A Lattice Formulation of Super Yang-Mills Theories with Exact Supersymmetry\textsuperscript{*)}

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We construct SU($N$) super Yang-Mills theories with extended supersymmetry on hypercubic lattices of various dimensions keeping one or two supercharges exactly. It is based on topological field theory formulation for the super Yang-Mills theories. Gauge fields are represented by compact unitary link variables, and the exact supercharges on the lattice are nilpotent up to gauge transformations. In particular, the lattice models are free from the vacuum degeneracy problem, which was encountered in earlier approaches. Thus, we do not need to introduce any supersymmetry breaking terms, and the exact supersymmetry is preserved wholly in the process of taking the continuum limit.

Among the models, we show that the desired continuum theories are obtained without any fine tuning of parameters for the cases $N = 2, 4, 8$ in two-dimensions. Also, the cases $N = 4, 8$ in three-dimensions are investigated, and a problem arising in four-dimensional models is discussed.

\section{Introduction}

Nonperturbative aspects in supersymmetric gauge theory are quite interesting not only from the field-theoretical point of view beyond the standard model, but also from the AdS/CFT duality between gauge theory and gravity in string theory\textsuperscript{4).}

A conventional approach to the nonperturbative study is lattice formulation, which enables numerical analysis for any observables not restricted to special operators with the BPS saturated or chiral properties. However, there has been difficulty on the lattice approach to supersymmetry, because of lack of infinitesimal translational invariance on the lattice and breakdown of the Leibniz rule\textsuperscript{5, 6).} In spite of the difficulty, it is possible to construct lattice models, which do not have manifest supersymmetry but flow to the desired supersymmetric theories in the continuum limit. One of the examples is $\mathcal{N} = 1$ super Yang-Mills (SYM) theory in four-dimensions whose field contents are gauge bosons and gauginos. Since in the theory the only relevant supersymmetry breaking operator is the gaugino mass, one can arrive at the supersymmetric continuum theory if the radiative corrections are not allowed to induce the relevant operator by symmetries realized in the lattice theory. Making use of domain wall or overlap fermions keeps discrete chiral symmetry on lattice, which is the symmetry excluding the fermion mass\textsuperscript{7).}

Supersymmetric theories with extended supersymmetry have some supercharges, which are not related to the infinitesimal translations and can be seen as fermionic internal symmetries. It is possible to realize a part of such supercharges as exact symmetry on lattice, and the exact supersymmetry is expected to play a key role to

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restore the full supersymmetry in the continuum limit with fine tuning of a few or no parameters.

For SYM theories with extended supersymmetry, Cohen, Kaplan, Katz and Ünsal recently proposed such a kind of various lattice models motivated by the idea of deconstruction. In these models, to generate the kinetic terms of the target theories and to stabilize noncompact bosonic zero-modes (the so-called radions), one has to add terms softly breaking the exact supersymmetry, which are tuned to vanish in large volume limit.

In many cases, the above ‘internal’ supersymmetries can be reinterpreted as the BRST symmetries in topological field theories. So, in order to construct lattice models respecting the ‘internal’ supersymmetry, it seems natural to start with the topological field theory formulation of the theory. In Ref. [13], Catterall discussed on a general formulation of lattice models based on the connection to topological field theory for supersymmetric theories without gauge symmetry.**)

Here, standing on the same philosophy, we construct lattice models for SYM theories with extended supersymmetry keeping one or two supercharges exactly. Our models are motivated by the topological field theory formulation of $\mathcal{N} = 2,4$ SYM theories, and free from the radion problems. The lattices have hypercubic structures, and the gauge fields are expressed as ordinary compact unitary variables on the lattice links.

In sections 2 and 3, we construct lattice models for two-dimensional $\mathcal{N} = 2,4$ SYM theories based on (balanced) topological field theory formulation, and discuss on renormalization near the continuum limit. In sections 4 and 5, starting naive lattice actions for four-dimensional $\mathcal{N} = 2,4$ SYM theories, we construct lattice models for $\mathcal{N} = 4,8$ in two-dimensions and for $\mathcal{N} = 8$ in three-dimensions. Section 6 is devoted to the summary and discussion on some future directions.

Throughout this paper, we focus on the gauge group $G = SU(N)$. At the points discussing continuum theories, notations of repeated indices in formulas are assumed to be summed. On the other hand, when treating lattice theories, we explicitly write the summation over the indices except the cases of no possible confusion.

§2. 2D $\mathcal{N} = 2$ SYM

2.1. Continuum Action

The action of $\mathcal{N} = 2$ SYM in two-dimensions can be written as the ‘topological field theory (TFT) form’

$$S_{2D\mathcal{N}=2} = Q \frac{1}{2g^2} \int d^2x \, \text{tr} \left[ \frac{1}{4} \eta [\phi, \bar{\phi}] - i\chi \Phi + \chi H - iv_\mu D_\mu \bar{\phi} \right], \quad (2.1)$$

where $\mu$ is the index for two-dimensional space-time. Bosonic fields are gauge fields $A_\mu$, complex scalars $\phi, \bar{\phi}$, and auxiliary field $H$. The other fields $\psi_\mu, \chi, \eta$ are

** For related works, see Refs. 14).

*) For some related works, see Refs. [10]. Also, for other attempts to lattice formulations of supersymmetry, see Refs. [11] [12] [13] [14].
fermionic, and \( \Phi = 2F_{12} \). \( Q \) is one of the supercharges of \( \mathcal{N} = 2 \) supersymmetry, and its transformation rule is given as

\[
QA_\mu = \psi_\mu, \quad Q\psi_\mu = iD_\mu \phi,
\]

\[
Q\phi = 0, \quad Q\chi = H, \quad QH = [\phi, \chi],
\]

\[
Q\phi = \eta, \quad Q\eta = [\phi, \bar{\phi}]. \tag{2.2}
\]

\( Q \) is nilpotent up to infinitesimal gauge transformations with the parameter \( \phi \). Note that the action has \( U(1)_R \) symmetry whose charge assignment is +2 for \( \phi \), -2 for \( \bar{\phi} \), +1 for \( \psi_\mu \), -1 for \( \chi \) and \( \eta \), 0 for \( A_\mu \) and \( H \).

2.2. Lattice Supersymmetry

We formulate the theory \((2.1)\) on the two-dimensional square lattice keeping the supersymmetry \( Q \). In the lattice theory, gauge fields \( A_\mu(x) \) are promoted to the compact unitary variables

\[
U_\mu(x) = e^{iaA_\mu(x)} \tag{2.3}
\]
on the link \( (x, x + \hat{\mu}) \). ‘a’ stands for the lattice spacing, and \( x \in \mathbb{Z}^2 \) the lattice site. All other variables are distributed at sites. Interestingly, the \( Q \)-transformation \((2.2)\) is extendible to the lattice variables preserving the property

\[
Q^2 = (\text{infinitesimal gauge transformation with the parameter } \phi) \tag{2.4}
\]
as follows:

\[
QU_\mu(x) = i\psi_\mu(x)U_\mu(x),
\]

\[
Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i \left( \phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x) \right),
\]

\[
Q\phi(x) = 0,
\]

\[
Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)],
\]

\[
Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)]. \tag{2.5}
\]

Also, \( QU_\mu(x) = -iU_\mu(x)\psi_\mu(x) \) follows from \( U_\mu(x)U_\mu(x)^\dagger = 1 \). All transformations except \( QU_\mu(x) \) and \( Q\psi_\mu(x) \) are of the same form as in the continuum case. Since \( QU_\mu(x)^\dagger = -iU_\mu(x)^\dagger\psi_\mu(x) \) follows from \( U_\mu(x)U_\mu(x)^\dagger = 1 \). All transformations expect \( QU_\mu(x) \) and \( Q\psi_\mu(x) \) are of the same form as in the continuum case. Since

\[
Q^2U_\mu(x) = i(Q\psi_\mu(x)U_\mu(x) - i\psi_\mu(x)(QU_\mu(x))
\]

\[
= \phi(x)U_\mu(x) - U_\mu(x)\phi(x + \hat{\mu}), \tag{2.6}
\]

if we assume the formula “\( QU_\mu(x) = \cdots \)”, the transformation \( Q\psi_\mu(x) \) is determined\(^*\). Then, happily \( Q^2\psi_\mu(x) = [\phi(x), \psi_\mu(x)] \) is satisfied, and the \( Q \)-transformation is consistently closed. Note that we use the dimensionless variables here, and that various quantities are of the following orders:

\[
\psi_\mu(x), \chi(x), \eta(x) = O(a^{3/2}), \quad \phi(x), \bar{\phi}(x) = O(a), \quad H(x) = O(a^2),
\]

\[
Q = O(a^{1/2}). \tag{2.7}
\]

\(^*\) The first term in the RHS does not vanish because \( i\psi_\mu(x)\psi_\mu(x) = -\frac{1}{2} \sum_{a,b,c} f^{abc} \psi_\mu(x)\psi_\mu(x)T^c \)

with \( f^{abc} \) being structure constants of the gauge group.
The first term in the RHS of \( Q\psi_\mu(x) = \cdots \) in (2.5) is of subleading order \( O(a^3) \) and irrelevant in the continuum limit.

2.3. Lattice Action

Once we have the \( Q \)-transformation rule closed among lattice variables, it is almost straightforward to construct the lattice action with the exact supersymmetry \( Q \):

\[
S_{2D N=2}^{\text{LAT}} = Q \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i\chi(x)(\Phi(x) + \Delta\Phi(x)) + \chi(x)H(x) \\
+ i \sum_{\mu=1}^2 \psi_\mu(x) \left( \bar{\phi}(x) - U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \right]
\]

(2.8)

where

\[
\Phi(x) = -i [U_{12}(x) - U_{21}(x)],
\]

(2.9)

\[
\Delta\Phi(x) = -r(2 - U_{12}(x) - U_{21}(x)).
\]

(2.10)

\( U_{\mu\nu} \) are plaquette variables written as

\[
U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + \hat{\mu})U_\mu(x + \hat{\nu})U_\nu(x)^\dagger.
\]

(2.11)

The action (2.8) is clearly \( Q \)-invariant from its \( Q \)-exact form, and is \( U(1)_R \) symmetric. It is an almost straightforward latticization of the continuum action (2.21) except the terms containing \( \Delta\Phi(x) \). We will explain a role of \( \Delta\Phi(x) \).

After acting \( Q \) in the RHS, the action takes the form

\[
S_{2D N=2}^{\text{LAT}} = \frac{1}{2g_0^2} \sum_x \text{tr} \left[ \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 + H(x)^2 - iH(x)(\Phi(x) + \Delta\Phi(x)) \\
+ \sum_{\mu=1}^2 \left( \phi(x) - U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^\dagger \right) \left( \bar{\phi}(x) - U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
- \frac{1}{4} \eta(x)[\phi(x), \eta(x)] - \chi(x)[\phi(x), \chi(x)] \\
- \sum_{\mu=1}^2 \psi_\mu(x)\psi_\mu(x) \left( \bar{\phi}(x) + U_\mu(x)\bar{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right) + i\chi(x)Q(\Phi(x) + \Delta\Phi(x)) \\
- i \sum_{\mu=1}^2 \psi_\mu(x) \left( \eta(x) - U_\mu(x)\eta(x + \hat{\mu})U_\mu(x)^\dagger \right) \right]
\]

(2.12)

In order to see the relevance of \( \Delta\Phi(x) \), let us consider the case without \( \Delta\Phi(x) \) in the action. After integrating out \( H(x) \), induced \( \Phi(x)^2 \) term yields the gauge kinetic term as the form

\[
\frac{1}{2g_0^2} \sum_x \sum_{\mu<\nu} \text{tr} \left[ -(U_{\mu\nu}(x) - U_{\nu\mu}(x))^2 \right]
\]

(2.13)
which is different from the standard Wilson action

\[ \frac{1}{2g_0^2} \sum_x \sum_{\mu<\nu} \text{tr} \left[ 2 - U_{\mu\nu}(x) - U_{\nu\mu}(x) \right]. \]  

(2.14)

In contrast with giving the unique minimum \( U_{\mu\nu}(x) = 1 \), the action (2.13) has many classical vacua

\[ U_{\mu\nu}(x) = \text{diag} (\pm 1, \cdots, \pm 1) \]  

(2.15)

up to gauge transformations, where any combinations of \( \pm 1 \) with \( ' - 1' \) appearing even times are allowed in the diagonal entries. Since the configurations (2.15) can be taken freely for each plaquette, it leads a huge degeneracy of vacua with the number growing as exponential of the number of the plaquettes. In order to see the dynamics of the model, we need to sum up contributions from all the minima, and the ordinary weak field expansion around a single vacuum \( U_{\mu\nu}(x) = 1 \) can not be justified.\(^*\)

Thus, we can not say anything about the continuum limit of the lattice model (2.12) without its nonperturbative investigations. In order to resolve the difficulty without affecting the \( Q \)-supersymmetry, we introduce the \( \Delta \Phi(x) \) terms with an appropriate choice of the parameter \( r = \cot \theta \):\(^{**}\)

\[ e^{i2\theta} \neq 1 \text{ for } \forall \ell = 1, \cdots, N. \]  

(2.16)

2.4. About Fermion Doublers

We expand the exponential of the link variable (2.3), and look at the kinetic terms in the action (2.12). Because in the bosonic sector no species doublers appear, in the fermionic sector also no doublers are expected due to the exact supersymmetry \( Q \) of (2.12). Let us see the fermionic sector explicitly.

After rescaling each fermion variable by \( a^{3/2} \) as indicated in (2.7), the fermion kinetic terms are expressed as

\[ S_f^{(2)} = \frac{a^4}{2g_0^2} \sum_{x,\mu} \text{tr} \left[ -\frac{1}{2} \Psi(x)^T \gamma_\mu (\Delta_\mu + \Delta^*_\mu) \Psi(x) - a\frac{1}{2} \Psi(x)^T P_\mu \Delta_\mu \Delta^*_\mu \Psi(x) \right], \]  

(2.17)

where fermions were combined as \( \Psi^T = (\psi_1, \psi_2, \chi, \frac{1}{2} \eta) \). The \( \gamma \)-matrices and \( P_\mu \) are given by

\[ \gamma_1 = -i\sigma_1 \otimes \sigma_1, \quad \gamma_2 = i\sigma_1 \otimes \sigma_3, \quad P_1 = \sigma_1 \otimes \sigma_2, \quad P_2 = \sigma_2 \otimes 1_2 \]  

(2.18)

with \( \sigma_i (i = 1, 2, 3) \) being Pauli matrices. Note that they all anticommute each other:

\[ \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, P_\nu\} = 0. \]  

(2.19)

\( \Delta_\mu \) and \( \Delta^*_\mu \) represent forward and backward difference operators respectively:

\[ \Delta_\mu f(x) \equiv \frac{1}{a} \left( f(x + \hat{\mu}) - f(x) \right), \quad \Delta^*_\mu f(x) \equiv \frac{1}{a} \left( f(x) - f(x - \hat{\mu}) \right). \]  

(2.20)

\(^*\) This kind of difficulty was already pointed out in Ref. 6).

\(^{**}\) For a discussion about how the degeneracy is removed, see Ref. 2).
The kernel of the kinetic terms (2.17) is written in the momentum space $-\frac{\pi}{a} \leq q_\mu < \frac{\pi}{a}$ as

$$D = \sum_{\mu=1}^{2} \left[ -i\gamma_\mu \frac{1}{a} \sin(q_\mu a) + 2P_\mu \frac{1}{a} \sin^2\left(\frac{q_\mu a}{2}\right) \right]. \quad (2.21)$$

It is easy to see that the kernel $D$ vanishes only at the origin $q_1 = q_2 = 0$, because using (2.19) we get

$$D^2 = \frac{1}{a^2} \sum_{\mu=1}^{2} \left[ \sin^2(q_\mu a) + 4\sin^4\left(\frac{q_\mu a}{2}\right) \right]. \quad (2.22)$$

Thus, the fermion kinetic terms contain no fermion doublers. The last term containing $P_\mu$ in (2.17) has a similar structure to the Wilson term, and plays a role of removing fermion doublers. Since the lattice action is $U(1)_R$ invariant and the fermion doublers are removed keeping the chiral $U(1)_R$, the model must break some assumptions of Nielsen-Ninomiya’s no go theorem. In fact, broken is the assumption “There exists a conserved charge $Q_F$ corresponding to the fermion number.”

When combining fermions into a two-component Dirac spinor as

$$\begin{align*}
\zeta &= \frac{1}{\sqrt{2}} \left( \psi_1 - i\psi_2 \right), \\
\bar{\zeta} &= \frac{1}{\sqrt{2}} \left( \psi_1 + i\psi_2, \chi - i\frac{1}{2}\eta \right),
\end{align*} \quad (2.23)$$

$Q_F$ corresponds to the $U(1)_J$ rotation: $\zeta \rightarrow e^{i\theta}\zeta$, $\bar{\zeta} \rightarrow e^{-i\theta}\bar{\zeta}$. The first term in (2.17), giving a naive fermion kinetic term on the lattice, is written as the combination $\zeta_\alpha\bar{\zeta}_\beta$, which is invariant under the $U(1)_J$. On the other hand, the last term containing $P_\mu$ takes the form:

$$\frac{a^4}{2g_0^2} \sum_x \frac{a}{2} \text{tr} \left[ \varepsilon_{\alpha\beta} \zeta_\alpha (\Delta_1 \Delta_1^* \Delta_2^* \Delta_2) \bar{\zeta}_\beta - \varepsilon_{\alpha\beta} \bar{\zeta}_\alpha (\Delta_1 \Delta_1^* \Delta_2^* \Delta_2^*) \zeta_\beta \right] \quad (2.24)$$

to break the $U(1)_J$ invariance.

2.5. Renormalization

At the classical level, the lattice action (2.8) leads to the continuum action (2.1) in the limit $a \rightarrow 0$ with $g^2 \equiv a^2g_0^{-2}$ kept fixed, and thus the $\mathcal{N} = 2$ supersymmetry and rotational symmetry in two-dimensions are restored. We will check whether the symmetry restoration persists against quantum corrections, i.e. whether symmetries of the lattice action forbid any relevant or marginal operators induced which possibly obstruct the symmetry restoration.

| $p = a + b + 3c$ | $\varphi^a \partial^b \psi^c$ |
|------------------|-------------------------------|
| 0                | 1                             |
| 1                | $\varphi$                     |
| 2                | $\varphi^2$                   |
| 3                | $\varphi^3$, $\psi\psi$, $\varphi\partial\varphi$ |
| 4                | $\varphi^4$, $\varphi^3\partial\varphi$, $(\partial\varphi)^2$, $\psi\partial\psi$, $\varphi\psi\psi$ |

Table 1. List of operators with $p \leq 4$. 
Assuming that the model has the critical point \( g_0 = 0 \) from the asymptotic freedom, we shall consider the renormalization effect perturbatively. The mass dimension of the coupling \( g^2 \) is two. For generic boson field \( \varphi \) (other than the auxiliary fields) and fermion field \( \psi \), the dimensions are 1 and 3/2 respectively. Thus, operators of the type \( \varphi^a \partial^b \psi^{2c} \) have the dimension \( p \equiv a + b + 3c \), where ‘\( \partial \)’ means a derivative with respect to the coordinates. From dimensional analysis, we can see that the operators receive the following radiative corrections up to some powers of possible logarithmic factors:

\[
\left( \frac{a^p - 4}{g^2} + c_1 a^{p-2} + c_2 a^p g^2 + \cdots \right) \int \! d^2 x \varphi^a \partial^b \psi^{2c},
\]

where \( c_1, c_2, \cdots \) are constants dependent on \( N \). The first, second and third terms in the parentheses represent the contributions at tree, one-loop and two-loop levels. It is easily seen from the fact that \( g^2 \) appears as an overall factor in front of the action and plays the same role as the Planck constant \( \hbar \). Due to the super-renormalizable property of two-dimensional theory, the relevant corrections terminate at the two-loop. From the above formula, it is seen that the following operators can be relevant or marginal in the \( a \to 0 \) limit: operators with \( p \leq 2 \) induced at the one-loop level and with \( p = 0 \) at the two-loop level. Operators with \( p \leq 4 \) are listed in Table I.

Since the identity operator does not affect the spectrum, we have to check operators of the types \( \varphi \) and \( \varphi^2 \) only. Gauge symmetry and \( U(1)_R \) invariance\(^*\) allow the operator \( \text{tr} \varphi \bar{\varphi} \), while it is forbidden by the supersymmetry \( Q \). Hence, no relevant or marginal operators except the identity are generated by radiative corrections, which means that in the continuum limit full supersymmetry and rotational symmetry are considered to be restored without any fine tuning.

§3. 2D \( \mathcal{N} = 4 \) SYM

3.1. Continuum Action

The action of \( \mathcal{N} = 4 \) SYM in two-dimensions can be written as the following 'Balanced Topological Field Theory (BTFT) form':

\[
S_{2D\mathcal{N}=4} = Q_+ Q_- F_{2D\mathcal{N}=4},
\]

\[
F_{2D\mathcal{N}=4} = \frac{1}{2g^2} \int \! d^2 x \text{tr} \left[ -iB\Phi - \psi_{+\mu} \psi_{-\mu} - \chi_{+} \chi_{-} - \frac{1}{4} \eta_{+} \eta_{-} \right].
\]

where \( Q_{\pm} \) are two of supercharges of the \( \mathcal{N} = 4 \) theory, and \( \Phi \equiv 2F_{12} \). Bosons are gauge fields \( A_\mu \ (\mu = 1, 2) \) and scalar fields \( B, C, \phi, \bar{\phi} \). Also, there are auxiliary fields \( \tilde{H}_\mu, H \). Other fields \( \psi_{\pm\mu}, \chi_{\pm}, \eta_{\pm} \) are fermions. Transformation rule of the supersymmetry \( Q_{\pm} \) is given by

\[
Q_+ A_\mu = \psi_{+\mu}, \quad Q_+ \psi_{+\mu} = i D_\mu \phi, \quad Q_- \psi_{+\mu} = i \frac{1}{2} D_\mu C - \tilde{H}_\mu,
\]

\[
Q_- A_\mu = \psi_{-\mu}, \quad Q_- \psi_{-\mu} = -i D_\mu \bar{\phi}, \quad Q_+ \psi_{-\mu} = i \frac{1}{2} D_\mu C + \tilde{H}_\mu.
\]

\(^*\) Note that the \( U(1)_R \) symmetry is not anomalous for \( G = SU(N) \) in the two-dimensions.
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\[ Q_+ \mu = [\phi, \psi_{-\mu}] - \frac{1}{2}[C, \psi_{+\mu}] - \frac{i}{2}D_\mu \eta_+, \]
\[ Q_- \mu = [\bar{\phi}, \psi_{+\mu}] + \frac{1}{2}[C, \psi_{-\mu}] + \frac{i}{2}D_\mu \eta_, \]  

(3.2)

\[ Q_+ B = \chi_+, \quad Q_+ \chi_+ = [\phi, B], \quad Q_- \chi_+ = \frac{1}{2}[C, B] - H, \]
\[ Q_- B = \chi_-, \quad Q_- \chi_- = -[\bar{\phi}, B], \quad Q_+ \chi_- = \frac{1}{2}[C, B] + H, \]
\[ Q_+ H = [\phi, \chi_-] + \frac{1}{2}[B, \eta_+] - \frac{1}{2}[C, \chi_+], \]
\[ Q_- H = [\bar{\phi}, \chi_+] - \frac{1}{2}[B, \eta_-] + \frac{1}{2}[C, \chi_-], \]  

(3.3)

\[ Q_+ C = \eta_+, \quad Q_+ \eta_+ = [\phi, C], \quad Q_- \eta_+ = -[\phi, \bar{\phi}], \]
\[ Q_- C = \eta_-, \quad Q_- \eta_- = -[\bar{\phi}, C], \quad Q_+ \eta_- = [\phi, \bar{\phi}], \]
\[ Q_+ \phi = 0, \quad Q_- \phi = -\eta_+, \quad Q_+ \bar{\phi} = \eta_-, \quad Q_- \bar{\phi} = 0. \]  

(3.4)

This transformation leads the following nilpotency of \( Q_\pm \) (up to gauge transformations):

\[ Q_+^2 = (\text{infinitesimal gauge transformation with the parameter } \phi), \]
\[ Q_-^2 = (\text{infinitesimal gauge transformation with the parameter } -\bar{\phi}), \]
\[ \{Q_+, Q_-\} = (\text{infinitesimal gauge transformation with the parameter } C). \]  

(3.5)

In this formulation, among the SU(4) internal symmetry of the \( N = 4 \) theory, its subgroup SU(2)\(_R\) is manifest which rotates \((Q_+, Q_-)\). Under the SU(2)\(_R\), each of \((\psi_{+\mu}^a, \psi_{-\mu}^a), (\chi_+^a, \chi_-^a), (\eta_+^a, -\eta_-^a)\) and \((Q_+, Q_-)\) transforms as a doublet, and \((\phi^a, C^a, -\bar{\phi}^a)\) as a triplet. Also, let us note a symmetry of the action (3.1) under exchanging the two supercharges \( Q_+ \leftrightarrow Q_- \) with

\[ \phi \rightarrow -\bar{\phi}, \quad \bar{\phi} \rightarrow -\phi, \quad B \rightarrow -B, \]
\[ \chi_+ \rightarrow -\chi_-, \quad \chi_- \rightarrow -\chi_+, \quad \tilde{H}_\mu \rightarrow -\tilde{H}_\mu, \]
\[ \psi_{\pm\mu} \rightarrow \psi_{\mp\mu}, \quad \eta_{\pm} \rightarrow \eta_{\mp}. \]  

(3.6)

3.2. Lattice Supersymmetry \( Q_\pm \)

Similarly to the \( N = 2 \) cases, it is possible to define the theory (3.1) on the square lattice preserving the two supercharges \( Q_\pm \). The transformation rule (3.2) is modified as

\[ Q_+ U_\mu (x) = i \psi_{+\mu} (x) U_\mu (x), \]
\[ Q_- U_\mu (x) = i \psi_{-\mu} (x) U_\mu (x), \]
\[ Q_+ \psi_{+\mu} (x) = i \psi_{+\mu} \psi_{+\mu} (x) - i \left( \phi (x) - U_\mu (x) \phi (x + \hat{\mu}) U_\mu (x)^\dagger \right), \]
\[ Q_- \psi_{-\mu} (x) = i \psi_{-\mu} \psi_{-\mu} (x) + i \left( \bar{\phi} (x) - U_\mu (x) \bar{\phi} (x + \hat{\mu}) U_\mu (x)^\dagger \right), \]
\[ Q_+ \psi_\mu(x) = \frac{i}{2} \{ \psi_\mu(x), \psi_{-\mu}(x) \} - \frac{i}{2} \left( C(x) - U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right) - \tilde{H}_\mu(x), \]
\[ Q_- \psi_{-\mu}(x) = \frac{i}{2} \{ \psi_{-\mu}(x), \psi_\mu(x) \} - \frac{i}{2} \left( C(x) - U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right) + \tilde{H}_\mu(x), \]
\[ Q_+ \tilde{H}_\mu(x) = -\frac{1}{2} \left[ \psi_{-\mu}(x), \phi(x) + U_\mu(x)\phi(x + \hat{\mu})U_\mu(x)^\dagger \right] \\
+ \frac{1}{4} \left[ \psi_\mu(x), C(x) + U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right] \\
+ \frac{i}{2} \left( \eta_+(x) - U_\mu(x)\eta_+(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
+ \frac{i}{2} \left[ \psi_\mu(x), \tilde{H}_\mu(x) \right] + \frac{1}{4} \left[ \psi_\mu(x)\psi_\mu(x), \psi_{-\mu}(x) \right], \]
\[ Q_- \tilde{H}_\mu(x) = -\frac{1}{2} \left[ \psi_\mu(x), \tilde{\phi}(x) + U_\mu(x)\tilde{\phi}(x + \hat{\mu})U_\mu(x)^\dagger \right] \\
- \frac{1}{4} \left[ \psi_{-\mu}(x), C(x) + U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger \right] \\
- \frac{i}{2} \left( \eta_-(x) - U_\mu(x)\eta_-(x + \hat{\mu})U_\mu(x)^\dagger \right) \\
+ \frac{i}{2} \left[ \psi_{-\mu}(x), \tilde{H}_\mu(x) \right] - \frac{1}{4} \left[ \psi_{-\mu}(x)\psi_{-\mu}(x), \psi_\mu(x) \right]. \tag{3.7} \]

The other transformations \[3.3, 3.4\] do not change the form under the latticization. Note that this modification keeps the nilpotency \[3.5\].

Making use of the \(Q_\pm\)-transformation rule in terms of lattice variables, we construct lattice actions with the exact supercharges \(Q_\pm\) as

\[ S^{\text{LAT}}_{2D N=4} = Q_+ Q - \frac{1}{2g_0^2} \sum_x \text{tr} \left[ -iB(x)(\Phi(x) + \Delta\Phi(x)) - \sum_{\mu=1}^2 \psi_\mu(x)\psi_{-\mu}(x) \right. \]
\[ \left. - \chi_+(x)\chi_-(x) - \frac{1}{4} \eta_+(x)\eta_-(x) \right], \tag{3.8} \]

where \(\Phi(x)\) and \(\Delta\Phi(x)\) are given by \[2.3\] and \[2.4\], respectively. Note that the lattice formulation retains the symmetries under \(\text{SU}(2)_R\) as well as the \(Q_+ \leftrightarrow Q_-\) exchange.

Similarly to the \(N = 2\) case, \(\Delta\Phi(x)\) removes the vacuum degeneracy and the fermion doublers do not appear. With respect to the renormalization argument, symmetries of the lattice action are sufficient to restore full supersymmetry and rotational invariance in the continuum limit. For instance, gauge invariance and \(\text{SU}(2)_R\) symmetry allow the operators \(\text{tr} (4\Phi\tilde{\phi} + C^2)\) and \(\text{tr} B^2\), but they are not admissible from the supersymmetry \(Q_\pm\). Thus, radiative corrections are not allowed to generate any relevant or marginal operators except the identity, which means the restoration of full supersymmetry and rotational invariance in the continuum limit.
§4. 3D \( \mathcal{N} = 4 \)

Also for \( \mathcal{N} = 2 \) theory in four-dimensions, we can write the action in the ‘TFT form’, and construct a naive lattice action as

\[
S_{4D\mathcal{N}=2}^{\text{LAT}} = Q \frac{1}{2g_0} \sum_x \left[ \frac{1}{4} \eta(x) \left[ \phi(x), \bar{\phi}(x) \right] - i \bar{\chi}(x) \cdot (\Phi(x) + \Delta \Phi(x)) + \bar{\chi}(x) \cdot \bar{H}(x) \right.
\]

\[
+ i \sum_{\mu=1}^4 \psi_{\mu}(x) \left( \bar{\phi}(x) - U_{\mu}(x) \bar{\phi}(x + \hat{\mu}) U_{\mu}^\dagger(x) \right) \left. \right]
\]

\[
Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)
\]

in order to arrive at the desired continuum theory at the quantum level.

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]

\[Q \sum_{\mu=1}^3 \text{tr} (\psi_{\mu} \bar{\phi}), \quad Q \text{tr} (\bar{\psi}_4 \bar{\phi}), \quad Q \sum_{A=1}^3 \text{tr} (\chi_A A_4)\]
§5. 3D $\mathcal{N} = 8$ and 2D $\mathcal{N} = 8$

We construct a naive lattice action for four-dimensional $\mathcal{N} = 4$ SYM as

$$S_{4D,N=4}^{\text{LAT}} = Q_+ Q_- \frac{1}{2g_0^2} \sum_x \text{tr} \left[ -i \vec{B}(x) \cdot (\vec{\Phi}(x) + \Delta \vec{\Phi}(x)) - \frac{1}{3} \varepsilon_{ABC} B_A(x) [B_B(x), B_C(x)] - \sum_{\mu=1}^4 \psi_+^\mu(x) \psi_-^\mu(x) - \vec{\chi}_+^\mu(x) \cdot \vec{\chi}_-^\mu(x) - \frac{1}{4} \eta_+^\mu(x) \eta_-^\mu(x) \right]. \quad (5.1)$$

Here, the same problem of the surplus modes occurs.

Considering the dimensional reduction with respect to the fourth direction, we obtain a lattice model for three-dimensional $\mathcal{N} = 8$ SYM which reproduces the desired theory in the classical continuum limit. Also, further reduction with respect to the third direction leads two-dimensional $\mathcal{N} = 8$ theory.

The result of the renormalization argument is as follows. For the three-dimensional $\mathcal{N} = 8$ model, required is the one parameter fine-tuning for the operator of the mass dimension three:

$$Q_+ Q_- \text{tr} [(B_1 + B_2 + B_3) A_4],$$

while the two-dimensional model of $\mathcal{N} = 8$ needs no fine-tuning of parameters.

§6. Summary and Discussions

We have constructed various lattice models for SYM theories of $\mathcal{N} = 2, 4, 8$ in two-dimensions and of $\mathcal{N} = 4, 8$ in three-dimensions, based on ‘(balanced) topological field theory form’ of the theories. The formulation exactly realizes a part of the supersymmetry and employs compact link variables for the gauge fields on hypercubic lattice. From the renormalization argument, we have shown that the desired continuum theories are obtained by fine-tuning three and one parameters for the three-dimensional $\mathcal{N} = 4$ and 8 theories respectively, while the two-dimensional theories require no tunings.

We have also seen that there exist surplus modes in four-dimensional naive lattice models for $\mathcal{N} = 2, 4$. It may be related to exact realization of the topological term $\text{tr} F \wedge F$ on the lattice which needs a nonabelian extension of the solution for the U(1) case.\textsuperscript{19}

Finally, we mention some future directions related to this work.

• The models for $\mathcal{N} = 8$ in two- and three-dimensions constructed here could be used for nonperturbative investigation of matrix string models.\textsuperscript{20}

• It is interesting to do actual simulative studies utilizing the formulation presented here. As a first step, it would be necessary to investigate the positivity of the fermion determinant.

• It would be possible to couple the lattice SYM models to matter fields by referring topological QCD formulation.\textsuperscript{21}
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