Mathematische Zeitschrift (2022) 301:3285–3343
https://doi.org/10.1007/s00209-022-03020-9

Spanning trees, cycle-rooted spanning forests on discretizations of flat surfaces and analytic torsion

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Received: 30 May 2021 / Accepted: 1 March 2022 / Published online: 8 April 2022
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Abstract
We study the asymptotic expansion of the determinant of the graph Laplacian associated to discretizations of a tileable surface endowed with a flat unitary vector bundle. By doing so, over the discretizations, we relate the asymptotic expansion of the number of spanning trees and the partition function of cycle-rooted spanning forests weighted by the monodromy of the unitary connection on the vector bundle, to the corresponding zeta-regularized determinants. As a consequence, we establish open problems 2 and 4, formulated by Kenyon in 2000. The spectral theory on discretizations of flat surfaces, Fourier analysis on discrete square and the analytic methods used in the proof of Ray–Singer conjecture lie in the core of our approach.

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1 Introduction

For a sequence of discretizations of a given surface, we study the asymptotic behaviour of
the number of spanning trees and of the partition function of cycle-rooted spanning forests
weighted by the monodromy of the unitary connection on the vector bundle, as the mesh of
the sequence of the discretizations goes to zero.

More precisely, by a spanning tree in a graph we mean a subtree covering all the vertices.
By a cycle-rooted spanning forest (CRSF in what follows) in a graph $G = (V(G), E(G))$
we mean a subset $S \subset E(G)$, spanning all vertices and with the property that each connected
component of $S$ has as many vertices as edges (in particular, it has a unique cycle).

The number of spanning trees on a finite graph $G$ is often called the complexity
of the graph, denoted here by $t(G)$. In what is now called the matrix-tree theorem, Kirchhoff showed
that for a finite graph $G$, the product of nonzero eigenvalues of the combinatorial Laplacian
$\Delta_G$, defined as a difference of degree and adjacency operators, is related to the complexity
of the graph as follows

$$t(G) = \frac{1}{\# V(G)} \det' \Delta_G := \frac{1}{\# V(G)} \prod_{\lambda \in \text{Spec}(\Delta_G) \setminus \{0\}} \lambda. \quad (1.1)$$

Forman in [34, Theorem 1] extended this result to the setting of a line bundle with a unitary
connection on a graph (in Sect. 2.1 we give a precise meaning for those notions). Kenyon
in [42, Theorems 8,9] generalized it to vector bundles of rank 2 endowed with $\text{SL}_2(\mathbb{C})$
connections.

To state Kenyon’s result, recall that for a finite graph $G$ and a vector bundle $V$ over $G$,
edowed with a connection $\nabla^V$, the twisted Laplacian is an operator, acting on $f \in \text{Map}(V(G), F)$ by

$$\Delta^V_G f(v) = \sum_{(v, v') \in E(G)} (f(v) - \phi_{v'v} f(v')), \quad \forall v \in V(G), \quad (1.2)$$

where $\phi_{v'v}$ is the parallel transport from $v'$ to $v$ associated to $\nabla^V$. In case if $V$ is a trivial line
bundle and $\nabla^V$ is the trivial connection, $\Delta^V_G$ gives the combinatorial Laplacian $\Delta_G$ on graph
$G$. See the end of Sect. 2.1 for the definition of vector bundles on graphs and related notions.

![A spanning tree and a CRSF on a square-grid graph approximating an annulus. The edges of the graph are not drawn, but they connect the nearest neighbors. The dotted component of the CRSF is non-contractible in the annulus and the non-dotted one is contractible.](a) A spanning tree (b) A CRSF)
For a Hermitian vector bundle \((F, hF)\) of rank 2 with a special unitary connection \(\nabla F\), Kenyon in [42, Theorems 8, 9], cf. also Kassel–Kenyon [39, Theorem 15], proved the following identity

\[
\sqrt{\det \Delta^F_G} = \sum_{T \in \text{CRSF}(G)} \prod_{\gamma \in \text{loops}(T)} \left(2 - \text{Tr}(w_\gamma)\right),
\]

where CRSF\((G)\) is the set of all CRSF’s on \(G\), the product is over all loops \(\gamma\) of the CRSF and \(\text{Tr}(w_\gamma)\) is the trace of the monodromy of \(\nabla F\), evaluated along \(\gamma\).

In this article, we consider a family \(\Sigma_n, n \in \mathbb{N}^*\), of graphs constructed as approximations of a given surface \(\Sigma\). Our goal is to understand the asymptotics of (1.1) and (1.3), as \(n \to \infty\), and to see how the geometry of \(\Sigma\) is reflected in it. From (1.1) and (1.3), it is equivalent to study the associated determinants on the graphs \(\Sigma_n\).

For rectangles \(\Sigma\), the problem of this article has received considerable attention in the past. This asymptotics is of some interest to statistical mechanics, as it corresponds to the asymptotics of the partition function in dimer model. In fact, by a result of Temperley [58], the dimer coverings on a square grid (with a removed corner) are in bijection with spanning trees on another grid.

For rectangles, the principal term of the asymptotics of (1.1) has been evaluated by Kasteleyn in [40], and it is related to the area of the rectangle. Then Fischer in [32] calculated the next term and related it to the perimeter of the rectangle. Further, Ferdinand in [26] obtained the asymptotic expansion up to the constant term and Duplantier-David in [25], cf. Theorem A.1, related the constant term to the zeta-regularized determinant of the Laplacian on the rectangle (we recall the definition later).

This article can be regarded as an extension of the result of Duplantier-David from rectangles to any surface tillable by equal squares, endowed with a vector bundle. Our study leads, in particular, to a complete and partial answers to Open problems 2 and 4 respectively from Kenyon [41, Sect. 8].

Let us describe our results more precisely. We consider in this article tileable surfaces \((\Sigma, g^T \Sigma)\) with conical singularities. This means that \(\Sigma\) can be tiled completely and without overlaps over subsets of positive Lebesgue measure by euclidean squares of area 1. In particular, the conical singularities \(\text{Con}(\Sigma)\) have angles \(k\pi, k \in \mathbb{N}^* \setminus \{2\}\), the boundary \(\partial \Sigma\) can be tiled by the boundaries of the tiles, and the angles of the boundary corners \(\text{Ang}(\Sigma)\) are of the form \(\frac{k\pi}{2}, k \in \mathbb{N}^* \setminus \{2\}\). For further reference, we denote by \(\text{Ang} \neq \pi/2(\Sigma)\) (resp. \(\text{Ang}=\pi/2(\Sigma)\)) the subset of \(\text{Ang}(\Sigma)\) corresponding to angles \(\neq \frac{\pi}{2}\) (resp. \(= \frac{\pi}{2}\)).

For example, if \(\Sigma\) is a torus, then it can be tiled by euclidean squares of the same size if and only if the ratio of its periods is rational. If \(\Sigma\) is a rectangular domain in \(\mathbb{C}\) with one boundary component, then it can be tiled by euclidean squares of the same size if and only if the ratios between the lengths of the sides of \(\Sigma\) are rational. In this case, the angles \(\frac{\pi}{2}\) are called convex, the angles \(\frac{3\pi}{2}\) are called concave and the arcs of the boundary meeting in the angle \(2\pi\) are called slits. Our surfaces generalize also the so-called pillowcase covers, which can be characterized as ramified coverings of \(\mathbb{C}\mathbb{P}^1\), branched over four points, cf. Zorich [62]. See Figs. 2, 3, 4, 7, 8 and Sect. 2.2 for other examples.

We fix a flat unitary vector bundle \((F, hF, \nabla F)\) on the compactification

\[
\overline{\Sigma} := \Sigma \cup \text{Con}(\Sigma).
\]

By flat we mean that the monodromies over the contractible loops in \(\overline{\Sigma}\) vanish, or equivalently \((\nabla F)^2 = 0\). By a unitary connection we mean a connection \(\nabla F\) preserving the metric \(hF\).
We fix a tiling of $\Sigma$. We construct a graph $\Sigma_1 = (V(\Sigma_1), E(\Sigma_1))$ by taking vertices $V(\Sigma_1)$ as the centers of the tiles and edges $E(\Sigma_1)$ in such a way that the resulting graph $\Sigma_1$ is the nearest-neighbor graph with respect to the flat metric on $\Sigma$. This means that an edge connects two vertices if and only if they are the closest neighbors with respect to the metric $g^{T\Sigma}$. Remark that $\Sigma_1$ (and $\Sigma_n$, $n \in \mathbb{N}^*$, constructed below) might have multiple edges due to some conical points of angles $\pi$, see Fig. 4. By an abuse of notation, we will nevertheless say that $\Sigma_n$ are graphs.

The vector bundle $F_1$ over $\Sigma_1$ and the Hermitian metric $h^{F_1}$ on $F_1$ are constructed by the restriction from $F$ and $h^F$. The connection $\nabla^{F_1}$ on $F_1$ is constructed as the parallel transport of $\nabla^F$ with respect to the straight path between the vertices. It is a matter of a simple verification to see that the fact that $(F, h^F, \nabla^F)$ is flat unitary implies that $(F_1, h^{F_1}, \nabla^{F_1})$ is unitary.

By considering regular subdivisions of tiles into $n^2$ equal squares, $n \in \mathbb{N}^*$, and repeating the same procedure, we construct a family of graphs $\Sigma_n = (V(\Sigma_n), E(\Sigma_n))$ with unitary vector bundles $(F_n, h^{F_n}, \nabla^{F_n})$ over $\Sigma_n$, for $n \in \mathbb{N}^*$. Note that we have a natural injection

$$V(\Sigma_n) \hookrightarrow \Sigma.$$  \hspace{1cm} (1.5)

For example, in case if $\Sigma$ is a rectangular domain in $\mathbb{C}$ with corners at integer points, then $\Sigma_n$ coincide with the subgraphs of $\frac{1+\sqrt{1+4n^2}}{2n} + \frac{1}{n^2}Z^2$, staying inside of $\Sigma$. See Fig. 2.

We denote by $(\nabla^F)^*$ the formal adjoint of $\nabla^F$ with respect to the $L^2$-metric induced by $g^{T\Sigma}$ and $h^F$. We denote by $\Delta^F_\Sigma$ the Laplacian on $(\Sigma, g^{T\Sigma})$, associated with $(F, h^F, \nabla^F)$, i.e.

$$\Delta^F_\Sigma := (\nabla^F)^* \nabla^F.$$  \hspace{1cm} (1.6)

It is easy to see that in a fixed flat unitary frame, $\Delta^F_\Sigma$ coincides with $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$.

In this paper we always consider $\Delta^F_\Sigma$ with Neumann boundary conditions on $\partial \Sigma$. In other words, for a normal $n$ to the boundary $\partial \Sigma$, the sections $f$ from the domain of our Laplacian satisfy

$$\nabla^F_n f = 0 \; \text{ over } \partial \Sigma.$$  \hspace{1cm} (1.7)
It is well-known that because of conical singularities and non-smoothness of the boundary, the Laplacian $\Delta^F_{\Sigma}$ is not necessarily essentially self-adjoint even after precising the boundary condition (1.7), cf. Cheeger [15] and Mooers [49]. Thus, to define the spectrum of $\Delta^F_{\Sigma}$, we are obliged to specify the self-adjoint extension of $\Delta^F_{\Sigma}$ we are working with. We choose the Friedrichs extension and, by an abuse of notation, we denote it by the same symbol $\Delta^F_{\Sigma}$. See Sect. 2.3 and [29] for more on Friedrichs extension and the motivation for this choice of a self-adjoint extension.

As in the case of smooth domains, $\text{Spec}(\Delta^F_{\Sigma})$ is discrete (cf. Proposition 2.4). We order it non-decreasingly as follows

$$\text{Spec}(\Delta^F_{\Sigma}) = \{\lambda_1, \lambda_2, \ldots\}. \quad (1.8)$$

The zeta-regularized determinant of $\Delta^F_{\Sigma}$ (also called the analytic torsion) is defined non-rigorously by the following non-convergent infinite product

$$\text{det}^\prime \Delta^F_{\Sigma} := \prod_{\lambda \in \text{Spec}(\Delta^F_{\Sigma}) \setminus \{0\}} \lambda. \quad (1.9)$$

More formally, by the Weyl’s law (cf. Corollary 2.6), the zeta-function $\zeta^F_{\Sigma}(s)$ is well-defined for $s \in \mathbb{C}, \text{Re}(s) > 1$, by the formula

$$\zeta^F_{\Sigma}(s) = \sum_{\lambda \in \text{Spec}(\Delta^F_{\Sigma}) \setminus \{0\}} \frac{1}{\lambda^s}. \quad (1.10)$$

As in the case of smooth manifolds, $\zeta^F_{\Sigma}$ extends meromorphically to $\mathbb{C}$ and 0 is a holomorphic point of this extension (cf. the end of Sect. 2.3). Following Ray–Singer [54, 55], we define the zeta-regularized determinant $\text{det}^\prime \Delta^F_{\Sigma}$ by

$$\text{det}^\prime \Delta^F_{\Sigma} := \exp \left( - (\zeta^F_{\Sigma})'(0) \right). \quad (1.11)$$

Clearly, by (1.10), the definition (1.11) corresponds formally to (1.9).

The value $\zeta^F_{\Sigma}(0)$ is also interesting and it can be evaluated as follows (cf. Proposition 2.7)

$$\zeta^F_{\Sigma}(0) = - \dim H^0(\Sigma, F) + \frac{\text{rk}(F)}{12} \left( \sum_{P \in \text{Con}(\Sigma)} \frac{4\pi^2 - \angle(P)^2}{2\pi \angle(P)} + \sum_{Q \in \text{Ang}(\Sigma)} \frac{\pi^2 - \angle(Q)^2}{2\pi \angle(Q)} \right). \quad (1.12)$$

where $H^0(\Sigma, F)$ is the vector space of flat sections of $(F, \nabla^F)$. In particular, up to some explicit contribution coming from the corners and the conical singularities, $\zeta^F_{\Sigma}(0)$ is a topological invariant.

Recall that the Catalan constant $G \in \mathbb{R}$ is defined as follows

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots. \quad (1.13)$$

We define the normalized logarithm $\tilde{\log}(\text{det}^\prime \Delta^F_{\Sigma_n})$, $n \in \mathbb{N}^*$ by

$$\tilde{\log}(\text{det}^\prime \Delta^F_{\Sigma_n}) := \log(\text{det}^\prime \Delta^F_{\Sigma_n}) - \frac{4G}{\pi} \cdot \text{rk}(F) \cdot A(\Sigma) \cdot n^2$$

$$- \frac{\log(\sqrt{2} - 1)}{2} \cdot \text{rk}(F) \cdot |\partial \Sigma| \cdot n + 2\zeta^F_{\Sigma}(0) \cdot \log(n), \quad (1.14)$$
where \( \det ' \Delta^F_{\Sigma_n} \) is the product of the non-zero eigenvalues of \( \Delta^F_{\Sigma_n} \), \( A(\Sigma) \) is the area of \( (\Sigma, g^T \Sigma) \) and \( |\partial \Sigma| \) is the perimeter respectively. Now, we can finally state our main result.

**Theorem 1.1** Consider a tileable surface \( (\Sigma, g^T \Sigma) \) with piecewise geodesic boundary, tiled by euclidean squares of area 1 as explained above. Construct a family of graphs \( \Sigma_n, n \in \mathbb{N}^* \) as above. We endow \( \Sigma \) with a flat unitary vector bundle \((F, h^F, \nabla^F)\) and \( \Sigma_n \) with the induced unitary vector bundles \((F_n, h^{F_n}, \nabla^{F_n})\).

Then there is a sequence \( A_n, n \in \mathbb{N}^* \), which depends only on the set of conical angles \( \angle(\text{Con}(\Sigma)) \) and the set of angles \( \angle(\text{Ang}^\neq \pi/2(\Sigma)) \), such that, as \( n \to \infty \), we have

\[
\tilde{\text{log}}(\det ' \Delta^F_{\Sigma_n}) - \text{rk}(F) \cdot A_n \to \text{log}(\det ' \Delta^F_{\Sigma}) - \frac{\text{log}(2) \cdot \text{rk}(F)}{16} \cdot \#\text{Ang}^\neq \pi/2(\Sigma). \tag{1.15}
\]

Moreover, the sequence \( A_n \) is additive under taking union of \( \angle(\text{Con}(\Sigma)) \), \( \angle(\text{Ang}^\neq \pi/2(\Sigma)) \), and for any \( k, K \in \mathbb{N} \), the following asymptotic bound holds

\[
\tilde{\text{log}}(\det ' \Delta^F_{\Sigma_n}) = o(\text{log}(n)), \quad \text{for } n = k! \cdot K, l \in \mathbb{N}. \tag{1.16}
\]

**Remark 1.2**

a) For rectangles, Theorem 1.1 was proved by Duplantier-David in \([25, (4.7)\) and (4.23)] by using an explicit expression for the spectrum, cf. Theorem A.1. For simply-connected rectangular domains in \( \mathbb{C} \) (and, consequently, trivial \((F, h^F, \nabla^F)\)), an expansion similar to (1.15) is due to Kenyon \([41, \text{Theorem 1}]\). The relation between the constant term and the analytic torsion, established here, was conjectured in \([41, \text{Sect. 8, Open problem 4}]\).

Since we work on multiply-connected domains and on surfaces of arbitrary genus, we also obtain an answer to \([41, \text{Sect. 8, Open problem 2}]\). See Sect. 2.6 for more relation with \([41]\).

b) The sequence \( A_n, n \in \mathbb{N}^* \) can be expressed as a sum of certain constants which depend additively on the sets \( \angle(\text{Con}(\Sigma)) \), \( \angle(\text{Ang}^\neq \pi/2(\Sigma)) \) and a sum of normalized logarithms \( \text{log}(\det ' \Delta^F_{\Sigma_n}) \), corresponding to a number of model spaces \( \Sigma \). For example, if we restrict our attention to rectangular domains in \( \mathbb{C} \) without slits, then there is only one model space \( \Sigma \), which is the \( \mathbb{L} \)-shape depicted in Fig. 2. If we allow slits in \( \Sigma \), this amounts to adding one more model space, which is the model slit, depicted in Fig. 3. See Sect. 2.4 and (2.64) for more on this.

c) We believe that for a certain constant \( c \), depending on \((\Sigma, g^T \Sigma)\) and \((F, h^F, \nabla^F)\), one can improve the bound (1.16) to \( c + o(1) \), as \( n \to \infty \). By Remark 1.2b) and (1.15), to prove such a statement, it is enough to do so only for a number of model cases\(^1\).

**Corollary 1.3** Let \((\Sigma, g^T \Sigma)\) be a torus with integer periods or a cylinder with integer height and circumference. We endow \( \Sigma \) with a flat unitary vector bundle \((F, h^F, \nabla^F)\). As \( n \to \infty \), we have

\[
\tilde{\text{log}}(\det ' \Delta^F_{\Sigma_n}) \to \text{log}(\det ' \Delta^F_{\Sigma}). \tag{1.17}
\]

**Proof** It follows directly from Theorem 1.1, and the fact that for \( \Sigma \) considered in this theorem, the sets \( \text{Con}(\Sigma), \text{Ang}(\Sigma) \) are empty. \( \Box \)

**Remark 1.4**

a) When \( \Sigma \) is a torus with integer periods (i.e. \( \Sigma = \mathbb{C}/a\mathbb{Z} + b\sqrt{-1}\mathbb{Z} \) for some \( a, b \in \mathbb{N}^* \)) and \((F, h^F, \nabla^F)\) is trivial, (1.17) has been obtained by Duplantier-David \([25, (3.18) \text{and (3.41)]}\). See also Chinta-Jorgenson-Anders \([16, 17] \) and Vertman \([60]\) for related results for tori in any dimension endowed with a trivial vector bundle. For

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\(^1\) Added in a proof: by a combination of Theorem 1.1 and the main result of the preprint by Izyurov-Khristoforov \([37]\), which was available half a year after the current article, we see that this is indeed correct.
tori with integer periods of any dimension, endowed with a line bundle, this has been proved by Friedli [35]. For cylinders with trivial vector bundles, Brankov–Priezzhev in [11, (46)] gave an explicit expression for the left-hand side of (1.17) without relating it to the analytic torsion.

b) Quite easily, our proof can be generalized to non-orientable surfaces by working with invariant subspaces with respect to the Deck transformations on the orientable double cover. Since our methods are local, this would entail (1.17) for Möbius bands and Klein bottles with integer periods. For trivial vector bundles, Brankov–Priezzhev in [11, (48)] and Izmailian–Oganesyan–Hu in [36] gave an explicit expression for the left-hand side of (1.17) for Möbius bands and Klein bottles respectively without identifying the result as the logarithm of the analytic torsion.

Directly from Theorem 1.1 and the fact that our normalization implies that \#V(\Sigma_n) = n^2 \cdot A(\Sigma), we obtain the following result.

**Theorem 1.5** Let (\Sigma, g^T \Sigma) and (\Sigma', g'^T \Sigma') be two tileable surfaces satisfying the assumptions of Theorem 1.1. Construct the graphs \Sigma_n and \Sigma'_n, n \in \mathbb{N}^* as in Theorem 1.1. Endow \Sigma (resp. \Sigma') with a flat unitary vector bundle (F, h_F, \nabla F) (resp. (G, h_G, \nabla G)) and induce unitary vector bundles (F_n, h_{F_n}, \nabla F_n) (resp. (G_n, h_{G_n}, \nabla G_n)) on \Sigma_n (resp. \Sigma'_n).

Assume that A(\Sigma) = A(\Sigma'), |\partial \Sigma| = |\partial \Sigma'|, \angle(\text{Con}(\Sigma)) = \angle(\text{Con}(\Sigma')), \angle(\text{Ang}(\Sigma)) = \angle(\text{Ang}(\Sigma')), \text{rk}(F) = \text{rk}(G) and dim H^0(\Sigma, F) = dim H^0(\Sigma', G). Then

\[
\lim_{n \to \infty} \frac{\det' \Delta_{\Sigma_n}^{F_n}}{\det' \Delta_{\Sigma_n}^{G_n}} = \frac{\det' \Delta_{\Sigma'}^{F}}{\det' \Delta_{\Sigma'}^{G}}.
\]

(1.18)

In particular, if we apply (1.18) for (F, h_F, \nabla F) and (G, h_G, \nabla G) trivial, we obtain

\[
\lim_{n \to \infty} \frac{t(\Sigma_n)}{t(\Sigma'_n)} = \frac{\det' \Delta_{\Sigma}}{\det' \Delta_{\Sigma'}}.
\]

(1.19)

**Remark 1.6** (a) For \Sigma = \Sigma' tori, Dubédat–Gheissari [24, Proposition 4] established (1.18) for non-trivial unitary line bundles (F, h_F, \nabla F), (G, h_G, \nabla G). In their setting, they have also shown in [24, Proposition 3] that under some mild assumptions on the approximation...
\( \Sigma_n, n \in \mathbb{N}^* \)—that the random walk on \( \Sigma_n \) converges to the Brownian motion on \( \Sigma \), as \( n \to \infty \)—the limit (1.18) does not depend on the choice of the approximation \( \Sigma_n \). According to [24, end of Sect. 1], the universality in this sense holds for all surfaces. This means that (1.18) holds for \( \Sigma = \Sigma', H^0(\Sigma, F) = 0 \) and \( H^0(\Sigma, G) = 0 \) for any discretizations \( \Sigma_n \) satisfying the above assumption. See also Dubédat [22, (5.39)] for a related result on the asymptotics of the Kasteleyn operators on tori.

(b) For \( \Sigma = \Sigma' \), and \( F, G \) of rank 2, endowed with special unitary connections, the fact that the limit (1.18) exists was proved by Kassel–Kenyon in [39, Theorem 17], see [39, Sect. 4.1] (cf. also Sect. 2.5). Their proof is very different from ours, and they don’t argue by the analogue of Theorem 1.1. Also, no relation between the value of this limit and the analytic torsion was given in [39].

As one application of Theorem 1.5, we see that (1.19) makes a connection between the maximization of the asymptotic complexity and maximization of the analytic torsion. In the realm of polygonal domains in \( \mathbb{C} \), the last problem has been considered by Aldana-Rowlett [1, Conjecture 1 and Theorem 5]. For maximization of the analytic torsion for flat tori in any dimension, see the article of Sarnak-Strömbergsson [57, Theorems 1, 2]. See also Osgood-Philips-Sarnak [51] for a similar maximization problem inside the class of all smooth metrics on Riemann surfaces.

Another interesting consequence of (1.18) is the conformal invariance of the left-hand side of (1.18) for \( \Sigma = \Sigma' \) and \( (F, \nabla F), (G, \nabla G) \), satisfying \( H^0(\Sigma, F) = H^0(\Sigma, G) = 0 \). We give more details about this interpretation in Sect. 2.7. Remark that if in addition we have \( \text{rk}(F) = \text{rk}(G) = 2 \), then the conformal invariance of the left-hand side of (1.18) in the setting of Remark 1.6b) has been proved by Kassel–Kenyon in [39, Theorem 17] by different methods.

As another application of Theorem 1.1, by relying on the work [39], as the mesh of the discretizations tends to zero, we calculate the limit of certain topological observables over the loops induced by the uniform measure on the set of non-contractible CRSF’s, see Sect. 2.5.

Generally speaking, this article falls into a study of the relationship between spectral invariants of a manifold and its discrete approximations. It is an active area of research, see Yves Colin de Verdière [18, 19], Dodziuk [20], Dodziuk–Patodi [21], Müller [50], Burago–Ivanov–Kurylev [12] for some interesting results. Readers interested in other applications of the analytic torsion can also consult recent articles Bismut [5], Bismut–Ma [6], Maillot–Rössler [47].

This article is organized as follows. In Sect. 2, we recall the necessary definitions and prove Theorem 1.1 modulo certain uniform bounds on the spectrum and zeta functions and modulo some Szegő-type asymptotic result on the mesh graphs. The results of our previous article [29] play the foremost role in this spectral approach to the problem. We then provide a relation between the main results of this paper, random geometry and the work of Kenyon [41]. In Sect. 3, we establish the needed bounds on the spectrum and on zeta functions. In Sect. 4, by doing Fourier analysis on discrete square, we establish the needed asymptotic result on the mesh graphs. In Appendix A.1 we recall the small-time asymptotic expansion of the heat kernel. Finally, in Appendix A.2 we recall the explicit evaluation of the analytic torsion for tori and rectangles and the already mentioned result of Duplantier-David. The results of this paper and [29] were announced in [31].
Notation. For a graph $G$, we denote by $V(G), E(G)$ the sets of vertices and edges of $G$ respectively. We denote by $\langle \cdot, \cdot \rangle_{L^2(G,F)}$ the $L^2$-scalar product on $\text{Map}(V(G), F)$ defined by
\[
\langle f, g \rangle_{L^2(G,F)} = \sum_{v \in V(G)} \langle f(v), g(v) \rangle_{hF}, \quad f, g \in \text{Map}(V(G), F).
\] (1.20)

Analogously, we denote by $\langle \cdot, \cdot \rangle_{L^2(G)}$ the $L^2$-scalar product on the set $\text{Map}(E(G), C)$. We denote by $\|\cdot\|_{L^2(G,F)}, \|\cdot\|_{L^2(G)}$ the associated norms. We denote by $d_G : \text{Map}(V(G), C) \to \text{Map}(E(G), C)$ the discrete differentiation operator, defined by
\[
(d_G f)(e) := f(h(e)) - f(t(e)), \quad f \in \text{Map}(V(G), C),
\] (1.21)
where $h(e), t(e)$ are the head and tail of $e$ respectively.

Similarly, for smooth sections $f, g$ of a Hermitian vector bundle $(\xi, h^\xi)$ over a compact Riemannian manifold $(X, g^TX)$, we define the $L^2$-scalar product
\[
\langle f, g \rangle_{L^2(X,\xi)} = \int_X \langle f(x), g(x) \rangle_{h^\xi} dv_{g^TX},
\] (1.22)
where $\langle \cdot, \cdot \rangle_{h^\xi}$ is the pointwise scalar product induced by $h^\xi$, and $dv_{g^TX}$ is the Riemannian volume form on $(X, g^TX)$. Similarly, we define the $L^2$-scalar product of differential forms with values in $\xi$. Finally, for $k \in \mathbb{N}$, we denote by
\[
\mathcal{O}^k(X, \xi) := \left\{ f \in \mathcal{O}^k(X, \xi) : \nabla^l f \in L^\infty(X, \xi), \text{ for } l \leq k \right\}.
\] (1.23)

2 Asymptotic of the discrete determinant

The main goal of this section is to prove Theorem 1.1. In Sect. 2.1, we recall some necessary definitions in relation with flat surfaces with conical singularities and vector bundles on the graphs. In Sect. 2.2, we construct a covering of tileable surfaces, which will be used in our proof of Theorem 1.1. In Sect. 2.3, we study the Laplacian and its zeta function on flat surfaces with conical singularities. In Sect. 2.4, we prove Theorem 1.1 modulo certain statements which will be proved later in this article. The results of our previous article [29] play the crucial role here. In Sect. 2.5, we discuss some of the applications of Theorem 1.1 to random geometry in the context of Kassel–Kenyon [39]. In Sect. 2.6, we discuss the relation between this work and the work of Kenyon [41]. Finally, in Sect. 2.7, we explain how anomaly formula of Bismut–Gillet–Soulé can be used to give an interpretation of the conformal invariance of observables as in Theorem 1.5.

2.1 Tileable surfaces and their discretizations

Here we recall the definition of flat surfaces, explain some properties of the discretizations of tileable surfaces and give a short introduction to vector bundles over graphs.

By Gauss-Bonnet theorem, the only closed Riemann surface admitting a flat metric has the topology of the torus. However, any Riemann surface can be endowed with a flat metric having a finite number of cone-type singularities. Let us explain this point more precisely.

A cone-type singularity is a Riemannian metric
\[
ds^2 = dr^2 + r^2 dt^2,
\] (2.1)
Fig. 4 Multiple edges in a discretization. The surface is obtained by identifying the edges along the directed lines. It has one conical point of angle $\pi$. The edges of the graph are dotted.

on the manifold

$$C_\theta := \{(r, t) : r > 0; t \in \mathbb{R}/\theta \mathbb{Z}\},$$

(2.2)

where $\theta > 0$. In what follows, when we speak of cones, we assume $\theta \neq 2\pi$ implicitly.

By a flat metric with a finite number of cone-type singularities we mean a metric defined away from a finite set of points such that there is an atlas for which the metric looks either like the standard metric on $\mathbb{R}^2$, or like the conical metric (2.1) on an open subset of (2.2).

For a flat surface $(\Sigma, g^T\Sigma)$ with a finite number of cone-type singularities, we denote by $\text{Con}(\Sigma)$ the conical points of the surface $\Sigma$, and by $\text{Ang}(\Sigma)$ the points where two different smooth components of the boundary meet (corners). We denote by $\angle : \text{Con}(\Sigma) \to \mathbb{R}$ the function which associates to a conical point its angle and by $\angle : \text{Ang}(\Sigma) \to \mathbb{R}$ the function which associates the interior angle between the smooth components of the boundary.

For the most part of this paper, we will be interested in tileable surfaces $\Sigma$. Those are flat surfaces with a finite number of cone-type singularities with a fixed tiling by equal euclidean squares. We normalize the metric $g^T\Sigma$ on $\Sigma$ so that the squares of the fixed tiling have area 1. Let us now set up some notations associated with discretization

Introduction. For $\theta > 0$, where $\theta$ is unitary with respect to the flat metric on $\Sigma_n$, we mean the choice of a positive-definite Hermitian metric $h_V$ on $V_n$ for each $v \in V(G)$ so that for any $v, v' \in V(G)$, the vector spaces $V_v$ and $V_{v'}$ are isomorphic. The set of sections $\Map(V(G), V)$ of $V$ is defined by

$$\Map(V(G), V) = \bigoplus_{v \in V(G)} V_v.$$ 

(2.3)

A connection $\nabla^V$ on a vector bundle $V$ is the choice for each (oriented) edge $e = (v, v') \in E(G)$ of an isomorphism $\phi_{v,v'} : V_v \to V_{v'}$ between the corresponding vector spaces, such that $\phi_{v,v'}^{-1} = \phi_{v',v}$. This isomorphism is called the parallel transport of vectors in $V_v$ to vectors in $V_{v'}$.

A Hermitian metric $h^V$ on the vector bundle $V$ is a choice of a positive-definite Hermitian metric $h_v$ on $V_v$ for each $v \in V(G)$. We say that a connection $\nabla^V$ is unitary with respect to $h^V$ if its parallel transports preserves $h^V$.

The Laplacian $\Delta^V_G$ on $(V, \nabla^V)$ is the linear operator $\Delta^V_G : \Map(V(G), V) \to \Map(V(G), V)$, defined for $f \in \Map(V(G), V)$ by (1.2). Remark that unlike Laplace-
Beltrami operator on a smooth manifold, we don’t use the metric to define the Laplacian (1.2).

Thus, in general, the operator $\Delta_G^V$ is not self-adjoint, see for example [42, Sect. 3.2]. However, if the connection is unitary with respect to $h^V$, then it becomes self-adjoint (cf. Kenyon [42, Sect. 3.3]).

Let’s represent the Laplacian $\Delta_G^V$ in the form (1.6). One can extend the definition of a vector bundle to the edges of $G$. A vector bundle $V'$ over $V(G) \oplus E(G)$ is by definition a collection of vector spaces $V_e$ for each edge $e \in E(G)$ as well as $V_v$ for each vertex $v \in V(G)$. Similarly, we extend the definition of Hermitian metric. A connection $\nabla^V$ on $V'$ is a choice of a connection $\nabla^V$ on $V$ as well as connection isomorphisms $\phi_{ve} : V_v \to V_e$ and $\phi_{e'} : V_e \to V_{e'}$ for each $v \in V(G)$ and $e \in E(G)$, and satisfying $\phi_{ve} = \phi_{e'}^{-1}$ and $\phi_{v'} = \phi_{e'} \circ \phi_{ve}$. Similarly to (2.3), we define

$$\text{Map}(E(G), V) := \oplus_{e \in E(G)} V_e.$$  \hfill (2.4)

Quite easily, for any vector bundle $V$ and a connection $\nabla^V$ on $V(G)$, we may extend it to a vector bundle $V'$ and a connection $\nabla'^V$ on $E(G) \oplus V(G)$. For $V$, endowed with a Hermitian metric $h^V$, for which the connection $\nabla^V$ is unitary, the vector bundles $V_e$, $e \in E(G)$ can be endowed with metrics and connections $\phi_{ve}$, $e \in E(G)$, and satisfying $\phi_{ve} = \phi_{e'}^{-1}$ and $\phi_{v'} = \phi_{e'} \circ \phi_{ve}$. Similarly to (2.3), we define the operator

$$\Delta_G^V = (\nabla_G^V)^* \nabla_G^V.$$  \hfill (2.7)
In general \((\nabla^V_G)^*\) is not the adjoint of \(\nabla^V_G\) with respect to the appropriate \(L^2\)-metrics (see (1.20)). But if the connection \(\nabla^V\) is unitary, it is indeed the case, cf. [42, Sect. 3.3]. In this article, all connections are unitary, and thus, by (2.7), the associated discrete Laplacians are self-adjoint and positive.

### 2.2 Covering a tileable surface by model surfaces

In this section we construct a covering of a tileable surface \(\Sigma\) by a finite number of model surfaces. We also construct a subordinate partition of unity in such a way that this partition of unity, regarded as functions on the covering gives rise to a finite number of isomorphism classes of partition sets endowed with a fixed function corresponding to the partition of unity in the cover. Those constructions will play an important role in our proof of Theorem 1.1.

To explain our construction better, let’s assume first that \(\Sigma\) is a rectangular domain in \(\mathbb{C}\) without slits and with corners at integer vertices. Then we can easily verify that up to making a homothety of factor 4 on \(\Sigma\), one can construct a covering \(U_\alpha, \alpha \in I\), of \(\Sigma\) so that each \(U_\alpha\) is either constructed from 4 tiles of \(\Sigma\), forming a square, or 12 tiles, forming the \(L\)-shape, \(A_{3\pi/2}\), as in Fig. 2.

Then, we see that for the number \(N_{A_{3\pi/2}}(\Sigma)\) of \(U_\alpha\), isomorphic to \(A_{3\pi/2}\), we have

\[
N_{A_{3\pi/2}}(\Sigma) = \# \left\{ Q \in \text{Ang}(\Sigma) : \angle(Q) = \frac{3\pi}{2} \right\}.
\]

(2.8)

Analogously, denote by \(N_{sq,i}(\Sigma), i = 0, 1, 2\) the number of \(U_\alpha\), which are isomorphic to a square sharing \(i\) sides with the boundary of \(\Sigma\). We have the following identity

\[
N_{sq,2}(\Sigma) = \# \text{Ang}_{=\pi/2}(\Sigma).
\]

(2.9)

Let’s now give an expression for \(N_{sq,i}(\Sigma)\) for \(i = 0, 1\). Clearly, if one puts the numbers on the tiles of the covering as it is shown in the Fig. 6, then after summing the numbers in the covering, corresponding to each tile, we get identically 1 over \(\Sigma\) (in other words, the functions depicted at Fig. 6 form \(L^\infty\)-partition of unity).

From Fig. 6, we see that the sum of the numbers in tiles corresponding to a square without common boundaries with \(\partial \Sigma\) is 1, with 1 common boundary - \(\frac{9}{4}\), with 2 common boundaries \(-\frac{9}{4}\), and the sum of tiles corresponding to an \(L\)-shape is \(\frac{27}{4}\). Also, the sum of the numbers in tiles sharing common side with \(\partial \Sigma\) in a square with 1 common boundary with \(\partial \Sigma\) is 1, with 2 common boundaries (taking into account the multiplicities) -3, and and in the \(L\)-shape -3. By this, we can compute the area \(A(\Sigma)\) in two ways: as the sum of tiles (as we have

---

**Fig. 6** The bold lines represent the boundary \(\partial \Sigma\), small squares represent the tiles of \(\Sigma\)
normalized our tiles to have area 1) and as the sum of all the entries in Fig. 6 (as the entries form a \( L^\infty \) partition of unity). Similarly, we can calculate \( |\partial \Sigma| \) in two different ways. We obtain from this

\[
|\partial \Sigma| = \left( N_{sq,1} + 3N_{sq,2} + 3N_A \frac{\pi}{2} \right)(\Sigma),
\]

\[
A(\Sigma) = \left( N_{sq,0} + \frac{3}{2}N_{sq,1} + \frac{9}{4}N_{sq,2} + \frac{27}{4}N_A \pi \right)(\Sigma).
\]

(2.10)

Now, similarly, a general tileable surface \( \Sigma \), considered up to a homothety of factor 4, can be covered by squares, tiled with 4 tiles of \( \Sigma \), the model angles \( A \frac{\pi}{2}, k \geq 3 \) of angle \( k\pi \frac{\pi}{2} \), the special case of which are the \( L \)-shape and slit considered in Figs. 2, 3, and the model cones \( C_k \pi, k = 1 \) or \( k \geq 3 \), of angle \( k\pi \frac{\pi}{2} \), depicted in Figs. 7, 8.

More precisely, let’s describe a model cone \( C_k \pi \) of angle \( \pi \). Consider a rectangle, which has ratios of sides \( 2 : 1 \) and which is tiled with 8 squares. Glue the intervals on one side of it as it is shown in Fig. 4. Endow the resulting surface \( C_k \pi \) with the tileable structure coming from 8 tiles and a metric induced by the standard metric on \( \mathbb{C} \). The resulting metric is flat with one conical angle \( \pi \) and with two corners with angles \( \frac{\pi}{2} \). The perimeter of the surface is equal to 8.

Now let’s describe the model cones \( C_{2k\pi} \) of angles \( 2k\pi \), \( k \in \mathbb{N}^* \{1\} \). Consider the \( k \)-covering

\[
\pi_k : C_{2k\pi} \rightarrow \mathbb{C},
\]

(2.11)

unramified everywhere except the origin, where the order of ramification is \( k \). Then we define

\[
C_{2k\pi} = \pi_k^{-1}([-2, 2] \times [-2, 2]).
\]

(2.12)

The surface \( C_{2k\pi} \) is endowed with the pull-back metric from \( \mathbb{C} \) and with the structure of the tileable surface coming from the 16-tile structure on \([-2, 2] \times [-2, 2]\), given by euclidean squares of area 1. In other words, \( C_{2k\pi} \) has 16k tiles, one conical angle \( 2k\pi \) and \( 4k \) angles of the boundary equal to \( \frac{\pi}{2} \). For an example, see Fig. 7.

Now let’s construct the model cones \( C_{(2k+1)\pi} \) of angles \( (2k + 1)\pi \), \( k \in \mathbb{N}^* \). Consider a surface \( C_{2k\pi} \) constructed in the previous step. Introduce a single cut on this surface such that its projection in \( \mathbb{C} \) coincides with the straight interval from 0 to 2. Consider a rectangle tiled with 8 squares as in the construction of \( C_{\pi} \). Glue the sides of the slits with the sides of

![Fig. 7](image-url) The model space \( C_{4\pi} \). It is obtained by gluing the edges of the same pattern.
the rectangle as it is shown in Fig. 8. Endow the resulting surface with the induced metric and tileable surface structure coming from corresponding structures on $C_{2k\pi}$ and $C_{\pi}$. The resulting surface $C_{(2k+1)\pi}$ has $16k + 8$ tiles and a flat metric with one conical angle $(2k + 1)\pi$ and $4k + 2$ angles of the boundary equal to $\frac{\pi}{2}$.

Finally, let’s describe the model angles $A_{k\pi}$ of angle $\frac{k\pi}{2}$, $k \geq 3$. Construct a surface $C_{k\pi}$ by the procedure above and introduce two cuts projecting to a vertical or to a horizontal intervals from 0 of length 2, so that the formed angles between those cuts coincide. Then the surface, obtained by deletion of those intervals, has 2 isomorphic connected components, which we denote by $A_{k\pi}$. The tileable surface structure and the metric is induced by the corresponding structures from $C_{k\pi}$. The surface $A_{k\pi}$ has $4k$ tiles, it is endowed with flat metric with no conical points; at the boundary it has one angle $\frac{k\pi}{2}$ and $k + 2$ angles $\frac{\pi}{2}$. For examples, see Figs. 2, 3.

Clearly, the formulas (2.8), (2.9) still hold, and the identities in the spirit of (2.10) continue to hold as well. We will only need a weak version of them, which we state below.

**Lemma 2.2** There are constants $c_0(\Sigma), c_1(\Sigma) \in \mathbb{R}$, which depend only on the sets $\angle(\text{Ang}(\Sigma))$ and $\angle(\text{Con}(\Sigma))$, for which the following holds

$$
N_{sq,0}(\Sigma) = A(\Sigma) - \frac{3}{2} |\partial \Sigma| + c_0(\Sigma),
$$

$$
N_{sq,1}(\Sigma) = |\partial \Sigma| + c_1(\Sigma).
$$

Moreover, $c_0(\Sigma)$ and $c_1(\Sigma)$ depend additively on $\angle(\text{Ang}(\Sigma))$ and $\angle(\text{Con}(\Sigma))$.

**Proof** The proof of (2.13) is identical to the proof of (2.10). \hfill \Box

Now let’s describe the choice of the subordinate partition of unity $\phi_\alpha$ associated with $U_\alpha$, which satisfy the assumptions described in the beginning of this section. Remark that an example of $L^\infty$ partition of unity of this form is given in Fig. 6.

We fix a function $\rho : [-1, 1] \to [0, 1]$, satisfying

$$
\rho(x) = \begin{cases} 
1, & \text{in the neighborhood of } x = 0, \\
0, & \text{in the neighborhood of } x = \pm 1.
\end{cases}
$$

(2.14)

$$
\rho(x) = \rho(-x),
$$

(2.15)
\[
\rho(x + 1/2) + \rho(1/2 - x) = 1, \quad \text{for } x \in [0, 1/2]. \tag{2.16}
\]

Now, assume that for \( \alpha \in I \), the open set \( U_\alpha \) corresponds to a square. Choose linear coordinates \( x, y \) with axes parallel to the boundaries of the tiles and normalize them so that they identify the square with \([-1, 1] \times [-1, 1]\). If \( U_\alpha \) doesn’t share a boundary with \( \partial \Sigma \), we define

\[
\phi_\alpha(x, y) := \phi_{sq,0}(x, y) := \rho(x)\rho(y). \tag{2.17}
\]

Now, assume that \( U_\alpha \) shares one boundary with \( \partial \Sigma \). Suppose without loosing the generality that this boundary corresponds to \( \{x = -1\} \). Then we define

\[
\phi_\alpha(x, y) := \phi_{sq,1}(x, y) := \begin{cases} 
\rho(x)\rho(y), & \text{for } x \geq 0, \\
\rho(y), & \text{for } x \leq 0.
\end{cases} \tag{2.18}
\]

Assume, finally, that \( U_\alpha \) shares two boundaries with \( \partial \Sigma \). Suppose without loosing the generality that those boundaries correspond to \( \{x = -1\} \) and \( \{y = -1\} \). Then we define

\[
\phi_\alpha(x, y) := \phi_{sq,2}(x, y) := \begin{cases} 
\rho(x)\rho(y), & \text{for } x \geq 0, y \geq 0, \\
\rho(y), & \text{for } x \leq 0, y \geq 0, \\
\rho(x), & \text{for } x \geq 0, y \leq 0, \\
1, & \text{for } x \leq 0, y \leq 0.
\end{cases} \tag{2.19}
\]

This finishes the description in case if \( U_\alpha \) is a square.

Now, to describe the choice of the function \( \phi_\alpha \) on model angles and cones, let’s define the function \( \phi_0 : [-2, 2] \times [-2, 2] \to [0, 1] \). For \((x, y) \in [-2, 2] \times [-2, 2]\), it satisfies the following

\[
\phi_0(x, y) = \phi_0(-x, y) = \phi_0(x, -y). \tag{2.20}
\]

Thus, it is enough to define it only in the quadrant \( x, y \geq 0 \). In this quadrant it is defined by

\[
\phi_0(x, y) := \phi_{sq,2}(x - 1, y - 1), \quad \text{for } (x, y) \in [0, 2] \times [0, 2]. \tag{2.21}
\]

Now let’s assume that \( U_\alpha \) is isometric with \( C_\pi \). Choose linear coordinates \( x, y \) with axes parallel to the boundaries of the tiles and normalize them so that they identify the rectangle in Fig. 4 with \([-2, 2] \times [0, 2]\). Normalize \( y \) so that the sides, which get glued, lie on the line \( \{y = 0\} \). Then we define the corresponding function \( \phi_\alpha \) to be equal to the restriction of \( \phi_0 \). Clearly, by (2.14) and (2.15), the function \( \phi_\alpha \) is well-defined and smooth on \( C_\pi \).

Now let’s assume that \( U_\alpha \) is isometric to \( \mathcal{C}_{2k\pi} \), \( k \in \mathbb{N}^* \). Then we define

\[
\phi_\alpha := \phi_{\mathcal{C}_{2k\pi}} := (\pi_k)^{-1} \phi_0, \tag{2.22}
\]

where \( \pi_k \) was defined in (2.11).

For \( U_\alpha \) isometric to \( \mathcal{C}_{(2k+1)\pi} \), \( k \in \mathbb{N}^* \), the function \( \phi_\alpha := \phi_{\mathcal{C}_{(2k+1)\pi}} \) is defined by gluing the corresponding functions on \( C_\pi \) and \( C_{2k\pi} \) through a pattern used in the definition of \( C_{(2k+1)\pi} \). This function is well-defined and smooth by (2.14) and (2.15).

For \( U_\alpha \) isometric to \( \mathcal{A}_{k\pi} \), \( k \geq 3 \), the function \( \phi_\alpha := \phi_{\mathcal{A}_{k\pi}} \) is defined by restriction of the corresponding function on \( C_{k\pi} \).

**Lemma 2.3** The functions \( \phi_\alpha, \alpha \in I \), form a partition of unity of \( \Sigma \), subordinate to \( U_\alpha \).

**Proof** It is a consequence of (2.15) and (2.16). \( \square \)
2.3 Zeta functions on flat surfaces

Here we study the Laplacian $\Delta^F_\Sigma$, (1.6), on a flat surface $(\Sigma, g^{T\Sigma})$ with conical singularities and piecewise geodesic boundary $\partial \Sigma$ endowed with a flat unitary vector bundle $(F, h^F, \nabla^F)$. We define the zeta function for the Friedrichs extension of the Laplacian, and study some of its properties.

There are many ways to motivate the choice of Friedrichs extension as the limiting self-adjoint extension of a sequence of discrete Laplacians. To name one, it is positive (cf. [56, Theorem X.23]), similarly to discrete Laplacians (see the end of Sect. 2.1). Moreover in [29], (cf. Theorems 2.9, 2.11), we have proved that the eigenvalues and the eigenvectors of Friedrichs extension can be obtained as limits of the eigenvalues and eigenvectors of the rescaled twisted discrete Laplacians $n^2 \cdot \Delta^F_{\Sigma_n}$, $n \in \mathbb{N}^*$. This was proved by establishing Harnack-type estimates for the discrete eigenvectors and by showing that the growth of the discrete eigenvectors near the the corners and conical singularities is as mild as the singularities near the corners and conical singularities of the sections from the domain of the Friedrichs extension.

The content of this section is most certainly not new, but we didn’t find a complete reference for all the results contained here, so we provide the details for the convenience of the reader.

We consider $\Delta^F_\Sigma$ as an operator acting on the functional space $\mathcal{C}_{0,N}(\Sigma, F)$ defined by

$$\mathcal{C}_{0,N}(\Sigma, F) := \left\{ f \in \mathcal{C}_0^\infty(\Sigma \setminus \text{Ang}(\Sigma)) : \nabla^F_n f = 0 \text{ over } \partial \Sigma \right\},$$

(2.23)

where $\nu$ is the normal to $\partial \Sigma$. Unlike in the case of a manifold with smooth boundary, the operator $\Delta^F_\Sigma$ is in general not essentially self-adjoint, cf. Cheeger [15] and Mooers [49].

Let $\text{Dom}_{\text{max}}(\Delta^F_\Sigma)$ denote the maximal closure of $\Delta^F_\Sigma$. In other words, for $u \in L^2(\Sigma, F)$, we have $u \in \text{Dom}_{\text{max}}(\Delta^F_\Sigma)$ if and only if $\Delta^F_\Sigma u \in L^2(\Sigma, F)$, where $\Delta^F_\Sigma u$ is viewed as a distribution.

Let’s denote by $H^1(\Sigma, F)$ the Sobolev space on $\Sigma$, defined as

$$H^1(\Sigma, F) = \left\{ u \in L^2(\Sigma, F) : \nabla^F u \in L^2(\Sigma, F) \right\}.$$

(2.24)

We denote by $\| \cdot \|_{H^1(\Sigma, F)}$ the norm on $H^1(\Sigma, F)$, given for $u \in H^1(\Sigma, F)$ by

$$\|u\|_{H^1(\Sigma, F)} = \|u\|_{L^2(\Sigma, F)} + \|\nabla^F u\|_{L^2(\Sigma, F)}.$$

(2.25)

For any densely defined positive symmetric operator, one can construct in a canonical way a self-adjoint extension, called Friedrichs extension, by the completion of the associated quadratic form, cf. [56, Theorem X.23]. Once the definition is unraveled, the domain $\text{Dom}_{\text{Fr}}(\Delta^F_\Sigma)$ of the Friedrichs extension of the Laplacian $\Delta^F_\Sigma$ on $\Sigma$ with Neumann boundary conditions on $\partial \Sigma$ satisfies

$$\text{Dom}_{\text{Fr}}(\Delta^F_\Sigma) \subset \text{Dom}_{\text{max}}(\Delta^F_\Sigma) \cap H^1(\Sigma, F).$$

(2.26)

The value of the Friedrichs extensions of the Laplacian $\Delta^F_\Sigma$ on $f \in \text{Dom}_{\text{Fr}}(\Delta^F_\Sigma)$ is defined in the distributional sense. By the definition of $\text{Dom}_{\text{max}}(\Delta^F_\Sigma)$, it lies in $L^2(\Sigma, F)$.

Next result is a simple consequence of Rellich-Kondrachov theorem and Green’s identity.
Proposition 2.4 (cf. [29, Proposition 2.3, Corollary 2.6]) The spectrum of $\Delta^F_\Sigma$ is discrete. Moreover, there is an isomorphism between $\ker \Delta^F_\Sigma$ and the space of flat sections of $F$, i.e.

$$\ker \Delta^F_\Sigma \simeq H^0(\Sigma, F).$$  \hspace{1cm} (2.27)

Consider now the heat operator $\exp(-t \Delta^F_\Sigma)$, $t > 0$. By Schwartz kernel theorem (cf. [4, Proposition 2.14]) and interior elliptic estimates, this operator has a smooth Schwartz kernel $\exp(-t \Delta^F_\Sigma)(x, y) \in \mathbb{R}$, defined for any $x, y \in \Sigma^\circ := \Sigma \setminus (\text{Con}(\Sigma) \cup \text{Ang}(\Sigma))$, also called the heat kernel. By definition, for any $f \in L^\infty(\Sigma)$, we have

$$\text{Tr}[f \cdot \exp(-t \Delta^F_\Sigma)] = \int_{\Sigma^\circ} f(x) \cdot \exp(-t \Delta^F_\Sigma)(x, x) dv_\Sigma(x).$$  \hspace{1cm} (2.28)

The proof of the following proposition is given in Appendix A.1 and it uses standard techniques from the heat kernel analysis and explicit construction of the heat kernel on some model spaces.

Proposition 2.5 For any $f \in \mathcal{C}^\infty(\Sigma)$, which is constant in the neighborhood of $\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$, and any $k \in \mathbb{N}$, as $t \to 0$, the following asymptotic expansion holds

$$\text{Tr}[f \cdot \exp(-t \Delta^F_\Sigma)] = \frac{\text{rk}(F)}{4\pi t} \int_{\Sigma} f(x) dv_\Sigma(x) + \frac{\text{rk}(F)}{8\sqrt{\pi} t^{1/2}} \int_{\partial \Sigma} f(x) dv_{\partial \Sigma}(x) + \sum_{P \in \text{Con}(\Sigma)} \frac{4\pi^2 - \triangle(P)^2}{2\pi \angle(P)} f(P) + \sum_{Q \in \text{Ang}(\Sigma)} \frac{\pi^2 - \angle(Q)^2}{2\pi \angle(Q)} f(Q) + o(t^k).$$  \hspace{1cm} (2.29)

where $\Gamma$ is the $\Gamma$-function. In particular, for any $k \in \mathbb{N}$, as $t \to 0$, we have

$$\text{Tr}[\exp(-t \Delta^F_\Sigma)] = \frac{A(\Sigma) \cdot \text{rk}(F)}{4\pi t} + \frac{|\partial \Sigma| \cdot \text{rk}(F)}{8\sqrt{\pi} t^{1/2}} + \sum_{P \in \text{Con}(\Sigma)} \frac{4\pi^2 - \angle(P)^2}{2\pi \angle(P)} + \sum_{Q \in \text{Ang}(\Sigma)} \frac{\pi^2 - \angle(Q)^2}{2\pi \angle(Q)} + o(t^k).$$  \hspace{1cm} (2.30)

As a consequence of Karamata’s theorem, cf. [4, Theorem 2.42], and Proposition 2.5, by repeating the proof of [4, Corollary 2.43], we obtain the following

Corollary 2.6 (Weyl’s law) The number $N^F(\lambda)$ of eigenvalues of $\Delta^F_\Sigma$, smaller than $\lambda$, satisfies

$$N^F(\lambda) \sim \frac{A(\Sigma) \text{rk}(F)}{4\pi} \lambda,$$  \hspace{1cm} (2.31)

as $\lambda \to +\infty$. In particular, for the $i$-th eigenvalue $\lambda_i$ of $\Delta^F_\Sigma$, as $i \to \infty$, we have

$$\lambda_i \sim \frac{4\pi i}{A(\Sigma) \text{rk}(F)}. \hspace{1cm} (2.32)$$

From Corollary 2.6 we deduce that the zeta function $\zeta^F(\lambda)$ from (1.10) is well defined for $s \in \mathbb{C}$, $\text{Re}(s) > 1$. Denote $\mu := \min(\text{Spec}(\Delta^F_\Sigma) \setminus \{0\})$. By Proposition 2.4, we deduce that for some $C > 0$ and any $t > 1$, we have

$$\left| \text{Tr}[\exp(-t \Delta^F_\Sigma)] - \dim \text{H}^0(\Sigma, F) \right| = \sum_{\lambda \in \text{Spec}(\Delta^F_\Sigma) \setminus \{0\}} \exp(-t\lambda).$$
\[
\leq \exp\left(-\frac{t\mu}{2}\right) \cdot \sum_{\lambda \in \text{Spec}(\Delta^F)} \exp(-\lambda/2) \leq C \exp\left(-\frac{t\mu}{2}\right).
\] (2.33)

By Proposition 2.5, (2.33) and standard properties of the Mellin transform (cf. [4, Lemma 9.35]), we deduce that \(\zeta^F(s)\) extends meromorphically to the whole complex plane \(\mathbb{C}\), and 0 is a holomorphic point of this extension. In particular, (1.11) makes sense.

**Proposition 2.7** The zero value \(\zeta^F(0)\) can be evaluated by the formula (1.12).

**Proof** It follows from the classical properties of the Mellin transform and Proposition 2.5. \(\square\)

For \(c > 0\), we denote by \(c\Sigma\) the surface \(\Sigma\) endowed with the flat metric \(c^2 \cdot g^{T\Sigma}\). The resulting surface is a flat surface with conical singularities. If \((\Sigma, g^{T\Sigma})\) can be embedded in \(\mathbb{C}\), then \(c\Sigma\) is just the homothety of \(\Sigma\) by a factor \(c\).

**Proposition 2.8** The analytic torsions of \(c\Sigma\) and \(\Sigma\) are related by

\[
\log \det'(\Delta^F_{\Sigma}) = \log \det'(\Delta^F_{\Sigma}) - 2 \log(c)\zeta^F_{\Sigma}(0).
\] (2.34)

**Proof** It follows trivially from (1.10) that the zeta-functions are related by

\[
\zeta^F_{\Sigma}(s) = c^{2s} \zeta^F_{\Sigma}(s).
\] (2.35)

We deduce Proposition 2.8 by (1.11) and (2.35). \(\square\)

### 2.4 Convergence of zeta-functions, a proof of Theorem 1.1

The main goal here is to prove Theorem 1.1. We conserve the notation from Theorem 1.1.

Let’s describe the main steps of the proof before going into details. First, we recall that in our previous paper [29, Theorem 1.1], cf. Theorem 2.9, we proved that the rescaled spectrum \(\text{Spec}(n^2 \cdot \Delta^F_{\Sigma_n}) = \{\lambda_1^n, \lambda_2^n, \cdots\}\),

of the discretization \(\Sigma_n\), ordered non-decreasingly, is asymptotically equal to the spectrum (1.8) of \(\Delta^F_{\Sigma}\). In this article, in Theorem 2.12, we prove a uniform linear lower bound on the eigenvalues of \(n^2 \cdot \Delta^F_{\Sigma_n}\). By this and Theorem 2.9, we prove in Corollary 2.13 that the discrete zeta-functions \(\zeta^F_{\Sigma_n}(s)\), associated to \(n^2 \cdot \Delta^F_{\Sigma_n}\), converge, as \(n \to \infty\), to \(\zeta^F_{\Sigma}(s)\) of \(\Delta^F_{\Sigma}\) on the half-plane \(\text{Re}(s) > 1\).

Then in Corollary 2.17, we show that up to some local contributions, depending on \(n \in \mathbb{N}^+\), this convergence extends to the whole complex plane \(\mathbb{C}\). To get the asymptotics of the logarithm of the determinant, now it suffices simply to take the derivative of the zeta function at 0. In our final step, Theorem 2.18, we study the asymptotic expansion of the derivative of at 0 of the introduced local contributions from Corollary 2.17, as \(n \to \infty\). This finishes our proof.

More precisely, in our previous paper, we’ve shown

**Theorem 2.9** [29, Theorem 1.1] For any \(i \in \mathbb{N}\), as \(n \to \infty\), the following limit holds

\[
\lambda_i^n \to \lambda_i,
\] (2.37)

where we use the notation from (1.8) and (2.36) for \(\lambda_i\) and \(\lambda_i^n\) respectively.
We also proved a similar statement for eigenvectors. Before describing it, recall that in [29, Sect. 3.2], we have defined a “linearization” $L_n : \text{Map}(V(\Sigma_n), F_n) \rightarrow L^2(\Sigma, F)$ functional. One should think of it as a sort of linear interpolation, which “blurs” the function near $\text{Con}(\Sigma) \cup \text{Ang}(\Sigma)$. In this article only the following property of $L_n$ will be used: for eigenvectors $f^n_i \in \text{Map}(V(\Sigma_n), F_n)$ of $n^2 \cdot \Delta^n_{\Sigma_n}$ corresponding to the eigenvalues $\lambda^n_i$, we have

**Proposition 2.10** [29, Proposition 3.8] For any $\phi \in \mathcal{C}^1(\Sigma)$ and $i, j \in \mathbb{N}$ fixed, as $n \rightarrow \infty$:

$$
\langle \phi L_n(f^n_i), L_n(f^n_j) \rangle_{L^2(\Sigma, F)} = \frac{1}{n^2} \langle \phi f^n_i, f^n_j \rangle_{L^2(\Sigma_n, F_n)} + o(1). \quad (2.38)
$$

Now, assume that $\lambda_i, i \in \mathbb{N}^*$ has multiplicity $m_i$ in $\text{Spec}(\Delta^F_{\Sigma})$. Let $f_{i,j} \in L^2(\Sigma, F)$, $j = 1, \ldots, m_i$ be the orthonormal basis of eigenvectors of $\Delta^F_{\Sigma}$ corresponding to $\lambda_i$. By Theorem 2.9, we conclude that there is a series of eigenvalues $\lambda^n_{i,j}, j = 1, \ldots, m_i$ of $n^2 \cdot \Delta^n_{\Sigma_n}$, converging to $\lambda_i$, as $n \rightarrow \infty$, and no other eigenvalue of $n^2 \cdot \Delta^n_{\Sigma_n}, n \in \mathbb{N}^*$, comes close to $\lambda_i$ asymptotically.

**Theorem 2.11** [29, Theorem 1.3] For any $i \in \mathbb{N}, n \in \mathbb{N}^*$ there are $f^n_{i,j} \in \text{Map}(V(\Sigma_n), F_n)$, $1 \leq j \leq m_i$, which are pairwise orthogonal, satisfy $\|f^n_{i,j}\|_{L^2(\Sigma_n, F_n)} = n^2$, and which are in the span of the eigenvectors of $n^2 \cdot \Delta^n_{\Sigma_n}$ corresponding to the eigenvalues $\lambda^n_{i,j}, j = 1, \ldots, m_i$, such that, as $n \rightarrow \infty$, in $L^2(\Sigma, F)$, the following limit holds

$$
L_n(f^n_{i,j}) \rightarrow f_{i,j}. \quad (2.39)
$$

Now, recall that we are trying to get the asymptotic expansion (1.15), which involves a product of terms, the number of which, $V(\Sigma_n) - \dim H^0(\Sigma, F)$, $n \in \mathbb{N}^*$, tends to infinity quite quickly, as $n \rightarrow \infty$. Thus, there is almost no chance to get Theorem 1.1 by studying simply the convergence of individual eigenvalues, as it would require much stronger convergence result compared to what we obtained in Theorem 2.9. Moreover, the analytic torsion is defined not through the normalized product of the first eigenvalues, but through the zeta-regularization procedure.

We are, consequently, obliged to work with some statistics of eigenvalues. The most important one in this paper is the corresponding zeta-function, defined for $s \in \mathbb{C}$ by

$$
\zeta_{\Sigma, n}^F(s) := \sum_{\lambda \in \text{Spec}(\Delta^F_{\Sigma_n}) \setminus \{0\}} \frac{1}{(n^2 \cdot \lambda)^s} = \frac{1}{\zeta(s).} \quad (2.40)
$$

Remark that Theorem 1.1 would follow if we were able to prove that the sequence of functions $\zeta_{\Sigma, n}^F(s), n \in \mathbb{N}^*$, converges to the function $\zeta_{\Sigma}^F(s)$, as $n \rightarrow \infty$, for any $s \in \mathbb{C}$ with all its derivatives. This is, of course, very optimistic (and false), as by Theorem 2.9 and (2.32), we see that $\zeta_{\Sigma, n}^F(s), n \in \mathbb{N}^*$ diverges for $s \leq 1$, as $n \rightarrow \infty$. Moreover, we know that the functions $\zeta_{\Sigma, n}^F(s), n \in \mathbb{N}^*$, are holomorphic over $\mathbb{C}$ and the function $\zeta_{\Sigma}^F(s)$ is only meromorphic.

Thus, to get a convergence in a reasonable sense, it would be nice to restrict ourselves to a subspace of $\mathbb{C}$, where no poles of $\zeta_{\Sigma}^F(s)$ appear. A very natural candidate for such subspace is $\{s \in \mathbb{C} : \text{Re}(s) > 1\}$, as over this set, the formula (1.10) holds.

Theorem 2.9 shows that for fixed $k \in \mathbb{N}$, the first $k$ terms of the discrete zeta function (2.40) converge to the the corresponding terms of the continuous zeta function (1.10). This is why it is important to bound the rest of the terms uniformly. In Sect. 3.1, we prove
Theorem 2.12 (Uniform weak Weyl’s law) There is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$ and any $i \in \mathbb{N}$, $i \leq V(\Sigma_n)$, we have

$$\lambda_i^n \geq Ci. \quad (2.41)$$

By Theorems 2.9, 2.12, (1.10) and Weyl’s law (2.32), we obtain the following

Corollary 2.13 For any $s \in \mathbb{C}$, $\text{Re}(s) > 1$, as $n \to \infty$, the following convergence holds

$$\zeta_{\Sigma_n}^F(s) \to \zeta_{\Sigma}(s). \quad (2.42)$$

Now, we try to nevertheless “extend” Corollary 2.13 to the whole complex plane. To overcome the fact that $\zeta_{\Sigma}^F(s)$ is only a meromorphic function, we use the parametrix construction.

In this step we were inspired by the approach used by Müller in his resolution of the Ray–Singer conjecture (now Cheeger-Müller theorem) in [50].2 A simple but basic identity in this approach is

$$\zeta_{\Sigma}^F(s) = \text{Tr}[\left((\Delta_{\Sigma}^F)^{-1}\right)^{-s}], \quad (2.43)$$

where $(\Delta_{\Sigma}^F)^{-s}$ is a power of $\Delta_{\Sigma}^F$, restricted to the vector space spanned by the eigenvectors corresponding to the non-zero eigenvalues.

Consider a covering of $\Sigma$ by a union of open sets $U_\alpha$, $\alpha \in I$, which are themselves tileable surfaces. Endow $U_\alpha$ with the restriction of the vector bundle $(F, h^F, V^F)$, which we denote by the same symbol by an abuse of notation. We take a subordinate partition of unity $\phi_\alpha$, $\alpha \in I$, of $\Sigma$. The normalized zeta function is defined for $s \in \mathbb{C}$, $\text{Re}(s) > 1$, by

$$\zeta_{\Sigma}^F_{\text{nor}}(s) := \text{Tr}\left[\left((\Delta_{\Sigma}^F)^{-1}\right)^{-s} - \sum_{\alpha \in I} \phi_\alpha \cdot (\Delta_{U_\alpha}^F)^{-s}\right], \quad (2.44)$$

where $(\Delta_{U_\alpha}^F)^{-s}$ are defined analogously to $(\Delta_{\Sigma}^F)^{-s}$, using Friedrichs extension of the Laplacian with Neumann boundary conditions, and they are viewed as operators acting on $L^2(\Sigma, F)$ by trivial extension. By Proposition 2.5, applied for $\Sigma$ and $U_\alpha$, and the classical properties of the Mellin transform, we see that the following proposition holds.

Proposition 2.14 The function $\zeta_{\Sigma}^F_{\text{nor}}(s)$ extends holomorphically to the whole complex plane $\mathbb{C}$.

Now, similarly, for $s \in \mathbb{C}$, we construct the normalized discrete zeta function

$$\zeta_{\Sigma_n}^F_{\text{nor}}(s) := \text{Tr}\left[\left((\Delta_{\Sigma_n}^F)^{-1}\right)^{-s} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha,n}^F)^{-s}\right], \quad (2.45)$$

where the powers $(\Delta_{\Sigma_n}^F)^{-s}$, $(\Delta_{U_\alpha,n}^F)^{-s}$ have to be understood as powers of the respective Laplacians, restricted to the vector spaces spanned by the eigenvectors corresponding to non-zero eigenvalues. The operators $(\Delta_{U_\alpha,n}^F)^{-s}$ are viewed as operators on $\text{Map}(V(\Sigma_n), F_n)$ by the obvious inclusion $V(U_{\alpha,n}) \hookrightarrow V(\Sigma_n)$, and $\phi_\alpha$ acts by the pointwise multiplications on the elements of $V(U_{\alpha,n}) \hookrightarrow U_\alpha$. By using methods of Müller [50], in Sect. 3.2, we prove the following

Theorem 2.15 For any compact $K \subset \mathbb{C}$, there is $C > 0$ such that for any $s \in K$, $n \in \mathbb{N}^*$:

$$\left|\zeta_{\Sigma_n}^F_{\text{nor}}(s)\right| \leq C. \quad (2.46)$$

2 See Bismut-Zhang [7] for the non-unitary version of this theorem.
Then, as in the proof of Corollary 2.13, but using Theorem 2.11, in Sect. 3.1, we prove

**Corollary 2.16** For \( s \in \mathbb{C} \), \( \text{Re}(s) > 1 \), as \( n \to \infty \), the following convergence holds

\[
\text{Tr}\left[ \phi_\alpha \cdot (n^2 \cdot \Delta_{U_{a,n}}^{F,\perp})^{-s} \right] \to \text{Tr}\left[ \phi_\alpha \cdot (\Delta_{U_a}^{F,\perp})^{-s} \right].
\]  

(2.47)

Recall a classical convergence result, stating that a sequence of uniformly locally bounded holomorphic functions converges on a connected domain if and only if it converges on some open subdomain. By this, Theorem 2.15, Proposition 2.14 and Corollaries 2.13, 2.16, we deduce

**Corollary 2.17** For any \( s \in \mathbb{C} \), as \( n \to \infty \), the following limit holds

\[
\zeta_{\Sigma_0}^{F_{n,\text{nor}}} (s) \to \zeta_{\Sigma}^{F,\text{nor}} (s).
\]  

(2.48)

To conclude, on the whole complex plane, the discrete zeta functions \( \zeta_{\Sigma_0}^{F_n} \), \( n \in \mathbb{N}^* \), do not converge to the continuous zeta function \( \zeta_{\Sigma}^F \), as \( n \to \infty \). However, we proved that by taking out the local terms from \( \zeta_{\Sigma_0}^{F_n} \), the continuous analogues of which “produce non-holomorphicity” of the function \( \zeta_{\Sigma_0}^F (s) \), the convergence holds on the whole complex plane. From now on and almost till the end of this section we will essentially focus on the understanding those local terms.

From Proposition 2.5 and the standard properties of the Mellin transform, similarly to Definition 1.9, we see that for \( f \) as in Proposition 2.5, the following quantity is well-defined

\[
\text{Tr}\left[ f \cdot \log(\Delta_{\Sigma_0}^{F_{n,\perp}}) \right] := -\frac{\partial}{\partial s} \text{Tr}\left[ f \cdot (\Delta_{\Sigma}^{F,\perp})^{-s} \right] \bigg|_{s=0}.
\]  

(2.49)

By Cauchy formula and Corollary 2.17, we see that, as \( n \to \infty \), we have

\[
\log \left( \det (n^2 \cdot \Delta_{\Sigma_0}^{F_{n,\perp}}) \right) - \sum_{\alpha \in I} \text{Tr}\left[ \phi_\alpha \cdot \log (n^2 \cdot \Delta_{U_{a,n}}^{F_{n,\perp}}) \right] \to \log \left( \det \left( \Delta_{\Sigma}^{F,\perp} \right) \right) - \sum_{\alpha \in I} \text{Tr}\left[ \phi_\alpha \cdot \log (\Delta_{U_a}^{F,\perp}) \right].
\]  

(2.50)

Remark that in (2.50), for the first time in our analysis we see the analytic torsion.

Now, in our final step we choose \( U_a \) and \( \phi_\alpha \) as in Sect. 2.2, so that the terms which appear in the sum of the left-hand side of (2.50) become relatively easy to handle. We use the notation from Sect. 2.2. As all the elements of the cover are contractible and the vector bundle \( (F, h^F, \nabla^F) \) is flat unitary, by taking a unitary flat frame, in the calculations over \( U_a \) we may and we will assume that \( (F, h^F, \nabla^F) \) restricts to a trivial vector bundle over the coverings.

Recall that \( c_0 (\Sigma) \) and \( c_1 (\Sigma) \) were defined in (2.13). If we specialize (2.50) for the covering from Sect. 2.2, and we use (2.13), we see that, as \( n \to \infty \), we have

\[
\log \left( \det (n^2 \cdot \Delta_{\Sigma_0}^{F_{n,\perp}}) \right) - \text{rk}(F) \cdot \left\{ N_{\text{sq},0} (\Sigma) \text{Tr}\left[ \phi_{\Sigma_0} \cdot \log (n^2 \cdot \Delta_{U_{\text{sq},n}}^{\perp}) \right] + N_{\text{sq},1} (\Sigma) \text{Tr}\left[ \phi_{\Sigma_1} \cdot \log (n^2 \cdot \Delta_{U_{\text{sq},n}}^{\perp}) \right] + \sum_{\alpha \in \mathfrak{L}(\text{Con}(\Sigma))} \text{Tr}\left[ \phi_{\alpha} \cdot \log (n^2 \cdot \Delta_{U_{\alpha,n}}^{\perp}) \right] \right\} + \text{rk}(F) \cdot \left( A(\Sigma) C_0 + |\beta \Sigma| C_1 + C_2 (\Sigma) \right).
\]  

(2.51)

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where \( C_0, C_1, C_2(\Sigma) \in \mathbb{R} \) are defined as follows

\[
C_0 = \Tr[\phi_{sq,0} \cdot \log(\Delta_{U_{sq}}^{1/2})],
\]

\[
C_1 = -\frac{3}{2} C_0 + \Tr[\phi_{sq,1} \cdot \log(\Delta_{U_{sq}}^{1/2})],
\]

\[
C_2(\Sigma) = c_0(\Sigma) C_0 + c_1(\Sigma) C_1 + N_{sq,2}(\Sigma) \Tr[\phi_{sq,2} \cdot \log(\Delta_{U_{sq}}^{1/2})]
\]

\[
+ \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \Tr[\phi_{\alpha} \cdot \log(\Delta_{U_{\alpha}}^{1/2})] + \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \Tr[\phi_{\alpha} \cdot \log(\Delta_{U_{\beta}}^{1/2})].
\]

(2.52)

We remark that by (2.13) and (2.52), the constants \( C_0, C_1 \) are universal, and the constant \( C_2(\Sigma) \) depends purely (and additively) on the sets \( \angle(\text{Ang}(\Sigma)) \) and \( \angle(\text{Con}(\Sigma)) \).

Next theorem studies the asymptotic expansion of the terms in the left-hand side of (2.51), which correspond to squares. By using Fourier analysis on \( n \times n \) mesh, in Sect. 4, we prove

**Theorem 2.18** Let \( U \) be a square with a tileable surface structure, containing 4 euclidean squares of area 1. Let \( U_n \) be a sequence of graphs, constructed from \( U \) as in Theorem 1.1. Consider a map \( \pi_U : U \to \tilde{U} \), from \( U \) to a torus \( \tilde{U} \), obtained by the identification of the opposite sides of \( U \). Assume that the function \( \phi : U \to \mathbb{C} \) is the pull-back of a smooth function on \( \tilde{U} \). Then there is a constant \( c(U, \phi) \in \mathbb{R} \) such that, as \( n \to \infty \), we have

\[
\Tr[\phi \cdot \log(n^2 \cdot \Delta_{U_{\alpha}}^{1/2})] = \left(2n^2 \log(n) + \frac{4Gn^2}{\pi}\right) \int_{U} \phi dv_{U}
\]

\[
+ \frac{\log(\sqrt{2} - 1)}{2} n \cdot \int_{\partial U} \phi dv_{\partial U} - \frac{\log(n)}{2} \phi(P) - c(U, \phi) + o(1),
\]

(2.53)

where \( P \) is some corner vertex of \( U \) and \( G \) is the Catalan constant, see (1.13).

Now, we plug Theorem 2.18 into (2.51) to see that what we get is more or less what we need to finish the proof of Theorem 1.1. The local contributions from the conical points and the corners of the boundary in (2.50) would constitute the sequence \( \{A_n, n \in \mathbb{N}^+\} \) from Theorem 1.1. The value \( \log(2) \) in (1.15) comes from Duplantier-David calculations (cf. Theorem A.1) and the fact that the undetermined constants, obtained in the course of the proof (as \( c(U, \phi) \) from (2.53)) are universal.

More precisely, we denote by \( I' \subset I \) the subset of indices \( \alpha \) such that \( U_\alpha \) corresponds to a square. We identify all \( U_\alpha \) with \( [-1, 1] \times [-1, 1] \) using the coordinates as in (2.17), and view all the functions \( \phi_{\alpha,0} \) as to be defined on \( [-1, 1] \times [-1, 1] \). Using this identification, we introduce the following function on the square \( [-1, 1] \times [-1, 1] \)

\[
\phi_{\alpha,0}^{\Sigma} : \sum_{\alpha \in I'} \phi_{\alpha,0}.
\]

(2.54)

We cannot apply Theorem 2.18 directly to the terms corresponding to the squares on the left-hand side of (2.51), as functions \( \phi_{sq,1} \) and \( \phi_{sq,2} \) do not satisfy the assumption of Theorem 2.18. But if one replaces them by their average with respect to the symmetry group of the square, they would satisfy this assumption by (2.14), (2.18), (2.19). The trace is unchanged under this procedure.
By this, Theorem 2.18 and (2.51), we see that, as \( n \to \infty \), the following holds

\[
\log \left( \det(n^2 \cdot \Delta_{\Sigma_n}^{F,\perp}) \right) - \text{rk}(F) \cdot \left( \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \text{Tr}[\phi_{c_\alpha} \cdot \log(n^2 \cdot \Delta_{\Sigma_n}^{c_\alpha,\perp})] \right)
+ \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \text{Tr}[\phi_{A\beta} \cdot \log(n^2 \cdot \Delta_{\Sigma_n}^{c_\alpha,\perp})] \right)
= \text{rk}(F) \cdot \left( 2n^2 \log(n) + \frac{4Gn^2}{\pi} \right) \int_U \phi_{c_a}^\Sigma dv_U + \text{rk}(F) \cdot \frac{\log(\sqrt{2} - 1)}{2} \cdot n \cdot \int_{\partial U} \phi_{c_a}^\Sigma dv_{\partial U}
- \text{rk}(F) \cdot \frac{\log(n)}{8} \cdot N_{sq,2}(\Sigma) + \log \left( \det \left( \Delta_{\Sigma}^F \right) \right)
- \text{rk}(F) \cdot \left( A(\Sigma)C_0' + |\partial \Sigma|C_1' + C_2'(\Sigma) \right) + o(1),
\]

(2.55)

where \( C_0', C_1', C_2'(\Sigma) \in \mathbb{R} \) are defined by

\[
C_0' := C_0 + c(U_{sq}, \phi_{sq,0}),
C_1' := C_1 - \frac{3}{2} c(U_{sq}, \phi_{sq,0}) + c(U_{sq}, \phi_{sq,1}),
C_2'(\Sigma) := C_2(\Sigma) + c_0(\Sigma)c(U_{sq}, \phi_{sq,0}) + c_1(\Sigma)c(U_{sq}, \phi_{sq,1}) + N_{sq,2}(\Sigma)c(U_{sq}, \phi_{sq,2}).
\]

Clearly, by (2.52) and (2.56), the constants \( C_0', C_1' \) are universal and the constant \( C_2'(\Sigma) \) depends purely on the sets \( \angle(\text{Ang}(\Sigma)) \) and \( \angle(\text{Con}(\Sigma)) \). Moreover, it depends additively on them.

Remark that since \( \phi_{c,\alpha,0}^\Sigma, \alpha \in I, \) form a partition of unity, we have

\[
\int_U \phi_{c_a}^\Sigma dv_U - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \int_U \phi_{c_\alpha}^\Sigma dv_U - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \int_U \phi_{c_\beta}^\Sigma dv_U
= A(\Sigma) - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} A(C_\alpha) - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} A(A_\beta),
\]

(2.57)

and similar relation holds between the integrals over the boundary and the perimeters. Now, we take the difference between (2.55), and (2.55) applied for \( C_\alpha \) and \( A_\beta \) for each \( \alpha \in \angle(\text{Con}(\Sigma)), \beta \in \angle(\text{Ang}^{\pi/2}(\Sigma)), \) and use the above observation (2.57), to see that, as \( n \to \infty \), we have

\[
\log \left( \det(n^2 \cdot \Delta_{\Sigma_n}^{F,\perp}) \right) - \text{rk}(F) \cdot \left( \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \log \left( \det(n^2 \cdot \Delta_{\Sigma_n}^{c_\alpha,\perp}) \right) \right)
+ \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \log \left( \det(n^2 \cdot \Delta_{\Sigma_n}^{c_\beta,\perp}) \right)
= \text{rk}(F) \cdot \left( 2n^2 \log(n) + \frac{4Gn^2}{\pi} \right) \cdot
\left( A(\Sigma) - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} A(C_\alpha) - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} A(A_\beta) \right)
\]
\[ + \text{rk}(F) \cdot \frac{\log(\sqrt{2} - 1)}{2} n \left( |\partial \Sigma| - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} |\partial C_{\alpha}| - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} |\partial A_{\beta}| \right) \]

\[ - \text{rk}(F) \cdot \frac{\log(n)}{8} \left( N_{\text{sq}, 2}(\Sigma) - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} N_{\text{sq}, 2}(C_{\alpha}) - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} N_{\text{sq}, 2}(A_{\beta}) \right) \]

\[ + \log \left( \det \Delta_{\Sigma}^{F} \right) - \text{rk}(F) \cdot \left( A(\Sigma) C_{0} + |\partial \Sigma| C_{1} + C_{2}''(\Sigma) \right) + o(1), \quad (2.58) \]

where \( C_{2}''(\Sigma) \in \mathbb{R} \) is defined as follows

\[ C_{2}''(\Sigma) := C_{2}(\Sigma) - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \left( C_{2}(C_{\alpha}) + A(C_{\alpha}) C_{0} + |\partial C_{\alpha}| C_{1} + \log(\det \Delta_{C_{\alpha}}) \right) \]

\[ - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \left( C_{2}(A_{\beta}) + A(A_{\beta}) C_{0} + |\partial A_{\beta}| C_{1} + \log(\det \Delta_{A_{\beta}}) \right). \]

\[ (2.59) \]

Clearly, by (2.56) and (2.59), the constant \( C_{2}''(\Sigma) \) depends only on the sets \( \angle(\text{Ang}(\Sigma)) \) and \( \angle(\text{Con}(\Sigma)) \). Moreover, it depends additively on them.

Note that by Proposition 2.4, and the fact that \#V(\Sigma_{n}) = A(\Sigma) n^{2} \) (which follows from our normalization on the area of the tiles), we have

\[ \log \left( \det(n^{2} \cdot \Delta_{\Sigma_{n}}^{F_{\perp}}) \right) = \log \left( \det \Delta_{\Sigma_{n}}^{F_{\perp}} \right) + 2 \log(n) \cdot \left( \text{rk}(F) A(\Sigma) n^{2} - \dim H^{0}(\overline{\Sigma}, F) \right). \]

\[ (2.60) \]

Also, remark that by Proposition 2.7, the fact that for \( \angle(Q) = \frac{\pi}{2} \), we have \( \frac{1}{12} \cdot \frac{\pi^{2} - \angle(Q)^{2}}{2\pi \cdot \angle(Q)} = \frac{1}{12} \)

and the fact that according to Proposition 2.7, in the sum below, there will be only the terms corresponding to the angles \( \angle(Q) = \frac{\pi}{2} \), which are in bijective correspondence with squares \( U_{\alpha} \) having 2 common sides with \( \partial \Sigma \), the following identity holds

\[ \left( 2\zeta_{\Sigma}^{F}(0) + 2 \dim H^{0}(\overline{\Sigma}, F) \right) - \text{rk}(F) \cdot \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \left( 2\zeta_{U_{\alpha}}(0) + 2 \right) \]

\[ + \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} \left( 2\zeta_{U_{\beta}}(0) + 2 \right) \]

\[ = \frac{\text{rk}(F)}{8} \left( N_{\text{sq}, 2}(\Sigma) - \sum_{\alpha \in \angle(\text{Con}(\Sigma))} N_{\text{sq}, 2}(C_{\alpha}) - \sum_{\beta \in \angle(\text{Ang}^{\pi/2}(\Sigma))} N_{\text{sq}, 2}(A_{\beta}) \right). \]

\[ (2.61) \]
Recall that the quantity \( \log(\det'\Delta_{\Sigma_n}^{F_n}) \), \( n \in \mathbb{N}^* \) was defined in (1.14). By (2.58), (2.60) and (2.61), we see that, as \( n \to \infty \), the following limit holds

\[
\log(\det'\Delta_{\Sigma_n}^{F_n}) - \operatorname{rk}(F) \cdot \left( \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \log(\det'\Delta_{c,\alpha,n}^{\kappa'}) \right) \]
\[
+ \sum_{\beta \in \angle(\text{Ang}^{\alpha}\Sigma(S))} \log(\det'\Delta_{\mathcal{A},\alpha,n}^{\kappa'}) \longrightarrow \log\det'\Delta_{\Sigma}^{F_n} - \operatorname{rk}(F) \cdot \left( A(\Sigma)C_0' + |\partial\Sigma|C_1' + C_2''(\Sigma) \right).
\]

Clearly, if we could prove that \( C_0', C_1' = 0 \), and that we can write

\[
C_2''(\Sigma) = \frac{\log(2) \cdot \#\text{Ang}^{\alpha}\Sigma(\Sigma)}{16} + C'''(\Sigma),
\]

where \( C'''(\Sigma) \) depends only on the sets \( \angle(\text{Ang}^{\alpha}\Sigma(\Sigma)) \), then (2.62) would imply (1.15) for \( A_n, n \in \mathbb{N}^* \), defined as follows

\[
A_n := \sum_{\alpha \in \angle(\text{Con}(\Sigma))} \log(\det'\Delta_{c,\alpha,n}^{\kappa'}) + \sum_{\beta \in \angle(\text{Ang}^{\alpha}\Sigma(\Sigma))} \log(\det'\Delta_{\mathcal{A},\alpha,n}^{\kappa'}) - C'''(\Sigma).
\]

Moreover, \( A_n \) from (2.64) depends only on the sets \( \angle(\text{Ang}^{\alpha}\Sigma(\Sigma)) \), and \( \angle(\text{Con}(\Sigma)) \), by (2.59), and it depends additively on them. Hence, to establish Theorem 1.1, it is enough to establish (2.63) and to prove \( C_0', C_1' = 0 \).

Let us prove that \( C_0', C_1' = 0 \). Recall that the surface \( c\Sigma, c > 0 \), was defined before Proposition 2.8. Clearly, for \( c \in \mathbb{N}^* \), the surface \( c\Sigma \) has a natural structure of a tileable surface, coming from \( \Sigma \). The number of tiles of \( c\Sigma \) is \( c^2 \) times the number of tiles of \( \Sigma \). Since \( \angle(\text{Con}(\Sigma)) = \angle(\text{Con}(c\Sigma)) \), \( \angle(\text{Ang}(\Sigma)) = \angle(\text{Ang}(c\Sigma)) \) and \( C_2''(\Sigma) \) depends only on those sets, we conclude that \( C_2''(\Sigma) = C_2''(c\Sigma) \). From this and (2.62), applied for \( \Sigma \) and \( c\Sigma \), by (1.14), we see that, for any \( c \in \mathbb{N}^* \), as \( n \to \infty \), we have

\[
\log(\det'\Delta_{c\Sigma_n}^{F_n}) - \log(\det'\Delta_{\Sigma_n}^{F_n}) \rightarrow \log(\det'\Delta_{c\Sigma}^{F_n}) - \log(\det'\Delta_{\Sigma}^{F_n}) + \operatorname{rk}(F) \cdot \left( (c^2 - 1)A(\Sigma)C_0' + (c - 1)|\partial\Sigma|C_1' \right).
\]

Remark that by the construction of the discretization \( \Sigma_n \) from Sect. 2.1, for any \( c \in \mathbb{N}^* \), we have the following isomorphism of graphs

\[
\Sigma_{cn} \simeq (c\Sigma)_n.
\]

By (1.14) and (2.66), we conclude that for any \( c \in \mathbb{N}^* \), \( n \in \mathbb{N}^* \), we have

\[
\log(\det'\Delta_{(c\Sigma)_n}^{F_n}) = \log(\det'\Delta_{\Sigma_n}^{F_n}) - 2\Delta_{\Sigma_n}^{F_n}(0) \log(c).
\]

From Proposition 2.8, (2.65) and (2.67), we deduce that for any \( c \in \mathbb{N}^* \), we have

\[
\log(\det'\Delta_{\Sigma_n}^{F_n}) - \log(\det'\Delta_{\Sigma_n}^{F_n}) \rightarrow \operatorname{rk}(F) \cdot \left( (c^2 - 1)A(\Sigma)C_0' + (c - 1)|\partial\Sigma|C_1' \right).
\]

\[\text{Springer}\]
From (2.68), from the fact that the constants $C'_0$, $C'_1$ are universal and from explicit calculations for rectangles due to Duplantier-David [25], (cf. Theorem A.1), we conclude that $C'_0$, $C'_1 = 0$.

Now, by Theorem A.1, we conclude that

$$C''_2([0, 1] \times [0, 1]) = \log(2) / 4.$$  

(2.69)

From this and the fact that $C''_2(\Sigma)$ depends additively on the sets $\angle(\text{Ang}(\Sigma))$ and $\angle(\text{Con}(\Sigma))$, we conclude that we have (2.63).

Now, since $C'_0$, $C'_1 = 0$, the limit (2.68) implies (1.16). The proof of Theorem 1.1 is finished.

2.5 Loop measure induced by CRSF’s

In this section we will describe a relation between our work and the study of loop measures induced by cycle-rooted spanning forests (later called CRSF’s, see Fig. 1b) for the definition of CRSF initiated by Kassel–Kenyon [39].

We fix $(\Sigma, g^T \Sigma)$, $(\Sigma_n, (F, h^F, \nabla^F), (F_n, h^{F_n}, \nabla^{F_n})$ as in Theorem 1.1. We denote by $\text{CRSF}_{\text{nonc}}(\Sigma_n)$ the subset of CRSF’s with non-contractible loops in $\Sigma$. We endow $\text{CRSF}_{\text{nonc}}(\Sigma_n)$ with the uniform probability measure $\mathbb{P}_{\Sigma_n}$. Let us recall first the following result.

**Theorem 2.19** (Kassel–Kenyon [39, Theorem 17]) For any flat special unitary vector bundle $(F, h^F, \nabla^F)$ of rank 2 over $\Sigma$ satisfying $H^0(\Sigma, F) = 0$, there is a constant $G(F) > 0$, such that we have

$$\lim_{n \to \infty} \mathbb{E}_{\Sigma_n} \left[ \prod_{\gamma \in \text{CRSF}} \left( 2 - \text{Tr}(w_{\gamma}) \right) \right] = G(F),$$

(2.70)

where the expected value is taken with respect to $\mathbb{P}_{\Sigma_n}$, the product is over all loops $\gamma$ of the CRSF and $\text{Tr}(w_{\gamma})$ is the trace of the monodromy of $\nabla^F$ evaluated along $\gamma$. Also, for any $(F, h^F, \nabla^F)$ as above, $G(F)$ depends only on the conformal type of the surface $(\Sigma, g^T \Sigma)$.

**Remark 2.20** In a similar situation when the loops are induced by double-dimers, an analogous result has been proved by Dubédat [23, Theorem 1].

The first result of this section gives a geometric meaning for $G(F)$.

**Theorem 2.21** There is a constant $Z(\Sigma) > 0$ such that for any flat special unitary vector bundle $(F, h^F, \nabla^F)$ of rank 2 over $\Sigma$, satisfying $H^0(\Sigma, F) = 0$, we have

$$G(F) = \frac{\sqrt{\text{det}' \Delta^F}}{Z(\Sigma)}.$$  

(2.71)

**Proof** Clearly, by Theorem 2.19, it suffices to prove that for any flat special unitary vector bundles $(F, h^F, \nabla^F), (G, h^G, \nabla^G)$ as in Theorem 2.21, we have

$$\lim_{n \to \infty} \frac{\mathbb{E}_{\Sigma_n} \left[ \prod_{\gamma \in \text{CRSF}} \left( 2 - \text{Tr}(w_{\gamma}^F) \right) \right]}{\mathbb{E}_{\Sigma_n} \left[ \prod_{\gamma \in \text{CRSF}} \left( 2 - \text{Tr}(w_{\gamma}^G) \right) \right]} = \frac{\sqrt{\text{det}' \Delta^F}}{\sqrt{\text{det}' \Delta^G}}.$$  

(2.72)
where $w^F$, $w^F_\gamma$ are the monodromies associated to $(F, h^F, \nabla^F)$ and $(G, h^G, \nabla^G)$ as in Theorem 2.19. But (2.72) is a consequence of (1.3) and Theorem 1.5. □

Let’s recall some notions from [39] and state an application of Theorem 2.21. We assume from now on that $\Sigma$ has at least one boundary component. Let $M$ be the space of flat special unitary vector bundles of rank 2 on $\Sigma$ modulo isomorphism. This space is compact, it is usually called the representation variety and it is given by

$$M = \text{Hom} (\pi_1(\Sigma), \text{SU}(2))/\text{SU}(2).$$

(2.73)

where the action of $\text{SU}(2)$ on $\text{Hom}(\pi_1(\Sigma), \text{SU}(2))$ is given by the conjugation. The space $M$ has a canonical Haar measure $\nu$ induced by the Haar measure on $\text{SU}(2)$, cf. [39, end of Sect. 4.5.1].

Recall that a finite lamination on $\Sigma$ is an isotopy class of a non contractible multiloop (i.e. a finite collection of pairwise disjoint simple non contractible closed curves). For $T \in \text{CRSF}_{\text{none}}(\Sigma_n)$, we denote by $[T]$ the lamination induced by its set of loops.

Fock-Goncharov in [33, Theorem 12.3] proved that, seen as functions on $M$, the functions

$$S_L(F) = \prod_{\gamma \in L} \text{Tr}(\omega_\gamma),$$

(2.74)

where $\omega_\gamma$ is the monodromy of $(F, \nabla^F)$, evaluated along $\gamma$, are linearly independent, when $L$ runs through all finite laminations. Hence the following functions are linearly independent as well

$$T_L(F) = \prod_{\gamma \in L} (2 - \text{Tr}(\omega_\gamma)).$$

(2.75)

We introduce a partial order $\prec$ on the laminations as in Basok–Chelkak [3, pp. 3, 8] (which is induced by the partial order of the intersection numbers of the laminations with the edges of a fixed triangulation) and choose an ordering of $T_L(F)$ consistent with this partial order. Let $P_L(F)$ be the corresponding Gram-Schmidt orthonormalization of $T_L(F)$ with respect to the $L^2$-product induced by the canonical Haar measure $\nu$ on functions over $M$. Define $A_{L,L'}$ by

$$P_L(F) = \sum_{L' \preceq L} A_{L,L'} \cdot T_{L'}(F).$$

(2.76)

Clearly, the coefficients $A_{L,L'}$ depend only on the topological type of $\Sigma$ and they satisfy $A_{L,L'} = 0$ for $L \prec L'$. Let us recall the following result.

**Theorem 2.22** (Kassel–Kenyon [39, Lemma 16, proof of Theorem 18 and end of Sect. 4.1])

We fix a tileable surface $(\Sigma, g^{T\Sigma})$. Assume also that $\Sigma$ has at least one boundary component. Then for any lamination $L$ on $\Sigma$, the integrals as well as the sum below converge

$$\sum_{L \preceq L'} A_{L',L} \cdot \int_M G(F) \cdot P_{L'}(F) d\nu.$$  \hspace{1cm} (2.77)

Moreover, in the notations of Theorem 2.19, for any lamination $L$, we have

$$\lim_{n \to \infty} P_{\Sigma_n}( [T] = L ) = \sum_{L \preceq L'} A_{L',L} \cdot \int_M G(F) \cdot P_{L'}(F) d\nu.$$  \hspace{1cm} (2.78)
Remark 2.23 Kassel–Kenyon in [39, proof of Theorem 18] in their proof assumed some bounds on the coefficients \( A_{L',L} \). Those bounds were later established by Basok–Chelkak [3, Theorem 4.9].

A combination of Theorems 2.21 and 2.22 gives us the following result.

**Corollary 2.24** In the notations of Theorem 2.22, we have the following identity

\[
\lim_{n \to \infty} \frac{\mathbb{P}_{\Sigma_n}([T] = L)}{Z(\Sigma)} = \frac{1}{Z(\Sigma)} \sum_{L \leq L'} A_{L',L} \cdot \int_{\mathcal{M}} \sqrt{\det \Delta^F_{\Sigma}} \cdot P_{L'}(F) d\nu.
\]

(2.79)

### 2.6 Dirichlet energy of the average height function and analytic torsion

In this section we relate our result with the main theorem from the paper of Kenyon [41], which we recall below for convenience of the reader.

**Theorem 2.25** (Kenyon [41, Theorem 1 and Corollary 2]) We fix a simply-connected rectangular domain \( \Omega \) in \( \mathbb{C} \), and denote by \( \Omega_n \) a sequence of subgraphs of \( \mathbb{Z}^2 \), approximating \( \Omega \). Let \( \partial V(\Omega_n) \subset V(\Omega_n) \) be the set of closest points to \( \partial \Omega \). We define \( \tilde{\log}(t(\Omega_n)) \), \( n \in \mathbb{N}^* \) as follows

\[
\tilde{\log}(t(\Omega_n)) := \log(t(\Omega_n)) - \frac{4G}{\pi} \cdot \#V(\Omega_n)
\]

\[ - \frac{\log(\sqrt{2} - 1)}{2} \cdot \#\partial V(\Omega_n) + \left( \frac{1}{2} + \frac{\#\text{Ang}^{\neq \pi/2}(\Omega)}{18} \right) \cdot \log(n). \]  

(2.80)

Then, as \( n \to \infty \), the following asymptotic bound holds

\[
\tilde{\log}(t(\Omega_n)) = o(\log(n)).
\]

Moreover, there is a universal constant \( C \in \mathbb{R} \) and a sequence \( D_n \), which depends only on the number of convex angles of \( \Omega \), such that, as \( n \to \infty \), the following limit holds

\[
\tilde{\log}(t(\Omega_n)) - D_n \to \int_{\Omega}^{\text{reg}} |\nabla h|^2 dxdy + C,
\]

(2.82)

where \( \int_{\Omega}^{\text{reg}} |\nabla h|^2 dxdy \) is the so-called regularized Dirichlet energy of the limiting average Thurston height function on \( \Omega \), [41, Sects. 2.3, 2.4], obtained by subtraction of the logarithmic divergent part from \( c_2(1/n) \log(n) + c_3(\Omega) \) in the notations of [41, Theorem 1].

**Remark 2.26** (a) This theorem continues to hold for approximations induced by \( \mathbb{Z}^2 + \frac{1}{n} \mathbb{Z}^2 + \frac{1}{2n} \mathbb{Z}^2 \).

(b) Kenyon asked in [41, Sect. 8, question 2] if the same asymptotics also holds for multiply-connected domains. By (1.1), Theorem 1.1 answers positively this question.

Now, remark first that for a simply-connected rectangular domain \( \Omega \) in \( \mathbb{C} \), by (1.12), we have

\[
\zeta(\Omega)(0) = -1 + \left( \frac{1}{4} + \frac{\#\text{Ang}^{\neq \pi/2}(\Omega)}{36} \right).
\]

(2.83)

By Remark 2.26a), (1.1) and (2.83), Theorems 1.1 and 2.25 are compatible. Moreover, we deduce the equality between the constant terms in (1.15) and (2.80), which gives a partial answer to Open problem 4 from Kenyon, [41, Sect. 8]. More precisely, we have
Corollary 2.27 There are universal constants $C, D \in \mathbb{R}$ such that for any simply-connected rectangular domain $\Omega$ in $\mathbb{C}$ with corners at rational vertices, we have
\begin{equation}
\int_{\Omega} |\nabla h|^2 dx dy = \log(\det' \Delta_{\Omega}) - \log(A(\Omega)) + D \cdot \# \text{Ang} \neq \pi/2(\Omega) + C. \tag{2.84}
\end{equation}

Remark 2.28 The conclusion holds without the requirement on the position of vertices as by Kenyon [41, Sect. 2.3] and methods similar to Bismut-Gillet-Soule [10, proof of Theorem 1.6] it is possible to see that both sides of the equation depend continuously on the variation of the domain.

2.7 Conformal invariance and anomaly formula

The main goal of this section is to prove the conformal invariance of the right-hand side of (1.18) for $\Sigma = \Sigma'$ and $(F, \nabla^F), (G, \nabla^G)$, satisfying $H^0(\Sigma, F) = H^0(\Sigma, G) = 0$. Remark that for $F, G$ of rank 2, this already follows from Theorems 2.19 and 2.21. In our approach, which is very different from the proof of Theorem 2.19, we will deduce the general case as a simple consequence of the generalized statement of the anomaly formula of Bismut-Gillet-Soule [10, Theorem 1.23].

Remark that in [8], [9], [10] authors consider smooth manifolds, and thus their results cannot be directly applied for manifolds with conical singularities and corners. However, considerable advances have been done to extend the anomaly formula for manifolds in this case, see Aurell-Salomonson [2], Kokotov-Korotkin [45], Aldana-Rowlett [1], Kalvin [38]. As none of those references treat fully the effect of angles in the anomaly formula, for brevity, we will assume in this section that $\partial \Sigma = \emptyset$.

We state below the main result of this section and an immediate corollary of it.

Theorem 2.29 Let $(\Sigma, g^{T\Sigma})$ be a flat surface with conical singularities and $\partial \Sigma = \emptyset$. Fix two Hermitian flat vector bundles $(F, \nabla^F, h^F), (G, \nabla^G, h^G)$ of the same rank, satisfying $H^0(\Sigma, F) = H^0(\Sigma, G) = 0$. Let $g^{T\Sigma}$ be a (smooth) metric on $\Sigma$ in the conformal class of $g^{T\Sigma}$. Then
\begin{equation}
\frac{\det' \Delta_{\Sigma}^F}{\det' \Delta_{\Sigma}^G} = \frac{\det' \Delta_{\Sigma}^F}{\det' \Delta_{\Sigma}^G}, \tag{2.85}
\end{equation}

where $\det' \Delta_{\Sigma}^F$ and $\det' \Delta_{\Sigma}^G$ are Ray–Singer analytic torsions associated to $g^{T\Sigma}, (F, h^F)$ and $g^{T\Sigma}, (G, h^G)$ respectively, defined as in (1.11).

Corollary 2.30 Let us fix a Riemann surface $\Sigma$ with $\partial \Sigma = \emptyset$ and two metrics with conical singularities $g_i^{T\Sigma}, i = 1, 2$ in the conformal class of $\Sigma$ (the singularities might lie at different points and have different angles). Fix two Hermitian flat vector bundles $(F, \nabla^F, h^F), (G, \nabla^G, h^G)$ of the same rank, satisfying $H^0(\Sigma, F) = H^0(\Sigma, G) = 0$. Then
\begin{equation}
\frac{\det' \Delta_{\Sigma,1}^F}{\det' \Delta_{\Sigma,1}^G} = \frac{\det' \Delta_{\Sigma,2}^F}{\det' \Delta_{\Sigma,2}^G}, \tag{2.86}
\end{equation}

where $\det' \Delta_{\Sigma,i}^F, \det' \Delta_{\Sigma,i}^G, i = 1, 2$, are Ray–Singer analytic torsions associated to $g_i^{T\Sigma}, (F, h^F)$ and $g_i^{T\Sigma}, (G, h^G)$ respectively.

To describe our proof of Theorem 2.29, we will need to formulate an analogue of the anomaly formula for the manifolds with conical singularities.
Let \((\Sigma, g^{T\Sigma})\) be a Riemann surface with conical singularities. We denote by \(T^{1,0}\Sigma \subset T\Sigma \otimes \mathbb{C}\) the holomorphic component of the tangent vector bundle corresponding to the \(\sqrt{-1}\)-eigenvectors of the induced action of the complex structure.

Let us define the conical norm \(\| \cdot \|_{\text{con}(p)}\) on \(T^{1,0}\Sigma|_P\) for \(P \in \Con(\Sigma)\) as follows. First, it is easy to verify using (2.1) that for any \(P \in \Con(\Sigma)\), in the neighborhood of it there is a (local) holomorphic coordinate \(z\), such that the Kähler form associated with \(g^{T\Sigma}\) can be written as follows

\[
|z|^{2\beta} \cdot \sqrt{-1} dz \wedge d\bar{z},
\]

for some \(\beta > -1\). The relation between the conical angle of \(P \in \Con(\Sigma)\) and the constant \(\beta(P)\), defined as in (2.87), is given by \(\beta(P) := \frac{\zeta(P)}{2\pi} - 1\). We define \(\| \cdot \|_{\text{con}(p)}\) on \(T^{1,0}\Sigma|_P\) by requiring

\[
\left\| \frac{\partial}{\partial \bar{z}} \right\|_{\text{con}(p)} = 1.
\]

An easy verification shows that the holomorphic coordinate as in (2.87) is well-defined up to a multiplication by a unimodular constant. Hence, the norm \(\| \cdot \|_{\text{con}(p)}\) is well-defined.

Remark the similarity between this norm and the Wolpert norm, defined in [61, Definition 2.7] for the generalization of this definition for Riemann surfaces with hyperbolic cusps with non-necessarily constant scalar curvature.

Following Quillen [53] and Bismut-Gillet-Soulé [10], let us recall the definition of Quillen norm. Consider the vector bundle \(\lambda(F) := \Lambda^{\max} H^0(\Sigma, F) - 1 \otimes \Lambda^{\max} H^1(\Sigma, F)\) and endow it with the Quillen norm, defined as follows

\[
\| \cdot \|_Q(g^{T\Sigma}, h^F) := \sqrt{\det \Delta_F} \cdot \| \cdot \|_{L^2},
\]

where \(\| \cdot \|_{L^2}\) is the norm on \(\lambda(F)\) induced by the \(L^2\)-scalar product, see (1.22). Similarly, by relying on the Ray–Singer definition of the determinant of the Laplacian, cf. (1.11), we define the Quillen norm \(\| \cdot \|_Q(g^{T\Sigma}, h^F)\) associated with a smooth metric \(g^{T\Sigma}\) over \(\Sigma\) and \(h^F\).

Let us now recall the definitions of Chern and Bott–Chern classes. The first Chern class of a Hermitian line bundle \((L, h^L)\) on \(\Sigma\) is a \((1, 1)\)-differential form on \(\Sigma\), defined by

\[
c_1(L, h^L) := \frac{\partial \bar{\partial} \log h^L(\upsilon, \nu)}{2\pi \sqrt{-1}},
\]

where \(\upsilon\) is any local holomorphic frame of \(L\). Clearly, \(c_1(L, h^L)\) is a well-defined closed differential form. By Chern–Weil theory, the associated de Rham class is given by \(c_1(L)\).

Recall that the Bott–Chern class \(\widetilde{\text{Td}}(L, h^L_1, h^L_2)\) of a line bundle \(L\) with Hermitian metrics \(h^L_1, h^L_2\) was defined in [8, Theorem 1.27], as a natural differential form, defined modulo \(\text{Im}(\partial) + \text{Im}(\bar{\partial})\), so that the following identity is satisfied

\[
\frac{\partial \bar{\partial}}{2\pi \sqrt{-1}} \widetilde{\text{Td}}(L, h^L_1, h^L_2) = \text{Td}(L, h^L_1) - \text{Td}(L, h^L_2).
\]

where \(\text{Td}\) is the Todd form. By [8, Theorem 1.27], we have the following identity

\[
\widetilde{\text{Td}}(L, h^L_1, h^L_2)^{[2]} = \frac{1}{12} \log(h^L_1/h^L_2) \left( c_1(L, h^L_1) + c_1(L, h^L_2) \right).
\]
where $\alpha^{[i]}$ means the degree $i$ component of a differential form $\alpha$. In what follows, when we write a Bott–Chern class, one should interpret it as a differential form, given by $(2.92)$.

**Theorem 2.31** (Anomaly formula for surfaces with conical singularities) Let $(Σ, g^TΣ)$ be a flat surface with conical singularities and $∂Σ = ∅$. We denote by $\| \cdot \|_{\text{con}(P)}$ the conical norm on $T^{1,0}Σ|_P$, $P \in \text{Con}(Σ)$, defined as in $(2.88)$. Let $g^TΣ$ be a Kähler metric over $Σ$. For $P \in Σ$, we denote by $\| \cdot \|_P$ the norm on $T^{1,0}Σ|_P$ induced by $g^TΣ$. Let $(F, ∇^F, h^F)$ be a Hermitian flat vector bundle over $Σ$, verifying $H^0(Σ, F) = 0$.

Then there is $E \in \mathbb{R}$, depending only on the set of conical angles $∠(\text{Con}(Σ))$, verifying

$$2 \log \left( \frac{\| \cdot \|_Q(g^TΣ, h^F)}{\| \cdot \|_Q(g^TΣ, h^F)} \right) = \text{rk}(F) \cdot \int_Σ \tilde{\text{td}}(TM, g^TΣ, g^TΣ) + \sum_{P \in \text{Con}(Σ)} \frac{\text{rk}(F)}{6} \cdot β(P) \cdot \log \left( \frac{\| \cdot \|_{\text{con}(P)}}{\| \cdot \|_P} \right) + \text{rk}(F) \cdot E. \quad (2.93)$$

Moreover, the constant $E$ is additive under taking union of $∠(\text{Con}(Σ))$.

**Remark 2.32** (a) The anomaly formula was firstly proved by Polyakov in [52] for trivial vector bundles over smooth Riemann surfaces. He used it to compute some integrals over moduli spaces of embedded surfaces which arise in mathematical physics. It was generalized by Bismut–Gillet–Soulé [10, Theorem 1.23] to compact Kähler manifolds of any dimension, for any vector bundle with a possible variation of the metric over it. In the setting of Riemann surfaces with conical singularities and angles, some related results were obtained by Aurell–Salomonson [2], Kokotov–Korotkin [45], Aldana-Rowlett [1], Calvin [38] and others.

For example, Kokotov–Korotkin in [45, Corollary 6], using tau-function, obtained a theorem analogous to Theorem 2.31 for trivial vector bundles over Riemann surfaces with two different half-translation structures in a given conformal class (in particular, instead of conical $g^TΣ$ and smooth $g^TΣ$, they consider two conical metrics $g_i^TΣ$, $i = 1, 2$, with angles $4π$). Using Burghelea–Friedlander–Kappeler type formula, see [13], the result of Kokotov–Korotkin was generalized by Kokotov in [44, Proposition 1] for Riemannian surfaces with conical singularities of the form $2π(k+1)$, $k \in \mathbb{N}^*$, without the explicit evaluation of the constant $E$. For more general conical singularities and trivial vector bundles, the statement analogous to Theorem 2.31 is the main statement from the article of Calvin [38], where author also explicitly evaluated the constant $E$.

Remark also that in the setting of non-compact Riemann surfaces with hyperbolic cusps singularities, a similar statement was proved in [30, Theorem B], see also [27, Theorem 1.8].

(b) As it will follow from our proof, which is based on the Calvin’s proof, the constant $E$ is given by a sum of the constants $C(β(P)), P \in \text{Con}(Σ)$, defined in [38, (1.2)].

(c) From the philosophy of [10], the assumption $H^0(Σ, F) = 0$ shouldn’t be necessary. It seems that our proof can be supplemented by a more precise analysis of the kernel of the Neumann jump operator in the case $H^0(Σ, F) \neq 0$ so that this assumption becomes redundant. For brevity, we prefer to omit those details.

Before proceeding with the proof of Theorem 2.31, let us deduce Theorem 2.29 first.

**Proof of Theorem 2.29** An easy verification shows that the $L^2$–scalar product, see (1.22), over 1-differential forms with values at $F$ is invariant under the conformal change of the metric.
Hence by (2.89) and our assumption $H^0(\Sigma, F) = 0$, we obtain
\[
\| \cdot \| Q (g^T \Sigma, h^F) = \frac{\sqrt{\det \Delta^F_{\Sigma}}}{\sqrt{\det \Delta^F_{\Sigma}}}.
\] (2.94)

We deduce Theorem 2.29 from Theorem 2.31, (2.94), and the fact that the right-hand sides of (2.93) coincide for $F$ and $G$ as in our theorem. \qed

**Proof of Theorem 2.31** The original proof of the anomaly formula in the smooth setting due to Bismut-Gillet-Soulé is adjacent with some technical difficulties in the setting of manifolds with conical singularities related to the fact that the Laplace operator is not essentially self-adjoint, and there is no apparent relation between the domains of the Friedrichs extensions of two conformally related conical metrics. Due to this, our proof will be based on the Burghelea-Friedlander-Kappeler type formula, see [13]. To explain it, let us recall the definition of the Neumann jump operator. Let $\epsilon > 0$ be such that in the $\epsilon$-neighborhood of every $P \in \text{Con}(\Sigma)$, the metric $g^T \Sigma$ has the form (2.1). We denote by $D^\Sigma_{P, \epsilon}$, $P \in \text{Con}(\Sigma)$ the $\epsilon$-neighborhood of $P$. Define

$$\Sigma_\epsilon := \Sigma \cup_{P \in \text{Con}(\Sigma)} D^\Sigma_{P, \epsilon}. \quad (2.95)$$

For $\lambda > 0$ and any $f \in C^\infty(\partial \Sigma_\epsilon, F)$, by the standard theory, there exists a unique solution $u(\lambda) \in C^0(\Sigma, F)$ to the Dirichlet problem

$$(\Delta^F_{\Sigma} + \lambda) u(\lambda) = 0, \quad \text{over } \Sigma \setminus \partial \Sigma_\epsilon, \quad u(\lambda)|_{\partial \Sigma_\epsilon} = f. \quad (2.96)$$

Following the definition for the trivial vector bundles, we introduce the Neumann jump operator $R^\Sigma_{\epsilon, \lambda}$ as follows

$$R^\Sigma_{\epsilon, \lambda}(f) := \nabla^F_n u(\lambda) + \nabla^F_{-n} u(\lambda), \quad (2.97)$$

where $n$ is the outward (for $\Sigma_\epsilon$) unit normal to $\partial \Sigma_\epsilon$. Similarly, we define the Neumann jump operator $R^\Sigma_{\epsilon}$ associated with $g^T \Sigma$.

The operator $R^\Sigma_{\epsilon, \lambda}$ is a first order elliptic classical pseudodifferential operator on $\partial \Sigma_\epsilon$ and for any $\lambda > 0$ it has empty kernel due to the fact that the kernel of $\Delta^F_{\Sigma} + \lambda$ is empty, cf. [13] and [38, before (2.15)]. Since the kernel of $\Delta^F_{\Sigma}$ coincides with $H^0(\Sigma, F)$ by Proposition 2.4 and $H^0(\Sigma, F)$ is empty by our assumption, we can reproduce the definition of Neumann jump operator for $\lambda = 0$ and the same conclusion would hold for the operator $R^\Sigma_{\epsilon, 0}$.

By repeating the proof of [13, Theorem A] (cf. [38, Lemma 2.7 and (2.23)] and [38, beginning of Sect. 2.3] for the references explaining why this reasoning also holds for surfaces with conical singularities and Friedrichs extensions), we deduce that for any $\lambda > 0$, the following identity holds

$$\det'(\Delta^F_{\Sigma} + \lambda) = \det'(R^\Sigma_{\epsilon, \lambda}) \cdot \det'(\Delta^F_{\Sigma_\epsilon} + \lambda) \cdot \prod_{P \in \text{Con}(\Sigma)} \det'(\Delta^F_{D^\Sigma_{P, \epsilon}} + \lambda), \quad (2.98)$$

where $\Delta^F_{\Sigma_\epsilon}$ (resp. $\Delta^F_{D^\Sigma_{P, \epsilon}}$) is the self-adjoint extension (resp. the Friedrichs extension) of the Laplacian with Dirichlet boundary conditions on $\partial \Sigma_\epsilon$, the determinants of $\Delta^F_{\Sigma} + \lambda$ and $\Delta^F_{D^\Sigma_{P, \epsilon}} + \lambda$ are defined using zeta-regularization as in (1.9) and the determinant of $R^\Sigma_{\epsilon, \lambda}$ is defined similarly, cf. [38, before (2.7)] for details.

Since the kernels of $\Delta^F_{D^\Sigma_{P, \epsilon}}$, $\Delta^F_{\Sigma_\epsilon}$ and $\Delta^F_{\Sigma}$ vanish (the last one is due to our assumption $H^0(\Sigma, F) = 0$ and Proposition 2.4, the first two due to the Dirichlet boundary conditions).
and the same holds for $R^\Sigma_{\epsilon,0}$ as described above, we deduce by the standard properties of the Mellin transform used in the definition of the zeta-regularization procedure, cf. [44, (2.20)], that the following limits hold

$$\lim_{\lambda \to 0} \det' (\Delta^F_\Sigma + \lambda) = \det' \Delta^F_\Sigma, \quad \lim_{\lambda \to 0} \det' (\Delta^F_{\Sigma e} + \lambda) = \det' \Delta^F_{\Sigma e},$$

$$\lim_{\lambda \to 0} \det' R^\Sigma_{\epsilon,0} = \det' R^\Sigma_{\epsilon,0}, \quad \lim_{\lambda \to 0} \det' (\Delta^F_{D_p,\epsilon} + \lambda) = \det' \Delta^F_{D_p,\epsilon}.$$  (2.99)

From (2.99), we conclude that

$$\det' \Delta^F_\Sigma = \det' R^\Sigma_{\epsilon,0} \cdot \Delta^F_{\Sigma e} \cdot \prod_{P \in \text{Con}(\Sigma)} \det' \Delta^F_{D_p,\epsilon}. \quad \text{(2.100)}$$

We now repeat the same procedure for $g^T \Sigma$ to obtain

$$\det' \Delta^F_\Sigma = \det' R^\Sigma_{\epsilon,0} \cdot \Delta^F_{\Sigma e} \cdot \prod_{P \in \text{Con}(\Sigma)} \det' \Delta^F_{D_p,\epsilon},$$  (2.101)

where we used the analogous notations for the objects associated with $g^T \Sigma$. We take the ratio between (2.100) and (2.101), use the fact that $\det' R^\Sigma_{\epsilon,0}$ is invariant under the conformal change of the metric, i.e. $\det' R^\Sigma_{\epsilon,0} = \det' R^\Sigma_{\epsilon,0}$, (for the proof for trivial $(F, h^F)$, see for example [38, Lemma 2.10.2]; the general case is proved analogously), to obtain

$$\frac{\det' \Delta^F_\Sigma}{\det' \Delta^F_{\Sigma e}} = \frac{\det' \Delta^F_{\Sigma e}}{\det' \Delta^F_{\Sigma e}} \cdot \prod_{P \in \text{Con}(\Sigma)} \frac{\det' \Delta^F_{D_p,\epsilon}}{\det' \Delta^F_{D_p,\epsilon}}. \quad \text{(2.102)}$$

Now, remark that $\Sigma e$ is a smooth manifold with smooth boundary. Hence the anomaly formula applies in this setting (for a reference, see for example [51, (1.17)], where this fact is proved for $(F, h^F)$ trivial; the general case is proved in analogous way). By applying the anomaly formula for $\Sigma e$ endowed with the metrics induced by $g^T \Sigma$ and $g^T \Sigma$, using the fact that $(F, h^F)$ is a Hermitian flat and the right-hand side of the anomaly formula is an integral of a local quantity, we conclude

$$\log \left( \frac{\det' \Delta^F_{\Sigma e}}{\det' \Delta^F_{\Sigma e}} \right) = \text{rk}(F) \cdot \log \left( \frac{\det' \Delta_{\Sigma e}}{\det' \Delta_{\Sigma e}} \right). \quad \text{(2.103)}$$

where $\det' \Delta_{\Sigma e}$, $\det' \Delta_{\Sigma e}$ are defined as $\det' \Delta_{\Sigma e}$, $\det' \Delta_{\Sigma e}$ for trivial $(F, h^F)$. Now, due to the fact that $D^\Sigma_{D_p,\epsilon}$ is contractible and $(F, h^F)$ is Hermitian flat, and, hence, can be trivialized over $D^\Sigma_{D_p,\epsilon}$, we conclude that

$$\log \left( \frac{\det' \Delta_{D_p,\epsilon}}{\det' \Delta_{D_p,\epsilon}} \right) = \text{rk}(F) \cdot \log \left( \frac{\det' \Delta_{D_p,\epsilon}}{\det' \Delta_{D_p,\epsilon}} \right). \quad \text{(2.104)}$$

Recall that by studying limit, as $\epsilon \to 0$, Kalvin proved in [38, proof of Theorem 1.1], that the following identity holds

$$\log \left( \frac{\det' \Delta_{\Sigma e}}{\det' \Delta_{\Sigma e}} \right) + \sum_{P \in \text{Con}(\Sigma)} \log \left( \frac{\det' \Delta_{D_p,\epsilon}}{\det' \Delta_{D_p,\epsilon}} \right).$$
\[
\begin{align*}
\mathcal{E} &= -\frac{1}{12\pi} \left( \int \Sigma K_{g^{T\Sigma}} \log \left( \frac{g^{T\Sigma}}{g^{T^{\Sigma}}} \right) dv_{g^{T\Sigma}} + \int \Sigma K_{g^{T\Sigma}} \log \left( \frac{g^{T\Sigma}}{g^{T^{\Sigma}}} \right) dv_{g^{T\Sigma}} \right) \\
&\quad + \sum_{P \in \text{Con}(\Sigma)} \frac{1}{6} \cdot \beta(P) \cdot \left( \frac{\phi(P)}{\beta(P) + 1} - \psi(P) \right) + E,
\end{align*}
\]

where \(E\) is an explicit constant verifying the description from the statement of Theorem 2.31, \(K_{g^{T\Sigma}}, K_{g^{T^{\Sigma}}}\) are the scalar curvatures of \(g^{T\Sigma}\) and \(g^{T^{\Sigma}}\) respectively and \(\phi_P, \psi_P\) are local functions, defined in a neighborhood of \(P\), so that for a fixed holomorphic coordinate \(z\) near \(P\), we can write

\[
g^{T\Sigma} = \exp(2\phi_P(z)) \cdot |z|^{2\beta(P)} \cdot |dz|^2, \quad g^{T^{\Sigma}} = \exp(2\psi_P(z)) \cdot |dz|^2.
\]

Remark that from the definitions, we have

\[
\log \left( \frac{\| \cdot \|_{\text{com}(P)}}{\| \cdot \|_P} \right) = \frac{\phi(P)}{\beta(P) + 1} - \psi(P).
\]

By Chern–Weil theory, we also have \(c_1(T^\Sigma, g^{T\Sigma}) = \frac{K_{g^{T\Sigma}}}{2\pi} dv_{g^{T\Sigma}}\). This with (2.92), (2.94) and (2.102)–(2.107) imply Theorem 2.31.

\section{Zeta functions: discrete and continuous}

In this section we prove some bounds on the spectrum of the graph approximations and on the respective zeta functions, which we used in Sect. 2.4. More precisely, this section is organized as follows. In Sect. 3.1 we prove Theorem 2.12 and its consequence, Corollary 2.16. In Sect. 3.2, modulo some statements about the uniform bound on the powers of the discrete Laplacian, we prove Theorem 2.15. Finally, in Sect. 3.3, we prove the left-out statements from Sect. 3.2. Throughout the whole section, we conserve the notation from Sect. 2.4.

\subsection{Uniform weak Weyl’s law, proofs of Theorem 2.12 and Corollary 2.16}

The goal of this section is to prove Theorem 2.12 and, as a consequence, Corollary 2.16.

The idea is to establish Theorem 2.12 by proving that the eigenvalues of \(n^2 \cdot \Delta^{F_n}_{\Sigma_n}\) are bounded from below by the eigenvalues of \(\Delta^F_\Sigma\) (up to some shifting and rescaling). Then Theorem 2.12 would follow from Weyl’s law, (2.32). The basic tool for proving such a bound is a construction of a map \(\mu_n\), acting on a subspace of \(\text{Map}(V(\Sigma_n), F_n)\) of uniformly bounded codimension (in \(n\)) with values in \(\mathcal{C}_{0,\infty}(\Sigma, F)\), in such a way that \(\mu_n\) preserves scalar products and scalar products associated with Laplacians \(\Delta^{F_n}_{\Sigma_n}\) and \(\Delta^F_\Sigma\) (see Theorem 3.2 for a precise statement).

To explain the construction of the map \(\mu_n\), consider a set \(\mathcal{F}\) of functions \(\rho : [-1; 1] \to \mathbb{R}\) satisfying (2.14), (2.15), (2.16) and

\[
\int_0^1 \rho(x)(1 - \rho(x))dx = 0.
\]

\textbf{Proposition 3.1} The set \(\mathcal{F}\) is not empty.
**Proof** Clearly, the assumptions (2.14), (2.15), (2.16) could be easily satisfied by a function \( \rho_1 : [-1, 1] \to \mathbb{R} \), having image inside of \([0, 1]\). For such a function, we have

\[
\int_0^1 \rho_1(x)(1 - \rho_1(x)) \, dx > 0. \tag{3.2}
\]

However, if one takes a function \( \rho_2 \) which satisfies (2.14), (2.15), (2.16), which has positive values over \([0, 1/2]\) and which takes value 4 on \([1/4, 1/3]\), it would satisfy

\[
\int_0^1 \rho_2(x)(1 - \rho_2(x)) \, dx < 1 - \frac{4 \cdot 3}{12} = 0. \tag{3.3}
\]

We conclude by (3.2) and (3.3) that there is \( t_0 \in [0, 1] \) such that the function \( \rho := t_0 \rho_1 + (1 - t_0) \rho_1 \) satisfies (3.1). However, since the set of functions, satisfying properties (2.14), (2.15), (2.16), is convex, we see that \( \rho \in \mathcal{F} \).

From (2.16) and (3.1), we see that for any \( \rho \in \mathcal{F} \), we have

\[
\int_0^1 \rho(x) \, dx = \frac{1}{2}, \quad \int_0^1 \rho(x)^2 \, dx = \frac{1}{2}. \tag{3.4}
\]

Let’s fix a function \( \rho \in \mathcal{F} \). Recall that the sets \( V_n(P), P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma), n \in \mathbb{N}^* \), were in Sect. 2.1. Introduce, for brevity, the set

\[
V_0(\Sigma_n) := \left\{ v \in V(\Sigma_n) : v \notin V_n(P), \text{ for any } P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma) \right\}. \tag{3.5}
\]

For \( P \in V_0(\Sigma_n) \), we define now a function \( \mu_P \in C_0^\infty(\Sigma) \) which will be constructed with the help of the fixed \( \rho \in \mathcal{F} \).

We fix linear coordinates \( x, y \), centered at \( P \), which have axes parallel to the boundaries of the tiles of \( \Sigma \). Assume \( P \) satisfies \( \text{dist}_\Sigma(P, \partial \Sigma) > \frac{1}{2n} \). Normalize the coordinates \( x, y \) in such a way that the coordinates identify the union of the 4 squares, formed by the edges and the vertices of \( \Sigma_n \) containing \( P \), with the square \([-1, 1] \times [-1, 1]\) as in Fig. 9a).

Over \( \{ (x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1 \} \), define

\[
\mu_P(x, y) := \rho(x)\rho(y), \tag{3.6}
\]

and extend it by zero to other values.

Now, assume \( P \) satisfies \( \text{dist}_\Sigma(P, \partial \Sigma) = \frac{1}{2n} \). Let’s assume that the boundary near \( P \) is parallel with the axis of \( x \)-coordinate. Normalize the coordinates \( x, y \) so that the coordinates identify the union of a rectangle and two squares, formed by the boundary, lines perpendicular to the axes and the vertices.

![Fig. 9](image)

**Fig. 9** Normalization of the coordinates \( x, y \), identifying colored rectangles with model ones
to the boundary emanating from the neighbors of $P$ and by the edges of a subgraph of $\Sigma_n$, containing $P$, with the rectangle $[-1, 1] \times [-1, \frac{1}{2}]$, so that $\{y > 0\}$ corresponds to a region containing $\partial \Sigma$ as in Fig. 9b). Over this rectangle, define
\[
\mu_p(x, y) := \begin{cases} 
\rho(x)\rho(y), & \text{for } y < 0, \\
\rho(x), & \text{for } y \geq 0,
\end{cases}
\] (3.7)
and extend it by zero to other values.

We define the functional $\mu_n : \text{Map}(V(\Sigma_n), F_n) \to \mathcal{C}_{0,N}^\infty(\Sigma, F)$ by
\[
\mu_n f = \sum_{P \in V(\Sigma_n)} \mu_p \cdot f(P),
\] (3.8)
where we implicitly used the parallel transport with respect to $\nabla^F$. Clearly, (2.14) and (3.8) ensure that the image of $\mu_n$ lies in $\mathcal{C}_{0,N}^\infty(\Sigma, F)$. Below we present the main result of this section.

**Theorem 3.2** For any $f \in \text{Map}(V(\Sigma_n), F_n)$ with $\text{supp} f \subset V(\Sigma)$, the following holds
\[
\frac{1}{n^2} \langle f, f \rangle_{L^2(\Sigma_n, F_n)} = \langle \mu_n(f), \mu_n(f) \rangle_{L^2(\Sigma, F)},
\] (3.9)
\[
\langle \Delta F_n^F f, f \rangle_{L^2(\Sigma_n, F_n)} = \frac{1}{\rho(x)^2} \langle \Delta^F \mu_n(f), \mu_n(f) \rangle_{L^2(\Sigma, F)}. \tag{3.10}
\]

The proof of Theorem 3.2 is a simple verification, and it is given in the end of this section. We will now explain how Theorem 3.2 could be used to prove Theorem 2.12. We define $t \in \mathbb{N}$ by
\[
t = \frac{2\text{rk}(F)}{\pi} \cdot \left( \sum_{P \in \text{Con}(\Sigma)} \angle(P) + \sum_{Q \in \text{Ang}(\Sigma)} \angle(Q) \right). \tag{3.11}
\]
Clearly, for any $n \in \mathbb{N}^*$, we have
\[
t = \text{rk}(F) \cdot \sum_{P \in \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)} \#V_n(P). \tag{3.12}
\]

Let’s see how Theorem 3.2 implies the following result

**Theorem 3.3** For any $n \in \mathbb{N}^*$, $i \in \mathbb{N}$, $i + t \leq \#V(\Sigma_n)$, the following inequality holds
\[
\left( \int_0^1 \rho'(x)^2 dx \right) \cdot \lambda_{i+t}^n \geq \lambda_i. \tag{3.13}
\]

**Proof** Consider the vector space $\mathcal{F}_{i+t}^n \subset \text{Map}(V(\Sigma_n), F_n)$, spanned by the first $i + t$ eigenvectors of $\Delta F_n$. Consider a subspace $\mathcal{F}_{i+t,0}^n \subset \mathcal{F}_{i+t}^n$ of functions, which take zero values on $Q \notin V(\Sigma_n)$. By (3.12),
\[
\dim \mathcal{F}_{i+t,0}^n \geq i. \tag{3.14}
\]
Now, construct a vector space $\mu_n(\mathcal{F}_{i+t,0}^n) \subset \mathcal{C}_{0,N}^\infty(\Sigma, F)$. By (2.14), (3.8), the value of $\mu_n(f)$, evaluated at $V(\Sigma_n)$, coincides with the value of $f$ at the evaluated point. Thus, we have
\[
\dim \mu_n(\mathcal{F}_{i+t,0}^n) = \dim \mathcal{F}_{i+t,0}^n. \tag{3.15}
\]
Clearly, Theorem 3.2 implies the following
\[
\sup_{f \in \mu_{n}(\mathcal{F}_{i+1}^{n})} \left\{ \frac{\langle \Delta_{\Sigma} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} \right\} = \int_{0}^{1} \rho'(x)^{2} dx \cdot \sup_{f \in \mathcal{F}_{i+1}^{n}} \left\{ \frac{\langle n^{2} \cdot \Delta_{\Sigma} F_{n} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} \right\}.
\]
(3.16)

However, we trivially have
\[
\sup_{f \in \mathcal{F}_{i+1}^{n}} \left\{ \frac{\langle n^{2} \cdot \Delta_{\Sigma} F_{n} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} \right\} \leq \sup_{f \in \mathcal{F}_{i+1}^{n}} \left\{ \frac{\langle n^{2} \cdot \Delta_{\Sigma} F_{n} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} \right\} = \lambda_{i+1}^{n}.
\]
(3.17)

Now, we use the characterization of the eigenvalues of \(\Delta_{\Sigma}^{F} \) through the Rayleigh quotient
\[
\lambda_{i} = \inf_{V \subset \text{Dom}_{F}(\Delta_{\Sigma}^{F})} \sup_{f \in V} \left\{ \frac{\langle \Delta_{\Sigma}^{F} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} : \dim V = i \right\}.
\]
(3.18)

By (3.14), (3.15), (3.18), we have
\[
\lambda_{i} \leq \sup_{f \in \mu_{n}(\mathcal{F}_{i+1}^{n})} \left\{ \frac{\langle \Delta_{\Sigma}^{F} f, f \rangle_{L^{2}(\Sigma, F)}}{\langle f, f \rangle_{L^{2}(\Sigma, F)}} \right\}.
\]
(3.19)

By (3.16), (3.17) and (3.19), we deduce Theorem 3.3.
\(\square\)

**Proof of Theorem 2.12** It follows from Theorem 3.3 and (2.32).
\(\square\)

**Proof of Corollary 2.16** For simplicity of the presentation, we assume that the spectrum of \(\Delta_{\Sigma}^{F} \) is simple, i.e. there is no multiple eigenvalues. Clearly, for any \(s \in \mathbb{C} \), we have
\[
\text{Tr} \left[ \phi_{\alpha} \cdot (n^{2} \cdot \Delta_{U_{\Sigma}^{F}, \Sigma})^{-s} \right] = \frac{1}{n^{2}} \sum_{i \geq 1} \langle \phi_{\alpha} f_{i}^{n}, f_{i}^{n} \rangle_{L^{2}(\Sigma, F_{n})} (\lambda_{i}^{n})^{-s}.
\]
(3.20)

Now, by Theorem 2.11, we know that in \(L^{2}(\Sigma)\), as \(n \to \infty\), we have
\[
L_{n}(f_{i}^{n}) \to f_{i},
\]
(3.21)

where \(f_{i}\) is the eigenvector of \(\Delta_{\Sigma}^{F}\) corresponding to the eigenvalue \(\lambda_{i} = \lim_{n \to \infty} \lambda_{i}^{n}\). By Proposition 2.10 and (3.21), we conclude that, as \(n \to \infty\), we have
\[
\frac{1}{n^{2}} \langle \phi_{\alpha} f_{i}^{n}, f_{i}^{n} \rangle_{L^{2}(\Sigma, F_{n})} \to \langle \phi_{\alpha} f_{i}, f_{i} \rangle_{L^{2}(\Sigma)}.
\]
(3.22)

From Theorems 2.9, 2.12, (2.32), (3.20) and (3.22), we obtain Corollary 2.16.
\(\square\)

**Proof of Theorem 3.2** Let’s establish that for \(P, Q \in V_{0}(\Sigma_{n})\), the following identity holds
\[
\langle \mu_{P}, \mu_{Q} \rangle_{L^{2}(\Sigma)} = \frac{1}{n^{2}},
\]
(3.23)

where \(\delta_{P, Q}\) is the Kronecker delta symbol. Clearly, (3.23) implies (3.9).

To prove (3.23), assume first that \(P, Q\) are not connected neither by an arc nor by a combination of a vertical and a horizontal arc. Then the supports of \(\mu_{P}\) and \(\mu_{Q}\) are disjoint and (3.23) holds.
Now, assume that \( P \) and \( Q \) are connected by a horizontal arc. We fix coordinates \( x, y \), which have axes parallel to the horizontal and vertical directions respectively. Assume that the coordinate \( x \) is normalized in such a way that \( x(P) = 0 \) and \( x(Q) = 1 \).

Assume that \( \text{dist}_\Sigma(P, \partial \Sigma) > \frac{1}{2n} \). Normalize the coordinate \( y \) in such a way that \( y \) takes values in \([-1, 1]\) on squares adjacent to \( P, Q \). Then by (3.1), we have

\[
\langle \mu_P, \mu_Q \rangle_{L^2(\Sigma)} = \frac{1}{n^2} \int_{-1}^{1} \left( \int_{0}^{1} \rho(x)(1 - \rho(x))dx \right) \rho(y)^2dy = 0. \tag{3.24}
\]

By a similar calculation, one can see that (3.23) holds if \( P \) and \( Q \) are connected by an arc or a combination of a vertical and a horizontal arc.

Finally, from (3.4), we see that for \( P \) satisfying \( \text{dist}_\Sigma(P, \partial \Sigma) > \frac{1}{2n} \), we have

\[
\langle \mu_P, \mu_P \rangle_{L^2(\Sigma)} = \frac{1}{n^2} \int_{-1}^{1} \left( \int_{-1}^{1} \rho(x)^2dx \right) \rho(y)^2dy = \frac{1}{n^2}. \tag{3.25}
\]

The same holds for \( P \), satisfying \( \text{dist}_\Sigma(P, \partial \Sigma) = \frac{1}{2n} \). From (3.24) and (3.25), we get (3.23).

Now let’s establish (3.10). Assume that \( z \in \Sigma \) lies in a square \( \Phi \), formed by the edges of \( \Sigma_n \), with vertices \( P, Q, R, S \in V(\Sigma_n) \). Assume \( P, Q \) and \( R, S \) share the same horizontal coordinate and \( P, R \) and \( Q, S \) share the same vertical coordinate. We fix linear coordinates \( x, y \), which have axes parallel to the horizontal and vertical directions respectively. We normalize \( x \) (resp. \( y \)) in such a way that \( x(P) = 0 \) and \( x(Q) = 1 \) (resp. \( y(P) = 0 \) and \( y(R) = 1 \)). Then by (2.16), we have

\[
\nabla_\varphi^F \mu_n(f)(z) = (f(P) - f(Q))\rho'(x)\rho(y) + (f(R) - f(S))\rho'(x)(1 - \rho(y)). \tag{3.26}
\]

From (3.1), (3.4), (3.26), we deduce that

\[
\langle \nabla_\varphi^F \mu_n(f), \nabla_\varphi^F \mu_n(f) \rangle_{L^2(\Phi, F)} = (f(P) - f(Q))^2 \cdot \int_{0}^{1} \rho'(x)^2dx \cdot \int_{0}^{1} \rho(y)^2dy
\]

\[
+ (f(R) - f(S))^2 \cdot \int_{0}^{1} \rho'(x)^2dx \cdot \int_{0}^{1} (1 - \rho(y))^2dy
\]

\[
= \frac{\int_{0}^{1} \rho'(x)^2dx}{2} \cdot \left( (f(P) - f(Q))^2 + (f(R) - f(S))^2 \right). \tag{3.27}
\]

Similar identity holds for the derivative with respect to \( y \)-variable, and we have

\[
\langle \nabla_\varphi^F \mu_n(f), \nabla_\varphi^F \mu_n(f) \rangle_{L^2(\Phi, F)} = \int_{0}^{1} \rho'(x)^2dx \cdot \left( (f(P) - f(R))^2 + (f(Q) - f(S))^2 \right). \tag{3.28}
\]

If we now repeat similar calculations for \( z \) lying near the boundary, by (2.7), we get (3.10).

\[\Box\]

### 3.2 Uniform bound on discrete zeta-functions, a proof of Theorem 2.15

The main goal of this section is to prove Theorem 2.15. We use the notation from Sect. 2.4.

The following two propositions form the core of the proof of Theorem 2.15, and they were inspired a lot by Müller [50]. Their proofs will be given in Sect. 3.3.
Proposition 3.4 Let’s fix two open subsets $U, V \subset \Sigma$ satisfying $\overline{U} \cap \overline{V} = \emptyset$. For any compact $K \subset \mathbb{C}$, there is $C > 0$ such that for any $s \in K$, $n \in \mathbb{N}^*$ and any functions $\phi, \psi \in \mathcal{C}^\infty(\Sigma) \cap L^\infty(\Sigma)$ with $\text{supp}(\phi) \subset U$, $\text{supp}(\psi) \subset V$, we have
\[
\left\| \phi \cdot (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^s \cdot \psi \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C \left\| \phi \right\|_{L^\infty(\Sigma)} \cdot \left\| \psi \right\|_{L^\infty(\Sigma)},
\]
where $\left\| \cdot \right\|_{L^2(\Sigma_n, F_n)}^0$ is the operator norm and we interpret the multiplication by functions $\phi, \psi$ by the multiplication of their restriction on $V(\Sigma_n)$.

Let $I, \phi_\alpha, U_\alpha, \alpha \in I$ be as in (2.44). Fix another family of functions $\psi_\alpha, \alpha \in I$, satisfying
\[
\psi_\alpha = 1 \quad \text{over} \quad V_\alpha \ni \text{supp}(\phi_\alpha) \quad \text{and} \quad \text{supp}(\psi_\alpha) \subset U_\alpha,
\]
for some open subsets $V_\alpha, \alpha \in I$.

Proposition 3.5 For any compact $K \subset \mathbb{C}$, there is $C > 0$ such that for any $n \in \mathbb{N}^*$, we have
\[
\left\| (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^{-s} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \cdot \psi_\alpha \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C.
\]

Proof of Theorem 2.15 First of all, by (3.30), we clearly have
\[
\text{Tr} \left[ \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \right] = \text{Tr} \left[ \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \cdot \psi_\alpha \right].
\]

Now, let’s denote by $P_{\Sigma_n}^F$ the orthogonal projection onto the functions from $\ker \Delta_{\Sigma_n}^{\bot}$. By the classical properties of the trace, we have
\[
\text{Tr} \left[ (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^{-s} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \cdot \psi_\alpha \right]
\leq \left\| (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^{-s+2} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \cdot \psi_\alpha \cdot (n^2 \cdot \Delta_{\Sigma_n}^{\bot} + P_{\Sigma_n}^F)^2 \right\|_{L^2(\Sigma_n, F_n)}^0.
\]

Since $\text{supp}(\psi_\alpha) \subset U_\alpha$, by the fact that $\Delta_{\Sigma_n}^{\bot}$ is “local”, for $n$ big enough, we have
\[
\psi_\alpha (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^2 = \psi_\alpha (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^2.
\]

By (3.34), we can write
\[
\phi_\alpha (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \psi_\alpha (n^2 \cdot \Delta_{\Sigma_n}^{\bot})^2 = \phi_\alpha (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s+2} \psi_\alpha + \phi_\alpha (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s+1} \left( (n^2 \cdot \Delta_{U_\alpha, n}^{\bot}), \psi_\alpha \right)
\]
\[
+ \phi_\alpha (n^2 \cdot \Delta_{U_\alpha, n}^{\bot})^{-s} \left[ (n^2 \cdot \Delta_{\Sigma_n}^{\bot}), (n^2 \cdot \Delta_{U_\alpha, n}^{\bot}), \psi_\alpha \right].
\]

Now, as $\psi_\alpha$ is smooth, we conclude that there is $C > 0$ such that for any $n \in \mathbb{N}^*$, we have
\[
\left\| (n^2 \cdot \Delta_{U_\alpha, n}^{\bot}), \psi_\alpha \right\|_{L^\infty(\Sigma)} \leq C,
\]
\[
\left\| \left[ (n^2 \cdot \Delta_{U_\alpha, n}^{\bot}), (n^2 \cdot \Delta_{\Sigma_n}^{\bot}), \psi_\alpha \right] \right\|_{L^\infty(\Sigma)} \leq C.
\]
Note that the supports of functions under the norms on the left-hand side of (3.36) is located in a ball of radius \( \frac{4 \pi}{n} \) around the support of \( \psi_\alpha(1 - \psi_\alpha) \). Moreover, by (3.30), we see that the functions \( \phi_\alpha \) and \( \left( n^2 \cdot \Delta F_{U_{\alpha,n}} \right) \) (resp. \( \left( n^2 \cdot \Delta F_{U_{\alpha,n}} \right) \), \( \left( n^2 \cdot \Delta F_{U_{\alpha,n}} \right) \)) satisfy the assumption of Proposition 3.4. By this, Proposition 3.4 and (3.36), we conclude that for any compact \( K \subset \mathbb{C} \), there is \( C > 0 \) such that for any \( s \in K \) and \( n \in \mathbb{N}^* \), we have

\[
\left\| \phi_\alpha \cdot (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s+1} \left( n^2 \cdot \Delta F_{U_{\alpha,n}} \right), \psi_\alpha \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C, \tag{3.37}
\]

By (3.35), (3.37), we see that there is \( C > 0 \) such that we have

\[
\left\| (n^2 \cdot \Delta F_{n,\perp})^{-s+2} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta F_{n,\perp})^{-s} \cdot \psi_\alpha \cdot (n^2 \cdot \Delta F_{n,\perp})^2 \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C. \tag{3.38}
\]

However, by Proposition 3.5, we conclude that there is \( C > 0 \) such that we have

\[
\left\| (n^2 \cdot \Delta F_{n,\perp})^{-s+2} - \sum_{\alpha \in I} \phi_\alpha \cdot (n^2 \cdot \Delta F_{n,\perp})^{-s+2} \cdot \psi_\alpha \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C. \tag{3.39}
\]

Now, by Theorem 2.12, we know that there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), we have

\[
\text{Tr} \left[ (n^2 \cdot \Delta F_{\Sigma_n} + P_{\Sigma_n} F_n)^{-2} \right] \leq C. \tag{3.40}
\]

We also clearly have

\[
\phi_\alpha (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} \psi_\alpha P_{\Sigma_n} F_n = \phi_\alpha (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} P_{\Sigma_n} F_n - \phi_\alpha (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} (1 - \psi_\alpha) P_{\Sigma_n} F_n. \tag{3.41}
\]

But since the flat sections of \( \Sigma_n \) restrict to flat sections on \( U_\alpha \), we have

\[
(n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} P_{\Sigma_n} F_n = 0. \tag{3.42}
\]

Also by Proposition 3.4 and (3.30), there is \( C > 0 \) such that we have

\[
\left\| \phi_\alpha (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} (1 - \psi_\alpha) P_{\Sigma_n} F_n \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C. \tag{3.43}
\]

We conclude by (3.41), (3.42) and (3.43) that there is \( C > 0 \) for any \( n \in \mathbb{N}^* \), such that

\[
\left\| \phi_\alpha \cdot (n^2 \cdot \Delta F_{U_{\alpha,n},\perp})^{-s} \cdot \psi_\alpha \cdot P_{\Sigma_n} F_n \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C \tag{3.44}
\]

By (3.32), (3.33), (3.38), (3.39), (3.40) and (3.44), we deduce Theorem 2.15.
3.3 Uniform bound on Laplacian powers, proofs of Propositions 3.4, 3.5

In this section we prove Propositions 3.4, 3.5. As in Sect. 3.2, here we were inspired a lot by Müller [50]. We conserve the notation from Sect. 3.2.

Let’s start with the proof of Proposition 3.4. The main ingredient is the following

**Proposition 3.6** Let open subsets $U, V \subset \Sigma$ be as in Proposition 3.4. Then there is $\epsilon > 0$ and $C > 0$ such that for any $t \in \mathbb{C}$, $|t| < \epsilon$, $n \in \mathbb{N}^*$ and functions $\phi, \psi \in C^{\infty}(\Sigma) \cap L^\infty(\Sigma)$ with $\text{supp}(\phi) \subset U$, $\text{supp}(\psi) \subset V$, we have

$$\left\| \phi \cdot \exp \left(-t(n^2 \cdot \Delta_{\Sigma_n}^F)^{1/2}\right) \cdot \psi \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C \|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)} \tag{3.45}$$

**Proof** Let’s define $A_n(t), B_n(t) \in \text{End}(\text{Map}(V(\Sigma_n), F_n))$, $t \in \mathbb{C}$, as follows

$$A_n(t) := \sum_{k=0}^{+\infty} \frac{(-t)^{2k} (n^2 \cdot \Delta_{\Sigma_n}^F)^k}{(2k)!}, \quad B_n(t) := \sum_{k=0}^{+\infty} \frac{(-t)^{2k+1} (n^2 \cdot \Delta_{\Sigma_n}^F)^{k+\frac{1}{2}}}{(2k+1)!} \tag{3.46}$$

By (1.2), the fact that the degree of every vertex is no bigger than 4 and the parallel transport is unitary, we have the following upper bound

$$\text{Spec}(\Delta_{\Sigma_n}^F) \subset [0, 8r_k(F)]. \tag{3.47}$$

By (3.47) the series (3.46) converge absolutely for $t \in \mathbb{C}$.

By the assumption on the supports of $\phi$ and $\psi$, there is $c > 0$, depending only on the sets $U, V$, such that for any $k < nc, k \in \mathbb{N}^*$, we have $\phi \cdot (\Delta_{\Sigma_n}^F)^k \cdot \psi = 0$, and, as a consequence, we have

$$\|\phi \cdot A_n(t) \cdot \psi\|_{L^2(\Sigma_n, F_n)}^0 \leq \sum_{k=0}^{+\infty} \|\phi \cdot \frac{(-t)^{2k} (n^2 \cdot \Delta_{\Sigma_n}^F)^k}{(2k)!} \cdot \psi\|_{L^2(\Sigma_n, F_n)}^0 \tag{3.48}$$

Now, by the upper bound on the spectrum of $\Delta_{\Sigma_n}^F$ from (3.47), the following bound holds

$$\left\| \phi \cdot \frac{(-t)^{2k} (n^2 \cdot \Delta_{\Sigma_n}^F)^k}{(2k)!} \cdot \psi \right\|_{L^2(\Sigma_n, F_n)}^0 \leq \frac{2^{2k} 4^{2k} n^{2k} r_k(F)^k}{(2k)!} \|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)} \tag{3.49}$$

By the Stirling’s bound, we have

$$(2k)! \geq \sqrt{2\pi} (2k)^{2k+\frac{1}{2}} \exp(-2k). \tag{3.50}$$

By (3.49) and (3.50), we conclude

$$\left\| \phi \cdot \frac{(-t)^{2k} (n^2 \cdot \Delta_{\Sigma_n}^F)^k}{(2k)!} \cdot \psi \right\|_{L^2(\Sigma_n, F_n)}^0 \leq \frac{2^{2k} 4^{2k} n^{2k} \exp(2k)r_k(F)^k}{(2k)^{2k}} \|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)} \tag{3.51}$$
Now, for any \( k \in \mathbb{N}^* \), satisfying \( k \geq cn \) for some \( c > 0 \), we have
\[
\frac{t^{2k} 4^{2k} n^{2k} \exp(2k)}{(2k)^{2k}} \leq t^{2k} 4^{2k} c^{-2k}.
\]
(3.52)

From (3.48), (3.51) and (3.52), we conclude that there is \( C > 0 \) such that for any \( t \in \mathbb{C} \), \(|t| \leq \frac{c}{\text{coker}(F)}\) and \( \phi, \psi \) as in the statement of the theorem we are proving, we have
\[
\|\phi \cdot A_0(t) \cdot \psi\|_{L^2(S_n, F_n)}^0 \leq C\|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)}.
\]
(3.53)

Now, note that \( \Delta_{S_n}^{F_n} \) is a positive self-adjoint operator (see (2.7) and the remark after), so for any \( t \in \mathbb{C} \), \( \text{Re}(t) \geq 0 \), the bound (3.45) trivially holds. From this and from the identity
\[
B_n(t) = \exp\left(-t(n^2 \cdot \Delta_{S_n}^{F_n})^{1/2}\right) - A_0(t),
\]
(3.54)
we conclude that there is \( C > 0 \) such that for any \( t \in \mathbb{C} \), \(|t| \leq \frac{c}{\text{coker}(F)}\), \( \text{Re}(t) \geq 0 \), we have
\[
\|\phi \cdot B_n(t) \cdot \psi\|_{L^2(S_n, F_n)}^0 \leq C\|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)}.
\]
(3.55)

Note, however, that \( B_n(t) = -B_n(-t) \). Thus, the bound (3.55) also holds for \( t \in \mathbb{C} \), \(|t| \leq \frac{c}{\text{coker}(F)}\) without the restriction on the real part of \( t \). From this, (3.53) and (3.54), we deduce Theorem 3.6.

\( \square \)

Let’s see now how Proposition 3.6 can be applied in the proof of Proposition 3.4.

**Proof of Proposition 3.4** Let \( P_{\Sigma_n}^{F_n} \) be as in the proof of Theorem 2.15. First, we clearly have
\[
(n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^s = (n^2 \cdot \Delta_{S_n}^{F_n, \perp})^s + P_{\Sigma_n}^{F_n},
\]
\[
\exp(-t(n^2 \cdot \Delta_{S_n}^{F_n})^{1/2}) = \exp(-t(n^2 \cdot \Delta_{S_n}^{F_n, \perp})^{1/2}) + \exp(-t) P_{\Sigma_n}^{F_n}.
\]
(3.56)
thus it is enough to prove the bound of Proposition 3.4 for \((n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^s\) instead of \((n^2 \cdot \Delta_{S_n}^{F_n, \perp})^s\).

For any \( k \in \mathbb{N} \) and \( s \in \mathbb{C} \), \( \text{Re}(s) > 0 \), by the definition of the \( \Gamma \)-function, we have
\[
(n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^{k-\frac{1}{2}} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^k \cdot \exp\left(-t(n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^{1/2}\right) dt.
\]
(3.57)

Let’s fix \( \epsilon > 0 \) as in the statement of Proposition 3.6. By Proposition 3.6, Cauchy theorem and (3.56), we conclude that for any \( l \in \mathbb{N} \), there is \( C > 0 \) such that for \( t \in \mathbb{C} \), \(|t| < \epsilon \) we have
\[
\left\|\phi \cdot (n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^{1/2} \exp\left(-t(n^2 \cdot \Delta_{S_n}^{F_n} + P_{\Sigma_n}^{F_n})^{1/2}\right) \cdot \psi\right\|_{L^2(S_n, F_n)}^0 \leq C\|\phi\|_{L^\infty(\Sigma)} \cdot \|\psi\|_{L^\infty(\Sigma)}.
\]
(3.58)

Since \( \Gamma(s)^{-1} \), \( s \in \mathbb{C} \) is a holomorphic function, and for any \( s \in \mathbb{C} \), \( \text{Re}(s) > 0 \), the integral \( \int_0^\infty t^{s-1} dt \) converges, by (3.58), we conclude that for any compact \( K \subset \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \), there is \( C > 0 \) such that for any \( s \in K \), we have
\[
\left\| \frac{1}{\Gamma(s)} \int_0^e t^{s-1} \phi \cdot (n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^k \exp \left( - t (n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right) \cdot \psi \, dt \right\|_{L^2(\Sigma_n, F_n)}^0 \\
\leq C \| \phi \|_{L^\infty(\Sigma)} \cdot \| \psi \|_{L^\infty(\Sigma)}.
\]
(3.59)

By integration by parts, we have
\[
\int_\epsilon^{+\infty} t^{s-1} \exp \left( - t (n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right) \cdot \exp \left( - t(n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right) \, dt \\
= (s-1) \cdots (s-2k) t^{-2k+s-1} \exp \left( - t(n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right) \, dt \\
+ \sum_{l=1}^{2k} (s-1) \cdots (s-l+1) e^{s-l} (n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{k-l/2} \cdot \exp \left( - t(n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right).
\]
(3.60)

Now, clearly Theorem 2.12 implies that there is \( \mu > 0 \) such that for any \( n \in \mathbb{N}^* \), we have
\[
\min \text{Spec}(n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F) > \mu.
\]
(3.61)

We conclude by (3.61) that for any \( K \subset \mathbb{C} \) there is \( C > 0 \) such that for any \( s \in K, k \in \mathbb{N}^* \):
\[
\left\| \int_\epsilon^{+\infty} t^{-2k+s-1} \exp \left( - t(n^2 \cdot \Delta_{\Sigma_n}^F + P_{\Sigma_n}^F)^{1/2} \right) \, dt \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C.
\]
(3.62)

Now, by (3.57) - (3.62), we deduce Proposition 3.4.

The main ingredient in the proof of Proposition 3.5 is the following statement. We borrow the notation from Proposition 3.5.

**Proposition 3.7** There is \( \epsilon > 0 \) and \( C > 0 \) such that for any \( t \in \mathbb{C}, |t| < \epsilon \) and \( n \in \mathbb{N}^* \), we have
\[
\left\| \exp \left( - t(n^2 \cdot \Delta_{\Sigma_n}^F)^{1/2} \right) - \sum_{\alpha \in I} \phi_{\alpha} \cdot \exp \left( - t(n^2 \cdot \Delta_{U_n}^F)^{1/2} \right) \cdot \psi_{\alpha} \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C.
\]
(3.63)

**Proof** The strategy of the proof is the same as in Proposition 3.6, so we only highlight the main steps. We define \( A_n^\alpha(t), B_n^\alpha(t) \in \text{End}(\text{Map}(V(\Sigma_n), F_n)) \), \( t \in \mathbb{C} \) similarly to \( A_n(t), B_n(t) \), from (3.46), only changing the Laplacian on \( \Sigma \) by the Laplacian on \( U_\alpha \).

By repeating the proof of (3.53), we see that there is \( C > 0 \) such that for any \( t \in \mathbb{C}, |t| < \epsilon \):
\[
\left\| A_n(t) - \sum_{\alpha \in I} \phi_{\alpha} \cdot A_n^\alpha(t) \cdot \psi_{\alpha} \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C.
\]
(3.64)

By using again positivity and identity (3.54), in the same way as in (3.55), we see by (3.64) that there is \( C > 0 \) such that for any \( t \in \mathbb{C}, |t| < \epsilon, \text{Re}(t) \geq 0 \), we have
\[
\left\| B_n(t) - \sum_{\alpha \in I} \phi_{\alpha} \cdot B_n^\alpha(t) \cdot \psi_{\alpha} \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C.
\]
(3.65)
However, since the operators $B_n(t), B_n^\alpha(t)$ satisfy $B_n(t) = -B_n(-t), B_n^\alpha(t) = -B_n^\alpha(-t)$, we see that (3.65) hold without the restriction on $\text{Re}(t)$. We deduce Theorem 3.7 by (3.54), (3.64) and (3.65).

Let’s see how Proposition 3.7 can be applied in the proof of Proposition 3.5.

**Proof of Proposition 3.5** Our proof is a repetition of the proof of Proposition 3.4, where we replace the use of Proposition 3.6 by the use of Proposition 3.7. Let’s just highlight the main steps.

Let’s fix $\epsilon > 0$ as in Proposition 3.7. By Proposition 3.7, Cauchy theorem and (3.56), we see that for any $l \in \mathbb{N}$, there is $C > 0$ such that

$$\left\| \left( n^2 \cdot \Delta_{\Sigma_n} \right)^{l/2} \exp \left( -t \left( n^2 \cdot \Delta_{\Sigma_n} \right)^{1/2} \right) - \sum_{\alpha \in I} \phi_{\alpha} \cdot \left( n^2 \cdot \Delta_{U_{\alpha,n}} \right)^{l/2} \exp \left( -t \left( n^2 \cdot \Delta_{U_{\alpha,n}} \right)^{1/2} \right) \cdot \psi_{\alpha} \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C. \quad (3.66)$$

Similarly to (3.59), but by using Proposition 3.7, we conclude that for any compact $K \subset \{ s \in \mathbb{C} : \text{Re}(s) > 0 \}$, and any $s \in K$, there is $C > 0$ such that

$$\left\| \frac{1}{\Gamma(s)} \left( \int_0^\epsilon t^{-1} n^2 \cdot \Delta_{\Sigma_n}^k \exp \left( -t \left( n^2 \cdot \Delta_{U_{\alpha,n}} \right)^{1/2} \right) dt - \sum_{\alpha \in I} \phi_{\alpha} \cdot \int_0^\epsilon t^{-1} n^2 \cdot \Delta_{U_{\alpha,n}}^k \exp \left( -t \left( n^2 \cdot \Delta_{U_{\alpha,n}} \right)^{1/2} \right) dt \cdot \psi_{\alpha} \right) \right\|_{L^2(\Sigma_n, F_n)}^0 \leq C. \quad (3.67)$$

Now, by (3.57), (3.60), (3.62), applied for $\Sigma_n$ and $U_{\alpha,n}$ and by (3.66), (3.67), we deduce Proposition 3.5.

**4 Fourier analysis on square**

The goal of this section is to prove Theorem 2.18. The main idea is to use discrete Fourier analysis on rectangle and then to recover the quantities from he right-hand side of (2.53) through the Fourier coefficients. This section is organized as follows. In Sect. 4.1 we recall the description of the spectrum and the eigenvectors of the Laplacian on a square-grid mesh graph, and we will study the multiplication operator in the basis of eigenvalues of the Laplacian. Then in Sect. 4.2, by using those results, the result of Duplantier-David from [25], cf. Theorem A.1, and some elementary but technical calculations, we prove Theorem 2.18.

**4.1 Spectrum of the discrete Laplacian on square**

In this section we recall the description of the spectrum and the eigenvectors of the Laplacian on the mesh graph. Then we study the multiplication operator in the basis of eigenvectors.

We fix $a, b \in \mathbb{N}^*$ and construct mesh graphs $A_{an \times bn}, n \in \mathbb{N}^* \text{ as } an \times bn \text{ subgraphs of } \mathbb{Z}^2$. Denote by $P_{A_{an \times bn}}$ the orthogonal projection onto the constant functions. The eigenvectors $f_{i,j}^n, 0 \leq i \leq an - 1, 0 \leq j \leq bn - 1$, of $n^2 \cdot \Delta_{A_{an \times bn}} + P_{A_{an \times bn}}$ are given by
and the corresponding eigenvalues $\lambda_{i,j}^n$ are

$$\lambda_{i,j}^n = \begin{cases} 1, & \text{for } i,j = 0, \\ 4n^2 \sin \left( \frac{\pi i}{2an} \right)^2 + 4n^2 \sin \left( \frac{\pi j}{2bn} \right)^2, & \text{otherwise}. \end{cases}$$

Denote by $\delta_{i,j}$ the the Kronecker symbol. We verify easily that

$$\left\| f_{i,j}^n \right\|_{L^2(A_{an\times bn})} = \frac{abn^2}{4} \gamma_{\delta_{i,0} + \delta_{j,0}}. \quad (4.3)$$

Now let’s fix a smooth function $\phi$, defined on the square $[0,a] \times [0,b]$, satisfying

$$\phi(x,y) = \phi(a-x, y) = \phi(x, b-y), \quad (x,y) \in [0,a] \times [0,b]. \quad (4.4)$$

Roughly, our goal now is to write down the multiplication of the restriction of $\phi$ on $V(A_{an\times bn})$ in the basis of the eigenvalues of $\Delta_{A_{an\times bn}}$. The resulting formula will be slightly cumbersome, so we will content ourselves with some remarks in this direction.

We first remark that by (4.4), only the terms of the form $\cos(\cdot) \cdot \cos(\cdot)$ appear in the Fourier coefficients of $\phi$. Thus, there are $a_{i,j} \in \mathbb{R}$, $i,j \in \mathbb{N}$, such that the following expansion holds

$$\phi(x,y) = \sum_{i,j=0}^{\infty} a_{i,j} \cos \left( \frac{2\pi i}{a} \right) \cos \left( \frac{2\pi j}{b} \right). \quad (4.5)$$

If in addition to (4.4), we require that the induced function $\tilde{\phi}$ on the torus $[0,a] \times [0,b]/(0,y) \sim (a,y), (x,0) \sim (x,b)$ is smooth, then by the standard techniques from Fourier analysis, we see that for any $c > 0$, there is $C > 0$ such that

$$a_{i,j} \leq C(i+j+1)^{-c}. \quad (4.6)$$

Recall that we view the graph $A_{an\times bn}$ injected in $\mathbb{C}$ as a subgraph of the nearest-neighbor graph on the vertices of $\frac{1+\sqrt{5}}{2n} + \frac{1}{n} \mathbb{Z}^2$, which stays inside of the square $\{z \in \mathbb{C}, 0 < \text{Re}(z) < a, 0 < \text{Im}(z) < b\}$. To simplify the notation, we denote the restriction of $\cos(\frac{2\pi i x}{a}) \cos(\frac{2\pi j y}{b})$ to $V(A_{an\times bn})$ by

$$g_{i,j}^n(r,s) := \cos \left( \frac{2\pi i}{an} \left( \frac{1}{2} + r \right) \right) \cos \left( \frac{2\pi j}{bn} \left( \frac{1}{2} + s \right) \right), \quad (4.7)$$

where $0 \leq r \leq an - 1, 0 \leq s \leq bn - 1$. In particular, for $i, j \in \mathbb{N}$, we have the following identities

$$g_{i,j}^n = -g_{i+an,j}^n = -g_{i,j+bn}^n. \quad (4.8)$$

From (4.7), we see

$$\langle g_{i,j}^n \cdot f_{k,l}^n, f_{k,l}^n \rangle_{L^2(V(A_{an\times bn}))} = \sum_{r=0}^{an-1} \sum_{s=0}^{bn-1} \cos \left( \frac{2\pi i}{an} \left( \frac{1}{2} + r \right) \right) \cos \left( \frac{2\pi j}{bn} \left( \frac{1}{2} + s \right) \right) \cdot \cos \left( \frac{2\pi k}{2an} \left( \frac{1}{2} + r \right) \right) \cos \left( \frac{2\pi l}{2bn} \left( \frac{1}{2} + s \right) \right)^2. \quad (4.9)$$
It is rather easy to verify that for any $0 \leq i, k \leq an - 1$, we have
\[
\sum_{r=0}^{an-1} \cos \left( \frac{2\pi i}{an} \left( \frac{1}{2} + r \right) \right) \cdot \cos \left( \frac{2\pi k}{2an} \left( \frac{1}{2} + r \right) \right) = \begin{cases} 
\frac{an}{2}, & \text{for } k = 0, \\
\frac{an}{4} \delta_{i,k} - \frac{an}{4} \delta_{i,an-k}, & \text{for } i \neq 0, k \neq 0.
\end{cases}
\] (4.10)

From this and (4.7), we see that the multiplication by $g^n_{i,j}$ is "sparse" unless $i$ or $j$ is equal to 0 modulo $an$ or $bn$ respectively. We will use this fact extensively in the next section.

### 4.2 Szego-type theorem, a proof of Theorem 2.18

The main goal of this section is to prove Theorem 2.18. As in Sect. 4.1, we fix $a, b \in \mathbb{N}^*$ and construct a family of graphs $A_{an \times bn}$, $n \in \mathbb{N}^*$. We fix a smooth function $\phi : [0, a] \times [0, b] \rightarrow \mathbb{R}$ satisfying (4.4) and such that the induced function $\hat{\phi}$ on the torus $[0, a] \times [0, b]/(0,y)\sim(a,y),(x,0)\sim(x,b)$ is smooth. In particular, the Fourier expansion of $\phi$ is of the form (4.5), and the coefficients $a_{i,j}$, $i, j \in \mathbb{N}$ satisfy the asymptotic bound (4.6).

Recall that the functions $f^n_{i,j}$, $g^n_{i,j} \in \text{Map}(V(A_{an \times bn}), \mathbb{R})$ were defined in (4.1) and (4.7) respectively. By (4.5), (4.7) and (4.8), we have
\[
\text{Tr}[\phi \cdot (n^2 \cdot \Delta^\perp_{A_{an \times bn}})] = a_{0,0} \sum_{i=0}^{an-1} \sum_{j=0}^{bn-1} \log (\lambda^n_{i,j}) \\
+ \sum_{i=1}^{an} \left( \sum_{k=0}^{+\infty} (-1)^k a_{i+k,an,0} \right) \text{Tr}[\delta_{i,0} \cdot \log (n^2 \cdot \Delta^\perp_{A_{an \times bn}})] \\
+ \sum_{j=1}^{bn} \left( \sum_{l=0}^{+\infty} (-1)^l a_{0,j+bn} \right) \text{Tr}[g^n_{0,j} \cdot \log (n^2 \cdot \Delta^\perp_{A_{an \times bn}})] \\
+ \sum_{i=1}^{an} \sum_{j=1}^{bn} \left( \sum_{k,l=0}^{+\infty} (-1)^{k+l} a_{i+k,an,j+bn} \right) \text{Tr}[\delta_{i,j} \cdot \log (n^2 \cdot \Delta^\perp_{A_{an \times bn}})].
\] (4.11)

We will now state the two main propositions of this section. Then we will explain how they are useful in proving Theorem 2.18 and then we will prove them.

**Proposition 4.1** There are $c_{1,\phi}^a, c_{2,\phi}^a \in \mathbb{R}$, such that as $n \rightarrow \infty$, we have
\[
\sum_{i=1}^{an} \left( \sum_{k=0}^{+\infty} (-1)^k a_{i+k,an,0} \right) \text{Tr}[\delta_{i,0} \cdot \log (n^2 \cdot \Delta^\perp_{A_{an \times bn}})] = -\left( bn \log (1 + \sqrt{2}) + \frac{\log(n)}{2} \right) \sum_{i=1}^{+\infty} a_{i,0} + c_{1,\phi}^a + o(n^{-1/3}).
\] (4.12)

\[
\sum_{j=1}^{bn} \left( \sum_{l=0}^{+\infty} (-1)^l a_{0,j+bn} \right) \text{Tr}[g^n_{0,j} \cdot \log (n^2 \cdot \Delta^\perp_{A_{an \times bn}})]
\]
\[= -\left(\alpha n \log(1 + \sqrt{2}) + \frac{\log(n)}{2}\right) \sum_{j=1}^{+\infty} a_{0,j} + \epsilon_{2,\phi} + o(n^{-1/3}). \tag{4.13}\]

**Proposition 4.2** There is \(c_{3,\phi} \in \mathbb{R}\) such that as \(n \to \infty\), we have
\[
\sum_{i=1}^{\alpha n} \sum_{j=1}^{bn} \left( \sum_{k,l=0}^{+\infty} (-1)^{k+l} a_{i+kan,j+lb^n} \right) \text{Tr} \left[ g_{i,j}^{*} \cdot \log(n^2 \cdot \Delta_{\tilde{A}_{kan \times lb^n}}) \right] = -\frac{\log(n)}{2} \sum_{i,j=1}^{+\infty} a_{i,j} + c_{3,\phi} + o(n^{-1/3}). \tag{4.14}\]

**Proof of Theorem 2.18** Theorem 2.18 is formulated on a rectangle \([0, a] \times [0, b]\) for \(a, b = 2\), but here we prove it for any \(a, b \in \mathbb{N^*}\). By averaging with respect to the symmetry group of rectangle, we see that we can assume that \(\phi\) satisfies (4.4).

We define \(c([0, a] \times [0, b], \phi) \in \mathbb{R}\) as follows
\[
c([0, a] \times [0, b], \phi) := c_{1,\phi} + c_{2,\phi} + c_{3,\phi} + a_{0,0} \cdot \left( \log\left( \det\Delta_{[0,a] \times [0,b]} \right) - \frac{\log(2)}{4} \right), \tag{4.15}\]

where \(c_{1,\phi}, c_{2,\phi}, c_{3,\phi}\) are defined as in Proposition 4.1, 4.2. By Propositions 4.1, 4.2, Theorem A1 and (4.11), the following expansion, as \(n \to \infty\), holds
\[
\text{Tr}\left[ \phi \cdot \log(n^2 \cdot \Delta_{\tilde{U}_n}) \right] = \left(2abn^2 \log(n) + \frac{4G}{\pi} abn^2 \right) a_{0,0}
+ \log(\sqrt{2} - 1)n \left(b \sum_{i=0}^{+\infty} a_{i,0} + a \sum_{j=0}^{+\infty} a_{0,j} \right)
- \frac{\log(n)}{2} \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} a_{i,j} + c([0, a] \times [0, b], \phi) + o(1). \tag{4.16}\]

Now, Theorem 2.18 follows from (4.16) and the following identities
\[
a_{0,0} = \frac{1}{ab} \int_{0}^{a} \int_{0}^{b} \phi(x, y) dx dy, \quad \sum_{i=0}^{+\infty} a_{i,0} = \frac{1}{b} \int_{0}^{b} \phi(0, y) dy, \tag{4.17}
\]
\[
+\sum_{j=0}^{+\infty} a_{0,j} = \frac{1}{a} \int_{0}^{a} \phi(x, 0) dy, \quad \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} a_{i,j} = \phi(0, 0),
\]
which follow trivially from (4.5).

From now on till the end of this section we will be proving Propositions 4.1, 4.2.

**Proposition 4.3** For any \(x \in \mathbb{R}\) and \(m \in \mathbb{N^*}\), we have
\[
\prod_{j=0}^{m-1} \left( \sin \left( \frac{\pi j}{2m} \right)^2 + x^2 \right) = \left| \frac{\left| x \right| \cdot \left( 1 + x^2 \right)^{-1/2}}{2^{2m}} \left( \sqrt{1 + x^2} + x \right)^{2m} - 1 \right| \left( \sqrt{1 + x^2} - x \right)^{2m} - 1. \tag{4.18}\]
\textbf{Proof} First, we have
\[
\left( \sin \left( \frac{\pi j}{2m} \right)^2 + x^2 \right) = \left( \sin \left( \frac{\pi j}{2m} \right) + \sqrt{-1}x \right) \left( \sin \left( \frac{\pi j}{2m} \right) - \sqrt{-1}x \right). \tag{4.19}
\]
Now, by Euleur’s identity, we have
\[
\sin \left( \frac{\pi j}{2m} \right) \pm \sqrt{-1}x = \frac{e^{-\frac{\pi j \sqrt{-1}}{2m}} + e^{\frac{\pi j \sqrt{-1}}{2m}}}{2} \mp 2x e^{-\frac{\pi j \sqrt{-1}}{2m}} - 1. \tag{4.20}
\]
By resolving quadratic equation, we deduce
\[
e^{\frac{\pi j \sqrt{-1}}{2m}} \mp 2x e^{-\frac{\pi j \sqrt{-1}}{2m}} - 1 = \left( e^{-\frac{\pi j \sqrt{-1}}{2m}} - (\pm x + \sqrt{1 + x^2}) \right) \cdot \left( e^{\frac{\pi j \sqrt{-1}}{2m}} - (\pm x - \sqrt{1 + x^2}) \right). \tag{4.21}
\]
We also clearly have
\[
\prod_{j=0}^{m-1} \left( \sin \left( \frac{\pi j}{2m} \right)^2 + x^2 \right) = |x| \cdot (1 + x^2)^{-1/2} \cdot \prod_{j=0}^{2m-1} \left( \sin \left( \frac{\pi j}{2m} \right)^2 + x^2 \right)^{1/2}. \tag{4.22}
\]
By (4.19), (4.20), (4.21) and (4.22), we conclude that
\[
\prod_{j=0}^{m-1} \left( \sin \left( \frac{\pi j}{2m} \right)^2 + x^2 \right) = \left| x \right| \cdot (1 + x^2)^{-1/2} \cdot \prod_{(\alpha, \beta) \in \{-1, 1\}^2} \prod_{j=0}^{2m-1} \left| \alpha x + \beta \sqrt{1 + x^2} - e^{\frac{\pi j \sqrt{-1}}{2m}} \right|^{1/2}. \tag{4.23}
\]
But by the definition of the roots of unity, we have
\[
\prod_{j=0}^{2m-1} \left( \alpha x + \beta \sqrt{1 + x^2} - e^{\frac{\pi j \sqrt{-1}}{m}} \right) = \left( \alpha x + \beta \sqrt{1 + x^2} \right)^{2m} - 1. \tag{4.24}
\]
From (4.23) and (4.24), we easily conclude that Proposition 4.3 holds. \qed

\textbf{Proposition 4.4} There is \( C > 0 \) such that for any \( n \in \mathbb{N}^q, 1 \leq q \leq n^{1/3}, 1 \leq j \leq n^{1/3}, \) we have
\[
\left| \sum_{l=0}^{bn-1} \left( \log(\lambda^n_{q,l}) - \log(\lambda^n_{an-q,l}) \right) + 2bn \log(1 + \sqrt{2}) + \log(n) - \log(e^{\pi q^2} - 1) - \log(1 - e^{-\pi q^2}) - \log \left( \frac{\pi q}{2a} \right) - \frac{\log(2)}{2} \right| \leq \frac{C}{n^{1/3}}, \tag{4.25}
\]
\[
\left| \sum_{k=0}^{an-1} \left( \log(\lambda^n_{k,j}) - \log(\lambda^n_{bn-j}) \right) + 2an \log(1 + \sqrt{2}) + \log(n) - \log(e^{\pi j^2} - 1) - \log(1 - e^{-\pi j^2}) - \log \left( \frac{\pi j}{2b} \right) - \frac{\log(2)}{2} \right| \leq \frac{C}{n^{1/3}}. \tag{4.26}
\]
Proof Clearly, the statement (4.26) is completely analogous to (4.25), so we will only concentrate on proving (4.25). From Proposition 4.3 and (4.2), for any \( n \in \mathbb{N}^* \), we have

\[
\sum_{l=0}^{bn-1} \log(\lambda_{q,l}^n) = 2bn \log(n) + \log\left( \sin\left( \frac{\pi q}{2an} \right) \right) \]

\[
- \frac{1}{2} \log \left( 1 + \sin\left( \frac{\pi q}{2an} \right)^2 \right) + \log\left( \left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 + \sin\left( \frac{\pi q}{2an} \right)^{2bn}} - 1 \right) \right) + \log\left( 1 - \left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 - \sin\left( \frac{\pi q}{2an} \right)^{2bn}} \right) \right). \tag{4.27}
\]

We now study the asymptotic expansion of the right-hand side of (4.27). By Taylor expansion, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\left| \sin\left( \frac{\pi q}{2an} \right) - \frac{\pi q}{2an} \right| \leq \frac{C}{n^2}. \tag{4.28}
\]

Similarly, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\left| \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 - 1} \right| \leq \frac{C}{n^{1/3}}. \tag{4.29}
\]

Hence, from (4.28) and (4.29), we obtain that

\[
\log\left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 + \sin\left( \frac{\pi q}{2an} \right)} \right) = 1 + \frac{\pi q}{2an} + O(n^{-4/3}). \tag{4.30}
\]

From (4.30), there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 + \sin\left( \frac{\pi q}{2an} \right)} \right)^{2bn} - e^{\pi q n} \leq \frac{C}{n^{1/3}}. \tag{4.31}
\]

As a consequence of (4.31), there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\log\left( \left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 + \sin\left( \frac{\pi q}{2an} \right)} \right)^{2bn} - 1 \right) - \log(e^{\pi q n} - 1) \leq \frac{C}{n^{1/3}}. \tag{4.32}
\]

Similarly, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\log\left( 1 - \left( \sqrt{1 + \sin\left( \frac{\pi q}{2an} \right)^2 - \sin\left( \frac{\pi q}{2an} \right)} \right)^{2bn} \right) - \log\left( 1 - e^{-\pi q n} \right) \leq \frac{C}{n^{1/3}}. \tag{4.33}
\]

From (4.27), (4.32) and (4.33), there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have

\[
\sum_{l=0}^{bn-1} \log(\lambda_{q,l}^n) - (2bn - 1) \log(n) - \log(e^{\pi q n} - 1) - \log(1 - e^{-\pi q n}) - \log\left( \frac{\pi q}{2a} \right) \leq \frac{C}{n^{1/3}}. \tag{4.34}
\]

Now, from Proposition 4.3, (4.2) and the identity \( \sin\left( \frac{\pi x}{2} \right) = \cos(x) \), we have

\[
\sum_{l=0}^{bn-1} \log(\lambda_{an-qn,l}^n) = 2bn \log(n) + \log\left( \cos\left( \frac{\pi q}{2an} \right) \right) - \frac{1}{2} \log\left( 1 + \cos\left( \frac{\pi q}{2an} \right)^2 \right).
\]
\[ + \log \left( \sqrt{1 + \cos \left( \frac{\pi q}{2an} \right)^2 + \cos \left( \frac{\pi q}{2an} \right)^{2bn}} \right) \]
\[ + \log \left( 1 - \left( \sqrt{1 + \cos \left( \frac{\pi q}{2an} \right)^2 - \cos \left( \frac{\pi q}{2an} \right)^{2bn}} \right) \right). \]
\tag{4.35}

We will now study the asymptotic expansion of (4.35). By Taylor expansion, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have
\[ \left| \cos \left( \frac{\pi q}{2an} \right) - 1 \right| \leq \frac{C}{n^{4/3}}. \]
\tag{4.36}

Similarly, there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have
\[ \left| \sqrt{1 + \cos \left( \frac{\pi q}{2an} \right)^2 - \sqrt{2}} \right| \leq \frac{C}{n^{4/3}}. \]
\tag{4.37}

From (4.36) and (4.37), we see that there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have
\[ \left| \log \left( \left( \sqrt{1 + \cos \left( \frac{\pi q}{2an} \right)^2 + \cos \left( \frac{\pi q}{2an} \right)^{2bn}} \right) - 1 \right) - 2bn \log(1 + \sqrt{2}) \right| \leq \frac{C}{n^{1/3}}. \]
\tag{4.38}

As \( |\sqrt{2} - 1| < 1 \), there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have
\[ \left| \log \left( 1 - \left( \sqrt{1 + \cos \left( \frac{\pi q}{2an} \right)^2 - \cos \left( \frac{\pi q}{2an} \right)^{2bn}} \right) \right) \right| \leq \frac{C}{n^{1/3}}. \]
\tag{4.39}

As a conclusion from (4.35), (4.38) and (4.39), we see that there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \( 0 \leq q \leq n^{1/3} \), we have
\[ \left| \sum_{l=0}^{bn-1} \log(\lambda_{an-q,l}^n) - 2bn \log(n) - 2bn \log(1 + \sqrt{2}) + \frac{\log(2)}{2} \right| \leq \frac{C}{n^{1/3}}. \]
\tag{4.40}

By combining (4.34) and (4.40), we deduce (4.25). \( \square \)

Now we are finally ready to give

**Proof of Proposition 4.1** Clearly, the statements (4.12) and (4.13) are analogous, so we will only concentrate on the proof of the former one.

First, by (3.47), the fact that \( \dim V_{A_{an \times bn}} = abn^2 \) and the fact that \( \|g_{i,j}^n\| < 1 \), for any \( 0 \leq i \leq an - 1 \) and any \( n \in \mathbb{N}^* \), we have
\[ \text{Tr} \left[ g_{i,0}^n \cdot \log(n^2 \cdot \Delta_{A_{an \times bn}}^+) \right] \leq abn^2 \log(8n^2 \text{rk}(F)). \]
\tag{4.41}

By (4.6) and (4.41), we see that for any \( c > 0 \) there is \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), the following bound holds
\[ \sum_{i=0}^{an} \left( \sum_{k=0}^{+\infty} \left( -1 \right)^k a_{q+k,0} \right) \text{Tr} \left[ g_{i,0}^n \cdot \log(n^2 \cdot \Delta_{A_{an \times bn}}^+) \right] \]
\[ - \sum_{q=1}^{n^{1/3}} a_{q,0} \text{Tr} \left[ g_{q,0}^n \cdot \log(n^2 \cdot \Delta_{A_{an \times bn}}^+) \right] \leq \frac{C}{n^c}. \]
\tag{4.42}
By the definition of trace and (4.2), we have
\[
\text{Tr}\left[g^n_{q,0} \cdot \log(n^2 \cdot \Delta^+_{A_{an \times bn}})\right] = \sum_{k=0}^{an-1} \sum_{l=0}^{bn-1} \frac{\log(\lambda^n_{k,l})}{\|f^n_{q,k,l}\|_{L^2(A_{an \times bn})}} \langle g^n_{q,0} \cdot f^n_{q,k,l}, f^n_{q,k,l} \rangle_{L^2(A_{an \times bn})}.
\] (4.43)

By (4.10), it is easy to conclude that for \(1 \leq q \leq an - 1\), the only non-vanishing terms from the sum in the right-hand side of (4.43) are those which correspond to \(k = q\) or \(k = an - q\). We study their contributions to the sum (4.43) separately.

By (4.3), (4.10), the contribution of the terms corresponding to \(k = q\) is given by
\[
\frac{b_n-1}{l=0} \frac{\log(\lambda^n_{q,l})}{\|f^n_{q,q,l}\|_{L^2(A_{an \times bn})}} \langle g^n_{q,0} \cdot f^n_{q,q,l}, f^n_{q,q,l} \rangle_{L^2(A_{an \times bn})} = \frac{1}{2} \sum_{l=0}^{bn-1} \log(\lambda^n_{q,l}).
\] (4.44)

Similarly, the contribution of the terms corresponding to \(k = an - q\) is given by
\[
\sum_{l=0}^{bn-1} \frac{\log(\lambda^n_{an-q,l})}{\|f^n_{an-q,l}\|_{L^2(A_{an \times bn})}} \langle g^n_{q,0} \cdot f^n_{an-q,l}, f^n_{an-q,l} \rangle_{L^2(A_{an \times bn})} = -\frac{1}{2} \sum_{l=0}^{bn-1} \log(\lambda^n_{an-q,l}).
\] (4.45)

By (4.43), (4.44) and (4.45), we conclude that for \(1 \leq q \leq an - 1\), we have
\[
\text{Tr}\left[g^n_{q,0} \cdot \log(n^2 \cdot \Delta^+_{A_{an \times bn}})\right] = \frac{1}{2} \sum_{l=0}^{bn-1} \left( \log(\lambda^n_{q,l}) - \log(\lambda^n_{an-q,l}) \right).
\] (4.46)

Now, by (4.25), (4.42) and (4.46), we see that (4.12) holds for
\[
\epsilon_{1,\phi}^{a,b} := \frac{1}{2} \sum_{q=1}^{\infty} a_{q,0} \left( \log(e^{\pi q^{2/3}} - 1) + \log(1 - e^{-\pi q^{2/3}}) + \log\left(\frac{\pi q}{2a}\right) + \frac{\log(2)}{2} \right),
\] (4.47)

which converges by (4.6), for any \(\phi\) as in the statement of Proposition 4.1.

Proof of Proposition 4.2 First, similarly to (4.42), we see that for any \(c > 0\) there is \(C > 0\) such that for any \(n \in \mathbb{N}^*\), the following bound holds
\[
\left| \sum_{q=1}^{an} \sum_{j=1}^{bn} \left( \sum_{k,l=0}^{+\infty} (-1)^{k+l} a_{q+kan,j+lbn} \right) \text{Tr}\left[g^n_{q,j} \cdot \log(n^2 \cdot \Delta^+_{A_{an \times bn}})\right] - \sum_{q,j=1}^{n^{1/3}} a_{q,j} \text{Tr}\left[g^n_{q,j} \cdot \log(n^2 \cdot \Delta^+_{A_{an \times bn}})\right] \right| \leq \frac{C}{nc}.
\] (4.48)

By (4.10), for any \(1 \leq q \leq an - 1\), \(1 \leq j \leq bn - 1\), we have
\[
\text{Tr}\left[g^n_{q,j} \cdot \log(n^2 \cdot \Delta^+_{A_{an \times bn}})\right] = \frac{1}{4} \left( \log(\lambda^n_{q,j}) - \log(\lambda^n_{an-q,j}) - \log(\lambda^n_{q,bn-j}) + \log(\lambda^n_{an-q,bn-j}) \right).
\] (4.49)

By (4.2), we see that there is \(C > 0\) such that for any \(n \in \mathbb{N}^*\) and \(1 \leq q, j \leq n^{1/3}\), we get
\[
\left| \log(\lambda^n_{q,j}) - \log(\lambda^n_{an-q,j}) - \log(\lambda^n_{q,bn-j}) + \log(\lambda^n_{an-q,bn-j}) \right|
\]
− \log \left( \frac{\pi^2 q^2}{a^2} + \frac{\pi^2 j^2}{b^2} \right) + \log(2) + 2 \log(n) \leq \frac{C}{n^{1/3}}. \quad (4.50)

By (4.48), (4.49) and (4.50), Proposition 4.2 holds for \( c_{a,b}^{3,\phi} \), defined as

\[ c_{a,b}^{3,\phi} := \frac{1}{4} \sum_{q,j=1}^{\infty} \left( \log \left( \frac{\pi^2 q^2}{a^2} + \frac{\pi^2 j^2}{b^2} \right) − \log(2) \right), \quad (4.51) \]

which converges by (4.6), for any \( \phi \) as in the statement of Proposition 4.1.

\[ \square \]

Acknowledgements The author would like to thank Dmitry Chelkak, Yves Colin de Verdière for related discussions and their interest in this article, and especially Xiaonan Ma for important comments and remarks. We also thank the anonymous referee for the important comments and the colleagues from Institute Fourier, Université Grenoble Alpes, where this article has been written, for their hospitality.

A Appendix: heat kernel and analytic torsion

The main goal of the appendix is to recall some folklore results about the heat kernel and the analytic torsion. More precisely, in Appendix A.1, we recall the asymptotics of the heat kernel on the surfaces with conical singularities. In Appendix A.2, we recall the Kronecker limit formula and its relation with the calculation of the analytic torsion of tori and square. We also recall the asymptotic expansion of the determinant of the discrete Laplacian on mesh graphs.

A.1 Small-time asymptotic expansion of the heat kernel

In this section we will prove Proposition 2.5. The proof is done by reducing the small-time asymptotic expansion of the heat kernel on the general flat surface to a number of model cases, for which the explicit calculations are possible.

First, let’s prove that for any \( l \in \mathbb{N}, \epsilon > 0 \) there are \( c, C > 0 \) such that for any \( x \in \Sigma \) satisfying \( \text{dist}_{\Sigma}(x, \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)) > \epsilon \), and any \( 0 < t < 1 \), we have

\[ \left\| \nabla_x \exp(-t \Delta_{\Sigma}^x)(x, \cdot) \right\|_{L^2(\Sigma \setminus B_{\Sigma}(x, \epsilon/2))} \leq C \exp \left( -\frac{c}{t} \right). \quad (A.1) \]

The estimate (A.1) can be proven by a variety of different methods. We do it by using finite propagation speed of solutions of hyperbolic equations and interior elliptic estimates.

More precisely, for \( r > 0 \), we introduce smooth even functions (cf. [46, (4.2.11)])

\[ K_{t,r}(a) := \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v \sqrt{2t} a) \exp \left( -\frac{v^2}{2} \right) \left( 1 - \psi \left( \frac{\sqrt{2tv}}{r} \right) \right) \frac{dv}{\sqrt{2\pi}}, \]

\[ G_{t,r}(a) := \int_{-\infty}^{+\infty} \exp(\sqrt{-1}v \sqrt{2t} a) \exp \left( -\frac{v^2}{2} \right) \psi \left( \frac{\sqrt{2tv}}{r} \right) \frac{dv}{\sqrt{2\pi}}, \quad (A.2) \]

where \( \psi : \mathbb{R} \rightarrow [0, 1] \) is a cut-off function satisfying

\[ \psi(u) = \begin{cases} 1 & \text{for } |u| < 1/2, \\ 0 & \text{for } |u| > 1. \end{cases} \quad (A.3) \]
Let \( \tilde{K}_{t,r}, \tilde{G}_{t,r} : \mathbb{R}^+ \to \mathbb{R} \) be the smooth functions given by \( \tilde{K}_{t,r}(a^2) = K_{t,r}(a) \), \( \tilde{G}_{t,r}(a^2) = G_{t,r}(a) \). Then the following identity holds
\[
\exp(-t \Delta^F_{\Sigma}) = \tilde{G}_{t,r}(\Delta^F_{\Sigma}) + \tilde{K}_{t,r}(\Delta^F_{\Sigma}). \tag{A.4}
\]

By the finite propagation speed of solutions of hyperbolic equations (cf. [46, Theorems D.2.1, 4.2.8]), the section \( \tilde{G}_{t,r}(\Delta^F_{\Sigma})(z, \cdot) \) depends only on the restriction of \( \Delta^F_{\Sigma} \) onto the ball \( B_{\Sigma}(z, r) \) of radius \( r \) around \( z \). Moreover, we have
\[
\text{supp} \tilde{G}_{t,r}(\Delta^F_{\Sigma})(z, \cdot) \subset B_{\Sigma}(z, r). \tag{A.5}
\]

Remark that in [46, Theorems D.2.1, 4.2.8], authors consider smooth manifolds, but as their reasoning essentially relies on the energy estimate, the proof of which is local and depends only on the validity of the Green’s identities, which hold in our setting according to [29, Proposition 2.4], it will hold in our setting as well. From (A.4) and (A.5), we get
\[
\exp(-t \Delta^F_{\Sigma})(z, y) = \tilde{K}_{t,r}(\Delta^F_{\Sigma})(z, y) \quad \text{if} \quad \text{dist}(z, y) > r. \tag{A.6}
\]

From (A.2), for any \( r_0 > 0 \) fixed, there exists \( c' > 0 \) such that for any \( m \in \mathbb{N} \), there is \( C > 0 \) such that for any \( t \in ]0, 1[ \), \( r > r_0 \), \( a \in \mathbb{R} \), the following inequality holds (cf. [46, (4.2.12)])
\[
|a|^m |K_{t,r}(a)| \leq C \exp(-c'r^2/t). \tag{A.7}
\]

Thus, by (A.7), for \( t \in ]0, 1[ \), \( r > r_0 \), \( a \in \mathbb{R}^+ \), we have
\[
|a|^m |\tilde{K}_{t,r}(a)| \leq C \exp(-c'r^2/t). \tag{A.8}
\]

Now, by (A.8), there exists \( c' > 0 \) such that for any \( k \in \mathbb{N} \), there is \( C > 0 \) such that for any \( t \in ]0, 1[ \) and \( r > r_0 \), we have
\[
\| (\Delta^F_{\Sigma})^k \tilde{K}_{t,r}(\Delta^F_{\Sigma}) \|^0_{L^2(\Sigma)} \leq C \exp(-c'r^2/t), \tag{A.9}
\]
where \( \| \cdot \|^0_{L^2(\Sigma)} \) is the operator norm between the corresponding \( L^2 \)-spaces. By interior elliptic estimates, applied in an \( \epsilon/2 \)-neighborhood of \( x \), we deduce that for any \( l \in \mathbb{N} \), there is \( C' > 0 \), such that for any \( t \in ]0, 1[ \) and \( r > r_0 \), we have
\[
\left\| \nabla^l_x \tilde{K}_{t,r}(\Delta^F_{\Sigma})(x, \cdot) \right\|^2_{L^2(\Sigma)} \leq C' \exp(-c'r^2/t). \tag{A.10}
\]
We get (A.1) from (A.6) and (A.10) by taking \( r = \epsilon/2 \).

Now, by using (A.1), we compare the small-time expansions of the heat kernels on \( \Sigma \) and on some model manifolds. To do so, we prove Duhamel’s formula, which, to simplify the presentation, we formulate in a vicinity of a conical point.

We fix \( P \in \text{Con}(\Sigma) \). We denote \( \alpha = \angle(P) \) and consider the infinite cone \( C_\alpha \), (2.2), with the induced metric (2.1). We denote by \( \Delta_{C_\alpha} \) the Friedrichs extension of the Riemannian Laplacian on \( C_\alpha \). Let \( \epsilon > 0 \) be such that \( B_{\Sigma}(P, \epsilon) \) is isometric to \( C_{\alpha,\epsilon} := B_{C_\alpha}(0, \epsilon) \). From now on, we identify those neighborhoods implicitly.

For \( x, y \in C_\alpha \) and \( t > 0 \), we define
\[
F(x, y, t) := \exp(-t \Delta^F_{\Sigma})(x, y) - \exp(-t \Delta_{C_\alpha})(x, y),
\]
\[
G(x, y, s) := \int_{C_{\alpha,\epsilon}} \exp(-(t-s) \Delta^F_{\Sigma})(x, z) \cdot \exp(-s \Delta_{C_\alpha})(z, y) dv_{C_\alpha}(z). \tag{A.11}
\]
Then by the definition of the heat kernel, we have
\[
\lim_{s \to t-} G(x, y, s) = \exp(-t \Delta_{C_a})(x, y), \quad \lim_{s \to 0+} G(x, y, s) = \exp(-t \Delta_{C_a}^E)(x, y). \tag{A.12}
\]

From (A.12), we deduce that
\[
\exp(-t \Delta_{C_a})(x, y) - \exp(-t \Delta_{C_a}^E)(x, y) = \int_0^t \frac{dG(x, y, s)}{ds}ds. \tag{A.13}
\]

However, by the definition of the heat kernel, we have
\[
\frac{dG(x, y, s)}{ds} = \int_{C_{a,s}} \left( \Delta_{\Sigma,x} \exp(-(t - s) \Delta_{C_a}^E)(x, z) \right) \cdot \exp(-s \Delta_{C_a})(z, y)dv_{C_{a,s}}(z) - \int_{C_{a,s}} \left( \Delta_{C_{a,s}x} \exp(-s \Delta_{C_a})(z, y) \right)dv_{C_{a,s}}(z), \tag{A.14}
\]

where by $\Delta_{\Sigma,x}$ and $\Delta_{C_{a,s}x}$ we mean the Laplace operators acting on variables $x$ and $z$ respectively. By the symmetry of the heat kernel, we have
\[
\Delta_{\Sigma,x} \exp(-(t - s) \Delta_{C_a}^E)(x, z) = \Delta_{\Sigma,z} \exp(-(t - s) \Delta_{C_a}^E)(x, z). \tag{A.15}
\]

Now, since both operators $\Delta_{C_a}^E$, $\Delta_{C_a}$ come from Friedrichs extension of the Riemannian Laplacian, for $x$ and $y$ fixed, the functions $\exp(-(t - s) \Delta_{C_a}^E)(x, \cdot)$ and $\exp(-s \Delta_{C_a})(\cdot, y)$ are in the domain of Friedrichs extension. By Green’s identity, cf. [29, Proposition 2.4], applied to $C_{a,\epsilon}$, and (A.13), (A.14), (A.15), we deduce
\[
\exp(-t \Delta_{C_a})(x, y) - \exp(-t \Delta_{C_a}^E)(x, y) = \int_0^t \int_{\partial C_{a,s}} \left( \frac{\partial}{\partial n_z} \exp(-s \Delta_{C_a})(z, y) \right)dv_{\partial C_{a,s}}(z)ds - \int_0^t \int_{\partial C_{a,s}} \left( \frac{\partial}{\partial n_z} \exp(-(t - s) \Delta_{C_a}^E)(x, z) \right) \cdot \exp(-s \Delta_{C_a})(z, y)dv_{\partial C_{a,s}}(z)ds. \tag{A.16}
\]

The formulas of type (A.16) are also known as Duhamel’s formulas (cf. [4, Theorem 2.48]).

From (A.1), (A.16), applied for $y = x$ and integrated over $\epsilon$ from $\epsilon_0/2$ to $\epsilon_0$, and Cauchy inequality, we deduce that there are $c, C > 0$ such that for any $0 < t < 1$, we have
\[
\int_{C_{a,\epsilon/2}} \left| \exp(-t \Delta_{C_a})(x, x) - \exp(-t \Delta_{C_a}^E)(x, x) \right|dv_{C_{a,\epsilon/2}}(x) \leq C \exp \left( -\frac{c}{t} \right). \tag{A.17}
\]

Similarly, for $Q \in \text{Ang}(\Sigma)$, we denote $\beta = \angle(Q)$ and consider the infinite angle $A_{\beta}$ with the induced metric (2.1). We denote by $\Delta_{A_{\beta}}$ the Friedrichs extension of the Riemannian Laplacian with Neumann boundary conditions on $\partial A_{\beta}$. We fix $\epsilon > 0$ in such a way that $B_{\Sigma}(\epsilon, Q)$ is isometric to $B_{A_{\beta}}(\epsilon, 0)$. Similarly to (A.17), we deduce that there are $c, C > 0$ such that for any $0 < t < 1$:
such that for any \( 0 < t < 1 \), we have

\[
\left| \int_{C_{\alpha,\epsilon/2}} \left( \exp(-t\Delta_{A_\beta})(x, x) - \exp(-t\Delta^F_{\Sigma})(x, x) \right) dv_{A_\beta,\epsilon/2}(x) \right| \leq C \exp\left(-\frac{c}{t}\right).
\]  
(A.18)

Now, let \( R \in \partial \Sigma \) satisfy \( \text{dist}_\Sigma(R, \text{Con}(\Sigma) \cup \text{Ang}(\Sigma)) > \epsilon \). We consider a half plane \( \mathbb{H} = \{ (x, y) \in \mathbb{R}^2 : y \geq 0 \} \) and identify \( 0 \in \mathbb{H} \) with \( R \). Then \( B_\Sigma(\epsilon, R) \) is isometric to \( B_{\mathbb{H}}(\epsilon, 0) \). We denote by \( \Delta_{\mathbb{H}} \) the self-adjoint extension of the standard Laplacian with Neumann boundary conditions on \( \partial \mathbb{H} \). Similarly to (A.17), we deduce that there are \( c, C > 0 \) such that for any \( 0 < t < 1 \), we have

\[
\left| \int_{B_{\mathbb{H}}(0, \epsilon/2)} \left( \exp(-t\Delta_{\mathbb{H}})(x, x) - \exp(-t\Delta^F_{\Sigma})(x, x) \right) dv_{\mathbb{H}}(x) \right| \leq C \exp\left(-\frac{c}{t}\right).
\]  
(A.19)

Finally, let \( R \in \Sigma \) satisfy \( \text{dist}_\Sigma(R, \text{Con}(\Sigma) \cup \partial \Sigma) > \epsilon \). We consider the real plane \( \mathbb{R}^2 \) and identify \( 0 \in \mathbb{R}^2 \) with \( R \). Then \( B_\Sigma(\epsilon, R) \) is isometric to \( B_{\mathbb{R}^2}(\epsilon, 0) \). We denote by \( \Delta_{\mathbb{R}^2} \) the self-adjoint extension of the standard Laplacian. Similarly to (A.17), we deduce that there are \( c, C > 0 \) such that for any \( 0 < t < 1 \), we have

\[
\left| \int_{B_{\mathbb{R}^2}(0, \epsilon/2)} \left( \exp(-t\Delta_{\mathbb{R}^2})(x, x) - \exp(-t\Delta^F_{\Sigma})(x, x) \right) dv_{\mathbb{R}^2}(x) \right| \leq C \exp\left(-\frac{c}{t}\right).
\]  
(A.20)

To conclude, the estimates (A.17) - (A.20) show that to study the asymptotic expansion of \( \text{Tr}\left[ \exp(-t\Delta^F_{\Sigma}) \right] \), as \( t \to 0 \), it is enough to consider the analogous problem for a number of model cases: \( \mathbb{R}^2, \mathbb{H}, C_\alpha \) for \( \alpha > 0 \) and \( A_\beta \) for \( \beta > 0 \).

We will start with the simplest one, which is \( \mathbb{R}^2 \). Here, we have the following identity

\[
\exp(-t\Delta_{\mathbb{R}^2})(x, y) = \frac{1}{4\pi t} \exp\left(-\frac{|x - y|^2}{4t}\right).
\]  
(A.21)

By (A.1), (A.16) and (A.21), we deduce that there are \( c, C > 0 \) such that for any \( x \in \Sigma \) satisfying \( \text{dist}_\Sigma(x, \text{Con}(\Sigma) \cup \partial \Sigma) > \epsilon \), and any \( 0 < t < 1 \), we have

\[
\left| \exp(-t\Delta^F_{\Sigma})(x, x) - \frac{1}{4\pi t} \right| \leq C \exp\left(-\frac{c}{t}\right).
\]  
(A.22)

Now, on the half-plane \( \mathbb{H} \) and \( z_1, z_2 \in \mathbb{H} \), we clearly have

\[
\exp(-t\Delta_{\mathbb{H}})(z_1, z_2) = \exp(-t\Delta_{\mathbb{R}^2})(z_1, z_2) + \exp(-t\Delta_{\mathbb{R}^2})(z_1, \overline{z}_2).
\]  
(A.23)

From (A.21) and (A.23), after an easy calculation, we see that for any \( k \in \mathbb{N} \) and for any \( f \in C_0^\infty(\mathbb{H}) \), there is \( C > 0 \) such that for \( 0 < t < 1 \), we have

\[
\left| \int_{\mathbb{H}} f(x) \exp(-t\Delta_{\mathbb{H}})(x, x) dv_{\mathbb{H}}(x) - \frac{1}{4\pi t} \int_{\mathbb{H}} f(x) dv_{\mathbb{H}}(x) \right| \leq C t^k.
\]  
(A.24)

Let’s now study the cone \( C_\alpha, \alpha > 0 \). Carslaw in [14] (cf. also Kokotov [43, Sect. 3.1]), gave an explicit formula for the heat kernel on \( C_\alpha \). By using this formula, Kokotov proved
in [43, Proposition 1, Remark 1] that for any \( \epsilon > 0 \), there are \( c, C > 0 \) such that for any \( 0 < t < 1 \):

\[
\left| \int_{B_{C_\alpha}(0,\epsilon)} \exp(-t \Delta_{C_\alpha})(x, x) dv_{C_\alpha}(x) - \frac{\alpha \epsilon^2}{8 \pi t} - \frac{1}{12} \frac{4\pi^2 - \alpha^2}{2\pi \alpha} \right| \leq C \exp \left( -\frac{c}{t} \right). \tag{A.25}
\]

Let’s finally study the angle \( A_\beta, \beta > 0 \). Recall that in [59, Theorem 1], Berg-Srisatkunarajah have obtained the small-time asymptotic expansion of the heat trace associated to the Friedrichs extension of the Laplacian on the angle endowed with Dirichlet boundary conditions. In [48, Proposition 2.1 and (4.2)], Mazzeo-Rowlett used this result, along with (A.25), and an interpretation of a cone through gluing of two angles: with Dirichlet and Neumann boundary conditions, to show that for any and \( \epsilon > 0 \), there are \( c, C > 0 \) such that for any \( 0 < t < 1 \), we have

\[
\left| \int_{B_{A_\beta}(0,\epsilon)} \exp(-t \Delta_{A_\beta})(x, x) dv_{A_\beta}(x) - \frac{\beta \epsilon^2}{8 \pi t} - \frac{2\epsilon}{4 \sqrt{\pi} \sqrt{t}} - \frac{1}{12} \frac{\pi^2 - \beta^2}{2\pi \beta} \right| \leq C \exp \left( -\frac{c}{t} \right). \tag{A.26}
\]

By (A.17)-(A.20), (A.22), (A.24), (A.25) and (A.26), we easily conclude that Proposition 2.5 holds.

### A.2 Kronecker limit formula and the analytic torsion

The goal of this section is to recall some explicit formulas for the analytic torsion and the asymptotic expansion of the determinants of discrete Laplacians on the mesh graphs. The material of this section is classical, and we include it here only to make this paper self-contained.

We start by considering the continuous side of the story. We fix \( a, b \in \mathbb{N}^* \) and consider a torus \( T_{a,b} := \mathbb{R}^2/(a\mathbb{Z} \times b\mathbb{Z}) \). The standard Laplacian \( \Delta_{\mathbb{R}^2} := -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \) descends to Laplacian \( \Delta_{T_{a,b}} \), acting on the functions on \( T_{a,b} \). Classically, the eigenvalues of \( \Delta_{T_{a,b}} \) are given by

\[
\text{Spec}(\Delta_{T_{a,b}}) = \left\{ (2\pi)^2 \left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 : n, m \in \mathbb{Z} \right\}. \tag{A.27}
\]

So the associated spectral zeta function \( \zeta_{T_{a,b}}(s) \) is given by

\[
\zeta_{T_{a,b}}(s) = (2\pi)^{-2s} \sum_{(m,n) \neq (0,0)} \frac{1}{(n/a)^2 + (m/b)^2} s. \tag{A.28}
\]

We rewrite it in the following form

\[
\zeta_{T_{a,b}}(s) = (ab)^s E(z, s), \tag{A.29}
\]

where \( z := \sqrt{-T_{a,b}} \), and

\[
E(z, s) = (2\pi)^{-2s} \sum_{(m,n) \neq (0,0)} \frac{\text{Im}(z)^s}{|nz + m|^{2s}}. \tag{A.30}
\]

The function \( E(z, s) \) is the Eisenstein series multiplied by \((2\pi)^{-2s}\) and thus it admits a meromorphic continuation to \( \mathbb{C} \). By Kronecker limit formula, it has the following expansion...
near $s = 0$:

$$E(z, s) = -1 - s \log \left( \frac{a}{b} \cdot v(e^{-\frac{2\pi a}{b}})^4 \right) + o(s),$$

(A.31)

where $v : D(1) \to \mathbb{C}$ is the Dedekind eta-function, defined by

$$v(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

(A.32)

By (A.29) and (A.31), we obtain

$$\log \det' \Delta_{[0,a] \times [0,b]} = -\zeta'_{[0,a] \times [0,b]}(0) = \log(ab) + \log \left( \frac{a}{b} \cdot v(e^{-\frac{2\pi a}{b}})^4 \right).$$

(A.33)

Remark that (A.33) goes in line with Duplantier-David [25, (3.18)] and Corollary 1.3.

Now, consider a square $[0,a] \times [0,b] \subset \mathbb{C}$. We consider the Friedrichs extension of the standard Laplacian $\Delta_{[0,a] \times [0,b]}$ endowed with Neumann boundary conditions. The eigenvalues of $\Delta_{[0,a] \times [0,b]}$ are given by

$$\text{Spec} \left( \Delta_{[0,a] \times [0,b]} \right) = \left\{ \pi^2 \left( \left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 \right) : n, m \in \mathbb{N} \right\}.$$  

(A.34)

So the associated spectral zeta function $\zeta_{[0,a] \times [0,b]}(s)$ is related to the zeta function of torus $\mathbb{T}_{a,b}$ by

$$\zeta_{[0,a] \times [0,b]}(s) = 4^{s-1} \zeta_{\mathbb{T}_{a,b}}(s) + \frac{1}{2} \left( \left( \frac{a}{\pi} \right)^{2s} + \left( \frac{b}{\pi} \right)^{2s} \right) \zeta(2s),$$

(A.35)

where $\zeta(s)$ is the Riemann zeta function. By using identities

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{\log(2\pi)}{2},$$

(A.36)

and (A.31), (A.33), we get the following expression

$$\log \det' \Delta_{[0,a] \times [0,b]} = -\zeta'_{[0,a] \times [0,b]}(0) = \frac{3}{4} \log(ab) + \frac{1}{4} \log \left( \frac{a}{b} \cdot v(e^{-\frac{2\pi a}{b}})^4 \right) + \frac{3}{2} \log(2).$$

(A.37)

We also obtain that $\zeta_{[0,a] \times [0,b]}(0) = -\frac{3}{4}$, which is compatible with (1.12).

Now, let us study the discrete side of the problem. As in Sect. 4.1, construct a family of graphs $A_{an \times bn}$, $n \in \mathbb{N}^*$. We use the notation as in Theorem 1.1. Using explicit formula (A.37) Duplantier-David in [25, (4.7) and (4.23)] proved the following theorem.

**Theorem A.1** As $n \to \infty$, the following asymptotic expansion holds

$$\log(\det' \Delta_{A_{an \times bn}}) = \frac{4G}{\pi} ab \cdot n^2 + \log(\sqrt{2} - 1) \cdot (a + b) \cdot n + \frac{\log(n)}{2}$$

$$+ \log(\det' \Delta_{F}) - \frac{\log(2)}{4} + o(1).$$

(A.38)

We see, in particular that Theorem 1.1 is compatible with Theorem A.1.
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