Massive Particles from Massless Spinors

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"Hark ye...the little lower layer. All visible objects, man, are but as paste-board masks... If man would strike, strike through the mask!" (Herman Melville; Moby Dick).

Abstract: Spinors are lightlike. How do they combine to make massive particles? We visit the zoo of Lagrangian singularities, or caustics, in spacetime projections from spin space- the phase space of lightlike, 8- spinor flows. We find that the species living there are the elementary particles. Codimension $J = (1, 2, 3, 4)$ phase singularities - vortex lines, sheets, tubes, and knots, are classified by the Coxeter groups generated by multiplicity $s$ reflections: "mass scatterings" off the vacuum spinors, that keep chiral pairs of matter envelopes confined to a timelike world tube, endowing a bispinor particle with mass. Using the volume in spin space as the action, the particle masses emerge in terms of the multiplicities, $s$: the number of null zigzags needed to close a cycle of mass scatterings. These mass values (calculated to lowest order in the vacuum intensity) are within a few percent of the observed masses for the leptons ($J = 1$) and hadrons ($J = 3$); but are up to 25 percent off for the mesons ($J = 2$).

1 From Spin Space to spacetime

Spinors live in the root space of vector and tensor fields, much as complex numbers live in the root space of real polynomials.

Left ($l \in L$) or Right ($r \in R$) chirality spinors have $L$ and $R$-handed $su(2)$ twists with spatial translation: they are Clifford ($C$)-analytic and conjugate $C$-analytic, respectively [BdS]. Complexified spinors [M.C.1] have an additional counter-clockwise or clockwise $u(1)$ phase advance with cosmic time, $T$: the log of the distance back to the big bang, in units of the compactification radius, $a_\#: T = a_\# \ln \gamma; \ \gamma (t) = \frac{a(t)}{a_\#}$.

Here $a(t)$ is the scale factor of our spatial hypersurface, $S_3[a(t)], as a function of Minkowsky time, or arctime, $t$: $|\Delta x| = \pm \Delta T = \Delta t$, the distance traveled by a photon. Arctime and cosmic time combine to make complex time, $z^0 \equiv t + iT$. A spinor is either analytic ($+$) or conjugate analytic ($-$) in complex time; the sign, $Sgn[\partial_\tau \theta^\tau]$ is its charge [MC.1][M.C.2]. Complex conjugation, $(z^0)^* \equiv t - iT$, gives charge reversal, C.
There are 8 Complex Clifford-algebra representations of the spin-isometry group of a 4-manifold: the Einstein group, E: 4 column spinors \( \psi_I = \{ l_+, r_+, l_-, r_- \} \), and 4 (provisionally independent) row spinors, \( \psi^I = \{ l^+, r^+, l^-, r^- \} \). \( PT \) reversal, \( PT : q \mapsto \overline{q}, z \mapsto z^* \) gives an equivalent representation, there are 2 spinors in each class:

\[
\begin{array}{cccc}
q & \ell^+ & r_- & r^+ \\
\overline{q} & \ell^- & r_+ & \ell^-
\end{array}
\]

(1)

where \( \overline{q}_a \equiv (q_0, -q_1, -q_2, -q_3) \) (notation as in [M.C.1], [M.C.2], [M.C.3]).

\( PT \)-conjugate spinors have the same helicities:

\[
(\ell^+, r_-) \subset \bigcirc (\text{left helicity})
\]

\[
(\ell^-, r_+) \subset \bigcirc (\text{right helicity}).
\]

The \( L \) and \( R \)-chirality spinors in an electron, \( e_- \equiv (l_+ \oplus r_-) \), have opposite helicities, but the same spins. They are thus counterpropagating. This explains their opposite boost dependence. For central bispinors, one propagates outward, in the direction of cosmic expansion, \( |\Delta x| = \Delta T > 0 \), and the other \( |\Delta x| = \Delta T < 0 \) propagates inward. Similarly, we identify direct sums of copropagating spinors of opposite chirality, but the same helicities, as neutrinos:

\[
\nu = (l_- \oplus r_+); e_- = (l_- \oplus r_-).
\]

Tensor products of 2 opposite-chirality spinors make null spin vectors: Photons. The null tetrads are "vacuum photons", of helicity \( \pm 1 \):

\[
q_l \equiv l_+ \otimes r^-; q_l \equiv l_- \otimes r; q_r \equiv l_+ \otimes r^+; q_r \equiv l_- \otimes r^-.
\]

Spin-1 sums of these make the Clifford tetrads of a moving Clifford algebra (C) frame:

\[
q_a \equiv \ell \otimes r \in CTM: \quad q_0 \equiv (q_l - q_r); \quad q_1 \equiv (q_l + q_r); \quad q_2 \equiv -i(q_r - q_l); \quad q_3 \equiv (q_l + q_r).
\]

(3)

The \( q_a \) are identified with the basis vectors, \( e_\alpha \), of a spacetime frame via the spin map, \( S \), whose pullback is the Dirac operator, \( D = S^* \). On compactified Minkowsky space, \( M_\# = S_1 \times S_3(a_\#) \),

\[
S = \frac{i}{2a_\#} q^\alpha \partial_\alpha e_\alpha \rightarrow (2a_\#) - 1 q_\alpha; \quad D = S^* : (2a_\#) e^\beta \leftrightarrow q^\beta.
\]

(4)

More generally, in moving frames in spin space and spacetime, the spin map reads

\[
S(x) \equiv [\partial_\alpha \zeta^\beta](x) q_\beta(x) e^\alpha(x) ; e_\alpha(x) \rightarrow [\partial_\alpha \zeta^\beta] q_\beta(x); \quad [\partial_\alpha \zeta^\beta](x) \equiv [d\zeta^\beta](x), \quad \text{where} \quad d \equiv e^\alpha d_\alpha.
\]

(5)

is the generalized exterior differential operator. The Jacobean determinant, \( |d\zeta^\beta|(x) \), is the 4-volume expansion factor; \( |d\zeta| = 0 \) at a singular (critical) point, \( x_c \) (we will suppress the generic point, \( x = (T, x^1, x^2, x^3) \) below).

General covariance says that the equations of motion must be covariant, and their action integrals invariant, under coordinated spin isometries of spacetime (external) and spin space (internal) frames.

Covariance is automatic if our spacetime is a horizontal local section of an 8-spinor bundle, 8; a different section for an observer in a different frame. We treat
the spinors here as the real, physical objects, and spacetime vector and tensor fields as their preimages under the spin map, $S$. We call this the

Spin principle, (P1). The 8-spinor bundle, $8$ is the real physical object. Spacetime, geometry, gauge, and matter fields, along with their interactions, all emerge as projections of $J$ chiral pairs of matter envelopes, $[\psi^I, d\psi^I, \bar{\psi}^I, \bar{\bar{\psi}}^I]$, selected from the $4$ left and $4$ right chirality spinors and their differentials. Their coupling constants are products of the remaining $(4 - J)$ vacuum pairs, chosen from the $4$ column spinors, $\psi_I = \{r_-, l_-, r_+, l_+\}$, and the $(4 - J)$ provisionally independent row spinors $\psi^I \equiv \{l^+, r^+, l^-, r^-\}$.

It takes $4$ spinors to make the pseudoscalar ”inner product,”

$$\psi^I [i q_2(x)] \psi^I \equiv \psi^I \tilde{i}[\tilde{l}(x) \otimes_2 \tilde{\bar{r}}(x)] \psi^I, \quad (6)$$

using $[q_2(x)]$ as a matrix in the moving spin frames. On a curved space, there is no $C$ scalar bilinear for like $\bar{\psi}\psi$, so no dualizing operation like $\bar{\bar{\psi}}\psi \epsilon = (\psi_2 - \psi_1)$ in flat space.

It takes $4$ spinors make the metric tensor (spin $2$); it takes $8$ spinors to make an $E$-invariant inner product; a $C$ scalar, $q_0$:

$$g_{\alpha\beta} = \frac{1}{2} [q_\alpha \otimes \bar{q}_\beta \otimes q_\beta \otimes \bar{q}_\alpha]; \quad q^\alpha g_{\alpha\beta}q^\beta. \quad (7)$$

Each spinor has conformal weight (dimension) $\frac{1}{2}$, so it takes $4$ spinors and $4$ spinor differentials to make the simplest $E$-invariant Lagrangian $4$ form, with no coupling constants: the $8$-spinor factorization of the bi-invariant Maurer-Cartan (M.C.) $4$ form,

$$\mathcal{L}_g = \left[ (\psi^3 d\psi_1) \wedge (d\psi^2 \psi_2) \wedge (\psi^3 d\psi_3) \wedge (d\psi^4 \psi_4) \right]^4, \quad (8)$$

where $[C^4]$ means ”the scalar part” of a $C$-valued $4$ form.

Here $d \equiv e^\alpha(x) \partial_\alpha(x)$ is the generalized (possibly path-dependent) exterior differential operator. It includes the derivatives of the moving spin frames, and reduces to the covariant derivative, coordinate wise:

$$d\psi_I \equiv d(\ell^I \psi_I) = l^I d\psi_I + dl^I \psi_I = l^I (\partial_\alpha + \Omega_\alpha) \psi_I e^\alpha \equiv l^I \nabla_\alpha \psi_I e^\alpha. \quad (9)$$

The action integral is the volume of the state, $\Psi \equiv (\psi_I, d\psi_I), (\psi^I, d\psi^I) \subset T^* \Sigma$, in spin space: the phase space of $8$-spinor flows. In complex coordinates on $T^* \Sigma$,

$$S_g = \int_{M_g} \left[ (\psi^I - i d\psi^I) \wedge (\psi_I + d\psi_I) \right]^4 \rightarrow PT \rightarrow \int_{M_g} \left[ (\psi^I d\psi_I) \right]^4, \quad (10)$$

in either the $PT$-symmetric ($PT_s$) case, or the $PT$-antisymmetric ($PT_a$) case, $\psi^I d\psi_I = \pm d\psi^I \psi_I$ Stationarizing $S_g$ gives a minimal $4$-surface, $\Psi \in T^* \Sigma$.

In order to be localized inside a compact world tube, $B_4$ the matter spinors must match the vacuum distribution on its boundary, $\gamma_3 \equiv \partial B_4$ [Taubes], [Uhlenbeck]. In the regular, geometrical-optics regime outside, $S_g$ yields the proper effective actions for electroweak ($PTa$, or charge-separated) and gravitostrong ($PTs$, or neutral) fields, $d\psi = \kappa \psi$. Here it agrees with Witten’s ”Weiss-Zumino $4$ form,” $Tr(g^{-1} d g)^4$, whose action is quantized over the boundary, $\gamma_4 \sim \partial B_5$, of a $5$-manifold [Witten1],
[Witten2]. Here, we find \( B_5 \sim \mathbb{C} \times S_3 \) embedded in the position-world velocity phase space, \( z^\alpha = x^\alpha + y^\alpha \), with complex time coordinate, \( z^\alpha = t + iT \).

Geometrical optics break down on boundary caustics, \( \gamma_{J} = \partial B_{J} \), where, the spin map \( S : TM \rightarrow T\Sigma \) becomes singular, and acquires a \( J \)-dimensional kernel. The domains these caustics enclose are branched covers, \( B_{J} \equiv *D^J \), with \( J \) extra bispinor sheets in spin space over each spacetime point \( x \in B_{J} \). These accommodate the wave functions of \( J \)-bispinor particles.

Caustics arise in optics, hydrodynamics, chemical reactions, acoustics, etc. as loci of partial focusing, or shock fronts [Arnold]. Joe Keller, Alan Newell [Newell], and others have used a powerful tool to look inside these apparent singularities: singular perturbation theory or multiscaling; defining a short spacetime scale inside the shock, and matching the inner solution to the outer one on the shock boundaries. We apply it to give a system of coupled envelope-modulation equations [Newell] to nonlinear waves in the 8 spinor medium: the spinfluid, and find that their caustics are the elementary particles. We outline the results below; details of the calculations appear in Part III [M.C. 4].

2 Singularities and Stratification

In the geometrical-optics (g.o.) regime, \( D^0 \), regular phase flows are created by nonsingular active-local (perhaps, path-dependent) Einstein transformations, \( (L(x), R(x)) \in E_A \), acting on the vacuum spinors, \( (\ell, \hat{r}) \), written as \( GL(2, \mathbb{C}) \) matrices column wise and row wise respectively:

\[
\ell(x) = \ell \exp \left[ \frac{i}{2s} \zeta_L^a q_a \right] \equiv \hat{\ell} L(x); \quad r(x) = \left[ \frac{i}{2s} \zeta_R^a \right] \hat{r} = R(x) \hat{r}.
\]

In the PT case, \( R(x) = L^{-1}(x) \), multiplying a spinor by the differential of the PT opposed spinor gives effective spin connections: \( C \)- algebra-valued 1 forms; or vector potentials [Keller]:

\[
\Omega_L \equiv -\ell^{-1} d\ell(x) = d\zeta_L = [\partial_a \zeta^a_L(x) q_a e^a, \quad \Omega^R \equiv (d\mathbf{r}) r^{-1}(x) = d\zeta_R = [\partial_a \zeta^a_R(x) q_a e^a].
\]

However, even for a regular initial distribution of spinor fields, codimension- \( J = (1, 2, 3, 4) \) singular Lagrangian \( \equiv *\gamma(4 - J) \), will form, shift, merge, annihilate, and recombine, like the projections of folds in a sheet to the bed. In addition to the regular stratum, \( \gamma^o \), where the projection from the Lagrangian submanifold of spin space solutions to the position-world velocity phase space,

\[
\pi : (\psi_I + i d\psi_I) \in \mathbb{C}T^* \Sigma \rightarrow x^a + iy^a \in \mathbb{C}T^* M,
\]

is 1 to 1, there will be codimension-\( J = (1, 2, 3, 4) \) branched covers, \( D^J \), where \( \pi \) is \( J + 1 \) to 1. Like the crisscrossing rays inside a kaleidoscope there are \( J + 1 \) world-velocity sheets, \( y^a \), over each spacetime point \( x^a \in B_{J} \), in the supports, \( B_{J} \), of \( J \)-bispinor particles. Each support is bounded by loci of partial focusing, boundary caustics,\( \gamma_{J} = \partial B_{J} \) : folds, cusps, tucks, swallow-tails and knots, where spin rays \( \psi_I d\psi_I = d\mathbf{r} \) branch or converge [ref Arnold]. Each \( (4 - J) \) brane, \( B_{J} \), carries a \( J \)-form matter current, \( *J_{\text{dual}} \) to the Clifford volume element contributed by the \( (4 - J) \) vacuum pairs. We call this complex of branes and currents the Spin (4,C) complex, or spinfoam.
A 3-dimensional example is a foam of soap bubbles, with the regular stratum, $B_3 = \ast D^0$ (the volumes), and singular strata, $\gamma_2 \subset \partial B_3, \gamma_1 \subset \partial B_2, \gamma_0 \subset \partial B_1$: the surfaces, edges, and vertices. Each stratum, $D^j$, carries a $J$-form current: density in volumes $\gamma_3$ pressure on surfaces $\gamma_2$, tension in line segments $\gamma_1$, and force on nodes $\gamma_0$.

A codimension-$J$ bifurcation occurs at the critical point, $x_c \in \gamma_{4-J}$, where the rank of the Jacobian matrix, $[d\zeta](x_c) \equiv [\partial_\alpha \zeta_\beta](x_c)$, drops by J. Here, $J + 1$ phase differentials become linearly dependent, to span only a $(4 - J)$-dimensional subspace.

If the Hessian, $[d^2 \zeta](x_c)$, is singular there too, $|\partial_\alpha \partial_\beta \zeta|(x_c) = 0$, $x_c$ is a degenerate critical point: a caustic, where rays $d\zeta_I$ merge or split, and there is a change in the topology of the orbits.

This is dynamical symmetry breaking. One tool to detect it is the

**Equivariant Branching lemma** (Michel’s ”theorem”): If the isotropy subgroup, $H \subset E$, that fixes a solution $\Psi_c$ contains just a single copy of the identity representation, then $\Psi_C$ is a possible direction for dynamical-symmetry breaking ref. [Sattinger].

Some corollaries are

1. the branched covers and boundary caustics stratify the base space, $M$, into orbits of $E$-group actions into isotropy subgroups, $H$:

$$M = \bigcup_{J=0}^{4} B_{4-J} \oplus \gamma_{4-J}.$$ 

2. Generically, as you cross a boundary caustic $\gamma_{3-J} \equiv \partial B_{4-J}$, where $|dd\zeta| = 0$, ker $d\zeta$ picks up generators one at a time

3. The boundary of each stratum consists of singular loci belonging to the next higher stratum, except where two caustics intersect. Here, their co-dimensions add:

$$\gamma_{4-J} \cap \gamma_{4-K} = \gamma_{4-M} : M = J + K.$$ 

Gluing conditions for splicing a compact ”bubble”, $\Omega^J \equiv (\psi^J d\psi_J)^J$ of $J$ matter-spinor pairs into the vacuum distribution give constraints on their integrals. As the ”neck” of the $J$-tube $\gamma_{4-J}$ joining the matter bubble and the vacuum background contracts to (or expands from) a single point, the matter wave functions must match the vacuum spinors there. This demands integral periods for $J$-form matter currents over compactified $J$ cycles: quantized topological charges. [Taubes], [Uhlenbeck].

The **Fibration Theorem** [Milnor] guarantees a complete set of $(4 - J)$ parallelizable fiber coordinates bridging the gaps between codimension-$J$ singular loci: the integral curves of the vacuum spin forms, $\tilde{\Omega}^{4-J}$ (Table I).
ments that multiply $\hat{\Omega}$ to contribute to the action. These make the Clifford line, surface, and volume elements quantumizing $\hat{\Omega}$ in a net action polynomial in the scale factor, $\gamma$.

uum spin forms, $\omega$.

It is the vacuum spinors, $\psi$.

The constraint that the Lagrangian density must be a C scalar assures that only the parts of $\hat{\Omega}^{J-1}$ both Clifford and Hodge dual to the matter forms, $\hat{\Omega}^J \equiv (\psi^J d\psi_J)^I$, to contribute to the action. These make the Clifford line, surface, and volume elements that multiply $\hat{\Omega}^J$ to fill out the E-invariant (C-scalar) 4-volume element,

$$|\langle d\zeta \rangle|^4 \sigma \omega 0 \wedge \omega 1 \wedge \omega 2 \wedge \omega 3 : \sim \gamma^4 (dx)^4.$$

Any C-dual contribution to $S_g$ must therefore be Hodge dual, as well, effectively quantizing $\hat{\Omega}^J$ against dual (perpendicular) cycles, $\gamma_{J-1}$, as well as over cycles $\gamma_J$ (e.g. quantization of electric, flux, $F_{\alpha\beta}$, over $S_2(\theta, \phi)$ [M.C. 2]).

These topological charges remain constant with cosmic expansion, while the vacuum spin forms, $\hat{\Omega}^{J-1}$ (table I) give a factor of $k^{J-1} \sim \gamma^{J-4}$ to the action contributed by the $D^J$ stratum. Integrating in the comoving frame, $E^\alpha = \gamma e^\alpha$ results in a net action polynomial in the scale factor, $\gamma$: the effective potential,

$$V(n, \gamma) = \int_{D_0} \hat{\Omega}^4 + \gamma \int_{D_1} \hat{\Omega}^3 \wedge (\psi^I d\psi_I) + \gamma^2 \int_{D_2} \hat{\Omega}^2 \wedge (\psi^I d\psi_I)^2 + \gamma^3 \int_{D_3} \hat{\Omega} \wedge (\psi^I d\psi_I)^3 + \gamma^4 \int_{D_4} (\psi^I d\psi_I)^4 = 16\pi^3 \left[ n_0 + n_1 \gamma + n_2 \gamma^2 + n_3 \gamma^3 + n_4 \gamma^4 \right],$$

where $n_J$ is the population of the $J$th stratum [M.C..3].

The polynomial $V(n, \gamma)$ can mimic the effect of the Higgs field by mixing positive-definite quadratic couplings in $\gamma^2$ with negative-definite quartic ones in $-\gamma^4$, to create a "Mexican hat" potential. But, unlike standard Q. F. T., the lepton, meson, hadron and atomic masses appear in a 4-term sequence, at $O(\gamma, \gamma^2, \gamma^3, \gamma^4)$, respectively.

The $\hat{\Omega}^3$ term contributes the 3-volume element in spin space to the Noether charge under complex-time ($z^0 \equiv t + iT$) translation, which includes the Jacobian determinant of the 3-space block of spin map, $S$:

$$|\langle d\zeta \rangle|^3 \sim s^3 e^1 \wedge e^2 \wedge e^3.$$

This gives quantization of both mass and charge:

$$\int_{B_3} [(\partial_\theta \theta^0) - i(\partial_\theta \theta^0)] e^1 \wedge e^2 \wedge e^3 = M + iQ.$$

It is the vacuum spinors, hiding the C 3-volume element $\hat{\Omega}^3$ that endow frequency, $\omega \equiv (d_\theta \theta^0)$, with mass: Mach's principle in action.

Table I: the vacuum spin forms,

Assuming the vacuum spinors all have the same amplitude, $k^\frac{4}{3},$

$$\hat{\Omega} = \pm \left( \frac{k^i}{3} \right) q_i e^a,$$

$$\hat{\Omega}^2 = \left( \frac{k^i}{3} \right)^2 q_i \left[ e_{jk} e^i \wedge e^k \mp e^0 \wedge e^i \right],$$

$$\hat{\Omega}^3 = \pm \left( \frac{k^i}{3} \right)^3 q_i e_{jk} e^i \wedge e^k \wedge e^0 \pm i e_{jk} q_i e^i \wedge e^k \wedge e^0,$$

$$\hat{\Omega}^4 = \left( \frac{k^i}{3} \right)^4 q_i \left[ \epsilon_{\alpha\beta\gamma} e^\alpha \wedge e^\beta \wedge e^\gamma \wedge e^\delta \right] = \frac{3}{2} \left( \frac{k^i}{3} \right) d^4 V,$$
We outline the results here; details of calculations appear in Part III [M.C.4].

From the Lagrangian submanifold in spin space to spacetime, "two timing and double crossing" Cohen calls spinfluid, ply it here to caustics in the front, and match the inner solution to the outer one on the shock boundaries. We approximate [Newell], and others, we defining a short spacetime scale, \( J \) rank of the spin map drops by \( \nu \) which turn out to be the elementary particles. We outline the results here; details of calculations appear in Part III [M.C.4].

First, we express each spinor field as a vacuum field, \( \varphi_I \equiv (\ell_{\pm}, r_{\pm}) \) of amplitude \( k^+ \sim \gamma^{-\frac{1}{2}} \), plus an envelope modulation:

\[
\ell_I(x, X) = k^\pm \ell_{\pm}(X) + \psi_{L,\pm}^\pm(x) = \gamma^{-\frac{1}{2}} \ell_{\pm}(X) + \psi_{L,\pm}(x); \\
r_{\pm}(x, X) = \gamma^{-\frac{1}{2}} (X) + \psi_{R,\pm}.
\]

In inflated regimes, like ours, \( \gamma \gg 1 \). In superdense regimes, \( \gamma \ll 1 \): the matter spinors are ripples riding on the vacuum: a deep ocean of dark energy. Since solutions are either symmetric or antisymmetric about the critical radius, \( a = a_{\#}; \gamma = 1 \), we can consider either case, and cover both [M.C.1]. Inserting ansatz (17), we obtain effective Lagrangians, \( L^J \), in which \( (4 - J) \) vacuum pairs couple \( J \) matter pairs. Varying with respect to \( \dot{\ell}_{\pm} \) or \( \dot{r}_{\pm} \) gives the massless Dirac equations. These say that the vacuum spinors are Clifford-analytic and conjugate-analytic respectively:

\[
\begin{align*}
\bar{D}\ell_{\pm} &\equiv q^\alpha (\partial_\alpha + \hat{\Omega}_\alpha^R) \ell_{\pm}(X) = O, \\
D\bar{r}_{\pm} &\equiv q^\alpha (\partial_\alpha + \hat{\Omega}_\alpha^L) \bar{r}_{\pm}(X) = O.
\end{align*}
\]

Covariantly constant (freely-falling) solutions, \( (\partial_\alpha + \Omega_\alpha) (\dot{\ell}_{\pm}, \dot{r}_{\pm}) = 0 \) define inertial spin frames. On \( M_{\#} \equiv S_l S_3(a_{\#}) \),

\[
\begin{align*}
\dot{\ell}_{\pm}(X) &= \dot{\ell}_{\pm}(0) \exp\left(\frac{1}{2a_{\#}} X^\alpha \sigma_\alpha^\pm\right); \\
\hat{\Omega}_\alpha^L &= \frac{i}{2a_{\#}} \sigma_\alpha^\pm e^{\alpha}, \\
\dot{r}_{\pm}(X) &= \dot{r}_{\pm}(0) \exp\left(\frac{1}{2a_{\#}} X^\beta \sigma_\beta^\pm\right); \\
\hat{\Omega}_\alpha^R &= \frac{i}{2a_{\#}} \sigma_\beta^\pm e^{\beta}.
\end{align*}
\]

For a given scale factor, \( \gamma \), the vacuum action is extremized when the inertial spinors span a hypercube in spin space.

Neutral combinations of vacuum spinors could be called "cosmological neutrinos", \( \nu_l = (\ell_+ \oplus \ell_-); \nu_r = (\ell_- \oplus \ell_+). \) More generally, left and right chirality moving spin frames, \( \ell_{\pm} \) and \( r_{\pm} \), are given by path-dependent, active-local \( (E_A) \) transformations on the inertial spinors [M.C. 1], [M.C. 2], [M.C. 3]. These vary on the cosmic scale, so \( \gamma \) beats of the logic clock, \( \Delta X^0 = \gamma \), elapse for each beat, \( \Delta x^0 = 1 \), of the local clock.
\[
\ell(X,x) \equiv \ell_\pm (X) L^\pm (x);
\]

\[
r(X,x) = \tilde{r}_\pm (X) \tilde{R}^\pm (x).
\] (19)

At \(O(\gamma)\), we obtain the massive Dirac system as our coupled-envelope equations. Dirac mass - chiral cross coupling - appears via a spin \((4, C)\) resonance; the 8-spinor analog of 4-wave mixing in nonlinear optics [M.C.5].

To contribute a \(C\) scalar 4 form \(\sigma_0 e^0 \wedge e^1 \wedge e^2 \wedge e^3\) to the action integral, a chiral pair of matter spinors must find 3 other pairs of vacuum spinors whose product meets the Bragg (solvability) conditions; the massive Dirac equations,

\[
\bar{D}\psi_I^T \equiv q^a (\partial_\alpha + \hat{\Omega}_I^L) \psi_I^T (X) = [2a_#]^{-1} \psi_I^R
\]

\[
D\psi_I^R \equiv q^a (\partial_\alpha + \hat{\Omega}_I^R) \psi_I^R (X) = [2a_#]^{-1} \psi_I^T.
\] (20)

The electron mass-the inverse of the critical diameter, 2\(a_#\), comes from the product of the 3 unbroken vacuum pairs; the \(\hat{\Omega}^3\) in Table I. If the vacuum spinors have different amplitudes, the scalar mass term is replaced by the term \(\psi^I \bar{\Omega}^{[3]}_J \psi_J\), in the lepton mass matrix. This is a rank-2 tensor product of the 6 remaining vacuum spinors C dual to \(\psi^I, \psi_J\); the ones needed to make the \(C\)-scalar (\(\sigma_0\)) term, at \(O(\gamma)\): \(\psi^I \bar{\Omega}^{[3]}_J \psi_J \in \{2, 6\} \in L^1\).

At \(O(\gamma^2)\), integration by parts gives wave equations in \((\bar{D}D + D\bar{D}) \equiv \Delta\) : Klein-Gordon (spin 1 or 0) equations, sourced in the baryon-current 3 form, \(J\), quantized over 3 -cycles:

\[
(\Delta + \hat{\Omega}^2)\bar{\Omega} = J; \quad \int_{B^3} J = \int_{B^3} [\partial_\alpha \bar{\partial}_\alpha] d\zeta^I \wedge d\zeta^2 \wedge d\zeta^3 = 8\pi^2 [2a_#]^{-1} B,
\] (21)

where \(B\) is the Baryon number. Again, the mass term, \(\hat{\Omega}^2\), comes from the vacuum energy. For photons, \(B = 0\).

At \(O(\gamma^3)\), the principal part is a system of 3 Euler equations, coupling each quark current, \(Q_I\), to the 2 others through the vacuum spinors:

\[
Q_I \in \{l \otimes r\}; Q^I \in r[r \otimes l]; \quad \bar{D}Q_I \sim [2a_#]^{-1} T^J_l Q_J Q^K.
\]

\[
DQ^I \sim [2a_#]^{-1} T^J_l Q^I Q^K.
\]

\(T\) is the rank -3 "moment of inertia" tensor, with eigenvalues \(I \equiv (p,q,r)\). Orbits lie on invariant tori or ellipsoids, and close for integer ratios \((p/q/r)\), with a frequency that is a common multiple, \(s = CM(p,q,r)\). Pythagoras would like this; it is the condition for a harmonious 3-note chord.

At \(O(\gamma^4)\), we obtain a class of exact solutions we call Spin(4,C) Vortices; "vortex atoms" with dense nuclei of matter currents flowing in the +\(T\) direction, outward from the big bang, and diffuse shells of returning currents, with charges +\(Z\) and -\(Z\), respectively [M.C.5]. Kelvin would like this.

Behind all this algebraic structure lives a simple physical picture: each chiral pair, \(q_I \equiv (l \otimes r)\), acts as a mirror for the other 3 chiral pairs, bootstrapping from noise a resonant s- cycle.
3 Reflection Varieties and Particle Masses

The Dirac operator, \( D = \sigma^a \partial_a : e^\beta \leftrightarrow q^\beta \), assigns a spacetime differential to an infinitesimal displacements in the Clifford algebra [BDS]; [G-M].

But on boundary caustics, \( \gamma_{3-J} \subset D B_{4-J} \) the spin map, \( S^* = d\zeta = [\partial_\alpha \zeta^\beta] \sigma_\beta e^\alpha \), becomes singular, with rank \((4-J)\). Here, some steps in internal phase no longer pull back to spacetime increments. Meanwhile, inside there are \( s \) bispinor sheets for each spin direction in the spin bundle over the particle support, \( B_{4-J} \).

For a volume element, \( e^\lambda \), to contribute to the action, the product of the \( 4 \) reflection operators in it must be a scalar. Physically, each cycle of \( 4 \) "interference gratings" \( \ell \otimes r \) — including the curved gratings involving matter spinors, must close to form a resonator, with a net loop transfer function proportional to the identity, \( \sigma_\alpha \). This closure constraint admits only a few sets of integers \{\( p, q, r, s \}\) characterizing the possible symmetry groups of singular loci and isophase contours for particle wave functions: the Coxeter groups, \( R_s \), [Coxeter] with their invariant polynomials in \( 4 \) complex variables, the Breiskorn varieties [Milnor]:

\[
R_s = (p, q, r)_s; \quad B(x, y, z, T) \equiv (x^p + y^q + z^r + T^{-s}) = \text{const.} \quad (22)
\]

These are \textit{fibred knots}. For example, \( (2, 2, 3)_3 \cap S_3(1) \) is a trefoil knot, with its isophase fibers, \( \ln f(x, y, z) = \text{const} \), making three vertical twists about the singular filament, \( f(x, y, z) \equiv x^2 + y^2 + z^3 = 0 \), over 2 horizontal circuits.

The isophase contours over position \( x^\alpha \in M \) seem to cross in the projections,

\[
\Pi : x^\alpha + iy^\alpha \rightarrow x^\alpha, \quad \zeta^\alpha = \theta^\alpha + i\phi^\alpha \rightarrow \theta^\alpha,
\]

from phase space to spacetime, like the crisscrossing rays in a 3D kaleidoscope. These apparent crossings are resolved by lifting via \( \Pi^{-1} \): i.e., by separating overlapping C-algebra valued wave vectors in the "quiver" of spin waves over \( x^\alpha \).

It turns out [Cox] that the Coxeter groups varieties \( (p, q, r)_s \) exhaust the topological types of \textit{resolvable singularities}. This is just one aspect of "the profound connections between the critical points of functions, quivers, caustics, wavefronts, regular polyhedra,... and the theory of groups generated by reflections" [Arnold 2]. The profound connections that are important here are

1. \( L \) or \( R \) multiplication by a \textit{spacelike} C vector gives a \( L \)- or \( R \)-\textit{helicity twist} about axis \( \ell \) or \( \bar{\ell} \) by angle \( \frac{\alpha}{2} \) or \( \frac{\bar{\alpha}}{2} \):

\[
\ell' = L(\lambda) \ell \equiv \exp \left( \frac{\lambda \ell}{2} \cdot \sigma \right) \ell; \quad r' = r \bar{R}(\rho) \equiv r \exp \left( \frac{i\rho \bar{\ell}}{2} \cdot \sigma \right) \Rightarrow q' = (\ell \otimes r)' = L(\ell \otimes r) \bar{R}.
\]

\( L \) or \( R \) action generated by a \textit{null} C vector gives an additional \( U(1) \) twist about the \( T \) axis:

\[
\ell' = L \ell = \exp \left( \frac{i\lambda}{2} \sigma^0 \pm i\ell \cdot \sigma \right), \quad r' = \bar{R} = \bar{r} \exp \frac{i\rho}{2} \left[ \pm \sigma^0 \pm i\bar{\ell} \cdot \sigma \right] \Rightarrow q' \equiv (\ell \otimes r)' = L(\ell \otimes r) \bar{R}.
\]

2. Conjugation by a spacelike C vector reflects a flag (a 3 vector, \( q \), and its normal frame) in a mirror with unit normal \( a \):

\[
q' = -a a^{-1} = [i a | a^{-1}] = [i a | (\ell \otimes r)][i a^{-1}] \Rightarrow \ell' = [i a]; \quad r' = r [i a^{-1}].
\]

An ordinary (period -2) reflection reverses the flag \( L \Rightarrow R \), preserves function values, but reverses differentials, creating a \textit{singularity} on the mirror plane. A domain
$B_3$ bounded by mirrors (like a laser cavity) becomes a resonator: it traps waves at its fundamental frequency or its harmonics to create a standing wave.

3. Reflections in mirror planes $P_\perp$ and $Q_\perp$ that intersect at dihedral angle $\theta$ give a rotation by $\theta$ around the (spacelike) axis. $a = P_\perp \cap Q_\perp$: $q' = l' \otimes \bar{r} = L (l \otimes \bar{r}) L^{-1}$; $L = \exp \left( \frac{i}{2} \bar{l} \cdot \sigma \right)$.

4. $L$ or $R$ action by the complex Clifford ($\mathbb{C}C$) vector, $\exp \frac{\varrho}{\pi} \left[ \pm \sigma_0 \pm i \hat{l} \cdot \sigma \right]$, gives a period-$p$ reflection. It takes $p$ repeated reflections to close a spatial cycle; this first happens for $\theta = \pi$, making an image with dihedral symmetry, $D_p$. Parallel mirrors a distance $\frac{1}{2}$ apart generate translations of $\Delta$.

5. On $[M_\#]_{\text{diag}} \equiv [S|a_\#\rangle \times S|a_\#\rangle]_{\text{diag}}$, multiple reflections in 3 planes that all intersect in one point form a 3D kaleidoscope in spin space. Its image is a discrete subgroup, $R_s \subset U(1) \times SU(2)$, provided that the three dihedral angles, $\left( \frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r} \right)$ and the multiplicity, $s$, obey the closure constraint: to commute, all 4 arguments above must be multiples of $\pi$:

$$R_P \equiv \exp \left( i \pi sp^{-1} P \right), \quad R_Q \equiv \exp \left( i \pi sq^{-1} Q \right), \quad R_R \equiv \exp \left( i \pi sr^{-1} R \right); \quad R_S \equiv \exp \left( i \pi sr^{-1} T \right); \quad R'_P = R'_Q = R'_R = R'_S = R_P R_Q R_R R_S = -1$$

$$= \exp \frac{i}{2} (n + 1) s \pi \Rightarrow s \left( p^{-1} + q^{-1} + r^{-1} - s^{-1} \right) = (n + 1) \frac{1}{2}.$$  

Here $(p, q, r)$ are integers, and $s$, a common multiple, $s = \text{C.M.} (p, q, r)$: the multiplicity, or Coxeter number of the Coxeter group. These are the $s$-fold covers of the Rotation, Dihedral, Tetrahedral, Octahedral, and Icosahedral groups [Coxeter],

$$(p, q, r)_s \subset \{ A_p, D_p, T, E_6, E_8 \}.$$  

The cycle of $s$ reflections closes after period $t = \tau$. For the odd spin structure [Geroch] this is the time it takes for a lightlike phase front, $\theta = \text{Const.}$, to circumnavigate a closed light cone, $[S|S_3(a_\#\rangle)]_{\text{diag}}$, twice. Patching principle $(P2)$ says that, for a periodic solution to match the vacuum on the boundary $\gamma_3 = \partial B_4$, the frequency, $\omega(s, n)$ inside the particle’s world tube must be an even harmonic of the vacuum frequency, $(2a_\#)^{-1}$:

$$\omega(s, n) = [2n(s/2)^3 + 1](2a_\#)^{-1} = [n(s/2)^3 + \frac{1}{2}](2a_\#)^{-1}.$$  

Each even increment, $\Delta n = 2$, adds a mass increment of

$$m = (s/2)^3(2a_\#)^{-1}.$$  

For $s = 2$, this is the mass of an electron, governed by the massive Dirac equations, (21). The critical radius, $a_\#$, is the time taken for a lightlike pulse to traverse the electron’s radius, in the (continuum) limit.

But isn’t each cycle of Bragg reflections a discrete process? Yes. But we must sum over all of them to get the wave equation for the matter fields; just as we sum over random walks to get the heat equation. The “random walk” underlying the Dirac system - the Dirac propagator - is the sum over all null zigzag histories connecting the initial and final states [Feynman], [Penrose], [Ord]. We explain, briefly here; see II for details [M.C.3].
So far, the nonlinear 8-spinor mixing has appeared as multilinear mode-mode coupling in spin space. In the spacetime picture, light-like spin rays propagate between localized scattering vertices.

Penrose calls a vertex where a L-chirality spinor reflects from a Bragg mirror into an oppositely-propagating R- chirality one (or vica-versa). A mass scattering; is a null zigzag a pair of mass scatterings, \(L \rightarrow R \rightarrow L\); the discrete version of a fold. To close a cycle of null zigzags, each chiral component must return to its original value. This happens only after a common multiple (c. m.) of the three binary reflection degrees, \(s = cm(p,q,r)\). But it takes only \(\frac{s}{2}\) reflections to restore a bispinor state; \(R_{\frac{s}{2}}: (\ell \oplus r) \rightarrow (r \oplus \ell)\), for \(\frac{s}{2}\) odd. The energy - the 3-volume in spin-space- is counted according to its multiplicity, \(s\): the number of spin-space sheets above the particle’s support.

In the simplest case, the free electron, \(e^- \equiv (l \oplus r) \in \langle 2, 2, 2 \rangle\), all 3 dihedral angles are \(\frac{\pi}{2}\). The 3 pairs of vacuum spinors which trap the matter pair inside a 3-cube form opposing pairs of corner-cube reflectors.

As we decrease one of the dihedral angles, we get a 3-cycle at \(\frac{\pi}{3}\); 3 sheets bounded by a tuck caustic ref. [Arnold]. But the cycle generated by both reflections doesn’t close up again until we reach their least-common multiple (lcm), \(2 \cdot 3 = 6\). giving a 6-fold cover of the reflection group: the Coxeter group, \(< 2,3,4 >_{6}\), with multiplicity 6. We identify this as the muon; and the next closed reflection cycle, \(< 2,3,4 >_{12} \) as the tauon. More generally,

A massive lepton, meson, or hadron is composed of \(J = 1,2,3\) pairs of oppositely-propagating bispinor pairs, trapped inside a timelike world tube by Bragg reflections off interference gratings with \((4-J)\) vacuum pairs on its boundary.

What is new here is that the reflection groups \(< p,q,r >_{s}\) of multiplicity \(s = (2, 3, 4, 5, 6, 12, 30)\) not only classify the elementary particles, but give their mass ratios, (24)

\[
\frac{m}{m_e} = \left(\frac{s}{2}\right)^3.
\]

These agree with the observed mass ratios (table III) within a few percent (except for the pions-which are off by \(\sim 25\%\) ).
Table III: Spin-J Resonances:

| Particle | Binary Group | Coxeter Numbers | $\frac{m}{m_e}$ |
|----------|--------------|-----------------|-----------------|
| $e^-$    | $D_2$        | $\langle 2, 2 \rangle_2$ | 11              |
| $e^-$    | $D_p$        | $\langle 2, 2, p \rangle_p$ |               |
| $e^-$    | $T$          | $\langle 2, 3, 3 \rangle_6$ |               |
| $\mu^-$  | $O$          | $\langle 2, 3, 4 \rangle_{12}$ | 216 207         |
| $\tau^-$ | $I$          | $\langle 2, 3, 5 \rangle_{39}$ | 3375 3478       |
| $\pi^-$  | $D_3 \otimes \bar{D}_4$ | $\langle 2, 2, 3 \rangle_3 \otimes \langle 2, 2, 4 \rangle_4$ | $d\bar{u}$ 216 275 |
| $\pi^-$  | $D_4 \otimes D_5$ | $\langle 2, 2, 5 \rangle_5 \otimes \langle 2, 2, 4 \rangle_4$ | $s\bar{u}$ 1000 975 |
| $\pi^-$  | $D_5 \otimes D_6$ | $\langle 2, 2, 5 \rangle_5 \otimes \langle 2, 2, 6 \rangle_6$ | $s\bar{c}$ 3375 3647 |
| $n_e$    | $D_6 \otimes \bar{D}_6$ | $\langle 2, 2, 6 \rangle_6 \otimes \langle 2, 2, 6 \rangle_6$ | $c\bar{c}$ 5832 5686 |
| $p^+$    | $D_4 \otimes D_4 \otimes D_3$ | $\langle 2, 2, 4 \rangle_4 \otimes \langle 2, 2, 3 \rangle_3 \otimes \langle 2, 2, 4 \rangle_4 [u, d]$ | $u$ 1728 1836 |

In the quantum calculation (III) we sum over histories in "imaginary time", $T$: all possible chains of null zigzags connecting the initial and final states [MC3].

Microscopically, it seems, the whole world, both outside and inside the world tubes of massive particles, resolves into a network of light-like spinors, and their scattering vertices: their multilinear interactions.

4 Conclusions and Open Question

Spin Principle Pl says that the 8-spinor bundle, $8$, is the physical reality; and that the action is just its volume in spin space. Our spacetime 4-fold, $M$, and the particle wave functions, $\Psi$, are horizontal and vertical projections of a minimal-surface in spin space: the spinfoam. The regular stratum, or vacuum, $D^o$, can be combed parallel locally by path-dependent phase differentials, $d\xi_I = \Psi^I d\Psi_I$, by spin connections: the vector potentials. Their spin curvatures, $\Psi^I \dd \Psi_I$, are the fields. If these carry a nontrivial flux (topological charge) over the boundary, it must enclose a singularity-at least, in the projection, $\pi: 8 \rightarrow M$: a caustic.

These are characterized by their symmetry groups in spin space, and there are only a few admissible types: the Coxeter groups, $(p, q, r)_s$. Their wave functions are Brieskorn varieties: fibred knots, whose isophase contours and normal rays ("lines of force") radiate and terminate on singularities. Their masses - i.e. their Noether time-translation charges, turn out to be $m = (s/2)^3$, in natural units of $2\alpha^{-1}$; the mass of the electron ($s = 2$):

$$\frac{m}{m_e} = \frac{(s/2)^3}{2\alpha^{-1}}$$

Why should the reflection groups -the same groups that classify resolvable singularities, regular polyhedra, Lie algebras, quivers, frieze patterns, honeycombs, crystals, and caustics- classify the elementary particles? Because they all arise from the generic structures of singularities in flows.
Like heat flow resolves into random walks, at the critical scale, $a_\#$, the 8–spinor flow resolves into a microhistory of null zigzags. In each discrete history, the multiplicity, $s$– the number of null zig-zags it takes to close a cycle- must be a common multiple of the reflection degrees $p, q,$ and $r$. This results in an image in spin space like that formed by light rays crisscrossing in a 3D kaleidoscope, with mirrors at angles \( \frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r} \).

What is subtle and beautiful about this picture is
1) how self-consistent cycles of $J$ chiral pairs of matter waves and $(4-J)$ vacuum pairs ”bootstrap” each other into existence as the radius passes through $\gamma = 1$, where $T = a_\#$; the critical radius for the inflationary phase transition (III).
2) How the $J$–dimensional critical modes that ”crystalize out” at $O(\gamma^{J+1})$, program the multilinear couplings of modes at the next shorter scale, much as a volume hologram couples input to output waves. This results in the ramification of patterns at smaller and smaller scales, much like the main sequence of wavenumber-doubling bifurcations leading to turbulence.

Is this what we’re seeing in the sequence of $l = (200, 400, 800...)$ modes in the Cosmic Microwave Background near the time of decoupling; or in the foam-like structure of incident $J = (1, 2, 3)$– branes in the large-scale distribution of galaxies?

Perhaps the regular background of vacuum spinors is the dark energy- the invisible Dirac sea, on which the wave functions of visible matter ride like waves on the surface of the ocean. Since the Dirac mass term is created by products of vacuum spinors, these might be called dark matter. This picture not only shows how the ”distant masses” endow particles with their rest masses, but closely approximates the measured particle masses.

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