ON REGULAR LIE ALGEBRAS

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Abstract. We study so called regular Lie algebras, i.e. Lie algebras in which each nonzero element is regular. We make a connection with an open problem whether any element of reduced trace zero in a simple associative algebra is a commutator.

INTRODUCTION

A Lie algebra is called regular, if for any its nonzero element \( x \), the characteristic polynomial of the adjoint linear map \( \text{ad}_x \) has the same minimal possible non-vanishing power. The study of regular Lie algebras was initiated in [28]. There we proved some elementary properties of those algebras over a field of characteristic zero, and established their connection with another classes of Lie algebras, anisotropic and minimal nonabelian. Here, after setting the necessary definitions and notation, and recalling the necessary facts in the preliminary §1 we continue to study regular Lie algebras, by extending results of [28] to include the case of positive characteristic; this is done in §2. The brief §3 contains a description of the real case. In §4 we discuss a connection with minimal non-abelian Lie algebras and minimal non-commutative associative algebras, and describe minimal non-regular algebras. In §5 we discuss an interesting open problem: whether any element of reduced trace zero in a simple associative algebra can be represented as a commutator of two elements, and suggest an attack on (a particular case of) this problem utilizing regular Lie algebras.

1. NOTATION, DEFINITIONS, AND RECOLLECTIONS

1. Ground field, general notation. Throughout this note, all algebras are assumed to be finite-dimensional, defined over an infinite field \( K \). In most of our results, we stipulate additional conditions on \( K \) (like being perfect, or not of small characteristic), and these conditions are always specified explicitly in the statements of theorems and lemmas. By \( \overline{K} \) we denote the algebraic closure of \( K \). Needless to say, all that will follow is nontrivial only if the ground field \( K \) is not algebraically closed (see the beginning of §3).

If \( S \) is a subalgebra of a Lie algebra \( L \), the normalizer of \( S \) in \( L \), i.e. the subalgebra \( \{ x \in L \mid [S,x] \subseteq S \} \), is denoted by \( N_L(S) \). The centralizer of an element \( x \in L \), i.e. the subalgebra \( \{ y \in L \mid [y,x] = 0 \} \), is denoted by \( C_L(x) \). By \( [L,L] \) and \( Z(L) \) are denoted the commutant and the center of \( L \), respectively. By \( \text{Der}(L) \), or, occasionally, by \( \text{Der}_K(L) \) if we want to stress over which ground field \( K \) we are working at the moment, we denote the Lie algebra of derivations of an algebra \( L \). If \( A \) is an associative algebra, by \( A(-) \) we will denote its “minus” algebra, i.e., the Lie algebra defined on the same vector space \( A \) subject to multiplication \( [a,b] = ab - ba \). The direct sum of algebras is denoted by \( \oplus \), while the direct sum of vector spaces (usually also subalgebras, but with not necessary trivial multiplication between them), is denoted by \( \bigoplus \).

Recall that a symmetric bilinear form \( \langle \cdot, \cdot \rangle : L \times L \to K \) on a Lie algebra \( L \) is called invariant, if \( \langle [x,y],z \rangle = \langle x,[y,z] \rangle \) for any \( x,y,z \in L \). The orthogonal complement of a subspace \( S \) of \( L \) with respect to \( \langle \cdot, \cdot \rangle \) is denoted by \( S^\perp \).

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2. **Fitting decomposition.** Following [5, Chapter VII, §1], we will call the set $X$ of elements of a Lie algebra $L$ almost commuting, if for any $x, y \in X$, there is an integer $n$ such that $(ad x)^n y = 0$. This is, essentially, equivalent to the existence of the Fitting decomposition of $L$ with respect to the adjoint action of $X$: $$L = L^0(X) + L^1(X),$$ where the Fitting 0-component is defined as $$L^0(X) = \{ y \in L \mid \text{for any } x \in X, \text{ there is integer } n \text{ such that } (ad x)^n(y) = 0 \},$$ and the Fitting 1-component is defined as $$L^1(X) = \sum_{x \in X} \left( \bigcap_{n \geq 0} (ad x)^n(L) \right).$$

3. **Regular Lie algebras.** Let $L$ be a Lie algebra. For any $x \in L$, consider the characteristic polynomial of the adjoint map $ad x$: $$\chi_{ad x}(t) = \det(t - ad x) = \sum_{i=0}^{dim L} a_i(x) t^i.$$ Recall ([5, Chapter VII, §2.2] or [13, Chapter III, §1]) that the rank of $L$, denoted by $rk L$, is the minimal number $r$ such that $a_r(x) \neq 0$ for some $x \in L$ (or, what is the same, the multiplicity of the eigenvalue zero of $ad x$). An element $x$ is called regular, if this minimum is attained for it, i.e., if $a_{rk L}(x) \neq 0$. An element $x \in L$ is regular if and only if the Fitting 0-component $L^0(x)$ is of the minimal possible dimension $rk L$.

A Lie algebra is called regular, if each its nonzero element is regular. (Note that this notion is entirely different from the notion of regular subalgebra in the classical works of Dynkin about subalgebras of simple Lie algebras over algebraically closed fields of characteristic zero).

4. **Anisotropic algebras.** Recall that a Lie algebra $L$ is called anisotropic, if for any $x \in L$, $ad x$ is a semisimple linear map. Anisotropic Lie algebras were studied in [9] and [17]–[19] (see also [28, §1]). We will use repeatedly the following powerful result due to Premet.

**Theorem A.** (*The ground field $K$ is perfect of characteristic $\neq 2, 3$.* If $L$ is an anisotropic centerless Lie algebra, then $L$ is sandwiched between the direct sum of forms of classical simple Lie algebras, and their derivation algebras. That is, $$S_1 \oplus \cdots \oplus S_n \subseteq L \subseteq \text{Der}(S_1) \oplus \cdots \oplus \text{Der}(S_n)$$ for some forms of classical simple Lie algebras $S_1, \ldots, S_n$.)

Here and below, by abuse of notation, we identify the algebra $S_i$ with the algebra of its inner derivations $ad(S_i)$.

In characteristic zero the statement is trivial, in characteristic $p > 5$ it as established in [17] Corollary 2, and in characteristic $p = 5$, in [18].

We also need to consider an auxiliary, a priori slightly more general condition, namely, that in a Lie algebra any nilpotent element is central. Such Lie algebras will be dubbed nilpotent-free. It is obvious that any anisotropic Lie algebra is nilpotent-free, and below we will see that over perfect fields these two notions coincide (Lemma 9).

5. **Minimal non-$\mathcal{P}$ algebras.** Let $\mathcal{P}$ be a property of Lie algebras. A Lie algebra is called minimal non-$\mathcal{P}$, if it does not satisfy $\mathcal{P}$, but any its proper subalgebra satisfies $\mathcal{P}$. We will encounter minimal non-abelian, minimal non-nilpotent, and minimal non-regular Lie algebras. The first two classes were studied extensively in the literature, see [9], [10], [11], [16], [19], [24], [27], and [28], among others; minimal non-regular Lie algebras will be discussed in §4.

The same notion can be considered for other classes of algebras. Here we will encounter minimal non-commutative associative algebras; such algebras were studied in [10].
6. **Field extensions, centroid, and central algebras.** If \(L\) is a Lie algebra over \(K\), \(K \subset F\) is a field extension, and \(L\) has also an \(F\)-vector space structure, compatible with \(K\), we can consider the Lie algebra \(L_F\) over \(F\), called *restriction over \(F\)* (not to be confused with the Lie algebra \(L \otimes_K F\) obtained by *extension* of the ground field from \(K\) to \(F\)). It is easy to see that the \(K\)-algebra \(L\) is regular (respectively, anisotropic) if and only if the \(F\)-algebra \(L_F\) is regular (respectively, anisotropic).

By \(\Omega(L)\) we denote the centroid of an algebra \(L\). An algebra \(L\) is called *central*, if \(\Omega(L) = K\). For a simple algebra \(L\), this is equivalent to the condition that the \(F\)-algebra \(L \otimes_K F\) remains to be simple under any extension \(F\) of the ground field \(K\).

If \(L\) is simple, then its centroid \(\Omega(L)\) is a field, an extension of the ground field \(K\), and we can consider the algebra \(L_{\Omega(L)}\) defined over \(\Omega(L)\). The latter algebra is a central \(\Omega(L)\)-algebra.

If \(L\) is simple and the ground field \(K\) is perfect, then being extended to a sufficiently large field \(F\) (for example, \(\Omega(L)\), or the algebraic closure \(\overline{K}\)), \(L\) is decomposed into the direct sum of a number of isomorphic copies of a central simple algebra, i.e. there is an isomorphism of \(F\)-algebras

\[
L \otimes_K F \simeq \overline{S}_1 \oplus \cdots \oplus \overline{S}_n,
\]

where \(\overline{S}_j \simeq \overline{S}\), are central simple Lie algebras over \(F\) (see, for example, [8 §2]). Of course, if \(L\) is central over \(K\), then \(n = 1\).

7. **Almost simple and classical algebras.** Recall that a Lie algebra \(L\) is called *almost simple*, if there is a simple subalgebra \(S\) of \(L\) such that \(L\) is sandwiched between \(S\) and its derivation algebra \(\text{Der}(S)\):

\[
S \subseteq L \subseteq \text{Der}(S).
\]

The simple algebra \(S\) is called the *simple constituent* of the almost simple Lie algebra \(L\). An almost simple Lie algebra is called *classical*, if its simple constituent is classical.

If \(S\) is central simple and classical, then we have \(\text{Der}(S) \simeq S\) in all the cases except when characteristic of the ground field \(p > 0\), and \(S\) is of type \(A_{kp-1}\) for some \(k\). To see what is happening in these exceptional cases, assume that \(S\) is split (which always can be achieved by passing to the algebraic closure of the ground field).

Recall that the algebra of \(n \times n\) matrices has a peculiarity if \(n\) divides \(p\); say, \(n = kp\). In this case, the \(kp \times kp\) identity matrix \(E\) has trace zero, so the Lie algebra \(\text{sl}_{kp}(K)\) of traceless matrices is no longer simple, but has the one-dimensional center linearly spanned by \(E\). The quotient algebra

\[
\text{psl}_{kp}(K) = \text{sl}_{kp}(K)/KE
\]
is simple. Similarly, we can consider the quotient algebra

\[
\text{pgl}_{kp}(K) = \text{gl}_{kp}(K)/KE
\]
which is semisimple and contains \(\text{psl}_{kp}(K)\) as an ideal of codimension 1. Moreover,

\[
[\text{pgl}_{kp}(K), \text{pgl}_{kp}(K)] = \text{psl}_{kp}(K),
\]

and

\[
\text{Der}(\text{psl}_{kp}(K)) \simeq \text{pgl}_{kp}(K)
\]
(see [21 Chapter V, §5] or [17 p. 152]), so \(\text{pgl}_{kp}(K)\) is almost simple.

8. **Forms of algebras over fields.** We will need a description of forms of simple classical Lie algebras of type \(A\), i.e., algebras of the kind \(\text{sl}_n(K)\) and \(\text{psl}_{kp}(K)\) (see, for example, [21 Chapter IV, §3]). Any central simple form of \(\text{sl}_n(K)\) is either of the kind \([A^{-1}, A^{-1}]\), where \(A\) is a central simple associative algebra, or of the kind \([S^{-}\langle A, J \rangle, S^{-}\langle A, J \rangle]\), where \(A\) is a simple associative algebra whose center is a quadratic extension of \(K\), \(J\) is involution on \(A\) of the second kind, and \(S^{-}\langle A, J \rangle = \{ a \in A | J(a) = -a \}\) is the Lie algebra of \(J\)-skew-symmetric elements of \(A\). In both cases \(A\) is of degree \(n\) over its center. If we are interested in forms of \(\text{psl}_{kp}(K)\), we should everywhere take the quotient by the central ideal \(K\mathbf{1}\), where \(\mathbf{1}\) is the unit in the respective associative algebra \(A\), getting in this way Lie algebras of the kind \([A^{-}, A^{-}]/K1\) and \([S^{-}\langle A, J \rangle, S^{-}\langle A, J \rangle]/K1\), respectively.
2. STRUCTURE AND PROPERTIES OF REGULAR LIE ALGEBRAS

The next three lemmas are elementary observations from [28, §2] readily following from the properties of regular elements, as expounded, for example, in [5, Chapter VII, §2.2], and are valid in any characteristic.

Lemma 1. Any nilpotent Lie algebra is regular.

Lemma 2. A subalgebra of a regular Lie algebra is regular.

Lemma 3. A non-semisimple regular Lie algebra is nilpotent.

Lemma 4. If I is a nontrivial ideal of a regular Lie algebra L, then L/I is nilpotent.

Proof. First note that L/I is a regular Lie algebra (as follows, for example, from [5, Chapter VII, §2.2, Proposition 8]).

Since for any element x ∈ L, the ideal I is an adx-invariant subspace of L, we have

\[\chi_{adx}(t) = \chi_{ad_{L/I}(x)}(t),\]

where ad_{I}x is the restriction of ad x to I, x is the image of x under the canonical homomorphism L → L/I, and ad_{L/I}(x) is its adjoint map acting on the algebra L/I.

Picking x ∈ I, we get from (3)

\[rkL = rkI + rkL/I,\]

On the other hand, if x ∈ I, x ≠ 0, then x̄ = 0, \(x̄(t) = t^{dimL/I},\) and (3) implies

\[rkL = rkI + dimL/I.\]

Consequently, rk L/I = dim L/I, and hence L/I is nilpotent.

□

Lemma 5. A semisimple regular Lie algebra is almost simple.

Proof. Let L be a semisimple regular Lie algebra. In characteristic zero, L is the direct sum of simple algebras, and by Lemma 4, L is simple.

In characteristic p > 0, by the Block theorem about the structure of semisimple Lie algebras,

\[\bigoplus_{i∈I}(S_i ⊗ O_n_i) ⊆ L ⊆ \bigoplus_{i∈I}(\text{Der}(S_i) ⊗ O_{n_i} + \text{Der}(O_{n_i}))\]

for some finite set of simple Lie algebras \(\{S_i\}_{i∈I}\) and some nonnegative integers \(\{n_i\}_{i∈I}\. Here

\[O_n = K[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)\]
denotes the reduced polynomial algebra in n variables. By Lemma 4, the direct sum \(\bigoplus_{i∈I}(S_i ⊗ O_{n_i})\) is regular, and by Lemma 4, it consists of a just one summand, i.e., \(|I| = 1\) and

\[S ⊗ O_n ⊆ L ⊆ \text{Der}(S) ⊗ O_n + \text{Der}(O_n)\]

for suitable S and n. Again, S ⊗ O_n is regular, and by Lemma 3, we have n = 0, i.e., \(O_n = K\) and (2) holds.

Now we will establish a connection of regular Lie algebras with another class of Lie algebras, namely, anisotropic Lie algebras.

Lemma 6. A subalgebra of an anisotropic Lie algebra is anisotropic.

Proof. Let S be a subalgebra in anisotropic Lie algebra L. For any x ∈ S, the restriction of the semisimple linear map ad x : L → L, to the invariant subspace S is also semisimple, thus x is semisimple element in S.

□

Lemma 7. A solvable nilpotent-free Lie algebra is abelian.

Proof. See the proof of [9] Proposition 1.2] (where a formally weaker assertion is stated – that a solvable anisotropic Lie algebra is abelian).

□
Lemma 8. An ideal of a nilpotent-free Lie algebra is nilpotent-free.

Proof. Let I be an ideal of a nilpotent-free Lie algebra L. If x ∈ I such that (ad x)^n(I) = 0 for some n, then

$$(\text{ad} x)^{n+1}(L) = (\text{ad} x)^n[L, x] \subseteq (\text{ad} x)^n(I) = 0,$$

and hence ad x = 0 . □

Lemma 9. (The ground field K is perfect). A Lie algebra is anisotropic if and only if it is nilpotent-free.

Proof. The “only if” part is obvious, so let us prove the “if” part. Let L be a nilpotent-free Lie algebra. In characteristic zero, Lemma 8 implies that the radical Rad(L) of L is nilpotent-free, and Lemma 7 implies that Rad(L) is abelian. Then for any x ∈ Rad(L), (ad x)^2 = 0, hence x is central and L is either reductive or abelian. But a reductive Lie algebra contains semisimple and nilpotent parts in the Jordan–Chevalley decomposition of each of its element (see, for example, [5] Chapter VII, §5.1, Proposition 2)). Thus any element in L is a sum of a semisimple and a central element, and hence is semisimple.

In the case of positive characteristic, consider the universal p-envelope $R(L)$ of L ([7 §1] or [25 Chapter 2, §5]). Let x ∈ R(L) be a nilpotent element. We may assume then that $x^{[p]^m} = 0$ for some n. As there is y ∈ L such that $x = y^{[p]^k}$ for some k, we have $y^{[p]} = 0$, thus y is a nilpotent element of L, hence y belongs to the center of L, and $x = y^{[p]^k}$ belongs to the center of R(L). Consequently, R(L) is nilpotent-free. Since R(L) is a Lie p-algebra, any its element decomposes into the sum of a semisimple and a p-nilpotent (and hence nilpotent) element (the so called Jordan–Chevalley–Seligman decomposition, see [21 Theorem V.7.2] or [25 Chapter 2, Theorem 3.5]). Then, as in the case of zero characteristic, R(L) is anisotropic. By Lemma 6 L is anisotropic. □

Lemma 10. (The ground field K is perfect). A non-nilpotent regular Lie algebra is anisotropic.

Proof. A non-nilpotent regular Lie algebra is obviously nilpotent-free (in fact, any nilpotent element vanishes), and by Lemma 9 is anisotropic. □

Theorem 1. (The ground field K is perfect of characteristic p ≠ 2, 3). A non-nilpotent regular Lie algebra L is a form of an almost simple classical Lie algebra. Moreover, if the simple constituent of L is central simple, then L is a form of either a simple classical Lie algebra, or of an algebra of the kind $\text{pgl}_k(K)$.

Proof. By Lemma 2 and Lemma 5 L is almost simple. In characteristic zero, almost simplicity is equivalent to simplicity, and any simple Lie algebra is a form of a classical one.

In the case of characteristic p > 3, by Lemma 10 L is anisotropic. Apply Theorem A. Since L is almost simple, in the inclusion (1) we have $n = 1$. Putting $S = S_1$ and passing to the algebraic closure of the ground field, we have

$$S \otimes_K \overline{K} \subseteq L \otimes_K \overline{K} \subseteq \text{Der}(S \otimes_K \overline{K}).$$

If S is central, then $S \otimes_K \overline{K}$ is simple. The only simple classical Lie algebras which have outer derivations, are Lie algebras $\text{psl}_{k^p}(\overline{K})$ for some nonnegative integer k. Hence either $L \otimes_K \overline{K} = S \otimes_K \overline{K}$, or $S \otimes_K \overline{K} \simeq \text{psl}_{k^p}(\overline{K})$. In the latter case, since $\text{psl}_{k^p}(\overline{K})$ is of codimension one in $\text{Der}(\text{psl}_{k^p}(\overline{K})) \simeq \text{pgl}_{k^p}(\overline{K})$, we have that $L \otimes_K \overline{K}$ is isomorphic to either $\text{psl}_{k^p}(\overline{K})$, or to $\text{pgl}_{k^p}(\overline{K}$). □

The following was proved as [28] Theorem 3) in the zero characteristic case. Comparing the proof below with the proof of [28] Theorem 3], the reader can appreciate how much more efforts are required to handle the case of positive characteristic.

Theorem 2. (The ground field K is perfect of characteristic ≠ 2, 3). Let L be a non-solvable Lie algebra with the trivial center. Then the following are equivalent:

(i) L is regular;
(ii) any proper subalgebra of L is regular;
(iii) any proper subalgebra of L is either almost simple, or abelian.
Proof. (i) \( \Rightarrow \) (ii) follows from Lemma 2.

(ii) \( \Rightarrow \) (iii): By Lemma 3, Lemma 5 and Lemma 10 any proper subalgebra of \( L \) is either almost simple anisotropic, or nilpotent. Our goal is to prove that any nilpotent subalgebra is abelian. We consider multiple possible cases.

Case 1. \( L \) is not semisimple. Let \( I \) be a nontrivial abelian ideal of \( L \).

Case 1.1. \( L \) contains a proper almost simple subalgebra \( S \). Since any element of \( I \) is nilpotent, we have \( S \cap I = 0 \). Further, \( S + I \) is a subalgebra of \( L \), which is not regular, and hence coincides with the whole \( L \). We have \( [L, L] = [S, S] + [S, I] \).

Case 1.1.1. \( [L, L] = L \). Then \( [S, S] = S \), i.e., \( S \) is perfect, and \( [S, I] = I \). Consider the action of \( S \) on \( I \) as a representation \( \rho \) of \( S \). For any \( x \in S \), the subalgebra \( Kx + I \) is a proper subalgebra of \( L \) which is not almost simple anisotropic, hence it is nilpotent, and \( x \) acts on \( I \) nilpotently. By the Engel theorem, \( S/\text{Ker}\rho \) is nilpotent. On the other hand, \( S/\text{Ker}\rho \) is a homomorphic image of a perfect Lie algebra, hence is also perfect. Consequently, \( S/\text{Ker}\rho = 0 \), i.e., \( [S, I] = 0 \), and \( I \) lies in the center of \( L \) and hence vanishes, a contradiction.

Case 1.1.2. \( [L, L] \) is a proper subalgebra of \( L \). It is not nilpotent, hence anisotropic, and thus \( [S, I] = 0 \), i.e., \( I \) lies in the center of \( L \), a contradiction.

Case 1.2. All proper subalgebras of \( L \) are nilpotent. Then, according to [11] Theorem 8.8, part 3], all proper subalgebras of \( L \) are abelian.

Case 2. \( L \) is semisimple. Repeating the reasoning in the proof of Lemma 5 based on the Block theorem, we get that \( L \) is almost simple: \( S \subseteq L \subseteq \text{Der}(S) \) for a simple Lie algebra \( S \).

Case 2.1. \( S \) is a proper subalgebra of \( L \). Then \( S \) is regular, and by Theorem [1] is an anisotropic form of a classical Lie algebra. Take an arbitrary nonzero element \( d \in \text{Der}(S) \) such that \( d \in L \) and \( d \notin S \). Now we use a reasoning from the proof of [9] Theorem 4.1, suitably modified for our purpose.

Let us prove by induction that \( \text{Ker}(d^n) \) is an abelian subalgebra of \( S \) for any \( n \). Indeed, if \( n = 1 \), we have that \( \text{Ker}(d) \) is a proper subalgebra of \( S \), and hence is either abelian, or almost simple. On the other hand, \( \text{Ker}(d) + Kd \) is a subalgebra of \( L \). Since \( d \) is a central element in this subalgebra, it is a proper subalgebra, and hence is regular. Since it contains a nonzero central element, it is nilpotent, and hence \( \text{Ker}(d) \) cannot be almost simple. This shows that \( \text{Ker}(d) \) is abelian.

Now suppose that \( \text{Ker}(d^n) \) is an abelian subalgebra of \( S \). For any \( x, y \in S \), we have

\[
\text{d}^{n+1}(x, y) = \sum_{i=0}^{n+1} \text{d}^{n+1-i}(x) \cdot \text{d}^i(y).
\]

If \( x, y \in \text{Ker}(d^{n+1}) \), then all terms at the right-hand side of this equality vanish, what shows that \( \text{Ker}(d^{n+1}) \) is a subalgebra of \( S \). On the other hand, similarly with the case \( n = 1 \), \( \text{Ker}(d^{n+1}) + Kd \) is a subalgebra of \( L \).

Case 2.1.1. \( \text{Ker}(d^{n+1}) + Kd \) is a proper subalgebra of \( L \). Then it is regular, and, since \( d \) is a nilpotent element in it, it is nilpotent. Thus \( \text{Ker}(d^{n+1}) \) is a nilpotent subalgebra of \( S \), and by Lemmas 6 and 7, \( \text{Ker}(d^{n+1}) \) is abelian.

Case 2.1.2. \( L = \text{Ker}(d^{n+1}) + Kd \). Then \( S = \text{Ker}(d^{n+1}) \), and both \( \text{Ker}(d) \) and \( \text{Im}(d) \) lie in \( \text{Ker}(d^n) \). Consequently,

\[
\dim S = \dim \text{Ker}(d) + \dim \text{Im}(d) \leq 2 \dim \text{Ker}(d^n).
\]

Let \( H \) be a maximal abelian subalgebra of \( S \) containing \( \text{Ker}(d^n) \). Since \( H \) is an ideal in its normalizer \( N_S(H) \), the latter is a proper abelian subalgebra of \( S \), and hence \( N_S(H) = H \), i.e. \( H \) is a Cartan subalgebra. Since \( S \) is a form of a classical simple Lie algebra, all its Cartan subalgebras are of the same dimension, equal to \( \text{rk} S \). Now from [6] we have \( \dim S \leq 2 \text{rk} S \). Passing to the algebraic closure of the ground field, and owing to the fact that the rank is preserved under field extensions, we have the same inequality \( \dim S \leq 2 \text{rk} S \) for a semisimple classical Lie algebra \( \overline{S} = S \otimes_K \overline{K} \) over an algebraically closed field \( \overline{K} \), which is, obviously, impossible.
Since \( \text{Ker}(d^n) \) is abelian for any \( n, d \) is not nilpotent. Since any nonzero element of \( S \) is not nilpotent too, we get that \( L \) is nilpotent-free, and by Lemma 9 \( L \) is anisotropic, and then by Lemma 7 any of its nilpotent subalgebras are abelian.

**Case 2.2.** \( L = S \), i.e., \( L \) is simple. Any proper nilpotent subalgebra of \( L \) is contained in a maximal nilpotent subalgebra \( H \). The latter is either maximal among all (not just nilpotent) subalgebras, or contained in an anisotropic almost simple subalgebra, and by Lemma 7 is abelian. If \( H \) is maximal, then its normalizer \( N_L(H) \) is a subalgebra of \( L \) containing \( H \), then either \( N_L(H) = L \) in which case \( H \) is an ideal of \( L \), a contradiction, or \( N_L(H) = H \), in which case \( H \) is a Cartan subalgebra of \( L \). To summarize: any nilpotent subalgebra of \( L \) is either abelian, or contained in a Cartan subalgebra of \( L \).

By [20, Corollary], any simple Lie algebra is either a form of a classical Lie algebra, or possesses absolute zero divisors, i.e., nonzero elements \( c \in L \) such that \( (ac)^2 = 0 \).

**Case 2.2.1.** \( L \) is a form of a classical simple Lie algebra. Then any Cartan subalgebra of \( L \), including \( H \), is abelian.

**Case 2.2.2.** \( L \) possesses an absolute zero divisor \( c \in L \). The centralizer \( C_L(c) \) is a subalgebra of \( L \) with a central element \( c \), hence it cannot be almost simple, and is nilpotent.

**Case 2.2.2.1.** \( C_L(c) \) is abelian. Let us prove by induction that

\[
[c, L, \ldots, L] \subseteq C_L(c)
\]

for any \( n \). For \( n = 1 \) this is true: \([c, L] \subseteq C_L(c)\). Suppose the inclusion (7) holds for some \( n > 1 \). By the Jacobi identity,

\[
[[[\ldots[c, L],\ldots, L], c], L] \subseteq [[[\ldots[c, L],\ldots, L], c], L] + [[[\ldots[c, L],\ldots, L], c], L].
\]

By the induction assumption, the first summand at the right-hand side of this inclusion vanishes, and the second summand is the product of two elements from \( C_L(c) \), and thus vanishes too. Thus the left-hand side vanishes, which is equivalent to

\[
[c, L, \ldots, L] \subseteq C_L(c).
\]

The sum of terms at the left-hand side of the inclusion (7) for all \( n \) is the ideal of \( L \) generated by \( c \), and thus coincides with \( L \). Consequently, \( L \subseteq C_L(c) \), a contradiction.

**Case 2.2.2.2.** \( C_L(c) \) is contained in a Cartan subalgebra \( H \). Consider the Fitting decomposition of \( L \) with respect to \( H: L = H + L_1 \). Since \( c \in H \), we have

\[
[H, c] + [L_1, c] = [L, c] \subseteq C_L(c) \subseteq H,
\]

what implies \([L_1, c] = 0 \) and \([L, c] = [H, c] \). The latter equality means that for any \( x \in L \) there is \( h \in H \) such that \([x, c] = [h, c] \), therefore \( x - h \in C_L(c) \), and \( x \in H + C_L(c) = H \), a contradiction.

(iii) \( \Rightarrow \) (i):

**Case 1.** \( L \) is not semisimple. Let \( I \) be a nontrivial abelian ideal of \( L \). Take \( x \in L \), \( x \notin I \), and consider a subalgebra \( I + Kx \). Obviously, this is a proper subalgebra of \( L \) which is not almost simple, and hence is abelian. This shows that \( I \) contained in the center of \( L \), and thus \( I = 0 \), a contradiction.

**Case 2.** \( L \) is semisimple. The reasonings based on the Block theorem, the same as in the proof of Lemma 5 show that \( L \) is almost simple. Fix a nonzero \( x \in L \), and let us prove by induction that \( \text{Ker}((ad x)^n) \) is an abelian subalgebra of \( L \) for any \( n \). The proof is very similar to the proof of the analogous statement in Case 2.1 of the implication (ii) \( \Rightarrow \) (iii) above.

For \( n = 1 \), \( \text{Ker}(ad x) \) is a subalgebra of \( L \) with a central element \( x \), hence it cannot be almost simple, and thus is abelian. Now suppose \( \text{Ker}((ad x)^n) \) is an abelian subalgebra of \( L \) for some \( n \). Using the generalized Leibniz formula (5) for the case \( d = ad x \), we get that \( \text{Ker}((ad x)^{n+1}) \) is a subalgebra of \( L \). Again, this subalgebra has a central element \( x \), hence cannot be almost simple, and thus is abelian.
Therefore, $\text{ad} x$ cannot be nilpotent, thus $L$ is nilpotent-free, and by Lemma 9 is anisotropic. By Theorem A, the simple constituent $S$ of $L$ is a form of a classical simple Lie algebra (and, in particular, all Cartan subalgebras of $S$ are of the same dimension $\text{rk} S$). Now the same reasoning as in the proof of the implication (iii) $\Rightarrow$ (i) in [28, Theorem 3] shows that $S$ is regular. If $L = S$ we are done, so assume $S \subsetneq L \subseteq \text{Der}(S)$. We can write

$$L = S + D,$$

where $D$ is a subspace of $\text{Der}(S)$, consisting of outer derivations of $S$. According to §1.7 we have $[\text{Der}(S), \text{Der}(S)] \subseteq S$, thus $D$ is an abelian subalgebra of $L$.

For any $x \in S$, we have

$$C_L(x) = \{ y + d \mid y \in S, d \in D, [y, x] = d(x) \}.$$ 

For each $d \in D$, the set $\{ y \in L \mid [y, x] = d(x) \}$ is an affine space over the vector subspace $C_S(x)$, and hence

$$\dim C_L(x) = \dim C_S(x) + \dim D = \text{rk} L + \dim D.$$ 

Any element $d \in L, d \notin S$ may be “fixed” by an inner derivation of $S$ to represent an outer derivation, so we may assume $d \in D$ for an appropriate decomposition (38). Then, since $[D, D] = 0$, we have $C_L(d) = \text{Ker}(d) + D$. Further, $\text{Ker}(d)$ is a subalgebra of $S$, and $\text{Ker}(d) + Kd$ is a subalgebra of $L$ with a central element $d$, thus $\text{Ker}(d) + Kd$ cannot be almost simple, and hence is abelian. Now, $N_L(\text{Ker}(d))$ is a subalgebra of $S$ having the abelian ideal $\text{Ker}(d)$. Note that according to §1.7, $\text{Ker}(d) \neq 0$, and hence $N_L(\text{Ker}(d))$ is abelian.

We also have:

$$[d(N_L(\text{Ker}(d))), \text{Ker}(d)] \subseteq d([N_L(\text{Ker}(d)), \text{Ker}(d)]) + [N_L(\text{Ker}(d)), d(\text{Ker}(d))]$$

$$\subseteq d(\text{Ker}(d)) = 0 \subset \text{Ker}(d),$$

what shows that $N_L(\text{Ker}(d))$ is invariant under $d$. Thus $N_L(\text{Ker}(d)) + Kd$ is a subalgebra of $L$ with an abelian subalgebra $N_L(\text{Ker}(d))$ of codimension 1. Clearly, $N_L(\text{Ker}(d)) + Kd$ cannot be almost simple, and hence is abelian. This means that $N_L(\text{Ker}(d)) \subseteq \text{Ker}(d)$, and hence $N_L(\text{Ker}(d)) = \text{Ker}(d)$. Thus $\text{Ker}(d)$ is a Cartan subalgebra of $S$, and hence $\dim \text{Ker}(d) = \text{rk} L$, and $\dim C_L(d) = \text{rk} L + \dim D$.

We see that centralizers of all nonzero elements in $L$ have the same dimension $\text{rk} L + \dim D$. Since $L$ is anisotropic, for any $x \in L$ the space $L^0(x)$ coincides with the centralizer $C_L(x)$, and therefore any nonzero element of $L$ is regular. □

3. **Real regular Lie algebras**

If the ground field is algebraically closed, the only regular Lie algebras are nilpotent ones. This follows from Lemma 3 Theorem 2 and an obvious fact that over an algebraically closed field any non-nilpotent Lie algebra contains the two-dimensional nonabelian subalgebra (consider an eigenvector of a non-nilpotent adjoint map).

A description of regular Lie algebras over $\mathbb{R}$, the field of real numbers, is almost as easy and readily follows from the developed structure theory. Recall that $su_2(\mathbb{R})$ denotes the simple 3-dimensional compact Lie algebra, a real form of $sl_2(\mathbb{C})$.

**Theorem 3.** (The ground field is $\mathbb{R}$). A Lie algebra is regular if and only if it is either nilpotent, or isomorphic to $su_2(\mathbb{R})$.

**Proof.** The “if” part follows from Lemma 1 and the fact that $su_2(\mathbb{R})$ is regular. The latter can be either verified directly, or follows from the fact that $su_2(\mathbb{R})$ is isomorphic to $Q(-1)/\mathbb{R}1$, where $Q$ is the quaternionic real division algebra (of degree 2), and Corollary 2 below.

The “only if” part: By Lemma 3 Lemma 5 and Lemma 10, a regular Lie algebra over a field of characteristic zero is either nilpotent, or simple anisotropic. Now the structure theory of real simple Lie algebras readily implies that a real simple anisotropic Lie algebra is compact (see, for example, [23, Theorem 2] or [26, Theorem 2]). Further, the structure of the compact form of a complex simple Lie
algebra (as expounded, for example, in [5, Chapter IX, §3, Proof of Proposition 2] or [13, Chapter IV, §7, Proof of Theorem 10]) implies that algebras of rank $> 1$ are not regular. Indeed, if
\[ g = h + \bigoplus_{\alpha \in R} Ce_\alpha \]
is a Cartan decomposition of a complex simple Lie algebra $g$ with Cartan subalgebra $h$ and the root system $R$, then the compact form of $g$ can be constructed as
\[ g_c = h_c + \bigoplus_{\alpha \in R_+} (\mathbb{R}u_\alpha + \mathbb{R}v_\alpha), \]
where
\[ h_c = i\{ h \in h \mid \alpha(h) \in \mathbb{R} \text{ for any } \alpha \in R \} \]
is the Cartan subalgebra of $g_c$, $u_\alpha = e_\alpha + e_{-\alpha}$, $v_\alpha = i(e_\alpha - e_{-\alpha})$, and $R_+$ is the set of positive roots.

If $\text{rk } g_c = \text{rk } g > 1$, the algebra $g_c$ has a lot of non-regular subalgebras: for example,
\[ h_c + \mathbb{R}u_\alpha + \mathbb{R}v_\alpha \]
for any fixed $\alpha \in R_+$, which is isomorphic to a split central extension of $su_2(\mathbb{R})$, or $\bigoplus_{\alpha \in R_+} \mathbb{R}u_\alpha$. Thus we are left with the compact Lie algebra of rank one, which is isomorphic to $su_2(\mathbb{R})$. \hfill \Box

4. **MINIMAL NON-ABELIAN, MINIMAL NON-REGULAR, AND MINIMAL NON-COMMUTATIVE ALGEBRAS**

In this section we discuss some ramifications and consequences of Theorem 2 and connection with the class of minimal nonabelian Lie algebras.

An alternative, more concrete, way to handle the almost simple case in the proof of implication (iii) $\Rightarrow$ (i) of Theorem 2 would be the following. Recall that the task there is boiled down to this: if $S$ is a regular simple Lie algebra, prove that its derivation algebra $\text{Der}(S)$ is also regular. By [13, Chapter X, §1, Theorem 5], for any $d \in \text{Der}(S)$ and $\omega \in \Omega(S)$, there is a derivation $\delta \in \text{Der}_K(\Omega(S))$ such that $[d, \omega] = \delta(\omega)$. But since the ground field $K$ is perfect, the finite field extension $K \subset \Omega(S)$ is separable, and $\text{Der}_K(\Omega(S)) = 0$. Therefore, $\text{Der}(S)$ carries a structure of a vector space over $\Omega(S)$, and $\text{Der}_\Omega(S) = \text{Der}(S)/\Omega(S)$. Since the fact whether the algebra is regular or not does not change under restriction over a larger field, to prove regularity of $\text{Der}(S)$, we can pass to the restriction over $\Omega(S)$, and assume that $S$ is central simple.

According to §17 we have $\text{Der}(S) \cong S$ only if characteristic of the ground field is $p > 0$, and $S$ is a form of the Lie algebra $\mathfrak{psl}_{kp}(K)$ for some $k$. According to §18 $S$ is isomorphic either to an algebra $[A^{-}, A^{-}] / K1$, where $A$ is a central simple associative algebra, or to an algebra $[S^{-}(A, J), S^{-}(A, J)] / K1$, where $A$ is a simple associative algebra whose center is a quadratic extension of $K$, and $J$ is involution on $A$ of the second kind; in both cases the degree of $A$ over its center is equal to $kp$.

In these two cases, we have $\text{Der}(S) \cong A^{-} / K1$, and $\text{Der}(S) \cong S^{-}(A, J) / K1$, respectively. In the first case, it is straightforward to see that if $[A^{-}, A^{-}] / K1$ is anisotropic, then $A$ is a division algebra. Then by a reasoning very similar to the proof of [22, Theorem 6] (reproduced as Theorem 5 below; the only difference is that we should deal with the algebra $[A^{-}, A^{-}] / K1$ instead of $A^{-} / K1$), $A$ is minimal noncommutative. Then by the same theorem $A^{-} / K1$ is regular, and we are done. The second case, involving $S^{-}(A, J)$ with involution of the second kind, is a bit more complicated, but could be treated similarly.

**Corollary 1** (to Theorem 2). *(The ground field $K$ is perfect of characteristic $\neq 2, 3$). A simple minimal non-abelian Lie algebra is regular.*

Over some fields $K$ the converse is true: trivially, over algebraically closed fields, where both classes of simple algebras are empty; or over $\mathbb{R}$, where both class of algebras consists of the single 3-dimensional nonsplit simple algebra, see Theorem 3. Generally, the converse is not true: take, for example, a division
algebra $D$ such that the Lie algebra $[D^{(-)}, D^{(-)}]^{\omega}$ is minimal nonabelian over the center $Z(D)$ (for example, take $D$ of prime degree over $Z(D)$, see [10, Theorem 5.2] or [11, Theorem 9.2]). Then $[D^{(-)}, D^{(-)}]^{\omega}$ is regular over $Z(D)$, and hence is regular over $K$. Assume now that there are “enough” intermediate fields $K \subset F \subset Z(D)$. Then we can choose $F$ such that the centralizer $C_D(F)$ is noncommutative, and hence the $K$-algebra $[D^{(-)}, D^{(-)}]^{\omega}$ contains a proper nonabelian Lie subalgebra $[C_D(F), C_D(F)]^{\omega}$.

Of course, the Lie algebra $[D^{(-)}, D^{(-)}]^{\omega}$ in this example is not central. We believe that there exist central simple regular Lie algebras which are not minimal nonabelian, but fail to provide an example. Probably, it can be found by refining some arguments in [10]. Note that according to [16, Theorem 1], over a local field of characteristic zero, no such central simple Lie algebras exist.

Now let us discuss the restrictions on $L$ in the condition of Theorem 2. Both restrictions – nonsolvability and triviality of the center – are essential. Examples of non-regular Lie algebras which are either solvable, or with a nontrivial center, all whose proper subalgebras are regular, are provided by the following description of minimal non-regular algebras.

**Theorem 4.** (The ground field $K$ is perfect of characteristic $p \neq 2, 3$.) A Lie algebra $L$ is minimal non-regular if and only if one of the following holds:

(i) $L$ is solvable minimal nilpotent;

(ii) $L$ is a split one-dimensional central extension of a minimal non-abelian simple Lie algebra;

(iii) $p > 0$, $L$ is not regular, is minimal non-abelian, and is a nonsplit central extension of a simple minimal non-abelian Lie algebra which over its centroid is isomorphic to a form of an algebra of the kind $\text{psl}_{kp}(K)$.

**Proof.** The “if” part is obvious, so let us prove the “only if” part. If $L$ is solvable, then by Lemma 3 all its proper subalgebras are nilpotent, and we are in part (i). So we may assume that $L$ is not solvable. By Theorem 2, $Z(L) \neq 0$.

**Case 1.** $L$ is minimal non-nilpotent. By [11, Theorem 8.8], $L$ is minimal non-abelian, and $L/Z(L)$ is simple. If $\overline{A}$ is a proper subalgebra of $L/Z(L)$, then the preimage of $\overline{A}$ in $L$ is a proper subalgebra of $L$, and hence is abelian. This shows that $L/Z(L)$ is minimal non-abelian. By [19, Lemma 7], or by [9, Theorem 4.1], any simple minimal non-abelian Lie algebra is anisotropic, and then by Theorem A, $L/Z(L)$ is a form of a classical Lie algebra.

If the central extension $0 \to Z(L) \to L \to L/Z(L) \to 0$ splits, then $L/Z(L)$ is a proper subalgebra of $L$, a contradiction. Therefore, this central extension does not split, and the second degree cohomology $H^2(L/Z(L), K)$ does not vanish. As cohomology is preserved under field extensions, $H^2(L/Z(L) \otimes_K \overline{\mathbb{K}}, \overline{\mathbb{K}}) \neq 0$. According to [4, Theorem 3.1], among classical Lie algebras, the second degree cohomology with trivial coefficients vanishes except for the case of the algebra $\text{psl}_{kp}(K)$, in which case $H^2(\text{psl}_{kp}(K), K)$ is one-dimensional. Therefore, $(L \otimes_K \overline{\mathbb{K}})/Z(L \otimes_K \overline{\mathbb{K}}) \cong L/Z(L) \otimes_K \overline{\mathbb{K}}$ is isomorphic to the direct sum of several copies of $\text{psl}_{kp}(\overline{\mathbb{K}})$ for some $k$, and hence $L/Z(L)$, being restricted over its centroid, is isomorphic to a form of $\text{psl}_{pk}(\overline{\mathbb{K}})$. We are in part (iii).

**Case 2.** $L$ contains a proper non-nilpotent subalgebra $S$. For any nonzero $z \in Z(L)$, $S + Kz$ is a subalgebra of $L$ with a central element $z$. If it is a proper subalgebra, then it is nilpotent, a contradiction. Hence $L = S + Kz$, and $Z(L) = Kz$ is one-dimensional. If $A$ is a proper subalgebra of $S$, then $A + Kz$ is a proper subalgebra of $L$ with a nonzero central element $z$, and hence $A + Kz$ is nilpotent. Therefore, $S$ is minimal non-nilpotent. Since $S$ is regular, by Theorem 1 it is almost simple, and since $S$ is minimal non-nilpotent, it is simple. Now by [19, Lemma 7], or by [11, Theorem 8.8, part 3], $S$ is minimal non-abelian, and we are in part (iii). □

Further comments about the conditions on Lie algebras arising in this theorem are in order. Solvable minimal non-nilpotent Lie algebras, i.e. algebras in part (i), are described in [11, Theorem 8.5], [24], and [27]. All of them are isomorphic to a semidirect sum of the kind $L = N + Kx$, where $N$ is nilpotent (of index 3 if the ground field is perfect), and $ad x$ acts on $N$ in a specific way.

Concerning part (iii): if $L/Z(L)$ is central, then it is a form of $\text{psl}_{pk}(\overline{\mathbb{K}})$ for some $k$. According to [4, Theorem 3.1], any nonsplit central extension of $\text{psl}_{pk}(\overline{\mathbb{K}})$ is one-dimensional, and is isomorphic
to $\mathfrak{sl}_{nk}(\mathbb{K})$. According to §11 in this case $L$ is isomorphic either to an algebra $[A(-), A(-)]$, where $A$ is a central simple associative algebra of degree $kp$, or to $[S^-(A, J), S^-(A, J)]$, where $A$ is a simple associative algebra with involution $J$ of the second kind. It is a natural and interesting question when algebras of this kind, and, generally, central simple forms of simple classical Lie algebras of all types, and close to them algebras, are regular. For types B–D this amounts to studying the Lie algebras $S^-(A, J)$, where $A$ is a central simple associative algebra with involution $J$ of the first kind, and is more or less straightforward. For exceptional types, the situation is more difficult. We hope to treat this question elsewhere.

A first step in this direction is

**Theorem 5.** Let $D$ be a central associative division algebra. Then the Lie algebra $D^{(-)}/K1$ is regular if and only if $D$ is a minimal non-commutative algebra.

**Proof.** This is [28, Theorem 6]. The proof given there, based on the double centralizer theorem, is characteristic-free. □

**Corollary 2.** If $D$ is a central associative division algebra of prime degree, then the Lie algebra $D^{(-)}/K1$ is regular.

**Proof.** According to [10, Corollary 1.2], $D$ is minimal non-commutative, and by Theorem 5, $D^{(-)}/K1$ is regular. □

### 5. Commutators

In 1936 Shoda proved that any $n \times n$ matrix of trace zero over a field of characteristic zero can be represented as a commutator of two matrices ([22, Satz 3]). Two decades later Albert and Muckenhoupt, [2], extended this result to matrices over an arbitrary field $K$. As matrices of trace zero form the Lie algebra $\mathfrak{sl}_n(K)$ under the commutator, this result can be formulated as follows: any element of $\mathfrak{sl}_n(K)$ can be represented as a commutator. Brown, [6], extended this result to all finite-dimensional simple Lie algebras of classical type.

It is natural to ask the same question for arbitrary finite-dimensional simple Lie algebras, in particular, for forms of the Lie algebra $\mathfrak{sl}_n(K)$. As indicated in §18 such forms are described in terms of simple associative algebras. In particular, if $A$ is a central simple associative algebra of degree $n$ over a field $K$, then $A \otimes K \cong M_n(K)$, the full matrix algebra over the algebraic closure $\overline{K}$, and we can define the reduced trace $\text{tr} : A \to K$ by the formula $\text{tr}(x) = \text{tr}(x \otimes 1)$ for $x \in A$, where at the right-hand side of the equality stands the usual trace in $M_n(\overline{K})$. Then

$$\{x \in A \mid \text{tr}(x) = 0\} = [A, A],$$

so any element of reduced trace zero in $A$ is a sum of commutators of elements of $A$. Then one can ask whether any element of reduced trace zero in $A$ is a commutator. This question was addressed by Amitsur and Rowen, [3]. It was proved there that if $A$ is a matrix algebra of order $n > 1$ over a division algebra, then any noncentral element of reduced trace zero (in particular, any element of reduced trace zero if characteristic of the ground field $K$ is zero), is a commutator, and further results were obtained for some particular cases of division algebras. However, in general the question remains open, and, as reported in [14, §2.1.1(v)], it is believed that in general the answer is negative.

Our contribution to this problem is the following

**Conjecture.** In a non-nilpotent regular Lie algebra having a nondegenerate symmetric invariant bilinear form, any element of a commutant is a commutator.

This conjecture would imply that in any central minimal noncommutative division algebra $D$ whose degree is not divided by the characteristic of the ground field $p$ (and, in particular, a central division algebra of prime degree different from $p$), any element of reduced trace zero is a commutator. Indeed: by Theorem 5, $D^{(-)}/K1$ is regular; since the degree of $D$ is not divided by $p$, $[D^{(-)}, D^{(-)}] \cong D^{(-)}/K1$; and
the Lie algebra $[D^{(-)},D^{(-)}]$, being a form of $\text{sl}_n(K)$, has a nondegenerate symmetric invariant bilinear form.

Over a perfect field of characteristic $\neq 2,3$, the conjecture is, essentially, about simple forms of classical Lie algebras. Indeed, by Theorem III we are talking in this case about almost simple Lie algebras: $S \subseteq L \subseteq \text{Der}(S)$, where $S$ is a simple form of a classical Lie algebra. By the same reasoning as at the beginning of §4 we can restrict this situation over the centroid $\Omega(S)$ of $S$: $\Omega(S) \subseteq L_{\Omega(S)} \subseteq \text{Der}(S_{\Omega(S)})$, and $L_{\Omega(S)}$ is not simple only in the case where $S_{\Omega(S)}$ is a form of $\text{psl}_{kp}(K)$ for some integer $k$, and $L_{\Omega(S)} = \text{Der}(S_{\Omega(S)})$ is a form of $\text{pgl}_{kp}(K)$. But neither $\text{psl}_{kp}(K)$, nor $\text{pgl}_{kp}(K)$ have a nondegenerate invariant symmetric bilinear form, hence the same is true for $\Omega(S)$-algebras $S_{\Omega(S)}$ and $L_{\Omega(S)}$, and then for $K$-algebras $S$ and $L$, so this case is excluded.

However, a possible attack on this conjecture is not to utilize specificity of forms of classical algebras, but general properties of Cartan subalgebras in regular Lie algebras, following the approach of [6] (where it is proved that any element in a classical simple Lie algebra is a commutator), and also of [12, Appendix 3], [1], and [15] (where the same is proved for the most real simple Lie algebras, including the compact ones). In all these treatises, it is proved – via judicious manipulation with the automorphism group in the classical (split) case, and with the corresponding Lie group in the real case – that any element in a simple Lie algebra $g$ under consideration is conjugate to an element lying in the Fitting 1-component with respect to a suitable Cartan subalgebra $h$. But $h = \mathfrak{g}(x)$ for some element $x \in h$, and then $g^1(h) = [g,x]$. Consequently, any element in the Fitting 1-component with respect to a Cartan subalgebra is a commutator, and thus any element of $g$ is a commutator (with the corresponding element $x$).

In the regular case, instead of manipulating with the automorphism group – which, in the case of an arbitrary field and an arbitrary form, appears to be problematic – we may try to use the flexibility of the choice of $h$ and $x$ (in fact, any nonzero $x$ will do!), and the presence of nondegenerate symmetric invariant bilinear form.

A first step might be provided by the following

**Lemma 11.** Let $L$ be a Lie algebra with a nondegenerate symmetric invariant bilinear form, and $X$ a set of almost commuting elements of $L$. Then $L^0(X) = L^1(X)$.

**Proof.** Let $\langle \cdot, \cdot \rangle$ be a nondegenerate symmetric invariant bilinear form on $L$. For any $x,y,z \in L$, and any $n$, we have

$$\langle y, (\text{ad}x)^n(z) \rangle = (-1)^n \langle (\text{ad}x)^n(y), z \rangle.$$  

If $y \in L^0(X)$, this equality implies that $y \in L^1(X)^\perp$ for any $x \in X$, and hence $y \in L^1(X)^\perp$. Therefore, $L^0(X) \subseteq L^1(X)^\perp$. Due to the Fitting decomposition, we have $\dim L^0(X) + \dim L^1(X) = \dim L$. On the other hand, since $\langle \cdot, \cdot \rangle$ is nondegenerate, $\dim L^0(X) + \dim L^0(X)^\perp = \dim L$. Consequently, $\dim L^0(X)^\perp = \dim L^1(X)$, and $L^0(X)^\perp = L^1(X)$.

Lemma [11] is just a slight generalization of [1] Lemma 2.1 and [15] Remark 5.1.

The suggested approach is illustrated by the following elementary

**Lemma 12.** In a simple regular Lie algebra of rank 1, having a nondegenerate symmetric invariant bilinear form, any element is a commutator.

**Proof.** Let $L$ be a simple regular Lie algebra of rank 1, with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$, and $x$ a nonzero element of $L$. Take a nonzero $y \in L$ such that $\langle x,y \rangle = 0$. Then $Kx$ is a Cartan subalgebra of $L$, and by Lemma [11] $x \in (Kx)^\perp = [L,y]$.

**Corollary 3.** In a quaternion division algebra, any element of reduced trace zero is a commutator.

**Proof.** If $Q$ is a quaternion division algebra, then elements of reduced trace zero form a Lie algebra $[Q^{(-)},Q^{(-)}]$, which satisfies the conditions of Lemma [12].

This elementary result is, of course, not new, and can be checked by simple direct computations. In [3] Corollary 0.9] another elementary proof, based on the existence of inseparable elements, is given.
In the general case, the task is to ensure, for any nonzero \( x \in L \), the existence of an element whose centralizer lies in \((Kx)^\perp\), a subspace of \( L \) of codimension 1. This seems to be highly plausible, but the proof is elusive so far.

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**REFERENCES**

[1] D. Akhiezer, *On the commutator map for real semisimple Lie algebras*, Moscow Math. J. 15 (2015), no.4, 609–613.
[2] A.A. Albert and B. Muckenhoupt, *On matrices of trace zero*, Michigan Math. J. 4 (1957), no.1, 1-3; reprinted in: A.A. Albert, *Collected Mathematical Papers*, Part 2, AMS, 1993.
[3] S.A. Amitsur and L.H. Rowen, *Elements of reduced trace 0*, Israel J. Math. 87 (1994), no.1-3, 161–179; reprinted in: *Selected Papers of S.A. Amitsur with Commentary*, Part 2, AMS, 2001.
[4] R.E. Block, *On the extensions of Lie algebras*, Canad. J. Math. 20 (1968), 1439–1450.
[5] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters 7-9*, Springer, 2005 (translated from the original French editions: Hermann, 1975 and Masson, 1982).
[6] G. Brown, *On commutators in a simple Lie algebra*, Proc. Amer. Math. Soc. 14 (1963), no.5, 763–767.
[7] A.S. Dzhumadil’daev, *Irreducible representations of strongly solvable Lie algebras over a field of positive characteristic*, Mat. Sbornik 123 (1984), no.2, 212–229 (in Russian); Math. USSR Sbornik 51 (1985), no.1, 207–223 (English translation).
[8] A. Elduque, *On Malcev modules*, Comm. Algebra 18 (1990), no.5, 1551–1561.
[9] R. Farnsteiner, *On ad-semisimple Lie algebras*, J. Algebra 83 (1983), no.2, 510–519.
[10] A.G. Gein, *Minimal noncommutative and minimal nonabelian algebras*, Comm. Algebra 13 (1985), no.2, 305–328.
[11] ______., *Lie algebras with constraints on subalgebras*, Ural State Univ., Sverdlovsk, 1989 (in Russian).
[12] K.H. Hofmann and S.A. Morris, *The Lie Theory of Connected Pro-Lie Groups*, EMS Publ. House, 2007.
[13] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962; reprinted by Dover, 1979.
[14] A. Kanel-Belov, B. Kunyavskii, and E. Plotkin, *Word equations in simple groups and polynomial equations in simple algebras*, Vestnik St. Petersburg Univ. Math. 46 (2013), no.1 (Vavilov Festschrift), 3–13; arXiv:1304.5052.
[15] J. Malkoun and N. Nahlus, *Commutators and Cartan subalgebras in Lie algebras of compact semisimple Lie groups*, J. Lie Theory 27 (2017), no.4, 1027–1032; arXiv:1602.03479.
[16] P. Müller, *On simple semiabelian p-adic Lie algebras*, Comm. Algebra 20 (1992), no.4, 1041–1049.
[17] A.A. Premet, *Lie algebras without strong degeneration*, Mat. Sbornik 129 (1986), no.1, 140–153 (in Russian); Math. USSR Sbornik 57 (1987), no.1, 151–163 (English translation).
[18] ______., *Inner ideals in modular Lie algebras*, Vesti Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 1986, no.5, 11–15 (in Russian).
[19] ______., *On Cartan subalgebras of Lie p-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no.4, 788–800 (in Russian); Math. USSR Izvestija 29 (1987), no.1, 145–157 (English translation).
[20] ______., *Absolute zero divisors in Lie algebras over a perfect field*, Doklady Akad. Nauk BSSR 31 (1987), no.10, 869–872 (in Russian).
[21] G.B. Seligman, *Modular Lie Algebras*, Springer-Verlag, 1967.
[22] K. Shoda, *Einige Sätze über Matrizen*, Japan. J. Math. 13 (1936), no.3, 361–365.
[23] I.M. Singer, *Uniformly continuous representations of Lie groups*, Ann. Math. 56 (1952), no.2, 242–247.
[24] E.L. Stitzinger, *Minimal nonnilpotent solvable Lie algebras*, Proc. Amer. Math. Soc. 28 (1971), no.1, 47–49.
[25] H. Strade and R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Marcel Dekker, 1988.
[26] M. Sugiuira, *On a certain property of Lie algebras*, Sci. Papers College Gen. Educ. Univ. Tokyo 5 (1955), 1–12.
[27] D.A. Towers, *Lie algebras all whose proper subalgebras are nilpotent*, Lin. Algebra Appl. 32 (1980), 61–73.
[28] P. Zusmanovich, *Lie algebras with given properties of subalgebras and elements*, Algebra, Geometry and Mathematical Physics (ed. A. Makhlfouf et al.), Springer Proc. Math. Stat. 85 (2014), 99–109; arXiv:1105.4284.

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