencies, $\omega \gtrsim 50 \Lambda_{\text{MS}} (\tilde{\gamma}_{\theta;1})$ or $\omega \gtrsim 80 \Lambda_{\text{MS}} (\tilde{\gamma}_{\chi;1})$.

We note that $\tilde{\gamma}_{\theta;e - 3p}$, $\tilde{\gamma}_{\chi;e - 3p}$ (eqs. (6.9), (6.12)) contain second and higher powers of $g^2$, so that they should gradually decrease with $\omega$ as dictated by asymptotic freedom (in analogy with fig. 3). On the other hand, if similar results were available in the “shear channel”, i.e. for 2-point correlators of the traceless part of the energy-momentum tensor, then the terms proportional to $e - 3p$ would allow to fix an unknown constant, denoted by $D$ in ref. [10], which is argued to be “saturated” at that level [10]. In that case there is no $g^4$ multiplying it, so the result is a constant. This implies that such terms require a special treatment in the context of sum rules, cf. sec. 7. The importance of determining such terms has recently been pointed out also in the context of a lattice investigation [20]. Unfortunately, determining the 2-point correlators in the shear channel is technically more demanding than the present analysis because of the more cumbersome Lorentz structures appearing in the numerator.

7. On sum rules

Starting from the spectral representation of a Euclidean correlator,

$$\tilde{G}(p_n) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega - ip_n},$$

(7.1)
which is generally valid in the presence of an ultraviolet regulator for spatial momenta, and setting $p_n \to 0$, we obtain

$$
\int_0^\beta d\tau G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega}.
$$

(7.2)

Adding the dependence on spatial coordinates (which were suppressed above), we thus obtain a “sum rule”, relating an integral over the spectral function to a Euclidean “susceptibility”.

Now, in practice, various complications can arise in the application of sum rules. One of them is of an infrared type, and particularly relevant for the “bulk channel” (our correlator $G_\theta$): indeed there must be a term $\propto \omega \delta(\omega)$ in $\rho_\theta$ because, if we set $p \to 0$ from the outset, the operator couples to a conserved charge $\int_x T_{00}$. Such a term does yield a contribution to both sides of the sum rule, however not to the transport coefficient of interest ($\lim_{\omega \to 0^+} \rho_\theta(\omega)/\omega$), so it would be wise to subtract it. In ref. [17], this was achieved by defining a modified spectral function $\Delta \rho_\theta$, which has no $\omega \delta(\omega)$ but nevertheless yields the same transport coefficient. If the problematic term is subtracted from the right-hand side, it must also be subtracted from the left-hand side, and this yields a modified sum rule (cf. appendix C of ref. [21]). However this extra contribution is of $O(g^8)$ and thus not directly visible in our $O(g^6)$ result.

Another possible complication, of an ultraviolet type, arises if we wish to remove the ultraviolet regularization before applying the sum rule. For instance, the vacuum part then grows as $\rho \propto \omega^4 \text{sign}(\omega)$, and both sides of eq. (7.2) are ill-defined. The problem becomes less severe if the vacuum parts are subtracted from both sides of eq. (7.2), imposing a relation between $\Delta G$ and $\Delta \rho$ instead. However, even such a subtraction may not be sufficient to remove a constant part, as is argued to be the case in the shear channel [21, 10]; then eq. (7.3) needs to contain a supplementary “contact term” on the right-hand side.

We now return to the case at hand, where only the former problem should be relevant [10, 17]. The left-hand side of the sum rule reads $\tilde{G}_\theta(0)$, $\tilde{G}_\chi(0)$ in our notation. Inspecting the master sum-integrals in eqs. (A.1)–(A.13), most are seen to vanish in the limit $P \to 0$; in fact only $I_a$ and $I_i$ give a contribution, and the latter can be reduced to the former through eq. (A.14). Inserting these values into eqs. (3.1), (3.2) we obtain

$$
\tilde{G}_\theta(0) = -8d_A c^2 g^6 N_c (D-2)^2 I_a = -8d_A b_0^2 g^6 N_c \int_{q,r} \frac{n_a(q) n_b(r)}{q \cdot r},
$$

(7.3)

$$
\tilde{G}_\chi(0) = 0,
$$

(7.4)

where in the final step we made use of eqs. (2.3), (A.23). Equation (7.3) agrees with eq. (B.4) and therefore confirms the sum rule of ref. [22], obtained also directly in lattice regularization [23], up to $O(g^6)$. Unfortunately, as mentioned, the infrared ambiguities discussed in refs. [21, 17] are of $O(g^8)$ and thus beyond the resolution of our computation.

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4Equation (7.4) can be derived for instance by Fourier transforming the known non-perturbative relation in eq. (1.1), which is unproblematic if $G(\tau)$ does not diverge at small distances.
8. Summary and outlook

Following ref. [11] and other recent works, we have analyzed the ultraviolet asymptotics of 2-point correlation functions of the trace of the energy-momentum tensor and of the topological charge density in finite-temperature Yang-Mills theory. The tool has been dimensionally regularized perturbation theory, pursued up to 2-loop order. We have considered both Euclidean (operators separated by a purely spatial separation; sec. 5) and Minkowskian correlators (operators separated by a Euclidean timelike separation, subsequently Fourier transformed and analytically continued; sec. 6).

Through our analysis, we have confirmed a number of expressions in recent literature, but simultaneously also refined the determination of the corresponding Wilson coefficients, particularly in the Euclidean domain (eqs. (5.9), (5.10), (5.12), (5.13)). In this case, we have also identified some inaccuracies in the literature; on the qualitative level, our expressions appear to be in a somewhat better accordance with lattice data than the earlier ones, because they show a qualitative difference between the two channels, with the terms proportional to the thermal part of the trace anomaly, \((e−3p)(T)\), coming with opposite signs. Unfortunately the perturbative series for the Wilson coefficients show slow convergence and it appears uncertain whether, even with the correct values inserted, they can lead to a quantitative agreement with lattice data. (The problem is worse in the case of the topological charge density correlator, cf. fig. 2.)

As an outlook, we believe that our results can be refined in a number of ways, some of them with reasonable prospects for phenomenological success. First of all, it would be interesting to determine the full \(r\)-dependence of the correlators up to distances \(r \sim 1/(\pi T)\) rather than the asymptotic expansions at \(r \ll 1/(\pi T)\) as in the present study. This would increase the range of applicability of the results and might simultaneously improve on the convergence of the weak-coupling expansion, both aspects facilitating a comparison with lattice data à la ref. [11]. However, we would suggest carrying out the computation after averaging over the time separation, \(\tau\), rather than setting it to zero, because this makes the correlators more analogous to the ones encountered in the context of heavy quarkonium physics [7], a problem of actual phenomenological significance. In any case, eqs. (3.1), (3.2) can serve as starting points for such investigations; furthermore, the asymptotic results for the time-averaged correlators can still be obtained from eqs. (4.1)–(4.4), simply by replacing the inverse Fourier transforms in eq. (5.1) by three-dimensional ones.

Second, it would be interesting to determine the full \(\omega\)-dependence of the spectral functions down to frequencies \(\omega \sim \pi T\), rather than carrying out an asymptotic expansion at \(\omega \gg \pi T\) as in the present study because, as mentioned around eq. (1.1), such information may be helpful in connection with lattice extractions of the corresponding transport coefficients. The full frequency dependence is believed to be quite non-trivial and display e.g. a large cancellation
of terms of $\mathcal{O}(g^4T^4)$ \cite{2} \cite{10}. Once more, eqs. (3.1), (3.2) can serve as starting points for this exercise; moreover the asymptotic expansions in eqs. (6.5), (6.6) could provide for useful crosschecks on the final results.

Third, for theoretical completeness, it would be interesting to determine the Wilson coefficients more accurately, particularly the missing $\mathcal{O}(g^6)$ terms in eqs. (5.11), (5.14), because this would allow us to estimate the renormalization scale relevant for the gauge coupling multiplying the trace anomaly. (Nevertheless, because of the reservations mentioned in the present study in the case of the other Wilson coefficients, it is not clear to us whether this could lead to a practically successful comparison with the lattice data of ref. \cite{11}.) Similarly, of course, it would in principle be nice to know all next-to-leading order corrections in the Minkowskian domain, relevant for eqs. (6.8), (6.9), (6.11), (6.12). All of these challenges necessitate at least 3-loop computations.

Fourth, it would be interesting to determine the dependence of the various Wilson coefficients on the number of massless quark flavours, $N_f$, not least because this offers a further crosscheck on the applicability of the Operator Production Expansion framework in the finite-temperature context. Partial results (including only “sea” quarks), and the problems encountered in this computation, are discussed in appendix C.

Finally, it would be interesting to repeat the study in the “shear” channel, i.e. for the traceless part of the energy-momentum tensor. The general techniques of sec. 3 as well as some of the actual sum-integrals and asymptotic expansions encountered in the present study, might play a role in that context as well.

We hope to return to some of these topics in future work.

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Appendix A. Bosonic master sum-integrals

The bosonic “master” sum-integrals appearing in our computation are defined as

\[ J_a \equiv \int \frac{P^2}{Q^2} , \quad (A.1) \]
\[ J_b \equiv \int \frac{P^4}{Q^2(Q - P)^2} , \quad (A.2) \]
\[ I_a \equiv \int \frac{1}{Q^2R^2} , \quad (A.3) \]
\[ I_b \equiv \int \frac{P^2}{Q^2R^2(R - P)^2} , \quad (A.4) \]
\[ I_c \equiv \int \frac{P^2}{Q^2R^4} , \quad (A.5) \]
\[ I_d \equiv \int \frac{P^4}{Q^2R^4(R - P)^2} , \quad (A.6) \]
\[ I_e \equiv \int \frac{P^2}{Q^2R^2(Q - R)^2} , \quad (A.7) \]
\[ I_f \equiv \int \frac{P^2}{Q^2(Q - R)^2(R - P)^2} , \quad (A.8) \]
\[ I_g \equiv \int \frac{P^4}{Q^2(Q - P)^2R^2(R - P)^2} , \quad (A.9) \]
\[ I_h \equiv \int \frac{P^4}{Q^2R^2(Q - R)^2(R - P)^2} , \quad (A.10) \]
\[ I_i \equiv \int \frac{(Q - P)^4}{Q^2R^2(Q - R)^2(R - P)^2} , \quad (A.11) \]
\[ I_{i'} \equiv \int \frac{4(Q \cdot P)^2}{Q^2R^2(Q - R)^2(R - P)^2} , \quad (A.12) \]
\[ I_j \equiv \int \frac{P^6}{Q^2R^2(Q - R)^2(Q - P)^2(R - P)^2} . \quad (A.13) \]

In fact there is some redundancy here, because the following relation can easily be established through changes of integration variables:

\[ I_i = I_a + I_e - I_f + I_{i'} . \quad (A.14) \]

Expanding in a small \((\pi T)^2/P^2\) as explained in sec. 3, the sum-integrals in eqs. (A.1),
[A.2] can be expressed as

\[ \mathcal{J}_a = P^2 \int_q \frac{n_B(q)}{q}, \quad (A.15) \]

\[ \mathcal{J}_b = \frac{P^{1-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)\epsilon(1 - 2\epsilon)} \left( 1 + 2P^2 \int_q \frac{n_B(q)}{q} + \frac{8}{P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \right) \int_q q n_B(q) + \mathcal{O} \left( \frac{1}{P^2} \right). \quad (A.16) \]

Introducing the shorthands

\[ S_1 = \frac{P^{1-4\epsilon} \Gamma^2(1 + \epsilon)\Gamma^4(1 - \epsilon)}{(4\pi)^{1-2\epsilon} \Gamma^2(1 - 2\epsilon)}, \quad (A.17) \]

\[ S_2 = \frac{\Gamma(1 + 2\epsilon)\Gamma^2(1 - 2\epsilon)}{\Gamma(1 - 3\epsilon)\Gamma^2(1 + \epsilon)\Gamma(1 - \epsilon)}, \quad (A.18) \]

\[ S_3 = \frac{P^{2-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \int_q \frac{n_B(q)}{q}, \quad (A.19) \]

\[ S_4 = \frac{(1 - 2\epsilon)P^2}{2} \int_{q,r} \frac{n_B(q) n_B(r)}{q r^3}, \quad (A.20) \]

\[ S_5 = \frac{P^{-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{1}{2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \int_q q n_B(q), \quad (A.21) \]

\[ S_6 = \int_{q,r} \frac{n_B(q) n_B(r)}{q r}, \quad (A.22) \]

we can write the expansions of the next-to-leading order sum-integrals as

\[ \mathcal{I}_a = S_6, \quad (A.23) \]

\[ \mathcal{I}_b = \frac{S_3}{\epsilon(1 - 2\epsilon)} + 2S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.24) \]

\[ \mathcal{I}_c = S_1, \quad (A.25) \]

\[ \mathcal{I}_d = -\frac{S_3}{\epsilon} + S_1 + \left[ \frac{2}{2 - \epsilon} + \frac{2(1 - \epsilon)(3 - 2\epsilon)}{(2 - \epsilon)P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \right] S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.26) \]

\[ \mathcal{I}_e = 0, \quad (A.27) \]

\[ \mathcal{I}_f = -\frac{S_3}{2\epsilon(1 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)} + \frac{3S_3}{\epsilon(1 - 2\epsilon)} + \frac{6(1 + \epsilon)S_5}{1 - 2\epsilon} + 3S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.28) \]

\[ \mathcal{I}_g = \frac{S_1 S_2}{\epsilon^2(1 - 2\epsilon)^2} + \frac{4S_3}{\epsilon(1 - 2\epsilon)} + \frac{16S_5}{\epsilon(1 - 2\epsilon)} + 4S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.29) \]

\[ \mathcal{I}_h = \frac{S_1 S_2}{2\epsilon^2(1 - 2\epsilon)(1 - 3\epsilon)} - \frac{(1 - 4\epsilon)S_3}{\epsilon(1 - 2\epsilon)} + \frac{2(1 + \epsilon)(2 + \epsilon)(1 + 4\epsilon)S_5}{3\epsilon(1 - 2\epsilon)} \]

\[ + \left[ \frac{4 - \epsilon}{2 - \epsilon} + \frac{4(1 - \epsilon)^2}{\epsilon(2 - \epsilon)P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \right] S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.30) \]
\[ \mathcal{I}_i = \frac{S_1 S_2}{3\epsilon^2 (1 - 3\epsilon)^2 (2 - 3\epsilon)} + \frac{2(1 + \epsilon) S_3}{\epsilon (1 - 2\epsilon)} + \frac{4(9 - 35\epsilon + 16\epsilon^2 - \epsilon^3 + 2\epsilon^4) S_5}{3\epsilon (1 - 2\epsilon) (3 - 2\epsilon)} + \left[ \frac{2(5 - 2\epsilon)}{2 - \epsilon} + \frac{4(1 - \epsilon)^2}{\epsilon (2 - \epsilon) P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \right] S_6 + \mathcal{O} \left( \frac{1}{P^2} \right), \quad (A.31) \]

\[ \mathcal{I}_j = \frac{S_1 (1 - S_2)}{\epsilon^3 (1 - 2\epsilon)} - \frac{2(3 + \epsilon) S_3}{\epsilon} - \frac{2(5 + \epsilon) (10 + 5\epsilon + \epsilon^2) S_5}{3\epsilon} + \left[ \frac{2(3 + \epsilon)}{2 - \epsilon} + \frac{20(1 - \epsilon)^2}{\epsilon (2 - \epsilon) P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \right] S_6 + \mathcal{O} \left( \frac{1}{P^2} \right). \quad (A.32) \]

Both structures proportional to \( P^2 \), namely \( S_3 \) appearing in almost every master sum-integral as well as \( S_4 \) appearing in \( \mathcal{I}_c \) and \( \mathcal{I}_d \), disappear from the final result for any \( \epsilon \). A useful relation allowing to change the basis in the terms proportional to \( S_6 \) is

\[ \frac{2}{\epsilon P^2} \left[ \frac{p^2}{3 - 2\epsilon} - (1 - 2\epsilon) p_n^2 \right] = \frac{2}{2 - \epsilon} + \frac{4(1 - \epsilon)^2}{\epsilon (2 - \epsilon) P^2} \left[ \frac{p^2}{3 - 2\epsilon} - p_n^2 \right]. \quad (A.33) \]

Appendix B. Basic thermodynamic functions

We recall here perturbative expressions for a number of thermodynamic potentials needed in our analysis. The results are needed up to order \( \mathcal{O}(g^6) \) in some cases and can be extracted from the explicit \( \mathcal{O}(g^6) \) results in ref. [24]; however, they could also be deduced through renormalization scale independence arguments already from the classic \( \mathcal{O}(g^2) \) results for the thermodynamic pressure given in refs. [25, 26].

As explained in the main body of the text, an essential role in the present study is played by the energy-momentum tensor of the thermalized system, which in the plasma rest frame takes the form \( \text{diag}(e, -p, -p, -p) \). We separate this into a traceless part, whose 00-component reads

\[ e - \frac{1}{4} (e - 3p) = \frac{3}{4} (e + p), \quad (B.1) \]

as well as a trace part, \( e - 3p \). At next-to-leading order, the combination appearing in the traceless part can be written as

\[ (e + p)(T) = \frac{8d_A}{3} \left[ \int q n_B(q) - \frac{3g^2 N_c}{2} \int_{q,r} \frac{n_B(q) n_B(r)}{q r} \right], \quad (B.2) \]

with \( d_A = N_c^2 - 1 \), whereas the leading-order expression for the trace anomaly reads

\[ (e - 3p)(T) = 2d_A g^4 b_0 N_c \int_{q,r} \frac{n_B(q) n_B(r)}{q r}, \quad (B.3) \]

with \( b_0 \) as defined in eq. [2.6]. Finally, the temperature dependence of the trace anomaly reads

\[ T^5 \frac{d}{dT} \left( \frac{e - 3p}{T^4} \right) = -8d_A g^6 b_0^2 N_c \int_{q,r} \frac{n_B(q) n_B(r)}{q r}. \quad (B.4) \]
In the expressions above, $n_B$ denotes the Bose distribution, $n_B(q) \equiv 1/(e^{\beta q} - 1)$, and the integrals can be carried out,

$$\int q n_B(q) = \frac{\pi^2 T^4}{30} + \mathcal{O}(\epsilon), \quad \int \frac{n_B(q)}{q} = \frac{T^2}{12} + \mathcal{O}(\epsilon), \quad (B.5)$$

but it is convenient to be able to recognize the integral representations as well.

Appendix C. On fermionic effects

We add here the contribution of $N_f$ massless “sea” quarks to the previous results for the trace anomaly and topological charge density correlators. By sea quarks we refer to effects originating from the fermionic contribution to the gluon self-energy; “valence” quarks would refer to the quark part of the energy-momentum tensor operator. It is often said that the quark contribution to the trace anomaly operator, which is proportional to $\bar{\psi} \gamma_\mu D_\mu \psi$ or $\bar{\psi} \gamma_\mu \overleftrightarrow{D}_\mu \psi$, vanishes in the chiral limit thanks to the equations of motion; in topological susceptibility there should be no valence quark contribution to start with. Below we find practically identical expressions for the two channels, however also structures in the Lorentz-symmetry violating part of the result which do not fit the form expected from the Operator Product Expansion. This could indicate that there is, after all, some mixing taking place and a non-zero valence quark contribution to be added at non-zero $T$; however, given the fair amount of work involved, we have not carried out a systematic investigation to clear up the issue.

With these reservations, the fermionic contributions to the correlators in eq. (2.8) read

$$\frac{\delta \tilde{G}_\theta(P)}{-4d_A c_\theta^2 g_5^0 N_f} = 4 \left[ \tilde{T}_a + \tilde{I}_i + 2(D - 1) \left[ -4 \tilde{T}_a + \tilde{T}_a - \tilde{T}_e \right] \right] + (D - 2) \left[ \tilde{T}_e + 2 \tilde{T}_e - 2 \tilde{T}_b \right] + (D - 4) \left[ 4 \tilde{T}_b - 2 \tilde{T}_t + \tilde{T}_f + \tilde{T}_h \right], \quad (C.1)$$

$$\frac{\delta \tilde{G}_\chi(P)}{16d_A c_\chi^2 g_5^0 (D - 3) N_f} = 4 \left[ \tilde{T}_a - \tilde{T}_a + \tilde{T}_i \right] - 2(D - 1) \left[ \tilde{T}_a + \tilde{T}_e \right] + (D - 2) \left[ \tilde{T}_e + 2 \tilde{T}_e - 2 \tilde{T}_d \right] + (D - 4) \left[ 4 \tilde{T}_b - 2 \tilde{T}_t + \tilde{T}_f + \tilde{T}_h \right]. \quad (C.2)$$
Here the following new master sum-integrals have been introduced:

\[
\begin{align*}
\tilde{I}_a &\equiv \int \frac{1}{Q^2 R^2} , \\
\tilde{I}_b &\equiv \int \frac{P^2}{Q^2 R^2 (R - Q)^2} , \\
\tilde{I}_c &\equiv \int \frac{P^2}{Q^2 R^2} , \\
\tilde{I}_d &\equiv \int \frac{P^4}{Q^2 R^2 (R - Q)^2} , \\
\tilde{I}_e &\equiv \int \frac{(R - P)^2}{Q^2 R^2 (Q - R)^2} = \tilde{I}_a + \tilde{I}_e , \\
\tilde{I}_f &\equiv \int \frac{P^2}{Q^2 (Q - R)^2 (R - P)^2} , \\
\tilde{I}_g &\equiv \int \frac{R^2}{Q^2 (Q - R)^2 (R - P)^2} = \tilde{I}_a - \tilde{I}_b + \tilde{I}_g , \\
\tilde{I}_h &\equiv \int \frac{2 R \cdot P}{Q^2 (Q - R)^2 (R - P)^2} , \\
\tilde{I}_i &\equiv \int \frac{4 (Q - P)^4}{Q^2 R^2 (Q - R)^2 (R - P)^2} = \tilde{I}_a + \tilde{I}_e - \tilde{I}_b + \tilde{I}_i' , \\
\tilde{I}_j &\equiv \int \frac{4 (Q - P)^2}{Q^2 R^2 (Q - R)^2 (R - P)^2} .
\end{align*}
\]

The sum-integral denoted by \( \int_{(Q)} \) goes over fermionic Matsubara momenta. As indicated this set is somewhat overcomplete (at least in the absence of a chemical potential, as we have assumed to be the case throughout).

We note that, like in the bosonic case, most of the master sum-integrals vanish in the limit \( P \to 0 \), relevant for the sum rule (cf. eqs. (7.3), (7.4)). Non-zero contributions arise only from
\(\tilde{I}_a\) and \(\tilde{T}_a\). In \(\delta G_\chi(0)\) these cancel; for \(\delta G_\theta(0)\) we obtain (inserting also \(c_\theta\) from eq. (2.3))

\[
\delta G_\theta(0) = 8d_A g^6 b_0^2 N_f (2\tilde{I}_a - \tilde{T}_a) = -\frac{5d_A g^6 b_0^2 N_f T^4}{72},
\]

where we made use of \(\Psi_{(q)}^1/Q^2 = -T^2/24\) and \(\Psi_{(r)}^1/R^2 = T^2/12\). This result agrees with the leading fermionic contribution to \(T^5d[(e - 3p)/T^4]/dT\), and therefore conforms with the sum rule of ref. [22] at \(O(g^6)\).

The large-momentum expansions of the master sum-integrals can be worked out like in the bosonic case (cf. sec. 3). Supplementing the structures in eqs. (A.17)–(A.22) with

\[
\tilde{S}_3 \equiv \frac{P^{2-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \int_q \frac{n_F(q)}{q},
\]

\[
\tilde{S}_4 \equiv \frac{(1 - 2\epsilon)P^2}{2} \int_{q,r} \frac{n_F(q) n_B(r)}{q r^3},
\]

\[
\tilde{S}_5 \equiv -\frac{P^{2-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{1}{P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \int_q n_F(q),
\]

\[
\tilde{S}_6 \equiv -\int_{q,r} \frac{n_F(q) n_B(r)}{q r},
\]

\[
\bar{S}_6 \equiv \int_{q,r} \frac{n_F(q) n_B(r)}{q r},
\]

where \(n_F\) denotes the Fermi distribution, \(n_F(q) \equiv 1/(e^{\beta q} + 1)\), we obtain the expansions

\[
\tilde{I}_a = \tilde{S}_6,
\]

\[
\tilde{T}_a = \bar{S}_6,
\]

\[
\tilde{I}_b = \frac{\tilde{S}_3}{\epsilon(1 - 2\epsilon)} + 2\tilde{S}_6 + O\left(\frac{1}{P^2}\right),
\]

\[
\tilde{I}_c = \tilde{S}_4,
\]

\[
\tilde{I}_d = \frac{-\tilde{S}_3}{\epsilon} + \tilde{S}_4 + \frac{2}{2 - \epsilon} + \frac{2(1 - \epsilon)(3 - 2\epsilon)}{(2 - \epsilon)P^2} \left( \frac{p^2}{3 - 2\epsilon} - p_n^2 \right) \tilde{S}_6 + O\left(\frac{1}{P^2}\right),
\]

\[
\tilde{I}_e = 0,
\]

\[
\tilde{I}_f = -\frac{S_1 S_2}{2\epsilon(1 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)} + \frac{S_3 + 2\tilde{S}_3}{\epsilon(1 - 2\epsilon)} + \frac{2(1 + \epsilon)(S_5 + 2\tilde{S}_5)}{1 - 2\epsilon} + \tilde{S}_6 + 2\tilde{S}_6 + O\left(\frac{1}{P^2}\right),
\]

where \(n^\alpha_{(q)}\) and \(n^\alpha_{(r)}\) are the momentum distributions of the photon and the quark, respectively.
\[ \bar{I}_t = -\frac{2S_1S_2}{3\epsilon(1-2\epsilon)(1-3\epsilon)(2-3\epsilon)} + \frac{2(S_3 + \bar{S}_3)}{\epsilon(1-2\epsilon)} + \frac{4\epsilon S_5 + 4(2 + \epsilon)\bar{S}_5}{1-2\epsilon} + 4\bar{S}_6 + O\left(\frac{1}{P^2}\right), \]

\[ \bar{I}_h = \frac{S_1S_2}{2\epsilon^2(1-2\epsilon)(1-3\epsilon)} + \frac{S_3 - 2(1-2\epsilon)\bar{S}_3}{\epsilon(1-2\epsilon)} + \frac{2(1+\epsilon)(2+\epsilon)[3S_5 - 2(1-2\epsilon)\bar{S}_5]}{3\epsilon(1-2\epsilon)} + 2\bar{S}_6 + \left[ \frac{\epsilon}{2-\epsilon} + \frac{4(1-\epsilon)^2}{\epsilon(2-\epsilon)P^2} \left( \frac{p^2}{3-2\epsilon} - p_n^2 \right) \right] \bar{S}_6 + O\left(\frac{1}{P^2}\right), \]

\[ \bar{I}_i = \frac{S_1S_2}{3\epsilon^2(1-3\epsilon)(2-3\epsilon)} + \frac{S_3 + (1+2\epsilon)\bar{S}_3}{\epsilon(1-2\epsilon)} + \frac{2(2+2\epsilon + \epsilon^2 - 2\epsilon^2)S_5}{\epsilon(1-2\epsilon)(3-2\epsilon)} - \frac{2(8 + \epsilon^2)\bar{S}_5}{3\epsilon} + \frac{2(5 - 2\epsilon)}{2-\epsilon} + \frac{4(1-\epsilon)^2}{\epsilon(2-\epsilon)P^2} \left( \frac{p^2}{3-2\epsilon} - p_n^2 \right) \bar{S}_6 + O\left(\frac{1}{P^2}\right). \]

All structures proportional to \( P^2 \), namely \( S_3, \bar{S}_3 \) and \( \bar{S}_4 \), cancel in the final results for any \( \epsilon \).

Inserting these expansions into eqs. (C.1), (C.2) and re-expressing the bare gauge coupling in terms of the renormalized one according to eq. (2.3), we obtain results analogous to eqs. (4.1), (4.2). Alas, many different thermal distributions appear and it is not easy to express the result in a concise way. We choose rather to insert explicit values,

\[ \int q \, n_\alpha(q) = \mu^{-2\epsilon} \frac{\pi^2 T^4}{30} \left\{ 1 + 2\epsilon \left[ \ln \frac{\bar{\mu}}{4\pi T} + 1 + \frac{\zeta'(3)}{\zeta(3)} \right] + O(\epsilon^2) \right\}, \]

\[ \int q \, n_\beta(q) = \mu^{-2\epsilon} \frac{T^2}{12} \left\{ 1 + 2\epsilon \left[ \ln \frac{\bar{\mu}}{4\pi T} + 1 + \frac{\zeta'(1)}{\zeta(-1)} \right] + O(\epsilon^2) \right\}, \]

\[ \int q \, n_F(q) = \mu^{-2\epsilon} \frac{7\pi^2 T^4}{240} \left\{ 1 + 2\epsilon \left[ \ln \frac{\bar{\mu}}{4\pi T} + 1 + \frac{\zeta'(3)}{\zeta(-3)} - \ln \frac{2}{7} \right] + O(\epsilon^2) \right\}, \]

\[ \int q \, n_F(q) = \mu^{-2\epsilon} \frac{T^2}{24} \left\{ 1 + 2\epsilon \left[ \ln \frac{\bar{\mu}}{4\pi T} + 1 + \frac{\zeta'(1)}{\zeta(-1)} - \ln 2 \right] + O(\epsilon^2) \right\}. \]

Thereby the fermionic contributions to eqs. (4.1), (4.2) become

\[ \frac{\delta \tilde{G}_\theta(P)}{4dA c_{\bar{\theta}} g^4 \mu^{2\epsilon}} = \frac{P^4}{(4\pi)^2} \left( \frac{\bar{\mu}}{P} \right)^{2\epsilon} \left\{ \frac{1}{\epsilon} + 1 + \ldots \right\} \times \frac{g^2 N_f}{(4\pi)^2} \frac{4}{3\epsilon} - \frac{g^2 N_f}{(4\pi)^2} \left( \frac{\bar{\mu}}{P} \right)^{4\epsilon} \left[ \frac{2}{3\epsilon^2} + \frac{3}{\epsilon} + \ldots \right] \]

\[ + \frac{g^2 N_f T^4}{45 P^2} \left( \frac{p^2}{3} - p_n^2 \right) \left[ \frac{5}{2} \ln \frac{\bar{\mu}}{4\pi T} - \frac{9}{2} \ln \frac{\bar{\mu}}{P} - 2 \ln 2 - \frac{151}{48} + \frac{5}{2} \frac{\zeta'(1)}{\zeta(-1)} - \frac{5}{2} \frac{\zeta'(3)}{\zeta(-3)} \right] \]

\[ - \frac{5g^2 N_f T^4}{144} + O\left( g^4, \frac{1}{P^2} \right), \]
\[
\frac{\delta G^\chi(P)}{-16d_AC^2 g^4 \mu^{2\epsilon}} = \frac{P^4}{(4\pi)^2} \left( \frac{\bar{\mu}}{P} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} - 1 + \ldots \right] \times \frac{g^2 N_f}{(4\pi)^2} \left( \frac{\bar{\mu}}{P} \right)^{4\epsilon} \left[ \frac{2}{3\epsilon^2} + \frac{5}{3\epsilon} + \ldots \right] 
\]
\[
+ \frac{g^2 N_f T^4}{45 P^2} \left( \frac{p^2}{3} - p_n^2 \right) \left[ \frac{5}{2} \ln \frac{\bar{\mu}}{4\pi T} - \frac{9}{2} \ln \frac{\bar{\mu}}{P} - 2 \ln 2 - \frac{151}{48} + \frac{5}{2} \zeta'(-1) - \frac{5}{2} \zeta'(-3) \right] 
\]
\[
+ \frac{5g^2 N_f T^4}{144} + \mathcal{O} \left( g^4, \frac{1}{P^2} \right). \tag{C.37}
\]

Inspecting the results, the terms on the last rows of eqs. (C.36), (C.37) are easy to understand: the leading-order quark contribution to the trace anomaly is

\[
\delta \left( \frac{e - 3p}{T^4} \right) = \frac{5d_AG^4b_0 N_f}{288}, \tag{C.38}
\]

and these terms amount to \(\mp 2\delta(e - 3p)/d_AG^2b_0\), in perfect accordance with the bosonic results of eqs. (4.3), (4.4). In contrast, the terms on the second rows of eqs. (C.36), (C.37) fit no simple pattern. In principle we might expect a fermionic contribution to the Wilson coefficient multiplying the leading bosonic \(e + p\), viz. \(8d_A \pi^2 T^4 / 90\), as well as direct fermionic effects to \(e + p\),

\[
\delta \left( \frac{e + p}{T^4} \right) = \frac{7N_c N_f \pi^2}{45} - \frac{5d_AG^2 N_f}{144}, \tag{C.39}
\]

the former multiplied by the next-to-leading order Wilson coefficient and the latter by the leading order one. However, there is no way to reproduce the leading fermionic contribution to \(e + p\) (the first term in eq. (C.39)) from the effects in eqs. (C.36), (C.37) (powers of \(g^2\) and/or group theory factors do not match) and, conversely, there is no way to understand the appearance of the temperature-dependent logarithms in eqs. (C.36), (C.37) in terms of the Operator Product Expansion contributions. Something is clearly missing and, as mentioned at the beginning, one possibility could be an unaccounted mixing with fermionic operators.

Let us stress that the problem only appears in the contributions proportional to \(e + p\), which vanish at zero temperature due to Lorentz symmetry (cf. the last lines of sec. 4).
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