ON LARGE SCALE PROPERTIES OF MANIFOLDS

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Abstract. We show that a space with a finite asymptotic dimension is embeddable in a non-positively curved manifold. Then we prove that if a uniformly contractible manifold $X$ is uniformly embeddable in $\mathbb{R}^n$ or non-positively curved $n$-dimensional simply connected manifold then $X \times \mathbb{R}^n$ is integrally hyperspherical. If a uniformly contractible manifold $X$ of bounded geometry is uniformly embeddable into a Hilbert space, then $X$ is stably integrally hyperspherical.

§1 Introduction

Gromov introduced several notions of largeness of Riemannian manifolds. For example, a manifold $X$ which is a universal cover of a closed aspherical manifold $M^n$ with the fundamental group $\Gamma = \pi_1(M^n)$, supplied with a $\Gamma$-invariant metric is large in a sense that it is uniformly contractible. We recall that $X$ is uniformly contractible if there is a function $S(r)$ such that every ball $B_r(x)$ of the radius $r$ centered at $x$ is contractible to a point in the ball $B_{S(r)}(x)$ for any $x \in X$. For the purpose of the Novikov conjecture and other related conjectures, it is important to show that universal covers of aspherical manifolds are also large in some cohomological sense. The weakest such property is called hypersphericity.

Definition [G-L]. An $n$-dimensional manifold $X$ is called hyperspherical if for every $\epsilon > 0$ there is a $\epsilon$-contracting proper map $f_\epsilon : X \to S^n$ of nonzero degree onto the standard unit $n$-sphere.

Here a continuous map $f : X \to S^n$ is proper if it has only one unbounded preimage. Gromov and Lawson proved the following [G-L]

Gromov-Lawson’s Theorem. An aspherical manifold with a hyperspherical universal cover cannot carry a metric of a positive scalar curvature.

The natural question appeared [G2]:

Problem 1. Is every uniformly contractible manifold hyperspherical?

Above we defined a notion of the rational hypersphericity. One can define integral hypersphericity by taking maps of degree one. Also the $n$-sphere can be replaced by $\mathbb{R}^n$. In that case we obtain the notion of a hypereuclidean manifold. In the integral case these two notions seems coincide. Also in the integral case the above question of Gromov has a negative answer [D-F-W]. Still there is no candidates for a rational counterexample and even new examples of Gromov [G4] leave a possibility for an affirmative answer.

The following definition is due to Gromov [G1]:

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Definition. A map \( f : X \to Y \) between metric spaces is called large scale uniform embedding if there are two functions \( \rho_1, \rho_2 : [0, \infty) \to [0, \infty) \) tending to infinity such that
\[
\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')).
\]
for all \( x, x' \in X \).

In Section 2 of this paper we prove the following:

Theorem 1. Suppose that a uniformly contractible manifold \( X \) admits a large scale uniform embedding into \( \mathbb{R}^n \). Then \( X \times \mathbb{R}^n \) is integrally hyperspherical.

Corollary. Let \( M \) be a closed aspherical manifold and assume that the fundamental group \( \Gamma = \pi_1(M) \) admits a coarsely uniform embedding into \( \mathbb{R}^n \) as a metric space with the word metric. Then \( M \) cannot carry a metric with a positive scalar curvature.

Some results of that type were already known to Gromov (see his remark (b), page 183 of [G3]). Also the Corollary follows from the theorems of Yu [Yu1],[Yu2].

The First Yu’s Theorem. If a finitely presented group \( \Gamma \) has a finite asymptotic dimension, in particular if it is coarsely uniformly embeddable in \( \mathbb{R}^n \), then the coarse Baum-Connes conjecture holds for \( \Gamma \).

The Second Yu’s Theorem. If a finitely presented group \( \Gamma \) can be uniformly in large scale sense embedded into a Hilbert space, then the coarse Baum-Connes conjecture holds for \( \Gamma \).

We note that The Second Yu’s theorem implies The First [H-R2]. Open Riemannian manifolds which obey the coarse Baum-Connes conjecture are of course large in some refined sense. This largeness is relevant to the hypersphericity or the hypereuclidianess but it is different. The best we can say [Ro1] that integrally (rationally) hypereuclidean manifolds satisfy the monicity part of the coarse (rational) Baum-Connes conjecture. We note that the monicity part of the coarse Baum-Connes is sufficient for the Gromov-Lawson conjecture about nonexistence of a positive scalar curvature metric on a closed aspherical Riemannian manifolds.

Both Yu’s theorems hold for all proper metric spaces with bounded geometry. We recall that a metric space \( X \) is called having a bounded geometry if for every \( \epsilon > 0 \) and every \( r > 0 \) there is \( c \) such that the \( \epsilon \)-capacity of every \( r \)-ball \( B_r(x) \) does not exceed \( c \). The latter means that a ball \( B_r(x) \) contains no more than \( c \) \( \epsilon \)-disjoint points.

Definition. An \( n \)-dimensional manifold \( X \) is called stably (integrally) hyperspherical if for any \( \epsilon > 0 \) there is \( m \) such that \( X \times \mathbb{R}^m \) admits an \( \epsilon \)-contracting map of nonzero degree (of degree one) onto the unit \( (n + m) \)-sphere \( S^{n+m} \).

We prove the following

Theorem 2. Suppose that \( X \) is a uniformly contractible manifold with bounded geometry and assume that \( X \) admits a coarsely uniform embedding into a Hilbert space. Then \( X \) is stably integrally hyperspherical.

It is unclear whether the stable hypersphericity implies the Gromov-Lawson conjecture. A positive answer to the following problem would give a simple argument for the implication: Stable Hypersphericity \( \Rightarrow \) Gromov-Lawson.
Problem 2. Does there exist a positive constant $c > 0$ such that the $K$-area of the unit $2n$-sphere is greater than $c$ for all $n$?

We recall the definition of $K$-area from [G3]. For an even dimensional Riemannian manifold $M$ the $K$-area is

$$K\text{-area}(M) = (\inf_X \{\|R(X)\|\})^{-1}$$

where the infimum is taken over all bundles $X$ over $M$ with some of the Chern numbers nonzero and with unitary connections, $R(X)$ is the curvature of $X$ equipped with the operator norm, i.e. $\|A\| = \sup_{\|x\|=1} \|Ax - x\|.$

The other way to establish the implication: Stable Hypersphericity $\Rightarrow$ Gromov-Lawson would be extending the notion of $K$-area on loop spaces and showing that the $K$-area of $\Omega^\infty \Sigma^\infty S^n$ is greater than zero.

One of the objectives of this paper is to give elementary proofs of Yu’s theorems with replacing the coarse Baum-Connes conjecture by the hypersphericity. It is accomplished with the first Yu’s Theorem. In Section 3 we prove The Embedding Theorem for asymptotic dimension. Then a slightly more general version of Theorem 1 completes the proof. The Theorem 2 can be considered as a version of the second Yu’s Theorem. Nevertheless the same set of corollaries would follow from it in the case of positive answer to Problem 2. Otherwise to obtain a proper analog of the second Yu’s theorem on this way one should do more elaborate differential geometry in the argument in order to avoid stabilizing with $R^m$.

§2 Proofs of Theorems 1 and 2

First we note that in both theorems it suffices to consider the case when $X$ is isometrically embedded in $\mathbb{R}^n$ (or $l_2$). Since a contractible $k$-manifold $X$ is homeomorphic to $\mathbb{R}^{k+1}$ after crossing with $\mathbb{R}$, without loss of generality we may assume that $X$ is homeomorphic to $\mathbb{R}^k$. Fix a homeomorphism $h: \mathbb{R}^k \to X$. We denote by $S^{k-1}_r$ the standard sphere in $\mathbb{R}^k$ of radius $r$ with the center at 0. Note that the family $h(S^{k-1}_r)$ tends to infinity in $X$ as $r$ approaches infinity. Denote by $N_\lambda(A)$ the $\lambda$-neighborhood of $A$ in an ambient space $W$. By $B^m_r$ we denote the standard $r$-ball in $\mathbb{R}^m$.

Lemma 1. Let $X$ be a uniformly contractible manifold with bounded geometry, $X$ is homeomorphic to $\mathbb{R}^k$ and $X$ is isometrically embedded in a metric space $W$. Then for any $\lambda > 0$ there is $r > 0$ such that the neighborhood $N_\lambda(h(S^{k-1}_r))$ in $W$ admits a retraction onto $h(S^{k-1}_r)$.

Proof. Let $S: \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone contractibility function on $X$. Clearly, $S(t) \geq t$. Since $X$ is a space of bounded geometry, there is a uniformly bounded cover $U$ on $X$ of finite multiplicity $m$ and with the Lebesgue number $> 4\lambda$ (see [H-R1] or [Dr]). If we squeeze every element $U \in U$ by $\lambda$ we get a cover $U'$ with the Lebesgue number $3\lambda$. For every $U \in U'$ we consider the open $\lambda$-neighborhood $ON_\lambda(U)$ in $W$. Then the cover $\tilde{U} = \{ON_\lambda(U) \mid U \in U'\}$ has the multiplicity $\leq m$.

Let $d$ be an upper bound for the diameters of elements of the cover $\tilde{U}$. Define $T(t) = 2S(4t)$. Let $T^l$ mean $l$ times iteration of $T$. We take $r$ such that $d(h(0), h(S^{k-1}_r)) \geq$
$T^{m+k}(d)$. Denote by $\bar{U}$ the restriction of $\bar{U}$ over $h(S_r^{k-1})$. Note that $\bar{U}$ covers the neighborhood $N_\lambda(h(S_r^{k-1})) = N$. Let $\nu : N \to N(\bar{U})$ be a projection to the nerve of the cover $\bar{U}$. We define a map $\phi : N(\bar{U}) \to X \setminus \{h(0)\}$ such that the restriction $\phi \circ \nu |_{h(S_r^{k-1})}$ is homotopic to the identity map $id_{h(S_r^{k-1})}$. Then by the homotopy extension theorem there is an extension $\beta : N \to X \setminus \{h(0)\}$ of the identity map $id_{h(S_r^{k-1})}$. Let $\gamma : \mathbb{R}^k \setminus \{0\} \to S_r^{k-1}$ be a retraction. We define a retraction $\alpha : N \to h(S_r^{k-1})$ as $h \circ \gamma \circ h^{-1} \beta$.

We define $\phi$ on the $l$-skeleton of $N(\bar{U})$ by induction on $l$ in such a way that the diameter of the image $\phi(\sigma^l)$ of every $l$-simplex $\sigma^l$ does not exceed $T^l(d)$. To do that, we choose points $x_U \in U \cap h(S_r^{k-1})$ for every $U \in \bar{U}$ and define $\phi(v_U) = x_U$ for every vertex $v_U$ in $N(\bar{U})$ corresponding to an open set $U \in \bar{U}$. For every edge $[v_U, v_U']$ we define $\phi$ on it in such a way that $diam(\phi([v_U, v_U'])) \leq S(d(x_U, x_U')) \leq S(2d) \leq T(d)$. Assume that $\phi$ is defined on the l-skeleton with the property that $diam(\phi(\sigma)) \leq T^l(d)$ for all simplices $\sigma$. If the boundary is connected, then for arbitrary $l + 1$-dimensional simplex $\Delta$ the image of the boundary $\phi(\partial \Delta)$ has the diameter $\leq 4T^l(d)$. Then we can extend $\phi$ over $\Delta$ with the diameter $\phi(\Delta) \leq 2S(4T^l(d)) = T(T^l(d)) = T^{l+1}(d)$. A map $\phi$ constructed on this way has the property that $d(x, \phi(x)) \leq 2(d + T^m(d)) \leq T^{m+1}(d)$.

Consider a small (with mesh smaller than $\frac{1}{4}T^{m+1}(d)$) triangulation $\tau$ on $h(S_r^{k-1})$ and the cellular complex $h(S_r^{k-1}) \times I$ defined by that structure. Using induction one can extend the map $id_{h(S_r^{k-1}) \times \{0\}} \bigwedge_{i \in \{0\}} \phi_{v_{h(S_r^{k-1}) \times \{1\}}} : h(S_r^{k-1}) \times \{0, 1\} \to X \setminus \{h(0)\}$ to a map $H : h(S_r^{k-1}) \times I \to X \setminus \{0\}$ with the diameter of the image of every $i$-dimensional cell less than $T^{m+i+1}(d)$. Then $H(h(S_r^{k-1})) \subset N_{T^{m+i+1}(d)}(h(S_r^{k-1})) \subset X \setminus \{h(0)\}$ by the choice of $r$. Thus, $H$ is a required homotopy. □

**Lemma 2.** Let $M$ be a closed smooth $l$-dimensional submanifold in the euclidean space $\mathbb{R}^n$ with a trivial tubular $\epsilon$-neighborhood $N_\epsilon(M)$. Then for any number $d > \epsilon$ there is a number $\mu$ such that the diagonal embedding $j = (1_M \Delta \mu 1_M) : M \to \mathbb{R}^n \times \mathbb{R}^n$ has a regular neighborhood $N$, $\nu : N \to j(M)$ such that:

1. $N$ admits a contracting map $h : N \to B_{d}^{2n-l}$ which is a homeomorphism on every fiber $\nu^{-1}(y)$;
2. $pr_1(N) \subset N_{2d}(M)$ where $pr_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the first factor.

**Proof.** Let $q : N_\epsilon(M) \to B_{d}^{n-1}$ be a trivialization of the tubular neighborhood $N_\epsilon(M)$. Let $\lambda$ be its Lipschitz constant. Take $\mu = \frac{\lambda \delta}{\epsilon}$. Also by $\mu$ we denote the map $\mu : \mathbb{R}^n \to \mathbb{R}^n$ which is a multiplication of vectors by $\mu$. We extend the embedding $j : M \to \mathbb{R}^n \times \mathbb{R}^n$ to the map $\tilde{j} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined as $\tilde{j}(x) = (x, \mu x)$. Thus, the map $\tilde{j}$ is a homothetic transformation of $\mathbb{R}^n$ to the space $L = \{(x, \mu x) | x \in \mathbb{R}^n\}$ with the homothety coefficient equal to $\sqrt{1 + \mu^2}$. Therefore $j(M)$ admits a trivial tubular $\delta$-neighborhood $N_\delta^1$ in $L$ with $\delta = \epsilon \sqrt{1 + \mu^2}$. We define $N$ as the product $N_\delta^{1} \times B_{d}^{n}$ isometrically realized in $L \times L^\perp$ where $L^\perp$ is the orthogonal complement of $L$ in $\mathbb{R}^n \times \mathbb{R}^n$. We consider the map $h_1 = \frac{d}{\epsilon}q \circ \tilde{j}^{-1} |_{N_\delta^{1}} : N_\delta^{1} \to B_{d}^{n-1}$. Note that $\frac{d}{\epsilon}(1/\sqrt{1 + \mu^2})$ is a Lipschitz constant for $h_1$. Therefore $1 = \frac{d}{\epsilon} \frac{\lambda}{\mu}$ is also a Lipschitz constant for $h_1$. Hence the map $h = h_1 \times id_{B_{d}^{n}} : N \to \mathbb{R}^n \times \mathbb{R}^n$. □
$B^n_d - l \times B^n_d$ is a short map. Let $p : B^n_d - l \times B^n_d \to B^n_d$ be the natural radial projection. We define $h = p \circ \tilde{h}$. Clearly, the condition 1) holds. Now let $y \in N$ be an arbitrary point, show that $pr_1(y) \in N_{2d}(M)$. First $y$ can be presented as $j(x) + w_1 + w_2$ where $x \in M$, $w_1 \in B_{\tilde{\delta}} \cong \nu_{\tilde{l} - 1}(j(x))$, $\nu_1 : N'_\delta \to j(M)$ is the natural projection, and $w_2 \in B^d_d \subset L^-$. Then $pr_1(j(x) + w_1 + w_2) = pr_1(j(x)) = pr_1(w_1) + pr_1(w_2) = x + u + pr_1 w_2$ where $w_1 = j(\tilde{u}) = (u, \mu u)$ and $u$ is a normal vector to $M$ at point $x$ of the length $\leq \epsilon$. Then $\|pr_1(y) - x\| = \|u + pr_1(w_2)\| \leq \|u\| + \|w_2\| \leq \epsilon + d \leq 2d$. Hence $pr_1(y) \in N_{2d}(M)$. □

Proof of Theorem 2. Let $\dim X = k$. For every $d > 0$ we construct a submanifold $V \subset X \times \mathbb{R}^n$ and a short map of degree one $f : (V, \partial V) \to (B^{k+n}_d, \partial B^{k+n}_d)$. Clearly, this would imply the integral hypersphericity of $X \times \mathbb{R}^n$. By Lemma 1 for large enough $r$ there is a retraction of the 2$d$-neighborhood $N_{2d}(h(S'^{k-1}_r))$ onto a curved $(k - 1)$-sphere $h(S'^{k-1}_r)$. Let $\alpha$ be the retraction. We may require that $\alpha$ as well as $h$ are smooth maps. We assume that $\mathbb{R}^n \subset S^n$ is compactified to the $n$-sphere and assume that $f : S^n \to B^k_r$ is a smooth extension of $h^{-1} \circ \alpha$. Let $x_0$ be a regular value of $f$ and assume that $f(\infty) \neq x_0$. Then the fiber $M = f^{-1}(x_0)$ is a closed $(n - k)$-dimensional manifold which admits a trivial tubular neighborhood $N_\epsilon(M)$ for some $\epsilon > 0$. We may assume that $\epsilon < d$. Then we apply Lemma 2 to obtain an embedding $j : M \to \mathbb{R}^n \times \mathbb{R}^n$ with a regular neighborhood $N$ with the properties (1)-(2) of the lemma. In view of condition (2) of Lemma 2 for large enough $R$ the neighborhood $N$ is contained in $N_{2d}(N) \times B^n_R$. Hence the boundary $\partial(h(B^k_r) \times B^n_R) = h(S'^{k-1}_r) \times B^n_R \cup h(B^k_r) \times \partial B^n_R$ does not intersect $N$. Note that the manifold $M$ is linked in $\mathbb{R}^n$ with $h(S'^{k-1}_r)$ with the linking number one. Also $M$ is linked with $\partial(h(B^k_r) \times B^n_R)$ with the linking number one. Since $j(M)$ is homotopic to $M$ inside $N_{2d}(M) \times \text{Int}(B^k_R)$, it follows that $j(M)$ is linked with $\partial(h(B^k_r) \times B^n_R)$ with the linking number one. Consider the intersection $V = X \times \mathbb{R}^n \cap N$. We may assume that $V$ is a manifold with boundary. Since $\partial V$ is homologous to $\partial(h(B^k_r) \times B^n_R)$ in $(\mathbb{R}^n \times \mathbb{R}^n) \setminus N$, the linking number of $\partial V$ and $j(M)$ is one. Since the intersection number of $V$ and $j(M)$ is one, the restriction $h \mid_V : (V, \partial V) \to (B^{n+k}_d, \partial B^{n+k}_d)$ of a short map $h : N \to B^{n+k}_d$ from Lemma 2 has degree one. □

Proof of Theorem 1. Let $\dim X = k$. Let $K$ be a triangulation on $X$ with diameters of simplices $\leq 1$. We change the original embedding of $X$ into $l_2$ to the piecewise linear which is the same for vertices. We present $X$ as a union of finite subcomplexes: $X = \cup K_i$, $K_i \subset K_{i+1}$. Then every complex $K_i$ lies in a finite dimensional euclidean space $\mathbb{R}^n_i \subset l_2$. Let $d$ be given. As in the proof of Theorem 1 we construct a submanifold $V \subset X \times \mathbb{R}^{n(d)}$ and a short map of degree one $f : (V, \partial V) \to (B^{k+n(d)}_d, \partial B^{k+n(d)}_d)$. By Lemma 1 for large enough $r$ there is a retraction $\alpha$ of the 2$d$-neighborhood $N_{2d}(h(S'^{k-1}_r))$ in $l_2$ onto $h(S'^{k-1}_r)$. There is $i$ such that $h(S'^{k-1}_r) \subset K_i$. Then we can work in $\mathbb{R}^{n_i}$ as in the prove of Theorem 1 and we get $n(d) = n_i$. □

REMARK. If we replace in the above argument the Hilbert space $l_2$ by the Banach space $l_\infty$ we will obtain the following condition on $X$: For every $\epsilon > 0$ there exist $m$ and
a submanifold with boundary \( W \subset X \times \mathbb{R}^m \) which admits an \( \varepsilon \)-contracting map onto the \( l_\infty \) unit ball \( f : (W, \partial W) \to (B^{n+m}_\infty, \partial B^{n+m}_\infty) \) of degree one. It is unclear if it would be possible to get a stable hypersphericity of \( X \) from this (see [G4], page 8).

§3 Embedding Theorem for asymptotic dimension

We recall that the asymptotic dimension \( asdim(X) \) of a metric space \( X \) is a minimal number \( n \), if exists, such that for any \( d > 0 \) there is a uniformly bounded cover \( \mathcal{U} \) of \( X \) which consists of \( n + 1 \) \( d \)-disjoint families \( \mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n \). A family of sets \( \mathcal{V} \) is \( d \)-disjoint if \( d(V, V') = \inf \{d(x, x') \mid x \in V, x' \in V' \} > d \). By \( N_r(A) \) we denote the \( r \)-neighborhood if \( r > 0 \) and the set \( A \setminus N_r(X \setminus A) \) if \( r \leq 0 \). Let \( \text{mesh} \mathcal{U} \) denote an upper bound for diameters of elements of a cover \( \mathcal{U} \) and let \( L(\mathcal{U}) \) denote the Lebesgue number of \( \mathcal{U} \).

**Proposition 1.** If a metric space \( X \) with a base point \( x_0 \) has \( asdim(X) \leq n \) then there is a sequence of uniformly bounded open covers \( \mathcal{U}_k \) of \( X \), each cover \( \mathcal{U}_k \) splits into a collection of \( n + 1 \) \( d_k \)-disjoint families \( \mathcal{U}_k = \mathcal{U}_k^0 \cup \cdots \cup \mathcal{U}_k^n \) such that

1. \( L(\mathcal{U}_k) > d_k \) and \( N_{-d_k}(U) \neq \emptyset \) for all \( U \in \mathcal{U}_k \),
2. \( d_k > 2^k m_{k-1} \) where \( m_{k-1} = \text{mesh}(\mathcal{U}_{k-1}) \),
3. For any \( m \in \mathbb{N} \) and every \( i \in \{0, \ldots, n\} \) there is \( l \) and \( U \in \mathcal{U}_i^l \) such that \( N_{-d_k}(U) \supset B_m(x_0) \),
4. For every \( i \in \{0, \ldots, n\} \) and \( k < l \) for any \( U \in \mathcal{U}_k^i \) and \( V \in \mathcal{U}_l^i \) with \( U \nsubseteq V \) there is the inequality \( d(U, V) \geq 4 \).

**Proof.** We construct it by induction on \( k \). We start with a cover \( \mathcal{U}_0 \) with \( d_0 > 2 \) and enumerate the partition \( \mathcal{U}_0 = \mathcal{U}_0^0 \cup \cdots \cup \mathcal{U}_0^n \) in such a way that \( d(x_0, X \setminus U) > d_0 \) for some \( U \in \mathcal{U}_0^0 \). We formulate the condition (3) in a concrete fashion:

(3) For \( l = m(n + 1) + i \) where \( i = l \mod n + 1 \) there is \( U \in \mathcal{U}_l^i \) such that \( N_{-d_k}(U) \supset B_m(x_0) \).

Assume that the family \( \{\mathcal{U}_k\} \) is constructed for all \( k \leq l \) such that the conditions (1), (2), (3)', (4) hold. We define \( d_{l+1} = 2^{l+2} m_l \) and consider a uniformly bounded cover \( \mathcal{U}_{l+1} \) with the Lebesgue number \( L(\mathcal{U}_{l+1}) > 2 d_{l+1} \) and with splitting in \( n + 1 \) \( d_{l+1} \)-disjoint families \( \mathcal{U}_{l+1} = \mathcal{U}_{l+1}^0 \cup \cdots \cup \mathcal{U}_{l+1}^n \). Then we may assume that for all elements \( U \in \mathcal{U}_{l+1} \) we have \( N_{-2d_{l+1}}(U) \neq \emptyset \). We just can delete all elements \( U \) from the cover \( \mathcal{U}_{l+1} \) which do not satisfy that property, and since \( L(\mathcal{U}_{l+1}) > 2 d_{l+1} \), still we will have a cover of \( X \). We enumerate families \( \mathcal{U}_{l+1}^0 \cup \cdots \cup \mathcal{U}_{l+1}^n \) in such a way that \( d(x_0, X \setminus U) > 2 d_{l+1} \) for some \( U \in \mathcal{U}_{l+1}^i \) for \( i = l + 1 \mod n + 1 \). For every \( U \in \mathcal{U}_{l+1}^i \) we define \( \tilde{U} = U \setminus \bigcup_{V \subset U; V \in \mathcal{U}_{l+1}^i \cup \cdots \cup \mathcal{U}_{l+1}^n} N_4(V) \).

We define \( \mathcal{U}_{l+1}^i = \{ \tilde{U} \mid U \in \mathcal{U}_{l+1}^i \} \). Next we check all the properties.

1. Take a point \( x \in X \), then there exists \( U \in \mathcal{U}_{l+1}^i \) such that \( B_{2d_{l+1}}(x) \subset U \).
Then \( B_{d_{l+1}}(x) \subset U \setminus N_{d_{l+1}}(X \setminus U) \subset U \setminus N_{m_{l+4}}(X \setminus U) \subset \tilde{U} \). Note that \( N_{-d_{l+1}}(\tilde{U}) = \tilde{U} \setminus N_{d_{l+1}}(X \setminus U) \supset U \setminus N_{d_{l+1}+m_{l+4}}(X \setminus U) \supset N_{2d_{l+1}}(U) = N_{-2d_{l+1}}(U) \neq \emptyset \).

2. This condition holds by the definition.

(3)'. This condition holds by the construction.
Theorem 3. Assume that $X$ is a metric space of bounded geometry with $\text{asdim}(X) \leq n$. Then $X$ can be uniformly embedded in a coarse sense in the product of $n+1$ locally finite trees.

Proof. Let $\mathcal{U}_k$ be a sequence of covers of $X$ from Proposition 1. Let $\mathcal{V}_i = \bigcup_k \mathcal{U}_k^i$. We define a map $\psi : \mathcal{V}_i \to \mathcal{V}_i$ by the following rule: $\psi(U)$ is the smallest $V \in \mathcal{V}_i$ with respect to the inclusion such that $V \neq U$ and $U \subset V$. The conditions 3) and 4) of Proposition 1 and disjointness of $\mathcal{U}_k^i$ for all $k$ imply that $\psi$ is well-defined. For every $i \in \{0, \ldots, n\}$ construct an oriented graph $T^i$ as follows. For every $U \in \mathcal{V}_i$ we consider an interval $I_U$ isometric with $[0, 2^k]$ and oriented from $2^k$ to 0. For every $V \in \psi^{-1}(U)$ we attach $I_V$ by the 0-end to an integer point $a_V = \min\{2^k, \sup\phi_U(V)\frac{2^k}{d_k}\}$ of $I_U$ where $\phi_U(x) = d(x, X \setminus U)$ and $[a]$ means the integer part of $a$.

We show that the graph $T^i$ is a locally finite tree. For every $U, V \in \mathcal{V}_i$ by the property 3) there exists $W \in \mathcal{V}_i$ such that $U \cup V \subset W$. This implies the connectedness of $T^i$. Since the orientation on $T^i$ defines a flow, i.e. every vertex is an initial point only for one arrow, it follows that every cycle in $T^i$ must be oriented. Oriented cycles in $T^i$ do not exist due to the inclusion nature of the orientation. Thus, $T^i$ is a tree. Note that for every nonzero vertex in $I_U$ only finitely many intervals are attached to it. It implies that if a vertex $v$ in $T^i$ is of infinite order, then $v$ must be the 0-vertex for all intervals involved. So a vertex $v$ of infinite order defines an infinite sequence $U_1 \subset U_2 \subset \cdots \subset U_m \subset \psi(U_{j+1}) = U_{j+1}$ and $a_{U,j} = 0$ for all $j$. Let $U_j \in \mathcal{U}_k^i$, then $2^{k_j} d_k d_j^{-1} d(x, X \setminus U_{j+1}) < 1$ for all $x \in U_j$. Hence $d(x, X \setminus U_{j+1}) < d_k^{-1}$ for all $x \in U_j$, i.e. $U_j \cap N_{-d_k(U_{j+1})} = \emptyset$. By the property 3) from Proposition 1 there is $U \in \mathcal{U}_k^i$ with $N_{-d_k}(U) \supset U_1$. Then the condition 4) implies that $U = U_j$ for some $j$ whence $l = k_j$. Therefore $U_1 \cap N_{-d_k}(U) = \emptyset$. Contradiction.

Next we define a map $p_i : X \to T^i$. By the condition 3) every point $x \in X$ is covered by some element $U \in \mathcal{U}_k^i$ for some $k$. Let $U$ containing $x$ be taken with the smallest $k$. We define $p_i(x) \in I_U$ as follows. Consider a map

$$\xi : \tilde{N}_{-d_k}(U) \cup \partial U \cup \bigcup_{V \in \psi^{-1}(U)} \partial V \to I_U = [0, 2^k]$$

defined as $\xi(\tilde{N}_{-d_k}(U)) = 2^k$, $\xi(\partial U) = 0$ and $\xi(\partial V) = a_V$. Show that $\xi$ is a short map. Let $y \in \partial V$ and $z \in \partial V'$, $V, V' \in \psi^{-1}(U)$ and $V \neq V'$. Then by the condition 4) $d(y, z) \geq 4$. Note that

$$|a_V - a_{V'}| \leq \frac{2^k}{d_k} d(y, X \setminus U) + m_{k-1} - d(z, X \setminus U) + 2 \leq \frac{2^k}{d_k} d(y, z) + \frac{2^k}{d_k} m_{k-1} + 2 \leq \frac{1}{4} d(y, z) + 3 \leq d(y, z).$$
Here we applied the condition 2). Now let \( y \in \partial V \) and \( z \in \partial U \). Then

\[ |\xi(y) - \xi(z)| = a_V \leq 2^k \left( \frac{d_k}{d_k}(d(y, X \setminus U) + m_{k-1}) \right) \leq \frac{1}{2m_{k-1}} d(y, z) + \frac{1}{2} \leq d(y, z) \]

provided \( d(y, z) > 1 \). Otherwise \( a_v < 1 \) and hence \( a_v = 0 \). The case when \( y \in \partial V \) and \( z \in \partial U \) is obvious. If \( y \in \partial V \) and \( z \in \partial \bar{V}_{d_k}(U) \) then

\[ |\xi(z) - \xi(y)| \leq 2^k - \frac{2^k}{d_k} (d(y, X \setminus U) - m_{k-1}) + 1 \leq \frac{2^k}{d_k} (d_k - d(y, X \setminus U)) + 2 \leq \frac{2^k}{d_k} (d(z, X \setminus U) - d(y, X \setminus U)) + 2 \leq d(y, z). \]

There exists a short extension \( \tilde{\xi}_U \to I_U \) of the map \( \xi \). We define \( p_i(x) = \tilde{\xi}_U(x) \). It easy to see that the map \( p_i \) is short.

Show that the diagonal product \( p = \Delta p_i : X \rightarrow \prod T^i \) is a uniform embedding. We consider \( l_1 \)-metric on \( \prod T^i \). Since each \( p_i \) is short the map \( p \) is Lipschitz. Clearly, \( \text{Dist}(p(x), p(x')) \geq \rho(d(x, x')) \) for the function \( \rho(t) = \inf\{\text{Dist}(p(x), p(x')) | d(x, x') \geq t\} \). Assume that \( \rho \) is bounded from above. Then there is a sequence of pairs \((x_k, x_k')\) of points with \( d(x_k, x_k') > m_k \) and with \( \text{Dist}(p(x_k), p(x_k')) \leq b \) for all \( k \). For any \( k \) there is an element \( U \in \mathcal{U}_k^i \) such that \( d(x_k, X \setminus U) > d_k \). Since \( d(x_k, x_k') > m_k \), it follows that \( x_k' \notin U \). Note that \( \tilde{\xi}_U(x_k) = 2^k \) and hence the distance between \( p_i(x_k) \) and \( p_i(x_k') \) in the graph \( T^i \) is greater than \( 2^k \). Therefore, \( \text{Dist}(p(x_k), p(x_k')) \geq 2^k \). Therefore \( \rho \) tends to infinity and \( p \) is a uniform embedding. \( \square \)

**Corollary.** Every metric space \( X \) with \( \text{asdim}(X) \leq n \) is coarsely isomorphic to a space \( Y \) of a linear type.

First we recall that according to Higson an asymptotically finite dimensional space \( Y \) has a linear type if in the above definition of \( \text{asdim} \) there exists a constant \( C \) such that for any \( d \) the number \( Cd \) is an upper bound on the size of the cover \( \mathcal{U} \).

**Proof.** Note that a tree and a finite product of trees are of linear type. Hence every subspace of a finite product of trees is of linear type. By Theorem 3 every asymptotically finite dimensional space \( X \) is coarsely isomorphic to a subset of a finite product of trees.

**Lemma 3.** Every locally finite tree is uniformly embeddable in a complete simply connected 2-dimensional manifold \( K(M) \) with negative curvature: \(-k_1 \leq K(M) \leq -k_2 < 0\).

**Proof.** Embed the tree into a plane. For every vertex \( v \) which is not the end point we consider the longest right-hand rule and left-hand rule paths from \( v \). If one of the paths is finite, then we end up in an end point of the tree. We attach a hyperbolic half-plane \( H_v \) by isometry between the union of these paths and an interval in \( \partial H_v \). For every end point vertex \( v \) we will get two half-planes \( H_{v_1} \) and \( H_{v_2} \) with corresponding intervals ended in \( v \). We attach them by isometries of corresponding rays in \( \partial H_{v_1} \). As the result we will get a plane with a piecewise hyperbolic metric on it with possible singularities only at vertices of the tree. We can approximate this metric by a smooth metric of strictly negative curvature.
Theorem 4. Every metric space \( X \) with asdim\((X) \leq n \) is uniformly embeddable in a \((2n + 2)\)-dimensional non-positively curved manifold \( W \) with asdim\((W) = 2n + 2 \).

**Proof.** We apply Theorem 3 and Lemma 3 to obtain an embedding of \( X \) into \( W = \prod M_i \) where each \( M_i \) is 2-dimensional negatively curved manifold. Then \( W \) is non-positively curved. By a theorem of Gromov [G1] the asymptotic dimension of \( M_i \) equals 2. Hence the asymptotic dimension of \( W \) is \( 2(n + 1) \) [D-J] 

**Problem 3.** Can \( 2n + 2 \) in the above theorem be improved to \( 2n + 1 \)?

Perhaps it is natural to ask whether every asymptotically \( n \)-dimensional metric space of bounded geometry is embeddable into \( (H^2)^{n+1} \), the product of \( n + 1 \) copies of the hyperbolic plane.

§4 On the First Theorem of Yu

The following lemma is a generalization of Lemma 2.

**Lemma 4.** Let \( M \) be a closed \( l \)-dimensional manifold smoothly embedded in a non-positively curved \( n \)-manifold \( W \) with a trivial tubular \( \epsilon \)-neighborhood \( N_\epsilon(M) \). Then for any \( d > 0 \) there is an embedding \( \gamma : N_\epsilon(M) \to \mathbb{R}^n \) such that the diagonal embedding \( j = (1 \Delta \gamma) : M \to W \times \mathbb{R}^n \) has a regular neighborhood \( N \) with the projection \( \nu : N \to j(M) \) such that:

1. there is a short map \( h : N \to B_d^{2n-l} \) such that the restriction \( h|_{\nu^{-1}(x)} : \nu^{-1}(x) \to B_d^{2n-l} \) is a homeomorphism for every \( x \in j(M) \);
2. \( pr_1(N) \subset N_{2d}(M) \) where \( pr_1 : W \times \mathbb{R}^n \to W \) is the projection onto the first factor.

This Lemma together with the above Embedding Theorem allows to prove the following:

**Theorem 5.** If a uniformly contractible manifold \( X \) has a finite asymptotic dimension, then there is \( n \) such that \( X \times \mathbb{R}^n \) is integrally hyperspherical.

The proof is exactly the same as in Theorem 1.

**Proof of Lemma 4.** Let \( TW \) be the tangent bundle of a non-positively curved complete simply connected \( n \)-dimensional Riemannian manifold \( W \). For every \( x \in W \) there is the exponential map \( e_x : \mathbb{R}^n \to W \) which takes a vector \( v \) to a point \( y = e_x(v) \) on the geodesic ray in the direction of \( v \) with \( d_W(x,y) = \|v\| \). Note that \( e_x \) is a homeomorphism for every \( x \). The visual sphere at infinity together with the exponential maps define a trivialization of \( TW \). A tubular \( \epsilon \)-neighborhood of a smooth submanifold \( M^l \subset W^n \) is a neighborhood \( N_\epsilon(M) \) with the projection \( p : N_\epsilon(M) \to M \) such that \( p^{-1}(x) = e_x(B_{\epsilon}^{n-l}) \) where \( B_{\epsilon}^{n-l} \subset N_\epsilon(M^l) \subset T_xW \) is an euclidean \( \epsilon \)-ball lying in the normal direction.

Using the embedding \( e_{\frac{\epsilon}{2}a}^{-1} |_{N_\epsilon(M)} : N_\epsilon(M) \to \mathbb{R}^n \) we define \( \gamma : N_\epsilon(M) \to \mathbb{R}^n \) to be a \( \mu \)-expanding map where \( \mu \) is a large number defined as follows. By the definition of a tubular neighborhood there is a lift \( \epsilon^{-1} : N_\epsilon(M) \to M \times B_{\epsilon}^{n-l} \subset TW \). Let \( q : \)
\( e^{-1}(N_e(M)) \to B^{n-l}_e \) be a trivialization. Let \( \lambda \) be a Lipschitz constant for the map \( q e^{-1} \). Note that \( e^{-1}(y) = (x, e^{-1}_x(y)) \in T_y W \) for \( y \in p^{-1}(x) \). Let \( N_{2d}(M) \) be a closed \( 2d \)-neighborhood of \( M \). The correspondence \( x \mapsto e^{-1}_x \mid_{N_{2d}(M)} \) defines a map \( \Phi : M \to C(N_{2d}(M), \mathbb{R}^n) \), where the functional space \( C(N_{2d}(M), \mathbb{R}^n) \) is supplied with the \( \sup \) norm \( \|f\| = \sup_{z \in N_{2d}(M)} \|f(z)\| \). Let \( \lambda > 1 \) be a Lipschitz constant of \( \Phi \). We take 
\[ \mu = \lambda \lambda d/\epsilon. \]
We define
\[ N = \bigcup_{x \in M} B_{2d}(x) \times \gamma(B_e(x)). \]
Here we use \( B_r \) to denote a ball of radius \( r \) in \( W \) and \( B_r \) for a ball in \( \mathbb{R}^n \). We define short maps \( h_1 : N \to B^{n}_{d} \) and \( h_2 : N \to B^{n-l}_{d} \) such that the sum \( (h_1 + h_2) : N \to B^{n}_d \times B^{n-l}_d \) satisfies the condition (1). We define \( h_1 \mid_{B_{2d}(x) \times \{y\}} = \frac{1}{2} e^{-1}_x \mid_{B_{2d}(x)} \) for every \( x \in M \) and every \( y \in \gamma(B_e(x)) \). Thus, \( h_1((u, \gamma(v))_x) = \frac{1}{2} e^{-1}_x(u) \) for \( u \in B_{2d}(x) \) and \( v \in B_e(x) \). We define \( h_2 = \frac{1}{2} q \circ e^{-1} \circ \gamma^{-1} \circ pr_2 \). Then we can estimate a Lipschitz constant for \( h_2 \) as the product \( \frac{1}{2} \lambda \mu^{-1} = \frac{1}{\lambda} \leq 1 \). Let \( z = (u, \gamma(v)) \) and \( z' = (u', \gamma(v')) \) be two points in \( N \). Then
\[
\|h_1(z) - h_1(z')\| = \frac{1}{2} \|e^{-1}_x(u) - e^{-1}_x(u')\| \leq \frac{1}{2} \lambda \mu^{-1}(u) - e^{-1}_x(u) + \frac{1}{2} \|e^{-1}_x(u) - e^{-1}_x(u')\| \leq \frac{1}{2} \lambda \mu^{-1}(u) + \frac{1}{2} \mu d(x, x') + \frac{1}{2} d(u, u') \leq \frac{1}{2} \lambda \mu^{-1}(u) + \frac{1}{2} \mu d(x, x') + \frac{1}{2} d(u, u') \leq \frac{1}{2} \lambda \mu^{-1}(u) + \frac{1}{2} \mu d(x, x') + \frac{1}{2} d(u, u') \leq \frac{1}{2} \lambda \mu^{-1}(u) \leq \frac{1}{2} \lambda \mu^{-1}(u).
\]
Then we define \( h \) as the composition of \( h_1 + h_2 \) and the natural projection of \( B^{n}_d \times B^{n-l}_d \) onto \( B^{n-l}_d \). The condition (1) holds. The condition (2) holds by the definition of \( N \). □

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