Numerical study of noise-induced transitions in nonlinear dynamics of optically injected semiconductor lasers

Chin-Hao Tseng¹, Jia-Han Yang², and Sheng-Kwang Hwang¹,3a)

¹ Department of Photonics, National Cheng Kung University, Tainan, Taiwan
² Graduate Institute of Opto-Mechatronics, National Chung Cheng University, Chia-Yi, Taiwan
³ Advanced Optoelectronic Technology Center, National Cheng Kung University, Tainan, Taiwan

a) skhwang@mail.ncku.edu.tw

Received July 11, 2021; Revised August 27, 2021; Published January 1, 2022

Abstract: This study numerically investigates two different noise-induced transitions of nonlinear dynamics in optically injected semiconductor lasers based on the Lang-Kobayashi laser model. Spontaneous emission noise is observed to induce dynamical transitions from stable injection locking to period-one dynamics and from period-one to period-two dynamics when operating points are close to the corresponding dynamical boundaries, respectively. Such transitions follow a smooth dynamical evolution as the noise level increases. The resulting noise-induced dynamical states exhibit features highly similar to their adjacent dynamical states that keep the same dynamical behaviors when subject to noise. Regions where the noise-induced dynamical transitions occur are identified.

Key Words: semiconductor lasers, nonlinear dynamics, spontaneous emission noise, dynamical transition

1. Introduction

Since the 1980s, a semiconductor laser subject to external optical injection has attracted much research interest due to its potential applications [1–20] and profound physics [21–25]. Tuning the power and frequency of the optical injection destabilizes the laser through a period-doubling route to chaos, which has been observed numerically [26] and experimentally [27]. A variety of different nonlinear dynamical states can therefore be induced, such as stable injection locking, period-one (P1) dynamics, period-two (P2) dynamics, and chaos. One critical parameter that influences the dynamical behavior of the laser is its intrinsic spontaneous emission noise. For example, Hwang et al. [21] have numerically observed a noise-induced transition from periodicity to chaos and quantitatively identified certain required characteristics leading to such an interesting dynamical phenomenon. This raises a question whether...
other types of noise-induced transitions would also occur in the laser system, such as a transition from P1 to P2 dynamics or from stable injection locking to P1 dynamics. This has not been reported yet in the literature and is therefore investigated numerically in this study using the typical Lang-Kobayashi laser model.

In nonlinear dynamical systems, noise can induce various dynamical phenomena, including noise-induced instabilities [28, 29], noise-induced order [30], and noise-induced chaos [31–33]. Noise-induced chaos was first observed by Crutchfield et al. using an anharmonic oscillator [31] and next studied through a logistic map [32]. It has been considered that when a system follows a period-doubling route to chaos, noise tends to induce chaos in certain parameter regions and smooth out the transition route across the boundary between different nonlinear dynamical states [34–36]. Simpson has experimentally observed a dynamical transition between stable injection locking and P1 dynamics in an optically injected semiconductor laser [37]. Since the spontaneous emission noise of a laser always exists in the real world, it is unclear whether such a dynamical transition is induced by noise. In addition, since the optical injection system follows a period-doubling route to chaos, the same as the anharmonic oscillator and logistic map mentioned above, it is wondered whether a dynamical transition across the boundary exists and follows a gradual and smooth dynamical evolution as the noise level increases. In this study, different extents of different operating conditions are considered, including spontaneous emission noise, optical injection power, and optical injection frequency, to investigate noise-induced dynamical transitions, in particular, from stable injection locking to P1 dynamics and from P1 to P2 dynamics.

The organization of this paper is arranged as follows. Chapter 2 presents the numerical model of an optically injected semiconductor laser based on normalized rate equations. In Chapter 3 and Chapter 4, the spontaneous emission noise level, optical injection power, and optical injection frequency are varied, respectively, to investigate noise-induced transitions from stable injection locking to P1 dynamics and from P1 to P2 dynamics, respectively. Finally, a conclusion is given in Chapter 5.

2. Numerical model

A schematic diagram of an optically injected laser system consisting of two single-mode semiconductor lasers, LD1 and LD2, respectively, is illustrated in Fig. 1. The free-running output of LD1 is unidirectionally injected into LD2 through an optical isolator. The output signal from LD2 is used to investigate the characteristics of noise-induced dynamical transitions under consideration.

The optically injected laser under study, i.e., LD2, can be mathematically described by the following normalized Lang-Kobayashi equations [38, 39]:

\[
\frac{da}{dt} = \frac{1}{2} \left[ \frac{\gamma_c \gamma_n}{\gamma_p J} \tilde{n} - \gamma_p (2a + a^2) \right] (1 + a) + \frac{\xi_i \gamma_c}{\gamma_p} \cos (\Omega t + \phi) + \mu F_a \\
\frac{d\phi}{dt} = -b \left[ \frac{\gamma_c \gamma_n}{\gamma_p J} \tilde{n} - \gamma_p (2a + a^2) \right] - \frac{\xi_i \gamma_c}{1 + a} \sin (\Omega t + \phi) + \frac{\mu F_\phi}{1 + a} \\
\frac{d\tilde{n}}{dt} = -\gamma_s \tilde{n} - \gamma_p \tilde{n}(1 + a)^2 - \gamma_s J(2a + a^2) + \frac{\gamma_p}{\gamma_c} \tilde{J}(2a + a^2)(1 + a)^2
\]

The normalized intracavity field amplitude and carrier density are described by \(a\) and \(\tilde{n}\), respectively. The phase difference between the injection field and LD2 is described by \(\phi\). Laser intrinsic parameters \(b, \gamma_c, \gamma_s, \gamma_n, \gamma_p\) present the linewidth enhancement factor, cavity decay rate, spontaneous carrier relaxation rate, differential carrier relaxation rate, and nonlinear carrier relaxation rate, respectively. The normalized bias current density of LD2 is expressed by \(\tilde{J}\). The normalized injection strength is described by \(\xi_i\), which is the field amplitude ratio between the optical injection from LD1 and the

![Fig. 1. Schematic diagram of an optically injected laser system.](image-url)
free-running LD2 and indicates the injection level that LD2 receives. The square of $\xi_i$ is proportional to the optical injection power from LD1. The detuning frequency described by $f_i = \Omega/2\pi$ is the frequency offset between the free-running LD1 and LD2. The spontaneous emission noise of LD2 is characterized by the normalized Langevin noise-source parameters $F_a$ and $F_\phi$ [40]:

$$\langle F_a(t)F_a(t') \rangle = \langle F_\phi(t)F_\phi(t') \rangle = \frac{R_{sp}}{2|A_0|^2} \delta(t - t')$$  \hspace{1cm} (4)$$

$$\langle F_a(t)F_\phi(t') \rangle = \langle F_\phi(t)F_a(t') \rangle = 0$$  \hspace{1cm} (5)$$

where $R_{sp}$ represents the fraction of the spontaneous emission noise into the laser mode, and $A_0$ represents the field amplitude of LD2 under a free-running state. In order to study how a different noise level affects a dynamical transition, a coefficient $\mu$ introduced before $F_a$ and $F_\phi$ is used to adjust the noise power of LD2. While $\mu = 0$ describes an ideal yet unrealistic situation where no spontaneous emission noise exists, $\mu = 1$ presents a practical case where the level of the spontaneous emission noise is experimentally measured. The optical spectrum is obtained by calculating the Fourier transform of the optical field $a(t)e^{j\phi(t)}$:

$$\int_{-\infty}^{\infty} a(t)e^{j\phi(t)} \cdot e^{j2\pi ft} dt$$  \hspace{1cm} (6)$$

The intensity of LD2 output $I(t) = (1 + a(t))^2$ is used to characterize its temporal behavior, and the microwave spectrum is obtained by carrying out the Fourier transform of $I(t)$:

$$\int_{-\infty}^{\infty} I(t) \cdot e^{j2\pi ft} dt$$  \hspace{1cm} (7)$$

The values of the laser intrinsic parameters, which are experimentally measured using the four-wave mixing method when the LD2 bias current is set at 40 mA, used for the numerical calculation here are $b = 3$, $\gamma_c = 5.36 \times 10^{11}$ s$^{-1}$, $\gamma_s = 5.96 \times 10^9$ s$^{-1}$, $\gamma_n = 7.53 \times 10^9$ s$^{-1}$, $\gamma_p = 1.91 \times 10^{10}$ s$^{-1}$, and $J = 1.222$, respectively [41]. Under this operating condition, the relaxation resonance frequency of the free-running LD2 is approximately 10.25 GHz. The fraction of the spontaneous emission noise into the laser mode, $R_{sp}$, is obtained to be $4.7 \times 10^{19}$ $V^2m^{-1}s^{-1}$ [40]. Equations (1)–(3) are solved through the well-known second-order Runge-Kutta method with a constant integration time step of 0.47 ps. The validity of the rate equations used in this study is supported by the experimental results under the same operation conditions [41, 42].

3. Noise-induced transition from P1 to P2 dynamics

For the given laser intrinsic parameters, the dynamical characteristics of an optically injected semiconductor laser are mainly determined by the two operational parameters, injection strength $\xi_i$ and detuning frequency $f_i$. Figure 2 presents a dynamical mapping as a function of these two operational parameters when $\mu = 0$, where different dynamical states are marked by different colors. For instance, stable injection locking is described by red. The regions of P1 and P2 dynamics are presented by yellow and blue, respectively. Periodic dynamics with periods higher than two are included in the regions of chaos, which are marked by black.

3.1 Dependence of dynamical transition on noise level

To investigate whether the spontaneous emission noise can induce a dynamical transition from P1 to P2 dynamics, two operating conditions that are next to each other across the boundary between the P1 and P2 regions are considered in the following demonstration. Figure 3(a) presents the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) of a P1 dynamical state at $(\xi_i, f_i) = (0.096, 15$ GHz) when $\mu = 0$. As shown in Fig. 3(a–ii), a regeneration of the optical injection appears at the offset frequency of 15 GHz owing to the injection pulling effect [43]. Meanwhile, oscillation sidebands emerge, which are equally separated from the regeneration by an oscillation frequency of $f_0 = 19.5$ GHz. Such an optical spectral feature indicates that the intensity of the LD2 output oscillates
Fig. 2. Dynamical mapping of the optically injected LD2. Regions of stable injection locking, P1 dynamics, P2 dynamics, and chaos are marked as red, yellow, blue, and black, respectively.

sinusoidally with a single period of $1/f_0 = 51.28$ ps, as shown in Fig. 3(a-i). Fourier transformation of the intensity time series gives rise to a microwave at $f_0 = 19.5$ GHz and its harmonics, as illustrated in Fig. 3(a-iii).

Figure 3(b) shows the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) at the same operating condition $(\xi_i, f_i) = (0.096, 15$ GHz) when the spontaneous emission noise is considered, i.e., $\mu = 1$. As Fig. 3(b-ii) demonstrates, while the spectral components observed in Fig. 3(a-ii) still appear with similar features, subharmonics are found to emerge in the midway between the spectral components when the spontaneous emission noise is considered. Such an optical spectral feature indicates that not only the intensity of the LD2 output oscillates with a period of $1/f_0 = 51.28$ ps but also the oscillating intensity is modulated with a period of around $2/f_0 = 102.56$ ps, as shown in Fig. 3(b-i). Fourier transformation of the intensity time series not only gives rise to a microwave at $f_0 = 19.5$ GHz and its harmonics, similar to those in Fig. 3(a-iii), but also leads to subharmonics at the midway between the spectral components, as illustrated in Fig. 3(b-iii). These features suggest that the spontaneous emission noise induces a dynamical transition from a P1 dynamical state to a P2 dynamical state. To quantitatively analyze and thus identify such a dynamical transition, the power ratio between the subharmonic at the lowest frequency and the background noise floor in the microwave spectrum is used and referred to as the relative intensity (RI) in this section.

To explore whether and to what extent the feature of the noise-induced P2 dynamical state is related to that of the P2 dynamical state, which is next to it and keeps the same P2 behavior when the spontaneous emission noise is considered, the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) at $(\xi_i, f_i) = (0.094, 15$ GHz) are presented in Fig. 3(c) with $\mu = 1$ (black curves). Compared to the same P2 dynamical state without spontaneous emission noise ($\mu = 0$), as the blue curves of Fig. 3(c-ii) and Fig. 3(c-iii) present, the spectral features of the black and blue curves are similar. These results show that the behavior of P2 dynamical state is similarly kept when the spontaneous emission noise is considered. As clearly observed, both the temporal and spectral features illustrated by the black curves of Fig. 3(c) are closely similar to those presented in Fig. 3(b). This indicates that the characteristics of the noise-induced P2 dynamical state are determined, to a large extent, by the P2 dynamical state that is next to it and keeps the same P2 behavior when the spontaneous emission noise is considered.
Fig. 3. Intensity time series (i), optical spectra (ii), and microwave spectra (iii) of the P1 dynamical state at \((\xi_i, f_i) = (0.096, 15 \text{ GHz})\) when \(\mu = 0\) (a) and \(\mu = 1\) (b), respectively, and of the P2 dynamical state at \((\xi_i, f_i) = (0.094, 15 \text{ GHz})\) when \(\mu = 1\) (black curve) and \(\mu = 0\) (blue curve) (c). The x-axes in (ii) are offset by the free-running frequency of LD2.

Fig. 4. (a)-(e) Microwave spectra under different values of \(\mu\) from \(10^{-4}\) to \(10^{-1}\), respectively. (f) RI as a function of \(\mu\) from \(10^{-4}\) to 1.

In order to study how different levels of the noise affect the dynamical transition from the P1 dynamical state to the P2 dynamical state, a progression of the microwave spectra at \((\xi_i, f_i) = (0.096, 15 \text{ GHz})\) as a function of \(\mu\) is presented in Figs. 4(a)-(e). By varying \(\mu\) from \(10^{-4}\) to \(10^{-1}\), RI gradually grows from 0 to around 30 dB. Figure 4(f) summarizes how RI evolves as \(\mu\) increases, and shows that RI starts to increase when \(\mu\) increases beyond \(3.16 \times 10^{-4}\) and saturates to around 30 dB when \(\mu = 10^{-1.5}\). The value of RI for the P2 dynamical state at \((\xi_i, f_i) = (0.094, 15 \text{ GHz})\) is found...
to be about 32.42 dB. This confirms that the characteristics of the noise-induced P2 dynamical state is determined, to a large extent, by the P2 dynamical state that is next to it and keeps the same P2 behavior when the spontaneous emission noise is considered. As Fig. 4 demonstrates, the dynamical transition from a P1 dynamical state to a P2 dynamical state follows a gradual and smooth change in its characteristics as the noise level increases.

3.2 Effects of $\xi_i$ and $f_i$ on dynamical transition

As demonstrated above, the spontaneous emission noise can induce a dynamical transition from a P1 dynamical state to a P2 dynamical state when the former appears close to the boundary separating the two different dynamical states. In this subsection, the injection strength $\xi_i$ and the detuning frequency $f_i$ are varied to study how far away a P1 dynamical state from the dynamical boundary should be in order for such a dynamical transition to happen.

Figure 5(a) presents RI as a function of the injection strength $\xi_i$ when $f_i = 15$ GHz and $\mu = 1$. As $\xi_i$ increases from 0.096 to 0.128 (black curve), RI decreases roughly from 30 to 4 dB at the fixed detuning frequency. On the other hand, when $\xi_i$ increases from 0.032 to 0.056 (blue curve), RI increases approximately from 1 to 27 dB. Comparing Fig. 5(a) with Fig. 2 shows that the dynamical transition from a P1 dynamical state to a P2 dynamical state depends solely on the extent of the proximity of the P1 dynamical state to the dynamical boundary separating the P1 and P2 regions, not on the injection strength.

Figure 5(b) shows RI as a function of the detuning frequency $f_i$ when $\xi_i = 0.08$ and $\mu = 1$. Under this operating condition, RI decreases approximately from 27 to 3 dB while $f_i$ increases from 18 to 25 GHz (black curve). On the other hand, RI increases roughly from 0 to 25 dB when $f_i$ increases from
4 to 11 GHz (blue curve). Comparing Fig. 5(b) with Fig. 2 again shows that the dynamical transition from a P1 dynamical state to a P2 dynamical state depends solely on the extent of the proximity of the P1 dynamical state to the dynamical boundary separating the P1 and P2 regions, not on the detuning frequency.

Figure 5(c) shows a map of RI as a function of the injection strength and detuning frequency, which summarizes effects of $\xi_i$ and $f_i$ on the noise-induced dynamical transition from a P1 dynamical state to a P2 dynamical state. The value of RI increases when a P1 dynamical state appears closer to the boundary separating the P1 and P2 regions. By referring to Fig. 4(f), Fig. 5(c) demonstrates that the noise-induced dynamical transition from a P1 dynamical state to a P2 dynamical state occurs only when the P1 dynamical state is highly close to the boundary, as the black region indicating a RI range between 30 and 35 dB presents.

4. Noise-induced dynamical transition from stable injection locking to P1 dynamics

To investigate whether the spontaneous emission noise can induce a dynamical transition from stable injection locking to P1 dynamics, two operating conditions that are next to each other across the boundary between the stable injection locking and P1 regions (i.e., the Hopf bifurcation boundary) are considered in the following demonstration. Figure 6(a) shows the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) of a stable injection locking state at $(\xi_i, f_i) = (0.272, 15 \text{ GHz})$ when $\mu = 0$. As shown in Fig. 6(a-ii), a regeneration of the optical injection appears at the offset frequency of 15 GHz owing to the injection pulling effect [43]. Such an optical spectral feature indicates that the intensity of the LD2 output is a constant, as shown in Fig. 6(a-i). Fourier transformation of the intensity time series results in no signal, as presented in Fig. 6(a-iii).

Fig. 6. Intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) of the stable injection locking state at $(\xi_i, f_i) = (0.272, 15 \text{ GHz})$ when $\mu = 0$ (a) and $\mu = 1$ (b), respectively, and the P1 dynamical state at $(\xi_i, f_i) = (0.271, 15 \text{ GHz})$ when $\mu = 1$ (black curve) and $\mu = 0$ (blue curve) (c). The x-axes in (ii) are offset by the free-running frequency of LD2.
Figure 6(b) shows the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) at the same operating condition \((\xi, f) = (0.272, 15 \text{ GHz})\) when \(\mu = 1\). As Fig. 6(b-ii) demonstrates, while the spectral component at 15 GHz observed in Fig. 6(a-ii) still appears with similar features, oscillation sidebands emerge, which are equally separated away from the regeneration by an oscillation frequency of \(f_0 = 29 \text{ GHz}\). Such an optical spectral feature indicates that the intensity of the LD2 output oscillates sinusoidally with a single period of \(1/f_0 = 34.48 \text{ ps}\), as shown in Fig. 6(b-i). Fourier transformation of the intensity time series gives rise to a microwave at \(f_0 = 29 \text{ GHz}\), as illustrated in Fig. 6(b-iii). These features suggest that the spontaneous emission noise induces a dynamical transition from a stable injection locking state to a P1 dynamical state. To quantitatively analyze and thus identify such a dynamical transition, the power ratio between the microwave and a reference noise level of \(-110 \text{ dB}\) in the microwave spectrum is used and referred to as the relative intensity (RI) in this section.

To investigate whether and to what extent the feature of the noise-induced P1 dynamical state is related to that of the P1 dynamical state, which is next to it and keeps the same P1 behavior when the spontaneous emission noise is considered, the intensity time series (i), optical spectrum (ii), and microwave spectrum (iii) at \((\xi, f) = (0.271, 15 \text{ GHz})\) are presented in Fig. 6(c) with \(\mu = 1\) (black curves). Compared to the same P1 dynamical state without spontaneous emission noise \((\mu = 0)\), as the blue curves of Fig. 6(c-ii) and Fig. 6(c-iii) show, the spectral features of the black and blue curves are similar. These results show that the behavior of the P1 dynamical state is similarly kept when the spontaneous emission noise is considered. As clearly observed, both temporal and spectral features illustrated by the black curves of Fig. 6(c) are closely similar to those presented in Fig. 6(b). This indicates that the characteristics of the noise-induced P1 dynamical state are determined, to a large extent, by the P1 dynamical state that is next to it and keeps the same P1 behavior when the spontaneous emission noise is considered.

### 4.1 Dependence of dynamical transition on noise level

In order to study how different levels of the noise affect the dynamical transition from the stable locking state to the P1 dynamical state, a progression of the microwave spectra at \((\xi, f) = (0.272, 15 \text{ GHz})\) as a function of \(\mu\) is presented in Figs. 7(a)-(e). By varying \(\mu\) from \(10^{-3}\) to \(10^{-1}\), RI gradually grows approximately from 21 to around 60 dB. Figure 7(f) summarizes how RI evolves as \(\mu\) increases, and shows that RI increases with \(\mu\) and saturates to around 73 dB when \(\mu\) increases beyond \(10^{-0.5}\). The value of RI for the P1 dynamical state at \((\xi, f) = (0.271, 15 \text{ GHz})\) is found to be around 73.42 dB. This confirms that the characteristics of the noise-induced P1 dynamical state are determined,
to a large extent, by the P1 dynamical state that is next to it and keeps the same P1 behavior when
the spontaneous emission noise is considered. As Fig. 7 demonstrates, the dynamical transition from
a stable injection locking state to a P1 dynamical state follows a gradual and smooth change in its
characteristics as the noise level increases.

4.2 Effects of $\xi_i$ and $f_i$ on dynamical transition

As demonstrated above, the spontaneous emission noise can induce a dynamical transition from a
stable injection locking state to a P1 dynamical state when the former appears close to the Hopf
bifurcation boundary separating the two different dynamical states. In this subsection, the injection
strength $\xi_i$ and the detuning frequency $f_i$ are adjusted to study how far away a stable injection
locking state from the Hopf bifurcation boundary should be in order for such a dynamical transition
to happen.

Figure 8(a) presents RI as a function of the injection strength $\xi_i$ when $f_i = 15$ GHz and $\mu = 1$.
As $\xi_i$ increases toward the Hopf bifurcation boundary from 0.272 to 0.304, RI decreases roughly from
72 to 51 dB. Comparing Fig. 8(a) with Fig. 2 shows that the dynamical transition from a stable
injection locking state to a P1 dynamical state depends solely on the extent of the proximity of the
stable injection locking state to the Hopf bifurcation boundary.

Figure 8(b) shows RI as a function of the detuning frequency $f_i$ when $\xi_i = 0.272$ and $\mu = 1$. Under
this operating condition, RI increases approximately from 45 to 72 dB while $f_i$ increases from 8 to
15 GHz. Comparing Fig. 8(b) with Fig. 2 again shows that the dynamical transition from a stable
injection locking state to a P1 dynamical state depends solely on the extent of the proximity of the
stable injection locking state to the Hopf bifurcation boundary.

Figure 8(c) shows a map of RI as a function of the injection strength and detuning frequency, which
summarizes effects of $\xi_i$ and $f_i$ on the noise-induced dynamical transition from a stable injection

Fig. 8. (a) RI as a function of $\xi_i$ at $f_i = 15$ GHz and $\mu = 1$. (b) RI as a
function of $f_i$ at $\xi_i = 0.272$ and $\mu = 1$. (c) The map of RI as a function of $\xi_i$
and $f_i$. A different gray scale shows a different RI range as indicated in the
figure.
locking state to a P1 dynamical state. The value of RI increases when a stable injection locking state appears closer to the Hopf bifurcation boundary. By referring to Fig. 7(f), Fig. 8(c) demonstrates that the noise-induced dynamical transition from a stable injection locking state to a P1 dynamical state occurs only when the stable injection locking state is highly close to the Hopf bifurcation boundary, as the black region indicating a RI range between 70 and 75 dB presents.

5. Conclusions
This study numerically investigates two different noise-induced transitions of nonlinear dynamics in an optically injected semiconductor laser based on the Lang-Kobayashi laser model. The spontaneous emission noise is observed to induce dynamical transitions from stable injection locking to P1 dynamics and from P1 to P2 dynamics when operating points are close to the corresponding dynamical boundaries, respectively. Such transitions follow a gradual and smooth dynamical evolution as the noise level increases. The resulting noise-induced dynamical states are found to exhibit features highly similar to their adjacent dynamical states that keep the same dynamical behaviors when subject to noise. Regions where these noise-induced dynamical transitions occur are identified.

Acknowledgments
We acknowledge the support from the Ministry of Science and Technology, Taiwan (MOST 106-2112-M-006-004-MY3, MOST 109-2112-M-006-018-MY3).

References
[1] K. Hirano, T. Yamazaki, S. Morikatsu, H. Okumura, H. Aida, A. Uchida, S. Yoshimori, K. Yoshimura, T. Harayama, and P. Davis, “Fast random bit generation with bandwidth-enhanced chaos in semiconductor lasers,” Optics Express, vol. 18, no. 6, pp. 5512–5524, 2010.
[2] Y. Akizawa, T. Yamazaki, A. Uchida, T. Harayama, S. Sunada, K. Arai, K. Yoshimura, and P. Davis, “Fast random number generation with bandwidth-enhanced chaotic semiconductor lasers at 8 × 50 Gb/s,” IEEE Photonics Technology Letters, vol. 24, no. 12, pp. 1042–1044, 2012.
[3] X.Z. Li and S.C. Chan, “Random bit generation using an optically injected semiconductor laser in chaos with oversampling,” Optics Letters, vol. 37, no. 11, pp. 2163–2165, 2012.
[4] Y.H. Hung, C.H. Chu, and S.K. Hwang, “Optical double-sideband modulation to single-sideband modulation conversion using period-one nonlinear dynamics of semiconductor lasers for radio-over-fiber links,” Optics Letters, vol. 38, no. 9, pp. 1482–1484, 2013.
[5] Y.H. Hung and S.K. Hwang, “Photonic microwave amplification for radio-over-fiber links using period-one nonlinear dynamics of semiconductor lasers,” Optics Letters, vol. 38, no. 17, pp. 3355–3358, 2013.
[6] X.Z. Li and S.C. Chan, “Heterodyne random bit generation using an optically injected semiconductor laser in chaos,” IEEE Journal of Quantum Electronics, vol. 49, no. 10, pp. 829–832, 2013.
[7] K.H. Lo, S.K. Hwang, and S. Donati, “Optical feedback stabilization of photonic microwave generation using period-one nonlinear dynamics of semiconductor lasers,” Optics Express, vol. 22, no. 15, pp. 18648–18661, 2014.
[8] Y.H. Hung and S.K. Hwang, “Photonic microwave stabilization for period-one nonlinear dynamics of semiconductor lasers using optical modulation sideband injection locking,” Optics Express, vol. 23, no. 5, pp. 6520–6532, 2015.
[9] R. Sakuraba, K. Iwakawa, K. Kanno, and A. Uchida, “Tb/s physical random bit generation with bandwidth-enhanced chaos in three-cascaded semiconductor lasers,” Optics Express, vol. 23, no. 2, pp. 1470–1490, 2015.
[10] K.L. Hsieh, Y.H. Hung, S.K. Hwang, and C.C. Lin, “Radio-over-fiber DSB-to-SSB conversion using semiconductor lasers at stable locking dynamics,” Optics Express, vol. 24, no. 9, pp. 9854–9868, 2016.
[11] J.P. Zhuang, X.Z. Li, S.S. Li, and S.C. Chan, “Frequency-modulated microwave generation with feedback stabilization using an optically injected semiconductor laser,” *Optics Letters*, vol. 41, no. 24, pp. 5764–5767, 2016.

[12] Y.H. Hung, J.H. Yan, K.M. Feng, and S.K. Hwang, “Photonic microwave carrier recovery using period-one nonlinear dynamics of semiconductor lasers for OFDM-RoF coherent detection,” *Optics Letters*, vol. 42, no. 12, pp. 2402–2405, 2017.

[13] K.L. Hsieh, S.K. Hwang, and C.L. Yang, “Photonic microwave time delay using slow- and fast-light effects in optically injected semiconductor lasers,” *Optics Letters*, vol. 42, no. 17, pp. 3307–3310, 2017.

[14] K.H. Lo, S.K. Hwang, and S. Donati, “Numerical study of ultrashortoptical- feedback-enhanced photonic microwave generation using optically injected semiconductor lasers at period-one nonlinear dynamics,” *Optics Express*, vol. 44, no. 13, pp. 3334–3337, 2019.

[15] C.H. Tseng, Y.H. Hung, and S.K. Hwang, “V- and W-band microwave generation and modulation using semiconductor lasers at period-one nonlinear dynamics,” *Optics Letters*, vol. 45, no. 24, pp. 6819–6822, 2020.

[16] S.K. Hwang, J.B. Gao, and J.M. Liu, “Noise-induced chaos in an optically injected semiconductor laser model,” *Physical Review E*, vol. 61, no. 5, pp. 5162–5170, 2000.

[17] X.Z. Li, J.P. Zhuang, S.S. Li, J.B. Gao, and S.C. Chan, “Dynamical characteristics of an optically injected semiconductor laser,” *Optics Express*, vol. 183, no. 1-4, pp. 765–784, 1997.

[18] T.B. Simpson, J.M. Liu, K.F. Huang, and K. Tai, “Nonlinearity induced by external optical injection in semiconductor lasers,” *Journal of Statistical Physics*, vol. 31, no. 1, pp. 87–106, 1983.
[31] J.P. Crutchfield and B.A. Huberman, “Fluctuations and the onset of chaos,” *Physics Letters A*, vol. 77, no. 6, pp. 407–410, 1980.

[32] J.P. Crutchfield, J.D. Farmer, and B.A. Huberman, “Fluctuations and simple chaotic dynamics,” *Physics Reports*, vol. 92, no. 2, pp. 45–82, 1982.

[33] J.B. Gao, S.K. Hwang, and J.M. Liu, “When can noise induce chaos?,” *Physical Review Letters*, vol. 82, no. 6, pp. 1132–1135, 1999.

[34] Z. Liu, Y.C. Lai, L. Billings, and I.B. Schwartz, “Transition to chaos in continuous-time random dynamical systems,” *Physical Review Letters*, vol. 88, no. 12, pp. 124101–1–124101-4, 2002.

[35] B. Xu, Y.C. Lai, L. Zhu, and Y. Do, “Experimental Characterization of Transition to Chaos in the Presence of Noise,” *Physical Review Letters*, vol. 90, no. 16, pp. 164101-1–164101-4, 2003.

[36] Y.C. Lai, Z. Liu, L. Billings, and I.B. Schwartz, “Noise-induced unstable dimension variability and transition to chaos in random dynamical systems,” *Physical Review E*, vol. 67, no. 2, pp. 026210-1–026210-17, 2003.

[37] T.B. Simpson, “Mapping the nonlinear dynamics of a distributed feedback semiconductor laser subject to external optical injection,” *Optics Communications*, vol. 215, pp. 135–151, 2003.

[38] T.B. Simpson and J.M. Liu, “Phase and amplitude characteristics of nearly degenerate four-wave mixing in Fabry-Perot semiconductor lasers,” *Journal of Applied Physics*, vol. 73, no. 5, pp. 2587–2589, 1993.

[39] J.M. Liu and T.B. Simpson, “Four-wave mixing and optical modulation in a semiconductor laser,” *IEEE Journal of Quantum Electronics*, vol. 30, no. 4, pp. 957–965, 1994.

[40] T.B. Simpson and J.M. Liu, “Spontaneous emission, nonlinear optical coupling, and noise in laser diodes,” *Optics Communications*, vol. 112, no. 1-2, pp. 43–47, 1994.

[41] S.K. Hwang, J.M. Liu, and J.K. White, “35-GHz intrinsic bandwidth for direct modulation in 1.3-μm semiconductor lasers subject to strong injection locking,” *IEEE Photonics Technology Letters*, vol. 16, no. 4, pp. 972–974, 2004.

[42] S.K. Hwang, J.M. Liu, and J.K. White, “Characteristics of period-one oscillations in semiconductor lasers subject to optical injection,” *IEEE Journal of Selected Topics in Quantum Electronics*, vol. 10, no. 5, pp. 974–981, 2004.

[43] S.C. Chan, “Analysis of an optically injected semiconductor laser for microwave generation,” *IEEE Journal of Quantum Electronics*, vol. 46, no. 3, pp. 421–428, 2010.