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H. Towsner, *Computability of Ergodic Convergence*. In André Nies (editor), Logic Blog, 2012, Part 1, Section 1, available at [http://arxiv.org/abs/1302.3686](http://arxiv.org/abs/1302.3686).

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References

Part 1. Computability theory
1. Nies and Stephan: randomness and $K$-triviality for measures
   1.1. A randomness notion for measures. We consider algorithmically
       defined randomness notions for finite measures on Cantor space $2^\mathbb{N}$
       (usually probability measures). We use the letters $\mu, \nu$ etc for finite measures,
       with $\lambda$
reserved to the uniform measure. Letters $\sigma, \tau$ denote binary strings, $Z, X, \ldots$

elements of $2^\mathbb{N}$, $[\sigma] = \{ Z : Z \succ \sigma \}$. So $\lambda[\sigma] = 2^{-|\sigma|}$.

This research interacts with a recent attempt to define ML-randomness for quantum states corresponding to infinitely many qubits [15]. (Probability measures correspond to the quantum states $\rho$ where the matrix $\rho[M_n]$ is diagonal for each $n$.) Here is the main definition, which was discussed during a meeting on effective dynamical systems in Toulouse March 2018, but is implicit in the earlier preprint [15]. We now have a paper on this [16].

**Definition 1.1.** A measure $\mu$ is called Martin-Löf absolutely continuous (ML-a.c. for short) if $\inf_m \mu(G_m) = 0$ for each ML-test $\langle G_m \rangle$.

It suffices to only consider descending ML-tests, because we can replace $\langle G_m \rangle$ by the ML-test $\hat{G}_m = \bigcup_{k > m} G_k$, and of course $\inf_m \mu(\hat{G}_m) = 0$ implies $\inf_m \mu(G_m) = 0$. So we can change the passing condition to $\lim_m G_m = 0$.

Also, just as for bit sequences, it suffices to only consider the usual universal ML-test $U_m = \bigcup_{c < m} G_{m+c+1}$. So Martin-Löf a.c. ness is a $\Pi^0_3$ property of measures.

Since $\bigcap_m U_m$ is the set $C$ of non-MLR bit sequences, we obtain

$\mu$ is Martin-Löf a.c. $\iff \mu(C) = 0$.

It follows that we can actually restrict the definition to any descending universal ML-test, such as $\langle R_k \rangle_{k \in \mathbb{N}}$ in the notation of [14, Ch. 3].

Recall that a Solovay test is a sequence $\langle S_n \rangle$ of uniformly $\Sigma^0_1$ sets such that $\sum_k S_k < \infty$. A bit sequence $Z$ passes such a test if $\forall k Z \notin S_k$. We say that a measure $\mu$ passes such a test if $\lim_k \mu(S_k) = 0$. For $Z \in 2^\mathbb{N}$, we let $\delta_Z$ denote the probability measure that is concentrated on $\{Z\}$.

**Fact 1.2.** (i) The uniform measure $\lambda$ is Martin-Löf a.c.

(ii) $\delta_Z$ is Martin-Löf a.c. iff $Z$ is ML-random.

(iii) Let $\mu = \sum c_k \delta_{Z_k}$, for a sequence $\langle c_k \rangle$ of reals in $[0, 1]$ with $\sum_k c_k = 1$. $\mu$ is Martin-Löf a.c. iff all the sets $Z_k$ are ML-random.

**Proof.** (i) and (ii) are immediate.

(iii) $\Rightarrow$: If $Z_k \in \bigcap G_m$ for a ML-test $\langle G_m \rangle$ then $\mu(\bigcap G_m) \geq \delta_k$, do $\mu$ is not Martin-Löf a.c. .

$\Leftarrow$: given a ML-test $\langle G_m \rangle$, note that the $Z_k$ pass this test as a Solovay test. Hence for each $r$, there is $M$ such that $Z_k \not\in G_m$ for each $k \leq r$ and $m \geq M$. This implies that $\mu(G_m) \leq \sum_{k>r} c_k$.

The well known fact that ML-tests are equivalent to Solovay tests generalises to measures. We use the following variant for measures of a result by Tejas Bhojraj that he proved in the quantum setting.

**Fact 1.3.** A measure $\mu$ is Martin-Löf a.c. iff $\mu$ passes each Solovay test.

**Proof.** Each ML-test is a Solovay test. So the implication from right to left is immediate. For the implication from left to right, suppose that $\langle S_k \rangle$ is a Solovay test and $\inf_k \mu(S_k) > \delta > 0$. We define a ML-test that $\mu$ fails at level $\delta/2$. Let $S_{k,t}$ denote the clopen set given by strings in $S_k$ of length $t$. By a minor modification of the standard proof (e.g. [14, Prop. 3.2.19]), let $G_{m,t}$ be the open set generated by strings $\sigma$ such that
[\sigma] \subseteq S_{k,t} \text{ for } 2^m - 1 \text{ many } k.

As in the standard proof one shows that \( \lambda G_{m,t} \leq 2^{-m}/\delta \). Then \( G_m = \bigcup_t G_{m,t} \) (after thinning) can be turned into a ML-test.

Given \( m \) we pick \( t \in \mathbb{N} \) sufficiently large so that for some set \( M \subseteq \{0, \ldots, t - 1\} \) of size \( 2^m \) we have \( \mu(S_{k,t}) > \delta \) for each \( k \in M \). We show that \( \mu(G_{m,t}) > \delta/2 \). Let \( \sigma \) range over strings of length \( t \). We have

\[
\sum_{[\sigma] \in G_{m,t}} \sum_{k \in M} \mu[\sigma] \leq 2^{m-1}\delta
\]

by definition of \( G_m \). Since \( 2^m \delta \leq \sum_{k \in M} \mu(S_{k,t}) \), this implies

\[
\sum_{[\sigma] \in G_m} \sum_{k \in M} \mu[\sigma] > 2^{m-1}\delta.
\]

Since \( |M| = 2^m \) this shows \( \mu G_{m,t} > \delta/2 \) as required. \( \square \)

For a measure \( \nu \) and string \( \sigma \) with \( \nu[\sigma] > 0 \) let \( \nu_\sigma \) be the localisation: \( \nu_\sigma(A) = 2^{-|\sigma|} \nu(A \cap [\sigma]) \). Clearly if \( \nu \) is ML-a.c. then so is \( \nu_\sigma \).

A set \( S \) of probability measures is called convex if \( \mu_i \in S \) for \( i \leq k \) implies that the convex combination \( \mu = \sum \alpha_i \mu_i \in S \), where the \( \alpha_i \) are reals in \([0, 1] \) summing up to 1. The extreme points of \( S \) are the ones that can only be written as convex combinations of length 1 of elements of \( S \).

**Proposition 1.4.** The Martin-Löf a.c. probability measures form a convex set. Its extreme points are the Dirac measures.

**Proof.** For convexity, suppose \( \langle G_m \rangle \) is a descending ML-test. Then

\[
\lim_{m} \mu_i(G_m) > 0 \text{ for each } i,
\]

and hence \( \lim_{m} \mu(G_m) > 0 \).

If \( \mu \) is a Dirac measure then it is an extreme point. Conversely, if \( \mu \) is not Dirac there is a least number \( t \) such that the decomposition

\[
\mu = \sum_{|\sigma| = t, \mu[\sigma] > 0} \mu[\sigma] \cdot \mu_\sigma
\]

is nontrivial. Hence \( \mu \) is not an extreme point. \( \square \)

1.2. **Initial segment complexity of a measure \( \mu \) as a \( \mu \)-average.** Let \( K(\mu_{|n}) = \sum_{|x|=n} K(x) \mu[x] \) be the \( \mu \)-average of all the \( K(x) \) over all strings \( x \) of length \( n \). In a similar way we define \( C(\mu_{|n}) \).

**Fact 1.5.** \( C(\lambda_{|n}) \geq + n \), and therefore \( K(\lambda_{|n}) \geq + n \)

**Proof.** Suppose \( d \) is chosen so that for each \( x \) we have \( C(x) \leq |x| + d \) (we can in fact ensure \( d = 1 \) with the right universal machine, see [14, Ch 2]).

\[
C(\lambda_{|n}) = \sum_{r=0}^{n+d} \sum_{x: |x|=n \wedge C(x) \geq r} 2^{-n} \\
\geq \sum_{r=0}^{n} \left[ \sum_{|x|=n} 2^{-n} - \sum_{|x|=n, C(x) < r} 2^{-n} \right] \\
\geq n + 1 - \sum_{r \leq n} 2^{-n+r} \geq n - 1.
\]

This does it. \( \square \)
We say that $\mu$ has complex initial segments if $K(\mu | n) \geq^+ n$. The analog of Levin-Schnorr fails for measures in both directions.

**Example 1.6.** There is a Martin-Löf a.c. measure $\mu$ such that $\sup(n - K(\mu | n)) = \infty$.

**Proof.** We let $\mu = \sum c_k \delta_{Z_k}$ where $Z_k$ is ML-random and $0^{n_k} \prec Z_k$ for a sequence $\langle c_k \rangle$ of reals in $[0, 1]$ that add up to 1, and a sufficiently fast growing sequence $n_k$. Such a $\mu$ is Martin-Löf a.c. by Fact 1.2.

For $n_k \leq n < n_{k+1}$ we have

$$K(\mu | n) \leq (\sum_{l=0}^{k} c_l) \cdot (n + 2 \log n) + \sum_{l=k+1}^{\infty} c_l \cdot 2 \log n \leq (1 - c_{k+1}) n + 2 \log n.$$

So if we ensure that $c_{k+1} \cdot n_k > k + 2 \log n_k$ we are good. For instance, we can let $c_k = 1/(k(k + 1))$ and $n_k = 2^{k+4}$.

We falsify the converse implication by the following.

**Theorem 1.7.** There are a random $X$ and a non-random $Y$ such that, for all $n$, $K(X | n) + K(Y | n) \geq^+ 2n$.

**Proof.** Let $X$ be a low Martin-Löf random set. There is strictly growing function $f$ such that the complement of the image of $f$ is a recursively enumerable set $E$ and $K(X | m) \geq m + 3n$ for all $m \geq f(n)$. Note that this function exists, as $X$ is low and Martin-Löf random and so, for all $n$, the maximal $m$ such that $K(X | m) \leq m + 3n$ can be found in the limit.

Now let $g(n) = \max\{m : f(m) \leq n\}$. By a result of Miller and Yu [12, Cor. 3.2], there is a Martin-Löf random $Z$ such that there exist infinitely many $n$ with $K(Z | n) \leq n + g(n)/2$. For this set $Z$, let $Y = \{n + f(n) : n \in Z\}$.

Note that $K(Z | n) \leq K(Y | n) + g(n) + K(g(n))$, as one can enumerate the set $E$ until there are, up to $n$, only $g(n)$ many places not enumerated and then one can reconstruct $Z | n$ from $Y | n$ and $g(n)$ and the last $g(n)$ bits of $Z$. As $Z$ is Martin-Löf random, $K(Z | n) \geq^+ n$ and so,

$$K(Y | n) \geq^+ n - g(n) - K(g(n)) \geq^+ n - 2g(n).$$

The definitions of $X, f, g$ give $K(X | n) \geq n + 3g(n)$. This shows that $K(X | n) + K(Y | n) \geq 2n$ for almost all $n$.

However, the set $Y$ is not Martin-Löf random, as there are infinitely many $n$ with $K(Z | n) \leq^+ n + g(n)/2$. Now $Y | n + g(n)$ can be computed from $Z | n$ and $g(n)$, as one needs only to enumerate $E$ until the $g(n)$ nonelements of $E$ below $n$ are found and they allow to see where the zeroes have to be inserted into the string $Z | n$ in order to obtain $Y | n + g(n)$. Note furthermore, that $K(g(n)) \leq g(n)/4$ for almost all $n$ and thus $K(Y | n + g(n)) \leq^+ n + 3/4 \cdot g(n)$ for infinitely many $n$, so $Y$ cannot be Martin-Löf random.

Note that the measure $\mu = (\delta_X + \delta_Y)/2$ has only two equal-weighted atoms and furthermore satisfies that one of these atoms is not Martin-Löf random. So every component of a universal Martin-Löf test has at least $\mu$-measure $1/2$. On the other hand, $K(\mu | n) \geq n$ for almost all $n$ by the preceding result. Thus one has the following corollary.
Corollary 1.8. There is a measure $\mu$ with complex initial segments which is not Martin-Löf a.c.

Proposition 1.9. Suppose that $\mu$ is a measure such that $K(\mu \mid n) \geq n + K(n) - r$ for infinitely many $n$. Then $\mu$ is Martin-Löf a.c.

Proof. Suppose that $\mu$ is not Martin-Löf a.c. So there is a ML-test $\langle G_d \rangle_{d \in \mathbb{N}}$ and $\epsilon > 0$ such that $\mu(G_d) > \epsilon$ for each $d$. If $x$ is a string of length $n$ such that $|x| \subseteq G_d$ then

$$K(x \mid n, d) \leq n - d.$$ 

To see this let $M$ be the machine that on a pair of auxiliary inputs $n, d$ gives a description of length $n - d$ for each such $x$ (so the descriptions for different $x$ are prefix free). It follows that for $x$ as above

$$K(x) \leq n + K(n) - d + 2 \log d.$$ 

Now view $G_d$ as given by an enumeration of strings, and choose $n$ large enough so that $\mu G_d^{\geq n} > \epsilon$, where $G_d^{\geq n}$ denotes the open set given by the strings in this enumeration of length at most $n$. Let $c$ be a constant such that $K(x) \leq n + K(n) + c$ for each $x$ of length $n$. We have

$$K(\mu \mid n) = \sum_{|x| = n} K(x)\mu[x]$$

$$= \sum_{|x| = n, |x| \subseteq G_d} K(x)\mu[x] + \sum_{|x| = n, |x| \notin G_d} K(x)\mu[x]$$

$$\leq n + K(n) + c - \epsilon d/2.$$ 

The last inequality holds because

$$\sum_{|x| = n, |x| \subseteq G_d} K(x)\mu[x] \leq \epsilon(n + K(n) + c - d + 2 \log d)$$

and

$$\sum_{|x| = n, |x| \notin G_d} K(x)\mu[x] \leq (1 - \epsilon)(n + K(n) + c).$$

Now given $r$ let $d = 2r/\epsilon$. By the above, for large enough $n$ we have $K(\mu \mid n) < n + K(n) - r$. So $\mu$ is not strongly Chaitin random.

Question 1.10. In analogy to the case of bit strings, does strong Chaitin randomness imply Martin-Löf a.c. ness relative to $\varnothing'$?

If the measure $\mu$ has an atom $A$ but is not Dirac then function $C(\mu \mid n)$ is not bounded from below by $n - c$ for any $c$. The reason is that when $c' = \mu(A)$ then for this atom, the function $n \mapsto c' \cdot (n - C(A \mid n))$ is not bounded by any constant and therefore it can go arbitrarily low: this would then make the average to be below $n - c$ for any given $c$ at infinitely many $n$.

A fan is a prefix-free set $V$ such that $|V|^{<} = 2^\mathbb{N}$. Note that $V$ is necessarily finite. The $\mu$-average length of $V$ is

$$\ell(\mu \mid V) = \sum_{\sigma \in V} |\sigma|\mu[\sigma].$$

Generalising the above, we let

$$K(\mu \mid V) = \sum_{\sigma \in V} K(\sigma)\mu[\sigma].$$
We say that $\mu$ has complex initial segments in the strong sense if $K(\mu|V) \geq^+ \ell(\mu|V)$ for each fan $V$. To be done: does this imply Martin-Löf a.c.?

1.3. Connection to ML-randomness of measures in $\mathcal{M}(2^\mathbb{N})$. A natural probability measure $P$ on the space $\mathcal{M}(2^\mathbb{N})$ of probability measures on Cantor space has been introduced implicitly in Mauldin and Monticino [11], and in Quinn Culver’s thesis [4] in the context of computability, where he shows that this measure is computable. Let $\mathcal{R} \subseteq [0,1]^{2^{\omega}}$ be the closed set of representations of probability measures; namely, $\mathcal{R}$ consists of those $X$ such that $X_\sigma = X_{\sigma_0} + X_{\sigma_1}$ for each string $\sigma$. $P$ is the unique measure on $\mathcal{R}$ such that for each string $\sigma$ and $r, s \in [0,1]$, we have

$$P(X_{\sigma_0} \leq r | X_\sigma = s) = \min(1, \frac{r}{s}).$$

That is, we choose $X_{\sigma_0}$ at random w.r.t. the uniformly distribution in the interval $[0, X_\sigma)$, and the choices made at different strings are independent.

Proposition 1.11. Every probability measure $\mu$ that is ML-random wrt to $P$ is Martin-Löf a.c.

The proof is based on two facts. For $G \subseteq 2^\mathbb{N}$ be open, for the duration of this proof let $\mu$ range over $\mathcal{M}(2^\mathbb{N})$ and let

$$r_G = \int \mu(G) dP(\mu).$$

Fact 1.12. $r_G = \lambda(G)$.

Proof. Clearly for each $n$ we have

$$\sum_{|\sigma|=n} r_{[\sigma]} = \int \sum_{|\sigma|=n} \mu([\sigma]) dP(\mu) = 1.$$

Further, $r_\sigma = r_\eta$ for $|\sigma| = |\eta| = n$ because there is a $P$-preserving transformation $T$ of $\mathcal{M}(2^\mathbb{N})$ such that $\mu([\sigma]) = T(\mu([\eta]))$. Therefore $r_{[\sigma]} = 2^{-|\sigma|}$.

If $\sigma, \eta$ are incompatible then $r_{[\sigma] \cup [\eta]} = r_{[\sigma]} + r_{[\eta]}$. Now it suffices to write $G = \bigcup_{i} [\sigma_i]$ where the strings $\sigma_i$ are incompatible, so that $\lambda G = \sum_i 2^{-|\sigma_i|}$. □

Fact 1.13. Let $\mu \in \mathcal{M}(2^\mathbb{N})$ and let $\langle G_m \rangle$ be a ML-test such that there is $\delta \in \mathbb{Q}^+$ with $\forall m \mu(G_m) > \delta$. Then $\mu$ is not ML-random w.r.t. $P$.

Proof. Observe that by the foregoing fact

$$\delta \cdot \mathbb{P}(\{\mu: \mu(G_m) \geq \delta\}) \leq \int \mu(G_m) dP(\mu) = \lambda(G_m) \leq 2^{-m}.$$

Let $G_m = \{\mu: \mu(G_m) > \delta\}$ which is uniformly $\Sigma^0_1$ in $\mathcal{M}(2^\mathbb{N})$. Fix $k$ such that $2^{-k} \leq \delta$; then $\langle G_{m+k} \rangle_{m \in \mathbb{N}}$ is a ML-test w.r.t. $P$ that succeeds on $\mu$. □

Culver shows that each ML-random $X$ for $P$ is non-atomic. So by Fact 1.12 the converse of Prop. 1.11 fails: not every Martin-Löf a.c. $X$ is ML-random with respect to $P$. 
1.4. SMB theorem. We recall some notation from the 2017 Logic Blog, Section 6.2, adapting some letter uses. $A^\infty$ denotes the space of one-sided infinite sequences of symbols in $A$. We can assume that this is the sample space, so that $X_n(\omega) = \omega(n)$. By $\mu$ we denote their joint distribution. A dynamics on $A^\infty$ is given by the shift operator $T$, which erases the first symbol of a sequence. A measure $\mu$ on $A^\infty$ is $T$-invariant if $\mu G = \mu T^{-1}(G)$ for each measurable $G$.

We consider the r.v. $h^\mu_n(Z) = -\frac{1}{n} \log \mu[Z|_n], \quad \text{(recall that log is w.r.t. base 2)}.$

Recall that $\mu$ is ergodic if every $\mu$ integrable function $f$ with $f \circ T = f$ is constant $\mu$-a.s. An equivalent condition that is easier to check is the following: for $u, v \in A^*$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{n-1} \mu([u] \cap T^{-k}[v]) = \mu[u]\mu[v].$$

For ergodic $\mu$, the entropy $H(\mu)$ is defined as $\lim_n H_n(\mu)$, where

$$H_n(\mu) = -\frac{1}{n} \sum_{|w|=n} \mu[w] \log \mu[w].$$

One notes that $H_{n+1}(\mu) \leq H_n(\mu) \leq 1$ so that the limit exists. Also note that $H_n(\mu) = \mathbb{E}h^\mu_n$.

The following says that in the ergodic case, $\mu$-a.s. the empirical entropy equals the entropy of the measure.

**Theorem 1.14 (SMB theorem).** Let $\mu$ be an ergodic invariant measure for the shift operator $T$ on the space $A^\infty$. Then for $\mu$-a.e. $Z$ we have $\lim_n h^\mu_n(Z) = H(\mu)$. If $\mu$ is computable, then the conclusion holds for $\mu$-ML-random $Z$ by results of Hochman (implicit) \cite{7} and Hoyrup \cite{9}. Recent work of A. Day extends this to spaces other than $A^\infty$ and amenable group actions. Here we keep the space but change the type of object. We say that a measure $\rho$ is $\mu$-Martin-Löf a.c. if $\rho(G_m) \to 0$ for each $\mu$-ML test $\langle G_m \rangle$. Here is a special case of Conjecture 6.3 in 2017 Logic Blog where the states $\rho, \mu$ when restricted to the matrix algebra $M_n$ are diagonal. Enough patience will suffice.

**Conjecture 1.15 (Effective SMB theorem for measures).** Let $\mu$ be a computable ergodic invariant measure for the shift operator $T$ on the space $A^\infty$. Suppose the measure $\rho$ is $\mu$-Martin-Löf a.c. Then $\lim_n \mathbb{E}\rho h^\mu_n = H(\mu)$.

1.5. $K$-triviality for measures.

**Definition 1.16.** A measure $\mu$ is called $K$-trivial if $K(\mu|_n) \leq^+ K(n)$.

For Dirac measures $\delta_A$ this is the same as saying that $A$ is $K$-trivial in the usual sense.

**Proposition 1.17.** Suppose $\mu$ is $K$-trivial. Then $\mu$ has atoms. In fact, $\mu$ is concentrated on its atoms.
Proof. For each $c$ there is a $d$ (in fact $d = O(2^c)$) such that for each $n$ there are at most $d$ strings $x$ of length $n$ with $K(x) \leq K(n) + c$. Since $\mu$ is non-atomic, there is $n$ so that for each $x$ of length $n$ we have $\mu[x] \leq 1/2d$. Note that there is a constant $b$ such that $K(x) \geq K(|x|) - b$ for each $x$. Then we have that in the $\mu$-average $\mu(K(|x|) \leq n) \geq x K(n) + c$ have total measure at most $1/2$, and each $K(x) \geq K(n) - b$. So the $\mu$-average is at least $K(n) + c/2$ up to a constant.

Proposition 1.18. For each order function $f$ there is a non-atomic measure $\mu$ such that $K(\mu | n) \leq^+ K(n) + f(K(n))$.

In fact, for each nondecreasing unbounded function $f$ which is approximable from above there is a non-atomic measure $\mu$ such that $K(\mu | n) \leq^+ K(n) + f(n)$.

Proof. There is a recursively enumerable set $A$ such that, for all $n$, $A$ has up to $n$ and up to a constant $f(n)/2$ non-elements. One let $\mu$ be the measure such that $\mu(x) = 2^{-m}$ in the case that all ones in $x$ are not in $A$ and $\mu(x) = 0$ otherwise, here $m$ is the number of non-elements of $A$ below $|x|$. One can see that when $\mu(x) = 2^{-m}$ then $x$ can be computed from $|x|$ and the string $b_0 b_1 \ldots b_{m-1}$ which describes the bits at the non-elements of $A$. Thus $K(x) \leq^+ K(|x|) + K(b_0 b_1 \ldots b_{m-1}) \leq^+ K(|x|) + 2m$. It follows that $K(\mu | n) \leq^+ K(n) + f(n)$, as the $\mu$-average of strings $x \in \{0, 1\}^n$ with $K(x) \leq^+ K(n) + f(n)$ is at most $K(n) + f(n)$ plus a constant.

REMARK. Note that when $f$ is a recursive order function or an order function which is approximable from above then there is a further order function $f'$ which is approximable from above such that $f'(n) \leq^+ f(K(n))$ for all $n$; one just chooses $f'(n) = \min \{f_s(K_s(m)) : m \geq n, s \geq 1\}$. Thus one can bring the above result into the form that for all recursive order functions $f$ there is a measure $\mu$ satisfying $K(\mu | n) \leq K(n) + f(K(n))$.

The $K$-trivial measure form a convex set. However it is not closed under infinite convex sums. One takes finite sets which pointwise converge to $\Omega$ and let the $c_k$ fall sufficiently slow so that at level $n$ there is still $(n + 2)^{-1/2}$ measure on $\Omega | n$ and therefore the corresponding $\mu$-average grows like the squareroot of $n$, and not like $K(n)$.

In more detail, let $A_k = \{\ell : \ell \in \Omega \wedge \ell < k\}$ and $c_k = (k + 1)^{-1/2} - (k + 2)^{-1/2}$. All sets $A_k$ are finite and thus $K$-trivial. Furthermore, the sum of all $c_k$ is 1.

Let $\mu = \sum_k c_k \cdot \delta_{A_k}$. Then $A_k(n) = \sum_{x \in \{0, 1\}^n} \mu(x) \cdot K(x) \geq (\sum_{m \geq n} c_m) \cdot K(\Omega | n) \geq (n + 2)^{-1/2} \cdot (n + 2) = \sqrt{n + 2}$ for almost all $n$ and thus the average grows faster than $K(n) + constant$. So the measure is not $K$-trivial.

We call a measure $\mu$ low for $K$ if for each $z$

$$\int K^X(z) d\mu(X) \geq^+ K(z).$$

Thus we form the $\mu$-average over all oracles. Clearly if $A$ is low for $K$ as a set then $\delta_A$ is low for $K$. Merkle and Yu have shown that $\lambda$ is low for $K$. So lowness for $K$ does not imply $K$-triviality. It would still be interesting to relate lowness for $K$ with $K$-triviality in the case of measures.
2. **Yu: A note on $\delta^1_2$**

Let $\delta^1_2$ be the least ordinal that cannot be presented by a $\Delta^1_2$-well ordering over $\omega$ and $\delta^{1,\omega}_2$ be the one relative to $x$. Define

$$\Delta_{12} = \{ x \mid \delta^{1,\omega}_2 = \delta^1_2 \}.$$

The following result must be well known but I have not found a reference.

**Proposition 2.1.** $\Delta_{12}$ is $\Delta^1_3$ but neither $\Sigma^1_2$ nor $\Pi^1_2$.

**Proof.** $x \in \Delta_{12}$ if and only if every $\Pi^1_2(x)$-singleton $z$ coding a well ordering of $\omega$ is bounded by a $\Pi^1_2$-singleton $z_0$ coding a well ordering of $\omega$ if and only if there is a real $s$ coding a well ordering of $\omega$ such that $L_s$ contains all the $\Pi^1_2(x)$-singletons and every real in $L_s$ is $\Delta^1_2$. So $\Delta_{12}$ is a $\Delta^1_3$-set.

Since every nonempty $\Sigma^1_2$-set contains a $\Delta^1_2$-member, we have that $\Delta_{12}$ is not $\Pi^1_2$. Now suppose that $\Delta_{12}$ is $\Sigma^1_2$. Then let $y \in L$ be a real computing all the $\Delta^1_2$-reals. Then every $y$-random is $\Delta^1_2$-random. By the assumption, $R = \{ r \mid r$ is $y$-random and $r \in \Delta_{12} \}$ is a $\Sigma^1_2(y)$-set. If $V$ contains an $L$-random real, then $R$ is not empty. So by Shoenfield’s absoluteness, $R$ is not empty and contains a real $r \in L$. But every real $r \in R$ must be $L$-random, a contradiction. \qed

**Part 2. Group theory and its connections with logic**

3. **Nies and Schlicht: the normaliser of a finite permutation group**

Let $G$ be a group. The group of inner automorphisms $\text{Inn}(G)$ forms a normal subgroup of $\text{Aut}(G)$. The quotient group is called the group of outer automorphisms, denoted by $\text{Out}(G)$. For instance, $\text{Out}(S_6)$ has 2 elements. For more examples, note that since $G^2$ is invariant, there is a canonical surjection $\text{Aut}(G) \to \text{Aut}(G^{ab})$ with kernel containing $\text{Inn}(G)$. In the case of $F_2$ equality holds, so that $\text{Out}(F_2) \cong GL_2(\mathbb{Z})$.

It is well-known that no cyclic group of odd order is of the form $\text{Aut}(G)$ for any group $G$. On the other hand, every finite group is the outer automorphism group of some group $N$ which can be chosen to be fundamental group of a closed hyperbolic 3-manifold (a result of Sayadoshi Kojima). See [2] for background.

Here is a simple (and known) fact. Given a finite group $G$ with domain $\{0, \ldots, n-1\}$, we think of $G$ as embedded into $S_n$ via the left regular representation $g \to \tau_g$ where $\tau_g(x) = gx$. (E.g. for $G = S_6$, we have $n = 720$.) Let $N_G$ denote the normaliser of $G$ in $S_n$.

**Proposition 3.1.** There is a canonical surjection $R: N_G \to \text{Aut}(G)$ mapping $G$ to $\text{Inn}(G)$, thereby showing that $N_G/G$ is isomorphic to $\text{Out}(G)$.

**Proof.** A canonical map $R: N_G \to \text{Aut}(G)$ is defined by

$$R(\phi)(g) = h \text{ if } \phi \tau_g \phi^{-1} = \tau_h.$$

Clearly $R(\phi)$ is an automorphism of $G$ for each $\phi \in N_G$. To check that $R$ is a homomorphism, note that for $\phi, \psi \in N_G$

$$R(\phi \psi)(g) = h \iff \phi \psi \tau_g \psi^{-1} \phi^{-1} = \tau_h.$$
Proof. Write \( k \) following kind. A closed subgroup \( G \) of the only finitely many \( n \)-orbits, for each \( n \). These are the automorphism groups of the \( \aleph_0 \)-categorical structures with domain \( \omega \). We say that a topological group \( G \) is quasi-oligomorphic if it is in a topological group isomorphism with an oligomorphic group.

4.1. The centre. Let \( G \) be a closed subgroup of \( S_\infty \). For \( p \in \omega \), by \( k_p(G) \) we denote the number of orbits of the natural action of \( G \) on \( \omega^p \); such orbits will be called \( p \)-orbits. (The parameter \( k_p(G) \) is denoted \( F_p^*(G) \) in [3].) For \( r \in \omega \) let \( k_2(G, r) \) denote the number of 2-orbits containing a pair of the form \( (r, t) \) (which only depends on the 1-orbit of \( r \)). Suppose that \( k_1(G) = n \) and let \( r_1, \ldots, r_n \in \omega \) represent the 1-orbits.

Fact 4.1. \(|C(G)| \leq \prod_{i \leq n} k_2(G, r_i)\). In particular, if \( G \) is 1-transitive then the size of the centre is at most \( k_2(G) \).

Proof. Write \( C = C(G) \). For any \( r \in \omega \), and \( c, d \in C \), if \( cr \neq dr \) then \( (r, cr) \) and \( (r, dr) \) are in different 2-orbits. Hence \(|Cr| \leq k_2(G, r)\).

Consider now the natural left action \( C \rtimes \prod_i C r_i \) (Cartesian product of sets). If \( c, d \in C \) are distinct then \( c r_i \neq d r_i \) for some \( i \) and \( g \in G \), so that \( cr_i \neq dr_i \). Hence \(|C| \leq \prod_{i \leq n} |C \rtimes r_i|\). This shows the required bound. \( \square \)

Greg Cherlin suggested another way to prove that for oligomorphic \( G \) the centre \( C(G) \) is finite (but without an explicit bound on its size). We may suppose \( G = \text{Aut}(M) \) for an \( \aleph_0 \)-categorical structure \( M \) with domain \( \omega \). Note that \( \Phi \in \text{Aut}(M) \) is definable as a binary relation on \( M \) iff \( \Phi \) is invariant under the natural action of \( G \) on \( M \), i.e., \( \Phi \in C(G) \). Since there are only finitely many definable binary relations, \( C(G) \) is finite.

We consider examples of oligomorphic groups \( G \) with a nontrivial centre.

1. For any finite abelian group \( A \), the natural action of \( A \times S_\infty \) on \( A \times \omega \) yields an 1-transitive oligomorphic group with centre \( A \). The number of 2-orbits is at least \(|A| \) because \( (an, a'n') \approx (bm, b'm') \) implies that \( a - b = a' - b' \). (In fact it is \( 2|A| \).)

2. Let \( M_1 \) be the structure with one equivalence relation \( E \) that has all classes of size 2; say, for \( x, y \in \omega \)

\[ x E y \iff x \mod 2 = y \mod 2. \]

Write \( C_k = \mathbb{Z}/k\mathbb{Z} \). We have \( G_1 := \text{Aut}(M_1) = C_2 \wr S_\infty \), where \( \wr \) denotes the unrestricted wreath product. Here \( S_\infty \) is viewed with its natural action on \( \omega \); so \( G \) is an extension of \( L = C_2^\omega \) by \( S_\infty \) with \( S_\infty \) acting by \( \phi \cdot f = f \circ \phi \), for \( f \in L \). (Note that \( L \) is the automorphism group of the structure where the individual equivalence classes are now distinct unary predicates. \( G \) is
the normaliser of $L$ in $S_{\infty}$.) The centre $C(G)$ consists of the identity and the automorphism that maps each element to the other one in its $E$ class. The centre of $G/C(G)$ is trivial.

3. Let $M_2$ be the structure with equivalence relations $E \subseteq F$ such that each $E$-class has size 2 and each $F$-class has size 4. Then the automorphism group of a single $F$-class is $C_2 \wr C_2$, and hence $G_2 := \text{Aut}(M_2) = (C_2 \wr C_2) \wr S_{\infty}$.

As before, the centre $C(G_2)$ consists of the identity and the automorphism that switches each element in its $E$ class. We have $G_2/C(G_2) \cong G_1$.

Similarly, for each $n$ there is an oligomorphic group $G_n = \text{Aut}(M_n)$ with a chain of $n$ higher centres.

4.2. The central quotient. The main purpose in this section is to show that for oligomorphic $G$, the central quotient $G/C(G)$ is quasi-oligomorphic. Some facts needed along the way hold in more generality.

Suppose a group $G$ acts on a set $X$ and $N \trianglelefteq G$. Write $\sim_N$ for the orbit equivalence relation of the subaction of $N$. Note that $G$ acts naturally on $X/\sim_N$ via $g \cdot [x] = [g \cdot x]$ (where $[x]$ is the $\sim_N$ class of $x$). Since elements of $N$ act as the identity, $G/N$ acts on $X/\sim_N$.

Suppose now $G$ initially acts faithfully on a set $Y$, say $Y = N$. Let $X = Y \times Y$ and let $G$ act on $X$ by the usual diagonal action. Let $N = C(G)$.

Fact 4.2. The action of $G/C(G)$ on $X/\sim_{C(G)}$ is faithful.

To see this, suppose $g \notin N = C(G)$. So grab $\eta \in G$ such that $(g\eta) \cdot w \neq (\eta g) \cdot w$.

Let $w' = \eta \cdot w$. Then $g \cdot (w, w') \not\sim_N (w, w')$, because any element $h$ of $G$ such that $h \cdot (w, w') = g \cdot (w, w')$ satisfies $(h\eta) \cdot w \neq (\eta h) \cdot w$, so that $h \notin C(G)$.

We now switch to topological setting. Given a Polish group $H$ with a faithful action $\gamma \colon H \times V \to V$, say for a countable set $V$, we obtain a monomorphism $\Theta_{\gamma} \colon H \to \text{Sym}(V)$ given by $\Theta_{\gamma}(g)(k) = \gamma(g, k)$. A Polish group action is continuous iff it is separately continuous (i.e. when one argument is fixed). In the case of an action on countable $V$ (with the discrete topology), the latter condition means that

(a) for each $k, n \in V$, the set $\{g \colon \gamma(g, k) = n\}$ is open.

So $\gamma$ is continuous iff $\Theta_{\gamma}$ is continuous.

Definition 4.3. We say that a faithful action $\gamma \colon H \times V \to V$ is strongly continuous if the embedding $\Theta_{\gamma}$ is topological.

Equivalently, the action is continuous, and for each neighbourhood $U$ of $1_H$, also $\Theta_{\gamma}(U)$ is open, namely,

(b) for each neighbourhood $U$ of $1_H$, there is finite set $B \subseteq V$ such that $\forall k \in B \gamma(g, k) = k$ implies $g \in U$.

Since $H$ is Polish, strong continuity of the action implies that $H$ is topologically isomorphic via $\Theta_{\gamma}$ to a closed subgroup of $\text{Sym}(V)$ (see e.g. [5] Prop. 2.2.1]).

Now consider the case that $Y = N$ and $G$ is a closed subgroup of $S_{\infty}$. Since $C(G)$ is closed, $H = G/C(G)$ is naturally a Polish group via the quotient topology: for $C \subseteq U \subseteq G$, the subgroup $U/C \leq H$ is declared to be open iff $U$ is open in $G$. (See e.g. [5] Prop. 2.2.10.)
Let $X = Y \times Y$ as above. Suppose that $V := X/\sim_{C(G)}$ is infinite (e.g. when $C(G)$ is finite), so through the action $\gamma$ above we obtain an (algebraic) embedding $\Theta_\gamma$ of $G/C(G)$ into $S_V$ (which can be identified with $S_\infty$).

**Claim 4.4.** Suppose that $G$ is a closed subgroup of $S_\infty$ that acts 1-transitively on $Y = \mathbb{N}$. Suppose that $C(G)$ is finite. Then $\Theta_\gamma : G/C(G) \rightarrow S_V$ is a topological embedding.

**Proof.** We check the conditions (a) and (b) above.

(a) Suppose that $k = [(w_0, w'_0)], n = [(w_1, w'_1)]$. Then $\gamma(g,k) = n$ iff there are $(v_0, v'_0) \sim_C (w_0, w'_0)$ and $(v_1, v'_1) \sim_C (w_1, w'_1)$ such that $g \cdot v_0 = v_1$ and $g \cdot v'_0 = v'_1$. Since $C(G)$ is finite this condition is open.

(b) An open neighbourhood of 1 is in the form $U/C$ wherehere $U \subseteq G$ is open and $C \leq U$. By definition of the topology on $G$, we may assume that $U = G_R C$ for some finite $R \subseteq Y$ (as usual $G_R$ is the pointwise stabiliser). Let $B = (R \times R)/\sim_C$. Consider $g = pC \in H$, where $p \in G$.

Suppose that $\forall k \in B \gamma(g,k) = k$. This means that for each $u, v \in R$, there is a $c_{u,v} \in C$ such that $p \cdot (u, v) = c_{u,v} \cdot (u, v)$. Since $G$ acts faithfully and 1-transitively on $Y$, for each $c, d \in C$, and each $y \in Y$, $c \cdot y = d \cdot y$ implies that $c = d$. Therefore given another pair $r, s \in R$, $c_{u,v} = c_{u,s} = c_{r,s}$. Let $c$ be this unique witness. Then $p \cdot u = c \cdot u$ for each $u \in R$, hence $p \in G_R C$ and therefore $g \in U/C$.

□

**Theorem 4.5.** Let $G$ be oligomorphic. The central quotient $G/C(G)$ is quasi-oligomorphic (i.e. homeomorphic to an oligomorphic group).

**Proof.** It is known (as pointed out to us by Todor Tsankov) that we may assume $G$ is 1-transitive. To see this, one shows that there is an open subgroup $W$ such that the left translation action $\gamma : G \acts G/W$ of $G$ on the left cosets of $W$ is faithful and oligomorphic, and the corresponding embedding into a copy of $S_\infty$ is topological. To get $W$, let $x_1, \ldots, x_k \in \omega$ represent the 1-orbits of $G$. Let $W$ be the pointwise stabiliser of $\{x_1, \ldots, x_k\}$. If $g \in G - \{1\}$ then there is $p \in G$ and $i \leq k$ such that $g \cdot (p \cdot x_i) \neq p \cdot x_i$. So $p^{-1}gp \notin W$, and hence $g \cdot pW \neq pW$. So the action is faithful. The rest is routine using (a) and (b) above.

Now we can apply Claim 4.3 recalling that $C(G)$ is finite. □

For any closed subgroup $G = \text{Aut}(M)$ of $S_\infty$ the higher centres are normal, so their orbit equivalence relations are $G$-invariant. If $G$ is oligomorphic they are definable in $M$. Hence the progression of higher centres has to stop at a finite stage for each oligomorphic group $G$.

4.3. **Conjugacy.** We show that conjugacy of oligomorphic groups is smooth. For a closed subgroup $G$ of $S_\infty$, let $V_G$ denote the orbit equivalence structure. For each $n$ this structure has a $2n$-ary relation symbol, denoting the orbit relation on $n$-tuples. (One could require here that the tuples have distinct elements.)

The following fact holds for non-Archimedean groups in general.

**Fact 4.6.** Let $G, H$ be closed subgroups of $S_\infty$.

$G, H$ are conjugate via $\alpha$ $\iff$ $V_G \cong V_H$ via $\alpha$. 
Proof. ⇒: Immediate.
⇐: Let $M_G$ be the canonical structure for $G$; namely there are $k_n \leq \omega$ many $n$-ary relation symbols, denoting the $n$-orbits. Let $M_H$ be the structure in the same language where the $n$-equivalence classes of $V_H$ are named so that $\alpha$ is an isomorphism $M_G \cong M_H$. Clearly $G = \text{Aut}(M_G)$ and $H = \text{Aut}(M_H)$. Further, $\alpha^{-1} \text{Aut}(M_H)\alpha = \text{Aut}(M_G)$. □

Proposition 4.7. Conjugacy of oligomorphic groups is smooth.

Proof. The map $G \to V_G$ is Borel because we can in a Borel way find a countable dense subgroup of $G$, which of course has the same orbits. Now apply Fact [4.6]. For countable structures $S$ in a fixed language, mapping $S$ to its theory $\text{Th}(S)$ is Borel. Since the theory can be seen as a real, for $\omega$-categorical structures, this shows smoothness. □

For corresponding structures $A, B$ with $\text{Aut}(A) = G, \text{Aut}(B) = H$, conjugacy of $G, H$ via $\alpha$ means that $\alpha(A)$ and $B$ have the same definable subsets. To see this, consider the case that $A$ is the canonical structure for $G$.

We note the following topological variation of Prop 3.1. The notation is introduced above.

Proposition 4.8. (i) $\text{Aut}(V_G)$ equals the normaliser $N_G$ of $G$ in $S_\infty$.
(ii) If $G$ is oligomorphic then $N_G/G$ as a topological group is profinite.

Proof. (i) $\subseteq$: Let $\alpha \in \text{Aut}(V_G), \beta \in G$. Clearly $\alpha$ maps $n$-orbits to $n$-orbits, so $\alpha(M_G)$ is a renumbering of the named $n$-orbits in $M_G$. Therefore $\beta^n \in \text{Aut}(M_G) = G$.
$\supseteq$: Let $V$ be an $n$-orbit, and let $\alpha \in N_G$. If $r, s \in \mathbb{N}^n$ and $r, s \in \alpha(V)$, choose $\beta \in G$ such that $\alpha \beta^{-1}(r) = s$. So $\alpha(V)$ is contained in an $n$-orbit $W$. By a similar argument, $\alpha^{-1}(W)$ is contained in an $n$-orbit. Therefore $\alpha(V) = W$ is an $n$-orbit. Hence $\alpha \in \text{Aut}(V_G)$.
(ii). Let $k_n$ be the number of $n$-orbits of $G$. Define a continuous homomorphism $\Theta : \text{Aut}(V_G) \to \prod_n S_{k_n}$ by $\Theta(\alpha) = f$ if $f(n)$ is the finite permutation describing the way $\alpha$ permutes $n$-orbits (numbered in some way). Clearly $G$ equals the kernel of $\Theta$, and $\Theta/G$ is therefore a topological embedding. Since the range is compact, its inverse is also continuous. A closed subgroup of a profinite group is again profinite. □

The converse of (ii) may fail: $N_G/G$ can be profinite, and even trivial, without $G$ being oligomorphic. For instance, there is a countable maximal-closed subgroup of $S_\infty$, e.g. $AGL_n(\mathbb{Q})$, the automorphism group of the structure $\mathbb{Q}^n$, $n \geq 2$ with the ternary function $f(x, y, z) = x + y - z$ (Kaplan and Simon). This structure is not $\omega$-categorical.

We don’t know at present whether every profinite group occurs that way.

4.4. $\omega$-categorical structures with essentially finite language. One says that a structure $M$ has essentially finite language if $M$ is interdefinable with a structure $\bar{M}$ over a language in a finite signature. (Interdefinable means same domain and same definable relations.). We present a basic fact that can be used to obtain an oligomorphic group that is not isomorphic to the automorphism group of such an $\omega$-categorical structure.

Lemma 4.9. The following are equivalent for a countable structure $M$.
(i) $M$ is interdefinable with a structure $N$ in finite language with maximum arity $k$, and quantifier elimination.

(ii) $M$ is $\omega$-categorical, and for each $n \geq k$, each $n$-orbit of $M$ is given by its projections to $k$-orbits.

Proof. (ii) implies (i): Let $G = \text{Aut}(M)$, and let $N$ be the orbit structure of $M$. Thus, $N$ is like $V_G$ above but has an $n$-ary predicate for each $n$-orbit. (Note that $N$ is a Fraisse limit. $V_G$ is a reduct of $N$, and as in the finite case above its automorphism group is the normaliser of $G$.)

(i) implies (ii): Clearly $N$ is $\omega$-categorical, as there are only finitely many $n$ types for each $n$. Each formula $\phi$ in $n \geq k$ variables is a Boolean combination of $q$-free formulas in $\leq k$ variables. If $\phi$ describes an $n$-orbit we can assume it is a conjunction of such formulas. A formula in $k' \leq k$ variables describes a finite union of $k'$-orbits. Hence the $n$-orbit is given by its projections: if two $n$ tuples have are in the same projection orbits then both or none satisfy $\phi$.

□

5. Kassabov and Nies: supershort first order descriptions in certain classes of finite groups

Nies and Tent [17] showed that every finite simple group $G$ has a first-order description (in the usual language of group theory) of length $O(\log(|G|))$. This result is near optimal for the whole class of finite simple groups because of the cyclic groups, using a counting argument together with the prime number theorem. We show that shorter descriptions can be obtained for certain natural classes of finite simple groups. This works for instance when the groups $G$ in the class have presentations of length $O(\log(|G|))$ and the diameter of the corresponding Cayley graph is also $O(\log)$. For instance, by this method the alternating groups $G$ can be described in length $O(\log \log |G|)$.

The following definition is from Nies and Tent [17].

**Definition 5.1.** Let $r: \mathbb{N} \to \mathbb{N}^+$ be an unbounded function. We say that an infinite class $C$ of finite $L$-structures is $r$-compressible if for each structure $G$ in $C$, there is a sentence $\phi$ in $L$ such that $|\phi| = O(r(|G|))$ and $\phi$ describes $G$.

For notational convenience, we will use the definition

$$\log m = \min\{r: 2^r \geq m\}.$$

**Theorem 5.2** ([17], Thm. 1.2). The class of finite simple groups is log-compressible.

The first-order formulas for generation developped in [17] will be used in the context of presentations with Cayley graphs of small diameter.

**Lemma 5.3** ([17], proof of Lemma 2.4). For each positive integers $k, v$, there exists a first-order formula $\delta_{v,k}(g; x_1, \ldots, x_k)$ of length $O(k+v)$ in the language of groups such that for each group $G$, $G \models \delta_{v,k}(g; x_1, \ldots, x_k) \iff g = w(x_1, \ldots, x_k)$ for some word $w$ in $F(x_1, \ldots, x_n)$ of length at most $2^v$. 

Proof. Let
de_{0,k}(g; x_1, \ldots, x_k) \equiv \bigvee_{1 \leq j \leq k} [g = x_j \lor g = x_j^{-1} \lor g = 1].

For \(i > 0\) let
de_{i,k}(g; x_1, \ldots, x_k) \equiv \exists u_i \exists v_i [g = u_i v_i \land
\forall w_i [(w_i = u_i \lor w_i = v_i) \rightarrow de_{i-1,k}(w_i; x_1, \ldots, x_k)]].

Then \(de_{i,k}\) has length \(O(k + i)\), and \(G |\equiv de_{i,k}(g; x_1, \ldots, x_k)\) if and only if \(g\) can be written as a product, of length at most \(2^i\), of \(x_1\)'s and their inverses. □

**Lemma 5.4.** Suppose that a finite simple group \(G\) has a presentation
\(\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle\) of length \(\ell\).

Also suppose that the diameter of the Cayley graph is bounded by \(2^v\), that is, each \(g \in G\) has the form \(w(x_1, \ldots, x_k)\) for some free group word of length at most \(2^v\).

There is a sentence \(\psi\) of length \(O(v + \ell)\) describing the structure \(\langle G, g \rangle\).

Proof. Let \(\psi\) be the formula
\[x_1 \neq 1 \land \bigwedge_{1 \leq i \leq n} r_i = 1 \land \forall y de_{k,v}(y; x_1, \ldots, x_k).\]

Replacing the \(x_1, \ldots, x_k\) by new constant symbols, the models of the sentence thus obtained are the nontrivial quotients of \(G\). Since \(G\) is simple, this sentence describes \(\langle G, g \rangle\). □

**Lemma 5.5.** Suppose \(S\) is a generating set of \(A_k\) containing a 3-cycle, say \((1, 2, 3)\). Then the Cayley graph of \(A_k\) with respect to \(S\) has diameter \(O(k^4)\).

Proof. \(A_k\) acts 1-transitively on the set of 3-cycles on \(\{1, \ldots, k\}\) by conjugation. Since the number of 3-cycles is \(O(k^3)\), each 3-cycle can be expressed by a word of length \(O(k^3)\) in the generating set. Any even permutation can be written as a product of at most \(k\) 3-cycles. □

**Proposition 5.6.** The classes of finite alternating/symmetric groups and of finite symmetric groups are \(\log \circ \log\)-compressible.

Proof. We want to describe each \(A_k\), and we may assume \(k > 2\). By [6, Cor 3.23], \(A_k\) has a presentation of length \(\ell = O(\log k) = O(\log \log |A_k|)\). By construction, one of the generators of \(A_k\) in the above presentation is a 3-cycle. Therefore the diameter of the Cayley graph is at most \(O(k^4)\) by Lemma 5.5. So we can apply Lemma 5.4 with \(v = O(\log k)\). (Actually a more careful look at the generation set gives that all 3-cycles can be expressed as words of length \(O(\log k)\) and the diameter of the Cayley graphs is \(O(k \log k)\).)

The case of symmetric groups is similar. One uses a transposition instead of 3-cycle. We need to take into account that the symmetric groups are not simple. Since the only nontrivial quotient has size 2, it suffices to require in the description that the group has at least 3 elements. □

**Proposition 5.7.** Fix a prime power \(q\). The class of groups \(\text{PSL}_n(q)\) is \(\log \circ \log\)-compressible.
Proof. The argument is similar to the case of alternating groups. By [6, Thm. A and Thm. 6.1], PSL$_n(q)$ has a presentation of length

$$\ell = O(\log q + \log n) = O_q(\log \log |\text{PSL}_n(q)|).$$

The generating set for this presentation contains a generating set for PSL$_2(q)$ with diameter in $O(\log q)$ and a generating set of $A_n$ with diameter in $O(n \log n)$. Thus, every elementary matrix in PSL$_n(q)$ can be expressed as a word of length at most $O(n \log n)$ and a generating set of $A_n$ with diameter in $O(n \log n)$. Finally, a row reduction argument gives at any matrix in PSL$_n(q)$ is a product of at most $n^2$ elementary matrices, which implies that the diameter of the Cayley graph is at most $O(n^3 \log n \log q)$ (this bound can be improved to $O(n^2 \log q + n^2 \log n)$ by more careful examination of all element in generating set). By Lemma 5.4 the groups PSL$_n(q)$ can be described by sentence of length $O_q(\log n) = O_q(\log \log |\text{PSL}_n(q)|)$.

□

5.1. Rank 1 groups. The result in [6, Thm. 4.36] gives a bound $O(\log q)$ for both length of presentation and diameter for groups such as S$L_2(q)$, PSU($3, q$) and Sz($q$). Since the size of these groups is polynomial in $q$, this doesn't help to get descriptions shorter than the ones in Theorem 5.2.

If we fix the characteristic and allow descriptions in second order logic, something can be done. Recall that second order logic allows quantification over relations and functions of arbitrary arity.

Proposition 5.8. Fix a prime $p$. The field GF($p^k$) has a second-order description of length $O(\log k)$

Proof. For $q = p^k$ the first-order sentence $\phi_q$ from [17, Section 4] describing GF($q$) says that the structure is a field of characteristic $p$ such that for all elements $x$ we have $x^{p^k} = x$ and there is some $x$ with $x^{p^{k-1}} \neq x$. Now in the second order version introduce function symbols $f_1, \ldots, f_k$ such that $f_1(x) = x^p$ and $f_{i+1}(x) = f_i(f_i(x))$. Thus $f_i(x) = x^{p^{2^i}}$. Given these we can express that $x^{p^k} = y$ in length $O(\log k)$ using the binary expansion of $k$. □

By the biinterpretability method described in [17, Section 5], short descriptions of the fields imply short descriptions of the finite simple groups defined over them. For Suzuki and Ree groups, we have $p = 2$ and $p = 3$, respectively.

Theorem 5.9. The classes of Suzuki groups Sz($2^{2l}$) and of small Ree groups $2G_2(3^{2l+1})$ are log log compressible in second order logic.

Part 3. Metric spaces and descriptive set theory

6. Nies and Schlicht: Scott relation in Polish metric spaces

For tuples $a, b$ in a Polish metric space, the Scott relation at level 1 is defined as usual: for each challenge $y$ on the left side there is response $z$ on the right side so that the enumerated metric spaces $a, y$ and $b, z$ are isometric; similar for the sides interchanged.
Proposition 6.1. There is a computable Polish metric space \((X, d)\) and a computable sequence \(\langle y_n^* \mid n \in \omega \rangle\) of distinct elements of \(X\) such that the set
\[
C_1 = \{ (m, n) \in \omega \times \omega \mid y_m^* \equiv_1 y_n^* \}
\]
is \(\Pi^1_2\)-complete.

Proof. Note that \(C_1\) is clearly a \(\Pi^1_2\) set. To prove that it is \(\Pi^1_2\)-complete, we first fix some notation. If \(A \subseteq X \times Y\) and \(n \in X\), let \(A_n\) denote the \(n\)-th slice of \(A\) and \(p(A) = \{ m \in X \mid \exists k (m, k) \in A \}\) its projection to the first coordinate. For \(X = \omega\), we say that \(A\) is universal for a point class \(\Gamma\) on \(Y\) if every set in \(\Gamma\) occurs as a slice.

Claim 6.2. There is a \(\Pi^1_1\)-universal set \(B \subseteq \omega \times \omega^\omega\) such that \(p(B)\) is \(\Sigma^1_2\)-complete.

Proof. Let \(B' \subseteq \omega \times \omega \times \omega^\omega\) be universal for \(\Pi^1_1\)-subsets of \(\omega \times \omega^\omega\). Let \(\pi : \omega \times \omega \to \omega\) be a computable bijection and \(\pi^*: \omega \times \omega \times \omega^\omega \to \omega \times \omega^\omega\) the induced bijection. It is easy to see that \(B = \pi^*(B')\) is universal for \(\Sigma^1_2\)-subsets of \(\omega^\omega\). Since the projection of \(B'\) to \(\omega \times \omega\) is universal for \(\Sigma^1_2\)-subsets of \(\omega\), it is \(\Sigma^1_2\)-complete. Since \(p(B')\) has the same 1-degree as \(p(B)\) it follows that the projection \(p(B)\) is \(\Sigma^1_2\)-complete as well.

We fix a \(\Pi^1_2\)-universal set \(B\) as in the previous claim. Then \(S = \omega \times \omega^\omega \setminus B\) is \(\Sigma^1_2\)-universal. It now follows that the equivalence relation on \(\omega\) defined by \((m, n) \in E \iff S_m = S_n\) is \(\Pi^1_2\)-complete as a set, since \(S_n \neq \omega^\omega \iff B_n \neq \emptyset \iff \exists x(n, x) \in B \iff n \in p(B)\) and \(p(B)\) is \(\Sigma^1_2\)-complete.

Question 6.3. Show that \(E\) is \(\Pi^1_2\)-complete as an equivalence relation.

Claim 6.4. We can associate in a computable way to each \(n \in \omega\) a Polish space \((Y_n, d_n)\) of diameter at most 1 and some \(y_n^* \in Y_n\) with distance set \(\{d_n(y_n^*, y) \mid y \in Y_n\} = \{0\} \cup S_n\).

Proof. We first define an auxiliary Polish metric space \((X, d_X)\). Let \(X = \{(0, 0)\} \cup (0, 1] \times \omega^\omega\). Let \(m_{r_0, r_1} = \min\{r_0, r_1\}\) for \(r_0, r_1 \in \mathbb{R}\) and let \(u\) be the standard ultrametric on \(\omega^\omega\). We define
\[
d_X((r_0, x_0), (r_1, x_1)) = |r_0 - r_1| + m_{r_0, r_1} u(x_0, x_1).
\]

To see that \(d_X\) is a metric on \(X\), first note that by the ultrametric inequality, \(u(x_0, x_2)\) is at most the maximum of \(u(x_0, x_1)\) and \(u(x_1, x_2)\). So if \(r_1 \geq r_0\) or \(r_1 \geq r_2\) then
\[
\Delta_0 = (m_{r_0, r_1} u(x_0, x_1) + m_{r_1, r_2} u(x_1, x_2)) - m_{r_0, r_2} u(x_0, x_2) \geq 0.
\]

We can hence assume that \(r_1 < r_0, r_2\) and additionally that \(r_0 \leq r_2\) by symmetry between \((r_0, x_0)\) and \((r_2, x_2)\). Again by the ultrametric inequality for \(\omega^\omega\). In both cases, we have
\[
\Delta_0 \geq (r_1 - r_0) u(x_0, x_2) \geq r_1 - r_0 = -(r_0 - r_1).
\]

By our assumption \(r_1 < r_0 \leq r_2\), we further have
\[
\Delta_1 = (|r_0 - r_1| + |r_1 - r_2|) - |r_0 - r_2| \geq r_0 - r_1.
\]

Hence \((X, d_X)\) satisfies the triangle inequality.

We now define the required spaces \((Y_n, d_n)\) as subspaces of \((X, d)\). By identifying \(\omega^\omega\) with the set of irrational numbers in \([0, 1]\), we let \(C_n \subseteq X\) be...
a closed set with \( S_n = p(C_n) \). Note that we can obtain \( C_n \) computably in \( n \), assuming that our universal sets are constructed in the usual way. Now let \( Y_n = \pi(C_n) \) and let \( y_n^* \) be an element that is identified with \( \pi(0,0) \). It is clear that \( y_n^* \) has the required distance set in \( (Y_n, d_n) \). \( \square \)

Now let \( Y = \bigcup_{n \in \omega} Y_n \) and \( d_Y \) the metric on \( Y \) given by the metrics \( d_n \) and \( d_Y(x,y) = 2 \) if \( x \in Y_m \) and \( y \in Y_n \) for some \( m \neq n \). Since player II wins \( G(y_m^*, y_n^*, 1) \) if and only if \( S_m = S_n \), it is now easy to see from the previous two claims that \( C_1 \) is \( \Pi^0_2 \)-complete. \( \square \)

Part 4. Model theory and definability

7. Nies and Schneider: Concrete presentations, isomorphism, and descriptions

7.1. Summary, mostly in layman’s terms. Mathematical structures are usually given by concrete presentations. A computer scientist might think of a graph as a concrete object stored in a computer, for instance an adjacency list, which is a list of all the vertices and all the edges. For another example, a set of generators together with a set of relators on them present a group.

What really counts is the essence of the structure, the structure “up to isomorphism”: think of the shape of the graph, or of the abstract group. Two concrete presentations that yield the same abstract structure are called isomorphic. Being concrete, the presentations can be used as input to some kind of computation. The question arises:

**Question 7.1.** How hard is it to tell whether two presentations are isomorphic?

It is still unknown whether one can decide efficiently that two concretely presented finite graphs are isomorphic (though Babai has recently shown that the decision problem is in pseudopolynomial time).

Related questions are the following. Given a reference class of infinite structures,

**Question 7.2.** which structures are determined within the class by their first order theory?

For the class of finitely generated groups, this property is called *quasi axiomatisable* (QA); see the last two chapters of the by now venerable survey [13]. For instance, abelian groups are QA. Even better,

**Question 7.3.** which structures can be described within the class by a single sentence in first-order logic?

In the same context, this property is called *quasi finitely axiomatisable*, or QFA [13]. Abelian groups are never QFA, but other very common groups are, e.g. the Heisenberg group \( UT_3(\mathbb{Z}) \) or the Baumslag-Solitar group \( \mathbb{Z}[1/n] \rtimes \mathbb{Z} \).

Even a description by the full first-order theory would necessarily only determine the essence of the structure (technically: isomorphic concrete structures are elementarily equivalent). It is interesting to study these questions especially in the setting of topological algebra and Lie algebras, where
not much has been done so far. The point is that first-order logic can only indirectly address the topology, because that is given by subsets.

7.2. Some more detail for mathematicians. As mentioned, we have to distinguish between concrete presentations of a structure, and the abstract structure “up to isomorphism”. Consider a finite presentation of a group:

\[ \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle. \]

This describes the concrete group \( F(x_1, \ldots, x_n)/N \), where \( N \) is the normal subgroup generated by \( r_1, \ldots, r_k \). Given two finite presentations, it is undecidable in general whether they describe isomorphic groups (Rabin).

A finite presentation of a group or Lie algebra, say, is a description of a concrete structure. We can also describe a structure merely up to isomorphism. If we want to do this, we have to pick some language from mathematical logic and a corresponding satisfaction relation. First-order logic has the additional advantage that it doesn’t look beyond the immediate structure as given by the elements and the relations among them (for instance, subsets of the structure are not allowed). This is a severe restriction, given that for instance in group theory, one frequently studies things like maximal subgroups etc. In first order logic, we can talk about particular ones if they are definable, e.g., the centre or the centraliser of an element. But we can’t quantify over the whole lot.

If the structure is finite, we can look for a short sentence, relative to the size of the structure, describing it; e.g. Nies and Tent [17] do this for (classes of) finite groups.

If the structure is infinite, we need the external information given by the reference class, because we can not describe it by a single first-order sentence. The class needs to at least prescribe the cardinality of the structure. For instance, we can describe the ordering of the rationals by a single sentence within the countable structures. A f.g. group is called QFA (for quasi-finitely axiomatisable) if, within the class of f.g. groups, it can be described by a single sentence in the language of group theory.

To reiterate, given a class \( C \) of concretely presented structures of the same signature, there are two interrelated types of question

(a) How complicated is the isomorphism relation between structures in \( C \)?

(b) Which structures can up to isomorphism be described within \( C \) by their theory?

(c) Or even by single a first-order sentence?

As for (a), the intuition may be that if a concrete structure has a complicated equivalence class under the isomorphism relation, it is hard to describe abstractly. The trivial upper bound for isomorphism is \( \Sigma^1_1 \) (assuming the class \( C \) itself is arithmetical). On the other hand, elementary equivalence is easier, namely hyperarithmetical, and in fact \( \Pi^0_1(\emptyset^e) \).

Question (b) is interesting in particular if some classification of structures in \( C \) is known; for instance, we will below consider the simple Lie algebras over \( \mathbb{C} \). In this case, one would try to prove that each structure in the class has a description by a first-order sentence. This means that the classification can be expressed in first-order language for the right signature, given the
7.3. Describing simple Lie algebras over $\mathbb{C}$ by a first-order sentence with an additional predicate. We describe simple Lie algebras over $\mathbb{C}$ by a first-order sentence in the language with $+, [, ]$ and the equivalence relation $E$ that vectors $x$ and $y$ span the same subspace.

There is a formula expressing that $x_1, \ldots, x_n$ generate $L$ as a vector space:

$$\delta(x_1, \ldots, x_n) \equiv \forall z \exists y_1, \ldots, y_n \left[ z = \sum_i y_i \wedge \bigwedge_i y_i Ex_i \right].$$

Let $R_n$ denote the free associative algebra in $n$ generators $x_1, \ldots, x_n$ over $\mathbb{C}$; it is generated as a $\mathbb{C}$-vector space by all the words in the generators. The usual commutator in $R_n$ is denoted $[a, b] = ab - ba$. With this commutator, $R_n$ becomes a Lie algebra. The free Lie algebra in $n$ variables over $\mathbb{C}$ is the Lie subalgebra of $R_n$ generated by $x_1, \ldots, x_n$.

Each finite-dimensional Lie algebra is finitely presented, because the multiplication table on the basis elements gives a finite presentation. So we have relators of the form $[x_i, x_j] = \sum c_k x_k$ where the $c_k$ are complex coefficients. (The situation is analogous to the case of finite groups.)

Using Cartan’s classification, Serre proved that each semisimple Lie algebra $L$ over $\mathbb{C}$ is finitely presented where coefficients are integers in $[-3, 3]$. This can be seen from the Cartan table; see Humphreys [10, Section 18.1] for the presentation, and also note that the $x_i$ and $y_i$ there generate $L$. Then there is $m$ depending on the dimension of $L$ such that only commutators of depth up to $m$ in the Lie generators $z_1, \ldots, z_k$ are needed to generate $L$ as a vector space. Thus the generators in $L$ satisfy the formula $\alpha_m(z_1, \ldots, z_k)$ saying that each commutator of depth $m + 1$ in those generators is a linear combination of the commutators of depth $m$.

To describe a simple $L$ within the Lie algebras over $\mathbb{C}$, we express that $L$ is non-trivial and that there are $z_1, \ldots, z_k$ satisfying the Serre relations, the formula $\alpha_m$, and, using the formulas $\delta(x_1, \ldots, x_n)$, that the commutators of depth $m$ in the $z_1, \ldots, z_k$ generate $L$ as a vector space. Since $L$ is simple, this sentence describes $L$.

To obtain a description in the language of Lie algebras we would need to define $E$ in terms of the Lie operations. First one would show that $E$ is invariant under automorphisms in the finite dimensional case.

8. NIES, SCHLICH AND TENT: BI-INTERPRETATIONS FOR $\omega$-CATEGORICAL STRUCTURES AND THEORIES

We discuss bi-interpretations of pairs of $\omega$-categorical theories. We begin with structures rather than theories, because definability is easier to grasp. Definability will always mean without parameters.

8.1. Interpreting in structures. Suppose that $L, K$ are first-order languages in countable signatures. Interpretations via first-order formulas of $L$-structures in $K$-structures are formally defined, for instance, in Hodges [8, Section 5.3]. Informally, an $L$-structure $A$ is interpretable in a $K$-structure
B if the elements of A can be represented by tuples in a definable k-ary relation D on B, in such a way that equality of A becomes a B-definable equivalence relation E on D, and the other atomic relations on A are also definable.

We think of the interpretation of A in B as a decoding function \( \Delta \). It decodes A from B using first-order formulas, so that \( A = \Delta(B) \) is an L-structure. Each L-formula \( \phi \) corresponds to a K-formula \( \psi' \) which is the saturation under E of a K-formula \( \psi \). We write \( \phi = \Delta(\psi') \).

For a structure A, recall that \( A_{eq} \) has a sort \( V = D/E \) for each definable equivalence relation E on A and definable \( E \)-closed \( D \subseteq A^n \), and besides the inherited ones has definable relations \( V_i, i < n \) between A and \( V \), given by

\[
S^V_i(av) \iff \exists v[a = y_i \land [y]_E = v].
\]

For instance if \( n = 1 \), we have the relation \( S^V_0 \) that \( v \) is the equivalence class of \( a \).

Clearly \( \text{Aut}(A) \) acts on \( A_{eq} \). The \( r \)-orbits on a sort \( V = D/E \) have the form \( [U]_{E^r} \) where \( U \) is an \( n \cdot r \)-orbit of A. So if \( \text{Aut}(A) \) is oligomorphic, there are only finitely many such \( r \)-orbits.

**Example 8.1.** Let E be an equivalence relation with all classes of size 2. Take unary predicates \( C_0, D_0, C_1, D_1 \) partitioning the domain, and let A be the structure where each E-class has exactly one element in \( C_i \) and one in \( D_i \), for some \( i \leq 1 \). Then A has 4 orbits, the sort \( A/E \) only has two orbits. For the first, U above can be either \( C_0 \) or \( D_0 \).

Throughout we will have \( \omega \)-categorical structures A, B with \( \text{Aut}(A) = G, \text{Aut}(B) = H \). There are various equivalent views of expressing interpretation of A in B.

(a) \( A = \Delta(B) \) for some interpretation \( \Delta \), as above
(b) A map \( \alpha: A \to B^{eq} \) with range contained in single sort, \( \Gamma \) sending relations \( \emptyset \)-definable in A to relations \( \emptyset \)-definable in \( B^{eq} \). This map extends canonically to a map \( \tilde{\alpha}: A_{eq} \to B_{eq} \).
(c) There exists a topological homomorphism \( h: \text{Aut}(B) \to \text{Aut}(A) \) such that the range of \( h \) is oligomorphic.

(a), (b) are merely reformulations of each other. For (a) to (c), let \( h(\beta) = \beta^n \mid_D / E \) which has oligomorphic range by remarks above.

For (c) to (a) see Hodges [8, Section 7.4].

8.2. **Bi-interpretations of structures.** There are several equivalent formulations. Fix structures A, B.

(a) \( A \cong \Delta(B), B \cong \Gamma(A) \), and some isomorphisms \( \gamma: A \cong \Delta(\Gamma(A)) \) and \( \delta: B \cong \Gamma(\Delta(B)) \) are definable in A, in B, respectively. (Thus, \( \gamma \) is described by a formula with \( 1+n \) free variables, where \( n \) is the product of the dimensions of the two interpretations.) If the structures are \( \omega \)-categorical, we can let \( \delta \) be the restriction of \( \gamma \) to the sort on which B is defined.

\[\text{An alternative definition (CITE EVANS) allows the range to be a subset of finitely many sorts.}\]
(b) $\alpha : A \rightarrow B^{eq}, \beta : B \rightarrow A^{eq}$, and the maps $\gamma : A \rightarrow A^{eq}$ given by $\gamma = \tilde{\beta} \circ \alpha$, and $\delta = \tilde{\alpha} \circ \beta$ analogous for $B$, are definable in the respective structure.

Note that $\gamma : A \rightarrow V$ for some sort $V$.

A bi-interpretation introduces a matching of orbits. Suppose $\alpha$ is $d$-dimensional, and $\beta$ is $e$-dimensional.

**Fact 8.2.** For each $n$-orbit $S$ of $A$, $\tilde{\alpha}(S)$ is an $n$-orbit of $B^{eq}$ (under the action of $\text{Aut}(B)$, on the sort which contains the range of $\alpha$).

**Proof.** As usual let $G = \text{Aut} A, H = \text{Aut} B$. For simplicity first let $n = 1$. Recall from Ahlbrandt/Ziegler [1] (detail in David Evans’ 2013 notes, Thm. 2.9) that the “dual” $\alpha^*: H \rightarrow G$ is a topological isomorphism, where

$$(\alpha^* h) w = \alpha^{-1}(h(\alpha(w))).$$

So as $h$ ranges over $H$, $\alpha^*(h)$ ranges over $G$. Let $S = G \cdot w$, then $\alpha(S) = H \cdot \alpha(w)$ as required. More generally, for each $n$ we have $\alpha(G \cdot (w_1, \ldots, w_n)) = H \cdot (\alpha(w_1), \ldots, \alpha(w_n))$. \hfill $\square$

### 8.3. Bi-interpretations of theories

We can also formulate biinterpretability for complete theories $S, T$, easiest in countable languages. Note that theories can be seen as infinite bit sequences and hence the set of theories carries the usual Cantor space topology. The complete theories form a closed set. To be $\omega$-categorical is an arithmetical property of theories, because by Ryll-Nardzewski this property is equivalent to saying that for each $n$, the Boolean algebra of formulas with at most $n$ free variables modulo $T$-equivalence is finite.

To fix some notation, the sorts in models of $S^{eq}, T^{eq}$ have the forms $C/E, D/F$, resp, where $C$ is an $r$-ary definable relation, $D$ is $s$-ary, and $E, F$ are definable equivalence relations. $\gamma, \delta$ each are $(1+n)$-ary, where $n = rs$. Given $\Gamma, \Delta$ as above, we express that for an arbitrary model $A$ of $S$ and $B := \Delta(A)$, we have $B \models T$, and $\gamma$ evaluated in $A$ induces an isomorphism of $A$ and $\Gamma(\Delta(A))$ (a structure with domain a sort of $A^{eq}$); similarly, $\delta$ is an isomorphism of $B$ and $\Delta(\Gamma(B))$. This can be expressed by two possibly infinite lists of sentences that have to be in $S$, and in $T$, respectively.

**Remark 8.3.** Since $B$’s domain is a sort of $A$ and $B$ is $\omega$-categorical, requiring that $\delta$ exists is actually redundant: $\delta$ can be chosen to be “$\Delta(\gamma)$”.

This means that we apply the interpretation $\Delta$ to the definable isomorphism $\gamma : A \cong \Gamma(\Delta(A))$, obtaining an isomorphism $\delta : \Delta(A) \cong \Delta(\Gamma(\Delta(A)))$, i.e. $\delta : B \cong \Delta(\Gamma(B))$. Clearly $\delta$ is invariant under the $\text{Aut}(B)$ action on $B^{eq}$. Hence $\delta$ is $B$-definable.

**Fact 8.4.** Suppose $S, T$ are bi-interpretable theories in the notation above. For each model $A$ of $S$, letting $B = \Delta(A)$, a model of $T$, we have that $A, B$ are bi-interpretable as models.

**Proof.** $\beta : B \rightarrow A^{eq}$ is the identity map on this concrete structure $B = \Delta(A)$. $\alpha : A \rightarrow B^{eq}$ we can therefore choose the same as $\gamma$, and $\tilde{\beta} \circ \alpha = \gamma$ is definable in $A$. $\tilde{\alpha} \circ \beta$ is the same as $\delta$, hence definable in $B$. \hfill $\square$
Remark 8.5. In the case of theories rather than structures, the matching of orbits in Fact 8.2 becomes a matching of types. Each $k$-type $\phi$ of $S$, i.e. an atomic formula with free variables $x_1, \ldots, x_k$, is given by a $k$ type $\psi'$ of $T^{eq}$ in the sense that $\phi = \Gamma(\psi')$, and therefore by a $k \cdot r$ type $\psi$ of $T$ whose saturation under the definable equivalence relation $F$ on $D \subseteq B^r$ is equivalent to $\psi'$ (note there could be various such types $\psi$). Similarly for types of $T$.

8.4. **Isomorphism of groups and bi-interpretability.** By isomorphisms of topological groups, we always mean topological isomorphisms. Two $\omega$-categorical structures are bi-interpretable iff their automorphism groups are isomorphic; [David Evans’ 2013 notes, Thm. 2.9. This was originally proved by Coquand.]

**Theorem 8.6.** Isomorphism of oligomorphic groups is Borel bi-reducible with bi-interpretability of $\omega$-categorical theories.

*Proof. $\leq B$: From oligomorphic $G$ we can in a Borel way determine a countable dense subgroup $\hat{G}$. The canonical structures for $G$ and $\hat{G}$ are equal. The canonical structure $M_G$ for $G$ can thus be Borel determined from $G$. From $M_G$ we can Borel determine the theory $\text{Th}(M_G)$. Then $G \cong H$ iff $\text{Th}(M_G)$ and $\text{Th}(M_H)$ are bi-interpretable by [David Evans’ 2013 notes, Thm. 2.9.]

$\geq B$: From a consistent theory $T$ in a countable signature, via the Henkin construction we can in a Borel way determine a model $M \models T$ with domain $\omega$. Let $F(T)$ be the automorphism group of such a model, which is a closed subgroup of $S_\infty$. Then for $\omega$-categorical theories $S, T$, we have that $S$ is bi-interpretable with $T$ iff $F(S)$ is isomorphic to $F(T)$ by [David Evans’ 2013 notes, Thm. 2.9.]

□

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