Definition of the Electromagnetic Field in the Broken-Symmetry Phase of the Electroweak Theory

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Abstract

In the broken-symmetry phase of the electroweak theory there is no unique definition of the electromagnetic field tensor in cases where the magnitude of the Higgs field differs from a constant value. The meaning of the electromagnetic field is therefore dubious near defects and during non-equilibrium stages of the electroweak phase transition. Nevertheless, by imposing a minimal set of natural requirements one is led to a specific, gauge-invariant definition that retains the familiar properties of an electromagnetic field. An electromagnetic vector potential is constructed whose curl (exterior derivative) in any gauge gives the electromagnetic field tensor. As is required, this vector potential transforms at most by a pure gradient under arbitrary SU(2) × U(1) gauge transformations. The flux of the magnetic field is expressed as a gauge-invariant line integral. Curiously, this provides a definition for magnetic flux in cases where the spatial region with broken symmetry is not simply connected and the magnetic field itself is not everywhere defined.
1. Introduction

It is well-known that the concept of an electromagnetic field has no meaning in the symmetric phase of the electroweak theory, since vector potentials may there be rotated into each other by gauge transformations. In contrast, one may expect that the electromagnetic field should become uniquely defined as soon as the SU(1)L×U(1)Y symmetry breaks to U(1)EM, because the photon field is then distinguished as the only vector field with zero electric charge and zero mass.

Despite this anticipated uniqueness, several gauge-invariant definitions of the electromagnetic field tensor are in common use. It was recently discovered by this author [1] that even those definitions that coincide when the Higgs magnitude \( \rho = (\Phi^\dagger \Phi)^{1/2} \) is constant, give different results when \( \rho \) has a space-time dependence. Unless this ambiguity can be resolved, statements about the strength, presence or absence of electromagnetic fields are meaningless when characterising field configurations that include a variation of the magnitude of the Higgs field, e.g. near (non-)topological defects, on the walls of expanding bubbles of the broken-symmetry phase in a first-order electroweak phase transition, or in the hot early universe subject to large thermal fluctuations of the Higgs field.

In the Higgs ground state, characterised by a constant Higgs field with magnitude \( \rho \equiv v \), the electromagnetic fields of everyday life are distinguished by a set of properties that uniquely set them apart from other strong and electroweak interactions:

A. An electromagnetic field is a long-range field, i.e. it always extends from point sources or line sources according to a power law without exponential suppression.

B. There are no magnetic charges or magnetic currents that can generate an electromagnetic field.

C. An electromagnetic field is never generated by an electrically neutral current.

In this Letter we show that the definition of the electromagnetic field tensor can be extended to a general gauge and space-time varying Higgs magnitude \( \rho(x) > 0 \) in such a way that all the above intuitive aspects of electromagnetic fields are preserved.

In section 2 we investigate the properties and physical consequences of various proposed gauge-invariant definitions of the electromagnetic field tensor in the electroweak
theory. By imposing the long-range force requirement (property A) one finds that all the
preexisting definitions are eliminated except the tensor $F_{\mu\nu}^{em}$ proposed in Ref. [1], given
here in eq. (7).

This argument does not prove per se the non-existence of other definitions that might
fulfil the requirement. It does, however, point to a field tensor which has the property C
and which satisfies the Bianchi identity, implying not only that property B is accommodated, but also that the field tensor may be written in any gauge as the curl of a vector
potential.

This vector potential is constructed in section 3 and in Appendix A. First, it is noted
that the electromagnetic vector potential in the unitary gauge is given by the usual massless field $A_{\mu}$, also when $\rho$ has a space-time dependence. By applying a general
gauge transformation to the vector fields, and expressing the SU(2) part of this gauge
transformation in terms of the Higgs field, a vector potential $A_{\mu}^{em}$ is constructed with
the property that it changes at most by a gradient under arbitrary SU(2)×U(1) trans-
formations and reduces to $A_{\mu}$ in the unitary gauge. Taking the curl of $A_{\mu}^{em}$ is shown to
produce $F_{\mu\nu}^{em}$.

Finally, in section 4 a gauge-invariant definition of magnetic flux is obtained as the line
integral of the electromagnetic vector potential $A_{\mu}^{em}$ along a closed curve. Curiously, this
provides an expression for the magnetic flux also in cases where the spatial region with
broken symmetry is not simply connected and the magnetic field itself is not everywhere
defined.

2. Gauge-Invariant Definitions of the
Electromagnetic Field Tensor

The mass eigenstates of the vector-boson fields are determined by the Lagrangian kinetic
term for the Higgs field, $(D^\mu \Phi)^\dagger D_\mu \Phi$, where

$$D_\mu = \partial_\mu - i(g/2)W^a_\mu \tau^a - i(g'/2)Y_\mu.$$  

With conventional definitions $A_\mu = \sin \theta_w W^3_\mu + \cos \theta_w Y_\mu$, $Z_\mu = \cos \theta_w W^3_\mu - \sin \theta_w Y_\mu$, and $W_\mu = (W^{1}_\mu - iW^{2}_\mu)/\sqrt{2}$ we can write the covariant derivative $D_\mu \Phi$ as

$$D_\mu \Phi = \begin{pmatrix}
\partial_\mu - ieA_\mu - i\gamma Z_\mu - i\frac{M_w}{v} W_\mu \\
-i\frac{M_w}{v} W^{\dagger}_\mu \\
\end{pmatrix} \Phi ,$$  

(1)
where \( \tan \theta_w = g'/g \), \( 2\gamma = g \cos 2\theta_w / \cos \theta_w \), \( e = g \sin \theta_w \), \( M_W^2 = g^2v^2/2 \) and \( M_Z = M_W / \cos \theta_w \).

In the unitary gauge the Higgs field is given by

\[
\Phi = \begin{pmatrix} 0 \\ \rho(x) \end{pmatrix}, \quad \rho(x) \geq 0,
\]

(2)

and it follows immediately from eq. (1) that only the field \( A_\mu \) is massless, while the fields \( W_\mu \) and \( Z_\mu \) acquire masses. What is usually not mentioned in textbooks is that this holds true regardless of the functional form of the magnitude \( \rho(x) \). Therefore, in a unitary gauge given by eq. (2), \( A_\mu \) is always the electromagnetic vector potential.

Although global field configurations may sometimes be expressed in the unitary gauge, this is usually not possible when the Higgs field \( \Phi \) has zeros. For example, in vortex solutions such as the electroweak \( W \)-string or \( Z \)-string [2], the winding of the Higgs isospin orientation is such that the vector fields become singular when transformed to the unitary gauge. Another example concerns the collision of expanding bubbles of the broken-symmetry phase in a first-order electroweak phase transition. In general the bubbles, having had no previous causal contact, will contain Higgs fields with different isospin orientation, and it would be awkward to describe them in a unitary gauge. Worse still, since field configurations similar to electroweak strings can be produced in such bubble collisions [3], imposing the unitary gauge would result in a loss of generality.

Consider therefore a general gauge with Higgs field \( \Phi = (\varphi_1(x), \varphi_2(x))^\top \). The field \( A_\mu \) defined in eq. (1) then couples to \( \varphi_1(x) \) and becomes massive. Evidently, at positions \( x \) where \( \varphi_1(x) \neq 0 \), \( A_\mu \) is no longer the photon field. Consequently, the electromagnetic field tensor \( F_{\mu\nu}^{em} \) can no longer have the same expression as in the unitary gauge. Instead, one must construct a gauge-invariant definition of \( F_{\mu\nu}^{em} \) whose value in any gauge coincides (except at points where \( \Phi = 0 \)) with that obtained by (locally) transforming all fields to the unitary gauge and evaluating it there. To this end, let us define a three-component unit isovector \( \hat{\varphi}^a = (\Phi^\dagger \tau^a \Phi) / (\Phi^\dagger \Phi) \), where \( \tau^a, a = 1 \ldots 3 \), are the Pauli spin matrices. Note that \( \hat{\varphi}^a \) is independent of the magnitude \( \rho \) and depends only on the isospin orientation of \( \Phi \).

Let us start by considering a gauge-invariant definition first proposed by Nambu [4] and subsequently used in investigations of the distribution of electromagnetic fields and
charges inside the electroweak sphaleron [3, 4]. It is given by

\[ F_{N}^{\mu\nu} := -\sin\theta_{W} \phi^{a} F_{a}^{\mu\nu} + \cos\theta_{W} F_{Y}^{\mu\nu}, \]

where \( F_{a}^{\mu\nu} = \partial_{\mu} W_{\nu}^{a} - \partial_{\nu} W_{\mu}^{a} + g\epsilon^{abc} W_{\mu}^{b} W_{\nu}^{c} \) and \( F_{Y}^{\mu\nu} = \partial_{\mu} Y_{\nu} - \partial_{\nu} Y_{\mu} \) are the SU(2) \( \text{L} \) and U(1) \( Y \) field tensors, respectively. In the unitary gauge (2) we have \( \phi^{3} = -\delta^{3} \) and \( F_{N}^{\mu\nu} \) reduces to

\[ F_{N}^{\mu\nu} = A_{\mu\nu} + i\epsilon(W_{\mu}^{\dagger} W_{\nu} - W_{\nu}^{\dagger} W_{\mu}) \]

with

\[ A_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \]

A peculiar, and in our opinion unattractive, property of this definition is that it admits finite-range electromagnetic fields. This becomes apparent in the unitary gauge if we consider a field configuration with \( A_{\mu\nu} = 0 \) containing a localised distribution of charged \( W \) bosons with \( W^{\dagger}_{\mu} W_{\nu} \neq 0 \). Although this is not a common-day occurrence, such a configuration of \( W \) fields is in principle realisable in nature. Because the \( W \) fields are massive, the electromagnetic fields given by \( F_{N}^{\mu\nu} \) are non-zero but decay exponentially away from their sources. This marks a departure from the usual, intuitive notion of electromagnetic fields as being long-range fields with a power-law behaviour. Moreover, since \( A_{\mu} \) is a massless field in the unitary gauge regardless of \( \rho(x) \), we are guaranteed of the existence of another field tensor, e.g. \( A_{\mu\nu} \), which always decays away from its sources according to a power law, and if this is not the electromagnetic field, one may have to consider a different name for the field with this property.

There is a clarifying analogy with Maxwell’s equations in a medium which corroborates this picture. We can identify the field tensor components \( F_{ij}^{N} \) with the magnetic intensity \( H \), whose sources are only the free, or external, currents. The magnetic field, or induction, \( B \) is in the unitary gauge identified with \( A_{ij} \) and is produced by all currents, including the magnetisation currents in the medium. These fields are related by

\[ H = B - 4\pi M, \]

where the magnetisation \( M \) can be identified with the \( W \)-boson terms.

To justify this analogy, let us consider a uniform magnetic intensity \( H = |H| \) in the \( x_{3} \) direction. If the external current is sufficiently high, so that \( H \) exceeds the critical value \( M_{W}^{2}/e \), the electroweak vacuum becomes unstable with respect to the production
of a condensate of $W$-boson pairs in the spin polarisation state $W_1 = -iW_2 \equiv W$. With $H = F_{12}^N$ and $B = A_{12}$ it follows that $B = H + 2e|W|^2$, and one finds that the $W$ bosons contribute a positive magnetisation $M = e|W|^2/(2\pi)$. As a consequence the vacuum is paramagnetic for $H > M_W^2/e$, as was shown in Ref. [7].

Let us return to the Gedanken experiment with the localised $W$-boson distribution. It follows from the field equations for $A_{\mu\nu}$ that, in order to obtain $A_{\mu\nu} = 0$, which was part of the premises, one would have to screen the current of the $W$ fields by means of an external current. If we now interpret $A_{ij}$ as the magnetic field, the mutual screening of the two currents provides a physical reason for $A_{ij}$ to be zero. Moreover, there is no contradiction between $A_{ij}$ being zero at large distances and being a long-range force with a power-law behaviour, because $A_{ij}$ is zero everywhere.

Armed with this new intuition, let us now investigate some alternative gauge-invariant definitions of the electromagnetic field tensor.

In order to obtain the full electromagnetic field $A_{\mu\nu}$ in the unitary gauge, i.e. not only the part generated by free currents, one would have to subtract the $W$-field terms of $F_{\mu\nu}^N$ in a gauge-invariant way. This problem was partly solved by Vachaspati, who proposed the following field tensor,

$$F^V_{\mu\nu} := -\sin \theta_w \hat{\phi}^a F^a_{\mu\nu} + \cos \theta_w F^Y_{\mu\nu} - \frac{i\sin \theta_w}{g} \frac{2}{\Phi} \left[ (D_\mu \Phi)^\dagger D_\nu \Phi - (D_\nu \Phi)^\dagger D_\mu \Phi \right].$$

The added term in the above expression cancels the quadratic terms in the $W$ field correctly when $\rho$ is constant, but introduces extraneous terms when $\rho$ has a space-time dependence. This becomes apparent in the unitary gauge, where $F^V_{\mu\nu}$ reduces to

$$F^V_{\mu\nu} = A_{\mu\nu} - 2\tan \theta_w (Z_\mu \partial_\nu \ln \rho - Z_\nu \partial_\mu \ln \rho).$$

Consider now the case $A_{ij} = 0$ with non-zero field components $Z_i$ and gradient of the Higgs magnitude $\partial_i \rho$, such that $\epsilon_{ijk} Z_j \partial_k \rho \neq 0$. This situation is characteristic of the interior of the electroweak $Z$-string solution. With the definition one would then conclude that there is a magnetic field present. However, as is shown in detail in Ref. [1], this definition would imply that electromagnetic fields can be generated by electrically neutral currents. These currents are obtained in the unitary gauge by taking the divergence of eq. (5), and consist of derivatives of the neutral fields $Z_\mu$ and $\rho$. 

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Because of the cylindrical symmetry of the Z-string solution, even a long-range electromagnetic field with power-law behaviour away from point or line sources would be zero outside the string (cf. a solenoid). For \( A_{\mu\nu} = 0 \), and for \( Z_{\mu\nu} \partial_\nu \rho \neq 0 \) inside some finite volume \( \mathcal{V} \) with less symmetry, one can show as before that the electromagnetic fields given by \( \mathcal{F}^{\mu\nu} \) are non-zero within \( \mathcal{V} \), but decay exponentially away from \( \mathcal{V} \), thereby departing from the expected behaviour of electromagnetic fields away from sources of generic geometry.

A definition of the electromagnetic field tensor that exhibits none of the above unattractive features was proposed in Ref. [1]:

\[
\mathcal{F}^{\text{em}}_{\mu\nu} \equiv -\sin \theta_w \hat{\phi}^a F^a_{\mu\nu} + \cos \theta_w F^Y_{\mu\nu} + \frac{\sin \theta_w}{g} \epsilon^{abc} \hat{\phi}^a (D_\mu \hat{\phi})^b (D_\nu \hat{\phi})^c ,
\]

(7)

where \( (D_\mu \hat{\phi})^a = \partial_\mu \hat{\phi}^a + g \epsilon^{abc} W^b_\mu \hat{\phi}^c \). It reduces to \( A_{\mu\nu} \) in the unitary gauge \([2]\) for arbitrary \( \rho = \rho(x) \). The fact that \( A_\mu \) is always a massless field in this gauge ensures that \( \mathcal{F}^{\text{em}}_{\mu\nu} \) is a long-range field with a power-law behaviour. In fact, \( \mathcal{F}^{\text{em}}_{\mu\nu} \) is the unique gauge-invariant realisation of \( A_{\mu\nu} \), since two gauge-invariant tensors that agree in one gauge have the same value in any gauge. The definition \((7)\) has appeared previously in Ref. \([3]\) and, for the Glashow-Georgi SO(3) model, in Ref. \([10]\).

Unlike \( \mathcal{F}^{\text{N}}_{\mu\nu} \) and \( \mathcal{F}^{\text{V}}_{\mu\nu} \), the electromagnetic field tensor \( \mathcal{F}^{\text{em}}_{\mu\nu} \) satisfies the Bianchi identity \( \mathcal{F}^{\text{em}}_{[\mu\nu,\alpha]} = 0 \) everywhere except on the worldlines of magnetic monopoles. Therefore, in the absence of monopoles, there is no magnetic charge or magnetic current. Furthermore, there are no contributions to \( \mathcal{F}^{\text{em}}_{\mu\nu} \) from electrically neutral currents. The electric current \( j^e_\nu \equiv \partial^\mu \mathcal{F}^{\text{em}}_{\mu\nu} \) and its properties were discussed at length in Ref. \([1]\).

An important consequence of the definition \((7)\) is that the electromagnetic field tensor receives contributions from gradients of the phases (i.e. isospin orientation) of the Higgs field, even when the vector potentials are zero. To better understand this aspect, consider a field configuration where all the vector potentials \( W^a_\mu \) and \( Y_\mu \) are zero, but the Higgs field is of the general form

\[
\Phi = \rho \left( \begin{array}{c} e^{i\alpha} \sin \omega \\ e^{-i\beta} \cos \omega \end{array} \right),
\]

(8)

where \( \rho, \alpha, \beta, \) and \( \omega \) are all functions of the space-time coordinates \( x^\mu \). This configuration can be transformed to the unitary gauge by means of a gauge transformation \( \Phi \rightarrow U \Phi \)
with \( U \in SU(2) \) defined by

\[
U = \begin{pmatrix}
 e^{-i\beta} \cos \omega & -e^{i\alpha} \sin \omega \\
 e^{-i\alpha} \sin \omega & e^{i\beta} \cos \omega
\end{pmatrix}.
\] (9)

We then obtain, in the unitary gauge, a Lie-algebra valued vector potential \( A_\mu = -i(\partial_\mu U)U^\dagger \) with components \( W_\mu^a = \text{Tr}(\tau^a A_\mu)/g \) and \( Y_\mu = \text{Tr} A_\mu/g' \). In this gauge the vector potential \( A_\mu = \sin \theta_w W_\mu^3 + \cos \theta_w Y_\mu \) becomes

\[
A_\mu = \frac{2 \sin \theta_w}{g} \left[ \sin^2 \omega \partial_\mu \alpha - \cos^2 \omega \partial_\mu \beta \right].
\] (10)

and its curl gives the electromagnetic field

\[
A_{\mu\nu} = \frac{2 \sin \theta_w}{g} \sin 2\omega (\omega_{[\mu\nu]} + \omega_{[\mu\beta\nu]}) \sin 2\omega \partial_\mu \alpha - \cos^2 \omega \partial_\mu \beta].
\] (11)

On the other hand, one can evaluate the field tensor \( F^\text{em}_{\mu\nu} \) of eq. (7) directly for the general Higgs configuration (8) and \( W_\mu^a = Y_\mu = 0 \). The last term of \( F^\text{em}_{\mu\nu} \) then takes into account the electromagnetic field associated with the space-time dependent Higgs isospin orientation, and reproduces the right-hand side of eq. (11). Such an electromagnetic field was not included in definition (3).

It is interesting to note that the expression (3) was derived by Nambu as the most general non-zero field configuration compatible with the two conditions \( \rho \equiv v \) and \( D_\mu \Phi = 0 \), which together define the Higgs vacuum. The purpose of his definition was to characterise the behaviour of fields far from defects, not in the interior of defects where these two conditions are violated. Indeed, in the unitary gauge the condition \( D_\mu \Phi = 0 \) implies that \( W_\mu = W_\mu^\dagger = 0 \), so if definition (3) is used only in the restricted case \( D_\mu \Phi = 0 \) as was intended, the field tensor \( F^N_{\mu\nu} \) reduces to \( A_{\mu\nu} \) in the unitary gauge.

3. “Gauge-Invariant” Vector Potential for the Electromagnetic Field

Because the preferred electromagnetic field tensor (7) satisfies the Bianchi identity \( F^\text{em}_{[\mu\nu,\alpha]} = 0 \), it is possible to construct a vector potential \( A^\text{em}_\mu \) for the electromagnetic field with the property that \( F^\text{em}_{\mu\nu} = \partial_\mu A^\text{em}_\nu - \partial_\nu A^\text{em}_\mu \). Since \( F^\text{em}_{\mu\nu} \) is gauge-invariant, the vector potential must transform at most by a pure gradient under arbitrary \( SU(2) \times U(1) \) gauge
transformations. Appendix A describes the method for constructing vector potentials
with this invariance. Here we shall state only the results.

A non-zero scalar-doublet Higgs field \( \Phi = (\varphi_1, \varphi_2)^\top \) can always be written \( \Phi = V(0, \rho)^\top \), where \( \rho = (\Phi^\dagger \Phi)^{1/2} \) is the magnitude and the isospin orientation is defined by
the SU(2)-valued function \( V \),

\[
V = \frac{1}{\rho} \left( (i\tau^2 \Phi)^* \Phi \right) = \frac{1}{\rho} \left( \varphi_2^* \varphi_1 \right. \left. - \varphi_1^* \varphi_2 \right).
\]

The vector potential \( A_{\mu}^{em} \) can then be expressed as

\[
A_{\mu}^{em} = \sin \theta_w \left\{ -\hat{\phi}^a W^a_{\mu} + \frac{i}{g} \text{Tr}(\tau^3 V^\dagger \partial_\mu V) \right\} + \cos \theta_w Y_\mu.
\]

As is shown in Appendix A, this expression for \( A_{\mu}^{em} \) is in fact invariant under SU(2) gauge
transformations, and changes by a pure gradient under U(1) transformations.

A more practical expression, for which the connection to eq. (7) is easier to establish,
is obtained by rewriting \( V \) in terms of the unit isovector \( \hat{\phi} = \{ \hat{\phi}^a \} \in S^2 \) and a U(1)
phase. One finds

\[
V = \begin{pmatrix} 1 - \hat{\phi}^3 & \hat{\phi}^1 - i\hat{\phi}^2 \\ -\hat{\phi}^1 - i\hat{\phi}^2 & 1 - \hat{\phi}^3 \end{pmatrix} \frac{e^{-i\xi\tau^3}}{\sqrt{2(1 - \hat{\phi}^3)}},
\]

where \( \xi \) is the phase of the lower component of \( \Phi \). By means of the unit-vector constraint
\( \hat{\phi}^a \hat{\phi}^a = 1 \) the vector potential \( A_{\mu}^{em} \) can now be written

\[
A_{\mu}^{em} = -\sin \theta_w \hat{\phi}^a W^a_{\mu} + \cos \theta_w Y_\mu - \frac{\sin \theta_w}{g} \frac{1}{1 - \hat{\phi}^3} e^{3ab} \hat{\phi}^a \partial_\mu \hat{\phi}^b.
\]

A term \( \partial_\mu (2 \sin \theta_w \xi / g) \) was omitted here, as \( A_{\mu}^{em} \) is defined only up to a gradient.

In eq. (15) the apparent singularity of \( A_{\mu}^{em} \) at \( \hat{\phi}^3 = 1 \), corresponding to Higgs fields
of the form \( \Phi = (\phi, 0)^\top, \phi \in C \), is merely an artefact of the coordinate system chosen
in eq. (14) and arises because there is no global decomposition of SU(2) as a Cartesian
product of \( S^2 \) and U(1). If one instead parametrises the isospin orientation of the Higgs
field by angles \( \alpha, \beta \) and \( \omega \) as in eq. (8), then the last term of eq. (15) becomes

\[
\frac{2 \sin \theta_w}{g} \sin^2 \omega \partial_\mu (\alpha + \beta),
\]

which is regular everywhere.
Although eqs. (15) and (7) look superficially similar, the proof that \( \mathcal{F}_{\mu \nu}^{em} = \partial_\mu A_{\nu}^{em} - \partial_\nu A_{\mu}^{em} \) requires some effort, and is deferred to Appendix B. Instead, let us here establish that \( A_{\mu}^{em} \) changes at most by a pure gradient under arbitrary gauge transformations.

First, under a U(1) transformation \( \Phi \rightarrow e^{i\theta} \Phi \), the only change is \( Y_\mu \rightarrow Y_\mu + 2 \partial_\mu \theta / g' \), which gives a pure gradient in \( A_{\mu}^{em} \). Continuing with the group SU(2) we consider, for simplicity, infinitesimal transformations defined by

\[
\begin{align*}
\Phi & \rightarrow (1 + \frac{i}{2} \omega^a(x) \tau^a) \Phi , \\
\hat{\phi}^a & \rightarrow \hat{\phi}^a - \epsilon^{abc} \omega_b \hat{\phi}^c , \\
W^a_\mu & \rightarrow W^a_\mu - \epsilon^{abc} \omega^b W^c_\mu + \frac{1}{g} \partial_\mu \omega^a .
\end{align*}
\]  

Inserting this into eq. (15) and expanding in \( \omega^a \), the linear terms are

\[
\begin{align*}
\frac{\sin \theta_w}{g} \left\{ \frac{1}{1 - \hat{\phi}^3} \left[ \omega^3_\mu - \hat{\phi}^a \omega^a_\mu - \omega^a \hat{\phi}^a \hat{\phi}^3_\mu + \hat{\phi}^3 \omega^a \hat{\phi}^a_\mu \right] + \frac{1}{(1 - \hat{\phi}^3)^2} \epsilon^{3ab} \epsilon^{3cd} \hat{\phi}^a \hat{\phi}^b \omega^c \hat{\phi}^d \right\} \\
= \partial_\mu \left[ \frac{\sin \theta_w}{g(1 - \hat{\phi}^3)} (\omega^3 - \hat{\phi}^a \omega^a) \right] .
\end{align*}
\]  

In the last step, leading again to a pure gradient, the constraint \( \hat{\phi}^a \hat{\phi}^a = 1 \) was used.

In the unitary gauge, where \( \Phi = (0, \rho)^T \) and \( \hat{\phi}^a \equiv -\delta^{a3} \), the electromagnetic potential \( A_{\mu}^{em} \) reduces to the usual expression \( A_\mu = \sin \theta_w W^3_\mu + \cos \theta_w Y_\mu \). This fact, together with the invariance property of \( A_{\mu}^{em} \) under gauge transformations, independently establishes that \( \mathcal{F}_{\mu \nu}^{em} = \partial_\mu A_{\nu}^{em} - \partial_\nu A_{\mu}^{em} \).

It is noteworthy that, although expressions similar to eq. (13) have occurred previously \[8, 5\], the crucial term with the matrix \( V \) was missing. This term is needed to cancel the inhomogeneous Maurer-Cartan term acquired by the vector potentials under gauge transformations. Expressions without this term are not gauge-invariant and also fail in general to project out the massless component of the gauge potential.
4. Gauge-Invariant Definition of Magnetic Flux

Because of the existence of a vector potential $A^{em}$ such that $F^{em} = d \wedge A^{em}$, we have by Stokes’ theorem that

$$\int_S F^{em}_{\mu\nu} \, dx^\mu \wedge dx^\nu = \oint_{\partial S} A^{em}_\mu \, dx^\mu,$$

where $S$ is an oriented surface with correspondingly oriented boundary $\partial S$. Therefore, the flux $\Phi_B$ of the magnetic field $B^i = \frac{1}{2} \epsilon^{ijk} F^{em}_{jk}$ across a surface $S$ with normal $n$ is given by

$$\Phi_B = \int_S B^i n^i \, dS = \oint_{\partial S} A^{em}_i \, dx^i.$$

This expression is invariant under $SU(2) \times U(1)$ gauge transformations and is useful for evaluating the magnetic flux in practical applications such as computer simulations of the production of magnetic fields in bubble collisions in the electroweak phase transition.

Numerical evaluation of the flux then requires only one-dimensional integration.

Let us now consider a rather curious example where the line integral of $A^{em}_i$ along the curve $\partial S$ has a well-defined value, but the magnetic field $B^i$ is ill-defined on a subset of the surface $S$. In such a case, Stokes’ theorem cannot strictly be said to hold, but the line integral in eq. (20) can nevertheless be used to define a magnetic flux.

Figure 1 shows a section in the plane of collision of three expanding spherical bubbles of the broken-symmetry phase in a first-order electroweak phase transition. If the Higgs field has different isospin orientation in each of the three bubbles, it can be shown that a magnetic field is produced in the collision [11, 12]. This magnetic field is initially given by the last term in eq. (7). At the particular instant depicted, the bubbles have recently joined at the midpoints of the line segments AB, BC and CA, but a triangle-like region at the center of the figure has not yet been reached by the expanding bubble walls and still contains the symmetric phase with $\Phi = 0$. In this region, $\hat{\phi}^a$ and the magnetic field $F^{em}_{ij}$ are undefined. Nevertheless, since $A^{em}_i$ is defined everywhere along the contour ABCA, the magnetic flux through the triangle bounded by this contour can be defined as

$$\Phi_B = \oint_{ABCA} A^{em}_i \, dx^i.$$  

In cases where the contour $\partial S$ encloses one or more string defects, the set where $\Phi = 0$ and $F^{em}_{ij}$ is ill-defined has measure zero, and both integrals in eq. (20) are well-defined.
Figure 1: Collision of three expanding bubbles in the electroweak phase transition. The shaded region corresponds to the broken-symmetry phase.

Except in pathological gauges, the components $A_{i}^{em}$ are continuous functions of time along the path ABCA. Therefore, the expression (21) will approach continuously the value of the surface integral of the magnetic field at the instant when the hole between the bubbles disappears and the region of broken symmetry becomes simply connected so that eq. (20) becomes valid.

5. Conclusions

We have shown that the definition of the electromagnetic field tensor can be extended to a general gauge and to a space-time varying Higgs magnitude $\rho(x)$ in such a way that the familiar properties of an electromagnetic field are retained. More precisely, the gauge-invariant field tensor $F_{\mu\nu}^{em}$ defined by eq. (7) is a long-range field with power-law behaviour away from point sources or line sources (A), satisfies the Bianchi identity everywhere except on the worldlines of magnetic monopoles (B), and is never generated
by an electrically neutral current (C).

These three properties do not uniquely define an electromagnetic field tensor. Even in the Higgs ground state with \( \rho \equiv v \), the choice \( F_{\mu\nu}^{\text{em}} = A_{\mu\nu} \) (unitary gauge), with \( A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), is a matter of long-standing convention. One may add, for example, \( \partial_\mu (A_{\nu\lambda} h^\lambda) \) to the field tensor, where \( h^\alpha \) is any tensor containing charged fields that is invariant under the unbroken U(1) symmetry, without affecting any of the properties A, B or C. Nonetheless, the conventional choice \( A_{\mu\nu} \) is the simplest field tensor that can be constructed from \( A_\mu \), and the tensor \( F_{\mu\nu}^{\text{em}} \) defined by eq. (7) is its unique gauge-invariant extension.

The field tensor \( F_{\mu\nu}^{\text{em}} \) has the important property that it receives contributions from gradients of the phases of the Higgs field also when the vector potentials are zero. This is as should be expected, because under a transformation to the unitary gauge these gradients are converted into non-zero vector potentials that contribute to \( A_{\mu\nu} \). The generation of electromagnetic fields from such Higgs gradients has important applications in cosmology, e.g. in electroweak bubble collisions [3, 11, 12], and may likewise have some significance in the interior of defects such as the electroweak sphaleron [5, 6].

Finally, we have constructed a “gauge-invariant” vector potential \( A_{\mu}^{\text{em}} \) for the electromagnetic field with the property that it transforms at most by a pure gradient under arbitrary SU(2) \( \times \) U(1) gauge transformations. The field tensor \( F_{\mu\nu}^{\text{em}} \) is given in any gauge by the curl of \( A_{\mu}^{\text{em}} \), which by Stokes’ theorem implies that magnetic flux can be expressed as a gauge-invariant line integral.

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Appendix A: SU(2)-invariant vector potentials

In this appendix we describe the method for constructing vector potentials invariant under SU(2) gauge transformations. In the unitary gauge, where \( \Phi = \Phi_0 \equiv (0, \rho)^T \), let the Lie-algebra valued vector potential be given by \( \vec{A}_\mu = (g/2) \bar{W}_\mu^a \tau^a + (g'/2) \bar{Y}_\mu \). The
vector potential in an arbitrary gauge with Higgs field \( \Phi = V \Phi_0 \) is then found by means of an SU(2) \( \times \) U(1) gauge transformation defined by the function \( U = V \exp(i\lambda Q) \), where \( V \) is given by eq. (12) and the charge operator \( Q = (1 + \tau^3)/2 \) satisfies \( Q\Phi_0 = 0 \). Under this transformation \( \bar{A}_\mu \rightarrow A_\mu = U\bar{A}_\mu U^\dagger - i(\partial_\mu U)U^\dagger \). Solving for \( \bar{A}_\mu \), we obtain

\[
\bar{A}_\mu = e^{-i\lambda Q}[V^\dagger A_\mu V + iV^\dagger \partial_\mu V]e^{i\lambda Q} - \partial_\mu \lambda Q . \tag{22}
\]

Because \( \bar{A}_\mu \) is the vector potential in the unitary gauge, it is, by definition, gauge-invariant, and so are its components \( \bar{W}^3_\mu = \text{Tr}(\tau^3\bar{A}_\mu)/g \) and \( \bar{Y}_\mu = \text{Tr}\bar{A}_\mu/g' \). Now write the vector potential in the new, arbitrary gauge as \( A_\mu = (g/2)W^a_\mu \tau^a + (g'/2)Y_\mu \). Using the identity \( V \tau^3 V^\dagger = -\hat{\phi}^a \tau^a \) one obtains

\[
\begin{align*}
\bar{W}^3_\mu &= -\hat{\phi}^a W^a_\mu + \frac{i}{g} \text{Tr}(\tau^3 V^\dagger \partial_\mu V) - \frac{1}{g} \partial_\mu \lambda , \\
\bar{Y}_\mu &= Y_\mu - \frac{1}{g'} \partial_\mu \lambda . \tag{23}
\end{align*}
\]

The gauge-invariant electromagnetic vector potential is given by its value in the unitary gauge, \( \bar{A}_\mu = \sin \theta W^3_\mu + \cos \theta W^1_\mu \). This differs from the expression (13) only by a pure-gradient term \(-\partial_\mu \lambda/e\), which may be omitted. Whereas \( V \) is uniquely determined by the Higgs field \( \Phi \), the function \( \lambda(x) \) is arbitrary and corresponds to the unbroken U(1) symmetry of electromagnetism.

**Appendix B: Curl of the electromagnetic vector potential**

In this appendix we prove that the curl of \( A^\text{em}_\mu \), given by eq. (15), equals \( F^\text{em}_{\mu\nu} \), expressed in eq. (7). The constraint \( \hat{\phi}^a \hat{\phi}^a = 1 \) will be employed repeatedly. We start by expanding \( F^\text{em}_{\mu\nu} \), noting that terms quadratic in the fields \( W^a_\mu \) cancel, and obtain a sum of four terms,

\[
F^\text{em}_{\mu\nu} = - \sin \theta \omega W^3_\mu + \cos \theta Y_\mu + \frac{\sin \theta}{g} \epsilon^{abc} \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c . \tag{24}
\]

Let us denote the terms of eq. (15) by \( A^{(i)}_\mu \), \( i = 1 \ldots 3 \), and those of eq. (24) by \( F^{(i)}_{\mu\nu} \), \( i = 1 \ldots 4 \). Taking the curl of the first term of \( A^\text{em}_\mu \), one obtains \( \partial_\mu A^{(1)}_\nu = F^{(1)}_{\mu\nu} + F^{(3)}_{\mu\nu} \).
Trivially, $\partial_\mu A^{(2)}_{\nu} = {\mathcal F}^{(2)}_{\mu\nu}$. It remains to show that $\partial_\mu A^{(3)}_{\nu} = {\mathcal F}^{(4)}_{\mu\nu}$. For simplicity set $\sin \theta_w / g = 1$, as it is a common factor, and consider

$$\partial_\mu A^{(3)}_{\nu} = -\frac{2}{1 - \partial^3} \epsilon^{3bc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c - \frac{1}{(1 - \partial^3)^2} \partial_\mu \hat{\phi}^3 \epsilon^{3bc} \partial_\nu \hat{\phi}^b \partial_\nu \hat{\phi}^c .$$

(25)

By means of the identity $\partial_\mu \hat{\phi}^3 \epsilon^{3bc} \partial_\nu \hat{\phi}^b \partial_\nu \hat{\phi}^c = -\epsilon^{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c + \epsilon^{3bc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c$ the right-hand side can be rearranged to give

$$\partial_\mu A^{(3)}_{\nu} = \frac{1}{(1 - \partial^3)^2} \left[ \epsilon^{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c + (\partial^3 - 2) \epsilon^{3bc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c \right] .$$

(26)

The identity $\epsilon^{3bc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c = \hat{\phi}^3 \epsilon^{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c \partial_\nu \hat{\phi}^c$ leads to the result $\partial_\mu A^{(3)}_{\nu} = \epsilon^{abc} \partial_\mu \hat{\phi}^b \partial_\nu \hat{\phi}^c = {\mathcal F}^{(4)}_{\mu\nu}$, which completes the proof. □

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