Holomorphic curves and continuation maps in Liouville bundles

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Abstract

We construct an unwrapped Floer theory for bundles of Liouville sectors. In particular, we construct a compatible collection of unwrapped Fukaya categories of fibers of a Liouville bundle, and prove that the natural two constructions of continuation maps in this setting behave compatibly. These constructions are exploited in [OT19] to construct homotopically coherent actions of Lie groups on wrapped Fukaya categories, thereby proving a conjecture from Teleman’s 2014 ICM address.

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1 Introduction

In this paper, we lay the groundwork for non-wrapped Floer theory in bundles of Liouville sectors. Specifically, given a Liouville bundle $E \to B$, we define a directed Fukaya category $\mathcal{O}_j$ for every simplex $j : \Delta^n \to B$ smoothly mapping to $B$. The totality of this data (i.e., the collection of these Fukaya categories), along with their compatibilities along face maps of simplices, is what one might call the (unwrapped, directed) Floer theory associated to a Liouville bundle. In [OT19], we will localize along continuation maps to study the full-fledged wrapped Floer theory of a Liouville bundle.

To motivate the study of Floer theory of bundles, let us note that many Floer-theoretic calculations have been successful precisely by exploiting symmetries. Many of the first computations of Lagrangian Floer cohomology arose by studying fixed points of antiholomorphic involutions, or by exploiting torus actions on toric manifolds. On the other hand, in homotopy theory, given a group action of $G$ on an object $Y$, it is often convenient to exhibit a family of $Y$ living over the classifying space $BG$. A standard way to do so is to construct a map $G \to \text{Aut}(Y)$, exhibit some universal $Y$-bundle living over $\text{Aut}(Y)$, and pull back this universal bundle along the map $G \to \text{Aut}(Y)$. One can perform this construction for any Liouville action of $G$ on a Liouville sector $M$, exhibiting a family of $M$ living over $BG$. Moreover, by combining our present work with the work in [OT20b], one can articulate the smoothness of such infinite-dimensional entities—at least, one can articulate enough smoothness to set up a Floer theory detecting the way in which the wrapped Fukaya categories of $M$ vary (locally constantly) over $BG$. The present work, combined with [OT20b] and [OT20a], culminate in [OT19] to realize this strategy. This opens the door not only to exploit symmetries of Liouville sectors, but also to study the homotopy fixed points of wrapped Fukaya categories.

We will say a little more on the motivation in a bit. For now, here is the main result of the present work:

**Theorem 1.1.** For every $j : \Delta^n \to B$, the unwrapped, directed Fukaya category $\mathcal{O}_j$ is an $A_\infty$-category. Moreover, one can arrange that for every commutative diagram

$$
\begin{tikzcd}
\Delta^n \ar[rr]^j \ar[dr] & & \Delta^n' \ar[dl] \\
& B \\
\end{tikzcd}
$$

where $\iota$ is an injective simplicial map, we have an induced fully faithful functor $\mathcal{O}_j \to \mathcal{O}_{j'}$.

Informally, an object of $\mathcal{O}_j$ is a brane $L_a$ contained in the fiber above some vertex $a$ of $\Delta^n$. To define hom$_{\mathcal{O}_j}(L_a, L_{a'})$ for two objects, we choose a Liouville connection along the edge of $\Delta^n$ from $a$ to $a'$, and we define the generators of hom to be given by parallel transport chords from $L_a$ to $L_{a'}$. We artificially impose an ordering by defining a partial order on the collection of objects, and we declare morphism complexes to be null when the target brane is not strictly larger than the domain brane. This directedness of $\mathcal{O}_j$ is imposed to avoid dealing with a further layer of perturbations and choices (which one must deal with if one is to achieve transversality in the non-directed setting).

The $A_\infty$ operations are defined by counting holomorphic disks mapping into $j^*E$. We mention here that it is a priori not at all obvious how to articulate this counting problem—one must set up this count in a way compatible with different choices of $j$ and $j'$ for Theorem 1.1 to hold. For this, we use a beautiful insight of [Sav13], where Savelyev exhibits an operadically compatible map, for

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1Note that we choose such a connection for every pair of objects; there is no global connection chosen.
all $d \geq 1$, between the moduli of $(d + 1)$-punctured holomorphic disks and the standard $d$-simplex. (See Section 3.1.)

To provide a more complete story of Floer theory in Liouville bundles, the present work also presents a careful, but fairly standard, consideration of continuation map methods in unwrapped Floer theory. Let us outline this aspect of our paper as well.

One standard way of defining a continuation map for Lagrangian Floer homology is by considering the pseudoholomorphic curve equation with moving boundary conditions:

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J_{(\rho(\tau), t)} \frac{\partial u}{\partial t} = 0 \\
u(\tau, 0) \in K, \quad u(\tau, 1) \in L_{\rho(1-\tau)}. \tag{1.1}
\end{cases}$$

Here, $\{J_{s,t}\}_{(s,t) \in [0,1]^2}$ is a family of almost complex structures, $L = \{L_s\}_{s \in [0,1]}$ is a Hamiltonian isotopy of branes, and $\rho : \mathbb{R} \to [0,1]$ is an elongation function. (See for example [Oh93].)

We note that, due to our Liouville setting, we will require that $L$ be non-negative at infinity so we may apply a (strong) maximal principle and ensure compactness of the relevant moduli spaces. (See [Oh01, Introduction] for an early appearance of such a discussion.) We also warn that, because of the non-negativity constraint, the continuation map is not usually an isomorphism in the unwrapped Fukaya category of a Liouville sector. Regardless, one obtains a chain map

$$h_\rho^\circ : CF^*(K, L) \to CF^*(K, L')$$

between the unwrapped Floer complexes.

Another standard way to construct a continuation map (given the same isotopy $\mathcal{L}$ as above) is to count holomorphic disks with one boundary puncture and moving boundary condition. Concretely, fix a point $z_0 \in \partial D^2$ and choose another elongation function $\chi : \partial D^2 \setminus \{z_0\} \to [0,1]$. We let

$${\mathcal M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi)$$

be the set of those maps

$$v : D^2 \setminus \{z_0\} \to M$$

satisfying:

$$\begin{cases}
\bar{\partial}_J v = 0, \\
\int_{D^2 \setminus \{z_0\}} |dv|^2 < \infty, \\
v(z) \in L_{\chi(z)} \quad \text{for} \quad z \in \partial D^2 \setminus \{z_0\}. \tag{1.2}
\end{cases}$$

The count of such disks defines an element

$$e_\chi^\circ \in CF^*(L, L').$$
Figure 1.3. A holomorphic disk with one boundary puncture and with a moving boundary condition given by a non-negative isotopy $\mathcal{L}$. The count of such disks gives rise to an element of $CF(L, L')$.

We prove (in the Liouville setting) that the two constructions above yield equivalent elements in cohomology after applying the $\mu^2$ operation:

**Theorem 1.4.** Let $M$ be a Liouville manifold and $[h^\rho_L]$ and $c^\chi_L$ be as above. Then we have that the

$$[h^\rho_L] = [\mu^2(c^\chi_L, -)]$$

as maps $HF^*(K, L) \to HF^*(K, L')$.

This compatibility is well-known to experts, but a detailed proof is not easily found in the literature. We prove it here in the Liouville setting, and reassure the readers that the analogous results can be proven in other settings with appropriate modifications (as necessary) taken to prove $C^0$ and energy estimates.

Along the way, we also provide details of the $C^0$-estimates and the energy estimates needed for the compactness study of moduli spaces of pseudoholomorphic sections; in proving a bundle analogue of Theorem 1.4, we also highlight the roles of nonnegativity of the isotopy and of the “pinchedness” (from below) of the curvature of Liouville bundles. (See Theorem 4.20 for the precise statement.)

1.1 More motivations

In [OT19], we use Theorem 1.1 to show that a Liouville action of a Lie group $G$ on a Liouville sector $M$ results in a homotopically coherent map from $G$ to the automorphism space of the wrapped Fukaya category of $M$, proving a conjecture of Teleman [Tel14]. The passage from $O_j$ to its wrapped counterpart $W_j$ eventually requires us to understand how continuation maps behave in the Liouville bundle setting—this requires Theorem 1.4 and its bundle version.

Let us remark why Theorem 1.4 is needed. The issue is that the strip definition of continuation maps does not naively define a map form $L$ to $L'$ in the directed, unwrapped category. Usually, one produces an element of $HF(L, L')$ out of a Hamiltonian isotopy by proving the naturality of

\[ \text{Note that in Figure 1.2 the moving boundary condition places } L \text{ near } \tau = \infty, \text{ and places } L' \text{ near } \tau = -\infty. \text{ (In particular, the isotopy evolves in the } -\partial/\partial \tau \text{ direction.)} \]
in the $K$ variable, then invoking the Yoneda embedding. But to geometrically interpret the Yoneda embedding, one needs a geometric interpretation of the unit map of an object—such an interpretation is unavailable in the directed Fukaya category, as the identity morphism is constructed by formal algebra, rather than by defining the actual endomorphism Floer complex of a brane.

Let us also warn readers that in the Liouville, non-wrapped setting, one does not expect continuation maps to be equivalences. This is because the non-negative directedness of Hamiltonian isotopies in the Liouville setting prevents the usual trick (of composing a continuation map with the continuation map of the reverse isotopy). Of course, as is now understood thanks to ideas of Abouzaid-Seidel and [GPS17], localizing with respect to non-negative continuation maps precisely recovers the wrapped theory.

Finally, we hope that our description of what choices are needed to specify a collection of unwrapped Fukaya categories living over a Liouville bundle, while exhibiting the existence of compatible choices (to simultaneously output the usual $A_\infty$-relations and to jive with face maps of the simplices $j$) may be of interest to readers looking to exploit Floer theory in the bundle setting.

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2 Liouville bundles and connections

2.1 Liouville domains

The notion of Liouville domain will not make a frequent appearance in our work; but it is a convenient stepping stone to the notion of Liouville manifold.

Definition 2.1. Fix a symplectic manifold $(M,\omega)$. A vector field $Z$ on $M$ is said to be a Liouville vector field if the Lie derivative of $\omega$ along $Z$ is $\omega$ itself:

\[ \mathcal{L}_Z \omega = \omega. \quad (2.1) \]

Given a Liouville vector field $Z$, its flow will be called the Liouville flow.

Definition 2.2. Given a Liouville vector field $Z$, let $\lambda$ be the 1-form defined by the equation

\[ \lambda = \omega(Z, \cdot). \quad (2.2) \]

We call $\lambda$ the Liouville form.

Remark 2.3. Fix a vector field $Z$ and its dual $\lambda$ as in Equation (2.2). Then Equation (2.1) is equivalent to the condition that $\lambda$ is an anti-derivative of $\omega$:

\[ \omega = d\lambda. \]

In particular, any symplectic manifold equipped with a Liouville vector field is an exact symplectic manifold. Conversely, given a 1-form $\lambda$ satisfying $d\lambda = \omega$, one sees that the dual vector field defined by (2.2) is automatically a Liouville vector field.

Definition 2.4. A Liouville domain is a compact symplectic manifold $W$ with boundary, equipped with a Liouville vector field $Z$ which points strictly outward along the boundary.
Remark 2.5. By the exactness witnessed in Remark 2.3, any Liouville domain $W$ must have non-empty boundary unless $W$ is 0-dimensional.

Notation 2.6 (The boundary $\partial_\infty W$ of a Liouville domain). Fix a Liouville domain $W$ (Definition 2.4). We let $\partial_\infty W$ be the boundary manifold.

Remark 2.7. Let $W$ be a Liouville domain. It follows that $\xi = \ker \lambda\mid_{\partial_\infty W}$ is a contact structure on $\partial_\infty W$.

Remark 2.8 (Co-orientation). Recall that a co-orientation on a contact manifold is a choice of 1-form whose kernel is equal to the contact distribution. We see that the boundary $\partial_\infty W$ of any Liouville domain is a contact manifold co-oriented by $\lambda\mid_{\partial_\infty W}$.

2.2 Symplectizations

Notation 2.9 (Symplectization $\text{SY}$ of a contact manifold). Given a co-oriented contact manifold $(Y, \alpha)$, its symplectization $\text{SY}$ is the manifold 

$$\text{SY} = \mathbb{R} \times Y = \{(s, y)\}.$$

We equip $\text{SY}$ with the Liouville form (Definition 2.2)

$$e^s \pi^* \alpha$$

where $\pi: \text{SY} \to Y$ is the projection map.

Notation 2.10 ($r$ and $s$). We will often use the change of coordinates

$$r = e^s.$$

2.3 Liouville manifolds

We now pass from the setting of a Liouville domain to a “manifold-with-conical-end” setting:

Notation 2.11. Let $M$ be a smooth manifold. We define an equivalence relation on the set of smooth 1-forms on $M$ as follows: We say $\theta \sim \theta'$ if and only if there exists a smooth, compactly supported function $f: M \to \mathbb{R}$ for which

$$\theta = \theta' + df.$$

We let $[\theta]_{\text{Liou}}$ denote the equivalence class of $\theta$.

Definition 2.12 (Liouville manifold). Fix the data of a pair $(M, [\theta]_{\text{Liou}})$, where $[\theta]_{\text{Liou}}$ is as in Notation 2.11. We say this pair is a Liouville manifold if for some (and hence any) choice $\theta \in [\theta]$, the pair $(M, \theta)$ is a completion of a Liouville domain.

Remark 2.13. Let us explain what we mean by a completion. We mean there exists a compact, co-oriented contact manifold $(Y, \alpha)$, and a map from the ‘positive half’ of $\text{SY}$

$$\iota: \mathbb{R}_{s \geq 0} \times Y \to M$$

such that

1. $\iota$ respects Liouville forms, i.e., $\iota^*(\theta) = e^s \pi^* \alpha$,  

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2. \( \iota \) is a diffeomorphism (of manifolds with boundary) onto its image, and

3. The complement \( M \setminus \iota(\mathbb{R}_{>0} \times Y) \) is a Liouville domain when equipped with (the restriction of) \( \theta \).

**Remark 2.14.** We have reserved \( M \) to denote possibly-non-compact exact symplectic manifolds, while \( W \) always denotes (compact) Liouville domains.

**Remark 2.15.** It is common to define a Liouville manifold as equipped with a choice of \( \theta \), rather than just of \([\theta]_{\text{Liou}} \). While we utilize particular choices of \( \theta \) to perform certain geometric constructions, a particular choice conceals the appropriate notion of automorphism. (See Definition 2.27.)

**Remark 2.16.** One may pass freely between a Liouville domain to a Liouville manifold (by completion), and vice versa (by choosing an \( \iota \) as in (2.3)). However, the notion of Liouville manifold will be more canonical—e.g., less choice-dependent—in our applications.

### 2.4 Liouville sectors

**Remark 2.17.** The notion of Liouville sector is due to [GPS17], and extends the notion of Liouville manifold to the setting with boundary.

Just as Liouville manifolds are naturally presented as completions of exact symplectic manifolds with boundary, a Liouville sector is naturally the completion of an exact symplectic manifold \( W \) with *corners*.

**Definition 2.18** (Liouville domain with convex boundary). Fix a compact exact symplectic manifold \((W, \theta)\) with corners. We let \( DW \) denote the entire boundary of \( W \)—i.e., the union of all faces and corners of \( M \).

We say the pair \((W, \theta)\) is a *Liouville domain with convex boundary* if the following are satisfied:

1. **(CB1)** (There are two kinds of boundary.) \( DW \) admits two smooth, codimension zero submanifolds-with-boundary \( \partial W \) and \( \partial_\infty W \) such that

   \[
   \partial_\infty W \cap \partial W
   \]

   is precisely the locus of corners of \( W \), and

   \[
   DW = \partial_\infty W \cup_{\partial_\infty W \cap \partial W} \partial W.
   \]

2. **(CB2)** (\( \partial_\infty W \) is contact.) We demand that the Liouville vector field \( Z \) is strictly outward-pointing with respect to \( \partial_\infty W \). In particular, \( \theta|_{\partial_\infty W} \) renders \( \partial_\infty W \) a (co-oriented) contact manifold with boundary.

3. **(CB3)** (\( \partial W \) is convex.) We demand that there exists a smooth function \( I : W \to \mathbb{R} \) satisfying \( ZI = \alpha I \) for some \( \alpha > 0 \) whose Hamiltonian flow along \( \partial W \) is strictly outward pointing. (See [GPS17, Definition 2.4].)

4. **(CB4)** (The \( \partial W \) boundary can be extended along \( Z \).) For simplicity, we will further assume that in some neighborhood of \( \partial_\infty W \), \( Z \) is contained in \( T(\partial W) \). (So near \( \partial_\infty W \), \( Z \) is tangent to \( \partial W \). One can always deform \( \theta \) so that this is the case).
Remark 2.19. Let $(W, \theta)$ be a Liouville domain with convex boundary (Definition 2.18). Using the notation from (CB1), one may informally think of $\partial W$ as the wall of $W$, while one may think of $\partial_{\infty}W$ as the ceiling. (There are no floors.)

Based on (CB4), the reader may imagine that the Liouville flow pushes the ceiling higher toward the sky, in a way such that the walls may similarly be extended upward. The Liouville flow may push on the walls inwards or outwards, but it only does so away from a neighborhood of the ceiling.

The reader should compare the following to the definition of Liouville domain (Definition 2.12). It is equivalent to Definition 2.4 of [GPS17].

Definition 2.20. Fix a pair $(M, [\theta]_{Liou})$ where $M$ is a smooth manifold with boundary. We say that $(M, [\theta]_{Liou})$ is a Liouville sector if, for some (and hence all) $\theta \in [\theta]_{Liou}$, the pair $(M, \theta)$ is the completion of a Liouville domain with convex boundary.

Remark 2.21. By a completion, we mean the data of a co-oriented contact manifold $(Y, \alpha)$ with boundary, and a map $\iota: \mathbb{R}_{s \geq 0} \times Y \to M$ such that the appropriate analogues of Remark 2.13 are satisfied. In particular, $\iota$ is a diffeomorphism of smooth manifolds with corners, and the restriction of $\theta$ exhibits the complement $M \setminus \iota(\mathbb{R}_{s \geq 0} \times Y)$ as a Liouville domain with convex boundary.

Remark 2.22. Let $(M, [\theta]_{Liou})$ be a Liouville sector. Then $M$ only has one “type” of boundary, $\partial M$, which one may think of as an extension of the wall $\partial W$ by the Liouville flow.

Remark 2.23. Henceforth, we use the term Liouville sector with the understanding that if a Liouville sector has empty boundary, then it is in particular a Liouville manifold.

2.5 Eventually conical branes

Definition 2.24. A subset $A \subset M$ is called conical near infinity if for some (and hence all) $\theta \in [\theta]_{Liou}$, and for some compact subset $K$, the complement $A \setminus K$ is closed under the positive Liouville flow.

There are standard decorations one should put on Liouville manifolds and their Lagrangians to obtain a $\mathbb{Z}$-graded, $\mathbb{Z}$-linear Fukaya category—for example, gradings and Pin structures. We assume these structures to be chosen throughout. To that end:

Definition 2.25. Let $M$ be a Liouville manifold. A brane is a conical-near-infinity Lagrangian $L \subset M$ equipped with the relevant brane decorations.

Because brane structures will not feature prominently in this work, we refer the reader to [Sei08] for the basics, and to Section 2.3 of [OT19] for how the structure group of a Liouville bundle changes as one demands different brane structure.

2.6 Non-negative isotopies

We recall the notion of a nonnegative exact Lagrangian isotopy.

Definition 2.26 (Non-negative isotopy). Fix an exact Lagrangian isotopy $j: L \times [0,1]_{t} \to M$ through conical-near-infinity Lagrangians. (In particular, this induces an isotopy of Legendrians inside $\partial_{\infty}M$.) We say this is a non-negative wrapping$^{3}$, or a non-negative isotopy if for some (and hence any) choice of Liouville form $\theta$ on $M$, we have the following outside a compact subset of $L$:

$$\theta(Dj(\partial_{t})) \geq 0.$$ 

Put another way, the flow of $L$ in $\partial_{\infty}M$ is non-negative with respect to the contact form induced by $\theta$.

$^{3}$In [GPS17], this notion is called a positive wrapping (see Definition 3.20 of loc. cit.).
2.7 Liouville automorphisms

**Definition 2.27** (Liouville automorphisms). Let $M_i$, $i=0,1$, be Liouville sectors. A **Liouville isomorphism** from $M_0$ to $M_1$ is a diffeomorphism $\phi: M_0 \to M_1$ satisfying

$$\phi^* [\theta_1]_{\text{Liou}} = [\theta_0]_{\text{Liou}}.$$

(See Notation 2.11.) If $M_0 = M_1$, we call $\phi$ a **Liouville automorphism**.

**Definition 2.28.** Let $M$ be a Liouville sector. We let

$$\text{Aut}^\sigma(M)$$

denote the topological group of Liouville automorphisms of $M$. It is topologized as a subspace of $C^\infty(M,M)$ with the strong Whitney topology.

**Warning 2.29.** The choice of $[\theta]_{\text{Liou}}$ is not explicit in the notation $\text{Aut}^\sigma(M)$.

2.8 Liouville bundles

**Definition 2.30** (Liouville bundle). Fix a Liouville manifold $M$. A **Liouville bundle** with fiber $M$ is the choice of a smooth $M$-bundle $p: E \to B$, together with a smooth reduction of the structure group from $\text{Diff}(M)$ to $\text{Aut}^\sigma(M)$.

**Remark 2.31.** Definition 2.30 applies when $p: E \to B$ is a smooth map of diffeological spaces (see [OT20b, Section 3.2]), or smooth manifolds with corners. By a **smooth** reduction of structure group, we mean that for an open cover, the specified transition maps $U_{\alpha\beta} \to \text{Aut}^\sigma(M)$ must be smooth (in the sense of the diffeology on $\text{Aut}^\sigma(M)$ and the diffeology of $B$).

**Remark 2.32.** If one is interested in a Liouville bundle with a structure group allowing one to trivialize brane structures over simplices, one should demand a smooth reduction of structure group not to $\text{Aut}^\sigma$, but to another structure group $\text{Aut}$. We refer the reader to Section 2.3 of [OT19] for possible other structure groups. We also note that the smoothness of reduction may now be tested by composing a map to $\text{Aut}$ with the natural projection $\text{Aut} \to \text{Aut}^\sigma$.

**Notation 2.33** ($\partial E$). Let $E \to B$ be a Liouville bundle whose fibers are Liouville manifolds, and suppose these fibers are all isomorphic to some Liouville manifold $M$. We denote by

$$\partial E \to B$$

the induced fiber bundle whose fibers are diffeomorphic to $\partial M$. Note that we use the symbol $\partial E$ regardless of whether the base $B$ has boundary, corners, et cetera.

**Remark 2.34** ($\Theta$). Let $E \to B$ be a Liouville bundle and suppose $E$ and $B$ are both smooth manifolds, possibly with corners. First let $B$ be smoothly contractible. Then there exists a global choice of 1-form

$$\Theta \in \Omega^1(E;\mathbb{R})$$

such that:

(\Theta 1) for every $b \in B$, the fiberwise restriction $\Theta|_{E_b}$ defines a 1-form on the fiber $E_b$ exhibiting $E_b$ as a Liouville completion.
By the paracompactness of $B$ and a partition of unity argument, we thus have a global 1-form $\Theta$ on $E \to B$ satisfying property $(\Theta 1)$ for arbitrary base manifolds $B$. We call $\Theta$ a fiberwise Liouville form for $E \to B$, or just a Liouville form as long as there is no danger of confusion.

**Remark 2.35.** Fix a Liouville bundle $E \to B$ where $B$ (and hence $E$) is a smooth manifold, possibly with corners. Then the space of $\Theta$ satisfying $(\Theta 1)$ is convex, and in particular, smoothly contractible.

**Example 2.36.** Fix a Liouville manifold $M$. There will be two classes of Liouville bundles of interest associated to $M$.

The first is the *universal* Liouville bundle. (See [OT20b, Section 2.7].) This is constructed from the universal principle bundle

$$E \overset{\text{Aut}}{\to} B \overset{\text{Aut}}{\to} M$$

by taking the induced principle $M$-bundle

$$E \overset{\text{Aut}}{\to} B$$

(whose structure group is canonically smoothly reduced to $\text{Aut}(M)$). The reader may appreciate that in, Remark 2.34 we assumed $B$ is a smooth manifold—in the example of $B = B \overset{\text{Aut}}{\to} M$, $B$ is not a manifold.

The second main example is given by taking a smooth map from an extended smooth simplex

$$j : |\Delta^n| \to B$$

where $|\Delta^n|$ is the affine hyperplane defined by the equation $\sum_{i=0}^{n} t_i = 1$. (We refer to [OT20b, Definition 2.5] for a discussion on why we use the extended smooth simplex.) We then pull back the bundle $E \to B$ to obtain a smooth Liouville bundle $j^* E \to |\Delta^n|$; in particular, one obtains another smooth Liouville bundle by restricting further to the standard $n$-simplex $|\Delta^n| \subset |\Delta^n|$.

### 2.9 Connections on bundles

**Definition 2.37** (Connection). Let $\pi : E \to B$ be a smooth fiber bundle. Recall that a (Ehresmann) connection is a choice of splitting

$$TE \cong HTE \oplus VTE$$

where $VTE = \ker(d\pi)$. As usual we will call $HTE$ the horizontal distribution (associated to the connection).

Fix a Liouville bundle $\pi : E \to B$ over a smooth manifold $B$, and equip $E$ with a choice of global 1-form $\Theta \in \Omega^1(E)$ as in Remark 2.34. Then one has a natural connection on $\pi : E \to B$, defined as follows:

**Definition 2.38.** The connection associated to $\Theta$ is the subbundle of $TE$ consisting of those tangent vectors $x$ for which

$$VTE \subset \ker(d\Theta(-,x)).$$

That is, any vertical tangent vector is annihilated when paired with $x$ using $d\Theta$.

In particular, any Liouville bundle equipped with $\Theta$ as in Remark 2.34 has a well-defined notion of parallel transport along smooth curves.
2.10 Almost complex structures

**Definition 2.39.** Let $E \to B$ be a Liouville bundle. Let $\mathcal{J}$ be a smooth choice of fiber-wise almost complex structures on $E$.

We say that $\mathcal{J}$ is **conical near infinity** if for some (hence any) choice of $\Theta$ (as in Remark 2.34), there exists a subset $K \subset E$, proper over $B$, such that the following holds:

1. For each $b \in B$, $K \cap E_b$ is a Liouville domain (exhibiting the fiber $E_b$ as the Liouville completion of $K \cap E_b$), and

2. Writing $E_b$ as the completion of $K \cap E_b$ with conical coordinate $r = e^s$, we have that

$$\Theta|_{E_b} \circ \mathcal{J}|_{E_b} = d(e^s)$$

for $s >> 0$.

**Remark 2.40.** If $\mathcal{J}$ is conical near infinity (Definition 2.39), it follows that along each fiber of $E \to B$, the Lie derivative of $\mathcal{J}|_{E_b}$ with respect to the Liouville flow vanishes outside some compact subset (for example, outside of $K \cap E_b$).

**Example 2.41.** If $B$ is a point, then a choice of $\mathcal{J}$ as in Definition 2.39 is a choice of conical-near-infinity almost-complex structure $\mathcal{J}$ on the fiber Liouville manifold, in the usual sense.

**Notation 2.42 ($\mathcal{J}$).** For every $b \in B$, let $S_b \subset B$ denote the Riemann surface containing $b$. Then over $E$ there is a natural bundle

$$\mathcal{J} \to E$$

whose fibers above $x \in E$ consist of almost-complex structures on the vector bundle

$$(d\pi)^{-1}(T_bS_b) \subset T_xE.$$ (2.5)

**Remark 2.43.** Here is another description of $\mathcal{J}$. Let $B = \mathcal{S}_{d+1} \to \mathcal{K}_{d+1}$ denote the projection map for the universal family of curves, and let $\mathcal{H} \subset TB$ denote the vertical tangent bundle of this projection. Fix further a Liouville form $\Theta$ on $\pi : E \to B$, so that we have an induced splitting $TE \cong HTE \oplus VTE$ as in (2.4). By the identification $HTE \cong \pi^*TB$, we have an induced subbundle $\pi^*\mathcal{H} \oplus VTE \subset TE$. $\mathcal{J}$ is the bundle whose global sections are choices of almost-complex structures on $\pi^*\mathcal{H} \oplus VTE$.

**Definition 2.44 ($\mathcal{J}$ Suitable for counting sections).** Let $B = \mathcal{S}_{d+1}$ and fix a Liouville bundle $\pi : E \to B$. Let $\mathcal{J}$ be the bundle from Notation 2.42. We say that a global section $\mathcal{J}$ of $\mathcal{J}$ is **suitable for counting sections** when the following are satisfied:

1. For every member of the universal family $S_r \subset \mathcal{S}_{d+1}$, let $E_r \to S_r$ denote the pulled back Liouville bundle. We demand that the projection map is holomorphic—that is,

$$d\pi \circ \mathcal{J}|_{E_r} = j_r \circ d\pi.$$

(Here, $j_r$ is the complex structure on $S_r$.)

2. $\mathcal{J}$ preserves the vertical tangent space $VTE$, and $\mathcal{J}|_{VTE}$ is a conical-near-infinity almost-complex structure for the bundle $E \to B$ as in Definition 2.39.
3. Finally, we demand that for some (and hence any) choice of global Liouville form $\Theta$ on $E$ as in Remark 2.34, there exists a subset $K \subset E$ (independent of $r \in \mathbb{R}_{d+1}$), proper over $B$, such that $\mathcal{J}(HTE_r) = HTE_r$. (Here, $HTE_r$ is the horizontal tangent space induced by pulling back the connection on $E$ to a connection on $E_r$.)

By abuse of notation, we will refer to $\mathcal{J}$ also as a choice of almost-complex structure. (Even though, strictly speaking, $\mathcal{J}$ only defines almost-complex structures on each $E_r$, and not on all of $E$.)

**Remark 2.45.** Let $\mathcal{J}$ be an almost-complex structure suitable for counting sections (Definition 2.44). Choose a splitting $T_xE_r \cong VT_xE \oplus T_{\pi(x)}S_r$, condition 1. says that $\mathcal{J}_x$ may be written as a block triangular matrix. Condition 3. says that, outside controlled, fiber-wise compact subset, $\mathcal{J}_x$ is block diagonal. In particular the space of $\mathcal{J}$ is seen to be contractible.

### 2.11 Defining functions and barriers on families

Let us recall from [GPS17] that if $M$ is a Liouville sector, there exists a smooth map

$$
\pi : \text{Nbhd}(\partial M) \to \mathbb{C}_{\mathbb{R} \geq 0}
$$

from a neighborhood of $\partial M$ to the complex numbers with positive real coordinate. This (possibly non-surjective) map satisfies the following:

1. The imaginary coordinate of $\pi$ defines a smooth, linear-near-infinity function $I$ whose Hamiltonian vector field is outward pointing at $\partial M$. (This is called a “defining function” in [GPS17].)

2. Moreover, there is a contractible space of almost-complex structures $J$ on $M$, compatible with the Liouville structure of $M$, such that $\pi$ is $J$-holomorphic.

**Remark 2.46.** This allows one to use a standard “barrier” type argument using the open mapping theorem to conclude the following: If a holomorphic curve $u : S \to M$ has boundary Lagrangians supported away from $\partial M$, then the image of $u$ must be bounded away from $\partial M$ as well. (See 2.10.1 of [GPS17].) This is the main utility of the definition of Liouville sector, and in particular of the defining function $I$. (Informally, while $I$ defines the imaginary coordinate of $\pi$, its negative Hamiltonian flow-time away from $\partial M$ defines the real coordinate—see the proof of Proposition 2.24 of [GPS17].)

**Remark 2.47.** Now if $E \to B$ is a Liouville bundle of Liouville sectors over a smooth manifold, a partition of unity argument defines a global function $\pi : \text{Nbhd}(\partial E) \to \mathbb{C}_{\mathbb{R} \geq 0}$ whose imaginary part restricts on each fiber to a defining function $I$. ($\partial E$ is defined in Notation 2.33.) By contractibility of the space of almost-complex structures, one can choose $\mathcal{J}$ on $E$ such that

$$
D\pi|_{\text{VTE}} \circ \mathcal{J} = j_{\mathbb{C}_{\mathbb{R} \geq 0}} \circ D\pi|_{\text{VTE}} \quad \text{on Nbhd}(\partial E).
$$

(Here $\text{VTE} = \ker D\pi$ is the vertical tangent bundle of $E$.) In particular, given any map $u : S \to E$ which is holomorphic with respect to $J$, the composite $\pi \circ u : S \to \mathbb{C}_{\mathbb{R} \geq 0}$ is holomorphic, and the same barrier argument as in Remark 2.46 shows that the image of $u$ must be bounded away from the boundary of each fiber Liouville sector.

In particular, if one has a priori $C^0$ bounds on the strip-like ends of $S$, then one has an a priori $C^0$ bound on $u$ given the boundary conditions.
3 Simplices, and families of disks

Notation 3.1 (Simplices). Fix an integer $d \geq 0$. We let $|\Delta^d|$ denote the standard topological $d$-dimensional simplex, given by the subset of those $(t_0, \ldots, t_d) \in \mathbb{R}^{d+1}$ satisfying $t_i \geq 0$ and $\sum t_i = 1$. More generally, given any linearly ordered set $A$, we let $|\Delta^A|$ denote the subset of $\mathbb{R}^A$ given by those $(t_a)_{a \in A}$ satisfying $t_a \geq 0$ and $\sum_{a \in A} t_a = 1$.

We will sometimes refer to $|\Delta^A|$ as the geometric realization of $A$.

The extended $d$-simplex is the space $|\Delta^d_e| \subset \mathbb{R}^{d+1}$ of those $(t_0, \ldots, t_d) \in \mathbb{R}^{d+1}$ satisfying $\sum t_i = 1$.

Definition 3.2 (ith vertex). Let $|\Delta^d|$ be a standard simplex. Given $i \in \{0, \ldots, d\}$, the $i$th vertex of $|\Delta^d|$ is the unique point whose $i$th coordinate is equal to 1. Likewise, if $|\Delta^d_e|$ is the extended simplex, the $i$th vertex is the same point (with $i$th coordinate 1 and other coordinate 0).

Remark 3.3 (Standard and extended simplices). Because the natural inclusion $|\Delta^n| \to |\Delta^d_e|$ is a smooth map (from a manifold with corners), it will make sense to pullback smooth objects living over an extended simplex to a standard simplex.

Finally, we will do our best to use the letter $h$ to denote maps from a standard simplex, and the letter $j$ to denote maps from the extended simplex:

$$h : |\Delta^n| \to B, \quad j : |\Delta^n_e| \to B.$$

We review Savelyev’s observation from [Sav13] that two fundamental objects of our fields—(i) universal families of holomorphic disks with $k+1$ boundary punctures, and (ii) standard simplices—have compatible operadic structures.

Let us explain how we use this observation. Our goal (Theorem 1.1) is to associate, for every smooth map $j : |\Delta^n_e| \to B$ and every Liouville bundle $E \to B$, a non-wrapped Fukaya category $\mathcal{O}_j$. This means that given $j$ and a Liouville bundle, we must associate a $d$-ary $A_{\infty}$ operation for every $(d+1)$-tuple of objects.

In a way we make explicit later, this is done by taking a map

$$|\Delta^d| \to |\Delta^n|$$

induced by the $(d+1)$-tuple of objects, and noticing that the $d$-simplex $|\Delta^d|$ itself can be (modulo a neighborhood of the boundary) identified with the total space of the universal family

$$S_{d+1} \to \mathbb{R}_{d+1}$$

of $(d+1)$-ary holomorphic disks (see Notation 3.4 and Remark 3.20). At the very end of the present section, we will choose such an identification $|\Delta^d| \approx S_{d+1}$ once and for all. (Here, we use $\approx$ rather than $\cong$ to indicate that this is an identification modulo boundary.)

Roughly speaking, we will then define the $d$-ary operation $m_d$ to be given by counts of holomorphic sections $u$ (with Lagrangian boundary conditions) from fibers of $S_{d+1}$ to the bundle obtained by pulling back $E$ along the composite $S_{d+1} \approx |\Delta^d| \to |\Delta^n| \to |\Delta^n_e| \to B$:

\[
\begin{array}{ccc}
E|_{S_r} & \text{\approx} & |\Delta^d| \to |\Delta^n| \to |\Delta^n_e| \to B
\end{array}
\]
Here, $S_r \subset S_{d+1}$ is a holomorphic disk with $d + 1$ boundary marked points; it is the fiber above $r \in \mathcal{R}_{d+1}$.

The reader may now appreciate that for such counts to satisfy the $A_\infty$-relations, one must impose some compatibilities on the structures chosen to define these operations—and especially the identifications $S_{d+1} \approx |\Delta^d|$—as one approaches the boundary moduli of nodal disks. To articulate these compatibilities, we will also be forced to choose maps
\[ \nu_{\beta} : \overline{S}_{d+1}^\beta \to |\Delta^n| \]
for each simplicial map $\beta : |\Delta^d| \to |\Delta^n|$. (See Notation 3.4 for the notation $\overline{S}_{d+1}^\beta$.) Moreover, we would later like these non-wrapped Fukaya categories to be functorial in the choice of $j$, meaning that if we have a simplicial inclusion $|\Delta^n_j| \subset |\Delta^n|$, the composite map $j' : |\Delta^n_j| \to |\Delta^n| \overset{\nu_{\beta}}{\to} B$, induces a functor $\mathcal{O}_{j'} \to \mathcal{O}_j$ of non-wrapped Fukaya categories. This imposes further compatibilities on our choices.

The main purpose of this section is to define what these compatibilities are in terms of the maps $\nu_{\beta}$, which for the special case of $\beta = \text{id}$ recovers the identifications $S_{d+1} \approx |\Delta^d|$. This is given in Definition 3.17. We record the existence of such choices in Proposition 3.18.

### 3.1 Universal families of curves and gluing along strip-like ends

**Notation 3.4** $(\overline{\mathcal{R}}, \overline{S}, \overline{S}^\beta)$. Let $\overline{\mathcal{R}}_{d+1}$ denote the compactified moduli space of holomorphic disks with $d + 1$ boundary punctures; we demand that one of these boundary punctures is distinguished, and we refer to it as the outgoing marked point, or the 0th marked point. Using the boundary orientation of a holomorphic disk, any other marked point may uniquely be labeled as the $i$th marked point for some $1 \leq i \leq d$.

We let $\overline{S}_{d+1} \to \overline{\mathcal{R}}_{d+1}$ denote the universal family of (possibly nodal) disks living over $\overline{\mathcal{R}}_{d+1}$. Note that a fiber is never compact; every disk—nodal or not—has boundary punctures.

Finally, we let $\overline{S}^0_{d+1} \subset \overline{S}_{d+1}$ denote the open subspace obtained by removing the nodal points of each fiber.

For any $r \in \overline{\mathcal{R}}_{d+1}$, we let $S_r \subset \overline{S}^0_{d+1}$ denote the fiber above $r$.

**Example 3.5.** If $d = 2$, then $\overline{\mathcal{R}}_{2+1}$ is homeomorphic to a single point. $\overline{S}_{2+1}$ is homeomorphic to a closed disk with three boundary points missing, as is $\overline{S}^0_{2+1}$.

If $d = 3$, then $\overline{\mathcal{R}}_{3+1}$ may be identified a closed unit interval $[0, 1]$. The universal family $\overline{S}_{3+1} \to [0, 1]$ has the property that the fiber over any element of the open interval $(0, 1)$ is homeomorphic to a closed disk minus four boundary points. Over either endpoint—0 or 1—the fiber is a wedge sum of two disks with two boundary points missing on each disk; in each fiber, the wedge point is the nodal point. Finally, the space $\overline{S}^0_{3+1}$ is obtained by removing exactly two points (the nodal points—one nodal point from each boundary element of $[0, 1]$) from $\overline{S}_{3+1}$.

More generally, $\overline{S}^0_{d+1}$ is obtained from $\overline{S}_{d+1}$ by removing $i$ wedge points (i.e., $i$ nodal points) from each fiber living over a codimension $i$ stratum of $\overline{\mathcal{R}}_{d+1}$.

**Choice 3.6** (Strip-like ends $\epsilon$). We assume we have chosen strip-like ends near the nodes and boundary marked points of each fiber of $\overline{S}^0_{d+1} \to \overline{\mathcal{R}}_{d+1}$. See Sections (8d), (9a), and (9c) of [Sei08].

We denote these strip like ends $\epsilon$ when necessary.

We assume we have also chosen diffeomorphisms $|\Delta^1| \approx [0, 1]$ once and for all, so that the strip like ends are biholomorphic embeddings
\[ \epsilon : [0, \infty) \times |\Delta^1| \to S_r \quad \text{or} \quad \epsilon : (-\infty, 0] \times |\Delta^1| \to S_r. \]

We denote by $\epsilon_i$ the strip-like end at the $i$th puncture.
**Notation 3.7 ($\alpha_i$).** Recall that every codimension one stratum of $\overline{\mathbb{R}}_{d+1}$ can be written as a direct product $\overline{\mathbb{R}}_{d+1} \times \overline{\mathbb{R}}_{d_1+1}$; indeed, for a given $d_1$, and for any $1 \leq i \leq d_1$, there is an $i$th wedging map

$$\alpha_i : \overline{\mathbb{R}}_{d+1} \times \overline{\mathbb{R}}_{d_1+1} \to \overline{\mathbb{R}}_{d_1}, \quad d_2 + d_1 - 1 = d$$

(3.1)

which glues the $0$th boundary vertex of a disk with $d_2 + 1$ marked points to the $i$th boundary vertex of a disk with $d_1 + 1$ marked points. For $r_1 \in \overline{\mathbb{R}}_{d_1+1}$ and $r_2 \in \overline{\mathbb{R}}_{d_2+1}$, we let $r_2 \circ_i r_1$ denote the image.

We will also write $S_{r_2} \circ_i S_{r_1}$ for the corresponding nodal disk.

Finally, because of our choice of strip-like ends, we can parametrize the corners of $\overline{\mathbb{R}}_{d+1}$; for instance, in codimension one, the maps from (3.1) extend to maps

$$\overline{\mathbb{R}}_{d+1} \times \overline{\mathbb{R}}_{d_1+1} \times [0, \epsilon) \to \overline{\mathbb{R}}_{d_1}, \quad d_2 + d_1 - 1 = d$$

(3.2)

(the dependence on $1 \leq i \leq d_1$ is suppressed in the above notation). See also Sections (9e) and (9f) of [Sei08].

Our main interest is in a lift of (3.2):

**Notation 3.8 ($\#_i$ and $\#_{i, \tau}$).** Let $\overline{\mathcal{S}}_{d_2, d_1} \to \overline{\mathbb{R}}_{d+1} \times \overline{\mathbb{R}}_{d_1+1} \times [0, \epsilon)$ denote the map obtained by pulling back $\overline{\mathcal{S}}_{d_1+1}$ and $\overline{\mathcal{S}}_{d_2+1}$ along the projections to $\overline{\mathbb{R}}_{d_1+1}$ and $\overline{\mathbb{R}}_{d_2+1}$, then taking the coproduct of these two pullbacks. Concretely, a fiber of $\overline{\mathcal{S}}_{d_2, d_1}$ over $(r_2, r_1, \tau)$ is the disjoint union $S_{r_2} \coprod S_{r_1}$. Then the gluing operation induced by the strip-like ends defines a map

$$\#_i : \overline{\mathcal{S}}_{d_2, d_1} \times [0, \epsilon) \to \overline{\mathcal{S}}_{d+1}^{i}, \quad d_2 + d_1 - 1 = d$$

(3.3)

where the restriction of $\#_i$ to time $\tau \in [0, \epsilon)$ will be denoted by $\#_{i, \tau}$.

**Remark 3.9.** Note the font $\overline{\mathcal{S}}_{d_2, d_1}$ rather than $\overline{\mathcal{S}}_{d_2, d_1}$; we use the former because Savelyev uses the latter font to indicate a different entity in [Sav13].

**Notation 3.10 ($\#_{i, \tau}$).** Let us describe $\#_{i, \tau}$ for the sake of establishing further notation. Fix $\tau \in [0, \epsilon)$ and elements $r_1 \in \overline{\mathcal{S}}_{d_1+1}, r_2 \in \overline{\mathcal{S}}_{d_2+1}$. Having fixed our strip-like ends long ago, the $i$th gluing operation identifies two open subsets of $S_{r_1}$ and $S_{r_2}$ to obtain a new disk $S_r$. The strip-like ends endow $S_r$ with a thick-thin decomposition, where we can holomorphically identify the “thin” region of $S_r$ with $(-\tau, \tau) \times |\Delta^1|$, and these thin regions are precisely the regions where the gluing operation has non-singleton fibers (i.e., this is the region over which $S_{r_2}$ and $S_{r_1}$ are glued). The gluing maps

$$S_{r_2} \coprod S_{r_1} \times \{\tau\} \to S_r$$

(where $r$ depends on $\tau$) define the maps $\#_{i, \tau}$. When $\tau = 0$, we have a map

$$\#_{i, 0} : S_{r_2} \coprod S_{r_1} \to S_{r_2} \circ_i S_{r_1}.$$  

(3.4)

### 3.2 Simplices and inserting posets

Now let us consider the simplicial analogue of the previous section’s constructions.

**Notation 3.11.** Fix an integer $d \geq 0$. We let $[d]$ denote the linear poset given by

$$[d] = \{0 < 1 < \ldots < d\}.$$  

It is the unique linear order with $d + 1$ elements (up to unique choice of order-preserving isomorphism).
Notation 3.12 $(A_2 \circ_i A_1)$. Let $A_2$ and $A_1$ be finite, non-empty, linearly ordered posets. We let $d_1 = \#A_1 - 1$. Then for any $1 \leq i \leq d_1$, we can construct a new linear poset $A_2 \circ_i A_1$ by gluing $A_2$ into $A_1$ as follows: Identify $\text{min} \ A_2$ with the $(i - 1)$st element of $A_1$, and identify $\text{max} \ A_2$ with the $i$th element of $A_1$. We see that $A_2 \circ_i A_1 \cong [d_2 + d_1 - 1]$ as posets.

We have a natural gluing map of posets

$$\#_i : A_2 \coprod A_1 \to A_2 \circ_i A_1.$$  \hfill (3.5)

By taking the geometric realization, we obtain a continuous map of topological spaces

$$|\Delta^{A_2}| \coprod |\Delta^{A_1}| \to |\Delta^{A_2 \circ_i A_1}|$$

which, by (canonically) identifying $A_2 \cong [d_2]$ and $A_1 \cong [d_1]$ as linear posets, is equivalent to a continuous map

$$\#_i : |\Delta^{d_2}| \coprod |\Delta^{d_1}| \to |\Delta^{d_2 + d_1 - 1}|.$$  \hfill (3.6)

Concretely, $\#_i$ simplicially includes $|\Delta^{d_2}|$ and $|\Delta^{d_1}|$ as subsimplices of $|\Delta^{d_2 + d_1 - 1}|$, and these inclusions overlap along the edge between the $i$th and $(d_2 + i)$th vertices of $|\Delta^{d_2 + d_1 - 1}|$.

Remark 3.13. (3.6) should be compared with the map (3.4) from Section 3.1.

3.3 Operadic compatibility

We start with fixing the following notation to avoid confusion arising from two integer-valued indices that will be used.

Notation 3.14 (The indices $n$ and $d$). Fix a Liouville bundle $E \to B$. We eventually want to define an $A_\infty$-category $O_j$ associated to any smooth map $j : |\Delta^n| \to B$; in particular, we must define the $A_\infty$-operations $\mu^d$ for the $A_\infty$-categories $O_j$. In this section, the integers $n$ and $d$ will be used precisely for these purposes.

Suppose we are given a simplicial map $\beta : |\Delta^d| \to |\Delta^n|$. (This is induced by a function $[d] \to [n]$, but this function need not be order-preserving.)\footnote{Savelyev uses the notation $u(m_1, \ldots, m_d, n)$, but the data of $m_1, \ldots, m_d, n$ is equivalent to the data of a single simplicial map $\beta : |\Delta^d| \to |\Delta^n|$, or equivalently, a map of sets $\beta : [d] \to [n]$.} We seek smooth maps

$$\nu_\beta : S_\circ_{d+1} \to |\Delta^n|$$

satisfying the following properties. (See the first row of Figure 3.15.)

(NS1) Fix any $0 \leq k \leq d$. For any $r \in S_{d+1}$, consider the fiber $S_r \subset S_{d+1}$. Then for the edge from $k - 1$ to $k$ in $|\Delta^d|$, the diagram

\[ [0, \infty) \times |\Delta^1| \; \xrightarrow{\epsilon_k} \; S_r \subset S_{d+1} \; \xrightarrow{\nu_\beta} \; |\Delta^n| \]

\[ \xrightarrow{\beta_{k-1,k}} |\Delta^1| \]

\[ \xrightarrow{\beta_{k-1,k}} |\Delta^1| \]
commutes. (Here, $\beta_{k-1,k}$ is the simplicial inclusion of the edge from the $(k - 1)$st vertex to the $k$th.) In plain English, this means that $\nu_{\beta}$ is compatible with the strip-like end parametrization by $|\Delta^1|$ near the $k$th puncture of $S_r$.

Note we are using Choice 3.6; also, when $k = 0$, the domain of $\epsilon_k$ should be $(-\infty, 0] \times [\Delta^1|$ as opposed to $[0, \infty) \times [\Delta^1]$.

(NS2) Now consider the boundary of $S_r$, and remove the images of the strip-like ends from this boundary. This results in $d + 1$ disconnected open intervals, and we enumerate them so that the $(k - 1)$st interval is contained in the boundary arc of $S_r$ beginning at the $(k - 1)$st marked point and ending at the $k$th marked point.

We demand that $\nu_{\beta}$ sends all of the $k$th interval to the vertex $\beta(k)$.

At this point, we see that as $r$ approaches a boundary stratum of $\overline{\mathbb{R}}_{d+1}$ (i.e., as disks degenerate), we would like the restrictions $\nu_{\beta}|_{S_r}$ to behave in a way compatible with the boundary faces of the $n$-simplex. See Figure 3.15. We make this compatibility (which Savelyev refers to as “natural” in [Sav13]) precise:

**Notation 3.16 ($*_i$).** Fix $d_2, d_1 \geq 2$ such that $d = d_2 + d_1 - 1$. Choose $1 \leq i \leq d_1$.

Then our fixed map $\beta : |\Delta^d| \to |\Delta^n|$ induces maps $\beta_1 : |\Delta^{d_1}| \to |\Delta^n|$ and $\beta_2 : |\Delta^{d_2}| \to |\Delta^n|$ so that the diagram

\[
|\Delta^{d_2}| \coprod |\Delta^{d_1}| \xrightarrow{\#_i} |\Delta^d| \xrightarrow{\beta} |\Delta^n|
\]

commutes. (Here, $\#_i$ is the map from (3.6).)

On the other hand, if we are given maps $\nu_{\beta_1} : \mathcal{F}_{d_1+1} \to |\Delta^n|$ and $\nu_{\beta_2} : \mathcal{F}_{d_2+1} \to |\Delta^n|$, the conditions (NS1) and (NS2) guarantee the existence of a (unique) map making the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{F}_{d_2+1} \coprod \mathcal{F}_{d_1+1} & \xrightarrow{\#_{i, r=0}} & \mathcal{F}_d \\
\downarrow{\nu_{\beta_2} \coprod \nu_{\beta_1}} & & \downarrow{\exists} \\
|\Delta^n| & & |\Delta^n|
\end{array}
\]

Here, the notation $\mathcal{F}_{d+1}|_{\mathcal{F}_{d_2+1} \circ \mathcal{F}_{d_1+1}}$ denotes the family $\mathcal{F}_{d+1}$ restricted to the image of the map $\circ_i$ from (3.1).

Extending the gluing parameter $\tau$ from 0 to an element of $[0, \epsilon)$, there is a neighborhood $U_{d_2, d_1, i} \supset \mathcal{F}_{d_2+1} \circ_i \mathcal{F}_{d_1+1}$ such that there is a unique extension $\nu_{\beta_2} \circ_i \nu_{\beta_1}$ making the diagram below commute:

\[
\begin{array}{ccc}
\mathcal{F}_{d_2+1} \coprod \mathcal{F}_{d_1+1} \times [0, \epsilon) & \xrightarrow{\#_i} & \mathcal{F}_d|_{U_{d_2, d_1, i}} \\
\downarrow{\nu_{\beta_2} \coprod \nu_{\beta_1}} & & \downarrow{\nu_{\beta_2} \circ_i \nu_{\beta_1}} \\
|\Delta^n| & & |\Delta^n|
\end{array}
\]

Explicitly, on the thin strips $|\Delta^1| \times [-\tau, \tau]$, we declare $\nu_{\beta_2} \circ_i \nu_{\beta_1}$ to equal the composition of the projection to $|\Delta^1|$ with the simplicial inclusion of the edge from the $(i - 1)$st vertex to the $i$th vertex.
Figure 3.15. An image of $\nu_\beta$, restricted to $S_r$ for various $r \in \mathbb{R}_{3+1}$, when $\beta$ is the identity $\beta : |\Delta^3| \to |\Delta^3|$. The blue arcs on the left-hand disk are images of the 1-simplices $\{t\} \times |\Delta^1| \subset (0, \infty) \times |\Delta^1|$ under strip-like parametrizations; they are sent to the blue edges indicated on the right-hand 3-simplices. The red thick edges on the disks are the “open intervals” referred to in the main text (in practice, these thick edges are labeled by Lagrangians $L_i$); these red edges are collapsed to the vertices labeled in red on the 3-simplices. (The edge labeled by $L_i$ is sent to the $i$th vertex.) In yellow is a drawing of the image of $\nu_\beta$ in the 3-simplex. In the bottom most image, the two components of a nodal disk are labeled by orange and yellow, and these are sent to two faces of the 3-simplex as indicated. Note the new green strip-like ends, and the newly highlighted green edge of the 3-simplex.
Hence it is natural to demand the following:

(NS3) For all $d_1, d_2, i$ as above, we demand that $\nu_\beta$ agrees with $\nu_{\beta_2} \star_i \nu_{\beta_1}$ on some neighborhood of $\mathcal{R}_{d_2+1} \circ_i \mathcal{R}_{d_1+1}$. That is,

$$\nu_\beta = \nu_{\beta_2} \star_i \nu_{\beta_1} \quad \text{on} \quad \mathcal{S}_{d+1}^0 |U_{d_2, d_1, i}$$

(possibly after replacing $U_{d_2, d_1, i} \supset \mathcal{R}_{d_2+1} \circ_i \mathcal{R}_{d_1+1}$ with some other neighborhood containing $\mathcal{R}_{d_2+1} \circ_i \mathcal{R}_{d_1+1}$, if necessary).

Finally, while we have fixed $n$ up until now, we demand that $\nu_\beta$ is functorial as the codomain of $\beta$ varies:

(NS4) Let $\alpha : [n] \to [n']$ be a map of posets; by abuse of notation, we also denote the induced simplicial map $\alpha : |\Delta^n| \to |\Delta^{n'}|$. Then for any $\beta : |\Delta^d| \to |\Delta^n|$, we demand

$$\alpha \circ \nu_\beta = \nu_{\alpha \circ \beta}.$$

**Definition 3.17** (Natural system). For every $n, d \geq 0$ and every simplicial map $\beta : |\Delta^d| \to |\Delta^n|$, choose a smooth map

$$\nu_\beta : \mathcal{S}^n_{d+1} \to |\Delta^n|.$$  

The collection $\{\nu_\beta\}$ is called a *natural system* if (NS1), (NS2), (NS3), and (NS4) above are satisfied.

A standard inductive argument shows the following:

**Proposition 3.18** (Proposition 3.4 of [Sav13]). Natural systems exist.

**Choice 3.19.** We will choose a natural system $\{\nu_\beta\}$ once and for all. (Note this is independent of any symplectic geometry or of any choice of Liouville bundle $E \to B$.)

**Remark 3.20.** One can prove that, given a natural system, each of the maps $\nu_\beta : \mathcal{S}^n_{d+1} \to |\Delta^n|$ (when $\beta = \text{id}$) is a degree one map on the interior; one can roughly think of $\nu_\beta$, then, as “homeomorphisms on the interior.” The naturality of the system says that these topological equivalences can be chosen in such a way that the gluing operations of disks are compatible with the insertion operation of simplices.

### 3.4 Collars on boundaries of simplices

We conclude this section with a final choice, made once and for all for every standard simplex $|\Delta^d| \subset \mathbb{R}^{d+1}$.

**Choice 3.21** (Collars of simplices). For every closed, codimension one face $F \subset |\Delta^d|$, we choose a small open neighborhood $U_F \subset |\Delta^d|$ together with a smooth retraction $\pi_F : U_F \to F$, which one thinks of as a projection map.

We choose the data of $(U_F, \pi_F)$ such that the following holds:

1. (The neighborhoods are mutually small.) If $A \subset |\Delta^d|$ is a closed subsimplex, let

$$U_A := \bigcap_{A \subset F} U_F.$$  

If $A, A' \subset |\Delta^d|$ are two closed subsimplices such that $A \cap A' = \emptyset$, we demand that

$$U_A \cap U_{A'} = \emptyset. \quad (3.7)$$
2. (Neighborhoods have controlled intersections.) Let $F$ and $F'$ be two codimension one faces with intersection $G = F \cap F'$. Then

$$U_F \cap F' \subset U_G. \quad (3.8)$$

**Remark 3.22.** Let us motivate the conditions. First, the collars are chosen so that we may trivialize certain data over the collars. (See for example (Θ3) of Choice 5.6.)

The intersection property (3.7) guarantees that these local trivializations do not enforce a global trivialization.

The intersection constraint (3.8) enables choices that are made inductively by dimension. For example, if we have already chosen data on lower-dimensional face $F'$ for which a trivialization does not extend beyond $U_G$, $U_F$ must be sufficiently small for us to be able to trivialize on $U_F$.

**Remark 3.23.** Note that these collars are independent of our choice of natural systems. The collar choices are likewise independent of any symplectic geometry. They obviously exist by a simple inductive argument on dimension.
4 Compactness

In this section, we study compactness properties of the moduli space of holomorphic sections of a Liouville bundle \( \pi : P \to \Sigma \) over a surface \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \). There are two fundamental analytical ingredients to establish: \( C^0 \)-estimates and energy estimates. These estimates imply Gromov compactness as usual, and establish both the \( A_\infty \) relations for \( \Theta_j \), and the existence and properties of continuation maps (which we study in the next section).

**Remark 4.1.** We reiterate that we are using cohomological conventions for our Floer complexes. So for example, given a morphism \( q \) from a brane \( L \) to another brane \( L' \), the differential \( \mu^1 q \) is computed by counting solutions to the equation
\[
\begin{cases}
\bar{\partial}_J u = 0 \\
u(-\infty) = p, \ u(\infty) = q \\
u(\tau, 0) \in L, \ u(\tau, 1) \in L'.
\end{cases}
\]
(4.1)

This is the same convention as in [Sei03].

4.1 Curvature

**Notation 4.2 \( (\Omega) \).** Fix a Liouville bundle \( E \to B \) with base \( B \), and fix a Liouville form \( \Theta \in \Omega^1(E) \) as in Remark 2.34; we have the associated two-form
\[
\Omega = d\Theta
\]
on \( E \).

Suppose that \( B = \Sigma \) is a Riemann surface. Then the curvature of the connection associated to \( \Theta \) (Definition 2.38) can be regarded as a 2-form on \( \Sigma \) with values in smooth functions of the fibers. More precisely:

**Proposition 4.3.** For any orientation-respecting volume form \( \omega_\Sigma \) on \( \Sigma \), we have
\[
\Omega|_{HTE} = f\pi^*\omega_\Sigma|_{HTE}
\]
(4.2)
where \( f : E \to \mathbb{R} \) is a smooth function.

For proofs, see Theorem 4.2.9 of [Oh15a] or (1.4) of [Sei03].

**Definition 4.4** (Compare with p.1007 [Sei03]). We say that the curvature of \( \Theta \) is non-negative if \( \Omega|_{HTE} \) is non-negative for the orientation on \( HTE \) (induced by that of \( \Sigma \)).

More generally, we say \( \Theta \) is \( (-C) \)-pinched (from below) if we have that
\[
\inf_{x \in E} f(x) \geq -C
\]
for some \( C \geq 0 \). (Here, the function \( f \) is as in (4.2).)

**Example 4.5.** \( \Theta \) has nonnegative curvature if and only if it is 0-pinched.

**Remark 4.6.** In all our choices of \( \Theta \), we can arrange for \( \Theta \) to be \( (-C) \)-pinched. This is for two reasons: First, \( \Theta \) has behavior controlled outside a set \( K \subset E \) that is proper over \( \Sigma \) (by assumption, \( K \) is a set whose complement is a fiberwise cylindrical region). Second, on a strip-like end of \( \Sigma \), \( \Theta \) is trivialized in the \( \tau \) direction (e.g., in the \( (-\infty, 0] \) direction of the strip \( (\infty, 0] \times [0, 1] \)), so one may extend \( K \) over a compactification of \( \Sigma \) (e.g., by filling in the punctures of \( \Sigma \) with chords).
4.2 On the interior of disks

We now establish that holomorphic curves with certain Lagrangian boundary conditions must be contained in some compact region of a Liouville bundle. A key result is Proposition 4.11, which shows that—if a holomorphic curve \( u : \Sigma \to E \) intersects the cylindrical region of \( E \)—the natural conical coordinate \( r = e^s \) is a subharmonic function on \( \Sigma \).

**Remark 4.7.** Recall that subharmonic functions behave like “convex up” functions, in that non-constant maxima are attained only along the boundary of the domain. Thus, knowing that the cylindrical coordinate of \( u \) is constrained along the interior of \( \Sigma \), by imposing appropriate boundary conditions on our Lagrangian family (to obtain constraints on the behavior of \( u \) along the boundary of \( \Sigma \)), we obtain the desired \( C^0 \) estimate in Theorem 4.20.

**Lemma 4.8.** Fix an conical-near-infinity conical almost-complex structure on \( E \to B \) (Definition 2.39). Let \( v : \Sigma \to E \) be any \((j,J)\)-holomorphic section. Then there exists some subset \( K \subset E \), proper over \( \Sigma \), outside of which we have

\[
\Delta (r \circ v) \omega_{\Sigma} = v^* \Omega.
\]  

(4.3)

**Proof.** We take \( K \) to be the same set as in the definition of conical-near-infinity almost-complex structure (Definition 2.39). We may then choose a global \( r \) coordinate.\(^5\) Using \( \Theta \circ J = dr \) and \( J \circ dv = dv \circ j \), we have

\[
d(r \circ v) = v^* dr = v^*(\Theta \circ J) = \Theta(J \circ dv) = \Theta(dv \circ j) = v^* \Theta \circ j.
\]

Therefore we have

\[
\Delta (r \circ v) \omega_{\Sigma} = -d(d(r \circ v) \circ j) = d(v^* \Theta) = v^* \Omega.
\]

This finishes the proof. \( \square \)

**Notation 4.9 \((l \text{ and } \omega_\beta)\).** Now let \( \beta \) be a positive 2-form on \( \Sigma \)—this means that

\[
\beta = l \omega_{\Sigma}
\]

for some positive function \( l : \Sigma \to \mathbb{R} \). We let

\[
\Omega_\beta = \Omega + \pi^* \beta.
\]

**Remark 4.10.** If \( \ell : \Sigma \to \mathbb{R} \) is sufficiently positive, then \( \Omega_\beta \) is a symplectic form. Moreover, given a conical-near-infinity choice of almost complex structure \( J \), choosing \( \ell \) sufficiently positive makes the projection \( E \to B \) a \((J,j)\)-holomorphic map (Lemma 2.1 of [Sei03]).

Consider the compatible metric

\[
g(X,Y) := \frac{1}{2} (\Omega_\beta(X,JY) + \Omega_\beta(Y,JX)).
\]

As usual, we have the identity

\[
|dv|^2 = |\partial_J v|^2 + |\overline{\partial_J} v|^2, \quad 2v^* \Omega_\beta = (|\partial_J v|^2 - |\overline{\partial_J} v|^2) \omega_{\Sigma}
\]

for any smooth section \( v \). We have more when \( v \) is holomorphic:

\(^5\)For example, by constructing an appropriate bundle of Liouville domains \( M_\Sigma \to \Sigma \), embedding \( M_\Sigma \hookrightarrow E \) over \( \Sigma \), then defining \( r \) by the Liouville flow time of the fiberwise contact boundaries.
Proposition 4.11. Suppose \( v : \Sigma \to E \) is a \((j, J)\)-holomorphic section. Then

\[
\Delta (r \circ v) = \frac{1}{2} |(dv)^v| \omega + f(v) \ell
\]

where \( f \) is the same function from (4.2) and \( \ell \) is from Notation 4.9.

In particular, if the pull-back connection of \( v^*E \) has nonnegative curvature, i.e., if \( f(v) \geq 0 \), and if \( \ell \) is large enough, then \( r \circ v \) is a subharmonic function on \( v^{-1}(E \setminus K) \). (Here, \( K \) is the same set as in Lemma 4.8.)

Proof. If \( \overline{\partial} J v = 0 \), then we obtain

\[
v^* \Omega_{\beta} = \frac{1}{2} |dv|^2 \omega_{\Sigma}. \tag{4.4}
\]

Splitting \( dv = (dv)^v + (dv)^h \) into horizontal and vertical components of \( dv \), we compute

\[
\frac{1}{2} |(dv)^h|^2 \omega_{\Sigma} = (f(v) + 1) \beta
\]

where \( f \) is the function given by (4.2). Rewriting (4.4) as

\[
v^* \Omega + v^* \pi^* \beta = \frac{1}{2} |(dv)^v|^2 \omega_{\Sigma} + (f(v) + 1) \beta \tag{4.5}
\]

and using \( \pi \circ v = \text{id}_\Sigma \), we obtain

\[
v^* \Omega = \frac{1}{2} |(dv)^v|^2 \omega_{\Sigma} + (f(v) + 1) \beta - (\pi \circ v)^* \beta = \frac{1}{2} |(dv)^v|^2 \omega_{\Sigma} + \ell f(v) \omega_{\Sigma}. \tag{4.6}
\]

Now combine Lemma 4.8 with (4.6). \( \square \)

4.3 Boundary conditions

The definitions below simply give names to the conditions we can guarantee in our set-up.

Definition 4.12. Let \( E \to \Sigma \) be a Liouville bundle, and fix strip-like ends of \( \Sigma \). We say that \( E \) is translation-invariant over the strip-like ends if, over the strip-like ends, \( E \) is equipped with an isomorphism to a pullback bundle

\[
p^* E_{[0,1]} \to E_{[0,1]}
\]

\[
R \times [0,1] \to [0,1]
\]

where \( R \) is either \((-\infty,0]\) (for incoming strips) or \([0,\infty)\) (for outgoing strips).

The following is a variation on Section 2.1 of Seidel’s work [Sei03].

Definition 4.13 (Lagrangian boundary conditions for bundles). Fix a Liouville bundle \( E \to \Sigma \) and strip-like ends on \( \Sigma \). Assume \( E \) is translation-invariant over the strip-like ends (Definition 4.12). A Lagrangian boundary condition suitable for our purposes is an \((n+1)\)-dimensional submanifold \( L \subseteq E|_{\partial \Sigma} \) equipped with the data of a smooth function \( K_L : L \to \mathbb{R} \), called a Liouville primitive, such that

1. \( L \) is fiberwise conical near infinity,
2. \( \pi|_L : L \to \partial \Sigma \) is a submersion,
3. \( \Theta|_L = dK_L \), and
4. \( K_L \) is fiberwise (affine) linear outside a subset \( K \subset E \) which is proper over \( \Sigma \), and
5. \( L \) is translation-invariant on the strip-like ends. (For example, for a strip-like end modeled on \((-\infty, 0]\), \( L \) can be obtained by pulling back a pair of branes \( L_0 \subset E_0, L_1 \subset E_1 \), along the projection \((-\infty, 0] \times [0, 1] \to [0, 1].\)"

**Remark 4.14.** The conditions of Definition 4.13 imply that for every \( z \in \partial \Sigma \), the fiber \( L_z \) is an exact Lagrangian submanifold of \( E_z \), and that \( K_L|_{L_z} \) is a Liouville primitive.

The following is the fibration version of non-negative wrapping.

**Definition 4.15.** Let \( L \) be a boundary condition as in Definition 4.13. We call \( L \) nonnegative (resp. nonpositive) relative to \( \partial \Sigma \) if \( \Theta(\xi) \geq 0 \) (resp. \( \Theta(\xi) \leq 0 \)) for all \( \xi \in T_x L \) whose projection to \( T\partial \Sigma \) compatible with the orientation of \( \partial \Sigma \).

Finally, in our setting, one can arrange for the following:

**Definition 4.16.** Let \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \), and let \( L \subset E \to \Sigma \) be a Lagrangian boundary condition as in Definition 4.13. Choose also a strip-like end for every \( z_i \), with \( z_0 \) incoming and others outgoing.

A connection induced by a Liouville form \( \Theta \) (as in Remark 2.34) will be called our kind of connection if it is trivial on \( \partial \Sigma \) outside the strip-like ends, and if the connection is invariant under the translation on the strip-like ends.

**Example 4.17.** For example, the conditions of Definition 4.16 hold on a strip-like end modeled after \((-\infty, 0]\times [0, 1] \to [0, 1]\).

**Remark 4.18.** In all our examples, \( E \to \Sigma \) is pulled back along a map \( \Sigma \to |\Delta^n| \) which collapses strip-like ends to a single edge of \(|\Delta^n|\), and collapses the rest of \( \partial \Sigma \) to vertices of \(|\Delta^n|\); as such, the connections pulled back from a bundle \( E' \to |\Delta^n| \) satisfy Definition 4.16.

**Remark 4.19.** If \( \Theta \) induces our kind of connection (Definition 4.16), and if for every connected component \( c_i \subset \partial \Sigma \), we choose some \( x_i \in c_i \) and a brane \( L_i \subset E_{x_i} \), one obtains a Lagrangian boundary condition \( L \) by parallel transport along \( \partial \Sigma \). \( L \) is non-negative in the sense of Definition 4.15.

### 4.4 The \( C^0 \) estimate

**Theorem 4.20.** Let \( (\Sigma, j) \) be a Riemann surface \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \) where \( \{z_0, \ldots, z_k\} \subset \partial D^2 \), and let \( \pi : E \to \Sigma \) be a Liouville bundle with translation invariance (Definition 4.12). Further choose:

- A non-negative boundary condition \( L \) as in Definitions 4.13 and 4.15.
- A conical-near-infinity almost-complex structure \( J \) on \( E \) for which \( E \to \Sigma \) is \((J,j)\)-holomorphic,
- A Liouville form \( \Theta \in \Omega^1(E) \) inducing our kind of connection \( \Theta \) (Definition 4.16) which is \((-C)\)-pinched (Definition 4.4), and
• For each \( i \in 1, \ldots, k \), a parallel transport chord \( x_i \) from the \((i-1)\)st boundary brane to the \(i\)th boundary brane, along with a parallel transport chord from \( L_0 \) to \( L_k \).

Suppose that \( v : \Sigma \to E \) is a \((j,J)\)-holomorphic section such that

• \( v(\partial \Sigma) \subset \mathcal{L} \), and
• \( v \) converges to the parallel transport chords \( x_i \) along the strip-like ends.

Then there exists some subset \( A \subset E \), proper over \( \Sigma \), and depending only on the \( x_i \), such that \( \text{image } v \subset A \).

**Proof.** By Proposition 4.11 we have

\[
\Delta (r \circ v) \omega_\Sigma = \left( \frac{1}{2} |(dv)^v|^2 + f(v) \right) \beta
\]

wherever the image of \( v \) is contained in the cylindrical region of \( E \). (That is, when \( v \) has image outside of the set \( K \) of the Proposition.)

We first consider the case of nonnegative curvature, i.e., \( f \geq 0 \). By the nonnegativity hypothesis, \( r \circ v \) is a subharmonic function on \( \Sigma \). Therefore it cannot have any local maximum at an interior point and we have only to check its boundary behavior.

We compute the radial derivative \( \frac{\partial (r \circ v)}{\partial \nu} \)—i.e., the derivative along an outward pointing boundary vector:

\[
\frac{\partial (r \circ v)}{\partial \nu} = dr \left( \frac{\partial v}{\partial \nu} \right) = \Theta \circ J \left( \frac{\partial v}{\partial \nu} \right).
\]

Since \( v \) is \((j,J)\) holomorphic, we have

\[
J \left( \frac{\partial v}{\partial \nu} \right) = J \circ dv \left( \frac{\partial}{\partial \nu} \right) = dv \left( J \frac{\partial}{\partial \nu} \right).
\]

Therefore if \( \partial/\partial \tau \) is any positively oriented tangent vector along \( \partial \Sigma \), we have

\[
\frac{\partial (r \circ v)}{\partial \nu} = -\Theta \left( \frac{\partial v}{\partial \tau} \right) \leq 0
\]

by the nonnegativity assumption of \( \mathcal{L} \).

Therefore the strong maximum principle implies that \( r \circ u \) cannot have boundary local maximum anywhere on \( \partial \Sigma \).

For the \((-C)\)-pinched case, we have \( \inf f \ell \geq -C \) and so the function \( r \circ v \) satisfies the differential inequality

\[
\Delta (r \circ v) \geq -C, \quad \frac{\partial (r \circ v)}{\partial \nu} \leq 0
\]

At this stage, we can apply the standard elliptic estimates (see for example [GT70, Theorem 3.7]).

Another more explicit way of proceeding is to consider the function \( g = (r \circ v) - \frac{C}{2} \ell^2 \) where \( t : \Sigma \to \mathbb{R} \) is the pull-back function of the standard coordinates \( (\tau, t) \) of \( \mathbb{R} \times [0, w] \) for some \( w > 0 \) via the *slit domain representation* of the conformal structure of \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \). (See [BKO19, Section 3.2], for example.)

Then \( g \) is a subharmonic function. We can apply the strong maximum principle to the function \( g \) to conclude

\[
\sup_{z \in \Sigma_{\text{end}}} g(z) \leq R_0
\]
satisfying $\frac{\partial (r \circ v)}{\partial v} \leq 0$ since $\frac{\partial}{\partial v} = 0$ along the boundary $\partial \Sigma$. Therefore we conclude

$$\sup_{z \in \Sigma_{\text{end}}} r \circ v(z) \leq R_0 + \frac{C}{2}.$$ 

This finishes the proof.

4.5 The energy estimate

Fix a non-negative Lagrangian boundary condition $\mathcal{L}$ (Definition 4.13 and 4.15). We have that

$$\Theta|_{\mathcal{L}} = dK_\mathcal{L} + \pi^*(\kappa_\mathcal{L})$$

(4.7)

for a one-form $\kappa_\mathcal{L} \in \Omega^1(\partial \Sigma)$ which vanishes on the strip-like ends. (In our case, $\kappa_\mathcal{L} = 0$ on all of $\partial \Sigma$, but we include $\kappa_\mathcal{L}$ in what follows for the interested reader.)

The action functional on the path space

$$\mathcal{P}(L_0, L_1) = \{ \gamma \in C^\infty([0,1], M) \mid \gamma(0) \in L_0, \gamma(1) \in L_1 \}$$

is given by

$$A_{L_0, L_1}(\gamma) = -\int \gamma^* \theta + K_{L_1}(\gamma(1)) - K_{L_0}(\gamma(1)).$$

Using (4.7), we also obtain

$$\int v^* \Omega = \sum_{e \in I^-} A_{L_0, L_1}(x_e) - \sum_{e \in I^+} A_{L_0, L_1}(x_e) + \int_{\partial \Sigma} \kappa_\mathcal{L}. \quad (4.8)$$

On the other hand, we derive from (4.5)

$$\frac{1}{2} |(dv)^v|^2 = v^* \Omega - f(v) \beta$$

and hence

$$\frac{1}{2} \int_{\Sigma} |(dv)^v|^2 = \sum_{e \in I^-} A_{L_0, L_1}(x_e) - \sum_{e \in I^+} A_{L_0, L_1}(x_e) + \int_{\partial \Sigma} \kappa_\mathcal{L} - \int_{\Sigma} f(v) \beta.$$ 

Here we would like to mention that both the integrals $\int_{\partial \Sigma} \kappa_\mathcal{Q}$ and $\int_{\Sigma} f(v) \beta$ are finite since $\kappa_\mathcal{Q} = 0$ and $f(v) = 0$ on the strip-like region of $\Sigma$. We also have

$$\frac{1}{2} \int_{W} |(dv)^h|^2 = \int_{W} (f(v) + 1) \beta = \int_{W} f(v) \beta + \int_{W} \beta$$

for any compact domain $W \subset \Sigma$.

We summarize the above discussion into the following uniform upper bound for the energy on any compact domain $W \subset \Sigma$ satisfying the property that $\kappa_\mathcal{Q} = 0 = f(v)$ on $\Sigma \setminus W$.

**Proposition 4.21.** Let $(\pi : E \to \Sigma, \Omega)$ and $\mathcal{L}$ be as above and $W \subset \Sigma$ be any given compact subdomain of $\Sigma$. Then

$$\frac{1}{2} \int_{W} |dv|^2 = \int_{\Sigma} v^* \Omega + \int_{W} \beta$$

(4.9)

for any $(j, J)$-holomorphic section $v : \Sigma \to E$. 27
Remark 4.22. The reason why we restrict to compact domain $W \subset \Sigma$ is that the form $\beta$ may not be integrable, unlike $f(v)\beta$. Moreover, the integrals above depend on the section $v$ and may not be uniformly bounded, mainly because the strip-like regions of $\Sigma = D^2 \setminus \{z_0, \ldots, z_k\}$ vary depending on the configuration of $\{z_0, \ldots, z_k\}$.

Remark 4.23. By requiring translation invariance of $\omega_\Sigma$ on the strip-like ends of $\Sigma$, we conclude that the full integral $\int_\Sigma \omega_\Sigma$ is infinite whenever there is at least one puncture on $\Sigma$. In choosing the 2-form $\beta = \ell \omega_\Sigma$, there are two competing interests:

- One one hand, we need the form $\Omega + \pi^* \beta$ to be nondegenerate,
- On the other hand, we wish to make the form $\beta$ have finite integral over $\Sigma$.

In general we cannot achieve both wishes simultaneously. This is the reason why we need to consider the horizontal energy on compact domains $W$ e.g., on $W = \Sigma \setminus \Sigma^{\text{end}}$.

Remark 4.24. However, because we are given a connection that is translation-invariant on the strip-like ends, when we restrict $v$ along a strip-like end, we may write

$$v(\tau, t) = (\tau, t, u(\tau, t))$$

where $u$ is a function satisfying

$$\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(\tau, t, u) \right) = 0.$$ 

This equation can be studied in the standard way of classical Floer theory.

Therefore with the uniform $C^0$-estimates at our disposal, we can apply the Gromov-Floer type of compactness arguments to the moduli space of pseudoholomorphic sections. (See [Sei03, Section 2.4] for some relevant details.)
5 Non-wrapped Fukaya categories in Liouville bundles

The present section is occupied by the construction of the $A\infty$ category $\mathcal{O}_j$ associated to a simplex $j : |\Delta^n| \to B$. As usual we have fixed a Liouville bundle $E \to B$.

5.1 Choice of objects

**Choice 5.1 (Lb and a partial ordering.).** For every point $b \in B$, we choose a countable collection $\mathbb{L}_b$ of eventually conical branes in the fiber $E_b$. We moreover choose a function $w : \mathbb{L}_b \to \mathbb{Z}_{\geq 0} = \{ n \in \mathbb{Z}, n \geq 0 \}$.

We will often abbreviate a pair $(L_b, w(L_b))$ in the graph of $w$ by $L^{(w)}$, omitting $b$, and omitting the dependence of $w$ on $L_b$.

For a fixed $b \in B$, the collection $\{L_b\}$ is countable, so we may choose the function $w$ so that given two branes $L$ and $L'$, $L$ and $L'$ are either transverse (in the fiber), or $L = L'$ and $w = w'$.

**Remark 5.2.** In [OT19], we will choose the ordering $w$ to encode cofinal sequences of non-negative wrappings of branes. We don’t need these details in the present work, so we refer the reader to Section 2.2 of [OT19] for more.

5.2 Choices of Floer data

**Remark 5.3.** Because we have already chosen a favorite collection of branes (Choice 5.1), any time we discuss a Lagrangian brane here, we assume that it equals $L^{(w)}$ for some $w \in \mathbb{Z}_{\geq 0}$ and some $L \in \mathbb{L}_b$ (for some $b \in B$). In particular, if two Lagrangians are in the same fiber $E_b$, they are either equal or transverse. This is not strictly necessary, but it will make certain things easier. See also Warning 5.8.

**Notation 5.4 ($\vec{L}$).** Recall we have fixed a Liouville bundle $E \to B$. Fix $d \geq 0$ and a smooth map $h : |\Delta^d| \to B$. We denote by $
abla = (L_0, \ldots, L_d)$ an ordered $(d + 1)$-tuple of branes with $L_i \subset h^*E$ contained above the $i$th vertex (Definition 3.2) of $|\Delta^d|$.

**Remark 5.5.** In later notation, $h$ will play the role of the composite $\beta \circ j$. (See Definition 5.18 and Remark 5.19.)

Consider an ordered $(d + 1)$-tuple $\vec{L}$ (Notation 5.4).

Because $|\Delta^d|$ is a smooth manifold with corners, one can construct a global 1-form $\Theta_\vec{L} \in \Omega^1(h^*E; \mathbb{R})$ realizing a Liouville structure on each fiber of $h^*E$. (Remark 2.34.) Using the natural system maps $\nu_{id} : \delta^0_{d+1} \to |\Delta^d|$, we may also choose almost-complex structures $\mathbb{J}$ on $\nu_{id}^*h^*E$ suitable for counting sections. (Definition 2.44.)

We now specify the choices we make to guarantee that the moduli spaces of holomorphic sections $\mathbb{S}_r \subset \delta^0_{d+1} \to |\Delta^d| \to h^*E$ are well-behaved moduli spaces.
Choice 5.6 (Liouville forms and almost complex structures suitable for counting sections). We begin with $d = 0$. Note that $\Theta_L$ is simply a choice of Liouville structure on the fiber of $E \to B$ determined by $h$, and likewise for $\mathbb{J}_L$. Given this choice of $\Theta_L$, we may assume that each brane $L_i$ admits a primitive $f_i : L_i \to \mathbb{R}$ such that $df_i = \Theta_L|L_i$ and such that $f_i$ has compact support for all $i = 0, \ldots, d$. (In other words, we require that $[f^* \Theta] = 0$ in $H^1_1(L; \mathbb{R})$). (Any brane admits a deformation so that this holds.)

We proceed inductively on $d$. Assume that for all $d' < d$, for all $h' : |\Delta^{d'}| \to B$, and for all $(d' + 1)$-tuples $\vec{L} = (L'_0, \ldots, L'_{d'})$, we have chosen $(\Theta_{L'}, \mathbb{J}_{L'})$ on $(h')^* E$.

Fix an ordered $(d + 1)$-tuple $\vec{L}$. We choose $(\Theta_{\vec{L}}, \mathbb{J}_{\vec{L}})$ on $h^*E$ subject to the following conditions:

(Θ1) (Constancy implies constancy.) If $h$ is constant, then so is $(\Theta_{\vec{L}}, \mathbb{J}_{\vec{L}})$. More concretely, if $h : |\Delta^d| \to B$ is constant, then for a constant map $p : |\Delta^d| \to v$ to some (hence every) point of $|\Delta^d|$, we have that $\Theta_{\vec{L}} = p^* \Theta_{\vec{L}}|((h^*E)_v)$ and $\mathbb{J}_{\vec{L}} = p^* (\mathbb{J}_{\vec{L}})|((h^*E)_v)$.

(Θ2) (Inductive step.) If $\vec{L}' \subset \vec{L}$ is an order-preserving inclusion, consider the induced map $|\Delta^{d'}| \to |\Delta^d|$. Then the data $(\Theta_{\vec{L}'}, \mathbb{J}_{\vec{L}'})$ is equal to the pullback of $(\Theta_{\vec{L}}, \mathbb{J}_{\vec{L}})$ along this induced map.

(Θ3) (Smooth collaring.) Recall the collaring choices from Choice 3.21. Suppose that $F \subset |\Delta^d|$ is a codimension one face. $F$ in particular determines an ordered $d$-tuple $\vec{L}' \subset \vec{L}$. We demand

$$\pi^*_F \Theta_{\vec{L}'} = \Theta_{\vec{L}}|U_F, \quad \pi^*_F \mathbb{J}_{\vec{L}'} = (\mathbb{J}_{\vec{L}})|U_F$$

where $\pi_F$ and $U_F$ are as in Choice 3.21.

(Θ4) (Transversality.) For any $0 \leq i < j \leq d$, let $\Pi_{ij}$ be the parallel transport along the simplicial edge from the $i$th vertex of $|\Delta^d|$ to the $j$th. (See Definition 2.38.) We demand that $\Pi_{ij}(L_i)$ and $L_j$ are transverse.

(Θ5) (Regularity.) Further, we demand that the associated linearized del-bar operators are regular, so that the holomorphic disk moduli spaces (see Definition 5.18) are smooth manifolds.

(Θ6) (Coherent barriers) Finally, we demand that there exists some neighborhood of $\partial h^* E$ such that, with respect to some global function $\pi : \text{Nbd}(\partial h^* E) \to \mathbb{C}_{\mathbb{R}e\geq 0}$ as in Remark 2.47, $\pi$ is $(\mathbb{J}_{\vec{L}})|_{VTE}$-holomorphic.

Remark 5.7. We may now further motivate the collaring choices made for simplices in Choice 3.21. If one chooses the above $\Theta_{\vec{L}}$ without collaring conditions, there is no guarantee that the $\Theta_{\vec{L}}$ glue smoothly along faces of a simplex.

Warning 5.8. The reader may be irked by an apparent incompatibility between (Θ1) and (Θ4). As stated, it is impossible to satisfy both conditions unless the branes $L_0, \ldots, L_d$ are a priori assumed transversal. This is the reason for Choice 5.1; see also Remark 5.3.

Remark 5.9. Recall we have fixed a natural system (Choice 3.19). We may pull back our choices $(\Theta_{\vec{L}}, \mathbb{J}_{\vec{L}})$ along the map $\mathbb{S}_r \subset \mathbb{S}^d_{d+1} \xrightarrow{v_r} |\Delta^d|$. Then by (NS1), along the strip-like ends, all our choices are translation-invariant. (Here, translation is by $[0, \infty)$ or by $(-\infty, 0]$ as parametrized by the strip-like end.)

---

6 When $d = 0$, a choice of $\mathbb{J}_{\vec{L}}$ suitable for counting sections is simply a choice of almost-complex structure $J$ on $E_b = E$ for which $J$ is compatible with the symplectic form and eventually cylindrical (Definition 2.39).

7 See also Warning 5.8 regarding compatibility with (Θ1).
Remark 5.10. We have used the notion of pulling back $J \mathcal{L}$—a choice of almost-complex structure suitable for counting sections (Definition 2.44)—in articulating the conditions of Choice 5.6. We note that this pullback is defined by utilizing the natural systems from Choice 3.19.

Proposition 5.11. There exist choices $\{(\Theta \mathcal{L}, J \mathcal{L})\}_{\mathcal{L}}$ satisfying all the conditions in Choice 5.6.

Proof. This follows from a standard argument using induction; see for example Lemma 3.8 and Section 3.3 of [Sav13]. Perhaps the main point to note in our present work is how to choose the $\Theta \mathcal{L}$ compatibly. Given the bundle $h^*E \to |\Delta^d|$, the space of 1-forms $\Theta$ on $h^*E$ for which the fiberwise restrictions are Liouville forms is a smoothly contractible space (for example, it is easy to see that the space is convex). This contractibility is a necessary ingredient in the inductive step, as one must extend $\Theta$ from the boundary of an $n$-simplex to its interior.

When one must also account for brane structures (such as a trivialization of $\det^2(TM)$—see Remark 2.32), the fact that the structure group has been reducted to $\text{Aut}$ as opposed to $\text{Aut}^0$ allows for this inductive step: By construction, the relevant trivialization from the boundary of an $n$-simplex extend to its interior. \hfill \Box

5.3 A non-wrapped Fukaya category over a simplex

Fix a smooth map $j : |\Delta^n_e| \to B$. (Note $|\Delta^n_e|$ is an extended simplex as in Notation 3.1.)

We define in this section the non-wrapped, directed Fukaya category $\mathcal{O}_j$ associated to $j$. The definition is inductive on $n$—we first define $\mathcal{O}_j$ for all $j$ having domain of dimension $\leq n$, then for those $j$ with domain having dimension $n+1$.

Remark 5.12. The reader will note that $\mathcal{O}_j$ only depends on the restriction of $j$ to the standard simplex $|\Delta^n| \subset |\Delta^n_e|$. The reason we insist on the domain of $j$ being the extended simplex $|\Delta^n_e|$ is to make use of homotopy-theoretic results concerning diffeological spaces; the technical reasons for this will not arise prominently in this paper, so we refer the reader to [OT20b].

Notation 5.13 ($b_i$ and $\mathbb{L}_{b_i}$). For every $0 \leq i \leq n$, let $b_i$ be the image of the $i$th vertex (Definition 3.2) of $|\Delta^n_e|$ under $j$.

Recall we have chosen a countable collection of branes and an order on these (Choice 5.1). In particular, $\mathbb{L}_{b_i}$ denote the countable collection of branes associated to $b_i$.

Definition 5.14 (Objects). An object of $\mathcal{O}_j$ is a pair $(i, L)$ where

- $i \in \{0, \ldots, n\}$ and
- $L \in \mathbb{L}_{b_i}$.

To emphasize the role of the ordering $w$ we have chosen, we will often write an object as a triple $(i, L, w)$

where $w = w(L)$. (See Choice 5.1.) We will also write this same object as

$L^{(w_i)}_i$

from time to time.
Notation 5.15 (Parallel transport $\Pi$). Fix a pair of objects $(L_0, i_0, w_0)$ and $(L_1, i_1, w_1)$. The integers $i_0$ and $i_1$ define a simplicial map $\beta : |\Delta^1| \to |\Delta^n| \subset |\Delta^m|$ sending the initial vertex of $|\Delta^1|$ to $i_0$ and the final vertex to $i_1$.

We let $h = j \circ \beta$. One also has an underlying ordered pair of branes $\bar{L} = (L_0, L_1)$. (The notation here is to be consistent with Notation 5.4.)

Because we have chosen $\Theta_E$ for $h^*E$ (Choice 5.6), we have a parallel transport taking the initial fiber of $h^*E$ (i.e., the fiber above the initial vertex of $|\Delta^1|$) to the final fiber of $h^*E$.

We let $\Pi_{i_0,i_1}$ denote this parallel transport.

We will render $\mathcal{O}_j$ to be directed in the $w$ index; this means that the morphism complex from $(i, L, w)$ to $(i', L', w')$ will be zero unless $w < w'$, or $(i, L, w) = (i', L', w')$ (in which case the morphism complex is just the ground ring $R$ in degree 0). Concretely:

Definition 5.16 (Morphisms). Fix two objects $(i_0, L_0, w_0)$ and $(i_1, L_1, w_1)$ of $\mathcal{O}_j$.

We define the graded abelian group

$$\text{hom}_{\mathcal{O}_j}((i_0, L_0, w_0), (i_1, L_1, w_1))$$

to be

$$\begin{cases} \bigoplus_{x \in \Pi_{i_0,i_1}(L_0^{(w_0)} \cap L_1^{(w_1)})} \mathfrak{o}_x [-|x|]. & w_0 < w_1 \\ R & (i_0, L_0, w_0) = (i_1, L_1, w_1) \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\Pi_{i_0,i_1}$ is the parallel transport map (Notation 5.15). We also note that $\mathfrak{o}_x$ is the orientation $R$-module of rank one associated to the intersection point $x$, and $|x|$ is the Maslov index associated to the brane data.

Remark 5.17. The set $x \in \Pi_{i_0,i_1}(L_0) \cap L_1$ is also in bijection with the set of flat sections of $h^*E \to |\Delta^1|$ (with respect to $\Theta_{(L_0,L_1)}$) beginning at $L_0^{(w_0)}$ and ending at $L_1^{(w_1)}$. (See Notation 5.15.)

Now we define the operation $\mu^d$ for $d \geq 1$.

Definition 5.18 ($\mu^d$ for the non-wrapped categories). As usual, fix a smooth map $j : |\Delta^n| \to B$. For $d \geq 1$, fix a collection

$$\bar{L} = \{(i_0, L_0, w_0), \ldots, (i_d, L_d, w_d)\}.$$

We may assume $w_0 < \ldots < w_d$ by Definition 5.16 (otherwise $\mu^d$ is forced to be 0).

Note that the integers $i_0, \ldots, i_d$ induce a simplicial map $\beta : |\Delta^d| \to |\Delta^n| \subset |\Delta^m|$ by sending the $a$th vertex of $|\Delta^d|$ to the $i_a$th vertex of $|\Delta^n|$. (This assignment, of course, need not be order-preserving.) Recall the map $\nu_\beta : \mathbb{S}^d_{d+1} \to |\Delta^n|$ as in Choice 3.19.

For a given collection of intersection points

$$x_a \in \Pi_{i_{a-1},i_a}(L_{a-1}^{(w_{a-1})} \cap L_a^{(w_a)}) \quad (a = 1, \ldots, d)$$

and

$$x_0 \in \Pi_{i_0,i_d}(L_0^{(w_0)} \cap L_d^{(w_d)}),$$

we define

$$\mathcal{M}(x_d, \ldots, x_1; x_0) \quad (5.1)$$
to be the moduli space of holomorphic sections \( u \)

\[
\begin{array}{c}
S_r \xrightarrow{\nu} S_{d+1} \xrightarrow{\nu_\beta} |\Delta^d| \xrightarrow{\beta} |\Delta^u| \subset |\Delta^n| \xrightarrow{\jmath} B
\end{array}
\]

satisfying the following boundary conditions:

1. Along the strip-line end near the \( a \)th puncture of \( S \), \( u \) converges to the parallel transport chord from \( L_{a-1}^{(u_a)} \) to \( L_a^{(u_a)} \) determined by \( x_a \).

2. Along the \( a \)th boundary arc of \( S \), but outside the strip-like ends, \( u \) is contained in the Lagrangian \( L_a^{(u_a)} \subset E_{b_{i_a}} \). Note this makes sense due to the canonical trivialization of \( E|_{\text{arc}} \cong E|_{b_{i_a} \times \text{arc}} \); this is a consequence of (NS2).

As usual, the brane structures on the \( L^{(w)} \) allow us to orient these moduli spaces, and predict their dimension based on the degrees of the \( x_a \). We define

\[
\mu^d(x_d, \ldots, x_1) = \sum_{x_0} \#M(x_d, \ldots, x_1; x_0) x_0
\]

where the number \( \#M \) is counted with sign. In case our branes are not \( \mathbb{Z} \)-graded, we as usual declare the \( x_0 \) coefficient of \( \mu^d \) to be zero when there is no zero-dimensional component of \( M(x_d, \ldots, x_1; x_0) \).

**Remark 5.19.** Given an ordered \((d+1)\)-tuple of objects in \( O_j \) with underlying branes \( \vec{L} \), consider the induced map \( \beta : |\Delta^d| \to |\Delta^u| \subset |\Delta^n| \). The \( A_\infty \)-operations are defined by moduli spaces depending only on \( h = j \circ \beta \). (This follows from Definition 5.18 and (\( \Theta1 \), (\( \Theta2 \). Note that \( h \) is the same \( h \) as in Notation 5.4.)

**Definition 5.20.** Fix \( j : |\Delta^n| \to B \). We let \( O_j \) denote the \( A_\infty \)-category where

- an object is the data of a brane \( L^{(w)} \) in one of the vertex-fibers (as in Definition 5.14),
- \( \text{hom}_{O_j}(L_0^{(w_0)}, L_1^{(w_1)}) \) is as in Definition 5.16,
- The operations \( \mu^d \) are as in Definition 5.18.

**Remark 5.21.** When a \( \mu^d \) operation involves an element of an endomorphism hom-complex \( \text{hom}_{O_j}(L, L) = R \), the operation is fully determined by demanding that the unit of the ring \( R \) be a strict unit of the \( A_\infty \)-category.

### 5.4 Proof of Theorem 1.1

**Proof.** Because we have already set up the painstaking details, the theorem will be a standard consequence of (i) regularity, (ii) Gromov compactness for holomorphic sections, and (iii) verifying that 1-dimensional moduli compactify in the usual way.

(i) Conditions (\( \Theta4 \)) and (\( \Theta5 \)) guarantee that our moduli are manifolds.

(ii) We established in Section 4 the estimates required for the standard Gromov compactness results for holomorphic sections in the Liouville setting.

(iii) Finally, we note that condition (\( \Theta2 \)), together with (NS3) and (NS4), allow us to compactify the \( d \)-ary moduli space using products of \( d' \)-ary moduli for \( d' < d \).

In particular, thanks to the operadic comparability conditions spelled out in (NS1) - (NS4), the usual Gromov-Floer compactification give rise to the following:
Proposition 5.22. Let $x = (x_d, \ldots, x_1)$ and consider the moduli space $\mathcal{M}(x; x_0)$ from (5.1). Then $\mathcal{M}(x; x_0)$ admits a compactification $\overline{\mathcal{M}}(x; x_0)$ whose boundary $\partial \overline{\mathcal{M}}(x; x_0) = \overline{\mathcal{M}}(x; x_0) \setminus \mathcal{M}(x; x_0)$ is naturally identified with the union
\[
\bigcup_{x \in L_i^{(w_i)}} \overline{\mathcal{M}}(x^1; x_0) \times \overline{\mathcal{M}}(x^2; x).
\]
Here, $0 \leq i \leq d - j$ and
\[
x^1 = (x_d, \ldots, x_{i+j+1}, \bar{x}, x_i, \ldots, x_1), \quad x^2 = (x_{i+j}, \ldots, x_{i+1}).
\]

As usual, this description of $\partial \overline{\mathcal{M}}(x; x_0)$—applied to the case of $\dim \overline{\mathcal{M}}(x; x_0) = 1$—guarantees that the $A_\infty$ relations hold.

To finish the proof of Theorem 1.1, we must only prove that an injective simplicial map $\iota : |\Delta^n| \to |\Delta^m'|$ induces a fully faithful functor $\mathcal{O}_j \to \mathcal{O}_{j'}$.

We define the functor to send an object $(i, L, w)$ to the object $(\iota(i), L, w)$. Because the morphism complexes from $(i, L, w)$ to $(i', L', w')$ depends only on the composite map
\[
|\Delta^1| \to |\Delta^n| \to |\Delta^m'| \to B
\]
(the first arrow is the simplicial map sending the initial vertex of $|\Delta^1|$ to $i$, and the terminal vertex to $i'$), that $\iota \circ j' = j$ means the morphism complexes admit a natural isomorphism
\[
\hom_{\mathcal{O}_j}((i, L, w), (i', L', w')) \cong \hom_{\mathcal{O}_{j'}}((\iota(i), L, w), (\iota(i'), L', w')).
\]
Finally, Property (Θ2) guarantees that the moduli spaces of holomorphic sections defining the $A_\infty$ operations are also in natural bijection. (See Remark 5.19.) Thus the functor $\mathcal{O}_j \to \mathcal{O}_{j'}$ is fully faithful.

Example 5.23. Suppose $j : |\Delta_c^0| = |\Delta^0| \to B$ is the data of a point of $b \in B$. If the ordering function $w$ of Choice 5.1 is chosen to yield cofinal sequences of non-negative wrappings (see Remark 5.19), cofinal sequences) $\mathcal{O}_j$ is equivalent to the non-wrapped category $\mathcal{O}$ that [GPS17] associates to the fiber $E_b$ above $b$. This is because the base case of $n = 0$ in Choice 5.6 implies that the boundary conditions for the holomorphic sections $u$ reduce to strip-like ends converging to intersection points $L_{a-1} \cap L_a$ in $M$. (When the pull-back bundle is canonically trivialized as $E_b \times |\Delta^d|$, sections are equivalent to maps $u : S \to E_b$.)

6 Continuation maps

Let $M$ be a Liouville manifold. If $L_0$ is a compact brane, any Hamiltonian isotopy from $L_0$ to $L_1$ induces an element in Floer cohomology $HF^*(L_0, L_1)$; this element is usually referred to as the continuation map, or sometimes the continuation element, associated to the isotopy.

Suppose $L_0$ is now a brane in a Liouville manifold. If $L_0$ is not compact and the Hamiltonian isotopy is not compactly supported, one must further impose the restriction that the isotopy be non-negative to construct the continuation map (Definition 2.26). Non-negativity yields the necessary $C^0$ and energy bounds to achieve Gromov compactness for moduli of disks (and continuation maps are constructed by counting holomorphic disks); see Theorem 4.20.

Throughout, we fix a Liouville manifold $M$ along with an exact Lagrangian isotopy of eventually conical branes
\[
\mathcal{L} : [0, 1] \times L_0 \to M
\]
in $M$. We assume $\mathcal{L}$ is non-negative, and we review two constructions of continuation elements associated to $\mathcal{L}$. 34
6.1 Using once-punctured disks

We assume that

1. this isotopy is non-negative,

2. For each \( s \), the image of the time \( s \) embedding \( L_s \) is an (eventually conical) brane (Definition 2.25), and

3. \( L_0 \) is transverse to \( L_1 \).

**Choice 6.1** (Choices for defining continuation map). Choose a marked point \( z_0 \in \partial D^2 \) and consider the Riemann surface with boundary \( D^2 \setminus \{z_0\} \). We equip \( D^2 \setminus \{z_0\} \) with a strip-like end near \( z_0 \). Further, we choose a function

\[
\chi : \partial D^2 \to [0, 1]
\]

such that \( \chi \) is weakly increasing (with respect to the boundary orientation on \( \partial D^2 \)), is locally constant outside a compact set, and is onto.

**Remark 6.2.** The non-negativity of the isotopy \( \mathcal{L} \) guarantees the usual \( C^0 \)-estimates.\(^8\) Thus the finite energy condition implies that as \( z \to z_0 \), the map \( u \) converges exponentially (with respect to the strip-like coordinates near \( z_0 \)) to a constant path supported at an intersection point \( x \in L_0 \cap L_1 \).

**Notation 6.3.** Remark 6.2 enables us to define the evaluation map

\[
ev_{z_0} : \mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi) \to L_0 \cap L_1.
\]

We denote

\[
\mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x) := \ev_{z_0}^{-1}(x)
\]

(6.2)

for each \( x \in L_0 \cap L_1 \).

**Proposition 6.4** (Theorem C.3.1 [Oh15b]). For a generic choice of isotopy \( \mathcal{L} \), \( \mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x) \) is a smooth manifold of dimension given by

\[
\dim \mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x) = \frac{n}{2} - \mu_\mathcal{L}(x)
\]

where \( \mu_\mathcal{L}(x) \) is the Maslov index of \( x \) relative to \( \mathcal{L} \).

**Remark 6.5.** The dimension count of Proposition 6.4 is compatible with the grading on \( \text{CF}^*(L_0, L_1) \), in the sense that

\[
|x| = \dim \mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi) = \frac{n}{2} - \mu_\mathcal{L}(x).
\]

(6.3)

**Remark 6.6.** This definition of the cohomological degree is adopted because we put the output at \(-\infty\) in the definition of the Floer moduli space. Another choice would be to take \( |x| \) to be the codimension instead of the dimension of the relevant moduli space if the output were put at \( \infty \).

**Construction 6.7.** We define a Floer cochain

\[
c_\mathcal{L}^\chi := \sum_{x \in L_0 \cap L_1; |x| = 0} n_\mathcal{L}^\chi(x) \langle x \rangle
\]

(6.4)

where \( n_\mathcal{L}^\chi(x) = \#(\mathcal{M}(D^2 \setminus \{z_0\}; \mathcal{L}^\chi)) \) (counted with sign as usual using orientations).

---

\(^8\)That is, the images of all the \( v \) are contained in an a-priori-determined compact subset of \( M \). See Section 4.
Proposition 6.8. The cochain $c^\chi_L$ is a cocycle. Moreover, its Floer cohomology class $[c^\chi_L] \in HF^0(L_0, L_1)$ is independent of the choice of $\chi$.

Proof of Proposition 6.8. We compute the matrix coefficient of the Floer coboundary $\mu^1(c^\chi_L)$. For each given $y \in L_0 \cap L_1$ with $|y| = 1$, we compute its coefficient in the linear expression of $\mu^1(c^\chi_L)$

$$\langle \mu^1(c^\chi_L), y \rangle = \sum_{x \in L_0 \cap L_1} n^\chi_L(x) \langle \mu^1(\langle x \rangle), \langle y \rangle \rangle = \sum_{x \in L_0 \cap L_1} \#M(y, x) n^\chi_L(x).$$

Here—by the standard compactness-and-gluing theorems—the last sum is nothing but the count of the boundary elements of compact one-dimensional manifold $M(D^2 \setminus \{z_0\}; \mathcal{L}, y)$; it hence vanishes. This proves $\langle \mu^1(c^\chi_L), y \rangle = 0$ for all $y$ with $|y| = 1$ and so $\mu^1(c^\chi_L) = 0$. Therefore $c^\chi_L$ defines a Floer cohomology class in $HF^0(L_0, L_1)$.

Now the standard compactness-cobordism argument proves the second statement noting that the space of elongation functions $\chi$ is contractible (and in particular, connected).

Definition 6.9 (The continuation element). Let $\mathcal{L} : [0, 1] \times L_0 \to M$ be a non-negative, exact Lagrangian isotopy from $L_0$ to $L_1$. We denote by

$$c^\chi_L \in HF^0(L_0, L_1)$$

the cohomology class associated to the cochain in Construction 6.7. We call it the continuation element associated to the isotopy.

6.2 Using strips

Choice 6.10 ($\rho$). We fix an elongation function $\rho : \mathbb{R} \to [0, 1]$ given by

$$\rho(\tau) = \begin{cases} 1 & \text{for } \tau \geq 1 \\ 0 & \text{for } \tau \leq 0 \end{cases}, \quad \rho'(\tau) > 0 \text{ for } 0 < \tau < 1.$$

Remark 6.11. For our purposes, any weakly monotone $\rho$ with value 0 near $-\infty$ and 1 near $\infty$ will suffice; we note that the space of such $\rho$ is contractible.

Notation 6.12. Given an exact Lagrangian isotopy $\mathcal{L}$ and an elongation function $\rho$ as in Choice 6.10, we denote by

$$\mathcal{L}^\rho : \tau \mapsto L_{\rho(\tau)}.$$

the induced $\mathbb{R}$-parametrized isotopy.

Choice 6.13. We also choose a smooth, 2-parameter family of eventually conical almost-complex structures on $M$ (Definition 2.39)

$$[0, 1] \times [0, 1] \to \{\text{Eventually conical } J\}, \quad (s, t) \mapsto J_{(s, t)}.$$

Construction 6.14 (Construction using strips). Fix an exact Lagrangian isotopy $\mathcal{L}$ and a brane $K$ such that $K \pitchfork L_i$ for $i = 0, 1$. The Floer continuation map

$$h^\rho_L : CF(K, L_0) \to CF(K, L_1)$$
is defined by counting isolated solutions of the following system:

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J(\rho(\tau), t) \frac{\partial u}{\partial t} &= 0 \\
u(\tau, 0) &\in K, \quad u(\tau, 1) \in L_{\rho(1-\tau)}.
\end{align*}
\] (6.6)

See Figure 1.2.

A similar argument to the proof Proposition 6.8 shows the following:

**Proposition 6.15.** $h_\rho^L$ is a chain map. Moreover, the map on cohomology

\[ [h_L] : HF(K, L_0) \rightarrow HF(K, L_1) \]

is independent of the choice of elongation function $\rho$.

**Remark 6.16.** Recall that in our constructions of the continuation elements and the Floer continuation map in the previous Subsections, we have used two different kind of elongation functions, $\chi$ and $\rho$ (Choices 6.1 and 6.10) where the domain of $\chi$ is $\partial D^2 \setminus \{z_0\}$ and the domain of $\rho$ is $\mathbb{R}$. Different choices of such functions define the same map in cohomology. However, when we study compatibility between the strip and disk definitions of continuation maps, which requires us to examine a family of moduli spaces, we will need to exhibit a compatibility between $\chi$ and $\rho$. We will again have some freedom in exhibiting this compatibility; see (6.15).

### 6.3 Recollections on $\overline{M}_4$

We start our proof with considering the configuration space $M_4$ of four boundary marked points of the unit disc modulo the action of $PSL(2, \mathbb{R})$. We denote an element thereof by an equivalence class $[z]$ of the tuple

\[ z = (z_0, z_1, z_2, z_3). \]

It is easy to see that $M_4$ is diffeomorphic to the open unit interval and its canonical compactification by stable curves, denoted by $\overline{M}_4$, is obtained by adding two points on the boundary of the open interval. Each of these two points represents a singular disc with two irreducible components.

More specifically, modulo the action of $PSL(2; \mathbb{R})$, we may assume $z_0 = -1 \in \partial D^2$. Then we consider a diffeomorphism $\tau : M_4 \rightarrow \mathbb{R}_{>0}$ given by the (real) cross ratio,

\[ \tau([1, z_1, z_2, z_3]) = \frac{w_2 - w_3}{w_1 - w_2}; \quad w_i = \log z_i \] (6.7)

where $w_i \in \partial \mathbb{H} \subset \mathbb{C}$. (Here we take the logarithm $w_i = \log z_i$ with respect to the branch cut along the positive real axis.)

**Notation 6.17 ($\varphi_r$).** For later use, for each $r \in (0, \infty)$ we denote by

\[ \varphi_r : D^2 \setminus \{z_0, z_1\} \rightarrow \mathbb{R} \times [0, 1] \]

the unique conformal map satisfying

\[
\begin{align*}
\varphi_r(z_0) &= -\infty, \\
\varphi_r(z_1) &= \infty, \\
\varphi_r(z_2) &= (r, 1), \\
\varphi_r(z_3) &= (-r, 1)
\end{align*}
\] (6.8)

for $r = \tau([z_0, z_1, z_2, z_3])$. 37
This realization on $\mathbb{R} \times [0,1]$ (of the unit disk’s boundary points as prescribed by elements of $\mathcal{M}_4$) will be important in the study of the continuation equation and its relationship with the continuation element.

The two boundary points of $\mathcal{M}_4$ represent singular curves of the types

$$r^{-1}(0) = (D^2, (z_0, z_1, \zeta)) \# (D^2, (\zeta, z_2, z_3)),$$
$$r^{-1}(\infty) = (D^2, (z_0, \zeta, z_3)) \# (D^2, (\zeta, z_1, z_2)).$$ (6.9)

Under the above diffeomorphism $r : \overline{\mathcal{M}}_4 \to [0, \infty]$, the unique conformal map $\varphi_r : D^2 \setminus \{z_0, z_1\} \to \mathbb{R} \times [0,1]$ respects degenerations of the domain and of the target and we can express the limit of the sequence $[z_0, z_1, z_2, z_3]$ in $\mathcal{M}_4$ at $r = \infty$ as a join of the morphisms between two stable curves

$$\psi : (D^2, (z_0, z_1, \zeta)) \to (\mathbb{R} \times [0,1], \{(0,1)\}),$$ (6.10)
$$\varphi : (D^2, \zeta) \to \Theta_-$$ (6.11)

where $\psi$ and $\varphi$ are uniquely defined by the symmetry (6.8) condition imposed on $\varphi_r$. We describe $\Theta_-$ in Notation 6.19 below.

**Figure 6.18.** The glued image for the composition of $\mu^2(c_L, -)$.

**Notation 6.19 ($\Theta_-$).** We denote by $\Theta_-$ the domain (equipped with the strip-like coordinates) of the relevant moduli spaces, and denote by $\Theta_- \# Z$ the nodal curve obtained by the obvious grafting. (See Figure 6.18 for the image of the grafted domain.) We mention that we have conformal equivalences $\Theta_- \cong D^2 \setminus \{z_0\}$ and $Z \cong D^2 \setminus \{z_0, z_1, z_2\}$. We take the following explicit model for $\Theta_-$. Consider the domain

$$\{z \in \mathbb{C} \mid |z| \leq 1, \text{Im} z \geq 0\} \cup \{z \in \mathbb{C} \mid |\text{Re} z| \leq 1, \text{Im} z \leq 0\}$$

and take its smoothing around $\text{Im} z = 0$ that keeps the reflection symmetry about the $y$-axis of the domain. Then we take

$$Z = \{z \in \mathbb{C} \mid 0 \leq \text{Im} z \leq 1\} \setminus \{(0,1)\}.$$ (6.12)

Again we equip $Z$ with a strip-like coordinate at $z = (0,1)$ that keeps the reflection symmetry.

**Remark 6.20.** We are using $\Theta_-$ to realize the degeneration of curves in (6.9) through a Riemann surface that is not only conformally equivalent to $D^2 \setminus \{z_0\}$ but also respects the symmetry of the kind (6.8). $\Theta_-$ will be useful in studying the degeneration of Floer moduli spaces involved in understanding how the Abouzaid functor $\mathcal{F}$ in [OT19] interacts with continuation maps.

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6.4 Proof of Theorem 1.4

In this subsection we give the proof of Theorem 1.4. We start with Proposition 6.15. Using the independence of the map \([h^g]\) on \(\rho\), we deform \(\rho\) through a suitably chosen one-parameter family \(\{\rho_r\}_{0 < r < 1}\) starting with \(\rho_1 = \rho\) whose construction is now in order. We would like to degenerate the moduli space \(M(K, L^\rho; x_-, x_+)\) to the one associated to the right-hand side of Theorem 1.4 at \(r \to 0\).

Applying \(\varphi_1\) (see Notation 6.17), let us study the collection of maps

\[
v : D^2 \setminus \{z_0, z_1\} \to M \quad \text{such that } u = v \circ \varphi_1^{-1} \text{ satisfies } (6.6)
\]

(6.13) satisfying the finite energy condition. Any such solution converges to \(x_\pm\) with \(x_+ \in K \cap L_0\) and \(x_- \in K \cap L_1\) as \(\tau \to \pm \infty\) respectively in the coordinate \((\tau, t) \in \mathbb{R} \times [0, 1]\). Given points \(x_-\) and \(x_+\), we denote by

\[
M(K, L^\chi; x_-, x_+).
\]

(6.14) the moduli space of (equivalence classes of) pairs \((v; z_0, z_1)\), where \(v\) is a solution of (6.13) converging to \(x_-\) and \(x_+\), modulo the biholomorphisms of \(D^2\). This is naturally isomorphic to \(M(K, L^\rho; x_-, x_+)\) if we set \(\chi = \rho \circ \varphi\) by definition. Therefore to make the following discussion consistent with the compactification of \(\mathcal{M}_4\), we will degenerate the moduli space (6.14) instead by deforming \(\chi\) used in Choice 6.1 through a one-parameter family of \(\chi\)'s parameterized by \(\mathcal{M}_4 \cong (0, \infty)\) via the diffeomorphism \(\tau : \mathcal{M}_4 \to \mathbb{R}_{\geq 0}\) as follows.

We first fix an elongation function \(\rho : \mathbb{R} \to [0, 1]\) and define

\[
\chi_r = \rho \circ \varphi_r
\]

(6.15) for \(r > 0\) and denote

\[
\rho_r = \rho \circ (\varphi_r \circ \varphi_1^{-1}) = \chi_r \circ \varphi_1^{-1}.
\]

Note that \(\rho_1 = \rho\). Then we introduce a parameterized moduli space of \((v; z)\) with \(z = (z_0, z_1, z_2, z_3)\) by adding two more marked points \(z_2, z_3\) to \((u, z_0, z_1)\). We have the natural fibration

\[
\tilde{f} : \mathcal{M}_4(K, L^\chi; x_-, x_+) \to \mathcal{M}_4
\]

whose fiber is given by \(\mathcal{M}_4(K, L^\chi; x_-, x_+)\) with \(\chi_r := \rho \circ \varphi_r\) for \(r = \tau([z_0, z_1, z_2, z_3])\): Each element of \(\mathcal{M}_4(K, L^\chi; x_-, x_+)\) is a pair

\[
(v; z_0, z_1, z_2, z_3) \quad \text{satisfying } \tau([z_0, z_1, z_2, z_3]) = r
\]

where \(v\) is defined on \(D^2 \setminus \{z_0, z_1\}\). Since adding two additional (free) marked points \(z_2, z_3\) increases the dimension by 2, we need to cut it down by putting a codimension 1 constraint on the location of each of \(v(z_2)\) and \(v(z_3)\). We do this by taking local codimension 1 slices transversal to the image of \(v\) at \(v(z_2)\) and \(v(z_3)\), respectively. For this purpose, we use the following lemma: Recall that in the situation of Theorem 1.4 the moduli space \(\mathcal{M}_4(K, L^\chi; x_-, x_+)\) is zero dimensional and compact for \(\chi = \chi_1\). In particular it consists of finitely many elements. We enumerate them by

\[
\mathcal{M}_4(K, L^\chi; x_-, x_+) = \{v^{(1)}, \ldots, v^{(N)}\}.
\]

Lemma 6.21. Suppose the moduli space \(M(K, L^\chi; x_-, x_+)\) is nonempty and regular for \(\chi = \chi_1\). Then we can choose marked points \(z^\ell = (z_0^\ell, z_1^\ell, z_2^\ell, z_3^\ell)\) so that

1. \(\tau([z_0^\ell, z_1^\ell, z_2^\ell, z_3^\ell]) = 1\),

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2. $v^{(\ell)}$ is immersed at $z_2^{\ell}$ and $z_3^{\ell}$ for all $\ell = 1, \ldots, N$.

3. there exists some $\delta_0 > 0$ such that

$$d \left( v^{(\ell)}(z_1^{\ell}), v^{(\ell')}(z_j^{\ell'}) \right) \geq \delta_0$$

for any pair $(\ell, i) \neq (\ell', j)$ with $j = 2, 3$ and $1 \leq \ell \leq N$.

The proof of this lemma—a simple application of the unique continuation of the image of pseudoholomorphic curves [FHS95, Oh97] and the implicit function theorem—is omitted.

We then pick a local transversal slice $S_i^{\ell} \subset M$ for each $i = 2, 3$ and $1 \leq \ell \leq N$ such that

- $\{S_i^{\ell}\}$ are pairwise disjoint for all $\ell$ and $i = 2, 3$,
- $v^{(\ell)}(z_i^{\ell}) \in S_i^{\ell} \cap L_{s_i^{\ell}}$ for all $\ell$ for each $i = 2, 3$,
- Both $S_i^{\ell} \cap v^{(\ell)}$ in $M$ and $(S_i^{\ell} \cap L_{s_i^{\ell}}) \ni \partial v^{(\ell)}$ hold at $z_i^0$ for $i = 2, 3$ where $s_i^{\ell} = \chi_{r^\ell}(z_i^\ell)$ with $r^\ell = r([z_i^0, z_1^\ell, z_2^\ell, z_3^\ell])$.

It follows from Condition 3 of Lemma 6.21 that by taking $S_i^{\ell}$ sufficiently small, we may also assume

$$d(S_2^{\ell}, S_3^{\ell}) > \frac{\delta_0}{2}$$

for all $\ell = 1, \ldots, N$. We denote these collection of $S_i^{\ell}$ by

$$S_i = \{S_i^{1}, \ldots, S_i^{N}\}, i = 2, 3.$$

**Proposition 6.22.** Suppose the moduli space $\mathcal{M}(K, \mathcal{L}^k; x_-, x_+)$ is nonempty and transversal. Let $S_i^{\ell}$ for $i = 2, 3$ be the collection of the local slices for the $\mathcal{M}(K, \mathcal{L}^k; x_-, x_+)$ chosen in Lemma 6.21 above. We define a subset of $\mathcal{M}_4(K, \mathcal{L}^k; x_-, x_+)$ by

$$\mathcal{M}_4^{S_2, S_3}(K, \mathcal{L}^k; x_-, x_+) = \bigcup_{i=1}^N S_i^{\ell} \ni \partial v^{(\ell)} \bigcup \bigcup_{i=1}^N \mathcal{M}_4(K, \mathcal{L}^k; x_-, x_+) \cap v^{-1}((0, 1))$$

where $z = (z_0, z_1, z_2, z_3)$. Then provided the isotopy $\{L_s\}$ is sufficiently small in fine $C^\infty$ topology, this moduli space is a smooth submanifold of $\mathcal{M}_4(K, \mathcal{L}^k; x_-, x_+) \to (0, \infty)$ of codimension 2 and so of one dimension. Moreover the same property also persists to the compactification

$$\overline{\mathcal{M}}_4^{S_2, S_3}(K, \mathcal{L}^k; x_-, x_+) \subset \overline{\mathcal{M}}_4(K, \mathcal{L}^k; x_-, x_+) \cap v^{-1}((0, 1)) :$$

1. $v$ is immersed at $z_2$ and $z_3$ for all $(v; z) \in \overline{\mathcal{M}}(K, \mathcal{L}^k_+; x_-, x_+)$,

2. $v(z_i) \in \cup_{i=1}^N S_i^{\ell}, i = 2, 3$.

Both $S_i^{\ell} \cap v$ in $M$ and $(S_i^{\ell} \cap L_{s_i^{\ell}}) \cap \partial v$ hold at $z_i$ for $i = 2, 3$ where $s_i = \chi_{r^i}(z_i)$ with $r = r([z_0, z_1, z_2, z_3])$ for all $r^{-1}((0, 1))$.

**Proof.** This is a standard local normal slice theorem; we refer readers to [FOOO09, p.424] for a proof and for the details of such a construction at the interior marked points. The current case of boundary marked points is the same except the following differences

- Here our slice $S_i^{\ell}$ is of codimension 1 in $M$ instead of codimension 2 in $M$,
• We want this slice theorem for the whole family of the moduli spaces $\overline{M}_{4}(K, L^{+}; x_{-}, x_{+}) \rightarrow r^{-1}([0, 1]) \subset \overline{M}^{4}$.

The $C^\infty$ smallness of the isotopy is required to ensure these properties for the whole family.

The $C^\infty$ smallness required in the proposition can be always achieved by breaking the given isotopy into a concatenation of smaller isotopies by choosing times

$$0 = t_{0} < t_{1} < t_{2} < \cdots < t_{N} = 1.$$ 

We also remark that for the proof of Theorem 1.4 will follow if we prove it for the portion on each interval $[t_{i}, t_{i+1}]$ of the given isotopy. With these being said, we will assume from now on that the isotopy is sufficiently $C^\infty$ small so that the global slices $S_{2}^{r}$, $S_{3}^{r}$ exist independently of $r \in [0, 1]$.

Then by construction we have the following obvious one-one correspondence

$$\mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) \cong \mathcal{M}(K, L^{+}; x_{-}, x_{+}) \cap (r)^{-1}((0, 1])$$

where we denote

$$\mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) := \mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) \cap (r)^{-1}(r).$$

Therefore we observe that there is a canonical diffeomorphism

$$\mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) \cong \mathcal{M}(K, L^{+}; x_{-}, x_{+})$$

induced by $v \mapsto v \circ \varphi_{r}^{-1}$ for the mapping part. We introduce a parameterized Floer continuation moduli space

$$\mathcal{M}^{\text{para}}(\mathcal{L}, L; x_{-}, x_{+}) = \prod_{r \in (0, 1]} \{r\} \times \mathcal{M}(K, L^{+}; x_{-}, x_{+})$$

defined on the domain $\mathbb{R} \times [0, 1]$. We have a natural fiberwise isomorphism

$$\tau : \mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) \rightarrow \mathcal{M}^{\text{para}}(K, L_{-}; x_{-}, x_{+})$$

given by

$$\tau(v; z_{0}, z_{1}, z_{2}, z_{3}) = (v \circ \varphi_{r}^{-1}, \tau([z_{0}, z_{1}, z_{2}, z_{3}]))$$

which makes the following commutative diagram commute,

$$\mathcal{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+}) \xrightarrow{\tau} \mathcal{M}^{\text{para}}(K, L; x_{-}, x_{+})$$

$$\downarrow \tau$$

$$\uparrow \tau$$

$$(\tau \circ \mathcal{ft})^{-1}((0, 1]) \rightarrow (0, 1]$$

where $\mathcal{ft}$ is the restriction of the forgetful map

$$\mathcal{ft} : \mathcal{M}_{4}(K, L^{+}; x_{-}, x_{+}) \rightarrow \mathcal{M}^{4}.$$ 

We note that the condition $v(z_{i}) \in S_{i}^{r} \cap L_{s_{i}}$ for $i = 2, 3$ in Proposition 6.22 and (6.16) imply the separating condition $d(v(z_{2}), v(z_{3})) \geq \frac{\delta_{p}}{2} > 0$ for all $v \in \overline{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+})$. Then it follows from the standard Gromov-Floer compactification and one-jet transversality that the compactified moduli space

$$\overline{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+})$$

carries its boundary consisting of the types

$$\overline{M}_{4}^{S_{2}, S_{3}}(K, L^{+}; x_{-}, x_{+})$$

classifying the boundary consisting of the types
Let us explain the notation.

1. $M_{4_s^{s3}}(K, \mathcal{L}; x_-, x_+)_{|r=0} \cong M_{3_s^{s3}}(\mathcal{L}; y) \# \mathcal{M}_3(K, L_0; y, x_-, x_+)$ with $y \in L_0 \cap L_1$,

2. $M_{4_s^{s3}}(K, \mathcal{L}; x_-, x_+)_{|r=1}$,

3. $M_{3_s^{s3}}(K, L_1; x_-, x') \# M_{3_s^{s3}}(K, \mathcal{L}; x', x_+)$ for some $x' \in K \cap L_1$ with $|x_-| = |x'| + 1$,

4. $M_{3_s^{s3}}(K, \mathcal{L}; x_-, x) \# M_{1_s^{s3}}(K, L_0; x, x_+)$ for some $x \in K \cap L_0$ with $|x| = |x_+| + 1$.

Let us explain the notation.

- $M_{3_s^{s3}}(\mathcal{L}; x)$ is the moduli space of equivalence classes of pairs $(w; \zeta, z_2, z_3)$ where $w$ is a function $D^2 \setminus \{\zeta\} \to (M, \mathcal{L})$ satisfying

$$w(z_2) \in \bigcup_{\ell=1}^N S_{2\ell}^t, \quad w(z_3) \in \bigcup_{\ell=1}^N S_{3\ell}^t$$

and

$$\begin{cases} \partial w = 0, \\
\lim_{z \to \zeta} w(z) = x, \\
w(z) \in L_{\chi}(z) \quad \text{for} \quad z \in \partial D^2 \setminus \{\zeta\}
\end{cases}$$

for $\chi := \rho \circ \varphi|_{\partial D^2 \setminus \{\zeta\}}$ where $\varphi$ is the map given in (6.11). The forgetful map $(w; \zeta, z_2, z_3) \mapsto w$ induces an isomorphism between $M_{3_s^{s3}}(\mathcal{L}; y)$ and the moduli space $M(D^2 \setminus \{\zeta\}; \mathcal{L}, y)$ studied in Section 6.1. Therefore the count of $M_{3_s^{s3}}(\mathcal{L}; y)$ gives rise to the operation $c_\mathcal{L}$.

- $M_3(K, L_0; L; y, x_-, x_+)$ is the moduli space whose count encodes the obvious coefficients in the usual $\mu^2$ operation.

- $M_{1_s^{s3}}(K, L_1; x_-, x') \cong M(K, L_1; x_-, x')$ and $M_{3_s^{s3}}(K, L_0; x, x_+) \cong M(K, L_0; x, x_+)$ are the similarly defined moduli of strips (with non-moving boundary conditions) whose counts encode the obvious coefficients in the usual $\mu^1$ operation. We note that one-jet transversality is used to establish that the bubbling of such type cannot occur at the interior parameter $0 < r < \infty$ by dimension counting.)

Now we define the map $\mathcal{H} : CF^*(K, L_0) \to CF^{*-1}(K, L_1)$ of degree $-1$ by

$$\mathcal{H}(z) = \sum_{|x| = |z| - 1} \# \left( \mathcal{M}_{4_s^{s3}}(K, \mathcal{L}; z, x) \right) \langle x \rangle.$$

We mention that $\dim M_{4_s^{s3}}(K, \mathcal{L}; z, x) = 0$ since $|z| = |x| - 1$. By summing up the sign counts of all the points in $\partial M_{4_s^{s3}}(K, \mathcal{L}; z, x_-, x_+)$ and utilizing

- The isomorphism $M_{4_s^{s3}}(K, \mathcal{L}; z, x_-, x_+)$ since $\rho_1 = \rho$,

- The definitions of $\mu^1$ and $\mu^2$ (which are standard),

- The definition of $c_\mathcal{L}$ (Construction 6.7),

- The definition of the continuation map $h_\mathcal{L}^0$ (Construction 6.14), and

- The definition of $\mathcal{H}$ just given,
we have proven the identity
\[ h_\rho^\mu(r^\chi_r, \ast) = \mu^1H + H\mu^1. \]
This proves that the two maps \( h_\rho^\mu, \mu^2(r^\chi_r, \ast) \) are chain-homotopic. By taking cohomology, we have finished the proof.

Remark 6.23. The scheme we use in the proof is in the same spirit as in the definition of the stable map topology given in [FO99]. There, convergence for a sequence of maps with unstable domain is defined by first stabilizing the domain by adding additional marked points, taking the limit, then finally forgetting the added extra marked points. (See also [Oh15b, Section 9.5.2] for an amplification of this trichotomy.) In the above proof, we take a choice of the minimal (and so optimal) number of additional marked points by taking suitable transversal normal slices for the convergence proof: This is guided by our goal to prove the identity spelled out in Theorem 1.4. All these steps are a part of a standard process in the study of moduli space of pseudoholomorphic curves in general—for example, in the construction of the Kuranishi structure and abstract perturbation of the moduli space of stable maps.

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