Quantifying magic for multi-qubit operations

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The development of a framework for quantifying “non-stabiliserness” of quantum operations is motivated by the magic state model of fault-tolerant quantum computation, and by the need to estimate classical simulation cost for noisy intermediate-scale quantum (NISQ) devices. The robustness of magic was recently proposed as a well-behaved magic monotone for multi-qubit states and quantifies the simulation overhead of circuits composed of Clifford+T gates, or circuits using other gates from the Clifford hierarchy. Here we present a general theory of the “non-stabiliserness” of quantum operations rather than states, which are useful for classical simulation of more general circuits. We introduce two magic monotonies, called channel robustness and magic capacity, which are well-defined for general n-qubit channels and treat all stabiliser-preserving CPTP maps as free operations. We present two complementary Monte Carlo-type classical simulation algorithms with sample complexity given by these quantities and provide examples of channels where the complexity of our algorithms is exponentially better than previous known simulators. We present additional techniques that ease the difficulty of calculating our monotonies for special classes of channels.

The Gottesman-Knill theorem showed that circuits comprised of stabiliser state preparations, Clifford gates, Pauli measurements, classical randomness and conditioning can be efficiently simulated by a traditional computer \cite{1, 2}. If a circuit involves a relatively small proportion of non-Clifford operations, simulation may be within the reach of a classical computer, albeit with a runtime overhead that is expected to scale exponentially with the amount of resource required. An important class of devices comprises so-called near-Clifford circuits where simulation may be feasible \cite{3, 4}.

There are two scenarios where near-Clifford circuits are relevant. As we enter the era of Noisy Intermediate Scale Quantum (NISQ) devices \cite{5}, many experiments proposed as demonstrators of quantum advantage may be near-Clifford so it is important to rigorously understand when a classical simulation is available. Furthermore, in the NISQ regime the need for classical simulation tools for benchmarking and verification becomes more pressing. The quantification of non-stabiliser resource is also of interest in the context of the magic state model of fault-tolerant quantum computation \cite{6, 7, 8}, the second scenario. Any device intended to provide quantum advantage must involve non-stabiliser operations. In circuits employing error-correcting codes, however, it is often not possible for the code to ‘natively’ implement non-Clifford gates fault-tolerantly \cite{8}. Instead, these gates are implemented indirectly by injection of so-called magic states. These are non-stabiliser states that must be prepared using the experimentally costly process of magic state distillation \cite{6, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33}, which is comprised of Clifford-dominated circuits.

Both of these scenarios motivate the development of a resource theory \cite{17, 26} where the class of free operations is generated by stabiliser state preparations and rounds of stabiliser operations as described above. For the case of odd d-dimensional qudits this problem is largely solved by the discrete phase space formalism \cite{27, 32}; odd dimension qudit stabiliser states are characterised by a positive discrete Wigner function. In Ref. \cite{33}, the discrete Wigner function was cast as a quasiprobability distribution, making a direct connection between the negativity of the distribution, and the complexity of calculating expectation values via a Monte Carlo-type simulation algorithm. However, the discrete phase space approach cannot be applied cleanly to qubits without excluding some Clifford operations from the free operations \cite{34, 35}. To retain all multi-qubit stabiliser...
channels as free operations, then, we must seek alternative approaches.

Howard and Campbell [36] introduced a scheme where density matrices are decomposed as real linear combinations of pure stabiliser state projectors. Non-stabiliser states sit outside the convex hull of the pure stabiliser states, so their decompositions necessarily contain negative terms and can again be viewed as quasiprobability distributions, with $\ell_1$-norm strictly larger than 1. The robustness of magic for a state, defined as the minimum $\ell_1$-norm over all valid decompositions, is a monotone under stabiliser operations and has several useful resource-theoretic properties. Alternative approaches include stabiliser rank methods, where the state vector is decomposed as a superposition of stabiliser states [37–40]. Exact and approximate stabiliser rank, and the associated quantity extent, are measures of magic for pure states. Here we are interested in measures naturally suited for applications to mixed states or general, noisy quantum channels. A stabiliser-based method to simulate noisy circuits by decomposition of states into Pauli operators was recently proposed in Ref. [41]. In this work we characterise the cost of quantum operations with respect to the resource theory of magic. Robustness of magic naturally quantifies the cost for a subclass of non-Clifford operations, namely gates from the third level of the Clifford hierarchy. It is less clear how the framework can be extended to more general quantum operations, and formalising this is one of our main aims.

In Ref. [3], Bennink et al. presented an algorithm in which completely positive trace-preserving (CPTP) maps are decomposed as quasiprobability distributions over a subset of stabiliser-preserving operations that we will call CPR. This subset supplements the Clifford unitaries with Pauli reset channels, in which measurement of some Pauli observable is followed by a conditional Clifford correction, so as to reset a state to a particular $+1$ Pauli eigenstate. While Bennink et al. showed that CPR spans the set of CPTP maps, there is no guarantee that all stabiliser-preserving CPTP maps can be found within its convex hull. Indeed, we will see in Section IV there exist channels that are stabiliser-preserving, but are nevertheless assigned a non-trivial cost by the algorithm of Ref. [3]. The implication is that decomposition in terms of elements of CPR is not the best strategy for simulating general non-stabiliser operations. An obvious extension of Ref. [3] is to replace CPR by the full set of stabiliser-preserving CPTP maps. The technical question to be answered is then how to correctly and concisely represent this set; how can we be sure that we have captured all possible stabiliser-preserving channels? This issue is addressed in Sections III and IV.

In this paper we introduce two magic monotones for channels: the channel robustness $R^*$ and the magic capacity $C$. Both are closely related to the robustness of magic for states. They are well-defined for general $n$-qubit channels and treat all stabiliser-preserving CPTP maps as free operations. We will see that these monotones give the sample complexity of two classical simulation algorithms. Other magic monotones have been proposed [17, 30, 31] but without known connections to classical simulation algorithms. Furthermore, we give several examples of channels where the simulation complexities of our approaches are exponentially faster (as a function of gate count) than other quasiprobability simulators such as the Bennink et al. simulator [3].

The paper is structured as follows. In Section II we review the properties of robustness of magic and give some definitions. Next, we summarise our main results in Section II before pinning down what we mean by stabiliser-preserving operations in Section III. Sections IV and V are chiefly concerned with proving important properties of our monotones. Two classical simulation algorithms, each related to one of our monotones, are described in Section VI. Finally, in Section VII we calculate the numerical values of our monotones for operations on up to five qubits, using techniques developed in Appendix E.

I. PRELIMINARIES

Let $\text{STAB}_n$ be the set of $n$-qubit stabiliser states. In an abuse of notation we will use $|\phi\rangle \in \text{STAB}_n$ to mean a pure state from this set, and $\rho \in \text{STAB}_n$ to mean the density matrix of a
state taken from the stabiliser polytope, the convex hull of pure stabiliser states. The pure states in \( \text{STAB}_n \) form an overcomplete basis for the set of \( 2^n \)-dimensional density matrices \( D_n \). We can therefore write the density matrix for any state as an affine combination of pure stabiliser state projectors \( \rho = \sum_j q_j |\phi_j\rangle\langle \phi_j| \) where \( |\phi_j\rangle \in \text{STAB}_n \), and \( \sum_j q_j = 1 \). In general, \( q_j \) can be negative. The robustness of magic is defined as the minimal \( \ell_1 \)-norm \( \| \vec{q} \|_1 = \sum_j |q_j| \) over all possible decompositions:

\[
\mathcal{R}(\rho) = \min_{\vec{q}} \left\{ \| \vec{q} \|_1 : \sum_j q_j |\phi_j\rangle\langle \phi_j| = \rho, |\phi_j\rangle \in \text{STAB}_n \right\}.
\]

(1)

In the definition above, the state of interest is expressed as a decomposition over pure stabiliser states. By collecting together all terms of the same sign, any state can instead be expressed in terms of a pair of mixed stabiliser states (Figure 1). An equivalent definition is then:

\[
\mathcal{R}(\rho) = \min_{\rho_\pm \in \text{STAB}_n} \{ 1 + 2p : (1 + p)\rho_+ - p\rho_- = \rho, p \geq 0 \}.
\]

(2)

FIG. 1. Schematic illustration of a density matrix \( \rho \in D_n \) decomposed as an affine combination of elements from the stabiliser polytope \( \text{STAB}_n \).

The robustness of magic is a well-behaved magic monotone, having the following properties:

1. **Convexity**: \( \mathcal{R} \left( \sum_j q_j \rho_j \right) \leq \sum_j |q_j| \mathcal{R}(\rho_j) \);
2. **Faithfulness**: If \( \rho \in \text{STAB}_n \), then \( \mathcal{R}(\rho) = 1 \). Otherwise \( \mathcal{R}(\rho) > 1 \);
3. **Monotonicity under stabiliser operations**: If \( \Lambda \) is a CPTP stabiliser-preserving operation, then \( \mathcal{R}(\Lambda(\rho)) \leq \mathcal{R}(\rho) \);
4. **Submultiplicativity under tensor product**: \( \mathcal{R}(\rho_A \otimes \rho_B) \leq \mathcal{R}(\rho_A)\mathcal{R}(\rho_B) \).

The quantity \( \mathcal{R} \) also has a clear operational meaning, quantifying the classical simulation cost in a Monte Carlo-type scheme that samples from a quasiprobability distribution over stabiliser states \[3, 33, 36\]. These algorithms estimate the expectation value of a Pauli observable after a stabiliser channel is applied to a non-stabiliser input state. The minimum number of samples required to achieve some stated accuracy scales with \( \mathcal{R}^2 \).

The robustness of magic can be calculated using standard linear programming techniques \[42\] (for example using the MATLAB package CVX \[43\]). The naive formulation of the linear program is practical on a desktop computer for up to five qubits (the number of stabiliser states increases super-exponentially with \( n \)). It was recently shown by Heinrich and Gross \[44\] that when states
FIG. 2. State injection gadget. A resource state $|U\rangle$ is consumed in order to implement the corresponding gate $U$. A Clifford correction $C$ is applied to qubit 1 conditioned on the outcome of a Pauli measurement on qubit 2. A single-qubit gate is shown, but the scheme can be generalised to multi-qubit gates from the third level of the Clifford hierarchy.

possess certain symmetries, the original optimisation problem can be mapped to a more tractable one, so that the robustness of magic can be calculated for up to 10 copies of a state.

The framework naturally extends to a subclass of non-stabiliser circuits: those that may be implemented by deterministic state injection \cite{36}, including all gates from the third level of the Clifford hierarchy (Figure 2). The canonical example is the T-gate, $T = \text{diag}(1, e^{i\pi/4})$, which can be implemented by consuming so-called magic states as a resource \cite{7}. The classical simulation overhead for implementing a gate is then the robustness of magic for the consumed resource state. Not all non-stabiliser operations can be implemented in this way, however.

Informally we say that an operation is stabiliser-preserving if it always maps stabiliser states to stabiliser states. To make this precise, define $\text{SP}_{n,m}$ to be the set of $n$-qubit operations $E$ such that $(E \otimes 1_m)\sigma \in \text{STAB}_{n+m}$ for all $\sigma \in \text{STAB}_{n+m}$, where $1_m$ is the identity map for an $m$-qubit Hilbert space. The set $\text{SP}_{n,0}$ is then the set of channels that map $n$-qubit stabiliser states to $n$-qubit stabiliser states. We say a channel is “completely” stabiliser-preserving if $E \in \text{SP}_{n,m}$ for all $m$.

II. OVERVIEW OF MAIN RESULTS

Our first result is a characterisation of the class of completely stabiliser-preserving operations, making use of the the well-known Choi-Jamiołkowski isomorphism \cite{45–47}.

**Theorem 1** (Completely stabiliser-preserving operations). *Given an $n$-qubit CPTP channel, for all $m > 0$, $E \in \text{SP}_{n,n+m}$ if and only if $E \in \text{SP}_{n,n}$. Furthermore, $E \in \text{SP}_{n,n}$ if and only if the Choi-state*

$$
\Phi_E = (E_A \otimes 1_n)|\Omega_n\rangle\langle\Omega_n|_{AB}, \quad \text{where} \quad |\Omega_n\rangle_{AB} = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle_A \otimes |j\rangle_B, \quad (3)
$$

*is a stabiliser state. Here, $|j\rangle$ are the $n$-qubit computational basis states.*

We prove this in section III. We take this to be the set of free operations in our resource theories.

Our first new monotone is channel robustness $R_\ast$. For an $n$-qubit CPTP channel $E$ this is defined as:

$$
R_\ast(E) = \min_{\Lambda_\pm \in \text{SP}_{n,n} \cap \text{CPTP}} \left\{ 2p + 1 : (1 + p)\Lambda_+ - p\Lambda_- = E, p \geq 0 \right\}, \quad (4)
$$

where $\Lambda_\pm$ are completely stabiliser-preserving and CPTP maps. To fully enumerate this class of maps, we notice that the associated Choi-state must satisfy two conditions: (i) $\Phi_E$ is a stabiliser state, and (ii) $\Phi_E$ satisfies the trace-preservation condition $\text{Tr}_A(\Phi_E) = \frac{1}{2^n}$. We can therefore write:

$$
R_\ast(E) = \min_{\rho_\pm \in \text{STAB}_{2^n}} \left\{ 2p + 1 : (1 + p)\rho_+ - p\rho_- = \Phi, p \geq 0, \text{Tr}_A(\rho_\pm) = \frac{1}{2^n} \right\}. \quad (5)
$$
This can now be calculated by linear program given access to a list of all stabiliser states (see Appendix C). Channel robustness satisfies the following:

1. **Faithfulness**: If \( \mathcal{E} \) is a CPTP channel, then \( R_+ (\mathcal{E}) = 1 \) if \( \mathcal{E} \) is completely stabiliser-preserving and strictly larger than 1 otherwise;

2. **Convexity**: \( R_+ \left( \sum_j q_j \mathcal{E}_j \right) \leq \sum_j |q_j| R_+ (\mathcal{E}_j) \);

3. **Submultiplicativity under composition**: \( R_+ (\mathcal{E}_2 \circ \mathcal{E}_1) \leq R_+ (\mathcal{E}_2) R_+ (\mathcal{E}_1) \);

4. **Submultiplicativity under tensor product**: \( R_+ (\mathcal{E}_A \otimes \mathcal{E}_B) \leq R_+ (\mathcal{E}_A) R_+ (\mathcal{E}_B) \).

As a special case, if \( \Lambda \) is a CPTP stabiliser channel, then

\[
R_+ (\Lambda \circ \mathcal{E}) \leq R_+ (\mathcal{E}) R_+ (\Lambda) = R_+ (\mathcal{E}),
\]

and similarly \( R_+ (\mathcal{E} \circ \Lambda) \leq R_+ (\mathcal{E}) \). This combines submultiplicativity under composition and faithfulness, to show that \( R_+ \) is suitably monotonically non-increasing under compositions with stabiliser channels. This is the sense in which channel robustness is a magic monotone for channels. We prove submultiplicativity in Section IV and show convexity and faithfulness in Appendix B.

The approach above is very close to the stabiliser decomposition of channels employed by Bennink et al. in Ref.\[3\]. The main difference is that Bennink et al. optimise their decomposition with respect to CPR, the set of Clifford unitaries supplemented by Pauli reset channels, rather than SP_{n,n}. The set CPR turns out to be a strict subset of the stabiliser-preserving CPTP maps, so \( R_+ \) is a lower bound to the \( \ell_1 \)-norm of any CPR decomposition (though the bound is tight in many cases). Just as the \( \ell_1 \)-norm in Bennink et al. quantifies the sample complexity of a classical simulation algorithm, we can construct a related algorithm where the runtime depends on \( R_+ \) in a similar way. We give the details of this algorithm in Section VIIA.

Before proceeding, let us reflect on the condition \( \text{Tr}_A (p_\pm) = \frac{1}{2^n} \) that enforces that the corresponding channels \( \mathcal{E}_\pm \) are trace-preserving. Dropping this condition would instead lead to \( R(\Phi_\mathcal{E}) \), the robustness of the Choi state. For gates from the third level of the Clifford hierarchy, deterministic state injection is always possible, and hence the resource cost of the gate \( U \) can be equated with the robustness of magic of the corresponding resource state. These resource states can always (by Clifford-equivalence) be taken to have the form \( |U\rangle \langle U| = (U_A \otimes I_B) |\Omega\rangle \langle \Omega| \). This is precisely the Choi state, so it is natural to ask if \( R(\Phi_\mathcal{E}) \) also quantifies non-stabiliserness for more general channels. We find that \( R(\Phi_\mathcal{E}) \) exhibits faithfulness, convexity and submultiplicativity under tensor product, but lacks submultiplicativity under composition. This arises from the fact that the decomposition of the Choi state corresponds to a decomposition of the channel into maps that are not necessarily trace-preserving. We illustrate this in Appendix A with counterexamples. Despite this shortcoming, we will see that \( R(\Phi_\mathcal{E}) \) is a useful quantity to compare to more well-behaved measures.

Our second new monotone is the **magic capacity**. Given an \( n \)-qubit channel \( \mathcal{E} \), it is natural to consider the largest possible increase in robustness of magic, over any possible input state. By analogy with the resource theories of entanglement \[43\] and coherence \[24\], we define the magic capacity as:

\[
\mathcal{C}(\mathcal{E}) = \max_{|\phi\rangle \in \text{STAB}_{2^n}} R([\mathcal{E} \otimes 1_n] |\phi\rangle \langle \phi|).
\]

Note that the definition of capacity involves forming a tensor product of an \( n \)-qubit channel \( \mathcal{E} \) with the \( n \)-qubit identity. This is necessary because there exist \( n \)-qubit channels that generate their maximum robustness when applied to part of an \( m \)-qubit state, where \( m > n \). Nevertheless, the \( n \)-qubit identity suffices for our definition; This is a consequence of Lemma 1 in Section III. The capacity has the following useful properties:
1. **Faithfulness**: If \( \mathcal{E} \) is a completely positive, trace-preserving (CPTP) channel, then \( C = 1 \) if \( \mathcal{E} \) is stabiliser-preserving (SP), and strictly larger than 1 otherwise;

2. **Convexity**: \( C \left( \sum_j q_j \mathcal{E}_j \right) \leq \sum_j q_j C(\mathcal{E}_j) \);

3. **Submultiplicativity under composition**: \( C(\mathcal{E}_1 \circ \mathcal{E}_2) \leq C(\mathcal{E}_1)C(\mathcal{E}_2) \);

4. **Submultiplicativity under tensor product**: \( C(\mathcal{E}_A \otimes \mathcal{E}_B) \leq C(\mathcal{E}_A)C(\mathcal{E}_B) \);

5. **Maximum increase in robustness**: \( \frac{\mathcal{R}(\mathcal{E} \otimes 1)\rho}{\mathcal{R}(\rho)} \leq C(\mathcal{E}), \forall \rho \).

In Section \( \text{V} \) we prove properties 3-5, leaving the proof of convexity and faithfulness to Appendix \( \text{D} \). We will also prove the following theorem relating magic capacity to channel robustness and \( \mathcal{R}(\Phi_\mathcal{E}) \).

**Theorem 2** (Sandwich Theorem). For any CPTP map \( \mathcal{E} \), the following inequalities hold:

\[
\mathcal{R}(\Phi_\mathcal{E}) \leq C(\mathcal{E}) \leq \mathcal{R}_*(\mathcal{E}).
\]

(7)

We are interested in whether or not these inequalities are tight. In Table \( \text{I} \) we summarise numerical results for a selection of diagonal gates. The results for these gates are presented in full in Section \( \text{VII} \).

| \( n \) | 2 | 3 | 4 | 5 |
|------|---|---|---|---|
| Multicontrol gates, \( t = 0 \) | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} < \mathcal{R} \) |
| Multicontrol gates, \( t \geq 1 \) | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} < \mathcal{R}_* \) | \( \mathcal{R}_\Phi < \mathcal{C} < \mathcal{R} \) |
| Random phase gates | \( \mathcal{R}_\Phi = \mathcal{C} = \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} \leq \mathcal{R}_* \) | \( \mathcal{R}_\Phi = \mathcal{C} \leq \mathcal{R}_* \) | - |

TABLE I. Tightness of bound given by Theorem 2, as determined by numerical estimation of diagonal gates, where \( \mathcal{R}_\Phi \) is the robustness of the Choi state, \( \mathcal{C} \) is the magic capacity, \( \mathcal{R}_* \) is the trace-preserving variant of \( \mathcal{R}_\Phi \). Here an equality indicates that in all cases investigated, values calculated were equal up to the precision of the solver. Multicontrol phase gates are taken to be those represented by unitaries of the form diag(1, ..., 1, exp[i\pi/2^t]).

The magic capacity also quantifies the sample complexity for a Monte Carlo-type classical simulation algorithm, presented in Section \( \text{VI B} \). This differs from previous algorithms such as Bennink et al. \[3\] in that a convex optimisation must be solved at each step. While this results in an increase in runtime per sample, it can be the case that \( \mathcal{C}(\mathcal{E}) \ll \mathcal{R}_*(\mathcal{E}) \), which can lead to an improvement in sample complexity over the algorithm of Section \( \text{VIA} \).

### III. COMPLETELY STABILISER-PRESERVING OPERATIONS

In this section, we justify setting \( \text{SP}_{n,n} \) as the class of free operations. We begin with an example channel \( \mathcal{E} \in \text{SP}_{n,0} \) that fails to be stabiliser-preserving when acting on part of a larger system. Consider the single-qubit channel \( \mathcal{E}_T \) defined by the Kraus operators \( \{ |0\rangle_T \langle 0|, |1\rangle_T \langle 1| \} \), where \( |T\rangle = T |+\rangle \) and \( |T\rangle = T |-\rangle \). Clearly, applied to any single-qubit state, the output will be some probabilistic mixture of \( |0\rangle \) and \( |1\rangle \), and so must have \( \mathcal{R} = 1 \), so \( \mathcal{E}_T \in \text{SP}_{n,0} \). But if \( \mathcal{E}_T \) is applied to one qubit in a Bell pair, we obtain:

\[
(\mathcal{E}_T \otimes 1) |\Phi+\rangle |\Phi+\rangle = \frac{1}{2}(|0T^*\rangle|0T^*\rangle + |1T^*_+\rangle|1T^*_+\rangle),
\]

(8)
where $|T^*\rangle = T^* \langle +|$, $|T_2^*\rangle = T^* \langle -|$. From this output state, we can deterministically recover a pure magic state on qubit 2 using only stabiliser operations, by making a $Z$-measurement on qubit 1 and then performing a rotation on qubit 2 conditioned on the outcome. The output state has robustness $R((E_T \otimes I) |\Phi^+_1 \rangle |\Phi^+_1 \rangle) = R(|T\rangle) = \sqrt{2}$.

So, there exist channels where $E \in SP_{n,m}$ but $E \notin SP_{n,m+1}$. To call a channel completely stabiliser-preserving, then, we need to be sure $E \otimes I_m$ remains stabiliser-preserving for all $m > 0$. We now show we only need tensor with the identity of the same dimension as the original channel.

**Lemma 1** (Maximum robustness achieved on 2n qubits). *Let $E$ be an n-qubit quantum channel. Then for $m > 0$, for any $|\phi\rangle \in STAB_{2n+m}$, there exists some state $|\psi\rangle \in STAB_{2n}$ such that:

$$R[(E_A \otimes I_{n+m}) |\phi\rangle |\phi\rangle_{AB}] = R[(E_A \otimes I_n) |\psi\rangle |\psi\rangle_{AB}].$$

**Proof.** Consider a $(2n + m)$-qubit stabiliser state $|\phi\rangle$, with partition $A|B$ between the first $n$ and last $n + m$ qubits. Ref. [49] shows that the state $|\phi\rangle_{AB}$ is local Clifford-equivalent to $p$ independent Bell pairs entangled across the partition $A|B$ (here “local” means with respect to the bipartition rather than per qubit). Since there are $n$ qubits in partition $A$, $p$ is at most $n$. So we have:

$$|\phi\rangle_{AB} = (I_n \otimes U_B) |\psi\rangle_{AB'} |\psi'\rangle_{B''},$$

where $U_B$ is a Clifford operation, $|\psi\rangle_{AB'} \in STAB_{2n}$, and $|\psi'\rangle_{B''} \in STAB_m$. So writing the channel corresponding to $U_B$ as $U_B$, for any $E$ on $n$ qubits, we know that:

$$R[(E_A \otimes I_{n+m}) |\phi\rangle |\phi\rangle_{AB}] = R[(I_n \otimes U_B)(E_A \otimes I_n) (|\psi\rangle |\psi\rangle_{AB'}) \otimes |\psi'\rangle |\psi'\rangle_{B''}].$$

Since $I_n \otimes U_B$ represents a (reversible) Clifford gate, by monotonicity of robustness of magic:

$$R[(E_A \otimes I_{n+m}) |\phi\rangle |\phi\rangle_{AB}] = R[(E_A \otimes I_n) (|\psi\rangle |\psi\rangle_{AB'}) \otimes |\psi'\rangle |\psi'\rangle_{B''}] = R[(E_A \otimes I_n) |\phi\rangle |\phi\rangle_{AB}],$$

where in the last line we used the fact that $|\psi'\rangle_{B''}$ is a stabiliser state, and hence does not contribute to the robustness. The state $|\psi\rangle |\psi\rangle_{AB}$ is a 2n-qubit state, so this proves the result.\hfill $\square$

This lemma allows us to prove the first claim of Theorem 1, which says that $E$ is completely stabiliser-preserving if and only if $E \in SP_{n,n}$. The inclusion $SP_{n,n} \subseteq SP_{n,n,m}$ is immediate since the stabiliser states are preserved under tracing out auxiliary systems. The interesting inclusion is $SP_{n,n,m} \subseteq SP_{n,n}$. Now suppose that $E \in SP_{n,n}$ and consider any $\sigma \in STAB_{2n+m}$. By Lemma 1 there exists some stabiliser state $\sigma' \in STAB_{2n}$ such that $R((E \otimes I_{n+m})\sigma) = R((E \otimes I_n)\sigma')$. But if $E \in SP_{n,n}$, then $(E \otimes I_n)\sigma'$ is a stabiliser state, so the robustness is equal to 1. By the faithfulness of robustness of magic, $(E \otimes I_{n+m})\sigma$ is a stabiliser state. Therefore, $E \in SP_{n,n}$ implies $E \in SP_{n,n,m}$. Next, we discuss a straightforward test for membership of this set, which does not require mechanically checking all possible input stabiliser states.

We can associate every $n$-qubit channel $E$ with a unique density operator on $2n$ qubits [47]:

$$\Phi_E = (E_A \otimes I_B) |\Omega\rangle |\Omega\rangle, \quad \text{where} \quad |\Omega\rangle_n = \frac{1}{\sqrt{2}^n} \sum_{j=0}^{2^n-1} |j\rangle_A \otimes |j\rangle_B.$$  

Here $|j\rangle$ label the computational basis states. We will also use the following property:

$$\Tr[A E(\rho)] = 2^n \Tr[\Phi_E (A \otimes \rho^T)], \quad \forall \rho, A.$$  

Consider the robustness of magic of the Choi state, $R(\Phi_E)$. We mentioned earlier that $R(\Phi_E)$ quantifies simulation cost for gates from the third level of the Clifford hierarchy. This motivates
us to consider its properties for more general operations, and it turns out that $R(\Phi_E)$ gives us our first criterion for completely stabiliser-preserving channels.

**Lemma 2** (Faithfulness of robustness of the Choi state). Consider the $n$-qubit CPTP channel $\mathcal{E}$. If $E \in \text{SP}_{n,n}$, then $R(\Phi_E) = 1$. Otherwise, $R(\Phi_E) > 1$.

*Proof.* The fact that $\mathcal{E} \in \text{SP}_{n,n}$ implies $R(\Phi_E) = 1$ is easy to see. Since $|\Omega_n\rangle|\Omega_n\rangle$ is itself a $2n$-qubit stabiliser state, $\mathcal{E} \in \text{SP}_{n,n}$ guarantees that $\Phi_E$ is a stabiliser state, so must have robustness 1. The implication in the other direction is less obvious; one might imagine there perhaps exist maps that fail to be submultiplicative under composition (see Appendix A). Rather, we use the faithfulness of the Choi state as a tool to give an alternative definition of the channel robustness as captured by Eq. [5].
IV. CHANNEL ROBUSTNESS

A natural extension of the algorithm of Bennink et al. is to replace CPR (the set of Clifford gates and Pauli reset channels) with $\text{SP}_{n,n}$. We therefore define the channel robustness as:

$$\mathcal{R}_*(\mathcal{E}) = \min_{\Lambda_{\pm} \in \text{SP}_{n,n}} \{2p + 1 : (1 + p)\Lambda_+ - p\Lambda_- = \mathcal{E}, p \geq 0\}. \quad (20)$$

To calculate this in practice, we decompose the Choi state $\Phi_{E}$ as per equation (5), adapting the robustness of magic optimisation problem from Ref. [36]. The details are given in Appendix C.

The channel robustness is convex and faithful with these properties inherited from the robustness of magic (see Appendix B for details). Here we discuss additional properties.

Submultiplicativity under composition: $\mathcal{R}_*(\mathcal{E}_2 \circ \mathcal{E}_1) \leq \mathcal{R}_*(\mathcal{E}_1)\mathcal{R}_*(\mathcal{E}_2)$. The channels $\mathcal{E}_1$ and $\mathcal{E}_2$ will have an optimal decomposition:

$$\mathcal{E}_j = (1 + p_j)\Lambda_{j,+} - p_j\Lambda_{j,-}, \quad (21)$$

where $\mathcal{R}_*(\mathcal{E}_j) = 1 + 2p_j$ and $\Lambda_{j,\pm}$ are CPTP maps and completely stabiliser preserving. Using these decompositions, we obtain that

$$\mathcal{E}_2 \circ \mathcal{E}_1 = (1 + q)\Lambda_+ - q\Lambda_-, \quad (22)$$

where

$$\Lambda_+ = (1 + q)^{-1}[(1 + p_2)(1 + p_1)\Lambda_{2,+} \circ \Lambda_{1,+} + p_2p_1\Lambda_{2,-} \circ \Lambda_{1,-}], \quad (23)$$

$$\Lambda_- = q^{-1}[p_2(1 + p_1)\Lambda_{2,-} \circ \Lambda_{1,+} + (1 + p_2)p_1\Lambda_{2,+} \circ \Lambda_{1,-}], \quad (24)$$

$$q = p_1 + p_2 + 2p_1p_2 \quad (25)$$

The set of CPTP completely stabiliser preserving channels is closed under composition and convex, so both $\Lambda'_{\pm}$ are in this set. Therefore, we have a valid decomposition for $\mathcal{E}_2 \circ \mathcal{E}_1$ that entails $\mathcal{R}_*(\mathcal{E}_2 \circ \mathcal{E}_1) \leq 1 + 2q$. One finds

$$1 + 2q = (1 + 2p_1)(1 + 2p_2) = \mathcal{R}_*(\mathcal{E}_1)\mathcal{R}_*(\mathcal{E}_2), \quad (26)$$

which completes the proof.

Submultiplicativity under tensor product: $\mathcal{R}_*(\mathcal{E}_A \otimes \mathcal{E}_B) \leq \mathcal{R}_*(\mathcal{E}_A)\mathcal{R}_*(\mathcal{E}_B)$. We treat tensor product as a special case of composition. For $n$-qubit $\mathcal{E}_A$ and $m$-qubit $\mathcal{E}_B$:

$$\mathcal{R}_*(\mathcal{E}_A \otimes \mathcal{E}_B) \leq \mathcal{R}_*(\mathcal{E}_A \otimes I_m)\mathcal{R}_*(I_n \otimes \mathcal{E}_B). \quad (27)$$

To complete the proof we will confirm that

$$\mathcal{R}_*(I_n \otimes \mathcal{E}) = \mathcal{R}_*(\mathcal{E} \otimes I_n) = \mathcal{R}_*(\mathcal{E}). \quad (28)$$

As noted earlier, we can write $|\Omega_{n+m}\rangle_{AA'BB'} = |\Omega_n\rangle_{AB} \otimes |\Omega_m\rangle_{A'B'}$, so that the Choi state for $\mathcal{E}_A \otimes I_m$ is given by:

$$\Phi_{E_A \otimes I_m} = (\mathcal{E}_A \otimes I_m \otimes I_{n+m})|\Omega_{n+m}\rangle_{A A' B B'} \langle \Omega_{n+m}|_{A A' B B'} = (\mathcal{E}_A \otimes I_m) |\Omega_n\rangle_{AB} \otimes |\Omega_m\rangle_{A'B'} \langle \Omega_n|_{AB} \otimes \langle \Omega_m|_{A'B'} = \Phi_{E_A} \otimes |\Omega_m\rangle_{A'B'}. \quad (29)$$

This state will have some optimal decomposition $\Phi_{E_A} = (1 + p)\rho_+ - p\rho_-$, where $\mathcal{R}_*(E_A) = 1 + 2p$,.
so that:

\[
\Phi_{\mathcal{E}A \otimes \mathbb{1}_m} = (1 + p)\rho_+ \otimes |\Omega_m\rangle\langle \Omega_m|_{A'B'} - p\rho_0 \otimes |\Omega_m\rangle\langle \Omega_m|_{A'B'}.
\]  

(30)

This is a valid, not necessarily optimal, stabiliser decomposition satisfying the trace condition, so we have

\[
\mathcal{R}_*(\mathcal{E}_A \otimes \mathbb{1}_B) \leq \mathcal{R}_*(\mathcal{E}_A).
\]  

(31)

This is enough to show submultiplicativity; for completeness, in Appendix B we will also show \(\mathcal{R}_*(\mathcal{E}) \leq \mathcal{R}_*(\mathcal{E} \otimes \mathbb{1})\) so that in fact we have equality.

V. MAGIC CAPACITY

We now turn to our second monotone, which quantifies the capacity of a channel to generate magic. Recall:

\[
\mathcal{C}(\mathcal{E}) = \max_{|\phi\rangle \in \text{STAB}_{2n}} \mathcal{R}[|\phi\rangle\langle \phi|] ,
\]  

(32)

where \(\mathcal{R}\) is the robustness of magic. Notice that we only need optimise over the pure stabiliser states. For mixed states or even non-stabiliser states, the capacity still captures the possible increase in robustness of magic by virtue of the maximum increase in robustness property:

\[
\frac{\mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n)\rho)}{\mathcal{R}(\rho)} \leq \mathcal{C}(\mathcal{E}).
\]  

(33)

Here we prove this property, using similar arguments to those deployed in [48]. Consider an \(n\)-qubit channel \(\mathcal{E}\). Any \(2n\)-qubit input state \(\rho\) will have an optimal stabiliser state decomposition \(\rho = \sum_j q_j |\phi_j\rangle\langle \phi_j|\), where \(\sum_j q_j = 1\), and such that \(\mathcal{R}(\rho) = \sum_j |q_j|\). By linearity we have:

\[
(\mathcal{E} \otimes \mathbb{1}_n)\rho = \sum_j q_j (\mathcal{E} \otimes \mathbb{1}_n) |\phi_j\rangle\langle \phi_j|.
\]  

(34)

By convexity of robustness of magic, we then have:

\[
\mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n)\rho) \leq \sum_j |q_j| \mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n) |\phi_j\rangle\langle \phi_j|).
\]  

(35)

The optimal pure stabiliser state \(|\phi_*\rangle\), satisfies:

\[
\mathcal{C}(\mathcal{E}) = \mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n) |\phi_*\rangle\langle \phi_*|) = \mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n) |\phi_j\rangle\langle \phi_j|)
\]  

for any \(j\). So we have:

\[
\mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n)\rho) \leq \mathcal{R}((\mathcal{E} \otimes \mathbb{1}_n) |\phi_*\rangle\langle \phi_*|) \sum_j |q_j| = \mathcal{C}(\mathcal{E})\mathcal{R}(\rho).
\]  

(36)

Rearranging we obtain inequality (33).

Submultiplicativity under composition: \(\mathcal{C}(\mathcal{E}_1 \circ \mathcal{E}_2) \leq \mathcal{C}(\mathcal{E}_1)\mathcal{C}(\mathcal{E}_2)\). Take the composition of two linear maps \(\mathcal{E}_1\) and \(\mathcal{E}_2\). There exists some stabiliser state \(\rho_* = |\phi_*\rangle\langle \phi_*|\) that achieves the optimal robustness:

\[
\mathcal{C}(\mathcal{E}_2 \circ \mathcal{E}_1) = \mathcal{R}((\mathcal{E}_2 \circ \mathcal{E}_1) \otimes \mathbb{1}_n|\phi_*\rangle\langle \phi_*|) = \mathcal{R}((\mathcal{E}_2 \otimes \mathbb{1}_n) \circ (\mathcal{E}_1 \otimes \mathbb{1}_n)\rho_*).
\]  

(37)
The operator \((\mathcal{E}_1 \otimes 1_n)\rho_s\) will have some optimal decomposition \((\mathcal{E}_1 \otimes 1_n)\rho_s = \sum_k q_{1k} |\phi_k\rangle\langle\phi_k|\) such that \(\mathcal{R}((\mathcal{E}_1 \otimes 1_n)\rho_s) = \sum_k |q_{1k}|^2\). So by linearity:

\[
(\mathcal{E}_2 \otimes 1_n) \circ (\mathcal{E}_1 \otimes 1_n)[\rho_s] = (\mathcal{E}_2 \otimes 1_n) \left[ \sum_k q_{1k} |\phi_k\rangle\langle\phi_k| \right] = \sum_k q_{1k}(\mathcal{E}_2 \otimes 1_n) |\phi_k\rangle\langle\phi_k|.
\]  

(38)

Then by convexity of robustness of magic:

\[
\mathcal{R}((\mathcal{E}_2 \otimes 1_n) \circ (\mathcal{E}_1 \otimes 1_n)[\rho_s]) = \mathcal{R}\left( \sum_k q_{1k}(\mathcal{E}_2 \otimes 1_n) |\phi_k\rangle\langle\phi_k| \right)
\leq \sum_k |q_{1k}| \mathcal{R}((\mathcal{E}_2 \otimes 1_n) |\phi_k\rangle\langle\phi_k|)
\leq \sum_k |q_{1k}| C(\mathcal{E}_2)
\leq \mathcal{R}(\mathcal{E}_1) C(\mathcal{E}_2),
\]  

(42)

where to go from (40) to (41) we used the fact that since \(|\phi_k\rangle\) are stabiliser states, \(\mathcal{R}((\mathcal{E}_2 \otimes 1_n) |\phi_k\rangle\langle\phi_k|)\) can be no larger than \(C(\mathcal{E}_2)\). Finally, using the fact that \(\mathcal{R}(\mathcal{E}_1 \otimes 1) \leq C(\mathcal{E}_1)\), we have \(\mathcal{C}(\mathcal{E}_2 \otimes 1) \leq C(\mathcal{E}_2) C(\mathcal{E}_1)\), completing the proof.

**Submultiplicativity under tensor product:** \(C(\mathcal{E}_A \otimes \mathcal{E}_B) \leq C(\mathcal{E}_A) C(\mathcal{E}_B)\). This follows directly from submultiplicativity under composition, since

\[
C(\mathcal{E}_A \otimes \mathcal{E}_B) = C((\mathcal{E}_A \otimes 1_m) \circ (1_n \otimes \mathcal{E}_B)) \leq C(\mathcal{E}_A \otimes 1_m) C(1_n \otimes \mathcal{E}_B).
\]  

(43)

We saw in Section III that any gains in robustness achievable by tensoring \(\mathcal{E}_A\) with the identity and acting on a larger Hilbert space are already taken care of by the \(\otimes 1_n\) in the definition \(\mathcal{C}\), so that \(C(\mathcal{E}_A \otimes 1_m) = C(\mathcal{E}_A)\) and \(C(1_n \otimes \mathcal{E}_B) = C(\mathcal{E}_B)\). Substituting this into inequality \(43\) gives the desired result.

### A. Sandwich theorem

We will now prove Theorem 2 which stated that \(\mathcal{R}(\Phi_\mathcal{E}) \leq C(\mathcal{E}) \leq \mathcal{R}_*(\mathcal{E})\), for any CPTP channel \(\mathcal{E}\).

**Proof.** By definition \(\Phi_\mathcal{E} = (\mathcal{E} \otimes 1_n) |\Omega_n\rangle\langle\Omega_n|\). But \(|\Omega_n\rangle\) is a stabiliser state, so \(\mathcal{R}(\Phi_\mathcal{E})\) can be no larger than \(\mathcal{R}((\mathcal{E} \otimes 1_n) |\phi_s\rangle\langle\phi_s|) = C(\mathcal{E})\), where \(|\phi_s\rangle\) is the stabiliser state that achieves the capacity, and so:

\[
\mathcal{R}(\Phi_\mathcal{E}) \leq C(\mathcal{E}).
\]  

(44)

Now suppose \(\mathcal{E} = (1+p)\Lambda_+ - p\Lambda_-\) is the optimal decomposition of \(\mathcal{E}\) into CPTP stabiliser-preserving maps, \(\Lambda_+ \in \text{SP}_{n,n}\), so that \(\mathcal{R}_*(\mathcal{E}) = 1 + 2p\). Then for any input stabiliser state \(\sigma \in \text{STAB}_{2n}\), we can write down a valid stabiliser decomposition of the output state:

\[
(\mathcal{E} \otimes 1_n)\sigma = (1+p)(\Lambda_+ \otimes 1_n)\sigma - p(\Lambda_- \otimes 1_n)\sigma.
\]  

(45)

In particular this is true for the stabiliser state \(\sigma_* = |\phi_s\rangle\langle\phi_s|\) that is optimal with respect to the capacity. But equation \(45\) could be a non-optimal decomposition, so its \(\ell_1\)-norm \(1 + 2p\) is at least as large as \(\mathcal{R}((\mathcal{E} \otimes 1_n)\sigma_*)\). So:

\[
C(\mathcal{E}) = \mathcal{R}((\mathcal{E} \otimes 1_n)\sigma_*) \leq 1 + 2p = \mathcal{R}_*(\mathcal{E}),
\]  

(46)
VI. CLASSICAL SIMULATION ALGORITHMS

Here we propose two classical simulation algorithms. The channel robustness $R_*$ relates to the runtime of our first simulator, which we call the static simulator. The magic capacity $C$ relates to the runtime of the second simulator, called the dynamic simulator. In both cases, we consider a circuit composed from a sequence of channels with $\{E_1, E_2, \ldots, E_L\}$ acting on an initial stabiliser state, which we take to be $|0^n\rangle$. The circuit ends with some final state $\rho = E_L \circ \cdots \circ E_2 \circ E_1(|0^n\rangle \langle 0^n|)$ and measurement of some Pauli observable $Z$. We assume that each channel acts non-trivially on a bounded number of qubits (e.g. 2 or 3) so we can evaluate the relevant monotones. Our goal is to estimate the expectation value $\text{Tr}[Z\rho]$ to within additive error. In the language of Ref. [50] our simulators will be poly-boxes.

Both our algorithms are inspired by previous methods that collect a large number of Monte Carlo samples that scales quadratically with the negativity of some quasiprobability distribution [3, 33, 36]. The static Monte Carlo simulator uses a precomputed, and therefore static, quasiprobability distribution. The dynamic Monte Carlo simulator recomputes optimal quasiprobability distributions at each step, which can lead to fewer samples required but with a higher runtime per sample. As such, there are subtle trade-offs in the runtime complexities.

Both our algorithms use that completely stabiliser preserving operations $\text{SP}_{n,n}$ acting on a stabiliser state can be classically efficiently simulated. This follows from the fact that given the Choi state $\Phi_E$ for an $n$-qubit channel, the channel may be implemented by performing a Bell measurement on $\Phi_E \otimes \sigma$, postselecting on the $\Omega$ outcome to obtain $(\mathbb{1}_n \otimes |\Omega\rangle \langle \Omega|) \Phi_E \otimes \sigma (\mathbb{1}_n \otimes |\Omega\rangle \langle \Omega|)$ and then tracing out the last $2n$ qubits [51]. This can be simulated using Gottesman-Knill when $\sigma$ is a stabiliser state and $E \in \text{SP}_{n,n}$. Curiously, it is unclear whether $\text{SP}_{n,n}$ can be physically realised using Clifford unitaries and Pauli measurements but without the use of postselection.

A. Static Monte Carlo

Consider a circuit where for every channel $E_j$ we find that the optimal decomposition w.r.t the channel robustness has the form

$$E_j = (1 + p_j)E_{j,0} - p_jE_{j,1},$$

where optimality means that $R_*(E_j) = 2p_j + 1$. Recall that $E_{j,k}$ are CPTP and stabiliser-preserving. The output of the computation is

$$\rho = \sum_{\vec{k}} p_{\vec{k}} \sigma_{\vec{k}},$$

where $\vec{k} \in \mathbb{Z}_2^L$ is a vector denoting the different $k$ values for each channel, and we introduce the shorthand

$$p_{\vec{k}} = \prod_{j: k_j = 0} (1 + p_j) \prod_{j: k_j = 1} (-p_j),$$

and also

$$\sigma_{\vec{k}} = (E_{1,k_L} \cdots E_{2,k_2} \circ E_{1,k_1}) |0^n\rangle \langle 0^n|.$$
This decomposition is a quasi-probability distribution with $\ell_1$-norm
\[ ||p||_1 = \sum_{\vec{k}} |p_{\vec{k}}| = \prod_j R_*(E_j). \tag{51} \]

Given this decomposition we can build a simulator by sampling from the renormalised distribution $P(\vec{k}) := |p_{\vec{k}}|/||p||_1$. It is crucial that we can efficiently sample from the probability distribution $|p_{\vec{k}}|/||p||_1$, but this is ensured by it being a product distribution (recall Eq. (49)). For each sample, we get some value of $\vec{k}$ and a corresponding stabiliser state $\sigma_{\vec{k}}$. We wish to evaluate $T[Z\sigma_{\vec{k}}]$. Let us first consider the special case when $\sigma_{\vec{k}}$ is a pure stabiliser state and furthermore is pure throughout the whole Clifford circuit. Then we are able to find $\sigma_{\vec{k}}$ using the Gottesman-Knill simulation method and evaluate the expectation value, all in polynomial time. For each sample of the simulator, the output is this expectation value adjusted by a factor $\text{sign}(p_{\vec{k}})||p||_1$ where sign is $\pm 1$ depending on the sign of the quasiprobability. Following standard arguments \cite{33,36}, the expectation of this simulator equals the actual expectation value but the variance is too large. The variance of the mean of $N$ samples will reduce as we take more samples and using Hoeffding inequalities one can show that any constant variance can be achieved using a number of samples proportional to $||p||_1^2$, which is related to our monotone (see Eq. (51)).

The last part of the above argument assumed that $\sigma_{\vec{k}}$ is a pure stabiliser state throughout the circuit. More generally it will be mixed and additional sampling is required. This can be performed iteratively. Before the $j$th circuit element, we have some pure stabiliser state $|\phi_{j-1}\rangle$ where $|\phi_0\rangle = |0^n\rangle$. The value of $k_j$ determines whether we apply $E_{j,0}$ or $E_{j,1}$. Each of these channels decomposes into a finite number of stabiliser-preserving (though not necessarily trace-preserving) maps with a single Kraus operator, as follows:
\[ E_{j,k} = \sum_i T_{i,j,k}. \tag{52} \]

At the $j$th circuit element, we have to calculate $T[|T_{i,j,k}(\phi_{j-1}\rangle\langle\phi_{j-1}|)]$ for each $i$, but there are a finite and bounded number of such calculations, since each $T_{i,j,k}$ corresponds to a pure stabiliser state projector in the decomposition of the Choi state $\Phi_{E_{i,j,k}}$. Since $E_{j,k}$ is trace-preserving we know these numbers form a proper probability distribution that we can sample from. Having to evaluate these probabilities adds a small additional runtime cost per sample, which will be linear in the number of circuit elements where $E_{j,k}$ is non-unitary. However, since this is a proper probability distribution this additional sampling does not increase the sample complexity of the simulator.

Suppose we want to compare the runtime of our simulator for a particular circuit with that of the Bennink et al. algorithm \cite{3}. Their decompositions are in terms of CPR, the set of Cliffords and Pauli reset channels, and an associated cost function is
\[ R_{\text{CPR}}(E) = \min_{\Lambda_j \in \text{CPR}} \left\{ \sum_j p_j\Lambda_j = E \right\}, \tag{53} \]

The sample complexity for simulating a given circuit element $E_j$ is proportional to $R_{\text{CPR}}(E_j)^2$. Since CPR $\subseteq$ SP, it must be the case that $R_* \leq R_{\text{CPR}}$, potentially leading to lower simulation sample complexity if there exist channels with $R_* < R_{\text{CPR}}$. We give here a simple toy example demonstrating a significant advantage to our static simulator.

Consider the single-qubit CPTP map $\Lambda_H$ defined by a $Z$-measurement followed by a Hadamard gate conditioned on the “-1” outcome. This has Kraus representation:
\[ K_1 = |0\rangle\langle 0|, \quad K_2 = |\rangle\langle 1|. \tag{54} \]

This is clearly a completely stabiliser-preserving map, so has channel robustness $R_*(\Lambda_H) = 1$. For a single qubit, CPR consists of the 24 Clifford gates, and 6 Pauli reset channels. Using this
set, we calculate $R_{\text{CPR}}(\Lambda_H) = 2$. Since $\Lambda_H \in \text{SP}_{0,1}$, this confirms that CPR is a strict subset of the completely stabiliser-preserving channels, and indicates that $R_{\text{CPR}}$ is not a monotone under stabiliser operations. We also note that the calculated value is larger than the robustness of magic for any single-qubit state, despite $\Lambda_H$ being a stabiliser operation. For a circuit containing $M$ uses of the channel $\Lambda_H$, the samples required for a CPR simulator would be proportional to $R_{\text{CPR}}(\Lambda_H)^{2M} = 4^M$. But for our simulator, $\Lambda_H$ can be simulated efficiently, as $R_*(\Lambda_H)^{2M} = 1$.

While the above example is quite artificial, a reduction in sample complexity is also achieved for channels where $R_*(\mathcal{E}) > 1$, but is strictly smaller than $R_{\text{CPR}}(\mathcal{E})$. Given a circuit decomposed as $\mathcal{E} = \mathcal{E}_L \circ \cdots \circ \mathcal{E}_2 \circ \mathcal{E}_1$, the sample complexity for the CPR simulator would be proportional to $\prod_{j=1}^M R_{\text{CPR}}(\mathcal{E}_j)^2$. It is always the case that $R_*(\mathcal{E}_j) \leq R_{\text{CPR}}(\mathcal{E}_j)$, so the sample complexity for our simulator will never be greater. Suppose we find that there are $M$ circuit elements such that $R_*(\mathcal{E}_j)/R_{\text{CPR}}(\mathcal{E}_j) \leq k$ for some constant $0 < k < 1$. Then we would find that using our simulator gives a reduction in sample complexity by a factor of $k^{2M}$. While our simulator sometimes incurs a modest increase in the runtime per sample, this must be weighed against a reduction in sample complexity that is exponential in the number of circuit elements where $R_*(\mathcal{E}_j) < R_{\text{CPR}}(\mathcal{E}_j)$. The obvious next question is whether there are any natural non-trivial examples where this happens. We show in Section VII that gate sequences subject to amplitude-damping noise provide one such case.

We also note that calculation of optimal CPR decompositions is only tractable for one- and two-qubit circuit elements, as the three-qubit case already involves a linear program with nearly 93 million variables [3]. For the most general quantum channels, we encounter a similar problem, as for three-qubit circuit elements, we in principle need to optimise over six-qubit stabiliser states. However, in Appendix [E] we show that for diagonal channels the problem can be greatly simplified, and the problem becomes tractable for operations on up to five qubits. This allows our algorithm to take advantage of the submultiplicativity of channel robustness; for example for diagonal channels where $R(\mathcal{E}^\otimes n) < R(\mathcal{E})^n$, it is advantageous to compose $n$ single-qubit circuit elements together as a single $n$-qubit circuit element, before running the linear program. We will see in Section VII that this strategy is useful for single-qubit $Z$-rotations.

### B. Dynamic Monte Carlo

In the previous simulator, all convex optimisations are calculated in advance. However, we have found examples of channels where $C(\mathcal{E}) < R_*(\mathcal{E})$. For such channels, the robustness of the output state $\mathcal{E}(\rho)$ will always be less than the $\ell_1$-norm of the decomposition of $\mathcal{E}$ into stabiliser-preserving CPTP channels. Our next simulator takes advantage of this. We shall describe it in more detail than the static simulator since it departs further from the standard quasiprobability approach. We present the simulator as pseudocode in Figure [3].

As in the previous simulator, we can represent the trajectory through the circuit by a vector $\vec{k}$, such that the output of the true quantum circuit would be $\rho = \sum_k q_k \sigma_{\vec{k}}$. The major difference is that $q_k$ cannot be written in the form of equation (49) with $p_j$ independent of $\vec{k}$, as the quasiprobabilities for each intermediate decomposition will depend on the stabiliser state sampled in the previous step. Consider what happens for the $j$th circuit element. After step (d)ii. of the algorithm, we have some stabiliser state $\sigma_{\vec{k}_j}$, where $\vec{k}_j$ labels the trajectory up to the $j$th element. Here we do not assume $\vec{k}_j$ is a binary vector; instead the elements of the vector label each pure stabiliser state. After steps (d)iii. and iv. we have a non-stabiliser state $\rho_{\vec{k}} = (\mathcal{E}_j \otimes 1)(\sigma_{\vec{k}_j})$, decomposed as:

$$
\rho_{\vec{k}_j} = \sum_{\vec{k}_{j+1}} q_{\vec{k}_{j+1}} \sigma_{\vec{k}_{j+1}}.
$$

(55)
Input: A sequence of $L$ quantum channels $\mathcal{E}_j$ and number of samples $N$.

Output: An estimate of an expectation value.

1. Set $i \leftarrow 0$ and $T \leftarrow 0$;

2. For $i \leq N$ do:
   (a) $i \leftarrow i + 1$;
   (b) Set $|\phi_0\rangle = |0\rangle^\otimes n$;
   (c) $R \leftarrow 1$;
   (d) For $1 \leq j \leq L$ do
      i. The channel $\mathcal{E}_j$ acts non-trivially on only $m$ qubits. Partition the qubits into three sets $A|B|C$ where $A$ is the set acted on by $\mathcal{E}_j$, $B$ is a set of any other $m$ qubits; and $C$ comprises the remaining qubits;
      ii. Find a Clifford $U = 1_A \otimes U_{B,C}$ that is local w.r.t $A|(B \cup C)$ such that $U |\phi_{j-1}\rangle = |\phi_{AB}^{j-1}\rangle \otimes |\phi_{j-1}^C\rangle$. This uses the efficient algorithm of Ref. [49].
      iii. Find $2m$-qubit density matrix $\rho_{AB}^{j-1} = \mathcal{E}_j \otimes 1 |\phi_{AB}^{j-1}\rangle \langle \phi_{AB}^{j-1}|$;
      iv. Solve convex optimisation to find $\mathcal{R}(\rho_{AB}^{j-1})$ and use optimal decomposition to build a quasiprobability distribution;
      v. Sample from the renormalised quasiprobability distribution to choose a stabiliser state $|\phi_{AB}^{j-1}\rangle$, and then set $|\phi_j\rangle = U^\dagger |\phi_{AB}^{j-1}\rangle \otimes |\phi_{j-1}^C\rangle$;
      vi. Replace $R \leftarrow R \times \mathcal{R}(\rho_{AB}^{j-1}) \times \lambda$ where $\lambda = \pm 1$ and denotes the phase of the sampled quasiprobability.
      vii. increment $j \leftarrow j + 1$ and loop;
   (e) Evaluate $E = \langle \phi_L | Z | \phi_L \rangle$
   (f) $T \leftarrow T + (R \times E)$.

Return: $T/N$.

FIG. 3. A classical simulator with sample complexity upper-bounded by $C$.

Here the summation is over all $(j+1)$-step trajectories consistent with the previous $j$-step trajectory labelled by $\vec{k}_j$. In steps (d)v. and vi. we will choose the stabiliser state $\sigma_{\vec{k}_{j+1}}$ with probability $|q_{\vec{k}_{j+1}}|/\mathcal{R}(\rho_{\vec{k}_j})$ and the variable $R$ picks up a factor $\lambda_{\vec{k}_{j+1}} \mathcal{R}(\rho_{\vec{k}_j})$, where $\lambda_{\vec{k}_{j+1}}$ is the sign of the corresponding quasiprobability.

The final state after the full sequence of quantum operations may be written:

$$\rho = \sum_{\vec{k}} p_{\vec{k}} R_{\vec{k}} \sigma_{\vec{k}}, \quad \text{where} \quad p_{\vec{k}} = \prod_{j=0}^{L-1} \frac{|q_{\vec{k}_{j+1}}|}{\mathcal{R}(\rho_{\vec{k}_j})}, \quad R_{\vec{k}} = \prod_{j=0}^{L-1} \text{sign}(q_{\vec{k}_{j+1}}) \mathcal{R}(\rho_{\vec{k}_j}). \quad (56)$$
The true expectation value for the observable $Z$ is given by:

$$\text{Tr}[Z\rho] = \sum_k p_k R_k E_k, \quad E_k = \text{Tr}[Z\sigma_k].$$  \hspace{1cm} (57)

A key difference with the previous simulator is that we never explicitly calculate the full distribution $p_k$. Nevertheless, each time the simulator samples, it produces output $R_k E_k$ with probability $p_k$. These probabilities exactly match the weightings in the above equation, so the simulator is an unbiased estimator. The number of samples required can be again derived using the Hoeffding inequalities, which depend on the maximum possible values of the output of each sample. Each output is bounded by $|R_k E_k| \leq |R_k| \leq \prod_j \mathcal{R}(\rho_{kj}) \leq \prod_j \mathcal{C}(\mathcal{E}_j)$. Therefore, the sample complexity is upper-bounded by order $\prod_j \mathcal{C}(\mathcal{E}_j)^2$.

Notice that for every sample, $L$ convex optimisations are performed, as well as $L$ steps involving the algorithm of Fattal et al. \cite{49}, which has runtime polynomial in the total number of qubits. If $\mathcal{C}(\mathcal{E}) = \mathcal{R}_s(\mathcal{E})$ then we would simply not use this method so that the only convex optimisations are in the preprocessing. However, $\mathcal{C}(\mathcal{E})$ now determines the sample complexity, so if $\mathcal{C}(\mathcal{E}) \ll \mathcal{R}_s(\mathcal{E})$ then the dynamic simulator may run much faster than the static simulator; here we have a trade-off of increase in per-sample runtime, versus a possibly exponential reduction in sample complexity. Indeed, since the sample complexity is typically the bottleneck, this approach would lead to significant improvements for some quantum channels. In the following section we investigate which types of channels may lead to an advantage.

**VII. NUMERICAL RESULTS**

**A. Single-qubit rotation with amplitude damping**

Consider the setting discussed in Section VI where a many-qubit circuit evolution $\mathcal{E}$ is decomposed as a series of few-qubit circuit elements $\mathcal{E} = \mathcal{E}_L \circ \cdots \circ \mathcal{E}_2 \circ \mathcal{E}_1$. Many implementations of quantum algorithms can be expected to involve single-qubit rotations about some Pauli axis. In near-term devices, the circuit will be subject to noise. Consider a simple model of a noisy computation where a noise channel $\Lambda$ acts between each unitary gate $U_j$, so the overall channel representing the circuit would be:

$$\mathcal{E} = \Lambda \circ U_L \circ \cdots \circ \Lambda \circ U_2 \circ \Lambda \circ U_1.$$  \hspace{1cm} (58)

Here we study the simulation cost for a single step in such a computation, comprised of a single-qubit rotation and a noise channel. Note that for intermediate steps in a circuit decomposition such as (58), we have a choice of ordering. We can take the circuit elements to be either $\Lambda \circ U_j$ or $U_j \circ \Lambda$. These choices are equivalent in terms of the output of the simulation, but could lead to different sample complexity depending on the cost function used. We studied circuit elements made up of a single-qubit Pauli $X$-rotation $U(\theta) = \exp(iX\theta)$ composed with an amplitude damping channel $\Lambda_p$ with noise parameter $p$, defined by Kraus operators:

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} (59)

We calculated channel robustness $\mathcal{R}_s$, magic capacity $\mathcal{C}$, and the Bennink et al. \cite{3} cost function $\mathcal{R}_{\text{CPR}}$, for both $\Lambda_p \circ U(\theta)$ and $U(\theta) \circ \Lambda_p$, and for a range of values of $p$ and $\theta$. For noise $p = 0.1$ we see that when the noise channel follows the gate, there is no difference between the three quantities (Figure 4, left panel). However, if the noise channel acts before the unitary, both our monotones show a reduced value, whereas $\mathcal{R}_{\text{CPR}}$ increases. This suggests that for this noise model, the better
strategy with respect to sample complexity would be to choose the ordering \( U(\theta) \circ \Lambda_p \), and use one of our simulators (subject to the caveats mentioned in the previous section). In Figure 4 (middle panel) we show how different levels of noise affect the channel robustness. We do not plot capacity since we find that \( \mathcal{R}_s(\mathcal{E}) = \mathcal{C}(\mathcal{E}) \) for this class of operation. We also compare channel robustness with the Choi state robustness (Figure 4, right panel). We find that \( \mathcal{R}(\Phi_\mathcal{E}) = \mathcal{R}_s(\mathcal{E}) \) for \( \theta \) up to approximately \( \pi/16 \), but \( \mathcal{R}(\Phi_\mathcal{E}) < \mathcal{R}_s(\mathcal{E}) \) for larger angles.

**FIG. 4.** Left: Comparison of \( \mathcal{C}(\mathcal{E}) \), \( \mathcal{R}_s(\mathcal{E}) \) and \( \mathcal{R}_{CP_R}(\mathcal{E}) \), where \( \mathcal{E} \) is a single-qubit \( X \)-rotation \( U(\theta) \) composed with an amplitude damping channel \( \Lambda_p \). We consider both possible orderings: noise after unitary \( (\Lambda_p \circ U(\theta)) \), and noise before unitary \( (U(\theta) \circ \Lambda_p) \). Middle: \( \mathcal{R}_s(U(\theta) \circ \Lambda_p) \) for several values of \( p \). Right: Comparison of channel robustness with robustness of Choi state for \( U(\theta) \circ \Lambda_p \) with \( p = 0.1 \).

## B. Multiqubit phase gates

Recall that from Theorem 2 we know \( \mathcal{R}(\Phi_\mathcal{E}) \leq \mathcal{C}(\mathcal{E}) \leq \mathcal{R}_s(\mathcal{E}) \). For the particular class of channel studied above, we saw \( \mathcal{C}(\mathcal{E}) = \mathcal{R}_s(\mathcal{E}) \) in all cases, and in the absence of noise we have \( \mathcal{R}(\Phi_\mathcal{E}) = \mathcal{C}(\mathcal{E}) = \mathcal{R}_s(\mathcal{E}) \). Under what conditions does this equality persist for multi-qubit operations? As explained earlier, to calculate each of our quantities for \( n \)-qubit channels, in general we must solve an optimisation problem over all \( 2^n \)-qubit stabiliser states. Since this problem is only tractable for up to 5-qubit states, in practice we are limited to studying two-qubit channels, in the most general case. However, it turns out the problem can be greatly simplified for certain types of operation. In particular, Appendix E shows how the problem size can be reduced for channels diagonal in the computational basis. This allows us to calculate values for diagonal operations up to 5 qubits, which we present here.

As a special case we consider multicontrol phase gates of the form:

\[
M_{t,n} = \text{diag}(\exp(i\pi/2^t), 1, 1, \ldots, 1), \quad t \in \mathbb{Z}
\]  

(60)

where \( n \) denotes the number of qubits. We note that by convention, controlled-phase gates typically apply the phase to the all-one state \( |1^n\rangle \), where \( 1^n = (1, \ldots, 1)^T \), but the form given above is Clifford-equivalent to the conventional version, and will be more convenient in the arguments of Appendix E. The family includes familiar gates such as \( CZ \) \((t = 0, n = 2)\), \( CCZ \) \((t = 0, n = 3)\), multicontrol-\( S \) \((t = 1)\) and multicontrol-\( T \) \((t = 2)\).

The main findings were that the inequalities are tight for the \( n = 2 \) and \( n = 3 \) cases, but that this does not persist for larger system sizes (Figure 5). The \( t = 0 \) case (the family of multicontrol-\( Z \) gates) turns out to be a special case (Figure 5, left panel). Here we find equality for all three quantities up to \( n = 4 \). For the \( t = 0, n = 5 \) case, \( \mathcal{R}(\Phi_{M_{0,5}}) = \mathcal{C}(M_{0,5}) \) holds, but \( \mathcal{R}_s(M_{0,5}) \) is strictly greater than both. Note also that for \( t = 0 \), all three quantities increase with each increment in \( n \).
FIG. 5. Comparison of quantities for multicontrol phase gates (see equation [60]). Middle: Multicontrol-S gates (t=1). Right: Multicontrol-T gates (t=2).

FIG. 6. Channel robustness against robustness of Choi state for random n-qubit diagonal gates, up to n = 4. Black line indicates \( R^\ast = R(\Phi) \). Each red dot represents the data point for an individual gate. Fewer points were calculated for larger n due to the increased time to calculate each value. 1000 data points were calculated for n = 2, 300 for n = 3, and 60 for n = 4.

The families of gates with \( t > 0 \) follow a pattern qualitatively similar to each other. The results for the \( t = 1 \) (multicontrol-S) and \( t = 2 \) (multicontrol-T) cases are shown in the middle and right panels of Figure 5. For \( n = 4, t > 0 \), the same situation holds as for \( n = 5, t = 0 \), as we find \( R(\Phi_{M_t,4}) = C(M_{t,4}) < R^\ast(M_{t,4}) \). At \( n = 5 \), all three quantities separate. In contrast with the multicontrol-Z, we see that \( R(\Phi_{M_{t,n}}) \) decreases as we go from four to five qubits, while \( C(M_{t,n}) \) levels off. We see similar behaviour for all non-zero values of \( t \) investigated numerically. Our current techniques limit us to five-qubit gates, but we have reason to believe that the capacity will remain level for \( n > 5 \), and we make the following conjecture, which we justify more fully in Appendix E 4.

**Conjecture 1.** For any fixed \( t \), the maximum increase in robustness of magic for \( M_{t,n} \) is achieved at some finite number of qubits \( n = K \) by acting on the state \( |+\rangle^{\otimes K} \). Therefore \( C(M_{t,n}) = R(M_{t,K} |+\rangle^{\otimes K}) \) for all \( n \geq K \).

We also numerically investigated the robustness of diagonal unitaries

\[
U = \sum_x e^{i\theta_x} |x\rangle\langle x|,
\]

with \( \theta_x \) chosen uniformly at random. We were particularly interested in understanding when
\( \mathcal{R}(\Phi) \leq C(\mathcal{E}) \leq \mathcal{R}_*(\mathcal{E}) \) is tight or loose. In Figure 6 we compare the Choi robustness with the channel robustness. For every 2-qubit gate tested we observed that \( \mathcal{R}(\Phi) = \mathcal{R}_*(\mathcal{E}) \) up to numerical precision. Whereas, for 3 and 4 qubit gates we typically saw that \( \mathcal{R}(\Phi) < \mathcal{R}_*(\mathcal{E}) \), though the gap is not often large. While the difference is slight for a single gate, these quantities influence the rate of exponential scaling when considering \( N \) uses of such a unitary and will lead to a large gap for modest \( N \).

We also compared the Choi robustness with the magic capacity but do not plot this data as it was equal for every random instance we observed. This is curious since in Figure 5 we clearly see that there do exist diagonal gates, the multicontrol phase gates, for which there is a gap between the Choi robustness and the magic capacity. While such gates exist, our random sampling of diagonal gates does not tend to provide such examples. We discuss this further in Appendix E 4.

Finally, we are also interested in the regularised channel robustness for single-qubit gates \( U \), defined as \( \left[ \mathcal{R}_*(U \otimes n) \right]^{1/n} \). This allows us to quantify the per-gate savings in sample complexity that can be achieved by grouping single-qubit rotations in \( n \)-qubit blocks. In Figure 7 we present results for qubit Z-rotations \( U(\theta) = \exp[iZ\theta] \), up to four qubits. We find that strict submultiplicativity is observed for all values of \( \theta \), with significant reductions between the \( n = 2 \) and \( n = 4 \) cases for a wide range of angles.

![Graph](image)

**FIG. 7.** Regularised channel robustness \( \left[ \mathcal{R}_*(U \otimes n) \right]^{1/n} \) plotted for Z-rotations \( U(\theta) = \exp[iZ\theta] \) and for \( n \) qubits, up to \( n = 4 \).

**VIII. CONCLUSIONS**

We have presented two new magic monotones for general quantum channels: the magic capacity \( \mathcal{C} \), which quantifies the ability of a channel to generate magic, and the channel robustness \( \mathcal{R}_* \), which is related to finding the minimal quasiprobability decomposition of a channel into stabiliser-preserving CPTP maps. Each of these monotones is directly related to the sample complexity for an associated Monte Carlo-type classical simulation algorithm. We found that for certain quantum channels, our static simulator would lead to an exponentially better sample complexity as compared to that for the algorithm due to Bennink et al. [3]. In particular we found a reduction in sample complexity for the case of a sequence of single-qubit rotations subject to amplitude-damping noise. Since our decompositions can be calculated for up to five qubits in the case of diagonal operations, our static simulator is also able to take advantage of the submultiplicativity of channel robustness under tensor product and composition.

For some channels, further improvements in sample complexity are possible using a different simulator that is related to the capacity. That is, we found the capacity can be strictly less
than channel robustness for certain multi-qubit entangling gates. This simulator has to introduce on-the-fly convex optimisation, however, so each sample will be more difficult to obtain.

For simulation of realistic quantum devices involving many qubits, one would need to decompose the circuit into a sequence of operations on smaller number of qubits, as described in Section VI in order that the associated optimisation problems are tractable. It is a non-trivial problem to decide what is the optimal way to block together the few-qubit operations making up a given many-qubit circuit: we saw in Section VII that the ordering of operations can make a difference to the sample complexity. We leave this problem for a future work.

Since our monotones are submultiplicative under tensor product and compositions, it is generally preferable to combine circuit elements where possible. In practice, to calculate our monotones, circuit elements can involve up to two qubits for the most general case, or at most five qubits for diagonal operations, where we can make use of the techniques described in Section E. In Ref. [44], Heinrich and Gross show that robustness of magic can be calculated for up to 10 copies of the resource states employed in the standard magic state model of fault-tolerant quantum computation. Their methods rely on both the permutation symmetry due to having multiple copies, and the stabiliser symmetries of the magic states considered. Another direction for future work could therefore be to investigate whether (and under what conditions) similar techniques can be applied to the channel picture to increase the number of qubits that can be involved in each circuit element.

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Appendix A: Properties of robustness of the Choi state

Here we confirm that the robustness of the Choi state, $R(\Phi)$ has the properties convexity and submultiplicativity under tensor product. We then give an example to show that it is not submultiplicative under composition.

**Convexity:** This follows immediately from convexity of robustness of magic. Consider a real
linear combination of $n$-qubit channels: $\mathcal{E} = \sum_k q_k \mathcal{E}_k$. The Choi state for $\mathcal{E}$ is:

\[
\Phi_{\mathcal{E}} = (\mathcal{E} \otimes \mathbb{1}_n) |\Omega_n\rangle\langle\Omega_n| = \sum_k q_k (\mathcal{E}_k \otimes \mathbb{1}_n) |\Omega_n\rangle\langle\Omega_n| = \sum_k q_k \Phi_{\mathcal{E}_k},
\]

where in the last line we identified $(\mathcal{E}_k \otimes \mathbb{1}_n) |\Omega\rangle\langle\Omega|$ as the Choi state for $\mathcal{E}_k$. Then by convexity of robustness of magic:

\[
\mathcal{R}(\Phi_{\mathcal{E}}) \leq \sum_k |q_k| \mathcal{R}(\Phi_{\mathcal{E}_k}),
\]

which shows $\mathcal{R}(\Phi_{\mathcal{E}})$ is convex in $\mathcal{E}$.

**Submultiplicativity under tensor product:** The maximally entangled state $|\Omega_{n+m}\rangle$ as defined by equation (14) can be factored as $|\Omega_{n+m}\rangle_{AA'BB'} = |\Omega_n\rangle_{A}\langle\Omega_m|_{A'}$. So the Choi state for a channel $\mathcal{E}_{AA'} = \mathcal{E}_A \otimes \mathcal{E}_{A'}$, where $\mathcal{E}_A$ and $\mathcal{E}_{A'}$ are respectively $n$-qubit and $m$-qubit channels, can be written:

\[
\Phi_{\mathcal{E}} = (\mathcal{E}_A \otimes \mathcal{E}_{A'} \otimes \mathbb{1}_{n+m}) |\Omega_{n+m}\rangle\langle\Omega_{n+m}|_{AA'BB'} = (\mathcal{E}_A \otimes \mathbb{1}_n) |\Omega_n\rangle\langle\Omega_n|_{AB} \otimes (\mathcal{E}_{A'} \otimes \mathbb{1}_m) |\Omega_m\rangle\langle\Omega_m|_{A'B'} = \Phi_{\mathcal{E}_A} \otimes \Phi_{\mathcal{E}_{A'}},
\]

Then by submultiplicativity of robustness of magic for states, we have:

\[
\mathcal{R}(\Phi_{\mathcal{E}_A \otimes \mathcal{E}_{A'}}) \leq \mathcal{R}(\Phi_{\mathcal{E}_A}) \mathcal{R}(\Phi_{\mathcal{E}_{A'}}),
\]

which is the desired property.

**Failure of submultiplicativity under composition:** Let $\mathcal{E}_1$ be the single-qubit $Z$-reset channel defined by Kraus operators $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and let $\mathcal{E}_2$ be the conditional channel defined by $\{|T\rangle\langle 0|, |1\rangle\langle 1|\}$, where $|T\rangle = T|+\rangle$. These channels respectively have Choi states $\Phi_{\mathcal{E}_1} = |0\rangle\langle 0| \otimes \frac{1}{2}$, and $\Phi_{\mathcal{E}_2} = \frac{1}{2}(|T\rangle\langle 0| + |1\rangle\langle 1|)$, with robustness of magic $\mathcal{R}(\Phi_{\mathcal{E}_1}) = 1$ and $\mathcal{R}(\Phi_{\mathcal{E}_2}) \approx 1.207$.

The composed channel $\mathcal{E}_2 \circ \mathcal{E}_1$ has a Kraus representation $\{|T\rangle\langle 0|, |T\rangle\langle 1|\}$, and so has a Choi state $\Phi_{\mathcal{E}_2 \circ \mathcal{E}_1} = |T\rangle\langle T| \otimes \frac{1}{2}$, with $\mathcal{R}(\Phi_{\mathcal{E}_2 \circ \mathcal{E}_1}) \approx 1.414 > \mathcal{R}(\Phi_{\mathcal{E}_2}) \mathcal{R}(\Phi_{\mathcal{E}_1})$. So it is not the case that the robustness of the Choi state is submultiplicative under composition.

More intuitively, such counterexamples arise for channels $\mathcal{E}$ where the stabiliser state $|\phi_+\rangle$ that results in maximal final robustness $\mathcal{R}(\Phi_{\mathcal{E} \otimes \mathbb{1}_n}) |\phi_+\rangle\langle \phi_+|$ is not the maximally entangled state $|\Omega_n\rangle$, as then we can always boost the output robustness by using a stabiliser-preserving operation to prepare $|\phi_+\rangle$ before applying $\mathcal{E}$.

**Appendix B: Properties of channel robustness**

**Faithfulness:** Suppose $\mathcal{E}$ is an $n$-qubit CPTP map. There are two cases:

(i) $\mathcal{E} \in \text{SP}_{n,n}$. In this case, $\Phi_{\mathcal{E}}$ is itself a mixed stabiliser state, and since $\mathcal{E}$ is trace-preserving it satisfies $\text{Tr}(\Phi_{\mathcal{E}}) = \mathbb{1}_n/2^n$. So $\Phi_{\mathcal{E}}$ is already trivially a decomposition of the correct form, with $p = 0$, so that $\mathcal{R}_*(\mathcal{E}) = 1 + 2p = 1$.

(ii) $\mathcal{E} \notin \text{SP}_{n,n}$. Then by Theorem 2, $\Phi_{\mathcal{E}}$ has $\mathcal{R}(\Phi_{\mathcal{E}}) > 1$. Since the definition of $\mathcal{R}_*$ is a restriction of $\mathcal{R}(\Phi)$, it must be the case that $\mathcal{R}_*(\Phi_{\mathcal{E}}) \leq \mathcal{R}_*(\mathcal{E})$. Therefore $\mathcal{R}_*(\mathcal{E}) > 1$.

**Convexity:** Suppose we have a set of Choi states $\Phi_{\mathcal{E}_j}$ corresponding to channels $\mathcal{E}_j$, with optimal decompositions:

\[
\Phi_{\mathcal{E}_j} = (1 + p_j)\rho_{j+} - p_j\rho_{j-},
\]

where each $\rho_{j\pm}$ separately satisfies the condition $\text{Tr}_A(\rho_{\pm}) = \frac{1}{2n}$, so that $\mathcal{R}_*(\mathcal{E}_j) = 1 + 2p_j$. Now
take a real linear combination of such channels:

$$\mathcal{E} = \sum_i q_i \mathcal{E}_i = \sum_{j \in P} q_j \mathcal{E}_j + \sum_{k \in N} q_k \mathcal{E}_k,$$

(B2)

where $P$ is the set of indices such that $q_j \geq 0$, and $N$ is the set such that $q_k < 0$. We assume that $\sum_i q_i = 1$ so that the trace of $\text{Tr}(\Phi_E) = 1$. Then the corresponding Choi state for channel $\mathcal{E}$ is:

$$\Phi_E = \sum_{j \in P} q_j \Phi_{\mathcal{E}_j} - \sum_{k \in N} |q_k| \Phi_{\mathcal{E}_k}$$

(B3)

$$= \sum_{j \in P} q_j [(1 + p_j)\rho_{j+} - p_j\rho_{j-}] - \sum_{k \in N} |q_k| [(1 + p_k)\rho_{k+} - p_k\rho_{k-}]$$

(B4)

$$= \left( \sum_{j \in P} q_j (1 + p_j)\rho_{j+} + \sum_{k \in N} |q_k| p_k\rho_{k-} \right) - \left( \sum_{j \in P} q_j p_j\rho_{j-} + \sum_{k \in N} |q_k|(1 + p_k)\rho_{k+} \right).$$

(B5)

Note that the terms inside the brackets all have positive coefficients, hence we can interpret as non-normalised mixtures over stabiliser states. To normalise them we can define:

$$\tilde{\rho}_+ = \frac{\sum_{j \in P} q_j (1 + p_j)\rho_{j+} + \sum_{k \in N} |q_k| p_k\rho_{k-}}{\sum_{j \in P} q_j (1 + p_j) + \sum_{k \in N} |q_k| p_k}$$

(B6)

and

$$\tilde{\rho}_- = \frac{\sum_{j \in P} q_j p_j\rho_{j-} + \sum_{k \in N} |q_k|(1 + p_k)\rho_{k+}}{\sum_{j \in P} q_j p_j + \sum_{k \in N} |q_k|(1 + p_k)}.$$ 

(B7)

Then writing:

$$\tilde{p} = \sum_{j \in P} q_j p_j + \sum_{k \in N} |q_k|(1 + p_k),$$

(B8)

one can check that:

$$1 + \tilde{p} = \sum_{j \in P} q_j (1 + p_j) + \sum_{k \in N} |q_k| p_k.$$ 

(B9)

This allows us to rewrite the Choi state as: $\Phi_E = (1 + \tilde{p})\tilde{\rho}_+ - \tilde{p}\tilde{\rho}_-$. Since $\tilde{\rho}_\pm$ are convex mixtures over stabiliser states satisfying $\text{Tr}_A(\rho_{j\pm}) = \frac{1}{2^n}$, they must satisfy the same condition. We also know that $\tilde{p} \geq 0$, so it is clear that the decomposition is in the form required for the definition of $\mathcal{R}_*$, except that it is not necessarily optimised to minimise $1 + 2\tilde{p}$. So we have:

$$\mathcal{R}_* \left( \sum_j q_j \mathcal{E}_j \right) \leq 1 + 2\tilde{p} = \sum_{j \in P} q_j (1 + p_j) + \sum_{k \in N} |q_k| + \sum_{j \in P} q_j p_j + \sum_{k \in N} |q_k|(1 + p_k)$$ 

(B10)

$$= \sum_{j \in P} |q_j|(1 + 2p_j) + \sum_{k \in N} |q_k|(1 + 2p_k)$$

(B11)

$$= \sum_i |q_i| \mathcal{R}_*(\mathcal{E}_j),$$

(B12)

which gives us the required result.

**Invariance under tensor with identity:** In Section IV we saw that $\mathcal{R}_*(\mathcal{E}_A \otimes 1) \leq \mathcal{R}_*(\mathcal{E}_A)$. We
now complete the proof that $\mathcal{R}_*(\mathcal{E}_A \otimes \mathbb{1}) = \mathcal{R}_*(\mathcal{E}_A)$ by showing that $\mathcal{R}_*(\mathcal{E}_A) \leq \mathcal{R}_*(\mathcal{E}_A \otimes \mathbb{1})$.

Consider an optimal decomposition for $\Phi_{\mathcal{E}_A \otimes \mathbb{1}_m} = (1 + p')\rho'_+ - p'\rho'_-$, such that $\mathcal{R}_*(\mathcal{E}_A \otimes \mathbb{1}_m) = 1 + 2p'$, where $\operatorname{Tr}_{A'A'}(\rho_{\pm}) = \mathbb{1}_{n+m}/2^{n+m}$. Here we do not assume that $\rho'_{\pm}$ are products across the partition $AB|A'B'$, as was the case in equation (30). However, we have just seen that $\Phi_{\mathcal{E}_A \otimes \mathbb{1}_m}$ can be written as a product, so that by tracing out systems $A'B'$ we obtain:

$$\Phi_{\mathcal{E}_A} = (1 + p')\operatorname{Tr}_{A'B'}(\rho'_+) - p'\operatorname{Tr}_{A'B'}(\rho'_-). \quad (\text{B13})$$

Partial trace of a stabiliser state remains a stabiliser state, so this is a stabiliser decomposition. We just need to check that the partial trace condition holds, so we want to show:

$$\operatorname{Tr}_A(\operatorname{Tr}_{A'B'}(\rho'_+)) = \operatorname{Tr}_{AA'B'}(\rho'_+) = \frac{\mathbb{1}_n}{2^m}, \quad (\text{B14})$$

but this is clearly the case from the fact that $\rho'_{\pm}$ were constrained such that $\operatorname{Tr}_{A'A'}(\rho_{\pm}) = \mathbb{1}_{n+m}/2^{n+m}$. Hence again we have a valid, not necessarily optimal decomposition and:

$$\mathcal{R}_*(\mathcal{E}_A) \leq 1 + 2p' = \mathcal{R}_*(\mathcal{E}_A \otimes \mathbb{1}_B). \quad (\text{B15})$$

Combining with inequality (31) we obtain the equality given in equation (28).

**Appendix C: Optimisation problem for channel robustness**

In Howard and Campbell [36], the optimisation problem for calculating robustness of magic for states was cast as follows:

$$\begin{align*}
\text{minimise} & \quad \|\vec{q}\|_1 \\
\text{subject to} & \quad A\vec{q} = \vec{b},
\end{align*}$$

where $\vec{q}$ is a vector of coefficients, $\vec{b}$ is the vector of Pauli expectation values for the target state $\Phi_{\mathcal{E}}$, and $A$ is a matrix whose columns are the Pauli vectors for the stabiliser states. For $n$-qubit channels, we have $2n$-qubit Choi states, so the number of generalised Paulis is $N_P = 4^{2n}$, and the number of stabiliser states is $N_S = 2^{2n}\prod_{j=1}^{2n}(2^j + 1)$ [39]. Then $\vec{b}$ has $N_P$ entries, $\vec{q}$ has $N_S$ entries, and the dimension of $A$ is $(N_P \times N_S)$. From this construction we can recover optimal decompositions of the form: $\Phi_{\mathcal{E}} = \sum_j q_j |\phi_j\rangle\langle\phi_j|$, where $\sum_j q_j = 1$ and $|\phi_j\rangle$ are the pure stabiliser states.

We want to restrict the problem to decompositions of the form:

$$\Phi_{\mathcal{E}} = (1 + p)\rho_+ - pp_- \quad (C1)$$

where $p \geq 0$ and $\rho_{\pm}$ correspond to trace-preserving channels, and can in general be mixed. Rather than enumerating all the extreme points of the set of stabiliser states corresponding to maps in $\text{SP}_{n,n}$, it is more convenient to retain the same $A$ matrix and modify the constraints. We still need to start from a finite set of extreme points, i.e. pure stabiliser states, so first rewrite as:

$$\Phi_{\mathcal{E}} = \sum_j q_{j+}\rho_j + \sum_j q_{j-}\rho_j = \sum_j p_{j+}\rho_j - \sum_j p_{j-}\rho_j \quad (C2)$$

where $q_{j+}$ are the positive quasiprobabilities, $q_{j-}$ are the negative quasiprobabilities, and $p_{j\pm} = |q_{j\pm}|$. In the Pauli vector picture we can write this as $\vec{b} = A\vec{p}_+ - A\vec{p}_-$, where all the entries of $\vec{p}_\pm$ are non-negative. We define a new variable vector $\vec{p}$ which will have twice the length of the
previous \( \vec{q} \), i.e. \( 2N_S \) entries:

\[
\vec{p} = \begin{pmatrix} \vec{p}_+ \\ \vec{p}_- \end{pmatrix}
\]  

(C3)

and define a new \((N_P \times 2N_S)\) matrix \( A' \) in block form, \( A' = (A - A) \). Then we have:

\[
A' \vec{p} = (A - A) \begin{pmatrix} \vec{p}_+ \\ \vec{p}_- \end{pmatrix} = A\vec{p}_+ - A\vec{p}_- = \vec{b}.
\]  

(C4)

So now we need to minimise \( \|\vec{p}\|_1 = \sum_j p_j \) subject to \( A' \vec{p} = \vec{b} \) and \( \vec{p} \geq 0 \).

Next, we need the trace-preserving condition. Provided \( E \) is CPTP, if one part of the decomposition is trace-preserving, then the other will be as well, so we only need enforce the constraint on one of \( \rho_+ \) or \( \rho_- \). Assume that we check \( \rho_+ \). The condition for a Choi state \( \Phi_{AB} = E_A \otimes 1_B (\ket{\Omega}_{AB} \bra{\Omega}) \) to be trace-preserving is:

\[
\text{Tr}_A(\Phi_{AB}) = \frac{1}{d},
\]  

(C5)

where \( d \) is the dimension of the subsystem. We need to convert this to a constraint on the vector \( \vec{b}_+ \) corresponding to \( \phi_+ \), which is given by \( \vec{b}_+ = A\vec{p}_+ \). First, note that all Paulis are traceless except for the identity \( P_0 = 1 \), so for the maximally mixed state:

\[
\bra{P_j} = \text{Tr} \left( P_j \frac{1}{d} \right) = \frac{\text{Tr}(P_j)}{d} = \delta_{j,0},
\]  

(C6)

so if the first entry in a Pauli vector is always \( (1) \), the maximally mixed state has Pauli vector:

\[
\vec{b}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]  

(C7)

where \( 0 \) is the zero vector. However, we need this to hold just for the reduced state on \( B \) rather than the full Pauli vector. Consider that if the whole state is written \( \Phi_{AB} = \sum_{j,k} r_{j,k} P_j \otimes P_k \) for some set of coefficients \( r_{j,k} \), then the expectation values are given by:

\[
\bra{P_l \otimes P_m} = \sum_{j,k} r_{j,k} \text{Tr}(P_l P_j \otimes P_m P_k) = \sum_{j,k} r_{j,k} d^2 \delta_{j,l} \delta_{m,k} = d^2 r_{l,m}.
\]  

(C8)

The reduced state is:

\[
\text{Tr}_A(\Phi_{AB}) = \sum_{j,k} r_{j,k} \text{Tr}_A[P_j \otimes P_k] = \sum_{j,k} r_{j,k} d \delta_{j,0} P_k = d \sum_k r_{0,k} P_k.
\]  

(C9)

and the entries of the reduced Pauli vector will be:

\[
\bra{P_m} = d \sum_k r_{0,k} \text{Tr}_A\{P_m P_k\} = d^2 r_{0,m} = \bra{P_0 \otimes P_m}.
\]  

(C10)

So for condition (C5) to hold for the reduced state on \( B \), we combine equations (C6) and (C10) to get:

\[
\bra{P_m} = \bra{P_0 \otimes P_m} = \delta_{m,0}.
\]  

(C11)

That is, we just need to look at the entries of \( \vec{b}_+ \) corresponding to Paulis of the form \( 1 \otimes P_j \). These
should all be zero except the first entry, which corresponds to \(\langle 1 \otimes 1 \rangle\). Note that \(\vec{b}_+ = A\vec{p}_+\) will in general not be normalised, but this does not matter, since we are only interested in whether or not entries are zero. We can use a binary matrix \(M\) to pick out the values of interest. As an example we consider the two-qubit case, and assume that the entries are ordered as:

\[
\vec{b}_+ = \begin{pmatrix}
    \langle 1 \otimes 1 \rangle \\
    \langle 1 \otimes X \rangle \\
    \langle 1 \otimes Y \rangle \\
    \langle 1 \otimes Z \rangle \\
    \langle X \otimes 1 \rangle \\
    \vdots \\
    \langle Z \otimes Z \rangle \\
\end{pmatrix}.
\]

\[\text{(C12)}\]

Here, we are only interested in the 2nd, 3rd and 4th entries. We form a new vector \(\vec{c}\) by left multiplying with \(M\):

\[
\vec{c} = M\vec{b}_+ = \begin{pmatrix}
    0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
    0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
    0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
    \langle 1 \otimes X \rangle \\
    \langle 1 \otimes Y \rangle \\
    \langle 1 \otimes Z \rangle \\
\end{pmatrix}.
\]

\[\text{(C13)}\]

Then the condition we need is just \(\vec{c} = 0\). To convert this to a condition on the \(2N_S\)-entry variable \(\vec{p} = \begin{pmatrix} \vec{p}_+ \\ \vec{p}_- \end{pmatrix}\), we first pad \(A\) with zeroes: \(A_+ = (A \ 0)\), where \(0\) is the \((NP \times N_S)\) zero matrix. We then have:

\[
\vec{b}_+ = A\vec{p}_+ + \vec{0}\vec{p}_- = (A \ 0) \begin{pmatrix} \vec{p}_+ \\ \vec{p}_- \end{pmatrix} = A_+\vec{p},
\]

\[\text{(C14)}\]

so that \(\vec{c} = M\vec{b}_+ = MA_+\vec{p}\). Therefore, our condition for trace-preserving \(\rho_+\) is \(MA_+\vec{p} = 0\). We can therefore specify the new optimisation problem as:

minimise \(\|\vec{p}\|_1 = \sum_j p_j\)

subject to \(A'\vec{p} = \vec{b},\)

\(\vec{p} \geq 0,\)

\(MA_+\vec{p} = 0\)

where \(A' = (A - A),\) and \(A_+ = (A \ 0),\) with \(A\) and \(\vec{b}\) having the same definitions as previously, \(0\) is the zero matrix with dimension the same as \(A\), and with \(M\) being the binary matrix that picks out the \(\langle 1 \otimes P_j \rangle\) entries from the vector \(A_+\vec{p}\). Most of this is straightforward to implement. The step that requires some care is in correctly constructing the matrix \(M\), as it will depend on the choice of ordering of Pauli operators in the construction of \(A\) and \(\vec{b}\). If the \(B\) subsystem has \(n\) qubits, then we will need to constrain \(4^n - 1\) non-trivial \(\langle 1 \otimes P_j \rangle\) expectation values to zero, so \(M\) should have dimension \(((4^n - 1) \times NP)\). If the Paulis are ordered as in the example given above for 2-qubit Choi states, then the construction is just \(M = \begin{pmatrix} 0 & 1' & 0 & \cdots & 0 \end{pmatrix},\) where \(1'\) is the \(((4^n - 1) \times (4^n - 1))\) identity, and \(0\) denotes a column of zeroes.
Appendix D: Properties of magic capacity

**Faithfulness:** For any $n$-qubit stabiliser-preserving CPTP channel $\Lambda$, if $\rho \in \text{STAB}_{2n}$ is a stabiliser state, then $(\Lambda \otimes 1_n)\rho$ is also a stabiliser state. So by the faithfulness of robustness of magic, $\mathcal{R}((\Lambda \otimes 1_n)\rho) = 1$ for any input stabiliser state $\rho \in \text{STAB}_{2n}$, and $\mathcal{C}(\Lambda) = 1$.

Suppose instead that $\mathcal{E}$ is non-stabiliser-preserving, but still CPTP. Then there exists at least one stabiliser state $\rho \in \text{STAB}_{2n}$ such that $(\mathcal{E} \otimes 1_n)\rho$ is a normalised state, but not a stabiliser state. Then by faithfulness of $\mathcal{R}$ when applied to states, $\mathcal{R}((\mathcal{E} \otimes 1_n)\rho) > 1$, and so $\mathcal{C}(\mathcal{E}) > 1$.

**Convexity:** Suppose we have a real linear combination of $n$-qubit CPTP maps $\mathcal{E}_k$:

$$\mathcal{E} = \sum_k q_k \mathcal{E}_k. \tag{D1}$$

There exists some optimal stabiliser state $\rho_*$ that achieves $\mathcal{C}(\mathcal{E}) = \mathcal{R}(\mathcal{E} \otimes 1(\rho_*))$. Then

$$\mathcal{R}((\mathcal{E} \otimes 1_n)\rho_*) = \mathcal{R}\left(\sum_k q_k [(\mathcal{E}_k \otimes 1_n)\rho_*]\right) \leq \sum_k |q_k| \mathcal{R}((\mathcal{E}_k \otimes 1_n)\rho_*), \tag{D2}$$

where the last line follows by convexity of the robustness of magic. But each robustness $\mathcal{R}((\mathcal{E}_k \otimes 1_n)\rho_*)$ can be no larger than $\mathcal{C}(\mathcal{E}_k)$. So we have:

$$\mathcal{C}\left(\sum_k q_k \mathcal{E}_k\right) \leq \sum_k |q_k| \mathcal{C}(\mathcal{E}_k). \tag{D4}$$

Appendix E: Calculating monotones for diagonal channels

1. Reducing the problem size

As mentioned earlier, the size of the optimisation problem for calculating our monotones (as well as $\mathcal{R}(\Phi_{\mathcal{E}})$) quickly becomes prohibitively large for $n$-qubit states, since the number of stabiliser states increases super-exponentially with $n$ (Table II). The issue is even worse than it first appears, since for an $n$-qubit channel we must in general consider $2n$-qubit stabiliser states. Direct calculation of either monotone is impractical for $n$-qubit channels with $n > 2$. This difficulty is aggravated when calculating the capacity as in principle we have to repeat the optimisation for every $(\mathcal{E} \otimes 1_n)\ket{\phi}\bra{\phi}$ such that $\ket{\phi} \in \text{STAB}_{2n}$. In some cases we can ameliorate these problems by looking for Clifford gates that commute with the channel of interest. Here we consider the case where $\mathcal{E}$ is a diagonal channel, meaning it has a Kraus representation where each Kraus operator

| $n$ | $N_S$ |
|-----|------|
| 1   | 6    |
| 2   | 60   |
| 3   | 1,080|
| 4   | 36,720|
| 5   | 2,423,520|
| 6   | 315,057,600|

TABLE II. Number of pure stabiliser states $N_S$ for number of qubits $n$.
is diagonal in the computational basis. This of course includes diagonal unitaries as a special case. One could likely reduce the problem size further by exploiting symmetries of channels using techniques similar to those used in Ref. [44], but we will not consider this strategy here.

It is straightforward to see how the problem can be simplified for calculating $\mathcal{R}(\Phi_E)$ and $\mathcal{R}_s(\mathcal{E})$. If $\mathcal{E}$ is diagonal, the operation $\mathcal{E} \otimes \mathbb{I}_n$ commutes with any sequence of CNOTs targeted on the last $n$ qubits. But the maximally entangled state $|\Omega_n\rangle$ can be written:

$$|\Omega_n\rangle = U_C(|+\rangle^\otimes n \otimes |0\rangle^\otimes n).$$

(E1)

Here $U_C = \otimes_{j=1}^n U_j$, where $U_j$ is the CNOT controlled on qubit $j$ and targeted on qubit $n+j$. By the monotonicity of robustness of magic, we immediately see that:

$$\mathcal{R}(\Phi_E) = \mathcal{R}[(\mathcal{E} \otimes \mathbb{I}_n)|\Omega_n\rangle\langle\Omega_n|] = \mathcal{R}[\mathcal{E}(|+\rangle\langle+|^\otimes n) \otimes |0\rangle\langle0|^\otimes n] = \mathcal{R}[\mathcal{E}(|+\rangle\langle+|^\otimes n)].$$

(E2)

For the channel robustness we would like to decompose $\mathcal{E}(|+\rangle\langle+|^\otimes n)$ in terms of states $\rho_{\pm} \in \text{STAB}_n$, but need to take care that the trace condition $\text{Tr}_A(\rho_{\pm}^\prime) = \mathbb{I}_n/2^n$ is satisfied for the equivalent $2n$-qubit Choi states $\rho_{\pm}^\prime$. In Appendix E 2 we show that the criterion is satisfied provided all diagonal elements of $\rho_{\pm}$ are equal to $1/2^n$. So for diagonal channels we can write:

$$\mathcal{R}_s(\mathcal{E}) = \min_{\rho_{\pm} \in \text{STAB}_n} \left\{ 1 + 2p : (1 + p)\rho_+ - p\rho_- = \mathcal{E}(|+\rangle\langle+|^\otimes n), p \geq 0, \langle x|\rho_{\pm}|x\rangle = \frac{1}{2^n}, \forall x \right\}.$$  

(E3)

So calculation of $\mathcal{R}_s(\mathcal{E})$ and $\mathcal{R}(\Phi_E)$ is tractable up to five qubits provided $\mathcal{E}$ is diagonal. We show in Appendix E 3 this is also true for the magic capacity.

2. Trace condition for diagonal channels

Consider that the Choi state for a diagonal channel has a decomposition

$$\Phi_E = U_C(\mathcal{E}(|+\rangle\langle+|^\otimes n) \otimes |0\rangle\langle0|^\otimes n)U_C^\dagger = (1 + p)\rho_+ - p\rho_-,$$

(E4)

where $U_C = \otimes_{j=1}^n U_j$ is the tensor product of CNOTs $U_j$ that are controlled on the $j$th qubit and targeted on the $n+j$th. Then

$$\mathcal{E}(|+\rangle\langle+|^\otimes n) \otimes |0\rangle\langle0|^\otimes n = (1 + p)\rho_+^\prime - p\rho_-^\prime,$$

(E5)

where $\rho_{\pm}^\prime$ are still stabiliser states since $U_C$ is Clifford. Now consider the stabiliser-preserving channel $\mathbb{I}_n \otimes \Lambda$ that resets the last $n$ qubits to $|0\rangle\langle0|^\otimes n$. Applying this to both sides of equation (E5) we get a new decomposition

$$\mathcal{E}(|+\rangle\langle+|^\otimes n) \otimes |0\rangle\langle0|^\otimes n = (1 + p)\rho_+'' \otimes |0\rangle\langle0|^\otimes n - p\rho_-'' \otimes |0\rangle\langle0|^\otimes n.$$  

(E6)

Then referring back to equation (E4), we obtain $\rho_{\pm} = U_C(\rho_{\pm}'' \otimes |0\rangle\langle0|^\otimes n)U_C^\dagger$. So, the trace-preserving condition becomes:

$$\frac{1}{2^n} = \text{Tr}_A(\rho_{\pm}) = \text{Tr}_A \left( U_C(\rho_{\pm}'' \otimes |0\rangle\langle0|^\otimes n)U_C^\dagger \right)$$  

(E7)

$$= \sum_x \langle x|A U_C(\rho_{\pm}'' \otimes |0\rangle\langle0|^\otimes n)U_C^\dagger |x\rangle_A,$$  

(E8)
where $|x\rangle$ are the computational basis states on subsystem $A$. Recalling that $U_C$ can be written as a tensor product of CNOTs $U_C = \otimes_{j=1}^{n} U_j$ one can check that this equation can be written:

$$\frac{1}{2^n} = \sum_x \langle x|_A \rho''_\pm |x\rangle_A |x\rangle_B. \quad (E9)$$

Therefore, the decomposition corresponds to a pair of trace-preserving channels provided that all diagonal elements of $\rho''_\pm$ are equal to $1/2^n$.

3. Magic capacity in the affine space picture

In this section we will make use of the formalism due to Dehaene and De Moor, in which stabiliser states are cast in terms of affine spaces and quadratic forms over binary vectors [52, 53], to prove the following theorem:

**Theorem 3** (Capacity for diagonal operations). Suppose the $n$-qubit channel $E_D$ is diagonal. Let:

$$|K\rangle = \frac{1}{|K|^{1/2}} \sum_{x \in K} |x\rangle, \quad (E10)$$

where $x \in \mathbb{F}_2^n$ are binary vectors and $K \subseteq \mathbb{F}_2^n$ is an affine space. Then:

$$C(E_D) = \max_K R(E_D(|K\rangle\langle K|)). \quad (E11)$$

That is, given an $n$-qubit channel $E$, provided the channel is diagonal, the capacity $C(E)$ may be calculated by optimisation over only the $n$-qubit states $|K\rangle$ as defined in equation (E10), rather than over all $2^n$-qubit stabiliser states.

We first review the formalism of Ref. [52]. Computational basis states $|x\rangle$ can be labelled by binary column vectors $x = (x_1, \ldots, x_n)^T \in \mathbb{F}_2^n$, so that $x_j \in \{0, 1\}$ relates to the $j$th qubit. Any pure $n$-qubit stabiliser state may be written:

$$|\mathcal{K}, q, d\rangle = \frac{1}{|\mathcal{K}|^{1/2}} \sum_{x \in \mathcal{K}} i^{d^T x} (-1)^q(x) |x\rangle, \quad (E12)$$

where $\mathcal{K} \subseteq \mathbb{F}_2^n$ is an affine space, $d$ is some fixed binary vector, and $q(x)$ has the form:

$$q(x) = x^T Q x + \lambda^T x. \quad (E13)$$

Here $Q$ is a binary, strictly upper triangular matrix, $\lambda$ is a vector, and addition is modulo 2. Conversely, any state that can be written in this way is a stabiliser state.

An affine space $\mathcal{K}$ is a linear subspace $\mathcal{L}$ shifted by some constant binary vector $h$, modulo 2: $\mathcal{K} = \mathcal{L} + h$. Every affine space is related in this way to exactly one linear subspace, and the dimension $k = \dim(\mathcal{K})$ of an affine space means the dimension of the corresponding subspace. Instead of enumerating all elements of an affine space, we can specify it by a shift vector $h$ and an $n \times k$ matrix where each column is one of the generators of the corresponding linear space:
Lemma 3 \(\text{(Equivalences for diagonal channels)}\). Suppose \(\mathcal{E}_D\) is a diagonal CPTP channel. Then:

1. All input stabiliser states with the same affine space \(\mathcal{K}\) result in the same final robustness:
   \[
   \mathcal{R}((\mathcal{E}_D \otimes \mathbb{1}) |\mathcal{K}, q, d\rangle\langle\mathcal{K}, q, d|) = \mathcal{R}((\mathcal{E}_D \otimes \mathbb{1}) |\mathcal{K}, q', d'\rangle\langle\mathcal{K}, q', d'|), \forall q, q', d, d'.
   \] (E16)

2. Given a 2n-qubit state \(|\phi\rangle \in \text{STAB}_{2n}\), there exists some n-qubit \(|\phi'\rangle \in \text{STAB}_n\) such that:
   \[
   \mathcal{R}((\mathcal{E}_D \otimes \mathbb{1}_n) |\phi\rangle\langle\phi|) = \mathcal{R}(\mathcal{E}_D(|\phi'\rangle\langle\phi'|)).
   \] (E17)

Proof. We first prove statement 1. Since robustness of magic is invariant under Clifford unitaries, we need to show that there exists a Clifford unitary \(U\) that converts \((\mathcal{E}_D \otimes \mathbb{1}) |\mathcal{K}, q, d\rangle\langle\mathcal{K}, q, d|\) to \((\mathcal{E}_D \otimes \mathbb{1}) |\mathcal{K}, q', d'\rangle\langle\mathcal{K}, q', d'|\). A suitable choice for \(U\) is one such that \(U |\phi_{K,q,d}\rangle = |\phi_{K,q',d'}\rangle\), and, crucially, that commutes with the channel \(\mathcal{E}_D\). Since \(\mathcal{E}_D\) is given to be diagonal, any diagonal Clifford \(U\) will suffice. The affine space \(\mathcal{K}\) remains unchanged, so we only need show there is always a diagonal Clifford that maps \(q \rightarrow q'\) and \(d \rightarrow d'\) for any \(q, q', d, d'\). That this is always possible is perhaps already evident from Ref. [52], but for completeness we give the argument here.

We can convert \(d\) to \(d'\) using appropriately chosen \(S_j\) gates, meaning the gate \(\text{diag}(1, i)\) acting on the \(j\)th qubit. Consider the action of \(S_j\) on a basis vector:

\[
S_j |x\rangle = \begin{cases} 
|x\rangle & \text{if } x_j = 0 \\
 i^T |x\rangle & \text{if } x_j = 1 
\end{cases}.
\] (E18)

If we define basis vector \(e_j\) so that it has 1 in the \(j\)th position and zeroes elsewhere, we can write the action of \(S_j\) as:

\[
S_j |x\rangle = i^T e_j \otimes e_j |x\rangle.
\] (E19)
Note that the form of this equation is independent of the value of $x$, so we can write:

\[
S_j |\phi_{K,q,d}\rangle = \frac{1}{|K|^{1/2}} \sum_{x \in K} i^{d^T x} (-1)^{q(x)} S_j |x\rangle
\]

(E20)

\[
= \sum_{x \in K} i^{(d^T + e_j^T) x} (-1)^{q(x)} S_j |x\rangle.
\]

(E21)

So, we can flip any bit of $d$ by applying the correct $S$ gate. The quadratic form $q(x)$ is left unchanged.

Now consider $q(x) = x^T Q x + \lambda^T x$, which we must convert to some other $q'(x) = x^T Q' x + \lambda'^T x$. We can use the same trick as above to convert any $\lambda$ to any other $\lambda'$, by replacing $S_j$ with the $Z_j$ gate, i.e. $\text{diag}(1, -1)$ acting on the $j$th qubit. For $Q$ we can use the controlled-$Z$ gate between the $j$th and $k$th qubit, which we denote $CZ_{jk}$. This has the following effect on a basis state:

\[
CZ_{jk} |x\rangle = (-1)^{x^T M_{jk} x} |x\rangle,
\]

(E22)

where $M_{jk}$ is the $n \times n$ matrix with a 1 in position $(j, k)$ and zeroes everywhere else. The set of all \{M_{jk}\} form a basis for $n \times n$ binary matrices, hence we can convert any $Q$ to any other $Q'$ by an appropriately chosen sequence of $CZ$ gates, leaving $d$ and $\lambda$ untouched. This completes the proof of statement 1.

Now to prove statement 2. From statement 1 any stabiliser state $|\phi\rangle$ is equivalent to:

\[
|K\rangle = \frac{1}{|K|^{1/2}} \sum_{x \in K} |x\rangle,
\]

(E23)

up to some diagonal Clifford, for some $K$. The strategy is to find a Clifford unitary $U$ that commutes with $E_D$, and converts the $2n$-qubit stabiliser state $|K\rangle$ to some product of two $n$-qubit states $|K'\rangle = |K'_A\rangle \otimes |K'_B\rangle$. Then we have:

\[
\mathcal{R}[E_D \otimes 1_n |K\rangle |K\rangle] = \mathcal{R}[(E_D \otimes 1_n)(|K'_A\rangle |K'_B\rangle \otimes |K'_A\rangle |K'_B\rangle)]
\]

\[
= \mathcal{R}[E_D (|K'_A\rangle |K'_A\rangle) \otimes |K'_B\rangle |K'_B\rangle] = \mathcal{R}[E_D (|K'_A\rangle |K'_A\rangle)]
\]

(E24)

where the last step follows as $|K'_B\rangle$ is a stabiliser state so makes no contribution to the robustness. The final state $|K'\rangle$ can be factored as $|K'_A\rangle \otimes |K'_B\rangle$ provided its generator $G'$ can be written in block matrix form as:

\[
G' = \begin{pmatrix} G'_A & 0 \\ 0 & G'_B \end{pmatrix},
\]

(E25)

where $G'_A$ and $G'_B$ have $n$ rows, and represent the generators for affine spaces $K'_A$ and $K'_B$.

We now show that we can always reach this form by a Clifford $U_C$ comprised of a sequence of CNOTs targeted on the last $n$ qubits. Such a sequence always commutes with $E_D \otimes 1_n$. Suppose we have some $2n \times k$ generator $G$ for an affine space $K$ with $k = \dim(K)$:

\[
G = \begin{pmatrix} G_A \\ G_B \end{pmatrix},
\]

(E26)

where $G_A$ and $G_B$ are each $n \times k$ submatrices. The full matrix $G$ will have rank $k$, and $G_A$ will have some rank $m \leq k$. Either $G_A$ is already full rank ($m = k$), or it can be reduced to the following form by elementary column operations, which is equivalent to multiplication on the right
by a $k \times k$ matrix $S$:

$$G_A \rightarrow G_A S = (G'_A \ 0), \quad (E28)$$

where $G'_A$ is $n \times m$ (and hence full column rank), and $0$ is $n \times (k - m)$. Multiplying $G$ on the right by $S$, we interpret as a change in the choice of generating set:

$$G \rightarrow GS = \begin{pmatrix} G_A S \\ G_B S \end{pmatrix} = \begin{pmatrix} G'_A \\ 0 \end{pmatrix} = \begin{pmatrix} G'_A \\ G'_B \end{pmatrix}. \quad (E29)$$

Now, apply the Clifford $U_C$ described by the matrix $C$ in equation (E15). This transforms the generator to:

$$G' = CGS = \begin{pmatrix} \mathbf{1} & 0 \\ M & 1 \end{pmatrix} \begin{pmatrix} G'_A \\ G'_B \end{pmatrix} = \begin{pmatrix} G'_A \\ MG'_A + G'_B \end{pmatrix}. \quad (E30)$$

Note that if $G_A$ was already full rank, the change of generating set is not necessary. If we can set the bottom-left submatrix to zero, then $U_C |K\rangle$ can be factored as described above. This is possible if there exists a binary matrix $M$ such that $MG'_A = G''_B$. But $G'_A$ has full column rank $m$, so there exists an $m \times n$ left-inverse $G''_A,\text{left}$ such that $G''_A,\text{left}G'_A = 1$, where $1$ is $m \times m$. Then we can set $M = G''_B G''_A,\text{left}$, so that:

$$MG'_A = G''_B G''_A,\text{left} = G''_B 1 = G''_B. \quad (E31)$$

Then $G' = CGS$ is in the form (E20), so $U_C |K\rangle = |K'\rangle \otimes |K''\rangle$, as required.

Lemma 3 shows that if $E_D$ is diagonal then for any 2n-qubit stabiliser state $|\phi\rangle$ we have that $\mathcal{R}((E_D \otimes 1_n) |\phi\rangle) = \mathcal{R}(E_D (|K\rangle\langle K|))$ for some $n$-qubit affine space $K$. This shows that the capacity can be calculated by maximising over just the representative states $|K\rangle$, proving Theorem 3. Table III illustrates the reduction in problem size. For example, whereas naively for a two-qubit channel we would need to calculate robustness for all 36,720 four-qubit stabiliser states, using the result above we only need check one stabiliser state for each of the 7 non-trivial affine spaces. Cases up to five qubits are now tractable using this method.

| $n$ | Stabiliser states | Total affine spaces | Non-trivial affine spaces |
|-----|------------------|---------------------|---------------------------|
| 2   | 60               | 11                  | 7                         |
| 3   | 1,080            | 51                  | 43                        |
| 4   | 36,720           | 307                 | 291                       |
| 5   | 2,423,520        | 2451                | 2419                      |

TABLE III. Number of $n$-qubit stabiliser states compared with number of affine spaces. By trivial affine spaces we mean those comprised of a single element, which correspond to computational basis states. Diagonal CPTP channels act as the identity on such states.

4. Dimension of affine space

Here we make further observations that will help interpret numerical results from Section VII.

Observation 1 (Dimension of affine space limits achievable robustness). Suppose $U$ is a diagonal unitary acting on $n$ qubits, and suppose $|\mathcal{K}\rangle$ is a stabiliser state associated with some affine space $\mathcal{K}$, $k = \dim(\mathcal{K})$. Then $\mathcal{R}(U |\mathcal{K}\rangle) = \mathcal{R}(U' |\phi\rangle)$ where $U' |\phi\rangle$ is a state on only $k$ qubits, and $U'$ is
some k-qubit unitary. Therefore $R(U \mid \mathcal{K})$ is upper-bounded by the maximum robustness achievable for a k-qubit state.

Proof. We prove the result by showing that there is a sequence of Clifford gates that takes $U \mid \mathcal{K}$ to the product of a k-qubit state and an $(n - k)$-qubit stabiliser state. We know from Lemma 3 that for diagonal unitaries, all states with same affine space result in the same robustness, so it is enough to consider the state:

$$|\mathcal{K}\rangle = \frac{1}{\sqrt{|\mathcal{K}|}} \sum_{x \in \mathcal{K}} |x\rangle. \quad (E32)$$

A diagonal unitary will map this to:

$$U |\mathcal{K}\rangle = \frac{1}{\sqrt{|\mathcal{K}|}} \sum_{x \in \mathcal{K}} e^{i\theta_x} |x\rangle, \quad (E33)$$

where $\{e^{i\theta_x}\}$ will be some subset of the diagonal elements of $U$. The affine space $\mathcal{K}$ will have a generator matrix of rank $k$. As we saw in Lemma 3, a sequence of elementary row operations on the generator matrix can be realised by a sequence of CNOT gates. So we can use Clifford gates to transform any rank $k$ generator matrix as:

$$G \rightarrow G' = AG = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (E34)$$

where $1$ is the $k \times k$ identity. Each element of $\mathcal{K}$ can be written $x = \sum_j g_j + h$, where $\sum_j g_j$ is some combination of columns of $G$, and $h$ is a fixed shift vector. The transformation $A$ corresponds to a sequence of CNOTs that we collect in a single Clifford unitary $U_A$, that acts on $n$-qubit computational basis states $|x\rangle$, where $x \in \mathcal{K}$, as follows:

$$U_A |x\rangle = |y(x)\rangle \otimes |h'\rangle, \quad (E35)$$

where $h'$ is an $(n - k)$-length vector, and $y(x)$ is a $k$-length vector given by:

$$\begin{pmatrix} y(x) \\ h' \end{pmatrix} = Ax = \sum_j Ag_j + Ah. \quad (E36)$$

Note that $y(x)$ is only defined for $x \in \mathcal{K}$, and that $h'$ is independent of $x$. Elements $x \in \mathbb{F}_2^n$ that are not in $\mathcal{K}$ could be mapped to a vector where the last $n - k$ bits are not $h'$, but these never appear as terms of $U |\mathcal{K}\rangle$. Since $U_A$ must preserve orthogonality, each $|x\rangle$, where $x \in \mathcal{K}$, maps to a distinct element of the k-qubit basis set $\{|y\rangle\}$. In fact, since $y$ are length $k$ and there are $2^k$ distinct elements, they must form the $k$-bit linear space $\mathcal{L}' = \mathbb{F}_2^k$. So we can write:

$$U_A U |\mathcal{K}\rangle = \frac{1}{\sqrt{|\mathcal{L}'|}} \sum_{y \in \mathcal{L}'} e^{i\theta_y} |y\rangle \otimes |h'\rangle \quad (E37)$$

$$= (U' |\mathcal{L}'\rangle) \otimes |h'\rangle, \quad (E38)$$

where $|\mathcal{L}'\rangle$ is a $k$-qubit stabiliser state, and $U'$ is the $k$-qubit diagonal unitary with $e^{i\theta_y(x)} = e^{i\theta_x}$ as the non-zero elements. The state $|h'\rangle$ is a stabiliser state, so cannot contribute to the robustness of $U_A U |\mathcal{K}\rangle$, and therefore $R(U |\mathcal{L}(\mathcal{K})) = R(U_A U |\mathcal{L}(\mathcal{K})) = R(U' |\mathcal{L}'\rangle)$, where $U' |\mathcal{L}'\rangle$ is a $k$-qubit state. \hfill \square

Recall that in Section VII, we found that highly structured examples of diagonal unitaries $U$ exist where $\mathcal{C}(U)$ is strictly larger than $\mathcal{R}(\Phi_U)$, whereas for all the random diagonal unitaries
sampled, we found them to be exactly equal. We can now explain this by a concentration effect, in conjunction with Observation \[1\]. The \( n \)-qubit random diagonal gates concentrate (with high probability) within a narrow range of values for the magic capacity, close to the maximum possible magic capacity for an \( n \)-qubit diagonal gate. If \( \mathcal{R}(\Phi_U) < C(U) \) then by Thm. \[3\] we must have that \( C(U) = R(U |K⟩⟨K| U^\dagger) \) for some affine space \( K \) of non-maximal dimension. However, \( U |K⟩⟨K| U^\dagger \) is Clifford equivalent to an \((n-1)\)-qubit stabiliser state acted on by a diagonal unitary. Then \( \mathcal{R}(U |K⟩⟨K| U^\dagger) \) would be upper bounded by the maximum \( C(\mathcal{E}) \) for \((n-1)\)-qubit diagonal unitaries. But if \( C(\mathcal{E}) \) is close to the maximum possible for \( n \)-qubit diagonal unitaries, then it is impossible for \( U |K⟩⟨K| U^\dagger \) to achieve the magic capacity.

Finally, we consider the special case of multi-control phase gates \( M_{t,n} \), as defined in equation \[60\]. Note that the gate \( M_{t,n} \) acts as the identity on states \( |K⟩ \) unless \( K \) contains the zero vector \( 0^n = (0, \ldots, 0)^T \), so if \( 0^n \notin K \), we get \( R(M_{t,n} |K⟩) = 1 \). But if \( 0^n \in K \), then \( K \) is a linear subspace. So for this type of gate, to find all possible values of \( R(M_{t,n} |K⟩) \) > 1 we need only consider linear subspaces. The following theorem implies that we actually only need solve one optimisation for each possible dimension of linear subspace rather than one for every linear subspace.

**Theorem 4.** Consider the \( n \)-qubit gate \( M_{t,n} \) defined by equation \[60\], and let \( \mathcal{L}_A \) and \( \mathcal{L}_B \) be linear subspaces such that \( \dim(\mathcal{L}_A) = \dim(\mathcal{L}_B) = k \). Then:

\[
\mathcal{R}(M_{t,n} |\mathcal{L}_A⟩) = \mathcal{R}(M_{t,n} |\mathcal{L}_B⟩).
\]

**Proof.** We largely repeat the arguments of Observation \[1\] for the special case where the phases are given by:

\[
\theta_x = \begin{cases} 
\pi/2^t & \text{if } x = 0^n \\
0 & \text{otherwise} 
\end{cases}
\]

(E40)

Since \( \dim(\mathcal{L}_A) = \dim(\mathcal{L}_B) \), their generator matrices \( G_A \) and \( G_B \) have the same rank. It follows from the arguments of Observation \[1\] that there exists an invertible \( C \), corresponding to a sequence of CNOT gates, such that \( G_B = CG_A \), and \( |\mathcal{L}_A⟩ = U_C |\mathcal{L}_A⟩ \), where \( U_C \) is a unitary Clifford operation.

If we consider instead the state \( M_{t,n} |\mathcal{L}_A⟩ \), which involves terms in the same basis vectors as \( |\mathcal{L}_A⟩ \), we just need to track what happens to the phase \( \exp(i\theta_0) \). Clearly, since any CNOT acts as the identity on \( |0^n⟩ \), we obtain:

\[
U_C M_{t,n} |\mathcal{L}_A⟩ = \frac{1}{2^k\sqrt{2}} \sum_{x \in \mathcal{L}_B} \exp(i\theta_x) |x⟩ = M_{t,n} |\mathcal{L}_B⟩
\]

(E41)

Since \( U_C \) is a reversible Clifford operation, by monotonicity of robustness of magic, equation (E39) follows.

From Theorem \[4\] then, to find \( C(M_{t,n}) \), we only need calculate \( \mathcal{R}(M_{t,n} |\mathcal{L}⟩) \) for a single representative subspace for each possible value of \( \dim(\mathcal{L}) \). Recall that for \( n \)-qubit stabiliser states \( |\mathcal{L}⟩ \), \( k = \dim(\mathcal{L}) \) can take integer values from 0 to \( n \). The states with \( k = 0 \) correspond to single computational basis states without superposition, so are unaffected by phase gates. That is, for \( n \)-qubit multicontrol phase gates we only have to calculate \( n \) robustnesses. Compare this to the number of optimisation problems we would need to solve without using the above observations (Table \[III\]).

We can go further. From Observation \[1\] we know that for a subspace with \( \dim(\mathcal{L}) = k < n \), it must be the case that \( M_{t,n} |\mathcal{L}⟩ \) is Clifford-equivalent to \((U′ |\mathcal{L}′⟩) ⊗ |h⟩′\) for the \( k \)-qubit state \( |\mathcal{L}′⟩ \) and \((n-k)\)-qubit computational basis state \( |h⟩′\), and some diagonal \( k \)-qubit unitary \( U′ \). By inspection of the phases given by equation (E40), \( U′ \) can only be the \( k \)-qubit multicontrol gate \( M_{t,k} \). This leads to the following statement:
Observation 2 \((n\text{-qubit multicontrol gates})\). For any fixed \(t\) and \(n\text{-qubit state }|\mathcal{L}\rangle\) where \(\dim(|\mathcal{L}\rangle) = k < n\), we have:

\[
R(M_{t,n} |\mathcal{L}\rangle) = R(M_{t,k} |\mathcal{L}'\rangle)
\]  

(E42)

where \(|\mathcal{L}'\rangle\) is the \(k\text{-qubit state with }|\mathcal{L}'\rangle = \mathbb{F}_2^k\).

| Linear subspace dimension, \(k\) | Number of qubits, \(n\) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1.414 | 1.414 | 1.414 | 1.414 |
| 2 | 1.849 | 1.849 | 1.849 | 1.849 |
| 3 | - | 2.195 | 2.195 | 2.195 |
| 4 | - | - | 2.264 | 2.264 |
| 5 | - | - | - | 2.195 |

TABLE IV. Final robustness after multicontrol-\(T\) gate applied to input stabiliser states \(|\mathcal{L}\rangle\) with \(k = \dim(|\mathcal{L}\rangle)\). In each column, the maximum robustness (i.e. the capacity) is highlighted red.

Observation 2 partially justifies our Conjecture 1 in Section VII that for fixed \(t\), the maximum increase in robustness achievable for \(M_{t,n}\), over any \(n\), is given by \(R\left(M_{t,K} |+\rangle^\otimes K\right)\), for some finite number of qubits \(K\). To unpack this claim further, let us consider the maximisation over input stabiliser states performed to calculate the capacity \(C\). In this appendix, we have seen that for the family of gates \(M_{t,n}\), we only need to calculate robustness for one representative input stabiliser state for each possible dimension of linear subspace; that is, for \(M_{t,n}\) there are only \(n\) robustnesses to calculate. In Table IV we present the relevant values for the family of multicontrol-\(T\) gates \((t = 2)\) and make two observations. First, looking across the rows of Table IV notice that the values for fixed \(k\) are constant with \(n\), assuming \(k \leq n\). Indeed, this is a generic feature of the \(M_{t,n}\) gates as formalised by Observation 2. Second, looking down the last column of Table IV we see that up until \(k = 4\), \(R(M_{t,n} |\mathcal{L}\rangle)\) increases with \(\dim(|\mathcal{L}\rangle)\), but at \(k = 5\) the value drops. With a little thought we can see that this is necessarily the case if \(R(\Phi_{M_{t,5}}) < C(M_{t,5})\); we saw earlier that for diagonal gates \(U\) the Choi state robustness is equal to \(R(U |+\rangle^\otimes n)\), and \(|+\rangle^\otimes n\) is a representative state for the \(k = n\) case.

Our current techniques limit us to five-qubit operations, so we are unable to confirm whether \(R(M_{t,n} |\mathcal{L}\rangle)\) continues to decrease with increasing \(\dim(|\mathcal{L}\rangle)\). An intuition for why a decrease is plausible goes as follows. A stabiliser state \(|\mathcal{L}\rangle\) with \(\dim(|\mathcal{L}\rangle) = k\) will have \(2^k\) equally weighted terms when written in the computational basis, so will have a normalisation factor of \(2^{-k/2}\). The non-stabiliser state \(M_{t,n} |\mathcal{L}\rangle\) is identical to \(|\mathcal{L}\rangle\) apart from the phase on the all-zero term \(|0\ldots0\rangle\). As \(k\) becomes large, the amplitude of the term \(\frac{e^{i\pi/2^t}}{2^{k/2}} |0\ldots0\rangle\) becomes very small, so that \(M_{t,n} |\mathcal{L}\rangle\) has high fidelity with the stabiliser state \(|\mathcal{L}\rangle\). We would therefore expect \(M_{t,n} |\mathcal{L}\rangle\) to have a small robustness if \(k\) is large.