Quantum precision thermometry with weak measurement

Arun Kumar Pati* and Chiranjib Mukhopadhyay†

Quantum Information and Computation Group, Harish-Chandra Research Institute, HBNI, Allahabad 211019, India

Sagnik Chakraborty‡ and Sibasish Ghosh§

Optics & Quantum Information Group, The Institute of Mathematical Sciences, HBNI, C. I. T. Campus, Taramani, Chennai - 600 113, India

As the minituarization of electronic devices, which are sensitive to temperature, grows apace, sensing of temperature with ever smaller probes is more important than ever. Genuinely quantum mechanical schemes of thermometry are thus expected to be crucial to future technological progress. We propose a new method to measure the temperature of a bath using the weak measurement scheme with a finite dimensional probe. The precision offered by the present scheme not only shows similar qualitative features as the usual Quantum Fisher Information based thermometric protocols, but also allows for flexibility over setting the optimal thermometric window through judicious choice of post selection measurements.

I. INTRODUCTION

Quantum mechanics lends different status to observable quantities and parameters of a system, the latter of which are not expressible through Hermitian operators. Among physical parameters, temperature of a system is perhaps one of the most ubiquitously useful. Temperature drives chemical reaction rates [1], control the generation of thermoelectric current in thermocouples [2], or heat flow between different baths [3]. Thus, thermometry, or the science of measuring temperature, is of paramount importance. This is especially relevant for modern technological applications and devices, where the thermal baths may themselves be relatively tiny. Nanoscale probes, which follow laws of quantum physics, are thus required to measure the temperature of such baths without disturbing them too much. Quantum thermometry thus aims at improving the technique of temperature sensing using small probes [4–10]. The analysis of quantum thermometry so far has focused on seeking to find and saturate the quantum Cramer Rao bound for various partially and fully thermalized configurations of systems [11, 12]. In addition to ensuring non-invasiveness, these studies show possible improvement in the precision due to quantum effects for non-equilibrium settings. Even in the steady state scenario, quantum enhancement in the precision using a quantum switch has been recently reported [13]. For observables, another technique to experimentally determine them quantitatively has recently come to the fore, which relies on a so called weak measurement scheme [14–17]. Weak measurement is a technique of ascertaining information about an observable through a weak interaction between the system and the measurement apparatus generated by the observable. It is then followed by a strong post-selection measurement on the system. One of the most well known facets of weak measurement scheme is the concept of the so called weak values of an observable, which are, in general, complex numbers. In the last few years, weak values have been used in the context of measuring quantities which may or may not have quantum mechanical observables associated with them, for example, the geometric phase [18], non Hermitian operators [19], density matrix corresponding to a quantum state [20–22], or the entanglement content of a quantum state [23, 24]. The weak value amplification technique has found recent physical application in observations of the spin Hall effect [25], photon trajectories [26], or the time delay between ultra fast processes [27] as well. Thus, it is reasonable to ask whether a weak measurement based scheme is a viable approach towards thermometry. In this paper, we answer this question in the affirmative.

In the present work, we outline how to measure temperature using weak values with finite dimensional probes and specifically concentrate on qubit probes. We show the presence of an optimal temperature window for the precision offered by this scheme, which is a feature repeated in the usual QFI based paradigm of quantum thermometry [11, 28–30]. One crucial advantage offered by the present scheme over the QFI based scheme is the ability to shift the optimal precision window while keeping the probe parameters fixed, and simply changing the post selected state. In addition, we argue that, the proposal based on the weak value is suitable for measurement of a very hot body. If the temperature of a system is very high, then prolonged contact with a measuring apparatus may damage the apparatus itself. On the other hand, in our weak measurement based scheme, the measuring apparatus is brought into contact with a thermalized qubit of the bath for a very short duration of time, thus potentially saving any damage to the apparatus. [31]

The organization of the paper is as follows. Section II briefly reviews the weak measurement scheme. The main theme of our paper, namely the protocol for measuring temperature through weak values, is described in Section III. This is followed by the detailed analysis of precision offered by the present scheme for a qubit probe in Section IV. We finally conclude with a few observations and outline some possible future developments.
II. WEAK MEASUREMENT

In theory of weak measurements, a quantum system $S$ is chosen to be in an initial state $|\psi_i\rangle_S$, called the pre-selected state, and it is made to interact with a measurement apparatus $M$, prepared in a state $|\phi\rangle_M$. The interaction is weak in strength, and generated by a Hamiltonian $H_{int} = g\hat{A} \otimes \hat{P}_s$, where $\hat{A}$ is a system observable, $\hat{P}_s$ is the momentum operator of $M$ and $g$ is a small positive number signifying the strength of the interaction. This is followed by a strong measurement on the system, called the post-selection measurement. The outcome of post-selection measurement that we focus on is along the state $|\psi_f\rangle$. Clearly, the post-selection is a selective measurement, as we ignore the other outcomes of the measurement.

The time-evolved state of $S + M$ before post-selection is given by

$$|\Psi\rangle = e^{-igA M \hat{P}_s} \left[ |\psi_i\rangle \otimes |\phi\rangle_M \right] \approx \left( I_S \otimes I_M - ig\hat{A} \otimes \hat{P}_s \right) \left[ |\psi_i\rangle \otimes |\phi\rangle_M \right],$$

up to first order in $g$. Note that the state $|\Psi\rangle_{SM} \equiv (I_S \otimes I_M - ig\hat{A} \otimes \hat{P}_s)[|\psi_i\rangle \otimes |\phi\rangle_M]$ is, in general, unnormalized. The reduced state of $|\Psi\rangle_{SM}$ on $M$ is

$$|\phi_f\rangle_M \approx \left( \langle \psi_f | \psi_i \rangle - ig \langle \psi_f | \hat{A} | \psi_i \rangle \hat{P}_s \right) |\phi\rangle_M,$$

which on normalizing looks like

$$|\phi_f\rangle_M \approx \left( 1 - igA_n \hat{P}_s \right) |\phi\rangle_M \approx e^{-igA_n \hat{P}_s} |\phi\rangle_M,$$

where $A_n \equiv \langle \psi_f | \hat{A} | \psi_i \rangle / \langle \psi_f | \psi_i \rangle$ is called as the weak value of $\hat{A}$. Note that $A_n$ can also be complex and take values beyond the eigenvalue range of the observable.

Since the weak value $A_n$ is, in general a complex quantity, one must experimentally observe the real as well as imaginary parts of the weak value. On measuring certain properties of $|\phi_f\rangle_M$, the real and imaginary parts of $A_n$ can be determined. These properties include shift in position and momentum values compared to that of $|\phi\rangle_M$, variance of momentum wave-function, rate of change of position wave-function and the strength of interaction [32]. Laguerre-Gaussian modes of an optical beam have also been used to measure the real and imaginary parts of a weak value [33]. Recently it has also been proposed and experimentally demonstrated that the weak value can be inferred from interference visibility and phase shifts [34].

III. ASSESSING TEMPERATURE THROUGH WEAK VALUES

Consider a $d$ dimensional quantum system $S$ under the action of a time-independent Hamiltonian $\hat{H}$ having non-degenerate energy eigenvalues $E_1, E_2, \ldots, E_d$ and the corresponding energy eigenstates $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_d\rangle$. Assume that $S$ is in contact with a heat bath of temperature $T$ and $S$ has reached the thermal equilibrium state $\rho_T \equiv e^{-\beta \hat{H}} / (\text{Tr}[e^{-\beta \hat{H}}]) = \left( \sum_{n=1}^{d} e^{-\beta E_n} |\psi_n\rangle \langle \psi_n| \right) / \left( \sum_{n=1}^{d} e^{-\beta E_n} \right)$. Here $\beta = 1/(k_B T)$, with $k_B$ being taken as unity. Prepare the measuring apparatus $M$ in a state $|\phi\rangle$, having position wave function $\phi(x)$. Note that here $M$ is considered to be a continuous-variable system, in general. We would like to perform measurement of an observable $\hat{A}$ on the system $S$.

Consider now evolution of $S + M$ under the action of an interaction Hamiltonian $\hat{H}_{int} = g\hat{A} \otimes \hat{P}_s$ for a small amount of time $\tau$. The interaction strength $g$ is also considered to be small – the regime of weak interaction between $S$ and $M$. Here $\hat{P}_s$ is the momentum observable of $M$ canonically conjugate to the position observable $\hat{X}$. We assume here that during the time interval $[0, \tau]$, $S$ and $M$ are under the action of only the Hamiltonian $\hat{H}_{int}$. This may be fulfilled in different ways: (i) We may decouple the system $S$ at time $t = 0$ from the heat bath (after $S$ achieves the thermal equilibrium state $\rho_T$) and thereby switch on the interaction Hamiltonian $\hat{H}_{int}$ for the time duration $[0, \tau]$. (ii) On the other hand, we may think of assuming here that the the strength $g$ of interaction is much higher than that of the system Hamiltonian $\hat{H}$, so that due to action of $\hat{H}_{int}$ for a small time span $\tau$, it is enough to consider the change in states of $S$ under the action of $\hat{H}_{int}$ only. At the end of the action of the interaction Hamiltonian, the joint state of $S + M$ becomes:

$$\rho_{SM}(\tau) = e^{-ig\tau \hat{A} \otimes \hat{P}_s} (\rho_T \otimes |\phi\rangle \langle \phi|) e^{ig\tau \hat{A} \otimes \hat{P}_s} \times (I_S \otimes I_M - ig\tau \hat{A} \otimes \hat{P}_s) \approx \rho_T \otimes |\phi\rangle \langle \phi| - ig\tau |\hat{A} \otimes \hat{P}_s| \rho_T \otimes |\phi\rangle \langle \phi| \hat{P}_s).$$

Now we post-select the state of $S$ to be $|\psi_f\rangle$. Then $M$ will get collapsed into the following (unnormalized) state:

$$\tilde{\rho}(\tau) = \langle \psi_f | \rho_T | \psi_f \rangle \rho_T |\psi_f\rangle \langle \psi_f| - ig\tau \left[ \langle \psi_f | \hat{A} \rho_T | \psi_f \rangle \hat{P}_s \rho_T |\psi_f\rangle \langle \psi_f| - \langle \psi_f | \rho_T \hat{A} | \psi_f \rangle \langle \psi_f| \hat{P}_s \right] \approx \langle \psi_f | \rho_T | \psi_f \rangle \times \eta(\tau),$$

where

$$\eta(\tau) = \rho_T |\psi_f\rangle \langle \psi_f|.$$
with the corresponding normalized collapsed state of $M$ being given by:

$$
\eta(\tau) = e^{-i\gamma A_{\tau}}|\phi\rangle \langle \phi| e^{i\gamma A_{\tau}}, \tag{6}
$$

and the corresponding weak value is given by:

$$
A_w = \frac{\langle \psi_f|\hat{A}\rho_T|\psi_f\rangle}{\langle \psi_f|\rho_T|\psi_f\rangle}. \tag{7}
$$

Using the value of $A_w$ together with a priori knowledge of $|\psi_f\rangle$, $\hat{A}$, and the energy eigen spectrum of the system Hamiltonian $\hat{H}$, one can, in principle, find out the value of the temperature $T$ – with the help of eqn. (7). Let us note in passing that the operator $A$ must commute with the relevant energy eigenbasis, else the weak value ceases to depend on the inverse temperature $\beta$, and thus, measuring the weak value furnishes no thermometric advantage.

**High temperature regime** - Let us now consider the case where the bath temperature is high, that is, $\beta \to 0$. Thus, we can replace $e^{-\beta} \approx 1 - \beta$. Now, assuming the spectral decomposition of $\hat{A}$ with eigenvalues $a_i$ and corresponding eigenstates $|a_i\rangle$ for $j = 1, 2, \ldots, d$, in conjunction with eqn. (7), helps us to obtain the following expression for the weak value.

$$
A_w = \frac{\sum_{j=1}^{d} a_i e^{-\beta E_i} \langle \psi_f|a_i\rangle \langle a_i|\psi_i\rangle \langle \psi_i|\psi_f\rangle}{\sum_{i=1}^{d} e^{-\beta E_i} \langle \psi_f|\psi_i\rangle^2}
$$

$$
\approx \frac{\langle \psi_f|\hat{A}|\psi_f\rangle - \beta \langle \psi_f|\hat{H}|\psi_f\rangle}{1 - \beta \langle \psi_f|\hat{H}|\psi_f\rangle}
$$

$$
\approx \left( \langle \psi_f|\hat{A}|\psi_f\rangle - \beta \langle \psi_f|\hat{H}|\psi_f\rangle \right) \left( 1 + \beta \langle \psi_f|\hat{H}|\psi_f\rangle \right)
$$

$$
\approx \langle \psi_f|\hat{A}|\psi_f\rangle + \beta \langle \psi_f|\hat{A}|\psi_f\rangle \times \langle \psi_f|\hat{H}|\psi_f\rangle - \langle \psi_f|\hat{H}|\psi_f\rangle. \tag{8}
$$

Inverting this expression, in the high temperature limit, the inverse temperature in expressible in terms of the weak value of the observable $A$ as

$$
\beta \approx \frac{A_w - \langle \psi_f|\hat{A}|\psi_f\rangle}{\langle \psi_f|\hat{A}|\psi_f\rangle \times \langle \psi_f|\hat{H}|\psi_f\rangle - \langle \psi_f|\hat{H}|\psi_f\rangle}. \tag{10}
$$

Let us now analyze the right hand side of the above result in further detail. We denote the standard deviation of an observable $O$ as $\Delta O$. According to Vaidman’s formula [35], $\Delta|\psi_f\rangle = \langle O|\psi_f\rangle + \Delta A|\psi_f\rangle$, which implies the following expression for the weak value

$$
A_w = \langle O \rangle + \Delta A \frac{\langle \psi_f|\rho_T|\psi_f\rangle}{\langle \psi_f|\rho_T|\psi_f\rangle}. \tag{11}
$$

Here $|\tilde{\psi}\rangle$ indicates that it is a state orthogonal to $|\psi\rangle$. Similarly applying Vaidman’s formula for the density operator $\rho_T$ yields $\rho_T = \langle \psi_f\rangle + \Delta \rho_T|\psi_f\rangle$, where $|\psi_f\rangle$ is another state perpendicular to $|\psi_f\rangle$. Plugging in this expression to the earlier equation for weak value of $A$ yields the following expression for the inverse temperature

$$
\beta = -\frac{\Delta A \Delta \rho_T}{\text{Cov}(A, H)} \frac{\langle \psi_f|\tilde{\psi}\rangle}{\langle \psi_f|\rho_T|\psi_f\rangle}. \tag{12}
$$

For a qubit state, the corresponding orthogonal state is unique. Hence, $|\tilde{\psi}\rangle = |\tilde{\psi}\rangle$, which, when plugged in yields the following expression

$$
\beta = -\frac{\Delta A \Delta \rho_T}{\text{Cov}(A, H)} \frac{\langle \psi_f|\tilde{\psi}\rangle}{\langle \psi_f|\rho_T|\psi_f\rangle}. \tag{13}
$$

The above equation may be of independent interest.

**Qubit case** - Let us now consider the generic situation in the case of the simplest non-trivial quantum system, which is a qubit, for arbitrary temperature. Consider $H = \sum_{j=1}^{d} E_j|\psi_j\rangle \langle \psi_j|$. If we take now $\hat{A} = -i|\psi_1\rangle \langle \psi_2| + i|\psi_2\rangle \langle \psi_1|$ and $\tilde{\psi}_f = (1/\sqrt{2})(|\psi_1\rangle + |\psi_2\rangle)$, then the weak value $A_w$ is given by:

$$
A_w \equiv \langle \psi_f|\hat{A}\rho_T|\psi_f\rangle/\langle \psi_f|\rho_T|\psi_f\rangle = i e^{-\beta E_1} - e^{-\beta E_2}. \tag{14}
$$

Note that here $\langle \psi_f|\tilde{\psi}_f\rangle = 0$, $\langle \psi_f|\hat{H}|\psi_f\rangle = (E_1 + E_2)/2$, $\langle \psi_f|\tilde{\psi}_f\rangle = i(E_1 - E_2)/2$. For the high temperature limit, we have the following (approximate) expression for the inverse temperature of the bath in the qubit case, which may be shown to be consistent with (10).

$$
\beta \approx -\frac{2iA_w}{E_2 - E_1}. \tag{15}
$$

Now, based upon the existing methods of identifying/measuring the weak value $A_w$ [32, 33], one can, in principle, get an estimate of the (inverse) temperature $\beta$ in eqn. (15). One of the advantages of the present scheme is that the measuring apparatus is brought into contact with the heat bath for a very short time, hence reducing the chance of damage to the apparatus.

**IV. PRECISION IN MEASUREMENT OF TEMPERATURE**

It is natural to wonder about the optimal temperature window where the present scheme works best, that is, most precisely. The usual procedure for determining the precision of quantum thermometers is through finding the corresponding quantum Cramer Rao bound. In this section, we adopt a different ansatz. We restrict to qubit systems for simplicity. However, the present analysis can be extended to higher dimensions in a similar fashion.

**A. Precision analysis for imperfect thermalization**

Let us assume that the initial pre-measurement state is not exactly a thermal state, but very close to it, and written in the following manner

$$
\rho_T^{(0)} = (1 - \delta)\rho_T + \delta |\chi(\theta, \phi)\rangle \langle \chi(\theta, \phi)|, \tag{16}
$$

where $0 \leq \delta \ll 1$, and $|\chi\rangle$ is a random pure qubit state with corresponding Bloch sphere parameters $\theta$ and $\phi$. Physically
this indicates imperfect thermalization, which is experimentally relevant, especially in situations where the thermalization timescale is not extremely fast compared to the time available for sensing. The corresponding weak value is denoted by \( A'_w \). Now, an experimentalist may use the formula (14) to find the apparent inverse temperature \( \tilde{\beta} \) of the bath as

\[
\tilde{\beta} = \frac{2}{E_2 - E_1} \arctanh(-iA'_w(\delta))
\]

(17)

It is of obvious practical interest to us that the inferred value of temperature does not change wildly if the thermalization is imperfect. In order to quantify this, we invoke the idea of quantifying a relative error, which is the difference between the temperature furnished from imperfect thermalization, and the genuine temperature of the bath, divided by the bath temperature, and scaled by the imperfection \( \delta \). The squared relative error introduced through imperfect thermalization may thus be taken to be \( |\beta - \beta|^2 / (\delta^2 \beta^2) \). It is this quantity we shall concentrate upon. To remove the effect of \(|\chi\rangle\), which may conceivably capture some information about the state in which the probe was initialized, we shall finally average over the pure states \(|\chi\rangle\). We will write \( E_2 - E_1 \) as the gap \( \Delta \) from here on. Performing a perturbation expansion for \( A'_w(\delta) \) around \( \delta = 0 \), and retaining terms up to first order in \( \delta \), the expression for \( \tilde{\beta} \) is given by

\[
\tilde{\beta} = \frac{2}{\Delta} \arctanh \left( c_1 \tan \frac{\beta \Delta}{2} - ic_2 \right),
\]

(18)

where \( c_1 = 1 - \frac{\delta \langle \psi | A | \psi \rangle}{\langle \psi | (A^2)^{1/2} | \psi \rangle} \), and \( c_2 = \frac{\delta \langle \psi | A | \psi \rangle \langle \psi | (A^2)^{1/2} | \psi \rangle}{\langle \psi | (A^2)^{1/2} | \psi \rangle} \). Note that, \( c_1 \) and \( c_2 \) are functions of the Bloch sphere angles \( \{\theta, \phi\} \) of the state \( \chi \). Thus, averaging over them, the resulting root mean squared relative error \( N_{\tilde{\beta}} \) for the particular weak measurement strategy adopted in the previous section, is written as a function of inverse temperature as

\[
N_{\tilde{\beta}}^2 = \frac{1}{4\pi} \int \int \sqrt{\frac{\beta - \beta'}{\beta' - \beta^2}} \frac{\sin \theta \sin \phi}{\beta^2 \delta^2} \sin \theta \sin \phi \frac{d\theta d\phi}{\beta^2 \Delta^2}\n
\]

Now, neglecting the second and the higher order coefficients of \( \delta \), we note that the expression for relative RMS error is written as

\[
N_{\tilde{\beta}}^2 = \frac{1}{4\pi} \int \int \frac{4|v_1 \tan \frac{\beta \Delta}{2} + iv_2|^2}{\Delta^2 \beta^2 (1 - \tan^2 \frac{\beta \Delta}{2})^2} \sin \theta \sin \phi \frac{d\theta d\phi}{\beta^2 \Delta^2}\n
\]

(20)

where \( v_1 = (1 - c_1) / \delta \), and \( v_2 = c_2 / \delta \). At this point, let us fix the observable \( A = -i |\psi_1\rangle \langle \psi_2 | + i |\psi_2\rangle \langle \psi_1 | \), as in the previous section, and assume the arbitrary post selected state \( |\psi_f\rangle = \cos(\xi/2)|\psi_1\rangle + e^{i\nu} \sin(\xi/2)|\psi_2\rangle \). We also assume, without loss of generality, that the energy gap \( \Delta = 1 \). The expression for root mean squared relative error \( N_{\tilde{\beta}} \) is given by

\[
\sqrt{\frac{(13 - \cos 2\nu) \cosh \beta - 4 \cos 2\nu \sin^2 \nu \cos^2 (\xi/2) - (3 + \cos 2\nu) \sin^2 \nu \cosh(\beta/2)}{3\beta [\cosh(\beta/2) + \cos \xi \sinh(\beta/2)] [1 - \tan^2(\beta/2)]}}.
\]

(21)

For specific choices of the post selected state, the above equation yields the expression for error, and consequently, the inverse quantity signifies the precision of measurement. For example, assuming \( |\psi_f\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle + |\psi_2\rangle) \), as in the previous section, the relative error reads as

\[
N_{\tilde{\beta}}^2 = \sqrt{\frac{1 + 4 \cosh \beta + 3 \cosh 2\beta}{3\beta^2}}.
\]

(22)

Fig. 2 shows that the precision, which is defined as the inverse of the relative error, attains a relatively narrow peak at a finite temperature, which determines the best operating window of the scheme. While the qubit thermometric schemes based on the QFI are different in conception than the present scheme, it is nonetheless noteworthy that the phenomenon of a narrow peak in the precision corresponding to an optimal temperature window is present in both the cases. We also note that for the QFI based analysis of optimal qubit thermometric probes with unit energy gap, the peak is situated at \( T \approx 0.41 \) [11, 28], which is obtained through solving the transcendental equation [11, 28] \( e^{1/T} = (1 + 2T)/(1 - 2T) \). In comparison, in the present scheme, for a specific post-selected state \( |\psi_f\rangle \), the location of the peak for optimal precision is ob-
FIG. 3. (color online) The temperature at which the optimal precision is achieved is plotted against the parameters $\xi$ and $\nu$ of the post selected state $|\psi_f\rangle$.

FIG. 4. (color online) The magnitude of precision at optimal temperature is plotted against the parameters $\xi$ and $\nu$ of the post selected state $|\psi_f\rangle$.

FIG. 5. (color online) The magnitude of precision is plotted against the parameters $\xi$ and $\nu$ of the post selected state $|\psi_f\rangle$ for (left) low temperature $T = 0.05$, and (right) high temperature $T = 100$.

in the previous section, the equation takes the form

$$\beta = \frac{3 \cosh \beta - 1}{3 \sinh \beta - 2 \tanh \frac{\beta}{2}},$$  \hspace{1cm} (23)

which has the solution $T \approx 0.79$. The temperatures corresponding to optimal precision for other choices of post-selected states are depicted in Fig 3. Interestingly, for every $|\psi_f\rangle$, the corresponding optimal temperature is significantly higher than 0.41. This indicates the possibility that the present scheme may be better than the QFI based one for the relevant temperature range. It is natural to wonder about the post-selected state $|\psi_f\rangle$ which corresponds to maximum precision. From Fig 4 as well as Fig 5, it may be concluded that for any arbitrary temperature, $|\psi_f\rangle$ close to $|\psi_1\rangle$ maximizes the precision. Here, let us add a note of caution, from the definition, the weak value is independent of the temperature when $|\psi_1\rangle$ is exactly $|\psi_f\rangle$, hence measuring the weak value furnishes no information about the temperature. Thus, we must take $|\psi_f\rangle$ to be extremely close to $|\psi_1\rangle$, but not identically equal.

B. Precision analysis for unsharp post-selection

Let us now consider another potential source of error in the present scheme, that is, the post selection may be unsharp. Let us assume the unsharp post-measurement qubit state in the form

$$\rho_f^{(\epsilon)} = (1 - \epsilon)|\psi_f\rangle\langle\psi_f| + \frac{\epsilon}{2}\mathbb{1}$$  \hspace{1cm} (24)

Thus, the corresponding perturbed weak value of the observable $A$ is given by

$$A_w' = \frac{\text{Tr}(\rho_f^{(\epsilon)} A \rho_f^{(\epsilon)})}{\text{Tr}(\rho_f^{(\epsilon)} \rho_f^{(\epsilon)})} \approx A_w \left[ 1 + \epsilon \frac{\text{Tr}(A \rho_f^{(\epsilon)}) - 1}{\langle \psi_f | \rho_f^{(\epsilon)} | \psi_f \rangle} \right] + O(\epsilon^2)$$  \hspace{1cm} (25)

Now, using (14), the corresponding expression for shifted inverse temperature is $\beta = \frac{2}{3} \arctanh(-\delta A_w^{(\epsilon)})$. From which, the formula for the squared relative error incurred is given by

$$N_{\beta}^2 = \frac{|\beta - \beta_0|^2}{\epsilon^2 \beta_0^2} = \frac{4}{\Delta^2 \beta_0^2 (1 - \tanh^2(\beta \Delta/2))^2} \frac{\text{Tr}(A \rho_f^{(\epsilon)}) - 1}{\langle \psi_f | \rho_f^{(\epsilon)} | \psi_f \rangle}.$$  \hspace{1cm} (26)

As in the previous section, if one chooses $A = -i(|\psi_1\rangle\langle\psi_2| + |\psi_2\rangle\langle\psi_1|)$, then, assuming $\Delta = 1$ without loss of generality, the corresponding expression for relative error reads as

$$N_{\beta} = \frac{4}{\beta \left[ 1 - \tanh^2 \frac{\beta}{2} \right] \left( 1 + \cos \xi \tanh \frac{\beta}{2} \right)},$$  \hspace{1cm} (27)

where $\xi$ is the polar angle of the pure post selection state $|\psi_f\rangle$. For the specific choice $|\psi_f\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ in the last section, this amounts to the following expression for precision, which is defined as the inverse of the error

$$1/N_{\beta}(\langle+\rangle) = \frac{4}{\beta \left[ 1 - \tanh^2 \frac{\beta}{2} \right]},$$  \hspace{1cm} (28)
in the low temperature regime.

ical scheme, the weak measurement protocol should work best in line with the intuition that, as a distinctly quantum mechanical scheme, the weak measurement protocol should work best in the higher temperature regime.

An interesting feature we observe in Fig. 7 is that there is a shift in the optimal temperature window towards the right, and higher temperatures is associated with the reduction in the optimal precision attainable through the present scheme. This is in line with the intuition that, as a distinctly quantum mechanical scheme, the weak measurement protocol should work best in the low temperature regime.

It can be seen from Fig. 6 that the precision, defined as the inverse of the relative error, attains a peak at some temperature, which determines the corresponding optimal temperature window for thermometry. Fig. 7 reveals that the optimal temperature for the scheme varies from $T_{\text{opt}} \approx 0.54$ to $T_{\text{opt}} \approx 1.12$. Solution of the following transcendental equation determines the location of the optimal temperature $T^*$ for any given post-selected state $|\psi_f\rangle$ with the polar angle $\xi$:

$$\cos \xi = \frac{\sinh \frac{1}{T} - T^*(1 + \cosh \frac{1}{T})}{T^* \sinh \frac{1}{T} - \cosh \frac{1}{T} + 2}$$

An interesting feature we observe in Fig 7 is that there is a shift in the optimal temperature window towards the right, and higher temperatures is associated with the reduction in the optimal precision attainable through the present scheme. This is in line with the intuition that, as a distinctly quantum mechanical scheme, the weak measurement protocol should work best in the low temperature regime.

V. DISCUSSIONS

We have presented a protocol for measuring temperature of a quantum mechanical bath through measuring weak values on a probe which is in thermal equilibrium with the bath. Our protocol relies on careful choice of Hamiltonian, which is generating the weak interaction, and the basis of post-selection measurement. Although our protocol is applicable to probes of any dimension, we restrict to qubit probes and compared our result with other thermometric schemes which employ POVM measurements on the probe and maximization of quantum Fisher information (QFI). We find that there is a narrow operational window of temperature values which can be optimally measured through our protocol, a feature characteristic to other thermometric schemes using QFI. Moreover, in our protocol the peak value of the optimal temperature window is shifted towards higher temperature compared to the QFI schemes. It may be helpful to investigate, in future works, the possible reason behind such a phenomenon. Interestingly, we find a trade off between shifting the optimal temperature window to higher temperatures and the optimal precision.

We expect that the present work will turn out to be useful in the context of improving techniques for precise and efficient temperature estimation of nanoscale thermodynamic systems. Several issues however, are left for future investigation. Firstly, a comprehensive description of the present scheme for higher dimensional probes, including harmonic oscillator probes, remains to be explored. In particular, we showed in the present work that the location of the optimal operating window depends on choice of post-selection, even for a probe with fixed energy spectrum. It is of some interest to determine whether the location of operating window can be made even more tunable by changing the post-selection in the higher dimensional cases. Additionally, some other questions emerge from the idea of weak thermometry. Firstly, in the usual QFI based approach to thermometry, optimizing the precision for a qudit probe leads to the optimal energy spectrum of the probe to be a highly unphysical one, namely, a gapped ground state and all the other energy eigenstates are energetically degenerate [11]. It is an open problem to figure out the form of the optimal energy spectrum of the probe in the arbitrary dimensional case, and check whether the corresponding Hamiltonian is experimentally feasible. On the basis of the present study, it is not feasible to fairly compare the magnitude of precision offered by the weak measurement scheme, and the magnitude of precision offered through the QFI scheme. Therefore, it may be useful to do a resource comparison between the two methods in the future. In particular, finding the information disturbance tradeoff [7] for thermometers working with weak measurement schemes is an interesting avenue of future work. Depending upon the choice of the actual physical systems for the probe, the operating windows for the precise measurement of temperature will have different signatures for systems under consideration, which we would like to explore in future.
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