A class of representations of Hecke algebras

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Abstract

A type of directed multigraph called a $W$-digraph is introduced to model the structure of certain representations of Hecke algebras, including those constructed by Lusztig and Vogan from involutions in a Weyl group. Building on results of Lusztig, a complete characterization of $W$-digraphs is given in terms of subdigraphs for dihedral parabolic subgroups. In addition, results are obtained relating graph-theoretic properties of $W$-digraphs (acyclicity, existence of sources or sinks, connectedness) to the structure of the corresponding $H$-module or its character.

0 Overview

Let $W$ be a Weyl group with set of fundamental generators $S$ and length function $\ell$, let $u$ be an indeterminate, and let $H$ be the Hecke algebra of $(W,S)$ over $\mathbb{Q}(u)$. (See the next section for a presentation of $H$.) Put $I = \{w \in W \mid w^{-1} = w\}$. In \cite{Lusz93}, Lusztig and Vogan construct an $H$-module $M$ with basis $\{m_w \mid w \in I\}$ indexed by $I$, on which the generator $T_s$ of $H$ acts according to the rule

$$T_s m_w = \begin{cases} 
    m_{sws} & \text{if } sw \neq ws, \ell(sw) > \ell(w), \\
    (u^2 - 1)m_w + u^2 m_{sws} & \text{if } sw \neq ws, \ell(sw) < \ell(w), \\
    um_w + (u + 1)m_w & \text{if } sw = ws, \ell(sw) > \ell(w), \\
    (u^2 - u - 1)m_w + (u^2 - u)m_{sw} & \text{if } sw = ws, \ell(sw) < \ell(w),
\end{cases}$$

for $s \in S$, $w \in I$. These expressions are given geometric interpretations in \cite{Lusz93}: when $u$ is replaced by a power $q$ of a prime number, each coefficient in the expansion of $T_s m_w$ evaluates to the number of $F_q$-rational points in a corresponding subset of a variety constructed from Borel subgroups in an algebraic group with Weyl group $W$. (See 1.1-1.6 of \cite{Lusz93} for the details of this construction, and Lusztig’s paper \cite{Lusz92} for an extension to arbitrary Coxeter groups.)

The present work originated in the author’s attempt to visualize the structure of the $H$-module $M$ described above. A directed multigraph $\Gamma$ can be constructed, with set of vertices $\{m_w \mid w \in I\}$, as follows. If $w \in I$, $s \in S$, $sw \neq ws$, and $\ell(w) < \ell(sw)$, then a solid edge $m_w \xrightarrow{s} m_{sw}$ is included in $\Gamma$, while if $sw = ws$ and $\ell(w) < \ell(sw)$, then a dashed edge $m_w \dashv \xrightarrow{s} \dashv m_{sw}$ is included. The result is an example of what will be called a $W$-digraph (see Definition \ref{defn:digraph}). In broad terms, the notion of $W$-digraph is similar to the notion of $W$-graph introduced by Kazhdan and Lusztig in \cite{KL80}: both give rise to graph-theoretic objects that encode the action of the generators $T_s$, $s \in S$, on an $H$-module. There are also combinatorial similarities: if a finite dimensional $H$-module $M$ affords both a $W$-digraph $\Gamma$ and a $W$-graph $\Psi$, then the number edges labeled $s \in S$ in $\Gamma$ is equal to the multiplicity of the eigenvalue $-1$ of $T_s$ on $M$ (see Lemma \ref{lem:mult}(i)), which in turn is equal to the number of vertices with label including $s$ in $\Psi$.

On the other hand, there are significant differences between the notions of $W$-digraph and $W$-graph, including the obvious structural differences: a $W$-digraph is directed rather
than undirected, can have two different types of edges (corresponding to commutation relations in the motivating example above) rather than one type, and has generators labeling edges rather than scalars. The encodings of the actions of generators for \(W\)-digraphs and \(W\)-graphs are necessarily different. Further, the class of modules afforded by \(W\)-digraphs need not coincide with the class of modules afforded by \(W\)-graphs. When \((W, S)\) is finite and \(S \neq \emptyset\), not all \(H\)-modules are afforded by \(W\)-digraphs, whereas every \(H\)-module is afforded by a \(W\)-graph over a suitable field of scalars (Gyoja, \cite{3}). Specifically, when \(S \neq \emptyset\), the sign representation \(T\) for \(s\) \(\in S\) has basis \(\{1\}\), whereas every \(H\)-module that is afforded by a \(W\)-graph with a single vertex labeled \(S\). In the other direction, an example can be given of an infinite \((W, S)\) and corresponding \(H\)-module that is afforded by a \(W\)-digraph but is not afforded by a \(W\)-graph (see Theorem \(1.12\) and Example \(7.1\)).

### 1 Statement of Results

The Coxeter system \((W, S)\) has a presentation of the form

\[
W = \langle s \in S \mid (rs)^{n(r, s)} = e \text{ for } r, s \in S, n(r, s) < \infty \rangle,
\]

where \(n(s, s) = 1\) and \(1 < n(r, s) = n(s, r) \leq \infty\) for \(r, s \in S, r \neq s\). The Hecke algebra \(H\) has basis \(\{T_w \mid w \in W\}\) satisfying

\[
T_sT_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w), \\
u^2T_{sw} + (u^2 - 1)T_w & \text{if } \ell(sw) < \ell(w)
\end{cases}
\]

for \(s \in S\). As a \(\mathbb{Q}(u)\)-algebra, \(H\) has generators \(\{T_s \mid s \in S\}\) satisfying the relations

\[
(T_s - u^2)(T_s + 1) = 0 \quad \text{if } s \in S,
\]

\[
T_sT_t = T_tT_s \quad \text{if } s, t \in S, 1 < n = n(s, t) < \infty
\]

(where the factors in the products of (1.2b) are alternately \(T_s\) and \(T_t\)). Moreover,

\[
T_xT_y = T_{xy} \quad \text{if } \ell(xy) = \ell(x) + \ell(y).
\]

**Definition 1.1.** Let \(S\) be a set. Let \(\Gamma = (\mathcal{V}, \mathcal{E})\) be a directed multigraph with set of vertices \(\mathcal{V} = \mathcal{V}(\Gamma)\) and set of edges \(\mathcal{E} = \mathcal{E}(\Gamma)\) such that each edge is either solid or dashed and is labeled by an element of \(S\), that is, has one of the forms

\[
\alpha \xrightarrow{s} \beta \quad \text{or} \quad \alpha \xleftarrow{s} \beta
\]

with \(\alpha, \beta \in \mathcal{V}, s \in S\). Then \(\Gamma\) is an \(S\)-labeled digraph if \(\Gamma\) has no loops and, for all \(s \in S\), every vertex of \(\Gamma\) occurs in exactly one edge labeled \(s\).

Examples of \(S\)-labeled digraphs appear in Figures \(1.1\) and \(1.2\).

Let \(\Gamma\) be an \(S\)-labeled digraph. Let \(M(\Gamma)\) be a vector space over \(\mathbb{Q}(u)\) with basis \(\mathcal{V}(\Gamma)\), and let \(gl(M(\Gamma))\) be the \(\mathbb{Q}(u)\)-algebra of all linear operators on \(M(\Gamma)\). For each \(s \in S\), define \(\tau_s \in gl(M(\Gamma))\) as follows: if \(\alpha \in \mathcal{V}(\Gamma)\), then

\[
\tau_s(\alpha) = \begin{cases} 
\beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma), \\
(u^2 - 1)\alpha + u^2\beta & \text{if } \alpha \xleftarrow{s} \beta \in \mathcal{E}(\Gamma), \\
u\alpha + (u + 1)\beta & \text{if } \alpha \xleftarrow{s} \beta \in \mathcal{E}(\Gamma), \\
(u^2 - u - 1)\alpha + (u^2 - u)\beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma).
\end{cases}
\]
Definition 1.2. An $S$-labeled digraph $\Gamma$ is a $W$-digraph if the mapping $T_s \mapsto \tau_s$ extends to a representation of $H$, that is, a homomorphism of $\mathbb{Q}(u)$-algebras $\rho : H \to \text{gl} (M(\Gamma))$.

Let $J \subseteq S$, so $(W_J, J)$ is a Coxeter system with $W_J = \langle J \rangle$ the associated parabolic subgroup of $W$. For $\Gamma$ an $S$-labeled digraph, denote by $\Gamma_J$ the subdigraph with the same set of vertices obtained from $\Gamma$ by removing all edges labeled by elements of $S \setminus J$. Thus $\Gamma_J$ is a $J$-labeled digraph. If $\Gamma$ is a $W$-digraph, then clearly $\Gamma_J$ is a $W_J$-digraph. Conversely, because of the presentation (1.2a), (1.2b) it is also clear that $\Gamma$ is a $W$-digraph if $\Gamma_J$ is a $W_J$-digraph whenever $J \subseteq S$, $|J| \leq 2$. Note also that $\Gamma$ is a $W$-digraph if and only if each connected component of $\Gamma$ is a $W$-digraph.

In Figures 1.3–1.10 several $J$-labeled digraphs are given with $J = \{s, t\}$. These multigraphs have $2m$ vertices, with $m \geq 2$ except for Figures 1.9–1.10. Also, $s' = s$ if $m$ is even, $s' = t$ if $m$ is odd, $t'$ is defined by $\{s', t'\} = \{s, t\}$, and any edge not shown has one of the forms $\alpha_{2j} \xrightarrow{s} \alpha_{2j+1}$, $\alpha_{2j-1} \xrightarrow{t} \alpha_{2j}$, $\beta_{2j-1} \xrightarrow{s} \beta_{2j}$, or $\beta_{2j} \xrightarrow{t} \beta_{2j+1}$.

Two $S$-labeled digraphs $\Gamma = (\mathcal{V}, \mathcal{E})$ and $\Gamma' = (\mathcal{V}', \mathcal{E}')$ are isomorphic if there is some bijection $\varphi : \mathcal{V} \to \mathcal{V}'$ such that for all $\alpha, \beta \in \mathcal{V}$ and $s \in S$, $\alpha \xrightarrow{s} \beta \in \mathcal{E}$ if and only if $\varphi(\alpha) \xrightarrow{s} \varphi(\beta) \in \mathcal{E}'$ and $\alpha \xrightarrow{-s} \beta \in \mathcal{E}$ if and only if $\varphi(\alpha) \xrightarrow{-s} \varphi(\beta) \in \mathcal{E}'$.

Theorem 1.3. Let $(W, S)$ be a Coxeter system. The following are equivalent.
(a) $\Gamma$ is a $W$-digraph.
(b) $\Gamma$ is an $S$-labeled digraph such that for all $s, t \in S$ with $1 < n = n(s, t) < \infty$, each connected component of $\Gamma_J$, $J = \{s, t\}$, is isomorphic to one of the $J$-labeled digraphs in Figures 1.3–1.10 with...
If $\Gamma$ is an $S$-labeled digraph, let $\Gamma_{\text{rev}}$ be the $S$-labeled digraph obtained by reversing the direction of all edges of $\Gamma$ while keeping their types and labels. For example, if $\Gamma$ is as in Figure 1.6, then $\Gamma_{\text{rev}}$ is isomorphic to the digraph in Figure 1.7. We can assume $M(\Gamma)$ and $M(\Gamma_{\text{rev}})$ are the same as vector spaces over $\mathbb{Q}(u)$ (since $V(\Gamma) = V(\Gamma_{\text{rev}})$), but the endomorphisms of $M(\Gamma)$ and $M(\Gamma_{\text{rev}})$ corresponding to an element of $S$ are different.

**Corollary 1.4.** If $\Gamma$ is an $S$-labeled digraph, then $\Gamma$ is a $W$-digraph if and only if $\Gamma_{\text{rev}}$ is a $W$-digraph.

For $\Gamma$ an $S$-labeled digraph, let $\Gamma_{\rightarrow}$ be the $S$-labeled digraph obtained from $\Gamma$ by replacing any dashed edge $\alpha \xrightarrow{s} \beta$ by the corresponding solid edge $\alpha \xrightarrow{s} \beta$. Let $\Gamma_{\text{dir}}$ be the directed multigraph obtained by removing all labels from $\Gamma_{\rightarrow}$. Let $\Gamma_{\text{undir}}$ be the (undirected) multigraph obtained from $\Gamma_{\text{dir}}$ by replacing each directed edge $\alpha \rightarrow \beta$ by an undirected edge $\alpha \longrightarrow \beta$. For example, with $\Gamma$ as in Figure 1.1, the associated graphs...
Theorem 1.5. If \( n(s,t) < \infty \) for all \( s,t \in S \) and \( \Gamma \) is a connected \( W \)-digraph, then the following hold.

(i) \( \Gamma \) can have at most one source and at most one sink.

(ii) If \( \Gamma \) has either a source or a sink, then \( \Gamma \) is acyclic.

(iii) If \((W,S)\) is finite, then \( \Gamma \) has both a source and a sink, and so is acyclic.

Corollary 1.6. If \((W,S)\) is finite, then any \( W \)-digraph is acyclic. Further, the number of sources (or sinks) in a finite \( W \)-digraph is equal to the number of its connected components.

Let \( \text{ind} \) and \( \text{sgn} \) be the linear characters of \( H \) determined by \( \text{ind}(T_w) = u_w = u^{2\ell(w)} \) and \( \text{sgn}(T_w) = \varepsilon_w = (-1)^{\ell(w)} \) for \( w \in W \), respectively. For \( \lambda \) a linear character of \( H \) and \( M \) an \( H \)-module, put \( M_\lambda = \{ v \in M \mid hv = \lambda(h)v \text{ for } h \in H \} \).

Theorem 1.7. If \( \Gamma \) is a \( W \)-digraph and \( \mathcal{V}(\Gamma) \) is finite, then the following hold.

(i) The number of connected components of \( \Gamma \) is equal to \( \dim M(\Gamma)_{\text{ind}} \).

(ii) If \( n(s,t) < \infty \) for all \( s,t \in S \), then the number of acyclic connected components of \( \Gamma \) is equal to \( \dim M(\Gamma)_{\text{sgn}} \).

Theorem 1.8. If \((W,S)\) is finite, \( J \subseteq S \), and \( \Gamma \) is a connected \( W \)-digraph, then \( \Gamma_J \) has at most \(|W:W_J|\) connected components.

Taking \( J = \emptyset \) gives the following.

Corollary 1.9. If \((W,S)\) is finite and \( \Gamma \) is a connected \( W \)-digraph, then \(|\mathcal{V}(\Gamma)| \leq |W|\).

The bound in Corollary 1.9 is always attained: see Example 4.5.

Theorem 1.10. Assume \( n(s,t) < \infty \) for \( s,t \in S \) and \( \Gamma \) is a connected \( W \)-digraph with a source or sink. Then for \( \alpha, \beta \in \mathcal{V}(\Gamma) \), any two directed paths from \( \alpha \) to \( \beta \) in \( \Gamma \) have the same number of edges.

If \( \Gamma \) is a \( W \)-digraph and \( \mathcal{V}(\Gamma) \) is finite, so \( M(\Gamma) \) is finite dimensional, let \( \chi_\Gamma \) be the character of \( H \) afforded by \( M(\Gamma) \).

Theorem 1.11. If \( \Gamma \) is a \( W \)-digraph and \( \mathcal{V}(\Gamma) \) is finite, then the following hold.

(i) If \( \sigma \) is the automorphism of \( Q(\Gamma) \) determined by \( \sigma u = -1/u \), then \( \chi_{\Gamma_{\text{rev}}}(T_w) = \sigma\chi_{\Gamma}(T_w^{-1}) \) for \( w \in W \).

(ii) If \( n(s,t) < \infty \) for \( s,t \in S \) and \( \Gamma \) is a cyclic, then \( \chi_{\Gamma_{\text{rev}}}(T_w) = \varepsilon_w u_w \chi_{\Gamma}(T_w^{-1}) \) for \( w \in W \).

In the case of an affine Weyl group, the following holds.

Theorem 1.12. If \((W_J,J)\) is finite for proper subsets \( J \) of \( S \), \( \Gamma \) is a finite, connected \( W \)-digraph, and \( M(\Gamma) \) affords a \( W \)-graph (as defined in \( [2] \)) over \( Q \), then \( \Gamma \) is acyclic.

The organization of this paper is as follows. Section 2 contains preliminary results, and Section 3 contains a proof of Theorem 1 and related results. Section 4 contains proofs of Theorems 1.5, 1.7, 1.8, and 1.10. Section 5 contains a proof of Theorem 1.11 and related results. Section 6 contains a proof of Theorem 1.12 and the last section has additional examples.
2 Preliminary results

Assume that \((W, S)\) is a Coxeter system and let \(\Gamma\) be an \(S\)-labeled digraph. Throughout this and later sections, the notation \(x \leq y\) is used to indicate the usual Bruhat order on \(W\) relative to \(S\) when \(x, y \in W\). For any \(s \in S\), we have

\[(2.1) \quad (\tau_s - u^2)(\tau_s + 1) = 0 \quad \text{in } \text{gl}(M),\]

where \(\tau_s\) is as in (1.3) and \(M = M(\Gamma)\) (see [6], 2.3). Indeed, suppose \(\alpha\) is connected to \(\beta\) by an edge of \(\Gamma\) labeled \(s\). Exchanging \(\alpha, \beta\) if necessary, we can assume this edge is directed from \(\alpha\) to \(\beta\). By (1.3), \(\tau_s\) leaves invariant the subspace with basis \(\{\alpha, \beta\}\), and the matrix of \(\tau_s\) acting on this subspace with respect to this basis is

\[
\begin{pmatrix}
0 & u^2 \\
1 & u^2 - 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
u & u^2 - u \\
u + 1 & u^2 - u - 1
\end{pmatrix}
\]

according to whether \(\alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma)\) or \(\alpha \xleftarrow{s} \beta \in \mathcal{E}(\Gamma)\). In either case, the eigenvalues are \(u^2\) and \(-1\), and thus (2.1) holds. Hence \(\Gamma\) is a \(W\)-digraph if and only if

\[(2.2) \quad \tau_s^n \tau_t \cdots = \tau_t \tau_s \cdots \quad \text{whenever } s, t \in S, \ 1 < n(s, t) < \infty.\]

Define \(T_s^\circ \in H\) by

\[T_s^\circ = (u + 1)^{-1}(T_s - u).\]

(This element is denoted \(\frac{T_s}{T_s^\circ}\) in [6], 2.2.) By (1.2a), both \(T_s\) and \(T_s^\circ\) are units in \(H\), with inverses given by

\[T_s^{-1} = u^{-2}(T_s - (u^2 - 1)), \quad (T_s^\circ)^{-1} = (u^2 - u)^{-1}(T_s - (u^2 - u - 1)).\]

The terminology used in the next definition will be justified by the remarks after Lemma 2.2.

**Definition 2.1.** Let \(M\) be an \(H\)-module. Then a subset \(X\) of \(M\) supports a \(W\)-digraph if \(X\) is linearly independent over \(\mathbb{Q}(u)\) and, for each \(\alpha \in X\) and \(s \in S\),

\[X \cap \{T_s \alpha, T_s^{-1} \alpha, T_s^\circ \alpha, (T_s^\circ)^{-1} \alpha\} \neq \emptyset.\]

**Lemma 2.2.** If \(M\) is an \(H\)-module and \(X \subseteq M\) supports a \(W\)-digraph, then the following hold.

(i) If \(s \in S\) and \(\alpha \in X\), then \(\alpha, T_s \alpha, T_s^{-1} \alpha, T_s^\circ \alpha, (T_s^\circ)^{-1} \alpha\) are distinct and \(X\) contains a unique element of \(\{T_s \alpha, T_s^{-1} \alpha, T_s^\circ \alpha, (T_s^\circ)^{-1} \alpha\}\).

(ii) The subspace of \(M\) spanned by \(X\) is an \(H\)-submodule of \(M\).

**Proof.** Suppose \(s \in S\) and \(\alpha \in X\). Put \(Y = \{T_s \alpha, T_s^{-1} \alpha, T_s^\circ \alpha, (T_s^\circ)^{-1} \alpha\}\). By (1.2a), there are unique \(\gamma, \delta \in M\) such that

\[\alpha = \gamma + \delta, \quad T_s \gamma = -\gamma, \quad T_s \delta = u^2 \delta.\]

Thus

\[T_s \alpha = -\gamma + u^2 \delta, \quad T_s^{-1} \alpha = -\gamma + \frac{1}{u^2} \delta, \quad T_s^\circ \alpha = -\gamma + \frac{u - 1}{u^2 - u} \delta, \quad (T_s^\circ)^{-1} \alpha = -\gamma + \frac{u + 1}{u^2 - u} \delta.\]
Since $X$ is linearly independent and $X$ contains $\alpha$ and at least one element of $Y$, it follows that $\gamma, \delta$ are linearly independent over $\mathbb{Q}(u)$. Therefore $\alpha, T_s\alpha, T_s^{-1}\alpha, T_s^0\alpha, (T_s^0)^{-1}\alpha$ are distinct. Also, since $\alpha, T_s\alpha, T_s^{-1}\alpha, T_s^0\alpha, (T_s^0)^{-1}\alpha$ are all in $\text{span} \{\gamma, \delta\}$, $X$ can contain at most one element of $Y$. Thus (i) holds. Further, since $X$ contains two elements of span $\{\gamma, \delta\}$, span $X$ contains span $\{\gamma, \delta\}$ by dimension, and thus $T_s\alpha \in $ span $X$. Since $\alpha \in X$ was arbitrary, we have $T_s$ span $X \subseteq $ span $X$. Thus span $X$ is an $H$-submodule of $M$ since $s \in S$ was arbitrary, so (ii) holds.

If $M$ is an $H$-module and $X \subseteq M$ supports a $W$-digraph, then we construct a directed multigraph $\Gamma$, as follows. If $\alpha, \beta \in X$ and $s \in S$, then

$$\alpha \xrightarrow{\tau_s} \beta \text{ is an edge of } \Gamma \text{ if } \beta = T_s\alpha,$$

$$\alpha \xrightarrow{-\tau_s} \beta \text{ is an edge of } \Gamma \text{ if } \beta = T_s^{-1}\alpha.$$

Then $\Gamma$ is a well-defined $S$-labeled digraph by Lemma 2.2. Moreover, from the definition of $T_s^0$, it is easily checked that $H$ acts on $M_0 = $ span $X$ according to

$$T_s\alpha = \tau_s(\alpha),$$

where $\tau_s$ is as in (1.3). Therefore $\Gamma$ is indeed a $W$-digraph with associated $H$-module $M_0$.

**Lemma 2.3.** Suppose $X$ is a linearly independent subset of an $H$-module $M$. Then $X$ supports a $W$-digraph if and only if for each $s \in S$, there exists a partition $P_s$ of $X$ such that, for all $U \in P_s$, there are $\alpha, \beta \in U$ such that $\alpha \neq \beta$, $U = \{\alpha, \beta\}$, and either $T_s\alpha = \beta$ or $T_s^0\alpha = \beta$.

**Proof.** First suppose $X$ supports a $W$-digraph. Let $s \in S$. For $\lambda \in X$, define $U_\lambda = \{\lambda, \mu\}$ where

$$X \cap \{T_s\lambda, T_s^{-1}\lambda, T_s^0\lambda, (T_s^0)^{-1}\lambda\} = \{\mu\}.$$

Then $\lambda \in X \cap \{T_s\mu, T_s^{-1}\mu, T_s^0\mu, (T_s^0)^{-1}\mu\}$, and so $U_\lambda = U_\mu$. By Lemma 2.2, $P_s = \{U_\lambda \mid \lambda \in X\}$ is a partition of $X$ satisfying the conditions above: if $U = U_\lambda$ and $\mu = T_s\lambda$ or $\mu = T_s^0\lambda$, then take $\alpha = \lambda$, $\beta = \mu$, and otherwise take $\alpha = \mu$, $\beta = \lambda$.

Conversely, suppose for each $s \in S$, a partition $P_s$ satisfying the conditions above exists. Let $\gamma \in X$. There is some $\delta \in X$ such that $U = \{\gamma, \delta\} \in P_s$. For this $\delta$ we either have $T_s^{\pm 1}\gamma = \delta$ or $(T_s^0)^{\pm 1}\gamma = \delta$. Thus $X$ supports a $W$-digraph.\qed

**Lemma 2.4.** Suppose $M$ is an $H$-module with basis $X$ supporting a $W$-digraph $\Gamma$, $v = \sum_{\gamma \in X} \lambda_\gamma \gamma \in M$, and $s \in S$. Then the following hold.

(i) $T_s v = u^2 v$ if and only if $\lambda_\beta = \lambda_\alpha$ whenever $\alpha \xrightarrow{\tau_s} \beta$ or $\alpha \xrightarrow{-\tau_s} \beta$ is an edge of $\Gamma$.

(ii) $T_s v = -v$ if and only if

$$\lambda_\beta = \begin{cases} -u^{-2}\lambda_\alpha \text{ whenever } \alpha \xrightarrow{\tau_s} \beta \in \mathcal{S}(\Gamma), \\ -(u + 1)(u^2 - u)^{-1}\lambda_\alpha \text{ whenever } \alpha \xrightarrow{-\tau_s} \beta \in \mathcal{S}(\Gamma). \end{cases}$$

**Proof.** With $P_s$ as in Lemma 2.3, $M$ is the direct sum of the subspaces span $\{\alpha, \beta\}$, $\{\alpha, \beta\} \in P_s$. Also, $T_s$ leaves each such subspace invariant, with eigenvalues $u^2$ and $-1$. Hence it suffices to show that for $\{\alpha, \beta\} \in P_s$, span $\{\alpha, \beta\}$ has a basis consisting of eigenvectors
for $T_s$ of the form $\lambda_\alpha \alpha + \lambda_\beta \beta$ with coefficients satisfying the relations of (i) and (ii). If $\alpha \overset{s}{\to} \beta$ is an edge of $\Gamma$, then
\[
T_s(\alpha + \beta) = \beta + (u^2 \alpha + (u^2 - 1)\beta) = u^2(\alpha + \beta)
\]
and
\[
T_s(\alpha - u^{-2}\beta) = \beta - u^{-2}(u^2 \alpha + (u^2 - 1)\beta) = -(\alpha - u^{-2}\beta),
\]
so the basis $\{\alpha + \beta, \alpha - u^{-2}\beta\}$ has the desired property. On the other hand, if $\alpha \overset{s}{\to} \beta$ is an edge of $\Gamma$, then
\[
T_s(\alpha + \beta) = ((u\alpha + (u + 1)\beta) + ((u^2 - u)\alpha + (u^2 - u - 1)\beta)
= u^2(\alpha + \beta)
\]
and
\[
T_s(\alpha - (u + 1)(u^2 - u)^{-1}\beta)
= ((u\alpha + (u + 1)\beta)
- (u + 1)(u^2 - u)^{-1}((u^2 - u)\alpha + (u^2 - u - 1)\beta)
= -(\alpha - (u + 1)(u^2 - u)^{-1}\beta),
\]
so $\{\alpha + \beta, \alpha - (u + 1)(u^2 - u)^{-1}\beta\}$ is an appropriate basis.

For the remainder of this section we assume $J = \{s, t\} \subseteq S$, $1 < n = n(s, t) < \infty$. For $0 \leq k \leq n$, define elements $s_k, t_k$ of $W_J$ by
\[
s_k = \cdots sts, \quad t_k = \cdots tst,
\]
with $k$ factors in each product, alternately $s$ and $t$. For example, $s_0 = e = t_0$, and $s_n = w_0 = t_n$ is the longest element of $W_J$. Define elements $\sigma_k$ of $H_J$ as follows:
\[
\sigma_k = \sum_{w \in W_J, t(w) = k} T_w.
\]
Thus $\sigma_0 = T_e$, $\sigma_n = T_{w_0}$, and $\sigma_k = T_{s_k} + T_{t_k}$ for $0 < k < n$.

**Lemma 2.5.** Suppose $a_0 = \sigma_k + \sum_{w \in W_J, t(w) < k} \gamma_w T_w \in H_J$, where $0 \leq k < n$ and $\gamma_w \in \mathbb{Q}(u)$ for $w \in W_J$. Suppose further that for $0 \leq j \leq n - k$, $S_j \in \{T_s, T_s^\circ\}$ and $T_j \in \{T_t, T_t^\circ\}$. Put $b_0 = a_0$, and define $a_1, \ldots, a_{n-k}, b_1, \ldots, b_{n-k}$ by
\[
a_{j+1} = \begin{cases} S_j a_j & \text{if } j \text{ is even}, \\ T_j a_j & \text{if } j \text{ is odd}, \end{cases}
\]
and
\[
b_{j+1} = \begin{cases} T_j b_j & \text{if } j \text{ is even}, \\ S_j b_j & \text{if } j \text{ is odd}. \end{cases}
\]
for $0 \leq j < n - k$. Then $X = \{a_0, a_1, \ldots, a_{n-k-1}, b_1, b_2, \ldots, b_{n-k}\}$ is linearly independent. Moreover, if $a_{n-k} = b_{n-k}$, then $X$ supports a $W_J$-digraph and $L = \text{span } X$ is a left ideal of $H_J$.

**Proof.** If $1 \leq j \leq n - k$ and $a_j$ is expressed as a linear combination of $\{T_w \mid w \in W_J\}$, then the unique $w \in W$ of maximal length such that $T_w$ appears with nonzero coefficient is given by
\[
w = \begin{cases} s_j s_k = s_{j+k} & \text{if } k \text{ is even}, \\ s_j t_k = t_{j+k} & \text{if } k \text{ is odd}. \end{cases}
\]
Similarly, if \(1 \leq j \leq n - k\), then the unique \(w \in W_J\) of maximal length such that \(T_w\) appears with nonzero coefficient in \(b_j\) is given by
\[
w = \begin{cases}
t_j t_k = t_{j+k} & \text{if } k \text{ is even}, \\
t_j s_k = s_{j+k} & \text{if } k \text{ is odd}.
\end{cases}
\]
Thus \(X\) is linearly independent.

Suppose \(a_{n-k} = b_{n-k}\). If \(n - k\) is even, then the partitions
\[
P_s = \{\{a_0, a_1\}, \{b_1, b_2\}, \ldots, \{b_{n-k-1}, b_{n-k}\}\},
\]
\[
P_t = \{\{b_0, b_1\}, \{a_1, a_2\}, \ldots, \{a_{n-k-1}, a_{n-k}\}\}
\]
satisfy the conditions of Lemma 2.3. On the other hand, if \(n - k\) is odd, then the partitions
\[
P_s = \{\{a_0, a_1\}, \{b_1, b_2\}, \ldots, \{a_{n-k-1}, a_{n-k}\}\},
\]
\[
P_t = \{\{b_0, b_1\}, \{a_1, a_2\}, \ldots, \{b_{n-k-1}, b_{n-k}\}\}
\]
satisfy the conditions of Lemma 2.3. Thus \(X\) supports a \(W_J\)-digraph, and \(L = \text{span } X\) is a left ideal of \(H_J\) by Lemma 2.2(ii).

For \(d \geq 0\), define a polynomial \(p_d(u) \in \mathbb{Q}[u]\) as follows: \(p_0(u) = 1\), and for \(d > 0\),
\[
p_d(u) = 1 + 2 \sum_{i=1}^{d-1} (-u^2)^i + (-u^2)^d.
\]
Thus \(p_1(u) = 1 - u^2\), \(p_2(u) = 1 - 2u^2 + u^4\), \(p_3(u) = 1 - 2u^2 + 2u^4 - u^6\). Let \(y \in W_J\). A straightforward induction argument based on 2.0.b and 2.0.c of [4] shows
\[
(2.3) \quad u^{2\ell(y)}T_{y^{-1}} = T_y + \sum_{x < y} p_{\ell(y) - \ell(x)}(u)T_x.
\]

For \(0 \leq j \leq n\), we define elements \(\bar{\varphi}_j, \bar{\eta}_j, \bar{\gamma}_j, \bar{\delta}_j\) of \(H_J\), as follows:
\[
\begin{align*}
\bar{\varphi}_j & = \sum_{i=0}^{j} p_{j-i}(u)\sigma_i \\
& = \sigma_j + (1 - u^2)\sigma_{j-1} + (1 - 2u^2 + u^4)\sigma_{j-2} + \cdots \\
& \quad + (1 - 2u^2 + 2u^4) + \cdots + 2(-u^2)^{j-1} + (-u^2)^j\sigma_0, \\
\bar{\eta}_j & = \bar{\varphi}_j + u\bar{\varphi}_{j-1} + u^2\bar{\varphi}_{j-2} + \cdots + u^j\bar{\varphi}_0, \\
\bar{\gamma}_j & = \bar{\varphi}_j - u\bar{\varphi}_{j-1} + u^2\bar{\varphi}_{j-2} + \cdots + (-u)^j\bar{\varphi}_0, \\
\bar{\delta}_j & = \frac{1}{2}(\bar{\eta}_j + \bar{\gamma}_j).
\end{align*}
\]
These elements are used in describing the constructions of Lusztig in the next section.

If \(0 < j < n\), then by (2.3) we have
\[
(2.4) \quad \bar{\varphi}_j = T_{s_j} + T_{t_j} + (1 - u^2)\sigma_{j-1} + (1 - 2u^2 + u^4)\sigma_{j-2} + \cdots \\
+ (1 - 2u^2) + \cdots + 2(-u^2)^{j-1} + (-u^2)^j\sigma_0 \\
= u^{2j}T_{s_j^{-1}} + T_{t_j} + u^{2j}T_{t_j^{-1}} + T_{s_j}.
\]
Lemma 2.6. If $0 < j \leq k$ and $j + k \leq n$, then

$$T_{s_k^{-1}\tilde{\varphi}_j} = u^{2j}T_{s_{k-j}^{-1}} + T_{s_{k+j}^{-1}}$$

and

$$T_{t_k^{-1}\tilde{\varphi}_j} = u^{2j}T_{t_{k-j}^{-1}} + T_{t_{k+j}^{-1}}.$$

Proof. Note $j < n$, so (2.4) applies to $\tilde{\varphi}_j$. Define $s^*, t^* \in \{s, t\}$ by $s_k = s^*_ks_{k-j}$ and $\{s^*, t^*\} = \{s, t\}$. Then

$$T_{s_k^{-1}\tilde{\varphi}_j} = T_{s_k^{-1}}T_{s_j^*-1}\left(u^{2j}T_{s_j^*-1}^{-1} + T_{t_j^*}\right)$$

$$= u^{2j}T_{s_k^{-1}} + T_{s_k^{-1}}T_{s_j^*-1}T_{t_j^*} = u^{2j}T_{s_k^{-1}} + T_{s_k^{-1}}$$

since $s_k^{-1}s_j^*-1t_j^* = s_k^{-1}$ and $\ell(s_k^{-1}) + \ell(s_j^*-1) + \ell(t_j^*) = \ell(s_k^{-1})$. Thus the first equation holds. The second follows by applying the automorphism $T_s \leftrightarrow T_t$ of $H_J$ to the first. 

Lemma 2.7. If $0 \leq j \leq k$ and $j + k \leq n$, then

(i) $T_{s_k^{-1}}\tilde{\eta}_j = \sum_{i=0}^{2j} u^iT_{s_{k-j-i}^{-1}}$ and $T_{t_k^{-1}}\tilde{\eta}_j = \sum_{i=0}^{2j} u^iT_{t_{k-j-i}^{-1}}$,

(ii) $T_{s_k^{-1}}\tilde{\gamma}_j = \sum_{i=0}^{2j} (-u)^iT_{s_{k-j-i}^{-1}}$ and $T_{t_k^{-1}}\tilde{\gamma}_j = \sum_{i=0}^{2j} (-u)^iT_{t_{k-j-i}^{-1}}$,

(iii) $T_{s_k^{-1}}\tilde{\delta}_j = \sum_{i=0}^{j} u^{2i}T_{s_{k-j-2i}^{-1}}$ and $T_{t_k^{-1}}\tilde{\delta}_j = \sum_{i=0}^{j} u^{2i}T_{t_{k-j-2i}^{-1}}$.

Proof. Observe that (i) holds when $k = 0$ because $\tilde{\eta}_0 = T_e$. Assume $\ell > 0$ and (i) holds when $0 \leq k < \ell$. Suppose $0 \leq j \leq \ell$ and $j + \ell \leq n$. If $j \leq \ell - 1$, then

$$T_{s_k^{-1}}\tilde{\eta}_j = T_sT_{t_{\ell-1}^{-1}}\tilde{\eta}_j = T_s\sum_{i=0}^{2j} u^iT_{t_{\ell-1}^{-1-i}} = \sum_{i=0}^{2j} u^iT_{s_{\ell-1}^{-i}}$$

by the induction hypothesis. On the other hand, if $j = \ell$, then

$$T_{s_k^{-1}}\tilde{\eta}_\ell = T_s^{-1}((\tilde{\varphi}_\ell + u\tilde{\eta}_{\ell-1}) = T_s^{-1}\tilde{\varphi}_\ell + uT_sT_{t_{\ell-1}^{-1}}\tilde{\eta}_{\ell-1}$$

$$= u^{2\ell}T_e + T_{s_{2\ell}^{-1}} + uT_s\sum_{i=0}^{2\ell-2} u^iT_{t_{2\ell-i-2}^{-1}}$$

$$= u^{2\ell}T_e + T_{s_{2\ell}^{-1}} + \sum_{i=0}^{2\ell-2} u^{i+1}T_{s_{2\ell-i-1}^{-1}} = \sum_{i=0}^{2\ell} u^iT_{s_{2\ell-i}^{-1}}$$

by Lemma 2.6. Thus the first equation of (i) holds in the case $k = \ell$. The second equation of (i) follows in the case $k = \ell$ by applying the automorphism $T_s \leftrightarrow T_t$ to the first equation. Thus (i) holds by induction.

Let $\zeta$ be the automorphism of $Q(u)$ determined by $\zeta(u) = -u$. Extend $\zeta$ to a semilinear automorphism of $H_J$ by defining $\zeta(\sum \alpha_u T_u) = \sum \zeta(\alpha_u) T_u$. Then $\zeta(\tilde{\eta}_m) = \tilde{\gamma}_m$, and so the formulas of (ii) are obtained by applying $\zeta$ to the formulas of (i). Finally, (iii) follows by averaging the formulas of (i) and (ii). 

10
3 Proof of Theorem 1.3

We begin this section by outlining constructions due to Lusztig (6, 2.4–2.10) of $H_J$-modules with bases supporting $W_J$-digraphs when $W_J$ is a finite dihedral group. The arguments given here, which differ somewhat from those in [6], are included for the sake of completeness.

Assume $J = \{s, t\}$, $1 < n = n(s, t) < \infty$. When arguing that the $\{s, t\}$-labeled digraphs in Figures 1.3, 1.10 are $W_J$-digraphs, we may as well assume $n = m$ for Figures 1.3, 1.10 $n = 2m - 1$ for Figures 1.6, 1.7, and $n = 2m - 2$ in Figure 1.8. Indeed, if $n, n'$ are positive integers and $n$ divides $n'$, then

$$\tau_s \tau_t \cdots = \tau_t \tau_s \cdots$$

implies

$$\tau_s \tau_t \cdots = \tau_t \tau_s \cdots$$

Put $s' = s$ if $m$ is even, $s' = t$ if $m$ is odd, and define $t'$ by $\{s', t'\} = \{s, t\}$. We consider cases.

**Case 1.** Figure 1.3 $n = m \geq 2$.

Define $\mu_0 = T_e$ and

$$\begin{align*}
\mu_1 &= T_s \mu_0, \\
\mu_2 &= T_t \mu_1, \\
& \vdots \\
\mu_{m-1} &= T_s \mu_{m-2}, \\
\mu_m &= T_t \mu_{m-1}, \\
\mu'_1 &= T_s \mu'_0, \\
\mu'_2 &= T_t \mu'_1, \\
& \vdots \\
\mu'_{m-1} &= T_s \mu'_{m-2}, \\
\mu'_m &= T_t \mu'_{m-1}.
\end{align*}$$

Then

$$\mu_m = T_{s^m} = T_{t^m} = \mu'_m,$$

and so by Lemma 2.5 $X = \{\mu_0, \mu_1, \ldots, \mu_{m-1}, \mu'_1, \mu'_2, \ldots, \mu'_m\} = \{T_w \mid w \in W_J\}$ supports a $W_J$-digraph. This $W_J$-digraph is isomorphic to the $J$-labeled digraph of Figure 1.3 via $\mu_j \leftrightarrow \alpha_j$ for $0 \leq j \leq m - 1$, $\mu'_j \leftrightarrow \beta_j$ for $1 \leq j \leq m$.

**Case 2.** Figure 1.4 $n = m \geq 2$.

Let $\nu_0 = T_e$, and define

$$\begin{align*}
\nu_1 &= T_s^2 \nu_0, \\
\nu_2 &= T_t \nu_1, \\
& \vdots \\
\nu_{m-1} &= T_s \nu_{m-2}, \\
\nu_m &= T_t \nu_{m-1}, \\
\nu'_1 &= T_s \nu'_0, \\
\nu'_2 &= T_t \nu'_1, \\
& \vdots \\
\nu'_{m-1} &= T_s \nu'_{m-2}, \\
\nu'_m &= T_t \nu'_{m-1}.
\end{align*}$$

Then

$$\begin{align*}
\nu_m &= T_{t^m} T_s^m = (u + 1)^{-1} (T_{s^m} u - u) (T_{t^m} u - u) (T_{s^m} u - u) (T_{t^m} u - u) \\
&= (u + 1)^{-1} (T_{s^m} u - u) T_{t^m} (u + 1)^{-1} (T_{s^m} u - u) T_{t^m} \\
&= (u + 1)^{-1} (T_{s^m} u - u) T_{t^m} T_{t^m} = T_{s^m} T_{t^m} = \nu'_m,
\end{align*}$$

so $X = \{\nu_0, \nu_1, \ldots, \nu_{m-1}, \nu'_1, \nu'_2, \ldots, \nu'_m\}$ is linearly independent, so is a basis for $H_J$, and supports a $W_J$-digraph by Lemma 2.5. This $W_J$-digraph is isomorphic to the $J$-labeled digraph of Figure 1.4 via $\nu_j \leftrightarrow \alpha_j$ for $0 \leq j \leq m - 1$, $\nu'_j \leftrightarrow \beta_j$ for $1 \leq j \leq m$.

**Case 3.** Figure 1.5 $n = m \geq 2$.

Interchanging $s$ and $t$ in the argument given for the previous case shows that the $J$-labeled multigraph in Figure 1.5 is a $W_J$-digraph.

**Case 4.** Figure 1.6 $n = 2m - 1$, $m \geq 2$.

Define an element $\eta_0$ of $H_J$ by

$$\eta_0 = \bar{\eta}_{m-1} = \bar{\varphi}_{m-1} + u \bar{\varphi}_{m-2} + u^2 \bar{\varphi}_{m-3} + \cdots + u^{m-1} \bar{\varphi}_0.$$
Suppose \( m \) is even. Then by part (i) of Lemma 2.7,
\[
T_{t_{m-1}}(T_s - u)\eta_0 = T_{s_m} \bar{\eta}_{m-1} - u T_{t_{m-1}} \bar{\eta}_{m-1} = T_{t_{m-1}} \bar{\eta}_{m-1} - u T_{t_{m-1}} \bar{\eta}_{m-1}
\]
\[
= \sum_{i=0}^{2m-2} u^i T_{t_{2m-i-1}} - u \sum_{i=0}^{2m-2} u^i T_{t_{2m-i-2}}
\]
\[
= T_{t_{2m-1}} - u^{2m-1} = T_{w_0} - u^n.
\]
On the other hand, if \( m \) is odd, then
\[
T_{t_{m-1}}(T_s - u)\eta_0 = T_{s_m} \bar{\eta}_{m-1} - u T_{t_{m-1}} \bar{\eta}_{m-1} = T_{s_m} \bar{\eta}_{m-1} - u T_{s_{m-1}} \bar{\eta}_{m-1}
\]
\[
= \sum_{i=0}^{2m-2} u^i T_{s_{2m-i-1}} - u \sum_{i=0}^{2m-2} u^i T_{s_{2m-i-2}}
\]
\[
= T_{s_{2m-1}} - u^{2m-1} = T_{w_0} - u^n.
\]
Hence
\[
T_{t_{m-1}}(T_s - u)\eta_0 = T_{w_0} - u^n = T_{s_{m-1}}(T_t - u)\eta_0,
\]
where the second equation follows by applying the automorphism \( T_s \leftrightarrow T_t \) to the first. Hence if we define
\[
\begin{align*}
\eta_1 &= T_s^r \eta_0, \eta_2 = T_t \eta_1, \ldots, \eta_{m-1} = T_s^r \eta_{m-2}, \eta_m = T_t \eta_{m-1}, \\
\eta'_1 &= T_t^s \eta_0, \eta'_2 = T_s \eta'_1, \ldots, \eta'_{m-1} = T_t^s \eta'_{m-2}, \eta'_m = T_s \eta'_{m-1},
\end{align*}
\]
then
\[
\eta_m = (u + 1)^{-1}(T_{w_0} - u^n) = \eta'_m.
\]
Therefore \( X = \{ \eta_0, \eta_1, \eta_2, \ldots, \eta_{m-1}, \eta'_1, \eta'_2, \ldots, \eta'_{m-1}, \eta'_m \} \) is a basis for a left ideal in \( H_J \) and \( X \) supports a \( W_{J_r} \)-digraph by Lemma 2.5. This \( W_{J_r} \)-digraph is isomorphic to the \( J \)-labeled digraph in Figure 1.7 via \( \eta_j \leftrightarrow \alpha_j \) for \( 0 \leq j \leq m - 1 \), \( \eta'_j \leftrightarrow \beta_j \) for \( 1 \leq j \leq m \).

**Case 5.** Figure 1.7 \( n = 2m - 1 \), \( m \geq 2 \).

Put
\[
\gamma_0 = \bar{\gamma}_{m-1} = \bar{\gamma}_{m-2} + u^2 \bar{\gamma}_{m-3} + \cdots + (-u)^{m-1} \bar{\gamma}_0.
\]
If \( m \) is even, then part (ii) of Lemma 2.7 gives
\[
(T_{t'} - u)T_{s_{m-1}} \gamma_0
\]
\[
= (T_s - u)T_{s_{m-1}} \bar{\gamma}_{m-1} = T_{s_m} \bar{\gamma}_{m-1} - T_{s_{m-1}} \bar{\gamma}_{m-1}
\]
\[
= T_{t_{m-1}} \bar{\gamma}_{m-1} - T_{s_{m-1}} \bar{\gamma}_{m-1} = \sum_{i=0}^{2m-2} (-u)^i T_{t_{2m-i-1}} - u \sum_{i=0}^{2m-2} (-u)^i T_{s_{2m-i-2}}
\]
\[
= \sum_{w \in W} (-u)^{n-\ell(w)} T_w.
\]
On the other hand, if \( m \) is odd, then
\[
(T_{t'} - u)T_{s_{m-1}} \gamma_0
\]
\[
= (T_t - u)T_{s_{m-1}} \bar{\gamma}_{m-1} = T_{s_m} \bar{\gamma}_{m-1} - u T_{s_{m-1}} \bar{\gamma}_{m-1}
\]
\[
= T_{s_m} \bar{\gamma}_{m-1} - u T_{t_{m-1}} \bar{\gamma}_{m-1} = \sum_{i=0}^{2m-2} (-u)^i T_{s_{2m-i-1}} - u \sum_{i=0}^{2m-2} (-u)^i T_{t_{2m-i-2}}
\]
\[
= \sum_{w \in W} (-u)^{n-\ell(w)} T_w.
\]
Therefore

\[(T_{t'} - u)T_{s_{m-1}} \gamma_0 = \sum_{w \in W} (-u)^{n-\ell(w)} T_w = (T_{s'} - u)T_{t_{m-1}} \gamma_0,\]

with the second equation following from the first by applying the automorphism \(T_s \leftrightarrow T_t\). Hence if we put

\[
\begin{cases}
  \gamma_1 = T_s \gamma_0, \gamma_2 = T_t \gamma_1, \ldots, \gamma_{m-1} = T_s \gamma_{m-2}, \gamma_m = T_t \gamma_{m-1}, \\
  \gamma'_1 = T_t \gamma_0, \gamma'_2 = T_s \gamma'_1, \ldots, \gamma'_{m-1} = T_t \gamma'_{m-2}, \gamma'_m = T_s \gamma'_{m-1},
\end{cases}
\]

then

\[\gamma_m = (u + 1)^{-1} \sum_{w \in W} (-u)^{n-\ell(w)} T_w = \gamma'_m.\]

Thus \(X = \{ \gamma_0, \gamma_1, \ldots, \gamma_{m-1}, \gamma'_1, \gamma'_2, \ldots, \gamma'_m \}\) is a basis for a left ideal of \(H_J\) supporting a \(W_J\)-digraph. Moreover, this \(W_J\)-digraph is isomorphic to the digraph of Figure 1.8 via \(\gamma_j \leftrightarrow \alpha_j\) for \(0 \leq j \leq m - 1, \gamma'_j \leftrightarrow \beta_j\) for \(1 \leq j \leq m\).

**Case 6. Figure 1.8** \(n = 2m - 2, m \geq 2\).

Define

\[\delta_0 = \tilde{\delta}_{m-2} = \frac{1}{2}(\tilde{\eta}_{m-2} + \tilde{\gamma}_{m-2}).\]

If \(m\) is even, then by part (iii) of Lemma 2.7 we have

\[
\begin{align*}
(T_{t'} - u)T_{t_{m-2}} (T_s - u) \delta_0 &= (T_s - u T_{s_{m-1}} - u T_{t_{m-1}} + u^2 T_{t_{m-2}}) \delta_{m-2} \\
&= (T_{t_{m-1}} - u T_{s_{m-1}} - u T_{t_{m-1}} + u^2 T_{s_{m-2}}) \delta_{m-2} \\
&= \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-2-2i}} - u \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-3-2i}} \\
&\quad - u \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-4-2i}} + u^2 \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-4-2i}} \\
&= \sum_{w \in W} (-u)^{n-\ell(w)} T_w.
\end{align*}
\]

On the other hand, if \(m\) is odd then

\[
\begin{align*}
(T_{t'} - u)T_{t_{m-2}} (T_s - u) \delta_0 &= (T_s - u T_{s_{m-1}} - u T_{t_{m-1}} + u^2 T_{t_{m-2}}) \tilde{\delta}_{m-2} \\
&= (T_{s_{m-1}} - u T_{t_{m-1}} - u T_{s_{m-1}} + u^2 T_{t_{m-2}}) \tilde{\delta}_{m-2} \\
&= \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-2-2i}} - u \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-3-2i}} \\
&\quad - u \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-4-2i}} + u^2 \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-4-2i}} \\
&= \sum_{w \in W} (-u)^{n-\ell(w)} T_w.
\end{align*}
\]

Therefore

\[
(T_{t'} - u)T_{t_{m-2}} (T_s - u) \delta_0 = \sum_{w \in W} (-u)^{n-\ell(w)} T_w = (T_{s'} - u)T_{s_{m-2}} (T_t - u) \delta_0,
\]

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with the second equation following from the first by applying the automorphism $T_s \leftrightarrow T_t$.

Thus if we define

$$
\begin{align*}
\delta_1 &= T_s^2 \delta_0, \\
\delta_2 &= T_t \delta_1, \\
&\vdots \\
\delta_{m-2} &= T_s T_{m-1}, \\
\delta_m &= T_s^2 \delta_{m-1}, \\
\delta'_1 &= T_t^2 \delta_0, \\
\delta'_2 &= T_t \delta'_1, \\
&\vdots \\
\delta'_{m-2} &= T_t \delta'_{m-1}, \\
\delta'_m &= T_t^2 \delta'_{m-1},
\end{align*}
$$

then

$$
\delta_m = (u + 1)^{-2} \sum_{k=0}^{n} (-u)^{n-k} \sigma_k = \delta'_m.
$$

Hence $X = \{\delta_0, \delta_1, \ldots, \delta_{m-1}, \delta'_1, \ldots, \delta'_m\}$ is a basis for a left ideal of $H_J$ that supports a $W_J$-digraph. This $W_J$-digraph is isomorphic to the $J$-labeled multigraph of Figure 1.8 via $\delta_j \leftrightarrow \alpha_j$ for $0 \leq j \leq m - 1$, $\delta'_j \leftrightarrow \beta_j$ for $1 \leq j \leq m$.

**Case 7.** Figure 1.9 or Figure 1.10 $m = 1, n \geq 2$ arbitrary.

Suppose $\Gamma$ is one of the $J$-labeled digraphs of Figures 1.9–1.10. Then with $M = \text{span} \{\alpha_0, \beta_1\}$, $T_s$ and $T_t$ induce the same operator $\tau_s = \tau_t$ on $M$. Thus the relation (2.2) holds automatically, and so $\Gamma$ is a $W_J$-digraph.

From the constructions above, it follows that (b) implies (a) in Theorem 1.3. To establish the converse, we can reduce to the case $S = J = \{s, t\}$, $1 < n = (s, t) < \infty$, $W = W_J$, $H = H_J$, and need only show that any connected $W$-digraph $\Gamma$ is isomorphic to one of the $J$-labeled digraphs of Figures 1.3–1.10 with $m$ and $n$ satisfying the appropriate divisibility conditions.

Let $X$ be the set of vertices of $\Gamma$, and let $M = \text{span} X$ be the associated $H$-module. If $\alpha \in X$, then $X \subseteq H\alpha$ because $\Gamma$ is connected, so $|X| = \dim M = \dim H\alpha \leq \dim H = 2m$.

Moreover, $|X|$ is even by Lemma 2.3. Since every vertex of $\Gamma$ is contained in exactly $|S| = 2$ edges, it follows that $\Gamma_{undir}$ is a simple cycle of size $2m$, where $1 \leq m \leq n$.

Let $\gamma_0$ be any vertex of $\Gamma$. Number the remaining vertices in such a way that $\Gamma$ has an edge from $\gamma_{i-1}$ to $\gamma_i$ or from $\gamma_i$ to $\gamma_{i-1}$ for $1 \leq i \leq 2m - 1$. Put $\gamma_2m = \gamma_0$, so $\Gamma$ also has an edge from $\gamma_{2m-1}$ to $\gamma_{2m}$ or from $\gamma_{2m}$ to $\gamma_{2m-1}$. We consider the subscript $j$ in $\gamma_j$ as an integer modulo $2m$.

Recall the linear characters $\lambda_1 = \text{ind}, \lambda_2 = \text{sgn} : H \rightarrow \mathbb{Q}(u)$ of $H$ are determined by

$$
\lambda_1(T_s) = \lambda_1(T_t) = u^2 \quad \text{and} \quad \lambda_2(T_s) = \lambda_2(T_t) = -1.
$$

If $n$ is even, there are two additional linear characters $\lambda_3, \lambda_4 : H \rightarrow \mathbb{Q}(u)$ given by

$$
\lambda_3(T_s) = u^2, \lambda_3(T_t) = -1 \quad \text{and} \quad \lambda_4(T_s) = -1, \lambda_4(T_t) = u^2.
$$

It is known that $H_C = \mathbb{C}(u) \otimes \mathbb{Q}(u) H$ is split semisimple over $\mathbb{C}(u)$, any irreducible representation of $H_C$ of dimension greater than 1 is two-dimensional, and the eigenvalues of $T_s$ and $T_t$ in any two-dimensional irreducible representation are $-1$ and $u^2$ (see [5], or [2], 8.3). Let $m_1, m_2, m_3, m_4$ be the number of summands in a direct sum decomposition of $M_C = \mathbb{C}(u) \otimes \mathbb{Q}(u) M$ into irreducible modules that afford $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, respectively (with $m_3 = m_4 = 0$ if $n$ is odd), and let $N$ be the number of two-dimensional irreducible summands. With $P_s$ as in Lemma 2.3, $T_s$ has eigenvalues $-1$ and $u^2$ on each subspace span $\{\alpha, \beta\}, \{\alpha, \beta\} \in P_s$, and thus $T_s$ has a total of $m$ eigenvalues $-1$ and $m$ eigenvalues $u^2$ on $M$. Since the same is true of $T_t$, we must have

$$
N = m_1 + m_2 + N = m_2 + m_3 + N = m_2 + m_4 + N = m,
$$

and so $m_1 = m_2$ and $m_3 = m_4$. By Lemma 2.3, the unique one-dimensional subspace $M_1$ of $M$ that affords the character $\lambda_1$ is spanned by $v_1 = \sum_{i=1}^{2m} \gamma_i$, and thus $m_1 = 1$. 

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Hence $m_2 = 1$, so there is a unique one-dimensional subspace $M_2$ of $M$ affording $\lambda_2$. Let $v_2 = \sum_{i=1}^{2m} \zeta_i \gamma_i$ be a nonzero element of $M_2$. By Lemma 2.4, we have

$$
\zeta_i = \begin{cases} 
\frac{-1}{u^2} \zeta_{i-1} & \text{if } \gamma_{i-1} \xrightarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xrightarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
-u^2 \zeta_{i-1} & \text{if } \gamma_{i-1} \xleftarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
-u + 1 & \text{if } \gamma_{i-1} \xrightarrow{t} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{t} \gamma_i \text{ is an edge of } \Gamma, \\
u^2 - u & \text{if } \gamma_{i-1} \xrightarrow{s} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{s} \gamma_i \text{ is an edge of } \Gamma, \\
-u^2 - u & \text{if } \gamma_{i-1} \xrightarrow{t} \gamma_i \text{ or } \gamma_{i-1} \xleftarrow{t} \gamma_i \text{ is an edge of } \Gamma.
\end{cases}
$$

for $1 \leq i \leq 2m$. If $m = 1$, then it follows that the edge joining $\gamma_0$ and $\gamma_1$ labeled $s$ must have the same type and direction as the edge joining $\gamma_0$ and $\gamma_1$ labeled $t$, and so $\Gamma$ is isomorphic to one of the $J$-labeled digraphs of Figure 19.10. We assume $m \geq 2$ for the remainder of the proof, so there is a unique edge joining $\gamma_i$ to $\gamma_{i-1}$ for $1 \leq i \leq 2m$. Further,

$$
\zeta_0 = \zeta_{2m} = \zeta_0 \prod_{i=1}^{2m} \zeta_i.
$$

and so $\prod_{i=1}^{2m} (\zeta_i / \zeta_{i-1}) = 1$. It follows that the number of edges of type $\gamma_{i-1} \rightarrow \gamma_i$ (labeled either $s$ or $t$) is equal to the number of edges of type $\gamma_{i-1} \leftarrow \gamma_i$, $1 \leq i \leq 2m$, and the number of edges of type $\gamma_{i-1} \rightarrow \gamma_i$ is equal to the number of edges of type $\gamma_{i-1} \leftarrow \gamma_i$, $1 \leq i \leq 2m$.

Next, we compute the coefficient $\kappa_j$ of $\gamma_j$ in the expression for $T_s T_t \gamma_j$ as a linear combination of $\{\gamma_1, \ldots, \gamma_{2m}\}$. These coefficients are given in Table 3.1, which is organized according to the types of edges joining $\gamma_j$ to the adjacent vertices $\delta, \varepsilon$ in $\Gamma$. (Either $\delta = \gamma_{j-1}$ and $\varepsilon = \gamma_{j+1}$ or $\delta = \gamma_{j+1}$ and $\varepsilon = \gamma_{j-1}$; the coefficient $\kappa_j$ is the same in either case.) The entries of this table are easily verified. For example, if $\gamma_{j-1} \xrightarrow{s} \gamma_j \xrightarrow{t} \gamma_{j+1}$ are edges in $\Gamma$, then

$$
T_s T_t \gamma_j = T_s (u \gamma_j + (u+1) \gamma_{j+1}) = u (u \gamma_j + (u+1) \gamma_{j-1}) + (u+1) T_s \gamma_{j+1},
$$

Table 3.1

| Edges in $\Gamma$ | Coefficient $\kappa_j$ | Edges in $\Gamma$ | Coefficient $\kappa_j$ |
|-------------------|------------------------|-------------------|------------------------|
| $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | 0 | $\delta \xleftarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | 0 |
| $\delta \xrightarrow{s} \gamma_j \xrightarrow{-} \gamma_j \xrightarrow{t} \varepsilon$ | $u(u^2 - 1)$ | $\delta \xleftarrow{s} \gamma_j \xrightarrow{-} \gamma_j \xrightarrow{t} \varepsilon$ | 0 |
| $\delta \xrightarrow{s} \gamma_j \xrightarrow{-} \gamma_j \xrightarrow{t} \varepsilon$ | 0 | $\delta \xleftarrow{s} \gamma_j \xrightarrow{-} \gamma_j \xrightarrow{t} \varepsilon$ | $u^2$ |
| $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | 0 | $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $(u^2 - 1)^2$ |
| $\delta \xleftarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | 0 | $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $(u^2 - 1)(u^2 - u - 1)$ |
| $\delta \xleftarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $u(u^2 - 1)$ | $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $(u^2 - 1)(u^2 - u - 1)$ |
| $\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $u(u^2 - 1)$ | $\delta \xleftarrow{s} \gamma_j \xrightarrow{t} \varepsilon$ | $(u^2 - 1)^2$ |
and so \( \kappa_j = u^2 \) since \( T_s \gamma_{j+1} \in \text{span}\{\gamma_{j+1}, \gamma_{j+2}\} \). On the other hand, if \( \Gamma \) has edges
\[ \gamma_{j+1} \xrightarrow{s} \gamma_j \xrightarrow{t} \gamma_{j-1} , \]
then
\[ T_s T_t \gamma_j = T_s(u \gamma_{j+1} + (u+1) \gamma_{j-1}) = u(u^2 \gamma_{j+1} + (u^2 - 1) \gamma_j) + (u + 1) T_s \gamma_{j-1}, \]
and so \( \kappa_j = u(u^2 - 1) \) because \( T_s \gamma_{j-1} \in \text{span}\{\gamma_{j-1}, \gamma_{j-2}\} \).

From Table 3.1 we see that the constant term in the trace \( \text{tr}(T_s T_t) = \sum_{j=1}^{2m} \kappa_j \) is equal to the number of sinks in \( \Gamma \). However, \( T_s T_t \) has values \( u^4 \) and 1 under \( \lambda_1 \) and \( \lambda_2 \), respectively, and value \(-u^2\) under both \( \lambda_3 \) and \( \lambda_4 \) if \( n \) is even. Also, \( T_s T_t \) has eigenvalues of the form \( e^{i\theta} u^2 \), \( e^{-i\theta} u^2 \) in any two-dimensional irreducible representation of \( H_\mathbb{C} \), where \( e^{i\theta} \) is a complex \( n \)-th root of unity ([5], Theorem 2, or [2], Theorem 8.3.1). Therefore the constant term of \( \text{tr}(T_s T_t) \) is \( m_2 = 1 \). Hence \( \Gamma \) has a unique sink \( \beta \), and so also a unique source \( \alpha \).

Renumbering the vertices if necessary, we can assume that \( \gamma_0 = \alpha \). Since \( \Gamma \) has a unique sink \( \beta \) and the number of edges of type \( \gamma_{i-1} \xrightarrow{s} \gamma_i \) is equal to the number of edges of type \( \gamma_{i-1} \leftarrow \gamma_i \), \( 1 \leq i \leq 2m \), and the number of edges of type \( \gamma_{i-1} \longrightarrow \gamma_i \) is equal to the number of edges of type \( \gamma_{i-1} \longleftarrow \gamma_i \), \( 1 \leq i \leq 2m \), it follows that \( \beta = \gamma_m \) is opposite to \( \alpha \).

Renumbering the vertices if needed, we can assume that \( \gamma_0 \) and \( \gamma_1 \) are connected by an edge labeled \( s \). Define \( \gamma'_j = \gamma_{2m-j} \) for \( 0 \leq j \leq m \), so \( \beta = \gamma'_m \). Then \( \Gamma \rightarrow \) has the form shown in Figure 3.1 (Since each edge of \( \Gamma \rightarrow \) arises from either a solid or a dashed edge in \( \Gamma \), there are \( 2^{2m} \) possible \( J \)-labeled digraphs \( \Gamma \) with this configuration.)

\[ \begin{array}{c}
\gamma_0 \\
\gamma_1 \\
\gamma'_1 \\
\vdots \\
\gamma_m \\
\gamma'_m
\end{array} \]

Figure 3.1 \( \Gamma \rightarrow \)

From the discussion above, we know that the number of edges in \( \Gamma \) of type \( \gamma_{i-1} \longrightarrow \gamma_i \) (labeled either \( s \) or \( t \), \( 1 \leq i \leq m \), is equal to the number of edges of type \( \gamma'_{i-1} \longrightarrow \gamma'_i \), \( 1 \leq i \leq m \). Also, from the description of the eigenvalues of \( T_s T_t \) above, \( \text{tr}(T_s T_t) \) must be an even function of \( u \). Let \( N_1 \) be the number of edges of the form \( \xi \longrightarrow \omega \) with \( \xi \) not a source, that is, \( \xi \neq \alpha = \gamma_0 \), and let \( N_2 \) be the number of edges \( \xi \longrightarrow \omega \) with \( \omega \) a sink, that is, \( \omega = \beta = \gamma'_m \). Then from Table 3.1, the coefficient of \( u^3 \) in \( \text{tr}(T_s T_t) \) is \( N_1 - N_2 \), and hence \( N_1 = N_2 \). Therefore any edge of type \( \xi \longrightarrow \omega \) that does not begin at \( \gamma_0 \) must end at \( \gamma'_m \). Hence \( \Gamma \) is isomorphic to one of the \( J \)-labeled digraphs in Figures 1.3, 1.8 via \( \gamma_j \leftrightarrow \alpha_j, 0 \leq j \leq m - 1 \), \( \gamma'_j \leftrightarrow \beta_j, 1 \leq j \leq m \).

Finally, let \( \tau_s \) and \( \tau_t \) be as in (1.3), and let \( \tilde{A}_s(u), \tilde{A}_t(u) \) be the \((2m) \times (2m)\) matrices over \( \mathbb{Q}[u] \) representing \( \tau_s \) and \( \tau_t \) with respect to the basis \( X \) for \( M \). Put \( A_s = \tilde{A}_s(1), A_t = \tilde{A}_t(1) \). Then
\[ A_s^2 = I = A_t^2, \]
\[ \cdots \cdot \cdot \cdot A_t A_s = \cdots \cdot \cdot \cdot A_s A_t, \]

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by (2.1), (2.2), and so \( s \mapsto A_s, \ t \mapsto A_t \) extends to a representation of groups \( W_J \to \text{GL}(2m, \mathbb{Q}) \). One checks that the characteristic polynomial of the matrix \( A_{st} = A_s A_t \) representing \( st \) is as given in Table 3.2. Hence the order of \( A_{st} \) as an element of \( \text{GL}(2m, \mathbb{Q}) \)

| Figure | characteristic polynomial of \( A_{st} \) |
|--------|------------------------------------------|
| Figure 1.3 | \((x^m - 1)^2\) |
| Figure 1.4 | \((-1)\cdot(x^{2m-1} - 1)\) |
| Figure 1.5 | \((-1)^2\cdot(x^{m-1} + 1)^2\) |
| Figure 1.6 | \((-1)^2\cdot(x^{m-1} + 1)^2\) |
| Figure 1.7 | \((-1)^2\cdot(x^{m-1} + 1)^2\) |
| Figure 1.8 | \((-1)^2\cdot(x^{m-1} + 1)^2\) |

is \( m \) in the case of Figures 1.3–1.5, \( 2m - 1 \) in the case of Figures 1.6–1.7, and \( 2m - 2 \) in the case of Figure 1.8. Since this order must divide \( n \), the proof is complete.

**Proof of Corollary 1.4.** Let \((W, S)\) be a Coxeter system, and let \( \Gamma \) be an \( S \)-labeled digraph. For \( J = \{s, t\} \) and \( 1 < n < \infty \), denote by \( \mathcal{F}_{J,n} \) the collection of all \( J \)-labeled digraphs \( C \) of Figures 1.3–1.10 for which \( m = |\mathcal{V}(C)| \) satisfies the divisibility conditions of Theorem 1.3. Then \( \Gamma \) is a \( W \)-digraph if and only if whenever \( J = \{s, t\} \subseteq S \) with \( 1 < n = n(s, t) < \infty \), each connected component of \( \Gamma_J \) is isomorphic to an element of \( \mathcal{F}_{J,n} \). It is easily seen that \( \mathcal{F}_{J,n} \) is invariant under \( C \mapsto C_{\text{rev}} \). Also, \( C \) is a connected component of \( \Gamma_J \) if and only if \( C_{\text{rev}} \) is a connected component of \( (\Gamma_J)_{\text{rev}} \). The assertion of the corollary follows.

Let \( w \mapsto w^* \) be an involutory automorphism of \((W, S)\), and let \( I_s = \{x \in W \mid x^* = x^{-1}\} \) be the set of twisted involutions of \( W \). By Lusztig [6], Theorem 0.1, there is an \( H \)-module \( M_s \) with basis \( X = \{m_w \mid w \in I_s\} \) and \( H \)-action determined by

\[
T_s m_w = \begin{cases} 
  m_{sws^*} & \text{if } sw \neq ws^* > w, \\
  (u^2 - 1)m_w + u^2 m_{sws^*} & \text{if } sw \neq ws^* < w, \\
  um_w + (u + 1)m_{sw} & \text{if } sw = ws^* > w, \\
  (u^2 - u - 1)m_w + (u^2 - u)m_{sw} & \text{if } sw = w^* < w.
\end{cases}
\]

(Here \( < \) and \( > \) refer to the usual Bruhat order on \((W, S)\).) The basis \( X \) for \( M_s \) then affords the \( W \)-digraph \( \Gamma_s \) defined by

\[
m_w \overset{s}{\longrightarrow} m_{sws^*} \in \mathcal{E}(\Gamma_s) \iff sw \neq ws^* > w
\]

and

\[
m_w \overset{s}{\longrightarrow} m_{sw} \in \mathcal{E}(\Gamma_s) \iff sw = ws^* > w.
\]

**Theorem 3.1.** Let \((W, S)\) be finite with longest element \( w_0 \), and let \( \Gamma_s \) be the \( W \)-digraph corresponding to the involutory automorphism \( w \mapsto w^* \) of \((W, S)\). Then \((\Gamma_s)_{\text{rev}} \) is isomorphic to the \( W \)-digraph \( \Gamma_{s^*} \) corresponding to the automorphism \( w \mapsto w^* = w_0 x^* w_0 w_0 = w_0 x^{-1} = (x w_0)^{-1} \).

**Proof.** Observe \( w_0^* = w_0 \) since \( w \mapsto w^* \) preserves lengths. Suppose \( x \in I_s \), so \( x^* = x^{-1} \). Then \( x w_0 \in I_{s^*} \) because

\[
(x w_0)^# = w_0 (x w_0)^* w_0 = w_0 x^* w_0 w_0 = w_0 x^{-1} = (x w_0)^{-1}.
\]
Likewise, if \( xw_0 \in I_\# \), then \( x \in I_\ast \). If \( x, y \in I_\ast \) and \( s \in S \), then

\[
m_x \xrightarrow{s} m_y \in \mathcal{E}(\Gamma_\ast) \iff x < y = xs^* \iff yw_0 < xw_0 = (syw_0)(w_0s^*w_0) = s(yw_0)s^# \iff m_yw_0 \xrightarrow{s} m_xw_0 \in \mathcal{E}(\Gamma_\#),
\]

and

\[
m_x \xleftarrow{s} m_y \in \mathcal{E}(\Gamma_\ast) \iff x < y = sx = xs^* \iff yw_0 < xw_0 = (syw_0)w_0 = (ys^*)w_0 = (yw_0)s^# \iff m_yw_0 \xleftarrow{s} m_xw_0 \in \mathcal{E}(\Gamma_\#).
\]

Therefore \((\Gamma_\ast)_{\text{rev}}\) is isomorphic to \(\Gamma_\#\) via the bijection \(m_x \mapsto m_xw_0\) on vertices. \(\square\)

**Corollary 3.2.** If \(w_0\) is central in \(W\), then \((\Gamma_\ast)_{\text{rev}}\) is isomorphic to \(\Gamma_\ast\).

**Example 3.3.** Suppose \(W = \langle r, s, t \rangle\) with \(n(r, s) = n(s, t) = 3\), \(n(r, t) = 2\) and \(w^* = w\) for \(w \in W\), so \(I_\ast\) is the set of involutions in \(W\) (including \(e\)). The corresponding \(W\)-digraph \(\Gamma_\ast\) is shown in Figure 3.2 (The vertices are labeled \(x\) rather than \(m_x\) for \(x \in I_\ast\).) If \(w \mapsto w^\#\) is the nonidentity graph automorphism of \(W\), the corresponding \(W\)-digraph \(\Gamma_\#\) is as shown in Figure 3.3. Note \((\Gamma_\ast)_{\text{rev}} \cong \Gamma_\#\).

**Example 3.4.** Suppose \(W = \langle r, s, t \rangle\) with \(n(r, s) = 3\), \(n(s, t) = 4\), \(n(r, t) = 2\). With \(w \mapsto w^* = w\), \(I_\ast\) is the set of involutions of \(W\). The corresponding \(W\)-digraph \(\Gamma_\ast\) takes the form shown in Figure 3.4. Note \((\Gamma_\ast)_{\text{rev}} \cong \Gamma_\#\).

**4 The proofs of Theorems [1.5, 1.7, 1.8, and 1.10]**

Throughout this section \((W, S)\) is a Coxeter system and \(\Gamma\) is a \(W\)-digraph. If \(\varepsilon = \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma)\) or \(\varepsilon = \alpha \xleftarrow{s} \beta \in \mathcal{E}(\Gamma)\), then we call \(\alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma_\ast)\) the
image of $\varepsilon$ in $\Gamma_{\rightarrow}$. Clearly there is a directed path from $\alpha$ to $\beta$ in $\Gamma$ if and only if there is a directed path from $\alpha$ to $\beta$ in $\Gamma_{\rightarrow}$.

Let $H_0$ be the 0-Hecke algebra of $(W,S)$ (see [8], or [1], Chapter IV, §2, Exercise 23, with $\lambda_s = -1$, $\mu_s = 0$ for $s \in S$). Thus $H_0$ is an associative algebra over $\mathbb{Q}$ with generating set $\{a_s \mid s \in S\}$ satisfying the presentation

$$a_s^2 = -a_s$$

for $s \in S$ and

$$a_s a_t a_s \cdots = a_t a_s a_t \cdots$$

if $s, t \in S$, $n(s,t) < \infty$. Also, $H_0$ has basis $\{a_w \mid w \in W\}$ with $a_e$ the identity element of $H_0$ and

$$a_s a_w = \begin{cases} a_{sw} & \text{if } sw > w, \\ -a_w & \text{if } sw < w. \end{cases}$$

(4.1)

It follows that for $x, y \in W$, there is $z \in W$ such that

$$a_x a_y = \pm a_z \quad \text{and} \quad \max\{\ell(x), \ell(y)\} \leq \ell(z),$$

and

$$a_x a_y = a_{xy} \iff \ell(x) + \ell(y) = \ell(xy).$$

If $(W,S)$ is finite and $w_0$ is the longest element of $W$, then

$$a_w a_{w_0} = (-1)^{\ell(w)} a_{w_0} a_w = a_{w_0} a_w$$

for $w \in W$.

Let $M = M(\Gamma)$ be the module afforded by $\Gamma$, so $M$ has basis $X = \mathcal{V}(\Gamma)$ over $\mathbb{Q}(u)$. Let $M_0$ be the $\mathbb{Q}$-subspace of $M$ with basis $X$. For $s \in S$, define a $\mathbb{Q}$-linear operator $(\tau_s)_0$ on $M_0$ by

$$(\tau_s)_0(\alpha) = \begin{cases} \beta & \text{if } \alpha \overset{s}{\rightarrow} \beta \in \mathcal{E}(\Gamma_{\rightarrow}), \\ -\alpha & \text{if } \alpha \text{ is a sink in } \Gamma_{\rightarrow}. \end{cases}$$
Notice that by (1.3), $(\tau_s)_0(\alpha)$ can be obtained by replacing the coefficients of the image $\tau_s(\alpha)$ expressed as a linear combination of the elements of $X$ with their values at $u = 0$. Since in $\text{gl}(M)$ we have

$$(\tau_s - u^2)(\tau_s + 1) = 0$$

and

$$\frac{n(s,t)}{\tau_s \tau_t \cdots} \cdot \frac{n(s,t)}{\tau_s \tau_t \cdots},$$

it follows that in $\text{gl}(M_0)$ we have

$$(\tau_s)_0)^2 = -(\tau_s)_0$$

and

$$\frac{n(s,t)}{\tau_s \tau_t \cdots} = \frac{n(s,t)}{\tau_s \tau_t \cdots}.$$ 

if $n(s,t) < \infty$. Hence $a_s \mapsto (\tau_s)_0$ defines a representation $\rho_0 : H_0 \to \text{gl}(M_0)$, giving $M_0$ the structure of an $H_0$-module. In particular, for $\alpha \in \mathcal{V}(\Gamma)$,

(4.2) \[ a_s \alpha = \begin{cases} \beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma), \\ -\alpha & \text{if } \alpha \text{ is a sink in } \Gamma_s. \end{cases} \]

**Lemma 4.1.** Assume $(W,S)$ is a Coxeter system, $\Gamma$ is a $W$-digraph, $X = \mathcal{V}(\Gamma)$, and $H_0$ acts on $M_0$ as described above. Then the following hold.

(i) If $\alpha \in X$ and $w \in W$, then $a_w \alpha \in X$ or $-a_w \alpha \in X$.

(ii) If $\beta \in X$, then there exists some $w \in W$ such that $\beta = \pm a_w \alpha$ if and only if there is a directed path from $\alpha$ to $\beta$ in $\Gamma$.

**Proof.** Since $a_w = a_{s_1}a_{s_2} \cdots a_{s_l}$ if $s_1s_2 \cdots s_l$ is a reduced expression for $w \in W$ by (1.1), an easy induction argument based on (1.2) establishes (i).

For (ii), we can argue with $\Gamma_\to$ in place of $\Gamma$. Suppose $\beta \in X$ and there is some directed path

(4.3) \[ \gamma_0 \xrightarrow{s_1} \gamma_1 \xrightarrow{s_2} \gamma_2 \xrightarrow{s_3} \cdots \xrightarrow{s_{k-1}} \gamma_{k-1} \xrightarrow{s_k} \gamma_k \]

in $\Gamma_\to$ with $\gamma_0 = \alpha$, $\gamma_k = \beta$. Define $y \in W$ by $\pm a_y = a_{s_k}a_{s_{k-1}} \cdots a_{s_2}a_{s_1}$. Then

$$\pm a_y \alpha = a_{s_k}a_{s_{k-1}} \cdots a_{s_2}a_{s_1} \gamma_0 = \gamma_k = \beta.$$ 

Conversely, assume $\beta = \pm a_w \alpha \in X$, where $w \in W$. Let $w = t_kt_{k-1} \cdots t_2t_1$ be a reduced expression for $w$ as a product of generators $t_k, \ldots, t_1 \in S$, so $a_w = a_{t_k} \cdots a_{t_2}a_{t_1}$ by (1.1). Put $\delta_0 = \alpha$ and $\delta_j = a_{t_j} \delta_{j-1}$ for $1 \leq j \leq k$, so $\beta = \pm \delta_k$. By (i), there are $\varepsilon_j \in \{-1,1\}$ such that $\alpha_j = \varepsilon_j \delta_j \in X$ for $0 \leq j \leq k$. Then for $1 \leq j \leq k$, $\alpha_{j-1} \neq \alpha_j$ if and only if $\alpha_j \xrightarrow{t_j} \alpha_{j-1} \in \mathcal{E}(\Gamma_\to)$. If $0 < j_1 < j_2 < \cdots < j_{\ell}$ are the values of $j$, $1 \leq j \leq k$, for which $\alpha_{j-1} \neq \alpha_j$, then

$$\alpha_0 \xrightarrow{t_{j_1}} \alpha_{j_1} \xrightarrow{t_{j_2}} \alpha_{j_2} \xrightarrow{t_{j_3}} \cdots \xrightarrow{t_{j_{\ell-1}}} \alpha_{j_{\ell-1}} \xrightarrow{t_{j_{\ell}}} \alpha_{j_{\ell}}$$

is a directed path in $\Gamma_\to$ from $\alpha$ to $\beta$. Thus (ii) holds. \qed

**Lemma 4.2.** Assume $(W,S)$ is a Coxeter system, $\Gamma$ is a $W$-digraph, $X = \mathcal{V}(\Gamma)$, and $H_0$ acts on $M_0$ as in (1.2). For $\omega \in X$, the following are equivalent.
(i) $\omega$ is a sink in $\Gamma$.

(ii) $a_s \omega = -\omega$ for all $s \in S$.

(iii) $a_w \omega = (-1)^{\ell(w)}\omega$ for all $w \in W$.

Moreover, if $(W, S)$ is finite, then (i)–(iii) are equivalent to

(iv) $\omega = \pm a_{w_0} \alpha$ for some $\alpha \in X$.

**Proof.** If $\omega$ is a sink in $\Gamma$, then $a_s \omega = -\omega$ for all $s \in S$ by (4.2). Thus (i) implies (ii).

Assume $a_s \omega = -\omega$ for all $s \in S$ and $w \in W$ has reduced expression $w = s_1 s_2 \cdots s_k$. Then by (4.1),

$$a_w \omega = a_{s_1} a_{s_2} \cdots a_{s_k} \omega = (-1)^k \omega = (-1)^{\ell(w)} \omega.$$ 

Hence (ii) implies (iii).

Suppose $a_w \omega = (-1)^{\ell(w)} \omega$ for all $w \in W$. Then $a_s \omega = -\omega$ for all $s \in S$, and thus $\omega$ must be a sink in $\Gamma$ by (4.2). Hence (iii) implies (i).

Suppose $(W, S)$ is finite and $\omega = \varepsilon a_{w_0} \alpha$, where $\alpha \in X$ and $\varepsilon \in \{-1, 1\}$. Then

$$a_s \omega = a_s (\varepsilon a_{w_0} \alpha) = \varepsilon (a_s a_{w_0}) \alpha = -\varepsilon a_{w_0} \alpha = -\omega$$

for any $s \in S$, and so $\omega$ is a sink in $\Gamma$. Conversely, if $\omega$ is a sink in $\Gamma$, then $a_{w_0} \omega = (-1)^N \omega = \pm \omega$ by (4.2), where $N = \ell(w_0)$, and thus $\omega = \pm a_{w_0} \omega$. Hence (i) and (iv) are equivalent.

Define relations $\equiv_{s, t}$ and $\equiv$ on the set of directed paths in $\Gamma$ as follows. If $\pi_1$ and $\pi_2$ are directed paths in $\Gamma$ and $s, t \in S$ satisfy $1 < n(s, t) < \infty$, then $\pi_1 \equiv_{s, t} \pi_2$ if there is some connected component $C$ of $\Gamma_{\{s, t\}}$ such that $\pi_1$ and $\pi_2$ both pass through the source $\sigma$ and the sink $\omega$ of $C$, and $\pi_2$ can be obtained from $\pi_1$ by replacing one of the directed paths from $\sigma$ to $\omega$ in $\Gamma_{\{s, t\}}$ by the other. Let $\equiv$ be the equivalence relation on directed paths in $\Gamma$ generated by the relations $\equiv_{s, t}$ for $s, t \in S$, $1 < n(s, t) < \infty$. Similar relations, also denoted $\equiv_{s, t}$ and $\equiv$, can be defined for directed paths in $\Gamma_\to$. It is clear that two directed paths in $\Gamma$ are in the same $\equiv$ equivalence class if and only if their images in $\Gamma_\to$ are in the same $\equiv$ equivalence class.

For $\alpha \in \mathcal{V}(\Gamma)$, denote by $[\alpha, \infty)$ the set of all $\beta \in \mathcal{V}(\Gamma)$ such that there exists a directed path in $\Gamma$ from $\alpha$ to $\beta$. Clearly if $\beta \in [\alpha, \infty)$ and $\gamma \in [\beta, \infty)$, then $\gamma \in [\alpha, \infty)$. For $\beta \in [\alpha, \infty)$, let $\mu(\alpha, \beta)$ be the minimum number of edges in a directed path from $\alpha$ to $\beta$ (with $\mu(\alpha, \alpha) = 0$).

**Lemma 4.3.** Suppose $(W, S)$ is a Coxeter system such that $n(s, t) < \infty$ for all $s, t \in S$, and $\Gamma$ is a $W$-digraph with source $\sigma$.

(i) If $\alpha \in [\sigma, \infty)$, then any two directed paths from $\sigma$ to $\alpha$ are in the same $\equiv$-equivalence class.

(ii) If $\alpha \in [\sigma, \infty)$, $\zeta \in \mathcal{V}(\Gamma)$, and $\alpha \in [\zeta, \infty)$, then $\zeta \in [\sigma, \infty)$.

(iii) If $\Gamma$ is connected, then $\mathcal{V}(\Gamma) = [\sigma, \infty)$.

**Proof.** We can argue with $\Gamma_\to$ in place of $\Gamma$. We prove (i) and (ii) simultaneously by induction on $\mu(\sigma, \alpha)$. If $\mu(\sigma, \alpha) = 0$, then $\alpha = \sigma$, so (i) holds because the only directed path from $\sigma$ to $\sigma$ is the empty path because $\sigma$ is a source. Also, if $\alpha = \sigma \in [\gamma, \infty)$, then there is a directed path from $\gamma$ to $\sigma$, so the path must be empty and $\gamma = \sigma \in [\sigma, \infty)$, and thus (ii) holds.

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Suppose \( \mu(\sigma, \alpha) = k > 0 \) and (i) and (ii) hold with \( \beta \) in place of \( \alpha \) whenever \( \beta \in [\sigma, \infty) \) and \( \mu(\sigma, \beta) < k \). Let \( \pi_1 \) be some directed path from \( \sigma \) to \( \alpha \) with \( k \) edges, and let \( \pi_2 \) be an arbitrary directed path from \( \sigma \) to \( \alpha \). For \( j = 1, 2 \), let \( \varepsilon_j \in \mathcal{E}(\Gamma_\rightarrow) \) be the last edge of \( \pi_j \) and let \( \rho_j \) be the remainder of the path \( \pi_j \), so \( \pi_j = \rho_j \varepsilon_j \), where juxtaposition indicates concatenation of paths. Thus \( \varepsilon_1 \) takes the form \( \beta \xrightarrow{s} \alpha \) for some \( s \in S \) and \( \beta \in \mathcal{V}(\Gamma) \) with \( \beta \in [\sigma, \infty) \) and \( \mu(\sigma, \beta) = k - 1 \). Also, \( \varepsilon_2 \) has the form \( \gamma \xrightarrow{t} \alpha \) for some \( t \in S \), \( \gamma \in \mathcal{V}(\Gamma) \). If \( t = s \), then \( \gamma = \beta \), so \( \rho_1 \equiv \rho_2 \) by (i) applied to \( \beta \), and thus \( \pi_1 \equiv \pi_2 \) as desired. Suppose \( t \neq s \). Let \( \tau \) be the source of the connected component \( C \) of \( (\Gamma_\rightarrow)\{s,t\} \) whose sink is \( \alpha \). (Note that \( C \) has a unique source by the classification of possible connected components of \( \Gamma_\{s,t\} \) given in Theorem 1.3.) Let \( \nu_1 \) (\( \nu_2 \)) be the directed path in \( C \) from \( \tau \) to \( \beta \) (\( \gamma \), respectively), so \( \nu_1 \varepsilon_1 \equiv_{s,t} \nu_2 \varepsilon_2 \). Since \( \beta \in [\tau, \infty) \), we have \( \tau \in [\sigma, \infty) \) by (ii) applied to \( \beta \), and so there is some directed path \( \rho \) from \( \sigma \) to \( \tau \) in \( \Gamma_\rightarrow \). (See Figure 4.1 in which edges represent directed paths in \( \Gamma_\rightarrow \)) We have \( \rho_1 \equiv \rho \nu_1 \) by (i) applied to \( \beta \), and

\[
\begin{array}{c}
\sigma \\
\rho_1 \\
\rho \\
\rho_2 \\
\tau \\
\nu_1 \\
\varepsilon_1 \\
\nu_2 \\
\varepsilon_2 \\
\gamma \\
\alpha
\end{array}
\]

Thus \( \rho \nu_1 \) has \( k - 1 \) edges. Since \( \nu_1 \) and \( \nu_2 \) have the same number of edges, it follows that \( \rho \nu_2 \) also has \( k - 1 \) edges, and thus \( \mu(\sigma, \gamma) \leq k - 1 \). Hence by (i) applied to \( \gamma \), we also have \( \rho \nu_2 \equiv \rho_2 \). Thus

\[
\pi_1 = \rho_1 \varepsilon_1 \equiv (\rho \nu_1) \varepsilon_1 = \rho (\nu_1 \varepsilon_1) \\
\equiv \rho (\nu_2 \varepsilon_2) = (\rho \nu_2) \varepsilon_2 \equiv \rho_2 \varepsilon_2 = \pi_2.
\]

Therefore (i) holds for \( \alpha \).

Now suppose \( \alpha \in [\delta, \infty) \). Let \( \psi \) be a directed path from \( \delta \) to \( \alpha \). Write \( \psi = \psi_0 \varepsilon_0 \), where \( \varepsilon_0 \in \mathcal{E}(\Gamma_\rightarrow) \) is the last edge of \( \psi \), so \( \varepsilon_0 \) has the form \( \phi \xrightarrow{r} \alpha \) for some \( r \in S \), \( \phi \in \mathcal{V}(\Gamma) \). If \( r = s \), then \( \phi = \beta \in [\delta, \infty) \), and hence \( \beta \in [\sigma, \infty) \) by (ii) applied to \( \beta \). Assume \( r \neq s \). Let \( \kappa \) be the source of the connected component of \( (\Gamma_\rightarrow)\{r,s\} \) whose sink is \( \alpha \). (See Figure 4.2 in which edges represent directed paths in \( \Gamma_\rightarrow \)) There exists some directed path from \( \sigma \) to \( \kappa \) by (ii) applied to \( \beta \). The argument given above for \( \gamma \) applies to show that

\[
\mu(\sigma, \phi) \leq k - 1.
\]

Since \( \phi \in [\delta, \infty) \), it follows that \( \delta \in [\sigma, \infty) \) by (ii) applied to \( \phi \). Hence (ii) holds for \( \alpha \), so the proof of (i) and (ii) is complete.
Finally, suppose \( \Gamma \) is connected. For \( \alpha \in \mathcal{V}(\Gamma) \), let \( \delta(\alpha) \) be the minimal number of edges in a path in \( \Gamma_{\text{undir}} \) from \( \sigma \) to \( \alpha \). We prove \( \alpha \in [\sigma, \infty) \) for all \( \alpha \in \mathcal{V}(\Gamma) \) by induction on \( \delta(\alpha) \). If \( \delta(\alpha) = 0 \), then \( \alpha = \sigma \in [\sigma, \infty) \). Suppose \( \delta(\alpha) = \ell > 0 \) and \( \gamma \in [\sigma, \infty) \) whenever \( \gamma \in \mathcal{V}(\Gamma) \) and \( \delta(\gamma) < \ell \). Let \( \beta \) --- \( \alpha \) be the last edge of a path in \( \Gamma_{\text{undir}} \) from \( \sigma \) to \( \alpha \) of length \( \ell \), so \( \delta(\beta) = \ell - 1 \) and \( \beta \in [\sigma, \infty) \). If \( \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma_\rightarrow) \) for some \( s \in S \), then \( \alpha \in [\sigma, \infty) \) because \( \beta \in [\sigma, \infty) \) and \( \alpha \in [\beta, \infty) \). On the other hand, if \( \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma_\rightarrow) \), then \( \beta \in [\alpha, \infty) \), and so \( \alpha \in [\sigma, \infty) \) by (ii) applied to \( \beta \). Hence \( \alpha \in [\sigma, \infty) \) for all \( \alpha \in \mathcal{V}(\Gamma) \), and therefore \( \mathcal{V}(\Gamma) = [\sigma, \infty) \). This completes the proof.

**Example 4.4.** Let \( W = W(A_3) = \langle r, s, t \rangle \), with \( n(r, s) = 3 = n(s, t), n(r, t) = 2 \), and let \( \Gamma \) be as in Figure 4.3. The directed paths \( \alpha_1 \xrightarrow{r} \alpha_2 \xrightarrow{s} \alpha_3 \xrightarrow{r} \alpha_4 \) and \( \alpha_2 \xrightarrow{t} \beta_2 \xrightarrow{s} \beta_3 \) from \( \alpha_2 \) to \( \beta_3 \) are not in the same \( \equiv \)-equivalence class (even though adjoining the edge \( \alpha_1 \xrightarrow{r} \alpha_2 \) to both does produce two equivalent paths). Therefore the conclusion of Lemma 4.3(i) does not apply to arbitrary directed paths in a W-digraph.

![Figure 4.3 Digraph for Example 4.4](image)

We now prove Theorems 1.5, 1.7, 1.8 and 1.10

**Proof of Theorem 1.3** Assume \( n(s, t) < \infty \) for all \( s, t \in S \) and \( \Gamma \) is a connected W-digraph. Since \( \Gamma_{\text{rev}} \) is also a connected W-digraph by Corollary 1.3, it is enough to prove the assertions involving sources. Suppose \( \sigma \) is a source of \( \Gamma \). Then \( \mathcal{V}(\Gamma) = [\sigma, \infty) \) by Lemma 4.3(iii). Hence if \( \gamma \neq \sigma \) is a vertex of \( \Gamma \), there must be some nonempty directed path in \( \Gamma \) from \( \sigma \) to \( \gamma \), and so \( \gamma \) cannot be a source. Thus \( \sigma \) is the unique source of \( \Gamma \), so part (i) of the theorem holds.

Suppose \( \Gamma \) has source \( \sigma \) but is not acyclic. Let \( \alpha \in \mathcal{V}(\Gamma) \) be contained in a nonempty directed circuit \( \rho \) in \( \Gamma \). Since \( \alpha \in [\sigma, \infty) \), there is some directed path \( \pi \) from \( \sigma \) to \( \alpha \). Then the directed paths \( \pi \) and \( \pi \rho \) from \( \sigma \) to \( \alpha \) are in different \( \equiv \) equivalence classes because their lengths are different, contradicting Lemma 4.3(i). Thus part (ii) of the theorem holds.

Finally, assume \( (W, S) \) is finite. By Lemma 4.2, \( \Gamma_{\text{rev}} \) has a sink. Thus \( \Gamma \) has a source, so part (iii) of the theorem holds.

**Proof of Theorem 1.7** Assume \( \mathcal{V}(\Gamma) \) is finite. For a linear character \( \lambda \) of \( H \), it is easily seen that \( M(\Gamma)_{\lambda} \) is the direct sum of \( M(C)_{\lambda} \) as \( C \) ranges over the connected components of \( \Gamma \). By Lemma 2.3(i), if \( C \) is a connected component of \( \Gamma \), then \( v \in M(C)_{\text{ind}} \) if and only if \( v \) is a scalar multiple of \( \sum_{\alpha \in \mathcal{V}(C)} \alpha \). Thus (i) holds.

Suppose now that \( n(s, t) < \infty \) for \( s, t \in S \). Let \( C \) be a connected component of \( \Gamma \). Assume \( C \) is acyclic. Then since \( \mathcal{V}(C) \) is finite, there must be a source \( \sigma \) in \( C \), and this source is unique by Theorem 1.3(i). Assign to each solid edge in \( C \) the weight \(-1/u^2\), and to each dashed edge in \( C \) assign the weight \(-u + 1)/(u^2 - u)\). For \( \alpha \in \mathcal{V}(C) \), let \( \mu_{\alpha} \) be the product of the weights of the edges of any directed path from \( \sigma \) to \( \alpha \) in \( C \); \( \mu_{\alpha} \) is well-defined by Lemma 4.3(i) since such products are constant on \( \equiv \)-equivalence classes. If
\(\alpha \stackrel{s}{\rightarrow} \beta\) is an edge of \(C\), then \(\mu_\beta = -\mu_\alpha/u^2\), while if \(\alpha \not\rightarrow s \rightarrow \beta\) is an edge of \(C\), then \(\mu_\beta = -(u + 1)\mu_\alpha/(u^2 - u)\). By Lemma 2.4(ii), \(v \in M(C)_{\text{sgn}}\) if and only if \(v\) is a scalar multiple of \(\sum_{\alpha \in \mathcal{V}(C)} \mu_\alpha \alpha\). Therefore \(\dim M(C)_{\text{sgn}} = 1\).

Conversely, suppose \(v = \sum_{\alpha \in \mathcal{V}(C)} \nu_\alpha \alpha \in M(C)_{\text{sgn}}\) is nonzero. Since at least one of the coefficients \(\nu_\alpha\) is nonzero and \(C\) is connected, all of the coefficients \(\nu_\alpha\) are nonzero by Lemma 2.4(ii). Assume

\[
\begin{align*}
\gamma_0 & \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_{k-1} \rightarrow \gamma_k \\
\end{align*}
\]

is a directed path in \(C_\rightarrow\), where \(k > 0, s_1, s_2, \ldots, s_k \in S\). For \(1 \leq j \leq k\) we have

\[
\nu_{\gamma_j} = \begin{cases} \\
-\frac{1}{u^2} \nu_{\gamma_{j-1}} & \text{if } \gamma_{j-1} \rightarrow s_j \rightarrow \gamma_j \in \mathcal{E}(C), \\
-\frac{u + 1}{u^2 - u} \nu_{\gamma_{j-1}} & \text{if } \gamma_{j-1} \rightarrow s_j \rightarrow \gamma_j \in \mathcal{E}(C), \\
\end{cases}
\]

If \(\gamma_0 = \gamma_k\), then

\[
\nu_{\gamma_0} = \nu_{\gamma_k} = \nu_{\gamma_0} \prod_{j=1}^{k} \frac{\nu_{\gamma_j}}{\nu_{\gamma_{j-1}}},
\]

and therefore \(\prod_{j=1}^{k} (\nu_{\gamma_j}/\nu_{\gamma_{j-1}}) = 1\), which is impossible since the product is equal to \(\pm u^{-2(k-j)}(u + 1)^j(u^2 - u)^{-j}\) for some \(j\) with \(0 \leq j \leq k\). Therefore \(C\) is acyclic, so the proof of (ii) is complete.

Proof of Theorem 1.8 Assume \((W, S)\) is a finite Coxeter system, \(\Gamma\) is a connected \(W\)-digraph, and \(J \subseteq S\). Let

\[
\Gamma_J = \bigcup_{i \in I} C_i
\]

be the decomposition of \(\Gamma_J\) into its connected components, indexed by some set \(I\). Let \(\sigma_i \in \mathcal{V}(\Gamma)\) be the source of \(C_i\). If \(\sigma\) is the source of \(\Gamma\), then in \(M(\Gamma)_0\) we have \(\sigma_i = \pm a_{x(i)} \sigma\) for some \(x(i) \in W\) by Lemma 4.3(iii) and Lemma 4.1(ii). Since \(\sigma_i\) is the source of \(C_i\), we have \(a_{s_i} \sigma_i = -\sigma_i\) for \(s \in J\), so \(a_{s} a_{x(i)} = -a_{x(i)}\), and thus \(sx(i) > x(i)\), for all \(s \in J\). Hence \(x(i)\) is in the set of distinguished right coset representatives \(X_J = \{w \in W \mid sw > w \text{ for } s \in J\}\) of \(W_J\) in \(W\). Since \(\sigma_i \neq \sigma_j\) when \(i \neq j\) are in \(I\), \(i \mapsto x(i)\) is an injection from \(I\) into \(X_J\).

Example 4.5. Let \((W, S)\) be a Coxeter system. Let \(\Gamma\) be the \(W\)-digraph defined by \(\mathcal{V}(\Gamma) = W\) and \(x \stackrel{s}{\rightarrow} y \in \mathcal{E}(\Gamma)\) if and only if \(x < s x = y\) for \(x, y \in W, s \in S\). Then the \(H\)-module \(M(\Gamma)\) afforded by \(\Gamma\) is isomorphic to the left regular module \(H\). Note that \(\Gamma\) is connected since if \(w = s_{k_1} s_{k_2} \cdots s_1\) is a reduced expression for \(w \in W\) and \(x_j = s_j s_{j-1} \cdots s_1\) for \(0 \leq j \leq k\) (with \(x_0 = e\)), then

\[
x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{k-1} \rightarrow x_k
\]

is a directed path from \(e\) to \(w\) in \(\Gamma\). When \((W, S)\) is finite, this example shows that the bound in Corollary 1.9 is always attained.

Proof of Theorem 1.10 Suppose \(n(s, t) < \infty\) for all \(s, t \in S\) and \(\Gamma\) is a connected \(W\)-digraph with source \(\sigma\). (The case in which \(\Gamma\) has a sink follows by applying the same reasoning to \(\Gamma_{\text{rev}}\).) Let \(\pi_1, \pi_2\) be two directed paths in \(\Gamma\) from \(\alpha\) to \(\beta\). Let \(\rho\) be some directed path from \(\sigma\) to \(\alpha\): such a path exists by Lemma 4.3(iii). By Lemma 4.3(i), the directed paths \(\rho \pi_1\) and \(\rho \pi_2\) from \(\sigma\) to \(\beta\) are in the same \(\equiv\) equivalence class, and thus have the same number of edges. Hence \(\pi_1\) and \(\pi_2\) have the same number of edges.
5 The proof of Theorem 1.11

Let \( \sigma \) be an automorphism of \( \mathbb{Q}(u) \), and let \( M \) be a vector space over \( \mathbb{Q}(u) \). Let \( \sigma M \) be the vector space over \( \mathbb{Q}(u) \) that has the same additive group as \( M \) and scalar multiplication \( (\alpha, v) \mapsto \alpha \ast_\sigma v \) given by

\[
\alpha \ast_\sigma v = (\sigma \alpha)v,
\]

where the scalar multiplication on the right hand side is that of \( M \). It is clear that if \( Y \subseteq M \), then \( Y \) is a basis (subspace) of \( M \) if and only if \( Y \) is a basis (subspace, respectively) of \( \sigma M \). Moreover, \( \text{gl}(M) = \text{gl}(\sigma M) \) since if \( \varphi : M \to M \) is an additive mapping, then

\[
\varphi(\alpha \ast_\sigma v) = \alpha \ast_\sigma \varphi(v) \iff \varphi((\sigma \alpha)v) = (\sigma \alpha)\varphi(v)
\]

for \( \alpha \in \mathbb{Q}(u) \), \( v \in M \).

**Lemma 5.1.** Let \((W, S)\) be a Coxeter system, and let \( M = M(\Gamma) \) be the \( H \)-module afforded by the \( W \)-digraph \( \Gamma \). Let \( \sigma \) be the automorphism of \( \mathbb{Q}(u) \) determined by \( \sigma u = -1/u \). For \( s \in S \), let \( \tau_s \in \text{gl}(M) \) be the operator \( v \mapsto T_s v \). Then \( T_s \mapsto \tau_s^{-1} \in \text{gl}(\sigma M) \) extends to a representation \( H \to \text{gl}(\sigma M) \). Moreover, as a basis for the \( H \)-module \( \sigma M \), \( X = \mathcal{V}(\Gamma) \) supports the \( W \)-digraph \( \Gamma_{\text{rev}} \).

**Proof.** Let \( s \in S \). Since \( (\tau_s - u^2)(\tau_s + 1) = 0 \) in \( \text{gl}(M) \), we have \( (\tau_s - u^{-2})(\tau_s + 1) = 0 \) in \( \text{gl}(\sigma M) \), and thus \( (\tau_s^{-1} - u^2)(\tau_s^{-1} + 1) = 0 \) in \( \text{gl}(\sigma M) \). Also, if \( s, t \in S \) and \( 1 < n(s, t) < \infty \), then

\[
\frac{n(s, t)}{\tau_s^{-1} \tau_t^{-1} \cdots} = \frac{n(s, t)}{(\cdots \tau_s \tau_t \cdots)^{-1}} = \frac{n(s, t)}{\tau_t^{-1} \tau_s^{-1} \cdots}.
\]

Therefore \( T_s \mapsto \tau_s^{-1} \) extends to a representation \( H \to \text{gl}(\sigma M) \).

Now suppose \( \alpha, \beta \in X \), \( s \in S \). If \( \alpha \xrightarrow{S} \beta \) is an edge of \( \Gamma \), then one checks that

\[
\tau_s^{-1}(\alpha) = (u^2 - 1) \ast_\sigma \alpha + u^2 \ast_\sigma \beta \quad \text{and} \quad \tau_s^{-1}(\beta) = \alpha
\]

in \( \sigma M \). On the other hand, if \( \alpha \xrightarrow{S} \beta \) is an edge of \( \Gamma \), then

\[
\tau_s^{-1}(\alpha) = (u^2 - u - 1) \ast_\sigma \alpha + (u^2 - u) \ast_\sigma \beta \quad \text{and} \quad \tau_s^{-1}(\beta) = (u + 1) \ast_\sigma \alpha + u \ast_\sigma \beta
\]

in \( \sigma M \). These relations show that the basis \( X \) for \( \sigma M \) supports the \( W \)-digraph \( \Gamma_{\text{rev}} \), so the proof is complete.

For a matrix \( A \) over \( \mathbb{Q}(u) \), denote by \( \sigma A \) the matrix obtained by applying the automorphism \( \sigma \) of \( \mathbb{Q}(u) \) to each entry of \( A \).

**Corollary 5.2.** Suppose \( \Gamma \) is a \( W \)-digraph, \( X = \mathcal{V}(\Gamma) \) is finite, and \( \sigma \) is the automorphism of \( \mathbb{Q}(u) \) determined by \( \sigma u = -1/u \). Let \( \rho \) and \( \rho_{\text{rev}} \) be the matrix representations relative to the basis \( X \) for the actions of \( H \) on \( M = M(\Gamma) \) and \( \sigma M \) according to the \( W \)-digraphs \( \Gamma \) and \( \Gamma_{\text{rev}} \), respectively. Then

\[
(5.1) \quad \rho_{\text{rev}}(T_w) = \sigma \rho(T_w^{-1})
\]

for \( w \in W \).

**Proof.** From the proof of Lemma 5.1, we have \( \rho_{\text{rev}}(T_s) = \sigma \rho(T_s^{-1}) \) for \( s \in S \). The assertion follows since if \( w \in W \) has reduced expression \( w = s_1 \cdots s_k \), then \( T_w = T_{s_1} T_{s_2} \cdots T_{s_k} \) and \( T_{w^{-1}} = T_{s_1}^{-1} T_{s_2}^{-1} \cdots T_{s_k}^{-1} \).
Next assume \( n(s, t) < \infty \) for \( s, t \in S \), \( \Gamma \) is an acyclic \( W \)-digraph, and \( \mathcal{V}(\Gamma) \) is finite. For \( \alpha \in X = \mathcal{V}(\Gamma) \), let \( \sigma_\alpha \) be the source in the connected component of \( \Gamma \) containing \( \alpha \), and let \( \mu(\alpha) \) be the number of edges in a directed path from \( \sigma_\alpha \) to \( \alpha \). (Thus \( \mu(\alpha) \) is well-defined by Lemma [13](i).) Put \( \varepsilon_\alpha = (-1)^{\mu(\alpha)} \) for \( \alpha \in X \), and define \( X' = \{ \varepsilon_\alpha \alpha \mid \alpha \in X \} \). Let \( \rho' \) be the matrix representation afforded by \( M(\Gamma) \) with basis \( X' \), and let \( \rho_{\text{rev}} \) be the matrix representation corresponding to \( M(\Gamma) \) with basis \( X \).

**Lemma 5.3.** If \( n(s, t) < \infty \) for \( s, t \in S \), \( \Gamma \) is an acyclic \( W \)-digraph, \( \mathcal{V}(\Gamma) \) is finite, and \( \rho_{\text{rev}} \) and \( \rho' \) are defined as above, then

\begin{equation}
(5.2) \quad \rho_{\text{rev}}(T_w) = \varepsilon_w u_w \rho'(T_w^{-1})^T \quad \text{for } w \in W.
\end{equation}

**Proof.** Let \( s \in S \). Suppose \( \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma) \), so also \( \beta \xrightarrow{s} \alpha \in \mathcal{E}(\Gamma_{\text{rev}}) \). Thus in \( M(\Gamma_{\text{rev}}) \) we have

\( T_s \alpha = (u^2 - 1)\alpha + u^2 \beta \) and \( T_s \beta = \alpha \),

so the matrix of \( T_s \) acting on the subspace with basis \( \{ \alpha, \beta \} \) is

\[
\begin{pmatrix}
    u^2 - 1 & 1 \\
    u^2 & 0
\end{pmatrix}.
\]

On the other hand, \( \varepsilon_\beta = -\varepsilon_\alpha \) and \( u_s T_s^{-1} = T_s - (u^2 - 1) \), so in \( M(\Gamma) \) we have

\[
\varepsilon_s u_s T_s^{-1} \varepsilon_\alpha \alpha = -\varepsilon_\alpha (T_s - (u^2 - 1)) \alpha = -\varepsilon_\alpha (\beta - (u^2 - 1)\alpha) = (u^2 - 1)\varepsilon_\alpha \alpha + \varepsilon_\beta \beta
\]

and

\[
\varepsilon_s u_s T_s^{-1} \varepsilon_\beta \beta = -\varepsilon_\beta (T_s - (u^2 - 1)) \beta = -\varepsilon_\beta ((u^2 - 1)\beta + u^2 \alpha - (u^2 - 1)\beta) = u^2 \varepsilon_\alpha \alpha,
\]

so the matrix of \( \varepsilon_s u_s T_s^{-1} \) acting on the subspace with basis \( \{ \varepsilon_\alpha \alpha, \varepsilon_\beta \beta \} \) is

\[
\begin{pmatrix}
    u^2 - 1 & u^2 \\
    1 & 0
\end{pmatrix}.
\]

Now suppose that \( \alpha \xrightarrow{s} \beta \in \mathcal{E}(\Gamma) \). Then in \( M(\Gamma_{\text{rev}}) \) we have

\( T_s \alpha = (u^2 - u - 1)\alpha + (u^2 - u)\beta \)

and

\( T_s \beta = (u + 1)\alpha + u \beta \),

so the matrix of \( T_s \) acting on the subspace with basis \( \{ \alpha, \beta \} \) is

\[
\begin{pmatrix}
    u^2 - u - 1 & u + 1 \\
    u^2 - u & u
\end{pmatrix}.
\]

In \( M(\Gamma) \) we have

\[
\varepsilon_s u_s T_s^{-1} \varepsilon_\alpha \alpha = -\varepsilon_\alpha (T_s - (u^2 - 1)) \alpha = -\varepsilon_\alpha (u\alpha + (u + 1)\beta - (u^2 - 1)\alpha) = (u^2 - u - 1)\varepsilon_\alpha \alpha + (u + 1)\varepsilon_\beta \beta
\]

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and
\[ \varepsilon_s u_s T_s^{-1} \varepsilon_\beta = -\varepsilon_\beta (T_s - (u^2 - 1)) \beta \]
\[ = -\varepsilon_\beta ((u^2 - u - 1)\beta + (u^2 - u)\alpha - (u^2 - 1)\beta) \]
\[ = (u^2 - u)\varepsilon_\alpha + \varepsilon_\beta \beta, \]
so the matrix of \( \varepsilon_s u_s T_s^{-1} \) acting on the subspace with basis \( \{\varepsilon_\alpha, \varepsilon_\beta \} \) is
\[
\begin{pmatrix}
  u^2 - u - 1 & u^2 - u \\
  u + 1 & u
\end{pmatrix}.
\]

To summarize, (5.2) holds when \( w = s \in S \). The general case follows since if \( w \in W \) has reduced expression \( w = s_1 s_2 \cdots s_k \), then \( T_w = T_{s_1} T_{s_2} \cdots T_{s_k} \).

\( \square \)

**Proof of Theorem 1.12** Part (i) of the theorem follows by taking traces in (5.1). Since \( \chi^\Gamma \) coincides with the character afforded by the matrix representation \( \rho' \), part (ii) of the theorem follows by taking traces in (5.2).

\( \square \)

### 6 The proof of Theorem 1.12

**Proof.** Suppose \((W, J, \Gamma)\) is finite for proper subsets \( J \) of \( S \) and \( \Gamma \) is a finite, connected \( W \)-digraph. Suppose further that \( M(\Gamma) \) is isomorphic to the module \( M(\Psi) \) afforded by a \( W \)-graph \( \Psi \) for \((W, S)\) (in the sense of [4]), and that \( \Gamma \) is not acyclic. For \( x \) in the set of vertices \( \mathcal{V}(\Psi) \) of \( \Psi \), let \( I_x \subseteq S \) be the associated set of generators. For \( \beta \in \mathcal{V}(\Gamma) \), let \( \text{In}(\beta) \) be the set of \( s \in S \) such that \( \Gamma \) as an edge of the form \( \alpha \xrightarrow{s} \beta \) or \( \alpha \xrightarrow{\sim} \beta \) for some \( \alpha \in \mathcal{V}(\Gamma) \). Let \( \chi^\Gamma = \chi^\Psi \) be the character of \( H \) afforded by \( M(\Gamma) \) or \( M(\Psi) \).

Put
\[
N^\Gamma_J = |\{ \beta \in \mathcal{V}(\Gamma) \mid \text{In}(\beta) = J \}|, \quad N^\Psi_J = |\{ x \in \mathcal{V}(\Psi) \mid I_x = J \}|
\]
for \( J \subseteq S \). Since \( M(\Gamma)_{\text{ind}} \cong M(\Psi)_{\text{ind}} \) is one-dimensional by Theorem 1.7(i), we must have \( N^\Psi(\emptyset) > 0 \). Also, since \( \Gamma \) is not acyclic, \( M(\Gamma)_{\text{sgn}} = \{ 0 \} = M(\Psi)_{\text{sgn}} \) by Theorem 1.7(ii), and therefore \( N^\Psi(S) = 0 \). Further, \( \Gamma \) has no sink by Theorem 1.5 and so also \( N^\Gamma(S) = 0 \).

For \( w \in W \), let \( J(w) \) be the minimal \( J \subseteq S \) such that \( w \in W_J \). Then
\[
\chi^\Psi(T_w)|_{u=0} = \varepsilon_w \{ x \in \mathcal{V}(\Psi) \mid J(w) \subseteq I_x \}.
\]
Since \( I_x \neq S \) for \( x \in \mathcal{V}(\Psi) \), we have
\[
0 < N^\Psi(\emptyset) = \sum_{x \in \mathcal{V}(\Psi)} \sum_{w \in W_I_x} \varepsilon_w = \sum_{w \in W, J(w) \neq S} \varepsilon_w |\{ x \in \mathcal{V}(\Psi) \mid J(w) \subseteq I_x \}|
\]
\[
= \sum_{w \in W, J(w) \neq S} \chi^\Psi(T_w)|_{u=0};
\]
with the sums finite by assumption. On the other hand, if \( J(w) \neq S \), then \( \Gamma(J(w)) \) is acyclic by Theorem 1.3(iii), so if \( \mathcal{V}(\Gamma) \) is ordered in a way consistent with directed paths in \( \Gamma(J(w)) \), then the matrix representing \( T_w \) acting on \( M(\Gamma) \), when evaluated at \( u = 0 \), is triangular. Moreover, the nonzero diagonal entries of this matrix are all equal to \( \varepsilon_w \), occurring in positions corresponding to those \( \beta \in \mathcal{V}(\Gamma) \) such that \( J(w) \subseteq \text{In}(\beta) \). Since \( \text{In}(\beta) \neq S \) for \( \beta \in \mathcal{V}(\Gamma) \) and \( \chi^\Gamma = \chi^\Psi \), it follows that
\[
0 < \sum_{w \in W, J(w) \neq S} \chi^\Gamma(T_w)|_{u=0} = \sum_{w \in W, J(w) \neq S} \varepsilon_w |\{ \beta \in \mathcal{V}(\Gamma) \mid J(w) \subseteq \text{In}(\beta) \}|
\]
\[
= \sum_{\beta \in \mathcal{V}(\Gamma)} \sum_{w \in W_{\text{In}(\beta)}} \varepsilon_w = N^\Gamma(\emptyset),
\]
and so \( \Gamma \) has a source. Therefore \( \Gamma \) is acyclic by Theorem 1.5. \( \square \)
7 Additional Examples

Let \((W, S)\) be a Coxeter system, let \(\gamma \mapsto \overline{\gamma}\) be the automorphism of \(\mathbb{Q}(u)\) determined by \(\overline{u} = u^{-1}\), and let \(h \mapsto \overline{h}\) be the ring automorphism \(\sum_{w \in W} \gamma_w T_w \mapsto \sum_{w \in W} \overline{\gamma}_w T_w^{-1}\) of \(H\). Following Lusztig \([6]\), define a bar operator on an \(H\)-module \(M\) to be an additive bijection \(\varphi : M \to M\) such that

\[
\varphi(hv) = \overline{h} \varphi(v) \quad \text{for } h \in H, \ v \in M.
\]

Let \(\Gamma^*\) be the \(W\)-digraph associated with an involutory automorphism \(w \mapsto w^*\) of \((W, S)\), as described before Theorem 3.1. Lusztig has shown that \(M(\Gamma^*)\) admits a unique bar operator that fixes the source of \(\Gamma^*\) \(([6], \text{Theorem 0.2})\). It can be shown that if \((W, S)\) is finite, then any \(H\)-module admits a bar operator. However, there need not be a bar operator if \((W, S)\) is infinite, as the next example shows.

Example 7.1. With \(W = W(\tilde{A}_2) = \langle r, s, t \rangle\), let \(\Gamma\) be as in Figure 7.1 and put \(w = tsr\). Suppose a bar operator \(\varphi\) exists on \(M(\Gamma)\). Let \(\alpha\) be the vertex in the lower left corner of Figure 7.1 so \(T_{tsr} \alpha = T_s T_r \alpha = \alpha\). Then \(T_{tsr} \varphi(\alpha) = \varphi(\alpha)\), so \(\varphi(\alpha) = T_{rst} \varphi(\alpha)\) is a fixed point of \(T_{rst}\). However, one checks that the characteristic polynomial of \(T_{rst}\) acting on \(M(\Gamma)\) is \((\lambda^2 + 1)(\lambda^2 - u^0)(\lambda - u^0)^2\), so a contradiction is obtained. Also, \(\Gamma\) provides an example in which the equation of Theorem 1.11(ii) fails: with \(y = w^{-1} = rst\), one checks that \(\chi_{\Gamma, w}(T_y) = \overline{\chi}_{\Gamma}(T_y^{-1}) = 2\) and \(\varepsilon_y u_y \chi_{\Gamma}(T_y^{-1}) = -2\). Moreover, \(M(\Gamma)\) does not afford a \(W\)-graph by Theorem 1.12.

Even if \((W, S)\) is finite and \(\Gamma\) is connected, there may not exist a bar operator on \(M(\Gamma)\) that fixes the source of \(\Gamma\), as the next example shows.

Example 7.2. Let \(W = W(B_3) = \langle r, s, t \rangle\), with \(n(r, s) = 3\), \(n(r, t) = 2\), \(n(s, t) = 4\). Let \(\Gamma\) be the \(W\)-digraph of Figure 7.2. Suppose \(M = M(\Gamma)\) admits a source-fixing bar operator \(\varphi : M \to M\). Since \(v_0\) is the source of \(\Gamma\) and \(v_4 = T_r T_s v_0\), we have

\[
\varphi(v_4) = T_r T_s v_0 = u^{-4}(T_r - (u^2 - 1))(T_s - (u^2 - 1))v_0 \\
= u^{-4} \left( v_1 - (u^2 - 1)v_2 - (u^2 - 1)v_8 + (u^2 - 1)^2 v_0 \right).
\]

On the other hand, \(v_4 = T_t T_s v_0\), so we also have

\[
\varphi(v_4) = T_t T_s v_0 = u^{-4}(T_t - (u^2 - 1))(T_s - (u^2 - 1))v_0 \\
= u^{-4} \left( v_1 - (u^2 - 1)v_2 - (u^2 - 1)v_1 + (u^2 - 1)^2 v_0 \right).
\]

Since (7.2) and (7.3) cannot simultaneously hold, a contradiction is reached. Thus \(M\) does not admit a source-fixing bar operator.
Let \((W, S)\) be finite, and let \(\Gamma\) be a finite \(W\)-digraph. By Theorem 1.11 we have \(\chi_{\Gamma}|_{u = -1} = \text{sgn}_W \cdot \chi_{\Gamma}|_{u = 1}\). Thus if \((W, S)\) has no connected components with exceptional characters in the sense of Gyoja [3], then \(\chi_{\Gamma}|_{u = 1} = \chi_{\Gamma}|_{u = -1}\) is self-associated, that is, \(\chi_{\Gamma}|_{u = 1} = \text{sgn}_W \cdot \chi_{\Gamma}|_{u = 1}\). In particular, if \((W, S)\) has no component of type \(H_3\), \(H_4\), \(E_7\), or \(E_8\), then \(\chi_{\Gamma}|_{u = 1}\) is self-associated. Our final example shows that if \((W, S)\) has an exceptional character, then \(\chi_{\Gamma}|_{u = 1}\) need not be self-associated.

**Example 7.3.** Let \(W = W(H_3) = \langle r, s, t \rangle\) with \(n(r, s) = 3\), \(n(s, t) = 5\), \(n(r, t) = 2\). The \(W\)-digraph \(\Gamma\) of Figure 7.3 affords the non-self-associated character \(\chi_{\Gamma}|_{u = 1} = 1_W + \text{sgn}_W + \chi_4^\prime\), where \(\chi_4^\prime\) is the irreducible character of degree 4 with value \(-4\) at the longest element of \(W\). Then \(\chi_{\Gamma|_{\text{rev}}}|_{u = 1} = \text{sgn}_W \cdot \chi_r|_{u = 1} \neq \chi_{\Gamma}|_{u = 1}\).

**References**

[1] N. Bourbaki. *Lie Groups and Lie Algebras, Chapters 4–6*. Springer, 2002.

[2] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras (London Mathematical Society Monographs)*. Oxford University Press, 2000.

[3] Akihiko Gyoja. On the existence of a \(W\)-graph for an irreducible representation of a Coxeter group. *J. Alg.*, 86:422–438, 1984.

[4] D. A. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.

[5] R. Kilmoyer and L. Solomon. On the theorem of Feit-Higman. *Journal of Combinatorial Theory, Series A*, 15(3):310 – 322, 1973.

[6] G. Lusztig. A bar operator for involutions in a Coxeter group. *Bulletin of the Institute of Mathematics Academia Sinica (N.S.*), 7(3):355–404, 2012.
[7] G. Lusztig and D. A. Vogan, Jr. Hecke algebras and involutions in Weyl groups. *Bulletin of the Institute of Mathematics Academia Sinica (N.S.)*, 7(3):323–354, 2012.

[8] P. N. Norton. 0-Hecke algebras. *J. Austral. Math. Soc. (Series A)*, 27:337–357, 1979.