Wick-Cutkosky model: an introduction

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Abstract

A general pedagogical survey of the basics of the Bethe-Salpeter equation and in particular Wick-Cutkosky model is given in great technical details.

Bethe-Salpeter amplitude

The Bethe-Salpeter approach to the relativistic two-body bound state problem assumes that all needed information about a bound state $|B>$ is contained in the B-S amplitude

$$\Phi(x_1, x_2; P_B) =<0|T\phi_1(x_1)\phi_2(x_2)|B>$$

and in its conjugate

$$\bar{\Phi}(x_1, x_2; P_B) =<B|T\phi_1^+(x_1)\phi_2^+(x_2)|0>.$$  

Here $P_B$ is the bound state 4-momentum and field operators are in the Heisenberg representation.

Note, that

$$<B|T\phi_1^+(x_1)\phi_2^+(x_2)|0> =<0|\bar{T}\phi_1(x_1)\phi_2(x_2)|B>^*$$

$\bar{T}$ being antichronological operator. So we can say that $\bar{\Phi}$ is obtained from $\Phi$ through time reversal.

It is clear that instead of individual $x_1, x_2$ coordinates more useful are variables which characterize two particle relative motion and the motion of the system as a whole.

The coordinate corresponding to the relative motion is of course $x = x_1 - x_2$. Then any linear independent combination $X = \eta_1 x_1 + \eta_2 x_2$ can
serve for a description of the system as a whole. Linear independence means \( \eta_1 + \eta_2 \neq 0 \). In order to have a correspondence to the nonrelativistic center of mass definition, we take

\[
\eta_1 = \frac{m_1}{m_1 + m_2}, \quad \eta_2 = \frac{m_2}{m_1 + m_2}.
\]

(3)

\( x_1 = X + \eta_2 x, \quad x_2 = X - \eta_1 x \) equalities and the fact that \( \hat{P} \) 4-momentum operator is the space-time translation operator enables us to write

\[
\phi_1(x_1) = e^{i\hat{P}X}\phi_1(\eta_2 x)e^{-i\hat{P}X}, \quad \phi_2(x_2) = e^{i\hat{P}X}\phi_2(-\eta_1 x)e^{-i\hat{P}X}.
\]

Inserting this expression into (1) we get

\[
<0|T\phi_1(x_1)\phi_2(x_2)|B> = \Theta(x_0) <0|\phi_1(x_1)\phi_2(x_2)|B> +
\]

\[
\Theta(-x_0) <0|\phi_2(x_2)\phi_1(x_1)|B> =
\]

\[
\Theta(x_0) <0|e^{i\hat{P}X}\phi_1(\eta_2 x)\phi_2(-\eta_1 x)e^{-i\hat{P}X}|B> +
\]

\[
\Theta(-x_0) <0|e^{i\hat{P}X}\phi_2(-\eta_1 x)\phi_1(\eta_2 x)e^{-i\hat{P}X}|B> =
\]

\[
e^{-iP_B X}\{\Theta(x_0) <0|\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|B> +
\]

\[
\Theta(-x_0) <0|\phi_2(-\eta_1 x)\phi_1(\eta_2 x)|B>\} =
\]

\[
e^{-iP_B X} <0|T\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|B> .
\]

So, if the reduced B-S amplitude is introduced

\[
\Phi(x; P_B) = (2\pi)^{3/2} <0|T\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|B> \tag{4}
\]

when

\[
\Phi(x_1, x_2; P_B) = (2\pi)^{-3/2}e^{-iP_B X}\Phi(x; P_B). \tag{5}
\]

Analogously

\[
\Phi(x_1, x_2; P_B) = (2\pi)^{-3/2}e^{iP_B X}\Phi(x; P_B). \tag{6}
\]

An equation for the B-S amplitude can be obtained with the help of 4-point Green’s function.
Equation for Green’s function

Let us consider 4-point Green’s function

\[ G(x_1, x_2; y_1, y_2) = <0|T\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)|0> . \]

Field operators here are in the Heisenberg representation, so we have the full Green’s function, which can be expanded into an infinite perturbation series. Let us rearrange the terms of this series: first of all we sum up self-energy insertions in propagators of \( \phi_1 \) and \( \phi_2 \) particles, this will give us the full propagators for them, then from the remaining diagrams we separate \( (\phi_1 + \phi_2) \)-two-particle irreducible ones, the sum of which will play the role of interaction operator (a diagram is \( (\phi_1 + \phi_2) \)-two-particle reducible, if by cutting one \( \phi_1 \) and one \( \phi_2 \) of inner lines it can be divided into two disconnected parts). More vividly this is illustrated graphically in Fig.1.

Here \( I \) is the sum of \( (\phi_1 + \phi_2) \)-two-particle irreducible diagrams without external propagators, as is shown in Fig.2.

The analytical expression of the above integral equation (Fig.1) looks like

\[ G(x_1, x_2; y_1, y_2) = \Delta_1(x_1 - y_1)\Delta_2(x_2 - y_2) + \]

\[ + \cdots \]
\[ \int dz_1 \, dz_2 \, dz'_1 \, dz'_2 \Delta_1(x_1 - z_1) \Delta_2(x_2 - z_2) I(z_1, z_2; z'_1, z'_2) G(z'_1, z'_2; y_1, y_2). \]  

We get the so-called ladder approximation if the full propagator \( \Delta(x - y) \) is replaced by the free propagator and in the interaction function \( I \) only the first single-particle-exchange term is left:

\[ \text{Figure 3: Ladder approximation.} \]

An appearance of the iterative solution of this equation explains the origin of the approximation name:

\[ \text{Figure 4: Iterative solution in the ladder approximation.} \]

**Momentum representation**

Eq. 7 is written in the coordinate space. As is well known, a relativistic particle can’t be localized in the space-time with arbitrary precision. Therefore in relativistic theory momentum space is often more useful than coordinate space.

In order to rewrite Eq. 7 in the momentum space, let us perform the Fourier transformation

\[ G(x_1, x_2; y_1, y_2) = (2\pi)^{-8} \int dp \, dq \, dP \, e^{-ipx} e^{iqy} e^{-iP(X-Y)} G(p, q; P). \]
Because of the translational invariance, $G$ depends only on coordinate differences. In the role of independent differences drawn up from the $x_1, x_2, y_1, y_2$ coordinates, we have taken $x = x_1 - x_2$, $y = y_1 - y_2$ and $X - Y = (\eta_1 x_1 + \eta_2 x_2) - (\eta_1 y_1 + \eta_2 y_2)$.

Let us clear up some technical details of the transition to the momentum space. First of all $\Delta_1(x_1 - y_1)\Delta_2(x_2 - y_2)$ product should be rewritten in the form of Eq. 8. Momentum space propagator is defined through the Fourier transform

$$\Delta(x) = (2\pi)^{-4} \int dp \, e^{-ipx} \Delta(p).$$

Then

$$\Delta_1(x_1 - y_1)\Delta_2(x_2 - y_2) = (2\pi)^{-8} \int dp_1 dp_2 e^{-ip_1(x_1 - y_1)} e^{-ip_2(x_2 - y_2)} \Delta_1(p_1)\Delta_2(p_2).$$

Let us make $p_1 = \eta_1 P + p$, $p_2 = \eta_2 P - p$ variable change in the integral with $\frac{D(p_1,p_2)}{D(P,p)} = \eta_1 + \eta_2 = 1$ unit Jacobian

$$\Delta_1(x_1 - y_1)\Delta_2(x_2 - y_2) = (2\pi)^{-8} \int dP \, dp \, e^{-iP(X-Y)} e^{-iPx} e^{ipy} \Delta_1(\eta_1 P + p)\Delta_2(\eta_2 P - p),$$

which, it is clear, can be rewritten also as

$$\Delta_1(x_1 - y_1)\Delta_2(x_2 - y_2) = (2\pi)^{-8} \int dP \, dp \, dq \, e^{-iP(X-Y)} e^{-iPx} e^{iqy} \delta(p - q) \Delta_1(\eta_1 P + p)\Delta_2(\eta_2 P - p).$$

Now let us transform $\Delta_1\Delta_2 IG$ integral

$$\Delta_1\Delta_2 IG = \int dz_1 dz_2 dz_1' dz_2' \Delta_1(x_1 - z_1)\Delta_2(x_2 - z_2) I(z_1, z_2; z_1', z_2'; y_1, y_2) = (2\pi)^{-24} \int dz_1 dz_2 dz_1' dz_2' dP \, dp \, dq \, dP' \, dp' \, dq' \, dP'' \, dp'' \, dq'' \times e^{-iP(X-Z)} e^{-iPx} e^{iqy} \delta(p - q') \Delta_1(\eta_1 P + p)\Delta_2(\eta_2 P - p) \times e^{-iP'(Z-Z')} e^{-ipz} e^{iq'y'} I(p', q''; P') e^{-iP''(Z'-Y)} e^{-ipz'} e^{iqy} G(p'', q; P'') = (2\pi)^{-24} \int dz \, dz' \, dZ \, dP \, dp \, dq \, dP' \, dp' \, dP'' \, dp'' \, dq'' \times \Delta_1(\eta_1 P + p)\Delta_2(\eta_2 P - p) I(p', q''; P') \times G(p'', q; P'') e^{-iz(p' - p)} e^{-iz'(p'' - q'')} e^{-iZ(P' - P)} e^{-iZ'(P'' - P')}.$$
Coordinate integrations will produce δ-functions and some 4-momentum integrals become trivial. At the end we get

\[ \Delta_1 \Delta_2 IG = (2\pi)^{-8} \int dP dp dq dq' \Delta_1(\eta_1 P + p) \Delta_2(\eta_2 P - p) I(p, q'; P) G(q', q; P) . \]

After all of these, Eq.7 easily is rewritten in the momentum space

\[ G(p, q; P) = \delta(p - q) \Delta_1(\eta_1 P + p) \Delta_2(\eta_2 P - p) + \Delta_1(\eta_1 P + p) \Delta_2(\eta_2 P - p) \int dq' I(p, q'; P) G(q', q; P) , \]

or

\[ [\Delta_1(\eta_1 P + p) \Delta_2(\eta_2 P - p)]^{-1} G(p, q; P) = \delta(p - q) + \int dq' I(p, q'; P) G(q', q; P) . \]

If the following definitions are introduced

\[ (A \cdot B)(p, q; P) = \int dq' A(p, q'; P) B(q', q; P) , \]

\[ K(p, q; P) = [\Delta_1(\eta_1 P + p) \Delta_2(\eta_2 P - p)]^{-1} \delta(p - q) , \]

this equation takes the form

\[ K \cdot G = 1 + I \cdot G . \]

Two-particle bound states contribution into Green’s function

Inserting \( \Sigma |n > < n| = 1 \) full set of states, the Green’s function can be rewritten as

\[ G(x_1, x_2; y_1, y_2) \equiv 0|T \phi_1(x_1) \phi_2(x_2) \phi_1^+(y_1) \phi_2^+(y_2)|0 > = \]

\[ \Theta(\min[(x_1)_0, (x_2)_0] - \max[(y_1)_0, (y_2)_0]) \times \]

\[ < 0|T \phi_1(x_1) \phi_2(x_2)|n > < n|T \phi_1^+(y_1) \phi_2^+(y_2)|0 > + \]

\[ \Theta(\min[(x_1)_0, (y_1)_0] - \max[(x_2)_0, (y_2)_0]) \times \]

\[ < 0|T \phi_1(x_1) \phi_1^+(y_1)|n > < n|T \phi_2(x_2) \phi_2^+(y_2)|0 > + \]

\[ \Theta(\min[(x_1)_0, (y_2)_0] - \max[(x_2)_0, (y_1)_0]) \times \]

\[ < 0|T \phi_1(x_1) \phi_2^+(y_2)|n > < n|T \phi_2(x_2) \phi_1^+(y_1)|0 > + \]
\[ \Theta(\min[(x_2)_0, (y_1)_0] - \max[(x_1)_0, (y_2)_0]) \times \]
\[ \langle 0 | T \phi_2(x_2) \phi_1^+(y_1) | n \rangle < \langle n | T \phi_1(x_1) \phi_2^+(y_2) | 0 \rangle + \]
\[ \Theta(\min[(x_2)_0, (y_2)_0] - \max[(x_1)_0, (y_1)_0]) \times \]
\[ \langle 0 | T \phi_2(x_2) \phi_1^+(y_2) | n \rangle < \langle n | T \phi_1(x_1) \phi_1^+(y_1) | 0 \rangle + \]
\[ \Theta(\min[(y_1)_0, (y_2)_0] - \max[(x_1)_0, (x_2)_0]) \times \]
\[ \langle 0 | T \phi_1^+(y_1) \phi_2^+(y_2) | n \rangle < \langle n | T \phi_1(x_1) \phi_2(x_2) | 0 \rangle \]  \(\text{(11)}\)

To separate the contribution from two-particle bound states \(|B\rangle\), let us substitute
\[ \sum |n\rangle < \langle n| \rightarrow \int dP \delta(P^2 - m_B^2) \Theta(P_0) |B\rangle < \langle B| . \]

\((\langle \vec{P}|\vec{Q} \rangle = 2P_0 \delta(\vec{P} - \vec{Q}) \) normalization is assumed).

An action of two annihilation operators over \(|B\rangle\) is needed to obtain the vacuum quantum numbers. Only in this case we get a nonzero matrix element in (11). Therefore two-particle bound state contribution into the Green’s function is
\[ B = \int dP \langle 0 | T \phi_1(x_1) \phi_2(x_2) | B \rangle < \langle B | T \phi_1^+(y_1) \phi_2^+(y_2) | 0 \rangle \times \]
\[ \delta(P^2 - m_B^2) \Theta(P_0) \Theta(\min[(x_1)_0, (x_2)_0] - \max[(y_1)_0, (y_2)_0]) . \]

The \(dP_0\) integration can be performed by using
\[ \delta(P^2 - m_B^2) \Theta(P_0) = \]
\[ \frac{\Theta(P_0)}{2P_0} \left\{ \delta(P_0 - \sqrt{\vec{P}^2 + m_B^2}) + \delta(P_0 + \sqrt{\vec{P}^2 + m_B^2}) \right\} = \]
\[ \frac{\Theta(P_0)}{2P_0} \delta(P_0 - \sqrt{\vec{P}^2 + m_B^2}) \]
and remembering (5) and (6):
\[ B = (2\pi)^{-3} \int \frac{d\vec{P}}{2\omega_B} \Phi(x; P_B) \bar{\Phi}(y, P_B) e^{-i\omega_B(x_0 - y_0)} e^{-i\vec{P}(\vec{x} - \vec{y})} \times \]
\[ \Theta(\min[(x_1)_0, (x_2)_0] - \max[(y_1)_0, (y_2)_0]) . \]

here \(P_B = (\omega_B, \vec{P})\) and \(\omega_B = \sqrt{\vec{P}^2 + m_B^2}; m_B\)-being the bound state mass.
The argument of \( \Theta \) function can be transformed as

\[
\min[(x_1)_0, (x_2)_0] - \max[(y_1)_0, (y_2)_0] = \\
\min[X_0 + \eta_2 x_0, X_0 - \eta_1 x_0] - \max[Y_0 + \eta_2 y_0, Y_0 - \eta_1 y_0] = \\
\begin{cases} 
X_0 - Y_0 - \eta_1 x_0 - \eta_2 y_0 , & \text{if } x_0 > 0, y_0 > 0 \\
X_0 - Y_0 - \eta_1 x_0 + \eta_1 y_0 , & \text{if } x_0 > 0, y_0 < 0 \\
X_0 - Y_0 + \eta_2 x_0 - \eta_2 y_0 , & \text{if } x_0 < 0, y_0 > 0 \\
X_0 - Y_0 + \eta_2 x_0 + \eta_1 y_0 , & \text{if } x_0 < 0, y_0 < 0 \\
\end{cases}
\]

\[
X_0 - Y_0 - \frac{1}{2} |x_0| - \frac{1}{2} |y_0| + \frac{1}{2} (\eta_2 - \eta_1) (x_0 - y_0) .
\]

Besides, let us use its integral representation

\[
\Theta(z) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k + i\epsilon} e^{-ikz},
\]

in which the variable change \( k \to P_0 - \omega_B \) is made. Then

\[
\Theta(X_0 - Y_0 - \frac{1}{2} |x_0| - \frac{1}{2} |y_0| + \frac{1}{2} (\eta_2 - \eta_1) (x_0 - y_0)) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dP_0}{P_0 - \omega_B + i\epsilon} \times
\]

\[
\exp \left\{ -i(P_0 - \omega_B) [X_0 - Y_0 - \frac{1}{2} |x_0| - \frac{1}{2} |y_0| + \frac{1}{2} (\eta_2 - \eta_1) (x_0 - y_0)] \right\}.
\]

inserting this into \( B \), we get

\[
B(x, y; X - Y) = (2\pi)^{-4} \int dP \Phi(x; P_B) \Phi(y; P_B) \frac{e^{-iP(X - Y)}}{2\omega_B(P_0 - \omega_B + i\epsilon)} \times
\]

\[
\exp \left\{ -i(P_0 - \omega_B) [-\frac{1}{2} |x_0| - \frac{1}{2} |y_0| + \frac{1}{2} (\eta_2 - \eta_1) (x_0 - y_0)] \right\} = \\
(2\pi)^{-12} \int dP \ dp \ dq \ e^{-iP(X - Y)} e^{-ipx} e^{iqy} \frac{\Phi(p; P_B) \Phi(q; P_B)}{2\omega_B(P_0 - \omega_B + i\epsilon)} \times
\]

\[
\exp \left\{ -i(P_0 - \omega_B) [-\frac{1}{2} |x_0| - \frac{1}{2} |y_0| + \frac{1}{2} (\eta_2 - \eta_1) (x_0 - y_0)] \right\} ,
\]

where in the reduced B-S amplitudes the transition to the momentum space was done:

\[
\Phi(x, P_B) = (2\pi)^{-4} \int dp e^{-ipx} \Phi(p, P_B)
\]

and

\[
\Phi(y, P_B) = (2\pi)^{-4} \int dq e^{iqy} \Phi(q, P_B).
\]
If we designate for brevity

\[ A(p, q; P) = \frac{\Phi(p; P_B)\Phi(q; P_B)}{2\omega_B(P_0 - \omega_B + i\epsilon)}, \]

\[ f(x_0, y_0) = \frac{1}{2}(\eta_2 - \eta_1)(x_0 - y_0) - \frac{1}{2}|x_0| - \frac{1}{2}|y_0| \]

then the momentum transform of B will be

\[ B(p, q, P) = \frac{1}{(2\pi)^4} \int e^{ipx}e^{-iqy}e^{iPX}B(x, y; X)dx
dy
dX = \]

\[ i(2\pi)^{-16} \int dx
dy
dX
dP' dp' dq' e^{i(p-p')x}e^{-i(q-q')y} \times \]

\[ e^{i(P-P')X}A(p', q'; P') \exp \{-i(P' - \omega_B)f(x_0, y_0)\} = \]

\[ i(2\pi)^{-12} \int dx
dy
dp' dq' e^{i(p-p')x}e^{-i(q-q')y}A(p', q'; P') \times \]

\[ \exp \{-i(P_0 - \omega_B)f(x_0, y_0)\} \simeq \frac{1}{(2\pi)^{-4}} A(p, q; P) \text{ when } P_0 \to \omega_B. \]

Therefore, near the point \( P_0 = \omega_B \)

\[ B(p, q; P) = \frac{i}{(2\pi)^4 2\omega_B(P_0 - \omega_B + i\epsilon)} \Phi(p; P_B)\Phi(q; P_B), \]

that is two-particle bound state contribution in \( G(p, q; P) \) has a pole at \( P_0 = \omega_B \). Other \( |n \rangle \rangle \) intermediate states will give a regular contribution at this point, if their masses differ from \( m_B \) (and we assume that this is the case). So

\[ G(p, q; P) \rightarrow \frac{i}{(2\pi)^4 2\omega_B(P_0 - \omega_B + i\epsilon)} \Phi(p; P_B)\Phi(q; P_B), \text{ when } P_0 \to \omega_B. \]  \( (12) \)

Bethe-Salpeter equation

Let us take \( p \neq q, \ P \to P_B \) in the equation (9) for Green’s function. Taking into account (12), we get

\[ [\Delta_1(\eta_1 P + p)\Delta_2(\eta_2 P - p)]^{-1} \frac{i}{(2\pi)^4 2\omega_B(P_0 - \omega_B + i\epsilon)} \int dq' I(p, q'; P_B) \frac{i}{(2\pi)^4 2\omega_B(P_0 - \omega_B + i\epsilon)} \Phi(q'; P_B) = \]

This implies the Bethe-Salpeter equation for the B-S amplitude

\[ [\Delta_1(\eta_1 P_B + p)\Delta_2(\eta_2 P_B - p)]^{-1}\Phi(p; P_B) = \int dq' I(p, q'; P_B)\Phi(q'; P_B). \]  \( (13) \)
Normalization condition

The normalization of the B-S amplitude cannot be determined from the homogeneous B-S equation. It can be obtained in such a way. From (10) \( G \cdot (K - I) = 1 \). Differentiating with respect to \( P_0 \), we find

\[
\frac{\partial G}{\partial P_0} (K - I) + G \left( \frac{\partial K}{\partial P_0} - \frac{\partial I}{\partial P_0} \right) = 0 ,
\]

that is

\[
\frac{\partial G}{\partial P_0} = -G \left( \frac{\partial K}{\partial P_0} - \frac{\partial I}{\partial P_0} \right) .
\]

Let us consider this equation near the point \( P_0 = \omega_B \), where Eq.12 for Green’s function is valid:

\[
-\frac{i}{(2\pi)^4} \Phi(p; P_B) \Phi(q; P_B) \frac{(2\pi)^4}{i} \frac{\Phi(p; P_B) \Phi(q; P_B)}{2\omega_B(P_0 - \omega_B + i\epsilon)} \times \\
\left[ \frac{\partial}{\partial P_0} (K - I) \right](q', q'') \frac{i}{(2\pi)^4} \frac{\Phi(q''; P_B) \Phi(q; P_B)}{2\omega_B(P_0 - \omega_B + i\epsilon)} ,
\]

which gives the following normalization condition

\[
\frac{i}{(2\pi)^4} \int dp dq \Phi(p; P_B) \left[ \frac{\partial}{\partial P_0} (K - I) \right]_{P_0=\omega_B} (p, q) \Phi(q; P_B) = 2\omega_B \equiv 2(P_B)_0
\]

or symbolically

\[
\frac{i}{(2\pi)^4} \Phi \left( \frac{\partial K}{\partial P_0} - \frac{\partial I}{\partial P_0} \right)_{P_0=\omega_B} \Phi = 2\omega_B . \tag{14}
\]

B-S equation in the ladder approximation

Let us consider the B-S equation in the ladder approximation for two scalar particles interacting via scalar quantum exchange. The interaction kernel in this case can be calculated according to standard Feynman rules: every vertex contributes \( ig \) and every scalar propagator

\[
\Delta(X) = (2\pi)^{-4} \int \frac{e^{-ipx} dp}{i[\mu^2 - p^2 - i\epsilon]} .
\]

Thus \([A \]

\[
I(x, y; X - Y) = -g_1 g_2 \Delta(x) \delta(x_1 - x_2) \delta(x_2 - y_2) = 
\]
\[-g_1 g_2 \Delta(x) \delta(X - Y + \eta_2(x - y)) \delta(X - Y - \eta_1(x - y))\,.

Let us note that
\[
\delta(X - Y + \eta_2(x - y)) \delta(X - Y - \eta_1(x - y)) = \\
\delta[\eta_1(x - y) + \eta_2(x - y)] \delta[X - Y - \eta_1(x - y)] = \delta(x - y) \delta(X - Y),
\]
that is
\[
I(x, y; X - Y) = -g_1 g_2 \Delta(x) \delta(x - y) \delta(X - Y),
\]
its momentum space image being
\[
I(p, q; P) = (2\pi)^{-4} \int e^{i p x} e^{-i q y} e^{-i P X} I(x, y; X) \, dx \, dy \, dX = \\
-g_1 g_2 (2\pi)^{-4} \int dx \, e^{i(p-q)x} \Delta(x),
\]
or
\[
I(p, q; P) = -g_1 g_2 (2\pi)^{-4} \Delta(p - q) = \frac{ig_1 g_2}{(2\pi)^4} \frac{1}{\mu^2 - (p-q)^2 - i\epsilon}. \quad (15)
\]

Let us substitute this into (13) and also let us take \(\vec{P}_B = 0\). As a result, we get the rest frame B-S equation in the ladder approximation
\[
\begin{aligned}
[m_1^2 + \vec{p}^2 - (\eta_1 P_0 + p_0)^2][m_2^2 + \vec{p}^2 - (\eta_2 P_0 - p_0)^2] \Phi(p; P_0) = \\
\frac{\lambda}{i\pi^2} \int dq \frac{\Phi(q; P_0)}{\mu^2 - (p-q)^2 - i\epsilon}, \quad \lambda = \frac{g_1 g_2}{16\pi^2}. \quad (16)
\end{aligned}
\]

This integral equation has a singular kernel and so standard mathematical tools are inapplicable to it. But the singularity in
\[
\frac{1}{\mu^2 - (p_0 - q_0)^2 + (\vec{p} - \vec{q})^2 - i\epsilon}
\]
disappears if we suppose that \(p_0\) and \(q_0\) are pure imaginary. The procedure of transition to such \(p_0, q_0\) is called "Wick rotation" and it assumes two things: the analytic continuation of \(\Phi(p; P_B)\) in the complex \(p_0\)-plane and a rotation of the \(dq_0\) integration contour from real to imaginary axis. During this rotation the integration contour, of course, should not hook on singularities of the integrand. So it is needed to study the analytical properties of the \(\Phi(p; P_B)\) amplitude.
Analytical properties of the B-S amplitude

Let us consider the reduced B-S amplitude

\[(2\pi)^{-3/2}\Phi(x; P) = <0|T\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|B> = \Theta(x_0) f(x; P) + \Theta(-x_0) g(x; P),\]

where

\[f(x; P) = <0|T\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|B>\]

and

\[g(x; P) = <0|T\phi_2(-\eta_1 x)\phi_1(\eta_2 x)|B> .\]

Each of them can be transformed by using a complete set of states

\[1 = \sum |n><n| \equiv \sum_\alpha \int dp |p, \alpha><p, \alpha|\]

(\(\alpha\) represents discrete quantum numbers of the state \(|n>|\) and \(p\) is its 4-momentum). Taking into account identities

\[\phi_1(\eta_2 x) = e^{i\eta_2 x \hat{P}} \phi_1(0)e^{-i\eta_2 x \hat{P}}\] \(\text{and}\)

\[\phi_2(-\eta_1 x) = e^{-i\eta_1 x \hat{P}} \phi_2(0)e^{i\eta_1 x \hat{P}},\]

we get

\[f(x; P) = \sum_\alpha \int dp <0|\phi_1(\eta_2 x)|p, \alpha><p, \alpha|\phi_2(-\eta_1 x)|B> = \sum_\alpha \int dp e^{-i(p-\eta_1 P)x} <0|\phi_1(0)|p, \alpha><p, \alpha|\phi_2(0)|B> ,\]

\[g(x; P) = \sum_\alpha \int dp <0|\phi_2(-\eta_1 x)|p, \alpha><p, \alpha|\phi_1(\eta_2 x)|B> = \sum_\alpha \int dp e^{-i(\eta_2 P-p)x} <0|\phi_2(0)|p, \alpha><p, \alpha|\phi_1(0)|B> ,\]

or

\[f(x; P) = \int dp e^{-i(p-\eta_1 P)x} f(p; P),\]

\[g(x; P) = \int dp e^{-i(\eta_2 P-p)x} g(p; P),\]

where we have designated

\[f(p; P) = \sum_\alpha <0|\phi_1(0)|p, \alpha><p, \alpha|\phi_2(0)|B> ,\]

\[g(p; P) = \sum_\alpha <0|\phi_2(0)|p, \alpha><p, \alpha|\phi_1(0)|B> .\]

The matrix element \(<0|\phi_1(0)|p, \alpha>\) is different from zero only then \(|p, \alpha>\) has the same quantum numbers as the \(\phi_1\) field quantum (otherwise \(\phi_1(0)|p, \alpha>\) will not have the vacuum quantum numbers). But among states
with quantum numbers of the first particle just this particle should have the smallest invariant mass, unless it will be not stable. So \( f(p; P) \neq 0 \) only then \( p_0 \geq \sqrt{m_1^2 + \vec{p}^2} \). Therefore
\[
f(x; P) = \int dpe^{-i(p-n_1\eta)^x}f(p; P)\Theta(p_0 - \sqrt{m_1^2 + \vec{p}^2}) = 
\int dqe^{-i\eta x}f(\eta_1 P + q; P)\Theta \left(q_0 + \eta_1 P_0 - \sqrt{m_1^2 + (\vec{q} + \eta_1 \vec{P})^2} \right).
\]
This last equation, if we designate
\[
\tilde{f}(q; P) = f(\eta_1 P + q; P), \quad \omega_+ = \sqrt{m_1^2 + (\vec{q} + \eta_1 \vec{P})^2} - \eta_1 P_0,
\]
can be rewritten as
\[
f(x; P) = \int d\vec{q} \int_{\omega_+}^{\infty} dq_0 e^{-iq x} \tilde{f}(q; P).
\]
Let us note
\[
\omega_+ \geq m_1 - \eta_1 P_0 = \frac{m_1}{m_1 + m_2}(m_1 + m_2 - P_0)
\]
In the rest frame \( P_0 \) equals to the bound state mass, therefore \( m_1 + m_2 > P_0 \). That is \( \omega_+ > 0 \) and only positive frequencies contribute in \( f(x; P) \).

Analogously, \( < 0|\phi_2(0)|p, \alpha > \neq 0 \) only then \( |p, \alpha > \) has quantum numbers of the second particle and stability of this particle means \( g(p; P) \equiv g(p; P)\Theta(p_0 - \sqrt{m_2^2 + \vec{p}^2}) \). So
\[
g(x; P) = \int dpe^{-i(p-n_2\eta)^x}g(p; P)\Theta(p_0 - \sqrt{m_2^2 + \vec{p}^2}) = 
\int dqe^{-i\eta x}g(\eta_2 P - q; P)\Theta \left(\eta_2 P_0 - q_0 - \sqrt{m_2^2 + (\eta_2 \vec{P} - \vec{q})^2} \right).
\]
If we designate \( \tilde{g}(q; P) = g(\eta_2 P - q; P), \quad \omega_- = \eta_2 P_0 - \sqrt{m_2^2 + (\eta_2 \vec{P} - \vec{q})^2}, \)
then
\[
g(x; P) = \int d\vec{q} \int_{\omega_-}^{-\infty} dq_0 e^{-iq x} \tilde{g}(q; P).
\]
\( \omega_- \leq \eta_2 P_0 - m_2 = \frac{m_2}{m_1 + m_2}(P_0 - m_1 - m_2) < 0, \) so only negative frequencies contribute in \( g(x; P) \) (in the bound system rest frame).

As a result we get
\[
\Phi(x; P) = (2\pi)^{3/2}\Theta(x_0) \int d\vec{q} \int_{\omega_+}^{\infty} dq_0 e^{-iq x} \tilde{f}(q; P) + 
(2\pi)^{3/2}\Theta(-x_0) \int d\vec{q} \int_{\omega_-}^{-\infty} dq_0 e^{-iq x} \tilde{g}(q; P).
\]
In the \( \vec{P} = 0 \) rest frame \( \omega_+ > 0, \quad \omega_- < 0, \) Therefore Eq.17 shows that from the \( x_0 > 0 \) positive half-line \( \Phi(x; P) \) can be analytically continued in the bottom half-plane (just then we will have falling exponent), and from the \( x_0 < 0 \) negative half-line – in the upper half-plane.
Analytical properties of the B-S amplitude in the momentum space

To deal with Eq.16, we need analytical properties of the $\Phi(p, P_0)$ amplitude. Let us consider therefore the momentum space amplitude $\Phi(p; P) = \int dx e^{ipx} \Phi(x; P)$. Because of Eq.17 we have

$$\Phi(p; P) = (2\pi)^{3/2} \int \omega^+ d\omega^+ d\omega_0 \int dx_0 e^{i(p_0 - q_0)x_0} \tilde{f}(q; P) +$$

$$+ (2\pi)^{3/2} \int \omega^- d\omega^- d\omega_0 \int dx_0 e^{i(p_0 - q_0)x_0} \tilde{g}(q; P).$$

Integrals over $dx$, $d\omega$ and $dx_0$ can be calculated, which gives

$$\Phi(p; P) = i(2\pi)^{9/2} \int_{\omega^+} d\omega_0 \frac{\tilde{f}(q_0, p; P)}{p_0 - q_0 + i\epsilon} - i(2\pi)^{9/2} \int_{-\omega^+} d\omega_0 \frac{\tilde{g}(q_0, p; P)}{p_0 - q_0 - i\epsilon}. \quad (18)$$

While integrating over $dx_0$ the following definition of the integrals was applied

$$\int_0^\infty dx_0 e^{i(p_0 - q_0)x_0} \equiv \int_0^\infty dx_0 e^{i(p_0 - q_0 + i\epsilon)x_0},$$

$$\int_{-\infty}^0 dx_0 e^{i(p_0 - q_0)x_0} \equiv \int_{-\infty}^0 dx_0 e^{i(p_0 - q_0 - i\epsilon)x_0}.$$

If now we try to continue $\Phi(p; P)$ analytically in the complex $p_0$-plane, everything will be O.K. except $p_0 - q_0 \pm i\epsilon$ case. To avoid these points, cuts should be assumed in the $p_0$-plane. In the rest frame we will have the following picture:

![Figure 5: Cuts in the $p_0$-plane.](image)

In the remaining $p_0$-plane $\Phi(p; P)$ will be an analytical function.
Wick rotation

In the r.h.s. of Eq.16 we have the following integral over $dq_0$

$$\int_{-\infty}^{\infty} dq_0 \frac{\Phi(q; P_0)}{\mu^2 - (p_0 - q_0)^2 + (\vec{p} - \vec{q})^2 - i\epsilon}.$$

Its integrand has singularities shown in Fig.6.

Therefore

$$\int_{C} dq_0 \frac{\Phi(q; P_0)}{\mu^2 - (p_0 - q_0)^2 + (\vec{p} - \vec{q})^2 - i\epsilon} = 0,$$

where the integration contour is shown in Fig.7.

Figure 6: Integrand singularities.

Figure 7: Integration contour.
The contributions from the two infinite quarter circles tends to zero. So Eq.19 means that the $dq_0$-integral over the real axis can be replaced by that over the $C_1$ contour, which follows the imaginary axis and goes around a $p_0 - \sqrt{\mu^2 + (\vec{p} - \vec{q})^2 + i\epsilon}$ pole (if $p_0 > 0$), or $p_0 + \sqrt{\mu^2 + (\vec{p} - \vec{q})^2 - i\epsilon}$ pole (if $p_0 < 0$).

Therefore

$$\left[ m_1^2 + \vec{p}^2 - (\eta_1 P_0 + p_0)^2 \right] \left[ m_2^2 + \vec{p}^2 - (\eta_2 P_0 - p_0)^2 \right] \Phi(p; P_0) = \frac{\lambda}{i\pi^2} \int \frac{d\vec{q}}{C_1} \int \frac{dq_0}{\mu^2 - (p_0 - q_0)^2 + (\vec{p} - \vec{q})^2 - i\epsilon} \Phi(q; P_0).$$

(20)

If now we begin to rotate $p_0$ counterclockwise, both sides of Eq.20 remain well defined. When we reach the imaginary axis $p_0 = ip_4$, the disposition of the integrand singularities with regard to the $C_1$ contour will be such:

![Figure 8: Disposition of the integrand singularities.](image)

As we see, the dangerous pole has left the $C_1$ contour loop, so this contour can be straightened and it will coincide the imaginary axis. Therefore Eq.20 becomes:

$$\left[ m_1^2 + \vec{p}^2 - (\eta_1 P_0 + ip_4)^2 \right] \left[ m_2^2 + \vec{p}^2 - (\eta_2 P_0 - ip_4)^2 \right] \Phi(ip_4, \vec{p}; P_0) = \frac{\lambda}{i\pi^2} \int \frac{d\vec{q}}{-i\infty} \int \frac{dq_0}{\mu^2 - (ip_4 - q_0)^2 + (\vec{p} - \vec{q})^2 - i\epsilon} \Phi(q; P_0).$$
Let us make $q_0 \rightarrow iq_4$ variable change in the integral over $dq_0$:

$$
\left[ m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 P_0)^2 \right] \left[ m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 P_0)^2 \right] \Phi(ip_4, \vec{p}; P_0) = \frac{\lambda}{\pi^2} \int dq_4 \int d\vec{q}_4 \frac{\Phi(iq_4, \vec{q}; P_0)}{\mu^2 + (p_4 - q_4)^2 + (\vec{p} - \vec{q})^2} ,
$$

If now introduce Euclidean 4-vectors $\tilde{p} = (\vec{p}, p_4)$, $\tilde{q} = (\vec{q}, q_4)$ and an amplitude $\tilde{\Phi} (\tilde{p}; P_0) = \Phi(ip_4, \vec{p}; P_0)$ (which is defined by the analytical continuation of the B-S amplitude), then the above equation can be rewritten as

$$
\left[ m_1^2 + \tilde{p}^2 + (p_4 - i\eta_1 P_0)^2 \right] \left[ m_2^2 + \tilde{p}^2 + (p_4 + i\eta_2 P_0)^2 \right] \tilde{\Phi}(\tilde{p}; P_0) = \frac{\lambda}{\pi^2} \int d\tilde{q} \frac{\tilde{\Phi}(\tilde{q}; P_0)}{\mu^2 + (\tilde{p} - \tilde{q})^2} .
$$

This is the Wick rotated, rest frame, ladder Bethe-Salpeter equation.

**Salpeter equation**

One of the main peculiarities, which distinguishes the B-S equation from the Schrödinger equation, is the presence of the relative time $x_0$ (in the momentum space the corresponding quantity is the relative energy $p_0$). It is clear that one of the physical sources for its appearance is the retardation of the interaction. If there are no other physical grounds behind the relative time, its effect should disappear in the instantaneous approximation, when the retardation of the interaction is neglected. Let us check this for an example of two interacting spinorial particles, for which the B-S equation looks like

$$
- \left[ m_1 - \gamma^{\mu}_{(1)}(\eta_1 P_\mu + p_\mu) \right] \left[ m_2 - \gamma^{\mu}_{(2)}(\eta_2 P_\mu - p_\mu) \right] \Psi(p; P) = \int dq I(p, q; P) \Psi(q; P) .
$$

in the l.h.s. instead of full propagators we have taken free ones, that is $\Delta(p) = \frac{1}{i(m - \vec{p})}$. Besides $\Psi(p; P) = \int dx e^{ipx} \Psi(x; P)$, and $\Psi(x; P)$ is a 16-component reduced B-S amplitude

$$
\Psi_{\alpha\beta}(x; P) = (2\pi)^{3/2} < 0|T\psi_{\alpha}(\eta_2 x)\psi_{\beta}(\eta_1 x)|B > .
$$

$\gamma_{(1)}$ matrices act on the first spinor index and $\gamma_{(2)}$ matrices on the second one.

The instantaneous approximation means that $I(p, q; P) \equiv I(\tilde{p}, \tilde{q}; P)$ does not depend on relative energy. Indeed, if this is the case, then

$$
I(x, y; X) = (2\pi)^{-8} \int dp \int dq \int dP e^{-ipx} e^{i\eta y} e^{-iP X} I(\tilde{p}, \tilde{q}; P) =
$$
(2\pi)^{-6}\delta(x_0)\delta(y_0) \int d\vec{p} \, d\vec{q} \, dPe^{i\vec{p} \cdot \vec{x}}e^{-i\vec{q} \cdot \vec{y}}e^{-iPX}I(\vec{p}, \vec{q}, P)

Let us multiply both sides of Eq.22 over \gamma^0_{(1)}\gamma^0_{(2)} and designate \tilde{I} = -\gamma^0_{(1)}\gamma^0_{(2)}I, then the equation in the rest frame \(P_\mu = (E, 0)\) becomes:

\[
[\frac{H_1(\vec{p}) - \eta_1 E - p_0]}{\frac{H_2(-\vec{p}) - \eta_2 E + p_0]}{\Psi(p; P) = \int dq \tilde{I}(\vec{p}, \vec{q}; P)\Psi(q; P),}
\]

where \(H(\vec{p}) = \vec{a} \cdot \vec{p} + \beta m\) is a conventional Dirac Hamiltonian. The right hand side of this equation can be rewritten as

\[
\int dq \tilde{I}(\vec{p}, \vec{q}; P)\Psi(q; P) = \int dq \tilde{I}(\vec{p}, \vec{q}; P)\Phi(\vec{q}; P),
\]

where

\[
\Phi(\vec{q}; P) = \int_{-\infty}^{\infty} dq_0 \Psi(q; P) = \int dq_0 \int dx e^{iqx} \Psi(x; P) =
\]

\[
2\pi \int dx \delta(x_0)e^{-i\vec{q} \cdot \vec{x}}\Psi(x_0, \vec{x}; P) = 2\pi \int d\vec{x} e^{-i\vec{q} \cdot \vec{x}}\Psi(0, \vec{x}; P).
\]

So \(\Phi(\vec{q}; P)\) is determined by a simultaneous B-S amplitude.

To express the l.h.s. of the Eq.23 also in terms of \(\Phi(\vec{q}; P)\), the following trick can be used. The projection operators

\[
\Lambda_{\pm}(\vec{p}) = \frac{1}{2}
\]

have a property \(\Lambda_{\pm}(\vec{p})H(\vec{p}) = \pm \sqrt{m^2 + \vec{p} \cdot \vec{p}}\Lambda_{\pm}(\vec{p})\). using this, Eq.23 can be replaced by a system

\[
\left[ \sqrt{m^2_1 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon \right] \left[ \sqrt{m^2_2 + \vec{p}^2} - \eta_2 E + p_0 + i\epsilon \right] \Psi_{++}(p; P) = \Lambda_+^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p}) \int dq \bar{I}(\vec{p}, \vec{q}; P)\Phi(\vec{q}; P),
\]

\[
\left[ \sqrt{m^2_1 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon \right] \left[ -\sqrt{m^2_2 + \vec{p}^2} + \eta_2 E + p_0 + i\epsilon \right] \Psi_{+-}(p; P) = \Lambda_+^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) \int dq \bar{I}(\vec{p}, \vec{q}; P)\Phi(\vec{q}; P),
\]

\[
\left[ -\sqrt{m^2_1 + \vec{p}^2} + \eta_1 E - p_0 + i\epsilon \right] \left[ \sqrt{m^2_2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon \right] \Psi_{-+}(p; P) = \Lambda_-^{(1)}(\vec{p})\Lambda_+^{(2)}(-\vec{p}) \int dq \bar{I}(\vec{p}, \vec{q}; P)\Phi(\vec{q}; P),
\]

\[
\left[ -\sqrt{m^2_1 + \vec{p}^2} + \eta_1 E - p_0 + i\epsilon \right] \left[ -\sqrt{m^2_2 + \vec{p}^2} - \eta_2 E + p_0 + i\epsilon \right] \Psi_{--}(p; P) = \Lambda_-^{(1)}(\vec{p})\Lambda_-^{(2)}(-\vec{p}) \int dq \bar{I}(\vec{p}, \vec{q}; P)\Phi(\vec{q}; P).
\]

\[\text{(23')}\]
Here $\Psi_{++}(p; P) = \Lambda_{+}^{(1)}(\vec{p})\Lambda_{+}^{(2)}(-\vec{p})\Psi(p; P)$ and so on. Note the substitution $m \to m - i\epsilon$ in the Feynman propagators (positive frequencies propagate forward in time, while negative frequencies - backward).

By means of residue theory we get

\[
\int_{-\infty}^{\infty} dp_0 \left[ \sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon \right]^{-1} \left[ \sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon \right]^{-1} =
\]

\[
-2\pi i \left[ E - \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2} \right]^{-1},
\]

\[
\int_{-\infty}^{\infty} dp_0 \left[ -\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 + i\epsilon \right]^{-1} \left[ -\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 + i\epsilon \right]^{-1} =
\]

\[
2\pi i \left[ E + \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} \right]^{-1},
\]

\[
\int_{-\infty}^{\infty} dp_0 \left[ \sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon \right]^{-1} \left[ -\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 + i\epsilon \right]^{-1} =
\]

\[
\int_{-\infty}^{\infty} dp_0 \left[ -\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 + i\epsilon \right]^{-1} \left[ \sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon \right]^{-1} = 0.
\]

Therefore for the $\Phi(\vec{q}; P)$ amplitude the following system holds

\[
\left[ E - \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2} \right] \Phi_{++}(\vec{p}; P) \equiv
\]

\[
\Lambda_{+}^{(1)}(\vec{p})\Lambda_{+}^{(2)}(-\vec{p})[E - H_1(\vec{p}) - H_2(-\vec{p})]\Phi(\vec{p}; P) =
\]

\[
\Lambda_{+}^{(1)}(\vec{p})\Lambda_{+}^{(2)}(-\vec{p})\frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P),
\]

\[
\left[ E + \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} \right] \Phi_{--}(\vec{p}; P) \equiv
\]

\[
\Lambda_{-}^{(1)}(\vec{p})\Lambda_{-}^{(2)}(-\vec{p})[E - H_1(\vec{p}) - H_2(-\vec{p})]\Phi(\vec{p}; P) =
\]

\[
-\Lambda_{-}^{(1)}(\vec{p})\Lambda_{-}^{(2)}(-\vec{p})\frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P),
\]

\[
\Phi_{+-}(\vec{p}; P) = \Phi_{-+}(\vec{p}; P) = 0.
\]

To rewrite the last two equations in the same form, as the first ones, note that

\[
E - \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} > 0, \quad E + \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2} > 0.
\]

Indeed, if $m_1 = m_2$, these inequalities are obvious. Let $m_1 > m_2$, then only

\[
E - \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} > 0
\]
inequality needs a proof. But
\[
\sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2} = \frac{m_1^2 - m_2^2}{\sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2}} \leq \frac{m_1^2 - m_2^2}{m_1 + m_2} = m_1 - m_2,
\]
and \( E > m_1 - m_2 \) is a stability condition for the first particle. Otherwise the first particle decay into the second antiparticle and \( |B| > \) bound state will be energetically permitted.

Because of the above mentioned inequalities, \( \Phi^+(\vec{p}; P) = \Phi^-(\vec{p}; P) = 0 \) equations are equivalent to
\[
0 = \left[ E - \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2} \right] \Phi^+(\vec{p}; P) \equiv \\
\Lambda_+^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) (E - H_1(\vec{p}) - H_2(-\vec{p})) \Phi(\vec{p}; P),
\]
\[
0 = \left[ E + \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2} \right] \Phi^-(\vec{p}; P) \equiv \\
\Lambda_-^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) (E - H_1(\vec{p}) - H_2(-\vec{p})) \Phi(\vec{p}; P).
\]

If we sum all four equations for \( \Phi(\vec{p}; P) \) and use
\[
\Lambda_+^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) + \Lambda_+^{(1)}(\vec{p}) \Lambda_-^{(2)}(-\vec{p}) + \Lambda_-^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) + \Lambda_-^{(1)}(\vec{p}) \Lambda_-^{(2)}(-\vec{p}) = 1,
\]
then we get the Salpeter equation
\[
[E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; E) = \\
[\Lambda_+^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) - \Lambda_-^{(1)}(\vec{p}) \Lambda_-^{(2)}(-\vec{p})] \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}; \vec{q}; P) \Phi(\vec{q}; E). \tag{24}
\]
So the relative energy is indeed excluded from the equation, but with the price of \( \Lambda_{++} - \Lambda_{--} \) operator introduction. To understand why relative energy has left such a trace, it is useful to compare with the nonrelativistic case.

**Bethe-Salpeter equation in the nonrelativistic theory**

To derive the nonrelativistic Bethe-Salpeter equation, one can use the fact that quantum field theory in many respects is similar to a second quantized many particle theory. In graphical representation, to the second quantized Hamiltonian
\[
H = \int d\vec{x} \Psi^+(\vec{x}, t) \left( -\frac{\Delta}{2m} \right) \Psi(\vec{x}, t) + \\
\frac{1}{2} \int d\vec{x} \, d\vec{y} \, \Psi^+(\vec{x}, t) \Psi^+(\vec{y}, t) V(|\vec{x} - \vec{y}|) \Psi(\vec{x}, t) \Psi(\vec{y}, t)
\]

...
there corresponds the free propagator
\[
\Delta(x - y) = \frac{i}{(2\pi)^4} \int dp \frac{e^{-ip(x-y)}}{p_0 - (p^2/2m) + i\epsilon}
\]
and the pair interaction (instantaneous) with the potential \(V\):
\[-i\delta(x_0 - y_0)V(|\vec{x} - \vec{y}|)\,.
\]
Everywhere in the derivation of the B-S equation, \(\phi(x)\) field operator can be replaced with the second quantized operator \(\Psi(x)\). As a result, we end with an equation for the following nonrelativistic Bethe-Salpeter amplitude
\[
\Phi(x; P) = (2\pi)^{3/2} < 0|T\Psi(\eta_2 x)\Psi(-\eta_1 x)|B >
\]
Namely, in the ladder approximation, the interaction operator is
\[
I(x_1, x_2; y_1, y_2) = -i\delta(x_0)V(|\vec{x}|)\delta(x_1 - y_1)\delta(x_2 - y_2) = -i\delta(x_0)V(|\vec{x}|)\delta(x - y)\delta(X - Y),
\]
with the corresponding expression in the momentum space
\[
I(p, q; P) = (2\pi)^{-4} \int dx \ dy \ dX e^{ipx} e^{-iqy} e^{iPX} I(x, y; X) =
\]
\[-i(2\pi)^{-4} \int e^{-i(\vec{p} - \vec{q})\cdot \vec{x}} V(|\vec{x}|) \ dx \vec{x} = \frac{-i}{(2\pi)^4} V(\vec{p} - \vec{q}).
\]
Besides, the momentum space free propagator looks like \(\Delta(p) = i/(p_0 - \vec{p}^2/2m + i\epsilon)\), therefore the B-S equation, in the rest frame and ladder approximation, will take the form
\[
\left[ (\eta_1 E + p_0) - \frac{\vec{p}^2}{2m} + i\epsilon \right] \left[ (\eta_2 E - p_0) - \frac{\vec{p}^2}{2m} + i\epsilon \right] \Phi(p; E) = i(2\pi)^{-4} \int dq \ V(\vec{p} - \vec{q})\Phi(q; E)
\]
(note that \(m_1 = m_2 \equiv m\) and so \(\eta_1 = \eta_2 = \frac{1}{2}\)).

Let us introduce integrated over \(dp_0\) amplitude \(\tilde{\Phi}(\vec{p}; E) = \int_{-\infty}^{\infty} dp_0 \ \Phi(q; E)\). For it the Salpeter equation holds
\[
\tilde{\Phi}(\vec{p}; E) = \left[ \int_{-\infty}^{\infty} \frac{dp_0}{[\eta_1 E + p_0 - \vec{p}^2/2m + i\epsilon][\eta_2 E - p_0 - \vec{p}^2/2m + i\epsilon]} \right] \times
\]
\[
\frac{i}{(2\pi)^4} \int d\vec{q} \, V(\vec{p} - \vec{q}) \tilde{\Phi}(\vec{q}; E).
\]

The square bracket integral can be evaluated via residue theory and it equals \(-2\pi i/(E - \vec{p}^2/2m)\). So we obtain the following equation

\[
\left(E - \frac{\vec{p}^2}{2m}\right) \tilde{\Phi}(\vec{p}; E) = (2\pi)^{-3} \int d\vec{q} \, V(\vec{p} - \vec{q}) \tilde{\Phi}(\vec{q}; E).
\] (25)

But this is just momentum space Schrödinger equation! Indeed, in configuration space

\[
\tilde{\Phi}(\vec{p}) = \int e^{-i\vec{p}\cdot\vec{x}} \tilde{\Phi}(\vec{x}) \, d\vec{x}, \quad V(\vec{p}) = \int e^{-i\vec{p}\cdot\vec{x}} V(|\vec{x}|) \, d\vec{x},
\]

we will have

\[
\left[-\frac{\Delta}{m} + V(|\vec{x}|)\right] \tilde{\Phi}(\vec{x}) = E \tilde{\Phi}(\vec{x}).
\]

Thus, in the nonrelativistic theory, the relative time can be excluded without any trace. So its introduction is purely formal.

**Physical meaning of the relative time**

What is the crucial peculiarity, which distinguishes the above given nonrelativistic model from the instantaneous approximation of the relativistic one? It is the propagator! There are no antiparticles in the nonrelativistic case. Therefore the propagator describes only forward propagation in time, that is we have retarded Green’s function: \(\Delta(x) = 0\), if \(x_0 < 0\). This boundary condition demands \(p_0 \rightarrow p_0 + i\epsilon\) prescription for the propagator poles. Let us see what will be changed in the Salpeter equation derivation if we replace the Feynman propagator \(i(\hat{p} + m)/(\vec{p}^2 - m^2 + i\epsilon)\) by the retarded Green’s function \(i(\hat{p} + m)/[(p_0 + i\epsilon)^2 - \vec{p}^2 - m^2]\). Instead of Eq.23' system, we will have

\[
\begin{align*}
\left[\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon\right] & \left[\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon\right] \Psi_{++}(p; P) = \Lambda_+^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \tilde{\Phi}(\vec{q}; P), \\
\left[-\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon\right] & \left[-\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon\right] \Psi_{--}(p; P) = \Lambda_+^{(1)}(\vec{p}) \Lambda_-^{(2)}(-\vec{p}) \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \tilde{\Phi}(\vec{q}; P), \\
\left[-\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon\right] & \left[\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon\right] \Psi_{+-}(p; P) = \Lambda_-^{(1)}(\vec{p}) \Lambda_+^{(2)}(-\vec{p}) \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \tilde{\Phi}(\vec{q}; P), \\
\left[\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon\right] & \left[-\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon\right] \Psi_{-+}(p; P) = \Lambda_-^{(1)}(\vec{p}) \Lambda_-^{(2)}(-\vec{p}) \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \tilde{\Phi}(\vec{q}; P),
\end{align*}
\]
\[
\left[-\sqrt{m_1^2 + \vec{p}^2} - \eta_1 E - p_0 - i\epsilon\right] \left[-\sqrt{m_2^2 + \vec{p}^2} - \eta_2 E + p_0 - i\epsilon\right] \Psi_-(p; P) = \\
\Lambda^{(1)}_-(\vec{p}) \Lambda^{(2)}_-(\vec{p}) \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P).
\]

(the propagators, which appear in the B-S equation, have \(\eta_1 P + p\) and \(\eta_2 P - p\) as their 4-momenta. Therefore the transition to the retarded Green’s function means the following replacements

\[
(\eta_1 P_0 + p_0) \to (\eta_1 P_0 + p_0) + i\epsilon, \quad (\eta_2 P_0 - p_0) \to (\eta_2 P_0 - p_0) + i\epsilon
\]

that is \(i\epsilon\) has the same sign, as \(E\). Integration over \(dp_0\) can be performed using

\[
\int_{-\infty}^{\infty} \frac{dp_0}{(a - p_0 - i\epsilon)(b + p_0 - i\epsilon)} = 2\pi i \frac{1}{a + b},
\]

and we get the system

\[
\left[E - \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2}\right] \Phi_+(\vec{p}; P) = \Lambda_+ [E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; P)
\]

\[
= \Lambda_+ \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P),
\]

\[
\left[E - \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2}\right] \Phi_-(\vec{p}; P) = \Lambda_- [E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; P)
\]

\[
= \Lambda_- \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P),
\]

\[
\left[E + \sqrt{m_1^2 + \vec{p}^2} - \sqrt{m_2^2 + \vec{p}^2}\right] \Phi_+(\vec{p}; P) = \Lambda_+ [E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; P)
\]

\[
= \Lambda_+ \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P),
\]

\[
\left[E + \sqrt{m_1^2 + \vec{p}^2} + \sqrt{m_2^2 + \vec{p}^2}\right] \Phi_-(\vec{p}; P) = \Lambda_- [E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; P)
\]

\[
= \Lambda_- \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; P).
\]

Summing these four equations, we get the Breit equation

\[
[E - H_1(\vec{p}) - H_2(-\vec{p})] \Phi(\vec{p}; E) = \frac{2\pi}{i} \int d\vec{q} \tilde{I}(\vec{p}, \vec{q}; P) \Phi(\vec{q}; E).
\]

which is a direct generalization of the two particle Schrödinger equation Eq.25. (The nonrelativistic Hamiltonian \(H(\vec{p}) = \vec{p}^2/2m\) is replaced by the Dirac Hamiltonian \(H(\vec{p}) = \vec{\alpha} \cdot \vec{p} + \beta m\) and \(-i(2\pi)^4 \tilde{I}(\vec{p}, \vec{q})\) plays the role of potential). Once again, the relative time disappears without any trace left.
Therefore, the second (and more important) source for the essential relative time dependence of the B-S amplitude is the existence of antiparticles, that is the possibility for particles to turn back in time and propagate backward. Because of retardation of the interaction, relative times of the order of bound system size will be significant, while the forward-backward motion in time makes essential configurations, for which individual routes in time are very different for bound state constituting particles, and so the relative time is large. Λ_{++} − Λ_{--} operator in the Salpeter equation just corresponds to the contribution of these configurations to the bound state amplitude.

Wick-Cutkosky model

This model corresponds to the B-S equation in the ladder approximation for two scalar particles interacting via massless quanta exchange. After the Wick rotation, the corresponding rest frame B-S equation takes the form

\[ [m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 E)^2][m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 E)^2]\Phi(p) = \frac{\lambda}{\pi^2} \int dq \frac{\Phi(q)}{(p - q)^2} \] (27)

where \( E \) is the bound state mass and \( p, q \) are Euclidean 4-vectors.

To investigate the mathematical structure of Eq.27, let us consider at first the simplest case \( m_1 = m_2 = m \) and \( E = 0 \) (although, in the rest frame, \( E = 0 \) is, of course, unphysical: massless bound state doesn’t have rest frame). Eq.27 then becomes

\[ (p^2 + m^2)^2\Phi(p) = \frac{\lambda}{\pi^2} \int dq \frac{\Phi(q)}{(p - q)^2} . \hspace{1cm} (28) \]

Let us show, that one of its solutions is \( \phi(p) = (p^2 + m^2)^{-3} \). We have

\[ A^{-1}B^{-1} = \frac{1}{A - B} \left( -\frac{1}{A} + \frac{1}{B} \right) = \]

\[ \frac{1}{\int_{0}^{1} \frac{dx}{[B + (A - B)x]^2}} = \frac{1}{\int_{0}^{1} \frac{dx}{[xA + (1 - x)B]^2}} , \]

besides

\[ A^{-n}B^{-m} = \frac{(-1)^{n+m}}{(n-1)!(m-1)!} \frac{\partial^{n-1}}{\partial A^{n-1}} \frac{\partial^{m-1}}{\partial B^{m-1}} (A^{-1}B^{-1}) , \]

and we get the following Feynman parameterization

\[ A^{-n}B^{-m} = \frac{(n + m - 1)!}{(n-1)!(m-1)!} \int_{0}^{1} \frac{x^{n-1}(1 - x)^{m-1}}{[xA + (1 - x)B]^{n+m}} dx . \hspace{1cm} (29) \]
In particular

\[ [(p - q)^2]^{-1}[q^2 + m^2]^{-3} = \]

\[ 3 \int_0^1 \frac{(1 - x)^2 dx}{[x(p - q)^2 + (1 - x)(q^2 + m^2)]^4} = \int_0^1 \frac{3(1 - x)^2 dx}{[(q - xp)^2 + (1 - x)(m^2 + xp^2)]^4}. \]

So

\[ \int dq \frac{\Phi(q)}{(p - q)^2} = \int dq \int_0^1 \frac{3(1 - x)^2 dx}{[(q - xp)^2 + (1 - x)(m^2 + xp^2)]^4} . \]

If now we use

\[ A^{-4} = - \frac{1}{3!} \frac{\partial^3}{\partial A^3} \left( \frac{1}{A} \right) = - \frac{1}{3!} \frac{\partial^3}{\partial A^3} \int_0^\infty e^{-\alpha A} d\alpha = \frac{1}{3!} \int_0^\infty \alpha^3 e^{-\alpha A} d\alpha , \]

then we get

\[ \int dq \int_0^1 \frac{3(1 - x)^2 dx}{[(q - xp)^2 + (1 - x)(m^2 + xp^2)]^4} = \]

\[ \frac{1}{2} \int_0^1 (1 - x)^2 dx \int_0^\infty \alpha^3 d\alpha \int dq \exp \{-\alpha[(q - xp)^2 + (1 - x)(m^2 + xp^2)]\} . \]

The Gaussian integral over \( dq \) equals

\[ \int dq \exp \{-\alpha[(q - xp)^2 + (1 - x)(m^2 + xp^2)]\} = \]

\[ \frac{\pi^2}{\alpha^2} \exp \{-\alpha(1 - x)(m^2 + xp^2)\} , \]

and we are left with an integral over \( d\alpha \) of the type

\[ \int_0^\infty \alpha e^{-\alpha A} d\alpha = - \frac{\partial}{\partial A} \int_0^\infty e^{-\alpha A} d\alpha = A^{-2} . \]

As a final result, we get

\[ \int dq \frac{\Phi(q)}{(p - q)^2} = \frac{\pi^2}{2} \int_0^1 \frac{dx}{[m^2 + xp^2]^2} = \frac{\pi^2}{2m^2(m^2 + p^2)} \]

Inserting this into Eq.28, we will see that \( \Phi(p) = (p^2 + m^2)^{-3} \) is indeed a solution and the corresponding eigenvalue is \( \lambda = 2m^2 \).

The most interesting thing about this solution is that we can indicate its analog in the nonrelativistic hydrogen atom problem.

The Schrödinger equation

\[ \left[ -\frac{\Delta}{2m} + V(\vec{r}) \right] \Psi(\vec{r}) = E \Psi(\vec{r}) \]
in the momentum space becomes an integral equation

\[(\vec{p}^2 - 2mE)\Psi(\vec{p}) = -\frac{2m}{(2\pi)^{3/2}} \int d\vec{q} \, V(\vec{p} - \vec{q})\Psi(\vec{q}) ,\]

where \(\Psi(\vec{q}) = (2\pi)^{-3/2} \int e^{-i\vec{p}\cdot\vec{r}} \Psi(\vec{r}) d\vec{r}\) and \(V(\vec{p}) = (2\pi)^{-3/2} \int e^{-i\vec{p}\cdot\vec{r}} V(\vec{r}) d\vec{r}\). 

\(\Delta(r^{-1}) = -4\pi\delta(\vec{r})\) identity indicates that \(V(\vec{r}) = -e^2/r\) Coulomb potential has \(V(\vec{p}) = -(4\pi/(2\pi)^{3/2})(e^2/\vec{p}^2)\) as its momentum space image. Therefore a nonrelativistic hydrogen atom is described by the equation:

\[(\vec{p}^2 + p_0^2)\Psi(\vec{p}) = \frac{me^2}{\pi^2} \frac{\Psi(\vec{q})}{(\vec{p} - \vec{q})^2} d\vec{q} ,\]

(30)

where \(p_0^2 = -2mE\) (we consider a discrete spectrum and therefore \(E < 0\)).

One of the solutions of this equation is \(\Psi(\vec{p}) = (\vec{p}^2 + p_0^2)^{-2}\). Indeed, in the similar way as above we get, after integrating over \(d\vec{q}\)

\[\int d\vec{q} [(\vec{p} - \vec{q})^2]^{-1}[\vec{p}^2 + p_0^2]^{-2} = \]

\[\pi^{3/2} \frac{1}{0} (1 - x) dx \int_0^\infty \sqrt{\alpha} \exp \left\{ -\alpha(1 - x)(p_0^2 + x\vec{p}^2) \right\} d\alpha . \]

Integral over \(d\alpha\) can be evaluated in such a way

\[\int_0^\infty \sqrt{\alpha} e^{-\alpha A} d\alpha = 2 \int_0^\infty t^2 e^{-t^2 A} dt = -2 \frac{\partial}{\partial A} \int_0^\infty e^{-t^2 A} dt = \]

\[-\frac{\partial}{\partial A} \frac{\sqrt{\pi}}{\sqrt{A}} = \frac{1}{2A\sqrt{A}} ,\]

therefore

\[\int \frac{\Psi(\vec{q})}{(\vec{p} - \vec{q})^2} d\vec{q} = \frac{\pi^2}{2} \int_0^1 \frac{dx}{(p_0^2 + x\vec{p}^2)(1 - x)(p_0^2 + x\vec{p}^2)} = \]

\[-2\pi^2 \frac{\partial^2}{\partial(p_0^2)^2} \int_0^1 \frac{p_0^2 + x\vec{p}^2}{1 - x} \frac{dx}{(p_0^2 + x\vec{p}^2)\sqrt{(1 - x)(p_0^2 + x\vec{p}^2)}} = \]

\[-2\pi^2 \frac{\partial^2}{\partial(p_0^2)^2} \int_1^\infty \frac{dt}{t^2 \sqrt{(p_0^2 + \vec{p}^2)t - \vec{p}^2}} = \frac{\pi^2}{2} \int_1^\infty [(p_0^2 + \vec{p}^2)t - \vec{p}^2]^{-3/2} dt = \]

\[\frac{\pi^2}{2(p_0^2 + \vec{p}^2)} \int p_0^2 \frac{x^{-3/2}}{p_0^2} dx = \frac{\pi^2}{p_0(p_0^2 + \vec{p}^2)} .\]
Inserting in Eq.30, we see that if \( \frac{m e^2}{p_0} = 1 \), then the equation is fulfilled. Thus \( \psi(\vec{p}) = (\vec{p}^2 + p_0^2)^{-\frac{1}{2}} \) corresponds to the \( E = -\frac{m e^4}{2} \) ground state of hydrogen atom.

If the analogy between the found solutions of Eq.28 and Eq.30 is not accidental, one can expect that the same methods, which are used in dealing with hydrogen atom, will be useful also for Eq.28 and maybe even for Eq.27. In particular, it is well known that the nonrelativistic hydrogen atom possesses a hidden symmetry and its study is more easily performed in the Fock space, where this symmetry becomes explicit. For the sake of simplicity, let us illustrate the Fock’s method on an example of a 2-dimensional hydrogen atom.

**Fock’s method for a 2-dimensional hydrogen atom**

The Schrödinger equation in the 2-dimensional momentum space looks like

\[
(p^2 - 2mE)\Psi(\vec{p}) = -\frac{m}{\pi} \int dq \ V(\vec{p} - \vec{q})\Psi(\vec{q}) .
\]

We have the following connection with the configuration space

\[
\Psi(\vec{p}) = \frac{1}{\pi} \int e^{-i\vec{p} \cdot \vec{r}} \Psi(\vec{r})d\vec{r} , \quad V(\vec{p}) = \frac{1}{\pi} \int e^{-i\vec{p} \cdot \vec{r}}V(\vec{r})d\vec{r} .
\]

For the Coulomb potential \( V(\vec{r}) = -\frac{e^2}{r} \) the above given integral diverges. But let us note that as bound state wave function is concentrated in a finite domain of space it should not feel a difference between Coulomb potential and a \((-\frac{e^2}{r})e^{-\alpha r}\) potential for sufficiently small \( \alpha \). Therefore, at least for bound states, the momentum space image of the Coulomb potential can be defined as

\[
V(\vec{p}) = \lim_{\alpha \to 0} \frac{1}{\pi} \int e^{-i\vec{p} \cdot \vec{r}} \left( -\frac{e^2}{r} \right) e^{-\alpha r}d\vec{r} =
\]

\[
-\frac{e^2}{\pi} \lim_{\alpha \to 0} \int_0^\infty e^{-\alpha r}dr \int_0^{2\pi} e^{-ipr\cos \Theta} d\Theta .
\]

Using

\[
\int_0^{2\pi} e^{-ipr\cos \Theta} d\Theta = 2\pi J_0(pr) , \quad \int_0^\infty e^{-\alpha r}J_0(pr)dr = \frac{1}{\sqrt{p^2 + \alpha^2}} ,
\]

we get

\[
V(\vec{p}) = -\frac{e^2}{\pi} \lim_{\alpha \to 0} \frac{1}{\sqrt{p^2 + \alpha^2}} = -\frac{e^2}{|p|} .
\]
So the Schrödinger equation for the 2-dimensional Coulomb problem, in the case of discrete spectrum $E < 0$, will be ($p_0^2 = -2mE$):

$$
(p^2 + p_0^2)\Psi(p) = \frac{m e^2}{\pi} \int \frac{\Psi(q)}{|p - q|} \, dq .
$$

(31)

The 2-dimensional momentum space can be mapped onto the surface of the 3-dimensional sphere (which we call the Fock space) by means of the stereographic projection. The stereographic projection transforms a 2-dimensional momentum $\vec{p} = p_x \vec{i} + p_y \vec{j}$ into a point on the surface of the 3-dimensional sphere where this surface is crossed by the line which connects the south pole of the sphere with the $(p_x, p_y)$ point in the equatorial plane. This is shown schematically in Fig. 9 below.
Let the radius of the sphere be $p_0$. If the polar coordinates of the vector $\vec{p}$ are $(p, \phi)$ when the point $P$ will have the spherical coordinates $(p_0, \Theta, \phi)$, where the angle $\Theta$, as it is clear from the Fig. 9, is determined by equation $|\vec{p}| = p_0 \operatorname{tg}(\Theta/2)$. Therefore

$$\cos \Theta = \frac{2 \cos^2 \frac{\Theta}{2} - 1}{2} = \frac{2}{1 + \operatorname{tg}^2(\Theta/2)} - 1 = \frac{p_0^2 - p^2}{p_0^2 + p^2}$$

and

$$\sin \Theta = \sqrt{1 - \cos^2 \Theta} = \frac{2p_0 p}{p_0^2 + p^2},$$

As for the Cartesian coordinates $P_x = p_0 \sin \Theta \cos \phi$, $P_y = p_0 \sin \Theta \sin \phi$, $P_z = p_0 \cos \Theta$ of the point $P$, we will have

$$P_x = \frac{2p_0^2 p_x}{p_0^2 + p^2}, \quad P_y = \frac{2p_0^2 p_y}{p_0^2 + p^2}, \quad P_z = \frac{p_0^2 - p^2}{p_0^2 + p^2}p_0. \quad (32)$$

Let $P$ and $Q$ be stereographic projections of the vectors $\vec{p}$ and $\vec{q}$, and let $\alpha$ be an angle between 3-dimensional vectors $\vec{P}$ and $\vec{Q}$. When, as $|\vec{P}| = |\vec{Q}| = p_0$, the distance between $P$ and $Q$ points in the 3-dimensional space will be $\sqrt{p_0^2 + p_0^2 - 2p_0^2 \cos \alpha} = 2p_0 \sin (\alpha/2)$. On the other hand, the square of this distance equals $(P_x - Q_x)^2 + (P_y - Q_y)^2 + (P_z - Q_z)^2$, therefore

$$\left(2p_0 \sin \frac{\alpha}{2}\right)^2 = \left(\frac{2p_0^2 p_x}{p_0^2 + p^2} - \frac{2p_0^2 q_x}{p_0^2 + q^2}\right)^2 + \left(\frac{2p_0^2 p_y}{p_0^2 + p^2} - \frac{2p_0^2 q_y}{p_0^2 + q^2}\right)^2 + \left(\frac{p_0}{p_0^2 + p^2} - \frac{p_0}{p_0^2 + q^2}\right)^2 =$$

$$\frac{4p_0^4}{(p_0^2 + p^2)(p_0^2 + q^2)} \left\{ p^2 \frac{p_0^2 + q^2}{p_0^2 + p^2} + q^2 \frac{p_0^2 + p^2}{p_0^2 + q^2} - 2 \vec{p} \cdot \vec{q} + p^4 \frac{p_0^2 + q^2}{p_0^2 + p^2} + q^4 \frac{p_0^2 + p^2}{p_0^2 + q^2} - 2 \frac{p^2 q^2}{p_0^2} \right\} =$$

$$\frac{4p_0^4}{(p_0^2 + p^2)(p_0^2 + q^2)} \left\{ p^2 \frac{p_0^2 + q^2}{p_0^2} + q^2 \frac{p_0^2 + p^2}{p_0^2} - 2 \vec{p} \cdot \vec{q} - 2 \frac{p^2 q^2}{p_0^2} \right\} =$$

$$\frac{4p_0^4}{(p_0^2 + p^2)(p_0^2 + q^2)}(\vec{p} - \vec{q})^2.$$ But

$$\frac{4p_0^4}{(p_0^2 + p^2)(p_0^2 + q^2)} = \frac{1}{(1 + \operatorname{tg}^2(\Theta/2))(1 + \operatorname{tg}^2(\Theta'/2))} = \cos^2 \frac{\Theta}{2} \sin^2 \frac{\Theta'}{2},$$

29
where $\Theta'$ spherical coordinate corresponds to the point $Q$. So we get

$$|\vec{p} - \vec{q}| = p_0 \sec \frac{\Theta}{2} \sec \frac{\Theta'}{2} \sin \frac{\alpha}{2}. \quad (33)$$

Now let us express $d\vec{p} = dp_x \, dp_y$ through the solid angle element $d\Omega = \sin \Theta \, d\Theta \, d\phi$.

$$d\Omega = -d(\cos \Theta) \, d\phi = -d\left(\frac{p_0^2 - p^2}{p_0^2 + p^2}\right) \, d\phi = \frac{4p_0^2}{(p_0^2 + p^2)^2} p \, dp \, d\phi = \left[ \frac{2p_0}{p_0^2 + p^2} \right] d\vec{p} = \left[ \frac{2}{p_0} \cos^2 \frac{\Theta}{2} \right]^2 d\vec{p}.$$  

Thus

$$d\vec{p} = \frac{p_0^2}{4} \sec^4 \frac{\Theta}{2} d\Omega. \quad (34)$$

With the help of Eq.33 and Eq.34, it is possible to rewrite the Schrödinger equation (31) in the Fock space

$$\sec^3 \frac{\Theta}{2} \Psi(\vec{p}) = \frac{m e^2}{4\pi p_0} \int \frac{\Psi(\vec{q}) \sec^3 \Theta'/2}{\sin \alpha/2} \, d\Omega',$$

or, if a new unknown function $\Phi(\Omega) = \sec^3 \frac{\Theta}{2} \Psi(\vec{p})$ is introduced

$$\Phi(\Omega) = \frac{m e^2}{4\pi p_0} \int \frac{\Phi(\Omega')}{{\sin \alpha/2}} \, d\Omega'.$$  

(35)

This is the Fock space Schrödinger equation. It is invariant with respect to a 3-dimensional rotational group $SO(3)$ (the angle $\alpha$ measures an angular distance between two points on the sphere surface and so it is unchanged under rotations of the sphere). In the momentum space this symmetry is hidden: Eq. 31 is explicitly invariant only under a smaller group $SO(2)$.

A solution of Eq. 35 is proportional to the spherical function $Y_{lm}$. Indeed, from the generating function of the Legendre polynomials

$$(1 - 2\epsilon \cos \alpha + \epsilon^2)^{-1/2} = \sum_{l=0}^{\infty} \epsilon^l P_l(\cos \alpha),$$

we will have for $\epsilon = 1$

$$\frac{1}{2 \sin (\alpha/2)} = \sum_{l=0}^{\infty} P_l(\cos \alpha) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\Omega)Y_{lm}^*(\Omega').$$
and therefore

\[ Y_{lm}(\Omega) = \frac{2l + 1}{8\pi} \int \frac{Y_{lm}(\Omega')}{\sin(\alpha/2)} \, d\Omega'. \tag{36} \]

So the general solution of Eq.35 has a form \( \Phi(\Omega) = \sum_{m=-l}^{l} a_m Y_{lm}(\Omega) \), if the eigenvalue condition \( (me^2/p_0) = l + (1/2) \) is fulfilled. The corresponding energy eigenvalues are \( E_l = -\frac{2me^4}{(2l+1)^2} \). Each of the energy levels is \((2l+1)\)-fold degenerate, and this degeneration is of course caused by the above mentioned hidden symmetry, which in the Fock space becomes explicit.

**Stereographic projection (general case)**

To generalize the Fock’s method for equations (28) and (30), one needs first of all a definition of the stereographic projection when the momentum space dimension is greater than two. But the above given definition of the stereographic projection admits in fact a trivial generalization for the \( n \)-dimensional momentum space.

Let a momentum space point \( p = (p_1, \ldots, p_n) \) is transformed into a \( P = (P_1, \ldots, P_{n+1}) \) point on the sphere surface by the stereographic projection. \( \vec{s}p \) and \( \vec{P} \) vectors are parallel because \( P \) lies on the line connecting the point \( p \) to the ”south pole” \( s = (0, \ldots, 0, -p_0) \) of the sphere. Therefore \( (P_1 - p_1, \ldots, P_n - p_n, P_{n+1}) = k(p_1, \ldots, p_n, p_0), \) where \( k \) is some constant. So \( P_i = (k+1)p_i, \ i = 1 \div n \) and \( P_{n+1} = kp_0 \). But the point \( P \) is on the sphere surface and therefore \( P_1^2 + \cdots + P_n^2 + P_{n+1}^2 = p_0^2 \). This enables us to determine \( k \) from the equation \( (k+1)^2 p^2 + k^2 p_0^2 = p_0^2 \) as \( k = \frac{p_0^2 - p^2}{p_0^2 + p^2} \).

Therefore a generalization of the Eq.32 is

\[ P_1 = \frac{2p_0^2 p_1}{p_0^2 + p^2}, \ldots, P_n = \frac{2p_0^2 p_n}{p_0^2 + p^2}, P_{n+1} = \frac{p_0^2 - p^2}{p_0^2 + p^2 p_0}. \tag{37} \]

It is also straightforward to generalize Eq.33

\[ |\vec{p} - \vec{q}| = p_0 \sec \frac{\Theta_n}{2} \sec \frac{\Theta'_n}{2} \sin \frac{\alpha}{2}. \tag{38} \]

(\( \vec{p} \) and \( \vec{q} \) are \( n \)-dimensional vectors, \( \alpha \) is the angle between \( \vec{P} \) and \( \vec{Q} \) \((n+1)\)-dimensional vectors). A proof of Eq.38 looks like this:

\[ 2p_0^2 \sin^2 \frac{\alpha}{2} = p_0^2 (1 - \cos \alpha) = p_0^2 - \vec{P} \cdot \vec{Q} = \]
\[ p_0^2 - \frac{4p_0^4\vec{p} \cdot \vec{q}}{(p_0^2 + p^2)(p_0^2 + q^2)} - \frac{(p_0^2 - p^2)(p_0^2 - q^2)}{(p_0^2 + p^2)(p_0^2 + q^2)}p_0^2 = \frac{2p_0^4(p^2 + q^2 - 2\vec{p} \cdot \vec{q})}{(p_0^2 + p^2)(p_0^2 + q^2)} = 2 \frac{p_0^2}{p_0^2 + p^2} \frac{p_0^2}{p_0^2 + q^2} (\vec{p} - \vec{q})^2 = 2 \cos^2 \frac{\Theta}{2} \cos^2 \frac{\Theta'}{2} (\vec{p} - \vec{q})^2. \]

Now it is necessary to generalize Eq.34. A connection between spherical and Cartesian coordinates in the \( n \)-dimensional space reads

\[
\begin{align*}
  r_1 &= r \sin \Theta_{n-1} \sin \Theta_{n-2} \ldots \sin \Theta_3 \sin \Theta_2 \cos \Theta_1, \\
  r_2 &= r \sin \Theta_{n-1} \sin \Theta_{n-2} \ldots \sin \Theta_3 \sin \Theta_2 \sin \Theta_1, \\
  r_3 &= r \sin \Theta_{n-1} \sin \Theta_{n-2} \ldots \sin \Theta_3 \cos \Theta_2, \\
  r_4 &= r \sin \Theta_{n-1} \sin \Theta_{n-2} \ldots \sin \Theta_4 \cos \Theta_3, \\
  r_n &= r \cos \Theta_{n-1}.
\end{align*}
\]

At that for the \( n \)-dimensional volume element \( d(n)\vec{r} = dr_1 \ldots dr_n \) we will have

\[
d(n)\vec{r} = r^{n-1} \, dr \, d^n\Omega = r^{n-1} \sin^{n-2} \Theta_{n-1} \sin^{n-3} \Theta_{n-2} \ldots \sin \Theta_2 \, dr \, d\Theta_1 \ldots d\Theta_{n-1}.
\]  

(39)

This equation can be proved by induction. As for \( n = 3 \) it is correct, it is sufficient to show that from its correctness there follows an analogous relation for the \((n + 1)\)-dimensional case. Let us designate \( r = \sqrt{r_1^2 + \ldots + r_{n+1}^2} \) and \( \rho = \sqrt{r_1^2 + \ldots + r_n^2} = r \sin \Theta_n \), when \( d^{(n+1)}\vec{r} = d(n)\vec{r} \, dr_{n+1} = \rho^{n-1} \, dr \, d(n)\Omega \), but

\[
dr_{n+1} \, d\rho = \frac{\partial(r_{n+1}, \rho)}{\partial(r, \Theta_n)} \, dr \, d\Theta_n = \begin{vmatrix} 
  \cos \Theta_n & \sin \Theta_n \\
  -r \sin \Theta_n & r \cos \Theta_n 
\end{vmatrix} \, dr \, d\Theta_n = r \, dr \, d\Theta_n,
\]

therefore

\[
d^{(n+1)}\vec{r} = (r \sin \Theta_n)^{n-1} \, rdr \, d\Theta_n \, d^n\Omega = r^n \, dr \sin^{n-1} \Theta_n \, d\Theta_n \, d(n)\Omega = r^n \, dr \, d^{(n+1)}\Omega.
\]

At a stereographic projection

\[
\sin \Theta_n = \frac{2p_0p}{p_0^2 + p^2} \quad \text{and} \quad \cos \Theta_n = \frac{p_0^2 - p^2}{p_0^2 + p^2},
\]

therefore

\[
d^{(n+1)}\Omega = \sin^{n-1} \Theta_n \, d\Theta_n \, d^n\Omega = -\sin^{n-2} \Theta_n \, d(\cos \Theta_n) \, d(n)\Omega = \]

\[
-\left(\frac{2p_0p}{p_0^2 + p^2}\right)^{n-2} \, \left(\frac{p_0^2 - p^2}{p_0^2 + p^2}\right) \, d(n)\Omega = \]

32
\[
\left( \frac{2^n}{p_0^n} \left( \frac{p_0^2}{p_0^2 + p^2} \right)^n \right)^{n-1} dpd(n)\Omega = \left( \frac{2}{p_0} \right)^n \cos^{2n} \frac{\Theta_n}{2} d^{(n)} \vec{p}.
\]

So, Eq.34 is generalized in such a way

\[
d^{(n)} \vec{p} = \left( \frac{p_0}{2} \right)^n \sec^{2n} \frac{\Theta_n}{2} d^{(n+1)} \Omega .
\]  

(40)

By using Eq.38 and Eq.40 it is possible to rewrite equations (28) and (30) in the Fock representation. In particular, Eq.30 for a hydrogen atom, after an introduction of the new \( \Phi(\Omega) = \sec^4 \frac{\Theta}{2} \Psi(\vec{p}) \) unknown function, takes the form

\[
\Phi(\Omega) = \frac{m e^2}{8 \pi^2 p_0} \int \frac{\Phi(\Omega')}{\sin^2 (\alpha/2)} d\Omega'
\]

(41)

where \( d\Omega \equiv d^{(4)} \Omega = \sin^2 \Theta_3 \sin \Theta_2 d\Theta_1 d\Theta_2 d\Theta_3 \).

As for the Eq.28, it can be rewritten, for the new \( \tilde{\Phi}(p) = \sec^6 \frac{\Theta}{2} \Phi(p) \) function, as

\[
\tilde{\Phi}(\Omega) = \frac{\lambda}{16 \pi^2 p_0} \int \frac{\tilde{\Phi}(\Omega')}{\sin^2 (\alpha/2)} d\Omega'
\]

(42)

here \( p_0 = m \) and \( d\Omega \equiv d^{(5)} \Omega = \sin^3 \Theta_4 \sin^2 \Theta_3 \sin \Theta_2 d\Theta_1 d\Theta_2 d\Theta_3 d\Theta_4 \). Note that, as it turned out, Eq.28 possesses a hidden \( SO(5) \) symmetry, which became explicit in the Fock space.

So, as we see, there is indeed a close analogy with the nonrelativistic hydrogen atom problem.

To solve equations (41) and (42), one needs a generalization of the \( Y_{lm} \) spherical functions for the \( n > 3 \) dimensions, as the example of Eq.35 suggests.

**Spherical functions in a general case**

It is well known that in a 3-dimensional space spherical functions are connected to solutions of the Laplace equation. Namely, \( \mathcal{Y}_{lm}(\vec{r}) = r^l Y_{lm}(\theta, \phi) \) harmonic polynomials (solid harmonics) are homogeneous polynomials of rank \( l \) with regard to the variables \( r_x, r_y, r_z \) and obey the Laplace equation \( \Delta \mathcal{Y}_{lm}(\vec{r}) = 0 \). For the multidimensional generalization we will use just this property of the spherical functions.
Let us define \( n \)-dimensional spherical function \( Y_{l_{n-1},\ldots,l_1}^{(n)}(\Theta_{n-1},\ldots,\Theta_1) \) by requirement that \( Y_{l_{n-1},\ldots,l_1}^{(n)}(\vec{r}) = r^{l_{n-1}} Y_{l_{n-1},\ldots,l_1}^{(n)}(\Theta_{n-1},\ldots,\Theta_1) \) turns up to be a harmonic polynomial, that is obeys the equation

\[
\Delta^{(n)} Y_{l_{n-1},\ldots,l_1}^{(n)}(\vec{r}) = 0 ,
\]

where \( \Delta^{(n)} = \frac{\partial^2}{\partial r^2} + \ldots + \frac{\partial^2}{\partial r_n^2} \) is \( n \)-dimensional Laplacian. For it we have the following decomposition into a radial and angular parts:

\[
\Delta^{(n)} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \delta^{(n)} ,
\]

\( \delta^{(n)} \) being the angular part of the Laplacian.

Let us prove Eq.44 by induction. For \( n = 3 \) it is correct. So let Eq.44 is fulfilled and consider

\[
\Delta^{(n+1)} = \Delta^{(n)} + \frac{\partial^2}{\partial r_{n+1}^2} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \delta^{(n)} + \frac{\partial^2}{\partial r_{n+1}^2} .
\]

here \( \rho = \sqrt{r^2_1 + \ldots + r^2_n} \). From \( r_{n+1} = r \cos \Theta_n \) and \( \rho = r \sin \Theta_n \) we get

\[
\frac{\partial}{\partial \rho} = \frac{\partial}{\partial r} + \frac{\partial}{\partial \Theta_n} \frac{\partial}{\partial \rho} = \sin \Theta_n \frac{\partial}{\partial r} + \frac{1}{r} \cos \Theta_n \frac{\partial}{\partial \Theta_n} ,
\]

\[
\frac{\partial}{\partial r_{n+1}} = \frac{\partial}{\partial r} \frac{\partial}{\partial r_{n+1}} + \frac{\partial}{\partial \Theta_n} \frac{\partial}{\partial r_{n+1}} = \cos \Theta_n \frac{\partial}{\partial r} - \frac{1}{r} \sin \Theta_n \frac{\partial}{\partial \Theta_n} ,
\]

and for the second derivatives

\[
\frac{\partial^2}{\partial \rho^2} = \sin^2 \Theta_n \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \cos^2 \Theta_n \frac{\partial^2}{\partial \Theta_n^2} - \frac{2}{r^2} \cos \Theta_n \sin \Theta_n \frac{\partial}{\partial \Theta_n} + \frac{2}{r} \cos \Theta_n \sin \Theta_n \frac{\partial^2}{\partial r \partial \Theta_n} + \frac{1}{r} \cos^2 \Theta_n \frac{\partial}{\partial r} ,
\]

\[
\frac{\partial^2}{\partial r_{n+1}^2} = \cos^2 \Theta_n \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \sin^2 \Theta_n \frac{\partial^2}{\partial \Theta_n^2} + \frac{2}{r^2} \cos \Theta_n \sin \Theta_n \frac{\partial}{\partial \Theta_n} - \frac{2}{r} \cos \Theta_n \sin \Theta_n \frac{\partial^2}{\partial r \partial \Theta_n} + \frac{1}{r} \sin^2 \Theta_n \frac{\partial}{\partial r} .
\]

Substitution of these expressions into \( \Delta^{n+1} \) gives

\[
\Delta^{(n+1)} = \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left[ \delta^{(n)} - (n-1) \cot \Theta_n \frac{\partial}{\partial \Theta_n} - \frac{\partial^2}{\partial \Theta_n^2} \right] .
\]
So the following recurrent relation holds

\[
\delta^{(n+1)} = \frac{\delta^{(n)}}{\sin^2 \Theta_n} - (n - 1) \cot \Theta_n \frac{\partial}{\partial \Theta_n} - \frac{\partial^2}{\partial \Theta_n^2}.
\]  (45)

From Eq.43 and the decomposition (44) it follows that a \( n \)-dimensional spherical function obeys an eigenvalue equation

\[
\delta^{(n)} Y_{l_{n-1}, \ldots, l_1}^{(n)}(\Theta_{n-1}, \ldots, \Theta_1) = l_{n-1}(l_{n-1} + n - 2) Y_{l_{n-1}, \ldots, l_1}^{(n)}(\Theta_{n-1}, \ldots, \Theta_1). \]  (46)

Eq.45 suggests that the spherical function has a factorized appearance

\[
Y_{l_{n-1}, \ldots, l_1}^{(n+1)}(\Theta_n, \ldots, \Theta_1) = f(\Theta_n) Y_{l_{n-1}, \ldots, l_1}^{(n)}(\Theta_{n-1}, \ldots, \Theta_1),
\]

Insertion of this into Eq.46 gives the following equation for \( f \):

\[
\left[ \frac{l_{n-1}(l_{n-1} + n - 2)}{\sin^2 \Theta_n} - (n - 1) \cot \Theta_n \frac{\partial}{\partial \Theta_n} - \frac{\partial^2}{\partial \Theta_n^2} \right] f(\Theta_n) = l_n(l_n + n - 1) f(\Theta_n).
\]

Since

\[
\cot \Theta_n \frac{\partial}{\partial \Theta_n} = -\cos \Theta_n \frac{\partial}{\partial \cos \Theta_n} \quad \text{and} \quad \frac{\partial^2}{\partial \Theta_n^2} = \frac{\partial}{\partial \Theta_n}(-\sin \Theta_n) \frac{\partial}{\partial \cos \Theta_n} =
\]

\[
-\cos \Theta_n \frac{\partial}{\partial \cos \Theta_n} - \sin \Theta_n \frac{\partial}{\partial \Theta_n} \frac{\partial}{\partial \cos \Theta_n} =
\]

\[
-\cos \Theta_n \frac{\partial}{\partial \cos \Theta_n} + \sin^2 \Theta_n \frac{\partial}{\partial (\cos \Theta_n)^2}
\]

this equation can be rewritten as

\[
(1 - x^2) \frac{d^2 f}{dx^2} - n x \frac{df}{dx} + \left[ l_n(l_n + n - 1) - \frac{l_{n-1}(l_{n-1} + n - 2)}{1 - x^2} \right] f = 0,
\]

where \( x = \cos \Theta_n \). Let us take \( f(x) = (1 - x^2)^{l_{n-1}/2} g(x) \), when the equation for \( g \) will be

\[
(1 - x^2) \frac{d^2 g}{dx^2} - [n + 2l_{n-1}] x \frac{dg}{dx} + (l_n - l_{n-1})(l_n + l_{n-1} + n - 1) g = 0.
\]

It should be compared with equation defining the Gegenbauer polynomials:

\[
(1 - x^2) \frac{d^2}{dx^2} C_N^{(\alpha)}(x) - (2\alpha + 1) x \frac{d}{dx} C_N^{(\alpha)}(x) + N(N + 2\alpha) C_N^{(\alpha)}(x) = 0.
\]
As we see, \( g(x) \) is proportional to \( C_{l_n-1}^{(l_{n-1}+l_n-1/2)}(x) \). Therefore a multidimensional spherical function can be defined through a recurrent relation

\[
Y_{l_n,l_{n-1}}^{(n+1)}(\Theta_n, \ldots, \Theta_1) = A_{l_n,l_{n-1}} \sin^{l_{n-1}} \Theta_n C_{l_n-1}^{(l_{n-1}+l_n-1/2)}(\cos \Theta_n) Y_{l_{n-1},l_1}^{(n)}(\Theta_{n-1}, \ldots, \Theta_1),
\]

where \( A_{l_n,l_{n-1}} \) constant is determined by the following normalization condition

\[
\int |Y_{l_n,l_{n-1}}^{(n+1)}|^2 \Omega^{(n+1)} = 1,
\]

which, if we take into account

\[
d^{(n+1)} = \sin^{n-1} \Theta_n d\Theta_n d^{(n)}\Omega = -\sin^{n-2} \Theta_n d(\cos \Theta_n) d^{(n)}\Omega
\]

can be rewritten as

\[
A_{l_n,l_{n-1}}^2 \int_{-1}^{1} \left[ C_{l_n-1}^{(l_{n-1}+l_n-1/2)}(x) \right]^2 (1 - x^2)^{(1/2)(n+2l_n-1-2)} dx = 1.
\]

But for the Gegenbauer polynomials we have

\[
\int_{-1}^{1} (1 - x^2)^{\alpha-1/2} [C_N^{(\alpha)}(x)]^2 dx = \frac{\pi 2^{1-2\alpha} \Gamma(N+2\alpha)}{N!(N+\alpha)[\Gamma(\alpha)]^2} ,
\]

therefore:

\[
A_{l_n,l_{n-1}}^2 = \frac{(l_n - l_{n-1})!(2l_n + n - 1)\Gamma(l_{n-1} + (n - 1)/2)]^2}{\pi 2^{3-n-2l_n-1} \Gamma(l_n + l_{n-1} + n - 1)}. \quad (48)
\]

This expression together with Eq.47 completely determines multidimensional spherical function.

Namely, 4-dimensional spherical function looks like

\[
Y_{nlm}(\Theta_3, \Theta_2, \Theta_1) = \frac{2^{2l+1}(n+1)(n-l)!(l)!^2}{\pi(n+l+1)!} (\sin \Theta_3)^l C_n^{(l+1)}(\cos \Theta_3) Y_{lm}(\Theta_2, \Theta_1), \quad (49)
\]

and a 5-dimensional one:

\[
Y_{Nnlm}(\Theta_4, \Theta_3, \Theta_2, \Theta_1) = \sqrt{\frac{2^{2n+1}(2N+3)(N-n)!\Gamma(n+3/2)!^2}{\pi(N+n+2)!}} \times (\sin \Theta_4)^n C_n^{(n+3/2)}(\cos \Theta_4) Y_{nlm}(\Theta_3, \Theta_2, \Theta_1), \quad (50)
\]

It is clear from these equations that \( l \leq n \leq N \). In general, as a lower index of the Gegenbauer polynomial coincides to the polynomial rank and so is not negative, from Eq.47 it follows the following condition on the quantum numbers \( l_n \geq l_{n-1} \).
For spherical functions the following addition theorem holds:

\[
\sum_{l_{n-2}=0}^{l_n-1} \cdots \sum_{l_2=0}^{l_3} \sum_{l_1=-l_2}^{l_2} Y_{l_{n-1} \ldots l_1}(\Theta_{n-1}, \ldots, \Theta_1)Y_{l_{n-1} \ldots l_1}^*(\Theta'_{n-1}, \ldots, \Theta'_1) = \frac{2^{l_{n-1}+n-2}}{4\pi^{n/2}} \Gamma \left(\frac{n-2}{2}\right) C_{l_{n-1}}^{(n-2)/2}(\cos \alpha),
\]

(51)

where \(\alpha\) is the angle between \(n\)-dimensional unit vectors, which are defined by spherical coordinates \((\Theta_1, \ldots, \Theta_{n-1})\) and \((\Theta'_1, \ldots, \Theta'_{n-1})\).

If \(n = 3\), Eq.51 gives the addition theorem for the 3-dimensional spherical functions \(Y_{lm}\), since \(C_{1/2}^{(1/2)}(\cos \alpha) = P_1(\cos \alpha)\). So let us try to prove Eq.51 by induction, that is suppose Eq.51 is correct and consider \((n + 1)\)-dimensional case:

\[
\sum_{l_{n-2} \ldots l_1} Y_{l_{n-1} \ldots l_1}^{(n+1)}(\Theta_n, \ldots, \Theta_1)Y_{l_{n-1} \ldots l_1}^{*(n+1)}(\Theta'_n, \ldots, \Theta'_1) = \sum_{l_{n-1}=0}^{l_n} A_{l_{n-1}}^2 \times \\
\sin^{l_{n-1}} \Theta_n \sin^{l_{n-1}} \Theta'_n C_{l_{n-1}-l_{n-1}}^{(l_{n-1}+n-1)/2}(\cos \Theta_n)C_{l_{n-1}-l_{n-1}}^{(l_{n-1}+n-1)/2}(\cos \Theta'_n) \times \\
\sum_{l_{n-2} \ldots l_1} Y_{l_{n-1} \ldots l_1}^{(n)}(\Theta_n-1, \ldots, \Theta_1)Y_{l_{n-1} \ldots l_1}^{*(n)}(\Theta'_n-1, \ldots, \Theta'_1) = \\
\frac{2^{n-3}(2l_n+n-1)\Gamma((n-2)/2)}{4\pi^{(n/2)+1}} \times \\
\sum_{l_{n-1}=0}^{l_n} 2^{l_{n-1}} \frac{(l_n-l_{n-1})!}{\Gamma(l_n+l_{n-1}+n-1)}(2l_n+n-2) \times \\
[\Gamma(l_{n-1}+(n-1)/2)]^2 \sin^{l_{n-1}} \Theta_n C_{l_{n-1}-l_{n-1}}^{(l_{n-1}+n-1)/2}(\cos \Theta_n) \times \\
\sin^{l_{n-1}} \Theta'_n C_{l_{n-1}-l_{n-1}}^{(l_{n-1}+n-1)/2}(\cos \Theta'_n)C_{l_{n-1}}^{(n-2)/2}(\cos \omega).
\]

\(\omega\) is the angle between \(n\)-dimensional unit vectors and it is connected with the angle \(\alpha\) between \((n + 1)\)-dimensional unit vectors as \(\cos \alpha = \cos \Theta_n \cos \Theta'_n + \sin \Theta_n \sin \Theta'_n \cos \omega\) (if \(\vec{e}\) is \((n + 1)\)-dimensional unit vector, determined by \((\Theta_1, \ldots, \Theta_n)\) angles and \(\vec{f} - n\)-dimensional unit vector, determined by \((\Theta_1, \ldots, \Theta_{n-1})\) angles, when \(\cos \alpha = \vec{e} \cdot \vec{e}'\), \(\cos \omega = \vec{f} \cdot \vec{f}'\), and \(\vec{e}' = \cos \Theta_n \vec{e}'' + \sin \Theta_n \vec{f}\)).

Using the addition theorem for the Gegenbauer polynomials

\[
C_n^{(\alpha)}(\cos \Theta \cos \Theta' + \sin \Theta \sin \Theta' \cos \phi) = \\
\sum_{m=0}^{n} 2^{2m}(2\alpha + 2m - 1) \frac{(n-m)!\Gamma(\alpha+m)\Gamma(2\alpha-1)}{\Gamma(\alpha)^2\Gamma(2\alpha+n+m)} \sin^m \Theta C_{n-m}^{(\alpha+m)}(\cos \Theta) \times \\
\sin^m \Theta' C_{n-m}^{(\alpha+m)}(\cos \Theta')C_{m}^{(\alpha-1)/2}(\cos \phi),
\]

37
we get:

\[
\sum_{l_n-1,\ldots,l_1} Y_{l_n\ldots l_1}^{(n+1)}(\Theta_n, \ldots, \Theta_1)Y_{l_n\ldots l_1}^{(n+1)*}(\Theta'_n, \ldots, \Theta'_1) =
\]

\[
2^{n-3} \frac{2l_n + n - 1}{4\pi^{(n-1)/2}} \frac{\Gamma((n-1)/2)\Gamma((n-2)/2)}{\Gamma(n-2)} C_{l_n}^{(n-1)/2}(\cos \alpha),
\]

but \(\Gamma(z)\Gamma(z + 1/2) = (\pi^{1/2}/2^{2z-1})\Gamma(2z)\) formula gives \(\Gamma((n-1)/2)\Gamma((n-2)/2) = (\pi^{1/2}/2^{n-3})\Gamma(n-2)\) and we end with the desired result:

\[
\sum_{l_n-1,\ldots,l_1} Y_{l_n\ldots l_1}^{(n+1)}(\Theta_n, \ldots, \Theta_1)Y_{l_n\ldots l_1}^{(n+1)*}(\Theta'_n, \ldots, \Theta'_1) =
\]

\[
2l_n + n - 1 \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}} C_{l_n}^{(n-1)/2}(\cos \alpha).
\]

In a 4-dimensional space the addition theorem for the spherical functions looks like

\[
\sum_{l=0}^{n} \sum_{m=-l}^{l} Y_{nlm}(\Theta_3, \Theta_2, \Theta_1)Y_{nlm}^*(\Theta'_3, \Theta'_2, \Theta'_1) =
\]

\[
\frac{n + 1}{2\pi^2} C_n^{(1)}(\cos \alpha) = \frac{n + 1}{2\pi^2} \frac{\sin (n + 1)\alpha}{\sin \alpha},
\]

Using this, we can prove, that a solution of Eq.41 is proportional to a spherical function \(Y_{nlm}(\Omega)\). Indeed, from the generating function of the Gegenbauer polynomials \((1 - 2\epsilon \cos \alpha + \epsilon^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)}(\cos \alpha)\epsilon^n\), we get in the \(\epsilon = \nu = 1\) case \(1/4 \sin^2 (\alpha/2) = \sum_{n=0}^{\infty} C_n^{(1)}(\cos \alpha)\) Note that it is not rigorously correct to take \(\epsilon = 1\) because in this case the series stops to be convergent. But

\[
\sum_{n=0}^{\infty} C_n^{(1)}(\cos \alpha) = \sum_{n=0}^{\infty} \frac{\sin (n + 1)\alpha}{\sin \alpha} = \lim_{N\to\infty} \frac{\sin ((N + 1)\alpha/2)\sin (N\alpha/2)}{\sin \alpha \sin (\alpha/2)}.
\]

of course a limit doesn’t exist, but

\[
\frac{\sin N+1\alpha \sin N\alpha}{N \sin \alpha \sin (\alpha/2)} = \cos \frac{\alpha}{2} - \cos (N + \frac{1}{2})\alpha = \frac{1}{4 \sin^2 (\alpha/2)} - \frac{\cos (N + \frac{1}{2})\alpha}{2 \sin \alpha \sin (\alpha/2)}
\]

and the second term oscillates more and more quickly as \(N\) increases. So its multiplication on any normal function and integration will give a result which tends to zero as \(N \to \infty\). Therefore \(1/4 \sin^2 (\alpha/2) = \sum_{n=0}^{\infty} C_n^{(1)}(\cos \alpha)\) relation can be used under integration without any loss of rigor. Now \(C_n^{(1)}(\cos \alpha)\) can be replaced by using the addition theorem and we get because of the orthonormality property of the spherical functions:

\[
\int \frac{Y_{nlm}(\Omega')}{\sin^2 \alpha/2} d\Omega' = \frac{8\pi^2}{n + 1} Y_{nlm}(\Omega).
\]
So a 4-dimensional spherical function is indeed a solution of the Eq. 41 and the corresponding eigenvalue is determined from the $p_0 = me^2/(n + 1)$ condition ($m$ being electron mass), which gives a well known expression for the hydrogen atom levels $E = -me^4/2(n + 1)^2$, $n = 0, 1, \ldots$.

Now it should be no surprise that a solution of the Eq. 42 is proportional to a 5-dimensional spherical function. To prove this, as the previous experience suggests, it is necessary to decompose $(1 - \cos \alpha)^{-1}$ into a series of Gegenbauer polynomials and afterwards, by using the addition theorem (51), replace them by the spherical functions.

Suppose $1/(1 - x) = \sum_{N=0}^{\infty} A_N C_N^{(3/2)}(x)$, when because of the orthogonality of the Gegenbauer polynomials and of the normalization

$$\int_{-1}^{1} (1 - x^2)[C_N^{(3/2)}(x)]^2 dx = \frac{2(N + 1)(N + 2)}{2N + 3}$$

we get

$$A_N = \frac{2N + 3}{2(N + 1)(N + 2)} \int_{-1}^{1} (1 + x)C_N^{(3/2)}(x)dx .$$

The integral can be evaluated by using relations

$$C_N^{(3/2)}(x) = \frac{d}{dx}P_{N+1}(x), \quad P_n(1) = 1, \quad \text{and} \quad \int_{-1}^{1} P_n(x)dx = 0 \text{ if } n \neq 0.$$

So

$$A_N = \frac{2N + 3}{(N + 1)(N + 2)}, \quad \frac{1}{1 - \cos \alpha} = \sum_{N=0}^{\infty} \frac{2N + 3}{(N + 1)(N + 2)} C_N^{(3/2)}(\cos \alpha) =$$

$$\sum_{Nnlm} \frac{8\pi^2}{(N + 1)(N + 2)} Y_{Nnlm}(\Omega)Y_{Nnlm}^*(\Omega'),$$

Therefore

$$\int \frac{Y_{Nnlm}(\Omega')}{\sin^2 \alpha/2} d\Omega' = \frac{16\pi^2}{(N + 1)(N + 2)} Y_{Nnlm}(\Omega) .$$

and eigenvalues of the Eq.42 will be $\lambda_N = (N + 1)(N + 2)m^2$ (here $m$ stands for the particle mass, not an index of the spherical function), with the corresponding solution

$$\Phi_N(\Omega) = \sum_{nlm} A_{nlm} Y_{Nnlm}(\Omega) .$$
Stereographic projection in the Wick-Cutkosky model

Let us return to Eq. 27 and consider again the equal mass case

\[
\begin{bmatrix}
 m^2 + (p + \frac{i}{2} P)^2 \\
 m^2 + (p - \frac{i}{2} P)^2
\end{bmatrix} \Phi(p) = \frac{\lambda}{\pi^2} \int dq \frac{\Phi(q)}{(p-q)^2},
\]

(52)

here \( P = (\vec{0}, M) \) - Euclidean 4-vector and \( M \) is the bound state mass.

Equation (52) exhibits explicitly only \( SO(3) \) symmetry (because it contains a fixed 4-vector \( P \)), but now we know that a higher symmetry can be hidden. To find out this, let us transform the 4-dimensional momentum space onto the 5-dimensional sphere surface by the stereographic projection. R.h.s. of the equation will be changed at that according to formulas (38) and (40)

\[
\frac{\lambda}{\pi^2} \int dq \frac{\Phi(q)}{(p-q)^2} = \frac{\lambda p_0^2}{16\pi^2} \sec^{-2} \frac{\Theta_4}{2} \int \frac{\sec^6 (\Theta'_4/2) \Phi(q)}{\sin^2 \alpha/2} d\Omega'.
\]

As for the l.h.s., we will have

\[
\begin{bmatrix}
 m^2 + (p + \frac{i}{2} P)^2 \\
 m^2 + (p - \frac{i}{2} P)^2
\end{bmatrix} = m^4 - \frac{1}{2} m^2 M^2 + \frac{1}{16} M^4 + 2 m^2 p^2 - \frac{1}{2} M^2 p^2 + p^4 + (p \cdot P)^2 =
\]

\[
\left( m^2 - \frac{1}{4} M^2 \right)^2 + 2 p_0^2 \tan^2 \frac{\Theta_4}{2} \left( m^2 - \frac{1}{4} M^2 \right) +
\]

\[
p_0^2 \tan^2 \frac{\Theta_4}{2} \left( p_0^2 \tan^2 \frac{\Theta_4}{2} + M^2 \cos^2 \Theta_3 \right).
\]

It is convenient to choose \( p_0 \) - 5-sphere radius as: \( p_0^2 = m^2 - (1/4) M^2 \), when

\[
\begin{bmatrix}
 m^2 + (p + \frac{i}{2} P)^2 \\
 m^2 + (p - \frac{i}{2} P)^2
\end{bmatrix} = p_0^2 \left[ p_0^2 \sec^4 \frac{\Theta_4}{2} +
\]

\[
M^2 \tan^2 \frac{\Theta_4}{2} \cos^2 \Theta_3 \right] = p_0^2 \sec^4 \frac{\Theta_4}{2} \left[ p_0^2 + \frac{M^2}{4} \sin^2 \Theta_4 \cos^2 \Theta_3 \right].
\]

and after introduction of the new \( F(\Omega) \equiv F(\Theta_1, \Theta_2, \Theta_3, \Theta_4) = \sec^6 \frac{\Theta_4}{2} \Phi(p) \) function, Eq. 52 takes the form

\[
\left[ p_0^2 + \frac{M^2}{4} \sin^2 \Theta_4 \cos^2 \Theta_3 \right] F(\Omega) = \frac{\lambda}{16\pi^2} \int \frac{F(\Omega')}{\sin^2 \alpha/2} d\Omega'.
\]

(53)
Note that \( \sin \Theta_4 \cos \Theta_3 = \tilde{P}_4/p_0 \) (\( \tilde{P} \) is that point, which corresponds to the \( p \) 4-vector in the stereographic projection), therefore Eq.53 is invariant under such rotations of the 5-dimensional coordinate system, which don’t change the 4-th axis. So Eq.52 possesses a hidden \( SO(4) \) symmetry. This symmetry was not explicit because \( p \)-momentum space is not orthogonal to the 4-th axis \( \vec{e}_4 \). But with the inverse stereographic projection, if we choose its pole on the axis \( \vec{e}_4 \), we get \( k \)-momentum space, which is orthogonal to \( \vec{e}_4 \). This is schematically shown in Fig.10:

![Diagram](image)

Figure 10: \( k \)-momentum space.

Therefore Eq.52, rewritten in terms of the \( k \)-variables, should become explicitly \( SO(4) \) invariant. Let us get this new equation. According to the last equality in (37)

\[
\sin \Theta_4 \cos \Theta_3 = \cos \gamma = \frac{\tilde{P}_4}{p_0} = \frac{p_0^2 - k^2}{p_0^2 + k^2}.
\]
Furthermore, because of Eq. 38

\[
\frac{1}{\sin^2 \alpha/2} = \frac{p_0^2}{(k - k')^2} \sec^2 \frac{\gamma}{2} \sec^2 \frac{\gamma'}{2},
\]

and according to Eq. 40

\[
d\Omega' = \frac{16}{p_0^4} \sec^{-8} \left( \frac{\gamma}{2} \right) dk'.
\]

The substitution all of these into the Eq. 53, after introduction of the new \( Q(k) = \cos \left( \frac{\gamma}{2} \right) F(\Omega) \) unknown function, will give

\[
\left[ p_0^2 + \frac{M^2}{4} \frac{(p_0^2 - k^2)^2}{(p_0^2 + k^2)^2} \right] \sec^4 \frac{\gamma}{2} Q(k) = \frac{\lambda}{\pi^2 p_0^2} \int \frac{Q(k')}{(k - k')^2} dk'.
\]

But

\[
\sec^4 \frac{\gamma}{2} = \left( 1 + \tan^2 \frac{\gamma}{2} \right)^2 = \left( 1 + \frac{k^2}{p_0^2} \right)^2 = \frac{1}{p_0^2} (p_0^2 + k^2)^2.
\]

and we finally get

\[
\left[ (p_0^2 + k^2)^2 + \frac{M^2}{4p_0^2} (p_0^2 - k^2)^2 \right] Q(k) = \frac{\lambda}{\pi^2} \int \frac{Q(k')}{(k - k')^2} dk'.
\] (54)

Since the equation just obtained is \( SO(4) \) invariant, its solution should have the form \( Q(k) = f(k^2) Y_{nlm}(\Omega) \), which enables to perform the angular integration in Eq. 54 and, as a result, to get a one-dimensional integral equation for \( f(k^2) \). Let us denote \( x = \sqrt{k^2} \) and let \( \alpha \) be the angle between \( k \) and \( k' \) 4-vectors, then

\[
\frac{1}{(k - k')^2} =
\left\{
\begin{array}{ll}
\frac{1}{x^2} \frac{1}{1 - 2 \frac{x^2}{x^2} \cos \alpha + \frac{x^2}{x^2}} = \frac{1}{x^2} \sum_{n' = 0}^{\infty} \left( \frac{x'}{x} \right)^{n'} C_{n'}^{(1)}(\cos \alpha), & \text{if } x > x' \\
\frac{1}{x^2} \frac{1}{1 - 2 \frac{x^2}{x^2} \cos \alpha + \frac{x^2}{x^2}} = \frac{1}{x^2} \sum_{n' = 0}^{\infty} \left( \frac{x}{x'} \right)^{n'} C_{n'}^{(1)}(\cos \alpha), & \text{if } x < x'
\end{array}
\right.
\]

\[
\sum_{n' = 0}^{\infty} C_{n'}^{(1)}(\cos \alpha) \left[ \frac{(x')^{n'}}{x^{n'+2}} \Theta(x - x') + \frac{x^{n'}}{(x')^{n'+2}} \Theta(x' - x') \right].
\]

According to the addition theorem

\[
C_{n'}^{(1)}(\cos \alpha) = \frac{2\pi^2}{n' + 1} \sum_{l'm'} Y_{nlm}(\Omega) Y_{n'l'm'}^{*}(\Omega'),
\]
therefore
\[
\frac{1}{(k - k')^2} = \sum_{n' l'm'} \frac{2\pi^2}{n' + 1} \left[ \frac{(x')^{n'}}{x'^{n'+2}} \Theta(x - x') + \frac{x'^{n'}}{(x')^{n'+2}} \Theta(x' - x) \right] Y_{n' l'm'}(\Omega) Y_{n' l'm'}^*(\Omega').
\]

The substitution of this and \( Q(k) = f(x) Y_{nlm}(\Omega) \) into Eq.54 will give (because of orthonormality of the spherical functions and \( dk' = |k'|^3 d|k'| d(4)\Omega' \))
\[
\left[ (p_0^2 + x^2)^2 + \frac{M^2}{4p_0^2} (p_0^2 - x^2)^2 \right] f(x) =
\frac{2\lambda}{n + 1} \int_0^\infty \frac{y^{n+3}}{x^{n+2}} \Theta(x - y) + \frac{x^n}{y^{n-1}} \Theta(y - x) \right] f(y) \, dy.
\]

Since
\[
(p_0^2 + x^2)^2 + \frac{M^2}{4p_0^2} (p_0^2 - x^2)^2 = (p_0^2 + x^2)^2 \left( 1 + \frac{M^2}{4p_0^2} \right) - 4\frac{M^2}{4p_0^2} p_0^2 x^2 =
\frac{(p_0^2 + x^2)^2 m^2}{p_0^2} - M^2 x^2 = \frac{m^2}{p_0^2} \left( p_0^2 + x^2 + \frac{p_0 M}{m} x \right) \left( p_0^2 + x^2 - \frac{p_0 M}{m} x \right)
\]
then this integral equation can be rewritten also as
\[
\left( p_0^2 + x^2 + \frac{p_0 M}{m} x \right) \left( p_0^2 + x^2 - \frac{p_0 M}{m} x \right) f(x) =
\frac{2\lambda p_0^2}{m^2(n + 1)} \int_0^\infty \frac{y^{n+3}}{x^{n+2}} \Theta(x - y) + \frac{x^n}{y^{n-1}} \Theta(y - x) \right] f(y) \, dy.
\]

or introducing \( s = x/p_0 \) undimensional variable:
\[
(1 + s^2 + \frac{M}{m} s) (1 + s^2 - \frac{M}{m} s) \ f(s) =
\frac{2\lambda}{m^2(n + 1)} \int_0^\infty \frac{r^{n+3}}{s^{n+2}} \Theta(s - r) + \frac{s^n}{r^{n-1}} \Theta(r - s) \right] f(r) \, dr.
\]

It is clear from this equation that for the \( f_n(s) \equiv f(s) \) function, if \( n \neq 0 \), we have \( f_n(0) = 0 \) boundary condition at the point \( s = 0 \).

Let us show that Eq.55 is equivalent to a second order differential equation. Let
\[
R(s, r) = \left[ \frac{r^{n+3}}{s^{n+2}} \Theta(s - r) + \frac{s^n}{r^{n-1}} \Theta(r - s) \right].
\]
Since \((d/ds)\Theta(s - r) = \delta(s - r)\), for the derivatives over \(s\) we will have:

\[
\frac{d}{ds} R(s, r) = -(n + 2) \frac{r^{n+3}}{s^n} \Theta(s - r) + n \frac{s^{n-1}}{r^{n-1}} \Theta(r - s),
\]

\[
\frac{d^2}{ds^2} R(s, r) = (n + 2)(n + 3) \frac{r^{n+3}}{s^{n+4}} \Theta(s - r) + n(n - 1) \frac{s^{n-2}}{r^{n-1}} \Theta(r - s) - 2(n + 1) \delta(s - r).
\]

Let us try to choose \(A\) and \(B\) coefficients such that

\[
\left[ \frac{d^2}{ds^2} + \frac{A}{s} \frac{d}{ds} + \frac{B}{s^2} \right] R(s, r) = -2(n + 1) \delta(s - r)
\]

which means the following system

\[(n + 2)A - B = (n + 2)(n + 3), \quad nA + B = -n(n - 1)\]

with \(A = 3\) and \(B = -n(n + 2)\) as the solution. Therefore

\[
\left[ \frac{d^2}{ds^2} + 3 \frac{d}{ds} - \frac{n(n + 2)}{s^2} \right] \left[ (1 + s^2)^2 - \frac{M^2}{m^2} s^2 \right] f(s) =
\]

\[-\frac{4\lambda}{m^2} \int_0^\infty \delta(s - r) f(r) dr = -\frac{4\lambda}{m^2} f(s).\]

Let us introduce the new variable \(t = s^2\) and the new unknown function \(\phi(t) = t[(1 + t^2) - (M^2/m^2)t]f(t)\). Because

\[
\frac{1}{s} \frac{d}{ds} = 2 \frac{d}{dt} \quad \text{and} \quad \frac{d^2}{ds^2} = \frac{d}{ds} 2s \frac{d}{ds} = 2 \frac{d}{dt} + 2s \frac{d}{ds} \frac{d}{dt} = 2 \frac{d}{dt} + 2t \frac{d^2}{dt^2}.
\]

We finally get

\[
\frac{d^2}{dt^2} \phi_n(t) + \left[ \frac{\lambda}{t[m^2(1 + t^2) - M^2t]} - \frac{n(n + 2)}{4t^2} \right] \phi_n(t) = 0. \quad (56)
\]

The asymptotic form of this equation (when \(n \neq 0\)) in the \(t \to 0\) or \(t \to \infty\) limit is

\[
\frac{d^2}{dt^2} \phi_n(t) - \frac{n(n + 2)}{4t^2} \phi_n = 0,
\]

with solutions \(t^{1+n/2}\) and \(t^{-n/2}\). At the origin \(\phi_n \sim t^{-n/2}\) behavior is not adequate, because then \(f_n \sim t^{-(n/2)-1}\), which contradicts to the \(f_n(0) = 0\) condition. Let us show that at the infinity, on the contrary, \(\phi_n \sim t^{1+n/2}\).
behavior is not good, because then $f_n \sim t^{(n/2) - 2} = s^{n-4}$, which contradicts to the integral equation (55). Indeed, when $s \to \infty$, left hand size of the equation will be of the order of $s^n$, and the r.h.s. of the order of

$$\frac{1}{s^{n+2}} \int_0^s r^{n+3} f_n(r) \, dr + s^n \int_s^\infty \frac{r^{n-4}}{r-1} \, dr = \frac{1}{2} s^{n-2} + \frac{1}{s^{n+2}} \int_0^s r^{n+3} f_n(r) \, dr .$$

Let $s_0$ be some big enough number, so that when $r > s_0$ we can use the asymptotic form of the solution. Then

$$\int_0^s r^{n+3} f_n(r) \, dr \approx \int_0^{s_0} r^{n+3} f_n(r) \, dr + \int_{s_0}^s r^{2n-1} \, dr =$$

$$\frac{1}{2n} (s^{2n} - s_0^{2n}) + \int_0^{s_0} r^{n+3} f_n(r) \, dr .$$

Therefore

$$\frac{1}{s^{n+2}} \int_0^s r^{n+3} f_n(r) \, dr \sim s^{n-2} .$$

So the r.h.s. of the Eq.55 turns out to be of the order of $s^{n-2}$ and it can’t be equal to the l.h.s.

As we see, Eq.56 should be accompanied by the following boundary conditions

$$\phi_n(t) \sim t^{1+n/2} , \text{ if } t \to 0 ,$$

$$\phi_n(t) \sim t^{-n/2} , \text{ if } t \to \infty . \quad (57)$$

In Eq.56 $t$-variable changes from 0 to $\infty$. For numeric calculations it is more convenient to have a finite interval. So let us introduce the new $z = (1-t)/(1+t)$ variable, which changes from $-1$ to 1. Using

$$\frac{d}{dt} = -\frac{2}{(1+t)^2} \frac{d}{dz} , \quad \frac{d^2}{dt^2} = \frac{d}{dt} \left( -\frac{2}{(1+t)^2} \frac{d}{dz} \right) =$$

$$\frac{4}{(1+t)^3} \frac{d}{dz} + \frac{4}{(1+t)^4} \frac{d^2}{dz^2} = \frac{1}{2} (1+z)^3 \frac{d}{dz} + \frac{1}{4} (1+z)^4 \frac{d^2}{dz^2}$$

we get

$$(1-z^2) \frac{d^2 \phi_n}{dz^2} + 2(1-z) \frac{d \phi_n}{dz} + \left[ \frac{\lambda}{m^2 - (M^2/4)(1-z^2)} - \frac{n(n+2)}{1-z^2} \right] \phi_n = 0 ,$$

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which can be somewhat simplified if we introduce the new $g_n(z) = (1+z)(1-z^2)^{n/2}\phi_n(z)$ function, for which the equation looks like

$$(1-z^2)\frac{d^2 g_n}{dz^2} + 2nz\frac{dg_n}{dz} + \left[\frac{\lambda}{m^2 - (M^2/4)(1-z^2)} - n(n+1)\right]g_n = 0 \quad (58)$$

Eq.57 indicates that at $z \to 1$ we should have $g_n \sim (1-z)^{n/2}(1-z)^{1+n/2} = (1-z)^{n+1}$, and at $z \to -1$ $g_n \sim (1+z)^{1+n/2}(1+z)^{n/2} = (1+z)^{n+1}$. Therefore the boundary conditions for the Eq.58 are $g_n(-1) = g_n(1) = 0$.

### Integral representation method

Transition from Eq.52 to an one-dimensional integral equation can be performed also in another way, which is not directly connected with the hidden symmetry of the equation. This time let us consider a general case of unequal masses and let $m_1 = m + \Delta, \ m_2 = m - \Delta$. Eq.27 takes the form

$$\Phi(p) = \frac{\lambda}{\pi^2} \left[\left((m+\Delta)^2 + (p-i\eta_1 P)^2\right)\left((m-\Delta)^2 + (p+i\eta_2 P)^2\right)\right]^{-1} \int dq \frac{\Phi(q)}{(p-q)^2} \quad (59)$$

Note, that the solution of the equation

$$\Phi(p) = \frac{\lambda}{\pi^2} (p^2 + m^2)^{-2} \int dq \frac{\Phi(q)}{(p-q)^2},$$

is, as we have seen earlier

$$\Phi(p) \sim \cos^6 \frac{\Theta_4}{2} Y_{Nnlm}(\Omega) \sim (1 + \cos \Theta_4)^3 \sin^n \Theta_4 \sin^l \Theta_3 Y_{lm}(\Theta_2, \Theta_1) \sim$$

$$(p^2 + m^2)^{-n-3}|\vec{p}| Y_{lm}(\Theta_2, \Theta_1) = \frac{\mathcal{Y}_{lm}(\vec{p})}{(p^2 + m^2)^{n+3}},$$

because

$$\cos \Theta_4 = \frac{m^2 - p^2}{m^2 + p^2}, \ \sin \Theta_4 = \frac{2m|p|}{m^2 + p^2} \ \text{and} \ \sin \Theta_3 = \frac{|\vec{p}|}{|p|}.$$  

(Here $|p| = \sqrt{p^2}$ is the length of the 4-vector $p = (\vec{p}, p_4)$.

Using the following parametric representation

$$\frac{1}{AB} = \frac{1}{2} \int_{-1}^{1} \frac{dz}{[(1/2)(A+B) + (1/2)(A-B)z]^2},$$

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we will have
\[
[(m - \Delta)^2 + (p + i\eta P)^2]^{-1}[(m + \Delta)^2 + (p - i\eta P)^2]^{-1} = \\
\frac{1}{2} \int_{-1}^{1} dz \left[ (1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})p \cdot P \right]^{-2} .
\]
(60)
Here $p_0^2 = m^2 - (1/4)M^2$, as earlier.

All of this gives a hint to check, how changes, when inserted in the r.h.s. of the Eq.59, the following function

$$
\Phi_{nlm}(p, z) = Y_{lm}(\vec{p}) \left[ (1 + \Delta^2 - 2\Delta z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})q \cdot P \right]^{-1} \times
$$

$$
\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} .
$$

First of all let us evaluate $\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2}$. According to Eq.29

$$
\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \int dq Y_{lm}(\vec{q}) \left[ (1 + \Delta^2 - 2\Delta z)p_0^2 + q^2 + i(z - \frac{\Delta}{m})q \cdot P \right]^{-1} \times
$$

$$
\left[ u\left( (1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z)p_0^2 + q^2 + i(z - \frac{\Delta}{m})q \cdot P \right) + (1 - u)(p - q)^2 \right]^{-1} .
$$

which can be transformed further as

$$
u \left[ (1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z)p_0^2 + q^2 + i(z - \frac{\Delta}{m})q \cdot P \right] + (1 - u)(p - q)^2 =
$$

$$
\left[ q + \frac{i}{2}u(z - \frac{\Delta}{m})P - (1 - u)p \right]^2 + u(1 - u)p^2 + \frac{1}{4}u^2(z - \frac{\Delta}{m})^2P^2 +
$$

$$
wp_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + iu(1 - u)(z - \frac{\Delta}{m})p \cdot P ,
$$

therefore let us introduce the new $k = q + \frac{i}{2}u(z - \frac{\Delta}{m})P - (1 - u)p$ integration variable:

$$
\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = (n + 3) \int \frac{1}{0} u^{n+2} du \int dk \ Y_{lm}(\vec{k} + (1 - u)\vec{p})[k^2 + u(1 - u)p^2 +
$$

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\[ \frac{1}{4}u^2(z - \frac{\Delta}{m})^2P^2 + up_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + iu(1 - u)(z - \frac{\Delta}{m})p \cdot P \]^{- (n+4)}.

For the solid harmonics the following equation holds

\[ Y_{lm}(\vec{a} + \vec{b}) = \sum_{k=0}^{l} \sum_{\mu=-k}^{k} Y_{l-k,m-\mu}(\vec{a})Y_{k\mu}(\vec{b}) \times \]

\[ \left[ \frac{4\pi(2l+1)(l+m)!(l-m)!}{(2k+1)(2l-2k+1)(k+\mu)!(k-\mu)!(l+m-k-\mu)!(l-m-k+\mu)!} \right]^{\frac{1}{2}}. \]

Let us decompose \( Y_{lm}((1-u)\vec{p} + \vec{k}) \) according to this relation and take into account that if \( l \neq 0 \), then \( \int Y_{lm}(\vec{k})d^{(3)}\Omega = 0 \). Therefore only the first term

\[ \sqrt{4\pi} Y_{lm}[(1-u)\vec{p}]Y_{00}(\vec{k}) = Y_{lm}[(1-u)\vec{p}] = (1-u)^l Y_{lm}(\vec{p}) \]

of this decomposition will give a nonzero contribution. Using also \( dk = \frac{1}{2}tdtd^{(4)}\Omega \), where \( t = k^2 \), and \( \int d^{(4)}\Omega = 2\pi^2 \), we will get

\[ \int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \pi^2(n + 3) Y_{lm}(\vec{p}) \int_0^1 u^{n+2}(1-u)^l du \times \]

\[ \int_0^\infty dt \left[ t + u(1-u)p^2 + \frac{1}{4}u^2(z - \frac{\Delta}{m})^2P^2 + \right. \]

\[ \left. up_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + iu(1 - u)(z - \frac{\Delta}{m})p \cdot P \right]^{- (n+4)}. \]

Over \( t \) the integral is of the type

\[ \int_0^\infty \frac{t dt}{[t + a]^{n+4}} = \int_0^\infty \frac{dt}{[t + a]^{n+3}} - a \int_0^\infty \frac{dt}{[t + a]^{n+4}} = \frac{1}{(n + 2)(n + 3)}a^{-(n+2)}. \]

That is

\[ \int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \frac{\pi^2}{n + 2} Y_{lm}(\vec{p}) \int_0^1 (1-u)^l \left[ (1-u)p^2 + \frac{1}{4}u(z - \frac{\Delta}{m})^2P^2 + \right. \]

\[ \left. p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + i(1 - u)(z - \frac{\Delta}{m})p \cdot P \right]^{- (n+2)} du . \]

or, after introduction of the new \( t = 1 - u \) integration variable:

\[ \int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \frac{\pi^2}{n + 2} Y_{lm}(\vec{p}) \int_0^1 t \left[ t^2 + i(z - \frac{\Delta}{m})p \cdot P - \right. \]

\[ \left. \cdots \right]. \]
\[
\frac{1}{4}(z - \frac{\Delta}{m})^2 p^2 \bigg) + p_0^2 \big(1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z \big) + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2 \bigg]^{-(n+2)} dt.
\]

Let us denote
\[
a = (1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})p \cdot P
\]
and
\[
b = (1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z)p_0^2 + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2.
\]
We have
\[
\int_0^1 \frac{t^l dt}{[t(a - b) + b]^{n+2}} = (-1)^l \frac{(n + 1 - l)!}{(n + 1)!} \frac{\partial^l}{\partial a^l} \int_0^1 \frac{dt}{[t(a - b) + b]^{n-l+2}} = \]
\[
(-1)^l \frac{(n + 1 - l)!}{(n + 1)!} \frac{\partial^l}{\partial a^l} \frac{1}{a - b} \left[ \frac{1}{b^{n-l+1}} - \frac{1}{a^{n-l+1}} \right] = \]
\[
(-1)^l \frac{(n - l)!}{(n + 1)!} \frac{\partial^l}{\partial a^l} \left( a^{n-l} + a^{n-l-1}b + a^{n-l-2}b^2 + \ldots + b^{n-l} \right) = \]
\[
(-1)^l \frac{(n - l)!}{(n + 1)!} \frac{\partial^l}{\partial a^l} \sum_{k=0}^{n-l} a^{-(n-l+1-k)} b^{-(k+1)} = \]
\[
\frac{(n - l)!}{(n + 1)!} \sum_{k=0}^{n-l} \frac{(n - k)!}{(n - l - k)!} a^{-(n-k+1)} b^{-(k+1)}.
\]
where \( n \geq l \) supposition was done. Thus
\[
\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \pi^2 \frac{(n - l)!}{(n + 2)!} \sum_{k=0}^{n-l} \frac{(n - k)!}{(n - l - k)!} \times \]
\[
\left\{ p_0^2 \big(1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z \big) + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2 \right\}^{-(k+1)} \times \]
\[
\mathcal{Y}_{lm}(\vec{p}) \frac{\mathcal{Y}_{lm}(\vec{p})}{[(1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})p \cdot P]^{n-k+1}},
\]
but
\[
\mathcal{Y}_{lm}(\vec{p}) \left\{ (1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})p \cdot P \right\}^{-(n-k+1)} =
\]
therefore
\[
\Phi_{n-k-2,lm}(p, z) ,
\]

\[
\int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} = \pi^2 \frac{n!}{(n + 2)!} \frac{(n - k)!}{(n - l - k)!} \times
\]
\[
\frac{\Phi_{n-k-2,lm}(p, z)}{[p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + \frac{1}{4}(z - \frac{\Delta}{m})^2P^2]^{k+1}}.
\]

The result is encouraging, because we have got a linear combination of \(\Phi_{nlm}\) functions, and therefore it can be expected that the solution of the Eq.59 is expressible through these functions. But, before this conclusion is done, we should check that nothing wrong happens after multiplication over the
\[
[(m + \Delta)^2 + (p - i\eta_1 P)^2]^{-1}[(m - \Delta)^2 + (p + i\eta_2 P)^2]^{-1}.
\]

According to formulas (60) and (29):
\[
[(m + \Delta)^2 + (p - i\eta_1 P)^2]^{-1}[(m - \Delta)^2 + (p + i\eta_2 P)^2]^{-1} \times
\]
\[
\left[(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z)p_0^2 + p^2 + i(z - \frac{\Delta}{m})p \cdot P\right]^{-(n-k+1)} =
\]
\[
\frac{1}{2}(n - k + 1)(n - k + 2) \int_{-1}^{1} dt \int_{0}^{1} dx \times
\]
\[
x(1 - x)^{n-k} \left[i x(t - z)p \cdot P + (1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z)p_0^2 + p^2 +
\right.
\]
\[
i(z - \frac{\Delta}{m})p \cdot P - 2\frac{\Delta}{m}x(t - z)p_0^2\right]^{-(n-k+3)} =
\]
\[
\frac{1}{2}(n - k + 1)(n - k + 2) \int_{-1}^{1} dt \int_{0}^{1} dx \times
\]
\[
x(1 - x)^{n-k} \left\{[1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}(xt + z - xz)]p_0^2 + p^2 + i(xt + z - xz - \frac{\Delta}{m})p \cdot P\right\}^{n-k+3},
\]

therefore
\[
\frac{\pi^2}{2} \frac{n!}{(n + 2)!} \frac{(n - k)!}{(n - l - k)!} \int_{-1}^{1} dt \times
\]
\[
\frac{\Phi_{nlm}(q, z)}{(p - q)^2} =
\]
\[
\frac{n!}{(n - k + 2)!} \frac{(n - l)!}{(n + 2)!} \frac{(n - k)!}{(n - l - k)!} \int_{-1}^{1} dt \times
\]
\[
\frac{1}{0} x(1-x)^{n-k} \Phi_{n-k,lm}(p, xt + z - xz) dx
\]
\[
\left[ p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2 \right]^{k+1}.
\]

Note, that when \(0 \leq x \leq 1\), then \(\min(z, t) \leq xt + z - xz \leq \max(z, t)\), therefore if the \(z\)-parameter of the \(\Phi_{nlm}(p, z)\) function changes in the \([-1 \leq z \leq 1]\) range, then \(-1 \leq xt + z - xz \leq 1\) and the above given equation can be rewritten as

\[
[(m + \Delta)^2 + (p - i\eta_1 P)^2]^{-1}[(m - \Delta)^2 + (p + i\eta_2 P)^2]^{-1} \int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} =
\]
\[
\frac{\pi^2}{2} \sum_{l=0}^{n-l} (n - l!)^2 l! \int dt \int dx (1-x)^{n-k} \int d\zeta \times
\]
\[
\frac{\delta(\zeta - xt - z + xz)}{\left[ p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2 \right]^{k+1}} \Phi_{n-k,lm}(p, \zeta).
\]

So we see that if \(\Phi_{nlm}(q, z)\) is substituted in the r.h.s. of the Eq.59, the result will be a superposition of the same kind functions. Therefore the solution of Eq.59 can be expressed in the form

\[
\Phi_{nlm}(p) = \sum_{k=0}^{n-l} \int_{-1}^{1} g_{nl}^{k}(z) \Phi_{n-k,lm}(p, z) dz.
\]

Then

\[
[(m + \Delta)^2 + (p - i\eta_1 P)^2]^{-1}[(m - \Delta)^2 + (p + i\eta_2 P)^2]^{-1} \int dq \frac{\Phi_{nlm}(q, z)}{(p - q)^2} =
\]
\[
\sum_{\nu=0}^{n-l} \int dz g_{nl}^{\nu}(z) [(m + \Delta)^2 + (p - i\eta_1 P)^2]^{-1} \times
\]
\[
[(m - \Delta)^2 + (p + i\eta_2 P)^2]^{-1} \int dq \frac{\Phi_{n-\nu,lm}(q, z)}{(p - q)^2} =
\]
\[
\sum_{\nu=0}^{n-l} \sum_{\tau=0}^{n-l-\nu} \frac{\pi^2}{2} \frac{(n - l - \nu)!(n - \nu - \tau + 2)!}{(n - \nu + 2)!(n - \nu - \tau - l)!} \times
\]
\[
\int dz g_{nl}^{\nu}(z) \int dt \int dx (1-x)^{n-\nu-\tau} \times
\]
\[
\int_{-1}^{1} d\zeta \frac{\delta(\zeta - xt - z + xz)}{\left[ p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m}z) + \frac{1}{4}(z - \frac{\Delta}{m})^2 P^2 \right]^{\tau+1}} \Phi_{n-\nu-\tau,lm}(p, \zeta).
\]
Let us denote \( k = \nu + \tau \) and take into account that

\[
\sum_{\nu=0}^{N-\nu} \sum_{\tau=0}^{\nu} A(\nu, \tau) = \sum_{k=0}^{N} \sum_{\nu=0}^{k} A(\nu, k - \nu)
\]

therefore Eq.59 can be rewritten as

\[
\sum_{k=0}^{n-l} \int_{-1}^{1} g_{nl}^{k}(\zeta) \Phi_{n-k,lm}(p, \zeta) d\zeta = \lambda \sum_{k=0}^{n-l} \sum_{\nu=0}^{k} \frac{(n-l-\nu)!(n-k+2)!}{(n-\nu+2)!(n-l-k)!} \int_{-1}^{1} dz g_{nl}^{\nu}(z) \int_{-1}^{1} dt \int_{-1}^{1} dx x(1-x)^{n-k} \times
\]

\[
\int_{-1}^{1} d\zeta \frac{\delta(\zeta - xt - z + xz) \Phi_{n-k,lm}(p, \zeta)}{[p_{0}^{2}(1 + \Delta_{m}^{2} - 2\Delta_{m}z) + \frac{1}{4}(z - \Delta_{m})^{2}P^{2}]^{k-\nu+1}},
\]

from this we get the following system of non-homogeneous integral equations for the \( g_{nl}^{k}(z) \) coefficient functions:

\[
g_{nl}^{k}(\zeta) = \lambda \sum_{\nu=0}^{k} \frac{(n-l-\nu)!(n-k+2)!}{(n-\nu+2)!(n-l-k)!} \int_{-1}^{1} dt \int_{0}^{1} dx x(1-x)^{n-k} \times
\]

\[
\int_{-1}^{1} d\zeta \frac{\delta(\zeta - xt - z + xz) g_{nl}^{\nu}(z) dz}{[p_{0}^{2}(1 + \Delta_{m}^{2} - 2\Delta_{m}z) + \frac{1}{4}(z - \Delta_{m})^{2}P^{2}]^{k-\nu+1}}, \tag{61}
\]

Note that we have interchanged integrations over \( dz \) and \( d\zeta \). If now integration over \( dz \) is further interchanged with integrations over \( dx \) and \( dt \), Eq.61 can be rewritten as (after \( z \leftrightarrow \zeta \) replacement)

\[
g_{nl}^{k}(z) = \lambda \sum_{\nu=0}^{k} \frac{(n-l-\nu)!(n-k+2)!}{(n-\nu+2)!(n-l-k)!} \times
\]

\[
\int_{-1}^{1} I(z, \zeta) g_{nl}^{\nu}(\zeta) d\zeta \frac{I(z, \zeta) g_{nl}^{\nu}(\zeta) d\zeta}{[p_{0}^{2}(1 + \Delta_{m}^{2} - 2\Delta_{m}z) + \frac{1}{4}(z - \Delta_{m})^{2}P^{2}]^{k-\nu+1}},
\]

where

\[
I(z, \zeta) = \int_{-1}^{1} dt \int_{0}^{1} dx x(1-x)^{n-k} \delta(z - xt - \zeta + x\zeta) = \int_{0}^{1} dx x(1-x)^{n-k} \int_{-1}^{1} \delta(z - xt - \zeta + x\zeta) dt .
\]
Let us introduce instead of $t$ the new $y = xt + (1 - x)\zeta$ integration variable:

$$I(z, \zeta) = \int_0^1 dx \, (1 - x)^{n-k} \int_{-x(1-x)\zeta}^{x+(1-x)\zeta} \delta(z - y)dy = \int \int_D (1 - x)^{n-k} \delta(z - y)dxdy .$$

The integration area $D = D_1 \cup D_2$ is shown in Fig. 11.

It is easy to calculate integrals over $D_1$ and $D_2$:

$$\int \int_{D_1} (1 - x)^{n-k} \delta(z - y)dxdy = \int \int_{D_2} (1 - x)^{n-k} \delta(z - y)dxdy .$$

$$= \int \frac{1}{\zeta} dy(1 - y) \left( \frac{1 - y}{1 - \zeta} \right)^{n-k+1} =$$

$$= \frac{1}{n - k + 1} \left( \frac{1 - z}{1 - \zeta} \right)^{n-k+1} \Theta(z - \zeta) ,$$

and

$$\int \int_{D_2} (1 - x)^{n-k} \delta(z - y)dxdy =$$

$$= \int \frac{\zeta}{1 + \zeta} dy(1 + y) \left( \frac{1 + y}{1 + \zeta} \right)^{n-k+1} =$$

$$= \frac{1}{n - k + 1} \left( \frac{1 + z}{1 + \zeta} \right)^{n-k+1} \Theta(\zeta - z) ,$$

53
Therefore
\[ I(z, \zeta) = \frac{1}{n-k+1} \left[ \left( \frac{1-z}{1-\zeta} \right)^{n-k+1} \Theta(z - \zeta) + \left( \frac{1+z}{1+\zeta} \right)^{n-k+1} \Theta(\zeta - z) \right] . \]

Let us introduce
\[ R(z, \zeta) = \frac{1-z}{1-\zeta} \Theta(z - \zeta) + \frac{1+z}{1+\zeta} \Theta(\zeta - z) \]
and define the \( \Theta \)-function at the origin as \( \Theta(0) = \frac{1}{2} \), then \( I(z, \zeta) = \frac{[R(z, \zeta)]^{n-k+1}}{n-k+1} \) and our system of integral equations can be rewritten as
\[ g^k_{nl}(z) = \frac{\lambda}{2} \sum_{\nu=0}^k \frac{(n-l-\nu)!(n-k)!}{(n-\nu+2)!(n-l-k)!} \int_{-1}^1 \[ \left( \frac{1}{p_0^2(1 + \Delta^2_{m^2} - 2\Delta_m \zeta) + \frac{1}{4}(\zeta - \Delta_m)^2 P^2} \right)^{k-\nu+1} , \]

In particular, \( g^0_{nl}(z) \equiv g_n(z) \) satisfies a homogeneous integral equation:
\[ g_n(z) = \frac{\lambda}{2(n+1)} \int_{-1}^1 \left[ \frac{[R(z, \zeta)]^{n+1}}{[R(z, \zeta)]^{n-1} g_n(z) d\zeta} \right] . \]

Note that \( R(1, \zeta) = R(-1, \zeta) = 0 \), if \( |\zeta| \neq 1 \). Therefore \( g_n(-1) = g_n(1) = 0 \). From Eq.63 \( \lambda \) eigenvalues are determined. Inverting \( \lambda = \lambda(M) \) dependence, we get a mass spectrum \( M = M(\lambda) \).

Let us see, to which second order differential equation is equivalent our integral equation (63). We have \( \frac{d}{dz} R(z, \zeta) = \Theta(\zeta - z) \frac{1}{1+\zeta} - \Theta(z - \zeta) \frac{1}{1-\zeta} \) and
\[ \frac{d}{dz} R(z, \zeta) = -\frac{2}{1-z^2} \delta(z - \zeta) . \]
Furthermore
\[ (1-z^2) \left[ \frac{dR}{dz} \right]^2 + 2zR \frac{dR}{dz} - R^2 = \left[ \frac{dR}{dz} \right]^2 - \left( R - z \frac{dR}{dz} \right)^2 = \]
\[ \left( 1-z \right) \frac{dR}{dz} + R \left( 1+z \right) \frac{dR}{dz} - R = -\frac{4}{1-\zeta^2} \Theta(z - \zeta) \Theta(\zeta - z) . \]
The r.h.s. differs from zero only at the \( z = \zeta \) point. So
\[ (1-z^2) \left[ \frac{dR(z, \zeta)}{dz} \right]^2 + 2zR(z, \zeta) \frac{dR(z, \zeta)}{dz} - R^2(z, \zeta) = \]
\[ -\frac{4}{1-\zeta^2} \Theta(z - \zeta) \Theta(\zeta - z) . \]
Since
\[
\left\{ (1 - z^2) \frac{d^2}{dz^2} + 2z(n-1) \frac{d}{dz} - n(n-1) \right\} R^n(z, \zeta) = \\
n(n-1)R^{n-2} \left\{ \left( \frac{dR}{dz} \right)^2 + 2zR \frac{dR}{dz} - R^2 \right\} + n(1 - z^2)R^{n-1}\frac{d^2R}{dz^2},
\]
and \( R(z, z) = 1 \), we get finally
\[
\left\{ (1 - z^2) \frac{d^2}{dz^2} + 2z(n-1) \frac{d}{dz} - n(n-1) \right\} R^n(z, \zeta) = \\
-2n\delta(z - \zeta) - \frac{4n(n-1)}{1-z^2}\Theta(z - \zeta)\Theta(\zeta - z). \tag{64}
\]

Using this equality, we can transform Eq.63 into a differential equation (note that if \( f(\zeta) \) is a normal function, without \( \delta(z - \zeta) \) type singularities, then
\[
\int_{-1}^{1} \Theta(z - \zeta)\Theta(\zeta - z) f(\zeta) d\zeta = 0
\]
\[
(1 - z^2) \frac{d^2}{dz^2} g_n + 2nz \frac{dg_n}{dz} - n(n+1)g_n +
\]
\[
\frac{\lambda}{p_0^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m} z) + \frac{1}{4}(z - \frac{\Delta}{m})^2P^2} g_n = 0. \tag{65}
\]

But
\[
p_0^2 \left( 1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m} \right) + \frac{1}{4}(z - \frac{\Delta}{m})^2P^2 =
\]
\[
\left( m^2 - \frac{1}{4}M^2 \right) \left( 1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m} \right) + \frac{1}{4}(z - \frac{\Delta}{m})^2M^2 =
\]
\[
m^2 \left( 1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m} \right) - \frac{1}{4}M^2(1 - z^2),
\]
therefore Eq.65 can be rewritten also as
\[
(1 - z^2) \frac{d^2}{dz^2} g_n + 2nz \frac{dg_n}{dz} +
\]
\[
\left[ \frac{\lambda}{m^2(1 + \frac{\Delta^2}{m^2} - 2\frac{\Delta}{m} z) - \frac{1}{4}M^2(1 - z^2)} - n(n+1) \right] g_n = 0. \tag{66}
\]

From this equation, when \( \Delta = 0 \), we get already known to us Eq.58. It turns out that even in a general case of unequal masses Eq.66 can be transformed into an equation of the Eq.58 type by suitable change of variables. To
guess this variable change, it is better to rewrite Eq.66 in the form of Eq.56. Introducing $\phi_n(z) = (1 + z)^{-1}(1 - z^2)^{-n/2}g_n(z)$ function, we get

$$(1 - z^2) \frac{d^2 \phi_n}{dz^2} + 2(1 - z) \frac{d \phi_n}{dz} +$$

$$\left[ \frac{\lambda}{m^2(1 + \frac{\Delta^2}{m^2} - 2 \frac{\Delta}{m} z)} - \frac{n(n + 2)}{4(1 - z^2)} \right] \phi_n = 0.$$ 

and after the $t = \frac{1 - z}{1 + z}$ variable change (let us note that $\frac{d}{dz} = -\frac{(1 + t)^2}{2} \frac{d}{dt}$ and $\frac{d^2}{dz^2} = \frac{(1 + t)^4}{4} \frac{d^2}{dt^2} + \frac{(1 + t)^3}{2} \frac{d}{dt}$):

$$\frac{d^2 \phi_n}{dt^2} + \frac{1}{t^2} \left[ \frac{\lambda}{m^2} \left[ \frac{(1 + t)^2}{t} \left( 1 + \frac{\Delta^2}{m^2} \right) - 2 \frac{\Delta}{m} \frac{1 - t^2}{t} \right] - M^2 \right] \phi_n = 0.$$ 

But

$$\frac{1}{t^2} \left[ \left( 1 + t \right)^2 \left( 1 + \frac{\Delta^2}{m^2} \right) - 2 \frac{\Delta}{m} \left( 1 - t^2 \right) \right] =$$

$$\frac{1}{t^2} \left[ \left( 1 + \frac{\Delta}{m} \right)^2 + \left( 1 - \frac{\Delta}{m} \right)^2 + 2t \left( 1 + \frac{\Delta^2}{m^2} \right) \right] =$$

$$\frac{1}{t^2} \left[ \left( 1 + \frac{\Delta}{m} \right) + \left( 1 - \frac{\Delta}{m} \right) \right]^2 + 4t \frac{\Delta^2}{m^2} =$$

$$4 \Delta^2 + m^2 \left( 1 - \frac{\Delta^2}{m^2} \right) \left[ 1 + \left( \frac{(1 + \frac{\Delta}{m})/(1 - \frac{\Delta}{m})}{t} \right)^2 \right],$$

therefore, if one more variable change $\tilde{t} = \frac{1 - \Delta/m}{1 - \Delta/m} t$ is made, we get an equation of the Eq.56 type:

$$\frac{d^2 \phi_n}{d\tilde{t}^2} + \left[ \frac{\lambda/ \left( 1 - \frac{\Delta^2}{m^2} \right)}{t[m^2(1 + \tilde{t})^2 - ((M^2 - 4 \Delta^2)/(1 - \frac{\Delta^2}{m^2})) \tilde{t}]} - \frac{n(n + 2)}{4 \tilde{t}^2} \right] \phi_n = 0.$$ 

(67)

As we see, it is enough to find $\lambda = F(M^2)$ spectrum for equal masses. Then the spectrum for the unequal masses case can be found from the relation

$$\frac{\lambda}{1 - \frac{\Delta^2}{m^2}} = F \left( \frac{M^2 - 4 \Delta^2}{1 - \frac{\Delta^2}{m^2}} \right).$$

(68)

Note, that $t \to \tilde{t}$ variable change implies the following transformation for the variable $z$:

$$z \to \tilde{z} = \frac{\frac{1 - \Delta}{1 + \Delta} z}{1 - \frac{\Delta}{m} z}.$$
Using Eq.64, we can transform Eq.62 also into a system of second order differential equations
\[
\begin{aligned}
\left\{(1 - z^2) \frac{d^2}{dz^2} + 2(n - k)z \frac{d}{dz} - (n - k)(n - k + 1)\right\} g_{nl}^k(z) + \\
\lambda \sum_{\nu=0}^{k} \frac{(n - l - \nu)!(n - k + 2)!}{(n - \nu + 2)!(n - l - k)!} g_{nl}^{\nu}(z) \\
\frac{1}{[m^2(1 + \Delta m^2 - 2\Delta m z) - \frac{1}{4} M^2(1 - z^2)]^{k-\nu+1}} = 0.
\end{aligned}
\]

**Variable separation in bipolar coordinates**

There exists one more method for Eq.27 investigation. It is based on the following idea: transform integral Wick-Cutkosky equation into a partial differential equation and try to separate variables in some special coordinate system.

Transition from the integral to the differential equation can be done by using the fact that in a 4-dimensional Euclidean space
\[
\Delta \frac{1}{x^2} = -4\pi^2 \delta(x).
\] (69)

Let us show that this is indeed correct. Derivatives of \(\frac{1}{x^2}\) are singular at the \(x = 0\) point. Therefore \(\Delta \frac{1}{x^2}\) should be specially defined at this point, for example, as
\[
\Delta \frac{1}{x^2} = \lim_{\epsilon \to 0} \frac{\Delta 1}{x^2 + \epsilon}.
\]

But \(\frac{\partial}{\partial x_i} \frac{1}{x^2 + \epsilon} = -\frac{2x_i}{(x^2 + \epsilon)^2}\) and
\[
\Delta \frac{1}{x^2 + \epsilon} = \frac{\partial^2}{\partial x_i \partial x_i} \frac{1}{x^2 + \epsilon} = -\frac{8}{(x^2 + \epsilon)^2} + \frac{8x^2}{(x^2 + \epsilon)^3} = -\frac{8\epsilon}{(x^2 + \epsilon)^3}.
\]

Therefore
\[
\lim_{\epsilon \to 0} \Delta \frac{1}{x^2 + \epsilon} = \begin{cases} 0, & \text{if } x \neq 0 \\ -\infty, & \text{if } x = 0 \end{cases}
\]

This suggests that \(\Delta \frac{1}{x^2}\) is proportional to \(-\delta(x)\). But
\[
-8\epsilon \int \frac{dx}{(x^2 + \epsilon)^3} = -16\epsilon \pi^2 \int_0^\infty \frac{r^3dr}{(r^2 + \epsilon)^3} = -8\epsilon \pi^2 \int_0^\infty \frac{tdt}{(t + \epsilon)^3} = -4\pi^2,
\]

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so Eq.69 holds.

Acting on the both sides of Eq.27 by the operator \( \Delta_p = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2} + \frac{\partial^2}{\partial p_4^2} \)
and using \( \Delta_p \frac{1}{(p-q)^2} = -4\pi^2 \delta(p-q) \), we get

\[
\Delta_p \left[ m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 M)^2 \right] \left[ m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 M)^2 \right] \Phi(p) = -4\lambda \Phi(p),
\]
or, after the introduction of a new function

\[
\Psi(p) = \left[ m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 M)^2 \right] \left[ m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 M)^2 \right] \Phi(p),
\]

\[
\Delta \Psi(p) + \frac{4\lambda}{\left[ m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 M)^2 \right] \left[ m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 M)^2 \right]} \Psi(p) = 0. \tag{70}
\]

So the partial differential equation is found. Now we should care about variable separation. It turns out that so called bipolar coordinates can be used for this goal.

On a plane, the bipolar coordinates \( \tau, \alpha \) are defined as follows: \( \alpha \) is the angle indicated in Fig.12 and \( \exp \tau = \frac{\Delta}{r_2} \).

![Figure 12: Bipolar coordinates.](image)

Because \( r_1 = \sqrt{(x+a)^2 + y^2} \) and \( r_2 = \sqrt{(x-a)^2 + y^2} \), this definition implies

\[
\tau = \frac{1}{2} \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}, \quad \alpha = \arccos \frac{x^2 + y^2 - a^2}{\sqrt{(x+a)^2 + y^2} \sqrt{(x-a)^2 + y^2}}.
\]

To inverse these relations, let us note that

\[
cosh \tau = \frac{1}{2} (e^\tau + e^{-\tau}) = \frac{x^2 + y^2 + a^2}{r_1 r_2}, \quad \sinh \tau = \sqrt{\cosh^2 \tau - 1} = \frac{2ax}{r_1 r_2},
\]

\[
\cos \alpha = \frac{x^2 + y^2 - a^2}{r_1 r_2},
\]

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\[
\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \frac{2ay}{r_1 r_2} \quad \text{and} \quad \cosh \tau \cos \alpha = \frac{2a^2}{r_1 r_2}.
\]

So

\[
x = \frac{a \sinh \tau}{\cosh \tau - \cos \alpha}, \quad y = \frac{a \sin \alpha}{\cosh \tau - \cos \alpha}.
\]

In a 4-dimensional Euclidean momentum space the bipolar coordinates are defined through

\[
P_1 = \frac{a \sin \alpha}{\cosh \tau - \cos \alpha} \sin \Theta \cos \varphi, \quad P_2 = \frac{a \sin \alpha}{\cosh \tau - \cos \alpha} \sin \Theta \sin \varphi,
\]

\[
P_3 = \frac{a \sin \alpha}{\cosh \tau - \cos \alpha} \cos \Theta, \quad P_4 = \frac{a \sinh \tau}{\cosh \tau - \cos \alpha}.
\]

(71)

It is convenient to choose the \(a\) parameter in such a way to have

\[
a^2 = m_1^2 - \eta_1^2 M^2 = m_2^2 - \eta_2^2 M^2.
\]

If \(\eta_i = \frac{m_i}{m_1 + m_2}\) (as it was assumed so far), this is impossible. But let us recall from the BS equation derivation that \(\eta_1\) and \(\eta_2\) are subject to only one \(\eta_1 + \eta_2\) condition. In other respects they are arbitrary. So let us demand

\[
m_1^2 - \eta_1^2 M^2 = m_2^2 - \eta_2^2 M^2,
\]

which gives

\[
\eta_1 = \frac{M^2 + m_1^2 - m_2^2}{2M^2}, \quad \eta_2 = \frac{M^2 - m_1^2 + m_2^2}{2M^2},
\]

\[
a = \frac{\sqrt{[(m_1 + m_2)^2 - M^2][M^2 - (m_1 - m_2)^2]}}{2M}.
\]

Now Eq.70 should be rewritten in the new coordinates. We have

\[
m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 M)^2 = a^2 + \frac{a^2 \sin^2 \alpha}{(\cosh \tau - \cos \alpha)^2} + \frac{a^2 \sinh^2 \tau}{(\cosh \tau - \cos \alpha)^2} - \frac{2i\eta_1 M \frac{a \sinh \tau}{\cosh \tau - \cos \alpha}}{\cosh \tau - \cos \alpha} = a^2 + a^2 \frac{1 - \cos^2 \alpha}{(\cosh \tau - \cos \alpha)^2} + a^2 \frac{\cosh^2 \tau - 1}{(\cosh \tau - \cos \alpha)^2} - \frac{2i\eta_1 M \frac{a \sinh \tau}{\cosh \tau - \cos \alpha}}{\cosh \tau - \cos \alpha} (a \cosh \tau - i\eta_1 M \sinh \tau).
\]

Analogously

\[
m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 M)^2 = \frac{2a}{\cosh \tau - \cos \alpha} (a \cosh \tau + i\eta_2 M \sinh \tau).
\]

Therefore

\[
[m_1^2 + \vec{p}^2 + (p_4 - i\eta_1 M)^2][m_2^2 + \vec{p}^2 + (p_4 + i\eta_2 M)^2] = \frac{4a^2}{b(\cosh \tau - \cos \alpha)^2} (a \cosh \tau - i\eta_1 M \sinh \tau)(a \cosh \tau + i\eta_2 M \sinh \tau).
\]
Now we should express the Laplacian in the bipolar coordinates. Denoting \( r = \frac{a \sin \alpha}{\cosh \tau - \cos \alpha} \), we have

\[
\Delta_p = \frac{\partial^2}{\partial p_4^2} + \frac{1}{r} \frac{\partial^2}{\partial r^2} r - \frac{\delta(3)}{r^2} = \frac{1}{r} \left( \frac{\partial^2}{\partial p_4^2} + \frac{\partial^2}{\partial r^2} \right) r - \frac{\delta(3)}{r^2}.
\]

Here \( \delta(3) \) is the angular part of the 3-dimensional Laplacian \( \Delta(3) = \frac{\partial^2}{\partial p_4^2} + \frac{\partial^2}{\partial p_3^2} + \frac{\partial^2}{\partial z^2} \). Let \( z = p_4 + ir \), then \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial p_4} + i \frac{\partial}{\partial r} \right) \), \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial p_4} - i \frac{\partial}{\partial r} \right) \), and \( \frac{\partial^2}{\partial p_4^2} + \frac{\partial^2}{\partial z^2} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \). So \( \alpha \) and \( \tau \) should be expressed through \( \bar{z} \) and \( z \)

\[
\tau = \frac{1}{2} \ln \left( \frac{(p_4 + a)^2 + r^2}{(p_4 - a)^2 + r^2} \right) = \frac{1}{2} \ln \left( \frac{(z + a)(\bar{z} + a)}{(z - a)(\bar{z} - a)} \right).
\]

As for \( \alpha \), using \( \arccos A = i \ln [A + \sqrt{A^2 - 1}] \), we get

\[
\alpha = \arccos \frac{p_4^2 + r^2 - a^2}{\sqrt{(p_4 + a)^2 + r^2 \sqrt{(p_4 - a)^2 + r^2}}} = \arccos \frac{zz^* - a^2}{\sqrt{(z^2 - a^2)(\bar{z}^2 - a^2)}} = \frac{i}{2} \ln \left( \frac{z + a)(\bar{z} - a)}{(z - a)(\bar{z} + a)} \right).
\]

It follows from the above given equations that

\[
\frac{\partial \alpha}{\partial z} = \frac{-ia}{z^2 - a^2}, \quad \frac{\partial \alpha}{\partial \bar{z}} = \frac{ia}{\bar{z}^2 - a^2}, \quad \frac{\partial \tau}{\partial z} = \frac{-a}{z^2 - a^2}, \quad \frac{\partial \tau}{\partial \bar{z}} = \frac{-a}{\bar{z}^2 - a^2}.
\]

Thus

\[
\frac{\partial}{\partial z} = \frac{\partial \alpha}{\partial z} \frac{\partial}{\partial \alpha} + \frac{\partial \tau}{\partial z} \frac{\partial}{\partial \tau} = -\frac{a}{z^2 - a^2} \left( \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \alpha} \right)
\]

and

\[
\frac{\partial}{\partial \bar{z}} = -\frac{a}{\bar{z}^2 - a^2} \left( \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \alpha} \right).
\]

Therefore

\[
\frac{\partial^2}{\partial p_4^2} + \frac{\partial^2}{\partial r^2} = \frac{4a^2}{(z^2 - a^2)(\bar{z}^2 - a^2)} \left( \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \alpha^2} \right),
\]

(note that \( \frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \alpha} = 2 \frac{\partial}{\partial (\tau - ia)} \) commutes with \( \frac{1}{z^2 - a^2} \), because \( \bar{z} = p_4 - ir = a \frac{\sinh \tau - \sinh (\tau - i \alpha)}{\cosh \tau - \cosh (\tau - i \alpha)} = \text{acoth} \frac{\tau + i \alpha}{2} \)). But

\[
(z^2 - a^2)(\bar{z}^2 - a^2) = (z + a)(\bar{z} + a)(z - a)(\bar{z} - a) =
\]

\[
[(p_4 + a)^2 + r^2][(p_4 - a)^2 + r^2] = \frac{4a^4}{(\cosh \tau - \cos \alpha)^2},
\]

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and for the Laplacian we finally get
\[
\Delta = \frac{(\cosh \tau - \cos \alpha)^3}{a^2 \sin \alpha} \left( \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \alpha^2} \right) \frac{\sin \alpha}{\cosh \tau - \cos \alpha} - \frac{(\cosh \tau - \cos \alpha)^2}{a^2 \sin^2 \alpha} \delta^{(3)}.
\]
This could be rewritten in a more convenient form by using
\[
\frac{\partial^2}{\partial \alpha^2} \sin \alpha = \sin \alpha \left( \frac{\partial^2}{\partial \alpha^2} + 2 \text{ctg} \alpha \frac{\partial}{\partial \alpha} - 1 \right)
\]
and Eq.45:
\[
\Delta = \frac{(\cosh \tau - \cos \alpha)^3}{a^2} \left( \frac{\partial^2}{\partial \tau^2} - \delta^{(4)} - 1 \right) \frac{1}{(\cosh \tau - \cos \alpha)}.
\]
So Eq.70 in the bipolar coordinates takes the form
\[
\left[ (\cosh \tau - \cos \alpha) \left( \frac{\partial^2}{\partial \tau^2} - \delta^{(4)} - 1 \right) \frac{1}{(\cosh \tau - \cos \alpha)} + \right.
\]
\[
\left. \frac{\lambda}{(a \cosh \tau - i \eta_1 M \sinh \tau)(a \cosh \tau + i \eta_2 M \sinh \tau)} \right] \Psi = 0.
\]
It is possible to separate variables in this equation. In particular, if we take
\[
\Psi(\tau, \alpha, \Theta, \varphi) = (\cosh \tau - \cos \alpha) f(\tau) Y_{nlm}(\alpha, \Theta, \varphi).
\]
and use \(\delta^{(4)} Y_{nlm} = n(n+2) Y_{nlm}\) equation, we get for the function \(f\) the following equation
\[
\frac{d^2 f}{d\tau^2} + \left[ \frac{\lambda}{(a \cosh \tau - i \eta_1 M \sinh \tau)(a \cosh \tau + i \eta_2 M \sinh \tau)} - (n+1)^2 \right] f = 0.
\]
(73)
To simplify, note that because
\[
\sinh 2\tau = 2 \sinh \tau \cosh \tau, \quad 1+\cosh 2\tau = 2 \cosh^2 \tau \quad \text{and} \quad \cosh 2\tau - 1 = 2 \sinh^2 \tau
\]
the following holds
\[
(a \cosh \tau - i \eta_1 M \sinh \tau)(a \cosh \tau + i \eta_2 M \sinh \tau) =
\]
\[
a^2 \cosh^2 \tau + \eta_1 \eta_2 M^2 \sinh^2 \tau + i M a (\eta_2 - \eta_1) \cosh \tau \sinh \tau =
\]
\[
\frac{1}{2} \left\{ (a^2 + \eta_1 \eta_2 M^2) \cosh 2\tau + i a M (\eta_2 - \eta_1) \sinh 2\tau - (\eta_1 \eta_2 M^2 - a^2) \right\}.
\]
But
\[
(a^2 + \eta_1 \eta_2 M^2)^2 - i^2 a^2 M^2 (\eta_2 - \eta_1)^2 =
\]
\[ a^4 \eta_1^2 \eta_2^2 M^4 + a^2 M^2 \eta_1^2 + \eta_1^2 M^2 = (a^2 + \eta_1^2 M^2)(a^2 + \eta_2^2 M^2) = m_1^2 m_2^2 , \]

therefore there exists a complex number \( \nu \) such that

\[
\cosh \nu = \frac{1}{m_1 m_2}(a^2 + \eta_1 \eta_2 M^2) \quad \text{and} \quad \sinh \nu = \frac{i}{m_1 m_2}aM(\eta_2 - \eta_1) .
\]

Besides

\[
\eta_1 \eta_2 M^2 - a^2 = \eta_1 \eta_2 M^2 - m_1^2 + \eta_1^2 M^2 = \eta_1 M^2 - m_1^2 = \frac{1}{2}(M^2 + m_1^2 - m_2^2) - m_1^2 = \frac{1}{2}(M^2 - m_1^2 - m_2^2) ,
\]

and, using \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \), we get

\[
(a \cosh \tau - i\eta_1 M \sinh \tau)(a \cosh \tau + i\eta_2 M \sinh \tau) = \frac{m_1 m_2}{2}
\[
\begin{cases}
\cosh (2\tau + \nu) - \frac{1}{2m_1 m_2}(M^2 - m_1^2 - m_2^2),
\end{cases}
\]

Therefore Eq.73, after the introduction of a new variable \( \sigma = \tau + \frac{\nu}{2} \), will look as

\[
\frac{d^2 f}{d\sigma^2} + \left[ \frac{\lambda}{m_1 m_2 \cosh^2 \sigma - \frac{1}{4}[M^2 - (m_1 - m_2)^2]} - (n + 1)^2 \right] f = 0 . \tag{74}
\]

Its form reveals the already known to us fact that the unequal mass case is equivalent to the equal mass problem with \( m^2 = m_1 m_2 \) and \( M'^2 = M^2 - (m_1 - m_2)^2 \).

Thus, it is sufficient to consider the following equation (for equal masses \( \nu = 0 \) and \( \sigma = \tau \))

\[
\frac{d^2 f}{d\tau^2} + \left[ \frac{\lambda}{m_2 \cosh^2 \tau - \frac{1}{4}M^2} - (n + 1)^2 \right] f = 0 . \tag{75}
\]

As is clear from the bipolar coordinates definition \( -\infty < \tau < \infty \), therefore Eq.75 should be supplied with boundary conditions at \( \tau \to \pm \infty \). These conditions follow from the initial integral equation, but instead we will show that Eq.75 is equivalent to the Eq.56. Indeed, let us introduce a new variable \( t = \sinh 2\tau + \cosh 2\tau = e^{2\tau} \), then \( \frac{\partial^2}{\partial \tau^2} = 4t \frac{\partial}{\partial t} + 4t^2 \frac{\partial^2}{\partial t^2} \). But \( (1 + t)^2 = (2 \cosh^2 \tau + 2 \cosh 2\tau \sinh 2\tau)^2 = 4 \cosh^2 \tau (\cosh 2\tau + \sinh 2\tau)^2 = 4(\cosh^2 \tau)t \), so \( \cosh^2 \tau = \frac{(1 + t)^2}{4t} \), and Eq.75 takes the form

\[
\frac{d^2 f}{dt^2} + \frac{1}{t} \frac{df}{dt} + \left\{ \frac{\lambda}{t[m^2(1 + t)^2 - M^2 t]} - \frac{(n + 1)^2}{4t^2} \right\} f = 0 ,
\]

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which after introduction of a new function $\phi = \sqrt{t} f$ coincides to the Eq.56. Therefore Eq.57 implies the following boundary conditions for the Eq.75

$$f(\tau) \sim e^{-(n+1)\tau}, \text{ when } \tau \to \pm \infty.$$ (76)

Appearance of the spherical function in Eq.72 indicates that the Wick-Cutkosky model possesses the $SO(4)$ hidden symmetry even for unequal masses. The variable separation is possible just because of this symmetry.

**Note about references**

Only the sources of this review are presented here. Exhaustive bibliography about the Bethe-Salpeter equation and in particular about the Wick-Cutkosky model can be found in [13].
References

[1] N. Nakanishi, Progr. Theor. Phys. Suppl. 43 (1969)
[2] D. Luriè, A. Macfarlane, Y. Takahashi, Phys. Rev. B140 (1965), 1091
[3] G.C. Wick, Phys. Rev. 96 (1954), 1124
[4] R.E. Cutkosky, Phys. Rev. 96 (1954), 1135
[5] S.S. Schweber, Annals of Phys. 20 (1962), 61
[6] E.E. Salpeter, Phys. Rev. 87 (1952), 328
[7] T. Shibuya, C.E. Wulfman, Am. J. Phys. 33 (1965), 570
[8] V.A. Fock, Z. Phys. 98 (1935), 145
[9] M. Levy, Proc. Roy. Soc. (London) A204 (1950), 145
[10] H.S. Green, Nuovo Cimento 5 (1957), 866
[11] Higher transcendental functions. McGraw-Hill, 1955. Bateman Manuscript Project, California Institute of Technology.
[12] A.O. Barut, R. Raczka, Theory of group representations and applications. World Scientific, 1986.
[13] Progr. Theor. Phys. Suppl. 95 (1988)