Non-Abelian vortices in the emergent U(2) gauge theory of the Hubbard model

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Abstract – By the spin-fermion formula, the Hubbard model on the honeycomb lattice is represented by a $U(2)$ gauge theory using the mean-field method; non-Abelian vortex solutions are constructed based on this theory. The quantization condition shows that the magnetic flux quanta are half-integer. There are 2$k$ bosonic zero modes for the $k$-winding vortex. For the fermions, there are 2 zero-energy states (ZESs) corresponding to the single elementary vortex. In the vortex core and on the edge, the system is in the semi-metal phase with a spin gap and in the insulator phase with a Néel order, and can be mapped to the superconductor in the class A and CI, respectively.

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Introduction. – In $(2+1)$-dimensional systems, electrons moving on the lattice can be well described by the Hubbard model. It is a long-standing research topic due to the richness of its phase diagram and its possible explanation of high-$T_c$ superconductors. The Hubbard model on the honeycomb lattice is also suitable to study novel materials like graphene, which has a great potential for applications. Previously, Hou, Chamon and Mudry (HCM) discussed the Abelian vortex in graphene-like structures where the vortex configuration of the order parameter is caused by the distortion on the Kekulé texture [1]. Herbut used the Hubbard model to consider vortices in the background Néel order, and found two orthogonal ZESs of fermions in the vortex core [2]. Nevertheless, the vortex configurations in the above cases are of Abelian type.

In recent years, vortices of non-Abelian type have been discovered in $N=2$ SQCD [3–5], which is indeed a breakthrough since the Abelian Abrikosov-Nielsen-Olesen (ANO) vortex has been constructed [6]. The similarity of the emergent gauge theory (EGT) of the Hubbard model and the bosonic truncation of the SQCD motivates us to construct the non-Abelian vortices in this theory. In this work, we present a complete investigation of the EGT of the Hubbard model in the low-energy limit [7]. By the mean-field theory method, we construct the non-Abelian vortex solutions in the theory. Non-Abelian vortices have non-trivial orientational zero modes, which induce net interactions between multi-vortices [8]. We hope that the non-Abelian vortices enable us to understand novel phenomenons in the condensed matter physics, especially the unconventional superconductor.

Emergent $U(2)$ gauge theory. – The honeycomb lattice has the special property that the occupied and empty states meet at two Fermi points in momentum space. Due to the bipartite property of the honeycomb lattice, one can introduce three Pauli matrices $\sigma^a$, $\tau^a$, and $\rho^a$ to act on the spin, the sublattice (pseudospin) and the valley spaces, respectively. The standard Hamiltonian of the Hubbard model is written as $H_t + H_U$, where $H_t$ describes the hopping of electrons, and $H_U$ describes the repulsive interaction of electrons. Expanding $H_t$ near the Fermi points, one obtains [9]

$$H_t = \frac{3t}{8\pi^2} \int d^2k C_{pia}^\dagger(k) (\tau^y_{pq} k_x + \tau^x_{pq} \rho^i_{qj} k_y) C_{qja}(k),$$

where $t$ is the hopping constant. The index $s = (\uparrow, \downarrow)$ denotes the spin, $p,q = (A,B)$ denote sublattices, and $i,j = (1,2)$ denote valleys. With the $H_U$ term, the Hubbard model has a very rich phase diagram.

The emergent gauge theory kicks in an elegant way to unify phases of the Hubbard model. Hermle constructed an $SU(2)$ gauge theory for the Hubbard model on the honeycomb lattice, where the electron operators were...
written in the slave-rotor formulation, and an $SU(2)$ algebra spin liquid (ASL) is found on the insulator side of the Mott transition [10], Sachdev et al. constructed an $SU(2)$ gauge theory of the Hubbard model in a different way [7,9], which will be presented in the following. By making use of the spin-fermion formula, the electron operators can be decomposed as

$$\begin{pmatrix} c_1^\dagger \\ c_2^\dagger \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = R \Psi,$$

(2)

where $R$ denotes the spin density wave (SDW) order. $\psi^\pm$ are the spinless fermion operators, the indices $\pm$ measure the spin-projection along the local spin reference axis [9]. The $R$ fields can also be interpreted as the “auxiliary boson” in the formulation of Kotliar and Ruckenstein (KR), which represents four boson operators corresponding to four atomic states per site [11]. In ref. [7], a Higgs-like potential has been constructed for $R$, which leads to symmetry breaking and is necessary for vortex solutions. In this manner, $R$ is generalized to an invertible complex $2 \times 2$ matrix, which represents a generic spin operator for arbitrary filling [12].

Let us show how the hidden gauge symmetry emerges from the spin-fermion formula. First, the $R$ matrix can be rotated by an $SU(2)$ global spin rotation from the left:

$$R \rightarrow V R, \quad \Psi \rightarrow \Psi, \quad C \rightarrow V C.$$

(3)

As this action is global, the spin rotation will be regarded as the “flavor” symmetry. Secondly, there is a local gauge redundancy, the theory is invariant under the gauge transformations

$$R \rightarrow R U^\dagger, \quad \Psi \rightarrow U \Psi, \quad C \rightarrow C,$$

(4)

where $U$ is the element of a gauge group. Sachdev et al. consider $U \in SU(2)$, here we argue that the system is invariant under $U(2)$, and the theory is similar to the Weinberg-Salam (WS) electro-weak theory, but limited to $2 + 1$ dimensions. This extra $U(1)$ gauge group is the key ingredient for the non-Abelian vortices, because the homotopy group is non-trivial for $U(1) \times SU(2)$, i.e.,

$$\pi_1\left(\frac{U(1) \times SU(2)}{Z_2}\right) = Z.$$

(5)

Instead, when the $SU(2)$ gauge group is considered, only the $Z_2$ vortex can be constructed, which is genuinely of Abelian type [13].

Non-Abelian vortices. – The Lagrangian of the emergent $U(2)$ gauge theory of the Hubbard model reads

$$\mathcal{L} = \bar{\Psi} \gamma_\mu (\partial_\mu + i e A_\mu^a) \Psi - \lambda \Phi^a \bar{\Phi} \sigma^a \Psi + \frac{1}{2} (\partial_\mu \Phi^a - 2 \epsilon_{abc} A_\mu^b \Phi^c)^2 + s (\Phi^a)^2 + u (\Phi^a)^4 + \text{Tr} [D_\mu R(D_\mu R)^\dagger] + \tilde{s} \text{Tr} (R^\dagger R) + \tilde{u} \text{Tr} (R^\dagger R)^2,$$

(6)

where the covariant derivative is defined to be $D_\mu R \equiv \partial_\mu R - i A_\mu^a \sigma^a [7]$. The index $a$ runs from 0 to 3, while $a, b$ take values from 1 to 3 in the following, “0” stands for the Abelian $U(1)$ part of the gauge group, and $a, b$ stand for the non-Abelian $SU(2)$ part. The representation of the algebra is that $\sigma^0 = 1_2$, $\sigma^a$ is the Pauli matrix, which satisfies $\text{Tr} (\sigma^a \sigma^b) = 2 \delta^{ab}$. Hence $\Phi^a$ transforms as the triplet of the $SU(2)$ group, $\rho^0$ is the third component of the Pauli matrices which acts on the valley space, see eq. (1). The theory has a set of symmetries $U(1)_g \otimes SU(2)_g \otimes SU(2)_s$, the transformation properties of the matter contents $\Psi, \Phi^a$ and $R$ are listed in table 1.

Table 1: The transformation properties of the fields.

| Matter content | $U(1)_g$ | $SU(2)_g$ | $SU(2)_s$ |
|---------------|----------|-----------|-----------|
| $\Psi$        | 1        | 2         | 1         |
| $\Phi^a$      | 0        | 3         | 1         |
| $R$           | 1        | 2         | 2         |

Table 2: The phase diagram of the system with $U(2)$ gauge theory [9], in which the VBS is the abbreviation of “valence bond solid”, and SG is the abbreviation of “spin gap”.

| $\langle \Phi^a \rangle$ | $\langle \Phi^a \rangle \neq 0$ |
|--------------------------|--------------------------|
| $(R) = 0$                | Semi-metal               |
| $(R) \neq 0$             | Insulator with VBS order |
| $(R) \neq 0$             | Semi-metal with SG       |
| $(R) \neq 0$             | Insulator with Néel order |

$$\forall (V_f(R) U^\dagger_f = (R), \quad V_f, U^\dagger_f \in SU(2)).$$

(8)
This means the spin rotation of $R$ can be equally done by the gauge rotation in the vacuum. So a combined color-flavor $SU(2)_{c+f}$ global symmetry remains.

With all the criteria above, at an energy scale below $v_1$, the effect action can be written as

$$S = S_\Phi + S_R,$$

where $2\bar{R}$

$$S_\Phi = \int d^3x \left\{ \left( \psi^\dagger \gamma_\mu D_\mu \psi + \frac{1}{2} m^2 \psi \sigma^a \psi \right) \right\},$$

$$S_R = \int d^3x \frac{1}{g^2} \left\{ [F_{\mu\nu}]^2 + |D_\mu R|^2 + V(R) \right\},$$

where $F_{\mu\nu} \equiv i[D_\mu, D_\nu]$, and $g$ is the coupling constant of gauge fields. The Yang-Mills term is added in consideration of relativistic dynamics. The static vortex configuration of the system is the minimal energy bound.

The action in eq. (11) is the non-Abelian generalization of the Abelian Higgs model. Setting $\beta = 2\pi/g^2$, which is the Landau-Ginzburg parameter classifying the types of superconductor. We will work in the BPS limit $\beta = 1$, which means $|\bar{u}| = g^2/2$. Performing the Bogomol'nyi completion, one obtains the BPS equations of the system

$$\frac{\partial}{\partial r} f_1 = 0, \quad \frac{\partial}{\partial r} f_2 = 0,$$

where $2\bar{D} = D_1 + iD_2$ and $z = x_1 + ix_2$ is the standard complex coordinate. The energy bound of the system is

$$E \geq - \int d^2x \left( F_{12} v_2^2 \right) = -2\pi v^2 \frac{\text{Tr}(\sigma^0)}{n_0},$$

where $k \in \mathbb{Z}$ is an integer, and $n_0 = 2$ is the greatest common divisor of $SU(2)$. The generators of $SU(2)$ are traceless, so it does not contribute to the bound energy. Moreover, the $U(1)$ part does contribute, and the minimal energy bound is the quantized magnetic flux. The winding number is $k/2$, which is a half-integer [15]. The value of $\text{Tr}(\sigma^0) = 2$ depends on the normalization condition, we choose $\text{Tr}(\sigma^0) = 2$ in the present paper. This can explain naturally the magnetic flux quanta found in certain superconductor materials.

Choosing the minimal winding, the ANO-like ansatz can be embedded in the upper left-hand corner in $R$, i.e.,

$$R = \begin{pmatrix} e^{i\theta} f_1(r) & 0 \\ 0 & f_2(r) \end{pmatrix},$$

which is named the $(1,0)$ vortex. Similarly, the winding term can also be embedded in the lower right-hand corner, which is named the $(0,1)$ vortex. Both cases have degenerate energy. Here $f_1(r)$ and $f_2(r)$ are the profile functions, the boundary conditions for them are $f_1(\infty) = f_2(\infty) = v_2/\sqrt{2}$, $f_1(0) = 0$ and $\partial_r f_2(0) = 0$. Requiring $D_1 R \rightarrow 0$ when $r \rightarrow \infty$, the ansatz for the gauge field $A_1$ is written as

$$A_1 = -\frac{1}{2} \epsilon_{ij} \frac{x_i}{r^2} \left[ (1 - g_1(r)) 1_2 + (1 - g_2(r)) \sigma^3 \right].$$

The profiles $g_{1,2}$ satisfy the boundary conditions $g_{1,2}(\infty) = 0$ and $g_{1,2}(0) = 1$. It is the combination of $U(1)$ and $SU(2)$ which cancels the divergence of $\partial_i R$. Substituting the ansatz into the eq. (12) and (13), one has the BPS equations for the profiles

$$\frac{d}{dr} f_1 - \frac{1}{r} (g_1 + g_2) = 0, \quad \frac{d}{dr} g_1 - g^2 (f_1^2 + f_2^2 - v_2^2) = 0,$$

where $r \in (1,0)$. These equations can be solved numerically.

In eq. (15), the configuration of the $(1,0)$ vortex breaks the global $SU(2)_{c+f}$ color-flavor symmetry down to $U(1)_{c+f} \subset SU(2)_{c+f}$, the Nambu-Goldstone-like modes arise, which are named as the internal orientational zero modes, expressed as [4]

$$\mathbb{C} P^1 \simeq SU(2)_{c+f} \big/ U(1)_{c+f}.$$ Other solutions can be generated by “color-flavor” transformations, i.e., $R \rightarrow U_{c+f} R U_{c+f}^\dagger$, $A_1 \rightarrow U_{c+f} A_1 U_{c+f}^\dagger$, where $U_{c+f}$ parameterizes the $\mathbb{C} P^1$. All solutions have the same energy and the same boundary conditions up to a regular gauge transformation. Thus, we have a moduli space. One can use the index theorem to calculate the dimension of the moduli space [3], which is found to be $\mathcal{I} = N_c N_f k/2 = 2k$. These bosonic zero modes stand for the remnant excitations of order parameters, and are massive modes in the bulk regime (where $R$ is in the vacuum). Therefore, they are localized in the vortex core [16]. At low energy, the dynamics of the zero modes can be well described by the $\mathbb{C} P^1$ sigma model [4,5]. Our analysis above is semi-classical. When quantum mechanism is considered, the $\mathbb{C} P^1$ sigma model will develop two vacua corresponding to the $(1,0)$ and $(0,1)$ elementary vortices, respectively [4].

Next, we will consider how many ZESs there are for the 8-component Dirac fermions. For Abelian vortices, $k$ ZESs of fermions in the vortex-fermion system correspond to the $k$ vortex background [17]. The HCM model discussed the Abelian vortex in Kekulé order parameters, see eq. (3) in ref. [1]. When the spin indices are summed, two ZESs are expected for 8-component Dirac fermions. Turning on the $H_U$ term, Herbut also found two orthogonal ZESs at the center of the vortex for the Hubbard model on the honeycomb lattice [2], but the vortex configuration is in the Néel order $N^a$, which exists in the spin space. The coincidence of the two instances is due to the fact that the Néel order $\vec{N} = (N_1, N_2, 0)$ in $H_U$ will turn into a Kekulé-like order $\Delta$, i.e., $\Delta = N_1 + i N_2$, when the Hamiltonian is expressed explicitly by using the conventional 4-component Dirac spinor $(v_a, v_b, v_a, v_b)^T$ as in ref. [1]. For our non-Abelian vortices, one of the two elementary vortex configurations $(1,0)$ and $(0,1)$ survives after considering the quantum effects, and can be set as the background vortex. Essentially, the $(1,0)$ or $(0,1)$
vortex is of Abelian type. Since \( R \) can represent the Néel order by the relation \( N^o r^a = R r^a R \), analogous zero-energy Dirac equations in Herbut’s case can be given for the Hamiltonian in eq. (1). So we expect two ZESs for the (1,0) or (0,1) vortex in the extended Hubbard model, and 2k ZESs for the k-winding vortex.

It is still an open question how the bosonic orientational zero modes interact with the fermions in the vortex core, a self-consistent way to treat this question is to draw on the Bogoliubov-de Gennes (B-dG) equation [18]. In the B-dG equations, the fermion zero modes depend non-trivially on the winding term, it can be inferred that the degree of freedom of the bosonic zero modes will not affect that of the fermion zero modes. It is shown that the fermion zero modes are well localized in the vortex core [17,18]. In ref. [18], singlet and triplet states of the unbroken symmetry are found for the non-Abelian vortices in high-density QCD, where the \( SU(3)_{c+L+R} \) symmetry is broken to \( SU(2)_{c+L+R} \times U(1)_{c+L+R} \). In our model, only the \( U(1)_{c+L+R} \) symmetry group exists after symmetry breaking, one singlet state can be constructed. We leave the B-dG equations of the extended Hubbard model for future work.

Discussion and conclusion. – It is interesting to notice that the vortex core and the vortex edge are in quite different phases. Far away from the vortex core, the system is in the insulator phase. In the vortex core, the system is in the superconductor phase [22]. \( \Psi \) turns to a massive fermion when \( \langle \Phi^o \rangle \neq 0 \). Integrating them out, \( S_\Psi \) will induce an effective non-Abelian Chern-Simons (CS) action [23], i.e.,

\[
S_{\text{CS}} = \frac{1}{4} \int \frac{d^3 x}{x} e^{\mu \nu \lambda} [A^\mu_{a} \partial_\nu A^\lambda_{a} + \frac{1}{4} \epsilon^{\mu \nu \lambda} A^\mu_{a} A^\nu_{b} A^\lambda_{c}],
\]

Combining with \( S_R \), the theory will turn into non-Abelian Chern-Simons-Higgs theory. The winding number of the case is found to be \( k/n_0 \) and \((k+q)/n_0 \) (\( q \) is a real number) for topological and non-topological vortices, respectively [24]. If \( \langle \Phi^o \rangle = 0 \), the \( \Psi \) fermions are massless. \( R \) has a spin gap, so the system is in the semi-metal phase with a spin gap.

The construction of non-Abelian vortices depends on the spin-fermion formula. In our analysis, we do not consider the physics in the valley and sublattice spaces. Thus, this decomposition can be applied to other types of lattices. Our discussion also holds for the 3-dimensional physics with \( x_3 \) as the direction of the vortex string.

In conclusion, we show that there is a local gauge redundancy in the emergent gauge theory of the Hubbard model, and the gauge group is \( U(2) \). Based on the effective Lagrangian in eq. (6), we construct the non-Abelian vortex solutions, and show that there are 2k orientational zero modes for the \( k \)-winding vortex. For the elementary vortices, there are 2k ZESs for the fermions. Far away from the vortex core, the system is in the insulator phase with Néel order. In the vortex core, the system is in the semi-metal phase with a spin gap.

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