Detecting Non-Commutativity of Unknown Hamiltonians via Extreme Gain of Correlations

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Non-commutativity is one of the most elementary non-classical features of quantum observables. Here we present a method to detect non-commutativity of Hamiltonians, which is applicable even if these Hamiltonians are completely unknown. We consider two probe objects individually interacting with the third object (mediator) but not with each other. If the probe-mediator Hamiltonians commute, we derive upper bounds on correlations between the probes that depend on the dimension of the mediator. We then demonstrate that these bounds can be violated with correlation dynamics generated by non-commuting Hamiltonians. An intuitive explanation is provided in terms of multiple exchanges of a virtual particle which lead to the excessive accumulation of correlations. A plethora of correlation quantifiers are helpful in our method, e.g. quantum entanglement, discord, mutual information, and even classical correlation, to detect non-classicality in the form of non-commutativity.

All classical observables are functions of positions and momenta. Since there is no fundamental limit on the precision of position and momentum measurement in classical physics, all classical observables are, in principle, measurable simultaneously. Quite differently, the uncertainty principle forbids simultaneous exact knowledge of quantum observables corresponding to position and momentum. The underlying non-classical feature is their non-commutativity: Any pair of non-commuting observables cannot be simultaneously measured to arbitrary precision, as first demonstrated by Robertson in his famous uncertainty relation [1]. Other examples of non-classical phenomena with underlying non-commutativity of observables include violations of Bell inequalities [2, 3] or, more generally, non-contextual inequalities (e.g. see [4]).

Here we describe a method to detect the non-classicality of interactions as characterised by the non-commutativity of Hamiltonians. The situation considered is depicted in Fig. 1. We emphasise that systems A and B do not interact directly, i.e., there is no Hamiltonian \( H_{AB} \), but only via a third object C, the mediator. We show that if \( H_{AC} \) and \( H_{BC} \) commute, correlations between A and B are bounded. We also show with concrete dynamics generated by non-commuting Hamiltonians that these bounds can be violated. The bounds derived depend solely on the dimensionality of C and not on the actual form of the Hamiltonians. Hence, they can remain unknown throughout the assessment. This is a desired feature, as experimenters usually do not reconstruct the Hamiltonians via process tomography. It also allows applications of the method to situations where the physics is not understood to the extent that reasonable Hamiltonians could be written down. Furthermore, the assessment does not depend on the initial state of the tripartite system and does not require any operations on the mediator. It is therefore applicable to a variety of experimental situations, Refs. [5, 7] provide concrete examples.

We begin by presenting the general bounds on the amount of correlations one can establish if the Hamiltonians commute. It will be shown that these bounds are quite generic and hold for a plethora of correlation quantifiers. We then calculate concrete bounds on exemplary correlations and show how they can be violated in a system of two fields coupled by a two-level atom. Finally, we discuss the origin of the violation in terms of “trotted” evolution, where a virtual particle is exchanged between A and B multiple times if the Hamiltonians do not commute but only once if they do commute.

General bounds.—Consider the setup illustrated in Fig. 1. An ancillary system C, with finite dimension \( d_C \), is mediating interactions between higher dimensional systems A and B. For simplicity we take \( d_A = d_B > d_C \). We assume that there is no direct interaction between A and B, such that the Hamiltonian of the whole tripartite system is of the form \( H_{AC} + H_{BC} \) (local Hamiltonians \( H_A, H_B, H_C \) included). Our bounds follow from a generalisation of the following simple observation. Consider, for the moment, the relative entropy of entanglement as the correlation quantifier [8]. If the Hamiltonians com-
mute, i.e., $[H_{AC}, H_{BC}] = 0$, then the evolution can be written as $U_{BC}U_{AC}$, or in reverse order, where for example $U_{AC} = \exp(-itH_{AC})$ and we set $h = 1$. Therefore, it is as if particle $C$ first interacted with $A$ and then with $B$, a scenario similar to Refs. [9][11]. There, the interactions are not necessarily unitary. Nevertheless, our bounds still apply, as we will show below. The first interaction can generate at most $\log(d_C)$ ebits of entanglement whereas the second, in the best case, can swap all this entanglement. At the end, particles $A$ and $B$ gain at most $\log(d_C)$ ebits. The bound is indeed independent of the form of interactions. Furthermore, it is intuitively clear, as this is just the “quantum capacity” of the mediator.

Now let us consider correlation quantifiers obtained in the so-called “distance” approach [8][12]. The idea is to quantify correlations $Q_{XY}$ in a state $\rho_{XY}$ as the shortest distance $D(\rho_{XY}, \sigma_{XY})$ from $\rho_{XY}$ to a set of states $\sigma_{XY} \in S$ without the desired correlation property, i.e., $Q_{XY} = \inf_{\sigma_{XY} \in S} D(\rho_{XY}, \sigma_{XY})$. For example, relative entropy of entanglement is given by the relative entropy of a state to the set of disentangled states $S$. It turns out that most of such quantifiers are useful for the task introduced here. The conditions we require are: (i) $S$ is closed under local operations $\Lambda_Y$ on $Y$; (ii) $D(\Lambda(\rho), \Lambda(\sigma)) \leq D(\rho, \sigma)$ (monotonicity); (iii) $D(\rho_{01}, \rho_{11}) \leq D(\rho_{01}, \rho_{12}) + D(\rho_{12}, \rho_{11})$ (triangle inequality). They are sufficient to prove the following theorem.

**Theorem 1.** Suppose a correlation $Q_{XY}$ satisfies properties (i) - (iii) listed above. If $[H_{AC}, H_{BC}] = 0$, then

$$Q_{AC:B}(t) \leq I_{AC:B}(0) + \sup_{|\psi\rangle} Q_{AC},$$

where $I_{AC:B}(0) = \inf_{\sigma_{AC} \otimes \sigma_B} D(\rho, \sigma_{AC} \otimes \sigma_B)$, $\rho$ is the initial tripartite state, and the supremum of $Q_{AC}$ is taken over pure states of $AC$.

**Proof.** See Appendix A.

Note that although relative entropy does not satisfy (iii) it still follows Theorem 1 cf. Lemma 3 in Appendix A. Correlations between the probes $A$ and $B$ are therefore bounded by the maximal achievable correlation with the mediator, $\sup_{|\psi\rangle} Q_{AC}$, and the additional term $I_{AC:B}(0)$ reduces to the usual mutual information if $D(\rho_{XY}, \sigma_{XY})$ is the relative entropy distance [12] and characterises the amount of total initial correlations between one of the probes and the rest of the system. Note that the bound is independent of time. This can be seen as a result of the effective description of such dynamics given by $U_{BC}U_{AC}$. The particle $C$ is exchanged between $A$ and $B$ only once, independently of the duration of dynamics. Furthermore, one can see from the proof of Theorem 1 that the conclusion also applies to any dynamics decomposable into a sequence of maps $\Lambda_{BC}\Lambda_{AC}$, which are not necessarily unitary.

In a typical experimental situation the initial state can be prepared as completely uncorrelated $\rho = \rho_A \otimes \rho_B \otimes \rho_C$, in which case Theorem 1 simplifies and the bound is given solely in terms of the “correlation capacity” of the mediator:

$$Q_{AC:B}(t) \leq \sup_{|\psi\rangle} Q_{AC}.$$  

(2)

Clearly, the same bound holds for initial states of the form $\rho = \rho_{AC} \otimes \rho_B$. In Appendix 1 we show that, with this initial state, Eq. (2) holds for any correlation quantifier that is monotonic under local operations $\Lambda_{BC}$, not necessarily based on the distance approach, e.g. any entanglement monotone.

For initial states that are close to $\rho = \rho_{AC} \otimes \rho_B$ one can utilise the continuity of the von Neumann entropy [13] and see that $I_{AC:B}(0)$ in Eq. (1) is indeed small. We can also ensure that the initial state is of the form $\rho = \rho_{AC} \otimes \rho_B$ by performing a correlation breaking channel on $B$ first. One example of such a channel is a measurement in the computational basis followed by a measurement in some complementary (say Fourier) basis. This implements the correlation breaking channel $(\rho_{AC} \otimes \Lambda_B)(\rho_{ABC}) = \rho_{AC} \otimes \frac{1}{d_B}$ [14]. In this way, our method does not require any knowledge of the initial state of the mediator or any operations on it, similar in spirit to the detection of quantum discord of inaccessible objects in Ref. [15]. We now move to concrete correlation quantifiers and their correlation capacities.

**Exemplary measures and bounds.**—We provide four correlation quantifiers which capture different types of correlations between quantum particles. All of them will be shown useful in detecting non-commutativity.

Mutual information is a measure of total correlations [16] and is defined as $I_{XY} = S_X + S_Y - S_{XY}$, where e.g. $S_X$ is the von Neumann entropy of subsystem $X$. It can also be seen as a distance-based measure with relative entropy as the distance and a set of product states $\sigma_X \otimes \sigma_Y$ as $S$ [12]. The supremum in Eq. (2) is attained by the state (recall that $d_A > d_C$):

$$|\psi\rangle = \frac{1}{\sqrt{d_C}} \sum_{j=1}^{d_C} |a_j\rangle|c_j\rangle,$$

(3)

where $|a_j\rangle$ and $|c_j\rangle$ form orthonormal bases. One finds $\sup_{|\psi\rangle} I_{AC} = 2 \log(d_C)$.

An interesting quantifier in the context of non-classicality detection is classical correlation in a quantum state. It is defined as mutual information of the classical state obtained by performing the best local von Neumann measurements on the original state $\rho$ [17], i.e., $C_{XY} = \sup_{\Pi_X \otimes \Pi_Y} I_{XY}(\Pi_X \otimes \Pi_Y(\rho))$, where $\Pi_X \otimes \Pi_Y(\rho) = \sum_{xy} |xy\rangle \langle xy| \rho |xy\rangle \langle xy|$, and $|x\rangle$, $|y\rangle$ form orthonormal bases. The supremum of mutual information over classical states of $AC$ is $\log(d_C)$. 
Quantum discord is a form of purely quantum correlations that contain quantum entanglement. It can be phrased as a distance-based measure. In particular, we consider the relative entropy of discord [12], also known as the one-way deficit [18]. It is an asymmetric quantity defined as $\Delta_{X|Y} = \inf_{\Pi_Y} S(\Pi_Y(\rho)) - S(\rho)$, where $\Pi_Y$ is a von Neumann measurement conducted on subsystem $Y$. The relative entropy of discord is maximised by the state (3) for which we have $\sup_{\rho} \Delta_{A|C} = \log(d_C)$.

Our last example is negativity, a computable entanglement monotone [19]. For a bipartite system negativity is defined as $N_{X,Y} = (||\rho^{TX}||_1 - 1)/2$, where $||\.||_1$ denotes the trace norm and $\rho^{TX}$ is a matrix obtained by partial transposition of $\rho$ with respect to $X$. Negativity is maximised by the state (3) and the supremum reads $\sup_{\rho} N_{A:C} = (d_C - 1)/2$.

Clearly, many other correlation quantifiers are suitable for our detection method because the assumptions behind Eqs. (1) and (2) are not demanding. In fact, one may wonder which correlations do not qualify for our method. A concrete example is geometric quantum discord based on $p$-Schatten norms with $p > 1$ as it may increase under local operations on BC [20] [21].

Violations.—We now demonstrate, with concrete dynamics generated by non-commuting Hamiltonians, that the bounds derived can be violated. We will next discuss the origin of this violation.

Consider a two-level atom $C$, i.e., $d_C = 2$, mediating interactions between two cavity fields $A$ and $B$. A similar scenario has been considered, for example, in Refs. [5] [21] [23]. The interaction between the atom and each cavity field is taken to follow the Jaynes-Cummings model:

$$H = g(\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-) + g(\hat{b} \hat{\sigma}_+ + \hat{b}^\dagger \hat{\sigma}_-),$$  

(4)

where $\hat{a}$ ($\hat{b}$) is the annihilation operator of field $A$ ($B$) while $\hat{\sigma}_+$ ($\hat{\sigma}_-$) is the raising (lowering) operator of the two-level atom. For simplicity, we have assumed that the interaction strengths between the two-level atom and the fields are the same. Note that $H$ is of the form $H_{AC} + H_{BC}$ with non-commuting components.

The resulting correlation dynamics are plotted in Fig. 2. Mutual information and negativity were calculated directly whereas, for classical correlation and the relative entropy of discord, we provide the lower bounds $\tilde{C}_{A:B}$ and $-S_{A|B}$, respectively. $\tilde{C}_{A:B}$ is calculated as the mutual information of the state resulting from projective local measurements in the Fock basis (no optimisation over measurements performed). The negative conditional entropy $-S_{A|B}$ is a lower bound on distillable entanglement [24] which in turn is a lower bound on the relative entropy of entanglement $E_{A:B}$ [25]. Therefore, we note the following chain of inequalities $-S_{A|B} \leq E_{A:B} \leq \Delta_{A|B} \leq I_{A:B}$, where the last two inequalities follow from [12]. Already these lower bounds can beat the limit set by commuting Hamiltonians and, therefore, all mentioned correlations can detect the non-commuting feature of the Jaynes-Cummings coupling. Also, the presence of entanglement in Fig. 2 (d) indicates non-classicality of the mediator, as characterised by nonzero discord $D_{ABC}$ [15].

It is apparent that the detection is easier (faster and with more pronounced violation) with a higher number of photons in the initial states of the cavity fields. We offer an intuitive explanation. Consider for example $|mn0\rangle$ as the initial state of $ABC$. By defining $\xi = (\hat{a} + \hat{b})/\sqrt{2}$, the Hamiltonian of Eq. (4) becomes $\sqrt{2}g(\xi \hat{\sigma}_+ + \xi^\dagger \hat{\sigma}_-)$ and it is straightforward to obtain the unitary evolution [26]. One will find that the quantum state of the fields oscillates incoherently between $\sum_{j=0}^{m+n} c_j(t)|j\rangle_A|m + n - j\rangle_B$ and $\sum_{j=0}^{m+n-1} d_j(t)|j\rangle_A|m + n - 1 - j\rangle_B$. Both of these states are superpositions of essentially $m + n$ bi-orthogonal terms giving rise to high entanglement and, therefore, also other forms of correlations.

Fig. 2 illustrates that different correlation quantifiers have different detection capabilities and it is not clear at this stage if there is a universal measure with which non-commutativity is detected, e.g. the fastest. For most initial states we studied mutual information detected the non-commutativity fastest, but there are exceptions, as seen by the black curve corresponding to the initial state $|101\rangle$. With this initial state mutual information never violates its bound, but negativity does.

![Fig. 2. Correlation dynamics with Jaynes-Cummings model (solid curves) and the corresponding bounds for the commuting Hamiltonians (dashed lines). In panel (a) we plot mutual information, in (b) the lower bound on classical correlation (see main text), in (c) the lower bound on the relative entropy of discord and in (d) negativity. In all cases, time is rescaled with the interaction strength $g$ and the initial state of $ABC$ is varied: $|110\rangle$ (red), $|101\rangle$ (black), $|210\rangle$ (green), and $|220\rangle$ (blue).](image-url)

Discussion.—Let us present the origin of the violation just observed. Since the total Hamiltonian is of the form $H_{AC} + H_{BC}$, the Suzuki-Trotter expansion of the result-
ing evolution is particularly illuminating:
\[ e^{i(H_{AC} + H_{BC})} = \lim_{n \to \infty} \left( e^{i\Delta t H_{BC}} e^{i\Delta t H_{AC}} \right)^n, \]
where \( \Delta t = t/n \). If Hamiltonians do not commute it is necessary to think about a sequence of pairwise interactions of \( C \) with \( A \) followed by \( C \) with \( B \), or in the reverse order. Each pair of interactions can only increase correlations up to the correlation capacity of the mediator, but their multiple use allows the accumulation of correlations beyond what is possible with commuting Hamiltonians. Recall that, in the latter case, we deal with only one exchange of system \( C \), independently of the duration of the dynamics. We stress that Trotterisation is just a mathematical tool and in the laboratory system \( C \) is continuously coupled to \( A \) and \( B \). It is rather as if a virtual particle \( C \) was transmitted multiple times between \( A \) and \( B \), interacting with each of them for a time \( \Delta t \).

Our results imply that the non-commutativity is a desired feature of interactions in the task of correlation distribution, provided we can wait for their accumulation. As a contrasting physical illustration, we consider the strong dipole-dipole interactions in our field-atom-field. As a contrasting physical illustration, we consider the strong dipole-dipole interactions in our field-atom-field.

Appendix C we prove that, with this coupling, the state of pairwise interactions of \( A \) and \( B \) is effectively given by a two-qubit separable state. This makes \( N_{AB}(t) = 0 \) and \( I_{AB}(t) \leq 1 \). It is worth noticing the counter-intuitive result that strong interactions produce bounded correlations between the probes, while the weak interactions (Jaynes-Cummings coupling) can increase the correlations above the bounds.

We also note the seemingly paradoxical effect that classical correlations can reveal non-classical features such as non-commutativity. In the virtual particle picture this is not surprising as, for non-commuting Hamiltonians, the particle is exchanged multiple times, leading to bigger accumulations of any kind of correlations. We also emphasise that we compute classical correlations in quantum states. It is an intriguing question if there exist dynamics, generated by non-commuting Hamiltonians, which keeps the state of \( A \) and \( B \) classical at all times and with correlations revealing the non-commutativity.

Finally, we wish to discuss a scenario where the three systems are open to their own local environments. We take the evolution following the master equation in Lindblad form
\[ \dot{\rho} = -i[H_{AC} + H_{BC}, \rho] + \sum_{X = A, B, C} L_X \rho, \]
where the last term in (7) describes the incoherent part of the evolution and operators \( Q_k^X \)'s act on system \( X \) only. We denote \( \mathcal{L}_{AC} = -i[H_{AC}, \cdot] + L_A + L_C \) and \( \mathcal{L}_{BC} = -i[H_{BC}, \cdot] + L_B \). One readily verifies that if \( [H_{AC}, H_{BC}] = 0 \) and \( [L_C, H_{BC}] = 0 \), we have \( [\mathcal{L}_{AC}, \mathcal{L}_{BC}] = 0 \). Note that, if one includes \( L_C \) in \( \mathcal{L}_{BC} \), the second condition becomes \( [L_C, H_{AC}] = 0 \). The corresponding evolution decomposes as \( \Lambda_{BC} \Lambda_{AC} \), or in reverse order. Therefore, our bounds apply accordingly. Their violation implies that either the Hamiltonians do not commute or the operators describing dissipative channels on \( C \) do not commute with \( H_{AC} \) and \( H_{BC} \). In particular, if \( C \) is kept isolated so that its noise can be ignored, the violation of our bounds is solely the result of the non-commutativity of the Hamiltonians.

Conclusions.—We introduced a method to detect the non-commutativity of Hamiltonians \( H_{AC} \) and \( H_{BC} \) in a scenario where \( C \) mediates interactions between \( A \) and \( B \). It requires no explicit form of the Hamiltonians or knowledge of the initial state of the tripartite system. The non-commutativity is detected by observing violation of certain bound on \( AB \) correlations (as measured by most correlation quantifiers). Furthermore, no operation on \( C \) is necessary at any time, which makes this strategy experimentally friendly. In particular, in addition to avoiding characterisation of the interactions, the physics of \( C \) can remain largely unknown — only its dimension should be identified.

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Appendix A: Proof of Theorem 1

For completeness let us begin with a useful lemma.

Lemma 2. For a measure of correlations \( Q_{X:Y} \) between parties \( X \) and \( Y \) that is non-increasing under local operations on \( Y \), the following properties hold:

1. \( Q_{X:Y} \) is invariant under local unitary on \( Y \).

2. \( Q_{X:Y} \) is invariant under tracing out uncorrelated systems on the side of \( Y \).

Proof. Since the correlation measure is non-increasing under local operations on \( Y \) we have \( Q_{X:Y}(\rho) \geq Q_{X:Y}(U_Y \rho U_Y^\dagger) \). Note that, if \( Q_{X:Y} \) is strictly decreasing under unitary \( U_Y \), there is always a unitary operation \( U_Y^\dagger \) that will revert the system and, hence, increase \( Q_{X:Y} \). Therefore, \( Q_{X:Y} \) has to be invariant under local unitary operations. In fact, this is true for all reversible
operations, including tracing out uncorrelated systems on the side of $Y$.

Our main theorem is proven as follows.

**Theorem 1.** Consider a correlation measure $Q_{X,Y} \equiv \inf_{\sigma_{XY} \in \mathcal{S}} D(\rho_{XY}, \sigma_{XY})$ satisfying the following properties:

(i) $\mathcal{S}$ is closed under local operation $\Lambda_Y$ on $Y$.

(ii) $D(\Lambda(\rho), \Lambda(\sigma)) \leq D(\rho, \sigma)$.

(iii) $D(\rho_0, \rho_1) \leq D(\rho_0, \rho_2) + D(\rho_2, \rho_1)$.

If $[H_{AC}, H_{BC}] = 0$, then

$$Q_{AB}(t) \leq I_{AC:B}(0) + \sup_{\rho_{AC}} Q_{AC}, \quad (A1)$$

where $I_{AC:B}(0) = \inf_{\sigma_{AC,\sigma_B}} D(\rho, \sigma_{AC} \otimes \sigma_B)$, $\rho$ is the initial tripartite state, and the supremum of $Q_{AC}$ is taken over pure states of $AC$.

**Proof.** Properties (i) and (ii), and the definition of $Q_{X,Y}$ as the shortest distance, imply that $Q_{X,Y}$ is non-increasing under local operations on $Y$. Accordingly, the properties proven in Lemma 2 apply. We have:

$$Q_{AB}(t) \leq Q_{ABC}(U_{BC}U_{AC}\rho U_{AC}^\dagger U_{BC}^\dagger) \quad (A2)$$

$$= Q_{ABC}(U_{AC}\rho U_{AC}^\dagger) \quad (A3)$$

$$= \inf_{\rho_{AC}} D(U_{AC}\rho U_{AC}^\dagger, \mu) \quad (A4)$$

$$\leq D(U_{AC}\rho U_{AC}^\dagger, \rho_{AC} \otimes \sigma_B) \quad (A5)$$

$$+ \inf_{\rho_{AC}} D(U_{AC}^\dagger \rho_{AC} \otimes \sigma_B, \mu) \quad (A6)$$

$$\leq D(\rho, \rho_{AC} \otimes \sigma_B) \quad (A7)$$

$$= I_{AC:B}(0) + Q_{AC}(U_{AC}^\dagger \rho_{AC} \otimes \sigma_B) \quad (A8)$$

where the steps are justified as follows. In line (A2) we used the fact that $Q_{X,Y}$ is non-increasing under local operations on $Y$ (tracing out $C$). In (A3) we used the invariance of $Q_{ABC}$ under local unitary $U_{BC}$. The next line, (A4), is the definition of $Q_{ABC}$ in terms of distance. The inequality of (A5) follows from the triangle inequality. Note that this distance does not depend on $\mu$. The inequality (A6) invokes property (ii) and the definition of $Q_{ABC}$ again. In (A7), we used the invariance of $Q_{ABC}$ under tracing out the uncorrelated system $\sigma_B$. For the final inequality, we note that a correlation measure that is non-increasing under local operations on at least one side must be maximal on pure states.

Note that our theorem still applies for operations of the form $\Lambda_{BC}\Lambda_{AC}$, not necessarily unitary. One can see that the only different step is in (A3) where now the equality becomes an inequality.

**Lemma 3.** The conclusion in Theorem 1 still follows for relative entropy as a distance measure.

**Proof.** Let us begin with an identity

$$S(\rho||\sigma_Y) = \text{tr}(\rho \log \rho - \rho \log \sigma_Y)$$

$$= \text{tr}(\rho \log \rho - \rho \log \rho_X \otimes \rho_Y) + \text{tr}(\rho \log \rho_X \otimes \rho_Y - \rho \log \sigma_Y \otimes \sigma_Y)$$

$$= S(\rho||\rho_X \otimes \rho_Y) + S(\rho_X||\sigma_X) + S(\rho_Y||\sigma_Y), \quad (A8)$$

where $\rho_X$ and $\rho_Y$ are the marginals of $\rho$ and we have used for example relation $\text{tr}(\rho \log \rho_X \otimes \sigma_Y) = \text{tr}(\rho \log \sigma_X \otimes \rho_Y)$. Although relative entropy satisfies (ii) it is well-known to not follow (iii). Therefore, starting from (A4), we have

$$\inf_{\rho \in S_{A,B,C}} S(U_{AC}\rho U_{AC}^\dagger, \mu)$$

$$\leq \inf_{\rho_{AC} \otimes \rho_B} S(U_{AC}\rho U_{AC}^\dagger, \mu)$$

$$= \inf_{\rho_{AC} \otimes \rho_B} S(U_{AC}\rho U_{AC}^\dagger, \rho_B')$$

$$+ S(\rho_B', \mu) \quad (A9)$$

$$= I_{AC:B}(0) + \inf_{\rho_{AC} \in S_{A,C}} S(\rho_{AC}') \quad (A10)$$

$$\leq I_{AC:B}(0) + \sup_{\rho_{AC}} Q_{AC}.$$

where $\rho_B'$ and $\rho_B$ are marginals of $U_{AC}\rho U_{AC}^\dagger$. The steps above are justified as follows. Line (A9) follows because the states $\rho_{AC} \otimes \rho_B \in S_{A,B,C}$ are a subset of all the states in $S_{A,B,C}$ which might not include the state closest to $U_{AC}\rho U_{AC}^\dagger$. We have used the identity (A3) in line (A10). The equality (A11) uses the definition of mutual information as the relative entropy from a state to its marginals and $\mu_B = \rho_B$. The last line follows as mutual information is invariant under local unitary operations and the correlation $Q_{AC}$ achieves supremum on pure states.

**Appendix B: Proof of (2) for correlations only monotonic under local operations $\Lambda_{BC}$**

**Theorem 4.** Suppose the initial state has the following form $\rho = \rho_{AC} \otimes \rho_B$. If $[H_{AC}, H_{BC}] = 0$, then

$$Q_{AB}(t) \leq \sup_{\rho_{AC}} Q_{AC} \quad (B1)$$

for all correlation measures, $Q$, non-increasing under local operations $\Lambda_{BC}$. 

\]
Proof. Since the Hamiltonians commute we can write the unitary evolution as \( U_{BC} U_{AC} \). For initial states of the form \( \rho_{AC} \otimes \rho_B \) we have the following chain of arguments:

\[
Q_{A:B}(t) = Q_{A:BC}(U_{AC} \rho U_{AC}^\dagger) = Q_{A:C}(U_{AC} \rho U_{AC}^\dagger) \leq \sup_{|\psi\rangle} Q_{A:C},
\]

where the steps are justified as follows. Since the action of tracing out the (in general correlated) system \( \sigma \) is a local operation on \( BC \), we obtain the first inequality. The first equality follows from property 1 of Lemma 2. As we start with the initial state \( \rho_{AC} \otimes \rho_B \) and \( U_{AC} \) does not act on \( B \), system \( B \) stays uncorrelated in \( U_{AC} \rho U_{AC}^\dagger \). Using property 2 of lemma 2 we have the second equality. Finally, the correlation \( Q_{A:C} \) is again maximal on pure states. \( \square \)

Appendix C: Proof of separability via dipole-dipole coupling for particular initial states

Let us define \( \xi = (\hat{a} + \hat{b})/\sqrt{2} \). The dipole-dipole Hamiltonian \( \hat{H} \) is reformulated as:

\[
\hat{H} = \hat{x} \xi + \hat{x} \xi^\dagger + 2 \hat{x} \xi - \sigma_x + \sigma_y + \sigma_z.
\]

where \( \hat{x} = \sigma_x + \sigma_y + \sigma_z \) and \( \{ \xi, \xi^\dagger \} = I \). The unitary evolution operator is given by:

\[
\hat{U}_t = e^{-i \hat{H} t} = \frac{1}{2} \left( (I - \sigma_x) e^{i \sigma_y t} + (I + \sigma_x) e^{-i \sigma_y t} \right),
\]

where \( \alpha = \sigma_x \hat{a} \) and \( \hat{D}(\alpha) = \exp(\sigma_x \hat{a} \dagger - \sigma_x \hat{a}) \). Given an initial state \( |mn\rangle \), the state at time \( t \) reads:

\[
|\psi_t\rangle = \frac{1}{4} \left( (d_{++}^{(mn)} |D_+^m \rangle |D_+^n \rangle + d_{+}^{(mn)} |D_-^m \rangle |D_-^n \rangle) |0\rangle - (d_{-+}^{(mn)} |D_-^m \rangle |D_+^n \rangle + d_{--}^{(mn)} |D_+^m \rangle |D_-^n \rangle) |1\rangle \right),
\]

Note that \( \langle D_+^m | D_-^n \rangle = 0 \) and \( \langle D_-^m | D_+^n \rangle = 1 \). The Laguerre polynomial which comes from the relation \( \langle n | \hat{D}(\alpha) | n \rangle = e^{-|\alpha|^2/2} L_n(|\alpha|^2) \). After tracing out the atomic mode \( C \), the state of the fields is effectively given by a two-qubit state:

\[
\frac{1}{16} \begin{pmatrix}
(d_{++}^{(mn)})^2 & 0 & 0 & d_{++}^{(mn)} d_{--}^{(mn)} \\
0 & (d_{+-}^{(mn)})^2 & d_{+-}^{(mn)} d_{-+}^{(mn)} & 0 \\
d_{+-}^{(mn)} d_{-+}^{(mn)} & 0 & (d_{--}^{(mn)})^2 & 0 \\
d_{++}^{(mn)} d_{--}^{(mn)} & 0 & 0 & (d_{--}^{(mn)})^2
\end{pmatrix},
\]

which is PPT, and hence separable [29]. The same result follows for the initial state \( |mn\rangle \).
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