FINITE MORPHIC $p$-GROUPS

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Abstract. According to Li, Nicholson and Zan, a group $G$ is said to be morphic if, for every pair $N_1, N_2$ of normal subgroups, each of the conditions $G/N_1 \cong N_2$ and $G/N_2 \cong N_1$ implies the other. Finite, homocyclic $p$-groups are morphic, and so is the nonabelian group of order $p^3$ and exponent $p$, for $p$ an odd prime. It follows from results of An, Ding and Zhan on self dual groups that these are the only examples of finite, morphic $p$-groups. In this paper we obtain the same result under a weaker hypothesis.

1. Introduction

A module $M$ over a ring is said to be morphic if whenever a submodule $N$ is a homomorphic image of $M$, so that there is an epimorphism $\varphi : M \to N$, then $M/N \cong \ker(\varphi)$. In other words, if $N_1$ and $N_2$ are submodules of $M$, then $M/N_1 \cong N_2$ if and only if $M/N_2 \cong N_1$. This condition, introduced by G. Ehrlich in 1976 [Ehr76], has been investigated by W.K. Nicholson and E. Sánchez Campos in [NSC04, NSC05], and by J. Chen, Y. Li and Y. Zhou in [CLZ06]. An extension to groups of results of Ehrlich was obtained by Li and Nicholson in [LN10].

The following analog for groups of the definition of Ehrlich was given by Li, Nicholson and L. Zan in [LNZ10].

Definition 1.1. A group $G$ is said to be morphic if, whenever $N$ is a normal subgroup of $G$, such that there is an epimorphism $\varphi : G \to N$, then $G/N \cong \ker(\varphi)$. In other words, $G$ is morphic if for every pair $N_1$ and $N_2$ of normal subgroups of $G$, each of the conditions $G/N_1 \cong N_2$ and $G/N_2 \cong N_1$ implies the other.

In [LNZ10] finite, nilpotent groups were considered, and it was shown that such a group is morphic if and only if each of its Sylow subgroups is morphic. This leads to the study of finite, morphic $p$-groups.

Clearly finite, homocyclic $p$-groups are morphic, and so is the nonabelian group of order $p^3$ and exponent $p$, for $p$ an odd prime. F. Aliniaeifard, Li and Nicholson conjectured that these are the only examples:

\begin{itemize}
  \item The first author gratefully acknowledges the support of the Department of Mathematics, University of Trento.
  \item The second author gratefully acknowledges the support of the Department of Mathematics, University of L’Aquila.
  \item The authors are members of GNSAGA—INdAM.
\end{itemize}
Conjecture 1.2 ([ALN13, Conjecture 3.5]). The only finite, morphic $p$-groups are the abelian, homocyclic $p$-groups, and the nonabelian group of order $p^3$ and exponent $p$, for $p$ an odd prime.

They were able to prove their conjecture for two-generated groups:

Theorem 1.3 ([ALN13, Theorem 2.10]). Let $G$ be a finite, morphic $p$-group.
If $G$ can be generated by two elements, then
1. either $G$ is abelian and homocyclic,
2. or $p$ is odd, and $G$ is isomorphic to the nonabelian group of order $p^3$ and exponent $p$.

Aliniaeifard, Li and Nicholson also proved that a finite, morphic $p$-group satisfies the following property:

Theorem 1.4 ([ALN13, Proposition 2.1]). Let $G$ be a finite, morphic $p$-group.
1. Every subgroup of $G$ is a homomorphic image of $G$.
2. Every homomorphic image of $G$ is isomorphic to a normal subgroup of $G$.

The latter result shows that a finite, morphic $p$-group belongs to the class of self dual groups, as introduced by A. E. Spencer [Spe72], and studied more recently by L. An, J. Ding and Q. Zhang [ADZ11]. A group is said to be self dual if the isomorphism classes of its subgroups and of its quotient groups coincide.

Now An, Ding and Zhang prove the following

Theorem 1.5 ([ADZ11, Corollary 7.2]). Let $G$ be a finite, self dual $p$-group. Then
1. either $G$ is abelian, or
2. $p$ is odd, and $G$ is the direct product of the nonabelian group of order $p^3$ and exponent $p$, by an elementary abelian group.

We observe here that a proof of Conjecture 1.2 can be rather straightforwardly obtained as a corollary of this result.

In this paper we study a class of finite $p$-groups that properly includes that of morphic $p$-groups, and is not contained in the class of self dual $p$-groups:

Definition 1.6. A finite $p$-group $G$ is said to be elementary abelian morphic, or ea-morphic for short, if, whenever $N$ is a normal subgroup of $G$, such that either $N$ is elementary abelian, or $G/N$ is elementary abelian, then there is an epimorphism $\varphi : G \rightarrow N$, and $G/N \cong \ker(\varphi)$.

Our main result is:

Theorem 1.7. Let $G$ be a finite, nonabelian ea-morphic $p$-group. Then $G$ is 2-generated.

Since finite, morphic $p$-groups are ea-morphic by Theorem 1.4, this result, together with Theorem 1.3, yields another proof of Conjecture 1.2.

In Section 2 we introduce a linear algebra setting, and use a counting argument to obtain a crucial estimate for the elementary abelian quotients of the derived subgroup. This estimate is closely related to the one in [ADZ11, Lemma 5.3],
but our treatment is self contained, and elementary, as it avoids the appeal made in [ADZ11 Theorem 5.2] to a substantial result in the text of N. Blackburn and B. Huppert [HB82 Theorem 9.8]. In Section 3 we complete the proof of Theorem 1.7 by comparing two series of normal subgroups on the top and on the bottom of the group.

Our notation is mainly standard. If $H$ is a subgroup of the group $K$, or $H$ is a subspace of the vector space $K$, we write $H \triangleleft K$ to indicate that $H$ is maximal in $K$.

We write $\mathbf{F}_p$ for the field with $p$ elements, $p$ a prime.

2. Morphic triples

We record the following immediate consequence of the second isomorphism theorem, which we will use repeatedly.

**Lemma 2.1.** If $N$ is a normal subgroup of the group $G$, then

$$(G/N)' \cong \frac{G'}{G' \cap N}.$$ 

Let $G \neq \{1\}$ be a finite, ea-morphic $p$-group, with minimal number of generators $d$, that is, $|G/\Phi(G)| = p^d$.

Let $N$ be a normal subgroup of $G$ of order $p$. Then for every maximal subgroup $M$ of $G$ one has $G/M \cong N$, and thus, according to Definition 1.6, also $G/N \cong M$. In particular we obtain

**Lemma 2.2** ([LNZ10 Theorem 37 (1)]). In a finite, morphic $p$-group, all maximal subgroups are isomorphic.

From now on, assume $G$ to be nonabelian, and take $N \leq G'$. Since $M \cong G/N$, Lemma 2.1 yields

$$|M'| = |G'/N| = \frac{|G'|}{p},$$

so that

$$|G'/M'| = p.$$ 

Consider the characteristic subgroup $K$ of $G$ defined by

$$K = \bigcap \{ M': M \triangleleft G \}.$$ 

$G'/K$ is isomorphic to a subgroup of $\prod_{M \leq G} G'/M'$, and thus it is elementary abelian.

Consider the map

$$\beta : G/\Phi(G) \times G/\Phi(G) \to G'/K$$

$$(a\Phi(G), b\Phi(G)) \mapsto [a, b]K.$$ 

$\beta$ is well defined, as we have

**Lemma 2.3.** $[G, \Phi(G)] \leq K$. In particular, $\Phi(G)' \leq K$. 

Proof. \((G/M')' = G'/M'\) has order \(p\). This yields first \([G, G'] \leq M'\) for all \(M \ll G\), so that \([G, G'] \leq K\). Also, if \(a, b \in G\), then for all \(M \ll G\) we have
\[
[a, b]^p \equiv [a, b]^p \equiv 1 \pmod{M'},
\]
as we have just seen that \([b, [a, b]] \in M'\). Therefore also \([G, G^p] \leq K\). \(\square\)

This also yields that \(\beta\) is bilinear, as
\[
[ab, c] = [a, c] \cdot [[a, c], b] \cdot [b, c] \equiv [a, c] \cdot [b, c] \pmod{K}.
\]
The \(\mathbb{F}_p\)-vector spaces \(V = G/\Phi(G)\) and \(W = G'/K\), with the map \(\beta\), thus satisfy the following definition.

**Definition 2.4.** Let \(V, W\) be vector spaces over \(\mathbb{F}_p\), and
\[
\beta : V \times V \to W
\]
\[(v_1, v_2) \mapsto [v_1, v_2]\]
be an alternating bilinear map. If \(U_1, U_2\) are subspaces of \(V\), write \([U_1, U_2]\) for the linear span of \(\beta(U_1, U_2)\), and shorten \([U_1, U_1]\) to \(U_1'\).

\((V, W, \beta)\) is said to be a **morphic triple** if the following conditions hold.

1. \(V' = W\).
2. For every \(U \ll V\) one has \(U' \ll W\).
3. \(\bigcap \{U' : U \ll V\} = \{0\}\).

Considering morphic triples alone is not sufficient to prove Theorem 1.7, as one can construct examples of morphic triples that are not associated to finite, morphic \(p\)-groups.

**Proposition 2.5.** Let \((V, W, \beta)\) be a morphic triple.

Let \(U \ll V\). Then there exist a unique \(T = T(U) \ll U\) which satisfies the following property.

For \(S \ll V\), the following are equivalent.

1. \(U' = S'\), and
2. \(S \geq T\).

**Proof.** Let \(a \in V \setminus U\). Consider the linear map \(\tau\)
\[
U \to W \to W/U'
\]
given by \(x \mapsto [a, x] + U'\). Since \(W = V' = \langle a, U \rangle' = [a, U] + U'\), this is a surjective linear map, with \(\dim(W/U') = 1\). Thus
\[
T = \mathcal{T}(U) = \ker(\tau) = \{x \in U : [a, x] \in U'\}
\]
is a maximal subspace of \(U\). By definition, \([a, T] \leq U'\).

\(T\) is easily seen to be independent of the choice of \(a \in V \setminus U\). Thus if \(T \ll S \ll V\), and \(S \neq U\), we may assume \(S = \langle a, T \rangle\). It follows that
\[
S' = [a, T] + T' \leq U' + T' \leq U'
\]
and thus \(S' = U'\), as they are both maximal subspaces of \(W\).
Suppose conversely that $U \neq S \ll V$, and $S' = U'$. Thus $S \cap U \ll U$, and if $S = \langle a, S \cap U \rangle$, then $a \notin U$, and we have $[a, S \cap U] \leq S' = U'$, so that $S \cap U = T(U)$. □

**Corollary 2.6.** For each $U \ll V$, the set

\[ \{ S \ll V : S' = U' \} \]

has $p + 1$ elements, namely the subspaces $S$ such that $T(U) \ll S \ll V$.

The set $\mathcal{M}$ of maximal subspaces of $W$ has

\[ 1 + p + \cdots + p^{e-1} \]

elements, where $e = \dim(W)$. Corollary 2.6 implies that the subset

\[ Z = \{ Z \ll W : Z = U' \text{ for some } U \ll V \} \]

of $\mathcal{M}$ has $(1 + p + \cdots + p^{d-1})/(1 + p)$ elements. Thus $d$ is even, and

\[ \frac{1 + p + \cdots + p^{d-1}}{1 + p} = 1 + p^2 + \cdots + p^{d-2}. \]

Clearly if $e < d - 1$ we have

\[ |\mathcal{M}| = 1 + p + \cdots + p^{e-1} < 1 + p^2 + \cdots + p^{d-2} = |Z|, \]

a contradiction. We have obtained

**Corollary 2.7.** In a morphic triple $(V, W, \beta)$, we have

\[ \dim(W) \geq \dim(V) - 1. \]

In particular, in a finite, nonabelian, ea-morphic $p$-group $G$ with minimal number of generators $d$ we have

\[ |G'/K| \geq p^{d-1}. \]

3. **Proofs**

We are now ready to prove Theorem 1.7.

Let $G$ be a finite, nonabelian, ea-morphic group, with minimum number of generators $d > 2$. We want to derive a contradiction.

By Definition 1.6, there is an epimorphism $G \to \Phi(G)$, and if $E$ is its kernel, so that $G/E \cong \Phi(G)$, we also have $G/\Phi(G) \cong E$. Since $|G/\Phi(G)| = p^d$, we have that $E$ is an elementary abelian, normal subgroup of $G$, of order $p^d$. Let

\[ p^t = |E \cap G'|, \]

so that $t \leq d$. Since $G/E \cong \Phi(G)$, Lemma 2.1 yields

\[ |\Phi(G)'| = \frac{|G'|}{|E \cap G'|} = \frac{|G'|}{p^t}. \]

In view of Lemma 2.3 and Corollary 2.7 we obtain

\[ |E \cap G'| = \frac{G'}{\Phi(G)'} = p^t, \quad \text{for } t \in \{d, d-1\}. \]
Let thus $L$ be an elementary abelian, normal subgroup of order $p^{d-1}$ contained in $G'$ (we take $L$ to be $E \cap G'$ if this has order $p^{d-1}$, otherwise if $E \leq G'$ we take $L$ to be a maximal subgroup of $E$ which is normal in $G$), and let
\[
\{ 1 \} = L_0 \lhd L_1 \lhd L_2 \lhd \ldots \lhd L_{d-1} = L \leq G',
\]
be a series of subgroups, each normal in $G$. In particular, $|L_i| = p^i$ for each $i$.

Consider an arbitrary series
\[
\Phi(G) = S_0 \lhd S_1 \lhd S_2 \lhd \ldots \lhd S_{d-1} \lhd S_d = G.
\]
(We will make a more precise choice of $S_d$ later.) In particular, each $S_i$ is normal in $G$, and $|G/S_i| = p^{d-i}$ for each $i$. Clearly for $i \geq 1$ one has $G/S_i \cong L_{d-i}$ as both groups are elementary abelian, of order $p^{d-i}$. By Definition 1.1 we also have
\[G/L_{d-i} \cong S_i, \quad \text{for all } i \geq 1.\]

For $i \geq 1$ we thus have from Lemma 2.1
\[|S_i'| = |G'/L_{d-i}| = \frac{|G'|}{p^{d-i}},\]
that is,
\[(3.1) \quad |G'/S_i'| = p^{d-i}.
\]

In particular, $|G'/S_1'| = p^{d-1}$. Since $S_1 = \langle a, \Phi(G) \rangle$ for some $a \in G$, Lemma 2.3 implies $S_1' \leq [\langle a \rangle, \Phi(G)] \Phi(G)' \leq K$, so that Corollary 2.7 yields $|G'/K| = p^{d-1}$ and $S_1' = K$. It follows that $S_i' \geq S_1' = K$ for each $i \geq 1$. We obtain from (3.1)
\[|S_i'/K| = |(G'/K)/(G'/S_i')| = p^{i-1}\]
for each $i \geq 1$.

In particular, $|S_2'/K| = p$ and $|S_3'/K| = p^2$. (This is the only point where we use the assumption $d > 2$.) Since $S_3 = \langle a, b, c, \Phi(G) \rangle$ for some $a, b, c \in G$, Lemma 2.3 yields $S_3' \leq \langle [a, b], [a, c], [b, c], K \rangle$. Since every element of the exterior square of a vector space of dimension 3 is a decomposable tensor, we may choose $a, b, c$ so that $[a, b] \in K$. Choosing $S_2 = \langle a, b, \Phi(G) \rangle$, Lemma 2.3 yields $S_2' \leq K$, a final contradiction.

Finally, we derive a proof of Conjecture 1.2.

If $G$ is a finite, abelian, morphic $p$-group, it is not difficult to see from Lemma 2.2 that $G$ must be homocyclic.

In view of Theorem 1.3, let thus assume that $G$ is nonabelian. Since morphic $p$-groups are $ea$-morphic, by Theorem 1.7 we conclude that $G$ is 2-generated, and by Theorem 1.4 we have the result.

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