INTEGRAL ESTIMATES OF CONFORMAL DERIVATIVES AND SPECTRAL PROPERTIES OF THE NEUMANN-LAPLACIAN

V. GOL’DSHITEIN, V. PCHELintSEV, A. UKhLOV

Abstract. In this paper we study integral estimates of derivatives of conformal mappings \( \varphi : \mathbb{D} \to \Omega \) of the unit disc \( \mathbb{D} \subset \mathbb{C} \) onto bounded domains \( \Omega \) that satisfy the Ahlfors condition. These integral estimates lead to estimates of constants in Sobolev-Poincaré inequalities, and by the Rayleigh quotient we obtain spectral estimates of the Neumann-Laplace operator in non-Lipschitz domains (quasidiscs) in terms of the (quasi)conformal geometry of the domains. Specifically, the lower estimates of the first non-trivial eigenvalues of the Neumann-Laplace operator in some fractal type domains (snowflakes) were obtained.

1. Introduction

1.1. Estimates of Conformal Derivatives. In the work [27] we obtained lower estimates of the first non-trivial eigenvalues of the Neumann-Laplace operator in the terms of integrals of complex derivatives (i.e. hyperbolic metrics) of conformal mappings \( \varphi : \mathbb{D} \to \Omega \). Let us recall that the classical Koebe distortion theorem [10] gives the following estimates of the complex derivatives in the case of univalent analytic functions (conformal homeomorphisms): \( \varphi : \mathbb{D} \to \Omega \) normalized so that \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 \):

\[
\frac{1 - |z|}{(1 + |z|)^3} \leq |\varphi'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}.
\]

The example of the Koebe function

\[
\varphi(z) = \frac{z}{(1 - z)^2}
\]

which maps the unit disc \( \mathbb{D} \) onto \( \Omega = \mathbb{C} \setminus (-\infty, -1/4] \) shows that these estimates don’t give even square integrability of the complex derivatives in arbitrary simply connected planar domains. But if \( \Omega \subset \mathbb{C} \) is a simply connected planar domain of finite measure then by simple calculation

\[
\iint_{\mathbb{D}} |\varphi'(z)|^2 \, dx \, dy = \iint_{\mathbb{D}} J(z, \varphi) \, dx \, dy = |\Omega| < \infty.
\]

(We identify the complex plane \( \mathbb{C} \) and the real plane \( \mathbb{R}^2 \): \( \mathbb{C} \ni z = x + iy = (x, y) \in \mathbb{R}^2 \).)

Hence in special classes of domains we have better integral estimates of the complex derivatives than by the Koebe distortion theorem.

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In the present work we study integral estimates of the complex derivatives in domains bounded by Jordan curves that satisfy the Ahlfors three point condition [3]. For this study we introduce a notion of hyperbolic (integral) $\alpha$-dilatation of $\Omega$:

$$Q(\alpha, \Omega) := \iint_{\mathbb{D}} |\varphi'(z)|^\alpha \, dxdy = \iint_{\Omega} \left| (\varphi^{-1})'(w) \right|^{2-\alpha} dudv.$$

The finiteness of the hyperbolic $\alpha$-dilatation and its convergence hyperbolic interval

$$HI(\Omega) := \{ \alpha \in \mathbb{R} : Q(\alpha, \Omega) < \infty \}.$$  

do not depend on choice of a conformal mapping $\varphi : \mathbb{D} \to \Omega$ and can be reformulated in terms of the hyperbolic metrics [14]. Namely

$$\iint_{\mathbb{D}} |\varphi'(z)|^\alpha \, dxdy = \iint_{\mathbb{D}} \left( \frac{\lambda_{\mathbb{D}}(z)}{\lambda_{\Omega}(\varphi(z))} \right)^\alpha \, dxdy = \iint_{\Omega} \left| (\varphi^{-1})'(w) \right|^{2-\alpha} dudv = \iint_{\Omega} \left( \frac{\lambda_{\mathbb{D}} (\varphi^{-1}(w))}{\lambda_{\Omega}(w)} \right)^{2-\alpha} dudv$$

where $\lambda_{\mathbb{D}}$ and $\lambda_{\Omega}$ are hyperbolic metrics in $\mathbb{D}$ and $\Omega$ [9]. Let us recall that the hyperbolic metrics generated by $\lambda_{\mathbb{D}} (\varphi^{-1}(w))$ are equivalent for different choice of conformal homeomorphism $\varphi : \Omega \to \mathbb{D}$ because any other conformal homeomorphism $\psi^{-1} : \Omega \to \mathbb{D}$ is a composition of $\varphi^{-1}$ and a Möbius homeomorphism (that is an isometry of the hyperbolic metric).

**Remark 1.1.** For any bounded simply connected domain $(-1.78, 2] \subset HI(\alpha, \Omega)$ [26, 28]. A more detailed discussion about the hyperbolic $\alpha$-dilatation and its convergence hyperbolic interval can be found in Appendix.

In [26] we proved that if a number $\alpha > 2$ belongs to $HI(\Omega)$ then $\Omega$ has finite geodesic diameter. By this reason we call domains that satisfy to a property $2 < \alpha \in HI(\Omega)$ as conformal $\alpha$-regular domains [14].

In this paper we obtain estimates of $Q(\alpha, \Omega)$ for a large class of conformal $\alpha$-regular domains (so-called quasidiscs) with the help of the exact inverse Hölder inequality for Jacobians of quasiconformal mappings. Using the estimates for $Q(\alpha, \Omega)$ we obtain estimates of constants for Sobolev-Poincaré inequalities and as a result we obtain lower estimates for first nontrivial eigenvalues of the Laplace operator with the Neumann boundary condition.

1.2. **Spectral Estimates of the Neumann-Laplacian.** Let $\Omega \subset \mathbb{C}$ be a bounded Lipschitz domain (an open connected set). The Neumann eigenvalue problem for the Laplace operator is:

$$\begin{cases} 
- \text{div} \, (\nabla u) = \mu u & \text{in} \, \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on} \, \partial \Omega.
\end{cases}$$

The weak statement of this spectral problem is as follows: a function $u$ solves the previous problem iff $u \in W^{1}_2(\Omega)$ and

$$\iint_{\Omega} \nabla u(x,y) \cdot \nabla v(x,y) \, dxdy = \mu \iint_{\Omega} u(x,y)v(x,y) \, dxdy$$

where $\mu$ is an eigenvalue and $v(x,y)$ is an eigenfunction corresponding to $\mu$.
for all \( v \in W^2_1(\Omega) \). This statement is correct in any bounded domain \( \Omega \subset \mathbb{C} \).

Let us give a short historical review about eigenvalues estimates for the Neumann-Laplace operator. The classical upper estimate of the first nontrivial Neumann eigenvalue

\[
\mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{p_{n/2}^2}{R^2}
\]

was proved by G. Szegö [39] for simply connected planar domains and by H. F. Weinberger [43] for domains in \( \mathbb{R}^n \). In this inequality \( p_{n/2} \) denotes the first positive zero of the function \( (t^{1-n/2}J_{n/2}(t))' \), and \( \Omega^* \) is an \( n \)-ball of the same \( n \)-volume as \( \Omega \) with \( R^* \) as its radius. In particular, if \( n = 2 \), \( p_1 = j'_1,1 \approx 1.84118 \) where \( j'_1,1 \) denotes the first positive zero of the derivative of the Bessel function \( J_1 \).

In 1961 G. Polya [35] obtained upper estimates for eigenvalues in so-called plane-covering domains. Namely, for the first nontrivial eigenvalue \( \mu_1(\Omega) \) it is:

\[
\mu_1(\Omega) \leq 4\pi \frac{|\Omega|}{|\Omega^*|}.
\]

The upper estimate of the Laplace eigenvalues, with the help of different techniques, were intensively studied in recent decades, see, for example, [5, 6, 7, 15, 32].

The lower estimates for the \( \mu_1(\Omega) \) for convex domains were obtained in the classical work [34]. It was proved that if \( \Omega \) is convex with diameter \( d(\Omega) \) (see, also [16, 19, 41]), then

\[
\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2}.
\]

Unfortunately in non-convex domains \( \mu_1(\Omega) \) can not be estimated in the terms of Euclidean diameters. It can be seen by considering a domain consisting of two identical squares connected by a thin corridor [12]. In [27] was proved, on the basis of the geometric theory of composition operators on Sobolev spaces [23, 40] with applications to the (weighted) Poincaré-Sobolev inequalities [22, 25], that if \( \Omega \subset \mathbb{C} \) be a conformal \( \alpha \)-regular domain, then the spectrum of Neumann-Laplace operator in \( \Omega \) is discrete, can be written in the form of a non-decreasing sequence

\[
0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq ... \leq \mu_n(\Omega) \leq ...
\]

and

\[
(1.1) \quad \frac{1}{\mu_1(\Omega)} \leq \frac{4}{\sqrt{\pi}^2} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha - 2}{\alpha}} Q(\alpha, \Omega)^{2/\alpha}.
\]

In the present article we suggest the approach which is based on the geometric theory of composition operators on Sobolev spaces and estimates of hyperbolic metric in quasidiscs. Recall that a domain \( \Omega \subset \mathbb{C} \) is called a \( K \)-quasidisc if it is the image of the unit disc \( D \) under a \( K \)-quasiconformal homeomorphism of the complex plane \( \mathbb{C} \) onto itself. Note that quasidiscs represent large class domains including fractal type domains like snowflakes. The Hausdorff dimensions of the quasidisc’s boundary can be any number in \([1, 2]\) [40].

Following [4] a homeomorphism \( \varphi : \Omega \to \Omega' \) between planar domains is called \( K \)-quasiconformal if it preserves orientation, belongs to the Sobolev class \( W^{1}_{2,\text{loc}}(\Omega) \) and its directional derivatives \( \partial_\xi \) satisfy the distortion inequality

\[
\max_\xi |\partial_\xi \varphi| \leq K \min_\xi |\partial_\xi \varphi| \text{ a.e. in } \Omega.
\]
If $\Omega$ is a $K$-quasidisc, then a conformal mapping $\varphi : \mathbb{D} \to \Omega$ allows $K^2$-quasiconformal reflection. It is well known that Jacobians of quasiconformal mappings satisfy the weak inverse Hölder inequality \cite{11}. On the basis of the weak inverse Hölder inequality and the estimates of the constants in doubling conditions for measures generated by Jacobians of quasiconformal mappings (Proposition 3.6) we obtain

**Theorem A.** Let $\Omega \subset \mathbb{C}$ be a $K$-quasidisc. Then the spectrum of Neumann-Laplace operator in $\Omega$ is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,$$

and

$$1 \leq \frac{1}{\mu_1(\Omega)} \leq \frac{K^2 C^2}{\pi} \left(\frac{2\alpha - 2}{\alpha - 2}\right)^{2\alpha - 2} \pi \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^4)^2}{2 \log 3} \right\} |\Omega|,$$

for $2 < \alpha < \frac{2K^2}{K^2 - 1}$, where

$$C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha} \frac{\alpha - 2}{\alpha - 1} \left(24\pi^2 K^2\right)^\alpha < 1.$$

The main technical problem of this estimate is evaluations of the quasiconformality coefficient $K$ for quasidiscs. For this aim we use an equivalent description of quasidiscs in the terms of the Ahlfors's 3-point condition. This description of quasidiscs allows to obtain the estimates for the specific fractal type domains.

**Theorem C.** Let $S_p \subset \mathbb{C}, \ 1/4 \leq p < 1/2$, be the Rohde snowflake. Then the spectrum of Neumann-Laplace operator in $S_p$ is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(S_p) < \mu_1(S_p) \leq \mu_2(S_p) \leq \ldots \leq \mu_n(S_p) \leq \ldots,$$

and

$$1 \leq \frac{1}{\mu_1(S_p)} \leq \frac{C^2_\alpha e^{4(1+e^{2\pi(16/(1-2p))^5})^2}}{240\pi} \left(\frac{2\alpha - 2}{\alpha - 2}\right)^{2\alpha - 2} \pi \exp \left\{ \frac{\pi^2 (2 + \pi^4)^2 e^{4(1+e^{2\pi(16/(1-2p))^5})^2}}{241 \log 3} \right\} |S_p|,$$

for $2 < \alpha < \frac{2K^2}{K^2 - 1}$, where

$$C_\alpha = \frac{10^6}{ [(\alpha - 1)(1 - \nu)]^{1/\alpha}}, \quad \nu = 10^{3\alpha} \frac{24\pi^2}{240\pi} e^{4(1+e^{2\pi(16/(1-2p))^5})^2} \left(1 + e^{2\pi(16/(1-2p))^5}\right)^\alpha \frac{\alpha - 2}{\alpha - 1} < 1.$$

The quasiconformal coefficient $K$ of the Rohde snowflake $S_p$ satisfying the condition

$$K < 2^{-10} \exp \left\{ (1 + e^{2\pi(16/(1-2p))^5})^2 \right\}.$$

On the basis of Theorem A and Theorem C we can assert that eigenvalues of the Laplace operator depend on (quasi)conformal geometry of planar domains.
2. Sobolev spaces and Poincaré-Sobolev inequalities

Let \( E \subset \mathbb{C} \) be a measurable set. For any \( 1 \leq p < \infty \) we consider the Lebesgue space of locally integrable functions with the finite norm

\[
\|f \|_{L^p(E)} := \left( \int_E |f(x,y)|^p \, dx \, dy \right)^{1/p} < \infty.
\]

We define the Sobolev space \( W^{1,p}(\Omega) \), \( 1 \leq p < \infty \), as a Banach space of locally integrable weakly differentiable functions \( f : \Omega \rightarrow \mathbb{R} \) equipped with the following norm:

\[
\|f \|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |f(x,y)|^p \, dx \, dy \right)^{1/p} + \left( \int_{\Omega} |\nabla f(x,y)|^p \, dx \, dy \right)^{1/p}.
\]

We also define the homogeneous seminormed Sobolev space \( L^{1,p}(\Omega) \), \( 1 \leq p < \infty \), of locally integrable weakly differentiable functions \( f : \Omega \rightarrow \mathbb{R} \) equipped with the following seminorm:

\[
\|f \|_{L^{1,p}(\Omega)} = \left( \int_{\Omega} |\nabla f(x,y)|^p \, dx \, dy \right)^{1/p}.
\]

Recall that the embedding operator \( i : L^{1,p}(\Omega) \rightarrow L^{1,\text{loc}}(\Omega) \) is bounded.

If \( f \in W^{1,1}(\Omega) \), \( 1 \leq p < \infty \), then for \( 0 \leq \kappa = 1/p - 1/q < 1/2 \) the Poincaré-Sobolev inequality

\[
\left( \int_{\Omega} |f(z) - f_{\Omega}|^q \, dz \, dy \right)^{1/q} \leq B_{q,p}(\Omega) \left( \int_{\Omega} |\nabla f(z)|^p \, dz \, dy \right)^{1/p}
\]

holds (see, for example, [20, 27]) with the constant

\[
B_{q,p}(\Omega) \leq \frac{2}{\pi \kappa} \left( \frac{1 - \kappa}{1/2 - \kappa} \right)^{1-\kappa}.
\]

This estimate is not applicable in the critical case \( p = 1 \) and \( q = 2 \). Now we obtain the upper estimate of the Poincaré constant in the Poincaré-Sobolev inequality for the Sobolev space \( W^{1,1}(\Omega) \) in this critical case. We use the following Gagliardo inequality for functions with compact support [18, 33]:

\[
\left( \int_{\Omega} |f(z)|^2 \, dz \, dy \right)^{1/2} \leq \frac{1}{2\sqrt{\pi}} \int_{\Omega} |\nabla f(z)| \, dz \, dy,
\]

where \( \Omega \subset \mathbb{C} \) be a bounded Lipschitz domain.

**Theorem 2.1.** Let \( f \in W^{1,1}(\Omega) \). Then for any \( r > 0 \) and any \( z_0 \in \Omega : \text{dist}(z_0, \partial \Omega) > 2r \), the following inequality

\[
\left( \int_{D(z_0, r)} |f(z) - f_{D(z_0, r)}|^2 \, dz \, dy \right)^{1/2} \leq \frac{3\sqrt{\pi^3}}{4} \int_{D(z_0, r)} |\nabla f(z)| \, dz \, dy
\]

holds.
Proof. We prove this inequality in the case $f_{D(z_0, r)} = 0$. In this case the inequity (2.2) can be rewritten as

$$
\left( \iint_{D(z_0, r)} |f(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi}}{4} \iint_{D(z_0, r)} |
abla f(z)| \, dx \, dy.
$$

(2.3)

The inequality (2.3) is invariant under translations and similarities and it is sufficient to prove one for the disc $D(0, 1)$.

Denote by $D(0, \delta)$ an open disc of the radius $\delta > 1$. Choose the cut function $\eta$ in the form

$$
\eta(z) = \begin{cases} 
1, & \text{if } |z| < 1, \\
0, & \text{if } |z| > \delta
\end{cases}
$$

and linear for $z \in \mathbb{R}_\delta$, where $\mathbb{R}_\delta := D(0, \delta) \setminus D(0, 1)$. Then

$$
|\nabla \eta(z)| = \begin{cases} 
\frac{1}{\delta - 1} & \text{if } z \in \mathbb{R}_\delta, \\
0 & \text{otherwise}.
\end{cases}
$$

Let $\widetilde{f} : \mathbb{R}_\delta \to \mathbb{R}$ be the extension function is defined by the rule

$$
\widetilde{f}(z) = f(w(z)) \eta(z), \quad w \in \mathbb{R}_\delta,
$$

where $w(z) = 1/z$ is the inversion in the unit circle $S(0, 1)$.

Define the extension operator on Sobolev spaces

$$
E : L^1_1(D(0, 1)) \to L^1_1(D(0, \delta))
$$

by the formula

$$
(Ef)(z) = \begin{cases} 
f(z) & \text{if } z \in D(0, 1), \\
\widetilde{f}(z) & \text{if } z \in \mathbb{R}_\delta.
\end{cases}
$$

In order to estimate the norm $\|E\|$ of the extension operator $E : L^1_1(D(0, 1)) \to L^1_1(D(0, \delta))$ we have

$$
\|E \mid L^1_1(D(0, \delta))\| = \iint_{D(0, 1)} |\nabla f(z)| \, dx \, dy + \iint_{\mathbb{R}_\delta} |\nabla \widetilde{f}(z)| \, dx \, dy.
$$

To estimate the second integral in the right side of this equality by elementary calculations we have

$$
|\nabla \widetilde{f}(z)| = |\nabla (f(w(z))\eta(z))| = |\nabla (f(w(z))) \cdot \eta(z) + f(w(z)) \cdot \nabla \eta(z)|
$$

$$
\leq |\nabla f(w(z))| + \frac{1}{\delta - 1} |f(w(z))|.
$$

Hence

$$
\iint_{\mathbb{R}_\delta} |\nabla \widetilde{f}(w(z))| \, dx \, dz \leq \iint_{\mathbb{R}_\delta} |\nabla f(w(z))| \, dx \, dy + \frac{1}{\delta - 1} \iint_{\mathbb{R}_\delta} |f(w(z))| \, dx \, dy.
$$
First consider the integral
\[
\iint_{R_\delta} |\nabla f(w(z))| \, dxdy = \iint_{R_\delta} |\nabla f|(w(z))|w'(z)| \, dxdy = \\
= \iint_{R_\delta} |\nabla f|(w(z))|w'(z)||J(z, w)||J(z, w)|^{-1} \, dxdy \\
\leq \sup_{z \in R_\delta} \frac{|w'(z)|}{|J(z, w)|} \iint_{R_\delta} |\nabla f|(w(z))|J(z, w)| \, dxdy \\
= \sup_{z \in R_\delta} \frac{|w'(z)|}{|J(z, w)|} \iint_{D_\delta} |\nabla f|(w) \, dudv,
\]
where \( w : R_\delta \to D_\delta, w(z) = 1/\|z\| \) and \( D_\delta = \{ w \in \mathbb{C} : 1/\delta < |z| < 1 \} \). We calculate the norm of the derivative of mapping \( w \) by the formula
\[
|w'(z)| = |w_z| + |w_\varphi|
\]
and the Jacobian of mapping \( w \) by the formula
\[
J(z, w) = |w_z|^2 - |w_\varphi|^2.
\]
Here
\[
w_z = \frac{1}{2} \left( \frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \quad \text{and} \quad w_\varphi = \frac{1}{2} \left( \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right).
\]
By elementary calculations
\[
w_z = 0 \quad \text{and} \quad w_\varphi = -\frac{1}{|z|^2}.
\]
Hence
\[
|w'(z)| = \frac{1}{|z|^2} \quad \text{and} \quad |J(z, w)| = \frac{1}{|z|^4}.
\]
Finally we get
\[
\iint_{R_\delta} |\nabla f(w(z))| \, dxdy \leq \sup_{z \in R_\delta} \|z\|^2 \iint_{D_\delta} |\nabla f|(w) \, dudv \leq \delta^2 \iint_{D(0,1)} |\nabla f|(w) \, dudv.
\]
In order to estimate the integral
\[
\iint_{R_\delta} |f(w(z))| \, dxdy
\]
we will use the change of variable formula.
We obtain
\[
\iint_{R_\delta} |f(w(z))| \, dxdy = \iint_{R_\delta} |f(w(z))||J(z, w)||J(z, w)|^{-1} \, dxdy \\
\leq \sup_{z \in R_\delta} \frac{1}{|J(z, w)|} \iint_{R_\delta} |f(w(z))||J(z, w)| \, dxdy \\
= \sup_{z \in R_\delta} \frac{1}{|J(z, w)|} \iint_{D_\delta} |f(w)| \, dudv \leq \delta^4 \iint_{D(0,1)} |f(w)| \, dudv.
\]
Using the following Poincaré–Sobolev inequality \( \| \nabla f(z) \|_{L^2(\Omega)} \leq \frac{d}{2} \int_{\Omega} |f(z)| \, dx \, dy \)

where \( \Omega \) be a convex domain with diameter \( d \) and \( f \in W^1_1(\Omega) \), finally we obtain

\[
\int_{B_s} |f(w(z))| \, dx \, dy \leq \delta^4 \int_{D(0,1)} |\nabla f(w)| \, du \, dv.
\]

Thus

\[
\int_{B_s} |\nabla \tilde{f}(z)| \, dx \, dy \leq \delta^2 \int_{D(0,1)} |\nabla f(w)| \, du \, dv + \delta^4 \int_{D(0,1)} |\nabla f(w)| \, du \, dv,
\]

and consequently

\[
\| E(f) \|_{L^1(D(0,\delta))} \leq \left( 1 + \delta^2 + \frac{\delta^4}{\delta - 1} \right) \int_{D(0,1)} |\nabla f(z)| \, dx \, dy
\]

\[
= \left( 1 + \delta^2 + \frac{\delta^4}{\delta - 1} \right) \| f \|_{L^1(D(0,1))}.
\]

Now, according to inequality (2.1) we obtain

\[
\left( \int_{D(0,1)} |f(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \left( \int_{D(0,\delta)} |Ef(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{\pi}} \int_{D(0,\delta)} |\nabla Ef(z)| \, dx \, dy
\]

\[
\leq \frac{\delta^4 + \delta^3 - \delta^2 + \delta - 1}{2\sqrt{\pi(\delta - 1)}} \int_{D(0,1)} |\nabla f(z)| \, dx \, dy \leq C(\delta) \int_{D(0,1)} |\nabla f(z)| \, dx \, dy.
\]

Taking the optimal \( \delta \approx 5/4 \) we have

\[
\left( \int_{D(0,1)} |f(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \frac{3\sqrt{\pi^3}}{4} \int_{D(0,1)} |\nabla f(z)| \, dx \, dy.
\]

3. The Doubling Condition and the Hölder Inequality

Here we recall necessary facts about conformal capacity. A well-ordered triple \((F_0, F_1; \Omega)\) of nonempty sets, where \( \Omega \) is an open set in \( \mathbb{C} \), and \( F_0, F_1 \) are closed subsets of \( \Omega \), is called a condenser on the complex plane \( \mathbb{C} \).

The value

\[
\text{cap}(E) = \text{cap}(F_0, F_1; \Omega) = \inf_{\Omega} \int |\nabla v|^2 \, dx \, dy,
\]
where the infimum is taken over all Lipschitz nonnegative functions $v: \overline{\Omega} \to \mathbb{R}$, such that $v = 0$ on $F_0$, and $v = 1$ on $F_1$, is called conformal capacity of the condenser $E = (F_0, F_1; \Omega)$. If the set of admissible functions is empty, then $\text{cap}(F_0, F_1; \Omega) = \infty$. For finite values of capacity $0 < \text{cap}(F_0, F_1; \Omega) < +\infty$ there exists a unique continuous weakly differentiable function $u_0$ (an extremal function) such that:

$$\text{cap}(F_0, F_1; \Omega) = \iint_{\Omega} |\nabla u_0|^2 \, dx \, dy.$$

Quasiconformal mappings can be characterized in capacitary terms (see, for example [24]). Namely, a homeomorphism $\varphi: \Omega \to \Omega', \Omega, \Omega' \subset \mathbb{C}$ is $K$-quasiconformal, if and only if

$$K^{-1} \text{cap}(F_0, F_1; \Omega) \leq \text{cap}(\varphi(F_0), \varphi(F_1); \Omega') \leq K \text{ cap}(F_0, F_1; \Omega)$$

for any condenser $E = (F_0, F_1; \Omega)$.

We will need the following estimate of the conformal capacity (see, for example [24, 42]).

**Lemma 3.1.** [24] Let $R > r > 0$. Then

$$\text{cap}\left(\overline{D(z_0, r)}, \mathbb{C} \setminus D(z_0, R); \mathbb{C}\right) = 2\pi (\log R/r)^{-1}.$$

Consider capacity estimates for Teichmüller type condensers in $\overline{\mathbb{C}}$ (see, for example [42], Lemma 5.32).

**Lemma 3.2.** Fix $0 < r < R$. Let $F_0$ and $F_1$ be continua in $\mathbb{C}$ such that

$$F_0 \cap S(0, \rho) \neq \emptyset \text{ and } F_1 \cap S(0, \rho) \neq \emptyset$$

for all $r < \rho < R$, where $S(0, \rho) \text{ is the circle of radius } \rho$. Then

$$\text{cap}(F_0, F_1; D(0, R) \setminus \overline{D(0, r)}) \geq \frac{2}{\pi} \log \frac{R}{r}.$$

Denote by $R_T(t) = \overline{\mathbb{C} \setminus \{[-1, 0] \cup [t, \infty]\}}$, $t > 1$, the Teichmüller ring in $\overline{\mathbb{C}}$.

**Lemma 3.3.** Let $R_T(t)$ be the Teichmüller ring in $\overline{\mathbb{C}}$. Then

$$\text{cap} R_T(t) = \frac{2\pi}{\log \Phi(t)},$$

where $\Phi$ satisfies the conditions

$$t + 1 \leq \Phi(t) < 32t, \, t > 1.$$

Using this capacity estimates we obtain estimates for a constant in the inverse Hölder inequality. We start from the weak Hölder inequality.

**Lemma 3.4.** Let $\varphi: \Omega \to \Omega'$ be a $K$-quasiconformal mapping of planar domains $\Omega, \Omega' \subset \mathbb{C}$. Then for every disc $D(z_0, r)$ such that $D(z_0, 2r) \subset \Omega$ the weak inverse Hölder inequality for the first generalized derivatives of $\varphi$

$$(3.1) \quad \left( \frac{1}{|D(z_0, r)|} \iint_{D(z_0, r)} |\varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq \frac{24\pi^2 K}{|D(z_0, 2r)|} \iint_{D(z_0, 2r)} |\varphi'(z)| \, dx \, dy$$

holds.
Proof. Derivatives of $K$-quasiconformal mappings satisfy the following inequality ([11], pp 274)

$$
\left( \frac{1}{|D(z_0, r)|} \int_{D(z_0, r)} |\varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}}
\leq 4K \left( \frac{1}{|D(z_0, 2r)|} \int_{D(z_0, 2r)} |\varphi(z) - \varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}}.
$$

Applying to the right side of this inequality the Poincaré-Sobolev inequality (2.2), we obtain the required inequality (3.1). □

Directly by Lemma 3.4 and Theorem 4.2 from [11] we get the following weak Hölder inequality:

**Theorem 3.5.** Let $\varphi : \Omega \to \Omega'$ be a $K$-quasiconformal mapping of planar domains $\Omega, \Omega' \subset \mathbb{C}$. Then for every disc $D(z_0, r)$ such that $D(z_0, 2r) \subset \Omega$ and for some $\sigma > 2$ the inequality

$$
(3.2) \left( \frac{1}{|D(z_0, r)|} \int_{D(z_0, r)} |\varphi'(z)|^\sigma \, dx \, dy \right)^{\frac{1}{\sigma}}
\leq C_\sigma \left( \frac{1}{|D(z_0, 2r)|} \int_{D(z_0, 2r)} |\varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}}
$$

holds, where

$$
C_\sigma = \frac{10^6}{[(\sigma - 1)(1 - \nu)]^{1/\sigma}}, \quad \nu = 10^{4\sigma} \frac{\sigma - 2}{\sigma - 1} \left( 24\pi^2 K \right)^{\sigma} < 1.
$$

Given Theorem 3.5 for further estimates of the left side of the inequality (3.2) we use the doubling property of measures generated by Jacobians of quasiconformal mappings (see, for example, [29]). Recall that a Borel measure $\mu$ on a set $\Omega$ is doubling if there exist a constant $C_\mu \geq 1$ so that the inequality

$$
\mu(D(z_0, 2r)) \leq C_\mu \cdot \mu(D(z_0, r)) < \infty
$$

hold for all discs $D(z_0, r)$ in $\Omega$.

In the following proposition we give an estimate of the constant $C_\mu$ in the measure doubling condition.

**Proposition 3.6.** Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a $K$-quasiconformal mapping. Then for any $z_0 \in \mathbb{C}$ and any $r > 0$ we have

$$
(3.3) \int_{D(z_0, 2r)} |J(z, \varphi)| \, dx \, dy \leq \exp \left\{ \frac{K\pi^2(2 + \pi^4)^2}{2 \log 3} \right\} \int_{D(z_0, r)} |J(z, \varphi)| \, dx \, dy.
$$
Proof. Because the inequality (3.3) is invariant under translations and similarities we can suppose that \( z_0 = 0 \) and radius \( r = 1 \). It is sufficient to show that
\[
(3.4) \quad |\varphi(D(0, 2))| \leq \exp \left\{ \frac{K \pi^2 (2 + \pi^4)^2}{2 \log 3} \right\} |\varphi(D(0, 1))|.
\]

Since \( \varphi \) is a quasiconformal mapping, \( \varphi(0) \) is an interior point of the open connected set \( U_0 = \varphi(D(0, 1)) \). Denote by \( \lambda_0 = \text{dist}(\varphi(0), \partial U_0) \) and by \( \lambda_1 = \max_{w \in \partial U_1} |\varphi(0) - w|, \ U_1 = \varphi(D(0, 2)) \).

For the proof of the inequality (3.4) we estimate the ratio
\[
\frac{|\varphi(D(0, 2))|}{|\varphi(D(0, 1))|} \leq \frac{\pi \lambda_1^2}{\pi \lambda_0^2} = \frac{\lambda_1^2}{\lambda_0^2},
\]
because the disc \( D(\varphi(0), \lambda_1) \) contains \( \varphi(D(0, 2)) \) and the disc \( D(\varphi(0), \lambda_0) \) is contained in \( \varphi(D(0, 1)) \). For this aim we use the capacity estimates and quasi-invariance of conformal capacity under quasiconformal mappings and consider continua of \( F_0 \) and \( F_1 \) where \( F_0 \) be a line segment of length \( \lambda_0 \) joining \( \varphi(0) \) to \( w_0 \in \partial U_0 \) and \( F_1 \) be a continuum in \( \mathbb{C} \setminus \mathbb{C} \cup \{\infty\} \) joining a point \( w_1 \) in \( \partial D(\varphi(0), \lambda_1) \cap U_1 \) to \( \infty \) in \( \mathbb{C} \setminus D(\varphi(0), \lambda_1) \).

Now we consider pre-images of continua \( \varphi^{-1}(F_0) \) and \( \varphi^{-1}(F_1) \) in order to obtain a lower estimate of the capacity of the condenser \( (\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0, \pi)) \).

Because capacity is invariant under rotations, without loss of generality we can suppose that \( z_0 = \varphi^{-1}(w_0) = (1, 0) \). Let \( z_1 = \varphi^{-1}(w_1) = (2, \theta_0) \), where \((1, 0)\) and \((2, \theta_0)\) are the polar coordinates.

Define the bi-Lipschitz mapping \( \psi : \mathbb{C} \rightarrow \mathbb{C} \) which maps \( z_0 \) to \( z_0 \) and \( z_1 \) to \((2, 0)\) by the rule
\[
\psi(\rho, \theta) = (\rho, \theta - (\rho - 1)\theta_0),
\]
where \((\rho, \theta)\) are polar coordinates on the complex plane \( \mathbb{C} \).

Now we estimate the Lipschitz coefficient \( L \) for the mapping \( \psi \). For this aim we represent \( \psi \) as
\[
\psi(x, y) = (\sqrt{x^2 + y^2} \cos t, \sqrt{x^2 + y^2} \sin t),
\]
where
\[
t = \arctan \frac{y}{x} - (\sqrt{x^2 + y^2} - 1)\theta_0, \quad 0 \leq \theta_0 \leq \pi,
\]
and we use the following expressions
\[
L = ||D\psi(x, y)||_0 \leq \frac{\sqrt{2}}{2} ||D\psi(x, y)||_e.
\]

Here \( D\psi(x, y) \) denotes the Jacobian matrix of the mapping \( \psi = \psi(x, y) \), \( || \cdot ||_0 \) and \( || \cdot ||_e \) denote the operator norm and the Euclidean norm respectively.

The Jacobian matrix has the form
\[
D\psi(x, y) = \begin{pmatrix}
x \cos t - y \sin t 
\sqrt{x^2 + y^2} + x\theta_0 \sin t & y \cos t - x \sin t \sqrt{x^2 + y^2} + y\theta_0 \sin t \\
\sqrt{x^2 + y^2} - x\theta_0 \cos t & \sqrt{x^2 + y^2} - y\theta_0 \cos t
\end{pmatrix}.
\]

A straightforward calculation yields
\[
||D\psi(x, y)||_e = \sqrt{2 + (x^2 + y^2)\theta_0^2}.
\]
Hence
\[
L \leq \frac{\sqrt{2}}{2} ||D\psi(x, y)||_e = \sqrt{1 + (x^2 + y^2)\theta_0^2} \leq \sqrt{1 + \frac{\pi^4}{4}}.
\]
Now we consider the ring 
\[ R = D(\frac{3}{2}, \frac{3}{2}) \setminus D(\frac{3}{2}, \frac{1}{2}). \]
Then by Lemma 3.2 using the monotonicity of the capacity and quasi-invariance under bi-Lipschitz mappings we obtain
\[ \text{cap}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0, \pi)) \geq \frac{1}{L^4} \text{cap}(E_0, E_1; R) \geq \frac{8 \log 3}{\pi (2 + \pi^4)^2}, \]
where \( E_0 = \psi(\varphi^{-1}(F_0)) \cap R, E_1 = \psi(\varphi^{-1}(F_1)) \cap R. \)

On the other hand, taking into account the capacitary definition of the quasi-conformal mapping and Lemma 3.1 we have
\[ \text{cap}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D(0, \pi)) \leq K \text{cap}(F_0, F_1; C) \leq K \text{cap}(D(\varphi(0), \lambda_0), C \setminus D(\varphi(0), \lambda_1); C) = \frac{2\pi K}{\log \lambda_1}. \]

Combining inequalities (3.5) and (3.6), we obtain
\[ 8 \log 3 \frac{\pi (2 + \pi^4)^2}{\pi (2 + \pi^4)^2} \leq \frac{2\pi K}{\log \lambda_1}. \]

By elementary calculations
\[ \lambda_1 \leq \exp \left( \frac{K \pi^2 (2 + \pi^4)^2}{4 \log 3} \right) \lambda_0. \]

In order to obtain the required inequality it is necessary to perform the following straightforward calculations
\[ |\varphi(D(0, 2))| \leq \pi \lambda_1^2 \leq \pi \exp \left( \frac{K \pi^2 (2 + \pi^4)^2}{2 \log 3} \right) \lambda_0 \leq \exp \left( \frac{K \pi^2 (2 + \pi^4)^2}{2 \log 3} \right) |\varphi(D(0, 1))|. \]

Thus
\[ \iint_{D(0, 2)} |J(z, \varphi)| \, dxdy \leq \exp \left( \frac{K \pi^2 (2 + \pi^4)^2}{2 \log 3} \right) \iint_{D(0, 1)} |J(z, \varphi)| \, dxdy. \]

□

For any planar \( K \)-quasiconformal homeomorphism \( \varphi : \Omega \to \Omega' \) the following sharp result is known: \( J(z, \varphi) \in L_{p, \text{loc}}(\Omega) \) for any \( 1 \leq p < \frac{K}{K-1} \) \((\S 21)\).

**Proposition 3.7.** For any conformal homeomorphism \( \varphi : \mathbb{D} \to \Omega \) of the unit disc \( \mathbb{D} \) onto a \( K \)-quasidisc \( \Omega \) derivatives \( \varphi' \in L_p(\mathbb{D}) \) for any \( 1 \leq p < \frac{2K^2}{K-1} \subset H(\Omega) \).

**Proof.** Any conformal homeomorphism \( \varphi : \mathbb{D} \to \Omega \) can be extended to a \( K^2 \) quasi-conformal homeomorphism \( \tilde{\varphi} \) of the whole plane to the whole plane by reflection. Hence \( \tilde{\varphi}' \) belongs to the class \( L_{p, \text{loc}}(\mathbb{C}) \) for any \( 1 \leq p < \frac{2K^2}{K-1} \subset H(\Omega) \). Therefore \( \varphi' \) belongs to the class \( L_p(\mathbb{D}) \). □

On the base of the weak Hölder inequality and the doubling condition we obtain integral estimates of complex derivatives of conformal mapping \( \varphi : \mathbb{D} \to \Omega \) in the unit disc onto a \( K \)-quasidisc \( \Omega \):
Theorem B. Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasidisc and $\varphi : \mathbb{D} \rightarrow \Omega$ be a conformal mapping. Suppose that $2 < \alpha < \frac{2K^2}{K^2-1}$. Then

\begin{equation}
\left( \iint_{D(0,1)} |J(z, \varphi)|^{\frac{\alpha}{2}} \, dx \, dy \right)^{\frac{1}{\alpha}} \leq C_\alpha^2 K^2 \pi^{\frac{\alpha}{2} - 1} \frac{4}{\pi} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^4)^2}{2 \log 3} \right\} \cdot |\Omega|.
\end{equation}

where

\[ C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu)]^{1/\alpha}}, \quad \nu = 10^4 \frac{\alpha - 2}{\alpha - 1} (24 \pi^2 K^2)^\alpha < 1. \]

Proof. Since $\varphi : \mathbb{D} \rightarrow \Omega$ is a conformal mapping, then by the inequality (3.2) using the equality $J(z, \varphi) = |\varphi'(z)|^2$ we have

\[ \left( \iint_{D(0,1)} |J(z, \varphi)|^\frac{\alpha}{2} \, dx \, dy \right)^{\frac{1}{\alpha}} = |D(0,1)|^\frac{1}{2} \left( \iint_{D(0,1)} |\varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} \leq C_\alpha^2 |D(0,1)|^\frac{1}{2} \left( \iint_{D(0,2)} |\varphi'(z)|^2 \, dx \, dy \right)^{\frac{1}{2}} = C_\alpha^2 \pi^{\frac{\alpha}{2} - 1} \frac{4}{\pi} \iint_{D(0,2)} |\varphi'(z)|^2 \, dx \, dy. \]

In the disc $D(0, 2)$ an extension of the conformal mapping $\varphi$ is a $K^2$-quasiconformal homeomorphism. Hence taking into account the inequality

\[ |\varphi'(z)|^2 \leq K^2 |J(z, \varphi)| \quad \text{for almost all } z \in D(0,2) \]

by the inequality (3.3) we get

\[ \frac{C_\alpha^2 \pi^{\frac{\alpha}{2} - 1}}{4} \iint_{D(0,2)} |\varphi'(z)|^2 \, dx \, dy \leq \frac{C_\alpha^2 K^2 \pi^{\frac{\alpha}{2} - 1}}{4} \iint_{D(0,2)} |J(z, \varphi)| \, dx \, dy \leq \frac{C_\alpha^2 K^2 \pi^{\frac{\alpha}{2} - 1}}{4} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^4)^2}{2 \log 3} \right\} \iint_{D(0,1)} |J(z, \varphi)| \, dx \, dy. \]

Thus, considering that

\[ \iint_{D(0,1)} |J(z, \varphi)| \, dx \, dy = |\Omega| \]

we have

\[ \left( \iint_{D(0,1)} |J(z, \varphi)|^{\frac{\alpha}{2}} \, dx \, dy \right)^{\frac{1}{\alpha}} \leq \frac{C_\alpha^2 K^2 \pi^{\frac{\alpha}{2} - 1}}{4} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^4)^2}{2 \log 3} \right\} \cdot |\Omega|. \]

□
4. ESTIMATES OF THE HYPERBOLIC $\alpha$-DILATATION AND THE FIRST NON-TRIVIAL EIGENVALUE OF THE NEUMANN–LAPLACE OPERATOR

In the work [27] was obtained the following estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in quasidiscs:

**Proposition 4.1.** Suppose a conformal homeomorphism $\varphi : \mathbb{D} \to \Omega$ maps the unit disc $\mathbb{D}$ onto a $K$-quasidisc $\Omega$. Then

$$
1/\mu_1(\Omega) \leq \frac{4}{\sqrt{\pi^2}} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha - 2}{\alpha}} Q(\alpha, \Omega)^{2/\alpha}
$$

for any $2 < \alpha < \frac{2K^2}{K^2 - 1}$.

By Proposition 4.1 and Theorem B we obtain the lower estimate of the first non-trivial eigenvalue of the Neumann–Laplace operator in $K$-quasidisc in terms of quasiconformal geometry of domains:

**Theorem A.** Let $\Omega \subset \mathbb{C}$ be a $K$-quasidisc. Then the spectrum of Neumann-Laplace operator in $\Omega$ is discrete, can be written in the form of a non-decreasing sequence

$$
0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq ... \leq \mu_n(\Omega) \leq ...
$$

and

$$
(4.1) \quad \frac{1}{\mu_1(\Omega)} \leq \frac{K^2C_\alpha^2}{\pi} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha - 2}{\alpha}} \exp \left\{ \frac{K^2\pi^2(2 + \pi^4)^2}{2\log 3} \right\} \cdot |\Omega|.
$$

for $2 < \alpha < \frac{2K^2}{K^2 - 1}$, where

$$
C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu)]^{1/\alpha}}, \quad \nu = 10^{4\alpha} \frac{\alpha - 2}{\alpha - 1} (24\pi^2 K^2)^\alpha < 1.
$$

**Proof.** By Proposition 4.1 for any $2 < \alpha < \frac{2K^2}{K^2 - 1}$ we have

$$
(4.2) \quad 1/\mu_1(\Omega) \leq \frac{4}{\sqrt{\pi^2}} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{\frac{2\alpha - 2}{\alpha}} Q(\alpha, \Omega)^{2/\alpha}.
$$

By Theorem B

$$
(4.3) \quad Q(\alpha, \Omega)^{2/\alpha} := \left( \iint_{D(0, 1)} |J(z, \varphi)|^{2/\alpha} \, dz \, d\varphi \right)^{\alpha} \leq C_\alpha^2 K^2 \pi^{\frac{2}{\alpha} - 1} \exp \left\{ \frac{K^2\pi^2(2 + \pi^4)^2}{2\log 3} \right\} \cdot |\Omega|.
$$

Combining inequalities (4.2) and (4.3) and performing a straightforward calculations we obtain the required inequality.

As an application of Theorem A, we obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the Laplace operator in the star-shaped and spiral-shaped domains.
A simply connected domain $\Omega^*$ is $\beta$-star-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps a unit disc $D$ onto $\Omega^*$ and satisfies the condition [17]:

$$\left| \arg \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |z| < 1.$$  

A simply connected domain $\Omega_*$ is $\beta$-spiral-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps a unit disc $D$ onto $\Omega_*$ and satisfies the condition [37, 38]:

$$\left| \arg e^{i\delta} \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |\delta| < \beta \pi/2, \quad |z| < 1.$$  

In [17] and [37, 38], respectively, it is shown that boundaries of domains $\Omega^*$ and $\Omega_*$ are $K$-quasicircles with $K = \cot^2(1 - \beta)\pi/4$.

Setting $\Omega = \Omega^*$ or $\Omega = \Omega_*$ Theorem A implies

$$\frac{1}{\mu_1(\Omega)} \leq \frac{C^2}{\pi} \frac{\cot^4(1 - \beta)\pi/4}{4} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{2n-2} \exp \left\{ \frac{\pi^2(2 + \pi^4)^2}{2 \log 3} \right\} \cdot |\Omega|,$$

where

$$C_\alpha = 10^6 \left( \frac{\pi}{\alpha - 4}\right)^{1/\alpha}, \quad \nu = 10^{4n} (24\pi^2 \cot^4(1 - \beta)\pi/4)^n \frac{2}{\alpha - 4} < 1.$$

5. Quasiconformal Reflection

Let $\Gamma$ be a Jordan curve on the Riemann sphere, and denote its complementary components by $\Omega$, $\Omega^*$. Suppose that there exists a sense-reversing quasiconformal mapping $\psi$ of the sphere onto itself which maps $\Omega$ onto $\Omega^*$ and keeps every point on $\Gamma$ fixed. Such mappings are called quasiconformal reflections.

Denote by $H$ the upper half-plane and by $H^*$ the lower half-plane of the complex plane $C$. Consider a conformal mapping $\varphi$ of $H$ on $\Omega$ and a conformal mapping $\varphi_*$ of $H^*$ to $\Omega^*$. Then a mapping $\varphi_*^{-1} \circ \psi \circ \varphi$ is a quasiconformal mapping of $H$ onto $H^*$ which induces a monotone mapping $h = \varphi_*^{-1} \circ \varphi$ of the real axis on itself [2].

In [2] L. Ahlfors and A. Beurling derived a necessary and sufficient condition for a boundary mapping $h$ to be restriction of a quasiconformal mapping of $H$ on itself (or on its reflection $H^*$). Without loss of generality it may be assumed that $h(\infty) = \infty$. Then $h$ admits a quasiconformal extension if and only if it satisfies a $M$-condition, namely an inequality

$$\frac{1}{M} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq M$$

which is to be fulfilled for all real $x$, $t$, $t > 0$, and with a constant $M \neq 0, \infty$. More precisely, if $h$ has a $K$-quasiconformal extension, then (5.1) holds with a $M = M(K) < e\pi K/16$, and if (5.1) holds, then $h$ has a $K$-quasiconformal extension such that $K = K(M) < M^2$.

In [3] L. Ahlfors proved that a Jordan curve $\Gamma$ admits a $K$-quasiconformal reflection if and only if $\Gamma$ satisfies the Ahlfors’s 3-point condition. In the following theorem, using the Ahlfors scheme, we calculate the upper estimates for the coefficient of quasiconformality $K$ for quasicircles given by the Ahlfors’s 3-point condition.
Theorem 5.1. Let a Jordan curve $\Gamma$ satisfies the Ahlfors’s 3-point condition: there exists a constant $C$ such that

$$ |\zeta_3 - \zeta_1| \leq C |\zeta_2 - \zeta_1| $$

for any three point on $\Gamma$ where $\zeta_3$ is between $\zeta_1$ and $\zeta_2$. Then $\Gamma$ to admit a $K$-quasiconformal reflection where $K$ depends only on $C$ and

$$ K < \frac{1}{210} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\}. $$

Proof. We will use the notations

$$ \alpha_1 = \text{arc} \zeta_1 \zeta_3, \quad \alpha_2 = \text{arc} \zeta_3 \zeta_2, \quad \beta_1 = \text{arc} \zeta_2 \infty, \quad \beta_2 = \text{arc} \zeta_1 \infty. $$

Thus

$$ \text{cap}(\alpha_1, \beta_1; \Omega) \cdot \text{cap}(\alpha_2, \beta_2; \Omega) = 1 \quad \text{and} \quad \text{cap}(\alpha_1, \beta_1; \Omega^*) \cdot \text{cap}(\alpha_2, \beta_2; \Omega^*) = 1. $$

Through the conformal mapping of $\Omega$, let $\zeta_1, \zeta_3, \zeta_2$ correspond to $x - t, x, x + t$. This means that

$$ \text{cap}(\alpha_1, \beta_1; \Omega) = \text{cap}(\alpha_2, \beta_2; \Omega) = 1. $$

Through the conformal mapping of $\Omega^*$, $\zeta_1, \zeta_3, \zeta_2$ correspond to $h(x - t), h(x), h(x + t)$.

In [3] Ahlfors proved that

$$ \text{cap}(\alpha_1, \beta_1; \Omega^*) \leq \pi \left( 1 + e^{2\pi C^5} \right)^2. $$

Now we obtain the lower estimate for the capacity $\text{cap}(\alpha_1, \beta_1; \Omega^*)$ using the capacity estimates for the Teichmüller condenser: denote by

$$ y = (h(x + t) - h(x))/ (h(x) - h(x - t)), $$

then by Lemma 3.3

$$ \text{cap} R_T(y) = \frac{2\pi}{\log \Phi(y)} > \frac{2\pi}{\log 32y}. $$

Combining inequalities (5.3) and (5.4), we obtain

$$ \frac{2\pi}{\log 32y} < \text{cap} R_T(y) = \text{cap}(\alpha_1, \beta_1; \Omega^*) \leq \pi \left( 1 + e^{2\pi C^5} \right)^2 $$

or

$$ \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq \frac{1}{32} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\}. $$

Hence $h$ satisfies an $M$-condition, i.e.

$$ \frac{1}{M(C)} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq M(C) $$

where

$$ M(C) = \frac{1}{32} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\}. $$

So, [2] there exist a $K$-quasiconformal reflection such that $K < M^2(C)$.

Finally we get

$$ K < \left( \frac{1}{32} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\} \right)^2 = \frac{1}{210} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\}. $$

□
Combining Theorem A and Theorem 5.1 we obtain

**Corollary 5.2.** Let a domain \( \Omega \subset \mathbb{R}^2 \) be bounded by a Jordan curve \( \Gamma \) satisfying the Ahlfors’s 3-point condition. Then

\[
\frac{1}{\mu_1(\Omega)} \leq \frac{C_\alpha^2 e^{2(1+e^{2\pi}C^\alpha)^2}}{2^{20}\pi} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^{2\alpha - 2} \exp \left\{ \frac{\pi^2(2 + \pi^4)^2 e^{2(1+e^{2\pi}C^\alpha)^2}}{2^{21}\log 3} \right\} \cdot |\Omega|,
\]

for \( 2 < \alpha < \frac{2K^2}{K^2 - 1} \), where

\[
C_\alpha = \frac{10^6}{((\alpha - 1)(1 - \nu))^{1/\alpha}}, \quad \nu = 10^{-4\alpha} \left( \frac{24\pi^2}{2^{20}} e^{2(1+e^{2\pi}C^\alpha)^2} \right)^\alpha \frac{\alpha - 2}{\alpha - 1} < 1.
\]

### 6. Examples in Fractal Type Domains

**Rohde snowflake.** In [36] S. Rohde constructed a collection \( S \) of snowflake type planar curves with the intriguing property that each planar quasicircle is bi-Lipschitz equivalent to some curve in \( S \).

Rohde’s catalog is

\[
S := \bigcup_{1/4 \leq p < 1/2} S_p
\]

where \( p \) is a snowflake parameter. Each curve in \( S_p \) is built in a manner reminiscent of the construction of the von Koch snowflake. Thus, each \( S \in S_p \) is the limit of a sequence \( S^n \) of polygons where \( S^{n+1} \) is obtained from \( S^n \) by using the replacement rule illustrated in Figure 6.1: for each of the \( 4^n \) edges of \( S^n \) we have two choices, either we replace \( E \) with the four line segments obtained by dividing \( E \) into four arcs of equal diameter, or we replace \( E \) by a similarity copy of the polygonal arc \( A_p \) pictured at the top right of Figure 6.1. In both cases \( E \) is replaced by four new segments, each of these with diameter \( (1/4)\text{diam}(E) \) in the first case or with diameter \( p \text{diam}(E) \) in the second case. The second type of replacement is done so that the "tip" of the replacement arc points into the exterior of \( S^n \). This iterative process starts with \( S^1 \) being the unit square, and the snowflake parameter, thus the polygon arc \( A_p \), is fixed throughout the construction.

![Figure 6.1. Construction of a Rohde-snowflake.](image)

The sequence \( S^n \) of polygons converges, in the Hausdorff metric, to a planar quasicircle \( S \) that we call a **Rohde snowflake** constructed with snowflake parameter.
Then $S_p$ is the collection of all Rohde snowflakes that can be constructed with snowflake parameter $p$.

In [31] established that each Rohde snowflake $S$ in $S_p$ is $C$-bounded turning with

$$C = C(p) = \frac{16}{1 - 2p}, \quad 1/4 \leq p < 1/2.$$  

A planar curve $\Gamma$ satisfies the $C$-bounded turning, $C \geq 1$, if for each pair of points $x, y$, on $\Gamma$, the smaller diameter subarc $\Gamma[x, y]$ of $\Gamma$ that joins $x, y$ satisfies

(6.1) $\text{diam}(\Gamma[x, y]) \leq C|x - y|.$

The $C$-bounded turning condition (6.1) is equivalent the Ahlfors’s 3-point condition (5.2) with the same constant $C$ [24].

The following theorem gives the lower estimates of the first non-trivial eigenvalue of the Neumann–Laplace operator in domains type a Rohde snowflakes:

**Theorem C.** Let $S_p \subset \mathbb{R}^2$, $1/4 \leq p < 1/2$, be the Rohde snowflake. Then the spectrum of Neumann–Laplace operator in $S_p$ is discrete, can be written in the form of a non-decreasing sequence

$$0 = \mu_0(S_p) < \mu_1(S_p) \leq \mu_2(S_p) \leq \ldots \leq \mu_n(S_p) \leq \ldots,$$

and

$$\frac{1}{\mu_1(S_p)} \leq \frac{C_\alpha^2 e^{4(1+\alpha^2((16/(1-2p))^{1/2})}}{2^{40} \pi^2} \left( \frac{2\alpha - 2}{\alpha - 2} \right)^\frac{2\alpha - 2}{\alpha} \times \exp \left\{ - \frac{\pi^2 (2 + \pi^4) e^{4(1+\alpha^2((16/(1-2p))^{1/2})}}{2^{41} \log 3} \right\} |S_p|,$$

for $2 < \alpha < \frac{2K^2}{K - 1}$, where

$$C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu)]^{1/\alpha}}, \quad \nu = 10^{14\alpha} \left( \frac{24\pi^2}{24^{\alpha}} e^{4(1+\alpha^2((16/(1-2p))^{1/2})}} \right)^{\frac{\alpha - 2}{\alpha} - 1} < 1.$$  

The proof of Theorem C immediately follows from the Corollary [5.2] given that any $L$-bi-Lipschitz planar homeomorphism is $K$-quasiconformal with $K = L^2$.

**Appendix**

Firstly we discuss a new notion of the hyperbolic $\alpha$-dilatation $Q(\alpha, \Omega)$ and its convergence hyperbolic interval $\text{HI}(\Omega)$ in connection with known results. Of course, quotients $Q(\alpha, \Omega)$ and $\text{HI}(\Omega)$ can be defined also in non bounded domains.

Recall definitions:

$$Q(\alpha, \Omega) := \iint_D |\varphi'(z)|^\alpha \, dxdy = \iint_{\Omega} \left| (\varphi^{-1})'(w) \right|^{2-\alpha} \, dudv$$

where $\varphi : \mathbb{D} \to \Omega$ is a Riemann conformal homeomorphism;

$$\text{HI}(\Omega) := \{ \alpha \in \mathbb{R} : Q(\alpha, \Omega) < \infty \}.$$  

Let us remark that by the definition

$$Q(2, \Omega) = |\Omega|,$$

i. e is finite for any domain of finite measure. We don’t know any simple interpretation of $Q(\alpha, \Omega)$ for a number $\alpha$ other than 2.
In these new terms of the Inverse Brennan’s conjecture [13] states: Let \( \Omega \subset \mathbb{C} \) be a simply connected planar domain with nonempty boundary, and \( \varphi : \mathbb{D} \to \Omega \) be a conformal homeomorphism. Then

\[ HI(\Omega) = (-2, 2/3). \]

The upper bound \( \alpha = 2/3 \) is proved, the lower bound is only conjectured.

By [26] for bounded domains it is proved that

\[ HI(\Omega) = (-2, 2]. \]

The upper bound \( \alpha = 2 \) is proved, the lower bound is only conjectured.

Note that for smooth bounded domains [13]

\[ HI(\Omega) = (-\infty, \infty). \]

In [27] we demonstrated that for any \( K \)-quasidisc

\[ 1 \leq \alpha < \frac{2K^2}{K^2 - 1} \subset HI(\Omega). \]

It can be reformulated in terms of the Ahlfors condition.

**Conjecture.** Interval \( HI(\Omega) \) is defined by the hyperbolic geometry of \( \Omega \).

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Department of Mathematics, Ben-Gurion University of the Negev, P.O.Box 653, Beer Sheva, 8410501, Israel
E-mail address: vladimir@math.bgu.ac.il

Department of Higher Mathematics and Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Ave. 30, Russia; Department of General Mathematics, Tomsk State University, 634050 Tomsk, Lenin Ave. 36, Russia
Current address: Department of Mathematics, Ben-Gurion University of the Negev, P.O.Box 653, Beer Sheva, 8410501, Israel
E-mail address: vpchelintsev@vtomske.ru

Department of Mathematics, Ben-Gurion University of the Negev, P.O.Box 653, Beer Sheva, 8410501, Israel
E-mail address: ukhlov@math.bgu.ac.il
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