The homotopy type of spaces of rational curves on a toric variety

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Abstract

Spaces of holomorphic maps from the Riemann sphere to various complex manifolds (holomorphic curves) have played an important role in several area of mathematics. In a seminal paper G. Segal investigated the homotopy type of holomorphic curves on complex projective spaces and M. Guest on compact smooth toric varieties. Recently Mostovoy and Villanueva, obtained a far reaching generalisation of these results, and in particular (for holomorphic curves) improved the stability dimension obtained by Guest. In this paper, we generalize their result to holomorphic curves, on certain non-compact smooth toric varieties.

1 Introduction

The motivation for this paper. For a complex manifold $X$, let $\text{Map}^*(S^2, X)$ (resp. $\text{Hol}^*(S^2, X)$) denote the space of all base point preserving continuous maps (resp. base point preserving holomorphic maps) from the Riemann sphere $S^2$ to $X$. The relation between the topology of the space $\text{Hol}^*(S^2, X)$ and that of the space $\text{Map}^*(S^2, X)$ has long been an object of study in several areas of mathematics and physics (e.g. [2], [3]). Since Segal’s seminal study [23] of the case $X = \mathbb{C}P^n$, a number of mathematicians have investigated this and various closely related problems. In particular, M. Guest [10] obtained the partial generalization of Segal’s result to the case of smooth compact toric varieties $X$. More recently, J. Mostovoy and E. Munguia-Villanueva [21] proved a far-reaching generalization of Guest’s result for the case of spaces of holomorphic maps from $\mathbb{C}P^m$ to a compact toric variety $X$ for $m \geq 1$. The homology stability dimension which they obtained is also an improvement on Guest’s result for the case $m = 1$. 

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In [17], the present authors studied this problem of the case \( m = 1 \) for a certain family of non-compact smooth toric subvarieties \( X_I \) of \( \mathbb{C}P^n \), and showed that the result of Mostovoy-Villanueva [21] can be extended to this case (with the same stability dimension).

In this paper, we shall prove that this result can be further extended to non-compact smooth toric varieties \( X \) which satisfy certain two conditions (see the conditions (1.12.1) and (1.12.2)). These conditions are satisfied for a wide range of smooth toric varieties (including all compact ones).

In fact, we will do better and show that under the certain condition “homotopy equivalence” can be replaced by “homotopy equivalence” (up to the same dimension).

The broad outline of our argument is analogous to Segal’s seminal paper [23] (a brief sketch of such an argument is given in [10, §5]). Namely, for a smooth toric variety \( X \), we first prove that there is a homotopy equivalence between certain limits of spaces \( \text{Hol}^D(S^2, X) \) of holomorphic maps, stabilized with respect to a suitably defined degree \( D \), and the double loop space \( \text{Map}^*(S^2, X) = \Omega^2 X \). We can refer to this as the stable result. The method used to prove it is a generalization of the scanning map technique used by Segal in [23]. In particular, we describe a generalization to the case of toric varieties of a fibration sequence that plays the key role in Segal’s argument (see Proposition 3.4).

Note that in [21] a quite different stabilization is used, which is based on the Stone-Weierstrass theorem. This stabilization has the advantage that it can used in the case of holomorphic maps from \( \mathbb{C}P^m \) to a compact toric variety \( X \) for any \( m \geq 1 \). However, the usefulness of the Stone-Weierstrass theorem is based on the fact that two holomorphic maps that are ‘uniformly close’, with respect to some metric, are actually homotopic. This is true when the metric on \( X \) is complete (e.g. when \( X \) is compact), but not for general \( X \). We are, therefore, unsure if our results can be extended to the case \( \text{Hol}^*(\mathbb{C}P^m, X) \), for \( m > 1 \). Even if this is possible, we believe that our generalization of Segal’s argument to the case of toric varieties is of some independent interest.

The second part of the paper is concerned with establishing “homology stability dimensions” for the inclusion map from \( \text{Hol}^D(S^2, X) \) to the double loop space \( \Omega^2 D_X \) of maps of degree \( D \). These stability dimensions depend both on the degree \( D \) and the toric variety \( X \). The method is based on an a modification of the Vassieiv spectral sequence [24] due to Mostovoy ([20], [19]). The stability dimensions in homology are obtained by identifying a stable region of this spectral sequence. By observing that under a certain condition (described later) these mapping spaces are simply connected, we can strengthen our results by replacing homology equivalences by homotopy equivalences (up to the same dimension).

**Basic definitions and notations.** A convex rational polyhedral cone \( \sigma \) in \( \mathbb{R}^n \) is a subset of \( \mathbb{R}^n \) of the form

\[
\sigma = \text{Cone}(m_1, \ldots, m_s) = \left\{ \sum_{k=1}^s \lambda_k m_k : \lambda_k \geq 0 \text{ for any } 1 \leq k \leq s \right\}
\]

for some finite set \( S = \{m_k\}_{k=1}^s \subset \mathbb{Z}^n \). The dimension of \( \sigma \) is the dimension of the smallest subspace which contains \( \sigma \). A convex rational polyhedral cone \( \sigma \) is called strongly convex.
if \( \sigma \cap (-\sigma) = \{0\} \). A face \( \tau \) of \( \sigma \) is a subset \( \tau \subset \sigma \) of the form \( \tau = \sigma \cap \{ x \in \mathbb{R}^n : L(x) = 0 \} \) for some linear form \( L \) on \( \mathbb{R}^n \), such that \( \sigma \subset \{ x \in \mathbb{R}^n : L(x) \geq 0 \} \). In this case, if we set \( \{ k : L(m_k) = 0 \} = \{ i_1, \ldots, i_s \} \), we easily see that \( \tau = \text{Cone}(m_{i_1}, \ldots, m_{i_s}) \) and so that a face \( \tau \) is also a strongly convex rational polyhedral cone.

A finite collection \( \Sigma \) of strongly convex rational polyhedral cones in \( \mathbb{R}^n \) is called a fan in \( \mathbb{R}^n \) if every face of an element of \( \Sigma \) belongs to \( \Sigma \) and the intersection of any two elements of \( \Sigma \) is a face of each.

An \( n \) dimensional irreducible normal variety \( X \) (over \( \mathbb{C} \)) is called a toric variety if it has a Zariski open subset \( T^n_\mathbb{C} = (\mathbb{C}^*)^n \) and the action of \( T^n_\mathbb{C} \) on itself extends to an action of \( T^n_\mathbb{C} \) on \( X \). The most significant property of a toric variety is the fact that it is characterized up to isomorphism entirely by its associated fan \( \Sigma \). We denote by \( X_\Sigma \) the toric variety associated to a fan \( \Sigma \) (see [8] in detail).

It is well known that there are no holomorphic maps \( \mathbb{C}P^1 = S^2 \to T^n_\mathbb{C} \) except the constant maps, and that the fan \( \Sigma \) of \( T^n_\mathbb{C} \) is \( \Sigma = \{0\} \). Hence, without loss of generality we always assume that \( X_\Sigma \neq T^n_\mathbb{C} \) and that any fan \( \Sigma \) in \( \mathbb{R}^n \) satisfies the condition \( \{0\} \subsetneq \Sigma \).

**Definition 1.1.** For such a fan \( \Sigma \) in \( \mathbb{R}^n \), let

\[
(1.1) \quad \Sigma(1) = \{ \rho_1, \ldots, \rho_r \}
\]
denote the set of all one dimensional cones in \( \Sigma \) for some positive integer \( r \). For each integer \( 1 \leq k \leq r \), we denote by \( n_k \in \mathbb{Z}^n \) the primitive generator of \( \rho_k \), such that \( \rho_k \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} \cdot n_k \). Note that \( \rho_k = \text{Cone}(n_k) = \mathbb{R}_{\geq 0} \cdot n_k \).

**Definition 1.2.** Let \( K \) be a simplicial complex on the index set \( [r] = \{1, 2, \ldots, r\} \)\(^1\) and let \( (X, \underline{A}) = \{(X_1, A_1), \ldots, (X_r, A_r)\} \) be a set of pairs of based spaces such that \( A_i \subset X_i \) for each \( i \).

(i) The polyhedral product \( Z_K(X, \underline{A}) \) of an \( r \)-tuple of pairs of spaces \( (X, \underline{A}) \) with respect to \( K \) is defined by \( Z_K(X, \underline{A}) = \bigcup_{\sigma \in K} (X, \underline{A})^\sigma \), where we set

\[
(1.2) \quad (X, \underline{A})^\sigma = \{ (x_1, \ldots, x_r) \in X_1 \times \cdots \times X_r : x_k \in A_k \text{ if } k \notin \sigma \}. 
\]

When \( (X_i, A_i) = (X, A) \) for each \( 1 \leq i \leq r \), we write \( Z_K(X, A) = Z_K(X, \underline{A}) \).

(ii) For each subset \( \sigma = \{i_1, \ldots, i_s\} \subset [r] \), let \( L_\sigma \) denote the coordinate subspace of \( \mathbb{C}^r \) defined by

\[
(1.3) \quad L_\sigma = \{ (x_1, \ldots, x_r) \in \mathbb{C}^r : x_{i_1} = \cdots = x_{i_s} = 0 \}.
\]

Let \( U(K) \) denote the complement of coordinate subspaces of type \( K \) given by

\[
(1.4) \quad U(K) = \mathbb{C}^r \setminus \bigcup_{\sigma \in I(K)} L_\sigma = \mathbb{C}^r \setminus \bigcup_{\sigma \subset [r] \mid \sigma \notin K} L_\sigma .
\]

\(^1\)Let \( K \) be some set of subsets of \( [r] \). Then the set \( K \) is called an abstract simplicial complex on the index set \( [r] \) if the following condition holds: if \( \tau \subset \sigma \) and \( \sigma \in K \), then \( \tau \in K \). In this paper by a simplicial complex \( K \) we always mean an abstract simplicial complex, and we always assume that a simplicial complex \( K \) contains the empty set \( \emptyset \).
where we set

\[(1.5)\]
\[I(K) = \{ \sigma \subseteq [r] : \sigma \notin K \}.\]

Note that \(U(K)\) is the Alexander dual of the space \(L(\Sigma) = \bigcup_{\sigma \in I(K)} L_\sigma\) in \(\mathbb{C}^r\), and it is easy to see that

\[(1.6)\]
\[U(K) = Z_K(\mathbb{C}, \mathbb{C}^r).\]

(iii) For a fan \(\Sigma\) in \(\mathbb{R}^n\) as in Definition \[\square\] let \(K_\Sigma\) denote the underlying simplicial complex of \(\Sigma\) defined by

\[(1.7)\]
\[K_\Sigma = \left\{ \{ i_1, \ldots, i_s \} \subseteq [r] : \langle n_{i_1}, n_{i_2}, \ldots, n_{i_s} \rangle \text{ span a cone in } \Sigma \right\}.\]

It is easy to see that \(K_\Sigma\) is a simplicial complex on the index set \([r]\).

Remark 1.3. The fan \(\Sigma\) is completely determined by the pair \((K_\Sigma, \{ n_k \}_{k=1}^r)\). In fact, if we set \(C(\sigma) = \text{Cone}(n_{i_1}, \ldots, n_{i_s})\) if \(\sigma = \{ i_1, \ldots, i_s \} \subseteq K_\Sigma\) and \(C(\emptyset) = \{0\}\), then it is easy to see that \(\Sigma = \{ C(\sigma) : \sigma \in K_\Sigma \}\).

\[\square\]

Definition 1.4 (\[\square\], Definition 6.27, Example 6.39). Let \(K\) be a simplicial complex on the index set \([r]\). Then we denote by \(Z_K\) and \(DJ(K)\) the moment-angle complex of \(K\) and the Davis-Januszkiewicz space of \(K\), respectively, which are defined by

\[(1.8)\]
\[Z_K = Z_K(D^2, S^1), \quad DJ(K) = Z_K(\mathbb{CP}^\infty, \ast).\]

Definition 1.5. Let \(\Sigma\) be a fan in \(\mathbb{R}^n\) as in Definition \[\square\]. Let \(G_\Sigma \subseteq T^r_\mathbb{C} = (\mathbb{C}^r)^r\) denote the subgroup defined by

\[(1.9)\]
\[G_\Sigma = \{ (\mu_1, \ldots, \mu_r) \in T^r_\mathbb{C} : \prod_{k=1}^r \langle \mu_k \rangle^{\langle n_k, m \rangle} = 1 \text{ for any } m \in \mathbb{Z}^n \},\]

where \(\langle u, v \rangle = \sum_{k=1}^n u_kv_k\) for \(u = (u_1, \ldots, u_n)\) and \(v = (v_1, \ldots, v_n) \in \mathbb{R}^n\).

Then consider the natural \(G_\Sigma\)-action on \(Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^r)\) given by coordinate-wise multiplication, i.e. \(\langle \mu_1, \ldots, \mu_r \rangle \cdot (x_1, \ldots, x_r) = (\mu_1x_1, \ldots, \mu_rx_r)\).

Let \(Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^r)/G_\Sigma = U(K_\Sigma)/G_\Sigma\) denote the corresponding orbit space.

Theorem 1.6 (\[\square\], Theorem 2.1; \[\square\], Theorem 3.1). Suppose that the set \(\{ n_k \}_{k=1}^r\) of all primitive generators spans \(\mathbb{R}^n\) (i.e. \(\sum_{k=1}^r \mathbb{R} \cdot n_k = \mathbb{R}^n\)).

(i) Then there is a natural isomorphism

\[(1.10)\]
\[X_\Sigma \cong Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^r)/G_\Sigma = U(K_\Sigma)/G_\Sigma.\]

(ii) If \(f : \mathbb{CP}^m \to X_\Sigma\) is a holomorphic map, there exists an \(r\)-tuple \(D = (d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 0})^r\) of non-negative integers satisfying the condition \(\sum_{k=1}^r d_k n_k = 0\) and homogenous
polynomials \( f_i \in \mathbb{C}[z_0, \cdots, z_m] \) of degree \( d_i \) \((i = 1, \cdots, r)\) such that polynomials \( \{f_i\}_{i \in \sigma} \) have no common root except \( 0 \in \mathbb{C}^{m+1} \) for each \( \sigma \in I(K_\Sigma) \) and that the diagram

\[
\begin{array}{ccc}
\mathbb{C}^{m+1} \setminus \{0\} & \xrightarrow{(f_1, \cdots, f_r)} & U(K_\Sigma) \\
\gamma_m \downarrow & & \downarrow q_\Sigma \\
\mathbb{CP}^m & \xrightarrow{f} & U(K_\Sigma)/G_\Sigma = X_\Sigma
\end{array}
\]

(1.11)

is commutative, where \( \gamma_m : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m \) denotes the canonical Hopf fibering and the map \( q_\Sigma \) is a canonical projection induced from the identification (1.10). In this case, we call this holomorphic map \( f \) as a holomorphic map of degree \( D = (d_1, \cdots, d_r) \) and we represent it as

\[
f = [f_1, \cdots, f_r].
\]

(iii) If \( g_i \in \mathbb{C}[z_0, \cdots, z_m] \) is a homogenous polynomial of degree \( d_i \) \((1 \leq i \leq r)\) such that \( f = [f_1, \cdots, f_r] = [g_1, \cdots, g_r] \), there exists some element \((\mu_1, \cdots, \mu_r) \in G_\Sigma\) such that \( f_i = \mu_i \cdot g_i \) for each \( 1 \leq i \leq r \). Thus, such \( r \)-tuple \((f_1, \cdots, f_r)\) of homogenous polynomials representing the holomorphic map \( f \) is uniquely determined up to \( G_\Sigma \)-action. \( \square \)

**Assumptions.** From now on, let \( \Sigma \) be a fan in \( \mathbb{R}^n \) satisfying the condition (1.11) as in Definition 1.1 and we shall assume that the following two conditions hold.

1.121) The set \( \{n_k\}_{k=1}^r \) of primitive generators spans \( \mathbb{Z}^n \) over \( \mathbb{Z} \), i.e.
\[
\sum_{k=1}^r \mathbb{Z} \cdot n_k = \mathbb{Z}^n.
\]

1.122) There is an \( r \)-tuple \( D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r \) such that \( \sum_{k=1}^r d_k n_k = 0 \).

**Remark 1.7.** (i) Note that the condition (1.121) always holds if \( X_\Sigma \) is a compact smooth toric variety (by Lemma 2.5 below).

(ii) If the condition (1.121) holds, one can easily see that the set \( \{n_k\}_{k=1}^r \) spans \( \mathbb{R}^n \) over \( \mathbb{R} \) and we have the isomorphism (1.10).

(iii) Let \( \Sigma \) denote the fan in \( \mathbb{R}^2 \) given by \( \Sigma = \{0, \text{Cone}(n_1), \text{Cone}(n_2)\} \) for the standard basis \( n_1 = (1, 0), \ n_2 = (0, 1) \). Then the toric variety \( X_\Sigma \) of \( \Sigma \) is \( \mathbb{C}^2 \) which has trivial homogenous coordinates. It is clearly a (simply connected) smooth toric variety, and the condition (1.121) also holds. However, in this case, \( \sum_{k=1}^2 d_k n_k = 0 \) iff \( (d_1, d_2) = (0, 0) \). Hence, it follows from (ii) of Proposition 1.6 that there are no holomorphic maps \( \mathbb{CP}^1 = S^2 \to X_\Sigma = \mathbb{C}^2 \) other than the constant maps. Assuming the condition (1.122) guarantees the existence of non-trivial holomorphic maps. Of course, it would be sufficient to assume that \( (d_1, \cdots, d_r) \neq (0, \cdots, 0) \) but if \( d_i = 0 \) for some \( i \), then the number \( d(D, \Sigma) \) (defined in (1.17)) is not a positive integer and our assertion (Theorem 1.9 below) is vacuous. For this reason, we will assume the condition (1.122). \( \square \)
Spaces of holomorphic maps. Let $X_\Sigma$ be a smooth toric variety and we make the identification $X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma$. Now consider a base point preserving holomorphic map $f = [f_1, \ldots, f_r] : \mathbb{C}P^m \to X_\Sigma$ for the case $m = 1$. In this situation, we identify $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \infty$ and choose the points $\infty$ and $[1, 1, \cdots, 1]$ as the base points of $\mathbb{C}P^1$ and $X_\Sigma$, respectively. Then by taking $z = \frac{m}{z_1}$ we can view $f_k$ as a monic polynomial $f_k(z) \in \mathbb{C}[z]$ of degree $d_k$ for each $1 \leq k \leq r$ with the complex variable $z$. Now we can define the space of holomorphic maps as follows.

**Definition 1.8.** (i) Let $P^d(\mathbb{C})$ denote the space of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{C}[z]$ of degree $d$, and we set

$$P^D = P^{d_1}(\mathbb{C}) \times P^{d_2}(\mathbb{C}) \times \cdots \times P^{d_r}(\mathbb{C}).$$

Note that there is an homeomorphism $P^d(\mathbb{C}) \cong \mathbb{C}^d$ by identifying $z^d + \sum_{k=1}^{d} a_k z^{d-k} \mapsto (a_1, \ldots, a_d) \in \mathbb{C}^d$.

(ii) For $r$-tuple $D = (d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers satisfying the condition (1.12), let $\text{Hol}^*_D(S^2, X_\Sigma)$ denote the space of all $r$-tuples $(f_1(z), \cdots, f_r(z)) \in P^D$ of monic polynomials satisfying the condition

$$\text{(†) the polynomials } f_{i_1}(z), \ldots, f_{i_s}(z) \text{ have no common root for any } \sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma), \text{ i.e. } (f_{i_1}(\alpha), \cdots, f_{i_s}(\alpha)) \neq (0, \cdots, 0) \text{ for any } \alpha \in \mathbb{C}.$$

By identifying $X_\Sigma = U(\mathcal{K}_\Sigma)/G_\Sigma$ and $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \infty$, define the natural inclusion map $i_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Map}^*_D(S^2, X_\Sigma) = \Omega^2 X_\Sigma$ by

$$i_D(f_1(z), \cdots, f_r(z))(\alpha) = \begin{cases} [f_1(\alpha), \cdots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1, 1, \cdots, 1] & \text{if } \alpha = \infty \end{cases}$$

where we choose the points $\infty$ and $[1, 1, \cdots, 1]$ as the base points of $S^2$ and $X_\Sigma$, respectively. Since the representation of polynomials in $P^D$ representing a base point preserving holomorphic map of degree $D$ is uniquely determined, the space $\text{Hol}^*_D(S^2, X_\Sigma)$ can be identified with the space of base point preserving holomorphic maps of degree $D$. Moreover, since $\text{Hol}^*_D(S^2, X_\Sigma)$ is connected, the image of $i_D$ is contained in a certain path-component of $\Omega^2 X_\Sigma$, which is denoted by $\Omega^2 D X_\Sigma$. Then we have the natural inclusion map

$$i_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Map}^*_D(S^2, X_\Sigma) = \Omega^2 D X_\Sigma.$$

(iii) We say that a set $\{n_{i_1}, \cdots, n_{i_s}\}$ is a a primitive collection if it does not span a cone in $\Sigma$ but any proper subset of it does.

Then define the integers $r_{\min}(\Sigma)$ and $d(D, \Sigma)$ by

$$r_{\min}(\Sigma) = \min \{s \in \mathbb{Z}_{ \geq 1} : \{n_{i_1}, \cdots, n_{i_s}\} \text{ is a primitive collection} \},$$

$$d(D, \Sigma) = (2 r_{\min}(\Sigma) - 3) d_{\min} - 2, \text{ where } d_{\min} = \min \{d_1, \cdots, d_r\}.$$  

(iv) A map $f : X \to Y$ is a homology equivalence through dimension $N$ (resp. a homotopy equivalence through dimension $N$) if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$ (resp. $f_* : \pi_k(X) \to \pi_k(Y)$) is an isomorphism for any $k \leq N$. 

6
The main results. The main result of this paper generalizes the results given in [17] and extend some result obtained in [21] as follows.

**Theorem 1.9.** Let \( X_\Sigma \) be a smooth toric variety such that the conditions (1.12.1) and (1.12.2) are satisfied. Then the inclusion map

\[
i_D : \text{Hol}_D^*(S^2, X_\Sigma) \to \Omega_D^2 X_\Sigma
\]

is a homotopy equivalence through dimension \( d(D, \Sigma) \) if \( r_{\min}(\Sigma) \geq 3 \) and a homology equivalence through dimension \( d(D, \Sigma) = d_{\min} - 2 \) if \( r_{\min}(\Sigma) = 2 \).

**Remark 1.10.** (i) If \( X_\Sigma \) is compact, we know that the map \( i_D \) is a homology equivalence through dimension \( d(D, \Sigma) \) by the result of Mostovoy-Villanueva [21]. However, their argument is based on the Stone-Weierstrass theorem which, as mentioned above, requires \( X_\Sigma \) to be compact (or at least to possess a complete metric).

(ii) If \( r_{\min}(\Sigma) \geq 3 \), Theorem 1.9 states that the map \( i_D \) is a homotopy equivalence through the dimension \( d(D, \Sigma) \). So the assertion of Theorem 1.9 is stronger than that of [21] even if \( X_\Sigma \) is compact (for \( m = 1 \)). Moreover, we conjecture that the map \( i_D \) is a homotopy equivalence through the same dimension even when \( r_{\min}(\Sigma) = 2 \). Although we cannot prove this, there are several reasons which support this conjecture (for example, see (ii) of Corollary 1.11). In fact, the conjecture is known to hold for certain non-compact toric varieties \( X_n \) ([17], Theorem 1.6]).

**Corollary 1.11.** Let \( X_\Sigma \) be a compact smooth toric variety and let \( D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r \) be an \( r \)-tuple of positive integers satisfying the condition (1.12.2). Let \( \Sigma(1) \) denote the set of all one dimensional cones in \( \Sigma \), and \( \Sigma_1 \) any fan in \( \mathbb{R}^n \) such that \( \Sigma(1) \subset \Sigma_1 \subset \Sigma \). Then \( X_{\Sigma_1} \) is a non-compact smooth toric subvariety of \( X_\Sigma \) and the inclusion map

\[
i_D : \text{Hol}_D^*(S^2, X_{\Sigma_1}) \to \Omega_D^2 X_{\Sigma_1}
\]

is a homotopy equivalence through the dimension \( d(D, \Sigma_1) \) if \( r_{\min}(\Sigma_1) \geq 3 \) and a homology equivalence through dimension \( d(D, \Sigma_1) = d_{\min} - 2 \) if \( r_{\min}(\Sigma_1) = 2 \).

Since the case \( X_\Sigma = \mathbb{CP}^n \) of Corollary 1.11 was treated in [17], we shall consider the case that \( X_\Sigma \) is the Hirzerbruch surface \( H(k) \).

**Example 1.12.** For an integer \( k \in \mathbb{Z} \), let \( H(k) \) be the Hirzerbruch surface defined by

\[
H(k) = \{([x_0 : x_1 : x_2], [y_1 : y_2]) \in \mathbb{CP}^2 \times \mathbb{CP}^1 : x_1y_1^k = x_2y_2^3 \} \subset \mathbb{CP}^2 \times \mathbb{CP}^1.
\]

Since there are isomorphisms \( H(-k) \cong H(k) \) for \( k \neq 0 \) and \( H(0) \cong \mathbb{CP}^2 \times \mathbb{CP}^1 \), without loss of generality we can assume that \( k \geq 1 \). Let \( \Sigma_k \) denote the fan in \( \mathbb{R}^2 \) given by

\[
\Sigma_k = \{ \text{Cone}(n_i, n_{i+1}) \ (1 \leq i \leq 3), \text{Cone}(n_4, n_1), \text{Cone}(n_j) \ (1 \leq j \leq 4), \{0\} \},
\]

where we set \( n_1 = (1, 0), \ n_2 = (0, 1), \ n_3 = (-1, k), \ n_4 = (0, -1) \).
It is easy to see that $\Sigma_k$ is the fan of $H(k)$ and that $\Sigma_k(1) = \{\text{Cone}(n_i) : 1 \leq i \leq 4\}$. Since $\{n_1, n_4\}$ and $\{n_2, n_4\}$ are the only primitive collections, $r_{\text{min}}(\Sigma_k) = 2$. Moreover, for a 4-tuple $D = (d_1, d_2, d_3, d_4) \in (\mathbb{Z}_{\geq 1})^4$ of positive integers, the equality $\sum_{k=1}^{3} d_k n_k = 0$ holds if and only if $(d_1, d_2, d_3, d_4) = (d_1, d_2, 1, kd_1 + d_2)$ and $d_{\text{min}} = \min\{d_1, d_2\} = \min\{d_1, d_2, d_3, d_4\} = \min\{d_1, d_2\}$. Hence, by Corollary 1.11 we have the following:

**Corollary 1.13.** Let $k \geq 1$ be a positive integer and $\Sigma$ be a fan in $\mathbb{R}^2$ such that $\Sigma_k(1) = \{\text{Cone}(n_i) : 1 \leq i \leq 4\} \subseteq \Sigma \subsetneq \Sigma_k$ as in Example 1.12. Then $X_\Sigma$ is a non-compact open smooth subvariety of $H(k)$. If $D = (d_1, d_2, d_1, kd_1 + d_2) \in (\mathbb{Z}_{\geq 1})^4$, the inclusion map

$$i_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \Omega^2_D X_\Sigma$$

is a homology equivalence through dimension $\min\{d_1, d_2\} - 2$. \qed

**Remark 1.14.** (i) There are 15 non isomorphic (as varieties) non-compact subvarieties $X_\Sigma$ of $H(k)$ which satisfy the assumption of Corollary 1.13.

(ii) Note that there is an isomorphism

$$(1.18) \quad \pi_2(X_\Sigma) \cong \mathbb{Z}^{r-n} \quad \text{(see Lemma 2.3 below)},$$

in general. So $(r - n)$ of the $r$ positive integers $\{d_k\}_{k=1}^r$ can be chosen freely. For example, in Example 1.12 $(r, n) = (4, 2)$ and $r - n = 4 - 2 = 2$. In this case, only two positive integers $d_1$, $d_2$ can be chosen freely and the other integers $d_3$ and $d_4$ are determined uniquely as $(d_3, d_4) = (d_1, kd_1 + d_2)$. \qed

This paper is organized as follows. In 2 we introduce the scanning map and prove the stability theorem for the scanning map (Theorem 2.12). In 3 we recall the basic properties of polyhedral products, determine the homotopy type of the space $E^\Sigma(\overline{U}, \partial U)$ (Lemma 3.3) and prove the existence of a Segal-type fibration sequence (Proposition 3.4). In 4 we prove our main stability result (Theorem 4.2) by using (Theorem 2.12). In 5 we recall the notion of a simplicial resolution and in 6 construct the (non-degenerate) Vassiliev spectral sequence and its truncated spectral sequence for computing the homology of $\text{Hol}^*_D(S^2, X_\Sigma)$. Finally in 7 we prove an unstable stability result (Theorem 7.1) and use it to prove our main results (Theorem 4.3 and Corollary 4.4).

2 The scanning map

First, we recall some known results.

**Lemma 2.1** ([4]; Corollary 6.30, Theorem 6.33, Theorem 8.9). Let $K$ be a simplicial complex on the index set $[r]$.

(i) The space $Z_K$ is 2-connected, and there is a fibration sequence

$$Z_K \longrightarrow DJ(K) \longrightarrow (\mathbb{C}P^\infty)^r.$$  

(ii) There is an $(S^1)^r$-equivariant deformation retraction

$$\text{ret} : Z_K(\mathbb{C}, \mathbb{C}^\ast) \longrightarrow Z_K.
Lemma 2.2 ([22]; (6.2) and Proposition 6.7). Let $X_\Sigma$ be a smooth toric variety such that the condition (1.12.1) holds. Then there is an isomorphism

$$G_\Sigma \cong \mathbb{T}^{r-n} = (\mathbb{C}^*)^{r-n},$$

and the group $G_\Sigma$ acts on the space $Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*)$ freely and there is a principal $G_\Sigma$-bundle

$$q_\Sigma : U(K_\Sigma) = Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to X_\Sigma.$$

Lemma 2.3. If the condition (1.12.1) is satisfied, the space $X_\Sigma$ is simply connected and $\pi_2(X_\Sigma) = \mathbb{Z}^{r-n}$.

Proof. By (1.12.1) and [8, Theorem 12.1.10] we easily see that the space $X_\Sigma$ is simply connected. By Lemma 2.1 and (2.2), $Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*)$ is 2-connected. Thus, by using the homotopy exact sequence of the principal $G_\Sigma$-bundle (2.4) and (2.3), we see that $\pi_2(X_\Sigma) = \mathbb{Z}^{r-n}$.

Recall the basic facts concerning the relation between a fan and a toric variety.

Definition 2.4. Let $\Sigma$ be a fan in $\mathbb{R}^n$. Then a cone $\sigma \in \Sigma$ is called smooth if it is generated by a subset of a basis of $\mathbb{Z}^n$.

Lemma 2.5 ([8]). Let $X_\Sigma$ be a toric variety determined by a fan $\Sigma$ in $\mathbb{R}^n$.

(i) $X_\Sigma$ is a smooth if and only if every cone $\sigma \in \Sigma$ is smooth.

(ii) $X_\Sigma$ is compact if and only if $\mathbb{R}^n = \bigcup_{\sigma \in \Sigma} \sigma$.

Now, we consider configuration spaces the scanning map.

Definition 2.6. For a positive integer $d \geq 1$ and a based space $X$, let $\text{SP}^d(X)$ denote the $d$-th symmetric product of $X$ defined by the orbit space

$$\text{SP}^d(X) = X^d/S_d,$$

where the symmetric group $S_d$ of $d$ letters acts on the $d$-fold product $X^d$ in the natural manner.

Remark 2.7. An element $\eta \in \text{SP}^d(X)$ may be identified with a formal linear combination $\eta = \sum_{k=1}^s n_k x_k$, where $x_1, \ldots, x_s$ are distinct points in $X$ and $n_1, \ldots, n_s$ are positive integers such that $\sum_{k=1}^s n_k = d$.

Definition 2.8. (i) When $A \subset X$ is a closed subspace, define the equivalent relation "$\sim$" on $\text{SP}^d(X)$ by

$$\eta_1 \sim \eta_2 \text{ if } \eta_1 \cap (X \setminus A) = \eta_2 \cap (X \setminus A) \text{ for } \eta_1, \eta_2 \in \text{SP}^d(X).$$

Define the space $\text{SP}^d(X, A)$ by the quotient space

$$\text{SP}^d(X, A) = \text{SP}^d(X)/\sim.$$
Note that the points in $A$ are ignored in $\text{SP}^d(X, A)$. If $A \neq \emptyset$, we have the natural inclusion $\text{SP}^d(X, A) \subset \text{SP}^{d+1}(X, A)$ by adding a point in $A$, and one can define the space $\text{SP}^\infty(X, A)$ by the union

$$\text{SP}^\infty(X, A) = \bigcup_{d=1}^\infty \text{SP}^d(X, A).$$

(ii) For each $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, let $E_D^\Sigma(X)$ denote the space

$$E_D^\Sigma(X) = \{(\xi_1, \cdots, \xi_r) \in \prod_{i=1}^r \text{SP}^{d_i}(X) : \bigcap_{i \in \sigma} \xi_i = \emptyset \text{ for any } \sigma \in I(K^\Sigma)\}.$$

If $A \subset X$ is a closed subspace and $A \neq \emptyset$, define the equivalence relation “$\sim$” on the space $E_D^\Sigma(X)$ by

$$(\xi_1, \cdots, \xi_r) \sim (\eta_1, \cdots, \eta_r) \quad \text{if} \quad \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A) \quad \text{for each } 1 \leq j \leq r.$$

Define the space $E_D^\Sigma(X, A)$ by the quotient space

$$E_D^\Sigma(X, A) = E_D^\Sigma(X)/\sim.$$

Then by adding points in $A$, we have the natural inclusion

$$E_D^{\Sigma(k)}(X, A) \subset E_D^{\Sigma(k+1)}(X, A)$$

for $D(k) = (d_1 + k, \cdots, d_r + k)$ if $k \geq 0$. So one can also define the space $E^\Sigma(X, A)$ by the union

$$E^\Sigma(X, A) = \bigcup_{k \geq 0} E_D^{\Sigma(k)}(X, A).$$

**Remark 2.9.** (i) It is easy to see that the space $E^\Sigma(X, A)$ does not depend on the choice of the $r$-tuple $D = (d_1, \cdots, d_r)$ and that the following equality holds:

$$E^\Sigma(X, A) = \{(\xi_1, \cdots, \xi_r) \in \text{SP}^\infty(X, A)^r : \bigcap_{i \in \sigma} \xi_i = \emptyset \text{ for any } \sigma \in I(K^\Sigma)\}.$$

(ii) Note that there is a natural homeomorphism $\text{Po}^d(\mathbb{C}) \cong \text{SP}^d(\mathbb{C})$ by identifying $\prod_{k=1}^s (z - \alpha_k)^{n_k} \mapsto \sum_{k=1}^s n_k \alpha_k$, where $\{\alpha_k\}_{k=1}^s$ are distinct points in $\mathbb{C}$ and $\{n_k\}_{k=1}^s$ are positive integers such that $\sum_{k=1}^s n_k = d$.

(iii) If the condition $(1.12.2)$ is satisfied, by using the above identification we have a natural homeomorphism

$$\text{Hol}_D^*(S^2, X^\Sigma) \cong E_D^\Sigma(\mathbb{C}). \quad \Box$$

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\textbf{Definition 2.10.} Let $a = (a_1, \cdots, a_r) \in (\mathbb{Z}_{\geq 1})^r$ be any fixed $r$-tuple of positive integers such that
\begin{equation}
\sum_{k=1}^{r} a_k n_k = 0. \tag{2.13}
\end{equation}

For each $E = (e_1, \cdots, e_r) \in (\mathbb{Z}_{\geq 1})^r$, let $U_E \subset \mathbb{C}$ denote the subset $U_E = \{w \in \mathbb{C} : \Re(w) < e_1 + \cdots + e_r\}$, and choose any $r$ points $\{x_j\}_{j=1}^{r} \subset U_{D+a} \setminus U_D$ such that $x_i \neq x_j$ if $i \neq j$. For any such a choice, we define the stabilization map $s_{D,\Sigma} : E_D^S(U_D) \rightarrow E_{D+a}^S(U_D+a)$ by
\begin{equation}
s_{D,\Sigma} : E_D^S(U_D) \longrightarrow E_{D+a}^S(U_D+a)
(\xi_1, \cdots, \xi_r) \longrightarrow (\xi_1 + a_1 x_1, \cdots, \xi_r + a_r x_r) \tag{2.14}
\end{equation}

We also have the stabilization map
\begin{equation}
s_D : \text{Hol}^*_D(S^2, X_{\Sigma}) \rightarrow \text{Hol}^*_{D+a}(S^2, X_{\Sigma}) \tag{2.15}
\end{equation}
defined as the composite of maps
\begin{equation}
\text{Hol}^*_D(S^2, X_{\Sigma}) \cong E_D^S(U_D) \overset{a \cdot \Sigma}{\longrightarrow} E_{D+a}^S(U_D+a) \cong \text{Hol}^*_{D+a}(S^2, X_{\Sigma}). \tag{2.16}
\end{equation}

Note that, although the map $s_D$ depends on the choice of the points $\{x_k\}_{k=1}^{r}$, its homotopy class does not.

Now we are ready to define the scanning map.

\textbf{Definition 2.11.} Let $\epsilon_0 > 0$ be any fixed sufficiently small number and let $U = \{w \in \mathbb{C} : |w| < 1\}$. For each $w \in \mathbb{C}$, let $U_w = \{x \in \mathbb{C} : |x - w| < \epsilon_0\}$. Then for an element $\eta = (\eta_1, \cdots, \eta_r) \in E_D^S(\mathbb{C})$, define a map $sc_D(\eta) : \mathbb{C} \rightarrow \Sigma^S(\mathbb{U}, \partial\mathbb{U})$ by
\begin{equation}
w \mapsto \eta \cap \overline{U}_w = (\eta_1 \cap \overline{U}_{w_1}, \cdots, \eta_r \cap \overline{U}_{w_r}) \in \Sigma^S(\overline{U}_w, \partial\overline{U}_w) \cong \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}}) \tag{2.17}
\end{equation}
for $w \in \mathbb{C}$, where we identify $(\overline{U}_w, \partial\overline{U}_w)$ with $(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}})$ in the canonical way. Since $\lim_{w \to \infty} sc(\eta)(w) = (\emptyset, \cdots, \emptyset)$, it naturally extends to a map
\begin{equation}
sc(\eta) : S^2 = \mathbb{C} \cup \infty \rightarrow \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}}) \tag{2.18}
\end{equation}
by taking $sc(\eta)(\infty) = (\emptyset, \cdots, \emptyset)$. Now we choose the point $\infty$ and the empty configuration $(\emptyset, \cdots, \emptyset)$ as the base-points of $S^2 = \mathbb{C} \cup \infty$ and $\Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}})$, respectively. Then the map $sc(\eta)$ is a base-point preserving map, and we obtain a map $sc : E_D^S(\mathbb{C}) \rightarrow \Omega^2 \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}})$. However, since $E_D^S(\mathbb{C})$ is connected, the image of the map $sc$ is contained some path-component of $\Omega^2 \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}})$, which we denote by $\Omega_D^2 \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}})$. Thus we have the map
\begin{equation}
sc_D : E_D^S(\mathbb{C}) \rightarrow \Omega_D^2 \Sigma^S(\overline{\mathbb{U}}, \partial\overline{\mathbb{U}}). \tag{2.18}
\end{equation}
If the condition (1.12.2) is satisfied, we can identify $\text{Hol}_D^*(S^2, X_\Sigma) = E_D^\Sigma(\mathbb{C})$ and we obtain the map

$$sc_D : \text{Hol}_D^*(S^2, X_\Sigma) \to \Omega^2_D E^\Sigma(\overline{U}, \partial \overline{U}).$$

We refer to this map (and others defined by the same method) as “the scanning map”. It is easy to see that there is a commutative diagram

$$\begin{array}{ccc}
\text{Hol}_D^*(S^2, X_\Sigma) & \xrightarrow{sc_D} & \Omega^2_D E^\Sigma(\overline{U}, \partial \overline{U}) \\
\downarrow s_D & & \downarrow \simeq \\
\text{Hol}_{D+k_\alpha}^*(S^2, X_\Sigma) & \xrightarrow{sc_{D+k_\alpha}} & \Omega^2_{D+k_\alpha} E^\Sigma(\overline{U}, \partial \overline{U})
\end{array}$$

Let $\text{Hol}_{D+\infty}^*(S^2, X_\Sigma) = \lim_{k \to \infty} \text{Hol}_{D+k_\alpha}^*(S^2, X_\Sigma)$ denote the colimit constructed from the maps $s_{D+k_\alpha}$. Then by using (2.20) we obtained the stabilized scanning map

$$S : \text{Hol}_{D+\infty}^*(S^2, X_\Sigma) = \lim_{k \to \infty} \text{Hol}_{D+k_\alpha}^*(S^2, X_\Sigma) \to \Omega^2_0 E(\overline{U}, \partial \overline{U}),$$

where we set $S = \lim_{k \to \infty} sc_{D+k_\alpha}$ and $\Omega^2_0 X$ denotes the path component of $\Omega^2 X$ which contains the constant map.

**Theorem 2.12.** The stabilized scanning map

$$S : \text{Hol}_{D+\infty}^*(S^2, X_\Sigma) = \lim_{k \to \infty} \text{Hol}_{D+k_\alpha}^*(S^2, X_\Sigma) \xrightarrow{\simeq} \Omega^2_0 E(\overline{U}, \partial \overline{U})$$

is a homotopy equivalence. \qed

**Proof.** The assertion can be proved by using Segal’s scanning method given in \cite{10}, Prop. 4.4 (cf. \cite{9}) and \cite{12}. \qed

### 3 Segal-type fibration sequences

In this section we recall the basic fact concerning the topology of polyhedral products and determine the homotopy type of the space $E^\Sigma(\overline{U}, \partial \overline{U})$. We also consider relation between the topologies of the spaces $E^\Sigma(\overline{U}, \partial \overline{U})$ and $X_\Sigma$, and construct some Segal-type fibration sequences (Proposition 3.4).

**Definition 3.1.** Let $(X, \ast)$ be a based space and let $I$ be a collection of some subsets of $[r] = \{1, 2, \ldots, r\}$. Then let $\vee^I X \subset X^r$ denote the subspace consisting of all $r$-tuples $(x_1, \ldots, x_r) \in X^r$ such that, for each $\sigma \in I$, there is some $j \in \sigma$ such that $x_j = \ast$.

**Lemma 3.2** (cf. \cite{17}, Lemma 6.3). If $K$ is a simplicial complex on the index set $[r]$ and $(X, \ast)$ is a based space, then $Z_K(X, \ast) = \vee^{I(K)} X$. 

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Proposition 3.4. There is a homotopy equivalence

\[ r_{\Sigma} : E^{S}(\overline{U}, \partial \overline{U}) \xrightarrow{\sim} DJ(K_{\Sigma}). \]

Proof. The proof is analogous to that of \cite{23} Prop. 3.1 and that of \cite{13} Lemma 7.10. Note that \( E^{S}(\overline{U}, \partial \overline{U}) \) is homeomorphic to the space

\[ E^{S}(S^{2}, \infty) = \{ (\eta_1, \ldots, \eta_r) \in SP^{\infty}(S^{2}, \infty)^{r} : \cap_{i \in \sigma} \eta_i = \emptyset \text{ for any } \sigma \in I(K_{\Sigma}) \}. \]

For each \( \epsilon > 0 \), let \( E^{S}_{\epsilon} \) denote the open subset of all \( r \)-tuples \( (\xi_1, \ldots, \xi_r) \in E^{S}(S^{2}, \infty) \) such that, for any \( \sigma \in I(K_{\Sigma}) \) there exists some \( i \in \sigma \) satisfying the condition \( \xi_i \cap \overline{U}(\epsilon) = \emptyset \). For each \( \epsilon > 0 \), radial expansion defines a deformation retraction \( r_{\epsilon} : E_{\epsilon}^{S} \xrightarrow{\sim} \vee^{I(K_{\Sigma})}SP^{\infty}(S^{2}, \infty) \) (in this case, if \( \xi_i \cap \overline{U}(\epsilon) = \emptyset \) and \( i \in \sigma \) (for any \( \sigma \in I(K) \)), then \( \xi_i \) gets retracted to \( \infty \)). Since \( E^{S}(S^{2}, \infty) = \bigcup_{\epsilon>0} E^{S}_{\epsilon} \), there is a deformation retraction \( E^{S}(S^{2}, \infty) \xrightarrow{\sim} \vee^{I(K_{\Sigma})}SP^{\infty}(S^{2}, \infty) \). Since there is a homeomorphism \( SP^{\infty}(S^{2}, \infty) \cong \mathbb{C}P^{\infty} \), the assertion follows from Lemma 3.3.  \( \square \)

Proposition 3.4. If \( \{ n_k \}_{k=1}^{\infty} \) spans \( \mathbb{R}^{n} \) and \( X_{\Sigma} \) is non-singular, there is a fibration sequence (up to homotopy)

\[ T_{\mathbb{C}}^{n} \to X_{\Sigma} \xrightarrow{p_{\Sigma}} DJ(K_{\Sigma}). \]

Proof. Let us write \( (K, G, G_{1}) = (K_{\Sigma}, T_{\mathbb{C}}^{n}, T_{\mathbb{C}}^{n}) \) and we identify \( X_{\Sigma} = Z_{K}(\mathbb{C}, \mathbb{C}^{*})/G_{\Sigma} \). Since \( G_{\Sigma} \cong T_{\mathbb{C}}^{n-1} \), there is a fibration sequence \( G_{\Sigma} \xrightarrow{i} T_{\mathbb{C}} = G_{1} \to T_{\mathbb{C}}^{n} = G \), where \( i \) denotes the inclusion. Note that \( G = T_{\mathbb{C}}^{n} \) acts naturally on \( X_{\Sigma} \) by the definition of toric variety and that we have a fibration sequence \( G \to X_{\Sigma} \to EG \times_{G} X_{\Sigma} \). Moreover, since the group \( G_{1} = T_{\mathbb{C}}^{n} \) also acts on \( Z_{K}(\mathbb{C}, \mathbb{C}^{*}) \) and we also obtain a fibration sequence \( G_{1} \to Z_{K}(\mathbb{C}, \mathbb{C}^{*}) \to EG_{1} \times_{G_{1}} Z_{K}(\mathbb{C}, \mathbb{C}^{*}) \). Hence, by using \cite{3} Lemma 2.1 we have the following homotopy commutative diagram

\[
\begin{array}{ccccccccc}
G_{\Sigma} & \xrightarrow{i} & G_{1} = T_{\mathbb{C}} & \xrightarrow{\pi} & G = T_{\mathbb{C}}^{n} \\
\| & & \downarrow & & \downarrow \\
G_{\Sigma} & \longrightarrow & Z_{K}(\mathbb{C}, \mathbb{C}^{*}) & \xrightarrow{p_{\Sigma}} & X_{\Sigma} = Z_{K}(\mathbb{C}, \mathbb{C}^{*})/G_{\Sigma} & \xrightarrow{p_{\Sigma}} & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ast & \longrightarrow & EG_{1} \times_{G_{1}} Z_{K}(\mathbb{C}, \mathbb{C}^{*}) & \longrightarrow & EG \times_{G} X_{\Sigma} & & \downarrow \\
\end{array}
\]
where all horizontal and vertical sequences are fibration sequences (up to homotopy equivalence). Thus, the map $E \pi \times q : E \pi_1 \times G_1 \mathbb{Z}_K(\mathbb{C}, \mathbb{C}^*) \longrightarrow E \pi \times G \mathbb{Z}(\mathbb{C}, \mathbb{C}^*)$ is a homotopy equivalence. However, it follows from [4, Theorem 6.29] that there is a homotopy equivalence $E \pi_1 \times G_1 \mathbb{Z}_K(\mathbb{C}, \mathbb{C}^*) \cong DJ(K)$. Thus, there is a homotopy equivalence $E \pi \times G \mathbb{Z}(\mathbb{C}, \mathbb{C}^*) \cong DJ(K)$ and the assertion follows.

**Example 3.5.** For a based space $(X, *)$, let $W_{n+1}(X)$ denote the fat wedge of $X$ defined by $W_{n+1}(X) = \{(x_0, \ldots, x_n) \in X^{n+1} : x_i = * \text{ for some } 0 \leq i \leq n\}$, and let $e_k$ ($1 \leq k \leq n$) be the standard basis of $\mathbb{R}^n$ given by

\[
e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_n = (0, 0, \ldots, 0, 1).
\]

It is known that the fan $\Sigma$ of $\mathbb{C}P^n$ is given by $\Sigma = \{\text{Cone}(S) : S \subset \{e_k\}_{k=0}^n\}$, where we set $e_0 = -\sum_{k=1}^n e_k^2$.

Hence, if $X_\Sigma = \mathbb{C}P^n$, we see that $DJ(K_\Sigma) = W_{n+1}(\mathbb{C}P^n)$ and the above fibration sequence (3.1) coincides with the following fibration sequence

\[
T^n_\Sigma \longrightarrow \mathbb{C}P^n \overset{p_\Sigma}{\longrightarrow} W_{n+1}(\mathbb{C}P^n+1).
\]

constructed by G. Segal in [23, §2],

**4 The stable result**

**Definition 4.1.** Let $a \in (\mathbb{Z}_{\geq 1})^r$ be an $r$-tuple of positive integers which satisfies the condition (2.13). Then it is easy to see that the following diagram is homotopy commutative:

\[
\begin{array}{c}
\text{Hol}_D^*(S^2, X_\Sigma) \\
\Downarrow s_n
\end{array} \xrightarrow{i_D} \begin{array}{c}
\Omega^2_D X_\Sigma \\
\cong \Omega^2_0 X_\Sigma
\end{array} \xrightarrow{\sim} \begin{array}{c}
\text{Hol}_{D+a}^*(S^2, X_\Sigma) \\
\Downarrow i_{D+a}
\end{array} \xrightarrow{i_{D+a}} \begin{array}{c}
\Omega^2_{D+a} X_\Sigma \\
\cong \Omega^2_0 X_\Sigma
\end{array}
\]

Hence we can stabilize the inclusion maps

\[i_{D+\infty} = \lim_{k \to \infty} i_{D+ka} : \text{Hol}_{D+\infty}^*(S^2, X_\Sigma) = \lim_{k \to \infty} \text{Hol}_{D+ka}^*(S^2, X_\Sigma) \to \Omega^2_0 X_\Sigma.\]

In this section, we shall prove the following result.

**Theorem 4.2.** The map $i_{D+\infty} : \text{Hol}_{D+\infty}^*(S^2, X_\Sigma) \cong \Omega^2_0 X_\Sigma$ is a homotopy equivalence.

\[\text{For } S = \emptyset, \text{ we set Cone}(\emptyset) = \{0\}.\]
\textbf{Definition 4.3.} (i) Let \( X \subset \mathbb{C} \) be an open set and let \( F(X) \) denote the space of tuples \((f_1(z), \ldots, f_r(z))\) of (not necessarily monic) polynomials satisfying the following two conditions:

(i-1) \( \sum_{k=1}^{r} \deg(f_k)n_k = 0 \).

(i-2) For each \( \sigma = \{i_1, \ldots, i_s\} \in I(K_\Sigma) \), the polynomials \( f_{i_1}(z), \ldots, f_{i_s}(z) \) have no common root in \( X \).

An element \((f_1(z), \ldots, f_r(z)) \in F(X)\), defines a map \( X \to U(K_\Sigma) \) and represents a map \( X \to X_\Sigma \).

(ii) Let \( U = \{w \in \mathbb{C} : |w| < 1\} \) and \( ev_0 : F(U) \to U(K_\Sigma) \) denote the map given by evaluation at 0, i.e. \( ev_0(f_1, \ldots, f_r) = (f_1(0), \ldots, f_r(0)) \).

(iii) Let \( \tilde{F}(U) \subset F(U) \) denote the subspace of all \((f_1(z), \ldots, f_r(z)) \in F(X)\) such that no \( f_i(z) \) is identically zero, and \( ev : \tilde{F}(U) \to U(K_\Sigma) \) the map given by the restriction \( ev = ev_0|\tilde{F}(U) \).

\textbf{Lemma 4.4.} \( ev : \tilde{F}(U) \xrightarrow{\sim} U(K_\Sigma) \) is a homotopy equivalence.

\textit{Proof.} Let \( i_0 : U(K_\Sigma) \to F(U) \) be the inclusion map given by viewing constants as polynomials. Clearly \( ev_0 \circ i_0 = \text{id} \). Let \( f : F(U) \times [0, 1] \to F(U) \) be the homotopy given by \( f((f_1, \ldots, f_i), t) = (f_{1,t}(z), \ldots, f_{r,t}(z)) \), where \( f_{i,t}(z) = f_i(tz) \). This gives a homotopy between \( i_0 \circ ev_0 \) and the identity map, and this proves that \( ev_0 \) is a homotopy equivalence. Since \( F(U) \) is an infinite dimensional manifold and \( \tilde{F}(U) \) is a subspace of \( F(U) \) of infinite codimension, the inclusion \( \tilde{F}(U) \to F(U) \) is a homotopy equivalence. Hence \( ev \) is also a homotopy equivalence. \( \square \)

\textit{Proof of Theorem 4.2.} Let \( U = \{w \in \mathbb{C} : |w| < 1\} \) as before and note that the group \( T_\mathbb{C} \) acts freely on the space \( \tilde{F}(X) \) by coordinate multiplication for \( X = U \) or \( \mathbb{C} \). Let \( \tilde{F}(X)/T_\mathbb{C} \) denote the corresponding orbit space. Let \( u : \tilde{F}(U) \to E^\Sigma(U, \partial U) \) denote the natural map which assigns to an \( r \)-tuple \([f_1(z), \ldots, f_r(z)] \in \tilde{F}(U)\) of polynomials the \( r \)-tuple of their roots which lie in \( U \). It is not difficult to see that the map \( u \) is a homotopy equivalence. Let \( scan : \tilde{F}(\mathbb{C}) \to \text{Map}(\mathbb{C}, \tilde{F}(U)) \) denote the map given by \( scan(f_1(z), \ldots, f_r(z))(w) = (f_1(z+w), \ldots, f_r(z+w)) \) for \( w \in \mathbb{C} \) and consider the diagram

\[
\begin{array}{ccc}
\tilde{F}(U) & \xrightarrow{ev} & U(K_\Sigma) \\
p \downarrow & & \\
\tilde{F}(U)/T_\mathbb{C} & \xrightarrow{u} & E^\Sigma(U, \partial U)
\end{array}
\]

where \( p : \tilde{F}(U) \to \tilde{F}(U)/T_\mathbb{C} \) denotes the natural projection map. Note that \( p \) is a \( T_\mathbb{C} \)-principal bundle projection. Now consider the diagram below

\[
\begin{array}{ccc}
\tilde{F}(\mathbb{C}) & \xrightarrow{\text{scan}} & \text{Map}(\mathbb{C}, \tilde{F}(U)) \\
p \downarrow & & \downarrow p^g \\
\tilde{F}(\mathbb{C})/T_\mathbb{C} & \xrightarrow{\text{scan}} & \text{Map}(\mathbb{C}, \tilde{F}(U)/T_\mathbb{C})
\end{array}
\]

\[
\begin{array}{ccc}
\text{Map}(\mathbb{C}, \tilde{F}(U)) & \xrightarrow{ev^g} & \text{Map}(\mathbb{C}, U(K_\Sigma)) \\
p^g \downarrow & & \downarrow p^g \\
\text{Map}(\mathbb{C}, \tilde{F}(U)/T_\mathbb{C}) & \xrightarrow{u^g} & \text{Map}(\mathbb{C}, E^\Sigma(U, \partial U))
\end{array}
\]

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induced from the above diagram. Observe that \( \text{Map}(C, \cdot) \) can be replaced by \( \text{Map}(S^2, \cdot) \) by extending from \( C \) to \( S^2 = C \cup \infty \) (as base point preserving maps). Thus by setting

\[
\begin{align*}
&j_D : \text{Hol}_D^*(S^2, X_{\Sigma}) \xrightarrow{\sim} \tilde{F}(C) \xrightarrow{\text{scan}} \text{Map}_D^*(S^2, \tilde{F}(U)) = \Omega_2^D \tilde{F}(U) \\
&j'_D : E^D_\Sigma(C) \xrightarrow{\sim} \tilde{F}(C)/T_C \xrightarrow{\text{scan}} \text{Map}_D^*(S^2, \tilde{F}(U)/T_C) = \Omega_2^D(\tilde{F}(U)/T_C)
\end{align*}
\]

we obtain the following commutative diagram, where the suffix \( D \) denotes the appropriate path component:

\[
\begin{array}{c}
\text{Hol}_D^*(S^2, X_{\Sigma}) \xrightarrow{j_D} \Omega_2^D \tilde{F}(U) \xrightarrow{\Omega^2_{\text{ev}}} \Omega_2^D U(K_{\Sigma}) \xrightarrow{\Omega^2_{q_{\Sigma}}} \Omega_2^D X_{\Sigma} \\
E^D_\Sigma(C) \xrightarrow{j'_D} \Omega_2^D(\tilde{F}(U)/T_C) \xrightarrow{\Omega^2_{\text{ev}}} \Omega_2^D E_{\Sigma}(U, \partial U)
\end{array}
\]

Note that the maps \( \Omega^2_{q_{\Sigma}}, \text{ev}, \Omega^2_{p} \) and \( u \) are homotopy equivalences. Moreover, from the definitions of the maps, one can see that the following two equalities hold (up to homotopy equivalence):

\[
(4.1) \quad \Omega^2_{q_{\Sigma}} \circ \Omega^2_{\text{ev}} \circ j_D = i_D, \quad \Omega^2_{u} \circ j'_D = \text{sc}_D.
\]

Hence, the maps \( i_D \) and \( \text{sc}_D \) are homotopic up to homotopy equivalences. Thus, if we replace \( D \) by \( D + ka \) and let \( k \to \infty \) then, by using Theorem 2.12 we see that the map \( i_{D+\infty} \) is a homotopy equivalence.

**Remark 4.5.** It is not easy to show that the following diagram is homotopy commutative:

\[
\begin{array}{c}
\tilde{F}(U) \xrightarrow{\text{ev}} U(K_{\Sigma}) \xrightarrow{q_{\Sigma}} X_{\Sigma} \\
\tilde{F}(U)/T_C \xrightarrow{u} E_{\Sigma}(U, \partial U) \xrightarrow{r_{\Sigma}} DJ(K_{\Sigma})
\end{array}
\]

Segal proved that the diagram (4.2) is homotopy commutative for the case \( X_{\Sigma} = \mathbb{C}P^n \) (see [23, Prop. 4.8]), and an analogous method can probably be used to show that this diagram is homotopy commutative for a general \( X_{\Sigma} \). However, the argument seems tedious. So we do not attempt to carry it out. Of course this result would give another proof of Theorem 4.2.

**5 Simplicial resolutions**

In this section, we summarize the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolutions ([19], [24]).
Definition 5.1. (i) For a finite set $v = \{v_1, \cdots, v_l\} \subset \mathbb{R}^N$, let $\sigma(v)$ denote the convex hull spanned by $v$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $X^\Delta$ and $h^\Delta : X^\Delta \to Y$ denote the space and the map defined by

$$X^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \quad h^\Delta(y, u) = y.$$  

The pair $(X^\Delta, h^\Delta)$ is called the simplicial resolution of $(h, i)$. In particular, it is called a non-degenerate simplicial resolution if for each $y \in Y$ any $k$ points of $i(h^{-1}(y))$ span $(k - 1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let $X_k^\Delta \subset X^\Delta$ be the subspace given by

$$X_k^\Delta = \{(y, u) \in X^\Delta : u \in \sigma(v), \quad v = \{v_1, \cdots, v_l\} \subset i(h^{-1}(y)), l \leq k\}.$$  

We make the identification $X = X_1^\Delta$ by identifying $x \in X$ with the pair $(h(x), i(x)) \in X_1^\Delta$, and we note that there is an increasing filtration

$$\emptyset = X_0^\Delta \subset X = X_1^\Delta \subset X_2^\Delta \subset \cdots \subset X_k^\Delta \subset \cdots \subset \bigcup_{k=0}^{\infty} X_k^\Delta = X^\Delta.$$  

Since the map $h^\Delta : X^\Delta \to Y$ is a proper map, it extends to the map $h^\Delta_+ : X^\Delta_+ \to Y_+$ between the one-point compactifications, where $X_+$ denotes the one-point compactification of a locally compact space $X$.

Lemma 5.2 ([21], [25] (cf. Lemma 3.3 in [18])). Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding.

(i) If $X$ and $Y$ are semi-algebraic spaces and the two maps $h, i$ are semi-algebraic maps, then the map $h^\Delta_+ : X^\Delta_+ \to Y_+$ is a homotopy equivalence.

(ii) There is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(X^\Delta, h^\Delta)$ of $(h, j)$ is non-degenerate.

(iii) If there is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(X^\Delta, h^\Delta)$ of $(h, j)$ is non-degenerate, the space $X^\Delta$ is uniquely determined up to homeomorphism. Moreover, there is a filtration preserving homotopy equivalence $q^\Delta : X^\Delta \to X^\Delta$ such that $q^\Delta|X = id_X$.  

Remark 5.3. In this paper we only need the weaker assertion that the map $h^\Delta_+$ is a homology equivalence. One can easily prove this result by the same argument as used in the second revised edition of Vassiliev’s book [21 Proof of Lemma 1 (page 90)].

Remark 5.4 ([24], [25]). Even for a surjective map $h : X \to Y$ which is not finite to one, it is still possible to construct an associated non-degenerate simplicial resolution. Recall that it is known that there exists a sequence of embeddings $\{i_k : X \to \mathbb{R}^N\}_{k \geq 1}$ satisfying the following two conditions for each $k \geq 1$ ([24], [25]).

(i) For any $y \in Y$, any $t$ points of the set $\tilde{i}_k(h^{-1}(y))$ span $(t - 1)$-dimensional affine subspace of $\mathbb{R}^N_k$ if $t \leq 2k$.
(ii) $N_k \leq N_{k+1}$ and if we identify $\mathbb{R}^{N_k}$ with a subspace of $\mathbb{R}^{N_{k+1}}$, then $\tilde{i}_{k+1} = \tilde{i} \circ \tilde{i}_k$, where $\tilde{i} : \mathbb{R}^{N_k} \hookrightarrow \mathbb{R}^{N_{k+1}}$ denotes the inclusion.

In this situation, in fact, a non-degenerate simplicial resolution may be constructed by choosing a sequence of embeddings $\{i_k : X \to \mathbb{R}^{N_k}\}_{k \geq 1}$ satisfying the above two conditions for each $k \geq 1$.

Let $X_k^\Delta = \{(y, u) \in Y \times \mathbb{R}^{N_k} : u \in \sigma(v), v = \{v_1, \ldots, v_l\} \subset i_k(h^{-1}(y)), l \leq k\}$. Then by identifying naturally $X_k^\Delta$ with a subspace of $X_{k+1}^\Delta$, define the non-degenerate simplicial resolution $\mathcal{X}^\Delta$ of $h$ as $\mathcal{X}^\Delta = \bigcup_{k \geq 1} X_k^\Delta$. \hfill \QED

**Definition 5.5.** Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, $j : X \to \mathbb{R}^N$ be a semi-algebraic embedding, and let $(\mathcal{X}^\Delta, h^\Delta : \mathcal{X}^\Delta \to Y)$ denote the associated non-degenerate simplicial resolution of $(h, j)$.

Let $k$ be a fixed positive integer and let $h_k : X_k^\Delta \to Y$ be the map defined by the restriction $h_k := h^\Delta|X_k^\Delta$. The fibers of the map $h_k$ are $(k-1)$-skeleton of the fibers of $h^\Delta$ and, in general, always fail to be simplices over the subspace $Y_k = \{y \in Y : \text{card}(h^{-1}(y)) > k\}$. Let $Y(k)$ denote the closure of the subspace $Y_k$. We modify the subspace $X_k^\Delta$ so as to make all the fibers of $h_k$ contractible by adding to each fibre of $Y(k)$ a cone whose base is this fibre. We denote by $X^\Delta(k)$ this resulting space and by $h_k^\Delta : X^\Delta(k) \to Y$ the natural extension of $h_k$.

Following [20], we call the map $h_k^\Delta : X^\Delta(k) \to Y the truncated (after the $k$-th term) simplicial resolution of $Y$. Note that that there is a natural filtration

$$X_0^\Delta \subset X_1^\Delta \subset \cdots \subset X_l^\Delta \subset X_{l+1}^\Delta \subset \cdots \subset X_k^\Delta \subset X_{k+1}^\Delta = X_{k+2}^\Delta = \cdots = X^\Delta(k),$$

where $X_0^\Delta = \emptyset$, $X_l^\Delta = X_l^\Delta$ if $l \leq k$ and $X_l^\Delta = X_k^\Delta$ if $l > k$.

**Lemma 5.6** ([20], cf. Remark 2.4 and Lemma 2.5 in [16]). Under the same assumptions and with the same notation as in Definition 5.5, the map $h_k^\Delta : X^\Delta(k) \to Y$ is a homotopy equivalence. \hfill \QED

### 6 The Vassiliev spectral sequence.

In this section, we always assume that $D = (d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ and $a = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 1})^r$ are $r$-tuples of positive integers which satisfy the conditions (1.12) and (2.13).

From now on, we identify the space $\text{Hol}_D^*(S^2, X_\Sigma)$ with the space consisting of all $r$-tuples $(f_1(z), \cdots, f_r(z)) \in \text{Po}^D$ of monic polynomials such that $f_i(z), \cdots, f_r(z)$ have no common root for any $\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)$ as in Definition 1.8. First, we construct the Vassiliev spectral sequence.

**Definition 6.1.** (i) Let $\Sigma_D$ denote the discriminant of $\text{Hol}_D^*(S^2, X_\Sigma)$ in $\text{Po}^D$ given by the complement

$$\Sigma_D = \text{Po}^D \setminus \text{Hol}_D^*(S^2, X_\Sigma)$$

$$= \{(f_1(z), \cdots, f_r(z)) \in \text{Po}^D : (f_1(x), \cdots, f_r(x)) \in L(\Sigma) \text{ for some } x \in \mathbb{C}\},$$
where we set

\[(6.1) \quad L(\Sigma) = \bigcup_{\sigma \in I(K\Sigma)} L_\sigma = \bigcup_{\sigma \subset [r], \sigma \notin K\Sigma} L_\sigma.\]

(ii) Let \(Z_D \subset \Sigma_D \times \mathbb{C}\) denote the tautological normalization of \(\Sigma_D\) consisting of all pairs \((F, x) = ((f_1(z), \ldots, f_r(z)), x) \in \Sigma_D \times \mathbb{C}\) satisfying the condition \((f_1(x), \ldots, f_r(x)) \in L(\Sigma)\). Projection on the first factor gives a surjective map \(\pi_D : Z_D \to \Sigma_D\).

**Remark 6.2.** Let \(\sigma_k \in [r]\) for \(k = 1, 2\). It is easy to see that \(L_{\sigma_1} \subset L_{\sigma_2}\) if \(\sigma_1 \supset \sigma_2\). Letting \(P r(\Sigma) = \{\sigma = \{i_1, \cdots, i_s\} \subset [r] : \{n_{i_1}, \cdots, n_{i_s}\}\) is a primitive collection\},

we see that

\[(6.2) \quad L(\Sigma) = \bigcup_{\sigma \in P r(\Sigma)} L_\sigma\]

and by using (1.16) we obtain the equality

\[(6.3) \quad \dim L(\Sigma) = 2(r - r_{\text{min}}(\Sigma)).\]

Our goal in this section is to construct, by means of the non-degenerate simplicial resolution of the discriminant, a spectral sequence converging to the homology of \(\text{Hol}^* D(S^2, X_\Sigma)\).

**Definition 6.3.** Let \((X^D, \pi_D^D : X^D \to \Sigma_D)\) be the non-degenerate simplicial resolution associated to the surjective map \(\pi_D : Z_D \to \Sigma_D\) with the natural increasing filtration as in Definition 5.1.

\[\emptyset = X^D_0 \subset X^D_1 \subset X^D_2 \subset \cdots \subset X^D = \bigcup_{k=0}^{\infty} X^D_k.\]

By Lemma 5.2 the map \(\pi_D^D : X^D \xrightarrow{\sim} \Sigma_D\) is a homotopy equivalence which extends to a homotopy equivalence \(\pi_D^D : X_+^D \xrightarrow{\sim} \Sigma_D^+,\) where \(X_+\) denotes the one-point compactification of a locally compact space \(X\). Since \(X^D_k/X^D_{k-1} \cong (X^D_k \setminus X^D_{k-1})_+\), we have a spectral sequence

\[\{E^{k,s}_{t,D}, d_t : E^{k,s}_{t,D} \to E^{k+t,s+1-t}_{t,D}\} \Rightarrow H^{k+s}(\Sigma_D, \mathbb{Z}),\]

where \(E^{k,s}_{t,D} = H^{k+s}(X^D_k \setminus X^D_{k-1}, \mathbb{Z})\) and \(H^k(X, \mathbb{Z})\) denotes the cohomology group with compact supports given by \(H^k_c(X, \mathbb{Z}) = H^k(X_+, \mathbb{Z}).\)

Let \(N(D)\) and \(N(a)\) denote the positive integers given by

\[(6.4) \quad N(D) = \sum_{k=1}^{r} d_k, \quad N(a) = \sum_{k=1}^{r} a_k.\]
Since there is a homeomorphism $\text{Po}^D \cong \mathbb{C}^{N(D)}$, by Alexander duality there is a natural isomorphism

\begin{equation}
\tilde{H}_k(\text{Hol}_D^d(S^2, X_\Sigma), \mathbb{Z}) \cong \tilde{H}^{2N(D) - k - 1}_c(\Sigma_D, \mathbb{Z}) \quad \text{for any } k.
\end{equation}

By reindexing we obtain a spectral sequence

\begin{equation}
\{ E^{i,j}_k : E^{i,j}_k \to E^{i,j}_{k+t,s+t-1} \} \Rightarrow H_{s-k}(\text{Hol}_D^d(S^2, X_\Sigma), \mathbb{Z}),
\end{equation}

where $E^{i,j}_k = \tilde{H}^{2N(D) + k - s - 1}_c(X_k^D \setminus X_{k-1}^D, \mathbb{Z})$.

Let $L_{k;\Sigma} \subset (\mathbb{C} \times L(\Sigma))^k$ denote the subspace defined by

$L_{k;\Sigma} = \{((x_1, s_1), \ldots, (x_k, s_k)) : x_j \in \mathbb{C}, s_j \in L(\Sigma), x_i \neq x_j \text{ if } i \neq j \}$.

The symmetric group $S_k$ on $k$ letters acts on $L_{k;\Sigma}$ by permuting coordinates. Let $C_{k;\Sigma}$ denote the orbit space

\begin{equation}
C_{k;\Sigma} = L_{k;\Sigma}/S_k.
\end{equation}

Note that $C_{k;\Sigma}$ is a cell-complex of dimension $2(1 + r - r_{\min}(\Sigma))k$ by (6.3).

**Lemma 6.4.** If $1 \leq k \leq d_{\min} = \min\{d_1, \ldots, d_r\}$, $X_k^D \setminus X_{k-1}^D$ is homeomorphic to the total space of a real affine bundle $\xi_{l,D,k}$ over $C_{k;\Sigma}$ with rank $l_{D,k} = 2N(D) - 2rk + k - 1$.

**Proof.** The argument is exactly analogous to the one in the proof of [1 Lemma 4.4]. Namely, an element of $X_k^D \setminus X_{k-1}^D$ is represented by $(F, u) = ((f_1, \ldots, f_r, u))$, where $F = (f_1, \ldots, f_r)$ is an $r$-tuple of monic polynomials in $\Sigma_D$ and $u$ is an element of the interior of the span of the images of $k$ distinct points $(x_1, \ldots, x_k) \in C_k(\mathbb{C})$ such that $F(x_j) = (f_1(x_j), \ldots, f_r(x_j)) \in L(\Sigma)$ for each $1 \leq j \leq k$, under a suitable embedding. Since the $k$ distinct points $(x_j)_{j=1}^k$, are uniquely determined by $u$, by the definition of the non-degenerate simplicial resolution, there are projection maps $\pi_{k,D} : X_k^D \setminus X_{k-1}^D \to C_{k;\Sigma}$ defined by

$((f_1, \ldots, f_r, u)) \mapsto \{(x_1, F(x_1)), \ldots, (x_k, F(x_k))\}$.

Now suppose that $1 \leq k \leq d_{\min}$. Let $c = \{(x_j, s_j)\}_{j=1}^k \in C_{k;\Sigma}$ be any fixed element and consider the fibre $\pi_{k,D}^{-1}(c)$. For each $1 \leq j \leq k$, we set $s_j = (s_{1,j}, \ldots, s_{r,j})$ and consider the condition

\begin{equation}
F(x_j) = (f_1(x_j), \ldots, f_r(x_j)) = s_j \iff f_i(x_j) = s_{t,j} \quad \text{for } 1 \leq t \leq r.
\end{equation}

In general, the condition $f_i(x_j) = s_{t,j}$ gives one linear condition on the coefficients of $f_i$, and determines an affine hyperplane in $\text{Po}^D(\mathbb{C})$. For example, if we set $f_i(z) = z^{d_i} + \sum_{t=0}^{d_i-1} a_{i,t} z^t$, then $f_i(x_j) = s_{t,j}$ for any $1 \leq j \leq k$ if and only if

\begin{equation}
\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{d_1-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{d_2-1} \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
1 & x_k & x_k^2 & \cdots & x_k^{d_k-1}
\end{bmatrix}
\begin{bmatrix}
a_{0,t} \\
a_{1,t} \\
\vdots \\
a_{d_t-1,t}
\end{bmatrix}
= \begin{bmatrix}
s_{t,1} - x_1^{d_1} \\
s_{t,2} - x_2^{d_2} \\
\vdots \\
s_{t,k} - x_k^{d_k}
\end{bmatrix}
\end{equation}
Since $1 \leq k \leq d_{\min}$ and $\{x_j\}_{j=1}^k \in C_k(\mathbb{C})$, it follows from the properties of Vandermonde matrices that the condition (6.9) gives exactly $k$ independent conditions on the coefficients of $f(z)$. Thus the space of polynomials $f(z)$ in $P^d(\mathbb{C})$ which satisfies (6.9) is the intersection of $k$ affine hyperplanes in general position and has codimension $k$ in $P^d(\mathbb{C})$. Hence, the fibre $\pi_{k,D}(c)$ is homeomorphic to the product of an open $(k-1)$-simplex with the real affine space of dimension $2 \sum_{i=1}^r (d_i - k) = 2N(D) - 2rk$. It is now easy to show that $\pi_{k,D}$ is a (locally trivial) real affine bundle over $C_{k;\Sigma}$ of rank $l_{D,k} = 2N(D) - 2rk + k - 1$. □

**Lemma 6.5.** If $1 \leq k \leq d_{\min}$, there is a natural isomorphism

$$E_{k,s}^{1:D} \cong \hat{H}_{k}^{2rk-s}(C_{k;\Sigma}, \pm \mathbb{Z}),$$

where the twisted coefficients system $\pm \mathbb{Z}$ comes from the Thom isomorphism.

**Proof.** Suppose that $1 \leq k \leq d_{\min}$. By Lemma 6.4, there is a homeomorphism $(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D)_+ \cong T(\xi_{D,k})$, where $T(\xi_{D,k})$ denotes the Thom space of $\xi_{D,k}$. Since $(2N(D) + k - s - 1) - l_{D,k} = 2rk - s$, by using the Thom isomorphism there is a natural isomorphism $E_{k,s}^{1:D} \cong \hat{H}^{2nd+k-s-1}(T(\xi_{D,k}), \mathbb{Z}) \cong \hat{H}_{k}^{2rk-s}(C_{k;\Sigma}, \pm \mathbb{Z})$. □

**Definition 6.6.** For an $r$-tuple $E = (e_1, \cdots, e_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers, let $N(E)$ denote the positive integer $N(E) = \sum_{k=1}^r e_k$ and let $U_E = \{w \in \mathbb{C} : \text{Re}(w) < N(E)\}$ as in Definition 2.10. □

Consider the stabilization map $s_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Hol}^*_D(S^2, X_\Sigma)$ of (2.15). It is easy to see that it extends to an open embedding

$$(6.10) \quad s_D : \mathbb{C}^{N(a)} \times \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Hol}^*_D(S^2, X_\Sigma).$$

Moreover, it also naturally extends to an open embedding $\tilde{s}_D : P^D_D \to P^D_D$ and by restriction we obtain an open embedding $\tilde{s}_D : \mathbb{C}^{N(a)} \times \Sigma_D \to \Sigma_{D+a}$. Since one-point compactification is contravariant for open embeddings, this map induces a map $\tilde{s}_{D+} : (\Sigma_{D+a})_+ \to (\mathbb{C}^{N(a)} \times \Sigma_D)_+ = S^{2N(a)}_+ \setminus \Sigma_{D+a}$.

Note that there is a commutative diagram

$$(6.11) \quad \begin{array}{c}
\hat{H}_k(\text{Hol}^*_D(S^2, X_\Sigma), \mathbb{Z}) \\
\text{Al} \downarrow \cong \text{Al} \uparrow \\
\hat{H}^{2N(D)-k-1}_{c}(\Sigma_D, \mathbb{Z}) \quad \tilde{s}_{D+}^* \quad \hat{H}^{2(N(D)+N(a))-k-1}_{c}(\Sigma_{D+a}, \mathbb{Z})
\end{array}$$

where $\text{Al}$ is the Alexander duality isomorphism and $\tilde{s}_{D+}$ denotes the composite of the suspension isomorphism with the homomorphism $(\tilde{s}_{D+})^*$,

$$\hat{H}^{M}_{c}(\Sigma_D, \mathbb{Z}) \cong \hat{H}^{M+2N(a)}_{c}(\mathbb{C}^{N(a)} \times \Sigma_D, \mathbb{Z}) \xrightarrow{(\tilde{s}_{D+})^*} \hat{H}^{M+2N(a)}_{c}(\Sigma_{D+a}, \mathbb{Z}),$$

where $M = 2N(D) - k - 1$. By the universality of the non-degenerate simplicial resolution ([19] pages 286-287), the map $\tilde{s}_D$ also naturally extends to a filtration preserving open
embedding \( \bar{s}_D : C^{N(a)} \times X^D \rightarrow X^{D+a} \) between non-degenerate simplicial resolutions. This induces a filtration preserving map \((\bar{s}_D)_+ : X^{D+a}_+ \rightarrow (C^{N(a)} \times X^D)_+ = S^{2N(a)} \wedge X^D_+ \), and thus a homomorphism of spectral sequences

\[
\{ \theta_{k,s}^t : E_{k,s}^{t,D} \rightarrow E_{k,s}^{t,(D+a)} \}, \quad \text{where}
\]

\[
\{ E_{k,s}^{t,D} = \tilde{d}^t : E_{k,s}^{t,D} \rightarrow E_{k+s,t+1}^{t,D} \} \quad \Rightarrow \quad H_{s-k}(\text{Hol}_D^t(S^2, X_S), \mathbb{Z}),
\]

\[
\{ E_{k,s}^{t,D+a} = \tilde{d}^t : E_{k,s}^{t,D+a} \rightarrow E_{k+s,t+1}^{t,D+a} \} \quad \Rightarrow \quad H_{s-k}(\text{Hol}_{D+a}^t(S^2, X_S), \mathbb{Z}),
\]

\[
E_{k,s}^{1,D} = \tilde{H}_c^{2N(D)+k-1-s}(X^D_k \setminus X^D_{k-1}), E_{k,s}^{1,D+a} = \tilde{H}_c^{2N(D+a)+k-1-s}(X^D_{k+a} \setminus X^D_{k-1}).
\]

**Lemma 6.7.** If \( 0 \leq k \leq d_{\min} \), \( \tilde{\theta}_{1,s}^1 : E_{k,s}^{1,D} \rightarrow E_{k,s}^{1,D+a} \) is an isomorphism for any \( s \).

**Proof.** Since the case \( k = 0 \) is clear, suppose that \( 1 \leq k \leq d_{\min} \). It follows from the proof of Lemma 6.3 that there is a homotopy commutative diagram of affine vector bundles

\[
\begin{array}{ccc}
\mathbb{C}^n \times (X^D_k \setminus X^D_{k-1}) & \longrightarrow & C_{k,S} \\
\downarrow & & \downarrow \\
X^D_{k+a} \setminus X^D_{k-1} & \longrightarrow & C_{k,S}
\end{array}
\]

Hence, we have a commutative diagram

\[
\begin{array}{ccc}
E_{k,s}^{1,D} & \longrightarrow & \tilde{H}_c^{2k-s}(C_{k,S}, \mathbb{Z}) \\
\downarrow \tilde{\theta}_{k,s}^1 & & \downarrow \\
E_{k,s}^{1,D+a} & \longrightarrow & \tilde{H}_c^{2k-s}(C_{k,S}, \mathbb{Z})
\end{array}
\]

and the assertion follows.

Now we consider the spectral sequences induced by the truncated simplicial resolutions.

**Definition 6.8.** Let \( X^\Delta \) denote the truncated (after the \( d_{\min} \)-th term) simplicial resolution of \( \Sigma_D \) with the natural filtration \( \emptyset = X^\Delta_0 \subset X^\Delta_1 \subset \cdots \subset X^\Delta_{d_{\min}} \subset X^\Delta_{d_{\min}+1} = X^\Delta_{d_{\min}+2} = \cdots = X^\Delta \), where \( X^\Delta_k = X^D_k \) if \( k \leq d_{\min} \) and \( X^\Delta_k = X^\Delta(d_{\min}) \) if \( k \geq d_{\min} + 1 \).

Similarly, let \( Y^\Delta \) denote truncated (after the \( d_{\min} \)-th term) simplicial resolution of \( \Sigma_{D+a} \) with the natural filtration \( \emptyset = Y^\Delta_0 \subset Y^\Delta_1 \subset \cdots \subset Y^\Delta_{d_{\min}} \subset Y^\Delta_{d_{\min}+1} = Y^\Delta_{d_{\min}+2} = \cdots = Y^\Delta \), where \( Y^\Delta_k = X^D_{k+a} \) if \( k \leq d_{\min} \) and \( Y^\Delta_k = Y^\Delta \) if \( k \geq d_{\min} + 1 \).

By using Lemma 5.6 and the same method as in [20 §2 and §3] (cf. [16, Lemma 2.2]), we obtain the following truncated spectral sequences

\[
\{ \tilde{E}_{k,s}^{t,D} = \tilde{d}^t : \tilde{E}_{k,s}^{t,D} \rightarrow \tilde{E}_{k+s,t+1}^{t,D+1} \} \quad \Rightarrow \quad H_{s-k}(\text{Hol}_D^t(S^2, X_S), \mathbb{Z}),
\]

\[
\{ \tilde{E}_{k,s}^{t,D+a} = \tilde{d}^t : \tilde{E}_{k,s}^{t,D+a} \rightarrow \tilde{E}_{k+s,t+1}^{t,D+a+1} \} \quad \Rightarrow \quad H_{s-k}(\text{Hol}_{D+a}^t(S^2, X_S), \mathbb{Z}),
\]
Proof. Since \( \Sigma \) is a smooth toric variety such that the conditions (1.12.1) and (1.12.2) are satisfied, first we prove the following key unstable result.

**Theorem 7.1.** The stabilization map \( s_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Hol}^*_{D+d}(S^2, X_\Sigma) \) is a homology equivalence through dimension \( d(D, \Sigma) = (2r_{\text{min}}(\Sigma) - 3)d_{\text{min}} - 2 \).

**Proof.** We write \( r_{\text{min}} = r_{\text{min}}(\Sigma) \), and consider the homomorphism \( \theta^t_{k,s} : E^t_{k,s} \to 'E^t_{k,s} \) of truncated spectral sequences given in (6.13). By using the commutative diagram (6.11) and the comparison theorem for spectral sequences, it suffices to prove that the positive integer \( d(D, \Sigma) \) has the following property:

\[
E^1_{k,s} = \tilde{H}^2_{c}(X_k^\Delta \setminus X_k^{\Delta-1}), \quad 'E^1_{k,s} = \tilde{H}^2_{c}(Y_k^\Delta \setminus Y_k^{\Delta-1}).
\]

By the naturality of truncated simplicial resolutions, the filtration preserving map \( \tilde{s}_D : \mathbb{C}^{N(\Sigma)} \times \mathcal{X}^{\Delta} \to \mathcal{X}^{\Delta+a} \) gives rise to a natural filtration preserving map \( \tilde{s}_D : \mathbb{C}^{N(\Sigma)} \times \mathcal{X}^{\Delta} \to \mathcal{X}^{\Delta} \) which, in a way analogous to (6.12), induces a homomorphism of spectral sequences

\[
(6.13) \quad \{ \theta^t_{k,s} : E^t_{k,s} \to 'E^t_{k,s} \}.
\]

**Lemma 6.9.** (i) If \( k < 0 \) or \( k \geq d_{\text{min}} + 2 \), \( E^1_{k,s} = 'E^1_{k,s} = 0 \) for any \( s \).

(ii) \( E^1_{0,0} = 'E^1_{0,0} = \mathbb{Z} \) and \( E^1_{0,s} = 'E^1_{0,s} = 0 \) if \( s \neq 0 \).

(iii) If \( 1 \leq k \leq d_{\text{min}} \), there are isomorphisms \( E^1_{k,s} \simeq 'E^1_{k,s} \simeq \tilde{H}^2_{c}(C_k; \mathbb{Z}) \).

(iv) \( E^1_{d_{\text{min}}+1,s} = 'E^1_{d_{\text{min}}+1,s} = 0 \) for any \( s \leq (2r_{\text{min}}(\Sigma) - 2)d_{\text{min}} - 1 \).

**Proof.** Let us write \( r_{\text{min}} = r_{\text{min}}(\Sigma) \). Since the proofs of both cases are identical, it suffices to prove the assertions for the case \( E^1_{k,s} \). Since \( X_k^\Delta = X_k^{\Delta} \) for any \( k \geq d_{\text{min}} + 2 \), the assertions (i) and (ii) are clearly true. Since \( X_k^\Delta = X_k^{\Delta} \) for any \( k \leq d_{\text{min}} \), (iii) easily follows from Lemma 6.5. Thus it remains to prove (iv). By Lemma 2.1,

\[
\dim(X_{d_{\text{min}}+1}^\Delta \setminus X_{d_{\text{min}}}^\Delta) = \dim(X_{d_{\text{min}}-1}^D \setminus X_{d_{\text{min}}-1}^D) + 1 = (l_D, d_{\text{min}} + 3d_{\text{min}} - 2r_{\text{min}}d_{\text{min}}).
\]

Since \( E^1_{d_{\text{min}}+1,s} = \tilde{H}^2_{c}(X_{d_{\text{min}}+1}^\Delta \setminus X_{d_{\text{min}}}^\Delta, \mathbb{Z}) \) and \( 2N(D) + d_{\text{min}} - s > \dim(X_{d_{\text{min}}+1}^\Delta \setminus X_{d_{\text{min}}}^\Delta) \)
\[ \iff s \leq (2r_{\text{min}} - 2)d_{\text{min}} - 1 \], we see that \( E^1_{d_{\text{min}}+1,s} = 0 \) for any \( s \leq (2r_{\text{min}} - 2)d_{\text{min}} - 1 \).

**Lemma 6.10.** If \( 0 \leq k \leq d_{\text{min}} \), \( \theta^t_{k,s} : E^t_{k,s} \to 'E^t_{k,s} \) is an isomorphism for any \( s \).

**Proof.** Since \( (X_k^\Delta, Y_k^\Delta) = (X_k^{\Delta}, X_k^{\Delta+a}) \) for \( k \leq d_{\text{min}} \), the assertion follows from Lemma 6.7.

7 The proofs of the main results

In this section we prove Theorem 1.9 and Corollary 1.11. For this purpose, from now on we always assume that \( X_\Sigma \) is a smooth toric variety such that the conditions (1.12.1) and (1.12.2) are satisfied. First we prove the following key unstable result.

**Theorem 7.1.** The stabilization map \( s_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \text{Hol}^*_{D+d}(S^2, X_\Sigma) \) is a homology equivalence through dimension \( d(D, \Sigma) = (2r_{\text{min}}(\Sigma) - 3)d_{\text{min}} - 2 \).

**Proof.** We write \( r_{\text{min}} = r_{\text{min}}(\Sigma) \), and consider the homomorphism \( \theta^t_{k,s} : E^t_{k,s} \to 'E^t_{k,s} \) of truncated spectral sequences given in (6.13). By using the commutative diagram (6.11) and the comparison theorem for spectral sequences, it suffices to prove that the positive integer \( d(D, \Sigma) \) has the following property:
Proof. By Lemma \[6.9\] \( E^1_{k,s} = \, E^1_{k,s} = 0 \) if \( k < 0 \), or if \( k \geq d_{\min} + 2 \), or if \( k = d_{\min} + 1 \) with \( s \leq (2r_{\min} - 2)d_{\min} - 1 \). Since \( (2r_{\min} - 2)d_{\min} - 1 - (d_{\min} + 1) = (2r_{\min} - 3)d_{\min} - 2 = d(D, \Sigma) \), we see that:

\[(*)_1 \text{ if } k < 0 \text{ or } k \geq d_{\min} + 1, \quad \theta_{k,s}^\infty \text{ is an isomorphism for all } (k, s) \text{ such that } s - k \leq d(D, \Sigma).\]

Next, we assume that \( 0 \leq k \leq d_{\min} \), and investigate the condition that \( \theta_{k,s}^\infty \) is an isomorphism. Note that the groups \( E^1_{k,1} \) and \( E^1_{k,s} \) are not known for \( (u, v) \in S_1 = \{(d_{\min} + 1, s) \in \mathbb{Z}^2 : s \geq (2r_{\min} - 2)d_{\min}\} \). By considering the differentials \( d^1 \)'s of \( E^1_{k,s} \) and \( 'E^1_{k,s} \), and applying Lemma \[6.10\] we see that \( \theta_{k,s}^2 \) is an isomorphism if \( (k, s) \notin S_1 \cup S_2 \), where

\[
S_2 = \{(u, v) \in \mathbb{Z}^2 : (u + 1, v) \in S_1 \} = \{(d_{\min}, v) \in \mathbb{Z}^2 : v \geq (2r_{\min} - 2)d_{\min}\}.
\]

A similar argument shows that \( \theta_{k,s}^2 \) is an isomorphism if \( (k, s) \notin \bigcup_{t=1}^3 S_t \), where \( S_3 = \{(u, v) \in \mathbb{Z}^2 : (u + 2, v + 1) \in S_1 \cup S_2 \} \). Continuing in the same fashion, considering the differentials \( d^t \)'s on \( E_{k,s}^t \) and \( 'E_{k,s}^t \), and applying the inductive hypothesis, we see that \( \theta_{k,s}^\infty \) is an isomorphism if \( (k, s) \notin \mathcal{S} := \bigcup_{t \geq 1} S_t = \bigcup_{t \geq 1} A_t \), where \( A_t \) denotes the set

\[
A_t = \left\{ (u, v) \in \mathbb{Z}^2 \mid \text{There are positive integers } l_1, \ldots, l_t \text{ such that,} \right. \\
\left. \begin{array}{c}
1 \leq l_1 < l_2 < \cdots < l_t, \ u + \sum_{j=1}^t l_j = d_{\min} + 1, \\
v + \sum_{j=1}^t (l_j - 1) \geq (2r_{\min} - 2)d_{\min}
\end{array} \right\}.
\]

Note that if this set was empty for every \( t \), then, of course, the conclusion of Theorem \[7.1\] would hold in all dimensions (this is known to be false in general). If \( A_t \neq \emptyset \), it is easy to see that

\[
a(t) = \min\{s - k : (k, s) \in A_t\} = (2r_{\min} - 2)d_{\min} - (d_{\min} + 1) + t = d(D, \Sigma) + t + 1.
\]

Hence, we obtain that \( \min\{a(t) : t \geq 1, A_t \neq \emptyset\} = d(D, \Sigma) + 2 \). Since \( \theta_{k,s}^\infty \) is an isomorphism for any \( (k, s) \notin \bigcup_{t \geq 1} A_t \), for each \( 0 \leq k \leq d_{\min} \), we have the following:

\[(*)_2 \text{ If } 0 \leq k \leq d_{\min}, \quad \theta_{k,s}^\infty \text{ is an isomorphism for any } (k, s) \text{ such that } s - k \leq d(D, \Sigma) + 1.\]

Then, by \((*)_1\) and \((*)_2\), we know that \( \theta_{k,s}^\infty : E_{k,s}^\infty \xrightarrow{\cong} 'E_{k,s}^\infty \) is an isomorphism for any \( (k, s) \) if \( s - k \leq d(D, \Sigma) \). This completes the proof of Theorem \[7.1\].

**Corollary 7.2.** The inclusion map \( i_D : \text{Hol}_{d}^\infty(S^2, X_\Sigma) \rightarrow \Omega^2_D X_\Sigma \) is a homology equivalence through dimension \( d(D, \Sigma) \).

**Proof.** The assertion easily follows from Theorem \[7.2\] and Theorem \[7.1\].

**Lemma 7.3.** The space \( \Omega^2_D X_\Sigma \) is \( 2(r_{\min}(\Sigma) - 2) \)-connected.
Hence, the map two spaces Holalance through dimension isomorphism. Note that Ω(D) = 2 was obtained. Now assume that r_{min}(Σ) ≥ 3. By Hurewicz Theorem, the Hurewicz homomorphism h : π_1(Ω(D)(S^2, X_Σ)) \xrightarrow{\cong} H_1(Ω(D)(S^2, X_Σ), \mathbb{Z}) is an isomorphism. Note that Ω(D)(S^2, X_Σ) is at least 2-connected (by Lemma 7.3). Since d(D, Σ) = 2(r_{min}(Σ) - 3)d_{min} - 2 ≥ 3d_{min} - 2 ≥ 1, by Corollary 7.2 the map i_D induces the isomorphism i_{D*} : H_1(Ω(D)(S^2, X_Σ), \mathbb{Z}) \xrightarrow{\cong} H_1(Ω(D)(X_Σ), \mathbb{Z}) = 0. Hence, π_1(Ω(D)(S^2, X_Σ)) = 0 and the case r_{min}(Σ) ≥ 3 was also obtained.

Now we can prove the main results (Theorem 1.9 and Corollary 1.11).

**Proof of Theorem 1.9.** By Corollary 7.2, it remains to prove that i_D is a homotopy equivalence through dimension d(D, Σ) if r_{min}(Σ) ≥ 3. Assume that r_{min}(Σ) ≥ 3. Note that two spaces Ω(D)(S^2, X_Σ) and Ω(D)(X_Σ) are simply connected by Lemma 7.3 and Lemma 7.4. Hence, the map i_D is a homotopy equivalence through dimension d(D, Σ).

**Proof of Corollary 1.11.** Let X_Σ be a compact smooth toric variety such that Σ(1) = \{Cone(n_k) : 1 ≤ k ≤ r\}, where \{n_k\}_{k=1}^r are primitive generators as in §1. Since X_Σ is a compact, by (ii) of Lemma 2.5 we easily see that the condition (1.12.1) is satisfied for X_Σ. Since Σ(1) ⊆ X_Σ, by using (i) of Lemma 2.5 we see that X_{Σ,1} is a non-compact smooth toric subvariety of X_Σ. Moreover, since Σ(1) ⊆ Σ(1) ⊆ X_Σ, we see that Σ(1) = Σ(1). Hence, the condition (1.12.1) holds for X_{Σ,1}, too. Thus, the assertion follows from Theorem 1.9.

**Corollary 7.5.** (i) If r_{min}(Σ) ≥ 3, the space Ω(D)(S^2, X_Σ) is 2(r_{min}(Σ) - 2)-connected.

(ii) If r_{min}(Σ) = 2 and d_{min} ≥ 3, i_{D*} : π_1(Ω(D)(S^2, X_Σ)) \xrightarrow{\cong} π_1(Ω(D)(X_Σ)) is an isomorphism.

**Proof.** Since the first assertion easily follows from Lemma 7.3 and Theorem 1.9, it suffices to prove the assertion (ii). Now assume that r_{min}(Σ) = 2. Then by Lemma 7.4, the Hurewicz homomorphism h : π_1(Ω(D)(S^2, X_Σ)) \xrightarrow{\cong} H_1(Ω(D)(S^2, X_Σ), \mathbb{Z}) is an isomorphism. Consider the commutative diagram

\[
\begin{array}{ccc}
π_1(Ω(D)(S^2, X_Σ)) & \xrightarrow{i_{D*}} & π_1(Ω(D)(X_Σ)) \\
\downarrow \cong & & \downarrow \cong \\
H_1(Ω(D)(S^2, X_Σ), \mathbb{Z}) & \xrightarrow{i_{D*}} & H_1(Ω(D)(X_Σ), \mathbb{Z})
\end{array}
\]

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If $d_{\min} \geq 3$, $d(D, \Sigma) = d_{\min} - 2 \geq 1$. Hence, by Theorem 1.9, the induced homomorphism $i_{D^*}$ on the homology is an isomorphism and the assertion (ii) follows from the above commutative diagram.

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