Nonparametric Estimation for Jump-Diffusion CIR Model

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Abstract: We study the nonparametric estimation for the intensity of Poisson random measure in jump-diffusion CIR model based on the low frequency observations. This is given in terms of the minimization of norms on a nonempty, closed and convex subset of some special Hilbert space. We establish the measurability of the estimator and derive its consistency and asymptotic risk bound.

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1 Introduction and main results

The Cox-Ingersoll-Ross model (CIR model) defined by the following stochastic differential equation (SDE) was firstly introduced by Cox et al. (1985) in the study of term structure of interest rate:

\[ dY(t) = b(\beta - Y(t))dt + \sqrt{2cY(t)}dB(t), \]

where \( \beta, b, c > 0 \) are given constants and \( \{B(t) : t \geq 0\} \) is a standard Brownian motion. Motivated by the study of jump risks which can not be ignored in the pricing of assets, Ahn and Thompson (1988) studied the following jump-diffusion process by adding a jump component into (1.1):

\[ dX(t) = b(\beta - X(t))dt + \sqrt{2cX(t)}dB(t) + \int_0^\infty zN(dt,dz), \]

where \( N(dt,dz) \) is a Poisson random measure on \((0, \infty)^2\) with Lévy intensity \( n(dz) \) and \( (1 \wedge z)n(dz) \) is a finite measure on \((0, \infty)\); more details can be seen Duffie et al. (2000, 2003). In this paper we always assume \( n(dz) \) is absolutely continuous with respect to Lebesgue measure, i.e. there exits a non-negative function \( k(z) \) satisfying \( n(dz) = k(z)dz \).

Actually, by Theorem 2.5 in Fu and Li (2010), the nonnegative solution \( \{X(t) : t \geq 0\} \) to (1.2) uniquely exists and is a continuous-state branching process with immigration (CBI processes) with transition semigroup \( (P_t)_{t \geq 0} \) given by

\[ \int_0^\infty e^{-ly}P_t(x,dy) = \exp \left\{ -xv_1(\lambda) - \int_0^t \psi(v_s(\lambda))ds \right\}, \]

where \( v(\lambda) \) is the unique positive solution to (1.1).
where

$$v_t(\lambda) = \frac{be^{-bt} \lambda}{b + c\lambda(1 - e^{-bt})}$$

and

$$\psi(z) = b\beta z + \int_0^\infty (1 - e^{-zu})n(du). \quad (1.4)$$

Otherwise, from Theorem 3.20 in Li (2011, p.66) we have the semigroup \( (P_t)_{t \geq 0} \) is ergodic, i.e. for any \( x \geq 0 \), \( P_t(x,\cdot) \) converges to a probability measure \( \eta \) on \( [0, \infty) \) as \( t \to \infty \) and the Laplace transform of \( \eta \) is given by

$$L_\eta(\lambda) = \exp \left\{ -\int_0^\infty \psi(v_s(\lambda))ds \right\}. \quad (1.5)$$

For any finite set \( \{t_1 < t_2 < \cdots < t_n\} \subset \mathbb{R} \) we can define the probability measure \( Q_{t_1,t_2,\cdots,t_n} \) on \( \mathbb{R}_+^n \) by

$$Q_{t_1,t_2,\cdots,t_n}(dx_1, dx_2, \cdots, dx_n) = \eta(dx_1)P_{t_2-t_1}(x_1, dx_2) \cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n). \quad (1.6)$$

Since \( \{Q_{t_1,t_2,\cdots,t_n} : t_1 < t_2 < \cdots < t_n \in \mathbb{R}\} \) is a consistent family, there is a stationary Markov process \( \{Z(t) : t \in \mathbb{R}\} \) with finite-dimensional distributions given by (1.6) and one-dimensional marginal distribution \( \eta \). From Remark 2.6 in Li and Ma (2015) and Birkhoff’s ergodic theorem, we have \( \{Z(t) : t \in \mathbb{R}\} \) is ergodic. By a fairly simple (continuous time) coupling argument, with out loss of generality we always assume \( X(t) \) defined by (1.2) is a stationary and ergodic process.

Before applying (1.2) into practical problems, the key preparation is estimating \((\beta, b, c)\) and \(n(dz)\). Since estimations for \((\beta, b, c)\) have been given by Huang et al. (2011), we just need to found some suitable estimations of \(n(dz)\) with other parameters known. There are a lot of works about parameter estimations for the standard CIR-model and a review had been given in Xu (2014) including the conditional least squares estimators (CLSEs) and the maximum likelihood estimators (MLEs) given by Overbeck and Rydén (1997) and Overbeck (1998). Here we only give a summary of some known works about nonparametric estimation for jump-diffusion processes. Unfortunately, limited works have been done in the nonparametric estimation in jump-diffusion CIR models compared with Lévy processes. Watteel and Kulperger (2003) proposed and implemented an approach for estimating the jump distribution of the Lévy processes by fixed spectral cut-off procedure. The penalized projection method was applied in Figueroa-López and Houdré (2006) to estimate the Lévy density on a compact interval separated from the origin, based on a continuous time observation of the sample path throughout a time interval \([0, T]\). Moreover, Figueroa-López (2009) used the projection method for discrete observations and provided minimum risks of estimation for smooth Lévy densities, as well as estimated on a compact interval separated from the origin. Comte and Genon-Catalot used a Fourier approach to construct an adaptive nonparametric estimators and provide bounds for the global \( L^2 \)- risk with high frequency data and low frequency data respectively; see Comte and Genon-Catalot (2009, 2010). Neumann and Reiss (2009) studied the nonparametric estimation for Lévy processes based on the empirical characteristic function. Jongbloed et al. (2005) considered a related low-frequency problem for the canonical function in Ornstein-Uhlenbeck processes driven by Lévy processes and a consistent estimator has been constructed.

In this work, based on the low frequency observations at equidistant time points \( \{k\Delta : k = 0, 1, \ldots\} \) of a single realization, we establish some nonparametric estimators for the Lévy density \( n(dz) \) by minimizing the norms of the elements of a closed and convex subset in some special
Hilbert space. For simplicity, we take $\Delta = 1$ and denote the observation be $\{X_k : k = 0, 1, \cdots \}$ but all the results presented below can be extended to the general case. We always assume all functions below are defined on $\mathbb{R}_+$. Let $\mu(dz) = (1 \wedge z)dz$ and

$$L(\mu) := \{f(z) : \mu(|f|) := \int_0^\infty |f(z)|\mu(dz) < \infty\}. \quad (1.7)$$

We say a real-valued function $f \in \mathcal{Y}_b$, if for any $-\infty < a < b < \infty$,

$$V_a^b(f) := \sup \sup_{n \in \mathbb{N}} \sum_{i=0}^{n-1} |f(x_{i+1} - f_{xi})| < \infty,$$

where $\mathcal{P}_n = \{p = \{x_0, \ldots, x_n\} : a = x_0 < \cdots < x_n = b\}$. We define the following convex subset of $L(\mu)$

$$K := \{k(z) \in L(\mu) \cap \mathcal{Y}_0 : k(z) \text{ is nonnegative and right-continuous}\}. \quad (1.8)$$

Otherwise, a linear operator $T$ is defined by

$$Tf(\lambda) := \int_0^1 \int_0^\infty (1 - e^{-v_\lambda(s)})f(z)dzds, \quad f(z) \in L(\mu). \quad (1.9)$$

Let

$$L_n(\lambda) = \frac{1}{n} \sum_{k=1}^n e^{-\lambda x_k + \lambda x_{k-1}v(\lambda) + D}, \quad (1.10)$$

where $v(\lambda) = v_1(\lambda)$ and $D = \frac{b^2}{c} \log(1 + c(1 - e^{-b})/b)$. By the ergodicity of $X(t)$ and the continuous mapping theorem, for any $\lambda > 0$ we have

$$g_n(\lambda) := -\ln(L_n(\lambda)) \overset{a.s.}{\longrightarrow} T\lambda(\lambda). \quad (1.11)$$

Let $\{k_R \geq 0 : R = 1, 2, \cdots \}$ be a increasing sequence of functions satisfying that $\mu(|\hat{k}_R|) \leq R$. For any $R \geq 1$ define

$$K_R := \{f \in K : f \leq k_R \text{ and } V_1^{i+1}(f) \vee V_1^{i-1}(f) \leq R \text{ for any } i = 1, 2, \cdots \}.$$ 

Here for any fixed $R \geq 1$, we establish the following estimator for $k(z)$:

$$\hat{k}_{R,n}(z) = \arg \min_{f \in K_R} \int_0^\infty |g_n(\lambda) - Tf(\lambda)|^2w(\lambda)d\lambda, \quad (1.12)$$

where $w(\lambda)$ is a bounded and non-negative weighted function with compact support, denote by $S_w$, and there exist $0 \leq a < b < \infty$ such that $[a, b] \subset S_w$. We give the main results in the following three theorems.

**Theorem 1.1** For any $R \geq 1$, we have $\hat{k}_{R,n}(z)$ is well defined, i.e. $\hat{k}_{R,n}(z)$ exists uniquely and is measurable.

**Theorem 1.2** If $k(\cdot) \in K_R$ for some $R > 0$, then the estimator $\hat{k}_{R,n}$ is strongly consistent. In details,

$$\mu(|\hat{k}_{R,n} - k|) \overset{a.s.}{\longrightarrow} 0. \quad (1.13)$$
Theorem 1.3 Suppose $z^2 n(dz)$ is a finite measure and $\exp\{ - \int_0^\infty \psi(v_s(\lambda)) ds \} < \infty$ for any $\lambda \in \left(-\frac{b e^{-b}}{c(1-e^{-c})}, \infty \right)$. If $k(\cdot) \in K_R$ for some $R > 0$, then there exists a constant $C > 0$ such that for $n$ large enough have

$$\sqrt{n} \mathbb{E} \{ |k_{R,n} - k| \} < C. \quad (1.14)$$

Remark 1.4 Conditions in Theorem 1.3 can be weakened; i.e., if $\int_0^\infty \psi(v_s(v(2a) - 2v(a))) ds < \infty$ for some $a > 0$, we can choose a weighted function $w(\lambda)$ with $S_w \subset [0, A]$.

This paper is organized as follows. In Section 2, we will prove Theorem 1.1. The consistency and asymptotic risk bound of estimator (Theorem 1.2 and 1.3) will be proved in Section 3.

**Notation:** In this paper, we denote $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{Q}$ be the set of all rational number. Moreover, $L_Q(\lambda)$ denotes the Laplace transform of the measure $Q$. $\xrightarrow{a.s.}$ and $\xrightarrow{d}$ mean converge almost surely and in distribution respectively. Similarly, $\equiv a.s.$ and $\equiv d$ mean equal almost surely and in distribution.

## 2 Existence, uniqueness and measurability

In this section, we will prove Theorem 1.1 by identifying the estimator defined by (1.12) exists uniquely and is measurable. Firstly, we recall a conclusion which can be found in many books about functional analysis.

**Lemma 2.1** If $S$ is a Banach space with norm $\| \cdot \|$, $M$ is a nonempty, closed, convex subset of $S$, then $M$ contains a unique element of smallest norm.

With this lemma we will give the most important theorem, which will guarantee the measurability of the estimators. Actually, least squares estimators (LSEs) and maximum likelihood estimators (MLEs) are just special cases of this theorem.

**Theorem 2.2** Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $S$ is a separable Banach space with norm $\| \cdot \|$ and Borel $\sigma$-algebra $\mathcal{F}$. Let $g$ be a measurable mapping from $(\Omega, \mathcal{F})$ to $(S, \mathcal{F})$ and $M \in \mathcal{F}$ be a nonempty, closed and convex subset. Then

$$h(\omega) := \arg \min_{f \in M} \| f - g(\omega) \| \quad (2.1)$$

is well defined and $\mathcal{F}$-measurable.

**Proof.** From Lemma 2.1, we have $h(\omega)$ exists uniquely. Now we prove it is $\mathcal{F}$-measurable. Let $m(\omega) = \min_{f \in M} \| f - g(\omega) \|$ which is $\mathcal{F}$-measurable. Indeed, since $M$ is separable, there exists a countable subset of $M$, denoted by $M_1 := \{ f_1, f_2, \cdots \}$ such that

$$m(\omega) = \lim_{n \to \infty} \min_{1 \leq i \leq n} \| f_i - g(\omega) \|.$$ 

Since $\| f_i - g(\omega) \|$ and $\min_{1 \leq i \leq n} \| f_i - g(\omega) \|$ are measurable for any $i, n \geq 1$, we have $m(\omega)$ is $\mathcal{F}$-measurable. Let $\mathcal{M} := \{ M \cap B : B \in \mathcal{F} \}$ and $\mathcal{G} = \{ A \in \mathcal{M} : h^{-1}(A) \in \mathcal{F} \}$, both of them are $\sigma$-algebras. We just prove $\mathcal{G}$ is a $\sigma$-algebra. Obviously, $\emptyset, M \in \mathcal{G}$. If $A \in \mathcal{G}$, then

$$h^{-1}(A^c) = \bigcup_{f \in A^c} \{ \omega : \| f - g(\omega) \| = m(\omega) \}.$$
\[
= \left( \bigcup_{f \in A} \{ \omega : \| f - g(\omega) \| = m(\omega) \} \right)^c \in \mathcal{F}
\]

For any \( \{A_n\}_{n=1}^{\infty} \in \mathcal{G} \), then
\[
h^{-1}\left( \bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{f \in \bigcup_{n=1}^{\infty} A_n} \{ \omega : \| f - g(\omega) \| = m(\omega) \}
= \bigcup_{n=1}^{\infty} \bigcup_{f \in A_n} \{ \omega : \| f - g(\omega) \| = m(\omega) \} \in \mathcal{F}.
\]

For any \( a > 0 \) and \( \xi \in M \), let \( A = \{ f \in M : \| f - \xi \| \leq a \} \). The desired result follows if we prove for any \( A \in \mathcal{G} \) have
\[
h^{-1}(A) = \bigcup_{f \in A} \{ \omega : \| f - g(\omega) \| = m(\omega) \} \in \mathcal{F}.
\]

Since \( M \) is separable, for any \( f \in A \) there exits a subset \( \{f_1, f_2, \ldots\} \subset A \) such that for any \( f \in A \) there exists an element \( f_n^i \) with \( \| f_n^i - f \| < 1/i \). Let
\[
B = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \{ \omega : \| f_n^i - g(\omega) \| \leq m(\omega) + 1/i \} \in \mathcal{F}.
\]

So it suffices to prove \( h^{-1}(A) = B \). Actually, for any \( \omega \in h^{-1}(A) \), there exists \( f(\omega) \in A \) such that \( \| f(\omega) - g(\omega) \| = m(\omega) \). Then for any \( i > 0 \), there exists \( f_n^i \) satifying \( \| f_n^i - f(\omega) \| < 1/i \). Thus
\[
m(\omega) \leq \| f_n^i - g(\omega) \| \leq \| f_n^i - f(\omega) \| + \| f(\omega) - g(\omega) \| \leq m(\omega) + 1/i,
\]
which means \( \omega \in B \) and \( h^{-1}(A) \subset B \). Otherwise, for any \( \omega \in B \) and \( i > 0 \), there exists \( f_n^i(\omega) \) such that
\[
m(\omega) \leq \| f_n^i(\omega) - g(\omega) \| \leq m(\omega) + 1/i,
\]
which means \( \| f_n^i(\omega) - g(\omega) \| \to m(\omega) \) as \( i \to \infty \). So there exists \( f(\omega) \in M \) such that \( \| f(\omega) - g(\omega) \| = m(\omega) \). Since \( S \) is a Hilbert space and \( A \) is a closed and convex ball, from Theorem 2.1 we have arg min \( f \in A \| f - g(\omega) \| \) exists uniquely. Otherwise, since
\[
\min_{f \in A} \| f - g(\omega) \| \geq \min_{f \in M} \| f - g(\omega) \| \quad \text{and} \quad \lim_{i \to \infty} \| f_n^i - g(\omega) \| = m(\omega),
\]
we have
\[
\| f(\omega) - g(\omega) \| = \min_{f \in A} \| f - g(\omega) \| = m(\omega).
\]

So \( f \in A \) and \( B \subset h^{-1}(A) \). Here we have finished this proof. \( \square \)

Define
\[
\mathcal{L}^2(w) := \left\{ f(\lambda) : \| f \|_w^2 := \int_0^\infty |f(\lambda)|^2 w(\lambda) d\lambda < \infty \right\},
\]
which is a Hilbert Space with inner product \( (f, g)_w = \int_0^\infty f(\lambda)g(\lambda)w(\lambda) d\lambda \) for any \( f, g \in \mathcal{L}^2(w) \). Let \( \Psi := \mathbb{T}K = \{ \mathbb{T}f(\lambda) : f \in K \} \), it is easy to see \( \Psi \subset \mathcal{L}^2(w) \). Indeed, for any \( f(z) \in K \) we have
\[
\mathbb{T}f(\lambda) = \int_{0}^{1} \int_{0}^{\infty} (1 - e^{-ze^{-b\exp(1-z)}}) f(z) dz ds
\]
\[
\leq \int_{0}^{1} \int_{0}^{\infty} (1 - e^{-ze^{-b\lambda}}) f(z) dz ds \leq \int_{0}^{\infty} (1 - e^{-ze^{-b\lambda}}) f(z) dz.
\]
Similarly, since \( V_i \in \{ \text{any} \} \) must exist, we have \( C > 0 \) is a constant independent to \( f(z) \).

### Lemma 2.3

For any two probability measures \( Q_1(\cdot) \) and \( Q_2(\cdot) \) on \( \mathbb{R}_+ \), we have \( Q_1 \overset{d}{=} Q_2 \) if and only if \( L_{Q_1}(\lambda) = L_{Q_2}(\lambda) \) on some interval \([\lambda_1, \lambda_2]\) with \( 0 \leq \lambda_1 < \lambda_2 \).

**Proof.** Sufficiency is obvious, we just need to prove necessity. Since \( L_{Q_1}(\lambda) \) and \( L_{Q_2}(\lambda) \) are analytic on this strip \( \{ \lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} : \lambda_1 \geq 0, |\lambda_2| < 1 \} \). By the assumption in this lemma and theorem in Brown and Churchill (2009, p.84), we have \( L_{Q_1}(\lambda) = L_{Q_2}(\lambda) \) on this strip. The desired result follows from \( L_{Q_1}(\lambda_1) = L_{Q_2}(\lambda_1) \) when we choose \( \lambda_2 = 0 \). \( \square \)

### Lemma 2.4

\( T : K \mapsto \Psi \) is a continuous bijection.

**Proof.** It is easy to see \( T \) is a one-to-one mapping. Indeed, for any \( k_1(z), k_2(z) \in K \) satisfying \( \|Tk_1(\lambda) - Tk_2(\lambda)\|_w = 0 \), we have \( Tk_1(\lambda) = Tk_2(\lambda) \) for any \( \lambda \in S_w \). Moreover, by the definition of CBI processes, there exist two probabilities \( Q_1(\cdot) \) and \( Q_2(\cdot) \) on \( \mathbb{R}_+ \) such that for \( i=1,2 \)

\[
L_{Q_i}(\lambda) = \exp \left\{ -\beta \int_0^1 v_n(\lambda) ds - Tk_i(\lambda) \right\}.
\]

By lemma 2.3, we have \( Q_1 \overset{d}{=} Q_2 \) and \( k_1(z) = k_2(z) \) almost everywhere. Thus \( \mu(|k_1 - k_2|) = 0 \). Since \( T \) is a linear operator, its continuity follows directly from its boundedness which have been proved in 2.2. \( \square \)

### Lemma 2.5

For any \( R \geq 1 \), \( K_R \) and \( TK_R \) are compact, convex subsets of \( L^1(\mu) \) and \( L^2(w) \) respectively.

**Proof.** Since \( T \) is continuous, \( TK_R \) is compact and its convexity follows from the convexity of \( K_R \) which is obvious. It suffices to prove \( K_R \) is compact. Since \( k_R(z) \) is integrable, so there must exist \( \{ z_j \}_{j=1}^\infty \), such that \( k_R(z_j) \to 0 \) as \( j \to \infty \). For any fixed sequence \( \{ k_n : n = 1, 2, \ldots \} \) in \( K_R \), we have \( k_n(z_j) \to 0 \) as \( j \to \infty \). There exist \( i_0 \in \mathbb{N} \) and \( z_0 \in [i_0, i_0 + 1] \) such that \( k_n(z_0) \leq k_R(z_0) < 1 \) for any \( n \geq 1 \). Without loss of generality, we assume \( i_0 = 1 \). For any \( i = 1, 2, \ldots \), since \( V_i^{i+1} k_n(z) \leq R \), we have \( 0 \leq k_n(z) \leq iR + 1 \) for any \( z \in [i, i+1] \). Similarly, since \( V_{(i+1)^{-1}}^{i+1} k_n(z) \leq R \), we have \( 0 \leq k_n(z) \leq iR + 1 \) for any \( z \in [(i+1)^{-1}, i^{-1}] \). Thus \( \{ k_n(z) : n = 1, 2, \ldots \} \) are uniformly bounded and have bounded variation on \( [(i+1)^{-1}, i^{-1}] \), which means there exist two sequences of nonnegative, monotone increasing and right-continuous functions \( \{ k_{n,1} : n = 1, 2, \ldots \} \) and \( \{ k_{n,2} : n = 1, 2, \ldots \} \) such that \( k_n = k_{n,1} - k_{n,2} \). Obviously, we have

\[
V_{(i+1)^{-1}}^{i+1} k_{n,1}(z) \leq 2(i+1)R \quad \text{and} \quad V_{(i+1)^{-1}}^{i+1} k_{n,2}(z) \leq 2(i+1)R.
\]
Without loss of generality, we assume that \( k_{n,2}(i+1)^{-1} = 0 \), so \( k_{n,2}(i+1) \leq 2(i+1)R \) and
\[
k_{n,1}(i+1)^{-1} \leq iR + 1, \quad k_{n,1}(i+1) \leq (3i+2)R + 1.
\]

Applying the diagonalization argument to \( k_{n,1}(z) \) and \( k_{n,2}(z) \), there exists a subsequence \( \{n^j\} \) such that \( k_{n^j,1} \) and \( k_{n^j,2} \) converge to some functions \( k_1 \) and \( k_2 \) on \( \mathbb{Q} \cap [(i+1)^{-1}, i+1] \) respectively. For any and \( l = 1, 2 \), define
\[
k^i_l(z) = \begin{cases} 
\inf \{k^i_1(x) : z \leq x \in \mathbb{Q} \cap [(i+1)^{-1}, i+1]\}, & \text{if } z \in [(i+1)^{-1}, i+1]; \\
0, & \text{otherwise}.
\end{cases}
\]

Define \( k^i = k^i_1 - k^i_2 \), then \( k^i \in K_R \) and \( k_{n^j}(z) \rightarrow k^i(z) \) at all continuity points of \( k^i \) as \( j \rightarrow \infty \).

Since \( k^i_1 \) and \( k^i_2 \) are right-continuous and nonnegative monotone increasing functions, so is \( k^i \) and the number of discontinuity points of \( k^i \) is at most countable. Thus \( k_{n^j} \rightarrow k^i \) almost everywhere as \( j \rightarrow \infty \). Applying the diagonalization argument again, for \( l = 1, 2 \) we can find a subsequence of \( \{k_{n^j,l} : j = 1, 2, \cdots\} \), denoted by \( \{k_{n^j+1,l} : j = 1, 2, \cdots\} \), such that the new subsequences converges to some functions \( k^{i+1}_l \) on \( \mathbb{Q} \cap [(i+2)^{-1}, i+2] \), then repeat the program above again and we get \( k_{n^j+1} \xrightarrow{a.s.} k^{i+1} \) on \( [(i+2)^{-1}, i+2] \). Obviously, \( k^i(z) = k^{i+1}(z) \) on \( [(i+1)^{-1}, i+1] \). Thus \( \{k^i : i = 1, 2, \cdots\} \) converges to some function \( k^i(z) \) almost everywhere on \( [0, \infty) \). Applying the diagonalization argument again and the dominated convergence theorem, we have \( \mu(|k_{n^j} - k^i|) \rightarrow 0 \) as \( i \rightarrow \infty \). Here we have proved that \( K_R \) is compact. \( \square \)

**Corollary 2.6** For any \( R \geq 1 \), the inverse operator of \( T \), \( T^{-1} : TK_R \rightarrow K_R \), is continuous.

**Proof of Theorem 2.2** Define
\[
\hat{g}_{R,n}(\lambda) = \arg \min_{g(\lambda) \in TK_R} \|g_n - g\|_w
\]
From Lemma 2.3 and Theorem 2.2, we have \( \hat{g}_{R,n}(z) \) is well defined, i.e. it exists uniquely and is \( (\mathcal{F}_n) \)-measurable. The desired results follows from \( \hat{k}_{R,n}(z) = T^{-1}\hat{g}_{R,n}(z) \) and Corollary 2.6. \( \square \)

### 3 Consistency and asymptotic risk bound

We will prove Theorem 1.2 in this section. The consistency of \( \hat{k}_{R,n}(z) \) comes directly from the following result.

**Corollary 3.1** \( L_n(\lambda) \xrightarrow{a.s.} e^{-Tk(\lambda)} \) uniformly on any compact subset, i.e. for any compact subset \( A \) of \( \mathbb{R}_+ \), we have
\[
\sup_{\lambda \in A} |L_n(\lambda) - e^{-Tk(\lambda)}| \xrightarrow{a.s.} 0.
\]

**Proof.** Obviously, \( e^{-Tk(\lambda)} \) is Laplace transform of the distribution \( \eta_0 \) of \( X(1) \), where \( X(t) \) is a CBI processes defined by (1.2) with \( X(0) = 0 \) and \( \beta = 0 \). For any \( n \geq 1 \), \( L_n(\lambda) \) is Laplace transform of some measure denoted by \( \mu_n \). Since \( L_n(\lambda) \xrightarrow{a.s.} L_{\eta_0}(\lambda) \) for any \( \lambda \in \mathbb{R}_+ \), we have \( \mu_n \rightarrow \eta \) weakly. The desired result follows from Lemma 7.6 in Sato (1999), i.e. \( L_n(\lambda) = L_{\mu_n}(\lambda) \rightarrow L_{\eta}(\lambda) \) uniformly on any compact subset. \( \square \)
Proof of Theorem 7.2. Recall \( \hat{g}_{R,n}(\lambda) \) defined in the proof of Theorem 1.1. From Corollary 3.1 and Theorem 7.6.3 in Chung (2001), we have
\[
\sup_{\lambda \in S_w} |L_n(\lambda) - e^{-Tk(\lambda)}| \xrightarrow{a.s.} 0 \quad \text{and} \quad \sup_{\lambda \in S_w} |g_n(\lambda) - Tk(\lambda)| \xrightarrow{a.s.} 0.
\]
By the definition of \( \hat{g}_{R,n} \) in (2.3),
\[
\|g_{R,n} - Tk(\lambda)\|_w \leq \|g_{R,n} - g_n\|_w + \|g_n - Tk(\lambda)\|_w \leq 2\|g_n - Tk(\lambda)\|_w \xrightarrow{a.s.} 0. \tag{3.1}
\]
The desired result follows from this result and Corollary 2.6. \( \square \)

Lemma 3.2 Suppose conditions in Theorem 7.3 hold, then there exists a constant \( C > 0 \) such that for \( n \) large enough have
\[
nE[\|g_{R,n} - Tk\|^2_w] < C.
\]
Proof. Let \( G_n(\lambda) = \sum_{k=1}^n \xi_k(\lambda) \), where
\[
\xi_k(\lambda) = \exp \left\{ -\lambda X_k + X_{k-1}v(\lambda) + \int_0^1 \psi(v(s)\lambda)ds \right\} - 1.
\]
It’s easily to prove \( \{G_n(\lambda)\}_{n=1}^\infty \) is a \((\mathcal{F}_n)\)-martingale. Obviously, we have
\[
E[\xi_k^2|\mathcal{G}_{k-1}] = E\left[\left( \exp \left\{ -\lambda X_k + X_{k-1}v(\lambda) + \int_0^1 \psi(v(s)\lambda)ds \right\} - 1 \right)^2 \right|\mathcal{G}_{k-1}]
\]
\[
= E\left[\exp \left\{ -2\lambda X_k + 2X_{k-1}v(\lambda) + 2\int_0^1 \psi(v(s)\lambda)ds \right\} \right|\mathcal{G}_{k-1}]
\]
\[
= \exp \left\{ -X_{k-1}(v(2\lambda) - 2v(\lambda)) - \int_0^1 \psi(v(s)2\lambda)ds + 2\int_0^1 \psi(v(s)\lambda)ds \right\} - 1. \tag{3.2}
\]
Define \( V_n = \sum_{k=1}^n E[\xi_k^2|\mathcal{G}_{k-1}] \). From (3.2) and ergodicity of \( X(t) \) we have
\[
\frac{1}{n}V_n \to E\left[\exp \left\{ -X_0(v(2\lambda) - 2v(\lambda)) - \int_0^1 \psi(v(s)2\lambda)ds + 2\int_0^1 \psi(v(s)\lambda)ds \right\} \right] - 1 = W(\lambda),
\]
where
\[
W(\lambda) = \exp \left\{ -\int_0^\infty \psi(v(s)\lambda(2\lambda) - 2v(\lambda))ds - \int_0^1 \psi(v(s)2\lambda)ds + 2\int_0^1 \psi(v(s)\lambda)ds \right\} - 1.
\]
It is easy to identify that \( W(\lambda) \) is continuous and bounded on any compact set. By the martingale central limitation theorem (see Durrett, 2010), we have
\[
h_n(\lambda) := \frac{1}{\sqrt{n}}G_n(\lambda) = \frac{1}{\sqrt{n}}\sum_{k=1}^n \xi_k(\lambda) \xrightarrow{d} N(0, W(\lambda)).
\]
Furthermore, by the definition of \( g_n \)
\[
g_n(\lambda) - Tk(\lambda) = \ln \left[ \frac{1}{n} \sum_{k=1}^n \exp \left\{ -\lambda X_k + X_{k-1}v(\lambda) + \int_0^1 \psi(v(s)\lambda)ds \right\} \right] = \ln \left[ h_n(\lambda)/\sqrt{n} + 1 \right].
\]
By the Skorohod’s representation theorem, see Billingsley (1999, p.70), there exist \( \{\tilde{h}_n(\lambda) : n = 1, 2, \cdots \} \) and \( \tilde{h}(\lambda) \) defined on \([0, 1], \mathcal{B}[0, 1], m) \), where \( m \) is Lebesgue measure, such that \( \tilde{h}_n(\lambda) \xrightarrow{d} \tilde{h}(\lambda) \xrightarrow{d} \tilde{h}(\lambda) \in (0, 1) \).
\( h_n(\lambda) \) for any \( n \geq 1 \), \( \tilde{h}(\lambda) \overset{d}{=} N(0, W(\lambda)) \) and \( \tilde{h}_n(\lambda) \overset{a.s.}{\to} \tilde{h}(\lambda) \). Define \( \tilde{H}_n(\lambda) = \ln[\tilde{h}_n(\lambda)/\sqrt{n} + 1] \), then

\[
\sqrt{n}(g_n(\lambda) - Tk(\lambda)) \overset{d}{=} \sqrt{n}\tilde{H}_n(\lambda) \overset{a.s.}{\to} \tilde{h}_n(\lambda) \overset{d}{=} N(0, W(\lambda)),
\]

thus \( \sqrt{n}(g_n(\lambda) - Tk(\lambda)) \overset{d}{=} N(0, W(\lambda)) \) as \( n \to \infty \) and

\[
\lim_{n \to \infty} E[n|g_n(\lambda) - Tk(\lambda)|^2] = W(\lambda).
\]

By Tonelli’s theorem and Fatou’s theorem, we have

\[
\limsup_{n \to \infty} nE[\|g_n - T^k\|^2_w] = \limsup_{n \to \infty} \int_0^{\infty} E[n|g_n(\lambda) - Tk(\lambda)|^2] w(\lambda)d\lambda
\leq \int_0^{\infty} \limsup_{n \to \infty} E[n|g_n(\lambda) - Tk(\lambda)|^2] w(\lambda)d\lambda
= \int_0^{\infty} W(\lambda)w(\lambda)d\lambda < \infty.
\]

From this and (3.1), there exists a constant \( C > 0 \) such that

\[
nE[\|\hat{g}_{R,n} - T^k\|^2_w] \leq 4nE[\|g_n - T^k\|^2_w] < C.
\]

Here we have finished this proof. \( \square \)

**Proof of Theorem 3.3** Obviously, we have

\[
\mu(|\hat{k}_{R,n} - k|) = \mu(|T^{-1}\hat{g}_{R,n} - k|) = \mu(|T^{-1}\hat{g}_{R,n} - Tk|).
\]

For any fixed \( R \geq 1 \), since \( T^{-1} \) is a continuous and linear bijection from \( TK_R \) to \( K_R \), from Corollary 2.12(c) in Rudin (1991, p49) we have

\[
\mu(|T^{-1}\hat{g}_{R,n} - Tk|) < C\|\hat{g}_{R,n} - Tk\|_w,
\]

where \( C > 0 \) is a constant determined by the norm of \( T^{-1} \). The desired result follows from this and Lemma 3.2. \( \square \)

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