QCD at low energy: a many-body approach

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Abstract. A review is given on recent results in the treatment of an arbitrary number of orbital levels in low energy QCD. For the pure quark part, analytic results for the dominant part of the Hamiltonian are presented. Possible extensions, including dynamic gluons, are discussed.

1. Introduction

QCD is considered as the correct theory for strong interactions \cite{1, 2}. Unfortunately, at low energy the coupling constant is very large and perturbative calculations are impossible. Up to now, only lattice-QCD \cite{3} is able to derive non-perturbative results from first principles. Lattice calculations require an enormous numerical effort and results, like the level ordering of the spectrum, are not well understood. As an alternative, phenomenological models were proposed \cite{4, 5, 6} (just to name a few) which have the advantage of being easy to apply and transparency in their physical interpretation, but with the disadvantage of involving several parameters. In \cite{7, 8} we presented a method on how to obtain analytical and semi-analytical (involving a simple numerical application) solution.

In this contribution we will shortly resume the results of \cite{7, 8}, concerning the semi-analytical solution including an arbitrary number of orbital levels for quarks and anti-quarks. Gluons are considered to be static. After that, we will discuss on possible methods on how include gluons dynamically.
2. The Hamiltonian and the basic Ingredients
The QCD Hamiltonian at low energy is given by [9]

\[ H = H_0 + H_{\text{int}}, \]
\[ H_0 = \int d^3x \psi^\dagger (-i \alpha \cdot \nabla + \beta m_0) \psi + \frac{1}{2} \int d^3x \Pi^2 - \frac{1}{2} \int d^3x A \cdot \nabla A + V_0 \int d^2xd^3y \rho_a(\vec{x}) \rho^a(\vec{y}), \]
\[ H_{\text{int}} = \int d^2x d^3y \rho_a(\vec{x}) \left( \frac{1}{2} K^{(0)}(\vec{x} - \vec{y}) - V_0 \right) \rho^a(\vec{y}) + \frac{1}{2} \int d^3x \left[ B^2 + A \cdot \nabla^2 A \right] - g \int d^2x \psi^\dagger \alpha \cdot A \psi + V_A + V_B + \frac{1}{2} \int d^3x d^3y \rho_a(\vec{x}) \left[ K^{ab}(\vec{x} - \vec{y}) \cdot A - \delta^{ab} K^{(0)}(\vec{x} - \vec{y}) \right] \rho_b(\vec{y}), \]  

(1)

where we divided the Hamiltonian in the dominant \( H_0 \) and a residual part \( H_{\text{int}} \). The \( H_0 \) part contains the kinetic and mass terms of the quark fields \( \psi \), the proposed kinetic gluon term and the color charge density interaction which is a remanent of the Coulomb-QCD interaction. The terms related to the Coulomb interaction are coupled by a certain low energy region (about 0 to 1.5 GeV), to which we will restrict. The other terms in the last two lines of eq. (1) corresponds to the cromo-magnetic and the quark-gluon terms, and the residual Coulomb-QCD interaction respectively. This residual Coulomb-QCD interaction corresponds to the difference between the real Coulomb-QCD kernel \( K^{ab}(\vec{x} - \vec{y}) \cdot A \) and its vacuum expectation value \( K^{(0)}(\vec{x} - \vec{y}) \delta^{ab} \).

In [7, 8] only the dominant part, \( H_0 \), of the Hamiltonian was considered. The \( K \) functions are kernels and some depend also on the gluon field \( A \). These functions can be expanded in the gluons fields and as shown in [10] it suffices to expand them up to third order in \( A \). The main reason is that at low energies the gluons fields represent effective gluons with a mass between 600 and 800 MeV, i.e., more gluons imply higher energies.

As basic ingredients we i) use \( SU_C(3) \) for the color part, ii) \( SU_F(2) \) for the flavor part and iii) static gluons only. The calculations are performed within the Coulomb gauge [1], which has the advantage of a reduced Hilbert space (no ghost contributions) but the price to pay is a complicated Hamiltonian. iv) The calculations are performed in a finite volume, of the size of a hadron. This approximation has later to be relaxed in order to study greater volumes. v) We use the Dirac picture [2], which allows easier access for many-body techniques coming from nuclear physics [11].

The degrees of freedom are described by the fermion creation operator

\[ b_{\alpha(N,l\frac{1}{2})j\lambda,cf}^\dagger = \sum_{m\sigma} (lm, \frac{1}{2} \sigma | j\lambda) b_{\alpha,Nlm,\sigma,cf}^\dagger, \]  

(2)

represented here in their coupled and decoupled form in spin. The first index \( \alpha = \pm \frac{1}{2} \) refer to states in the upper (+ sign) and lower (− sign) orbital level. \( N \) refers to the number of oscillation quanta of the orbital level (i.e., we use the three dimensional harmonic oscillator as the basis), \( l \) is the orbital angular momentum, \( j \) is the total spin with projection \( \lambda \) and \( c, f \)
are the components of color and flavor respectively. Under raising and lowering indices, these operators satisfy [12, 13]

\[ b^\dagger_a^{(N,l,\frac{1}{2})} l^\lambda_{cf} = (-1)^{\frac{3}{2} - \alpha} (-1)^{j - \lambda} (-1)^{\lambda_C} (-1)^{\frac{3}{2} - f} b^\dagger_{-\alpha(N,l,\frac{1}{2}) j - \lambda; cf - f}, \]

where \((-1)^{\lambda_C}\) is a short hand notation for the phase of color-hypercharge \(Y_C\), color-isospin \(I_C\) and projection of the color-isospin \(T_{ZC}\) [12].

3. Analytic and semi-analytic solutions for \(H_0\)

Expressed in terms of the fermion creation and annihilation operators, restricting to the pure quark part, the potential and the kinetic energy of \(H_0\) acquire respectively the form

\[ V = 2V_0 C_2(SU(3)), \]
\[ K = \widetilde{K}_+ + \widetilde{K}_- = \sum_{N,j} \left( \widetilde{K}_N^+ + \widetilde{K}_N^- \right), \]

where \(C_2(SU(3))\) is the second order Casimir operator of \(SU_C(3)\), with eigenvalues \((\lambda_C^2 + \lambda_C \mu_C + \mu_C^2 + 3\lambda_C + 3\mu_C)\) and

\[ \widetilde{K}_+ = \sum_{N,N',j,\lambda, cf} k_{NN'}^{j} \left[ b^\dagger_{\frac{1}{2}(N,j+\frac{1}{2})} l^\lambda_{cf} b_{\frac{1}{2}(N', j-\frac{1}{2} + \frac{1}{2})} l^\lambda_{cf} \right] + b^\dagger_{\frac{1}{2}(N', j-\frac{1}{2} + \frac{1}{2})} l^\lambda_{cf} b_{\frac{1}{2}(N,j+\frac{1}{2})} l^\lambda_{cf} \]
\[ \widetilde{K}_- = \sum_{N,N',j,\lambda, cf} k_{NN'}^{j} \left[ b^\dagger_{\frac{1}{2}(N,j+\frac{1}{2})} l^\lambda_{cf} b_{\frac{1}{2}(N', j-\frac{1}{2} + \frac{1}{2})} l^\lambda_{cf} \right] + b^\dagger_{\frac{1}{2}(N', j-\frac{1}{2} + \frac{1}{2})} l^\lambda_{cf} b_{\frac{1}{2}(N,j+\frac{1}{2})} l^\lambda_{cf} \]

The factors \(k_{NN'}^{j}\) can be retrieved from [8].

The terms in (5) connect orbital levels at negative energy to the ones at positive energy, changing the orbital angular momentum. In Fig. 1 the situation is illustrated, indicating only the positive energy levels, in representation also for the negative energy levels. The terms of the kinetic energy connect only states with the same total spin \(j\). As a consequence it suffices to restrict to an arbitrary but fixed \(j\). A solution obtained for an arbitrary but fixed \(j\) implies many solutions for arbitrary \(j\).

Restricting to two or three levels (positive and negative energy) we were able to obtain analytic results, rewriting the Hamiltonian in such a way that it exhibits an \(SU(2)\) structure of pseudo-spin. This could not be repeated for more than three orbital levels.

For more than three levels, one has to apply two successive unitary transformations. The first one mixes, for a given \(j\), only states with the same orbital spin, leading to a partial diagonalization. The matrix elements of this transformation are denoted by \(\alpha_{Nk}^{j}\), for orbital spin \(l = j + \frac{1}{2}\), and \(\beta_{Nk}^{j}\), for orbital spin \(l = j - \frac{1}{2}\). The index \(N\) denotes the main oscillator quantum number and \(k\) the new orbital index. The new quark creation and annihilation operators are then a combination of all quark creation and annihilation operators of the same orbital spin, \(l\), but different \(N\). The second unitary transformation is a BCS transformation which diagonalizes the orbital index. For more details, please consult [8].
Figure 1. Illustration of the effect of the Hamiltonian on the basis states. The operators, which
appear in the kinetic energy, can be divided into columns of operators, related to a given spin
$j$, which commute with those of different columns. Only the positive energy states are plotted.
The ellipses indicate which states are connected through the interaction terms appearing in the
kinetic energy. Columns of different $j$ are separated by a vertical dashed line.

For the Hamiltonian, we finally obtained the analytic result

\[
K_{BCS} = \sum_k \left\{ \epsilon_{bkj+\frac{1}{2},k} b_{k}^{\dagger} b_{j+\frac{1}{2},k} + \epsilon_{dkj+\frac{1}{2},k} d_{k}^{\dagger} d_{j+\frac{1}{2},k}
\right. \\
\left. +\epsilon_{bkj-\frac{1}{2},k} b_{k}^{\dagger} b_{j-\frac{1}{2},k} + \epsilon_{dkj-\frac{1}{2},k} d_{k}^{\dagger} d_{j-\frac{1}{2},k} \right\},
\]

having introduced the notation $A B = \sum_{\lambda c f} A_{\lambda c f} B^{\lambda c f}$. The Hamiltonian is a function in terms
of new creation and annihilation operators, where $b_{k}^{\dagger}$ describe quarks and $d_{k}^{\dagger}$ anti-quarks.
The numerical part involves the solution of bi-quadratic equations for the unknown variable
$\alpha_{N,k}$ and $\beta_{N,k}$. The solution of the bi-quadratic equations is obtained numerically, which is the origin
of the notation semi-analytic.

The spectrum of (6) is trivial, because only number operators are involved. Some examples
for hadronic states are

\[
\left[ b_{j+\frac{1}{2},k}^{\dagger} \otimes d_{j+\frac{1}{2},k'}^{\dagger} \right]_{J=0}^{(0,0)cF} |BCS\rangle,
\]

\[
\left[ b_{j+\frac{1}{2},k}^{\dagger} \otimes d_{j+\frac{1}{2},k'}^{\dagger} \right]_{J=0}^{(0,0)cF} |BCS\rangle,
\]

with the energy $\left( \epsilon_{bkj+\frac{1}{2}} + \epsilon_{dkj+\frac{1}{2}} \right)$. Because each creation operator represent a complicated sum
into the former quark-antiquark operators, they represent a parton, where each parton is given by a valence quark (anti-quark) and a sea of quark-antiquarks pairs.

In Fig. 2 the behavior of the lowest one-particle excitations is shown, as a function in the number of orbitals taken into account. The energy of a meson state is then the sum of two one-particle energies. For a large number of orbital levels the lowest meson excitation is about 150 MeV for each parton.

This puts the first meson excitation at 300 MeV. There is still a degeneration related to the coupling to different flavors, i.e., the $F = \frac{1}{2}$, $J^{π}0^-$ multiplet is degenerate with the $F = \frac{1}{2}$, $J^{π}1^-$ multiplet. The degeneration will be lifted, once the interaction terms of the Hamiltonian are taken into account.

The diagonalization of the quark part represents an important step towards the complete diagonalization of the QCD Hamiltonian. It represents a prediagonalization of the dominant part of the QCD Hamiltonian. Note that it refers to the diagonalization of the single-particle levels. Including the interaction terms we will diagonalize again only the single-particle levels. The prediagonalization represents a good chance that the new off-diagonal matrix elements (in the single-particle space) are not too large, as was the case of the kinetic terms of the quarks part), and that a convergence can be reached not including too many single particle quark and gluon states.

4. Including gluons
In this section we discuss possible methods to include dynamically gluons into the treatment. The discussion will not be complete because it is an on-going investigation. Nevertheless, it will show that standard many-body technique, leaned from nuclear physics [11], may suffice to obtain a complete treatment of QCD at low energy.
4.1. Coherent state method
For the gluons and quarks-antiquarks we use respectively the following coherent states [14, 15]

\[ |\gamma \rangle_g = N_g e^{\sum_\mu \gamma_\mu \beta_\mu^\dagger} |0\rangle_g , \]
\[ |\alpha \rangle_q = N_q e^{\sum_\alpha \alpha^\dagger_\mu b_\mu + \sum_\alpha \alpha^\dagger_\alpha d_\alpha} |0\rangle_q , \]  

(8)

where \( b_\mu^\dagger, d_\mu^\dagger \) and \( \beta_\nu^\dagger \) are short-hand notations for the quark-antiquark and gluons respectively, \( \bar{\mu} \) is the adjoint to \( \mu \) and \( N_g, N_q \) are normalization factors for the gluon and quark part respectively. The variables \( \alpha^\dagger_k \) \((k=a, b)\) are Grassmann variables [15] (in this reference one also can find a detailed description on the mathematical properties).

In (8) care has to be taken in the quark sector. Denoting by \( \mathcal{N} \) the total degrees of freedom of the quark sector, the exponential expansion stops at the power of \( \mathcal{N} \), i.e., the exponential in (8) in reality are finite sum. Further below we will shortly discuss the error involved, using still the exponential notation.

The expectation of the creation and annihilation operators with respect to these coherent states are

\[ g(\beta_\nu^\dagger) = \gamma_\nu , \quad g(\beta_\nu^\dagger) = \gamma_\nu , \]
\[ q(b_\mu^\dagger) = \alpha^\dagger_\mu , \quad q(b_\mu^\dagger) = \alpha^\dagger_\mu , \]
\[ q(d_\mu^\dagger) = \alpha^\dagger_\mu , \quad q(d_\mu^\dagger) = \alpha^\dagger_\mu . \]  

(9)

The expressions in (9) are only approximate. The coherent state for quarks is in reality not an exponential but, with \( X = \sum_\mu \alpha^\dagger_\mu b_\mu + \sum_\mu \alpha^\dagger_\mu d_\mu \),

\[ e^X |0\rangle = \sum_{n=0}^{\mathcal{N}} \frac{X^n}{n!} |0\rangle . \]  

(10)

Applying a fermion annihilation to it results in a sum from \( n = 0 \) to \( n = (\mathcal{N} - 1) \). The error involved originates from the last term, with \( n = \mathcal{N} \), which has to added when we write the final expression in terms of an exponential. The error is proportional to \( 1/\mathcal{N}! \). Assuming that the variables of the coherent state are of the order of one and that \( \mathcal{N} \) is at least 48 (two spin degrees of freedom, two for flavor, three for color and four orbital states, two at positive and two at negative energy), the error involved is negligible.

The next step is to expand the creation and annihilation operators around these expectation values [15, 16], i.e., around a new vacuum,

\[ \beta_\nu^\dagger = \gamma_\nu + \delta \beta_\nu^\dagger , \quad \beta_\nu = \gamma_\nu + \delta \beta_\nu^\dagger , \]
\[ b_\mu^\dagger = \alpha^\dagger_\mu + \delta b_\mu^\dagger , \quad b_\mu = \alpha^\dagger_\mu + \delta b_\mu^\dagger , \]
\[ d_\mu^\dagger = \alpha^\dagger_\mu + \delta d_\mu^\dagger , \quad d_\mu = \alpha^\dagger_\mu + \delta d_\mu^\dagger . \]  

(11)

This mapping conserves the commutation relations of the gluon operators and the anti-commutation relations of the quark operators.

The final step is to apply this mapping to the Hamiltonian and expand this operator up to second order in the creation and annihilation operators. If we denote in general by \( c_i^\dagger \) and \( c^\dagger \) the creation and annihilation operator for both sectors (fermion and boson), fermions and bosons, and exclude third order interaction, the general structure of the Hamiltonian will be

\[ H \approx V(\alpha, \gamma) + \sum_{i,j} \left( A_{ij} c_i^\dagger c_j + B_{ij} c_i^\dagger c_j + B_{ij} c_i c_j \right) . \]  

(12)
This type of Hamiltonian can be solved using BCS and RPA [11] methods. However, in general, also third order interaction terms appear and will need some special attention.

This is as far as we got. In the future we will apply the steps proposed and investigate the possibility to solve the Hamiltonian (12).

4.2. Mixed treatment for quark and gluons

Instead of applying the coherent state method [14] to both sectors, quarks and gluons, one can use it for the gluon sector only. The advantage is that no Grassmann variables have to be dealt with. Also, the gluon sector is the most complicated part, involving large powers in the gluon fields, and a quadratic expansion in $\delta \beta^\dagger_\nu$ and $\delta \beta^\nu$ will simplify the structure of the interacting part. Solving the Hamiltonian as a function in the coherent state parameters $\gamma_\nu$, will then determine the optimal values for $\gamma_\nu$ minimizing the ground state.

The dynamical part of the gluons are then introduced by expanding around the $\gamma_\nu$ values (see (11), introducing the $\delta \beta^\dagger_\nu$ and $\delta \beta^\nu$ boson operators.

In what follows, we discuss shortly the structure of the pure gluon part (without any mixing to the quark sector), which would be the result of the before mentioned expansion. Expanding the pure gluon sector up to second order terms in $\delta \beta^\dagger_\nu$ and $\delta \beta^\nu$, the structure of the gluon Hamiltonian is as follows

$$H_g(\delta \beta^\dagger, \delta \beta) = \sum_v E_v \delta \beta^\dagger_v \delta \beta^v + \sum_{vw} V_{vw} \delta \beta^\dagger_v \delta \beta^w$$

$$+ \frac{1}{2} \sum_{vw} W_{vw} \left( \delta \beta^\dagger_v \delta \beta^\dagger_w + \delta \beta^v \delta \beta^w \right),$$

with $v$ and $w$ is a short hand notation for all possible indices. One possibility is to apply BCS or RPA [11]. For RPA one maps this Hamiltonian to a new one which depends on new boson creation and annihilation operators $\Gamma^\dagger_n, \Gamma^n$, i.e.,

$$H(\Gamma^\dagger, \Gamma) = \sum_n \Omega_n \left( \Gamma^\dagger_n \Gamma^n + \text{const} \right),$$

with

$$\Gamma^\dagger_n = \sum_v \left[ X_{v,n} \beta^\dagger_v - Y_{v,n} \beta^v \right],$$

with $X_{v,n}$ and $Y_{v,n}$ being some functions to be determined [11]. This procedure does nothing else than contracting a new vacuum, due to the effect of the residual interaction, around which a harmonic expansion is applied.

Involving in addition the quark fields require more involved considerations. One possibility is to diagonalize the pure quark sector, as given above, and separately the pure gluon sector, as indicated in this section, and afterwards to include the quark-gluon mixing terms, already expanded up to second order on the gluon fields.

5. Conclusions

Methods to treat QCD at low energies were reviewed. Taking into account only static gluons and restricting to a finite volume, semi-analytic results were obtained, involving an arbitrary number of orbital quark states.

In the next step the gluons will be included dynamically. We reviewed a couple of procedures, with great prospects. They show that probably standard many-body techniques, applied in the right order, will be sufficient to determine the structure of the hadron spectrum at low energy.
Note, that the QCD Hamiltonian, used here, depends only on two parameters, namely the coupling constant $g$ and the size of the volume. The latter can be removed, considering the limit of large volumes.

This gives us hope to possibly derive the structure of phenomenological models in future.

This is still an ongoing investigation and the future will show if we are on the right track.

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