ASYMPTOTIC PROFILES OF STEADY STATES FOR A DIFFUSIVE SIS EPIDEMIC MODEL WITH SPONTANEOUS INFECTION AND A LOGISTIC SOURCE

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(Communicated by Junping Shi)

Abstract. Spatial heterogeneity and movement of population play an important role in disease spread and control in reality. This paper concerns with a spatial Susceptible-Infected-Susceptible epidemic model with spontaneous infection and logistic source, aiming to investigate the asymptotic profiles of the endemic steady state (whenever it exists) for large and small diffusion rates. We firstly establish uniform upper bound of solutions. By studying the local and global stability of the unique constant endemic equilibrium when spatial environment is homogeneous, we apply the well-known Leray-Schauder degree index formula to confirm the existence of endemic steady state. Our theoretical results suggest that spontaneous infection and varying total population strongly enhance the persistence of disease spread in the sense that disease component of the endemic steady state will not approach zero whenever the large and small diffusion rates of the susceptible or infected population is used. This gives new insights and aspects for infectious disease modeling and control.

1. Introduction. Since the classical work of Allen et al. [1], mathematical modeling and investigating joint effect of the spatial environmental heterogeneity and the movement of individuals in a diffusive Susceptible-Infected-Susceptible (SIS) model have attracted many researchers in recent years, see recent a few publications along this line, such as [2, 3, 5, 13, 14, 15, 16, 17] and the references therein. Mathematical analysis for these models is necessary to give us a detailed information and enhances our understanding the spatial transmission of a disease. In particular, with considering distinct dispersal rates of susceptible and infected individuals, some results on disease dynamics have been explored. We summarize some previous studies in the literature on this topic:

- With the standard incidence infection mechanism, recent studies [1, 14] explored that the asymptotical profiles of the endemic steady state by using large and
small diffusion rates. Under the assumption that low-risk site (the low-risk site contains the positions where the disease transmission rate is less than the recovery rate) is not empty, the main contributions in [1, 14] lie in that controlling the diffusion rate of the susceptible individuals can help to control the disease, while controlling the diffusion rate of the infectious individuals can not eliminate the disease.

• With the mass action mechanisms instead of standard incidence, Wu and his collaborators [17] have studied a diffusive SIS model of the same structure as in [1, 14], and explored the asymptotic profiles of the endemic steady state for small and large diffusion rates. The main results shown in [17] reveals that the disease may be controlled by limiting the movement of the susceptible individuals but with an extra requirement on the total population. In strong contrast to the results in [1, 14], they shown that if the total population is large enough, controlling the diffusion rate of the susceptible individuals cannot eradicate the infection, and inversely, limiting the diffusion rate of the infected individuals can eliminate the disease in certain area.

• With total population varying in contrast to that in [1, 14, 17], Li et al. [9] analyzed an spatial SIS model with linear source. When a small or large diffusion rate of susceptible and infected individuals is used, they continued to investigate the asymptotic profile of endemic steady state. Furthermore, Li et al. [10] extended model in [9] to be one with logistic source. By investigating the asymptotical profiles of the endemic steady state by using large and small diffusion rates, “Varying total population can enhance persistence of infectious disease” are obtained in [9, 10], which is a serious situation that the disease can not be controlled by limiting the diffusion rates of the susceptible and infected individuals.

• With considering “spontaneous” social infection (see e.g. [7]) in addition to disease transmission, Tong and Lei [16] extended the diffusive SIS model in [1] by adding the effect of spontaneous infection and investigated the asymptotic profile of endemic steady state. “Spontaneous” social infection is a infection mechanism (by incorporating an term hereby susceptible individuals become infected individuals) that differ from disease transmission. The results in [16] reveal that spontaneous infection can enhance disease persistence. This brings new insight into disease control.

• With considering “spontaneous” social infection with linear source, A recent work [18] further investigated the effects of the movement and spatial heterogeneity on disease transmission. The theoretical results (consistent with those in [16]) reveals that the disease can not be controlled in the sense that the disease component of the endemic steady state will not approach zero whenever the dispersal rates of the susceptible or infected population is small or large.

Inspired by the above works, this paper is a continuation of [1, 9, 10, 14, 17] to investigate the effect of spatial environmental heterogeneity and the movement of individuals in a diffusive SIS model by incorporating “spontaneous” social infection with logistic source. Would incorporation of “spontaneous” social infection and logistic source in [1] lead to any new phenomenon in disease transmission. Answering such a question would give new understanding and implications into disease transmission and control. This constitutes one motivation of the present paper. To this end, we consider the host individuals living in an open and bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$. We imposed on the homogeneous Neumann boundary conditions, which means that no population flux crosses the boundary $\partial \Omega$. Let $S(x,t)$ and $I(x,t)$ be the populations of susceptible and infectious individuals
The L^2. Uniform boundedness of solutions to

or small dispersal rates of the susceptible or infected population is considered.

component of the endemic steady state will not approach zero whenever the large

and EE

(3.8), the local and global stability of constant EE when all parameters, \( \Lambda, b, \beta, \gamma \)

are positive H"older continuous functions on \( \overline{\Omega} \).

According the standard theory for parabolic equations, we can obtain that (1.1)

has a unique classical solution \((S, I) \in C^{2,1}(\Omega \times \mathbb{R}^+)\), where \( \mathbb{R}^+ = (0, \infty) \). Further,

with the aid of the strong maximum principle and the Hopf boundary lemma for

parabolic equations (see e.g., [6, Proposition 13.1]), both \( S(x, t) \) and \( I(x, t) \) are positive for \( x \in \Omega \) and \( t \in (0, \infty) \). Denote by \( EE \) the componentwise positive

solution \((S, I) \in C^2(\Omega) \times C^2(\Omega) \) to (3.1) if it exists. Then \( S(x) > 0 \) and \( I(x) > 0 \) for all \( x \in \Omega \), directly follows from the strong maximum principle and Hopf boundary

lemma for elliptic equations.

For convenience of notations, we denote

\[
\Psi^* = \max_{x \in \Omega} \Psi(x) \in (0, \infty) \quad \text{and} \quad \Psi_* = \min_{x \in \Omega} \Psi(x) \in (0, \infty),
\]

where \( \Psi(x) = a(x), b(x), \beta(x), \eta(x), \gamma(x) \), respectively. We point out here that un-

like models in [1, 9, 10, 14, 17], adoption of spontaneous infection term in (1.1) makes

model (1.1) on longer have a obvious definition of the basic reproductive number.

The reason stems from the fact that the rate of spontaneous infection from suscep-

tible individuals directly contributes to the migration of infected individuals, which

is adopted independently of neighbors mean. On the other hand, incorporating the

spontaneous infection term makes (1.1) has a non-zero steady state if the disease

transmission between susceptible and infected individuals is ignored.

The rest of this paper is organized as follows. In section 2, we shall establish the

uniform bounds of solutions to (1.1). Section 3 is devoted to investigating the exis-
tence of EE by the topological degree arguments. To this end, we need to establish

the estimate of upper bounds and lower bounds for any positive solution \((S, I) \) to

(3.8), the local and global stability of constant EE when all parameters, \( \Lambda, b, \beta, \gamma \)

and \( \eta \) are positive constants and then apply the well-known Leray-Schauder degree

index formula. In section 4, by letting dispersal rates of the susceptible or infected

population to be small or large, we investigate the asymptotic behavior of EE. Our

results suggest that the disease can not be controlled in the sense that the disease

component of the endemic steady state will not approach zero whenever the large

or small dispersal rates of the susceptible or infected population is considered.

2. Uniform boundedness of solutions to (1.1). The following result establishes

the \( L^\infty \) estimates of solutions to (1.1).
Theorem 2.1. There exists a positive constant $C_1$, depending on initial data, such that the solution $(S, I)$ of (1.1) satisfies

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \forall t \geq 0.$$  \hfill (2.1)

Furthermore, there exists a positive constant $C_2$, independent of initial data, such that

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \forall t \geq T,$$  \hfill (2.2)

for some large time $T > 0$.

Proof. We prove this theorem by using the method of mathematical induction, step by step.

**Step 1:** There exists a positive constant $C$, independent of initial data, such that

$$\|S(\cdot, t)\|_{L^1(\Omega)} + \|I(\cdot, t)\|_{L^1(\Omega)} \leq C, \forall t > T,$$  \hfill (2.3)

for some large time $T > 0$.

In fact, this assertion can be easily obtained by usual analysis. From (1.1), we can get

$$\frac{d}{dt} \int_{\Omega} (S + 2I)dx = \int_{\Omega} \left( \beta SI + \eta S \right) dx - \int_{\Omega} \gamma I dx + \int_{\Omega} (aS - bS^2)dx$$

$$\leq (\beta^* + \eta^*) \int_{\Omega} Sdx - \gamma_* \int_{\Omega} I dx + \int_{\Omega} (a^* S - b_* S^2)dx$$

$$\leq (\beta^* + \eta^* + a^* + \epsilon) \int_{\Omega} Sdx - \gamma_* \int_{\Omega} I dx + \int_{\Omega} \left( aS - \frac{c^2}{4} \right) dx$$

$$= \frac{b_* \epsilon^2 |\Omega|}{4} + (\beta^* + \eta^* + a^* - \epsilon b_*) \int_{\Omega} Sdx - \gamma_* \int_{\Omega} I dx.$$

Selecting $\epsilon > 0$ such that

$$\beta^* + \eta^* + a^* - \epsilon b_* = -\frac{\gamma_*}{2},$$

which implies that

$$\int_{\Omega} (S + 2I)dx \leq e^{\frac{\gamma_*}{2}t} \int_{\Omega} (S_0(x) + 2I_0(x))dx + \frac{b_* \epsilon^2 |\Omega|}{2 \gamma_*} (1 - e^{-\frac{\gamma_*}{2}t}), \forall t \geq 0.$$

This completes the proof of **Step 1**.

**Step 2:** We assume that

$$\|S(\cdot, t)\|_{L^{k-1}(\Omega)} + \|I(\cdot, t)\|_{L^{k-1}(\Omega)} \leq C_{k-1}, \forall t > T.$$  \hfill (2.4)

In the following, we will use the inequality,

$$\int_{\Omega} S^{k-1}dx \leq C_{k-1}.$$  \hfill (2.4)

**Step 3:** We multiply the both sides of first and second equations of (1.1) by $S^{k-1}$ and $I^{k-1}$, respectively, to get

$$\frac{1}{k} \frac{d}{dt} \int_{\Omega} S^k dx + d_s(k - 1) \int_{\Omega} S^{k-2} |\nabla S|^2 dx$$

$$= \int_{\Omega} \left[ -\beta \frac{SI}{S + I} - \eta S^k + \gamma IS^{k-1} + (aS - bS^2)S^{k-1} \right] dx,$$
Collecting terms with respect to \( S \)

Setting \( I \)

To estimate \( EE \)

Adding the two resulting equations, we obtain

\[
\frac{1}{k} \frac{d}{dt} \int \Omega (S^k + I^k)dx + (k - 1) \int \Omega (d_S S^{k-2} |\nabla S|^2 + d_I I^{k-2} |\nabla I|^2)dx \\
\leq \int \Omega \left[ \frac{\beta}{S + I} I^{k-1} S + \eta S I^{k-1} - \gamma I^k + \gamma IS^{k-1} + (aS - bS^2)S^{k-1} \right] dx \\
\leq \int \Omega \left[ \beta S^{k-1}S + \eta S I^{k-1}S - \gamma S I^k + \gamma S IS^{k-1} + a^* S^k - b_s S^{k-1} \left( \epsilon_1 S - \frac{\epsilon_3}{4} \right) \right] dx.
\]

To estimate \( I^{k-1}S \) and \( IS^{k-1} \), we apply Young’s inequality:

\[
ab \leq \epsilon a^p + C_\epsilon b^q,
\]

where \( C_\epsilon = \epsilon^{-\frac{q}{p}} \), and \( \frac{1}{p} + \frac{1}{q} = 1 \). Setting \( p = k/(k - 1) \) and \( q = k \), we obtain

\[
\int \Omega (S^{k-1}S)dx \leq \int \Omega \left[ \epsilon_2 I^k + C_\epsilon S^k \right] dx.
\]

Setting \( p = k \) and \( q = k/(k - 1) \), we obtain

\[
\int \Omega (IS^{k-1})dx \leq \int \Omega \left[ \epsilon_3 I^k + C_\epsilon S^k \right] dx.
\]

Collecting terms with respect to \( S^k \) and \( I^k \), we arrive at

\[
\frac{1}{k} \frac{d}{dt} \int \Omega (S^k + I^k)dx \leq \left[ (\beta^* + \eta^*) \epsilon_2 + \gamma^* \epsilon_3 - \gamma_* \right] \int \Omega I^k dx \\
+ \left[ (\beta^* + \eta^*) C_{\epsilon_2} + \gamma^* C_{\epsilon_3} + a^* - b_s \epsilon_1 \right] \int \Omega S^k dx \\
+ \frac{\epsilon^2 b_s}{4} \int \Omega S^{k-1}dx.
\]

By choosing suitable \( \epsilon_1, \epsilon_2, \epsilon_3 \), we can obtain that \( (\beta^* + \eta^*) \epsilon_2 + \gamma^* \epsilon_3 - \gamma_* = -\gamma_*/2 \)

and \( (\beta^* + \eta^*) C_{\epsilon_2} + \gamma^* C_{\epsilon_3} + a^* - b_s \epsilon_1 = -\gamma_*/2 \). Further, due to (2.4), we arrive at

\[
\frac{1}{k} \frac{d}{dt} \int \Omega (S^k + I^k)dx \leq -\frac{\gamma_*}{2} \int \Omega (S^k + I^k)dx + \frac{\epsilon^2 b_s}{4} C_{k-1}^{k-1}, \quad t \geq 0,
\]

where the positive constant \( C_{k-1}^{k-1} \) is independent of the initial data for large \( t \).

**Step 4:** Applying the assertions in [8, Theorem 1] or [4, Lemma 2.1] with \( q = 2 \), we can conclude that if \( p_0 > \frac{q}{p} \), any solution of (1.1) is bounded, which is dependent, or independent of the initial data for large \( t \). This completes the proof. \( \square \)

3. **Existence of the EE.** This section is devoted to establishing a sufficient condition for the existence of EE. We suppose that \( a(x), b(x), \beta(x), \gamma(x) \) and \( \eta(x) \) are positive Hölder continuous functions on \( \Omega \).

Due to (1.1), the equilibrium problem satisfies the following elliptic system

\[
\begin{cases}
-d_S \Delta S = a(x)S - b(x)S^2 - \beta(x) \frac{SI}{S + I} - \eta(x)S + \gamma(x)I, & x \in \Omega, \\
-d_I \Delta I = \beta(x) \frac{SI}{S + I} + \eta(x)S - \gamma(x)I, & x \in \Omega, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]
In the following, we first aim to give the priori bounds of \((S, I)\) to (3.1).

3.1. The \(L^p\) bounds for any solutions of \((S, I)\) to (3.1).

**Lemma 3.1.** Let \(p\) be any positive constant. Then the solution of equilibrium problem (3.1) satisfies

\[
\int_{\Omega} S^p dx \leq C_p \quad \text{and} \quad \int_{\Omega} I^p dx \leq C_p,
\]

where \(C_p\) is a positive constant, and independent of \(d_S, d_I > 0\).

**Proof.** Integrating both sides of the first two equations of (3.1) and adding them up, we have

\[
0 = \int_{\Omega} (a(x)S - b(x)S^2) dx \leq \int_{\Omega} (a^* S - b^* S^2) dx \leq \int_{\Omega} (\mu - \nu S^2) dx.
\]

Here we used the fact that \(a^* S - b^* S^2 \leq \mu - \nu S^2, \forall S > 0\), for some positive constants \(\mu\) and \(\nu\). It follows that

\[
\int_{\Omega} S^2 dx \leq \frac{\mu}{\nu} |\Omega|. \tag{3.2}
\]

For \(k > 0\), multiplying both sides of the second equation of (3.1) by \(I^k\), and then employing integration by parts, we get

\[
0 \leq kd_I \int_{\Omega} I^{k-1} |\nabla I|^2 dx = \int_{\Omega} \left[ \frac{\beta SI^{k+1}}{S^2 + I} + \eta S I^k - \gamma I^{k+1} \right] dx.
\]

It follows that

\[
\gamma_s \int_{\Omega} I^{k+1} dx \leq \gamma I^{k+1} dx \leq \int_{\Omega} \frac{\beta}{S + I} SI^k dx + \int_{\Omega} \eta SI^k dx \leq (\beta^* + \eta^*) \int_{\Omega} SI^k dx.
\]

Letting \(\epsilon_4 = \frac{\gamma_s}{2(\beta^* + \eta^*)}\), by using Young’s inequality, we have

\[
(\beta^* + \eta^*) \int_{\Omega} SI^k dx \leq (\beta^* + \eta^*) \epsilon_4 \int_{\Omega} I^{k+1} dx + (\beta^* + \eta^*) C_{\epsilon_4} \int_{\Omega} S^{k+1} dx. \tag{3.3}
\]

Hence

\[
\gamma_s \int_{\Omega} I^{k+1} dx \leq \frac{\gamma_s}{2} \int_{\Omega} I^{k+1} dx + C \int_{\Omega} S^{k+1} dx. \tag{3.4}
\]

From (3.2) and letting \(k = 1\), we can obtain

\[
\int_{\Omega} I^2 \leq C. \tag{3.5}
\]

For simplicity, here and in what follows, we allow positive constant \(C\) to vary from line to line.

Multiplying both sides of the first equation (3.1) by \(S^k\), and then employing integration by parts, we have

\[
0 \leq kd_S \int_{\Omega} S^{k-1} |\nabla S|^2 dx
= \int_{\Omega} a S^{k+1} dx - \int_{\Omega} b S^{k+2} dx - \int_{\Omega} \frac{\beta S^{k+1} I}{S + I} dx - \int_{\Omega} \eta S^{k+1} dx + \int_{\Omega} \gamma IS^k dx.
\]

It then follows from Young’s inequality that

\[
b_S \int_{\Omega} S^{k+2} \leq a_S \int_{\Omega} S^{k+1} + \gamma_S \int_{\Omega} IS^k \leq (a_S + \gamma_S \epsilon_5) \int_{\Omega} S^{k+1} + \gamma S C_{\epsilon_5} \int_{\Omega} I^{k+1} dx. \tag{3.6}
\]
Letting $k = 1$ and with aid of (3.2) and (3.5), we get
\[ \int_{\Omega} S^3 dx \leq C. \]  
(3.7)

Letting $k = 2$ in (3.4). We have
\[ \int_{\Omega} I^3 dx \leq C. \]

Operating the above iteration procedure, we can complete the proof. \square

3.2. The existence for any solutions of $(S, I)$ to (3.1). In this subsection, by the topological degree argument, we investigate the existence of positive solutions to (3.1). To this end, we construct the following auxiliary system

\[-d_S \Delta S = \bar{a}S - \bar{b}S^2 - \bar{\beta} \frac{SI}{S+I} - \bar{\eta}S + \bar{\gamma}I, \quad x \in \Omega,\]
\[-d_I \Delta I = \bar{\beta} \frac{SI}{S+I} + \bar{\eta}S - \bar{\gamma}I, \quad x \in \Omega,\]
\[ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega,\]

(3.8)

where $\bar{a}(x) := \theta a(x) + (1 - \theta) a_0$, $\bar{b}(x) := \theta b(x) + (1 - \theta) b_0$, $\bar{\beta}(x) := \theta \beta(x) + (1 - \theta) \beta_0$, $\bar{\eta}(x) := \theta \eta(x) + (1 - \theta) \eta_0$, $\bar{\gamma}(x) := \theta \gamma(x) + (1 - \theta) \gamma_0$, $a_0, b_0, \beta_0, \eta_0$ are positive constants and the parameter $\theta \in [0, 1]$. It is easy to see that when $\theta = 1$, problem (3.8) becomes steady state problem (3.1).

**Theorem 3.2.** Assume that
\[ \min_{x \in \Omega} \left( \bar{a} - \bar{\eta} - \bar{\beta} \right) > 0. \]  
(3.9)

Then, the steady state problem (3.1) admits at least one positive solution.

**Proof.** We prove this result by proving the following claims, step by step.

**Claim 1:** Upper bounds and lower bounds for any positive solution $(S, I)$ to (3.8).

By using similar arguments in Lemma 3.1, we can also conclude that
\[ ||S||_{L^\infty(\Omega)} \leq C, \quad ||I||_{L^\infty(\Omega)} \leq C. \]  
(3.10)

We next prove that
\[ S(x), \ I(x) > \frac{1}{C}, \ \forall x \in \bar{\Omega}, \]  
(3.11)

for sufficiently large number $C$. Letting
\[ S(x_0) = \min_{x \in \Omega} S(x) \text{ and } I(x_1) = \min_{x \in \Omega} I(x). \]

Inserting them into the first two equations of (3.8), respectively, and with the help of the maximum principle (see e.g., [11]), yield
\[ \begin{cases}
\bar{a}(x_0)S(x_0) - \bar{b}(x_0)S^2(x_0) - \frac{\bar{\beta}(x_0)S(x_0)I(x_0)}{S(x_0) + I(x_0)} - \bar{\eta}(x_0)S(x_0) + \bar{\gamma}(x_0)I(x_0) \leq 0,
\bar{\beta}(x_1)S(x_1)I(x_1) - \bar{\eta}(x_1)S(x_1) + \bar{\gamma}(x_1)I(x_1) \leq 0.
\end{cases} \]  
(3.12)

Consequently, we have
\[ \bar{a}(x_0)S(x_0) + \bar{\gamma}(x_0)I(x_0) \leq \bar{b}(x_0)S^2(x_0) + \frac{\bar{\beta}(x_0)S(x_0)I(x_0)}{S(x_0) + I(x_0)} + \bar{\eta}(x_0)S(x_0) \]  
(3.13)
and
\[ \bar{\gamma}(x_1)I(x_1) \geq \bar{\eta}(x_1)S(x_1). \] (3.14)

By (3.13), we have
\[ \bar{a}(x_0)S(x_0) \leq \bar{b}(x_0)S^2(x_0) + \bar{\beta}(x_0)S(x_0) + \bar{\eta}(x_0)S(x_0), \]
which indicates that
\[ S(x) \geq S(x_0) \geq \frac{\bar{a}(x_0) - \bar{\beta}(x_0) - \bar{\eta}(x_0)}{b^*} \geq \min_{x \in \Omega} \{ \bar{a} - \bar{\beta} - \bar{\eta} \} > 0. \] (3.15)

By (3.14) and (3.15), we get
\[ I(x) \geq I(x_1) \geq \frac{\bar{\eta}(x_1)}{\bar{\gamma}(x_1)}S(x_1) \geq \frac{\bar{\eta}(x_1)}{\bar{\gamma}(x_1)}S(x_0) > 0. \] (3.16)

This proving (3.11).

Claim 2: Local and global stability of constant EE.

Recall that when \( \theta = 0 \), the unique positive constant steady state of (3.8) becomes
\[ \hat{S}_0 = \frac{a_0}{b_0}, \quad \hat{I}_0 = a_0 \left( \beta_0 + \eta_0 - \gamma_0 \right) + \frac{\sqrt{(\beta_0 + \eta_0 - \gamma_0)^2 + 4\gamma_0\eta_0}}{2\gamma_0} b_0. \]

Note that with \( (a_0, b_0, \beta_0, \eta_0, \gamma_0) \) replaced by \( (a, b, \beta, \eta, \gamma) \), the unique positive constant steady state of (3.8) can be obtained by elementary computation,
\[ \hat{S} = \frac{a}{b}, \quad \hat{I} = a \left( \beta + \eta - \gamma \right) + \frac{\sqrt{(\beta + \eta - \gamma)^2 + 4\eta}}{2\gamma} b. \]

The linearly stability of \( (\hat{S}, \hat{I}) \) directly follows from [18, Theorem 3.1]. Further we can determine global stability of the EE of (1.1) by constructing Lyapunov functional,
\[ L(t) := \int_\Omega \left[ \int (1 - \frac{\hat{S}^2}{S^2}) dS + \int (1 - \frac{\hat{I}^2}{I^2}) dI \right] dx. \]

We calculate the derivative of \( L(t) \) as follows,
\[ \dot{L}(t) := \int_\Omega \left[ \int (1 - \frac{\hat{S}^2}{S^2}) dS + \int (1 - \frac{\hat{I}^2}{I^2}) dI \right] dx, \]

By using equalities: \( a = b\hat{S} \) and \( \gamma = \frac{\beta\hat{S}}{S + I} + \frac{\eta\hat{S}}{I} \), we then calculate the derivative of \( L(t) \) as follows,
\[ \frac{dL(t)}{dt} = \int_\Omega \left[ \frac{S^2 - \hat{S}^2}{S^2} dS + \frac{I^2 - \hat{I}^2}{I^2} dI \right] dx + \int_\Omega \frac{S^2 - \hat{S}^2}{S^2} (aS - b\hat{S}^2) dx \]
\[ + \int_\Omega \left[ \frac{S^2 - \hat{S}^2}{S^2} \left( \frac{-\beta SI}{S + I} - \eta S + \gamma I \right) + \frac{I^2 - \hat{I}^2}{I^2} \left( \frac{\beta SI}{S + I} + \eta S - \gamma I \right) \right] dx \]
\[ = - \int_\Omega dS \left( \frac{2\hat{S}^2}{S^3} |\nabla S|^2 + dI \left( \frac{2\hat{I}^2}{I^3} |\nabla I|^2 \right) \right) + \int_\Omega \left( \frac{S + \hat{S})(S - \hat{S})}{S} b(\hat{S} - S) dx \right) \]
\[ + \int_\Omega \left( \frac{\beta SI}{S + I} + \eta S - \gamma I \right) \left[ \frac{I^2 - \hat{I}^2}{I^2} - \frac{S^2 - \hat{S}^2}{S^2} \right] dx \]
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\[ - \int_{\Omega} \left[ d_{S} \frac{2S^{2}}{S^{3}} |\nabla S|^{2} + d_{I} \frac{2I^{2}}{I^{3}} |\nabla I|^{2} \right] dx - \int_{\Omega} \eta SI \left( \frac{i}{I} - \frac{i}{S} \right)^{2} \left( \frac{\hat{I}}{I} + \frac{\hat{S}}{S} \right) dx \\
- \int_{\Omega} \frac{\beta SI}{(S + I)(S + I)} \left( \frac{\hat{I}}{I} - \frac{\hat{S}}{S} \right)^{2} \left( \frac{\hat{I}}{I} + \frac{\hat{S}}{S} \right) dx - \int_{\Omega} b(S + \hat{S})(\hat{S} - S)^{2} dx \leq 0, \forall t > 0. \]

As similar arguments as in [18, Theorem 3.2], the global stability of the constant EE of (1.1) directly follows.

Claim 3: Existence of positive solution to (3.8). Let

\[ \Theta = \{ (S, I) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \frac{1}{C} < S(x), I(x) < C, \forall x \in \Omega \}. \]

Noticing that (3.8) has no positive solution \((S, I) \in \partial \Theta. \) For \( \theta \in [0, 1], \) let us define the following operator

\[ A(\theta, (S, I)) = (-\Delta + I)^{-1} (F_{1}(\theta, (S, I)), F_{2}(\theta, (S, I))), \]

where

\[ F_{1}(\theta, (S, I)) = S + d_{S}^{-1} \left\{ \bar{a}S - \bar{b}S^{2} - \frac{\beta SI}{S + I} + \bar{\gamma}I - \bar{\eta}S \right\}, \]

and

\[ F_{2}(\theta, (S, I)) = I + d_{I}^{-1} \left\{ \frac{\beta SI}{S + I} - \bar{\gamma}I + \bar{\eta}S \right\}. \]

Here \((-\Delta + I)^{-1}\) is the inverse operator of \(-\Delta + I\) with Neumann boundary condition. It is easy to see that \( A \) is a compact operator defined on \([0, 1] \times \Theta, (S, I) \neq A(\theta, (S, I)), \forall \theta \in [0, 1] \) and \((S, I) \notin \partial \Theta. \) With these considerations, the topological degree \( \text{deg}(I - A(\cdot, \cdot)) \) is well-defined, which is also independent of \( \theta \in [0, 1]. \)

We next investigate the existence of fixed point of the operator \( A(1, \cdot) \) in \( \Theta \) which is equivalent to the existence of positive solutions to (3.8). From the well-known Leray-Schauder degree index formula (see, for example, [12, Theorem 2.8.1]), we have

\[ \text{deg}(I - A(0, \cdot), \Theta) = \text{index}(I - A(0, \cdot), (S_{*}, I_{*})) = 1. \]

Further, from the homotopy invariance of the Leray-Schauder degree, we can conclude that

\[ \text{deg}(I - A(1, \cdot), \Theta) = \text{deg}(I - A(0, \cdot), \Theta) = 1, \]
i.e., \( A(1, \cdot) \) admits at least one fixed point in \( \Theta, \) which in turn implies that (3.1) admits at least one positive solution.

4. Asymptotic profiles of the EE. With consideration of large or small diffusion rates of susceptible and infected individuals, this section is devoted to investigating the asymptotic behavior of the EE of (1.1). Before going into details, we first consider the following system,

\[
\begin{aligned}
&d\Delta u = a(x)u - b(x)u^2 + k, \quad x \in \Omega, \\
&\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(4.1)

The following result comes from [10, Lemma 5.1], we omit the proof here.
Lemma 4.1 ([10], Lemma 5.1). Let $d, k > 0$ be constants, and $a(\cdot), b(\cdot) \in C^2(\Omega)$. Then system (4.1) admits a unique positive solution, denoted by $u_d$. Furthermore, $u_d \to g(x)$ uniformly on $\bar{\Omega}$ as $d \to 0$, where $g(x)$ is the unique positive root of $h(\tau) := a(\tau) - b(\tau)\tau^2 + k = 0$.

4.1. The case of $d_S \to 0$.

Theorem 4.2. Assume that (3.9) holds. Fixing $d_I > 0$, up to a subsequence of $d_S \to 0$, then the positive solution $(S_{d_S}, I_{d_S})$ of (3.1) satisfies

$$(S_{d_S}, I_{d_S}) \to (S^+, I^+) \text{ uniformly on } \bar{\Omega},$$

where $S^+(x) = G(x, I^+(x))$ is the unique positive root of $f(\zeta) = 0$ with

$$f(\zeta) = -b(x)\zeta^3 + (a(x) - \theta(x) - b(x)I^+\zeta^2 + (a(x) - \beta(x) - \theta(x) + \gamma(x))I^+\zeta + \gamma(x)(I^+)^2,$$

and $I^+$ is a positive solution to

$${\begin{cases} -d_I \Delta I^+ + \gamma(x) I^+ = \frac{\beta(x)G(x, I^+(x))I^+}{G(x, I^+(x)) + I^+} + \eta(x)G(x, I^+(x)), & x \in \Omega, \\ \frac{\partial I^+}{\partial n} = 0, & x \in \partial \Omega. \end{cases}} \quad (4.2)$$

Proof. We prove this result by proving the following claims, step by step.

Claim 1: Upper bounds and lower bounds for any positive solution $(S, I)$ to (3.1).

From Lemma 3.1 and the $L^p$ estimate to the second equation of (3.1), we have

$$\|I\|_{W^{2,p}(\Omega)} \leq C.$$ 

Picking $p$ sufficiently large such that $p > n$, and using the Sobolev embedding theorem, yield

$$W^{2,p}(\Omega) \hookrightarrow C^{1+\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{p}.$$ 

Since $\Omega$ is a bounded domain, we have $\max_{x \in \Omega} I(x) \leq C$. Letting

$$S(x_2) = \max_{x \in \Omega} S(x).$$ 

From the first equation of (3.1), with the help of the maximum principle \[11\], we have

$$0 \leq a(x_2)S(x_2) - b(x_2)S^2(x_2) - \frac{\beta(x_2)S(x_2)I(x_2)}{S(x_2) + I(x_2)} - \eta(x_2)S(x_2) + \gamma(x_2)I(x_2),$$

which in turn implies that

$$\eta_s S(x_2) \leq \eta(x_2)S(x_2) \leq a(x_2)S(x_2) - b(x_2)S^2(x_2) + \gamma(x_2)I(x_2)$$

$$\leq a^* S(x_2) - b_* S^2(x_2) + \gamma^* I(x_2) \leq \frac{a^*}{4b_*} S(x_2) + \gamma^* I(x_2).$$

It follows that

$$S(x) \leq \max_{x \in \Omega} S(x) = S(x_2) \leq \frac{a^*}{4b_*} S(x_2) + \frac{\gamma^* \max_{x \in \Omega} I(x)}{\eta_s}, \quad \forall x \in \Omega.$$ 

Consequently, we have $\|I\|_{L^\infty(\Omega)} \leq C$ and $\|S\|_{L^\infty(\Omega)} \leq C$, for sufficiently large number $C$. 
By using the similar arguments as those in Theorem 3.2, with \((\bar{a}, \bar{b}, \bar{b}, \bar{b})\) replaced by \((a, b, \beta, \eta, \gamma)\) in (3.12)-(3.16), we can obtain that
\[
S(x), \quad I(x) > \frac{1}{C}, \quad \forall x \in \bar{\Omega}, \tag{4.3}
\]
for sufficiently large number \(C\).

**Claim 2:** Convergence of \(I\) when \(d_I > 0\) is fixed and \(d_S \to 0\). Noticing that \(I\) satisfies
\[
\begin{aligned}
-d_I \Delta I + \gamma(x)I &= \beta(x) \frac{SI}{S+I} + \eta(x)S, & x \in \Omega, \\
\frac{\partial I}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned} \tag{4.4}
\]
By Claim 1, we have
\[
\left\| \frac{\beta I}{S+I} S + \eta S \right\|_{L^p(\Omega)} \leq C, \quad \forall p > 1.
\]
With the help of the standard \(L^p\)-estimate for elliptic equations, we can obtain that
\[
\|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall p \geq 1.
\]
Further, by using the Sobolev embedding theorem, we have
\[
\|I\|_{C^{1+\alpha}(\Omega)} \leq C \quad \text{for} \quad \alpha = 1 - \frac{n}{p},
\]
where \(p\) is large enough \((p > n)\) such that \(0 < \alpha < 1\). It follows that there exists a subsequence of \(d_S \to 0\), say \(d_n := d_{S,n}\), satisfying \(d_n \to 0\) as \(n \to \infty\), and a corresponding positive solution \((S_n, I_n) = (S_{d_n}, I_{d_n})\) of (3.1), such that
\[
I_n \to I^+(x) \quad \text{uniformly on} \quad \bar{\Omega}, \quad \text{as} \quad n \to \infty, \tag{4.5}
\]
where \(I^+ \in C^1(\Omega)\) and \(I^+ \geq 0\). Hence, from (3.11),
\[
I_n \to I^+(x) > 0 \quad \text{uniformly on} \quad \bar{\Omega}, \quad \text{as} \quad n \to \infty. \tag{4.6}
\]
**Claim 3:** Convergence of \(S\) when \(d_I > 0\) is fixed and \(d_S \to 0\).

From the first equation of (3.1), \(S_n\) should satisfies the following equation
\[
\begin{aligned}
-d_n \Delta S_n &= a(x)S_n - b(x)S_n^2 - \beta(x) \frac{S_n I_n}{S_n + I_n} - \eta(x)S_n + \gamma(x)I_n, \\
\frac{\partial S_n}{\partial n} &= F_1(x, S_n, I_n), & x \in \Omega, \\
\frac{\partial S_n}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned} \tag{4.7}
\]
Since (4.6), we can choose small \(\epsilon_0 > 0\) such that for all large \(n\),
\[
0 < I^+ - \epsilon_0 \leq I_n \leq I^+ + \epsilon_0, \quad x \in \bar{\Omega}. \tag{4.8}
\]
It follows that
\[
F_1(x, S_n, I_n) \leq a(x)S_n - b(x)S_n^2 - \beta(x) \frac{S_n(I^+ - \epsilon_0)}{S_n + (I^+ - \epsilon_0)} - \eta(x)S_n + \gamma(x)(I^+ + \epsilon_0) = \frac{G_\epsilon(x, S_n(x))}{S_n + (I^+ - \epsilon_0)},
\]
with
\[
G_\epsilon(x, \zeta) = -b \zeta^3 - b(I^+ - \epsilon_0)\zeta^2 + (a - \eta)\zeta^2 + (a - \beta - \eta)(I^+ - \epsilon_0)\zeta + \gamma(I^+ + \epsilon_0)\zeta + \gamma(I^+ - \epsilon_0)(I^+ + \epsilon_0).
\]
We rewrite \(G_{\epsilon_0}(x, \zeta)\) as \(G_{\epsilon_0}(\zeta)\) for each fixed \(x \in \Omega\), and calculate the derivative of \(G_{\epsilon_0}(\zeta)\) with respect to \(\zeta > 0\) as follows:

\[
\begin{align*}
G'_{\epsilon_0}(\zeta) &= -3b\zeta^2 - 2b(I^+ + \epsilon_0)\zeta + 2(a - \eta)\zeta + (a - \beta - \eta)(I^+ - \epsilon_0) + \gamma(I^+ + \epsilon_0), \\
G''_{\epsilon_0}(\zeta) &= -6b\zeta - 2b(I^+ + \epsilon_0) + 2(a - \eta) + 6b.
\end{align*}
\]

It follows from \(G''_{\epsilon_0}(\zeta) < 0\) that \(G''_{\epsilon_0}(\zeta)\) is strictly decreasing. As a consequence, \(G''_{\epsilon_0}(\zeta)\) changes sign at most once for \(\zeta \in (0, \infty)\). We distinguish two cases.

**Case 1:** \(G''_{\epsilon_0}(\zeta)\) changes sign exactly once.

In this case, we must have \(G''_{\epsilon_0}(0) > 0\) as \(G''_{\epsilon_0}(\infty) = -\infty\). Thanks to (3.9), we can easily check that \(G'_{\epsilon_0}(0) > 0\) and \(G'_{\epsilon_0}(\infty) = -\infty\). It follows that the exists a positive value \(\zeta^*\) such that \(G_{\epsilon_0}(\zeta)\) is strictly increasing on \((0, \zeta^*)\) and strictly decreasing on \((\zeta^*, \infty)\) with a positive maximum value for \(\zeta \in (0, \infty)\). Further, we can check that \(G'_{\epsilon_0}(0) > 0\) and \(G_{\epsilon_0}(\infty) = -\infty\). We then conclude that \(G_{\epsilon_0}(\zeta) = 0\) has a unique positive root.

**Case 2:** \(G''_{\epsilon_0}(\zeta)\) does not change sign.

In this case, we have \(G''_{\epsilon_0}(\zeta) < 0\) for all \(\zeta \in (0, \infty)\). It follows that \(G'_{\epsilon_0}(\zeta)\) is strictly decreasing with respect to \(\zeta \in (0, \infty)\). Due to (3.9), we have that \(G'_{\epsilon_0}(0) > 0\) and \(G'_{\epsilon_0}(\infty) = -\infty\). It follows that there exists a positive value \(\zeta^{**}\) such that \(G'_{\epsilon_0}(\zeta) > 0\) on \((0, \zeta^{**})\) and \(G'_{\epsilon_0}(\zeta) < 0\) on \((\zeta^{**}, \infty)\), which changes sign exactly once on \(\zeta \in (0, \infty)\). It then follows from the fact \(G_{\epsilon_0}(0) > 0\) and \(G_{\epsilon_0}(\infty) = -\infty\), we can conclude that \(G_{\epsilon_0}(\zeta)\) is strictly increasing and then decreasing, and admits a positive maximum value at \(\zeta^{**}\).

Both of the cases indicate that \(G_{\epsilon_0}(\zeta) = 0\) admits a unique positive root, denoted by \(G_{\epsilon_0}(x, I^+)\). Without loss of generality, we denote \(G_{\epsilon_0}(\zeta) = (G_{\epsilon_0}(x, I^+) - \zeta)G^+_{\epsilon_0}(\zeta)\) with \(G^+_{\epsilon_0}(\zeta) > 0\) for all \(x \in \Omega\) and \(\zeta > 0\).

As to (4.7), we next construct the following auxiliary problem

\[
\begin{cases}
-d_0 \Delta W = \left(\frac{G_{\epsilon_0}(x, I^+) - W}{W + (I^+ - \epsilon_0)}\right) G^+_{\epsilon_0}(W), & x \in \Omega, \\
\frac{\partial W}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\tag{4.9}
\]

Noticing that \(S_\alpha\) is a lower solution of (4.9). Let \(M\) being a sufficiently large positive constant. Then \(M\) is an upper solution of (4.9). By the method of upper/lower solution, (4.9) has at least one solution \(\bar{W}_n\) satisfying \(S_\alpha \leq \bar{W}_n \leq M\) on \(\tilde{\Omega}\). Further from the maximum principle, any positive solution \(W_n\) of (4.9) satisfies

\[
\min_{x \in W} G_{\epsilon_0}(x, I^+) \leq \min_{x \in \tilde{\Omega}} \bar{W}_n \leq \max_{x \in \tilde{\Omega}} \bar{W}_n \leq \max_{x \in W} G_{\epsilon_0}(x, I^+).
\]

By using the results in Lemma 4.1, together with \(G_{\epsilon_0}(x, I^+) > 0\) and \(G^+_{\epsilon_0}(W) > 0\) on \(\tilde{\Omega}\), it follows that any positive solution \(\bar{W}_n\) of (4.9) satisfies the following estimate,

\[
W_n \rightarrow G_{\epsilon_0}(x, I^+) \text{ uniformly on } \tilde{\Omega}, \text{ as } n \rightarrow \infty,
\]

which in turn implies that

\[
\limsup_{n \rightarrow \infty} S_\alpha(x) \leq G_{\epsilon_0}(x, I^+) \text{ uniformly on } \tilde{\Omega}. \tag{4.10}
\]
In light of (4.8), given any small $\epsilon_0 > 0$, choosing sufficiently large $n$, we yield
\[
F_1(x, S_n, I_n) \geq a(x)S_n - b(x)S_n^2 - \beta(x)\frac{S_n(I^+ + \epsilon_0)}{S_n + (I^+ + \epsilon_0)} - \eta(x)S_n + \gamma(x)(I^+ - \epsilon_0)
\]

\[
= \frac{G^{\epsilon_0}(x, S_n(x))}{S_n + (I^+ + \epsilon_0)}.
\]

with
\[
G^{\epsilon_0}(x, \zeta) = - b\zeta^3 - b(I^+ + \epsilon_0)\zeta^2 + (a - \eta)\zeta^2 + (a - \beta - \eta)(I^+ + \epsilon_0)\zeta + \gamma(I^+ + \epsilon_0)(I^+ - \epsilon_0).
\]

Similar arguments as above, we have
\[
\liminf_{n \to \infty} S_n(x) \geq G^{\epsilon_0}(x, I^+) \text{ uniformly on } \bar{\Omega}.
\] (4.11)

where $G^{\epsilon_0}(x, I^+)$ is the unique positive root of $G^{\epsilon_0}(x, \zeta) = 0$. It follows from (4.10), (4.11) and
\[
\lim_{\epsilon_0 \to 0} G_{\epsilon_0}(x, I^+) = \lim_{\epsilon_0 \to 0} G^{\epsilon_0}(x, I^+) = G(x, I^+(x)),
\]

that
\[
S_n(x) \to G(x, I^+(x)) \text{ uniformly for } x \in \bar{\Omega}, \text{ as } n \to \infty.
\]

This completes the proof. \□

4.2. The case of $d_I \to 0$.

**Theorem 4.3.** Assume that (3.9) holds. Fixing $d_S > 0$ and let $d_I \to 0$. Then the positive solution of $(S, I)$ of (3.1) satisfies
\[
(S, I) \to (S_+(x), I_+(x)) \text{ uniformly on } \bar{\Omega},
\]
where
\[
I_+(x) = \frac{(\beta + \eta - \gamma) + \sqrt{(\beta + \eta - \gamma)^2 + 4\gamma \eta}}{2\gamma} S_+(x)
\] (4.12)

and $S_+(x)$ is the unique positive solution of
\[
\begin{cases}
-d_S \Delta S_+ = a(x)S_+ - b(x)S_+^2 - \beta(x)\frac{S_+I_+}{S_+ + I_+} - \eta(x)S_+ + \gamma(x)I_+, & x \in \Omega, \\
\frac{\partial S_+}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\] (4.13)

**Proof.** Recall that $S$ solves
\[
\begin{cases}
-d_S \Delta S = a(x)S - b(x)S^2 - \beta(x)\frac{SI}{S + I} - \eta(x)S + \gamma I, & x \in \Omega, \\
\frac{\partial S}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\] (4.14)

By Lemma 3.1, and with aid of the standard $L^p$-estimate for elliptic equations, we can obtain that
\[
\|S\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 < p < \infty.
\]

Further using the Sobolev embedding theorem, we have
\[
\|S\|_{C^{1,\alpha}(\Omega)} \leq C \quad \text{for} \quad \alpha = 1 - \frac{n}{p},
\]
where $p$ is large enough ($p > n$) such that $0 < \alpha < 1$. It follows that there exists a subsequence of $d_I \to 0$, say $d_I^n = d_{I,n} \to 0$ with $d_I^n \to 0$ as $n \to \infty$, the corresponding positive solution sequence $(S_n, I_n)$ of (3.1) with $d_I = d_I^n$ satisfies
\[
S_n \to S_+ \text{ uniformly on } \bar{\Omega}, \quad \text{as } n \to \infty,
\] (4.15)
where $S_+ \in C^1(\Omega)$ and $S_+ > 0$ on $\Omega$ due to (3.11). Then, given any small $\epsilon^* > 0$, choosing sufficiently large $n$, we yield

$$0 < S_+ - \epsilon^* \leq S_n \leq S_+ + \epsilon^*, \quad x \in \bar{\Omega}.$$ 

Note that $I_n$ satisfies

$$\begin{cases}
-d_t \Delta I_n = \frac{\beta(x)S_n I_n}{S_n + I_n} + \eta(x)S_n - \gamma(x)I_n, & x \in \Omega, \\
\frac{\partial I_n}{\partial n} = 0, & x \in \partial \Omega.
\end{cases} \tag{4.16}$$

It follows that

$$\begin{align*}
\frac{\beta(x)S_n I_n}{S_n + I_n} + \eta(x)S_n - \gamma(x)I_n \\
\leq \frac{\beta(x)(S_+ + \epsilon^*)I_n}{(S_+ + \epsilon^*) + I_n} + \eta(x)(S_+ + \epsilon^*) - \gamma(x)I_n \\
= -\gamma(x)I_n^2 + (\beta(x) + \eta(x) - \gamma(x))(S_+ + \epsilon^*)I_n + \eta(x)(S_+ + \epsilon^*)^2 \\
= \frac{(H^1_{+; \epsilon^*}(x, S_+(x)) - I_n)(I_n - H^1_{+; \epsilon^*}(x, S_+(x)))}{H^1_{+; \epsilon^*}(x, S_+(x)) - I_n} \gamma(x),
\end{align*}$$

where

$$H^1_{+; \epsilon^*}(x, S_+(x)) = \frac{1}{2\gamma} \left\{ (\beta + \eta - \gamma)(S_+ + \epsilon^*) \\
\pm \sqrt{(\beta + \eta - \gamma)^2(S_+ + \epsilon^*)^2 + 4\gamma(S_+ + \epsilon^*)^2} \right\},$$

and $H^1_{+; \epsilon^*}(x, S_+(x)) > 0$, $H^1_{-; \epsilon^*}(x, S_+(x)) < 0$. For given large $n$, we next consider the following auxiliary problem

$$\begin{cases}
-d_n \Delta \omega = \frac{(H^1_{+; \epsilon^*}(x, S_+(x)) - \omega)(\omega - H^1_{+; \epsilon^*}(x, S_+(x)))}{\omega + (S_+ + \epsilon^*)} \gamma(x), & x \in \Omega, \\
\frac{\partial \omega}{\partial n} = 0, & x \in \partial \Omega.
\end{cases} \tag{4.17}$$

It is easy to verify that $I_n$ is a lower solution of (4.17). Let $M$ being a sufficiently large positive constant. Then $M$ is an upper solution of (4.17). Thus by the method of upper/lower solution, (4.17) has at least one solution $\tilde{\omega}_n$ satisfying $I_n \leq \tilde{\omega}_n \leq M$ on $\bar{\Omega}$. Together with the fact that $H^1_{+; \epsilon^*}(x, S_+) > 0$ and $H^1_{-; \epsilon^*}(x, S_+) < 0$ on $\bar{\Omega}$, we know that any positive solution $\tilde{\omega}_n$ of (4.17) satisfies

$$\tilde{\omega}_n \to H^1_{+; \epsilon^*}(x, S_+(x)) \quad \text{uniformly on } \bar{\Omega}, \quad \text{as } n \to \infty,$$

it follows that

$$\limsup_{n \to \infty} I_n(x) \leq H^1_{+; \epsilon^*}(x, S_+(x)) \quad \text{uniformly on } \bar{\Omega}. \tag{4.18}$$

In light of (4.15), given any small $\epsilon^* > 0$, choosing sufficiently large $n$, we yield

$$\begin{align*}
\frac{\beta(x)S_n I_n}{S_n + I_n} + \eta(x)S_n - \gamma(x)I_n \\
\geq \frac{\beta(x)(S_+ - \epsilon^*)I_n}{(S_+ - \epsilon^*) + I_n} + \eta(x)(S_+ - \epsilon^*) - \gamma(x)I_n
\end{align*}$$
where
\[
H^2_{\pm}(x, I^{**}) = \frac{1}{2\gamma} \left\{ (\beta + \eta - \gamma)(S_+ - \epsilon^*) \\
\pm \sqrt{\beta + \eta - \gamma)^2(S_+ - \epsilon^*)^2 + 4\gamma(S_+ - \epsilon^*)^2} \right\},
\]
and \(H^2_{-}(x, S_+) > 0, H^2_{-}(x, S_+) < 0\). For given large \(n\), we next consider the following auxiliary problem
\[
\begin{cases}
-d_n\Delta \omega = \frac{(H^2_{+}(x, S_+) - \omega)(\omega - H^2_{-}(x, S_+))}{\omega + (S_+ - \epsilon^*)} \gamma(x), & x \in \Omega, \\
\frac{\partial \omega}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]
(4.19)

It is easy to verify that 0 is a lower solution of (4.19) and \(S_n\) is an upper solution of (4.19). Thus by the method of upper/lower solution, (4.19) admits at least one positive solution, denote by \(\tilde{\omega}_n\), such that \(0 \leq \tilde{\omega}_n \leq S_n\). Further, we have
\[
\liminf_{n \to \infty} I_n(x) \geq H^2_{+}(x, I^{**}(x)) \text{ uniformly on } \bar{\Omega}. \quad (4.20)
\]

By the arbitrariness of \(\epsilon^*\), we can obtain that
\[
\lim_{\epsilon^* \to 0} H^1_{+}(x, S_+) = \lim_{\epsilon^* \to 0} H^2_{+}(x, S_+) = I_+(x),
\]
where \(I_+(x)\) is given by (4.12). This, together with (4.18) and (4.20), we have
\[
I_n(x) \to I_+(x) \text{ uniformly for } x \in \bar{\Omega}, \text{ as } n \to \infty.
\]

Further, \(S_+\) satisfies (4.13). This completes the proof. \(\square\)

4.3. The case of \(d_S \to \infty\). We now determine the asymptotic behavior of positive solutions of (3.1) when \(d_S \to \infty\).

**Theorem 4.4.** Assume that (3.9) holds. Fixing \(d_I > 0\), up to a subsequence of \(d_S \to \infty\), then the positive solution \((S, I)\) of (3.1) satisfies
\[
(S, I) \to (S^\infty, I^\infty) \text{ uniformly on } \bar{\Omega},
\]
where \(S^\infty = \int_\Omega a(x)dx / \int_\Omega b(x)dx\) is a positive constant and \(I^\infty > 0\) on \(\Omega\), and \((S^\infty, I^\infty)\) solves
\[
\begin{cases}
\int_\Omega \left[ a(x)S^\infty - b(x)(S^\infty)^2 - \eta(x)S^\infty - \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \gamma(x)I^\infty \right] dx = 0, \\
d_I \Delta I^\infty = \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \eta(x)S^\infty - \gamma(x)I^\infty, \quad x \in \Omega, \\
\frac{\partial I^\infty}{\partial n} = 0, \quad x \in \partial \Omega.
\end{cases}
\]
(4.21)
Proof. Recall that the first equation of (3.1) satisfies

\[
\begin{aligned}
-\Delta S &= \frac{1}{d_S} \left[ a(x)S - b(x)S^2 - \beta(x) \frac{SI}{S+I} - \eta(x)S + \gamma I \right], & x \in \Omega, \\
\frac{\partial S}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned}
\] (4.22)

According to the standard $L^p$-estimate for elliptic equations, and Lemma 3.1, we can obtain that

\[\|S\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 < p < \infty.\]

Further, for sufficiently large $p$, the Sobolev embedding theorem ensures that

\[\|S\|_{C^{1+\alpha}(\Omega)} \leq C \quad \text{for} \quad \alpha = 1 - \frac{n}{p}, \quad \text{for some} \quad 0 < \alpha < 1.\]

It follows that there exists a subsequence of $d_S$, labeled by $d_S^2$ with $d_S^2 \to \infty$ as $n \to \infty$ such that the corresponding positive solution $(S_n, I_n) := (S_{d_S^2}, I_{d_S^2})$ of (3.1) for $d_S = d_S^2$ satisfies $S_n \to S^\infty$ in $C^1(\Omega)$ as $n \to \infty$, where $S^\infty > 0$ on $\Omega$ due to (3.11). On the other hand, $S^\infty$ satisfies

\[
\begin{aligned}
-\Delta S^\infty &= 0, & x \in \Omega, \\
\frac{\partial S^\infty}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned}
\] (4.23)

Consequently, $S^\infty > 0$ on $\Omega$ must be a positive constant. Integrating the second equation of (4.21) then adding it to the first equation of (4.21), we have $S^\infty = \int_\Omega a(x)dx / \int_\Omega b(x)dx$.

Similar arguments as those in Claim 2 of Theorem 4.2, it follows from the second equation of (3.1) that

\[I_n \to I^\infty \quad \text{in} \quad C^1(\Omega), \quad \text{as} \quad n \to \infty,\]

where the nonnegative function $I^\infty \in C^1(\Omega)$ satisfies

\[
\begin{aligned}
-\frac{d_I}{d_n} \Delta I^\infty &= \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \eta(x)S^\infty - \gamma(x)I^\infty, & x \in \Omega, \\
\frac{\partial I^\infty}{\partial n} &= 0, & x \in \partial \Omega.
\end{aligned}
\] (4.24)

Furthermore, due to (3.11), we can claim that $I^\infty > 0$ on $\bar{\Omega}$. Finally, from (4.22), it is easily checked that

\[
\int_{\Omega} \left[ a(x)S_n - b(x)S_n^2 - \beta(x) \frac{S_n I_n}{S_n + I_n} - \eta(x)S_n + \gamma I_n \right] dx = 0.
\]

Consequently, $(S^\infty, I^\infty) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ solves (4.21).

4.4. The case of $d_I \to \infty$. This subsection is devoted to the asymptotic behavior of positive solutions of (3.1) when $d_I \to \infty$.

**Theorem 4.5.** Assume that (3.9) holds. Fixing $d_S > 0$, up to a subsequence of $d_I \to \infty$, then the positive solution $(S, I)$ of (3.1), satisfies

\[(S, I) \to (S^\infty, I^\infty) \quad \text{uniformly on} \quad \bar{\Omega},\]
where $I_\infty$ is a positive constant and $S_\infty > 0$ on $\bar{\Omega}$, and $(S_\infty, I_\infty)$ solves

$$\begin{cases}
-d_S \Delta S_\infty = a(x)S_\infty - b(x)S_\infty^2 - \eta(x)S_\infty - \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \gamma(x)I_\infty, & x \in \Omega, \\
\partial S_\infty / \partial n = 0, & x \in \partial \Omega, \\
\int_{\Omega} \left[ \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \eta(x)S_\infty - \gamma(x)I_\infty \right] dx = 0.
\end{cases}$$

(4.25)

**Proof.** Recall that the second equation of (3.1) satisfies

$$\begin{cases}
-\Delta I = \frac{1}{d_I} \left[ \frac{\beta(x)SI}{S + I} + \eta(x)S - \gamma(x)I \right], & x \in \Omega, \\
\partial I / \partial n = 0, & x \in \partial \Omega.
\end{cases}$$

(4.26)

The standard $L^p$-estimate for elliptic equations and Lemma 3.1 determine that

$$\|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 < p < \infty.$$  

Further, for sufficiently large $p$, the Sobolev embedding theorem ensures that

$$\|I\|_{C^{1+\alpha}(\Omega)} \leq C \text{ for } \alpha = 1 - \frac{n}{p},$$

for some $0 < \alpha < 1$.

It follows that there exists a subsequence of $d_S$, labeled by $d_n^3$ with $d_n^3 \to \infty$ as $n \to \infty$ such that the corresponding positive solution $(S_n, I_n) := (S_{d_n^3}, I_{d_n^3})$ of (3.1) for $d_S = d_n^3$ satisfies $I_n \to I_\infty$ in $C^1(\bar{\Omega})$ as $n \to \infty$, where $I_\infty > 0$ on $\bar{\Omega}$ due to (3.11). On the other hand, $I_\infty$ satisfies

$$\begin{cases}
-\Delta I_\infty = 0, & x \in \Omega, \\
\partial I_\infty / \partial n = 0, & x \in \partial \Omega.
\end{cases}$$

(4.27)

Consequently, $I_\infty > 0$ on $\bar{\Omega}$ must be a positive constant. With the help of Theorem 4.3, it follows from the first equation of (3.1) that

$$S_n \to S_\infty \text{ in } C^1(\bar{\Omega}), \quad \text{as } n \to \infty,$$

where the nonnegative function $S_\infty \in C^1(\bar{\Omega})$ satisfies

$$\begin{cases}
-d_S \Delta S_\infty = a(x)S_\infty - b(x)S_\infty^2 - \eta(x)S_\infty - \frac{\beta(x)S_\infty I_\infty}{S_\infty + I_\infty} + \gamma(x)I_\infty, & x \in \Omega, \\
\partial S_\infty / \partial n = 0, & x \in \partial \Omega.
\end{cases}$$

(4.28)

Furthermore, due to (3.11), we can claim that $S_\infty > 0$ on $\bar{\Omega}$. Finally, from (4.26), $(S_\infty, I_\infty)$ satisfies

$$\int_{\Omega} \left[ \frac{\beta(x)S_n I_n}{S_n + I_n} + \eta(x)S_n - \gamma(x)I_n \right] dx = 0,$$

this proves (4.25).

**Acknowledgments.** We thank the anonymous referee for valuable comments which have led to a substantial improvement in the revision.
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Received September 2019; revised November 2019.

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