RESTRICTION THEOREMS FOR μ-(SEMI)STABLE FRAMED SHEAVES

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Abstract. We provide a generalization of Mehta-Ramanathan restriction theorems to framed sheaves: we prove that the restriction of a μ-semistable framed sheaf on a nonsingular projective irreducible variety of dimension \(d \geq 2\) to a general hypersurface of sufficiently high degree is again μ-semistable. The same holds for μ-stability under some additional assumptions.

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1. Introduction

In [5], Donaldson proved that the moduli space of gauge-equivalence classes of framed \(SU(r)\)-instantons with instanton charge \(n\) on \(S^4\) is isomorphic to the moduli space of isomorphism classes of vector bundles on \(\mathbb{CP}^2\) of rank \(r\) and second Chern class \(n\) that are trivial along a fixed line \(l_\infty\), and with a fixed trivialization there. It is an open subset \(\mathcal{M}^{reg}(r, n)\) of the moduli space \(\mathcal{M}(r, n)\) of framed sheaves on \(\mathbb{CP}^2\), that is, the moduli space parametrizing pairs \((E, \alpha)\), modulo isomorphism, where \(E\) is a torsion free sheaf on \(\mathbb{CP}^2\) of rank \(r\) and \(c_2(E) = n\), locally trivial in a neighborhood of \(l_\infty\), and \(\alpha : E|_{l_\infty} \overset{\sim}{\rightarrow} O_{l_\infty}^{\oplus r}\) is the framing at infinity. \(\mathcal{M}(r, n)\) is a nonsingular quasi-projective variety of dimension \(2rn\). Moreover, it admits a description in terms of linear data, the so-called ADHM data (see, for example, Chapter 2 in

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In some sense we can look at $\mathcal{M}(r, n)$ as a partial compactification of $\mathcal{M}^{\text{reg}}(r, n)$. There exists another type of partial compactification $\mathcal{M}^{\text{Uh}}(r, n)$ of the latter moduli space, called Uhlenbeck-Donaldson compactification: using linear data and geometric invariant theory it is possible to construct a projective morphism

$$\pi_r : \mathcal{M}(r, n) \to \mathcal{M}^{\text{Uh}}(r, n) = \bigcup_{i=0}^{n} \mathcal{M}^{\text{reg}}(r, n - i) \times \text{Sym}^i(\mathbb{C}^2)$$

such that the restriction to the "locally free" part is an isomorphism with its image (see Chapter 3 in [17]).

The moduli spaces $\mathcal{M}(r, n)$ can be regarded as higher-rank generalizations of Hilbert schemes of $n$-points on the complex affine plane. From this point of view, the previous morphism is a higher-rank generalization of Hilbert-Chow morphism for Hilbert schemes of points on the complex affine plane.

Because of the relation with moduli space of framed instantons, since Nekrasov’s partition function was introduced in [20], the moduli space $\mathcal{M}(r, n)$ has been studied quite intensively (see, e.g., [11, 17, 18, 19, 4]) and the geometry of moduli spaces of framed sheaves on the complex projective plane is quite well known.

In [3,9] Huybrechts and Lehn laid the foundations of a systematic theory of framed sheaves on varieties of arbitrary dimension (they used different names to denote the same object like, e.g., stable pairs, framed modules, etc). Let $X$ be a nonsingular, projective, irreducible variety of dimension $d$ defined over an algebraically closed field $k$ of characteristic zero. A framed sheaf is a pair $(E, \alpha)$ where $E$ is a coherent sheaf on $X$ and $\alpha$ is a morphism from $E$ to a fixed coherent sheaf $F$. They define a generalization of Gieseker semistability (resp. $\mu$-semistability) for framed sheaves that depends on a polarization and a rational polynomial $\delta$ of degree $d - 1$ with positive leading coefficient (resp. a rational number $\delta_1$).

Let us fix a numerical polynomial $P$ of degree $d$ and denote by $\mathcal{M}^{ss}_\delta(X; F, P)$ (resp. $\mathcal{M}^{ss}_\delta(X; F, P)$) the contravariant functor from the category of Noetherian $k$-schemes of finite type to the category of sets, that associates to every scheme $T$ the set of isomorphism classes of families of semistable (resp. stable) framed sheaves with Hilbert polynomial $P$ parametrized by $T$. The main result in their papers is the following:

**Theorem** (Huybrechts, Lehn). There exists a projective scheme $\mathcal{M}^{ss}_\delta(X; F, P)$ that corepresents the functor $\mathcal{M}^{ss}_\delta(X; F, P)$. Moreover, there is an open subscheme $\mathcal{M}^{\text{reg}}_\delta(X; F, P)$ of $\mathcal{M}^{ss}_\delta(X; F, P)$ that represents the functor $\mathcal{M}^{\text{reg}}_\delta(X; F, P)$, i.e., $\mathcal{M}^{\text{reg}}_\delta(X; F, P)$ is a fine moduli space for stable framed sheaves.

If $X$ is a surface, we can extend the original definition of framed sheaves on $\mathbb{CP}^2$ with framing along a fixed line in the following way: let $F$ be a coherent sheaf on $X$, supported on a big and nef curve $D$, such that $F$ is a Gieseker-semistable locally free $\mathcal{O}_D$-module. A $(D, F)$-framed sheaf is a pair $(E, \alpha)$ where $E$ is a coherent sheaf of positive rank where ker $\alpha$ is nonzero and torsion free, $E$ is locally free in a neighborhood of $D$ and $\alpha|_D$ is an isomorphism. It is possible to prove that there exists a rational polynomial $\delta$ such that there is an open subset in $\mathcal{M}^{\text{reg}}_\delta(X; F, P)$ parametrizing $(D, F)$-framed sheaves on $X$ with Hilbert polynomial $P$. Moreover if the surface $X$ is rational and $D$ is a smooth irreducible big and nef curve of genus zero, the moduli space of $(D, F)$-framed sheaves is a nonsingular quasi-projective variety (see [2]). It is possible to generalize the definition of $(D, F)$-framed sheaves to varieties of arbitrary dimension (see Definition [2]).
Leaving aside the results on the representability of the moduli functor $\mathcal{M}_s^{(s)}(X; F, P)$ discussed, a complete theory of framed sheaves and a study of the geometry of their moduli spaces is missing in the literature. In the present paper we fill one of the gaps of the theory, by providing a generalization of the Mehta-Ramanathan restriction theorems:

**Theorem.** Let $X$ be a nonsingular, projective, irreducible variety of dimension $d$, defined over an algebraically closed field $k$ of characteristic zero, endowed with a very ample line bundle $\mathcal{O}_X(1)$. Let $F$ be a coherent sheaf on $X$ supported on a divisor $D_{fr}$. Let $E = (E, \alpha: E \to F)$ be a framed sheaf on $X$ of positive rank with nontrivial framing. If $E$ is $\mu$-semistable with respect to $\delta_1$, then there is a positive integer $a_0$ such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in U_a$ the divisor $D$ is smooth, meets transversely the divisor $D_{fr}$ and $E|_D$ is $\mu$-semistable with respect to $a\delta_1$.

The same statement holds with “$\mu$-semistable” replaced by “$\mu$-stable” under the following additional assumptions: the framing sheaf $F$ is a locally free $\mathcal{O}_{D_{fr}}$-module and $E$ is a $(D_{fr}, F)$-framed sheaf on $X$.

Mehta-Ramanathan restriction theorems are very useful as they often allow one to reduce a problem from a higher-dimensional variety to a surface or even to a curve, as for example happens with the proof of Hitchin-Kobayashi correspondence (see Chapter VI in [11]).

The classical Mehta-Ramanathan restriction theorems are also used in the algebro-geometric construction of the Uhlenbeck-Donaldson compactification of moduli space of $\mu$-stable vector bundles on a nonsingular projective surface (see [14] and [10], Section 8.2). In the same way, our framed version of these theorems is used in a work of Bruzzo, Markushevich and Tikhomirov in the construction of the Uhlenbeck-Donaldson compactification for framed sheaves (see [3]). In this way they provide a generalization of the morphism $\pi_r$ (see formula (1)) to an arbitrary smooth projective complex surface.

The main difficulty in the generalization of Mehta-Ramanathan restriction theorems has been the lack of some basic tools in the theory of framed sheaves. In this paper, we complete the study of the semistability condition for framed sheaves. We construct the (relative) Harder-Narasimhan filtration, used in the proof of the first restriction theorem. We also define the Jordan-Hölder filtration and construct the (extended) socle, used in the proof of the second restriction theorem. We want to emphasize that the the second theorem is not proved in the same generality as the first one. This is due to some technical problems: for example, in general it is impossible to define a natural framing on the double dual of the underlying sheaf of a framed sheaf; moreover, under the assumption that $F$ is an arbitrary coherent sheaf, if a framed sheaf is simple, it is no more true that it is remains simple upon the restriction to a (general) divisor. To circumvent these difficulties, we had to strengthen the hypotheses.

All the results we proved from Section 3 to Section 8 are stated for the semistability condition. Obviously, they hold also for the $\mu$-semistability condition (cf. Remark 64 and the subsequent theorem).

Each section of the paper starts with a summary which describes when the results in the framed case coincide with the corresponding ones in the nonframed case or when there are unexpected phenomena. We refer to the book [10] of Huybrechts and Lehn for the nonframed case.
Conventions. Let $E$ be a coherent sheaf on a Noetherian scheme $Y$. The support of $E$ is the closed set $\text{Supp}(E) := \{ x \in Y \mid E_x \neq 0 \}$. Its dimension is called the dimension of $E$ and is denoted by $\dim(E)$. The sheaf $E$ is pure if for all nontrivial coherent subsheaves $E' \subset E$, we have $\dim(E') = \dim(E)$. Let us denote by $T(E)$ the torsion subsheaf of $E$, i.e., the maximal subsheaf of $E$ of dimension less or equal to $\dim(E) - 1$.

Let $Y$ be a projective scheme over a field. Recall that the Euler characteristic of a coherent sheaf $E$ is $\chi(E) := \sum (-1)^i \dim H^i(Y, E)$. Let $P(E, n) := \chi(E \otimes \mathcal{O}(n))$ be the Hilbert polynomial of $E$ and $\det(E)$ its determinant line bundle (cf. Section 1.1.17 in [10]). The degree of $E$, $\deg(E)$, is the integer $c_1(\det(E)) \cdot H^{d-1}$, where $H \in |\mathcal{O}(1)|$ is a hyperplane section.

By Lemma 1.2.1 in [10], the Hilbert polynomial $P(E)$ can be uniquely written in the form

$$P(E, n) = \sum_{i=0}^{\dim(E)} \beta_i(E) \frac{n^i}{i!},$$

where $\beta_i(E)$ are rational coefficients. Moreover for $E \neq 0$, $\beta_{\dim(E)}(E) > 0$. We call $\text{hat-slope}$ the quantity

$$\hat{\mu}(E) = \frac{\beta_{\dim(E)-1}(E)}{\beta_{\dim(E)}(E)}.$$

For a polynomial $P(n) = \sum_{i=0}^{t} \beta_i n^i / i!$ with nonzero leading coefficient $\beta_t$, we define $\hat{\mu}(P) = \beta_{t-1} / \beta_t$.

Let $Y \to S$ be a morphism of finite type of Noetherian schemes. If $T \to S$ is an $S$-scheme, we denote by $Y_T$ the fibre product $T \times_S Y$ and by $p_T : Y_T \to T$ and $p_Y : Y_T \to Y$ the natural projections. If $E$ is a coherent sheaf on $Y$, we denote by $E_T$ its pull-back to $Y_T$. We use similar notation for morphisms between coherent sheaves on $Y$.

For $s \in S$ we denote by $Y_s$ the fibre $\text{Spec}(k(s)) \times_S Y$. For a coherent sheaf $E$ on $Y$, we denote by $E_s$ its pull-back to $Y_s$. Often, we shall think of $E$ as a collection of sheaves $E_s$ parametrized by $s \in S$. If $\alpha : E \to F$ is a morphism of coherent sheaves on $Y$, $\alpha_s$ denotes its pull-back to $Y_s$.

Whenever a scheme has a base field, we assume that the latter is an algebraically closed field $k$ of characteristic zero.

A polarized variety of dimension $d$ is a pair $(X, \mathcal{O}_X(1))$, where $X$ is a nonsingular, projective, irreducible variety of dimension $d$, defined over $k$, and $\mathcal{O}_X(1)$ a very ample line bundle. The canonical line bundle of $X$ is denoted by $\omega_X$ and its associated class of linear equivalence of divisors by $K_X$.

Let $E$ be a coherent sheaf on $X$. By the Hirzebruch-Riemann-Roch theorem the coefficients of the Hilbert polynomial of $E$ are polynomial functions of its Chern classes, in particular

$$P(E, n) = \deg(X) \text{rk}(E) \frac{n^d}{d!} + \left( \deg(E) - \text{rk}(E) \frac{\deg(\omega_X)}{2} \right) \frac{n^{d-1}}{(d-1)!} + \text{terms of lower order in } n.$$
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2. Preliminaries on framed sheaves

In this section we introduce the notion of framed sheaf and morphism of framed sheaves. Moreover for such objects we introduce some invariants, like the framed Hilbert polynomial and the framed degree. When the framing is zero, a framed sheaf is just its underlying coherent sheaf and these notions coincide with the classical ones (see Section 1.2 of [10]).

Let \((X, \mathcal{O}_X(1))\) be a polarized variety of dimension \(d\). Fix a coherent sheaf \(F\) on \(X\) and a polynomial \(\delta(n) = \delta_1 n^{d-1} - \delta_2 (d-2)! + \cdots + \delta_d \in \mathbb{Q}[n]\) with \(\delta_1 > 0\). We call \(F\) framing sheaf and \(\delta\) stability polynomial.

Definition 1. A framed sheaf on \(X\) is a pair \(E := (E, \alpha)\), where \(E\) is a coherent sheaf on \(X\) and \(\alpha : E \to F\) is a morphism of coherent sheaves. We call \(\alpha\) framing of \(E\).

For any framed sheaf \(E = (E, \alpha)\), we define the function \(\epsilon(\alpha)\) by

\[
\epsilon(\alpha) := \begin{cases} 
1 & \text{if } \alpha \neq 0, \\
0 & \text{if } \alpha = 0.
\end{cases}
\]

The framed Hilbert polynomial of \(E\) is

\[
P(E, n) := P(E, n) - \epsilon(\alpha)\delta(n),
\]

and the framed degree of \(E\) is

\[
\deg(E) := \deg(E) - \epsilon(\alpha)\delta_1.
\]

We call Hilbert polynomial of \(E\) the Hilbert polynomial \(P(E)\) of \(E\). If \(E\) is a \(d\)-dimensional coherent sheaf, we define the rank of \(E\) as the rank of \(E\). The reduced framed Hilbert polynomial of \(E\) is

\[
p(E, n) := \frac{P(E, n)}{\text{rk}(E)}
\]

and the framed slope of \(E\) is

\[
\mu(E) := \frac{\deg(E)}{\text{rk}(E)}.
\]

If \(E'\) is a subsheaf of \(E\) with quotient \(E'' = E/E'\), the framing \(\alpha\) induces framings \(\alpha' := \alpha|_{E'}\) on \(E'\) and \(\alpha'' := \alpha|_{E''}\) on \(E''\), where the framing \(\alpha''\) is defined in the following way: \(\alpha'' = 0\) if \(\alpha' \neq 0\), else \(\alpha''\) is the induced morphism on \(E''\). With this convention the framed Hilbert polynomial of \(E\) behaves additively:

\[
P(E) = P(E', \alpha') + P(E'', \alpha'')
\]
and the same happens for the framed degree:

\[ \deg(E) = \deg(E', \alpha') + \deg(E'', \alpha''). \]

**Notation:** If \( E = (E, \alpha) \) is a framed sheaf on \( X \) and \( E' \) is a subsheaf of \( E \), then we denote by \( E' \) the framed sheaf \((E', \alpha')\) and by \( E/E' \) the framed sheaf \((E'', \alpha'')\).

Thus we have a canonical framing on subsheaves and on quotients. The same happens for subquotients, indeed we have the following result.

**Lemma 2** (Lemma 1.12 in [9]). Let \( H \subset G \subset E \) be coherent sheaves and \( \alpha \) a framing of \( E \). Then the framings induced on \( G/H \) as a quotient of \( G \) and as a subsheaf of \( E/H \) agree. Moreover

\[
P\left(\frac{E/H}{G/H}\right) = P\left(\frac{E}{G}\right) \quad \text{and} \quad \deg\left(\frac{E/H}{G/H}\right) = \deg\left(\frac{E}{G}\right).
\]

Now we introduce the notion of a morphism of framed sheaves.

**Definition 3.** Let \( E = (E, \alpha) \) and \( G = (G, \beta) \) be framed sheaves. A morphism of framed sheaves \( \varphi: E \rightarrow G \) between \( E \) and \( G \) is a morphism of the underlying coherent sheaves \( \varphi: E \rightarrow G \) for which there is an element \( \lambda \in k \) such that \( \beta \circ \varphi = \lambda \alpha \). We say that \( \varphi: E \rightarrow G \) is injective (surjective) if the morphism \( \varphi: E \rightarrow G \) is injective (surjective).

**Remark 4.** Let \( E = (E, \alpha) \) be a framed sheaf. If \( E' \) is a subsheaf of \( E \) with quotient \( E'' = E/E' \), then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & E' \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
F & \longrightarrow & E \\
\downarrow{\lambda} & & \downarrow{\mu} \\
F & \longrightarrow & F
\end{array}
\]

where \( \lambda = 0, \mu = 1 \) if \( \alpha' = 0 \), and \( \lambda = 1, \mu = 0 \) if \( \alpha' \neq 0 \). Thus the inclusion morphism \( i \) (the projection morphism \( q \)) induces a morphism of framed sheaves between \( E' \) and \( E \) (\( E \) and \( E/E' \)). Note that in general an injective (surjective) morphism \( E \rightarrow G \) between the underlying sheaves of two framed sheaves \( E = (E, \alpha) \) and \( G = (G, \beta) \) does not lift to a morphism \( E \rightarrow G \) of the corresponding framed sheaves.  

**Lemma 5** (Lemma 1.5 in [9]). Let \( E = (E, \alpha) \) and \( G = (G, \beta) \) be framed sheaves. The set \( \mathrm{Hom}(E, G) \) of morphisms of framed sheaves is a linear subspace of \( \mathrm{Hom}(E, G) \). If \( \varphi: E \rightarrow G \) is an isomorphism, then the factor \( \lambda \) in the definition can be taken in \( k^* \). In particular, the isomorphism \( \varphi_0 = \lambda^{-1} \varphi \) satisfies \( \beta \circ \varphi_0 = \alpha \). Moreover, if \( E \) and \( G \) are isomorphic, then their framed Hilbert polynomials and their framed degrees coincide.

**Proposition 6.** Let \( E = (E, \alpha) \) and \( G = (G, \beta) \) be framed sheaves. If \( \varphi \) is a nontrivial morphism of framed sheaves between \( E \) and \( G \), then

\[
P\left(\frac{E}{\ker \varphi}, \alpha''\right) \leq P(\text{Im} \varphi, \beta') \quad \text{and} \quad \deg\left(\frac{E}{\ker \varphi}, \alpha''\right) \leq \deg(\text{Im} \varphi, \beta').
\]

**Proof.** Consider a morphism of framed sheaves \( \varphi \in \mathrm{Hom}(E, G) \), \( \varphi \neq 0 \). There exists \( \lambda \in k \) such that \( \beta \circ \varphi = \lambda \alpha \). Note that \( E/\ker \varphi \simeq \text{Im} \varphi \) hence their Hilbert polynomials and their degree coincide. It remains to prove that \( \epsilon(\alpha'') \geq \epsilon(\beta') \). If \( \lambda = 0 \), then \( \beta' = 0 \) and therefore \( \epsilon(\beta') = 0 \leq \epsilon(\alpha'') \). Assume now \( \lambda \neq 0 \): \( \alpha = 0 \) if and only if \( \beta|_{\text{Im} \varphi} = 0 \), hence \( \epsilon(\beta') = 0 = \epsilon(\alpha'') \). If \( \alpha \neq 0 \), then also \( \alpha'' \neq 0 \). Indeed if \( \alpha'' = 0 \), then \( \alpha||_{\ker \varphi} \neq 0 \); this implies that \( \lambda|_{\ker \varphi} = (\beta \circ \varphi)|_{\ker \varphi} = 0 \) and therefore \( \lambda = 0 \), but this is in contradiction with our previous assumption. Thus, if \( \lambda \neq 0 \) and \( \alpha \neq 0 \) then we obtain \( \epsilon(\beta') = 1 = \epsilon(\alpha'') \). \(\square\)
Remark 7. Let $E = (E, \alpha)$ and $G = (G, \beta)$ be framed sheaves and $\varphi : E \to G$ a nontrivial morphism of framed sheaves. By the previous proposition, we get

$$P(E) = P(\ker \varphi, \alpha') + P(\ker \varphi, \beta') \leq P(\ker \varphi, \alpha') + P(\Im \varphi, \beta').$$

The inequality may be strict. This phenomenon does not appear in the nonframed case and depends on the fact that in general the isomorphism $E/\ker \varphi \cong \Im \varphi$ does not induce an isomorphism $E/\ker \varphi \cong (\Im \varphi, \beta')$ (there are examples where indeed it does not). △

3. Semistability

In this section we give a generalization to framed sheaves of Gieseker’s (semi)stability condition for coherent sheaves (see Definition 1.2.4 in [10]). Comparing with the classical case, the (semi)stability condition for framed sheaves has an additional parameter $\delta$, which is a polynomial with rational coefficients. This definition was given in Huybrechts and Lehn’s article [8]; we only had to modify it for the case of torsion sheaves. The necessity to handle torsion sheaves is due to the fact that even if we want to work only with framed sheaves $E = (E, \alpha)$ with $E$ torsion free, the graded factors of the framed Harder-Narasimhan or Jordan-Hölder filtrations of $E$ may be torsion. We will also present examples where the underlying coherent sheaf of a semistable framed sheaf is not necessarily torsion free, and examples of non-semistable framed sheaves $(E, \alpha)$ with $E$ Gieseker-semistable (see Example 10).

Recall that there is a natural ordering of rational polynomials given by the lexicographic order of their coefficients. Explicitly, $f \leq g$ if and only if $f(m) \leq g(m)$ for all $m \gg 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for all $m \gg 0$.

We shall use the following convention: if the word “(semi)stable” occurs in any statement in combination with the symbol $(\leq)$, then two variants of the statement are asserted at the same time: a “semistable” one involving the relation “$\leq$” and a “stable” one involving the relation “$<$”.

We now give a definition of semistability for framed sheaves $E = (E, \alpha)$ of positive rank.

Definition 8. A framed sheaf $E = (E, \alpha)$ of positive rank is said to be (semi)stable with respect to $\delta$ if and only if the following conditions are satisfied:

(i) $\rk(E)P(E') (\leq) \rk(E')P(E)$ for all subsheaves $E' \subset \ker \varphi$,

(ii) $\rk(E)(P(E') - \delta) (\leq) \rk(E')P(E)$ for all subsheaves $E' \subset E$.

Lemma 9 (Lemma 1.2 in [9]). Let $E = (E, \alpha)$ be a framed sheaf of positive rank. If $E$ is (semi)stable with respect to $\delta$, then $\ker \alpha$ is torsion free.

Proof. Let $T(\ker \alpha)$ denote the torsion sheaf of $\ker \alpha$. By the semistability condition, we get

$$\rk(E)P(T(\ker \alpha), n) (\leq) \rk(T(\ker \alpha))P(E, n - \delta(n))$$

for $n \gg 0$. Since $\rk(T(\ker \alpha)) = 0$, we get $P(T(\ker \alpha), n) (\leq) 0$ for $n \gg 0$. On the other hand, if $T(\ker \alpha) \neq 0$, then the leading coefficient of $P(T(\ker \alpha), n)$ is positive. Thus we get a contradiction and therefore $T(\ker \alpha) = 0$. □

Example 10. Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension $d$ and $D = D_1 + \cdots + D_l$ an effective divisor on $X$, where $D_1, \ldots, D_l$ are distinct prime divisors. Consider the short exact sequence associated to the line bundle $\mathcal{O}_X(-D)$:

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \xrightarrow{\alpha} i_*(\mathcal{O}_Y) \to 0,$$
where \( Y := \text{Supp}(D) = D_1 \cup \cdots \cup D_l \). Recall that
\[
P(i_*(O_Y)) = \deg(Y) \frac{n^{d-1}}{(d-1)!} + \text{terms of lower degree in } n.
\]
Let \( \delta(n) \in \mathbb{Q}[n] \) be a polynomial of degree \( d-1 \) such that \( \delta > P(i_*(O_Y)) \). Then we get
\[
P(O_X, n) - \delta(n) < P(O_X, n) - P(i_*(O_Y), n) = P(O_X(-D), n) < P(O_X, n).
\]
Thus in this way we have the framed sheaf \((O_X, \alpha : O_X \to i_*(O_Y))\) is not semistable with respect to \( \delta \). We thus have obtained an example of a framed sheaf which is not semistable with respect to a fixed \( \delta \) but the underlying coherent sheaf is Gieseker-semistable. It is possible to construct examples of semistable framed sheaves whose underlying coherent sheaves are not Gieseker-semistable, how we will see in Example 43.

On the other hand, it is easy to check that the framed sheaf \((O_X(-D) \oplus i_*(O_Y), \alpha : O_X(-D) \oplus i_*(O_Y) \to i_*(O_Y))\) is semistable with respect to \( \delta := P(i_*(O_Y)) \) and the underlying coherent sheaf has a nonzero torsion subsheaf. \( \Delta \)

**Definition 11.** A framed sheaf \( E = (E, \alpha) \) of positive rank is geometrically stable with respect to \( \delta \) if for any base extension \( X \times_{\text{Spec}(k)} \text{Spec}(K) \to X \), the pull-back \( f^*(E) := (f^*(E), f^*(\alpha)) \) is stable with respect to \( \delta \).

A stable framed sheaf may not be geometrically stable. The two notions coincide only for a particular class of framed sheaves of positive rank, as we will show in Section 7.1.

We have the following characterization of the semistability condition in terms of quotients:

**Proposition 12.** Let \( E = (E, \alpha) \) be a framed sheaf of positive rank. Then the following conditions are equivalent:

(a) \( E \) is semistable with respect to \( \delta \).

(b) For any surjective morphism of framed sheaves \( \varphi : E \to (Q, \beta) \), one has \( \text{rk}(Q)p(E) \leq P(Q, \beta) \).

**Proof.** By using Proposition 6, the assertion follows from the same arguments as in the non-framed case (see Proposition 1.2.6 in [10]). \( \square \)

In the papers by Huybrechts and Lehn, one finds two different definitions of the (semi)stability of rank zero framed sheaves. In [8], they use the same definition for the framed sheaves of positive or zero rank, and with that definition, all framed sheaves of rank zero are automatically semistable but not stable (with respect to any stability polynomial \( \delta \)). According to Definition 1.1 in [9], the semistability of a rank zero framed sheaf depends on the choice of a stability polynomial \( \delta \), but all semistable framed sheaves of rank zero are automatically stable. Now we give a new definition of the (semi)stability for rank zero framed sheaves which singles out exactly those objects which may appear as torsion components of the Harder-Narasimhan and Jordan-Hölder filtrations.

**Definition 13.** Let \( E = (E, \alpha) \) be a framed sheaf with \( \text{rk}(E) = 0 \). If \( \alpha \) is injective, we say that \( E \) is semistable\(^1\). Moreover, if \( P(E) = \delta \) we say that \( E \) is stable with respect to \( \delta \).

\(^1\)For torsion sheaves, the definition of semistability of the corresponding framed sheaves does not depend on \( \delta \).
Lemma 14 (Lemma 2.1 in [8]). Let $E = (E, \alpha)$ be a framed sheaf with ker $\alpha$ nonzero and $\alpha$ surjective. If $E$ is (semi)stable with respect to $\delta$, then
\[ \delta (\leq) P(E) - \frac{\text{rk}(E)}{\text{rk}(\text{ker } \alpha)} (P(E) - P(F)). \]

If $F$ is a torsion sheaf, then $\delta (\leq) P(F)$ and in particular $\delta_1 (\leq) \deg(F)$.

Proof. By the (semi)stability condition, we get $\text{rk}(E) P(\text{ker } \alpha) (\leq) \text{rk}(\text{ker } \alpha) P(E) = \text{rk}(\text{ker } \alpha) (P(E) - \delta)$.

Since $\text{rk}(\text{ker } \alpha) > 0$ by Lemma 9, we obtain $\delta (\leq) P(E) - \text{rk}(E) P(\text{ker } \alpha)$.

Since $P(E) - P(\text{ker } \alpha) = P(\text{Im } \alpha) = P(F)$, we obtain the assertion. Moreover, if $F$ is a torsion sheaf, then $\text{rk}(\text{Im } \alpha) = 0$. Therefore $\text{rk}(\text{ker } \alpha) = \text{rk}(E)$ and
\[ \delta (\leq) P(E) - P(\text{ker } \alpha) = P(F). \]

In particular, by formula (2) we obtain $\delta_1 (\leq) \deg(F)$. □

4. Characterization of semistability by means of framed saturated subsheaves

Let $E = (E, \alpha)$ be a framed sheaf, and assume that ker $\alpha$ is nonzero and torsion free. In this section we would like to answer the following question: to verify if $E$ is (semi)stable or not, do we need to check the inequalities (i) and (ii) in the Definition 8 for all subsheaves of $E$? Or, can we restrict our attention to a smaller family of subsheaves of $E$? For Gieseker’s (semi)stability condition, this latter family consists of saturated subsheaves of $E$ (see Proposition 1.2.6 in [10]). In the framed case, we need to enlarge this family because of the framing, as we explain in what follows.

Definition 15. Let $E$ be a coherent sheaf. The saturation of a subsheaf $E' \subset E$ is the minimal subsheaf $\bar{E}' \subset E$ containing $E'$ such that the quotient $E/\bar{E}'$ is pure of dimension $\dim(E)$ or zero.

Now we generalize this definition to framed sheaves:

Definition 16. Let $E = (E, \alpha)$ be a framed sheaf where ker $\alpha$ is nonzero and torsion free. Let $E'$ be a subsheaf of $E$. The framed saturation $\bar{E}'$ of $E'$ is the saturation of $E'$ as subsheaf of

- ker $\alpha$, if $E' \subset \text{ker } \alpha$.
- $E$, if $E' \nsubset \text{ker } \alpha$.

Remark 17. Let $\bar{E}'$ be the framed saturation of $E' \subset E$. In the first case described in the definition, if $\text{rk}(E') < \text{rk}(\text{ker } \alpha)$, the quotient $Q = E/\bar{E}'$ is a coherent sheaf of positive rank, with nonzero induced framing $\beta$, and fits into an exact sequence
\[ 0 \longrightarrow Q' \longrightarrow Q \overset{\beta}{\longrightarrow} \text{Im } \alpha \longrightarrow 0, \]
where $Q' = \text{ker } \beta$ is a torsion free quotient of ker $\alpha$. If $\text{rk}(E') = \text{rk}(\text{ker } \alpha)$, then $\bar{E}' = \text{ker } \alpha$ and $Q = E/\text{ker } \alpha$. In the second case, $Q$ is a torsion free sheaf with zero induced framing. Moreover

- $\text{rk}(E') = \text{rk}(\bar{E}')$, $P(E') \leq P(\bar{E}')$ and $\deg(E') \leq \deg(\bar{E}')$. 

• \( P(\mathcal{E}') \leq P(\tilde{\mathcal{E}}') \) and \( \deg(\mathcal{E}') \leq \deg(\tilde{\mathcal{E}}') \).  

\[ \triangle \]

**Example 18.** Let us consider the framed sheaf \((\mathcal{O}_X, \alpha : \mathcal{O}_X \to i_*(\mathcal{O}_Y))\) on \( X \), defined in Example 10. Since \( \ker \alpha = \mathcal{O}_X(-D) \), the saturation of \( \mathcal{O}_X(-D) \) (as subsheaf of \( \mathcal{O}_X \)) is \( \mathcal{O}_X \) but the framed saturation of \( \mathcal{O}_X(-D) \) is \( \mathcal{O}_X(-D) \).  

\[ \triangle \]

We have the following characterization:

**Proposition 19.** Let \( \mathcal{E} = (E, \alpha) \) be a framed sheaf where \( \ker \alpha \) is nonzero and torsion free. Then the following conditions are equivalent:

(a) \( E \) is semistable with respect to \( \delta \).

(b) For any framed saturated subsheaf \( E' \subset E \) one has \( P(E', \alpha') \leq \rk(E')p(E) \).

(c) For any surjective morphism of framed sheaves \( \varphi : \mathcal{E} \to (Q, \beta) \), where \( \alpha = \beta \circ \varphi \) and \( Q \) is one of the following:

- a coherent sheaf of positive rank with nonzero framing \( \beta \) such that \( \ker \beta \) is nonzero and torsion free,
- a torsion free sheaf with zero framing \( \beta \),
- \( Q = E/\ker \alpha \),

one has \( \rk(Q)p(E) \leq P(Q, \beta) \).

**Proof.** The implication \((a) \Rightarrow (b)\) is obvious. By Remark 17, \( P(\mathcal{E}') \leq P(\tilde{\mathcal{E}}') \leq \rk(\tilde{\mathcal{E}}')p(\mathcal{E}) = \rk(E')p(\mathcal{E}) \), where \( E' \) is the framed saturation of \( \mathcal{E}' \), thus \((b) \Rightarrow (a)\). Finally, the framed sheaf \( \mathcal{Q} \) has the properties asserted in condition \((c)\) if and only if \( \ker \varphi \) is a framed saturated subsheaf of \( \mathcal{E} \), hence \((b) \iff (c)\).  

\[ \square \]

**Corollary 20.** Let \( \mathcal{E} = (E, \alpha) \) and \( \mathcal{G} = (G, \beta) \) be framed sheaves of positive rank with the same reduced framed Hilbert polynomial \( p \).

1. If \( \mathcal{E} \) is semistable and \( \mathcal{G} \) is stable, then any nontrivial morphism \( \varphi : \mathcal{E} \to \mathcal{G} \) is surjective.

2. If \( \mathcal{E} \) is stable and \( \mathcal{G} \) is semistable, then any nontrivial morphism \( \varphi : \mathcal{E} \to \mathcal{G} \) is injective.

3. If \( \mathcal{E} \) and \( \mathcal{G} \) are stable, then any nontrivial morphism \( \varphi : \mathcal{E} \to \mathcal{G} \) is an isomorphism. Moreover, in this case \( \Hom(\mathcal{E}, \mathcal{G}) \simeq k \). If in addition \( \alpha \neq 0 \), or equivalently, \( \beta \neq 0 \), there is a unique isomorphism \( \varphi_0 \) with \( \beta \circ \varphi_0 = \alpha \).

**Definition 21.** Let \( \mathcal{E} \) be a framed sheaf. We say that \( \mathcal{E} \) is **simple** if \( \End(\mathcal{E}) \simeq k \).

As in the unframed case, a stable framed sheaf of positive rank is simple.

### 5. Maximal destabilizing framed subsheaf

Let \( \mathcal{E} = (E, \alpha) \) be a framed sheaf with \( \ker \alpha \) nonzero and torsion free. If \( \mathcal{E} \) is not semistable with respect to \( \delta \), then there exist destabilizing subsheaves of \( \mathcal{E} \). In this section we would like to find the maximal one (with respect to the inclusion) and show that it has some interesting properties. Because of the framing, it is possible that this subsheaf has rank zero or it is not saturated and we want to emphasize that this kind of situations are not possible in the nonframed case (see Lemma 1.3.5 in [10]).
Proposition 22. Let $E = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. If $E$ is not semistable with respect to $\delta$, then there is a framed saturated subsheaf $G \subset E$ such that for any subsheaf $E' \subset E$ one has

$$\text{rk}(E') P(G) \geq \text{rk}(G) P(E')$$

and in case of equality, one has $E' \subset G$.

Moreover, the framed sheaf $G$ is uniquely determined and is semistable with respect to $\delta$.

Definition 23. We call $G$ the maximal destabilizing framed subsheaf of $E$.

Proof of Proposition 22. On the set of nontrivial subsheaves of $E$ we define the following order relation $\preceq$: let $G_1$ and $G_2$ be nontrivial subsheaves of $E$, $G_1 \preceq G_2$ if and only if $G_1 \subseteq G_2$ and $\text{rk}(G_2) P(G_1) \leq \text{rk}(G_1) P(G_2)$. Since any ascending chain of subsheaves stabilizes, for any subsheaf $E'$, there is a subsheaf $G'$ such that $E' \subset G' \subset E$ and $G'$ is maximal with respect to $\preceq$.

First assume that there exists a subsheaf $E'$ of rank zero with $P(E') > 0$, that is, $P(E') > \delta$. Let $T(E)$ be the torsion subsheaf of $E$. Then $P(T(E)) \geq P(E') > \delta$. Hence $E' \preceq T(E)$. Moreover, there are no subsheaves $G \subset E$ of positive rank such that $T(E) \preceq G$. Indeed, should that be the case, by the definition of $\preceq$ we would obtain $P(T(E)) - \delta \leq 0$, in contradiction with the previous inequality. Thus we choose $G := T(E)$. Since $\ker \alpha |_{E'} = 0$, $G$ is semistable.

From now on assume that for every rank zero subsheaf $T \subset E$ we have $P(T, \alpha') \leq 0$. Let $G \subset E$ be a $\preceq$-maximal subsheaf with minimal rank among all $\preceq$-maximal subsheaves. Note that $\text{rk}(G) > 0$. Suppose there exists a subsheaf $H \subset E$ with $\text{rk}(H) P(G) < P(H)$. By hypothesis we have $\text{rk}(H) > 0$. From $\preceq$-maximality of $G$ we get $G \preceq H$ (in particular $H \neq E$). Now we want to show that we can assume $H \subset G$ by replacing $H$ with $G \cap H$.

If $H \nsubseteq G$, then the morphism $\varphi: H \to E \to E/\alpha$ is nonzero. Moreover $\ker \varphi = G \cap H$. The sheaf $I = \text{Im} \varphi$ is of the form $J/\alpha$ with $G \subsetneq J \subset E$ and $\text{rk}(J) > 0$. By the $\preceq$-maximality of $G$ we have $p(J) < p(G)$, hence we obtain

$$\text{rk}(G) P(I) = \text{rk}(G)(P(J) - P(G)) < \text{rk}(J) P(G) - \text{rk}(G) P(G) = \text{rk}(I) P(G),$$

and therefore

$$\text{rk}(G) P(I) < \text{rk}(I) P(G).$$

Now we want to prove the following:

Claim: The sheaf $G \cap H$ is a nontrivial subsheaf of positive rank of $E$.

Proof. Assume that $G \cap H = 0$. In this case, we get $H \cong I$; moreover this isomorphism lifts to an isomorphism $\mathcal{H} \cong I$ of the corresponding framed sheaves and therefore $\varphi$ lifts to a morphism of framed sheaves $\varphi$ between $\mathcal{H}$ and $E/G$. By Proposition 6, $P(\mathcal{H}) \leq P(I)$ and using formula (6) one has

$$\text{rk}(H) P(G) < \text{rk}(G) P(\mathcal{H}) \leq \text{rk}(G) P(I) < \text{rk}(H) P(G),$$

which is absurd.

The rank of $G \cap H$ is positive, indeed if we assume that $\text{rk}(G \cap H) = 0$, then we have $\text{rk}(I) = \text{rk}(H)$ and again by Proposition 6 and formula (6) we get

$$\text{rk}(G) P(G \cap H, \alpha') = \text{rk}(G) P(\mathcal{H}) - \text{rk}(G) P(\mathcal{H}/G \cap H) \geq \text{rk}(G) P(\mathcal{H}) - \text{rk}(G) P(I) > \text{rk}(G) P(I) - \text{rk}(H) P(G) > 0$$
hence $G \cap H$ is a rank zero subsheaf of $E$ with $P(G \cap H, \alpha') > 0$, but this is in contradiction with the hypothesis. □

By the following computation:
\[
\text{rk}(G \cap H) (p(G \cap H, \alpha') - p(H)) = \text{rk}(H) (p(H) - p(H/\alpha' \cap H))
\]
\[
> \text{rk}(I) (p(H) - p(I)) > \text{rk}(I) (p(H) - p(G)) > 0
\]
we get $p(H) < p(G \cap H, \alpha')$, hence from now on we can consider a subsheaf $H \subset G$ such that $H$ is $\leq$-maximal in $G$, $\text{rk}(H) > 0$ and
\[
p(G) < p(H).
\]
Let $H' \subset E$ be a sheaf that contains $H$ and is $\leq$-maximal in $E$. In particular, one has
\[
p(G) < p(H) \leq p(H').
\]
By $\leq$-maximality of $H$ and $G$, we have $H' \not\subset G$. Then the morphism $\psi: H' \to E \to E/G$ is nonzero and $H \subset \ker \psi$. As before
\[
p(H') < p(\ker \psi, \alpha').
\]
Thus we have $H \subset H' \cap G = \ker \psi$ and $p(H) < p(\ker \psi, \alpha')$, hence $H \leq \ker \psi$. This contradicts the $\leq$-maximality of $H$ in $G$. Thus for all subsheaves $H \subset E$, we have $\text{rk}(H)p(G) \geq P(H)$.

If there is a subsheaf $H \subset E$ of rank zero such that $P(H) = 0$ and $H \not\subset G$, then by using the same argument as before, we get $P(H \cap G, \alpha') > 0$, but this is in contradiction with the hypothesis. So there are no subsheaves $H \subset E$ of rank zero such that $P(H) = 0$ and $H \not\subset G$. If there is a subsheaf $H \subset E$ of positive rank such that $p(G) = p(H)$, then $H \subset G$. In fact, if $H \not\subset G$ then we can replace $H$ by $G \cap H$ and using the same argument as before we obtain $p(G) = p(H) < p(H \cap G, \alpha')$ and this is absurd. □

**Minimal destabilizing framed quotient.** Let $E = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Suppose that $E$ is not semistable with respect to $\delta$.

**Remark 24.** If the rank of the framing sheaf $F$ is zero, we further assume that $\ker \alpha$ is not the maximal destabilizing framed subsheaf. △

Let $T_1$ be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that
- $Q$ is torsion free,
- the induced framing on $\ker q$ is nonzero,
- $p(Q) < p(E)$.

Let $T_2$ be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that
- $Q$ has positive rank,
- the induced framing on $\ker q$ is zero,
- As in Remark 17, $Q$ fits into an exact sequence of the form [5],
- $p(Q) < p(E)$.

By Proposition 19 the set $T_1 \cup T_2$ is nonempty. For $i = 1, 2$ define an order relation on $T_i$ as follows: if $Q_1, Q_2 \in T_i$, we say that $Q_1 \sqsubseteq Q_2$ if and only if $p(Q_1) \leq p(Q_2)$ and $\text{rk}(Q_1) \leq \text{rk}(Q_2)$ in the case $p(Q_1) = p(Q_2)$. 

Let us consider the relation $\sqsupset$ defined in the following way: for $Q_1, Q_2 \in T_i$, we have $Q_1 \sqsupset Q_2$ if and only if $Q_1 \subseteq Q_2$ and $p(Q_1) < p(Q_2)$ or $\text{rk}(Q_1) < \text{rk}(Q_2)$ in the case $p(Q_1) = p(Q_2)$. Let $Q_\leftarrow_i$ be a $\sqsupset$-minimal element in $T_i$, for $i = 1, 2$. Define

$$Q_- := \begin{cases} Q_1^- & \text{if } p(Q_1^-) < p(Q_2^-) \text{ or if } p(Q_2^-) = p(Q_1^-) \text{ and } \text{rk}(Q_1^-) \leq \text{rk}(Q_2^-), \\ Q_2^- & \text{if } p(Q_2^-) < p(Q_1^-) \text{ or if } p(Q_2^-) = p(Q_1^-) \text{ and } \text{rk}(Q_2^-) < \text{rk}(Q_1^-). \end{cases}$$

We call $Q_-$ the minimal destabilizing framed quotient. By easy computations one can prove the following:

**Proposition 25.** The sheaf $G := \ker(E \to Q_-)$ is the maximal destabilizing framed subsheaf of $E$.

By the uniqueness of the maximal destabilizing framed subsheaf, $Q_-$ is unique.

6. Harder-Narasimhan filtration

In this section we construct the Harder-Narasimhan filtration for a framed sheaf. We adapt the techniques used by Harder and Narasimhan in the case of vector bundles on curves (see [7]). When the framing sheaf has rank zero, the rank of the kernel of the framing is equal to the rank of the sheaf and because of this fact we get a more involved characterization of the Harder-Narasimhan filtration than in the nonframed case (as one can see in Proposition 30). The characterization of the Harder-Narasimhan filtration when the framing sheaf has positive rank is similar to the nonframed case (see Theorem 1.3.4 in [10]).

In this section we consider separately the case in which the rank of the framing sheaf $F$ is zero or positive.

In the first case, we can have two types of torsion sheaves as graded factors of the Harder-Narasimhan filtration of a framed sheaf $(E, \alpha)$: the torsion subsheaf $T(E)$ of $E$ and the quotient $F/\ker \alpha$. In the second case, the only torsion sheaf that can appear as a graded factor of the Harder-Narasimhan filtration is the torsion subsheaf.

Consider first the case $\text{rk}(F) = 0$.

**Definition 26.** Let $F$ be a coherent sheaf of rank zero and $E = (E, \alpha : E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. A Harder-Narasimhan filtration for $E$ is an increasing filtration of framed saturated subsheaves

$$HN_{\bullet}(E) : 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_l(E) = E$$

which satisfies the following conditions

(A) the quotient sheaf $gr_i^{HN}(E) := HN_i(E)/HN_{i-1}(E)$ with the induced framing $\alpha_i$ is a semistable framed sheaf with respect to $\delta$ for $i = 1, 2, \ldots, l$.

(B) The quotient $E^{HN}_{/HN_{i-1}(E)}$ has positive rank, the kernel of the induced framing is nonzero and torsion free and the subsheaf $gr_i^{HN}(E)$ is the maximal destabilizing framed subsheaf of $(E^{HN}_{/HN_{i-1}(E)}, \alpha', \alpha'')$ for $i = 1, 2, \ldots, l - 1$.

**Lemma 27.** Let $F$ be a coherent sheaf of rank zero and $E = (E, \alpha : E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Suppose that $E$ is not semistable (with respect to $\delta$). Let $G$ be the maximal destabilizing framed subsheaf of $E$. If $G \neq \ker \alpha$, then for every rank zero subsheaf $T$ of $F/G$, we get $P(T, \beta') \leq 0$, where $\beta$ is the induced framing on $F/G$. 

Theorem 28. Let \( T \subset E/G \) be a rank zero subsheaf with \( P(T, \beta') > 0 \). The sheaf \( T \) is of the form \( E/G \), where \( G \subset E' \) and \( \text{rk}(E') = \text{rk}(G) \), hence we obtain \( p(E') > p(G) \), therefore \( E' \) contradicts the maximality of \( G \).

\[ \square \]

Theorem 28. Let \( E \) be a coherent sheaf of rank zero and \( \mathcal{E} = (E, \alpha: E \to F) \) a framed sheaf where \( \ker \alpha \) is nonzero and torsion free. Then there exists a unique Harder-Narasimhan filtration for \( \mathcal{E} \).

Proof. Existence. If \( \mathcal{E} \) is a semistable framed sheaf with respect to \( \delta \), we put \( l = 1 \) and a Harder-Narasimhan filtration is

\[ \text{HN}_0(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset \text{HN}_1(\mathcal{E}) = E \]

Else there exists a subsheaf \( E_1 \subset E \) such that \( E_1 \) is the maximal destabilizing framed subsheaf of \( \mathcal{E} \). If \( E_1 = \ker \alpha \), a Harder-Narasimhan filtration is

\[ \text{HN}_0(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset \ker \alpha \subset \text{HN}_2(\mathcal{E}) = E \]

Otherwise, by Lemma 27, \( E' \) is a framed sheaf with \( \ker \alpha' \neq 0 \) torsion free and no rank zero destabilizing framed subsheaves. If \( E'/E_1, \alpha'' \) is a semistable framed sheaf, a Harder-Narasimhan filtration is

\[ \text{HN}_0(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset E_1 \subset \text{HN}_2(\mathcal{E}) = E \]

Else there exists a subsheaf \( E'_2 \subset E/E_1 \) of positive rank such that \( E'_2 \) is the maximal destabilizing framed subsheaf of \( (E/E_1, \alpha'') \). We denote by \( E_2 \) its pre-image in \( E \). Now we apply the previous argument to \( E_2 \) instead of \( E_1 \). Thus we can iterate this procedure and we obtain a finite length increasing filtration of framed saturated subsheaves of \( E \), which satisfies conditions (A) and (B).

Uniqueness. The uniqueness of the Harder-Narasimhan filtration follows from the uniqueness of the maximal destabilizing framed subsheaf.

Remark 29. By construction, for \( i > 0 \) at most one of the framings \( \alpha_i \) is nonzero and all but possibly one of the factors \( gr_i^{\text{HN}}(\mathcal{E}) \) are torsion free. In particular if \( \text{rk}(gr_i^{\text{HN}}(\mathcal{E})) = 0 \), then \( gr_i^{\text{HN}}(\mathcal{E}) = T(E) \) and \( \alpha_i \neq 0 \); if \( \text{rk}(gr_i^{\text{HN}}(\mathcal{E})) = 0 \), then \( gr_i^{\text{HN}}(\mathcal{E}) = E/\ker \alpha \) and \( \alpha_i \neq 0 \). \[ \triangle \]

Now we want to relate condition (B) in Definition 26 with the framed Hilbert polynomials of the pieces of the Harder-Narasimhan filtration. In particular we get the following.

Proposition 30. Let \( F \) be a coherent sheaf of rank zero and \( \mathcal{E} = (E, \alpha: E \to F) \) a framed sheaf where \( \ker \alpha \) is nonzero and torsion free. Suppose there exists a filtration of the form \( [7] \) satisfying condition (A). Then condition (B) is equivalent to the following:

(B') the quotient \( (E/\text{HN}_i(\mathcal{E}), \alpha'') \) is a framed sheaf where \( \ker \alpha'' \) is nonzero and torsion free for \( j = 1, 2, \ldots, l - 2 \), it has no rank zero destabilizing framed subsheaves, and

\[ \text{rk}(gr_{i+1}^{\text{HN}}(\mathcal{E}))(gr_i^{\text{HN}}(\mathcal{E}), \alpha_i) > \text{rk}(gr_{i+1}^{\text{HN}}(\mathcal{E}))P(gr_{i+1}^{\text{HN}}(\mathcal{E}, \alpha_{i+1}) \]

for \( i = 1, \ldots, l - 1 \).

Proof. The arguments used to prove this proposition are similar to the one used in the proof of the analogous result for vector bundles on curves (see Lemma 1.3.8 in \[7\]). \[ \square \]
Now we turn to the case in which the rank of $F$ is positive. First, we give the following definition.

**Definition 31.** Let $F$ be a coherent sheaf of positive rank and $\mathcal{E} = (E, \alpha : E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. A **Harder-Narasimhan filtration** for $\mathcal{E}$ is an increasing filtration of framed saturated subsheaves

$$\text{HN}^n(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset \text{HN}_1(\mathcal{E}) \subset \cdots \subset \text{HN}_l(\mathcal{E}) = E$$

which satisfies the following conditions

(A) the quotient sheaf $gr^\text{HN}_i(\mathcal{E}) := \text{HN}_i(\mathcal{E})/\text{HN}_{i-1}(\mathcal{E})$ with the induced framing $\alpha_i$ is a semistable framed sheaf with respect to $\delta$ for $i = 1, 2, \ldots, l$.

(B) the quotient $(F/\text{HN}_i(\mathcal{E}), \alpha'')$ is a framed sheaf where $\ker \alpha''$ is nonzero and torsion free for $i = 1, \ldots, l - 1$, it has no rank zero destabilizing framed subsheaves, and

$$\text{rk}(gr_i^\text{HN}(\mathcal{E}))P(gr_i^\text{HN}(\mathcal{E}), \alpha_i) > \text{rk}(gr_{i+1}^\text{HN}(\mathcal{E}))P(gr_{i+1}^\text{HN}(\mathcal{E}), \alpha_{i+1}).$$

In this case one can prove results similar to those stated in Lemma 27, Theorem 28 and Proposition 30. In particular we get the following:

**Theorem 32.** Let $F$ be a coherent sheaf of positive rank and $\mathcal{E} = (E, \alpha : E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Then there exists a unique Harder-Narasimhan filtration for $\mathcal{E}$.

We conclude this section by proving a result for the maximal destabilizing framed subsheaf of a framed sheaf. This result holds for a framing sheaf of any rank.

**Definition 33.** Let $B$ be a torsion free sheaf on $X$. The **maximal reduced Hilbert polynomial** of $B$ is the reduced Hilbert polynomial of the maximal Gieseker-destabilizing subsheaf of $B$. We denote it by $p_{\text{max}}(B)$.

**Lemma 34.** Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank and $B$ a torsion free sheaf with zero framing. Suppose that $p(\mathcal{E}) > p_{\text{max}}(B)$. Then $\text{Hom}(\mathcal{E}, (B, 0)) = 0$.

**Proof.** Let $\varphi \in \text{Hom}(\mathcal{E}, (B, 0))$, $\varphi \neq 0$. Let $j$ be minimal such that $\varphi(E) \subset \text{HN}_j(B)$. Then there exists a nontrivial morphism of framed sheaves $\tilde{\varphi} : \mathcal{E} \to gr_j^\text{HN}(B)$. By Propositions 6 and 19 we get

$$p(\mathcal{E}) \leq p(\mathcal{E}/\ker \tilde{\varphi}) \leq p(\text{Im} \tilde{\varphi}) \leq p(gr_j^\text{HN}(B)) \leq p_{\text{max}}(B)$$

and this is a contradiction with our assumption.

**Proposition 35.** Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Assume that $\mathcal{E}$ is not semistable with respect to $\delta$. Let $G$ be the maximal destabilizing framed subsheaf of $\mathcal{E}$. Then

$$\text{Hom}(G, \mathcal{E}/G) = 0.$$ 

**Proof.** We have to consider separately four different cases.

**Case 1:** $G = \ker \alpha$. In this case by definition of morphism of framed sheaves, we get $\text{Hom}(G, \mathcal{E}/G) = 0$.

**Case 2:** $\alpha|_G = 0$ and $\text{rk}(G) < \ker \alpha$. In this case $\text{Hom}(G, \mathcal{E}/G) = \text{Hom}(G, \ker \alpha/G)$. Recall that $G$ is a Gieseker-semistable sheaf and $\ker \alpha/G$ is a torsion free sheaf; moreover from the maximality of $G$ follows that $p_G > p(T/G)$ for all subsheaves $T/G \subset \ker \alpha/G$, hence $p_{\text{min}}(G) = p(G) > p_{\text{max}}(\ker \alpha/G)$ and by Lemma 1.3.3 in [10] we obtain the assertion.
Case 3: \( \alpha|_G \neq 0 \) and \( \text{rk}(G) > 0 \). In this case \( E/G \) is a torsion free sheaf and the induced framing is zero. From the maximality of \( G \) it follows that \( p(G) > p(T/G) \) for all subsheaves \( T/G \subset E/G \), so we can apply Lemma 34 and we get the assertion.

Case 4: \( G = T(E) \). Let \( \varphi : T(E) \to E/T(E) \). Since \( \text{rk}(\text{Im} \varphi) = 0 \) and \( E/T(E) \) is torsion free, we have \( \text{Im} \varphi = 0 \) and therefore we obtain the assertion. \( \square \)

7. Jordan-H"older filtration

By analogy with the study of Gieseker-semistable coherent sheaves, we will define Jordan-H"older filtrations for framed sheaves. Because of the framing, one needs to use Lemma 2 in the construction of the filtration. Moreover, in general we cannot extend the notions of socle and the extended socle for stable torsion free sheaves to the framed case, because, for example, the sum of two framed saturated subsheaves may not be framed saturated, hence we construct these objects only for a smaller family of framed sheaves having some extra properties.

Definition 36. Let \( E = (E, \alpha) \) be a semistable framed sheaf of positive rank \( r \). A Jordan-H"older filtration of \( E \) is a filtration

\[ E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E \]

such that all the factors \( E_i/E_{i-1} \) together with the induced framings \( \alpha_i \) are stable with framed Hilbert polynomial \( P(E_i/E_{i-1}, \alpha_i) = \text{rk}(E_i/E_{i-1})p(E) \).

Proposition 37 (Proposition 1.13 in [9]). Jordan-H"older filtrations always exist. The framed sheaf

\[ \text{gr}(E) = (\text{gr}(E), \text{gr}(\alpha)) = \bigoplus_i (E_i/E_{i-1}, \alpha_i) \]

does not depend on the choice of the Jordan-H"older filtration.

Remark 38. By construction for \( i > 0 \) all subsheaves \( E_i \) are framed saturated and the framed sheaves \( (E_i, \alpha') \) are semistable with framed Hilbert polynomial \( \text{rk}(E_i)p(E) \). In particular \( (E_1, \alpha') \) is a stable framed sheaf. Moreover at most one of the framings \( \alpha_i \) is nonzero and all but possibly one of the factors \( E_i/E_{i-1} \) are torsion free. \( \triangle \)

Lemma 39. Let \( E = (E, \alpha) \) be a semistable framed sheaf of positive rank \( r \). Then there exists at most one subsheaf \( E' \subset E \) such that \( \alpha|_{E'} \neq 0 \), \( \mathcal{E}' \) is a stable framed sheaf and \( P(\mathcal{E}') = \text{rk}(\mathcal{E}')p(E) \).

Proof. Suppose that there exist \( E_1 \) and \( E_2 \) subsheaves of \( E \) such that \( \alpha|_{E_i} \neq 0 \), the framed sheaf \( E_i \) is stable (with respect to \( \delta \)) and \( P(E_i) = r_ip(E) \), where \( r_i = \text{rk}(E_i) \), for \( i = 1, 2 \). So we have \( P(E_i) = r_ip(E) + \delta \) for \( i = 1, 2 \). Let \( E_{12} = E_1 \cap E_2 \). Suppose that \( E_{12} \neq 0 \) and \( \alpha|_{E_{12}} \neq 0 \).

Denote by \( r_{12} \) the rank of \( E_{12} \). Since \( E_i \) is stable, \( P(E_{12}) - \delta < r_{12}p(E) \). Consider the exact sequence

\[ 0 \longrightarrow E_{12} \longrightarrow E_1 \oplus E_2 \longrightarrow E_1 + E_2 \longrightarrow 0. \]

The induced framing on \( E_1 + E_2 \) by \( \alpha \) is nonzero; we denote it by \( \beta \).

\[
P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_{12}) = r_1p(E) + \delta + r_2p(E) + \delta - P(E_{12}) > \text{rk}(E_1 + E_2)p(E) + \delta
\]

and therefore

\[
P(E_1 + E_2, \beta) = P(E_1 + E_2) - \delta > \text{rk}(E_1 + E_2)p(E),
\]
but this is a contradiction, because $E$ is semistable. Now consider the case $\alpha|_{E_{12}} = 0$. By similar computations, we obtain

$$P(E_1 + E_2, \beta) = P(E_1 + E_2) - \delta > \text{rk}(E_1 + E_2)p(E) + \text{rk}(E_1 + E_2)\delta > \text{rk}(E_1 + E_2)p(E),$$

but this is absurd. Thus $E_{12} = 0$ and therefore $E_1 + E_2 = E_1 \oplus E_2$. In this case we get

$$P(E_1 + E_2, \beta) = P(E_1 + E_2) - \delta = P(E_1) + P(E_2) - \delta = r_1p(E) + \delta + r_2p(E) + \delta - \delta = \text{rk}(E_1 + E_2)p(E) + \delta > \text{rk}(E_1 + E_2)p(E),$$

but this is not possible. □

**Remark 40.** Let $E = (E, \alpha)$ be a semistable framed sheaf of positive rank $r$. If there exists $E' \subset E$ such that $\text{rk}(E') = 0$ and $P(E') = \delta$, then $E' = T(E)$, indeed from $P(T(E)) \geq P(E')$ follows that $P(T(E)) \geq \delta$. Since $E$ is semistable, we have $P(T(E)) = \delta$ and so $E' = T(E)$. △

By using similar computations as before, one can prove:

**Lemma 41.** Let $E = (E, \alpha)$ be a semistable framed sheaf of positive rank. Let $E_1$ and $E_2$ be two different subsheaves of $E$ such that $P(E_i) = \text{rk}(E_i)p(E)$ for $i = 1, 2$. Then $P(E_1 + E_2, \alpha') = \text{rk}(E_1 + E_2)p(E)$ and $P(E_1 \cap E_2, \alpha') = \text{rk}(E_1 \cap E_2)p(E)$.

### 7.1. Framed sheaves that are locally free along the support of the framing sheaf

In this section we assume that $F$ is supported on a divisor $D$ and is a locally free $\mathcal{O}_D$-module.

**Definition 42.** Let $E = (E, \alpha)$ be a framed sheaf on $X$. We say that $E$ is $(D, F)$-framable if $E$ is locally free in a neighborhood of $D$ and $\alpha\big|_D$ is an isomorphism. We call $E$ also $(D, F)$-framed sheaf.

Recall that in general for a framed sheaf $E = (E, \alpha)$ where $\text{ker } \alpha$ is nonzero and torsion free, the torsion subsheaf of $E$ is supported on $\text{Supp}(F)$. Therefore if $E$ is $(D, F)$-framable, $E$ is torsion free.

**Example 43.** Let $\mathbb{C}P^2$ be the complex projective plane and $\mathcal{O}_{\mathbb{C}P^2}(1)$ the hyperplane line bundle. Let $l_\infty$ be a fixed line in $\mathbb{P}^2$ and $i: l_\infty \to \mathbb{C}P^2$ the inclusion map. The torsion free sheaves of rank $r$ on $\mathbb{C}P^2$, trivial along a fixed line $l_\infty$ are — in the language we introduced before — $(l_\infty, \mathcal{O}_l_{\infty})$-framed sheaves of rank $r$ on $\mathbb{C}P^2$. Let $\mathcal{M}(r, n)$ be the moduli space of $(l_\infty, \mathcal{O}_l_{\infty})$-framed sheaves of rank $r$ and second Chern class $n$ on $\mathbb{C}P^2$. This moduli space is nonempty for $n \geq 1$ as one can see from the description of this space through ADHM data (see, e.g., Chapter 2 in [17]). Let $[(E, \alpha)]$ be a point in $\mathcal{M}(r, 1)$: the sheaf $E$ is a torsion free sheaf of rank $r$ with second Chern class one. By Proposition 9.1.3 in [13], $E$ is not Gieseker-semistable. On the other hand, the framed sheaf $(E, \alpha)$ is stable with respect to a suitable choice of $\delta$ (see the proof of Theorem 3.1 in [2]). Thus we have proved that there exist semistable framed sheaves such that the underlying coherent sheaves are not Gieseker-semistable. △

**Lemma 44.** Let $E = (E, \alpha)$ be a semistable $(D, F)$-framed sheaf. Let $E_1$ and $E_2$ be two different framed saturated subsheaves of $E$ such that $p(E_i) = p(E)$, for $i = 1, 2$. Assume that $\alpha|_{E_1} = 0$. Then $E_1 + E_2$ is a framed saturated subsheaf of $E$ such that $\text{gr}(E_1 + E_2, \alpha') = \text{gr}(E_1) \oplus \text{gr}(E_2)$. 


Proof. Since $E$ is $(D, F)$-framable, the quotient $E/E_i$ is torsion free for $i = 1, 2$, hence $E/(E_1 + E_2)$ is torsion free as well and therefore $E_1 + E_2$ is framed saturated.

By Lemma 41, $p(E_1 + E_2, \alpha') = p(E)$. Moreover we can always start with a Jordan-Hölder filtration of $E_1$ and complete it to one of $(E_1 + E_2, \alpha')$, hence we get $gr(E_1) \subset gr(E_1 + E_2, \alpha')$ (as framed sheaves) for $i = 1, 2$. Let $G_\bullet : 0 = G_0 \subset G_1 \subset \cdots \subset G_{t-1} \subset G_t = E_1$ be a Jordan-Hölder filtration for $E_1$ and $H_\bullet : 0 = H_0 \subset H_1 \subset \cdots \subset H_s \subset H_{s+1} = E_2$ a Jordan-Hölder filtration for $E_2$. Consider the filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{t-1} \subset G_t = E_1 \subset E_1 + H_p \subset \cdots \subset E_1 + H_{t-1} \subset H_t = E_1 + E_2$$

where $p = \min \{ i \mid H_i \not\subset E_1 \}$. We want to prove that this is a Jordan-Hölder filtration for $(E_1 + E_2, \alpha')$. It suffices to prove that $E_1 + H_j/E_1 + H_{j-1}$ with its induced framing $\gamma_j$ is stable for $j = p, \ldots, t$ (we put $H_{p-1} = 0$). First note that by Lemma 41, we get $P(E_1 + H_j, \alpha') = \rk(E_1 + H_j)p(E)$ and $P(E_1 + H_{j-1}, \alpha') = \rk(E_1 + H_{j-1})p(E)$, hence

$$P(E_1 + H_j/E_1 + H_{j-1}, \gamma_j) = \rk(E_1 + H_j/E_1 + H_{j-1})p(E).$$

Since $E/E_1 + H_{j-1}$ is torsion free, $\rk(E_1 + H_j/E_1 + H_{j-1}) > 0$. Let $T/E_1 + H_{j-1}$ be a subsheaf of $E_1 + H_j/E_1 + H_{j-1}$. We have

$$P(T/E_1 + H_{j-1}, \gamma_j) = P(T, \alpha') - P(E_1 + H_{j-1}, \alpha') \leq \rk(T)p(E) - \rk(E_1 + H_{j-1})p(E) = \rk(T/E_1 + H_{j-1})p(E) = \rk(T/E_1 + H_{j-1})p(E_1 + H_j/E_1 + E_{j-1}, \gamma_j),$$

so the framed sheaf $(E_1 + H_j/E_1 + H_{j-1}, \gamma_j)$ is semistable. Moreover we can construct the following exact sequence of coherent sheaves

$$0 \longrightarrow E_1 \cap H_j/E_1 \cap H_{j-1} \longrightarrow H_j/H_{j-1} \xrightarrow{\varphi} E_1 + H_j/E_1 + H_{j-1} \longrightarrow 0.$$

Recall that the induced framing on $E_1$ is zero, hence the induced framing on $E_1 \cap H_j/E_1 \cap H_{j-1}$ is zero and therefore the morphism $\varphi$ induces a surjective morphism between framed sheaves $\varphi : (H_j/H_{j-1}, \beta_j) \longrightarrow (E_1 + H_j/E_1 + H_{j-1}, \gamma_j)$.

Since $(H_j/H_{j-1}, \beta_j)$ is stable, by Corollary 20 the morphism $\varphi$ is injective, hence it is an isomorphism. 

Now we introduce the extended framed socle of a semistable $(D, F)$-framed sheaf, which plays a similar role of the maximal destabilizing framed subsheaf of a framed sheaf of positive rank.

**Definition 45.** Let $E = (E, \alpha)$ be a semistable $(D, F)$-framed sheaf. We call framed socle of $E$ the subsheaf of $E$ given by the sum of all framed saturated subsheaves $E' \subset E$ such that the framed sheaf $E' = (E', \alpha(E'))$ is stable with reduced framed Hilbert polynomial $p(E') = p(E)$.

Let $E = (E, \alpha)$ be a semistable $(D, F)$-framed sheaf. Consider the following two conditions on framed saturated subsheaves $E' \subset E$:

(a) $p(E') = p(E)$,

(b) each component of $gr(E')$ is isomorphic (as a framed sheaf) to a subsheaf of $E$.

Let $E_1$ and $E_2$ two different framed saturated subsheaves of $E$ satisfying conditions (a) and (b). By previous lemmas the subsheaf $E_1 + E_2$ is a framed saturated subsheaf of $E$ satisfying conditions (a) and (b) as well.
Definition 46. For a semistable \((D,F)\)-framed sheaf \(\mathcal{E} = (E, \alpha)\), we call extended framed socle the maximal framed saturated subsheaf of \(E\) satisfying the above conditions (a) and (b).

Proposition 47. Let \(G\) be the extended framed socle of a semistable \((D,F)\)-framed sheaf \(\mathcal{E} = (E, \alpha)\). Then

1. \(G\) contains the framed socle of \(E\).
2. If \(E\) is simple and not stable, then \(G\) is a proper subsheaf of \(E\).

Proof. (1) It follows directly from the definition.

(2) Let \(E \bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E\) be a Jordan-Hölder filtration of \(E\). If \(E = G\), then the framed sheaf \((E/E_{l-1}, \alpha_l)\) is isomorphic (as framed sheaf) to a proper subsheaf \(E' \subset E\) with induced framing \(\alpha'\). The composition of morphisms of framed sheaves

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E/E_{l-1} \\
\downarrow & & \downarrow \sim \\
F & \xrightarrow{\nu} & F
\end{array}
\]

induces a morphism \(\varphi : E \to E\) that is not a scalar endomorphism of \(E\). \(\square\)

Corollary 48. A \((D,F)\)-framed sheaf \(\mathcal{E} = (E, \alpha)\) is stable with respect to \(\delta\) if and only if it is geometrically stable.

Proof. By using the previous proposition and the same arguments as the unframed case (cf. Lemma 1.5.10 and Corollary 1.5.11 in [10]), we get the assertion. \(\square\)

8. Relative Harder-Narasimhan filtration

In this section we would like to construct a flat family of minimal destabilizing framed quotients associated to a framed sheaf. The construction in the framed case is somehow more complicated than in the nonframed case (see Theorem 2.3.2 in [10]), as one can see in what follows. Moreover, we restrict ourselves to the case in which the rank of the framing sheaf is zero; the positive rank case is similar.

Let \(g : Y \to S\) be a morphism of finite type of Noetherian schemes.

Definition 49. A flat family of coherent sheaves on the fibres of the morphism \(g\) is a coherent sheaf \(F\) on \(Y\), which is flat over \(S\).

Let \(S\) be an integral \(k\)-scheme of finite type, \(f : \mathcal{X} \to S\) a smooth projective morphism with equidimensional irreducible fibres and \(\mathcal{O}_X(1)\) an \(f\)-ample line bundle\(^2\). It follows that for any \(s \in S\) the pair \((\mathcal{X}_s, \mathcal{O}_X(1)_s)\) is a polarized variety of a fixed dimension \(d\).

Fix a flat family of sheaves \(F\) of rank zero on the fibres of \(f\) and a rational polynomial \(\delta\) of degree \(d - 1\) and positive leading coefficient \(\delta_1\).

Definition 50. A flat family of framed sheaves of positive rank on the fibres of the morphism \(f\) consists of a framed sheaf \(\mathcal{E} = (E, \alpha : E \to F)\) on \(\mathcal{X}\), where \(\alpha_s \neq 0\) and \(\text{rk}(E_s) > 0\) for all \(s \in S\) and \(E\) and \(\text{Im} \alpha\) are flat families of coherent sheaves on the fibres of \(f\).

\(^2\)An \(f\)-ample line bundle is a line bundle on \(\mathcal{X}\) such that the restriction to any fibre \(\mathcal{X}_s\) is ample for any \(s \in S\).
Remark 51. By flatness of $E$ and $\text{Im} \alpha$, we have that also $\ker \alpha$ is $S$-flat. △

Let us consider a flat family $\mathcal{E} = (E, \alpha)$ of framed sheaves of positive rank $r$ on the fibres of $f$ such that $P(\text{Im} \alpha_s) \geq \delta$ for $s \in S$. From now on we fix $S$, $f: \mathcal{X} \to S$, $F$, $\delta$ and $\mathcal{E} = (E, \alpha)$ as introduced above, unless otherwise stated.

The direction we choose to obtain a flat family of minimal destabilizing framed quotients is the following: first we construct a universal quotient (with fixed Hilbert polynomial) such that the induced framing is either nonzero at each fibre or zero at a generic fibre. In this way, generically, not only the Hilbert polynomial of that quotient is constant along the fibres, but also its framed Hilbert polynomial. Later we need to find a numerical polynomial such that the universal quotient with this polynomial as Hilbert polynomial gives the minimal destabilizing framed quotient at each fibre.

Relative framed Quot functor. Let $S$, $f: \mathcal{X} \to S$, $F$, $\delta$ and $\mathcal{E} = (E, \alpha)$ be as introduced before.

Let $P(n) \in \mathbb{Q}[n]$ be a numerical polynomial. Define the contravariant functor from the category of Noetherian $S$-schemes of finite type to the category of sets

$$\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}): (\text{Sch}/S) \to (\text{Sets})$$

in the following way:

- For an object $T \to S$, $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E})(T \to S)$ is the set consisting of the quotients

  (modulo isomorphism) $E_T \xrightarrow{q} Q \to 0$ such that

  (i) $Q$ is $T$-flat,

  (ii) the Hilbert polynomial of $Q_t$ is $P$ for all $t \in T$,

  (iii) there is an induced morphism $\tilde{\alpha}: Q \to F_T$ such that $\tilde{\alpha} \circ q = \alpha_T$.

- For an $S$-morphism $g: T' \to T$, $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E})(g)$ is the map that sends $E_T \to Q$ to $E_{T'} \to g_\mathcal{X}^*Q$, where $g_\mathcal{X}: X_{T'} \to X_T$ is the induced morphism by $g$.

This functor is a subfunctor of the relative Quot functor $\text{Quot}^P_{\mathcal{X}/S}(E)$, that is representable by a projective $S$-scheme $\pi: \text{Quot}^P_{\mathcal{X}/S}(E) \to S$ (cf. Theorem 2.2.4 in [10]).

The property (iii) in the definition is closed, indeed by using the same arguments as in the proof of Theorem 1.6 in [24], in particular results (iii) and (iv), one can see that property (iii) is equivalent to the vanishing of some regular functions on $\text{Quot}^P_{\mathcal{X}/S}(E)$. Hence, the functor $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E})$ is representable by a closed subscheme $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}) \subset \text{Quot}^P_{\mathcal{X}/S}(E)$. We denote by $\pi_{fr}$ the composition

$$\pi_{fr}: \text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}) \hookrightarrow \text{Quot}^P_{\mathcal{X}/S}(E) \xrightarrow{\pi} S.$$  

Roughly speaking, $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E})$ parametrizes all the quotients $E_s \xrightarrow{q} Q$, for $s \in S$, such that the induced framing on $\ker q$ is zero.

The universal object on $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}) \times_S \mathcal{X}$ is the pull-back of the universal object on $\text{Quot}^P_{\mathcal{X}/S}(E) \times_S \mathcal{X}$ with respect to the morphism $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}) \times_S \mathcal{X} \to \text{Quot}^P_{\mathcal{X}/S}(E) \times_S \mathcal{X}$, induced by the closed embedding $\text{FQuot}^P_{\mathcal{X}/S}(\mathcal{E}) \hookrightarrow \text{Quot}^P_{\mathcal{X}/S}(E)$.
Let $s \in S$ and $q \in \pi_{fr}^{-1}(s)$ be $k$-rational points corresponding to the commutative diagram on $X_s$

$$
0 \longrightarrow K \xrightarrow{i} E_s \xrightarrow{q} Q \longrightarrow 0
$$

One has the following result about the tangent space of $\pi_{fr}^{-1}(s)$ at $q$:

**Proposition 52.** The kernel of the linear map $(d\pi_{fr})_q : T_q \text{FQuot}_{X/S}(E) \to T_s S$ is isomorphic to the linear space $\text{Hom}(K, \text{ker}\alpha_s/K) = \text{Hom}(K, \mathbb{Q})$.

**Proof.** It suffices to use the same techniques of the proof of the corresponding result for $\pi$ (see Proposition 4.4.4 in [22]). $\square$

Now we have a tool for constructing a flat family of quotients (with a fixed Hilbert polynomial) of $E$ such that the induced framing is nonzero in each fibre. Using the relative Quot scheme $\text{Quot}_{X/S}(E)$ associated to $E$, one can construct a flat family of quotients such that the induced framing is generically zero.

**Boundedness result.** Let $Y \to S$ be a projective morphism of Noetherian schemes and denote by $\mathcal{O}_Y(1)$ a line bundle on $Y$, which is very ample relative to $S$.

**Definition 53.** A family $\mathcal{G}$ of isomorphism classes of coherent sheaves on the fibres of the morphism $Y \to S$ is bounded if there is an $S$-scheme $T$ of finite type and a coherent $\mathcal{O}_Y T$-sheaf $E$ such that the given family is contained in the set $\{E_t | t$ is a closed point in $T\}$.

Now we recall some characterizations of the property of boundedness which will be useful in the sequel.

**Lemma 54** (Lemma 2.5 in [6]). Let $L$ be a coherent sheaf on $Y$ and $\mathcal{G}$ the set of isomorphism classes of quotient sheaves $G$ of $L_s$ for $s$ running over the points of $S$. Suppose that the dimension of $Y_s$ is $\leq r$ for all $s$. Then the coefficient $\beta_r(G)$ is bounded from above and from below, and $\beta_{r-1}(G)$ is bounded from below. If $\beta_{r-1}(G)$ is bounded from above, then the family of coherent sheaves $G/\text{F}(G)$ is bounded.

**Corollary 55.** Let $E$ be a flat family of coherent sheaves on the fibres of a projective morphism $Y \to S$. Then the family of torsion free quotients $Q$ of $E_s$ for $s \in S$ with hat-slopes bounded from above is a bounded family.

**Theorem 56** (Theorem 2.1 in [6]). The following properties of a family $\mathcal{G}$ of isomorphism classes of coherent sheaves on the fibres of $Y \to S$ are equivalent:

(i) The family is bounded.

(ii) The set of Hilbert polynomials $\{P(G)\}_{G \in \mathcal{G}}$ is finite and there is a coherent sheaf $E$ on $Y$ such that all $G \in \mathcal{G}$ admit surjective morphisms $E_s \to G$.

From this result and Corollary 55 it follows that there are only a finite number of rational polynomials corresponding to Hilbert polynomials of destabilizing quotients $Q$ of $E_s$ for $s \in S$. Thus it is possible to find the “minimal” polynomial that will be the Hilbert polynomial of the minimal destabilizing quotient of $E_s$ for a generic point $s \in S$. Now we want to use the same argument in the framed case.
Let \( S, f : X \to S, F, \delta \) and \( E = (E, \alpha : E \to F) \) be as before.

Let \( \mathcal{F}_1 \) be a family of quotients \( E_s \xrightarrow{q} Q, \) for \( s \in S, \) such that

- \( \ker \alpha_s \) is torsion free,
- \( \ker q \not\subseteq \ker \alpha_s, \)
- \( Q \) is torsion free and \( \hat{\mu}(Q) < \hat{\mu}(E_s). \)

**Proposition 57.** The family \( \mathcal{F}_1 \) is bounded.

**Proof.** The family \( \mathcal{F}_1 \) is contained in the family of torsion free quotients of \( E, \) with hat-slopes bounded from above, hence it is bounded by Corollary 55 and Theorem 56. \( \square \)

Let \( \mathcal{F}_2 \) be a family of quotients

\[
\begin{array}{c}
E_s \\
\downarrow \alpha_s \\
F_s
\end{array} \xrightarrow{q} Q \xrightarrow{\alpha} 0
\]

for which

- \( \ker \alpha_s \) is a torsion free sheaf,
- \( Q \) fits into an exact sequence

\[
0 \to Q' \to Q \xrightarrow{\alpha} \text{Im} \alpha_s \to 0
\]

where \( Q' = \ker \alpha \) is a nonzero torsion free quotient of \( \ker \alpha_s, \)
- \( \hat{\mu}(Q) < \hat{\mu}(E_s) + \delta_1. \)

**Proposition 58.** The family \( \mathcal{F}_2 \) is bounded.

**Proof.** Since a family given by extensions of elements from two bounded families is bounded (cf. Proposition 1.2 in [6]), it suffices to prove that every element in \( \mathcal{F}_2 \) is an extension of two elements that belong to two bounded families. By definition of flat family of framed sheaves, the families \( \{\ker \alpha_s\}_{s \in S} \) and \( \{\text{Im} \alpha_s\}_{s \in S} \) are bounded.

So it remains to prove that the family of quotients \( Q' \) is bounded. Since the family \( \{\ker \alpha_s\} \) is bounded, there exists a coherent sheaf \( G \) on \( X \) such that we have surjective morphisms \( G_s \to \ker \alpha_s \) (see Theorem 56), hence the compositions \( G_s \to \ker \alpha_s \to Q' \) are surjective as well.

By Lemma 54, the coefficient \( \beta_d(Q') \) is bounded from above and from below and the coefficient \( \beta_{d-1}(Q') \) is bounded from below, hence \( \hat{\mu}(Q') \) is bounded from below.

Moreover, since \( \{E_s\} \) is a bounded family, the coefficients of their Hilbert polynomials are uniformly bounded from above and from below, hence \( \hat{\mu}(E_s) \) is uniformly bounded from above and from below. Therefore, from the inequality \( \hat{\mu}(Q) < \hat{\mu}(E_s) + \delta_1, \) it follows that \( \hat{\mu}(Q) \) is uniformly bounded from above. By using the fact that also \( \{\text{Im} \alpha_s\} \) is a bounded family, we obtain that \( \hat{\mu}(Q') \leq A\hat{\mu}(Q) + B \) for some positive constants \( A, B, \) hence we get that \( \hat{\mu}(Q') \) is uniformly bounded from above.

 Altogether, \( \hat{\mu}(Q') \) is uniformly bounded from above and below, and by Lemma 54, the family of quotients \( Q' \) is bounded. \( \square \)
Relative minimal destabilizing framed quotient and Harder-Narasimhan filtration. By using the same arguments as in the nonframed case (see Proposition 2.3.1 in [10]), we can prove the following:

**Proposition 59.** Let $S$, $f : X \to S$, $F$, $\delta$ and $E = (E, \alpha : E \to F)$ be as in the assumptions of this section. The set of points $s \in S$ such that $(E_s, \alpha_s)$ is (semi)stable with respect to $\delta$ is open in $S$.

Now we can prove the following generalization to the relative case of the theorem of the minimal destabilizing framed quotient.

**Theorem 60.** Let $S$, $f : X \to S$, $F$, $\delta$ and $E = (E, \alpha : E \to F)$ be as in the assumptions of this section. Then there is an integral k-scheme $T$ of finite type, a projective birational morphism $g : T \to S$, a dense open subscheme $U \subset T$ and a flat quotient $Q$ of $E_T$ such that for all points $t$ in $U$, $(E_t, \alpha_t)$ is a framed sheaf of positive rank with $\ker \alpha_t \neq 0$ torsion free and $Q_t$ is the minimal destabilizing framed quotient of $E_t$ with respect to $\delta$ or $Q_t = E_t$.

Moreover, the pair $(g, Q)$ is universal in the sense that if $\bar{g} : \bar{T} \to S$ is any dominant morphism of k-integral schemes and $\bar{Q}$ is a flat quotient of $E_{\bar{T}}$ satisfying the same property as $Q$, then there is a unique $S$-morphism $h : T \to \bar{T}$ such that $h_E^* \bar{Q} = Q$.

**Proof.** Let $P$ be the Hilbert polynomial of $E$ and $r$ its rank. Denote by $p$ the rational polynomial $P/r$.

For $i = 1, 2$ let $A_i \subset \mathbb{Q}[u]$ be the set consisting of polynomials $P''$ such that there is a point $s \in S$ and a surjection $E_s \to E''$, where $P_{E''} = P''$ and $E''$ belongs to the family $\mathcal{F}_i$ (introduced the page before). By Propositions 57 and 58 and Theorem 56, the sets $A_1$ and $A_2$ are finite. Denote by $r''$ the leading coefficient of $P''$ and by $p''$ the rational polynomial $P''/r''$ (and similarly for other polynomials). Let

$$B_1 = \left\{ p'' \in A_1 \mid p'' < p - \frac{\delta}{r} \right\},$$

$$B_2 = \left\{ p'' \in A_2 \mid p'' - \frac{\delta}{r''} \leq p - \frac{\delta}{r} \right\}.$$  

The set $B_1 \cup B_2$ is nonempty. We define an order relation on $B_1$: $P_1 \sqsubseteq P_2$ if and only if $p_1 \leq p_2$ and $r_1 \leq r_2$ in the case $p_1 = p_2$. We define an order relation on $B_2$: $P_1 \sqsubseteq P_2$ if and only if $p_1 - \frac{\delta}{r_1} \leq p_2 - \frac{\delta}{r_2}$ and $r_1 \leq r_2$ in the case of equality.

Let $C_1$ be the set of polynomials $P'' \in B_1$ such that $\pi(\text{Quot}^{P''}_{\pi/s}(E)) = S$ and for any $s \in S$ one has $\pi^{-1}(s) \not\subset \text{FQuot}^{P''}_{\pi/s}(E)$. Let $C_2$ be the set of polynomials $P'' \in B_2$ such that $\pi_{fr}(\text{FQuot}^{P''}_{\pi/s}(E)) = S$. Note that $C_1 \cup C_2$ is nonempty. Now we want to find a polynomial $P_\ast$ in $C_1 \cup C_2$ that is the Hilbert polynomial of the minimal destabilizing framed quotient of $E_s$ for a general point $s \in S$.

Now we introduce a new order relation, slightly more restrictive than the one introduced before. Let us consider the relation $\sqsubseteq$ defined in the following way: for $P_1, P_2 \in B_1$ we have $P_1 \sqsubseteq P_2$ if and only if $P_1 \sqsubseteq P_2$ and $p_1 < p_2$ or $r_1 < r_2$ in the case $p_1 = p_2$. In a similar way we can define $\sqsubseteq$ for polynomials in $B_2$. Let $P_\ast$ be a $\sqsubseteq$-minimal polynomial among all polynomials of $C_i$ for $i = 1, 2$. Denote by $r'_\ast$ the leading coefficient of $P'_\ast$ and by $p'_\ast$ the polynomial $P'_\ast/r'_\ast$.

Consider the following cases:
subscheme of $S$. Note that the set $(\bigcup_{s} \pi(\text{Quot}^{p''}_{X/s}(E))) \cup (\bigcup_{p'' \leq p^2} \pi_{fr}(\text{FQuot}^{p''}_{X/s}(E)))$

is a proper closed subscheme of $S$. Let $U_-$ be its complement. Let $U_{tf}$ be the dense open subscheme of $S$ consisting of points $s$ such that $\ker \alpha_s$ is torsion free. Put $V = U_- \cap U_{tf}$.

Suppose that $P_- \in C_2$, the other case is similar. By definition of $P_-$ the projective morphism $\pi_{fr}: \text{FQuot}^{P_-}_{X/s}(E) \to S$ is surjective. For any point $s \in S$ the fiber of $\pi_{fr}$ at $s$ parametrizes possible quotients of $E_s$ with Hilbert polynomial $P_-$. If $s \in V$, then any such quotient is a minimal destabilizing framed quotient by construction of $V$. Recall that the minimal destabilizing framed quotient is unique by Proposition \ref{unique_minimal} this implies that $\pi_{fr}|_U: U := \pi_{fr}^{-1}(V) \to V$ is bijective. By the same arguments as in the nonframed case (see Proposition 3 in \cite{12}), that quotient is defined over the residue field $k(s)$, hence for $t \in U$, $s = \pi_{fr}(t)$ one has $k(s) \cong k(t)$. Let $t \in \pi_{fr}^{-1}(s)$ be a point corresponding to a diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & K \\
& \downarrow{i} & \phantom{\downarrow{\alpha}} \downarrow{\bar{\alpha}} & \phantom{\downarrow{\alpha}} \downarrow{q} \\
& E_t & \longrightarrow & Q \\
& \phantom{\downarrow{\alpha}} \downarrow{\alpha_t} & & \phantom{\downarrow{\alpha}} \downarrow{\bar{\alpha}} \\
& F_t & \longrightarrow & 0
\end{array}
$$

By Proposition \ref{tangent_space}, the Zariski tangent space of $\pi_{fr}^{-1}(s)$ at $t$ is $\text{Hom}(\mathcal{K}, Q)$. Moreover, $K$ is the maximal destabilizing framed subsheaf of $\mathcal{E}_t$, hence $\text{Hom}(\mathcal{K}, Q) = 0$ by Proposition \ref{maximal_unramified} and therefore $\Omega_{U/V} = 0$, hence $\pi_{fr}|_U: U \to V$ is unramified. Since $\pi_{fr}$ is projective and $V$ is integral, $\pi_{fr}|_U$ is an isomorphism. Now let $T$ be the closure of $U$ in $\text{FQuot}^{P_-}_{X/s}(E)$ with its reduced subscheme structure and $g := \pi_{fr}|T: T \to S$ is a projective birational morphism. We put $Q$ equal to the pull-back on $X_T$ of the universal quotient on $\text{FQuot}^{P_-}_{X/s}(E) \times_S X$.

The proof of the universality of the pair $(g, Q)$ is similar to that for the case of torsion free sheaves (second part of Theorem 2.3.2 in \cite{14}), since to prove this part of the theorem we need only the universal property of $\text{FQuot}^{P_-}_{X/s}(E)$ or $\text{Quot}^{P_-}_{X/s}(E)$. \hfill $\Box$

By using the same arguments as in the nonframed case, from the previous theorem one can construct the relative version of the Harder-Narasimhan filtration:

**Theorem 61.** Let $S, f: \mathcal{X} \to S, F, \mathcal{E} = (E, \alpha: E \to F)$ and $\delta$ be as in the assumptions of this section. Then there is an integral $k$-scheme $T$ of finite type, a projective birational morphism $g: T \to S$ and a filtration

$$
\text{HN}_i(\mathcal{E}): 0 = \text{HN}_0(\mathcal{E}) \subset \text{HN}_1(\mathcal{E}) \subset \cdots \subset \text{HN}_l(\mathcal{E}) = ET
$$

such that the following holds:

- The factors $\text{HN}_i(\mathcal{E})/\text{HN}_{i-1}(\mathcal{E})$ are $T$-flat for all $i = 1, \ldots, l$, and
• there is a dense open subscheme $U \subset T$ such that $(\text{HN}_t(\mathcal{E}))_t = g_t^*\text{HN}_t(\mathcal{F}_{g(t)})$ for all $t \in U$.

Moreover, the pair $(g, \text{HN}_t(\mathcal{E}))$ is universal in the sense that if $\bar{g}: \bar{T} \to S$ is any dominant morphism of $k$-integral schemes and $E_\bullet$ is a filtration of $E_{\bar{T}}$ satisfying these two properties, then there is an $S$-morphism $h: \bar{T} \to T$ such that $h^*_\bar{T}(\text{HN}_t(\mathcal{E})) = \bar{E}_\bullet$.

9. $\mu$-(semi)stability

In this section we give a generalization to framed sheaves of the Mumford-Takemoto (semi)stability condition for torsion free sheaves (see Definition 1.2.12 in [10]). Also in this case one can construct examples of framed sheaves that are semistable with respect to this new condition but the underlying coherent sheaves are not $\mu$-semistable and vice versa.

**Definition 62.** A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is $\mu$-(semi)stable with respect to $\delta_1$ if and only if $\ker \alpha$ is torsion free and the following conditions are satisfied:

(i) $\text{rk}(E) \deg(E') (\leq) \text{rk}(E') \deg(\mathcal{E})$ for all subsheaves $E' \subset \ker \alpha$,

(ii) $\text{rk}(E)(\deg(E') - \delta_1) (\leq) \text{rk}(E') \deg(\mathcal{E})$ for all subsheaves $E' \subset E$ with $\text{rk}(E') < \text{rk}(E)$.

One has the usual implications among different stability properties of a framed module of positive rank:

$\mu$ - stable $\Rightarrow$ stable $\Rightarrow$ semistable $\Rightarrow$ $\mu$ - semistable.

**Definition 63.** Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\text{rk}(E) = 0$. If $\alpha$ is injective, we say that $\mathcal{E}$ is $\mu$-semistable.\(^3\) Moreover, if the degree of $E$ is $\delta_1$, we say that $\mathcal{E}$ is $\mu$-stable with respect to $\delta_1$.

**Remark 64.** Most results of Sections 3 - 8 still hold if one replaces (semi)stability by $\mu$-(semi)stability. For example, this is the case for Theorem 60, Propositions 19, 22, 25, 47, Corollaries 20, 48.

Below we state separately the theorem on the existence of the $\mu$-Harder-Narasimhan and $\mu$-Jordan-Hölder filtrations. We start by specifying the definitions of these notions.

**Definition 65.**

1. Let $F$ be a coherent sheaf of rank zero and $\mathcal{E} = (E, \alpha: E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. A $\mu$-Harder-Narasimhan filtration is a filtration of framed saturated subsheaves

$\text{HN}_t(\mathcal{E}) : 0 = \text{HN}_0(\mathcal{E}) \subset \text{HN}_1(\mathcal{E}) \subset \cdots \subset \text{HN}_l(\mathcal{E}) = E$

which satisfies the following conditions

• the quotient sheaf $g_i^{\text{HN}}(\mathcal{E}) := \text{HN}_i(\mathcal{E})/\text{HN}_{i-1}(\mathcal{E})$ with the induced framing $\alpha_i$ is a $\mu$-semistable framed sheaf for $i = 1, 2, \ldots, l$.

• the quotient $(E/\text{HN}_j(\mathcal{E}), \alpha'')$ is a framed sheaf where $\ker \alpha''$ is nonzero and torsion free for $j = 1, 2, \ldots, l-2$, it has no rank zero $\mu$-destabilizing framed subsheaves, and

$\text{rk}(g_i^{\text{HN}}(\mathcal{E})) \deg(g_i^{\text{HN}}(\mathcal{E}), \alpha_i) > \text{rk}(g_{i+1}^{\text{HN}}(\mathcal{E})) \deg(g_{i+1}^{\text{HN}}(\mathcal{E}), \alpha_{i+1})$

for $i = 1, \ldots, l - 1$.

\(^3\)For torsion sheaves, the definition of $\mu$-semistability of the corresponding framed sheaves does not depend on $\delta_1$. 

(2) Let $\mathcal{E} = (\mathcal{E}, \alpha)$ be a $\mu$-semistable framed sheaf of positive rank $r$. A $\mu$-Jordan-Hölder filtration is a filtration of framed saturated subsheaves

$$E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that all the factors $E_i/E_{i-1}$ together with the induced framings $\alpha_i$ are $\mu$-stable with framed degree $\deg(E_i/E_{i-1}, \alpha_i) = \text{rk}(E_i/E_{i-1})\mu(\mathcal{E})$.

**Theorem 66.**

1. Let $F$ be a coherent sheaf of rank zero and $\mathcal{E} = (\mathcal{E}, \alpha: E \to F)$ a framed sheaf where $\ker \alpha$ is nonzero and torsion free. Then there exists a unique $\mu$-Harder-Narasimhan filtration.
2. Let $\mathcal{E} = (\mathcal{E}, \alpha)$ be a $\mu$-semistable framed sheaf of positive rank $r$. Then there exist $\mu$-Jordan-Hölder filtrations. Moreover, the graded framed sheaf

$$\text{gr}(\mathcal{E}) = (\text{gr}(\mathcal{E}), \text{gr}(\alpha)) = \bigoplus_i (E_i/E_{i-1}, \alpha_i)$$

does not depend on the choice of the $\mu$-Jordan-Hölder filtration.

For $i \geq 0$, let us denote by $\text{Coh}_i(X)$ the full subcategory of $\text{Coh}(X)$ whose objects are sheaves of dimension less or equal to $i$.

Let $\text{Coh}_{d,d-1}(X)$ be the quotient category $\text{Coh}_d(X)/\text{Coh}_{d-1}(X)$. In Section 1.6 of [10], Huybrechts and Lehn define the notion of $\mu$-Jordan-Hölder filtration for $\mu$-semistable sheaves $E$ in the category $\text{Coh}_{d,d-1}(X)$. For a $\mu$-semistable torsion free sheaf $E$, the graded object associated to a $\mu$-Jordan-Hölder filtration is uniquely determined only in codimension one. In our case, we construct $\mu$-Jordan-Hölder filtrations by using filtrations in which every term is a framed saturated subsheaf of the next term. In this way, the graded object is uniquely determined. Thus, when the framing of a $\mu$-semistable framed sheaf is zero, our definition of $\mu$-Jordan-Hölder filtration does not coincide with the nonframed one given by Huybrechts and Lehn.

**10. Restriction theorems**

In this section we generalize the Mehta-Ramanathan restriction theorems to framed sheaves. We limit our attention to the case in which the framing sheaf $F$ is a coherent sheaf supported on a divisor $D_{fr}$. In the framed case the results depend also on the parameter $\delta_1$. Moreover the proofs are somehow more complicated than in the nonframed case (see, e.g., Section 7.2 in [10]) because of the presence of the framing.

$\mu$-semistable case. We provide a generalization of Mehta-Ramanathan’s restriction theorem for $\mu$-semistable torsion free sheaves (Theorem 6.1 in [15]).

**Theorem 67.** Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension $d$. Let $F$ be a coherent sheaf on $X$ supported on a divisor $D_{fr}$. Let $\mathcal{E} = (\mathcal{E}, \alpha: E \to F)$ be a framed sheaf on $X$ of positive rank with nontrivial framing. If $\mathcal{E}$ is $\mu$-semistable with respect to $\delta_1$, there exists a positive integer $a_0$ such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in U_a$ the divisor $D$ is smooth, meets transversely the divisor $D_{fr}$ and $\mathcal{E}|_D$ is $\mu$-semistable with respect to $a\delta_1$. 

In order to prove this theorem, we need some preliminary results: for a positive integer \( a \), let \( |\mathcal{O}_X(a)| \) be the complete linear system of hypersurfaces of degree \( a \) and let \( \tilde{Z}_a := \{(D, x) \in |\mathcal{O}_X(a)| \times X | x \in D\} \) be the incidence variety with its natural projections

\[
\begin{array}{c}
\tilde{Z}_a \\
\downarrow \tilde{p} \\
|\mathcal{O}_X(a)|
\end{array}
\]

**Remark 68.** It is possible to give a schematic structure on \( \tilde{Z}_a \) so that \( \tilde{p} \) is a projective morphism with equidimensional fibres (see Section 3.1 in [10]). Moreover one can prove the following property (see Section 2 in [13]):

\[
\text{(10)} \quad \text{Pic}(\tilde{Z}_a) = q^*(\text{Pic}(X)) \oplus \tilde{p}^*(\text{Pic}(|\mathcal{O}_X(a)|)).
\]

We restrict ourselves to the dense open subset \( \Pi_a \) of \( |\mathcal{O}_X(a)| \) consisting of smooth irreducible divisors. From now on, we shall denote the fibre product \( \tilde{Z}_a \times |\mathcal{O}_X(a)| \Pi_a \) by \( Z_a \). Thus the corresponding morphism \( p: Z_a \to \Pi_a \) is a smooth projective morphism with equidimensional irreducible fibres and the property (10) holds for \( \Pi_a \) instead of \( |\mathcal{O}_X(a)| \). We denote by \( q \) the induced morphism \( Z_a \to X \).

For all \( D \in \Pi_a \), the Hilbert polynomials of the restrictions \( E|_D, F|_D \) and \( \text{Im} \alpha|_D \) are independent from \( D \), indeed, e.g., the Hilbert polynomial of \( E|_D \) is \( P(E|_D, n) = P(E, n) - P(E, n-a) \). Since \( \Pi_a \) is a reduced scheme, by Proposition 2.1.2 in [10] \( q^*F \) is a flat family of sheaves of rank zero on the fibres of \( p \) and \( (q^*E, q^*\alpha) \) is a flat family of framed sheaves of positive rank on the fibres of \( p \). For any \( a \) and for general \( D \in \Pi_a \) the restriction \( \ker \alpha|_D \) is again torsion free (see Corollary 1.1.14 in [10]), hence the set \( \{ C \in \Pi_a | \ker \alpha|_C \text{ is torsion free} \} \subset \Pi_a \) is nonempty. Since \( E \) is \( \mu \)-semistable with respect to \( \delta_1 \), \( \deg(\text{Im} \alpha) \geq \delta_1 \), hence \( \deg(\text{Im} \alpha|_D) = a \deg(\text{Im} \alpha) \geq a \delta_1 \) for an integer \( a > 0 \). According to the Theorem 60 which states the existence of the relative minimal \( \mu \)-destabilizing framed quotient with respect to \( a \delta_1 \), there are a dense open subset \( V_a \subset \Pi_a \) and a \( V_a \)-flat quotient on \( Z_{V_a} := V_a \times_{\Pi_a} Z_a \)

\[
\begin{array}{c}
(q^*E)|_{Z_{V_a}} \\
\downarrow q_a \\
(q^*\alpha)|_{Z_{V_a}}
\end{array}
\]

\[
\begin{array}{c}
(q^*F)|_{Z_{V_a}} \\
\end{array}
\]

with a morphism \( \tilde{\alpha}_a: Q_a \to (q^*F)|_{Z_{V_a}} \), such that for all \( D \in V_a \) the framed sheaf \( (E|_D, \alpha|_D) \) has positive rank, and \( \ker \alpha|_D \) is torsion free; moreover, \( Q_a|_D \) is a coherent sheaf of positive rank, \( \tilde{\alpha}_a|_D \) is the framing induced by \( \alpha|_D \) and \( (Q_a|_D, \tilde{\alpha}_a|_D) \) is the minimal \( \mu \)-destabilizing framed quotient of \( (E|_D, \alpha|_D) \). Let \( Q \) be an extension of \( \det(Q_a) \) to some line bundle on all of \( Z_a \). Then \( Q \) can be uniquely decomposed as \( Q = q^*L_a \otimes p^*M \) with \( L_a \in \text{Pic}(X) \) and \( M \in \text{Pic}(\Pi_a) \). Note that \( \deg(Q_a|_D) = a \deg(L_a) \) for \( D \in V_a \).

Let \( U_a \subset V_a \) be the dense open set of points \( D \in V_a \) such that \( D \) meets transversely the divisor \( D_{fr} \).
Let $\deg(a)$, $r(a)$ and $\mu_{fr}(a)$ denote the degree, the rank and the framed slope of the minimal $\mu$-destabilizing framed quotient of $(E|_D, \alpha|_D)$ for a general point $D \in \Pi_a$. By construction of the relative minimal $\mu$-destabilizing framed quotient, the quantity $\epsilon(\bar{\alpha}_a|D)$ is independent of $D \in V_a$, so we denote it by $\epsilon(a)$. Then we have $1 \leq r(a) \leq \text{rk}(E)$ and
\[
\frac{\mu_{fr}(a)}{a} = \frac{\deg L_a - \epsilon(a)\delta_1}{r(a)} \in \frac{\mathbb{Z}}{\delta_1(r(\text{rk}(E)!))} \subset \mathbb{Q},
\]
where $\delta_1 = \delta/\delta'$. Let $l > 1$ be an integer, $a_1, \ldots, a_l$ positive integers and $a = \sum_i a_i$. We would like to compare $r(a)$ (resp. $\mu_{fr}(a)/a$) with $r(a_i)$ (resp. $\mu_{fr}(a_i)/a_i$) for all $i = 1, \ldots, l$. To do this, we use the following result, which allows us to compare the rank and the framed degree of $Q_{a_i}$ in a generic fibre with the same invariants of a “special quotient” of $(q^*E)|_{Z_{Va}}$.

**Lemma 69** (Lemma 7.2.3 in [10]). Let $l > 1$ be an integer, $a_1, \ldots, a_l$ positive integers, $a = \sum_i a_i$, and let $D_i \in U_{a_i}$ be divisors such that $D = \sum_i D_i$ is a divisor with normal crossings. Then there is a smooth locally closed curve $C \subset \Pi_a$ containing the point $D \in \Pi_a$ such that $C \setminus \{D\} \subset U_a$ and $Z_C = C \times_{\Pi_a} Z_a$ is smooth in codimension 2.

**Remark 70.** If $D_1 \in U_{a_1}$ is given, one can always find $D_i \in U_{a_i}$ for $i \geq 2$ such that $D = \sum_i D_i$ is a divisor with normal crossings. △

**Lemma 71.** Let $a_1, \ldots, a_l$ be positive integers, with $l > 1$, and $a = \sum_i a_i$. Then $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$ and in case of equality $r(a) \geq \max\{r(a_i)\}$.

**Proof.** Let $D_i$ be divisors satisfying the requirements of Lemma 69 and let $C$ be the curve with the properties of 69. Over $V_a$ there is the quotient
\[
\begin{array}{cccc}
(q^*E)|_{Z_{Va}} & q_a & Q_a \\
\downarrow & \downarrow & \\
(q^*[\alpha])|_{Z_{Va}} & (q^*[\alpha])|_{Z_{Va}} & Z_{Va} \\
\end{array}
\]
(11)

Now we have to consider two cases:

1. there exists a nonzero framing $\bar{\alpha}_a$ on $Q_a$ such that $(q^*\alpha)|_{Z_{Va}} = \bar{\alpha}_a \circ q_a$,
2. $\ker q_a|_{D'} \not\subset \ker \alpha|_{D'}$ for all $D' \in V_a$.

In the first case, $\bar{\alpha}_a|_{D'} \neq 0$ for all $D' \in V_a$. The restriction of diagram (11) to $Z_{Va \cap C}$ is

\[
\begin{array}{cccc}
0 & \rightarrow & K & \rightarrow \bigl(\frac{q^*E}{Z_{Va \cap C}}\bigr) & q_a|_{Z_{Va \cap C}} & Q_a|_{Z_{Va \cap C}} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & & (q^*[\alpha])|_{Z_{Va \cap C}} & & & \rightarrow & \\
& & & (q^*[\alpha])|_{Z_{Va \cap C}} & & & \rightarrow & \\
& & & (q^*[F]|_{Z_{Va \cap C}} & \bar{\alpha}_a|_{Z_{Va \cap C}} & \rightarrow & 0 \\
\end{array}
\]

Since the morphism $Z_{Va \cap C} \rightarrow Z_C$ is flat (because it is an open embedding), $\ker(q^*[\alpha]|_{Z_{Va \cap C}}) = (\ker q^*[\alpha]|_{Z_C})|_{Z_{Va \cap C}}$ and we can extend the inclusion $K \subset \ker q^*[\alpha]|_{Z_{Va \cap C}}$ to an inclusion $K_C \subset \ker q^*[\alpha]|_{Z_{Va \cap C}}$. \n\n}
Since $V_a \cap C = C \setminus \{D\}$, in this way we extend $Q_a|_{Z_{V_a\cap C}}$ to a $C$-flat quotient $Q_C$ of $q^*E|_{Z_C}$ and we get the following commutative diagram

\[
\begin{array}{c}
(q^*E)|_{Z_C} \\ (q^*\alpha)|_{Z_C} \\ (q^*F)|_{Z_C}
\end{array} \xrightarrow{g_C} \begin{array}{c}
Q_C \\
\tilde{\alpha}_C \\
\end{array}
\]

Therefore $\tilde{\alpha}_C|_c \neq 0$ for all $c \in C$. By the flatness of $Q_C$ we obtain $P(Q_C|_c, n) = P(Q_C|_D, n)$ for all $c \in C \setminus \{D\}$. Hence $\text{rk}(Q_C|_D) = r(a)$ and $\text{deg}(Q_C|_D) = \text{deg}(a)$, therefore $\mu(Q_C|_D, \tilde{\alpha}_C|_D) = \mu_{fr}(a)$. Let $\bar{Q} = Q_{C|D}/T'(Q_C|_D)$, where $T'(Q_C|_D)$ is the sheaf that to every open subset $U$ associates the set of sections $f$ of $Q_C|_D$ in $U$ such that there exists $n > 0$ for which $I^n_D$, $f = 0$, where $I_D$ is the ideal sheaf associated to $D$. Roughly speaking, $T'(Q_C|_D)$ is the part of the torsion subsheaf $T(Q_C|_D)$ of $Q_C|_D$ that is not supported in the intersection $D \cap D_{fr}$. By the transversality of $D_i$ with respect to $D_{fr}$, we have $T'(Q_C|_D) \subset \text{ker} \tilde{\alpha}_C|_D$, hence there is a nonzero induced framing $\tilde{\alpha}$ on $Q$. Moreover, $\text{rk}(\bar{Q}|_{D_i}) = \text{rk}(\bar{Q}) = \text{rk}(Q_C|_D) = r(a)$. So

$$\mu_{fr}(a) = \mu(Q_C|_D, \tilde{\alpha}_C|_D) \geq \mu(\bar{Q}, \tilde{\alpha}).$$

The sequence

$$0 \rightarrow \bar{Q} \rightarrow \bigoplus_i \bar{Q}|_{D_i} \rightarrow \bigoplus_i \bigoplus_{i < j} \bar{Q}|_{D_i \cap D_j} \rightarrow 0$$

is exact modulo sheaves of dimension $d - 3$ (the kernel of the morphism $\bar{Q} \rightarrow \bigoplus_i \bar{Q}|_{D_i}$ is zero because the divisors $D_i$ are transversal with respect to the singular set of $Q$). By the same computations as in the proof of Lemma 7.2.5 in [10] we have

$$\mu(\bar{Q}) = \sum_i \left( \mu(\bar{Q}|_{D_i}) - \frac{1}{2} \sum_{j \neq i} \left( \frac{\text{rk}(\bar{Q}|_{D_i \cap D_j})}{\text{rk}(a)} - 1 \right) a_i a_j \right).$$

For every $i$ and $j \neq i$ we define also the sheaf $T_{ij}(\bar{Q}|_{D_i})$ as the sheaf on $D_i$ that to every open subset $U$ associates the set of sections $f$ of $\bar{Q}|_{D_i}$ in $U$ such that there exists $n > 0$ for which $I^n_{D_j}, f = 0$. Note that $T_{ij}(\bar{Q}|_{D_i}) \subset \text{ker} \tilde{\alpha}|_{D_i}$. We define $Q_i = \bar{Q}|_{D_i}/\bigoplus_{j \neq i} T_{ij}(\bar{Q}|_{D_i})$. By construction $\text{rk}(Q_i) = \text{rk}(\bar{Q})$, there exists a nonzero induced framing $\alpha_i$ on $Q_i$, and

$$\mu(Q_i) = \mu(\bar{Q}|_{D_i}) - \sum_{j \neq i} \left( \frac{\text{rk}(\bar{Q}|_{D_i \cap D_j})}{\text{rk}(a)} - 1 \right) a_i a_j.$$

Therefore $\mu(\bar{Q}) \geq \sum_i \mu(Q_i)$, and

$$\mu_{fr}(a) \geq \mu(\bar{Q}, \tilde{\alpha}) \geq \sum_i \mu(Q_i, \alpha_i).$$

By definition of minimal $\mu$-destabilizing framed quotient, we have $\mu(Q_i, \alpha_i) \geq \mu_{fr}(a_i)$, hence $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$. 


Consider the second case. On the restriction to \( Z_{V_a \cap C} \) there is the quotient:

\[
\begin{array}{c}
(q^*E)|_{Z_{V_a \cap C}} \xrightarrow{q} Q_a|_{Z_{V_a \cap C}} \\
\downarrow (q^*\alpha)|_{Z_{V_a \cap C}} \\
(q^*F)|_{Z_{V_a \cap C}}
\end{array}
\]

By definition of \( Q_a \), we get \( \ker q|_{D'} \not\subset \ker \alpha|_{D'} \) for all points \( D' \in V_a \cap C \), hence \( \ker q \not\subset \ker(q^*\alpha)|_{Z_{V_a \cap C}} \). As before, we can extend \( Q_a|_{Z_{V_a \cap C}} \) to a \( C \)-flat quotient:

\[
\begin{array}{c}
(q^*E)|_{Z_C} \xrightarrow{q_C} Q_C \\
\downarrow (q^*\alpha)|_{Z_C} \\
(q^*F)|_{Z_C}
\end{array}
\]

Since \( \ker q_C \) and \( \ker(q^*\alpha)|_{Z_C} \) are \( C \)-flat, also \( \ker q_C \cap \ker(q^*\alpha)|_{Z_C} \) is \( C \)-flat. Moreover for all points \( D' \in V_a \cap C \) we have \( (\ker q_C \cap \ker(q^*\alpha)|_{Z_C})|_{D'} = \ker q|_{D'} \cap \ker \alpha|_{D'} \), hence by flatness we get \( \ker q_C|_{D'} \not\subset \ker \alpha|_{D'} \) for all points \( D' \in C \). As before, by flatness of \( Q_C \) we have that \( \rk(Q_C|_{D'}) = r(a) \) and \( \deg(Q_C|_{D'}) = \deg(a) \); moreover the induced framing on \( Q_C|_{D'} \) is zero, hence \( \mu(Q_C|_{D'}) = \mu_{fr}(a) \). Let \( \bar{Q} = Q_{\bar{c}|_{D'}}/\tau(Q_{\bar{c}|_{D'}}) \) and \( Q_i = Q_{\bar{c}|_{D}}/\tau(Q_{\bar{c}|_{D}}) \). Using the same computations as in the proof of Lemma 7.2.5 in [10], we obtain \( \mu(\bar{Q}) \geq \sum \mu(Q_i) \). As before, we get \( \mu_{fr}(a) = \mu(Q_C|_{D'}) \geq \mu(\bar{Q}) \geq \sum \mu(Q_i) \geq \sum \mu_{fr}(a_i) \).

Now let us consider the case \( \mu_{fr}(a) = \sum \mu_{fr}(a_i) \). In both cases, if we denote by \( \alpha_i \) the induced framing on \( Q_i \), from this equality, follows that \( \mu(Q_i, \alpha_i) = \mu_{fr}(a_i) \) and \( \rk(\bar{Q}|_{D_i \cap D_j}) = r(a) \). Since \( \mu_{fr}(a_i) \) is the framed-slope of the minimal \( \mu \)-destabilizing framed quotient, we have that \( r(a) = \rk(Q_i) \geq r(a_i) \) for all \( i \).

Using the same arguments of Corollary 7.2.6 in [10], we can prove:

**Corollary 72.** \( r(a) \) and \( \mu_{fr}(a)/a \) are constant for \( a \gg 0 \).

If \( \mu_{fr}(a)/a = \mu_{fr}(a_i)/a_i \) and \( r(a) = r(a_i) \) for all \( i \), then \( Q_i \) is the minimal \( \mu \)-destabilizing framed quotient of \( E|_{D_i} \), hence \( Q_C|_{D_i} \) differs from the minimal \( \mu \)-destabilizing framed quotient of \( E|_{D_i} \) only in dimension \( d - 3 \), in particular their determinant line bundles as sheaves on \( D_i \) are equal. From this argument it follows:

**Lemma 73.** There is a line bundle \( L \in \Pic(X) \) such that \( L_a \simeq L \) for all \( a \gg 0 \).

**Proof.** The proof is similar to that of Lemma 7.2.7 in [10].

By Corollary 72 and Lemma 73, \( \varepsilon(a) \) is constant for \( a \gg 0 \). In this way, we proved that for any \( a \gg 0 \) and \( V_a \)-flat family \( Q_a \) (introduced before), any extension of the determinant line bundle \( \det(Q_a) \) is of the form \( q^*L \otimes p^*M \) for some line bundle \( M \in \Pic(V_a) \). Moreover \( \deg(Q_a|_D) = a \deg(L) \) for any \( D \in V_a \).
Proof of Theorem 7.2.1. Suppose the theorem is false: we have to consider separately two cases: \( \epsilon(a) = 1 \) and \( \epsilon(a) = 0 \) for \( a \gg 0 \). In the first case we have
\[
\frac{\deg(L) - \delta_1}{r} < \mu(\mathcal{E})
\]
and \( 1 \leq r \leq \text{rk}(E) \), where \( r = r(a) \) for \( a \gg 0 \). We want to construct a rank \( r \) quotient \( Q \) of \( E \), with nonzero induced framing \( \beta \) and \( \det(Q) = L \). Thus
\[
\mu(Q) < \mu(\mathcal{E})
\]
and therefore we obtain a contradiction with the hypothesis of \( \mu \)-semistability of \( \mathcal{E} \) with respect to \( \delta_1 \). Let \( a \) be sufficiently large, \( D \in U_a \) and the minimal \( \mu \)-destabilizing framed quotient
\[
\begin{align*}
E|_D \xrightarrow{q_D} Q_D \\
\alpha|_D \downarrow \downarrow \beta_D \\
F|_D
\end{align*}
\]
Put \( K_D = \ker \beta_D \) and \( L_{K_D} = \det(K_D) \). By Proposition \[19\] (for \( \mu \)-semistability), \( Q_D \) fits into an exact sequence
\[
(12) \quad 0 \rightarrow K_D \rightarrow Q_D \rightarrow \text{Im} \alpha|_D \rightarrow 0
\]
with \( K_D \) torsion free quotient of \( \ker \alpha|_D \). So there exists an open subscheme \( D' \subset D \) such that \( K_D|_{D'} \) is locally free of rank \( r \) and \( D \setminus D' \) is a closed subset of codimension two in \( D \). Consider the restriction of the sequence (12) on \( D' \)
\[
0 \rightarrow K_D|_{D'} \rightarrow Q_D|_{D'} \rightarrow \text{Im} \alpha|_{D'} \rightarrow 0.
\]
By Proposition V-6.9 in [11], we have a canonical isomorphism
\[
L_{K_D}|_{D'} \otimes \det(\text{Im} \alpha|_{D'}) = \det(Q_D|_{D'}) = L|_{D'}.
\]
If we denote by \( \bar{L} \) the determinant bundle of \( \text{Im} \alpha \), we get \( L_{K_D}|_{D'} = L|_{D'} \otimes \bar{L}|_{D'} = (L \otimes \bar{L})|_{D'} \). Therefore \( L_{K_D} = (L \otimes \bar{L})|_D \). So we have a morphism \( \sigma_D : \Lambda^r \ker \alpha|_D \rightarrow (L \otimes \bar{L})|_D \) which is surjective on \( D' \) and morphisms
\[
D' \rightarrow \text{Grass}(\ker \alpha, r) \rightarrow \mathbb{P}(\Lambda^r \ker \alpha)
\]
By Serre’s vanishing theorem and Serre duality, one has for \( i = 0, 1 \)
\[
\text{Ext}^i(\Lambda^r \ker \alpha, (L \otimes \bar{L})^\vee(-a)) = H^{d-i}(X, \Lambda^r \ker \alpha \otimes (L \otimes \bar{L})^\vee \otimes \omega_X^\vee(a))^\vee = 0
\]
for all \( a \gg 0 \) (since \( d \geq 2 \)), hence
\[
\text{Hom}(\Lambda^r \ker \alpha, L \otimes \bar{L})^\vee = \text{Hom}(\Lambda^r \ker \alpha|_D, (L \otimes \bar{L})|_D).
\]
So for \( a \) sufficiently large, the morphism \( \sigma_D \) extends to a morphism \( \sigma : \Lambda^r \ker \alpha \rightarrow L \otimes \bar{L} \). The support of the cokernel of \( \sigma \) meets \( D \) in a closed subscheme of codimension two in \( D \), hence there is an open subscheme \( X' \subset X \) such that \( \sigma|_{X'} \) is surjective, \( X \setminus X' \) is a closed subscheme of codimension two and \( D' = X' \cap D \). So we have a morphism \( X' \rightarrow \mathbb{P}(\Lambda^r \ker \alpha) \) and we want that it factorizes through \( \text{Grass}(\ker \alpha, r) \). Using the same arguments of the final part of proof of Theorem 7.2.1 in [10], for sufficiently large \( a \) that morphism factorizes, hence we get a rank \( r \) locally free quotient \( \ker \alpha|_{X'} \rightarrow K_{X'} \) such that \( \det(K_{X'}) = (L \otimes \bar{L})|_{X'} \). So we can extend \( K_{X'} \) to a rank \( r \) coherent quotient \( K \) of \( \ker \alpha \) such that \( \det(K) = L \otimes \bar{L} \).
Let $G = \ker(\ker \alpha \to K)$. We have the following commutative diagram

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & & \downarrow \\
0 & G & \ker \alpha & K & 0 \\
\downarrow & & & \downarrow \\
0 & G & E & Q & 0 \\
\downarrow & & & \downarrow \\
0 & \Im \alpha & \Im \alpha & 0 \\
\downarrow & & & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

We have that the determinant of $Q$ is canonically isomorphic to $\det(K) \otimes \bar{L} = L$, so $Q$ destabilizes $E$ and this contradicts the hypothesis.

In the second case we have

$$\frac{\deg(L)}{r} < \mu(E).$$

Let $a$ be sufficiently large, $D \in U_a$ and the minimal $\mu$-destabilizing framed quotient

$$
E|_D \xrightarrow{q_D} Q_D
$$

with $\ker q_D \not\subset \ker |_D$. By Proposition 19 (for $\mu$-semistability), $Q_D$ is torsion free, hence there exists an open subscheme $D' \subset D$ such that $D \setminus D'$ is a closed set of codimension two in $D$ and $Q_D|_{D'}$ is locally free of rank $r$. Moreover $\ker q_D|_{D'} \not\subset \ker |_{D'}$. Using the same techniques as in the last part of the proof of Theorem 7.2.1 in [10], we extend $Q_D|_{D'}$ to a quotient $Q_{X'}$ of $X'$ which is locally free of rank $r$ with $\det(Q_{X'}) = L|_{X'}$. By construction we have $\ker(E|_{X'} \to Q_{X'}) \not\subset \ker |_{X'}$, hence in this way we obtain a quotient $Q$ of $E$ with $\det(Q) = L$ and zero induced framing, such that $Q$ destabilizes $E$. \hfill \Box

$\mu$-stable case. Now we want to prove the following generalization of Mehta-Ramanathan’s restriction theorem for $\mu$-stable torsion free sheaves (Theorem 4.3 in [10]).

**Theorem 74.** Let $(X, \mathcal{O}_X(1))$ be a polarized variety of dimension $d$. Let $F$ be a coherent sheaf on $X$ supported on a divisor $D_{fr}$, which is a locally free $\mathcal{O}_{D_{fr}}$-module. Let $E = (E, \alpha: E \to F)$ be a $(D_{fr}, F)$-framed sheaf on $X$. If $E$ is $\mu$-stable with respect to $\delta_1$, then there is a positive integer $a_0$ such that for all $a \geq a_0$ there is a dense open subset $W_a \subset |\mathcal{O}_X(a)|$ such that for all $D \in W_a$ the divisor $D$ is smooth, meets transversely the divisor $D_{fr}$ and $E|_D$ is $\mu$-stable with respect to $a\delta_1$.

The techniques to prove this theorem are quite similar to the ones used before. By Proposition 47 a $\mu$-semistable $(D_{fr}, F)$-framed sheaf which is simple but not $\mu$-stable has a proper extended framed socle. Thus we first show that the restriction is simple and we use the
Proposition 75. Let $\mathcal{E} = (E, \alpha)$ be a $\mu$-stable $(D_{fr}, F)$-framed sheaf. For $a \gg 0$ and general $D \in |\mathcal{O}_X(a)|$ the restriction $\mathcal{E}|_D = (E|_D, \alpha|_D)$ is a simple $(D_{fr} \cap D, F|_D)$-framed sheaf on $D$.

To prove this result, we need to define the double dual of a framed sheaf. Let $E = (E, \alpha)$ be a $(D_{fr}, F)$-framed sheaf; we define a framing $\alpha^{\vee\vee}$ on the double dual of $E$ in the following way: $\alpha^{\vee\vee}$ is the composition of morphisms

$$E^{\vee} \longrightarrow E^{\vee\vee}|_{D_{fr}} \cong E|_{D_{fr}} \xrightarrow{\alpha|_{D_{fr}}} F|_{D_{fr}}.$$

Then $\alpha$ is the framing induced on $E$ by $\alpha^{\vee\vee}$ by means of the inclusion morphism $E \hookrightarrow E^{\vee\vee}$.

We denote the framed sheaf $(E^{\vee\vee}, \alpha^{\vee\vee})$ by $E^{\vee\vee}$.

Lemma 76. Let $E = (E, \alpha)$ be a $\mu$-stable $(D_{fr}, F)$-framed sheaf. Then the framed sheaf $E^{\vee\vee} = (E^{\vee\vee}, \alpha^{\vee\vee})$ is $\mu$-stable.

Proof. Consider the exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow A \longrightarrow 0$$

where $A$ is a coherent sheaf supported on a closed subset of codimension at least two. Thus $\text{rk}(E^{\vee\vee}) = \text{rk}(E)$ and $\text{deg}(E^{\vee\vee}) = \text{deg}(E)$. Moreover, since $\alpha = \alpha^{\vee\vee}|_E$, we have $\mu(E^{\vee\vee}) = \mu(E)$. Let $G$ be a subsheaf of $E^{\vee\vee}$ and denote by $G'$ its intersection with $E$. So $\text{rk}(G) = \text{rk}(G')$, $\text{deg}(G) = \text{deg}(G')$ and $\alpha|_{G'} = \alpha^{\vee\vee}|_G$. Thus we obtain

$$\mu(G, \alpha^{\vee\vee}|_G) = \mu(G', \alpha|_{G'}) < \mu(E) = \mu(E^{\vee\vee}).$$

□

Definition 77. A $d$-dimensional coherent sheaf $G$ on $X$ is reflexive if the natural morphism $G \rightarrow G^{\vee\vee}$ is an isomorphism.

Recall the following result (see the proof of Lemma 7.2.9 in [10]):

Lemma 78. Let $G$ be a reflexive sheaf. For $a \gg 0$ and $D \in |\mathcal{O}_X(a)|$ the homomorphism $\text{End}(G) \rightarrow \text{End}(G|_D)$ is surjective.

Proof of Proposition 75. For arbitrary $a$ and general $D \in |\mathcal{O}_X(a)|$ the sheaf $E|_D$ is torsion free on $D$ and $E^{\vee\vee}|_D$ is reflexive on $D$, moreover the double dual of $E|_D$ (as sheaf on $D$) is $E^{\vee\vee}|_D$ (cf. Section 1.1 in [10]). In addition, $E|_D$ is locally free in a neighborhood of $D_{fr} \cap D$ and the restriction of the framing $\alpha|_D$ to $D_{fr} \cap D$ is an isomorphism. We have injective homomorphisms

$$\delta : \text{End}(E) \longrightarrow \text{End}(E^{\vee\vee}),$$

$$\delta_D : \text{End}(E|_D) \longrightarrow \text{End}(E^{\vee\vee}|_D).$$
Let \( \varphi \in \text{End}(\mathcal{E}) \): the image \( \varphi^{\vee\vee} = \delta(\varphi) \) of \( \varphi \) is an element of \( \text{End}(E^{\vee\vee}, \alpha^{\vee\vee}) \), indeed if \( \alpha \circ \varphi = \lambda \alpha \), then we can define an endomorphism of \( E^{\vee\vee} \) in the following way:

\[
\begin{array}{ccc}
E^{\vee\vee} & \overset{\varphi^{\vee\vee}}{\longrightarrow} & E^{\vee\vee} \\
\downarrow \vert_{D_{fr}} & & \downarrow \vert_{D_{fr}} \\
E^{\vee\vee}|_{D_{fr}} & \overset{\varphi^{\vee\vee}|_{D_{fr}}}{\longrightarrow} & E^{\vee\vee}|_{D_{fr}} \\
\downarrow \simeq & & \downarrow \simeq \\
E|_{D_{fr}} & \overset{\varphi|_{D_{fr}}}{\longrightarrow} & E|_{D_{fr}} \\
\downarrow \vert_{D_{fr}} & & \downarrow \vert_{D_{fr}} \\
F|_{D_{fr}} & \overset{\alpha|_{D_{fr}}}{\longrightarrow} & F|_{D_{fr}} \\
\end{array}
\]

In the same way it is possible to prove that for \( \varphi \in \text{End}(\mathcal{E}|_D) \), \( \delta_D(\varphi) \) is an element of \( \text{End}(E^{\vee\vee}|_D) \). So the homomorphisms

\[
\delta : \text{End}(\mathcal{E}) \longrightarrow \text{End}(E^{\vee\vee}), \\
\delta_D : \text{End}(\mathcal{E}|_D) \longrightarrow \text{End}(E^{\vee\vee}|_D)
\]

are injective. Therefore it suffices to show that \( E^{\vee\vee}|_D \) is simple for \( a \gg 0 \) and general \( D \).

By Lemma 76 \( E^{\vee\vee} \) is \( \mu \)-stable, hence by point (3) of Corollary 20 it is simple. By Lemma 78 the homomorphism \( \chi : \text{End}(E^{\vee\vee}) \rightarrow \text{End}(E^{\vee\vee}|_D) \) is surjective for \( a \gg 0 \) and general \( D \).

Since for \( \varphi \in \text{End}(E^{\vee\vee}) \), \( \chi(\varphi) \) is an element of \( \text{End}(E^{\vee\vee}|_D) \), we have that the map

\[
\chi|_{\text{End}(E^{\vee\vee})} : \text{End}(E^{\vee\vee}) \rightarrow \text{End}(E^{\vee\vee}|_D)
\]

is also surjective. Thus \( \text{End}(\mathcal{E}|_D) = \text{End}(E^{\vee\vee}|_D) \simeq k \).

Remark 79. Since \( \mathcal{E} \) is \( \mu \)-stable with respect to \( \delta_1 \), we have \( \deg(\text{Im} \alpha) > \delta_1 \), hence \( \deg(\text{Im} \alpha|_D) = a \deg(\text{Im} \alpha) > a \delta_1 \) for a positive integer, hence \( \ker \alpha|_D \) is not framed \( \mu \)-destabilizing for all \( D \in \Pi_a \). \( \square \)

Let \( a_0 \geq 3 \) be an integer such that for all \( a \geq a_0 \) and a general \( D \in \Pi_a \), the restriction \( \mathcal{E}|_D \) is \( \mu \)-semistable with respect to \( a \delta_1 \) and simple (cf. Proposition 73). Suppose that for an integer \( a \geq a_0 \), the framed sheaf \( \mathcal{E}|_D \) is not \( \mu \)-stable with respect to \( a \delta_1 \) for a general divisor \( D \). Then \( \mathcal{E}|_{D_\eta} \) is not geometrically \( \mu \)-stable for the divisor \( D_\eta \) associated to the generic point \( \eta \in |\mathcal{O}_X(a)| \), i.e., the pull-back to some extension of \( k(\eta) \) is not \( \mu \)-stable (cf. Corollary 48). Hence \( \mathcal{E}|_{D_\eta} \) is not \( \mu \)-stable. Since \( \mathcal{E}|_{D_\eta} \) is simple, by Proposition 47 the extended socle of \( \mathcal{E}|_{D_\eta} \) is a proper \( \mu \)-destabilizing framed subsheaf. Consider the corresponding quotient sheaf \( Q_\eta \), with induced framing \( \beta_\eta \); we can extend it to a coherent quotient \( q^*E \rightarrow Q_a \) over all of \( Z_a \).

Let \( W_a \) be the dense open subset of points \( D \in \Pi_a \) such that

- \( D \) meets transversely the divisor \( D_{fr} \) and \( E|_D \) is torsion free,
- \( Q_a \) is flat over \( W_a \) and \( \epsilon((\tilde{\alpha}_a)|_D) = \epsilon(\beta_\eta) \), where we denote by \( \tilde{\alpha}_a \) the induced framing on \( Q_a \).

Thus \( Q_a|_D \) is a coherent sheaf of positive rank such that with the induced framing is a \( \mu \)-destabilizing framed quotient for all \( D \in W_a \).

Proof of Theorem 74. Assume that the theorem is false: for all \( a \geq a_0 \) and general \( D \in \Pi_a \), \( \mathcal{E}|_D \) is not \( \mu \)-stable with respect to \( a \delta_1 \). Thus one can construct for any \( a \geq a_0 \) a coherent
quotient $q^* E \to Q_a$ and a dense open subset $W_a \subset \Pi_a$ such that $Q_a|_D$, with the induced framing, is a $\mu$-destabilizing framed quotient for all $D \in W_a$.

Using the same arguments of proof of Theorem 7.2.8 in [10], one can prove that there is a line bundle $L$ on $X$ and an integer $0 < r < \text{rk}(E)$ such that for $a \gg 0$ and for general $D \in W_a$

$$\mu(Q_a|_D, \alpha_a|_D) = \frac{\deg(L|_D) - \epsilon(a) \delta_1}{r} = a\left(\frac{\deg(L) - \epsilon(a) \delta_1}{r}\right) = \mu(E|_D, \alpha|_D) = a\mu(E, \alpha),$$

hence

$$\frac{\deg(L) - \epsilon(a) \delta_1}{r} = \mu(E, \alpha).$$

Using the arguments at the end of the proof of the restriction theorem for $\mu$-semistable framed sheaves, one can show that this suffices to construct a $\mu$-destabilizing framed quotient $E \to Q$ for sufficiently large $a$. This contradicts the assumptions of the theorem. $\square$

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