DUNKL POSITIVE DEFINITE FUNCTIONS

JAMEL EL KAMEL AND KHALED MEHREZ

Abstract. We introduce the notion of Dunkl positive definite and strictly positive definite functions on $\mathbb{R}^d$. This done by the use of the properties of Dunkl translation. We establish the analogue of Bochner’s theorem in Dunkl setting. The case of radial functions is considered. We give a sufficient condition for a function to be Dunkl strictly positive definite on $\mathbb{R}^d$.

keywords: Positive definite functions; Dunkl transform; Dunkl translation.

MSC (2000) 42A38, 42B08, 42B15, 33D15, 47A05

1. Introduction

In classical analysis a complex valued continuous function is said positive definite (resp. strictly positive definite) on $\mathbb{R}$, if for every distinct real numbers $x_1, x_2, ..., x_n$ and every complex numbers $z_1, z_2, ..., z_n$ not all zero, the inequality

$$\sum_{j=1}^{n} \sum_{k=1}^{n} z_j \overline{z_k} f(x_j - x_k) \geq 0 \ (\text{resp.} > 0)$$

hold true. (see [7])

In 1930, the class of positive definite functions is fully characterized by Bochner’s theorem [1], the function $f$ being positive definite if and only if it is the Fourier transform of a nonnegative finite Borel measure on the real line $\mathbb{R}$.

In this work, we introduce the analogue of positive definite function in Dunkl setting. This done by the use of the properties of the Dunkl translation. We establish a version of Bochner’s theorem in Dunkl setting. We give a sufficient condition for a function in $A_\kappa(\mathbb{R}^d)$ to be Dunkl strictly positive definite.

Our paper is organized as follows: In section 2, we present some preliminaries results and notations that will be useful in the sequel. In section 3, we give some properties of the Dunkl transform, the Dunkl translation and the Dunkl convolution. In section 4, we introduce the notion of the Dunkl positive definite functions in studying their properties, some examples are given. We prove that if $\varphi \in A_\kappa(\mathbb{R}^d)$ is Dunkl positive definite, then the Dunkl transform of $\varphi$ is nonnegative and $\varphi$ is bounded. The case of radial function is considered. We state a version of Bochner’s theorem in Dunkl setting. As application, we are interested with the Dunkl heat kernel , and we get a new equality for the modified Bessel function. The section 5 is devoted to Dunkl strictly positive definite functions.
2. Notations and preliminaries

Let \( R \) be a fixed root system in \( \mathbb{R}^d \), \( G \) the associated finite reflexion group, and \( R_+ \) a fixed positive subsystem of \( R \), normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for all \( \alpha \in R_+ \), where \( \langle x, y \rangle \) denotes the usual Euclidean inner product.

For a non zero \( \alpha \in \mathbb{R}^d \), let us define the reflexion \( \sigma_\alpha \) by
\[
\sigma_\alpha x = x - 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in \mathbb{R}^d.
\]

Let \( \kappa \) be a nonnegative multiplicity function \( \alpha \mapsto \kappa_\alpha \) defined on \( R_+ \) with the property that \( \kappa_\alpha = \kappa_\beta \) where \( \sigma_\alpha \) is conjugate to \( \sigma_\beta \) in \( G \). The weight function \( h_\kappa \) est defined by
\[
(1) \quad h_\kappa(x) = \prod_{\alpha \in R_+} | \langle x, \alpha \rangle |^{\kappa_\alpha}, \quad x \in \mathbb{R}^d.
\]

This is a nonegative homogeneous function of degree \( \gamma_\kappa = \sum_{\alpha \in R_+} \kappa_\alpha \), which is invariant under the reflexion group \( G \).

Let \( T_i \) denote Dunkl’s differential-difference operator defined in [2] by
\[
(2) \quad T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} < \alpha, e_i >, \quad 1 \leq i \leq d,
\]
where \( \partial_i \) is the ordinary partial derivative with respect to \( x_i \), and \( e_1, e_2, ..., e_d \) are the standard unit vectors of \( \mathbb{R}^d \). It was proved in [2] that \( T_1, T_2, ..., T_d \) commute. Therefore we can naturally define \( P(T) \) for any polynomial \( P \), where \( T = (T_1, T_2, ..., T_d) \).

Let \( P_n^d \) denote the space of homogeneous polynomials of degree \( n \) in \( d \) variables. The operators \( T_i, 1 \leq i \leq d \) map \( P_n^d \) to \( P_{n-1}^d \). The intertwining operator \( V_\kappa \) is linear and determined uniquely as
\[
(3) \quad V_\kappa P_n^d \subset P_{n-1}^d, \quad V_\kappa 1 = 1, \quad T_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d.
\]

The Dunkl kernel \( E_\kappa \) associated with \( G \) and \( \kappa \) is defined by
\[
(4) \quad E_\kappa(x, y) = V_\kappa (e^{\langle \cdot, y \rangle}) (x), \quad x, y \in \mathbb{R}^d.
\]

**Proposition 1.** (see[8]) Let \( y \in \mathbb{C}^d \). Then the function \( f = E_\kappa(., y) \) is the unique solution of the system
\[
(5) \quad T_i f = \langle e_i, y \rangle f, \quad \text{for all} \ 1 \leq i \leq d,
\]
which is real-analytic in \( \mathbb{R}^d \) and satisfies \( f(0) = 1 \).

We collect some further properties of the Dunkl kernel \( E_\kappa \)

**Proposition 2.** (see[4],[8]) For \( x, y \in \mathbb{C}^d, \lambda \in \mathbb{C} \)

1. \( E_\kappa(x, y) = E_\kappa(y, x) \),
2. \( E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y) \),
3. \( E_\kappa(x, y) = E_\kappa(\overline{x}, \overline{y}) \)
(4) $|E_\kappa(-ix, y)| \leq 1.$
(5) $|E_\kappa(x, y)| \leq e^{||x|| ||y||}$.
(6) Let $\nu(z) = z_1^2 + \ldots + z_d^2$, $z \in \mathbb{C}$. For $z, \omega \in \mathbb{C}^d$,
\[
c_\kappa \int_{\mathbb{R}^d} E_\kappa(x, z) E_\kappa(x, \omega) h_\kappa^2(x) e^{-\frac{||x||^2}{4}} dx = e^{\frac{\nu(z) + \nu(\omega)}{4}} E_\kappa(z, \omega),
\]
where $c_\kappa$ denotes the Mehta-type constant defined by
\[
c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\frac{||x||^2}{4}} dx.
\]
In particular, the function
\[
E_\kappa(x, y) = V_\kappa^{(x)}(e^{<x,y>}), \ x, y \in \mathbb{R}^d,
\]
plays the role of $e^{i<x,y>}$ in the ordinary Fourier analysis. Throughout this paper, we fix the values of $\gamma$ and $\lambda$ as
\[
\gamma := \gamma_\kappa = \sum_{\alpha \in \mathbb{R}_+} k_\alpha \quad \text{and} \quad \lambda := \gamma + \frac{d - 2}{2}.
\]
Let us recall some classical functional spaces:
- $C(\mathbb{R}^d)$ the set of continuous functions on $\mathbb{R}^d$ and $C_0(\mathbb{R}^d)$ its subspace of continuous functions on $\mathbb{R}^d$ vanishing at infinity.
- $S(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable functions on $\mathbb{R}^d$ which are rapidly decreasing as their derivatives.
- $L^p(\mathbb{R}^d, h^2_\kappa)$, $1 \leq p < \infty$, the space of measurable functions on $\mathbb{R}^d$ such that
\[
\|f\|_{\kappa,p} = \left( \int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{1}{p}} < \infty.
\]

3. Harmonic analysis related to the Dunkl operator

In this section, we present some properties of the Dunkl transform, the Dunkl translation and the Dunkl convolution studied and developed in great detail in [4,6,10,11]. The Dunkl transform is defined for $f \in L^1(\mathbb{R}^d, h^2_\kappa)$ by
\[
D_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y) dy, \ x \in \mathbb{R}^d.
\]
If $\kappa = 0$, then $V_\kappa = \text{id}$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line.

In fact, in this case,
\[
E_\kappa(x, -iy) = \Gamma \left( \kappa + \frac{1}{2} \right) \left( \frac{|xy|}{2} \right)^{-\kappa + \frac{1}{2}} \left[ J_{\kappa - \frac{1}{2}}(|xy|) - i \text{sign}(xy) J_{\kappa + \frac{1}{2}}(|xy|) \right],
\]
where $J_\alpha$ denotes the usual Bessel function of first kind and order $\alpha$. 

Let us recall some classical functional spaces:
- $C(\mathbb{R}^d)$ the set of continuous functions on $\mathbb{R}^d$ and $C_0(\mathbb{R}^d)$ its subspace of continuous functions on $\mathbb{R}^d$ vanishing at infinity.
- $S(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable functions on $\mathbb{R}^d$ which are rapidly decreasing as their derivatives.
- $L^p(\mathbb{R}^d, h^2_\kappa)$, $1 \leq p < \infty$, the space of measurable functions on $\mathbb{R}^d$ such that
\[
\|f\|_{\kappa,p} = \left( \int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{1}{p}} < \infty.
\]
Theorem 1. (see [10])

1. For \( f \in L^1(\mathbb{R}^d, h_k^2) \), we have \( D_\kappa f \in C_0(\mathbb{R}^d) \), and
   \[
   \| D_\kappa f \|_{C_0} \leq \| f \|_{\kappa, 1}.
   \]

2. When both \( f \) and \( D_\kappa f \in L^1(\mathbb{R}^d, h_k^2) \), we have the inversion formula
   \[
   f(x) = c_\kappa \int_{\mathbb{R}^d} D_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y) dy \quad a.e.
   \]

3. The Dunkl transform \( D_\kappa \) is an isomorphism of the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \) onto itself, and \( D_\kappa^2 f(x) = f(-x) \).

4. The Dunkl transform \( D_\kappa \) on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to an isometry of \( L^2(\mathbb{R}^d, h_k^2) \).

5. If \( f, g \in L^2(\mathbb{R}^d, h_k^2) \) then
   \[
   \int_{\mathbb{R}^d} D_\kappa f(y) g(y) h_\kappa^2(y) dy = \int_{\mathbb{R}^d} f(y) D_\kappa g(y) h_\kappa^2(y) dy.
   \]

Let \( y \in \mathbb{R}^d \) be given. The Dunkl translation operator \( f \mapsto \tau_y f \) is defined in \( L^2(\mathbb{R}^d, h_k^2) \) by the equation
\[
D_\kappa(\tau_y f)(x) = E_\kappa(iy, x) D_\kappa f(x), \quad x \in \mathbb{R}^d.
\]
The above definition gives \( \tau_y f \) as an \( L^2 \) function.

Let
\[
\mathcal{A}_\kappa(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d, h_k^2) : D_\kappa f \in L^1(\mathbb{R}^d, h_k^2) \right\}.
\]
Note that \( \mathcal{A}_\kappa(\mathbb{R}^d) \) is contained in the intersection of \( L^1(\mathbb{R}^d, h_k^2) \) and \( L^\infty \) and hence is a subspace of \( L^2(\mathbb{R}^d, h_k^2) \). For \( f \in \mathcal{A}_\kappa(\mathbb{R}^d) \) we have
\[
(9) \quad \tau_y f(x) = \int_{\mathbb{R}^d} E_\kappa(ix, y) E_\kappa(-iy, \xi) D_\kappa f(\xi) h_\kappa^2(\xi) d\xi, \quad \forall x \in \mathbb{R}^d.
\]

Theorem 2. (see [10]) Assume that \( f \in \mathcal{A}_\kappa(\mathbb{R}^d) \) and \( g \in L^1(\mathbb{R}^d, h_k^2) \) is bounded. Then
\[
(11) \quad \int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi) d\xi = \int_{\mathbb{R}^d} f(\xi) \tau_{-y} f(\xi) h(\xi) d\xi.
\]

Theorem 3. (see[10]) Let \( f \in \mathcal{A}_\kappa(\mathbb{R}^d) \) be a radial and nonnegative function. Then \( T_y f \geq 0 \), \( T_y f \in L^1_{\alpha}(\mathbb{R}^d) \) and
\[
\int_{\mathbb{R}^d} T_y f(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(y) dy.
\]
The Dunkl convolution operator is defined on \( L^2(\mathbb{R}^d, h_k^2) \) by: for \( f, g \in L^2(\mathbb{R}^d, h_k^2) \),
\[
(13) \quad f \ast \kappa g(x) = \int_{\mathbb{R}^d} f(y) \tau_x g(y) h_\kappa^2(y) dy,
\]
where \( g^\vee(y) = g(-y) \).

Note that as \( \tau_x g^\vee \in L^2(\mathbb{R}^d, h_k^2) \), the above convolution is well defined. We can also write the definition as

\[
(14) \quad f \ast_ \kappa g(x) = \int_{\mathbb{R}^d} D_\kappa f(\xi) D_\kappa g(\xi) E_\kappa(ix, \xi) h_k^2(\xi) d\xi.
\]

**Theorem 4.** (see [9,10,11])

1. Let \( f, g \in L^2(\mathbb{R}^d, h_k^2) \), then
   (a) \( D_\kappa(f \ast_ \kappa g) = D_\kappa f \ast_ \kappa D_\kappa g \).
   (b) \( f \ast_ \kappa g = g \ast_ \kappa f \).

2. Let \( f \in L^2(\mathbb{R}^d, h_k^2) \) and \( g \in L^1 \cap L^2(\mathbb{R}^d, h_k^2) \), then \( f \ast_ \kappa g \in L^2(\mathbb{R}^d, h_k^2) \) and

\[
(15) \quad \| f \ast_ \kappa g \|_{\kappa, 2} \leq \| g \|_{\kappa, 1} \| f \|_{\kappa, 2}.
\]

4. **Dunkl Positive definite Functions**

**Definition 1.** A continuous function \( \varphi \) of \( L^2(\mathbb{R}^d, h_k^2) \) is said Dunkl positive definite (resp. strictly Dunkl positive definite) if for every finite distinct real numbers \( x_1, \ldots, x_n \), and every complex numbers \( \alpha_1, \ldots, \alpha_n \), not all zero, the inequality

\[
\sum_{j=1}^n \sum_{k=1}^n \alpha_j \overline{\alpha_k} \tau_{x_j}(\varphi)(x_k) \geq 0, \quad \text{(resp.} > 0)\]

holds true. Where \( \tau_x \) denotes the Dunkl translation.

From definition [1] we can read of the elementary properties of a Dunkl positive definite function.

**Proposition 3.** (Properties of Dunkl positive definite functions)

1. A nonnegative finite linear combination of Dunkl positive definite functions is Dunkl positive definite.
2. Let \( \varphi \) be a Dunkl positive definite function, then
   (a) The function \( \tau_x \varphi(x) \geq 0 \), for all \( x \in \mathbb{R}^d \). In particular, \( \varphi(0) \geq 0 \).
   (b) \( \overline{\varphi(-x)} = \varphi(x) \), for all \( x \in \mathbb{R}^d \).

**Proof.**

1. The first property is immediate consequence of the definition [1]
2. The second property follows by choosing \( n = 1, \alpha_1 = 1 \) and \( x_1 = x \) in the definition [1]
3. In the definition [1] let \( n = 2, x_1 = 0, \alpha_1 = 1, \alpha_2 = c \) and \( x_2 = x \), then

\[
\varphi(0) + |c|^2 \tau_x \varphi(x) + c \overline{\varphi(-x)} + \overline{c} \varphi(x) \geq 0.
\]

Setting \( c = 1 \) and \( c = i \), respectively, we deduce that \( \varphi(x) + \overline{\varphi(-x)} \) and \( i(\varphi(-x) - \varphi(x)) \) must be reals. This can only be satisfied when \( \varphi(-x) = \varphi(x) \).
Corollary 1. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a Dunkl positive definite function, then $D_\kappa(\varphi)$ is real.

Proof. For $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$D_\kappa(\varphi)(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(y, ix) \varphi(y) h_\kappa^2(y) dy.$$ 

Hence

$$\overline{D_\kappa(\varphi)(x)} = c_\kappa \int_{\mathbb{R}^d} E_\kappa(y, -ix) \overline{\varphi(y)} h_\kappa^2(y) dy.$$ 

Since $\overline{E_\kappa(x, y)} = E_\kappa(\overline{x}, \overline{y})$ for $x, y \in \mathbb{C}^d$, we obtain

$$\overline{D_\kappa(\varphi)(x)} = c_\kappa \int_{\mathbb{R}^d} E_\kappa(y, -ix) \overline{\varphi(y)} h_\kappa^2(y) dy = c_\kappa \int_{\mathbb{R}^d} E_\kappa(-y, ix) \overline{\varphi(-y)} h_\kappa^2(y) dy.$$ 

So, by proposition 3, we have

$$\overline{\varphi(-x)} = \varphi(x),$$

and $E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y)$, for any $\lambda \in \mathbb{C}$ we obtain

$$D_\kappa(\varphi)(x) = D_\kappa(\varphi)(x).$$

Which completes the proof. $\blacksquare$

We begin by seeking sufficient conditions for a function to be Dunkl positive definite.

Theorem 5. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a nonnegative function, then $D_\kappa(\varphi)$ is Dunkl positive definite.

Proof. For $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$\tau_y(D_\kappa(\varphi))(x) = \int_{\mathbb{R}^d} E_\kappa(-iy, \xi) E_\kappa(ix, \xi) \varphi(-\xi) h_\kappa^2(\xi) d\xi.$$ 

Thus,

$$\sum_{j=1}^n \sum_{l=1}^n \alpha_j \alpha_l \tau_{x_j}(D_\kappa(\varphi))(x_l) = \int_{\mathbb{R}^d} \left[ \sum_{j=1}^n \sum_{l=1}^n \alpha_j \alpha_l E_\kappa(-ix_j, \xi) E(i\xi, \xi) \right] \varphi(-\xi) h_\kappa^2(\xi) d\xi$$

$$= \int_{\mathbb{R}^d} \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \xi) \left[ \sum_{l=1}^N \alpha_l E_\kappa(-ix_l, \xi) \right] \varphi(-\xi) h_\kappa^2(\xi) d\xi$$

$$= \int_{\mathbb{R}^d} \left[ \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \xi) \right] \varphi(-\xi) h_\kappa^2(\xi) d\xi \geq 0.$$ 

Which completes the proof. $\blacksquare$
Example 1. For $t > 0$, the function

$$F_t(x) = e^{-t\|x\|^2}$$

is Dunkl positive definite.

Indeed, put

$$G_t(x) = \frac{c_\kappa}{(4t)^{\gamma+\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}.$$

Thus, $G_t$ is nonnegative function of $L^1(\mathbb{R}^d, h_\kappa^2)$. Moreover, (see [9])

(16) $$F_t(x) = D_\kappa(G_t)(x).$$

Since $F_t(x) \in L^1(\mathbb{R}^d, h_\kappa^2)$, we conclude by theorem 5.

Example 2. Consider the modified Bessel function of the second kind of order $\alpha$ defined by

$$K_\alpha(x) = \int_0^{+\infty} e^{-x \cosh(t)} \cosh(\alpha t) dt, \quad x > 0.$$  

Using the integral representation [5,(7.12.24)]

$$2K_\alpha(ax) = a^\alpha \int_0^{+\infty} t^{-1-\alpha} e^{-\frac{a^2}{2} t + \frac{a^2}{2} u^2} dt,$$

by setting $a = r$, $x = 1$ and substituting $u = 2t$, and using $K_\alpha = K_{-\alpha}$, we have

(17) $$K_\alpha(r) = r^{-\alpha} 2^{\alpha-1} \int_0^{+\infty} u^{\alpha-1} e^{-u} e^{-r^2 u} du.$$  

Now, putting

$$\Phi(y) = \frac{1}{(1 + \|y\|_2^2)^p}, \quad y \in \mathbb{R}^d,$$

with $p \in \mathbb{N}$, such that $p > \frac{d}{2} + \gamma + 1$.

Since $p > \frac{d}{2} + \gamma + 1$, the function $\Phi$ is in $(L^1 \cap L^2)(\mathbb{R}^d, h_\kappa^2)$. From, the integral representation of the gamma function, for $p > 0$, we have

$$\Gamma(p) = \int_0^{+\infty} t^{p-1} e^{-t} dt$$

$$= s^p \int_0^{+\infty} u^{p-1} e^{-su} du.$$  

Let $s = 1 + \|y\|_2^2$, then we get

$$\Phi(y) = \frac{1}{\Gamma(p)} \int_0^{+\infty} u^{p-1} e^{-u} e^{-\|y\|_2^2 u} du.$$
Thus

\[ D_\kappa(\Phi)(\omega) = c_\kappa \int_{\mathbb{R}^d} \Phi(x) E_\kappa(x, -i\omega) h_\kappa^2(x) dx \]

\[ = \frac{c_\kappa}{\Gamma(p)} \int_{\mathbb{R}^d} \int_0^\infty u^{p-1} e^{-u} e^{-\|y\|^2/2} E_\kappa(y, -i\omega) h_\kappa^2(y) dy du \]

\[ = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-u} \left[ c_\kappa \int_{\mathbb{R}^d} e^{-\|y\|^2/2} E_\kappa(y, -i\omega) h_\kappa^2(y) dy \right] du \]

\[ = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-u} D_\kappa(F_u(\cdot))(\omega) du \]

\[ = \frac{c_\kappa}{\Gamma(p)2^{\gamma+\frac{d}{2}}} \int_0^\infty u^{p-\gamma-\frac{d}{2}} e^{-u} e^{-\|\omega\|^2/4u} du \]

Using the relation (17) we obtain

\[ (18) \quad D_\kappa(\Phi)(\omega) = \frac{c_\kappa}{2p-1} \| \omega \|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\| \omega \|_2). \]

Since for \( \alpha > 0 \), the even function \( x^\alpha K_\alpha(x) \) is positive and belongs to \( L^1([0, +\infty[, x^{2\alpha+1}dx) \), by the inversion formula and theorem 5 we deduce that \( \Phi(y) = \frac{1}{(1+\|y\|^2)^p} \) is a Dunkl positive definite function.

**Example 3.** Let \( \varphi \in L^2(\mathbb{R}^d, h_\kappa^2) \) be a continuous function. We consider the functions \( \gamma_t, \ t > 0 \), defined by

\[ \gamma_t(y) = \sum_{j=1}^N \alpha_j \tau_{x_j} (G_t(y)), \ y \in \mathbb{R}^d \]

where \( \alpha_j \in \mathbb{C}, \ x_j \in \mathbb{R}^d \) for all \( 1 \leq j \leq N \) and \( G_t \) is the function definite in example1.

If \( \langle \varphi \ast_\kappa \gamma_t, \gamma_t \rangle \geq 0 \), then \( \varphi \) is Dunkl positive definite.

Indeed, by the definition of the generalized translation operator and equation (16), we have

\[ D_\kappa(\gamma_t)(\omega) = \frac{1}{c_\kappa} \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \omega) e^{-2\|\omega\|^2}, \]
which leads to
\[
D_\kappa (\gamma_t \star_{\kappa} \gamma^\vee_t) (\omega) = |D_\kappa (\gamma_t)|^2 (\omega).
\]
\[
= \frac{1}{c_\kappa^2} \left| \sum_{j=1}^{N} \alpha_j E_{\kappa}(-ix_j, \omega) \right|^2 e^{-2t||\omega||^2}
\]
\[
= \frac{1}{c_\kappa} \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} E_{\kappa}(-ix_j, \omega) E_{\kappa}(-ix_l, \omega) e^{-2t||\omega||^2}
\]
\[
= D_\kappa \left( \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} \tau_{x_j} [\tau_{-x_l} G_{2t}(\cdot)](\omega) \right).
\]
(19)

Since the Dunkl transform is a topological automorphism of the Schwartz space $S(\mathbb{R}^d)$, then
\[
\gamma_t \star_{\kappa} \gamma^\vee_t (\gamma) = \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} \tau_{x_j} [\tau_{-x_l} G_{2t}(\cdot)](\omega),
\]
i.e.
\[
\gamma_t \star_{\kappa} \gamma^\vee_t (\omega) = \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} \tau_{x_j} (\Gamma_{\kappa}(2t, x_l, \cdot))(\omega),
\]
where $\Gamma_{\kappa}$ is the Dunkl type heat kernel. Thus,
\[
\int_{\mathbb{R}^d} \varphi(y) \gamma_t \star_{\kappa} \gamma^\vee_t(y) h_{2}^2(y)dy = \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_k} \int_{\mathbb{R}^d} \varphi(y) \tau_{x_j} \Gamma_{\kappa}(2t, x_l, y) h_{2}^2(y)dy,
\]
\[
= \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} \int_{\mathbb{R}^d} \tau_{x_j} \varphi(y) \Gamma_{\kappa}(2t, x_l, y) h_{2}^2(y)dy.
\]

By theorem 4.7 in [9], we have
\[
\lim_{t \to 0} \int_{\mathbb{R}^d} \varphi(y) \gamma_t \star_{\kappa} \gamma^\vee_t(y) h_{2}^2(y)dy = \sum_{j,l=1}^{N} \alpha_j \overline{\alpha_l} \tau_{x_j} \varphi(x_l).
\]

Which completes the proof.

**Proposition 4.** Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$. If $\varphi$ is Dunkl positive definite function and $f \in L^2 (\mathbb{R}^d, h_{2}^2)$, then
\[
< \varphi \star_{\kappa} f, f > \geq 0.
\]

**Proof.** Since $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, and $f \in L^2 (\mathbb{R}^d, h_{2}^2)$, then $\varphi \star_{\kappa} f \in L^2 (\mathbb{R}^d, h_{2}^2)$, and
\[
\varphi \star_{\kappa} f(x) = \int_{\mathbb{R}^d} \tau_{x} \varphi(y) f(y) h_{2}^2(y)dy.
\]
Since \( \varphi \) is Dunkl positive definite function, then \( \overline{\varphi} = \varphi \), so

\[
\varphi \ast_{\kappa} f(x) = \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y)f(y)h_{\kappa}^2(y)dy.
\]

Thus, for \( f \in L^2(\mathbb{R}^d, h_{\kappa}^2) \),

\[
< \varphi \ast_{\kappa} f, f > = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y)f(y)f(x)h_{\kappa}^2(y)dyh_{\kappa}^2(x)dx.
\]

Let \( f \in \mathcal{S}(\mathbb{R}^d) \), its known that, for \( \epsilon > 0 \), there exists a closed cube \( W \subseteq \mathbb{R}^d \), such that

\[
\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y)f(y)f(x)h_{\kappa}^2(y)dyh_{\kappa}^2(x)dx - \int_W \int_W \tau_x(\overline{\varphi})(y)f(y)f(x)h_{\kappa}^2(y)dyh_{\kappa}^2(x)dx \right| < \frac{\epsilon}{2}.
\]

But the double integral over the cubes is the limit of Riemannian sums. Hence, we can find \( x_1, \ldots, x_N \in \mathbb{R}^d \) and weights \( \omega_1, \ldots, \omega_N \) such that

\[
\left| \int_W \int_W \tau_x(\overline{\varphi})(y)f(y)f(x)h_{\kappa}^2(y)dyh_{\kappa}^2(x)dx - \sum_{j,l=1}^N \tau_{x_j}(\varphi)(x_l)f(x_l)\omega_jf(x_l)\omega_l \right| < \frac{\epsilon}{2}.
\]

This means that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y)f(y)f(x)h_{\kappa}^2(y)dyh_{\kappa}^2(x)dx > \sum_{j,l=1}^N \tau_{x_j}(\varphi)(x_l)f(x_l)\omega_jf(x_l)\omega_l - \epsilon.
\]

Letting \( \epsilon \) tend to zero and using that \( \varphi \) is Dunkl positive definite function shows that (20) is true for all \( f \in L^2(\mathbb{R}^d, h_{\kappa}^2) \).

Corollary 2. Let \( \varphi \in A_{\kappa}(\mathbb{R}^d) \) be a Dunkl positive definite function, we define \( \lambda : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C} \) by

\[
(21) \quad \lambda(\gamma) = \int_{\mathbb{R}^d} \varphi(x)D_{\kappa}^{-1}(\gamma)(x)h_{\kappa}^2(x)dx.
\]

If \( \gamma = |\psi|^2 \) with \( \psi \in \mathcal{S}(\mathbb{R}^d) \) and even, then \( \lambda(\gamma) \) is nonnegative.

Proof. Put \( f = D_{\kappa}^{-1}(\psi) \). Since \( \psi \) is even, then \( f \) and \( D_{\kappa}(f) \) are even, and \( D_{\kappa}(f)(x) = \psi(x) \).

Thus,

\[
\gamma(x) = \psi(x)\overline{\psi(x)} = D_{\kappa}f(x)\overline{D_{\kappa}f(x)} = D_{\kappa}f(x)D_{\kappa}\overline{f}(x) = D_{\kappa}(f \ast_{\kappa} \overline{f})(x).
\]
Then,
\[
\lambda(\gamma) = \int_{\mathbb{R}^d} \varphi(x) D^{-1}_\kappa(\gamma)(x) h^2_{\kappa}(x) dx
\]
\[
= \int_{\mathbb{R}^d} \varphi(x) \left( f * \tau f \right) (-x) h^2_{\kappa}(x) dx
\]
\[
= \int_{\mathbb{R}^d} \varphi(x) \left[ \int_{\mathbb{R}^d} \tau_x f(y) \overline{f(y) h^2_{\kappa}(y)} dy \right] h^2_{\kappa}(x) dx
\]
\[
= \int_{\mathbb{R}^d} \overline{f(y)} \left[ \int_{\mathbb{R}^d} \varphi(x) \tau_x f(y) h^2_{\kappa}(x) dx \right] h^2_{\kappa}(y) dy
\]
\[
= \int_{\mathbb{R}^d} \overline{f(y)} (\varphi \ast_{\kappa} f) (-y) h^2_{\kappa}(y) dy
\]
\[
= \int_{\mathbb{R}^d} \overline{f(-y)} (\varphi \ast_{\kappa} f) (y) h^2_{\kappa}(y) dy
\]
\[
= \int_{\mathbb{R}^d} \overline{f(y)} (\varphi \ast_{\kappa} f) (y) h^2_{\kappa}(y) dy
\]
\[
= \int_{\mathbb{R}^d} \overline{f(y)} (\varphi \ast_{\kappa} f) (y) h^2_{\kappa}(y) dy
\]
\[
= \langle \varphi \ast_{\kappa} f, f \rangle \geq 0.
\]

**Proposition 5.** Let \( \varphi \in \mathcal{A}_\kappa(\mathbb{R}^d) \). If \( \varphi \) is a Dunkl positive definite function, then
\( D_{\kappa} \varphi(x) \geq 0, \forall x \in \mathbb{R}^d \).

**Proof.** For \( \varphi, f \in \mathcal{A}_\kappa(\mathbb{R}^d) \), we have
\[
\int_{\mathbb{R}^d} |D_{\kappa}(f)|^2(\xi) D_{\kappa}(\varphi)(\xi) h^2_{\kappa}(\xi) d\xi = \int_{\mathbb{R}^d} D_{\kappa}(f)(\xi) \overline{D_{\kappa}(f)(\xi)} D_{\kappa}(\varphi)(\xi) h^2_{\kappa}(\xi) d\xi
\]
\[
= c^2_{\kappa} \int_{\mathbb{R}^d} D_{\kappa}(\varphi)(\xi) \left[ \int_{\mathbb{R}^d} f(x) E_{\kappa}(x, -i\xi) h^2_{\kappa}(x) dx \right]
\]
\[
\times \left[ \int_{\mathbb{R}^d} \overline{f(y)} E_{\kappa}(y, i\xi) h^2_{\kappa}(y) dy \right] h^2_{\kappa}(\xi) d\xi
\]
\[
= c^2_{\kappa} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \overline{f(y)} \left[ \int_{\mathbb{R}^d} E_{\kappa}(x, -i\xi) E_{\kappa}(y, i\xi) D_{\kappa}(\varphi)(\xi) h^2_{\kappa}(\xi) d\xi \right]
\]
\[
\times h^2_{\kappa}(x) dx h^2_{\kappa}(y) dy
\]
\[
= c^2_{\kappa} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \overline{f(y)} \tau_x \varphi(y) h^2_{\kappa}(x) dx h^2_{\kappa}(y) dy
\]
\[
= c^2_{\kappa} \langle \varphi \ast_{\kappa} f, f \rangle \geq 0.
\]
Since \( \varphi \) is Dunkl positive definite then \( D_\kappa \varphi \) is real. Since the last inequality holds for an arbitrary function \( f \in \mathcal{A}_\kappa(\mathbb{R}^d) \), we conclude. ■

**Corollary 3.** Let \( \varphi \in \mathcal{A}_\kappa(\mathbb{R}^d) \). If \( \varphi \) is a Dunkl positive definite function, then \( \varphi \) is bounded and
\[
|\varphi(x)| \leq \varphi(0), \quad \forall x \in \mathbb{R}^d.
\]

**Proof.** In definition 1 let \( n = 2, \alpha_1 = |\varphi(x)|, \alpha_2 = -\varphi(x) \), \( x_1 = 0 \) and \( x_2 = x \), we have
\[
\varphi(0)|\varphi(x)|^2 - \varphi(-x)|\varphi(x)|\varphi(-x)|\varphi(x)| + |\varphi(x)|^2 \tau_x \varphi(x) \geq 0.
\]
Since \( \varphi(-x) = \varphi(x) \), we obtain
\[
|\varphi(x)|^2 [\varphi(0) - 2|\varphi(x)| + \tau_x \varphi(x)] \geq 0
\]
i.e
\[
(22) \quad |\varphi(x)| \leq \frac{1}{2}(\varphi(0) + \tau_x \varphi(x)).
\]

Furthermore, by the definition of Dunkl translation, and since \( \varphi \) is Dunkl positive definite function, we have
\[
\tau_x \varphi(x) = |\tau_x \varphi(x)| = \left| \int_{\mathbb{R}^d} |E_\kappa(ix, \xi)|^2 D_\kappa \varphi(\xi) h^2_\kappa(\xi) d\xi \right|
\]
\[
\leq \int_{\mathbb{R}^d} |D_\kappa \varphi(\xi)| h^2_\kappa(\xi) d\xi
\]
\[
= \int_{\mathbb{R}^d} D_\kappa \varphi(\xi) h^2_\kappa(\xi) d\xi
\]
\[
= \varphi(0).
\]

The relations (22) and (23) lead to
\[
|\varphi(x)| \leq \varphi(0).
\]

■

**Corollary 4.** Let \( \varphi_1, \varphi_2 \in \mathcal{A}_\kappa(\mathbb{R}^d) \). If \( \varphi_1, \varphi_2 \) are Dunkl positive definite functions, then the convolution product \( \varphi_1 \ast_\kappa \varphi_2 \) is also.

**Proof.** For \( \varphi_1, \varphi_2 \in \mathcal{A}_\kappa(\mathbb{R}^d) \), we have
\[
\varphi_1 \ast_\kappa \varphi_2 \in L^1(\mathbb{R}^d, h^2_\kappa),
\]
and
\[
D_\kappa (\varphi_1 \ast_\kappa \varphi_2) = D_\kappa \varphi_1 D_\kappa \varphi_2 \in L^1(\mathbb{R}^d, h^2_\kappa).
\]
Now, for every complex numbers $\alpha_1, \ldots, \alpha_n$ and for every distinct real numbers $x_1, \ldots, x_n$, we have
\[
\sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \alpha_l r_{x_j} (\varphi_1 \ast_{\kappa} \varphi_2) (x_l) = \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} \alpha_j E_{\kappa} (ix_j, \xi) \right|^2 D_\kappa (\varphi_1 \ast_{\kappa} \varphi_2) (\xi) h_{\kappa}^2 (\xi) d\xi
\]
\[
= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} \alpha_j E_{\kappa} (ix_j, \xi) \right|^2 D_\kappa (\varphi_1) (\xi) D_\kappa (\varphi_2) (\xi) h_{\kappa}^2 (\xi) d\xi
\]
\[
\geq 0,
\]
where the last inequality follows from Proposition 5. \hfill \blacksquare

**Proposition 6.** Let $\varphi \in \mathcal{A}_\kappa (\mathbb{R}^d)$ be a radial and Dunkl positive definite function. If $f \in \mathcal{A}_\kappa (\mathbb{R}^d)$ is a positive radial function, then the product $\varphi D_\kappa f$ is a Dunkl positive definite function.

**Proof.** Since $\varphi, f \in \mathcal{A}_\kappa (\mathbb{R}^d)$, and radials we have
\[
\varphi D_\kappa f \in \mathcal{A}_\kappa (\mathbb{R}^d),
\]
and radial function. Thus,
\[
\sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \alpha_l r_{x_j} (\varphi D_\kappa f) (x_l) = \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} \alpha_j E_{\kappa} (ix_j, \xi) \right|^2 D_\kappa (\varphi D_\kappa f) (\xi) h_{\kappa}^2 (\xi) d\xi
\]
\[
= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} \alpha_j E_{\kappa} (ix_j, \xi) \right|^2 D_\kappa (\varphi \ast_{\kappa} f) (\xi) h_{\kappa}^2 (\xi) d\xi.
\]
Moreover, by the definition of Dunkl convolution, we can write
\[
(24) \quad D_\kappa \varphi \ast_{\kappa} f (x) = \int_{\mathbb{R}^d} D_\kappa \varphi (t) r_t \bar{f} (t) h_{\kappa}^2 (t) dt.
\]
From proposition 5 and theorem 3.4 in [10], we have
\[
D_\kappa \varphi \ast_{\kappa} f (x) \geq 0.
\]
Which completes the proof. \hfill \blacksquare

**Corollary 5.** Let $\varphi_1, \varphi_2 \in \mathcal{A}_\kappa (\mathbb{R}^d)$ are radials. If $\varphi_1, \varphi_2$ are Dunkl positive definite functions, then the product $\varphi_1 \varphi_2$ is also.

**Proof.** Let $\psi = D_\kappa \varphi_2$, then $D_\kappa \psi = \varphi_2$, and since $\varphi_2$ is radial, we have $\psi$ is radial. So, by proposition 5, we have
\[
\psi \geq 0.
\]
By proposition 6, we conclude. \hfill \blacksquare
In the following we state a version of Bochner’s theorem in Dunkl setting and we establish a necessary and sufficient condition for a function to be a Dunkl positive definite.

**Theorem 6. (Bochner)** Let \( \varphi \in A_\kappa(\mathbb{R}^d) \). Then, \( \varphi \) is Dunkl positive definite, if and only if, there exist a nonnegative function \( \psi \in A_\kappa(\mathbb{R}^d) \) such that

\[
\varphi = D_\kappa \psi.
\]  

Proof. Since \( \varphi \) is Dunkl positive definite function, we have \( \varphi(-x) = \varphi(x) \), and \( D_\kappa \varphi \) is even real function (see corollary \( \Pi \)). By the inversion formula we have

\[
\varphi(x) = D^2_\kappa \varphi(x), \quad \forall x \in \mathbb{R}^d.
\]

Let

\[
\psi(x) = D_\kappa \varphi(x).
\]

By proposition \( \Pi \), we deduce that \( \psi \) is nonnegative function of \( A_\kappa(\mathbb{R}^d) \).

Inversely, since \( \psi \) is nonnegative function and belong to \( A_\kappa(\mathbb{R}^d) \), by theorem \( \Pi \) we deduce that \( \varphi = D_\kappa \psi \) is Dunkl positive definite function. \( \blacksquare \)

4.1. Applications.

**Proposition 7.** Let \( \varphi \in A_\kappa(\mathbb{R}^d) \) be a radial function. If \( \varphi \) is Dunkl positive definite function, then there exist a nonnegative radial function \( \psi \in A_\kappa(\mathbb{R}^d) \) such that

\[
\begin{align*}
1) & \quad \tau_y \psi \geq 0, \\
2) & \quad \tau_y \psi \in L^1(\mathbb{R}^d, h^2_\kappa),
\end{align*}
\]

and

\[
\| \tau_y \psi \|_{1,\kappa} = \| \psi \|_{1,\kappa} = \varphi(0).
\]

Proof. Bochner’s theorem asserts that there exist a nonegative function \( \psi \) such that

\[
\varphi = D_\kappa \psi.
\]

Since \( \varphi \) is radial, then \( \psi = D_\kappa \varphi \) is radial, nonnegative and belongs to \( A_\kappa(\mathbb{R}^d) \).

Using theorem 3.4 in [10], we get

\[
\begin{align*}
(i) & \quad \tau_y \psi \geq 0, \quad \tau_y \psi \in L^1(\mathbb{R}^d, h^2_\kappa), \\
(ii) & \quad \| \tau_y \psi \|_{1,\kappa} = \| \psi \|_{1,\kappa} = \varphi(0).
\end{align*}
\]

**Corollary 6.** For \( t > 0 \) and \( x \in \mathbb{R}^d \), we have

\[
\begin{align*}
\Gamma_\kappa(t, x, y) & \geq 0, \quad \forall y \in \mathbb{R}^d \\
\Gamma_\kappa(t, x, y) & \in L^1(\mathbb{R}^d, h^2_\kappa(y)).
\end{align*}
\]
and

$$
\int_{\mathbb{R}^d} \Gamma_\kappa(t, x, y) h_\kappa^2(y) dy = 1.
$$

Where \( \Gamma_\kappa(t, x, y) \) is the Dunkl type heat kernel defined by

$$
\Gamma_\kappa(t, x, y) = \frac{c_\kappa}{(4t)^{\gamma+d/2}} e^{-\left(\frac{\|x\|^2+\|y\|^2}{4t}\right)} E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).
$$

Proof. For \( t > 0 \), the function \( \varphi(x) = e^{-t\|x\|^2} = F_t(x) \) is Dunkl positive definite function (see example 1), radial and belongs to \( A_\kappa(\mathbb{R}^d) \). Moreover,

$$
\psi(x) = D_\kappa(\varphi)(x) = \frac{c_\kappa}{(4t)^{\gamma+d/2}} e^{-\frac{\|x\|^2}{4t}}.
$$

Then

$$
\Gamma_\kappa(t, x, y) = \tau_x(\psi)(y) = \tau_x(D_\kappa\varphi)(y).
$$

By the last proposition \( \Gamma_\kappa(t, x, y) \) is nonnegative, belongs to \( L^1(\mathbb{R}^d, h_\kappa^2) \), and we have

$$
\int_{\mathbb{R}^d} \Gamma_\kappa(t, x, y) h_\kappa^2(y) dy = \int_{\mathbb{R}^d} \tau_x(\psi)(y) h_\kappa^2(y) dy
$$

$$
= \int_{\mathbb{R}^d} \psi(y) h_\kappa^2(y) dy
$$

$$
= \int_{\mathbb{R}^d} D_\kappa\varphi(y) h_\kappa^2(y) dy
$$

$$
= \varphi(0) = 1.
$$

\[\square\]

**Corollary 7.** For \( p \geq \gamma + \frac{d}{2} + 1 \), let \( K_\alpha \) be the modified Bessel of the second kind and order \( \alpha \), then

$$
\int_{\mathbb{R}^d} \tau_y \left( \| x \|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\| x \|) \right) h_\kappa^2(\xi) d\xi = \int_{\mathbb{R}^d} \| x \|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\| x \|) h_\kappa^2(x) dx = \frac{\Gamma(p)}{c_\kappa 2^{p-1}}.
$$

Proof. Let \( p \geq \gamma + \frac{d}{2} + 1 \), be an integer. Put \( \varphi(y) = \frac{1}{(1+\|y\|^2)^p} \), then \( \varphi \) is Dunkl positive definite function (see example 2), we have

$$
\psi(\lambda) = D_\kappa(\varphi)(\lambda) = \frac{c_\kappa}{\Gamma(p) 2^{p-1}} \| \lambda \|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\| \lambda \|).
$$
By the last proposition we have

\[ \int_{\mathbb{R}^d} \tau_x(\psi)(y)h_\kappa^2(y)dy = \int_{\mathbb{R}^d} \psi(y)h_\kappa^2(y)dy \]
\[ = \int_{\mathbb{R}^d} D_\kappa \varphi(y)h_\kappa^2(y)dy \]
\[ = \varphi(0) = 1. \]

\[ \blacksquare \]

5. Strictly Dunkl Positive Definite Functions

Lemma 1. Let \( U \subseteq \mathbb{R}^d \) is open. Suppose that \( x_1, \ldots, x_n \in \mathbb{R}^d \), are pairwise distinct and that \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \). If \( \sum_{j=1}^n \alpha_j E_\kappa (ix_j, \omega) = 0 \), for all \( \omega \in U \), then \( \alpha \equiv 0 \).

Proof. Suppose that

\[ \sum_{j=1}^n \alpha_j E_\kappa (ix_j, \omega) = 0, \ \forall \omega \in U. \]

Since \( z \rightarrow E_\kappa (y, z) \) is analytic on \( \mathbb{C} \), by analytic continuation, we get

\[ \sum_{j=1}^n \alpha_j E_\kappa (ix_j, \omega) = 0, \ \forall \omega \in \mathbb{R}^d. \]

Let \( f \) be a \( C^\infty \) function with compactly supported, we know that

\[ D_\kappa (\tau_x f)(\lambda) = E_\kappa (-ix, \lambda) D_\kappa f(\lambda). \]

Then

\[ \sum_{j=1}^n \alpha_j E_\kappa (ix_j, \omega) = D_\kappa \left( \sum_{j=1}^n \alpha_j \tau_x f \right)(\lambda) = 0. \]

Since for all \( j \in \{1, \ldots, n\} \), \( \tau_x \) is \( C^\infty \) function with compactly supported, then we get

\[ (27) \quad \sum_{j=1}^n \alpha_j \tau_x f(\lambda) = 0, \ \forall \lambda \in \mathbb{R}^d. \]

If the support of \( f \) is contained in the ball around zero with radius \( \epsilon < \min_{j \neq k} || x_k || - || x_j || \), we have (see [10] proposition 3.13), \( \tau_x f \) is supported in

\[ \{ x, \ || x || \leq \epsilon + || x_j || \} . \]

Thus

\[ \tau_x f(x_k) = 0, \ \forall k \neq j; \ \tau_x f(x_j) \neq 0, \ \forall j, k \in \{1, \ldots, n\} \]

Using (27), we obtain

\[ \alpha_j \tau_x f(x_j) = 0, \ \forall j \in \{1, \ldots, n\} . \]
We conclude.

**Theorem 7.** Let \( \varphi \in A_\kappa(\mathbb{R}^d) \), be a nonidentically zero and Dunkl positive definite function. Then \( \varphi \) is Dunkl strictly positive definite.

**Proof.** Let \( \varphi \in A_\kappa(\mathbb{R}^d) \) be nonidentically zero and Dunkl positive definite function. Suppose that there exist distinct reals points \( x_1, \ldots, x_n \) and complex numbers \( \alpha_1, \ldots, \alpha_n \) not all zero, such that

\[
\sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \alpha_l \tau_{x_j} \varphi(x_l) = 0.
\]

By (11), we get

\[
\int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} \alpha_j E_\kappa(-ix_j, \xi) \right|^2 D_\kappa \varphi(\xi) h_\kappa^2(\xi) d\xi = 0.
\]

Since \( \varphi \) is Dunkl positive definite and belongs to \( A_\kappa(\mathbb{R}^d) \), we have \( D_\kappa \varphi \) is nonnegative continuous function. Then

\[
\left| \sum_{j=1}^{n} \alpha_j E_\kappa(-ix_j, \xi) \right|^2 D_\kappa \varphi(\xi) = 0, \forall \xi \in \mathbb{R}^d.
\]

Since \( D_\kappa \varphi \) is nonidentically zero, then there exist an open \( U \subset \mathbb{R}^d \) such that

\[
D_\kappa \varphi(\xi) \neq 0, \forall \xi \in U.
\]

Thus

\[
\left| \sum_{j=1}^{n} \alpha_j E_\kappa(-ix_j, \xi) \right| = 0, \forall \xi \in U.
\]

Using, lemma \( \blacksquare \) we get

\[
\alpha_j = 0, \forall j \in \{1, \ldots, n\}.
\]

We conclude. \( \blacksquare \)

**Example 4.** The functions \( \varphi(x) = e^{-t\|x\|^2}, t > 0 \), and \( \psi(x) = \frac{1}{(1+\|x\|^2)^p}, p \geq \gamma + \frac{d}{2} + 1 \) are Dunkl strictly positive definite functions.

**References**

[1] S. Bochner, Integral transform and their application. Applied Math. Sciences 25. Springer-verlag. New York, Berlin, Heidelberg, Tokyo.

[2] C. F. Dunkl, Differential-difference operators associated to reflexion groups. Trans. Amer. Math. Soc., 311 (1989), 167-183.

[3] C. F. Dunkl, Integral kernels with reflexion group invariance. Can. J. Math., 43 (1991), 1213-1227.
[4] C. F. Dunkl, Hankel transform associated to finite reflexion groups. Contemp. Math., 138 (1992), 123-138.
[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, Higher transcendental functions, volume 2, McGraw-Hill, New York, 1954.
[6] M. F. E. de Jeu, The Dunkl transform. Invent. Math., 113 (1993), 147-162.
[7] M. Ky Fan, Les fonctions définies positives et les fonctions complément monotones, Memorial Sciences Mathmatiques, Paris (1950).
[8] M. Rösler, Dunkl operator: Theory and Applications. In orthogonal polynomials and special functions (Leuven, 2002), Vol. 1817 of lecture notes in Maths, 93-135. Springer, Berlin, (2003).
[9] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators, Comm. Math. Phys. 192, (1998) 519-542.
[10] S. Thangavelu, Yuan Xu, Convolution operator and maximal function for the Dunkl transform, J. Anal. Math. 97 (2005) 25–55.
[11] K. Trimèche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, Integral Transforms Spec. Funct. 13 (2002), 17–38.

JAMEL EL KAMEL. DÉPARTEMENT DE MATHEMATIQUES FSM. MONASTIR 5000, TUNISIA.
E-mail address: jamel.elkamel@fsm.rnu.tn

KHALED MEHEREZ. DÉPARTEMENT DE MATHEMATIQUES IPEIM. MONASTIR 5000, TUNISIA.
E-mail address: k.mehrez@yahoo.fr