Boosted and Differentially Private Ensembles of Decision Trees

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Abstract

Boosted ensemble of decision tree (DT) classifiers are extremely popular in international competitions, yet to our knowledge nothing is formally known on how to make them also differential private (DP), up to the point that random forests currently reign supreme in the DP stage. Our paper starts with the proof that the privacy vs boosting picture for DT involves a notable and general technical tradeoff: the sensitivity tends to increase with the boosting rate of the loss, for any proper loss. DT induction algorithms being fundamentally iterative, our finding implies non-trivial choices to select or tune the loss to balance noise against utility to split nodes. To address this, we craft a new parameterized proper loss, called the $M_{\alpha}$-loss, which, as we show, allows to finely tune the tradeoff in the complete spectrum of sensitivity vs boosting guarantees. We then introduce objective calibration as a method to adaptively tune the tradeoff during DT induction to limit the privacy budget spent while formally being able to keep boosting-compliant convergence on limited-depth nodes with high probability. Extensive experiments on 19 UCI domains reveal that objective calibration is highly competitive, even in the DP-free setting. Our approach tends to very significantly beat random forests, in particular on high DP regimes ($\varepsilon \leq 0.1$) and even with boosted ensembles containing ten times less trees, which could be crucial to keep a key feature of DT models under differential privacy: interpretability.
1 Introduction

The past decade has seen considerable growth of the subfield of machine learning (ML) tackling the augmentation of the classical models with additional constraints that are now paramount in applications (Agarwal et al., 2019; Kaplan et al., 2019; Alistarh et al., 2017; Drumond et al., 2018; Jacob et al., 2018; Jagielski et al., 2019). One challenge posed by such constraints is the potentially risky design process for new approaches: it may not be hard to modify the state of the art to accommodate for the new constraint(s), but if not cared for enough, the modification may come at a hefty price tag for accuracy. Differential privacy (DP) is a very good example of a now popular constraint, which essentially proceeds by randomizing parts of the whole process to reduce the output’s sensitivity to local changes in the input (Dwork & Roth, 2014). DP possesses a toolbox of simple randomisation mechanisms that can allow for simple modifications of ML algorithms to make them private. However, a careful optimization of the utility (accuracy) under the DP constraints typically requires rethinking the training process, as exemplified by the output perturbation mechanism to train kernel machines in Chaudhuri et al. (2011).

There is to date no such comparable achievement in the case of Decision Trees (DTs) induction, a crucial problem to address: decision trees have been popular in machine learning for decades (Breiman et al., 1984; Quinlan, 1993), they are widely used, in particular for tabular data, and recognised for their accuracy, interpretability, and efficiency; they are virtually present in almost every Kaggle competition Andriushchenko & Hein (2019), with extremely popular implementations like Chen & Guestrin (2016); Ke et al. (2017). On the DP side, there is to our knowledge no extension of boosting properties to DP. We attribute the fact that random forests (RFs) currently "reign supreme" in DP (Fletcher & Islam, 2019, Section 6) as more a consequence of the lack of formal results for boosting rather than following from any negative result.

**Our first contribution** shows a tradeoff to address to solve this problem. On the accuracy side, it has been known for a long time that the curvature of the Bayes risk used conditions the convergence rate in the boosting model (Kearns & Mansour, 1996; Nock & Nielsen, 2004). In this paper, we first investigate the privacy side and show that the sensitivity of the splitting criterion has the same dependence on the curvature: in few words, faster rate goes along with putting more noise to pick the split. Since the total privacy budget spent grows with the size of the tree, there is therefore a nontrivial tradeoff to solve between rate and noise injection to get sufficient accuracy under DP budget constraints.

**Our second contribution** brings a nail to hammer for this tradeoff: a new proper loss, properness being the minimal requirement that Bayes rule achieves the optimum of the loss. This loss, that we call Ma-loss, admits parameter $\alpha \in [0,1]$ which finely tunes the boosting convergence vs privacy budget tradeoff. As $\alpha \to 1$, boosting rate converges to the optimal rate while as $\alpha \to 0$, sensitivity converges to the minimum. In addition, we provide the full picture of boosting rates for the Ma-loss, of independent interest since generalizing the results of Kearns & Mansour (1996).

**Our third contribution** brings a possible hammer for this nail. We show how to *tune* the loss during induction to limit the privacy budget spent while keeping the *same* boosting rates as in the noise-free case for a subtree of the tree with the same root, with a guaranteed probability. As the training sample increase in size, all else being equal, this probability...
converges to 1 and the subtree converges to the full boosted tree. This technique, that we nickname **objective calibration**, picks at the beginning of the induction a splitting criterion with optimal boosting convergence, thus paying significant privacy budget, and then reduces the budget spent as we split deeper nodes, thus also reducing convergence. Ultimately, the budget converges to the smallest splitting budget as the tree converges to consistency on training.

**Our fourth contribution** provides extensive experiments on 19 UCI domains (Dua & Graff, 2017). An extensive comparison of our approach with two SOTA RFs reveals that our approach tends to very significantly beat RFs, even with ensembles more than ten times smaller. Our results display the benefits of combining boosting with DP, as well as the fact that objective calibration happens to be competitive also in the noise-free case.

The rest of this paper follows the order of contributions: after some definition in Section §2, the tradeoff between privacy and accuracy is developed in Section §3, the $M_\alpha$-loss is presented in §4, results on boosting with the $M_\alpha$-loss are given in §5, objective calibration is presented in §6, experiments are summarized in §7 and a last Section, §8, concludes the paper. In order not to laden the main body’s content, all proofs and considerably more detailed experiments have been pushed to an appendix (App.), available from pp 18 (proofs) and from pp 50 (experiments).

## 2 Definitions

▷ **Batch learning**: most of our notations from Nock & Williamson (2019). We use the shorthand notations $[n] = \{1, 2, ..., n\}$ for $n \in \mathbb{N}$, and $z + z \cdot [a, b] = [z' + za, z' + zb]$ for $z \geq 0, z' \in \mathbb{R}, a \leq b \in \mathbb{R}$. We also let $\mathbb{R} = [-\infty, \infty]$. In the batch supervised learning setting, one is given a training set of $m$ examples $S = \{(x_i, y_i), i \in [m]\}$, where $x_i \in X$ is an observation ($X$ is called the domain: often, $X \subseteq \mathbb{R}^n$) and $y_i \in Y = \{-1, 1\}$ is a label, or class. The objective is to learn a classifier, i.e. a function $h : X \rightarrow \mathbb{R}$ which belongs to a given set $\mathcal{H}$. The first class of models we consider are decision trees (DTs). A (binary) DT $h$ makes a recursive partition of a domain. There are two types of nodes: internal nodes are indexed by a binary test and leaves are indexed by a real number. The depth of a node (resp. a tree) is the minimal path length from the root to the node (resp. the maximal node depth). Thus, depth(root) is zero. The classification of some $x \in X$ is achieved by taking the sign of the real number whose leaf is reached by $x$ after traversing the tree from the root, following the path of the tests it satisfies. The other types of classifiers we consider are linear combinations of base classifiers, now hugely popular when base classifiers are DTs, after the advents of bagging (Breiman, 1996) and boosting (Friedman et al., 2000).

▷ **Losses**: the goodness of fit of some $h$ on $S$ is evaluated by a given loss. There are two dual views of losses to train domain-partitioning classifiers (like DTs) and linear combinations of base classifiers (Nock & Nielsen, 2009). Both views start from the definition of a loss for class probability estimation, $\ell_{cpe} : Y \times [0, 1] \rightarrow \mathbb{R}$,

$$
\ell_{cpe}(y, u) \equiv [y = 1] \cdot \ell_1(u) + [y = -1] \cdot \ell_{-1}(u),
$$

where $[\cdot]$ is Iverson’s bracket. Functions $\ell_1, \ell_{-1}$ are called partial losses; we refer to Reid & Williamson (2010) for the additional background on partial losses. We consider symmetric
losses for which \( \ell_1(u) = \ell_{-1}(1 - u), \forall u \in [0, 1] \) (Nock & Nielsen, 2008) (in particular, this assumes that there is no class-dependent misclassification loss). For example, the square loss has \( \ell_{sq}^2(u) = (1/2) \cdot (1 - u)^2 \) and \( \ell_{sq}^{-1}(u) = (1/2) \cdot u^2 \). The log loss has \( \ell_{log}^0(u) = -\log u \) and \( \ell_{log}^{-1}(u) = -\log(1 - u) \). The 0/1 loss has \( \ell_{0/1}^0(u) = [\pi \geq 1/2] \) and \( \ell_{0/1}^{-1}(u) = [\pi \leq 1/2] \). All these losses are symmetric. The associated (pointwise) Bayes risk is

\[
\bar{L}(\pi) \doteq \inf_u \mathbb{E}_{Y \sim B(\pi)}[\ell_{cpe}(Y, u)],
\]

(2)

where \( B(\pi) \) denotes a Bernoulli for picking label \( Y = 1 \). Most DT induction algorithms follow the greedy minimisation of a loss which is in fact a Bayes risk (Kearns & Mansour, 1996). For example, up to a multiplicative constant that plays no role in its minimisation, the square loss gives Gini criterion, \( L_{sq}(\pi) = (1/2) \cdot \pi(1 - \pi) \) (Breiman et al., 1984); the log loss gives the information gain, \( L_{log}(u) = -\pi \log(\pi) - (1 - \pi) \log(1 - \pi) \) (Quinlan, 1993) and the 0/1 loss gives the empirical risk \( L_{0/1}(u) = \min\{\pi, 1 - \pi\} \). To follow Kearns & Mansour (1996), we assume wlog that all Bayes risks are normalized so that \( L(1/2) = 1 \), which is the maximum for any symmetric proper loss (Nock & Nielsen, 2008), and \( L(0) = L(1) = 0 \) (the loss is fair, Reid & Williamson (2010)). Any Bayes risk is concave (Reid & Williamson, 2010). So, if \( h \) is a DT, then the loss minimized to greedily learn \( h \), \( F(h; S) \), can be defined in general as:

\[
F(h; S) \doteq \mathbb{E}_S[L(q(\ell(x_i)))],
\]

(3)

where \( \ell(.) \) is the leaf reached by \( x_i \) in \( H \) and \( q(\ell) = \hat{p}[Y = 1|\ell] \) is the relative proportion of class 1 in the examples reaching \( \ell \). To ensure that a real valued classification is taken at each leaf of \( h \), the predicted value for leaf \( \lambda \) is

\[
h(\ell) \doteq -L'(q(\ell)) \in \mathbb{R}.
\]

(4)

Function \(-L' \) is called the canonical link of the loss (Buja et al., 2005; Nock & Williamson, 2019; Reid & Williamson, 2010). If the loss is non differentiable, the canonical link is obtained from any selection of its subdifferential.

If \( H \) is a linear combination of base classifiers, we adopt the convex dual formulation of (negative) the Bayes risk which, by the property of Bayes risk, admits a domain that can be the full \( \mathbb{R} \) (Boyd & Vandenberghe, 2004). In this case, we replace (3) by the following loss:

\[
F(h; S) \doteq \mathbb{E}_S((-L)^*(-y_ih(x_i))],
\]

(5)

where \( \ast \) denotes the Legendre conjugate of \( F \), \( F^*(z) = \sup_{z' \in \text{dom}(F)}\{zz' - F(z')\} \) (Boyd & Vandenberghe, 2004). Losses like (5) are sometimes called balanced convex losses (Nock & Nielsen, 2008) and belong to a broad class of losses also known as margin losses (Masnadi-Shirazi & Vasconcelos, 2015, Section 2.3). The most popular losses are particular cases of (5), like the square or logistic losses (Masnadi-Shirazi & Vasconcelos, 2015). It can be shown that if a DT has its outputs mapped to \( \mathbb{R} \) following the canonical link (1), then minimizing (5) to learn the DT is equivalent to minimizing (3), which therefore make both views equivalent (Nock & Nielsen, 2009 Theorem 3). Finally, the empirical risk of \( H \), \( \varepsilon_{0/1}(H) \), is (5) in which the inside brackets is predicate \( y_ih(x_i) < 0 \).

\[1\] Not to be confused with the general notation of a loss for class probability estimation, \( \ell_{cpe} \).
Differential privacy (DP) essentially relies on randomized mechanisms to guarantee that neighbor inputs to an algorithm \( M \) should not change too much its distribution of outputs \( \text{Dwork et al., 2006} \). In our context, \( M \) is a learning algorithm and its input is a training sample (omitting additional inputs for simplicity) and two training samples \( S \) and \( S' \) are neighbors, noted \( S \approx S' \) iff they differ by at most one example. The output of \( M \) is a classifier \( h \).

**Definition 1** Fix \( \varepsilon \geq 0 \). \( M \) gives \( \varepsilon \)-DP if \( p[M(S) = h] \leq \exp(\varepsilon) \cdot p[M(S') = h], \forall S \approx S', \forall h \), where the probabilities are taken over the coin flips of \( M \).

The smaller \( \varepsilon \), the more private the algorithm. Privacy comes with a price which is in general the noisification of \( M \). A fundamental quantity that allows to finely calibrate noise to the privacy parameters relies on the sensitivity of a function \( f(.) \), defined on the same inputs as \( M \), which is just the maximal possible difference of \( f \) among two neighbor inputs. Assuming \( \text{Im}(f) \subseteq \mathbb{R}^n \), the global sensitivity of \( f(.) \), \( \Delta_f \), is \( \Delta_f = \max_{S \approx S'} \| f(S) - f(S') \|_1 \) \( \text{Dwork et al., 2006} \). DP offers two standard tools to devise general mechanisms with \( \varepsilon \)-DP guarantees, one to protect real values and the other to protect a choice in a fixed set \( \text{Dwork & Roth, 2014 McSherry & Talwar, 2007} \). The former, the Laplace mechanism, adds Lap\((b)\) noise to a real-valued input, with \( b = \Delta_f / \varepsilon \) is the scale parameter. The latter is the exponential mechanism: let \( \{g : g \in \mathcal{G}\} \) denote a set of alternatives and \( f : \mathcal{R} \rightarrow \mathbb{R} \) a function that scores each of them (the higher, the better), whose values depend of course on \( S \). The exponential mechanism outputs \( g \in \mathcal{G} \) with probability \( \propto \exp(\varepsilon f(g)/(2\Delta_f)) \), thus tending to favor the highest scores. Finally, the composition theorem, particularly useful when training \( h \) is iterative like for DTs, states that the sequential application of \( \varepsilon_i \)-DP mechanisms \( i = 1, 2, ... \), provides \((\sum_i \varepsilon_i)\)-DP \( \text{Dwork et al., 2006} \).

### 3 The Privacy vs Boosting Dilemma for DT

Let \( \Lambda(h) \) denote the set of leaves of tree \( h \). Let \( w \in (0,1]^m \) denote a set of non-normalized weights over the training sample \( S \). Because \( h \) produces a partition of \( S \), we rewrite the loss \( \text{Friedman & Schuster, 2010} \) as \( w(S) \cdot F(h;S) = \sum_{\lambda \in \Lambda(h)} f_L(h,\lambda,S) \) with \( \text{Friedman & Schuster, 2010} \)

\[
f_L(h,\lambda,S) = w(\lambda) \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right),
\]

and \( w(S) = 1^\top w, w^1(\lambda) = \sum_i [i \in \lambda \land (y_i = 1)] \cdot w_i \), \( w(\lambda) = \sum_i [i \in \lambda] \cdot w_i \) and \( i \in \lambda \) is the predicate "observation \( x_i \) reaches leaf \( \lambda \) in \( h \)". Following \( \text{Friedman & Schuster, 2010} \), we want to compute the sensitivity of \( f \),

\[
\Delta_L(h,\lambda) = \sup_{S \approx S'} |f_L(h,\lambda,S') - f_L(h,\lambda,S)|
\]

(we sometimes note \( \Delta_L \) to save readability). To compute it, we need a definition from convex analysis, perspectives.

\footnote{We multiply both sides by \( w(S) \) to follow \( \text{Friedman & Schuster, 2010} \). \( w(S) \) is indeed constant when growing a tree and does not influence the exponential mechanism.}
Definition 2  (Maréchal, 2005a,b) Given closed convex function $f$, the perspective of $f$, noted $\tilde{f}(x,y)$ is:

$$\tilde{f}(x,y) \doteq y \cdot f(x/y) \ , \ if \ y > 0 \ ,$$

and otherwise $\tilde{f}(x,y) \doteq f^0+(x)$ if $y = 0$ and $\tilde{f}(x,y) \doteq +\infty$ if $y < 0$. Here, $f^0+$ is the recession function of $f$.

To save notations, we extend this notion to Bayes risks, that are concave, and therefore write for short $\hat{L} \doteq -\left(-\frac{L}{L'}\right)$.

Theorem 3  $\Delta_L(h,\lambda) \leq \max\{3, 1 + \hat{L}(1, m + 1)\}$.

Theorem 3 generalizes SOTA in two ways, first because only up to 4 Bayes risks were covered (Friedman & Schuster, 2010), and second because classical analyses have $w$ uniform (which precludes boosting). We now show that the variation of a perspective transform of a Bayes risk is linked to its weight (or curvature, Reid & Williamson (2010)).

Lemma 4  For any twice differentiable $L$, for any $m$, there exists $a \in [0, 1/(m+1)]$ such that

$$(\hat{L}')(1, m + 1) = (m + 1)^{-2} \cdot (-L'')(a) .$$

The proof of Theorem 3 includes the proof that the bound is in fact almost tight as some neighboring samples admit $\Delta_L(h,\lambda) = \hat{L}(1, m)$, so the variation in DP budget with $m$ is directly linked to $\hat{L}$. In other words, the larger the weight ($-L''$, Reid & Williamson (2010)), the more expensive becomes DP with $m$ when relying on $\Delta_L$ as sensitivity measure — such as the exponential mechanism in Friedman & Schuster (2010). It turns out that it has long been known that boosting’s convergence works the exact same way: the larger the weight, the better is the rate guaranteed under boosting-compliant assumptions (Kearns & Mansour, 1996). Since the top-down induction of a greedy tree gradually spends privacy budget to split each node, the boosting vs privacy dilemma is thus to guarantee fast enough convergence — because it also saves budget as we converge in less iterations — while keeping the privacy budget within required bounds. We now give an example of the budget required for popular Bayes risks using Theorem 3. $L^{\text{Mat}}(u) = 2\sqrt{u(1-u)}$ is Bayes risk of Matsushita loss (Nock & Nielsen, 2008, 2009), which guarantees optimal boosting convergence (Kearns & Mansour, 1996) and thus, as expectable, is the most "expensive" DP-wise.

Lemma 5  $\forall L \in \{L^{\text{Mat}}, L^{\text{Log}}, L^{\text{sq}}, L^{0/1}\}$, we have $\Delta_L \leq \max\{3, 1 + \Delta_L^*(m)\}$ where $\Delta_L^{*\text{Mat}} = 2\sqrt{m}$, $\Delta_L^{*\text{Log}} = (1 + \log(m+1)) \cdot \log^{-1} 2$, $\Delta_L^{*\text{sq}} = 4m/(m + 1)$, $\Delta_L^{*0/1} = 2$.

We note that all our bounds are within 1 of the bounds known for $L^{\text{Log}}, L^{\text{sq}}, L^{0/1}$ (Friedman & Schuster, 2010), so the generality of Theorem 3 (all applicable Bayes risks, non-uniform weights over examples) comes at reduced price.
\[
\psi^{(\alpha)}(u) \in \alpha \cdot \frac{2u - 1}{\sqrt{u(1-u)}} - 2(1-\alpha) \cdot \begin{cases} 
1 & \text{if } u < 1/2 \\
-1 & \text{if } u = 1/2 \\
-1 & \text{if } u > 1/2
\end{cases},
\]

\[
\psi^{-1}(z) = \frac{1}{2} \left( 1 + \left\lceil z \not\in 2(1-\alpha) \cdot [-1,1] \right\rceil \right) \cdot \frac{\sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1-\alpha) \right)^2}}{\sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1-\alpha) \right)^2}},
\]

\[
F^{(\alpha)}(z) = 1 - \frac{z}{2} + \left\lceil z \not\in 2(1-\alpha) \cdot [-1,1] \right\rceil \cdot \left( \frac{\sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1-\alpha) \right)^2}}{\sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1-\alpha) \right)^2}} - \alpha \right)
\]

Figure 1: Left: canonical link \(\psi^{(\alpha)}\), inverse canonical link \(\psi^{-1}(z)\) and convex surrogate \(F^{(\alpha)}\) for the \(M_{\alpha}\)-loss. Right: plots of Bayes risk \(L^{(\alpha)}(q)\), sensitivity \(\Delta^{(\alpha)} \doteq \Delta L_{(\alpha)}(m)\), \(\psi^{-1}(z)\) and \(F^{(\alpha)}(u)\) for the \(M_{\alpha}\)-loss, for various \(\alpha\)s (colors).

4 The \(M_{\alpha}\)-loss

In the boosting vs DP picture, there are two extremal losses. The 0/1 loss is the one that necessitates the smallest DP budget (Lemma 5) but achieves the poorest convergence guarantee [Kearns & Mansour, 1996, Section 5.1]. On the other side of the spectrum, Matsushita loss guarantees the optimal convergence rate [Kearns & Mansour, 1996; Nock & Nielsen, 2004] but necessitates a considerable DP budget (Lemmata 4, 5). We address the challenge of tuning the convergence rate vs DP budget by creating a new proper symmetric loss, allowing to stand anywhere in between these extremes via a simple tunable parameter \(\alpha\).

\textbf{Definition 6} The \(M_{\alpha}\)-loss is defined for any \(\alpha \in [0,1]\) by the following partial losses, for \(y \in \{-1,1\}\):

\[
\ell^{(\alpha)}_{y}(u) = 2\alpha \cdot \left( \frac{1-u}{u} \right)^{y/2} + 2(1-\alpha) \cdot \left[ yu \leq y/2 \right].
\]

It is easy to check that the \(M_{\alpha}\)-loss is proper (strictly if \(\alpha > 0\)) and symmetric, as well as its Bayes risk is a convex combination of those of the 0/1 and Matsushita losses:

\[
L^{(\alpha)}(u) = \alpha \cdot L_{\text{Mat}}(u) + (1-\alpha) \cdot L_{0/1}(u).
\]

It is also not hard to show that the sensitivity intrapolates between both losses’ sensitivities using Lemma 5.

\textbf{Corollary 7} The sensitivity \(\Delta^{(\alpha)} \doteq \Delta L_{(\alpha)}(m)\) satisfies

\[
\Delta^{(\alpha)} = 3 + 2\alpha(\sqrt{m} - 1).
\]

Because of the 0/1 loss is not differentiable, getting the inverse canonical link and the convex surrogate is trickier.

\textbf{Theorem 8} The canonical link \(\psi^{(\alpha)}\), inverse canonical link \(\psi^{-1}(z)\) and surrogate \(F^{(\alpha)}\) of the \(M_{\alpha}\)-loss are as in Fig. 1.
5 Boosting with the $\alpha$-loss

\textbf{Boosting decision trees}: We know from the last Section that the $\alpha$-loss allows, by tuning $\alpha$, to continuously change the sensitivity of the criterion between the minimal ($\alpha = 0$) and a maximal one ($\alpha = 1$). We are now going to show that the criterion allows as well to intrapolate between optimal boosting regime ($\alpha = 1$) and a "minimal" convergence guarantee ($\alpha = 0$), thereby completing the boosting vs privacy picture for the $\alpha$-loss. We first tackle the induction of a single DT as in [Kearns & Mansour (1996)]. Boosting start by formulating a \textbf{Weak Learning Assumption} (WLA) which gives a weak form of correlation with labels for the elementary block of a classifier. In the case of a DT, such a block is a split. So, consider leaf $\lambda$ and a test $g : \mathcal{X} \to \{0, 1\}$ that splits the leaf in two, the examples going to the left (for which $g = 0$) and those going to the right (for which $g = 1$). The relative weight of positive examples reaching $\lambda$ is $q \in (0, 1)$, where $q \neq 0, 1$ ensures that the leaf is not pure. Define the balanced weights at leaf $\lambda$ to be (a) $\tilde{w}_i = 0$ if $x_i \not\in \lambda$, else (b) $\tilde{w}_i = 1/(2q)$ if $y_i = +1$, else (c) $\tilde{w}_i = 1/(2(1-q))$. Let $\tilde{w}_\lambda$ denote the complete distribution and $g^{+/−} \in \{-1, 1\}^X$ as $g^{+/−} = −1 + 2g$. We adopt the \textbf{edge notation} $\eta(\tilde{w}, h) = \sum_i \tilde{w}_iy_ih(x_i)$ for any $h \in \mathbb{R}^X$. Suppose $\gamma > 0$ a constant.

\textbf{Definition 9 (WLA for DT)} Split $g$ at leaf $\lambda$ satisfies the $\gamma$-WLA iff $|\eta(\tilde{w}_\lambda, g^{+/−})| \geq \gamma$.

Definition 9 is not the same as [Kearns & Mansour (1996)], but it is equivalent (App., §14) and in fact more convenient for our framework. A random split would not satisfy the WLA so the WLA enforces the existence of splits at least moderately correlated with the class. Top-down DT induction usually does not proceed by optimizing the split based on the WLA, but in fact it can be shown that the WLA \textit{implies} good splits according to top-down DT induction criteria [Kearns & Mansour (1996), Section 5.3]. So we let $h_\ell$ denote the current DT with $\ell$ leaves and $\ell − 1$ internal nodes. We grow it to get $h_{\ell+1}$ by minimizing $L(\alpha)$ as:

$$L^{(\alpha)}(h_{\ell+1}) = \alpha_\ell \cdot L^{\text{mat}}(h_{\ell+1}) + (1 - \alpha_\ell) \cdot L^{\text{err}}(h_{\ell+1}),$$

where $h_{\ell+1}$ is $h_\ell$ with a leaf $\lambda \in \Lambda(h_\ell)$ replaced by a split. Noting $K \geq 0$ a constant, $w = \sum_i w_i$ and

$$\tilde{w}_\ell = \left(\frac{1}{w}\right) \cdot \sum_i w_i \mathbb{1}[i \in \lambda_\ell] \in [0, 1]$$

(10)

the total \textit{normalized} weight of the examples reaching $\lambda_\ell$, we say that the sequence of $\alpha$ is $K$-monotonic iff $\alpha_\ell \leq \alpha_{\ell-1} \cdot \exp(K\tilde{w}_\ell(1 - \alpha_{\ell-1}))$ for any $\ell > 2$ (and $\alpha_1 \in [0, 1]$). Since the parameter in the exp is $\geq 0$, $K$-monotonicity prevents the sequence from growing too fast.

\textbf{Theorem 10} Suppose all splits satisfy the $\gamma$-WLA and the sequence of $\alpha$ is $(\gamma^2/16)$-monotonic. Then $\forall \xi \in (0, 1]$, the empirical risk of $h_\ell$ satisfies $\varepsilon_{\ell+1}(h_\ell) \leq \xi$ as long as

$$\sum_{\ell=1}^L \tilde{w}_\ell \alpha_\ell \geq \left(\frac{16}{\gamma^2}\right) \cdot \log\left(\frac{1}{\xi}\right).$$

(11)
It is worth remarking that this is indeed a generalization of [Kearns & Mansour (1996)] : suppose \( \alpha_t = \alpha \) constant (which is \((1/16)\)-monotonic \( \forall \gamma \geq 0 \)) and we pick at each iteration the heaviest leaf to split. We thus have \( \tilde{w}_t \geq 1/\ell \), assuming further it satisfies the WLA. Since \( \sum_{t=1}^{L} (1/\ell) \geq \int_1^{L+1} dz/z = \log(L+1) \), (11) is guaranteed if

\[
L \geq \left( \frac{1}{\xi} \right)^{\frac{16}{\alpha \gamma^2}},
\]

which, for \( \alpha = 1 \), is in fact the square root of the bound in [Kearns & Mansour (1996)] Theorem 10 and is thus significantly better. Rather than a quantitative improvement, we were seeking for a qualitative one as [Kearns & Mansour (1996)] pick the heaviest leaf to split, which means using DP budget to find it. To see how we can get essentially the same guarantee without this contraint, suppose instead that we split all current leaves, all of them satisfying the WLA. Since \( \sum_{\lambda} \tilde{w}(\lambda) = 1 \) (those weights are normalized), once we remark that it takes one split for the root, then two, then four and so on to fully split the current leaves, \( L \) boosting iterations guarantee a full split up to depth \( O(\log(L)) \), which delivers the same condition as (12) with an eventual change in the exponent constant. Our result is also a generalization of [Kearns & Mansour (1996)] since it allows to tune \( \alpha \) during learning, which is important for us (§ 6).

\[\text{Boosting linear combinations of classifiers:}\] we now consider that we build a Linear Combination (LC) of classifiers, \( H_T \doteq \sum_{t=1}^{T} \beta_t h_t \), where \( h_t : X \rightarrow \mathbb{R} \) is a real valued classifier — this could be a DT or any other applicable classifier. We tackle the problem of achieving boosting-compliant convergence when building \( H_T \), which means we have a WLA on each \( h_t \). We also assume \( \exists M > 0 \) such that \( |h_t| \leq M \). Let \( w_t \in [0,1]^m \) an \textit{unnormalized} weight vector on \( \mathcal{S} \), \( t \) denoting the iteration number from which \( h_t \) is obtained. Noting \( \tilde{w}_t = (1/m) \cdot \sum_i w_{ti} \in [0,1] \) the expected unnormalized weight at iteration \( t \), we also let \( \tilde{w}_t = (1/(m\tilde{w}_t)) \cdot w_t \) denote the \textit{normalized} weight vector at iteration \( t \). The WLA is as follows.

**Definition 11** (WLA for LC) \( h_t \) obtained at iteration \( t \) satisfies the \( \gamma \)-WLA iff \( |\eta(\tilde{w}_t, h_t)| \geq \gamma M \).

Remark that this definition is similar to Definition 9, since \( g^{1/2} = 1 \). All the crux is now how to get the weight vectors \( w_t \) so that we can prove a boosting-compliant convergence rate using the \( M \)-loss. We do so using a standard mechanism, which consists in initializing \( w_1 = (1/2) \cdot 1 \) (unnormalized) and then using the mirror update of the \( M \)-loss to update weights after \( h_t \) has been received:

\[
w_{(t+1)i} \leftarrow \psi_i^{(\alpha)}(\alpha^{-1} (\beta_i \eta_i h_t(x_i) + \psi_i^{(\alpha)}(w_{ti}))), \tag{13}
\]

where \( \beta_t \) is a leveraging coefficient for \( h_t \) in the final classifier, taken to be \( \beta_t \leftarrow a\tilde{w}_t \cdot \eta(\tilde{w}_t, h_t) \), where \( a \) is a constant chosen beforehand anywhere in interval \((\alpha/M^2) \cdot [1 - \pi, 1 + \pi], \pi \in [0,1]\) quantifying the freedom in choosing \( a \). This is sufficient to complete the description of the algorithm (also given in \textit{in extenso} in App., § [15]).

\[\]
Theorem 12  Suppose all $h_t$ satisfy the $\gamma$-WLA. Then $\forall \xi \in [0, 1]$, we have $\varepsilon_{0/1}(H_T) \leq \xi$ as long as:

$$\sum_{t=1}^{T} \tilde{w}_t^2 \geq \frac{2(1 - \xi)}{(1 - \pi^2)\gamma^2\alpha}. \text{ (14)}$$

This Theorem has a very similar flavour on boosting conditions as we had in Theorem 10 for DTs but its dependence on $\xi$ is comparatively misleading. What Theorem 12 indeed tells us is boosting for LC is efficient under the WLA as long as $w_t$ is "large" enough in $[0, 1]$. The weight update in (13) meets the classical boosting property that an example has its weight directly correlated to classification: the better, the smaller its weight ($Cf$ the plot of $\psi^{(\alpha)}$ in Figure 1). Hence, as classification gets better, the sum on the LHS of (14) increases at smaller rate and if $\xi$ is too small, this means a potentially larger number of iterations to meet (14).

6 Privacy and boosting: objective calibration

We have so far described the complete picture of DP for DT with any noisification mechanism that relies on the sensitivity of a Bayes risk, and the complete but noise-free boosting picture for the $\alpha$-loss for DT and LC. We now assemble them. In an iterative boosted combination of DT, two locations of privacy budget spending can make the full classifier meet DP: (a) node splitting in trees, (b) leaf predictions in trees. The protection of the leveraging coefficients $\beta_t$ can be obtained in two ways: either we multiply each leaf prediction by $\beta_t$, then replace $\beta_t \leftarrow 1$ and then carry out (b), or use the faster but more conservative approach to just do (b) e.g. with the Laplace mechanism from which follows the protection of $\beta_t$ ($\S$ 5). We do not carry out pruning as boosting alone can be sufficient for good generalization, see e.g. Schapire et al. (1998, Section 2.1), Bartlett & Mendelson (2002, Theorems 16, 17), and pruning also requires privacy budget (Fletcher & Islam, 2019, $\S$ 3.5). The public information is the attribute domain, which is standard (Fletcher & Islam, 2019), and we consider that each continuous attributes is regularly quantized using a public number $v_q$ of values. This makes sense for many common attributes like age, percentages, $\$-$value, and this can contribute to ease interpretation; this also has three technical justifications: (1) a private approaches requires budget, (2) $v_q$ allows to tightly control the computational complexity of the whole DT induction, (3) boosting does not require exhaustive split search provided $v_q$ is not too small (more in App., $\S$ 17.1).

▷ Private induction of a DT: objective calibration. The overall privacy budget $\varepsilon$ is split in two proportions: $\beta_{\text{tree}}$ for node splitting (a) and $\beta_{\text{pred}} = 1 - \beta_{\text{tree}}$ leaves’ predictions (b). The basis of our approach to split nodes is the nice — but never formally analyzed — trick of Friedman & Schuster (2010) which consists in using the exponential mechanism to choose splits. Let $\mathcal{S}$ denote the whole set of splits. The probability to pick $g \in \mathcal{S}$ to split leaf $\lambda \in \Lambda(h_\ell)$ is:

$$p_{\exp}((g, \lambda)) \propto \exp \left( -\frac{\varepsilon(h_\ell, \lambda)w(S)F(h_\ell \oplus (g, \lambda))}{2\Delta^*_{\mathcal{L}(\alpha)}(m)} \right), \text{ (15)}$$
where notation $h_\ell \oplus (g, \lambda)$ refers to decision tree $h_\ell$ in which leaf $\lambda$ is replaced by split $g$, $w(S) \cdot F(h_\ell \oplus (g, \lambda))$ is the unnormalized Bayes risk (Section 3) and $\Delta_{x(h_\ell)}(m)$ is given in Lemma 7. $\varepsilon(h_\ell, \lambda)$ is the fraction of the total privacy budget allocated to the split. So far, all recorded approaches consider uniform budget spending (Fletcher & Islam, 2019) but such a strategy is clearly oblivious to the accuracy vs privacy dilemma as explained in Section 3. We now introduce a more sophisticated approach exploiting our result, allowing to bring strong probabilistic guarantees on boosting while being private. The intuition behind is simple: the "support" (total unnormalized weight) of a node is monotonic decreasing on any root-to-leaf path. Therefore, we should typically increase the budget spent in low-depth splits because (i) it impacts more examples and (ii) it increases the likelihood of picking the splits that meet the WLA in the exponential mechanism (15). Consequently, we also should pick $\alpha$ larger for low-depth splits, to increase the early boosting rate and drive as fast as possible the empirical risk to the minimum, yet monitoring the dependency of the exponential mechanism in $\alpha$ to control the probability of picking the splits that meet the WLA. This may look like a quite intricate set of dependences between privacy and boosting, but here is a solution that matches all of them. If we denote $h_1$ the tree reduced to a leaf from which $h_\ell$ was built, $\text{depth}(\cdot)$ as the depth of a node, $d$ the maximal depth of a tree and $T$ the number of trees in the combination, then we let:

$$\alpha_\ell = \frac{\varepsilon_{0/1}(h_\ell)}{\varepsilon_{0/1}(h_1)} (\in [0, 1]),$$

(16)

$$\varepsilon(h_\ell, \lambda) = \frac{\beta_{\text{tree}}}{T d^{2 \text{depth}(\lambda)}} \cdot \varepsilon.$$  

(17)

The choice of $\alpha_\ell$ makes it decreasing along every path from the root: while we split the root using Matsushita loss ($\alpha = 1$), which guarantees optimal boosting rate, we gradually move in deeper leaves to using more of the Bayes risk of the 0/1 loss, which may reduce the rate but reduces privacy budget used as well. Referring to objective perturbation which noisifies the loss (Chaudhuri et al, 2011), we call our method that tunes the loss objective calibration (O.C).

We formally analyze O.C. First, remark that the total budget spent for one tree is $\beta_{\text{tree}} \varepsilon / T$, which fits in the global budget $\varepsilon$. To develop the boosting picture, we build on the $\gamma$-WLA. We first remark that for any $h$, leaf $\lambda \in \Lambda(h)$ and split $g \in \mathcal{S}$, there exists $u \geq 0$ such that

$$L^{(\alpha)}(h) - L^{(\alpha)}(h \oplus (g, \lambda)) = u \cdot \frac{\gamma^2 \alpha \tilde{w}(\lambda)}{16} \cdot L^{(\alpha)}(h).$$

(18)

This is a simple consequence of the concavity of any Bayes risk. Interestingly, for all splits that satisfy the WLA, it can be shown that we can pick $u \geq 1$ (App., § 16). Let us denote $\mathcal{S}_{\text{wla}} \subseteq \mathcal{S}$ the whole set of such boosting amenable splits, and let $\mathcal{S}_{\text{lazy}} = \mathcal{S} \setminus \mathcal{S}_{\text{wla}}$ denote the remaining splits. The exponential mechanism might of course pick splits in $\mathcal{S}_{\text{lazy}}$ but let us assume that there is at least a small "gap" between those splits and those of $\mathcal{S}_{\text{wla}}$, in such a way that for any split in $\mathcal{S}_{\text{lazy}}$, (18) holds only for $u \leq \delta$ for some $\delta < 1$. This property always holds for some $\delta < 1$ but let us assume that this $\delta$ is a constant, just like the $\gamma$ of the WLA, and call it the $\delta$-Gap assumption. Let $N(h)$ denotes the set of nodes of $h$, including leaves in
The tree-efficiency of \( \nu \in N(h) \) in \( h \) is defined as

\[
J(\nu, h) = \frac{8 \bar{w}(\nu) \varepsilon_{\alpha, \eta}(h)^2}{2^{\log h(\nu)}} \in [0, 1],
\]

where \( \bar{w}(\nu) \) is the normalized weights of examples reaching \( \nu \). Let \( L \) be a subset of indexes of the leaves split from \( h_1 \) to create a depth-\( d \) tree, with unnormalized weights \( w \). Each element \( \ell \) refers to a couple \((\lambda_\ell, h_\ell)\) where \( h_\ell \) is the tree in which \( \lambda_\ell \) was replaced by a split.

**Theorem 13** Suppose the exponential mechanism is implemented with \( \alpha_\ell \) and \( \varepsilon_\ell \) as in (16), (17). Suppose \( Td \leq \log m, m \geq 3 \) and both the \( \gamma \)-WLA and \( \delta \)-Gap assumptions hold. Suppose that \( \forall \ell \in L \),

\[
|G_{wla, \ell}| \geq |G_{\lambda, \ell}| \cdot \exp \left( -\Omega \left( J(\lambda_\ell, h_\ell) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right) \right).
\]

Then, for any \( \xi > 0 \), if

\[
\min_{\ell \in L} J(\lambda_\ell, h_\ell) = \Omega \left( \frac{\log m}{\varepsilon \sqrt{m} \log |L|} \right),
\]

then with probability \( \geq 1 - \xi \), all splits chosen by the exponential mechanism to split the leaves indexed in \( L \) satisfy the WLA.

The proof (App., §16), explicits all hidden constants. We insist on the message that Theorem 13 carries about the exponential mechanism: under the WLA/Gap assumptions and a size constraint on each \( G_{wla, \ell} \) (which by the way authorises it to be reasonably smaller than \( G_{\lambda, \ell} \)), the exponential mechanism has essentially no negative impact on boosting with high probability. This, we believe, is a very strong incentive in favor of the exponential mechanism as designed in [Friedman & Schuster] (2010). Finally, the condition \( Td \leq \log m \) could be replaced by a low-degree polylog but even without doing so, it actually fits well in a series of experimental work (Fletcher & Islam, 2019), for example Mohammed et al. (2015) (\( Td = 4 \)), [Friedman & Schuster] (2010) (\( Td = 5 \)), Fletcher & Islam (2015) (\( Td = 20 \)).

**Remark 14** Theorem 13 reveals another reason why we should indeed put emphasis on boosting on low-depth nodes: for any node of \( h \), if its tree efficiency is above a threshold, then so is the case for all nodes along a shortest path from this node to the root of \( h \). Hence the largest set \( L \) for which (20) holds corresponds to a subtree of \( h \) with the same root.

To get a simple idea of how (20) vanishes with \( m \), remark that \( |L| \leq 2^{d+1} - 1 \). Condition (189) is therefore satisfied if for example \( \varepsilon, \xi, d \) are related to \( m \) as

\[
\frac{\log(m)}{\sqrt{m}} = o(\varepsilon),
\]

\[
d, \log \frac{1}{\xi} = o \left( \frac{\sqrt{m}}{\log m} \right),
\]

and in this case the constraint on \( \min_{\ell \in L} J(\lambda_\ell, h) \) in (190) vanishes with \( m \). This makes that strong privacy regimes can fit to Theorem 13, e.g. with \( \varepsilon = \log^{1+\varepsilon}(m) / \sqrt{m} \) for \( c > 0 \) a constant.
| \( \varepsilon \) | 0.01 | 0.1 | 1 | 10 | 25 |
|-----------------|------|-----|---|----|----|
| (O.C, 0.1, 1.0) | (14,3,5) | (13,2,6) | (9,3,9) | (5,4,11) | (8,6,6) |

Table 1: Summary, on the 19 domains, of the \# of domains for which one strategy for \( \alpha \) in \{objective calibration (O.C), 0.1, 1, 0\} leads to the best result (ties lead to sums > 19).

Figure 2: UCI domain banknote: in each plot, \( x \) depicts test errors and \( y \) a cumulated \% of runs of BDPE_{\alpha} having test error at most \( x \). In each pane (left, right), the left plot is without DP and the right plot is with DP. Left pane: comparison of BDPE_{\alpha} for three strategies on \( \alpha \) (see text). Right pane: mean \pm stddev for the number of leaves in the related trees (see text).

\[ \triangleright \textbf{Private predictions at the leaves}: \] because our trees output real values, we use the Laplace mechanism. This fits well with the WLA using \( |h_t| \leq M \) (Definition 11), for the sensitivity of the mechanism (Dwork & Roth, 2014).

## 7 Experiments

We have performed 10-folds stratified CV experiments on 19 UCI domains, detailed in App., Section 17.3, ranging from \( m \cdot n < 3 \ 000 \) to \( m \cdot n > 200 \ 000 \). We have compared our approach, BDPE_{\alpha}, to two state of the art implementation of RFs based on Fletcher & Islam (2017) but replacing the smooth sensitivity by the global sensitivity (Definition 1). RFs have the appealing property for DP that privacy budget needs only be spent at the leaves: we have tried both the Laplace (RF-L) and the exponential (RF-E) mechanisms (see App., § 17.2) with RFs containing \( T = 21 \) trees to prevent ties. We have performed three kinds of experiments: (i) check that BDPE_{\alpha} performs well and complies with the boosting theory in the privacy-free case, (ii) compare the various flavours of BDPE_{\alpha} in the private case, (iii) compare BDPE_{\alpha} vs RFs in the private case. We ran BDPE_{\alpha}, both private and not private, for all combinations of \( T \in \{2,5,10,20\}, \alpha \in \{0.1,1.0,\text{O.C}\}, \text{depth} \in \{1,2,3,4,5,6\}, \varepsilon \in \{0.01,0.1,1.0,10.0,25.0\}, \beta_{\text{tree}} \in \{0.1,0.5,0.9\}, \) and even more parameters (see App., § 17.1), for a total number of boosting experiments alone that far exceeds the million ensemble models learned. When there is no DP constraint, we add in BDPE_{\alpha} the test of whether a leaf is pure – i.e. is not reached by examples of both classes – before attempting to split it (we do not split further pure leaves). When there is DP however, we do not make the test in order not
to spend privacy budget, and so $BDPE_\alpha$ builds trees in which all leaves are at the required depth. The App., § 17 gives the experiments in greater details, summarized here.

- **BDPE$_\alpha$, with and without noise**: Figure 2, left pane, displays a picture that can be observed more or less over all domains: $BDPE_\alpha$ with $\alpha = 1$ tends to obtain better results than with $\alpha = 0.1$, which complies with Theorems 10 and 12 and with the boosting theory more generally [Kearns & Mansour, 1996]. Three additional observations emerge: (a) objective calibration (O.C) is competitive without noise, (b) this also holds with noise, which we believe indicate a good compromise between convergence rates and safekeeping privacy budget in $BDPE_\alpha$ (Section 3) and contributes to experimentally validate our theory in Section 6; (c) DP curves display predictable degradations due to noise, but on many domains nisation still gives interesting results compared to the noise-free setting: in banknote for example (Figure 2), more than 2/3 of the private runs with O.C get test error $\leq 20\%$, an upperbound test error for noise-free boosting.

- **BDPE$_\alpha$ in various privacy regimes**: Table 1 is an extremal experiments which looks at the best models that can be learned under DP under various $\varepsilon$. The picture that seems to emerge is that objective calibration is the best technique for high privacy demand, which we take as a good sign given our theory (Section 6). Obviously, the experiments aggregate a number of parameters for each $\varepsilon$, such as $T, d, \beta_{tree}$, so to really get the best of a regime for $\alpha$, one should be able to have clues on how to fix those other parameters. It turns out that the experiments display that this should be possible. In particular, for each domain, the value $\beta_{tree}$ does not seem to significantly matter to get the best results but the model size parameters seem to matter a lot more: for each domain, there is a particular regime of $d, T$ that tends to give the top DP results (like rather deep trees for banknote, Figure 2). App., Section 17.3 presents the whole list details. This, we believe, is important, in particular for domains where $T$ is small like page, as some RFs approaches fit huge sets reducing interpretability [Fletcher & Islam, 2019] (Table 1).

- **BDPE$_\alpha$ vs (RF-L and RF-E)**: a Table (4, given in App., § 17.3) computes over all 19 domains the % of runs where $BDPE_\alpha$ beats RFs, among all runs for which one approach statistically significantly ($p_{val} < 0.01$) beats the other. The scale heavily tips in favor of $BDPE_\alpha$ when it boosts $T = 20$ trees: O.C and $\alpha = 1.0$ are significantly superior than RF-E and RF-L on more than 80% of such cases (less than 4% of the differences are not significant). This means two things: first, for these strategies of $BDPE_\alpha$, there is not much care needed to optimize some parameters of $BDPE_\alpha$ ($d, \beta_{pred}$) to get to or beat SOTA, which is good news; second, this suggests that we can compete with RFs on much smaller trees, which is indeed displayed in the left pane of Table 4 where $BDPE_\alpha$ fits less than ten times trees than RFs, and still beat those in a majority of cases, which is good news for interpretability. When we drill down into the results as a function of $\varepsilon$, we observe that $BDPE_\alpha$ tends to be especially good against RFs for high privacy regimes (e.g. $\varepsilon = 0.01$).

- **BDPE$_\alpha$ in the $v_q = 10$ vs $v_q = 50$ regime**: the previous summarizes experiments for a regular quantization with $v_q = 10$ of the continuous attributes. Our experiments (App., Section 17.3) also contain a summary of the comparisons for $BDPE_\alpha$ when we rather use $v_q = 50$. Notice that multiplying by five the potential number of splits significantly affects the time complexity of the algorithm. The results display that the impact varies as a function of the domain at hand. There can be significant improvements: qsar and winewhite are two domains for which $v_q = 50$ buys more than 2% improvement for objective calibration, a clear
winner among all tested strategies for \( \alpha \). On banknote, the improvement is more in favor of \( \alpha = 1.0 \). On winered, there is no significant improvement for the best strategy and apart from a seemingly better "concentration" of more than 3/4 of the runs of objective calibration towards its best results with \( v_q = 50 \), there is no apparent gain otherwise.

8 Conclusion

While boosted ensemble of DTs have long shown their accuracy in international competitions, to our knowledge nothing is known on how to fit them in a differentially private framework while keeping some of the boosting guarantees, a setting in which random forests have been reigning supreme. In this paper, we first establish the existence of a nontrivial tradeoff to push boosting methods in a differentially private framework. To address this tradeoff, we first create a tunable proper canonical loss, whose boosting rate and sensitivity can be controlled up to optimal boosting rate, or minimal sensitivity. We then show guaranteed boosting rates for both the induction of DTs and ensembles using this loss, of independent interest. We introduce objective calibration as a way to dynamically tune this loss and make the most of boosting under a given privacy budget with high probability. Experiments reveal that our approach manages to significantly beat random forests, that the best private models tend to be learned by objective calibration, and that our technique appears all the better on high privacy regimes.

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# Appendix

## 9 Table of contents

| Appendix on proofs | Pg 19 |
|--------------------|-------|
| Proof of Theorem 3  | Pg 19 |
| Proof of Lemma 4    | Pg 28 |
| Proof of Lemma 5    | Pg 29 |
| Proof of Theorem 8  | Pg 29 |
| Proof of Theorem 10 | Pg 33 |
| Proof of Theorem 12 | Pg 39 |
| Proof of Theorem 13 | Pg 42 |

| Appendix on experiments | Pg 50 |
|-------------------------|-------|
| Implementation          | Pg 50 |
| General setting         | Pg 50 |
| Domain summary Table    | Pg 51 |
| UCI transfusion         | Pg 53 |
| UCI banknote            | Pg 55 |
| UCI breastwisc          | Pg 57 |
| UCI ionosphere          | Pg 59 |
| UCI sonar               | Pg 61 |
| UCI yeast               | Pg 63 |
| UCI winered             | Pg 65 |
| UCI cardiotocography    | Pg 67 |
| UCI creditcardsmall     | Pg 69 |
| UCI abalone             | Pg 71 |
| UCI qsar                | Pg 73 |
| UCI page                | Pg 75 |
| UCI mice                | Pg 77 |
| UCI hill+noise          | Pg 79 |
| UCI hill+nonoise        | Pg 81 |
| UCI firmteacher         | Pg 83 |
| UCI magic               | Pg 85 |
| UCI eeg                 | Pg 87 |

Summary in $d, T$ for the best DP results in $\text{BDPE}_a$ | Pg 89 |
Summary of the comparison $\text{BDPE}_a$ vs RFs with DP | Pg 90 |
Summary comparison $v_q = 10$ vs $v_q = 50$ ($M = 10$) | Pg 90 |
Appendix on Proofs

10 Proof of Theorem 3

The proof is split in three parts. The two first being the following two Lemmata.

Lemma 15 Fix \( u \geq 0 \). \( \mathcal{L}(u,v) \) is non-decreasing over \( v \geq u \).

Proof We know that \(-\mathcal{L}\) is convex and therefore \( D_{-\mathcal{L}}(a\|b) \) is non negative, \( D_{-\mathcal{L}} \) being the Bregman divergence with generator \(-\mathcal{L}\) (Nock & Nielsen 2009). We obtain, with \( a = 0, b = u/v \),

\[
D_{-\mathcal{L}}(a\|b) = L\left(\frac{u}{v}\right) - L(0) - \left(0 - \frac{u}{v}\right) \cdot (-\mathcal{L})' \geq 0,
\]

where \((-\mathcal{L})' \in \partial(-\mathcal{L})(u/v)\), \( \partial \) denoting the subdifferential. Simplifying (\( L(0) = 0 \)) yields

\[
L\left(\frac{u}{v}\right) + \frac{u}{v} \cdot (-\mathcal{L})' \geq 0. \tag{21}
\]

We then remark that

\[
\partial_v \mathcal{L}(u,v) = \left\{ L\left(\frac{u}{v}\right) + \frac{u}{v} \cdot (-\mathcal{L})' \mid L' \in \partial L\left(\frac{u}{v}\right) \right\}. \tag{22}
\]

Therefore, \( \mathcal{L}(u,v) \) is non-decreasing when \( v \geq u \), and we obtain the statement of Lemma 15.

The next Lemma shows a few more facts about \( \mathcal{L} \).

Lemma 16 The following holds true:

(A) \( \mathcal{L} \) is not decreasing (resp. not increasing) over \([0, 1/2]\) (resp. \([1/2, 1]\));

(B) For any \( 0 \leq p \leq q \leq 1/2 \), or any \( 1/2 \leq q \leq p \leq 1 \), we have

\[
0 \leq L(q) - L(p) \leq L(|q - p|). \tag{23}
\]

(C) Suppose \( m \geq 2 \). For any \( 0 < v \leq m + 1 \), \( 0 < u \leq \min\{1, v\} \), \( \mathcal{L}(u,v) \leq \mathcal{L}(1, m + 1) \)

(D) For any \( x \geq 2 \)

\[
L\left(\frac{1}{2}\right) - L\left(\frac{1}{2} - \frac{1}{x}\right) \leq \frac{2}{x}. \tag{24}
\]

Proof A fact that we will use repeatedly hereafter is the fact that a concave function sits above all its chords. We first prove (A): if \( \mathcal{L} \) were decreasing somewhere on \([0, 1/2]\), there would be some \( a \geq b, a, b \in [0, 1/2] \) such that \( \mathcal{L}(a) > \mathcal{L}(b) \). Since \( \mathcal{L}(a) \leq \mathcal{L}(1/2), (a, \mathcal{L}(a)) \) sits below the chord \((b, \mathcal{L}(b)), (1/2, 1)\), which is impossible. The case \([1/2, 1]\) is obtained by symmetry.

We now prove (B). We prove it for the case \( 0 \leq p \leq q \leq 1/2 \), the other following from the
symmetry of $L$. Non-negativity follows from (A) and the fact that $L(0) = 0$. The right inequality follows from the concavity of $L$: indeed, since $L(0) = 0$, this inequality is equivalent to proving
\[ \frac{L(q) - L(p)}{q - p} \leq \frac{L(q - p) - L(0)}{q - p - 0}, \tag{25} \]
which since $p \geq 0$, is just stating that slopes of chords that intersect $L$ at points of constant difference between abscissae do not increase, i.e. $L$ is concave.

We prove (C). To get the result, we just need to write:
\[ \bar{L}(u, v) = v \cdot L \left( \frac{u}{v} \right) \]
\[ \leq (m + 1) \cdot L \left( \frac{1}{m + 1} \right) \]
\[ \leq (m + 1) \cdot L \left( \frac{1}{m + 1} \right), \tag{26} \]
\[ \leq \frac{1}{m + 1} \cdot L \left( \frac{x}{m + 1} \right), \tag{27} \]
where Ineq. (26) follows from Lemma 15 and $v \leq m$. Ineq. (27) follows from $u/m \leq 1/m \leq 1/2$. We finally prove (D). We have
\[ L(x) \geq 2x, \forall x \in [0, 1/2], \tag{28} \]

since $y = 2x$ is a chord for $L$ over $[0, 1/2]$ and $L$ is concave (it therefore sits over its chords).

We get for any $x \geq 2$,
\[ L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} - \frac{1}{2}x \right) = 1 - L \left( \frac{1}{2} - \frac{1}{2}x \right) \]
\[ \leq 1 - 2 \left( \frac{1}{2} - \frac{1}{2}x \right) \]
\[ = \frac{2}{x}, \]
as claimed. We obtain the statement of Lemma 16.

We now embark on the proof of Theorem 3. Let us fix for short
\[ \Delta = \left| f_{L}(h, \lambda, S) - f_{L}(h, \lambda, S') \right| \]
\[ = \left| w'(\lambda) \cdot L \left( \frac{w'(\lambda)}{w(\lambda)} \right) - w(\lambda) \cdot L \left( \frac{w(\lambda)}{w(\lambda)} \right) \right|, \tag{29} \]
and let us assume without loss of generality that samples contain at least two examples (otherwise $\Delta = 0$). The only eventual difference between $S$ and $S'$ that can make $\Delta > 0$ is on a weight and / or class change for the switched example. So, for some $\delta \leq 1$, we consider the following cases.

**Case A**: the total weight in leaf $\lambda$ changes vs it does not change
\[ A_1 = "w'(\lambda) = w(\lambda)" ; \quad A_2 = "w'(\lambda) = w(\lambda) + \delta". \tag{30} \]
Case B: the total weight for class 1 in leaf $\lambda$ changes vs it does not change

$$B_1 \equiv "w^1(\lambda) = w^1(\lambda) + \delta" \quad ; \quad B_2 \equiv "w^1(\lambda) = w^1(\lambda)".$$  

(31)

And we also consider different cases depending on the relationship between the weight of class 1 and the total weight in leaf $\lambda$ in $S$: $\exists u \in (w(\lambda)/2) \cdot [-1, 1]$ such that

$$w^1(\lambda) = \frac{w(\lambda)}{2} + u.$$  

(32)

We also suppose wlog that $\delta > 0$ (otherwise, we permute $S$ and $S'$, which does not change $\Delta$ because of $|.|$). We also remark that if we prove the result for $u \leq 0$, then because of the symmetry of $L$, we get the result for $u \geq 0$ as well – this just amounts to reasoning on negative examples instead of positive examples, changing notations but not the reasoning.

$\iff$ Case $A_1 \land B_1 \land (u \leq -\delta/2)$. Because of the constraint on $u$, either $w(\lambda) + \delta \leq w(\lambda)/2$ (if $u \leq -\delta$), or when $u \in (-\delta, -\delta/2]$, we have both $w(\lambda)/w(\lambda) \leq 1/2$, $(w(\lambda) + \delta)/w(\lambda) > 1/2$ and $(w(\lambda) + \delta)/w(\lambda) - (1/2) \leq (1/2) - w(\lambda)/w(\lambda)$. So,

$$L \left( \frac{w(\lambda) + \delta}{w(\lambda)} \right) \geq L \left( \frac{w(\lambda)}{w(\lambda)} \right),$$  

(33)

and therefore using $A_1, B_1$, we get

$$\Delta = w(\lambda) \cdot \left| L \left( \frac{w(\lambda) + \delta}{w(\lambda)} \right) - L \left( \frac{w(\lambda)}{w(\lambda)} \right) \right| = w(\lambda) \cdot \left( L \left( \frac{w(\lambda) + \delta}{w(\lambda)} \right) - L \left( \frac{w(\lambda)}{w(\lambda)} \right) \right).$$

We now have two sub-cases,

$\bullet$ If $(w(\lambda) + \delta)/w(\lambda) \leq 1/2$, then we directly get

$$\Delta \leq w(\lambda) \cdot L \left( \frac{\delta}{w(\lambda)} \right) = \bar{L}(\delta, w(\lambda)) \leq m \cdot L \left( \frac{1}{m} \right).$$  

(34)

(35)

(34) holds because of Lemma 16 (B). Ineq. (35) follows from Lemma 16 (C).

$\bullet$ If $(w(\lambda) + \delta)/w(\lambda) > 1/2$, then we know that since $w(\lambda)/w(\lambda) \leq 1/2$,

$$\frac{1}{2} - \frac{w(\lambda)}{w(\lambda)} \leq \frac{\delta}{w(\lambda)},$$  

(36)

and so

$$\Delta = w(\lambda) \cdot \left( L \left( \frac{w(\lambda) + \delta}{w(\lambda)} \right) - L \left( \frac{w(\lambda)}{w(\lambda)} \right) \right) \leq w(\lambda) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} - v \right) \right),$$

21
with therefore

\[ v = \frac{w(\lambda) - 2w^1(\lambda)}{2w(\lambda)} \leq \frac{\delta}{w(\lambda)}. \]  

(37)

We get

\[ \Delta \leq w(\lambda) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} - \frac{\delta}{w(\lambda)} \right) \right) \]

\[ \leq w(\lambda) \cdot \left( \frac{2\delta}{w(\lambda)} \right) \]

\[ = 2\delta \leq 2, \]  

(38)

because of Lemma 16 (D) \((x \doteq w(\lambda)/\delta \geq 2), \delta \leq 1, L(1/2) = 1\) and Lemma 15.

↪ Case \(A_1 \land B_1 \land (u \in (-\delta/2, \delta/2))\). We now obtain:

\[ \Delta = w(\lambda) \cdot \left( L \left( \frac{w^1(\lambda)}{w(\lambda)} \right) - L \left( \frac{w^1(\lambda) + \delta}{w(\lambda)} \right) \right) \]

\[ \leq w(\lambda) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{w^1(\lambda) + \delta}{w(\lambda)} \right) \right) \]

\[ = w(\lambda) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} + v \right) \right) \]

with

\[ v = \frac{2w^1(\lambda) + 2\delta - w(\lambda)}{2w(\lambda)}. \]  

(39)

we remark that \(v \geq 0\) because it is equivalent to

\[ w^1(\lambda) \geq \frac{w(\lambda)}{2} - \delta, \]  

(40)

which indeed holds because \(u > -\delta/2 \geq -\delta\) (we recall \(\delta > 0\)). We also obviously have \(v \geq 1/2\), so using the symmetry of \(L\) around \(1/2\), we get

\[ \Delta \leq w(\lambda) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} - v \right) \right) \]

\[ \leq w(\lambda) \cdot \left( \frac{2w^1(\lambda) + 2\delta - w(\lambda)}{w(\lambda)} \right) \]

\[ = 2w^1(\lambda) + 2\delta - w(\lambda) \]

\[ = 2(u + \delta) \leq 3\delta \leq 3. \]  

(41)

follows from Lemma 16 (D).

↪ Case \(A_1 \land B_1 \land (u \geq \delta/2)\). Since \(L\) is symmetric around \(1/2\), this boils down to case \((u \leq -\delta/2)\) with the negative examples.

↪ Case \(A_1 \land B_2\). In this case, \(\Delta = 0 \leq L(1, m)\).
\[ \Rightarrow \text{Case } A_2 \land B_1 \land \left( u \leq -\frac{\delta}{2}, \frac{w(\lambda)}{2w(\lambda)+\delta} \right). \] In this case, there is no class flip, just a change in weight. We first show that because of the constraint on \( u \),

\[
L \left( \frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \right) \geq L \left( \frac{w^1(\lambda)}{w(\lambda)} \right).
\] (43)

A sufficient condition for this to happen is \( (w^1(\lambda) + \delta)/(w(\lambda) + \delta) \leq 1/2 \), which, after reorganising yields

\[
w^1(\lambda) \leq \frac{w(\lambda)}{2} - \frac{\delta}{2},
\] (44)

and so is covered by the fact that \( u \leq -\delta/2 \), or, given the symmetry of \( L \), can also be achieved if the following conditions are met:

\[
\frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \geq \frac{1}{2},
\] (45)

\[
\frac{w^1(\lambda)}{w(\lambda)} \leq \frac{1}{2},
\] (46)

\[
\frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} - \frac{1}{2} \leq \frac{1}{2} - \frac{w^1(\lambda)}{w(\lambda)}.
\] (47)

To satisfy all these inequalities, we need, respectively, \( u \geq -\delta/2 \), \( u \leq 0 \) and

\[
u \leq -\frac{\delta}{2} \cdot \frac{w(\lambda)}{2w(\lambda) + \delta},
\] (48)

all of which are then implied if

\[
u \in -\frac{\delta}{2} \left( 1, \frac{w(\lambda)}{2w(\lambda) + \delta} \right),
\] (49)

which, together with the previous case results in the Case condition on \( u \). We therefore have:

\[
\Delta = (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \right) - w(\lambda) \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\] (50)

We now have two sub-cases:

- If

\[
\frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \leq \frac{1}{2},
\] (51)

then since we have as well \( w^1(\lambda)/w(\lambda) \leq (w^1(\lambda) + \delta)/(w(\lambda) + \delta) \) and \( L \) is non decreasing over \([0, 1/2]\), we get directly from Lemma 16 (B),

\[
\Delta \leq (w(\lambda) + \delta) \cdot L \left( \frac{\delta}{w(\lambda) + \delta} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\] (52)
We also remark that

\[
(w(\lambda) + \delta) \cdot L \left( \frac{\delta}{w(\lambda) + \delta} \right) \leq (m + 1) \cdot L \left( \frac{\delta}{m + 1} \right)
\]

\[
\leq (m + 1) \cdot L \left( \frac{1}{m + 1} \right),
\]

respectively because of Lemma 15 and \(1/(m + 1) \leq 1/2\), which is in the regime where \(L\) is non decreasing. We get from (52) that

\[
\Delta \leq \tilde{L}(1, m + 1) + 1. \tag{53}
\]

- If

\[
\frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} > \frac{1}{2}, \tag{54}
\]

then we can use (47) and get

\[
\Delta = (w(\lambda) + \delta) \cdot \left( L \left( \frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \right) - L \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq (w(\lambda) + \delta) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
= (w(\lambda) + \delta) \cdot \left( L \left( \frac{1}{2} \right) - L \left( \frac{1}{2} - v \right) \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right), \tag{55}
\]

with

\[
v = 1 - \frac{w^1(\lambda)}{w(\lambda)}. \tag{56}
\]

Remark that

\[
\frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} = \frac{w^1(\lambda)}{w(\lambda)} + \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)}. \tag{57}
\]

so to get both (54) and (46), we need

\[
\frac{w^1(\lambda)}{w(\lambda)} \geq 1 - \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)}, \tag{58}
\]

which implies

\[
v \leq \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)}, \tag{59}
\]

and so (55) and Lemma 16 (D) yields

\[
\Delta \leq (w(\lambda) + \delta) \cdot \left( \frac{2\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
= 2\delta \cdot \left( 1 - \frac{w^1(\lambda)}{w(\lambda)} \right) + \delta \leq 3\delta \leq 3. \tag{60}
\]
\( \leftrightarrow \text{Case } A_2 \wedge B_1 \wedge \left( u \in \left( -\frac{\delta}{2}, \frac{w(\lambda)}{2w(\lambda)+\delta}, \frac{\delta}{2} \cdot \frac{w(\lambda)}{2w(\lambda)+\delta} \right) \right) \). In this case, we have

\[
L \left( \frac{w^1(\lambda)+\delta}{w(\lambda)+\delta} \right) \leq L \left( \frac{w^1(\lambda)}{w(\lambda)} \right),
\]

and therefore, by virtue of the triangle inequality,

\[
\Delta = \left| (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda)+\delta}{w(\lambda)+\delta} \right) - w(\lambda) \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right|
\]

\[
= (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda)+\delta}{w(\lambda)+\delta} - \frac{w^1(\lambda)}{w(\lambda)} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq (w(\lambda) + \delta) \cdot L \left( \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right).
\]

We have two sub-cases.

- \( w^1(\lambda)/w(\lambda) \geq 1/2 \). In this case, we apply Lemma 16 (B) and get

\[
\Delta \leq (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda)+\delta}{w(\lambda)+\delta} - \frac{w^1(\lambda)}{w(\lambda)} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
= (w(\lambda) + \delta) \cdot L \left( \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right).
\]

Fixing \( u = \frac{\delta(w(\lambda)-w^1(\lambda))}{w(\lambda)} \leq \delta \) and \( v = w(\lambda) + \delta \), we remark that \( u \leq v \) and \( v \leq m + 1 \) so we can apply Lemma 16 (C) and get

\[
\Delta \leq L(1, m + 1) + \delta \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq L(1, m + 1) + \delta \leq L(1, m + 1) + 1.
\]

- \( w^1(\lambda)/w(\lambda) < 1/2 \). In this case, we remark that 16 implies \((w^1(\lambda)+\delta)/(w(\lambda)+\delta) > 1/2\), in which case since we still get 57, to get \( w^1(\lambda)/w(\lambda) < 1/2 \), we must have

\[
\frac{w^1(\lambda)+\delta}{w(\lambda)+\delta} \leq \frac{1}{2} + \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)},
\]

and combining this with the fact that (i) \( L \) is maximum in 1/2 and non increasing afterwards,
and symmetric around 1/2,

\[
\Delta \leq (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{1}{2} \right) - \mathcal{L} \left( \frac{w^1(\lambda) + \delta}{w(\lambda) + \delta} \right) \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{1}{2} \right) - \mathcal{L} \left( \frac{1}{2} + \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
= (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{1}{2} \right) - \mathcal{L} \left( \frac{1}{2} - \frac{\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq (w(\lambda) + \delta) \cdot \left( \frac{2\delta(w(\lambda) - w^1(\lambda))}{w(\lambda)(w(\lambda) + \delta)} \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
= 2\delta \cdot \left( 1 - \frac{w^1(\lambda)}{w(\lambda)} \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq 3\delta \leq 3. \quad (65)
\]

We have used Lemma 16(D) in (65).

\( \iff \) Case \( A_2 \land B_1 \land \left( u \geq \frac{\delta}{2} \cdot \frac{w^1(\lambda)}{2w(\lambda) + \delta} \right) \). Since \( \mathcal{L} \) is symmetric around 1/2, this boils down to case \( u \leq -\frac{\delta}{2} \cdot \frac{w^1(\lambda)}{2w(\lambda) + \delta} \) with the negative examples.

\( \iff \) Case \( A_2 \land B_2 \land \left( u \leq \frac{\delta}{2} \cdot \frac{2w^1(\lambda)}{2w(\lambda) + \delta} \right) \). This time, we can immediately write, independently from the condition on \( u \),

\[
\Delta = \left| (w(\lambda) + \delta) \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) - w(\lambda) \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right|
\]

\[
= \left| (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) - \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right|
\]

\[
\leq \left| (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) - \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) \right| + \delta \cdot \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right)
\]

\[
\leq \left| (w(\lambda) + \delta) \cdot \left( \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) - \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) \right| + \delta. \quad (67)
\]

We first examine the condition under which

\[
\mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) \leq \mathcal{L} \left( \frac{w^1(\lambda)}{w(\lambda)} \right). \quad (68)
\]

Again, \( u \leq 0 \) is a sufficient condition. Otherwise, if therefore \( w^1(\lambda)/w(\lambda) \geq 1/2 \), then we need

\[
\frac{w^1(\lambda)}{w(\lambda) + \delta} < \frac{1}{2},
\]

\[
\frac{1}{2} - \frac{w^1(\lambda)}{w(\lambda) + \delta} \geq \frac{w^1(\lambda)}{w(\lambda)} - \frac{1}{2}; \quad (69)
\]

the latter constraint is equivalent to

\[
\frac{w^1(\lambda)}{w(\lambda)} \leq \frac{w(\lambda) + \delta}{2w(\lambda) + \delta}, \quad (70)
\]

26
and therefore
\[ \frac{u}{w(\lambda)} - \frac{1}{2} \leq \frac{w(\lambda) + \delta}{2w(\lambda) + \delta} - \frac{1}{2} = \frac{\delta}{2w(\lambda) + \delta}, \tag{71} \]

which leads to our constraint on \( u \) and gives
\[ \Delta \leq (w(\lambda) + \delta) \cdot \left( L\left(\frac{w^1(\lambda)}{w(\lambda)}\right) - L\left(\frac{w^1(\lambda)}{w(\lambda) + \delta}\right)\right) + \delta. \tag{72} \]

We have two sub-cases.
- \( w^1(\lambda)/w(\lambda) \leq 1/2 \). In this case, we get directly from Lemma 16 (B),
\[ \Delta \leq (w(\lambda) + \delta) \cdot L\left(\frac{w^1(\lambda)}{w(\lambda) + \delta}\right) + \delta \leq \tilde{L}(1, m+1) + 1, \tag{73} \]
where we have used Lemma 16 (C) with \( u = w^1(\lambda)\delta/w(\lambda) \leq 1 \) and \( v = w(\lambda) + \delta \leq m + 1 \). We also check that \( u \leq \delta \leq v \).
- \( w^1(\lambda)/w(\lambda) \geq 1/2 \). In this case, we remark that
\[ \frac{w^1(\lambda)}{w(\lambda)} = \frac{w^1(\lambda)}{w(\lambda) + \delta} + \frac{w^1(\lambda)\delta}{w(\lambda)(w(\lambda) + \delta)}, \tag{74} \]
and since we need \( w^1(\lambda)/(w(\lambda) + \delta) \leq 1/2 \) (otherwise, (68) cannot hold), then it implies
\[ \frac{w^1(\lambda)}{w(\lambda) + \delta} \geq \frac{1}{2} - \frac{w^1(\lambda)\delta}{w(\lambda)(w(\lambda) + \delta)}, \tag{75} \]
and so the fact that \( L \) is non-decreasing before 1/2 and Lemma 16 (D) yield
\[ \Delta \leq (w(\lambda) + \delta) \cdot L\left(\frac{1}{2}\right) - L\left(\frac{w^1(\lambda)}{w(\lambda) + \delta}\right) + \delta \leq (w(\lambda) + \delta) \cdot L\left(\frac{1}{2}\right) - L\left(\frac{1}{2} - \frac{w^1(\lambda)\delta}{w(\lambda)(w(\lambda) + \delta)}\right) + \delta \leq (w(\lambda) + \delta) \cdot \frac{2w^1(\lambda)\delta}{w(\lambda)(w(\lambda) + \delta)} + \delta = \frac{2w^1(\lambda)\delta}{w(\lambda)} + \delta \leq 3\delta \leq 3. \tag{76} \]

To complete the proof of the Case, suppose now that
\[ L\left(\frac{w^1(\lambda)}{w(\lambda) + \delta}\right) \geq L\left(\frac{w^1(\lambda)}{w(\lambda)}\right), \tag{77} \]
which therefore imposes
\[
\frac{w^1(\lambda)}{w(\lambda)} \geq \frac{w^1(\lambda)}{w(\lambda) + \delta} \geq \frac{1}{2},
\]  
(78)
so using Lemma 16 (B) yields
\[
\Delta \leq (w(\lambda) + \delta) \cdot \left( L \left( \frac{w^1(\lambda)}{w(\lambda) + \delta} - L \left( \frac{w^1(\lambda)}{w(\lambda)} \right) \right) \right) + \delta
\leq (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda)}{w(\lambda)} - \frac{w^1(\lambda)}{w(\lambda) + \delta} \right) + \delta
= (w(\lambda) + \delta) \cdot L \left( \frac{w^1(\lambda) \delta}{w(\lambda)(w(\lambda) + \delta)} \right) + \delta
\leq \tilde{L}(1, \delta + 1) + 1, \tag{79}
\]
where we have used Lemma 16 (C) with \( u = \frac{w^1(\lambda) \delta}{w(\lambda)} \leq 1 \) and \( v = \frac{w(\lambda) + \delta}{m + 1} \).

We also check that \( u \leq \delta \leq v \).

\( \leftarrow \) Case \( A_2 \land B_2 \land \left( u > \frac{\delta}{2} \cdot \frac{2w(\lambda)}{2w(\lambda) + \delta} \right) \). Since \( L \) is symmetric around \( 1/2 \), this boils down to case \( u \leq \frac{\delta}{2} \cdot \frac{2w(\lambda)}{2w(\lambda) + \delta} \) with the negative examples.

We can now finish the upperbound on \( \Delta \) by taking all bounds in (35), (38), (42), (53), (60), (63), (66), (73), (76) and (79):
\[
\Delta \leq \max \{ \tilde{L}(1, m) \cdot 2, 3, 1 + \tilde{L}(1, m + 1) \} = \max \{ 3, 1 + \tilde{L}(1, m + 1) \}, \tag{80}
\]
as claimed, using Lemma 16 (C).

**Remark:** We can prove that \( \Delta = \tilde{L}(1, m) \) can be realized: consider set \( S \) with \( m \) examples with unit weight, 1 of which each is from the positive class class. In \( S' \), we flip this class. We get:
\[
\Delta = m \cdot \tilde{L}\left( \frac{1}{m} \right) - m \cdot \tilde{L}\left( \frac{0}{m} \right) = m \cdot \tilde{L}\left( \frac{1}{m} \right) = \tilde{L}(1, m), \tag{81}
\]
as claimed (since \( L(0) = 0 \)).

11 Proof of Lemma 4

We perform a Taylor expansion of \( L \) up to second order and obtain:
\[
L(0) = L\left( \frac{1}{x} \right) + \left( 0 - \frac{1}{x} \right) \cdot L'\left( \frac{1}{x} \right) \]
\[
+ \left( 0 - \frac{1}{x} \right)^2 \cdot L''(a),
\]
for some $a \in [0, x]$. There remains to see that $J = \hat{L}'(1, x)$ (eq. 22 in the Appendix), fix $x = m + 1$ and reorder given $L(0) = 0$.

### 12 Proof of Lemma 5

We have

$$L^{\text{Mat}}(1, m + 1) = (m + 1) \cdot 2\sqrt{\frac{1}{m + 1} \cdot \frac{m}{m + 1}} = 2\sqrt{m}, \tag{82}$$

as claimed.

We have (we make the distinction $\log$ base-2 and $\ln$ base-$e$)

$$L^{\text{log}}(1, m + 1) = (m + 1) \cdot \left(-\frac{1}{m + 1}\log \frac{1}{m + 1} - \frac{m}{m + 1}\log \frac{m}{m + 1}\right)$$

$$= \log(m + 1) + m\log \frac{m + 1}{m}$$

$$\leq \log(m + 1) + \frac{1}{\ln 2}. \tag{83}$$

The last inequality follows from Friedman & Schuster (2010, Claim 1).

We have

$$L^{\text{sq}}(1, m + 1) = (m + 1) \cdot \frac{4}{m + 1} \cdot \frac{m}{m + 1}$$

$$= \frac{4m}{m + 1}, \tag{84}$$

as claimed.

Finally, we have

$$L^{0/1}(1, m + 1) = (m + 1) \cdot 2\min\left\{\frac{1}{m + 1}, \frac{m}{m + 1}\right\}$$

$$= 2, \tag{85}$$

as claimed.

### 13 Proof of Theorem 8

That Matsushita’s $\alpha$-loss is symmetric is a direct consequence of its definition. It is proper because it is a convex combination of two proper losses, Matsushita loss and the 0/1-loss

---

4In the main body, log is base-$e$ by default.
We consider the topmost condition in (90). Reorganising, we want

\[ L^{(\alpha)}(u) = 2 \cdot (\alpha \cdot \sqrt{u(1-u)} + (1 - \alpha) \cdot \min\{u, 1-u\}). \]  

We get the canonical link in the subdifferential of negative the pointwise Bayes risk:

\[ \psi^{(\alpha)}(u) = -\partial L^{(\alpha)}(u) = \alpha \cdot \frac{2u - 1}{\sqrt{u(1-u)}} - 2(1 - \alpha) \cdot \left\{ \begin{array}{ll} 1 & \text{if } u < 1/2 \\ [-1, 1] & \text{if } u = 1/2 \\ -1 & \text{if } u > 1/2 \end{array} \right., \]  

and we immediately get the weight function from the fact that \( w^{(\alpha)} = -L^{(\alpha)''} \) [Reid & Williamson (2010) Theorem 6]. We get the corresponding convex surrogate of the proper loss by taking the convex conjugate of negative the pointwise Bayes risk:

\[ \lambda_{\alpha}(z) = \sup_{u \in [0,1]} \{ zu + 2 \cdot (\alpha \cdot \sqrt{u(1-u)} + (1 - \alpha) \cdot \min\{u, 1-u\}) \}. \]  

We remark that if \( z < 0 \) then the sup is going to be attained for \( u \) closer to 0 than 1 (thus \( u \leq 1/2 \)), and if \( z > 0 \), it is the opposite: the sup is going to be attained for \( u \) closer to 1 than to 0 (thus \( u \geq 1/2 \)). If \( z = 0 \), the sup is trivially going to hold for \( u = 1/2 \) (that is, \( \lambda_{\alpha}(0) = 1/2 \)).

**Case 1:** \( \alpha = 0 \) – when \( z < -2 \) (resp. \( z > 2 \)), the sup is attained for \( u = 0 \) (resp. \( u = 1 \)). Otherwise, the sup is attained for \( u = 1/2 \). Hence

\[ \lambda_{\alpha}(z) = \left\{ \begin{array}{ll} 0 & \text{if } z < -2 \\ 1 + \frac{z}{2} & \text{if } z \in 2 \cdot [-1, 1] \\ z & \text{if } z > 2 \end{array} \right.. \]  

**Case 2:** \( \alpha \neq 0 \) – Let us find the values of \( z \) for which the argument \( u = 1/2 \) in (88), that is we want to find \( z \) such that

\[ \left\{ \begin{array}{l} zu + 2\alpha \sqrt{u(1-u)} + 2(1 - \alpha)u \leq 1 + \frac{z}{2}, \forall u \in [0, 1/2] \\ zu + 2\alpha \sqrt{u(1-u)} + 2(1 - \alpha)(1-u) \leq 1 + \frac{z}{2}, \forall u \in [1/2, 1] \end{array} \right.. \]  

We consider the topmost condition in (90). Reorganising, we want \( 2\alpha \sqrt{u(1-u)} \leq 1 + (z/2) - (z + 2(1 - \alpha))u \) for \( u \in [0, 1/2] \). Fix \( z = -2(1 - \alpha) + \delta \), which gives the condition

\[ 2\sqrt{u(1-u)} \leq 1 + \frac{\delta}{\alpha} \cdot (1-u), \forall u \in [0, 1/2]. \]  

This condition obviously holds when \( \delta \geq 0 \), and it is in fact violated when \( \delta < 0 \) because the LHS can be made as close as desired to 1. So the topmost condition holds for \( z \geq -2(1 - \alpha) \). Regarding the bottommost condition, we now want \( 2\alpha \sqrt{u(1-u)} \leq 1 - 2(1 - \alpha) + (z/2) - (z - 2(1 - \alpha))u \) for \( u \in [1/2, 1] \), which, after letting \( z = 2(1 - \alpha) + \delta \), gives equivalently

\[ 2\sqrt{u(1-u)} \leq 1 - \frac{\delta}{\alpha} \cdot \left( u - \frac{1}{2} \right), \forall u \in [1/2, 1]. \]
While the condition trivially holds when $\delta \leq 0$, it is in fact violated when $\delta > 0$ because the LHS can be made as close as desired to 1. To summarize, the trivial argument $u = 1/2$ giving us (88) is obtained when $z \in [-2(1-\alpha), +\infty) \cap (-\infty, 2(1-\alpha)] = 2(1-\alpha) \cdot [-1, 1]$, and we get

$$\lambda_\alpha(z) = 1 + \frac{z}{2} \text{ if } z \in 2(1-\alpha) \cdot [-1, 1],$$

(93)

which, we also remark, gives the mid condition in (89) when $\alpha \to 0$.

Now, when $z \not\in 2(1-\alpha) \cdot [-1, 1]$, we can differentiate (88) to find the argument $u$ realising the max. Let

$$h_-(u) = (z + 2(1-\alpha)) \cdot u + 2\alpha \cdot \sqrt{u(1-u)}$$

$$= \alpha \cdot \left[ Z_- u + 2\sqrt{u(1-u)} \right],$$

$$h_+(u) = 2(1-\alpha) + (z - 2(1-\alpha)) \cdot u + 2\alpha \cdot \sqrt{u(1-u)}$$

$$= 2(1-\alpha) + \alpha \cdot \left[ Z_+ u + 2\sqrt{u(1-u)} \right],$$

(94)

with $Z_- = (z + 2(1-\alpha))/\alpha$, $Z_+ = (z - 2(1-\alpha))/\alpha$. We compute $\max_{[0,1/2]} h_-(u)$ and $\max_{[1/2,1]} h_+(u)$, granted that the max of the two will give us the convex conjugate. Let us focus on $h_-(u)$. The argument $u$ we seek satisfies, after derivating $g_-(u)$,

$$Z_- + \frac{1-2u}{\sqrt{u(1-u)}} = 0,$$

(95)

i.e. $1-2u = -Z_- \sqrt{u(1-u)}$, or $1 - (4 + Z_-^2)u + (4 + Z_-^2)u^2 = 0$, which brings the solution $u^*(z)$,

$$u^*(z) = \frac{4 + Z_-^2 \pm |Z_-| \sqrt{4 + Z_-^2}}{2(4 + Z_-^2)} = \frac{1}{2} \pm \frac{|Z_-|}{2\sqrt{4 + Z_-^2}} = \frac{1}{2} - \frac{|Z_-|}{2\sqrt{4 + Z_-^2}},$$

(96)
because we maximize \( g_- \) in \([0, 1/2]\). We get:

\[
\begin{align*}
    h_-(u^*(z)) &= \frac{\alpha Z_-}{2} - \frac{\alpha|Z_-|Z_-}{2\sqrt{4 + Z_-^2}} + 2\alpha \sqrt{\frac{1}{4} - \frac{Z_-^2}{4(4 + Z_-^2)}} \\
    &= \frac{\alpha Z_-}{2} - \frac{\alpha|Z_-|Z_-}{2\sqrt{4 + Z_-^2}} + \alpha \sqrt{1 - \frac{Z_-^2}{4 + Z_-^2}} \\
    &= \frac{\alpha Z_-}{2} - \frac{\alpha|Z_-|Z_-}{2\sqrt{4 + Z_-^2}} + \frac{2\alpha}{\sqrt{4 + Z_-^2}} \\
    &= \alpha \cdot \left( \frac{Z_-}{2} + \frac{4 - |Z_-|Z_-}{2\sqrt{4 + Z_-^2}} \right) \\
    &= \frac{z + 2(1 - \alpha)}{2} + \frac{4\alpha^2 - |z + 2(1 - \alpha)|(z + 2(1 - \alpha))}{2\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}} \\
    &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 - |z + 2(1 - \alpha)|(z + 2(1 - \alpha))}{2\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}} \\
    &= h_+(z). \tag{97}
\end{align*}
\]

We now focus on \( h_+(u) \). It is straightforward to check that \((95)\) still holds but with \( Z_+ \) replacing \( Z_- \) and

\[
u^*(z) = \frac{1}{2} + \frac{|Z_+|}{2\sqrt{4 + Z_+^2}} \geq 1/2, \tag{98}
\]

leading to

\[
\begin{align*}
    h_+(u^*(z)) &= 2(1 - \alpha) + \frac{\alpha Z_+}{2} + \frac{\alpha|Z_+|Z_+}{2\sqrt{4 + Z_+^2}} + 2\alpha \sqrt{\frac{1}{4} - \frac{Z_+^2}{4(4 + Z_+^2)}} \\
    &= 2(1 - \alpha) + \frac{z - 2(1 - \alpha)}{2} + \frac{4\alpha^2 - |z - 2(1 - \alpha)|(z - 2(1 - \alpha))}{2\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}} \\
    &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 - |z - 2(1 - \alpha)|(z - 2(1 - \alpha))}{2\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}} \\
    &= h_+(z). \tag{99}
\end{align*}
\]

To finish up, we need to compute \( \lambda_\alpha(z) = \max\{h_-(z), h_+(z)\} \) for \( z \notin 2(1 - \alpha) \cdot [-1, 1] \).

**Case 2.1:** \( z < -2(1 - \alpha) \) — In this case,

\[
\begin{align*}
    h_-(z) &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 + (z + 2(1 - \alpha))^2}{2\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}}, \\
    &= 1 - \alpha + \frac{z}{2} + \frac{\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}}{2}; \\
    h_+(z) &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 - (z - 2(1 - \alpha))^2}{2\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}}, \tag{100}
\end{align*}
\]

and it is easy to check that \( h_-(z) > h_+(z) \).
Case 2.1: \( z > 2(1 - \alpha) \) — In this case,
\[
\begin{align*}
  h_\star^-(z) &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 - (z + 2(1 - \alpha))^2}{2\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}}, \\
  h_\star^+(z) &= 1 - \alpha + \frac{z}{2} + \frac{4\alpha^2 + (z - 2(1 - \alpha))^2}{2\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}}, \\
  &= 1 - \alpha + \frac{z}{2} + \frac{\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}}{2}.
\end{align*}
\]
(101)

and it is easy to check that \( h_\star^+(z) > h_\star^-(z) \).

To summarize \textbf{Case 2}, we get the convex conjugate and surrogate loss for Matsushita \( \alpha \)-entropy:
\[
\lambda_\alpha(z) = \begin{cases} 
1 - \alpha + \frac{z}{2} + \frac{\sqrt{4\alpha^2 + (z + 2(1 - \alpha))^2}}{2} & \text{if } z < -2(1 - \alpha) \\
1 + \frac{z}{2} & \text{if } z \in 2(1 - \alpha) \cdot [-1, 1] \\
1 - \alpha + \frac{z}{2} + \frac{\sqrt{4\alpha^2 + (z - 2(1 - \alpha))^2}}{2} & \text{if } z > 2(1 - \alpha)
\end{cases}
\]
(102)

which can be further simplified to
\[
\lambda_\alpha(z) = 1 + \frac{z}{2} + \|z\| \notin 2(1 - \alpha) \cdot [-1, 1] \cdot \left( \sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1 - \alpha) \right)^2} - \alpha \right),
\]
(103)

and the convex surrogate is just by definition
\[
F^{(\alpha)}(z) = \lambda_\alpha(-z),
\]
(104)
as claimed. We also get the inverse canonical link by differentiating \( \lambda_\alpha \), giving
\[
\psi^{\alpha^{-1}}(z) = \lambda_\alpha'(z)
\]
\[
= \frac{1}{2} \cdot \left( 1 + \|z\| \notin 2(1 - \alpha) \cdot [-1, 1] \cdot \text{sign}(z) \cdot \frac{|z|}{2} - (1 - \alpha) \right) \cdot \sqrt{\alpha^2 + \left( \frac{|z|}{2} - (1 - \alpha) \right)^2}
\]
(105)

This achieves the proof of Theorem 8.

14 Proof of Theorem 10

The proof proceeds in two steps. First we give some notations and explain why our WLA in Definition 9 is equivalent to Kearns & Mansour (1996, Section 3). We then proceed to the proof itself.
Figure 3: Notations used in our proof of Theorem 10: leaf $\lambda$ in tree $h$ is replaced by subtree indexed by binary subtree with root test $g : \mathbb{R} \rightarrow \{0, 1\}$ and two new leaves $\lambda_0$ and $\lambda_1$ in grown tree $h \oplus (g, \lambda)$. The total proportion of examples reaching $\lambda$ (and therefore subject to test $g$) is $w$; the relative proportion of those for which $g(.) = 0$ (resp. $g(.) = 1$) is $1 - \tau$ (resp. $\tau$). The relative proportion of positive examples in $\lambda$ (resp. $\lambda_0$; resp. $\lambda_1$) is $q$ (resp $p$; resp. $r$).

▷ Notations and the Weak Learning Assumption: recall that our objective is to minimise

$$L^{(\alpha)}(h) = \sum_{\lambda \in \Lambda} w(\lambda) L^{(\alpha)}(q(\lambda)),$$  \hspace{1cm} (106)

where $h$ is a tree and $\Lambda$ is its set of leaves. Note also that $\sum_{\lambda} w(\lambda) = w(S)$, which is not normalized. Even when un-normalizing makes no difference, we are going to stick to Kearns & Mansour (1996)'s setting and assume that our loss in (106) is normalized (thus divided by $w(S)$). We shall remove this assumption at the end of the proof.

We have alleviated the boosting iteration index in $w$, so that $w(\lambda) = \sum_i w_i \cdot [i \in \lambda]$. $q(\lambda) \in [0, 1]$ is the relative proportion of positive examples reaching leaf $\lambda$,

$$q(\lambda) = (1/w(\lambda)) \cdot \sum_i [(i \in \lambda) \land (y_i = +1)] \cdot w_i.$$  \hspace{1cm} (107)

It should be clear at this stage that because we spend part of our DP budget each time we learn a split in a tree, we need to minimise (106) as fast as possible under the weakest possible assumptions. Boosting gives us a very convenient framework to do so. Notations used are now simplified as summarized in Figure 3, so that for example $q = q(\lambda)$.

We first review the weak learning assumption (WLA) for decision trees as carried out in Kearns & Mansour (1996), which imposes a weak correlation between split $g$ and the labels.
of the examples reaching \( \lambda \) for the split to meet the WLA. This correlation is measured not with respect to the current weights \( w \) but to a distribution restricted to leaf \( \lambda \) and giving equal weight to positive and negative examples: let

\[
\begin{align*}
  w_{\lambda,i} & = w_i \cdot \left\{ \begin{array}{ll}
  0 & \text{if } i \not\in \lambda \\
  \frac{1}{2q} & \text{if } (i \in \lambda) \land (y_i = +1) \\
  \frac{1}{2(1-q)} & \text{if } (i \in \lambda) \land (y_i = -1)
\end{array} \right. 
\end{align*}
\]

(108)

**Definition 17** (Weak learning assumption, Kearns & Mansour (1996)) Fix \( \gamma > 0 \). Split \( g \) at leaf \( \lambda \) satisfies the \( \gamma \)-weak learning assumption (WLA for short, omitting \( \gamma \)) iff

\[
\left| \sum_i w_{\lambda,i} \cdot \left[ ((g(x_i) = 0) \land (y_i = +1)) \lor ((g(x_i) = 1) \land (y_i = -1)) \right] \right| - \frac{1}{2} \geq \gamma.
\]

(109)

It is not hard to check that, provided the splits are closed under negation (that is, if \( g \) is a potential split then so is \( \neg g \)), then Definition [17] is equivalent to the weak hypothesis assumption of Kearns & Mansour (1996, Lemma 2). To better see the correlation, define \( g^{+/-} = -1 + 2g \in \{-1,1\} \). Then it is not hard to check that

\[
\sum_i w_{\lambda,i} \cdot \left[ ((g(x_i) = 0) \land (y_i = +1)) \lor ((g(x_i) = 1) \land (y_i = -1)) \right] 
= \frac{1}{2} \cdot \sum_i w_{\lambda,i} \cdot (1 - y_i g^{+/-}(x_i)) 
= \frac{1}{2} \left( 1 - \sum_i w_{\lambda,i} \cdot y_i g^{+/-}(x_i) \right),
\]

so the WLA is equivalent to \( |\sum_i w_{\lambda,i} \cdot y_i g^{+/-}(x_i)| \geq 2\gamma \), that is, using the edge notation \( \eta(w,h) = \sum_i w_i y_i h(x_i) \) with \( h : X \to \mathbb{R} \) and \( w \) defines a discrete distribution over the training sample \( S \), we can reformulate the weak learning assumption as: split \( g \) at leaf \( \lambda \) satisfies the \( \gamma \)-WLA if \( |\eta(w_\lambda,g^{+/-})| \geq \gamma \), which is Definition [0] and is therefore equivalent to Definition [17] up to a factor 2 in the weak learning guarantee.

\[ \triangleright \text{Proof of the Theorem:} \] we now embark on the proof of Theorem [10]. The proof follows the same schema as Kearns & Mansour (1996) with some additional details to handle the change of \( \alpha \) in the course of training a DT. We first summarize the high-level details of the proof. Denote \( h \oplus (g,\lambda) \) tree \( h \) in which a leaf \( \lambda \) has been replaced by a split indexed with some \( g : \mathbb{R} \to \{0,1\} \) satisfying the weak learning assumption (Figure [3]). The decrease in \( L(.) \), \( \Delta = L(h) - L(h \oplus (g,\lambda)) \), is lowerbounded as a function of \( \gamma \) and then used to lowerbound the number of iterations (each of which is the replacement of a leaf by a binary subtree) to get to a given value of \( L(.) \). It follows that \( \Delta = \omega(\lambda) \cdot \Delta_{L^{(\alpha)}(q,\tau,\delta)} \), with

\[
\Delta_{L^{(\alpha)}(q,\tau,\delta)} = L^{(\alpha)}(q) - (1 - \tau)L^{(\alpha)}(q - \tau\delta) - \tau L^{(\alpha)}(q + (1 - \tau)\delta)
\]

(110)

with \( \delta = \gamma q(1-q)/(\tau(1-\tau)) \) with \( \tau \) denoting the relative proportion of examples for which \( g = +1 \) in leaf \( \lambda \), following Kearns & Mansour (1996). We thus have

\[
\tau \approx \frac{\sum_i w_i \cdot [i \in \lambda] \land (g(x_i) = 1)}{\sum_i w_i \cdot [i \in \lambda]}.
\]

(111)
Figure 4: Sequence of key parameters for the induction of a DT, which leads to tree $h_{t+1}$ after having split leaf $\lambda_t$ in $h_t$. $\alpha_t$ is the parameter chosen for the $M\alpha$-loss.

We also introduce normalized weights with notation $\tilde{w}_i = w_i / w(S)$, so the total normalized weight of examples reaching leaf $\lambda_t$ can also be denoted with the tilda: $\tilde{w}(\lambda) = \sum_i \tilde{w}_i \cdot [i \in \lambda]$. We now let $h_\ell$ denote the current DT with $\ell$ leaves and $\ell - 1$ internal nodes, the first tree being thus the single root leaf $h_1$. We obtain $h_{\ell+1}$ by splitting a leaf $\lambda_\ell \in \Lambda(h_\ell)$, chosen to minimize

$$L^{(\alpha_\ell)}(h_{\ell+1}) = \alpha_\ell \cdot L^\text{Mat}(h_{\ell+1}) + (1 - \alpha_\ell) \cdot L^\text{err}(h_{\ell+1})$$

over all possible leaf splits in $\Lambda(h_\ell)$. Figure 4 summarizes the whole process of getting $h_{\ell+1}$ from $h_\ell$.

Lemma 18 Suppose the sequence of $\alpha_\ell$ satisfies:

$$\alpha_\ell \leq \alpha_{\ell-1} \cdot \exp \left( \frac{\gamma^2 \tilde{w}_\ell}{16} \cdot (1 - \alpha_{\ell-1}) \right), \forall \ell > 0,$$

with $\tilde{w}_\ell$ the total normalized weight of examples reaching leaf $\lambda_\ell$ split at iteration $\ell$. Then for any $\xi \in (0, 1]$, the empirical risk of $h_L$ satisfies $\varepsilon_{o/1}(h_L) \leq \xi$ as long as

$$\sum_{\ell=1}^L \tilde{w}_\ell \alpha_\ell \geq \frac{16}{\gamma^2} \cdot \log \frac{1}{\xi}.$$  

Proof We first need a technical Lemma, in which we replace $\alpha_\ell$ by $\alpha$ for the sake of readability.

Lemma 19 (Equivalent of Kearns & Mansour (1996, Lemma 13) for $\Delta_L^{(\alpha)}$) Fix $\alpha \in [0, 1]$. If $\gamma < 0.2$ and $q$ is sufficiently small, then $\Delta_L^{(\alpha)}$ is minimized by $\tau \in [0.4, 0.6]$.

Proof We have

$$\Delta_L^{(\alpha)}(q, \tau, \delta) = \alpha \cdot \Delta_L^\text{Mat}(q, \tau, \delta) + (1 - \alpha) \cdot \Delta_L^\text{err}(q, \tau, \delta).$$

Suppose without loss of generality that $p \leq q \leq r$. It follows that if $r \leq 1/2$ or $p \geq 1/2$, $\Delta_L^\text{err}(q, \tau, \delta) = 0$ so we get the result directly from Kearns & Mansour (1996, Lemma 13). Otherwise, we have two cases.
Case 1: \( q \leq 1/2, r > 1/2 \). In this case,
\[
\Delta_{L}^{err}(q, \tau, \delta) = 2q - 2(1 - \tau)(q - \tau \delta) - 2\tau(1 - (q + (1 - \tau)\delta))
= 2\tau \cdot (2q + 2(1 - \tau)\delta - 1)
= 2\tau \cdot \left( 2q + \frac{2\gamma q(1 - q)}{\tau} - 1 \right)
= 2\tau (2q - 1) + 4\gamma q(1 - q),
\]
under the additional condition (for \( r > 1/2 \))
\[
\tau < \frac{2\gamma q(1 - q)}{1 - 2q} \sim 0.4\gamma q.
\]
We get \( \partial \Delta_{L}^{err}(q, \tau, \delta)/\partial \tau = 2(2q - 1) \) and so
\[
\frac{\partial \Delta_{L}^{(\alpha)}(q, \tau, \delta)}{\partial \tau} = \alpha \cdot \frac{\partial \Delta_{L}^{Mat}(q, \tau, \delta)}{\partial \tau} + 2(1 - \alpha)(2q - 1)
\leq \alpha \cdot \frac{\partial \Delta_{L}^{Mat}(q, \tau, \delta)}{\partial \tau}
\]
(115)
since \( q \leq 1/2 \), and it comes from Kearns & Mansour (1996) that \( \partial \Delta_{L}^{(\alpha)}(q, \tau, \delta)/\partial \tau \leq 0 \) for \( \tau \leq 0.4 \), and under the condition of their Lemma (\( q \) is sufficiently small, \( \gamma < 0.2 \)), then (115) precludes \( \tau \geq 0.6 \) on Case 1.

Case 2: \( q \geq 1/2, p < 1/2 \). In this case, we remark that \( \Delta_{L}^{(\alpha)} \) is invariant to the change \( p \mapsto 1 - p, q \mapsto 1 - q, r \mapsto 1 - r \), which brings us back to Case 1.

The following Lemma brings the key brick to the proof of Lemma 18.

Lemma 20 Using notations of Figure 3, suppose the split put at left \( \lambda_{\ell} \) in \( h_{\ell} \) satisfies the \( \gamma \)-Weak Learning Assumption and furthermore the sequence of \( \alpha \)s satisfies (112). Then we have
\[
L^{(\alpha_{\ell})}(h_{\ell+1}) \leq \left( 1 - \frac{\gamma^{2} \bar{w}_{\ell}\alpha_{\ell}}{16} \right) \cdot L^{(\alpha_{\ell-1})}(h_{\ell}).
\]
(117)

Remark: the key result for Matsushita’s loss in Kearns & Mansour (1996, Theorem 10) follows from the particular case of Lemma 20 for \( \alpha_{\ell} = 1, \forall \ell \) (for which condition (112) obviously holds for any \( \gamma \) and \( \bar{w}_{\ell} \)).

Proof We use the notations of Figures 3 and 4. As long as the split satisfies the \( \gamma \)-Weak Learning Assumption, we get from the proof of Kearns & Mansour (1996, Theorem 10)
\[
L^{Mat}(h_{\ell+1}) \leq \left( 1 - \frac{\gamma^{2} \bar{w}_{\ell}}{16} \right) \cdot L^{Mat}(h_{\ell}),
\]
(118)
further noting that the use of Lemma 19 is "hidden" in this bound, but proceeds as in the proof of Kearns & Mansour (1996, Theorem 10). We remind that if we tune \( \alpha \) then by definition
\[
L^{(\alpha_{\ell})}(h_{\ell+1}) = \alpha_{\ell} \cdot L^{Mat}(h_{\ell+1}) + (1 - \alpha_{\ell}) \cdot L^{err}(h_{\ell+1}),
\]
\[
L^{(\alpha_{\ell-1})}(h_{\ell}) = \alpha_{\ell-1} \cdot L^{Mat}(h_{\ell}) + (1 - \alpha_{\ell-1}) \cdot L^{err}(h_{\ell}).
\]
Now we have, successively because of (118) and $L^{\text{err}}(h_{\ell+1}) \leq L^{\text{err}}(h_{\ell})$ (error cannot increase as the partition of $X$ achieved by $h_{\ell+1}$ is finer than that of $h_{\ell}$),

\[
L^{(\alpha_{\ell})}(h_{\ell+1}) \leq \alpha_{\ell} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + (1 - \alpha_{\ell}) \cdot L^{\text{err}}(h_{\ell+1}) \\
\leq \alpha_{\ell} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + (1 - \alpha_{\ell}) \cdot L^{\text{err}}(h_{\ell}) \\
= \alpha_{\ell} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + Q \cdot L^{\text{err}}(h_{\ell}) \\
+ (1 - \alpha_{\ell-1}) \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot L^{\text{err}}(h_{\ell}),
\]

(119)

with

\[Q \doteq \alpha_{\ell-1} - \alpha_{\ell} + \frac{\gamma^2 \hat{w}_{\ell}}{16} \cdot \alpha_{\ell}(1 - \alpha_{\ell-1}).\]

(120)

Now, if

\[\alpha_{\ell} \leq \frac{\alpha_{\ell-1}}{1 - \frac{\gamma^2 \hat{w}_{\ell}}{16} \cdot (1 - \alpha_{\ell-1})},\]

(121)

then $Q \geq 0$. Since $L^{\text{err}}(h_{\ell}) \leq L^{\text{Mat}}(h_{\ell})$,

\[\alpha_{\ell} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + Q \cdot L^{\text{err}}(h_{\ell}) \\
\leq \alpha_{\ell} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + Q \cdot L^{\text{Mat}}(h_{\ell}) \\
= \left(\alpha_{\ell} - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16} + \alpha_{\ell-1} - \alpha_{\ell} + \frac{\gamma^2 \hat{w}_{\ell}}{16} \cdot \alpha_{\ell}(1 - \alpha_{\ell-1})\right) \cdot L^{\text{Mat}}(h_{\ell}) \\
= \alpha_{\ell-1} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}),
\]

and so, assembling with (119), we get

\[L^{(\alpha_{\ell})}(h_{\ell+1}) \leq \alpha_{\ell-1} \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot L^{\text{Mat}}(h_{\ell}) + (1 - \alpha_{\ell-1}) \cdot \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot L^{\text{err}}(h_{\ell}) \\
= \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot (\alpha_{\ell-1} \cdot L^{\text{Mat}}(h_{\ell}) + (1 - \alpha_{\ell-1}) \cdot L^{\text{err}}(h_{\ell})) \\
= \left(1 - \frac{\gamma^2 \hat{w}_{\ell} \alpha_{\ell}}{16}\right) \cdot L^{(\alpha_{\ell-1})}(h_{\ell}),
\]

(122)

which achieves the proof of Lemma 20 once we use the fact that $1 - z \leq \exp(-z)$ on the denominator of (121), which yields a lower-bound on its right-hand side and thus a sufficient condition of this inequality to hold, which, after simplification, is (112) and the definition of $\Gamma$-monotonicity in the main file. Notice finally that the first split, on $h_1$ to get $h_2$ ($t \doteq 1$)
Figure 5: Second derivative of the convex surrogate \( F^{(\alpha)} \), for various values of \( \alpha \). The color code follows Figure 1 in the main file.

introduces a dependence on \( \alpha_0 \in [0, 1] \) to compute the \( \text{Mo}_0 \)-loss of the root leaf. Since \( L^{(\alpha)}(q) \leq L^{\text{Mat}}(q), \forall q \in [0, 1] \), we just pick \( \alpha_0 = 1 \), which implies complete freedom to pick \( \alpha_1 \in [0, 1] \) under \( \Gamma \)-monotonicity.

To finish the proof of Lemma 18 we use the fact that \( 1 - z \leq \exp(-z) \) and unravel (117): after \( L \) iterations of boosting, under the conditions of Lemma 20 we get

\[
 L^{(\alpha)}(h_L) \leq \exp\left(-\frac{\gamma^2}{16} \cdot \sum_{\ell=1}^{L} \tilde{w}_\ell \alpha_\ell \right),
\]  

from which, since \( \alpha_\ell \in [0, 1], \forall \ell \), we have the empirical risk of \( h_L, \varepsilon_{0/1}(h_L) \), satisfy \( \varepsilon_{0/1}(h_L) = L^{\text{err}}(h_L) \leq L^{(\alpha)}(h_L) \) and a sufficient condition for \( \varepsilon_{0/1}(h_L) \leq \xi \) is thus

\[
\sum_{\ell=1}^{L} \tilde{w}_\ell \alpha_\ell \geq \frac{16}{\gamma^2} \cdot \log \frac{1}{\xi},
\]

which is the statement of Lemma 18.

Remark that Lemma 18 is Theorem 10 with normalized weights. If we consider unnormalized weights in \( L^{(\alpha)} \) then we need to multiply the right hand side of (123) by \( w(S) \), but we also have in this case \( \varepsilon_{0/1}(h_L) \leq L^{(\alpha)}(h_L)/w(S) \), which in fact does not change the statement for normalized weights. We also remark that the Weak Learning Assumption is not affected by this change in normalization, so we get the statement of Theorem 10 for unnormalized weights as well.

15 Proof of Theorem 12

We first display in Algorithm \( \text{M}^{\alpha}\text{-boost} \) the complete pseudo-code of our approach to boosting using the \( \text{M}^{\alpha} \)-loss. In stating the algorithm, we have simplified notations; in
Algorithm 1 Mo-boost

**Input** sample $S = \{(x_i, y_i), i = 1, 2, ..., m\}$, number of iterations $T$, loss and update parameters

$$
\begin{align*}
\alpha & \in (0, 1) \\
\pi & \in [0, 1) \\
a & \in \frac{\alpha}{M^2} \cdot [1 - \pi, 1 + \pi];
\end{align*}
$$

(125)

Step 1 : let $w_i = 1/2, \forall i = 1, 2, ..., m$; // initial weights

Step 2 : for $t = 1, 2, ..., T$

Step 2.1 : let $h_t \leftarrow \text{WL}(S, w_t)$ // weak classifier

Step 2.2 : let $\beta_t \leftarrow (a/m) \cdot \sum_i w_i y_i h_t(x_i)$ // leveraging coefficient

Step 2.3 : for $i = 1, 2, ..., m$, let

$$
\begin{align*}
w_{(t+1)i} & \leftarrow \psi(\alpha)^{-1} \left(-\beta_t y_i h_t(x_i) + \psi(\alpha)(w_{ti})\right) \ (\in [0, 1]) ;
\end{align*}
$$

(126)

Return $H_T = \sum_t \beta_t h_t$.

particular we can indeed check that the leveraging coefficient of $h_t$ satisfies:

$$
\beta_t = a\tilde{w}_t \eta(\tilde{w}_t, h_t).
$$

(127)

We make use of the same proof technique as in [Nock & Williamson (2019, Theorem 7)]. We sketch here the main steps. A first quantity we define is:

$$
\begin{align*}
X &= \mathbb{E}_S \left[(y_i H_t(x_i) - y_i H_{t+1}(x_i))F^{(\alpha)'}(y_i H_t(x_i))\right] \\
&= \beta_t \mathbb{E}_S \left[-y_i h_t(x_i) \cdot -\psi(\alpha)^{-1} (-y_i H_t(x_i))\right] \\
&= \beta_t \mathbb{E}_S \left[w_i y_i h_t(x_i)\right] \\
&= \beta_t \cdot \frac{1}{m} \cdot \sum_i w_i y_i h_t(x_i) \\
&= a\tilde{w}_t^2 \eta^2(\tilde{w}_t, h_t).
\end{align*}
$$

(128)  (129)  (130)

(128) holds because of (104) and the fact that $H_{t+1}(x_i) = H_t(x_i) + y_i h_t(x_i)$ by definition. (129) holds because of the definition of $w_{ti}$ and (130) is just a rewriting using the distribution of examples in $S$. A second quantity we define is

$$
Y(Z) = \mathbb{E}_S \left[(y_i H_t(x_i) - y_i H_{t+1}(x_i))^2 F^{(\alpha)''}(z_i)\right],
$$

(132)

where $Z \doteq \{z_1, z_2, ..., z_m\} \subset \mathbb{R}^m$. We then need to compute the second derivative of $F^{(\alpha)}$,
which we find to be (Figure 5)

\[
F^{(\alpha)''}(z) = \begin{cases} 
0 & \text{if } z \in 2(1 - \alpha) \cdot (-1, 1) \\
\frac{4\alpha^2}{(4\alpha^2 + |z| - 2(1-\alpha)^2)^2} & \text{if } z \notin 2(1 - \alpha) \cdot [-1, 1] \\
\text{undefined} & \text{if } z \in 2(1 - \alpha) \cdot \{-1, 1\} 
\end{cases}
\quad (133)
\]

from which we easily find

\[\sup_z F^{(\alpha)''} = \frac{1}{2\alpha}, \quad (134)\]

and therefore for any \(Z \subset \mathbb{R}^m\),

\[
Y(Z) \leq \frac{1}{2\alpha} \cdot \mathbb{E}_S \left[ (y_i H_t(x_i) - y_i H_{t+1}(x_i))^2 \right]
= \frac{1}{2\alpha} \cdot \mathbb{E}_S \left[ (a \eta_t \cdot h_t(x_i))^2 \right]
\leq \frac{a^2 \tilde{\eta}^2 \eta_t^2 (\tilde{w}_t, h_t) M^2}{2\alpha}.
\quad (135)
\]

We then get from the proof of Nock & Williamson (2019, Theorem 7) and (131), (135) that there exists a set \(Z \subset \mathbb{R}^m\) such that

\[
\mathbb{E}_S \left[ F^{(\alpha)}(y_i H_t(x_i)) \right] - \mathbb{E}_S \left[ F^{(\alpha)}(y_i H_{t+1}(x_i)) \right] \geq X - Y(Z)
\geq a \tilde{\eta}_t^2 \eta^2 (\tilde{w}_t, h_t) - \frac{a^2 \tilde{\eta}^2 \eta_t^2 (\tilde{w}_t, h_t) M^2}{2\alpha}
\left(1 - \frac{a M^2}{2\alpha}\right) \cdot a \eta_t^2.
\quad (136)
\]

Suppose

\[a \in \frac{\alpha}{M^2} \cdot [1 - \pi, 1 + \pi] \quad (137)\]

for some \(\pi \in [0, 1]\). We then have:

\[
\mathbb{E}_S \left[ F^{(\alpha)}(y_i H_t(x_i)) \right] - \mathbb{E}_S \left[ F^{(\alpha)}(y_i H_{t+1}(x_i)) \right] \geq \frac{(1 - \pi^2) \alpha}{2M^2} \cdot \eta_t^2,
\quad (138)
\]

so after combining \(T\) classifiers in the linear combination, we get

\[
\mathbb{E}_S \left[ F^{(\alpha)}(y_i H_T(x_i)) \right] \leq F^{(\alpha)}(0) - \frac{(1 - \pi^2) \alpha}{2M^2} \cdot \sum_{t=1}^{T} \tilde{\eta}_t^2 \eta_t^2 (\tilde{w}_t, h_t)
= 1 - \frac{(1 - \pi^2) \alpha}{2M^2} \cdot \sum_{t=1}^{T} \tilde{\eta}_t^2 \eta_t^2 (\tilde{w}_t, h_t).
\quad (139)
\]

To summarize, if the sequence of edges satisfies

\[
\frac{1}{M^2} \cdot \sum_{t=1}^{T} \tilde{\eta}_t^2 \eta_t^2 (\tilde{w}_t, h_t) \geq \frac{2(1 - \xi)}{(1 - \pi^2) \alpha},
\quad (140)
\]

41
then
\[ \mathbb{E}_s \left[ F^{(\alpha)}(y_i H_T(x_i)) \right] \leq \xi. \] (141)

Since for any \( \alpha > 0 \), \( F^{(\alpha)} \) is strictly decreasing and non-negative, for any \( \theta \geq 0 \), if \( \mathbb{P}_s \left[ \left[ y_i H_T(x_i) \leq \theta \right] \right] > \xi \), then
\[ \mathbb{E}_s \left[ F^{(\alpha)}(y_i H_T(x_i)) \right] > \xi F^{(\alpha)}(\theta) + (1 - \xi) \inf_z F^{(\alpha)}(z) \geq \xi F^{(\alpha)}(\theta). \] (142)

Hence, we get from (140) and (141) that if the sequence of edges satisfies
\[ \sum_{t=1}^{T} \tilde{w}_t \eta^2(\tilde{w}_t, h_t) \geq \frac{2M^2(1 - \xi F^{(\alpha)}(\theta))}{(1 - \pi^2)\alpha}, \] (143)
then \( \mathbb{E}_s \left[ F^{(\alpha)}(y_i H_T(x_i)) \right] \leq \xi F^{(\alpha)}(\theta) \) and so
\[ \mathbb{E}_s \left[ \left[ y_i H_T(x_i) \leq \theta \right] \right] \leq \xi. \] (144)

There remains to remark that \( \varepsilon_{0/1}(H_T) \leq \mathbb{E}_s \left[ \left[ y_i H_T(x_i) \leq 0 \right] \right] \), and therefore pick \( \theta = 0 \) for which \( F^{(\alpha)}(\theta) = 1 \). Under the \( \gamma \)-WLA, we note that
\[ \tilde{w}_t^2 \eta^2(\tilde{w}_t, h_t) \geq \tilde{w}_t^2 \gamma^2 M^2, \]
and so, to summarise, under the \( \gamma \)-WLA, if the sequence of expected weights satisfies
\[ \sum_{t=1}^{T} \tilde{w}_t^2 \geq \frac{2(1 - \xi)}{(1 - \pi^2)\gamma^2\alpha}, \] (145)
then \( \varepsilon_{0/1}(H_T) \leq \xi \). This ends the proof of Theorem 12.

16 Proof of Theorem 13

We first prove a preliminary result used in the main file.

**Lemma 21** For any \( \alpha \in [0, 1] \), any split \( g \) on leaf \( \lambda \) that satisfies the \( \gamma \)-Weak Learning Assumption on \( h_\ell \) yields
\[ L^{(\alpha \ell)}(h_\ell \oplus (g, \lambda)) \leq \left(1 - \frac{\gamma^2 \alpha \ell \tilde{w}(\lambda)}{16}\right) \cdot L^{(\alpha \ell)}(h_\ell). \] (146)

**Proof** As long as split \( g \) on leaf \( \lambda \) satisfies the \( \gamma \)-Weak Learning Assumption, we get from the proof of Kearns & Mansour (1996, Theorem 10)
\[ L^\text{Mat}(h_\ell \oplus (g, \lambda)) \leq \left(1 - \frac{\gamma^2 \tilde{w}_\ell}{16}\right) \cdot L^\text{Mat}(h_\ell), \] (147)

42
Then the tree efficiency is strictly decreasing along this path: the normalized weight of examples reaching which depends on \( \nu \). This ends the proof of Lemma 21 \( \blacksquare \)

Notations are as follows: \( \mathcal{G} \) denotes the complete set of possible splits and

\[
\kappa \doteq \frac{\varepsilon}{2 \Delta^*_I \ell_0(m)},
\]

which depends on \( \varepsilon, m, \alpha, \lambda \) (See Corollary 7 in the main file). \( \mathcal{N}(h) \) denotes the set of nodes of \( h \), including leaves in \( \Lambda(h) \).

**Definition 22** For any node \( \nu \in \mathcal{N}(h) \), let \( \text{depth}(\nu) \) denote its depth in \( h \) and \( \tilde{w}(\nu) \in [0, 1] \) the normalized weight of examples reaching \( \nu \). The tree-efficiency of \( \nu \) in \( h \) is:

\[
J(\nu, h) \doteq \frac{8 \tilde{w}(\nu) \varepsilon \alpha / h_0^2}{2 \Delta^*_I \ell_0(h)} \in [0, 1].
\]

The following Lemma gives a key property of the tree efficiency of a node.

**Lemma 23** (Tree efficiency is root-to-node decreasing) For any decision tree \( h \), consider any path of nodes \( \nu_1, \nu_2, \ldots, \nu_k \in \mathcal{N}(h) \) where \( \nu_1 \) is the root of \( h \) and \( \text{depth}(\nu_{i+1}) = \text{depth}(\nu_i) + 1, \forall i \). Then the tree efficiency is strictly decreasing along this path: \( J(\nu_i, h) > J(\nu_{i+1}, h), \forall i \).
$L^{(\alpha)}(h) - L^{(\alpha)}(h \oplus (g, \lambda)) = u \cdot \frac{\gamma^2 \omega(\lambda)}{16} \cdot L^{(\alpha)}(h)$

$u = 0 \quad u \leq \delta \quad u \geq 1 \quad \neq \emptyset$ from WLA

$G_{\text{WLA}}$ $G_{\text{LZ}}$ $\emptyset$

Figure 6: In the $\delta$-Gap model of boosting, the total set of potential splits $G$ contains two subsets in regard to the current leaf that is being split, $\lambda$. A subset $G_{\text{WLA}}$ contains all splits that guarantee a moderate decrease in the Bayes risk – this set is guaranteed non empty under the Weak Learning Assumption (Lemma 21). Another set, $G_{\text{LZ}}$, contains all the other splits, supposed to yield a decrease in the Bayes risk at least smaller by factor $\delta < 1$. In the main file, we have assumed for simplicity that we can fix $\delta = \gamma$ but the proof of Theorem 13 below relaxes this assumption.

We now prove Theorem 13. We consider two cases, starting first with the simplified case of a single split and then investigate a set of splits.

▷ Single split: notation $h \oplus (g, \lambda)$ indicates decision tree $h$ in which leaf $\lambda \in \Lambda(h)$ is replaced by split $g \in G$. It follows from [10] that the probability to pick split $g$ for leaf $\lambda \in h$ following the exponential mechanism, $p_{\text{exp}}((g, \lambda))$,

$$p_{\text{exp}}((g, \lambda)) = \frac{1}{Z} \cdot \exp \left( -\kappa \cdot w(S) \cdot F(h \oplus (g, \lambda)) \right),$$

(153)

where $Z = \sum_{g' \in G} \exp \left( -\kappa w(S) \cdot F(h \oplus (g, \lambda)) \right)$. Notice that the part in the sum in $F(h \oplus (g, \lambda))$ that does not depend on $\lambda$ can be factored thanks to the exp, which allows us to simplify

$$p_{\text{exp}}((g, \lambda)) = \frac{1}{Z} \cdot \exp \left( -\kappa \cdot \left[ w(\lambda \wedge g) \cdot L \left( \frac{w(\lambda \wedge g)}{w(\lambda \wedge g)} \right) + w(\lambda \wedge \neg g) \cdot L \left( \frac{w(\lambda \wedge \neg g)}{w(\lambda \wedge \neg g)} \right) \right] \right) \propto \exp \left( \kappa \cdot \left[ L^{(\alpha)}(h) - L^{(\alpha)}(h \oplus (g, \lambda)) \right] \right)$$

(154)

and $Z$ is the normalization coefficient modified accordingly. Suppose, $h$ and $\lambda$ being fixed, that we have two subsets, $G_{\text{WLA}}$ and $G_{\text{LZ}}$, such that

$$L^{(\alpha)}(h) - L^{(\alpha)}(h \oplus (g, \lambda)) \geq \frac{\gamma^2 \omega(\lambda)}{16} \cdot L^{(\alpha)}(h), \forall g \in G_{\text{WLA}},$$

(155)

$$L^{(\alpha)}(h) - L^{(\alpha)}(h \oplus (g, \lambda)) \leq \frac{\delta^2 \gamma^2 \omega(\lambda)}{16} \cdot L^{(\alpha)}(h), \forall g \in G_{\text{LZ}},$$

(156)
where we remind that $\hat{w}(\lambda)$ is the total normalized weight of examples reaching leaf $\lambda$ (10).

Assuming $\mathcal{G} = \mathcal{G}_{wla} \cup \mathcal{G}_{lazy}$ and letting $\rho = |\mathcal{G}_{wla}|/|\mathcal{G}_{lazy}|$, we get

$$
\frac{p_{\text{exp}}(g \in \mathcal{G}_{wla}|\lambda)}{p_{\text{exp}}(g \in \mathcal{G}_{lazy}|\lambda)} \geq \rho \cdot \exp \left( \frac{(1 - \delta^2)\gamma^2\alpha\varepsilon \hat{w}(\lambda)}{32} \cdot \frac{L^{(a)}(h)}{\Delta^*_L^{(a)}(m)} \right).
$$

(157)

We want $p_{\text{exp}}(g \in \mathcal{G}_{wla}|\lambda) \geq \exp(-\xi)$ for some $\xi > 0$. From (157), this shall be the case if

$$
\frac{(1 - \delta^2)\gamma^2\alpha\varepsilon \hat{w}(\lambda)}{32} \cdot \frac{L^{(a)}(h)}{\Delta^*_L^{(a)}(m)} \geq \log \left( \frac{1}{\exp(\xi)} - 1 \right) - \log \rho
\]

$$
= \log \frac{1}{\exp(\xi)} - \log \rho,
$$

(158)

where $F^{\log}$ is the convex surrogate of the log-loss. This can also be inverted to get all $\xi$s for which this applies using the fact that $F^{\log}$ is strictly decreasing, as

$$
\xi \geq F^{\log} \left( \frac{(1 - \delta^2)\gamma^2\alpha\varepsilon \hat{w}(\lambda)}{32} \cdot \frac{L^{(a)}(h)}{\Delta^*_L^{(a)}(m)} + \log \rho \right).
$$

(159)

**Sequence $\mathcal{L}$ of split**: we now index quantities $\lambda_\ell$ (replacing notation $\hat{w}(\lambda_\ell)$ by $\hat{w}_\ell$ to follow Theorem 10), $h_\ell, \rho_\ell, \alpha_\ell, \xi_\ell, \varepsilon_\ell$. In particular, the exponential mechanism to pick $g \in \mathcal{G}$ to split $\lambda_\ell$ in $h_\ell$ now becomes

$$
p_{\text{exp}}((g, \lambda_\ell)) \propto \exp \left( -\varepsilon_\ell w(S) F(h_\ell \oplus g, \lambda_\ell) \right).
$$

(160)

We constrain the analysis to indexes $\ell$ in a specific set $\mathcal{L}$ of size $|\mathcal{L}|$. We get that for any $\xi_\ell$, \[ \xi_\ell \geq F^{\log} \left( \frac{(1 - \delta^2)\gamma^2}{32} \cdot \frac{\alpha_\ell\varepsilon_\ell \hat{w}_\ell L^{(a)}(h_\ell)}{\Delta^*_L^{(a)}(m)} + \log \rho_\ell \right) \Rightarrow p_{\text{exp}}(g \in (\mathcal{G}_{wla})_\ell | \lambda_\ell) \geq \exp(-\xi_\ell).
\] (161)
with the simplifying assumption that \( \forall \ell, G = (G_{\text{wla}})_\ell \cup (G_{\text{lazy}})_\ell \). Because of Theorem 11 whenever the sequence of \( \alpha_\ell \) is \( \gamma^2/16 \)-monotonic, letting

\[
Q = \frac{(1 - \delta^2) \gamma^2}{32}, \quad A_\ell = \frac{\varepsilon_\ell \bar{w}_\ell \alpha_\ell L^{(\alpha_\ell)}(h_\ell)}{L^*_{\ell}(m)},
\]

(162)

if furthermore \( \xi_\ell \geq F_{\text{log}}(QA_\ell + \log \rho_\ell), \forall \ell \), then with probability \( \geq \exp(-\sum_\ell \xi_\ell) \), all splits in \( \mathcal{L} \) satisfy the \( \gamma \)-WLA and therefore the boosting condition in (11) is met. In other words, the use of the exponential mechanism to make splits differentially private does not endanger at all convergence with high probability. We now have two competing objectives in a differentially private induction of a top-down decision tree:

(i) we need to pick the \( \varepsilon_\ell \)'s so as to match the total privacy budget allowed for the induction of a single tree,

\[
\frac{\beta_{\text{tree}} \varepsilon}{T} \leq \sum_\ell \varepsilon_\ell,
\]

(163)

(composition theorem).

(ii) we want to find \( \xi_\ell, \ell = 1, 2, \ldots, L \) such that we have simultaneously, for some \( \xi > 0 \),

\[
\sum_\ell \xi_\ell \leq \log \frac{1}{1 - \xi},
\]

(164)

\[
\xi_\ell \geq F_{\text{log}}(QA_\ell + \log \rho_\ell), \forall \ell,
\]

(165)

because then we can lowerbound the probability that all splits chosen comply with the WLA:

\[
p_{\exp}(\land \ell (g \in (G_{\text{wla}})_\ell | \lambda_\ell)) \geq 1 - \xi,
\]

(166)

Note that, in particular for the first tree induced, \( w(S) = m/2 = \Omega(m) \) and in all cases, \( w(S) \leq m = O(m) \), so suppose \( w(S) = \xi'm \) with \( \xi' \in (0, 1) \) a constant.\(^5\) We have

\[
L^{(\alpha_\ell)}(h_\ell) = \sum_{\lambda_\ell \in \Lambda(h_\ell)} w(\lambda_\ell) \cdot L^{(\alpha_\ell)} \left( \frac{w^1(\lambda_\ell)}{w(\lambda_\ell)} \right)
\]

\[
= w(S) \cdot \sum_{\lambda_\ell \in \Lambda(h_\ell)} \frac{w(\lambda_\ell)}{w(S)} \cdot L^{(\alpha_\ell)} \left( \frac{w^1(\lambda_\ell)}{w(\lambda_\ell)} \right)
\]

\[
\geq 2\xi' m \cdot \varepsilon_{0/1}(h_\ell).
\]

(167)

Then we can refine and lowerbound

\[
A_\ell = \frac{\varepsilon_\ell \bar{w}_\ell \alpha_\ell w(S) \cdot L^{(\alpha_\ell)}(h_\ell)}{3 + 2\alpha_\ell (\sqrt{m} - 1)}
\]

\[
\geq \varepsilon_\ell \bar{w}_\ell \cdot \frac{2\alpha_\ell m \xi' \varepsilon_{0/1}(h_\ell)}{3 + 2\alpha_\ell (\sqrt{m} - 1)}.
\]

\(^5\)The boosting weight update \([13]\) prevents zero / unit weights if the number of boosting iterations \( T \ll \infty \).
Suppose we fix\(^6\)
\[
\alpha_\ell = \frac{\varepsilon_{0/1}(h_\ell)}{\varepsilon_{0/1}(h_1)} \quad (\in [0, 1]),
\]
which, since \(\varepsilon_{0/1}(h_\ell)\) is non increasing, is therefore \(\gamma^2/16\)-monotonic as a sequence. We get
\[
A_\ell \geq \xi_\ell \varepsilon_\ell \tilde{w}_\ell \cdot \frac{4m\varepsilon_{0/1}(h_\ell)^2}{3\varepsilon_{0/1}(h_1) + 4\varepsilon_{0/1}(h_\ell)(\sqrt{m} - 1)}.
\]
Define for \(r \geq 0\)
\[
t(z) = \frac{4z^2}{3r + 4z},
\]
We can check that if \(z \geq (3qr)/(4(1 - q))\) for some \(q > 0\), then \(t(z) \geq qz\). Now,
\[
A_\ell \geq \xi_\ell \varepsilon_\ell \tilde{w}_\ell \cdot \frac{4m\varepsilon_{0/1}(h_\ell)^2}{3 + 4\varepsilon_{0/1}(h_\ell)\sqrt{m}}
\]
\[
= \xi_\ell \varepsilon_\ell \tilde{w}_\ell \cdot \frac{4z^2}{3 + 4z}
\]
for \(z = \varepsilon_{0/1}(h_\ell)\sqrt{m}\). We get
\[
A_\ell \geq \xi_\ell \varepsilon_\ell \tilde{w}_\ell \varepsilon_{0/1}(h_\ell)^2 \sqrt{m},
\]
provided \(\varepsilon_{0/1}(h_\ell)\sqrt{m} \geq (3\varepsilon_{0/1}(h_\ell)\xi')/(4(1 - \varepsilon_{0/1}(h_\ell))))\), which simplifies in
\[
m \geq \frac{9\xi'^2}{16(1 - \varepsilon_{0/1}(h_\ell))^2},
\]
and since \(\xi' \leq 1, \varepsilon_{0/1}(h_\ell) \leq 1/2\), holds whenever
\[
m \geq \frac{9}{4}.
\]
We then have
\[
\Phi^{\log}(QA_\ell + \log \rho_\ell) \leq \Phi^{\kappa\log}\left(\frac{(1 - \delta^2)\gamma^2 \xi_\ell \varepsilon_\ell \tilde{w}_\ell \varepsilon_{0/1}(h_\ell)^2}{32} \cdot \sqrt{m} + \log \rho_\ell\right).
\]
Suppose
\[
m \geq 3,
\]
which implies \(173\). Fix now
\[
\varepsilon_\ell = \frac{\beta_{\text{tree}}}{T d^{\log\text{th}(\lambda_\rho)}} \cdot \varepsilon,
\]
\[
Td \leq \log m.
\]
\(^6\)We note that \(\varepsilon_{0/1}(h_\ell) \leq 1/2, \forall h_\ell\).
We recall that $d$ is the maximum depth of a tree and $T$ is the number of trees in the boosted combination. $Td$ is therefore a proxy for the maximal number of tests in trees to classify an observation.

\[
F^{\log}(QA_\ell + \log \rho_\ell) \leq F^{\log}(\frac{\beta_{\text{tree}}(1 - \delta^2)\gamma^2 \xi_\ell \varepsilon}{32Td} \cdot \frac{\bar{w}_\ell \varepsilon_{0/1}(h_\ell)^2 \sqrt{m}}{2^{\text{depth}(\lambda_\ell)}} + \log \rho_\ell)
\]

\[
\leq F^{\log}(\frac{\beta_{\text{tree}}(1 - \delta^2)\gamma^2 \xi_\ell}{256} \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} + \log \rho_\ell),
\]

with

\[
J(\lambda_\ell, h) \doteq \frac{8 \bar{w}_\ell \varepsilon_{0/1}(h_\ell)^2}{2^{\text{depth}(\lambda_\ell)}} \in [0, 1].
\]

Suppose now that

\[
\log \rho_\ell \geq -\frac{\beta_{\text{tree}}(1 - \delta^2)\gamma^2 \xi_\ell}{256} \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m},
\]

which is equivalent to

\[
\frac{|S_{\text{wla}}|}{|S_{\text{lazy}}|} \geq \exp \left( -\frac{\beta_{\text{tree}}(1 - \delta^2)\gamma^2 \xi_\ell}{256} \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right),
\]

or

\[
|S_{\text{wla}}| \geq \frac{|S|}{1 + \exp \left( \frac{\beta_{\text{tree}}(1 - \delta^2)\gamma^2 \xi_\ell}{256} \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right)}
\]

and thus $S_{\text{wla}}$ cannot be vanishing (or at least too fast as a function of $m$) with respect to $S$. This implies

\[
F^{\log}(QA_\ell + \log \rho_\ell) \leq F^{\log}(Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m}),
\]

with

\[
Q' \doteq \frac{\beta_{\text{tree}}\xi_\ell'(1 - \delta^2)\gamma^2}{256} \in (0, 1/256].
\]

Notice that $Q' = \theta(1)$, i.e. it is a constant. The concavity of $\log$ yields

\[
\sum_{\ell \in \mathcal{L}} F^{\log}(Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m}) \leq |\mathcal{L}| \log \left( 1 + E_{\mathcal{L}} \exp \left( -Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right) \right)
\]

and so if we pick

\[
\xi_\ell \doteq F^{\log}(Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m}),
\]

48
then a sufficient condition to have \[164\] is

\[
\mathbb{E}_\mathcal{L} \exp \left( -Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right) \leq \left( \frac{1}{1 - \xi} \right)^{\frac{1}{|\mathcal{L}|}} - 1. \tag{186}
\]

We also have \( \forall \xi \in [0, 1], |\mathcal{L}| \geq 1, \)

\[
\left( \frac{1}{1 - \xi} \right)^{\frac{1}{|\mathcal{L}|}} - 1 \geq \frac{\xi}{|\mathcal{L}|}, \tag{187}
\]

so to get \[186\] it is sufficient that

\[
\mathbb{E}_\mathcal{L} \exp \left( -Q' \cdot J(\lambda_\ell, h) \cdot \frac{\varepsilon \sqrt{m}}{\log m} \right) \leq \frac{\xi}{|\mathcal{L}|}. \tag{188}
\]

which is ensured if

\[
\min_{\ell \in \mathcal{L}} J(\lambda_\ell, h) \geq \frac{1}{Q'} \cdot \frac{\log m}{\varepsilon \sqrt{m}} \frac{\log |\mathcal{L}|}{\xi} = \Omega \left( \frac{\log m}{\varepsilon \sqrt{m}} \log \frac{|\mathcal{L}|}{\xi} \right). \tag{189}
\]

This ends the proof of Theorem \[13\].

**Remark:** Notice that \(|\mathcal{L}| \leq 2^{d+1} - 1\), so we get that condition \[189\] is satisfied if for example

\[
\min_{\ell \in \mathcal{L}} J(\lambda_\ell, h) = \Omega \left( \frac{\log m}{\varepsilon \sqrt{m}} \cdot \left( d + \log \frac{1}{\xi} \right) \right). \tag{190}
\]

As long as for example

\[
\frac{\log m}{\sqrt{m}} = o(\varepsilon), \tag{191}
\]

\[
d, \log \frac{1}{\xi} = o \left( \frac{\sqrt{m}}{\log m} \right), \tag{192}
\]

then the constraint on \( \min_{\ell \in \mathcal{L}} J(\lambda_\ell, h) \) in \[190\] will vanish.
17 Appendix on Experiments

17.1 General setting

▷ Public information is as follows. First, the attribute domain is public, which is standard in the field (Fletcher & Islam, 2019). Several authors have tried to compute the threshold information for continuous attributes in a private way (Fletcher & Islam, 2019; Friedman & Schuster, 2010). This is not necessarily a good approach: it requires privacy budget, it can require weakening privacy and does not necessarily buys improvements (Fletcher & Islam, 2019, Section 3.2.2). Since the attribute domain is public, there is a simple alternative that does not suffer most of these workarounds: the regular quantisation of the domain using a public number of values. This particularly makes sense e.g. for many commonly used attribute classes like age, percentages, $p$-value, mileages, distances, or for any attribute for which the key segments are known from the specialists, such as in life sciences or medical domain. This also has three technical justifications: (1) a private approaches requires budget, (2) \( v_q \) allows to tightly control the computational complexity of the whole DT induction, and most importantly (3) boosting does not require exhaustive split search. It indeed just assumes the WLA, which essentially requires \( v_q \) not too small, even more if the tree is not too deep.

▷ Parameters for \( \text{BDPE}_\alpha \). We ran out approach, both private and not private, for all combinations of \( T \in \{2, 5, 10, 20\} \), \( \alpha \in \{0.1, 1.0, \text{O.C}\} \) (O.C = Objective Calibration), \( d \in \{1, 2, 3, 4, 5, 6\} \). Finally, we have tried a quantisation in \( v_q \in \{10, 50\} \) values, for all numeric attributes (Section 6 in the main file). In order not to give a potential advantage to noise-free boosting in its tests that would not come from the absence of noise, we also use this regular quantisation for the noise-free boosting tests of our approach.

For the private version, in addition to all these combinations, we considered \( \varepsilon \in \{0.01, 0.1, 1.0, 10.0, 25.0\} \) and \( \beta_{\text{tree}} \in \{0.1, 0.5, 0.9\} \). For the private trees, after having noisified the leaf predictions, we clamp the output values of the private trees to a maximal \( M \in \{1, 10, 100\} \), which is another parameter. In the private setting, once the depth is fixed, all tree induced have each of their leaves at the same depth: this means that we even split leaves that are pure if they are below the required depth, to prevent using DP budget to test for purity (which we do when there is no DP, as we do not split pure leaves in this case).

Altogether, this represents more than 1.3 million (ensemble) models learned using our approach. Obviously, increasing \( v_q \) tends to improve accuracy but significantly increases time complexity for \( \text{BDPE}_\alpha \), in particular to split the nodes, a task carried out repeatedly for both the non private but also for the exponential mechanism in the differential privacy case, adding an further computational burden in this case. Because of the size of the experiments, we report here the results obtained for \( M = 10, v_q = 10 \), which seems to lead to a good compromise between accuracy and execution time.
17.2 Implementation

We give here a few details on the implementation.

**Boosting**: For boosting algorithms, we clamp the value \( q(\ell) \in [\zeta, 1 - \zeta] \) with \( \zeta = 10^{-4} \) to prevent infinite predictions and NaNs via the link function. Then the value is noisified if DP, and if DP, after that, the maximal value is clamped to a maximum value, \( M \). Since in theory weights cannot be 0 or 1 when \( \alpha \neq 0 \) but numerical precision errors can result in 0 or 1 weights in exceptional cases, we replace such weights by a corresponding value in \( \{\zeta', 1 - \zeta'\} \).

**Random forests**: A random decision forest is an ensemble of random decision trees (Fan et al., 2003). A random decision tree is constructed by choosing the split features purely at random. Fletcher & Islam (2017) showed that this independence of the training data can be favourable for learning differentially private classifier, as the construction of the tree does not incur any privacy costs.

We implemented random decision forest based on the ideas from those papers. However, instead of smooth sensitivity, we use global sensitivity, not just to rely on the exact same definition of sensitivity: our code was written with federated learning in mind, and, as smooth sensitivity is data dependent, it is an open problem if you can cooperatively compute smooth sensitivity over distributed datasets without leaking information. Since privacy is spent at the leaves’ predictions, we have implemented two mechanisms to make those private: the exponential mechanism using the class counts, and the Laplace mechanism, still on the class counts, splitting evenly the privacy budget among the leaves prior to applying each mechanism. We refer to the two random forest approaches as RF-E and RF-L, respectively for the exponential and Laplace mechanisms.

17.3 Additional experimental results

Domain summary Table
| Domain                  | $m$  | $n$ |
|-------------------------|------|-----|
| Transfusion             | 748  | 4   |
| Banknote                | 1372 | 4   |
| Breast wisc             | 699  | 9   |
| Ionosphere              | 351  | 33  |
| Sonar                   | 208  | 60  |
| Yeast                   | 1484 | 7   |
| Wine-red                | 1599 | 11  |
| Cardiotocography (*)    | 2126 | 9   |
| CreditCardSmall (**)    | 1000 | 23  |
| Abalone                 | 4177 | 8   |
| Qsar                    | 1055 | 41  |
| Wine-white              | 4898 | 11  |
| Page                    | 5473 | 10  |
| Mice                    | 1080 | 77  |
| Hill+noise              | 1212 | 100 |
| Hill+nonoise            | 1212 | 100 |
| Firmteacher             | 10800| 16  |
| Magic                   | 19020| 10  |
| EEG                     | 14980| 14  |

Table 2: UCI domains considered in our experiments ($m =$ total number of examples, $n =$ number of features), ordered in increasing $m \times n$. (*) we used features 13-21 as descriptors; (**) we used the first 1 000 examples of the UCI domain.
Results for $v_q = 10, M = 10$

Due to the excessive number of files/plots, results on a subset of the domains are shown here. Contact the authors for a more comprehensive non-ArXiv version of the paper.

▷ UCI transfusion

| without DP | with DP |
|------------|---------|
| ![Graph](depth) | ![Graph](depth) |
| ![Graph](#leaves) | ![Graph](#leaves) |

Figure 8: UCI domain transfusion: $x =$ test error values, $y =$ cumulated expected depth (left plots) or number of leaves (right plots) for the models having test error $\leq x$, aggregated over all runs ($\pm$ standard deviation) – the vertical black bar depicts the test error of the default class. **Left panel:** w/o DP; **Left panel:** with DP; values are aggregated over all varying parameters (left: $\alpha$; right: $\alpha, \varepsilon, [\beta_\text{tree}|\beta_\text{pred}]$).

| performances wrt $\alpha$ | performances wrt $\varepsilon$s (with DP) |
|---------------------------|------------------------------------------|
| ![Graph](w/o DP) | ![Graph](full) |
| ![Graph](with DP) | ![Graph](crop) |

Figure 9: UCI domain transfusion: $x =$ test error values and $y =$ aggregated percentage of runs having error no less than $x$ – the vertical black bar depicts the test error of the default class; **Left pane:** performances as a function of $\alpha$ (O.C = objective calibration), without (left plot) or with DP (right plot); **Right pane:** performances as a function of $\varepsilon$, either displaying the full plot (left plot) or a crop over the best results (right plot). The crop panel is indicated in the left plot.
Figure 10: UCI domain transfusion: $x =$ test error values and $y =$ aggregated percentage of runs having error no less than $x$ – the vertical black bar depicts the test error of the default class; top row: performances as a function of $\alpha$ showing the full plot for each value of $\varepsilon$; bottom row: crop of the best results from the top row (the crop panel is indicated in the left plot).
> UCI banknote

| without DP | with DP |
|------------|---------|
| ![Graph](image1) | ![Graph](image2) |
| depth | #leaves | depth | #leaves |

Figure 11: UCI domain banknote, conventions identical as in Figure 8.

| performances wrt $\alpha$s | performances wrt $\varepsilon$s (with DP) |
|-----------------------------|------------------------------------------|
| w/o DP | with DP | full | crop |
| ![Graph](image3) | ![Graph](image4) | ![Graph](image5) | ![Graph](image6) |

Figure 12: UCI domain banknote, conventions identical as in Figure 9.
Figure 13: UCI domain banknote, conventions identical as in Figure 10.
## UCI breastwisc

| depth | #leaves | depth | #leaves |
|-------|---------|-------|---------|

Figure 14: UCI domain `breastwisc`, conventions identical as in Figure 8.

| performances wrt $\alpha$ | performances wrt $\varepsilon$ (with DP) |
|-----------------------------|------------------------------------------|
| w/o DP                      | full                                      |
| with DP                     | crop                                      |

Figure 15: UCI domain `breastwisc`, conventions identical as in Figure 9.
Figure 16: UCI domain \texttt{breastwisc}, conventions identical as in Figure 10.
Figure 17: UCI domain *ionosphere*, conventions identical as in Figure 8.

Figure 18: UCI domain *ionosphere*, conventions identical as in Figure 9.
Figure 19: UCI domain ionosphere, conventions identical as in Figure 10.
Figure 20: UCI domain sonar, conventions identical as in Figure 8.

Figure 21: UCI domain sonar, conventions identical as in Figure 9.
\[ \varepsilon = 0.01 \quad \varepsilon = 0.1 \quad \varepsilon = 1 \quad \varepsilon = 10 \quad \varepsilon = 25 \]

Figure 22: UCI domain \textit{sonar}, conventions identical as in Figure 10.
> UCI yeast

| without DP | with DP |
|------------|---------|
| ![Graph](image1.png) | ![Graph](image2.png) |
| depth | #leaves | depth | #leaves |

Figure 23: UCI domain yeast, conventions identical as in Figure 8

| performances wrt $\alpha$ | performances wrt $\varepsilon$ (with DP) |
|---------------------------|------------------------------------------|
| w/o DP | with DP | full | crop |
| ![Graph](image3.png) | ![Graph](image4.png) | ![Graph](image5.png) | ![Graph](image6.png) |

Figure 24: UCI domain yeast, conventions identical as in Figure 9
Figure 25: UCI domain yeast, conventions identical as in Figure [10]
Figure 26: UCI domain winered, conventions identical as in Figure 8.

Figure 27: UCI domain winered, conventions identical as in Figure 9.
Figure 28: UCI domain \textit{winered}, conventions identical as in Figure 10.
UCI cardiotocography

| without DP | with DP |
|------------|---------|
| ![Graph](image1) | ![Graph](image2) |
| depth | #leaves | depth | #leaves |

Figure 29: UCI domain cardiotocography, conventions identical as in Figure 8.

| performances wrt αs | performances wrt εs (with DP) |
|---------------------|--------------------------------|
| w/o DP | with DP | full | crop |
| ![Graph](image3) | ![Graph](image4) | ![Graph](image5) | ![Graph](image6) |

Figure 30: UCI domain cardiotocography, conventions identical as in Figure 9.
Figure 31: UCI domain cardiotocography, conventions identical as in Figure 10.
\textbf{UCI creditcardsmall}

| without DP | with DP |
|------------|---------|
| depth      | depth   |
| #leaves    | #leaves |

Figure 32: UCI domain \textit{creditcardsmall}, conventions identical as in Figure 8.

| performances wrt $\alpha$ | performances wrt $\varepsilon$ (with DP) |
|----------------------------|-----------------------------------------|
| w/o DP                     | full                                    |
| with DP                    | crop                                    |

Figure 33: UCI domain \textit{creditcardsmall}, conventions identical as in Figure 9.
Figure 34: UCI domain \texttt{creditcardsmall}, conventions identical as in Figure 10.
### UCI abalone

| without DP | with DP |
|------------|---------|
| depth | #leaves | depth | #leaves |

Figure 35: UCI domain abalone, conventions identical as in Figure 8.

| performances wrt $\alpha$s | performances wrt $\varepsilon$s (with DP) |
|-----------------------------|------------------------------------------|
| w/o DP | with DP | full | crop |

Figure 36: UCI domain abalone, conventions identical as in Figure 9.
Figure 37: UCI domain abalone, conventions identical as in Figure 10
Figure 38: UCI domain \texttt{qsar}, conventions identical as in Figure 8.

Figure 39: UCI domain \texttt{qsar}, conventions identical as in Figure 9.
Figure 40: UCI domain qsar, conventions identical as in Figure 10.
Figure 41: UCI domain page, conventions identical as in Figure 8.

Figure 42: UCI domain page, conventions identical as in Figure 9.
Figure 43: UCI domain page, conventions identical as in Figure 10.
UCI mice

Figure 44: UCI domain mice, conventions identical as in Figure 8.

Figure 45: UCI domain mice, conventions identical as in Figure 9.
Figure 46: UCI domain mice, conventions identical as in Figure 10.
UIC hill+noise

Figure 47: UCI domain hill+noise, conventions identical as in Figure 8.

Figure 48: UCI domain hill+noise, conventions identical as in Figure 9.
Figure 49: UCI domain \texttt{hill+noise}, conventions identical as in Figure 10.
UCI hill+nonoise

Figure 50: UCI domain hill+nonoise, conventions identical as in Figure 8.

Figure 51: UCI domain hill+nonoise, conventions identical as in Figure 9.
Figure 52: UCI domain hill+noise, conventions identical as in Figure 10.
> UCI firmteacher

|                  | without DP | with DP |
|------------------|------------|---------|
| depth            | ![Graph](image1.png) | ![Graph](image2.png) |
| #leaves          | ![Graph](image3.png) | ![Graph](image4.png) |

Figure 53: UCI domain firmteacher, conventions identical as in Figure 8.

|                  | performances wrt $\alpha$s | performances wrt $\varepsilon$s (with DP) |
|------------------|-----------------------------|------------------------------------------|
| w/o DP           | ![Graph](image5.png)       | ![Graph](image6.png)                    |
| with DP          | ![Graph](image7.png)       | ![Graph](image8.png)                    |
| full             | ![Graph](image9.png)       | ![Graph](image10.png)                   |
| crop             | ![Graph](image11.png)      | ![Graph](image12.png)                   |

Figure 54: UCI domain firmteacher, conventions identical as in Figure 9.
Figure 55: UCI domain \texttt{firmteacher}, conventions identical as in Figure 10.
> UCI magic

|                      | without DP | with DP |
|----------------------|------------|---------|
| ![Graph](image1)    | ![Graph](image2) |         |

|                      | depth | #leaves |
|----------------------|-------|---------|
| ![Graph](image3)    | ![Graph](image4) |         |

Figure 56: UCI domain magic, conventions identical as in Figure 8

|                      | performances wrt \( \alpha \) | performances wrt \( \varepsilon \) (with DP) |
|----------------------|---------------------------------|-----------------------------------------------|
|                      | w/o DP | with DP | full | crop |
| ![Graph](image5)    | ![Graph](image6) | ![Graph](image7) | ![Graph](image8) | ![Graph](image9) |

Figure 57: UCI domain magic, conventions identical as in Figure 9
Figure 58: UCI domain magic, conventions identical as in Figure 10.
Figure 59: UCI domain eeg, conventions identical as in Figure 8.

Figure 60: UCI domain eeg, conventions identical as in Figure 9.
Figure 61: UCI domain eeg, conventions identical as in Figure 10.
Table 3: Localisation of each of the 19 domains in terms of the model complexity parameters \((d,T)\) allowing to get the best DP results, as observed from the "results*" file (see above, Section 17.2).

### Summary in \(d,T\) for the best DP in BDPE\(_{\alpha}\)

Table 3 roughly summarizes the optimal regimes for \(d\) (depth) and \(T\) (number of trees) for the best DP results in BDPE\(_{\alpha}\).
Summary of the comparison $\text{BDPE}_\alpha$ vs RFs with DP

Summary comparison $v_q = 10$ vs $v_q = 50$ ($M = 10$)
Table 4: Comparison of BDPE_{\alpha} vs two SOTA random forest (RFs) approaches, each inducing $T = 21$ random trees. For each domain and each depth value in $\{2, 4, 6\}$, we compute the number of runs where one algorithm significantly (evaluated with a Student’s $t$ test and all counts get p-value $p < 0.01$) beats the other and then compute the percentage of those where BDPE_{\alpha} is the lead, for several values of $\alpha$ and a number of trees $T \in \{2, 20\}$ (left and right tables, resp.) for BDPE_{\alpha}. 

| Domain | $T = 2$ trees for BDPE_{\alpha} | $T = 20$ trees for BDPE_{\alpha} |
|--------|------------------------------|-------------------------------|
|        | $% \text{wins vs RF-L}$ | $% \text{wins vs RF-E}$ | $% \text{wins vs RF-L}$ | $% \text{wins vs RF-E}$ |
|        | $\text{O.C.} \ 0.1$ | $\text{O.C.} \ 1.0$ | $\text{O.C.} \ 0.1$ | $\text{O.C.} \ 1.0$ |
| Math-Lin.| 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Math-NoLin.| 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Word-Asc.| 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Word-Asc.| 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Word-Asc.| 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| Overall | 53 | 55 | 57 | 58 | 61 | 64 | 82 | 53 | 83 | 85 | 60 | 85 |

91
Figure 62: Extract of the comparison between quantization in $v_q = 10$ vs $v_q = 50$ values for continuous attributes, for both the overall privacy results (left subtable) and results as a function of $\alpha$ for high privacy regime ($\varepsilon = 0.01$, right subtable). Conventions follow Figures 8 and 9. The vertical black line is the test error of the majority class.