CLOSURE AND INTERIOR OPERATORS OF THE CATEGORY OF POSITIVE TOPOLOGIES

JOAQUÍN LUNA-TORRES

Abstract. We define and study the notions of closure \( C \) operators and interior \( I \) operators of the category \( CCov \) of convergent covers which appears in positive topologies. The main motivation of this paper is to construct the concrete categories \( C-CCov \), of \( CCov \)-spaces, and \( I-CCov \), of \( CCov \)-spaces and deduce that they are topological categories.

0. Introduction

Closure operators have been used intensively in Algebra (for example, G. Birkhoof, R. Pierce) and topology (for instance, K. Kuratowski, E. Čech). Category theory provides a variety of notions which expand on the lattice-theoretic concept of closure operator most notably through the notion of reflective subcategory (for example, P. Freyd, J. F. Kennison, H. Herrlich). The notions of Grothendieck topology and Lawvere-Tierney topology (see [7]) provide standard tools in Sheaf-and Topos Theory and are most conveniently described by particular closure operators. Both lattice-theoretic and categorical views play an important role Theoretic Computer Science. For a topological space it is well-known that, for example, the associated closure and interior operators provide equivalent descriptions of the topology; but this is not always true in other categories, consequently it makes sense to define and study separately these operators. The main motivation of this paper is to construct

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Programa de Matemáticas, Universidad Distrital Francisco José de Caldas, Bogotá D. C., Colombia (retired professor).
the concrete categories $\mathcal{C}$-$\mathbf{Cov}$, of $\mathbf{Cov}$-spaces, and $\mathbf{I}$-$\mathbf{Cov}$, of $\mathbf{Cov}$-spaces and deduce that they are topological categories.

After the construction of the coframe $\mathfrak{S}_{\infty}(S)$, of all subobjects $T = (T^c; \triangleleft T^c)$ of a convergent cover $S = (S; \triangleleft)$ (in other words, $\mathfrak{S}_{\infty}(S)^{op}$ is a frame), we introduce, in section 2, the notion of closure operator $\mathcal{C}$ of the category $\mathbf{Cov}$ as a version of the closure operator studied by D. Dikranjan and W. Tholen [6]; further, we present a notion of closed subobjects different from the one allowed in (4); and finally, in that section, a reflective subcategory of $\mathfrak{S}_{\infty}(L)$ is constructed.

We shall be concerned, in section 3, with a version on the category $\mathbf{CCov}$ of the interior operator studied in [8], farther we present a notion of open subobjects at last, in that section, a coreflective subcategory of $\mathfrak{S}_{\infty}(L)$ is constructed.

1. Preliminaries

For a comprehensive account on the the categories of positive topologies we refer to F. Ciraulo and G. Sambin [4]; and T. Coquand, G. Sambin, J. Smith and S. Valentín [5], from whom we take the following notions:

Formal Topology is a way to approach Topology by means of intuitionistic and predicative tools only. The original definition given in [10] is now known to correspond to overt (or open) locales, in the sense that every formal topology is a predicative presentation of an overt locale and the category of formal topologies is (dually) equivalent to the full subcategory of the category of locales whose objects are overt. A deep rethinking of the foundations of constructive topology has led G. Sambin to a two-sided generalization of the notion of a convergent cover. The structure of a convergent cover can be enriched by means of a second relation, called a positivity relation, which is used to speak about some particular sub-topologies (overt weakly closed sublocales). This enrichment produces a larger category (positive topologies) in which the category of convergent covers (locales) embeds as a
reflective subcategory. The two generalizations can be combined together to obtain an extension of the category of suplattices.

1.1. Predicative presentations of neighborhoods.

A concrete topological space is a triple \( X \equiv (X, S, N) \) where \( X \) is a set of concrete points, \( S \) is a set of observables, \( N \) is a map from \( S \) into subsets of \( X \), called the neighborhood map, which satisfies

\[
(B_1) \quad X = \bigcup_{a \in S} N(a)
\]

\[
(B_2) \quad (\forall a, b \in S) \ (\forall x \in X) \ (x \in N(a) \cap N(b) \implies (\exists c \in S) \ (x \in N(c) \& N(c) \subseteq N(a) \cap N(b))
\]

Note that this description re-establishes a balance between the side of points: the concrete side, and the side of observables, or formal basic neighbourhoods, which is called the formal side. Note that \((B_2)\) is just a rigorous writing of the usual condition stating that if \( N(a) \) and \( N(b) \) are two neighbourhoods of \( x \) then there exists a neighborhood \( N(c) \) of \( x \) which is contained both in \( N(a) \) and \( N(b) \) and this is all what we need to obtain closure under intersection.

Now, a map \( N : S \to \mathcal{P}(X) \) is a propositional function with two arguments \( N(x)(a) \) \( \text{prop} \ [x : X, a : S] \), that is a binary relation, written in a more suggestive way as

\[ x \vdash \text{prop} \ [x : X, a : S] \]

and read it “\( x \) lies in \( a \)” or “\( x \) forces \( a \)”.

It is convenient to use also a few abbreviations:

\[ x \vdash U \equiv (\exists b \in U) (x \vdash b) \]

\[ \text{ext}(a) \equiv \{ x : X \mid x \vdash a \} \]

\[ \text{ext}(U) \equiv \bigcup_{a \in U} \text{ext}(a) \]

Hence \( x \vdash a \) is the same as \( x \in \text{ext}(a) \) and \( x \vdash U \) is the same as \( x \in \text{ext}(U) \); thus the map \( N \) coincides with \( \text{ext} \).

Then \((B_1)\) and \((B_2)\) can be rewritten as
\[(B_1) \ (\forall x \in X)(\exists a \in S) \ x \models a\]
\[(B_2) \ (\forall a, b \in S) \ ext(a) \cap ext(b) \subseteq ext(a \downarrow b)\]

where \(a \downarrow b \equiv \{ c : S \mid ext(c) \subseteq ext(a) \& ext(c) \subseteq ext(b) \}\)

1.2. **Basic cover.** In [3], the authors show that Sup-lattices can be characterized as pairs \((L; \bigvee)\) where \(\bigvee\) is an infinitary operation on \(L\) satisfying

(i) \(\bigvee \{ x \} = x\) for every \(x \in L\);
(ii) \(\bigvee_{i \in I}(\bigvee U_i) = \bigvee(\bigcup_{i \in I} U_i)\) for every family \((U_i)_{i \in I}\) of subsets of \(L\).

Now, they define \(x \leq y\) putting \(\bigvee \{ x, y \} = y\). This partial order induce a relation \(\prec\) between subsets, where the intended meaning of \(U \prec W\) is that \(\bigvee U \leq \bigvee W\).

Recalling that \(a = \bigvee \{ a \}\), the characterizing property of joins can be written in terms of \(\prec\) as \((\forall a \in U)(\{ a \} \prec V)\) iff \(U \prec V\). For any preorder \(\prec\), they define a relation between elements and subsets by putting

\[a \triangleleft U \equiv \{ a \} \prec U.\]

In general, the opposite of a set-based suplattice need not be set-based. In the set-based case all the information about the suplattice under consideration can be coded by means of a cover relation on the base:

Let \(S\) be a set. A small relation \(\triangleleft\) between elements and subsets of \(S\) is called a (basic) **cover** if

1. \(a \in U \Rightarrow a \triangleleft U\)
2. \((a \triangleleft U \& (\forall u \in U)(u \triangleleft V) \Rightarrow a \triangleleft V)\) for every \(a \in S\) and \(U; V \subseteq S\).

The motivating example is given by a set-based suplattice with base \(S\), where \(a \triangleleft U\) is taken to mean \(a \leq \bigvee U\).

A base for the suplattice (least upper bound lattice) \((L; \leq)\) is a set \(S \subseteq L\) such that

\(\bigvee \{ a \in S \mid a \leq p \} = p\) for all \(p\) in \(L\). This is called a set-based suplattice.
A basic cover \((S; \sqsubset)\) has to be understood as a presentation of a set-based suplattice, where \(S\) plays the role of a set of codes for the base. Indeed, any cover \((S; \sqsubset)\) can be extended to a preorder \(U \sqsubset V\) on \(\mathcal{P}(S)\) defined by \((\forall u \in U)(u \sqsubset V)\). This induces an equivalence relation \(=\) on \(\mathcal{P}(S)\) where \(U = V\) if and only if \(U \sqsubset V\). Such a suplattice has a base, namely the set \(\{[a] | a \in S\}\). (Here we have adopted a convention we are going to use quite often: for readability’s sake, we denote a singleton by its unique element.)

To complete the illustration, one should note that:

\[ (i) \text{ the cover induced by a set-based suplattice } L \text{ presents a suplattice which is isomorphic to } L, \text{ the isomorphism being given by the two mappings } x \mapsto \{a \in S \mid a \leq x\} \text{ and } [U] \mapsto \bigvee U; \]

\[ (ii) \text{ the cover associated to the suplattice presented by a cover } (S; \sqsubset) \text{ is isomorphic to } (S; \sqsubset) \text{ itself, according to the definition of morphism given below.} \]

Note that each set-based suplattice can be presented by several covers; all of them are going to be isomorphic to each other, according to the notion of morphism we are going to introduce below.

1.3. **Morphisms between basic covers.** Let \(S_1 = (S_1; \sqsubset_1)\) and \(S_2 = (S_2; \sqsubset_2)\) be two basic covers. A small relation \(\mathfrak{R} \subseteq S_1 \times S_2\) respects the covers if

\[ U \sqsubset_2 V \Rightarrow \mathfrak{R}^{-} U \sqsubset_1 \mathfrak{R}^{-} V \text{ for all } U; V \subseteq S_2 \]

where \(\mathfrak{R}^{-} W = \{a \in S_1 \mid (\exists w \in W)(a \mathfrak{R} w)\}\). A morphism between \(S_1\) and \(S_2\) is an equivalence class of relations between \(S_1\) and \(S_2\) which respect the covers, where two relations \(\mathfrak{R}\) and \(\mathfrak{R}'\) are equivalent if \(\mathfrak{R}^{-} W =_{S_1} \mathfrak{R}'^{-} W\) for every \(W \subseteq S_2\).

The previous definition has a very natural meaning: a morphism between two covers is just a presentation of a suplattice homomorphism between the corresponding suplattices.
Basic covers and their morphisms form a category, called $\text{BCov}$, which is dual to the category $\text{SL}$ of suplattices, impredicatively. The previous discussion says that

$$\text{BCov}((S_1; \triangleleft_1); (S_2; \triangleleft_2)) = \text{SL}(\mathcal{P}(S_2)/=\triangleleft_2; \mathcal{P}(S_1)/=\triangleleft_1).$$

1.4. Convergent basic cover. A basic cover is convergent if its corresponding suplattice is a frame. A morphism between convergent covers is a morphism of basic covers whose corresponding suplattice homomorphism is, in fact, a frame homomorphism (preserves finite meets). The resulting category will be called $\text{CCov}$. Impredicatively, $\text{CCov}$ is dual to the category $\text{Frm}$ of frames and hence equivalent to the category $\text{Loc}$ of locales.

An explicit description of convergent covers and their morphisms is the following:

A basic cover $(S; \triangleleft)$ is convergent if and only if

$$\star \ a \triangleleft U \& a \triangleleft V \Rightarrow a \triangleleft U \downarrow V \quad \text{for every } a \in S \text{ and } U; V \subseteq S,$$

where $U \downarrow V = \{b \in S \mid b \triangleleft u \& b \triangleleft v \text{ for some } (u; v) \in U \times V\}$. In this case, $[U] \wedge [W] = [U \downarrow W]$.

A morphism $\alpha : (S_1; \triangleleft_1) \to (S_2; \triangleleft_2)$ between convergent covers is a morphism of basic covers such that

- $S_1 \triangleleft_1 \alpha \triangleleft_2 S_2$ and
- $(\alpha \downarrow U)\downarrow_1 (\alpha \downarrow V) \triangleleft_1 \alpha \downarrow (U \downarrow_2 V)$ for every $U; V \subseteq S_2$.

1.5. Basic and positive topology. It is convenient to use the symbol $\check{\cap}$ for inhabited intersection, that is, $U \check{\cap} V \overset{\text{def.}}{=} (\exists a \in S)(a \in U \& a \in V)$ for $U; V \subseteq S$.

An element $x$ of a locale $L$ is positive if $(x \leq \bigvee Y) \Rightarrow (Y \check{\cap} L)$ for every $Y \subseteq L$. With classical logic, $x$ is positive if and only if $x \neq 0$. In the language of formal topology this notion is translated as follows, which requires some impredicativity:

Given a (convergent) cover $(S; \triangleleft), a \in S$ is said to be positive if $(a \triangleleft U) \rightarrow (U \check{\cap} S)$ for every $U \subseteq S$. $\text{Pos}$ is the subset of positive elements of $S$. A subset $U \subseteq S$ is said to be positive if $U \check{\cap} \text{Pos}$.

A convergent cover $(S; \triangleleft)$ is overt if $a \triangleleft \{a\} \cap \text{Pos}$ for every $a \in S$.

$(S; \triangleleft)$ is overt if and only if $[U] = [U \cap \text{Pos}]$ for every $U \subseteq S$. 6
Overt locales are usually defined in an equivalent way, as follows: The category of locales has a terminal object which, as a frame, is the power \( \mathcal{P}(1) \) of the singleton \( 1 = \{0\} \). This corresponds to the convergent cover \( (1; \in) \). It can be thought of the elements of \( \mathcal{P}(1) \) as propositions modulo logical equivalence (that is, truth values).

For each convergent cover \( (S; \langle \rangle) \) there exists a unique (up to equivalence) morphism \( \mathfrak{a} : (S; \langle \rangle) \to (1; \in) \) between convergent covers (put \( \mathfrak{a}^{-1} 0 = S \)). As a frame homomorphism \( \mathcal{P}(1) \to \mathcal{P}(S)/=\mathfrak{a} \) it maps a proposition \( p \) to the equivalence class \( \{a \in S \mid p\} \).

A basic cover \( (S; \langle \rangle) \) equipped with a compatible positivity relation is called a **basic topology**. A convergent cover equipped with a compatible positivity relation is called a **positive topology**.

## 2. Closure Operators

In this section we shall be concerned with a version on the category \( \text{CCov} \) of the closure operator studied by D. Dikranjan and W. Tholen [6].

It is important to remember that impredicatively, \( \text{CCov} \) is dual to the category \( \text{Frm} \) of frames and hence equivalent to the category \( \text{Loc} \) of locales.

Our first aim in this section is to describe a special class of subobjects of a set \( S \) in such a way that they will be sublocales of \( \mathcal{P}(S) \), and with them we shall get a coframe.

**Lemma 2.1.** Let \( S \) be a set and let \( T \) be a subset of \( S \). then

\[
\mathcal{P}_*(T) = \{ V \cup T^c \mid V \in \mathcal{P}(T) \}
\]

is a sublocale of \( \mathcal{P}(S) \).

( Here \( T^c \) is the complement of \( T \) in \( S \))
Proof. Clearly $\mathcal{P}_s(T)$ is a complete lattice satisfying the distributivity law of arbitrary joins and finite meets, and for every $V \cup T^c \in \mathcal{P}_s(T)$ and every $U \in \mathcal{P}(S)$, $(U^c \cap T^c) \cup V \in \mathcal{P}(T)$ and $U^c \cap T^c \subseteq T^c$, therefore $U \rightarrow (V \cup T^c)$ is an element of $\mathcal{P}_s(T)$ (here “$\rightarrow$” denote the Heyting implication). □

It is clear that $\bigwedge \mathcal{P}_s(T) = T^c$, therefore it is important to use $T^c$ in the definition of subobject.

**Definition 2.2.** A subobject $\mathcal{T} = (T; \triangleleft_T)$ of a convergent cover $\mathcal{S} = (S; \triangleleft)$ consists of

1. The complement $T^c$ of a subset $T$ of $S$;
2. A convergent cover $\triangleleft_{T^c} \subseteq T \times \mathcal{P}_s(T)$ obtained as follows: if $a \in T^c$ and $a \triangleleft U$ for $U \subseteq S$, then $a \triangleleft_{T^c} (U \cup T^c)$.

Note that since $U \cup T^c = (U \cap T) \cup (U \cap T^c) \subseteq (U \cap T) \cup U^c$, then $(U \cup T^c) \in \mathcal{P}_s(T)$.

From now on, we shall denote by $\mathcal{S}_{\mathcal{C}}(\mathcal{S})$ the coframe of all subobjects $\mathcal{T} = (T^c; \triangleleft_{T^c})$ of $\mathcal{S} = (S; \triangleleft)$ (i.e. $\mathcal{S}_{\mathcal{C}}(\mathcal{S})^{\text{op}}$ is a frame).

**Definition 2.3.** A closure operator $\mathcal{C}$ of the category $\mathcal{CCov}$ is given by a family $\mathcal{C} = (c_s)_{S \in \mathcal{CCov}}$ of maps $c_s : \mathcal{S}_{\mathcal{C}}(S) \rightarrow \mathcal{S}_{\mathcal{C}}(S)$ such that

1. (Extension) $\mathcal{T} \subseteq c_s(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{S}_{\mathcal{C}}(S)$;
2. (Monotonicity) If $\mathcal{U} \subseteq \mathcal{T}$ in $\mathcal{S}_{\mathcal{C}}(S)$, then $c_s(\mathcal{U}) \subseteq c_s(\mathcal{T})$
3. (Lower bound) $c_s(\Phi) = \Phi$, where $\Phi = (\emptyset, \triangleleft_{\emptyset})$.

**Definition 2.4.** An $\mathcal{C}$-space is a pair $(S, c_s)$ where $S$ is an object of $\mathcal{CCov}$ and $c_s$ is a closure map on $S$.

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1 Since $U \cup V^c \in \mathcal{P}_s(S)$ for every $U \in \mathcal{P}(S)$, $(\emptyset, \triangleleft_{\emptyset})$ is a convergent coverage for each $V \subseteq S$, provided that $(S; \triangleleft)$ is convergent.

2 Since $U \cup T^c \in \mathcal{P}_s(T)$ for every $U \in \mathcal{P}(S)$, $(T, \triangleleft_T)$ is a convergent cover for every $T \subseteq S$, whenever $(S; \triangleleft)$ is convergent.
Remark 2.5. If \( \mathfrak{s} \subseteq S_1 \times S_2 \) is a representative of an equivalence class of relations between \( S_1 = (S_1, \triangleleft_1) \) and \( S_2 = (S_2, \triangleleft_2) \) which respect the covers, we denote by

- \( \mathfrak{s}^{-1} \) the inverse of \( \mathfrak{s} \);
- \( \mathfrak{s}^{*} \) the direct image of \( \mathfrak{s} \) defined by
  \[
  (\forall X \subseteq S_1)(\mathfrak{s}^{*}(X) = \{ t \in S_2 | (\exists s \in X)((s, t) \in \mathfrak{s}) \});
  \]
- \( \mathfrak{s}^{*} \) the inverse image of \( \mathfrak{s} \) is the direct image of \( \mathfrak{s}^{-1} \).

Definition 2.6. A morphism \( \mathfrak{s} : L \rightarrow S \) of \( \text{CCov} \) is said to be \( \mathcal{C} \)-continuous if

\[
\mathfrak{s}^{*}(c_L(T)) \subseteq c_S(\mathfrak{s}^{*}(T))
\]
for all \( T \in \mathcal{S}_{\Delta}(L) \).

Note that, in the presence of requirement \( (C_2) \), the continuity condition can equivalently be expressed as

\[
c_L(\mathfrak{s}^{*}(U)) \subseteq \mathfrak{s}^{*}[c_S(U)]
\]
for all \( U \in \mathcal{S}_{\Delta}(S) \). Indeed, from \( (1) \), we have

\[
\mathfrak{s}^{*}(c_L(\mathfrak{s}^{*}(U))) \subseteq c_S(\mathfrak{s}^{*}(\mathfrak{s}^{*}(U))) \subseteq c_S(U).
\]

consequently, \( c_L(\mathfrak{s}^{*}(U)) \subseteq \mathfrak{s}^{*}[c_S(U)] \).

Proposition 2.7. Let \( \mathfrak{s} : L \rightarrow M \) and \( \mathfrak{t} : M \rightarrow N \) be two \( \mathcal{C} \)-continuous morphisms of \( \text{CCov} \) then \( \mathfrak{t} \circ \mathfrak{s} \) is an \( \mathcal{C} \)-continuous morphism of \( \text{CCov} \).

Proof. Since \( \mathfrak{s} : L \rightarrow M \) is \( \mathcal{C} \)-continuous, we have

\[
\mathfrak{s}^{*}[c_L(T)] \subseteq c_M(\mathfrak{s}^{*}(T))
\]
for all \( T \in \mathcal{S}_{\Delta}(S) \), it follows that

\[
\mathfrak{t}^{*}[\mathfrak{s}^{*}[c_L(T)]] \subseteq \mathfrak{t}^{*}[c_M(\mathfrak{s}^{*}(T))]
\]
now, by the \( \mathcal{G} \)-continuity of \( \mathcal{L} \),

\[
\mathcal{L} \rightarrow [c_M (\mathcal{A} \rightarrow (T))] \subseteq c_N (\mathcal{L} \rightarrow [\mathcal{A} \rightarrow (T)])
\]

therefore

\[
\mathcal{L} \rightarrow [\mathcal{A} \rightarrow (c_L (T))] \subseteq c_N (\mathcal{L} \rightarrow [\mathcal{A} \rightarrow (T)]),
\]

that is to say

\[
(\mathcal{L} \cdot \mathcal{A}) \rightarrow [c_L (T)] \subseteq c_N ((\mathcal{L} \cdot \mathcal{A}) \rightarrow (T))
\]

\[\square\]

As a consequence we obtain

**Definition 2.8.** The category \( \mathcal{G} \text{-CCov} \) of \( \text{CCov} \) – spaces comprises the following data:

1. **Objects:** Pairs \((S, c_S)\) where \( S = (S, \prec) \) is an object of \( \text{CCov} \) and \( c_S \) is a closure map on \( S \).

2. **Morphisms:** Morphisms of \( \text{CCov} \) which are \( \mathcal{G} \)-continuous.

2.1. The lattice structure of all closure operators. For the category \( \text{CCov} \) we consider the collection

\[
\mathcal{G}l(\text{CCov})
\]

of all closure operators on \( \text{CCov} \). It is ordered by

\[
\mathcal{G} \leq \mathcal{D} \iff c_S (T) \subseteq d_S (T),
\]

for all \( T \in \mathcal{S}_\text{ee}(S) \) and for all \( S \) object of \( \text{CCov} \)

This way \( \mathcal{G}l(\text{CCov}) \) inherits a lattice structure from \( \mathcal{S}_\text{ee}(S) \):

**Proposition 2.9.** Every family \( \mathcal{G} (\lambda) \lambda \in \Lambda \) in \( \mathcal{G}l(\text{CCov}) \) has a join \( \bigvee \mathcal{G} (\lambda) \lambda \in \Lambda \) and a meet \( \bigwedge \mathcal{G} (\lambda) \lambda \in \Lambda \) in \( \mathcal{G}l(\text{CCov}) \). The discrete closure operator

\[
\mathcal{G} D = (c_{DL}) L \in \text{CCov} \quad \text{with} \ c_{DL} (T) = T \quad \text{for all} \ T \in \mathcal{S}_\text{ee}(L)
\]

is the least element in $\mathcal{C}(\mathbf{Cov})$, and the trivial closure operator

$$\mathcal{C}_T = (c_{\mathcal{L}})_T \in \mathbf{Cov} \text{ with } c_{\mathcal{L}}(T) = \begin{cases} \mathcal{L} & \text{for all } T \in \mathfrak{S}_{\mathfrak{S}}(\mathcal{L}), \ S \neq \Phi \\ \Phi & \text{if } T = \Phi \end{cases}$$

is the largest one.

Proof. For $\Lambda \neq \emptyset$, let $\hat{\mathcal{C}} = \bigvee_{\lambda \in \Lambda} \mathcal{C}_\lambda$, then

$$\hat{\mathcal{C}} = \bigvee_{\lambda \in \Lambda} c_{\lambda, \mathcal{L}},$$

for all $\mathcal{L}$ object of $\mathbf{Cov}$, satisfies

- $T \subseteq \hat{c}_{\mathcal{L}}(T)$, because $T \subseteq c_{\lambda, \mathcal{T}}(T)$ for all $T \in \mathfrak{S}_{\mathfrak{S}}(\mathcal{L})$ and for all $\lambda \in \Lambda$.
- If $R \subseteq T$ in $\mathfrak{S}_{\mathfrak{S}}(\mathcal{L})$ then $c_{\lambda, \mathcal{L}}(R) \subseteq c_{\lambda, \mathcal{L}}(T)$ for all $\lambda \in \Lambda$, therefore $\hat{c}_{\mathcal{L}}(R) \subseteq \hat{c}_{\mathcal{L}}(T)$.
- Since $c_{\lambda, \mathcal{L}}(\Phi) = \Phi$ for all $\lambda \in \Lambda$, we have that $\hat{c}_{\mathcal{L}}(\Phi) = \Phi$.

Similarly $\bigwedge_{\lambda \in \Lambda} \mathcal{C}_\lambda$, $\mathcal{C}_D$ and $\mathcal{C}_T$ are closure operators. \(\square\)

**Corollary 2.10.** For every object $\mathcal{L}$ of $\mathbf{Cov}$

$$\mathfrak{C}(\mathcal{L}) = \{ c_\mathcal{L} \mid c_\mathcal{L} \text{ is a closure map on } \mathcal{L} \}$$

is a complete lattice.

### 2.2. Initial closure operators.

Let $\mathfrak{C}-\mathbf{Cov}$ be the category of $\mathbf{Cov}$-spaces. Let $(\mathcal{M}, c_\mathcal{M})$ be an object of $\mathfrak{C}-\mathbf{Cov}$ and let $\mathcal{L}$ be an object of $\mathbf{Cov}$. For each morphisms $\mathfrak{A} : \mathcal{L} \to \mathcal{M}$ of $\mathbf{Cov}$ we define on $\mathcal{L}$ the map

$$c_{\mathcal{L}} \mathfrak{A} := \mathfrak{A}^* \cdot c_\mathcal{M} \cdot \mathfrak{A}^*$$
Proposition 2.11. Equation (3) define a map of the closure operator \( C \) for which the morpism \( s : L \rightarrow M \) in \( \mathbb{C} \mathbb{C} \mathbb{V} \) is \( C \)-continuous.

Proof.

\((C_1)\) (Extension) Let \( L \) be in \( \mathbb{C} \mathbb{C} \mathbb{V} \), then since \( c_M \) is a closure map on \( M \) it follows that \( \mathcal{A} \rightarrow(T) \subseteq c_M(\mathcal{A} \rightarrow(T)) \), which is equivalent to saying that \( \mathcal{T} \subseteq (\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow)(\mathcal{T}) = c_{\mathcal{L}}(\mathcal{T}); \)

\((C_2)\) (Monotonicity) \( \mathcal{R} \subseteq \mathcal{T} \) in \( \mathcal{S}_{\mathbb{C}}(L) \), implies \( \mathcal{A} \rightarrow(\mathcal{R}) \subseteq \mathcal{A} \rightarrow(\mathcal{T}) \), then we have

\[(c_M \cdot \mathcal{A} \rightarrow)(\mathcal{R}) \subseteq (c_M \cdot \mathcal{A} \rightarrow)(\mathcal{T}),\]

consequently

\[(\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow)(\mathcal{R}) \subseteq (\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow)(\mathcal{T}).\]

\((C_3)\) (Lower bound) \( (\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow)(\Phi) \subseteq (\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow)(\Phi).\)

Finally,

\[\mathcal{A} \rightarrow(c_{\mathcal{L}}(\mathcal{T})) = (\mathcal{A} \rightarrow \cdot c_M \cdot \mathcal{A} \rightarrow(\mathcal{T})) \subseteq c_M(\mathcal{A} \rightarrow(\mathcal{T}))\]

for all \( \mathcal{T} \in \mathcal{S}_{\mathbb{C}}(L) \).

\( \square \)

It is clear that \( c_{\mathcal{L}} \) is the coarsest closure map on \( L \) for which the morphism \( \mathcal{A} \) is \( \mathcal{C} \)-continuous; more precisely
Proposition 2.12. Let $(\mathcal{M}, c_{\mathcal{M}})$ and $(\mathcal{N}, c_{\mathcal{N}})$ be objects of $\mathcal{C}\text{-CCov}$, and let $\mathcal{L}$ be an object of $\mathcal{CCov}$. For each morphism $\mathcal{t}: \mathcal{N} \rightarrow \mathcal{L}$ in $\mathcal{CCov}$ and for the $\mathcal{C}$-continuous morphism $\mathcal{s}: (\mathcal{L}, c_{\mathcal{L}}) \rightarrow (\mathcal{M}, c_{\mathcal{M}})$, $\mathcal{t}$ is $\mathcal{C}$-continuous if and only if $\mathcal{s} \cdot \mathcal{t}$ is $\mathcal{C}$-continuous.

Proof. Suppose that $\mathcal{s} \cdot \mathcal{t}$ is $\mathcal{C}$-continuous, i.e.

$$(\mathcal{s} \cdot \mathcal{t})^{-}(c_{\mathcal{N}}(U)) \subseteq c_{\mathcal{M}}((\mathcal{s} \cdot \mathcal{t})^{-}(U))$$

for all $U \in \mathcal{S}_{\mathcal{CC}}(\mathcal{N})$.

It follows that

$$\begin{align*}
\mathcal{t}^{-}[c_{\mathcal{N}}(U)] & \subseteq \mathcal{s}^{-}[c_{\mathcal{M}}(\mathcal{s}^{-}(\mathcal{t}^{-}(U)))] \\
& \subseteq (\mathcal{s}^{-} \cdot c_{\mathcal{M}} \cdot \mathcal{s}^{-})(\mathcal{t}^{-}(U)) \\
& = c_{\mathcal{L}}(\mathcal{t}^{-}(U)),
\end{align*}$$

i.e. $\mathcal{t}$ is $\mathcal{C}$-continuous. $\square$

As a consequence of corollary (2.10), proposition (2.11) and proposition (2.12) (cf. [2]), we obtain

Theorem 2.13. The forgetful functor $U: \mathcal{C}\text{-CCov} \rightarrow \mathcal{CCov}$ is topological, i.e. the concrete category $(\mathcal{C}\text{-CCov}, U)$ is topological.

2.3. Closed and dense subobjects. In this section we introduce a notion of closed subobjects different from the one allowed in (I).

Definition 2.14. An subobject $\mathcal{T}$ of a convergent cover $\mathcal{S}$ is called

- $\mathcal{C}$-closed (in $\mathcal{S}$) if $c_{\mathcal{S}}(\mathcal{T}) = \mathcal{T}$;
- $\mathcal{C}$-dense (in $\mathcal{S}$) if $c_{\mathcal{S}}(\mathcal{T}) = \mathcal{S}$. 

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It is easy to verify that for the Kuratowski closure operator $K$ of $\text{Top}$, $K$-closed and $K$-dense for a subspace inclusion $M \rightarrow X$ means closed and dense in the usual topological sense, respectively.

The $C$-continuity condition \([1]\) implies that $C$-closedness is preserved by inverse images, and that $C$-denseness is preserved by images:

**Proposition 2.15.** Let $\mathfrak{a} : \mathcal{L} \rightarrow \mathcal{M}$ be a morphism in $\text{CCov}$,

1. If $\mathcal{V}$ is $C$-closed in $\mathcal{M}$, then $\mathfrak{a}^{-1}(\mathcal{V})$ is $C$-closed in $\mathcal{L}$,
2. If $\mathcal{U}$ is $C$-dense in $\mathcal{L}$, and $\mathfrak{a}^{-1}(\mathcal{L}) = \mathcal{L}$, then $\mathfrak{a}^{-1}(\mathcal{U})$ is $C$-dense in $\mathcal{M}$.

**Proof.**

1. If $c_{\mathcal{M}}(\mathcal{V}) = \mathcal{V}$ then $c_{\mathcal{L}}(\mathfrak{a}^{-1}(\mathcal{V})) \subseteq \mathfrak{a}^{-1}(c_{\mathcal{M}}(\mathcal{V})) = \mathfrak{a}^{-1}(\mathcal{V})$.
2. If $c_{\mathcal{L}}(\mathcal{U}) = \mathcal{L}$ and $\mathfrak{a}^{-1}(\mathcal{L}) = \mathcal{L}$, then $\mathcal{L} = \mathfrak{a}^{-1}(\mathcal{L}) = \mathfrak{a}^{-1}(c_{\mathcal{L}}(\mathcal{U})) \subseteq c_{\mathcal{M}}(\mathfrak{a}^{-1}(\mathcal{U}))$.

2.4. A reflective subcategory of $\text{Scc}(\mathcal{L})$. For every $\mathcal{L}$ object of $\text{CCov}$, let $\text{Scc}(\mathcal{L})^c$ denote the collection of $C$-closed subobjects of $\mathcal{L}$.

Since for every $\mathcal{L} \in \text{CCov}$, the inclusion $i : \text{Scc}(\mathcal{L})^c \rightarrow \text{Scc}(\mathcal{L})$ preserves all meets, it has a left Galois adjoint\(^3\)

4. $\mathfrak{R}_L : \text{Scc}(\mathcal{L}) \rightarrow \text{Scc}(\mathcal{L})^c$ defined by $\mathfrak{R}_L(\mathcal{T}) = \bigcap \{ \mathcal{V} \in \text{Scc}(\mathcal{L}) \mid \mathcal{T} \subseteq i(\mathcal{V}) \}$.

**Proposition 2.16.** The family $\mathfrak{R} = (\mathfrak{R}_L)_{\mathcal{L} \in \text{CCov}}$ of maps \([4]\) is another closure operator of the category $\text{CCov}$.

**Proof.** Let $\mathcal{L}$ be an object of $\text{CCov}$. Then

\[
(C_1) \quad \mathcal{T} \subseteq \mathfrak{R}_L(\mathcal{T}) \text{ for all } \mathcal{T} \in \text{Scc}(\mathcal{L}), \text{ because } \mathcal{T} \subseteq \mathcal{V} \text{ for all } \mathcal{V} \in \text{Scc}(\mathcal{L});
\]

\(^3\)We use The Galois Adjunction Theorem in CZF; see \([\text{I}]\)
(C_2) If \( S \subseteq T \) in \( \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \), then
\[
\mathcal{R}_\mathcal{L}(S) = \bigcap \{ V \in \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \mid S \subseteq i(V) \}
\subseteq \bigcap \{ V \in \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \mid T \subseteq i(V) \}
= \mathcal{R}_\mathcal{L}(T);
\]

(C_3) Clearly, we have \( \mathcal{R}_\mathcal{L}(\Phi) = \Phi \).

Additionally, it is interesting to note that
\[
\mathcal{R}_\mathcal{L}(\mathcal{R}_\mathcal{L}(T)) = \bigcap \{ V \in \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \mid \mathcal{R}_\mathcal{L}(T) \subseteq i(V) \} = \mathcal{R}_\mathcal{L}(T);
\]
in other words, \( \mathcal{R} \) is an idempotent closure operator of the category \( \mathcal{C}_{\mathcal{C}} \). \( \square \)

Corollary 2.17. \( \mathcal{S}_{\mathcal{C}}(\mathcal{L})^\mathcal{E} \) is a reflective subcategory of \( \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \).

Proof. As we have already seen, for every \( \mathcal{L} \in \mathcal{C}_{\mathcal{C}} \), the closure map \( \mathcal{R}_\mathcal{L} : \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \rightarrow \mathcal{S}_{\mathcal{C}}(\mathcal{L})^\mathcal{E} \) is left adjoint of the inclusion morphism \( i : \mathcal{S}_{\mathcal{C}}(\mathcal{L})^\mathcal{E} \hookrightarrow \mathcal{S}_{\mathcal{C}}(\mathcal{L}) \). \( \square \)

3. Interior Operators

We shall be concerned in this section with a version on the category \( \mathcal{C}_{\mathcal{C}} \) of the interior operator studied in [8].

Definition 3.1. An interior operator \( I \) of the category \( \mathcal{C}_{\mathcal{C}} \) is given by a family \( I = (i_S)_{S \in \mathcal{C}_{\mathcal{C}}} \) of maps \( i_S : \mathcal{S}_{\mathcal{C}}(S) \rightarrow \mathcal{S}_{\mathcal{C}}(S) \) such that

(I_1) (Contraction) \( i_S(T) \subseteq T \) for all \( T \in \mathcal{S}_{\mathcal{C}}(S) \);

(I_2) (Monotonicity) If \( R \subseteq T \) in \( \mathcal{S}_{\mathcal{C}}(S) \), then \( i_S(R) \subseteq i_S(T) \);

(I_3) (Upper bound) \( i_S(S) = S \).

Definition 3.2. An \( I \)-space is a pair \( (S, i_S) \) where \( S \) is an object of \( \mathcal{C}_{\mathcal{C}} \) and \( i_S \) is an interior operator on \( S \).
Remark 3.3. If \( s \subseteq S_1 \times S_2 \) is a representative of an equivalence class of relations between \( S_1 \) and \( S_2 \) which respect the covers, we denote by

- \( s^{-1} \) the inverse of \( s \);
- \( s^\rightarrow \) the direct image of \( s \) defined by
  \[
  (\forall X \subseteq S_1)(s^\rightarrow(X) = \{ t \in S_2 \mid (\exists s \in X)((s, t) \in s) \});
  \]
- \( s^\leftarrow \) the inverse image of \( s \) is the direct image of \( s^{-1} \).

Definition 3.4. A morphism \( \alpha : L \rightarrow M \) of \( \text{CCov} \) is said to be \( I \)-continuous if

\[
(5) \quad \alpha^\leftarrow(i_M(T)) \subseteq i_L(\alpha^\rightarrow(T))
\]

for all \( T \in \mathcal{S}_{\text{cc}}(S) \).

Proposition 3.5. Let \( \alpha : L \rightarrow M \) and \( \epsilon : M \rightarrow N \) be two \( I \)-continuous morphisms of \( \text{CCov} \) then \( \epsilon \circ \alpha \) is an \( I \)-continuous morphism of \( \text{CCov} \).

Proof. Since \( \epsilon : M \rightarrow N \) is \( I \)-continuous, we have

\[
\epsilon^\leftarrow(i_N(W)) \subseteq i_M(\epsilon^\rightarrow(W))
\]

for all \( T \in \mathcal{S}_{\text{cc}}(N) \), it follows that

\[
\alpha^\leftarrow(\epsilon^\leftarrow(i_N(W))) \subseteq \alpha^\leftarrow(i_M(\epsilon^\rightarrow(W)));
\]

now, by the \( I \)-continuity of \( \alpha \),

\[
\alpha^\leftarrow(i_M(\epsilon^\rightarrow(W))) \subseteq i_L(\alpha^\rightarrow(\epsilon^\rightarrow(W)));
\]

therefore

\[
\alpha^\leftarrow(\epsilon^\leftarrow(i_N(W))) \subseteq i_L(\alpha^\rightarrow(\epsilon^\rightarrow(W)));
\]

that is to say

\[
(\epsilon \circ \alpha)^\leftarrow(i_N(W)) \subseteq i_L((\epsilon \circ \alpha)^\rightarrow(W)) \]

\( \square \)

As a consequence we obtain
Definition 3.6. The category \textbf{I-C Cov} of \textit{I}-spaces comprises the following data:

(1) **Objects**: Pairs \((S, i_S)\) where \(S\) is an object of \textbf{C Cov} and \(i_S\) is an interior operator on \(S\).

(2) **Morphisms**: Morphisms of \textbf{C Cov} which are \(I\)-continuous.

3.1. The lattice structure of all interior operators. For the category \textbf{C Cov} we consider the collection \(\text{Int}(\text{C Cov})\) of all interior operators on \textbf{C Cov}. It is ordered by

\[ I \leq J \iff i_S(T) \subseteq j_S(T), \text{ for all } T \in \mathcal{S}_{\text{cc}}(S) \text{ and all } S \text{ object of } \textbf{C Cov}. \]

This way \(\text{Int}(\text{C Cov})\) inherits a lattice structure from \(\mathcal{S}_{\text{cc}}(S)\):

**Proposition 3.7.** Every family \((I_\lambda)_{\lambda \in \Lambda}\) in \(\text{Int}(\text{C Cov})\) has a join \(\bigvee_{\lambda \in \Lambda} I_\lambda\) and a meet \(\bigwedge_{\lambda \in \Lambda} I_\lambda\) in \(\text{Int}(\text{C Cov})\). The discrete interior operator

\[ I_D = (i_D S)_{S \in \text{C Cov}} \text{ with } i_D S(T) = T \text{ for all } T \in \mathcal{S}_{\text{cc}}(S) \]

is the largest element in \(\text{Int}(\text{C Cov})\), and the trivial interior operator

\[ I_T = (i_T S)_{S \in \text{C Cov}} \text{ with } i_T S(T) = \begin{cases} \Phi & \text{for all } T \in \mathcal{S}_{\text{cc}}(S) \\ S & \text{if } T = S \end{cases} \]

is the least one.

**Proof.** For \(\Lambda \neq \emptyset\), let \(\hat{I} = \bigvee_{\lambda \in \Lambda} I_\lambda\), then

\[ \hat{i}_S = \bigvee_{\lambda \in \Lambda} i_{\lambda S}, \]

for all \(S\) object of \textbf{C Cov}, satisfies

- \(\hat{i}_S(T) \subseteq T\), because \(i_{\lambda S}(T) \subseteq T\) for all \(T \in \mathcal{S}_{\text{cc}}\) and for all \(\lambda \in \Lambda\).
- If \(R \leq T\) in \(\mathcal{S}_{\text{cc}}\) then \(i_{\lambda S}(R) \subseteq i_{\lambda S}(T)\) for all \(\lambda \in \Lambda\), therefore \(\hat{i}_S(R) \subseteq \hat{i}_S(T)\).
- Since \(i_{\lambda S}(S) = S\) for all \(\lambda \in \Lambda\), we have that \(\hat{i}_S(S) = S\).
Similarly \( \bigwedge_{\lambda \in \Lambda} I_{\lambda}, I_D \) and \( I_T \) are interior operators.

\[ \square \]

**Corollary 3.8.** For every object \( S \) of \( \mathbf{CCov} \)

\[ \text{Int}(S) = \{ i_S \mid i_S \text{ is an interior operator on } S \} \]

is a complete lattice.

### 3.2. Initial interior operators.

Let \( \mathbf{I-CCov} \) be the category of \( I \)-spaces. Let \((M, i_M)\) be an object of \( \mathbf{I-CCov} \) and let \( \mathcal{L} \) be an object of \( \mathbf{CCov} \). For each morphism \( \mathcal{I} : \mathcal{L} \to \mathcal{M} \) in \( \mathbf{CCov} \) we define on \( \mathcal{L} \) the operator

\[
\begin{array}{ccc}
\mathfrak{S}_{\text{ce}}(\mathcal{L}) & \xrightarrow{\mathcal{I}} & \mathfrak{S}_{\text{ce}}(\mathcal{M}) \\
\bigwedge & & \bigwedge \\
\mathfrak{S}_{\text{ce}}(\mathcal{L}) & \xleftarrow{\mathcal{I}} & \mathfrak{S}_{\text{ce}}(\mathcal{M})
\end{array}
\]

\[ i_{\mathcal{L}} := \mathcal{I} \cdot i_M \cdot \mathcal{I}. \]

**Proposition 3.9.** The operator \( i_{\mathcal{L}} \) is an interior operator on \( \mathcal{L} \) for which the morphism \( \mathcal{I} \) is \( I \)-continuous.

**Proof.**

\( (I_1) \) (Contraction) \( i_{\mathcal{L}}(T) = \mathcal{I} \cdot i_M \cdot \mathcal{I}^{-1}(T) \subseteq \mathcal{I} \cdot \mathcal{I}^{-1}(T) \subseteq \mathcal{I}^{-1}(T) \) for all \( T \in \mathfrak{S}_{\text{ce}}(\mathcal{S}) \);

\( (I_2) \) (Monotonicity) \( R \subseteq T \) in \( \mathfrak{S}_{\text{ce}}(\mathcal{S}) \), implies \( \mathcal{I}^{-1}(R) \subseteq \mathcal{I}^{-1}(T) \), then \( i_M \cdot \mathcal{I}^{-1}(R) \subseteq i_M \cdot \mathcal{I}^{-1}(T) \), consequently \( \mathcal{I} \cdot i_M \cdot \mathcal{I}^{-1}(R) \subseteq \mathcal{I} \cdot i_M \cdot \mathcal{I}^{-1}(T) \);

\( (I_3) \) (Upper bound) \( i_{\mathcal{L}}(S) = \mathcal{I} \cdot i_M \cdot \mathcal{I}^{-1}(S) = S \).
Finally,

\[ \mathcal{A} \left( i_M(T) \right) \subseteq \mathcal{A} \left( i_M \cdot \mathcal{A}^{-} \cdot \mathcal{A}^{-}(T) \right) \]

\[ = \left( \mathcal{A} \cdot i_M \cdot \mathcal{A}^{-} \right) \cdot \mathcal{A}^{-}(T) \]

\[ = i_{L_f}(f^{-1}(T)), \]

for all \( T \in \mathcal{S}_{\infty}(S) \). \( \square \)

It is clear that \( i_{L_f} \) is the coarsest interior operator on \( L \) for which the morphism \( \mathcal{A} \) is \( I \)-continuous; more precisely

**Proposition 3.10.** Let \( (N, i_N) \) and \( (M, i_M) \) be objects of \( I\text{-Cov} \), and let \( L \) be an object of \( CCov \). For each morphism \( \zeta : N \to L \) in \( CCov \) and for \( \mathcal{A} : (L, i_{L_f}) \to (M, i_M) \) an \( I \)-continuous morphism, \( \zeta \) is \( I \)-continuous if and only if \( \mathcal{A} \cdot \zeta \) is \( I \)-continuous.

**Proof.** Suppose that \( \mathcal{A} \cdot \zeta \) is \( I \)-continuous, i.e.

\[ (\mathcal{A} \cdot \zeta)^{-}(i_M(T)) \subseteq i_N((\mathcal{A} \cdot \zeta)^{-}(T)) \]

for all \( T \in \mathcal{S}_{\infty}(M) \). Then, for all \( R \in \mathcal{S}_{\infty}(L) \), we have

\[ \zeta^{-}(i_{L_f}(R)) = \zeta^{-}(\mathcal{A}^{-} \cdot i_M \cdot \mathcal{A}^{-}(R)) = (\mathcal{A} \cdot \zeta)^{-}(i_M(\mathcal{A}^{-}(R))) \]

\[ \subseteq i_N((\mathcal{A} \cdot \zeta)^{-}(\mathcal{A}^{-}(R))) = i_N(\mathcal{A}^{-} \cdot \mathcal{A}^{-} \cdot \mathcal{A}^{-}(R)) \]

\[ \subseteq i_N(\zeta^{-}(S)), \]

i.e. \( \zeta \) is \( I \)-continuous. \( \square \)

As a consequence of corollary 3.8, proposition 3.9 and proposition 3.10 (cf. [2]), we obtain

**Theorem 3.11.** The forgetful functor \( U : I\text{-CCov} \to CCov \) is topological, i.e. the concrete category \( (I\text{-CCov}, U) \) is topological.
3.3. **Open subobjects.** In this section we introduce a notion of open subobjects different from the one allowed in ([4]).

**Definition 3.12.** An subobject \( T \) of a convergent cover \( S \) is called \( \mathcal{I} \)-open (in \( S \)) if \( i_S(T) = T \);

It is easy to verify that for the Kuratowski interior operator \( i \) of \( \text{Top} \), \( i \)-open for a subspace inclusion \( M \rightarrow X \) means open in the usual topological sense.

The \( \mathcal{I} \)-continuity condition \( \square \) implies that \( \mathcal{I} \)-openness is preserved by inverse images:

**Proposition 3.13.** Let \( \mathfrak{a} : L \rightarrow M \) be a morphism in \( \text{CCov} \). If \( V \) is \( \mathcal{I} \)-open in \( M \), then \( \mathfrak{a}^{-1}(V) \) is \( \mathcal{I} \)-open in \( L \).

**Proof.** If \( i_M(V) = V \) then \( \mathfrak{a}^{-1}(V) = \mathfrak{a}^{-1}(i_M(V)) \subseteq i_L(\mathfrak{a}^{-1}(V)) \). In other words, \( \mathfrak{a}^{-1}(V) \) is \( \mathcal{I} \)-open in \( L \). \( \square \)

3.4. **A coreflective subcategory of \( \mathcal{S}_{\text{cc}}(L) \).** For every \( L \in \text{CCov} \), let \( \mathcal{S}_{\text{cc}}(L)^{\mathcal{D}} \) denote the collection of \( \mathcal{I} \)-open subobjects of \( L \).

Since for every \( L \in \text{CCov} \), the inclusion \( j : \mathcal{S}_{\text{cc}}(L)^{\mathcal{D}} \rightarrow \mathcal{S}_{\text{cc}}(L) \) preserves all joins, it has a right Galois adjoint:

\[
\mathcal{R}_L : \mathcal{S}_{\text{cc}}(L) \rightarrow \mathcal{S}_{\text{cc}}(L)^{\mathcal{D}} \text{ defined by } \mathcal{R}_L(T) = \bigcup \{ V \in \mathcal{S}_{\text{cc}}(L)^{\mathcal{D}} | j(V) \subseteq T \}.
\]

**Proposition 3.14.** The family \( \mathcal{R} = (\mathcal{R}_L)_L \in \text{CCov} \) of maps ([7]) is another interior operator of the category \( \text{CCov} \).

**Proof.** Let \( L \) be an object of \( \text{CCov} \). Then

\[(I_1) \mathcal{R}_L(T) \subseteq T \text{ for all } T \in \mathcal{S}_{\text{cc}}(L), \text{ because } V \subseteq T \text{ for some } V \in \mathcal{S}_{\text{cc}}(L)^{\mathcal{D}};\]

We use The Galois Adjunction Theorem in CZF; see [1].
(I_2) If \( S \subseteq T \) in \( \mathcal{G}_{\text{cc}}(\mathcal{L}) \), then
\[
\mathcal{R}_L(S) = \bigcup \{ V \in \mathcal{G}_{\text{cc}}(\mathcal{L}) \mid j(V) \subseteq S \} \\
\subseteq \bigcup \{ V \in \mathcal{G}_{\text{cc}}(\mathcal{L}) \mid j(V) \subseteq T \} \\
= \mathcal{R}_L(T);
\]

(I_3) Clearly, we have \( \mathcal{R}_L(\mathcal{L}) = \mathcal{L} \).

Additionally, it is interesting to note that
\[
\mathcal{R}_L(\mathcal{R}_L(T)) = \bigcup \{ V \in \mathcal{G}_{\text{cc}}(\mathcal{L}) \mid \mathcal{R}_L(T) \subseteq j(V) \} = \mathcal{R}_L(T);
\]
in other words, \( \mathcal{G} \) is an idempotent interior operator of the category \( \text{CCov} \). \( \square \)

**Corollary 3.15.** \( \mathcal{G}_{\text{cc}}(\mathcal{L})^\mathcal{D} \) is a reflective subcategory of \( \mathcal{G}_{\text{cc}}(\mathcal{L}) \).

**Proof.** As we have already seen, for every \( \mathcal{L} \in \text{CCov} \), the interior map \( \mathcal{R}_L : \mathcal{G}_{\text{cc}}(\mathcal{L}) \to \mathcal{G}_{\text{cc}}(\mathcal{L})^\mathcal{D} \) is left adjoint of the inclusion morphism \( j : \mathcal{G}_{\text{cc}}(\mathcal{L})^\mathcal{D} \hookrightarrow \mathcal{G}_{\text{cc}}(\mathcal{L}) \). \( \square \)

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Email address: joaquin.luna@sequoia-space.com