THE KAPLANSKY TEST PROBLEMS FOR $\aleph_1$-SEPARABLE GROUPS

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Abstract. We answer a long-standing open question by proving in ordinary set theory, ZFC, that the Kaplansky test problems have negative answers for $\aleph_1$-separable abelian groups of cardinality $\aleph_1$. In fact, there is an $\aleph_1$-separable abelian group $M$ such that $M$ is isomorphic to $M \oplus M \oplus M$ but not to $M \oplus M$. We also derive some relevant information about the endomorphism ring of $M$.

Introduction

Kaplansky [15, pp. 12f] posed two test problems in order to “know when we have a satisfactory [structure] theorem. ... We suggest that a tangible criterion be employed: the success of the alleged structure theorem in solving an explicit problem.” The two problems were:

(I) If $A$ is isomorphic to a direct summand of $B$ and conversely, are $A$ and $B$ isomorphic?

(II) If $A \oplus A$ and $B \oplus B$ are isomorphic, are $A$ and $B$ isomorphic?

In fact, he says ([15, p. 75]) that he invented the problems “to show that Ulm’s theorem [a structure theory for countable abelian $p$-groups] could really be used”. For some other classes of abelian groups, such as finitely-generated groups, free groups, divisible groups, or completely decomposable torsion-free groups, the existence of a structure theory leads to an affirmative answer to the test problems. On the other hand, negative answers are taken as evidence of the absence of a useful classification theorem for a given class; Kaplansky says “I believe their defeat is convincing evidence that no reasonable invariants exist” ([15, p. 75]). Negative answers to both questions have been proven, for example, for the class of uncountable abelian $p$-groups and for the class of countable torsion-free abelian groups.

Of particular interest is the method developed by Corner (cf. [1], [2],[4]) which, by realizing certain rings as endomorphism rings of groups, provides negative answers to both test problems (for a given class) as special cases of an even more extreme pathology. More precisely, Corner’s method — where applicable — yields, for any positive integer $r$, an abelian group $G_r$ (in the class) such that for any positive integers $m$ and $k$, the direct sum of $m$ copies of $G_r$ is isomorphic to the direct sum of $k$ copies of $G_r$ if and only if $m$ is congruent to $k$ mod $r$. (See, for example, [3] or [4], Thm 91.6, p. 145.) Then we obtain negative answers to both test problems by letting $A = G_2 \cong G_2 \oplus G_2 \oplus G_2$ and $B = G_2 \oplus G_2$. 

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Our focus here is on the class of $\aleph_1$-separable abelian groups (of cardinality $\aleph_1$). We will prove, in ordinary set theory (ZFC), that both test problems have negative answers by deriving the Corner pathology:

**Theorem 0.1.** For any positive integer $r$ there is an $\aleph_1$-separable group $M = M_r$ of cardinality $\aleph_1$ such that for any positive integers $m$ and $k$, $M^m$ is isomorphic to $M^k$ if and only if $m$ is congruent to $k$ mod $r$.

(Here $M^m$ denotes the direct sum of $m$ copies of $M$.) We do not determine the endomorphism ring of $M$, even modulo an ideal. However, we can derive a property of the endomorphism ring of $M$ which is sufficient to imply the Corner pathology: see section 3.

A group $M$ is called $\aleph_1$-separable [10, p. 184] (respectively, strongly $\aleph_1$-free) if it is abelian and every countable subset is contained in a countable free direct summand of $M$ (resp., contained in a countable free subgroup $H$ which is a direct summand of every countable subgroup of $M$ containing $H$). Obviously, an $\aleph_1$-separable group is strongly $\aleph_1$-free, so a negative answer to one of the test problems for the class of $\aleph_1$-separable groups implies a negative answer to the problem for the class of strongly $\aleph_1$-free groups. (It is independent of ZFC whether these classes are different for groups of cardinality $\aleph_1$: the weak Continuum Hypothesis ($2^{\aleph_0} < 2^{\aleph_1}$) implies that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are not $\aleph_1$-separable; on the other hand, Martin’s Axiom (MA) plus the negation of the Continuum Hypothesis ($\neg$CH) implies that every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is $\aleph_1$-separable; cf. [16].) This group $G$ cannot be $\aleph_1$-separable since the endomorphism ring of an $\aleph_1$-separable group has too many idempotents. However, Thomé ([20] and [21]) showed that ZFC plus V = L (Gödel’s Axiom of Constructibility) implies the Corner pathology for $\aleph_1$-separable groups of cardinality $\aleph_1$; he did this by constructing an $\aleph_1$-separable $G$ such that End($G$) is a split extension of $A$ by $I$ (in the sense of [3, p. 277]), where $I$ is the ideal of endomorphisms with a countable image.

It follows from known structure theorems for the class of $\aleph_1$-separable groups of cardinality $\aleph_1$ under the hypothesis MA + $\neg$CH that the Dugas-Göbel and Thomé realization results are not theorems of ZFC (cf. [5] or [17]). The fact that there are positive structure theorems for the class of $\aleph_1$-separable groups assuming MA + $\neg$CH or the stronger Proper Forcing Axiom (PFA) — see, for example, [8] or [18] — led to the question of whether the Kaplansky test problems could have affirmative answers for this class assuming, say, PFA. Thomé [21] gave a negative answer to the second test problem in ZFC, using a result of Jónsson [14] for countable torsion-free groups; however, till now, the first test problem as well as the Corner pathology were open (in ZFC).

Our construction of the Corner pathology involves a direct construction of the pathological group $M$ using a tree-like ladder system and a “countable template” which comes from the Corner example for countable torsion-free groups. A key role is played by a paper of Göbel and Goldsmith [13] which — while it does not itself
prove any new results about the Kaplansky test problems for strongly \( \aleph_1 \)-free or \( \aleph_1 \)-separable groups — provides the tools for creating a suitable template from the Corner example.

1. The countable template

Fix a positive integer \( r \). For this \( r \), let \( A = A_r \) be the countable ring constructed by Corner in \([8]\). (See also \([11, p. 146] \).) Specifically, \( A \) is the ring freely generated by symbols \( \rho_i \) and \( \sigma_i \) \((i = 0, 1, ..., r)\) subject to the relations

\[
\rho_i \sigma_i = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\sum_{i=0}^{r} \sigma_i \rho_i = 1.
\]

Then \( A \) is free as an abelian group, and \( \sigma_0 \rho_0, \ldots, \sigma_r \rho_r \) are pairwise orthogonal idempotents. Moreover, if \( M \) is a right \( A \)-module, then \( M = M \sigma_0 \rho_0 \oplus M \sigma_1 \rho_1 \oplus \ldots \oplus M \sigma_r \rho_r \), and \( M \sigma_i \rho_i \cong M \) because \( \sigma_i \rho_i \sigma_i : M \to M \sigma_i \rho_i \) and \( \rho_i \sigma_i \rho_i : M \sigma_i \rho_i \to M \) are inverses; therefore \( M \cong M^{r+1} \).

Our construction will work for any countable torsion-free ring \( A \) whose additive subgroup is free; but hereafter \( A \) will denote the ring \( A_r \) just defined.

Corner shows that there is a torsion-free countable abelian group \( G \) whose endomorphism ring is \( A \); thus \( G \) is an \( A \)-module and hence \( G \cong G^{r+1} \). Furthermore, he shows that \( G^\ell \) is not isomorphic to \( G^n \) if \( 1 \leq \ell < n \leq r \), and hence \( G^{m} \) is not isomorphic to \( G^k \) if \( m \) is not congruent to \( k \) mod \( r \). We shall require these and further properties of \( G \), which we summarize in the following:

**Proposition 1.1.** There are countable free \( A \)-modules \( B \subseteq H \) such that \( G \cong H/B \) and \( B \) is the union of a chain of free \( A \)-modules, \( B = \bigcup_{n \in \omega} B_n \), such that \( B_0 = 0 \) and for all \( n \in \omega \), \( H/B_n \) and \( B_{n+1}/B_n \) are free \( A \)-modules of rank \( \omega \). Moreover, for any positive integers \( m \) and \( k \), if \( m \) is not congruent to \( k \) mod \( r \), then \( G^m \oplus \mathbb{Z}^{(\omega)} \) is not isomorphic to \( G^k \oplus \mathbb{Z}^{(\omega)} \).

The main work in proving Proposition 1 will be done in two lemmas from \([8]\). For the first one, we give a revised proof (cf. \([13, p. 343]\)). We maintain the notation above.

**Lemma 1.2.** The group \( G \) is the union, \( G = \bigcup_{n \geq 1} G_n \), of an increasing chain of free \( A \)-modules.

**Proof.** By \([1, p. 699]\) \( G \) is the pure closure \( \langle G_1 \rangle_\hat{A} \) of a free \( A \)-module \( G_1 = \bigoplus_{i \in I} e_i A \oplus A \) containing \( A \). Here \( \hat{A} \) is the natural, or \( \mathbb{Z} \)-adic, completion of \( A \) (cf. \([1, p. 692]\)). We will define inductively \( G_n = \bigoplus_{i \in I} e_i A \oplus A \) such that \( G_n \supseteq G_{n-1} \) and for all \( i \in I \), \( n e_{i,n} + A = e_{i,n-1} + A \). Let \( e_{i,1} = e_i \) for all \( i \in I \). If \( G_{n-1} \subseteq G \) has been defined for some \( n > 1 \), then since \( A \) is dense in \( \hat{A} \), there exists \( e_{i,n} \in \hat{A} \) such that \( n e_{i,n} + A = e_{i,n-1} + A \); say \( n e_{i,n} = e_{i,n-1} + a_i \). By the definition of \( G \), \( e_{i,n} \in G \). We need to show that \( \{ e_{i,n} : i \in I \} \cup \{ 1 \} \) is \( A \)-linearly independent. Suppose that \( \Sigma_{i \in I} e_{i,n} a_i + 1 \cdot c_0 = 0 \) for some \( a_i, c_i \in A \). Then \( \Sigma_{i \in I} n e_{i,n} a_i + n c_0 = 0 \), so \( \Sigma_{i \in I} n e_{i,n-1} a_i + 1 \cdot (\Sigma_{i \in I} a_i c_i + n c_0) = 0 \). By the \( A \)-linear independence of \( \{ e_{i,n-1} : i \in I \} \cup \{ 1 \} \), we can conclude that each \( c_i \) equals 0 and hence also \( c_0 \) equals 0. This completes the definition of \( G_n \).
It remains to prove that \( G \subseteq \bigcup_{n \geq 1} G_n \). Let \( g \in G \setminus G_1 \). For some \( n > 1 \), \( ng \in G_1 \). We claim that \( g \in G_n \). Since \( ng \in G_{n-1} \), \( ng = \Sigma_{i \in \ell} e_{i,n-1} c_i + c_0 \) for some \( c_i, c_0 \in A \). Then

\[
ng = \Sigma_{i \in \ell} (ne_{i,n} - a_i) c_i + c_0 = n \Sigma_{i \in \ell} e_{i,n} c_i + a'
\]

for some \( a' \in A \). Since \( A \) is pure in \( \mathcal{A} \), \( a' = n a'' \) for some \( a'' \in A \). Thus \( g = \Sigma_{i \in \ell} e_{i,n} c_i + a'' \in G_n \).

\[\Box\]

The second lemma is proved in [3, Lemma 2.5] generalizing a result in [9, Lemma XII.1.4]. We state it here for the sake of completeness.

**Lemma 1.3.** Let \( G \) be a countable \( A \)-module which is the union, \( G = \bigcup_{n \geq 1} G_n \), of an increasing chain of free \( A \)-modules, then there exist countable free \( A \)-modules \( B \subseteq H \) such that \( G \cong H/B \) and \( B \) is the union of a chain of free \( A \)-modules, \( B = \bigcup_{n \geq 1} B_n \), such that for all \( n \geq 1 \), \( H/B_n \) and \( B_{n+1}/B_n \) are free \( A \)-modules. \( \square \)

**Proof of Proposition 1.1.** The existence of \( H, B \), and the \( B_n \) is now an immediate consequence of Lemmas 1.2 and 1.3. All that is left to show is that if \( m \) is not congruent to \( k \) mod \( r \), then \( G^m \oplus \mathbb{Z}^{(\omega)} \) is not isomorphic to \( G^k \oplus \mathbb{Z}^{(\omega)} \). Since \( G^m \) is not isomorphic to \( G^k \), it is enough to show that \( R_\mathbb{Z}(G^l \oplus \mathbb{Z}^{(\omega)}) = G^l \) for any \( l \in \omega \). Here \( R_\mathbb{Z}(N) \) is the \( \mathbb{Z} \)-radical of \( N \), that is, \( R_\mathbb{Z}(N) = \cap \{ \ker(\varphi) : \varphi : N \to \mathbb{Z} \} \). (See, for example, [3, pp. 289f].) To show that \( R_\mathbb{Z}(G^l \oplus \mathbb{Z}^{(\omega)}) = G^l \) it is enough to show that \( \text{Hom}(G^l, \mathbb{Z}) = 0 \), or, equivalently, \( \text{Hom}(G, \mathbb{Z}) = 0 \). This follows from Observation 2.7 of [3], but we give here a self-contained argument based on the notation of Lemma 1.2. Suppose \( \psi \in \text{Hom}(G, \mathbb{Z}) \); we can regard \( \psi \) as an endomorphism of \( G \) by identifying \( \mathbb{Z} \) with the subgroup \( \langle 1 \rangle \) of \( A \subseteq G \) which is generated by the unit 1 of \( A \). Since the endomorphism ring of \( G \) is \( A \), there is \( a \in A \) such that \( \psi(g) = ga \) for all \( g \in G \). By considering \( \psi(1) = 1a = a \), we see that \( a \in \langle 1 \rangle \). Now consider \( \psi(e_i) \) for any \( e_i \); since \( \psi(e_i) = e_ia \) and since \( e_i A \cap \langle 1 \rangle = \{ 0 \} \) we see that \( a = 0 \). \( \square \)

2. The Main Construction

Fix a positive integer \( r \) and let \( A, H, B, B_n \) and \( G \) be as in Proposition 1.1. For each \( n \in \omega \), fix a basis \( \{ b_{n,i} + B_n : i \in \omega \} \) of \( B_{n+1}/B_n \) (as \( A \)-module). Also, fix a set of representatives \( \{ h_i : i \in \omega \} \) for \( H/B \) where \( h_0 = 0 \); thus each coset \( h + B \) equals \( h_i + B \) for a unique \( i \in \omega \).

Fix a stationary subset \( E \) of \( \omega_1 \) consisting of limit ordinals and a ladder system \( \{ \eta_\delta : \delta \in E \} \). That is, for every \( \delta \in E \), \( \eta_\delta : \omega \to \delta \) is a strictly increasing function whose range is cofinal in \( \delta \); we shall also choose \( \eta_\delta \) so that its range is disjoint from \( E \). Furthermore, we choose a ladder system which is tree-like, that is, for all \( \delta, \gamma \in E \) and \( n, m \in \omega \), \( \eta_\delta(n) = \eta_\gamma(m) \) implies that \( m = n \) and \( \eta_\delta(l) = \eta_\gamma(l) \) for all \( l < n \) (cf. [3, pp. 368, 386]).

Inductively define free \( A \)-modules \( M_\beta (\beta < \omega_1) \) as follows: if \( \beta \) is a limit ordinal, \( M_\beta = \bigcup_{\alpha < \beta} M_\alpha \); if \( \beta = \alpha + 1 \) where \( \alpha \notin E \), let

\[
M_\beta = M_\alpha \oplus \bigoplus_{i \in \omega} x_{\alpha,i} A.
\]
If $\beta = \delta + 1$ where $\delta \in E$, define an embedding $i_\delta : B \rightarrow M_\delta$ by sending the basis element $b_{n,i}$ to $x_{\eta_\delta(n),i}$. Essentially $M_{\delta+1}$ will be defined to be the pushout of

\[
\begin{array}{c}
M_\delta \\
\uparrow i_\delta \\
B & \hookrightarrow H
\end{array}
\]

but we will be more explicit in order to avoid the necessity of identifying isomorphic copies. Let $y_{\delta,i} = 0$ and let \( \{ y_{\delta,i} : i \in \omega \setminus \{ 0 \} \} \) be a new set of distinct elements (not in $M_\delta$). Then define $M_{\delta+1}$ to be $\{ y_{\delta,i} + u : u \in M_\delta, i \in \omega \}$ where the operations on $M_{\delta+1}$ extend those on $M_\delta$ and are otherwise determined by the rules

\[
y_{\delta,i} + y_{\delta,j} = y_{\delta,k} + i_\delta(b) \quad \text{if} \quad h_i + h_j = h_k + b
\]

\[
y_{\delta,i}a = y_{\delta,\ell} + i_\delta(b) \quad \text{if} \quad h_i = h_\ell + b
\]

where $b \in B$ and $a \in A$. Then there is an embedding $\theta_\delta : H \rightarrow M_{\delta+1}$ extending $i_\delta$ which takes $h_i$ to $y_{\delta,i}$ and induces an isomorphism of $H/B$ with $M_{\delta+1}/M_\delta$.

This completes the inductive definition of the $M_\beta$. Let $M = \bigcup_{\beta < \omega_1} M_\beta$. Note that it follows from the construction that every element of $M$ has a unique presentation in the form

\[
\sum_{j=1}^\alpha y_{\delta_j,n_j} + \sum_{\ell=1}^t x_{\alpha_\ell,i_\ell} a_\ell
\]

where $\delta_1 < \delta_2 < ... < \delta_\alpha$ are elements of $E$, $n_j \in \omega \setminus \{ 0 \}$, $\alpha_\ell \in \omega_1 \setminus \{ E \}$, $i_\ell \in \omega$, $a_\ell \in A$, and the pairs $(\alpha_\ell, i_\ell)$ ($\ell = 1, ..., t$) are distinct.

Since $M$ is constructed to be an $A$-module, $M$ is isomorphic to $M^{r+1}$. We claim that

(\dagger) $M$ is $\aleph_1$-separable; in fact for all $\alpha < \omega_1$, $M_{\alpha+1}$ is a free direct summand of $M$.

Assuming this for the moment, we can show that

(\dagger\dagger) $M^m$ is not isomorphic to $M^k$ if $m$ is not congruent to $k$ mod $r$.

In brief this is because $M^m$ and $M^k$ are not quotient-equivalent (cf. [I, pp. 251f]) since for all $\delta \in E$, $(M_{\delta+1}/M_\delta)^m \oplus \mathbb{Z}^\omega$ is not isomorphic to $(M_{\delta+1}/M_\delta)^k \oplus \mathbb{Z}^\omega$ by Proposition [I.1]. In more detail, if there is an isomorphism $\varphi : M^m \rightarrow M^k$, then there is a closed unbounded subset $C$ of $\omega_1$ such that for $\beta \in C$, $\varphi[M^m\beta] = M^k\beta$. Since $E$ is stationary in $\omega_1$, there exist $\delta \in C \cap E$; choose $\beta > \delta$ such that $\beta \in C$. Then $\varphi$ induces an isomorphism of $M_{\beta}^m/M_{\beta}^m$ with $M_{\beta}^k/M_{\beta}^k$. Since $M_\beta/M_{\beta+1}$ is free (of infinite rank) by (\dagger), we can conclude that

\[
(M_{\beta+1}/M_\delta)^m \oplus \mathbb{Z}^\omega \cong (M_{\beta+1}/M_\delta)^k \oplus (M_{\beta}^m/M_{\beta}^m) \cong M_{\beta}^k/M_{\beta}^m \cong M_{\beta}^k/M_{\beta}^k
\]

\[
\cong (M_{\beta+1}/M_\delta)^k \oplus (M_{\beta}^k/M_{\beta}^k) \cong (M_{\beta+1}/M_\delta)^k \oplus \mathbb{Z}^\omega
\]

which contradicts Proposition [I.1].

We are left with the task of proving (\dagger). First we shall show that each $M_{\alpha+1}$ is a direct summand of $M$ by defining a projection $\pi_{\alpha}$ of $M$ onto $M_{\alpha+1}$ (that is, $\pi_{\alpha}|M_{\alpha+1}$ is the identity). For every integer $k$ there is a projection $\rho_k : H \rightarrow B_{k+1}$ since $H/B_{k+1}$ is free. Given $\alpha$, for each $\delta \in E$ with $\delta > \alpha$, let $k_\delta$ be the maximal integer $k$ such that $\eta_\delta(k) \leq \alpha$. For each $\delta \in E$, we let $\pi_{\alpha}$ act like $\rho_{k_\delta}$ on the isomorphic copy, $\theta_\delta[H]$, of $H$. More precisely, for each element $z$ of $\theta_\delta[H]$, 

\[
\pi_{\alpha}(z) = \theta_\delta(z) \quad \text{if} \quad z \in \theta_\delta[H]
\]
define $\pi_\alpha(z)$ to be $\theta_\delta(\rho_k(\theta_\delta^{-1}(z)))$; if $\nu \notin \bigcup \{\text{ran}(\eta_m) : \delta \in E\}$ and $\nu > \alpha$, define $\pi_\alpha(x_{\nu,i}) = 0$. Extend to an arbitrary element of $M$ by additivity; this will define a homomorphism on $M$ provided that $\pi_\alpha$ is well-defined. It is easy to see, using the unique representation of elements, that the question of well-definition reduces to showing that the definition of $\pi_\alpha(x_{\beta,i})$ for $x_{\beta,i} \in \theta_\delta[M]$ is independent of $\delta$. If $\beta \leq \alpha$, then $\pi_\alpha(x_{\beta,i}) = x_{\beta,i}$. Say $\beta > \alpha$ and $\beta = \eta_\delta(n) = \eta_\delta'(n)$; by the tree-like property, $\eta_\delta(m) = \eta_\delta'(m)$ for all $m \leq n$, and hence $k_\delta = k_\gamma$. Hence $\pi_\alpha(x_{\beta,i})$ is well-defined because $\rho_k = \rho_\gamma$, and thus $\theta_\delta(\rho_k(\theta_\delta^{-1}(x_{\beta,i}))) = \theta_\gamma(\rho_\gamma(\theta_\gamma^{-1}(x_{\beta,i})))$.

It remains to prove that each $M_{\beta}$ is $\aleph_\alpha$-free (as abelian group). Since $A$ is free as abelian group, it suffices to show that $M_{\beta+1}$ is a free $A$-module for every $\beta \in E$. We will inductively define $S_n$ so that

$$B = \bigcup_{n \in \omega} S_n \cup \{x_{\nu,i} : \nu \in \delta \setminus (E \cup \bigcup \{\text{ran}(\eta_\mu) : \mu \in E \cap (\delta + 1)\}), i \in \omega\}$$

is an $A$-basis of $M_{\beta+1}$. Let $S_0$ be the image under $\theta_\delta$ of a basis of $H$. Fix a bijection $\psi : \omega \to E \cap \delta$; also, for convenience, let $\psi(-1) = \delta$. Suppose that $S_m$ has been defined for $m \leq n$ so that $\bigcup_{m \leq n} S_m$ is $A$-linearly independent and generates $\bigcup \{\theta_\psi(m)[H] : -1 \leq m < n\}$. Let $\gamma = \psi(n)$ and let $k = k_\gamma$ be maximal such that $\eta_\gamma(k) = \eta_\psi(m)(k)$ for some $-1 \leq m < n$. Notice that $\{x_{\psi(m),i} : \ell \leq k, i \in \omega\}$ is contained in the $A$-submodule generated by $\bigcup_{m \leq n} S_m$. Since $H/B_{\beta+1}$ is $A$-free, we can write $H = B_{\beta+1} \oplus C_{\gamma}$ for some $A$-free module $C_{\gamma} = \ker(\rho_k)$; let $S_{n+1}$ be the image under $\theta_\gamma$ of a basis of $C_{\gamma}$. This completes the inductive construction. One can then easily verify that $B$ is an $A$-basis of $M_{\beta+1}$; indeed, the fact that $\bigcup_{m \leq n} S_m$ is $A$-linearly independent can be proved by induction on $n$, using the unique representation of elements of $M$ to show that if $\sum_{i=1}^r z_i a_i \in \langle \bigcup_{m \leq n} S_m \rangle$, where $z_1, \ldots, z_r$ are distinct elements of $S_{n+1}$, then $a_i = 0$ for all $i = 1, \ldots, r$.

3. The endomorphism ring of $M$

While we cannot show that $\text{End}(M)$ is a split extension of $A$ by an ideal, we can obtain enough information about $\text{End}(M)$ to imply the negative results on the Kaplansky test problems. (A similar idea is used in [19, p. 118].)

The ring $A$ is naturally a subring of $\text{End}(M)$. We say that $A$ is algebraically closed in $\text{End}(M)$ when every finite set of ring equations with parameters from $A$ (i.e., polynomials in several variables over $A$) which is satisfied in $\text{End}(M)$ is also satisfied in $A$.

**Proposition 3.1.** If $A = A_r$ is as in section 1, and $A$ is algebraically closed in $\text{End}(M)$, then for any positive integers $m$ and $k$, $M^m$ is isomorphic to $M^k$ if and only if $m$ is congruent to $k$ mod $r$.

**Proof.** Since $M$ is an $A$-module, $M \cong M^{r+1}$. If $M^\ell$ is isomorphic to $M^n$ where $1 \leq \ell < n \leq r$, then $\sum_{i=1}^\ell M \sigma_i \rho_i \cong \sum_{i=1}^n M \sigma_i \rho_i$, so by Lemma 2 of [2], there are elements $x$ and $y$ of $\text{End}(M)$ such that $xy = \sum_{i=1}^\ell \sigma_i \rho_i$ and $yx = \sum_{i=1}^n \sigma_i \rho_i$. So by hypothesis, such elements $x$ and $y$ exist in $A$. We then obtain a contradiction as in [2, p. 45].

**Proposition 3.2.** If $M$ is defined as in section 2, then $A$ is algebraically closed in $\text{End}(M)$.
Proof. For any $\sigma \in \text{End}(M)$, there is a closed unbounded subset $C_\sigma$ of $\omega_1$ such that for all $\alpha \in C_\sigma$, $\sigma[M_\alpha] \subseteq M_\alpha$. For any $\sigma_1, ..., \sigma_n$ in $\text{End}(M)$, choose $\alpha < \beta$ in $C_{\sigma_1} \cap ... \cap C_{\sigma_n}$ so that also $\alpha \in E$. Then each $\sigma_i$ induces an endomorphism, also denoted $\sigma_i$, of $M_\beta/M_\alpha$. The endomorphism ring of $M_\beta/M_\alpha$ is $\text{End}(G \oplus \mathbb{Z}(\omega))$ and restriction to $G$ defines a natural homomorphism, $\pi$, of $\text{End}(G \oplus \mathbb{Z}(\omega))$ onto $\text{End}(G) \cong A$ because $\text{Hom}(G, \mathbb{Z}(\omega)) = 0$. If $\sigma_i = a \in A$ (regarded as an element of $\text{End}(M)$), then $\pi(a) = a$. Hence if $\sigma_1, ..., \sigma_m$ satisfy some ring equations over $A$, then so do $\pi(\sigma_1), ..., \pi(\sigma_m)$.

Propositions 3.1 and 3.2 provide an alternative proof of (††).

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