Connected edge Detour global domination number of a graph

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Abstract
In this paper, we introduce the concept of connected edge detour global domination number of a graph. A subset D of the vertex set V(G) of a connected graph G is called a connected edge detour global dominating set if D is an edge detour global dominating set and the induced subgraph <D> is connected. The connected edge detour global domination number γcedg(G) of G is the minimum cardinality taken over all connected edge detour global dominating sets in G. A connected edge detour global dominating set of cardinality γcedg(G) is called a γcedg-set of G. We determine γcedg(G) for some standard and special graphs and its properties are studied.

Keywords
Edge detour global domination number, connected edge detour global domination number.

AMS Subject Classification
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Contents
1 Introduction ........................................... 1580
2 Connected Edge Detour Global Domination Number of a Graph ........................................... 1580
References ........................................... 1582

1. Introduction

By a graph G = (V, E), we consider a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n, m respectively. Edge Detour Global Dominating graphs were introduced and studied by Punitha Tharani and Ferdina [12]. For underlying definition and results, see references [1-14].

Theorem 1.1. For any connected graph of order n ≥ 2. Then, 2 ≤ d(G) ≤ γd(G) ≤ n.

Theorem 1.2. Let G be a graph of order n. Then γd(G) = n if and only if G contains only end and full vertices.

Theorem 1.3. For the path graph Pn, γd(G) (Pn) = ⌈n+1n⌉ for 2, n ≥ 5.

Theorem 1.4. For the complete graph Kn, γcedg(G) = n, n ≥ 2.

2. Connected Edge Detour Global Domination Number of a Graph

Definition 2.1. A subset D of V of a connected graph G = (V, E) is called a connected edge detour global dominating set of G if D is an edge detour global dominating set and the induced subgraph <D> is connected. The Connected edge detour global domination number γcedg(G) of G is the minimum cardinality taken over all connected edge detour global dominating sets in G. A connected edge detour global dominating set of cardinality γcedg(G) is called a γcedg-set of G.

Example 2.2. Consider the graph G given in Figure 1.
Here, D1 = {v1, v4, v6}, D2 = {v1, v4, v5}, D3 = {v1, v3, v5} are γd-sets of G and so γd(G) = 3. Now D5 = {v1, v2, v3, v4}, D6 = {v1, v2, v3, v6}, D7 = {v1, v2, v5, v6} are γcedg-set of G. Then γcedg(G) = |D5| = |D6| = |D7| = 4.
Theorem 2.4. Let $G$ be a connected graph of order $n$. Then $2 \leq \gamma_{edg}(G) \leq n$.

Proof. Let $D$ be an edge detour global dominating set. Every set $D$ needs at least two vertices so that $\gamma_{edg}(G) \geq 2$. Again, every connected edge detour global dominating set is an edge detour global dominating set, $\gamma_{edg}(G) \geq \gamma_{edg}(G)$ since the set of all vertices of $G$ is always a connected edge detour global dominating set. Therefore $n \geq \gamma_{edg}(G)$. Hence $2 \leq \gamma_{edg}(G) \leq \gamma_{edg}(G) \leq n$.

Remark 2.5. For a connected graph $G$ with $n \geq 2$.

(i) $\gamma(G) \leq \gamma_{edg}(G)$.

(ii) $\gamma_{edg}(G) \leq \gamma_{edg}(G)$.

(iii) Strict inequality is also true in the above relation.

(iv) From the above Example 2.2 $n = 6$, $\gamma_{edg}(G) = 3$, $\gamma_{edg}(G) = 4$, the bound (Theorem 2.4) is sharp.

Observation 2.6. (i) Path $P_n$ of order $n(n \geq 2)$, $\gamma_{edg}(P_n) = |V(P_n)|$.

(ii) Cycle $C_n$ of order $n(n \geq 3)$, $\gamma_{edg}(C_n) = |V(C_n)| - 2$.

(iii) Complete graph $K_n$ of order $n(n \geq 2)$, $\gamma_{edg}(K_n) = |V(K_n)|$.

(iv) Complete bipartite graph $K_{m,n}$.

$$\gamma_{edg}(K_{m,n}) = \begin{cases} 2 & \text{if } m = n = 1 \\ |V(K_{m,n})| - m + 1 & \text{if } n \geq 2, m = 1 \\ 3 & \text{if } m, n \geq 2 \end{cases}$$

(v) Star graph $K_{1,n}$, $\gamma_{edg}(K_{1,n}) = |V(K_{1,n})|$.

(vi) Bistar graph $B_{n,n}$, $\gamma_{edg}(B_{n,n}) = 2n + 2$.

(vii) Wheel graph $W_n(n \geq 5)$, $\gamma_{edg}(W_n) = 3$.

Theorem 2.7. Every $\gamma_{edg}$-set of a connected graph $G$ contains all the pendant vertices of $G$.

Proof. Let $D$ be a connected edge detour global dominating set of $G$. Then every set $D$ contains all the pendant vertices, since the pendant edges lie only in the detour joining the corresponding pendant vertex with some other vertex.

Theorem 2.8. Every $\gamma_{edg}$-set of a connected graph $G$ contains all the vertices of $G$ has degree $n - 1$.

Proof. Let $w$ be a vertex of a connected graph $G$ has degree $n - 1$. Then the vertex $w$ belongs to every dominating set in the complement $G$ of $G$. since $w$ is dominate itself in $G$. Then all the full vertices of $G$ belong to the global dominating set of $G$. Hence, every $\gamma_{edg}$-set contains all the vertices.

Theorem 2.9. Let $G$ be a connected graph of order $n \geq 2$ and $D$ be a $\gamma_{edg}$-set of $G$. Then for any cut vertex $x$ of $G$, every component of $G - x$ contains an element of $D$.

Proof. Let $x$ be a cut vertex of a connected graph $G$ and $D$ be a connected edge detour global dominating set. Let $H$ be one of the components of $G - x$. Suppose no vertex of $D$ belongs to $H$. Then any pendant vertex of $G$ does not belong to $H$ (by Theorem 2.7). Therefore, $H$ has at least one edge, say $u_1u_{i+1}$. Since $D$ is a $\gamma_{edg}$-set, there exists vertices $u_i, w \in D$ such that $u_iu_{i+1}$ lies on some $u - w$ detour. $P: u = u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n = w$ in $G$ or both the ends $u_i$ and $u_{i+1}$ of the edge $u_iu_{i+1}$ are in $D$. Suppose that $u_iu_{i+1}$ lies on the detour $P$. Let $P_i$ be the subpath of $P$, say $u_i - u_i$ and $P_b$ be the subpath of $P$, say $u_i - w$. Since $x$ is a cut vertex of $G$, then $x$ belongs to both $P_a$ and $P_b$ so that $P$ is not a detour, which is a contradiction to the fact. Suppose that $u_i$ and $u_{i+1}$ are in $D$, then $H$ contains vertices of $D$, which is again a contradiction.

Theorem 2.10. Every $\gamma_{edg}$-set of a connected graph $G$ contains all the cut vertices of $G$.

Proof. Let $x$ be a cut vertex of a connected graph $G$ of order $n \geq 2$ and $D$ be a connected edge detour global dominating set of $G$. Then $G - x$ has more than one component, say $G_1, G_2, \ldots, G_i(i \geq 2)$. Then $\gamma_{edg}$-set $D$ contains at least one vertex from each component $G_i(1 \leq k \leq i)$ of $G - x$ (by Theorem 2.9). Since induced subgraph $<D>$ is connected it follows that $x \in D$.

Corollary 2.11. Every $\gamma_{edg}$-set of a connected graph $G$ contains pendant vertices, full vertices and cut vertices of $G$.

Proof. The proof follows from Theorem 2.7, 2.8 and 2.10.

Corollary 2.12. For any tree $T$ of $n$ vertices, $\gamma(T) = |V(T)|, n \geq 2$.

Proof. The proof follows from Corollary 2.11.

Corollary 2.13. Let $G$ be any connected graph with $l$ pendant vertices, $m$ full vertices and $n$ cut vertices. Then $\gamma(G) \leq n$. $m + n \leq \gamma(G) \leq n$.

Proof. The proof follows from Theorem 2.4 and Corollary 2.11.

Theorem 2.14. For $3 \leq j \leq n$, there exists a connected graph $G$ of order $n$ with $\gamma(G) = j$. 

\[ \square \]
The graph $G$ is shown in Figure 2.

If $j = n$, Let $G = P_n$. Then by Observation 2.6 (i), $\gamma_{ced}(G) = j$.

**Case 2.** If $3 < j < n$, Let $G = W_n$. Then by Observation 2.6 (vii), $\gamma_{ced}(G) = j$.

**Case 3.** $3 < j < n$, Let $G$ be a connected graph obtained from $W_{n-j+3}$. Let $\gamma(G) = \{v_1, v_2, v_3, \ldots, v_{n-j+2}, w_1, w_2, \ldots, w_{j-3}\}$. The graph $G$ is shown in Figure 2.

Let $V\{W_{n-j+3}\} = \{v_1, v_2, v_3, \ldots, v_{n-j+2}\}$ and $w_1, w_2, \ldots, w_{j-3}$ be the new vertices which are joining to $v_2$. Now we have to prove that $\gamma_{ced}(G) = j$. Then the set $D = \{w_1, w_2, \ldots, w_{j-3}\}$ together with a cut vertex $v_2$ is a subset of every $\gamma_{ced}$-set $G$. It is clear that $D$ is a global dominating set but not an edge detour set of $G$. Let $D' = D \cup \{v, v_{j-3}\}$. Then every edge of $G$ lies on a detour joining a pair of vertices of $D'$. Clearly, the set $D'$ is $\gamma_{ced}$-set and $D'$ is connected. Therefore, $D'$ is a connected edge detour global dominating set of minimum cardinality,

\[
|D'| = |D \cup \{v, v_{j-3}\}| \\
= |D| + |\{v, v_{j-3}\}| \\
= |\{w_1, w_2, \ldots, w_{j-3}\}| + |v_2| + |\{v, v_{j-3}\}| \\
= j - 3 + 1 + 2 = j.
\]

Hence $\gamma_{ced}(G) = j$. \qed

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