Azumaya structure on D-branes
and resolution of ADE orbifold singularities revisited:
Douglas-Moore vs. Polchinski-Grothendieck

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Abstract
In this continuation of [L-Y1] and [L-L-S-Y], we explain how the Azumaya structure on D-branes together with a netted categorical quotient construction produces the same resolution of ADE orbifold singularities as that arises as the vacuum manifold/varieties of the supersymmetric quantum field theory on the D-brane probe world-volume, given by Douglas and Moore [D-M] under the string-theory contents and constructed earlier through hyper-Kähler quotients by Kronheimer and Nakajima. This is consistent with the moral behind this project that Azumaya-type structure on D-branes themselves – stated as the Polchinski-Grothendieck Ansatz in [L-Y1] – gives a mathematical reason for many originally-open-string-induced properties of D-branes.

Key words: D-brane probe, Polchinski-Grothendieck Ansatz, Azumaya scheme, morphism, orbifold, resolution of singularity, ADE orbifold singularity, moduli stack, netted categorical quotient.

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In memory of Professor Sidney Coleman, 1937 - 2007.

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†Special tribute from C.-H.L. I was very lucky to get the chance to attend the one-year quantum field theory course, Physics 253a and Physics 253b, given by Prof. Coleman, fall 2000 - spring 2001 at Harvard, after having studied his book, “Aspects of Symmetry”, and heard of his legendary QFT lectures/course from Jacques Distler and Brian Greene on various occasions. The lecture notes of the whole year (705 pp.) are very impressive and the style of giving stories/anecdotes from his reflections/witness of the development of quantum field theory and particle physics up to 1970s along with the lectures is very unique. I still remember the discussion with him, when he lectured on the asymptotic freedom of nonabelian gauge field theories near the end of that academic year, on Wilson’s theory-space and renormalization group flow. These latter two notions, together with rigidity of supersymmetry, are very fundamental to understanding stringy dualities and should be regarded as providing a master moduli space that unifies the various moduli spaces in mathematics. They still await mathematicians to unravel.
0. Introduction and outline.

In this continuation of [L-Y1] and [L-L-S-Y], we explain how the Azumaya structure on D-branes together with a netted categorical quotient construction produces the same resolution of ADE orbifold singularities as that arises as the vacuum manifold/variety of the open-string-induced supersymmetric quantum field theory on the D-brane probe world-volume, given by Douglas and Moore [D-M] under the string-theory contents and constructed earlier through hyper-Kähler quotients by Kronheimer and Nakajima ([Kr], [K-N], [Na]).

Azumaya-type structure on D-branes.

([Po2: vol. I: Sec. 8.7; vol. II: Chap. 13], [L-Y1: Sec. 1, Sec. 2], and [L-L-S-Y: Sec. 1].) Originally, a D-brane in string theory is by definition an embedding of a manifold/variety/cycle in the space-time that serves as a boundary condition for open-strings moving in the space-time. This operational definition allows string-theorists to deduce properties of, quantum field theories on, and dynamics of D-branes. When a few D-branes are stacked together, open-string dictates that a certain noncommutative geometry emerges. This noncommutativity feature when viewed from Grothendieck’s construction/notion of a “geometry” and their morphisms (cf. [Ha]), taking into account the fact that unital associative rings are more natural to do geometries from local to global via gluing, says that:

Polchinski-Grothendieck Ansatz [D-brane: noncommutativity] 1 ([L-Y1: Sec. 2.2].) A D-brane (or D-brane world-volume) X carries an Azumaya-type noncommutative structure locally associated to a function ring of the form \( M_r(R) \) for some \( r \in \mathbb{Z}_{\geq 1} \) and (possibly noncommutative) ring \( R \). Here, \( M_r(R) \) is the \( r \times r \) matrix-ring over \( R \).

Based on this ansatz, a D-brane in a space(-time) \( Y \) in the sense of Polchinski and others in the decade 1986-1995 can be rephrased prototypically as a morphism \( \varphi : (X^{Az}, \mathcal{E}) \to Y \) from an Azumaya manifold/scheme \( X^{Az} \) with a fundamental module \( \mathcal{E} \) to \( Y \), cf. [L-Y1: Sec. 1.1, Sec. 1.2, Definition 2.2.3] and [L-L-S-Y: Sec. 1, Sec. 2.1, Definition 2.1.2].

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1 For string-theorists concerning the naming of this ansatz: This ansatz is nothing more than a rephrasing of a few related paragraphs in Polchinski’s lecture notes or textbooks ([Po2: vol. I, Sec. 8.7]) in terms of Grothendieck’s aspect of local geometries versus coordinate rings, cf. [Ha] and [L-Y1: Sec. 2.2]. Together with the fact that Polchinski was an early pioneer on D-branes and has made a special contribution to the understanding of the role of D-branes in string theories [Po1], we decided to use this name from the very start of the project. A thought came to us after [L-Y1] as whether this is the best name to reflect the content of this ansatz. Note that this ansatz has nothing to do with supersymmetry. It reflects actually only the nature of the enhancement of open-string-induced massless spectrum on stacked D-branes, particularly the scalar fields thereupon that describe deformations of D-branes in space-time. This stringy feature on D-branes is in turn a reflection of the fact that the tension of a fundamental string is a constant in nature and, hence, in particular the mass of a fundamental open string and of the spectrum it creates on D-branes are proportional to its length. Thus, after seeking the advice of a string theorist - who himself is also a figure on the study of branes before 1995 and who agreed with our naming -, we fixed on our original name: one string-theorist vs. one mathematician, with each giving a revolutionary change of the landscape of their respective field. This naming also takes into account that, since 1995, the realization of D-“branes” has broadened considerably and this ansatz addresses only the region of the related Wilson’s theory-space from string theory where a D-brane remains a brane. See [L-Y1] for more comments. Special thanks to Lubos Motl for comments that came to my attention accidentally, which propelled me to re-think about this naming while preparing this manuscript and, hence, led to this footnote. — Sincerely, C.-H.L. 2008.12.21.
Deformations of such morphisms reproduce the pattern of gauge-Higgsing/gauge-un-Higgsing behaviors of D-branes ([L-Y1: Remark 3.2.4, Sec. 4], [L-L-S-Y: Remark 2.1.7, Figure 2-1-1], and [Liu1]) and the moduli space $\mathcal{M}^{D\text{-brane}}(Y)$ of such morphisms reveals a feature as a master moduli space that simultaneously incorporates several standard moduli spaces in commutative algebraic geometry ([L-Y1: Sec. 4], [L-L-S-Y: Sec. 4], and [Liu2]). This latter feature is consistent with the very robust and versatile nature/role of D-branes to serve as a medium/broker/catalyst for various stringy dualities since Polchinski’s work [Po1] in year 1995.

In this sequel to [L-Y1] and [L-L-S-Y], we proceed to justify the third known feature of D-branes, namely the ability to resolve singularities of the target space(-time), along the line of the Polchinski-Grothendieck Ansatz.

D-brane probe resolution of ADE orbifold singularities à la Douglas and Moore.

In the work [D-M] of Douglas and Moore\(^2\), the $d = 6, N = 1$ supersymmetric effective field theory (SQFT) on the D5-branes world-volume $X$ that is embedded in the product space-time $\mathbb{M}^{5+1} \times (\mathbb{C}^2/\mathbb{Z}_r)$ as $\mathbb{M}^{5+1} \times \{0\}$ is studied in detail. Here $\mathbb{M}^{5+1}$ is the 6-dimensional Minkowski space-time and $\mathbb{C}^2/\mathbb{Z}_r$ is the quotient of $\mathbb{C}^2$ by a $\mathbb{Z}_r$-action via a group-embedding $\mathbb{Z}_r \hookrightarrow SU(2)$. The massless multiplets of this $d = 6, N = 1$ SQFT on the brane consists of:

1. [closed-superstring-induced sector] those from the Kaluza-Klein compactification/reduction of a chosen $d = 10$ superstring theory on the internal $\mathbb{C}^2/\mathbb{Z}_r$, identifying the embedded D5-brane world-volume with the $d = 6$ effective space-time,

2. [open-superstring-induced sector] those from open strings with one or both end-points attached to the D-brane.

The twisted sectors from orbifolding via the $\mathbb{Z}_r$-action are taken into account. In particular, the scalar fields that describe the deformations of this D-brane, sitting at the orbifold singularity, are contained in the hypermultiplets of the theory. The combinatorial type of this field content on the D-brane world-volume is coded in a quiver diagram.

The Lagrangian for these supermultiplets that governs the low-energy dynamics of the D-brane includes a Born-Infeld term, a term for hypermultiplets, a Chern-Simons term, and the SUSY completion of these terms. The space of vacua for this D-brane - computed from

- the potential for the hypermultiplets,

- a path-integral manipulation to integrate out the auxiliary D-fields of the vector multiplets in the Fayet-Iliopoulos terms,

- a condensation of the scalar fields from the NS-NS sector in the Kaluza-Klein compactification

- gives a resolution of the singular space $\mathbb{C}^2/\mathbb{Z}_r$ for a generic choice of the vacuum expectation value (vev) for the condensation. In other words, the geometry of the inner space $\mathbb{C}^2/\mathbb{Z}_r$ at an ultrashort distance in string theory as seen by this D5-brane, transverse to $\mathbb{C}^2/\mathbb{Z}_r$ at the orbifold singularity, can be different from the original $\mathbb{C}^2/\mathbb{Z}_r$ to begin with. In particular, this

\(^2\) Readers are referred to the references of [D-M], [J-M], [D-G-M], [G-L-R] for more literatures on related stringy works and notions that influence the development. Our work proceeds specifically with [D-M] in mind as the inner/compactified part of their setting is a D0-brane moving on an orbifold. The latter is the case we will study.
geometry seen by the D-brane can be smooth. This gives rise to the phenomenon of *D-brane probe resolution of singularities* of a space.

A generalization of [D-M] to all ADE orbifolds \( \mathbb{C}^2/\Gamma \) is given later in the work [J-M] of Johnson and Myers from a slightly different D-brane aspect but with similar conclusion on the D-brane probe resolution of ADE orbifold singularities. Further studies along this line, e.g. [D-G-M] and [G-L-R], were made for other spaces with orbifold singularities.

**Geometric-invariant-theory (GIT) quotient vs. netted categorical quotient.**

The construction of the vacuum manifold/variety in Douglas-Moore [D-M] matches with the hyper-Kähler quotient construction of Kronheimer and Nakajima ([Kr], [K-N]; see also [Na]). In the algebrao-geometric setting, this is closely related to Mumford’s geometric-invariant-theory (GIT) quotient construction ([M-F-K]).

In general, for an algebraic group \( G \) acting on a scheme \( Z \) with a categorical quotient \( Z/\sim \), the stabilizers of the action can be too big to allow the application of the GIT construction to produce other quotient spaces from \( Z \). One can try to enhance \( Z \) to another \( G \)-scheme \( \tilde{Z} \) with \( G \)-equivariant morphism \( \pi : \tilde{Z} \to Z \) to reduce the stabilizers and perform the GIT construction on \( \tilde{Z} \). When it works, different choices of \( \tilde{Z} \) with a \( G \)-linearized line bundle \( \tilde{L}^x \) on \( \tilde{Z} \) give rise then to a net of \( G \)-invariant \( \pi(\tilde{Z}^{ss}(\tilde{L}^x)) \subset Z \), whose categorical quotients form now a net of quotient spaces with a natural morphism to \( Z/\sim \). Here, \( \tilde{Z}^{ss}(\tilde{L}^x) \) is the semistable locus of \( \tilde{Z} \) with respect to the line bundle \( \tilde{L} \) on \( \tilde{Z} \) with the \( G \)-linearization \( \chi \). We will call this procedure a *netted categorical quotient construction* on the \( G \)-scheme \( Z \). In good cases, one can apply this to produce a net of birational models for the (usually bad/singular) scheme \( Z/\sim \).

Readers are referred to [M-F-K] and [Ne] for detailed discussions on moduli problems, orbit spaces, and the GIT construction.

**D-brane probe and birational geometry.**

Associated to a combinatorial type (e.g. dimension and the number of susy’s) of quantum field theories is the *Wilson’s theory-space* \( S_{\text{Wilson}} \) that parameterizes all the quantum field theories of the given combinatorial type. In simple cases, \( S_{\text{Wilson}} \) is locally parameterized by the tuple of coupling constants in the Lagrangian for the quantum field theories. For convenience, one may also add in the space on which the condensation of fields takes value. The vacuum manifold/variety of a quantum field theory depends on the coupling constants and the condensation values and, hence, on where we are on \( S_{\text{Wilson}} \). Moving around in \( S_{\text{Wilson}} \) may give rise to a web/net of vacuum manifolds/varieties of different topologies. *Walls* can form on \( S_{\text{Wilson}} \) that locally separates quantum field theories of the same combinatorial type but of different details, e.g. with different topologies of the vacuum manifold/variety. This gives a *phase structure*\(^3\) on \( S_{\text{Wilson}} \).

Applying this picture to the SQFT on a D-brane probe as in [D-M] can give rise to a web/net of birational models of the singular space in question by taking the vacuum manifold associated to a [theory] \( \in S_{\text{Wilson}} \). See, e.g., [G-L-R] for an example of flips-and-flops of vacuum manifolds of D-brane probes.

\(^3\)It should be noted that \( S_{\text{Wilson}} \) can be non-smooth and with several irreducible components. In such a case, besides the transitions due to crossing the walls inside each irreducible component, there are also transitions due to moving from one irreducible component of \( S_{\text{Wilson}} \) to another.
As we are studying D-branes along the line of the Polchinski-Grothendieck Ansatz, the D-brane moduli space arises from the moduli stack $\mathcal{M}^{D\text{-brane}}(Y)$ of morphisms from Azumaya spaces with a fundamental module to a string target-space $Y$. An important guiding/natural question is thus:

**Q. [D-brane probe resolution]**

*Can one extract birational models, in particular resolutions, of $Y$ from $\mathcal{M}^{D\text{-brane}}(Y)$?*

In this work, we will give an affirmative explicit answer to this question for $Y$ being an ADE orbifold $[C^2/\Gamma]$.

**Remark 0.1 [naturality]** It should be noted that the production of a web/net of birational models of an open-string target space/variety from a sub-moduli space of the D-brane probe moduli space in the sense of [L-Y1] and [L-L-S-Y] by the netted categorical quotient construction is a very general outcome of the Azumaya nature on D-branes. In particular, it applies not just to the case of ADE orbifold singularities. One should think of this setting as an extension of the application of the *functor of points* associated to a given target-space to a special noncommutative, namely the Azumaya type, source geometry. Specifically for D0-branes, allowing the rank of the fundamental/Chan-Paton module on D0-brane probes to increase, this then generalizes the notion of *arcs* and *jet schemes* of a target-space in commutative algebraic geometry. From this last point of view, the close tie of D-branes, along the line of the Polchinski-Grothendieck Ansatz, with birational geometry and resolution of singularities is very natural and anticipated.

**Convention.** Standard notations, terminology, operations, facts in (1) (super)string theory; (2) (commutative) algebraic geometry; (3) (commutative) stacks; (4) descent theory can be found respectively in (1) [Po2]; (2) [Ha]; (3) [L-MB]; (4) [SGA1] and [Vi].

- All schemes, algebraic stacks are Noetherian over $C$. All coherent sheaves of modules on an algebraic stack are Cartesian.

- To make the discussion more down to earth, the $Y$ in a *presentation* $P : Y \rightarrow \mathcal{Y}$ of an algebraic stack $\mathcal{Y}/S$ over a base scheme $S$ will be chosen by definition to be a scheme $/S$, instead of an algebraic space $/S$; we will call $Y$ also directly as an *atlas* of $\mathcal{Y}$. Similarly, for open charts, . . . , etc..

- For linear spaces, e.g. $C^2$, $\text{End}(C^r)$, . . . , we occasionally identify them with their canonically associated affine variety for simplicity of notations. Similarly, for their algebraic subsets.

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4 String-theorists should not misunderstand us as saying that what string theorists did based on supersymmetric quantum field theory setting is just trivial. Completely the opposite: what string theorists have taught mathematicians are highly non-trivial! Rather, what we mean to say is that the Polchinski-Grothendieck Ansatz together with mathematical naturality alone are enough to “foresee” what could/should happen. The fact that the mathematical outcome based solely on this and the string-theoretical outcome can be arranged to agree tells us that this ansatz does characterize a very fundamental nature of D-branes. Indeed, from the Grothendieck’s point of view of the contravariant equivalence of the category of functions rings (commutative or not) and the category of local geometries, there is no other way/opt thing can be than the emergence of an Azumaya-type noncommutative structure on stacked D-branes themselves if D-branes have to behave as open strings would dictate. From Grothendieck’s viewpoint, it is the noncommutative structure on a D-brane itself that will in turn enable it to probe the noncommutativity, if any, of the string target space(-time); cf. [L-Y1: Sec. 2.2] and [L-L-S-Y: Figure 1-2].
According to the Polchinski-Grothendieck Ansatz, a D0-brane can be modelled prototypically by an Azumaya point with a fundamental module of type $r$, $(\text{Spec } \mathbb{C}, \text{End}(\mathbb{C}^r), \mathbb{C}^r)$. When the target space $Y$ is commutative, the surrogates involved are commutative $\mathbb{C}$-sub-algebras of the matrix algebra $M_r(\mathbb{C}) = \text{End}(\mathbb{C}^r)$. This part already contains an equal amount of information/richness/complexity as the moduli space of 0-dimensional coherent sheaves of length $r$. When the target space is noncommutative, more surrogates to the Azumaya point will be involved. Allowing $r$ to go to $\infty$ enables Azumaya points to probe “infinitesimally nearby points” to points on a scheme to arbitrary level/order/depth. In (commutative) algebraic geometry, a resolution of a scheme $Y$ comes from a blow-up. In other words, a resolution of a singularity $p$ of $Y$ is achieved by adding an appropriate family of infinitesimally nearby points to $p$. Since D-branes with an Azumaya-type structure are able to “see” these infinitesimally nearby points via morphisms therefrom to $Y$, they can be used to resolve singularities of $Y$. Thus, from the viewpoint of Polchinski-Grothendieck Ansatz, the Azumaya-type structure on D-branes is why D-branes have the power to “see” a singularity of a scheme not just as a point, but rather as a partial or complete resolution of it. Such effect should be regarded as a generalization of the standard technique in algebraic geometry of probing a singularity of a scheme by arcs of the form $\text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^r))$, which leads to the notion of jet-schemes in the study of singularity and birational geometry.
Outline.

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1 D-branes on a (commutative) algebraic stack.

A formulation of D-branes on an algebraic stack that follows the Polchinski-Grothendieck Ansatz and extends \([L-Y1]\) and \([L-L-S-Y]\) is given in this section.

Coherent sheaves on algebraic stacks and flatness.

The notion of

- **dimension** of an algebraic stack,
- **(Cartesian) coherent sheaves** on an algebraic stack and their support, pull-back and push-forward

are defined in \([L-MB]\). For the notion of the support \(\text{Supp} \mathcal{F}\) of a coherent sheaf \(\mathcal{F}\), we will use instead the one that goes through the ideal sheaf of annihilators of the coherent sheaf on an algebraic stack that generalizes the notion of a scheme-theoretic support of a coherent sheaf on a scheme. The **dimension of a coherent sheaf** \(\mathcal{F}\) on an algebraic stack \(\mathcal{X}\) is defined to be the dimension of the support \(\text{Supp} \mathcal{F}\) of \(\mathcal{F}\) on \(\mathcal{X}\).

**Explanation/Definition 1.1. [Property P of \(O\)-module on algebraic stack]** \([\text{Lau}]\)

Let \(P\) be a property of \(O\)-modules on schemes satisfying the following two conditions:

1. **(pull-back)** If an \(O\)-module \(\mathcal{F}\) on a scheme \(X\) has the property \(P\), then for any smooth morphism of schemes \(X' \rightarrow X\), the pull-back \(\mathcal{F}'\) of \(\mathcal{F}\) on \(X'\) also has the property \(P\).

2. **(descent)** An \(O\)-module \(\mathcal{F}\) on a scheme \(X\) has the property \(P\) as soon as there exists a smooth and surjective morphism of schemes \(X' \rightarrow X\) such that the pull-back \(\mathcal{F}'\) of \(\mathcal{F}\) on \(X'\) has the property \(P\).

Then the property \(P\) makes sense for \(O\)-modules on algebraic stacks and it is enough to check it on any presentation of the algebraic stack.

**Definition 1.2. [flatness of coherent module on \(\mathcal{X}/\mathcal{Y}\)].** Given a \((1-)\)morphism \(F : \mathcal{X} \rightarrow \mathcal{Y}\) of algebraic stacks and a coherent \(O_{\mathcal{X}}\)-module \(\mathcal{M}\) on \(\mathcal{X}\), we say that \(\mathcal{M}\) is flat over \(\mathcal{Y}\) if for every open chart \(u : U \rightarrow \mathcal{Y}\) (in the smooth-étale topology/site) of \(\mathcal{Y}\), the pull-back \((\tilde{u} \circ v)^* \mathcal{M}\) on every \(V\) is flat over \(U\), where \(v : V \rightarrow U \times_{u,\mathcal{Y},F} \mathcal{X}\) is an open chart of \(U \times_{u,\mathcal{Y},F} \mathcal{X}\) and

\[
\begin{array}{ccc}
V & \xrightarrow{v} & U \times_{u,\mathcal{Y},F} \mathcal{X} \\
\downarrow & & \downarrow \tilde{u} \\
U & \xrightarrow{u} & \mathcal{Y} \\
& \searrow & \searrow F \\
& & \mathcal{X}
\end{array}
\]

(Note that both \(U\) and \(V\) are schemes by our convention.)

This notion plays a fundamental role in the setting of D-branes on an algebraic stack studied in the current work.

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\(^5\)As this definition is not given explicitly in \([L-MB]\), we particularly like to thank Gérard Laumon for the very careful and authoritative explanation of this to us. Thanks also to Andrew Tolland for a discussion.
D-branes on an algebraic stack à la Polchinski-Grothendieck Ansatz.

In [L-Y1: Sec. 1] and [L-L-S-Y: Sec. 2.1] the notion of morphisms from an Azumaya scheme with a fundamental module \((X^{Az}, E)\) to a projective variety \(Y\) is developed from Grothendieck’s fundamental principle of doing geometries and morphisms from local to global via gluing, together with the natural requirement that the composition of morphisms \(X_1 \to X_2 \to X_3\) should be a morphism \(X_1 \to X_3\). This gives us a prototype definition of a D-brane on \(Y\) along the line of Polchinski-Grothendieck Ansatz. An algebraic stack and morphisms between them can be constructed from local to global with generalized notion of coverings and gluing via smooth morphisms. Thus, one can repeat [L-Y1] and [L-L-S-Y: Sec. 2.1] to give a fundamental treatment of the notion of morphisms from Azumaya schemes with a fundamental module to an algebraic stack \(Y\) and, hence, the notion of D-branes on the stack \(Y\) following the Polchinski-Grothendieck Ansatz.

In [L-L-S-Y: Sec. 2.2] it is next observed that the above fundamental treatment for target a projective variety can be recast into an equivalent Azumaya-without-Azumaya-’n-morphism-without-morphism setting. In this latter setting, a morphism \(\varphi : (X^{Az}, \mathcal{E}) \to Y\), where \(X^{Az} = (X, \mathcal{O}_X^{Az} = \text{End}_{\mathcal{O}_X} \mathcal{E})\) is completely coded by a coherent \(\mathcal{O}_{X \times Y}\)-module \(\bar{\mathcal{E}}\) on \((X \times Y)/X\) that is flat over \(X\) and of relative dimension 0. With the notion of flatness of coherent sheaves on an algebraic stack over another algebraic stack being set up in the previous theme, in this work we take the shorter second route to define prototypically a D-brane on an algebraic stack \(Y\) as a morphism from an Azumaya scheme with a fundamental module to \(Y\) via a direct generalization of the Azumaya-without-Azumaya-’n-morphism-without-morphism setting of [L-L-S-Y: Sec. 2.2], as follows:

**Definition 1.3. [D-brane on algebraic stack à la Polchinski-Grothendieck Ansatz].** A D-brane with the underlying domain (commutative) scheme \(X\) on a target algebraic stack \(Y\) à la Polchinski-Grothendieck Ansatz is defined to be a coherent \(\mathcal{O}_{X \times Y}\)-module \(\mathcal{E}\) on \(X \times Y\) that

(i) is flat over \(X\),

(ii) is of relative dimension 0 with respect to \(X\), and

(iii) \(\mathcal{E} := pr_1^* \bar{\mathcal{E}}\) is a locally-free coherent \(\mathcal{O}_X\)-module, where \(pr_1 : X \times Y \to X\) is the projection map to \(X\).

Two D-branes, represented respectively by \(\bar{\mathcal{E}}_1\) on \(X_1 \times Y\) and \(\bar{\mathcal{E}}_2\) on \(X_2 \times Y\) under the above setting, are said to be isomorphic if there exit a (\(\mathbb{C}\)-)scheme-isomorphism \(h : X_1 \sim X_2\) and an \(\mathcal{O}_{X_1 \times Y}\)-module-isomorphism \(\tilde{h} : h^* \bar{\mathcal{E}}_2 \sim \bar{\mathcal{E}}_1\).

Note that for general \(Y\) in Definition 1.3, Condition (i) and Condition (ii) together do not imply Condition (iii). The latter thus has to be imposed additionally.

**Remark 1.4. [recovering morphism from Azumaya scheme with fundamental module to \(Y\) from \(\bar{\mathcal{E}}\).]** (Cf. [L-L-S-Y: Sec. 2.2 and Figure 2-2-1].) (1) The \(\bar{\mathcal{E}}\) in the above definition reproduces

- an underlying Azumaya scheme with a fundamental module

\[
(X^{Az}, \mathcal{E}) := (X, \mathcal{O}_X^{Az} := \text{End}_{\mathcal{O}_X} \mathcal{E}, \mathcal{E}),
\]

where \(\mathcal{E} = pr_1^* \bar{\mathcal{E}}\) in the Definition,
a morphism

\[ \varphi : (X^A, \mathcal{E}) \rightarrow \mathcal{Y} \]

via the restriction of the projection map \( pr_2 : X \times \mathcal{Y} \rightarrow \mathcal{Y} \) to \( \text{Supp} \, \bar{\mathcal{E}} \).

The surrogate \( X_\varphi \) of \( X^A \) associated to \( \varphi \) is given by \( \text{Spec} (pr_1^*(\mathcal{O}_{\text{Supp} \, \bar{\mathcal{E}}})) \). This is a scheme that is finite over \( X \). In terms of \( X_\varphi \), \( \varphi \) is represented by a scheme-morphism \( \hat{f}_\varphi : \hat{X}_\varphi \rightarrow \mathcal{Y} \), where \( \hat{X}_\varphi \) is a scheme with a built-in smooth surjective morphism \( \hat{X}_\varphi \rightarrow X_\varphi \), depending on both \( \bar{\mathcal{E}} \) and the choice of \( P \).

(2) The image D-brane with a Chan-Paton module on the stack \( \mathcal{Y} \) is defined/given by the push-forward \( \varphi^* \mathcal{E} := pr_2^* \bar{\mathcal{E}} \). The support \( \text{Supp}(\varphi_* \mathcal{E}) \) on \( \mathcal{Y} \) is by definition the underlying “brane” on \( \mathcal{Y} \).

**Lemma 1.5.** [pull-back to atlas]. Continuing Definition 1.3, a coherent \( \mathcal{O}_{X \times \mathcal{Y}} \)-module \( \bar{\mathcal{E}} \) on \( X \times \mathcal{Y} \) is flat over \( X \) if and only if there exists an atlas \( P : \mathcal{Y} \rightarrow \mathcal{Y} \) such that \((\text{Id}_X \times P)^* \bar{\mathcal{E}} \) on \( X \times \mathcal{Y} \) is flat over \( X \). The latter holds if and only if \((\text{Id}_X \times P)^* \bar{\mathcal{E}} \) on \( X \times \mathcal{Y} \) is flat over \( X \) for every atlas \( P : \mathcal{Y} \rightarrow \mathcal{Y} \).

**Proof.** This follows from the definition of flat modules on schemes, [Ha: Proposition 9.2(d)], and the descent of morphisms of coherent sheaves under a smooth morphism (cf. [SGA1] and [Vi]), and Explanation/Definition 1.1.

**The case of an orbifold target.**

An orbifold \( \mathcal{Y} \) is a special Deligne-Mumford stack for which a local chart is of the form \((U, \Gamma_U)\) and \( \mathcal{Y} \) is locally modelled on Deligne-Mumford stacks of the form \([U/\Gamma_U] \), the quotient stack of \( U \) by \( \Gamma \), where \( \Gamma_U \) is a finite group that acts on the scheme \( U \). The topological language of Thurston [Th: Chap. 13] on orbifolds is directly adaptable to the algebro-geometric language of Deligne-Mumford stacks. In particular:

**Definition/Lemma 1.6.** [orbifold structure group at point]. Let \( p \in \mathcal{Y} \) be a geometric point of the orbifold \( \mathcal{Y} \) and \((U, \Gamma_U)\) be an orbifold chart of \( \mathcal{Y} \) that contains \( p \). Define the orbifold structure group \( \Gamma_p \) of \( \mathcal{Y} \) at \( p \) to be the stabilizer \( \text{Stab}(p) \) of the \( \Gamma \)-action on \( U \). This is well-defined up to group-isomorphisms as, up to group-isomorphisms, \( \Gamma_p \) is independent of the choice of \((U, \Gamma_U)\) that contains \( p \).

**Definition/Lemma 1.7.** [orbifold-length]. Given a coherent 0-dimensional \( \mathcal{O}_\mathcal{Y} \)-module on an orbifold \( \mathcal{Y} \), let \( |\text{Supp} \, \mathcal{F}| = \{p_1, \ldots, p_k\} \) be the set of geometric points in the support \( \text{Supp} \, \mathcal{F} \) of \( \mathcal{F} \) and \( l_i \) be the length of \( \mathcal{F} \) at \( p_i \). Define the orbifold-length \( \text{orbi.l}(\mathcal{F}) \) of \( \mathcal{F} \) on \( \mathcal{Y} \) to be

\[
\text{orbi.l}(\mathcal{F}) = \sum_{i=1}^{k} \frac{l_i}{|\Gamma_p|}.
\]

This is invariant under flat deformations of \( \mathcal{F} \).

\( ^6 \)For readers unfamiliar with stacks: In terms of Deligne-Mumford stack language, this means that \( \{p\} \times \mathcal{Y} \) \( U \) is non-empty, where the fibered product is taken with respect to the built-in morphisms \( \{p\} \rightarrow \mathcal{Y} \) and \( U \rightarrow \mathcal{Y} \).
Remark 1.8. [fractional orbifold-length]. For an orbifold \( Y \) with the orbifold structure group \( \Gamma \), being trivial on an open dense sub-orbifold of \( Y \), a \( \mathcal{O}_Y \)-module \( F \) with non-integer orbifold \( \mathcal{O}_F \) contains a submodule \( F' \) that is supported on some orbifold-point(s) with nontrivial structure group in such away that \( F' \) cannot be deformed away from this/these point(s). In other words, \( F \) has an unmovable/trapped direct summand supported at point(s) with nontrivial orbifold structure group.

Remark 1.9. [orbifold Euler characteristic]. The notion of orbifold-length is a special case of the notion of orbifold Euler characteristic for a coherent \( \mathcal{O}_Y \)-module on an orbifold \( Y \).

2 D-brane probe resolution of ADE orbifold singularities revisited à la Polchinski-Grothendieck Ansatz.

We now address the main theme of the current work: extracting a resolution for the variety \( \mathbb{A}^2/\Gamma \) from a D-brane probe moduli space in the sense of Polchinski-Grothendieck Ansatz.

The moduli stack \( \mathcal{M}^D_1([\mathbb{A}^2/\Gamma]) \) of D0-branes on \([\mathbb{A}^2/\Gamma]\).

Let \( \Gamma \) be a finite subgroup of \( SU(2) \) acting on \( \mathbb{A}^2 = \text{Spec} \mathbb{C}[z_1, z_2] \) via the (anti-)action on generators of the function ring, \( r = |\Gamma| \) be the order of \( \Gamma \), and \([\mathbb{A}^2/\Gamma]\) be the orbifold associated to the group action. Note that \((\gamma_2 \gamma_1) \circ z_i = \gamma_1 \circ (\gamma_2 \circ z_i)\), \( i = 1, 2 \). By construction, \([\mathbb{A}^2/\Gamma]\) is a Deligne-Mumford stack with an atlas \( P : \mathbb{A}^2 \to [\mathbb{A}^2/\Gamma] \). The fibered product

\[
\begin{array}{ccc}
\mathbb{A}^2 \times_P [\mathbb{A}^2/\Gamma], & \xrightarrow{pr_2} & \mathbb{A}^2 \\
\downarrow & & \downarrow \\
\mathbb{A}^2 & \xrightarrow{pr_1} & [\mathbb{A}^2/\Gamma]
\end{array}
\]

from the Isom-functor construction defines a morphism \( \mathbb{A}^2 \times \Gamma \to \mathbb{A}^2 \) that recovers the \( \Gamma \)-action on \( \mathbb{A}^2 \). From our setting Definition 1.3 along the Polchinski-Grothendieck Ansatz, a D0-brane on \([\mathbb{A}^2/\Gamma]\) of stacky type 1 is, by definition, given by a 0-dimensional coherent sheaf on \([\mathbb{A}^2/\Gamma]\) of orbifold-length 1. It follows from Lemma 1.5 and the above discussion that this coherent \( \mathcal{O}_{[\mathbb{A}^2/\Gamma]} \)-module on \([\mathbb{A}^2/\Gamma]\) is identical, via pull-back versus descent, to a 0-dimensional coherent \( \Gamma \)-\( \mathcal{O}_{\mathbb{A}^2} \)-module \( \tilde{E} \) of length \( r \) on \( \mathbb{A}^2 \). It follows that:

\footnote{An anti-action of \( \Gamma \) is by definition an action of \( \Gamma^\circ \), where \( \Gamma^\circ \) is the group that has the same elements as \( \Gamma \) but with \( \gamma_1 \gamma_2 \) in \( \Gamma^\circ \) defined to be \( \gamma_2 \gamma_1 \) in \( \Gamma \). In this work, there is no chance of ambiguity of whether \( \Gamma \) acts or anti-acts as all the actions or anti-actions involved are induced from the \( \Gamma \)-action on \( \mathbb{A}^2 \) or its lifting. We thus call either directly as an action.}
Lemma 2.1. [moduli stack of D0-branes on \([\mathbb{A}^2/\Gamma]\)]. The moduli stack \(\mathcal{M}_1^{D0}(\mathbb{A}^2/\Gamma)\) of D0-branes of stacky type 1 on the orbifold \([\mathbb{A}^2/\Gamma]\) is an Artin stack, given by the stack of coherent \(\Gamma\)-\(\mathcal{O}_{\mathbb{A}^2}\)-modules of length \(r\) on \(\mathbb{A}^2\), where \(r = |\Gamma|\).

Cf. [L-L-S-Y: Sec. 3.1].

A digression: the equivalent Azumaya-'n-morphism setting

While in this work the Azumaya-without-Azumaya-'n-morphism-without-morphism setting of D-branes along the Polchinski-Grothendieck Ansatz is adopted to circumvent the otherwise necessary but tedious - albeit more fundamental - presentation for Sec. 1, it is instructive to see what morphism a geometric point of \(\mathcal{M}_1^{D0}(\mathbb{A}^2/\Gamma)\) really corresponds to. This recovers then the hidden equivalent Azumaya-'n-morphism setting of D-branes on the orbifold \([\mathbb{A}^2/\Gamma]\).

Recall Remark 1.4 that generalizes [L-L-S-Y: Sec. 2.2]. Let

\[
[\mathbb{A}^2/\Gamma]^\bullet := \text{Spec } \mathbb{C} \times [\mathbb{A}^2/\Gamma] = [\mathbb{A}^2/\Gamma] \xrightarrow{pr_2} [\mathbb{A}^2/\Gamma]
\]

be the projection maps. Here we use \([\mathbb{A}^2/\Gamma]^\bullet\) to distinguish the product (over \(\text{Spec } \mathbb{C}\)) of the domain scheme \(\text{Spec } \mathbb{C}\) and the target orbifold \([\mathbb{A}^2/\Gamma]\) from the target orbifold \([\mathbb{A}^2/\Gamma]\) for conceptual clarity. Then, for an \(\mathcal{O}_{[\mathbb{A}^2/\Gamma]^\bullet}\)-module \(\tilde{E}\) of orbifold-length 1, \(pr_1^*\tilde{E} \simeq \mathcal{C}'\), where recall that \(r = |\Gamma|\). Thus, a geometric point of \(\mathcal{M}_1^{D0}(\mathbb{A}^2/\Gamma)\) corresponds to a morphism

\[
\varphi : (pt^{\mathcal{A}_r}; \mathcal{C}') := (\text{Spec } \mathbb{C}, \text{End}(\mathcal{C}'), \mathcal{C}') \rightarrow [\mathbb{A}^2/\Gamma]
\]

from the Azumaya \(\mathbb{C}\)-point of type \(r\) with a fundamental module to \([\mathbb{A}^2/\Gamma]\). Furthermore, as \(orbi.l(\tilde{E}) = 1\), \(\text{Supp } \tilde{E}\) contains only one geometric point \(p\) of \([\mathbb{A}^2/\Gamma]\). There are only two situations: either \(\Gamma_p\) is trivial or \(\Gamma_p = \Gamma\). For convenience, denote by \(0\) the geometric point in \(\mathbb{A}^2\) associated to the ideal \((z_1, z_2)\) for \(\mathbb{A}^2 = \text{Spec } (\mathbb{C}[z_1, z_2])\) and let \(U = \mathbb{A}^2 - \{0\}\). The geometric point in \([\mathbb{A}^2/\Gamma]\) associated to \(0 \in \mathbb{A}^2\) will also be denoted by \(0\).

Case \(a\): \(\Gamma_p\) is trivial.

This happens when \(p\) lies in the open dense sub-orbifold \([U/\Gamma]\) of \([\mathbb{A}^2/\Gamma]\). In this case:

- The length \(l_p\) of \(\tilde{E}\) at \(p\) is equal to 1.
- \(\text{Supp } \tilde{E}\) is a 0-dimension closed sub-orbifold \(\{p_1, \cdots, p_r\}/\Gamma\) of \([\mathbb{A}^2/\Gamma]\), where \(\Gamma\) acts on the (0-dimensional reduced) scheme \(\{p_1, \cdots, p_r\}\) effectively and transitively.
- The surrogate \(pt_\varphi\) of \(pt^{\mathcal{A}_r}\) associated to \(\varphi\) is given by \(\{p_1, \cdots, p_r\}\), which is \(\Gamma\)-isomorphic to the disjoint union \(\text{Spec } (\prod \mathbb{C}) = \prod \text{Spec } \mathbb{C}\) of \(r\)-many \(\mathbb{C}\)-points, equipped with an effective and transitive \(\Gamma\)-action modelled on the left multiplication of \(\Gamma\) on itself. Here \(\prod \mathbb{C}\) is the product ring of \(\mathbb{C}\)'s and is canonically embedded in \(\text{End}(\mathcal{C}')\) as a \(\mathbb{C}\)-sub-algebra.

String-theorists are highly recommended to compare the several very concrete/explicit geometric pictures of D-branes - all following from the Polchinski-Grothendieck Ansatz - in this theme with whatever stringy geometric picture(s) you may have had for Douglas-Moore [D-M].
The morphism \( \varphi \) is thus represented by an embedding of \( \Gamma \)-schemes:

\[
\hat{f}_\varphi : \pt_\varphi = \{p_1, \ldots, p_r\} \longrightarrow \mathbb{A}^2,
\]

where \( \mathbb{A}^2 \) is the built-in atlas of the orbifold \([\mathbb{A}^2/\Gamma]\) regarded as a Deligne-Mumford stack.

The fundamental module \( C^r = \pr_1^* \hat{E} \) on \( \pt^\mathbb{A} \) is naturally a \( \Gamma - \mathcal{O}_{\pt_\varphi} \)-module. In terms of the latter, it is isomorphic to \( \mathcal{O}_{\pt_\varphi} = \bigoplus_{i=1}^r \mathcal{O}_{p_i} \). Note that in the current case, \( \Gamma \) acts on the fundamental module \( C^r \) via the regular representation. The image D0-brane with Chan-Paton module on \([\mathbb{A}^2/\Gamma]\) that corresponds to \( \varphi \) is given by \( \varphi^* \hat{E} := \pr_2^* \hat{E} \), which is \( \hat{E} \) itself after identifying \([\mathbb{A}^2/\Gamma]^\bullet \) with \([\mathbb{A}^2/\Gamma]\) canonically. It is represented by the \( \Gamma - \mathcal{O}_{\mathbb{A}^2} \)-module \( \hat{f}_\varphi^* \mathcal{O}_{\pt_\varphi} = \bigoplus_{i=1}^k \mathcal{O}_{\hat{f}_\varphi(p_i)} \) on the atlas \( \mathbb{A}^2 \) of \([\mathbb{A}^2/\Gamma]\).

**Case (b) :** \( \Gamma_p = \Gamma \).

This happens when \( p = 0 \). In this case,

- The length \( l_p \) of \( \hat{E} \) at \( p \) is equal to \( r \) ( = |\( \Gamma \)|).
- \( \text{Supp} \hat{E} \) is a 0-dimension closed sub-orbifold \([Z/\Gamma]\) of \([\mathbb{A}^2/\Gamma]\). Here \( \Gamma \) acts on the 0-dimensional connected scheme \( Z \) and \( Z = \text{Spec} A \) for some local Artin \( \Gamma \)-\( \mathbb{C} \)-algebra \( A \) of length \( \leq r \).
- The surrogate \( pt_\varphi \) of \( pt^\mathbb{A} \) associated to \( \varphi \) is given by the \( \Gamma \)-scheme \( Z \).

The morphism \( \varphi \) is thus represented by an embedding of \( \Gamma \)-schemes:

\[
\hat{f}_\varphi : \pt_\varphi = Z \longrightarrow \mathbb{A}^2,
\]

where \( \mathbb{A}^2 \) is the built-in atlas of \([\mathbb{A}^2/\Gamma]\).

Again, the fundamental module \( C^r = \pr_1^* \hat{E} \) on \( pt^\mathbb{A} \) is naturally a \( \Gamma - \mathcal{O}_{pt_\varphi} \)-module, denoted by \( \hat{E} \). However, it can happen that \( \hat{E} \) is not isomorphic to \( \mathcal{O}_{pt_\varphi} \). Note also that \( \Gamma \) now acts on the fundamental module \( C^r \), but possibly as a direct sum of irreducible representations, cf. Figure 2-1: \( \varphi_4 \). The image D0-brane with Chan-Paton module on \([\mathbb{A}^2/\Gamma]\) that corresponds to \( \varphi \) is given by \( \varphi^* \hat{E} := \pr_2^* \hat{E} \), which is \( \hat{E} \) itself after identifying \([\mathbb{A}^2/\Gamma]^\bullet \) with \([\mathbb{A}^2/\Gamma]\) canonically. It is represented by the \( \Gamma - \mathcal{O}_{\mathbb{A}^2} \)-module \( \hat{f}_\varphi^* \hat{E} \) on the atlas \( \mathbb{A}^2 \) of \([\mathbb{A}^2/\Gamma]\). Its support \( \text{Supp} (\hat{f}_\varphi^* \hat{E}) \) on \( \mathbb{A}^2 \) is a representation of the image D0-brane \( \text{Supp} (\varphi^* \hat{E}) \) (a sub-orbifold) on \([\mathbb{A}^2/\Gamma]\).

Cf. Figure 2-1.
Figure 2-1. Examples of morphisms from an Azumaya point with a fundamental module \((\text{Spec} \mathbb{C}, \text{End}(\mathbb{C}^r), \mathbb{C}^r)\), which models an intrinsic D0-brane according to the Polchinski-Grothendieck Ansatz, to the orbifold \([\mathbb{A}^2/\Gamma]\) are shown. Morphism \(\varphi_1\) is in Case (a) while morphisms \(\varphi_2, \varphi_3, \varphi_4\) are in Case (b). The image D0-brane under \(\varphi_i\) on the orbifold \([\mathbb{A}^2/\Gamma]\) is represented by a 0-dimensional \(\Gamma\)-subscheme of length \(\leq r\) on the atlas \(\mathbb{A}^2\) of \([\mathbb{A}^2/\Gamma]\).
D0-branes on $\mathbb{A}^2$.

In this theme we recast [L-Y1: Sec. 4.3] as a preparation to understanding $\mathfrak{M}^D_0([\mathbb{A}^2/\Gamma])$ further.

A 0-dimensional coherent $\mathcal{O}_{\mathbb{A}^2}$-module $\mathcal{F}$ of length $r$ on $\mathbb{A}^2$ with a specified isomorphism $\iota : \mathbb{C}^r \cong H^0(\mathcal{F})$ can be identified with a pair $(m_1, m_2)$ of commuting $r \times r$ matrices/\(\mathbb{C}\), and vice versa, as follows:

1. **From $(\mathcal{F}, \iota)$ to $(m_1, m_2)$:** Given $(\mathcal{F}, \iota)$ as said, the $\mathbb{C}[z_1, z_2]$-module structure on $H^0(\mathcal{F})$ and, hence, on $\mathbb{C}^r$ via $\iota : \mathbb{C}^r \cong H^0(\mathcal{F})$, gives uniquely a pair $(m_1, m_2)$ as said by taking the representation of $(z_1, z_2)$ on $\mathbb{C}^r$.

2. **From $(m_1, m_2)$ to $(\mathcal{F}, \iota)$:** Given $(m_1, m_2)$ as said, the specification $z_1 \mapsto m_1$ and $z_2 \mapsto m_2$ defines a $\mathbb{C}$-algebra homomorphism $\varphi_{(m_1, m_2)} : \mathbb{C}[z_1, z_2] \to \text{End}(\mathbb{C}^r)$. This realizes $\mathbb{C}^r$ as a $\mathbb{C}[z_1, z_2]$-module and, hence, an $\mathcal{O}_{\mathbb{A}^2}$-module $\mathcal{F}$, together with an isomorphism $\iota : \mathbb{C}^r \cong H^0(\mathcal{F})$. Note that $\text{Supp} \mathcal{F} = \text{V}(\text{Ker} \varphi_{(m_1, m_2)}) \simeq \text{Spec}(1, m_1, m_2)$, where $(1, m_1, m_2)$ is the subalgebra of $\text{End}(\mathbb{C}^r)$ generated by $1$, $m_1$, and $m_2$.

These two operations are inverse to each other. One should think of $(\mathcal{F}, \iota)$ in Item (1) and $\varphi_{(m_1, m_2)}$ in Item (2) as defining a morphism from the fixed/ rigidified Azumaya point with the fundamental module ($\text{Spec} \mathbb{C}, \text{End}(\mathbb{C}^r)$, $\mathbb{C}^r$) to (the fixed) $\text{Spec} \mathbb{C}[z_1, z_2] = \mathbb{A}^2$. The rigidification is given by $\iota$ in Item (1) and by expressing the fundamental module on the Azumaya point explicitly as $\mathbb{C}^r$ in Item (2).

Recall from [L-Y1: Sec. 4.3] and the references quoted ibidem the **commuting scheme**

$$C_2 M_r(\mathbb{C}) = \{(m_1, m_2) : m_1 m_2 = m_2 m_1\} \subset \text{End}(\mathbb{C}^r) \times \text{End}(\mathbb{C}^r)$$

with the scheme structure from the standard scheme structure $\mathbb{A}^2$ on $\text{End}(\mathbb{C}^r)$. This is an irreducible variety of dimension $r^2 + r$. The universal commuting pair of endomorphisms of $\mathbb{C}^r$ is given by a section $s$ of $\mathcal{O}_{C_2 M_r(\mathbb{C})} \otimes (\text{End}(\mathbb{C}^r) \oplus \text{End}(\mathbb{C}^r))$. The morphism

$$P_{r, \mathbb{A}^2} : C_2 M_r(\mathbb{C}) \longrightarrow \mathfrak{M}^D_0(\mathbb{A}^2)$$

defined by the composition of the correspondences

$$(m_1, m_2) \mapsto s(m_1, m_2) \mapsto (\mathcal{F}, \iota) \mapsto \mathcal{F}$$

realizes $C_2 M_r(\mathbb{C})$ as an atlas of the Artin stack $\mathfrak{M}^D_0(\mathbb{A}^2)$ of D0-branes of type $r$ on $\mathbb{A}^2$. The defining $\text{GL}_r(\mathbb{C})$-action on $\mathbb{C}^r$ induces a $\text{GL}_r(\mathbb{C})$-action on $C_2 M_r(\mathbb{C})$ via

$$(m_1, m_2) \xrightarrow{g} (g m_1 g^{-1}, g m_2 g^{-1}) .$$

An orbit of the latter action corresponds to an isomorphism class of morphisms from the *unfixed* Azumaya point/\(\mathbb{C}\) with a fundamental module, $(\text{pt}^\mathbb{A}^2, E) := (\text{Spec} \mathbb{C}, \text{End} E, E)$, where $E \cong \mathbb{C}^r$ abstractly, to (the fixed) $\mathbb{A}^2$.

**Lemma 2.2.** [closed orbit in $C_2 M_r(\mathbb{C})$]. $\text{GL}_r(\mathbb{C}) \cdot (m_1, m_2)$ is a closed orbit in $C_2 M_r(\mathbb{C})$ if and only if $m_1$ and $m_2$ are simultaneously diagonalizable. The closure $\text{GL}_r(\mathbb{C}) \cdot (m_1, m_2)$ of every orbit $\text{GL}_r(\mathbb{C}) \cdot (m_1, m_2)$ in $C_2 M_r(\mathbb{C})$ contains a unique closed orbit.
Proof. $GL_r(\mathbb{C}) \cdot (m_1, m_2)$ can be represented by $(m_1, m_2)$ with $m_1$ and $m_2$, say, upper simultaneously triangulated. The $t \to 0$ limit $(m_0^1, m_0^2)$ of such $(m_1, m_2)$ by the 1-parameter subgroup $\text{Diag}(1, t^{-1}, \ldots, t^{-(n-1)})$ is then diagonal. Up to the $GL_r(\mathbb{C})$-action, $(m_0^1, m_0^2)$ is uniquely determined by the orbit $GL_r(\mathbb{C}) \cdot (m_1, m_2)$ as the former is determined by the characteristic polynomials $\det(\lambda I - m_1)$ and $\det(\lambda I - m_2)$ of $m_1$ and $m_2$ up to simultaneous permutations.

\begin{corollary}
\textbf{[categorical quotient of $\mathcal{M}^{D_0}_r(\mathbb{A}^2)$].} The categorical quotient of $\mathcal{M}^{D_0}_r(\mathbb{A}^2)$, which by definition is the categorical quotient of the atlas $C_2 M_r(\mathbb{C})$ by the above $GL_r(\mathbb{C})$-action, is given by the symmetric product $S^r(\mathbb{A}^2)$ of $\mathbb{A}^2$.
\end{corollary}

\textbf{Remark 2.4.} A point in $S^r(\mathbb{A}^2)$ corresponds to an isomorphism class of morphisms from $(pt^{k_1}, E)$ to $\mathbb{A}^2$ whose associated surrogates are reduced. The above discussion also realizes $S^r(\mathbb{A}^2)$ as $\text{Spec} \left( R(C_2 M_r(\mathbb{C}))^{GL_r(\mathbb{C})} \right)$, where $R(C_2 M_r(\mathbb{C}))^{GL_r(\mathbb{C})}$ is the ring of $GL_r(\mathbb{C})$-invariant functions in the function/coordinate ring $R(C_2 M_r(\mathbb{C}))$ of $C_2 M_r(\mathbb{C})$; see [Va] for a discussion in general dimensions and [L-Y] for more references.

Once having the categorical quotient of $C_2 M_r(\mathbb{C})$ under the $GL_r(\mathbb{C})$-action, and hence of the stack $\mathcal{M}^{D_0}_r(\mathbb{A}^2)$, it is natural to attempt to follow [K] and [M-F-K], and the related discussion in [Na] to consider a geometric-invariant-theory setting on the $GL_r(\mathbb{C})$-variety $C_2 M_r(\mathbb{C})$ to produce birational models of $S^r(\mathbb{A}^2)$. The would-be procedure goes as follows: Let $L$ be the trivial line bundle $\mathcal{O}_{C_2 M_r(\mathbb{C})}$ on $C_2 M_r(\mathbb{C})$ with a linearization of the $GL_r(\mathbb{C})$-action through a character $\chi : GL_r(\mathbb{C}) \to \mathbb{C}^\times$ via the determinant function $\det$ to some positive power: $g \cdot ((m_1, m_2), z) = (g \cdot (m_1, m_2), \chi(g)\cdot z)$. A would-be birational model of $S^r(\mathbb{A}^2)$ is then obtained by taking the categorical quotient $(C_2 M_r(\mathbb{C}))^{ss}(\chi)/\sim$ of the $GL_r(\mathbb{C})$-action on the semistable locus $C_2 M_r(\mathbb{C})^{ss}(\chi)$ in $C_2 M_r(\mathbb{C})$ with respect to the linearization on $L$ specified by $\chi$. However, this won’t work here as $C_2 M_r(\mathbb{C})^{ss}(\chi) = \emptyset$. The reason is that the stabilizer of the $GL_r(\mathbb{C})$-action on $C_2 M_r(\mathbb{C})$ at a point on a principal orbit is isomorphic to $(\mathbb{C}^\times)^n$; this is too big to allow the $\chi$-linearized $L$ to have $GL_r(\mathbb{C})$-invariant sections except the zero-section. This is why we need to employ a netted categorical quotient construction, instead of a direct GIT-quotient construction. We will use a construction of Nakajima in [Na] to guide our netted categorical quotient construction relevant to our goal.

To remedy the above issue, consider instead the total space $E$ of the universal fundamental module $\mathcal{O}_{C_2 M_r(\mathbb{C})} \otimes \mathbb{C}$ on $C_2 M_r(\mathbb{C})$ and employ the above GIT setting to $E$:

- \textbf{the $GL_r(\mathbb{C})$-action on $E$:} The $GL_r(\mathbb{C})$-action on $C_2 M_r(\mathbb{C})$ lifts to a $GL_r(\mathbb{C})$-action on $E$
  $$((m_1, m_2; v) \mapsto (g m_1 g^{-1}, g m_2 g^{-1}; g v)).$$

The stabilizer of $(m_1, m_2; v) \in a$ principal orbit in $E$ is now trivial.

- \textbf{the line bundle $L$ and its linearization:} Let $L$ be the trivial line bundle $\mathcal{O}_E$ with the linearization specified by a character $\chi$ as in the previous discussion:
  $$((m_1, m_2; v), z) \mapsto (g \cdot (m_1, m_2; v), \chi(g) \cdot z)).$$

Thus, the above GIT-constructive setting of [K] and [M-F-K] is now more likely to be applicable to the $GL_r(\mathbb{C})$-variety $E$ and, indeed, the detail is worked out by Nakajima in [Na].
Theorem 2.5. [Hilbert scheme \((\mathbb{A}^2)^r\) from GIT-quotient]. ([Na: Theorem 1.9 and Lemma 3.25.]) The GIT-quotient \(E \sslash \chi GL_r(\mathbb{C}) := E^{ss}(\chi)/GL_r(\mathbb{C})\) of \(E\) with respect to \(L\) with the linearization specified by \(\chi\) is canonically isomorphic to the Hilbert scheme \((\mathbb{A}^2)^r\) of 0-dimensional subschemes of length \(r\) on \(\mathbb{A}^2\).

Lemma 2.6. [netted categorical quotient on \(C_2M_r(\mathbb{C})\)]. ([L-Y1: Proposition 4.3.3.]) Let \(\pi : E \rightarrow C_2M_r(\mathbb{C})\) be the defining projection morphism. Then

\[
\pi(E^{ss}(\chi))/GL_r(\mathbb{C}) \simeq E^{ss}(\chi)/GL_r(\mathbb{C}) \simeq (\mathbb{A}^2)^r
\]

canonicaly.

This is a rephrasing of [L-Y1: Proposition 4.3.3]; we give another proof below that fits better the current setting. Note that, as \(C_2M_r(\mathbb{C})\) is irreducible and \(\pi : E \rightarrow C_2M_r(\mathbb{C})\) is an open morphism, \(\pi(E^{ss}(\chi))\) is open and dense in \(C_2M_r(\mathbb{C})\).

Proof. Let \(R(E)\) be the coordinate ring of the affine variety \(E\) and \(R(E)^{\chi,n}\) be the \(\mathbb{C}\)-subspace of \(R(E)\) that consists of \(f \in R(E)\) that satisfies \(f(g \cdot (m_1, m_2; v)) = \chi(g)^n f(m_1, m_2; v)\). Then, by definition, \((m_1, m_2; v) \in E^{ss}(\chi)\) if there exists an \(f \in R(E)^{\chi,n}\) with \(n \geq 1\) such that \(f(m_1, m_2; v) \neq 0\). In our case, as \(\chi\) is a positive power of the determinant function on \(GL_r(\mathbb{C})\) and the stabilizer \(Stab(\cdot)\) is an open subset of an affine space, such an \(f\) can exist only if \(Stab(m_1, m_2; v)\) is trivial. Thus \(E^{ss}(\chi)\) is identical to the stable locus \(E^s(\chi)\) of \(GL_r(\mathbb{C})\)-action on \(E\) and the quotient \(E^{ss}(\chi) \to E^{ss}(\chi)/GL_r(\mathbb{C})\) is a geometric quotient; indeed, a \(GL_r(\mathbb{C})\)-bundle.

For convenience, express \(E\) canonically as \(C_2M_r(\mathbb{C}) \times \mathbb{C}^r\) in the analytic language/notation. Then, for \((m_1, m_2, v) \in E^{ss}(\chi)\), the projection map \(GL_r(\mathbb{C}) \cdot (m_1, m_2; v) \rightarrow \mathbb{C}^r\) is a bundle map over \(\mathbb{C}^r \setminus \{0\}\) with fiber the inhomogeneous general linear group \(IGL_{r-1}(\mathbb{C})\) (i.e. the affine transformation group of the vector space \(\mathbb{C}^{r-1}\)). This shows that for any \((m_1, m_2; v) \in E^{ss}(\chi),

\[
\dim(GL_r(\mathbb{C}) \cdot (m_1, m_2; v) \cap (C_2M_r(\mathbb{C}) \times \{v\})) = r^2 - r
\]

and that, for any fixed \(v_0 \neq 0\),

\[
E^{ss}(\chi)/GL_r(\mathbb{C}) = (E^{ss}(\chi) \cap (C_2M_r(\mathbb{C}) \times \{v_0\}))/IGL_{r-1}(\mathbb{C}).
\]

Identify \(Stab(v_0)\) of the \(GL_r(\mathbb{C})\)-action on \(\mathbb{C}^r\) with \(IGL_{r-1}(\mathbb{C})\). As, for \((m_1, m_2, v_0) \in E^{ss}(\chi),

\[
IGL_{r-1}(\mathbb{C}) \cdot (m_1, m_2; v_0) = GL_r(\mathbb{C}) \cdot (m_1, m_2; v_0) \cap (C_2M_r(\mathbb{C}) \times \{v_0\})
\]

\[
\simeq \pi(IGL_{r-1}(\mathbb{C}) \cdot (m_1, m_2; v_0))
\]

\[
\subset \pi(GL_r(\mathbb{C}) \cdot (m_1, m_2; v_0)) \subset GL_r(\mathbb{C}) \cdot (m_1, m_2),
\]

and \(IGL_{r-1}(\mathbb{C}) \cdot (m_1, m_2; v_0)\) has dimension \(r^2 - r\), the orbit \(GL_r(\mathbb{C}) \cdot (m_1, m_2)\) in \(C_2M_r(\mathbb{C})\) must also have dimension \(r^2 - r\) and the dimension of \(Stab(m_1, m_2)\) must be \(r\). As \(Stab(m_1, m_2; v_0)\) is trivial, this implies that \(Stab(m_1, m_2) \cdot v_0\) is open dense in \(\mathbb{C}^r\). Translating this to any \((m_1', m_2', v_0) \in IGL_{r-1}(\mathbb{C}) \cdot (m_1, m_2; v_0)\) via the \(IGL_{r-1}(\mathbb{C})\)-action, this implies that

\[
GL_r(\mathbb{C}) \cdot (m_1, m_2; v_0) \subset \pi^{-1}(\pi(IGL_{r-1}(\mathbb{C}) \cdot (m_1, m_2; v_0))) = \pi(IGL_{r-1} \cdot (m_1, m_2; v_0) \times \mathbb{C}^r).
\]

Since \(\pi(GL_r(\mathbb{C}) \cdot (m_1, m_2; v_0)) = GL_r(\mathbb{C}) \cdot (m_1, m_2),\) one concludes that

\[
\pi(GL_r(\mathbb{C}) \cdot (m_1, m_2; v_0) \cap (C_2M_r(\mathbb{C}) \times \{v_0\})) = GL_r(\mathbb{C}) \cdot (m_1, m_2).
\]
It follows that the isomorphism \( \pi : \mathbb{E}^{ss}(\chi) \cap \langle C_{2}M_{r}(\mathbb{C}) \times \{v_{0}\} \rangle \xrightarrow{\sim} \pi(\mathbb{E}^{ss}(\chi)) \) is equivariant under the group homomorphism \( IGL_{r-1}(\mathbb{C}) = Stab(v_{0}) \hookrightarrow GL_{r}(\mathbb{C}) \) with isomorphic orbit-spaces. The lemma follows.

\[ \square \]

**Remark 2.7.** [characterization of \( \pi(\mathbb{E}^{ss}(\chi)) \)]. The image \( \pi(\mathbb{E}^{ss}(\chi)) \) in \( C_{2}M_{r}(\mathbb{C}) \) consists of \((m_{1}, m_{2})\) such that there exists a \( v \in \mathbb{C}^{\gamma} \) with \( \mathbb{C}[m_{1}, m_{2}] \cdot v = \mathbb{C}^{\gamma} \). This happens if and only if the subalgebra \( \{1, m_{1}, m_{2}\} \) of \( End(\mathbb{C}^{\gamma}) \) is isomorphic to \( \mathbb{C}^{\gamma} \) as \( \mathbb{C} \)-vector spaces (and, hence, as \( \langle 1, m_{1}, m_{2}\rangle \)-modules as well).

**Remark 2.8.** The induced morphism \( \mathbb{A}^{2}[\gamma] \to S^{r}(\mathbb{A}^{2}) \) through the construction is the Hilbert-Chow morphism.

**D0-branes on \([\mathbb{A}^{2}/\Gamma]\) and an atlas for \( \mathcal{M}_{1}^{D0}([\mathbb{A}^{2}/\Gamma]) \).**

The \( \Gamma \)-action on \( \mathbb{C}[z_{1}, z_{2}] \) induces a \( \Gamma \)-action on \( C_{2}M_{r}(\mathbb{C}) \) via

\[
\gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto (m_{1}, m_{2}) \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{t} =: \gamma \odot (m_{1}, m_{2}).
\]

In terms of analytic expression, regard \( C_{2}M_{r}(\mathbb{C}) \) as a subset in the vector space \( End(\mathbb{C}^{\gamma}) \otimes \mathbb{C}^{2} \); then, the \( \Gamma \)-action on \( \mathbb{C}^{2} \) induces a \( \Gamma \)-action on \( End(\mathbb{C}^{\gamma}) \otimes \mathbb{C}^{2} \) that leaves \( C_{2}M_{r}(\mathbb{C}) \) invariant. This gives the above action. Here, \( \mathbb{C}^{2} \) is identified with the vector space \( \text{Span} \{z_{1}, z_{2}\} \), with \( \mathbb{A}^{2} = \text{Spec Sym}^{*}(\mathbb{C}^{2}) \). Note that the \( \Gamma \)- and the \( GL_{r}(\mathbb{C}) \)-action on \( C_{2}M_{r}(\mathbb{C}) \) commute:

\[
\gamma \odot ((g \cdot (m_{1}, m_{2}))) = g \cdot (\gamma \odot (m_{1}, m_{2}))
\]

for \((m_{1}, m_{2}) \in C_{2}M_{r}(\mathbb{C}), g \in GL_{r}(\mathbb{C}), \) and \( \gamma \in \Gamma \). In terms of this action, a rigidified \( \Gamma \)-\( \mathcal{O}_{\mathbb{A}^{2}} \)-module \( (\mathcal{F}, \iota) \) on the \( \Gamma \)-variety \( \mathbb{A}^{2} \), where \( \iota : \mathbb{C}^{\gamma} \xrightarrow{\sim} H^{0}(\mathcal{F}) \), is given by a triple \((m_{1}, m_{2}; \rho)\), where \( \rho : \Gamma^{0} \to GL_{r}(\mathbb{C}) \) is a representation of \( \Gamma^{0} \), that satisfies the \( \mathcal{O}_{\mathbb{A}^{2}} \)-linearity condition of the \( \Gamma \)-action on \( \mathcal{F} \), now expressed as

\[
\gamma \odot (m_{1}, m_{2}) = \rho(\gamma) \cdot (m_{1}, m_{2}) \quad \text{for all } \gamma \in \Gamma.
\]

Here \( \Gamma^{0} \) is the dual group of \( \Gamma \), which has the same elements as in \( \Gamma \) but with \( \gamma_{1}\gamma_{2} \) in \( \Gamma^{0} \) defined to be \( \gamma_{2}\gamma_{1} \) in \( \Gamma \). The gluing condition of \( (\mathcal{F}, \iota) \) gives a correspondence \( \rho : \Gamma^{0} \to GL_{r}(\mathbb{C}) \) while the cocycle condition on the gluing imposes further that \( \rho \) is a group-homomorphism.

Let \( \text{Rep}_{\Gamma^{0}}(\mathbb{C}^{\gamma}) \subset \prod GL_{r}(\mathbb{C}) \) be the representation variety of \( \Gamma^{0} \) into \( GL_{r}(\mathbb{C}) \). This is a \( GL_{r}(\mathbb{C}) \)-scheme under \( (Ad_{g} : \rho)(\gamma) = g \rho(\gamma) g^{-1} \) for \( \gamma \in \Gamma \), where \( Ad_{g} \) is the adjoint action of \( GL_{r}(\mathbb{C}) \) on itself. The McKay correspondence gives a bijection between the set of equivalence classes of irreducible representations of \( \Gamma \) and the set of vertices of the extended Dynkin diagram associated to \( \Gamma \). Any other representation of \( \Gamma \) is a direct sum of these irreducible representations. It follows that \( \text{Rep}_{\Gamma^{0}}(\mathbb{C}^{\gamma}) \) consists of finitely many \( GL_{r}(\mathbb{C}) \)-orbits. Let

\[
C_{2}\Gamma M_{r}(\mathbb{C}) := \{ (m_{1}, m_{2}; \rho) : \gamma \odot (m_{1}, m_{2}) = \rho(\gamma) \cdot (m_{1}, m_{2}) \text{ for all } \gamma \in \Gamma \}
\]

\[
\subset C_{2}M_{r}(\mathbb{C}) \times \text{Rep}_{\Gamma^{0}}(\mathbb{C}^{\gamma})
\]
with the subscheme structure the same as the subscheme structure defined by these algebraic constraint equations on $C_2 M_r (\mathbb{C}) \times \text{Rep}_{\Gamma^r}(C^r)$. The tautological $\Gamma \cdot \mathcal{O}_{C_2 \Gamma M_r (\mathbb{C}) \times \mathbb{A}^2}$-module on $C_2 \Gamma M_r (\mathbb{C}) \times \mathbb{A}^2$ defines a unique morphism

$$P_{1,[\mathbb{A}^2/\Gamma]} : C_2 \Gamma M_r (\mathbb{C}) \rightarrow \mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma]).$$

This gives an atlas of the Artin stack $\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])$. Change of rigidifications of $\Gamma \cdot \mathcal{O}_{\mathbb{A}^2}$-modules on $\mathbb{A}^2$ induces a $GL_r (\mathbb{C})$-action on $C_2 \Gamma M_r (\mathbb{C})$ by

$$(m_1, m_2 ; \rho) \mapsto (gm_1 g^{-1}, gm_2 g^{-1} ; \text{Ad}_g \cdot \rho).$$

There is a bijection between the set of $GL_r (\mathbb{C})$-orbits in $C_2 \Gamma M_r (\mathbb{C})$ and the set of isomorphism classes of $\Gamma \cdot \mathcal{O}_{\mathbb{A}^2}$-modules of length $r$ on the $\Gamma$-variety $\mathbb{A}^2$.

Resolution of the ADE orbifold singularity of $\mathbb{A}^2/\Gamma$ via $\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])$.

We now proceed to extract a $GL_r (\mathbb{C})$-invariant locus from the atlas $C_2 \Gamma M_r (\mathbb{C})$ of $\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])$ whose geometric quotient gives the minimal smooth resolution of $\mathbb{A}^2/\Gamma$.

Recall the $\Gamma$-action on $C_2 M_r (\mathbb{C})$. It leaves $\pi (E^\text{as}(\chi))$ invariant and descends to the natural $\Gamma$-action on $\pi (E^\text{as}(\chi))/GL_r (\mathbb{C})$ via the canonical isomorphism with $(\mathbb{A}^2)^r$.

Theorem 2.9. [$\Gamma$-invariant subscheme]. (Ginzburg-Kapranov, Ito-Nakamura, [Na: Theorem 4.1 and Theorem 4.4].) Let $((\mathbb{A}^2)^r)^{\Gamma}$ be the fixed-point locus of the $\Gamma$-action on $(\mathbb{A}^2)^r$. Then

$$(\mathbb{A}^2)^r)^{\Gamma} = W \amalg (a \text{ finite set of points}),$$

where $W$ is the minimal smooth resolution of $\mathbb{A}^2/\Gamma \simeq (S^r (\mathbb{A}^2))^{\Gamma}$ via the Hilbert-Chow morphism.

Let $\tilde{Z} \subset W \times \mathbb{A}^2$ be the universal subscheme on $(W \times \mathbb{A}^2)/W$ associated to $W$. Then $\mathcal{O}_{\tilde{Z}}$ is a $W$-family of $\Gamma \cdot \mathcal{O}_{\mathbb{A}^2}$-modules of length $r$ on $\mathbb{A}^2$ and, hence, defines a morphism $p^W : W \rightarrow \mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])$. Consider the fibered product

$$\begin{array}{ccc}
C_2 \Gamma M_r (\mathbb{C}) \times_{P_{1,[\mathbb{A}^2/\Gamma]},\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma]), p^W} W & \xrightarrow{pr_2} & W \\
\downarrow{pr_1} & & \downarrow{p^W} \\
C_2 \Gamma M_r (\mathbb{C}) & \xrightarrow{P_{1,[\mathbb{A}^2/\Gamma]}} & \mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])
\end{array}$$

and let

$$C_2 \Gamma M_r (\mathbb{C})^o := pr_1 \left( C_2 \Gamma M_r (\mathbb{C}) \times_{P_{1,[\mathbb{A}^2/\Gamma]},\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma]), p^W} W \right) \subset C_2 \Gamma M_r (\mathbb{C}).$$

Then, it follows from Theorem 2.9 and the surjectivity of $pr_2$ in the above fibered product diagram that:

Corollary 2.10. [D-brane probe resolution of orbifold singularity]. The categorical quotient $C_2 \Gamma M_r (\mathbb{C})^o/ GL_r (\mathbb{C})$ is a geometric quotient and is isomorphic to $W$. Thus, the moduli stack $\mathcal{M}^{\text{D0}}_1 ([\mathbb{A}^2/\Gamma])$ of D0-branes on the orbifold $[\mathbb{A}^2/\Gamma]$ of stacky type 1 à la Polchinski-Grothendieck Ansatz contains a substack whose associated coarse moduli space is isomorphic to the minimal resolution $\mathbb{A}^2/\Gamma$ of the variety $\mathbb{A}^2/\Gamma$ with an ADE orbifold singularity.
Remark 2.11. [characterization of $C_2\Gamma M_r(C)^\circ$]. It follows from Lemma 2.6 that the forgetful morphism $C_2\Gamma M_r(C) \to C_2M_r(C)$ sends $C_2\Gamma M_r(C)^\circ$ onto $\pi(\mathcal{E}^{\text{ss}}(\chi))$. It follows from [Na: Theorem 4.4] and Remark 2.7 that a geometric point in $C_2\Gamma M_r(C)^\circ/GL_r(C)$ corresponds to an isomorphism class of morphisms

$$\varphi : (pt^{A^2}, C^r) := ((\text{Spec} \ C, \text{End}(C^r)), C^r) \longrightarrow [A^2/\Gamma]$$

from the Azumaya point with the fundamental module to the orbifold that satisfy:

- the induced $\Gamma$-action/representation on the Chan-Paton module $C^r$ is the regular representation;

- there exists a $\Gamma$-fixed element $v_0 \in C^r$ such that $H^0(\mathcal{O}_{pt}) \cdot v_0 = C^r$, where $pt_{\varphi}$, a 0-dimensional scheme of length $r$, is the surrogate of $pt^{A^2}$ associated to $\varphi$.

And vice versa. In particular, any morphism $\varphi : (pt^{A^2}, C^r) \to [A^2/\Gamma]$ of stacky type 1 whose image is not the orbifold-point with structure group $\Gamma$ satisfies the above conditions and its isomorphism class is contained in $C_2\Gamma M_r(C)^\circ/GL_r(C)$.

Remark 2.12. [movable vs. unmovable morphism/D-branes]. Morphisms associated to geometric points in $C_2\Gamma M_r(C)^\circ$ are movable in the sense that they can be deformed to have the image D-brane anywhere on $[A^2/\Gamma]$. However, in general, there are morphisms from D0-branes to $[A^2/\Gamma]$ that are trapped at the orbifold-point that corresponds to the singular point of $A^2/\Gamma$. Here, “trapped” means that these morphisms cannot be deformed to make the image D-brane move away from this orbifold point. It is unknown to us whether such trapped/unmovable D-branes at the singularity have any stringy significance/implication.
References

[Ar1] M. Artin, Grothendieck topologies, lecture notes at Harvard, 1962.
[Ar2] ———, On Azumaya algebras and finite dimensional representations of rings, J. Alg. 11 (1969), pp. 532 - 563.

[D-M] M.R. Douglas and G.W. Moore, D-branes, quivers, and ALE instantons, arXiv:hep-th/9603167

[D-G-M] M.R. Douglas, B.R. Greene, D.R. Morrison, Orbifold resolution by D-branes, Nucl. Phys. B506 (1997), pp. 84 - 106. (arXiv:hep-th/9704151)

[G-L-R] B.R. Greene, C.I. Lazaroiu, and M. Raugas, D-branes on non-abelian threefold quotient singularities, Nucl. Phys. B553 (1999), pp. 711 - 749. (arXiv:hep-th/9811201)

[Ha] R. Hartshorne, Algebraic geometry, GTM 52, Springer, 1977.

[J-M] C.V. Johnson and R.C. Myers, Aspects of type IIB theory on ALE spaces, Phys. Rev. D55 (1997), pp. 6382 - 6393. (arXiv:hep-th/9610140)

[Ki] A.D. King, Moduli of representations of finite dimensional algebras, Quarterly J. Math. 45 (1994), pp. 515 - 530.

[Kr] P.B. Kronheimer, The construction of ALE spaces as a hyper-Kähler quotient, J. Diff. Geom. 29 (1989), pp. 665 - 683.

[K-N] P.B. Kronheimer and H. Nakajima, Yang-Mills instantons on ALE gravitational instantons, Math. Ann. 288 (1990), pp. 263 - 307.

[Lau] G. Laumon, private communications.

[Liu1] C.-H. Liu, Polchinski’s D-branes from Grothendieck’s viewpoint, long lecture given at Yau’s Seminar, Harvard University, October, 2007.

[Liu2] ———, D-strings as a master object for curves, mini-course given at the workshop “Algebraic Geometry and Physics, Hangzhou, 2008”, September, 2008.

[L-L-S-Y] S. Li, C.-H. Liu, R. Song, S.-T. Yau, Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves. arXiv:0809.2121 [math.AG].

[L-MB] G. Laumon and L. Moret-Bailly, Champs algébriques, Ser. Mod. Surveys Math. 39, Springer, 2000.

[L-Y1] C.-H. Liu and S.-T. Yau, Azumaya-type noncommutative spaces and morphism therefrom: Polchinski’s D-branes in string theory from Grothendieck’s viewpoint, arXiv:0709.1515 [math.AG].

[L-Y2] ———, manuscript in preparation.

[Mo] D.R. Morrison, Open strings and noncommutative algebraic geometry, talk given at the workshop “Algebraic Geometry and Physics, Hangzhou, 2008”, September, 2008.

[M-F-K] D. Mumford, J. Fogarty, and F. Kirwan, Geometric invariant theory, 3rd ed., Springer, 1994.

[Na] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, Univ. Lect. Ser. 18, Amer. Math. Soc., 1999.

[Ne] P.E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute Lect. 51, Springer, 1978.

[Po1] J. Polchinski, Dirichlet-branes and Ramond-Ramond charges, Phys. Rev. Lett. 75 (1995), pp. 4724 - 4727. (arXiv:hep-th/9510017)

[Po2] ———, String theory, vol. I: An introduction to the bosonic string; vol. II: Superstring theory and beyond, Cambridge Univ. Press, 1998.

[S-GA1] A. Grothendieck and M. Raynaud, Séminaire de géométrie algébrique du Bois Marie 1960-61: Revêtements étalés et groupe fondamental (SGA1), Lect. Notes Math. 224, Springer, 1971.

[Th] W.P. Thurston, The geometry and topology of three-manifolds, Princeton mimeographed lecture notes, 1979.

[Va] F. Vaccarino, Linear representations, symmetric products and the commuting scheme, math.AG/0602660.

[Vi] A. Vistoli, Grothendieck topologies, fibered categories and descent theory, in Fundamental algebraic geometry: Grothendieck’s FGA explained, B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, A. Vistoli, pp. 1 - 104, Math. Surv. Mono. 123, Amer. Math. Soc., 2005.