Vortex condensation in a model of random $\phi^4$-graphs

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Abstract

We consider a soluble model of large $\phi^4$-graphs randomly embedded in one compactified dimension; namely the large-order behaviour of finite-temperature perturbation theory for the partition function of the anharmonic oscillator. We solve the model using semi-classical methods and demonstrate the existence of a critical temperature at which the system undergoes a second-order phase transition from $D = 1$ to $D = 0$ behaviour. Non-trivial windings of the closed loops in a graph around the compactified time direction are interpreted as vortices. The critical point has a natural interpretation as the temperature at which these vortices condense and disorder the system. We show that the vortex density increases rapidly in the critical region indicating the breakdown of the dilute vortex gas approximation at this point. We discuss the relation of this phenomenon to the Berezinskii-Kosterlitz-Thouless transition in the $D = 1$ matrix model formulated on a circle.
1 Introduction

Models of random graphs have applications to theories of polymers and random surfaces. An example of particular interest is the $D = 1$ Hermitian matrix model [1, 2, 3, 4] formulated on a circle of radius $R$ which is equivalent to matrix quantum mechanics at a finite temperature $k_B T = 1/2\pi R$. The perturbative expansion of the matrix path-integral for the case of a quartic potential generates a sum over $\phi^4$ Feynman graphs, each one dual to the quadrandulation of a random surface; these surfaces being randomly embedded in $S^1$. In the double-scaling limit, the resulting continuum theory can be interpreted as an XY model coupled to two-dimensional quantum gravity or as non-critical string theory in one compactified embedding dimension (an extra dimension corresponding to the Liouville mode also arises). An important difference between a model of graphs embedded on the circle and one formulated on the real line, is that the former admits vortex configurations where a closed loop in the graph wraps around the compactified time direction. Gross and Klebanov [2] have argued that these vortices are suppressed at large radius but, at critical value of the radius $R_c$, the system undergoes a Berezinskii-Kosterlitz-Thouless (BKT) [5] transition to a high-temperature phase in which the vortices condense and disorder the embedding coordinate. They find that the embedding coordinate is completely randomized for $R < R_c$ and effectively decouples leading to zero-dimensional behaviour in the high-temperature phase.

Unfortunately the $D = 1$ matrix model on a circle cannot be solved in the double scaling limit by the usual methods which are effective in the case of a non-compact target space. The reason is that the integral over the angular degrees of freedom of the matrix field, which can be done exactly on the real line to yield a quantum mechanical problem involving the eigenvalues only, is no longer tractable on the circle. Indeed this difficulty seems to be precisely due to the presence of vortices in the latter model: at low temperature the model can be solved approximately by considering only the $U(N)$ singlet sector, however this appears to be equivalent to neglecting the vortex contribution [1].

Similar vortex configurations occur in any model of random graphs embedded on a circle. In general, these graphs need not correspond directly to discretized random surfaces as they do for the matrix model. In this letter we present such a model which exhibits vortex condensation accompanied by an abrupt transition from $D = 1$ to $D = 0$ behaviour and has the additional virtue of being exactly soluble. The model involves an ensemble of large $\phi^4$-graphs randomly embedded in a single compactified dimension and is equivalent to the large-order behaviour of finite-temperature perturbation theory for the anharmonic oscillator. The graphs we consider carry no index structure, which means that there is no unique prescription for attaching plaquettes and thereby defining a discretized surface. Thus, it will not be possible to draw any definite conclusions for one-dimensional string theory from our results. Nevertheless, the model exhibits a non-trivial critical behaviour which is similar to that expected at the BKT transition point of the $D = 1$ matrix model and therefore merits study.

1 Boulatov and Kuzakov [6] have considered the adjoint sector of the theory and have shown that this corresponds to the contribution of a single vortex-antivortex pair.
The large-order behaviour of perturbation theory for the quantum anharmonic oscillator defined by the Hamiltonian,
\[ H = \left( p^2 + \phi^2 \right)/2 + g\phi^4, \] (1.1)
is determined by the instanton solution \[ 7, 8 \], which is the classical motion of a particle in the inverted potential \( V = -x^2/2 + x^4/4 \) (see Figure 1). At finite temperature \( T \), the relevant classical motion is constrained to have period \( \beta = 1/k_B T \). It is clear that there is a minimum period for which there exists a non-trivial instanton solution \( x(t) \). The period is that of an infinitesimal simple harmonic motion about the quadratic minimum of \( V \) and is determined by the curvature of the potential well at that point. Below this critical period, the large-order behaviour of perturbation theory is governed by the trivial solutions \( x(t) = \pm 1 \). Although this phenomenon has no special significance for the anharmonic oscillator itself \[ 2 \], we will argue that it has a natural interpretation as a second-order phase transition in the corresponding model of randomly embedded graphs. In particular, we will show that the specific heat is sharply peaked at the critical temperature and exhibits a finite discontinuity at this point. In addition we will show that density of free vortices undergoes a crossover in the vicinity of the critical point from a low temperature regime of tightly bound vortex-antivortex pairs to a high temperature regime populated by free (anti-)vortices. Thus we will interpret the critical temperature as the point at which vortices condense and disorder the system.

In Section 2, we define the partition function of the model as a sum over Feynman graphs with a fixed large number of vertices and identify the vortex configurations. In Section 3 we solve the model using semi-classical methods as described above and exhibit the critical behaviour. In the process, we give some new analytic results for the large-order behaviour of perturbation theory for the anharmonic oscillator at finite temperature. Section 4 is devoted to a discussion of these results.

## 2 The model

The partition function for the anharmonic oscillator defined by the Hamiltonian (1.1), at a finite temperature \( T \), is given by
\[ Z(\beta, g) = \text{Tr} \left[ e^{-\beta H} \right] = \int \mathcal{D}\phi \exp \left[ -\int_0^\beta dt \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi^2 + g\phi^4 \right] \] (2.2)
with \( \beta = 1/k_B T \). Finite-temperature perturbation theory is generated by expanding the partition function as an asymptotic series in the coupling constant \( g \),
\[ Z(\beta, g) \sim \sum_{k=0}^\infty Z_k(\beta) g^k \] (2.3)
The coefficients \( Z_k \) are given by the sum of all connected \( \phi^4 \) vacuum graphs \( G \) with \( k \) vertices and the Feynman rules instruct us to assign a time coordinate \( t_i \in [0, \beta] \) to each vertex and a

\[^2\text{The occurrence of the minimum period was noted in this context in [8].}\]
finite-temperature propagator,
\[ D_{ij} = \sum_{m=-\infty}^{\infty} \exp \left[ -|t_i - t_j + m\beta| \right] \] (2.4)
to each link \( \langle ij \rangle \) and then integrate over each \( t_i \). The propagator \( D_{ij} \) has a geometrical interpretation as a sum over the topologically inequivalent ways of embedding the link \( \langle ij \rangle \) in the \( S^1 \) obtained by identifying the endpoints of the time interval \([0, \beta]\) (see Figure 2). Each embedding of a link \( \langle ij \rangle \), is weighted by a factor \( \exp(-l\beta) \) in the summation (2.4), where \( l \) is the length (in units of \( \beta \)) of the image of \( \langle ij \rangle \) in the target space.

Applying the Feynman rules and summing over all graphs with \( k \) vertices we have,
\[ Z_k(\beta) = (-1)^k \sum_{G} S(G) \int_{0}^{\beta} \ldots \int_{0}^{\beta} \prod_{i=1}^{k} dt_i \prod_{\langle ij \rangle} \sum_{m_{ij}=-\infty}^{\infty} \exp \left[ -|t_i - t_j + m_{ij}\beta| \right] \] (2.5)

For each graph \( G \), a choice of the time coordinate \( \{t_i\} \) for each vertex and of a winding number \( m_{ij} \) for each link specifies an embedding of \( G \) in \( S^1 \). The coefficient \( Z_k \) is a sum/integral over all such embeddings of all \( k \)-vertex vacuum graphs \( G \), each one weighted by the symmetry factor \( S(G) \) and by \( \exp(-L\beta) \) where \( L \) is the total length of the image of \( G \) in the target space. In what follows, we will consider \( Z_k(\beta) \) itself, for fixed large \( k \), as defining the partition function of a theory of graphs randomly embedded in \( S^1 \), where the “action” of a “configuration” (ie a specific embedding) is just its target-space length. From this point of view, \( k \to \infty \) is the bulk limit of the model, and we can define the usual thermodynamic quantities. In this limit, the number of graphs grows as \( n_k \sim (16)^k(k-1)! \) and we define a normalized partition function \( \hat{Z}_k = Z_k/n_k \) which can be thought of as the contribution of a typical graph. The free energy per vertex is defined as,
\[ f = -\lim_{k \to \infty} \frac{1}{k} \log |\hat{Z}_k(\beta)| \] (2.6)
and the corresponding specific heat is given by \( c_v = -\beta^2 \partial^2 f / \partial \beta^2 \). In Section 3 we will give the exact solution of the model in the thermodynamic limit by applying the standard semi-classical analysis of large orders in perturbation theory to this finite temperature case. In particular, we will show that \( f \) and \( c_v \) remain finite as \( k \to \infty \) and that the latter exhibits a discontinuity as a function of temperature indicating a second-order phase transition. It is convenient to express the partition function, \( Z_k(\beta) \), as a Laplace transform,
\[ Z_k(\beta) = \int_{0}^{\infty} dL \hat{Z}_k(L) \exp \left[ -L\beta \right] \] (2.7)
The inverse Laplace transform, \( \hat{Z}_k(L) \), is the corresponding microcanonical partition function for configurations of fixed target-space length \( L \).

For each configuration, it is possible to assign a vortex number \( m_{\ell} \) to every closed loop \( \ell \) in \( G \). Specifically, each term obtained by expanding the product of propagators,
\[ \prod_{\langle ij \rangle \in \ell} \sum_{m_{ij}=-\infty}^{\infty} \exp \left[ -|t_i - t_j + m_{ij}\beta| \right] \] (2.8)
is characterized by a vortex number $m_\ell = \sum_{(ij) \in \ell} m_{ij}$ which labels the homotopy class of the corresponding embedding of $\ell$ in $S^1$. Negative values of $m_\ell$ correspond to anti-vortices and, by analogy with the two-dimensional XY-model, we interpret loops for which $m_\ell = 0$ as tightly-bound vortex-antivortex pairs. The total vortex number of a configuration is given by summing over all closed loops in $G$, $M = \sum_\ell m_\ell$. Because the sum over configurations (2.5) is symmetric under $m_{ij} \to -m_{ij}$, vortices and anti-vortices occur with equal statistical weight and therefore $\langle M \rangle = 0$.

For low temperatures, the summation for each propagator in (2.3) is dominated by the $m_{ij} = 0$ term. An isolated (anti-)vortex necessitates at least one of the $m_{ij}$ being non-zero and is thus suppressed by a factor of $\exp(-\beta)$. Thus, at low temperature, we expect that the links with non-zero $m_{ij}$ in a typical configuration are well separated and therefore that most vortices and antivortices are tightly bound in pairs. The unbinding of a vortex-antivortex pair is accompanied by an increase in the target-space length of the configuration by two units. Thus, at least in this dilute low-temperature ground state, the expected number of free (anti-)vortices increases linearly with the expectation value of the quantity $L$. In addition, when the corresponding length per vertex $\rho = \langle L \rangle / k$ becomes $O(1)$, it follows that the free vortices have become dense and the dilute gas approximation is no longer valid. In this sense we will refer to $\rho$ as a “vortex density”.

For any given configuration, $L$ is also the corresponding action and so the vortex density in the thermodynamic limit is given by the internal energy per vertex, $\rho = \partial f / \partial \beta$ (this formula can be obtained by differentiating (2.7)). In the next section, we calculate $\rho$ explicitly and show that the system leaves the dilute regime in the vicinity of the critical temperature. This suggests that the phase transition corresponds to a disordering of the system by the free vortices.

3 Semi-classical solution for $k \to \infty$

In this section we follow the standard approach to the large order behaviour of perturbation theory for the anharmonic oscillator [7] (for a pedagogical exposition see [8]). Although this problem is frequently formulated at finite temperature as a way of regulating the infra-red divergences which occur at zero temperature, the explicit evaluation of the temperature dependence of large-order behaviour given below appears to be new. The coefficients $Z_k(\beta)$ in (2.3) can be expressed in terms of the imaginary part of the original partition function (2.2) for the anharmonic oscillator as,

$$Z_k(\beta) = \frac{1}{\pi} \int_{-\infty}^{0} dg \frac{\text{Im} Z(g)}{g^{k+1}}$$

reflecting the existence of a cut in the complex $g$-plane along the negative real axis. For $k \to \infty$ the integral is dominated by the $g \to 0$ region of the integrand. In this region, the functional integral (2.2) for $Z(g)$ can be evaluated accurately in the saddle-point approximation. The saddle-point solutions, $x(t) = 2 \sqrt{-g} \phi(t)$, are given by the appropriate periodic classical trajectories of a particle moving in the potential, $V = -x^2/2 + x^4/4$ with turning points $x_- \leq x_+$ and total energy $E = V(x_\pm)$ (see Figure 1). At finite temperature the relevant trajectories have period $\beta$. As discussed in the Introduction, there is a minimum period $\beta_c$ for which a non-trivial solution.
exists. Expanding the potential in the vicinity of the positive minimum \( x_c = 1 \) we find,

\[
V = -\frac{1}{4} + \frac{1}{2} \omega^2 (x - x_c)^2 + O((x - x_c)^3)
\]

(3.10)

with \( \omega = \sqrt{2} \). Thus as the total energy of the particle approaches a critical value \( E_c = -1/4 \), its motion becomes an infinitesimal harmonic oscillation about the well-bottom with period \( \beta_c = 2\pi/\omega = \sqrt{2}\pi \). Although \( n \)-instanton corrections corresponding to paths with periods \( \beta/n \) can occur at low temperature, these solutions do not occur for \( \beta < 2\beta_c \).

The relevant instanton solutions, \( x(t) \) satisfy the classical equation of motion,

\[
\ddot{x} - x + x^3 = 0
\]

(3.11)

with boundary conditions \( x(0) = x(\beta) \). For \( \beta > \beta_c \), there are two one-parameter families of such solutions generated by time-translations and reflections, \( x \rightarrow -x \). For definiteness, we will consider the particular solution such that \( x(0) > 0 \) is the turning point of the motion \( x_- \). This solution is given as a Jacobian elliptic function [9],

\[
x(t) = x_\pm \text{dn} \left[ \frac{x - \sqrt{2}/2}{x_\pm \sqrt{2}}, k \right]
\]

(3.12)

with imaginary modulus, \( k = \sqrt{1 - x_\pm^2/x^2} \).

The resulting saddle-point expression for \( \text{Im}Z(g) \) depends on \( \beta \) as [8],

\[
\text{Im}Z(g) \sim -\frac{\beta}{4} \left( \frac{1}{2\pi g} \frac{\partial E}{\partial \beta} \right)^{\frac{1}{2}} \exp \left( -\frac{I(\beta)}{g} \right)
\]

(3.13)

where \( I(\beta) \) is the classical action of the trajectory, which is conveniently determined by the relation

\[
\frac{\partial I(\beta)}{\partial \beta} = -\frac{E(\beta)}{4}
\]

(3.14)

together with the condition \( I(\beta_c) = \beta_c/16 \) for the action of the limiting infinitesimal harmonic motion. The pre-exponential factor in (3.13) is given by the determinant of gaussian fluctuations about the saddle-point and is proportional to \( 1/\sqrt{g} \) reflecting the contribution of the zero-mode corresponding to time-translation of the instanton solution. Adopting the convention of [8], all functional determinants considered (see also (3.22) below) are normalized with respect to the zero coupling result.

The formulae (3.14,3.13) demand that we find the dependence of the turning point energy \( E = V(x_\pm) \) on the period \( \beta \). In particular we are interested in the behaviour of \( E(\beta) \) and its derivatives in the critical region \( (\beta - \beta_c) << 1 \). The inverse function, \( \beta(E) \), can be expressed as a first integral of the classical equation of motion (3.11),

\[
\beta(E) = 2 \int_{x_-}^{x_+} \frac{dx}{\sqrt{2(E - V(x))}} = \int_{E_c}^{E} dV \frac{dW}{V} \frac{1}{\sqrt{2(E - V)}}
\]

(3.15)
where \( W(E) = x_+ - x_- \) is the width of the well at energy \( E < 0 \). The above relation (Abel’s integral equation) can be solved for \( W \) by noting that the second integral in (3.15) is proportional to the Riemann-Liouville integral \[ 10 \] which defines the fractional derivative \( D^{\gamma} \). In particular, using the composition rule for fractional derivatives \[ 10 \] \( D^{\delta} D^{\gamma} = D^{\delta+\gamma} \), we have \( \beta(E) \propto (D^{-\frac{1}{2}} DW)(E) = (D^{\frac{1}{2}} W)(E) \), which implies that \( W(E) \propto (D^{-\frac{1}{2}} \beta)(E) \) or, reintroducing numerical factors,

\[
W(E) = \frac{1}{\pi} \int_{E_c}^{E} dE' \frac{\beta(E')}{\sqrt{2(E - E')}}
\]  (3.16)

However, in the present case of a quartic potential, \( W(E) \) is given explicitly by,

\[
W(E) = \sqrt{1 + \sqrt{E - E_c}} - \sqrt{1 - \sqrt{E - E_c}}
\]  (3.17)

Comparing (3.16) and (3.17) we find that \( \beta(E) \) is analytic for all \( E < 0 \) and has a Taylor series,

\[
\beta(E) = \beta_c + \beta_c \sum_{n=1}^{\infty} a_n (E - E_c)^n
\]  (3.18)

with coefficients,

\[
a_n = \left( \frac{1}{2n + 1} \right) / \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k + 1}
\]  (3.19)

Standard theorems on implicit functions imply that \( E \) is an infinitely differentiable function of \( \beta \) in some neighbourhood of \( \beta_c \). The Taylor series for \( E(\beta) \) in powers of \( (\beta - \beta_c) \) can be derived by inverting the series (3.18). Although the general term in this series cannot be expressed in closed form, the coefficients can be generated systematically using the algorithm described in the appendix. For the present purpose, we will require only the leading terms,

\[
E(\beta) = -\frac{1}{4} + \frac{4}{3\beta_c} (\beta - \beta_c) + O((\beta - \beta_c)^2)
\]  (3.20)

For \( \beta \to \infty \), \( E(\beta) \) has a double expansion in powers of \( \beta \) and \( e^{-\beta} \). A straightforward calculation yields,

\[
E(\beta) = -16e^{-\beta} - 64(3\beta - 10)e^{-2\beta} + O(\beta^2 e^{-3\beta})
\]  (3.21)

The first term in this series yields the standard zero-temperature result \[ 8 \] when substituted for \( E \) in (3.13). We computed \( E(\beta) \) numerically for the whole range \( \beta_c < \beta < \infty \) by solving equation (3.15), the resulting graph is shown in Figure 3. We also computed \( I(\beta) \) by integrating our numerical solution for \( E(\beta) \).

\[ ^3 \]This integral can also be thought of as arising from the standard Duhamel integral \[ 11 \] by explicit differentiation.
For $\beta < \beta_c$, the only solutions of the equation of motion are the trivial ones $x(t) = \pm 1$. The corresponding contribution to the partition function is,

$$\text{Im} Z(g) \sim 2 \left( \frac{\sqrt{2} \sinh \beta}{\sinh \sqrt{2} \beta} \right)^{\frac{1}{2}} \exp \left( -\frac{\beta}{16g} \right)$$

(3.22)

where the determinant prefactor is independent of $g$ as the trivial saddle-point has no zero-mode. Applying equation (3.9), we find that $\beta_c$ separates two “phases” characterized by different asymptotic behaviour of $Z_k$. For $\beta > \beta_c$ we have,

$$Z_k \sim (-1)^k \frac{1}{2} \left( \frac{1}{2\pi} \frac{\partial E}{\partial \beta} \right)^{\frac{1}{2}} (I(\beta))^{-\left(k + \frac{1}{2}\right)} \Gamma \left(k + \frac{1}{2}\right)$$

(3.23)

whereas for $\beta < \beta_c$,

$$Z_k \sim (-1)^k 2 \left( \frac{\sqrt{2} \sinh \beta}{\sinh \sqrt{2} \beta} \right)^{\frac{1}{2}} \left( \frac{\beta}{16} \right)^{-k} \Gamma(k)$$

(3.24)

In the low-temperature phase, $Z_k$ exhibits the same $I^{-k} k^{1/2} (k-1)!$ growth, as the zero-temperature result. In contrast, the large-order behaviour of perturbation theory in the high-temperature phase is identical (up to a $k$-independent prefactor) to that of the perturbative expansion of the ordinary integral,

$$Z^* = \int_0^\infty dy \exp \left[ -\frac{\beta}{4} \left( \frac{y^2}{2} + gy^4 \right) \right]$$

(3.25)

In this sense, the large-order behaviour of perturbation theory exhibits a transition from $D = 1$ to $D = 0$ behaviour at $\beta = \beta_c$.

Equations (3.23) and (3.24) constitute an exact solution for the partition function of the model of random graphs considered in Section 2 in the thermodynamic limit. Although the partition function itself grows wildly with $k$ reflecting the $k!$ growth in the number of Feynman diagrams, the free energy per vertex of a typical graph (2.6) remains finite as does the corresponding specific heat. For $\beta > \beta_c$ we find,

$$c_v = \frac{\beta^2}{4} \left[ \frac{\partial E}{\partial \beta} \frac{1}{I(\beta)} + \frac{E^2(\beta)}{4I^2(\beta)} \right]$$

(3.26)

while for $\beta < \beta_c$, $c_v = 1$. Using equations (3.14, 3.20, 3.20) we find that $c_v \rightarrow 19/3$ as $\beta \downarrow \beta_c$. The resulting discontinuity in the specific heat at $\beta = \beta_c$ implies that the system undergoes a second-order phase transition at this point. The result of a numerical evaluation of $c_v$ is shown in Figure 4. As expected for a second-order phase transition, the specific heat has a narrow peak in the critical region.
In Section 2 we argued that the density of free vortices is given by the expectation value of the target-space length per vertex $L/k$ which is exactly the internal energy $\partial f/\partial \beta$. Hence, for $\beta > \beta_c$, we have $\rho = -E(\beta)/4I(\beta)$ while for $\beta < \beta_c$, $\rho = 1/\beta$. The corresponding graph of the vortex density against temperature is shown in Figure 5. At low temperature, the vortex density is exponentially suppressed,

$$\rho \sim \exp\left(-\frac{1}{k_B T}\right) \quad (3.27)$$

This implies that the low-temperature ground state of the system can be thought of as a dilute gas of tightly-bound vortex-antivortex pairs. In the high-temperature phase the vortex density is $O(1)$ and increases linearly with temperature indicating a breakdown of the dilute gas approximation. The numerical results shown in Figure 5 reveal a rapid crossover between these two regimes in a narrow region near the critical temperature $k_B T_c = 1/\sqrt{2\pi}$. In particular, the vortex density increases by a factor of about 6.35 in the interval $0.75 T_c < T < T_c$. This supports an interpretation of the critical temperature as the point at which vortex-antivortex pairs unbind and disorder the system. There is a qualitative similarity between our numerical results for $\rho$ and an evaluation of the vortex density in a Monte-Carlo simulation of the two-dimensional XY model presented in [12].

4 Conclusions

In this paper we have presented a novel interpretation for the large-order behaviour of finite-temperature perturbation theory. In particular we have shown that a fixed large order in the perturbation series for the partition function of an anharmonic oscillator defines a non-trivial statistical mechanical model with a sensible thermodynamic limit. The model exhibits two phases separated by a second-order phase transition and it is important to establish the universality class to which this critical behaviour belongs. Although this transition has the same interpretation as the BKT transition in the two-dimensional XY model, the latter is a transition of infinite order having no discontinuities in the derivatives of the free energy.

The behaviour of $Z_k$ for $\beta < \beta_c$ is identical to the large-order behaviour of the perturbative expansion of an ordinary integral. The perturbation series for the integral $Z^*$ is just a sum over all $\phi^4$ graphs with no embedding dimension. This suggests that, in the high-temperature phase of the model, the embedding coordinate is completely randomized by the gas of free vortices and decouples leaving an effectively zero-dimensional theory. This is exactly the behaviour expected at the BKT transition point of the $D = 1$ matrix model [2]. However, due to the unconventional procedure of summing over all Feynman graphs with equal weight irrespective of genus, there is no direct correspondence between our model and the double-scaling limit of the $D = 1$ matrix model. In addition, we note that the estimates for the critical temperature of the matrix model given in [2, 3] differ from the critical temperature found here by a factor of two. On the other hand, the occurrence of vortices is a purely local phenomenon on any graph or discretized surface and it is expected that the BKT transition should occur independently at each order in the genus expansion. Hence one might expect the same critical behaviour to appear even in the unweighted
sum over different genera considered here. Thus it is possible that our model exhibits the same universal behaviour as the XY model coupled to two-dimensional quantum gravity.

The model presented in this paper has several unusual features. As in the matrix models, the sum over configurations is generated by the perturbative expansion of a functional integral and can be evaluated using semi-classical methods. However, in our case, the thermodynamic limit of large graphs is taken explicitly rather than by tuning the coupling constant to a critical value (indeed the latter procedure is not available here as the perturbation series has zero radius of convergence). In principle we can obtain a continuum theory only at the critical temperature $\beta = \beta_c$. Finally we note that the model considered here can be generalised to any theory where the large-order behaviour of perturbation theory may be evaluated by saddle-point methods.

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Appendix

In this appendix we give a general formula for the inversion of a power series of the form,

\[ w = f(z) = \sum_{n=0}^{\infty} a_n z^n \] (4.28)

which provides an algorithm for generating the coefficients of the series (3.20) for \( E(\beta) \). We consider the inversion of (4.28) in an open neighbourhood of \( z = 0 \); accordingly we demand that \( f'(0) \neq 0 \) and set \( f(0) = w_0 \). The inverse series for \( z(w) \) can be written as,

\[ z = \frac{1}{a_1} (w - w_0) + \sum_{m=2}^{\infty} \frac{(w - w_0)^m}{m!} \frac{(-1)^m+1(m-2)}{a_1^{2m-1}} \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \delta(\sum l_i - m + 1, 0) C_{l_1} \cdots C_{l_m} \] (4.29)

where

\[ C_l = \begin{vmatrix} a_2 & a_1 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_t & a_{t-1} & a_{t-2} & \ldots & a_1 \\ a_{t+1} & a_t & a_{t-1} & \ldots & a_2 \end{vmatrix} \] (4.30)

More explicitly, if

\[ w = w_0 + az + bz^2 + cz^3 + dz^4 + \ldots \] (4.31)

(4.29) gives,

\[ z = \frac{1}{a} (w - w_0) - \frac{b}{a^3} (w - w_0)^2 - \frac{1}{a^5} (ac - 2b^2)(w - w_0)^3 - \frac{1}{a^7} (a^2 d - 5abc + 5b^3)(w - w_0)^4 + \ldots \] (4.32)

Figure Captions

Figure 1. The potential \( V(x) \).

Figure 2. The geometrical interpretation of the finite-\( T \) propagator (2.4) as a sum over topologically inequivalent embeddings of the link \( \langle ij \rangle \) in \( S^1 \).

Figure 3. A graph of \( E \) as a function of \( \beta/\beta_c \).
Figure 4. A graph of $c_v$ as a function of $k_B T = 1/\beta$.

Figure 5. A graph of $\rho$ as a function of $k_B T$. 