On Transforming Narrowing Trees into Regular Tree Grammars Generating Ranges of Substitutions

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The grammar representation of a narrowing tree for a syntactically deterministic conditional term rewriting system and a pair of terms is a regular tree grammar that generates expressions for substitutions obtained by all possible innermost-narrowing derivations that start with the pair and end with particular non-narrowable terms. In this paper, under a certain syntactic condition, we show a transformation of the grammar representation of a narrowing tree into another regular tree grammar that overapproximately generates the ranges of ground substitutions generated by the grammar representation. In our previous work, such a transformation is restricted to the ranges w.r.t. a given single variable, and thus, the usefulness is limited. We extend the previous transformation by representing the range of a ground substitution as a tuple of terms, which is obtained by the coding for finite trees. We show a precise definition of the transformation and prove that the language of the transformed regular tree grammar is an overapproximation of the ranges of ground substitutions generated by the grammar representation. We leave an experiment to evaluate the usefulness of the transformation as future work.

1 Introduction

Conditional term rewriting [25, Chapter 7] is known to be more complicated than unconditional term rewriting in the sense of analyzing properties, e.g., operational termination [17], confluence [29], and reachability [5]. A popular approach to the analysis of conditional rewriting is to transform a conditional term rewriting system (a CTRS, for short) into an unconditional term rewriting system (a TRS, for short) that is in general an overapproximation of the CTRS in terms of reduction. This approach enables us to use existing techniques for the analysis of TRSs. For example, a CTRS is operationally terminating if the unravelled TRS [18, 25] is terminating [4]. To prove termination of the unravelled TRS, we can use many techniques for proving termination of TRSs (cf. [25]). On the other hand, it is not so easy to analyze reachability which is relevant to, e.g., (in)feasibility of conditions.

Let us consider to prove confluence of the following syntactically deterministic 3-CTRS [25, Example 7.1.5] defining the \( \text{gcd} \) operator over the natural numbers represented by \( 0 \) and \( s \):

\[
\mathcal{R}_1 = \begin{cases} 
  x < 0 \rightarrow \text{false}, & 0 - s(y) \rightarrow 0, \\
  0 < s(y) \rightarrow \text{true}, & x - 0 \rightarrow x, \\
  s(x) < s(y) \rightarrow x < y, & s(x) - s(y) \rightarrow x - y, \\
  \text{gcd}(x, x) \rightarrow x, & \text{gcd}(s(s(x), s(y))) \rightarrow \text{gcd}(x - y, s(y)) \quad \text{if } y < x \rightarrow \text{true}, \\
  \text{gcd}(s(x), 0) \rightarrow s(x), & \text{gcd}(s(s(x), s(y))) \rightarrow \text{gcd}(s(s(x), y - x)) \quad \text{if } x < y \rightarrow \text{true}. 
\end{cases}
\]

\[\text{gcd}(s(x), 0) \rightarrow s(x), \quad \text{gcd}(0, s(y)) \rightarrow s(y), \quad \text{gcd}(s(x), s(y)) \rightarrow \text{gcd}(s(x), y - x) \quad \text{if } y < x \rightarrow \text{true}, \]

\[\text{gcd}(s(s(x), s(y))) \rightarrow \text{gcd}(x - y, s(y)) \quad \text{if } y < x \rightarrow \text{true}, \]

\[\text{gcd}(s(s(x), s(y))) \rightarrow \text{gcd}(s(s(x), y - x)) \quad \text{if } x < y \rightarrow \text{true}. \]

This work was partially supported by JSPS KAKENHI Grant Number JP17H01722.
A transformational approach in [11][10] does not succeed in proving confluence of $\mathcal{R}_1$. On the other hand, a direct approach to reachability analysis to prove infeasibility of the conditional critical pairs (i.e., non-existence of substitutions satisfying conditions), which is implemented in some confluence provers, does not prove confluence of $\mathcal{R}_1$ well, either. Let us consider the critical pairs of $\mathcal{R}_1$:

\[
\begin{align*}
&\langle \text{gcd}(x-y, s(y)), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
&\langle \text{gcd}(s(x), x-y), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
&\langle \text{gcd}(s(x), y-x), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
&\langle \text{gcd}(x-y, s(y)), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
&\langle \text{gcd}(s(x), x-y), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
&\langle \text{gcd}(s(x), y-x), \text{gcd}(s(x), y-x) \rangle \not\exists \text{ } \langle x \not\rightarrow y \rangle \\
\end{align*}
\]

Note that the above critical pairs are symmetric because they are caused by overlaps at the root position only. An operationally terminating CTRS is confluent if all critical pairs of the CTRS are infeasible (cf. [1][3]). Operational termination of $\mathcal{R}_1$ can be proved, e.g., AProVE [8]. To prove infeasibility of the critical pairs above, it suffices to show both (i) non-existence of terms $t$ such that $t < t \rightarrow_{\mathcal{R}_1}^* \text{true}$, and (ii) non-existence of terms $t_1, t_2$ such that $t_1 < t_2 \rightarrow_{\mathcal{R}_1}^* \text{true}$ and $t_2 < t_1 \rightarrow_{\mathcal{R}_1}^* \text{true}$. Thanks to the meaning of $<$, it would be easy for a human to notice that such terms $t, t_1, t_2$ do not exist. However, it is not so easy to mechanize a way to show non-existence of $t, t_1, t_2$. In fact, confluence provers for CTRSs, CoCon [23], Co3 [20], and CoScart [9], based on e.g., transformations of CTRSs into TRSs or reachability analysis for infeasibility of conditional critical pairs, failed to prove confluence of $\mathcal{R}_1$ (see Confluence Competition 2016, 2017, and 2018). In addition, a semantic approach in [16][15] cannot prove confluence of $\mathcal{R}_1$ using AGES [12], which is based on generating logical models of order-sorted first-order theories—non-existence of $t_1, t_2$ above cannot be proved via its web interface with default parameters. Timbuk 3.2 [7], which is based on tree automata techniques [6], cannot prove infeasibility of $x < y \rightarrow \text{true}$, $y < x \rightarrow \text{true}$ w.r.t. the rules for $<$ under the default use.

The non-existence of a term $t$ with $t < t \rightarrow_{\mathcal{R}_1} \text{true}$ can be reduced to the non-existence of substitutions $\theta$ such that $x < x \rightarrow_{\theta, \mathcal{R}_1}^* \text{true}$, where $\rightarrow$ denotes the narrowing step [14]—for example, $x < y \rightarrow_{\theta, \mathcal{R}_1} \text{true}$, where $\rightarrow_{\theta, \mathcal{R}_1}$ denotes the narrowing step. In addition, the non-existence of such substitutions can be reduced to the emptiness of the set of substitutions, i.e., the emptiness of $\{ \theta \mid x < x \rightarrow_{\theta, \mathcal{R}_1}^* \text{true} \}$. From this viewpoint, for a pair of terms, the enumeration of substitutions obtained by narrowing would be useful in analyzing rewriting that starts with instances of the pair. To analyze sets of substitutions derived by innermost narrowing, narrowing trees [23] are useful. For example, infeasibility of conditional critical pairs of some normal 1-CTRS can be proved by using the grammar representation of a narrowing tree [21]. Simplification of the grammar representation implies the non-existence of substitutions satisfying the conditional part of a critical pair. However, there are some examples (shown later) for which the simplification method in [21] does not succeed in converting grammar representations to those explicitly representing the empty set.

In this paper, under a certain syntactic condition, we show a transformation of the grammar representation of a narrowing tree into a regular tree grammar [2] (an RTG, for short) that overapproximates the ranges of ground substitutions generated by the grammar representation. The aim of the transformation is to simplify grammar representations as much as possible together with the existing one in [21].

Let $\mathcal{R}$ be a syntactically deterministic 3-CTRS (a 3-SDCTRS, for short) that is a constructor system, $s$ a basic term, and $t$ a constructor term, where basic terms are of the form $f(u_1, \ldots, u_n)$ with a defined

1. http://cops.uibk.ac.at/results/?y=2018&c=CTRS
symbol \( f \) and constructor terms \( u_1, \ldots, u_n \). A narrowing tree \([23, 21]\) of \( \mathcal{R} \) with the root pair \( s \rightarrow t \) is a finite representation that defines the set of substitutions \( \theta \) such that the pair \( s \rightarrow t \) narrows to a particular ground term \( u_T \) consisting of a special binary symbol \& and a special constant \( \top \) by innermost narrowing \( i_{\theta, \mathcal{R}}^{s \rightarrow t} \) with a substitution \( \theta \) (i.e., \( (s \rightarrow t) \stackrel{i_{\theta, \mathcal{R}}^{s \rightarrow t}}{\longrightarrow} u_T \) and thus \( \theta s \rightarrow_{\theta, \mathcal{R}} t \)). Note that \( \rightarrow \) is considered a binary symbol, \( (x \rightarrow x) \rightarrow \top \) is assumed to be included in \( \mathcal{R} \), and \( \stackrel{i_{\theta, \mathcal{R}}^{s \rightarrow t}}{\longrightarrow} \) denotes the constructor-based rewriting step which applies rewrite rules to basic terms. Such a narrowing tree can be the enumeration of substitutions obtained by innermost narrowing of \( \mathcal{R} \) to ground terms consisting of \& and \( \top \). The idea of narrowing trees has been extended to finite representations of SLD trees for logic programs \([24]\).

Using narrowing trees, it is easy to see that there is no substitution \( \theta \) such that \( x < x \rightarrow_i^{s \rightarrow t} \top \), true, and hence the above four critical pairs with \( x < x \rightarrow \top \) are infeasible. Let us now consider to prove infeasibility of \( x < y \rightarrow \top \), \( y < x \rightarrow \top \). A narrowing tree for \( x < y \rightarrow \top \) \& \( y < x \rightarrow \top \) can be represented by the following grammar representation \([23, 21]\) that can be considered an RTG (see Section 4):

\[
G_{x<y->true&<x->true} \rightarrow \Gamma_{x<y->true} \& \Gamma_{y<x->true} \\
\Gamma_{x<y->true} \rightarrow \{ x \mapsto 0, \ y \mapsto s(y_2) \} \\
\quad \text{REC}(\Gamma_{x<y->true}, \{ x \mapsto x, \ y \mapsto y \}) \bullet \{ x \mapsto s(x), \ y \mapsto s(y) \} \\
\Gamma_{y<x->true} \rightarrow \text{REC}(\Gamma_{x<y->true}, \{ x \mapsto y, \ y \mapsto x \})
\]

We denote by \( G_1 \) the RTG with the initial non-terminal \( \Gamma_{x<y->true&<x->true} \), the other non-terminals \( \Gamma_{x<y->true}, \Gamma_{y<x->true} \), and the above production rules. We also denote by \( P_1 \) the set of the above production rules, i.e., \([1]\). Substitutions are considered constants, and the RTG generates terms over \&, \( \varnothing \), \( \circ \), \( \bullet \), REC, and substitutions. Thebinary symbols \( \bullet \) and \& are interpreted by standard composition and parallel composition \([13, 26]\), respectively. Parallel composition \( \uparrow \) of two substitutions returns a most general unifier of the substitutions if the substitutions are unifiable (see Definition 4.2). For example, \( \{ y' \mapsto a, y \mapsto a \} \uparrow \{ y' \mapsto a, y \mapsto a \} \) returns \( \{ y' \mapsto a, y \mapsto a \} \) and \( \{ y' \mapsto a, y \mapsto a \} \uparrow \{ y' \mapsto a \} \) fails. The symbol REC is used for recursion, which is interpreted as standard composition of a renaming and a substitution recursively generated. To simplify the discussion in the remainder of this section, following the meaning of the operators, we simplify the rules of \( \Gamma_{x<y->true} \) and \( \Gamma_{y<x->true} \) as follows:

\[
\Gamma_{x<y->true} \rightarrow \{ x \mapsto 0, \ y \mapsto s(y_2) \} \mid \Gamma_{x<y->true} \bullet \{ x \mapsto s(x), \ y \mapsto s(y) \} \\
\Gamma_{y<x->true} \rightarrow \Gamma_{y<x->true} \bullet \{ x \mapsto y, \ y \mapsto x \}
\]

In our previous work \([21]\), to show the emptiness of the set of substitutions generated from e.g., \( \Gamma_{x<y->true} \& \Gamma_{y<x->true} \), we transform the grammar representation to an RTG that overapproximately generates the ranges of ground substitutions w.r.t. a single variable. For example, for \( x \), the production rules of (2) is transformed into the following ones:

\[
\Gamma_{x<y->true} \rightarrow 0 \mid s(\Gamma_{x<y->true}) \quad \Gamma_{y<x->true} \rightarrow \Gamma_{x<y->true} \\
\Gamma_{x<y->true} \rightarrow s(A) \mid s(\Gamma_{x<y->true}) \quad A \rightarrow 0 \mid s(A) \mid true \mid false
\]

Note that non-terminal \( A \) generates arbitrary ground constructor terms. Since we focus on \( x \) only, non-terminals \( \Gamma_{x<y->true} \) and \( \Gamma_{y<x->true} \) generate \( \{ s^n(a) \mid n \geq 0, \ a \in \{ 0, true, false \} \} \) and \( \{ s^n(a) \mid n > 0, \ a \in \{ 0, true, false \} \} \), respectively, and we cannot prove that there is no substitution generated from \( \Gamma_{x<y->true} \& \Gamma_{y<x->true} \).

In this paper, we aim at showing that there is no substitution generated by (2) from the initial non-terminal \( \Gamma_{x<y->true&<y<x->true} \), i.e., showing that \( L(G_1; \Gamma_{x<y->true}) \cap L(G_1; \Gamma_{y<x->true}) = \emptyset \). To this end,
under a certain syntactic condition, we show a transformation of the grammar representation of a narrowing tree into an RTG that overapproximately generates the ranges of ground substitutions generated by the grammar representation (Section 5). More precisely, using the idea of coding for tuples of ground terms [2, Section 3.2.1] (see Figure 1), we extend a transformation in [21] w.r.t. a single variable to two variables. It is straightforward to further extend the transformation to three or more variables. We do not explain how to, given a constructor 3-SDCTRS, construct (the grammar representation of) a narrowing tree, and concentrate on how to transform a grammar representation into an RTG that generates the ranges of ground substitutions generated by the grammar representation.

**Outline of Our Approach** Using the rules of [2], we briefly illustrate the outline of the transformation. Roughly speaking, we apply the coding for tuples of terms to the range of substitutions, e.g., x ↠ 0, y ↠ s(y2) for \{x ↠ 0, y ↠ s(y2)\}. The rules for \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\) are transformed into

\[
\begin{align*}
\Gamma_{x<\mathbf{y}\rightarrow\mathbf{false}}^{(y,x)} & \rightarrow 0s(A\perp) \\
\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)} & \rightarrow ss(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)})
\end{align*}
\]

where the non-terminal \(\perp A\) generates ground terms obtained by applying the coding to \(\perp\) and ground constructor terms. The coding of \(s(x)\) and \(s(y)\) is \(ss(xy)\). Variables \(x, y\) are instantiated by substitutions generated from \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\), and hence we replaced \(xy\) by \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\). The rule for \(\Gamma_{y<\mathbf{x}\rightarrow\mathbf{true}}\) is transformed into

\[
\Gamma_{y<\mathbf{x}\rightarrow\mathbf{true}}^{(y,x)} \rightarrow \Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)}
\]

Since \(x, y\) are swapped by \(\{x ↠ y, y ↠ x\}\), we generate a new non-terminal \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)}\) and its rules as well as the above rules:

\[
\begin{align*}
\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)} & \rightarrow s0(A\perp) \\
\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)} & \rightarrow ss(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)})
\end{align*}
\]

where the non-terminal \(A\perp\) generates ground terms obtained by applying the coding to ground constructor terms and \(\perp\). Every ground term generated from \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(y,x)}\) contains \(0s\), and every ground term generated from \(\Gamma_{y<\mathbf{x}\rightarrow\mathbf{true}}^{(x,y)}\) contains \(s0\). Neither \(0s\) nor \(s0\) is shared by the languages of \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}^{(x,y)}\) and \(\Gamma_{y<\mathbf{x}\rightarrow\mathbf{true}}^{(x,y)}\), and hence there is no substitution which corresponds to an expression generated from \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\&\&y<\mathbf{x}\rightarrow\mathbf{true}\). For this reason, we can transform \(\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\&\&y<\mathbf{x}\rightarrow\mathbf{true}\) of [1] into

\[
\Gamma_{x<\mathbf{y}\rightarrow\mathbf{true}}\&\&y<\mathbf{x}\rightarrow\mathbf{true} \rightarrow \emptyset
\]

which means that there exist no constructor substitution \(\theta\) satisfying the condition \(x < y \rightarrow \mathbf{true} \& y < x \rightarrow \mathbf{true}\) under the constructor-based rewriting.

One may think that tuples of terms are enough for our goal. However, substitutions are generated by standard compositions, and tuples makes us introduce composition of tuples. For example, the range of \(\sigma = \{x ↠ f(x', g(a)), y ↠ f(y', a)\}\) is represented as a tuple \(\text{tup}_2(f(x', g(a)), f(y', a))\), where \(\text{tup}_2\) is a binary symbol for tuples of two terms. To apply \(\theta = \{x' ↠ g(a), y' ↠ f(a, a)\}\) to the tuple, we reconstruct.
a tuple from \(\text{tup}_2 (f(x', g(a)), f(y', a))\) and \(\theta\). On the other hand, the coding of terms makes us avoid the reconstruction and use standard composition of substitutions to compute the range of composed substitution. For example, \(\sigma\) and \(\theta\) can be represented by \(\{xy \mapsto ff(x'y', ga(a_\perp))\}\) and \(\{x'y' \mapsto gf(aa, a_\perp)\}\), respectively, where both \(xy\) and \(x'y'\) are considered single variables.

Using the rules for \(\Gamma_{x < y \rightarrow \text{true}}\) of (2), we further show that the weakness of the above approach of using tuples. Let us try to transform the rules of \(\Gamma_{x < y \rightarrow \text{true}}\) into an RTG that generates \(\{\text{tup}_2(s^m(0), s^n(a)) \mid 0 \leq m < n, a \in \{0, \text{true}, \text{false}\}\}\). The first rule \(\Gamma_{x < y \rightarrow \text{true}} \rightarrow \{x \mapsto 0, y \mapsto s(y_2)\}\) is transformed into \(\Gamma_{x < y \rightarrow \text{true}} \rightarrow \text{tup}_2(0, s(A))\) with the rules of \(A\) above. The second rule \(\Gamma_{x < y \rightarrow \text{true}} \rightarrow \Gamma_{x < y \rightarrow \text{true}} \bullet\{x \mapsto s(x), y \mapsto s(y)\}\) is transformed into \(\Gamma_{x < y \rightarrow \text{true}} \rightarrow \text{tup}_2(s(\Gamma_{x < y \rightarrow \text{true}}^x), s(\Gamma_{x < y \rightarrow \text{true}}^y))\) with the rules of \(\Gamma_{x < y \rightarrow \text{true}}\) and \(\Gamma_{y < x \rightarrow \text{true}}\) above. These rules generates not only terms in \(\{\text{tup}_2(s^m(0), s^n(a)) \mid 0 \leq m < n, a \in \{0, \text{true}, \text{false}\}\}\) but also other terms, e.g., \(\text{tup}_2(s(0), s(0))\). The term \(\text{tup}_2(s(0), s(0))\) should not be generated because the term can be a common element generated by \(\Gamma_{x < y \rightarrow \text{true}}\) and \(\Gamma_{y < x \rightarrow \text{true}}\) and we cannot prove \(\Gamma_{x < y \rightarrow \text{true}} \& \Gamma_{y < x \rightarrow \text{true}}\) does not generate any substitution.

## 2 Preliminaries

In this section, we recall basic notions and notations of term rewriting \([1, 25]\) and regular tree grammars \([2]\). Familiarity with basic notions on term rewriting \([1, 25]\) is assumed.

### 2.1 Terms and Substitutions

Throughout the paper, we use \(V\) as a countably infinite set of variables. Let \(F\) be a signature, a finite set of function symbols \(f\) each of which has its own fixed arity, denoted by \(\text{arity}(f)\). We often write \(f/n \in F\) instead of “an \(n\)-ary symbol \(f \in F\)”, and so on. The set of terms over \(F\) and \(V\) (\(\subseteq V\)) is denoted by \(T(F, V)\), and the set of ground terms, is abbreviated to \(T(F)\). The set of variables appearing in any of terms \(t_1, \ldots, t_n\) is denoted by \(\text{Var}(t_1, \ldots, t_n)\). We denote the set of positions of a term \(t\) by \(\text{Pos}(t)\).

For a term \(t\) and a position \(p\) of \(t\), the subterm of \(t\) at \(p\) is denoted by \(t|_p\). The function symbol at the root position \(e\) of a term \(t\) is denoted by \(\text{root}(t)\). Given terms \(s, t\) and a position \(p\) of \(s\), we denote by \(s[t|_p]s\) the term obtained from \(s\) by replacing the subterm \(s|_p\) at \(p\) by \(t\).

A substitution \(\sigma\) is a mapping from variables to terms such that the number of variables \(x\) with \(\sigma(x) \neq x\) is finite, and is naturally extended over terms. The domain and range of \(\sigma\) are denoted by \(\text{Dom}(\sigma)\) and \(\text{Ran}(\sigma)\), respectively. The set of variables in \(\text{Ran}(\sigma)\) is denoted by \(\forall \text{Ran}(\sigma): \forall \text{Ran}(\sigma) = \bigcup_{x \in \text{Dom}(\sigma)} \text{Var}(\sigma x)\). We may denote \(\sigma\) by \(\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}\) if \(\text{Dom}(\sigma) = \{x_1, \ldots, x_n\}\) and \(\sigma(x_i) = t_i\) for all \(1 \leq i \leq n\). The identity substitution is denoted by \(\text{id}\). The set of substitutions that range over a signature \(F\) and a set \(V\) of variables is denoted by \(\text{Subst}(F, V): \text{Subst}(F, V) = \{\sigma \mid \sigma\) is a substitution, \(\forall \text{Ran}(\sigma) \subseteq T(F, V)\}\). The application of a substitution \(\sigma\) to a term \(t\) is abbreviated to \(\sigma t\), and \(\sigma t\) is called an instance of \(t\). Given a set \(V\) of variables, \(\sigma|_V\) denotes the restricted substitution of \(\sigma\) w.r.t. \(V\): \(\sigma|_V = \{x \mapsto \sigma x \mid x \in \text{Dom}(\sigma) \cap V\}\). A substitution \(\sigma\) is called a renaming if \(\sigma\) is a bijection on \(V\). The composition \(\theta \cdot \sigma\) (simply \(\theta\sigma\)) of substitutions \(\sigma\) and \(\theta\) is defined as \((\theta \cdot \sigma)(x) = \theta(\sigma(x))\). A substitution \(\sigma\) is called idempotent if \(\sigma \sigma = \sigma\) (i.e., \(\text{Dom}(\sigma) \cap \forall \text{Ran}(\sigma) = \emptyset\)). A substitution \(\sigma\) is called more general than \(\theta\) a substitution \(\theta\), written by \(\sigma \leq \theta\), if there exists a substitution \(\delta\) such that \(\delta \sigma = \theta\). A finite set \(E\) of term equations \(s \approx t\) is called unifiable if there exists a unifier of \(E\) such that \(\sigma s = \sigma t\) for all term equations \(s \approx t\) in \(E\). A most general unifier (mgu) of \(E\) is denoted by \(\text{mgu}(E)\) if \(E\) is unifiable. Terms \(s\) and \(t\) are called unifiable if \(s \approx t\) is unifiable. The application of a substitution \(\theta\) to \(E\), denoted by \(\theta E\), is defined as \(\theta E = \{\theta s \approx \theta t \mid s \approx t \in E\}\).
2.2 Conditional Rewriting

An oriented conditional rewrite rule over a signature $\mathcal{F}$ is a triple $(\ell, r, c)$, denoted by $\ell \rightarrow r \leftarrow c$, such that the left-hand side $\ell$ is a non-variable term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, the right-hand side $r$ is a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and the conditional part $c$ is a sequence $s_1 \rightarrow t_1, \ldots, s_k \rightarrow t_k$ of term pairs ($k \geq 0$) where $s_1, t_1, \ldots, s_k, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. In particular, a conditional rewrite rule is called unconditional if the conditional part is the empty sequence (i.e., $k = 0$), and we may abbreviate it to $\ell \rightarrow r$. Variables in $\text{Var}(r, c) \setminus \text{Var}(\ell)$ are called extra variables of the rule. An oriented conditional term rewriting system (a CTRS, for short) over $\mathcal{F}$ is a set of oriented conditional rewrite rules over $\mathcal{F}$. A CTRS is called an (unconditional) term rewriting system (a TRS, for short) if every rule $\ell \rightarrow r \leftarrow c$ in the CTRS is unconditional and satisfies $\text{Var}(\ell) \supseteq \text{Var}(r)$. The reduction relation $\rightarrow^\ast_R$ of a CTRS $\mathcal{R}$ is defined as $\rightarrow^\ast_R = \bigcup_{n \geq 0} \rightarrow^{(n)}_R$, where $\rightarrow^{(0)}_R = \emptyset$, and $\rightarrow^{(i+1)}_R = \{ (s[\sigma]_p, s[\sigma]_p) \mid s \in \mathcal{T}(\mathcal{F}, \mathcal{V}), \ell \rightarrow r \leftarrow s_1 \rightarrow t_1, \ldots, s_k \rightarrow t_k \in \mathcal{R}, \sigma s_1 \rightarrow^{(i)}_R \sigma t_1, \ldots, \sigma s_k \rightarrow^{(i)}_R \sigma t_k \}$ for $i \geq 0$. To specify the position where the rule is applied, we may write $\rightarrow_{p, \mathcal{R}}$ instead of $\rightarrow_R$. The underlying unconditional system $\{ \ell \rightarrow r \mid \ell \rightarrow r \leftarrow c \in \mathcal{R} \}$ of $\mathcal{R}$ is denoted by $\mathcal{R}_u$. A term $t$ is called a normal form (of $\mathcal{R}$) if $t$ is irreducible w.r.t. $\mathcal{R}$. A substitution $\sigma$ is called normalized (w.r.t. $\mathcal{R}$) if $\sigma x$ is a normal form of $\mathcal{R}$ for each variable $x \in \text{Dom}(\sigma)$. A CTRS $\mathcal{R}$ is called Type 3 (3-CTRS, for short) if for every rule $\ell \rightarrow r \leftarrow c \in \mathcal{R}$ satisfies that $\text{Var}(r) \subseteq \text{Var}(\ell, c)$. $\text{Var}(s) \subseteq \text{Var}(\ell, t_1, \ldots, t_{i-1})$ for all $1 \leq i \leq k$.

The sets of defined symbols and constructors of a CTRS $\mathcal{R}$ over a signature $\mathcal{F}$ are denoted by $\mathcal{D}_R$ and $\mathcal{C}_R$, respectively: $\mathcal{D}_R = \{ \text{root}(\ell) \mid \ell \rightarrow r \leftarrow c \in \mathcal{R} \}$ and $\mathcal{C}_R = \mathcal{F} \setminus \mathcal{D}_R$. Terms in $\mathcal{T}(\mathcal{C}_R, \mathcal{V})$ are called constructor terms of $\mathcal{R}$. A substitution in $\text{Subst}(\mathcal{C}_R, \mathcal{V})$ is called a constructor substitution of $\mathcal{R}$. A term of the form $f(t_1, \ldots, t_0)$ with $f/\mathcal{N} \in \mathcal{D}_R$ and $t_1, \ldots, t_0 \in \mathcal{T}(\mathcal{C}_R, \mathcal{V})$ is called basic. A CTRS $\mathcal{R}$ is called a constructor system if for every rule $\ell \rightarrow r \leftarrow c$ in $\mathcal{R}$, $\ell$ is basic. A 3-DCTRS $\mathcal{R}$ is called syntactically deterministic (an SDCTRS, for short) if for every rule $\ell \rightarrow r \leftarrow s_1 \rightarrow t_1, \ldots, s_k \rightarrow t_k \in \mathcal{R}$, every $t_i$ is a constructor term or a ground normal form of $\mathcal{R}_u$.

A CTRS $\mathcal{R}$ is called operationally terminating if there are no infinite well-formed trees in a certain logical inference system [17]—operational termination means that the evaluation of conditions must either successfully terminate or fail in finite time. Two terms $s$ and $t$ are said to be joinable, written as $s \downarrow^\ast_R t$, if there exists a term $u$ such that $s \rightarrow^\ast_R u \leftarrow^\ast_R t$. A CTRS $\mathcal{R}$ is called confluent if $t_1 \downarrow^\ast_R t_2$ for any terms $t_1, t_2$ such that $t_1 \leftarrow^\ast_R s \rightarrow^\ast_R t_2$.

2.3 Innermost Conditional Narrowing

We denote a pair of terms $s, t$ by $s \rightarrow t$ (not an equation $s \approx t$) because we analyze conditions of rewrite rules and distinguish the left- and right-hand sides of $s \rightarrow t$. In addition, we deal with pairs of terms as terms by considering $\rightarrow$ a binary function symbol. For this reason, we apply many notions for terms to pairs of terms without notice. For readability, when we deal with $s \rightarrow t$ as a term, we often bracket it such as $(s \rightarrow t)$. As in [19], any CTRS in this paper is assumed to implicitly include the rule $(x \rightarrow x) \rightarrow T$ where $T$ is a special constant. The rule $(x \rightarrow x) \rightarrow T$ is used to test structural equivalence between two terms $t_1, t_2$ by means of $t_1 \rightarrow t_2$.

To deal with a conjunction of pairs $e_1, \ldots, e_k$ of terms ($e_i$ is either $s_i \rightarrow t_i$ or $T$) as a term, we write $e_1 \& \cdots \& e_k$ by using an associative binary symbol $\&$. We call such a term an equational term. Unlike [23], to avoid $\&$ to be a defined symbol, we do not use any rule for $\&$, e.g., $(\top \& x) \rightarrow x$. Instead of derivations ending with $T$, we consider derivations that end with terms in $\mathcal{T}(\{T, \&\})$. We assume that none of $\&$, $\rightarrow$, or $T$ is included in the range of any substitution below.

In the following, for a constructor 3-SDCTRS $\mathcal{R}$, a pair $s \rightarrow t$ of terms is called a goal of $\mathcal{R}$ if the
left-hand side $s$ is either a constructor term or a basic term and the right-hand side $t$ is a constructor term. An equational term is called a goal clause of $\mathcal{R}$ if it is a conjunction of goals for $\mathcal{R}$. Note that for a goal clause $T$, any instance $\theta T$ with $\theta$ a constructor substitution is a goal clause.

**Example 2.1** The equational term $x < y \rightarrow \text{true} \land y < x \rightarrow \text{true}$ is a goal clause of $\mathcal{R}_1$.

The narrowing relation [27] [14] mainly extends rewriting by replacing matching with unification. This paper follows the formalization in [22], while we use the rule $(x \rightarrow x) \rightarrow \mathcal{T}$ instead of the corresponding inference rule. Let $\mathcal{R}$ be a CTRS. A goal clause $S = U \land s \rightarrow t \land S'$ with $U \in \mathcal{T}(\{\top, \&\})$ is said to conditionally narrow into an equational term $T$ at an innermost position, written as $S \xrightarrow{i} T$, if there exist a non-variable position $p$ of $(s \rightarrow t)$, a variant $\ell \rightarrow r \leftarrow C$ of a rule in $\mathcal{R}$, and a constructor substitution $\sigma$ such that $\text{Var}(\ell, r, C) \cap \text{Var}(S) = \emptyset$, $(s \rightarrow t)|_p$ is basic, $(s \rightarrow t)|_p$ and $\ell$ are unifiable, $\sigma = \text{mgu}(\{(s \rightarrow t)|_p \approx \ell\})$, and $T = U \land \sigma C \land \sigma((s \rightarrow t)[r]|_p) \land \sigma S'$. Note that all extra variables of $\ell \rightarrow r \leftarrow C$ remain in $T$ as fresh variables which do not appear in $S$. We assume that $\text{Var}(S) \cap \text{Varan}((s \rightarrow t)|_p) = \emptyset$ (i.e., $\sigma|_{\text{Var}(s \rightarrow t)|_p}$ is idempotent) and $\text{Var}((s \rightarrow t)|_p) \subseteq \text{Dom}(\sigma)$. We write $S \xrightarrow{i} \sigma|_{\text{Var}(s \rightarrow t)|_p} T$ to make the substitution explicit. An innermost narrowing derivation $T_0 \xrightarrow{i_1} \sigma_1, R \cdots \xrightarrow{i_n} \sigma_n, R T_n$ (and $T_0 \xrightarrow{i_1} \sigma_1, R \cdots \xrightarrow{i_n} \sigma_n, R T_n$) denotes a sequence of narrowing steps $T_0 \xrightarrow{i_1} \sigma_1, R \cdots \xrightarrow{i_n} \sigma_n, R T_n$ with $\sigma = (\sigma_1 \cdots \sigma_n)|_{\text{Var}(T_0)}$ an idempotent substitution.

When we consider two (or more) narrowing derivations $S_1 \xrightarrow{i_1} \sigma_1, R T_1$ and $S_2 \xrightarrow{i_2} \sigma_2, R T_2$, we assume that $\text{Varan}(\sigma_1) \cap \text{Varan}(\sigma_2) = \emptyset$.

Innermost narrowing is a counterpart of constructor-based rewriting (cf. [22]). Following [22], we define constructor-based conditional rewriting on goal clauses as follows: for a goal clause $S = U \land s \rightarrow t \land S'$ with $U \in \mathcal{T}(\{\top, \&\})$, we write $S \xrightarrow{c} T$ if there exist a non-variable position $p$ of $(s \rightarrow t)$, a rule $\ell \rightarrow r \leftarrow C$ in $\mathcal{R}$, and a constructor substitution $\sigma$ such that $(s \rightarrow t)|_p$ is basic, $(s \rightarrow t)|_p = \sigma \ell$, and $T = U \land \sigma C \land (s \rightarrow t)[\sigma r]|_p \land \sigma S'$.

**Theorem 2.2** ([21]) Let $\mathcal{R}$ be a constructor SDCTRS, $T$ a goal clause, and $U \in \mathcal{T}(\{\top, \&\})$.

- If $T \xrightarrow{i} \sigma, R U$, then $\sigma T \xrightarrow{c} \sigma, R U$ (i.e., $\sigma S \xrightarrow{c} \sigma T$ for all goals $s \rightarrow t$ in $T$).
- For a constructor substitution $\theta$, if $\theta T \xrightarrow{c} U$, then there exists an idempotent constructor substitution $\sigma$ such that $T \xrightarrow{c} \sigma, R U$ and $\sigma \leq \theta$.

**Example 2.3** Consider $\mathcal{R}_1$ in Section 1 again. The following is an instance of innermost conditional narrowing of $R_1$:

$$
\begin{align*}
(gcd(s^4(0), y) & \rightarrow z) \land (s(0) < z \rightarrow \text{true}) \\
\xrightarrow{1} & (y < s(1)) \land \mathcal{R}_1 \land (y_1 < s^3(0, y_1) \rightarrow \text{true}) \land (gcd(s^3(0, y_1) \rightarrow z) \land (s(0) < z \rightarrow \text{true}) \\
\xrightarrow{2} & (y < s(0)) \land \mathcal{R}_1 \land (true \rightarrow \text{true}) \land (gcd(s^3(0) - s(0), s^2(0)) \rightarrow z) \land (s(0) < z \rightarrow \text{true}) \\
\xrightarrow{id,R_1} & \mathcal{T} \land (gcd(s^3(0, s^2(0)) \rightarrow z) \land (s(0) < z \rightarrow \text{true}) \\
\xrightarrow{id,R_1} & \mathcal{T} \land (s^2(0) \rightarrow z) \land (s(0) < z \rightarrow \text{true}) \\
\xrightarrow{id,R_1} & \mathcal{T} \land \mathcal{T} \land (s(0) < s^2(0) \rightarrow \text{true}) \\
\xrightarrow{id,R_1} & \mathcal{T} \land \mathcal{T} \land (true \rightarrow \text{true}) \\
\xrightarrow{id,R_1} & \mathcal{T} \land \mathcal{T} \land \mathcal{T}
\end{align*}
$$
The following constructor-based rewriting derivation corresponds to the above narrowing derivation:

\[(\gcd(s^1(0), s^2(0)) \rightarrow s^2(0)) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} (s(0) < s^3(0) \rightarrow \text{true}) \land (\gcd(s^3(0) - s(0), s^2(0)) \rightarrow s^2(0)) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} (\text{true} \rightarrow \text{true}) \land (\gcd(s^3(0) - s(0), s^2(0)) \rightarrow s^2(0)) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} \top \land (\gcd(s^3(0) - s(0), s^2(0)) \rightarrow s^2(0)) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} \top \land (s^2(0) \rightarrow s^2(0)) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} \top \land (s^2(0) \rightarrow \text{true}) \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} \top \land (s(0) < s^2(0) \rightarrow \text{true})\]

\[\xrightarrow{\mathcal{R}_1} \top \land \top \land \top\]

### 2.4 Regular Tree Grammars

A regular tree grammar (an RTG, for short) is a quadruple \(G = (S,N,F,P)\) such that \(F\) is a signature, \(N\) is a finite set of non-terminals (constants not in \(F\)), \(S \subset N\), and \(P\) is a finite set of production rules of the form \(A \rightarrow \beta\) with \(A \in N\) and \(\beta \in T(F \cup N)\). Given a non-terminal \(S' \in N\), the set \(\{t \in T(F) \mid S' \rightarrow_p t\}\) is the language generated by \(G\) from \(S'\), denoted by \(L(G, S')\). The initial non-terminal \(S\) is not so relevant in this paper. A regular tree language is a language generated by an RTG from one of its non-terminals. The class of regular tree languages is equivalent to the class of recognizable tree languages which are recognized by tree automata. This means that the intersection (non-emptiness problem) for regular tree languages is decidable.

**Example 2.4** The RTG \(G_2 = (X, \{X, X'\}, \{0/0, s/1\}, \{X \rightarrow 0, X \rightarrow s(X'), X' \rightarrow s(X)\})\) generates the sets of even and odd numbers over 0 and \(s\) from \(X\) and \(X'\), respectively: \(L(G_2, X) = L(G_2) = \{s^{2n}(0) \mid n \geq 0\}\) and \(L(G_2, X') = \{s^{2n+1}(0) \mid n \geq 0\}\).

### 3 Coding of Tuples of Ground Terms

In this section, we introduce the notion of coding of tuples of ground terms [2] Section 3.2.1. To simplify discussions, we consider pairs of terms.

Let \(F\) be a signature. We prepare the signature \(F' = (F \cup \{\bot\})^2\), where \(\bot\) is a new constant. For symbols \(f_1, f_2 \in F\), we denote the function symbol \((f_1, f_2) \in F'\) by \(f_1 f_2\), and the arity of \(f_1 f_2\) is \(\max(\text{arity}(f_1), \text{arity}(f_2))\). The coding of pairs of ground terms, \([\ldots, \ldots]\), is recursively defined as follows:

- \([f(s_1, \ldots, s_m), g(t_1, \ldots, t_n)] = \text{fg}([s_1, t_1], \ldots, [s_m, t_m], [\bot, t_{m+1}], \ldots, [\bot, t_n])\) if \(m \leq n\),
- \([f(s_1, \ldots, s_m), g(t_1, \ldots, t_n)] = \text{fg}([s_1, t_1], \ldots, [s_n, t_n], [s_{n+1}, \bot], \ldots, [s_m, \bot])\) if \(m > n\),
- \([f(s_1, \ldots, s_m), \bot] = \text{f}([s_1, s_2], \ldots, [s_m, \bot])\), and
- \([\bot, g(t_1, \ldots, t_n)] = \text{g}([\bot, t_1], \ldots, [\bot, t_n])\).

Note that \(P\text{os}([t_1, t_2]) = P\text{os}(t_1) \cup P\text{os}(t_2)\). Note also that for \(i = 1, 2\) and for \(p \in P\text{os}(t_i)\), if \(p \notin P\text{os}(t_i)\), then \(\bot\) is complemented for \(t_i\). As described in [2] Section 3.2.1, the basic idea of coding is to stack function symbols as illustrated in Figure 1.

**Example 3.1** As in Figure 1, \([f(g(a), g(a)), f(f(a,a),a)] = \text{ff}(ga(a,\bot), ga(a,\bot))\).
4 Grammar Representations for Sets of Idempotent Substitutions

In this section, we briefly introduce grammar representations that define sets of idempotent substitutions. We follow the formalization in [21], which is based on success set equations in [23]. Since substitutions derived by narrowing steps are assumed to be idempotent, we deal with only idempotent substitutions which introduce only fresh variables not appearing in any previous term.

In the following, a renaming \( \xi \) is used to (partially) rename a particular term \( t \) w.r.t. a set \( X \) of variables with \( X \subseteq \text{Var}(t) \cap \text{Dom}(\xi) \) by assuming that \( \xi|_X \) is injective on \( X \) (i.e., for all variables \( x, y \in X \), if \( x \neq y \) then \( \xi x \neq \xi y \)) and \( \text{VRan}(\xi|_X) \cap (\text{Var}(t) \setminus X) = \emptyset \). For this reason, we write \( \xi|_X \) instead of \( \xi \), and call \( \xi|_X \) a renaming for \( t \) (simply, a renaming).

We first introduce terms to represent idempotent substitutions computed using composition operators \( \cdot \) and \( \uparrow \). We prepare the signature \( \Sigma \) consisting of the following symbols [21]:

- a finite number of idempotent substitutions which are considered constants, \( \emptyset \) (basic elements)
- a constant \( \emptyset \), \( \emptyset \) (the empty set/non-existence)
- an associative binary symbol \( \cdot \), \( \cdot \) (standard composition)
- an associative binary symbol \( \& \), and \( \& \) (parallel composition)
- a binary symbol \( \text{REC} \). \( \text{REC} \) (recursion with renaming)

We use infix notation for \( \cdot \) and \( \& \), and may omit brackets with the precedence such that \( \cdot \) has a higher priority than \( \& \).

We deal with terms over \( \Sigma \) and some constants used for non-terminals of grammar representations, where we allow such constants to only appear in the first argument of \( \text{REC} \). Note that a term without any constant may appear in the first argument of \( \text{REC} \). Given a finite set \( \mathcal{N} \) of constants (\( \Sigma \cap \mathcal{N} = \emptyset \)), we denote the set of such terms by \( \mathcal{T}(\Sigma \cup \mathcal{N}) \). We assume that each constant in \( \mathcal{N} \) has a term \( t \) (possibly a goal clause) as subscript such as \( \Gamma_t \). For an expression \( \text{REC}(\Gamma_t, \delta) \), the role of \( \Gamma_t \) is to generate substitutions (more precisely, terms in \( \mathcal{T}(\Sigma) \)) from \( \Gamma_t \), e.g., recursively, and the role of \( \delta \) is to connect such substitutions with other substitutions if necessary, where the application of \( \delta \) to some term results in \( t \). For this reason, we restrict the second argument of \( \text{REC} \) to renamings, and for each term \( \text{REC}(\Gamma_t, \delta) \), we require \( \delta \) to be an idempotent renaming (i.e., \( \text{Dom}(\delta) \cap \text{VRan}(\delta) = \emptyset \) and \( \delta \) is injective on \( \text{Dom}(\delta) \)) such that \( \text{VRan}(\delta) \subseteq \text{Var}(t) \), and \( \text{Dom}(\delta) \cap (\text{Var}(t) \setminus \text{VRan}(\delta)) = \emptyset \).

**Example 4.1 ([27])** The following are terms in \( \mathcal{T}(\Sigma) \):

- \( \{y \mapsto 0\} \cdot \{x \mapsto s(y)\} \),
- \( \{x' \mapsto s(y)\} \cdot \{x \mapsto x', y \mapsto y\} \),
- \( \{z \mapsto 0\} \cdot \{x \mapsto s(z)\} \),
- \( \emptyset \), \( \emptyset \) (basic elements)
- \( \emptyset \) (the empty set/non-existence)
- \( \cdot \) (standard composition)
- \( \& \) (parallel composition)
- \( \text{REC} \) (recursion with renaming)

We use infix notation for \( \cdot \) and \( \& \), and may omit brackets with the precedence such that \( \cdot \) has a higher priority than \( \& \).

Next, we recall parallel composition \( \uparrow \) of idempotent substitutions [13 26], which is one of the most important key operations to enable us to construct finite narrowing trees. Given a substitution \( \theta = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\} \), we denote the set of term equations \( \{x_1 \approx t_1, \ldots, x_n \approx t_n\} \) by \( \hat{\theta} \).
Definition 4.2 (parallel composition ↑[26]) Let \( \theta_1 \) and \( \theta_2 \) be idempotent substitutions. Then, we define \( \uparrow \) as follows: \( \theta_1 \uparrow \theta_2 = \text{mgu}(\theta_1 \cup \theta_2) \) if \( \theta_1 \) & \( \theta_2 \) is unifiable, and otherwise, \( \theta_1 \uparrow \theta_2 = \text{fail} \). Note that we define \( \theta_1 \uparrow \theta_2 = \text{fail} \) if \( \theta_1 \) or \( \theta_2 \) is not idempotent. Parallel composition is extended to sets \( \Theta_1, \Theta_2 \) of idempotent substitutions in the natural way: \( \Theta_1 \uparrow \Theta_2 = \{ \theta_1 \uparrow \theta_2 \mid \theta_1 \in \Theta_1, \theta_2 \in \Theta_2, \theta_1 \uparrow \theta_2 \neq \text{fail} \} \).

We often have two or more substitutions that can be results of \( \theta_1 \uparrow \theta_2 \) (\( \neq \text{fail} \)), while most general unifiers are unique up to variable renaming. To simplify the semantics of grammar representations for substitutions, as a result of \( \theta_1 \uparrow \theta_2 \) (\( \neq \text{fail} \)), we adopt an idempotent substitution \( \sigma \) such that \( \text{Dom}(\theta_1) \cup \text{Dom}(\theta_2) \subseteq \text{Dom}(\sigma) \). Note that most general unifiers we can adopt as results of \( \theta_1 \uparrow \theta_2 \) under the convention are still not unique, while they are unique up to variable renaming.

Example 4.3 ([21]) The parallel composition \( \{ x \mapsto s(z), y \mapsto z \} \uparrow \{ x \mapsto w \} \) may return \( \{ x \mapsto s(z), y \mapsto z, w \mapsto s(z) \} \), but we do not allow \( \{ x \mapsto s(y), z \mapsto y, w \mapsto s(y) \} \) as a result because \( y \) appears in the range. On the other hand, \( \{ x \mapsto s(z), y \mapsto z \} \uparrow \{ x \mapsto y \} = \text{fail} \).

A key of construction of narrowing trees (and their grammar representations) is compositionality of innermost narrowing (cf. [21]): \( S_1 \uparrow \uparrow \uparrow_{\sigma_1, \mathcal{R}} S_2 \) if and only if \( S_1 \uparrow \uparrow_{\sigma_1, \mathcal{R}} T_1, S_2 \uparrow \uparrow_{\sigma_1, \mathcal{R}} T_2, T = T_1 \uparrow \downarrow T_2 \), and \( \sigma = \sigma_1 \uparrow \sigma_2 \). To compute a substitution derived by innermost narrowing from a goal clause \( S_1 \uparrow \sigma_2 \), we compute substitutions \( \sigma_1 \) and \( \sigma_2 \) derived by innermost narrowing from \( S_1 \) and \( S_2 \), respectively, and then compute \( \sigma_1 \uparrow \sigma_2 \). When we compute \( \sigma_1 \uparrow \sigma_2 \) from two narrowing derivations \( S_1 \uparrow \uparrow_{\sigma_1, \mathcal{R}} T_1 \) and \( S_2 \uparrow \uparrow_{\sigma_2, \mathcal{R}} T_2 \), we assume that \( \mathcal{V} \text{Ran}(\sigma_1) \cap \mathcal{V} \text{Ran}(\sigma_2) = \emptyset \). To satisfy this assumption explicitly in the semantics for \( \mathcal{T}(\Sigma) \), we introduce an operation \( \text{fresh}_\delta(\cdot) \) of substitutions to make a substitution introduce only variables that do not appear in \( \text{Dom}(\delta) \cup \mathcal{V} \text{Ran}(\delta) \): for substitutions \( \sigma, \delta \), we define \( \text{fresh}_\delta(\sigma) \) by \( (\xi \cdot \sigma)|_{\text{Dom}(\sigma)} \) where \( \xi \) is a renaming such that \( \text{Dom}(\xi) = \mathcal{V} \text{Ran}(\sigma) \) and \( \mathcal{V} \text{Ran}(\xi) \cap (\text{Dom}(\delta) \cup \mathcal{V} \text{Ran}(\delta) \cup \text{Dom}(\sigma)) = \emptyset \). The subscript \( \delta \) of \( \text{fresh}_\delta(\cdot) \) is used to specify freshness of variables—we say that a variable \( x \) is \( \text{fresh} \) w.r.t. a set \( X \) of variables if \( x \notin X \).

A term \( e \) in \( \mathcal{T}(\Sigma) \) defines a substitution. The semantics of terms in \( \mathcal{T}(\Sigma) \) is inductively defined as follows [21]:

- \( \llbracket \theta \rrbracket = \theta \) if \( \theta \) is a substitution,
- \( \llbracket e_1 \cdot e_2 \rrbracket = \llbracket e_1 \rrbracket \cdot \llbracket e_2 \rrbracket \) if \( \llbracket e_2 \rrbracket \neq \text{fail} \) and \( \llbracket e_1 \rrbracket \neq \text{fail} \),
- \( \llbracket e_1 \& e_2 \rrbracket = (\theta_1 \uparrow \theta_2)|_{\text{Dom}(\theta_1) \cap \text{Dom}(\theta_2)} \) if \( \llbracket e_1 \rrbracket \neq \text{fail} \) and \( \llbracket e_2 \rrbracket \neq \text{fail} \), where \( \theta_1 = \llbracket e_1 \rrbracket \) and \( \theta_2 = \text{fresh}_{\theta_1}(\llbracket e_2 \rrbracket) \),
- \( \llbracket \text{REC}(e, \delta) \rrbracket = (\text{fresh}_\delta(\llbracket e \rrbracket) \cdot \delta)|_{\text{Dom}(\delta)} \) if \( \llbracket e \rrbracket \neq \text{fail} \) and \( \mathcal{V} \text{Ran}(\delta) \subseteq \text{Dom}(\llbracket e \rrbracket) \), and
- otherwise, \( \llbracket e \rrbracket = \text{fail} \) (e.g., \( \llbracket \varnothing \rrbracket = \text{fail} \)).

Notice that \( \Gamma_i \), a non-terminal used in an RTG, is not included in \( \mathcal{T}(\Sigma) \), and thus, \( \llbracket \Gamma_i \rrbracket \) is not defined. Since \( \uparrow \) may fail, we allow to have \( \text{fail} \), e.g., \( \llbracket \{ y \mapsto s(x) \} \cdot \{ x \mapsto y \} \} = \{ x \mapsto y \} \) & \{ z \mapsto 0 \} \} = \text{fail} \). The number of variables appearing in an RTG defined below is finite. However, we would like to use RTGs to define infinitely many substitutions such that the maximum number of variables we need cannot be fixed. To solve this problem, in the definition of \( \llbracket \text{REC}(e, \delta) \rrbracket \), we introduced the operation \( \text{fresh}_\delta(\cdot) \) that makes all variables introduced by \( \llbracket e \rrbracket \) \( \text{fresh} \) w.r.t. \( \text{Dom}(\delta) \cup \mathcal{V} \text{Ran}(\delta) \). In [23], this operation is implicitly considered, but in [21], \( \text{REC} \) is explicitly introduced to the syntax in order to convert terms in \( \mathcal{T}(\Sigma) \) precisely. To assume \( \mathcal{V} \text{Ran}(\llbracket e_1 \rrbracket) \cap \mathcal{V} \text{Ran}(\llbracket e_2 \rrbracket) = \emptyset \) for \( \llbracket e_1 \rrbracket \& \llbracket e_2 \rrbracket \), we also introduced \( \text{fresh}_{\theta_1}(\cdot) \) in the case of \( \llbracket e_1 \rrbracket \& \llbracket e_2 \rrbracket \).

\(^2\) For \( \mathcal{V} \text{Ran}(\xi) \), we choose variables not appearing in any substitutions in \( \Sigma \).
On Transforming Narrowing Trees into RTGs Generating Ranges of Substitutions

The semantics of terms in $\mathcal{T}(\Sigma)$ is naturally extended to subsets of $\mathcal{T}(\Sigma)$ as follows: for a set $L \subseteq \mathcal{T}(\Sigma)$, $[[L]] = \{[[e]] | e \in L, [[e]] \neq \text{fail}\}$.

Example 4.4 (27) The expressions in Example 4.1 are interpreted as follows:

- $[[\{y \mapsto 0\} \bullet \{x \mapsto s(y)\}]] = \{y \mapsto 0\} \bullet \{x \mapsto s(y)\} = \{x \mapsto s(0), y \mapsto 0\}$,
- $[[\{x' \mapsto s(y)\} \bullet \{x \mapsto x'\} \cup \{x \mapsto s(z)\}]]$
  $= \left(\{x \mapsto s(y), x' \mapsto s(y)\} \cup \text{fresh}_{\{x \mapsto s(y), x' \mapsto s(y)\}}(\{x \mapsto s(z)\})\right)_{(x,x')}$
  $= \left(\{x \mapsto s(z'), x' \mapsto s(z')\}\right)_{(x,x')}$
  $= \{x \mapsto s(z'), x' \mapsto s(z')\}$,
- $[[\{\emptyset \cup \{y \mapsto z\}\} \bullet \{x \mapsto s(y)\}]] = \text{fail}$ (since $[[\emptyset]] = \text{fail}$ and then $[[\emptyset \cup \{y \mapsto z\}]] = \text{fail}$), and
- $[[\text{REC}((x \mapsto 0, y \mapsto s(y')), \{x' \mapsto x, y' \mapsto y\}) \bullet \{y \mapsto s(x')\}]]$
  $= \left(\text{fresh}_{\{x \mapsto 0, y \mapsto s(y')\}}(\{x \mapsto 0, y \mapsto s(y')\}) \bullet \{x' \mapsto x, y' \mapsto y\}\right)_{(x',y')}$
  $= \left(\{x \mapsto 0, y \mapsto s(y')\} \bullet \{x' \mapsto x, y' \mapsto y\}\right)_{(x',y')}$
  $= \{x \mapsto 0, y \mapsto s(y'), y \mapsto s(0)\}$.

To define sets of idempotent substitutions, we adopt RTGs. In the following, we drop the third component from grammars constructed below because the third one is fixed to $\Sigma$ with a finite number of substitutions that are clear from production rules. A substitution-set grammar (SSG) for a term $t_0$ is an RTG $G = (\Gamma_{t_0}, \mathcal{N}, \mathcal{P})$ such that $\mathcal{N}$ is a finite set of non-terminals $\Gamma_i$, $\Gamma_{t_0} \in \mathcal{N}$, and $\mathcal{P}$ is a finite set of production rules of the form $\Gamma_i \rightarrow \beta$ with $\beta \in \mathcal{T}(\Sigma \cup \mathcal{N})$. Note that $L(G, \Gamma_i) = \{e \in \mathcal{T}(\Sigma) | \Gamma_i \rightarrow^* G e\}$ for each $\Gamma_i \in \mathcal{N}$, and the numbers of variables appearing in $L(G, \Gamma_i)$ is finite. The set of substitutions generated by $G$ from $\Gamma_i \in \mathcal{N}$ is $[[L(G, \Gamma_i)]]$, i.e., $[[L(G, \Gamma_i)]] = \{[[e]] | e \in L(G, \Gamma_i), [[e]] \neq \text{fail}\}$. Note that the number of variables in $\bigcup_{\theta \in [[L(G, \Gamma_i)]]} \text{Ran}(\theta)$ may be infinite because of the interpretation for REC.

Example 4.5 The RTG $G_1$ in Section 4 is an SSG for a term $\Gamma_{x<y\rightarrow \text{true} \land y<x\rightarrow \text{true}}$. We have that

$L(G_1, \Gamma_{x<y\rightarrow \text{true}}) = \left\{ x \mapsto 0, y \mapsto s(y_2), \begin{array}{l}\text{REC}((x \mapsto 0, y \mapsto s(y_2)), \{x_3 \mapsto x, y_3 \mapsto y\}) \bullet \{x \mapsto s(x_3), y \mapsto s(y_3)\}, \hfill \\
\text{REC}((x \mapsto 0, y \mapsto s(y_2)), \{x_3 \mapsto x, y_3 \mapsto y\}) \bullet \{x \mapsto s(x_3), y \mapsto s(y_3)\}, \\
\text{REC}((x \mapsto 0, y \mapsto s(y_2)), \{x_3 \mapsto x, y_3 \mapsto y\}) \bullet \{x \mapsto s(x_3), y \mapsto s(y_3)\}, \\
\vdots \end{array} \right\}$

and $[[L(G_1, \Gamma_{x<y\rightarrow \text{true}})]] = \{x \mapsto s^m(0), y \mapsto s^m(a) | 0 \leq m < n, a \in \{0, \text{true}, \text{false}\}\}$.

5 Transforming SSGs into RTGs Generating Ranges of Substitutions

In this section, given a goal clause $T$ and two variables $x_1, x_2$ appearing in $T$, we show a transformation of an SSG $G = (\Gamma_{t_0}, \mathcal{N}, \mathcal{P})$ into an RTG $G'$ such that $L(G', \Gamma_{T(x_1,x_2)}^{x_1,x_2}) \supseteq \{[[\xi \theta x_1, \xi \theta x_2] | \theta \in L(G, \Gamma_T), \xi \in \text{Subst}(\Gamma), \text{Var}(\xi x_1, \xi x_2) \subseteq \text{Dom}(\xi)]\}$, where $\text{C}$ is a set of constructors we deal with. Note that $T$ does not

Note that $\{x \mapsto s(y), x' \mapsto s(y)\} \cup \{x \mapsto s(z)\} = \{x \mapsto s(z'), x' \mapsto s(z'), y \mapsto s(z')\}$.
have to be $T_0$. The transformation is an extension of the transformation in [21, Section 7] and applicable to SSGs satisfying a certain syntactic condition shown later. In the following, we aim at showing that $L(G_1, \Gamma_{x<\text{cy}=\text{true}}) \cap L(G_1, \Gamma_{y<\text{cy}=\text{true}}) = \emptyset$. We use $C$ as a set of constructors unless noted otherwise.

Let $G$ be an SSG $(\Gamma_0, \mathcal{N}, P)$ and $T$ a goal clause such that $\Gamma_T \in \mathcal{N}$. We denote by $P|_{\Gamma_T}$ the set of production rules that are reachable from $\Gamma_T$. We assume that any rule in $P|_{\Gamma_T}$ is of the following form:

$$\Gamma_{T'} \rightarrow \theta_1 | \cdots | \theta_m | \text{REC}(\Gamma_{T_1}, \delta_1) \cdot \theta_{m+1} | \cdots | \text{REC}(\Gamma_{T_n}, \delta_n) \cdot \theta_{m+n}$$

where $\forall \text{Ran}(\delta_j) = \text{Var}(T_j) \setminus \{x\}$ for all $1 \leq j \leq n$, and $\theta_1, \ldots, \theta_{m+n}$ are idempotent substitutions such that $\text{Dom}(\theta_j) = \text{Var}(T_j)$ for all $1 \leq j \leq m + n$. Note that $\Gamma_{T'} \rightarrow \text{REC}(\Gamma_{T'}, \delta)$ is considered $\Gamma_{T'} \rightarrow \text{REC}(\Gamma_{T'}, \delta) \cdot \theta_{m+i}$ with $1 \leq i \leq n$, we assume that for all variables $x, y$ in $T'$ and for each position $p \in \text{Pos}(\delta \theta_{m+i} x) \cap \text{Pos}(\delta \theta_{m+i} y)$, all of the following hold:

- if $(\delta \theta_{m+i} x)|_p \in \text{Var}(T_i)$, then $(\delta \theta_{m+i} y)|_p \in \text{Var}(T_i) \cup \mathcal{T}(\mathcal{C}, \mathcal{V} \setminus \text{Var}(T_i))$,
- if $(\delta \theta_{m+i} y)|_p \in \text{Var}(T_i)$, then $(\delta \theta_{m+i} x)|_p \in \text{Var}(T_i) \cup \mathcal{T}(\mathcal{C}, \mathcal{V} \setminus \text{Var}(T_i))$.

This assumption implies that for such $x, y$, and $p$, the terms $(\delta \theta_{m+i} x)|_p$ and $(\delta \theta_{m+i} y)|_p$ satisfy one of the following:

- both are rooted by function symbols,
- both are variables in $\text{Var}(T_i)$,
- one is a variable in $\text{Var}(T_i)$ and the other is a term in $\mathcal{T}(\mathcal{C}, \mathcal{V} \setminus \text{Var}(T_i))$, or
- both are terms in $\mathcal{T}(\mathcal{C}, \mathcal{V} \setminus \text{Var}(T_i))$.

For example, both $P|_{\Gamma_{x<\text{cy}=\text{true}}}$ and $P|_{\Gamma_{y<\text{cy}=\text{true}}}$ satisfy the above assumption.

Our idea of extending the previous transformation is the use of coding; Roughly speaking, for $\Gamma_{T'} \rightarrow \text{REC}(\Gamma_{T'}, \delta) \cdot \theta_{m+i}$ with $1 \leq i \leq n$ and for all variables $x, y$ in $T'$, we apply coding to $\delta \theta_{m+i} x$ and $\delta \theta_{m+i} y$. A variable in $\text{Var}(T_i)$, which is instantiated by substitutions generated from $\Gamma_{T_i}$, may prevent us from constructing a finite number of production rules (see Example 5.3 below). For this reason, we expect any variable $\boxed{x}$ in $\text{Var}(\delta \theta_{m+i} x, \delta \theta_{m+i} y) \cap \text{Var}(T_i)$ to be coded with

- $\perp$ (the case where the precondition “$p \in \text{Pos}(\delta \theta_{m+i} x) \cap \text{Pos}(\delta \theta_{m+i} y)$’’ does not hold),
- another variable in $\text{Var}(\delta \theta_{m+i} x, \delta \theta_{m+i} y) \cap \text{Var}(T_i)$ (the case where (b) above holds), or
- a constructor term without any variable in $\text{Var}(T_i)$ (the case where (c) above holds).

**Definition 5.1** We denote the set of constructor terms appearing in substitutions in $P$ by $\text{Patterns}(P)$, where such constructor terms are instantiated with a non-terminal $A$ introduced during the transformation below: $\text{Patterns}(P) = \{ \{x \mapsto A \mid x \in \text{Var}(t)\} \mid t \text{ appears in } P, s \in \text{VRan}(\theta), t \subseteq s \}$\footnote{In defining SSGs, we only required that $\forall \text{Ran}(\delta_j) \subseteq \text{Var}(T_j)$, but to make the transformation below precise, we require that $\forall \text{Ran}(\delta_j) = \text{Var}(T_j)$. This requirement is not restrictive because SSGs for narrowing trees satisfy this requirement because $\delta_j$ connects $T_i$ with a renamed variant which has no shared variable with $T_j$.} We denote the set of variables appearing in $\mathcal{N}$ by $\text{Vars}(\mathcal{N})$: $\text{Vars}(\mathcal{N}) = \bigcup_{\mathcal{T} \subseteq \mathcal{N}} \text{Var}(\mathcal{T})$. The RTG obtained from $G$ and variables $x_1, x_2$ in $T$, denoted by $\text{Ran}(\mathcal{G}, T, x_1, x_2)$, is $\langle T(x_1, x_2), \mathcal{N'}, \mathcal{N}_A, \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_{\mathcal{AA}} \cup \mathcal{P}_{\mathcal{A}_L} \cup \mathcal{P}_{\mathcal{A}_R} \rangle$ such that $\mathcal{N}_A = \{ \{x \mapsto A \mid x \in \text{Var}(t)\} \mid t \text{ appears in } P, s \in \text{VRan}(\theta), t \subseteq s \}$\footnote{This is the case where either (a) or (d) holds.} and $\mathcal{P}_{\mathcal{AA}} = \{ \{x \mapsto A \mid x \in \text{Var}(t)\} \mid t \text{ appears in } P, s \in \text{VRan}(\theta), t \subseteq s \}$\footnote{The current definition of $\text{Patterns}(P)$ is not well optimized and $\text{Patterns}(P)$ may include some terms that are not necessary for the transformation. However, for readability, we adopt this simpler definition.}.
\( \mathcal{N}' = \{ \Gamma_{T'}^{(x,y)}, \Gamma_{T'}^{(x,t)}, \Gamma_{T'}^{(y)} \mid x,y \in \text{Vars}(\mathcal{N}), \Gamma_{T'} \in \mathcal{N}', t \in \text{Patterns}(\mathcal{P}) \cup \{ \bot \} \}, \)

\( \mathcal{N}_A = \{ AA, A \bot, \bot A \}, \)

\( \mathcal{P}_1 = \{ \Gamma_{T'}^{(t,t)} \to u \mid \Gamma_{T'} \to \theta \in \mathcal{P}, \Gamma_{T'}^{(t,t)} \in \mathcal{N}', \xi_A = \{ x \mapsto A \mid x \in \text{Var}(\theta t_1, \theta t_2) \}, u \in \langle \xi_A \theta t_1, \xi_A \theta t_2 \rangle \mathcal{T} \}, \)

\( \mathcal{P}_2 = \{ \Gamma_{T'}^{(t,t)} \to u \mid \Gamma_{T'} \to \text{REC}(\Gamma_{T''}), \delta \in \mathcal{P}, \Gamma_{T'}^{(t,t)} \in \mathcal{N}', \xi_A = \{ x \mapsto A \mid x \in \text{Var}(\delta t_1, \delta t_2) \setminus \text{Var}(\mathcal{T}'') \}, u \in \langle \xi_A \delta t_1, \xi_A \delta t_2 \rangle \mathcal{T} \mathcal{T}' \}, \)

\( \mathcal{P}_{AA} = \{ AA \to u \mid f/m, g/n \in \mathcal{C}, u \in \langle f(A, \ldots, A), g(A, \ldots, A) \rangle \mathcal{T} \}, \)

\( \mathcal{P}_{A\bot} = \{ A \bot \to u \mid f/m \in \mathcal{C}, u \in \langle f(A, \ldots, A), \bot \rangle \mathcal{T} \}, \)

\( \mathcal{P}_{\bot A} = \{ \bot A \to u \mid g/n \in \mathcal{C}, u \in \langle \bot, g(A, \ldots, A) \rangle \mathcal{T} \}, \)

where \( \langle \cdot, \cdot \rangle \mathcal{P} \), which takes a goal clause \( T' \) and two terms in \( \mathcal{T}(\mathcal{F} \cup \{ A \}, \text{Var}(T')) \) as input and returns a set of terms in \( \mathcal{T}(\mathcal{F} \cup \mathcal{N}' \cup \mathcal{N}_A) \), is recursively defined as follows:

\( \langle x, y \rangle \mathcal{T}' = \{ \Gamma_{T'}^{(x,y)} \}, \) where \( x,y \in \mathcal{V}, \)

\( \langle x, t \rangle \mathcal{T}' = \{ \Gamma_{T'}^{(x,t)} \}, \) where \( x \in \mathcal{V} \) and \( t \in \text{Patterns}(\mathcal{P}), \)

\( \langle x, \bot \rangle \mathcal{T}' = \{ \Gamma_{T'}^{(x,\bot)} \}, \) where \( x \in \mathcal{V}, \)

\( \langle t, y \rangle \mathcal{T}' = \{ \Gamma_{T'}^{(A,y)} \}, \) where \( y \in \mathcal{V} \) and \( t \in \text{Patterns}(\mathcal{P}), \)

\( \langle \bot, y \rangle \mathcal{T}' = \{ \Gamma_{T'}^{(\bot,y)} \}, \) where \( y \in \mathcal{V}, \)

\( \langle A, A \rangle \mathcal{T}' = \{ AA \}, \)

\( \langle A, \bot \rangle \mathcal{T}' = \{ A \bot \}, \)

\( \langle \bot, A \rangle \mathcal{T}' = \{ \bot A \}, \)

\( \langle \bot, g(t_1, \ldots, t_n) \rangle \mathcal{T}' = \{ \bot g(u_1, \ldots, u_n) \mid 1 \leq i \leq n, u_i \in \langle \bot, t_i \rangle \mathcal{T} \}, \)

\( \langle f(s_1, \ldots, s_m), \bot \rangle \mathcal{T}' = \{ f\bot(u_1, \ldots, u_m) \mid 1 \leq i \leq m, u_i \in \langle s_i, \bot \rangle \mathcal{T} \}, \)

\( \langle A, g(t_1, \ldots, t_n) \rangle \mathcal{T}' = \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n) \mid f/m \in \mathcal{C}, m < n, 1 \leq i \leq m, u_i \in \langle A, t_i \rangle \mathcal{T}, 1 \leq j \leq n - m, u_{m+j} \in \langle \bot, t_{m+j} \rangle \mathcal{T} \} \cup \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_m) \mid f/m \in \mathcal{C}, m \geq n, 1 \leq i \leq n, u_i \in \langle A, t_i \rangle \mathcal{T}, 1 \leq j \leq m - n, u_{n+j} \in \langle \bot, t_{n+j} \rangle \mathcal{T} \}, \)

\( \langle f(s_1, \ldots, s_m), A \rangle \mathcal{T}' = \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n) \mid g/n \in \mathcal{C}, m < n, 1 \leq i \leq m, u_i \in \langle s_i, A \rangle \mathcal{T}, 1 \leq j \leq n - m, u_{m+j} \in \langle A, t \rangle \mathcal{T} \} \cup \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_m) \mid g/n \in \mathcal{C}, m \geq n, 1 \leq i \leq n, u_i \in \langle s_i, A \rangle \mathcal{T}, 1 \leq j \leq m - n, u_{n+j} \in \langle \bot, t \rangle \mathcal{T} \}, \)

\( \langle f(s_1, \ldots, s_m), g(t_1, \ldots, t_n) \rangle \mathcal{T}' = \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n) \mid 1 \leq i \leq m, u_i \in \langle s_i, t_i \rangle \mathcal{T}, 1 \leq j \leq n - m, u_{m+j} \in \langle \bot, t_{m+j} \rangle \mathcal{T} \} \text{ if } m < n, \) and

\( \langle f(s_1, \ldots, s_m), g(t_1, \ldots, t_n) \rangle \mathcal{T}' = \{ fg(u_1, \ldots, u_m, u_{m+1}, \ldots, u_m) \mid 1 \leq i \leq n, u_i \in \langle s_i, t_i \rangle \mathcal{T}, 1 \leq j \leq m - n, u_{n+j} \in \langle s_{n+j}, \bot \rangle \mathcal{T} \} \text{ if } m \geq n. \)

Note that the non-terminal AA generates \( \{ [t_1, t_2] \mid t_1, t_2 \in \mathcal{T}(\mathcal{C}) \} \), the non-terminal A \( \bot \) generates \( \{ [t_1, \bot] \mid t_1 \in \mathcal{T}(\mathcal{C}) \} \), and the non-terminal \( \bot A \) generates \( \{ [\bot, t_2] \mid t_2 \in \mathcal{T}(\mathcal{C}) \} \). Note also that we generate only production rules that are reachable from \( \Gamma_{T'}^{(x,y)} \), and drop from \( \mathcal{N}' \cup \mathcal{N}_A \) non-terminals not appearing in the generated production rules.
Example 5.2 Consider \( G_1 = (\Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}}, \{\Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}}, \Gamma_{x<y \rightarrow \text{true}}, \Gamma_{y<x \rightarrow \text{true}}\}, P_1) \) in Section 1. We have that

- \( \text{Patterns}(P_1) = \{0, s(A), \perp\} \), and
- \( \text{Vars}(\{\Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}}, \Gamma_{x<y \rightarrow \text{true}}, \Gamma_{y<x \rightarrow \text{true}}\}) = \{x, y\} \).

Let us focus on \( \Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}} \) and \( x, y \). Since neither \( \Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}} \) nor \( \Gamma_{x<y \rightarrow \text{true}} \) is reachable from \( \Gamma_{x<y \rightarrow \text{true}} \) by \( P_1 \), when we construct the RTG \( \text{Ran}(G_1, \Gamma_{x<y \rightarrow \text{true}}, x, y) \), we do not take into account \( \Gamma_{x<y \rightarrow \text{true} \land y<x \rightarrow \text{true}}, \Gamma_{y<x \rightarrow \text{true}} \), and their rules. The RTG \( \text{Ran}(G_1, \Gamma_{x<y \rightarrow \text{true}}, x, y) = (\Gamma_{x<y \rightarrow \text{true}}', N' \cup N_A, P'_1 \cup P'_2 \cup P_{AA} \cup P_{\ominus \perp} \cup P_{\perp}) \) is constructed as follows:

- \( N' = \{\Gamma_{x<y \rightarrow \text{true}}, \Gamma_{y<x \rightarrow \text{true}}, \Gamma_{x<y \rightarrow \text{true}}\} \),
- \( N_A = \{AA, A \perp, \perp A\} \),
- \( P'_1 = \{\Gamma_{x<y \rightarrow \text{true}} \rightarrow 0s(\perp A)\} \), because \( \Gamma_{x<y \rightarrow \text{true}} \rightarrow \{x \mapsto 0, y \mapsto s(y_2)\} \in P_1 \) and \( (0, s(A)) \rightarrow = \{0s(\perp A)\} \),
- \( P'_2 = \{\Gamma_{x<y \rightarrow \text{true}} \rightarrow ss(\Gamma_{x<y \rightarrow \text{true}})\} \), because \( \Gamma_{x<y \rightarrow \text{true}} \rightarrow \text{REC}(\Gamma_{x<y \rightarrow \text{true}}, \{x_3 \mapsto x, y_3 \mapsto y\}) \bullet \{x \mapsto s(x_3), y \mapsto s(y_3)\} \in P_1 \) and \( (s(x), (s(y))) \rightarrow = \{ss(\Gamma_{x<y \rightarrow \text{true}})\} \),
- \( P_{AA} = \{AA \rightarrow u \mid u \in \{0, 0s(\perp A), 0\text{true}, 0\text{false}, s0(\perp A), ss(\perp A), s\text{true}(\perp A), s\text{false}(\perp A), \text{true0}, \text{true} \perp, \text{false} \perp, \text{false0}, \text{false} \perp, \text{falsefalse} \} \},
- \( P_{\ominus \perp} = \{ \ominus A \rightarrow u \mid u \in \{0 \perp, s(\perp A), \perp \text{true}, \perp \text{false} \} \} \), and
- \( P_{\perp} = \{ \perp A \rightarrow u \mid u \in \{0 \perp, s(\perp A), \perp \text{true}, \perp \text{false} \} \}. \)

For \( \Gamma_{y<x \rightarrow \text{true}} \) and \( x, y \), we add \( \Gamma_{x<y \rightarrow \text{true}} \rightarrow \Gamma_{y<x \rightarrow \text{true}} \) to the above production rules. Rules that are not reachable from \( \Gamma_{x<y \rightarrow \text{true}} \) or \( \Gamma_{y<x \rightarrow \text{true}} \) can be dropped from \( \text{Ran}(G_1, \Gamma_{x<y \rightarrow \text{true}}, x, y) \), obtaining an RTG, denoted by \( G_4 \), with the following production rules:

\[
\begin{align*}
\Gamma_{x<y \rightarrow \text{true}} & \rightarrow 0s(\perp A) | ss(\Gamma_{x<y \rightarrow \text{true}}) & A \perp & \rightarrow 0 \perp | s(\perp A) | \text{true} \perp | \text{false} \\
\Gamma_{x<y \rightarrow \text{true}} & \rightarrow s0(A \perp) | ss(\Gamma_{x<y \rightarrow \text{true}}) & A \ominus \perp & \rightarrow 0 \ominus | s(\perp A) | \perp \text{true} | \perp \text{false} \\
\Gamma_{y<x \rightarrow \text{true}} & \rightarrow \Gamma_{x<y \rightarrow \text{true}} & & 
\end{align*}
\]

It is easy to see that

- \( L(G_4, \Gamma_{x<y \rightarrow \text{true}}) \subseteq T(\{0s, ss, 0 \perp, s \perp, \text{true} \perp, \text{false} \perp\}) \),
- \( L(G_4, \Gamma_{y<x \rightarrow \text{true}}) \subseteq T(\{s0, ss, 0 \perp, s \perp, \text{true} \perp, \text{false} \perp\}) \),

and hence, there is no shared constant between the two sets. This means that

\[
L(G_4, \Gamma_{x<y \rightarrow \text{true}}) \cap L(G_4, \Gamma_{y<x \rightarrow \text{true}}) = \emptyset
\]

and hence

\[
[L(G_1, \Gamma_{x<y \rightarrow \text{true}})] \cap [L(G_1, \Gamma_{y<x \rightarrow \text{true}})] = \emptyset.
\]

Note that the emptiness problem of RTGs is decidable, and hence we can decide the emptiness problem of \( L(G_4, \Gamma_{x<y \rightarrow \text{true}}) \cap L(G_4, \Gamma_{y<x \rightarrow \text{true}}) \).
The following example illustrates both why not all SSGs can be transformed and why we adopt the assumption.

**Example 5.3** Let $G_5$ be the following SSG which does not satisfy the assumption:

$$(\Gamma_{x\to y}, \{ \Gamma_{x\to y} \}, \{ \Gamma_{x\to y} \to \{ x \mapsto 0, y \mapsto 0 \} | \text{REC}(\Gamma_{x\to y}, \{ x' \mapsto x, y' \mapsto y \}) \bullet \{ x \mapsto 0, y \mapsto 0 \} \})$$

The domains of substitutions generated by $G_5$ w.r.t. $x, y$ is $\{(s^n(0), s^{2n}(0)) \mid n \geq 0\}$ which is not recognizable. This implies that there is no RTG generating this set, while every substitution appearing in $G_5$ preserves linearity.

Let us now apply $\mathcal{R}$ to $G_5$, while $G_5$ does not satisfy the assumption. To generate rules from $\Gamma_{x\to y} \to \text{REC}(\Gamma_{x\to y}, \{ x' \mapsto x, y' \mapsto y \}) \bullet \{ x \mapsto 0, y \mapsto 0 \}$, we need to compute $(s(x), s(s(y)))_{\Gamma_{x\to y}}$, resulting in $ss((x, s(y))_{\Gamma_{x\to y}}$). The first argument $x$ of $(x, s(y))_{\Gamma_{x\to y}}$ cannot be instantiated any more without $\Gamma_{x\to y}$. Then, let us define $(x, s(y))_{\Gamma_{x\to y}} = \Gamma_{x\to y}^{(x,s(y))}$. Then, the non-terminal $\Gamma_{x\to y}^{(x,s(y))}$ is not generated in computing the set of non-terminals $(\mathcal{A}' \cup \mathcal{N}_A)$ in Definition 5.1. Let us now add $\Gamma_{x\to y}^{(x,s(y))}$ into the set of non-terminals, and generate rules for $\Gamma_{x\to y}^{(x,s(y))}$ from $\Gamma_{x\to y} \to \text{REC}(\Gamma_{x\to y}, \{ x' \mapsto x, y' \mapsto y \}) \bullet \{ x \mapsto 0, y \mapsto 0 \}$. Then, we need non-terminal $\Gamma_{x\to y}^{(x,s(y))}$. In summary, we need infinitely many non-terminals and their production rules. The assumption enables us to avoid such a case.

Finally, we show correctness of the transformation in Definition 5.1 i.e., that $L(\mathcal{R}(G, T, x_1, x_2))$ is an overapproximation of the ranges of ground substitutions obtained from $\|L(G, T)\|$ w.r.t. $x_1, x_2$. We first show some auxiliary lemmas, and then show the main theorem.

**Lemma 5.4** Let $T$ be a goal clause, $t_1, t_2 \in \mathcal{T}(C, V)$, $\theta \in \text{Subst}(C)$, $\xi \in \text{Subst}(C)$ such that $\text{Dom}(\theta) \cap \text{Dom}(\xi) = \emptyset$ and $\text{Var}(\theta) \cup \text{Var}(\xi) = \text{Var}(t_1, t_2)$. Note that $\theta \cup \xi = \theta \xi = \xi \theta$. Let $\xi_A = \{ x \mapsto A \mid x \in \text{Dom}(\xi) \}$ and $u \in (\xi_A(t_1), \xi_A(t_2))_T$. Suppose that for all positions $p \in \text{Pos}(t_1) \cap \text{Pos}(t_2)$, both of the following hold:

- if $t_1|_p \in \text{Dom}(\theta)$, then $t_2|_p \in \text{Dom}(\theta) \cup \mathcal{T}(C, \text{Dom}(\xi))$, and
- if $t_2|_p \in \text{Dom}(\theta)$, then $t_1|_p \in \text{Dom}(\theta) \cup \mathcal{T}(C, \text{Dom}(\xi))$.

Then, all of the following hold:

(a) $\text{Pos}(\{ \theta \xi t_1, \theta \xi t_2 \}) \supseteq \text{Pos}(t_1, t_2) \supseteq \text{Pos}(u)$ (i.e., $\text{Pos}(t_1) \cup \text{Pos}(t_2) \supseteq \text{Pos}(u)$),

(b) for any position $p \in \text{Pos}(t_1) \cap \text{Pos}(t_2)$, all of the following hold:

- if $t_1|_p = x \in \text{Dom}(\theta)$ and $t_2|_p = y \in \text{Dom}(\theta)$, then $([t_1, t_2]|_p = xy$ i.e., $([\theta \xi t_1, \theta \xi t_2]|_p = [\theta x, \theta y]$) and $u|_p = \Gamma_T^{(x,y)}$
- if $t_1|_p = x \in \text{Dom}(\theta)$ and $t_2|_p = y \in \text{Dom}(\xi)$, then $([t_1, t_2]|_p = xy$ i.e., $([\theta \xi t_1, \theta \xi t_2]|_p = [\theta x, \xi y]$) and $u|_p = \Gamma_T^{(x,\xi)}$
- if $t_1|_p = x \in \text{Dom}(\xi)$ and $t_2|_p = y \in \text{Dom}(\theta)$, then $([t_1, t_2]|_p = xy$ i.e., $([\theta \xi t_1, \theta \xi t_2]|_p = [\xi x, \theta y]$) and $u|_p = AA$
- if $t_1|_p = x \in \text{Dom}(\xi)$ and $t_2|_p = y \in \text{Dom}(\xi)$, then $([t_1, t_2]|_p = xy$ i.e., $([\theta \xi t_1, \theta \xi t_2]|_p = [\xi x, \xi y]$) and $u|_p = \Gamma_T^{(x,\xi)}$.
\[ \begin{align*}
\text{if } t_1 \upharpoonright p &= x \in \text{Dom}(\xi) \text{ and } \text{root}(t_2 \upharpoonright p) = g \in C, \text{ then root}((\xi t_2)) \upharpoonright p = xg (\text{i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\xi x, \xi t_2]) \upharpoonright p, u \upharpoonright p \in (A, \xi (t_2)), \xi (t_2)) \text{ and there exists a term } t'_1 \in T(C, V) \text{ and a term } u' \in C, \text{ such that } t'_1 \leq \xi x, u' = [\xi x, (t_2), \xi (t_2)].
\end{align*} \]

- For all \( q \in \text{Pos}(t_1) \cap \text{Pos}(t_2) \), \( \xi (t_1') \upharpoonright q = A \text{ if and only if } t_2 \upharpoonright pq = A \)

- For all \( q \in \text{Pos}(t_2) \), \( \xi (t_1') \upharpoonright q = A \upharpoonright p \)

where \( \xi (t_1') = \{ x \mapsto A \mid x \in \text{Dom}(t_1') \} \).

- If \( \text{root}(t_1 \upharpoonright p) = f \in C \text{ and } t_2 \upharpoonright p = y \in \text{Dom}(\xi), \text{ then root}((\xi t_2)) \upharpoonright p = fy \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\xi x, \xi t_2]) \upharpoonright p, u \upharpoonright p \in (A, \xi (t_1), \xi (t_1')) \text{, and there exists a term } t'_2 \in T(C, V) \text{ and a term } u' \in C, \text{ such that } t'_2 \leq \xi y, u' = [t_2, \xi x, t_2').
\]

- For all \( q \in \text{Pos}(t_2) \cap \text{Pos}(t_1) \), \( \xi (t_1') \upharpoonright q = A \text{ if and only if } t_1 \upharpoonright pq = A \)

- For all \( q \in \text{Pos}(t_2) \setminus \text{Pos}(t_1) \), \( \xi (t_1') \upharpoonright q = A \)

where \( \xi (t_1') = \{ x \mapsto A \mid x \in \text{Dom}(t_1') \} \).

- If \( \text{root}(t_1 \upharpoonright p) = f \in C \text{ and root}(t_2 \upharpoonright p) = g \in C, \text{ then root}((\xi t_2)) \upharpoonright p = \text{root}((\xi t_1), \xi (t_2)) \upharpoonright p = \text{root}(u \upharpoonright p) = fg \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\xi x, \xi t_2]) \upharpoonright p, u \upharpoonright p \in (A, \xi (t_1), \xi (t_1')) \).

(c) For any position \( p \in \text{Pos}(t_1) \cap \text{Pos}(t_2) \), both of the following hold:

- If \( t_1 \upharpoonright p = x \in \text{Dom}(\theta) \), then \( ([t_1, t_2]) \upharpoonright p = x \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\theta_1 t_1, \perp]) \upharpoonright p \) and \( u \upharpoonright p = \Gamma_x \).

- If \( t_1 \upharpoonright p = x \in \text{Dom}(\xi) \), then \( ([t_1, t_2]) \upharpoonright p = x \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\theta_1 t_1, \perp]) \upharpoonright p \) and \( u \upharpoonright p = \perp \).

- If \( \text{root}(t_1 \upharpoonright p) = f \in C, \text{ then root}((\xi t_2)) \upharpoonright p = \text{root}((\xi t_1), \xi (t_2)) \upharpoonright p = \text{root}(u \upharpoonright p) = f \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\theta_1 t_1, \perp]) \upharpoonright p \).

and

(d) For any position \( p \in \text{Pos}(t_2) \setminus \text{Pos}(t_1) \), both of the following hold:

- If \( t_2 \upharpoonright p = y \in \text{Dom}(\theta) \), then \( ([t_1, t_2]) \upharpoonright p = y \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\perp, \theta_1 y]) \upharpoonright p \) and \( u \upharpoonright p = \Gamma_y \).

- If \( t_2 \upharpoonright p = y \in \text{Dom}(\xi) \), then \( ([t_1, t_2]) \upharpoonright p = y \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\perp, \theta_1 y]) \upharpoonright p \) and \( u \upharpoonright p = \perp \).

- If \( \text{root}(t_2 \upharpoonright p) = g \in C, \text{ then root}((\xi t_2)) \upharpoonright p = \text{root}((\xi t_1), \xi (t_2)) \upharpoonright p = \text{root}(u \upharpoonright p) = g \perp \text{ (i.e.}, ([\theta_1 t_1, \theta_2 t_2]) \upharpoonright p = ([\perp, \theta_1 t_2]) \upharpoonright p \).

Proof. By definition, the claim (a) is trivial. The claims (b)–(d) can be proved by induction on the length of \( p \). For the claims (b), (c), and (d), we make a case distinction depending on what \( t_1 \upharpoonright p \) and \( t_2 \upharpoonright p \) are, what \( t_1 \upharpoonright p \) is, and what \( t_2 \upharpoonright p \) is, respectively. \(\)
• if $t_2 \rho \in V \cap \text{Dom}(\theta)$, then $t_1 \rho \in \text{Dom}(\theta) \cup T(C, \text{Dom}(\xi))$.

Then, there exists a term $u \in \langle \xi_A t_1, \xi_A t_2 \rangle_T$, a context $C[] \in T((C \cup \{\bot\})^2 \cup \{\square\})$, terms $S_1, \ldots, S_n$, and non-terminals $\Gamma_1, \ldots, \Gamma_n$ such that $[\theta \xi t_1, \theta \xi t_2] = C[S_1, \ldots, S_n]$, $u = C[\Gamma_1, \ldots, \Gamma_n]$, and for all $1 \leq i \leq n$, all of the following hold:

- $S_i = xy$ if and only if $\Gamma_i = \Gamma_T^{(x,y)}$,
- $S_i = [\theta x, \xi(t_1|p)]$ if and only if $\Gamma_i = \Gamma_T^{(x,\xi(t_1|p))}$ for some $p \in \mathcal{P}(t_2)$,
- $S_i = [\theta x, \bot]$ if and only if $\Gamma_i = \Gamma_T^{(x,\bot)}$,
- $S_i = [\xi(t_1|p), \theta y]$ if and only if $\Gamma_i = \Gamma_T^{(\xi(t_1|p),y)}$ for some $p \in \mathcal{P}(t_1)$,
- $S_i = [\bot, \theta y]$ if and only if $\Gamma_i = \Gamma_T^{(\bot,y)}$,
- $S_i = [\xi(t_1|p), \xi(t_2|p)]$ for some $p \in \mathcal{P}(\xi t_1) \cap \mathcal{P}(\xi t_2)$ if and only if $\Gamma_i = AA$,
- $S_i = [\xi(t_1|p), \bot]$ for some $p \in \mathcal{P}(\xi t_1) \setminus \mathcal{P}(\xi t_2)$ if and only if $\Gamma_i = A \bot$, and
- $S_i = [\bot, \xi(t_2|p)]$ for some $p \in \mathcal{P}(\xi t_2) \setminus \mathcal{P}(\xi t_1)$ if and only if $\Gamma_i = \bot A$.

**Proof.** Using Lemma [5.4] this lemma can be proved by structural induction on $t_1, t_2$. □

**Lemma 5.6** Let $\mathcal{G}$ be an SSG $(\Gamma_0, \mathcal{N}, \mathcal{P})$, $\Gamma_T \in \mathcal{N}$, $x_1, x_2 \in \text{Var}(T)$, $\text{Ran}(\mathcal{G}, T, x_1, x_2)$ be constructed, and $\mathcal{G}' = \text{Ran}(\mathcal{G}, T, x_1, x_2)$. Let $t_1, t_2 \in T(C, V)$, $\xi \in \text{Subst}(C)$ with $\text{Dom}(\xi) \supseteq \text{Var}(t_1, t_2)$, and $\xi_A = \{x \mapsto A \mid x \in \text{Var}(t_1, t_2)\}$. Then, all of the following hold:

- there exists a term $u \in \langle \xi_A t_1, \xi_A t_2 \rangle_T$ such that $u \rightarrow_{\mathcal{G}}^* [\xi t_1, \xi t_2]$,
- there exists a term $u \in \langle \xi_A t_1, \bot \rangle_T$ such that $u \rightarrow_{\mathcal{G}}^* [\xi t_1, \bot]$, and
- there exists a term $u \in \langle \bot, \xi_A t_2 \rangle_T$ such that $u \rightarrow_{\mathcal{G}}^*[\bot, \xi t_2]$.

**Proof.** Using the definition of $\mathcal{P}_{AA}, \mathcal{P}_{A \bot}$, and $\mathcal{P}_{\bot A}$, and Lemma [5.4] this lemma can be proved by structural induction on $t_1, t_2$. □

**Theorem 5.7** Let $\mathcal{G}$ be an SSG $(\Gamma_0, \mathcal{N}, \mathcal{P})$, $\Gamma_T \in \mathcal{N}$, $x_1, x_2 \in \text{Var}(T)$, and $\text{Ran}(\mathcal{G}, T, x_1, x_2)$ be constructed (i.e., $\mathcal{P}_T$ satisfies the assumption). Then,

$$L(\text{Ran}(\mathcal{G}, T, x_1, x_2)) \supseteq \{[\xi \theta t_1, \xi \theta t_2] \mid \theta \in [L(\mathcal{G}, \Gamma_T)]\} = \text{Subst}(\mathcal{C}), \text{Var}(\theta t_1, \theta t_2) = \text{Dom}(\xi)\}.$$

**Proof.** Let $\mathcal{G}' = \text{Ran}(\mathcal{G}, T, x_1, x_2)$. It suffices to show that for all $\Gamma_T \in \mathcal{N}$, $t_1, t_2 \in \text{Var}(T') \cup \text{Patterns}(\mathcal{P}) \cup \{\bot\}$ with $\{t_1, t_2\} \cap \mathcal{V} \neq \emptyset$, and $e \in L(\mathcal{G}, \Gamma_T)$ with $\theta = [e]$, we have $[\xi \theta t_1, \xi \theta t_2] \in L(\mathcal{G}', \Gamma_T^{(t_1, t_2)})$ for all substitutions $\xi \in \text{Subst}(\mathcal{C})$ with $\text{Dom}(\xi) = \text{Var}(\xi t_1, \xi t_2)$. We prove this claim by induction on the length of derivations from $\Gamma_T$ to $e$. We make a case distinction depending on which rule is applied at the first step.

- The case where $\Gamma_T \rightarrow \theta$ is applied. By construction, we have the following production rule $\Gamma_T^{(t_1, t_2)} \rightarrow u \in \mathcal{G}'$ for each $u \in \langle \xi_A t_1, \xi_A t_2 \rangle_T$, where $\xi_A = \{x \mapsto A \mid x \in \text{Var}(\xi_A \theta t_1, \xi_A \theta t_2)\}$. Then, the claim follows from Lemma [5.6].
The remaining case where $\Gamma_T' \rightarrow \text{REC}(\Gamma_T', \delta) \bullet \sigma$ is applied. Suppose that $\Gamma_T' \rightarrow_{G'} e'$ and $\theta = \left[ \text{REC}(e', \delta) \bullet \sigma \right]$. Let $\theta' = \left[ e' \right]$. Then, $\theta = (\theta')^{\left| \text{Dom}(\delta) \right| \sigma}$. By construction, we have the following production rule $\Gamma_{(t_1,t_2)}^{(t_1,t_2)} \rightarrow u \in G'$ for each $u \in \langle \xi_A \delta \sigma t_1, \xi_A \delta \sigma t_2 \rangle_T$, where $\xi_A = \{ x \mapsto A \mid x \in \text{Var}(\delta \sigma t_1, \delta \sigma t_2) \setminus \text{Var}(T') \}$. By the assumption, $\text{Var}(T') = \text{VRan}(\delta)$, and thus, $\text{Dom}(\xi_A) \cap \text{VRan}(\delta) = \emptyset$. Since $\xi_A$ is a ground substitution, we have that $\xi_A \delta \sigma t_i = \delta \xi_A \mid \text{VRan}(\sigma) \setminus \text{Dom}(\delta) \sigma t_i$. Since $\xi$ is a ground substitution and $\text{Dom}(\xi) \supseteq \text{VRan}(\delta) = \text{VRan}(\xi') \mid \text{Dom}(\delta) \sigma$, we have that

- $\xi \theta t_1 = \xi' (\theta')^{\left| \text{Dom}(\delta) \sigma \right| \xi \theta t_1}$, and
- $\xi_A \delta \sigma t_i = \xi_A \delta \xi_A \mid \text{VRan}(\sigma) \setminus \text{Dom}(\delta) \sigma t_i$,

and hence

$$\langle \xi_A \delta \sigma t_1, \xi_A \delta \sigma t_2 \rangle_{T''} = \langle \xi_A \delta \xi_A \mid \text{VRan}(\sigma) \setminus \text{Dom}(\delta) \sigma t_1, \xi_A \delta \xi_A \mid \text{VRan}(\sigma) \setminus \text{Dom}(\delta) \sigma t_2 \rangle_{T''}.$$  

It follows from Lemma 5.5 that there exists a term $u \in \langle \xi_A \delta \sigma t_1, \xi_A \delta \sigma t_2 \rangle_{T''}$, a context $C[\Sigma] \in T((\Sigma \cup \{ \bot \})^2 \cup \{ \square \})$, terms $S_1, \ldots, S_n$, and non-terminals $\Gamma_1, \ldots, \Gamma_n$ such that $[\xi \theta t_1, \xi \theta t_2] = C[S_1, \ldots, S_n]$, $u = C[\Gamma_1, \ldots, \Gamma_n]$, and for all $1 \leq i \leq n$, all of the following hold:

- $S_i = xy$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}$,
- $S_i = [\xi \theta x, \xi \theta (t_1) \mid p]$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$ for some $p \in \text{Pos}(t_1)$,
- $S_i = [\xi \theta x, \bot]$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$,
- $S_i = [\xi \theta (t_1) \mid p], \xi \theta y$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$ for some $p \in \text{Pos}(t_1)$,
- $S_i = [\bot, \xi \theta y]$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$,
- $S_i = [\xi \theta (t_1) \mid p], \bot$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$ for some $p \in \text{Pos}(\xi \theta t_1) \cap \text{Pos}(\xi \theta t_2)$ if and only if $\Gamma_i = A A$,
- $S_i = [\xi \theta (t_1) \mid p], \bot$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$ for some $p \in \text{Pos}(\xi \theta t_1) \cap \text{Pos}(\xi \theta t_2)$ if and only if $\Gamma_i = A A$, and
- $S_i = [\bot, \xi \theta (t_2) \mid p]$ for some $p \in \text{Pos}(\xi \theta t_2)$ if and only if $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}(t_1)$ for some $p \in \text{Pos}(\xi \theta t_1)$ if and only if $\Gamma_i = \bot A$.

In the case where $\Gamma_i = \Gamma_{(x,y)}^{(x,y)}$, $\Gamma_{(x,y)}^{(x,y)}$, $\Gamma_{(x,y)}^{(x,y)}$, or $\Gamma_{(x,y)}^{(x,y)}$, it follows from the induction hypothesis that $\Gamma_i \rightarrow_{G'} S_i$. In the remaining case where $\Gamma_i = AA, \bot A$, or $\bot A$, it follows from Lemma 5.6 that $\Gamma_i \rightarrow_{G'} S_i$. Therefore, we have that $\Gamma_{(t_1,t_2)}^{(t_1,t_2)} \rightarrow_{G'} u = C[\Gamma_1, \ldots, \Gamma_n] \rightarrow_{G'} C[S_1, \ldots, S_n] = [\xi \theta t_1, \xi \theta t_2]$, and hence, $[\xi \theta t_1, \xi \theta t_2] \in L(\mathcal{G}', \Gamma_{(t_1,t_2)}^{(t_1,t_2)}).$ \hfill $\square$

The converse inclusion (i.e., $L(\text{Ran}(\mathcal{G}, T, x_1, x_2)) \subseteq \{ [\xi \theta x_1, \xi \theta x_2] \mid \theta \in \left[ L(\mathcal{G}, \Gamma_T) \right] \}$) does not hold in general (cf. [21, Example 31]).

### 6 Conclusion

In this paper, under a certain syntactic condition, we showed a transformation of the grammar representation of a narrowing tree into an RTG that overapproximates the ranges of ground substitutions generated by the grammar representation. We showed a precise definition of the transformation and proved that the language of the transformed RTG is an overapproximation of the ranges of ground substitutions generated by the grammar representation. We will make an experiment to evaluate the usefulness of the transformation in e.g., proving confluence of CTRSs.

The syntactic assumption in Section 5 is a sufficient condition to, given an SSG, obtain an RTG that generates the ranges of ground substitutions generated by the SSG. It is not known yet whether the assumption is a necessary condition or not. We will try to clarify this point.
As stated in Section 5, the converse inclusion of Theorem 5.7, \(L(Ran(G, T, x_1, x_2)) \subseteq \{[\xi \theta x_1, \xi \theta x_2] | \theta \in [[L(G, \Gamma_T)], \ldots]\}, does not hold in general. However, the converse inclusion must hold for an SSG such that all substitutions in the SSG preserve linearity, i.e., for any substitution \(\sigma\) in the SSG, \(\sigma x\) is linear for all \(x \in Dom(\sigma)\), and \(Var(\sigma x) \cap Var(\sigma y) = \emptyset\) for all \(x, y \in Dom(\sigma)\) such that \(x \neq y\). We will prove this conjecture and try to find other sufficient conditions for the converse inclusion.

Acknowledgements We gratefully acknowledge the anonymous reviewers for their useful comments and suggestions to improve the paper.

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