Emergent Discrete Space in a Generic Lifshitz Model

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Abstract
We have considered the possibility of Spontaneous Symmetry Breaking in momentum space in a generic Lifshitz scalar model, which is a non-relativistic model with higher spatial derivative terms. We have explicitly shown that a spatially varying ground state, (termed as a flowing ground state), minimizes the energy as compared to a constant valued condensate. We have studied stability property as well as the fluctuation spectra above the flowing ground state. The most interesting observation is that this flowing ground state requires a discretization of space coordinate.
In recent years there has been a paradigm shift in the way of thinking of High Energy physics community. We are prepared even to give up Lorentz invariance to construct a UV complete quantum theory of Gravity as long as it does not contradict observations. There are indications from diverse areas that Lorentz invariance possibly breaks down at length scales of the order of Planck length. In this perspective Horava has proposed a conventional field theory for gravity \cite{1} where spatial higher derivative terms make the theory UV complete. Indeed, it was known for long time that higher derivative gravity has better high energy behavior but covariant higher derivative terms necessarily introduce ghost excitations \cite{2}. The novelty of Horava’s formulation \cite{1} is that higher time derivatives are simply ignored thereby avoiding ghosts. A precursor to this type of non-relativistic theory - Lifshitz model of scalars \cite{3} - is quite well known in Condensed Matter physics. In the present paper we plan to study an interesting aspect of higher derivative generic Lifshitz models which have so far been overlooked. The higher derivative terms can induce an instability leading to phase transition and generating a space dependent ground state. This is applicable in the Horava-Lifshitz theory of gravity as suggested earlier by us \cite{4}. 

Furthermore the ground state turns out to be discrete. This can be relevant in any Quantum Gravity theory because it is generally expected that Quantum Gravity resides in a qualitatively different, possibly discrete space (or spacetime). (For a recent study on quantum field theory in discrete space see for example \cite{5}).

The above scenario might be compared to the celebrated Landau theory \cite{6} of liquid solid phase transition where Spontaneous Symmetry Breaking (SSB) leads to a less symmetric state, the solid, from a more symmetric state, the liquid. A similar thing happens here: SSB generates a spatially inhomogeneous less symmetric vacuum condensate having a discrete nature. The higher derivative terms, a signature of the Lifshitz scalar model, is essential in inducing the inhomogeneous condensate. In a series of works, Alexander and Mctague \cite{7} have shown in an essentially model independent way how crystalline lattice structure emerges from liquid. Rabinovici et.al, have exploited these ideas in the context of String
Theory compactification \[3\]. Very recently somewhat similar ideas have been suggested by Wilczek and by Shapere and Wilczek in \[9\].

Let us briefly recall SSB in a conventional scalar theory. The Lagrangian

\[ L = \frac{(\dot{\phi})^2}{2} - (a\frac{(\phi')^2}{2} + c\phi^2 + \lambda\phi^4), \]

leads to the energy,

\[ E = \int dx \left( \frac{(\dot{\phi})^2}{2} + (a\frac{(\phi')^2}{2} + c\phi^2 + \lambda\phi^4) \right). \]

The minimum energy ground state will obviously correspond to a spacetime constant \( \phi \) obtained from the solution of \( (\partial E)/(\partial \phi) = 0 \) yielding \( \phi_S = 0 \), \( \phi_{BS} = \pm \sqrt{(-c)/2\lambda} \) with the corresponding energies \( E_S = 0 \), \( E_{BS} = -c^2/(8\lambda) \). \( S, BS \) stand for Symmetric and Broken Symmetric phases respectively. For the ground state to exist \( c < 0 \), \( \lambda > 0 \) so that \( E_{BS} < E_S \) indicating that the symmetry broken phase has a lower energy. Incidentally, the symmetry in question that is broken is the \( \phi \rightarrow -\phi \) reflection symmetry that either of the ground states \( \phi_{BS} \) fail to preserve.

Several well known features are worth pointing out for contrasting and comparing with the SSB phenomenon we are going to present later. (i) For the Lorentz invariant scalar theory one needs to have \( a = 1 \) and for a conventional massive theory \( c > 1 \). However for SSB to occur in the relativistic theory with positive \( \lambda \) we need \( c < 0 \). But this not a problem since after shifting \( \phi \) appropriately for one of the BS ground states the resulting scalar gets the correct sign mass term. (ii) Both \( \phi_S, \phi_{BS} \) are solutions of the equation of motion. (ii) From a momentum space point of view with \( \phi(x) = \int dk \varphi(k)exp(ikx) \) the S and BS ground states with constant \( \phi \) indicate that \( \varphi(k) \) has support at \( k = 0 \) only that is \( \varphi(k) \sim \delta(k) \). This suggests that, to construct a theory with a variable ground state \( \phi(x) \) we must look for a model where the Fourier transform \( \varphi(k) \) has a support at some non-zero momentum \( k = \bar{k} \).

Keeping in mind the structure of \( \phi \)-terms in \[2\], for SSB to occur in momentum space we will need at least a fourth order (space) derivative term that is quadratic in \( \phi \). In the rest of the paper we have precisely constructed and studied such a model in the context of Lifshitz
We emphasize that our present Lifshitz model can act as a toy model for Horava theory of gravity. After this brief recapitulation of SSB in conventional scalar \( \phi \) let us move to the arena of Lifshitz scalar field theory. We posit a generic Lifshitz Lagrangian as,
\[
L_{\text{Lif}} = \frac{\dot{\phi}^2}{2} - \left( a \frac{\phi'}{2} + b \frac{\phi''}{2} + c \frac{\phi^2}{2} + \lambda \frac{\phi^4}{2} \right),
\]
where the \( b \)-term represents a higher spatial derivative term. Clearly we have sacrificed Lorentz invariance. Again we look for the ground state. The static energy is
\[
E_{\text{Lif}} = \int dx \left( a \frac{\phi'}{2} + b \frac{\phi''}{2} + c \frac{\phi^2}{2} + \lambda \frac{\phi^4}{2} \right).
\]
Once again a constant \( \phi = \tilde{\phi} \) can be a possible ground state with
\[
\tilde{E} = \int dx \left( c \frac{\phi^2}{2} + \lambda \frac{\phi^4}{2} \right).
\]
Minimizing \( \tilde{E} \) leads to a normal phase with \( \tilde{\phi}_S = 0, \tilde{E}_S = 0 \) or a broken symmetry phase \( \tilde{\phi}_{BS} = \sqrt{-c/2\lambda}, \tilde{E}_{BS} = -c^2/(8\lambda) \). Since \( \lambda \) has to be positive for the ground state to exist \( c < 0 \) for the broken symmetry phase. As far as constant-\( \phi \) ground state is concerned this is same as the analysis below [2] with the \( b \)-term having no effect but indeed, this is not what we are after.

Now we come to the more interesting possibility, that of a space dependent \( \phi_{BS}(x) \) ground state. We introduce a Fourier transform: \( \phi(x) = \int dk \varphi(k) e^{i k x} \). The derivative part of the energy in momentum space gives
\[
E_{\text{Lif}}(\text{derivative}) = \int dk \left( a \frac{k^2}{2} + b \frac{k^4}{2} \right) \varphi(k) \varphi(-k) = 2 \int dk \left( a \frac{k^2}{2} + b \frac{k^4}{2} \right) (\varphi(k))^2,
\]
where in the last step we have assumed \( \varphi(k) = \varphi(-k) \) mainly for the sake of convenience. Clearly for ground state to exist \( b > 0 \) and minimizing with respect to \( k \) for constant \( \varphi \) we find
\[
k_N = 0 \ for \ a > 0, \ b > 0, \ \tilde{E}(k_N) = \tilde{E}_S = -\frac{c^2}{8\lambda}.
\]
\[ k_{BS} \equiv \bar{k} = \sqrt{-\frac{a}{2b}} \text{ for } a < 0, \ b > 0, \]  

which in turn lead to \( \varphi_S(k) = \bar{\varphi} \delta(k) \) and \( \varphi_{BS}(k) = \bar{\varphi} \delta(k - \bar{k}) \). These indicate a constant \( \phi_S = \bar{\varphi} \) in the first case (that we have already studied above) and a spatially varying \( \phi_{BS}(x) = \bar{\varphi} \exp(i\bar{k}x) \) in the second case which is of interest to us. Since we have already assumed \( \varphi(k) = \varphi(-k) \), let us consider the broken symmetry condensate to be,

\[ \phi_{BS}(x) = \bar{\varphi} \cos(\bar{k}x). \]  

We consider \( \phi_{BS}(x) \) to be a candidate solution for the ground state in the broken symmetry phase of the Lifshitz model. We still need to find \( \bar{\varphi} \). We compute this by putting back \( \phi_{BS}(x) \) of (8) in to the energy (4) and minimizing it. The energy becomes,

\[ \bar{E} = \int dx \frac{1}{2} \left[ (ak^2 + b\bar{k}^4 + c)\phi^2 \cos^2(\bar{k}x) + \lambda(\bar{\varphi})^4 \cos^4(\bar{k}x) \right] \]

\[ = \int dx \left[ (-\frac{a^2}{8b} + \frac{c}{2})\phi^2 \cos^2(\bar{k}x) + \frac{\lambda}{2}(\bar{\varphi})^4 \cos^4(\bar{k}x) \right]. \]  

We consider an approximation \( \cos^2(\bar{k}x) \approx \cos^4(\bar{k}x) \) which is reasonable for small \( \bar{k}x \) and obtain,

\[ \bar{E} = \int dx \left[ \left(-\frac{a^2}{8b} + \frac{c}{2}\right)\phi^2 + \frac{\lambda}{2}(\bar{\varphi})^4 \right] \cos^2(\bar{k}x). \]  

Minimizing the functional with respect to \( \bar{\varphi} \)

\[ (c - \frac{a^2}{4b})\bar{\varphi} + 2\lambda\bar{\varphi}^3 = 0 \]  

the solutions of the resulting equation are

\[ \bar{\varphi}_S = 0, \ \bar{\varphi}_{BS} = \sqrt{\frac{1}{2\lambda} \left( \frac{a^2}{4b} - c \right)}. \]  

For the non-zero solution, since \( \lambda > 0 \), the parameters have to satisfy the inequality \( \frac{a^2}{4b} - c > 0 \).

Note that, although not essential, we can restrict to \( c < 0 \) to include SSB with a constant:

\[ \text{Explicitly } \int_{-L}^{L} \cos^2(\bar{k}x)dx = L + \frac{\sin(2\bar{k}L)}{2\bar{k}}, \ \int_{-L}^{L} \cos^4(\bar{k}x)dx = \frac{1}{4}(3L + 2\frac{\sin(2\bar{k}L)}{\bar{k}} + \frac{\sin(4\bar{k}L)}{4\bar{k}}). \]  

Hence to \( O(\bar{k}L) \) the integrals are equal to \( 2L \).
\( \phi \) ground state (see below \( \square \)), \( \tilde{\varphi}_{BS} = \sqrt{-c/2\lambda} \), \( \tilde{E}_{BS} = -c^2/(8\lambda) \), for which the inequality is automatically satisfied. Thus, for the Lifshitz scalar a possible broken symmetry space dependent solution is

\[
\phi_{BS}(x) = \sqrt{\frac{1}{2\lambda}} \left( \frac{a^2}{4b} - c \right) \cos \left( \sqrt{\frac{-a}{2b}} x \right).
\]

This is our principal result. Since the state has a non-zero momentum \( \bar{k} \) it is termed as a flowing state following \([10]\).

Before proceeding further let us summarize. For the generic Lifshitz model we can have three possible ground states: the symmetric \( \phi = 0 \) phase, the broken symmetry constant \( \phi \) phase and finally the broken symmetry variable \( \phi \) phase. We are interested in the last one.

In rest of the paper we establish the claim of \( \phi_{BS}(x) \) being a viable ground state by ensuring three crucial points: (i) \( \tilde{\varphi}_{BS}(x) \) is a solution of the equation of motion. This condition leads to the space discretization. (ii) Energy of the space dependent ground state \( \phi_{BS} \) can be lower than the energy of the constant \( \phi_{S} \) ground state otherwise the \( \phi_{BS} \) ground state will not be selected by the system. (iii) The ground state with non-zero momentum \( \bar{k} \) will be stable against decay into small fluctuations.

(i) The equation of motion is derived as,

\[
\ddot{\phi}(x, t) - a\phi''(x, t) + b\phi''''(x, t) + c\phi(x, t) + 2\lambda\phi^3(x, t) = 0.
\]

Substitution of \( \phi_{BS}(x) \) from \( \square \) in \( \square \) yields

\[
\frac{1}{\sqrt{2\lambda}} \left( \frac{a^2}{4b} - c \right)^{3/2} \cos(\bar{k}x) \left( \cos^2(\bar{k}x) - 1 \right) = 0.
\]

Since the other factors are non-vanishing, (note that \( \cos(\bar{k}x) = 0 \rightarrow \phi = 0 \) state), a possible condition emerges on the space coordinate itself:

\[
\cos^2(\bar{k}x) - 1 = 0 \rightarrow \cos(\bar{k}x) = \pm 1, \quad x_n = \pi n = \pi \sqrt{\frac{2b}{-a}} n, \quad n = 0, 1, 2, ...
\]

This shows that the discretization of space comes about quite naturally with the lattice spacing being \( \Delta x = \pi/\bar{k} = \pi \sqrt{2b/(-a)} \). This is the most important outcome of our result
the variable condensate. Clearly a necessary condition for this to occur is the presence of the higher derivative \( b \)-term in (3). Hence at length scales \( \sim (\bar{k})^{-1} \) the ground state will consist of alternate values of \( \pm \bar{\varphi} \) at the discrete points \( x_n \). However at larger length (or low energy) the ground state will have as a smoother periodic behavior. In the Quantum Gravity context \( (\bar{k})^{-1} \) can be identified as the Planck length.

(ii) The energy of the condensate ground state is now computed from (10)

\[
\bar{E}(\bar{k}) = -\frac{c^2}{8\lambda} (\frac{a^2}{4bc} - 1)^2 = -|\bar{E}(k_N)| (\frac{a^2}{4bc} - 1)^2.
\]

Since we have already restricted to \( c < 0 \), it is true that \( |\bar{E}(\bar{k})| > |\bar{E}(k_N)| \) and hence the variable \( \phi_{BS}(x) \) energy is lower than the constant \( \bar{\varphi} \) ground state. Alternatively if we do not restrict \( c \) to be negative, we know \( \frac{a^2}{4bc} - 1 > 0 \) (see below (12) and hence to satisfy the above we just require \( \frac{a^2}{4bc} - 1 > 1 \).

Figure 1: In Figure we show that the condensate ground state can be lower than the constant \( \varphi \) ground state.

(iii) We consider a small fluctuation above the flowing ground state, \( \phi(x, t) = \bar{\varphi}\cos(\bar{k}x) + \delta\phi(x, t) \) and the Lagrangian, quadratic in \( \delta\phi(x, t) \) becomes,

\[
\mathcal{L}(\mathcal{O}(\delta\phi)^2) = (\dot{\delta}\varphi)^2 - [a(\delta\phi')^2 + b(\delta\phi'')^2 + c(\delta\phi)^2 + 6\lambda\bar{\varphi}^2(\delta\phi)^2]
\]

In the \( \lambda \)-term above, we have replaced \( \phi \) by \( \bar{\varphi} \) as it is already \( \mathcal{O}(\delta\phi)^2 \). To derive the minimal stability condition we introduce a time-independent Fourier transform \( \delta\phi = \sum_k \delta\phi_k e^{ikx} \)
and find,

\[ E(\delta\phi) = \sum_{k} (ak^2 + bk^4 + c + 6\lambda \bar{\phi}^2)\phi_{-k}\phi_k. \] (19)

The energy will be positive if the function within the bracket is positive. Since \(bk^4\) is positive definite, the positivity condition reduces to

\[ ak^2 + 3\left(\frac{a^2}{4b} - \frac{2c}{3}\right) > 0 \rightarrow - |a| k^2 + 3\left(\frac{a^2}{4b} - \frac{2c}{3}\right) > 0, \] (20)

and finally

\[ k^2 < 3\frac{|a|}{4b} - \frac{2c}{|a|} \rightarrow k^2 < \frac{3}{2}k^2 - \frac{2c}{|a|}. \] (21)

Again for \(c < 0\) this boils down to \(k^2 < \frac{3}{4}\bar{k}^2 + \frac{2|c|}{|a|}\). Physically this means that so long as the excitation velocities are below (of the order of) \(\bar{k}\) the excitation energies are positive and thus they are not capable of reducing the flowing ground state energy thus making the latter stable [10].

Next, following [10], we compute the spectra of the small fluctuations above the condensate ground state \(\phi_{BS}(x,t) = \bar{\phi}_{BS}(x) + \delta\phi_{BS}(x,t)\). The equation of motion gives

\[ (\delta\phi_{BS}) - a(-\bar{k}^2\bar{\phi} + (\delta\phi_{BS})''') + b(\bar{k}^4\bar{\phi} + (\delta\phi_{BS})''') + c(\bar{\phi} + \delta\phi_{BS}) + 2\lambda \bar{\phi}^2(\bar{\phi} + 3\delta\phi_{BS}) = 0. \] (22)

In the above we have used the condition (16). By substituting \(\bar{\phi}\) and \(\bar{k}\) and cancelling out the inhomogeneous terms, we obtain

\[ (\delta\phi_{BS}) - a(\delta\phi_{BS})''' + b(\delta\phi_{BS})'''' + (c + 6\lambda \bar{\phi}^2)(\delta\phi_{BS}) = 0. \] (23)

Consider a wave solution of the form

\[ \delta\phi_S(x,t) = Ae^{-i\omega_k t+ikx} + Be^{i\omega_k t-ikx}. \] (24)

The energy spectra is

\[ \omega_k = \pm \sqrt{ak^2 + bk^4 + \left(\frac{3a^2}{4b} - 2c\right)}. \] (25)
For small $k$ we can drop $bk^4$ so long as $k^2 < \frac{3}{2} \bar{k}^2 + \frac{2|c|}{|\alpha|}$. (since $a$ is negative). The modes, for small $k$, are not gapless. A possible reason might be that although a continuous symmetry, i.e. translation invariance, is being broken still it is not completely lifted since the symmetry broken phase retains a periodic structure in the infrared and an ordered discrete structure in the ultraviolet.

To conclude, in this paper, we have considered exhaustively various possibilities of SSB in momentum space that can be induced by higher spatial derivative terms in a Lifshitz scalar model. We have shown that a flowing ground state condensate, with a periodic spatial variation is certainly possible that can have the lowest ground state energy with respect to other conventional ground state of pure vacuum or having a constant valued condensate. The stability of this novel ground state against decay into small disturbances is discussed. Spectra of fluctuations lying above the flowing condensate have an unconventional spectrum.

Finally, the most interesting observation is that presence of this flowing ground state demands a discretization of the space coordinate. As we have discussed above, the ground state will look like a discrete set of spin half particles with positive/negative values of the condensate amplitude $\pm \bar{\phi}$ at the prescribed discrete one dimensional lattice. At long wavelength the ground state will appear to be wavelike.

From our earlier work we know that the recently conjectured Horava-Lifshitz gravity theory, under certain restrictions, can be reduced to the generic Lifshitz form that we have analyzed here. Infact quite generally one expects space(time) to be discrete at Planck scales so the present work can be extended to higher dimensions (which is straightforward) to generate a discrete space or maybe spacetime in the context of Horava gravity.

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