Scaling of crossing probabilities for the $q$-state Potts model at criticality

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Abstract

We present study of finite-size scaling and universality of crossing probabilities for the $q$-state Potts model. Crossing probabilities of the Potts model are similar ones in percolation problem. We numerically investigated scaling of $\pi_s$ - the probability of a system to percolate only in one direction for two-dimensional site percolation, the Ising model, and the $q$-state Potts model for $q = 3, 4, 5, 6, 8, 10$. We found the thermal scaling index $y = \frac{1}{\nu}$ for $q < 4$. In contrast, $y \neq \frac{1}{\nu}$ for $q = 4$.

I. INTRODUCTION.

In recent years the scaling and universality of the $q$-state Potts model and percolation are subject of intensive study [1–5]. There are two kinds of quantities of interest in percolation problem: cluster distribution functions and crossing (spanning) probabilities. With $n_s(p)$ - the number of clusters of size $s$ per lattice site, the $m(p) = s_{\text{max}}n_{s_{\text{max}}}(p) = p(1 - \sum s_{\neq s_{\text{max}}}n_s(p))$ is probability of a lattice site to belong to ”infinite” cluster (the cluster of maximum size) and $\chi(p) = \sum_{s \neq s_{\text{max}}} s^2 n_s(p)$ is the mean size of a finite cluster. For an infinite lattice near the percolation point $p_c$, $\chi(p) \sim |p - p_c|^{-\gamma}$, $m \sim (p - p_c)^{\beta}$, $p > p_c$ [6]. This quantities corresponds to magnetization $m(T)$ and magnetic susceptibility $\chi(T)$ of the Potts model.

Critical properties of percolation may be found from the study of crossing probabilities $\pi(p; L)$. Crossing probabilities were used by Reynolds, Stanley and Klein for renormalization group study of percolation [7,8]. Universality of this crossing probabilities as functions of an aspect ratio $r$ in the critical point was found in [9]. The exact results for $\pi_h(r)$ and $\pi_{hv}(r)$ in the percolation point was developed by Cardy [10] and Watts [11] for different $r$.

Scaling properties of the function $\pi_h(p; L)$ - the probability, that a system percolates in the horizontal direction for two- and three-dimensional lattices was investigated by Hu, Lin, Chen [12–14]. They show, that $\pi_h(p)$ is a universal function of scaling argument $x = (p - p_c) L^{\frac{1}{\nu}}$, where $p_c$ - critical point, $L$ - lattice size, $\nu$ - correlation length exponent.

Universality of the crossing probability $\pi_h(\beta)$ for Ising model was investigated by Langlands et. al. [15].

In this paper we present results of numerical investigations of the crossing probability $\pi_s(\beta; L)$ (the probability of a system to percolate only in the one, horizontal either vertical,
direction) for the $q$-state Potts model. We found, that $\pi_s(\beta; L)$ is a universal function of the scaling variable $(\beta - \beta_c)L^\nu$, where for the Ising model and the $q = 3$ Potts model the thermal scaling power $y = \frac{1}{\nu}$. Further, the limit of the crossing probability at the critical point $\pi_s(p_c; L)$ is nonvanishing for the Ising model and for the Potts model with $q = 3, 4$. In the case of the Potts model $q = 5, 6, 8, 10$ percolation probabilities goes to zero, when the system size $L$ tends to infinity.

The crossing probability in the critical point behaves as $\pi_s(\beta_c; L) \simeq A - a \log(L)$ for $L < \xi_c$, while the dependence is exponential $\pi_s(\beta_c; L) \simeq \bar{A} \exp(-\bar{a}L)$ for $L > \xi_c$, where $\xi_c$ is the correlation length in the critical point.

II. CROSSING PROBABILITIES.

A. Crossing probabilities for percolation.

Let us consider the site percolation on the square lattice. Each site is occupied with probability $p$ and is empty with probability $1 - p$. Let us denote by $\omega$ the sample, i.e. the fixed distribution of occupied sites on the lattice. The full set of samples $\Omega$ consists of $2^{L^2}$ configurations.

Let us introduce indicator functions of crossing $I_h(\omega)$ in horizontal and $I_v(\omega)$ in vertical directions [10]. For example, if the cluster spans the sample horizontally then $I_h(\omega) = 1$ otherwise $I_h(\omega) = 0$. Let us $S(\omega)$ - the number of occupied sites in the configuration $\omega$.

The crossing probabilities for the site percolation on the square lattice are introduced by

$$\pi_h(p; L) = \sum_{\omega \in \Omega} P_{site}(\omega)I_h(\omega) = \sum_{\omega \in \Omega} p^{S(\omega)}(1 - p)^{L^2 - S(\omega)} I_h(\omega)$$

(1)

$$\pi_v(p; L) = \sum_{\omega \in \Omega} P_{site}(\omega)I_v(\omega) = \sum_{\omega \in \Omega} p^{S(\omega)}(1 - p)^{L^2 - S(\omega)} I_v(\omega)$$

(2)

$$\pi_{hv}(p; L) = \sum_{\omega} P_{site}(\omega)I_h(\omega)I_v(\omega) = \sum_{\omega \in \Omega} p^{S(\omega)}(1 - p)^{L^2 - S(\omega)} I_hI_v(\omega)$$

(3)

$$\pi_s(p; L) = \sum_{\omega \in \Omega} p^{S(\omega)}(1 - p)^{L^2 - S(\omega)} [I_h(\omega)(1 - I_v(\omega)) + (1 - I_h(\omega))I_v(\omega)]$$

(4)

The function $\pi_h(p; L)$, defined by expression (1), is the probability, that the lattice with linear size $L$ percolates in the horizontal direction when the occupation probability is $p$. The function $\pi_v(p; L)$ is the probability, that the lattice percolates in the vertical direction, $\pi_{hv}(p; L)$ is the probability, that the lattice percolates in both directions, $\pi_s(p; L)$ is the probability, that the lattice percolates in the only direction (horizontal or vertical). It is clear, that on the square lattice $\pi_h(p; L) = \pi_v(p; L)$ and $\pi_h(p; L) = \frac{1}{2}\pi_s(p; L) + \pi_{hv}(p; L)$. The example of functions $\pi_s(p; L)$ for different $L$ is shown on the Fig 1 a.)
B. The \(q\)-state Potts model as a correlated site-bond percolation.

In the \(q\)-state Potts model the spin variable \(\sigma\) takes values from the set \(\{1, 2, \ldots, q\}\). The probability \(P(\omega)\) of spin configuration \(\omega\) is defined by

\[
P_{\text{Potts}}(\omega) = \exp(-\beta H(\omega))/Z(\beta), \quad Z(\beta) = \sum_\omega P_{\text{Potts}}(\omega), \quad H = -J \sum_{<i,j>} \delta_{\sigma_i,\sigma_j},
\]

where \(H(\omega)\) - the Hamiltonian of the Potts model, \(Z(\beta)\) is the partition function, and \(<i,j>\) is the sum over all neighbor sites. We put through this paper the coupling constant \(J = 1\) and Boltzmann factor \(k_B = 1\).

Fortuin and Kastelleyn propose mapping of \(q\)-state Potts model onto the site-bond correlated percolation [17]. It was used by Swendsen and Wang for their Monte-Carlo cluster algorithm [18]. The single-cluster algorithm was proposed by Wolff [19]. Thermodynamical quantities can be expressed in terms of clusters in correlated site-bond percolation [20,21].

Let the bond between neighbor sites with same values of spin variable \(\sigma_i = \sigma_j\) is closed \(b_{i,j} = 1\) with probability \(r = 1 - \exp(-\beta)\) and opened \(b_{i,j} = 0\) with probability \(1 - r\), and the bond between neighbor sites with different values of spin variables is always opened.

Define the chosen configuration of closed and opened bonds by \(\upsilon\). Then joint distribution of the spin configuration \(\omega\) and the bond configuration \(\upsilon\) is (see [22,21])

\[
P_{\text{Potts}}(\omega, \upsilon) = Z^{-1}(\beta) \prod_{<i,j>} [ (1 - r)\delta_{b_{i,j},0} + r\delta_{\sigma_i,\sigma_j}\delta_{b_{i,j},1} ], \quad Z(\beta) = \sum_\omega \sum_\upsilon P(\omega, \upsilon)
\]

where \(P_{\text{Potts}}(\omega) = \sum_\upsilon P_{\text{Potts}}(\omega, \upsilon)\) is the probability of spin configuration \(\omega\) and \(Z(\beta)\) is partition function for \(q\)-state Potts model. To each spin configuration \(\omega\) in accordance with (6) corresponds a bond configuration \(\upsilon\), when bonds between neighbor sites with different values of spin variables are always opened, and the bonds between sites with equal values of spin variables are closed with probability \(r = 1 - \exp(-\beta)\) and opened with probability \(1 - r\). The subset of sites, connected by closed bonds, called ”physical” cluster. If we assign to spin variables in each ”physical” cluster the random value from the set \(\{1, 2, \ldots, q\}\), we obtain the new spin configuration. This procedure describes the Swendsen-Wang cluster algorithm, used in this paper.

C. Crossing probabilities for the \(q\)-state Potts model.

For the percolation problem the crossing probabilities \(\pi_h(p; L), \pi_v(p; L), \pi_{hv}(p; L), \pi_s(p; L)\) are defined as indicator functions \(I_h(\omega), I_v(\omega), I_{hv}(\omega), I_s(\omega)\) averaged over all site (or bond) configurations in accordance with (1), (2), (3), (4). Crossing probabilities are probabilities for a system with linear size \(L\) to percolate according chosen rules at probability \(p\) of a site (or bond) to be occupied.

For a \(q\)-state Potts model statistical weights of site-bond configuration (in terms of Fortuin-Kasteleyn mapping) are defined by expressions (6), therefore it is natural to define the crossing probabilities through the indicator functions \(I_h(\omega, \upsilon), I_v(\omega, \upsilon)\) averaged over all site and bond configurations.
FIG. 1. Crossing probabilities for percolation and Ising model.

$\pi_s(p; L)$ for percolation (a) and for the Ising model (b)

$\tilde{\pi}_s(x)$ - normalized crossing probabilities for percolation (c) and for the Ising model (d)
\[ \pi_h(\beta; L) = \sum_{\omega} \sum_{v} P_{\text{Potts}}(\omega, v) I_h(\omega, v) \]  

(7)

where \( \pi_h(\beta; L) \) is a probability, that at least one percolating in the horizontal direction "physical" cluster exists, \( \beta = \frac{1}{T} \) is the inverse temperature. By the same way others crossing probabilities are defined.

III. SCALING OF CROSSING PROBABILITY FUNCTION \( \pi_s(\beta; L) \) FOR THE TWO-DIMENSIONAL \( q \)-STATE POTTS MODEL.

A. Features of approximation.

We use the Swendsen-Wang algorithm to generate the different spin configurations. For each spin configuration we generate the bond configuration. Then we decompose the lattice into independent clusters of connected sites using Hoshen-Kopelman [23] algorithm. After that we analyze crossing properties \( I_h(\omega, v) \), \( I_v(\omega, v) \) of this configuration. We average indicator functions \( I_h(\omega, v) \), \( I_v(\omega, v) \) over \( N \) configurations.

\[ \pi_h(\beta) = \frac{1}{N} \sum_{l=1}^{N} I_h(\omega_l), \quad \pi_s(\beta) = \frac{1}{N} \sum_{l=1}^{N} \left[ I_h(\omega_l) (1 - I_v(\omega_l)) + I_v(\omega_l) (1 - I_h(\omega_l)) \right] \]  

(8)

We compute crossing probabilities for 40 – 50 values of \( \beta \) in the interval \( [\beta_c - d\beta(L), \beta_c + d\beta(L)] \), where the width of the interval was proportional \( d\beta \sim L^{-y} \) - see Fig. 1 b). The total number of configurations for every value of \( \beta \) are \( N = 2 \times 10^4 - 15 \times 10^4 \). It should be noted, that \( \beta_c(q = 2) = \log(\sqrt{2} + 1) = 2\beta_c(I_{\text{ising}}) = 0.8813735870... \).

For percolation we generate site configurations by "grand canonical" method [24] - every site is occupied with probability \( p \) and is empty with probability \( 1 - p \). For each concentration \( p \) we generate \( N = 5 \times 10^5 \) configurations. The number of analyzed configurations is the same for small and large lattice sizes. We use four-point shift-register random number generator with maximum length 9689 [25]. The crossing probabilities are not self-averaging by definition (8), and the variance of \( \pi_h \) and \( \pi_s \) not depend upon lattice size.

Then we approximate crossing probabilities \( \pi_s(\beta; L) \) for different \( q \) by the Gauss function

\[ \pi_s(L, \beta) \simeq A(L) \exp \left( -\frac{1}{2} (\beta - \beta_c(L))^2 B^2(L) \right) \]  

(9)

Normalized crossing probabilities \( \tilde{\pi}_s(x) \) looks like Gaussian - see. Fig. 1 c)-d). On this figures we placed normalized crossing probabilities \( \tilde{\pi}_s(x) = \frac{1}{\sqrt{2\pi A(L)}} \pi_s((\beta - \beta_c(L))B(L); L) \) where \( x = (\beta - \beta_c(L))B(L) \). For normalization we use parameters \( A(L), B(L), \beta_c(L) \), which we obtained as a result of approximation of the numerical data by the expression (9).

We dont approve, that the crossing probability is Gauss function, although they look similar Fig. 1 c), Fig. 1 d). We use this approximation like convenient method to compute the amplitude, the inverse variance and the maximum location of this function.

As a result of approximations, for every investigated value of \( q \) and \( L \) we get the set of values \( A(L), B(L), \beta_c(L) \).
B. Scaling of crossing probabilities.

We plot $A$ as a function of $L$ on Fig. 2. For $q \leq 4$ we expect finite size corrections to limit values in the form $aL^{-x}$, as it was shown for percolation [26]. We found, that the functions $A(L)$ for $q = 2, 3, 4$ could be well approximated by

$$A(L) \simeq A_0 + aL^{-x}, \quad q \leq 4$$  \hfill (10)

Results of approximation plotted on Fig. 2 by lines. Second, third and fourth rows of the Table I represents result for amplitude $A_0$, and scaling terms $a$ and $x$ for $q \leq 4$. In the Table I below a row with values of $q$ placed results of approximation for this $q$. The case $q = 4$ is intermediate of models with phase transitions of the first order $q > 4$ and the second order $q \leq 4$ [27].

We can expect, that for $q > 4$ the crossing probability tends to zero with $L$ increasing, because the correlation length in the critical point $\xi_c$ is finite [28]. Values of correlation length $\xi_c$ for $q > 4$ in sixth row of Table I are taken from [28]. We can also expect the different behavior of $A(L)$ for cases $L < \xi_c$ and $L > \xi_c$. For $q = 5, 6, 8, 10$ behavior of $A(L)$ have another form - see Fig. 3.

$$A(L) \simeq A_0 - a \log(L), \quad q > 4, \quad L < \xi_c$$  \hfill (11)

$$A(L) \simeq A_0 \exp(-aL), \quad q > 4, \quad L > \xi_c$$  \hfill (12)

On this figure we use the logarithmic scale for $L$ axis, and we see, that points for $q = 5, 6$ lay on straight lines. So we approximate data for $q = 5, 6$ by the logarithmic low (11). We put results of approximation into seventh and eighth rows of the Table I. Also we plot results of approximation on the Fig. 3 by lines.
FIG. 3. Amplitudes $A(L; q)$ of crossing probabilities $\pi_s(\beta; L)$ in log($L$) scale.

But the data for $q = 8, 10$ looks like straight lines, when we use the logarithmic scale for $A(L)$ axis, while the $L$ axis use the normal scale - see Fig. 4. Therefore we approximate $A(L)$ for $q = 8, 10$ by the exponential low (12). Results of approximation for $q = 8, 10$ by the exponential formula (12) are placed in seventh and eights rows of Table I and are marked by stars *, and plotted on Fig. 3, Fig. 4 by lines.

We see, that for $q = 5, 6$ the correlation length is greater, then the linear size of our lattices $L = 8, \ldots, 256$. But for the case $q = 10$ we examine region $L > \xi_c$. For $q = 8$ the value $\xi_c = 23.8$ and only two sizes $L = 8, 16$ are smaller, than $\xi_c$.

We make additional calculations to check this approximation formulas (11) (12). We compute the crossing probability $\pi_s(\beta; L)$ for $q = 5, 6, 8, 10$ direct in critical points $\beta_c(q) = \log(\sqrt(q) + 1)$ [27]. Results of this calculation presented on Fig. 5 and Fig. 6. This figures corresponded to figures Fig. 3, Fig. 4 respectively.

The picture, which we can see in this figures, allowed us to make the following conclusion: for the case $q > 4$ the amplitude $A(L)$ of crossing probability goes to zero by the logarithmic low (11) - see Fig. 3, Fig. 4, when $L < \xi_c$ goes by the exponential low (12) - see Fig. 4, Fig. 5, when $L > \xi_c$. For the case $q \leq 4$ the amplitude $A(L)$ goes to finite value by the power low (10).

The expression for the inverse dispersion $B$ in (9) follows from the scaling relation (for the case of the Gauss function (9), parameter $B$ is the inverse dispersion $B = \frac{1}{W}$). The universality of the function $\pi_s(\beta; L)$ implies, that the inverse dispersion is proportional the lattice size $L$ in the power of $y = \frac{1}{\nu}$ at least for cases of the second order phase transitions, when the critical index $\nu$ of correlation length is defined - see formula (13). Behavior of parameter $B(L)$ as a function of linear size $L$ is shown on the Fig. 7 in log-log scales. The points for different $q$ lay on straight lines in accordance with (13)

$$\frac{1}{W(L)} = B(L) \simeq bL^y$$ (13)
In eleventh, twelfth, fourteenth, fifteenth rows Table I placed results for $b$ and $y$ of approximation of inverse width $B(L)$ (13). In the tenth row we put analytical values of $\frac{1}{\nu}$ for $q \leq 4$. We see, that for $q < 4$ the scaling parameter $y$ is equal the inverse critical index $\nu$ in agreement with scaling relations $\pi_s(\beta; L) = f(L^{\frac{1}{\nu}}(\beta - \beta_c(L)))$. But for $q = 4$ we see, that $y \neq \frac{1}{\nu}$. For $q > 4$ we obtain values of thermal scaling exponents $y$, but it is unclear, how to relate it to another critical exponents.

For the locations $\beta_c(L)$, maximum of the function $\pi_s(\beta; L)$, we expect deviations from limit values in the power form $L^{-z}$. For the percolation, the power $z$ seems to be equal $z = 1 + \frac{1}{\nu} = 1.75$ (26). But the same deviation of maxima for magnetic susceptibility and specific heat of the Ising model is proportional to $L^{1 + \frac{1}{\nu}}$ (29,30). So we considered $z$ as a free parameter and use for approximation formula (14).

$$\beta_c(L) \simeq \beta_c(\infty) + cL^{-z} \quad (14)$$

In seventeenth and twenty second rows of the Table I we put the precise values of critical points. For $q = 2, 3, 4, 5, 6, 8, 10$ we use expression $\beta_c(q) = \log(\sqrt{q}+1)$. We use data from (26) for the critical point for percolation.

The position of the maximum of the crossing probability $\pi_{P_s}(\beta; L)$ on the lattice with linear size $L$ goes to the critical point by power low (14). We placed parameters of approximation $\beta_c(\infty), c, z$ in eighteenth, nineteenth, twentieth, twenty-third, twenty-fourth and twenty-fifth rows in the Table I.

There in good agreement between results of approximation and analytical values of critical temperature. This values deviates only in fifth and fourth digits after the decimal point.
IV. SUMMARY.

- crossing probabilities for $q$-state Potts model are universal functions of scaling variable $(\beta - \beta_c)L^{\frac{1}{\nu}}$, looks like the Gauss function $A \exp(- (\beta - \beta_c(L))^2 B^2 L^{\frac{2}{\nu}})$.

- locations of maxima of crossing probabilities $\beta_c(L; q)$ goes to phase transition points $\beta(\infty; q) = \log(\sqrt{q} + 1)$ when $L$ goes to $\infty$.

- for the case $q < 4$ the thermal scaling index $y = \frac{1}{\nu}$, where $\nu$ is the critical index of correlation length. For $q = 4$ the thermal index $y \neq \frac{1}{\nu}$. For $q \leq 4$ the limit of the crossing probability in the critical point is nonzero.

- for $q > 4$ crossing probabilities in critical points go to zero by the logarithmic low, when $L < \xi_c$ and by the exponential low, when $L > \xi_c$, where $\xi_c$ is the correlation length in the critical point.

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FIG. 6. Crossing probability at critical points $\pi_s(\beta_c; L)$ in log($\pi_s$) scale.

FIG. 7. The inverse width $B(L; q) = \frac{1}{\nu(L)}$ of crossing probabilities.
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TABLE I. Results of approximation by (10)-(14).

|   | $q$ | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
|---|----|--------|--------|--------|--------|
| 1 | $A_0$ | 0.3538(3) | 0.3059(8) | 0.2610(12) | 0.17(5) |
| 2 | $a$ | -0.73(60) | 0.36(23) | 0.32(13) | 0.17(3) |
| 3 | $x$ | -2.1(4) | 1.48(33) | 1.20(20) | 0.3220(29) |

|   | $q$ | $q = 5$ | $q = 6$ | $q = 8$ | $q = 10$ |
|---|----|--------|--------|--------|--------|
| 4 | $\xi(\beta_c)$ - see. [28] | 2512.1 | 158.9 | 23.9 | 10.6 |
| 5 | $A_0$ | 0.277(3) | 0.301(15) | 0.198(27)* | 0.231(7)* |
| 6 | $a$ | 0.0269(8) | 0.048(4) | 0.035(3)* | 0.076(1)* |

|   | $q$ | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
|---|----|--------|--------|--------|--------|
| 7 | $\frac{1}{\beta}$ | 0.75 | 1.0 | 1.2 | 1.5 |
| 8 | $y$ | 0.7343(28) | 0.9988(19) | 1.1945(26) | 1.337(11) |
| 9 | $b$ | 2.131(29) | 0.961(8) | 0.766(8) | 0.68(3) |

|   | $q$ | $q = 5$ | $q = 6$ | $q = 8$ | $q = 10$ |
|---|----|--------|--------|--------|--------|
| 10 | $\beta_c$ exact | $p_c = 0.592746$ | 0.881374** | 1.005053 | 1.098612 |
| 11 | $\beta_c(\infty)$ | $p_c(\infty) = 0.592731(28)$ | 0.881267(28) | 1.004990(19) | 1.098600(22) |
| 12 | $c$ | -0.333(12) | -0.309(17) | -0.527(24) | -0.74(4) |
| 13 | $z$ | 1.57(12) | 1.148(17) | 1.277(14) | 1.391(18) |

|   | $q$ | $q = 5$ | $q = 6$ | $q = 8$ | $q = 10$ |
|---|----|--------|--------|--------|--------|
| 14 | $\beta_c$ exact | 1.174359 | 1.238226 | 1.342454 | 1.426062 |
| 15 | $\beta_c(\infty)$ | 1.17440(6) | 1.23828(5) | 1.3427(5) | 1.42648(18) |
| 16 | $c$ | -1.114(24) | -1.30(9) | -1.5(8) | -1.89(15) |
| 17 | $z$ | 1.542(7) | 1.608(25) | 1.68(19) | 1.78(3) |

* - approximation by exponent - formula (12)

* - $\beta_c(q = 2) = \log(\sqrt{2} + 1) = 2\beta(Ising) = 0.881373587...$