The $L^p$ boundedness of the wave operators for matrix Schrödinger equations∗†

Ricardo Weder‡
Departamento de Física Matemática,
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas.
Universidad Nacional Autónoma de México,
Apartado Postal 20-126, Ciudad de México, 01000, México.

Abstract

We prove that the wave operators for $n \times n$ matrix Schrödinger equations on the half line, with general selfadjoint boundary condition, are bounded in the spaces $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, for slowly decaying selfadjoint matrix potentials, $V$, that satisfy $\int_0^\infty (1 + x) |V(x)| \, dx < \infty$. Moreover, assuming that $\int_0^\infty (1 + x^\gamma) |V(x)| \, dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}^+, \mathbb{C}^n)$, and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. We also prove that the wave operators for $n \times n$ matrix Schrödinger equations on the line are bounded in the spaces $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$, assuming that the perturbation consists of a point interaction at the origin and of a potential, $V$, that satisfies the condition $\int_{-\infty}^{\infty} (1 + |x|) |V(x)| \, dx < \infty$. Further, assuming that $\int_{-\infty}^{\infty} (1 + |x|) \gamma |V(x)| \, dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}, \mathbb{C}^n)$, and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$. We obtain our results for $n \times n$ matrix Schrödinger equations on the line from the results for $2n \times 2n$ matrix Schrödinger equations on the half line.

1 Introduction.

In this paper we consider the wave operators for the matrix Schrödinger equation on the half line with general selfadjoint boundary condition,

$$\begin{align*}
\frac{\partial}{\partial t} u(t,x) &= \left( \frac{\partial^2}{\partial x^2} + V(x) \right) u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^+, \\
\quad u(0,x) &= u_0(x), \quad x \in \mathbb{R}^+, \quad (1.1) \\
-B^\dagger u(t,0) + A^\dagger \frac{d}{dx} u(t,0) &= 0. \quad (1.2)
\end{align*}$$

Here $\mathbb{R}^+ := (0, +\infty)$, $u(t,x)$ is a function from $\mathbb{R} \times \mathbb{R}^+$ into $\mathbb{C}^n$, $A, B$ are constant $n \times n$ matrices, the potential $V$ is a $n \times n$ selfadjoint matrix-valued function of $x$,

$$V(x) = V^\dagger(x), \quad x \in \mathbb{R}^+. \quad (1.3)$$

The dagger designates the matrix adjoint. Let us denote by $M_n$ the set of all $n \times n$ matrices. We assume that $V$ is in the Faddeev class $L^1_1(\mathbb{R}^+, M_n)$, i.e. that it is a Lebesgue measurable $n \times n$ matrix-valued function and,

*2010 AMS Subject Classifications: 34L10; 34L25; 34L40; 47A40; 81U99.
†Research partially supported by projects PAPIIT-DGAPA UNAM IN103918 and IN 106321, and SEP-CONACYT CB 2015, 254062.
‡Fellow, Sistema Nacional de Investigadores. Electronic mail: weder@unam.mx. Home page: https://www.iimas.unam.mx/rweder/rweder.html
where by \(|V|\) we denote the matrix norm of \(V\). The more general selfadjoint boundary condition at \(x = 0\) has been extensively studied. It can be written in many equivalent ways. See, [7] and [9], [23], [25], and [26]. For other formulations of the general selfadjoint boundary condition see [37]. In this paper we use the parametrization of the boundary condition given in [7], and Section 3.4 of Chapter 3 of [9]. We write the boundary condition as in (1.2), with constant matrices \(A\) and \(B\) satisfying,

\[
B^\dagger A = A^\dagger B, \tag{1.5}
\]

and

\[
A^\dagger A + B^\dagger B > 0. \tag{1.6}
\]

We prove that the wave operators for the \(n \times n\) matrix Schrödinger equation on the half line (1.1) with the general selfadjoint boundary condition (1.2), (1.5), and (1.6), are bounded in the spaces \(L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty\). For this purpose, we suppose that the potential satisfies (1.3) and (1.4). Moreover, assuming that \(\int_0^\infty (1 + x^\gamma)|V(x)|\,dx < \infty, \gamma > \frac{5}{2}\), and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in \(L^1(\mathbb{R}^+, \mathbb{C}^n)\), and in \(L^\infty(\mathbb{R}^+, \mathbb{C}^n)\). We also prove that the wave operators for the \(n \times n\) Schrödinger equation on the line, with a point interaction at the origin and a potential, are bounded in \(L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty\). We assume that the potential, that we denote by \(V\), is selfadjoint, i.e., \(V(x) = V(x)^\dagger\), and

\[
\int_{\mathbb{R}} (1 + |x|^\gamma)|V(x)|\,dx < \infty. \tag{1.7}
\]

Further, assuming that \(\int_{-\infty}^\infty (1 + |x|^\gamma)|V(x)|\,dx < \infty, \gamma > \frac{5}{2}\), and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in \(L^1(\mathbb{R}, \mathbb{C}^n)\), and in \(L^\infty(\mathbb{R}, \mathbb{C}^n)\). We obtain the boundedness of the wave operators on the line for a \(n \times n\) matrix Schrödinger equation from the boundedness of the wave operators for a \(2n \times 2n\) matrix Schrödinger equation on the half line.

In the scalar case there are several results on the boundedness of the wave operators on the line. Recall that in the scalar case the potential is generic if the zero energy Jost solutions from the left and from the right are linearly independent, and that it is exceptional if the zero energy Jost solutions from the left and from the right are linearly dependent. In the exceptional case the stationary Schrödinger equation on the line (5.84) with zero energy, \(k^2 = 0\), has a bounded solution, that is called a zero-energy resonance, or a half-bound state. In [41] it was proven that the wave operators are bounded in \(L^p(\mathbb{R}), 1 < p < \infty\), under the assumption,

\[
\int_{\mathbb{R}} (1 + |x|^\gamma)|V(x)|\,dx < \infty, \tag{1.8}
\]

with \(\gamma > 3/2\) in the generic case and \(\gamma > 5/2\) in the exceptional case. Furthermore, in [41] it was proven that in the exceptional case if the Jost solution from the left at zero energy tends to one as \(x \to -\infty\), then, the wave operators are bounded in \(L^1(\mathbb{R})\) and in \(L^\infty(\mathbb{R})\). The paper [41] used a constructive proof that allowed to obtain a detailed low-energy expansion, but that was somehow more demanding concerning the decay of the potential. In [20] the boundedness of the wave operators in \(L^p(\mathbb{R}), 1 < p < \infty\), was proven assuming that (1.8) holds with \(\gamma = 3\) in the generic case and \(\gamma = 4\) in the exceptional case, and that moreover, \(\frac{d}{dx}V(x)\) satisfies (1.8) with \(\gamma = 2\), both in the generic and the exceptional cases. The boundedness of the wave operators in \(L^p(\mathbb{R}), 1 < p < \infty\), was proven in [15] assuming (1.8) with \(\gamma = 1\), in the generic case and with \(\gamma = 2\) in the exceptional case. Furthermore, in [16] the boundedness of the wave operators in \(L^p(\mathbb{R}), 1 < p < \infty\), was proven for a potential that is the sum of a regular potential that satisfies (1.8) with \(\gamma > 3/2\) and of a singular potential that is a sum of Dirac delta functions. In [14] the boundedness of the
wave operators was proven for the discrete Schrödinger equation on the line. There is a very extensive literature on the $L^p$-boundedness of the wave operators and on the related problem of dispersive estimates. For surveys see [19] and [38], and [43] for recent results. In these papers also the results in the multidimensional case are discussed.

The matrix Schrödinger equations find their origin at the very beginning of quantum mechanics. They are important in the description of particles with internal structure like spin and isospin, in atoms, molecules and in nuclear physics, and also in the study systems of particles. A well known example is the Pauli equation, that is the equation for half-spin particles. For further applications and references see [2]-[5], [13], [17], [27], [33] and [35].

Since a number of years there is a renewal of the interest in matrix Schrödinger equations due to the importance of these equations for quantum graphs. For example, see [10]-[12], [22], [25, 26], and [28]-[32], as well as the references quoted there. The matrix Schrödinger equation with a diagonal potential corresponds to a star graph. Such a quantum graph describes the dynamics of many connected very thin quantum wires that form a star-graph, that is, a graph with only one vertex and a finite number of edges of infinite length. This situation appears, for example, in the design of elementary gates in quantum computing, in quantum wires, and in nanotubes for microscopic electronic devices. In these cases strings of atoms can form a star-shaped graph. The analysis of the most general boundary condition at the vertex is important in the applications to problems in physics. A relevant example is the Kirchoff boundary condition. A quantum graph is an idealization of wires with a small cross section that meet at vertices. The graph is connected very thin quantum wires that form a star-graph, that is, a graph with

provided that the strong limits exist. The operator $H$ is the perturbed Hamiltonian, and the operator $H_0$ is the unperturbed Hamiltonian. The wave operators $W_{\pm}(H, H_0)$ are said to be complete if their range is equal to $\mathcal{H}_{ac}(H)$. Moreover, for any pair $H, H_0$ of self-adjoint operators in a Hilbert space the wave operators are defined as follows,

$$W_{\pm}(H, H_0) := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} P_{ac}(H_0),$$

provided that the strong limits exist. The operator $H$ is the perturbed Hamiltonian, and the operator $H_0$ is the unperturbed Hamiltonian. The wave operators $W_{\pm}(H, H_0)$ are said to be complete if their range is equal to $\mathcal{H}_{ac}(H)$. In the theory of scattering, the scattering solutions to the interacting Schrödinger equation

$$i \frac{\partial}{\partial t} u(t) = H u(t), \quad u(0) = \varphi,$$  \hspace{1cm} (1.9)

are defined as $e^{-itH} \varphi$, with $\varphi \in \mathcal{H}_{ac}(H)$. It is a purpose of scattering theory to compare the behavior for large times of the scattering solutions $e^{-itH} \varphi$ with the scattering solutions for free Schrödinger equation

$$i \frac{\partial}{\partial t} v(t) = H_0 v(t), \quad v(0) = \psi,$$  \hspace{1cm} (1.10)

with Hamiltonian, $H_0$, that are given by $e^{-itH_0} \psi$, with $\psi \in \mathcal{H}_{ac}(H_0)$. If the wave operators exist and are complete, all the scattering solutions $e^{-itH} \varphi$, to the interacting Schrödinger equation behave for large positive and negative times as scattering solutions for the free Schrödinger equation,

$$\lim_{t \to \pm \infty} \| e^{-itH} \varphi - e^{-itH_0} W_{\pm}(H, H_0)^* \varphi \| = 0, \quad \varphi \in \mathcal{H}_{ac}.$$

Furthermore, the wave operators fulfill the important intertwining relations,

$$f(H) P_{ac}(H) = W_{\pm}(H, H_0) f(H_0) P_{ac}(H_0) W_{\pm}(H, H_0)^*,$$  \hspace{1cm} (1.11)
where $f$ is a Borel function. For these results see [36]. The intertwining relations allow us to obtain important properties of $f(H) P_{ac}(H)$ from those of $f(H_0) P_{ac}(H_0)$. Let us explain. Assume that the wave operators $W_{\pm}(H, H_0)$ are bounded in a Banach space, $Y$, and the adjoints $W_{\pm}(H, H_0)^*$ are bounded in a Banach space, $X$. Then, if $f(H_0) P_{ac}(H_0)$ is bounded from $X$ into $Y$, it follows from (1.11) that also $f(H) P_{ac}(H)$ is bounded between the same spaces, and furthermore,

$$\|f(H) P_{ac}(H)\|_{B(X,Y)} \leq C \|f(H_0) P_{ac}(H_0)\|_{B(X,Y)},$$

for some constant $C$ and where $B(X,Y)$ denotes the Banach space of bounded operators from $X$ into $Y$. In the applications $H_0$ is often a constant coefficients operator and $f(H_0)$ is a Fourier multiplier. It is usually a simple matter to obtain important dispersive estimates, like the $L^p - L^{p'}$, estimates, $\frac{1}{p} + \frac{1}{p'} = 1, 1 \leq p \leq \infty$, and the Strichartz estimates for the free Schrödinger equation (1.10) with Hamiltonian $H_0$, and then (1.12) gives us these estimates for the interacting Schrödinger equation (1.9) with Hamiltonian $H$. These dispersive estimates play a crucial role in the study of initial value problems and in the scattering theory of nonlinear dispersive equations, like the nonlinear Schrödinger equation, and also in other problems, like the stability of soliton solutions. See [19] and [38].

The wave operators are singular integral operators in the spectral representation of the unperturbed operator. On this point see [18] and Section 1 in Chapter 4 of [42] where this question is discussed. As singular integral operators are bounded in $L^p$ spaces, the wave operators are bounded in $L^p$ spaces in the spectral representation of the unperturbed operator. I thank D. R. Yafaev for calling this fact to my attention. Note that in this paper we consider the related, but different, problem of the boundedness of the wave operators in $L^p$ spaces in the configuration representation.

The paper is organized as follows. In Section 2 we introduce the notation that we use. In Section 3 we state our results on the boundedness of the wave operators on the half line. In Section 4 we state our results on the boundedness of the wave operators on the line. In Section 5 we mention the results on the scattering theory of matrix Schrödinger equations that we need and we give the proofs of our theorems.

## 2 Notation

We denote by $\mathbb{R}^+$ the positive half line, $(0, \infty)$, and we designate by $\mathbb{C}$ the complex numbers. For a vector $Y \in \mathbb{C}^n$ we denote by $Y^T$ its transpose. By $\langle \cdot, \cdot \rangle$ we designate the scalar product in $\mathbb{C}^n$. We introduce the following convenient notation. For any vector $Y = (y_1, y_2, \ldots, y_{2n})^T \in \mathbb{C}^{2n}$ we denote $Y_+ := (y_1, y_2, \ldots, y_n)^T \in \mathbb{C}^n$ the vector with the first $n$ components of $Y$, and $Y_- := (y_{n+1}, y_{n+2}, \ldots, y_{2n})^T \in \mathbb{C}^n$ the vector with the last $n$ components of $Y$. Further, we use the notation $Y = (Y_+, Y_-)^T$.

We denote the entries of a $n \times m$ matrix $M$ by $M_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m$. By $0_n$ and $I_n$, $n = 1, 2, \ldots$, we designate the $n \times n$ zero and identity matrices, respectively. By $|M|$ we denote the norm of a matrix $M$. We designate by $L^p(U, \mathbb{C}^n)$, $1 \leq p \leq \infty$, where $U = \mathbb{R}^+$ or $U = \mathbb{R}$, the Lebesgue spaces of $\mathbb{C}^n$ valued functions defined on $U$. Let us denote by $C^\infty_0(U, \mathbb{C}^n)$ the space of all infinitely differentiable functions defined on $U$ and that have compact support. We designate by $L^p(U, M_n)$, $1 \leq p \leq \infty$, the Lebesgue space of $n \times n$ matrix valued functions defined on $U$. Further, we designate by $L^1_\gamma(U, M_n)$, $\gamma > 0$, the Lebesgue space of $n \times n$ matrix valued functions defined on $U$ such that

$$\int_U (1 + |x|^\gamma) |V(x)| \, dx < \infty.$$ 

For an integer $m \geq 1$, $H^{(m)}(U, \mathbb{C}^n)$, where $U = \mathbb{R}^+$ or $U = \mathbb{R}$, is the standard Sobolev space of $\mathbb{C}^n$ valued functions (see [1] for the definition and the properties of these spaces). By $H^{(m,0)}(\mathbb{R}^+, \mathbb{C}^n), m \geq 1$, we denote the closure of
$C_0^\infty (\mathbb{R}^+, \mathbb{C}^n)$ in the space $H^m (\mathbb{R}^+, \mathbb{C}^n)$. Note that the functions in $H^{(m,0)} (\mathbb{R}^+, \mathbb{C}^n)$, as well as their derivatives of order up to $m - 1$, are zero at $x = 0$.

The Fourier transform, and the inverse Fourier transform are designated by,

$$\mathcal{F}f(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) \, dx, \quad \mathcal{F}^{-1}f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(k) \, dk.$$  

For any set $O \subset \mathbb{R}$, we denote by $\chi_O$ the characteristic function of $O$.

For any operator $G$ in a Banach space $X$ we denote by $D[G]$ the domain of $G$. Further, for a densely defined operator $G$ in a Banach space we denote by $G^\dagger$ its adjoint. For any selfadjoint operator, $H$, in a Hilbert space and for any Borel set $O$ we designate by $E(O; H)$ the spectral projector of $H$ for $O$.

We designate by $\mathcal{E}_{\text{even}}$ the extension operator from $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, to even functions in $L^p(\mathbb{R}, \mathbb{C}^n)$ as follows

$$(\mathcal{E}_{\text{even}}Y)(x) := \begin{cases} Y(x), & x > 0, \\ Y(-x), & x \leq 0. \end{cases}$$

Clearly, $\mathcal{E}_{\text{even}}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^n)$ into $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$.

Moreover, we denote by $\mathcal{E}_{\text{odd}}$ the extension operator from $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, to odd functions in $L^p(\mathbb{R}, \mathbb{C}^n)$ in the following way

$$(\mathcal{E}_{\text{odd}}Y)(x) := \begin{cases} Y(x), & x > 0, \\ -Y(-x), & x \leq 0. \end{cases}$$

We have that $\mathcal{E}_{\text{odd}}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^n)$, into $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$.

We denote by $\mathcal{R}$ the restriction operator from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, given by,

$$(\mathcal{R}Y)(x) := Y(x), \quad x > 0.$$  

We have that $\mathcal{R}$ is bounded from $L^p(\mathbb{R}, \mathbb{C}^n)$, into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$.

For any integrable $n \times n$ matrix valued function, $G(x), x \in \mathbb{R}$, we denote by $Q(G)$ the operator of convolution by $G(x)$,

$$(Q(G)Y)(x) := \int_{\mathbb{R}} G(x - y) Y(y) \, dy = \int_{\mathbb{R}} G(y) Y(x - y) \, dy.$$  

Since $G$ is integrable, the operator $Q(G)$ is bounded in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$. For any $n \times n$ matrix valued measurable function $K(x, y)$ defined for $x, y \in \mathbb{R}^+$, we denote by $K(K)$ the operator,

$$K(K)Y(x) := \int_{\mathbb{R}^+} K(x, y) Y(y) \, dy.$$  

The operator $K(K)$ is bounded in $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, provided that the following two conditions are satisfied,

$$\sup_{x \in \mathbb{R}^+} \int_{\mathbb{R}^+} |K(x, y)| \, dy < \infty, \quad \sup_{y \in \mathbb{R}^+} \int_{\mathbb{R}^+} |K(x, y)| \, dx < \infty. \quad (2.1)$$

The Hilbert transform, $\mathcal{H}$, is defined as follows,

$$(\mathcal{H}Y)(x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{Y(y)}{x - y} \, dy,$$

where PV means the principal value of the integral. As is well known [39, 40], the Hilbert transform is a bounded operator in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$. 

5
3 The wave operators on the half line

To define the wave operators we take as unperturbed Hamiltonian $H_0$ the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal differential operator $-\frac{d^2}{dx^2}$ with the Neumann boundary condition, $\frac{d}{dx}Y(0) = 0$, see Section 5 below and Sections 3.3 and 3.5 of Chapter three of [9]. This choice is motivated by the theory of quantum graphs [25, 26]. Note that the spectrum of $H_0$ is absolutely continuous and that it coincides with $[0, \infty)$. The perturbed Hamiltonian, $H$, is the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal differential operator $-\frac{d^2}{dx^2} + V(x)$ with the boundary condition

$$-B^1Y(0) + A^1\frac{d}{dx}Y(0) = 0,$$

where the constant matrices $A, B$ satisfy (1.5) and (1.6), and the potential $V$ fulfills (1.3) and (5.2). For the definition of $H$ see Section 5 below and Sections 3.3 and 3.5 of Chapter three of [9].

The wave operators, $W_\pm(H, H_0)$, are defined as follows,

$$W_\pm(H, H_0) := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0},$$

(3.2)

since $P_{ac}(H_0) = I$. It is proven in Section 4.4 of Chapter four of [9] that the wave operators $W_\pm(H, H_0)$ exist, and are complete.

Our result in the case of $L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$, is the following theorem.

**THEOREM 3.1.** Suppose that $V$ fulfills (1.3) and (1.4) and that the constant matrices $A, B$ satisfy (1.5), and (1.6). Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^n)$ we have,

$$W_\pm(H, H_0)Y = \sum_{j=1}^{3} W_\pm^{(j)}Y,$$

(3.3)

where,

$$W_\pm^{(1)}Y := (I + K(K))R\left(\frac{-i}{2} \mathcal{H}\mathcal{E}_{even}Y + \frac{1}{2} \mathcal{E}_{even}Y\right),$$

(3.4)

$$W_\pm^{(2)}Y := (I + K(K))R\left(\frac{+i}{2} (\mathcal{H}S\mathcal{E}_{even}Y) + \frac{1}{2} S\mathcal{E}_{even}Y\right),$$

(3.5)

$$W_\pm^{(3)}Y := (I + K(K))R\left(\frac{+i}{2} (\mathcal{H}Q\mathcal{E}_{even}Y) + \frac{1}{2} Q\mathcal{E}_{even}Y\right).$$

(3.6)

Furthermore, the wave operators $W_\pm(H, H_0)$ restricted to $L^2(\mathbb{R}^+, \mathbb{C}^n) \cap L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$, and equations (3.3)-(3.6) hold for all $Y \in L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$. Moreover, the adjoints of the wave operators $W_\pm(H, H_0)^\dagger$, restricted to $L^2(\mathbb{R}^+, \mathbb{C}^n) \cap L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$, extend uniquely to bounded operators on $L^p(\mathbb{R}^+, \mathbb{C}^n), 1 < p < \infty$. The $n \times n$ matrix valued function $K(x, y), x, y \in \mathbb{R}^+$ is defined in (5.12). Moreover, the quantity $S_\infty$ is defined in (5.16), and the $n \times n$ matrix valued function $F_\pm(x), x \in \mathbb{R}$ is defined in (5.17).

Our result in the case of $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$ is stated in the next theorem. We first prepare a convenient notation, where the scattering matrix $S(k)$ is defined in (5.15).

$$P_+(x) := \frac{1}{\sqrt{2\pi}} \left(\mathcal{F}}\chi_{\mathbb{R}^+}(k) (S(-k) - S_\infty)\right)(x), P_-(x) := \frac{1}{\sqrt{2\pi}} \left(\mathcal{F}^{-1} \chi_{\mathbb{R}^+}(k) (S(k) - S_\infty)\right)(x).$$

(3.7)
Theorem 3.2. Suppose that $V$ fulfills (1.3), that $V \in L^1_\gamma(\mathbb{R}^+, M_n)$, $\gamma > \frac{1}{2}$, that the constant matrices $A, B$ satisfy (1.5), and (1.6), and that $S(0) = S_\infty = I_n$. Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^n)$ we have,

$$W_\pm(H, H_0)Y = Y + K(K)Y + RQ(P_\pm)E_{even}Y + K(K)RQ(P_\pm)E_{even}Y.$$  

Furthermore, the wave operators $W_\pm(H, H_0)$ and $W_\pm(H, H_0)^\dagger$ restricted to $L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, respectively to $L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R})$, extend to bounded operators on $L^1(\mathbb{R}^+)$ and to bounded operators on $L^\infty(\mathbb{R})$. The $n \times n$ matrix valued function $K(x, y), x, y \in \mathbb{R}^+$ is defined in (5.12). Moreover, the scattering matrix, $S(k), k \in \mathbb{R}$, is defined in (5.15), the quantity $S_\infty$ is defined in (5.16), and the $n \times n$ matrix valued functions $P_\pm(x)$ are defined in (3.7).

In Remark 5.8 we prove by means of a counter example that the condition $S(0) = S_\infty = I_n$ is necessary for the boundedness of the wave operators on $L^1(\mathbb{R}^+)$ and on $L^\infty(\mathbb{R})$.

Remark 3.3. It follows from equation (3.10.37) in page 197 of [9] that $S_\infty = I_n$ if and only if there are no Dirichlet boundary conditions in the diagonal representation of the boundary matrices given in (5.3), (5.4), and (5.5). Further, by Theorem 3.8.13 in page 137 and Theorem 3.8.14 in pages 138-139 of [9], $S(0) = I_n$ if and only if the geometric multiplicity, $\mu$, of the eigenvalue zero of the zero energy Jost matrix, $J(0)$, (see (5.11)) is equal to $n$. Moreover, by Remark 3.8.10 in pages 129-130 of [9] the geometric multiplicity of the eigenvalue zero of $J(0)$ is equal to $n$ if and only if there are $n$ linearly independent bounded solutions to the Schrödinger equation (5.1)with zero energy, $k^2 = 0$, that satisfy the boundary condition (3.1). This corresponds to the purely exceptional case where there are $n$ linearly independent half-bound states or zero-energy resonances. We provide below a simple example of this situation, with a non-trivial potential. Consider the scalar case, $n = 1$, with the potential,

$$V(x) = \begin{cases} 
0, & x > 1, \\
1, & 0 < x < 1.
\end{cases}$$

The Jost solution (see (5.10)) is computed in Example 6.4.1 in pages 536-538 of [9]. It is given by,

$$f(k, x) = \begin{cases} 
\frac{1}{2} \left( 1 + \frac{k}{\sqrt{1 - \theta^2}} \right) e^{ikx} + \frac{1}{2} \left( 1 - \frac{k}{\sqrt{1 - \theta^2}} \right) e^{-ikx}, & 0 \leq x \leq 1,
\end{cases}$$

We take the boundary matrices, $A = -\sin \theta, B = \cos \theta$, with $\theta = \arctan \coth 1$. The boundary condition is $\cos \theta Y(0) + \sin \theta Y'(0) = 0$. The Jost function is given by, $J(k) = f(k, 0) \cos \theta + f'(k, 0) \sin \theta$. We have, $J(0) = 0$. Then $S(0) = 1$, and as we have the Robin boundary condition, $S_\infty = 1$. Of course, these results can be obtained by explicit computation.

4 The wave operators on the line

We obtain our results on the line proving that a $2n \times 2n$ matrix Schrödinger equation on the half line is unitarily equivalent to a $n \times n$ matrix Schrödinger equation on the line with a point interaction at $x = 0$. For this purpose we follow Section 2.4 in Chapter 2 of [9]. Let us denote by $U$ the unitary operator from $L^2(\mathbb{R}^+, \mathbb{C}^{2n})$ onto $L^2(\mathbb{R}, \mathbb{C}^n)$, defined as follows,

$$Y(x) = UZ(x) := \begin{pmatrix} Z_+(x), & x \geq 0, \\
Z_-(x), & x < 0,
\end{pmatrix}$$

where $Z = (Z_+, Z_-)^T$, with $Z_+, Z_- \in L^2(\mathbb{R}^+, \mathbb{C}^n)$. Let us take as potential the diagonal matrix

$$V(x) := \begin{pmatrix} V_+(x) & 0_n \\
0_n & V_-(x)
\end{pmatrix},$$
where \( V_+, V_- \) are selfadjoint \( n \times n \) matrix-valued functions that belong to \( L^1_1(\mathbb{R}^+, M_n) \). Under the action of the unitary transformation \( U \) the Hamiltonian in the half line, \( H \), is unitarily transformed into the Hamiltonian on the line, \( H_\mathbb{R} \), as follows,

\[
H_\mathbb{R} := U H U^\dagger, \quad D[H_\mathbb{R}] := \{ Y \in L^2(\mathbb{R}, \mathbb{C}^n) : U^\dagger Y \in D[H] \}.
\]

The operator \( H_\mathbb{R} \) is a selfadjoint realization in \( L^2(\mathbb{R}, \mathbb{C}^n) \) of the formal differential operator \(-\frac{d^2}{dx^2} + V(x)\) where the selfadjoint \( n \times n \) matrix valued potential, \( V \), is given by,

\[
V(x) = \begin{cases} V_+(x), & x \geq 0, \\ V_-(x), & x < 0. \end{cases}
\]

Note that \( V \in L^1_1(\mathbb{R}, M_n) \). The boundary condition (3.1) satisfied by the functions in the domain of \( H_\mathbb{R} \) implies that the functions in the domain of \( H_\mathbb{R} \) fulfill a transmission condition at \( x = 0 \). To compute this transmission condition it is convenient to write the matrices \( A, B \) in (3.1) in the following way,

\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},
\]

where \( A_j, B_j, j = 1, 2 \), are \( n \times 2n \) matrices. Hence, (3.1) implies that the functions in the domain of \( H_\mathbb{R} \) satisfy the following transmission condition at \( x = 0 \),

\[
-B_1^Y(0^+) - B_2^Y(0^-) + A_1^Y \frac{d}{dx} Y(0^+) - A_2^Y \frac{d}{dx} Y(0^-) = 0.
\]

Remark that \( u(t,x) \) is a solution of the problem (1.1), (1.2) if and only if \( v(t,x) := U u(t,x) \) is a solution of the following \( n \times n \) matrix equation on the line,

\[
\begin{cases} i \frac{\partial}{\partial t} v(t,x) \left( -\frac{\partial^2}{\partial x^2} + V(x) \right) v(t,x), & t \in \mathbb{R}, x \in \mathbb{R}, \\ v(0,x) = v_0(x) := U u_0(x), & x \in \mathbb{R}, \\ -B_1^v(t,0^+) - B_2^v(t,0^-) + A_1^v \frac{\partial}{\partial x} v(t,0^+) - A_2^v \frac{\partial}{\partial x} v(t,0^-) = 0. \end{cases}
\]

Below we give an example. Let \( A, B \) be the following matrices,

\[
A = \begin{bmatrix} 0_n & I_n \\ 0_n & I_n \end{bmatrix}, \quad B = \begin{bmatrix} -I_n & \Lambda \\ I_n & 0_n \end{bmatrix},
\]

where \( \Lambda \) is a selfadjoint \( n \times n \) matrix. It is easy to check that these matrices satisfy (1.5, 1.6). The transmission condition in (4.5) is given by,

\[
v(t,0^+) = v(t,0^-) = v(t,0), \quad \frac{\partial}{\partial x} v(t,0^+) - \frac{\partial}{\partial x} v(t,0^-) = \Lambda v(t,0).
\]

This transmission condition is a Dirac-delta point interaction at \( x = 0 \) with coupling matrix \( \Lambda \). In the particular case \( \Lambda = 0 \), the functions \( v(t,x) \) and \( \frac{\partial}{\partial x} v(t,x) \) are continuous at \( x = 0 \) and we have the matrix Schrödinger equation on the line without a point interaction at \( x = 0 \).

Let us denote by \( H_{0,\mathbb{R}} \) the Hamiltonian (4.2) with the potential \( V \) identically zero and with the boundary condition given by the matrices (4.6) with \( \Lambda = 0 \). Note that \( H_{0,\mathbb{R}} \) is the standard selfadjoint realization of the formal differential operator \(-\frac{d^2}{dx^2}\) with domain, \( D[H_{0,\mathbb{R}}] := H^{(2)}(\mathbb{R}, \mathbb{C}^n) \). In particular, \( H_{0,\mathbb{R}} \) is absolutely continuous and its spectrum consists of \([0, \infty)\). We define the wave operators on the line as follows,

\[
W_\pm(H_\mathbb{R}, H_{0,\mathbb{R}}) := \lim_{t \to \pm \infty} e^{itH_\mathbb{R}} e^{-itH_{0,\mathbb{R}}}.
\]

Using Theorem 3.1 and the unitary transformation (4.2) we prove the following theorem, on the boundedness of the wave operators on \( L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty \).
THEOREM 4.1. Let $H_{\mathbb{R}}$ be the Hamiltonian (4.2), with the transmission condition (4.4), and where $V(x), x \in \mathbb{R}$, is a $n \times n$ selfadjoint matrix-valued function, i.e., $\mathcal{V}(x) = \mathcal{V}^\dagger(x)$ and, moreover, $V$ satisfies (1.7). Then, the wave operators $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})$ exist and are complete. Moreover, $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})$, and the adjoint wave operators $W(H_{\mathbb{R}},H_{0,\mathbb{R}})^\dagger$, restricted to $L^2(\mathbb{R},\mathbb{C}^n) \cap L^p(\mathbb{R},\mathbb{C}^n), 1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R},\mathbb{C}^n), 1 < p < \infty$.

Theorem 4.1 generalizes the results obtained in [41, 20, 15, 16] to the case of general point interactions at $x = 0$, and to potentials that satisfy (1.8) with $\gamma = 1$.

Below we state our theorem on the boundedness of the wave operators on the line in $L^1(\mathbb{R},\mathbb{C}^n)$, and in $L^\infty(\mathbb{R},\mathbb{C}^n)$.

THEOREM 4.2. Let $H_{\mathbb{R}}$ be the Hamiltonian (4.2), with the transmission condition (4.4), and where $V(x), x \in \mathbb{R}$, is a $n \times n$ selfadjoint matrix-valued function, i.e., $\mathcal{V}(x) = \mathcal{V}^\dagger(x)$, and moreover, $V \in L^1_{\nu}(\mathbb{R},M_n), \gamma > \frac{\pi}{2}$. Assume that $S_{\mathbb{R}}(0) = S_{\mathbb{R},\infty} = I_{2n}$. Then, the wave operators $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})$ and $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})^\dagger$ restricted to $L^2(\mathbb{R},\mathbb{C}^n) \cap L^1(\mathbb{R},\mathbb{C}^n)$, respectively to $L^2(\mathbb{R},\mathbb{C}^n) \cap L^\infty(\mathbb{R},\mathbb{C}^n)$, extend to bounded operators on $L^1(\mathbb{R},\mathbb{C}^n)$ and to bounded operators on $L^\infty(\mathbb{R},\mathbb{C}^n)$. The scattering matrix on the line, $S_{\mathbb{R}}(k), k \in \mathbb{R}$, is defined in (5.100) and the quantity $S_{\mathbb{R},\infty}$ in (5.102).

REMARK 4.3. In the case where the matrices $A, B$ are equal to the matrices in (4.6) and $\Lambda = 0$, there is no point interaction at $x = 0$. The scattering theory in this situation has been studied in [6], and the references quoted there. In this case $S_{\mathbb{R}}(0) = S_{\mathbb{R},\infty} = I_{2n}$ in the purely exceptional case where there are $n$ linearly independent bounded solutions to the Schrödinger equation (5.84) with zero energy, $k^2 = 0$ (in this case we say that there are $n$ linearly independent zero-energy resonances, or half-bound states), and if moreover, the zero-energy Jost solution from the left, $f_I(0,x)$, satisfies, $\lim_{x \to -\infty} f_I(0,x) = I_n$. For the definition of $f_I(k,x), k \in \mathbb{R}$, see (5.85).

As mentioned above, in Theorem 1.1 of [41] we proved that in the scalar case, $n = 1$, and without point interactions, the wave operators $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})$ and $W_\pm(H_{\mathbb{R}},H_{0,\mathbb{R}})^\dagger$ extend to bounded operators on $L^1(\mathbb{R},\mathbb{C})$ and to bounded operators on $L^\infty(\mathbb{R},\mathbb{C})$ in the exceptional case, and assuming that $\lim_{x \to -\infty} f_I(0,x) = 1$, for potentials that satisfy (1.8) with $\gamma > \frac{\pi}{2}$. Theorem 4.2 generalizes this result of [41] to the case where there is a general point interaction.

5 Scattering theory and the $L^p$-boundedness of the wave operators

5.1 Scattering theory for the matrix Schrödinger equation on the half line.

We study the following the stationary matrix Schrödinger equation on the half line

$$-rac{d^2}{dx^2}Y(x) + V(x)Y(x) = k^2Y(x), \ x \in \mathbb{R}^+.$$  \ (5.1)

In this equation $k^2$ is the complex-valued spectral parameter, the $n \times n$ matrix valued potential $V(x)$ satisfies (1.3) and moreover,

$$V \in L^1(\mathbb{R}^+,M_n).$$  \ (5.2)

The solution $Y$ that appears in (5.1) is either a column vector with $n$ components, or a $n \times n$ matrix-valued function. As we already mentioned, the general selfadjoint boundary condition at $x = 0$ can be expressed in terms of two constant $n \times n$ matrices $A$ and $B$ as in (3.1), where the matrices $A$ and $B$ fulfill (1.5), (1.6).

Actually, there is a simpler equivalent form of the boundary condition (3.1). In fact, in [8], and in Section 3.4 of Chapter three of [9], it is given the explicit steps to go from any pair of matrices $A$ and $B$ appearing in the selfadjoint boundary condition (3.1), and that satisfy (1.5), and (1.6) to a pair $\tilde{A}$ and $\tilde{B}$, given by

$$\tilde{A} = - \text{diag}\{\sin \theta_1, ..., \sin \theta_n\}, \ \tilde{B} = \text{diag}\{\cos \theta_1, ..., \cos \theta_n\},$$  \ (5.3)
Further, there are \( n \) unitarily equivalent. The case \( \theta_j = \pi \) corresponds to the Dirichlet boundary condition and the case \( \theta_j = \pi/2 \) corresponds to the Neumann boundary condition. In the general case, there are \( n_N \leq n \) values with \( \theta_j = \pi/2 \) and \( n_D \leq n \) values with \( \theta_j = \pi \). Further, there are \( n_M \) remaining values, where \( n_M = n - n_N - n_D \) such that those \( \theta_j \)-values lie in the interval \( (0, \pi/2) \) or \( (\pi/2, \pi) \). It is proven in [8], and in Section 3.4 of Chapter three of [9], that for any pair of matrices \((A, B)\) that satisfy (1.5, 1.6) there is a pair of matrices \((\tilde{A}, \tilde{B})\) as in (5.3), a unitary matrix \( M \) and two invertible matrices \( T_1, T_2 \) such

\[
A = M \tilde{A} T_1 M^\dagger T_2, \quad B = M \tilde{B} T_1 M^\dagger T_2.
\]

As we will see, the Hamiltonians with the boundary condition given by matrices \( A, B \) and with the matrices \( \tilde{A}, \tilde{B} \), are unitarily equivalent.

We define the matrix-valued Jost solution \( f \) as in equations (5.3). We denote

\[
\Theta := \text{diag} \{ \cot \theta_1, \ldots, \cot \theta_n \},
\]

where \( \cot \theta_j = 0 \), if \( \theta_j = \pi/2 \), or \( \theta_j = \pi \), and \( \cot \theta_j = \cot \theta_j \), if \( \theta_j \neq \pi/2, \pi \). Suppose that the potential \( V \) satisfies (1.3) and (5.2). The following quadratic form is closed, symmetric and bounded below,

\[
\begin{align*}
&h(Y, Z) := \left( \frac{d}{dx} Y, \frac{d}{dx} Z \right)_{L^2(\mathbb{R}^+, \mathbb{C}^n)} - \langle M \Theta M^\dagger Y(0), Z(0) \rangle + \langle VY, Z \rangle_{L^2(\mathbb{R}^+, \mathbb{C}^n)}, \\
&Q(h) := \mathcal{H}^{(A, B)}(\mathbb{R}^+, \mathbb{C}^n),
\end{align*}
\]

where by \( Q(h) \) we denote the domain of \( h \) and,

\[
\mathcal{H}^{(A, B)}(\mathbb{R}^+, \mathbb{C}^n) := M \tilde{H}^{(1)}(\mathbb{R}^+, \mathbb{C}^n) \subset \mathcal{H}^{(1)}(\mathbb{R}^+, \mathbb{C}^n). 
\]

We denote by \( H_{A,B,V} \) the selfadjoint bounded below operator associated to \( h \) [24]. The operator \( H_{A,B,V} \) is the selfadjoint realization of \(-\frac{d^2}{dx^2} + V(x)\) with the selfadjoint boundary condition (3.1). When there is no possibility of misunderstanding we will use the notation \( H \), i.e., \( H \equiv H_{A,B,V} \). It is proven in Section 3.6 of Chapter three of [9] that,

\[
H_{A,B,V} = MH_{\tilde{A}, \tilde{B}, M^\dagger V} M^\dagger. 
\]

In the next proposition we introduce the Jost solution given in [5]. See also Sections 3.1 and 3.2 of [9].

**PROPOSITION 5.1.** Suppose that the potential \( V \) satisfies (5.2). For each fixed \( k \in \mathbb{C}^+ \setminus \{0\} \) there exists a unique \( n \times n \) matrix-valued Jost solution \( f(k,x) \) to equation (5.1) satisfying the asymptotic condition

\[
f(k,x) = e^{ikx} (I + o(1)), \quad x \to +\infty.
\]

Moreover, for any fixed \( x \in [0, \infty) \), \( f(k,x) \) is analytic in \( k \in \mathbb{C}^+ \) and continuous in \( k \in \mathbb{C}^+ \setminus \{0\} \). If, moreover, \( V \) satisfies (1.4) the Jost solution also exists at \( k = 0 \), and for each fixed \( x \in [0, \infty) \), \( f(k,x) \) is continuous in \( k \in \mathbb{C}^+ \). Furthermore if (1.4) holds, for \( k \in \mathbb{C}^+ \setminus \{0\} \), the \( o(1) \) in (5.10) can be replaced by \( o \left( \frac{1}{k} \right) \).
Using the Jost solution and the boundary matrices $A$ and $B$ satisfying (1.5)-(1.6), we construct the Jost matrix $J(k)$,

$$J(k) = f(-k^*,0)^\dagger B - f'(-k^*,0)^\dagger A, \quad k \in \mathbb{C}^+, \quad (5.11)$$

where the asterisk denotes complex conjugation. For the following result see [5], and also Theorem 3.8.1 in page 114 of [9].

**PROPOSITION 5.2.** Suppose that the potential $V$ satisfies (1.3) and (1.4). Then, the Jost matrix $J(k)$ is analytic for $k \in \mathbb{C}^+$, continuous for $k \in \mathbb{C}^+$ and invertible for $k \in \mathbb{R} \setminus \{0\}$.

Let $K(x,y)$ be defined as follows

$$K(x,y) = (2\pi)^{-1} \int_{-\infty}^{\infty} [f(k,x) - e^{ikx}I]e^{-iky}dk, \quad x, y \geq 0. \quad (5.12)$$

We introduce the following quantities,

$$\sigma(x) = \int_{x}^{\infty} |V(y)| dy, \quad \sigma_1(x) = \int_{x}^{\infty} y|V(y)| dy, \quad x \geq 0.$$

Remark that for potentials satisfying (1.4), both $\sigma(0)$ and $\sigma_1(0)$ are finite, and furthermore, $\int_{0}^{\infty} \sigma(x) dx = \sigma_1(0) < \infty$.

The following proposition is given in [5]. See also Proposition 3.28 in pages 77-78 of [9].

**PROPOSITION 5.3.** Suppose that the potential $V$ satisfies (1.3) and (1.4). Then, we have.

1. The matrix $K(x,y)$ is continuous in $(x,y)$ in the region $0 \leq x \leq y$, and is related to the potential via

$$K(x,x^+) = \frac{1}{2} \int_{x}^{\infty} V(z) dz, \quad x \in [0, +\infty).$$

2. The matrix $K(x,y)$ satisfies,

$$K(x,y) = 0, \quad y < x, x, y \in [0, +\infty),$$

$$|K(x,y)| \leq \frac{1}{2} e^{\sigma_1(x)} \sigma(x) \left( \frac{x + y}{2} \right), \quad x, y \in \mathbb{R}^+. \quad (5.13)$$

3. The Jost solution $f(k,x)$ has the representation

$$f(k,x) = e^{ikx}I + \int_{x}^{\infty} e^{iky}K(x,y) dy. \quad (5.14)$$

The scattering matrix, $S(k)$, is a $n \times n$ matrix-valued function of $k \in \mathbb{R}$ that is given by

$$S(k) = -J(-k)J(k)^{-1}, \quad k \in \mathbb{R}. \quad (5.15)$$

In the exceptional case where $J(0)$ is not invertible the scattering matrix is defined by (5.15) only for $k \neq 0$. However, it is proven in [7], and in Theorem 3.8.14 in page 138 of [9], that for potentials satisfying (1.3) and (1.4) the limit $S(0) := \lim_{k \to 0} S(k)$ exists in the exceptional case and, moreover, a formula for $S(0)$ is given.

It is proven in [8] and in Theorem 3.10.6 in pages 196-197 of [9]) that the following limit exist,

$$S_{\infty} := \lim_{|k| \to \infty} S(k). \quad (5.16)$$
Let us denote by $F_s$ the following quantity, that up to the factor $1/\sqrt{2\pi}$ is the inverse Fourier transform of $S(k) - S_\infty$,

$$F_s(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - S_\infty] e^{iky} dk, \quad y \in \mathbb{R}. \quad (5.17)$$

The following theorem is proven in [34].

**THEOREM 5.4.** Suppose that the potential $V$ satisfies (1.3) and (1.4). Then,

$$F_s \in L^1(\mathbb{R}). \quad (5.18)$$

In terms of the Jost solution $f(k,x)$ and the scattering matrix $S(k)$ we construct the physical solution, [8] and equation (2.2.29) in page 26 of [9],

$$\Psi(k,x) = f(-k,x) + f(k,x) S(k), \quad k \in \mathbb{R}. \quad (5.19)$$

The physical solution $\Psi(k,x)$ is the main input to construct the generalized Fourier maps, $F^\pm$, for the absolutely continuous subspace of $H$, that are defined in equation (4.3.44) in page 284 of [9] (see also Proposition 4.3.4 in page 287 of [9]),

$$(F^\pm Y)(k) = \sqrt{\frac{1}{2\pi}} \int_0^\infty (\Psi(\mp k,x))^\dagger Y(x) dx, \quad (5.20)$$

for $Y \in L^1(\mathbb{R}^+, \mathbb{C}^n) \cap L^2(\mathbb{R}^+, \mathbb{C}^n)$.

We have (see equation (4.3.46) in page 284 of [9]),

$$\|F^\pm Y\|_{L^2(\mathbb{R}^+, \mathbb{C}^n)} = \|E(\mathbb{R}^+; H)Y\|_{L^2(\mathbb{R}^+, \mathbb{C}^n)}. \quad (5.21)$$

Thus, the $F^\pm$ extend to bounded operators on $L^2(\mathbb{R}^+, \mathbb{C}^n)$ that we also denote by $F^\pm$.

The following results on the spectral theory of $H$ are proven in Theorem 3.11.1 in pages 199-200, Theorem 4.3.3 in page 284 and Proposition 4.3.4 in page 287, of [9].

**THEOREM 5.5.** Suppose that the potential $V$ satisfies (1.3) and (5.2), and that the constant matrices $A,B$ fulfill (1.5), and (1.6). Then, the Hamiltonian $H$ has no positive eigenvalues and the negative spectrum of $H$ consists of isolated eigenvalues of multiplicity smaller or equal than $n$, that can accumulate only at zero. Furthermore, $H$ has no singular continuous spectrum and its absolutely continuous spectrum is given by $[0, \infty)$. The generalized Fourier maps $F^\pm$ are partially isometric with initial subspace $\mathcal{H}_{ac}(H)$ and final subspace $L^2(\mathbb{R}^+, \mathbb{C}^n)$. Moreover, the adjoint operators are given by

$$(F^\pm Z)(x) = \sqrt{\frac{1}{2\pi}} \int_0^\infty \Psi(\mp k,x) Z(k) dk, \quad (5.22)$$

for $Z \in L^1(\mathbb{R}^+, \mathbb{C}^n) \cap L^2(\mathbb{R}^+, \mathbb{C}^n)$. Furthermore,

$$F^\pm H (F^\pm)^\dagger = \mathcal{M}, \quad (5.23)$$

where $\mathcal{M}$ is the operator of multiplication by $k^2$. If, in addition, $V \in L^1(\mathbb{R}^+, M_n)$, zero is not an eigenvalue and the number of eigenvalues of $H$ including multiplicities is finite.

Note that by (5.21) $(F^\pm)^\dagger F^\pm$ is the orthogonal projector onto $\mathcal{H}_{ac}(H)$,

$$(F^\pm)^\dagger F^\pm = P_{ac}(H). \quad (5.24)$$
We denote by $F_0$ the cosine transform,
\[(F_0 Y)(k) := \sqrt{\frac{2}{\pi}} \int_0^\infty dx \cos(kx) Y(x), \ Y \in L^2(\mathbb{R}^+).\] (5.25)

Actually, $F_0$ coincides with the generalized Fourier maps for $H_0$ given by Theorem 5.5.

The following theorem, proven in Theorem 4.4.3 in page 297 of [9], gives the stationary formulae for the wave operators.

**THEOREM 5.6.** Suppose that $V$ satisfies (1.3) and (5.2). Then, the wave operators $W_\pm(H,H_0)$ exist and are complete. Further, the following stationary formulae hold,
\[W_\pm = (F^\pm )^\dagger F_0.\] (5.26)

### 5.2 $L^p$-boundedness of the wave operators in the half-line

We prepare the following proposition.

**PROPOSITION 5.7.** Suppose that $Y \in L^2(\mathbb{R}, \mathbb{C}^n)$. Then,
\[F^{-1}(\chi_{\mathbb{R}^+}(k)(FY)(k))(x) = \frac{i}{2} (HY)(x) + \frac{1}{2} Y(x), \ x \in \mathbb{R},\] (5.27)

and
\[F(\chi_{\mathbb{R}^+}(k)(F^{-1}Y)(k))(x) = \frac{-i}{2} (HY)(x) + \frac{1}{2} Y(x), x \in \mathbb{R}.\] (5.28)

**Proof:** The proof is an immediate consequence of equations (3) and (4) in page 60 of [21].

**Proof of Theorem 3.1:** Let us first take $Y \in C_0^\infty(\mathbb{R}^+)$. Note that $\mathcal{E}_{\text{even}} Y \in C_0^\infty(\mathbb{R})$. By (5.25)
\[(F_0 Y)(k) = (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k), \ k \in \mathbb{R}^+.\] (5.29)

By (5.14), (5.19), (5.22), (5.26), and (5.29)
\[(W_\pm Y)(x) := \sum_{j=1}^6 T^{(j)}_\pm (x),\] (5.30)

where
\[T^{(1)}_\pm (x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikx} \chi_{\mathbb{R}^+}(k) (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) dk,\] (5.31)
\[T^{(2)}_\pm (x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz K(x, z) \int_{\mathbb{R}} e^{\pm ikz} \chi_{\mathbb{R}^+}(k) (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) dk,\] (5.32)
\[T^{(3)}_\pm (x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbb{R}^+}(k) S_\infty(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) dk,\] (5.33)
\[T^{(4)}_\pm (x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz K(x, z) \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) S_\infty(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) dk,\] (5.34)
\[T^{(5)}_\pm (x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbb{R}^+}(k) (S(\mp k) - S_\infty(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) dk,\] (5.35)
and
\[ T^{(6)}_{\pm}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dz K(x, z) \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) (S(\mp k) - S_{\infty})(\mathcal{FE}_{\text{even}}Y)(k) \, dk. \] (5.36)

Observe that
\[ (\mathcal{FE}_{\text{even}}Y)(k) = (F^{-1}\mathcal{E}_{\text{even}}Y)(k), \quad k \in \mathbb{R}. \] (5.37)

It follows from Proposition 5.7, and (5.30)-(5.37) that (3.3) holds for \( Y \in C_0^\infty(\mathbb{R}^+) \). Finally, approximating \( Y \in L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 < p < \infty \), by a sequence of functions in \( C_0^\infty(\mathbb{R}^+) \), it follows that equation (3.3)- (3.6) hold for all \( Y \in L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 < p < \infty \), and that the wave operators \( W_\pm(H, H_0) \) extend to bounded operators on \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 < p < \infty \). Here we used that \( Q(F) \) is bounded in \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \), since by Theorem 5.4 \( F \in L^1(\mathbb{R}, \mathbb{C}^n) \), that \( K(K) \) is bounded in \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \), because by (5.13), equations (2.1) hold, and that \( \mathcal{H} \) is bounded in \( L^p(\mathbb{R}, \mathbb{C}^n) \), \( 1 < p < \infty \) [39], [40]. The operator \( \mathcal{R} \) is clearly bounded from \( L^p(\mathbb{R}, \mathbb{C}^n) \) into \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). The wave operator \( W_\pm(H, H_0) \) extend to bounded operators on \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 < p < \infty \), by duality.

\[ \square \]

**Proof of Theorem 3.2:** Let us first take \( Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n) \). Recall that \( \mathcal{E}_{\text{even}}Y \in C_0^\infty(\mathbb{R}) \), and that (5.29) holds. By (5.31), (5.33), \( S_{\infty} = I_n \), and since \( \mathcal{FE}_{\text{even}}Y \) is an even function,
\[ T^{(1)}_{\pm}(x) + T^{(3)}_{\pm}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikz} \chi_{\mathbb{R}^+}(k) (\mathcal{FE}_{\text{even}}Y)(k) \, dk + \]
\[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) (\mathcal{FE}_{\text{even}}Y)(k) \, dk = \mathcal{E}_{\text{even}}Y(x) = Y(x), \quad x \geq 0, \] (5.38)

where in the second integral in the middle equation in (5.38) we made the change of variable of integration \( k \to -k \), and we used, \( \mathcal{FE}_{\text{even}}Y(k) = F^{-1}\mathcal{E}_{\text{even}}Y(k) \), and \( S_{\infty} = I_n \). We similarly prove, using (5.32) and (5.34),
\[ T^{(2)}_{\pm}(x) + T^{(4)}_{\pm}(x) = (K(K)Y)(x). \] (5.39)

Hence, by (5.30), (5.38), and (5.39),
\[ (W_\pm Y)(x) = Y(x) + (K(K)Y)(x) + T^{(5)}_{\pm}(x) + T^{(6)}_{\pm}(x). \] (5.40)

By (5.35), (5.36) and the convolution theorem of the Fourier transform,
\[ T^{(5)}_{\pm}(x) = (Q(P_\pm)\mathcal{E}_{\text{even}}Y)(x), \quad x \geq 0, \] (5.41)
and,
\[ T^{(6)}_{\pm}(x) = (K(K)RQ(P_\pm)\mathcal{E}_{\text{even}}Y)(x), \] (5.42)
where \( P_\pm \) is defined in (3.7). Equation (3.8) for \( Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n) \) follows from (5.40), (5.41) and (5.42). Moreover, as \( \mathcal{R} \) is bounded from \( L^p(\mathbb{R}, \mathbb{C}^n) \) into \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \), \( \mathcal{E}_{\text{even}} \) is bounded from \( L^p(\mathbb{R}^+, \mathbb{C}^n) \) into \( L^p(\mathbb{R}, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \), \( K(K) \) is bounded in \( L^p(\mathbb{R}^+, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \), because by (5.13) equations (2.1) hold. Moreover, by Schwarz inequality,
\[ \|P_\pm\|_{L^1(M^n)} \leq \|(1 + |x|^2)^{-1/2}\|_{L^2(\mathbb{R}^+)} \|\mathcal{E}_{\text{even}}\|_{L^2(M^n)} \leq C \|\chi_{\mathbb{R}^+}(k) (S(\mp k) - S_{\infty})\|_{H^1(\mathbb{R}, M^n)}. \] (5.43)

By the definition of \( S(k) \) in (5.15) and by Proposition 3.2.4 in page 65, Theorem 3.81 in page 114 and Theorem 3.9.15 in pages 189-190, of [9] \( S(k) \) is differentiable for \( k \in \mathbb{R} \), with continuous derivative for \( k \in \mathbb{R} \setminus \{0\} \). Then, since \( S(0) = S_\infty \), and as by Proposition A.3 \( S(k) - S_{\infty} \in H^1(\mathbb{R}^+, M^n) \) we have that \( \chi_{\mathbb{R}^+}(k) (S(\mp k) - S_{\infty}) \in H^1(\mathbb{R}^+, M^n) \). Hence, by (5.43), \( P_\pm \in L^1(\mathbb{R}^+, M^n) \), and then, \( Q(P_\pm) \) is a bounded operator in \( L^p(\mathbb{R}, \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). Hence, (3.8) holds for
all \( Y \in L^2(\mathbb{R}^+, \mathbb{C}^n) \) and, moreover, the wave operators \( W_{\pm}(H, H_0) \) extend to bounded operators in \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and in \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \).

We now prove that the adjoint wave operators \( W_{\pm}(H, H_0)^\dagger \) extend to bounded operators in \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and in \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \). By (3.8),

\[
(W_{\pm}^\dagger Y)(x) = Y(x) + (K^\dagger(K)Y)(x) + (\mathcal{E}^\dagger_{\text{even}} Q(P_{\pm})^\dagger \mathcal{R}^\dagger Y)(x) + (\mathcal{E}^\dagger_{\text{even}} Q(P_{\pm})^\dagger \mathcal{R}^\dagger K(K)^\dagger Y)(x).
\]

(5.44)

We have,

\[
\begin{align*}
K(K)^\dagger Y(x) &= \int_0^x K^\dagger(y, x) Y(y) \, dy.
\end{align*}
\]

By (5.13), equation (2.1) holds, and then \( K(K)^\dagger \) is bounded in \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and in \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \). Further,

\[
Q^\dagger(P_{\pm}) Y(x) = \int_{-\infty}^{\infty} P_{\pm}^\dagger(y - x) Y(y) \, dy,
\]

and as \( P_{\pm}^\dagger \in L^1(\mathbb{R}, \mathbb{C}^n) \), it follows that \( Q^\dagger(P_{\pm}) \) in bounded \( L^1(\mathbb{R}, \mathbb{C}^n) \) and in \( L^\infty(\mathbb{R}, \mathbb{C}^n) \). Moreover, \( \mathcal{E}^\dagger_{\text{even}}(Y(x) = Y(x) + Y(-x) \), and then, \( \mathcal{E}^\dagger_{\text{even}} \) is bounded from \( L^1(\mathbb{R}, \mathbb{C}^n) \) into \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and from \( L^\infty(\mathbb{R}, \mathbb{C}^n) \) into \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \).

Furthermore,

\[
\mathcal{R}^\dagger Y(x) = \begin{cases} Y(x), & x \geq 0, \\ 0, & x < 0, \end{cases}
\]

and it follows that \( \mathcal{R}^\dagger \) is bounded from \( L^1(\mathbb{R}, \mathbb{C}^n) \) into \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and from \( L^\infty(\mathbb{R}, \mathbb{C}^n) \) into \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \). Then, by (5.44) the adjoint wave operators \( W_{\pm}(H, H_0)^\dagger \) extend to bounded operators in \( L^1(\mathbb{R}^+, \mathbb{C}^n) \) and in \( L^\infty(\mathbb{R}^+, \mathbb{C}^n) \).

\[
\square
\]

**REMARK 5.8.** The condition \( S_\infty = S(0) = I_n \) is actually necessary in Theorem 3.2, as the following example shows. Consider the scalar case, \( n = 1 \), with \( V = 0 \), Dirichlet boundary condition, \( Y(0) = 0 \), and boundary matrices, \( B = -1, A = 0 \). In this case by equation (3.7.5) in page 113 of [9], \( S_\infty = S(0) = -1 \). Further by equation (4.3.8) in page 277 of [9] the generalized Fourier maps are given by,

\[
(F^\pm Y)(k) := \mp 2i \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin kx Y(x) \, dx.
\]

(5.45)

Hence, by (5.26) and (5.29)

\[
W_{\pm}(H, H_0) Y = \pm \mathcal{R} \mathcal{F}^{-1} \text{sign } k \mathcal{F} \mathcal{E}_{\text{even}} Y.
\]

(5.46)

Moreover, we have

\[
\mathcal{F}^{-1} \chi_{\mathbb{R}^+}(k) \mathcal{F} \mathcal{E}_{\text{even}} Y = \mathcal{F} \chi_{\mathbb{R}^+}(k) \mathcal{F}^{-1} \mathcal{E}_{\text{even}} Y.
\]

(5.47)

Then, by (5.27), (5.28), (5.46), (5.47),

\[
W_{\pm}(H, H_0) Y = \mp i \mathcal{R} \mathcal{H} \mathcal{E}_{\text{even}} Y.
\]

(5.48)

Finally, since the Hilbert transform is not bounded in \( L^1(\mathbb{R}, \mathbb{C}) \) and in \( L^\infty(\mathbb{R}, \mathbb{C}) \) [39], [40], it follows that \( W_{\pm}(H, H_0) \) are not bounded \( L^1(\mathbb{R}^+, \mathbb{C}) \) and in \( L^\infty(\mathbb{R}^+, \mathbb{C}) \).

### 5.3 The \( L^p \)-boundedness of the wave operators on the line

**Proof of Theorem 4.1:** We first prepare some results. Let us denote by \( H_1 \) the Hamiltonian \( H_{A,B,V} \) with the matrices given in (4.6) with \( \Lambda = 0 \), and with the potential \( V \) identically zero. Note that
\[ H_{0,R} = U H_1 U^\dagger. \] (5.49)

By Theorem 5.6 the wave operators \( W_\pm(H_1, H_0) \), exist and are complete. Then, by Proposition 3 in page 18 of [36] the wave operators \( W_\pm(H_0, H_1) \), also exists and are complete, and furthermore,

\[ W_\pm(H_0, H_1) = W_\pm(H_1, H_0)^\dagger. \] (5.50)

Then, by Theorem 3.1 the wave operators \( W_\pm(H_0, H_1) \), restricted to \( L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty \), extend uniquely to bounded operators in \( L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty \). Further, by the chain rule, see Proposition 2 in page 18 of [36],

\[ W_\pm(H, H_1) = W_\pm(H_0, H_0) W_\pm(H_0, H_1). \] (5.51)

Hence, as \( W_\pm(H, H_0) \), and \( W_\pm(H_0, H_1) \), restricted to \( L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty \), if follows that also \( W_\pm(H, H_1) \), restricted to \( L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty \), extend uniquely to bounded operators in \( L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty \). Finally, by (4.2) and (5.49),

\[ W_\pm(H_R, H_0,R) = U W_\pm(H, H_1) U^\dagger, \] (5.52)

and, as \( U \) is bounded from \( L^p(\mathbb{R}^+, \mathbb{C}^{2n}), \) into \( L^p(\mathbb{R}, \mathbb{C}^n), \) and \( U^\dagger \) is bounded from \( L^p(\mathbb{R}, \mathbb{C}^n), \) into \( L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 \leq p \leq \infty, \) we have that the \( W_\pm(H_R, H_0,R) \) restricted to \( L^2(\mathbb{R}, \mathbb{C}^n) \cap L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty, \) extend uniquely to bounded operators in \( L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty. \) Finally, by duality the adjoint wave operators \( W_\pm(H_R, H_{0,R})^\dagger \) extend uniquely to bounded operators in \( L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty. \)

\[ \square \]

We now proceed to prove that the wave operators \( W_\pm(H_R, H_{0,R}) \) are bounded in \( L^1(\mathbb{R}, \mathbb{C}^n), \) and in \( L^\infty(\mathbb{R}, \mathbb{C}^n), \) as stated in Theorem 4.2. We first prepare some results.

Let us denote by \( A_1, B_1 \) the matrices (4.6) with \( \Lambda = 0. \) Then, \( H_1 = H_{A_1,B_1,0}. \) Let \( \tilde{A}_1, \tilde{B}_1 \) be the matrices related to \( A_1, B_1 \) as in (5.3), (5.4), and (5.5) for some invertible matrices \( T_{1,1}, T_{2,1} \) and some unitary matrix \( M_1. \) Hence,

\[ A_1 = M_1 \tilde{A}_1 T_{1,1} M_1^\dagger T_{2,1}, \quad B_1 = M_1 \tilde{B}_1 T_{1,1} M_1^\dagger T_{2,1}. \]

To simplify the notation we denote, \( \tilde{H}_1 := H_{\tilde{A}_1, \tilde{B}_1,0}. \) Applying (5.9) to \( H_1 \) and \( \tilde{H}_1 \) we obtain,

\[ H_1 = M_1 \tilde{H}_1 M_1^\dagger. \] (5.53)

Let us denote by \( F_1^\pm, \) respectively \( \tilde{F}_1^\pm, \) the generalized Fourier maps for \( H_1, \) and for \( \tilde{H}_1 \) defined in (5.20). Then, by (5.53) we get (see equation (4.3.35) in page 282 of [9])

\[ F_1^\pm = M_1 \tilde{F}_1^\pm M_1^\dagger. \] (5.54)

By (5.26), (5.50), (5.51) and as \( F_0 = F_0^\dagger = F_0^{-1}, W_\pm(H, H_1) = (F^\pm)^\dagger F_1^\pm, \) and using (5.54) we prove,

\[ W_\pm(H, H_1) = (F^\pm)^\dagger M_1 \tilde{F}_1^\pm M_1^\dagger. \] (5.55)

To use (5.55) to study the boundedness of the wave operators we need to compute explicitly the unitary matrix \( M_1. \) For this purpose, we first introduce the following unit vectors in \( \mathbb{C}^{2n}, \)

\[ Y^{(j)} := (0, \ldots, \frac{1}{\sqrt{2}}, 0, 0, \ldots, -\frac{1}{\sqrt{2}}, 0, \ldots, 0)^T, \quad j = 1, 1, \ldots, n, \] (5.56)
with components that take the value $\frac{1}{\sqrt{2}}$ at the component $j$, the value $-\frac{1}{\sqrt{2}}$ at the component $j + 1$, $1 \leq j \leq n$, and all the other components are zero. Further, we define the following unit vectors in $\mathbb{C}^{2n}$

$$Y^{(j)} := (0, \ldots, \frac{1}{\sqrt{2}}, 0, 0, \ldots, \frac{1}{\sqrt{2}}, 0, \ldots, 0)^T, \quad j = n + 1, \ldots, 2n,$$

(5.57)

with components that take the value $\frac{1}{\sqrt{2}}$ at the component $j - n$, the value $\frac{1}{\sqrt{2}}$ at the component $j$, $n + 1 \leq j \leq 2n$, and all the other components are zero.

**PROPOSITION 5.9.** Let $A_1, B_1$ be the matrices (4.6) with $\Lambda = 0$, and let $\tilde{A}_1, \tilde{B}_1$ be the matrices related to $A_1, B_1$ as in (5.3), (5.4), and (5.5) for some invertible matrices $T_{1,1}, T_{2,1}$ and some unitary matrix $M_1$. Then,

1. $$\tilde{A}_1 = \begin{cases} 0_n & 0_n \\ 0_n & -I_n \end{cases}.$$  \hspace{1cm} (5.58)

2. $$\tilde{B}_1 = \begin{cases} -I_n & 0_n \\ 0_n & 0_n \end{cases}. \hspace{1cm} (5.59)

3. The boundary conditions (5.4) are given by,

$$Y_j(0) = 0, \quad j = 1, \ldots, n, \quad Y_j'(0) = 0, \quad j = n + 1, \ldots, 2n.$$  \hspace{1cm} (5.60)

That is to say, the first $n$ components of $Y$ satisfy the Dirichlet boundary condition, and the last $n$ components fulfill the Neumann boundary condition.

4. The unitary matrix, $M_1$, is given by,

$$M_1 = \left\{ Y^{(1)} Y^{(2)} \ldots Y^{(2n)} \right\}.$$  \hspace{1cm} (5.61)

5. The invertible matrices $T_{1,1}$, and $T_{2,1}$ are given by,

$$T_{1,1} = \begin{cases} -I_n & 0_n \\ 0_n & iI_n \end{cases}.$$  \hspace{1cm} (5.62)

$$T_{2,1} = \begin{cases} -I_n & iI_n \\ I_n & iI_n \end{cases}. \hspace{1cm} (5.63)

**Proof:** We use the notation of the proof of Proposition 3.4.5 in pages 101-103 of [9]. We denote, $E := \sqrt{A_1^\dagger A_1 + B_1^\dagger B_1}$, and $U := (B_1 - iA_1)E^{-2}(B_1^\dagger - iA_1^\dagger)$. Then, by (4.6) and a simple computation, we get,

$$U = \begin{cases} 0 & -I_n \\ -I_n & 0 \end{cases}.$$  \hspace{1cm} (5.64)

It is easily verified that the $Y^{(j)}, j = 1, \ldots, n$ are eigenvectors of $U$ with eigenvalues one, and that the vectors $Y^{(j)}, j = n + 1, \ldots, 2n$ are eigenvectors of $U$ with eigenvalue minus one. Then, the columns of $M_1$ are an orthonormal system of eigenvectors of $U$, and in consequence $M_1$ diagonalizes $U$, as required in equation (3.4.39) in page 101 of [9], $M_1^\dagger U M_1 = \text{diag}\{1, \ldots, 1, -1, \ldots, -1\}$, is the matrix with the first $n$ diagonal entries equal to one, the second $n$ diagonal entries equal to minus one and all other entries equal to zero. This proves that item (4) is satisfied. Using the notation of equation (3.4.41) in page 102 of [9] with $P = I_{2n}$ we get $M_1^\dagger U M_1 = \text{diag}\{e^{\theta_1}, \ldots, e^{\theta_{2n}}\} = \text{diag}\{1, -1, \ldots, -1\}, 0 < \theta_j \leq \pi, j = 1, \ldots, 2n$. Then, $\theta_1 = \theta_2 = \cdots = \theta_n = \pi$, and $\theta_{n+1} = \theta_{n+2} = \cdots = \theta_{2n} = \pi$.
π/2, and items (1), (2) and (3) hold. That item (5) holds is immediate since by the definition of $T_{1,1}$ and $T_{2,1}$ in page 103 of [9], we have $T_{1,1} := (B_1 + iA_1)^{-1}$, and $T_{2,1} := B_1 + iA_1$.

Let us denote $\tilde{\psi}_1^+(k, x) := \tilde{\psi}_1(-k, x)$, and $\tilde{\psi}_1^-(k, x) := \tilde{\psi}_1(k, x)$, where $\tilde{\psi}_1(k, x)$ is the physical solution of $\tilde{H}_1$. By equations (4.3.6) and (4.3.7) in page 277 of [9],

$$\tilde{\psi}_1^+(k, x) = \{\pm 2i \sin kx, \ldots, \pm 2i \sin kx, 2 \cos kx, \ldots, 2 \cos kx\},$$

is the diagonal matrix with the first $n$ diagonal components equal to $\pm 2i \sin kx$, and the last $n$ diagonal components equal to $2 \cos kx$. Further, by equation (4.3.8) in page 277 of [9],

$$(\tilde{\mathbf{F}}_1^Y)(k) = \sqrt{\frac{1}{2\pi}} \int_0^\infty \tilde{\psi}_1^+(k, x)^\dagger Y(x) \, dx.$$  

By (5.64) and (5.65),

$$(\tilde{\mathbf{F}}_1^Y)(k) = \mathcal{R} \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T.$$  

We denote,

$$W_{\pm, \mathcal{M}_1}(H, H_1) := \mathcal{M}_1^\dagger W_{\pm}(H, H_1) \mathcal{M}_1,$$  

$$S_{\mathcal{M}_1}(k) := \mathcal{M}_1^\dagger S(k) \mathcal{M}_1, \quad k \in \mathbb{R},$$

and

$$S_{\infty, \mathcal{M}_1} := \mathcal{M}_1^\dagger S_{\infty} \mathcal{M}_1, \quad k \in \mathbb{R}.$$  

Using (5.19), (5.22), (5.55), and (5.66)-(5.69) we prove,

$$(W_{\pm, \mathcal{M}_1} Y)(x) := \sum_{j=1}^6 J_j^{(j)}(x),$$

where

$$J_1^{(1)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikx} \chi_{\mathbb{R}^+}(k) \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T \, dk,$$

$$J_2^{(2)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} K(x, z) \mathcal{M}_1 \left( \int_{\mathbb{R}} e^{\pm ikz} \chi_{\mathbb{R}^+}(k) \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T \, dk \right),$$

$$J_3^{(3)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} S_{\mathcal{M}_1}(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k))^T \, dk,$$

$$J_4^{(4)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) S_{\infty, \mathcal{M}_1} \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T \, dk,$$

$$J_5^{(5)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) \left( S_{\mathcal{M}_1}(\mp k) - S_{\infty, \mathcal{M}_1} \right) \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T \, dk,$$

and

$$J_6^{(6)}(x) := \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) \left( S_{\mathcal{M}_1}(\mp k) - S_{\infty, \mathcal{M}_1} \right) \left( (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T \, dk.$$  

We denote,

$$P_{\pm, \mathcal{M}_1}(x) := \mathcal{M}_1^\dagger P_{\pm}(x) \mathcal{M}_1,$$  

where $P_{\pm}$ is defined in (3.7).
THEOREM 5.10. Suppose that $V$ fulfills (1.3), that $V \in L^1((\mathbb{R}^+, M_{2n}))$, $\gamma > \frac{3}{2}$, that the constant matrices $A, B$ are given by (4.6) with $\Lambda = 0$, and that
\[ S(0) = S_\infty = \left\{ \begin{array}{cc} 0_n & I_n \\ I_n & 0_n \end{array} \right\}, \quad (5.78) \]
where, $0_n$, and $I_n$ are, respectively, the $n \times n$ zero matrix and the $n \times n$ identity matrix.

Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^{2n})$ we have,
\[ W_\pm(H, H_1)Y = Y + K(K)Y + M_1RQ(P_{\pm,M_1}) \left( -\mathcal{E}_{\text{odd}}(M_1^2Y)^+, \mathcal{E}_{\text{even}}(M_1^2Y)^- \right)^T + K(K)M_1RQ(P_{\pm,M_1}) \left( -\mathcal{E}_{\text{odd}}(M_1^2Y)^+, \mathcal{E}_{\text{even}}(M_1^2Y)^- \right)^T. \quad (5.79) \]

Furthermore, the wave operators $W_\pm(H, H_1)$ and $W_\pm(H, H_1)^\dagger$ restricted to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^1(\mathbb{R}^+, \mathbb{C}^{2n})$, respectively to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, extend to bounded operators on $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and to bounded operators on $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. The $2n \times 2n$ matrix valued function $K(x, y), x, y \in \mathbb{R}^+$ is defined in (5.12). Moreover, the $2n \times 2n$ matrix valued function $P_{\pm,M_1}(x)$ is defined in (5.77). The scattering matrix $S(k), k \in \mathbb{R}$, is defined in (5.15) and the quantity $S_\infty$ in (5.16).

Proof of Theorem 5.10 : By (5.61), (5.68), (5.69), (5.78),
\[ S_{M_1}(0) = S_{\infty,M_1} = \left\{ \begin{array}{cc} -I_n & 0_n \\ 0_n & I_n \end{array} \right\}. \quad (5.80) \]

Then, by (5.70)-(5.77), and (5.80),
\[ W_{\pm,M_1}(H, H_1)Y = Y + M_1^2K(K)M_1Y + RQ(P_{\pm,M_1}) \left( -\mathcal{E}_{\text{odd}}Y^+, \mathcal{E}_{\text{even}}Y^- \right)^T + M_1^2K(K)M_1RQ(P_{\pm,M_1}) \left( -\mathcal{E}_{\text{odd}}Y^+, \mathcal{E}_{\text{even}}Y^- \right)^T. \quad (5.81) \]

Equation (5.79) follows from (5.67) and (5.81). By (5.77),
\[ \|P_{\pm,M_1}\|_{L^1(\mathbb{R}, M_{2n})} = \|P_{\pm}\|_{L^1(\mathbb{R}, M_{2n})}. \quad (5.82) \]

As we already proved in the proof of Theorem 3.2 that $P_{\pm} \in L^1(\mathbb{R}, M_{2n})$ it follows from (5.82) that $P_{\pm,M_1} \in L^1(\mathbb{R}, M_{2n})$. Hence, $Q(P_{\pm,M_1})$ is bounded in $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. We already proved in the proof Theorem 3.2 $K(K)$, is bounded in $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, that $\mathcal{R}$ is bounded from $L^1(\mathbb{R}, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. Clearly, $\mathcal{E}_{\text{even}}$ is bounded from $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. Moreover, it is clear that $\mathcal{E}_{\text{odd}}$ is also bounded from $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. Then, by (5.79) the wave operators $W_{\pm}(H, H_1)$ extend to bounded operators on $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and to bounded operators on $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. By (5.81) and taking adjoints we obtain,
\[ W_{\pm,M_1}^\dagger Y = Y + M_1^2K(K)^\dagger M_1Y + \left( -\mathcal{E}_{\text{odd}}^\dagger(Q(P_{\pm,M_1})^\dagger \mathcal{R}Y)^+, \mathcal{E}_{\text{even}}^\dagger(Q(P_{\pm,M_1})^\dagger \mathcal{R}Y)^- \right)^T + M_1^2K(K)^\dagger M_1RQ(P_{\pm,M_1}) \left( -\mathcal{E}_{\text{odd}}^\dagger(Q(P_{\pm,M_1})^\dagger \mathcal{R}Y)^+, \mathcal{E}_{\text{even}}^\dagger(Q(P_{\pm,M_1})^\dagger \mathcal{R}Y)^- \right)^T. \quad (5.83) \]

We already proved in the proof of Theorem 3.2 that $K(K)^\dagger$ is bounded in $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. The fact that $\mathcal{E}_{\text{even}}^\dagger$ and $\mathcal{E}_{\text{odd}}^\dagger$ are bounded from $L^1(\mathbb{R}, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, and that $\mathcal{R}^\dagger$ is bounded from $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$, follows immediately. Further, as $\|P_{\pm,M_1}\|_{L^1(\mathbb{R}, M_{2n})} = \|P_{\pm,M_1}\|_{L^1(\mathbb{R}, M_{2n})}$, and since we already proved, $P_{\pm,M_1} \in L^1(\mathbb{R}^+, M_{2n})$, we obtain,
\[ P_{\pm,M_1}^\dagger \in L^1(\mathbb{R}^+, M_{2n}), \quad (5.84) \]
and then as
\[ Q(P_{\pm,M_1})^\dagger Y(x) = \int_{-\infty}^{\infty} P_{\pm,M_1}^\dagger (y - x) Y(y) dy. \]
it follows that $Q_{±}(P_{\mathcal{M}_{1}})\dagger$ is bounded in $L^{1}(\mathbb{R}, \mathbb{C}^{2n})$ and in $L^{∞}(\mathbb{R}, \mathbb{C}^{2n})$. Finally, it follows from (5.83) that the wave operators $W_{±}(H, H_{1})\dagger$ extend to bounded operators on $L^{1}(\mathbb{R}^{+}, \mathbb{C}^{2n})$ and to bounded operators on $L^{∞}(\mathbb{R}^{+}, \mathbb{C}^{2n})$, and by (5.67) this also holds for the wave operators $W_{±}(H, H_{1})$.

Using the unitary transformation $U$ given in (4.1) we obtain our result on the boundedness of the wave operators on the line, from Theorem 5.10. However, since Theorem 5.10 involves $S(0)$, and $S_{∞}$, we first introduce some concepts from the stationary scattering theory of matrix Schrödinger operators on the line that we quote from [6]. Under the assumption that $V \in L^{1}_{1}(\mathbb{R}, \mathbb{C}^{n})$, the Jost solution from the left, $f_{l}(k, x), x \in \mathbb{R}, k \in \mathbb{C}^{+}$, is the $n \times n$ matrix-valued solution to the Schrödinger equation on the line,

$$-rac{d^{2}}{dx^{2}}Y(x) + V(x)Y(x) = k^{2}Y(x), \quad x \in \mathbb{R},$$

that satisfies

$$f_{l}(k, x) = e^{ikx}[I_{n} + o(1)], f'_{l}(k, x) = e^{ikx}[iI_{n} + o(1)], x \rightarrow \infty.$$  \hspace{1cm} (5.85)

Further, for $k \in \mathbb{R} \setminus \{0\}$, $f_{l}(k, x)$ fulfills,

$$f_{l}(k, x) = a_{l}(k)e^{ikx} + b_{l}(k)e^{-ikx} + o(1), \quad x \rightarrow -\infty.$$ \hspace{1cm} (5.86)

Similarly, Jost solution from the right, $f_{r}(k, x), x \in \mathbb{R}, k \in \overline{\mathbb{C}^{+}}$, is the $n \times n$ matrix-valued solution to the Schrödinger equation (5.84), such that,

$$f_{r}(k, x) = e^{-ikx}[I_{n} + o(1)], f'_{r}(k, x) = e^{-ikx}[-iI_{n} + o(1)], \quad x \rightarrow -\infty.$$ \hspace{1cm} (5.87)

Moreover, for $k \in \mathbb{R} \setminus \{0\}$, $f_{r}(k, x)$ fulfills,

$$f_{r}(k, x) = a_{r}(k)e^{-ikx} + b_{r}(k)e^{ikx} + o(1), \quad x \rightarrow \infty.$$ \hspace{1cm} (5.88)

The transmission coefficient from the left, $T_{l}(k)$, and the transmission coefficient from the right, $T_{r}(k)$, are defined by

$$T_{l}(k) := \frac{1}{a_{l}(k)}, \quad T_{r}(k) := \frac{1}{a_{r}(k)}.$$ \hspace{1cm} (5.89)

The reflection coefficient from the left, $L(k)$, and the reflection coefficient from the right, $R(k)$, are given by,

$$L(k) := \frac{b_{l}(k)}{a_{l}(k)}, \quad R(k) := \frac{b_{r}(k)}{a_{r}(k)}.$$ \hspace{1cm} (5.90)

The physical solution from the left, $\Psi_{l}(k, x)$, is defined as,

$$\Psi_{l}(k, x) := T_{l}(k) f_{l}(k, x).$$ \hspace{1cm} (5.91)

Then, $\Psi_{l}(k, x)$ satisfies,

$$\Psi_{l}(k, x) = \left\{ \begin{array}{ll}
T(k)e^{ikx} + o(1), x \rightarrow \infty, \\
e^{ikx} + e^{-ikx}L(k) + o(1), x \rightarrow -\infty.
\end{array} \right.$$ \hspace{1cm} (5.92)

The physical solution from the left corresponds to a scattering process where a particle is incident from the left with unit amplitude, it is reflected with amplitude $L(k)$ and it is transmitted with amplitude $T_{l}(k)$. Similarly, the physical solution from the right, $\Psi_{r}(k, x)$, is defined as,

$$\Psi_{r}(k, x) := T_{r}(k) f_{r}(k, x).$$ \hspace{1cm} (5.93)
Hence, \( \Psi_r(k, x) \) satisfies,
\[
\Psi_r(k, x) = \begin{cases} 
  e^{-ikx} + e^{ikx} R(k) + o(1), & x \to \infty, \\
  T_r(k) e^{-ikx} + o(1), & x \to -\infty.
\end{cases}
\] (5.94)

The physical solution from the right corresponds to a scattering process where a particle is incident from the right with unit amplitude, it is reflected with amplitude \( R(k) \) and it is transmitted with amplitude \( T_r(k) \).

The scattering matrix on the line, \( S_R(k) \), is defined as follows,
\[
S_R(k) := \begin{pmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{pmatrix}.
\] (5.95)

Using our results, we can directly define the physical solutions from the left and from the right from the physical solution \( \Psi(k, x), k \in \mathbb{R} \setminus \{0\} \), given in (5.19), by means of our unitary transformation \( U \), given in (4.1). We proceed as follows. Let us denote by \( \Psi^{(1)}(k, x) \) the \( 2n \times n \) matrix with the first \( n \) columns of \( \Psi(k, x) \), and let \( \Psi^{(2)}(k, x) \) be the \( 2n \times n \) matrix with the second \( n \) columns of \( \Psi(k, x) \). Then, by (5.10) and (5.19),
\[
\begin{pmatrix} U \Psi^{(1)} \end{pmatrix}_{ij} = \begin{cases} 
  e^{-ikx} \delta_{i,j} + \frac{e^{ikx}}{S} \{S\}_{i,j}(k) + o(1), & x \to \infty, 1 \leq i, j \leq n, \\
  e^{-ikx} \{S\}_{n+i,j}(k) + o(1), & x \to -\infty, 1 \leq i, j \leq n.
\end{cases}
\] (5.96)

Further, by (5.94) and (5.96), we define,
\[
\{\Psi_r(k, x)\}_{i,j} := \begin{pmatrix} U \Psi^{(1)} \end{pmatrix}_{ij}, \{T_r\}_{i,j}(k) := \{S\}_{n+i,j}(k), R_{i,j}(k) := \{S\}_{i,j}(k), 1 \leq i, j \leq n.
\] (5.97)

Moreover,
\[
\begin{pmatrix} U \Psi^{(2)} \end{pmatrix}_{ij} = \begin{cases} 
  e^{ikx} \{S\}_{i,n+j}(k) + o(1), & x \to \infty, 1 \leq i, j \leq n, \\
  e^{ikx} \delta_{i,j} + e^{-ikx} \{S\}_{n+i,n+j}(k) + o(1), & x \to -\infty, 1 \leq i, j \leq n.
\end{cases}
\] (5.98)

Then, by (5.92) and (5.98), we define,
\[
\{\Psi_l(k, x)\}_{i,j} := \begin{pmatrix} U \Psi^{(2)} \end{pmatrix}_{ij}, \{T_l\}_{i,j}(k) := \{S\}_{i,n+j}(k), L_{i,j}(k) := \{S\}_{n+i,n+j}(k), 1 \leq i, j \leq n.
\] (5.99)

Moreover, by (5.97), (5.99), we can directly define the scattering matrix on the line from the scattering matrix on the half line as follows,
\[
S_R(k) := \begin{pmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{pmatrix}, \text{ where, } \{T_r\}_{i,j}(k) := \{S\}_{n+i,j}(k),
\] (5.100)
\[
R_{i,j}(k) := \{S\}_{i,j}(k), \{T_l\}_{i,j}(k) := \{S\}_{i,n+j}(k), L_{i,j}(k) := \{S\}_{n+i,n+j}(k), 1 \leq i, j \leq n.
\]

Note that, by (5.100)
\[
S(0) = S_{\infty} = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} \iff S_R(0) = S_{R, \infty} = \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix},
\] (5.101)

where we denote,
\[
S_{R, \infty} = \lim_{|k| \to \infty} S_R(k).
\] (5.102)

**Proof of Theorem 4.2:** By (4.2) and (5.49)
\[
W_{\pm}(H_R, H_{0,R}) = U W_{\pm}(H, H_1) U^\dagger.
\] (5.103)

Then, the theorem follows from Theorem 5.10 and (5.101). \( \square \)
A The scattering matrix for potentials in $L^1_\gamma(\mathbb{R}^+, M_n), 2 \leq \gamma \leq 3$

In this appendix we always assume that $V \in L^1_\gamma(\mathbb{R}^+, M_n), 2 \leq \gamma \leq 3$. By the definition of $S(k)$ in (5.15) and by Proposition 3.2.4 in page 65, Theorem 3.81 in page 114 and Theorem 3.9.15 in pages 189-190 of [9] $S(k)$ is differentiable for $k \in \mathbb{R}$, with continuous derivative for $k \in \mathbb{R} \setminus \{0\}$, provided that $V \in L^1_\gamma(\mathbb{R}^+)$. We denote by $\dot{S}(k)$ the derivative of $S(k)$. Moreover, by Theorem 3.10.6 in page 196-197 of [9],

$$S(k) - S_\infty = O\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \quad (A.1)$$

We now consider the high-energy behavior of the derivative of $S(k)$.

**PROPOSITION A.1.** suppose that (1.3) is satisfied, that $V \in L^1_\gamma(\mathbb{R}^+)$, and that the constant matrices $A, B$ satisfy (1.5), and (1.6). Then,

$$\dot{S}(k) = O\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \quad (A.2)$$

**Proof:** As in [9] we denote, $m(k, x) := e^{-ikx}f(k, x)$. By Proposition 3.9.1 in pages 155-156 of [9] $\dot{m}(k, x)$, and $\dot{m}'(k, x)$ exist for $x \in [0, \infty)$ and $k \in \mathbb{C}^+$, they are analytic in $k \in \mathbb{C}^+$ and continuous in $k \in \mathbb{C}^+$ for each $x \in [0, \infty)$, and they are continuous in $x \in [0, \infty)$ for each $k \in \mathbb{C}^+$. Moreover, by equation (3.2.30) in page 56 of [9],

$$m(k, 0) = I + O\left(\frac{1}{|k|}\right), \quad k \to \infty \text{ in } \mathbb{C}^+. \quad (A.3)$$

Further, by equations (3.9.3) and (3.9.4) in page 156, and equations (3.9.15) in page 157 and (3.9.17) in page 158 of [9], and (A.3)

$$\dot{m}(k, 0) = O\left(\frac{1}{|k|}\right), \dot{m}'(k, 0) = O(1), \quad k \to \infty \text{ in } \mathbb{C}^+. \quad (A.4)$$

By (5.11)

$$\dot{J}(k) = -\dot{m}(-k, 0)\dot{B} - i\dot{m}(-k, 0)^\dagger A + ik\dot{m}(-k, 0)\dot{A} + \dot{m}'(-k, 0)\dot{A}, \quad k \in \mathbb{R}. \quad (A.5)$$

Then, by (A.3)-(A.5)

$$\dot{J}(k) = O(1)A + O\left(\frac{1}{|k|}\right), \quad |k| \to \infty \text{ in } \mathbb{R}. \quad (A.6)$$

Further, by (5.15)

$$\dot{S}(k) = \left(\dot{J}(-k)J_0(-k)^{-1}\right) \left(J_0(-k)J_0(k)^{-1}\right) \left(J_0(k)J(k)^{-1}\right) +
\left(J(-k)J_0(-k)^{-1}\right) \left(J_0(-k)J_0(k)^{-1}\right) \left(J_0(k)J(k)^{-1}\right) \left(\dot{J}(k)J_0(k)^{-1}\right) \left(J_0(k)J(k)^{-1}\right). \quad (A.7)$$

By (A.6) and equations (3.7.11) and (3.7.12) in page 113 of [9],

$$\dot{J}(-k)J_0(-k)^{-1} = O\left(\frac{1}{|k|}\right), \quad |k| \to \infty \text{ in } \mathbb{R}. \quad (A.8)$$

Moreover, by equations (3.6.3) in page 110, and (3.7.3), (3.7.4) in page 113 of [9],

$$J_0(-k)J_0(k)^{-1} = O(1), \quad k \to \infty \text{ in } \mathbb{C}. \quad (A.9)$$

Further, by equations (3.10.17) and (3.10.18) in page 194 of [9],

$$J(k)J_0(k)^{-1} = I + O\left(\frac{1}{|k|}\right), J_0(k)J(k)^{-1} = I + O\left(\frac{1}{|k|}\right), \quad k \to \infty \text{ in } \mathbb{C}^+. \quad (A.10)$$
Finally, by (A.7)-(A.10) we obtain,
\[ \dot{S}(k) = O\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \]  
(A.11)

□

We now study the low-energy behavior of \( \dot{S}(k) \).

PROPOSITION A.2. suppose that (1.3) holds and that the constant matrices \( A, B \) satisfy (1.5), and (1.6). Then,

a) In the generic case where \( J(0) \) is invertible, if \( V \in L^1_\gamma(\mathbb{R}^+) \),
\[ \dot{S}(k) = O(1), \quad |k| \to 0. \]  
(A.12)

b) In the exceptional case where \( J(0) \) is not invertible, if \( V \in L^1_\gamma(\mathbb{R}^+), 2 \leq \gamma \leq 3 \),
\[ \dot{S}(k) = O(|k|^{\gamma-3}), \quad |k| \to 0. \]  
(A.13)

Proof: As in [9] we denote,
\[ m(k,x) := e^{-ikx}f(k,x). \]
Note that \( m(0,x) = f(0,x) \). By equations (3.2.13), (3.2.14), (3.2.15) and (3.2.16) in page 54 of [9], and since,
\[ |e^z - 1| \leq C\frac{|z|}{1+|z|}, \quad z \in \mathbb{C}, \]  
(A.14)
we have
\[ |m(k,x)| \leq C, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+, \]  
(A.15)
provided that \( V \in L^1_\gamma(\mathbb{R}^+) \). By equation (3.9.15) in page 157 of [9],
\[ |\dot{m}(k,x)| \leq C, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+. \]  
(A.16)

By equation (3.9.17) in page 158 of [9],
\[ |\dot{m}'(k,x)| \leq C, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+. \]  
(A.17)

Further, by (A.15) and (A.16)
\[ |m(k,x) - m(0,x)| \leq C \min[|k|, 1], \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+. \]  
(A.18)

Moreover, by equation (3.9.239) in page 190 of [9],
\[ S(k) = S(0) + k\dot{S}(0) + o(|k|), \quad k \to 0. \]  
(A.19)

By (5.15),
\[ \dot{S}(k) = \dot{J}(-k)J(k)^{-1} + J(-k)\dot{J}(k)J(k)^{-1} - \dot{J}(-k)J(k)^{-1} - S(k)\dot{J}(k)J(k)^{-1}. \]  
(A.20)

If \( J(0) \) is invertible, (A.12) follows from Proposition 5.2, (A.5), (A.15), (A.16), (A.17), and the first equality in (A.20). This proves item a). Let us prove b). By equation (3.9.3) in page 156 of [9]
\[ \dot{m}(k,x) = \dot{m}_0(k,x) + \frac{1}{2ik} \int_x^\infty dy \left[ e^{2ik(y-x)} - 1 \right] V(y) \dot{m}(k,y), \]  
(A.21)
where,
\[ \dot{m}_0(k,x) := \frac{1}{2ik^2} \int_x^\infty dy \left[ e^{-2ik(y-x)} - 1 + 2ik(y-x) \right] e^{2ik(y-x)} V(y) m(k,y). \] (A.22)

Further, taking the limit as \( k \to 0 \) in (A.21) and (A.22), and using (A.15) and (A.16) we obtain
\[ \dot{m}(0,x) = \dot{m}_0(0,x) + \int_x^\infty dy \left( y - x \right) V(y) \dot{m}(0,y), \] (A.23)
with,
\[ \dot{m}_0(0,x) := i \int_x^\infty dy \left( y - x \right)^2 e^{2ik(y-x)} V(y) m(0,y). \] (A.24)

Note that
\[ |e^z - 1 - z| \leq C \frac{|z|^2}{1 + |z|}, \quad z \in \mathbb{C}, \] (A.25)
and
\[ |e^z - 1 - z^2 - z| \leq C \frac{|z|^3}{1 + |z|}, \quad z \in \mathbb{C}. \] (A.26)

It follows from (A.14), (A.18), (A.22), (A.24), (A.25) and (A.26) that,
\[ |\dot{m}_0(k,x) - \dot{m}_0(0,x)| \leq C \min \{|k|^{-2}, 1\}, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \] (A.27)

By equations (3.9.6) and (3.9.7) in page 156 of [9],
\[ \dot{m}(k,x) = \sum_{j=0}^\infty \dot{m}_j(k,x), \] (A.28)
where
\[ \dot{m}_j(k,x) := \frac{1}{2ik} \int_x^\infty dy \left[ e^{2ik(y-x)} - 1 \right] V(y) \dot{m}_{j-1}(k,x), \quad j \geq 1, \] (A.29)
and the series in (A.28) is uniformly convergent. Taking the limit as \( k \to 0 \) in (A.28) and (A.29) we get,
\[ \dot{m}(0,x) = \sum_{j=0}^\infty \dot{m}_j(0,x), \] (A.30)
where
\[ \dot{m}_j(0,x) := \int_x^\infty dy \left( y - x \right) V(y) \dot{m}_{j-1}(0,x), \quad j \geq 1. \] (A.31)

By (A.28) and (A.30),
\[ \dot{m}(k,x) - \dot{m}(0,x) = \dot{m}_0(k,x) - \dot{m}_0(0,x) + \sum_{j=1}^\infty (\dot{m}_j(k,x) - \dot{m}_j(0,x)), \] (A.32)
and by (A.29) and (A.31), for \( j \geq 1, \)
\[ \dot{m}_j(k,x) - \dot{m}_j(0,x) = \int_x^\infty dy V(y) \left( \frac{1}{2ik} \left[ e^{2ik(y-x)} - 1 \right] \dot{m}_{j-1}(k,x) - (y - x) \dot{m}_{j-1}(0,x) \right). \] (A.33)

By equation (3.9.11) in page 156 of [9]
\[ |\dot{m}_0(k,x)| \leq C, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \] (A.34)

Further, by equation (3.9.12) in page 157 of [9],
\[ |\dot{m}_j(k,x)| \leq \int_x^\infty dy \left| V(y) \right| |\dot{m}_{j-1}(k,y)|, \quad j \geq 1, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \] (A.35)
Then, by (A.25), and (A.33), for \( j \geq 1 \),
\[
|\dot{m}_j(k,x) - \dot{m}_j(0,x)| \leq C \int_x^\infty dy \left(1 + y\right)^2 |V(y)| \left|k\right| |\dot{m}_{j-1}(k,y)| + |\dot{m}_{j-1}(k,y) - \dot{m}_{j-1}(0,y)|. \tag{A.36}
\]
Without loss of generality we can take the constant \( C \) in (A.27), (A.34), and (A.36) bigger or equal than one. Then, using (A.27), (A.34), (A.35), and (A.36), we prove by mathematical induction that for \( j \geq 0 \)
\[
|\dot{m}_j(k,x) - \dot{m}_j(0,x)| \leq \min\{\left|k\right|^{-2}, 1\} \left( j + 1 \right) C^{j+1} \frac{1}{j!} \left[ \int_x^\infty dy \left(1 + y\right)^2 |V(y)| \right]^{j}, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+. \tag{A.37}
\]
Then, by (A.32) and (A.37)
\[
|\dot{m}(k,x) - \dot{m}(0,x)| \leq C \min\{\left|k\right|^{-2}, 1\} e^{C \int_x^\infty dx \left(1 + y\right)^2 |V(y)|} \left[ 1 + C \int_x^\infty dy \left(1 + y\right)^2 |V(y)| \right] \leq C_1 \min\{\left|k\right|^{-2}, 1\}, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+.
\]
for a constant \( C_1 \).

Furthermore, taking the derivative with respect to \( k \) in both sides of equation (3.2.7) in page 53 of [9] we get
\[
\dot{m}'(k,x) = -2i \int_x^\infty dy \left(y - x\right) e^{2ik(y-x)} V(y) m(k,y) - \int_x^\infty dy e^{2ik(y-x)} V(y) \dot{m}(k,y).
\tag{A.39}
\]
Taking the limit as \( k \to 0 \), in (A.39) and using (A.15) and (A.16) we obtain,
\[
\dot{m}'(0,x) = -2i \int_x^\infty dy \left(y - x\right) V(y) m(0,y) - \int_x^\infty dy V(y) \dot{m}(0,y).
\tag{A.40}
\]
By (A.14), (A.15), (A.16), (A.18), (A.38), (A.39), and (A.40), it follows that,
\[
|\dot{m}'(k,x) - \dot{m}'(0,x)| \leq C \min\{\left|k\right|^{-2}, 1\}, \quad k \in \mathbb{C}^+, x \in \mathbb{R}^+.
\tag{A.41}
\]
Furthermore, by (A.5), (A.16), (A.18), (A.38), and (A.41),
\[
\dot{J}(k) - \dot{J}(0) = O \left(\left|k\right|^{-2}\right), \quad \left|k\right| \to 0 \text{ in } \mathbb{R}.
\tag{A.42}
\]
By equation (3.9.237) in page 189 of [9],
\[
\dot{J}(k)^{-1} = \frac{1}{k} \mathcal{M} + \mathcal{E}_1 + o(1), \quad k \to 0 \text{ in } \mathbb{C}^+,
\tag{A.43}
\]
where \( \mathcal{M} \) and \( \mathcal{E}_1 \) are constant matrices. Then, by (A.19), the second equality in (A.20), (A.42), and (A.43),
\[
\dot{S}(k) = \frac{1}{k} \mathcal{N} + O \left(\left|k\right|^{-3}\right), \quad k \to 0,
\tag{A.44}
\]
where,
\[
\mathcal{N} := \dot{J}(0) \mathcal{M} - S(0) \dot{J}(0) \mathcal{M}.
\tag{A.45}
\]
In the case \( \gamma = 2 \), (A.44) gives us (A.13). When, \( 2 < \gamma \leq 3 \), from (A.44) we obtain, for \( \varepsilon > 0 \),
\[
S(\varepsilon) = S(1) - \int_\varepsilon^1 dk \dot{S}(k) = S(1) + \ln \varepsilon \mathcal{N} + O(1), \varepsilon \downarrow 0.
\tag{A.46}
\]
However, (A.46) is compatible with (A.19) only if \( \mathcal{N} = 0 \). Hence, by (A.44),
\[
\dot{S}(k) = O \left(\left|k\right|^{-3}\right), \quad k \to 0.
\tag{A.47}
\]
This concludes the proof of (A.13). □

The results above give us the following proposition
PROPOSITION A.3. Suppose that $V$ fulfills (1.3) and that the constant matrices $A, B$ satisfy (1.5), and (1.6). Then, $S(k) - S_\infty \in H^{(1)}(\mathbb{R}, M_n)$, provided that in the generic case, where $J(0)$ is invertible, $V \in L^1_2(\mathbb{R}^+)$, and in the exceptional case, where $J(0)$ is not invertible, $V \in L^1_\gamma(\mathbb{R}^+), \gamma > \frac{5}{2}$.

Proof: Since $S(k)$ is differentiable for $k \in \mathbb{R}$, with continuous derivative for $k \in \mathbb{R} \setminus \{0\}$, and it satisfies (A.1) (A.2), (A.12), and (A.13), it follows that it belongs to $H^{(1)}(\mathbb{R}^+, M_n)$.

\[ \square \]

Acknowledgement

This paper was partially written while I was visiting the Institut de Mathématique d’Orsay, Université Paris-Sud. I thank Christian Gérard for his kind hospitality.

References

[1] R. A. Adams and J.J.F. Fournier, *Sobolev Spaces. Second Edition*, Elsevier/Academic Press, Amsterdam, 2003.

[2] Z.S. Agranovich and V. A. Marchenko, Reconstruction of the potential energy from the scattering matrix, Usp. Mat. Nauk (N.S.) 12, 143-145 (1957, in russian)

[3] Z.S. Agranovich and V. A. Marchenko, Re-establishment of the potential from the scattering matrix for a system of differential equations, Dokl. Acad. Nauk SSSR (N.S.) 113, 951-954 (1957, in russian).

[4] Z.S. Agranovich and V. A. Marchenko, Construction of tensor forces from scattering data, Dokl. Acad. Nauk SSSR (N.S.) 118, 1055-1058 (1958, in russian).

[5] Z. S. Agranovich and V. A. Marchenko, *The inverse Problem of Scattering Theory*. Translated from the Russian by B. D. Seckler, Gordon and Breach, New York-London, 1963.

[6] T. Aktosun, M. Klaus, and C. Van der Mee, Small energy asymptotics of the scattering matrix for the matrix Schrödinger equation on the line, J. Math. Phys. 42, 4627-4652 (2001).

[7] T. Aktosun, M. Klaus, and R. Weder, Small-energy analysis for the self-adjoint matrix Schrödinger operator on the half line, J. Math. Phys. 52, 102101 (2011).

[8] T. Aktosun and R. Weder, High-energy analysis and Levinson’s theorem for the self-adjoint matrix Schrödinger operator on the half line, J. Math. Phys. 54, 112108 (2013).

[9] T. Aktosun, and R. Weder, *Direct and Inverse Scattering for the Matrix Schrödinger Equation*, Applied Mathematical Sciences 203, Springer Verlag, New York, 2021.

[10] G. Berkolaio and P. Kuchment, *Introduction to Quantum Graphs*, Mathematical Surveys and Monographs 186, AMS, Providence, RI, 2013.

[11] J. Behrndt and A. Luger, On the number of negative eigenvalues of the Laplacian on a metric graph, J. Phys. A 43, 474006 (2010).

[12] J. Boman and P. Kurasov, Symmetries of quantum graphs and the inverse scattering problem, Adv. Appl. Math. 35, 58–70 (2005).
13. K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd edn., Springer, New York, 1989.

14. S. Cuccagna, Lp continuity of wave operators in Z, J. Math. Anal. Applications, **354**, 594-605 (2009).

15. P. D’Ancona, and L. Fanelli, Lp-boundedness of the wave operator for the one dimensional Schrödinger operator, Comm. Math. Phys. **268**, 415-438 (2006).

16. V. Duchene, J. Marzuola, and M.I. Weinstein, Wave operator bounds for one-dimensional Schrödinger operators with singular potentials, J. Math. Phys. **52**, 013505 (2011).

17. L. D. Faddeev, The inverse problem in the quantum theory of scattering, Usp. Mat. Nauk **14**, 57-119 (1959, in Russian) [J. Math. Phys. **4**, 72-104 (1963) (English translation)].

18. L. D. Faddeev, On the Friedrichs model in the theory of perturbations of the continuous spectrum, Trudy Math. Inst. Steklov **73**, 292-313 (1964).

19. L. Fanelli, Dispersive equations in quantum mechanics, Rend. Mat. Appl. (7) **28**, 237–384 (2008).

20. A. Galtbayar, and K. Yajima, The Lp continuity of wave operators for one dimensional Schrödinger operators, J. Math. Sci. Univ. Tokyo, **7**, 221-240, 2000.

21. I. M. Gelfand and G. E. Shilov, *Generalized Functions Volume I*, Academic Press, New York, 1964.

22. B. Gutkin and U. Smilansky, Can one hear the shape of a graph?, J. Phys. A **34**, 6061–6068 (2001).

23. M. S. Harmer, The Matrix Schrödinger Operator and Schrödinger Operator on Graphs, Ph.D. thesis, University of Auckland, New Zealand, 2004.

24. T. Kato, *Perturbation Theory of Linear Operators. Second Edition*, Springer, Berlin, 1976.

25. V. Kostrykin and R. Schrader, Kirchhoff’s rule for quantum wires, J. Phys. A **32**, 595–630 (1999).

26. V. Kostrykin and R. Schrader, Kirchhoff’s rule for quantum wires. II: The inverse problem with possible applications to quantum computers, Fortschr. Phys. **48**, 703–716 (2000).

27. M. G. Krein, On the theory of accelerants and S-matrices of canonical differential systems, Dokl. Acad. Nauk SSSR **111**, 1167-1180 (1956, in Russian).

28. P. Kuchment, Quantum graphs. I. Some basic structures, Waves Random Media **14**, S107–S128 (2004).

29. P. Kuchment, Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs, J. Phys. A **38**, 4887–4900 (2005).

30. P. Kurasov and F. Stenberg, On the inverse scattering problem on branching graphs, J. Phys. A **35**, 101–121 (2002).

31. P. Kurasov and M. Nowaczyk, Inverse spectral problem for quantum graphs, J. Phys. A **38**, 4901–4915 (2005).

32. P. Kurasov and M. Nowaczyk, Geometric properties of quantum graphs and vertex scattering matrices, Opusc. Math. **30**, 295–309 (2010).

33. L. D. Landau and E. M. Lifschitz, *Quantum Mechanics Non-Relativistic Theory*, 3rd edn., Pergamon Press, New York, 1989.
[34] I. Naumkin and R. Weder, $L^p - L^{p'}$ estimates for matrix Schrödinger equations, J. Evol. Equ. 21, 891–919 (2021), arXiv 1906.07846 [math-ph] (2019).

[35] R. G. Newton, Connection between the $S$-matrix and the tensor force, Phys Rev. D 100, 412-428 (1955).

[36] M. Reed and B. Simon, Methods of Modern Mathematical Physics III: Scattering Theory, Academic Press, San Diego, 1979.

[37] F. S. Rofe-Beketov and A. M. Kholkin, Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillation Properties. World Scientific, Singapore, 2005.

[38] W. Schlag, Dispersive estimates for Schrödinger operators: a survey, Mathematical Aspects of Nonlinear Dispersive Equations, 255-285, Ann. Math. Stud. 163, Princeton Univ. Pres, NJ, 2007.

[39] C. Sadowsky, Interpolation of Operators and Singular Integrals, Marcel Dekker, 1979, New York.

[40] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, NJ, 1970.

[41] R. Weder, The Wkp continuity of the Schrödinger wave operators on the line, Comm. Math. Phys. 208, 507-520 (1999).

[42] D. R. Yafaev, Mathematical Scattering Theory General Theory, Translations of Mathematical Monographs 105, Amer. Math. Soc., Providence, Rhode Island, 1992.

[43] K. Yajima, The $L_p$ boundedness of the wave operators for two dimensional Schrödinger operators with threshold singularities, arXiv:2008.07906v2 [math.AP] (2021).