Existence of rational curves on algebraic varieties, minimal rational tangents, and applications

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Introduction

Over the last two decades, the study of rational curves on algebraic varieties has met with considerable interest. Starting with Mori’s landmark works [Mor79, Mor82] it has become clear that many of the varieties met daily by the algebraic geometer contain rational curves, and that a variety at hand can often be studied by looking at the rational curves it contains. Today, methods coming from the study of rational curves on algebraic varieties are applied to a broad spectrum of problems in higher-dimensional algebraic geometry, ranging from uniqueness of complex contact structures to deformation rigidity of Hermitian symmetric manifolds.

In this survey we would like to give an overview of some of the recent progress in the field, with emphasis on methods developed in and around the DFG Schwerpunkt “Globale Methoden in der komplexen Geometrie”. Accordingly, there is a large body of important work that we could not cover here. Among the most prominent results of the last years is the breakthrough work of Graber, Harris and Starr, [GHS03], where it is shown that the base of the rationally connected fibration is itself not covered by rational curves. Another result not touched in this survey is the recent progress toward the abundance conjecture, by Boucksom, Demailly, Păun and Peternell, [BDPP04] —see the article of Jahnke-Peternell-Radloff in this volume instead.

Outline of the paper. We start this survey in Chapter 1 by reviewing criteria that can be used to show that a given variety is covered by rational curves. After mentioning Mori’s results, we discuss foliated varieties in some detail and present a recent criterion that contains Miyaoka’s fundamental characterization of uniruledness, [Miy87], as a special case. Its proof is rather elementary, and a number of known results follow as simple corollaries. Keel-McKernan’s work [KMcK99] on rational curves on quasi-projective varieties will be briefly discussed.

We will then, in Chapter 2 discuss the geometry of (higher-dimensional) varieties that are covered by rational curves, and present some ideas how minimal degree rational curves can be used to study these spaces. The hero of these sections is the “variety of minimal rational tangents”, or VMRT. In a nutshell, if $X$ is a projective variety covered by rational curves, and $x \in X$ a general point, then the VMRT is the subvariety of $\mathbb{P}(T_X|_x)$ which contains the tangent directions to minimal degree rational curves that pass through $x$. If $X$ is covered by lines, this is a very classical object that has been studied by Cartan and Fubini in the past. In the general case, the VMRT is an important variety, similar perhaps to a conformal structure, whose projective geometry as a subset of $\mathbb{P}(T_X|_x)$ encodes much of the information on the underlying space $X$ and determines $X$ to a large extent.

In Chapters 3–5 we will apply the methods and results of Chapters 1 and 2 in three different settings. To start, we discuss the moduli space of stable rank-two vector bundles on a curve in Chapter 3. There exists a classical construction of rational curves on these spaces, the so-called “Hecke-curves”. These have been used to answer a large number of questions about moduli of vector bundles. We name a few of the applications and sketch a proof for a result that helps to give an upper bound for the multiplicities of divisors at general points of the moduli space; the result bears perhaps some resemblance with the classical Riemann singularity theorem.

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1 see [Ara05] for a good introduction
The existence results of Chapter 1 can also be used to study varieties for which it is known *a priori* that they are not covered by rational curves. We conclude this paper by giving two examples in Chapters 4 and 5.

Chapter 4 deals with deformations of surjective morphisms $f : X \to Y$, where the target is not covered by rational curves, or at least not rationally connected. It turns out that there exists a natural refinement of Stein factorization which factors $f$ via an intermediate variety $Z$, and that the existence results of Chapter 1 can be used to show that the associated component of the deformation space $\text{Hom}(X, Y)$ is essentially the automorphism group of $Z$.

In Chapter 5 we apply the results of Chapter 1 to the study of families of canonically polarized varieties. Generalizing Shafarevich’s hyperbolicity conjecture, it has been conjectured by Viehweg that the base of a smooth family of canonically polarized varieties is of log general type if the family is of maximal variation. Using Keel-McKernan’s existence results for rational curves on quasi-projective varieties, we relate the variation of a family to the logarithmic Kodaira dimension of the base and sketch a proof for an affirmative answer to Viehweg’s conjecture for families over surfaces.

Unless explicitly mentioned, we always work over the complex number field.

**Other references.** The reader who is interested in a broader perspective will obviously want to consult the standard reference books [Kol96] and [Deb01]. Hwang’s important survey [Hwa01] explains by way of examples how rational curves can be employed to study Fano manifolds of Picard number one. The article [AK03] contains an excellent introduction to the deformation theory of rational curves and rational connectivity, also in the more general setting of varieties defined over non-closed fields.
Part 1

Existence of rational curves
Existence of rational curves

The modern interest in rational curves on projective varieties started with Shigefumi Mori’s fundamental work [Mor79]. In his proof of the Hartshorne-Frankel conjecture, he devised a new method to prove the existence of rational curves on manifolds whose tangent bundle possess certain positivity properties. Though not explicitly formulated like this, the following results appear in his papers.

**Theorem 1.1 ([Mor79, Mor82]).** Let $X$ be a complex-projective manifold. If $X$ is Fano, i.e., if $-K_X$ is ample, then $X$ is uniruled, i.e., covered by rational curves. More precisely, if $x \in X$ is any point, then there exists a rational curve $\ell \subset X$, such that $x \in \ell$, and $-K_X \cdot \ell \leq \dim X + 1$. □

**Theorem 1.2 ([Mor82]).** Let $X$ be a complex-projective manifold. If $K_X$ is not nef, then $X$ contains rational curves. More precisely, if $C \subset X$ is a curve with $K_X \cdot C < 0$, and $x \in C$ any point, then there exists a rational curve $\ell \subset X$ that contains $x$. □

We refer to [CKM88] or [Deb01] for an accessible introduction. These results, and the subsequently developed “minimal model program” allowed, in dimension three, to give a positive answer to a long-standing conjecture attributed to Mumford that characterizes manifolds covered by rational curves as those without pluricanonical forms.

**Conjecture 1.3.** A projective manifold $X$ is covered by rational curves if and only if $\kappa(X) = -\infty$.

In higher dimensions, the conjecture is still open, although the recent result of Boucksom, Demailly, Peternell and Păun is considered a serious step forward.

The first Chern class of $T_X$ used in Theorems 1.1 and 1.2 is, however, a rather coarse measure of positivity. For instance, if $Y$ is a Fano manifold and $Z$ a torus, then the tangent bundle of the product $X = Y \times Z$ splits into a direct sum $T_X = F \oplus G$ where $F = p^*_1(T_Y)$ has positivity properties and identifies tangent directions to the rational curves contained in $X$. In general, rather than looking at $K_X$, one would often like to deduce the existence of rational curves from positivity properties of subsheaves $\mathcal{F} \subset T_X$ and relate the geometry of those curves to that of $\mathcal{F}$. The most fundamental result in this direction is Miyaoka’s criterion of uniruledness. In order to state Miyaoka’s result, we recall the theorem of Mehta-Ramanathan for normal varieties.

**Theorem 1.4 (Mehta-Ramanathan, Flenner, [MR82, Fle84]).** Let $X$ be a normal variety of dimension $n$, and $L_1, \ldots, L_{n-1} \in \text{Pic}(X)$ be ample line bundles. If $m_1, \ldots, m_{n-1} \in \mathbb{N}$ are large enough and $H_i$ are general elements of the linear systems $|m_i \cdot L_i|$, then the curve

$$C = H_1 \cap \cdots \cap H_{n-1}$$

is smooth, reduced and irreducible, and the restriction of the Harder-Narasimhan filtration of $T_X$ to $C$ is the Harder-Narasimhan filtration of $T_X|_C$. □
1. EXISTENCE OF RATIONAL CURVES

Figure 1.1. The points $x$ and $y$ are joined by a chain of rational curves of length $k$.

We refer to [Lan04b, Lan04a] for a discussion and an explicit bound for the $m_i$. A detailed account of slope, semistability and of the Harder-Narasimhan filtration of vector bundles on curves is found in [Ses82].

Definition 1.5. We call a curve $C \subset X$ as in Theorem 1.4 a general complete intersection curve in the sense of Mehta-Ramanathan.

Miyaoka’s result then goes as follows.

Theorem 1.6 ([Miy87, thm. 8.5]). Let $X$ be a normal projective variety, and $C \subset X$ a general complete intersection curve in the sense of Mehta-Ramanathan. Then $\Omega^1_X|_C$ is a semi-positive vector bundle unless $X$ is uniruled. □

We refer the reader to [MP97] for a detailed overview of Miyaoka’s theory of foliations in positive characteristic. The relation between negative directions of $\Omega^1_X|_C$ and tangents to rational curves has been studied in [Kol92, sect. 9]. We give a full account in Section 2.

Outline of the section. In Section 1, we study the case where $X$ is a complex manifold and $F \subset T_X$ is a (possibly singular) foliation. The main result—which appeared first in the preprint [BM01] of Bogomolov and McQuillan—gives a criterion to guarantee that the leaves of $F$ are compact and rationally connected. Miyaoka’s characterization of uniruledness, Theorem 1.6, and the statements of [Kol92, sect. 9] follow as immediate corollaries. Apart from a simple vanishing theorem for vector bundles in positive characteristic, the proof employs only standard techniques of Mori theory that are well discussed in the literature. In particular, it will not be necessary to make any reference to the more involved properties of foliations in characteristic $p$. We also mention a sufficient condition to ensure that all leaves of a given foliation are algebraic.

In Sections 2 and 3 we discuss the relation with the rationally connected quotient. The results of Section 1 are applied to show that $\mathbb{Q}$-Fano varieties with unstable tangent bundles always admit a sequence of partial rational quotients naturally associated to the Harder-Narasimhan filtration of the tangent bundle.

We will later need to discuss an analog of Mumford’s Conjecture 1.3 for quasi-projective varieties. Here the logarithmic Kodaira dimension takes the role of the regular Kodaira dimension, and rational curves are replaced by $\mathbb{C}$ or $\mathbb{C}^*$. For surfaces, this setup has been studied by Keel and McKernan. We recall their results in Section 4.

1. Rationally connected foliations

In the previous section we have mentioned that positivity properties of $T_X$ imply the existence of rational curves in $X$. Here, we will study how Mori’s ideas can be applied to foliations on complex varieties. We recall the notion of rational connectivity and fix notation first.
1. RATIONALLY CONNECTED FOLIATIONS

**Definition 1.7.** A normal variety $X$ is rationally chain connected if any two general points $x, y \in X$ can be joined by a chain of rational curves, as shown in Figure 1.1. The variety $X$ is rationally connected if for any two general points $x, y \in X$ there exists a single rational curve that contains both.

**Remark 1.8.** For smooth varieties, the two notions rationally chain connected and rationally connected agree, see [Deb01, sect. 4.7].

**Definition 1.9.** In this survey, a foliation $\mathcal{F}$ on a normal variety $X$ is a saturated, integrable subsheaf of $T_X$. A leaf of $\mathcal{F}$ is a maximal $\mathcal{F}$-invariant connected subset of the set $X^0$ where both $X$ and $\mathcal{F}$ are regular. A leaf is called algebraic if it is open in its Zariski closure.

The main result of this section asserts that positivity properties of $\mathcal{F}$ imply algebraicity and rational connectivity of the leaves. In particular, it gives a criterion for a manifold to be covered by rational curves.

**Theorem 1.10.** Let $X$ be a normal complex projective variety, $C \subset X$ a complete curve which is entirely contained in the smooth locus $X_{\text{reg}}$, and $\mathcal{F} \subset T_X$ a (possibly singular) foliation which is regular along $C$. Assume that the restriction $\mathcal{F}|_C$ is an ample vector bundle on $C$. If $x \in C$ is any point, the leaf through $x$ is algebraic. If $x \in C$ is general, the closure of the leaf is rationally connected.

The statement appeared first in the preprint [BM01] by Bogomolov and McQuillan. Below we sketch a simple proof which recently appeared in [KST05].

**Remark 1.11.** In Theorem 1.10, if $x \in C$ is any point, it is not generally true that the closure of the leaf through $x$ is rationally connected —this was wrongly claimed in [BM01] and in the first preprint versions of [KST05].

The classical Reeb stability theorem for foliations [CLN85, thm. IV.3], the fact that rationally connected manifolds are simply connected [Deb01, cor. 4.18], and the openness of rational connectivity [KMM92a, cor. 2.4] immediately yield the following.

**Theorem 1.12 ([KST05, thm. 2]).** In the setup of Theorem 1.10, if $\mathcal{F}$ is regular, then all leaves are rationally connected submanifolds.

In fact, a stronger statement holds that guarantees that most leaves are algebraic and rationally connected if there exists a single leaf through $C$ whose closure does not intersect the singular locus of $\mathcal{F}$, see [KST05, thm. 28].

The following characterization of rational connectivity is a straightforward corollary of Theorem 1.10.

**Corollary 1.13.** Let $X$ be a complex projective variety and let $f : C \to X$ be a curve whose image is contained in the smooth locus of $X$ and such that $T_X|_C$ is ample. Then $X$ is rationally connected.

1 Höring has independently obtained similar results, [Hör05].
THEOREM 1.14 (Campana, and Kollár-Miyaoka-Mori). Let $X$ be a smooth projective
variety. Then there exists a rational map $q : X \dasharrow Q$, with the following properties.

(1) The map $q$ is almost holomorphic\(^2\), i.e., there exists an open set $X^0 \subset X$ such
that $q|_{X^0}$ is a proper morphism.

(2) If $X_F \subset X$ is a general fiber of $q$, then $X_F$ is a compact, rationally chain
connected manifold, i.e., any two points in $X_F$ can be joined by a chain of rational
curves.

(3) If $X_F \subset X$ is a general fiber of $q$ and $x \in X_F$ a general point, then any rational
curve that contains $x$ is automatically contained in $X_F$.

Properties (1)–(3) define $q$ uniquely up to birational equivalence. □

DEFINITION 1.15. Any map $q : X \dasharrow Q$ for which properties (1)–(3) of Theo-
rem 1.14 hold is called a maximally rationally chain connected fibration of $X$, or MRCC
fibration, for short.

Let $X$ be a normal projective variety, and let $q : X \dasharrow Q$ be the rational map defined
through the maximally rationally chain connected fibration of a desingularization of $X$.
We call $q$ the rationally connected quotient of $X$.

REMARK 1.16. If $X$ is normal, the rationally connected quotient is again defined
uniquely up to birational equivalence.

REMARK 1.17. Rational chain connectivity is not a birational invariant. For instance,
a cone over an elliptic curve is rationally chain connected, while the ruled surface obtained
by blowing up the vertex is not. It is therefore important in Definition 1.15 to pass to a
desingularization.

In the proof of Theorem 1.10, we will need two important properties of the rationally
connected quotient.

THEOREM 1.18 (Universal property of the maximally rationally chain connected fi-
bribation, [Kol96, thm. IV.5.5]). Let $X_1$, $X_2$ be projective manifolds, and $f_X : X_1 \dasharrow X_2$
dominant. If $q_i : X_i \dasharrow Q_i$ are the maximally rationally chain connected fibrations, then
there exists a commutative diagram as follows.

\[
\begin{array}{ccc}
X_1 & \dasharrow & X_2 \\
| & f_X & | \\
|q_1| & |q_2| \\
\uparrow & \uparrow & \uparrow \\
Q_1 & \dasharrow & Q_2 \\
f_Q & & \\
\end{array}
\]

□

THEOREM 1.19 (Graber-Harris-Starr, [GHS03]). If $X$ is a normal projective variety
and $q : X \dasharrow Q$ the rationally connected quotient, then $Q$ is not covered by rational
curves. □

1.2. Sketch of proof of Theorem 1.10.

\(^2\)[Deb01] and [Kol96] use the word “fibration” for an almost holomorphic map.
1. RATIONALLY CONNECTED FOLIATIONS

Step 1: Reduction to the case where \( C \) is transversal to \( \mathcal{F} \). Following an idea of Bogomolov and McQuillan, we consider a non-constant morphism \( \nu : \tilde{C} \to C \) from a smooth curve \( \tilde{C} \) of positive genus \( g(\tilde{C}) > 0 \). Let \( Y \) denote the product \( X \times \tilde{C} \) with projections \( p_1 \) and \( p_2 \) and consider the following diagram, depicted in Figure 1.2.

It is obviously enough to show Theorem 1.10 for the variety \( Y \), the curve \( C' := \sigma(\tilde{C}) \) and the foliation \( \mathcal{F}_Y := p_1^*(\mathcal{F}) \subset T_Y|_{C'} \subset T_Y \), which is ample along \( C' \). The advantage lies in the smoothness of \( C' \), and in the transversality of \( \mathcal{F}_Y \) to \( C' \). Both properties are required for the next step.

Step 2: Algebraicity of the leaves. Since \( C' \) is everywhere transversal to \( \mathcal{F}_Y \), we can apply the classical Frobenius theorem: there exists an analytic submanifold \( W \subset Y \) which contains \( C' \) and has the property that its fibers over \( \tilde{C} \) are analytic open sets of the leaves of \( \mathcal{F}_Y \). Let \( \overline{W} \) be the Zariski closure of \( W \).

Using the transversality of \( C' \) and \( \mathcal{F}_Y \) and the fact that \( \mathcal{F}_Y|_C \) is ample, a theorem of Hartshorne, [Har68, thm. 6.7], asserts that \( \dim \overline{W} = \dim W \). Accordingly, every leaf is algebraic. Step 2 again follows [BM01], but see also [Bos01, thm. 3.5].

Step 3: Setup of notation. Replacing \( X \) by a desingularization of the normalization of \( \overline{W} \), we are reduced to prove Theorem 1.10 under the following extra assumptions: \( X \) is smooth, \( C \) is a smooth curve of genus \( g(C) > 0 \), and there exists a morphism \( \pi : X \to C \) such that

- \( \pi \) has connected fibers and is smooth along \( C \),
- the foliation \( \mathcal{F} \) is the foliation associated to \( \pi \), i.e. \( \mathcal{F} = T_{X|C} \) wherever \( \pi \) is smooth, and
• \( \pi \) admits a section, \( \sigma : C \to X \).

We will have to show that the general \( \pi \)-fiber is rationally connected. To this end, consider the rationally connected quotient \( q : X \to Z \) of \( X \). The universal property of the maximally rationally chain connected fibration, Theorem 1.18, then yields a diagram as follows.

\[
\begin{array}{ccc}
X & \overset{q}{\to} & Z \\
\downarrow \sigma & & \downarrow \\
C & \overset{\beta}{\to} & \tau
\end{array}
\]

We finally fix a very ample line bundle \( H \) on \( Z \), and we denote by \( H_X \) its pull-back to \( X \). We can then consider \( q \) as the rational map associated to a certain linear subsystem of \( H^0(X, H_X) \).

Observe that to prove Theorem 1.10, it suffices to show that \( \dim Z = 1 \). Namely, if \( \dim Z = 1 \), then \( \pi \) is itself a rationally connected quotient, and its general fiber will therefore be rationally chain connected, hence rationally connected.

We assume the contrary and suppose that \( \dim Z > 1 \). Below we will then show the following.

**Proposition 1.20.** Assume that \( \dim(Z) \geq 2 \). Then \( Z \) is uniruled with curves of \( H_Z \)-degrees at most \( d := 2 \deg \sigma^*(H_X) \cdot \dim X \).

This clearly contradicts Theorem 1.19, and concludes the proof of Theorem 1.10.

**Step 4: Strategy of proof for Proposition 1.20 and Theorem 1.10.** Assume for the moment that the morphism \( \sigma \) admits a large number of deformations. More precisely, assume that there exists a component \( \mathcal{H} \subset \text{Hom}(C, X) \) that contains \( \sigma \), and an open set \( \Omega \subset \mathcal{H} \) such that the following holds:

1. If \( T \subset X \) is any set of codimension \( \leq 2 \), then the set of morphisms that avoid \( T \), \( A := \{ \tau \in \Omega \mid \tau^{-1}(T) = \emptyset \} \), is a non-empty open set in \( \Omega \).
2. If \( \tau \in \Omega \) and \( x \in C \) a point, then the evaluation morphism associated to the set \( \Omega_{\tau(x)} = \{ \tau' \in \Omega \mid \tau'(x) = \tau(x) \} \) still dominates \( X \).

If (1) and (2) hold true, choosing \( T \) to be the indeterminacy locus of \( q \) the natural morphism \( A \to \text{Hom}(C, Z) \) provides a family \( C \subset \text{Hom}(C, Z) \) verifying:

- The evaluation morphism associated to \( C \) dominates \( Z \), and
- if \( x \in C \) and \( \tau \in C \) are general points, then the evaluation morphism associated to the set \( C_x = \{ \tau' \in C \mid \tau'(x) = \tau(x) \} \) still dominates \( Z \).

If \( \dim Z \geq 2 \), we can then use Mori’s Bend-and-Break argument.

**Definition 1.21.** Let \( f : C \to Z \) be any morphism and \( B \subset C \) a subscheme of finite length. The space of morphisms that agree with \( f \) on \( B \) is denoted as \( \text{Hom}(C, Z, f|_B) \).

**Theorem 1.22** (Mori’s Bend-and-Break, [MM86], [Kol91, prop. 3.3]). Let \( Z \) be a projective variety and let \( H_Z \) be a nef \( \mathbb{R} \)-divisor on \( Z \). Let \( C \) be a smooth, projective and irreducible curve, \( B \subset Z \) a finite subscheme and \( f : C \to Z \) a non-constant morphism. Assume that \( Z \) is smooth along \( f(C) \). If \( \dim(f) \text{Hom}(C, Z, f|_B) \geq 1 \) then there exists a rational curve \( R \) on \( Z \) meeting \( f(B) \) and such that

\[
H_Z \cdot R \leq \frac{2H_Z \cdot C}{\#B}.
\]

If \( H_Z \) is ample, the above inequality can be made strict. \( \square \)
Remark 1.23. Theorem 1.22 works for varieties defined over algebraically closed fields of arbitrary characteristic.

In our situation, Theorem 1.22 with $B = (*)$ and $f$ a general element of $C$ would show that $Z$ is uniruled and hence prove Proposition 1.20 by contradiction with Theorem 1.19 —for this, a simpler version of Mori’s Bend-and-Break would suffice, but we will need the full force of Theorem 1.22 later.

Unfortunately, the ampleness of the normal bundle of $C$ in $X$ does not guarantee the existence of large family of deformations of $\sigma$ that satisfy conditions (1) and (2). We can circumvent this problem using reduction modulo $p$.

Step 5: Reduction modulo $p$. In view of Mori’s standard argument using reduction modulo $p$, see [Mor79], [Deb01] or [CKM88], it is enough to prove Proposition 1.20 over an algebraically closed field of characteristic $p$, for $p$ large enough. We use a subindex $k$ to denote the reductions modulo $p$ of all the objects defined above, and let $F$ denote the $k$-linear Frobenius morphism.

Step 6: A vanishing result in characteristic $p$. We briefly recall the language of $\mathbb{Q}$-twisted vector bundles, as explained in [Laz04, II, 6.2]. This notion is a generalization of the concept of a $\mathbb{Q}$-divisor to higher rank. It allows us to make a finer use of the positivity of a vector bundle.

We identify rational numbers $\delta$ with numerical classes $\delta \cdot [P] \in N^1_\mathbb{Q}(C)$, where $P$ is a point in $C$. For every $\delta \in \mathbb{Q}$, the $\mathbb{Q}$-twist $E(\delta)$ is defined as the ordered pair of $E$ and $\delta$. A $\mathbb{Q}$-twisted vector bundle is said to be ample if the class $c_1(O_{C}(E)(1)) + \pi^*(\delta)$ is ample on the projectivized bundle $P_C(E)$, where $\pi$ denotes the natural projection. One defines the degree $\deg(E(\delta)) := \deg(E) + \text{rank}(E)\delta$. A quotient of $E(\delta)$ is a $\mathbb{Q}$-twisted vector bundle of the form $E'(\delta)$ where $E'$ is a quotient of $E$. Pull-backs of $\mathbb{Q}$-twisted vector bundles are defined in the obvious way.

We can now formulate the following vanishing result in characteristic $p$ that will be used later on.

Proposition 1.24 ([KST05, prop. 9]). Let $C_k$ be a curve defined over an algebraically closed field of characteristic $p > 0$. Let $E_k$ be a vector bundle of rank $r$ over $C_k$, and $\delta$ a positive rational number. Assume that $E_k(-\delta)$ is ample and that the “vanishing threshold”

$$b_p(\delta) := p\delta - 2g(C) + 1$$

is non-negative. Let $F : C_k[1] \rightarrow C_k$ be the $k$-linear Frobenius morphism. Then for every subscheme $B \subset C_k[1]$ of length smaller than or equal to $b_p(\delta)$ we have

$$H^1(C_k[1], F^*(E_k) \otimes I_B) = \{0\}.$$

Further, $F^*(E_k) \otimes I_B$ is globally generated.

Proof. To prove both vanishing and global generation, it is enough to show that

$$H^1(C_k[1], F^*(E_k) \otimes I_B) = \{0\} \text{ for } \#(B) \leq b_p(\delta) + 1.$$

Since $F$ is finite, the pull-back $F^*(E_k(-\delta)) = F^*(E_k)(-p\delta)$ is also ample, and so is every quotient of rank one. In particular,

$$\text{Hom}_{C_k}(F^*(E_k), \mathcal{O}_{C_k[1]}(B) \otimes \omega_{C_k[1]}) = \{0\} \text{ if } \deg(\mathcal{O}_{C_k[1]}(B) \otimes \omega_{C_k[1]}) \leq p\delta.$$

The proof is concluded by applying Serre duality.
Step 7: Proof in characteristic $p$. Back to the proof of Theorem 1.10. We would like to apply the vanishing result of Proposition 1.24 to describe the space of relative deformations of

$$\tau := \sigma_k \circ F : C_k[1] \to X_k,$$

over $C_k$. To this end, recall the following standard description of the Hom-scheme.

**Theorem 1.25.** Let $H := \text{Hom}_{C_k}(C_k[1], X_k)$ be the space of relative deformations of $\tau$, and let $\nu \in H$ be any element.

1. If $H^1(C_k[1], \nu^*(T_{X_k}[C_k])) = 0$, then $H$ is smooth at $\nu$, and has dimension $H^0(C_k[1], \nu^*(T_{X_k}[C_k]))$.

2. Let $T \subset X_k$ be any set of codimension $\leq 2$. If $\tau^*(T_{X_k}[C_k])$ is globally generated and $H^1(C_k[1], \tau^*(T_{X_k}[C_k])) = 0$, then the set of morphisms whose images avoid $T$ is a non-empty open subset of $H$.

3. If $B \subset C_k$ is any subscheme of finite length, if $\nu^*(T_{X_k}[C_k]) \otimes I_B$ is globally generated and $H^1(C_k[1], \nu^*(T_{X_k}[C_k]) \otimes I_B) = 0$, then the images of morphisms $\nu^i$ that agree with $\nu$ along $B$ dominate $X$.

By Hartshorne’s characterization of ampleness, [Laz04, Thm. 6.4.15], the ampleness of $\sigma^*(T_{X_k}/C_k)$ is equivalent to the ampleness of the $\mathbb{Q}$-twisted vector bundle $\sigma^*(T_{X_k}/C_k)(-1/\dim X)$. Note also that the ampleness of $\sigma^*(T_{X_k}/C_k)(-1/\dim X)$ is preserved by the general reduction modulo $p$, for $p$ sufficiently large.

Apply Proposition 1.24 to the vector bundle $\sigma_k^*(T_{X_k}/C_k)(-1/\dim X)$. Observe that by semicontinuity, the vanishing and global generation results obtained in Proposition 1.24 extend to general deformations of $\tau = \sigma_k \circ F$. Theorem 1.25 then immediately yields the following.

**Corollary 1.26.** There is an open neighborhood $\Omega \subset \text{Hom}_{C_k}(C_k[1], X_k)$ of $\tau$ such that

1. If $[\nu] \in \Omega$ is any morphism and $B \subset C_k[1]$ any subscheme of length $\#(B) \leq b_p(1/\dim X)$, then the relative deformations of $\nu$ over $C_k$ fixing $B$ dominate $X_k$.

2. If $T \subset X$ is the set of fundamental points of the birational map $q$ then the subset

$$\Omega^0 = \{[\nu] \in \Omega \mid (\nu)^{-1}(T_k) = \emptyset\}$$

of morphisms whose images avoid $T_k$ is again open in $\text{Hom}_{C_k}(C_k[1], X_k)$.

In particular, Corollary 1.26 states the following: given a general element $[\nu] \in \Omega^0$ and a subscheme $B \subset C_k[1]$ of length $\#(B) \leq b_p(1/\dim X)$, the deformations of $q \circ \nu$ that fix $B$ dominate $Z_k$. Using Mori’s Bend-and-Break, Theorem 1.22, we obtain that $Z_k$ is uniruled in curves of $H_{Z,k}$-degree at most

$$2 \deg(q \circ \nu)^*(H_{Z,k})/[b_p(1/\dim X)] = 2 \deg \nu^*(H_X)/[b_p(1/\dim X)]$$

$$= 2p \cdot \deg \sigma^*(H_X)/[b_p(1/\dim X)].$$

The proof of Proposition 1.20 and Theorem 1.10 is finished if we note that this number is smaller than or equal to $d$, for $p >> 0$. □
2. An effective version of Miyaoka’s criterion

As an immediate corollary of Theorem 1.10, we deduce an effective version of Miyaoka’s characterization of uniruledness. It asserts that positive parts in the restriction of $T_X$ to a general complete intersection curve are tangent to rational curves. More precisely, we use the following definition.

**Definition 1.27.** Let $X$ be a normal projective variety, and $C \subset X$ a subvariety which is not contained in the singular locus of $X$, and not contained in the indeterminacy locus of the rationally connected quotient $q : X \dashrightarrow Q$. If $\mathcal{F} \subset T_X|_C$ is any subsheaf, we say that $\mathcal{F}$ is vertical with respect to the rationally connected quotient, if $\mathcal{F}$ is contained in $T_X|_Q$ at the general point of $C$.

The effective version of Miyaoka’s criterion is then formulated as follows.

**Corollary 1.28.** Let $X$ be a normal complex projective variety and $C \subset X$ a general complete intersection curve. Assume that the restriction $T_X|_C$ contains an ample locally free subsheaf $\mathcal{F}_C$. Then $\mathcal{F}_C$ is vertical with respect to the rationally connected quotient of $X$.

This statement appeared first implicitly in [Kol92, chap. 9], but we believe there are issues with the proof, see [KST05, rem. 23]. To our best knowledge, the argument presented here gives the first complete proof of this important result.

2.1. Vector bundles over complex curves. The proof of Corollary 1.28 relies on a number of facts about the Harder-Narasimhan filtration of vector bundles on curves, which are known to the experts. For lack of an adequate reference we include full proofs here. To start, we show that any vector bundle on a smooth curve contains a maximally ample subbundle.

**Proposition 1.29.** Let $C$ be a smooth complex-projective curve and $E$ a vector bundle on $C$, with Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E.$$  

Let $\mu_i := \mu(E_i/E_{i-1})$ be the slopes of the Harder-Narasimhan quotients. Suppose that $\mu_1 > 0$ and let $k := \max\{i | \mu_i > 0\}$. Then $E_i$ is ample for all $1 \leq i \leq k$ and every ample subsheaf of $E$ is contained in $E_k$.

**Definition 1.30.** In the setup of Proposition 1.29, the bundle $E_k$ is called the maximal ample subbundle of $E$.

**Proof of Proposition 1.29.** Hartshorne’s characterization of ampleness, [Har71, thm. 2.4], says that $E_i$ is ample iff all its quotients have positive degree. Dualizing, we have to prove that every subbundle of $E_i^\vee$ has negative degree, or, equivalently, that its maximal destabilizing subsheaf has negative slope, see [HL97, 1.3.4]. This, however, holds because the uniqueness of the Harder-Narasimhan filtration of $E_i^\vee$, [HL97, 1.3.5], implies that

$$\mu_{\max}(E_i^\vee) = -\mu_i < 0.$$  

To show the second statement, let $F \subset E$ be any ample subsheaf of $E$ and set

$$j := \min\{i | F \subset E_i, 1 \leq i \leq r\}.$$  

We need to check that $j \leq k$. By the definition of $j$ and the ampleness of $F$, the image of $F$ in $E_j/E_{j-1}$ has positive slope. The semi-stability of $E_j/E_{j-1}$ therefore implies $\mu_j > 0$ and $j \leq k$. \qed
Proposition 1.29 says that the first few terms in the Harder-Narasimhan filtration are ample. The following, related statement will be used in the proof of Corollary 1.28 to construct foliations on $X$.

**Proposition 1.31.** In the setup of Proposition 1.29, the vector bundles $E_j \otimes (E_i/E_i)^\vee$ are ample for all $0 < j \leq i < r$. In particular, if $E_i$ is any ample term in the Harder-Narasimhan Filtration of $E$, then $\text{Hom}(E_i, E_i/E_i)$ and $\text{Hom}(E_i \otimes E_i, E_i/E_i)$ are both zero.

**Remark 1.32.** If $X$ is a polarized manifold whose tangent bundle contains a subsheaf of positive slope, Proposition 1.31 shows that the first terms in the Harder-Narasimhan filtration of $T_X$ are special foliations in the sense of Miyaoka, [Miy87, sect. 8]. By [Miy87, thm. 8.5], this already implies that $X$ is dominated by rational curves that are tangent to these foliations.

**Proof of Proposition 1.31.** As a first step, we show that the vector bundle

$$F_{i,j} := (E_j/E_{j-1}) \otimes (E_i/E_i)^\vee$$

is ample. Assume not. Then, by Hartshorne’s ampleness criterion [Har71, prop. 2.1(ii)], there exists a quotient $A$ of $F_{i,j}$ of degree $\deg_C A \leq 0$. Equivalently, there exists a non-trivial subbundle

$$\alpha : B \to F_{i,j}^\vee = (E_j/E_{j-1})^\vee \otimes (E_i/E_i)$$

with $\deg_C B \geq 0$. Replacing $B$ by its maximally destabilizing subbundle, if necessary, we can assume without loss of generality that $B$ is semistable. In particular, $B$ has non-negative slope $\mu(B) \geq 0$. On the other hand, we have that $(E_j/E_{j-1})$ is semistable. The slope of the image of the induced morphism

$$B \otimes (E_j/E_{j-1}) \to (E_i/E_i)$$

will thus be larger than $\mu_{\text{max}}(E_i/E_i) = \mu(E_{i+1}/E_i)$. This shows that $\alpha$ must be zero, a contradiction which proves the amplitude of $F_{i,j}$.

With this preparation we will now prove Proposition 1.31 inductively. If $j = 1$, then the above claim and the statement of Proposition 1.31 agree. Now let $1 < j \leq i < r$ and assume that the statement was already shown for $j - 1$. Then consider the sequence

$$0 \to E_{j-1} \otimes (E_i/E_i)^\vee \to E_j \otimes (E_i/E_i)^\vee \to (E_j/E_{j-1}) \otimes (E_i/E_i)^\vee \to 0$$

But then also the middle term is ample, which shows Proposition 1.31.

**2.2. Proof of Corollary 1.28.** We will show that the sheaf $\mathcal{F}_C$, which is defined only on the curve $C$ is contained in a foliation $\mathcal{F}$ which is regular along $C$ and whose restriction to $C$ is likewise ample. Corollary 1.28 then follows immediately from Theorem 1.10.

An application of Proposition 1.29 to $E := T_X|_C$ yields the existence of a locally free term $E_i \subset T_X|_C$ in the Harder-Narasimhan filtration of $T_X|_C$ which contains $\mathcal{F}_C$ and is ample. The choice of $C$ then guarantees that $E_i$ extends to a saturated subsheaf $\mathcal{F} \subset T_X$. The proof is thus finished if we show that $\mathcal{F}$ is a foliation, i.e. closed under the Lie bracket. Equivalently, we need to show that the associated O’Neill tensor\(^3\)

$$N : \mathcal{F} \otimes \mathcal{F} \to T_X/\mathcal{F}$$

\(^3\)The Lie bracket is of course not $\mathcal{O}_X$-linear. However, an elementary computation show that $N$ is well-defined and linear.
vanishes. By Proposition 1.31, the restriction of the bundle
\[ \text{Hom} \left( F \otimes F, T_X / F \right) \cong (F \otimes F) \otimes T_X / F \]
to \( C \) is anti-ample. In particular,
\[ N|_C \in H^0 \left( C, \text{Hom} \left( F \otimes F, T_X / F \right) \right) = \{0\} . \]
Ampleness is an open property, \([\text{Gro66}, \text{cor. 9.6.4}]\), so that the restriction of \( N \) to deformations \((C_t)_{t \in T}\) of \( C \) stays zero for most \( t \in T \). Since the \( C_t \) dominate \( X \), the claim follows. This ends the proof of Corollary 1.28. \( \square \)

3. The stability of the tangent bundle, and partial rationally connected quotients

Recall that a complex variety \( X \) is called \( \mathbb{Q} \)-Fano if a sufficiently high multiple of the anticanonical divisor \(-K_X\) is Cartier and ample. The methods introduced above immediately yield that \( \mathbb{Q} \)-Fano varieties whose tangent bundles are unstable allow sequences of rational maps with rationally connected fibers.

**Corollary 1.33.** Let \( X \) be a normal complex \( \mathbb{Q} \)-Fano variety and \( L_1, \ldots, L_{\dim X - 1} \in \text{Pic}(X) \) be ample line bundles. Let
\[ \{0\} = E_{-1} = E_0 \subset E_1 \subset \cdots \subset E_m = T_X \]
be the Harder-Narasimhan filtration of the tangent sheaf with respect to \( L_1, \ldots, L_{\dim X - 1} \) and set
\[ k := \max \{0 \leq i \leq m \mid \mu(E_i/E_{i-1}) > 0\} . \]
Then \( k > 0 \), and there exists a commutative diagram of dominant rational maps
\[
\begin{array}{c}
X \\
q_1 \downarrow \downarrow \downarrow q_2 \\
\uparrow \\
Q_1 \rightarrow \cdots \rightarrow \rightarrow Q_k \\
\end{array}
\]
with the following property: if \( x \in X \) is a general point, and \( F_i \) the closure of the \( q_i \)-fiber through \( x \), then \( F_i \) is rationally connected, and its tangent space at \( x \) is exactly \( E_i \), i.e., \( T_{F_i}|_x = E_i|_x \).

**Proof.** Let \( C \subset X \) be a general complete intersection curve with respect to \( L_1, \ldots, L_{\dim X - 1} \). Since \( c_1(T_X) \cdot C > 0 \), Proposition 1.29 implies \( k > 0 \) and that the restrictions \( E_i|_C, \ldots, E_k|_C \) are ample vector bundles. We have further seen in Theorem 1.10 that the \((E_i)_{1 \leq i \leq k}\) give a sequence of foliations with algebraic and rationally connected leaves.

To end the construction of Diagram (2), let \( q_i : X \rightarrow \text{Chow}(X) \) be the map that sends a point \( x \) to the \( E_i \)-leaf through \( x \), and let \( Q_i := \text{Image}(q_i) \). \( \square \)

**Remark 1.34.** Corollary 1.33 also holds in the more general setup where \( X \) is a normal variety whose anti-canonical class is represented by a Weil divisor with positive rational coefficients.
3.1. Open Problems. It is of course conjectured that the tangent bundle of a Fano manifold $X$ with $b_2(X) = 1$ is stable. We are therefore interested in a converse to Corollary 1.33 and ask the following.

**Question 1.35.** Given a $\mathbb{Q}$-Fano variety and a sequence of rational maps with rationally connected fibers as in Diagram (2), when does the diagram come from the unstability of $T_X$ with respect to a certain polarization? Is Diagram (2) characterized by universal properties?

**Question 1.36.** To what extent does Diagram (2) depend on the polarization chosen?

**Question 1.37.** If $X$ is a uniruled manifold or variety, is there a polarization such that the rational quotient map comes from the Harder-Narasimhan filtration of $T_X$?

4. Rational curves on quasi-projective manifolds

Quasi-projective varieties appear naturally in a number of settings, e.g., as moduli spaces or modular varieties. In this setup, it is often not reasonable to ask if a given variety $S^\circ$ contains complete rational curves. Instead, one is interested in hyperbolicity properties of $S^\circ$, i.e., one asks if there are non-constant morphisms $\mathbb{C} \to U$, or if $S^\circ$ is dominated by images of such morphisms—we refer to [Siu04] for a general discussion, and to Chapter 5 for a very brief overview of the hyperbolicity question for moduli of canonically polarized manifolds and for applications.

In this respect, a famous conjecture attributed to Miyanishi suggests that the following logarithmic analog of Conjecture 1.3 holds true.

**Conjecture 1.38 (Miyanishi).** Let $S^\circ$ be a smooth quasi-projective variety. Then $\kappa(S^\circ) = -\infty$ if and only if $S^\circ$ is dominated by images of $\mathbb{C}$.

We briefly recall the definition of the logarithmic Kodaira dimension.

**Definition 1.39.** Let $S^\circ$ be a smooth quasi-projective variety and $S$ a smooth projective compactification of $S^\circ$ such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. The logarithmic Kodaira dimension of $S^\circ$, denoted by $\kappa(S^\circ)$, is defined as the Kodaira-Iitaka dimension $\kappa(K_S + D)$ of the line bundle $O_S(K_S + D) \in \text{Pic}(S)$.

The variety $S^\circ$ is called of log general type if $\kappa(S^\circ) = \dim S^\circ$, i.e., if the divisor $K_S + D$ is big.

It is a standard fact in logarithmic geometry that a compactification $S$ with the described properties exists, and that the logarithmic Kodaira dimension does not depend on the choice of the compactification, [Iit82, chap. 11].

Conjecture 1.38 was studied by Miyanishi and Tsunoda in [MT83, MT84] and a number of further papers, and by Zhang [Zha88]. Complete results for surfaces were obtained by Keel-McKernan, as follows.

**Theorem 1.40 ([KMcK99, thm. 1.1]).** Let $S^\circ$ be a smooth quasi-projective variety of dimension at most two. Then $\kappa(S^\circ) = -\infty$ if and only if $S^\circ$ is dominated by images of $\mathbb{C}^\ast$.

Keel-McKernan also give conditions that guarantee that $U$ is dominated by images of $\mathbb{C}^\ast$.

---

4But see [KMcK99, cor. 5.9] for criteria that can sometimes be used if $S^\circ$ can be compactified by a finite number of points.

5See [GZ94] for a partial case.
Theorem 1.41 ([KMcK99, prop. 1.4]). Let $S$ be a normal projective surface, let $D \subset S$ be a reduced curve, and set $U := S \setminus (D \cup \text{Sing}(S))$. Consider the following conditions:

1. $K_S + D$ is numerically trivial, but not log-canonical.
2. $K_S + D$ is numerically trivial, and $D \neq \emptyset$.
3. $K_S$ is numerically trivial, $D = \emptyset$, and $S$ has a singularity which is not a quotient singularity.

If any of the above hold, then $U$ is dominated by images of $\mathbb{C}^\ast$. \hfill \square

4.1. Open Problems. Theorems 1.40 and 1.41 were shown in [KMcK99] using deformation theory on non-separated algebraic spaces and a rather involved case-by-case analysis of possible curve configurations on $S$. The analysis alone covers more than a hundred pages, and a generalization to higher dimensions seems out of the question. We would therefore like to pose the following problem.

Problem 1.42. Find a more conceptual proof of Theorem 1.40, perhaps using methods introduced in Section 1.
Part 2

Geometry of uniruled varieties
CHAPTER 2

Geometry of rational curves on projective manifolds

In Chapter 1 we have reviewed criteria to guarantee that a given variety is covered by rational curves. In this chapter, we assume that we are given a variety $X$ that is covered by rational curves, and study the geometry of curves on $X$ in more detail.

Outline of the chapter. In Section 1 we review a number of known concepts concerning rational curves on $X$. In particular, we give the definition of a family of rational curves and fix the notation that will be used later on.

Section 2 deals with the locus of singular curves of a dominating family of minimal rational curves. We apply the results of [Keb02b] in Section 3 to ensure the existence and finiteness of the tangent morphism, which maps every minimal rational curve passing by a point $x$ to its tangent direction at $x$. The image, i.e., the set of tangent directions to which there exists a minimal degree rational curve, is called variety of minimal rational tangents, or VMRT. Recently, Hwang and Mok have shown that the general minimal degree rational curve on $X$ is uniquely determined by its tangent direction at a point. This result gives more information about the VMRT. We sketch their argument in Section 4.

We have claimed in the introduction that the VMRT determines the geometry of $X$ to a large degree. In order to substantiate this claim, we mention, in Section 5, a number of results in that direction. One particular result, an estimate for the minimal number of rational curves required to connect two points on a Fano manifold, is explained in Section 6 in more detail.

1. The space of rational curves, setup of notation

We begin by defining some well known concepts about parameter spaces of rational curves in projective varieties. We refer the reader to [Kol96, II.2] for a detailed account.

Definition 2.1. Let $X$ be a normal complex projective variety, and $\text{RatCurves}(X) \subset \text{Chow}(X)$ be the quasi-projective subvariety whose points correspond to irreducible and generically reduced rational curves in $X$. Let $\text{RatCurves}^n(X)$ be its normalization and $U$ the normalization of the universal family over $\text{RatCurves}(X)$. We obtain a diagram as follows.

(3) $\begin{array}{ccc}
U & \xrightarrow{\iota} & X \\
\pi \downarrow & & \downarrow \\
\text{RatCurves}^n(X) & & 
\end{array}$

Remark 2.2. The normalization morphism $\text{RatCurves}^n(X) \to \text{Chow}(X)$ is finite and generically injective, but not necessarily injective. It is therefore possible that points in $\text{RatCurves}^n(X)$ are not in 1:1 correspondence with actual curves in $X$. We use the notation $[\ell]$ to denote points in $\text{RatCurves}^n(X)$, and $\ell$ for the associated curves.
A standard cohomological argument for families of irreducible and generically reduced rational curves shows that the morphism $\pi$ is in fact a $\mathbb{P}^1$-bundle. The space of rational curves is often described in terms of the Hom-scheme. The following theorem establishes the link.

**Theorem 2.3 ([Kol96, II thm. 2.15]).** There exists a diagram as follows

\[
\begin{array}{ccc}
\text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1 & \xrightarrow{\text{quotient by natl. action of Aut}(\mathbb{P}^1)} & U \\
\downarrow & & \downarrow \\
\text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X) & \xrightarrow{\text{quotient by natl. action of Aut}(\mathbb{P}^1)} & \text{RatCurves}^n(X)
\end{array}
\]

where $\text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X) \subset \text{Hom}(\mathbb{P}^1, X)$ is the scheme parametrizing birational morphisms from $\mathbb{P}^1$ to $X$ and $\text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X)$ its normalization.

**Definition 2.4.** A maximal family of rational curves is an irreducible component $H \subset \text{RatCurves}^n(X)$. A maximal family $H$ is called dominating, if $\iota|_{\pi^{-1}(H)}$ dominates $X$. A dominating family $H$ is a dominating family of rational curves of minimal degrees if the degrees of the associated rational curves on $X$ are minimal among all dominating families.

**Example 2.5.** Let $X \subset \mathbb{P}^3$ be a general cubic surface. It is classically known that $X$ contains 27 lines, and that there exists a dominating family of conics, i.e., rational curves of degree 2, on $X$. In this case, the family of conics would be a dominating family of rational curves of minimal degree.

Example 2.5 shows that a dominating family of rational curves of minimal degrees needs not be proper —there are sequences of conics on $X$ whose limit cycle is a union of two lines. It is, however, true that given a sequence of rational curves on any $X$, the limit cycle is composed of rational curves. This observation immediately gives the following properness statement for families of curves of minimal degrees.

**Theorem 2.6 ([Kol96, IV cor. 2.9]).** Let $H \subset \text{RatCurves}^n(X)$ be a dominating family of rational curves of minimal degrees. If $x \in X$ is a general point and $\ell \in H_x$ any element, with normalization $\eta: \mathbb{P}^1 \rightarrow \ell$, then the following holds.

**Definition 2.7.** Let $H \subset \text{RatCurves}^n(X)$ be a dominating family of rational curves. If $H$ is proper, it is often called unsplit. If for a a general point $x \in X$ the associated subspace $H_x$ is proper, the family is called generically unsplit.

One of the key tools in the description of rational curves on a manifold $X$ is an analysis of the restriction of $T_X$ to the rational curves in question —recall that any vector bundle on $\mathbb{P}^1$ can be written as a sum of line bundles. The following proposition summarizes the most important facts.

**Proposition 2.8 ([Deb01, Prop. 4.14], [Kol96, IV Cor. 2.9]).** Let $X$ be a smooth complex projective variety and $H \subset \text{RatCurves}^n(X)$ be a dominating family of rational curves of minimal degrees. If $x \in X$ is a general point and $[\ell] \in H_x$ any element, with normalization $\eta: \mathbb{P}^1 \rightarrow \ell$, then the following holds.
• $H$ is smooth at $[\ell]$.
• The normalization $\tilde{H}_x$ of $H_x$ is smooth.
• $\ell$ is free, i.e. the vector bundle $\eta^* (T_X)$ is nef.
• If $[\ell]$ is a general point of $H_x$, then $\ell$ is standard, i.e., there exists a number $p$ such that
  \[ \eta^* (T_X) \cong \mathcal{O}_{\mathbb{P}^1} (2) \oplus \mathcal{O}_{\mathbb{P}^1} (1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \dim(X) - 1 - p. \]

\[ \square \]

1.1. Open Problems.

PROBLEM 2.9. To what extent are dominating families of rational curves of minimal degrees characterized by their generic unsplitness?

PROBLEM 2.10. Let $X$ be smooth and $H$ be a dominating family of rational curves of minimal degrees. Assume that for general $x \in X$, the space $H_x$ is positive dimensional. Is it true that $H_x$ is irreducible? See [KK04, sect. 5.1] for a partial case.

2. Singular rational curves

Let $X \subset \mathbb{P}^n$ be a projective manifold that is covered by lines. If $x \in X$ is any point, we can consider the space $C_x \subset \mathbb{P}(T_X|_x)$ of tangent directions that are tangent to lines through $x$. The so-defined variety of minimal rational tangents is a very classical and important object that has been studied in the past by Cartan and Fubini, and a number of other projective differential geometers.

We will, in this section, give a similar construction for families of rational curves of minimal degrees. The main obstacle is that these curves may be singular, which makes it difficult to properly define tangents to them. It is, however, well understood that minimal degree curves have only mild singularities at the general point of $X$.

DEFINITION 2.11. We say that a curve $C$ is immersed, if its normalization morphism $\tilde{C} \to C$ has rank 1 at every point.

THEOREM 2.12 ([Keb02b, thm. 3.3]). Let $X$ be normal and $H \subset \text{RatCurves}^n(X)$ a dominating family of rational curves of minimal degrees. Further, let $x \in X$ be a general point, and consider the closed subvarieties

\[ H^\text{Sing}_x := \{ [\ell] \in H_x \mid \ell \text{ is singular} \} \text{ and } \]

\[ H^\text{Sing,x}_x := \{ [\ell] \in H_x \mid \ell \text{ is singular at } x \}. \]

Then the following holds.

1. The space $H^\text{Sing}_x$ has dimension at most one, and the subspace $H^\text{Sing,x}_x$ is at most finite. Moreover, if $H^\text{Sing,x}_x$ is not empty, the associated curves are immersed.
2. If there exists a line bundle $L \in \text{Pic}(X)$ that intersects the curves with multiplicity 2 then $H^\text{Sing}_x$ is at most finite and $H^\text{Sing,x}_x$ is empty.

The main idea in the proof of Theorem 2.12 is the observation that an arbitrary family of singular rational curves, like any family of higher genus curves, is hardly ever projective —see [Keb02c] for worked examples. An analysis of the projectivity condition yields the statement.

2.1. Sketch of proof of Theorem 2.12. As usual, we subdivide the proof into several steps. We will only give an idea how to show that the subspace $H^\text{Sing,x}_x$ is at most finite.
2. GEOMETRY OF RATIONAL CURVES ON PROJECTIVE MANIFOLDS

~\text{\P^1-bundle} ~\xrightarrow{\alpha} ~\tilde{U} ~\xrightarrow{\tilde{\pi}} ~U' ~\xrightarrow{\beta} ~H'

family of plane cubics

family of very singular curves

\text{FIGURE 2.1. Replacing singular curves by plane cubics}

\textbf{Step 1: Dimension count.} We assume that \( H_x^{\text{Sing},x} \) is not empty because otherwise there is nothing to prove. A technical dimension count—which we are not going to detail in this sketch—shows that

\[ \dim H_x^{\text{Sing}} \geq \dim H_x^{\text{Sing},x} + 1. \]

Thus, the assumption implies that \( \dim H_x^{\text{Sing}} \geq 1 \). We fix a proper 1-dimensional subfamily \( H' \subset H_x^{\text{Sing}} \).

\textbf{Step 2: A partial resolution of singularities.} Recall that \( H' \subset \text{RatCurves}^n(X) \) has a natural morphism into the Chow variety of \( X \). Let \( \pi : U' \to H' \) be the pull-back of the universal family. We aim to replace \( \tilde{U}' \) by a family where all fibers are singular plane cubics. For that, consider the normalization diagram.

\[ \tilde{U} \xrightarrow{\eta} U' \xrightarrow{\pi} H' \]

After performing a series of finite base changes, if necessary, we can assume that the following holds:

1. \( H' \) is smooth.
2. \( \tilde{U} \) is a \( \P^1 \)-bundle over \( H' \)—see [Kol96, thm. II.2.8].
3. There exists a curve \( s \subset U'_x^{\text{Sing}} \) contained in the singular locus of \( U' \) such that \( \pi|_s \) is an isomorphism. For this, let \( s \) be the normalization of a suitable component of \( U_x^{\text{Sing}} \).
4. There exists a subscheme \( \tilde{s} \subset \eta^{-1}(s) \) whose restriction to all \( \tilde{\pi} \)-fibers is of length 2. For this, let \( \tilde{s} \) be the normalization of a curve in \( \text{Hilb}_2(\eta^{-1}(s)/H') \) and note that the relative \( \text{Hilb} \)-functor commutes with base change.
We would like to extend the diagram (4) to

where all fibers of \( \hat{\pi} \) are rational curves with a single cusp or node, i.e., isomorphic to a plane cubic. Figure 2.1 depicts this setup. Here we explain only how to do this locally.

Knowing that \( \tilde{U} \) is a \( \mathbb{P}^1 \)-bundle over \( H' \), we find an (analytic) open set \( V \subset H' \) with coordinate \( v \), identify an open subset of \( \tilde{\pi}^{-1}(V) \) with \( V \times \mathbb{C} \), choose a bundle coordinate \( u \) and write

\[
\tilde{s} = \{ u^2 = f(v) \}
\]

where \( f \) is a function on \( V \). We would then define \( \alpha \) to be

\[
\alpha : \quad V \times \mathbb{C} \quad \rightarrow \quad V \times \mathbb{C}^2 \\
(v, u) \quad \mapsto \quad (v, u^2 - f(v), u(u^2 - f(v)))
\]

A direct calculation shows that these locally defined morphisms glue together to give a global morphism \( \alpha : \tilde{U} \rightarrow \hat{U} \), that a morphism \( \beta : \hat{U} \rightarrow U' \) exists and that the induced map \( \hat{\pi} := \pi \circ \beta \) has the desired properties.

**Step 3: Ruling out several cases.** In order to conclude in the next step, we have to rule out several possibilities for the geometry of \( \tilde{U} \). We do this in every case by a reduction to the absurd.

**Step 3a: The case where all curves are immersed.** If all curves associated with \( H' \) are immersed, then the construction outlined above will automatically give a family \( \tilde{U} \) where all fibers are isomorphic to nodal plane cubics —see Figure 2.2. Let \( \sigma_{\infty} \subset \tilde{U} \) be the section which is contracted to the point \( x \in X \) (drawn as a solid line) and consider the preimage of the singular locus \( \alpha^{-1}(\tilde{U}_{\text{Sing}}) \). After another finite base change, if necessary, we may assume that this set decomposes into two disjoint components...
\[ \alpha^{-1}(\tilde{U}_{\text{Sing}}) = \sigma_0 \cup \sigma_1, \] drawn as dashed lines. That way we obtain three sections \( \sigma_0, \sigma_1 \) and \( \sigma_{\infty} \) in \( \tilde{U} \), where \( \sigma_{\infty} \) can be contracted to a point and \( \sigma_0, \sigma_1 \) are disjoint. But then it follows from an elementary calculation with intersection numbers that either \( \sigma_0 = \sigma_{\infty} \) or that \( \sigma_1 = \sigma_{\infty} \). This however, is impossible, because then the Stein factorization of \( \iota \circ \eta \) would contract both \( \sigma_0 \) and \( \sigma_1 \), but ruled surfaces allow at most a single contractible section.

**Remark 2.13.** This setup has already been considered by several authors. See e.g. [CS95].

**Step 3b: The case where no curve is immersed.** If none of the curves associated with \( H' \) is immersed, then the curves \( s \) and \( \hat{s} \) in the construction can be chosen so that \( \tilde{U} \to H' \) is a family of cuspidal plane cubics, see Figure 2.3. Again, let \( \sigma_{\infty} \subset \tilde{U} \) be the section which can be contracted and let \( \sigma_0 \) be the preimage of the singularities. That way we obtain two sections.

In order to obtain a third one, remark that if \( C \subset \mathbb{P}^2 \) is a cuspidal plane cubic and \( H \in \text{Pic}(C) \) a line bundle of positive degree \( k > 0 \), then there exists a unique point\(^1\) \( y \in C \), contained in the smooth locus of \( C \), such that \( \mathcal{O}_C(ky) \cong H \). Thus, using the pull-back of an ample line bundle \( L \in \beta^* \text{Pic}(X) \), we obtain a third section \( \sigma_1 \) which is disjoint from \( \sigma_0 \). Now conclude as above. This time, however, it is not obvious that neither \( \sigma_0 \) nor \( \sigma_1 \) coincides with \( \sigma_{\infty} \). Actually, this is true because \( x \in X \) was chosen to be a general point. The proof of this is rather technical therefore omitted.

**Step 4: End of proof.** To end the proof, we argue by contradiction and assume that \( \dim H_{x_{\text{Sing}}}^{\text{Sing},x} \geq 1 \). We have seen in Step 1 that this implies \( \dim H_{x_{\text{Sing}}}^{\text{Sing}} \geq 2 \). The argumentation of Step 3 implies that points of \( \dim H_{x_{\text{Sing}}}^{\text{Sing}} \geq 2 \) correspond to both immersed and non-immersed singular rational curves in \( X \).

Elementary deformation theory, however, shows that the closed subfamily of non-immersed curves is always of codimension at least one. Thus, the subfamily \( H_{x_{\text{Sing},ni}}^{\text{Sing},ni} \subset H_{x_{\text{Sing}}}^{\text{Sing}} \) of non-immersed curves is proper and positive-dimensional. This again has been ruled out in Step 3b.

---

\(^1\)If \( H = \mathcal{O}_{\mathbb{P}^2}(1)|_C \), this is the classical inflection point. In general, this will be the hyperosculating point associated with the embedding given by \( H \).
2.2. Open Problems.

**Problem 2.14.** Examples show that Theorem 2.12 is optimal for normal varieties $X$. It is not clear to us if Theorem 2.12 can be improved if $X$ is smooth.

3. The tangent morphism and the variety of minimal rational tangents

We apply Theorem 2.12 to show the existence and finiteness of the tangent morphism, an important tool in the study of uniruled manifolds that encodes the infinitesimal behavior of $H$ near a general point $x \in X$.

**Definition 2.15.** Given a dominating family $H$ of rational curves of minimal degrees on $X$, and a general point $x \in X$ we consider a rational tangent map

$$t_x : H_x \longrightarrow \mathbb{P}(T_{X | x}^\vee)$$

associating to a curve through $x$ its tangent direction at $x$.

**Theorem 2.16** ([Keb02b, thm. 3.4]). With the notation as above, let $x \in X$ be a general point, let $\tilde{H}_x$ be the normalization of $H_x$, and let

$$\tau_x : \tilde{H}_x \longrightarrow \mathbb{P}(T_{X | x}^\vee)$$

be the composition of $t_x$ with the normalization morphism. Then $\tau_x$ is a finite morphism, called tangent morphism.

**Proof.** Consider the pull-back of Diagram (3) from page 25,

$$\begin{array}{ccc}
U_x & \xrightarrow{\iota_x} & X \\
\downarrow{\pi_x} & & \downarrow \\
\tilde{H}_x & & 
\end{array}$$

Theorem 2.12 asserts that the preimage $\iota_x^{-1}(x)$ contains a reduced section $\sigma_\infty \cong \tilde{H}_x$, and at most finitely many points. Since all curves are immersed at $x$, the tangent morphism of $\iota_x$ gives a nowhere vanishing morphism of vector bundles,

$$T\iota_x : T_{U_x | \tilde{H}_x} |_{\sigma_\infty} \rightarrow \iota_x^*(T_{X | x}) \quad .$$

The tangent morphism is then given by the projectivization of (5).

Assuming that $\tau_x$ is not finite, Equation (5) asserts that we can find a curve $C \subset \tilde{H}_x$ such that $N_{\sigma_\infty, U_x}$ is trivial along $C$. But $\sigma_\infty$ can be contracted, and the normal bundle must thus be negative. \[\square\]

With these preparations, we can now introduce one of the central objects of this survey, the variety of minimal rational tangents, or VMRT.

**Definition 2.17.** If $x \in X$ is a general point, we call the image $C_x := \tau_x(\tilde{H}_x) \subset \mathbb{P}(T_{X | x}^\vee)$ the variety of minimal tangents of $H$ at $x$. The subvariety $C := \text{closure of } \bigcup_{x \text{ general in } X} C_x \subset \mathbb{P}(T_X^\vee)$ is called the total variety of minimal rational tangents of $H$. 
**Remark 2.18.** The projectivized tangent map of the evaluation morphism yields a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\iota} & X \\
\rho, \text{ projection} & \downarrow & \downarrow \\
C & \subset & \mathbb{P}(T_X^{\vee}) \\
\end{array}
\]

that we will later also use to describe the tangent map \(\tau_x\) at a general point. We call \(\tau\) the **global tangent map**.

The variety of minimal rational tangents has been extensively studied by several authors, including Hwang and Mok. It can be computed in a number of examples of uniruled varieties, such as Fano hypersurfaces, rational homogeneous spaces and moduli spaces of vector bundles. We refer the reader to [Hwa01] for examples, but see also Chapter 3 below.

**Proposition 2.19 (Hwa01, prop. 1.5).** Suppose that there exists an embedding \(X \subset \mathbb{P}^N\) such that the curves associated with points of \(H\) are lines. If \(x \in X\) is a general point, then \(\tau_x\) is an embedding and \(C_x\) is smooth.

If \([\ell] \in H_x\) is any line through \(x\), then the projective tangent space to \(C_x\) at \(\mathbb{P}(T_x|_x)\) is exactly \(\mathbb{P}\left((T_X|_x^{\vee})^{\vee}\right)\), where \(T_X|_x^{\vee} \subset T_X|_x\) is the maximal ample subbundle.

In general, there is a direct and well understood relation between the splitting type of \(T_X|_x^{\vee}\) and the Zariski tangent space of \(C_x\). See [Hwa01, sect. 1] for details.

### 3.1. Open Problems.

**Problem 2.20.** In all smooth examples that we are aware of, the variety \(C_x\) is smooth, and has good projective-geometrical properties. What can be said in general? See [Hwa01] for more detailed lists of problems.

### 4. Birationality of the tangent morphism

We will later see that the projective geometry of the VMRT \(C_x \subset \mathbb{P}(T_X^{\vee}|_x)\) encodes a lot of the geometrical properties of \(X\). One of the main difficulties in the applications is that the variety \(C_x\) of minimal rational tangents might be singular or reducible. To overcome this difficulty one often studies the tangent morphism \(\tau_x : H_x \to C_x\). Since \(H_x\) is smooth, one asks if \(\tau_x\) is injective, and if it has maximal rank —this question can sometimes be answered in the examples.

In general, it has been shown by Hwang and Mok that the normalization of \(C_x\) is smooth. This is a direct consequence of Proposition 2.8 and the main result of [HM04].

**Theorem 2.21 ([HM04, thm. 1]).** With the same notation as above, the global tangent map \(\tau\) is generically injective. In particular, the normalization of \(C_x\) is smooth for general \(x \in X\).

See [KK04] for related criteria to guarantee that \(\tau_x\) is in fact injective.

**Remark 2.22.** A line in \(\mathbb{P}^n\) is specified by giving a point \(x \in \mathbb{P}^n\) and a tangent direction at \(x\). Theorem 2.21 says that a similar statement holds for minimal degree curves. See [Keb02b, sect. 3.3] for the related question if a minimal degree curve can be specified by two points.
Theorem 2.21 was known to be true in the case where \( C_x = \mathbb{P}(T_X^x) \), where it follows as a by-product of a characterization of \( \mathbb{P}^n \).

**Theorem 2.23 (CMSB02, Keb02a).** Let \( X \) be an irreducible normal projective variety of dimension \( n \). Let \( H \subset \text{RatCurves}^n(X) \) be a dominating family of rational curves of minimal degrees and assume that \( C = \mathbb{P}(T_X^x) \). Then there exists a finite morphism \( \mathbb{P}^n \to X \), étale over \( X \setminus \text{Sing}(X) \) that maps lines in \( \mathbb{P}^n \) to curves parametrized by \( H \). In particular, \( X \cong \mathbb{P}^n \) if \( X \) is smooth. \( \square \)

Hwang and Mok have applied the theory of differential systems to \( C \) in order to reduce the general case to that of Theorem 2.23. We give a very rough sketch of the proof and refer to [HM04] for details.

### 4.1. Sketch of proof of Theorem 2.21.

**Step 1: Differential systems in uniruled varieties.**

**Definition 2.24.** A distribution on \( X \) is a saturated subsheaf of \( T_X \). Given a distribution \( D \), its Cauchy characteristic distribution is the integrable subdistribution \( \text{Ch}(D) \subset D \) whose fiber at the general point \( x \in X \) is

\[
\text{Ch}(D)_x := \{ v \in D_x; N(v, D_x) = 0 \}
\]

where \( N \) denotes the O'Neill tensor, i.e., the \( \mathcal{O}_X \)-linear map \( N : D \otimes D \to T_X/D \) induced by the Lie bracket.

On \( C \subset \mathbb{P}(T_X^x) \) we can consider a natural distribution \( P \) defined in the general point \( \alpha \in C \) by:

\[
P_\alpha := (T\rho)^{-1}(T_{C_x,\alpha}),
\]

where \( T_{C_x,\alpha} \subset T_X^x |_x \) is the tangent space to the affine cone of \( C_x \) along the ray \( C \cdot \alpha \subset T_X^x \) determined by \( \alpha \). The following proposition is the technical core of Theorem 2.21. It is based in a detailed study of the distribution \( P \) and its relation with the family \( H \), which is beyond the purpose of this survey. We refer to [HM04] for a detailed account.

**Proposition 2.25.** With notation as above, the following holds.

1. The distribution \( \text{Ch}(P) \subset T_C \) contains the tangent directions of the images in \( C \) of the curves parametrized by \( H \). More precisely, if \( [\ell] \in H \) is a general point, then the morphism \( \tau|_{\pi^{-1}(\ell)} : \mathbb{P}^1 \to C \) is immersive and its image is tangent to \( \text{Ch}(P) \).

2. Let \( y \in C \) be a general point and \( S \) the associated leaf of \( \text{Ch}(P) \). Then \( S \) is algebraic, and there exists a dense open set \( S_0 \) such that \( W := \pi(S_0) \subset X \) is quasi-projective, and such that

\[
\pi|_{S_0} : S_0 \to W
\]

is a bundle of projective spaces, isomorphic to \( \mathbb{P}(T_W^y) \).

3. Let \( W \subset X \) be the Zariski closure of \( W \). The subvariety

\[
H_W := \{ [\ell] \in H \mid \ell \subset W \}
\]

is a dominating family of rational curves of minimal degrees in \( W \). \( \square \)

The subschemes of the form \( \overline{W} \) are called Cauchy subvarieties of \( X \) with respect to \( H \).

---

2 Strictly speaking, we have defined the notion of a family of rational curves only for normal varieties because the notion of the Chow-variety and its universal property is a little delicate for non-normal spaces. Although \( \overline{W} \) need not be normal, we ignore this (slight) complication in this sketch for simplicity.
Step 2: End of sketch of proof. Let \( x \in X \) be a general point and \( c \in C_x \subset \mathbb{P}(T_X|_x) \) be a general minimal rational tangent. We assume to the contrary and take two general curves \([\ell_1]\) and \([\ell_2]\) \( \in H_T \) that have \( c \) as tangent direction. Let \( S \) be the leaf of \( \text{Cl}(\mathcal{P}) \) that contains \( c \) and let \( W \subset X \) be the corresponding Cauchy subvariety. By Proposition 2.25,(1), \( W \) contains both \( \ell_1 \) and \( \ell_2 \).

By Proposition 2.25.(2) and (3), the tangent map associated with \( H_W \),

\[
\tau_{W,x} : H_{W,x} \dashrightarrow \mathbb{P}(T_W|_x) \subset \mathbb{P}(T_X|_x),
\]

is surjective. Theorem 2.23 then applies to the normalization of \( W \) and asserts that \( \tau_W \) must be birational. By general choice, \( \ell_1 \) and \( \ell_2 \) must then be equal, a contradiction.  

5. The importance of the VMRT

Given a uniruled variety \( X \) and a dominating family \( H \subset \text{RatCurves}^p(X) \) of rational curves of minimal degrees, we have claimed that geometry of \( X \) is determined to a large degree by the projective geometry of the associated VMRT \( C \subset \mathbb{P}(T_X^X) \). In this section we would like to name a few results that support the claim. Section 6 discusses one particular example in more detail.

In view of the Minimal Model Program, we restrict ourselves to Fano manifolds of Picard number 1. In this case, Hwang and Mok have shown under some technical assumptions, that \( X \) is completely determined by the family of minimal rational tangents over an analytic open set. This “Cartan-Fubini” type result is stated as follows.

**Theorem 2.26 ([HM01]).** Let \( X \) and \( X' \) be Fano manifolds of Picard number 1 defined over the field of the complex numbers, and let \( H \) and \( H' \) be dominating families of minimal rational curves in \( X \) and \( X' \), respectively. Let \( C_x \) and \( C'_x \) denote the associated VMRT at \( x \in X \) and \( x' \in X' \). Assume that \( C_x \) is positive-dimensional and that the Gauss map of the embedding \( C_x \subset \mathbb{P}(\Omega_X|_x) \) is finite for general \( x \in X \). Then any biholomorphic map \( \phi : V \to V' \) between analytic open neighborhoods \( V \subset X \) and \( V' \subset X' \) inducing an isomorphism between \( C_x \) and \( C'_x \), for all \( x \in V \) can be extended to a biholomorphic map \( \Phi : X \to X' \).

A detailed proof of Theorem 2.26 is given in the survey article [Hwa01]. There it is also discussed under what conditions even stronger results can be expected.

The VMRT has also been used to attack the following problems.

**Stability of the tangent bundle:** For Fano manifolds of Picard number 1, this property can be easily restated in terms of projective properties of \( C_x \subset \mathbb{P}(T_X^X) \) for general \( x \), which can be checked in some cases, see e.g. [Hwa98, Hwa02] for low-dimensional Fano manifolds and moduli spaces of vector bundles, or [Keb05] for contact manifolds.

**Deformation rigidity:** The VMRT have been used by Hwang and Mok to prove deformation rigidity of various types of varieties and morphisms. See, e.g., [Hwa97, Hwa01] or [HM98].

**Uniqueness of contact structures:** It has been shown in [Keb01] that contact structures on Fano manifolds of Picard number 1 are unique since their projectivization at a point coincides with the linear span of the VMRT.

**The Remmert–Van de Ven / Lazarsfeld problem:** Theorem 2.26 can be used to classify smooth images of surjective morphisms from rational homogeneous spaces of Picard number 1, see [HM99].
6. Higher secants and the length of a uniruled manifold

As before, let $X$ be a complex projective manifold and $H \subset \text{RatCurves}^n(X)$ be a dominating family of rational curves of minimal degrees. If $b_2(X) = 1$, then $X$ is rationally connected —see Definition 1.7 on page 11. In particular, if $x, y \in X$ are any two general points, there exists a connected chain of rational curves $\ell_1, \ell_2, \ldots, \ell_k \in H$ such that $x \in \ell_1$, $y \in \ell_k$ —see figure 1.1 on page 10.

The number $k$, i.e., the minimal length of chains of $H$-curves needed to connect two general points is called the length of $X$ with respect to $H$. The length is an important invariant; among other applications, it was used by Nadel in the proof of the boundedness of the degrees of Fano manifolds of Picard number one, [Nad91].

The main aim of this section is to introduce an effective method that allows to compute the length for a number of interesting varieties. We relate the length of $X$ to the projective geometry of the variety of minimal rational tangents. The length of $X$ can then be computed in situations where the secant defect of the VMRT is known.

We will employ these results in Chapter 3 below to give a bound on the multiplicities of divisors at a general point of the moduli of stable bundles of rank two on a curve.

6.1. Statement of result. To formulate the result precisely, let us fix our assumptions first. For the remainder of the present section, let $X$ be a complex projective manifold of arbitrary Picard number and $H \subset \text{RatCurves}^n(X)$ a dominating family of rational curves of minimal degrees, as in Section 1. If $x \in X$ is a general point, assume additionally that the space $H_x$ is irreducible.

We also need to consider a few following auxiliary spaces.

**Definition 2.27.** Consider the irreducible subvarieties
\[
\text{loc}^1(x) := \text{closure of } \bigcup_{\text{general } C \in H_x} C, \quad \text{and}
\text{loc}^{k+1}(x) := \text{closure of } \bigcup_{\text{general } C \in H_y} C,
\]
\[
\text{general } y \in \text{loc}^k(x)
\]
Set $d_k := \dim(\text{loc}^k(x))$; we call it the $k$-th spanning dimension of the family $H$. Finally, let $C_x \subset \mathbb{P}(T_x|_x)$ be the tangent cone to $\text{loc}^k(x)$ at $x$.

**Remark 2.28.** Even though $C_x$ is assumed to be irreducible, $C_x^1$ can have several components. This might be the case if some of this curves associated with points in $H_x$ are nodal, see Figure 6.1.

It is clear that the spanning dimensions of $H$ do not depend on the choice of the general point $x$. The tangent cones $C_x^k \subset \mathbb{P}(T_x|_x)$ have pure dimension $\dim C_x^k = d_k - 1$ —we refer to [Har95, lect. 20] for an elementary introduction to tangent cones.

**Remark 2.29.** The spanning dimensions are natural invariants of the pair $(X, H)$. In many situations, however, there is a canonical choice of $H$, so that we can actually view the $d_k$ as invariants of $X$.

**Definition 2.30.** If there exists a number $l > 0$ such that $d_l = \dim(X)$ and $d_{l-1} < \dim(X)$, we call $X$ rationally connected by $H$-curves, write $\text{length}_H(X) = l$ and say that $X$ has length $l$ with respect to $H$.

---

3The irreducibility assumption is posed for simplicity of exposition. It holds for all examples we will encounter. See [KK04, thm. 5.1] for a general irreducibility criterion.
FIGURE 6.1. Nodal curves causing reducibility of $C^1_x$

REMARK 2.31 ([Nad91]). If there exists a number $k$ such that $d_k = d_{k+1}$, it is clear from the definition that $\text{loc}^k(x) = \text{loc}^{k+1}(x) = \text{loc}^{k+2}(x) = \cdots$. In particular, if $X$ is rationally connected by $H$-curves, then $\text{length}_H(X) \leq \dim X$.

DEFINITION 2.32 ([CC02, Zak93]). Let $S^kC_x$ be the $k$-th secant variety of $C_x$, i.e., the closure of the union of $k$-dimensional linear subspaces of $\mathbb{P}(T_X \big|_x)$ determined by general $k + 1$ points on $Y$.

With this notation, the main result is formulated as follows.

THEOREM 2.33 ([HK05, thms. 3.11–14]). Let $X$ be as above. Then the following holds.

1. We have $C_x \subset C^1_x \subset S^1C_x$. If none of the curves of $H$ has nodal singularities at $x$ then $C_x = C^1_x$.
2. For each $k \geq 1$, we have $S^kC_x \subset C^{k+1}_x$. In particular, $d_{k+1} \geq \dim(S^kC_x) + 1$.
3. If $X$ admits an embedding $X \hookrightarrow \mathbb{P}^N$ such that the curves parametrized by $H$ are mapped to lines, then $S^1C_x = C^{2}_x$, and so $d_2 = \dim(S^1C_x) + 1$.

We will give a rough sketch of a proof below. In view of Theorem 2.33, one might be tempted to conjecture that $C^{k+1}_x = S^kC_x$. Remark 2.28 and the following remark show that this is not the case.

REMARK 2.34. The inequality in Theorem 2.33.(2) can be strict. For an example, let $X \not\cong \mathbb{P}^n$ be a Fano manifold with Picard number one which carries a complex contact structure, and $H$ a dominating family of rational curves of minimal degrees. In this setup, $C_x$ is linearly degenerate. In particular, we have that

$$\dim(S^kC_x) + 1 < \dim X = \text{length}_H(X)$$

for all $k$.

Generalizing Nadel’s product theorem, [Nad91], the knowledge of the length can be used to bound multiplicities of sections of vector bundles at general points of $X$, as follows.

PROPOSITION 2.35 ([HK05, prop. 2.6]). indexNadel, Alan M!product theorem!generalization Assume that $X$ is rationally connected by curves of the family $H$ and let $V$ be a vector bundle on $X$. Consider a general curve $C \in H$, let $\nu : \mathbb{P}^1 \to C$ be its normalization, and write

$$\nu^*(V) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r), \quad \text{with} \ a_1 \geq \cdots \geq a_r.$$ 

4See e.g. [Bea99, KPSW00] and the references therein for an introduction to complex contact manifolds.
6. Higher Secants and the Length of a Uniruled Manifold

If \( x \in X \) is a general point and \( \sigma \in H^0(X, V) \) any non-zero section, then the order of vanishing of \( \sigma \) at \( x \) satisfies \( \text{mult}_x(\sigma) \leq \text{length}_H(X) \cdot a_1. \] □

6.2. Sketch of Proof of Theorem 2.33.(1). Let \( x \in X \) be a general point. The first inclusion, \( C_x \subset C^1_x \), is immediate from the definition.

For the other inclusion, let again \( \tilde{H}_x \) be the normalization of \( H_x \) and consider the pull-back of Diagram (3) from page 25,

\[
\begin{array}{ccc}
U_x & \longrightarrow & X \\
\downarrow_{\pi_x} & & \downarrow_{\text{evaluation}} \\
\tilde{H}_x & & \\
\end{array}
\]

If none of the curves of \( H_x \) has nodal singularities at \( x \), Theorem 2.12 asserts that the preimage \( \iota^{-1}_x(x) \) is exactly a reduced section \( \sigma_{\infty} \cong \tilde{H}_x \). By the universal property of blowing up, the evaluation morphism \( \iota_x \) factors via the blow-up of \( X \) at \( x \) and the equality \( C^1_x = C_x \) follows.

If some of the curves in \( H_x \) are nodal, \( \iota^{-1}_x(x) \) contains the section \( \sigma_{\infty} \) and finitely many points. A somewhat technical analysis of the tangent morphism \( T \iota_x \), and a comparison between positive directions in the restriction of \( T_X \) to a nodal curve and the Zariski tangent space to the VMRT then yields the inclusion \( C^1_x \subset S^1C_x \).

6.3. Sketch of Proof of Theorem 2.33.(2). Here we only give an idea of why the first secant variety of the VMRT, \( S^1C_x \) is contained in the tangent cone \( C^2_x \) to the locus of length-2 chains of rational curves.

If \( v \) and \( w \in C_x \) are any two general elements, we need to show that the line in \( \mathbb{P}(T_X|_{\ell_v}) \) through \( v \) and \( w \) is contained in the tangent cone to \( \text{loc}^2(x) \). To this end, let \( [\ell_v] \) and \( [\ell_w] \in H_x \) be two rational curves that have \( v \) and \( w \) as tangent directions, respectively.

By general choice of \( v \) and \( w \), the curves \( \ell_v \) and \( \ell_w \) are smooth at \( x \).

Consider a small unit disk \( \Delta \subset \ell_v \), centered about \( x \). By general choice of \( x \), we can find a smooth holomorphic arc \( \gamma : \Delta \rightarrow H \) such that

- \( \gamma(0) = [\ell_w] \),
- for any point \( y \in \Delta \subset \ell_v \subset X \), the curve \( \gamma(y) \) contains \( y \), i.e., \( \gamma(y) \in H_y \).

The situation is described in Figure 6.2 above. The unions of the curves \( \gamma(\Delta) \) then forms a surface \( S \subset \text{loc}^2(x) \). By general choice, it can be seen that \( S \) is smooth at \( x \) and contains both \( v \) and \( w \) are tangent directions —see Figure 6.2. As a consequence, we obtain that the
line in $\mathbb{P}(T_{X|_x}^\vee)$ through $v$ and $w$ is also contained in $\mathbb{P}(T_{S|_x}^\vee)$. Since $\mathbb{P}(T_{S|_x}^\vee) \subset C^2_x$, this shows the claim.

6.4. Sketch of Proof of Theorem 2.33.(3). Let $v \in C^2_x$ be any element in the tangent cone to $\text{loc}^2(x)$. We need to show that $v$ is contained in the first secant variety to $C_x$, i.e., that $v \in S^1C_x$.

Recall from [Har95, lect. 20] that the tangent cone $C^2_x$ to $\text{loc}^2(x)$ is set-theoretically exactly the union of the tangent lines to holomorphic arcs $\gamma : \Delta \to \text{loc}^2(x)$ that are centered about $x$. Recall also that if the arc $\gamma$ is not smooth at 0, then the tangent line at 0 is the limit of the tangent lines to $\gamma$ at points where $\gamma$ is smooth.

We can thus find an arc $\gamma : \Delta \to \text{loc}^2(x)$ with $\gamma(0) = x$ and tangent $v$. Recall that $\text{loc}^2(x)$ is the locus of chains of rational curves of length 2 that contain $x$. Replacing $\Delta$ by a finite covering, if necessary, we can then find arcs

$$G : \Delta \to H_x \quad \text{and} \quad F : \Delta \to H$$

such that for all $t \in \Delta$ the curves $G(t) \cup F(t)$ form a chain of length two that contains both $x$ and $\gamma(t)$—this construction is depicted in Figure 6.3. Further, we consider arcs, defined for general $t$ as follows.

$$g : \Delta \to \text{loc}^1(x) \quad \quad P : \Delta \to \text{Grass}(2, \mathbb{P}^N)$$

$$t \to G(t) \cap F(t) \quad \quad t \to \text{plane spanned by } G(t) \text{ and } F(t)$$

Observe that this makes $g$ and $P$ well-defined because the target varieties are proper.

We distinguish two cases.

Case (i), $G(0) \neq F(0)$: In this case, $x = g(0) = G(0) \cap F(0)$, and $[v] \in \mathbb{P}(T_{P(0)|_x}^\vee)$. Since for general $t \in \Delta$, the line $\mathbb{P}(T_{P(t)|_x}^\vee)$ is contained in the secant variety $S^1C_{g(t)}$, we have $\mathbb{P}(T_{P(0)|_x}^\vee) \subset S^1C_x$. This shows the claim.

Case (ii), $G(0) = F(0)$: Since curves of $H$ are lines by assumption, Proposition 2.19 applies to the point $y = g(0)$ and the line $\ell = G(0)$.

Since $\mathbb{P}(T_{P(t)|_y}^\vee)$ is secant to $C_{g(t)}$ for general $t$, we obtain that $\mathbb{P}(T_{P(0)|_y}^\vee)$ is secant to $C_y$. Moreover, since $\ell = G(0) = F(0)$, the line $\mathbb{P}(T_{P(0)|_y}^\vee)$ lies in projective tangent space to $C_y$ at the point corresponding to $[\ell]$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{arcs.png}
\caption{Acs in $\text{loc}^2(x)$}
\end{figure}
To end, observe that the plane $P(0)$ is tangent to $X$ all along $\ell$. Its tangent directions determine a subbundle of $T_X|_{\ell}$. This implies $\mathbb{P}(T_{P(0)}|_\ell) \subset \mathbb{P}(T_X|_{\ell})$ and concludes the proof.

6.5. Open Problems.

Problem 2.36. In view of Remarks 2.28 and 2.34, is it possible to improve on Theorem 2.33 and find a formula that computes the spanning dimensions in all cases? Perhaps one needs to take into account how the VMRT (or its linear span) deforms as the base point changes.
Examples of uniruled varieties: moduli spaces of vector bundles

In this chapter we apply the results of Chapter 2 to one particular example of a uniruled variety, namely the moduli space of stable vector bundles on a curve. We first recall a classic construction of rational curves on the moduli space, and then show by example how the understanding of the VMRT can be used to study geometric properties of the moduli space.

1. Setup of notation. Definition of Hecke curve

Let $C$ be a smooth projective curve of genus $g \geq 3$ and $L$ be a line bundle of $C$ of degree $d$. We will denote by $M^r(L)$, or simply by $M^r$ if there is no possible confusion, the moduli scheme of semistable vector bundles of rank $r$ and determinant $L$. We can assume without loss of generality that $0 \leq d \leq r - 1$, otherwise twist with a line bundle. For simplicity, let us assume that $(r, d) = 1$, in which case $M^r$ is a smooth Fano manifold of Picard number one. We remark that most of the arguments and constructions presented here work the same for the general case.

**Definition 3.1.** Given two integers $k$ and $l$, we say that a vector bundle $E$ of rank $r$ is $(k, l)$-stable if every locally free subsheaf $F \subset E$ verifies:

$$\frac{\deg(F) + k}{\text{rank}(F)} < \frac{\deg E + k - l}{\text{rank}(E)}.$$ 

Let $M^r(k, l) \subset M^r$ the subset parameterizing $(k, l)$-stable vector bundles of rank $r$.

**Remark 3.2.** The subsets $M^r(k, l)$ are open in $M^r$. For instance $M^r(0, 0) \subset M^r$ is the open set parametrizing stable bundles.

Narasimhan and Ramanan showed that the subset $M^r(1, 1)$ is non-empty for $g \geq 3$ —see [NR78, prop. 5.4] for a precise statement. For every $(1, 1)$-stable bundle $E$ they constructed a 1-dimensional family deformations of $E$ that corresponds to a rational curve in $M^r$. The so-constructed curves are called Hecke curves. We briefly review their definition below.

**Construction of the Hecke curves.** Let $E$ be a vector bundle of rank $r$ and let $x$ be any point of $C$. Every element $p = [\alpha] \in \mathbb{P}(E_x)$ provides an exact sequence of $\mathcal{O}_C$-modules,

$$(6) \quad 0 \to E^p \xrightarrow{\phi} E \to \mathcal{O}_x \to 0,$$

which is induced by the map $\alpha : E_x \to \mathbb{C}$. The kernel $E^p$ depends only on the class $p = [\alpha]$ and it is called the elementary transform of $E$ at $p$. Let $H_p(E) := (E^p)^\vee$ be its dual.
EXAMPLES OF UNIRULED VARIETIES: MODULI SPACES OF VECTOR BUNDLES

Consider the associated projective bundles, elementary transforms can be described in terms of blow-ups and blow-downs. We have depicted the case \( r = 2 \) in Figure 1.1.

Dualizing the exact sequence (6) we obtain the following.

\[
0 \to E^\vee \to H_p(E) \to \mathcal{O}_x \to 0
\]

(7)

Sequence (7) expresses \( E^\vee \) as the elementary transformation of \( H_p(E) \) at the point \( p' = \text{coker } \phi^\vee_x \in \mathbb{P}(H_p(E)_x) \), that is \( H_{p'}(H_p(E)) = E \). We can then consider the \( H_q(H_p(E)) \) as deformations of \( E \) parametrized by \( q \in \mathbb{P}(H_p(E)_x) \).

It is easy to see that if \( E \) is \((k, l)\)-stable, then \( H_p(E) \) is \((l - 1, k)\)-stable —see [NR78, lem. 5.5]. In particular, when \( E \) is \((1, 1)\)-stable, then every element of the form \( H_q(H_p(E)) \) is stable. A line \( L \subset \mathbb{P}(H_p(E)_x) \) through \( p' \) will therefore determine a rational curve through \([E]\) in \( M^r \). We denote this curve by \( C(E, p, L) \).

DEFINITION 3.3. A curve of the form \( C(E, p, L) \) constructed as above is called a Hecke curve through \([E]\).

2. Minimality of the Hecke curves

Notice that lines in \( \mathbb{P}(H_p(E)_x) \) through \( p' \) are in one-to-one correspondence with the set of hyperplanes in \( \mathbb{P}(E_x) \) that contain \( p \), i.e., with \( \mathbb{P}(T_{\mathbb{P}(E_x)}|_p) \). The next proposition states basic properties of Hecke curves that have been originally proved by Narasimhan and Ramanan—we refer the interested reader to [NR78, sect. 5] for details.

PROPOSITION 3.4. With the same notation as above, the following holds.

- [NR78, 5.13]: The map that sends \((p, L)\) to the Hecke curve \( C(E, p, L) \) is injective for every \([E]\) in \( M^r(1, 1) \). In particular, Hecke curves through \([E]\) are naturally parametrized by the points of the projectivization of the relative tangent bundle \( \mathbb{P}(T_{\mathbb{P}(E), C}) \).

- [NR78, 5.9, 5.15, 5.16]: Hecke curves are free smooth rational curves of anti-canonical degree \( 2r \) —see Proposition 2.8 on page 26 for the notion of a free rational curve.

□

DEFINITION 3.5. Set \( H_E := \mathbb{P}(T_{\mathbb{P}(E), C}) \), and let \( \mathcal{O}(1) \) be the associated tautological line bundle. Further, let \( \rho : H_E \to C \) be the natural projection.
If \([E] \in M^r(1,1)\) is any point, we can view \(H_E\) as a subset of \(\text{RatCurves}^M(M^r)\). Let \(H \subset \text{RatCurves}^M(M^r)\) be the closure of the union of the \(H_E\) for all \([E] \in M^r(1,1)\).

Minimality of the Hecke curves was shown by Hwang in [Hwa00, prop. 9] for the case \(r = 2\). Recently Sun has proven that Hecke curves are minimal curves in \(M^r\) for any \(r\), and that the converse is also true:

**Theorem 3.6 ([Sun05, thm. 1]).** Any rational curve \(C \subset M^r\) passing through a general point of \(M^r\) has anticanonical degree at least \(2r\). If \(g \geq 3\), then the anticanonical degree of \(C\) is \(2r\) if and only if it is a Hecke curve. □

In particular, Sun shows that the variety \(H\) is a dominating family of rational curves of minimal degrees in \(M^r\). If \([E] \in M^r(1,1)\) is a general point, the associated subspace of curves through \([E]\) is exactly \(H_E\).

3. VMRT associated to \(M^r\)

The variety of minimal rational tangents to \(M^r\) at \([E] \in M^r(1,1)\) is the image of the tangent map:

\[
\tau_E : H_E \rightarrow \mathbb{P}(T^\vee_{M^r,[E]}) \rightarrow \mathbb{P}(\mathcal{T}_{C(E,p,L),[E]})
\]

We would like to understand \(\tau_E\) in two ways, namely

- in terms of the Kodaira-Spencer map associated to Hecke curves, as studied by Narasimhan and Ramanan in [NR78], and
- in terms of linear systems on \(H_E\).

For the first item, recall the standard description of the tangent space to the moduli scheme,

\[
T_{M^r,[E]} \cong H^1(C, \text{ad}(E)) \cong H^0(C, K_C \otimes \rho^* \mathcal{O}(1)),
\]

where \(\text{ad}(E)\) is the sheaf of traceless endomorphisms of \(E\). The morphism \(\tau_E\) is then described as follows.

**Proposition 3.7 ([NR78, Lemma 5.10]).** The Kodaira-Spencer map \(T_{L,p'} \rightarrow H^1(C, \text{ad}(E))\) coincides, up to sign, with the composition of the following two morphisms.

- The natural map \(\alpha : T_{L,p'} \rightarrow E_x\), and
- the connecting morphism \(\beta : E_x \rightarrow H^1(E \otimes E^\vee)\) of the exact sequence (7) tensored with \(E\). □

For the other description of \(\tau_E\), we mention another result of Hwang.

**Theorem 3.8 ([Hwa02, thms. 3-4], [Hwa00, prop. 11]).** With the same notation as above, the morphism \(\tau_E\) is given by the complete linear system associated to the line bundle \(\rho^*(K_C) \otimes \mathcal{O}(1)\). If moreover \(g > 2r + 1\) then \(\tau_E\) is an embedding. □

By the minimality of Hecke curves, Theorem 2.16 implies that \(\tau_E\) is a finite map, that is, that the line bundle \(\rho^*(K_C) \otimes \mathcal{O}(1)\) is ample.

4. Applications

The description of the VMRT given in Section 3 has interesting consequences. For instance, Hwang has used the projective properties of the tangent morphism \(\tau_E\) to deduce the following properties of the tangent bundle to the moduli space.
Theorem 3.9 ([Hwa00, thm. 1], [Hwa02, cor. 3]). Let $M^2$ be the moduli space of stable bundles of rank 2 with a fixed determinant of odd degree over an algebraic curve of genus $g \geq 2$. Then the tangent bundle of $M^2$ is stable. Let $M^r$ be the moduli space of semistable bundles of rank $r$ with fixed determinant and $(M^r)^0$ its smooth locus. Then $T_{(M^r)^0}$ is simple for $g \geq 4$. □

The above description of the variety of minimal rational tangents also allows to deduce some of its projective properties, for instance its secant defect. This has been done in the case $r = 2$.

Proposition 3.10 ([HK05, prop. 6.10]). Let $M^2$ be the moduli space of stable bundles of rank 2 with a fixed determinant of odd degree over an algebraic curve of genus $g \geq 4$. The variety of minimal rational tangents at a general point of $M^2$ has no secant defect. □

This result, combined with Proposition 2.35 from page 36, provides a bound on the multiplicity of divisors in the moduli space, perhaps similar in spirit to the classical Riemann singularity theorem.

Corollary 3.11 ([HK05, cor. 6.12]). With the same notation as above, let $x \in M^2$ be a general point, and $L$ be the ample generator of $\text{Pic}(M^2)$, and $D \in |mL|$, $m \geq 1$ be any divisor. Then

$$\text{mult}_x(D) \geq 2m(g - 1).$$

□

The fact that the only minimal rational curves at the general point are Hecke curves has also very important corollaries, as pointed out by Sun. To begin with, it allows us to state a Torelli-type theorem for moduli spaces:

Theorem 3.12 ([Sun05, cor. 1.3]). Let $C$ and $C'$ be two smooth projective curves of genus $g \geq 4$. Let $M^r$ and $(M')^r$ be two irreducible components of the moduli schemes of vector bundles of rank $r$ over $C$ and $C'$, respectively. If $M^r \cong (M')^r$ then $C \cong C'$. □

Second, it can be used to describe the automorphism group of $M^r$.

Theorem 3.13 ([Sun05, cor. 1.4]). Let $C$ be a smooth projective curve of genus $g \geq 4$, $L$ a line bundle on $C$. If $r > 2$, then the group of automorphisms of $M^r(L)$ is generated by:

- automorphisms induced by automorphisms of $C$, and
- automorphisms of the form $E \mapsto E \otimes L'$, where $L'$ is an $r$-torsion element of $\text{Pic}^0(C)$.

For $r = 2$, we need additional generators of the form $E \mapsto E' \otimes L'$ where $L'$ is a line bundle verifying $(L')^\otimes 2 \cong L^\otimes 2$. □
Part 3

Consequences of non-uniruledness
Deformations of surjective morphisms

Let \( f : X \to Y \) be a surjective morphism between normal complex projective varieties. A classical problem of complex geometry asks for a criterion to guarantee the (non-)existence of deformations of the morphism \( f \), with \( X \) and \( Y \) fixed. More generally, one is interested in a description of the connected component \( \text{Hom}_f(X, Y) \subset \text{Hom}(X, Y) \) of the space of morphisms.

For instance, if \( X \) is of general type, it is well-known that the automorphism group is finite. It is more generally true that surjective morphisms between projective manifolds \( X, Y \) of general type are always infinitesimally rigid so that the associated connected components of \( \text{Hom}(X, Y) \) are reduced points. Similar questions were discussed in the complex-analytic setup by Borel and Narasimhan, [BN67]. We will show here how Miyaoka’s characterization of uniruledness, or the existence result for rational curves contained in Theorem 1.10/Corollary 1.28 can be used to give a rather satisfactory answer in the projective case. Before giving an idea of the methods employed, we will first, in Sections 1 and 2, state and explain the result.

1. Description of \( \text{Hom}_f(X, Y) \) if \( Y \) is not uniruled

If \( f : X \to Y \) is as above, it is obvious that \( f \) can always be deformed if the target variety \( Y \) has positive-dimensional automorphism group; this is because the composition morphism

\[
\begin{align*}
f^* : \text{Aut}^0(Y) & \to \text{Hom}_f(X, Y) \\
g & \mapsto g \circ f
\end{align*}
\]

is clearly injective. One could naïvely hope that all deformations of \( f \) come from automorphisms of the target. While this hope does not hold true in general, we will show, however, that it is almost true: if \( Y \) is not uniruled, \( f \) always factors via an intermediate variety \( Z \) whose automorphism group is positive-dimensional and induces all deformations of \( f \). More precisely, the following holds.

**Theorem 4.1** ([HKP03, thm. 1.2]). Let \( f : X \to Y \) be a surjective morphism between normal complex-projective varieties, and assume that \( Y \) is not uniruled. Then there exists a factorization of \( f \),

\[
X \overset{\alpha}{\longrightarrow} Z \overset{\beta}{\longrightarrow} Y,
\]

such that:

1. the morphism \( \beta \) is unbranched away from the singularities of \( X \) and \( Y \), and
The natural morphism
\[ \frac{\text{Aut}^0(Z)}{\text{Deck}(Z/Y)} \rightarrow \text{Hom}_f(X, Y) \]
\[ g \mapsto \beta \circ g \circ \alpha \]
is an isomorphism of schemes, where Deck(Z/Y) is the group of Deck transformations, i.e., relative automorphisms.

In particular, \( f \) deforms unobstructedly, and the associated component \( \text{Hom}_f(X, Y) \) is a smooth abelian variety.

The following is an immediate corollary whose proof we omit for brevity.

**Corollary 4.2 ([HKP03, cor. 1.3]).** In the setup of Theorem 4.1, if the target variety \( Y \) is smooth, then \( Y \) admits a finite, étale covering of the form \( T \times W \), where \( T \) is a torus of dimension \( \dim T = h^0(X, f^*(T_Y)) \). Additionally, we have
\[ \dim \text{Hom}_f(X, Y) \leq \dim Y - \kappa(Y), \]
where \( \kappa(Y) \) is the Kodaira dimension.

\[ \square \]

We will later, in section 3 give an idea of the proof of Theorem 4.1.

**2. Description of \( \text{Hom}_f(X, Y) \) in the general case**

If \( Y \) is rationally connected, partial descriptions of the Hom-scheme are known — the results of [HM03, thm. 1] and [HM04, thm. 3] assert that whenever \( Y \) is a Fano manifold of Picard number 1 whose variety of minimal rational tangents is finite, or not linear, then all deformations of \( f \) come from automorphisms of \( Y \). This covers all examples of Fano manifolds of Picard number one that we have encountered in practice.

If \( Y \) is covered by rational curves, but not rationally connected, we consider the rationally connected quotient \( q_Y : Y \dashrightarrow Q_Y \) which is explained in Definition 1.15. It is shown in [KP05] that \( f \) can be factored via an intermediate variety \( Z \), in a manner similar to that of Theorem 4.1, such that a covering of \( \text{Hom}_f(X, Y) \) decomposes into
- an abelian variety, which comes from the automorphism group of \( Z \), and
- the space of deformations that are relative over the rationally connected quotient, i.e., \( H^1_{\text{vert}} := \{ f' \in \text{Hom}_f(X, Y)_{\text{red}} \mid q_Y \circ f' = q_Y \circ f \} \).

To formulate the result precisely, we recall a result that yields a factorization of \( f \) and may be of independent interest.

**Theorem 4.3 ([KP05, thm. 1.4]).** Let \( f : X \rightarrow Y \) be a surjective morphism between normal projective varieties. Then there exists a factorization

(9)
\[ \begin{array}{ccc}
X & \overset{\alpha}{\longrightarrow} & Z \\
\downarrow{\beta} & & \downarrow{\beta'} \\
Y & \overset{f}{\longrightarrow} & Y
\end{array} \]

where \( \beta \) is finite and étale in codimension one\(^1\), and where the following universal property holds: for any factorization \( f = \beta' \circ \alpha' \), where \( \beta' : Z' \rightarrow Y \) is finite and étale in codimension 1, there exists a morphism \( \gamma : Z \rightarrow Z' \) such that \( \beta = \beta' \circ \gamma \).

\[ \square \]

\(^1\)I.e., where \( \beta \) is finite and étale away from a set of codimension two.
3. Sketch of proof of Theorem 4.1

It follows immediately from the universal property that the factorization (9) is unique up to isomorphism. We call (9) the maximally étale factorization of \( f \). The maximally étale factorization can be seen as a natural refinement of the Stein factorization. More precisely, we can say that a surjection \( f : X \to Y \) of normal projective varieties decomposes as follows.

\[
\begin{array}{ccccccc}
X & \xrightarrow{\text{conn. fibers}} & W & \xrightarrow{\text{finite}} & Z & \xrightarrow{\text{max. étale}} & Y \\
\end{array}
\]

The paper [KP05] discusses the maximally étale factorization in more detail. Its stability under deformations of \( f \) is shown [KP05, sect. 1.B], and a characterization in terms of the positivity of the push-forward sheaf \( f_* (\mathcal{O}_X) \) is given, [KP05, sect. 4].

The main result is then formulated as follows.

**Theorem 4.4 ([KP05, thm. 1.10]).** In the setup of Theorem 4.3, let \( T \subset \text{Aut}^0(Z) \) be a maximal compact abelian subgroup. Then there exists a normal variety \( \tilde{H} \) and an étale morphism

\[
T \times \tilde{H} \to \text{Hom}_f^0(X, Y)
\]

that maps \( \{ e \} \times \tilde{H} \) to the preimage of \( H^f_{\text{vert}} \). If \( Y \) is smooth or if \( f \) is itself maximally étale, then \( \{ e \} \times \tilde{H} \) surjects onto the preimage of \( H^f_{\text{vert}} \). \( \square \)

In Theorem 4.4, we discuss the maximal compact abelian subgroup of an algebraic group. Recall the following basic fact of group theory.

**Remark 4.5.** Let \( G \) be an algebraic group. Then there exists a maximal compact abelian subgroup, i.e., an abelian variety \( T \subset G \) which is a subgroup and such that no intermediate subgroup \( T \subset S \subset G, T \neq S \), is an abelian variety. A maximal compact abelian subgroup is unique up to conjugation.

**Remark 4.6.** In the setup of Theorem 4.4, it need not be true that \( \text{Aut}^0(Z) \) is itself an abelian variety. Unlike Theorem 4.1, Theorem 4.4 does not make any statement about the scheme structure of \( \text{Hom}_f(X, Y) \).

While the methods used to show Theorems 4.1 and 4.4 are obviously related, the fact that the rational quotient is generally not a morphism, and the lack of a good parameter space for rational maps makes the proof of Theorem 4.4 technically more involved. We have thus decided to restrict to a sketch of a proof of Theorem 4.1 only.

3. Sketch of proof of Theorem 4.1

We give only a rough idea of a proof for Theorem 4.1 — our main intention is to show how the existence result for rational curves comes into the picture. The interested reader is referred to the rather short original article [HKP03], and perhaps to the more detailed survey [Keb04].

**Step 1: simplifying assumption.** As we are only interested in a presentation of the core of the argumentation, we will show Theorem 4.1 under the simplifying assumption that the surjective morphism \( f : X \to Y \) is a finite morphism between complex-projective manifolds.
4. DEFORMATIONS OF SURJECTIVE MORPHISMS

Step 2: The tangent map to \( f^\circ \). Consider again the natural morphism \( f^\circ : \text{Aut}^0(Y) \to \text{Hom}_f(X,Y) \), as introduced in Equation (8) on page 47. The universal properties of \( \text{Hom}(X,Y) \) and of the automorphism group \( \text{Aut}^0(Y) \) yield the following description of the tangent map \( T f^\circ \) at the point \( e \in \text{Aut}^0(Y) \),

\[
T f^\circ|_e : T_{\text{Aut}(Y)}|_e \to T_{\text{Hom}|f}.
\]  

Namely, the natural identifications

\[
T_{\text{Aut}(Y)}|_e \cong H^0(Y, T_Y) \quad \text{and} \quad T_{\text{Hom}|f} \cong H^0(X, f^*(T_Y))
\]

associate the tangent morphism (10) with the pull-back map

\[
T f^\circ|_e = f^* : H^0(Y, T_Y) \to H^0(X, f^*(T_Y)).
\]

The following observation is now immediate:

**Lemma 4.7.** The morphism (10) is injective. If the pull-back morphism (11) is surjective, i.e., if any infinitesimal deformation of \( f \) comes from a vector field on \( Y \), then (10) yields an isomorphism of Zariski tangent spaces. \( \square \)

Similar considerations also yield a description of the tangent morphism at an arbitrary point \( g \in \text{Aut}^0(Y) \).

\[
T f^\circ|_g = (g \circ f)^* : H^0(Y, T_Y) \to H^0(X, (g \circ f)^*(T_Y)).
\]

Since \( g : Y \to Y \) has maximal rank everywhere, it can be shown that the rank of \( T f^\circ \) is constant on all of \( \text{Aut}^0(Y) \). Using that \( \text{Aut}^0(Y) \) is smooth, an elementary argumentation then yields the following.

**Corollary 4.8.** If the pull-back morphism (11) is surjective, then the composition map \( f^\circ : \text{Aut}^0(Y) \to \text{Hom}_f(X,Y) \) is isomorphic. In this case, the proof of Theorem 4.1 is finished by setting \( Z := Y \). \( \square \)

**Step 3, central step in the proof: construction of a covering.** Corollary 4.8 allows us to assume without loss of generality that there exists a section \( \sigma \in H^0(X, f^*(T_Y)) \) that does not come from a vector field on \( Y \). Under this assumption, we will then construct an unbranched covering of \( Y \), which factors the surjection \( f \). The main tool is the following negativity theorem of Lazarsfeld for the push-forward sheaf \( f_* \mathcal{O}_X \) which can, in our setup, be seen as a competing statement to Corollary 1.28 of page 17, the characterization of uniruledness.

**Theorem 4.9 ([Laz80], [PS00, thm. A]).** The trace map \( \text{tr} : f_* \mathcal{O}_X \to \mathcal{O}_Y \) yields a natural splitting

\[
f_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{E}^\nu,
\]

where \( \mathcal{E} \) is a vector bundle with the following positivity property: if \( C \subset Y \) is any curve not contained in the branch locus of \( f \), then \( \mathcal{E}|_C \) is nef. The curve \( C \) intersects the branch locus of \( f \) if and only if \( \text{deg}(\mathcal{E}|_C) > 0 \). \( \square \)

**Corollary 4.10.** Choose an ample line bundle \( H \in \text{Pic}(Y) \) and let \( C \subset Y \) be an associated general complete intersection curve in the sense of Mehta-Ramanathan, as in Definition 1.5 of page 10. Then the following holds:

- If \( X \to Z \xrightarrow{\beta} Y \) is a factorization of \( f \) with \( \beta \) étale, then \( \beta_* \mathcal{O}_Z \subset f_* \mathcal{O}_Y \) is a subbundle which is closed under the multiplication map

\[
\mu : f_* \mathcal{O}_X \otimes f_* \mathcal{O}_X \to f_* \mathcal{O}_X,
\]

and satisfies \( \text{deg}(\beta_* \mathcal{O}_Z)|_C = 0 \).
Conversely, if \( F \subset f_*(O_X) \) is a subbundle that is closed under multiplication, and \( \deg(F|_C) = 0 \), then \( f \) factors via \( Z := \text{Spec}(F) \), and \( Z \) is étale over \( Y \). □

Observe that the projection formula gives an identification

\[
H^0(X, f^*(T_Y)) = H^0(Y, f_*(f^*(T_Y))) = H^0(Y, E^\vee \otimes T_Y)
\]

Since \( \sigma \) does not come from a vector field on \( Y \), the section \( \sigma \) is not contained in the component \( H^0(Y, T_Y) \). Thus, we obtain a non-trivial morphism \( \sigma : E \to T_Y \).

Now choose \( H \) and \( C \) as in Corollary 4.10. Lazarsfeld’s Theorem 4.9 then implies that \( \text{Image}(\sigma)|_C \) is nef. On the other hand, Corollary 1.28 asserts that \( \text{Image}(\sigma)|_C \) cannot be ample. In summary we have the following.

**Lemma 4.11.** The restricted vector bundle \( E|_C \) is nef, but not ample. □

**Definition 4.12.** Let \( \mathcal{V}_C \subset E|_C \) be the maximal ample subbundle, as discussed in Proposition 1.29 and Definition 1.30. Further, let \( \mathcal{F}_C \subset E^\vee|_C \) be the kernel of the associated morphism \( E^\vee|_C \to \mathcal{V}_C^\vee \).

**Remark 4.13.** The bundle \( \mathcal{F}_C \) is dual to the quotient \( E|_C / \mathcal{V}_C \). In particular, \( \mathcal{F}_C \) is nef and has degree 0.

With these preparations we will now construct the factorization of \( f \). We construct the factorization first over \( C \), and then extend it to all of \( Y \).

**Lemma 4.14.** The sub-bundle \( \mathcal{O}_C \oplus \mathcal{F}_C \subset f_*(O_X)|_C \) is closed under the multiplication map

\[
\mu : (f_*(O_X) \otimes f_*(O_X))|_C \to f_*(O_X)|_C,
\]

because the associated morphism

\[
\mu' : (\mathcal{O}_C \oplus \mathcal{F}_C) \oplus (\mathcal{O}_C \oplus \mathcal{F}_C) \to \mathcal{O}_C \oplus E^\vee|_C / \mathcal{O}_C \oplus \mathcal{F}_C
\]

is necessarily trivial. □

By Corollary 4.10, the subbundle \( \mathcal{O}_C \oplus \mathcal{F}_C \subset f_*(O_X)|_C \) induces a factorization of the restricted morphism \( f|_C : f^{-1}(C) \to C \) via an étale covering of \( C \). In order to extend this covering from \( C \) to all of \( Y \), if suffices to extend the maximal ample subbundle \( \mathcal{V}_C \subset E|_C \) to a subbundle \( V \subset E \) that has the property that the restriction \( \mathcal{V}|_{C'} \subset E|_{C'} \) to any complete intersection curve \( C' \subset Y \) is exactly the maximal ample subbundle of \( E|_{C'} \). But since the maximal ample subbundle is by definition a term of the Harder-Narasimhan filtration of \( E|_C \), Theorem 1.4 of Flenner and Mehta-Ramanathan, exactly asserts that this is possible.

**Corollary 4.15.** If the pull-back morphism \( H^0(Y, T_Y) \to H^0(X, f^*(T_Y)) \) is not surjective, then there exists a factorization of \( f \),

\[
X \xrightarrow{\alpha} Y^{(1)} \xrightarrow{\beta} Y,
\]

where \( \beta \) is étale. □
Step 4: End of proof. Assuming that the pull-back map $H^0(Y, T_Y) \to H^0(X, f^*(T_Y))$ was not surjective, we have in Step 3 constructed a factorization of $f$ via an étale covering $\beta : Y^{(1)} \to Y$. The étalé obviously implies $f^*(T_Y) = \alpha^*(T_{Y^{(1)}})$. We ask again if the pull-back morphism

$$\alpha^* : H^0(Y^{(1)}, T_{Y^{(1)}}) \to H^0(X, f^*(T_Y))$$

is surjective.

Yes $\rightarrow$: Following the considerations of Step 2, the proof is finished if we set $Z := Y^{(1)}$.

No $\rightarrow$: repeat Step 3, using the morphism $\alpha : X \to Y^{(1)}$ rather than $f : X \to Y$.

Repeating this procedure, we construct a sequence of étale coverings

$$X \rightarrow Y^{(d)} \rightarrow Y^{(d-1)} \rightarrow \cdots \rightarrow Y^{(1)} \rightarrow Y.$$

The sequence, however, must terminate after finitely many steps, simply because the number of leaves is finite. The same line of argumentation that led to Corollary 4.8 shows that we can end the proof is we set $Z = Y^{(d)}$. This finishes the proof of Theorem 4.1 —under the simplifying assumption that $f$ is a finite morphism between complex manifolds.

4. Open Problems

It is not quite clear to us if projectivity or if complex number field is really used in an essential manner. We would therefore like to ask the following.

QUESTION 4.16. Does Theorem 4.1 hold in positive characteristic?

QUESTION 4.17. Does it hold for Kähler manifolds or complex spaces?

Theorem 4.1 can also be interpreted as follows: all obstructions to deformations of surjective morphisms come from rational curves in the target. Is it possible to make this statement precise?
Families of canonically polarized varieties

Let \( B^o \) be a smooth quasi-projective curve, defined over an algebraically closed field of characteristic 0 and \( q > 1 \) a positive integer. In his famous paper [Sha63], Shafarevich considered the set of families of curves of genus \( q \) over \( B^o \). More precisely, he considered isomorphism classes of smooth proper morphisms \( f : S \to B^o \) whose fibers are connected curves of genus \( q \). He conjectured the following.

**Finiteness conjecture:** There are only finitely many isomorphism classes of non-isotrivial families of smooth projective curves of genus \( q \) over \( B^o \) — recall that a family is called isotrivial if any two fibers are isomorphic.

**Hyperbolicity conjecture:** If \( \kappa(B^o) \leq 0 \), then no such families exist.

These conjectures, which later played an important role in Faltings’ proof of the Mordell conjecture, were confirmed by Parshin [Par68] for projective bases \( B^o \) and by Arakelov [Ara71] in general. We refer the reader to the survey articles [Vie01] and [Kov03] for a historical overview and references to related results.

It is a natural and important question whether similar statements hold for families of higher dimensional varieties over higher dimensional bases. Families over a curve have been studied by several authors in recent years and they are now fairly well understood — the strongest results known were obtained in [VZ01, VZ02], and [Kov02]. For higher dimensional bases, however, a complete picture is still missing and no good understanding of subvarieties of the corresponding moduli stacks is available. As a first step toward a better understanding, Viehweg conjectured the following:

**Conjecture 5.1 ([Vie01, 6.3]).** Let \( f^o : X^o \to S^o \) be a smooth family of canonically polarized varieties. If \( f^o \) is of maximal variation, then \( S^o \) is of log general type — see Definition 1.39 on page 20 for the logarithmic Kodaira dimension and log general type.

For the reader’s convenience, we briefly recall the definition of variation that was introduced by Kollár and Viehweg.

**Definition 5.2.** Let \( f^o : X^o \to S^o \) be a family of canonically polarized varieties and \( \mu_{f^o} : S^o \to \mathbf{M} \) the induced map to the corresponding moduli scheme. The variation of \( f^o \) is defined as \( \text{Var}(f^o) := \dim \mu_{f^o}(S^o) \).

The family \( f^o \) is called isotrivial if \( \text{Var}(f^o) = 0 \). It is called of maximal variation if \( \text{Var}(f^o) = \dim S^o \).

1. Statement of result

Using a result of Viehweg-Zuo and Keel-McKernan’s proof of the Miyanishi conjecture in dimension two, we describe families of canonically polarized varieties over quasi-projective surfaces. We relate the variation of the family to the logarithmic Kodaira dimension of the base and give an affirmative answer to Viehweg’s Conjecture 5.1 for families over surfaces.
THEOREM 5.3 ([KK05, thm. 1.4]). Let \( S^o \) be a smooth quasi-projective surface and \( f^o : X^o \to S^o \) a smooth non-isotrivial family of canonically polarized varieties, all defined over \( \mathbb{C} \). Then the following holds.

1. If \( \kappa(S^o) = -\infty \), then \( \text{Var}(f^o) \leq 1 \).
2. If \( \kappa(S^o) \geq 0 \), then \( \text{Var}(f^o) \leq \kappa(S^o) \).

In particular, Viehweg’s Conjecture holds true for families over surfaces.

REMARK 5.4. Notice that in the case of \( \kappa(S^o) = -\infty \) one cannot expect a stronger statement. For an easy example take any non-isotrivial smooth family of canonically polarized varieties over a curve \( g : Z \to C \), set \( X := Z \times \mathbb{P}^1 \), \( S^o := C \times \mathbb{P}^1 \), and let \( f^o := g \times \text{id}_{\mathbb{P}^1} \) be the obvious morphism. Then we clearly have \( \kappa(S^o) = -\infty \) and \( \text{Var}(f) = 1 \).

2. Open problems

Theorem 5.3 and its proof seem to suggest that the logarithmic Kodaira dimension of a variety \( S^o \) gives an upper bound for the variation of any family of canonically polarized varieties over \( S^o \), unless \( \kappa(S^o) = -\infty \). We would thus like to propose the following generalization of Viehweg’s conjecture.

CONJECTURE 5.5 ([KK05, conj. 1.6]). Let \( f^o : X^o \to S^o \) be a smooth family of canonically polarized varieties. Then either \( \kappa(S^o) = -\infty \) and \( \text{Var}(f^o) < \dim S^o \), or \( \text{Var}(f^o) \leq \kappa(S^o) \).

3. Sketch of proof of Theorem 5.3

Again we give only an incomplete proof of Theorem 5.3. We restrict ourselves to the case where \( \kappa(S^o) = 0 \) and show only that \( \text{Var}(f^o) \leq 1 \). We will then, at the end of the present section, give a rough idea of how isotriviality can be concluded.

3.1. Setup of Notation. Let \( S \) be a compactification of \( S^o \) as in Definition 1.39, and let \( D := S \setminus S^o \) be the boundary divisor, which has at worst simple normal crossings. We assume \( \kappa(S^o) = \kappa(K_S + D) = 0 \).

A part of the argumentation involves the log minimal model of \( (S, D) \) — we refer to [KM98] and [Mat02] for details on the log minimal model program for surfaces. If \( \kappa(S^o) \neq -\infty \), we denote the birational morphism from \( S \) to its logarithmic minimal model by

\[ \phi : (S, D) \to (S\lambda, D\lambda), \]

where \( D\lambda \) is the cycle-theoretic image of \( D \) in \( S\lambda \). We briefly recall a few important facts of minimal model theory in dimension two that can be found in the standard literature, e.g. [KM98] or [Mat02].

PROPOSITION 5.6. The minimal model has the following properties:

1. The pair \( (S\lambda, D\lambda) \) has only log-canonical singularities and \( S\lambda \) itself has only log-terminal singularities. In particular, \( S\lambda \) has only quotient singularities and is therefore \( \mathbb{Q} \)-factorial.
2. The log-canonical divisor \( K_{S\lambda} + D\lambda \) is nef.
3. The log Kodaira dimension remains unchanged, i.e., \( \kappa(K_{S\lambda} + D\lambda) = \kappa(K_S + D) \).
4. Logarithmic Abundance in Dimension 2: The linear system \( |n(K_{S\lambda} + D\lambda)| \) is basepoint-free for sufficiently large and divisible \( n \in \mathbb{N} \). \( \square \)

□
3. SKETCH OF PROOF OF THEOREM 5.3

3.2. A result of Viehweg and Zuo. Before starting the proof we recall an important result of Viehweg and Zuo that describes the sheaf of logarithmic differentials on the base of a family of canonically polarized varieties in our setup.

THEOREM 5.7 ([VZ02, thm. 1.4(i)]). In the setup introduced above, there exists an integer $n > 0$ and an invertible subsheaf $\mathcal{A} \subset \text{Sym}^n \Omega^1_S(\log D)$ of Kodaira dimension $\kappa(\mathcal{A}) \geq \text{Var}(f^\circ)$.

3.3. Reduction to the uniruled case. A surface $S$ with logarithmic Kodaira dimension zero need of course not be uniruled. Using the result of Viehweg-Zuo we can show, however, that any family of canonically polarized varieties over a non-uniruled surface with Kodaira dimension zero is isotrivial.

PROPOSITION 5.8. If $S$ is not uniruled, then $\text{Var}(f^\circ) = 0$.

We prove Proposition 5.8 using three lemmas that describe the sheaves of logarithmic differentials on the minimal model $S_\lambda$.

LEMMA 5.9. If $n \in \mathbb{N}$ is sufficiently large and divisible, then
\[ \mathcal{O}_{S_\lambda}(n(K_{S_\lambda} + D_\lambda)) = \mathcal{O}_{S_\lambda}. \]
In particular, the log-canonical $\mathbb{Q}$-divisor $K_{S_\lambda} + D_\lambda$ is numerically trivial.

PROOF. Equation (12) is an immediate consequence of the assumption $\kappa(S^\circ) = 0$ and the logarithmic abundance theorem in dimension 2, which asserts that the linear system $|n(K_{S_\lambda} + D_\lambda)|$ is basepoint-free.

LEMMA 5.10. If $\kappa(S) \geq 0$, then $S_\lambda$ is $\mathbb{Q}$-Gorenstein, $K_{S_\lambda}$ is numerically trivial and $D_\lambda = \emptyset$.

PROOF. Lemma 5.9 together with the assumption that $|nK_S| \neq \emptyset$ for large $n$ imply that $\phi$ contracts all irreducible components of $D$, and all divisors in any linear system $|nK_S|$, for all $n \in \mathbb{N}$. The claim follows.

LEMMA 5.11. Assume that $\kappa(S) \geq 0$ and $\text{Var}(f^\circ) \geq 1$. If $H \in \text{Pic}(S_\lambda)$ is any ample line bundle, then the (reflexive) sheaf of differentials $(\Omega^1_{S_\lambda})^{\vee\vee}$ has slope $\mu_H((\Omega^1_{S_\lambda})^{\vee\vee}) > 0$, but it is not semistable with respect to $H$.

PROOF. Fix a sufficiently large number $m > 0$ and a general curve $C_\lambda \in |mH|$. Theorem 1.4 ensures that if $(\Omega^1_{S_\lambda})^{\vee\vee}$ is semistable, then so is its restriction $\Omega^1_{S_\lambda}|_{C_\lambda}$.

By general choice, $C_\lambda$ is contained in the smooth locus of $S_\lambda$ and stays off the fundamental points of $\phi^{-1}$. The birational morphism $\phi$ will thus be well-defined and isomorphic along $C := \phi^{-1}(C_\lambda)$. Lemma 5.10 then asserts that
\[ \mu_H((\Omega^1_{S_\lambda})^{\vee\vee}) = \frac{K_{S_\lambda} \cdot C_\lambda}{2} = 0, \]
which shows the first claim.

Similarly, Lemma 5.10 implies that $\text{codim}_{S_\lambda} \phi(D) \geq 2$, and so $C$ is disjoint from $D$. The unstability of $(\Omega^1_{S_\lambda})^{\vee\vee}$ can therefore be checked using the identifications
\[ (\Omega^1_{S_\lambda})^{\vee\vee}|_{C_\lambda} \cong \Omega^1_{S_\lambda}|_{C_\lambda} \cong \Omega^1_S|_C \cong \Omega^1_S(\log D)|_C. \]
Since symmetric powers of semistable vector bundles over curves are again semistable [HL97, cor. 3.2.10], in order to prove Lemma 5.11, it suffices to show that there exists a
number \( n \in \mathbb{N} \) such that \( \text{Sym}^n \Omega^1_S(\log D)|_C \) is not semistable. For that, use the identifications (13) to compute
\[
\deg_C \text{Sym}^n \Omega^1_S(\log D)|_C = const^+ \cdot \deg_C \Omega^1_S|_C \\
= const^+ \cdot \deg_{C_\lambda} (\Omega^1_{S_\lambda})^{\vee \vee}|_{C_\lambda} \quad \text{Isomorphisms (13)} \\
= const^+ \cdot (K_{S_\lambda} \cdot C_\lambda) = 0. \quad \text{Lemma 5.10}
\]
Hence, to prove unstability it suffices to show that \( \text{Sym}^n \Omega^1_S(\log D)|_C \) contains a subsheaf of positive degree.

Theorem 5.7 implies that there exists an integer \( n > 0 \) such that \( \text{Sym}^n \Omega^1_S(\log D) \) contains an invertible subsheaf \( \mathcal{A} \) of Kodaira dimension \( \kappa(\mathcal{A}) \geq 1 \). But by general choice of \( C_\lambda \), this in turn implies that \( \deg_C (\mathcal{A}|_C) > 0 \), which shows the required unstability. This ends the proof of Lemma 5.11.

With these preparations, the proof of Proposition 5.8 is now quite short.

**Proof of Proposition 5.8.** We argue by contradiction and assume to the contrary that both \( S \) is not uniruled, i.e., \( \kappa(S) \geq 0 \), and \( \text{Var}(f^o) \geq 1 \). Again, let \( H \in \text{Pic}(S_\lambda) \) be any ample line bundle.

Lemma 5.11 implies that \( \Omega^1_{S_\lambda}|_{C_\lambda} \) has a subsheaf of positive degree or, equivalently, that it has a quotient of negative degree. On the other hand, Corollary 1.28 then asserts that \( S \) is uniruled, leading to a contradiction. This ends the proof of Proposition 5.8.

**3.4. Images of \( \mathbb{C}^* \) on \( S^0 \), end of proof.** In view of Proposition 5.8, to prove that \( \text{Var}(f^o) \leq 1 \), we can assume without loss of generality that \( \text{Var}(f^o) > 0 \) and therefore \( S \) is uniruled. Since families of canonically polarized varieties over \( \mathbb{C}^* \) are always isotrivial, [Kov00, thm. 0.2], the result then follows from the following proposition.

**Proposition 5.12.** If \( \text{Var}(f^o) > 0 \), then \( S^0 \) is dominated by images of \( \mathbb{C}^* \). In particular \( \text{Var}(f^o) \leq 1 \).

As a first step in the proof of Proposition, recall the following fact. If \( X \) is a projective manifold, and \( H \subset \text{RatCurves}^n(X) \) is an irreducible component of the space of rational curves such that the associated curves dominate \( X \), it is well understood that a general point of \( H \) corresponds to a free curve, i.e., a rational curve whose deformations are not obstructed (cf. [KMM92a, 1.1], [Kol96, II Thm. 3.11], see also Prop. 2.8). In particular, if \( E \subset X \) is an algebraic set of codimension \( \text{codim}_X E \geq 2 \), then the subset of curves that avoid \( E \),
\[
H' := \{ \ell \in H \mid E \cap \ell = \emptyset \},
\]
is Zariski-open, not empty, and curves associated with \( H' \) still dominate \( X \). A similar statement, which we quote as a fact without giving a proof, also holds if \( X \) is a surface with mild singularities, and for quasi-projective varieties that are dominated by \( \mathbb{C}^* \) rather than complete rational curves.

**Proposition 5.13 (Small Set Avoidance, [KK05, prop. 2.7]).** Let \( X \) be a smooth projective surface, and \( E \subset X \) a divisor with simple normal crossings. Assume that \( X \setminus E \) is dominated by images of \( \mathbb{C}^* \), and let \( F \subset X \setminus E \) be any finite set. Then \( X \setminus (E \cup F) \) is also dominated by images of \( \mathbb{C}^* \).

**Lemma 5.14.** In the setup of Proposition 5.12, the quasi-projective surface \( S_\lambda \setminus (\text{Sing}(S_\lambda) \cup D_\lambda) \) is dominated by images of \( \mathbb{C}^* \).

**Proof.** We aim to apply Theorem 1.41(2), and so we need to show that
• the log-canonical divisor $K_{S_\lambda} + D_\lambda$ is numerically trivial, and that
• the boundary divisor $D_\lambda$ is not empty.

The numerical triviality of $K_{S_\lambda} + D_\lambda$ has been shown in Lemma 5.9 above. To show that $D_\lambda \neq \emptyset$, we argue by contradiction, and assume that $D_\lambda = \emptyset$. Set

$$S_\lambda^0 := S_\lambda \setminus \phi(\text{exceptional set of } \phi).$$

Then $S_\lambda^0$ is the complement of a finite set and $\phi^{-1}|_{S_\lambda^0}$ is a well-defined open immersion. Let $f_\lambda := \phi \circ f$. Then $X := X^0|_{f_\lambda^{-1}(S_\lambda^0)} \rightarrow S_\lambda^0$ is a smooth family of canonically polarized varieties. Consider the following diagram:

$$\begin{array}{ccc}
X & \xrightarrow{f_\lambda} & S_\lambda \\
\downarrow & & \downarrow \alpha \\
\hat{X} := X \times_{S_\lambda} \hat{S} & \xleftarrow{\beta} & \hat{S}
\end{array}$$

where $\alpha$ is the index-one-cover described in [KM98, 5.19] or [Rei87, sect. 3.5], and $\beta$ is the minimal desingularization of $\hat{S}_\lambda$ composed with blow-ups of smooth points so that $\beta^{-1}(\hat{S}_\lambda \setminus \alpha^{-1}(S_\lambda^0))$ is a divisor with at most simple normal crossings.

By Lemma 5.9, $K_{\hat{S}_\lambda}$ is torsion. Since $\alpha$ is étale in codimension one this implies that $K_{\hat{S}_\lambda}$ is trivial. Furthermore, $\hat{S}_\lambda$ has only canonical singularities: we have already noted in Proposition 5.6 that the singularities of $S_\lambda$ are log-terminal, i.e., they have minimal discrepancy $> -1$. Then by [KM98, Prop. 5.20] the minimal discrepancy of the singularities of $S_\lambda$ is also $> -1$, and as $K_{\hat{S}_\lambda}$ is Cartier, the discrepancies actually must be integral and hence $\geq 0$, cf. [KM98, proof of Cor. 5.21]. Consequently,

$$K_{\hat{S}_\lambda} = \beta^*(K_{\hat{S}_\lambda}) + (\text{effective and } \beta\text{-exceptional}).$$

This in turn has two further consequences:

i) $\kappa(K_{\hat{S}_\lambda}) = 0$. In particular, $\hat{S}_\lambda$ is not uniruled.

ii) If we set $\hat{S}_\lambda^0 := (\alpha \circ \beta)^{-1}(S_\lambda^0)$ and $\hat{X}_\lambda^0 := \hat{f}^{-1}(\hat{S}_\lambda^0)$ then $\hat{X}_\lambda^0 \rightarrow \hat{S}_\lambda^0$ is again a smooth family of canonically polarized varieties. Letting $\hat{D} := \hat{S}_\lambda \setminus \hat{S}_\lambda^0$ then $\hat{D}$ is exactly the $\beta$-exceptional set, and (14) implies that

$$\kappa(\hat{S}_\lambda^0) = \kappa(K_{\hat{S}_\lambda} + \hat{D}) = 0.$$

In particular, Proposition 5.8 applies to $\hat{f} : \hat{X} \rightarrow \hat{S}$ and shows that $\text{Var}(\hat{f}|_{\hat{X}_\lambda^0}) = 0$.

This is a contradiction and thus ends the proof of Lemma 5.14. $\square$

Observe that Lemma 5.14 does not immediately imply Proposition 5.12. The problem is that the boundary divisor $D \subset S$ can contain connected components that are contracted by $\phi$ to points. These points do not appear in the cycle-theoretic image divisor $D_\lambda$, and it is a priori possible that all morphisms $\mathbb{C}^* \rightarrow S_\lambda \setminus \text{Sing}(S_\lambda)$ contain these points in the image. Taking the strict transforms would then give morphisms $\mathbb{C}^* \rightarrow S \setminus \phi^{-1}(\text{Sing}(S_\lambda) \cup D_\lambda)$, but $\phi^{-1}(\text{Sing}(S_\lambda) \cup D_\lambda) \neq D$. An application of Proposition 5.13 will solve this problem.
5. FAMILIES OF CANONICALLY POLARIZED VARIETIES

PROOF OF PROPOSITION 5.12. If \( \phi(D) \subseteq D_\lambda \cup \operatorname{Sing}(S_\lambda) \), i.e., if all connected components of \( D \) are either mapped to singular points, or to divisors, Lemma 5.14 immediately implies Proposition 5.12. Likewise, if \( S_\lambda \) was smooth, Proposition 5.13 on small set avoidance would imply that almost all curves in the family stay off the isolated zero-dimensional components of \( \phi(D) \), and Proposition 5.12 would again hold. In the general case, where \( S_\lambda \) is singular, and \( d_1, \ldots, d_n \) are smooth points of \( S_\lambda \) that appear as connected components of \( \phi(D) \), a little more argumentation is required.

If \( D' \) is the union of connected components of \( D \) which are contracted to the set of points \( \{d_1, \ldots, d_n\} \subseteq S_\lambda \), it is clear that the birational morphism \( \phi : S \to S_\lambda \) factors via the contraction of \( D' \), i.e., there exists a diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\phi} & S_\lambda \\
\downarrow{\alpha} & & \downarrow{\beta} \\
S' & & S_\lambda
\end{array}
\]

where \( S' \) is smooth, and \( \alpha \) maps the connected components of \( D' \) to points \( d'_1, \ldots, d'_k \in S' \) and is isomorphic outside of \( D' \).

Now, if \( D'' := D \setminus D' \), the above argument shows that \( S' \) is dominated by rational curves that intersect \( \alpha(D'') \) in two points. Since \( S' \) is smooth, Proposition 5.13 applies and shows that almost all of these curves do not intersect any of the \( d'_i \). In summary, we have seen that most of the curves in question intersect \( \alpha(D) \) in two points. This completes the proof of Proposition 5.12. \( \Box \)

3.5. Sketch of further argument. We have seen above that \( S^o \) is dominated by images of \( \mathbb{C}^* \), which implies \( \operatorname{Var}(f^o) \leq 1 \). If \( S^o \) is connected by \( \mathbb{C}^* \), i.e., if there exists an open subset \( \Omega \subseteq S^o \) such that any two points \( x, y \in \Omega \) can be joined by a chain of \( \mathbb{C}^* \), then it is clear that \( \operatorname{Var}(f^o) = 0 \), and Theorem 5.3 is shown in case \( \kappa(S^o) = 0 \).

Since \( \dim S^o = 2 \), we can thus assume without loss of generality that a general point of \( S^o \) is contained in exactly one curve which is an image of \( \mathbb{C}^* \). Blowing up points on the boundary \( D \) if necessary, we can thus assume that \( S \) is a birationally ruled surface, with a map \( \pi : S \to C \) to a curve \( C \), and that the boundary \( D \) is a divisor that intersects the general \( \pi \)-fiber \( F \) in exactly two points, which implies that the restriction of the vector bundle \( \Omega^1_S(\log D) \) to \( F \) is trivial. If \( \operatorname{Var}(f^o) > 0 \), the result of Viehweg-Zuo, Theorem 5.7, then implies that the restriction of \( \Omega^1_S(\log D) \) to non-fiber components of \( D \) cannot be stable of degree zero. However, a detailed analysis of the self-intersection graph of \( D \), and the standard description of the restriction of \( \Omega^1_S(\log D) \) to components of \( D \) shows both stability and zero-degree.
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