EFFICIENT SUPERIMPOSITION RECOVERING ALGORITHM

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ABSTRACT

In this article, we address the issue of recovering latent transparent layers from superimposition images. Here, we assume we have the estimated transformations and extracted gradients of latent layers. To rapidly recover high-quality image layers, we propose an Efficient Superimposition Recovering Algorithm (ESRA) by extending the framework of accelerated gradient method. In addition, a key building block (in each iteration) in our proposed method is the proximal operator calculating. Here we propose to employ a dual approach and present our Parallel Algorithm with Constrained Total Variation (PACTV) method. Our recovering method not only reconstructs high-quality layers without color-bias problem, but also theoretically guarantees good convergence performance.

Index Terms— Superimposition recovering, proximal operator, optimization

Assume we have the extracted gradients and transformations of each latent transparent layer, the reconstruction step is the final crucial part for reflection separation. We assume the variations of transmitted layers in each mixtures conform to a parametric transformation \( f(x, \theta) \) (\( x \) is the pixel coordinates) with different parameters \( \theta_i \). Here we propose an Efficient Superimposition Recovering Algorithm (ESRA) to fast recover the high quality latent layers.

1. EFFICIENT SUPERIMPOSITION RECOVERING ALGORITHM

With estimated transformation parameters \( \theta_i \), we align the transmitted layers by warping mixtures \( I_i \) with \( f^{-1}(x, \theta_i) \). Then our mixing model is rewritten as:

\[
I_i(f_i^{-1}(\phi)) = a_{i1}L_i^t(\phi) + a_{i2}L_r^{\theta_i}(f_i^{-1}(\phi)), \quad i = 1, \ldots, m.
\]

(1)

Here \( L_i^t \) is the latent transmitted layer, \( L_r^{\theta_i} \) is the reflected layer in \( i \)th mixtures, \( a_{i1}, a_{i2} \) is the mixing coefficients. With this new mixing model, the influence of parametric transformations \( f(x, \theta_i) \) can be ignored in the intermediate recovering process. For simplicity, we use \( I_i(\phi) \) to represent \( I_i(f_i^{-1}(\phi)) \). \( L_1(x) \) and \( L_{i+1}(\phi) \) denote \( L_i^t(\phi) \) and \( a_{i2}L_r^{\theta_i}(f_i^{-1}(\phi)) \), respectively. Let \( E^i(\phi) \) stand for the extracted gradients from \( L_i^t(\phi) \). To recover high quality latent image layers, we propose to employ \( L_1 \) penalty on the extracted gradients and nonnegative constraints on the layers’ intensities along with the \( L_2 \) loss of the mixing model. Thus our recovering objective function is written as:

\[
\min_{0 \leq \ell^{vec} \leq 1} F(l^{vec}) = \lambda \sum_{x,i=1}^{m+1} |\nabla L_i^t(x) - E^i(x)|
\]

\[
+ \sum_{x,i=1}^{m+1} \frac{1}{2} (I_i(\phi) - a_{i1}L_1(\phi) - L_{i+1}(\phi))^2
\]

(2)

where \( \lambda \) is a large vector containing all pixel values in all latent layers. The first \( L_1 \) term enforces the agreement between reconstructed layer gradients and extracted layer gradients, while the second \( L_2 \) term tends to satisfy our mixing mode. Since the extracted gradients are nonzero at very few coordinates, the \( L_1 \) norm term not only prefers layers with sparse gradients but also avoids over-smooth results. \( \lambda \) is a trade off coefficient.

To solve the nonsmooth convex optimization model (2) efficiently, we denote

\[
f(l^{vec}) = \sum_{x,i=1}^{m+1} \frac{1}{2} (I_i(\phi) - a_{i1}L_1(\phi) - L_{i+1}(\phi))^2, \quad \text{s.t.} \ 0 \leq l^{vec} \leq 1,
\]

\[
g(l^{vec}) = \lambda \sum_{x,i=1}^{m+1} |\nabla L_i^t(x) - E^i(x)|.
\]

(3)

Here \( g(l^{vec}) \) is the \( \ell_1 \) penalty on the extracted gradients and \( f(l^{vec}) \) corresponds to the \( L_2 \) loss and nonnegative constraints. \( f(l^{vec}) \) can be formulated in the following matrix form:

\[
f(l^{vec}) = \frac{1}{2} ||A(l^{vec}) - b||^2, \quad \text{s.t.} \ 0 \leq l^{vec} \leq 1,
\]

\[
\text{where } A = \begin{bmatrix} a_{i1} & I \\ \\ \vdots & \ddots \\ a_{m1} & I \end{bmatrix}, \quad b = \begin{bmatrix} \text{vec}(I_1) \\ \vdots \\ \text{vec}(I_m) \end{bmatrix},
\]

(4)

where \( f(l^{vec}) \) is continuously differentiable and \( \nabla f(l^{vec}) = A^\top(A(l^{vec}) - b) \), of which Lipschitz constant \( L(f) = \lambda_{\max}(A^\top A) = \sum \alpha_i^2 + 1 \), and \( f \in \mathbb{R}^{hw \times w} \) is the unit matrix. We note the objective function in (2) is a composite function of a differential term \( f(l^{vec}) \) and a non-differential term \( g(l^{vec}) \). Denote
\( P_{L\nu,k} \) is the first order Taylor expansion of \( f(l_{k-1}^{ve}) \) at \( l_{k-1}^{ve} \), with the squared Euclidean distance between \( l_{k}^{ve} \) and \( l_{k-1}^{ve} \) as the regularization term. The traditional gradient descent algorithm obtains the solution at the \( k \)-th iteration \( (k \geq 1) \) by \( l_{k}^{ve} = \arg \min P_{L\nu,k}(l_{k}^{ve}) + g(l_{k}^{ve}) \) with a proper step size \( L_k \) (greater than \( L(f) \)). Here we propose to employ the accelerated gradient descent [3,4] to solve the reconstruction problem, named Efficient Superimposition Recovering Algorithm (ESRA). Here we generate a solution at the \( k \)-th iteration \( (k \geq 1) \) by computing the following proximal operator \[ l_{k}^{ve} = \text{arg min}_{0 \leq t \leq L} P_{L\nu,k}(t_{k}^{ve}) + g(t_{k}^{ve}), \] where \( Y_k = l_{k}^{ve} \) and \( Y_{k-1}^{ve} = \sum_{i=1}^{m+1} \langle \lambda \nabla L_i(x) - E_i(x) \rangle \). As illustrated in (10), finding \( l_{k}^{ve} \) is to solve following \( m+1 \) separable problems with constrained total variation in parallel:

\[
\min_{0 \leq L \leq 1} \sum_{i=1}^{m+1} \frac{1}{2} \left\| L(x) - d_{i}(x) \right\|^2 + \beta |\nabla L(x) - E(x)|. \tag{11}
\]

Here \( \beta = \lambda / L_i \), and \( L, d, E \) represent \( L^i, d_i, E_i \), respectively. Similar with the image denoising problem [4,5], we propose a dual approach to solve (11) and give some notation in order:

- \( P \) is the set of matrix-pairs \( (p,q) \) where \( p \in \mathbb{R}^{(h-1)\times w} \) and \( q \in \mathbb{R}^{h \times (w-1)} \) that satisfy \( |p_{i,j}| \leq 1 \) and \( |q_{i,j}| \leq 1 \) \( \forall i,j \). And we assume \( p_{i,j} = q_{i,j} = q_{i,w} = 0 \), for every \( i = 1, \ldots, h, j = 1, \ldots, w \).

- The linear operator \( \mathcal{L} : \mathbb{R}^{(h-1)\times w} \times \mathbb{R}^{h \times (w-1)} \to \mathbb{R}^{hw} \) is defined by the formula \( \mathcal{L}(p,q)_{i,j} = p_{i-1,j} + q_{i,j-1} - p_{i,j} \) \( \forall i,j \).

- The operator \( \mathcal{L}^T : \mathbb{R}^{hw} \to \mathbb{R}^{(h-1)\times w} \times \mathbb{R}^{h \times (w-1)} \) which is adjoint to \( \mathcal{L} \) is given by \( \mathcal{L}^T(L) = (p,q) \), where \( p_{i,j} = L_{i+1,j} - L_{i,j} \) and \( q_{i,j} = L_{i,j+1} - L_{i,j} \).

- \( P_C \) is the orthogonal projection operator on the convex closed set \( C = \{ L : 0 \leq L \leq 1 \} \).

Equipped with these notation, we derive a dual problem of (11), and give following proposition to state the relation between the primal and dual optimal solutions.

**Proposition 1.** Let \( (p,q) \in P \) be the optimal solution of the problem

\[
\min_{(p,q) \in P} \{ H(p,q) \equiv \frac{1}{2} \left\| \mathcal{H}(d - \beta \mathcal{L}(p,q)) \right\|^2 + \left\| d - \beta \mathcal{L}(p,q) \right\|^2 + \beta \left( \text{Tr}(p^T E_1) + \text{Tr}(q^T E_2) \right) \}.
\]

where \( H_C(L) = L - P_C(L) \) for every \( L \in \mathbb{R}^{hw} \). Then the optimal solution of (11) is given by \( L = P_C(d - \beta \mathcal{L}(p,q)) \).

**Proof.** First note the following relation holds true:

\[
|x| = \max_{p \in P} \{ px : |p| \leq 1 \}. \tag{13}
\]

Hence, we can give

\[
\sum_{k} \left\| \nabla k L - E_k \right\| = \max_{(p,q) \in P} T(L,p,q), \tag{14}
\]
where,
\[
T(L, p, q) = \sum_{i=1}^{h-1} \sum_{j=1}^{w-1} [p_{i,j}(L_{i+1,j} - L_{i,j} - E_{1,j}) + q_{i,j}(L_{i,j+1} - L_{i,j} - E_{2,j})] + \sum_{i=1}^{h-1} p_{i,w}(L_{i+1,w} - L_{i,w} - E_{1,w}) + \sum_{j=1}^{w-1} p_{h,j}(L_{h,j+1} - L_{h,j} - E_{2,h}).
\]

With this notation we have
\[
T(L, p, q) = Tr(L(p) \top L) - Tr(p \top E_1) - Tr(q \top E_2).
\]

Thus the original problem (11) becomes
\[
\min_{0 \leq L \leq 1} \max_{(p,q) \in P} \left\{ \frac{1}{2} ||L - d||^2 + \beta [Tr(L(p) \top L)] - Tr(p \top E_1) - Tr(q \top E_2) \right\}.
\]

Since the objective function is convex in $L$ and concave in $p, q$, we can exchange the order of the minimum and maximum and get
\[
\max_{(p,q) \in P} \min_{0 \leq L \leq 1} \left\{ \frac{1}{2} ||L - d||^2 + \beta [Tr(L(p) \top L)] - Tr(p \top E_1) - Tr(q \top E_2) \right\}.
\]

and which can be written as
\[
\max_{(p,q) \in P} \min_{0 \leq L \leq 1} \left\{ \frac{1}{2} ||L - (d - \beta L(p,q))||^2 - ||d - \beta L(p,q)||^2 + ||d||^2 - \beta [Tr(p \top E_1) + Tr(q \top E_2)] \right\}.
\]

Thus the optimal solution of the inner minimization problem is
\[
L = P_{[0 \leq L \leq 1]}(d - \beta L(p,q)).
\]

And last, we plug the above expression for $L$ back into (19) and ignore the constant term, we obtain the dual problem is
\[
\min_{(p,q) \in P} \left\{ H(p, q) \equiv \frac{1}{2} (-||H_C(d - \beta L(p,q))||^2 + ||d - \beta L(p,q)||^2) + \beta [Tr(p \top E_1) + Tr(q \top E_2)] \right\},
\]

which is the same as (12).

\[\square\]

What’s more, given (12), we can easily have following lemma.

**Lemma 1.** The objective function $H$ of (12) is continuously differentiable and its gradient is given by
\[
\nabla H(p, q) = -\beta L \top P_C(d - \beta L(p,q)) + \beta (E_1, E_2).
\]

And let $L(H)$ be the Lipschitz constant of $\nabla H(p, q)$, then $L(H) \leq 8\beta^2$.

\[\square\]

**Proof.** Consider the function $s : \mathbb{R}^{hw} \to \mathbb{R}$ defined by
\[
s(L) = ||H_C(L)||^2.
\]

Then the dual function (12) can be written as:
\[
H(p, q) = \frac{1}{2} (-||s(d - \beta L(p,q)) + ||d - \beta L(p,q)||^2) + \beta [Tr(p \top E_1) + Tr(q \top E_2)].
\]

Obviously, $s(\cdot)$ is continuously differentiable and its gradient is given by
\[
\nabla s(L) = 2(L - P_C(L)).
\]

Therefore,
\[
\nabla H(p, q) = \frac{1}{2} \nabla \left( -||s(d - \beta L(p,q)) + ||d - \beta L(p,q)||^2) + \beta (E_1, E_2) \right)
\]

\[
= \frac{1}{2} \beta L \top \nabla s(d - \beta L(p,q)) - 2(d - \beta L(p,q)) \right) + \beta (E_1, E_2)
\]

\[
= -\beta L \top P_C(d - \beta L(p,q)) + \beta (E_1, E_2)
\]

Then for every two pairs of matrices $(p_1, q_1), (p_2, q_2)$ where $p_i \in \mathbb{R}^{(h-1) \times w}$ and $q_i \in \mathbb{R}^{w \times (w-1)}$ for $i = 1, 2$, we have
\[
||\nabla H(p_i, q_i) - \nabla H(p_2, q_2)|| \\
\leq \beta ||L \top P_C(d - \beta L(p_1, q_1)) - L \top P_C(d - \beta L(p_2, q_2))|| \\
\leq \beta^2 ||L \top P_C(d - \beta L(p_1, q_1)) - P_C(d - \beta L(p_2, q_2))|| \\
\leq \beta^2 ||L \top P_C(d - \beta L(p_1, q_1)) - (p_1, q_1) - (p_2, q_2)|| \\
= \beta^2 ||L \top P_C(d - \beta L(p_1, q_1)) - (p_1, q_1) - (p_2, q_2)||
\]

Here the above inequalities follow from the non-expansiveness property of the orthogonal projection operator and property of linear operators $L, L \top$. And from (4), we have $||\nabla H(x)|| \leq \sqrt{8} ||x||$. Therefore, implying that $||\nabla H|| \leq \sqrt{8}$ and hence $L(H) \leq 8\beta^2$.

With definition of $H(p,q)$ and $\nabla H(p,q)$, fast gradient projection (FGP) is applied on the dual problem (12). And the complexity of each iteration in FGP is $O(hw)$. Above all, our proposed Parallel Algorithm with Constrained Total Variation (PACTV) is using FGP to solve the $m + 1$ dual problems (12) in parallel. Then we concatenate the optimal $L_i^*$ ($i = 1, \ldots, m+1$) and resize them into vector form to achieve $\hat{v}_{vec}$.

Given above proposition and lemma, we can use the fast gradient projection (FGP) on dual problem (12). Fast gradient projection (FGP) is outlined in Algorithm 2. Here $P_{\mathcal{P}}(p,q)$ means projecting the matrix-pair $(p,q)$ on the set $\mathcal{P}$. And finally we achieve the optimal solution of (11). Then our recovering method ESRA is outlined in Algorithm 1.

In our implementations, we set the total iteration number of ESRA is 100 and FGP tolerance is 0.0001, and we also
Algorithm 2: FGP\((b, \beta, N, E_1, E_2)\)

**Input**: \(d \in \mathbb{R}^{h \times w}\), \(\beta = \lambda/L_s\), \(N\) is the total number of iterations, \(E_1, E_2\).

1. **Step 0**: Take \((\tilde{p}_1, \tilde{q}_1) = (p_0, q_0) = (0_{(h-1) \times w}, 0_{w \times (w-1)}), t_1 = 1\).

2. **Step k**: For \(n \geq 1\), Compute

\[
(p_k, q_k) = P_P[(\tilde{p}_k, \tilde{q}_k) - \frac{1}{8\beta^2} \nabla H(\tilde{p}_k, \tilde{q}_k)],
\]

\[
t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},
\]

\[
(\tilde{p}_{k+1}, \tilde{q}_{k+1}) = (p_k, q_k) + (\frac{t_k - 1}{t_{k+1}})(p_k - p_{k-1}, q_k - q_{k-1})
\]

**Output**: \(L^*\) An optimal solution of (11) up to a tolerance.

set \(L_s = 2L(f)\) to ensure a constant stepsize. The initial value of \(l^{vec}\) is zero. The final recovered reflected layers of (2) should be warped with \(f_i\) and enhance the intensity by 2 to be visible. Our recovering method launches a general optimization framework and can be extended to solve other reconstruction problems in [5, 6].

3. REFERENCES

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