A BLOWUP ALTERNATIVE RESULT FOR FRACTIONAL NON-AUTONOMOUS EVOLUTION EQUATION OF VOLTERRA TYPE

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Abstract. In this article, we consider a class of fractional non-autonomous integro-differential evolution equation of Volterra type in a Banach space $E$, where the operators in linear part (possibly unbounded) depend on time $t$. Combining the theory of fractional calculus, operator semigroups and measure of noncompactness with Sadovskii’s fixed point theorem, we firstly proved the local existence of mild solutions for corresponding fractional non-autonomous integro-differential evolution equation. Based on the local existence result and a piecewise extended method, we obtained a blowup alternative result for fractional non-autonomous integro-differential evolution equation of Volterra type. Finally, as a sample of application, these results are applied to a time fractional non-autonomous partial integro-differential equation of Volterra type with homogeneous Dirichlet boundary condition. This paper is a continuation of Heard and Rakin [13, J. Differential Equations, 1988] and the results obtained essentially improve and extend some related conclusions in this area.

1. Introduction. Evolution equations describe time dependent processes as they occur in physics, chemistry, economy, biology or other sciences. Mathematically, they appear in quite different forms, such as parabolic or hyperbolic partial differential equations, integro-differential equations, delay or difference differential equations or more general functional differential equations. When treating such classes of evolution equations, it is usually assumed that the partial differential operators depend on time $t$ on account of this class of operators appears frequently in the applications, for the details please see Pazy [23] and Tanabe [25]. As a result, it is significant and interesting to investigate non-autonomous evolution equations, i.e., the differential operators in the main parts of the considered problems are dependent of time $t$. In 1988, Heard and Rakin [13] considered the following Volterra integro-differential equation

\[
\begin{align*}
  u'(t) + A(t)u(t) &= f(t, u(t)) + \int_{t_0}^{t} q(t-s)g(s, u(s))ds, \quad t \geq t_0 \geq 0, \\
  u(t_0) &= u_0
\end{align*}
\]

in a Banach space $E$, where for each $t \geq 0$, the linear operator $-A(t)$ is the infinitesimal generator of an analytic semigroup in $E$, the nonlinear function $f :$
[0, +∞) \times E \to E satisfies a Hölder condition of the form
\[
\|f(t, y_1) - f(s, y_2)\| \leq C|t - s|^\eta + \|y_1 - y_2\|_E^\nu,
\]
\(\|\cdot\|\) denotes the norm on Banach space \(E\), \(\|\cdot\|_\mu\) denotes the graph norm on \(E_\mu = D(A^\mu(0))\), \(\eta, \gamma, \nu\) are positive constants satisfying \(0 < \eta, \mu, \gamma < 1\), and the nonlinear map \(g\) is Lipschitz continuous on the domain of \(A(0)\) into \(E\), with respect to the graph norm of \(A(0)\). Also, the uniqueness of solution is proved under the restriction that the space \(E\) is a Hilbert space and \(\gamma = 1\).

Fractional calculus is a mathematical topic more than 300 years. The concept of noninteger derivative and integral is a generalization of the traditional integer-order differential and integral calculus. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In recent years, fractional differential equations especially fractional evolution equations have attracted great interest because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance and other areas of science and engineering. For more details, see [2, 4, 6, 8, 9, 11, 12], [18]-[22], [28]-[32] and the references therein for more comments and citations.

Recently, in 2011, Rashid and Al-Omari [24] studied the local and global existence of mild solutions to a class of fractional semilinear impulsive Volterra type integro-differential evolution equation in Banach space \(E\)
\[
\begin{cases}
  C D^\alpha_t u(t) + A u(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, & t \geq 0, \\
  \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, 2, \ldots, m, \\
  u(0) = u_0
\end{cases}
\tag{1.2}
\]
by using the fixed point technique, where \(C D^\alpha_t\) is the standard Caputo’s fractional time derivative of order \(\alpha \in (0, 1]\), \(-A\) is assumed to be an infinitesimal generator of a compact \(C_0\)-semigroup \(T(t)\) \((t \geq 0)\), the nonlinear maps \(f, g : I \times E \to E, I = [0, a], 0 < a < \infty\), are continuous, \(q : I \to \mathbb{R}\) and \(u_0 \in E\). \(I_k : E \to E, 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} := a, \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), u(t_k^+)\) and \(u(t_k^-)\) represent respectively the right and left limits of \(u(t)\) at \(t = t_k\). In 2017, Gou and Li [12] generalized to the case that the \(C_0\)-semigroup \(T(t)\) \((t \geq 0)\) generated by \(-A\) is non-compact, and obtained the local and global existence of mild solution for fractional impulsive Volterra type integro-differential evolution equation (1.2) with non-compact semigroup in Banach space \(E\).

We noticed that the papers [12] and [24] are all devoted to investigating the local and global existence of mild solution for fractional Volterra type integro-differential evolution equation under the situation that the differential operators in the main parts are independent of time \(t\), which means that the fractional Volterra type integro-differential evolution equations under considerations are autonomous. As we pointed out in the first paragraph, it is significant and interesting to investigate non-autonomous evolution equations, which means that the differential operators in the main parts of the considered problems are dependent of time \(t\). In fact, fractional non-autonomous evolution equations have been studied by several authors in recent years. For example, El-Borai [8] investigated the existence and continuous dependence of fundamental solutions for a class of linear fractional non-autonomous evolution equations in 2004. In 2010, El-Borai, El-Nadi and El-Akabawy [9] give
some conditions to ensure the existence of resolvent operator for a class of fractional non-autonomous evolution equations with classical Cauchy initial condition. In [22, 32], the authors investigated the nonlinear fractional non-autonomous reaction-diffusion equation with delay by using appropriate fixed point theory.

Motivated by the above mentioned aspects, in this paper we will combine these earlier works and extend the study to the following fractional non-autonomous integro-differential evolution equation (FNEE) of Volterra type in Banach space $E$

$$
\begin{align*}
C D_t^\alpha u(t) + A(t)u(t) &= f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, \quad t \geq 0, \\
u(0) &= u_0,
\end{align*}
$$

(1.3)

which is more general than those in many previous publications, where $C D_t^\alpha$ is the standard Caputo’s fractional time derivative of order $0 < \alpha \leq 1$, $A(t)$ is a family of closed linear operators defined on a dense domain $D(A)$ in Banach space $E$ into $E$ such that $D(A)$ is independent of $t$, $f$, $g : [0, +\infty) \times E \to E$ are continuous nonlinear functions, the integral kernel function $q : [0, T) \to E$ is locally integrable for $0 < T \leq +\infty$, $u_0 \in E$. One can easily see that when $\alpha = 1$, then the problem (1.3) will degrade into (1.1) studied by Heard and Rakin in [13].

Let us point out that the work of this paper has three wedges; Firstly, we extended the study of first order non-autonomous integro-differential evolution equation of Volterra type to fractional ones. Secondly, we extended the study of fractional autonomous integro-differential evolution equation of Volterra type to non-autonomous ones. Lastly, we proved that fractional non-autonomous integro-differential evolution equation of Volterra type (1.3) exists a mild solution $u \in C([0, T_{max}), E)$ on a maximal existence interval $[0, T_{max})$ and satisfies the blowup alternative under weaker conditions on the nonlinear items.

In this article, by using the famous Sadovskii’s fixed point theorem, a new estimation technique of the measure of noncompactness and a piecewise extended method, we obtained a blowup alternative result for fractional non-autonomous integro-differential evolution equation of Volterra type (1.3). The results obtained in this paper are generalizations of related results. As the readers can see, the hypotheses on the nonlinear items in our theorems are reasonably weak and different from those in many previous papers such as [22, 24, 32], and the proofs provided are concise. One will see that even for the case $\alpha = 1$, Theorem 4.1 below essentially extends the main results of Heard and Rakin [13]; as far as the mild solution of non-autonomous integro-differential evolution equation of Volterra type is concerned, by dropping the Hölder continuity of the nonlinear items from the hypotheses. Moreover, even for corresponding fractional autonomous integro-differential evolution equation of Volterra type, the results here are new.

The rest of this article is organized as follows. We provide in Section 2 some definitions, notations and necessary preliminaries. The local existence of mild solutions for fractional non-autonomous integro-differential evolution equation of Volterra type is obtained in Section 3. In Section 4, we proved that the fractional autonomous integro-differential evolution equation of Volterra type (1.3) exists a mild solution $u \in C([0, T_{max}), E)$ on a maximal existence interval $[0, T_{max})$ and satisfies the blowup alternative. An concrete example is given to illustrate the feasibility of our main results in the last section.
2. Preliminaries. We begin with this section by giving some notations. Let $E$ be a Banach space with norm $\| \cdot \|$. Throughout this work, we use $I$ to denote the closed subset of the interval $[0, +\infty)$. We denote by $C(I, E)$ the Banach space of all continuous functions from interval $I$ into $E$ equipped with the supremum norm $\|u\|_C = \sup_{t \in I} \|u(t)\|$, and by $L(E)$ the Banach space of all bounded linear operators from $E$ to $E$ endowed with the topology defined by operator norm. Let $L^1(I, E)$ be the Banach space of all $E$-value Bochner integrable functions defined on $I$ with the norm $\|u\|_1 = \int_I \|u(t)\| dt$.

Secondly, we introduce some basic definitions about the Riemann-Liouville integral and Caputo derivative of fractional order.

**Definition 2.1** ([20]). The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function $f \in L^1([0, +\infty), \mathbb{R})$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$  

Here and elsewhere $\Gamma(\cdot)$ denotes the Gamma function.

**Definition 2.2** ([20]). The Caputo fractional derivative of order $\alpha$ with the lower limit zero for a function $f : [0, +\infty) \to \mathbb{R}$, which is at least $n$-times differentiable can be defined as

$$C^D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I_t^{n-\alpha} f(t),$$  

where $n-1 < \alpha < n$, $n \in \mathbb{N}$.

**Remark 2.3.** If $f$ is an abstract function with values in $E$, then the integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner’s sense. A measurable function $u : [0, +\infty) \to E$ is Bochner integrable if $\|u\|$ is Lebesgue integrable.

Thirdly, we give the proper definition of of mild solutions for FNEE (1.3). Throughout this paper, we assume that the linear operator $-A(t)$ satisfies the following conditions:

(A1) For any $\lambda$ with $\Re \lambda \geq 0$, the operator $\lambda I + A(t)$ exists a bounded inverse operator $[\lambda I + A(t)]^{-1}$ in $L(E)$ and

$$\| [\lambda I + A(t)]^{-1} \| \leq \frac{C}{|\lambda| + 1},$$  

where $C$ is a positive constant independent of both $t$ and $\lambda$;

(A2) For any $t, \tau, s \in I$, there exists a constant $\gamma \in (0, 1]$ such that

$$\| [A(t) - A(\tau)] A^{-1}(s) \| \leq C |t - \tau|^\gamma,$$  

where the constants $\gamma$ and $C > 0$ are independent of both $t$, $\tau$ and $s$.

From Henry [15], Pazy [23] and Temam [26], we know that the assumption (A1) means that for each $s \in I$, the operator $-A(s)$ generates an analytic semigroup $e^{-tA(s)} (t > 0)$, and there exists a positive constant $C$ independent of both $t$ and $s$ such that

$$\| A^n(s) e^{-tA(s)} \| \leq \frac{C}{t^n},$$  

for all $t > 0$ and $n = 1, 2, \ldots$.
where \( n = 0, 1, t > 0, s \in I \). Furthermore, in assumption (A1), if we choose \( \lambda = 0 \) and \( t = 0 \), then there exists a positive constant \( C \) independent of both \( t \) and \( \lambda \) such that
\[
\|A^{-1}(0)\| \leq C.
\]

By the above discussion and \([8, \text{Theorem 2.6}]\), we can get the definition of mild solutions for FNEE (1.3).

**Definition 2.4.** By a mild solution of FNEE (1.3) on \([0, +\infty)\), we mean a continuous function \( u \) defined from \([0, +\infty)\) into \( E \) satisfying
\[
u(t) = u_0 + \int_0^t \psi(t - \eta, \eta)U(\eta)A(0)u_0 d\eta
- \int_0^t \psi(t - \eta, \eta)\int_0^\eta q(\eta - \tau)g(\tau, u(\tau))d\tau d\eta
+ \int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s)\left[f(s, u(s)) + \int_0^s q(s - \tau)g(\tau, u(\tau))d\tau\right]dsd\eta,
\]
where the operators \( \psi(t, s), \varphi(t, \eta) \) and \( U(t) \) are defined by
\[
\psi(t, s) = \alpha \int_0^\infty \theta^{\alpha - 1}\xi_{\alpha}(\theta)e^{-t^\alpha A(s)}d\theta, \tag{2.2}
\]
\[
\varphi(t, \eta) = \sum_{k=1}^{\infty} \varphi_k(t, \eta), \tag{2.3}
\]
and
\[
U(t) = -A(t)A^{-1}(0) - \int_0^t \varphi(t, s)A(s)A^{-1}(0)ds, \tag{2.4}
\]
\( \xi_{\alpha} \) is a probability density function defined on \([0, +\infty)\) such that it’s Laplace transform is given by
\[
\int_0^\infty e^{-\theta x}\xi_{\alpha}(\theta)d\theta = \sum_{i=0}^{\infty} \frac{(-x)^i}{\Gamma(1 + \alpha i)}, \quad 0 < \alpha \leq 1, \ x > 0,
\]
\[
\varphi_1(t, \eta) = [A(t) - A(\eta)]\psi(t - \eta, \eta),
\]
\[
\varphi_{k+1}(t, \eta) = \int_\eta^t \varphi_k(t, s)\varphi_1(s, \eta)ds, \quad k = 1, 2, \cdots.
\]

The following properties about the operators \( \psi(t, s), \varphi(t, \eta) \) and \( U(t) \) will be needed in our argument.

**Lemma 2.5** ([8]). The operator-valued functions \( \psi(t - \eta, \eta) \) and \( A(t)\psi(t - \eta, \eta) \) are continuous in uniform topology about the variables \( t \) and \( \eta \), where \( t \in I, 0 \leq \eta \leq t - \epsilon \) for any \( \epsilon > 0 \), and
\[
\|\psi(t - \eta, \eta)\| \leq C(t - \eta)^{\alpha - 1}, \tag{2.5}
\]
where \( C \) is a positive constant independent of both \( t \) and \( \eta \). Furthermore,
\[
\|\varphi(t, \eta)\| \leq C(t - \eta)^{\gamma - 1}, \tag{2.6}
\]
and
\[ \|U(t)\| \leq C(1 + t^r). \] (2.7)

Next, we introduce the definition for Kuratowski measure of noncompactness, which will be used in the proof of our main results.

**Definition 2.6** ([3, 7]). The Kuratowski measure of noncompactness \( \mu(\cdot) \) defined on bounded set \( S \) of Banach space \( E \) is
\[ \mu(S) := \inf \{ \delta > 0 : S = \bigcup_{k=1}^{n} S_k \text{ and } \text{diam}(S_k) \leq \delta \text{ for } k = 1, 2, \ldots, n \}. \]

The following properties about the Kuratowski measure of noncompactness are well known.

**Lemma 2.7** ([3, 7]). Let \( E \) be a Banach space and \( U, V \subset E \) be bounded. The following properties are satisfied:

(i) \( \mu(U) = 0 \) if and only if \( U \) is compact, where \( U \) means the closure hull of \( U \);
(ii) \( \mu(\lambda U) = |\lambda|\mu(U) \), where \( \lambda \in \mathbb{R} \);
(iii) \( \mu(U) = \mu(\overline{U}) = \mu(\text{conv } U) \), where \( \text{conv } U \) means the convex hull of \( U \);
(iv) \( \mu(U \cup V) = \max\{\mu(U), \mu(V)\} \);
(v) \( U \subset V \) implies \( \mu(U) \leq \mu(V) \);
(vi) \( \mu(U + V) \leq \mu(U) + \mu(V) \), where \( U + V = \{x \mid x = y + z, y \in U, z \in V\} \);
(vii) If the map \( F : D(F) \subset E \to X \) is Lipschitz continuous with constant \( k \), then \( \mu(F(S)) \leq k \mu(S) \) for any bounded subset \( S \subset D(F) \), where \( X \) is another Banach space.

In this article, we denote by \( \mu(\cdot) \) and \( \mu_C(\cdot) \) the Kuratowski measure of noncompactness on the bounded set of \( E \) and \( C(I, E) \) respectively. For any \( B \subset C(I, E) \) and \( t \in I \), set \( D(t) = \{ u(t) \mid u \in D \} \), then \( D(t) \subset E \). If \( D \subset C(I, E) \) is bounded, then \( D(t) \) is bounded in \( E \) and \( \mu(D(t)) \leq \mu_C(D) \). For more details about the properties of Kuratowski measure of noncompactness, we refer to the monographs [3] and [7].

The following lemmas are needed in our argument.

**Lemma 2.8** ([5, 17]). Let \( E \) be a Banach space, and let \( D \subset C(I, E) \) be bounded. Then there exists a countable set \( D_0 \subset D \), such that \( \mu(D) \leq 2\mu(D_0) \).

**Lemma 2.9** ([14]). Let \( E \) be a Banach space. If \( D = \{ u_n \}_{n=1}^{\infty} \subset C(I, E) \) is a countable set and there exists a function \( m \in L^1(I, \mathbb{R}^+) \) such that for every \( n \in \mathbb{N} \),
\[ \|u_n(t)\| \leq m(t), \quad \text{a.e. } t \in I. \]

Then \( \mu(D(t)) \) is Lebesgue integrable on \( I \), and
\[ \mu\left( \left\{ \int_I u_n(t)dt \mid n \in \mathbb{N} \right\} \right) \leq 2\int_I \mu(D(t))dt. \]

**Lemma 2.10** ([3]). Let \( E \) be a Banach space, and let \( D \subset C(I, E) \) be bounded and equicontinuous. Then \( \mu(D(t)) \) is continuous on \( I \), and \( \mu_C(D) = \max_{t \in I} \mu(D(t)) \).
3. Local existence of mild solutions. In this section, we prove the local existence of mild solutions for fractional non-autonomous integro-differential evolution equation (FNNE) of Volterra type with initial time \( t_0 \in (0, \infty) \) and initial value \( x_0 \in E \)

\[
\begin{cases}
C D_t^\alpha u(t) + A(t) u(t) = f(t, u(t)) + \int_0^t q(t-s) g(s, u(s)) \, ds, & t \geq t_0, \\
u(t_0) = x_0
\end{cases}
\]

(3.1)
on interval \( J := [t_0, t_0 + h] \), where \( h = h(t_0, \|x_0\|) > 0 \) is a constant will be specified later.

**Theorem 3.1.** Assume that the nonlinear functions \( f, g : [t_0, t_0 + 1] \times E \to E \) are continuous, map bounded sets in \( [t_0, t_0 + 1] \times E \) into bounded sets in \( E \) and satisfy the following condition:

**(H_f)** For any \( R > 0 \), there exist positive constants \( L_f = L_f(R, 1) \) and \( L_g = L_g(R, 1) \) such that for any equicontinuous and countable set \( D \subset B_R = \{ u \in E : \|u\| \leq R \}, \)

\[
\mu(f(t, D)) \leq L_f \mu(D), \quad \mu(g(t, D)) \leq L_g \mu(D), \quad t \in [t_0, t_0 + 1].
\]

Then for every \( x_0 \in E \), there exists a positive constant \( h = h(t_0, \|x_0\|) \leq 1 \) such that FNNE (3.1) has a mild solution \( u \in C([t_0, t_0 + h], E). \)

**Proof.** Define an operator \( Q : C(J, E) \to C(J, E) \) by

\[
(Qu)(t) = x_0 + \int_{t_0}^t \psi(t-\eta, \eta) f(\eta, u(t)) \, d\eta + \int_{t_0}^t \psi(t-\eta, \eta) \int_0^\eta q(\eta - \tau) g(\tau, u(\tau)) \, d\tau \, d\eta
+ \int_{t_0}^t \int_0^\eta \psi(t-\eta, \eta) \varphi(\eta, s) f(s, u(s)) \, ds \, d\eta
+ \int_0^t q(s - \tau) g(\tau, u(\tau)) \, d\tau \, ds, \quad t \in J.
\]

(3.2)

By direct calculus we know that the operator \( Q \) is well defined. From Definition 2.4 and (3.2), it is easy to verify that the mild solution of FNNE (3.1) on \( J \) is equivalent to the fixed point of the operator \( Q \) defined by (3.2).

Denote

\[
R(t_0) = 2\|x_0\| + 1, \quad L_f(t_0) = L_f(R, 1), \quad L_g(t_0) = L_g(R, 1),
\]

\[
M_f(t_0) = \sup \{ \|f(t, u(t))\| : \|u(t)\| \leq R(t_0), t_0 \leq t \leq t_0 + 1 \},
\]

\[
M_g(t_0) = \sup \{ \|g(t, u(t))\| : \|u(t)\| \leq R(t_0), t_0 \leq t \leq t_0 + 1 \},
\]

\[
q^* = \sup_{t_0 \leq s \leq t_0 + 1} \int_0^s \|q(t-s)\| \, ds,
\]

\[
N = 4C \left( \frac{1}{\alpha} + \frac{2C B(\alpha, \gamma)}{\alpha + \gamma} \right) (L_f + 2q^* L_g),
\]

\[
M = C^2 \left( \frac{1}{\alpha} + B(\alpha, \gamma) + 1 \right) \|A(t_0)x_0\| + \left( \frac{C}{\alpha} + \frac{C^2 B(\alpha, \gamma)}{\alpha + \gamma} \right) \left( M_f(t_0) + q^* M_g(t_0) \right),
\]

where

\[
B(\alpha, \gamma) = \int_0^1 t^{\alpha-1} (1-t)^{\gamma-1} \, dt
\]
is the Beta function. We firstly prove that the operator \( Q \) defined by (3.2) maps the bounded closed convex set \( \Omega = \{ u \in C(J,E) : \| u(t) \| \leq R(t), t \in J \} \) into itself. Set

\[
h = h(t_0, \| x_0 \|) := \min \left\{ 1, \left( \frac{\| x_0 \| + 1}{M} \right)^{\frac{1}{\beta}}, \left( \frac{1}{N} \right)^{\frac{1}{\gamma}} \right\}.
\]

For any \( u \in \Omega \) and \( t \in J \), by Lemma 2.5, (3.2) and (3.3), we get that

\[
\| (Qu)(t) \|
\leq \| x_0 \| + \left\| \int_{t_0}^{t} \psi(t - \eta, \eta) U(\eta) A(t_0) x_0 d\eta \right\|
+ \left\| \int_{t_0}^{t} \psi(t - \eta, \eta) \left[ f(\eta, u(\eta)) + \int_{0}^{\eta} q(\eta - \tau) g(\tau, u(\tau)) d\tau \right] d\eta \right\|
+ \left\| \int_{t_0}^{t} \int_{t_0}^{\eta} \psi(t - \eta, \eta) \varphi(\eta, s) \left[ f(s, u(s)) + \int_{0}^{s} q(s - \tau) g(\tau, u(\tau)) d\tau \right] ds d\eta \right\|
\leq \| x_0 \| + C^2 \int_{t_0}^{t} (t - \eta)^{\alpha - 1}(1 + \eta^\beta) d\eta \| A(t_0) x_0 \|
+ C \left( M_f(t_0) + q^* M_g(t_0) \right) \int_{t_0}^{t} (t - \eta)^{\alpha - 1} d\eta
+ C^2 \left( M_f(t_0) + q^* M_g(t_0) \right) \int_{t_0}^{t} \int_{t_0}^{\eta} (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} ds d\eta
\leq \| x_0 \| + C^2 h^\alpha \left( \frac{1}{\alpha} + h^\beta B(\alpha, \gamma + 1) \right) \| A(t_0) x_0 \|
+ Ch^\alpha \left( M_f(t_0) + q^* M_g(t_0) \right) + \frac{C^2 \beta B(\alpha, \gamma) h^\alpha + \gamma}{\alpha + \gamma} \left( M_f(t_0) + q^* M_g(t_0) \right)
\leq \| x_0 \| + M h^\alpha \leq R(t_0),
\]

which means that \( Qu \in \Omega \). Therefore, we have proved that the operator \( Q \) maps \( \Omega \) to \( \Omega \).

Secondly, we prove that \( Q : \Omega \to \Omega \) is a continuous operator. To this end, let \( \{ u_n \}_{n=1}^{\infty} \subset \Omega \) be a sequence such that \( \lim_{n \to +\infty} u_n = u \) in \( \Omega \). By the continuity of the second variable for the nonlinear functions \( f \) and \( g \), we get that for any \( t \in J \)

\[
\lim_{n \to +\infty} \| f(t, u_n(t)) - f(t, u(t)) \| = 0, \quad \lim_{n \to +\infty} \| g(t, u_n(t)) - g(t, u(t)) \| = 0.
\]

(3.4) combined with the boundedness of nonlinear functions \( f \) and \( g \), one gets that for every \( t \in J \)

\[
\left\| f(t, u_n(t)) + \int_{0}^{t} q(t - s) g(s, u_n(s)) ds - f(t, u(t)) - \int_{0}^{t} q(t - s) g(s, u(s)) ds \right\|
\leq \left\| f(t, u_n(t)) - f(t, u(t)) \right\| + \int_{0}^{t} \| q(t - s) \| ds \sup_{s \in [0,t]} \| g(s, u_n(s)) - g(s, u(s)) \|
\to 0 \quad \text{as} \quad n \to +\infty.
\]

By again Lemma 2.5, (3.2) and (3.3), we have

\[
\| (Qu)_n(t) - (Qu)(t) \|
\leq \left\| \int_{t_0}^{t} \psi(t - \eta, \eta) \left[ f(\eta, u_n(\eta)) + \int_{0}^{\eta} q(\eta - \tau) g(\tau, u_n(\tau)) d\tau \right] d\eta \right\|
\leq \left\| \int_{t_0}^{t} \psi(t - \eta, \eta) \left[ f(\eta, u(\eta)) + \int_{0}^{\eta} q(\eta - \tau) g(\tau, u(\tau)) d\tau \right] d\eta \right\|
\leq C \| A(t_0) x_0 \|
\leq R(t_0),
\]

where the last inequality holds because \( Qu \in \Omega \). Therefore, \( (Qu)_n \) converges to \( Qu \) in \( \Omega \).
\[-f(\eta, u(\eta)) - \int_0^\eta q(\eta - \tau)g(\tau, u(\tau))d\tau \]
\[+ \left\| \int_{t_0}^t \int_{0}^\eta \psi(t - \eta, \eta)\varphi(\eta, s) \left[ f(s, u_n(s)) + \int_0^s q(s - \tau)g(\tau, u_n(\tau))d\tau - f(s, u(s)) - \int_0^s q(s - \tau)g(\tau, u(\tau))d\tau \right] d\eta \right\|
\[\leq C \int_{t_0}^t (t - \eta)^{\alpha - 1} \left\| f(\eta, u_n(\eta)) + \int_0^\eta q(\eta - \tau)g(\tau, u_n(\tau))d\tau \right\| d\eta \]
\[+ C^2 \int_{t_0}^t \int_{t_0}^\eta (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} \left\| f(s, u_n(s)) + \int_0^s q(s - \tau)g(\tau, u_n(\tau))d\tau \right\| d\eta \]
\[\leq 2 \left[ M_f(t_0) + q^* M_g(t_0) \right] (t - \eta)^{\alpha - 1}. \quad \text{(3.6)}\]

From the boundedness of nonlinear functions \(f\) and \(g\), we know that for every \(t \in J\) and \(t_0 \leq \eta \leq t\)
\[(t - \eta)^{\alpha - 1} \left\| f(\eta, u_n(\eta)) + \int_0^\eta q(\eta - \tau)g(\tau, u_n(\tau))d\tau \right\|
\[-f(\eta, u(\eta)) - \int_0^\eta q(\eta - \tau)g(\tau, u(\tau))d\tau \right\|
\[\leq 2 \left[ M_f(t_0) + q^* M_g(t_0) \right] (t - \eta)^{\alpha - 1}. \quad \text{(3.7)}\]

By again the boundedness of nonlinear functions \(f\) and \(g\) combined with proper integral transformation and the definition of Beta function, we get that for every \(t \in J\), \(t_0 \leq \eta \leq t\) and \(t_0 \leq s \leq \eta\)
\[\int_{t_0}^t \int_{t_0}^\eta (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} \left\| f(s, u_n(s)) + \int_0^s q(s - \tau)g(\tau, u_n(\tau))d\tau \right\| d\eta \]
\[\leq 2 \int_{t_0}^t \int_{t_0}^\eta (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} \left[ M_f(t_0) + q^* M_g(t_0) \right] d\eta \]
\[= 2B(\alpha, \gamma) \left[ M_f(t_0) + q^* M_g(t_0) \right] \int_{t_0}^t (t - \eta)^{\alpha + \gamma - 1} d\eta. \quad \text{(3.8)}\]

From the fact that the functions \(\eta \to 2 \left[ M_f(t_0) + q^* M_g(t_0) \right] (t - \eta)^{\alpha - 1}\) and \(\eta \to 2B(\alpha, \gamma) \left[ M_f(t_0) + q^* M_g(t_0) \right] (t - \eta)^{\alpha + \gamma - 1}\) are Lebesgue integrable for any \(\eta \in [t_0, t]\) and \(t \in J\), combined with (3.5)-(3.8) and the Lebesgue dominated convergence theorem, we know that for every \(t \in J\),
\[\|(Q u_n)(t) - (Qu)(t)\| \to 0 \quad \text{as} \quad n \to \infty,\]
which implies that
\[\|Q u_n - Qu\|_C \to 0 \quad \text{as} \quad n \to \infty.\]

Therefore, the operator \(Q : \Omega \to \Omega\) is continuous.
In what follows, we prove that the operator \( Q : \Omega \to \Omega \) is equicontinuous. For any \( u \in \Omega \) and \( t_0 \leq t_1 < t_2 \leq t_0 + h \), we get from (3.2) that
\[
(Qu)(t_2) - (Qu)(t_1) \\
\leq \int_{t_1}^{t_2} \psi(t_2 - \eta, \eta)U(\eta)A(t_0)x_0 d\eta \\
+ \int_{t_0}^{t_1} [\psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta)]U(\eta)A(t_0)x_0 d\eta \\
+ \int_{t_1}^{t_2} \psi(t_2 - \eta, \eta) \left[ f(\eta, u(\eta)) + \int_{0}^{\eta} q(\eta - \tau)g(\tau, u(\tau)) d\tau \right] d\eta \\
+ \int_{t_0}^{t_1} [\psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta)] \left[ f(\eta, u(\eta)) + \int_{0}^{\eta} q(\eta - \tau)g(\tau, u(\tau)) d\tau \right] d\eta \\
+ \int_{t_1}^{t_2} \int_{0}^{\eta} \psi(t_2 - \eta, \eta) \varphi(\eta, s) \left[ f(s, u(s)) + \int_{0}^{s} q(s - \tau)g(\tau, u(\tau)) d\tau \right] d\eta ds \\
+ \int_{0}^{t_1} \int_{0}^{\eta} \psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta) \varphi(\eta, s) \left[ f(s, u(s)) + \int_{0}^{s} q(s - \tau)g(\tau, u(\tau)) d\tau \right] d\eta ds \\
\leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\]
which means that
\[
\| (Qu)(t_2) - (Qu)(t_1) \| \leq \sum_{i=1}^{6} \| I_i \|.
\]

Therefore, in order to prove that the operator \( Q : \Omega \to \Omega \) is equicontinuous, we only need to check \( \| I_i \| \) tend to 0 independently of \( u \in \Omega \) when \( t_2 - t_1 \to 0 \), \( i = 1, 2, \cdots, 6 \).

For \( I_1 \), from Lemma 2.5 and the fact that the function \( \eta \to (t_2 - \eta)^{\alpha - 1}(1 + \eta^\gamma) \) is Lebesgue integrable we get that
\[
\| I_1 \| \leq C^2 \| A(t_0)x_0 \| \int_{t_1}^{t_2} (t_2 - \eta)^{\alpha - 1}(1 + \eta^\gamma) d\eta \to 0 \quad \text{as} \quad t_2 - t_1 \to 0.
\]

For \( t_1 = t_0 \) and \( t_0 < t_2 \leq t_0 + h \), it is easy to see that \( \| I_2 \| = 0 \). For \( t_1 > t_0 \) and \( \epsilon > 0 \) small enough, by Lemma 2.5 and the facts that the function \( \eta \to [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}](1 + \eta^\gamma) \) is Lebesgue integrable and operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for \( t_0 < t_2 \leq t_0 + h \) and \( t_0 \leq \eta \leq t - \epsilon \), we have
\[
\| I_2 \| \leq \sup_{\eta \in [t_0, t_1 - \epsilon]} \| \psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta) \| \cdot C \| A(t_0)x_0 \| \int_{t_0}^{t_1 - \epsilon} (1 + \eta^\gamma) d\eta \\
+ C^2 \| A(t_0)x_0 \| \int_{t_1 - \epsilon}^{t_1} [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}](1 + \eta^\gamma) d\eta \\
\to 0 \quad \text{as} \quad t_2 - t_1 \to 0 \quad \text{and} \quad \epsilon \to 0.
\]

For \( I_3 \), by Lemma 2.5 and the boundedness of nonlinear functions \( f \) and \( g \), one get that
\[
\| I_3 \| \leq C \int_{t_1}^{t_2} (t_2 - \eta)^{\alpha - 1} \left( M_f(t_0) + q^* M_g(t_0) \right) d\eta
\]
\[
C \frac{1}{\alpha} \left( M_f(t_0) + q^* M_g(t_0) \right) (t_2 - t_1)^\alpha \rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0.
\]

For \( t_1 = t_0 \) and \( t_0 < t_2 \leq t_0 + h \), it is easy to see that \( \|I_4\| = 0 \). For \( t_1 > t_0 \) and \( \epsilon > 0 \) small enough, from the boundedness of nonlinear functions \( f \) and \( g \), Lemma 2.5 and the facts that the function \( \eta \rightarrow [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}] \) is Lebesgue integrable and operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for \( t_0 \leq t \leq t_0 + h \) and \( t_0 \leq \eta \leq t - \epsilon \), we have

\[
\|I_4\| \leq \sup_{\eta \in [t_0, t_0 - \epsilon]} \| \psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta) \| \cdot (t_1 - t_0 - \epsilon) \left( M_f(t_0) + q^* M_g(t_0) \right) \\
+ C \left( M_f(t_0) + q^* M_g(t_0) \right) \int_{t_1 - \epsilon}^{t_1} [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}] d\eta \\
\rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0 \quad \text{and} \quad \epsilon \rightarrow 0.
\]

For \( I_5 \), by the boundedness of nonlinear functions \( f \) and \( g \), Lemma 2.5 and the fact that the function \( \eta \rightarrow (t_2 - \eta)^{\alpha - 1} (\eta - t_0)^{\gamma} \) is Lebesgue integrable, we know that

\[
\|I_5\| \leq C^2 \int_{t_1}^{t_2} \int_{t_0}^{t_0} (t_2 - \eta)^{\alpha - 1} (\eta - s)^{\gamma - 1} \left( M_f(t_0) + q^* M_g(t_0) \right) ds d\eta \\
\leq C^2 \gamma \int_{t_1}^{t_2} (t_2 - \eta)^{\alpha - 1} (\eta - t_0)^{\gamma} d\eta \\
\rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0.
\]

For \( t_1 = t_0 \) and \( t_0 < t_2 \leq t_0 + h \), it is easy to see that \( \|I_6\| = 0 \). For \( t_1 > t_0 \) and \( \epsilon > 0 \) small enough, by Lemma 2.5, the boundedness of nonlinear functions \( f \) and \( g \), the facts that the function \( \eta \rightarrow [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}] \) is Lebesgue integrable as well as the operator-valued function \( \psi(t - \eta, \eta) \) is continuous in uniform topology about the variables \( t \) and \( \eta \) for \( t_0 \leq t \leq t_0 + h \) and \( t_0 \leq \eta \leq t - \epsilon \), we know that

\[
\|I_6\| \leq \sup_{\eta \in [t_0, t_1 - \epsilon]} \| \psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta) \| \\
\cdot C \int_{t_0}^{t_1 - \epsilon} \int_{t_0}^{\eta} (\eta - s)^{\gamma - 1} \left( M_f(t_0) + q^* M_g(t_0) \right) ds d\eta \\
+ C^2 \int_{t_1 - \epsilon}^{t_1} \int_{t_0}^{\eta} [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}] \\
\cdot (\eta - s)^{\gamma - 1} \left( M_f(t_0) + q^* M_g(t_0) \right) ds d\eta \\
\leq \sup_{\eta \in [t_0, t_1 - \epsilon]} \| \psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta) \| \\
\cdot \frac{C (t_1 - t_0 - \epsilon)^{\gamma + 1}}{\gamma (\gamma + 1)} \left( M_f(t_0) + q^* M_g(t_0) \right) \\
+ C^2 \Gamma(\gamma) \left( M_f(t_0) + q^* M_g(t_0) \right) \int_{t_1 - \epsilon}^{t_1} [(t_2 - \eta)^{\alpha - 1} + (t_1 - \eta)^{\alpha - 1}] d\eta \\
\rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0 \quad \text{and} \quad \epsilon \rightarrow 0.
\]

Therefore, \( \| (Qu)(t_2) - (Qu)(t_1) \| \) tends to zero independently of \( u \in \Omega \) as \( t_2 - t_1 \rightarrow 0 \), which means that the operator \( Q : \Omega \rightarrow \Omega \) is equicontinuous.
Let $B = \overline{co} Q(\Omega)$, where $\overline{co}$ means the closure of convex hull. Then one can easily to verify that the operator $Q$ maps $B$ into itself and $B \subset C(J, E)$ is equicontinuous. Next, we prove that the operator $Q : B \rightarrow B$ is a condensing operator. For any $D \subset B$, by Lemma 2.8, we know that there exists a countable set $D_0 = \{u_n\} \subset D$, such that
\[
\mu_C(Q(D)) \leq 2\mu_C(Q(D_0)). \tag{3.9}
\]
By the equicontinuity of $B$, we know that $D_0 \subset B$ is also equicontinuous. Therefore, by (3.2), Lemma 2.9 and the assumption $(H_f)$ one get that
\[
\mu(Q(D_0)(t)) \leq \mu \left( x_0 + \int_{t_0}^{t} \psi(t - \eta)U(\eta)A(t_0)\eta d\eta \right)
\]
\[
+ \mu \left( \left\{ \int_{t_0}^{t} \psi(t - \eta, \eta) \left[ f(\eta, u_n(\eta)) + \int_{0}^{\eta} q(\eta - \tau)g(\tau, u_n(\tau))d\eta \right] d\eta \right\} \right)
\]
\[
+ \mu \left( \left\{ \int_{t_0}^{t} \int_{t_0}^{\eta} \psi(t - \eta, \eta) \varphi(\eta, s)[f(s, u_n(s))
\]
\[
+ \int_{0}^{s} q(s - \tau)g(\tau, u_n(\tau))d\tau \right] dsd\eta \right\} \right)
\]
\[
\leq 2C \int_{t_0}^{t} (t - \eta)^{\alpha - 1} \left[ \mu \left\{ \left\{ f(\eta, u_n(\eta)) \right\} + 2 \int_{0}^{\eta} q(\eta - \tau)\mu \left\{ \left\{ g(\tau, u_n(\tau)) \right\} \right\} d\tau \right] d\eta \right.
\]
\[
+ 4C^2 \int_{t_0}^{t} \int_{t_0}^{\eta} (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} \left[ \mu \left\{ \left\{ f(s, u_n(s)) \right\} \right\} \right]
\]
\[
+ 2 \int_{0}^{s} q(s - \tau)\mu \left\{ \left\{ g(\tau, u_n(\tau)) \right\} \right\} d\tau \right] dsd\eta \right)
\]
\[
\leq 2C \int_{t_0}^{t} (t - \eta)^{\alpha - 1} \left[ L_f \mu(D_0(\eta)) + 2 \int_{0}^{\eta} q(\eta - \tau)L_g\mu(D_0(\tau))d\tau \right] d\eta \right.
\]
\[
+ 4C^2 \int_{t_0}^{t} \int_{t_0}^{\eta} (t - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1} \left[ L_f \mu(D_0(s)) \right]
\]
\[
+ 2 \int_{0}^{s} q(s - \tau)L_g\mu(D_0(\tau))d\tau \right] dsd\eta \right)
\]
\[
\leq 2C (L_f + 2q^*L_g) \int_{t_0}^{t} (t - \eta)^{\alpha - 1} d\eta \cdot \mu_C(D)
\]
\[
+ 4C^2 B(\alpha, \gamma)(L_f + 2q^*L_g) \int_{t_0}^{t} (t - \eta)^{\alpha + \gamma - 1} d\eta \cdot \mu_C(D)
\]
\[
\leq 2Ch^\alpha \left[ \frac{1}{\alpha} + \frac{2CB(\alpha, \gamma)}{\alpha + \gamma} \right] (L_f + 2q^*L_g) \cdot \mu_C(D). \tag{3.10}
\]
Because $Q(D_0) \subset D$ is bounded and equicontinuous, from Lemma 2.10 we know that
\[
\mu_C(Q(D_0)) = \max_{t \in [t_0, t_0 + h]} \mu(Q(D_0)(t)). \tag{3.11}
\]
Therefore, by (3.3) and (3.9)-(3.11), we get that
\[
\mu_C(Q(D)) \leq 4Ch^\alpha \left[ \frac{1}{\alpha} + \frac{2CB(\alpha, \gamma)}{\alpha + \gamma} \right] (L_f + 2q^*L_g) \cdot \mu_C(D) < \mu_C(D),
\]
which means that $Q : \Omega \to \Omega$ is a condensing operator. It follows from the famous Sadovskii’s fixed point theorem that the operator $Q$ has at least one fixed point $u \in \Omega$, which is just a mild solution of FNEE (3.1) on interval $[t_0, t_0 + h]$. This completes the proof of Theorem 3.1.

4. Main results. In this section, based on the local existence of mild solutions for FNEE (3.1) on interval $[t_0, t_0 + h]$ (see Theorem 3.1), we will prove that given any $u_0 \in E$, FNEE (1.3) exists a mild solution $u \in C([0, T_{max}), E)$ on a maximal existence interval $[0, T_{max})$ and satisfies the blowup alternative: either $T_{max} = +\infty$ (i.e. $u$ is a global solution) or else $T_{max} < +\infty$ and $\|u(t)\| \to +\infty$ as $t \to T_{max}^{-}$ (i.e. $u$ blows up in finite time).

**Theorem 4.1.** Assume that the nonlinear functions $f, g : [0, +\infty) \times E \to E$ are continuous, map bounded sets in $[0, +\infty) \times E$ into bounded sets in $E$ and satisfy the following condition:

$(H_f)$ \(\forall R > 0 \text{ and } T > 0, \text{ there exist positive constants } L_f = L_f(R, T) \text{ and } L_g = L_g(R, T) \text{ such that for any equicontinuous and countable set } D \subset B_R = \{u \in E : \|u\| \leq R\}, \mu(f(t, D)) \leq L_f \mu(D), \quad \mu(g(t, D)) \leq L_g \mu(D), \quad t \in [0, T].\)

Then for every $u_0 \in E$, FNEE (1.3) exists a mild solution $u \in C([0, T_{max}), E)$ on a maximal existence interval $[0, T_{max})$. If $T_{max} < +\infty$ then $\lim_{t \to T_{max}} \|u(t)\| = +\infty$.

**Proof.** By Theorem 3.1 we proved in Section 3, we know that there exists a constant $0 < h_0 \leq 1$ such that FNEE (1.3) has a mild solution $u \in C([0, h_0], E)$ on interval $[0, h_0]$. Moreover, $u$ can be extended to a large interval $[0, h_0 + h_1]$ with $0 < h_1 \leq 1$ by defining $u(t) = v(t)$ on $[h_0, h_0 + h_1]$, where $v(t)$ is the mild solution of the initial value problem

\[
\begin{cases}
C \mathcal{D}^\alpha_t v(t) + A(t)v(t) = f(t, v(t)) + \int_0^t q(t - s)g(s, v(s))ds, \quad h_0 \leq t \leq h_0 + h_1, \\
v(0) = u(h_0),
\end{cases}
\tag{4.1}
\]

where $h_1$ depends only on $\|u(h_0)\|$, $M_f(h_0)$, $M_g(h_0)$ and $R(h_0)$. Hence, repeating the above procedure and using the methods of steps, we can prove that $u$ can be extended to a maximal existence interval $[0, T_{max})$, namely $u \in C([0, T_{max}), E)$ is a mild solution of FNEE (1.3).

In the following, we show that the mild solution $u$ of FNEE (1.3) blows up in finite time, i.e. if $T_{max} < +\infty$ then $\lim_{t \to T_{max}} \|u(t)\| = +\infty$. To do so we first prove that $T_{max} < +\infty$ implies $\limsup_{t \to T_{max}} \|u(t)\| = +\infty$. In fact, if $T_{max} < +\infty$ and $\limsup_{t \to T_{max}} \|u(t)\| < +\infty$, then there exists a constant $0 < R(T_{max}) < +\infty$ such that $\sup_{0 \leq t < T_{max}} \|u(t)\| \leq R(T_{max})$. Denote

\[
\begin{align*}
M_f(T_{max}) &= \sup_{0 \leq t \leq T_{max} + 1} \|f(t, u(t))\| : \|u(t)\| \leq R(T_{max}), \\
M_g(T_{max}) &= \sup_{0 \leq t \leq T_{max} + 1} \|g(t, u(t))\| : \|u(t)\| \leq R(T_{max}), \\
q^*(T_{max}) &= \sup_{0 \leq t \leq T_{max} + 1} \int_0^t \|q(t - s)\|ds.
\end{align*}
\tag{4.2}
\]

For $0 < t' < t'' < T_{max}$ and $\epsilon > 0$ small enough, by (3.2), (4.2), Lemma 2.5 and the fact that the operator-valued function $\psi(t - \eta, \eta)$ is continuous in uniform topology
about the variables $t$ and $\eta$ for $0 \leq t < T_{\text{max}}$ and $0 \leq \eta \leq t - \epsilon$, we know that

\[
\|u(t'') - u(t')\| \\
\leq C^2 \|A(0)u_0\| \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1}(1 + \eta^\gamma)d\eta \\
+ C \|A(0)u_0\| \int_0^{t'} \|\psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta)\|(1 + \eta^\gamma)d\eta \\
+ C \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1}d\eta \\
+ \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \int_0^{t'} \|\psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta)\|d\eta \\
+ C^2 \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \int_{t'}^{t''} \int_0^\eta (t'' - \eta)^{\alpha - 1}(\eta - s)^{\gamma - 1}dsd\eta \\
+ C \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \\
\cdot \int_0^{t'} \int_0^\eta \|\psi(t_2 - \eta, \eta) - \psi(t_1 - \eta, \eta)\|(\eta - s)^{\gamma - 1}dsd\eta \\
\leq C^2 \|A(0)u_0\| \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1}(1 + \eta^\gamma)d\eta \\
+ C \|A(0)u_0\| \int_{t'-\epsilon}^{t'} [(t'' - \eta)^{\alpha - 1} + (t' - \eta)^{\alpha - 1}](1 + \eta^\gamma)d\eta \\
+ \frac{C}{\alpha} \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] (t'' - t')^\alpha \\
+ \sup_{\eta \in [0, t' - \epsilon]} \|\psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta)\| \\
\cdot (t' - \epsilon) \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \\
+ C \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \int_{t'-\epsilon}^{t'} [(t'' - \eta)^{\alpha - 1} + (t' - \eta)^{\alpha - 1}]d\eta \\
+ \frac{C^2}{\gamma} \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \int_{t'}^{t''} (t'' - \eta)^{\alpha - 1}\eta^\gamma d\eta \\
+ \sup_{\eta \in [0, t' - \epsilon]} \|\psi(t'' - \eta, \eta) - \psi(t' - \eta, \eta)\| \\
\cdot \frac{(t' - \epsilon)^{\gamma + 1}}{\gamma(\gamma + 1)} \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \\
+ C^2 \Gamma(\gamma) \left[M_f(T_{\text{max}}) + q^*(T_{\text{max}})M_g(T_{\text{max}})\right] \\
\cdot \int_{t'-\epsilon}^{t'} [(t'' - \eta)^{\alpha - 1} + (t' - \eta)^{\alpha - 1}]I_{\gamma^2}^1d\eta \\
\to 0 \text{ as } t', t'' \to T_{\text{max}}^{-} \text{ and } \epsilon \to 0.
Therefore, by Cauchy criteria we know that \( \lim_{t \to T_{\text{max}}} u(t) = u(T_{\text{max}}) \) exists. Set \( \lim_{t \to T_{\text{max}}} u(t) = u_1 \). By Theorem 3.1 we know that the following fractional non-autonomous integro-differential evolution equation (FNEE) of Volterra type

\[
C D_t^\alpha u(t) + A(t)u(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, \quad t \geq T_{\text{max}}, \tag{4.3}
\]

exists a mild solution on \([T_{\text{max}}, T_{\text{max}} + h]\), where \( h > 0 \) is a constant. This means that the mild solution \( u \) of FNEE (1.3) can be extended beyond \( T_{\text{max}} \), which contradicts with \([0, T_{\text{max}}]\) is a maximal existence interval. Therefore, the assumption \( T_{\text{max}} < +\infty \) implies that \( \limsup_{t \to T_{\text{max}}} \|u(t)\| = +\infty \). To conclude the proof we will show that \( T_{\text{max}} < +\infty \) implies \( \lim_{t \to T_{\text{max}}} \|u(t)\| = +\infty \). If this is not true then there exist a sequence \( t_n \uparrow T_{\text{max}} \) and a constant \( K > 0 \) such that \( \|u(t_n)\| \leq K \) for all \( n \). Set

\[
M_f = \sup \{ \|f(t, u(t))\| : \|u(t)\| \leq K + 1, 0 \leq t \leq T_{\text{max}} \},
\]

\[
M_g = \sup \{ \|g(t, u(t))\| : \|u(t)\| \leq K + 1, 0 \leq t \leq T_{\text{max}} \}.
\]

By the fact that \( t \to \|u(t)\| \) is continuous and \( \limsup_{t \to T_{\text{max}}} \|u(t)\| = +\infty \) one can find a sequence \( \{h_n\} \) \((0 < h_n < T_{\text{max}} - t_n)\) such that \( \|u(t)\| \leq K + 1 \) for \( t_n \leq t \leq t_n + h_n \) and \( \|u(t_n + h_n)\| = K + 1 \). By Lemma 2.5, (4.4) and direct calculation, we know that

\[
K + 1
\]

\[
= \|u(t_n + h_n)\|
\]

\[
\leq \|u(t_n)\| + \left\| \int_{t_n}^{t_n + h_n} \psi(t_n + h_n - \eta, \eta)U(\eta)A(t_n)u(t_n) d\eta \right\|
\]

\[
+ \left\| \int_{t_n}^{t_n + h_n} \psi(t_n + h_n - \eta, \eta) \left[ f(\eta, u(\eta)) + \int_0^\eta q(\eta - \tau)g(\tau, u(\tau)) d\tau \right] d\eta \right\|
\]

\[
+ \left\| \int_{t_n}^{t_n + h_n} \int_{t_n}^\eta \psi(t_n + h_n - \eta, \eta) \varphi(\eta, s) f(s, u(s)) ds d\eta \right\|
\]

\[
+ \left\| \int_0^\cdot g(s - \tau)g(\tau, u(\tau)) d\tau d\eta \right\|
\]

\[
\leq \|u(t_n)\| + C^2 \int_{t_n}^{t_n + h_n} (t_n + h_n - \eta)^{\alpha - 1} (1 + \eta^\gamma) d\eta : \|A(t_n)\| \|u(t_n)\|
\]

\[
+ C \left( M_f + q^*(T_{\text{max}})M_g \right) \int_{t_n}^{t_n + h_n} (t_n + h_n - \eta)^{\alpha - 1} d\eta
\]

\[
+ C^2 \left( M_f + q^*(T_{\text{max}})M_g \right) \int_{t_n}^{t_n + h_n} \int_{t_n}^\eta (t_n + h_n - \eta)^{\alpha - 1} (\eta - s)^{\gamma - 1} ds d\eta
\]

\[
\leq K + C^2 K h_n^\alpha \left( \frac{1}{\alpha} + (h_n)^\gamma B(\alpha, \gamma + 1) \right) \|A(t_n)\|
\]

\[
+ \frac{C h_n^\alpha}{\alpha} \left( M_f + q^*(T_{\text{max}})M_g \right)
\]
\[ + \frac{C^2B(\alpha, \gamma)(h_n)^{\alpha+\gamma}}{\alpha + \gamma} \left( M_f + q'(T_{\max})M_g \right) \rightarrow K \quad \text{as} \quad n \rightarrow \infty, \]

which is a contradiction. Therefore, we have proved that if \( T_{\max} < +\infty \) then
\[
\lim_{t \to T_{\max}^-} \|u(t)\| = +\infty. \quad \Box
\]

5. **An application.** In this section, we present an example to indicate how our abstract result can be applied to concrete problems. Throughout this section, we let \( E = L^2([0, \pi], \mathbb{R}) \) be a Banach space with the \( L^2 \)-norm \( \| \cdot \|_2 \). As an application, we consider the following time fractional non-autonomous partial integro-differential equation of Volterra type with homogeneous Dirichlet boundary condition

\[
\left\{ \begin{array}{ll}
\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) - a(x,t) \frac{\partial^2}{\partial x^2}u(x,t) = \frac{e^{-t}}{2 + |u(x,t)|} \\
+ \int_0^t e^{-(t-s)} \cos(u(x,s))ds, \quad x \in [0, \pi], \ t \geq 0, \\
u(0,t) = u(\pi,t) = 0, \quad t \geq 0, \\
u(x,0) = \varphi(x), \quad x \in [0, \pi],
\end{array} \right. \quad (5.1)
\]

where \( \frac{\partial^\alpha}{\partial t^\alpha} \) is the Caputo fractional order partial derivative of order \( \alpha \) with \( 0 < \alpha \leq 1 \), the coefficient of heat conductivity \( a(x,t) \) is continuous on \([0, \pi] \times [0, +\infty)\) and it is uniformly Hölder continuous in \( t \), which means that for any \( t_1, t_2 \in [0, +\infty) \), there exist a constant \( 0 < \gamma \leq 1 \) and a positive constant \( C \) independent of \( t_1 \) and \( t_2 \), such that

\[
|a(x, t_2) - a(x, t_1)| \leq C|t_2 - t_1|^\gamma, \quad x \in [0, \pi], \quad (5.2)
\]

We consider the operator \( A(t) : D(A) \subset E \to E \) defined by

\[
D(A) = \left\{ u \in E : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in E, \ u(0) = u(\pi) = 0 \right\}, \quad A(t)u = -a(x,t) \frac{\partial^2 u}{\partial x^2}. \quad (5.3)
\]

Then it is well known from [10] that \( -A(s) \) generates an analytic semigroup \( e^{-tA(s)} \) in \( E \). By (5.2) and (5.3) one can easily verify that the linear operator \( -A(t) \) satisfies the assumptions (A1) and (A2).

For any \( t \geq 0 \), define

\[
u(t) = u(\cdot, t), \quad q(t-s) = e^{-|t-s|} \quad \text{for} \quad 0 \leq s \leq t < +\infty, \]

\[
f(t, u(t)) = \frac{e^{-t}}{2 + |u(\cdot, t)|}, \quad g(t, u(t)) = \cos(u(\cdot, t)).
\]

Then the time fractional non-autonomous partial integro-differential equation of Volterra type with homogeneous Dirichlet boundary condition (5.1) can be transformed into the abstract form of time fractional non-autonomous integro-differential evolution equation of Volterra type (1.3).

**Theorem 5.1.** The time fractional non-autonomous partial integro-differential equation of Volterra type with homogeneous Dirichlet boundary condition (5.1) exists a mild solution \( u \in C([0, T], L^2([0, \pi], \mathbb{R})) \) on a maximal existence interval \([0, T)\) for \( T > 0 \). If \( T < +\infty \) then \( \lim_{t \to T^-} \|u(t)\|_2 = +\infty \).
Proof. By the definitions of nonlinear functions $f$ and $g$ one can easily to verify that the nonlinear functions $f, g : [0, +\infty) \times L^2([0, \pi], \mathbb{R}) \to L^2([0, \pi], \mathbb{R})$ are continuous, map bounded sets in $[0, +\infty) \times L^2([0, \pi], \mathbb{R})$ into bounded sets in $L^2([0, \pi], \mathbb{R})$. Furthermore, from the definition of nonlinear functions $f$ and $g$, we know that $f(t, u)$ and $g(t, v)$ is Lipschitz continuous about the variables $u$ and $v$ with Lipschitz constants $k_f = 1/4$ and $k_g = 1$, respectively. Therefore, by Lemma 2.7 (vii) we know that the assumption $(H_f)$ is satisfied with positive constants

$$L_f = \frac{1}{4}, \quad L_g = 1.$$  

In addition, the definition of $q(t - s)$ means that $q : [0, +\infty) \to L^2([0, \pi], \mathbb{R})$ is locally integrable. Therefore, all the assumptions of Theorem 4.1 are satisfied. Hence, from Theorem 4.1 we know that the time fractional non-autonomous partial integro-differential equation of Volterra type with homogeneous Dirichlet boundary condition (5.1) exists a mild solution $u \in C([0, T), L^2([0, \pi], \mathbb{R}))$ on a maximal existence interval $[0, T)$ for $T > 0$, and if $T < +\infty$ then $\lim_{t \to T^-} \|u(t)\|_2 = +\infty$. This completes the proof of Theorem 5.1. □

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