Construction of a measure of noncompactness in Sobolev spaces with an application to functional integral-differential equations

Mahnaz Khanehgir1 · Reza Allahyari1 · Nayereh Gholamian2

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Abstract
In this paper, first we introduce a measure of noncompactness in the Sobolev space $W^{k,1}(\Omega)$ and then, as an application, we study the existence of solutions for a class of the functional integral-differential equations using Darbo’s fixed point theorem associated with this new measure of noncompactness. Further, two examples are presented to verify the effectiveness and applicability of our main results.

Keywords Darbo’s fixed point theorem · Integral-differential equation · Measure of noncompactness · Sobolev space

Mathematical Subject Classification 45J05 · 47H08 · 47H10

Introduction

Sobolev spaces [11], i.e., the class of functions with derivatives in $L^p$, play an outstanding role in the modern analysis. In the last decades, there has been increasing attempts to study these spaces. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces. They also highlighted in approximation theory, calculus of variation, differential geometry, spectral theory etc.

On the other hand, integral-differential equations (IDE) have a great deal of applications in some branches of sciences. It arises especially in a variety of models from applied mathematics, biological science, physics and another phenomenon, such as the theory of electrodynamics, electromagnetic, fluid dynamics, heat and oscillating magnetic, etc. [9, 12, 18, 21, 24]. There have appeared recently a number of interesting papers [2, 6, 10, 19, 22, 23, 27] on the solvability of various integral equations with help of measures of noncompactness.

The first such measure was defined by Kuratowski [25]. Next, Banas’ et al. [8] proposed a generalization of this notion which is more convenient in the applications. The technique of measures of noncompactness is frequently applicable in several branches of nonlinear analysis, in particular the technique turns out to be a very useful tool in the existence theory for several types of integral and integral-differential equations. Furthermore, they are often used in the functional equations, fractional partial differential equations, ordinary and partial differential equations, operator theory and optimal control theory [1, 3, 7, 13, 15–17, 26, 28, 29]. The most important application of measures of noncompactness in the fixed point theory is contained in the Darbo’s fixed point theorem [4, 5].

Now, in this paper, we introduce a new measure of noncompactness in the Sobolev space $W^{k,1}(\Omega)$ as a more effective approach. Then, we study the problem of existence of solutions of the functional integral-differential equation

$$u(x) = p(x) + q(x)u(x) + \int_{\Omega} k(x,y)g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy.$$ 

We provide some notations, definitions and auxiliary facts which will be needed further on.
Throughout this paper, $\mathbb{R}_+$ indicates the interval $[0, +\infty)$ and for the Lebesgue measurable subset $D$ of $\mathbb{R}$, $m(D)$ denotes the Lebesgue measure of $D$. Moreover, let $L^1(D)$ be the space of all Lebesgue integrable functions $f$ on $D$ equipped with the standard norm $\|f\|_{L^1(D)} = \int_D |f(x)|\,dx$.

Let $(E, \| \cdot \|)$ be a real Banach space with zero element 0. The symbol $B(x, r)$ stands for the closed ball centered at $x$ with radius $r$ and put $B^* = B(0, r)$. Denote by $\mathcal{M}_E$ the family of nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets of $E$. For a nonempty subset $X$ of $E$, the symbols $\overline{X}$ and $\text{Conv}X$ will denote the closure and the closed convex hull of $X$, respectively.

**Definition 1.1** [8] A mapping $\mu : \mathcal{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions:

1° The family $\ker \mu = \{ X \in \mathcal{M}_E \mid \mu(X) = 0 \}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(\text{Conv}X) = \mu(X)$.

5° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

6° If $\{ X_n \}$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n \to \infty} \mu(X_n) = 0$ then the set $X_{\infty} = \bigcap_{n=1}^{\infty} X_n$ is nonempty. A measure of noncompactness $\mu$ is said to be regular if it additionally satisfies the following conditions:

7° $\mu(X \cup Y) = \max\{ \mu(X), \mu(Y) \}$.

8° $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

9° $\mu(\lambda X) = |\lambda| \mu(X)$ for $\lambda \in \mathbb{R}$.

10° $\ker \mu = \mathcal{N}_E$.

In what follows, we recall the well known Darbo’s fixed point theorem.

**Theorem 1.2** [13] Let $\Omega$ be a nonempty, bounded, closed and convex subset of a Banach space $E$ and let $F : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property

$$\mu(FX) \leq k \mu(X),$$

(1)

for any nonempty subset $X$ of $\Omega$, where $\mu$ is a measure of noncompactness defined in $E$. Then, $F$ has a fixed point in the set $\Omega$.

**Construction of a measure of noncompactness in Sobolev spaces**

In this section, we introduce a measure of noncompactness in the Sobolev space $W^{k, 1}(\Omega)$.

Let $\Omega$ be a subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $W^{k, 1}(\Omega)$ the space of functions $f$ which, together with all their distributional derivatives $D^j f$ of order $|x| \leq k$, belong to $L^1(\Omega)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, i.e., each $\alpha_j$ is a nonnegative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}.$$

Then, $W^{k, 1}(\Omega)$ is equipped with the complete norm

$$\|f\|_{k, 1} = \max_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^1(\Omega)}.$$

We present the following theorem which characterizes the compact subsets of the Sobolev spaces.

**Theorem 2.1** [20] A subset $F \subset W^{k, 1}(\mathbb{R}^n)$ is totally bounded if, and only if, the following holds:

(i) $F$ is bounded, i.e., there is some $M$ so that

$$\int |D^\alpha f(x)|\,dx < M, \quad f \in F, \quad |x| \leq k.$$

(ii) For every $\varepsilon > 0$ there is some $R$ so that

$$\int_{|x| > R} |D^\alpha f(x)|\,dx < \varepsilon, \quad f \in F, \quad |x| \leq k.$$

(iii) For every $\varepsilon > 0$ there is some $\rho > 0$ so that

$$\int_{|y| < \rho} |D^\alpha f(x + y) - D^\alpha f(x)|\,dx < \varepsilon, \quad f \in F, \quad |x| \leq k, \quad \|y\|_{\mathbb{R}^n} < \rho.$$

Now, we are going to describe a measure of noncompactness in $W^{k, 1}(\Omega)$.

**Theorem 2.2** Suppose $1 \leq k < \infty$ and $U$ is a bounded subset of $W^{k, 1}(\Omega)$. For $u \in U, \varepsilon > 0$ and $0 \leq |x| \leq k$, let

$$\omega^T(u, \varepsilon) = \sup_{h \in H, \|h\|_{L^1(\Omega)} \leq \varepsilon} \left\{ \|T_h Du - D^\alpha u\|_{L^1(B_T)} : h \in H, \|h\|_{L^1(\Omega)} \leq \varepsilon \right\},$$

$$\omega^T(U, \varepsilon) = \sup_{u \in U} \left\{ \omega^T(u, \varepsilon) : u \in U \right\},$$

$$\omega^T(U) = \lim_{\varepsilon \to 0} \omega^T(U, \varepsilon),$$

$$d(U) = \lim_{T \to \infty} \sup_{u \in U} \{ \|D^\alpha u\|_{L^1(\Omega)} : u \in U, 0 \leq |x| \leq k \},$$

where $B_T = \{ a \in \Omega : \|a\|_{L^1(\Omega)} \leq T \}$ and $T_h u(t) = u(t + h)$. Then $\omega_0 : \mathcal{M}_E \to \mathbb{R}$ given by

$$\omega_0(U) = \omega(U) + d(U)$$

(2)

defines a measure of noncompactness in $W^{k, 1}(\Omega)$.

**Proof** Take $U \in \mathcal{M}_E$ such that $\omega_0(U) = 0$. Fix arbitrary $\varepsilon$ such that $0 \leq |x| \leq k$. Let $\eta > 0$ be arbitrary, since $\omega_0(U) = 0$, ...
Thus, there exists small enough $\delta > 0$ and large enough $T > 0$ such that $\omega^T(U, \delta) < \eta$. This implies that

$$\| T_h D^2 u - D^2 u \|_{L^1(B_t)} < \eta$$

for all $u \in U$ and $h \in \Omega$ such that $\|h\|_{X^e} < \delta$. Since $\eta > 0$ was arbitrary, we obtain

$$\lim_{h \to 0} \| T_h D^2 u - D^2 u \|_{L^1(\Omega)} = 0$$

Using again the fact that $\omega_0(U) = 0$ we have

$$\lim_{T \to \infty} \sup \{ \| D^2 u \|_{L^1(\Omega; B_t)} : u \in U \} = 0,$$

and for $\varepsilon > 0$ there exists large enough $T > 0$ such that $\| D^2 u \|_{L^1(\Omega; B_t)} < \varepsilon$ for all $u \in U$.

It follows then from Theorem 2.1 that $U$ is totally bounded. Thus, 1° holds.

2° is obvious by the definition of $\omega_0$.

Now, we check that condition 3° holds. For this purpose, suppose that $U \in \mathfrak{U}_{\text{aff}}(\Omega)$ and $\{ u_n \} \subset U$ such that $u_n \to u$ in $U$ in $W^{k,1}(\Omega)$. From the definition of $\omega^T(U, \varepsilon)$, we have

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)} \leq \omega^T(U, \varepsilon),$$

for any $n \in \mathbb{N}$, $T > 0$ and $h \in \Omega$ with $\|h\|_{X^e} < \varepsilon$. Letting $n \to \infty$, we get

$$\| T_h D^2 u - D^2 u \|_{L^1(B_t)} \leq \omega^T(U, \varepsilon),$$

for any $T > 0$ and $h \in \Omega$ with $\|h\|_{X^e} < \varepsilon$. Hence

$$\lim_{T \to \infty, \varepsilon \to 0} \| \omega^T(U, \varepsilon) \|_{L^1(\Omega; B_t)} \leq \lim_{T \to \infty, \varepsilon \to 0} \| \omega^T(U, \varepsilon) \|_{L^1(\Omega; B_t)}.$$

This concludes that $\omega(U) \leq \omega_0(U)$. Similarly, we can show that

$$d(\overline{U}) \leq d(U),$$

and thus

$$\omega_0(\overline{U}) \leq \omega_0(U).$$

From (3) and 2° we obtain $\omega_0(\overline{U}) = \omega_0(U)$.

4° follows directly from $D^2[\text{Conv}(U)] = \text{Conv}(D^2 U)$ and hence is omitted.

The proof of condition 5° can be obtained by using the equality

$$D^2(\lambda u_1 + (1 - \lambda) u_2) = \lambda D^2 u_1 + (1 - \lambda) D^2 u_2,$$

for all $\lambda \in [0, 1], u_1 \in X$ and $u_2 \in Y$.

It remains only to verify 6°, suppose that $\{ U_n \}$ is a sequence of closed and nonempty sets of $\mathfrak{U}_{\text{aff}}(\Omega)$ such that $U_{n+1} \subset U_n$ for $n = 1, 2, \ldots$, and $\lim_{n \to \infty} \omega_0(U_n) = 0$. Now, for any $n \in \mathbb{N}$, take $u_n \in U_n$ and set $G = \{ u_n \}$.

Claim: $G$ is a compact set in $W^{k,1}(\Omega)$.

Let $\varepsilon > 0$ be fixed, since $\lim_{n \to \infty} \omega_0(U_n) = 0$, there exists sufficiently large $m_1 \in \mathbb{N}$ such that $\omega_0(U_{m_1}) < \varepsilon$. Hence, we can find small enough $\delta_1 > 0$ and large enough $T_1 > 0$ such that $\omega^T(U_{m_1}, \delta_1) < \varepsilon$ and $d(U_{m_1}) < \varepsilon$. Therefore,

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)} < \varepsilon,$$

and

$$\| D^2 u_n \|_{L^1(\Omega; B_t)} < \varepsilon,$$

for all $n > m_1, 0 \leq |x| \leq k$ and $h \in \Omega$ with $\|h\|_{X^e} < \delta_1$. Thus we have

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega)} \leq \| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)} + \| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)}$$

$$\leq \| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)} + \| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)}$$

$$+ \| D^2 u_n \|_{L^1(\Omega; B_t)} < 3\varepsilon.$$

On the other hand, we know that the set $\{ u_1, u_2, \ldots, u_{m_1} \}$ is compact, hence there exist $\delta_2 > 0$ and $T_2 > 0$ such that

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega; B_t)} < \varepsilon,$$

for all $n = 1, 2, \ldots, m_1$, $0 \leq |x| \leq k$ and $h \in \Omega$ with $\|h\|_{X^e} < \delta_2$. Furthermore,

$$\| D^2 u_n \|_{L^1(\Omega; B_t)} < \varepsilon,$$

which implies that

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega)} < 3\varepsilon,$$

for all $n = 1, 2, \ldots, m_1$. Thus,

$$\| T_h D^2 u_n - D^2 u_n \|_{L^1(\Omega)} < 3\varepsilon,$$

and

$$\| D^2 u_n \|_{L^1(\Omega; B_t)} < \varepsilon < 3\varepsilon,$$

for all $n \in \mathbb{N}$, $\|h\|_{X^e} < \min\{\delta_1, \delta_2\}$ and $T = \max\{T_1, T_2\}$. Therefore, all the hypotheses of Theorem 2.1 are satisfied, that completes the proof of the claim.

Using the above claim, there exists a subsequence $\{ u_{n_k} \}$ and $u_0 \in W^{k,1}(\Omega)$ such that $u_{n_k} \to u_0$. Since $u_n \in U_n$, $U_{n+1} \subset U_n$ and $U_n$ is closed for all $n \in \mathbb{N}$, we yield
that finishes the proof of 6°.

We now investigate the regularity of $\omega_0$.

**Theorem 2.3**  The measure of noncompactness $\omega_0$ defined in (2) is regular.  

**Proof**  Suppose that $X,Y \in \mathfrak{M}_{\mathbb{R}^0}(\Omega)$. First, notice that for all $\epsilon > 0$, if $\lambda \in \mathbb{R}$ and $T > 0$ we have

$$\alpha^T(X \cup Y, \epsilon) = \max\{\alpha^T(X, \epsilon), \alpha^T(Y, \epsilon)\},$$

$$\alpha^T(X+Y, \epsilon) \leq \alpha^T(X, \epsilon) + \alpha^T(Y, \epsilon),$$

$$\alpha^T(\lambda X, \epsilon) = |\lambda| \alpha^T(X, \epsilon),$$

$$\sup_{uv \in X+Y} \|D^2u\|_{L^1(\Omega;B_1)} \leq \sup_{uv \in X} \|D^2u\|_{L^1(\Omega;B_1)} + \sup_{uv \in Y} \|D^2u\|_{L^1(\Omega;B_1)}$$

Then, the hypotheses 7°–9° hold. Next, we show that 10° holds. Take $U \in \mathfrak{M}_{\mathbb{R}^0}(\Omega)$. Thus, the closure of $U$ in $W^{k,1}(\Omega)$ is compact. By Theorem 2.1, for all $|x| \leq k$ and for all $\epsilon > 0$, there exists $T > 0$ such that $\|D^2u\|_{L^1(\Omega;B_1)} < \epsilon$ for all $u \in U$, and there exists $\delta > 0$ such that $\|D^2u\|_{L^1(\Omega;B_1)} < \epsilon$ for all $h \in \Omega$ with $\|h\|_{\mathbb{R}^0} < \delta$. Then, for all $u \in U$ we have

$$\alpha^T(u, \delta) = \sup \{\|D^2u\|_{L^1(\Omega;B_1)} : h \in \Omega, \quad \|h\|_{\mathbb{R}^0} < \delta\} \leq \epsilon.$$

Therefore,

$$\alpha^T(U, \delta) = \sup \{\|D^2u\|_{L^1(\Omega;B_1)} : u \in U\} = 0.$$

It yields that

$$\omega(U) = \lim_{T \to \infty} \lim_{\delta \to 0} \alpha^T(U, \delta) = 0.$$

Furthermore,

$$d(U) = \lim_{T \to \infty} \sup \{\|D^2u\|_{L^1(\Omega;B_1)} : u \in U\} = 0.$$
continuous and nondecreasing function \( \zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that
\[
|g(x, u_0, u_1, \ldots, u_{n+1})| \leq a(x)\zeta\left(\max_{0 \leq i \leq n+1} |u_i|\right).
\] (8)

(iii) \( k : \Omega \times \Omega \rightarrow \mathbb{R} \) satisfies the Carathéodory conditions and has a derivative of order 1 with respect to the first argument. Moreover, there exist \( g_1, g_2 \in W^{1,1}(\Omega) \) and \( g_2 \in L^\infty(\Omega) \) such that
\[
|k(x, y)| \leq g_1(x)g_2(y), \quad |k(x_1, y) - k(x_2, y)| \leq g_2(y)|g'(x_1) - g'(x_2)|,
\]
and
\[
\left| \frac{\partial k}{\partial x_i}(x, y) \right| \leq g_1(x)g_2(y), \quad \left| \frac{\partial k}{\partial y_i}(x_1, y) - \frac{\partial k}{\partial y_i}(x_2, y) \right| \leq g_2(y)|g'(x_1) - g'(x_2)|,
\]
for almost \( x, y, x_1, x_2 \in \Omega \) and \( 1 \leq i \leq n \).

(iv) There exists a positive solution \( r_0 \) of the inequality
\[
\left\| \frac{1}{m(\Omega)} \right\| ||u||_{1,1} \leq r.
\] (9)

(v) \( T : W^{1,1}(\Omega) \rightarrow L^1(\Omega) \) is a continuous operator such that for any \( x \in W^{1,1}(\Omega) \) we have
\[
\|T(x)\|_{L^1(\Omega)} \leq \|x\|_{1,1}.
\]

Then, the functional integral-differential equation
\[
u(x) = p(x) + g(x)u(x) + \int_\Omega k(x, y)g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy
\] (10)
has at least one solution in the space \( W^{1,1}(\Omega) \).

**Proof** We define the operator \( F : W^{1,1}(\Omega) \rightarrow W^{1,1}(\Omega) \) by
\[
F(u)(x) = p(x) + g(x)u(x) + \int_\Omega k(x, y)g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy.
\]

Obviously, \( F(u) \) is measurable for any \( u \in W^{1,1}(\Omega) \). Also, for any \( x \in \Omega \) we have
\[
\frac{\partial (F(u))}{\partial x_i}(x) = \frac{\partial p}{\partial x_i}(x) + \frac{\partial g}{\partial x_i}(x)u(x) + g(x)\frac{\partial u}{\partial x_i}(x)
\]
\[
+ \int_\Omega \frac{\partial k}{\partial x_i}(x, y)g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy
\]
and \( Fu \) has measurable derivatives. We show that, \( Fu \in W^{1,1}(\Omega) \). Using our assumptions, for arbitrarily fixed \( x \in \Omega \), we have
\[
|Fu(x)| \leq |p(x)| + |g(x)||u(x)| + \int_\Omega |k(x, y)|g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy.
\]

According to the Jensen’s inequality, we deduce
\[
]\|Fu\|_{L^1(\Omega)} \leq \|p\|_{L^1(\Omega)} + \|g\|_u\|u\|_{L^1(\Omega)} + Mm(\Omega)\|g_1\|_{L^1(\Omega)} \|g_2\|_{L^\infty}^{-1}\zeta\left(\frac{1}{m(\Omega)}\right)\|u\|_{1,1}.
\]

By the same argument as above,
\[
\left| \frac{\partial (F(u))}{\partial x_i}(x) \right| \leq \left| \frac{\partial p}{\partial x_i}(x) \right| + \left| \frac{\partial g}{\partial x_i}(x) \right| |u(x)| + \left| \frac{\partial u}{\partial x_i}(x) \right| \left| \int_\Omega \frac{\partial k}{\partial x_i}(x, y)g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y))dy\right|
\]
and
\[
\left\| \frac{\partial (F(u))}{\partial x_i}\right\|_{L^1(\Omega)} \leq \left\| \frac{\partial p}{\partial x_i}(x) \right\|_{L^1(\Omega)} + \left\| \frac{\partial g}{\partial x_i}(x) \right\|_u\|u\|_{L^1(\Omega)} + \left\| \frac{\partial u}{\partial x_i}(x) \right\|_{L^1(\Omega)} + \left\| g_1\right\|_{L^1(\Omega)} \|g_2\|_{L^\infty}^{-1}\zeta\left(\frac{1}{m(\Omega)}\right)\|u\|_{1,1}.
\]

Thus, we obtain
\[
\|Fu\|_{1,1} \leq \|p\|_{1,1} + L\|u\|_{1,1} + Mm(\Omega)\|g_1\|_{L^1(\Omega)} \|g_2\|_{L^\infty}^{-1}\zeta\left(\frac{1}{m(\Omega)}\right)\|u\|_{1,1}.
\] (11)

Due to (11) and using condition (iv), we derive that \( F \) is a mapping from \( B_{r_0} \) into \( B_{r_0} \). Now, we show that the map \( F \) is continuous. Let \( \{u_m\} \) be an arbitrary sequence in \( W^{1,1}(\Omega) \) which converges to \( u \in W^{1,1}(\Omega) \). By Lemma 3.1 there is a subsequence \( \{u_{m_k}\} \) which converges to \( u \) a.e., \( \left\| \frac{\partial u_{m_k}}{\partial x_1}\right\| \) converges to \( \left\| \frac{\partial u}{\partial x_1}\right\| \) a.e., \( \{Tu_{m_k}\} \) converges to \( Tu \) a.e. and there is \( h \in L^1(\Omega) \), \( h \geq 0 \), such that
\[
\max\{|u_m(y)|, \left| \frac{\partial u_{m_k}}{\partial x_1}(y) \right|, \left| \frac{\partial u_{m_k}}{\partial x_2}(y) \right|, \ldots, |Tu_{m_k}(y)| \} \leq h(y)
\]
for a.e. \( y \in \Omega \).

Since \( u_{m_k} \rightarrow u \) almost everywhere and \( g \) satisfies the Carathéodory conditions, it follows that.
g(y, u_m(y), \frac{\partial u_m}{\partial x_1}(y), \ldots, Tu_m(y)) \rightarrow g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, Tu(y)),$

for almost all \( y \in \Omega \).

From condition (ii) we have

\[ g(y, u_m(y), \frac{\partial u_m}{\partial x_1}(y), \ldots, Tu_m(y)) \leq a(y)\zeta(h(y)) \quad \text{for a.e. } y \in \Omega. \]

As a consequence of the Lebesgue’s Dominated Convergence Theorem, it yields that

\[
\int g(y, u_m(y), \frac{\partial u_m}{\partial x_1}(y), \ldots, Tu_m(y)) \, dy \rightarrow \int g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, Tu(y)) \, dy,
\]

for almost all \( y \in \Omega \). Inequality (12) and condition (iii) imply that

\[
\| Fu_m - Fu \|_{L^1(\Omega)} \rightarrow 0 \quad \text{and} \quad \left\| \frac{\partial Fu_m}{\partial x_i} - \frac{\partial Fu}{\partial x_i} \right\|_{L^1(\Omega)} \rightarrow 0 \quad \text{as} \quad k \to \infty \quad (1 \leq i \leq n).
\]

Therefore, \( F : W^{1,1}(\Omega) \to W^{1,1}(\Omega) \) is continuous.

To finish, the proof we have to verify that condition (1) is satisfied. We fix arbitrary \( T \) and \( \varepsilon > 0 \). Let \( U \) be a nonempty and bounded subset of \( B_m \). Choose \( u \in U \) and \( x, h \in B_T \) with \( \| h \|_{L^\infty} \leq \varepsilon \), then we have

\[
\int_{B_T} | Fu(x) - Fu(x + h) | \, dx
\]

\[
\leq \int_{B_T} | p(x) - p(x + h) | \, dx
\]

\[
+ \int_{B_T} | q(x) - q(x + h) | \| u(x) \| \, dx
\]

\[
+ \int_{B_T} | q(x + h) | \| u(x) - u(x + h) \| \, dx
\]

\[
+ \int_{B_T} \int_{\Omega} | k(x, y) - k(x + h, y) | \| g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)) | \, dy \, dx
\]

\[
\leq \omega^T(p, \varepsilon) + \int_{B_T} | q(x) - q(x + h) | \| u(x) \| \, dx + \lambda \omega^T(U, \varepsilon)
\]

\[
+ Mm(\Omega) \| g_2 \|_{L^\infty} \zeta \left( \frac{1}{m(\Omega)} \| u \|_{L^1(\Omega)} \right) \omega^T(g_3, \varepsilon).
\]

(13)

Obviously, \( \omega^T(p, \varepsilon) \rightarrow 0 \), \( \omega^T(g_3, \varepsilon) \rightarrow 0 \) and by continuity of \( q \),

\[
\int_{B_T} | q(x) - q(x + h) | \| u(x) \| \, dx \rightarrow 0,
\]

as \( \varepsilon \to 0 \). Then the right hand side of (13) tends to \( \lambda \omega^T(U) \) as \( \varepsilon \to 0 \).

By a similar argument and using condition (i), for each \( i = 1, \ldots, n \), we get

\[
\int_{B_T} \frac{\partial (Fu)}{\partial x_i}(x) - \frac{\partial (Fu)}{\partial x_i}(x + h) \, dx
\]

\[
\leq \int_{B_T} \frac{\partial q}{\partial x_i}(x) - \frac{\partial q}{\partial x_i}(x + h) \, dx + \int_{B_T} \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_i}(x + h) \, dx
\]

\[
+ \int_{B_T} \frac{\partial (Fu)}{\partial x_i}(x) - \frac{\partial (Fu)}{\partial x_i}(x + h) \, dx
\]

\[
+ \int_{B_T} \left( \frac{\partial k}{\partial x_i}(x, y) - \frac{\partial k}{\partial x_i}(x + h, y) \right) \| g(y, u(y), \frac{\partial u}{\partial x_1}(y), \ldots, \frac{\partial u}{\partial x_n}(y), Tu(y)) \| \, dy \, dx
\]

\[
\leq \omega^T(p, \varepsilon) + \int_{B_T} \left| \frac{\partial q}{\partial x_i}(x) - \frac{\partial q}{\partial x_i}(x + h) \right| \| u(x) \| \, dx
\]

\[
+ \lambda \omega^T(U, \varepsilon)
\]

\[
+ Mm(\Omega) \| g_2 \|_{L^\infty} \zeta \left( \frac{1}{m(\Omega)} \| u \|_{L^1(\Omega)} \right) \omega^T(g_3, \varepsilon).
\]

(14)

Applying the same reasoning as above, the right hand side of (14) tends to \( \lambda \omega^T(U) \) as \( \varepsilon \to 0 \), too. Regarding to (13) and (14), and since \( u \) is an arbitrary element of \( U \), then \( \omega^T(FU) \leq \lambda \omega^T(U) \). Letting \( T \to \infty \), we deduce

\[
\omega(FU) \leq \lambda \omega(U).
\]

(15)

Next, let us fix an arbitrary number \( T > 0 \). Then, taking into account our hypotheses, for an arbitrary function \( u \in U \) we derive

\[
\| Fu \|_{L^1(\Omega, \mathbb{R}^n)} \leq \| p \|_{L^1(\Omega, \mathbb{R}^n)} + \| q \|_{L^1(\Omega, \mathbb{R}^n)} + Mm(\Omega) \| g_1 \|_{L^1(\Omega, \mathbb{R}^n)} \| g_2 \|_{L^\infty(\Omega, \mathbb{R}^n)}
\]

\[
\zeta \left( \frac{1}{m(\Omega)} \| u \|_{L^1(\Omega)} \right)
\]

\[
\leq \lambda d(U).
\]

(16)

Now, since

\[
\| p \|_{L^1(\Omega, \mathbb{R}^n)} \to 0, \quad \| g_1 \|_{L^1(\Omega, \mathbb{R}^n)} \to 0 \quad \text{as} \quad T \to \infty,
\]

then

\[
\lim_{T \to \infty} \| Fu \|_{L^1(\Omega, \mathbb{R}^n)} \leq \lambda d(U).
\]

Similarly,
\[ \lim_{t \to -\infty} \left\{ \left\| \frac{\partial(Fu)}{\partial x_i} \right\|_{L^1(\Omega, \mathbb{R})} : i = 1, \ldots, n \right\} \leq \lambda d(U). \]

These relations imply that
\[ d(FU) \leq \lambda d(U). \]  
\[ (16) \]

Finally, from (15) and (16) we conclude that \( \omega_0(FU) \leq 2\omega_0(U) \).

According to Theorem 1.2, we obtain that the operator \( F \) has a fixed point \( x \) in \( B_\theta \), and thus functional integral-differential equation (10) has at least one solution in the space \( W^{1,1}(\Omega) \).

Now, we present two examples which verify the effectiveness and applicability of Theorem 3.3.

**Example 3.4** Consider the following functional integral-differential equation
\[
\begin{align*}
  u(x_1, x_2, x_3) &= \sqrt[3]{x_3} + e^{-(x_1+x_2+x_3)} u(x_1, x_2, x_3) \\
  &+ \int_0^1 \int_0^1 \int_0^1 \frac{e^{-(x_1+x_2+x_3)}}{(y_1 + 1)^3(y_2 + 2)^2(y_3 + 5)} \\
  &\times \cos(u_1 y_1 u_2 y_2 x_3) + \frac{\partial u}{\partial x_1}(y_1, y_2, y_3) + \frac{\partial u}{\partial x_2}(y_1, y_2, y_3) \\
  &+ \frac{\partial u}{\partial x_3}(y_1, y_2, y_3) \frac{1}{2} u(y_1, y_2, y_3) dy_1 dy_2 dy_3.
\end{align*}
\]  
\[ (17) \]

Eq. (17) is a special case of Eq. (10) with
\[ \Omega = [0, 1] \times [0, 1] \times [0, 1], p(x_1, x_2, x_3) = \sqrt[3]{x_3}, \]
\[ q(x_1, x_2, x_3) = e^{-(x_1+x_2+x_3)} + \cos(y_1 u_0 u_1 + y_2 u_2 + y_3 u_3 + u_4), T u = \frac{1}{2} u, \]
\[ k(x_1, x_2, x_3, y_1, y_2, y_3) = e^{-(y_1+y_2+y_3+1)} \\
  \times \frac{1}{(y_1 + 1)^3(y_2 + 2)^2(y_3 + 5)}, \]
\[ g_1(x_1, x_2, x_3) = g_3(x_1, x_2, x_3) = e^{-(x_1+x_2+x_3+1)}, \]
\[ g_2(x_1, x_2, x_3) = \frac{1}{(y_1 + 1)^2(y_2 + 2)(y_3 + 5)}. \]

It is easy to see that \( p \in W^{1,1}(\Omega) \), \( q \in BC^1(\Omega) \) and \( \lambda = 2e^{-1} \). Also, \( g \) satisfies Carathéodory conditions and if we define \( a(x_1, x_2, x_3) = \xi(x) = 1 \), then condition (ii) of Theorem 3.3 holds. We observe that \( g_1, g_3 \in L^1(\Omega), g_2 \in L^\infty(\Omega) \) and \( k \) satisfies condition (iii). Moreover, it can be easily shown that each number \( r \geq 4 \) satisfies the inequality in condition (iv), i.e.,
\[ \|p\|_{L_1, 1} + \lambda r + M\|g_1\|_{L^1(\Omega)} \|g_2\|_{L^\infty(\Omega)} \leq 1 + 2e^{-1}r \]
\[ + \frac{(1 - e^{-1})^3}{10} \leq r. \]

Thus, as the number \( r_0 \) we can take \( r_0 = 4 \). Consequently, all the conditions of Theorem 3.3 are satisfied. Hence the functional integral-differential equation (17) has at least one solution in the space \( W^{1,1}(\Omega) \).

**Example 3.5** Consider the following functional integral-differential equation
\[
u(x) = \frac{u(x)}{x + 2} + \int_0^1 \frac{\sqrt[3]{y^3 u(y) + 3u^2(y) + u^3(y)}}{1 + u^2(y) e^{|u^2(y)+1|}} dy.
\]  
\[ (18) \]

Eq. (18) is a special case of Eq. (10) with
\[ \rho(x) = 0, q(x) = \frac{1}{x + 2}, k(x, y) = e^{x-y}, T(u) = 0, \Omega = [0, 1] \]
and
\[ g(y, u_0, u_1, u_2, u_3, u_4) = \frac{\sqrt[3]{y^3 u_0 + 3u_2 + u_4}}{1 + u_2^2 e^{|u_2^2+1|}}.T(u) = 0 \]

It is easy to see that \( q \in BC^1(\Omega) \) and \( \lambda = \frac{3}{4} \). Also, \( g \) satisfies Carathéodory conditions and if we define \( a(x) = \sqrt{5} \) and \( \xi(x) = \sqrt{x} \), then condition (ii) of Theorem 3.3 holds. Moreover, \( k \) is continuous and has a continuous derivative of order 1 with respect to the first argument. On the other hand, \( g_1(x) = g_3(x) = e^x \) and \( g_2(x) = e^{-x} \). It can be easily shown that each number \( r \geq 10 \) satisfies the inequality in condition (iv), i.e.,
\[ \|p\|_{L_1, 1} + \lambda r + M\|g_1\|_{L^1(\Omega)} \|g_2\|_{L^\infty(\Omega)} \leq \frac{3}{4}r + \sqrt{5}(e - 1) \]
\[ (1 - e^{-1}) \sqrt{r} \leq r. \]

Hence, as the number \( r_0 \) we can take \( r_0 = 10 \). Consequently, all the conditions of Theorem 3.3 are satisfied. It implies that the functional integral-differential equation (18) has at least one solution in the space \( W^{1,1}(\Omega) \).

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