HEEGAARD DISTANCE OF THE LINK COMPLEMENTS IN $S^3$

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ABSTRACT. We show that, for any integers, $g \geq 3$ and $n \geq 2$, there exists a link in $S^3$ such that its complement has a genus $g$ Heegaard splitting with distance $n$.

1. Introduction

The Heegaard distance (distance) was introduced by Hempel [7] to measure the complexity of 3-manifolds using Heegaard splittings, which generalizes the notion of Casson and Gordon’s strong irreducibility of Heegaard splittings. Using this notion, Hempel showed that for any integer $g \geq 2$ and $n > 0$, there is a 3-manifold with Heegaard splitting of genus $g$ that admits the distance at least $n$, that is, there exist 3-manifolds with high distance. In line with this result, Evans [4] and Yoshizawa [23] used combinatorial techniques to construct 3-manifolds of high distance. Lustig and Yoriah [13] introduced the fat train tracks to construct 3-manifolds of high distance.

The topology of the underlying 3-manifold places constraints on the Heegaard distance. If a 3-manifold is Haken, Hartshorn [5] showed that its Heegaard distance is bounded above by the double of the genus of an incompressible closed surface. However, Li [11] proved that closed non-Haken manifolds admit high Heegaard distance.

When restricting to the knot complements in $S^3$, Minsky, Moriah and Schleimer [19] proved the existence of high distance knot, that is, a knot in $S^3$ that its exterior has a genus $g$ Heegaard splitting of arbitrarily high distance for any $g > 1$. In [2], Campisi and Rathbun generalized the result to knots in 3-manifolds.

On the other hand, the exactness of Heegaard distance of 3-manifolds has been studied in [8, 10, 20]. Ido, Jang, Kobayashi [8] showed that, for any integers $n \geq 2$ and $g \geq 2$, there exists a genus-$g$ Heegaard splitting of a closed 3-manifold with distance exactly $n$. Johnson [10] proved that, for every pair of positive integers $d \geq 4$, $g \geq 2$ with $d$ even, there is a compact, connected, closed, orientable three-manifold $M$ with both a genus $g$ Heegaard surface $\Sigma$ such that $d(\Sigma) = d$, and a separating, two-sided, closed, embedded, incompressible surface of genus $\frac{1}{2}d$. Qiu, Zou, Guo [20] showed that, for any integers $n \geq 1$ and $g \geq 2$, there is a closed 3-manifold $M^n_g$ which admits a distance-$n$ Heegaard splitting of genus $g$ unless $(g, n) = (2, 1)$.

In the setting of bridge splitting of links in 3-manifolds, Ido, Jang, Kobayashi [9] was able to show that, for any integers $n \geq 2$, $g \geq 0$ and $b \geq 1$ except for $(g, b) = (0, 1)$ and $(0, 2)$, there exists a $(g, b)$-bridge splitting of a link in some 3-manifold with distance exactly $n$.

Comparable to these results, we show the exactness of Heegaard distance for the link complements in $S^3$. Our result lies in the intersection of high distance knot in [2, 19] and the exactness of Heegaard distance of 3-manifolds in [8, 9, 10, 20].
ambient 3-manifold is $S^3$ and the link complements in $S^3$ achieve the exact Heegaard distance.

The tools we will utilize in the paper are not new, and the result might be known to some experts. The construction of geodesics can be found in [8]. The extension to the link complements relies on the result of high distance knot [19] and a similar argument of exact Heegaard distance [8, 20].

We start off with two compression bodies of genus $g \geq 3$, each of which is obtained from attaching one 2-handle to the $S \times [0,1]$ along a separating curve. The curve is a meridian of one handlebody in the complement of $S$ embedded in $S^3$ in a standard way. The union of two compression bodies along the common boundary surface $S$ can be embedded in $S^3$. It follows that the disk graph of each compression body contains the unique separating meridian. This gives us a good control of the geodesic realizing the distance between the compression bodies. To achieve the exact distance, we will adopt Ido, Jang and Kobayashi’s approach to construct geodesics such that the two ends are meridians.

**Theorem 1.1.** Let $S$ be a closed oriented surface of genus $g \geq 3$ embedded in $S^3$. For any integer $n \geq 0$, there exists a compact oriented 3-manifold $V_0 \cup_\bar{S} W_0$ obtained by the union of two compression bodies $V_0, W_0$ with Heegaard distance $n$, and it can be embedded in $S^3$.

The meridian realizing the distance divides the genus $g$ surface $S$ into a one-holed torus and a genus $g-1$ surface with one boundary component. Inspired by Minsky, Moriah and Schleimer’s work [19], we can attach a genus $g-1$ handlebody that minuses a knot to push the disk graph far away except for the meridian that realizes the distance.

**Theorem 1.2.** For any integers $g \geq 3$ and $n \geq 2$, there exists a link in $S^3$ such that its complement has a genus $g$ Heegaard splitting with distance $n$.

The paper is organized as follows. In Section 2, some relevant definitions and results about curve complex, Heegaard splitting and Heegaard distance are given. In Section 3, we construct geodesics in the curve complex using a method from [8], subject to some constraints. Then, we prove the Theorem 1.1 in Section 4. In Section 5, we prove the main result, Theorem 1.2 using the result of high distance knot from [19] and the argument of exact Heegaard distance from [20].

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2. Preliminaries

2.1. Curve complex. Let $S = S_{g,b}$ be a compact oriented surface of genus $g$ and $b$ boundary components. The complexity of surface $S$ is defined as $\xi(X) = 3g + b - 3$. A simple closed curve in $S$ is essential if it does not bound a disk or an annulus in $S$.

The curve complex is a simplicial complex such that each vertex is represented by the isotopy class of an essential simple closed curve, and $n+1$ vertices form an $n$-simplex of $C(S)$ if their representatives can be realized disjointly. It was introduced by Harvey [6] to study the mapping class group. The information of curve complex is encoded in its 1-skeleton, which is called curve graph. Throughout, we only consider
the curve graph instead of curve complex, and we use the same notation $C(S)$ for the curve graph.

The arc and curve complex $\mathcal{AC}(S)$ of a compact oriented surface $S$ with boundary can be defined similarly. Each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc (each endpoint is allowed to move freely in its boundary) or an essential simple closed curve on $S$, and $n+1$ vertices form an $n$-simplex of $\mathcal{AC}(S)$ if their representatives can be realized disjointly. The notation $\mathcal{AC}(S)$ will also be used as the 1-skeleton of the arc and curve complex, and it is called the arc and curve graph.

For any two vertices $x$ and $y$ in the $C(S)$, the distance $d_{C(S)}(x,y)$ is the minimal number of edges in $C(S)$ joining $x$ and $y$. For any two subsets $A$ and $B$, define $d_{C(S)}(A,B) = \min\{d_{C(S)}(x,y)|x \in A, y \in B\}$. A geodesic in the curve graph $C(S)$ is a sequence of vertices $\Gamma = (\gamma_i)$ such that $d_{C(S)}(\gamma_i, \gamma_j) = |i - j|$ for all $i, j$. These notions can be defined on the arc and curve graph $\mathcal{AC}(S)$ similarly. A metric space is $\delta$-hyperbolic, if for each geodesic triangle in the metric space, each side lies in the $\delta$-neighborhood of the other two sides.

**Theorem 2.1.** (Masur-Minsky [14] Theorem 1.1) Let $S$ be a compact oriented surface with $\xi(S) \geq 1$, the curve graph $C(S)$ is a $\delta$-hyperbolic metric space with infinite diameter for some $\delta$, where $\delta$ depends on the surface.

**2.2. Disk graph.** Let $M$ be a compact oriented 3-manifold, and suppose the closed oriented surface $S_g$ is a boundary component of $M$. Denote

$$D(M, S_g) = \{[\partial D] : (D, \partial D) \subset (M, S_g) \text{ is an essential disk}\} \subset C(S_g),$$

where $[*]$ denotes the isotopy class of a simple closed curve $*$ in the surface $S_g$. As a full subgraph of $C(S_g)$, $D(M, S_g)$ is called the disk graph. It is the 1-skeleton of disk complex introduced by McCullough [18].

**2.3. Heegaard splitting.** A handlebody $V$ of genus $g$ is a 3-manifold homeomorphic to a regular neighborhood of a connected graph in $S^3$, whose boundary is a closed oriented surface of genus $g$. A Heegaard splitting of genus $g$ for a closed 3-manifold $M$ is $M = V \cup_\partial W$, where $V$ and $W$ are two handlebodies of genus $g$. The common boundary $S = \partial V = \partial W$ is a closed oriented surface of genus $g$ in $M$, which is called the Heegaard surface of the Heegaard splitting.

A compression body $V$ is a compact oriented 3-manifold obtained from $S \times [0,1]$ and a 0-handle $B$ by attaching 1-handles to $S \times \{0\} \cup \partial B$, where $S$ is a closed oriented surface, but it is not necessarily connected or nonempty. The negative (inner) boundary is $S \times \{1\} \subset V$ and the positive (outer) boundary $\partial_+ V = \partial V - \partial_- V$. By convention, a handlebody is a compression body with $\partial_- V = \emptyset$.

Dually, a compression body $V$ is obtained by attaching some 2-handles to $S \times \{1\}$ as the boundary of $S \times [0,1]$, and 3-handles to cap off any 2-spheres created by the attachment of the 2-handles. The positive (outer) boundary $\partial_+ V$ is the boundary component $S \times \{0\}$. The negative (inner) boundary $\partial_- V = \partial V - \partial_+ V$. If $\partial_+ V = \emptyset$, then it is a handlebody.

For a compact oriented 3-manifold $M$ with boundary, the Heegaard splitting of genus $g$ is $M = V \cup_\partial W$, where $V$ and $W$ are compression bodies. $S = \partial_+ V = \partial_+ W$ is a closed oriented surface of genus $g$ embedded in $M$.

A Heegaard splitting $M = V \cup_\partial W$ is called reducible if there exists a pair of essential embedded disks $(D, D') \subset (V, W)$ with $\partial D = \partial D' \subset S$. Otherwise, it is
irreducible. A Heegaard splitting is called weakly reducible if there exists a pair of essential embedded disks \( (D, D') \subset (V, W) \) with \( \partial D \cap \partial D' = \emptyset \). Otherwise, it is strongly irreducible. The weak reducibility and strong reducibility were introduced by Casson and Gordon [3].

2.4. Heegaard distance. An essential curve on \( \partial_s V \) is a meridian of the compression body \( V \) if it bounds an essential disk in \( V \). If the closed oriented surface \( S = \partial_s V \), then the subcomplex \( \mathcal{D}(V, S) \) of \( \mathcal{C}(S) \) spanned by the meridians is the disk graph of the compression body. In the case of compression body and handlebody, the disk graph is denoted by \( \mathcal{D}(V) \) instead of \( \mathcal{D}(V, S) \) for simplicity.

For a Heegaard splitting \( M = V \cup S W \) with the Heegaard surface \( S \), Hempel [7] defined the Heegaard distance (distance) of a Heegaard splitting to be

\[
d_{\mathcal{C}(S)}(V, W) := d_{\mathcal{C}(S)}(\mathcal{D}(V), \mathcal{D}(W))
\]

The Heegaard splitting \( M = V \cup S W \) is reducible if and only if \( d_{\mathcal{C}(S)}(V, W) = 0 \), and it is weakly reducible if and only if \( d_{\mathcal{C}(S)}(V, W) \leq 1 \).

2.5. Subsurface projection. A subsurface \( X \) is called an essential subsurface of a compact connected oriented surface \( S = S_{g,b} \), if each boundary component of \( X \) is essential in \( S \). Define a map \( \pi_A : \mathcal{C}(S) \rightarrow \mathcal{P}(\mathcal{AC}(X)) \), where \( \mathcal{P}(\mathcal{AC}(X)) \) is the power set of the arc and curve graph \( \mathcal{AC}(X) \). Take any \( \alpha \in \mathcal{C}(S) \), then consider the representative of \( \alpha \) such that it intersects \( X \) minimally, \( \pi_A(\alpha) \) is the set of all isotopy classes of \( \alpha \cap X \) relative to the boundary of \( X \). Note that \( \pi_A(\alpha) \) is empty when \( \alpha \) can be realized disjointly from \( X \). We say \( \alpha \) cuts \( X \) if \( \alpha \cap X \neq \emptyset \), and \( \alpha \) misses \( X \) if \( \alpha \cap X = \emptyset \).

The image of \( \pi_A \) is in the arc and curve graph \( \mathcal{AC}(X) \), and there is a natural way to send them back to the curve graph \( \mathcal{C}(X) \). We can define \( \pi_0 : \mathcal{P}(\mathcal{AC}(X)) \rightarrow \mathcal{C}(X) \) as follows. If \( \alpha \) is in the \( X \), then \( \pi_0(\pi_A(\alpha)) = \alpha \) in \( \mathcal{AC}(X) \) and \( \pi_0(\pi_A(\alpha)) = \alpha \) in \( \mathcal{C}(X) \). Otherwise, \( \pi_A(\alpha) = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) is a collection of isotopy classes of essential properly embedded arcs in \( X \). The set \( \pi_0(\pi_A(\alpha)) \) is the isotopy classes of the essential components of \( \partial N(\alpha \cup \partial X) \) in \( X \), where \( N(\alpha \cup X) \) is a regular neighborhood of \( \alpha \cup \partial X \) in \( X \). The composition \( \pi_0 \circ \pi_A = \pi_X : \mathcal{C}(S) \rightarrow \mathcal{C}(X) \) is called the subsurface projection.

Lemma 2.2. (Mazur-Minsky [15] Lemma 2.2) Suppose that the complexity \( \xi(X) > 1 \), and \( d_{\mathcal{AC}(X)}(\alpha, \beta) \leq 1 \) for \( \alpha, \beta \in \mathcal{AC}(X) \), then \( d_{\mathcal{C}(X)}(\pi_0(\alpha), \pi_0(\beta)) \leq 2 \).

The subsurface projection \( \pi_X \) has a strong contraction property.

Theorem 2.3. (Bounded Geodesic Image Theorem, Masur-Minsky [15], Theorem 3.1) Let \( X \) be an essential subsurface of \( S \) with \( \xi(X) \neq 0 \) and let \( \Gamma = (\gamma_i)_{i \in I} \) be a geodesic in \( \mathcal{C}(S) \). If each \( \gamma_i \) cuts \( X \), then there is a constant \( M \) depending only on the surface so that \( d_{\mathcal{C}(X)}(\pi_X(\Gamma)) \leq M \).

The constant \( M \) can be taken to be independent of the surface [22].

Let \( F \) be a boundary component of a compact orientable 3-manifold \( M \). A simple closed curve \( \gamma \) on an oriented closed surface \( F \) is disk-busting if it intersects every simple closed curve in the disk graph \( \mathcal{D}(M, F) \), that is, \( F - \gamma \) is incompressible in \( M \). If \( M \) is an I-bundle over a compact surface \( S \) with boundary \( \partial S \), then the boundary \( \partial M \) can be decomposed into two parts, vertical boundary and horizontal boundary. The vertical boundary \( \partial_v M \) is the I-bundle restricted to the \( \partial S \). The horizontal boundary \( \partial_h M = \partial M - \partial_v M \) is the portion of \( \partial M \) that is transverse to the I-fibers.
The following theorem of the subsurface projection of disk complex was proved independently by Li [12], Mazur and Schleimer [17].

**Theorem 2.4.** (Li [12] Theorem 1, Mazur-Schleimer [17] Theorem 12.1) Let $M$ be a compact orientable and irreducible 3-manifold and $F$ a component of $\partial M$. Suppose $\partial M - F$ is incompressible in $M$. Let $D$ be the disk complex for $F$. Let $S$ be a compact connected subsurface of $F$ and suppose that every component of $\partial S$ is disk-busting. Then either

1. $M$ is an I-bundle over a compact surface, and $S$ is a component of the horizontal boundary of this I-bundle, and the vertical boundary of this I-bundle is a single annulus, or
2. the image $\pi_A(D)$ of the disk complex lies in a ball of radius 3 in $\mathcal{AC}(S)$, in particular, $\pi_A(D)$ has diameter at most 6 in $\mathcal{AC}(S)$. Moreover, $\pi_S(D)$ has diameter at most 12 in $\mathcal{C}(S)$.

In the end, we recall a main property of pseudo-Anosov maps.

**Theorem 2.5.** (Masur-Minsky [14] Proposition 4.6) For a surface $S_{g,b}$ with the complexity $\xi(S_{g,b}) > 1$, there exists $c > 0$ such that, for any pseudo-Anosov $h \in \text{Mod}(S_{g,b})$, any $\gamma \in \mathcal{C}(S_{g,b})$ and any $n \in \mathbb{Z}$,

$$d_{\mathcal{C}(S_{g,b})}(h^n(\gamma),\gamma) \geq c|n|.$$ 

A well known method to construct pseudo-Anosov maps on $S_{g,b}$ is Thurston’s construction. A filling pair on $S_{g,b}$ is a pair of curves $\alpha$ and $\beta$ such that each complement $S_{g,b} - \alpha \cup \beta$ is either a disk or an annulus.

**Theorem 2.6.** (Thurston [21]) If $\alpha$ and $\beta$ is a filling pair on $S_{g,b}$, then the composition of Dehn twists $T_\alpha \circ T_\beta^{-1}$ is a pseudo-Anosov map.

### 3. Construction of geodesics

In this section, we will construct geodesics of exact distance in the curve graph $\mathcal{C}(S_g)$ in certain conditions. First, we state two criteria that have been used to extend the geodesics from [8].

**Proposition 3.1.** (Ido-Jang-Kobayashi [8] Proposition 4.1) For an integer $n \geq 4$, suppose that $[\alpha_0,\alpha_1,\ldots,\alpha_n]$ is a path in the curve graph $\mathcal{C}(S_g)$ satisfying the following:

1. $[\alpha_0,\cdots,\alpha_{n-2}]$ and $[\alpha_{n-2},\alpha_{n-1},\alpha_n]$ are geodesics in $\mathcal{C}(S_g)$.
2. $\text{diam}_{\mathcal{C}(S_{n-2})}(\pi_{X_{n-2}}(\alpha_{n-4}),\pi_{X_{n-2}}(\alpha_n)) \geq 4n$, where $X_{n-2} = \overline{S - N(\alpha_{n-2})}$.

Then $[\alpha_0,\alpha_1,\cdots,\alpha_n]$ is a geodesic in $\mathcal{C}(S_g)$.

**Proposition 3.2.** (Ido-Jang-Kobayashi [8] Proposition 4.4) For an integer $n \geq 3$, suppose that $[\alpha_0,\alpha_1,\cdots,\alpha_n]$ is a path in the curve graph $\mathcal{C}(S_g)$ satisfying the following:

1. $[\alpha_0,\cdots,\alpha_{n-1}]$ and $[\alpha_{n-2},\alpha_{n-1},\alpha_n]$ are geodesics in $\mathcal{C}(S_g)$.
2. the union $\alpha_{n-2} \cup \alpha_{n-1}$ is nonseparating in $S_g$, and $\text{diam}_{\mathcal{C}(S')}(\pi_{S'}(\alpha_0),\pi_{S'}(\alpha_n)) \geq 2n$, where $S' = \overline{S - N(\alpha_{n-2} \cup \alpha_{n-1})}$.

Then $[\alpha_0,\alpha_1,\cdots,\alpha_n]$ is a geodesic in $\mathcal{C}(S_g)$.

With these two propositions, we will be able to construct the geodesics in the $\mathcal{C}(S_{g \geq 3})$, where $S_{g \geq 3}$ is the boundary of a handlebody. The proof given below is similar to the construction of the geodesics in [8]. The difference is that we also need to take account of the meridians when we choose the curves.
Lemma 3.3. Let $V$ be one handlebody in the standard Heegaard splitting $S^3 = V \cup_3 W$ of $S^3$. For any positive integer $n$, there exists a geodesic $[\alpha_0, \alpha_1, \cdots, \alpha_n]$ in the curve graph $C(S)$ such that $|\alpha_{i-2} \cap \alpha_i| = 1$ for any positive even number $i \leq n$. The curve $\alpha_k$ is a meridian if $k$ is divisible by 4. Moreover, $\alpha_k$ is a meridian if $k$ is odd and $k < n$. If $n$ is odd, then $|\alpha_n \cap \alpha_{n-2}| = 1$.

![Figure 1](image)

**Figure 1.** A geodesic $[\alpha_0, \alpha_1, \alpha_2]$ on the boundary surface of a genus $g \geq 3$ handlebody $V$ with $|\alpha_0 \cap \alpha_2| = 1$; $\alpha_0$ and $\alpha_1$ are meridians of $V$.

Proof. First, let us consider the case when $n$ is even with $n \geq 4$. Let $\alpha_0$, $\alpha_1$ and $\alpha_2$ be nonseparating simple closed curves on $S$ such that $\alpha_1$ is disjoint from $\alpha_0$ and $\alpha_2$ and $|\alpha_0 \cap \alpha_2| = 1$, see the Figure 1. Notice that $\alpha_0$ and $\alpha_1$ are meridians of $V$ and $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length 2 in $C(S)$. Let $X_2 = S - N(\alpha_2)$ be the closure of the complement of regular neighborhood of $\alpha_2$, then one can choose a partial pseudo-Anosov map $\varphi_2 : S \rightarrow S$ such that $\varphi_2$ fixes $\alpha_2$ and $\text{diam}_{C(X_2)}(\pi_{X_2}(\alpha_0), \pi_{X_2}(\varphi_2(\alpha_0))) \geq 4n$. The existence of such partial pseudo-Anosov map is justified by the Theorem 2.5. Furthermore, one can choose a pseudo-Anosov map that can be extended over the handlebody $V$.

Since $|\alpha_0 \cap \alpha_2| = 1$ and $\alpha_0$ is meridian, then the boundary curve $\alpha' = \partial N(\alpha_2 \cup \alpha_0)$ is a separating meridian. The $\alpha'$ separates the handlebody into one solid torus and a handlebody of genus $g - 1$. On the handlebody of genus $g - 1 \geq 2$, there exists a filling pair, as illustrated in the Figure 2. One can choose a filling pair in the positive boundary $S_{g-1}$ such that only one subarc of one meridian intersects $\alpha'$ twice. To make the pair fill the subsurface $X_2$, one can replace the subarc of the meridian with the arc that passes over the $\alpha_2$ once. The resulting new curve is denoted as $\gamma$, then $\partial N(\alpha_2 \cup \gamma)$ is a meridian. Together with the other meridian, they are a filling pair of the subhandlebody $X_2$.

Let the filling pair of meridians of $X_2$ be $\alpha$, $\beta$, and define the map $\varphi_2 = T_{\alpha} \circ T_{\beta}^{-1}$. By Thurston’s construction, $\varphi_2$ is a pseudo-Anosov map, and it can be extended over the handlebody $V$. Iterate it if needed to satisfy $\text{diam}_{C(X_2)}(\pi_{X_2}(\alpha_0), \pi_{X_2}(\varphi_2(\alpha_0))) \geq 4n$.

Both $\alpha_3 = \varphi_2(\alpha_1)$ and $\alpha_4 = \varphi_2(\alpha_0)$ are meridians, and $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length 2 with $|\alpha_2 \cap \alpha_4| = 1$. By the Proposition 3.1, $[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length 4.

The $\alpha_4$ is a nonseparating meridian of $V$, the disk bounded by $\alpha_4$ cuts $V$ into a handlebody of genus $g - 1$, as illustrated in the Figure 3. Again, on the genus $g - 1$
handlebody, there exists a filling pair of meridians, the subsurface $X_4 = S - N(\alpha_4)$ can be obtained from removal of two disks in two disk complements of a filling pair of meridians. Similar as before, one can construct a pseudo-Anosov $\varphi_4$ on $X_4$, and it can be extended over $V$. Let $\alpha_5 = \varphi_4(\alpha_3)$ and $\alpha_6 = \varphi_4(\alpha_2)$, then $[\alpha_0, \alpha_1, \cdots, \alpha_5, \alpha_6]$ is a geodesic of length 6.

Continue in this way, we can construct a geodesic $[\alpha_0, \alpha_1, \cdots, \alpha_n]$ of even length. Assume that $[\alpha_0, \alpha_1, \cdots, \alpha_i]$ is a geodesic with $|\alpha_{i-2} \cap \alpha_i| = 1$ for even $i < n$. Let $X_i = S - N(\alpha_i)$, then we can take a partial pseudo-Anosov map $\varphi_i : S \to S$ such that $\varphi_i$ fixes $\alpha_i$ and $\text{diam}_{C(X_i)}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(\varphi_i(\alpha_{i-2}))) \geq 4n$. The pseudo-Anosov map $\varphi_i$ can be chosen to be able to be extended over the handlebody $V$ as we did before. If $i$ is not divisible by 4, then $\alpha_i$ is not a meridian. The $\alpha_{i-2}$ is a meridian and $|\alpha_i \cap \alpha_{i-2}| = 1$, so the boundary curve $\partial N(\alpha_i \cup \alpha_{i-2})$ is meridian, and it bounds a disk that cuts $V$ into a solid torus and a handlebody of genus $g - 1$. Hence, we end up with the case in the Figure 2. Similarly, the other case is illustrated in the Figure 3.

Denote $\alpha_{i+1} = \varphi_i(\alpha_{i-1})$ and $\alpha_{i+2} = \varphi_i(\alpha_{i-2})$, then $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length 2. Again, by the Proposition 3.1, the extended path $[\alpha_0, \alpha_1, \cdots, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic with $|\alpha_i \cap \alpha_{i+2}| = 1$. The construction yields a geodesic $[\alpha_0, \alpha_1, \cdots, \alpha_n-1, \alpha_n]$ of length $n$ being even. The curves are all meridians except for the ones $\alpha_{4m-2}$, where $m$ is a positive integer. Moreover, $|\alpha_0 \cap \alpha_2| = 1$ and $|\alpha_{i-2} \cap \alpha_i| = 1$ for any positive even number $i \leq n$.

Next, we discuss the geodesics of odd length. Suppose that $n - 1$ is even with $n \geq 3$, let $[\alpha_0, \alpha_1, \cdots, \alpha_{n-1}]$ be a geodesic constructed as the previous case. Since $\alpha_0$, $\alpha_1$ and $\alpha_2$ are nonseparating, then all curves in the geodesic are non-separating.
by construction. By construction, $|\alpha_{n-3} \cap \alpha_{n-1}| = 1$, and $\alpha_{n-2}$ is disjoint from $\alpha_{n-3} \cup \alpha_{n-1}$, then $\alpha_{n-2} \cup \alpha_{n-1}$ is nonseparating. Let $S' = S - N(\alpha_{n-2} \cup \alpha_{n-1})$, the Theorem 2.1 states that the curve graph $C(S')$ has infinite diameter. Then there exists $\gamma'$ in $S'$ with $d_{C(S')} (\gamma', \pi_{S'}(\alpha_0)) > 2n + 2$. Since the genus $g \geq 3$, we can find $\gamma''$ in $S'$ with $d_{C(S')} (\gamma'', \gamma') \leq 2$ and that $\gamma''$ cuts off a pair of pants $P$ with $\partial N(\alpha_{n-2}) \subset \partial P$.

**Figure 3.** Remove two open disks in two distinct disk complements of the filling pair of meridians on $S_{g-1}$ to obtain the subsurface.

**Figure 4.** Construction of $\gamma$, $\gamma''$ and $\alpha_n$. 
There exists a curve (not a meridian) $\alpha_n \in \mathcal{C}(S)$ with $|\alpha_n \cap \alpha_{n-2}| = 1$, $\alpha_n \cap \alpha_{n-1} = \emptyset$ and $\pi_{\mathcal{S}'}(\alpha_n) = \gamma''$. By the triangle inequality, we have
\[
\text{diam}_{\mathcal{C}(S')}(\pi_{\mathcal{S}'}(\alpha_0), \pi_{\mathcal{S}'}(\alpha_n)) = \text{diam}_{\mathcal{C}(S')}(\pi_{\mathcal{S}'}(\alpha_0), \gamma'') \\
\geq \text{diam}_{\mathcal{C}(S')}(\pi_{\mathcal{S}'}(\alpha_0), \gamma') - d_{\mathcal{C}(S')}(\gamma'', \gamma') \\
> (2n + 2) - 2 = 2n.
\]
Since $\alpha_n \cap \alpha_{n-2} = 1$ and $\alpha_n \cap \alpha_{n-1} = \emptyset$, then $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$, see the Figure 4. By the Proposition 3.2, $[\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

4. PROOF OF THEOREM 1.1

Let $S$ be a closed oriented surface, and $a$ is a simple closed curve on $S$. The compression body $S[a]$ is obtained from $S \times [0, 1]$ by attaching a $2$-handle along $a$ onto the boundary $S \times \{1\}$. The disk complex of $S[a]$ is described as follows.

**Proposition 4.1.** (Biringer-Vlamis [1] Proposition 2.5) Suppose that $S$ is a closed, orientable surface and $a$ is a simple closed curve on $S$. If $S$ is a torus or $a$ is separating,
\[
D(S[a]) = \{a\},
\]
while if the genus $g(S) \geq 2$ and $a$ is nonseparating, then
\[
D(S[a]) = \{a\} \cup \{B(a, b) : b \in \mathcal{C}(S), i(a, b) = 1\} \\
= \{a\} \cup \{\partial T : T \subset S, a \text{ punctured torus with } a \subset T\}.
\]

The $B(a, b)$ is the band sum of $a$ and $b$, that is, $B(a, b) = \partial N(a \cup b)$. Let $S^3 = V \cup_S W$ be the standard genus $g \geq 3$ Heegaard splitting. Using the Lemma 1, we can find two meridians in the two handlebodies $V$ and $W$ with exact distance. The proof of the Theorem 1.1 is followed by a sequence of propositions.

**Proposition 4.2.** For any integer $g \geq 3$, there exists a genus $g$ Heegaard splitting $V_0 \cup_S W_0$ with distance $n - 2$, where $n$ is any positive integer divisible by 4. The $V_0$ and $W_0$ are compression bodies and $V_0 \cup_S W_0$ can be embedded in $S^3$.

**Proof.** Suppose a positive integer $n$ is divisible by 4, let $[\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n]$ be a geodesic constructed in the Lemma 1, then $|\alpha_0 \cap \alpha_2| = 1$, $|\alpha_2 \cap \alpha_4| = 1$ and $\alpha_n$ is a meridian in $V$. Let $\beta = \partial N(\alpha_0 \cup \alpha_2)$. Since $|\alpha_0 \cap \alpha_2| = 1$ and $\alpha_2$ is a meridian of the handlebody $W$, then $\beta$ is a meridian of $W$, and it is a meridian of $V$ as well. Similarly, $\gamma = \partial N(\alpha_{n-2} \cup \alpha_n)$ is a meridian of $V$. So we have the following diagram.

\[
\alpha_0 \longrightarrow \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \cdots \longrightarrow \alpha_{n-2} \longrightarrow \alpha_{n-1} \longrightarrow \alpha_n.
\]

Take the trivial compression body $S \times [0, 1]$, where $S$ is the Heegaard splitting surface in the standard Heegaard splitting. Let $W_0 = S[\beta]$ and $V_0 = S[\gamma]$, then $V_0 \subset V$ and $W_0 \subset W$, and $V_0 \cup_S W_0 \subset V \cup_S W$ can be embedded in $S^3$. Note that the geodesic segment $[\alpha_2, \cdots, \alpha_{n-2}]$ has distance $n - 4$ in curve graph $\mathcal{C}(S)$. By the triangle inequality,
\[
d_{\mathcal{C}(S)}(\beta, \gamma) \leq d_{\mathcal{C}(S)}(\beta, \alpha_2) + d_{\mathcal{C}(S)}(\alpha_2, \alpha_{n-2}) + d_{\mathcal{C}(S)}(\alpha_{n-2}, \gamma) = 1 + (n - 4) + 1 = n - 2.
\]

On the other hand, by the Proposition 4.1, $\beta$ is the unique meridian in $W_0$, and $\gamma$ is a unique meridian in $V_0$. Since $d_{\mathcal{C}(S)}(\alpha_0, \beta) = 1$ and $d_{\mathcal{C}(S)}(\gamma, \alpha_n) = 1$, then
\[
d_{\mathcal{C}(S)}(\alpha_0, \alpha_n) \leq d_{\mathcal{C}(S)}(\alpha_0, \beta) + d_{\mathcal{C}(S)}(\beta, \gamma) + d_{\mathcal{C}(S)}(\gamma, \alpha_n) \leq 1 + d_{\mathcal{C}(S)}(\beta, \gamma) + 1.
\]
It follows that
\[
d_{C(S)}(\beta, \gamma) \geq d_{C(S)}(\alpha_0, \alpha_n) - 2 = n - 2.
\]

Then we have \(d_{C(S)}(V_0, W_0) = n - 2\). \(\square\)

**Proposition 4.3.** For any integer \(g \geq 3\), there exists a genus \(g\) Heegaard splitting \(V_0 \cup_S W_0\) with distance \(n\), where \(n\) is any positive integer divisible by 4. The \(V_0\) and \(W_0\) are compression bodies and \(V_0 \cup_S W_0\) can be embedded in \(S^3\).

**Proof.** The geodesic \([\alpha_0, \alpha_1, \alpha_2]\) consists of the \(\alpha_0\) as a meridian of \(V\) and \(\alpha_2\) as a meridian of \(W\). Let \([\beta_0, \beta_1, \beta_2]\) be a geodesic in the curve graph \(C(S)\) such that \(\beta_0 = \alpha_2\) and \(\beta_2 = \alpha_0\), but \(\beta_1 \neq \alpha_1\), see the Figure 5.

\[
\begin{align*}
V & \\
\alpha_0 & \quad \alpha_1 \\
\beta_0 & \quad \beta_1 \\
W & \\
\beta_2 & \quad \alpha_2
\end{align*}
\]

Figure 5. \(S^3 = V \cup_S W\) is the standard Heegaard splitting. \(\beta_0 = \alpha_2\), \(\beta_2 = \alpha_0\) and \(\beta_1 \neq \alpha_1\).

By the same construction, one will be able to extend the geodesic \([\beta_0, \beta_1, \beta_2]\) to \([\beta_0, \beta_1, \beta_2, \beta_3, \beta_4]\) with \(|\beta_0 \cap \beta_2| = 1\) and \(|\beta_2 \cap \beta_4| = 1\). Since \(\alpha_0 = \beta_2\) and \(\alpha_2 = \beta_0\), then \(\partial N(\alpha_0 \cup \alpha_2) = \partial N(\beta_0 \cup \beta_2)\). Choose the geodesic \([\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n]\) as before with \(n\) divisible by 4, and let \(\beta = \partial N(\beta_2 \cup \beta_4)\), \(\gamma = \partial N(\alpha_{n-2} \cup \alpha_n)\), then
Hence, \( d_{C(S)}(\beta, \gamma) \leq d_{C(S)}(\beta, \beta_2 = \alpha_0) + d_{C(S)}(\alpha_0, \alpha_{n-2}) + d_{C(S)}(\alpha_{n-2}, \gamma) = 1 + (n - 2) + 1 = n. \)

Let \( W_0 = S[\beta] \) and \( V_0 = S[\gamma] \). By the Proposition 4.1, \( \beta \) is the unique meridian in \( W_0 \), and \( \gamma \) is the unique meridian in \( V_0 \), we have \( d_{C(S)}(\beta_4, \beta) = 1 \) and \( d_{C(S)}(\gamma, \alpha_n) = 1 \). It follows that

\[
d_{C(S)}(\beta_4, \alpha_n) \leq d_{C(S)}(\beta_4, \beta) + d_{C(S)}(\beta, \gamma) + d_{C(S)}(\gamma, \alpha_n) \leq 1 + d_{C(S)}(\beta, \gamma) + 1.
\]

Rearrange the inequality, we obtain

\[
d_{C(S)}(\beta, \gamma) \geq d_{C(S)}(\beta_4, \alpha_n) - 2 = (n + 2) - 2 = n.
\]

Then we have \( d_{C(S)}(V_0, W_0) = n. \)

To sum up, the previous two propositions prove the Theorem 1.1 for all even natural numbers. In the following, we want to show it also holds for the odd natural number \( n \geq 3 \).

**Proposition 4.4.** For any integer \( g \geq 3 \), there exists a genus \( g \) Heegaard splitting \( V_0 \cup_S W_0 \) with distance \( n - 1 \), where \( n \) is any positive integer divisible by 4. The \( V_0 \) and \( W_0 \) are compression bodies and \( V_0 \cup_S W_0 \) can be embedded in \( S^3 \).

**Proof.** Using the Lemma 1, we can find a geodesic \( [\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}] \) such that the geodesic segment \( [\alpha_0, \alpha_1, \ldots, \alpha_n] \) is of length \( n \) divisible by 4, with \( |\alpha_0 \cap \alpha_2| = 1 \) and \( |\alpha_{n-1} \cap \alpha_{n+1}| = 1 \). Let \( \beta = \partial N(\alpha_0 \cup \alpha_2) \), and \( \gamma = \partial N(\alpha_{n-1} \cup \alpha_{n+1}) \), then \( \beta \) is the meridian of \( W \) and \( \gamma \) is a meridian of \( V \). The reason is that \( \alpha_2 \) is a meridian of \( W \) and \( \alpha_{n+1} \) is a meridian of \( V \). We can construct the compression bodies \( W_0 = S[\beta] \) and \( V_0 = S[\gamma] \). The diagram illustrates the path

\[
\begin{array}{cccccccccc}
\alpha_0 & \beta & \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-2} & \Gamma & \alpha_{n-1} & \alpha_{n+1}.
\end{array}
\]

Rearrange the inequality, we obtain

\[
d_{C(S)}(\beta, \gamma) \geq d_{C(S)}(\beta_4, \alpha_n) - 2 = (n + 2) - 2 = n.
\]

Then we have \( d_{C(S)}(V_0, W_0) = n - 1. \)

**Proposition 4.5.** For any integer \( g \geq 3 \), there exists a genus \( g \) Heegaard splitting \( V_0 \cup_S W_0 \) with distance \( n + 1 \), where \( n \) is any positive integer divisible by 4. The \( V_0 \) and \( W_0 \) are compression bodies and \( V_0 \cup S W_0 \) can be embedded in \( S^3 \).
Proof. The proof is similar as the even case. Take the geodesic in the above proposition and extend the geodesic on the other end by distance 2. The following path between $\beta$ and $\gamma$ realizes the distance.

$$\begin{align*}
\alpha_0 & \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \ldots \rightarrow \alpha_{n-1} \rightarrow \alpha_n \rightarrow \alpha_{n+1}. \\
\beta_4 & \rightarrow \beta_3 \rightarrow \beta_2 \rightarrow \beta_1 \rightarrow \beta_0
\end{align*}$$

Similarly, we construct two compression bodies $V_0 = S[\gamma]$ and $W_0 = S[\beta]$. Then, $d_{C(S)}(V_0, W_0) = 2 + (n - 1) = n + 1$. \hfill $\square$

The remaining cases that $n = 0$ and $n = 1$ are obvious. One can take two separating meridians in $V$ and $W$ that are same or disjoint. The resulting compression bodies has distance 0 or 1. With the preceding propositions, we complete the proof of Theorem 1.1.

5. **Proof of Theorem 1.2**

In this section, we utilize the Theorem 1.1 to show the exact distance of the link complement with a genus $g \geq 3$ Heegaard splitting. Let us recall Minsky, Moriah and Schleimer’s work on the high distance knot.

**Theorem 5.1.** (Minsky-Moriah-Schleimer [19] Theorem 3.1) For any pair of integers $g > 1$ and $n > 0$, there is knot $K \subset S^3$ and a genus $g$ splitting $S \subset E(K)$ having distance greater than $n$.

In a nutshell, one can take the standard genus $g$ Heegaard splitting $S^3 = V \cup_S W$, and let $D \subset V$ be a disk that cuts $V$ into a solid torus $X$ and a handlebody $Y$ of genus $g - 1$. Let $K_0$ be the core of $X$, then the complement $V_0 = V - N(K_0)$ is a compression body. It follows that $E(K_0) = V_0 \cup_S W$. The strategy is using a particular train track to construct a pseudo-Anosov map $\Phi$ that can be extended over $V$. Denote $V_n = \Phi^n(V_0)$ and $K_n = \Phi^n(K_0) \subset V \subset S^3$, then $V_n \cup_S W$ is a genus $g$ Heegaard splitting of the knot exterior $E(K_n) = S^3 - N(K_n)$. Then,

$$d_{C(S)}(D_{V_n}, D_W) \rightarrow \infty,$$

as $n \rightarrow \infty$.

The iteration of $\Phi$ is a pseudo-Anosov map that can be extended over handlebody $V$. For any number $M$, there is a pseudo-Anosov map $\Phi$ such that $\Phi(V_0) \cup_S W$ is a Heegaard splitting of some knot exterior $E(K)$, and the distance $d_{C(S)}(\Phi(V_0), W) > M$.

Our main goal in this section is to prove the Theorem 1.2.

**Proof of Theorem 1.2.** For $n \geq 2$, using the Theorem 1.1, one can find the disk $D_1$ in $V$ and disk $D_2$ in $W$ such that the meridians $\partial D_1$ and $\partial D_2$ realize the distance $n$. By the construction, we can let $V_0 = S[\partial D_1]$ and $W_0 = S[\partial D_2]$, then the Heegaard distance $d_{C(S)}(V_0, W_0) = n$. The two compression bodies are illustrated in the Figure 6. Note that the figures are only for illustration, as $\partial D_1$ and $\partial D_2$ are supposed to be intersecting. In $V_0, W_0$, the meridians $\partial D_1$ and $\partial D_2$ divide the surface $S$ into two subsurfaces, one of which is a one-holed torus. Denote them as $S_1, S_2$ and $S_3, S_4$, and we assume $S_1$ and $S_3$ are one-holed tori, as shown in the Figures 7 and 8. In addition,
we denote the corresponding negative boundaries as \( \{F_1, F_2\} \) and \( \{F_3, F_4\} \). As the genus \( g(S) \geq 3 \), then the genus \( g(F_2) \geq 2 \) and \( g(F_4) \geq 2 \). The proof is analogous to that of the Proposition 3.1 in [20] and Proposition 5.1 in [8].

The disk \( D_1 \) cuts \( V_0 \) into \( F_1 \times [0, 1] \) and \( F_2 \times [0, 1] \), and \( D_2 \) cuts \( W_0 \) into \( F_3 \times [0, 1] \) and \( F_4 \times [0, 1] \). In the compression body \( V_0 \), we identify \( F_i = F_i \times \{0\} \) for \( i = 1, 2 \). Let \( f_{F_i} = S_i \cup D_1 \to F_i \) be the homeomorphism such that \( f_{F_i}(x \times \{0\}) = x \times \{1\} \). The homeomorphism \( f_{F_i} \) induces an isomorphism on the curve graphs, that is,

\[
d_{C(F_i)}(f_{F_i}(\alpha), f_{F_i}(\beta)) = d_{C(S_i \cup D_1)}(\alpha, \beta),
\]

for any two essential simple closed curves \( \alpha \) and \( \beta \) on \( S_i \cup D_1 \).

Let \( l : S_i \to S_i \cup D_1 \) be the inclusion map, then

\[
d_{C(S_i \cup D_1)}(l(\alpha), l(\beta)) \leq d_{C(S_i)}(\alpha, \beta),
\]

for any two essential simple closed curves \( \alpha \) and \( \beta \) on \( S_i \). Define

\[
P_{F_i} = f_{F_i} \circ l \circ \pi_S : S \to F_i
\]
to be a map either between surfaces or the induced map between curve graphs, where \( \pi_S \) is the subsurface projection. Since \( d_{C(S)}(\partial D_1, \partial D_2) = n \geq 2 \), then

\[
diam_{C(F_i)}(P_{F_i}(\partial D_2)) \leq diam_{C(S)}(\pi_S(\partial D_2)) \leq 2.
\]

As a 3-manifold embedded in \( S^3 \), \( V_0 \cup W_0 \) has two non-torus boundary components \( F_2 \) and \( F_4 \). The complement of \( V_0 \cup W_0 \) in \( S^3 \) with the boundary \( F_2 \) is a handlebody \( X_{g-1} \) of genus \( g-1 \). It follows that \( V_0 \cup W_0 \subset S^3 - X_{g-1} \), where \( S^3 - X_{g-1} \) is a handlebody of genus \( g-1 \). It induces an inclusion of the disk graph, \( \mathcal{D}(V_0 \cup W_0, F_2) \subset \mathcal{D}(S^3 - X_{g-1}, F_2) \).
Figure 7. The meridian $\partial D_1$ cuts the positive (outer) boundary $S$ of the compression body $V_0$ into two subsurface $S_1$ and $S_2$. The gluing map is the standard identification of the boundary of handlebody $X_{g-1}$ onto $F_2$.

Inside the handlebody $X_{g-1}$, we take a core as a trivial knot $K_0$. Let $\tilde{V}_0 = X_{g-1} - K_0$. Note that $P_{F_2}(\partial D_2)$ has diameter 2 in the curve graph $C(F_2)$, then

$$d_{C(F_2)}(P_{F_2}(\partial D_2), D(S^3 - X_{g-1}, F_2)) \leq N_1$$

for some constant $N_1$. Let $M$ be the upper bound in the Bounded Geodesic Image Theorem 2.3. By the Theorem 5.1, there exists a pseudo-Anosov map $\Phi : F_2 \rightarrow F_2$ such that $\Phi$ can be extended over the handlebody $X_{g-1}$, and

$$d_{C(F_2)}(D(\Phi(\tilde{V}_0), F_2), D(S^3 - X_{g-1}, F_2)) > M + N_1 + 2.$$

By the triangle inequality,

$$d_{C(F_2)}(D(\Phi(\tilde{V}_0), F_2), P_{F_2}(\partial D_2)) > M.$$

**Claim 1.** The distance $d_{C(S)}(V_{F_2}, W_0) = n$, where $V_{F_2} = V_0 \cup_{F_2} \Phi(\tilde{V}_0)$.

**Proof.** Suppose not, then the distance $d_{C(S)}(V_{F_2}, W_0) = k < n$. Since $W_0$ contains a unique disk $D_2$, then there is an essential disk $D \neq D_1$ in $V_{F_2}$ such that $d_{C(S)}(\partial D, \partial D_2) = k < n$. It means there is a geodesic $\{\alpha_0 = \partial D, \alpha_1, \cdots, \alpha_k = \partial D_2\}$ in curve graph $C(S)$. Then for each $\alpha_i$, we have $\alpha_i \cap \partial D_1 \neq \emptyset$, for any $1 \leq i \leq k$. Suppose there is some $\alpha_i$ such that $\alpha_i \cap \partial D_1 = \emptyset$. It follows that

$$n = d_{C(S)}(\partial D_1, \partial D_2)$$

$$\leq d_{C(S)}(\partial D_1, \alpha_i) + d_{C(S)}(\alpha_i, \partial D_2)$$

$$\leq 1 + (k - i) \leq k < n,$$
which is a contradiction. Moreover, \( \alpha_0 = \partial D \) is not in \( S_1 \). Otherwise, \( D \subset F_1 \times [0,1] \), then \( D \) is inessential. Hence, we show that \( \alpha_i \) cuts \( S_2 \) for all \( 0 \leq i \leq k \).

By the Bounded Geodesic Image Theorem 2.3, we have \( d_{C(S_2)}(\pi_{S_2}(\partial D), \pi_{S_2}(\partial D')) \leq M \). It implies that \( d_{C(F_2)}(\pi_{S_2}(\partial D), \pi_{S_2}(\partial D')) \leq M \) and \( d_{C(F_2)}(P_{F_2}(\partial D), P_{F_2}(\partial D')) \leq M \). Assume that \( D \) and \( D' \) intersect minimally. By the innermost disk argument, we can assume that \( D \cap D' \) has no loop components.

**Case i:** \( |D \cap D'| = 0 \). Since \( D \) is not isotopic to \( D' \), then \( P_{F_2}(\partial D) \) bounds an essential disk in the \( \Phi(\tilde{V}_0) \), then

\[
d_{C(F_2)}(P_{F_2}(\partial D), P_{F_2}(\partial D')) \leq M,
\]

which means that

\[
d_{C(F_2)}(\Phi(\tilde{V}_0), F_2), P_{F_2}(\partial D')) \leq M.
\]

It is contradictory to the choice of \( \Phi(\tilde{V}_0) \).

**Case ii:** \( |D \cap D'| \neq 0 \). Let \( a \) be an outermost arc of \( D \cap D' \) on the disk \( D \). It implies that \( a \) and a subarc \( b \subset \partial D_1 \), bounds another disk \( D' \) such that \( D' \cap D_1 = a \).

As we know that \( D_1 \) cuts \( V_{F_2} \) into \( \Phi(\tilde{V}_0) \) and \( F_1 \times [0,1] \). Note that \( D' \subset \Phi(\tilde{V}_0) \), then an essential simple closed curve in \( P_{F_2}(\partial D) \) bounds an essential disk in \( \Phi(\tilde{V}_0) \). By the case i, it is impossible.

Next, let’s look at the compression body \( W_0 \), which contains the unique essential disk \( D_2 \). The \( D_2 \) cuts \( W_0 \) into two submanifolds \( F_3 \times [0,1] \) and \( F_4 \times [0,1] \). \( \partial D_2 \) cuts \( S \) into \( S_3 \) and \( S_4 \), where \( S_3 \) is a one-holed torus. Identify \( F_1 = F_1 \times \{ 1 \} \) and \( S_i \cup D_2 = F_i \times \{ 0 \} \), for \( i = 3, 4 \). Similarly, let \( f_{F_i} : S_i \cup D_2 \to F_i \) be the homeomorphism such that \( f_{F_i}(x \times \{ 0 \}) = x \times \{ 1 \} \). For any two essential simple closed curves \( \alpha \) and \( \beta \) on \( S_i \),

\[
d_{C(S_i \cup D_2)}(f_{F_i}(\alpha), f_{F_i}(\beta)) = d_{C(S_i \cup D_2)}(\alpha, \beta).
\]

The isomorphism between the curve graphs is also denoted as \( f_{F_i} \). Let \( l : S_i \to S_i \cup D_2 \) be the inclusion map, then for any two essential simple closed curves \( \alpha \) and \( \beta \) on \( S_i \),

\[
d_{C(S_i \cup D_2)}(l(\alpha), l(\beta)) \leq d_{S_i}(\alpha, \beta).
\]

Let \( \pi_{S_i} \) be the subsurface projection, we define \( P_{F_i} = f_{F_i} \circ l \circ \pi_{S_i} : S \to F_i \).

The Heegaard surface \( S \) cuts \( S^3 \) into two handlebodies \( V \) and \( W \). Suppose that \( V_{F_2} \subset V \), then \( V_{F_2} \) has two torus boundary components. One is the \( F_1 \), and the other comes from the boundary of the regular neighborhood of the knot \( \Phi(K_0) \). Both torus boundary components are incompressible in \( V_{F_2} \). Since \( d_{C(S)}(V_{F_2}, W_0) = n \geq 2 \), and \( W_0 \) has the unique disk \( D_2 \), then the meridian \( \partial D_2 = \partial S_4 \) is disk-busting in \( S \). Note that \( V_{F_2} \) is not an I-bundle over some compact surface. Using the Theorem 2.4, we have \( \text{diam}_{C(S)}(\pi_{S_4}(C(V_{F_2}, S))) \leq 12 \). It follows that

\[
\text{diam}_{C(S)}(P_{F_2}(C(V_{F_2}, S))) \leq \text{diam}_{C(S)}(\pi_{S_4}(C(V_{F_2}, S))) \leq 12.
\]

The surface \( F_4 \) bounds two handlebodies in \( S^3 \). Denote the one that does not contain the 3-manifold \( V_{F_2} \cup S_1 \) as \( Y_{g-1} \), then \( V_{F_2} \cup S_1 \) lies in \( S^3 - Y_{g-1} \). Since \( P_{F_2}(C(V_{F_2}, S)) \) is bounded, there exists a constant \( N_3 \) such that

\[
d_{C(F_4)}(P_{F_4}(C(V_{F_2}, S))), D(S^3 - Y_{g-1}, F_4)) \leq N_2.
\]

Let a trivial knot \( K'_0 \) be a core of the handlebody \( Y_{g-1} \) and \( \tilde{W}_0 = Y_{g-1} - K'_0 \). Once again, by the Theorem 5.1, for the constant \( M + N_2 + 12 \), there is a pseudo-Anosov map \( \Psi : F_4 \to F_4 \) that can be extended over the handlebody \( Y_{g-1} \), and it satisfies

\[
d_{C(F_4)}(C(\Psi(\tilde{W}_0), F_4)), D(S^3 - Y_{g-1}, F_4)) > M + N_2 + 12.
\]
Using the triangle inequality, we have
\[ d_{C(F_4)}(D(\Psi(\tilde{W}_0), F_4), P_{F_4}(D(V_{F_2}, S))) > M. \]

**Claim 2.** The distance \( d_{C(S)}(V_{F_2}, W_{F_4}) = n \), where \( W_{F_4} = W_0 \cup F_4 \Psi(\tilde{W}_0) \).

**Proof.** Suppose not, then the distance \( d_{C(S)}(V_{F_2}, W_{F_4}) = k < n \). It means there is a geodesic \( \{ \alpha_0 = \partial E, \alpha_1, \ldots, \alpha_k = \partial E' \} \) in curve graph \( C(S) \), where \( E \) is an essential disk in \( V_{F_2} \) and \( E' \) is an essential disk in \( W_{F_4} \). Note that \( D_2 \) is an essential disk in \( W_{F_4} \). Then for each \( \alpha_i \), we have \( \alpha_i \cap \partial D_2 \neq \emptyset \), for any \( 0 \leq i \leq k - 1 \). Suppose there is some \( \alpha_i \) such that \( \alpha_i \cap \partial D_2 = \emptyset \). It follows that
\[ n = d_{C(S)}(\partial D_1, \partial D_2) \leq d_{C(S)}(\partial E, \partial D_2) \leq d_{C(S)}(\partial E, \alpha_i) + d_{C(S)}(\alpha_i, \partial D_2) \leq i + 1 \leq k < n, \]
which is a contradiction. Moreover, \( \partial E' \) is not in \( S_3 \). Otherwise, \( E' \subset F_3 \times [0, 1] \), then \( E' \) is inessential. Hence, we show that all \( \alpha_i \) cut \( S_4 \) for \( 0 \leq i \leq k \). By the Bounded Geodesic Image Theorem 2.3, we have \( d_{C(S_s)}(\pi_{S_4}(\partial E), \pi_{S_4}(\partial E')) \leq M \). It yields that \( d_{C(S_{s\cup D_2})}(\pi_{S_4}(\partial E), \pi_{S_4}(\partial E')) \leq M \) and \( d_{C(F_4)}(P_{F_4}(\partial E), P_{F_4}(\partial E')) \leq M \). Assume that \( E' \) and \( D_2 \) intersect minimally. By the innermost disk argument, we can assume that \( E' \cap D_2 \) has no loop components.

**Case i:** \( |E' \cap D_2| = 0 \). Since \( E' \) is not isotopic to \( D_2 \) and \( \partial E' \) is not in \( S_3 \), then \( P_{F_4}(\partial E') = \partial E' \in D(\Psi(\tilde{W}_0), F_4) \). We know that \( \partial E \in D(V_{F_2}, S) \), then the inequality
\[ d_{C(F_4)}(D(\Psi(\tilde{W}_0), F_4), P_{F_4}(D(V_{F_2}, S))) \leq M, \]

**FIGURE 8.** The meridian \( \partial D_2 \) cuts the positive (outer) boundary \( S \) into two subsurface \( S_3 \) and \( S_4 \). The gluing map is the standard identification of the boundary of handlebody \( Y_{g-1} \) onto \( F_4 \).
follows from \( d_{C(F_4)}(P_{F_4}(\partial E), P_{F_4}(\partial E')) \leq M \). The inequality contradicts to the choice of \( \Psi(\tilde{W}_0) \).

**Case ii:** \( |E' \cap D_2| \neq 0 \). Let \( a \) be an outermost arc of \( E' \cap D_2 \) on the disk \( E' \). It implies that \( a \) and a subarc \( b \subset \partial D_2 \), bounds another disk \( D'' \) such that \( D'' \cap D_2 = a \). As we know that \( D_2 \) cuts \( W_{F_4} \) into \( \Psi(\tilde{W}_0) \) and \( F_3 \times [0,1] \). Note that \( D'' \subset \Psi(\tilde{W}_0) \), then an essential simple closed curve in \( P_{F_4}(\partial E') \) bounds an essential disk in \( \Psi(W_0) \). Again, it is impossible by the **Case i**. □

Hence, the distance \( d_{C(S^3)}(V_{F_2}, W_{F_4}) = n \), where \( V_{F_2} \cup S W_{F_4} = S^3 - N(K) \), and \( K \) is a link of four components. This completes the proof of the Theorem 1.2. □

**Remark 5.2.** More link components can be added in the \( \Psi(\tilde{W}_0) \), and it does not change the distance.

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