The Maldacena conjecture and Rehren duality

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Abstract

We analyse the implications of the fact that there are two claims for a dual to \(N=4\) superconformal SU\(N\) Yang-Mills theory (SCYM), the Maldacena conjecture and the theorem of Rehren. While the Maldacena dual is conjectured to be a non-perturbative string theory for large string coupling \(g_s\) and small \(N\), the Rehren dual is an ordinary quantum field theory on \(AdS_5\) for all values of the parameters. We argue that as a result, if we accept the Maldacena conjecture, one of the following statements must be true: 1) SCYM does not satisfy the axioms of algebraic quantum field theory for finite \(N\) because its observables do not obey the causal structure of conformal Minkowski space; 2) String theory on \(AdS_5 \times S^5\) is not a quantum theory of gravity in 10 dimensions because it is dual to an ordinary quantum field theory on \(AdS_5\) whose causal structure remains fixed for all values of its couplings; or 3) there is no consistent quantisation of string theory on \(AdS_5 \times S^5\) for finite string scale \(l_s\) and \(g_s\).

In evaluating the evidence for each of these conclusions we point out that many of the tests of the Maldacena conjecture can be explained by a weaker form of an \(AdS/CFT\) correspondence.

1 Introduction

In 1997 Maldacena \(^1\) made a bold conjecture that string theory on \(AdS_5 \times S^5\) is actually equivalent under a certain duality relation to a quantum field theory on the four dimensional Minkowski spacetime which is a time-like boundary

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for AdS. This was quickly reinforced by arguments coming from a number of directions [2, 3], among them the fact that it seemed to offer striking confirmation of the holographic principle. It was seen also to be closely related to older ideas of Polyakov on the role of additional dimensions in conformally invariant theories as well as old results from studies of supergravity on AdS [4].

This led to a very large number of papers, many of which do confirm the existence of a correspondence between the predictions of $\mathcal{N}=4$ Super-Yang-Mills theory and linearised supergravity on a background which is $AdS_5 \times S^5$. Some of the checks are quite striking in their precision. There are also very interesting results that show that Yang-Mills theory with fewer supersymmetries are related by the same kind of correspondence to other supergravity theories, or to linearised supergravity on different backgrounds.

Among the papers inspired by the Maldacena conjecture are a series by Rehren [5, 6, 7] proving a theorem which states that any algebraic quantum field theory on an $AdS$ space of any dimension is dual to an explicitly constructible, conformal, algebraic field theory on the boundary. This appears to be highly relevant to Maldacena’s conjecture. In the limit in which gravity is weak, Rehren’s duality seems to confirm the predictions of the Maldacena conjecture, and may even be said to provide a simple explanation for some of the results which support the Maldacena conjecture.

It would seem remarkable if there were two independent dualities between field theories on AdS and its boundary. However, the exact relationship between the theorem and the conjecture is not completely clear. The issue, which it is the purpose of this note to explore, is that the duality proved by Rehren connects the $\mathcal{N}=4$ supersymmetric Yang Mills theory to an ordinary local quantum field theory on the fixed $AdS_5$ background for all values of coupling and for all $SU(N)$. This is puzzling as the Maldacena conjecture appears to require that the bulk theory should be a string theory when $g_s \approx g^2_{YM}$ is large and $N$ is small. But if we use both dualities we are able to predict that this string theory is dual to an ordinary quantum field theory on the fixed $AdS_5$ background, as both are dual to the supersymmetric Yang-Mills theory. If true, this is surprising, as it seems to imply that a string theory is dual to an ordinary quantum field theory in a space of the same dimension, not one lower. It also seems to imply that the causal structure does not become dynamical once gravity and quantum gravitational effects become important.

In this note we offer what seems to us to be the most logical reading of the possible implications of the existence of both Rehren’s theorem and the conjecture of Maldacena. We find that present knowledge does not allow us to distinguish between several alternatives, each of which has non-trivial consequences.

2 The Maldacena conjectures

The Maldacena conjecture [1, 2, 3, 4] identifies IIB string theory on $AdS_5 \times S^5$ with $\mathcal{N}=4$ superconformal-Yang-Mills theory with gauge group $SU(N)$ (from
now on SCYM). We will distinguish between weak and strong versions of the conjecture. First we list the relevant constants of the theory and their proposed relations:

| Symbol | Description                                      |
|--------|--------------------------------------------------|
| $g_s$  | string coupling constant                        |
| $l_s$  | string scale                                    |
| $G_N$  | Newton’s constant                               |
| $g_{YM}$ | Yang-Mills coupling constant                     |
| $N$    | dimension of gauge group SU($N$)                 |
| $R$    | radius of curvature of $AdS_5 \times S^5$       |

This leads to the following relations:

$$
\lambda_{tH} = g_{YM}^2 N \approx g_s N \approx R^4/l_s^4,
$$

where $\lambda_{tH}$ denotes the ’t Hooft coupling.

It helps to divide the Maldacena conjecture into several forms, from weaker to stronger. The weak form of the Maldacena conjecture can be stated as:

**M1:** In the limit $N \to \infty$, $g_s$ small, $\lambda_{tH}$ large, SCYM on the 4-d Minkowski-space boundary $M_4$ of $AdS_5 \times S^5$ becomes equivalent to type IIB supergravity on $AdS_5 \times S^5$, defined by a power series expansion with fixed metric on $AdS_5 \times S^5$.

Note that this is necessarily a duality with supergravity, *defined in terms of a power series expansion around the fixed $AdS_5 \times S^5$ metric.* To any finite order this can be considered a field theory on $AdS_5 \times S^5$ since the limit of very small $g_s$ implies that string theory may be neglected. We may note that this weak form of the Maldacena conjecture is confirmed by many calculations, there is also no conflict with Rehren’s results, as we will describe below. We must also emphasize that the relationship is not necessarily a duality, at this level there is no conjecture that all of type IIB supergravity is contained in SCYM.

The next strongest form of the Maldacena conjecture concerns the case where we consider the full nonlinearities in classical gravitational effects, while still suppressing quantum gravity and string effects. We do this by considering arbitrary $\lambda_{tH}$ while still keeping $g_s$ very small. We then have what we may call the “medium” Maldacena conjecture:

**M1.5:** SCYM with arbitrary $N$ but $g_{YM}$ very small must be equivalent to classical supergravity on spacetimes which are *asymptotically* $AdS_5 \times S^5$.

The strong form comes from considering arbitrary values of the parameters $N$ and $g_{YM}$:

**M2:** For all values of $N$ and $g_{YM}$, SCYM on $M_4$ is dual to type IIB string theory on spacetimes which are *asymptotically* $AdS_5 \times S^5$.

In the medium and strong forms of the conjecture we can only require the spacetime on which the string theory lives to be asymptotically $AdS_5 \times S^5$ since we are no longer studying weak gravitational and matter degrees of freedom.
propagating on a fixed causal structure given by the metric on $AdS_5 \times S^5$. This is necessary if the theory is to describe structures like black holes where the deviations from a fixed $AdS_5 \times S^5$ are arbitrarily strong.

Indeed, the idea behind the strong Maldacena conjecture is that the Yang-Mills theory “effectively sums over all spacetimes” which are asymptotic to $AdS_5 \times S^5$, thus providing a non-perturbative definition of string theory. Furthermore, different pure and statistical states of SCYM are believed to correspond, in the classical limit, to different asymptotically $AdS_5 \times S^5$ spacetimes. For example time dependent metrics, such as those with gravitational waves are believed to correspond to non-equilibrium or time dependent states of the SCYM theory.

Hence, any theory which has general relativity or supergravity as a classical limit ($\hbar \to 0, G_N$ fixed) — which is a minimal requirement for a quantum theory of gravity — cannot have a fixed causal structure, otherwise we come into conflict with basic predictions of general relativity (now well confirmed), such as light bending, gravitational lensing etc.

In summary, we require:

**BI:** A quantum theory of gravity must be background independent so that its classical limit is general relativity or a generalisation thereof, whose metric and hence causal structure are dynamical. Such a theory may be constrained by boundary conditions that depend on a metric asymptotically, in the same sense that general relativity can be formulated in terms of asymptotically flat boundary conditions. But the causal structure in the bulk cannot be fixed, and must be different with different solutions or quantum histories of the spacetime.

Thus, we should make explicit the following, which is usually assumed in the statement of the Maldacena conjecture:

**M3:** Type IIB string theory on spacetimes which are asymptotically $AdS_5 \times S^5$ for all values of $l_s$ and $g_s$ is a quantum theory of gravity, in the sense that it has the full solution space of supergravity with asymptotically $AdS_5 \times S^5$ as a classical limit and hence it satisfies the property **BI**.

### 3 Algebraic field theory and AdS/CFT

Rehren’s papers \[5, 6, 7\] assume that SCYM is an algebraic quantum field theory (AQFT). This is the most general structure for a quantum field theory that incorporates relativistic covariance and local causality (c.f. \[8\] for a review and references). The basic objects in AQFT are the localised algebras of observables. More specifically an AQFT is completely specified by an assignment of an algebra $A(O)$ to every open region $O$ of spacetime. Furthermore these algebras have to comply with covariance, locality and isotony as is captured in the following axioms:
1. The algebras localised at spacetime regions related by a space-time symmetry transformation \( g \) (in our case this will be \( SO(2, 4) \)) are unitarily equivalent i.e.: \( A(gO) = U(g)A(O)U(g^{-1}) \), where \( U(g) \) is a representation of the symmetry group.

2. Two observables localised at space-like distances commute.

3. An observable localised in a region \( O \) is localised in a larger region \( O' \supset O \), i.e.: \( O' \supset O \Rightarrow A(O') \supset A(O) \).

It has been shown that this local algebra structure suffice to recover standard quantum field theory. In particular states arise as the carriers of a chosen representation of the net of algebras. Point fields operators \( \phi(x) \) can be recovered, roughly, by taking the intersections of the algebras localised around \( x \):

\[
\{ \phi(x) \} = \cap_{x \in O} A(O),
\]

where the above has to be understood in an appropriate technical sense to be meaningful. The \( AdS/CFT \) correspondence relies on the fact that the isometry group of \( AdS_{1,d} \) in \( d+1 \) dimensions and the conformal group of Minkowski spacetime are both \( SO(2,d) \). Furthermore the boundary of \( AdS_{1,d} \) is \( d \)-dimensional conformal Minkowski space \( M_d \) and the restriction of the \( AdS \) group to the boundary acts like the \( d \)-dimensional Minkowski conformal group.

Remarkably, Rehren \[5, 6, 7\] has been able to show that this symmetry property is in fact sufficient to establish a duality between any conformal AQFT on \( M_d \) and an AQFT on \( AdS_{1,d} \), and also, conversely, a duality between any AQFT on \( AdS_{1,d} \) and a conformal AQFT on the boundary. Note that this duality is not a conjecture but a rigorous theorem of algebraic quantum field theory. The duality is unique and provides an explicit map between observables of the two theories. Furthermore, the duality does not refer to string theory or quantum gravity and hence is not related to standard versions of the holographic principle.

The idea is to map observable algebras on \( M_d \) to algebras on \( AdS_{1,d} \) in such a way that causality, covariance and isotony are preserved. This gives us a unique net of algebras on \( AdS_{1,d} \), which in turn defines a quantum field theory with the causal structure of \( AdS \). Since the observable algebras for the theory on \( M_d \) and the theory on \( AdS_{1,d} \) are the same it follows that they will have the same representations and thus that the state spaces of the two theories will coincide - hence the theories are dual. But crucially the space-time localisation of the algebras is very different in the two theories. This leads to different interpretations of the observables. In particular, the Hamiltonian on \( AdS_{1,d} \), given by the generator of time translations, is identified with the linear combination of translations and conformal transformations in the 0-direction of \( M_d \): \( \frac{1}{2}(P^0 + K^0) \). This allows for the matching of the degrees of freedom in the bulk and the boundary theory.

In more detail, the correspondence is given as follows. Elementary regions in \( M_d \) are double cones \( K \), i.e. the intersections of a future and past-directed light
cone. These are causally complete convex regions. Each double cone uniquely determines a wedge region $W$ in $AdS_{1,d}$ which is the causal completion of $K$ in $AdS$, i.e. all points from which one can receive signals from a point in $K$ and send signals to a different point in $K$. Given a net of algebras on $M_d$ this allows us to construct a net on $AdS_{1,d}$ via: $A(W) = A(K)$ if $K$ corresponds to $W$. The key to Rehren’s result is to show that this correspondence preserves causality, covariance and isotony. Given the algebras defined on the wedges $W$ we can now associate an algebra to any region $O$ in $AdS_{1,d}$ via:

$$A(O) = \cap_{O \subset W} A(W).$$

This defines a unique AQFT on $AdS_{1,d}$ which is dual to the original AQFT on $M_d$.

It is important to note that the above duality is between local algebras. Point-like field operators, in general, are not mapped to other field operators. Indeed, an important property of the above map is that genuine field operators on $M_d$ are dual to observables in $AdS_{1,d}$ that are attached to the boundary. Bulk localised observables in $AdS_{1,d}$ on the other hand are mapped to ‘extended’ observables on the boundary. Such observables cannot be described in terms of fields and could, for example, correspond to Wilson loops in a boundary Yang-Mills theory.

Schroer [9] has used this property of the correspondence to suggest that the locality structure of a field theory on $AdS$ dual to a SCYM theory on the boundary is not compatible with that of string theory or (linearised) supergravity. We will not investigate this further, instead we focus on the fact that any SCYM theory on $M_4$ is dual, via Rehren’s map, to a theory with the fixed causal structure of $AdS_5$.

## 4 Consequences

SCYM is a conformal quantum field theory on 3 + 1 dimensional Minkowski space. In this section we assume that the observables of SCYM obey the casual structure of $M_4$ and hence satisfy the axioms of algebraic field theory. Let us make this explicit,

**AQFT: $N = 4$ supersymmetric Yang Mills theory satisfies the axioms of algebraic quantum field theory for all $SU(N)$ and all values of the gauge coupling.**

In this section we draw out the implications of the above in conjunction with the various versions of the Maldacena conjecture.

Rehren duality allows us to construct uniquely an algebraic quantum field theory (the Rehren dual in the following) on 4+1 dimensional AdS space which is dual to SCYM on its 3+1 dimensional Minkowski space boundary. This duality is valid for any values of the parameters $N$ and $g_{YM}$. Furthermore, by definition, the AQFT on $AdS$ is defined with respect to a fixed causal structure which is
reflected in the commutation relations between any two observables. Hence the theory will not satisfy property BI. This leads to the following conclusion:

**Conclusion 1:** Any theory that is dual to SU(N) superconformal Yang-Mills theory on $M_4$ is also dual to an ordinary field theory defined on the fixed $AdS_5$ background.

The Maldacena conjectures relates the SCYM on $M_4$ to a theory on a manifold $X_5$ which is asymptotically $AdS_5 \times S^5$. The above implies that this theory is dual — via a duality $\pi = (\text{Rehren duality}) \circ (\text{Maldacena duality})$ — to a field theory with the fixed *global* causal structure of $AdS_5$. This is illustrated in figure 1. We now examine the implications for each version of the Maldacena conjecture.

**M1:** If we assume conjecture M1 then there is no contradiction since linearised supergravity in the $g_s \to 0$ limit is a field theory with a fixed background. In this case the Maldacena duality and the Rehren duality could coincide. Essentially the representation theory of the supersymmetric extension of SO(4,2) includes both the states of SCYM and the linearised modes of supergravity on $AdS_5 \times S^5$.

**M1.5:** Conjecture M1.5 can be refuted since it implies that the space of asymptotically $AdS$ solutions to classical general relativity can be described equivalently by an ordinary field theory on $AdS$ with a fixed metric and causal structure. But this is contradicted by the fact that in general relativity the causal structure varies dynamically, which has been observed in many light bending experiments.

**M2:** If we assume conjecture M2 then type IIB string theory cannot satisfy property BI. Therefore M3 fails and string theory cannot be a theory of quantum gravity.

It is worth mentioning that these conclusions need not be in contradiction with the so called “uniqueness” theorems that show, subject to various assumptions, that any theory whose linearisation is general relativity has the interactions given by general relativity or some extension of it. First, we should note
that this theorem has been proven only under restricted assumptions \[10\] and explicit counterexamples are known in 3 and 5 dimensions \[11\]. So it is possible that these theorems do not apply in the special case under consideration.

However, even if this is not the case, it must be emphasised that the existing uniqueness theorems only establish the forms of the interactions and gauge symmetry order by order in a power series expansion of the classical theory on a fixed background. Whether such a power series expansion is equivalent to the full non-linear gravitational theory is an additional issue, not addressed by the existing theorems. It should further be stressed that the expansions in question are generally asymptotic rather than converging power series. Thus, it is possible that $M_1$ may hold to some finite order of an expansion in small disturbances around the fixed $AdS_5 \times S^5$ background, but that still $M_{1.5}$ and $M_2$ fail. In this case the Rehren and Maldacena dualities could agree to any finite order, but SCYM would only give a construction of a classical gravitational theory good to a finite order in an expansion in small amplitude excitations propagating on the fixed $AdS_5 \times S^5$ background. This would agree with many of the checks which have been performed, but there is then no necessity that $M_2$ holds or that the Maldacena duality gives a non-perturbative or background independent definition of string theory in terms of the SCYM theory.

There is also the possibility that $M_2$ could fail because string theory on $AdS_5 \times S^5$ fails to exist as a quantum theory for all values of $l_s$ and $g_s$. Despite the large literature on the Maldacena conjecture it has proved so far surprisingly difficult to construct the quantum string theory on $AdS_5 \times S^5$, even in the case of the free string. The present situation \[12\] is that classical actions for string theory on $AdS_5 \times S^5$ have been constructed and it has been shown that their $l_s \to 0$ limits are described by the superparticle on $AdS_5 \times S^5$. Hence, one can deduce that the quantisation of the $l_s \to 0$ limit reproduces the spectrum of linearised supergravity on $AdS_5 \times S^5$. However for finite $l_s$ the worldsheet action is non-linear and there has been as yet no successful quantisation of the theory, even at the level of the free string. Hence the spectrum of the free string is not known and it is not known for certain that there actually is a consistent string theory on $AdS_5 \times S^5$.

To summarise the logic, we reach the conclusion either that AQFT fails or at least one of the conclusions reached in this section must hold. In particular, if AQFT holds then $M_{1.5}$ and $M_2$ or $M_3$ must fail.

5 Conformal Induction

Given that a large number of results have been offered in support of one or another version of the Maldacena conjecture, the reader may be tempted to conclude from the foregoing that AQFT fails. However before reaching that conclusion we should check whether the evidence adduced in favour of a version of the Maldacena conjecture can be explained by a weaker hypothesis. This is what we discuss in this section. We may note that the arguments that follow provide one possible resolution of the conflict between Rehren duality and the
Maldacena conjecture. The arguments of the previous section hold regardless of whether we chose to adopt this explanation.

Witten [2] and Gubser, Klebanov and Polyakov [3] have offered a correspondence between a quantum theory on a \(d+1\) dimensional spacetime \(X_{d+1}\), which is asymptotically anti-DeSitter and a conformal field theory on the spacetime \(M_d\) which is the conformal boundary of \(X_{d+1}\). This version of the AdS/CFT correspondence is considerably weaker and more general than the conjecture of Maldacena.

Let us review the prescription, which allows us to construct the correlation functions of a conformal field theory on \(M_d\) by evaluating the correlation functions of fields of the \(d+1\) dimensional theory in the limit in which the fields are taken to the boundary.

More precisely, let \(\phi\) be a field of the \(d+1\) dimensional theory and \(\mathcal{O}\) an operator of the \(d\) dimensional conformal field theory, which is chosen to correspond to it. Then the CFT is defined by the following formula [2]

\[
\langle e^{\int_{\mathcal{M}} \phi \mathcal{O}} \rangle_{\text{CFT}} = Z(\phi)
\]

where \(Z(\phi)\) is the partition function on \(X_{d+1}\) with \(\phi\) constrained to approach fixed values \(\phi_0\) on the boundary \(M_d\).

We say the boundary theory is induced by the bulk theory through its definition in terms of correlation functions. We call the boundary theory the conformal image of the bulk theory.

As already mentioned conformal induction is a general procedure that requires only that the bulk theory be a well defined (Euclidean or Lorentzian) quantum theory on a fixed manifold. There is no necessary relationship with gravity, supersymmetry\(^1\) or string theory. If the partition function \(Z(\phi)\) exists and the fields \(\phi\) have well defined boundary values then the symmetries of the asymptotically AdS manifold are sufficient to show that the field theory defined by (1) will satisfy the symmetries of a conformal quantum field theory.

We note however, that in the case that the bulk field theory is defined on a Lorentzian manifold and \(M_d\) is a Minkowski spacetime, there may be an obstruction to conformal induction. Additional consistency conditions must be satisfied if the induced theory is to satisfy unitarity and locality. This follows from the possibility that events on the boundary which are spacelike with respect to the metric on the boundary may be causally connected through causal geodesics in the bulk. This can occur, for example if there are closed time like curves in the bulk. Some examples of this phenomena relevant for supersymmetric theories are described in [3]. Hence we need the causal consistency condition (CCC), which states that two events are causally related in the bound-

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\(^1\) Supersymmetry may be necessary in some cases to guarantee that the bulk partition function exists, but it need not be necessary in all cases. There are indeed low dimensional examples of conformal induction in which supersymmetry plays no role such as in the case of 2+1 gravity.

\(^2\) The induced theory is conformal because the asymptotically AdS metric defines the boundary metric only up to a conformal factor.
ary Minkowski spacetime if and only if they are causally related in the metric of
the bulk theory.

We come now to the crucial point, which is that there is no reason that the
above procedure, which is very general, should provide us with a duality between
theories. In particular, conformal induction is a one way process. There is no
known general procedure for reconstructing the bulk theory from the induced
boundary theory, nor should the relationship between the bulk and the boundary
theories be unique. Conformal induction involves only the boundary data of the
bulk quantum theory. There is no requirement that all observable quantities
of the bulk are computable in the conformal image. This applies especially to
observables which are completely localised in the bulk.

In the case of a Lorentzian bulk manifold $\text{AdS}_{d+1}$ we can ask about the
relation between conformal induction and Rehren duality. These need not neces-
sarily be related, however this would result in there being two different bound-
ary conformal field theories corresponding to a given theory on $\text{AdS}_{d+1}$. More
interesting is the possibility that the conformal image and the Rehren dual co-
cide. In this case the conformal induction would provide a true duality with
an inverse. But as described at the end of section 3 the Rehren dual maps
bulk localised observables to extended objects in the boundary. It is unclear
how this can be achieved via the conformal induction procedure. More likely is
the possibility [14] that the conformal image is precisely the sub-theory of the
Rehren dual that contains the true field observables, which according to Rehren
duality are related to bulk observables attached to the boundary.

5.1 Conformal Induction and the Maldacena conjecture

We now turn to the implications of the above for the Maldacena conjecture. This
is examined most easily by formulating the conformal induction conjecture:

CI: The correlation functions of observables in SCYM theory on the Minkowski
space boundary $M_4$ can be evaluated via equation (1) using the partition
function of the type IIB string theory on the bulk manifold, which is
asymptotically $\text{AdS}_5 \times S^5$.

As with the Maldacena conjecture this can also be formulated in weaker versions
in terms of linearised supergravity theory on $\text{AdS}_5 \times S^5$, in which it holds only
in the limits of $N \to \infty$, $g_s$ small, $\lambda_{\text{H}}$ large.

The essential point is that, as explained above, CI is weaker than any of the
versions of the Maldacena conjecture, in particular it does not posit a duality.
For example, M2 is equivalent to the conjecture CI together with the claim
that there is a unique inverse to the conformal induction procedure, so that

\footnote{It is clear from the Penrose diagram of $\text{AdS}$ that data on the time like boundary is not
sufficient to determine a classical solution to the field equations, data must be given as well
at past infinity.}

\footnote{We may note that in the Lorentzian case the CCC must be imposed as a restriction on
the sum over manifolds, otherwise the induced theory will violate causality on the boundary
Minkowski spacetime.}
all observable quantities in IIB string theory can be computed from the SCYM theory.

This implies, however, that any tests of the relation between SCYM and IIB string theory that can be explained by the hypothesis CI are not sufficient to provide support of the much stronger Maldacena conjecture. Let us examine some of the evidence that has been adduced:

- The matching of $N$ point functions between classical scattering in linearised supergravity on $AdS_5 \times S^5$ and correlation functions of SCYM. This is clearly explained by CI to any finite order in a power series expansion in $g_s$ around $AdS_5 \times S^5$. Note that in this case we also have the possibility that the conformal image is the Rehren dual or a subset of it.

- Various results regarding the scaling behaviour of field theories in four dimensions in terms of the behaviour of classical supergravity in asymptotically $AdS$ spacetimes. These are very useful but again are explained by conformal induction and so offer no independent support for an equivalence of the kind postulated in the Maldacena conjecture.

- Various results in 2+1 dimensions where there are no local degrees of freedom. Given the special nature of gravitational physics in 2+1 dimensions these illuminate but cannot be taken as strong evidence for conjectures about higher dimensional theories.

- The matching \([2]\) (up to an overall constant) of the entropy of large black holes in asymptotically $AdS_5$ spacetimes with thermal states of SCYM. Again this only provides evidence for CI. In this case the boundary manifold is $S^1 \times S^3$ so that the induced boundary theory is a thermal quantum field theory defined at the temperature given by the inverse period of the $S^1$.

- However, in addition the entropies of two different phases of SCYM have been argued to match the entropies of field theories on different asymptotically $S^1 \times S^3$ manifolds \([3]\). This suggests that, in the Euclidean case, fluctuations around different bulk manifolds are necessary to match supergravity to all the physics of SCYM. This appears at present to be the strongest evidence for $M_{1.5}$ and $M_2$, or at least for a form of CI in which the partition function of a full quantum theory of gravity is required to induce the full physics of SCYM. To the extent that the only known candidate for such a theory whose semiclassical limit reproduces supergravity is string theory, this provides evidence for a correspondence between string theory and SCYM at the level of CI.

This is an incomplete list, but it shows that care must be taken in evaluating the significance of various results offered as evidence for the Maldacena conjecture.
6 Summary

We have seen that the Maldacena conjecture in conjunction with Rehren’s theorem leads to a duality $\pi$ between string theory and a quantum field theory on a fixed $AdS_5$ background. This is consistent in the linearised supergravity limit and hence with conjecture M1 but there appears to be a problem once $g_s$ and/or $\lambda^{-1}_{\text{YM}}$ are allowed to take on arbitrary values. The arguments sketched above have led to the conclusion that one of the following must be true:

1. **AQFT** fails for finite $N$ and $g_{YM}$. In this case Rehren’s theorem is not relevant and there is no contradiction with the strong Maldacena conjecture. However, the axioms of AQFT are very general. Unless SCYM fails altogether to be a good quantum theory, it seems the only way it could fail to be an AQFT is if it violates causality on the four dimensional Minkowski space.

2. **M2** fails, possibly because there is no interacting quantum string theory on $AdS_5 \times S^5$ for finite $l_s$.

3. **M3** fails. In this case quantum string theory exists on $AdS_5 \times S^5$ for finite $g_s$ but it is not a quantum theory of gravity, because property **BI** fails. This comes about because the combination of M2 and the Rehren theorem implies it can be expressed in terms of a dual description based on an AQFT on the fixed metric and causal structure of $AdS_5 \times S^5$.

Further, we noted that many of the tests carried out on the $AdS/CFT$ correspondence can be explained by the hypothesis **CI**. Since **CI** is logically weaker and more general than any form of the Maldacena conjecture, these tests do not provide independent evidence for any part of the Maldacena conjectures that goes beyond **CI**. In particular, this leaves the possibility that SCYM captures some information about certain observables in supergravity, without there being a duality or equivalence of the theories.

This leaves us with the question of the status of M1, as it is the one form of the Maldacena conjecture that does not run into conflict with the conjunction of AQFT and Rehren’s theorem. But as M1 involves only a theory on the fixed background $AdS_5 \times S^5$ it is then possible to ask whether it may be in fact a consequence of Rehren’s theorem, i.e. whether the conformal image of linearised supergravity is given by the Rehren dual or a subset of it. This seems plausible since it would be highly remarkable if there were two independent dualities between SCYM and a field theory on $AdS$. If this were the case the connection to a gravitational theory might be in some sense accidental.

Indeed, while Rehren’s theorem demonstrates explicitly that two theories living on spaces of different dimensions can be dual, it also implies that this duality need not involve gravity or holography. In the present case the duality would be a consequence only of the fact that the supersymmetric extension of $SO(4,2)$ has representations that include both the states of SCYM and the linearised supergravity modes expanded around $AdS_5 \times S^5$. The correspondence
would be no less useful as a tool to understand SCYM theory, even if it could not then be used to give a definition of a quantum theory of gravity away from the limits of large $N$ and $\lambda_{\text{H}}$ and small $g_{YM}^2$. To conclude, so far as we know, present knowledge does not suffice to determine which of the three alternatives mentioned above is correct. More work will be required to decide the issue.

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