Pairings in Hopf-cyclic cohomology of algebras and coalgebras with coefficients.

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Abstract

This paper is concerned with the theory of cup-products in Hopf-type cyclic cohomology of algebras and coalgebras. Here we give detailed proofs of the statements, announced in our previous paper [21]. We show that the cyclic cohomology of a coalgebra can be obtained from a construction involving noncommutative Weil algebra. Then we use a generalization of Quillen and Crainic’s construction (see [18] and [7]) to define the cup-product. We discuss the relation of the introduced cup-product and $S$-operations on cyclic cohomology. After this we describe the relation of this type of product and bivariant cyclic cohomology. In the last section we briefly discuss the relation of our constructions with that of [15].

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1 Definitions and notation

Throughout the paper $\mathcal{H}$ will denote a fixed Hopf algebra with invertible antipode over a fixed characteristic zero field $k$ and $\mathcal{M}$ – a stable anti-Yetter-Drinfeld module over $\mathcal{H}$. We assume that $\mathcal{M}$ is a left $\mathcal{H}$-comodule and right module. Throughout the text we shall use the standard (Sweedler’s) notation with superscripts for all the comultiplications and coactions, e.g. $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for all $h \in \mathcal{H}$ and $\Delta_M(m) = m^{(-1)} \otimes m^{(0)}$ for every $m \in \mathcal{M}$. Under this assumptions the anti-Yetter-Drinfeld condition takes form

$$ (mh)^{(-1)} \otimes (mh)^{(0)} = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)}, $$

and the stability of $\mathcal{M}$ means that $m^{(0)}m^{(-1)} = m$ for all $m \in \mathcal{M}$.

Now recall the definition of the algebras and coalgebras cyclic cohomology with coefficients in stable anti Yetter-Drinfeld modules.

Let $C$ be a coalgebra over the same field. Suppose $\mathcal{H}$-acts on $C$ from the left $\mathcal{H} \otimes C \rightarrow C$, in a way that respects the coalgebraic structure of $C$, i.e. $(hc)^{(1)} \otimes (hc)^{(2)} = h^{(1)}c^{(1)} \otimes h^{(2)}c^{(2)}$ for all $h \in \mathcal{H}$, $c \in C$.

Recall the definitions of the Hopf-type cyclic cohomology of $C$ with coefficients in $\mathcal{M}$, $HC_{\mathcal{H}}^*(C, \mathcal{M})$, given in [1].

First of all, one considers paracocyclic module $C^*(C, \mathcal{M})$:

$$ C^n(C; \mathcal{M}) = \mathcal{M} \otimes C^{\otimes n+1}, $$

and the cyclic operations are defined by the formulas

$$ \delta_i(m \otimes c_0 \otimes \cdots \otimes c_n) = \begin{cases} m \otimes c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes \cdots \otimes c_n, & 0 \leq i \leq n, \\ m^{(0)} \otimes c_0^{(1)} \otimes c_1 \otimes \cdots \otimes c_n \otimes m^{(-1)}c_0^{(2)}, & i = n + 1, \end{cases} $$

$$ \sigma_i(m \otimes c_0 \otimes \cdots \otimes c_n) = m \otimes c_0 \otimes \cdots \otimes c(c_i) \otimes \cdots \otimes c_n, $$

$$ \tau_n(m \otimes c_0 \otimes \cdots \otimes c_n) = m^{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes m^{(-1)}c_0. $$

Recall that “paracocyclic” means that all the usual cocyclic relations are satisfied, probably except for $\tau_n^{n+1} = 1$. $C$ being a $\mathcal{H}$-module, one can extend the action of the Hopf algebra to the tensor power of $C$ diagonally and consider the factor-module $C_{\mathcal{H}}^*(C, \mathcal{M})$, $C^n_{\mathcal{H}}(C, \mathcal{M}) = \mathcal{M} \otimes_{\mathcal{H}} (C^{\otimes n+1})$. Now it is easy to show that the paracocyclic operations, introduced above, can be pulled down to $C_{\mathcal{H}}^*(C, \mathcal{M})$ iff $\mathcal{M}$ is anti-Yetter-Drinfeld. And if $\mathcal{M}$ is also stable, then $C_{\mathcal{H}}^*(C, \mathcal{M})$ with the operations restricted on it from $C^*(C, \mathcal{M})$ is cocyclic.

By definition Hopf-type cyclic (respectively, periodic) cohomology of $C$ with coefficients in $\mathcal{M}$, $HC_{\mathcal{H}}^*(C, \mathcal{M})$ (resp. $HP_{\mathcal{H}}^*(C, \mathcal{M})$) is the cyclic (resp. periodic) cohomology of the cocyclic module $C_{\mathcal{H}}^*(C, \mathcal{M})$. This means that one introduces the mixed complex with differentials $b$ and
$B$, associated in a usual way with the cocyclic module $C^*_H(C, \mathcal{M})$ (see, e.g. [14]) and takes the cohomology of the corresponding total complex (resp., periodic super-complex).

Similarly, let $A$ be a (left) Hopf-module algebra over $\mathcal{H}$, i.e. there’s an action $\mathcal{H} \otimes A \to A$, such that for all $a, b \in A$ and all $h \in \mathcal{H}$ one has $h(ab) = h^{(1)}(a)h^{(2)}(b)$. The following construction is also taken from [1].

Consider the (paraco)cyclic module $C^n(A, \mathcal{M})$:

\[(6)\quad C^n(A, \mathcal{M}) = \text{Hom}(\mathcal{M} \otimes A^{\otimes n+1}, k)\]

where $\text{Hom}(A, B)$ is the space of $k$-linear homomorphisms from $A$ to $B$. The (paraco)cyclic operations on $C^n(A, \mathcal{M})$ are given by the following formulae (where for the sake of brevity we have substituted commas for the tensor product signs

\[(7)\quad \delta_i f(m, a_0, \ldots, a_{n+1}) = \begin{cases} f(m, a_0, \ldots, a_ia_{i+1}, \ldots, a_{n+1}), & 0 \leq i \leq n, \\ f(m^{(0)}, S^{-1}(m^{(-1)}))a_0, \ldots, a_n), & i = n + 1, \end{cases}\]

\[(8)\quad \sigma_i f(m, a_0, \ldots, a_{n-1}) = f(m, a_0, \ldots, 1, \ldots, a_{n-1}), \quad 0 \leq i \leq n,\]

where 1 stands in the $i$-th place and

\[(9)\quad \tau_n f(m, a_0, \ldots, a_n) = f(m^{(0)}, S^{-1}(m^{(-1)}))a_0, \ldots, a_n).\]

Again, all the cyclic relations except $\tau^{n+1}_n = 1$ are satisfied. One passes from para-cocyclic to cocyclic module by taking space $\mathcal{H}$-linear homomorphisms $\text{Hom}_H(\mathcal{M} \otimes A^{\otimes n+1}, k)$, where we let $\mathcal{H}$ act on $k$ on the left via the counit and the left $\mathcal{H}$-action on $\mathcal{M} \otimes A^{\otimes n+1}$ is given by

$h \cdot (m \otimes a_0 \otimes \cdots \otimes a_n) = mS(h^{(1)}) \otimes h^{(2)}a_0 \otimes \cdots \otimes h^{(n+2)}a_n$.

The (co)cyclic module which is obtained in this way is denoted $C^n_H(A, \mathcal{M})$ and its cyclic (resp. Hochschild, resp. periodic) cohomology are called the Hopf-type cyclic (resp. Hochschild, resp. periodic) cohomology of $A$ with coefficients in $\mathcal{M}$. They are denoted by $HC^*_H(A, \mathcal{M})$ (resp. $H^*_H(A, \mathcal{M})$, resp. $HP^*_H(A, \mathcal{M})$).

Suppose now that $C$ acts on $A$ so that

\[(10)\quad c(ab) = c^{(1)}(a)c^{(2)}(b),\]

and

\[(11)\quad (h(c))(a) = h(c(a))\]

for all $a, b \in A$, $c \in C$, $h \in \mathcal{H}$. In paper [15] there was defined a pairing

\[(12)\quad HC^p_H(C, \mathcal{M}) \otimes HC^q_H(A, \mathcal{M}) \rightarrow HC^{p+q}(A),\]

extending to higher dimensions the Connes-Moscovici characteristic map (see [8], [9] and [1]) $\gamma : HC^*_H(C, \mathcal{M}) \to HC^*_H(A, \mathcal{M})$ constructed for an equivariant trace $\gamma : \mathcal{M} \otimes A \to k$ (in Connes’ and Moscovici’s papers this map was defined only for the 1-dimesional modules $\mathcal{M} = \mathbb{K}^\delta$ for a modular pair in involution $(\sigma, \delta)$).
On the other hand when \( M = \sigma^d \) the map \( \gamma \) was generalized to higher equivariant traces by Crainic in [7]. Methods used in that paper are quite different from those of [15]. In the paper [21] the authors gave a brief outline of the recipe which allows one extend the methods used by Crainic to obtain another construction of the pairing, similar to [12]. We shall denote this pairing by \( \sharp' \).

The present paper is devoted to, first of all, giving a detailed proofs of the statements only formulated and/or sketched in [21]. In particular, we discuss the S-operation relations, verified by the map \( \sharp' \). Second, we propose another construction of a pairing, similar to \( \sharp \), this time following the methods of the book [14].

2 Non-commutative Weil algebra and its cohomologies

Let \( C \) be a coalgebra over a characteristic zero field \( k \). Recall the definition of the Weil algebra of \( C \) given at [21], which is a straightforward generalization of the definition in [7], where it is given for Hopf algebras.

**Definition 1.** One calls the “Weil algebra of a coalgebra \( C \)” the free differential graded algebra (without unit) generated by elements \( i_c, \ \deg i_c = 1 \) and \( w_c, \ \deg w_c = 2 \), where both symbols are linear in \( c \in C \). The differential is given by

\[
\begin{align*}
\partial i_c &= w_c - i_c(1) i_c(2) \\
\partial w_c &= w_c(1) i_c(2) - i_c(1) w_c(2).
\end{align*}
\]

Weil algebra of a coalgebra \( C \) is denoted by \( W(C) \). And the symbol \( I(C) \) is reserved for the canonical ideal in \( W(C) \) generated by the elements \( w_c, c \in C \).

For better understanding of this algebra and its properties it is convenient to consider a different description of it.

First of all, remind that cobar resolution of a coalgebra \( C \) is the tensor algebra \( F(C) = \bigoplus_{i=0}^{\infty} C^\otimes n \), equipped with the differential

\[
\delta_F([c_1| \ldots |c_n]) = \sum_{i=1}^{n} (-1)^i [c_1| \ldots |c_i(1)| c_i(2)| \ldots |c_n].
\]

Here we have used | instead of the tensor product sign for the sake of brevity. (Observe that this differential turns \( F(C) \) into a differential graded algebra with respect to the usual free tensor product in it.)

**Proposition 1.** The Weil algebra \( W(C) \) of a coalgebra \( C \) is isomorphic to the free differential algebra \( \Omega(F(C)) \) generated by algebra \( F(C) \). Moreover, the differential \( \partial \) of \( W(C) \) under this isomorphism is equal to the sum of the free differential \( d \) in \( \Omega(F(C)) \) and \( \delta \) the natural extension of \( \delta_F \) from \( F(C) \) to \( \Omega(F(C)) \).

**Proof.** The isomorphism \( \Phi \) in question identifies \( i_c \) with \( [c] \) and \( w_c \) with \( d[c] \) (recall that both algebras that we consider are free, so it is enough to define all maps on their generators). One easily checks that this map commutes with the differentials. For instance, \( \partial i_c = w_c - i_c(1) i_c(2) \) is sent by this map to \( d[c] - [c(1)|c(2)] \), which is equal to \( (d + \delta)[c] \). \( \square \)
We shall denote by $W(C)_2$ the factor space $W(C)/[W(C), W(C)]$ of $W(C)$ with respect to the commutator subspace $[W(C), W(C)]$ (i.e. the subspace spanned by the graded commutators $[\omega_1, \omega_2] = \omega_1\omega_2 - (-1)^{|\omega_1||\omega_2|}\omega_2\omega_1$, $\omega_1, \omega_2 \in W(C)$).

Let now $C$ be a left Hopf-module coalgebra over a Hopf algebra $H$, i.e. there’s an action $H \otimes C \to C$ of $H$ on $C$ satisfying the relation $(hc)(1) \otimes (hc)(2) = h(1)c(1) \otimes h(2)c(2)$. Let $M$ be a left comodule over $H$. Consider the “crossed-product” $W(C)$-bimodule $W(C, M) = M \ltimes W(C)$, which is isomorphic to $\mathcal{M} \otimes W(C)$ as vector space, and in which the right action of $W(C)$ is given by multiplication in right leg of the tensor product and the left action is defined by the formula

\[
\alpha \cdot (m \otimes \beta) = m^{(0)} \otimes S^{-1}(m^{(-1)})(\alpha)\beta.
\]

In this formula we let $H$ act on $W(C)$ diagonally, i.e. so that $h(\alpha\beta) = h(1)(\alpha)h(2)(\beta)$ and we define the action on the generators as $h(i_c) = i_{hc(c)}$ and similarly for $w_c$.

One easily checks that the map $\partial_M = 1 \otimes \partial$ is a differentiation of the graded bimodule $W(C, \mathcal{M})$ with respect to the differential $\partial$ on $W(C)$, that is $\partial_M(\alpha \cdot \omega) = \partial\alpha \cdot \omega + (-1)^{|\alpha|}\alpha \cdot \partial_M\omega$ and $\partial_M(\omega \cdot \alpha) = \partial_M\omega \cdot \alpha + (-1)^{|\alpha|}\omega \cdot \partial\alpha$ for all $\alpha \in W(C)$, $\omega \in W(C, \mathcal{M})$. Clearly, $\partial^2_M = 0$, so $(W(C, \mathcal{M}), \partial_M)$ and its commutant $W(C, \mathcal{M})_2 = W(C, \mathcal{M})/[W(C, \mathcal{M}), W(C)]$ with induced differential $\partial_M$ are chain complexes.

Now suppose that $\mathcal{M}$ is a right module and left comodule, verifying the anti-Yetter-Drinfeld condition (see (1)). We shall denote by $W^H(C, \mathcal{M})$ the complex $\mathcal{M} \otimes_H W(C)$ with differential induced from $W(C)$ as above (we again let $H$ act on $W(C)$ diagonally by the same formula as above). In this case the formula

\[
m \otimes_H \alpha \beta = m^{(0)} \otimes_H S^{-1}(m^{(-1)})(\beta)\alpha
\]

determines a well-defined equivalence relation on $W^H(C, \mathcal{M})$ and we denote the resulting factor-complex of $W^H(C, \mathcal{M})$ w.r.t. these relations by $W^H(C, \mathcal{M})_2$ and $\partial_M$ will stand for the differential in this complex.

Finally consider the ideal $I(C) \subset W(C)$ (the ideal, generated by the curvatures $w_c$) and its powers. We shall denote by $W_n(C)$ the factor-algebra $W_n(C) = W(C)/I^{n+1}(C)$ and by $W_n(C)_2$ its commuted version $W_n(C)_2 = W_n(C)/[W_n(C), W_n(C)] = W_n(C)/[W(C), W_n(C)]$. Similarly $W^H_n(C, \mathcal{M}) = \mathcal{M} \otimes_H W(C)/I^{n+1}(C)$ and $W^H_n(C, \mathcal{M})_2$ will denote the factorization of $W^H_n(C, \mathcal{M})$ by the set of relations, similar to (13).

Remark 1. In fact, there’s a different way to identify the algebra $W(C)$ with $\Omega(F(C))$. Namely, consider the map $\Phi'$, sending $i_c$ to $[c]$ and $w_c$ to $d[c] + [c_{(1)}][c_{(2)}]$ (its inverse sends $[c]$ to $i_c$ and $d[c]$ to $w_c - i_{c(1)}i_{c(2)}$). In this case the differential $\partial$ on $W(C)$ corresponds to $d$ on $\Omega(F(C))$, which is easily checked by a simple calculation. One direct consequence of this observation is the following universal property of $W(C)$: for any unital differential algebra $\Omega$ and any linear map $f$ from $C$ to the space of degree 1 elements in $\Omega$, there is a unique map of differential graded algebras $W(C) \to \Omega$, which coincides with $f$ in degree 1.

### 2.1 Cohomology of $W^H(C, \mathcal{M})$ and $W^H(C, \mathcal{M})_2$

The identification of proposition 1 allows one to introduce a structure of double complex on $W(C)$ simply by transfering it from $\Omega(F(C))$. This amounts to introducing a bigrading on the
free generators $i_c$, $w_c$ of $W(C)$ so that bideg $i_c = (0, 1)$ and bideg $w_c = (1, 1)$ and presenting the differential $\partial$ as the sum of two "partial" differentials, $\delta$ defined by $\delta(i_c) = -i_c(1)i_c(2)$, $\delta(w_c) = w_c(1)i_c(2) - i_c(1)w_c(2)$ and $d, di_c = w_c, dw_c = 0$.

This observation simplifies the calculation of cohomology of $W(C)$ and of its commutant space $W(C)_2$ (compare [13], §3). Actually, $W(C)$ being a bicomplex, the same is true for $W(C)_2$ and we can use spectral sequence arguments to compute the cohomology in both cases. For example, consider the spectral sequence abutting to the cohomology of $W(C)$ (resp. $W(C)_2$) which starts from its $d$-cohomology. One sees that $W(C)$ with differential $d$ is direct sum of tensor powers of the complex $L(C)$, which is equal to $C$ in dimensions 1 and 2 and zero elsewhere with differential equal to the identity map from $C$ to itself. In the case of $W(C)_2$ tensor powers are replaced by their "cyclic" variant, i.e. by tensor powers, factorized by the action of cyclic groups. As the cohomology of such complex $L(C)$ vanishes, one concludes that $d$-cohomologies of $W(C)$ and $W(C)_2$ vanish, so the spectral sequence collapses and we conclude that the cohomology of $W(C)$ and $W(C)_2$ (with respect to the differential $\partial$) also vanish. Moreover from these considerations one obtains a contracting homotopy $H$ for the $d$-cohomology of $W(C)$, which sends the elements $w_c$ to $i_c$ and $i_c$ to 0 and is extended to other elements by the Leibnitz rule. Now one can rephrase the reasoning dealing with $W(C)_2$ by saying that $H$ commutes with the action of cyclic group and hence descends to a contracting homotopy $H_2$ on $W(C)_2$.

Similar arguments allow one to calculate the cohomology of the Weil complex of $C$ in the presence of coefficients’ module $\mathcal{M}$. It is clear that the bigrading on $W(C)$ induces bigradings on both complexes $W(C, \mathcal{M})$ and $W(C, \mathcal{M})_2$ so that the differentials $\partial_{\mathcal{M}}$ and $\partial_{\mathcal{M}_2}$ take the form of the sum of two differentials, induce by $\delta$ and $d$. Now the contracting homotopy $H$ on $W(C)$ above gives rise to homotopies $\mathcal{M}H$ and $\mathcal{M}H_2$ of these complexes, so that their $d$-cohomologies vanish. By the spectral sequence argument we conclude that their $\partial_{\mathcal{M}}$-cohomologies also vanish. The same argument applies to $W^H(C, \mathcal{M})$ and $W^H(C, \mathcal{M})_2$ (to see this it is enough to observe that the contracting homotopy $H$ commutes with the action of $\mathcal{H}$, which is clear, since it is completely determined by its values on the free generators of $W(C)$) so we obtain the following result:

**Proposition 2.**

\[(17)\quad H^*(\overline{W}^H(C, \mathcal{M})_2, \tilde{\partial}_{\mathcal{M}}) = 0,\]

where $\overline{W}^H(C, \mathcal{M})_2$ denotes the factor-complex of $W^H(C, \mathcal{M})$ by the subspace generated by $1 \in W(C)$.

One can also use the isomorphism of remark 1 to calculate the $\partial$-cohomology of $W(C)$, $W(C)_2$ and $W^H(C, \mathcal{M})$. Actually this approach makes the arguments easier, since it allows one to avoid considering the spectral sequence. In fact, in the case of $W(C)$ and $W^H(C, \mathcal{M})$ this is just the consequence of the well-known fact that the universal differential calculus of an algebra is acyclic (presence of coefficients adds very little to the proof of this statement). And for $W(C)_2$ one can use the isomorphism $\Phi'$ and apply the Goodwillie’s ([14]) and Karoubi’s ([17]) theorems, to show that $H(W(C)_2, \partial) \cong H_{DR}(F(C)) = 0$, where $H_{DR}(F(C))$ is the non-commutative de Rham cohomology. In other words, we use the fact that the de Rham cohomology of a graded algebra without zero degree component vanishes.
However, the map \( \Phi \) does not respect the bigrading, hence it is very hard to understand, where it sends the ideal \( I(C) \subset W(C) \) and its powers. So this identification is not very convenient if one wants to describe the cohomology of \( W_n(C) \) and \( W^H_n(C, M) \), which is our next purpose.

But before we can proceed we need to give few definitions, similar to those given in the paper [1].

Let \( A \) be a left Hopf-module algebra. Dually to the Hopf-type cyclic cohomology of \( A \) with coefficients in \( M \) defined in [1], we define its Hopf-type cyclic homology as the homology of the cyclic module \( C^H_n(A, M) \), where \( C^H_n(A, M) = M \otimes_A A^{\otimes n+1} \) and the cyclic operations are induced from the following operations on \( M \otimes A^{\otimes n+1} \):

\[
\partial_i(m \otimes (a_0, a_1, \ldots, a_n)) = \begin{cases} m \otimes (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), & 0 \leq i \leq n, \\ m^{(0)} \otimes (S^{-1}(m(-1))(a_n)a_1, \ldots, a_{n-1}), & i = n. \end{cases}
\]

\[
\sigma_i(m \otimes (a_0, a_1, \ldots, a_n)) = m \otimes (a_0, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)
\]

\[
\tau_n(m \otimes (a_0, a_1, \ldots, a_n)) = m^{(0)} \otimes (S^{-1}(m(-1))(a_n), a_1, \ldots, a_{n-1}).
\]

One easily checks that the anti-Yetter-Drinfeld condition guarantees that these formulas can be pulled down to \( C^H_n(A, \mathcal{M}) \) and that if \( \mathcal{M} \) is stable (i.e. \( m^{(0)} m(-1) = m \) for all \( m \in \mathcal{M} \)) then \( \tau^{n+1} = 1 \). We shall denote the corresponding cyclic homology by \( HC^H_n(A, \mathcal{M}) \).

Similarly to the coefficient-free case one can define cyclic homology of an algebra with the help of non-commutative differential forms and various operators on them. Put \( \Omega_H(A, \mathcal{M}) = \mathcal{M} \otimes_{\mathcal{H}} \Omega(A) \). As in the coefficientless case, one defines the Karoubi operator \( \kappa \) and the differentials \( b \) and \( B \) on \( \Omega_H(A, \mathcal{M}) \) turning it into a mixed complex, whose cyclic homology would coincide with \( HC^H_n(A, \mathcal{M}) \).

Also similarly to the coefficient-free case, one can introduce the non-commutative de Rham homology of \( A \) as the homology of the complex \( \Omega_H(A, \mathcal{M}) \) defined as the factorization of \( \Omega_H(A, \mathcal{M}) \) modulo the relations similar to (16). It is possible to prove a Karoubi-type theorem identifying this homology with the image of the Connes’ \( S \)-operator in \( HC^H_n(A, \mathcal{M}) \).

It is evident that \( W(C)_1 = \overline{\Omega}(F(C)) \) and \( W^H(C, M)_1 = \overline{\Omega}_H(F(C), M) \), and one can regard the reasoning involving the \( d \)-cohomology of \( W(C)_1 \) and \( W^H(C, M)_1 \) as a particular case of the following simple proposition

**Proposition 3.** The non-commutative de Rham cohomology of a free \( \mathcal{H} \)-module algebra (including the case when one introduces coefficients in a anti-Yetter-Drinfeld module) vanishes.

Observe that in order to define this type of homology and to prove this statement one does not need the stability conjecture (however if \( \mathcal{M} \) is not stable, it is not clear whether one can interprete this sort of homology à la Karoubi’s theorem).

For later use we shall need a better knowledge of the structure of the cyclic homology (with coefficients) of a free algebra.

**Proposition 4.** Let \( V \) be a \( \mathcal{H} \)-module. Then the Hopf-type Hochschild homology of the free (tensor) algebra \( F = T(V) \) with coefficients in a SAYD module \( \mathcal{M} \) vanishes in dimensions greater than 1. Moreover, the mixed complex associated to \( F \) (see above) is quasi-isomorphic to the following super \((\mathbb{Z}/2\mathbb{Z}\text{-graded})\) complex,

\[
X_H(F, \mathcal{M}) : \mathcal{M} \otimes_{\mathcal{H}} F \xrightarrow{bd_M} \Omega^1_H(F, \mathcal{M})_2,
\]
and the reduced (i.e. mod $\mathbb{k}$) cyclic homology of $F$ with arbitrary coefficients is

$$\overline{HC^H}(F, \mathcal{M}) = \begin{cases} (\mathcal{M} \otimes_{\mathcal{H}} \bar{F})_z, & * = 0, \\ 0, & * \geq 1 \end{cases}$$

Here $\Omega^1_h(F, \mathcal{M})_z = \Omega^1_h(F, \mathcal{M}) / \{m \otimes_{\mathcal{H}} \omega f - m^{(0)} \otimes_{\mathcal{H}} S^{-1}(m^{(-1)})(f)\omega\}$ for all $m \in \mathcal{M}$, $\omega \in \Omega^1(F)$, $f \in F$ and $(\mathcal{M} \otimes_{\mathcal{H}} \bar{F})_z$ is defined similarly ($\bar{F}$ is just $\oplus_{n \geq 1} V^{\otimes n}$). Observe that due to the dimension this space is equal to $\overline{\Omega^1_h}(F, \mathcal{M})$. The differentials in this complex are defined similarly to the coefficientless case: $d_{\mathcal{M}}$ is the universal differential $1_{\mathcal{M}} \otimes_{\mathcal{H}} d : F \to \Omega^1_h(F, \mathcal{M})$ followed by the natural projection, and $b_{\mathcal{M}}$ is given by the formula

$$b_{\mathcal{M}}(m \otimes_{\mathcal{H}} adb) = m \otimes_{\mathcal{H}} ab - m^{(0)} \otimes_{\mathcal{H}} S^{-1}(m^{(-1)})(b)a.$$  

The complex $X_h(F, \mathcal{M})$ is in fact the first level of the Hodge tower of the mixed complex $(\Omega_h(F, \mathcal{M}), b, B)$, see [14] and section 3.1 below.

**Proof.** is essentially the same as that in the absence of coefficients, see e.g. [14], §3.1. First one shows that the Hochschild complex of $F$ is quasi-isomorphic to $C^{\text{small}}(F, \mathcal{M})$,

$$\ldots \rightarrow 0 \rightarrow 0 \rightarrow \mathcal{M} \otimes_{\mathcal{H}} (F \otimes V) \xrightarrow{b} \mathcal{M} \otimes_{\mathcal{H}} F,$$

where the nontrivial map $b$ is given by $m \otimes_{\mathcal{H}} (f \otimes v) \mapsto m \otimes_{\mathcal{H}} fv - m^{(0)} \otimes_{\mathcal{H}} S^{-1}(m^{(-1)})(v)f$ (here and below $v$ will denote an element from $V$ and we omit the tensors, while speaking about $F$). The quasi-isomorphism is given by the evident inclusion of $C^{\text{small}}(F, \mathcal{M})$ into the Hochschild complex of $F$ with coefficients in $\mathcal{M}$, $CH^H(F, \mathcal{M})$ and the projection $\phi$ from $CH^H(F, \mathcal{M})$ to $C^{\text{small}}(F, \mathcal{M})$ defined as follows: $\phi_0 = \text{id}$, and

$$\phi_1(m \otimes_{\mathcal{H}} (f \otimes v_1 \ldots v_n)) = \sum_{i=1}^n m^{(0)} \otimes_{\mathcal{H}} (S^{-1}(m^{(-1)})(v_{i+1} \ldots v_n))fv_1 \ldots v_{i-1} \otimes v_i,$$

and $\phi_1(m \otimes_{\mathcal{H}} (f \otimes 1)) = 0$. The contracting homotopy from $CH^H(F, \mathcal{M})$ to the image of $\phi$ is given by $h_0 = 0$, and in higher dimensions we put $h_n(m \otimes_{\mathcal{H}} (f_0 \otimes \ldots \otimes f_{n-1} \otimes v)) = 0$ ($v \in V$) and then extend $h_n$ by the recursive formula

$$h_n(m \otimes_{\mathcal{H}} (f_0 \otimes \ldots \otimes f_{n-1} \otimes f_n v)) =
\begin{align*}
&h_n(m^{(0)} \otimes_{\mathcal{H}} (S^{-1}(m^{(-1)})(v)f_0 \otimes \ldots \otimes f_n)) + (-1)^n m \otimes_{\mathcal{H}} (f_0 \otimes \ldots \otimes f_n v). 
\end{align*}$$

After this one shows that the cyclic bicomplex of $F$ (with coefficients $\mathcal{M}$) is quasiequivalent to

\begin{align}
0 & \xleftarrow{b} \mathcal{M} \otimes_{\mathcal{H}} (F \otimes V) \xrightarrow{\gamma} \mathcal{M} \otimes_{\mathcal{H}} F \\
\downarrow & \quad \quad \quad \downarrow b \\
\mathcal{M} \otimes_{\mathcal{H}} (F \otimes V) & \xleftarrow{\gamma} \mathcal{M} \otimes_{\mathcal{H}} F \\
\downarrow b & \\
\mathcal{M} \otimes_{\mathcal{H}} F
\end{align}
for the map \( \gamma \) given by the equation \( \gamma(m \otimes_H f) = \phi_1(m \otimes_H (1 \otimes f)) \). It is an easy exercise to see that this bicomplex is nothing but the expansion of \( X^H(F, \mathcal{M}) \). The last statement (concerning the the reduced homology of a free algebra) follows easily from the structure of the complex.

\[ \square \]

**Remark 2.** Observe that essentially the same proof can be used to show that the Hochschild homology (and cohomology) of a free algebra \( F = T(V) \) of an \( \mathcal{H} \)-module \( V \) with coefficients in \( \mathcal{M} \) and a \( \mathcal{H} \)-equivariant bimodule \( N \) over \( F \) (that is in a bimodule, on which \( \mathcal{H} \) acts so that the actions of \( \mathcal{H} \) on \( N \) and \( F \) agree) vanishes in degrees exceeding 1. This is a generalization of discussion in \([3]\) (see the proof that the free algebras are quasi-free in the cited paper). However we don’t know whether it is possible to exclude the explicit formulas for homotopies etc. from our proof, as it is done in \([3]\). This is due to the fact that to our knowledge there’s no description of the Hochschild homology and cohomology in terms of the derived functors in the \( \mathcal{H} \)-equivariant setting (and especially when coefficient module \( \mathcal{M} \) appear).

### 2.2 Homology of \( W^H_n(C, \mathcal{M})_z \) and \( S \)-operations

Let us now denote the cohomology of the complex \( W^H_n(C, \mathcal{M})_z \) by \( H^*_H(C, \mathcal{M}; n) \). Note that obviously \( H^*_H(C, \mathcal{M}; 0) \cong HC^*_H(C, \mathcal{M}) \). Now the following statement is a straightforward generalization of the Theorem 7.1 in \([7]\).

**Theorem 5.** There are canonical isomorphisms \( \alpha_n : H^*_H(C, \mathcal{M}; n) \cong HC^*_H(C, \mathcal{M}) \), \( n \geq 0 \). Moreover, under this identification the homomorphism induced by the canonical projection \( \pi_n : W^H_n(C, \mathcal{M})_z \to W^H_{n-1}(C, \mathcal{M})_z \) coincides with the \( S \)-operation in the cyclic cohomology.

**Proof.** The proof can be obtained as a word to word repetition of that in \([7]\). Here we shall give a slightly modified version of this reasoning.

First of all, we observe that similarly to the cited paper, in our case it is enough to prove the non-\( \mathcal{H} \)-equivariant, coefficientless version of this theorem. To see this, just notice that all the statements below commute with the \( \mathcal{H} \)-action. (However for the convenience of the reader in the end we shall show, what is to be changed in the general case.)

Further we consider the case \( n = 0 \). As it is observed in the last sentence preceding this theorem, the statement holds in this particular case (actually \( W_0(C)_z \) is isomorphic to \( C^*_e(C) \)[1], and hence \( H^*(C; 0) = HC^*(C) \)).

In order to proceed from \( n = 0 \) to \( n \geq 1 \), we use the following proposition, similar to the Lemma 8.2 of \([7]\):

**Lemma 6.** Let \( I_z^{(n)}(C) \) denote the subspace of \( W(C)_z \) spanned by the elements containing exactly \( n \) factors of type \( w_c \). Then

1. there is isomorphism \( p : H^*(W_n(C)_z) \cong H^*(I_z^{(n)}(C)/\text{Im} d, \delta) \), compatible with \( S \)-operation;
2. the isomorphism \( p \) identifies the map \( \pi_n^*: H^*(W_n(C)_z) \to H^*(W_{n-1}(C)_z) \) with the coboundary operation in the long exact sequence, associated to the short exact sequence

\[
0 \longrightarrow (I_z^{(n-1)}(C)/\text{Im} d, \delta) \overset{d}{\longrightarrow} (I_z^{(n)}(C), \delta) \longrightarrow (I_z^{(n)}(C)/\text{Im} d, \delta) \longrightarrow 0,
\]

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(observe that the dimension of an element is shifted by $d$) i.e. the following diagram commutes

$$
\begin{array}{ccc}
H^*(W_n(C)_\natural) & \xrightarrow{\pi^*_n} & H^*(W_{n-1}(C)_\natural) \\
\downarrow p & & \downarrow p \\
H^*(I^{(n)}_2(C)/\text{Im}d, \delta) & \xrightarrow{\delta^*_n} & H^*(I^{(n-1)}_2(C)/\text{Im}d, \delta).
\end{array}
$$

This lemma reduces our theorem to an equivalent statement about the complex $(I^{(n)}_2(C)/\text{Im}d, \delta)$. Now we recall that $W(C)$ is isomorphic to $\Omega(F(C))$ — the universal differential calculus of the cobar-resolution of $C$, and that the differential $d$ in $W(C)$ corresponds to the universal differential in $\Omega(F(C))$. So, we are to compare the cyclic coalgebra cohomology of $C$, $HC^*(C)$ with the homology of the complex $(\Omega^*(F(C))/\text{Im}d, \delta)$.

To this end we recall from §2.6 [14] and [17], §2.12-2.14, that the space $\Omega^*(A)/\text{Im}d$, which is the same as $\Omega_n^*(A)_{ab}/\text{Im}d$ in the cited paper (here $\Omega_n^*(A)$ denotes the normalized cyclic complex of an algebra $A$, i.e. complex, calculating the mod $k$ homology of $A$) is isomorphic to $\Omega_n^*(A)/\text{Im}b$ ($b$ is the Hochschild differential). Thus we are to compare the homology of $(\Omega_n^*(F(C))/\text{Im}b, \delta)$, where $\delta$ is the intrinsic differential in $F(C)$, with $HC^*(C)$.

So suppose $n \geq 1$ and consider the complex $(\Omega_n^*(F(C))/\text{Im}b, \delta)$. We shall define an isomorphism $\varphi : H^*(\Omega_n^*(F(C))/\text{Im}b, \delta) \xrightarrow{\approx} H^{*+2}(\Omega_n^*(F(C))/\text{Im}b, \delta)$. Let $[x]$ be an element in its homology, represented by a (co)cycle $x \in \Omega_n^*(F(C))/\text{Im}b$. We can choose an element $x'$ in $\Omega_n^*(F(C))$, equal to $x$ modulo the image of $b$. Then since $x$ is a cocycle in $(\Omega_n^*(F(C))/\text{Im}b, \delta)$, we conclude that $\delta x' \in \text{Im}b$. Choose $w' \in \Omega_{n+1}^*(F(C))$ so that $\delta x' = bw'$. Let $w$ be the image of $w'$ under the natural projection $\Omega_{n+1}^*(F(C)) \to \Omega_n^*(F(C))/\text{Im}b$. Then (all the equalities are strict and not mod $b$ since $b^2 = 0$)

$$b(\delta w) = b(\delta w') = \delta(bw') = \delta \delta x' = 0.\]

Thus $\delta w \in \text{Kerb}$. But the complex $\Omega_n^*(F(C))$ is acyclic in dimensions $* \geq 1$ (see proposition [8]), so $\text{Kerb} = \text{Im}b$, and we see that $\delta w = 0$. Put $\varphi([x]) = [w]$.

Let us show, that $\varphi$ is well-defined. There were three ambiguities in its definition. First when we chose a representative cocycle $x$ in $[x]$, second when we passed from $x$ to $x'$ and finally, when we found $w'$. Now if $\tilde{x}$ is a different representative of $[x]$ and $\tilde{x}'$ is its preimage in $\Omega_n^*(F(C))$, then $x' - \tilde{x}' = \delta \alpha + b \beta$ for some $\alpha \in \Omega_n^*(F(C))$, $\beta \in \Omega_{n+1}^*(F(C))$. Then

$$\delta x' - \delta \tilde{x}' = \delta (\delta \alpha + b \beta) = b \delta \beta = b \delta \beta.\]

Now let $\tilde{w}'$ verify the equation $b \tilde{w}' = \delta \tilde{x}'$. It follows from [28], that $b(w' - \tilde{w}') = b \delta \beta$. From the acyclicity of $\Omega_n^*(F(C))$ with respect to $b$ it follows that $w' - \tilde{w}' = \delta \beta + b \gamma$ for some $\gamma \in \Omega_{n+2}^*(F(C))$. Hence the difference $w - \tilde{w}$, where $\tilde{w}$ is the image of $\tilde{w}'$ under the natural projection mod $\text{Im}b$, is equal to the image of $\delta \beta$ under this projection. So $[w] = [\tilde{w}]$.

In order to show, that $\varphi$ is an isomorphism, we shall construct its inverse $\psi$. To this end observe that since the differential graded (nonunital) algebra $F(C)$ is acyclic, so are all the complexes $(\Omega_n^*(F(C)), \delta)$ and $(\Omega_n^*(F(C)), \delta)$ for all $n \in \mathbb{N}$. Now we can define the map $\psi$ as follows. Take a representative $y$ of $[y] \in H^*(\Omega_{n+1}^*(F(C))/\text{Im}b, \delta)$, then $by \in \Omega_n^*(F(C))$ is a
well-defined element (since \( b^2 = 0 \)) and \( \delta y = -b\delta y = 0 \), since \( y \) is a cocycle. By the virtue of acyclicity of \((\mathcal{C}_n^\lambda(F(C)), \delta)\) there exists an element \( v' \in \mathcal{C}_n^\lambda(F(C)) \), such that \( \delta v' = by \). Thus \( v' \) is a modImb cocycle in \((\mathcal{C}_n^\lambda(F(C)), \delta)\) and we put \( \psi([v]) = [v] \), where \( v \in \mathcal{C}_n^\lambda(F(C))/\text{Imb} \) is the projection of \( v' \). The proof that \( \psi \) is well-defined is as easy as before.

Now we can show that \( \psi \circ \varphi = \varphi \circ \psi = 1 \). In fact if \([w] = \varphi([x])\), then \( bw = bw' = \delta x' \) and by the very definition of \( \psi([w]) = [x] \) (since \( x' \) projects in \( x \) modulo \( \text{Imb} \)). Vice-versa, if \( \psi([y]) = [v] \), then \( \delta v' = by \) (where \( v' \) is the preimage of the cocycle \( v \) representing \([v]\) under the natural projection) and, again by the definition of \( \varphi \), \( \varphi([v]) = [y] \). Thus \( \varphi \) is an isomorphism and we put \( \alpha_n = p^{-1} \circ \varphi^n \).

Now it only remains to show, that the iterations of \( \varphi \) send \( S \)-operations on the cyclic cohomology of a coalgebra to the projection \( \pi_n \), or, more accurately, that

\[
S = \sigma^{-1} \circ \psi^{n-1} \circ p \circ \pi_1^* \circ p^{-1} \circ \varphi^n \circ \sigma. \tag{11}
\]

Here \( \sigma : H^*(C) \rightarrow H^{*+1}(W_0(C)_\mathbf{z}) \) is the natural isomorphism induced by the dimension shift and \( p \) is the isomorphism of the lemma \( \mathbb{1} \). In the future we shall omit the “suspension” \( \sigma \), since it doesn’t change the proof, except, for probably, the introduction of some signs.

Once again we start with the case \( n = 1 \). We have to compare the morphism \( S \) with \( p \circ \pi_1 \circ p^{-1} \circ \varphi \). First of all we recall from \((ii)\) Lemma \( \mathbb{1} \) that \( p \circ \pi_1 \circ p^{-1} = \delta_1^* \), the coboundary operation of the exact sequence \( \mathbb{2} \) for \( n = 1 \), so we are reduced to showing that \( S = \delta_1^* \circ \varphi \).

Consider the exact sequence, which is used to define the \( S \)-operation in cohomology of \( F(C)_\mathbf{z} \):

\[
0 \longrightarrow F(C)_\mathbf{z} \xrightarrow{d} \Omega^1(F(C))_\mathbf{z} \xrightarrow{b} F(C) \xrightarrow{\pi} F(C)_\mathbf{z} \longrightarrow 0, \tag{22}
\]

where, for an algebra \( R \), \( \Omega^1(R)_\mathbf{z} \overset{\text{def}}{=} \Omega^1(R)/[\Omega^1(R), R] \) (this space was denoted \( \Omega^1(R) \) above). This sequence is exact, because \( F(C) \) is a free algebra, and the map \( S \) is defined as the result of the diagram chasing: for a cocycle \( x \in F(C)_\mathbf{z} \) one finds \( x' \in F(C) \) that surjects on \( x \), then \( \exists \delta(x') = 0 \) and hence \( \delta x' = by \). Once again \( b\delta y = 0 \) and one can find \( z \) in \( F(C)_\mathbf{z} \) such that \( dz = \delta y \). The map \( S \) is then given by \( x \mapsto z \). Observe now that the sequence \( \mathbb{2} \) is the result of splicing together two short exact sequences:

\[
0 \longrightarrow \Omega^1(F(C))_\mathbf{z}/\text{Imd} \longrightarrow F(C) \xrightarrow{\pi} F(C)_\mathbf{z} \longrightarrow 0 \tag{23}
\]

and

\[
0 \longrightarrow F(C)_\mathbf{z} \xrightarrow{d} \Omega^1(F(C))_\mathbf{z} \longrightarrow \Omega^1(F(C))_\mathbf{z}/\text{Imd} \longrightarrow 0. \tag{24}
\]

So the map \( S \) is equal to the composition of the coboundary operations in the long exact sequences associated to \( \mathbb{23} \) and \( \mathbb{24} \).

But \( I(0) \cong F(C), I(1) = \Omega^1(F(C)) \), and consequently \( I^0_\mathbf{z} = F(C)_\mathbf{z}, I^1_\mathbf{z} = \Omega^1(F(C))_\mathbf{z}, \)

\( I^1_\mathbf{z}/\text{Imd} = \Omega^1(F(C))_\mathbf{z}/\text{Imd} = \mathcal{C}_n^\lambda(F(C))/\text{Imb} \). Now it is evident, that \( \varphi \) is equal to the coboundary operation of \( \mathbb{23} \) (remark that it is obviously an isomorphism because \( F(C) \) is acyclic) and \( \delta_1^* \) — to that of \( \mathbb{24} \). The proof in case \( n = 1 \) is finished.

\( ^1 \)This is not quite true: below we shall have to introduce some scalar coefficients into the game in order to make the diagram commute.
The general case is reduced to the one we have just considered by the following observations. First, the Hochschild homology of \( F(C) \) vanishes in dimensions \( \geq 2 \) and hence the maps \( \phi \) and \( \psi \) can be extended from \( I_2^{(n)}/\text{Im}d = C_0^\lambda(F(C))/\text{Im}b \) to \( I_2^{(n)} \) \( \cong \left( C_n(F(C))/\text{Im}b \right)/\text{Im}(1 - t) \)

where \( t \) is the cyclic operator (see [17], §2.12-2.16) when \( n \geq 1 \).

Now an easy diagramm chasing shows that (up to a scalar multiple) these extensions will commute with the maps in the exact sequences [22] for \( n \) and \( n + 1 \) respectively. To this end consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I_2^{(n)}/\text{Im}d & \overset{d}{\longrightarrow} & I_2^{(n)} & \overset{b}{\longrightarrow} & \hat{\Omega}^n(F(C)) & \longrightarrow & I_2^{(n)}/\text{Im}d & \longrightarrow & 0 \\
\downarrow \frac{n}{n + 1} & & \downarrow -sb & & \downarrow sd & & \uparrow & & \downarrow & & 0 \\
0 & \longrightarrow & I_2^{(n)}/\text{Im}d & \overset{b}{\longrightarrow} & \hat{\Omega}^{n-1}(F(C)) & \longrightarrow & I_2^{(n)} & \longrightarrow & I_2^{(n)}/\text{Im}d & \longrightarrow & 0.
\end{array}
\]

Here we identify \( I_2^{(n)}/\text{Im}d \) with \( \Omega^n(F(C))/\text{Im}d \) and \( I_2^{(n)} \) with \( \Omega^n/\text{Im}b + \text{Im}(1 - \kappa) \) (see [2] §3 for reference), \( \hat{\Omega}^n(F(C)) \) is an abbreviation for \( \Omega^n(F(C))/\text{Im}d + \text{Im}(1 - \kappa) \) and \( s \) is the standard contracting homotopy for the universal differential \( d \) of \( \Omega(F(C)) \). This diagram is commutative because of the identities

\[
-sbd = -\frac{1}{n + 1} sdbN_n = -\frac{1}{n + 1} sbB = \frac{1}{n + 1} sBb = \frac{1}{n + 1} sdN_{n-1}b = \frac{n}{n + 1} (b - db) = \frac{n}{n + 1} b, \\
\]

\[
sdb = (id - ds)b = b - dsb = d(-sb), \\
\]

\[
\]

It follows that we have a commutative diagram for boundary homomorphisms. The boundary morphism of the top row is \( \pi \circ \phi \) and that of the bottom row is equal to \( \varphi \circ \pi \). So we obtain the equality

\[
\frac{n}{n + 1} \pi_{n+1} \circ \varphi_n = \varphi_{n-1} \circ \pi_n.
\]

Thus we see that \( \varphi \) commutes with \( \pi \). Since \( \psi \) is the inverse of \( \varphi \), the same is true for \( \psi \).

So we conclude that \( \varphi \), rescaled, if necessary, so as to eliminate the factor \( \frac{n}{n + 1} \) is an isomorphism of the exact sequences (\( \varphi^{-1} = \psi \), or its rescaling), in particular, \( \delta_{n+1} = \varphi \circ \delta_{n} \circ \psi \). So,

\[
\sigma^{-1} \circ \psi^{n-1} \circ p \circ \pi^n \circ p^{-1} \circ \varphi^n \circ \sigma = \sigma^{-1} \circ \psi^{n-1} \circ \delta^n \circ \varphi^n \circ \sigma \\
= \sigma^{-1} \circ \psi^{n-2} \circ \delta^n \circ \varphi^{n-1} \circ \sigma \\
= \ldots = \\
\]

\[
= \sigma^{-1} \circ \delta^n \circ \varphi \circ \sigma = S
\]

Finally, let us briefly describe, what changes should be made in this reasoning to make it work in the general situation, when the coefficients appear.

First of all, observe that an analog of lemma[4] holds with \( I_2^{(n)}(C) \) replaced with \( I_2^{(n)}(C, M) \) the subspace of \( W_H(C, M) \) spanned by the elements, containing exactly \( n \) factors of type \( w_c \). Further, the isomorphism \( \overline{C}_n(A)_{ab}/\text{Im}d = \overline{C}_n^\lambda(A)/\text{Im}b \) can be extended to \( \overline{C}_n^H(A, M)_{ab}/\text{Im}d = \overline{C}_n^H(A, M)_{ab}/\text{Im}b \)
that of $I$ following statements:

In future we shall need to know not only the homology of $H$-module algebras $A$. Finally, observe that the properties of the cyclic complexes used in definitions of $\varphi$ and $\psi$ and in the proof of the $S$-operation relation, such as the acyclicity of the complexes $(\overline{C}_n^\lambda(F(C)), b)$ and $(\overline{C}_n^\lambda(F(C)), \delta)$, hold for $(\overline{C}_n^\lambda(F(C), M), b)$ and $(\overline{C}_n^\lambda(F(C), M), \delta)$.

**Remark 3.** In general, the following diagram shows that the composition $\pi \circ \varphi$ coincides (up to scalar multiplier) with $S$-operation described in [7]

\[
0 \longrightarrow \Omega^n/b, d \longrightarrow \Omega^{n+1}/b, (1 - \kappa) \longrightarrow \Omega^n/d, (1 - \kappa) \longrightarrow \Omega^n/b, d \longrightarrow 0
\]

Here $t$ is a cyclic permutation on $W(C)$

\[
t(\omega) = (-1)^{|a||\omega|} \omega a, \quad \omega \in W(C), \quad a = i_c \text{ or } a = w_c, \quad c \in C,
\]

and $N(x) = \sum_{i=0}^{p} t^i(x)$, where $p$ is $\delta$-degree of the element $x$.

**Remark 4.** In future we shall need to know not only the homology of $W_n^H(C, M)_2$, but also that of $I_n^H(C, M)_2$, which is equal to $I_{n+1}^H(C, M)/I_n^H(C, M), I(C)]$. Here $I_{n+1}^H(C, M)$ (resp. $I_n^H(C, M)$) is the subspace of $W^H(C, M)$, generated by the $n + 1$-st (resp. $n$-th) power of the ideal $I(C)$. But similarly to the proof of theorem 6.7 of [7] one shows first that $I_n^H(C, M)_2$ is quasiisomorphic to $I_{n+1}^H(C, M)_2 = I_{n+1}^H(C, M)/I_n^H(C, M) \cap [W^H(C, M), W(C)]$ and from the long exact sequence of the three-term exact sequence

\[
0 \longrightarrow I_{n+1}^H(C, M)_2 \longrightarrow W^H(C, M)_2 \longrightarrow W^H(C, M)_2 \longrightarrow 0
\]

where in the middle stands an acyclic complex, we find that $H^*(I_{n+1}^H(C, M)_2) \cong H^*(I_n^H(C, M)_2) \cong H^{*+1}(W_n^H(C, M)_2) = H_{n+1}^H(C, M; n) \cong HC_{n}^{n-2-2n}(C, M)$. We shall denote the resulting isomorphism by $\beta_n$.

Observe, that although the maps $\alpha_n$, $\beta_n$ and their inverses are defined only on the level of cohomology, one can write down explicit formulas for them on the level of the chain complexes. To this end one shall need to choose the contracting homotopies of all the acyclic complexes that appeared in the proof and use an explicit isomorphism $\overline{C}_{\lambda}(A)/Imd \cong \overline{C}_{\lambda}(A)/Imb$. This is precisely what is done in the the section 8 of [7] with the help of the maps $\varphi_0$, $\varphi_1$, $\varphi_0$, etc.

There is one more way to define the $S$-operations in terms of the Weil complex cohomology. To this end consider the $X$-complex of $W(C)$ or rather the $X_{\lambda}$-complex with coefficients $M$, see above. Operations on this super-complex commute with the differential $\partial$ on $W(C, M)$, so by the same computations as in Corollaries 6.9 and 6.10 and theorem 7.9 of [7] we obtain the following statements:

**Proposition 7.** For all $n$ there are long exact sequences of complexes:

1. \(CC(W_n^H(C, M)) : \ldots \longrightarrow W_n^H(C, M; b) \longrightarrow^{t-1} W_n^H(C, M) \longrightarrow^{N} W_n^H(C, M; b) \longrightarrow \ldots\)

2. \(0 \longrightarrow W_n^H(C, M)_2 \longrightarrow^{N} W_n^H(C, M; b) \longrightarrow^{t-1} W_n^H(C, M) \longrightarrow^{N} W_n^H(C, M; b) \longrightarrow \ldots\)

3. \(0 \longrightarrow I_n^H(C, M)_2 \longrightarrow^{N} I_n^H(C, M; b) \longrightarrow^{t-1} I_n^H(C, M) \longrightarrow^{N} I_n^H(C, M; b) \longrightarrow \ldots\)
Here we use the notation of previous remark and the following agreements: \( t \) is the twisted cyclic permutation on \( W^H(C, M) \), \( t(m \otimes xa) = (-1)^{|a||x|}m(0) \otimes S^{-1}(m(-1))(a)x \) for all \( x \in W(C) \) and generator \( a \) of \( W(C) \); \( N = 1 + t + t^2 + \ldots + t^{p-1} \) for all elements of tensor degree \( p \); and \( W^H_n(C, M; b) \) is the complex \( W^H_n(C, M) \) with differential \( b = \partial + b_1 \),

\[
b_1(m \otimes xa) = (-1)^{|a|}t(a \delta(x)).
\]

In particular the bicomplexes \((28)\) and \((29)\) compute the cohomology of \( W^H_n(C, M) \) and \( I^H_n(C, M) \) respectively. The \( S \)-operation in the cohomology of \( W^H_n(C, M) \) and \( I^H_n(C, M) \) is given by the shift of these complexes.

**Proposition 8.** In the previous notation, there exist short exact sequences of complexes

\[
\begin{align*}
(30) & \quad 0 \rightarrow W^H_n(C, M) \rightarrow W^H_n(C, M; b) \rightarrow W^H_n(C, M) \rightarrow 0 \\
(31) & \quad 0 \rightarrow I^H_n(C, M) \rightarrow I^H_n(C, M; b) \rightarrow I^H_n(C, M) \rightarrow 0.
\end{align*}
\]

The \( S \)-operation in the cohomology of \( W^H_n(C, M) \) and \( I^H_n(C, M) \) is given by the cup-product with the \( \text{Ext}^2 \)-class, determined by the sequences \((30), (31)\).

It is easy to see that both methods give the same result, i.e. the \( S \)-operation doesn’t depend on the way we define it. To see this, observe, that the sequence \((30)\) is the version of \((24)\) with \( F(C) \) replaced with \( W_n(C) \) (for a while we omit the coefficients from our notations). As in the lemma \([5]\) we can replace \( W_n(C) \) by \( I(n)(C) \) in this exact sequence. Then as we have shown in the end of the proof of theorem \([5]\) the cup product with the class of this exact sequence is equal to the composition of the map \( \varphi \) and \( \delta_n^* \), which we have identified with \( S \).

### 3 Pairings

The purpose of this section is to define the pairing of \( HC^*_H(C, M) \) and \( HC^*_H(A, M) \) (\( A \) is an \( \mathcal{H} \)-module algebra on which \( C \) acts on the left in a way, described in the section 1). Our approach to this question will be again a suitable generalization of the one used in paper \([7]\). In the next section we shall compare this construction with a construction, generalizing the traditional way of introducing multiplications and comultiplications in cyclic homology.

#### 3.1 Equivariant X-complexes with coefficients

Before we define the pairing of the cyclic cohomologies of algebras and coalgebras we shall briefly discuss the way one generalizes the Cuntz-Quillen tower of \( X \)-complexes to embrace the cyclic homology and cohomology with coefficients. Below we shall need the description of cyclic cohomology of an \( \mathcal{H} \)-module algebra based on these ideas.

First of all we shall work in the category of (left) \( \mathcal{H} \)-module algebras \( A^H \) and their morphisms. If \( A \in \text{Ob} A^H \) and \( M \) is a (stable) anti-Yetter-Drinfeld module over \( \mathcal{H} \), then one can define the \textit{universal differential calculus of} \( A \) \textit{with values in} \( M \) as it is done in previous section and the Hochschild and cyclic operators on it. As before, we shall denote the module of \( M \)-valued differential forms by \( \Omega_H(A, M) \) and consider the operators \( b \) and \( B \). One defines the
Hodge filtration on $\Omega_H(A, \mathcal{M})$ and the associated Hodge tower of supercomplexes by the same formulas as in [2]:

\begin{equation}
F^n \Omega_H(A, \mathcal{M}) = b^n \Omega_H^{n+1}(A, \mathcal{M}) \oplus \bigoplus_{k>n} \Omega^k_H(A, \mathcal{M})
\end{equation}

(32)

\begin{equation}
\theta \Omega_H(A, \mathcal{M}) = (\Omega_H(A, \mathcal{M})/F^n \Omega_H(A, \mathcal{M})).
\end{equation}

(33)

The tower $\theta \Omega_H(A, \mathcal{M})$ is a special tower, and similar to the coefficientless case we have the following identities (here $\Omega = \Omega_H(A, \mathcal{M})$)

\begin{equation}
H_\nu(F^{n-1} \Omega/F^n \Omega) = \begin{cases} 
HH^{\nu}_n(A, \mathcal{M}), & \nu = n + 2\mathbb{Z}, \\
0, & \nu = n - 1 + 2\mathbb{Z}, 
\end{cases}
\end{equation}

(34)

\begin{equation}
H_\nu(\Omega/F^n \Omega) = \begin{cases} 
HC^{\nu}_n(A, \mathcal{M}), & \nu = n + 2\mathbb{Z}, \\
HD^{\nu}_n(A, \mathcal{M}), & \nu = n - 1 + 2\mathbb{Z}, 
\end{cases}
\end{equation}

\begin{equation}
H_\nu(\Omega) = HP_\nu(A, \mathcal{M}).
\end{equation}

Here $HH_n$, $HC_n$, $HD_n$ and $HP_\nu$ are respectively Hochschild, cyclic, (non commutative) de Rham homology and the $\mathbb{Z}/2\mathbb{Z}$ graded periodic homology of $A$, and $\bar{\Omega} \overset{\text{def}}{=} \varprojlim \Omega/F^n \Omega$. Similarly, one retrieves the cyclic (periodic, Hochschild) cohomology of $A$ (with coefficients in $\mathcal{M}$) by considering the dual complex $\text{Hom}(\theta \Omega, k)$. The first level $\Omega/F^1 \Omega$ of the Hodge tower is called the $X$-complex of $A$ (with coefficients in $\mathcal{M}$). It is precisely the super-complex $X_H(A, \mathcal{M})$ that appeared in the proposition [4].

Similarly to the cited paper of Cuntz and Quillen one can define a filtration on $X_H(R, \mathcal{M})$ associated to an ideal $I$ in algebra $R$ ($I$ should be stable under the action of $\mathcal{H}$):

\begin{equation}
F^{2n+1}_I X_H(R, \mathcal{M}) : \mathcal{M} \otimes_H I^{n+1} \Rightarrow \mathcal{F} (\mathcal{M} \otimes_H (I^{n+1} d R + I^n d I)), \\
F^{2n}_I X_H(R, \mathcal{M}) : \mathcal{M} \otimes_H I^{n+1} + [\mathcal{M} \otimes_H I^n, R] \Rightarrow \mathcal{F} (\mathcal{M} \otimes_H (I^n d R)).
\end{equation}

(35)

In this formulas and below we denote by $[\cdot, \cdot] : \mathcal{M} \otimes_H (I^n \otimes R) \to \mathcal{M} \otimes_H R$ the image of the map $[,] : \mathcal{M} \otimes_H (I^n \otimes R) \to \mathcal{M} \otimes_H R$ given by the formula

\begin{equation}
[,] (m \otimes_H (x \otimes r)) = m \otimes_H (x r) - m^{(0)} \otimes_H (S^{-1}(m^{-1})(r)x).
\end{equation}

(36)

for all $m \in \mathcal{M}$, $x \in I^n$, $r \in R$. This map is well-defined, because $\mathcal{M}$ is a AYD-module. Similarly the subscript $\mathcal{F}$ or the same sign in front of an expression denotes the factorization by the space of commutators.

Then we put $X^p_H(R, I; \mathcal{M}) = X_H(R, \mathcal{M})/F^p_I X_H(R, \mathcal{M})$:

\begin{equation}
X^{2n+1}_H(R, I; \mathcal{M}) : \mathcal{M} \otimes_H (R/I^{n+1}) \Rightarrow \Omega^1_H(R, \mathcal{M}) \mathcal{F} (\mathcal{M} \otimes_H (I^{n+1} d R + I^n d I)), \\
X^{2n}_H(R, I; \mathcal{M}) : \mathcal{M} \otimes_H R/\mathcal{M} \otimes_H I^{n+1} + [\mathcal{M} \otimes_H I^n, R] \Rightarrow \Omega^1_H(R, \mathcal{M}) \mathcal{F} (\mathcal{M} \otimes_H (I^n d R)).
\end{equation}

(37)

We apply this construction to $R = RA \overset{\text{def}}{=} \Omega^{even}(A) = \bigoplus_{i \geq 0} \Omega^{2i}(A)$, equipped with Fedosov product, $\omega \circ \omega' = \omega \omega' + (-1)^{[\omega]} d \omega d \omega'$, and the ideal $I = IA = \bigoplus_{i > 0} \Omega^{2i}(A)$ (the action of $\mathcal{H}$ on
RA is defined via its action on \(A\). Thus we obtain a tower \(\mathcal{X}_R(A, \mathcal{M}) = \mathcal{X}_R(RA, IA; \mathcal{M}) = (\mathcal{X}_R^{p}(RA, IA; \mathcal{M}))\) of supercomplexes, which is again a special tower and its homology (resp. cohomology) verifies the same equations \(\mathcal{X}_R\). Moreover this tower is homotopy equivalent to \(\theta \mathcal{N}_R(A, \mathcal{M})\). So the cyclic-type homology of \(A\) (with coefficients in \(\mathcal{M}\)) is given by the formulas, similar to \(\mathcal{X}_R\) with \(\mathcal{X}_R(A, \mathcal{M})\) instead of \(\theta \mathcal{O}\). In particular

\[
H_\nu(\mathcal{X}_R^p(A, \mathcal{M})) = \begin{cases} 
HC^p_\nu(A, \mathcal{M}), & \nu = n + 2\mathbb{Z}, \\
HD^p_\nu(A, \mathcal{M}), & \nu = n - 1 + 2\mathbb{Z}, 
\end{cases}
\]

\[
H_\nu(\mathcal{X}_R(A, \mathcal{M})) = HD^p_\nu(A, \mathcal{M}),
\]

where \(\mathcal{X}_R(A, \mathcal{M}) = \lim_\leftarrow \mathcal{X}_R^p(A, \mathcal{M})\). Further one can extend the definition of quasi-free algebras given in \(\mathcal{X}_R\) to the category \(\mathcal{A}^H\), by saying that algebras in this category are quasi-free, if they verify the conditions, similar to those, listed in \(\mathcal{X}\) Prop. 3.3 (see also \(\mathcal{X}\) Prop. 7.1), only this time all the morphisms should be in \(\mathcal{A}^H\) (the only problem is with the condition, concerning the homology dimension with respect to the Hochschild cohomology, see remark following the proposition \(\mathcal{X}\)). Then one can show, that for any exact sequence of \(\mathcal{H}\)-algebras with quasi-free \(\mathcal{R}\)

\[
0 \rightarrow I \rightarrow R \rightarrow A \rightarrow 0,
\]

which is splittable as a sequence of \(\mathcal{H}\)-modules (that is we suppose that one can choose a linear splitting \(\rho : A \rightarrow \mathcal{R}\), such that \(\rho(h(a)) = h(\rho(a))\)), the tower \(\mathcal{X}_R(R, I; \mathcal{M})\) is equivalent to \(\mathcal{X}_R(A, \mathcal{M})\). So one can use any quasi-free extension of \(A\) to calculate cyclic (co)homology of \(A\). The extension

\[
0 \rightarrow IA \rightarrow RA \rightarrow A \rightarrow 0
\]

is the universal quasi-free extension of \(A\) (this means that for any extension \(\mathcal{X}\) of \(A\) there is a map \((RA, IA) \rightarrow (R, I)\) covering the identity map on \(A\)).

### 3.2 Hopf-type cyclic cohomology and higher traces

In view of the statements formulated in the section \(\mathcal{X}\) one can give the following description of the cyclic cohomology of an algebra \(A\). First of all, as we have already said, \(H^\nu(\mathcal{X}_R^p(R, I; \mathcal{M})) = HC^p_\nu(A, \mathcal{M}), \nu = n + 2\mathbb{Z}\) for any quasi-free extension \(\mathcal{X}\) of \(A\). Thus for \(p = 2n\) all the cohomology classes of \(A\) are representable by linear functionals \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H I^{n}], R] \rightarrow k\), vanishing on the image of \(b : \Omega^1_H(R, \mathcal{M})/\mathcal{M} \otimes H (I^n dR) \rightarrow \mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H I^{n}], R].\) The image of \(b\) coincides with the image of \([\cdot]_I : \mathcal{M} \otimes H (R \otimes R) \rightarrow \mathcal{M} \otimes H R\), projected to \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H I^{n}], R\), so one can say that \(HC^{2n}_H(A, \mathcal{M})\) is generated by the linear functionals on \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H R, R].\)

Further observe that two such functionals give the same class in \(HC^{2n}_H(A, \mathcal{M})\), iff their difference is equal to the composition of \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H R, R] \rightarrow \Omega^1_H(R, \mathcal{M})/\mathcal{M} \otimes H (I^n (dR))\) and \(\text{a linear functional on the space of 1-forms. Let } \tau_0 \text{ and } \tau_1 \text{ be the functionals on } \mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} + [\mathcal{M} \otimes H R, R]\) defining the same cohomology class and \(\tau_0, \tilde{\tau}\) be the functionals on \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n} + [\mathcal{M} \otimes H R, R] \rightarrow \mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n} + [\mathcal{M} \otimes H R, R],\) equal to the composition of \(\tau_0, \tilde{\tau}\) with the natural projection \(\mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n+1} \rightarrow \mathcal{M} \otimes H R/\mathcal{M} \otimes H I^{n} + [\mathcal{M} \otimes H R, R].\) Similarly let \(\tilde{T}\) be the functional on \(\Omega^1_H(R, \mathcal{M})/\mathcal{M} \otimes H (I^n dR)^{-1}\) (here \(\mathcal{M} \otimes H I^n dR)^{-1}\) denotes
the preimage of $z(\mathcal{M} \otimes_{\mathcal{H}} I^n dR)$ in $\Omega^1_{\mathcal{H}}(R, \mathcal{M})$ equal to the composition of $T$ with the natural projection $\Omega^1_{\mathcal{H}}(R, \mathcal{M})/(\mathcal{M} \otimes_{\mathcal{H}} I^n dR)^{-1} \to \Omega^1_{\mathcal{H}}(R, \mathcal{M})/z(\mathcal{M} \otimes_{\mathcal{H}} (I^n dR))$. Then the condition that $T$ makes $\tau_0$ and $\tau_1$ cohomologous is equivalent to the equation $\tilde{\tau}_1 - \tilde{\tau}_0 = T d$.

There is a nice way to interpret this condition: consider the semi-direct product $L = R \oplus \Omega^1(R)$ of $R$ and $\Omega^1(R)$ (the multiplication is given by the formula

$$(a, \omega) \cdot (b, \omega') = (ab, a\omega' + \omega b).$$

Then $L$ is a $\mathcal{H}$-module algebra, $\mathcal{M} \otimes L$ is a $L$-bimodule and (left) $\mathcal{H}$-module and $I^{n+1} \oplus (I^n dR + I^{n-1} dR I + \cdots + dRI^n)$ is an ideal in $L$ (it is the $n+1$ power of $J = I \oplus \Omega^1(R)$). Consider the 1-parameter family of homomorphisms of vector spaces $1 \otimes u_t : \mathcal{M} \otimes_{\mathcal{H}} R \to \mathcal{M} \otimes_{\mathcal{H}} L$, extending the 1-parameter family of homomorphisms of algebras in $A^H$ (i.e. commuting with the action of $\mathcal{H}$) $u_t : R \to L, u_t(r) = (r \oplus t dr), t \in [0, 1]$. One can easily check that $u_t(I^{n+1}) \subseteq J^{n+1}$. Define linear functional $\tilde{T} : \mathcal{M} \otimes_{\mathcal{H}} L/J^{n+1}$ by the formula $\tilde{T}(m \otimes_{\mathcal{H}} (r \oplus x dy)) = \tilde{\tau}_0(m \otimes_{\mathcal{H}} r) + \tilde{T}(m \otimes_{\mathcal{H}} x dy), m \in \mathcal{M}, r \in R, xdy \in \Omega^1(R)$. Then clearly $\tilde{T}$ vanishes on the space of commutators $[\mathcal{M} \otimes_{\mathcal{H}} L, L]$ and on $J^{n+1}$ and verifies the formulas $\tilde{T} u_0 = \tilde{\tau}_0$ and $\tilde{T} u_1 = \tilde{\tau}_1$.

Contrarily, if there exists a 1-parameter family of morphisms $v_t : (R, I) \to (R', I')$ in $A^H$ for a $\mathcal{H}$-module algebra $R'$ and an $\mathcal{H}$-module ideal $I'$ in $R'$, and if there is a linear functional $T'$, vanishing on $(I')^{n+1}$ and $[\mathcal{M} \otimes_{\mathcal{H}} R', R']$, such that $T' v_0 = \tilde{\tau}_0$, $T' v_1 = \tilde{\tau}_1$ (here we write $v_t$ instead of $1 \otimes_{\mathcal{H}} v_t : \mathcal{M} \otimes_{\mathcal{H}} R \to \mathcal{M} \otimes_{\mathcal{H}} R'$), then the formula

$$T(m \otimes_{\mathcal{H}} x dy) = \int_0^1 T'(m \otimes_{\mathcal{H}} (v_t(x) \dot{v}_t(y)))$$

defines a linear functional on $\Omega^1_{\mathcal{H}}(R, \mathcal{M})$, which vanishes on the preimage of $z(\mathcal{M} \otimes_{\mathcal{H}} I^n dR)$ and on the commutators $[\mathcal{M} \otimes_{\mathcal{H}} \Omega^1(R), R]$ and such that $\tilde{\tau}_1 - \tilde{\tau}_0 = T d$.

Summarizing these observations, we define $\mathcal{M}$-twisted $\mathcal{H}$-equivariant higher even traces of order $n$ on $A$ (or even $\mathcal{M}$-traces for short) as $\mathcal{M}$-twisted traces on $R$-bimodule $\mathcal{M} \otimes_{\mathcal{H}} (R/I^{n+1})$ (defined for arbitrary extension $[R, I]$ of $A$). Here we say that a linear functional on $\mathcal{H}$-bimodule $M$ over an $\mathcal{H}$-algebra $R$ is an $\mathcal{M}$-twisted trace, if it vanishes on the space of commutators $[\mathcal{M} \otimes_{\mathcal{H}} M, R]$ (see formula (36)). We say that two such traces $\tau$ and $\tau'$ defined for different extensions $(R, I)$ and $(R', I')$ are equivalent, if there exist a third extension $(R'', I'')$ and maps $f : R'' \to R, f' : R'' \to R'$ of extensions, such that $\tau f = \tau' f'$. Since $(\mathcal{R}, I)$ is a universal extension, we can assume that (mod the equivalence relation) all traces are functionals on the same linear space. We say that two traces (on the same extension $(R, I)$) are homotopic, if there exists another extension $(R', I')$ of $A$ (in $A^H$), a degree $n$ even $\mathcal{M}$-trace $T'$ on it and a 1-parameter family of homomorphisms $v_t : (R, I) \to (R', I')$, such that $T' v_0 = \tilde{\tau}_0$, $T' v_1 = \tilde{\tau}_1$. Then

**Proposition 9.** There is a 1-1 correspondence between the cohomology classes in $HC^2_{\mathcal{H}}(A, \mathcal{M})$ and the homotopy classes of even $\mathcal{M}$-traces of degree $n$ on $A$.

Similarly the degree $2n+1$ cohomology of $A$ can be described in terms of the odd part of the cohomology of $\mathcal{X}^2_{\mathcal{H}}(R, I; \mathcal{M})$ for a quasi-free extension of $A$. This description would involve the equivalence relation cyclic 1-cocycles on $\mathcal{H}$-module algebra $R/I^{n+1}$ with coefficients in $\mathcal{M}$. However it is more convenient to use the following ideas similar to the discussion following the Proposition 9.5 of [2].
Let \((R, I)\) be an extension of \(A\) in \(\mathcal{A}^k\). One calls \(\mathcal{M}\)-twisted \(\mathcal{H}\)-equivariant odd higher trace of degree \(n\) on \(A\) an \(I\)-adic \(\mathcal{M}\)-twisted trace on \(\mathcal{M} \otimes_\mathcal{H} I^{n+1}\), i.e. a linear functional on \(\mathcal{M} \otimes_\mathcal{H} I^{n+1}\) vanishing on the space of commutators \([\mathcal{M} \otimes_\mathcal{H} I^{n+1}, I]\) (we shall denote the factor-space \(\mathcal{M} \otimes_\mathcal{H} I^{n+1}/[\mathcal{M} \otimes_\mathcal{H} I^{n+1}, I]\) by \((\mathcal{M} \otimes_\mathcal{H} I^{n+1})_\sharp\)). Two odd traces \(\tau\) on \((R, I)\) and \(\tau'\) on \((R', I')\) are equivalent, if they coincide on an extension \((R'', I'')\), which maps to \((R, I)\) and \((R', I')\). Two odd traces on the same extension \((R, I)\) are homotopic, if their difference is equal to the restriction of an ordinary trace on \(R\). Then

**Proposition 10.** There is a 1-1 correspondence between the the cohomology classes in \(H^{2n+1}_\mathcal{H}(A, \mathcal{M})\) and homotopy classes of equivalence classes of odd \(\mathcal{M}\)-traces of degree \(n\) on \(A\).

### 3.3 Crainic-type pairing

Now we can define the pairing. Let \(\mathcal{H}\)-module algebra \(A\) be at the same time a \(C\)-module algebra in a manner described in section 1, equations \((\mathbf{10}), (\mathbf{11})\). Let \((R, I)\) be an extension of \(A\), in which \(R\) is an \(\mathcal{H}\)-module and \(C\)-module algebra and \(I\) — ideal in \(R\), stable under the action of \(C\) and \(\mathcal{H}\). We assume that the action of \(\mathcal{H}\) and \(C\) on \(R\) verify the same conditions \((\mathbf{10})\) and \((\mathbf{11})\) and all the maps in the sequence \((\mathbf{38})\) are morphisms of \(\mathcal{H}\)- and \(C\)-modules. We also suppose, that the exact sequence \((\mathbf{38})\) is splittable not only as the sequence of \(\mathcal{H}\)-modules, but also as the sequence of \(C\)-modules. There is exists a section \(\rho\) of the epimorphism \(R \rightarrow A\), such that \(\rho(h(a)) = h\rho(a)\) and \(\rho(c(a)) = c\rho(a)\) for all \(h \in \mathcal{H}, c \in C\). One should think of the universal extension \(R = RA, I = IA\) of \(A\) as the model example of such extensions.

First of all we interpret the map \(\rho\) as a map from \(C\) to \(\text{Hom}(B(A), R)\). Here we let an element \(c \in C\) go to a map \(\rho(c) : A \rightarrow R, \rho(c)(a) = \rho(c(a))\) and \(B(A)\) stands for the bar-resolution of the algebra \(A\) (recall that as a linear space \(B(A) = \bigoplus_{n \geq 1} A^\otimes n\)). The space \(\text{Hom}(B(A), R)\) is a DG algebra w.r.t. the differential induced from the standard codifferential on \(B(A)\) and the cup-product of maps from coalgebra \(B(A)\) to the algebra \(R\): for \(\varphi \in \text{Hom}(A^{\otimes p}, R), \psi \in \text{Hom}(A^{\otimes q}, R)\) we put

\[
(39) \quad \varphi \cup \psi : A^{\otimes p+q} \rightarrow R, \quad \varphi \cup \psi(a_1, \ldots, a_{p+q}) = \varphi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_{p+q}).
\]

So we can in a unique way extend the map \(\rho\) to the map of DG algebras \(\rho^\sharp : W(C) \rightarrow \text{Hom}(B(A), R)\) (recall that this is the universal property of the Weil algebra, see remark \(\mathbf{11}\)). We begin the description of our pairing construction with the case of even cohomology classes in \(H^*_\mathcal{H}(A, \mathcal{M})\).

It is easy to see that the map \(\rho^\sharp\) sends elements of the form \(\omega_c \in W(C)\) to \(I\) and intertwines the action of \(\mathcal{H}\) on \(W(C)\) and \(\text{Hom}(B(A), R)\), if we define the latter via the action of \(\mathcal{H}\) on \(R\). So it gives the maps \(\mathcal{M} \otimes_\mathcal{H} \rho^\sharp_n : \mathcal{M} \otimes_\mathcal{H} W_n(C) \rightarrow \mathcal{M} \otimes_\mathcal{H} \text{Hom}(B(A), R/I^{n+1}) \subseteq \text{Hom}(B(A), \mathcal{M} \otimes_\mathcal{H} R/I^{n+1})\) and \(\mathcal{M} \otimes_\mathcal{H} \rho^\sharp_I : I^{n+1}_\mathcal{H}(C, \mathcal{M}) \rightarrow \text{Hom}(B(A), \mathcal{M} \otimes_\mathcal{H} I^{n+1})\). Finally, observe that \(\mathcal{M} \otimes_\mathcal{H} \rho^\sharp_n\) and \(\mathcal{M} \otimes_\mathcal{H} \rho^\sharp_I\) send the commutators’ subspace \([\mathcal{M} \otimes_\mathcal{H} W(C), W(C)]\) and \([\mathcal{M} \otimes_\mathcal{H} I(C)^n, I(C)]\) to the corresponding commutators’ subspace of morphisms in \(\mathcal{M} \otimes_\mathcal{H} \text{Hom}(B(A), R)\) and \(\text{Hom}(B(A), R)\), \([\mathcal{M} \otimes_\mathcal{H} \text{Hom}(B(A), R), \text{Hom}(B(A), R)]\) \(\subseteq \text{Hom}(B(A)^\sharp, [\mathcal{M} \otimes_\mathcal{H} R, R])\) and similarly with the ideal \(I\). Here \(B(A)^\sharp\) denotes the cocommutator subspace \((\Leftrightarrow \text{subspace of cyclically invariant tensors})\) in \(B(A)\). Thus we obtain maps of (co)chain complexes

\[
(40) \quad \tilde{\rho}_n : W_n(C, \mathcal{M})_z \rightarrow \text{Hom}(B(A)^\sharp, (\mathcal{M} \otimes_\mathcal{H} R/I^{n+1})_z)
\]
Let $\tau$ be an even $\mathcal{M}$-trace on $A$ of degree $n$, then $[\tau]$ will denote the corresponding cohomology class in $H^{2n}_A(A, \mathcal{M})$. We shall define the product of a class $[\omega] \in H^p(W^p_n(C, \mathcal{M})_z) = H^{-p-2n-1}_A(C, \mathcal{M})$ represented by cocycle $\omega \in W^p_n(C, \mathcal{M})_z$ with $[\tau]$ by the following rule
\begin{equation}
[\omega]|^p[\tau] = [\tau \circ \tilde{\rho}_n(\omega) \circ N].
\end{equation}
Here $N : B(A) \to B(A)^{\otimes 2}$, $N = 1 + t + \cdots + t^p$ — the universal cocontra $\tau$ on $A^{\otimes p} \subset B(A)$, $N : A^{\otimes p} \to (A^{\otimes p})^{\otimes 2}$. Thus, the map on the right can be regarded as cyclic cochain of degree $p - 1$ on $A$ (the degree 1 shift is due to the fact, that the grading in cyclic complex is defined so that degree $k$ is given to linear functionals on $A^{\otimes k+1}$). It is closed, because $\omega$ is (and $\tilde{\rho}_n$ commutes with differentials).

**Proposition 11.** The operation (42) is well-defined on the level of cohomology, i.e. it does not depend on the choice of representative $\omega$ of class $[\omega]$ in $H^p(W^p_n(C, \mathcal{M})_z)$, of trace $\tau$ representing an element in $H^{2n}_A(A, \mathcal{M})$ and of the splitting $\rho$ of extension (38).

**Proof.** As far as the independence of the class of $[\tau \circ \tilde{\rho}_n(\omega) \circ N]$ on the choice of $\omega$ is concerned, it follows from the fact that $\tilde{\rho}_n$ is a map of chain complexes.

If $\tau_0 \sim \tau_1$, then $\tau_0$ and $\tau_1$ are homotopic; choose a polynomial 1-parameter family of homomorphisms $u_t : R \to L$ and a degree $n$ $\mathcal{M}$-trace $T$ on $L$, connecting $\tau_1$ and $\tau_1$. Then one can consider instead of $u_t$ a map $U : R \to L[t]$. In this way we obtain from the splitting $\rho$ a map $W(C) \to Hom(B(A), \Omega(\mathbb{R}^1) \otimes L)$. Extending this map in a manner similar to what we had above, and using the trace $\int_0^1 \otimes T$ on $\Omega^1(\mathbb{R}^1) \otimes \mathcal{M} \otimes L / J^{n+1}$ we obtain a homotopy between $[\tau_0 \circ \tilde{\rho}_n(\omega) \circ N]$ and $[\tau_1 \circ \tilde{\rho}_n(\omega) \circ N]$.

Similarly, if $\rho^0$ and $\rho^1$ are two splittings of the sequence (38), we combine them in the following way $\rho = (1 - t)\rho^0 + t\rho^1$, $t \in [0, 1]$. As before we use this map to obtain a homomorphism $W(C) \to Hom(B(A), \Omega(\mathbb{R}^1) \otimes R)$, and use the trace $\int_0^1 \otimes T$ to define an element connecting $[\tau \circ \tilde{\rho}_n^0(\omega) \circ N]$ and $[\tau \circ \tilde{\rho}_n^1(\omega) \circ N]$.

Further let $\tau'$ be an odd $\mathcal{M}$-trace of degree $n$, and $[\tau'] \in H^{2n+1}_A(A, \mathcal{M})$ the cohomology class. We define its product with the class $[\omega] \in H^p(I^p_n(C, \mathcal{M})_z) = H^{-p-2n-2}_A(C, \mathcal{M})$ (see remark, following the theorem 5) as the result of the composition, similar to (42):\begin{equation}
[\omega]|^p[\tau'] = [\tau' \circ \tilde{\rho}_n(\omega) \circ N].
\end{equation}

**Proposition 12.** The operation (43) is well-defined on the level of cohomology.

**Proof.** The only thing that needs proof is the independence of (43) of the choice of representative $\tau'$ in the class $[\tau']$ (independence of the choice of splitting $\rho$ can be proven similarly to the even case). But $\tau'$ and $\tilde{\tau}$ are cohomologous iff their difference is equal to the restriction of a $\mathcal{M}$-trace $T$ on $R$. Using this trace instead of $\tau'$ and $\tilde{\tau}'$, we can define linear functional $T \circ \tilde{\rho}_n(\omega) \circ N$ on $B_p(A)^{\otimes 2}$, equal to the difference $\tau' \circ \tilde{\rho}_n(\omega) \circ N - \tilde{\tau}' \circ \tilde{\rho}_n(\omega) \circ N$. But $T$ being a trace on the greater algebra $R$, we can define $T \circ \tilde{\rho}_n(\omega) \circ N$ for arbitrary $\omega' \in W^p(C, \mathcal{M})_z$, in particular we can replace $I^p_n(C, \mathcal{M})_z$ with $I^p_n(C, \mathcal{M})_z$ (see remark following the proof of theorem 5).
think of the original element \( \omega \) as of a cocycle in \( \tilde{H}_{n+1}(C, \mathcal{M})_R \subseteq W^H(C, \mathcal{M})_R \). The complex \( W^H(C, \mathcal{M})_R \) being acyclic, there is an element \( \alpha \in W^H(C, \mathcal{M})_R \) for which \( d\alpha = \omega \). Then the functional \( T \circ \tilde{\rho}_{n,I}(\alpha) \circ N \) satisfies the equation \( (T \circ \tilde{\rho}_{n,I}(\alpha) \circ N)d = \tau' \circ \tilde{\rho}_{n,I}(\omega) \circ N - \tau'' \circ \tilde{\rho}_{n,I}(\omega) \circ N \).

Combining this construction and the result of the previous section we obtain the desired map:

\[
HC^p_H(C, \mathcal{M}) \otimes HC^q_H(A, \mathcal{M}) \xrightarrow{\tau} HC^{p+q}(A),
\]

defined for even \( q = 2n \) by the formula

\[
[x] \otimes [y] \mapsto \tau(\rho_n(\alpha^{-1}_n(x))(Nz))
\]

and for odd \( q = 2n + 1 \)

\[
[x] \otimes [y] \mapsto \tau'(\rho_n(\beta^{-1}_n(x))(Nz)),
\]

where \( \tau \), (resp. \( \tau' \)) are the even (resp. odd) \( \mathcal{M} \)-traces, representing the class \([y] \in HC^p_H(A, \mathcal{M})\) and \( \alpha_n, \beta_n \) are the isomorphisms \( H^*(W^H_{n+1}(C, \mathcal{M})) \cong HC^{n-2}_{H}(C, \mathcal{M}) \) and \( H^*(\tilde{H}_{n+1}(C, \mathcal{M})) \cong HC^{n-2}_{H}(C, \mathcal{M}) \) from theorem [S] and remark [T] and \( z \) is the argument — element from \( B(A) \).

Our next purpose is to determine the relation of the cup-product \( \tau' \) of equations (44)-(45) with \( S \)-operation on cyclic cohomologies involved. To this end we shall again generalize the methods, used in [I].

Let once again \( \rho \) be a \( C \)- and \( H \)-linear splitting of the extension \( B(A) \). We consider it as a map from \( C \) to \( \text{Hom}(B(A), R) \) and extend to a homomorphism of DG algebras \( \rho^\#: W(C) \rightarrow \text{Hom}(B(A), R) \). Consider the universal modules of 1-forms \( \Omega^1(W(C)) \), \( \Omega^1(R) \) and \( \Omega^1(\text{Hom}(B(A), R)) \) of algebras \( W(C) \), \( R \) and \( \text{Hom}(B(A), R) \) and the universal comodule of 1-forms \( \Omega_1(B(A)) \) on coalgebra \( B(A) \) (see [M]). First of all, we extend \( \rho^\# \) to the map \( \rho^\#_1 : \Omega^1(W(C)) \rightarrow \Omega^1(\text{Hom}(B(A), R)) \) commuting with differentials and such that the following diagramm commutes

\[
\begin{array}{ccc}
W(C) & \xrightarrow{\rho^\#} & \text{Hom}(B(A), R) \\
\downarrow d_u & & \downarrow d_u \\
\Omega^1(W(C)) & \xrightarrow{\rho^\#_1} & \Omega^1(\text{Hom}(B(A), R)).
\end{array}
\]

Here \( d_u \) denotes the universal differential for algebras \( W(C) \) and \( \text{Hom}(B(A), R) \). On the other hand, composition of a map \( \phi \in \text{Hom}(B(A), R) \) with the universal differential \( d_u : R \rightarrow \Omega^1(R) \) and with universal codifferential \( d^\#: \Omega^1(B(A)) \rightarrow B(A) \) are derivatives on \( \text{Hom}(B(A), R) \) with values in \( \text{Hom}(B(A), \Omega^1(R)) \) and \( \text{Hom}(\Omega_1(B(A)), R) \) (observe that both these spaces are modules over \( \text{Hom}(B(A), R) \)). By the universal property of \( \Omega^1 \) there’s a map of \( \text{Hom}(B(A), R) \)-modules \( \xi : \Omega^1(\text{Hom}(B(A), R)) \rightarrow \text{Hom}(B(A), \Omega^1(R)) \oplus \text{Hom}(\Omega_1(B(A)), R) \) such that \( \xi \circ d_u \) is a derivative on \( \text{Hom}(B(A), R) \).

Let us restrict the image of the map \( \xi \) to the first summand and consider its composition with \( \rho_1^\#: \Omega^1(W(C))_R \rightarrow \text{Hom}(\Omega_1(B(A)) \oplus R_2) \) (in other words that commutators in \( \Omega^1(W(C)) \) send cocommutators in
Ω₁(B(A)) to commutators in R. Moreover if we put ρ∗ = 1 ∘ (ρ⁺) : W(C) → Hom(B(A), R) then one can check by a straightforward computation that X(ρ⁺) = (ρ⁺, ρ⁻) is a map of supercomplexes of chain complexes X(ρ⁺) : X(W(C)) → Hom(X(B(A)), R).

Further consider the odd part of the filtration in X(W(C)) induced ideal I(C). The observation that elements of the type w_c in W(C) are sent by ρ⁺ to Hom(B(A), I) shows that X(ρ⁺)(F_{I(C)}^2 X(W(C))) ⊆ Hom(X(B(A)), I^{n+1}). Thus we obtain a collection of maps X^{2n+1}(ρ⁺) : X^{2n+1}(W(C), I(C)) → Hom(X(B(A)), (R/I^{n+1})₂). Finally one can introduce into this construction the coefficients module M, so that all the commutators become M-twisted commutators. We obtain a map

\[X^{2n+1}(\rho⁺) : X^{2n+1}(W(C), I(C); M) → Hom(X(B(A)), (M ⊗ ℋ R/I^{n+1})₂)\].

On the other hand one has the following

**Proposition 13.** The total (bigraded) complex of \(X^{2n+1}_H(W(C), I(C); M)\) is isomorphic to that of the sequence [28].

The proof of this statement is a word-to-word repetition of Theorem 7.9 of [7], so we omit it.

But the S-operations in \(HC^*_H(C, M)\) are given by diagram chasing in the sequence [28] and similarly the S-operations in \(HC^*_H(A, M)\) are related to diagram chasing in the total complex of \(X(B(A))\). Thus we see that, if \(q = deg [y]\) is even, one has the equation \(S([x]⁺[y]) = S[x]⁺[y]\). The same is true for odd \(q\). To see this, just observe that instead of dividing out by an ideal we can restrict to it thus obtaining a map from the exact sequence [28].

In order to find the relation with S-operations in \(HC^*_H(A, M)\) we recall that, if \(τ : (M ⊗ ℋ R/I^{n+1})₂ → k\) represents \([y] ∈ HC^{2n+1}_H(A, M)\), then the class \(S[y]\) is represented by the composition \(τ^p = τ ∘ p_{n+1}\) of \(σ\) with projection \(p_{n+1} : (M ⊗ ℋ R/I^{n+2})₂ → (M ⊗ ℋ R/I^{n+1})₂\). Thus we have \(([x]⁺S[y]) = τ^p(ρ_{n+1}(α⁻¹_{n+1}(x))(b))\) (here \(b ∈ B(A)\) is the argument of the cyclic cochain). But obviously \(p_{n+1}(ρ_{n+1}(ω)) = ρ_n(π_{n+1}(ω))\) for all \(ω ∈ W^H_{n+1}(C, M)₂\) (\(π_{n+1} : W^H_{n+1}(C, M)₂ → W^H_{n+1}(C, M)₂\) is the natural projection). So from the second statement of the theorem [4] we obtain \(τ^p(ρ_{n+1}(α⁻¹_{n+1}(x))(b)) = τ(ρ_n(π_{n+1} ∘ α⁻¹_{n+1}(x))(b)) = τ(ρ_n(α⁻¹_{n+1}(Sx))(b))\), and hence \([x]⁺S[y] = S[x]⁺[y] = S([x]⁺[y])\) (the last equality was proven earlier). Similarly one obtains the same equalities for the odd case.

We sum up the results of this section in the following theorem

**Theorem 14.** There is a pairing

\[HC^*_H(C, M) ⊗ HC^*_H(A, M) → HC^{p+q}(A)\]

of the Hopf-type cyclic cohomology of ℋ-module coalgebra C and C- and ℋ-module algebra A with coefficients in a SAYD-module M. This pairing satisfies the following relation with S-operation

\(S([x]⁺[y]) = S[x]⁺[y] = [x]⁺S[y]\).

### 4 Relation with the bivariant theory

In this section we give a nice interpretation of the construction described above in terms of a cohomology theory, closely related with the bivariant cyclic cohomology (see [6], [2] and [4]).
In effect, we manage to construct a map from the cohomology $HC^*(C, \mathcal{M})$ to the bivariant cohomology $HC^*(\mathcal{X}, X_{\mathcal{M}}(R, I; \mathcal{M}))$ of two special towers (see §2 of [2] for the definitions and relation of this cohomology with that of [6]). Then we show that the cup-product of section 3 is equal to the composition of this map with the usual composition product in bivariant theory. We also use this result to construct a dual cup-product of cyclic theories.

4.1 The mapping complex

For the most part of this section we assume that the algebra $A$ is finite-dimensional (as $k$-vector space). First of all observe that under this condition $\text{Hom}(A, V) \cong V \otimes \text{Hom}(A, k)$ for any vector space $V$. The isomorphism is given by the map

\begin{equation}
F : V \otimes \text{Hom}(A, k) \to \text{Hom}(A, V), \quad F(v \otimes \varphi)(a) = \varphi(a)v, \quad v \in V, \varphi \in \text{Hom}(A, k), \quad a \in A.
\end{equation}

The same is true if we replace $A$ with its bar-resolution, i.e.

\begin{equation}
\text{Hom}(B(A), V) \cong V \otimes \text{Hom}(B(A), k)
\end{equation}

as differential graded vector spaces. The differential on both sides is induced from the differential in the bar-resolution. To see this recall that $B(A) = \bigoplus_{n \geq 1} A^{\otimes n}$ and every tensor product in this sum is a finite-dimensional space. Let us denote the chain complex $\text{Hom}(B(A), k)$ by $B$. Then $B$ is in fact a DG algebra, multiplication being defined as the cup-product of cochains for $\varphi \in \text{Hom}(A^{\otimes p}, k)$, $\psi \in \text{Hom}(A^{\otimes q}, k)$ we put

$$\varphi \cup \psi : A^{\otimes p+q} \to k, \quad \varphi \cup \psi(a_1, \ldots, a_{p+q}) = \varphi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_{p+q})$$

(compare with [39]). If $V = R$ is an algebra then one can check that the isomorphism (48) is in fact an isomorphism of DG algebras, it is only necessary to show that the extension of map $F$ in (47) commutes with multiplication, which is clear.

Now consider the map $\rho^x$ from the previous section (recall that $\rho^x$ is induced by a splitting $\rho$ of the exact sequence [38]). In the view of the previous discussion it can be interpreted as a homomorphism of the free DG algebra $W(C)$ to $B \otimes R$. In §14 of [2] it is shown that in case of usual (non-differential) algebras this homomorphism can be extended to a unique (up to a homotopy) homomorphism of $X$-complexes: $X(\rho^x) : X(W(C)) \to X(R) \otimes X(B)$. The following result is a direct generalization of the corresponding theorems in the cited paper.

**Proposition 15.** The map $X(\rho^x)$ defined by the methods of §14 [2] is a map of super-complexes of cochain complexes. This map sends the $p$-th term of filtration [33] on $X(W(C))$, associated to the ideal $I(C) \subset W(C)$, to the subcomplex $X(B) \otimes F_{p-1}X(R)$. If all the algebras are $H$-module algebras, then the map $X(\rho^x)$ can be raised to the homomorphism of $H$-equivariant $X$-complexes with coefficients: $X_H(\rho^x; \mathcal{M}) : X_H(W(C), \mathcal{M}) \to X(H(R, \mathcal{M}) \otimes X(B)$.

**Proof.** In order to prove this statement we first recall the construction of $X(\rho^x)$ from [2].

First of all one considers the free product $F = R \ast B$ of the algebras $B$ and $R$. Let $J$ be the ideal in $F$ generated by the commutators $[b, r]$, $b \in B$, $r \in R$. Define the algebra $R \# B$ as the factor of $F$ by the square of the ideal $J$. Then $R \# B$ is a square-zero extension of $R \otimes B \cong F/J$ (we shall denote the image of $J$ in $R \# B$ by the same letter $J$). Since $W(C)$ is a quasi-free (in
fact, free) algebra, there’s a lifting of \( \rho^\# \) to a homomorphism \( \rho^\#: W(C) \to R \# B \) and one can use this lifting to define a map of super-complexes \( X(W(C)) \to X(R \# B) \).

On the other hand, there’s an isomorphism of super-complexes \( \mathcal{X}^2(F, J) \cong X(R) \otimes X(B) \). Thus we obtain a morphism:

\[
X(W(C)) \to X(R \# B) = X(F/J^2) \to \mathcal{X}^2(F, J) \cong X(R) \otimes X(B).
\]

The second map in this composition is the natural projections \( F/J^2 \to F/J^2 + [J, F] \) and \( \Omega^1(F/J^2)_2 \to \Omega^1(F)_2/(JdF) \).

Now the first statement of our proposition follows from the simple observation that all the algebras and modules in this construction can be made differential graded and all morphisms will commute with differentials. For instance one induces the grading on \( F = R \# B \) from that on \( B \) and similarly the differential. Moreover since \( W(C) \) is free as a DG algebra, one can choose the map \( W(C) \to R \# B \) to be a homomorphism of differential algebras.

In order to prove the second statement, let us denote in spite of a slight abuse of notation by the same symbol \( I \) the ideals generated by \( I \subseteq R \) in \( F \) and \( R \# B \). Let \( \hat{\rho}^\#: W(C) \to R \# B \) denote the homomorphism, covering \( \hat{\rho}^\#: W(C) \to R \otimes B = F/J \) (\( \hat{\rho}^\# \) exists, because \( W(C) \) is free DG algebra). Then \( \hat{\rho}^\#(I(C)) \subseteq I + J \) and hence \( \rho^\#(I_n+1(C)) \subseteq I_nJ + I_n^{-1}JI + \cdots + IJI^{n-1} + JI^n + I^{n+1} \) since \( J^2 = 0 \) in \( R \# B \). Similarly \( \rho^\#([I_n(C), W(C)]) \subseteq [I_n + I_n^{-1}J + I_n^{-2}JI + \cdots + JI^{n-1}, F] \).

In the even degree the isomorphism

\[
\phi_3^\#: \mathcal{X}^2(F, J) \cong X(R) \otimes X(B)
\]

is induced by the following map (see [3], Prop. 1.4 and [2], Prop. 14.1):

\[
\phi_0: R \# B \to R \otimes B \oplus \Omega^1(R) \otimes \Omega^1(B).
\]

It is given on the generators \( r \in R, b \in B \) by the formulas

\[
\phi_0(r) = r \otimes 1, \quad \phi_0(b) = 1 \otimes b
\]

and extended to the whole algebra \( F = R \# B \) in a way to obtain a homomorphism of algebras with respect to the following multiplication on the right

\[
(\xi_0 \otimes \eta_0) \circ (\xi_1 \otimes \eta_1) \equiv \xi_0 \xi_1 \otimes \eta_0 \eta_1 + (-1)^{|\xi_1|} \xi_0 d\xi_1 \otimes d\eta_0 \eta_1 (\text{mod } \Omega^2(R) \otimes \Omega^2(B)).
\]

Composing this map with the natural projection \( \Omega^1(R) \otimes \Omega^1(B) \to \Omega^1(R)_2 \otimes \Omega^1(B)_2 \) (we denote this composition by \( \phi_{0,2} \)) we obtain the following inclusions

\[
\phi_{0,2}\rho^\# \left( (F_0^{2n+1}X(W(C)))_+ \right) = \phi_{0,2} \left( \rho^\#(I_n+1(C)) \right)
\]

\[
\subseteq \phi_{0,2}(I_nJ + I_n^{-1}JI + \cdots + IJI^{n-1} + JI^n + I^{n+1})
\]

\[
\subseteq I^{n+1} \otimes B \oplus \Omega^1(B)
\]

\[
\subseteq (I^{n+1} + [I^n, R]) \otimes B \oplus \Omega^1(B)
\]

\[
= (F_0^{2n}X(R) \otimes X(B))_+
\]
since the terms containing $J$ give no input to the first summand and $\phi_0([r, b]) = dr \otimes db$. Similarly on the even terms of filtration we obtain

$$
\phi_{0,2}\rho^\#\left((F_{I(C)}^{2n})X(W(C))_+\right) = \phi_{0,2}(\rho^\#(I^{n+1}(C) + [I^n(C), W(C)]))
\subseteq \phi_{0,2}(I^n J + I^{n-1} J I + \cdots + IJ I^{n-1} + J I^n + I^{n+1})
\subseteq (I^{n+1} \otimes B \oplus \sharp(I^n dR) \otimes \Omega^1(B)_z)
\subseteq ([I^n, R] \otimes B \oplus \sharp(I^n dR) \otimes \Omega^1(B)_z),
$$

since the terms of type $[J, F]$ are sent to the commutators subspace $[\Omega^1(R), R] \otimes \Omega^1(B)$ by $\phi_0$ (see definition). So

$$
\phi_{0,2}\rho^\#\left((F_{I(C)}^{2n})X(W(C))_+\right) \subseteq I^n \otimes B \oplus \sharp(I^n dR + I^{n-1} dI) \otimes \Omega^1(B)_z
= (F_{I}^{2n-1}X(R) \otimes X(B))_+.
$$

Similarly the odd part of the isomorphism $\phi_2$ (equation (49)) is given by the following sequence of isomorphisms (Prop. 14.1 of [2]): first one has

$$
\Omega^1(F) \cong F \otimes_R \Omega^1(R) \otimes R F \oplus F \otimes_B \Omega^1(B) \otimes_B F
$$

which is induced by the universal properties of $\Omega^1(F)$ and the free product. Further one tensors this equality on both sides by $F/J$ to obtain

$$
\Omega^1(F)/(JdF + dFJ) \cong B \otimes \Omega^1(R) \otimes B \oplus R \otimes \Omega^1(B) \otimes R
$$

and finally, we factor-out the commutators obtaining

$$
(X^2(F, J))_- = \Omega^1(F)_z/\sharp(JdF) \cong \Omega^1(R)_z \oplus R \oplus \Omega^1(B)_z = (X(R) \otimes X(B))_-.
$$

We shall denote the resulting homomorphism by $\phi_{1,2}$. Then we have the following chain of inclusions (by abuse of notation we shall denote the natural extension of $\rho^\#$ to $\Omega^1(W(C))_z$ by the same symbol)

$$
\phi_{1,2}\rho^\#\left((F_{I(C)}^{2n})X(W(C))_+\right) = \phi_{1,2}\rho^\#(\sharp(I^n(C)dW(C)))
\subseteq \phi_{1,2}(\sharp((I^n + I^{n-1}J + \cdots + JI^{n-1})dF))
\subseteq \phi_{1,2}(\sharp(I^n dF))
\subseteq \sharp(I^n dR) \otimes B \oplus I^n \otimes \Omega^1(B)_z
\subseteq \sharp(I^n dR + I^{n-1}dI) \otimes B \oplus I^n \otimes \Omega^1(B)_z
= (F_{I}^{2n-1}X(R) \otimes X(B))_-.
$$

Since the odd part of the $2n + 1$ term of filtration is equal to a subspace of all 1-forms, generated by $dI^{n+1}$, we have

$$
\phi_{1,2}\rho^\#\left((F_{I(C)}^{2n+1})X(W(C))_+\right) \subseteq \phi_{1,2}(\sharp(Fd(I^{n+1} + I^n J + \cdots + JI^n)))
\subseteq \phi_{1,2}(\sharp(I^{n+1}dF + I^n dJ)).
$$

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The first summand gives elements in \( \tilde{\omega}(I^{n+1}dR) \otimes B \oplus I^{n+1} \otimes \Omega^1(B) \); and the second one — in \( \tilde{\omega}(I^n dR) \otimes B \oplus [I^n, R] \otimes \Omega^1(B) \). The last inclusion follows from the construction of the map \( \phi_{1,3} \); and the fact that \( J \) is generated by the commutators \([r, b] \). So

\[
\phi_{1,3} \rho^\# \left( (F_{I(C)}^{2n+1}X(W(C)))_\cdot \right) \subseteq \phi_{1,3} \left( \tilde{\omega}(I^{n+1}dF + I^n dJ) \right)
\subseteq \left( \tilde{\omega}(I^{n+1}dR) \otimes B \oplus I^{n+1} \otimes \Omega^1(B) \right)_2 \\
+ \left( \tilde{\omega}(I^n dR) \otimes B \oplus [I^n, R] \otimes \Omega^1(B) \right)_2
\subseteq \tilde{\omega}(I^n dR) \otimes B \oplus (I^{n+1} + [I^n, R]) \otimes \Omega^1(B)_2
= (F_{I(C)}^{2n} X(R) \otimes X(B))_\cdot.
\]

Finally in order to prove the last statement, one introduces the action of \( \mathcal{H} \) on \( F \) via its action on \( R \). Then it is a matter of direct calculation to show that one can extend all the definitions to the \( \mathcal{H} \)-equivariant complexes with coefficients in \( \mathcal{M} \). For instance, one should use the maps \( 1_{\mathcal{M}} \otimes \mathcal{H} \rho^\# \) and \( 1_{\mathcal{M}} \otimes \mathcal{H} \rho^\# \) instead of \( \rho^\# \) and \( \rho^\# \), regard the map \( \phi_0 \) as a map of \( \mathcal{H} \)-modules, and modify the definition of \( \phi_{1,3} \). Observe that the image of \( F_{I(C)}^p X^\mathcal{H}(W(C), \mathcal{M}) \) is included in \( F_{I(C)}^{p-1} X^\mathcal{H}(R, \mathcal{M}) \otimes X(B) \), just like in the non-equivariant case.

\[ \square \]

Remark 5. One can show as a further generalization of §14, [2] (prop. 14.3) that the map \( X(W(C)) \to X(R) \otimes X(B) \) defined above is a natural extension of the map \( X(W(C)) \to R \otimes X(B) \), induced by the map \( X(W(C)) \to X(R \otimes B) \) and the natural projection \( \Omega^1(R \otimes B) \to R \otimes \Omega^1(B) \). Similarly in the equivariant case our map is an extension of \( X(W(C)) \to (\mathcal{M} \otimes X(R, I)) \otimes X(B) \). And on the \( 2n+1 \)-st term of filtration our map is an extension of \( F_{I(C)}^{2n+1} X(W(C)) \to I^{n+1} \cap [R, R] \otimes X(B) \), while on the \( 2n \)-th term \( F_{I(C)}^{2n} X(W(C)) \to I^{n+1} \cap [I^n, I] \otimes X(B) \), since \( \tilde{\omega}(I^{n+1}dR) \subseteq \tilde{\omega}(I^n dI) \) and hence \( \tilde{\omega}(I^{n+1}dR + I^n dI) = \tilde{\omega}(I^n dI) \).

From the proposition [15], it follows, that one can define the maps of super-complexes of cochain complexes for all \( p \geq 1 \):

\[
(50) \quad \mathcal{X}_p^\mathcal{H} : \mathcal{X}_p^\mathcal{H}(W(C), I(C); \mathcal{M}) \to \mathcal{X}_p^\mathcal{H}(R, I; \mathcal{M}) \otimes X(B).
\]

Since we have assumed that \( A \) is finite-dimensional, then similarly to the discussion in the beginning of this section we have the isomorphism of towers of complexes \( \mathcal{X}_\mathcal{H}(R, I; \mathcal{M}) \otimes X(B) \) and \( \text{Hom}(X(B(A)), \mathcal{X}_\mathcal{H}(R, I; \mathcal{M})) \) given by the level-wise isomorphisms

\[
(51) \quad \mathcal{X}_\mathcal{H}^p(R, I; \mathcal{M}) \otimes X(B) \cong \text{Hom}(X(B(A)), \mathcal{X}_\mathcal{H}^p(R, I; \mathcal{M})).
\]

Let us denote the tower of super-complexes \( \{ \text{Hom}(X(B(A)), \mathcal{X}_\mathcal{H}^p(R, I; \mathcal{M})) \} \) by \( \mathcal{X}_\mathcal{H}(A; (R, I; \mathcal{M})) \). We come to the following proposition

**Proposition 16.** There exists a morphism of towers of super-complexes of cochain complexes \( \tilde{\mathcal{X}}_{\mathcal{H}}^p : \mathcal{X}_\mathcal{H}(W(C), I(C); \mathcal{M}) \to \mathcal{X}_\mathcal{H}(A; (R, I; \mathcal{M})) \) induced by a \( \mathcal{H} \)- and \( C \)-linear splitting \( \rho \) of the exact sequence [3,8]. By this we mean that there exists a series of morphisms \( \tilde{\mathcal{X}}_{\mathcal{H}}^p_{\rho, \mathcal{M}}, p \geq 1 \),

\[
(52) \quad \tilde{\mathcal{X}}_{\mathcal{H}}^p_{\rho, \mathcal{M}} : \mathcal{X}_\mathcal{H}^p(W(C), I(C); \mathcal{M}) \to \text{Hom}(X(B(A)), \mathcal{X}_\mathcal{H}^{p-1}(R, I; \mathcal{M})), p \geq 1,
\]
which commute with the natural projections of complexes on both sides:

$$
\begin{align*}
\mathcal{X}_H^p(W(C), I(C); \mathcal{M}) & \xrightarrow{\tilde{X}_H^p\rho_M} \text{Hom}(X(B(A)), \mathcal{X}_H^{p-1}(R, I; \mathcal{M})) \\
\downarrow & \\
\mathcal{X}_H^{p-1}(W(C), I(C); \mathcal{M}) & \xrightarrow{\tilde{X}_H^{p-1}\rho_M} \text{Hom}(X(B(A)), \mathcal{X}_H^{p-2}(R, I; \mathcal{M})).
\end{align*}
$$

Moreover the homotopy classes of these morphisms does not depend on the choice of the splitting $\rho$.

Before we prove this statement, we need to explain in what way we understand the homotopy between the maps of super complexes in this setting. If $Y = \{(Y_0^n)_{n=0}^{\infty}, b_0, b_1\} = ((Y^n_1)_{n=0}^{\infty}, d)\}$ is a supercomplex of cochain complexes, then we define its cohomology as the shifted by $-1$ cohomology of the following periodic double complex

\[
\begin{array}{ccccccc}
\cdots & & & & & & \\
& d_0 & & d_0 & & d_0 & \\
Y_0^0 & \xrightarrow{b_0} & Y_1^0 & \xrightarrow{b_1} & Y_0^1 & \xrightarrow{b_0} & Y_1^1 & \cdots \\
& d_0 & & d_1 & & d_0 & \\
Y_0^1 & \xrightarrow{b_0} & Y_1^1 & \xrightarrow{b_1} & Y_0^2 & \xrightarrow{b_0} & Y_1^2 & \cdots \\
& d_0 & & d_1 & & d_0 & \\
Y_0^2 & \xrightarrow{b_0} & Y_1^2 & \xrightarrow{b_1} & Y_0^3 & \xrightarrow{b_0} & Y_1^3 & \cdots \\
& & & & & & \\
\end{array}
\] (53)

Clearly any map $X \rightarrow Y$ of two super-complexes of cochain complexes defines a map of bi-complexes (53) and hence of their total complexes. The proposition 16 states that the induced maps of total complexes are homotopic as maps of cochain complexes.

**Proof.** The first and the second statements of this proposition are evident. (They follow from the inclusions of the terms of filtration, Prop. 15.) In order to proof that the maps $\tilde{X}_H^p\rho_M$ and $\tilde{X}_H^p\rho'_M$ defined for two different splittings $\rho$ and $\rho'$ are homotopic, we consider as in section 3.3 linear combination of splittings $t\rho + (1-t)\rho'$ as a map from $C$ to the algebra of $R$-valued polynomials $R[t]$. We extend this map to a homomorphism $\tilde{X}(W(C)) \rightarrow \text{Hom}(X(B(A)), X(\Omega^*(\mathbb{R}^1) \otimes R))$ and similarly in $\mathcal{H}$-equivariant setting and with coefficients module $\mathcal{M}$. Finally we consider the corresponding maps of total complexes (53) and integrate the forms on $\mathbb{R}^1$ that belong to the image of this map from 0 to 1. This gives the desired homotopy. \qed

Let us conclude this subsection with a brief discussion of corrections that should be made in our construction, if the algebra $A$ is not finite-dimensional.

First of all if $A$ can be represented as an inverse limit of its finite-dimensional subalgebras $A_\lambda$, then everything is clear: one can obtain all the maps that we need as direct and inverse limits of the corresponding maps for the subalgebras $A_\lambda$. In general $A$ can not be represented in this way. However its bar-resolution $B(A)$ is a direct limit of its finite-dimensional differential subcoalgebras $C_\lambda, \lambda \in \Lambda$. Moreover one can choose the subcoalgebras $C_\lambda$ so that $C_\lambda \subseteq C_\chi, \lambda \leq \chi$. 

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Then $B = \text{Hom}(B(A), k)$ is an inverse limit of its finite-dimensional subalgebras $B_\lambda = \text{Hom}(C_\lambda, k)$. For each of these subalgebras there’s a map

$$X_H(R, \mathcal{M}) \otimes X(B_\lambda) \to \text{Hom}(X(C_\lambda), X_H(R, \mathcal{M})).$$

These maps commute with natural projections $B_\lambda \to B_\nu$ and thus define a map

$$X_H(R, \mathcal{M}) \otimes \lim X(B_\lambda) \to \lim(\text{Hom}(X(C_\lambda), X_H(R, \mathcal{M}))) \otimes X(B_\lambda) \to \lim\text{Hom}(X(C_\lambda), X_H(R, \mathcal{M})).$$

Combining the map $W^H(C, \mathcal{M}) \to \text{Hom}(B(A), R)$ with natural projections $\text{Hom}(B(A), R) \to \text{Hom}(C_\lambda, R) \cong B_\lambda \otimes R$ and passing to the $X$-complexes we obtain a collection of maps

$$X_H(W(C), \mathcal{M}) \to X_H(R, \mathcal{M}) \otimes X(B_\lambda), \ \lambda \in \Lambda,$$

which verify the same properties with respect to the filtration as the map of Prop. 15 and commute with projections in the inverse system. Thus we obtain a map

$$X_H(W(C), \mathcal{M}) \to \lim\text{Hom}(X(C_\lambda), X_H(R, \mathcal{M})).$$

Finally we observe that if $C_\lambda$ are chosen as described above, there’s a map

$$\lim\text{Hom}(X(C_\lambda), X_H(R, \mathcal{M})) \to \text{Hom} \left( \lim\text{Hom}(X(C_\lambda), X_H(R, \mathcal{M})) \right)$$

$$\to \text{Hom}(X(\lim X(C_\lambda), X_H(R, \mathcal{M})) = \text{Hom}(X(B(A)), X_H(R, \mathcal{M})).$$

### 4.2 Towers of supercomplexes and bivariant cohomology

It is natural to regard the tower $X_H(A; (R, I; \mathcal{M}))$ as complex, determining a (version of) the bivariant cyclic cohomology theory of algebra $A$ with values in $A$ (with coefficients $\mathcal{M}$). Below we shall develope this analogy and show how one obtains a map of two towers of super-complexes from elements of $X_H(A; (R, I; \mathcal{M}))$. However this construction of bivariant cohomology is somewhat different from those, given in [2] and [3]. We are not going to discuss here in details the relation of these two cohomology theories (that of the definition [2] below and the one defined in the cited papers).

First observe that $X(B(A))$ is a super-complex of chain complexes (i.e. $X(B(A)) = \{X_0 \leadsto X_1\}$ where both $X_i$, $i = 0, 1$ are complexes with differentials of degrees $-1$). We shall denote the structure maps of this complex by $\beta : B(A) \to \Omega_1(B(A))^2$, $\delta_1 : \Omega_1(B(A))^2 \to B(A)$ and the symbols $\delta_0$ and $\delta_1$ will denote the differentials induced from the standard differential in $B(A)$. One can construct the 2-periodic bicomplex $biX(B(A))$ in a manner similar to [53]. Consider the total complex $Q = \text{Tot}(biX(B(A)))$. Now cyclic cohomology $HC^p(A)$ of $A$ is equal to the $(p - 2)$-dimensional cohomology of $Q$ (the first column of $biX(B(A))$, which is just $B(A)$, is acyclic and the $n$-th tensor power of $A$ corresponds to the $n - 1$ degree in cyclic homology).

On the other hand, we can associate to $X(B(A))$ a tower of 2-periodic bicomplexes $X^p(A)$, $p \geq 0$, $X^p(A) = \{X^p(A)_{i,j}\}$, $i, j \in \mathbb{Z}$, $j \geq 1$, where

$$X^p(A)_{i,j} = \begin{cases} 0, & j > p + 1 \\ X_\nu(B(A))_{p+1}/\partial_\nu X_\nu(B(A))_{p+2}, & j = p + 1, \ \nu = i + 2\mathbb{Z} \\ X_\nu(B(A))_{j} & 1 \leq j < p + 1, \ \nu = i + 2\mathbb{Z} \end{cases}$$

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(here $\partial_i$ denotes the differential in the $i$-th column). For example $X^1(A)$ looks as follows

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_2/\partial_0B_3 & \bar{\Omega}(B_2)/\partial_1\bar{\Omega}(B_3) & B_2/\partial_0B_3 & \bar{\Omega}(B_2)/\partial_1\bar{\Omega}(B_3) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
B_1 & \bar{\Omega}(B_1) & B_1 & \bar{\Omega}(B_1) \\
\end{array}
\]

Here we abbreviated $B_p(A)$ to $B_p$ and $\Omega_1(B_p(A))$ to $\bar{\Omega}(B_p)$, $\beta'$, $\delta'^{\prime}$ and $\delta'_i$ denote the maps induced by $\beta'$, $\delta'^{\prime}$ and $\delta'_i$ on factor-spaces. The leftmost column at this diagram corresponds to $i = 0$ and the lowest row is indexed by $j = 1$. The maps $X^{p+1}(A) \rightarrow X^p(A)$, $p \geq 0$ are given by the natural projections of columns.

Observe now that the total complex of $X^p(A)$ is 2-periodic and we can finally define the tower of super-complexes $X(B(A))$ by the formula

\[(56) \quad X^p_{\nu+2\mathbb{Z}}(B(A)) = \text{Tot}_\nu(X^p(A)), \quad \nu = 0, 1.\]

Here $\text{Tot}_\nu(X^p(A)) = \bigoplus_{i+j=\nu} X^p(A)_{i,j}$. The differentials in $X^p(B(A))$ are induced from $\text{Tot}_\nu(X^p(A))$. Then we have the following proposition.

**Proposition 17.** The tower $X(B(A))$ is a special tower of super-complexes, whose cohomology is given by equation

\[H_\nu(X^p(B(A))) = \begin{cases} 
HC_p(A), & \nu = p + 2\mathbb{Z} \\
HD_p(A), & \nu = p - 1 + 2\mathbb{Z}
\end{cases}\]

and the $S$-operation in the cyclic cohomology of $A$ is induced by the natural projections $X^{p+2} \rightarrow X^p$. Further the periodic cohomology of $A$ is given by the formula

\[H_\nu(\hat{X}(B(A))) = HP_\nu(A)\]

where $\hat{X}(B(A)) = \lim \sup X^p(B(A))$. In particular the supercomplex $\hat{X}(B(A))$ with its natural filtration is homotopy equivalent to the inverse limit of the tower $X_A$ of Cuntz and Quillen.

**Proof.** In order to prove the first equality we observe that $H_{p+2\mathbb{Z}}(X^p(B(A))) = H_{p+2}(Q)$, which is equal, as we have remarked, to $HC_p(A)$. In fact $X^p_{p+2\mathbb{Z}}(B(A)) = Q_{p+2}/(B(A) \cap Q_{p+2} + \partial_1\Omega_1(B_{p+2}(A))^2)$, and we recall, that $B(A)$ is acyclic.

In order to prove two other identities we observe that the bicomplex $X^p(A)$ is the total complex of $X(B(A))/F^p(A)$, where $F^0(A) \supset F^1(A) \supset \cdots \supset F^p(A) \supset \cdots \supset X(B(A))$ is a decreasing filtration,

\[(57) \quad F^p(A)_0 = \partial_0B_{p+2}(A) \oplus \bigoplus_{k \geq p+2} B_k(A), \quad F^p(A)_1 = \partial_1\Omega_1(B_{p+2}(A))^2 \oplus \bigoplus_{k \geq p+2} \Omega_1(B_k(A))^2.\]

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Now it is evident that the homology of $\text{gr}^p\mathcal{X}(B(A)) = F^{p-1}(A)/F^p(A)$ is equal to
\begin{equation}
H_{p+2\mathbb{Z}}(\text{gr}^p\mathcal{X}(B(A))) = H\mathcal{H}_p(A), \quad H_{p-1+2\mathbb{Z}}(\text{gr}^p\mathcal{X}(B(A))) = 0.
\end{equation}
Hence the tower $\mathcal{X}(B(A))$ is special. This observation also proves the formulas for the periodic and de Rham cohomology of $\mathcal{X}(B(A))$.

Finally the statement, concerning the supercomplex $\tilde{\mathcal{X}}(B(A))$ comes from the explicit description of this complex as the super complex, associated with the bicomplex $b\mathcal{X}(B(A))$. So the complex $\tilde{\mathcal{X}}(B(A))$ is equivalent (in fact, isomorphic) to the standard complex computing the periodic cyclic homology, which is known to be homotopy equivalent to that of Cuntz and Quillen.

\textbf{Remark 6.} Explicitly complexes $\mathcal{X}^p(B(A))$ are given by the following formulas: if $p = 2k$

\begin{align}
\mathcal{X}^{2k}(B(A))_0 &= \Omega_1(B_{2k+1}(A))^2/\partial_1\Omega_1(B_{2k+2}(A))^2 \oplus B_{2k}(A) \\
&\quad \oplus \Omega_1(B_{2k-1}(A))^2 \oplus B_{2k-2}(A) \oplus \cdots \oplus \Omega_1(B_1(A))^2,
\end{align}

\begin{align}
\mathcal{X}^{2k}(B(A))_1 &= B_{2k+1}(A)/\partial_1 B_{2k+2}(A) \oplus \Omega_1(B_{2k}(A))^2 \\
&\quad \oplus B_{2k-1}(A) \oplus \Omega_1(B_{2k-2}(A))^2 \oplus \cdots \oplus B_1(A),
\end{align}

and if $p = 2k + 1$

\begin{align}
\mathcal{X}^{2k+1}(B(A))_0 &= B_{2k+2}(A)/\partial_1 B_{2k+3}(A) \oplus \Omega_1(B_{2k+1}(A))^2 \\
&\quad \oplus B_{2k}(A) \oplus \Omega_1(B_{2k-2}(A))^2 \oplus \cdots \oplus \Omega_1(B_1(A))^2,
\end{align}

\begin{align}
\mathcal{X}^{2k+1}(B(A))_1 &= \Omega_1(B_{2k+2}(A))^2/\partial_1 \Omega_1(B_{2k+3}(A))^2 \oplus B_{2k+1}(A) \\
&\quad \oplus \Omega_1(B_{2k}(A))^2 \oplus B_{2k-2}(A) \oplus \cdots \oplus B_1(A).
\end{align}

In a dual way, given a super-complex of cochain complexes $Y^0 \Rightarrow Y^1$ (i.e. a supercomplex in which both summands $Y^0$ and $Y^1$ are positively graded complexes $Y^\nu = \{Y^\nu_p\}_{p \geq 1}$ with differentials $\partial^\nu$ of degree +1, $\nu = 0, 1$, see above) one can associate to it an \textit{inverse tower} of supercomplexes:

\begin{equation}
Y^\nu = \text{Tot}^\nu\tilde{Y}_p, \; \nu = 0, 1
\end{equation}

where $\tilde{Y}_p, \; p \geq 0$ is a tower of bicomplexes, $\tilde{Y}_p = \{\tilde{Y}_p^{i,j}\}$

\begin{equation}
\tilde{Y}_p^{i,j} = \begin{cases}
0, & j > p \\
\text{Ker} \partial^\nu Y^\nu_p, & j = p, \; \nu = i + 2\mathbb{Z} \\
Y^\nu_j, & 0 \leq j < p, \; \nu = i + 2\mathbb{Z}.
\end{cases}
\end{equation}

Here the term \textit{inverse tower} means that this time the structure maps act in the positive direction, $Y^p \rightarrow Y^{p+1}$, i.e. their direction is opposite to that in $\mathcal{X}$-tower. Explicitly the $p = 2k$-th level of this tower is given by

\begin{align}
Y^0_{2k} &= \text{Ker} \partial^0(Y^0_{2k}) \oplus Y^1_{2k-1} \oplus Y^0_{2k-2} \oplus \cdots \oplus Y^0_0, \\
Y^1_{2k} &= \text{Ker} \partial^1(Y^1_{2k}) \oplus Y^0_{2k-1} \oplus Y^1_{2k-2} \oplus \cdots \oplus Y^1_0,
\end{align}
and the \( p = (2k + 1) \)-st level by

\[
\begin{align*}
\mathcal{Y}^0_{2k+1} &= \text{Ker}\partial(Y^1_{2k+1}) \oplus Y^0_{2k} \oplus Y^1_{2k-1} \oplus \cdots \oplus Y^0_0, \\
\mathcal{Y}^1_{2k+1} &= \text{Ker}\partial(Y^0_{2k+1}) \oplus Y^1_{2k} \oplus Y^0_{2k-1} \oplus \cdots \oplus Y^1_0.
\end{align*}
\]

(67) (68) (69)

In particular one can apply this construction to \( \mathcal{X}^k_\mathcal{H}(A; (R, I; \mathcal{M})) \). In this case \( Y^0_p = \text{Hom}(B_{p+1}(A), \mathcal{X}^k(R, I; \mathcal{M})_0) \oplus \text{Hom}(\Omega_1(B_{p+1}(A))^j, \mathcal{X}^k(R, I; \mathcal{M})_1) \) and \( Y^1_p = \text{Hom}(B_{p+1}(A), \mathcal{X}^k(R, I; \mathcal{M})_1) \oplus \text{Hom}(\Omega_1(B_{p+1}(A))^j, \mathcal{X}^k(R, I; \mathcal{M})_0) \). Since the differentials \( \partial' \) in this complex are given by dualizing the corresponding differentials \( \partial \) in \( X(B(A)) \), we conclude that the following statement holds

**Proposition 18.** Let \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \) denote the \( p \)-th level of the tower, associated to \( \mathcal{X}^k_\mathcal{H}(A; (R, I; \mathcal{M})) \). Then there is an isomorphism of supercomplexes

\[
\mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \to \text{Hom}(\mathcal{X}^p(B(A)), \mathcal{X}^k_\mathcal{H}(R, I; \mathcal{M})).
\]

Here on the right hand side stands the mapping complex of two supercomplexes.

Observe that the collection of supercomplexes \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \) is naturally a tower in both directions, i.e. there are maps \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \to \mathcal{Y}^k_{p+1}(A, R, I; \mathcal{H}, \mathcal{M}) \) and \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \to \mathcal{Y}^k_{p-1}(A, R, I; \mathcal{H}, \mathcal{M}) \), commuting with differentials. Consider the diagram (we shall call it a bitower, associated to \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \))

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathcal{Y}^2_0 & \to & \mathcal{Y}^2_1 & \to & \mathcal{Y}^2_2 & \to & \mathcal{Y}^2_3 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{Y}^1_0 & \to & \mathcal{Y}^1_1 & \to & \mathcal{Y}^1_2 & \to & \mathcal{Y}^1_3 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{Y}^0_0 & \to & \mathcal{Y}^0_1 & \to & \mathcal{Y}^0_2 & \to & \mathcal{Y}^0_3 & \to & \cdots \\
\end{array}
\]

(70)

where \( \mathcal{Y}^k_p \) stands for \( \mathcal{Y}^k_p(A, R, I; \mathcal{H}, \mathcal{M}) \). Let \( T_n\mathcal{Y}, \ n \in \mathbb{Z} \) denote the subdiagram of (70), such that \( \mathcal{Y}^k_p \in T_n\mathcal{Y} \) iff \( p - k \geq n \). Now we can give the following definition

**Definition 2.** The \( n \)-dimensional bivariant cyclic cohomology of the tower \( \mathcal{X}^k_\mathcal{H}(A; (R, I; \mathcal{M})) \) is

\[
HC^n(A, R, I; \mathcal{H}, \mathcal{M}) \overset{\text{def}}{=} H_{n+2\mathbb{Z}}(\lim\limits_{\to k} \lim\limits_{\to p} (\mathcal{Y}^k_p \in T_n\mathcal{Y}))).
\]

(71)

This definition is inspired by discussions in [3] and [2]. Recall (14), that for two pro-vector spaces \( V = \{V_i\} \) and \( W = \{W_j\} \) one defines the space of homomorphisms \( V \to W \) by the formula

\[
\text{Hom}(V, W) = \lim\limits_{\to i} \lim\limits_{\to j} \text{Hom}(V_i, W_j).
\]
Further, given two towers of supercomplexes $\mathcal{X}$ and $\mathcal{X}'$ (for instance, one can take the canonical towers associated to two algebras), one defines their bivariant cohomology (2) by formula

$$H^n_c(\mathcal{X}, \mathcal{X}') \overset{\text{def}}{=} H_{n+2\mathbb{Z}}(\text{Hom}_n(\hat{\mathcal{X}}, \hat{\mathcal{X}}')),$$

where $\hat{\mathcal{X}} = \lim_{\leftarrow} \mathcal{X}^p$ and

$$\text{Hom}_n(\hat{\mathcal{X}}, \hat{\mathcal{X}}') = \left\{ f : \mathcal{X} \to \mathcal{X}' | \forall m, \ f(F^m+n\hat{\mathcal{X}}) \subset F^m\hat{\mathcal{X}}' \right\}.$$  

Here $F^n\hat{\mathcal{X}}$ is the natural filtration on the inverse limit of towers.

Now the following proposition is an evident consequence of our definitions and Proposition 17 (its last statement):

**Proposition 19.** There’s a natural map of complexes

$$\lim_{\leftarrow} \lim_{\leftarrow} \lim_{\leftarrow} (\mathcal{Y}^k_\mathcal{P} \in T_n\mathcal{Y}) \to \text{Hom}_n(\hat{\mathcal{X}}(B(A)), \hat{\mathcal{X}}_\mathcal{H}(R, I; \mathcal{M})), $$

inducing (since $\hat{\mathcal{X}}(B(A))$ is equivalent to $\hat{\mathcal{X}}_A$) a map of cohomology $HC^*(A, R, I; \mathcal{H}, \mathcal{M}) \to HC^n(\mathcal{X}_A, \mathcal{X}_\mathcal{H}(R, I; \mathcal{M}))$.

In fact one can make the following conjecture

**Conjecture 20.** There’s an isomorphism of homology theories

$$HC^*(A, R, I; \mathcal{H}, \mathcal{M}) \cong HC^*(\mathcal{X}_A, \mathcal{X}_\mathcal{H}(R, I; \mathcal{M}))$$

induced by the map (74) of Proposition 19.

However we shall not need this statement in full generality.

In fact the construction described above can be applied to any tower of supercomplexes of cochain complexes. Let $X(k)$ be such a tower, $k \in \mathbb{N}$ is the level of the tower and for every $k$, $X(k) = \{X^p(k)\}$, $p \in \mathbb{N}$, $\nu = 0, 1$. To each supercomplex (of chain complexes) $X(k)$ we associate an inverse tower of supercomplexes $\mathcal{Y}_p(X(k)) = \mathcal{Y}^\nu_p(k)$ (we write $\mathcal{Y}^\nu$ to distinguish this case from $\mathcal{Y}^\nu_p(A, R, I; \mathcal{H}, \mathcal{M})$). Finally we define the cyclic cohomology of the tower $X(k)$ by the formula similar to (71):

$$HC^n(X(k)) \overset{\text{def}}{=} H_{n+2\mathbb{Z}}(\lim_{\leftarrow} \lim_{\leftarrow} \lim_{\leftarrow} (\mathcal{Y}^k_\mathcal{P} \in T_n\mathcal{Y})). $$

In particular let us apply this construction to the tower $\mathcal{X}_\mathcal{H}(W(C), I(C); \mathcal{M}) = \{\mathcal{X}^k_\mathcal{H}(W(C), I(C); \mathcal{M})\}$ of supercomplexes of cochain complexes. Then it is clear that for a fixed $k$ the cohomology of the $\mathcal{Y}$-tower $\mathcal{Y}_p(\{\mathcal{X}^k_\mathcal{H}(W(C), I(C); \mathcal{M})\})$ associated to $\mathcal{X}^k_\mathcal{H}(W(C), I(C); \mathcal{M})$ coincides with the cohomology of the bicomplex constructed from $\mathcal{X}^k_\mathcal{H}(W(C), I(C); \mathcal{M})$, so that the injection $\mathcal{Y}_p \to \mathcal{Y}_{p+2}$ gives the $S$-operation in this cohomology. The following proposition can be proven by mere inspection of definitions (take care of the dimension shift in our definitions of towers associated to $\mathcal{X}_\mathcal{H}(W(C), I(C); \mathcal{M})$ and $X(B(A)))$.  

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Proposition 21. The map $\tilde{\Theta}_H^p \rho_M$ (formula (52)) determines a map of towers

\[
\gamma_p(\{X^k_H(W(C), I(C); M)\}) \to \gamma_{p-1}^{k-1}(A, R, I; H, M).
\]

and hence a map of cohomology $H^*(\{X^k_H(W(C), I(C); M)\}) \to H^*(A, R, I; H, M)$ (on the left here stands the homology of the double tower, associated with $W(C)$, see the discussion preceding this proposition).

Consider now an element $[x] \in HC^m_H(C, M)$. With the help of the isomorphisms $\alpha_n$ (Theorem 5) we can associate to it a series of elements $\alpha_n^{-1}([x]) \in H^{m+2n+1}(C, M; n) = H^{m+2n+1}(X^k_H(W(C), I(C); M))$ and cocycles $\{\alpha_n^{-1}([x])\}$. From the discussion of $S$-operations in the complexes $W^H_k(C, M)_r$ and supercomplex $X^k_H(W(C), I(C); M)$ (see propositions \(\text{[7]}\) and \(\text{[8]}\) and the paragraph above), we conclude, that the cocycles $\{\alpha_n^{-1}([x])\}$ fit together. In fact it was shown that on the cochain level the $S$-operator defined via the diagram (52) (which gives the even horizontal arrows in diagram (70)) coincides with the projections $\pi_n : W^H_k(C, M)_r \to W^H_{n-1}(C, M)_r$ (which give the even vertical arrows in (70)). Thus we obtain a cocycle in the limit complex $\lim_{k \to p} \gamma_p^{k-1}(A)$ (where $\gamma$ is the $\gamma$-bitower associated to $X^k_H(W(C), I(C); M)$). Parity of this cocycle is equal to the parity of $m$. Finally since the set of all odd numbers is cofinal in $\mathbb{N}$ it follows that this collection determines an element in $H^m(X^k_H(W(C), I(C); M))$.

In this way we obtain a map

\[
HC^m_H(C, M) \to H^*(X^k_H(W(C), I(C); M)).
\]

In effect one can make one more conjecture

Conjecture 22. Thed map \(\text{[7]}\) is an isomorphism of the cohomology groups.

However we shall not need this statement.

4.3 The cup-products

Now we are going to define the pairing in Hopf-type cohomology of algebra $A$ and coalgebra $C$ using the bivariant theories described in this section. In effect we shall use the maps \(\text{[4]}, \text{[7]}, \text{[6]}\) (and the corresponding map in cohomology, see Prop. 21) and the composition product in bivariant theory. Indeed if $(R, I)$ is a quasi-free $\mathcal{H}$- and $C$-equivariant extension of $A$ then

$HC^p_H(A, M) = H^p(\text{Hom}(X^p_H(R, I; M), \mathbb{k})) = H^q(X^p_H(R, I; M), \mathbb{k})$ (the last group is the group of bivariant cohomology of tower $X^p_H(R, I; M)$ and the constant tower $\mathbb{k}$). Consider the following composition of maps

\[
HC^p_H(C, M) \otimes HC^q_H(A, M) \xrightarrow{\text{[7]}} H^p(\text{Hom}(X^p_H(W(C), I(C); M))) \otimes HC^q_H(A, M)
\]

\[
\xrightarrow{\text{[4]}} HC^p(A, R, I; H, M) \otimes HC^q_H(A, M) \xrightarrow{\text{[4]}} HC^p(\text{Hom}(X^p_A, X^p_H(R, I; M))) \otimes HC^q_H(A, M)
\]

\[
\xrightarrow{\text{[4]}} HC^p_A(X^p_A, X^p_H(R, I; M)) \otimes H^q(X^p_H(R, I; M), \mathbb{k}) \to HC^{p+q}(X^p_A, \mathbb{k}) = HC^{p+q}(A).
\]

The last map in this composition is just the composition product, defined in \(\text{[2]}\). Thus we obtain another cup-product

\[
HC^p_H(C, M) \otimes HC^q_H(A, M) \xrightarrow{\text{[7]}} HC^{p+q}(A).
\]
It is convenient to write down the cup-product \( y^{♯}_{n} \) in the terms of maps \( \tilde{\alpha}^{2n+1}_{\rho_{\mathcal{M}}}_{H} \). We start with the case of even degree elements in \( HC_{H}^{*}(A, \mathcal{M}) \). For instance, let \( [x] \in HC^{p}(C, \mathcal{M}) \) and \( [y] \in HC^{2n}(A, \mathcal{M}) \). Let \( y \in \text{Hom}_{0}(\mathcal{X}_{H}^{2n}(RA, IA; (\mathcal{M})) \otimes k) \) be a cocycle representing \( [y] \) and \( x \) be a cocycle in \([x]\). Then as it is mentioned in discussion following the proof of the theorem \( \square \) the isomorphism \( \alpha_{n} \) can be raised to the level of cochains. So we obtain the following map

\[
y^{♯}_{n}x = y \circ \tilde{\alpha}^{2n+1}_{\rho_{\mathcal{M}}}(\alpha_{n}^{-1}(x)) : X(B(A)) \to k.
\]

From the properties of all the maps that appear in this formula it follows that \( y^{♯}_{n}x \) is a cocycle and that the cohomology class of this cocycle does not depend on the choices made in its construction (cf. Prop. \( \square \)).

In the case of the odd degree elements in \( HC_{H}^{*}(A, \mathcal{M}) \), we can proceed in the following way. First we observe that the rows of \( T \) for the bitower \( \square \) for \( \mathcal{X}_{k} = \mathcal{X}_{H}^{p}(W(C), I(C); (\mathcal{M})) \) can be regarded as the (inverse) towers of the complex \( X \) and the complex \( X \) up to a shift of dimension this is the periodic complex of cochain complexes, which is defined by the same formula as above. Now it is clear that the rows of \( T \) for the bitower \( \square \) for \( \mathcal{X}_{k} = \mathcal{X}_{H}^{p}(W(C), I(C); (\mathcal{M})) \) can be regarded as the (inverse) towers of the complex \( X \) and the complex \( X \) up to a shift of dimension this is the periodic complex of cochain complexes, which is defined by the same formula as above. Now it is clear that the rows of \( T \) for the bitower \( \square \) for \( \mathcal{X}_{k} = \mathcal{X}_{H}^{p}(W(C), I(C); (\mathcal{M})) \). Here, in analogy with p. 384 of \( \square \) we put for an inverse tower \( X \) \( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \) \( \Rightarrow \).

Thus instead of the elements \( \alpha_{n}^{-1}(x) \in X_{H}^{2n+1}(W(C), I(C); (\mathcal{M})) \) one can consider the corresponding elements \( \beta_{n}^{-1}(x) \) in the \( 2n+1 \)-st level of filtration. They fit together to define an element in the limit of bitower \( F_{I(C)}X_{H}(W(C); (\mathcal{M})) \). On the other hand, this bitower is mapped by the map \( X_{H}(\rho^{♯}, (\mathcal{M})) \) constructed in the Proposition \( \square \) to \( \text{Hom}(X(B), F_{I(C)}^{2n+1}X_{H}(W(C); (\mathcal{M}))) \). Finally we need to remark that \( 2n+1 \) degree cohomology classes of \( A \) correspond to even cocycles on \( F_{I(C)}^{2n+1}X_{H}(W(C); (\mathcal{M})) \). Since all the maps in our construction of the cup-product \( ♯^{♯} \) evidently commute with the homomorphisms relating the towers \( F_{I(C)}X_{H}(W(C); (\mathcal{M})) \) and \( \mathcal{X}_{H}^{k}(W(C), I(C); (\mathcal{M})) \) and similar homomorphisms for all the other towers that appear in the construction of \( \square \) we conclude that the construction involving \( F_{I(C)}^{*}X_{H}(W(C), (\mathcal{M}) \) instead of \( X_{H}(W(C), I(C); (\mathcal{M})) \) yields the same results.

Thus for an odd-degree class \( [y] \) in \( HC_{H}^{2n+1}(A, \mathcal{M}) \) we can choose a representing it even cocycle \( \tilde{y} \) in \( F_{I(C)}^{2n}X_{H}(W(C); (\mathcal{M})) \); then the class \( [x]^{♯}_{n}[y] \) is represented by

\[
\tilde{y}^{♯}_{n}x = y \circ X_{H}(\rho^{♯}, (\mathcal{M}))(\beta_{n}^{-1}(x)) : X(B(A)) \to k.
\]

**Proposition 23.** The cup-product \( ♯^{♯} \) coincides with \( ♯^{♯} \).
Proof. From the remark it follows that the map \( \tilde{X}_{\mathcal{H}}^{2n+1}(\rho^3) : \mathcal{X}_\mathcal{H}^{2n+1}(W(C), I(C); \mathcal{M}) \to \text{Hom}(X(B(A)), (\mathcal{M} \otimes \mathcal{H} R/I^{n+1})) \).

Thus from (SU) we see that in the case of even degree classes \([y], [x]\)\(^{2n}[y] = [x][y]\).

In the same way the odd-degree case follows from (S2) and the simple observation that the natural projection \( X_\mathcal{H}(R, \mathcal{M}) \to (\mathcal{M} \otimes \mathcal{H} R)_2 \) sends \( F^n_1 X_\mathcal{H}(R, \mathcal{M}) \) to \( I^n_2(C, \mathcal{M})_2 \) and allows identify the even cocycles in \( F^n_1 X_\mathcal{H}(R, \mathcal{M}) \) with the odd \( \mathcal{M} \)-traces on \( A \) .

As a simple consequence of the theory developed in this section one can derive the following statement

**Corollary 24.** There exists a cap-product

\[
(83) \quad HC_p(A) \otimes HC_q^q(C, \mathcal{M}) \to HC_{p-q}^H(A, \mathcal{M}).
\]

This product verifies the next equations with respect to \( S \)-operations and the cup-product \( \# \)

\[
(84) \quad S([z] \# [x]) = S[z] \# x = [z] \# S[x],
\]

\[
(85) \quad \langle [z] \# x, [y] \rangle = \langle [z], [x] \# [y] \rangle.
\]

Here \( \langle, \rangle \) is the pairing between the cyclic homology and cohomology of algebra \( A \) (cohomology with coefficients in \( \mathcal{M} \) on the left hand side).

**Proof.** The definition of \( \# \) is based on the fact that \( HC_*(A) = H_*(\mathbb{K}, \mathcal{X}_A) \) and \( HC^H_*(A, \mathcal{M}) = H_*(\mathbb{K}, \mathcal{X}_H(R, I; \mathcal{M})) \) and is similar to that of \( \#'' = \#' \), (S8). In fact one uses the following composition of maps

\[
(86) \quad HC_p(A) \otimes HC_q^q(C, \mathcal{M}) \xrightarrow{(76)} HC_p(A) \otimes H^q(\mathcal{X}_\mathcal{H}(W(C), I(C); \mathcal{M}))
\]

\[
\xrightarrow{(84)} HC_p(A) \otimes HC^q(A, R, I; \mathcal{H}, \mathcal{M}) \xrightarrow{(85)} HC_p(A) \otimes HC^q(\mathcal{X}_A, \mathcal{X}_H(R, I; \mathcal{M}))
\]

\[
= H_p(k, \mathcal{X}_A) \otimes HC^q(\mathcal{X}_A, \mathcal{X}_H(R, I; \mathcal{M})) \to HC^{-p+q}(\mathcal{X}_A, k) = HC_{p-q}(A).
\]

And the equalities (S4), (S5) follow from the standard properties of the composition pairing in bivariant cyclic cohomology.

\[ \square \]

## 5 Other constructions and comparison theorem

In this section we introduce a new pairing construction in Hopf-cyclic cohomology with coefficients which generalizes the Quillen’s and Crainic’s constructions (see [18] and [7]) and show that it coincides with cup-product defined by Khalkhali and Rangipour in [15].

So let \( C \) be a coalgebra acting on an algebra \( A \), in a way, intertwining the actions of \( \mathcal{H} \) on both sides. Observe that any action of a coalgebra \( C \) on an algebra \( A \) can be regarded as a map

\[
\rho_C : C \to \text{Hom}(A, A), \quad \rho_C(c)(a) = c(a), \quad c \in C, a \in A.
\]

Due to the universal property of Weil algebra \( W(C) \), it induces a homomorphism of bigraded bidifferential algebras

\[
\rho_C : W(C) \to \text{Hom}(BA, \Omega),
\]

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where coalgebra $BA$ is the bar resolution of $A$ and $\Omega$ is the abbreviation of $\Omega A$, the universal differential algebra generated by $A$ (the universal differential calculus of $A$). The homomorphism $\rho_C$ intertwines the differential $\delta$ of $BA$ with the differential $\delta_F$ in $W(C)$ and differential $d$ of $\Omega A$ with $d$ in $W(C)$. If $A$ is an $\mathcal{H}$-algebra then $\rho_C$ is an $\mathcal{H}$-linear map. There’s an $\mathcal{H}$-module structure on $\text{Hom}(BA, \Omega)$, given by the action of $\mathcal{H}$ on $\Omega A$.

For any SAYD-module $\mathcal{M}$ and an integer $n$ we have a map $\rho_{C,\mathcal{M}}$ induced by $\rho_C$:

$$\rho_{C,\mathcal{M}} : (\mathcal{M} \otimes_{\mathcal{H}} (W(C)^n / \text{Im } d))_z \rightarrow \text{Hom}(BA^z, (\mathcal{M} \otimes_{\mathcal{H}} (\Omega^n / \text{Im } d))_z).$$

Given a closed graded $(\mathcal{H}, \mathcal{M})$-trace of degree $n$ on $\Omega A$ in the sense of [15], i.e. a linear functional $\int : (\mathcal{M} \otimes_{\mathcal{H}} (\Omega^n / \text{Im } d))_z \rightarrow k$, one obtains a chain map of complexes

$$\int \circ \rho_{C,\mathcal{M}} : ((\mathcal{M} \otimes_{\mathcal{H}} (W(C)^n / \text{Im } d))_z, \delta_F) \rightarrow (\text{Hom}(BA^z, k), \delta)$$

and hence a map in cohomology

$$(\int \circ \rho_{C,\mathcal{M}})_* : HC^* (C; \mathcal{M}; n) = H^* ((\mathcal{M} \otimes_{\mathcal{M}} (W(C)^n / \text{Im } d))_z, \delta_F)$$

$$\rightarrow H^* (\text{Hom}(BA^z, k), \delta) = HC^{*+1} (A).$$

Taking into account theorem 5 and [15, lemma 3.2] that identifies closed $(\mathcal{H}, \mathcal{M})$-traces with Hopf-cyclic cocycles of $A$, we obtain a pairing:

$$(\rho_{C,\mathcal{M}})_* : H^m_H (C, \mathcal{M}) \otimes H^n_H (A, \mathcal{M}) \rightarrow HC^{m+n} (A).$$

Let us show now that the construction defined above coincides with cup product of [15].

Denote $W = W(k)$ the Weil algebra of the ground field. One always has a map of bigraded bidifferential algebras

$$\rho = \rho_k : W \rightarrow \text{Hom}(BA, \Omega A).$$

On the other hand, there is a morphism of bigraded bidifferential algebras

$$\sigma : W \rightarrow \text{Hom}_H (\Omega C, W(C)),$$

induced by map $\Omega C \xrightarrow{\rho} \Omega C_0 = C \xrightarrow{id} C \subset W(C)$, where $\Omega C$ is universal differential coalgebra of $C$ (see previous sections) and differentials in $\text{Hom}_H (\Omega C, W(C))$ are given by formulas $df = [d, f]$ and $\delta f = \delta \circ f$. Finally, thanks to universal property of $\Omega A$ one has a homomorphism of differential graded algebras

$$\tau : \Omega A \rightarrow \text{Hom}_H (\Omega C, \Omega A),$$

induced by map $\tau : A \rightarrow \text{Hom}_H (C, A)$, $\tau(a)(c) = c(a)$. These homomorphisms give the following diagram (recall that $\Omega$ is an abbreviation for $\Omega A$)

$$
\begin{array}{cccccccc}
W & \xrightarrow{\rho} & \text{Hom}(BA, \Omega) & \xrightarrow{\tau} & \text{Hom}(BA, \text{Hom}_H (\Omega C, \Omega)) & \rightarrow & \text{Hom}_H (\Omega C \otimes BA, \Omega) \\
\| & & & & & & \\
W & \xrightarrow{\sigma} & \text{Hom}_H (\Omega C, W(C)) & \xrightarrow{\rho_C} & \text{Hom}_H (\Omega C, \text{Hom}(BA, \Omega)) & \rightarrow & \text{Hom}_H (\Omega C \otimes BA, \Omega).
\end{array}
$$
The diagram is commutative because for the generator $i = i_1$ of $W$ the upper and the lower row lead to the same map $c \otimes a \mapsto c(a)$. After factorization we get another diagram of complexes with differential $\delta$.

\[
\begin{array}{ccc}
\text{Hom}(BA^2, \Omega A_2/ \text{Im } d) & \xrightarrow{\tau} & \text{Hom}(BA^2, \text{Hom}((\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d}, (\mathcal{M} \otimes \mathcal{H} \Omega A)_z/ \text{Im } d)) \\
\rho & & \downarrow \\
W_z/ \text{Im } d & & \text{Hom}((\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d} \otimes BA^2, (\mathcal{M} \otimes \mathcal{H} \Omega A)_z/ \text{Im } d) \\
\sigma & & \downarrow \\
& & \text{Hom}((\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d}, \text{Hom}(BA^2, (\mathcal{M} \otimes \mathcal{H} \Omega A)_z/ \text{Im } d)) \\
\rho_{G,M} & & \\
\text{Hom}((\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d}, (\mathcal{M} \otimes \mathcal{H} W(C))_z/ \text{Im } d).
\end{array}
\]

Here $(\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d}$ is the space of closed graded $(\mathcal{H}, \mathcal{M})$-cotraces on $\Omega C$ in the sense of [15], i.e. the set of elements $\xi = \sum_i m_i \otimes \theta_i \in \mathcal{M} \otimes \mathcal{H} \Omega C$ such that $(\text{id} \otimes d)\xi = 0$ and

\[
\sum (-1)^{||\theta_i^{(1)}||_g(2)} m_i^{(0)} \otimes \theta_i^{(2)} \otimes m_i^{(-1)} \theta_i^{(1)} = \sum_i m_i \otimes \theta_i^{(1)} \otimes \theta_i^{(2)}.
\]

Let $\int$ be a closed graded $(\mathcal{H}, \mathcal{M})$-trace of degree $n$ on $\Omega A$ and $\xi$ be a closed graded $(\mathcal{H}, \mathcal{M})$-cotrace of degree $m$ on $\Omega C$. Evaluating the upper map of the diagram on $\int$ and $\xi$, we obtain a homomorphism

\[
(\int \cup \xi) \circ \rho : W_z/ \text{Im } d \rightarrow \text{Hom}(BA^2, \mathbb{k}),
\]

which is the composition of $\rho$ and cup product cocycle

\[
\int \cup \xi : \Omega A_2^{n+m}/ \text{Im } d \rightarrow \text{Hom}((\mathcal{M} \otimes \mathcal{H} \Omega C)^{\leq d}_m, (\mathcal{M} \otimes \mathcal{H} \Omega A)_z^{n}/ \text{Im } d) \xrightarrow{\int \text{ev}_{\xi}} \mathbb{k}
\]

defined in [15]. Take element $cs_{m+n} = iw^{m+n} \in W$. It is a cocycle in $W_z/ \text{Im } d$ and the map

\[
\rho(cs_{m+n}) : BA_{n+m+1} \rightarrow \Omega A^{n+m}, \ a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n+1} \mapsto a_1a_2\cdots a_{m+n+1}
\]

just identifies closed traces on $\Omega A$ with cyclic cocycles of $A$. Thus, the cup product of $\int$ and $\xi$ defined in [15] can be expressed as composition cocycle

\[
\int \circ \text{ev}_{\xi} \circ \tau \circ \rho(cs_{m+n}) : BA_{m+n+1}^2 \rightarrow \mathbb{k}.
\]

On the other hand, the lower row of the diagram shows that the cup product of $\int$ and $\xi$ coincides with the pairing we defined of $\int$ and the element $\sigma(cs_{m+n})(\xi) \in (\mathcal{M} \otimes \mathcal{H} W(C))_z/ \text{Im } d)$. Remark that due to the obvious inclusion

\[
\Omega_m C = C \otimes (\ker \varepsilon)^m \subset C^\otimes m+1 \subset W(C)
\]

$\xi$ can be regarded as element in $(\mathcal{M} \otimes \mathcal{H} W(C))_z/ \text{Im } d$ and in fact it is a cocycle in this complex. Thus, in order to show that the cup-product of [15] coincides with the pairing construction of this section it is sufficient to check the following proposition.
Proposition 25. \(\alpha_n[\sigma(cs_{m+n})(\xi)] = \frac{m+1}{m+n+1}\xi\) in \(HC^m_H(C, M)\)

Proof. We shall prove this identity by induction on \(n\). If \(n = 0\) then we have

\[\sigma(cs_m)(c_0 \otimes c_1 \otimes \cdots \otimes c_m) = c_0c_1 \cdots c_m\]

for every \(c_0 \otimes c_1 \otimes \cdots \otimes c_m \in \Omega_mC\). So the identity is true. In order to establish the step of induction consider the following diagram

\[
\begin{array}{c}
0 \\ \downarrow \text{ev}_\xi \\
W^{n+1}/b, d -b- W^n/d, (1-\kappa) -ev_\xi-
\end{array}
\]

\[
\begin{array}{c}
(M \otimes_H W(C)^{n+1})/b, d -b- (M \otimes_H W(C)^n)/d, (1-\kappa)
\end{array}
\]

\[
\begin{array}{c}
\dashrightarrow W^n/b, d -ev_\xi-
\end{array}
\]

\[
\begin{array}{c}
(\mathcal{M} \otimes_H W(C)^n)/b, d -ev_\xi-
\end{array}
\]

It induces a commutative diagram for boundary maps:

\[
H^*(W^n / \text{Im } d, \delta) \xrightarrow{\varphi} H^*(W^{n+1}_2 / \text{Im } d, \delta) \\
\text{ev}_\xi \downarrow \quad \text{ev}_\xi \\
H^*_H(C, M; n) \xrightarrow{\varphi} H^*_H(C, M; n+1)
\]

Show now that \(\varphi[cs_n] = \frac{n+1}{n+2}[cs_{n+1}] \in H^*(W^{n+1}/ \text{Im } d, \delta)\). Indeed, we have the equality in \(W/ \text{Im}(1-\kappa)\)

\[
d\left(\sum_{k=0}^{n-1} i^2 w^k i w^{n-k-1}\right) = \sum_{k=0}^{n-1} (w i w^k i w^{n-k-1} - i w^{k+1} i w^{n-k-1} + i^2 w^n) = (n+1)i^2 w^n - iw^n i,
\]

that implies \(iw^n i = (n+1)i^2 w^n\) in \(W/ \text{Im } d + \text{Im}(1-\kappa)\) and

\[
b(iw^{n+1}) = -iw^n i - i^2 w^n = -(n+2)i^2 w^n, \\
\delta(iw^n) = -iw^n i = -(n+1)i^2 w^n.
\]

Hence, the equality holds \(\varphi[\sigma(cs_{n+1})(\xi)] = \frac{n+1}{n+2}[\sigma(cs_n)(\xi)]\) and

\[
\alpha_{n+1}[\sigma(cs_{m+n+1})(\xi)] = \alpha_n \circ \varphi[\sigma(cs_{m+n+1})(\xi)] = \frac{m+n+1}{m+n+2} \alpha_n[\sigma(cs_{m+n})(\xi)] \\
= \frac{m+n+1}{m+n+2} \cdot \frac{m+1}{m+n+1}[\xi] = \frac{m+1}{m+n+2}[\xi].
\]

\(\square\)
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