When to carry out analytic continuation?

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Abstract

This paper discusses the analytic continuation in the thermal field theory by using the theory of $\eta - \xi$ spacetime. Taking a simple model as example, the $2 \times 2$ matrix real-time propagator is solved from the equation obtained through continuation of the equation for the imaginary-time propagator. The geometry of the $\eta - \xi$ spacetime plays important role in the discussion.

Key words: continuation, Spacetime, Thermal Field Theory
It is well known that the finite temperature field theory (FTFT) has two formalisms: the imaginary-time formalism and the real-time formalism (Landsman 1987). The imaginary-time formalism is characterized by the periodicity of imaginary-time which leads to discrete imaginary energy in the imaginary-time thermal Green functions (Matsubara 1955, Fetter and Walecka 1965), and the real-time formalism is characterized by the doubling of the degrees of freedom which causes the real-time thermal Green functions to have $2 \times 2$ matrix structures (Niemi and Semenoff 1984, Umezawa et al 1982). This paper discusses a problem concerning the connection of the two formalisms, i.e. the analytic continuation of thermal propagators. Although there is not direct analytic continuation between the imaginary-time thermal propagators and the $2 \times 2$ form real-time thermal propagators even in the most simple case, such a relation does exist between the equations for propagators. It is interesting to find that the difference between these two situations is easily explained by the geometrical features of a spacetime with $S^1$ topology, named the $\eta - \xi$ spacetime (Gui 1988, Gui 1990, Gui 1992, Gui 1993) rather than by the Minkowskian spacetime.

The theory of $\eta - \xi$ spacetime is constructed in order to provide a unique geometrical background for FTFT. The most important parts of the $\eta - \xi$ spacetime are its Euclidean section and Lorentzian section. The Euclidean section has a $S^1$ topology, which makes quantum fields satisfy the periodicity for imaginary-time, and the Euclidean propagators in the $\eta - \xi$ spacetime correspond to the imaginary-time thermal propagators. An interesting fact about the Lorentzian section is that its geometrical structure is very much similar to those of the Rindler spacetime and black hole. The infinities of the Minkowskian spacetime become "horizons" on the Lorentzian section which lead to the doubling of degrees of freedom of fields, and the vacuum propagator in the $\eta - \xi$ spacetime is equal to the $2 \times 2$ matrix real-time thermal propagator in the Minkowskian spacetime. It was suggested (Gui 1993, Zuo and Gui 1995) that the field theories on the Euclidean section and Lorentzian section correspond to the imaginary-time formalism and real-time formalism of FTFT, respectively. The special geometrical structures of the $\eta - \xi$ spacetime shall also affect the relation between the two formalisms of FTFT, e.g. the situation in which direct analytic continuation can be carried out.

This paper first gives a brief description of the structures of the $\eta - \xi$ spacetime and some relations to be used. Then by discussing the procedures of solving the equations for propagators
on the Euclidean section and on the Lorentzian section respectively, it explains how the geometry influences the feasibility of analytic continuation.

The four dimensional $\eta-\xi$ spacetime can be regarded as the maximal analytic complex extension of $S^1 \times R^3$ manifold (Gui 1990). It has the following complex metric:

$$ds^2 = \frac{1}{\alpha^2(\xi^2 - \eta^2)}(-d\eta^2 + d\xi^2) + dy^2 + dz^2$$  \hspace{1cm} (1)$$

where $\alpha = 2\pi/\beta$ is a real constant and $\eta, \xi, y, z$ are complex variables. If we limit $\xi, y, z$ to be real and $\eta$ to be a pure imaginary variable $i\sigma$, the Euclidean section of the $\eta-\xi$ spacetime is obtained:

$$ds^2 = \alpha^{-2}(\xi^2 + \sigma^2)^{-1}(d\sigma^2 + d\xi^2) + dy^2 + dz^2$$  \hspace{1cm} (2)$$

which under the transformation

$$\sigma = \alpha^{-1}e^{\alpha x} \sin \alpha \tau$$

$$\xi = \alpha^{-1}e^{\alpha x} \cos \alpha \tau$$  \hspace{1cm} (3)$$

becomes a flat Euclidean spacetime

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2$$  \hspace{1cm} (4)$$

The metric (2) is singular at $\sigma = \xi = 0$, so it describes an Euclidean spacetime with $S^1 \times R^3$ topology. The periodicity of polar angle $\alpha \tau$ naturally supplies the periodicity of imaginary-time in FTFT. Now continue $\sigma$ to $\sigma e^{-i\theta}$. The singularity becomes

$$\xi^2 + \sigma^2 e^{-2i\theta} = 0$$  \hspace{1cm} (5)$$

which requires $\sigma = \xi = 0$ for all values of $\theta$ except $0$ and $\pi/2$. $\theta = 0$ is just the case of the Euclidean section. When $\theta = \pi/2$, the $\sigma$ coordinate changes into a real $\eta$ coordinate and the resulted $\eta-\xi$ hyperplane is the Lorentzian section of the $\eta-\xi$ spacetime. The singularities on the Lorentzian section are described by

$$\xi^2 - \eta^2 = 0$$  \hspace{1cm} (6)$$

which divide the Lorentzian section into four disjointed regions I, II, III, IV. This structure resembles that of the Schwarzschild spacetime, thus the singularities (6) are also called ”horizons".
Each of the regions is identified with a four dimensional Minkowskian spacetime. One can see this from the transformation

\[ \eta = \alpha^{-1}e^{\alpha x} \sinh \alpha t \quad \xi = \alpha^{-1}e^{\alpha x} \cosh \alpha t \]  

(7)

which transforms region I of the Lorentzian section into

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \]

(8)

Similarly, regions II, III, IV are transformed by

\[ \eta = -\alpha^{-1}e^{\alpha x} \sinh \alpha t \quad \xi = -\alpha^{-1}e^{\alpha x} \cosh \alpha t \]

\[ \eta = \alpha^{-1}e^{\alpha x} \cosh \alpha t \quad \xi = \alpha^{-1}e^{\alpha x} \sinh \alpha t \]

\[ \eta = -\alpha^{-1}e^{\alpha x} \cosh \alpha t \quad \xi = -\alpha^{-1}e^{\alpha x} \sinh \alpha t \]  

(9)

respectively. The appearance of singularities (6) and the existence of several regions make it possible for the Lorentzian section to explain the doubling of degrees of freedom. While the original degrees of freedom are provided by region I, the additional degrees of freedom can be supplied by region II.

There is a relation between the transformations of regions I and II:

\[ \eta = -\alpha^{-1}e^{\alpha x} \sinh \alpha t = \alpha^{-1}e^{\alpha x} \sinh \alpha(t - i\beta/2) \]

\[ \xi = -\alpha^{-1}e^{\alpha x} \cosh \alpha t = \alpha^{-1}e^{\alpha x} \cosh \alpha(t - i\beta/2) \]  

(10)

Using Minkowskian coordinates \((t, x, y, z)\), the relation (10) becomes the relation between a point \(a_1(t_1, x_1, y_1, z_1)\) in region I and its reflected point \(a_2(t_2, x_2, y_2, z_2)\) in region II:

\[ t_2 = t_1 - i\beta/2 \quad x_2 = x_1 \quad y_2 = y_1 \quad z_2 = z_1 \]  

(11)

i.e. their Minkowskian coordinates differ only in an imaginary-time interval \(i\beta/2\).

Another important relation is that the direction of time \(t\) in region II is against the time direction in region I. The time-like Killing fields on the Lorentzian section are defined by \(\text{Gui} 1990\):

\[ \left( \frac{\partial}{\partial \lambda} \right)^a = \varepsilon \alpha (\xi \eta^a + \eta \xi^a) \]  

(12)
where $\varepsilon = 1$ and -1 for region I and II respectively. It is natural to choose the Killing parameter $\lambda$ as the time coordinate, which coincide with Minkowskian time coordinate $t$ in region I, and $-t$ in region II.

Using the above relations, one can see how the geometry of the $\eta - \xi$ spacetime gets into effect on the problem of analytic continuation. The following discussion takes the example of the massless free scalar field in two dimensional $\eta - \xi$ spacetime. Simple as it is, it makes one touch directly the physical and geometrical essence and avoid difficult technical details. The equation for propagators of this field on the Euclidean section is

$$\left( \frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \xi^2} \right) D_I(A - A') = -(-g_E)^{-1/2} \delta(A - A') \quad (13)$$

where $g_E$ stands for the determinant of metric of the Euclidean section and $D_I$ is the imaginary-time propagator. Under transformation (3), the equation becomes:

$$\left( \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial x^2} \right) D_I(A - A') = -\delta(A - A') \quad (14)$$

in which the points $A$ and $A'$ have the coordinates $(\tau, x)$ and $(\tau', x')$ respectively. Since the Euclidean section has $S^1$ topology, the propagators naturally satisfy the periodicity boundary condition:

$$D_I(\tau - \tau') = D_I(\tau - \tau' + \beta) \quad (15)$$

which is just the KMS condition (Kubo 1957, Martin and Schwinger 1959). Using this condition, the imaginary-time thermal propagator in momentum space is routinely obtained (Fetter and Walecka 1965):

$$D_I(\omega_n, k) = \frac{1}{\omega_n^2 + k^2} \quad (16)$$

where $\omega_n = 2\pi n/ - i\beta$.

Now continue the equation (13) to the equation on the Lorentzian section. This is done by continue $\sigma$ to $i\eta$:

$$\left[ -\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \xi^2} \right] D_R(A - A') = -(-g_L)^{-1/2} \delta(A - A') \quad (17)$$

where $g_L$ stands for the determinant of metric of the Lorentzian section. It is noticed that, since the whole Lorentzian section is obtained after a continuation of the Euclidean section, the points
A and $A'$ can be located in every one of the four regions. However, the regions III and IV are spacelike with respect to regions I and II, so one only need to consider the cases that A and $A'$ are located in regions I and II. Using the Minkowskian coordinates $(t, x)$, the equation (17) can be transformed into

$$\left[ \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right] D_R(A - A') = -\delta(A - A')$$

(18)

Since both of A and $A'$ can be located in each of the two regions, there are four cases. If one perform a Fourier transformation of equation (18), there will be four expressions for the field modes. Using the suffix 1, 2 to stand for coordinates in regions I and II respectively, one can write the expressions of field modes as:

$$e_{11} = \exp\{ik(x_1 - x'_1) - ik_0(t_1 - t'_1)\}$$  
$$e_{12} = \exp\{ik(x_1 - x'_2) - ik_0(t_1 - t'_2)\}$$  
$$e_{21} = \exp\{ik(x_2 - x'_1) - ik_0(t_2 - t'_1)\}$$  
$$e_{22} = \exp\{ik(x_2 - x'_2) - ik_0(t_2 - t'_2)\}$$

(19)

It shall be noted that $e_{12}$ and $e_{21}$ are only formally like plane waves. But we can view the coordinates $t$ and $x$ in these two expressions as functions of the coordinates $\eta$ and $\xi$, thus $e_{12}$ and $e_{21}$ represent the field modes on the whole Lorentzian section which relate different regions.

The Fourier coefficients $D(k, k_0)$ shall be different for different field modes. Hence the transformed equation takes a form of matrix:

$$\int dkdk_0 (k^2 - k_0^2) \begin{pmatrix} D_{11} e_{11} & D_{12} e_{12} \\ D_{21} e_{21} & D_{22} e_{22} \end{pmatrix} = -\int dkdk_0 \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

(20)

By the use of coordinate relation

$$t_2 = t_1 - i\beta/2$$  
$$x_2 = x_1$$

(21)

one rewrites the matrix on the right side of (20) as

$$\begin{pmatrix} e^{ik(x_1 - x'_1) - ik_0(t_1 - t'_1)} & e^{ik(x_1 - x'_1) - ik_0(t_1 - t'_1 + i\beta/2)} \\ e^{ik(x_1 - x'_1) - ik_0(t_1 - t'_1 - i\beta/2)} & e^{ik(x_1 - x'_1) + ik_0(t_1 - t'_1)} \end{pmatrix}$$

(22)

A sign is changed in 2-2 component because the direction of time in region I points against the one in region II.
The term $i\beta/2$ in the off-diagonal components is explained as thermal factor caused by the geometrical structure of the Lorentzian section. Since regions I and II are separated on the Lorentzian section by the "horizons", they can only be connected by complex paths which run along half circles on the Euclidean section and result in the imaginary-time interval $i\beta/2$. Physically, the field modes $e_{12}$ and $e_{21}$ can not be completely measured by the observers in region I, to whom the information in region II is screened by the "horizons". It is just this lost of information that makes the observers in region I find a finite temperature. In this explanation, the multiple-region geometrical structure, and thus the doubling of degrees of freedom, are the origins of thermal factor. One notices that this geometry-induced thermal effect is quite similar to the gravity-induced Hawking-Unruh effects (Hawking 1974, Unruh 1975).

Using similar coordinate relation on the left side of (20), one gets four equations:

\[
\begin{align*}
(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})D_{11}(t-t', x-x') &= -\delta(t-t', x-x') \\
(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})D_{12}(t-t' + i\beta/2, x-x') &= -\delta(t-t' + i\beta/2, x-x') \\
(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})D_{21}(t-t' - i\beta/2, x-x') &= -\delta(t-t' - i\beta/2, x-x') \\
(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2})D_{22}(t'-t, x-x') &= -\delta(t'-t, x-x')
\end{align*}
\]

(23)

Here the suffixes 1 for coordinates are omitted. By solving these equations one get

\[
D_{\beta}(k_0, k) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}
\]

(24)

where

\[
D_{11} = \frac{1}{k_0^2 - k^2 + i\epsilon} - \frac{2\pi i\delta(k_0^2 - k^2)}{e^{i\beta k_0} - 1}
\]

(25)

\[
D_{12} = \frac{2\pi i e^{-\beta k_0/2} \delta(k_0^2 - k^2)}{1 - e^{-\beta k_0}}
\]

(26)

and

\[
D_{21} = D_{12} \quad D_{22} = -D_{11}^*
\]

(27)

Thus the solution of the equation (17), which is obtained by continuation of the equation for the imaginary-time thermal propagator, is just the $2 \times 2$ matrix real-time thermal propagator.
It may seem contradictory that propagators with so diverse forms that they can not be directly continued to each other are solutions of equations which can be mutually obtained by continuation. This can be explained by reviewing the roles of different geometrical structures of the Euclidean and Lorentzian sections in the procession of solving the equations.

Since each of the equations (13) and (17) holds on the whole Euclidean section or Lorentzian section respectively, and no additional singularity is met while the sections rotate to each other, the analytic continuation between the equations is feasible. But direct analytic continuation between the thermal propagators is hindered by the different singularities on the two sections, which impose different requirements in the courses of solving these equations. On one hand, the singularity \( \sigma = \xi = 0 \) results in the \( S^1 \) topology of the Euclidean section and thus the periodicity boundary condition (15), which requires the propagators on this section to be transformed into Fourier series. On the other hand, the Lorentzian section is divided into four disjointed regions by the "horizons" (6), which cause different expressions of field modes and thus give four matrix components of the thermal propagator.

One remembers an early work (Dolan and Jackiw 1974) which tried to get the real-time thermal propagators through direct analytic continuation of the imaginary-time thermal propagators. However, the result was only the 1-1 component of the \( 2 \times 2 \) real-time thermal propagator, which leads to difficulties in calculations.

In view of the analysis in this paper, it is also clear to see why the early continuation was not complete. In fact, there was an implicit premise for that attempt. It was supposed that the background spacetime for the real-time formalism is the Minkowskian spacetime. This corresponds to only a region of the Lorentzian section of the \( \eta - \xi \) spacetime, hence it can not provide the doubling of degrees of freedom. The 1-1 component of the propagator is just the case that both \( A \) and \( A' \) are located in region I, while the other cases are ignored in that attempt. Since the singularities on the Lorentzian section, which play important role in solving the equation for the real-time propagator, had not been considered, it is not strange that the \( 2 \times 2 \) matrix real-time thermal propagator could not be obtained.

It shall be noticed that this paper takes as example a very simple model, and its extension to more practical calculation is far from straightforward. However, by discussing the influence of
spacetime geometry on the feasibility of analytic continuation, it suggests a new intuitive way of looking at this problem.

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