A remark on capillary surfaces in a 3-dimensional space of constant curvature

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Abstract

We generalize a theorem by J. Choe on capillary surfaces for arbitrary 3-dimensional spaces of constant curvature. The main tools in this paper are an extension of a theorem of H. Hopf due to S.-S. Chern and two index lemmas by J. Choe.

1 History

A well known theorem due H. Hopf [4] state that a CMC immersion of the sphere into the Euclidean three-dimensional space is a round sphere. In 1982, in Rio de Janeiro, in occasion of a International Congress at IMPA, S. S. Chern [2] showed a generalization of this theorem when the ambient space has constant sectional curvature. Recently, J. Choe [3], using sufficient hypothesis, generalized Hopf’s Theorem for immersion of the closed disk in $R^3$. Following these lines of ideas, the first result in this paper can be stated as:

**Theorem A.** Let $S$ be a CMC immersed compact $C^{2+\alpha}$ surface of disk type in a 3-dimensional ambient space $M^3$ of constant curvature ($C^{2+\alpha}$ surface means $C^{2+\alpha}$ up to and including $\partial S$ and $\partial S$ is $C^{2+\alpha}$ up to and including its vertices). Suppose that the regular components of $\partial S$ are lines of curvature. If the number of vertices with angle $< \pi$ is less than or equal to 3, then $S$ is totally umbilic.

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This kind of theorem is motivated by the study of capillary surfaces. In fact, J. Nitsche, in 1995, showed that a regular capillary immersion (see definition 3.1) of the closed disk in the sphere is either a plane disk or a piece of a round sphere. In [3] this result was obtained for capillary immersion without strong regularity assumptions. In 1997 Ros-Souam [7] showed a version of Nitsche’s theorem for ambient space with constant sectional curvature. They used Chern’s extension of Hopf’s theorem. This motivated us to formulate the theorem below:

**Theorem B.** Let $U \subset M^3$ be a domain of a 3-dimensional space of constant curvature bounded by totally umbilic surfaces. If $S$ is a capillary surface in $U$ of disk type which is $C^{2+\alpha}$ and $S$ has less than 4 vertices with angle $< \pi$, then $S$ is totally umbilic.

The paper is organized as follows. In section 2 we recall the context of Chern’s work [2], state the main theorems needed here and briefly define the concept of rotation index of the lines of curvature at umbilic points. Finally, we prove the theorems A and B in the section 3.

2 Some Lemmas

In this section we fix some notation and briefly sketch the proof of the main tools used here: S.-S. Chern’s generalization of Hopf’s theorem and the two index lemmas by J. Choe.

Let $M^3$ be a 3-dimensional manifold of constant curvature $c$. Following Chern [2], if $X : S \rightarrow M$ is an immersed surface and $p \in S$, we can fix an orthonormal local frame $e_1, e_2, e_3$ such that $e_3$ is the unit normal vector to $S$ at $x$, supposing $S$ orientable ($x \in S$ is a point near to $p$). If $\theta_i$ denotes the coframe ($i = 1, 2, 3$), then $\theta_3 = 0$. The first and second fundamental forms are $I = \theta_1^2 + \theta_2^2$ and $II = h_{11}\theta_1^2 + 2h_{12}\theta_1\theta_2 + h_{22}\theta_2^2$, respectively.

Recall that the invariants $H = \frac{1}{2}(h_{11} + h_{22})$ and $\tilde{K} = h_{11}h_{22} - h_{12}^2$ are the mean curvature and the total curvature of $S$, where $S$ has the induced Riemannian metric. By the structure equations (see [2]), we have that the Gaussian curvature is $K = \tilde{K} + c$. With this setting, we recall the definitions:  

**Definition 2.1.** $S$ is totally umbilical (resp. totally geodesic) if $II - H \cdot I = 0$ (resp. if $I I = 0$).

Defining $\phi = \theta_1 + i\theta_2$, we have a complex structure on $S$. Note that $II - H \cdot I = \frac{1}{2}(h_{11} - h_{22})(\theta_1^2 - \theta_2^2) + 2h_{12}\theta_1\theta_2$ is a trace zero form and it is the real part of the complex 2-form $\Phi = \tilde{H}\phi^2$, where $\tilde{H} = \frac{1}{2}(h_{11} - h_{22}) - h_{12}i$. 

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Since $\Phi$ is uniquely determined by $II - H \cdot I$ and $II - H \cdot I$ is associated to $S$, $\Phi$ is a globally defined 2-form, independent of a choice of local frames.

In the work [4], H. Hopf shows that, in the case $M = \mathbb{R}^3$, i.e., $c = 0$, if the mean curvature is constant, then $\Phi$ is holomorphic on $S$. However, a more general fact is true, as proved by Chern:

**Lemma 2.2 (Theorem 1 of Chern [2]).** If $H \equiv \text{const.}$, then $\Phi$ is a holomorphic 2-form on $S$.

The holomorphicity of $\Phi$ was used by Hopf to prove that an immersed sphere $f : S^2 \to \mathbb{R}^3$ of constant mean curvature (CMC) is round. Indeed, this follows from a standard result about Riemann surfaces which says that, except by the trivial 2-form $\Phi = 0$, there is no holomorphic 2-form on a compact Riemann surface of zero genus. With the same argument, as a corollary of 2.2 Chern was able to conclude that:

**Corollary 2.3 (Theorem 2 of Chern [2]).** If $f : S^2 \to M^3$ is an CMC immersed sphere and $M^3$ has constant mean curvature then $f$ is totally umbilic.

On the other hand, in the case of surfaces with boundary (with ambient space $M = \mathbb{R}^3$), Choe extends Hopf’s arguments to study capillary surfaces. In order to make the ideas of Hopf works in his case, Choe introduce a natural concept of rotation index of the lines of curvature at umbilic points (including boundary points). Now, as a preliminary work, we consider Choe’s notion of rotation index in the context of a general ambient space $M^3$ of constant curvature.

Consider a point $p \in \partial S$. Let $\psi : D_h \to S$ be a conformal parametrization of a neighborhood of $p$ in $S$, where $D_h = \{(x, y) \in D : y \geq 0\}$ is a half disk and the diameter $l$ of $D_h$ is mapped into $\partial S$. Let $F$ be the line field on $D_h$ obtained by pulling back (by $\psi$) the lines of curvature of $S$. If $\psi(l)$ is a line of curvature of $S$, we can extend $F$ to a line field on $D$ by reflection through the diameter $l$. For simplicity, the extension of $F$ to $D$ is denoted by $F$. At this point, it is natural define the rotation index of the lines of curvature at an umbilic point $p \in \partial S$ to be half of the index of $F$ at $\psi^{-1}(p)$. Clearly this definition is independent of the choice of the parametrization $\psi$. However, the definition only makes sense if we show that the umbilic points on $\partial S$ are isolated. But this fact follows from an easy argument:

Following Choe [3], the equation of the lines of curvature, in complex coordinates is given by

$$\Im(\Phi) = 0,$$
where $\Im z$ denotes the imaginary part of $z$.

So the rotation index of the lines of curvature is

$$r = \frac{1}{2\pi} \delta(\arg \phi) = -\frac{1}{4\pi} \delta(\arg \Phi),$$

where $\delta$ is the variation as one winds once around an isolated umbilic point $p$. In particular, if $p$ is an interior point of $S$ (i.e., $p \notin S$) and is a zero of order $n$ of $\Phi$, then $\delta(\arg \Phi) = 2\pi n$. Consequently,

$$r = -\frac{n}{2} \leq -\frac{1}{2}.$$  \hfill (1)

Suppose now that $\Phi$ has a zero (resp., pole) of order $n > 0$ (resp., $-n > 0$) at a boundary umbilic point $p$. Then,

$$r = \frac{1}{2} \left[ -\frac{1}{4\pi} \delta(\arg \Phi) \right] = -\frac{n}{4} \hfill (2)$$

With these equations in mind, Choe proves the following lemma, which compares interior umbilic points and boundary umbilic points.

**Lemma 2.4.** Let $S$ be a CMC immersed $C^{2+\alpha}$ surface (up to and including the boundary $\partial S$). Suppose that $\partial S$ consist of $C^{2+\alpha}$ curves (up to and including some possible singular points called vertices). If the regular components of $\partial S$ are lines of curvature, then:

1. The boundary umbilic points of $S$ are isolated;

2. The boundary umbilic points which are not vertices have, at most, rotation index $-1/4$;

3. The vertices of $S$ with angle $< \pi$ have rotation index $\leq 1/4$ and the vertices with angle $> \pi$ have rotation index $\leq -1/4$.

The proof of this lemma is a straightforward consequence (with only minor modifications) of Choe’s proof of lemma 2 in [3].

Now we are in position to prove the main results of this paper.

### 3 Proof of the theorems

**Proof of theorem** Fix $\psi : D \to S$ a conformal parametrization and $F$ the pull-back under $\psi$ of the lines of curvature of $S$. Since $\partial S$ are lines of curvature, we can apply the Poincaré-Hopf theorem (even in the case that
is a parametrization of a boundary point) to conclude that, if the number of singularities of $F$ is finite, the sum of rotation indices is equal to 1. Let $A$ be the set of such singularities. Suppose that $A$ is finite. Using equation \ref{eq:rotation} lemma \ref{lemma:rotation} and, by hypothesis, the number of vertices with angle is $< \pi$ is $\leq 3$, we get the estimate:

$$\sum r(p) \leq 3/4,$$

a contradiction with Poincaré-Hopf’s theorem.

Therefore, $A$ is infinite. In particular, since the number of vertices is finite, we have an infinite set $A - \{\text{vertices}\} \subset \{\text{zeros of } \Phi\}$. But $\Phi$ is holomorphic. In particular, this implies that $A = S$, so $S$ is totally umbilic.

We point out that, as remarked by Choe \cite{Choe}, Remark 1, the condition on the number of vertices with angle $< \pi$ is necessary. In fact, a rectangular region in a cylinder $N = S \times \mathbb{R}^1 \subset \mathbb{R}^3$ bounded by two straight lines and two circles provides a counter-example with 4 vertices with angle $\pi/2$ and rotation index 1/4.

Before starting the proof of the second main result, we recall the definition:

**Definition 3.1.** A capillary surface $S$ in a domain $U$ of a 3-dimensional space $M^3$ of constant curvature is a CMC immersed surface which meets $\partial U$ along $\partial S$ at a constant angle.

As an immediate consequence of theorem \ref{thm:CMC}, we have the theorem \ref{thm:capillary}:

**Proof of theorem \ref{thm:capillary}**. This follows from theorem \ref{thm:CMC} and the Terquem - Joachimsthal theorem \cite{Terquem} that says:

“If $C = S_1 \cap S_2$ is a line of curvature of $S_1$, then $C$ is also a line of curvature of $S_2$ if and only if $S_1$ intersect $S_2$ at a constant angle along $C$.”

A result for capillary hypersurfaces with the same flavor of theorem \ref{thm:capillary} was also obtained by Choe (see \cite{Choe}, theorem 3). However, these arguments do not work \textit{a priori} for more general ambient spaces than $\mathbb{R}^n$ since the following fact (valid only in $\mathbb{R}^n$) is used: if $X$ denotes the position vector on $S$ from a fixed point and $H$ is the mean curvature vector, then $\Delta X = H$. In particular, it is an open question if there exists \textit{unbalanced} capillary hypersurfaces in the conditions of theorem 3 of Choe.\footnote{J. Choe pointed out to one of the authors that, in fact, there exists a generalization of theorem 3 of \cite{Choe} to be published elsewhere.}
To finish this paper we point out that another kind of generalization of the theorem 3 of Choe cited above is obtained by replacing the mean curvature by the higher order curvatures $H_r$. In this direction, Choe showed in [3], theorem 4, that if an immersed hypersurface $S \subset \mathbb{R}^{n+1}$ has constant mean curvature and $H_r$ is constant for some $r \geq 2$, then $S$ is a hypersphere. Moreover, we can replace the constancy of the mean curvature by $S$ is embedded, as Ros proved [6]. Furthermore, for general ambients of constant curvature and supposing only that $H_r/H_l$ is constant, Koh-Lee [5] were able to get the same result. Recently, Alencar-Rosenberg-Santos [1] proved a result in this direction with an extra hypothesis on the Gauss image of $S$ (with ambient space $\mathbb{S}^{n+1}$).

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**References**

[1] Alencar H., Rosenberg H. and Santos W., On the Gauss map of hypersurfaces with constant scalar curvature in spheres, Preprint March 31, 2003.

[2] Chern S.-S., On surfaces of constant mean curvature in a three-dimensional space of constant curvature, *Geometric dynamics*, Lect. Notes in Math., 1007 (1983) 104–108.

[3] Choe J., Sufficient conditions for constant mean curvature surfaces to be round, *Math. Ann.* 323 (2002) 143–156.

[4] Hopf H., Differential Geometry in the Large, *Lecture Notes in Math.* 1000, Springer-Verlag, 1989.

[5] Koh S.-E. and Lee S.-W., Addendum to the paper: Sphere theorem by means of the ratio of mean curvature functions, *Glasgow Math. J.* 43 (2001) 275–276.

[6] Ros A., Compact hypersurfaces with constant higher order mean curvatures, *Revista Mat. Ibero.* 3 (1987) 447–453.

[7] Ros A., Souam R., On stability of capillary surfaces in a ball, *Pacific J. Math.* 178 (1997) 345–361.
[8] Spivak M., A comprehensive introduction to differential geometry, *Publish or Perish*, Vol.III, Berkeley, 1979.

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