DISCRETE INTEGRABLE SYSTEMS GENERATED BY HERMITE-PADÉ APPROXIMANTS

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Abstract. We consider Hermite-Padé approximants in the framework of discrete integrable systems defined on the lattice $\mathbb{Z}^2$. We show that the concept of multiple orthogonality is intimately related to the Lax representations for the entries of the nearest neighbor recurrence relations and it thus gives rise to discrete integrable systems. In particular, we show how to construct classes of global solutions of the discrete integrable systems in question.

1. Introduction

Nowadays modern technologies allow us to handle an enormous amount of information. As a consequence of this development, it is in many instances more advantageous to face the analysis of discrete data rather than continuous data. For this reason we are witnessing that the world in this century requires more and more the understanding of discrete models and that is why we decided to concentrate our attention on studying discrete models that take their origin in orthogonality, one of the basic mathematical concepts. Mathematically speaking, the discrete models we are going to consider are systems of difference equations. Recent advances in a number of mathematical fields reveal that discrete systems are in many respects even more fundamental than continuous ones (for instance see [21], [24]).

In this paper we follow the streamline of discrete integrable systems (see [9], [10]) and our main interest is in discrete systems on $\mathbb{Z}^2$ represented by a field of square invertible matrices

$$L_{n,m}, M_{n,m} \in \mathbb{C}^{d \times d}, \quad n, m \in \mathbb{Z},$$

which satisfy the discrete zero curvature (or form a Lax pair) condition on $\mathbb{Z}^2$:

$$L_{n,m+1}M_{n,m} - M_{n+1,m}L_{n,m} = 0.$$ 

The elements of the discrete system (1.1) are transition matrices which define the evolution of the wave function $\Psi_{n,m}$

$$\Psi_{n+1,m}(z) = L_{n,m}(z)\Psi_{n,m}(z), \quad \Psi_{n,m+1}(z) = M_{n,m}(z)\Psi_{n,m}(z),$$

and the condition (1.2) describes the consistency or integrability of the equations (1.3). In turn, the relations (1.2) represent a nonlinear system of difference equations.

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Our findings are mainly inspired by the connection between discrete integrable systems, orthogonal polynomials, and Padé approximants (see [17, 25]). For example, the discrete dynamics

\[ x^k \, d\mu(x), \quad k \in \mathbb{Z}_+ \]

of the measure \( d\mu \) supported on \([0, +\infty)\) generates a family of orthogonal polynomials \( \{P_n^{(k)}(x)\} \), \( \deg P_n^{(k)} = n \). These polynomials also appear in the Padé table and some nearest neighbours of this table are related

\[
P_{n+1}^{(k)}(x) = xP_n^{(k+1)}(x) - V_n^{(k)}P_n^{(k)}, \quad P_{n+1}^{(k)}(x) = xP_n^{(k+2)}(x) - W_n^{(k)}P_n^{(k+1)},
\]

where the coefficients of the relations are expressed by means of the Hankel determinants

\[
V_n^{(k)} = \frac{S_n^{(k+1)}S_n^{(k)}}{S_n^{(k+1)}S_n^{(k+1)}}, \quad W_n^{(k)} = \frac{S_n^{(k+1)}S_n^{(k+1)}}{S_n^{(k)}S_n^{(k+2)}}, \quad S_n^{(k+1)} = \begin{vmatrix} s_k & \ldots & s_{n+k} \\ \vdots & \ddots & \vdots \\ s_{n+k} & \ldots & s_{2n+k} \end{vmatrix},
\]

and \( s_j = \int_0^\infty x^j \, d\mu(x) \). Finally, the consistency of these relations gives the discrete zero curvature condition (1.2) for the discrete system (1.1) of 2 \( \times \) 3 Hermite-Padé rational approximants

\[
L_{n,k} = \begin{pmatrix} -V_n^{(k)} & x \\ -V_n^{(k)} & x + W_n^{(k)} - V_n^{(k+1)} \end{pmatrix}, \quad M_{n,k} = \frac{1}{x} \begin{pmatrix} 0 & x \\ -V_n^{(k)} & x + W_n^{(k)} \end{pmatrix}.
\]

Recall that in the theory of Padé approximants this discrete system becomes the quotient-difference algorithm, and in integrable systems theory it leads to the discrete-time Toda equation (see, e.g., [27]).

In the present paper we introduce a new class of discrete integrable systems of 3 \( \times \) 3 matrices (1.1)–(1.2). The construction of these systems is based on the theory of Hermite-Padé rational approximants, which were introduced by Hermite [14] in connection to his outstanding proof of the transcendence of \( e \). These days this theory is known to play an important role in various fields ranging from number theory [4, 6, 29] to random matrix theory [16, 7].

To proceed, let us briefly consider the concept of Hermite-Padé rational approximants (for details, see the surveys [5, 28]). Let \( \vec{f} = (f_1, f_2) \) be a vector of Laurent series at infinity

\[
f_j(z) = \sum_{k=0}^{\infty} \frac{s_{j,k}}{z^{k+1}}, \quad j = 1, 2.
\]

The Hermite-Padé rational approximants (of type II)

\[
\pi_{\vec{n}} = \begin{pmatrix} Q^{(1)}_{\vec{n}} \\ P_{\vec{n}} \end{pmatrix}, \quad P_{\vec{n}} = \begin{pmatrix} Q^{(2)}_{\vec{n}} \\ P_{\vec{n}} \end{pmatrix}
\]

for the vector \( \vec{f} \) and multi-index \( \vec{n} = (n_1, n_2) \in \mathbb{N}^2 \) are defined by

\[
\deg P_{\vec{n}} \leq |\vec{n}| = n_1 + n_2,
\]

\[
f_j(z)P_{\vec{n}}(z) - Q^{(j)}_{\vec{n}}(z) =: R^{(j)}_{\vec{n}}(z) = O \left( \frac{1}{z^{n_j+1}} \right), \quad z \to \infty,
\]

where the \( Q^{(j)}_{\vec{n}} \) are polynomials, for \( j = 1, 2 \). This definition is equivalent to a homogeneous linear system of equations for the coefficients of the polynomial \( P_{n_1,n_2} \).
Hermite-Padé polynomials (1.6) satisfy a system of recurrence relations. The notion of perfect systems was introduced by Mahler [18]. For perfect systems, the solution has full degree \( \deg P_n = n + 2 \), the multi-index \((n_1, n_2)\) is called normal and the polynomial \( P_n \) can be normalized to be monic.

Clearly these polynomials can be put in a table \( \{P_{n,m}\} \). If all indices of this table are normal, then the system of functions (1.5) is called a perfect system. The polynomial

\[
\begin{align*}
L_{n,m} &= \begin{pmatrix} x + \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_4 & 0 & 0 \\
\alpha_5 & 0 & 1 \end{pmatrix}, \\
M_{n,m} &= \begin{pmatrix} x + \beta_1 & \beta_2 & \beta_3 \\
\beta_4 & 1 & 0 \\
\beta_5 & 0 & 0 \end{pmatrix},
\end{align*}
\]

with \( (\alpha_2)_{0,m} = (\alpha_4)_{0,m} = (\beta_5)_{n,0} = 0 \) for all \( n, m \geq 0 \) and for the rest of the indices \((n, m)\) we have \((\alpha_j)_{n,m}, (\beta_j)_{n,m} \neq 0, j = 2, \cdots, 5\). Both sets of coefficients of the relations (1.7) and of entries of the matrices (1.8) can be represented by the power series coefficients of the perfect system of functions (1.5).

(1.9) \( (a, b, c, d)_{n,m} \leftarrow \{s_{j,k}\}_{j=1,2} \rightarrow (\alpha_1, \cdots, \alpha_5, \beta_1, \cdots, \beta_5)_{n,m} \)

(details will be given below). Our main result is the following.

**Theorem 1.1.** The zero curvature condition (1.2) holds for a family of \( 3 \times 3 \) transition matrices \( L_{n,m} \) and \( M_{n,m} \) of the form (1.8) if and only if there is a perfect system of two functions (1.5) such that \( P_{n,m} \) are the Hermite-Padé polynomials with the coefficients of the recurrence relations (1.7) corresponding to (1.9).

To give more details on the equivalence of discrete integrable system (1.8) and recurrence relations for the Hermite-Padé polynomials (1.7) we also highlight the following.

**Theorem 1.2.** The discrete Lax pair equations (1.2) for the matrices \( L_{n,m}, M_{n,m} \) of the form (1.8) are equivalent to the nonlinear system of difference equations for the coefficients of the recurrence relations (1.7)

\[
\begin{align*}
c_{n,m+1} &= c_{n,m} + \frac{(a + b)_{n+1,m} - (a + b)_{n,m+1}}{(c - d)_{n,m}}, \\
d_{n,m+1} &= d_{n,m} + \frac{(a + b)_{n+1,m} - (a + b)_{n,m+1}}{(c - d)_{n,m}}, \\
a_{n,m+1} &= a_{n,m} \frac{(c - d)_{n,m}}{(c - d)_{n-1,m}}, \\
b_{n+1,m} &= b_{n,m} \frac{(c - d)_{n,m}}{(c - d)_{n,m-1}},
\end{align*}
\]

with initial \( (c, a)_{n,0}, (d, b)_{0,m} \) and boundary conditions \( a_{0,m} = 0 = b_{n,0} \).
The Hermite-Padé approximants are intimately related to the notion of multiple orthogonal polynomials. If the coefficients of the Laurent series (1.5) are the moments of positive measures \( \mu_1 \) and \( \mu_2 \) supported on \( \mathbb{R} \)

\[
\sum_{k=0}^{\infty} s_{j,k} z^{k+1} = \int_{\mathbb{R}} \frac{d\mu_j(x)}{z-x}, \quad s_{j,k} = \int_{\mathbb{R}} x^k d\mu_j(x),
\]

then the Hermite-Padé denominators \( P_{n_1,n_2} \) from (1.6) satisfy

\[
\int_{\mathbb{R}} P_{n_1,n_2}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2.
\]

Polynomials defined by the system of orthogonality relations (1.12) are called multiple orthogonal polynomials. The idea of this concept is the following: given two measures \( \mu_1, \mu_2 \), we split the orthogonality relation between these measures and aim to find a monic polynomial \( P_{n_1,n_2} \) of degree \( \deg P_{n_1,n_2} = n_1 + n_2 \) that is orthogonal to the first \( n_1 \) monomials with respect to one measure and to the first \( n_2 \) monomials with respect to the other one.

The multiple orthogonal polynomials (i.e., the Hermite-Padé polynomials) inherit all the remarkable properties for Hermite-Padé approximants, like existence of the monic polynomials of full degree for the normal indices and the recurrence relations (1.7), which were obtained for the first time in [30] for the multiple orthogonal polynomials. In the context of our paper we use these polynomials to generate a general class of perfect systems for which the corresponding tables of multiple orthogonal polynomials exist entirely.

Structure of the paper. The following two sections serve as an introduction to the topic. In particular we give in Section 2 more explanations about general discrete integrable systems. Then, in Section 3 we consider some known 2 × 2 matrix relations from the theory of orthogonal polynomials and continued fractions, i.e., the theory that concerns classical diagonal Padé approximants. A generalization of these relations to the 3 × 3 matrix case for Hermite-Padé approximants and multiple orthogonal polynomials, which plays a decisive role for establishing the connection to discrete integrable systems represented by 3 × 3 matrices, is presented in Section 4. Particularly, in that section we give and prove several propositions, which lead to a proof of Theorems 1.1, 1.2 (Subsections 4.1, 4.2) and provide the reader with a generic class of perfect systems such as Angelesco and Nikishin systems (Subsection 4.3).

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2. The generic Lax representations

Here we recall some basic notions in the theory of discrete integrable systems following [1], [10] (see also [17], [25]).

Let us consider a regular square lattice \( \mathbb{Z}^2 \), that is the set of all pairs \((n, m)\) of integer numbers \( n \) and \( m \). The main object of our study are wave functions
Ψ_{n,m} defined on all the vertices \((n,m)\) of \(\mathbb{Z}^2\) and having their values in \(\mathbb{C}^{k \times k}\) (for simplicity, we restrict ourselves here to the cases \(k = 2\) and \(k = 3\)). The wave function \(\Psi_{n,m}\) depends on a complex parameter \(z\), which is interpreted as the spectral parameter. We assume that for any oriented edge the values of the wave function at the vertices that this edge connects are related via transition matrices \(L_{n,m}\) and \(M_{n,m}\) as follows
\[
\Psi_{n+1,m}(z) = L_{n,m}(z)\Psi_{n,m}(z), \quad \Psi_{n,m+1}(z) = M_{n,m}(z)\Psi_{n,m}(z).
\]
We always require that the transition matrices are invertible and therefore one has
\[
\Psi_{n,m}(z) = L_{n,m}^{-1}(z)\Psi_{n+1,m}(z), \quad \Psi_{n,m}(z) = M_{n,m}^{-1}(z)\Psi_{n,m+1}(z).
\]
It is clear that the value of the wave function must not depend on the path one takes to get to the corresponding vertex. Thus, in order that the wave function \(\Psi_{n,m}\) is well defined, the following zero curvature condition must be satisfied
\[
(2.1) \quad L_{n,m+1}M_{n,m} - M_{n+1,m}L_{n,m} = 0, \quad n, m \in \mathbb{Z}.
\]
As is known, the zero curvature condition is equivalent to integrability. Thus, a discrete system that admits the representation \((2.1)\) is called integrable \([1]\), \([9]\), \([10]\).

Before going further, let us take a careful look at the zero curvature condition. Observe at first that one can rewrite \((2.1)\) as
\[
L_{n,m}^{-1}M_{n+1,m}^{-1}L_{n,m+1}M_{n,m} = I.
\]
Next, from the following picture
\[
\begin{array}{c}
(n,m+1) \quad L_{n,m+1} \quad (n+1,m+1) \\
M_{n,m} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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we see that the zero curvature condition implies that the product of the transition matrices along the oriented simple square path on \(\mathbb{Z}^2\) beginning at the vertex \((n, m)\) is the identity matrix. This observation can be immediately extended to the case of domino paths by reducing them to the just considered simplest case.
Now it is clear how to generalize this statement to the case of any closed oriented path on $\mathbb{Z}^2$. Thus the condition (2.1) means that if one fixes a closed oriented path on the lattice $\mathbb{Z}^2$, then the product of the transition matrices in the order they appear along the path must be the identity matrix. Note that this property resembles the Cauchy theorem for holomorphic functions and, therefore, it can be considered as its noncommutative multiplicative analogue for functions on $\mathbb{Z}^2$. The relation (2.1) is also called the Lax representation and this is one of the ways to say that the underlying discrete system is integrable.

It turns out that different types of wave functions appear in the theory of orthogonal polynomials and they are very useful to achieve a big variety of goals. However, it has to be pointed out that the discrete integrable systems and wave functions, that have their origin in orthogonality, are naturally defined on $\mathbb{N}^2$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. However, one can appropriately extend them to $\mathbb{Z}^2$ or even to $\mathbb{Z}^2_+$ depending on the needs.

3. ORTHOGONAL POLYNOMIALS VIA $2 \times 2$ MATRIX POLYNOMIALS

In this section we show how wave functions naturally originate in the classical theory of orthogonal polynomials and continued fractions. The classical concepts of this section will be generalized in Section 4 in order to get new discrete integrable systems, whose Lax pairs are expressed via $3 \times 3$ matrices.

3.1. The Schur-Euclid algorithm. Suppose we are given a nontrivial Borel measure $d\mu$ on the real line $\mathbb{R}$. Assume also that all the moments of the measure $d\mu$ are finite. Then the Schur algorithm, which is a straightforward generalization of Euclid’s algorithm, leads to the following continued fraction

$$\varphi(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \sim -\frac{1}{z - a_0 - b_0^2 \varphi(z)},$$

where $b_j^2 > 0$ and $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots$. This continued fraction is called a J-fraction.

It is natural to consider continued fractions as infinite sequences of linear fractional transformations. In particular, in the case of the J-fraction, one has the following sequence

$$\varphi_j(z) = T_j(\varphi_{j+1}(z)) = -\frac{1}{z - a_j + b_j^2 \varphi_{j+1}(z)}, \quad j \in \mathbb{Z}_+,$$

with the initial condition $\varphi_0 = \varphi$. Also, it is well known that a linear fractional transformation can be represented as a $2 \times 2$ matrix, i.e.

$$T_j \mapsto W_j(z) = \begin{pmatrix} 0 & -\frac{1}{b_j} \\ b_j & z - d_j \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$

Let us now introduce matrices corresponding to the approximants for the J-fraction, that is, the finite truncations of the continued fraction:

$$W_{[n, 0]}(z) = W_0(z)W_1(z) \ldots W_n(z), \quad n \in \mathbb{Z}_+.$$
Before showing how to construct a set of a wave functions and transition matrices on \( \mathbb{Z}^2 \) let us see what the elements of the matrix polynomial \( W_{[n,0]} \) are. To this end, let us put
\[
\begin{pmatrix}
-Q_0 \\
P_0
\end{pmatrix} := \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\quad \begin{pmatrix}
-Q_{j+1}(z) \\
P_{j+1}(z)
\end{pmatrix} := W_{[j,0]}(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad j \in \mathbb{Z}_+.
\]
Then taking into account the relation \( W_{[j,0]}(z) = W_{[j-1,0]}(z)W_j(z) \) we also have that
\[
W_{[j,0]}(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = W_{[j-1,0]}(z) \begin{pmatrix} 0 \\ b_j \end{pmatrix} = \begin{pmatrix}
-b_j Q_j(z) \\
b_j P_j(z)
\end{pmatrix}, \quad j \in \mathbb{N}.
\]
So, the matrix \( W_{[j,0]} \) has the following form
\[
W_{[j,0]}(z) = \begin{pmatrix}
-b_j Q_j(z) & -Q_{j+1}(z) \\
b_j P_j(z) & P_{j+1}(z)
\end{pmatrix}, \quad j \in \mathbb{Z}_+.
\]
Furthermore, rewriting the relation entrywise
\[
\begin{pmatrix}
-Q_{j+1}(z) \\
P_{j+1}(z)
\end{pmatrix} = W_{[j-1,0]}(z)W_j(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{b_j} W_{[j-1,0]}(z) \begin{pmatrix} -1 \\ z - a_j \end{pmatrix}, \quad j \in \mathbb{N},
\]
we see that the polynomials \( P_j, Q_j \) are solutions of the following three-term recurrence relation
\[
(3.2) \quad b_{j-1} u_{j-1}(z) + a_j u_j(z) + b_j u_{j+1}(z) = z u_j(z), \quad j \in \mathbb{N},
\]
with the initial conditions
\[
P_0(z) = 1, \quad P_1(z) = \frac{z - a_0}{b_0},
\]
\[
Q_0(z) = 0, \quad Q_1(z) = \frac{1}{b_0}.
\]
Thus the entries of the matrix \( W_{[n,0]} \) are orthogonal polynomials of the first and second kind and the corresponding orthogonality measure is \( d\mu \). It is worth mentioning that such \( 2 \times 2 \) matrix polynomials are extensively used in the theory of moment problems [2] and also show that this theory is a particular case of the theory of canonical systems [22, Chapter 8] (see also [23]).

To proceed with the wave function, first of all note that we can define it as follows
\[
\Psi_{n,0}(z) = W_{[n,0]}(z), \quad n \in \mathbb{Z}_+.
\]
Next, we can extend it to \( \mathbb{Z}_+^2 \) by the following rule
\[
\Psi_{n,m}(z) = W_{[n+m,0]}(z), \quad n, m \in \mathbb{Z}_+,
\]
and, consequently, the transition matrices are
\[
L_{n,m} = W_{n+m}, \quad M_{n,m} = W_{n+m}, \quad n, m \in \mathbb{Z}_+.
\]
Finally, it remains to extend the wave function to the entire lattice \( \mathbb{Z}^2 \) by the symmetry
\[
(3.3) \quad \Psi_{-n,m} = \Psi_{n,m}, \quad \Psi_{n,-m} = \Psi_{n,m}, \quad \Psi_{-n,-m} = \Psi_{n,m}, \quad n, m \in \mathbb{Z}_+.
\]
In this case, the corresponding Lax representations become a trivial identity and thus they do lead to trivial discrete integrable systems. However, it is very useful to keep this observation in mind in order to see what happens for various generalizations of orthogonal polynomials.
3.2. Riemann-Hilbert problems. Here we consider a different interpretation of the Schur-Euclid algorithm in the context of Riemann-Hilbert problems, which turned out to be quite efficient for asymptotic analysis. Recall that in [13] a fascinating characterization of orthogonal polynomial in terms of a Riemann-Hilbert problem was found. We will explain this characterization here briefly. To this end, let us consider a weight function \( w \) on \( \mathbb{R} \) that is smooth and has sufficient decay at \( \pm \infty \) so that all the moments \( \int_{\mathbb{R}} x^k w(x) \, dx \) exist. Then the Riemann-Hilbert problem (RHP) consists of the following: find a \( 2 \times 2 \) matrix valued function \( Y_n(z) = Y(z) \) such that

- (i) \( Y(z) \) is analytic for \( z \in \mathbb{C} \setminus \mathbb{R} \).
- (ii) \( Y \) possesses continuous boundary values for \( x \in \mathbb{R} \) denoted by \( Y_+(x) \) and \( Y_-(x) \), where \( Y_+(x) \) and \( Y_-(x) \) are the limiting values of \( Y(z') \) as \( z' \) approaches \( x \) from above and below, respectively, and

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}.
\]

- (iii) \( Y(z) \) has the following asymptotic behavior at infinity:

\[
Y(z) = \left( I + \mathcal{O}\left( \frac{1}{z} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.
\]

Before giving the solution of this RHP for \( Y \), let us recall that the monic orthogonal polynomials \( \pi_n(z) = z^n + \ldots \) satisfy the following three term recurrence relation:

\[
z \pi_j(z) = \pi_{j+1}(z) + a_j \pi_j(z) + b_{j-1}^2 \pi_{j-1}(z), \quad j \in \mathbb{Z}_+.
\]

According to [13], the matrix valued function \( Y(z) \) given by

\[
Y(z) = \left( \begin{array}{c} \pi_n(z) \\ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_n(x)w(x)}{x-z} \, dx \end{array} \right)
\]

is the unique solution of the RHP for \( Y \). Here \( \gamma_n \) is the leading coefficient of the corresponding orthonormal polynomial. Now, as in the previous section, one can define a function \( \Psi_{n,m} \) on \( \mathbb{Z}^2 \). To see that this function is a wave function, we need to be able to construct transition matrices. To this end, let us observe that \( \det Y \) is an analytic function in \( \mathbb{C} \setminus \mathbb{R} \) which has no jump on the real axis. Therefore, \( \det Y \) is an entire function. Its behavior near infinity is \( \det Y(z) = 1 + \mathcal{O}(1/z) \). Thus by Liouville’s theorem we find that \( \det Y = 1 \). Consequently, one can consider the matrix

\[
L_{n,0} = Y_{n+1}^{-1} Y_n^{-1}.
\]

Clearly \( L_{n,0} \) is an analytic function on \( \mathbb{C} \setminus \mathbb{R} \), and since \( Y_n \) and \( Y_{n+1} \) have the same jump matrix on \( \mathbb{R} \) we see that \( L_{n,0} \) has no jump on \( \mathbb{R} \). Hence \( L_{n,0} \) is an entire matrix function. We write the asymptotic condition in the following form

\[
Y_n(z) = \left( I + \frac{A(n)}{z} + \mathcal{O}(1/z^2) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},
\]

where \( A(n, m) \) is the \( 2 \times 2 \) matrix coefficient of \( 1/z \) in the \( \mathcal{O}(1/z) \) term. After some calculations and using Liouville’s theorem, we find that

\[
L_{n,0} = \begin{pmatrix} z + A_{1,1}(n+1) - A_{1,1}(n) & -A_{1,2}(n) \\ A_{2,1}(n+1) & 0 \end{pmatrix}, \quad n \in \mathbb{Z}_+.
\]
Remark 3.1. In fact we haven’t fully used the Riemann-Hilbert problem to recover the wave function and transition matrices in this case. What we actually exploited is the fact that the solution admits the following factorization

\[ Y_n(z) = R_n(z) \left( \int_0^1 \frac{w(x) \, dx}{z-x} \right), \]

where \( R_n(z) \) is a matrix polynomial that has the form

\[ R_n = L_{n-1,0} \ldots L_{0,0}. \]

Basically, \( R_n \) has a structure similar to that of \( W_{n,0} \) (see formula (3.1)). Moreover, \( R_n \) coincides with \( W_{n,0} \) up to a constant factor and the inversion. Now, we can clearly see that what we really need here is the Cauchy transform \( \int \frac{w(x) \, dx}{x-z} \) and its asymptotic behavior at infinity in order to have (3.5). Therefore, it is clear that one can repeat all the steps for any Borel measure with finite moments of all orders. In other words, we have arrived at the Schur-Euclid algorithm:

(i) we start with the function

\[ Y(z) = Y_0(z) = \left( \int_0^1 \frac{d\mu(z)}{z-x} \right), \]

where \( d\mu \) is a probability Borel measure with finite moments of all orders;

(ii) having constructed \( Y_n \), we look for the transition matrix \( L_{n,0} \) of the form (3.7) such that the function

\[ Y_{n+1} = L_{n,0} Y_n \]

obeys the asymptotic condition (3.5).

Let us emphasize that the transition matrix in step (ii) is uniquely determined due to the construction.

We will see in the next section that Riemann-Hilbert problems admit generalizations in higher dimensions. Thus, they can serve as a tool to develop the multidimensional Schur-Euclid algorithm.

4. HERMITE-PADÉ AND MULTIPLE ORTHOGONAL POLYNOMIALS

Here we present a discrete integrable system associated with a family of Hermite-Padé approximants and multiple orthogonal polynomials.

4.1. Two-dimensional recurrence relations. We begin by recalling a generalization of orthogonal polynomials to Hermite-Padé polynomials \( P_{n,m} \) for two functions \( (f_1, f_2) \), which are analytic in a neighbourhood of infinity. It follows from the Cauchy theorem applied to (1.6) that Hermite-Padé polynomials satisfy the orthogonality relations

\[ \oint_{\Gamma} P_{n_1,n_2}(z) z^k f_j(z) \, dz = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2, \]

where the contour \( \Gamma := \partial \Omega \) is the boundary of a domain \( \Omega \ni \infty \) in which the functions \( f_j \in H(\Omega) \), \( j = 1, 2 \) have holomorphic (analytic and single-valued) continuations. We note that the orthogonality relations (4.1) are non-Hermitian. They actually become Hermitian when the functions \( (f_1, f_2) \) are the Cauchy transforms (1.12) of positive measures \( d\mu_1(x) \), \( d\mu_2(x) \) with compact support on the real line.
In this case, the coefficients of the Laurent series \( (1.5) \) for \((f_1, f_2)\) can be considered as the moments of \(d\mu_1(x), d\mu_2(x)\):

\[
s_k^{(j)} = \oint \frac{z^k f_j(z)}{z} \, dz \quad \rightarrow \quad s_k^{(j)} = \int x^k \, d\mu_j(x), \quad j = 1, 2.
\]

Using the determinant of the coefficients \( s_k^{(j)} \)

\[
S_{n,m} = \begin{vmatrix}
1 & s_0^{(1)} & s_1^{(1)} & \cdots & s_{n-1}^{(1)} & s_0^{(2)} & s_1^{(2)} & \cdots & s_{m-1}^{(2)} \\
& s_1^{(1)} & s_0^{(1)} & & s_1^{(2)} & s_0^{(2)} & & \cdots & \\
& & \ddots & & & \ddots & & \cdots & \\
& & & s_{n+m-1}^{(1)} & s_{n+m}^{(1)} & s_{2n+m-1}^{(2)} & s_{n+m}^{(2)} & \cdots & s_{n+2m-2}^{(2)}
\end{vmatrix},
\]

we can write a formula for the Hermite-Padé polynomials

\[
P_{n,m}(x) = \frac{1}{S_{n,m}} \begin{vmatrix}
1 & s_0^{(1)} & s_1^{(1)} & \cdots & s_{n-1}^{(1)} & s_0^{(2)} & s_1^{(2)} & \cdots & s_{m-1}^{(2)} \\
& s_1^{(1)} & s_0^{(1)} & & s_1^{(2)} & s_0^{(2)} & & \cdots & \\
& & \ddots & & & \ddots & & \cdots & \\
& & & s_{n+m-1}^{(1)} & s_{n+m}^{(1)} & s_{2n+m-1}^{(2)} & s_{n+m}^{(2)} & \cdots & s_{n+2m-2}^{(2)}
\end{vmatrix} x
\]

provided that \( S_{n,m} \) is nonvanishing. The latter case is a criterion of normality of the index \((n, m)\). In this paper we assume that all multi-indices are normal and we investigate the nearest-neighbor recurrence relations.

In [31] a matrix Riemann-Hilbert problem formulation for multiple orthogonal was proposed. Here we slightly generalize this approach for the case of Hermite-Padé polynomials. We can formulate the following Riemann-Hilbert problem: find a \(3 \times 3\) matrix function \(Y\) such that

(i) \(Y\) is analytic on \(\mathbb{C} \setminus \Gamma\), (i.e. \(Y \in H(\Omega)\) and \(Y \in H(\mathbb{C} \setminus \Gamma)\)),

(ii) the continuous limits \(Y_+(x) := \lim_{\Omega \ni \xi \to x \in \Gamma} Y(\xi), Y_-(x) := \lim_{\mathbb{C} \setminus \Gamma \ni \xi \to x \in \Gamma} Y(\xi)\) exist and

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & f_1(x) & f_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \Gamma,
\]

(iii) for \(z \to \infty\) one has

\[
Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^{n+m} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-m} \end{pmatrix}.
\]

Following [13, 31] it is easy to show that this Riemann-Hilbert problem has a unique solution in terms of the Hermite-Padé when \((n, m), (n-1, m)\) and \((n, m-1)\) are normal indices, i.e.,

\[
Y = \begin{pmatrix}
P_{n,m} & C_1(P_{n,m}) & C_2(P_{n,m}) \\
c_1(n, m)P_{n-1,m} & c_1c_1(P_{n-1,m}) & c_1c_2(P_{n-1,m}) \\
c_2(n, m)P_{n,m-1} & c_2c_1(P_{n,m-1}) & c_2c_2(P_{n,m-1})
\end{pmatrix}
\]

where the Cauchy transform is used

\[
C_j(P) = \frac{1}{2\pi i} \oint \frac{P(x)f_j(x)}{x - z} \, dx, \quad j = 1, 2.
\]
and the constants $c_1$ and $c_2$ are given by

$$\frac{-2\pi i}{c_1(n,m)} = \oint_{\Gamma} P_{n-1,m}(x)x^{n-1}f_1(x)\,dx, \quad \frac{-2\pi i}{c_2(n,m)} = \oint_{\Gamma} P_{n,m-1}(x)x^{m-1}f_2(x)\,dx.$$ 

One of the natural outcomes of representing the Hermite-Padé polynomials in the form of Riemann-Hilbert problems is that the nearest-neighbor recurrence relations come out in a natural way.

**Proposition 4.1.** Suppose all multi-indices $(n, m) \in \mathbb{Z}_+^2$ are normal. Then the Hermite-Padé polynomials satisfy the system of recurrence relations (1.7).

**Proof.** As we already mentioned in the introduction, the recurrence relations (1.7) for multiple orthogonal polynomials were obtained in [30]. Here, we will follow the same approach. Actually, the proof presented in [30] uses the Riemann-Hilbert problem but, as we see later, the main ingredient of that proof is a factorization similar to the one revealed in Remark 3.1. Basically, the proof goes along the same lines as the construction of the wave function from the Riemann-Hilbert problem in the case of orthogonal polynomials (see Section 3.2).

Let us start by making the standard observation that $\det Y$ is an analytic function in $\mathbb{C} \setminus \mathbb{R}$ with no jump on the contour $\Gamma$. Hence $\det Y$ is an entire function and its behavior near infinity is $\det Y(z) = 1 + O(1/z)$. Thus, by Liouville’s theorem we find that $\det Y = 1$. We can therefore consider the matrix

$$L_{n,m} = Y_{n+1,m}^{-1}Y_{n,m},$$

where the subscript $(n, m)$ is used for the solution (4.4) of the Riemann-Hilbert problem with the polynomial $P_{n,m}$ in the entry of the first row and the first column of $Y_{n,m}$. Clearly $L_{n,m}$ is an analytic function on $\mathbb{C} \setminus \mathbb{R}$, and since $Y_{n,m}$ and $Y_{n+1,m}$ have the same jump matrix on $\mathbb{R}$ we see that $L_{n,m}$ has no jump on $\mathbb{R}$. Hence $R_1$ is an entire matrix function. If we write the asymptotic condition (4.4) as

$$Y_{n,m}(z) = \left(I + \frac{A(n,m)}{z} + O(1/z^2)\right) \begin{pmatrix} z^{n+m} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-m} \end{pmatrix},$$

where $A(n,m)$ is the $3 \times 3$ matrix coefficient of $1/z$ in the $O(1/z)$ term of (4.4), then after some calculus and in view of Liouville’s theorem we find

$$L_{n,m} = \begin{pmatrix} z + A_{1,1}(n+1,m) & -A_{1,2}(n,m) & -A_{1,3}(n,m) \\ A_{2,1}(n+1,m) & 0 & 0 \\ A_{3,1}(n+1,m) & 0 & 1 \end{pmatrix},$$

where $A_{i,j}(n,m)$ is the entry on row $i$ and column $j$ of $A(n,m)$. We can therefore write

$$Y_{n+1,m} = L_{n,m}Y_{n,m}.$$ 

In a similar way we also have

$$Y_{n,m+1} = M_{n,m}Y_{n,m},$$

with

$$M_{n,m} = \begin{pmatrix} z + A_{1,1}(n,m+1) & -A_{1,2}(n,m) & -A_{1,3}(n,m) \\ A_{2,1}(n,m+1) & 1 & 0 \\ A_{3,1}(n,m+1) & 0 & 0 \end{pmatrix}.$$
Proof. We take a function
Hermite-Padé polynomials of the functions $f_Y$ and \((4.8)\) we only used the fact that the relations \((4.11)\) and \((4.10)\) give \((4.12)\). Finally, we notice that to prove \((4.7)\) whose entries are related to the coefficients of the recurrence relations normalizing factors in \((4.11)\) are matrices \((1.8)\) matrices are given by \((4.5)\) are normal. Then there exists a wave function

\[ P_{n+1,m}(x) = (x - c_{n,m})P_{n,m}(x) - a_{n,m}P_{n-1,m}(x) - b_{n,m}P_{n,m-1}(x), \]

and \((4.8)\) gives the second relation in \((1.7)\)

\[ P_{n,m+1}(x) = (x - d_{n,m})P_{n,m}(x) - a_{n,m}P_{n-1,m}(x) - b_{n,m}P_{n,m-1}(x). \]

\[ \square \]

4.2. Discrete integrable systems represented by $3\times 3$ matrices. Another immediate consequence of the reformulation of Hermite-Padé approximation in terms of Riemann-Hilbert problems is a bridge between the corresponding recurrence relations and discrete integrable system whose transition matrices are $3 \times 3$ matrices.

**Proposition 4.2.** Let $(f_1, f_2)$ be a perfect system., i.e., all the multi-indices $(n, m)$ are normal. Then there exists a wave function \((4.5)\) on $\mathbb{Z}_+^2$ and its transition matrices are given by \((1.8)\):

\[ L_{n,m} = \begin{pmatrix} x + \alpha_{n,m}^{(1)} & \alpha_{n,m}^{(2)} & \alpha_{n,m}^{(3)} \\ \alpha_{n+1,m}^{(4)} & 0 & 0 \\ \alpha_{n+1,m}^{(5)} & 0 & 1 \end{pmatrix}, \quad M_{n,m} = \begin{pmatrix} x + \beta_{n,m}^{(1)} & \alpha_{n,m}^{(2)} & \alpha_{n,m}^{(3)} \\ \alpha_{n,m+1}^{(4)} & 1 & 0 \\ \alpha_{n,m+1}^{(5)} & 0 & 0 \end{pmatrix}, \]

whose entries are related to the coefficients of the recurrence relations \((1.7)\) for the Hermite-Padé polynomials of the functions $f_1$ and $f_2$ as follows:

\[ c_{n,m} = -\alpha_{n,m}^{(1)}, \quad d_{n,m} = -\beta_{n,m}^{(1)}, \quad a_{n,m} = -\alpha_{n,m}^{(4)}\alpha_{n,m}^{(2)}, \quad b_{n,m} = -\alpha_{n,m}^{(5)}\alpha_{n,m}^{(3)}. \]

**Proof.** We take a function $Y$ of the form \((4.5)\), then \((4.7)\) and \((4.8)\) give us transition matrices $L_{n,m}$, $M_{n,m}$ of the form \((4.6)\) and \((4.9)\). Taking into account that the normalizing factors in \((4.11)\) are

\[ c_1(n, m) = A_{2,1}(n, m) \quad \text{and} \quad c_2(n, m) = A_{3,1}(n, m), \]

the relations \((4.11)\) and \((4.10)\) give \((4.12)\). Finally, we notice that to prove \((4.7)\) and \((4.8)\) we only used the fact that $Y$ admits the following factorization

\[ Y(z) = R(z) \begin{pmatrix} 1 & \int_{1}^{z} \frac{f_1(x)}{x} \, dx & \int_{1}^{z} \frac{f_2(x)}{x} \, dx \\ 0 & 1 \end{pmatrix}, \]

where $R$ is a matrix polynomial.

\[ \square \]

**Remark 4.3.** Amazingly, the algorithm behind this provides us with the tool for factorizing matrix polynomials. At the same time, it should be stressed that the described scheme to find transition matrices can be considered as a two-dimensional generalization of the Schur-Euclid algorithm that can be used to define two-dimensional continued fractions:
(i) we start with the function
\[ Y(z) = Y_0(z) = \begin{pmatrix} 1 & \int_{\Gamma} f_1(x) \, dx & \int_{\Gamma} f_2(x) \, dx \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
where \( f_1 \) and \( f_2 \) are Laurent series (1.5);
(ii) having constructed \( Y_{n,m} \), we look for the transition matrices \( L_{n,m} \) and \( M_{n,m} \) of the form (4.6) and (4.9), such that the functions
\[ Y_{n+1,m} = L_{n,m} Y_{n,m}, \quad Y_{n,m+1} = M_{n,m} Y_{n,m}, \]
obey the corresponding asymptotic condition (4.4).

Note that the transition matrices in step (ii) are uniquely determined due to the construction.

Now we simplify the zero curvature condition
\[ 0 = L_{n,m+1} M_{n,m} - M_{n+1,m} L_{n,m}, \]
to the form of (1.10).

Proof of Theorem 1.2. In [30] the consistency condition for the recurrence coefficients of (1.7) was obtained in the following form:

\[
\begin{align*}
\frac{d_{n+1,m} - d_{n,m}}{c_{n,m+1} - c_{n,m}} &= c_{n,m+1} - c_{n,m} \\
\frac{b_{n+1,m} - b_{n,m+1} + a_{n+1,m} - a_{n,m+1}}{c_{n,m+1} - c_{n,m}} &= \det \begin{pmatrix} d_{n+1,m} & d_{n,m} \\ c_{n,m+1} & c_{n,m} \end{pmatrix} \\
\frac{a_{n,m+1}}{c_{n,m+1} - c_{n,m}} &= \frac{c_{n,m} - d_{n,m}}{c_{n,m+1} - c_{n,m}} \\
\frac{a_{n,m}}{c_{n,m} - d_{n,m}} &= \frac{c_{n-1,m} - d_{n-1,m}}{c_{n,m} - d_{n,m}} \\
\frac{b_{n+1,m}}{c_{n,m+1} - c_{n,m}} &= \frac{c_{n,m} - d_{n,m}}{c_{n,m+1} - c_{n,m}} \\
\frac{b_{n,m}}{c_{n,m+1} - c_{n,m}} &= \frac{c_{n-1,m} - d_{n-1,m}}{c_{n,m} - d_{n,m}}
\end{align*}
\]

Using the first equation in (4.13), we subtract the columns of the determinant of the second equation in (4.13). We thus obtain the first two equations of (1.10). The third and fourth equations of (1.10) and (4.13) are the same. \( \square \)

Remark 4.4. There are other systems related to the orthogonality concepts for which the consistency leads to non-trivial zero curvature conditions [17], [25], [26].

It turns out that the consistency conditions (4.13) (or, equivalently, the zero curvature condition) are also sufficient for a sequence of Hermite-Padé polynomials to exist and correspond to a perfect system of functions. To complete the proof of Theorem 1.2 it remains to prove the following result.

Proposition 4.5. Suppose that the zero curvature condition
\[ 0 = L_{n,m+1} M_{n,m} - M_{n+1,m} L_{n,m}, \]
holds for a family of invertible matrices \( L_{n,m} \) and \( M_{n,m} \) of the form (4.6) and (4.9). Then there are two functions \( f_1 \) and \( f_2 \) such that the polynomials \( P_{n,m} \) satisfying the corresponding relations (1.7) are the Hermite-Padé polynomials for \( f_1 \) and \( f_2 \).
Proof. To determine the functions we first construct the polynomials \( P_{n,0} \) and \( P_{0,m} \). This can be done since they satisfy ordinary three-term recurrence relations. So these polynomials are orthogonal polynomials due to the Favard theorem. Let \( f_1 \) be the function corresponding to \( P_{n,0} \) and let \( f_2 \) be the function for \( P_{0,m} \). Next, the consistency conditions (4.13) allow us to define \( Y_{n,m} \) in a unique way for all pairs \( (n,m) \in \mathbb{Z}_+^2 \). Due to the asymptotic condition (4.4), the first column of \( Y_{n,m} \) consists of Hermite-Padé polynomials. At the same time, these polynomials coincide with \( P_{n,m} \). Some more details on how to reconstruct the sequence \( P_{n,m} \) from the marginal orthogonal polynomials are given in [12]. □

To conclude this subsection, note that we are again dealing with the wave function \( \Psi_{n,m} \) that coincides with \( Y_{n,m} \) for \( (n,m) \in \mathbb{Z}_+^2 \) and is extended to \( \mathbb{Z}^2 \) by the symmetry (3.3).

4.3. Systems of measures that generate discrete integrable systems. One of the main obstacles to construct a table of multiple orthogonal polynomials is to ensure that each index is normal, that is, the corresponding determinant \( S_{n,m} \) is non-zero. This issue was addressed for the first time by K. Mahler [18], who coined the notion of perfect systems. To be more precise, a system of two measures is called perfect if each index in the corresponding table is normal, i.e. \( S_{n,m} \neq 0 \) for all \( n, m \in \mathbb{Z}_+ \). In this section we give two rather general classes of perfect systems. In turn, these systems give rise to an infinite number of examples of discrete integrable systems.

4.3.1. Angelesco systems. A. Angelesco considered in [3] the following systems of measures. Let \( \Delta_1 \) and \( \Delta_2 \) be disjoint bounded intervals on the real line and \( \mu_1 \) and \( \mu_2 \) be a system of measures such that \( \text{supp} \mu_j = \Delta_j \).

Fix \( \vec{n} \in \mathbb{Z}_+^2 \) and consider the multiple orthogonal polynomials of the so called Angelesco system \((\hat{\mu}_1, \hat{\mu}_2)\) relative to \( \vec{n} \); here \( \hat{\mu} \) denotes the Cauchy transform of \( \mu \):

\[
\hat{\mu}(z) = \int \frac{d\mu(x)}{z - x}.
\]

By construction, we have that

\[
\int x^{\nu} P_{\vec{n}}(x) d\mu_j(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \quad j = 1, 2.
\]

Therefore, \( P_{\vec{n}} \) has \( n_j \) simple zeros in the interior (with respect to the Euclidean topology of \( \mathbb{R} \)) of \( \Delta_j \). As a consequence, since the intervals \( \Delta_j \) are disjoint, \( \deg P_{\vec{n}} = |\vec{n}| \) and Angelesco systems are perfect.

Unfortunately, Angelesco’s paper received little attention and such systems reappeared many years later in [19] where E.M. Nikishin deduced some of their formal properties.

4.3.2. Nikishin systems. Another class of systems for which the perfectness was proved only recently was introduced by E.M. Nikishin [20].

To get an idea about these systems let us consider two non intersecting bounded intervals \( \Delta_1, \Delta_2 \) on the real line. Suppose we are given two measures \( \sigma_1 \) and \( \sigma_2 \) supported on \( \Delta_1 \) and \( \Delta_2 \), respectively. With these two measures we define a third one in the following way

\[
d(\sigma_1, \sigma_2)(x) = \hat{\sigma}_2(x) d\sigma_1(x);
\]

\[
\int x^{\nu} P_{\vec{n}}(x) d\sigma_j(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \quad j = 1, 2.
\]

Therefore, \( P_{\vec{n}} \) has \( n_j \) simple zeros in the interior (with respect to the Euclidean topology of \( \mathbb{R} \)) of \( \Delta_j \). As a consequence, since the intervals \( \Delta_j \) are disjoint, \( \deg P_{\vec{n}} = |\vec{n}| \) and Angelesco systems are perfect.

Unfortunately, Angelesco’s paper received little attention and such systems reappeared many years later in [19] where E.M. Nikishin deduced some of their formal properties.

4.3.2. Nikishin systems. Another class of systems for which the perfectness was proved only recently was introduced by E.M. Nikishin [20].

To get an idea about these systems let us consider two non intersecting bounded intervals \( \Delta_1, \Delta_2 \) on the real line. Suppose we are given two measures \( \sigma_1 \) and \( \sigma_2 \) supported on \( \Delta_1 \) and \( \Delta_2 \), respectively. With these two measures we define a third one in the following way

\[
d(\sigma_1, \sigma_2)(x) = \hat{\sigma}_2(x) d\sigma_1(x);
\]
that is, one multiplies the first measure by a weight formed by the Cauchy transform of the second measure. Thus, we have arrived to the notion of a Nikishin system. A system of two measures \((\mu_1, \mu_2)\) of the form

\[ d\mu_1(x) = d\sigma_1(x), \quad d\mu_2(x) = \int \frac{d\sigma_1(t)}{x-t} d\mu_1(x) = d\langle \sigma_1, \sigma_2 \rangle \]

is called a Nikishin system. Let us emphasize here that the measures from a Nikishin system have the same support, which is a totally different situation than in the case of an Angelesco system. The notion of Nikishin systems can be generalized to any finite number of measures \([8]\). Let us mention here that it was a long standing problem to prove that a general Nikishin system \((\mu_1, \mu_2, \ldots, \mu_p)\) is perfect for \(p \geq 2\). This fact was finally proved in the remarkable paper \([11]\).

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