SPINS AND CHARGES IN GRASSMANN SPACE and
KÄHLER SPINORS IN SPACE OF DIFFERENTIAL FORMS

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Abstract

One of us got spins and charges of not only scalars and vectors but also of
spinors out of fields, which are antisymmetric tensor fields. Kähler got spins
of spinors out of differential forms, which again are antisymmetric tensor fields.
Using our simple Grassmann formulation of spins and charges of either spinors
or vectors and comparing it to the Dirac-Kähler formulation of spinors, we gen-
eralize the Dirac-Kähler approach to vector internal degrees of freedom and to
charges of either spinors or vectors and tensors and point out how at all spinors
can appear in both approaches.

1 Introduction.

Kähler[1] has shown how to use differential forms to describe the spin of fermions.
One of us[2] has shown how a space of anticommuting coordinates can be used
to describe spins and charges of not only fermions but also of bosons, unifying
spins and charges for either fermions or for bosons.

In the present talk we point out the analogy and nice relations between the two

1The invited talk, presented on International Workshop on Lorentz group, CPT and Neutrinos,
Zacatecas, 23-26 June, 1999.
different ways of achieving the appearance of spin one half degrees of freedom when starting from pure vectors and tensors. We comment the necessity of appearance of four copies of Dirac fermions in both approaches. Comparing carefully the two approaches we generalize the Kähler approach to describe also integer spins as well as charges for either spinors or vectors, unifying spins and charges.

2 Dirac equations in Grassmann space.

What we call quantum mechanics in Grassmann space is the model for going beyond the Standard Model with extra dimensions of ordinary and anticommuting coordinates, describing spins and charges of either fermions or bosons in an unique way. In a $d$-dimensional space-time the internal degrees of freedom of either spinors or vectors and scalars come from the odd Grassmannian variables $\theta^a$, $a \in \{0, 1, 2, 3, 5, \ldots, d\}$.

We write wave functions describing either spinors or vectors in the form

$$\Phi(\theta^a) = \sum_{i=0,1,\ldots,3,5,\ldots,d} \sum_{\{a_1 < a_2 < \ldots < a_i\} \in \{0,1,\ldots,3,5,\ldots,d\}} \alpha_{a_1,a_2,\ldots,a_i} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_i},$$

where the coefficients $\alpha_{a_1,a_2,\ldots,a_i}$ depend on commuting coordinates $x^a$, $a \in \{0, 1, 2, 3, 5, \ldots, d\}$. The wave function space spanned over Grassmannian coordinate space has the dimension $2^d$. Completely analogously to usual quantum mechanics we have the operator for the conjugate variable $\theta^a$ to be

$$p_{\theta^a} = -i \overrightarrow{\partial}_a.$$

The right arrow tells that the derivation has to be performed from the left hand side. These operators then obey the odd Heisenberg algebra, which written by means of the generalized commutators

$$\{A, B\} := AB - (-1)^{n_{AB}} BA,$$

where

$$n_{AB} = \begin{cases} +1, & \text{if A and B have Grassmann odd character} \\ 0, & \text{otherwise,} \end{cases}$$

takes the form

$$\{p^\theta_a, p^\theta_b\} = 0 = \{\theta^a, \theta^b\}, \quad \{p^\theta_a, \theta^b\} = -i\eta^{ab}.$$
Here $\eta^{ab}$ is the flat metric $\eta = \text{diag}\{1, -1, -1, \ldots\}$. We may define the operators

$$\tilde{a}^a := i(p^a - i\theta^a), \quad \tilde{\tilde{a}}^a := -(p^a + i\theta^a),$$

for which we can show that the $\tilde{a}^a$'s among themselves fulfill the Clifford algebra as do also the $\tilde{\tilde{a}}^a$'s, while they mutually anticommute:

$$\{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab}, \quad \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\} = 0.$$  

(6)

We could recognize formally either $\tilde{a}^a p_a |\Phi\rangle = 0$, or $\tilde{\tilde{a}}^a p_a |\Phi\rangle = 0$ (8) as the Dirac-like equation, because of the above generalized commutation relations. Applying either the operator $\tilde{a}^a p_a$ or $\tilde{\tilde{a}}^a p_a$ on the two equations we get the Klein-Gordon equation $p^a p_a |\Phi\rangle = 0$, where we define $p_a = i\partial_a$.

One can, however, check that none of the two equations (8) have solutions which would transform as spinors with respect to the the generators for the Lorentz transformations, when taken in analogy with the generators of the Lorentz transformations in ordinary space ($L^{ab} = x^a p^b - x^b p^a$)

$$S^{ab} := \theta^a p^b - \theta^b p^a.$$  

(9)

We can write, however, these generators as the sum

$$S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \quad \tilde{S}^{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \quad \tilde{\tilde{S}}^{ab} := -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b].$$

(10)

with $[A, B] := AB - BA$ and recognize that the solutions of the two equations (8) now transform as spinors with respect to either $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$.

One also can easily see that the untilded, the single tilded and the double tilded $S^{ab}$ obey the $d$-dimensional Lorentz generator algebra $\{M^{ab}, M^{cd}\} = -i(M^{ad} \eta^{bc} + M^{bc} \eta^{ad} - M^{ac} \eta^{bd} - M^{bd} \eta^{ac})$, when inserted for $M^{ab}$.

We shall present this approach in more details in section 4 when pointing out the similarities between this approach and the Kähler approach and generalizing the Kähler approach.

3 Kähler formulation of spinors.

Kähler formulates spinors in terms of wave functions which are superpositions of the p-forms in the $d = 4$ - dimensional space. The 0-forms are scalars, the 1-forms are defined as dual vectors to the (local) tangent spaces, the higher
p-forms are defined as antisymmetrized cartesian (exterior) products of the one-form spaces. A general linear combination of forms is then written

\[ u = u_0 + u_1 + \ldots + u_d, \quad u_p = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p} \, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}. \] (11)

One can define the exterior product \( \wedge \) and the Clifford product \( \vee \) among the forms. The exterior product has the property of making the product of a p-form and a q-form to be a (p+q)-form, if a p-form and a q-form have no common differentials. The Clifford product \( dx^a \vee \) on a p-form is either a \( p+1 \) form, if a p-form does not include a one form \( dx^a \), or a \( p-1 \) form, if a one form \( dx^a \) is included in a p-form.

Kähler found how the Dirac equation could be written as an equation \([1]\) (Eq. (26.6) in the Kähler’s paper)

\[-i \delta u = m \vee u, \quad \text{with} \quad \delta u = \sum_{i=1}^{3} dx^i \vee \frac{\partial u}{\partial x^i} - dt \vee \frac{\partial u}{\partial t}.\] (12)

with \( u \) defined in Eq.(11). The symbol \( \delta \) denotes the inner differentiation, \( a \in \{0, 1, 2, 3\} \) and \( m \) means the electron mass.

For a free massless particle living in a d dimensional space-time Eq.(12) can be rewritten in the form

\[ dx^a \vee p_a \, u = 0, \quad a = 0, 1, 2, 3, 5, \ldots, d. \] (13)

The wave function describing the state of the spin one half particle is packed into the exterior algebra function \( u \).

4 Parallelism between the two approaches.

We demonstrate the parallelism between the Kähler\([1]\) and the one of us\([2]\) approach in steps, first paying attention on spin \( \frac{1}{2} \) only, as Kähler did. Using simple and transparent definitions of the exterior and interior product in Grassmann space, we generalize the Kähler approach first by defining the two kinds of \( \delta \) (Eq.(12)) operators on the space of p-forms and accordingly three kinds of the generators of the Lorentz transformations, two of the spinorial and one of the vectorial character. We try to put clearly forward how the spinorial degrees of freedom emerge out of vector objects like the 1-forms or \( \theta^a \)'s. We then generalize the p-forms to describe not only spins but also charges of spin \( \frac{1}{2} \) and spin 0 and 1 objects, unifying also in the space of forms spins and charges.
4.1 Dirac-Kähler equation and Dirac equation in Grassmann space for massless particles.

We present here, side by side, the operators in the space of differential forms and in Grassmann space: the “exterior” product
\[ dx^a \wedge dx^b \wedge \cdots, \quad \theta^a \theta^b \cdots, \]
the operator of “differentiation”
\[ -i e^a, \quad p^\theta a = -i \frac{\overrightarrow{\partial}}{\partial \theta^a}, \]
and the two superpositions
\[ dx^a \tilde{\nabla} := dx^a \wedge + e^a, \quad \tilde{a}^a := i (p^\theta a - i \theta^a), \]
\[ dx^a \tilde{\tilde{\nabla}} := i (dx^a \wedge - e^a), \quad \tilde{\tilde{a}}^a := -(p^\theta a + i \theta^a). \]

The superposition with the sign \( \tilde{} \) is the one used by Kähler (Eqs. (12)). One easily finds (see Eqs. (6,7)) the commutation relations, understood in the generalized way of Eq. (3)
\[ \{dx^a \tilde{\nabla}, dx^b \tilde{\nabla}\} = 2 \eta^{ab}, \quad \{\tilde{a}^a, \tilde{b}^b\} = 2 \eta^{ab}, \]
\[ \{dx^a \tilde{\tilde{\nabla}}, dx^b \tilde{\tilde{\nabla}}\} = 2 \eta^{ab}, \quad \{\tilde{\tilde{a}}^a, \tilde{\tilde{b}}^b\} = 2 \eta^{ab}. \]

Since \( \{e^a, dx^b \wedge\} = \eta^{ab} \) and \( \{e^a, e^b\} = 0 = \{dx^a \wedge, dx^b \wedge\} \), while \( \{-ip^\theta a, \theta^b\} = \eta^{ia} \tilde{\theta}^b \) and \( \{p^\theta a, p^\theta b\} = 0 = \{\theta^a, \theta^b\} \), it is obvious that \( e^a \) plays in the p-form formalism the role of the derivative with respect to a differential 1-form, similarly as \( ip^\theta a \) does with respect to a Grassmann coordinate.

We find for both approaches the Dirac-like equations:
\[ dx^a \tilde{\nabla} p_a u = 0, \quad \tilde{a}^a p_a \Phi(\theta^a) = 0, \]
\[ dx^a \tilde{\tilde{\nabla}} p_a u = 0, \quad \tilde{\tilde{a}}^a p_a \Phi(\theta^a) = 0. \]

Taking into account the above definitions it follows that
\[ dx^a \tilde{\nabla} p_a dx^b \tilde{\nabla} p_b u = p^a p_a u = 0, \quad \tilde{a}^a p_a \tilde{b}^b p_b \Phi(\theta^b) = p^a p_a \Phi(\theta^b) = 0. \]

We see that either \( dx^a \tilde{\nabla} p_a u = 0 \) or \( dx^a \tilde{\tilde{\nabla}} p_a u = 0 \), similarly as either \( \tilde{a}^a p_a \Phi(\theta^a) = 0 \) or \( \tilde{\tilde{a}}^a p_a \Phi(\theta^a) = 0 \) can represent the Dirac-like equation.

Both, \( dx^a \tilde{\nabla} \) and \( dx^a \tilde{\tilde{\nabla}} \) define the algebra of the \( \gamma^a \) matrices and so do both \( \tilde{a}^a \) and \( \tilde{\tilde{a}}^a \). One would thus be tempted to identify
\[ \gamma^a_{\text{naive}} := dx^a \tilde{\nabla}, \quad \text{or} \quad \gamma^a_{\text{naive}} := \tilde{\tilde{a}}^a. \]
But there is a large freedom in defining what to identify with the gamma-matrices, because except when using $\gamma^0$ as a parity operation, one has an even number of gamma matrices occurring in the physical applications such as

construction of currents $\bar{\psi}\gamma^a\psi$ or for the Lorentz generators on spinors $\frac{i}{4} [\gamma^a,\gamma^b]$. Then all the gamma matrices can be multiplied by some factor provided it does disturb neither their algebra nor their even products. This freedom might be used to solve, what seems a problem:

Having an odd Grassmann character, neither $\tilde{a}^a$ nor $\tilde{\tilde{a}}^a$ and similarly neither $dx^a \tilde{\bigvee}$ nor $dx^a \tilde{\tilde{\bigvee}}$ should be recognized as the Dirac $\gamma^a$ operators, since they would change, when operating on polynomials of $\theta^a$ or on superpositions of p-form objects of an odd Grassmann character to objects of an even Grassmann character. One would, however, expect - since Grassmann odd fields second quantize to fermions, while Grassmann even fields second quantize to bosons - that the $\gamma^a$ operators do not change the Grassmann character of the wave functions so that the canonical quantization of Grassmann odd fields then automatically assures the anticommuting relations between the operators of the fermionic fields.

We may propose that accordingly

\begin{equation}
\text{either } \tilde{\gamma}^a := \ i \ dx^0 \tilde{\bigvee} \ dx^a \tilde{\bigvee}, \quad \text{or } \tilde{\gamma}^a = i \ \tilde{a}^a \tilde{a}^a \quad (21)
\end{equation}

are recognized as the Dirac $\gamma^a$ operators operating on the space of p-forms or polynomials of $\theta^a$’s, respectively, since they both have an even Grassmann character and they both fulfill the Clifford algebra $\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}$. (The role of $\tilde{\bigvee}$ and $\tilde{\bigvee}$ can in either the Kähler case or the case of polynomials in Grassmann space, be exchanged.)

The two definitions of gamma-matrices ((21), (20)) make only a difference when $\gamma^0$-matrix is used alone. This $\gamma^0$-matrix has to simulate the parity reflection which is

\begin{equation}
\text{either } \tilde{d}x \rightarrow -\tilde{d}x, \quad \text{or } \tilde{\theta} \rightarrow -\tilde{\theta}. \quad (22)
\end{equation}

The “ugly” gamma-matrix identifications (21) indeed perform this operation. Kähler did not connect eveness and oddness of the forms with the statistics. He used the “naive” gamma-matrix identifications (20). The same can be said for the Becher-Joos (3) paper.

4.2 Generators of Lorentz transformations.

We are presenting the generators of the Lorentz transformations of spinors for both approaches

\begin{equation}
M^{ab} = L^{ab} + S^{ab}, \quad L^{ab} = x^a p^b - x^b p^a. \quad (23)
\end{equation}

The two approaches differ in the definition of the generators of the Lorentz transformations in the internal space $S^{ab}$. While Kähler suggested the definition for
spin $\frac{1}{2}$ particles

$$S^{ab} = dx^a \wedge dx^b, \quad S^{ab}u = \frac{1}{2}((dx^a \wedge dx^b) \vee u - u \vee (dx^a \wedge dx^b)), \quad (24)$$

in the Grassmann case the two kinds of the operators $S^{ab}$ for spinors can be defined, presented in Eqs. (10), with the properties

$$[\tilde{S}^{ab}, \tilde{a}^c] = i(\eta^{ac}\tilde{a}^b - \eta^{bc}\tilde{a}^a), \quad [\tilde{S}^{ab}, \tilde{\tilde{a}}^c] = i(\eta^{ac}\tilde{\tilde{a}}^b - \eta^{bc}\tilde{\tilde{a}}^a), \quad [\tilde{S}^{ab}, \tilde{\tilde{a}}^c] = 0 = [\tilde{\tilde{S}}^{ab}, \tilde{a}^c]. \quad (25)$$

Following the approach in Grassmann space one can also in the Kähler case define two kinds of the Lorentz generators for spinors, which (both) simplify Eq.(24)

$$\tilde{S}^{ab} = -\frac{i}{4}[dx^a \wedge + e^a, dx^b \wedge + e^b], \quad \tilde{\tilde{S}}^{ab} = \frac{i}{4}[dx^a \wedge - e^a, dx^b \wedge - e^b], \quad \tilde{S}^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]. \quad (26)$$

The above definition enables us to define also in the Kähler case the generators of the Lorentz transformations of the vectorial character

$$S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a), \quad S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = \theta^a p^b - \theta^b p^a. \quad (27)$$

The operator $S^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a)$, being applied on differential p-forms, transforms vectors into vectors.

### 4.3 Four copies of Weyl bi-spinors in Kähler or in approach in Grassmann space and vector representations.

In the case of $d = 4$ one may arrange the space of $2^d$ vectors into four copies of two Weyl spinors, one left ($<\tilde{\Gamma}^{(4)}> = -1$, $\Gamma^{(4)} = i\frac{(2)}{4}\epsilon_{abcd}S^{ab}S^{cd}$) and one right ($<\tilde{\Gamma}^{(4)}> = 1$) handed (we have made a choice of $\tilde{\Gamma}$), in such a way that they are at the same time the eigen vectors of the operators $\tilde{S}^{12}$ and the $\tilde{S}^{03}$ and have either an odd or an even Grassmann character. These vectors are in the Kähler approach the superpositions of p-forms and in the one of us approach the polynomials of $\theta^m$’s, $m \in \{0, 1, 2, 3\}$. The two Weyl vectors of one copy of the Weyl bi-spinors are connected by the $\tilde{\gamma}^m$ (Eq.(21)) operators.

Analysing the irreducible representations of the group $SO(1, 3)$ with respect to the generator of the Lorentz transformations of the vectorial type $\tilde{\Gamma}$ (Eqs.(27)) one finds for $d = 4$ two scalars (a scalar and a pseudo scalar), two three vectors (in the $SU(2) \times SU(2)$ representation of $SO(1, 3)$ denoted by $(1, 0)$ and $(0, 1)$ representation, respectively, with $<\Gamma^{(4)} = \pm 1>$) and two four vectors.
4.4 Generalization to extra dimensions.

It has been suggested\cite{2} that the Lorentz transformations in the space of $\theta^a$’s in $d - 4$ dimensions manifest themselves as generators for charges observable at the end for the four dimensional particles. Since both the extra dimension spin degrees of freedom and the ordinary spin degrees of freedom originate from the $\theta^a$’s or the forms we have a unification of these internal degrees of freedom.

Let us take as an example the model\cite{2} which has $d = 14$ and at first - at the high energy level - $SO(1, 13)$ Lorentz group, but which should be broken (in two steps) to first $SO(1, 7) \times SO(6)$ and then to $SO(1, 3) \times SU(3) \times SU(2)$.

5 Appearance of spinors.

One of course wonders about how it is at all possible that the Dirac equation appears for a spinor field out of models with only scalar, vector and tensor objects! It only can be done by exchanging the Lorentz generators $S^{ab}$ by the $\tilde{S}^{ab}$ say (or the $\tilde{S}^{ab}$ if we choose them instead), see equations (10, 26). This indeed means that one of the two kinds of operators fulfilling the Clifford algebra and anticommuting with the other kind - it has been made a choice of $dx^a \tilde{\gamma}$ in the Kähler case and $\tilde{a}^a$ in the approach of one of us - are put to zero in the operators of Lorentz transformations; as well as in all the operators representing physical quantities. The use of $dx^0 \tilde{\gamma}$ or $\tilde{a}^0$ in the operator $\tilde{\gamma}^0$ is the exception only used to simulate the Grassmann even parity operation $\tilde{dx}^a \rightarrow -\tilde{dx}^a$ and $\tilde{\theta} \rightarrow -\tilde{\theta}$, respectively.

In (2) the $\tilde{a}^a$’s are argued away on the ground of the action.

Acknowledgement

This work was supported by Ministry of Science and Technology of Slovenia. One of the authors (H. B. Nielsen) would like to thank the funds CHRX-CT94-0621, INTAS 93-3316, INTAS-RFBR 95-0567.

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