Generalized Boltzmann equation for a trapped Bose-condensed gas using the Kadanoff-Baym formalism *

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Abstract

Using the Kadanoff-Baym non-equilibrium Green’s function formalism, we derive kinetic equations for the non-condensate atoms at finite temperatures which include the effect of binary collisions between atoms. The effect of collisions is included using the second-order self-energy given by the Beliaev (gapless) approximation. We limit our discussion to finite temperatures where we can use the single-particle Hartree-Fock spectrum for the excited atoms. In this limit, we can neglect the off-diagonal propagators ($\tilde{g}_{12}$ and $\tilde{g}_{21}$). As expected, this leads to the kinetic equations and collision integrals used in recent work by Zaremba, Nikuni, and Griffin (ZNG) [1]. We also derive a consistent equation of motion for the condensate wavefunction, involving a finite-temperature generalization of the well-known Gross-Pitaevskii equation which includes a dissipative term, as well as the mean field of the non-condensate.

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I. INTRODUCTION

In a trapped, weakly-interacting Bose gas at temperatures low compared to the Bose-Einstein transition temperature $T_{BEC}$, the fraction of atoms that are excited out of the condensate is very small (only a few percent in contrast to 90% in liquid He4). As a result, the dynamics of the trapped Bose gas is completely described by an equation of motion for the macroscopic wave-function $\Phi(r, t)$. This is the famous time-dependent Gross-Pitaevskii equation (GP) 2

$$\frac{i}{\hbar} \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{1}{2m} \nabla^2 + U_{ext}(r) + gn_c(r, t) \right] \Phi(r, t),$$

(1)

where $n_c(r, t) = |\Phi(r, t)|^2$ is the non-equilibrium density of the atoms in the condensate and $U_{ext}(r)$ is a parabolic trap potential. In this article, we set $\hbar = 1$. For a discussion of the properties of a dilute Bose gas at low temperatures, the two-body interaction $v(r - r')$ can be treated using the s-wave approximation:

$$v(r) = g\delta(r), \quad g = 4\pi a/m.$$  

(2)

One can see that the GP equation describes the motion of the condensate moving in the Hartree mean-field produced by the other atoms in the condensate. The GP equation is a closed equation for $\Phi(r, t)$. It provides a very accurate description of the static and dynamic properties of a trapped Bose gas at low temperatures $T \leq 0.4T_{BEC}$, as confirmed by many experiments in the last few years 2.

At finite temperatures, however, the number of atoms thermally excited out of the condensate becomes significant and the GP theory is no longer valid. The simplest way to include the effect of the excited atoms on the condensate is to add the additional Hartree-Fock mean field $V_{HF} = 2g\tilde{n}(r, t)$ produced by the non-condensate atoms (here $\tilde{n}$ is the local non-condensate
density). One immediately sees this new equation is no longer closed since it involves the dynamics of the non-condensate atoms.

To find the time-dependent non-condensate density, we need a quantum Boltzmann equation for the distribution function of the non-condensate atoms $f(p, r, t)$. A simple kinetic equation for a trapped Bose gas which has been used in recent studies [1,3] can be written in the following Boltzmann-like form

$$\left[ \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla_r U(r, t) \cdot \nabla_p \right] f(p, r, t) = \left[ \frac{\partial f(p, r, t)}{\partial t} \right]_{\text{coll}}. \quad (3)$$

Here, the thermally excited atoms are assumed to be well described by the single-particle spectrum $\frac{p^2}{2m} + U(r, t)$, where

$$U(r, t) \equiv U_{\text{ext}}(r) + 2g [n_c(r, t) + \tilde{n}(r, t)] \quad (4)$$

includes the self-consistent Hartree-Fock dynamic mean field involving the total time-dependent local density $n(r, t)$. The right-hand side of (3) describes the effect of collisions between atoms on the evolution of the distribution function $f(p, r, t)$. In trapped Bose-condensed gases, it is the sum of two parts

$$\left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} = C_{12}[f] + C_{22}[f]. \quad (5)$$

Here, $C_{22}$ denotes the part of the collision integral that describes two-body collisions between non-condensate atoms. Above $T_{\text{BEC}}$, this is the only term present. $C_{12}$ describes collisions between non-condensate atoms which involve one condensate atom. The role of $C_{12}$ is crucial since it couples the condensate and non-condensate degrees of freedom.

The kinetic equation (3) is valid only in the semiclassical limit: it assumes that the thermal energy is much greater than the spacing between the trap SHO energy levels ($k_B T \gg \hbar \omega_0$, where $\omega_0$ is the harmonic well frequency).
and the average interaction energy \( k_B T \gg \epsilon_n \)). ZNG have given a detailed derivation \[1\] of (3) at finite temperatures using the approach of Kirkpatrick and Dorfman \[4\], who originally considered a uniform gas. ZNG have used this kinetic equation to derive generalized two-fluid hydrodynamic equations of the kind first discussed by Landau in 1941.

Kane and Kadanoff (KK) \[5\] first used the Kadanoff-Baym (KB) formalism \[6\] as a microscopic basis for the derivation of the Landau two-fluid hydrodynamic equations for a uniform Bose fluid. In the present work, we use the KB approach to derive the kinetic equation (3) for a trapped Bose gas, obtaining the same collision integrals as derived by ZNG. This is the first step in more systematic derivation of kinetic equations for a trapped Bose gas. A more detailed and extended account of the present calculations will be published elsewhere \[7\].

II. EQUATIONS OF MOTION FOR NON-EQUILIBRIUM GREEN’S FUNCTIONS

In terms of quantum field operators, the many-body Hamiltonian \( \hat{K} = \hat{H} - \mu_0 \hat{N} \) describing interacting Bosons confined by an external harmonic potential \( U_{ext}(r) \) is given by:

\[
\hat{K} = \int dr \psi^\dagger(r) \left[ -\frac{1}{2m} \nabla_r^2 + U_{ext}(r) - \mu_0 \right] \psi(r) + \frac{1}{2} \int dr dr' \psi^\dagger(r) \psi^\dagger(r') \psi(r - r') \psi(r).
\]

We separate out the condensate part of the field operator in the usual fashion \[8\]

\[
\psi(r) = \langle \psi(r) \rangle_t + \tilde{\psi}(r),
\]

where \( \langle \tilde{\psi}(r) \rangle = 0 \) and \( \langle \psi(r) \rangle_t = \Phi(r, t) \) is the Bose macroscopic wavefunction. The non-condensate (or excited atom component) field operators \( \tilde{\psi}(r) \) and
\( \tilde{\psi}(\mathbf{r}) \) satisfy the usual Bose commutation relations.

In a Bose-condensed system, the finite value of \( \Phi(\mathbf{r}, t) \) leads to finite values of the off-diagonal (or anomalous) propagators \( \langle \tilde{\psi}(1)\tilde{\psi}(1') \rangle \) and \( \langle \tilde{\psi}^\dagger(1)\tilde{\psi}^\dagger(1') \rangle \). These must be dealt with on an equal basis with the diagonal (or normal) propagators, and thus we must work with a \( 2 \times 2 \) matrix single-particle Green’s function defined by \[ 5, 9 \]

\[
\hat{g}(1, 1'; U) = -i \begin{pmatrix}
\langle T\psi(1)\psi^\dagger(1') \rangle & \langle T\psi(1)\psi(1') \rangle \\
\langle T\psi^\dagger(1)\psi^\dagger(1') \rangle & \langle T\psi^\dagger(1)\psi(1') \rangle
\end{pmatrix}.
\] (8)

Here, \( T \) represents the time-ordering operation and we use the usual KB abbreviated notation, \( 1 \equiv (\mathbf{r}, t) \) and \( 1' \equiv (\mathbf{r}', t') \). We define \( \hat{g}^< \) and \( \hat{g}^> \) by

\[
\hat{g}(1, 1'; U) = \hat{g}^>(1, 1'; U) \quad t_1 > t_1' \\
= \hat{g}^<(1, 1'; U) \quad t_1 < t_1'.
\] (9)

Using (8), the matrix propagator in (8) splits into two parts

\[
\hat{g}(1, 1'; U) = \hat{g}(1, 1'; U) + \hat{h}(1, 1'; U).
\] (10)

Here \( \hat{g} \) is identical to (8) except that it involves the non-condensate part of the field operators, while the non-condensate part is given by

\[
\hat{h}(1, 1'; U) \equiv -i \begin{pmatrix}
\Phi(1)\Phi^*(1') & \Phi(1)\Phi(1') \\
\Phi^*(1)\Phi^*(1') & \Phi^*(1)\Phi(1')
\end{pmatrix},
\] (11)

with \( \langle \psi^\dagger(\mathbf{r}) \rangle_t \equiv \Phi^*(\mathbf{r}, t) \).

A very convenient way of generating the equations of motion for \( \hat{g} \) and \( \Phi \) is to use functional derivatives with respect to weak external fields \[ 3, 9 \]. The latter are described by

\[
H'(t_1) = \frac{1}{2} \int d\mathbf{r}_1 d2\psi^\dagger(1)U(1, 2)\psi(2) + \int d\mathbf{r}_1 \left[ \psi^\dagger(1)\eta_{ext}(1) + \psi(1)\eta^*_{ext}(1) \right],
\] (12)

where \( U(1, 2) \) is an external generating scalar field non-local in space and time. It represents a perturbation in which a particle is removed from the system at
point 1 and added at 2. The symmetry-breaking fields $\eta_{ext}$ and $\eta_{ext}^*$ describe particle creation and destruction [9,10]. Higher-order Green’s functions can all be expressed as functional derivatives of single-particle Green’s functions with respect to these fields.

It is useful to define the matrix inverse of $\hat{\tilde{g}}$ by

$$\hat{\tilde{g}}^{-1}(1,1';U) \equiv \hat{g}_0^{-1}(1,1') - U(1,1') - \hat{\Sigma}(1,1';U).$$

(13)

Following the Kane-Kadanoff (KK) analysis [5,6], the Dyson-Beliaev equations of motion for the real-time non-condensate propagators $\hat{\tilde{g}}(1,1')$ can be conveniently written in the following $2 \times 2$ matrix form

$$\int d\bar{1} \left[ \hat{g}_0^{-1}(\bar{1},\bar{1}) - \hat{\Sigma}^{HF}(\bar{1},\bar{1}) \right] \hat{g}^\leq(\bar{1},1')$$

$$= \int_{-\infty}^{t_1} d\bar{1}\hat{\Pi}(\bar{1},\bar{1})g^\leq(\bar{1},1') - \int_{-\infty}^{t_1'} d1\hat{\Sigma}^\geq(1,\bar{1})\hat{a}(\bar{1},1'),$$

(14)

and

$$\int d\bar{1}\hat{g}^\geq(1,\bar{1}) \left[ \hat{g}_0^{-1}(\bar{1},1') - \hat{\Sigma}^{HF}(\bar{1},1') \right]$$

$$= \int_{-\infty}^{t_1} d\bar{1}\hat{a}(1,\bar{1})\hat{\Sigma}^\leq(\bar{1},1') - \int_{-\infty}^{t_1'} d1\hat{g}^\leq(1,\bar{1})\hat{\Gamma}(\bar{1},1').$$

(15)

Here $\hat{a}(1,1')$ and $\hat{\Gamma}(1,1')$ are defined by the matrix elements

$$a_{\alpha\beta}(1,1') \equiv \hat{g}_{\alpha\beta}^>(1,1') - \hat{g}_{\alpha\beta}^<(1,1')$$

$$\Gamma_{\alpha\beta}(1,1') \equiv \Sigma_{\alpha\beta}^>(1,1') - \Sigma_{\alpha\beta}^<(1,1').$$

(16)

The spectral density $a_{\alpha\beta}(1,1')$ will play a crucial role in our later discussion. In the above equations and elsewhere, integration over $d\bar{1}$ means integration over the coordinates $(r_1,t_1)$ and a trace over the matrix index $\alpha_1$; and $\delta(11') \equiv \delta(r - r')\delta(t - t').$

We have explicitly split the self-energy which is involved in (14) and (13) into two parts [3,8]

$$\hat{\Sigma}(1,1') = \hat{\Sigma}^{HF}(1,1') + \hat{\Sigma}_c(1,1').$$

(17)
The Hartree-Fock self-energy is given by, using (2)

\[
\hat{\Sigma}^{HF}(11') = g \begin{pmatrix} 2n(1), & m(1) \\ m^*(1), & 2n(1) \end{pmatrix} \delta(11'),
\]

and \(\Sigma_c\) is the “collisional” part of the self-energy. The total density is given by \(n(1) \equiv ig_{11}(1,1^+) + |\Phi(1)|^2 = \tilde{n}(1) + n_c(1)\) and the total anomalous density by \(m(1) \equiv ig_{12}(1,1) + |\Phi(1)|^2\). In addition, we define

\[
\hat{\Sigma}_c(1,1') = \hat{\Sigma}^{>}(1,1') \quad t_1 > t_1' \\
= \hat{\Sigma}^{<}(1,1') \quad t_1 < t_1'.
\]

In (14) and (15), the inverse of the 2 × 2 matrix non-interacting Bose gas propagator \(\hat{g}_0(1,1')\) is defined by

\[
\hat{g}_0^{-1}(1,1') = \left[ i\tau_3 \frac{\partial}{\partial t_1} + \frac{\nabla^2}{2m} - U_{ext}(r_1) + \mu_0 \right] \delta(1,1').
\]

We note that the equations in (14) and (15) already have the “structure” of the kinetic equation in (3). The Hartree-Fock part of the self-energy is included into the left-hand side of (14) and (15), giving the mean-field contribution to the “streaming” term. The second order self-energy describing binary collisions is included on the right-hand side of (14) and (15) and it will eventually be shown to give rise to the collision integrals in (5).

The specific form of \(\Sigma_c\) will depend on the approximation that we use. In this paper, we work with the second-order self-energy given by the Beliaev (gapless) approximation [9,11]. The advantage of the Beliaev approximation is that the non-condensate Green’s function exhibits the correct spectrum (phonon-like in the long-wavelength, uniform gas limit). In contrast, “conserving approximations” are based on a functional, from which both self-energy \(\Sigma\) and the source \(\eta\) functions can be derived by functional differentiation. The resulting single-particle Green’s function can be used to generate a density response function whose spectrum is guaranteed to satisfy conservation
laws \[9,11–13\], even though the Green’s function has an energy gap in the long-wavelength limit. Kinetic equations for a trapped Bose gas based on a “conserving approximation” will be derived elsewhere \[7\].

The equation for the condensate can be written as \[9\]

\[
\int d\tilde{\eta}_0^{-1}(1, \tilde{1})\hat{G}_{1/2}(\tilde{1}) = \sqrt{-i\hat{\eta}(1)} + \sqrt{-i\hat{\eta}_{ext}(1)}, \tag{21}
\]

where the so-called condensate source function \(\eta\) is defined by the three-field correlation function

\[
\sqrt{-i\hat{\eta}(1)} \equiv \frac{1}{2} \int d2\sqrt{-i\nu(12)}\langle T\Psi(1)\Psi^\dagger(2)\Psi(2)\rangle. \tag{22}
\]

Here \(\Psi\) is defined as

\[
\Psi(1) \equiv \begin{pmatrix} \psi(1) \\ \psi^*(1) \end{pmatrix}. \tag{23}
\]

We have introduced the 2-component order parameter \(\hat{G}_{1/2}(1) \equiv \sqrt{-i}\langle\Psi(1)\rangle\), and \(\hat{\eta}_{ext}\) describes the external particle-source fields defined in \[12\], with

\[
\hat{\eta}_{ext}(1) \equiv \begin{pmatrix} \eta_{ext}(1) \\ \eta^*_{ext}(1) \end{pmatrix}. \tag{24}
\]

The external generating fields \(U\) and \(\eta_{ext}\) will be left implicit in the rest of this paper.

### III. BELIAEV APPROXIMATION FOR THE SELF-ENERGY

The self-energy in the Beliaev (gapless) approximation is defined by \[9\]

\[
\sqrt{-i}\frac{\delta\hat{\eta}(1)}{\delta\hat{G}_{1/2}(\tilde{1}')} \equiv \hat{\Sigma}(11'). \tag{25}
\]

To derive the second-order Beliaev approximation for the self-energy, we need to find the second-order expression for the source function \(\hat{\eta}\). For clarity, in
this section we will use a general interatomic potential \( v \), rather than the \( s \)-wave approximation given by (2). We express the three-point function in (22) in terms of functional derivatives in the following way \([9,14]\)

\[
\sqrt{-i\eta}(1) = iv(12) \left[ \frac{1}{2} \left[ \tilde{G}(22) + G_{1/2}(2) \tilde{G}^\dagger_{1/2}(2) \right] \tilde{G}_{1/2}(1) + \tilde{G}(12) G_{1/2}(2) \right] + i\delta \tilde{G}_{1/2}(1) \frac{\delta U(22)}{\delta \tilde{G}_{1/2}(1)},
\]

(26)

where we recall that repeated matrix labels are summed over. \( \tilde{G}(1,1') \) is the imaginary-time non-condensate propagator, from which one obtains the real-time propagator \( \tilde{g}(1,1') \) used in the KB formalism. Equation (26) can also be written as (from now on, the matrix nature of the quantities is left implicit):

\[
\sqrt{-i\eta}(1) = iv(12) \left[ \frac{1}{2} \left[ \tilde{G}(22) + G_{1/2}(2) \tilde{G}^\dagger_{1/2}(2) \right] G_{1/2}(1) + \tilde{G}(12) G_{1/2}(2) \right] + i\delta \tilde{G}(12) \tilde{G}_{1/2}(1) \tilde{G}(22) + 2v(12) G_{1/2}(2) \tilde{G}(12) \tilde{G}^\dagger_{1/2}(2).
\]

(27)

We approximate \( \Sigma \) in (27) by the full Hartree-Fock-Bogoliubov (HFB) first-order self-energy (denoted as HF for simplicity)

\[
\Sigma_{HF}(11') = \frac{i}{2} v(12) \left[ G_{1/2}(2) G^\dagger_{1/2}(2) + \tilde{G}(22) \right] \delta(11') + iv(11') \left[ G_{1/2}(1) G^\dagger_{1/2}(1') + \tilde{G}(11') \right].
\]

(28)

If one recalls that \( G_{1/2}(1) \equiv \sqrt{-i\Phi}(1) \) and the definitions for \( m(1) \) and \( n(1) \) given below (18), one can immediately see that for the contact interaction (2), (28) reduces to (18). Using (28), we obtain the following second-order expression for the source function

\[
\sqrt{-i\eta}(1) = \frac{i}{2} v(12) \left[ G_{1/2}(1) G_{1/2}(2) G^\dagger_{1/2}(2) + G_{1/2}(1) \tilde{G}(22) \right] + iv(12) G_{1/2}(2) \tilde{G}(12) - \frac{1}{2} v(13) v(24) \tilde{G}(32) \tilde{G}(23) \tilde{G}(14) G_{1/2}(4) - v(13) v(24) \tilde{G}(14) \tilde{G}(43) \tilde{G}(32) G_{1/2}(2).
\]

(29)

To obtain a second-order expression for the self-energy, we use (29) in (25).

In this gapless approximation, we note that \( \tilde{G} \) and \( G_{1/2} \) are not independent; using (13), one finds
Using (25), we obtain the second order Beliaev collisional part of the self-energy

\[\hat{\Sigma}_c(1, 1') = -\frac{1}{2}v(13)v(21')\tilde{G}(11') \left[ \tilde{G}(23)\tilde{G}(32) + \tilde{G}(23)h(32) + h(23)\tilde{G}(32) \right] \]

\[\quad - v(13)v(21')\tilde{G}(12) \left[ \tilde{G}'(23)h(31') + h(23)\tilde{G}(31') + \tilde{G}(23)\tilde{G}(31') \right] \]

\[\quad - \frac{1}{2}v(13)v(21') \left[ h(11')\tilde{G}(23)\tilde{G}(32) + 2h(12)\tilde{G}(23)\tilde{G}(31') \right].\] (31)

We now use these results to find, in a consistent manner, explicit expressions for the collision integrals in our kinetic equation and an equation of motion for the condensate \[\tilde{G}\].

**IV. GENERALIZED KINETIC EQUATIONS**

To derive a generalized kinetic equation, we first gauge transform (14) and (15) to the local rest frame to remove the phase of the macroscopic wavefunction \[\tilde{G}\]. They remain unchanged in form when \(g_0^{-1}\) is replaced by (compare with (20))

\[g_0^{-1}(1, 1') = \left[ i\tau_3 \frac{\partial}{\partial t_1} - \frac{\partial \theta(1)}{\partial t_1} + \frac{1}{2}(\nabla_1 + i\tau_3\nabla_1\theta(1))^2 - U_{\text{ext}}(r_1) + \mu_0 \right] \delta(1, 1').\] (32)

We recall that the superfluid velocity \(v_s(R, T)\) and the local chemical potential \(\mu(R, T)\) are defined by \[\epsilon\]

\[mv_s(R, T) \equiv \nabla_R \theta(R, T)\]

\[\frac{\partial \theta(R, T)}{\partial T} \equiv - \left[ \mu(R, T) - \mu_0 + \frac{1}{2}mv_s^2(R, T) \right].\] (33)

where, in the lab frame, the condensate wavefunction is given by \(\Phi(R, T) = \sqrt{n_c(R, T)}e^{i\theta(R, T)}\).

To illustrate how this method works, in this brief report we neglect the off-diagonal anomalous Green’s functions \(\tilde{g}_{12}\) and \(\tilde{g}_{21}\). The same approach
can be used when we include the off-diagonal propagators \[7\]. In this approximation, the \(2 \times 2\) matrix equations in (14) and (15) reduce to scalar equations for the \(\tilde{g}_{11}^<\) component

\[
\begin{bmatrix}
\frac{i}{\hbar} \frac{\partial}{\partial t_1} - \frac{\partial \theta(1)}{\partial t_1} + \frac{1}{2} [\nabla_1 + i m v_s(1)]^2 - U(1) + \mu_0
\end{bmatrix} \tilde{g}_{11}^<(\bar{1}, 1') = \int_{-\infty}^{t_1} d\bar{1} \Gamma_{11}(1, \bar{1}) \tilde{g}_{11}^<(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \Sigma_{11}^<(1, \bar{1}) a_{11}(\bar{1}, 1')
\] (34)

and

\[
\begin{bmatrix}
-\frac{i}{\hbar} \frac{\partial}{\partial t_1'} - \frac{\partial \theta(1')}{\partial t_1'} + \frac{1}{2} [\nabla_1' - i m v_s(1')]^2 - U(1') + \mu_0
\end{bmatrix} \tilde{g}_{11}^<(\bar{1}, 1') = \int_{-\infty}^{t_1} d\bar{1} a_{11}(1, \bar{1}) \Sigma_{11}^<(\bar{1}, 1') - \int_{-\infty}^{t_1'} d\bar{1} \tilde{g}_{11}^<(\bar{1}, 1') \Gamma_{11}(1, \bar{1})
\] (35)

The effective self-consistent Hartree-Fock dynamic mean field \(U(1)\) is given by (4).

Following KB [6], we find it very useful to express correlation functions in terms of relative and center-of-mass coordinates defined by

\[
\begin{align*}
\mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_1', \quad t = t_1 - t_1'; \quad \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_1'}{2}, \quad T = \frac{t_1 + t_1'}{2}.
\end{align*}
\] (36)

Correlation functions (like \(\tilde{g}, \Sigma, \) etc.) are dominated by the small values of relative coordinates \((\mathbf{r}, t)\) (or high momenta and frequencies in the Fourier transforms), but vary slowly as functions of the center-of-mass coordinates \((\mathbf{R}, T)\). Using these key properties of correlation functions to simplify the equations, we write (34) and (35) in terms of the center-of-mass and relative coordinates to obtain (see Ref. 7 for details)

\[
\begin{align*}
&\left[i \frac{\partial}{\partial T} + \left[ \mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right] (\mu(\mathbf{R}, T) - U(\mathbf{R}, T)) + \mathbf{i} \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \\
+ \frac{1}{m} \nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{r}} + \mathbf{i} \left[ \left( \mathbf{r} \cdot \nabla_{\mathbf{R}} + t \frac{\partial}{\partial T} \right) \mathbf{v}_s(\mathbf{R}, T) \right] \cdot \nabla_{\mathbf{r}} + \mathbf{i} \mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} \right] \tilde{g}_{11}^>(\mathbf{r}, t) = \int d\mathbf{r} d\mathbf{t} \left[ \tilde{g}_{11}^>(\mathbf{r}, \tilde{t}) \Sigma_{11}^<([\mathbf{r} - \mathbf{t}, \tilde{t}; \mathbf{R}, T] - \tilde{g}_{11}^<(\mathbf{r}, \tilde{t}) \Sigma_{11}^>([\mathbf{r} - \mathbf{t}, \tilde{t}; \mathbf{R}, T]).
\end{align*}
\] (37)

The \((\mathbf{R}, T)\) dependence of \(\tilde{g}_{11}(\mathbf{r}, t; \mathbf{R}, T)\) is left implicit. The double Fourier transform of (37) is given by
\[
\frac{\partial}{\partial T} + \nabla_p [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_R - \nabla_R [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_p \\
+ \frac{\partial}{\partial T} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega} \tilde{g}_{11}^>(\mathbf{p}, \omega; \mathbf{R}, T) \\
= \tilde{g}_{11}^>(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_{11}^>(\mathbf{p}, \omega; \mathbf{R}, T) - \tilde{g}_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T) \Sigma_{11}^<(\mathbf{p}, \omega; \mathbf{R}, T). \tag{38}
\]

Here, we have defined the “normal” HF single-particle energy by

\[
\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{\mathbf{p}^2}{2m} + U(\mathbf{R}, T) - \mu(\mathbf{R}, T). \tag{39}
\]

In the so-called Thomas-Fermi approximation \cite{2}, we note that this energy reduces to

\[
\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{\mathbf{p}^2}{2m} + g_n c(\mathbf{R}, T). \tag{40}
\]

Using (38) to calculate the spectral density as defined in (16), we find that the self-energies \(\Sigma_{11}^<\) cancel out and we are left with

\[
\frac{\partial}{\partial T} + \nabla_p [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_R - \nabla_R [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_p \\
+ \frac{\partial}{\partial T} [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \frac{\partial}{\partial \omega} a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) = 0. \tag{41}
\]

This equation will prove very important in deriving a kinetic equation for the distribution function for the non-condensate atoms. One may explicitly verify that the quasiparticle approximation for the spectral density given by

\[
a_{11}(\mathbf{p}, \omega; \mathbf{R}, T) = (2\pi)\delta(\omega - \mathbf{p} \cdot \mathbf{v}_s - \tilde{\epsilon}_p(\mathbf{R}, T)) \tag{42}
\]

satisfies (11). We note that the assumption that the correlation functions \(\tilde{g}^>(\mathbf{r}, t; \mathbf{R}, T)\) are peaked at small values of \(\mathbf{r}\) and \(t\) implies that large values of \(\mathbf{p}\) and \(\omega\) determine \(\tilde{g}^>(\mathbf{p}, \omega; \mathbf{R}, T)\), where the particle-like spectrum (39) is a good approximation.

Following Ref \cite{3}, we introduce a quasiparticle distribution function \(f\) in the following way (see (14))

\[
\tilde{g}_{11}^>(\mathbf{p}, \omega; \mathbf{R}, T) \equiv a_{11}(\mathbf{p}, \omega; \mathbf{R}, T)f(\mathbf{p}, \mathbf{R}, T) \\
\tilde{g}_{11}^>(\mathbf{p}, \omega; \mathbf{R}, T) \equiv a_{11}(\mathbf{p}, \omega; \mathbf{R}, T)[1 + f(\mathbf{p}, \mathbf{R}, T)]. \tag{43}
\]
We recall that the usual Wigner distribution function is related to the KB diagonal single-particle Green’s function by

\[ f(p, R, T) \equiv \int d\mathbf{r} e^{-i\mathbf{p} \cdot \mathbf{r}} \tilde{g}_{11}^{<}(\mathbf{r}, t = 0; R, T). \] (44)

Using (43) in conjunction with the explicit result for \( a_{11} \) in (42), we see that the quasiparticle distribution function in (43) reduces to the Wigner distribution function (44). This is correct in the high-temperature limit only, where the HF quasiparticle spectrum reduces to the Hartree-Fock single-particle excitation spectrum and the quasiparticle distribution function is equivalent to the particle distribution function.

Using (41), (42), and (43) and after integration over \( \omega \), (38) reduces to a kinetic equation for the distribution function \( f \), namely

\[
\frac{\partial}{\partial T} + \nabla_p [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_R - \nabla_R [\tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s] \cdot \nabla_p \] \( f(p, R, T) \)

\[ = (1 + f) \Sigma_{11}^{<}(p, \omega = \tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s; R, T) - f \Sigma_{11}^{>}(p, \omega = \tilde{\epsilon}_p + \mathbf{p} \cdot \mathbf{v}_s; R, T), \] (45)

where \( f \equiv f(p, R, T) \). A generalized quantum kinetic equation of this kind has been derived by Stoof [3] using the related Keldysh formalism and path integrals.

V. COLLISION INTEGRALS IN THE SECOND-ORDER BELIAEV APPROXIMATION

Neglecting the off-diagonal Green’s functions, the Fourier transform of the collisional part of the self-energy \( \Sigma^{<} \), given by (31), is given by the following expression

\[
\Sigma_{11}^{<}(p, \omega) = 2g^2 \int \frac{d\mathbf{p}_1 d\omega_1}{(2\pi)^4} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(p + p_1 - p_2 - p_3)
\times \left[ \tilde{g}_{11}^{<}(p_1, \omega_1) \tilde{g}_{11}^{<}(p_2, \omega_2) \tilde{h}_{11}(p_3, \omega_3) + \tilde{g}_{11}^{<}(p_1, \omega_1) \tilde{h}_{11}(p_2, \omega_2) \tilde{g}_{11}^{<}(p_3, \omega_3) + \tilde{h}_{11}(p_1, \omega_1) \tilde{g}_{11}^{<}(p_2, \omega_2) \tilde{g}_{11}^{<}(p_3, \omega_3) \right] \] (46)
Here and elsewhere, the \((R, T)\) dependence of the self-energies and the Green’s functions is left implicit. Using (42), (45), and (46), one can evaluate the collision integral \(C_{22}\) which involves collisions between excited atoms (i.e., the parts of the self-energy in (46) which involve three propagators for non-condensate atoms). We obtain

\[
C_{22}[f] = 2g^2 \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \delta(\tilde{\epsilon}_p + \tilde{\epsilon}_{p_1} - \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3}) \delta(p + p_1 - p_2 - p_3) \\
\times [(1 + f)(1 + f_1)f_2 f_3 - f f_1 (1 + f_2)(1 + f_3)],
\]

(47)

where \(f \equiv f(p, R, T)\) and \(f_i \equiv f(p_i, R, T)\). \(C_{22}\) describes collisions where 2 excited atoms are scattered into 2 excited atoms, hence the subscript 22.

The collision integral \(C_{12}[f]\) involving collisions between normal atoms and one condensate atom comes from the terms involving \(h_{11}\) in (46). One can show that for slowly varying external disturbances \([3]\), one has

\[
h_{11}(p, \omega; R, T) = n_c(R, T) \delta(p) \delta(\omega)(2\pi)^4.
\]

(48)

Using (48), \(C_{12}[f]\) is found to be given by

\[
C_{12}[f] = 4\pi g^2 n_c(R, T) \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \left[ \delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3) \right] \\
\delta(\tilde{\epsilon}_{p_1} - \tilde{\epsilon}_{p_2} - \tilde{\epsilon}_{p_3}) \delta(p_1 - p_2 - p_3) \left[ [(1 + f_1)f_3 f_2 - f f_1 (1 + f_2)(1 + f_3)] \right].
\]

(49)

\(C_{12}\) describes collisions where we go from 1 excited and one condensate atom to 2 excited atoms, hence the subscript 12. Inserting (47) and (49) into (45), we arrive at a kinetic equation identical to that derived by ZNG \([1]\), namely

\[
\left[ \frac{\partial}{\partial T} + \nabla p \left[ \tilde{\epsilon}_p + p \cdot v_s \right] \cdot \nabla R - \nabla p \left[ \tilde{\epsilon}_p + p \cdot v_s \right] \cdot \nabla R \right] f(p, R, T) = C_{22}[f(p, R, T)] + C_{12}[f(p, R, T)].
\]

(50)

The left-hand side of (50) is the expected streaming term in the frame of reference in which the superfluid velocity is zero (ZGN have derived the equivalent of (50) in the lab frame). For a uniform gas at finite temperatures, (50) was
first obtained by Kirkpatrick and Dorfman [4] using a completely different approach.

VI. GENERALIZED GROSS-PITAEVSKII EQUATION

To derive an equation of motion for the order parameter, we first write equation (21) for $\Phi(r,t)$ in the local rest frame, namely

$$i \frac{\partial}{\partial t} \Phi(r,t) - \frac{\partial \theta}{\partial t} + \frac{1}{2m} \left[ \nabla \Phi + imv_s(1) \right]^2 + \mu_0 - U_{ext}(r) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) = \int_{-\infty}^{t} d\bar{t} [S_{11}^{>} - S_{11}^{<}] (r - \bar{r}, t - \bar{t}; (r + \bar{r})/2, (t + \bar{t})/2) \Phi(\bar{r}, \bar{t}). \quad (51)$$

We assume, as usual, that the $S_{11}$ correlation function related to (22) is dominated by small values of the relative coordinates $(r - \bar{r}, t - \bar{t})$, and therefore we can approximate $S_{11}^{>}$ in (51) by $S_{11}^{<}(r - \bar{r}, t - \bar{t}; r, t)$. For the same reason, we can approximate the macroscopic wavefunction in (51) by $\Phi(\bar{1}) \approx \Phi(1) \equiv \sqrt{n_c(r,t)}$. Hence, we approximate (51) by

$$i \frac{\partial}{\partial t} \Phi(r,t) - \frac{\partial \theta}{\partial t} + \frac{1}{2m} \left[ \nabla \Phi + imv_s(1) \right]^2 + \mu_0 - U_{ext}(r) - g(2\tilde{n}(1) + n_c(1)) \right] \Phi(1) = \int_{-\infty}^{t} d\bar{r} d\bar{t} [S_{11}^{>} - S_{11}^{<}] (r - \bar{r}, t - \bar{t}; r, t) \Phi(r,t). \quad (52)$$

We can rewrite (52) as (renaming $(r,t) \rightarrow (R,T)$)

$$i \frac{\partial}{\partial T} - \frac{\partial \theta}{\partial T} + \frac{1}{2m} \left[ \nabla \Phi + imv_s(R,T) \right]^2 + \mu_0 - U_{ext}(R) - g(2\tilde{n}(R,T) + n_c(R,T)) \right] \Phi(R,T) = \Phi(R,T) \int \frac{dp d\omega}{(2\pi)^2} [S_{11}^{>} - S_{11}^{<}] (p, \omega; R,T) \int_{-\infty}^{T} d\bar{\omega} \Phi(\bar{R} - \bar{p} - \omega(t - \bar{t})) \quad (53)$$

In the Beliaev approximation, $S_{11}(p,\omega; R,T)$ is found to be given by [7]

$$S_{11}^{>}(p,\omega; R,T) = 2g^2 \int \frac{dp d\omega}{(2\pi)^2} (2\pi)^4 \delta(\omega + \omega_1 - \omega_2 - \omega_3) \delta(p + p_1 - p_2 - p_3) \times \tilde{g}_{11}^{<}(p_1, \omega_1; R,T) \tilde{g}_{11}^{>}(p_2, \omega_2; R,T) \tilde{g}_{11}^{>}(p_3, \omega_3; R,T). \quad (54)$$

In evaluating the right-hand side of (53), we have used the following identity
\[
\lim_{\delta \to 0^+} \int_{-\infty}^{T} dt e^{-i(\omega+i\delta)(T-t)} \simeq \pi \delta(\omega) + iP \left( \frac{1}{\omega} \right),
\]

in conjunction with (12) and (13). We finally obtain a generalized Gross-Pitaevskii equation in the following form

\[
i \partial \sqrt{n_c(R, T)} \left[ \frac{\partial \theta(R, T)}{\partial T} - \frac{1}{2m} [\nabla_R + imv_s(R, T)]^2 - \mu_0 + U_{ext}(R) + g [2\tilde{n}(R, T) + n_c(R, T)] - iR(R, T) \right] \sqrt{n_c(R, T)}. \]

The new dissipative term \( R \) in the GP equation is related to the \( C_{12} \) collision term in the kinetic equation (50), namely

\[
R(R, T) \equiv \int \frac{dp}{(2\pi)^3} \frac{C_{12}[f(p, R, T)]}{2n_c(R, T)}. \quad (57)
\]

This term describes the damping of condensate amplitude fluctuations due to collisions with the atoms in the thermal cloud. The appearance of the dissipative term in (56) is expected since the \( C_{12} \) collisions change the number of atoms in the condensate and hence modify the magnitude of the condensate macroscopic wavefunction. The real part of the right-hand side of (53) is omitted since we only work to first order in the interaction as far as renormalized energies are concerned. We note that if we transform back into the lab frame (recall that in the lab frame \( \Phi = \sqrt{n_c} e^{i\theta} \)), (56) reduces to the time-dependent generalized Gross-Pitaevskii equation for \( \Phi(R, T) \) discussed by ZNG.

**VII. CONCLUDING REMARKS**

In this brief report, we have shown how the elegant and powerful Kadanoff-Baym formalism can be used in a trapped Bose gas to derive a generalized Gross-Pitaevskii equation for the condensate wavefunction as well as a quantum kinetic equation for distribution function for the non-condensate atoms [7].
We have limited ourselves to discussing the case of finite temperatures, where the single-particle spectral density can be approximated by (42). In another paper, we will show how this formalism can be extended to deal with the case of low temperatures, where both the diagonal and the off-diagonal components of the spectral density $a_{\alpha\beta}$ must be kept and they exhibit the Bogoliubov-Popov spectrum in place of the particle-like spectrum in (12).

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