Characterizing Nöbeling spaces

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Abstract

It is shown that Nöbeling spaces are uniquely determined by the universal extension and embedding properties.

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1 Introduction

All spaces are assumed to be separable metrizable. A complete \( n \)-dimensional metric space \( X \) is said to be an \( n \)-dimensional Nöbeling space if the following properties are satisfied:

(i) \( X \) is an absolute extensor in dimension \( n \), that is every map \( f : A \rightarrow X \) from a closed subset \( A \) of a space \( Y \) of \( \text{dim} \leq n \) extends over \( Y \);

(ii) every map \( f : Y \rightarrow X \) from a complete metric space \( Y \) of \( \text{dim} \leq n \) can be arbitrarily closely approximated by a closed embedding, that is for every open cover \( U \) of \( X \) there is a closed embedding \( g : Y \rightarrow X \) which is \( U \)-close to \( f \) (\( U \)-close means that for every \( y \in Y \) there is an element of \( U \) that contains both \( f(y) \) and \( g(y) \)).

The characterization theorem for Nöbeling spaces says that:

**Theorem 1.1** Every two Nöbeling spaces of the same dimension are homeomorphic.\(^1\)

This result is well-known in dimension 0. One-dimensional Nöbeling spaces were characterized in [3]. Our main goal is to prove the characterization theorem in all dimensions. This paper is a continuation of [4] where key ingredients and techniques for proving Theorem 1.1 were established. The general pattern of the proof of Theorem 1.1 follows Toruńczyk’s and Bestvina’s proofs of the characterization theorems for the Hilbert cube [6], the Hilbert space [7] and the Menger universal compacta [1]. Let us remind some facts from [4] and outline key steps of the proof of Theorem 1.1.

Throughout the paper a manifold means a manifold with (possibly empty) boundary and a triangulated space means a locally finite simplicial complex which we identify

\(^1\)This theorem was also obtained by A. Nagorko [5] using a different approach. The results of [5] were announced in July 2005 in Bedlewo, Poland.
with the underlying space. For a triangulated space we consider only triangulations compatible with the PL-structure of the space. All triangulated manifolds are assumed to be combinatorial. We will work with the following model of Nöbeling spaces.

By a rational map between triangulated spaces we mean a PL-map that sends points with rational barycentric coordinates to points with rational barycentric coordinates. Two triangulations of a space are said to be rationally equivalent if the identity map is a rational map with respect to these triangulations (it is easy to see that if a PL-homeomorphism is rational in one direction then it is rational in the opposite direction as well). Let $M$ be a triangulated space. Every triangulation of $M$ which is rationally equivalent to the given triangulation of $M$ is said to be a rational triangulation and the class of all rational triangulations is said to be the rational structure of $M$. Denote by $M(k)$ the subspace of $M$ which is the complement of the union of all the triangulated spaces of dim $\leq k$ which are rationally embedded in $M$.

Let us state the following important fact leaving its proof to the reader.

**Theorem 1.2** Let $M$ be a triangulated $m$-dimensional manifold, let $k \geq 0$ be an integer and let $n = m - k - 1$. If $M$ is $(n-1)$-connected and $m \geq 2n + 1$ then $M(k)$ is an $n$-dimensional Nöbeling space.

A space $M(k)$ satisfying the assumptions of Theorem 1.2 is called a Nöbeling space modeled on a triangulated manifold.

A subset $A$ of a space $X$ is called a Z-set if $A$ is closed in $X$ and the identity map of $X$ can be arbitrarily closely approximated by a map $f : X \to X$ with $f(X) \cap A = \emptyset$. Note that if $X$ is an $n$-dimensional Nöbeling space modeled on a manifold $M$ and $A \subset X$ is a Z-set in $X$ then $X \setminus A$ is also an $n$-dimensional Nöbeling space modeled on the manifold $N = M \setminus$ the closure of $A$ in $M$ (the rational structure of $N$ is defined such that the inclusion is a rational map). The following version of a Z-set unknotting theorem was proved in [4].

**Theorem 1.3 (Unknotting Theorem [4])** Let $X_1$ and $X_2$ be $n$-dimensional Nöbeling spaces and let $A_1$ and $A_2$ be Z-sets in $X_1$ and $X_2$ respectively such that $X_1 \setminus A_1$ and $X_2 \setminus A_2$ are homeomorphic to $n$-dimensional Nöbeling spaces modeled on triangulated manifolds. If $A_1$ and $A_2$ are homeomorphic then any homeomorphism between $A_1$ and $A_2$ can be extended to a homeomorphism between $X_1$ and $X_2$.

A subset of a space $X$ is called a $\sigma$-Z-set if it is a countable union of Z-sets. In Section 2 we prove the following version of a resolution theorem which establishes a connection between general Nöbeling spaces and Nöbeling spaces modeled on triangulated manifolds.

**Theorem 1.4 (Resolution Theorem)** Let $Y$ be an $n$-dimensional Nöbeling space. Then there is a subspace $X \subset Y$ such that $X$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold and $Y \setminus X$ is a $\sigma$-Z-set in $Y$.

In Section 4 we prove the following version of a shrinking theorem which is the most important step in proving Theorem 1.1.
Theorem 1.5 (Shrinking Theorem) Let $X$ be a subspace of an $n$-dimensional Nöbeling space $Y$ such that $X$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold and $Y \setminus X$ is a $\sigma$-$Z$-set in $Y$. Then for every covers $\mathcal{V}_X$ and $\mathcal{V}_Y$ of $X$ and $Y$ by sets open in $X$ and $Y$ respectively there is a homeomorphism $h : X \to X$ such that for every $x \in X$ the point $h(x)$ is $\mathcal{V}_Y$-close to $x$ and for every $y \in Y$ there are a neighborhood $G$ of $y$ in $Y$ and a set $V \in \mathcal{V}_X$ such that $h^{-1}(G \cap X) \subset V$.

Theorems 1.3, 1.4 and 1.5 imply Theorem 1.1. Indeed, assume that $X$ and $Y$ satisfy the assumptions of Theorem 1.5. Since $Y \setminus X$ is a $\sigma$-$Z$-set in $Y$ we have that $X$ is dense in $Y$. Then by Bing’s Shrinking Criterion and Theorem 1.5 the inclusion of $X$ into $Y$ can be arbitrarily closely approximated by a homeomorphism between $X$ and $Y$. Thus by Theorem 1.4 we get that every $n$-dimensional Nöbeling space is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold. By Theorem 1.3 every two $n$-dimensional Nöbeling spaces modeled on triangulated monifolds are homeomorphic, and Theorem 1.1 follows.

As we already mentioned this paper is a continuation of [4]. Our presentation heavily relies on [4] and we assume that the reader is closely familiar with the results and the constructions of [4].

2 Unknotting and resolution theorems

For an open subset $V$ of a dense subset $D$ of a space $X$, by the extension of $V$ to $X$ we understand the largest open set $V'$ in $X$ such that $V = D \cap V'$. Similarly, by the extension to $X$ of a collection $\mathcal{V}$ of open subsets of $D$ we mean the collection consisting of the extensions to $X$ of the elements of $\mathcal{V}$.

For a collection $\mathcal{V}$ of subsets of $X$ denote $\text{st}^0(\mathcal{V}) = \mathcal{V}$, $\text{st}(\mathcal{V}) = \text{st}^1(\mathcal{V}) = \text{st}(\mathcal{V}, \mathcal{V})$ and define by induction $\text{st}^{n+1}(\mathcal{V}) = \text{st}(\text{st}^n(\mathcal{V}))$. We write $\mathcal{V} \prec \mathcal{U}$ if $\mathcal{V}$ refines a collection $\mathcal{U}$ of subsets of $X$.

A collection $\mathcal{V}$ of subsets of $X$ is said to be an $(n-1)$-refinement of a collection $\mathcal{U}$ of subsets of $X$, written $\mathcal{V} \prec_{n-1} \mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$ and the inclusion of $V$ into $U$ induces the zero-homomorphism of the homotopy groups in $\dim \leq n-1$.

A map $f : X \to Y$ is said to be $UV^{n-1}$ if $f(X)$ is dense in $Y$ and for every $y \in Y$ and every neighborhood $U$ of $y$ there is a smaller neighborhood $V$ of $y$ such that the inclusion of $f^{-1}(V)$ into $f^{-1}(U)$ induces the zero-homomorphism of the homotopy groups in $\dim \leq n-1$.

A map $f : X \to Y$ is said to be a $Z$-embedding if $f$ is a closed embedding of $X$ into $Y$ and $f(X)$ is a $Z$-set in $Y$.

The following proposition whose proof is left to the reader presents one of the basic properties of $UV^{n-1}$-maps on Nöbeling spaces.

Proposition 2.1 Suppose that $f : X \to Y$ is a $UV^{n-1}$-map from an $n$-dimensional Nöbeling space $X$ and $\mathcal{U}$ is an open cover of $Y$. Then there is an open cover $\mathcal{V}$ of $Y$
having the property: given a closed subset $A$ of a complete space $B$ of \( \dim \leq n \), a $Z$-embedding $\phi_A : A \to X$ of $A$ into $X$ and a map $\psi_B : B \to Y$ such that $\psi_B | A$ and $f \circ \phi_A$ are $V$-close, the map $\phi_A$ extends to a $Z$-embedding $\phi_B : B \to X$ of $B$ into $X$ such that $\psi_B$ and $f \circ \phi_B$ are $U$-close.

The following version of a $Z$-set unknotting theorem generalizes Theorem 1.3.

**Theorem 2.2** Let $f_i : X_i \to Y, i = 1, 2$ be $UV^{n-1}$-maps from $n$-dimensional Nöbeling spaces $X_i$ modeled on triangulated manifolds and let $U$ be an open cover of $Y$. Then there is an open cover $V$ of $Y$ such that for every pair of homeomorphic $Z$-sets $A_i \subset X_i$ and a homeomorphism $h_A : A_1 \to A_2$ such that the maps $f_i|A_1$ and $f_2 \circ h_A$ are $V$-close, the homeomorphism $h_A$ extends to a homeomorphism $h : X_1 \to X_2$ such that the maps $f_1$ and $f_2 \circ h$ are $U$-close.

For proving Theorem 2.2 we need the following simple modification of Proposition 3.1 of [4].

**Proposition 2.3** Let $X$ be an $n$-dimensional Nöbeling space, $A$ a $Z$-set in $X$, $C$ a cover of $X \setminus A$ that properly approaches $A$ and $W$ an open cover of $X$ such that $C$ is an $(n-1)$-refinement of $W$. Then for every $C \in C$ there is an open set $C \subset V_C \subset X \setminus A$ such that the inclusion $C \subset V_C$ induces the zero-homomorphism of the homotopy groups in $\dim \leq n - 1$, $V = \{V_C : C \in C\}$ properly approaches $A$ and $V$ refines $\text{st}^3 W$.

**Proof.** The only change in the proof of Proposition 3.1 of [4] that we need to make is to assume that the sets $G$ and $G_C$ are contained in elements of $W$, the maps $e_n$ are $W$-close to the identity map of $X$ and $U$ refines $W$. Then $\{Y_C : C \in C\}$ refines $\text{st}^3 W$ and therefore $V$ can be chosen so that $V$ refines $\text{st}^3 W$. \( \square \)

**Proof of Theorem 2.2.** The proof is similar to the proof of the unknotting theorem in [4]. The homeomorphism between $X_1$ and $X_2$ constructed in the proof of Theorem 1.2 of [4] will have the required properties if the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ are constructed so that for every $P \in \mathcal{P}_1$ there is an element of $\mathcal{U}$ which contains both $f_1(P \cap X_1)$ and $f_2(\mu(P) \cap X_2)$. In order to gain this additional control on $\mathcal{P}_1$ and $\mathcal{P}_2$ we need to make the following adjustments in the proof of Proposition 3.2 of [4].

To avoid any possible confusion with the notation of the proof of Proposition 3.2 of [4] let us re-denote $V$ and $U$ by $V_Y$ and $U_Y$, $f_1$ and $f_2$ by $f_{X_1}$ and $f_{X_2}$ and $h$ by $h_X$ respectively. Now we can adopt the notation of Proposition 3.2 of [4].

Let $\omega$ be the number of times that we use the constructions 2.7 and 2.9 of [4] for improving connectivity of $\mathcal{P}_2$ in the proof of 3.2 of [4]. Note that $\omega$ is finite and depends only on the dimension $n$. Because $f_{X_i}$ is $UV^{n-1}$ we can choose a finite sequence $Y^j, j = 1, \ldots, 2\omega$ of open covers of $Y$ such that $Y^{2\omega}$ refines $U_Y$ and for $X^j_i = f^{-1}_{X_i}(Y^j)$ we have $\text{st}^5(X^j_i)$ is an $(n-1)$-refinement of $X^{j+1}_i, j = 1, \ldots, 2\omega - 1$. Set $V_Y = Y^{j_1}$.

Fix a pair of $Z$-sets $A_i \subset X_i$ and a homeomorphism $h_{A_i} : A_1 \to A_2$ such that $f_{X_2} \circ h_{A_i}$ and $f_{X_1}|A_1$ are $V^3$-close. Recall that in Proposition 3.2 of [4] $X_i$ is considered
as a subspace of a space $Y$ that can be represented as $Y = M_i \cup X_i$ such that $M_i$ is a triangulated manifold, $X_i \setminus A_i = M_i(k_i)$, $A_i = Y_i \setminus M_i$ and $A_i$ is closed in $Y_i$. Following the proof of Proposition 3.2 of [4] we make the following adjustments.

We can assume that $M'_1$ and a partition $\mathcal{P}_1$ of $M'_1$ are chosen so that the extension $\mathcal{Y}_1^j$ of $\mathcal{X}_1^j$ to $Y_1$ covers $M'_1$ and $\mathcal{P}_1$ refines $\mathcal{Y}_1^1$. Since $f_{X_i}$ is $UV^{n-1}$, we can assume by Proposition 2.1 that the map $g : X_2 \to X_1$ is chosen so that $f_{X_2} : X_2 \to Y$ and $f_{X_1} \circ g : X_2 \to Y$ are $\mathcal{Y}_1^1$-close. Then, it is easy to see that $M'_2$ and the initial partition $\mathcal{P}_2$ of $M'_2$ and can be constructed so that the extension $\mathcal{Y}_2^j$ of $\mathcal{X}_2^j$ to $Y_2$ covers $M'_2$ and the one-to-one correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is $\mathcal{Y}_1^1$-agreeable with $f_{X_1}$ and $f_{X_2}$. The last property means that for every $P \in \mathcal{P}_1$ there is an element of $\mathcal{Y}_1^1$ which contains both $f_{X_1}(P \cap X_1)$ and $f_{X_2}(\mu(P) \cap X_2)$. Note that since every map from a sphere of dim $\leq n - 1$ into an open subset $G$ of $Y_i$ can be homotoped inside $G$ into $G \cap X_i$ we have that $\text{st}^5(\mathcal{Y}_1^j)$ is an $(n - 1)$-refinement of $\mathcal{Y}_1^{j+1}$, $j = 1, \ldots, 2\omega - 1$.

Our next step is to analyze the procedure of improving connectivity of $\mathcal{P}_2$. Recall that, in order to simplify the notation, in the beginning of each step of the procedure we replace $M_2$ and $\mathcal{P}_2$ by the output $M'_2$ and $\mathcal{P}'_2$ of the previous step (the modifications of $M_2$ and $\mathcal{P}_2$) and we also use $\mu$ to denote the one-to-one correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ which is the composition of $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ with the natural correspondence between $\mathcal{P}_2$ and its modification $\mathcal{P}'_2$.

Thus we replace $M_2$ by $M'_2$ for our initial partition $\mathcal{P}_2$ and now we assume that at some step of the procedure $M_2$ and $\mathcal{P}_2$ are already constructed so that $\text{st}(\mathcal{P}_2)$ refines $\mathcal{Y}_2^j$ and the one-to-one correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is $\mathcal{Y}_1^j$-agreeable with $f_{X_2}$ and $f_{X_1}$.

The following adjustments should be made in the part the proof of Proposition 3.2 of [4] where the construction 2.7 of [4] is applied. Choose $C$ so that $C$ refines $\mathcal{Y}_2^j$. Then by Proposition 2.3 the cover $\mathcal{V}$ of $M_2(k_2)$ can be chosen so that $\mathcal{V}$ refines $\text{st}^3(\mathcal{X}_2^{j+1})$. It implies that $\mathcal{V}$ refines $\mathcal{Y}_2^{j+1}$ and hence $\mathcal{W} = \text{st}(C, \mathcal{V})$ refines $\text{st}(\mathcal{Y}_2^{j+1})$. Then the needed modification of $\mathcal{P}'_2$ which is (the output of 2.7 of [4]) can be constructed so that it will refine $\text{st}^2(\mathcal{W})$ and hence it will refine $\text{st}^3(\mathcal{Y}_2^{j+1})$ and, as a result, will refine $\mathcal{Y}_2^{j+2}$ as well.

Now we will adjust the part of the proof of Proposition 3.2 of [4] where the construction 2.9 of [4] is applied. Once again we assume that $M_2$ and $\mathcal{P}_2$ are constructed so that $\text{st}(\mathcal{P}_2)$ refines $\mathcal{Y}_2^j$ and the one-to-one correspondence $\mu : \mathcal{P}_1 \to \mathcal{P}_2$ is $\mathcal{Y}_1^j$-agreeable with $f_{X_2}$ and $f_{X_1}$. Then $\mu^{-1}(\text{st}(\mathcal{P}_2))$ refines $\text{st}(\mathcal{Y}_1^j)$ and therefore by Proposition 2.3 the cover $\mathcal{V}$ of $M_1(k_1)$ can be chosen so that $\mathcal{V}$ refines $\mathcal{Y}_1^{j+1}$. Then $\mathcal{H} = \text{st}(\mathcal{V}, \mu^{-1}(\text{st}(\mathcal{P}_2)))$ refines $\text{st}(\mathcal{Y}_1^{j+1})$ and hence $\mu(\mathcal{H})$ refines $\mathcal{Y}_2^{j+1}$. Now $\mathcal{W}$ can be chosen so that $\mathcal{W}$ refines $\mathcal{Y}_2^{j+1}$ and therefore the modification of $\mathcal{P}_2$ will refine $\text{st}^2(\mathcal{Y}_2^{j+1})$ and, as a result, will refine $\mathcal{Y}_2^{j+2}$ as well.

In both 2.7 and 2.9 of [4] the modification $\mathcal{P}'_2$ of $\mathcal{P}_2$ can be constructed such that for every element $P \in \mathcal{P}_2$, its modification $P' \in \mathcal{P}'_2$ intersects $P$. Then we get that $\mu : \mathcal{P}_1 \to \mathcal{P}'_2$ is $\mathcal{Y}_1^{j+2}$-agreeable with $f_{X_1}$ and $f_{X_2}$ because $\mu$ was $\mathcal{Y}_1^j$-agreeable before the constructions. In addition, applying a homeomorphism $h : M_2 \cup A_2 \to M_2 \cup A_2$ sufficiently close to the identity map as it is described in the proof of Proposition 3.2 of [4] we can replace $M'_2$ and $\mathcal{P}'_2$ by $h(M'_2)$ and $h(\mathcal{P}'_2)$ respectively and assume that $\mathcal{P}'_2$ is a rational decomposition of $M'_2$. 
Thus denoting the final modification of $\mathcal{P}_2$ again by $\mathcal{P}_2$ we get that $\mu : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is $\mathcal{Y}^{2\omega}$-agreeable with $f_{X_1}$ and $f_{X_2}$. Recall that $\mathcal{Y}^{2\omega}$ refines $\mathcal{U}_Y$. Now we can construct a homeomorphism $h_X : X_1 \rightarrow X_2$ which carries $P \cap X_1$ onto $\mu(P) \cap X_2$ for every $P \in \mathcal{P}_1$. Then $f_{X_2} \circ h_X$ and $f_{X_1}$ are $\mathcal{U}_Y$-close and the theorem is proved.

Proposition 2.4 Let $f : X \rightarrow Y$ be a $UV^{n-1}$-map from an $n$-dimensional Nöbeling space $X$ modeled on a triangulated manifold to a complete space $Y$ and let $g_A : A \rightarrow Y$ be a map from a $Z$-set $A \subset X$. Then there is a $UV^{n-1}$-map $g : X \rightarrow Y$ such that $g|A = g_A$.

Proof. Fix a complete metric in $Y$ with distances bounded by $1/8$. Set $\mathcal{Y}_0 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3$ = the trivial cover $Y$ consisting of only one set $Y$. Using Theorem 2.2 choose a sequence of open covers $\mathcal{Y}_i$ of $Y$ such that

1. mesh$\mathcal{Y}_i \leq 1/2^i$;
2. st$^2\mathcal{Y}_{i+1} \subset \mathcal{Y}_i$;
3. $f^{-1}(\mathcal{Y}_{i+1}) \sim_{n-1} f^{-1}(\mathcal{Y}_i)$;
4. for every homeomorphism $h_B : B_1 \rightarrow B_2$ of $Z$-sets in $X$ such that $f|B_1$ and $f \circ h_B$ are $\mathcal{Y}_{i+1}$-close, $h_B$ extends to a homeomorphism $h : X \rightarrow X$ such that $f$ and $f \circ h$ are $\mathcal{Y}_i$-close.

Set $A_0 = A$ and $h_0$ = the identity map of $X$. We are going to construct for every $i$ a homeomorphism $h_i : X \rightarrow X$ such that

5. $f \circ h_i$ and $f$ are $\mathcal{Y}_i$-close;
6. $f \circ h^i|A$ and $g_A$ are $\mathcal{Y}_{i+3}$-close where $h^i = h_i \circ \cdots \circ h_0 : X \rightarrow X$.

Assume that the construction is completed up to the index $i$. Proceed to $i + 1$ as follows. By Proposition 2.1 choose a $Z$-embedding $h_B : B = h^i(A) \rightarrow X$ such that $f \circ h_B$ and $g_A \circ (h^i)^{-1}|B$ are $\mathcal{Y}_{i+4}$-close. By (6), $f|B$ and $g_A \circ (h^i)^{-1}|B$ are $\mathcal{Y}_{i+3}$-close and hence by (2) $f \circ h_B$ and $f|B$ are $\mathcal{Y}_{i+2}$-close. Then by (4) there is a homeomorphism $h_{i+1} : X \rightarrow X$ such that $f \circ h_{i+1}$ and $f$ are $\mathcal{Y}_{i+1}$-close and $h_{i+1}|B = h_B$. Note that the $\mathcal{Y}_{i+4}$-closeness of $f \circ h_B$ and $g_A \circ (h^i)^{-1}|B$ is equivalent to the $\mathcal{Y}_{i+4}$-closeness of $f \circ h^{i+1}|A$ and $g_A$. The construction is completed.

Denote $g_i = f \circ h^i : X \rightarrow Y$. By (5) we have that

7. $g_i$ and $g_{i+1}$ are $\mathcal{Y}_{i+1}$-close

and by (3)

8. $g_j^{-1}(\mathcal{Y}_{i+1}) \sim_{n-1} g_j^{-1}(\mathcal{Y}_i)$.

Define $g = \lim g_i : X \rightarrow Y$. By (6) we have $g|A = g_A$ and by (2), (7) and (8) we have

$g^{-1}(\mathcal{Y}_{i+5}) \sim g_{i+5}^{-1}(\mathcal{Y}_{i+3}) \sim_{n-1} g_{i+5}^{-1}(\mathcal{Y}_{i+2}) \sim g^{-1}(\mathcal{Y}_i)$.

This implies that $g$ is $UV^{n-1}$ and the proposition follows.

Proposition 2.5 Let $A$ be a $\sigma$-$Z$-set in an $n$-dimensional Nöbeling space $X$ modeled on a triangulated manifold. Then $X \setminus A$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold.

Proof. Represent $A = \bigcup_{i=1}^\infty A_i$ where $A_i \subset A_{i+1}$ and $A_i$ is a $Z$-set in $X$. Set $X_0 = X$, $A_0 = \emptyset$ and $X_i = X \setminus A_i$. Note that by Theorem 1.3, $X_i$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold. Also note that the
inclusion of $X_{i+1}$ into $X_i$ is a $UV^{n-1}$-map. Fix a complete metric $d$ on $X$. Using Theorem 2.2 construct for every $i$ an open cover $\mathcal{V}_i$ of $X_i$ and a homeomorphism $f_i : X_i \to X_{i+1}$ such that

1. mesh$\mathcal{V}_i < 1/2^i$;
2. $\mathcal{V}_i$ properly approaches $A_i$;
3. st$\mathcal{V}_{i+1} \prec \mathcal{V}_i$;
4. $f_i$ is $\mathcal{V}_i$-close to the identity map of $X_i$;
5. mesh$(f_i^0)^{-1}(\mathcal{V}_i) < 1/2^i$

where $f_i^0 = id$ and $f_i^0 = f_{i-1} \circ \ldots f_0 : X = X_0 \to X_i$.

Denote $f = \lim_{i \to \infty} f_i^0 : X \to X$ and let us show that $f(X) = X \setminus A$ and $f$ is a homeomorphism between $X$ and $X \setminus A$.

Let $x \in X$. By (3) and (4), $f(x)$ and $f_i^0(x)$ are st$\mathcal{V}_i$-close and hence by (2), $f(x)$ is not in $A_i$. Thus $f(x) \in X \setminus A$.

Take $y \in X \setminus A$ and let $x_i \in X_i$ be such that $f_i^0(x_i) = y$. By (3) and (4), $f_i^0(x_i)$ and $f_{i+1}^0(x_i)$ are $\mathcal{V}_i$-close and hence by (1) and (5), $d(x_i, x_{i+1}) \leq 1/2^i$. Denote $x = \lim x_i$. By (3) and (4), $f(x_i)$ and $f_{i}^0(x_i)$ are st$\mathcal{V}_i$-close and hence $f(x) = y$. Thus we showed that $f(X) = X \setminus A$.

Now take $x_1, x_2 \in X$ such that $d(x_1, x_2) > 1/2^{i-100}$ for some $i > 100$. Then by (3) and (5), $f_i^0(x_1)$ and $f_i^0(x_2)$ are not st$\mathcal{V}_i$-close. By (3) and (4), $f(x)$ and $f_i^0(x)$ are st$\mathcal{V}_i$-close for every $x \in X$ and hence $f(x_1)$ and $f(x_2)$ are not st$\mathcal{V}_i$-close. This implies that $f$ is a homeomorphism between $X$ and $X \setminus A$. \qed

**Proposition 2.6** Let $X$ be an $n$-dimensional Nöbeling space modeled on a triangulated manifold. Then $X$ can be split into closed subsets $X = \bigcup_{i \in \mathbb{Z}} X_i$ such that $\{X_i : i \in \mathbb{Z}\}$ is a locally finite cover of $X$, $X_i \cap X_j = \emptyset$ if $|i - j| > 1$, $X_i$ and $A_i = X_i \cap X_{i-1}$ are homeomorphic to $n$-dimensional Nöbeling spaces modeled on triangulated manifolds and the sets $A_i$ and $A_{i+1}$ are $Z$-sets in $X_i$.

**Proof.** By Theorems 1.2 and 1.3 we can assume that $X$ is the $n$-dimensional Nöbeling space $R^{2n+2}(n+1)$ modeled on the $(2n+2)$-dimensional Euclidean space $R^{2n+2}$. By a reasoning similar to the one used in the interpretation of Nöbeling spaces given in the introduction (Section 1) of [4] we can represent $X$ as $X = R^{2n+2} \setminus K$ where $K$ is the union of the rational planes of dim $\leq n + 1$ (an $m$-dimensional plane of $R^{2n+2}$ is rational if it is spanned by $m + 1$ points with rational coordinates). Consider $R^{2n+2}$ as the product $R^{2n+2} = R^{2n+1} \times R$ and denote by $p : R^{2n+2} = R^{2n+1} \times R \to R$ and $q : R^{2n+2} = R^{2n+1} \times R \to R^{2n+1}$ the projections.

Choose a discrete sequence of irrational numbers $a_i \in R$ indexed by the integers $i \in \mathbb{Z}$ such that $a_i < a_{i+1}$, $\lim_{i \to +\infty} a_i = -\infty$ and $\lim_{i \to -\infty} a_i = \infty$. Denote $X_i = X \cap p^{-1}([a_i, a_{i+1}])$ and $A_i = X \cap p^{-1}(a_i)$. Since $X_i \setminus (A_i \cup A_{i+1}) = X \cap p^{-1}((a_i, a_{i+1}))$, $X_i \setminus (A_i \cup A_{i+1})$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold.

Note that $A_i = p^{-1}(a_i) \setminus K \subset p^{-1}(a_i) \cap q^{-1}(R^{2n+1}(n))$ and $p^{-1}(a_i) \cap K$ is a countable union of planes of dim $\leq n$. Also note that the intersection of a plane of dim $\leq n$ in $R^{2n+1}$ with $R^{2n+1}(n)$ is a $Z$-set in $R^{2n+1}(n)$. Thus $A_i$ can be considered as a subspace of
Let $X$ be an $n$-dimensional Nöbeling space modeled on a triangulated manifold. We get by Theorem 1.3 that $X$ is also homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold. The proposition is proved.

**Theorem 2.7** Let a complete space $Y$ of dim $Y \leq n$ be an absolute extensor in dimension $n$. Then there is a $UV^{n-1}$-map $f : X \rightarrow Y$ from an $n$-dimensional Nöbeling space $X$ modeled on a triangulated manifold.

**Proof.** Split $X$ as described in Proposition 2.6. For every $i$ choose a homeomorphism $h_{A_i} : A_i \rightarrow A_0$ such that $h_{A_0} = id_{A_0}$. Consider a closed embedding $e : Y \rightarrow A_0$ of $Y$ into $A_0$ and a continuous retraction $r : A_0 \rightarrow e(Y)$, and denote $f_{A_i} = e^{-1} \circ r \circ h_{A_i} : A_i \rightarrow Y$. Note that $f_{A_i} = f_{A_0} \circ h_{A_i}$. By Proposition 2.4 there is a $UV^{n-1}$-map $g_{X_i} : X_i \rightarrow A_i$ such that $g_{X_i}|A_i$ is the identity map of $A_i$ and $g_{X_i}|A_{i+1} = h_{A_i}^{-1} \circ e \circ f_{A_0} \circ h_{A_i} : A_{i+1} \rightarrow A_i$. Set $f_X = f_{A_0} \circ g_{X_i} : X_i \rightarrow Y$ and note that $f_X|A_{i+1} = f_{A_{i+1}}$ and $f_X|A_i = f_{A_i}$. Then the maps $f_X$ define the corresponding map $f : X \rightarrow Y$.

Let us show that $f$ is a $UV^{n-1}$-map. Take open sets $V$ and $U$ in $Y$ such that $V \subset U$ and the inclusion of $V$ into $U$ induces the zero-homomorphism of the homotopy groups in dim $\leq n - 1$. Consider a map $\psi : S \rightarrow f^{-1}(V)$ from a sphere $S$ of dim $\leq n - 1$ and let us show that $\psi$ is null-homotopic inside $f^{-1}(U)$. Since $S$ is compact there are $i < j$ such that $\psi(S) \subset X_i \cup X_{i+1} \cup \cdots \cup X_j$.

Recall that $g_{X_i} : X_j \rightarrow A_j$ is a $UV^{n-1}$-retraction onto $A_j \subset X_j$ and note that $f^{-1}(V) \cap X_j = g_{X_i}^{-1}(W)$ for $W = f_{A_j}^{-1}(V)$. Then by Proposition 2.1, for any map $\phi : B \rightarrow X_j$ from a space of $B$ of dim $\leq n - 1$ such that $\phi(B) \subset f^{-1}(V) \cap X_j$, $\phi$ can be homotoped inside $f^{-1}(V) \cap X_j$ and relative to $\phi^{-1}(A_j)$ to the map $g_{X_j} \circ \phi : B \rightarrow A_j$.

Thus we can homotope $\psi$ inside $f^{-1}(V)$ into a map to $X_i \cup \cdots \cup X_{j-1}$ and, proceeding by induction on $j$, finally homotope $\psi$ inside $f^{-1}(V)$ into a map to $A_i$. Now $\psi$ can be homotoped inside $f^{-1}(V)$ to a map to $g_{X_{i-1}} \circ \psi : S \rightarrow g_{X_{i-1}}(A_i) = h_{A_{i-1}}^{-1}(e(Y)) \subset A_{i-1}$. Because of the similarity between $A_{i-1}$ and $A_0$ we may assume that $A_{i-1} = A_0$. Thus without loss of generality we may assume that $\psi : S \rightarrow e(V) = f^{-1}(V) \cap e(Y) \subset A_0$. Then $\psi$ is null-homotopic inside $e(U) \subset f^{-1}(U)$ and the theorem follows. □

Let $f : X \rightarrow Y$ be a map. We say that a point $x \in X$ is a regular point of $f$ if the collection $\{f^{-1}(U) : f(x) \in U, U \text{ is open in } Y\}$ is a base of $x$ in $X$. Let $A$ be a closed subset of $Y$. Denote by $X \cup_f A$ the disjoint union of $X \setminus f^{-1}(A)$ and $A$, and define the topology of $X \cup_f A$ such that the topology of $X \setminus f^{-1}(A)$ is preserved and for every $U$ open in $Y$, the set $(U \cap A) \cup f^{-1}(U \setminus A)$ is open in $X \cup_f A$. Then the space $X \cup_f A$ is separable metrizable (since so are $X$ and $Y$), the induced function $f' : X' = X \cup_f A \rightarrow Y$ defined by $f'(a) = a$ if $a \in A$ and $f'(x) = f(x)$ if $x \in X \setminus f^{-1}(A)$ is continuous and every point of $A$ is a regular point of $f'$.

A proof of the following proposition is left to the reader.

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Proposition 2.8 Let $f : X \to Y$ be a $UV^{n-1}$-map of $n$-dimensional Nöbeling spaces $X$ and $Y$ and let $A \subset Y$ be a $Z$-set in $Y$. Then $X' = X \cup_f A$ is an $n$-dimensional Nöbeling space, $A$ is a $Z$-set in $X'$ and the induced map $f' : X' \to Y$ is $UV^{n-1}$.

Theorem 2.9 Let $f : X \to Y$ be a $UV^{n-1}$-map from an $n$-dimensional Nöbeling space $X$ modeled on a triangulated manifold to an $n$-dimensional Nöbeling space $Y$. Then there is a $UV^{n-1}$-map $g : X \to Y$ for which there is a $\sigma$-$Z$-set $A \subset Y$ such that $g^{-1}(A)$ is a $\sigma$-$Z$-set in $X$ and $g^{-1}(Y \setminus A)$ consists of regular points of $g$.

Proof. Fix complete metrics in $X$ and $Y$. Take a sequence of open covers $\mathcal{X}_i$, $i = 1, 2, \ldots$ and $Z$-embeddings $\psi_i : X \to X, i = 2, 3, \ldots$ such that mesh$\mathcal{X}_i \leq 1/2^i$, $\psi_i$ is $X'_i$-close to the identity map and the $Z$-sets $A_i = \psi_i(X)$ are pair-wise disjoint.

Set $f_i = f$, $A_1 = \emptyset$ and let $\mathcal{Y}_1$ be an open cover of $Y$ with mesh$\mathcal{Y}_1 \leq 1/2$. We are going to construct for every $i$ an open cover $\mathcal{Y}_i$ and a $UV^{n-1}$-map $f_i : X \to Y$ such that:

1. mesh$\mathcal{Y}_i \leq 1/2^i$ and st$_{\mathcal{Y}_i}$ $\mathcal{Y}_{i+1} \prec_{n-1} \mathcal{Y}_i$;
2. $f_i$ and $f_{i+1}$ are $\mathcal{Y}_i$-close;
3. $f_i^{-1}(f_i(A_i)) = A_i$, $f_i|A_i$ is a $Z$-embedding of $A_i$ in $Y$ and $f_i|A_j = f_j|A_j$ for $i > j$;
4. every point of $A_1 \cup \cdots \cup A_i$ is a regular point of $f_i$ and the family $\mathcal{A}_i = \{f_i^{-1}(U) \cap (A_1 \cup \cdots \cup A_i) \neq \emptyset, U \in \mathcal{Y}_i\}$ refines $\mathcal{X}_i$.

Assume that the construction for the indices $\leq i$ is completed. Proceed to $i + 1$ as follows.

Let $\mathcal{U} = \mathcal{Y}_i$. By Theorem 2.2 there is an open cover $\mathcal{V}$ of $Y$ such that the conclusions of Theorem 2.2 hold for $f$ replaced by the map $f_i : X \to Y$.

Approximate $f_i$ by a $Z$-embedding $\phi : X \to Y$ such that $\phi(X)$ does not intersect $f_i(A_1 \cup \cdots \cup A_i)$ and $\phi$ is $\mathcal{V}$-close to $f_i$. Denote $B = \phi(A_{i+1})$ and $X' = X \cup_f B$, and let $f' : X' \to Y$ be the map induced by $f_i$. By Proposition 2.8 and Theorem 1.3 the space $X'$ is homeomorphic to an $n$-dimensional Nöbeling space modeled on a triangulated manifold. Then by Theorem 2.2 one can choose a homeomorphism $h' : X \to X'$ such that $f' \circ h'$ is arbitrarily close to $f_i$ and $h'(a) = a$ for $a \in A_1 \cup \cdots \cup A_i$. In particular $h'$ can be chosen so that $f' \circ h'$ is $\mathcal{V}$-close to $f_i$. Then again by Theorem 2.2 there is a homeomorphism $h : X \to X$ such that $h(a) = (h')^{-1}(\phi(a))$ for $a \in A_{i+1}$, $h(a) = a$ for $a \in A_1 \cup \cdots \cup A_i$ and $f_{i+1} = f' \circ h \circ h$ is $\mathcal{U}$-close to $f_i$. Note that every point of $A_1 \cup \cdots \cup A_{i+1}$ is a regular point of $f_{i+1}$.

Choose an open cover $\mathcal{Y}_{i+1}$ of $Y$ such that the properties (1), (2) and (4) are satisfied. The construction is completed.

Define $g = \lim f_i : X \to Y$. The properties (1) and (2) imply that

5. $g^{-1}(\mathcal{Y}_{i+3}) \prec_{f_{i+3}} f_{i+3}^{-1}(\mathcal{Y}_{i+2}) \prec_{f_i} f_i^{-1}(\mathcal{Y}_{i+1}) \prec g^{-1}(\mathcal{Y}_i)$.

Since $\mathcal{Y}_{i+2} \prec_{n-1} \mathcal{Y}_{i+1}$ and $f_{i+3}$ is $UV^{n-1}$ we have $f_i^{-1}(\mathcal{Y}_{i+2}) \prec_{n-1} f_i^{-1}(\mathcal{Y}_{i+1})$ and therefore $g$ is $UV^{n-1}$. The properties (3-5) imply that for every $i$, $g|(A_1 \cup \cdots \cup A_i)$ is a $Z$-embedding of $A_1 \cup \cdots \cup A_i$ into $Y$ and every point of $A_1 \cup \cdots \cup A_i$ is a regular point of $g$. Denote by $C$ the set of all regular points of $g$. Then $C$ is $G_\delta$ in $X$ and since $A_i \subset C$ for every $i$ we have that $X \setminus C$ is a $\sigma$-$Z$-set in $X$. Hence by Proposition 2.1 one can choose a map.
Thus by Theorems 2.7 and 2.9 there is a $\sigma$-Z-set in $Y$ and the conclusions of the theorem hold for $A = Y \setminus g(C)$. □

Proof of Theorem 1.4. By Theorems 2.7 and 2.9 there is a $UV^{n-1}$-map $g : X' \to Y$ from an $n$-dimensional Nöbeling space $X'$ modeled on a triangulated manifold such that there is a $\sigma$-Z-set $A$ in $Y$ for which $g^{-1}(A)$ is a $\sigma$-Z-set in $X'$ and $g$ embeds $X = X' \setminus g^{-1}(A)$ into $Y$. By Proposition 2.5, $X$ is homeomorphic to an $n$-dimensional Nöbeling space and the theorem follows. □

3 Auxiliary constructions and properties

3.1 A few general properties

Let $M$ be an $m$-dimensional triangulated manifold with $m \geq 2n+1$. Set $k = m - n - 1$. Fix a triangulation $\mathcal{T}$ of $M$ and embed $M$ into a Hilbert space $H$ by a map which is linear on each simplex of $\mathcal{T}$. Let $K$ be a countable union of planes in $H$ of dim $\leq k$ and denote $X = M \setminus K$ (it is shown in Introduction of [4] that $M(k)$ admits such a representation). It is clear that $M \cap K$ can be represented as a countable union of simplexes of dim $\leq k$ PL-embedded in $M$.

Let $f : B^q \to \text{Int} M$ be a PL-embedding of a $q$-dimensional ball $B^q$ with $q \leq n$. Then $f$ can be arbitrarily closely approximated by a PL-embedding $f' : B^q \to \text{Int} M$ such that $f'(B^q) \subset X$. To show that identify $B^q$ with $f(B^q)$ and extend the embedding $B^q \subset \text{Int} M$ to a PL-embedding $B^m = B^q \times B^{m-q}$ of an $m$-dimensional ball $B^m$ with $B^q = B^q \times O$. Since $K$ is a countable union of simplexes of dim $\leq k$ PL-embedded in $M$ one can choose a point $a \in B^{m-q}$ arbitrarily close to $O$ such that $B^q \times a \subset X$. This way we get the required approximation of $f$.

Let $\mathcal{P}$ be a decomposition of $M$. Then there is an open subset $M'$ of $M$ containing $X$ such that the finite intersections of $\mathcal{P}$ restricted to $M'$ have dense subsets lying in $X$. Indeed, take a subdivision $\mathcal{T}'$ of $\mathcal{T}$ such that $\mathcal{T}'$ underlies $\mathcal{P}$. Then one can easily verify that for every simplex $\Delta$ of $\mathcal{T}'$ having an open subset lying outside $X$, the entire simplex $\Delta$ lies outside $X$. Remove from $M$ all the simplexes of $\mathcal{T}'$ lying outside $X$ and get the open subset $M'$ with the required properties.

Let $F$ be a PL-subcomplex of $M$ such that $\mathcal{P}$ forms a partition on $M \setminus F$. Then for every open subset $M'$ of $M$ containing $X$ we have that for every finite intersection $P$ of $\mathcal{P}$ the inclusion $(P \cap M') \setminus F \subset P \setminus F$ induces an isomorphism of the homotopy groups in co-dimensions $\geq m - n + 1$ (=in dimensions $\leq \dim (P \setminus F) - (m - n + 1)$). Indeed, let $P$ be a finite intersection of $\mathcal{P}$ with $t = \dim P \setminus F \geq m - n + 1$. Since $M \setminus X$ is a countable union of simplexes of dim $\leq k$ PL-embedded in $M$, we have that $(P \setminus F) \setminus X$ is also a countable union of simplexes of dim $\leq k$ PL-embedded in $P \setminus F$. Then every map of a sphere of dim $\leq t - (m - n + 1)$ into $P \setminus F$ can be homotoped into $(P \setminus F) \cap X \subset (P \setminus F) \cap M'$. Now let $f : S \to (M' \cap P) \setminus F$ be a map from a sphere of dim $\leq t - (m - n + 1)$. Since $(P \setminus F) \setminus X$ is a countable union of simplexes of dim $\leq k$ PL-embedded in $P \setminus F$ we have that $P \setminus (X \cup F \cup f(S))$ is also a countable union of simplexes of dim $\leq k$ PL-
embedded in $P \setminus F$. Then if $f$ is null-homotopic in $P \setminus F$ we get that $f$ is null-homotopic in $(P \setminus F) \cap (X \cup f(S))$. Thus $f$ is null-homotopic in $(M' \cap P) \setminus F$ and we are done.

In particular we get that if $P$ forms an $l$-co-connected partition on $M \setminus F$ with $l \geq m - n + 1$ then $P$ restricted to $M' \setminus F$ is also $l$-co-connected. Note that the correspondence $\mu$ sending $P \in \mathcal{P}$ to $P \cap M'$ induces an $n$-matching between $P$ restricted to $M \setminus F$ and $P$ restricted to $M' \setminus F$, see 3.3.

Let $X = M(k)$. We leave to the reader to verify that for every PL-subcomplex $L$ of $M$ with $\dim L \leq n$, $X \cap L$ is a $Z$-set in $X$.

### 3.2 Creating intersections

Let $M$ be a triangulated $(l-1)$-co-connected manifold such that $m = \dim M \geq 2q+1$, $q = m-l+2$. Assume that $F$ is a PL-subcomplex with $\dim F \leq l-2$, $P$ is a decomposition of $M$ such that $P$ forms an $(l-1)$-co-connected partition on $M \setminus F$. Consider $t+1$ distinct elements $P_0, \ldots, P_t$ of $P$, $t = m-l+2$, such that their intersection is empty but the intersection of any $t$ of them is not empty.

Let us show how the construction 2.5 of [4] can be used to create an intersection of $P_0, \ldots, P_t$. Use the notation of 2.5 of [4] and assume that $\partial_\mu = \emptyset$. Then for every simplex $\Delta'$ of $\Delta$, $S_\mu \ast \Delta' = \Delta'$ and we start the construction for 0-dimensional simplexes $\Delta'$ with any $f_{\Delta'} : \Delta' \rightarrow \text{Int}(P(\Delta') \cap U)$. Now we can follow 2.5 of [4] to construct $f_{\partial \Delta'}$ and after that to modify $P$ to $P^\pi$. It is easy to see that the modifications of $P_0, \ldots, P_t$ will intersect on $M \setminus F^\pi$.

Now suppose that we have a (possibly countable) collection $C = \{(P_0, \ldots, P_t) : P_i \in \mathcal{P}\}$ of $(t+1)$-tuples of elements of $\mathcal{P}$ of the type described above and we need to create an intersection for every $(t+1)$-tuple from $C$. Assume that $W$ is an open cover of $M$ such that for every $(t+1)$-tuple $(P_0, \ldots, P_t) \in C$ there is a set $W \in W$ such that $P_0 \cup \cdots \cup P_t \subset W$ and the inclusion $P_0 \cup \cdots \cup P_t \subset W$ induces the zero homomorphism of the homotopy groups in $\dim \leq m-l+1$.

Then the construction 2.7 of [4] applies to modify $M$ to an open subset $M'$ of $M$, $F$ to a PL-subcomplex of $F'$ of $M'$, each element $P$ of $\mathcal{P}$ to a PL-subcomplex $P'$ of $M'$ such that the following properties are satisfied: $\dim M \setminus M' \leq q$, $\dim F' \leq l-2$, $P' = \{P' : P \in \mathcal{P}\}$ is a decomposition of $M'$ which forms an $(l-1)$-co-connected partition on $M' \setminus F'$, $P' \subset \text{st}(P, \text{st}W)$ for each $P \in \mathcal{P}$, all the finite intersections of $P$ restricted to $M \setminus F$ are preserved in $P'$ restricted to $M' \setminus F'$ (via the natural correspondence between $\mathcal{P}$ and $\mathcal{P}'$) and the only new finite intersections of $P'$ that are created on $M' \setminus F'$ are for the $(t+1)$-tuples in $C$.

### 3.3 A $t$-matching of partitions

Let $M_1$ and $M_2$ be triangulated manifolds and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be partitions of $M_1$ and $M_2$ respectively. A one-to-one correspondence $\mu : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is said to be a $t$-matching if for any $P_0, P_1, \ldots, P_t \in \mathcal{P}_1$ we have that $P_0 \cap \cdots \cap P_t \neq \emptyset$ if and only if $\mu(P_0) \cap \cdots \cap \mu(P_t) \neq \emptyset$.  

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In other words, \( \mu \) is a \( t \)-matching if it preserves the intersections in co-dimensions \( \leq t \). Thus \( \mu \) is a matching if it is a \( t \)-matching for every \( t = 0, 1, 2, \ldots \).

Denote by \( F_i \) the union of all finite intersections of \( \mathcal{P}_i \) of \( \dim \leq \dim M_i - t - 1 \). Then \( \mu \) induces a matching of partitions when \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are restricted to \( M_1 \setminus F_1 \) and \( M_2 \setminus F_2 \) respectively if and only if \( \mu \) is a \( t \)-matching between \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

### 3.4 Improving the level of matching of partitions

Let \( M_i, i = 1, 2 \) be \((l_i - 1)\)-co-connected triangulated manifolds such that \( m_i = \dim M_i \geq 2(m_i - l_i + 2) + 1 \) and \( m_1 - l_1 = m_2 - l_2 \). Denote \( t = m_i - l_i + 1 \). Assume that \( \mathcal{P}_1 \) is an \((l_i - 1)\)-co-connected partition of \( M_1 \), \( F_2 \) is a PL-subcomplex of \( M_2 \), \( \dim F_2 \leq l_2 - 2 \), \( \mathcal{P}_2 \) is a decomposition of \( M_2 \) forming an \( l_2\)-co-connected partition of \( M_2 \setminus F_2 \) and \( \mu : \mathcal{P}_1 \to \mathcal{P}_2 \) is a one-to-one correspondence such that \( \mu \) induces a \( t \)-matching between \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) restricted to \( M_2 \setminus F_2 \). Let us show how to turn \( \mu \) into a \((t + 1)\)-matching simultaneously with improving connectivity of \( \mathcal{P}_2 \).

By \( M'_2, F'_2 \) and \( \mathcal{P}'_2 \) we denote the modifications of \( M_2, F_2 \) and \( \mathcal{P}_2 \) and we always assume that \( M'_2 \) is an open subset of \( M_2 \), \( M_2 \setminus M'_2 \) is PL-presented in \( M_2 \), \( F'_2 \) is a PL-subcomplex of \( M'_2 \), \( \mathcal{P}'_2 \) is a decomposition of \( M'_2 \) which forms a partition on \( M'_2 \setminus F'_2 \). By \( \mu' : \mathcal{P}_1 \to \mathcal{P}'_2 \) we denote the correspondence sending \( P \in \mathcal{P}_1 \) to the modification of \( \mu(P) \) in \( \mathcal{P}'_2 \).

Apply the construction 2.8 (improving the total connectivity of a partition) of [4] to modify \( M_2 \), \( F_2 \) and \( \mathcal{P}_2 \) to \( M'_2, F'_2 \) and \( \mathcal{P}'_2 \) such that \( \dim M_2 \setminus M'_2 \leq m_2 - l_2 \), \( \dim F'_2 \leq l_2 - 2 \) and \( \mathcal{P}_2 \) is \((l_2 - 1)\)-co-connected on \( M'_2 \setminus F'_2 \).

The next step is to create the missing intersections of \( \mathcal{P}'_2 \) in \( \dim = m - t - 1 \). Denote \( \mathcal{C} = \{(P_0, \ldots, P_{t+1}) : P_0, \ldots, P_{t+1} \) are distinct elements in \( \mathcal{P}'_2 \) such that \( P_0 \cap \cdots \cap P_{t+1} \cap (M'_2 \setminus F'_2) = \emptyset \) and \( \mu'^{-1}(P_0) \cap \cdots \cap \mu'^{-1}(P_{t+1}) \neq \emptyset \}. \) Apply 3.2 to modify \( M'_2, F'_2 \) and \( \mathcal{P}'_2 \) to create intersections for the \((t + 2)\)-tuples in \( \mathcal{C} \).

Let us make \( \mu' \) induce a matching. We will say that a non-empty finite intersection \( P = P_0 \cap \cdots \cap P_s \) of distinct elements in \( \mathcal{P}_1 \) is brought from \( \mathcal{P}'_2 \) if \((\mu'(P_0) \cap \cdots \cap \mu'(P_s)) \cap (M'_2 \setminus F'_2) \neq \emptyset \). Similarly we say that a finite intersection \( P = P_0 \cap \cdots \cap P_s \) of distinct elements in \( \mathcal{P}_2 \) is brought from a finite intersection of \( \mathcal{P}_1 \) if \((P_0 \cap \cdots \cap P_s) \cap (M'_2 \setminus F'_2) \neq \emptyset \) and \( \mu'^{-1}(P_0) \cap \cdots \cap \mu'^{-1}(P_s) \neq \emptyset \).

Denote \( F_1 = \) the union of the finite intersections of \( \mathcal{P}_1 \) of \( \dim < m - t - 1 \) which are not brought from \( \mathcal{P}_2 \):

\( F_2^+ = \) the union of the finite intersections of \( \mathcal{P}_2 \) which are not brought from the finite intersections of \( \mathcal{P}_1 \) of \( \dim < m - t - 1 \)

\( F_2^{++} = \) the union of finite intersections of \( \mathcal{P}_2 \) which are not brought from the finite intersections of \( \mathcal{P}_1 \) of \( \dim = m - t - 1 \)

Since \( F_2^+ \) and \( F_2^{++} \) are unions of finite intersections of \( \mathcal{P}_2 \) then for every finite intersection \( P \) of \( \mathcal{P}'_2 \) such that \((P \setminus F'_2) \setminus (F_2^+ \cup F_2^{++}) \neq \emptyset \) we have \((P \setminus F'_2) \setminus (F_2^+ \cup F_2^{++}) \subset \partial (P \setminus F'_2)\), see 2.2 of [4]. Hence replacing \( F'_2 \) by the union \( F'_2 \cup F_2^+ \cup F_2^{++} \) we get that \( \mathcal{P}'_2 \) remains to be
an \((l_2 - 1)-\)co-connected partition on \(M'_2 \setminus F'_2\), \(\dim F'_2 \leq l_2 - 2\) and every finite intersection of \(\mathcal{P}'_2\) (on \(M'_2 \setminus F'_2\)) is brought from \(\mathcal{P}_1\) by the correspondence \(\mu'\). Similarly we conclude that \(\mathcal{P}_1\) restricted to \(M_1 \setminus F_1\) is \((l_1 - 1)-\)connected.

Thus \(\mu'\) becomes a matching of partitions when \(\mathcal{P}_1\) and \(\mathcal{P}'_2\) are restricted to \(M_1 \setminus F_1\) and \(M'_2 \setminus F'_2\) respectively. Since \(\dim F_1 \leq m_1 - t - 2 = l_1 - 3\) and \(M_1\) is \((l_1 - 1)-\)co-connected we get that \(M_1 \setminus F_1\) is also \((l_1 - 1)-\)co-connected. Then by 2.3 of [4] we get that \(M'_2 \setminus F'_2\) is \((l_2 - 1)-\)co-connected. Now we can apply the construction 2.9 (absorbing simplexes) of [4] to modify \(M'_2, F'_2\) and \(\mathcal{P}'_2\) in order to reduce the dimension of \(F'_2\) to \(\dim \leq l_2 - 3\) leaving the other characteristics of \(M'_2, F'_2\) and \(\mathcal{P}'_2\) unchanged.

Thus we finally get that \(\mu'\) is a \((t + 1)-\)matching between \(\mathcal{P}_1\) and \(\mathcal{P}'_2\) restricted to \(M'_2 \setminus F'_2\), \(\dim M_2 \setminus M'_2 \leq m_2 - l_2\), \(\dim F'_2 \leq l_2 - 3\) and \(\mathcal{P}'_2\) is \((l_2 - 1)-\)connected on \(M'_2 \setminus F'_2\).

The procedure described above can be used iteratively as follows. Assume that \(\dim M_i \geq 2n + 1\), \(M_i\) is \((n - 1)-\)connected \((= (m_i - n + 1)-\)co-connected\), \(\mathcal{P}_1\) is an \((m_1 - n + 1)-\)co-connected partition of \(M_1\) and \(\mathcal{P}_2\) is a partition of \(M_2\) which admits a 0-matching \(\mu : \mathcal{P}_1 \to \mathcal{P}_2\). Then repeating inductively the above procedure we can modify \(M_2, F_2, \mathcal{P}_2\) and \(\mu\) to \(M'_2, F'_2, \mathcal{P}'_2\) and \(\mu' : \mathcal{P}_1 \to \mathcal{P}'_2\) such that \(\dim M_2 \setminus M'_2 \leq n\), \(\dim F'_2 \leq m_2 - n - 1\), \(\mathcal{P}'_2\) is \((m_2 - n + 1)-\)co-connected on \(M'_2 \setminus F'_2\) and \(\mu'\) induces an \(n\)-matching between \(\mathcal{P}_1\) and \(\mathcal{P}'_2\) restricted to \(M'_2 \setminus F'_2\).

### 3.5 A remark on improving connectivity of intersections

In the construction 2.4 of [4] we consider a decomposition \(\mathcal{P}\) of a triangulated \(m\)-dimensional manifold \(M\) and a PL-subcomplex \(F\) such that \(\mathcal{P}\) forms a partition on \(M \setminus F\). We fix a finite intersection \(P\) of whose connectivity has to be improved, take a PL-embedding \(f\) of a sphere \(S_P\) into \(\text{Int}(P \setminus F)\), extend \(f\) to a PL-embedding \(f_{\partial\Delta}\) of a larger \((q - 1)\)-dimensional sphere \(S^{q - 1}\), identify \(f_{\partial\Delta}(S^{q - 1})\) with the boundary \(\partial B^q\) of a \(q\)-dimensional ball \(B^q\) and finally (using 4.2 of [4]) observe that the PL-embedding of \(\partial B^q\) can be extended to a PL-embedding of an \(m\)-dimensional ball \(B^m = B^q \times B^{m - q}\) \(\subset \text{Int} M\) such that \(B^m \cap F = (B^q \cap F) \times B^{m - q}\) and for every \(P' \in \mathcal{P}\), \(B^m \cap P' = (B^q \cap P') \times B^{m - q}\).

Now it is clear that the original embedding of \(\partial B^q = \partial B^q \times O\) can be replaced by any embedding \(\partial B^q \times a\), \(a \in \text{Int} B^{m - q}\). Let \(X = M(k)\) and \(m - q > k\). Then, by 3.1, \(M \setminus X\) is a countable union of simplexes of \(\dim \leq k\) PL-embedded in \(M\) and hence \(a \in B^{m - q}\) can be chosen to be arbitrarily close to \(O\) and such that \(B^q \times a \subset X\). Thus without loss of generality we can replace the original embedding \(f_{\partial\Delta}\) by an arbitrarily close embedding whose image is contained in \(X\).

### 3.6 A remark on absorbing simplexes

Let \(M_i, i = 1, 2\) be an \(l_i\)-co-connected \(m_i\)-dimensional manifold such that \(m_1 - l_1 = m_2 - l_2 \geq 0\) and \(m_i \geq 2(m_i - l_i + 1) + 1\). Let \(F_i\) be a PL-subcomplex of \(M_i\) with \(\dim F_i \leq l_i - 1\) and let \(\mathcal{P}_i\) be a decomposition of \(M_i\) such that \(\mathcal{P}_i\) forms an \(l_i\)-co-connected
partition on $M_1 \setminus F_1$ and there is a one-to-one correspondence $\mu : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ which induces a matching of partitions when $\mathcal{P}_1$ and $\mathcal{P}_2$ are restricted to $M_1 \setminus F_1$ and $M_2 \setminus F_2$ respectively.

We want to reduce the dimension of $F_2$ to $l_2 - 2$. Following the construction 2.9 (absorbing simplexes) of [4] fix a sufficiently fine triangulation of $M_2$ underlying $\mathcal{P}_2$ and $F_2$. Assume such that $\mathcal{A}$ is a cover $M_1$ such that the elements of $\mathcal{A}$ are unions of elements of $\mathcal{P}_1$ and for every simplex $\Delta \subset F_2$ of dim $= l_2 - 1$ there is an element $\Delta \in \mathcal{A}$ such that $\mu^{-1}(\text{st}(\Delta, \mathcal{P}_2)) \subset \Delta$ and the inclusion $\mu^{-1}(\text{st}(\Delta, \mathcal{P}_2)) \setminus F_1 \subset A \setminus F_1$ induces the zero-homomorphism of the homotopy groups in dimensions $\leq m_1 - l_1$ ($= $co-dimensions of $l_1$).

Then the inclusion $\text{st}(\Delta, \mathcal{P}_2) \setminus F_2 \subset \mu(A) \setminus F_2$ induces the zero-homomorphism of the homotopy groups in dim $\leq m_1 - l_2$, see 2.3 of [4]. Now it is easy to derive from 2.9 of [4] that the construction of absorbing the $(l_2 - 1)$-dimensional simplexes of $F_2$ can be carried out such that the modification $\mathcal{P}_2'$ of $\mathcal{P}_2$ will refine $\mu(\text{st}^2 \mathcal{A})$ and for every $P \in \mathcal{P}_2$, the modification of $P$ will be contained in $\text{st}(P, \mu(\text{st}^2 \mathcal{A}))$.

It is also easy to derive from 2.9 of [4] that if $K$ is a PL-subcomplex of $M_2$ such that dim $K \leq l_2 - 2$ and $K \cap F_2 = \emptyset$ then the modification $\mathcal{P}_2'$ of $\mathcal{P}_2$ can be constructed so that $K$ will be contained in the modification $\mathcal{P}_2'$ of $M_2$ and $\mathcal{P}_2$ will be left unmodified on a small neighborhood of $K$.

### 3.7 A remark on digging holes for improving connectivity

In the construction 4.2 (digging holes for improving connectivity) of [4] we extend a PL-embedding of an $(q - 1)$-dimensional sphere $S^{q-1}$ into a manifold $M$ to a PL-embedding of a ball $B^m = B^q \times B^{m-q}$ such that the ball $B^q$ bounds $S^{q-1}$ and $B^m$ has special properties with respect to a decomposition $\mathcal{P}$ of $M$. It is noted in the end of 4.2 of [4] that if $S^{q-1}$ can be contracted to a point in an open subset $W$ of $M$ then $B^m$ can be chosen to be in $W$. We will need a slightly improved control on $B^m$. Namely, if $B^q_\#$ is any $q$-dimensional ball with $\partial B^q_\# = S^{q-1}$ and $\phi_\# : B^q_\# \rightarrow M$ is a map which is the identity map on $S^{q-1}$ then the ball $B^q_\#$ can be chosen to be arbitrarily close to $\phi_\#(B^q_\#)$ in the sense that $B^q_\#$ can be chosen to be $B^q = \phi(B^q_\#)$ for a sufficiently close approximation of $\phi_\#$ by a PL-embedding $\phi : B^q \rightarrow M$ such that $\phi$ is the identity map on $S^{q-1}$. This can be done as follows.

The case $m = 3$ and $q = 1$ can be visualized directly. Assume that $m \geq 4$ (actually we can always assume that $m \geq 4$ restricting ourselves to Nöbeling spaces modeled on manifolds of dim $\geq 4$). The space $R^m$ in the beginning of 4.2 of [4] can be chosen so that $\phi_\#(B^q_\#) \subset R^m$. Then after replacing the original embedding of $R^m$ by $e : R^m \rightarrow e(R^m) = R^m \subset M$ we also have $\phi_\#(B^q_\#) \subset R^m$.

Approximate $\phi_\#$ by a PL-embedding $\phi'_\# : B^q_\# \rightarrow R^m$ such that $\phi'_\#$ coincides with $\phi_\#$ on $\partial B^q_\#$ and $\phi'_\#$ sends a small neighborhood of $\partial B^q_\#$ into $B^q_\#$. Then the block bundle $\eta$ can be chosen to be so close to $S^{q-1}$ that $E(\eta) \cap \phi'_\#(B^q_\#) = E(\eta) \cap B^q_\#$. Since $m \geq 4$, it follows from Unknotting Theorem (see Theorem 10.1 of [2]) that there is a PL-homeomorphism $e' : R^m \rightarrow R^m$ such that $e'$ does not move the points of $E(\eta)$ and $e'$ sends $\phi'_\#(B^q_\#)$ onto $B^q_\#$.

Now replace the embedding $e$ by $e' \circ e : R^m \rightarrow R^m \subset M$. Following the construction 4.2 of [4] we can choose the point $a \in B^m_1 - q$ to be arbitrarily close to the center $O$ of
Then the homeomorphism \( \psi : M \to M \) can be chosen to be arbitrarily close to the identity map. Thus the final embedding of \( B^q \) can be constructed to be arbitrarily close to \( \phi_\#(B^q_\#) \) and hence it can be made close to \( \phi_\#(B^q_\#) \).

### 3.8 A remark on moving to a rational position

The reasoning of construction 2.11 (moving to a rational position) of [4] applies to show the following property. Assume \( P \) is a decomposition of a triangulated manifold \( M \) which is represented as the union \( M = U \cup V \) of two open subsets \( U \) and \( V \) such \( P \) restricted to \( V \) is a rational decomposition of \( V \). Then there is a PL-homeomorphism \( h : M \to M \) such that \( h(P) \) is a rational decomposition, \( h(x) = x \) for every \( x \in X \setminus U \) and \( h \) can be chosen to be arbitrarily close to the identity map.

Indeed, let a rational triangulation \( T \) of \( M \) be such that st(\( U \setminus V, T \)) \( \cap \) st(\( V \setminus U, T \)) = \( \emptyset \). Embed \( M \) into a Hilbert space \( H \) by a map which is linear on the simplexes of \( T \). Take a subdivision \( T' \) of \( T \) such that the simplexes of \( T' \) are linear in \( H \), \( T' \) underlies \( P \) and the simplexes of \( T' \) contained in st(\( V \setminus U, T \)) are rational. Similarly to 2.11 of [4], we can define a map \( h' \) from the vertices of \( T' \) to rational points in \( M \) such that \( h'(v) \) and \( v \) are sufficiently close for every vertex \( v \) of \( T' \).

### 3.9 A remark on the discretization of maps’ images

Assume that in the construction 2.6 of [4] the finite intersections of the decomposition \( P \) have dense subsets on \( X = M(k) \) where \( k = \dim M - r - 1 \). Then the PL-subcomplex \( R \) of \( M \) can be chosen to be rational. Indeed, using 3.8 the set \( R \) can be moved to a rational subcomplex with the needed properties. Note that, since the finite intersections of \( P \) are dense on \( X \), every rationally presented subset of \( M \) of dim \( \leq k \) is nowhere dense on the finite intersections of \( P \).

This implies that if in the constructions 2.7 and 2.9 of [4] the decomposition \( P \) is rational such that the finite intersections of \( P \) have dense subsets on \( X = M(n) \) with \( \dim M = m \geq 2n + 1 \) and \( n \geq q \) then in 2.9 of [4] the sets \( R^{0\Delta} \) and \( R \) can be chosen to be rationally presented and in 2.9 of [4] the set \( R \) can be chosen to be a rational subcomplex.

### 3.10 A subdivision of a partition

Let \( P \) be a partition of a triangulated manifold \( M \). A partition \( P_\# \) of \( M \) is said to be a subdivision of \( P \) if \( P_\# \) refines \( P \).

For every open cover \( \mathcal{V} \) of \( M \) there is a subdivision \( P_\# \) of \( P \) such that \( P_\# \) refines \( \mathcal{V} \) and every finite intersections of \( P_\# \) is contractible. Indeed, arrange the elements of \( P \) into a sequence \( P^1, P^2, \ldots \). Take a triangulation \( T \) of \( M \) such that \( T \) underlies \( P \) and st\( T \) refines \( \mathcal{V} \). Denote by \( T_i \) the \( i \)-th barycentric subdivision of \( T \). Define \( P_i \) as the collection
\{\text{st}(v, \mathcal{T}_{i+1}) \cap P^i : v \text{ is a vertex of } \mathcal{T}_i \text{ and } v \in P^i\}. \text{ Then } \mathcal{P}_\# = \bigcup_i \mathcal{P}_\#^i \text{ is a partition of } M \text{ having the required properties. Note that } \mathcal{P}_\# \text{ will be a rational partition if } \mathcal{P} \text{ is a rational partition and we choose } \mathcal{T} \text{ to be a rational triangulation.}

Assume that \( \mathcal{P} \) and \( \mathcal{P}_\# \) are \( l \)-co-connected partitions of \( M \) such that \( \mathcal{P}_\# \) is a subdivision of \( \mathcal{P} \). Assume that each \( P_\# \in \mathcal{P}_\# \) is modified to \( P'_\# \) such that \( \mathcal{P}_\#' = \{P'_\# : P_\# \in \mathcal{P}_\#\} \) is an \( l \)-co-connected partition of \( M \) and the correspondence \( \mu_\# : \mathcal{P}_\# \to \mathcal{P}_\#' \) defined by \( \mu_\#(P_\#) = P'_\# \) is a matching. For every \( P \in \mathcal{P} \) define \( P' = \mu(P) = \bigcup\{P'_\# : P_\# \subset P\} \) and \( \mathcal{P}' = \{P' : P \in \mathcal{P}\} \). Then by 2.2 and 2.3 of [4] imply that \( \mathcal{P}' \) is an \( l \)-co-connected partition of \( M \) and \( \mu \) is a matching between \( \mathcal{P} \) and \( \mathcal{P}' \). We will call \( \mathcal{P}' \) the modification of \( \mathcal{P} \) induced by the modification \( \mathcal{P}_\#' \).

3.11 A radial modification

Let a cube \( B^m = B^q \times B^{m-q} \) be PL-embedded in an \( m \)-dimensional triangulated manifold \( M \). We consider \( B^m \) with the linear and coordinate structure induced by a PL-embedding of \( B^m \) in the Euclidean space \( R^m = R^q \times R^{m-q} \) as the unit cube \( B^m = [-1, 1]^m \) with \( B^q = B^m \cap R^q \) and \( B^{m-q} = B^m \cap R^{m-q} \). For \( b \in B^m \), \(|b| \) stands for the maximal absolute value of the coordinates of \( b \).

Let \( r \) be the radial projection \( r : B^m \setminus O \to \partial B^m \) and \( B^m_r = \frac{1}{2} B^m = \{|b| \leq 1/2 : b \in B^m\} \). The radial modification \( P^* \) of a PL-subcomplex \( P \) of \( M \) is the set
\[ P^* = P \text{ if } P \cap B^m = \emptyset \text{ and } P^* = (P \setminus \text{Int}B^m) \cup (r^{-1}(P \cap \partial B^m)) \text{ if } P \cap B^m \neq \emptyset \]
and the radial modification of a decomposition \( \mathcal{P} \) of \( M \) is the decomposition \( \mathcal{P}^* = \{P^* : P \in \mathcal{P}\} \) of \( M \setminus O \).

3.12 A property of the radial modification

Adopt the notation and the assumptions of 3.11. Let \( p : B^m \to B^q \) be the projection to \( B^q \). Assume that the decomposition \( \mathcal{P} \) of \( M \) is such that for every \( P \in \mathcal{P} \), \( P \cap B^m = (B^q \cap P) \times B^{m-q} \).

Suppose that \( f : X \to Y \) is a map from a dense subspace \( X \) of \( M \) such that \( B^q \subset X \) and every \( x \in B^q \) is a regular point of \( f \) (see Section 2) and suppose that \( \mathcal{E} \) is an open cover of \( M \).

Fix a triangulation \( \mathcal{T} \) of \( B^q \) such that \( \text{st}^2 \mathcal{T} \) refines \( \mathcal{E} \). Denote \( \mathcal{B} = \{\Delta \times B^{m-q} : \Delta \in \mathcal{T}\} \). Replacing (if needed) \( B^{m-q} \) by a smaller cube assume that
\[(*) \text{ for every two disjoint simplexes } \Delta_1 \text{ and } \Delta_2 \text{ of } \mathcal{T} \text{ there are disjoint neighborhoods of } \Delta_1 \times B^{m-q} \subset V_i \text{ in } M \text{ such that the closures of } f(V_1 \cap X) \text{ and } f(V_2 \cap X) \text{ do not intersect in } Y.\]

Let us show that there is a PL-homeomorphism \( h : M \to M \) such that
\[(**) h \text{ does not move the points of } (M \setminus \text{Int}B^m) \cup B^q, \text{ } p(h(c)) = p(c) \text{ for every } c \in B^m \text{ and } h \text{ has the property: for every } y \in Y \text{ such that } y \text{ belongs to the closure.}\]
of $f(h(B^m \setminus \text{Int}B^*_m) \cap X)$ there are a neighborhood $G$ of $y$ and $E \in \mathcal{E}$ such that for every $x$ in $h^{-1}((B^m \setminus \text{Int}B^*_m) \cap f^{-1}(G))$ we have that $p(r(x)) \in E$ and for every $P \in \mathcal{P}$, $P$ intersects $E$ if $P^*$ intersects $h^{-1}((B^m \setminus \text{Int}B^*_m) \cap f^{-1}(G))$.

Denote $\mathcal{B}^* = \{B^* : B \in \mathcal{B}\}$. For $c = (a, b) \in B^m$ denote by $a$ and $b$ the coordinates of $c$ in $B^a$ and $B^{m-a}$ respectively. Note that if $c = (a, b), c' = (a', b') \in B^m \setminus O$ are such that $a' = a$ and $|b'| = |b|$ then $p(r(c)) = p(r(c'))$. Hence if $c = (a, b), c' = (a', b') \in B^m \setminus O$ are such that $a' = a$ and $|b'| = |b|$ then for every $B^* \in \mathcal{B}^*$ we have that $c \in B^*$ if and only if $c' \in B^*$. This means that for every $a \in B^a$ and $B^* \in \mathcal{B}^*$ the intersection $B^* \cap (a \times B^{m-a})$ is concentric about $a$ (with respect to the norm $|.|$).

For disjoint $B^*_1, B^*_2 \in \mathcal{B}^*$ denote $\epsilon(B^*_1, B^*_2) = \inf\{|c_1 - c_2| : c_i \in B^*_i \setminus \text{Int}B^*_m\}$ and let $\epsilon = \min\{\epsilon(B^*_1, B^*_2) : B^*_1 \in \mathcal{B}^*, B^*_1 \cap B^*_2 = \emptyset\}$. Clearly $\epsilon > 0$. Let $n$ be a natural number such that $n > 2/\epsilon$. Since $B^a$ consists of regular points of $f$ we can find $0 < \delta_1 < \delta_2 < \cdots < \delta_n < 1$ such that for $C^+_i = \{c = (a, b) \in B^m : |b| \geq \delta_i\}$ and $C^-_i = \{c = (a, b) \in B^m : |b| \leq \delta_i\}$ the closure of $f(C^+_i \cap X)$ does not intersect the closure of $f(C^-_{i-1} \cap X)$ for every $1 < i \leq n$, and $\delta_n$ is so close to 0 that for every $y$ in the closure of $f(C^-_n \cap X)$ there is a neighborhood $G$ of $y$ such that $\text{diam}(f^{-1}(G)) \leq \epsilon/2$. Define the piece-wise linear function $g : [0, 1] \longrightarrow [0, 1]$ such that $g(0) = 0, g(1) = 1$ and $g(\delta_i) = i/(n+1), 1 \leq i \leq n$.

Define a PL-homeomorphism $h : M \longrightarrow M$ such that $h$ does not move the points of $(M \setminus \text{Int}B^m) \cup B^a$, for every $c \in B^m$, $p(h(c)) = p(c)$ and for every $c = (a, b) \in B^m$ such that $|a| \leq n/(n+1)$, $h(c) = (a, g(|b|)b)$. Denote $C^- = \{c \in B^m : |c| \leq n/(n+1)\}$ and $C^+ = \{c \in B^m : |c| \geq n/(n+1)\}$.

It is easy to see that for every disjoint $B^*_1$ and $B^*_2$ in $\mathcal{B}$, the closures $f((h((B^*_1 \cap C^-) \setminus \text{Int}B^*_m) \cap X)$ and $f((h((B^*_2 \cap C^-) \setminus \text{Int}B^*_m) \cap X)$ do not intersect in $Y$. It is also easy to see that since $h$ leaves the $B^a$-coordinate of the points in $B^m$ unchanged we can choose $n$ to be so large and hence $C^+$ to be so close to $\partial B^m$ that (*) will imply that for every disjoint $B_1$ and $B_2$ in $\mathcal{B}$, the closures $f(h(B^*_1 \cap C^+) \cap X)$ and $f(h(B^*_2 \cap C^+) \cap X)$ do not intersect in $Y$.

Thus for every disjoint $B^*_1$ and $B^*_2$ in $\mathcal{B}^*$ we have that the closures $f(h((B^*_1 \setminus \text{Int}B^*_m) \cap X)$ and $f(h(B^*_2 \setminus \text{Int}B^*_m) \cap X)$ do not intersect in $Y$.

Take $y$ in the closure of $f(h(B^m \setminus \text{Int}B^*_m) \cap X)$. The last property implies that there is a neighborhood $G$ of $y$ such that $h^{-1}(f^{-1}(G) \cap (B^m \setminus B^*_m))$ is contained in an element $C$ of $\text{st}^2\mathcal{B}^*$. Since $r(C)$ is contained in an element of $\text{st}^2\mathcal{B}$ and $\text{st}^2\mathcal{T}$ refines $\mathcal{E}$ there is $E \in \mathcal{E}$ such that $p(r(C))) \subset E$. Thus we get that if for $P \in \mathcal{P}$ the radial modification $P^*$ of $P$ in $\mathcal{P}^*$ intersects $h^{-1}(f^{-1}(G) \cap (B^m \setminus \text{Int}B^*_m))$ then $P$ intersects $E$. Clearly for every $x \in h^{-1}(f^{-1}(G) \cap (B^m \setminus \text{Int}B^*_m))$, $p(r(x)) \in E$ and (***) follows.

### 3.13 Combining radial and black hole modifications

Let $M$ be an $m$-dimensional triangulated manifold, $F$ a PL-subcomplex of $M$ and $\mathcal{P}$ a decomposition of $M$ which forms a partition on $M \setminus F$. Assume that $B^m = B^a \times B^{m-a} \subset M$ is a ball constructed for creating or improving connectivity of a finite intersection of $\mathcal{P}$, see 2.5 and 4.2 of [4] and 3.2 of this section.
Assume that $B^m$ is represented as in 3.11. For a PL-subcomplex $A$ of $M$ define $A' = (A^* \setminus \text{Int}B^m) \cup A$, where $A^*$ is the radial modification of $A$ and $A_* = \frac{1}{2}(A \cap B^m) = \{\frac{1}{2}a : a \in A \cap B^m\}$. It is easy to see that $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a decomposition of $M$ which forms a partition on $M \setminus \text{Int}^m$ and we preserve in $\mathcal{P}'$ restricted to $M \setminus \text{Int}B^m$ the connectivity properties that we have in $\mathcal{P}$ restricted to $M \setminus \text{Int}B^m$. Notice that $\mathcal{P}^*$ and $\mathcal{P}'$ both restricted to $M \setminus \text{Int}B^m$ coincide and $\mathcal{P}'$ restricted to $B^m$ is an exact copy of $\mathcal{P}$ restricted to $B^m$. Then the block hole modification used in see 2.5 of [4] for improving connectivity can be carried out in $B^m$ instead of $B^m$ leaving $\mathcal{P}'$ on $M \setminus \text{Int}B^m$ as the radial modification. Thus we can shift the block hole modification to $B^m$ combining it with the radial modification.

4 Proof of Theorem 1.5

Assume that $X$ is homeomorphic to $M(k)$ of an $m$-dimensional $(n - 1)$-connected triangulated manifold $M$ with $m \geq 2n + 1$ and $k = m - n - 1$. Let $\mathcal{P}$ be a decomposition of an open subset $M_\#$ of $M$ containing $X$. We say that $y \in Y$ is a $\mathcal{P}$-regular point if there is neighborhood $G$ of $y$ in $Y$ such that $G \cap X$ is contained in an element of $\text{st}\mathcal{P}$, and we say that $y \in Y$ is $\mathcal{P}$-singular otherwise. Note that the set of $\mathcal{P}$-singular points is closed in $Y$ and does not meet $X$ and therefore it is a $Z$-set in $Y$.

Assume that $Z' \subset Z \subset Y$, $Z'$ and $Z$ are homeomorphic to $n$-dimensional Nöbeling spaces, $Z$ is a $Z$-set in $Y$, $Z'$ is closed in $Z$, a map $g_Z : Z \to Z'$ is a $UV^{n-1}$-retraction and a map $g_Y : Y \to Z$ is a retraction. Consider an open cover $\mathcal{W}_Z'$ of $Z'$ and denote by $\mathcal{W}_M$ the extension of $g_Y^{-1}(g_Z^{-1}(\mathcal{W}_{Z'}))|X$ to $M$. Replacing $M$ by an open subset of $M$ containing $X$ we assume that $\mathcal{W}_M$ covers $M$.

Note that each time when we replace $M$ by a smaller open set containing $X$ we automatically replace everything that was already defined on $M$ (sets, covers, decompositions etc.) by its restriction to that smaller open set.

By a nice partition we mean a rational $(m - n + 1)$-co-connected partition with no non-empty intersections of dim $\leq m - n - 1$. Note that every non-empty finite intersection of a nice partition has a dense subset lying in $X$.

By a neighborhood we always mean an open neighborhood. If $Q$ is an open subset in one of the spaces $X, Y$ or $M$ then the closure $\text{cl}Q$ of $Q$ is considered with respect to that space where $Q$ is open.

**Proposition 4.1 (Shifting Singularities)** Assume that $X$ and $Y$ as in Theorem 1.5 and $M$, $Z'$, $Z$, $g_Z$, $g_Y$, $\mathcal{W}_{Z'}$, and $\mathcal{W}_M$ as above.

Then for every open neighborhood $H$ of $Z$ in $Y$ there is a smaller neighborhood $Z \subset Q_Z \subset H$ of $Z$ in $Y$ such that for every nice partition $\mathcal{P}$ of an open subset $M_\#$ of $M$ such that $X \subset M_\#$ and the set of $\mathcal{P}$-singular points of $Y$ is contained in $Q_Z$ we can do the following:

for every neighborhood $Q_{Z'}$ of $Z'$ in $Y$ and a ball $K$ of dim $\leq n$ PL-embedded in $M$ such that $K \subset X$, the manifold $M_\#$ can be modified to an open subset $M'_\#$ of $M_\#$ and each $P \in \mathcal{P}$ to a PL-subcomplex $P'$ of $M'_\#$ such that
(1) $X \subset M'_\#, \mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a nice partition of $M'_\#$; the correspondence $P \mapsto P'$ is a matching between $\mathcal{P}$ and $\mathcal{P}'$;

(2) $P \cap ((X \setminus H) \cup K) = P' \cap ((X \setminus H) \cup K)$ for every $P \in \mathcal{P}$ (that is $\mathcal{P}$ and $\mathcal{P}'$ coincide on $(X \setminus H) \cup K$);

(3) the set of $\mathcal{P}'$-singular points of $Y$ is contained in $Q_{Z'}$; and

(4) for every $P \in \mathcal{P}$, $P'$ is contained in $\text{st}(P, W_M)$.

Let us show how Theorem 1.5 can be derived from Proposition 4.1.

**Proof of Theorem 1.5.** Consider the space constructed in Proposition 2.6 and denote $Z_i = \cup_{j \geq i} X_j$. Let $h_i : A_i \rightarrow A_0$ be a homeomorphism such that $h_0$ is the identity map. By Proposition 2.4, extend $h_i$ and $h_{i+1}$ to a $UV^n-1$-map $\psi_i : X_i \rightarrow A_0$ and define the $UV^n-1$-retraction $g_i^Z : Z_i \rightarrow Z_{i+1}$ by $g_i^Z(x) = h_{i+1}^{-1}(\psi_i(x))$ for $x \in X_i$ and the map $g_i^A : Z_i \rightarrow A_0$ by $g_i^A(x) = \psi_j(x)$ for $x \in X_j$, $j \geq i$. Note that $g_i^Z = g_{i+1}^- A \circ g_i^Z$.

Let $\mathcal{V}'_Y$ and $\mathcal{V}'_X$ be open covers of $Y$ and $X$ respectively such that $\text{st}^2 \mathcal{V}'_Y$ refines $\mathcal{V}_Y$ and $\text{st}^2 \mathcal{V}'_X$ refines $\mathcal{V}_X$. Replacing $M$ by an open subset of $M$ containing $X$ we assume that the extension $\mathcal{V}'_Y$ of $\mathcal{V}'_Y|X$ to $M$ and the extension $\mathcal{V}'_X$ of $\mathcal{V}'_X$ to $M$ both cover $M$.

Take a nice partition $\mathcal{P}$ of $M$ such that $\text{st}^2 \mathcal{P}$ refines both $\mathcal{V}'_Y$ and $\mathcal{V}'_X$. Denote by $T(\mathcal{P})$ the set of $\mathcal{P}$-singular points of $Y$ and recall that $T(\mathcal{P})$ is a $Z$-set in $Y$. Embed $A_0$ in $Y$ as a $Z$-set in $Y$ such that $T(\mathcal{P}) \subset A_0$. Let $\mathcal{V}'_Y$ be an open cover of $Y$ such that $\text{st}^2 \mathcal{V}'_Y$ refines $\mathcal{V}'_Y$. Consider a sequence $W_i^A$ of open covers of $A_0$ such that $\text{st}^2 W_i^A$ refines $W_i^A$ and $W_0^A$ refines $\mathcal{V}'_Y$. Via the map $g_0^A : Z_0 \rightarrow A_0$ extend the embedding of $A_0$ to a $Z$-embedding $Z_0 \subset Y$ such that $(g_0^A)^{-1}(W_0^A)$ refines $\mathcal{V}'_Y$. Let $g_Y : Y \rightarrow Z_0$ be any retraction and let a neighborhood $H$ of $Z_0$ be such that $W_i^Y = g_Y^{-1}(g_0^A)^{-1}(W_0^A))$ restricted to $H$ refines $\text{st}^2 \mathcal{V}'_Y$. Then $\text{st}^2 \mathcal{V}'_Y$ restricted to $H$ refines $\mathcal{V}'_Y$.

Choose a neighborhood $H_i$ of $Z_i$ in $Y$ such that $H_0 = H$, the closure of $H_{i+1}$ is contained in $H_i$ and $\cap_{i=0}^{\infty} H_i = \emptyset$. Denote $H^M_i = \text{the extension of } H_i$ to $M$. Replacing $M$ by an open subset of $M$ containing $X$ we assume that the closure $\text{cl} H^M_i$ of $H^M_i$ is contained in $H^M_i$ and $\cap_{i=0}^{\infty} H^M_i = \emptyset$. Denote $\mathcal{P}_0 = \mathcal{P}$, $M_0 = M$, $W_0^M = \text{the extension of } W_0^A$ to $M$ and replacing $M$ by an open subset of $M$ containing $X$ assume that $W_0^M$ covers $M_0$. For every $i > 0$ fix an open subset $M_i$ of $M$ such that $X \subset M_i$, $M_{i+1} \subset M_i$ and the extension $W_i^M_i$ of $W_i^M = g_Y^{-1}(g_0^A)^{-1}(W_0^A))$ to $M_i$ covers $M_i$. Denote $g_0^Y = g_Y$, $g_i^Y = g_{i-1}^Y \circ \cdots \circ g_0^Y \circ g_Y : Y \rightarrow Z_i$ for $i > 0$ and $W_i^Z = (g_i^A)^{-1}(W_i^A)$. Note that $W_i^Z$ is an open cover of $Z_{i+1}$ and $W_i^Y = (g_i^Y)^{-1}(g_i^Z)^{-1}(W_i^Z))$.

By Proposition 4.1 there is a neighborhood $Q_i$ of $Z_i$ such that the conclusions of Proposition 4.1 are satisfied for $M,H, Z, Z_i, g_Y, g_Z, W_{Z'}$ and $W_M$ replaced by $M_i, H_i, Z_i, Z_{i+1}, g_i^Y, g_i^Z, W_i^Z$ and $W_i^M$ respectively. Note that $Q_i \subset H_i$.

Let $K_i \subset X$ be a sequence of balls PL-embedded in $M$ which will be chosen later. Recall that $T(\mathcal{P}_i) \subset Z_0$ and therefore $T(\mathcal{P}_0) \subset Q_0$. Set $M_0 = M_0$ and apply Proposition 4.1 to construct for every $i$ a nice partition $\mathcal{P}_i$ of an open subset $M'_i$ of $M$ such that $X \subset M'_i \subset M_i$, the set $T(\mathcal{P}_i)$ of $\mathcal{P}_i$-singular points of $Y$ is contained in $Q_i$, and $\mathcal{P}_{i+1}$ and $M'_{i+1}$ are produced by Proposition 4.1 as the output $\mathcal{P}'$ and $M'_\#$ when the input $\mathcal{P}_i$, $M'_i$, $Q_{Z'}$ and $K$ corresponds to $\mathcal{P}_i$, $M'_i$, $Q_{i+1}$ and $K_i$ respectively with the other parameters as...
above. Note that replacing $M'_{i}$ by $M'_{i} \cap M_{i+1}$ we indeed can assume that $M'_{i+1} \subset M_{i+1}$. Let $\mu^{i}: P_{i} \rightarrow P_{i+1}$ be the matching sending each element to its modification and let $\mu_{i}: \mathcal{P} \rightarrow \mathcal{P}_{i}$ be the composition of the corresponding chain of matchings. Denote $P_{i} = \mu_{i}(P)$ for $P \in \mathcal{P}$. Since $\mu_{i}$ is a matching, $\mu_{i}(P)$ is well defined for every finite intersection $P$ of $\mathcal{P}$.

Define $M' = \cup_{i=0}^{\infty}(M'_{i} \setminus \text{cl}H_{i}^{M})$, $P' = \cup_{i=0}^{\infty}(P_{i} \setminus \text{cl}H_{i}^{M})$ for $P \in \mathcal{P}$. It is clear that $M'$ is an open subset of $M$ containing $X$, $Y$ has no $\mathcal{P}'$-singular point and $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$ is a rational partition of $M'$. However we can say nothing about the connectivity of $\mathcal{P}'$ and it may well happen that $P' = \emptyset$ for $P \in \mathcal{P}$. In order to cope with this problem we need to choose the balls $K_{i}$ in the following manner. Take a bijection $\alpha: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ such that for $\alpha(s) = (j, t, l)$ we have $s \geq \max\{j, t, l\}$. Arrange the non-empty finite intersections of $\mathcal{P}$ into a sequence $P^{0}, P^{1}, P^{2}, \ldots$. For every $j$ and $t$ such that $\mu_{j}(P_{t}) \setminus \text{cl}H_{j}^{M} \neq \emptyset$ choose a countable collection $\mathcal{F}(j, t) = \{f(j, t, l) : l = 0, 1, 2, \ldots\}$ of maps from spheres of dim $\leq \text{dim}P_{t} - m + n - 1$ into $(\mu_{j}(P_{t}) \setminus \text{cl}H_{j}^{M}) \cap X$ such that $\mathcal{F}(j, t)$ generates the homotopy groups of $\mu_{j}(P_{t}) \setminus \text{cl}H_{j}^{M}$ in dim $\leq \text{dim}P_{t} - m + n - 1$. For $j$ and $t$ such that $\mu_{j}(P_{t}) \setminus \text{cl}H_{j}^{M} = \emptyset$ define $\mathcal{F}(j, t) = \emptyset$. Set $i(0) = 0$ and for each $s$ define by induction a set $K_{s}' \subset X$ and $i(s) \geq s$ as follows. Let $\alpha(s) = (j, t, l)$. If $\mathcal{F}(j, t) = \emptyset$ define $K_{s}'$ to be any singleton in $\mu_{i(s)}(P_{t}) \cap X$. If $\mathcal{F}(j, t) \neq \emptyset$ then by 3.1 let $K_{s}'$ be a ball of dim $\leq n$ PL-embedded in $\mu_{i(s)}(P_{t})$ such that $K_{s}' \subset X$ and the map $f(j, t, l)$ is null-homotopic in $(\mu_{i(s)}(P_{t}) \setminus \text{cl}H_{i(s)}^{M}) \cup K_{s}'$. Recall that $i(s) \geq s \geq j$ and therefore the image of $f(j, t, l)$ is contained in $\mu_{i(s)}(P_{t}) \setminus \text{cl}H_{i(s)}^{M}$. Since $\cap_{i=0}^{\infty}\text{cl}H_{i}^{M} = \emptyset$ there is $i(s + 1) > i(s)$ such that $K_{s}' \cap \text{cl}H_{i(s+1)}^{M} = \emptyset$. Now define $K_{i} = K_{i}'$ for $i(s) \leq i < i(s + 1)$. Note that $K_{i}$ is already defined after $\mathcal{P}_{i}$ is constructed and therefore the construction of $\mathcal{P}_{i+1}$ is well-defined. Then, by (2) of Proposition 4.1, we get that $P'$ will be non-empty for every non-empty $P \in \mathcal{P}$, $\mathcal{P}'$ is a nice partition of $M'$ and the correspondence $\mu: \mathcal{P} \rightarrow \mathcal{P}'$ defined by $\mu(P) = P'$ is a matching of partitions.

Let $P \in \mathcal{P}$. Recall that $P = P_{0}$. By (4) of Proposition 4.1 we have that $P_{i+1} \subset \text{st}(P_{i}, W_{i}^{M})$ and since $\text{st}^{2}W_{i+1}$ refines $W_{i}^{M}$ we have that $P' \subset \text{st}(P_{0}, \text{st}W_{0}^{M}) = \text{st}(P, \text{st}W_{0}^{M})$. Since $P \setminus H_{0}^{M} = P_{0} \setminus H_{0}^{M} = P' \setminus H_{0}^{M}$ we get $P' \subset (P \setminus H_{0}^{M}) \cup (\text{st}(P_{0}, \text{st}W_{0}^{M}) \cap H_{0}^{M})$ and since $\text{st}W_{0}^{Y}$ restricted to $H_{0}^{M}$ refines $H_{0}^{M}$ we have that $P' \subset \text{st}(P, \text{st}W_{0}^{Y})$. Recall that $\mathcal{P}$ refines $\mathcal{V}_{Y}^{M}$ and $\text{st}\mathcal{V}_{Y}'$ refines $\mathcal{V}_{Y}$. Hence for every $P \in \mathcal{P}$ there is an element of $\mathcal{V}_{Y}$ which contains both $P \cap X$ and $P' \cap X$.

Using Theorem 1.3 construct for every finite intersection $P$ of $\mathcal{P}$ (by induction on dim $P$) a homeomorphism $h_{P}: P \cap X \rightarrow P' \cap X$ such that $h_{P}$ agree on common intersections. Then the homeomorphisms $h_{P}$ define the corresponding homeomorphism $h: X \rightarrow X$ such that for every $P \in \mathcal{P}$, $h(P \cap X) = P' \cap X$ and hence $x$ and $h(x)$ are $\mathcal{V}_{Y}$-close for every $x \in X$.

Recall that st$\mathcal{P}$ restricted to $X$ refines $\mathcal{V}_{X}$ and $Y$ has no $\mathcal{P}'$-singular point. Then for every $y$ in $Y$ there is a neighborhood $G$ such that $G \cap X$ is contained in an element of st$\mathcal{P}'$ and hence $h^{-1}(G \cap X)$ is contained in an element of st$\mathcal{P}$. Therefore $h^{-1}(G \cap X)$ is contained in an element of $\mathcal{V}_{X}$ and the theorem follows.

For proving Proposition 4.1 we need the following
Lemma 4.2 Let $X$ and $Y$ be as in Theorem 1.5. Then there are a $Z$-embedding $g : X \rightarrow Y$ of $X$ into $Y$ and a continuous retraction $f : Y \rightarrow g(X)$ such that both $g$ and $f$ can be chosen to be arbitrarily close to the inclusion of $X$ into $Y$ and the identity map of $Y$ respectively.

Proof. Let us first show that there are a $Z$-embedding $\phi : X \rightarrow X$ and a continuous retraction $r : X \rightarrow \phi(X)$ such that both $\phi$ and $r$ can be chosen to be arbitrarily close to the identity map of $X$. Indeed, take any topological copy $X'$ of $X$ such that $X'$ is $Z$-embedded in $X$ and let $\psi_1 : X_1 = X \rightarrow X'$ be any homeomorphism. By Proposition 2.4 there is a $U^nV^{n-1}$ retraction $\psi_2 : X_2 = X \rightarrow X'$. Take any $Z$-embedding $\phi : X_1 \rightarrow X_1$ which is sufficiently close to the identity map. Then, by Theorem 2.2, the homeomorphism $\psi_1 \circ \phi^{-1} : X_1 = \phi(X_1) \rightarrow A_2 = X' \subset X_2$ extends to a homeomorphism $h : X_1 \rightarrow X_2$ such that the retraction $r = h^{-1} \circ \psi_2 \circ h : X_1 \rightarrow \phi(X_1)$ is close to the identity map of $X_1$.

Note that the inclusion of $X$ into $Y$ is a $U^nV^{n-1}$-map. Choose a $Z$-embedding $\xi : \phi(X) \rightarrow Y$ to be so close to the inclusion of $\phi(X)$ into $Y$ that, by Proposition 2.1, the map $\xi^{-1} : (\xi(\phi(X))) \rightarrow \phi(X)$ extends to a map $\eta : Y \rightarrow X$ which is close to the identity map of $Y$. Then $f : Y \rightarrow \xi(\phi(X))$ defined by $f(y) = \xi(\eta(y)))$ is close to the identity map of $Y$ provided $r$ is close to the identity map of $X$. Denote $g = \xi \circ \phi$ and we are done. \[\square\]

Proof of Proposition 4.1. Fix a (sufficiently large) natural number $\omega$ which will be determined later and which depends only on $n$. We assume that indices involving $i$ and $j$ run from 1 to $\omega$. Thus, for example, if we write $i + 1$ or $j - 2$ as an index we assume that $1 \leq i + 1 \leq \omega$ and $1 \leq j - 2 \leq \omega$. All the properties with indices involving $i$ and $j$ are automatically restricted to the cases when the indices make sense (= remain within the range from 1 to $\omega$).

Recall that $Z'$ and $Z$ are Nöbeling spaces and $g_Z$ is $U^nV^{n-1}$. Then there are open covers $W_Z'(i)$ of $Z'$ and neighborhoods $Q_Z(i)$ of $Z$ in $Y$ such that st$^2W_Z'(i) \prec_{n-1} W_Z'(i + 1)$, $W_Z'(\omega) \prec W_Z'$, $Q_Z(i) \subset Q_Z(i + 1)$, $Q_Z(i) \subset H$ and for the open covers $W_Y(i) = g_Y^{-1}(g_Z^{-1}(W_Z'(i)))$ of $Y$ we have that st$W_Y(i)$ restricted to $Q_Z(i)$ is an $(n - 1)$-refinement of $W_Y(i + 1)$ restricted $Q_Z(i + 1)$.

The sets $Q_Z(i)$ can be constructed starting from $Q_Z(\omega)$ and choosing for each $i = \omega - 1, \ldots, 1$ the set $Q_Z(i)$ to be so close to $Z$ that every map $\alpha$ from a sphere $S$ of dim $\leq n - 1$ to $Q_Z(i)$ can be $W_Y(i)$-closely homotoped inside $Q_Z(i + 1)$ to the map $g_Y \circ \alpha : S \rightarrow Z$.

We are going to show that $Q_Z = Q_Z(1)$ satisfies the conclusions of the proposition.

Let $M_\#$ be an open subset of $M$ containing $X$, $P$ a nice partition of $M_\#$, $Q_Z'$ a neighborhood of $Z'$ in $Y$ and $K$ a ball of dim $\leq n$ PL-embedded in $M$ such that $K \subset X$. Denote by $T(P)$ the set of the $P$-singular points of $Y$ and assume that $T(P)$ is contained in $Q_Z$. Since our choice of $Q_Z$ does not depend on $M$ we can replace $M$ by $M_\#$ and assume that $P$ is a partition of $M$. Note that without loss of generality we can always replace $M$ by any open subset of $M$ containing $X$ (with the automatic replacement of $P$ by the restriction of $P$ to that subset).
Let $Q_{Z'}$ be a neighborhood of $Z'$ in $Y$. Since $Z'$ is a $Z$-set in $Y$ there are neighborhoods $Q_{Z'}(i)$ of $Z'$ in $Y$ so that $Q_{Z'}(i) \subset Q_{Z'}$, $Q_{Z'}(i) \subset Q_Z(1)$, $clQ_{Z'}(i) \subset Q_{Z'}(i + 1)$, $\mathcal{W}_Y(i)$ restricted to $Q_{Z'}(i)$ is an $(n - 1)$-refinement of $\mathcal{W}_Y(i + 2)$ restricted to $Q_{Z'}(i + 2)$ and for every $i > j$ we have that $\mathcal{W}_Y(i)$ restricted to $Q_{Z'}(i + 2) \setminus clQ_{Z'}(j)$ is an $(n - 1)$-refinement of $\mathcal{W}_Y(i + 2) \setminus clQ_{Z'}(j - 2)$ and for every point in $Z'$ there is a $\mathcal{W}_Y(i)$-close point contained in $Q_{Z'}(i) \setminus clQ_{Z'}(j)$.

The sets $Q_{Z'}(i)$ can be constructed starting from $Q_{Z'}(\omega)$. Assume that $Q_{Z'}(\omega), \ldots Q_{Z'}(i + 1)$ are already constructed. Construct $Q_{Z'}(i)$ as follows. The first property of $Q_{Z'}(i)$ can be taken care of in a way similar to the one used for constructing $Q_Z(i)$. In order to satisfy the second property choose a $Z$-embedding $\beta : Y \to Y$ such that $\beta(Y) \cap Z' = \emptyset$ and $\beta$ is so close to the identity map of $Y$ that there is a neighborhood $Q_{Z'}(i)$ of $Z'$ such that $clQ_{Z'}(i) \subset Q_{Z'}(i + 1) \setminus \beta(Y)$ and every map from a sphere of dim $\leq n - 1$ to $Y \setminus clQ_{Z'}(i + 1)$ can be sufficiently closely homotoped in $Y \setminus clQ_{Z'}(i)$ to a map to $\beta(Y)$. If $\beta$ is close enough to the identity map of $Y$ then $Q_{Z'}(i)$ has the required properties.

Denote by $Q^M_Z(i)$, $Q^M_{Z'}(i)$ and $\mathcal{W}^M_Z(i)$ the extensions to $M$ of $Q_Z(i)$, $Q_{Z'}(i)$ and $\mathcal{W}_Y(i)$ all restricted to $X$. Replacing $M$ by an open subset of $M$ containing $X$ assume that $\mathcal{W}^M_Y(i)$ cover $M$, $clQ^M_Z(i) \subset Q^M_Z(i + 1)$ and $clQ^M_{Z'}(i) \subset Q^M_{Z'}(i + 1)$.

Let $A$ be a space of dim $\leq n$ and $B$ a closed subset of $A$. Consider maps $\alpha : A \to Y$ and $\beta : B \to Y$ and let $\delta > 5n + 5$. From the properties of $Q_Z(i)$, $Q_{Z'}(i)$ and $\mathcal{W}_Y(i)$ one can derive the following. If the image of $\beta$ is contained in $Q_Z(i)$ and in an element of $\mathcal{W}_Y(i)$ then $\beta$ can be extended to $\beta' : L \to Y$ such that the image of $\beta'$ is contained in $Q_Z(i + \delta)$ and in an element of $\mathcal{W}_Y(i + \delta)$. If the image of $\beta$ is contained in $Q_{Z'}(i) \setminus clQ_{Z'}(i + \delta)$ then $\beta$ extends to a map into $Q_{Z'}(i + 3\delta) \setminus clQ_{Z'}(i + 2\delta)$. If the image of $\beta$ is contained in $Q_{Z'}(i + 3\delta) \setminus clQ_{Z'}(i + 2\delta)$ then $\alpha$ is any extension of $\beta$ then there is an extension $\alpha' : L \to Q_{Z'}(i + 3\delta) \setminus clQ_{Z'}(i + 2\delta)$ of $\beta$ such that $\alpha$ and $\alpha'$ are $\mathcal{W}_Y(i + 3\delta)$-close.

**Constructing an initial subdivision $P_\#$ and an initial modification $P_\#(1)$.**

Since $T(P) \subset Q_Z(1)$ and $T(P) \cap K = \emptyset$ there are open neighborhoods $Q_T(i)$ of $T(P)$ in $Y$ such that $clQ_T(i) \subset Q_Z(1) \setminus K$ and $clQ_T(i) \subset Q_T(i + 1)$. Then there are open covers $\mathcal{W}_T(i)$ of $Y$ such that $st^2 \mathcal{W}_T(i) \prec_{n - 1} \mathcal{W}_T(i + 1)$, $st(clQ_T(j), \mathcal{W}_T(i)) \cap st(Y \setminus Q_T(i + 1), \mathcal{W}_T(i)) = \emptyset$, $st(clQ_T(j), \mathcal{W}_T(i)) \cap st(Y \setminus Q_Z(j + 1), \mathcal{W}_T(i)) = \emptyset$, $\mathcal{W}_T(\omega) \prec \mathcal{W}_Y(1)$, and $st^2 \mathcal{W}_T(\omega)$ restricted to $X \setminus Q_T(1)$ refines $stP$.

Replacing $M$ by an open subset of $M$ containing $X$ assume that the extension to $M$ of $\mathcal{W}_T(1)|X$ covers $M$. By 3.10 there are $(m - n + 1)$-co-connected rational partitions $\mathcal{D}(i)$ of $M$ such that $\mathcal{D}(i)$ is a subdivision of $\mathcal{D}(i + 1)$, $st^3 \mathcal{D}(i) \prec_{n - 1} st \mathcal{D}(i + 1)$, $\mathcal{D}(\omega)$ is a subdivision of $\mathcal{P}$ and $\mathcal{D}(\omega)$ refines the extension to $M$ of $\mathcal{W}_T(1)|X$. Deleting from $M$ the finite intersections of $\mathcal{D}(i)$ of dim $\leq m - n - 1$ we assume that $\mathcal{D}(i)$ are nice partitions of $M$.

By Lemma 4.2 there is a $Z$-embedding $g : X \to Y$ and a continuous retraction $f : Y \to g(X)$ such that $g$ and $f$ are $\mathcal{W}_T(1)$-close to the inclusion of $X$ into $Y$ and the identity map of $Y$ respectively. Denote $F_Y(D) = f^{-1}(g(D \cap X))$ for $D \in \mathcal{D}(1)$ and $F_Y = \{F_Y(D) : D \in \mathcal{D}(1)\}$. Note that $F_Y$ is a closed locally finite cover of $Y$. Replacing $M$ by an open subset of $M$ containing $X$ we assume that for $F_M = \{F_M(D) : D \in \mathcal{D}(1)\}$,
where $F_M(D) = \text{the closure of } F_Y(D) \cap X$ in $M$, we have that $\mathcal{F}_M$ is a locally-finite cover of $M$ such that $F_M(D_1) \cap F_M(D_2) = \emptyset$ provided $D_1 \cap D_2 = \emptyset, D_1, D_2 \in \mathcal{D}(1)$.

Let $\mathcal{P}_\#$ be a rational $(m - n + 1)$-co-connected partition of $M$ such that $\mathcal{P}_\#$ is a subdivision of $\mathcal{D}(1)$ and $\text{st}(F_M(D_1), \mathcal{P}_\#) \cap \text{st}(F_M(D_2), \mathcal{P}_\#) = \emptyset$ provided $D_1 \cap D_2 = \emptyset, D_1, D_2 \in \mathcal{D}(1)$. Deleting from $M$ the finite intersections of $\mathcal{P}_\#$ of dim $\leq m - n - 1$ we assume that $\mathcal{P}_\#$ is a nice partition of $M$.

For every $P \in \mathcal{P}_\#$ that intersects $X \setminus Q_T(\omega)$ define $P(1) = P$. Arrange into a sequence $P_1, P_2, \ldots$ the elements of $\mathcal{P}_\#$ that do not intersect $X \setminus Q_T(\omega)$ and define by induction on $s$ the set $P_s(1)$ as follows: $P_1(1) =$ the union of the elements of $\mathcal{P}_\#$ that do not intersect $X \setminus Q_T(\omega)$ and do intersect $F_M(P_1)$, and $P_{s+1}(1) =$ the union of the elements of $\mathcal{P}_\#$ that do not intersect $(P_1(1) \cup \ldots \cup P_s(1)) \cup (X \setminus Q_T(\omega))$ and do intersect $F_M(P_{s+1})$.

Define $\mathcal{P}_\#(1) = \{P(1) : P \in \mathcal{P}_\#\}$. Clearly $\mathcal{P}_\#(1)$ covers $M$. However we may have that $P_s(1) = \emptyset$ for some $P_s$. To avoid such situation modify the elements of $\mathcal{P}_\#(1)$ as follows. For every $P_s$ such that $P_s(1) = \emptyset$ choose a point $x_s$ in $X$ such that $x_s$ is contained in $\text{Int} P_x$, of some $P_x \in \mathcal{P}_\#$, $x_s$ is $\mathcal{P}_\#$-close to some point of $F_M(P_x)$ and the set of all the points $x_s$ is discrete in $X$. Replacing $M$ by a smaller open subset containing $X$ we assume that the set of $x_s$ is discrete in $M$ as well. Let $B_{x_s}$ be an $m$-dimensional PL-ball rationally embedded in $\text{Int} P_x$, such that $x_s \in \text{Int} B_{x_s}, \text{Int} P_x \setminus B_{x_s} \neq \emptyset$ and the collection of the balls $B_{x_s}$ is discrete in $M$. Then cutting the interior of the balls $B_{x_s}$ from the elements of $\mathcal{P}_\#(1)$ and defining $P_s(1) = B_{x_s}$ we get that $\mathcal{P}_\#(1)$ is a rational partition of $M$ for which $\mu_i : \mathcal{P}_\# \rightarrow \mathcal{P}_\#(1)$, defined by $\mu_i(P) = P(1)$, is a $0$-matching.

Define $\mathcal{A}(i)$ as the cover of $M$ consisting of the elements of $\text{st}\mathcal{D}(i)$ and the sets $\text{st}(W \cap X, \mathcal{D}(i))$ for $W \in \mathcal{W}_T(i)$ such that $W \cap (Y \setminus Q_T(\omega - i)) \neq \emptyset$. From the properties of $\mathcal{W}_T(i)$ and $\mathcal{D}(i)$ it follows that

$$
(C_1) \quad \text{each element of } \mathcal{A}(i) \text{ is a union of elements of } \mathcal{P}_\#, \mathcal{A}(i) \text{ refines } \text{st}\mathcal{P} \text{ for } i \leq \omega - 3, \mathcal{A}(i) \text{ refines } \mathcal{W}_Y(i + 2), \text{st}\mathcal{A}(i) \prec_{n-1} \mathcal{A}(i + 2) \text{ and } \text{st}(\text{cl}Q_Z(i), \mathcal{A}(i)) \subset Q_Z(i + 2).
$$

**Constructing modifications $\mathcal{P}_\#(i)$.** We are going to construct modifications $\mathcal{P}_\#(i)$ of $\mathcal{P}_\#$ and rational subcomplexes $F(i)$ of $M$ such that $\mathcal{P}_\#(i)$ is a rational decomposition of $M$ forming a partition on $M \setminus F(i)$ and $\mathcal{P}_\#$ admits a one-to-one correspondence $\mu_i : \mathcal{P}_\# \rightarrow \mathcal{P}_\#(i)$ to $\mathcal{P}_\#(i)$ where $P(i) = \mu_i(P)$ is the modification of $P \in \mathcal{P}_\#$ in $\mathcal{P}_\#(i)$.

Note that during the construction we will often replace $M$ by smaller open subsets containing $X$. Our goal is to get in the end of the construction a nice partition $\mathcal{P}(i)$ for which $\mu_i$ is a matching. We will carry out the construction such that for every $i$ there will exist $j \geq i$ for which the following important condition holds.

$$
(C_2) \quad F(i) \cap ((X \setminus Q_Z(j)) \cup K) = \emptyset, \mathcal{P}_\#(i) \text{ and } \mathcal{P}_\# \text{ coincide on a neighborhood in } X \text{ of } (X \setminus Q_Z(j)) \cup K \text{ (that is the intersections of } P \text{ and } P(i) \text{ with that neighborhood coincide for every } P \in \mathcal{P}_\#, P(i) = P \text{ if } P \in \mathcal{P}_\# \text{ is contained in } M \setminus Q_Z(j), \text{ for every } P \in \mathcal{P}_\# \text{ there is an element of } \mathcal{W}_Y(j) \text{ containing both } P \cap X \text{ and } P(i) \cap X, \text{ and for every}
$$
Let us verify that the initial modification \( P_#(1) \) satisfies \((C_2)\) with \( F(1) = \emptyset \) and \( j = 4 \). The condition concerning \( K \) is satisfied because \((X \setminus Q_Z(4)) \cup K \subset X \setminus \text{cl}Q_T(\omega)\). It is also clear that \( P(1) = P \) if \( P \subset M \setminus Q^M_Z(4) \). Since \( f \) and \( g \) are \( W_T(1) \)-close to the inclusion of \( X \) into \( Y \) and the identity map of \( Y \) respectively we have for \( P \in P_# \) that \( P \cap X \) and \( P(1) \cap X \) are contained in an element of \( st^4W_T(1) \) and, since \( W_T(1) \) refines \( W_T(1) \), \( P \cap X \) and \( P(1) \cap X \) are contained in an element of \( W_T(4) \).

Let \( y \in Y \) be such that \( f(y) \in Y \setminus Q_T(\omega - 1) \). Then for every element \( G \) of \( W_T(1) \) containing \( y \) we have that \( \mu_1^{-1}(st(G \cap X, P_#(1))) \) is contained in an element of \( st^6W_T(1) \) which intersects \( Y \setminus Q_T(\omega - 4) \) and therefore it is contained in an element of \( A(4) \).

Now let \( y \in Y \) be such that \( f(y) \in Q_T(\omega - 1) \). Take a neighborhood of \( y \) of the form \( G = \) the interior in \( Y \) of \( g^{-1}(V) \) where \( V = st(f(y), g(D(1)|X)) \). Then no element of \( P_# \) intersecting \( Q_T(\omega) \) will intersect \( G \) and hence for \( P \in P_# \), \( P(1) \) intersects \( G \) only if \( P \subset st(g^{-1}(f(y)), st^2D(1)) \). Thus \( \mu_1^{-1}(st(G \cap X, P_#(1))) \) is contained in an element of \( st^2D(1) \) and therefore it is contained in \( A(4) \). Condition \((C_2)\) has been verified for \( P_#(1) \).

The purpose of the modifications \( P_#(i) \) is to gradually improve the level of connectivity and the level of matching to \( P_# \). It will be done following the construction 3.4. This construction is a combination of 2.7 and 2.9 of [4] with a use of 3.2 for creating intersections.

Note that we do not need to worry about the rationality of \( P_#(i) \) and \( F(i) \) inside the constructions themselves. Indeed, assume that \( P_#(i) \) and \( F(i) \) are constructed such that \((C_2)\) holds. By 3.1 we may assume that every finite intersection of \( P_#(i) \) has a dense subset in \( X \). Now replacing \( M \) again by an open subset containing \( X \) and we can choose by 3.8 a PL-homeomorphism \( h : M \rightarrow M \) to be sufficiently close to the identity map of \( M \) such that \( h \) does not move the points on the extension to \( M \) of a neighborhood of \((X \setminus Q_Z(j)) \cup K \) in \( X \) on which \( P_#(i) \) and \( P_# \) coincide, \( h \) moves \( P_#(i) \) to a rational decomposition and \( F(i) \) to a rational subcomplex of \( M \) such that \( P_#(i) \) and \( F(i) \) replaced by \( h(P_#(i)) \) and \( h(F(i)) \) respectively will satisfy \((C_2)\) with \( j \) replaced by \( j + 1 \).

Let us analyze separately the constructions involved in 3.4. We always assume that the finite intersections of \( P_#(i) \) have dense subsets on \( X \). Fix \( 0 \leq s \leq n - 1 \), \( i \) and \( j \), set \( l = m - s + 1 \), \( q = s + 1 \) and assume that \((C_2)\) and the following condition hold for \( i, j, s \) and \( l \).

\[(C_3)\]
\[\dim F(i) \leq l - 2, \ P_#(i) \text{ is } l\text{-co-connected on } M \setminus F(i), \ \mu_i \text{ induces an } s\text{-matching between } P_# \text{ and } P_#(i) \text{ restricted to } M \setminus F(i).\]

Let \( G \) be a cover of \( Y \setminus Q_Z(j) \) by open sets in \( Y \) such that for every \( G \in G \),
\(\mu_{-1}(st(G \cap X, \mathcal{P}_{\#}(i)))\) is contained in an element of \(A(j)\). Let \(\mathcal{E}_Y\) be the collection of open subsets of \(Y\) consisting of the elements of \(\mathcal{G}\) and the set \(Q_{Z}(j)\). Denote by \(\mathcal{E}\) the extension to \(M\) of \(\mathcal{E}_Y\) restricted to \(X\). Replacing \(M\) by an open subset containing \(X\) assume that \(\mathcal{E}\) covers \(M\). Choosing \(\mathcal{G}\) to be sufficiently fine assume that \(st(clQ_{Z}^{M}(j), st^{0}\mathcal{E}) \subset Q_{Z}^{M}(j+1)\).

1). Improving connectivity of intersections. Fix \(0 \leq t \leq m - l + 1\) and assume that the finite intersection of \(\mathcal{P}_{\#}(i)\) restricted to \(M \setminus F(i)\) of \(\dim > m - t\) are \((l - 1)\)-co-connected. We are going to improve the connectivity of the intersections of \(\dim = m - t\) using the construction 2.7 of [4].

Adopt the notation of 2.7 of [4] with \(\mathcal{P}\) and \(F\) replaced by \(\mathcal{P}_{\#}(i)\) and \(F(i)\) respectively. Recall that in 2.7 of [4] we choose for every finite intersection \(P\) of \(\mathcal{P}_{\#}(i)\) such that \(\dim P \setminus F(i) = m - t\) and \(P \setminus F(i)\) is not \((l - 1)\)-co-connected, a countable collection \(\mathcal{F}_P\) of PL-embeddings of an \((q - t - 1)\)-dimensional sphere into \(P \setminus F(i)\) such that \(\mathcal{F}_P\) represents all the elements of the \((q - t - 1)\)-dimensional homotopy group of \(P \setminus F(i)\). After that we extend all the embeddings in \(\mathcal{F}_P\) for all \(P\) to a collection \(\mathcal{F}^{0\Delta}\) of PL-embeddings \(f_{\partial\Delta}: \partial B^q \rightarrow IntM\) of a \(q\)-dimensional PL-ball \(B^q\) such that the images of the embeddings in \(\mathcal{F}^{0\Delta}\) form a discrete family in \(M \setminus R^{0\Delta}\) for a PL-presented subset \(R^{0\Delta}\) of \(M\) with \(\dim R^{0\Delta} \leq q - 1\) (here we may consider \(\Delta\) and \(\partial\Delta\) only as a part of the notation). By 3.5 we can assume that \(R^{0\Delta}\) is rationally presented and replace \(M\) by \(M \setminus R^{0\Delta}\). By 3.5 we assume that \(f_{\partial\Delta}(\partial B^q) \subset X\) for every \(f_{\partial\Delta} \in \mathcal{F}^{0\Delta}\).

Recall that \(st(clQ_{Z}^{M}(j), \mathcal{P}_{\#}) \cap st(M \setminus Q_{Z}^{M}(j+1), \mathcal{P}) = \emptyset\). By \((C_2)\), the elements of \(\mathcal{P}_{\#}\) that are contained in \(M \setminus Q_{Z}^{M}(j)\) are left unchanged in \(\mathcal{P}_{\#}(i)\). Then, since \(\mathcal{P}_{\#}\) is a nice partition, the finite intersections of \(\mathcal{P}_{\#}(i)\) that are not \((l - 1)\)-co-connected intersect \(Q_{Z}^{M}(j)\). The embeddings in \(\mathcal{F}^{0\Delta}\) needed for improving the connectivity of a finite intersection \(P\) of \(\mathcal{P}_{\#}(i)\) lie in \(st(P, \mathcal{P}_{\#}(i))\). Thus we may assume that the images of the embeddings in \(\mathcal{F}^{0\Delta}\) refine \(st\mathcal{P}_{\#}(i)\) restricted to \(Q_{Z}^{M}(j+1)\) and therefore they will refine \(st\mathcal{W}_{Y}(j+1)\) restricted to \(Q_{Z}(j+1)\).

Consider \(B^q\) as the unit cube \(B^q = [-1, 1]^q\) in the \(q\)-dimensional Euclidean space and let \(r: B^q \setminus O \rightarrow \partial B^q\) be the radial projection. Extend each \(f_{\partial\Delta}\) to a map \(f^{+}_{\partial\Delta}: B^q \setminus Int_{\frac{1}{2}}B^q \rightarrow X\) such that \(f^{+}_{\partial\Delta} \circ r\) and \(f_{\partial\Delta}\) are both \(\mathcal{W}_{Y}(j)\) and \(\mathcal{E}_Y\)-close, the images of the maps \(f^{+}_{\partial\Delta}\) are contained in \(Q_{Z}(j+3)\) and the images of the maps \(f^{-}_{\partial\Delta}\) restricted to \(\partial_{\frac{3}{4}}B^q\) form a discrete family in \(Y\) (recall that \(Y \setminus X\) is a \(\sigma\)-\(Z\)-set in \(Y\)). Note that the images of the maps \(f^{+}_{\partial\Delta}\) refine \(\mathcal{W}_{Y}(j+3)\).

Set \(\delta = 10n + 10\) and let \(r' : \frac{3}{4}B^q \setminus O \rightarrow \partial_{\frac{3}{4}}B^q\) be the radial projection. Denote

\[
B_{0}(f^{+}_{\partial\Delta}) = r'^{-1}[(f^{+}_{\partial\Delta})^{-1}[clQ_{Z}(j + 3\delta) \setminus Q_{Z}(j + 2\delta)] \cap \partial_{\frac{3}{4}}B^q] \cup \frac{1}{2}B^q,
\]

\[
B_{-}(f^{-}_{\partial\Delta}) = r'^{-1}[(f^{-}_{\partial\Delta})^{-1}[clQ_{Z}(j + 3\delta) \cap \partial_{\frac{3}{4}}B^q] \setminus \text{Int}_{\frac{1}{2}}B^q],
\]

\[
B_{+}(f^{+}_{\partial\Delta}) = r'^{-1}[(f^{+}_{\partial\Delta})^{-1}[Y \setminus Q_{Z}(j + 2\delta)] \cap \partial_{\frac{3}{4}}B^q] \setminus \text{Int}_{\frac{1}{2}}B^q.
\]

By the properties of \(Q_{Z}(j)\), \(Q_{Z}(j)\) and \(\mathcal{W}_{Y}(j)\) the maps \(f^{+}_{\partial\Delta}\) can be extended to maps \(f^{+}_{\Delta}: B^q \rightarrow X\) such that the images of \(f^{+}_{\Delta}\) are contained in \(Q_{Z}(j + 5\delta)\) and refine \(\mathcal{W}_{Y}(j+5\delta)\), the images of \(f^{-}_{\Delta}\) restricted to \(\frac{3}{4}B^q\) form a discrete family in \(Y\) and for every \(f^{+}_{\Delta}\) we have that the image of \(f^{+}_{\Delta}\) restricted to \(B_{0}(f^{+}_{\partial\Delta})\) is contained in \(Q_{Z}(j + \delta) \setminus clQ_{Z}(j + \delta),\)

\[25\]
the image of $f_\Delta^*$ restricted to $B_+(f_{\partial \Delta}^*)$ is contained in $Y \setminus \text{cl}Q_{Z'}(j + \delta)$ and the image of $f_\Delta^*$ restricted to $B_-(f_{\partial \Delta}^*)$ are contained in $Q_{Z'}(j + 4\delta)$.

By 3.1 we can replace $M$ by an open subset containing $X$ and assume that the images of the maps $f_{\partial \Delta}^*$ restricted to $\partial B^a$ form a discrete family in $M$. By 2.6 of [4], the maps $f_{\Delta}^*$ can be arbitrarily closely approximated by maps $f_{\Delta}'' : B^a \to \text{Int}M \setminus R^\Delta$ such that $f_{\Delta}''$ coincides with $f_{\Delta}^*$ on $\partial B^a$ and the images of $f_{\partial \Delta}^*$ are contained and form a discrete family in $M \setminus R^\Delta$ for some PL-presented subset $R^\Delta$ of $M$ with $\dim R^\Delta \leq n$. By 3.9 assume that $R^\Delta$ is rationally presented and replace $M$ by $M \setminus R^\Delta$. Then by 3.7 each map $f_{\Delta}''$ can be arbitrarily closely approximated by a PL-embedding $f_\Delta : B^a \to \text{Int}M$ such that $f_\Delta$ coincides with $f_{\partial \Delta}^*$ on $\partial B^a$, $f_\Delta$ extends to a PL-embedding $f_B$ of $B^m = B^a \times B^{m-q}$ into $M$ with the properties described in 4.2 of [4]. Then if $f_\Delta$ is sufficiently close to $f_{\Delta}^*$ we can choose $f_B$ so that the images of $f_B$ form a discrete family in $M$ and the sets $f_B((\frac{1}{4}B^a) \times B^{m-q}) \cap X$ form a discrete family in $Y$. By the reasoning similar to the one of 3.5 there is a point $a \in \text{Int}B^{m-q}$ arbitrarily close to $O \in B^{m-q}$ such that $f_B(B^a \times a) \subset X \setminus K$. Replacing $B^{m-q}$ by a small cube centered at $a$ we assume that $f_B(B^a) \subset X \setminus K$, and $f_B$ and $f_B \circ p$ are $\mathcal{E}$-close where $p : B^m \to B^a$ is the projection.

Thus without loss of generality we may assume that the images of $f_\Delta$ are contained in $X$ and all the above properties hold when $f_{\Delta}^*$, $f_{\partial \Delta}^*$ and $f_{\partial \Delta}$ are replaced by $f_\Delta$, $f_\Delta^*$ restricted to $B^a \setminus \text{Int}\frac{1}{3}B^a$ and $f_\Delta$ restricted to $\partial B^a$ respectively with the only change that because the images of $f_\Delta$ may slightly move and the distance between $f_{\partial \Delta}^* \circ r$ and $f_{\partial \Delta}$ may slightly increase on the set $B^a \setminus \text{Int}\frac{1}{3}B^a$ where we do not have the discreteness in $Y$ of the images we will require that the images of $f_\Delta$ are contained in $Q_{Z}(j + 6\delta)$ and refine $\mathcal{W}_Y(j + 6\delta)$, and $f_{\partial \Delta}^* \circ r$ and $f_{\partial \Delta}$ are st$\mathcal{W}_Y(j)$ and st$\mathcal{E}_Y$-close.

In addition, for each $f_B$ we can replace $B^{m-q}$ by a smaller cube and assume that $f_B(B_+(f_{\partial \Delta}^*) \times B^{m-q}) \subset M \setminus \text{cl}Q_{Z'}^{M}(j + \delta)$ and $f_B(B_+(f_{\partial \Delta}^*) \times B^{m-q}) \subset Q_{Z'}^{M}(j + 4\delta)$, the images of $f_B$ are contained in $Q_{Z'}^{M}(j + 6\delta)$ and refine $\mathcal{W}_Y^{M}(j + 6\delta)$, and the images of $f_B$ are outside some neighborhood of $K$ in $M$.

For a given embedding $f_B$ identify $B^m = B^a \times B^{m-q}$ with $f_B(B^m)$. Clearly replacing $B^{m-q}$ by a smaller cube we may assume that (*)& of 3.12 is satisfied with $f$ being the inclusion of $X$ into $Y$. Now apply the radial modification to $\mathcal{P}_\#(i)$ as described in 3.11, and let $h : M \to M$ be a PL-homeomorphism satisfying (**) of 3.12 with $\mathcal{P}$ replaced by $\mathcal{P}_\#(i)$ and $f$ replaced by the inclusion of $X$ into $Y$. Note that now $r$ stands for the radial projection $r : B^m \setminus O \to \partial B^m$ which extends the radial projection for $B^a$ previously denoted by $r$.

Since the images of $f_B$ are discrete in $M$ this procedure can be done independently for each $f_B$. Let us denote by $\mathcal{P}_\#^*(i)$ the result of all the radial modifications (note that we also modify $F(i)$) and let us denote again by $h$ the resulting PL-homeomorphism of $M$ for all embeddings $f_B$.

Let $y \in Y \setminus \text{cl}Q_{Z'}(j + 4\delta)$. Let us show that

$$(C_4)$$

there are a neighborhood $G$ of $y$ in $Y$ and an element $E$ in $\mathcal{E}$ lying in $M \setminus \text{cl}Q_{Z'}^{M}(j + 1)$ such that for every $P \in \mathcal{P}_\#(i)$, $h(P^*) \cap X$ intersects $G$ only if $P$ intersects $\text{st}(E, \text{st}^5E)$
where $P^* \in \mathcal{P}_#(i)$ is the modification of $P$.

Denote $F_Y = \text{the closure in } Y \text{ of the union of } f_B((\frac{3}{4}B^q) \times B^{m-q}) \cap X \text{ for all } f_B$ and recall that the family of the sets $f_B((\frac{3}{4}B^q) \times B^{m-q}) \cap X$ is discrete in $Y$.

Consider first the case $y \notin F_Y$. Take a neighborhood $G$ of $y$ in $Y$ such that $G \cap F_Y = \emptyset$ and $G$ is contained in an element $E_Y$ of $\mathcal{E}_Y$ lying in $Y \setminus \text{cl}Q_{Z'}(j + 1)$. Let $E \in \mathcal{E}$ be the extension of $E_Y \cap X$ to $M$ and let $P \in \mathcal{P}_#(i)$ be such that $h(P^*) \cap X \text{ intersects } G$. If $P \cap X \text{ intersects } G$ then clearly $P \text{ intersects } E$. Assume that $P \cap X \text{ does not intersect } G$. Take $x \in G \cap (h(P^*) \cap X)$. Then there is $f_B$ such that $x \in f_B(B^m)$. Identify $f_B(B^m)$ with $B^m$ and note that $r(h^{-1}(x)) \in P$ and, $x$ and $r(h^{-1}(x))$ are st$E$-close because $G \cap F_Y = \emptyset$. Hence $P \text{ intersects } st(E, st^3\mathcal{E}) \text{ and } (C_4)$ follows.

Now let $y \in F_Y$ and $y \in Y \setminus \text{cl}Q_{Z'}(j + 4\delta)$. Then there is (only one) $f_B$ such that $y \in f_B(B^m)$ and it belongs to the closure in $Y$ of $f_B((\frac{3}{4}B^q) \times B^{m-q}) \cap X$. Identify $B^m$ with $f_B(B^m)$. Take a neighborhood $G$ of $y$ in $Y$ such that $G \subseteq Y \setminus \text{cl}Q_{Z'}(j + 4\delta)$ and $G$ satisfies (**) of 3.12 for an element $E \in \mathcal{E}$. Clearly $G$ can be replaced by any smaller neighborhood of $y$.

Assume $y$ does not belong to the closure in $Y$ of $\partial B^m \cap X$. Replace $G$ by a smaller neighborhood and assume that $G \cap X \subseteq B^m$. Let $x \in (G \cap X) \cap (\frac{3}{4}B^q) \times B^{m-q}, x' = p(r(h^{-1}(x))), x'' = x' \in \frac{3}{4}B^q \text{ and } x'' = \frac{3}{4}x' \text{ otherwise (note that } O \text{ does not belong to } G \cap X \text{ since } (\frac{1}{2})B\times B^{m-q} \subseteq Q, (j + 4\delta))$. From the properties of $f_B$ it follows that $x'' \in B_* f_B((\frac{3}{4}B^q) \subseteq M \setminus \text{cl}Q_{Z'}(j + \delta)$. By (**) of 3.12 we have that $x'' \in E$. Then, since $x'$ and $x''$ are st$E$-close, we have that $E \in E \setminus \text{cl}Q_{Z'}(j + 1)$ and $(C_d)$ follows.

Now assume that $y$ belongs to the closure of $\partial B^m \cap X$ in $Y$. Replace $G$ by a smaller neighborhood of $y$ and assume that $G$ does not intersect $F_Y \setminus (B^m \cap X)$ and is contained in an element of $\mathcal{E}_Y$. Take a point $x \in G \cap B^m$ which is so close to $\partial B^m$ that $p(x)$ and $p(h^{-1}(x))$ are $\mathcal{E}$-close. Then $G \cap X$ is contained in $st(E, st\mathcal{E})$. Let $P \in \mathcal{P}_#(i)$ be such that $P^* \cap (G \cap X) \neq \emptyset$. By (**) of 3.12 we have that $h(P^*) \cap X \text{ intersects } B^m \cap (G \cap X)$ then $P$ intersects $E$. If $P^*$ does not intersect $B^m$ then by the reasoning similar to the one applied in the case $y \notin F_Y$ we conclude that $P$ intersects st$(E, st\mathcal{E})$. Since $G \cap X$ is contained in $st(E, st\mathcal{E})$ and $G \subseteq Y \setminus \text{cl}Q_{Z'}(j + \delta)$ we have that $E \subseteq M \setminus \text{cl}Q_{Z'}(j + 1)$ and $(C_d)$ follows. Thus $(C_d)$ has been verified in all cases.

Now we are ready to construct $\mathcal{P}_#(i + 1)$ and $F(i + 1)$. Fix $f_B$ and identify $B^m$ with $f_B(B^m)$. Modify $\mathcal{P}_#(i)$ on $B^m$ as described in 3.13 and combine this modification with the block hole modification shifted to $B^m = \frac{1}{2}B^m$. Denote by $\mathcal{P}_#^*(i)$ and $F^*(i)$ the result of all these modifications for all $f_B$. Note that $\mathcal{P}_#^*(i)$ and $F^*(i)$ coincide on $\Omega \cap \Omega$ and $\Omega$ is contained in $Q_{Z'}(j + 4\delta)$. Define $P^*_#(i + 1) = h(P^*_#(i))$ and $F(i + 1) = h(F^*(i))$. Note that in $(C_d)$, st$(E, st\mathcal{E}) \subseteq M \setminus Q_{Z'}(j + 4\delta)$ and st$(\text{cl}Q_{Z'}(j), st\mathcal{E}) \subseteq M \setminus Q_{Z'}(j + 1)$. Then $(C_d)$ implies that for every $y \in Y \setminus \text{cl}Q_{Z'}(j + 4\delta)$ there is a neighborhood $y$ in $Y$ such that $\mu^{-1}_{j+1}(st(G \cap X, P_#(i + 1)))$ refines $st\mathcal{A}(j)$.

Since the images of $f_B$ are contained in $Q_{Z'}^M(j + 6\delta)$, do not intersect $K$ and form a discrete family in $M$ we have that $F(i + 1) \subseteq Q_{Z'}^M(j + 6\delta)$ and there is a neighborhood of $(M \setminus Q_{Z'}^M(j + 6\delta)) \cup K$ in $M$ on which $\mathcal{P}_#(i)$ and $\mathcal{P}_#(i + 1)$ coincide. Thus we finally get that $\mathcal{P}_#(i + 1)$ and $F(i + 1)$ satisfy $(C_2)$ with $j$ replaced by $j + 6\delta$, $\mathcal{P}_#(i + 1)$ and
$F(i + 1)$ satisfy $(C_3)$ and all the intersections of $\dim \geq m - t$ of $\mathcal{P}_#(i + 1)$ restricted to $M \setminus F(i + 1)$ are $(l - 1)$-co-connected.

2). Creating intersections. Assume that $\mathcal{P}_#(i)$ is $(l - 1)$-co-connected on $M \setminus F(i)$ and satisfies $(C_2)$ and $(C_3)$. Following 3.4 we need to create the missing intersections of $\mathcal{P}_#(i)$. Let $P = P_0 \cap \cdots \cap P_{s+1}$ be a non-empty intersection of distinct elements of $\mathcal{P}_#$ such that $(P_0(i) \cap \cdots \cap P_{s+1}(i)) \cap (M \setminus F(i)) = \emptyset$. Since $\mathcal{P}_#$ and $\mathcal{P}_#(i)$ coincide on $M \setminus Q_M^2(j)$ we have that $P_0(i), \ldots, P_{s+1}(i) \subset Q_M^2(j + 1)$. Since for every $P \in \mathcal{P}_#$ there is an element of $\mathcal{W}_X^M(j)$ containing both $P$ and $P(i)$, the union of $P_0(i), \ldots, P_{s+1}(i)$ is contained in an element of $\mathcal{W}_X^M(j + 2)$. Then, by 3.2, the maps $\mathcal{F}^{\partial \Delta}$ can be chosen so that the images of $\mathcal{F}^{\partial \Delta}$ are contained in $Q_M^2(j + 2)$ and refine $\mathcal{W}_X^M(j + 2)$. The rest of the construction is identical to the previous construction of improving connectivity of intersections.

3). Absorbing simplexes. This is the last construction in 3.4. Assume that $\mathcal{P}_#(i)$ is $(l - 1)$-co-connected on $M \setminus F(i)$, $\mathcal{P}_#(i)$ satisfies $(C_2)$ and $(C_3)$ and all the intersections of $\mathcal{P}_#(i)$ (restricted to $M \setminus F(i)$) are brought from $\mathcal{P}_#$ (by $\mu_i$). Following 3.4 add to $F(i)$ all the finite intersections of $\mathcal{P}_#(i)$ which are not brought from the finite intersections of $\mathcal{P}_#$ of $\dim \leq m - s - 1$. Since $\mathcal{P}_#$ and $\mathcal{P}_#(i)$ coincide on $M \setminus Q_M^2(j)$ we have $F(i) \subset Q_M^2(j + 1)$. Since $\mathcal{P}_#$ and $\mathcal{P}_#(i)$ coincide on a neighborhood of $K$ in $M$ (because they coincide on a neighborhood of $K$ in $X$) we have that $F(i) \cap K = \emptyset$.

Following 2.9 of [4] fix a sufficiently fine triangulation of $M$ underlying $\mathcal{P}_#(i)$ and $F(i)$ We will first absorb the $(m - s - 1)$-simplexes of $F(i)$ lying in $M \setminus Q_M^2(j)$. Replacing $M$ by an open subset containing $X$ and the triangulation of $M$ by a finer triangulation assume that for a simplex $\Delta$ of $\dim = m - s - 1$ lying in $F(i) \setminus Q_M^2(j)$ we have that $\mu_i^{-1}(\text{st}(\Delta, \mathcal{P}_#(i)))$ is contained in an element $A$ of $\mathcal{A}(j)$. Take an element $A'$ of $\mathcal{A}(j + 1)$ such that $A \subset A'$ and the inclusion of $A$ into $A'$ induces the zero-homomorphism of the homotopy groups in $\dim \leq n - 1$. Recall that $\mathcal{P}_#$ underlies both $A$ and $A'$. Denote $F = \text{the union of the finite intersections of } \mathcal{P}_# \text{ of } \dim < m - s - 1 \text{ which are not brought from } \mathcal{P}_#(i) \text{ restricted to } M \setminus F(i)$ Then $\mu_i$ induces a matching between $\mathcal{P}_#$ restricted to $M \setminus F$ and $\mathcal{P}_#(i)$ restricted to $M \setminus F(i)$. Since $F < m - s - 1$ we have that $\mathcal{P}_#$ is $(l - 1)$-co-connected the inclusion $A \setminus F \subset A' \setminus F$ induces the zero-homomorphism of the homotopy groups in co-dimensions $\geq l - 1$.

Hence, by 3.6, the construction for absorbing all simplexes of $\dim = m - s - 1$ lying in $M \setminus Q_M^2(j)$ can be carried out such that the modification $\mathcal{P}_#(i + 1)$ of $\mathcal{P}_#(i)$ will refine $\mathcal{B} = \mu_i(\text{st}^2 \mathcal{A}(j + 1))$ and for every $P(i) \in \mathcal{P}_#(i)$, $P(i + 1) \subset \text{st}(P(i), \mathcal{B})$ where $P(i + 1) \in \mathcal{P}_#(i + 1)$. Note that we also modify $M$, however by 3.9 the PL-subcomlex of $\dim \leq n - 1$ needed to be removed from $M$ can be rational and hence $M$ can be replaced by an open subset containing $X$. Also note that by 3.6 we can assume that $\mathcal{P}_#(i + 1)$ coincides with $\mathcal{P}_#(i)$ on a small neighborhood of $K$ in $M$. Clearly $\mathcal{P}_#(i + 1)$ coincides with $\mathcal{P}_#(i)$ outside $\text{st}(Q_M^2(j + 1), \mathcal{B})$. Hence $\mathcal{P}_#(i + 1)$ refines $\mathcal{W}_X^M(j + 10)$ and coincides with $\mathcal{P}_#(i + 1)$ on $M \setminus Q_M^2(j + 10)$.

Let $y \in M \setminus Q_M^2(j)$ and let $G$ be neighborhood of $y$ in $Y$ such that that $\mu_i^{-1}(\text{st}(G \cap X, \mathcal{P}_#(i)))$ is contained in an element of $\mathcal{A}(j)$. Then $\mu_i^{-1}(\text{st}(G \cap X, \mathcal{P}_#(i + 1))) \subset \mu_i^{-1}(\text{st}(G \cap X, \mathcal{P}_#(i)))$. 

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Constructing the final modification $\mathcal{P}'$. The construction of the modifications of $\mathcal{P}_\#$ ends up with a rational decomposition $\mathcal{P}_\#(i)$ and a rational subcomplex $F(i)$ of $M$ such that $\mathcal{P}_\#(i)$ and $F(i)$ satisfy (C2) for some $j$, $\mathcal{P}_\#(i)$ forms an $(m-n+1)$-co-connected partition on $M \setminus F(i)$, dim $F(i) \leq m-n-1$ and $\mu_i$ is an $n$-matching between $\mathcal{P}_\#$ and $\mathcal{P}_\#(i)$ restricted to $M \setminus F(i)$. Removing $F(i)$ and the finite intersections of $\mathcal{P}_\#(i)$ of dim $\leq m-n-1$ from $M$ we can assume that $\mathcal{P}_\#(i)$ is a nice partition of $M$ and $\mu_i$ is a matching. Note that the number of the modifications of $\mathcal{P}_\#$ needed to get $\mathcal{P}_\#(i)$ depends only on $n$ and the increase in the value of $j$ for each modification is bounded by a number depending only on $n$. Thus we can estimate the maximal value of $j$ and assign this value to $\omega$ in the beginning of the proof.

Following 3.10 for each $P \in \mathcal{P}$ define $P' = \cup \{\mu_i(P_\#) : P_\# \in \mathcal{P}_\# \cap P \}$ and let $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$. Since $\mathcal{P}$, $\mathcal{P}_\#$ and $\mathcal{P}_\#(i)$ are nice partitions, $\mathcal{P}_\#$ is a subdivision of $\mathcal{P}$ and $\mu_i$ is a matching between $\mathcal{P}_\#$ and $\mathcal{P}_\#(i)$ we conclude that $\mathcal{P}$ is a nice partition of $M$ and $\mu : \mathcal{P} \rightarrow \mathcal{P}'$ defined by $\mu(P) = P'$ is a matching. From (C2) it follows that $P' \subset st(P, W^M_Y(j))$ and, since $W^M_Y(j)$ refines $W_M$ we get $P' \subset st(P, W_M)$. It also follows from (C2) that $\mathcal{P}$ and $\mathcal{P}'$ coincide on a neighborhood of $(M \setminus Q^M_Z(j)) \cup K$ and since $Q_Z(j) \subset H$ the requirement (2) of the proposition is satisfied.

Let us verify the requirement (3). Take a point $y \in Y \setminus Q_Z^r$. Then $y \in Y \setminus Q_Z(j)$. By (C2) there is a neighborhood $G$ of $y$ in $Y$ such that $\mu_i^{-1}(st(G \cap X, \mathcal{P}_\#(i)))$ is contained in an element of $\mathcal{A}(j)$ and hence, by (C1), $\mu_i^{-1}(st(G \cap X, \mathcal{P}_\#(i)))$ is contained in an element of $st\mathcal{P}$. Then, since $\mu_i$ is a matching and $\mathcal{P}_\#$ is a subdivision of $\mathcal{P}$ we have that $\mu_i(\mu_i^{-1}(st(G \cap X, \mathcal{P}_\#(i)))) = st(G \cap X, \mathcal{P}_\#(i))$ is contained in an element of $st\mathcal{P}'$ and therefore $y$ is not a $\mathcal{P}'$-singular point. The requirement (3) has been verified. □
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