Quark confinement mechanism for baryons

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Abstract

The confinement mechanism proposed earlier and then successfully applied to meson spectroscopy by the author is extended over baryons. For this aim the wave functions of baryons are built as tensorial products of those corresponding to the 2-body problem underlying the confinement mechanism of two quarks. This allows one to obtain the Hamiltonian of the quark interactions in a baryon and, accordingly, the possible energy spectrum of the latter. Also one may construct the electric and magnetic form factors of baryon in a natural way which entails the expressions for the root-mean-square radius and anomalous magnetic moment. To illustrate the formalism in the given Chapter for the sake of simplicity only symmetrical baryons (i.e., composed from three quarks of the same flavours) $\Delta^{++}$, $\Delta^-$, $\Omega^-$ are considered. For them the masses, the root-mean-square radii and anomalous magnetic moments are expressed in an explicit analytical form through the parameters of the confining SU(3)-gluonic field among quarks and that enables one to get a number of numerical estimates for the mentioned parameters from experimental data. We also discuss chiral limit for the baryons under consideration and estimate the purely gluonic contribution to their masses. Further the problem of masses in particle physics is shortly discussed within the framework of the given approach. Finally, a few remarks are made about the so-called Yang-Mills Millennium problem and a possible way for proving it is outlined.

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1 Introduction

The present Chapter is a natural continuation of our previous papers on studying both heavy quarkonia and the pseudoscalar meson nonet within the framework of the (quark) confinement mechanism proposed earlier by the author. Global strategy may consist in reconsidering the whole spectroscopy of both mesons and baryons from the positions of the mentioned mechanism so, e.g., exploring the pseudoscalar meson nonet is in essence only the first step in the given direction. But an extension of the approach over baryons requires a relativistic 3-body problem to be considered in contrast to mesons where we deal with a relativistic 2-body problem as follows from the standard quark model (SQM). It is clear, however, that the confinement mechanism under discussion (which in essence describes the mentioned relativistic 2-body problem) should also occupy a fitting place in the 3-body constructions. Let us, therefore, shortly outline the main features of the approach suggested.

In [1, 2, 3] for the Dirac-Yang-Mills system derived from QCD-Lagrangian an unique family of compatible nonperturbative solutions was found and explored, which could pretend to describing confinement of two quarks. The applications of the family to the description of both the heavy quarkonia spectra [4, 5, 6, 7] and a number of properties of pions, kaons, \( \eta \) and \( \eta' \)-mesons [8, 9, 10, 11, 12, 13] showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia. At this moment it can be described in the following way.

The next main physical reasons underlie linear confinement in the mechanism under discussion. The first one is that gluon exchange between quarks
is realized with the propagator different from the photon-like one, and existence and form of such a propagator is a direct consequence of the unique confining nonperturbative solutions of the Yang-Mills equations \[2, 3\]. The second reason is that, owing to the structure of the mentioned propagator, quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see Refs. \[2, 3\]) but, because of the fact that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form a linear confining magnetic colour field of enormous strengths, which leads to confinement of quarks. This is by virtue of the fact that just the magnetic part of the mentioned propagator is responsible for a larger portion of gluon concentrations at large distances since the magnetic part has stronger infrared singularities than the electric one. In the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra) \[2, 3\] which describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave the hadron (meson) until some gluons obtain additional energy (due to an external reason) to rush out. So we also deal with the confinement of gluons.

The approach under discussion equips us with the explicit wave functions for every two quarks (meson or quarkonium). The wave functions are parametrized by a set of real constants \(a_j, b_j, B_j\) describing the mentioned nonperturbative confining SU(3)-gluonic field (the gluon condensate) and they are nonperturbative modulo square integrable solutions of the Dirac equation in the above confining SU(3)-field and also depend on \(\mu_0\), the reduced mass of the current masses of quarks forming meson. It is clear that under the given approach just constants \(a_j, b_j, B_j, \mu_0\) determine all properties of any meson (quarkonium), i. e., the approach directly appeals to quark and gluonic degrees of freedom as should be according to the first principles of QCD. Also it is clear that the constants mentioned should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. \[4, 5, 6, 7, 8, 9, 10, 11, 12, 13\] for both heavy quarkonia and the pseudoscalar meson nonet. It is clear, however, that baryons constitute no less physically important class of hadrons. But according to SQM (see, e. g., \[17\]) baryons are composed from three quarks, generally speaking, with different flavours. As a consequence, we are faced with a relativistic 3-body problem.

Under the circumstances the main aim of the present Chapter is to de-
velop a certain approach to baryons based on the above confinement mechanism. For the sake of simplicity we shall here restrict ourselves to the symmetrical baryons (i.e., composed from three quarks of the same flavours) \( \Delta^{++}, \Delta^{-}, \Omega^{-} \) inasmuch as in accordance with SQM \( \Delta^{++} = uuu, \Delta^{-} = ddd, \Omega^{-} = sss \).

Section 2 contains a survey of main relations underlying description of any mesons (quarkonia) in our approach (the 2-body problem). Section 3 takes into account the considerations of Section 2 when constructing possible wave functions of baryons in our approach. Section 4 uses the obtained baryonic wave functions to build masses, the electric form factors, the root-mean-square radii \( <R> \) and the anomalous magnetic moments of the symmetrical baryons under consideration in an explicit analytic form. Section 5 gives estimates for parameters of the confining SU(3)-gluonic field for baryons under discussion while Section 6 deals with discussion about chiral limit for baryons under studying. In Section 7 the problem of masses in particle physics is shortly discussed within the framework of approach to the chiral symmetry breaking in QCD proposed recently by the author. In Section 8 we make a few remarks about the so-called Yang-Mills Millennium problem and outline a possible way for proving it. Section 9 is devoted to a number of concluding remarks.

Appendices A and B contain the detailed description of main building blocks for meson wave functions in the approach under discussion, respectively: eigenspinors of the Euclidean Dirac operator on 2-sphere \( S^2 \) and radial parts for the modulo square integrable solutions of Dirac equation in the confining SU(3)-Yang-Mills field while Appendix C recalls a few facts about tensorial products of Hilbert spaces necessary to construct the baryonic wave functions. At last, Appendix D supplements Section 2 with a proof of the uniqueness theorem from that Section in the case of SU(3)-Yang-Mills equations.

Further we shall deal with the metric of the flat Minkowski spacetime \( M \) that we write down (using the ordinary set of local spherical coordinates \( r, \vartheta, \varphi \) for the spatial part) in the form

\[
d s^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2).
\]

(1)

Besides, we have \(|\delta| = |\det(g_{\mu\nu})| = (r^2 \sin \vartheta)^2 \) and \( 0 \leq r < \infty, 0 \leq \vartheta < \pi, 0 \leq \varphi < 2\pi \).

Throughout the Chapter we employ the Heaviside-Lorentz system of units with \( \hbar = c = 1 \), unless explicitly stated otherwise, so the gauge coupling constant \( g \) and the strong coupling constant \( \alpha_s \) are connected by the
relation \( \frac{g^2}{4\pi} = \alpha_s \). Further, we shall denote by \( L_2(F) \) the set of the modulo square integrable complex functions on any manifold \( F \) furnished with an integration measure, then \( L_2^n(F) \) will be the \( n \)-fold direct product of \( L_2(F) \) endowed with the obvious scalar product while \( \dagger \) and \( * \) stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac \( \gamma \)-matrices conforms to the so-called standard representation and is the same as in Ref. [8]. At last \( \otimes \) means tensorial product of matrices and \( I_n \) is the unit \( n \times n \) matrix so that, e.g., we have

\[
I_3 \otimes \gamma^\mu = \begin{pmatrix}
\gamma^\mu & 0 & 0 \\
0 & \gamma^\mu & 0 \\
0 & 0 & \gamma^\mu
\end{pmatrix}
\]

for any Dirac \( \gamma \)-matrix \( \gamma^\mu \) and so forth.

When calculating we apply the relations

\[
1 \text{ GeV}^{-1} \approx 0.1973269679 \text{ fm}, \quad 1 \text{ s}^{-1} \approx 0.658211915 \times 10^{-24} \text{ GeV}, \quad 1 \text{ V/m} \approx 0.2309956375 \times 10^{-23} \text{ GeV}^2, \\
1 \text{ T} = 4\pi \times 10^{-7} \text{H/m} \quad \text{v} \text{A/m} \approx 0.6925075988 \times 10^{-15} \text{ GeV}^2, \quad \mu_N = 3.152451236 \times 10^{-17} \text{ GeV}/\text{T} \approx 0.455226197 \times 10^{-4} \text{ GeV}^{-1}.
\]

Finally, for the necessary estimates we shall employ the \( T_{00} \)-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

\[
T_{\mu\nu} = - F^a_{\mu\alpha} F^{a\beta}_\nu g^{\alpha\beta} + \frac{1}{4} F^a_{\beta\gamma} F^{a\delta}_\alpha g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu}.
\]  

(2)

2 Survey of main relations for the 2-body problem

2.1 Preliminaries

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied at the whole length in Refs. [1, 2, 3]. Referring for more details to those references, let us briefly describe and specify only the relations necessary to us in the present necessary to us in the present necessary to us in the present Chapter.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the SU(3)-Yang-Mills equations for the gluonic field

\[
A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu \quad (\lambda_a \text{ are the known Gell-Mann matrices,} \\
\mu = t, r, \vartheta, \varphi, \ a = 1, \ldots, 8) \quad \text{and looks as follows}
\]

\[
A_{1t} \equiv A^3_t + \frac{1}{\sqrt{3}} A^8_t = - \frac{a_1}{r} + A_1,
\]
\[ A_{2t} \equiv -A_3^2 + \frac{1}{\sqrt{3}} A_8^8 = -\frac{a_2}{r} + A_2, \]
\[ A_{3t} \equiv -\frac{2}{\sqrt{3}} A_8^8 = \frac{a_1 + a_2}{r} - (A_1 + A_2), \]
\[ A_{1\phi} \equiv A_3^3 + \frac{1}{\sqrt{3}} A_8^8 = b_1 r + B_1, \]
\[ A_{2\phi} \equiv -A_3^3 + \frac{1}{\sqrt{3}} A_8^8 = b_2 r + B_2, \]
\[ A_{3\phi} \equiv -\frac{2}{\sqrt{3}} A_8^8 = -(b_1 + b_2)r - (B_1 + B_2) \quad (3) \]

with the real constants \( a_j, A_j, b_j, B_j \) parametrizing the family. The word unique should be understood in the strict mathematical sense. In fact in Ref. [2] the following theorem was proved (see also appendix D):

The unique exact spherically symmetric (nonperturbative) solutions (depending only on \( r \) and \( r^{-1} \)) of SU(3)-Yang-Mills equations in Minkowski spacetime consist of the family of (3).

Additionally we impose the Lorentz condition on the sought solutions. The latter condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e.g., Ref. [41]), so we should impose the given condition that can be written in the form \( \text{div}(A) = 0 \), where the divergence of the Lie algebra valued 1-form \( A = A_\mu dx^\mu = A_\mu^a \lambda_a dx^\mu \) is defined by the relation (see, e.g., Refs. [40])

\[ \text{div} A = *(d*A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu) \]

with the Hodge star operator * (see Appendix D). It should be emphasized that, from the physical point of view, the Lorentz condition reflects the fact of transversality for gluons that arise as quanta of SU(3)-Yang-Mills field when quantizing the latter (see, e.g., Ref. [41]).

It should be noted that solution (3) was found early in Ref. [1] but its uniqueness was proved just in Ref. [2] (see also Ref. [3]). Besides, in Ref. [2] (see also Ref. [8]) it was shown that the above unique confining solutions (3) satisfy the so-called Wilson confinement criterion [14]. Up to now nobody contested the above results so if we want to describe interaction between quarks by spherically symmetric SU(3)-fields then they can be only those from the above theorem. On the other hand, the desirability of spherically symmetric (colour) interaction between quarks at all distances naturally follows from analysing the \( p\bar{p} \)-collisions (see, e.g., Ref. [16]) where
one observes a Coulomb-like potential in events which can be identified with scattering quarks on each other, i.e., actually at small distances one observes the Coulomb-like part of solution (3). Under this situation, a natural assumption will be that the quark interaction remains spherically symmetric at large distances too but then, if trying to extend the Coulomb-like part to large distances in a spherically symmetric way, we shall inevitably come to the solution (3) in virtue of the above theorem.

Now one should say that the similar unique confining solutions exist for all semisimple and non-semisimple compact Lie groups, in particular, for \( \text{SU}(N) \) with \( N \geq 2 \) and \( \text{U}(N) \) with \( N \geq 1 \) \cite{2,3}. Explicit form of solutions, e.g., for \( \text{SU}(N) \) with \( N = 2, 4 \) can be found in Ref.\cite{3} but it should be emphasized that components linear in \( r \) always represent the magnetic (colour) field in all the mentioned solutions. Especially the case \( \text{U}(1) \)-group is interesting which corresponds to usual electrodynamics. Under this situation, as was pointed out in Refs.\cite{2,3}, there is an interesting possibility of indirect experimental verification of the confinement mechanism under discussion. Indeed the confining solutions of Maxwell equations for classical electrodynamics point out the confinement phase could be in electrodynamics as well. Though there exist no elementary charged particles generating a constant magnetic field linear in \( r \), the distance from particle, after all, if it could generate this electromagnetic field configuration in laboratory then one might study motion of the charged particles in that field. The confining properties of the mentioned field should be displayed at classical level too but the exact behaviour of particles in this field requires certain analysis of the corresponding classical equations of motion. Such a program has been recently realized in Ref.\cite{46}. Motion of a charged (classical) particle was studied in the field representing magnetic part of the mentioned solution of Maxwell equations and it was shown that one deals with the full classical confinement of the charged particle in such a field: under any initial conditions the particle motion is accomplished within a finite region of space so that the particle trajectory is near magnetic field lines while the latter are compact manifolds (circles). Those results might be useful in thermonuclear plasma physics (for more details see \cite{46}).

As has been repeatedly explained in Refs.\cite{2,3,4,5,6,8}, parameters \( A_{1,2} \) of solution (3) are inessential for physics in question and we can consider \( A_1 = A_2 = 0 \). Obviously we have \( \sum_{j=1}^{3} A_{jt} = \sum_{j=1}^{3} A_{j\phi} = 0 \) which reflects the fact that for any matrix \( T \) from \( \text{SU}(3) \)-Lie algebra we have \( \text{Tr} T = 0 \). Also, as has been repeatedly discussed by us earlier (see, e. g., Refs.\cite{2,3}), from the above form it is clear that the solution (3) is a configuration describing the electric Coulomb-like colour field (components \( A_{t}^{3,5,8} \)) and the
magnetic colour field linear in $r$ (components $A_\varphi^{3,8}$) and we wrote down the solution (3) in the combinations that are just needed further to insert into the Dirac equation (4).

For the sake of completeness one should note that the similar unique confining solutions exist for all semisimple and non-semisimple compact Lie groups, in particular, for SU($N$) with $N \geq 2$ and U($N$) with $N \geq 1$ [2, 3]. Explicit form of solutions, e.g., for SU($N$) with $N = 2, 4$ can be found in Ref. [3] but it should be emphasized that components linear in $r$ always represent the magnetic colour field in all the mentioned solutions.

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors $\Psi_j$ representing the $j$th colour component of the meson, so $\Psi$ may describe the relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac equation is written as follows

$$i\partial_t \Psi \equiv i \begin{pmatrix} \partial_t \Psi_1 \\ \partial_t \Psi_2 \\ \partial_t \Psi_3 \end{pmatrix} = H \Psi \equiv \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix},$$

(4)

where the Hamiltonian $H_j$ is

$$H_j = \gamma^0 \left[ \mu_0 - i\gamma^1 \partial_r - \frac{1}{r} \gamma^2 \gamma^1 \gamma^2 \right] - \frac{i\gamma^3}{r \sin \vartheta} \left( \partial_\varphi + \frac{1}{2} \sin \vartheta \gamma^1 \gamma^3 + \frac{1}{2} \cos \vartheta \gamma^2 \gamma^3 \right) - g \gamma^0 \left( \gamma^1 A_{j^1} + \gamma^3 \frac{1}{r \sin \vartheta} A_{j^2} \right)$$

(5)

with the gauge coupling constant $g$ while $\mu_0$ is a mass parameter and one should consider it to be the reduced mass which is equal to $m_q m_{q_2}/(m_q + m_{q_2})$ with the current quark masses $m_{q_k}$ (rest energies) of the corresponding quarks forming a meson (quarkonium).

Then the unique nonperturbative modulo square integrable solutions of (4) are at $j = 1, 2, 3$ (with Pauli matrix $\sigma_1$)

$$\Psi_j = e^{-i\omega_j t} \psi_j \equiv e^{-i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r) \Phi_j(\vartheta, \varphi) \\ F_{j2}(r) \sigma_1 \Phi_j(\vartheta, \varphi) \end{pmatrix}$$

(6)

9
with the 2D eigenspinor \( \Phi_j = \left( \Phi_{j1}, \Phi_{j2} \right) \) of the Euclidean Dirac operator \( D_0 \) on the unit sphere \( S^2 \), while the coordinate \( r \) stands for the distance between quarks. The explicit form of \( \Phi_j \) is discussed in Appendix A. We can call the quantity \( \omega_j \) the relative energy of the \( j \)th colour component of a meson (while \( \psi_j \) is the wave function of a stationary state for the \( j \)th colour component), but we can see that if we want to interpret (4) as an equation for eigenvalues of the relative motion energy, i.e., to rewrite it in the form \( H \psi = \omega \psi \) with different flavours then the energy spectrum of the meson will be given by \( \epsilon = m_{q_1} + m_{q_2} + \omega \) with the current quark masses \( m_{q_k} \) (rest energies) of the corresponding quarks. On the other hand for determination of \( \omega_j \) the following quadratic equation can be obtained \([1, 2, 3]\)

\[
\begin{align*}
[g^2 a_j^2 + (n_j + \alpha_j)^2] \omega_j^2 - 2(\lambda_j - gB_j)g^2 a_j b_j \omega_j + \\
[(\lambda_j - gB_j)^2 - (n_j + \alpha_j)^2 g^2 b_j^2 - \mu_0^2(n_j + \alpha_j)^2 = 0 ,
\end{align*}
\]

which yields

\[
\omega_j = \omega_j(n_j, l_j, \lambda_j) =
\]

\[
\frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + \Lambda_j^2) \mu_0^2 + g^2 b_j^2(n_j^2 + 2n_j \alpha_j)}}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2},
\]

\[
j = 1, 2, 3 ,
\]

where \( a_3 = -(a_1 + a_2), b_3 = -(b_1 + b_2), B_3 = -(B_1 + B_2), \Lambda_j = \lambda_j - gB_j, \alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}, n_j = 0, 1, 2, ..., \) while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on a unit sphere with \( l_j = 0, 1, 2, ... \). It should be noted that in the papers \([1, 2, 3, 4, 5, 6, 8]\) we used the ansatz (6) with the factor \( e^{i \omega_j t} \) instead of \( e^{-i \omega_j t} \) but then the Dirac equation (4) would look as \(-i \partial_t \Psi = H \Psi \) and in equation (7) the second summand would have the plus sign while the first summand in numerator of (8) would have the minus sign. In the papers \([9, 10]\) we returned to the conventional form of writing Dirac equation and this slightly modified the equations (7)–(8). In the given Chapter we conform to the same prescription as in Refs. \([3, 10]\).

In line with the above we should have \( \omega = \omega_1 = \omega_2 = \omega_3 \) in energy spectrum \( \epsilon = m_{q_1} + m_{q_2} + \omega \) for any meson (quarkonium) and this at once imposes two conditions on parameters \( a_j, b_j, B_j \) when choosing some experimental value for \( \epsilon \) at the given current quark masses \( m_{q_1}, m_{q_2} \).
The general form of the radial parts of (6) is considered in Appendix B. Within the given Chapter we need only the radial parts of (6) at $n_j = 0$ (the ground state) that are [see (B.5)]

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{g b_j}{\beta_j} \right), P_j = g b_j + \beta_j,$$

$$F_{j2} = i C_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{g b_j}{\beta_j} \right), Q_j = \mu_0 - \omega_j$$

with $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$, while $C_j$ is determined from the normalization condition $\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3}$. Consequently, we shall gain that $\Psi_j \in L^2_2(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum $\epsilon$.

### 2.2 Nonrelativistic and the weak coupling limits

It is useful to specify the nonrelativistic limit (when $c \to \infty$) for spectrum (8). For this one should replace $g \to g/\sqrt{\hbar c}$, $a_j \to a_j/\sqrt{\hbar c}$, $b_j \to b_j/\sqrt{\hbar c}$, $B_j \to B_j/\sqrt{\hbar c}$ and, expanding (8) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) = \pm \mu_0 c^2 \left[ 1 \mp \frac{g^2 a_j^2}{2\hbar^2(n_j + |\lambda_j|)^2} z^2 \right] + \frac{\lambda_j g^2 a_j b_j}{\hbar(n_j + |\lambda_j|)^2} \mp \mu_0 \frac{g^3 B_j a_j^2 f(n_j, \lambda_j)}{\hbar^3(n_j + |\lambda_j|)^7} z + O(z^2),$$

(10)

where $f(n_j, \lambda_j) = 4\lambda_j n_j (n_j^2 + \lambda_j^2) + \frac{\lambda_j}{\lambda_n} \left( n_n^4 + 6n_n^2 \lambda_n^2 + \lambda_n^4 \right)$.

As is seen from (10), at $c \to \infty$ the contribution of linear magnetic colour field (parameters $b_j, B_j$) to the spectrum really vanishes and the spectrum in essence becomes the purely nonrelativistic Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm \sqrt{\mu_0^2 + g^2 b_j^2}$. At last, one should specify the weak coupling limit of (8), i.e., the case $g \to 0$. As is not complicated to see from (8), $\omega_j \to \pm \mu_0$ when $g \to 0$. But then quantities $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} \to 0$ and wave functions of (9) cease to be the modulo square integrable ones at $g = 0$, i.e., they cease to describe relativistic bound states. Accordingly, this means that the equation (8) does not make physical meaning at $g = 0$.

We may seemingly use (8) with various combinations of signes ($\pm$) before the second summand in numerators of (8) but, due to (10), it is reasonable to
take all signs equal to plus which is our choice within the Chapter. Besides, as is not complicated to see, radial parts in the nonrelativistic limit have the behaviour of form $F_{j1}, F_{j2} \sim r^{l_j+1}$, which allows one to call quantum number $l_j$ angular momentum for the $j$th colour component though angular momentum is not conserved in the field (3) [1, 3]. So, for mesons under consideration we should put all $l_j = 0$.

2.3 Chiral limit

There is one more interesting limit for relation (8) – the chiral one, i.e., the situation when $m_{q1}, m_{q2} \to 0$ which entails $\mu_0 \to 0$ and (8) reduces to (at $j = 1, 2, 3$)

$$\\(\omega_j)_{\text{chiral}} = \frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j) g |b_j| \sqrt{n_j^2 + 2n_j \alpha_j}}{n_j^2 + 2n_j \alpha_j + \Lambda_j^2}, \quad (8')$$

which mathematically signifies that Dirac equation in the field (3) possesses a nontrivial spectrum of bound states even for massless fermions. Physically this gives us a possible approach to the problem of chiral symmetry breaking in QCD [12]: in chirally symmetric world masses of mesons are fully determined by the confining SU(3)-gluonic field between (massless) quarks and not equal to zero. Accordingly chiral symmetry is a sufficiently rough approximation holding true only when neglecting the mentioned SU(3)-gluonic field between quarks and no additional mechanism of the spontaneous chiral symmetry breaking connected to the so-called Goldstone bosons is required. As a result, e.g., masses of mesons from pseudoscalar nonet have a purely gluonic contribution and we have considered it in [12, 13] (see also sect. 6).

One can note that for to be the nonzero chiral limit of (8) the crucial role belongs to the colour magnetic field linear in $r$ [parameters $b_{1,2}$ from solution (3)] inasmuch as chiral limit is equal exactly to zero when $b_{1,2} = 0$. On the contrary, when parameters $a_{1,2}$ the Coulomb colour electric part of solution (3) are equal to zero the chiral limit may be nonzero at $b_{1,2} \neq 0$, as is seen from (8') except for the case $n_j = 0$ when both parts of SU(3)-gluonic field (3) are important for confinement and mass generation in chiral limit.

2.4 Eigenspinors with $\lambda = \pm 1$

Finally it should be noted that spectrum (8) is degenerated owing to the degeneracy of eigenvalues for the Euclidean Dirac operator $D_0$ on the unit sphere $S^2$. Namely, each eigenvalue of $D_0 \lambda = \pm (l + 1), l = 0, 1, 2..., \text{has}$
multiplicity $2(l+1)$, so we have $2(l+1)$ eigenspinors orthogonal to each other.

Ad referendum we need eigenspinors corresponding to $\lambda = \pm 1$ ($l = 0$) so here is their explicit form [see (A.16)]

$$\lambda = -1 : \Phi = \frac{C}{2} \left( e^{i\frac{\varphi}{2}} e^{-i\frac{\varphi}{2}} \right) e^{i\varphi/2},$$

or

$$\Phi = \frac{C}{2} \left( e^{i\frac{\varphi}{2}} e^{-i\frac{\varphi}{2}} \right) e^{-i\varphi/2},$$

$$\lambda = 1 : \Phi = \frac{C}{2} \left( e^{-i\frac{\varphi}{2}} e^{i\frac{\varphi}{2}} \right) e^{i\varphi/2},$$

or

$$\Phi = \frac{C}{2} \left( e^{-i\frac{\varphi}{2}} e^{i\frac{\varphi}{2}} \right) e^{-i\varphi/2}$$

(11)

with the coefficient $C = 1/\sqrt{2\pi}$ (for more details, see Appendix A).

2.5 Meson energy (mass)

Within the present Chapter we shall use relations (8) at $n_j = 0 = l_j$ so energy (mass) of a meson is given by $\mu = m_{q_1} + m_{q_2} + \omega$ with $\omega = \omega_j(0,0,\lambda_j)$ for any $j = 1,2,3$ whereas

$$\omega = \frac{g^2 a_1 b_1}{\Lambda_1} + \frac{\alpha_1 \mu_0}{|\Lambda_1|} = \frac{g^2 a_2 b_2}{\Lambda_2} + \frac{\alpha_2 \mu_0}{|\Lambda_2|}$$

$$= \frac{g^2 a_3 b_3}{\Lambda_3} + \frac{\alpha_3 \mu_0}{|\Lambda_3|} = \mu - m_{q_1} - m_{q_2}$$

(12)

and, as a consequence, the corresponding meson wave functions of (6) are represented by (9) and (11).

2.6 Choice of quark masses and the gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant $g$. We take the current quark masses used in [9, 10, 12] and they are $m_u = 2.25$ MeV, $m_d = 5$ MeV, $m_s = 107.5$ MeV. Under the circumstances, the reduced mass $\mu_0$ of (5) will be equal to $m_u/2$, $m_d/2$, $m_s/2$ accordingly. As to the gauge coupling constant $g = \sqrt{4\pi\alpha_s}$, it should be noted that recently some attempts have been made to generalize the standard formula for $\alpha_s = \alpha_s(Q^2) = 12\pi/[(33 - 2n_f)\ln(Q^2/\Lambda^2)]$ ($n_f$ is number of quark flavours) holding true.
at the momentum transfer $\sqrt{Q^2} \to \infty$ to the whole interval $0 \leq \sqrt{Q^2} \leq \infty$. We shall employ one such a generalization used in Refs. [18]. It is written as follows ($x = \sqrt{Q^2}$ in GeV)

$$\alpha(x) = \frac{12\pi}{(33 - 2n_f)} \frac{f_1(x)}{\ln \frac{x^2 + f_2(x)}{\Lambda^2}}$$

(13)

with

$$f_1(x) = 1 + \left( \left( \frac{(1 + x)(33 - 2n_f)}{12} \ln \frac{m^2}{\Lambda^2} - 1 \right)^{-1} + 0.6x^{1.3} \right)^{-1},$$

$$f_2(x) = m^2(1 + 2.8x^2)^{-2},$$

wherefrom one can conclude that $\alpha_s \to \pi = 3.1415...$ when $x \to 0$, i.e., $g \to 2\pi = 6.2831...$. We used (13) at $m = 1$ GeV, $\Lambda = 0.234$ GeV, $n_f = 3$, $x = 1232.0$ MeV or $x = 1672.45$ MeV to obtain $g \approx 3.367244706$ or $g \approx 2.841355055$ necessary for our further computations at the mass scale of $\Delta^{++}$, $\Delta^-$, or $\Omega^-$ respectively.

### 2.7 Electric form factor and the root-mean-square radius

For each meson (quarkonium) with the wave function $\Psi = (\Psi_j)$ of (6) we can define the electromagnetic current $J^\mu = \overline{\Psi}(I_3 \otimes \gamma^\mu)\Psi = (\Psi^\dagger \Psi, \Psi^\dagger(I_3 \otimes \alpha)\Psi) = (\rho, J)$, $\alpha = \gamma^0 \gamma$. The electric form factor $f(K)$ is the Fourier transform of $\rho$

$$f(K) = \int \Psi^\dagger \Psi e^{-iKr}d^3x = \sum_{j=1}^{3} \int \Psi_j^\dagger \Psi_j e^{-iKr}d^3x =$$

$$\sum_{j=1}^{3} f_j(K) = \sum_{j=1}^{3} \int (|F_{j1}|^2 + |F_{j2}|^2)\Phi_j^\dagger \Phi_j \frac{e^{-iKr}}{r^2} d^3x,$$

$$d^3x = r^2 \sin \vartheta dr d\vartheta d\varphi$$

(14)

with the momentum transfer $K$. At $n_j = 0 = l_j$, as is easily seen, for any spinor of (11) we have $\Phi_j^\dagger \Phi_j = 1/(4\pi)$, so the integrand in (14) does not depend on $\varphi$ and we can consider vector $K$ to be directed along z-axis.
Then \( K r = K r \cos \vartheta \) and with the help of (9) and relations (see Ref. [20]):

\[
\int_0^\infty r^{\alpha - 1} e^{-p r} dr = \Gamma(\alpha) p^{-\alpha}, \quad \text{Re} \alpha, p > 0, \quad \int_0^\infty r^{\alpha - 1} e^{-p r} \left( \frac{\sin (K r)}{\cos (K r)} \right) dr = \\
\Gamma(\alpha)(K^2 + p^2)^{-\alpha/2} \left( \frac{\sin (\arctan (K/p))}{\cos (\arctan (K/p))} \right), \\
\text{Re} \alpha > -1, \quad \text{Re} p > |\text{Im} K|, \quad \Gamma(\alpha + 1) = \alpha \Gamma(\alpha), \\
\int_0^\pi e^{-i K r \cos \vartheta} \sin \vartheta d\vartheta = 2 \sin (K r)/(K r),
\]

we shall obtain

\[
f(K) = \sum_{j=1}^3 f_j(K) = \sum_{j=1}^3 \frac{(2\beta_j)^{2\alpha_j + 1}}{6\alpha_j} \cdot \frac{\sin [2\alpha_j \arctan (K/(2\beta_j))]}{K(K^2 + 4\beta_j^2)^{\alpha_j}} \cdot \frac{1}{\left[ 3\Gamma(2\alpha_j + 1) \right]},
\]

wherefrom it is clear that \( f(K) \) is a function of \( K^2 \), as should be, and we can determine the root-mean-square radius of meson (quarkonium) in the form

\[
<r> = \sqrt{\sum_{j=1}^3 \left( \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2} \cdot \frac{K^2}{6} \right) + O(K^4)},
\]

When calculating (15) also the fact was used that by virtue of the normalization condition for wave functions we have \( C_j^2[P_j^2(1 - gb_j/\beta_j)^2 + Q_j^2(1 + gb_j/\beta_j)^2] = (2\beta_j)^{2\alpha_j + 1}/[3\Gamma(2\alpha_j + 1)] \).

It is clear, we can directly calculate \(<r>\) in accordance with the standard quantum mechanics rules as

\[
<r> = \sqrt{\int r^2 \Psi^\dagger \Psi d^3x} = \sqrt{\sum_{j=1}^3 \int r^2 \Psi_j^\dagger \Psi_j d^3x}
\]

and the result will be the same as in (16). So we should not call \(<r>\) of (16) the charge radius of meson (quarkonium) - it is just the radius of meson (quarkonium) determined by the wave functions of (6) (at \( n_j = 0 = l_j \)) with respect to strong interaction, i.e., radius of confinement. Now we should note the expression (15) to depend on 3-vector \( K \). To rewrite it in the form holding true for any 4-vector \( Q \), let us recall that according to general considerations (see, e.g., Ref. [21]) the relation (15) should correspond to the
so-called Breit frame where $Q^2 = -K^2$ [when fixing the metric by (1)] so it is not complicated to rewrite (15) for arbitrary $Q$ in the form

$$f(Q^2) = \sum_{j=1}^{3} f_j(Q^2) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin[2\alpha_j \arctan(\sqrt{Q^2/(2\beta_j)})]}{\sqrt{|Q^2|(4\beta_j^2 - Q^2)^{\alpha_j}}}$$

which passes on to (15) in the Breit frame.

### 2.8 Magnetic moment

We can define the volumetric magnetic moment density by

$$m = q(r \times J)/2 = q[(yJ_z - zJ_y)i + (zJ_x - xJ_z)j + (xJ_y - yJ_x)k]/2$$

with the meson charge $q$ and $J = \Psi^\dagger(I_3 \otimes \alpha)\Psi$. Using (6) we have in the explicit form

$$J_x = \sum_{j=1}^{3} (F^*_1F_{j2} + F^*_2F_{j1}) \frac{\Phi_j^\dagger \Phi_j}{r^2},$$

$$J_y = \sum_{j=1}^{3} (F^*_1F_{j2} - F^*_2F_{j1}) \frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2},$$

$$J_z = \sum_{j=1}^{3} (F^*_1F_{j2} - F^*_2F_{j1}) \frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2}$$

with Pauli matrices $\sigma_{1,2,3}$. Magnetic moment of meson (quarkonium) is

$$M = \int_V m d^3x,$$

where $V$ is the volume of the meson (quarkonium) (the ball of radius $< r >$). Then at $n_j = l_j = 0$, as is seen from (9), (11), $F^*_j = F_{j1}$, $F^*_j = -F_{j2}$, $\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j = 0$ for any spinor of (11) which entails $J_x = J_y = 0$, i.e., $m_z = 0$ while $\int_V m_{x,y} d^3x = 0$ because of turning the integral over $\varphi$ to zero, which is easy to check. As a result, the magnetic moments of mesons (quarkonia) with the wave functions of (6) (at $l_j = 0$) are equal to zero, as should be according to experimental data [17].

### 2.9 Magnetic form factor and anomalous magnetic moment

Following Refs. [9][10] we can, however, define magnetic form factor $F(K)$ of a meson if noticing that

$$\int_V \sqrt{m_x^2 + m_y^2 + m_z^2} d^3x \neq$$
\[
\sqrt{\left(\int_V m_x d^3x\right)^2 + \left(\int_V m_y d^3x\right)^2 + \left(\int_V m_z d^3x\right)^2} = M = 0.
\]

Under the circumstances we should define \( F(K) \) as the inverse Fourier transform of \( \sqrt{m_x^2 + m_y^2 + m_z^2} \)

\[
\sqrt{m_x^2 + m_y^2 + m_z^2} = \frac{q}{(2\pi)^3} \int F(K) e^{iK_r} d^3K,
\]

so that

\[
F(K) = \frac{1}{q} \int \sqrt{m_x^2 + m_y^2 + m_z^2} e^{-iK_r} d^3x =
\]

\[
\frac{|q|}{q} \int r |J_z| \sin \vartheta |e^{-iK_r} d^3x =
\]

\[
\frac{1}{2\pi q} \int \frac{|q|}{q} \sum_{j=1}^3 |F_{j1}| |F_{j2}| \frac{\sin^2 \vartheta}{r} e^{-iK_r} d^3x
\]

(19)

for any spinor of (11) so the integrand in (19) does not again depend on \( \varphi \).

Then \( K_r = Kr \cos \vartheta \) and using the relation

\[
\int_0^\pi e^{-iKr \cos \vartheta} \sin^3 \vartheta d\vartheta =
\]

\[4[\sin (Kr)/(Kr)^3 - \cos (Kr)/(Kr)^2],\]

we shall obtain

\[
F(K) =
\]

\[
2 \left| q \right| \int_0^\infty r \sum_{j=1}^3 C_j^2 P_j Q_j r^{2\alpha_j} e^{-2\beta_j r} \left( 1 - \frac{g^2 b^2}{4} \right) \times
\]

\[
\left[ \frac{\sin (Kr)}{(Kr)^3} - \frac{\cos (Kr)}{(Kr)^2} \right] dr
\]

\[
= \frac{1}{3} \left| q \right| \sum_{j=1}^3 (2\beta_j)^{2\alpha_j + 1} \frac{P_j Q_j}{\beta_j^2 + Q_j^2} \cdot \frac{1}{K^2(K^2 + 4\beta_j^2)^{\alpha_j}} \cdot
\]

\[
\left\{ \frac{\sin [(2\alpha_j - 1) \arctan (K/(2\beta_j))] [(2\alpha_j - 1) \arctan (K/(2\beta_j))] - \cos [2\alpha_j \arctan (K/(2\beta_j))]}{(2\alpha_j - 1)K(K^2 + 4\beta_j^2)^{-1/2}} \right\}
\]
with \( P_j = P_j(1 - g_1 \alpha_j) \), \( Q_j = Q_j(1 + g_1 \alpha_j) \). It is clear from (20) that \( F(K) \) is a function of \( K^2 \) and we can rewrite (20) for arbitrary 4-vector \( Q \) by the same manner as was done for electric form factor \( f(K) \) in (17).

Now we can define the anomalous magnetic moment for a meson by the relation
\[
\mu_a = F(K = 0) = \frac{2 |q|}{3 q} \sum_{j=1}^{3} \frac{2 \alpha_j + 1}{6 \beta_j} \cdot \frac{P_j Q_j}{P_j^2 + Q_j^2} \left( 1 - \frac{2 \alpha_j^2 + 5 \alpha_j + 3}{10 \beta_j^2} \cdot \frac{K^2}{6} \right) + O(K^4) \tag{20}
\]

as follows from (20). It should be noted that a possibility of existence of anomalous magnetic moments for mesons is permanently discussed in literature (see, e.g., Refs. [22, 23, 24] and references therein) though there is no experimental evidence in this direction [17]. Another matter is baryons with their nonzero anomalous magnetic moments and the latter are some essential characteristics of baryons. So that the relation (21) will be very useful to us below.

### 2.10 Remarks about the relativistic two-body problem

It is well known (see any textbook on quantum mechanics, e.g., Ref. [26]) that nonrelativistic two-body problem, when the particles interact with potential \( V(r_1, r_2) \) depending only on \( |r_2 - r_1| = r \), reduces to the motion of one particle with reduced mass \( m = m_1 m_2/(m_1 + m_2) \) in potential \( V(r) \), where \( r \) becomes the ordinary spherical coordinate from triplet \((r, \vartheta, \varphi)\). I.e., the bound states of two particles are the bound states of the particle with mass \( m \) in potential \( V(r) \) and they are the modulo square integrable solutions of the corresponding Schrödinger equation. Another matter is relativistic two-body problem. As we emphasized in Refs. [2, 3], up to now it has no single-valued statement. But if noting that the most fundamental results of nonrelativistic quantum mechanics (hydrogen atom and so on) are connected with potentials \( V(r) \) which are a part of the electromagnetic field \( A = (A_0, \mathbf{A}) \), i.e. \( V(r) = A_0, \mathbf{A} = 0 \) then one may propose
some formulation of the relativistic two-body problem. Really, now $V(r)$ additionally obeys the Maxwell equations and if we want to generalize the corresponding Schrödinger equation to include an interaction with arbitrary electromagnetic field for $A \neq 0$ then the answer is known: this is the Dirac equation with the replacement $\partial_\mu \rightarrow \partial_\mu - igA_\mu$, $g$ is a gauge coupling constant. Indeed, when going back in nonrelativistic limit at the light velocity $c \rightarrow \infty$ the magnetic field $A$ vanishes because, as is well known, in the world with $c = \infty$ there exist no magnetic fields (see any elementary textbook on physics). At the same time $V(r) = A_0$ does not vanish and remains the same as in nonrelativistic case and the Dirac equation turns into the Schrödinger equation (see Ref. [26]). But then we can see that $m$ in the above Dirac equation should consider the same reduced mass as before since in nonrelativistic limit we again should come to the standard formulation of two-body problem through effective particle with reduced mass $m$. So we can draw the conclusion that if an electromagnetic field is a combination of electric $V(r) = A_0$ field between two charged elementary particles and some magnetic field $A = A(r)$ (which may be generated by the particles themselves and also depends only on $r$, distance between particles) then there are certain grounds to consider the given (quantum) relativistic two-body problem to be equivalent to the one of motion for one particle with usual reduced mass in the mentioned electromagnetic field. As a result, we can use the Dirac equation for finding possible relativistic bound states for such a particle implying that this is really some description of the corresponding two-body problem. Under this situation we should remark the following.

1. Although for simplicity we talked about electromagnetic field but everything holds true for any Yang-Mills field, in particular, for SU(3)-gluonic field while the Maxwell equations are replaced by the Yang-Mills ones.

2. There arises the question: whether the Maxwell or Yang-Mills equations possess solutions with spherically symmetric $A_0(r), A(r)$? The answer is given by the uniqueness theorem (see subsection 2.1 and Appendix D).

3. The Dirac equation in such a field has the nonperturbative spectrum (8) and the latter should be treated as the nonperturbative interaction energy of two quarks and $r$ as the distance between quarks and so on (see Section 2). So eq. (4) should be understood just in this manner. In line with the above we should have $\omega = \omega_1 = \omega_2 = \omega_3$ (with $\omega_j$ of (8)) in energy spectrum $\epsilon = m_{q1} + m_{q2} + \omega$ for any meson (quarkonium) and this at once imposes two conditions on parameters $a_j, b_j, B_j$ of solution (3) when choosing some experimental value for $\epsilon$ at the given current quark masses $m_{q1}, m_{q2}$. At last, the Dirac equation in question is obtained from QCD lagrangian (with one flavor) if mass parameter in the latter is taken to be equal to the above
reduced mass. Therefore, we can say that meson wave functions (6) are the nonperturbative solutions of the Dirac-Yang-Mills system directly derived from QCD-lagrangian.

4. To summarize, there exist good physical and mathematical grounds for formulation of the above relativistic two-body problem that has a correct nonrelativistic limit and is Lorentz and gauge invariant. It is clear that all the above considerations can be justified only by comparison with experimental data but now we obtain some intelligible programme of further activity which has been partly realized in many our papers cited above.

Much of the above was discussed in Refs. [2, 3] but perhaps in other words.

3 Wave functions of baryons

3.1 Preliminaries

In accordance with SQM [17] baryons are composed from three quarks, generally speaking, with different flavours, so a baryon can be represented in the schematic form shown in Fig. 1, where \( r_i \) are distances between the corresponding quarks. As a consequence, we are faced with a relativistic 3-body problem. Under the circumstances the main task is to obtain an interaction Hamiltonian \( H \) of the 3-quark system if taking that every two quarks interact through the gluonic field described by solution (3), in general, with different constants \( a_j, b_j, B_j \) for each pair of quarks. Then energy levels of a baryon will be given by relation \( \epsilon = m_{q_1} + m_{q_2} + m_{q_3} + \omega \) with the current quark masses \( m_{q_k} \) (rest energies) of the quarks constituting the baryon while the interaction energy \( \omega \) should be subject to equation \( H \psi = \omega \psi \) with a baryonic wave function \( \psi \) describing relative motion (bound state) of three quarks.

To construct wave functions of baryons let us at first make a remark concerning the picture of interaction between two quarks which was obtained when applying the confinement mechanism in question to description of heavy quarkonia and mesons from pseudoscalar nonet [4, 5, 7, 8, 9, 10, 11, 12, 13].
3.2 Qualitative picture of quark interaction

Let us recall that, according to Refs. [3, 8], one can confront the field (3) with the $T_{00}$-component (the volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

$$T_{00} \equiv T_{tt} = \frac{E^2 + H^2}{2} =$$

$$\frac{1}{2} \left( \frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta} \right) \equiv \frac{A}{r^4} + \frac{B}{r^2 \sin^2 \vartheta}$$

with electric $E$ and magnetic $H$ colour field strengths and with real $A > 0$, $B > 0$. One can also introduce magnetic colour induction $B = (4\pi \times 10^{-7}\text{H/m})H$, where $H$ in A/m.

We should now note that one can calculate energy $E$ of gluon condensate conforming to solution (3) in a volume $V$ through relation $E = \int_V T_{00} r^2 \sin \vartheta drd\vartheta d\varphi$ with $T_{00}$ of (22) but one should take into account that classical $T_{00}$ has a singularity along $z$-axis ($\vartheta = 0, \pi$) and we have to introduce some angle $\vartheta_0$ so $\vartheta_0 \leq \vartheta \leq \pi - \vartheta_0$. As well as in Refs. [2, 8, 12], we may consider $\vartheta_0$ to be a parameter determining some cone $\vartheta = \vartheta_0$ so the quark emits gluons outside of the cone. Now if there are two quarks $Q_1, Q_2$ and each of them emits gluons outside of its own cone $\vartheta = \vartheta_1, 2$ (see Figs. 2, 3) then we have soft gluons (as mentioned in section 1) in regions I, II and between quarks.

Accordingly, we shall have some region $V$ with gluon condensate between quarks $Q_1, Q_2$ and its vertical projection is shown in Fig. 2. Another projection of $V$ onto a plane perpendicular to the one of Fig. 2 is sketched out in Fig. 3.

To clarify a physical meaning of the quantities $r_{1,2}$ in Figs. 2, 3, let us recall an analogy with classical electrodynamics where is well known (see, e.g., [25]) that the notion of classical electromagnetic field (a photon condensate) generated by a charged particle is applicable only at distances much greater than the Compton wavelength $\lambda_c = 1/m$ for the given particle.
with mass $m$. Within the QCD framework the parameter $\Lambda_{QCD}$ plays a similar part (see, e.g., Ref. [17]). Namely, the notion of classical SU(3)-gluonic field (a gluon condensate) is not applicable at the distances much less than $1/\Lambda_{QCD}$. In accordance with subsection 2.6 we took $\Lambda_{QCD} = \Lambda = 0.234$ GeV which entails $1/\Lambda \sim 0.8433$ fm so one may consider $r_{1,2} \sim 0.08$ fm.

If taking into account that only the values of $\vartheta_{1,2}, \varphi_{1,2}$ between 0 and $90^\circ$ are of physical meaning then it should be noted that the estimates of $\vartheta_{1,2}, \varphi_{1,2}$ obtained in Refs. [8, 12] show that $\vartheta_{1,2} \to \pi/2, \varphi_{1,2} \to 0$ with increasing the quark masses so gluon condensate (a classical SU(3)-gluonic field) is mainly concentrated near the axis connecting quarks. Under the circumstances we may represent a baryon composed from three pairs of the interacting quarks in the schematic form of Fig. 4, where the shaded areas schematically stand for soft gluons between quarks. As a consequence, we may suppose main interaction of quarks in a baryon to be concentrated along the axes joining quarks in each pair while one may practically neglect possible interaction of any pair of quarks as a whole with third remaining quark. When approaching all three quarks the latter interaction should not increase due to asymptotical freedom so the given picture may in fact hold true at the scales of baryon. In what follows we consider that is the case and then we can construct the baryonic wave functions in the next way.

### 3.3 Baryonic wave functions and the interaction Hamiltonian

In view of the above it is natural to characterize interaction of each pair of quarks in a baryon by the Hamiltonian of form (4)–(5) and, accordingly, relative motion of quarks in the pair by spherical coordinates $r_i, \vartheta_i, \varphi_i$, where $i = 1, 2, 3$ correspond to pairs $q_1q_2, q_1q_3, q_2q_3$ respectively (see Fig. 1, 4). Then, in accordance with (4)–(5) we can introduce the Hamiltonians $H^{(i)} = (H_j^{(i)})$ with colour index $j = 1, 2, 3$. In their turn, $H^{(i)}$ will depend on $a_j^{(i)}, b_j^{(i)}, B_j^{(i)}$, i.e., the conforming parameters of solution (3) describing interaction of the $i$th pair of quarks [cf. (5)] while $\mu_0$ will be the reduced mass of the corresponding quark pair. Each $H^{(i)}$ acts in Hilbert space $L^2_1(\mathbb{R}^3)$ which possesses the basis consisting of the eigenfunctions of $H^{(i)}$ and having the form of (6) (with the corresponding change of notations and omitting the phase factor $e^{-i\omega_j t}$). In what follows we denote such a basis by $\psi^{(i)} = (\psi_j^{(i)})$.
according to (6). In virtue of smallness of baryon sizes we may put the time variables \( t_i \) for each quark pair to be equal, i.e., \( t_1 = t_2 = t_3 \) and then the stationary states of a baryon can be chosen in the form of tensorial product of \( \psi^{(i)} \) (see Appendix C)

\[
\psi = \psi^{(1)} \otimes \psi^{(2)} \otimes \psi^{(3)} ,
\]

so \( \psi \) will be a basis in the Hilbert space \( \mathcal{H} = L^2_2(\mathbb{R}^3) \otimes L^2_2(\mathbb{R}^3) \otimes L^2_2(\mathbb{R}^3) \) and also the eigenfunctions for operator in \( \mathcal{H} \) (with identical operator \( I \))

\[
H = H^{(1)} \otimes I \otimes I + I \otimes H^{(2)} \otimes I + I \otimes I \otimes H^{(3)}
\]

with the eigenvalues \( \omega = \omega^{(1)}_j + \omega^{(2)}_j + \omega^{(3)}_j \), where \( \omega^{(i)}_j \) is an eigenvalue for operator \( H^{(i)}_j \) (see subsection 2.1 and Appendix C), i.e.,

\[
H\psi = H(\psi^{(1)} \otimes \psi^{(2)} \otimes \psi^{(3)}) = \omega \psi = (\omega^{(1)}_j + \omega^{(2)}_j + \omega^{(3)}_j)\psi .
\]

It is clear that operator \( H \) is just the sought interaction Hamiltonian for quarks in a baryon. As a result, the energy levels of the baryon will be given by relation \( \epsilon = m_{q_1} + m_{q_2} + m_{q_3} + \omega \) with the current quark masses \( m_{q_k} \) (rest energies) of the quarks constituting the baryon and the interaction energy \( \omega \) of (25).

Under the situation, the stay probability of a baryon in the volume element \( dV_1 dV_2 dV_3 \) of the configuration space with \( dV_i = r_i^2 \sin \vartheta_i dr_i d\vartheta_i d\varphi_i \) will be equal to

\[
dP = (\psi^{+(1)}\psi^{(1)})(\psi^{+(2)}\psi^{(2)})(\psi^{+(3)}\psi^{(3)})dV_1 dV_2 dV_3
\]

in accordance with (C.3), so

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dP = 1 .
\]

Having integrated \( dP \) over the angles \( \vartheta_i, \varphi_i \) (with taking into account the properties of the eigenspinors of the Euclidean Dirac operator on two-sphere \( S^2 \), see Appendix A), we obtain the probability of that the quarks in the baryon are at the distances \( r_i \) from each other (see Fig. 4)

\[
dW = \prod_{i=1}^{3} \left( \sum_{j=1}^{3} (|F^{(i)}_{j1}|^2 + |F^{(i)}_{j2}|^2) dr_i \right) .
\]
4 Main characteristics of symmetrical baryons

Obviously, we should choose a few quantities that are the most important from the physical point of view to characterize baryons under consideration and then we should evaluate the given quantities within the framework of our approach. In the circumstances let us settle on the ground state energy (mass), the root-mean-square radius and the anomalous magnetic moment. All three magnitudes are essentially nonperturbative ones, and can be calculated only by nonperturbative techniques.

4.1 Ground state energy (mass)

One should evidently take into account that for symmetrical baryons in question from physical point of view we have all the grounds of considering the interaction in pairs \(q_1 q_2, q_1 q_3, q_2 q_3\) (see Figs. 1, 4) to be the same because \(q_1 = q_2 = q_3\) and, as a result, the corresponding parameters \(a_j^{(i)}, b_j^{(i)}\) of the gluonic field from solution (3) describing interaction of the \(i\)th pair of quarks should be equal: \(a_j^{(i)} = a_j, b_j^{(i)} = b_j, B_j^{(i)} = B_j\). Then, in accordance with (12) and (25) we obtain the symmetrical baryon mass as

\[
\mu = 3(m_q + \omega), \quad \omega = \omega_j = \frac{g^2 a_j b_j}{\Lambda_j} + \frac{\alpha_j \mu_0}{|\Lambda_j|},
\]

or, in more explicit form,

\[
\omega = \frac{g^2 a_1 b_1}{\lambda_1 - g B_1} + \mu_0 \frac{\sqrt{(\lambda_1 - g B_1)^2 - g^2 a_1^2}}{|\lambda_1 - g B_1|} = \frac{g^2 a_2 b_2}{\lambda_2 - g B_2} + \mu_0 \frac{\sqrt{(\lambda_2 - g B_2)^2 - g^2 a_2^2}}{|\lambda_2 - g B_2|} = \frac{g^2 (a_1 + a_2)(b_1 + b_2)}{\lambda_3 + g(B_1 + B_2)} + \mu_0 \frac{\sqrt{\lambda_3 + g(B_1 + B_2)^2 - g^2(a_1 + a_2)^2}}{|\lambda_3 + g(B_1 + B_2)|} (30)
\]

with the reduced mass \(\mu_0 = m_q/2\) while in what follows we take the eigenvalues of the Euclidean Dirac operator on \(S^2\) \(\lambda_j = 1, j = 1, 2, 3\).

4.2 Baryon radius

We can note that the root-mean-square radius of meson from (16) can be defined by the relation

\[
<r>^2 = -6\frac{\partial^2 F}{\partial K^2}(K = 0)
\]

(31)
with form factor $F(K)$ of (15). Obviously for symmetrical baryons it is natural to put $<r_1>^2=<r_2>^2=<r_3>^2=r_0^2$, where $<r_i>^2$ conforms to the corresponding quark pair (see Figs. 1, 4). But then simple geometric considerations give rise to the baryon radius $< R >= r_0/\sqrt{3}$ which can be defined by the relation analogous to (31)

$$< R >^2 = -6 \frac{\partial^2 F_E}{\partial K^2} (K = 0)$$

if putting the baryon electric form factor $F_E(K) = \frac{1}{q} \sum_{j=1}^{3} F_i(K)$, where form factor $F_i(K)$ corresponds to $i$th quark pair (see Figs. 1, 4), which entails the expression

$$< R > = \frac{1}{\sqrt{3}} \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}$$

for symmetrical baryons.

### 4.3 Anomalous magnetic moment

Also it is natural to define the baryon magnetic form factor by the relation $F_M(K) = \frac{1}{3} \sum_{j=1}^{3} F_i(K)$, where form factor $F_i(K)$ of (20) corresponds to $i$th quark pair (see Figs. 1, 4), which gives rise to the expression for the baryon anomalous magnetic moment

$$\mu_a = F_M(K = 0) = \frac{2 |q|}{3} \sum_{j=1}^{3} \frac{2\alpha_j + 1}{6\beta_j} \cdot \frac{\mathcal{P}_j \mathcal{Q}_j}{\mathcal{P}_j^2 + \mathcal{Q}_j^2},$$

$$\mathcal{P}_j = P_j(1 - gb_j/\beta_j), \mathcal{Q}_j = Q_j(1 + gb_j/\beta_j)$$

in the case of symmetrical baryons while $q$ is electric charge of $i$th quark pair.

### 5 Numerical estimates

#### 5.1 $\Delta^{++}$

At computation we took mass $\mu = 1232.0$ MeV [17] while the anomalous magnetic moment is experimentally restricted within limits $3.7 \leq \mu_a \leq 7.5$ [17] in units of nuclear magneton $\mu_N$, so under calculation we took $\mu_a =$
Table 1: Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for $\Delta^{++}$

| $g$   | $\mu_0$ (MeV) | $a_1$       | $a_2$       | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$   | $B_2$   |
|-------|---------------|-------------|-------------|-------------|-------------|---------|---------|
| 3.36724 | 1.12500       | 0.550635    | -0.764655   | -0.175434   | -0.170501   | 1.09500 | -0.7800 |
| 3.36724 | 1.12500       | 0.404093    | 0.203740    | -0.306289   | 0.514750    | 1.3200  | -0.5700 |
| 3.36724 | 1.12500       | 0.432052    | -0.136707   | 0.318374    | -0.780394   | -0.8400 | -0.58500|

$(3.7 + 7.5)/2 = 5.6$. As for the radius $< R >$, no experimental values of it exist [17] but if taking into account that in the case of proton with mass 938 MeV we have $< R > \approx 0.875$ fm [17], then when considering the symmetrical baryons to be more compact than proton we should put, for example, $0.3$ fm $\leq < R > \leq 0.7$ fm to make computation more sensible. Some numerical results are adduced in Tables 1 and 2 and each line of Table 2 corresponds to the conforming one of Table 1 while the values of $\mu_a$ are expressed in units of $\mu_N$. 

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Table 2: Theoretical and experimental mass, radius and anomalous magnetic moment for $\Delta^{++}$

| Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $< R >$ (fm) | Experim. $< R >$ (fm) | Theoret. $\mu_a$ | Experim. $\mu_a$ |
|----------------------|----------------------|----------------------|----------------------|----------------|----------------|
| 1232.0               | $\mu = 3(m_u + \omega_j(0, 0, 1)) = 1232.0$ | 0.696633             | -                    | 5.37931        | 5.6            |
| 1232.0               | $\mu = 3(m_u + \omega_j(0, 0, 1)) = 1232.0$ | 0.516086             | -                    | 5.77838        | 5.6            |
|                      | $\mu = 3(m_d + \omega_j(0, 0, 1)) = 1232.0$ | 0.356102             | -                    | 6.08430        | 5.6            |

Table 3: Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for $\Delta^-$

| $g$ | $\mu_0$ (MeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|-----|---------------|-------|-------|-------------|-------------|-------|-------|
| 3.36724 | 2.500 | -0.708466 | 0.160912 | -0.187350 | 0.332727 | -0.8100 | -0.1500 |
| 3.36724 | 2.500 | 0.343724  | 0.192053 | -0.178990 | 0.362950 | 0.8100  | -0.28500 |
| 3.36724 | 2.500 | 0.205870  | -0.372948 | -0.254934 | -0.215958 | 0.73500 | -0.37500 |

5.2 $\Delta^-$

We here used mass $\mu = 1232.0$ MeV [17] but, in virtue of absence of the generally accepted experimental value of $\mu_a$ [17] the value $\mu_a = -3.68$ from [28] was accepted. A number of numerical results is given by Tables 3 and 4.

Table 4: Theoretical and experimental mass, radius and anomalous magnetic moment for $\Delta^-$

| Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $< R >$ (fm) | Experim. $< R >$ (fm) | Theoret. $\mu_a$ | Experim. $\mu_a$ |
|----------------------|----------------------|----------------------|----------------------|----------------|----------------|
| 1232.0               | $\mu = 3(m_d + \omega_j(0, 0, 1)) = 1232.0$ | 0.698664             | -                    | -3.68497       | -3.68         |
| 1232.0               | $\mu = 3(m_d + \omega_j(0, 0, 1)) = 1232.0$ | 0.519021             | -                    | -3.95427       | -3.68         |
| 1232.0               | $\mu = 3(m_d + \omega_j(0, 0, 1)) = 1232.0$ | 0.358006             | -                    | -3.69648       | -3.68         |
Table 5: Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for $\Omega^-$

| $g$      | $\mu_0$ (MeV) | $a_1$   | $a_2$   | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$  | $B_2$  |
|----------|---------------|---------|---------|-------------|-------------|--------|--------|
| 2.841363 | 53.7500       | -0.753910 | 0.628741 | -0.311036   | -0.250568   | -1.30500 | 1.45500 |
| 2.841363 | 53.7500       | 0.231041  | 0.207007 | -0.194473   | 0.432236    | 0.6600  | -0.28500 |
| 2.841363 | 53.7500       | -0.406736 | 0.369405 | -0.560202   | -0.307761   | -1.27500 | 1.15500 |

5.3 $\Omega^-$

In this case the generally accepted values $\mu = 1672.45$ MeV and $\mu_a = -2.02$ [17] were employed. Tables 5 and 6 contain some numerical results.
Table 6: Theoretical and experimental mass, radius and anomalous magnetic moment for $\Omega^-$

| Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $<R>$ (fm) | Experim. $<R>$ (fm) | Theoret. $\mu_a$ | Experim. $\mu_a$ |
|----------------------|----------------------|----------------------|----------------------|-----------------|------------------|
| 1672.45              | $\mu = 3(m_s + \omega_j(0,0,1)) = 1672.43$ | 0.586809             | -                    | -1.82490        | -2.02            |
| 1672.45              | $\mu = 3(m_s + \omega_j(0,0,1)) = 1672.43$ | 0.430573             | -                    | -2.07908        | -2.02            |
| 1672.45              | $\mu = 3(m_s + \omega_j(0,0,1)) = 1672.43$ | 0.332305             | -                    | -2.02136        | -2.02            |

6 Chiral limit and concluding remarks

6.1 Chiral limit

Having obtained estimates for symmetrical baryons we can address to the chiral symmetry breaking problem in QCD whose possible resolution within the framework of the above confinement mechanism has been discussed in [12]. As was mentioned in subsection 2.3, merits of case consists in that the Dirac equation in the field (3) possesses a nontrivial spectrum of bound states even for massless fermions [see relation (8')]. As a result, mass of any meson remains nonzero in chiral limit when masses of quarks $m_q \to 0$ and meson masses will be expressed only through the parameters of the confining SU(3)-gluonic field of (3). This purely gluonic residual mass of meson should be interpreted as a gluonic contribution to the meson mass.

Physically this gives us a possible approach to the problem of chiral symmetry breaking in QCD [12]: in chirally symmetric world masses of mesons are fully determined by the confining SU(3)-gluonic field between (massless) quarks and not equal to zero. Accordingly chiral symmetry is a sufficiently rough approximation holding true only when neglecting the mentioned SU(3)-gluonic field between quarks and no additional mechanism of the spontaneous chiral symmetry breaking connected to the so-called Goldstone bosons is required. Referring for more details to [12], we can here only note that masses of symmetrical baryons have also a purely gluonic contribution and we may be interested in what part of their masses is obligatory to that contribution.

Indeed, in chiral limit $m_{q_1}, m_{q_2}, m_{q_3} \to 0$ we, according to (30), obtain $\mu_{chiral} = 3\omega$ with

$$\omega \approx \frac{g^2a_1b_1}{\lambda_1 - gB_1} \approx \frac{g^2a_2b_2}{\lambda_2 - gB_2} \approx \frac{g^2(a_1 + a_2)(b_1 + b_2)}{\lambda_3 + g(B_1 + B_2)} \neq 0.$$  (35)
We can see that in chiral limit the baryon masses are completely determined only by the parameters \( a_j, b_j, B_j \) of SU(3)-gluonic field among quarks, i.e. by interaction among quarks, and those masses have the purely gluonic nature. Accordingly, one can use the parameters \( g, a_j, b_j, B_j \) adduced in Tables 1, 3, 5 to compute \((\mu)_{\text{chiral}}\) and then find the ratio \( gl = (\mu)_{\text{chiral}}/\mu \) which in fact represents the sought gluonic contribution to the baryon masses. Then, for example, for the data of first lines of Tables 1, 3, 5 one can easy obtain 
\[
\mu \approx 99.254 \%, 98.315 \%, 72.131 \%,
\]
respectively for \( \Delta^{++}, \Delta^{-}, \Omega^{-} \).

Similar numbers hold true also for the other data from Tables 1, 3, 5.

So, even in chirally symmetric world, e.g., the symmetric baryons would have nonzero masses that would be determined only by SU(3)-gluonic interaction among massless quarks, i.e. those masses would have a purely gluonic nature. Moreover, since gluons are verily relativistic particles then the most part of masses for baryons under discussion is conditioned by relativistic effects, as is seen from the above estimates. Further discussion of the proposed chiral symmetry breaking mechanism can be found in [12, 13].

6.2 Concluding remarks

It should be noted that modern baryon spectroscopy in theoretical aspects can not be considered satisfactory enough: the main theoretical tools for describing dynamics of quarks in baryons here are miscellaneous potential models (see, e.g., early and recent reviews [29, 30]). Quark interaction at such a description is modelled mainly by either oscillator potential or the confining potential of form \( a/r + br \). But, as follows from the uniqueness theorem of Section 2 (see also Appendix D), such potentials cannot describe any gluon configuration between quarks. It would be possible if the mentioned potentials were solutions of Yang-Mills equations directly derived from QCD-Lagrangian since, from the QCD-point of view, any gluonic field should be a solution of Yang-Mills equations (as well as any electromagnetic field is by definition always a solution of Maxwell equations). According to the above uniqueness theorem neither oscillator potential nor the confining potential of form \( a/r + br \) are not solutions of Yang-Mills equations. In particular, Coulomb and linear parts belong to different parts of the YM-potentials (see (3)), accordingly, to the colour electric and colour magnetic parts so they cannot be united in one component of form \( a/r + br \). So, we draw the conclusion (mentioned as far back as in Refs. [4, 5] and elaborated more in detail in Ref. [7]) that the potential approach seems to be inconsistent.

Another matter is our confinement mechanism proposed in [1, 2, 3] which
gives new possibilities from the first principles of QCD immediately appealing to the quark and gluonic degrees of freedom. Indeed, the words quark and gluonic degrees of freedom make exact sense: gluons come forward in the form of bosonic condensate described by parameters $a_j$, $b_j$, $B_j$ from the unique exact solution (3) of the Yang-Mills equations while quarks are represented by their current masses $m_q$.

Consequently, the given mechanism is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system directly derived from QCD-Lagrangian and, as a result, the approach is itself nonperturbative, relativistic from the outset, admits self-consistent nonrelativistic limit and may be employed for any meson (quarkonium). Applications of the approach in Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] allow one to speak about the fact that the confinement mechanism elaborated in [1, 2, 3] gives new possibilities for considering many old problems of hadronic (meson) physics (such as nonperturbative computation of decay constants, masses and radii of mesons, chiral symmetry breaking, the origin of the so-called scale $\Lambda_{QCD}$ [47] and so forth) from the first principles of QCD immediately appealing to the quark and gluonic degrees of freedom.

At the same time, as the results of the present Chapter show, this mechanism may be extended over baryons in a reasonable way. We hope further study of baryons with the help of the given approach to shed new light on the internal structure of baryons.

7 Problem of masses in particle physics

7.1 Preliminaries

As is known [17], the generally accepted standard model with one Higgs doublet asserts that the masses of fundamental fermions (quarks and leptons) are acquired through the Higgs mechanism so for their masses $m_i$ we obtain (without taking mixings into account) $m_i = f_i v / \sqrt{2}$, where the vacuum Higgs condensate $v \approx 246$ GeV and $i$ stands for quark and lepton flavours. But little is known about the coupling constants $f_i$ and much may be elucidated only with discovering Higgs bosons. The same holds true for the gauge bosons $W^\pm, Z$ where masses $m_W = ev/(2 \sin \theta_W), m_Z = ev/(\sin 2 \theta_W)$ with the so-called weak angle $\theta_W$ so that $\sin^2 \theta_W \approx 0.23$ and $e$ is the elementary electric charge. If taking into account that the mass of Higgs boson $m_H = \lambda v$ with a self-interaction constant $\lambda$ then it is clear that masses of all the abovementioned particles are proportional to $m_H$, and, consequently, the discovery of Higgs boson will not completely resolve the puzzle of origin.
of masses in particle physics – the question will remain where the mass $m_H$ comes from not speaking already about the nature of the above miscellaneous constants $f_i$ and $\lambda$.

At present, to our mind, one can single out two most promising approaches to a possible resolution for the mentioned problems: technicolour theories and preon models. Under the circumstances let us shortly outline how both these directions might be estimated from the point of view of our confinement mechanism and the chiral symmetry breaking one based on the latter and discussed above and in [12].

### 7.2 Technicolour theories

Referring for more details concerning those models to both early references [31, 32, 33] and modern status of them (see, e.g., [34]) let us note the following. The main idea of acquiring masses, e.g., for $W^\pm$ and $Z$ bosons, consists in that a new set of the so-called techniquarks is postulated at the energy scale of order 1 TeV which interact with each other through the technigluons and it makes the massless techipions exist as Goldstone bosons. The latter give masses to $W^\pm$ and $Z$ after spontaneous symmetry breaking. It should be noted, however, those massless technipions appear as a result of violating chiral symmetry connected with technicolour QCD on the analogy with chiral symmetry breaking in usual QCD. But, as we have discussed in [12] and in Section 6, the hypothetical mechanism for chiral symmetry breaking in standard QCD with appearance of Goldstone bosons (pions) seems to fail because of pions can never be massless inasmuch as they have nonzero masses even in chirally symmetric world due to gluons. The same will also perfectly hold true for technipions which would always have nonzero masses due to technigluons since technicolour QCD should manifest the confinement mechanism similar to our one in usual QCD. Therefore, technicolour theories look rather doubtful from the point of view of our confinement mechanism.

### 7.3 Preon models

Another cardinal approach to the problem of masses is connected with the preon models (see, e.g., [35] and references therein). Under this approach quarks, leptons and gauge vector bosons are suggested to be composed of stable spin-1/2 preons, for example, existing in three flavours and being combined according to simple rules. The main theoretical objection to preon theories is the mass paradox which arises by virtue of the Heisenberg’s uncertainty principle. Scattering experiments have shown [16] that quarks and
leptons are point-like up to the scales of order $10^{-3}$ fm which corresponds to a preon mass of order 197 GeV (due to the uncertainty principle) if the preon is confined to a box of such a size, i.e. its mass will approximately be $0.4 \times 10^5$ times greater than, e.g., that of $d$-quark. Thus, the preon models are faced with a mass paradox: how could quarks or electrons be made of smaller particles that would have masses of many orders of magnitude greater than the fundamental fermion masses? The paradox might be resolved by the rather dubious postulate about a large binding force between preons cancelling their mass-energies. Our confinement mechanism points out the more physically acceptable way of overcoming these obstacles. If the interaction among preons is described by a QCD-like theory based on, e.g., SU(N)-group with $N \geq 2$ then, according to our results [2,3], such theories should also manifest confinement to generate masses described by relations similar to (8) and (8'). This signifies that preons might possess small masses or be just massless and, as a result, mass paradox would be removed.

Among fundamental fermions one should pay special attention to electron. Within the framework of preon models the most natural composition for it should be the one of three preons [35], so electron might be described by triplet $(\beta, \beta, \delta)$, where preons $\beta$ and $\delta$ carry electric charges $-2e/3, e/3$, respectively, with elementary charge $e$ [35]. If supposing that interaction among them is described by the gauge group SU(3) through pre-gluons [35] then we formally come to the same scheme which was under exploration for baryons in the given chapter. Under the circumstances we can find the parameters in solution (3) which will correspond to the observed mass, some radius (of order of the classical electron radius $r_e \sim 10^{15}$ m) and (anomalous) magnetic moment for electron. In those calculations one can consider the mentioned preons to be massless, as is said above. We hope to study this interesting task elsewhere.

8 Remarks about the Yang-Mills Millennium problem

Our approach to the quark confinement mechanism is closely connected with the so-called Yang-Mills Millennium problem (see [36] for more details). Its formulation looks as follows:

**Yang-Mills Existence and Mass Gap:** Prove that for any compact simple gauge group $G$, quantum Yang-Mills theory on $\mathbb{R}^4$ exists and has a mass gap $\Delta > 0$. 
It is not complicated to see that our results on the confinement mechanism [in particular, the solution (3)] to a certain degree clear the way to obtain a proof of the above problem. So long as group SU(3) belongs to the class of groups $G$ from the formulation of problem then let us demonstrate possible opportunities with an example of the SU(3) gauge group and then make some remarks concerning the common case.

Let us note the following. From mathematical point of view the Yang-Mills field are the connections in vector (or, which is equivalent, in principal) bundles over some manifold $M$. When $M$ is the Minkowski space the situation is simplified so long as all the bundles over $M$ with topology $\mathbb{R}^4$ are trivial. When quantising Yang-Mills field we obtain, from physical point of view, quanta of that field, e.g., photons or gluons, which realize an interaction among the corresponding (matter) particles able to interchange by those quanta. But from mathematical point of view the mentioned particles are described by the cross-sections of vector bundles over the given space-time, in particular, over $\mathbb{R}^4$. As a result, mathematically, connections do not live separately from cross-sections while, physically, the sources of any quanta are the matter particles.

Under the circumstances when discussing the above millennium problem we should take into account also the matter fields rather than restrict ourselves to the pure Yang-Mills theory and further we should pass on to the pure Yang-Mills case by some limiting procedure. Then, returning to group SU(3), we have seen that there exists the exact solution (3) of the Yang-Mills equations which is unique in a certain sense (see Appendix D) and spectrum of bound states in such a field is given by the relation (8). The latter formula cannot be obtained without including spinor fields as cross-sections of the corresponding bundle into consideration.

We now especially are interested in the limiting case of (8), namely in the relation (8'). In it all the information about matter (spinor) fields vanished since that information was related only with mass parameters in the conforming Dirac equation. As a consequence, the relation (8') should be interpreted as the pure gluonic energy. Obviously, the latter only depends on quantities $a_j$, $b_j$, $B_j$ created by group SU(3) and is quantized in virtue of availability of the nonnegative integer numbers $n_j$. Also from considerations of Section 2 it is clear that the next relation should be fulfilled

\[ (\omega_1)_{\text{chiral}} = (\omega_2)_{\text{chiral}} = (\omega_3)_{\text{chiral}}, \]  

which gives an additional number of restrictions on the parameters $a_j$, $b_j$, $B_j$. Further we can see that there is a mass gap $\Delta$ defined at $n_j = 0$, $l_j = 0$
from the relation
\[ \Delta = \frac{a_1 b_1}{1 - g B_1} = \frac{a_2 b_2}{1 - g B_2} = \frac{(a_1 + a_2)(b_1 + b_2)}{1 + g(B_1 + B_2)}. \] (37)

Of course, \( \Delta \) can be positive or negative depending from a choice of parameters \( a_j, b_j \), but parameters \( a_j, b_j \) should not be equal to 0. I. e., we obtained a quantum theory for SU(3)-group asserting that any reasonable energy connected with its quanta (gluons) is quantized and has a mass gap. As an application of those properties we have already discussed the chiral symmetry breaking mechanism in Sections 2 and 6 and in Refs. [12, 13].

Passing on to the general case, we can note that (see Appendix D) the obtained results may be extended over all semisimple compact Lie groups since for them the corresponding Lie algebras possess just the only Cartan subalgebra. Also we can talk about the compact non-semisimple groups, for example, U(\( N \)). In the latter case additionally to Cartan subalgebra we have centrum consisting from the matrices of the form \( \alpha I_N \) (\( I_N \) is the unit matrix \( N \times N \)) with arbitrary constant \( \alpha \). For all those groups there exists their own uniqueness theorem, as follows from considerations in Appendix D, one can evaluate spectrum of bound states (examples for groups SU(2) and SU(4) can be found in [3]) and chiral limit that entails results about quantizing energy of the corresponding quanta and gives a mass gap for all those groups. To get the formulas in an explicit form one should take a concrete realization of the conforming Lie algebras.

The most relevant physical cases are of course U(1)- and SU(3)-ones (QED and QCD). In particular, the U(1)-case allows us to build the classical model of confinement (see Section 2 and Ref. [46]).

9 Conclusion

The main idea of quark confinement may be borrowed from classical electrodynamics. Indeed, let us recall the well-known case of motion of a charged particle in the homogeneous magnetic field (see, e.g., Ref. [25]). In the latter case the particle moves along helical curve with lead of helix \( h = 2\pi mv \cos \alpha/(q H \sqrt{1 - v^2}) \) and radius \( R = mv \sin \alpha/(q B \sqrt{1 - v^2}) \), where \( \alpha \) is an angle between vectors of the particle velocity \( \mathbf{v} \) and magnetic induction \( \mathbf{B} \), \( q \) is particle charge, \( m \) is particle mass. As a consequence, the homogeneous magnetic field does not give rise to the full confinement of the particle since the latter may go to infinity along the helical curve. The situation is not changed at quantum level as well: there exist no bound
states in the homogeneous magnetic field [27]. But if estimating module of $B$ at $R \sim 10^{-15} \text{m} = 1 \text{fm}$ for electron then $B$ will be of order $10^{23} \text{T}$. I. e., if considering that quarks are confined by a colour magnetic field that they themselves create then one needs a colour magnetic field between quarks with $B$ of such an order and that field should not allow to quark to go to infinity. It is clear this field should be a solution of the Yang-Mills equations. Really, if speaking about Minkowski spacetime then searching for classical solutions of the Yang-Mills equations makes sense because at large distances between quarks the latter are surrounded with a huge number of gluons that are emitted by both quarks and gluons themselves. Under this situation it is quite plausible that confinement of quarks arises due to certain properties of such gluonic clouds while the latter should be described just by classical solutions of the Yang-Mills equations.

As was shown in Refs. [1, 2, 3], the necessary solution has the form (3) for group SU(3) and is unique in a certain sense. This result allowed us to propose a quark confinement mechanism which was successfully applied to meson spectroscopy. Indeed, as follows from (16) at $|b_j| \to \infty$ we have

$$< r > \sim \sqrt{\sum_{j=1}^{3} \frac{1}{(g|b_j|)^r}}$$

so in the strong magnetic colour field when $|b_j| \to \infty$,

$$< r > \to 0,$$

while the meson wave functions of (6) and (9) behave as $\Psi_j \sim e^{-g|b_j|r}$, i. e., just the magnetic colour field of (3) provides two quarks with confinement. This situation also holds true at classical level [46].

The given Chapter extends this mechanism over baryons and further study of that approach will be connected with analysis of concrete baryons with its help.

10 Appendix A

We here represent some results about eigenspinors of the Euclidean Dirac operator on two-sphere $S^2$ employed in the main part of the Chapter.

When separating variables in the Dirac equation (4) there naturally arises the Euclidean Dirac operator $D_0$ on the unit two-dimensional sphere $S^2$ and we should know its eigenvalues with the corresponding eigenspinors. Such a problem also arises in the black hole theory while describing the so-called twisted spinors on Schwarzschild and Reissner-Nordström black holes and it was analysed in Refs. [3, 37, 38], so we can use the results obtained therein for our aims. Let us adduce the necessary relations.
The eigenvalue equation for corresponding spinors $\Phi$ may look as follows

$$
\mathcal{D}_0 \Phi = \lambda \Phi. \tag{A.1}
$$

As was discussed in Refs. [37, 38] the natural form of $\mathcal{D}_0$ (arising within applications) in local coordinates $\vartheta, \varphi$ on the unit sphere $S^2$ looks as

$$
\mathcal{D}_0 = -i\sigma_1 \left[ i\sigma_2 \partial_\vartheta + i\sigma_3 \frac{1}{\sin \vartheta} \left( \partial_{\varphi} - \frac{1}{2} \sigma_2 \sigma_3 \cos \vartheta \right) \right] = \sigma_1 \sigma_2 \partial_\vartheta + \frac{1}{\sin \vartheta} \sigma_1 \sigma_3 \partial_{\varphi} + \frac{\cot \vartheta}{2} \sigma_1 \sigma_2 \tag{A.2}
$$

with the ordinary Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

so that $\sigma_1 \mathcal{D}_0 = -\mathcal{D}_0 \sigma_1$.

The equation (A.1) was explored in Refs. [37, 38]. Spectrum of $\mathcal{D}_0$ consists of the numbers $\lambda = \pm (l + 1)$ with multiplicity $2(l + 1)$ of each one, where $l = 0, 1, 2, \ldots$. Let us introduce the number $m$ such that $-l \leq m \leq l + 1$ and the corresponding number $m' = m - 1/2$ so $|m'| \leq l + 1/2$. Then the conforming eigenspinors of operator $\mathcal{D}_0$ are

$$
\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \Phi_{\mp \lambda} = \frac{C}{2} \left( \frac{P^k_{m' - 1/2} \pm P^k_{m'1/2}}{P^k_{m' - 1/2} \mp P^k_{m'1/2}} \right) e^{-im'\varphi} \tag{A.3}
$$

with the coefficient $C = \sqrt{\frac{l + 1}{2\pi}}$ and $k = l + 1/2$. These spinors form an orthonormal basis in $L^2(S^2)$ and are subject to the normalization condition

$$
\int_{S^2} \Phi^* \Phi d\Omega = \int_0^\pi \int_0^{2\pi} (|\Phi_1|^2 + |\Phi_2|^2) \sin \vartheta d\vartheta d\varphi = 1. \tag{A.4}
$$

Further, owing to the relation $\sigma_1 \mathcal{D}_0 = -\mathcal{D}_0 \sigma_1$ we, obviously, have

$$
\sigma_1 \Phi_{\mp \lambda} = \Phi_{\pm \lambda}. \tag{A.5}
$$

As to functions $P^k_{m'n'}(\cos \vartheta) \equiv P^k_{m',n'}(\cos \vartheta)$ then they can be chosen by miscellaneous ways, for instance, as follows (see, e. g., Ref. [39])

$$
P^k_{m'n'}(\cos \vartheta) = \ldots
$$
with the orthogonality relation at \( m', n' \) fixed

\[
\int_0^\pi P^*_{m'n'}(\cos \vartheta) P^{k'}_{m'n'}(\cos \vartheta) \sin \vartheta d\vartheta = \frac{2}{2k+1} \delta_{kk'}.
\]

It should be noted that square of \( D_0 \) is

\[
D_0^2 = -\Delta_{S^2} I_2 + \sigma_2 \sigma_3 \frac{\cos \vartheta}{\sin^2 \vartheta} \partial_\varphi + \frac{1}{4 \sin^2 \vartheta} + \frac{1}{4},
\]

while laplacian on the unit sphere is

\[
\Delta_{S^2} = \frac{1}{\sin \vartheta} \partial_\vartheta \sin \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2 = \partial_\vartheta^2 + \cot \vartheta \partial_\vartheta + \frac{1}{\sin^2 \vartheta} \partial_\varphi^2,
\]

so the relation (A.8) is a particular case of the so-called Weitzenböck-Lichnerowicz formulas (see Refs. [40]). Then from (A.1) it follows \( D_0^2 \Phi = \lambda^2 \Phi \) and, when using the ansatz \( \Phi = P(\vartheta) e^{-im' \varphi} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} e^{-im' \varphi}, P_{1,2} = P_{1,2}(\vartheta) \), the equation \( D_0^2 \Phi = \lambda^2 \Phi \) turns into

\[
\begin{pmatrix}
-\partial_\vartheta^2 - \cot \vartheta \partial_\vartheta + \frac{m^2 + \frac{1}{4}}{\sin^2 \vartheta} + \frac{m' \cos \vartheta}{\sin^2 \vartheta} \sigma_1
\end{pmatrix} P = \\
\left( \lambda^2 - \frac{1}{4} \right) P,
\]

wherefrom all the above results concerning spectrum of \( D_0 \) can be derived [37, 38].

When calculating the functions \( P^k_{m'n'}(\cos \vartheta) \) directly, to our mind, it is the most convenient to use the integral expression [39]

\[
P^k_{m'n'}(\cos \vartheta) = \frac{1}{2\pi} \sqrt{\frac{(k-m')!(k+n')!}{(k+m')!(k+n')!}} \cos \left( \frac{k^2 - m'n'}{2} \right) \sin \left( \frac{n'}{2} \right) \times
\]

\[
\int_0^{2\pi} \left( e^{i\varphi/2} \cos \frac{\vartheta}{2} + ie^{-i\varphi/2} \sin \frac{\vartheta}{2} \right)^{k-n'}
\]
\[
\left( i e^{i \varphi/2} \sin \frac{\vartheta}{2} + e^{-i \varphi/2} \cos \frac{\vartheta}{2} \right)^{k+n'} e^{im' \varphi} \, d\varphi
\] (A.11)

and the symmetry relations \((z = \cos \vartheta)\)

\[
P_{m'n'}^k(z) = P_{n'm'}^k(z), \quad P_{m',-n'}^k(z) = P_{-m',n'}^k(z),
\]

\[
P_{m'n'}^k(z) = P_{-m',-n'}^k(z),
\]

\[
P_{m'n'}^k(-z) = i^{2k-2m'-2n'} P_{m',-n'}^k(z).
\] (A.12)

In particular

\[
P_{kk}^k(z) = \cos^{2k} (\vartheta/2), \quad P_{k,-k}^k(z) = i^{2k} \sin^{2k} (\vartheta/2),
\]

\[
P_{k0}^k(z) = \frac{i^k \sqrt{(2k)!}}{2^k k!} \sin^k \vartheta
\]

\[
P_{kn'}^k(z) = i^{k-n'} \frac{(2k)!}{(k-n')!(k+n')!} \times
\]

\[\sin^{k-n'} (\vartheta/2) \cos^{k+n'} (\vartheta/2).
\] (A.13)

**Eigenspinors with \( \lambda = \pm 1, \pm 2 \)**

If \( \lambda = \pm(l+1) = \pm 1 \) then \( l = 0 \) and from (A.3) it follows that \( k = l+1/2 = 1/2, |m'| \leq 1/2 \) and we need the functions \( P_{m',\pm 1/2}^{1/2} \) that are easily evaluated with the help of (A.11)–(A.13) so the eigenspinors for \( \lambda = -1 \) are

\[
\Phi = \frac{C}{2} \left( \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \right) e^{i \varphi/2}
\]

\[
\Phi = \frac{C}{2} \left( \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \right) e^{-i \varphi/2},
\] (A.14)

while for \( \lambda = 1 \) the conforming spinors are

\[
\Phi = \frac{C}{2} \left( \cos \frac{\vartheta}{2} - i \sin \frac{\vartheta}{2} \right) e^{i \varphi/2},
\]

\[
\Phi = \frac{C}{2} \left( - \cos \frac{\vartheta}{2} + i \sin \frac{\vartheta}{2} \right) e^{-i \varphi/2}
\] (A.15)

with the coefficient \( C = \sqrt{1/(2\pi)} \).
It is clear that (A.14)–(A.15) can be rewritten in the form

\[ \lambda = -1 : \Phi = \frac{C}{2} \left( \begin{array}{c} e^{i\varphi/2} \\ e^{-i\varphi/2} \end{array} \right), \]

or

\[ \Phi = \frac{C}{2} \left( \begin{array}{c} e^{i\varphi/2} \\ -e^{-i\varphi/2} \end{array} \right) e^{-i\varphi/2}, \]

\[ \lambda = 1 : \Phi = \frac{C}{2} \left( \begin{array}{c} e^{-i\varphi/2} \\ e^{i\varphi/2} \end{array} \right) e^{i\varphi/2}, \]

or

\[ \Phi = \frac{C}{2} \left( \begin{array}{c} e^{-i\varphi/2} \\ e^{i\varphi/2} \end{array} \right) e^{-i\varphi/2}, \]

so the relation (A.5) is easily verified at \( \lambda = \pm 1 \).

In studying vector mesons and excited states of heavy quarkonia eigen-
spinors with \( \lambda = \pm 2 \) may also be useful. Then \( k = l + 1/2 = 3/2 \), \( |m'| \leq 3/2 \) and we need the functions \( P_{m', \pm 1/2}^{3/2} \) that can be evaluated with the help of (A.11)–(A.13). Computation gives rise to

\[ P_{3/2, -1/2}^{3/2} = -\frac{\sqrt{3}}{2} \sin \vartheta \sin \frac{\vartheta}{2} = P_{-3/2, 1/2}^{3/2}, \]

\[ P_{3/2, 1/2}^{3/2} = i\frac{\sqrt{3}}{2} \sin \vartheta \cos \frac{\vartheta}{2} = P_{-3/2, -1/2}^{3/2}, \]

\[ P_{1/2, -1/2}^{3/2} = -\frac{i}{4} \left( \sin \frac{\vartheta}{2} - 3 \sin \frac{3\vartheta}{2} \right) = P_{-1/2, 1/2}^{3/2}, \]

\[ P_{1/2, 1/2}^{3/2} = \frac{1}{4} \left( \cos \frac{\vartheta}{2} + 3 \cos \frac{3\vartheta}{2} \right) = P_{-1/2, -1/2}^{3/2}, \]

and according to (A.3) this entails eigenspinors with \( \lambda = 2 \) in the form

\[ \frac{C}{2} \frac{\sqrt{3}}{2} \sin \vartheta \left( \begin{array}{c} e^{-i\varphi/2} \\ e^{i\varphi/2} \end{array} \right) e^{i3\varphi/2}, \frac{C}{8} \left( \frac{3e^{-i\varphi/2} + e^{i\varphi/2}}{3e^{i\varphi/2} + e^{-i\varphi/2}} \right) e^{i\varphi/2}, \]

\[ \frac{C}{8} \left( \frac{-3e^{-i\varphi/2} - e^{i\varphi/2}}{3e^{i\varphi/2} + e^{-i\varphi/2}} \right) e^{-i\varphi/2}, \frac{C}{2} \frac{\sqrt{3}}{2} \sin \vartheta \left( \begin{array}{c} e^{-i\varphi/2} \\ e^{i\varphi/2} \end{array} \right) e^{-i3\varphi/2} \]

with \( C = 1/\sqrt{\pi} \), while eigenspinors with \( \lambda = -2 \) are obtained in accordance with relation (A.5).
11 Appendix B

We here adduce the explicit form for the radial parts of meson wave functions from (6). At \( n_j = 0 \) they are given by

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{Y_j}{Z_j} \right),
\]

\[
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{Y_j}{Z_j} \right),
\]

(B.1)

while at \( n_j > 0 \) they are given by

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \times
\]

\[
\left[ \left( 1 - \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j} (r_j) + \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{2\alpha_j+1} (r_j) \right],
\]

\[
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \times
\]

\[
\left[ \left( 1 + \frac{Y_j}{Z_j} \right) L_{n_j}^{2\alpha_j} (r_j) - \frac{P_j Q_j}{Z_j} r_j L_{n_j-1}^{2\alpha_j+1} (r_j) \right]
\]

(B.2)

with the Laguerre polynomials \( L_n^\rho (r_j) \), \( r_j = 2 \beta_j r \), \( \beta_j = \sqrt{\beta_0^2 - \omega_j^2 + g^2 b_j^2} \) at \( j = 1, 2, 3 \) with \( b_3 = -(b_1 + b_2) \), \( P_j = gb_j + \beta_j \), \( Q_j = \mu_0 - \omega_j \), \( Y_j = P_j Q_j \alpha_j + (P_j^2 - Q_j^2) a_j / 2 \), \( Z_j = P_j Q_j \Lambda_j + (P_j^2 + Q_j^2) a_j / 2 \) with \( a_3 = -(a_1 + a_2) \), \( \Lambda_j = \lambda_j - g B_j \) with \( B_3 = -(B_1 + B_2) \), \( \alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2} \), while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of Euclidean Dirac operator \( D_0 \) on unit two-sphere with \( l_j = 0, 1, 2, ... \) (see Appendix A) and quantum numbers \( n_j = 0, 1, 2, ... \) are defined by the relations

\[
n_j = \frac{gb_j Z_j - \beta_j Y_j}{\beta_j P_j Q_j},
\]

(B.3)

which entails the spectrum (8). Further, \( C_j \) of (B.1)–(B.2) should be determined from the normalization condition

\[
\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{1}{3}.
\]

(B.4)

As a consequence, we shall gain that in (4) \( \Psi_j \in L^4_2(\mathbb{R}^3) \) at any \( t \in \mathbb{R} \) and, accordingly, \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) may describe relativistic bound states in the
field (3) with the energy spectrum (8). As is clear from (B.3) at $n_j = 0$ we have $g b_j / \beta_j = Y_j / Z_j$ so the radial parts of (B.1) can be rewritten as

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left( 1 - \frac{g b_j}{\beta_j} \right),$$

$$F_{j2} = i C_j Q_j r^{\alpha_j} e^{-\beta_j r} \left( 1 + \frac{g b_j}{\beta_j} \right).$$

(B.5)

More details can be found in Refs. [1, 3].

12 Appendix C

We here recall some facts about tensorial products of Hilbert spaces necessary in the main part of the Chapter. For more information see, e.g., Ref. [39] and references therein.

For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, by definition, their tensorial product $\mathcal{H}_1 \otimes \mathcal{H}_2$ consists from every possible linear combinations of elements of the form $\vec{x}_1 \otimes \vec{x}_2$, where $\vec{x}_1 \in \mathcal{H}_1$, $\vec{x}_2 \in \mathcal{H}_2$. The obvious relations take place

$$(\alpha \vec{x}_1 + \beta \vec{x}_2) \otimes \vec{y} = \alpha (\vec{x}_1 \otimes \vec{y}) + \beta (\vec{x}_2 \otimes \vec{y}),$$

(C.1)

$$x \otimes (\alpha \vec{y}_1 + \beta \vec{y}_2) = \alpha (x \otimes \vec{y}_1) + \beta (\vec{x} \otimes \vec{y}_2)$$

(C.2)

with arbitrary complex numbers $\alpha, \beta$.

Scalar product in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by the equality

$$(\vec{x}_1 \otimes \vec{y}_1, \vec{x}_2 \otimes \vec{y}_2) = (x_1, x_2)(y_1, y_2),$$

(C.3)

where $(x_1, x_2), (y_1, y_2)$ are the scalar products, respectively, in $\mathcal{H}_1$ and $\mathcal{H}_2$. Under the situation, if $\{\vec{e}_i\}$ is an orthonormal basis in $\mathcal{H}_1$ and $\{\vec{f}_j\}$ is an orthonormal basis in $\mathcal{H}_2$ then $\vec{h}_{ij} = e_i \otimes f_j$ is an orthonormal basis in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Further, if $A$ is a linear operator in $\mathcal{H}_1$ and $B$ is a linear operator in $\mathcal{H}_2$ then tensorial (Kronecker) product $A \otimes B$ acts in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and is defined by relation $(A \otimes B)(\vec{x}_1 \otimes \vec{x}_2) = A\vec{x}_1 \otimes B\vec{x}_2$ with the properties

$$A \otimes (\alpha B_1 + \beta B_2) = \alpha A \otimes B_1 + \beta A \otimes B_2,$$

(C.4)

$$(\alpha A_1 + \beta A_2) \otimes B = \alpha A_1 \otimes B + \beta A_2 \otimes B,$$

(C.5)

$$A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2).$$

(C.6)
In a similar way, the Kronecker sum of $A$ and $B$ is defined by $A \oplus B = A \otimes I + I \otimes B$ with identical operator $I$. Under the circumstances, let $\lambda_i$ be the eigenvalues of $A$ and $\mu_j$ be those of $B$ (listed according to multiplicity). Then the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ while those of $A \oplus B$ will be $\lambda_i + \mu_j$.

All the above is directly generalized to the arbitrary number of Hilbert spaces. For example, the operator $A_1 \otimes I \otimes I \otimes A_2 \otimes I \otimes I \otimes A_3$ has the eigenvalues $\lambda_i + \mu_j + \nu_k$ in space $H_1 \otimes H_2 \otimes H_3$ if $\lambda_i$, $\mu_j$, $\nu_k$ are the eigenvalues of $A_1$, $A_2$ and $A_3$, respectively, in $H_1$, $H_2$, $H_3$.

13 Appendix D

The facts adduced here have been obtained in Refs. [2, 3] and we concisely give them only for completeness of discussion in Section 2.

To specify the question, let us note that in general the Yang-Mills equations on a manifold $M$ can be written as

$$d \ast F = g (\ast F \wedge A - A \wedge \ast F),$$

where a gluonic field $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ [$\lambda_a$ are the known Gell-Mann matrices, $\mu = t, r, \theta, \varphi$ (in the case of spherical coordinates), $a = 1, \ldots, 8$], the curvature matrix (field strength) $F = dA + gA \wedge A = F^a_\mu \lambda_a dx^\mu \wedge dx^\nu$ with exterior differential $d$ and the Cartan’s (exterior) product $\wedge$, while $\ast$ means the Hodge star operator conforming to a metric on manifold under consideration, $g$ is a gauge coupling constant.

The most important case of $M$ is Minkowski spacetime and we are interested in the confining solutions $A$ of the SU(3)-Yang-Mills equations. The confining solutions were defined in Ref. [1] as the spherically symmetric solutions of the Yang-Mills equations (1) containing only the components of the SU(3)-field which are Coulomb-like or linear in $r$. Additionally we impose the Lorentz condition on the sought solutions. The latter condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e.g., Ref. [41]), so we should impose the given condition that can be written in the form div($A$) = 0, where the divergence of the Lie algebra valued 1-form $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ is defined by the relation (see, e.g., Refs. [40])

$$\text{div} A = \ast (d \ast A) = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu \nu} A_\nu).$$

It should be emphasized that, from the physical point of view, the Lorentz condition reflects the fact of transversality for gluons that arise as quanta of SU(3)-Yang-Mills field when quantizing the latter (see, e.g., Ref. [41]).
We shall use the Hodge star operator action on the basis differential 2-forms on Minkowski spacetime with local coordinates \( t, r, \vartheta, \varphi \) in the form

\[
\begin{align*}
* (dt \wedge dr) &= -r^2 \sin \vartheta d\vartheta \wedge d\varphi, \\
* (dt \wedge d\varphi) &= -\frac{1}{\sin \vartheta} dr \wedge d\vartheta, \\
* (dr \wedge d\varphi) &= -\frac{1}{\sin \vartheta} dt \wedge d\vartheta, \\
* (d\vartheta \wedge d\varphi) &= \frac{1}{r^2 \sin \vartheta} dt \wedge dr,
\end{align*}
\]

so that on 2-forms \(*^2 = -1\). More details about the Hodge star operator can be found in [10].

The most general ansatz for a spherically symmetric solution is

\[
A = A_t(r) dt + A_r(r) dr + A_\vartheta(r) d\vartheta + A_\varphi(r) d\varphi.
\]

But then the Lorentz condition (D.2) for the given ansatz gives rise to

\[
\sin \vartheta \partial_r (r^2 A_r) + \partial_\vartheta (\sin \vartheta A_\vartheta) = 0,
\]

which yields \( A_r = \frac{C}{r^2} - \frac{\cot \vartheta}{r} \int A_\vartheta(r) dr \) with a constant matrix \( C \). But the confining solutions should be spherically symmetric and contain only the components which are Coulomb-like or linear in \( r \), so one should put \( C = A_\vartheta(r) = 0 \). Consequently, the ansatz \( A = A_t(r) dt + A_\varphi(r) d\varphi \) is the most general spherically symmetric one.

For the latter ansatz we have \( F = dA + g A \wedge A = -\partial_r A_t dt \wedge dr + \partial_\vartheta A_\varphi dr \wedge d\varphi + g[A_t, A_\varphi] dt \wedge d\varphi \), where \([\cdot, \cdot]\) signifies matrix commutator.

Then, according to (D.3), we obtain

\[
* F = (r^2 \sin \vartheta) \partial_r A_t d\vartheta \wedge d\varphi - \frac{1}{\sin \vartheta} \partial_r A_\varphi dt \wedge d\vartheta -
\]

\[
\frac{g}{\sin \vartheta} [A_t, A_\varphi] dr \wedge d\vartheta,
\]

which entails

\[
d * F = \sin \vartheta \partial_r (r^2 \partial_r A_t) dr \wedge d\vartheta \wedge d\varphi + \frac{1}{\sin \vartheta} \partial_r^2 A_\varphi dt \wedge dr \wedge d\vartheta,
\]

while

\[
* F \wedge A - A \wedge * F =
\]

\[
\left( r^2 \sin \vartheta [\partial_r A_t, A_t] - \frac{1}{\sin \vartheta} [\partial_r A_\varphi, A_t] \right) dt \wedge d\vartheta \wedge d\varphi
\]

\[
- \frac{g}{\sin \vartheta} \left( [A_t, A_\varphi] dt \wedge dr \wedge d\vartheta
\]

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\[ +[[A_t, A_\varphi], A_\varphi] \, dr \wedge d\vartheta \wedge d\varphi \] . \tag{D.6}

Under the circumstances the Yang-Mills equations (D.1) are tantamount to the conditions

\[ \partial_r (r^2 \partial_r A_t) = -\frac{g^2}{\sin^2 \vartheta} [[A_t, A_\varphi], A_\varphi], \tag{D.7} \]

\[ \partial_r^2 A_\varphi = -g^2 [[A_t, A_\varphi], A_t], \tag{D.8} \]

\[ r^2 \sin \vartheta [\partial_r A_t, A_t] - \frac{1}{\sin \vartheta} [\partial_r A_\varphi, A_\varphi] = 0. \tag{D.9} \]

The key equation is (D.7) because the matrices \( A_t, A_\varphi \) depend on merely \( r \) and (D.7) can be satisfied only if the matrices

\[ A_t = A_t^a \lambda_a \quad \text{and} \quad A_\varphi = A_\varphi^a \lambda_a \]

belong to the so-called Cartan subalgebra of the SU(3)-Lie algebra. Let us remind that, by definition, a Cartan subalgebra is a maximal abelian subalgebra in the corresponding Lie algebra, i.e., the commutator for any two matrices of the Cartan subalgebra is equal to zero (see, e.g., Ref. \[45\]). For SU(3)-Lie algebra the conforming Cartan subalgebra is generated by the Gell-Mann matrices \( \lambda_3, \lambda_8 \) which are

\[ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{D.10} \]

Under the situation we should have

\[ A_t = A_t^3 \lambda_3 + A_t^8 \lambda_8 \quad \text{and} \quad A_\varphi = A_\varphi^3 \lambda_3 + A_\varphi^8 \lambda_8, \]

then \( [A_t, A_\varphi] = 0 \) and we obtain

\[ \partial_r (r^2 \partial_r A_t) = 0, \partial_r^2 A_\varphi = 0, \tag{D.11} \]

while (D.9) is identically satisfied and (D.11) gives rise to the solution (3) with real constants \( a_j, A_j, b_j, B_j \) parametrizing the solution which proves the uniqueness theorem of Section 2 for the SU(3) Yang-Mills equations.

The more explicit form of (3) is

\[ A_t^3 = [(a_2 - a_1)/r + A_1 - A_2]/2, \]

\[ A_t^8 = [A_1 + A_2 - (a_1 + a_2)/r]\sqrt{3}/2, \]

\[ A_\varphi^3 = [(b_1 - b_2)/r + B_1 - B_2]/2, \]

\[ A_\varphi^8 = [(b_1 + b_2)/r + B_1 + B_2]\sqrt{3}/2. \tag{D.12} \]

Clearly, the obtained results may be extended over all SU(\( N \))-groups with \( N \geq 2 \) and even over all semisimple compact Lie groups since for them
the corresponding Lie algebras possess just the only Cartan subalgebra. Also we can talk about the compact non-semisimple groups, for example, $U(N)$. In the latter case additionally to Cartan subalgebra we have centrum consisting from the matrices of the form $\alpha I_N$ ($I_N$ is the unit matrix $N \times N$) with arbitrary constant $\alpha$.

The most relevant physical cases are of course $U(1)$- and $SU(3)$-ones (QED and QCD). In particular, the $U(1)$-case allows us to build the classical model of confinement (see Section 2 and Ref. [10]).

At last, it should also be noted that the nontrivial confining solutions obtained exist at any gauge coupling constant $g$, i.e. they are essentially nonperturbative ones.

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