ERRATUM TO “C*-ALGEBRAS ASSOCIATED WITH INTEGRAL DOMAINS AND CROSSED PRODUCTS BY ACTIONS ON ADELE SPACES” BY J. CUNTZ AND X. LI

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1. Introduction

In [Cu-Li], we had computed the K-theory for C*-algebras associated with rings of integers in number fields. Unfortunately, there was a miscalculation in [Cu-Li, § 6.4, case c)], where the case of number fields with roots of unity +1, −1 and with an even strictly positive number of real places was treated (i.e. the case where # \{v_R\} ≥ 2 even). In this case the final result for the K-theory of the ring C*-algebra \( \mathcal{A}[\mathcal{o}] \) of the ring of integers \( \mathcal{o} \) of our number field should not be \( K_*(\mathcal{A}[\mathcal{o}]) \cong \Lambda^*(\Gamma) \oplus (\mathbb{Z}/2\mathbb{Z}) \otimes \Lambda^*(\Gamma) \), but \( K_*(\mathcal{A}[\mathcal{o}]) \cong \Lambda^*(\Gamma) \). This means that the torsion-free part in [Cu-Li, § 6.4, case c)] was determined correctly, but the torsion part was not computed correctly. The correct computation shows that the K-theory of the ring C*-algebra is torsion-free.

On the whole, the correct final result is the following (compare [Cu-Li, § 6]):

Let \( K \) be a number field with roots of unity \( \mu = \{\pm 1\} \). Choose a free abelian subgroup \( \Gamma \) of \( K^\times = \mu \times \Gamma \). We obtain for the K-theory of the ring C*-algebra \( \mathcal{A}[\mathcal{o}] \) attached to the ring of integers \( \mathcal{o} \) of \( K \):

\[
K_*(\mathcal{A}[\mathcal{o}]) \cong \begin{cases} K_0(C^*(\mu)) \otimes \mathbb{Z} \Lambda^*(\Gamma) & \text{if } \# \{v_R\} = 0, \\
\Lambda^*(\Gamma) & \text{if } \# \{v_R\} \geq 1. \end{cases}
\]

The distinction between the formulas in the two different cases corresponds to a natural identification on the level of generators. As abstract groups one obtains the same K-theory independently of the number of real embeddings.

2. The correct computation

Let us first of all explain what went wrong in our original computation in [Cu-Li, § 6.4, case c)]: Let \( \theta \in \text{Aut}(C_0(\mathbb{R})) \) be the flip, i.e. \( \theta(f)(x) = f(-x) \)

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for all \( f \in C_0(\mathbb{R}) \). By equivariant Bott periodicity, we know that
\[
K_i(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}^2 & \text{if } i = 0, \\
\{0\} & \text{if } i = 1.
\end{cases}
\]

In the first part of the proof of [Cu-Li, Lemma 6.4], we have claimed that the automorphism \( \text{id} \otimes \theta \) of \( C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \) acts as \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) in K-theory (in [Cu-Li], \( \text{id} \otimes \theta \) is denoted by \( \beta_{[1,-1]} \)). This however cannot be true. The reason is that using the Pimsner-Voiculescu sequence, we would obtain as an immediate consequence that \( K_0(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \rtimes_{\text{id} \otimes \theta} \mathbb{Z} \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \). But as Lemma 2.1 below shows, the correct result is \( K_0(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}) \cong \mathbb{Z} \).

In the first part of the proof of [Cu-Li, Lemma 6.4], we had considered the number field \( K = \mathbb{Q}[\sqrt{2}] \) with ring of integers \( \mathcal{O} = \mathbb{Z} + \mathbb{Z}\sqrt{2} \). The problem in our original computation was that we have assumed that in this particular case, the element \([u^1] \times [u^2] \) is part of a \( \mathbb{Z} \)-basis for \( G_{inf} \subseteq K_0(C^*(\mathcal{O} \rtimes \mu)) \) (in the terminology of [Cu-Li, Lemma 6.1]). But this is not the case, only up to finite index. This is why [Cu-Li, Lemma 6.4] is false.

Here is now the correct computation:

**Lemma 2.1.** \( K_i(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}) \cong \mathbb{Z} \) for \( i = 0, 1 \).

**Proof.** The first step is the following simple observation:
\[
(1) \quad C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \\
\cong (C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \\
\cong (C_0(\mathbb{R}) \otimes C_0(\mathbb{R})) \rtimes_{\theta \otimes \text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \\
\cong ((C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}) \otimes C_0(\mathbb{R})) \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \\
\cong [C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}] \otimes [C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}].
\]

To get from the second to the third line, we just made use of the automorphism \( (\mathbb{Z}/2\mathbb{Z})^2 \cong (\mathbb{Z}/2\mathbb{Z})^2 \) given by \( t_1 \mapsto t_1t_2, \ t_2 \mapsto t_2 \). Here \( t_1 \) and \( t_2 \) are the generators of the two copies of \( \mathbb{Z}/2\mathbb{Z} \).

Since \( K_0(C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \) and \( K_1(C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}) \cong \{0\} \) (see [Cu-Li] § 3.3, Equation (12)), we deduce
\[
(2) \quad K_i(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } i = 0, \\
\{0\} & \text{if } i = 1.
\end{cases}
\]

Now consider the automorphism \( (\text{id} \otimes \theta)^* \) of \( C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z} \) which is dual to the action of the second copy of \( \mathbb{Z}/2\mathbb{Z} \). Under the isomorphism (1), \( (\text{id} \otimes \theta)^* \) corresponds to the automorphism \( \theta \otimes \hat{\theta} \), where \( \hat{\theta} \) is the automorphism
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on \( C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z} \) dual to \( \theta \). Since \( \hat{\theta} \) is either \( \text{id} \) or \( -\text{id} \) on \( K_0(C_0(\mathbb{R}) \rtimes_{\theta} \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \), we conclude that

(3) \( ((\text{id} \otimes \theta)^*)_* = \text{id} \) on \( K_0(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \).

Plugging (2) and (3) into the exact sequence from [Bla, Theorem 10.7.1], which connects the \( K \)-theory of the crossed products by \( \mathbb{Z} \) and by \( \mathbb{Z}/2 \) induced by \( \text{id} \otimes \theta \) respectively, we obtain

\[
K_i(C_0(\mathbb{R}^2) \rtimes_{\theta \otimes \theta} \mathbb{Z}/2\mathbb{Z} \rtimes_{\text{id} \otimes \theta} \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}
\]

for \( i = 0, 1 \).

□

With this lemma, the computation of the \( K \)-theory of the ring \( C^* \)-algebras follows the same line of arguments as in [Cu-Li]. Let us explain this briefly using the same notations as in the introduction and as in [Cu-Li, § 6.4, case c)]. Combining [Cu-Li] (4) with [Cu-Li Corollary 4.2] and using a refined version of [Cu-Li, Lemma 6.3], it is straightforward to see that the \( K \)-theory of \( \mathcal{A}[\sigma] \) coincides with the \( K \)-theory of \( C_0(\mathcal{A}_\infty) \rtimes K^\times \). As in [Cu-Li § 6.4, case c)], let \( K^\times = \mu \times \Gamma \) and choose a \( \mathbb{Z} \)-basis \( \{p, p_1, p_2, \ldots\} \) of \( \Gamma \), with \( p \in \mathbb{Z}_{>0} \). We can arrange that \( \# \{v_\mathbb{R}: v_\mathbb{R}(p_1) < 0\} \) is odd and \( \# \{v_\mathbb{R}: v_\mathbb{R}(p_i) < 0\} \) is even for all \( i > 1 \). Let \( \Gamma_m = \langle p, \ldots, p_m \rangle \) and \( \Gamma'_m = \langle p, p_2, \ldots, p_m \rangle \). An iterative application of the Pimsner-Voiculescu sequence gives

\[
K_*(C_0(\mathcal{A}_\infty) \rtimes (\mu \times \Gamma_m)) \cong \Lambda^*(\Gamma'_m)
\]

and thus

\[
K_*(\mathcal{A}[\sigma]) \cong \Lambda^*(\Gamma).
\]

REFERENCES

[Bla] B. Blackadar, *K-theory for Operator Algebras*, vol. 5 of Mathematical Sciences Research Institute Publications, CUP, Cambridge, 2nd edition (1986).
[Cu-Li] J. Cuntz and X. Li, *\( C^* \)-algebras associated with integral domains and crossed products by actions on adele spaces*, Journal of Noncommutative Geometry 5 (2011), 1–37.