Convergence of the Formal Expansion for $\lambda_d(p)$
of the Monomer-Dimer Problem for Small $p$

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February 7, 2011

Abstract

Shmuel Friedland and the author recently presented a formal expansion for $\lambda_d(p)$ of the monomer-dimer problem. Herein we prove that if the terms in the expansion are rearranged as a power series in $p$, then for sufficiently small $p$ this series converges.

In a series of papers the author presented a formal asymptotic expansion for $\lambda_d$ of the dimer problem, in inverse powers of $d$. See [1]. The expansion is as follows

$$\lambda_d \sim \frac{1}{2} \ln(2d) - \frac{1}{2} + \sum_{k=1} c_k \frac{d}{d^k}$$ (1)

computed through the $k = 3$ term as

$$\lambda_d \sim \frac{1}{2} \ln(2d) - \frac{1}{2} + \frac{1}{8d} + \frac{5}{96d^2} + \frac{5}{64d^3}.$$ (2)

In a recent paper, [2], Shmuel Friedland and the author extended this work to yield a formal asymptotic expansion for $\lambda_d(p)$ of the dimer-monomer problem

$$\lambda_d(p) \sim \frac{1}{2} (p \ln(2d) - p \ln p - 2(1-p) \ln(1-p) - p) + \sum_{k=1} c_k(p) \frac{d}{d^k}$$ (3)

computed through the $k = 3$ term as

$$\lambda_d(p) \sim \frac{1}{2} (p \ln(2d) - p \ln p - 2(1-p) \ln(1-p) - p) + \frac{1}{8d} + \frac{(2p^3 + 3p^4)}{96} \frac{1}{d^2} + \frac{(-5p^4 + 12p^5 + 8p^6)}{192} \frac{1}{d^3}.$$ (4)
For given \( d \) we rearrange the expansion in (3) as a power series in \( p \)

\[
\lambda_d(p) \sim \frac{1}{2} (p \ln(2d) - p \ln p - 2(1 - p) \ln(1 - p) - p) + \sum_{k=2} a_k(d) p^k. \tag{5}
\]

We see from (4) that

\[
a_2(d) = \frac{1}{8d} \quad \tag{6}
a_3(d) = \frac{1}{48 d^2} \quad \tag{7}
a_4(d) = \frac{1}{32 d^2} - \frac{5}{192 d^3}. \quad \tag{8}
\]

We have here used the fact that \( c_k(p) \) of equation (3) is a sum of powers \( p^s \) where \( k < s \leq 2k \), see Lemma 4 and Theorem 2, and thereby getting these values from equation (4). Yi bu zuo er bu xiu, moreover using the fact that we know first six \( \bar{J}_i \) in the development below we can actually calculate two further values

\[
a_5(d) = \frac{1}{16 d^3} - \frac{39}{640 d^4} \quad \tag{9}
a_6(d) = \frac{1}{24 d^3} - \frac{1}{32 d^4} - \frac{19}{1920 d^5}. \quad \tag{10}
\]

It is the primary goal of this paper to show that if \( p \) is small enough (0 \( \leq p < p_0 \), \( p_0 \) independent of \( d \)) the sum in (5) converges, see Theorem 4 below. (Throughout the paper we are not careful about getting the best value of \( p_0 \); with any improvements we could make to the current procedure the value we get for \( p_0 \) would still be anemic.)

We will assume familiarity with Section 5 of [2], and use many of the formulae therefrom. \( \lambda_d(p) \) is determined, by a complicated computation, from the infinite sequence of cluster expansion kernels

\[
\bar{J}_1, \bar{J}_2, \bar{J}_3, \ldots \quad \tag{11}
\]

defined in equations (5.21), (5.23). (We will not indicate herein that such (5.–) equation comes from [2].) The first six \( \bar{J}_i \) have been computed and are listed in (5.25) – (5.30). From (5.17) and (5.31) an infinite sequence of auxiliary quantities

\[
\alpha_1, \alpha_2, \ldots \quad \tag{12}
\]

are computed from the \( \bar{J}_i \). An easy computation from (5.17) and (5.31) leads to the nice expression

\[
\alpha_k = (\bar{J}_k p^k) \cdot \frac{1}{(1 - 2 \sum i \alpha_i) 2^k} \cdot \left( 1 - 2 \sum i \alpha_i / p \right)^k. \tag{13}
\]
which replaces (5.31).

We view the $\alpha_k$ as determined from (13) by recursive iteration. Later working with bounds on the $J_k$ we will study values of $p$ for which iterations converge to a solution of (13).

From (5.10), (5.11), and (5.12) we have that

$$\lambda_d(p) = S + \lim_{N \to \infty} \frac{1}{N} \ln Z^*$$

(14)

where we have defined

$$S = \frac{p}{2} \ln(2d) - \frac{p}{2} \ln p - (1 - p) \ln(1 - p) - \frac{p}{2}.$$  (15)

Now from (5.32), (5.31), and (5.17) we may easily compute

$$\lambda_d(p) = S + \sum \alpha_i - \sum_{k=2}^{1} \frac{1}{k} \left( 2 \sum_i i \alpha_i \right)^k + \frac{1}{2}p \sum_{k=2}^{1} \frac{1}{k} \left( 2 \sum_i i \alpha_i / p \right)^k.$$  (16)

Equations (13), (15), and (16) are our master equations. All our results below concern solutions of these equations, we do not address here whether such solutions actually correspond to a computation of the monomer-dimer partition function as

$$\sum \text{covers} \sim e^{N \lambda_d(p)}$$  (17)

although certainly this is the case.

We state the information in equation (5.22) as a lemma.

**Lemma 1.** $J_k$ is a sum of inverse powers of $d$, $(1/d)^s$, with

$$\frac{k}{2} \leq s < k$$

(18)

**Lemma 2.** At the first iteration of equation (13) $\alpha_k$ is a sum of powers of $p$ and $(1/d)$, $p^i (1/d)^j$, with

$$i = k$$

$$\frac{i}{2} \leq j < i$$

(19)

**Lemma 3.** At the end of any number of iterations of equation (13) $\alpha_k$ is a sum of terms $p^i (1/d)^j$ with

$$i \geq k$$

$$\frac{i}{2} \leq j < i$$

(20)

**Lemma 4.** Substituting the $\alpha_k$ as satisfying (20) into (16) one finds $\lambda_d(p) - S$ is a sum of terms $p^i (1/d)^j$ satisfying (20).
These lemmas are easily proven by studying the evolution of powers of \( p \) and \( (1/d) \) through the iterations and expansions.

One may consider the formal expansion of \( \alpha_k \) after an infinite number of iterations of (13), and its substitution into (16), yielding an infinite formal expansion for \( \lambda_d(p) - S \). These also are a sum of terms \( p^i (1/d)^j \) satisfying (20).

We reorganize our formal expansions as a power series in \( p \).

\[
\alpha_k = \sum_{s=k} p^s f_{k,s} \tag{21}
\]

\[
\lambda_d(p) = S + \sum_{s=2} p^s g_s \tag{22}
\]

The \( f_{k,s} \) and \( g_s \) are built up of powers of \( (1/d), (1/d)^i \) satisfying

\[
\frac{s}{2} \leq i < s \tag{23}
\]

We now consider working with a fixed value of \( d \), and assume we have a bound on the \( J_k \)

\[
|J_k| \leq B^k, \quad k = 1, 2, \ldots \tag{24}
\]

for some \( B \). Under these circumstances we set up the machinery to use the contraction mapping principle. On any formal infinite polynomial in \( p \)

\[
f = \sum a_ip^i \tag{25}
\]

we define a norm \( |f| \)

\[
|f| \equiv \sum |a_i p^i| \tag{26}
\]

This norm has the properties

\[
P1) \quad |cf| = |c||f| \tag{27}
\]

\[
P2) \quad |f + g| \leq |f| + |g| \tag{28}
\]

\[
P3) \quad |fg| \leq |f||g| \tag{29}
\]

for scalar \( c \) and polynomials \( f \) and \( g \).

We denote the sequence of \( \alpha_k \), as in (12), by \( \alpha \), and define a norm on \( \alpha \)

\[
\|\alpha\| = \sum_{k} 2^k |\alpha_k| \tag{30}
\]

We find an \( \varepsilon, 0 < \varepsilon < 1/2 \), small enough so that

\[
\frac{1}{2} \frac{1}{(1 - 2\varepsilon)^2} (1 + 2\varepsilon) \leq 1 \tag{31}
\]
and
\[ \frac{6\varepsilon}{1 - 2\varepsilon} \leq 1. \] (32)

We then require \( p > 0 \) to be small enough that
\[ p^{k-1}B^k \leq \varepsilon \frac{1}{8^k}, \quad k = 2, 3, \ldots \] (33)

Working with this choice of \( \varepsilon \) and \( p \) we define the complete metric space \( S \) on which we establish a contraction mapping
\[ S = \{ \alpha = \{ \alpha_k \} | \| \alpha \| \leq p\varepsilon \}. \] (34)

We rewrite (13) as
\[ \alpha_k = f_k(\alpha), \quad k = 2, 3, \ldots \] (35)
or
\[ \alpha = f(\alpha). \] (36)

Conditions (31) and (33) ensure that \( f \) carries \( S \) into \( S \). With the further condition (32) one establishes that \( f \) is a contraction.

**Theorem 1.** With the conditions on \( p \) and \( \varepsilon \) above, there is a unique solution of (36) in \( S \), exactly the one obtained by iteration of (13).

Substituting this solution into (16) one obtains the expression for \( \lambda_d(p) \). We collect the properties of this quantity.

**Theorem 2.** For \( 0 < p \leq p_0 \), \( p_0 \) determined by (33),
\[ \lambda_d(p) = \frac{p}{2} \ln(2d) - \frac{p}{2} \ln p - (1 - p) \ln(1 - p) - \frac{p}{2} + \sum_{s=2}^{s} p^s g_s \] (37)

where \( g_s \) is a polynomial in \((1/d)^i\) with powers \((1/d)^i\) satisfying
\[ \frac{s}{2} \leq i < s. \] (38)

The sum in (37) is absolutely convergent, \( g_s \) is a polynomial in \( \bar{J}_1, \bar{J}_2, \ldots, \bar{J}_s \), and is determined by a finite number of iterations of (13) substituted into (16). One need only keep the finite number of terms throughout whose power of \( p \) is less than or equal to \( s \) to get \( g_s \).

We content ourselves with presenting the proof that the \( f \) of (36) maps \( S \) into \( S \). We look at the mapping of (35) carrying \( \alpha_k \) into \( \alpha'_k \)
\[ \alpha'_k = f_k(\alpha) \] (39)
and we wish to prove if $\alpha$ is in $S$ then $\alpha'$ is in $S$. Parallel to (13) we have

$$\alpha' = (\bar{J}_k p^k) \cdot \frac{1}{(1 - 2 \sum i\alpha_i)^k} \left(1 - 2 \sum i\alpha_i/p\right)^k. \quad (40)$$

We take the $|\cdot|$ norm of both sides using $P1, P2, P3$ of (27)–(29).

By (33), (24), and (30),

$$|\alpha'_k| \leq p\varepsilon \frac{1}{8k} \left(\frac{1}{1 - 2 \sum i|\alpha_i|}\right)^{2k} \left(1 + 2 \sum i|\alpha_i|/p\right)^k \quad (41)$$

$$\leq p\varepsilon \frac{1}{2k} \left(\frac{1}{(1 - 2||\alpha||)^2}\right) \left(1 + 2||\alpha||/p\right)^k \quad (42)$$

and since $\alpha \in S$

$$\leq p\varepsilon \frac{1}{2k} \left(\frac{1}{(1 - 2\varepsilon p)^2}\right) \left(1 + 2\varepsilon\right)^k \quad (43)$$

using (31)

$$\leq p\varepsilon \frac{1}{2k} \frac{1}{2^k} \quad (44)$$

Or

$$2^k |\alpha'_k| \leq p\varepsilon \frac{1}{2^k} \quad (45)$$

so that

$$||\alpha'|| = \sum 2^k|\alpha'_k| \leq p\varepsilon \sum 2^k \frac{1}{2^k} \leq p\varepsilon \frac{1}{2} \quad (46)$$

and thus $\alpha' \in S$ as was to be proved.

**Theorem 3.** There is a value of $B_0$ that ensures

$$|\bar{J}_n| \leq B_0^n, \quad n = 1, 2, \ldots$$

for all values of $d$.

**Theorem 4.** There is a value $p_0$ (independent of $d$) such that for $0 \leq p < p_0$ the series for $\lambda_d(p)$ in (5) converges.

Theorem 4 follows from Theorem 3 by the development above.

We turn to Theorem 3. In fact we will see $B_0 = 4\varepsilon$ works. We could follow the general cluster expansion formalism as given in [3] and [4]. However in this case it is more elementary to work from the ideas in [5], and especially the appendix to [5], due to David Brydges.
Now we require the reader to have some familiarity both with [5] and either [1] or Section 5 of [2]. Fortunately these are all rather short.

We consider an elegant generalization of the setup in [5]. We replace the configuration space of a single particle, $\mathbb{R}^3$, with individual configurations, points $x \in \mathbb{R}^3$, by the space of two element subsets of $\mathbb{Z}^3$, with individual elements $\{i, j\}$, subsets of $\mathbb{Z}^3$. The sum over one dimensional configurations, is changed from

$$\int dx$$

to

$$\sum_{\{i, j\}} v(i, j)$$

where $v$ is as in (5.6) of [2] or (10) of [1]. Thus we are using the $v$’s to weight the points of the new configuration space. Of the potentials in [5] we keep only $V_r$, given in the Appendix of [5], in eq (A1). It is constructed from $v_r$ a two-body potential as follows

$$v_r(\{i, j\}, \{k, l\}) = \begin{cases} 0 & \{i, j\} \cap \{k, l\} = \emptyset \\ +\infty & \text{otherwise} \end{cases}.$$  

(47)

Then $u(\{i, j\}, \{k, l\})$ as defined in (A2) of [5] becomes

$$u(\{i, j\}, \{k, l\}) = \begin{cases} 0 & \{i, j\} \cap \{k, l\} = \emptyset \\ -1 & \text{otherwise} \end{cases}.$$  

(48)

A natural generalization of (6) of [5] is given by

$$\|u\| = \sup_{\{k, l\}} \left( \sum_{\{i, j\}} |v(k, l)| |u(\{i, j\}, \{k, l\})| \right).$$  

(49)

It is easy to see from the definition of $v(i, j)$ that

$$\|u\| \leq 4$$  

(50)

since

$$\sum_{j} |v(i, j)| \leq 2.$$  

(51)

The generalization of (56) of [5] easily leads to

$$|\tilde{J}_n| \leq c^n 4^n.$$  

(52)

For $d = 1$ the expansion in [6] holds for all $0 \leq p \leq 1$, as was noted at the end of [2]. We may expect this is true for all $d$? The methods of the current paper do not get near this result. But the result we have encourages research to address this question. For that matter is $\lambda_d(p)$ analytic in both $p$ and $1/d$ for $|1/d| < 1$, $|p| < 1$? Or on the other hand perhaps the result of this paper is essentially the best one can do!
References

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