UNBOUNDED OPERATORS HAVING SELF-ADJOINT OR NORMAL POWERS AND SOME RELATED RESULTS

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Abstract. We show that a densely defined closable operator $A$ such that the resolvent set of $A^2$ is not empty, is necessarily closed. This result is then extended to the case of a polynomial $p(A)$. We also generalize a recent result by Sebestyén-Tarczay concerning the converse of a result by J. von Neumann. Other interesting consequences are also given, one of them being a proof that if $T$ is a quasinormal (unbounded) operator such that $T^n$ is normal for some $n \geq 2$, then $T$ is normal. By a recent result by Pietrzycki-Stochel, we infer that a closed subnormal operator such that $T^n$ is normal, must be normal.

Another remarkable result is the fact that a hyponormal operator $A$, bounded or not, such that $A^p$ and $A^q$ are self-adjoint for some co-prime numbers $p$ and $q$, is self-adjoint. It is also shown that an invertible operator (bounded or not) $A$ for which $A^p$ and $A^q$ are normal for some co-prime numbers $p$ and $q$, is normal. These two results are shown using Bézout’s theorem in arithmetic.

Notation

First, we assume that readers have some familiarity with the standard notions and results in operator theory (see e.g. [17] and [25] for some background). We do recall most of the needed notions though. First, note that in this paper all operators are linear.

Let $H$ be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators defined from $H$ into $H$.

If $S$ and $T$ are two linear operators with domains $D(S) \subset H$ and $D(T) \subset H$ respectively, then $T$ is said to be an extension of $S$, written $S \subset T$, when $D(S) \subset D(T)$ and $S$ and $T$ coincide on $D(S)$.

The product $ST$ and the sum $S + T$ of two operators $S$ and $T$ are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}$$

and

$$D(S + T) = D(S) \cap D(T).$$

When $D(T) = H$, we say that $T$ is densely defined. In such case, the adjoint $T^*$ exists and is unique.

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An operator $T$ is called closed if its graph is closed in $H \oplus H$. $T$ is called closable if it has a closed extension. If $T$ is densely defined, then $T$ is closable iff $T^*$ is densely defined. The smallest closed extension of $T$ is called its closure, noted $\overline{T}$.

We say that $T$ is symmetric if
\[\langle Tx, y \rangle = \langle x, Ty \rangle, \forall x, y \in \mathcal{D}(T).\]

If $T$ is densely defined, we say that $T$ is self-adjoint when $T = T^*$; normal if $T$ is closed and $TT^* = T^*T$. Observe that a densely defined operator $T$ is symmetric iff $T \subset T^*$.

A symmetric operator $T$ is called positive if
\[\langle Tx, x \rangle \geq 0, \forall x \in \mathcal{D}(T).\]

Recall that the absolute value of a closed $T$ is given by $|T| = \sqrt{T^*T}$ where $\sqrt{T}$ designates the unique positive square root of $T^*T$, which is self-adjoint and positive by the closedness of $T$. It is also known that $D(T) = D(|T|)$ when $T$ is closed and densely defined.

We say that $B \in B(H)$ commutes with a linear operator $A$ with domain $D(A) \subset H$ when $BA \subset AB$, that is when $BAx = ABx$ for all $x \in D(A) \subset D(AB)$.

Let $A$ be an injective operator (not necessarily bounded) from $D(A)$ into $H$. Then $A^{-1} : \text{ran}(A) \to H$ is called the inverse of $A$, with $D(A^{-1}) = \text{ran}(A)$.

If the inverse of an unbounded operator is bounded and everywhere defined (e.g. if $A : D(A) \to H$ is closed and bijective), then $A$ is said to be boundedly invertible. In other words, such is the case if there is a $B \in B(H)$ such that
\[AB = I \text{ and } BA \subset I.\]

If $A$ is boundedly invertible, then it is closed. Recall also that $T + S$ is closed (resp. closable) if $S \in B(H)$ and $T$ is closed (resp. closable), and that $ST$ is closed (resp. closable) if e.g. $S$ is closed (resp. closable) and $T \in B(H)$.

Based on the bounded case and the previous definition, we say that an unbounded $A$ with domain $D(A) \subset H$ is right invertible if there exists an everywhere defined $B \in B(H)$ such that $AB = I$; and we say that $A$ is left invertible if there is an everywhere defined $C \in B(H)$ such that $CA \subset I$. Clearly, if $A$ is left and right invertible simultaneously, then $A$ is boundedly invertible.

The spectrum of unbounded operators is defined as follows: Let $A$ be an operator. The resolvent set of $A$, denoted by $\rho(A)$, is defined by
\[\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective and } (\lambda I - A)^{-1} \in B(H)\}.\]

The complement of $\rho(A)$, denoted by $\sigma(A)$, i.e.
\[\sigma(A) = \mathbb{C} \setminus \rho(A)\]

is called the spectrum of $A$.

Clearly, $\lambda \in \rho(A)$ iff there is a $B \in B(H)$ such that
\[(\lambda I - A)B = I \text{ and } B(\lambda I - A) \subset I.\]

Also, recall that if $A$ is a linear operator which is not closed, then $\sigma(A) = \mathbb{C}$. Recall also that
\[\sigma(A^n) = [\sigma(A)]^n\]
when $A$ is closed ($n \in \mathbb{N}$).
Let us now recall some rudimentary facts about matrices of non-necessarily bounded operators. Let $H$ and $K$ be two Hilbert spaces and let $A : H \oplus K \to H \oplus K$ (we may also use $H \times K$ instead of $H \oplus K$) be defined by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \in L(H)$, $A_{12} \in L(K, H)$, $A_{21} \in L(H, K)$ and $A_{22} \in L(K)$ are not necessarily bounded operators. If $A_{ij}$ has a domain $D(A_{ij})$ with $i, j = 1, 2$, then

$$D(A) = (D(A_{11}) \cap D(A_{21})) \times (D(A_{12}) \cap D(A_{22}))$$

is the natural domain of $A$. So if $(x_1, x_2) \in D(A)$, then

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}.$$ 

As is customary, we allow the abuse of notation $A(x_1, x_2)$. Readers should also be careful when dealing with products of matrices of (unbounded) operators as they may encounter some issues with their domains.

Also, recall that the adjoint of $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is not always $\begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}$ (even when all domains are dense including the main domain $D(A)$) as known counterexamples show. Nonetheless, e.g.

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \text{ and } \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & D^* \\ C^* & 0 \end{pmatrix}$$

if $A$, $B$, $C$ and $D$ are all densely defined. See e.g. [35] for proofs of other results which will be used below, and for more about unbounded operator matrices.

In the end, we recall some definitions of unbounded non-normal operators. A densely defined operator $A$ with domain $D(A)$ is called hyponormal if

$$D(A) \subset D(A^*) \text{ and } \|A^*x\| \leq \|Ax\|, \forall x \in D(A).$$

A densely defined linear operator $A$ with domain $D(A) \subset H$, is said to be subnormal when there are a Hilbert space $K$ with $H \subset K$, and a normal operator $N$ with $D(N) \subset K$ such that

$$D(A) \subset D(N) \text{ and } Ax = Nx \text{ for all } x \in D(A).$$

A quasinormal operator $A$ is a closed densely defined one such that $AA^*A = A^*A^2$ (or $AA^*A \subset A^*A^2$ as in say [10]).

A linear operator $A : D(A) \subset H \to H$ is said to be paranormal if

$$\|Ax\|^2 \leq \|A^2x\| \|x\|$$

for all $x \in D(A^2)$.

As in the bounded case, we have (see e.g. [10], [11], [14] and [34]):

Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal $\subset$ Paranormal.

Observe that a hyponormal operator is necessarily closable, hence so are quasinormal and subnormal operators. Paranormal operators, however, need not be closable. See [4] for a counterexample. In fact, in [20], the writer found a densely defined paranormal operator $A$ such that downright $D(A^*) = \{0\}$. 

1. Introduction

As is known, if $A$ is a closed operator, then $A^2$ need not be closed. There are known counterexamples, see e.g. [18]. On the other hand, there are closable operators having closed squares, e.g. if $\mathcal{F}_0$ denotes the restriction of the $L^2(\mathbb{R})$-Fourier transform to the dense subspace $C_0^\infty(\mathbb{R})$ (the space of infinitely differentiable functions with compact support), then it is well known that

$$D(\mathcal{F}_0^2) = \{0\}.$$  

Hence $\mathcal{F}_0$ is unclosed whilst $\mathcal{F}_0^2$ is trivially closed on $\{0\}$. In fact, there is an unclosable operator $A$, which is everywhere defined and such that $A^2 = 0$ on all of $H$, see again [18].

If $A$ is a closable operator such that $A^2$ is self-adjoint, then $A^2$ need not be self-adjoint. A simple example is to take $A$ to be the restriction of the identity operator $I$ (on $H$) to some dense (non-closed) subspace $D$ of $H$. Denote this restriction by $I_D$. Then $\overline{A^2} = I$ fully on $H$ and so $\overline{A^2}$ is self-adjoint. However, $A^2$ is not self-adjoint for $A^2 = I_D$ and so $A^2$ is not even closed.

What about the converse? I.e. if $A$ is closable and $A^2$ is self-adjoint, then could it be true that $A^2$ is self-adjoint? A positive answer can be obtained if one comes to show that if $A$ is a closable operator with a self-adjoint square, then $A$ is closed. This question, and other related and more general ones are investigated in the present paper.

There are known conditions for which $p(A)$ is closed whenever $A$ is closed, where $p$ is a complex polynomial in one variable of degree $n$ say. For instance, if $A$ is a closed operator in some Hilbert (or even Banach) space with domain $D(A)$ such that $\sigma(A) \neq \mathbb{C}$, then $p(A)$ is closed on $D(A^n)$. See e.g. pp. 347-348 in [5]. Also, K. Schmüdgen showed in [24] that if $A$ is a densely defined closed and symmetric operator, then $p(A)$ is closed. J. Stochel proved this result in the case of unbounded subnormal operators in [31]. A related paper to some of our results (especially Theorem 3.3) is [33]. In fact, in an unpublished work yet [34], it is shown that if $A$ is a paranormal closed operator, then $A^n$ is closed for every $n \in \mathbb{N}$. It is one of our aims to investigate the converse of some of these results.

Before announcing an interesting application of some of our results, recall first a well-known result by the legendary J. von Neumann stating that if $T$ is a densely defined closed operator, then both $TT^*$ and $T^*T$ are self-adjoint (and positive). Amazingly, no one had studied the converse until very recently, i.e. until [27]. The outcome is quite striking. Indeed, in the previous reference, the writers Sebestyén-Tarcsey discovered that if $TT^*$ and $T^*T$ are both self-adjoint, then $T$ must be closed. Then Gesztesy-Schmüdgen provided in [9] a simpler proof based on a technique using matrices of unbounded operators. Notice that Gesztesy-Schmüdgen’s proof only work for complex Hilbert spaces while the original proof by Sebestyén-Tarcsey works also for real Hilbert spaces just as good. To end this remark, Sebestyén-Tarcsey also gave a proof of their result using block operator matrices as well as some other results (see e.g. Theorem 8.1 in [30]). This result with the two different proofs is referred to here as the Sebestyén-Tarcsey-Gesztesy-Schmüdgen reversed von Neumann theorem. Note that the self-adjointness of only one of $TT^*$ and $T^*T$ is not sufficient to guarantee the closedness of $T$. This was already noted in [26]. Herein, we show that if $T$ is a densely defined closable such that $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, then $T$ is necessarily closed.
Other consequences are also given. For example, we show that if $T$ is a quasi-normal (unbounded) operator such that $T^n$ is normal for some $n \geq 2$, then $T$ must be normal. By a recent result by Pietrzycki-Stochel in [23], we deduce that a closed subnormal operator such that $T^n$ is normal for some $n$, is necessarily normal. These results are closely related to others which have been of some interest recently (see [3] and [22]).

Other remarkable results are shown by invoking Bézout’s theorem in arithmetic. For instance, if a hyponormal operator $A$, bounded or not, is such that $A^p$ and $A^q$ are self-adjoint for some co-prime numbers $p$ and $q$, then it is self-adjoint. By the same token, it is shown that an invertible operator (bounded or not) $A$ for which $A^p$ and $A^q$ are normal for some co-prime numbers $p$ and $q$, is normal. Another application of Bézout’s theorem is: If a boundedly invertible $T$ commutes with both $A^p$ and $A^q$, for some relatively prime numbers $p$ and $q$, then $T$ commutes with $A$.

2. Main Results: The case of monomials

We choose to deal first with the case of squares for it is closely related to the important notion of square roots. Then we give the generalizations to $p(A)$ as long as these generalizations are known to hold for us.

**Theorem 2.1.** Let $A$ be a closable densely defined operator with domain $D(A) \subset H$ (where $H$ could also be a Banach space here) such that $\sigma(A^2) \neq \mathbb{C}$. Then $A$ is closed.

**Proof.** Let $\lambda \in \mathbb{C} \setminus \sigma(A^2)$. Then $A^2 - \lambda I$ is boundedly invertible. Let $\alpha$ be complex and such that $\alpha^2 = \lambda$ and write

$$A^2 - \lambda I = (A - \alpha I)(A + \alpha I) = (A + \alpha I)(A - \alpha I).$$

Letting $B$ to be the bounded inverse of $A^2 - \lambda I$, we see that

$$I = (A^2 - \lambda I)B = (A - \alpha I)(A + \alpha I)B.$$

Since $A$ is closable, so is $A + \alpha I$. Hence $(A + \alpha I)B$ too is closable. But

$$H = D[(A - \alpha I)(A + \alpha I)B] \subset D[(A + \alpha I)B],$$

whereby $(A + \alpha I)B$ becomes everywhere defined. Therefore, $(A + \alpha I)B$ must be in $B(H)$ by invoking the closed graph theorem. In other words, $A - \alpha I$ is right invertible, i.e. it possesses an everywhere defined bounded right inverse. Now, by the first displayed formula, we see that $A - \alpha I$ is one-to-one as well. Thus, $A - \alpha I$ is bijective. So, let $C$ be a left inverse (a priori not necessarily bounded) of $A - \alpha I$.

Clearly, $C$ must be defined on all of $H$. Moreover,

$$C(A - \alpha I) \subset I \implies C(A - \alpha I)(A + \alpha I)B \subset (A + \alpha I)B \implies C \subset (A + \alpha I)B$$

and so $C = (A + \alpha I)B$ as they are both defined on all of $H$. This actually says that $A - \alpha I$ is boundedly invertible. Accordingly, $A - \alpha I$ is closed or merely $A$ is closed, as needed. □

The following simple consequence seems to have some interest.

**Corollary 2.2.** If $A$ is a closable (unclosed) operator such that $A^2$ is closed, then $\sigma(A^2) = \mathbb{C}$. 
Proof. If $\sigma(A^2) \neq \mathbb{C}$, Theorem 2.1 yields the closedness of $A$ which is a contradiction. 

□

Remark. The assumption $\sigma(A^2) \neq \mathbb{C}$ made in Theorem 2.1 may not just be dropped. A simple counterexample is to consider a closable non-closed $A$ such that $A^2$ is closable but unclosed. Then $\sigma(A^2) = \mathbb{C}$. An explicit example would be to take $A = I_D$, the identity operator restricted to some dense subspace $D$.

Remark. The closability is indispensable for the result to hold. For example and as alluded to above, there are non-closable everywhere defined operators $T$ (i.e. $D(T) = H$) such that $T^2 = 0$ everywhere on $H$. Clearly

$$\sigma(T^2) = \{0\} \neq \mathbb{C}$$

and yet $T$ is not even closable.

Similarly, there are everywhere defined unclosable operators $T$ such that $T^2 = I$ on all of $H$, and hence $\sigma(T^2) = \{1\}$. These examples may be consulted in [18].

Corollary 2.3. Let $A$ be a closable densely defined operator such that $A^2$ is self-adjoint. Then $A$ is closed.

Proof. Recall that a self-adjoint operator has always a real spectrum (see [2] for a new proof). Since $A^2$ is self-adjoint, $\sigma(A^2) \subset \mathbb{R}$. Now, apply Theorem 2.1. □

It was shown in [28] (and also in [30]) that a symmetric operator having a self-adjoint (and positive!) square must be self-adjoint as well. As a consequence of Corollary 2.3, we have a different proof of this result.

Proposition 2.4. Let $A$ be a symmetric operator (not necessarily densely defined) such that $A^2$ is self-adjoint. Then $A$ too is self-adjoint.

Proof. First, as $A^2$ is self-adjoint, it is densely defined, and hence so is $A$. So $A$ is a densely defined symmetric operator, i.e. it becomes closable. Let us show that $A$ is self-adjoint. Indeed, since $A \subset A^*$, we have

$$A \subset A^* = A^*.$$

Hence

$$A^2 \subset A^2 \subset A^* A.$$

By the self-adjointness of both $A^2$ and $A^* A$ as well as a known maximality argument, it ensues that

$$A^2 = A^* A.$$

Theorem 3.2 in [8] now intervenes and gives the self-adjointness of $A$. But Corollary 2.3 gives the closedness of $A$, that is, $A$ is self-adjoint. □

In other language, we have re-shown that a symmetric square root of a self-adjoint (necessarily positive!) is itself self-adjoint. Next, we generalize this result to normal operators.

Corollary 2.5. Let $A$ be a symmetric operator such that $A^2$ is normal. Then $A$ is self-adjoint.
Proof. That $A$ is densely defined is clear. Since $A \subset A^*$, we have

$$A^2 \subset (A^*)^2 \subset (A^2)^*.$$  

In other words, $A^2$ is symmetric. But, and as is known, symmetric operators that are normal are self-adjoint. Therefore, $A^2$ is self-adjoint, and so $A$ too is self-adjoint by the foregoing results. \hfill \Box

Remark. The above proof is algebraic. There is also a spectral proof. Indeed, since $A$ is symmetric, it follows that $A^2$ is positive. However, a normal operator with a positive spectrum is self-adjoint. Then Proposition 2.4 does the remaining job.

The following result contains some important conclusions (observe that it generalizes Proposition 2.4).

**Theorem 2.6.** Let $A$ be a linear operator with domain $D(A)$ such that $A^2$ is self-adjoint and $D(A) \subset D(A^*)$. Then $A$ is closed, $D(A) = D(A^*)$, $(A^*)^2 = A^2$ and $D(AA^*) = D(A^*A)$.

Proof. As above, $A$ is densely defined. Since $D(A) \subset D(A^*)$, it follows that $A$ is closable. Hence $A$ is closed by Corollary 2.3. So, it only remains to show that $D(A) = D(A^*)$. To this end, observe that

$$(A^*)^2 \subset (A^2)^* = A^2.$$

Since $D(A) \subset D(A^*)$, we get

$$(2) \quad D(AA^*) \subset D([A^*]^2) \subset D(A^2) \subset D(A^*A).$$

By Theorem 9.4 in [37], we obtain $D(\sqrt{AA^*}) \subset D(\sqrt{A^*A})$ for $AA^*$ and $A^*A$ are self-adjoint and positive. But $\sqrt{AA^*} = |A^*|$ and $\sqrt{A^*A} = |A|$. So, by the closedness of both $A$ and $A^*$, we finally infer that

$$D(A^*) = D(|A^*|) \subset D(|A|) = D(A)$$

thereby $D(A) = D(A^*)$.

To prove the other claimed properties, start with the inclusion $(A^*)^2 \subset A^2$ which was obtained above. Since $A^2$ is self-adjoint, $\emptyset \neq \sigma(A^2) \subset \mathbb{R}$. So, let $\lambda \in \sigma(A^2)$ and so $\lambda = \mu^2$ for some $\mu \in \sigma(A)$. Hence $\overline{\mu} \in \sigma(A^*)$. Therefore,

$$\lambda = \overline{\mu} = \overline{\mu}^2 \in [\sigma(A^*)]^2 = \sigma(A^2).$$

That is, $\rho(A^*) \subset \rho(A^2)$. On the other hand, $\rho(A^2) \neq \emptyset$ as it is easy to see that $\sigma(A^2) \neq \mathbb{C}$. So let $\alpha \in \rho(A^2)$ and write

$$(A^*)^2 - \alpha I \subset A^2 - \alpha I.$$  

Since $(A^*)^2 - \alpha I$ is onto and $A^2 - \alpha I$ is one-to-one, by a simple maximality result (see Lemma 1.3 in [25]), it follows that

$$(A^*)^2 - \alpha I = A^2 - \alpha I$$

or merely $(A^*)^2 = A^2$. Finally, as $D(A) = D(A^*)$ and $(A^*)^2 = A^2$, Inclusions (2) become

$$D(AA^*) = D([A^*]^2) = D(A^2) = D(A^*A),$$

marking the end of the proof. \hfill \Box

**Corollary 2.7.** Let $A$ be an unbounded hyponormal operator such that $A^2$ is self-adjoint and positive. Then $A$ too is self-adjoint.
Proof. Since $A$ is closable and $A^2$ is self-adjoint, it follows that $A$ is closed. Recall also that closed hyponormal operators having a real spectrum are self-adjoint (see the proof of Theorem 8 in [6]). Let $\lambda \in \sigma(A)$ and so $\lambda^2 \in \sigma(A^2)$, i.e. $\lambda^2 \geq 0$ because $A^2$ is positive. Thus $\lambda$ must be real. Consequently, $A$ is self-adjoint. \hfill \box

**Proposition 2.8.** Let $A$ be an unbounded quasinormal operator with $A^2 = A^*2$. Then $A$ is normal.

**Proof.** We may write $|A|^4 = A^*AA^*A = A^*2A^2 = A^4$ and $|A^*|^4 = AA^*AA^* = A^*A^2A^* = A^*4$. Since $A^2 = A^{*2}$, we have $A^4 = A^{*4}$, thereby $|A|^4 = |A^*|^4$. Upon passing to the unique positive square root, we obtain $|A|^2 = |A^*|^2$ or $A^*A = AA^*$. Since $A$ is already closed, the normality of $A$ follows, as needed. \hfill \box

**Remark.** If $A \in B(H)$, then $A^2$ is self-adjoint if and only if $A^2 = A^{*2}$. This is not always the case when $A$ is closed and densely defined. Indeed, in [7] we found a closed densely defined operator $T$ such that $D(T^2) = D(T^{*2}) = \{0\}$ (and so $T^2$ cannot be self-adjoint).

What about assuming that $D(T^2)$ is dense? This is still not enough. Indeed, consider two self-adjoint operators $A$ and $B$ such that $AB = BA$ on some common core but $A$ and $B$ do not commute strongly. This is obviously not trivial and it is some kind of a Nelson-like counterexample. To get the appropriate example, we choose Schmüdgen’s example (see Example 5.5 in [25]). Then set $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ with $D(T) = D(B) \oplus D(A)$. Then $T$ is closed. Moreover, $T^* = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$ with $D(T^*) = D(A) \oplus D(B)$. Hence $T^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \begin{pmatrix} BA & 0 \\ 0 & AB \end{pmatrix} = T^{*2}$ because $D(T^2) = D(T^{*2}) = D(AB) \oplus D(AB)$. Finally, $T^2$ cannot be self-adjoint for if it were, then this would lead to the strong commutativity of $A$ and $B$, which is impossible.

The next result generalizes some known results in the bounded case. Its proof is based upon a quite interesting paper by M. Uchiyama ([36]) on quasinormality (both bounded and unbounded operators). Unfortunately, many operator theorists were unaware of it as could be guessed by looking at some papers which appeared after Uchiyama’s, where some of his results were re-shown.

Before giving the alluded result, recall that when $T^n$ is densely defined, then in general $T^{*n} \subset (T^n)^*$ even when $T$ belongs to some nice class. For example, Jabłoński et al. constructed in [10] a quasinormal operator $T$ such that $T^{*n} \subset (T^n)^*$ for all $n \geq 2$. So, the next lemma seems to have some interest.
Lemma 2.9. Let $T$ be a quasinormal operator such that $T^n$ is densely defined and $D[(T^n)^*] = D(T^n)$ (e.g. when $T^n$ is normal). Then

$$(T^n)^* = T^n.$$  

Proof. By the general theory, we already have that $T^{*n} \subset (T^n)^*$. So, we only have to show that $D[(T^n)^*] \subset D(T^{*n})$. Since $T$ is quasinormal, $U|T| \subset |T|U$ where $T = U|T|$ is the usual polar decomposition in terms of partial isometries (see Proposition 1 in [32], in fact $U|T| = |T|U$ as in [36]). Hence $T^* = |T|U^*$ and $U^*|T| \subset |T|U^*$. Since $T$ is quasinormal, $|T^n| = |T|^n$ (see e.g. [10]). Moreover, $T^n$ is closed for $T$ is subnormal and closed (see [31]).

Therefore,

$$D(T^{*n}) = D[(|T|U^n)] \supset D(U^{*n}|T|^{n}) = D(|T|^n) = D(|T^n|) = D(T^n) = D[(T^n)^*],$$

as wished.  

We are now in a position to state and prove a result on quasinormal $n$th roots of normal operators (see [23] for a related result).

Theorem 2.10. Let $T$ be a quasinormal (unbounded) operator such that $T^n$ is normal for some $n \geq 2$. Then $T$ is normal.

Proof. By Corollary 3.1 in [36], we know that

$$T^{*n}T^n = (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n}$$

whenever $T$ is quasinormal and for all $n \geq 2$.

A little digression, it was unclear to us which order relation was used in the above inequalities given that M. Uchiyama did not indicate the one he was using. However, in our case, it does not affect our result given the assumptions we made in the theorem. Indeed, all operators in the inequalities are self-adjoint (this includes $T^nT^{*n}$ for by Lemma 2.9 $(T^n)^* = T^{*n}$). A look at Page 230 in [25] will shed light on this particular point.

Let us finish the proof now. By the normality of $T^n$, Lemma 2.9 and the above inequalities, we may write

$$T^n(T^n)^* = (T^n)^*T^n = T^{*n}T^n = (T^*T)^n \geq (TT^*)^n \geq T^nT^{*n} = T^n(T^n)^*.$$  

Therefore,

$$(T^*T)^n = (TT^*)^n = T^n(T^n)^* = (T^n)^*T^n.$$  

Passing to the unique positive $n$th root yields $|T|^2 = |T^*|^2$, that is, it yields the normality of $T$, as suggested.

The next consequence might be known to some specialists, however, we believe that it is not documented anywhere. Besides, the proof is extremely easy once we fall back on a very recent result by P. Pietrzycki and J. Stochel, whose proof is somewhat involved.

Corollary 2.11. Let $T$ be a closed subnormal (unbounded) operator such that $T^n$ is normal for some $n \geq 2$. Then $T$ is normal.

Proof. Since $T^n$ is normal, it is quasinormal. By Theorem 1.4 in [23], we know that $T$ must be quasinormal. Theorem 2.10 then gives the normality of $T$.  

□
As mentioned above, we present a generalization of the Sebestyén-Tarsesay-Gesztesy-Schmüdgen reversed von Neumann theorem. Remember in passing that when $TT^*$ and $T^*T$ are self-adjoint, then $TT^*$ and $T^*T$ are closed and $\sigma(TT^*), \sigma(T^*T) \subset \mathbb{R}$.

**Theorem 2.12.** Let $T$ be a densely defined closable such that $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$. Then $T$ is necessarily closed.

**Proof.** Just write
\[
A = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}
\]
which is closable. Then
\[
A^2 = \begin{pmatrix} TT^* & 0 \\ 0 & T^*T \end{pmatrix}
\]

By the assumption $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, we see that neither $\sigma(TT^*) = \mathbb{C}$ nor $\sigma(T^*T) = \mathbb{C}$. Therefore, $TT^*$ and $T^*T$ are both closed. Hence $A^2$ too is closed. Since $\sigma(A^2) \neq \mathbb{C}$, Theorem 2.1 entails the closedness of $A$ which, in turn, yields the closedness of $T$. \hfill \Box

Next, we give two simple characterizations of (unbounded) normal operators.

**Corollary 2.13.** Let $T$ be a densely defined closable linear operator. Then $T$ is normal $\iff TT^* = T^*T$ and $\sigma(T^*T) \neq \mathbb{C}$.

**Proof.** If $T$ is normal, then $TT^* = T^*T$ and $T$ is closed. Hence $T^*T$ is self-adjoint and so $\mathbb{C} \neq \sigma(T^*T) \subset \mathbb{R}$. Conversely, as $\sigma(T^*T) \neq \mathbb{C}$, $T^*T$ is closed. Hence $TT^*$ too is closed. Theorem 2.12 then yields the closedness of $T$ because $\sigma(TT^*) \cup \sigma(T^*T) = \sigma(T^*T) \neq \mathbb{C}$. \hfill \Box

By calling on Proposition 1.1 in [31], we have the following result:

**Corollary 2.14.** Let $T$ be a densely defined closable linear operator such that $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$. Then either $T$ is normal or $TT^* \not\subset T^*T$ or $T^*T \not\subset TT^*$.

**Proof.** As $\sigma(TT^*) \cup \sigma(T^*T) \neq \mathbb{C}$, we know that $T$ must be closed. Hence by Proposition 1.1 in [31], either $TT^* = T^*T$ (i.e. $T$ is normal) or $TT^* \not\subset T^*T$ or $T^*T \not\subset TT^*$. \hfill \Box

Before giving another related result, we give an example.

**Example 2.15.** ([16]) Consider the following two operators defined by
\[
Af(x) = e^{2x}f(x) \quad \text{and} \quad Bf(x) = (e^{-x} + 1)f(x)
\]
on their respective domains
\[
D(A) = \{ f \in L^2(\mathbb{R}) : e^{2x}f \in L^2(\mathbb{R}) \}
\]
and
\[
D(B) = \{ f \in L^2(\mathbb{R}) : e^{-2x}f, e^{-x}f \in L^2(\mathbb{R}) \}.
\]
Then obviously $A$ is self-adjoint while $B$ closable without being closed.

Now, the operator $BA$ defined by $BAf(x) = (e^{2x} + e^{x})f(x)$ on
\[
D(BA) = \{ f \in L^2(\mathbb{R}) : e^{2x}f, e^{x}f \in L^2(\mathbb{R}) \}
\]
is plainly self-adjoint. Readers may also check that $AB$ is not self-adjoint.

Inspired by this example, we have:
Proposition 2.16. It is impossible to find two unbounded linear operators $A$ and $B$, one of them is closable (without being closed) and the other is self-adjoint, such that both $BA$ and $AB$ are self-adjoint.

Proof. Let $B$ be a solely closable operator with domain $D(B)$, and let $A$ be a self-adjoint operator with domain $D(A)$. Then set

$$T = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

which is defined on $D(T) = D(A) \oplus D(B)$. Clearly, $T$ is only closable (i.e. $T$ is not closed). Now,

$$T^2 = \begin{pmatrix} BA & 0 \\ 0 & AB \end{pmatrix}.$$

So if $BA$ and $AB$ were both self-adjoint, it would ensue that $T^2$ is self-adjoint which, given the closability of $T$, would yield the closedness of $T$ (by Corollary 2.3). Since this is absurd, at least one of $AB$ and $BA$ must be non-self-adjoint. □

There remains to investigate the very related question: Is there a densely defined, closable and unclosed operator $A$ such that $A^2$ is closed, densely defined, and obeys $\sigma(A^2) = \mathbb{C}$? By looking closely at the foregoing results, we actually see that the condition $\sigma(A^2) = \mathbb{C}$ is superfluous.

A simple example is already available in the case of the absence of the density of $D(A^2)$, as alluded to in the introduction. So, things are more interesting when $D(A^2)$ is dense, and there is a counterexample in this case too. But, to get the sought example, we need a densely defined closed operator $T$ such that $D(T^2) = D(T)$. Recall now that S. Ōta ([21]) introduced the concept of an unbounded idempotent operator: Let $T$ be a non-necessarily bounded operator with a dense domain $D(T)$. We say that $T$ is idempotent if $T^2$ is well defined and

$$T^2 = T$$
on D(T).

Surprisingly, S. Ōta did not provide any example of an unbounded closed idempotent operator even though he did provide other examples of unclosable idempotents. So, let us supply such an example:

Example 2.17. ([19]) Let $B$ be an unbounded closed operator with domain $D(B) \subset H$ and let $I$ be the identity operator on all of $H$. Define

$$T = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$$

with $D(T) = H \times D(B)$. Then $T$ is densely defined, closed and unbounded. Since

$$D(T^2) = \{(x, y) \in H \times D(B) : (x + By, 0) \in H \times D(B)\} = D(T),$$

we see that

$$T^2 = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix} = T.$$

With this example at hand, we may now construct a densely defined, closable and unclosed operator $A$ such that $A^2$ is densely defined and closed (hence necessarily $\sigma(A^2) = \mathbb{C}$):
Example 2.18. ([19]) Let $T$ be a densely defined, unbounded and closed operator $T$ such that $D(T^2) = D(T) \subset H$, and consider the identity operator on $H$ restricted to $D(T)$, noted $I_{D(T)}$. Next, let

$$A = \begin{pmatrix} 0 & T \\ I_{D(T)} & 0 \end{pmatrix}$$

where $D(A) = D(T) \times D(T)$. Clearly, $A$ is closable but unclosed. Let $0_{D(T)}$ and $T_{D(T^2)}$ designate the restrictions of the zero operator and of $T$ to the subspaces $D(T)$ and $D(T^2)$ respectively. We then have

$$A^2 = \begin{pmatrix} 0 & T \\ I_{D(T)} & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ I_{D(T)} & 0 \end{pmatrix} = \begin{pmatrix} T & 0_{D(T)} \\ 0_{D(T)} & T_{D(T^2)} \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

Thus, $A^2$ is patently closed on the dense $D(T) \times D(T)$, as wished.

Corollary 2.19. If $T = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$, then $\sigma(T) = \mathbb{C}$ for any unbounded closed (even self-adjoint) operator $B$, where $I \in B(H)$ is the usual identity.

Proof. Let $A$ be as in Example 2.18 and so $A^2 = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$. Hence $\sigma(A^2) = \sigma(T)$.

But, we already observed above that $\sigma(A^2) = \mathbb{C}$, i.e. $\sigma(T) = \mathbb{C}$, as needed. \qed

We would like to generalize Corollary 2.7 to the case $A^n$ where $n \geq 3$. A direct generalization is obviously false. For instance, if $A = e^{2\pi i/3}I$, then $A$ is unitary and non-self-adjoint, yet $A^3 = I$ is obviously positive. If $A^n$ and $A^{n+1}$ are both positive and self-adjoint for some $n$, then $A$ is self-adjoint and positive whenever it is hyponormal (non necessarily bounded). After having shown this result, a better version came to our minds. It is interesting to see how elementary number theory may be called on to show results in (unbounded) operator theory.

Theorem 2.20. Let $A$ be a non-necessarily bounded hyponormal operator with domain $D(A) \subset H$. If $A^p$ and $A^q$ are two self-adjoint operators, where $p$ and $q$ are two co-prime numbers, then $A$ is self-adjoint.

Proof. By a similar method as in the proof of Theorem 2.1 or a consequence of Theorem 3.1 below, $A$ is closed. Let $\lambda$ be in $\sigma(A)$. Hence $\lambda^p \in \mathbb{R}$ and $\lambda^q \in \mathbb{R}$. If $\lambda = 0$, then $\sigma(A) = \{0\}$, whereby $A$ becomes self-adjoint.

Now, let $\lambda \neq 0$. By Bézout’s theorem in arithmetic, we know that $ap + bq = 1$ for some integers $a$ and $b$. Therefore, $\lambda^{ap}, \lambda^{bq} \in \mathbb{R} - \{0\}$, and so $\lambda^{ap+bq} \in \mathbb{R} - \{0\}$. In other words, $\lambda \in \mathbb{R} - \{0\}$. So in all cases, $\lambda$ must be real. This says that $A$ has a real spectrum, and whence $A$ becomes self-adjoint. \qed

Remark. A similar result holds by replacing "self-adjoint" by "positive" in both the assumption and the conclusion, in the case of bounded operators; and by replacing "self-adjoint" by "self-adjoint and positive" in the unbounded case.

Remark. The assumption $p$ and $q$ being relatively prime numbers is essential. To show its importance, consider as above $A = e^{2\pi i/3}I$. Then both $A^3$ and $A^6$ are positive, yet $A$ is not even self-adjoint.

The idea of the proof of the previous theorem may be applied to show some similar results. As is known, square roots of normal operators need not be normal, even when $\dim H = 2$.要求一个方根为正常操作符是可逆的，但仍然是不帮助的（见《Theorem 3 in [13]》）。The next result may therefore be useful.
Theorem 2.21. Let $A$ be a boundedly invertible non-necessarily bounded operator with domain $D(A)$. If $A^p$ and $A^q$ are normal, where $p$ and $q$ are two co-prime numbers, then $A$ is normal.

Proof. As above, we have by Bézout’s theorem that $ap + bq = 1$ for some integers $a$ and $b$. Necessarily, one of $ap$ and $bq$ has to be negative, and WLOG assume it is $bq$. Since $A$ is boundedly invertible, we have $A^{-1}A \subset I$ and $AA^{-1} = I$. Hence

$$A^{bq}A^p \subset A = A^pA^{bq}.$$ 

Since $A$ is boundedly invertible, $A^{bq} \in B(H)$. Since $A^p$ and $A^q$ are also normal, $A^{bq}A^p$ and $A^pA^{bq}$ remain normal. In other words, the bounded and normal $A^{bq}$ commutes with the normal $A^p$. Hence, by say Theorem 2.2 in [15],

$$A^pA^{bq} = A$$

is normal, as needed. \(\square\)

Mutatis mutandis, a similar result holds for self-adjoint as well as self-adjoint positive operators. Consulting Lemma 3.1 in [13] comes in handy, and so we omit the proof.

Theorem 2.22. Let $A$ be a boundedly invertible non-necessarily bounded operator with domain $D(A)$. If $A^p$ and $A^q$ are self-adjoint (resp. self-adjoint and positive), where $p$ and $q$ are two co-prime numbers, then $A$ is self-adjoint (resp. self-adjoint and positive).

Remark. To see why the condition "$p$ and $q$ being co-prime numbers" may not just dropped, it suffices to take an invertible non-normal square root $A$ of the identity matrix, and then $A^2 = A^4 = I$. An explicit example would be $A = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$.

It is well known that if $S$ is a positive self-adjoint operator which commutes with some $R \in B(H)$, i.e. $RS \subset SR$, then $R\sqrt{S} \subset \sqrt{S}R$ where $\sqrt{S}$ designates the unique positive self-adjoint square root of $S$. See e.g. [29] for a new proof.

What about arbitrary roots? The answer is negative even on finite-dimensional spaces. For example, consider a $2 \times 2$ matrix, noted $T$. Then $T$ commutes with $A^2 = A^4 = I$, and we can obviously choose $T$ such that it does not commute with an invertible square root of $I$ (for example the matrix $A$ in the preceding remark). The same $T$ also commute with $A^4$. So, a judicious choice of exponents must be made if we want a positive result.

Proposition 2.23. Let $A$ be a non-necessarily bounded operator which is boundedly invertible, and let $T \in B(H)$. If $T$ commutes with both $A^p$ and $A^q$, i.e. $TA^p \subset A^pT$ and $TA^q \subset A^qT$ for some relatively prime numbers $p$ and $q$, then $TA \subset AT$, i.e. $T$ commutes with $A$.

Proof. The proof iterum uses Bézout’s theorem. So, $ap + bq = 1$ for some integers $a$ and $b$ (choose $bq$ to be negative). Since $TA^p \subset A^pT$ and $TA^q \subset A^qT$, it follows that $TA^{ap} \subset A^{ap}T$ and $TA^{aq} = A^{aq}T$ for $A^{aq}, T \in B(H)$. Hence

$$TA^{ap} \subset A^{ap}T \implies TA^{ap}A^{bq} \subset A^{ap}TA^{aq} = A^{ap}A^{aq}T.$$ 

As in the proof of Theorem 2.21 we derive $A^{ap}A^{bq} = A$. Therefore, $TA \subset AT$, as wished. \(\square\)
3. Main Results: The case of polynomials

Now, we treat the more general case of \( p(A) \) where \( p \) is a complex polynomial.

**Theorem 3.1.** Let \( p \) be a complex polynomial of one variable of degree \( n \). Assume that \( A \) is a closable operator in a Hilbert (or Banach) space such that \( (p(A) \) is densely defined and) \( \sigma[p(A)] \neq \mathbb{C} \). Then \( A \) is closed.

**Remark.** Observe that the closedness of \( p(A) \) is tacitly assumed because \( \sigma[p(A)] \neq \mathbb{C} \).

**Proof.** Let \( \lambda \) be in \( \mathbb{C} \setminus \sigma[p(A)] \). We may also assume that the leading coefficient of \( p(A) \) equals 1. By the fundamental theorem of Algebra, we know that there are complex numbers \( \mu_1, \mu_2, \ldots, \mu_n \) such that

\[
p(z) = (z - \mu_1)(z - \mu_2) \cdots (z - \mu_n)
\]

where \( z \in \mathbb{C} \). Hence

\[
p(A) - \lambda I = (A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I).
\]

The previous is in effect a full equality for

\[
D[p(A) - \lambda I] = D[(A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I)] = D(A^n)
\]

(which may be checked using a proof by induction). Since \( p(A) - \lambda I \) is boundedly invertible, we have that

\[
(A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I)B = I
\]

for some \( B \in B(H) \). But \( (A - \mu_2 I) \cdots (A - \mu_n I)B \in B(H) \). Indeed, since \( A \) is closable, so is \( A - \mu_n I \) and so \((A - \mu_n I)B \) is closable. Hence \((A - \mu_n I)B \in B(H) \) for \( D[(A - \mu_n I)B] = H \).

Now, \((A - \mu_{n-1})(A - \mu_n I)B \in B(H) \) by using a similar argument. By induction, it may then be shown that \((A - \mu_2 I) \cdots (A - \mu_n I)B \in B(H) \). Thus, \( A - \mu_1 I \) is right invertible. Since

\[
(A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I) = (A - \mu_2 I)(A - \mu_3 I) \cdots (A - \mu_n I)(A - \mu_1 I),
\]

it is seen that \( A - \mu_1 I \) is one-to-one. Finally, as in the proof of Theorem 2.1, we may conclude that \( A \) is closed, as suggested. \( \square \)

As in the case of \( p(z) = z^2 \), we have:

**Corollary 3.2.** Let \( p \) be a complex polynomial of one variable. Assume that \( A \) is a closable non-closed operator in a Hilbert (or Banach) space such that \( p(A) \) is closed. Then

\[
\sigma[p(A)] = \mathbb{C}.
\]

Next, we generalize Proposition 2.4.

**Theorem 3.3.** Let \( p \) be a polynomial of one variable of degree \( n \). Assume that \( A \) is a symmetric (not necessarily densely defined) operator in a Hilbert space \( H \) such that \( p(A) \) is self-adjoint. Then \( A \) is self-adjoint.

**Proof.** Since \( A \) is symmetric, to show it is self-adjoint we may instead show that \( \text{ran}(A - \mu I) = H \) and \( \text{ran}(A - \overline{\mu} I) = H \) where \( \overline{\mu} \) is the complex conjugate of \( \mu \) (see Proposition 3.11 in [25]).
Let $\lambda \in \mathbb{C} \setminus \sigma[p(A)]$. As above, we may assume that the leading coefficient of $p(A)$ is equal to 1. Write
\[ p(A) - \lambda I = (A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I) \]
where $\mu_i, i = 1, \cdots, n$ are complex numbers. As above, it can be shown that $A - \mu_1 I$ is right invertible, i.e., it is surjective. Since $p(A) - \lambda I$ is boundedly invertible, so is $[p(A) - \lambda I]^*$. But
\[ [p(A) - \lambda I]^* = p(A) - \overline{\lambda} I \]
as $p(A)$ is self-adjoint. Since $A$ is symmetric and using some standard properties, we see that
\[
(A - \overline{\mu_1} I)(A - \overline{\mu_2} I) \cdots (A - \overline{\mu_n} I) \subset (A^* - \overline{\mu_1} I)(A^* - \overline{\mu_2} I) \cdots (A^* - \overline{\mu_n} I)
\]
as both sides have the same domain, namely $D(A^*)$. Thus, $A - \overline{\mu} I$ too is surjective. Accordingly, $A$ is self-adjoint.

\textbf{Corollary 3.4.} Let $p$ be a polynomial of one variable of degree $n$ with real coefficients. Assume that $A$ is a symmetric (not necessarily densely defined) operator in a Hilbert space $H$ such that $p(A)$ is normal. Then $A$ is self-adjoint.

\textit{Proof.} Since $A$ is symmetric, so is $p(A)$ because $p$ has real coefficients. Hence
\[
\{p(A) \subset p(A^*) \subset [p(A)]^*,
\]
i.e. $p(A)$ is symmetric. Given its normality, it becomes self-adjoint and so $A$ is self-adjoint by Theorem 3.3. \hfill \Box

Amazingly, M. Uchiyama showed that a symmetric quasinormal operator is self-adjoint (the last corollary in [36]). So, the next consequence is stated without proof.

\textbf{Corollary 3.5.} Let $p$ be a polynomial of one variable of degree $n$ with real coefficients. Assume that $A$ is a symmetric (not necessarily densely defined) operator in a Hilbert space $H$ such that $p(A)$ is quasinormal. Then $A$ is self-adjoint.

The results above may not be generalized to functions $f(A)$ even when the latter is well defined. Let us give a counterexample.

\textbf{Example 3.6.} Let $A$ be a densely defined symmetric (hence closable) unclosed operator such that $A^* A$ is self-adjoint. Then $|A|^2$ is self-adjoint while $A$ is not closed. An explicit realization (inspired by one which appeared in [36]) reads:

Let $T = i \frac{d}{dx}$ be defined on $H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \}$. Then $T$ is self-adjoint (hence $T^* T = T^2$ is self-adjoint). Set $A = T|_{H^2(\mathbb{R})}$ where $H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R}) \}$. Then $A$ is not closed and $T^* = A^*$. Since
\[
D(A^* A) = D(T^2) = H^2(\mathbb{R}),
\]
it follows that $A^* A = T^2$. Thus, since $T^2$ is self-adjoint so is $A^* A$. 

\textbf{Example 3.6.} Let $A$ be a densely defined symmetric (hence closable) unclosed operator such that $A^* A$ is self-adjoint. Then $|A|^2$ is self-adjoint while $A$ is not closed. An explicit realization (inspired by one which appeared in [36]) reads:

Let $T = i \frac{d}{dx}$ be defined on $H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \}$. Then $T$ is self-adjoint (hence $T^* T = T^2$ is self-adjoint). Set $A = T|_{H^2(\mathbb{R})}$ where $H^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R}) \}$. Then $A$ is not closed and $T^* = A^*$. Since
\[
D(A^* A) = D(T^2) = H^2(\mathbb{R}),
\]
it follows that $A^* A = T^2$. Thus, since $T^2$ is self-adjoint so is $A^* A$. 

Conjectures

(1) If it can be shown that a closed hyponormal operator having a quasinormal power is necessarily quasinormal, then a similar idea as in Corollary 2.11 may be applied to show that a closed hyponormal operator having a normal power is automatically normal. This result is perhaps known to a few specialists, but we could not find it anywhere. In addition, if a proof exists somewhere, then the one we have proposed here would be simpler.

(2) Let \( A \) be a closed densely defined paranormal operator, with \( \sigma(A) \subset \mathbb{R} \). Must \( A \) be self-adjoint? If this is true, then Theorem 2.20 could be improved by replacing the hyponormality assumption by a paranormality one.

(3) Powers of (unbounded) paranormal operators are normal (as in say \([34]\)), and so are inverses of paranormal operators (as in \([1\]) or in \([34]\)). So can Theorem 2.21 be generalized to closed paranormal operators?

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