Chaotic dynamics of two beams described by the kinematic hypothesis of the third approximation in the case of small clearance

O A Saltykova\textsuperscript{1,2}, I V Papkova\textsuperscript{1} and V.A.Krysko\textsuperscript{1}

\textsuperscript{1}Saratov State Technical University, Politehnicheskaya str, 77, Saratov, Russia
\textsuperscript{2}Cybernetic Institute, National Research Tomsk Polytechnic University, 634050 Tomsk, Lenin Avenue, 30, Russia

Abstract. The chaotic dynamics of the contact interaction of two beams described by the third-approximation hypothesis (the Pelekh–Sheremetev model) are studied in the paper. There is a small clearance between the beams. One of the beams is subjected to the action of a transversal harmonic load. Contact pressure is determined by Kantor’s method. The mathematical model of the beam structure taking into account the geometric nonlinearity and contact interaction. The system of partial differential equations reduces to the ordinary differential equations by the finite differences method with the approximation of the second order. The resulting system is solved by the Runge–Kutta methods of various orders. The chaotic vibrations of two beams were investigated by the methods of nonlinear dynamics. The reliability of the obtained results was grounded. The Lyapunov exponents are calculated by three different algorithms – Kantz, Wolf, Rosenstein. The 2D and 3D phase portraits and the Fourier power spectra, the Poincaré pseudo section were constructed. The phenomenon of frequency synchronization has been detected.

Keywords – Contact interaction, 3d approximation beam, nonlinear dynamics, finite difference method, Runge–Kutta methods, geometric nonlinearity.

1. Introduction

Beams and beam structures are widely used in modern industry, machine building, rocket engineering, etc. Often such structures are subjected to various external dynamic influences. The study of nonlinear dynamics and contact interaction of beam structures is a very important issue at the present stage development of science. In the Russian and foreign scientific literature we can find a large number of works devoted to the study of Euler–Bernoulli [1], Timoshenko [2], Pelekh–Sheremetev [3, 4] beams models, but there are no papers on the contact interaction of beams of third-order. A third-order mechanics model is used in this paper. The hypothesis of the third approximation is commonly called the Reddy hypothesis [5] in the foreign literature. But this theory was first described in [6]. It was shown in [4]. The Pelekh–Sheremetev hypothesis makes it possible to maximize the model to a three-dimensional and to obtain the most accurate results. In 1964 two Ukrainian scientists M.P. Sheremetev and B.L. Pelekh [6] proposed a third-order theory. This theory was applied to the calculation of beams of plates and shells in publications [7]. Twenty-seven years later, in the works of Levinson [8] and Reddy [5, 9], this model was reopened. Thus, the following situation has developed. In the Russian scientific press and in publications of scientists from several countries of Eastern Europe, the approximation of the deflection function by a polynomial of the third degree is associated with the names of M.P. Sheremetev and B.L. Pelekh. In the other works this theory is called the Reddy-Levinson theory. We will adhere to the historical name – the Pelekh–Sheremetev model.

The equations describing the dynamic processes are very difficult, so it is not possible to find an analytical exact solution of the nonlinear dynamics problems. The solution by numerical methods of such problems is the correct solution this problem. But the question about the reliability of the obtained solutions is appears. The numerical errors that accumulate when solving problems by numerical methods are often mistaken for chaotic vibrations.
We will try to determine the truth of chaos for problems of nonlinear dynamics and contact interaction of beams, depending on applied numerical methods and initial conditions.

We will follow the chaos definition according to the Gulik work [10] in our studies. Chaos exists when there is a significant dependence on the initial conditions, or the function has a positive Lyapunov exponent at each point of the domain of its definition, and therefore is not ultimately periodic.

As the initial conditions, we assume: kinetic hypotheses, boundary and initial conditions, the number of intervals of integration in the finite differences method, the methods for solving the Cauchy problem in the form of Runge–Kutta methods, the time step in solving dynamic problems.

To reduce the infinite-dimensional problem to the Cauchy problem, we used the finite–difference method with second-order approximation. The parameters of this method and the method itself remain constant throughout the work.

In the present paper, we propose a comprehensive study to reveal the truth of chaotic vibrations in the numerical solution of the problem of nonlinear vibrations of two beams structure described by the third approximation kinematic hypothesis with a gap under the action of a transverse alternating load:

1. Ensure complete convergence of the results for \( n \) and \( n \times 2 \) obtained by the finite difference method.
2. Check the convergence of the results depending on the used method for solve the Cauchy problem.
3. The time histories, phase portraits, Fast Fourier transform (FFT), displacement diagram, and Poincare’s section are constructed for each step in the spatial coordinate and each method for solving the Cauchy problem.
4. For each step in the spatial coordinate, the highest Lyapunov exponent is calculated using three methods: Kants [11], Wolf [12], and Rosenstein [13].

The final decision on the validity of chaotic vibrations and on the convergence of the results is taken in the case when all the points of complex study described were checked.

The gap between the beams is a tenth of the beam thickness. The beams displacement are small and can be described within the framework of the linear theory of elasticity. However, geometric nonlinearity according to T. von Karman model is taking into account.

2. Formulation of the problem

We will study the nonlinear dynamics and contact interaction of two beams with a gap. The beam 1 is under the action of a transverse alternating load, and the beam 2, comes into motion due to contact with the beam 1. Both beams are described by the kinematic hypothesis of the third approximation.

The contact interaction of the beams is taken into account by the B.Ya. Cantor model [14].

We are focused on the investigation of two beams structure being as 2D object in space \( \mathbb{R}^2 \) with rectangular co-ordinates introduced in the following way. In the body of beam 1 a certain arbitrary reference curve \( z = 0 \) is fixed; the OX goes along main curvature of the reference curve, whereas the axis OZ is directed to the reference curvature center. In the given coordinates the beam structure as a 2D object \( \Omega \) is defined as \( \Omega = \{ x \in [0, a] : -h \leq z \leq h + 3h \} \), \( 0 \leq t \leq \infty \), where \( [0, a] \) defines a straight beam line.

![Figure 1. The settlement scheme](image)

The equations of beams motion, as well as the boundary and initial conditions, are obtained from the Hamilton-Ostrogradskiy principle.

For construction of a Pelekh–Sheremetev beam mathematical problem, we will formulate the hypothesis [4, 6] (the third approximation hypothesis):
– cross sections don’t remain flat and perpendicular to the deformed axis of a beam;
– a turn and a normal curvature for a beam we will set in a form: \( u^2 = u + z \gamma + z^2 \gamma T + z^3 \gamma T^2 \), \( w^2 = w \);
– the inertial components associated with the rotation of the sections are taken into account.

It is necessary to introduce the term \((-1)^i K (w_i - w_2 - h_i) \Psi\) in the equations, where \( i = 1, 2 \) is a beam’s number for simulating the contact interaction according to the Cantor model. The function \( \Psi = \frac{1}{2} [1 + \text{sign}(w_1 - h_k - w_2)] \) describes the contact pressure between beams. We begin with detection of contact between beams, i.e., \( \Psi = 1\) if \( w_1 > w_2 + h_k \) and there is a contact between beams; otherwise \( \Psi = 0 \) [14]. \( K \) is stiffness coefficient of a contact zone, \( h_k \) denotes the clearance between the beams. The contact problem is solved at each time step of the computational procedure.

3. Theory

After application of a variation principle of virtual work we receive system of nonlinear partial differential equations for two Pelekhe-Shcheremetev beams in the displacements taking into account energy dissipation:

\[
\begin{align*}
\left[ \frac{1}{\lambda^2} \frac{\partial^3 \gamma_{w}}{\partial x^3} + \frac{1}{5} \frac{\partial^4 w_i}{\partial x^4} \right] &+ \frac{k^2}{E_1} \left[ \frac{\partial \gamma_{w}}{\partial x} + \frac{\partial^2 w_i}{\partial x^2} \right] + \\
\frac{1}{\lambda^2} \left[ L_4(w_i, u_i) + L_4(w_i, u_i) + \frac{3}{2} L_2(w_i, w_i) \right] &- (-1)^i K (w_i - w_2 - h_i) \Psi - \frac{\partial^2 w_i}{\partial t^2} - \varepsilon_1 \frac{\partial w_i}{\partial t} = 0;
\end{align*}
\]

\[(1) \]

\[i = 1, 2 - \text{beam’s number}, \]

\[L_4(w_i, u_i) = \frac{\partial^2 w_i}{\partial x^2} \frac{\partial u_i}{\partial x}, \quad L_2(w_i, w_i) = \frac{\partial^2 w_i}{\partial x^2} \left( \frac{\partial w_i}{\partial x} \right)^2, \quad L_3(w_i, u_i) = \frac{\partial w_i}{\partial x} \frac{\partial^2 u_i}{\partial x^2}, \quad L_4(w_i, w_i) = \frac{\partial w_i}{\partial x} \frac{\partial^2 w_i}{\partial x^2} \]

are the nonlinear operators, \( \gamma_{w} \) – the transverse shear function, \( w_i, u_i \) – the deflection and displacement functions of the beams, respectively. We must add boundary and initial conditions to the system of differential equations (1). In equations (1) bars are omitted.

Equations (1) contain the fourth derivative for \( w_i \). This fact improves the convergence of the applied numerical methods to solve the problem and facilitates the proof of the existence theorem for the solution, in comparison with the second-approximation hypothesis.

The system of governing equations (1), together with the boundary and initial conditions are reduced to the non-dimensional form using the following relations:

\[
\begin{align*}
\bar{w} &= \frac{w}{2h}, \quad \bar{u} = \frac{au}{(2h)^2}, \quad \bar{x} = \frac{x}{a}, \quad \bar{\lambda} = \frac{a}{(2h)}, \quad \bar{q} = q \frac{a^4(1-\nu^2)}{(2h)^4 E}, \\
\bar{t} &= \frac{t}{\tau}, \quad \tau = \frac{a}{c}, \quad c = \sqrt{\frac{Eg}{(1-\nu^2)\rho}}, \quad \bar{\varepsilon} = \frac{\varepsilon_1}{a}, \quad \bar{\gamma}_{w} = \frac{\gamma_{w}a}{(2h)}. \end{align*}
\]

(2)

The resulting system of nonlinear partial differential equations (1), together with the boundary and initial conditions, reduces to a system of ordinary differential equations by the finite differences method with approximation \( O(c^2) \). Where \( c \) is a step along the spatial coordinate. In each grid node we obtain the following system of ordinary differential equations. The Cauchy problem is solved by Runge–Kutta type methods. In the work was used the Runge–Kutta methods of the second, fourth and eighth orders are compared in explicit and implicit form (the second and fourth orders Runge–Kutta method (rk2 and rk4), the fourth-order Runge–Kutta–Fehlberg method (Rkf45), the 4th order Kesh–
Karp Runge-Kutta method (rkck), the 8th Prince–Dormand Runge–Kutta method (rk8pd), the implicit 2nd and 4th order Runge–Kutta method (rk2imp and rk4imp)). The explicit method is characterized by the fact that the matrix of coefficients has a lower triangular form (including the zero main diagonal) – in contrast to the implicit method, where the matrix has an arbitrary form. The methods rkf45, rkck, rk8pd provide for automatic step change, as well as the possibility of controlling the integration error. A software package has been created for solve the task posed depending on the control parameters \([q_0, \omega_p]\). Much attention was paid to the problem of non-penetration of structural elements into each other. As noted above, we study the problems nonlinear, so there is a question about the reliability of the results.

4. Experimental results
To carry out a numerical experiment, we put: \(\omega_p = 5.1, \quad q_0 = 5000, \quad h_k = 0.1, \quad \lambda = 6/2h = 50, \quad \varepsilon_1 = 1.\)

The \(\omega_p\) is a natural beam frequency. Beams are rigidly fixed from both ends. The boundary conditions for clamping both ends of the beams:

\[
\begin{align*}
\frac{\partial^2 w_i(0,t)}{\partial x^2} &= \frac{\partial^2 w_i(1,t)}{\partial x^2} = 0; \\
\frac{\partial w_i(0,t)}{\partial x} &= \frac{\partial w_i(1,t)}{\partial x} = 0; \\
16 \frac{\partial^2 \gamma_{ai}(0,t)}{\partial x^2} - \frac{\partial^3 w_i(0,t)}{\partial x^3} &= 0; \\
16 \frac{\partial^2 \gamma_{ai}(1,t)}{\partial x^2} - \frac{\partial^3 w_i(1,t)}{\partial x^3} &= 0;
\end{align*}
\]

(3)

Initial conditions:

\[
\begin{align*}
\frac{\partial w_i(x,t)}{\partial t} \bigg|_{t=0} &= u_i(x,t) \bigg|_{t=0}, \quad \gamma_{ai}(x,t) \bigg|_{t=0} = 0, \\
\frac{\partial^2 w_i(x,t)}{\partial t^2} \bigg|_{t=0} &= \frac{\partial u_i(x,t)}{\partial t} \bigg|_{t=0} = \frac{\partial \gamma_{ai}(x,t)}{\partial t} \bigg|_{t=0} = 0.
\end{align*}
\]

(4)

The following harmonic load is applied on the beam 1:

\[
q = q_0 \sin(\omega_p t)
\]

(5)

Where \(q_0\) is the amplitude, \(\omega_p\) is the excitation frequency.

We study the convergence of results at the different \(n = 40; 80; 120; 240; 360; 400; 440\) in this paper. \(n\) is a number of points of division of an interval. We used the Prince–Dormand Runge–Kutta method of the 8th order for solving the Cauchy problem. The time histories for \(n = 40; 80; 120; 240\) are very different from the time histories for \(n = 360; 400\). Since \(n = 360; 400\) the time histories for the first and second beams are coincide. In Figure 2 we give the time histories \(w_i(0.5,t)\), \(t \in [0.500,0.506]\) in case \(n = 40; 360; 400\) for beams 1 and 2. The short dotted line corresponds to \(n = 40\), the long dotted line corresponds to \(n = 360\), and the solid line corresponds to \(n = 400\).
We note that it is not possible to ensure complete convergence with respect to the time histories for chaotic vibrations for beam 2, in contrast to beam 1. As a result, the convergence of chaotic signals is achieved. It was considered sufficient to show convergence with respect to the FFT for chaotic vibrations[15]. By signals, it was impossible to achieve convergence. Further, the convergence of the signals was investigated depending on the type of the Runge–Kutta method. In Figure 3 we give the time histories \( w_i(0.5,t), t \in [500..506] \) obtained by the methods rk2 (dashed line) and rk4 (solid line).

For the beam 1, the results of the Runge-Kutta methods of the second, fourth and eighth orders coincided completely. For beam 2 the convergence is worse, but the difference in the values of the deflection does not exceed 0.5%. The 8th order Prince–Dormand Runge–Kutta method chosen as the reference.

According to the above algorithm, we further investigate the dynamic characteristics of beams for different number of partitions along the spatial coordinate. In Table 1, we give graphs of FFT, 2D phase portraits, and Poincare’s section for both beams.
Table 1. Dynamic characteristics of beams.

| n  | № | FFT | 2D Phase portrait $w(w')$ | Poincare’s section |
|----|----|-----|--------------------------|--------------------|
| 1  | 40 | ![FFT](image1) | ![2D Phase portrait](image2) | ![Poincare’s section](image3) |
| 2  | 2  | ![FFT](image4) | ![2D Phase portrait](image5) | ![Poincare’s section](image6) |
| 1  | 360, 400 | ![FFT](image7) | ![2D Phase portrait](image8) | ![Poincare’s section](image9) |

5. Discussion of the results

The FFT for any n have a chaotic component. With an increase in n, the noise harmonics become smaller in the signal. For $n = 240$ appears the bifurcation frequency $\omega_p / 2$. A further increase in n leads to the appearance of a linearly dependent frequency $\omega_1 = \omega_p / 4$ for both beams.

2D phase portraits for beam 1 represent, for a small n = 40, a continuous spot, for beam 2 there are two attracting sets, and for large values n = 360; 400 is a thickened oval for the beam 1 and a spot for the beam 2. The Poincare’s section qualitatively changes and becomes more pronounced with increasing number of segments of the partitions along the spatial coordinate. This indicates a decrease in the chaotic component in the signal.

For $n = 360$ and $n = 400$ all dynamic characteristics are completely coincide, both for the first and for the second beam. On FFT are present bifurcation frequencies $\omega_p, \omega_p / 2, \omega_p / 4$. There is a synchronization of beam vibrations at the frequencies of the first and second bifurcations. With increasing n, the noise component in the signal decreases. This can be caused by the error of numerical
methods, which can be mistaken for true chaos. However, the signal remains chaotic, which is confirmed by Lyapunov exponents and other dynamics characteristics. The values of the highest Lyapunov exponent for all \( n \) are calculated by the Wolff, Rosenstein, and Kant's methods. The values are obtained on the basis of solutions of the Cauchy problem by the 8th-order Renge–Kutta method. Different methods of calculating Lyapunov's indices must be used to determine the true chaos. When the number of divisions of the beam is \( n = 40; 80, 120; 240, 360, 400 \) segments in the finite differences method, the Lyapunov exponents converge at any decimal point to the second decimal point. The values for the highest Lyapunov exponent are mostly positive. That is, we are dealing with the true chaotic vibrations of the investigated beam structure.

6. Conclusions
The reliability of the solution of chaotic dynamics problem and the contact interaction of two beams described by the Pelekh–Sheremetev kinematic hypothesis is proved and justified in the paper. A complex study of the nonlinear dynamics of the contact interaction of beams is carried out. The choice of the number of partitions with respect to the spatial coordinate \( (n = 400) \) is justified. The choice of the method for solving the Cauchy problem (the Runge–Kutta method of the 8th order of Prince–Dormand) is justified.

The values of the highest Lyapunov exponent are calculated for the three methods (Kantz, Wolf, and Rosenstein). The sign of the highest exponent coincides for each method, which allows us to speak about the truth of the chaotic vibrations. The phenomenon of frequency synchronization has been detected. It was noted that the presence of the fourth derivative of the deflection function with respect to the spatial coordinate \( x \) in the beam motion equations promotes rapid convergence in solving the finite difference method, in contrast to the equations derived with S.P.Timoshenko hypothesis, in which the highest derivative has a second order.

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