A GENERAL APPROACH TO WEIGHTED $L^p$ RELLICH TYPE INEQUALITIES RELATED TO GREINER OPERATOR

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Abstract. In this paper we exhibit some sufficient conditions that imply general weighted $L^p$ Rellich type inequality related to Greiner operator without assuming a priori symmetric hypotheses on the weights. More precisely, we prove that given two nonnegative functions $a$ and $b$, if there exists a positive supersolution $\vartheta$ of the Greiner operator $\Delta_k$ such that

$$\Delta_k \left( a \, |\Delta_k \vartheta|^{p-2} \, \Delta_k \vartheta \right) \geq b \vartheta^{p-1}$$

almost everywhere in $\mathbb{R}^{2n+1}$, then $a$ and $b$ satisfy a weighted $L^p$ Rellich type inequality. Here, $p > 1$ and $\Delta_k = \sum_{j=1}^n \left( X_j^2 + Y_j^2 \right)$ is the sub-elliptic operator generated by the Greiner vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2k y_j |z|^{2k-2} \frac{\partial}{\partial z}, \quad Y_j = \frac{\partial}{\partial y_j} - 2k x_j |z|^{2k-2} \frac{\partial}{\partial z}, \quad j = 1, \ldots, n,$$

where $(z,l) = (x,y,l) \in \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $|z| = \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}$ and $k \geq 1$. The method we use is quite practical and constructive to obtain both known and new weighted Rellich type inequalities. On the other hand, we also establish a sharp weighted $L^p$ Rellich type inequality that connects first to second order derivatives and several improved versions of two-weight $L^p$ Rellich type inequalities associated to the Greiner operator $\Delta_k$ on smooth bounded domains $\Omega$ in $\mathbb{R}^{2n+1}$.

1. Introduction. There has been intensive research over the years related to inequalities involving integrals of a function and its derivatives motivated by their applications to problems in analysis, mathematical physics, spectral theory, geometry and quantum mechanics. In 1954, Rellich [26] proved the following inequality which bears his name:

$$\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} \, dx,$$

where $u \in C_0^\infty (\mathbb{R}^n \setminus \{0\})$, $n \geq 5$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp. Since then this remarkable inequality has evoked the interest of many mathematicians and has been exhaustively analyzed in many different directions, see [9, 11, 16, 1, 6, 27, 14, 22, 10, 12, 23, 21] and the references therein. For instance, Davies and Hinz [11]

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generalized the inequality (1) to the $L^p$ case and also considered various types of weighted Rellich inequalities. Gazzola, Grunau and Mitidieri [16] obtained some improvements of (1) in smooth bounded domains of $\mathbb{R}^n$ containing the origin by adding some nonnegative remainder terms.

On the other hand, attached to the inequality (1), there is also a similar Rellich inequality that connects first to second order derivatives. To be more precise, Terrikas and Zographopoulos showed in [27] that for any $u \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 5$ one has

$$
\int_{\mathbb{R}^n} |\Delta u|^2 \, dx \geq \frac{n^2}{4} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx
$$

and the constant $\frac{n^2}{4}$ is sharp. We also mention here that Adimurthi and Santra [2] proved an $L^p$ version of (2) in the unit ball $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$ for only radial functions. Afterwards, Musina [23] established a weighted $L^p$ analogue of (2) in the Euclidean space $\mathbb{R}^n$ for only radial functions again.

A large amount of work concerning these inequalities has been also developed in sub-elliptic settings and this is because they have applications to singular problems in the theory of partial differential equations. In the case of sub-Riemannian space $\mathbb{R}^{2n+1}$ defined by the Greiner vector fields

$$
X_j = \frac{\partial}{\partial x_j} + 2k y_j |z|^{2k-2} \frac{\partial}{\partial l} \quad Y_j = \frac{\partial}{\partial y_j} - 2k x_j |z|^{2k-2} \frac{\partial}{\partial l}, \quad j = 1, \ldots, n,
$$

where $(z, l) = (x, y, l) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, $|z| = \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}$ and $k \geq 1$, Ahmetolan and Kombe [3, 4] proved the following weighted Rellich type inequalities:

$$
\int_{\mathbb{R}^{2n+1}} \rho^{\alpha} \left( \frac{\rho}{|z|} \right)^{4k-2} |\Delta_k u|^2 \, dzdl \geq C_{Q, \alpha} \int_{\mathbb{R}^{2n+1}} \rho^{\alpha} \left( \frac{|z|}{\rho} \right)^{4k-2} u^2 \, dzdl
$$

(4)

and

$$
\int_{\mathbb{R}^{2n+1}} \rho^{\alpha} \left( \frac{\rho}{|z|} \right)^{4k-2} |\Delta_k u|^2 \, dzdl \geq \frac{(Q - \alpha)^2}{4} \int_{\mathbb{R}^{2n+1}} \rho^{\alpha} |\nabla_k u|^2 \, dzdl
$$

(5)

for all $u \in C_0^\infty(\mathbb{R}^{2n+1}\setminus\{(0, 0)\})$ with the sharp constants $C_{Q, \alpha} := \frac{(Q + \alpha - 4)^2 (Q - \alpha)^2}{16}$ and $\frac{(Q - \alpha)^2}{4}$. Here, $\nabla_k = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is the sub-elliptic gradient, $Q = 2n + 2k$ is the homogeneous dimension and $\rho = \left(|z|^{4k} + l^2\right)^{1/4k}$ is the gauge induced by the fundamental solution for the sub-elliptic operator $\Delta_k = \sum_{j=1}^n (X_j^2 + Y_j^2)$.

Lian [19], among the other results, extended the inequality (4) to the $L^p$ form which states that for every $u \in C_0^\infty(\mathbb{R}^{2n+1}\setminus\{(0, 0)\})$ and $1 < p < \frac{Q + \alpha}{2}$ one has

$$
\int_{\mathbb{R}^{2n+1}} \rho^{\alpha} \left( \frac{\rho}{|z|} \right)^{(2p-2)(4k-1)} |\Delta_k u|^p \, dzdl \geq C_{Q, p, \alpha} \int_{\mathbb{R}^{2n+1}} \rho^{\alpha} \left( \frac{|z|}{\rho} \right)^{4k-2} \frac{|u|^p}{\rho^{2p}} \, dzdl,
$$

where $C_{Q, p, \alpha} := \frac{(Q+1-p)(Q+\alpha-2p)}{p^2}$ and the positive constant $C_{Q, p, \alpha}$ is sharp. Recently, Ahmetolan and Kombe [4] have studied on some sharp Rellich type inequalities with two-weight functions related to the Greiner operator $\Delta_k$.

In this paper, we first provide some sufficient conditions that imply general weighted $L^p$ Rellich type inequality related to Greiner operator without assuming a priori symmetric hypotheses on the weights. To be explicit, we show that
given two nonnegative functions \(a\) and \(b\) if there exists a positive supersolution \(\vartheta\) of the given sub-elliptic operator \(\Delta_k\) such that

\[
\Delta_k \left( a |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \right) \geq b \vartheta^{p-1}
\]

almost everywhere (a.e.) in \(\mathbb{R}^{2n+1}\), then for every \(u \in C_0^\infty(\mathbb{R}^{2n+1})\) and \(p > 1\) one has

\[
\int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dzdl \geq \int_{\mathbb{R}^{2n+1}} b |u|^p \, dzdl.
\]

We should point out here that the method we use is quite practical and constructive to obtain both known and new weighted Rellich type inequalities. We shall demonstrate these cases by giving many explicit examples of Rellich type inequalities including radial, logarithmic and particularly non-radial weights (see Applications of Theorem 3.2).

The other principal result of this paper can be roughly described as follows: if \(Q_p > Q^+ > p > 1\) and \(\alpha \in \mathbb{R}\), then we have

\[
\int_{\mathbb{R}^{2n+1}} \rho^\alpha \left( \frac{p}{|z|} \right)^{(2k-1)} |\Delta_k u|^p \, dzdl \geq \left( \frac{Q_p - Q - \alpha}{p} \right)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\nabla_k u|^p \, dzdl
\]

for any \(u \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{(0,0)\})\) with \(\left( \frac{Q_p - Q - \alpha}{p} \right)^p\) being the sharp constant. This result is a weighted \(L^p\) generalization of the inequality (2) to the Greiner operator and also seems to be new even in the case of Euclidean space \(\mathbb{R}^n\).

On the other hand, we present a general method that can be used to deduce new results on two-weight \(L^p\) Rellich type inequalities with remainder terms from a particular nonlinear partial differential inequality on smooth bounded domains \(\Omega\) in sub-Riemannian space \(\mathbb{R}^{2n+1}\) generated by the Greiner vector fields (3). We also exhibit several concrete examples to illustrate our result for different weights (see Applications of Theorem 5.1).

2. Preliminaries and notations. In this section we shall introduce some notations and preliminary facts that will be needed in the sequel. The generic point is \(w = (z, l) = (x, y, l) \in \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) with \(n \geq 1\). The generalized Greiner operator is of the form

\[
\Delta_k = \sum_{j=1}^n \left( X_j^2 + Y_j^2 \right),
\]

where

\[
X_j = \frac{\partial}{\partial x_j} + 2ky_j \frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j \frac{\partial}{\partial l}
\]

for \(j = 1, \ldots, n\) and \(k \geq 1\). We remind that when \(k = 1\), \(\Delta_k\) becomes the sub-Laplacian \(\Delta_{H^n}\) on the Heisenberg group \(H^n\); see [13]. If \(k = 2, 3, \ldots\), \(\Delta_k\) is the Greiner operator; see [17]. Also we note that the vector fields (7) are neither left nor right invariant and for \(k > 1, k \notin \mathbb{Z}\) they do not satisfy the Hörmander’s condition and not smooth. The sub-elliptic gradient associated with \(\Delta_k\) is as follows

\[
\nabla_k = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)
\]

and the family of dilation is expressed by

\[
\delta_\lambda(z, l) = (\lambda z, \lambda^{2k} l), \quad \lambda > 0.
\]

The change of variable formula for the Lebesgue measure yields that

\[
d\delta_\lambda(z, l) = \lambda^Q \, dzdl = \lambda^Q \, dw,
\]
where
\[ Q = 2n + 2k \]
is the homogeneous dimension with respect to the dilation \( \delta_\lambda \) and \( dw = dzdl \) denotes the Lebesgue measure on \( \mathbb{R}^{2n+1} \).

For \( w = (z,l) \in \mathbb{R}^{2n} \times \mathbb{R} \), we define the norm
\[ \rho = \rho(z,l) = \left( |z|^{2k} + l^2 \right)^{\frac{1}{2k}}, \]
where we have set \( |z| := \sqrt{|x|^2 + |y|^2} \). Here \(|\cdot|\) is the standard Euclidean norm. We remark that \( \rho \) is positive, smooth away from the origin and symmetric. The norm function \( \rho \) is also closely related to the fundamental solution of sub-elliptic operator \( \Delta_k \) at the origin, see [7, 8]. Namely, if \( Q > 2 \) then the function \( u := \rho^{2-Q} \) satisfies the relation
\[ -\Delta_k u = \ell \delta_0 \quad \text{on} \quad \mathbb{R}^{2n+1} \]
in weak sense, where \( \delta_0 \) is the Dirac distribution at 0 and \( \ell \) is a positive constant.

For a differentiable real valued function \( u : \mathbb{R}^{2n+1} \rightarrow \mathbb{R} \), the \( p \)-degenerate sub-elliptic operator \( \Delta_{k,p} \) associated with the vector fields (7) is given by
\[ \Delta_{k,p} u = \nabla_k \cdot \left( |\nabla_k u|^{p-2} \nabla_k u \right), \quad p > 1. \]
If \( p = 2 \), it coincides with \( \Delta_k \) in (6). It is immediate to check that \( \Delta_{k,p} \) is a homogeneous partial differential operator of degree \( p \) with respect to the anisotropic dilations (8), that is, \( \Delta_{k,p} \circ \delta_\lambda = \lambda^p \delta_\lambda \circ \Delta_{k,p} \).

Now let us mention, without proofs, some useful facts which we shall use throughout the computations in this paper. An evident calculation yields
\[ \nabla_k \rho = \frac{|z|^{2k-2}}{\rho^{2k-1}} \left( x |z|^{2k} + y |z|^{2k} - x \right). \]
Hence we immediately get
\[ |\nabla_k \rho| = \frac{|z|^{2k-1}}{\rho^{2k-1}}. \]
A function \( u \) on \( \mathbb{R}^{2n+1} \) is said to be radial when \( u \) has the form \( u = u(\rho) \). If \( u \) is a radial function, then we have
\[ |\nabla_k u(\rho)| = |\nabla_k \rho| |u'(\rho)| \]
and
\[ \Delta_k u(\rho) = |\nabla_k \rho|^2 \left[ u''(\rho) + (Q - 1) \frac{u'(\rho)}{\rho} \right] \]
at every point \( w \in \mathbb{R}^{2n+1} - \{0\} \). In particular, when \( u(\rho) = \rho^\alpha \) we obtain
\[ |\nabla_k \rho^\alpha| = |\alpha| |\nabla_k \rho| \rho^{\alpha-1} \]
and
\[ \Delta_k \rho^\alpha = \alpha (Q + \alpha - 2) |\nabla_k \rho|^2 \rho^{\alpha-2}. \]
where \( \alpha \in \mathbb{R} \). Moreover, one can derive the following identities:
\[ \nabla_k \rho^\alpha \cdot \nabla_k |z|^\beta = \alpha \beta |\nabla_k \rho|^2 |z|^\beta \rho^{\alpha-2} \]
and
\[ \Delta_k (\rho^\alpha |z|^\beta) = \alpha (Q + 2\beta + \alpha - 2) |\nabla_k \rho|^2 |z|^\beta \rho^{\alpha-2} + \beta (Q + \beta - 2k - 2) |z|^{\beta-2} \rho^\alpha. \]
with \( \alpha, \beta \in \mathbb{R} \) and \( n \geq 1 \). We also note that the gauge \( \rho \) is infinite harmonic in \( \mathbb{R}^{2n+1} - \{0\} \), that is, \( \rho \) is solution of the following equation:

\[
\nabla_k (|\nabla_k \rho|^2) \cdot \nabla_k \rho = 0.
\]

Let \( B_R = \{ w \in \mathbb{R}^{2n+1} \mid \rho(w) < R \} \), \( \partial B_R = \{ w \in \mathbb{R}^{2n+1} \mid \rho(w) = R \} \) and call these sets, respectively, \( \rho \)-ball and \( \rho \)-sphere centered at the origin with radius \( R \). Let \( D = B_{R_2} \setminus \overline{B_{R_1}} \) be an annulus with \( 0 \leq R_1 < R_2 \leq +\infty \). In order to compute some radial integrals, we use the spherical transformation

\[
w = (x, y, l) := \Phi (\rho, \theta, \theta_1, \ldots, \theta_{2n-1})
\]
defined as in [24]:

\[
x_1 = \rho \sin^{\frac{n-k}{2}} \theta \cos \theta_1,
\]
\[
y_1 = \rho \sin^{\frac{n-k}{2}} \theta \sin \theta_1 \cos \theta_2,
\]
\[
x_2 = \rho \sin^{\frac{n-k}{2}} \theta \sin \theta_1 \sin \theta_2 \cos \theta_3,
\]
\[
y_2 = \rho \sin^{\frac{n-k}{2}} \theta \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4,
\]
\[
\vdots
\]
\[
x_n = \rho \sin^{\frac{n-k}{2}} \theta \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{2n-2} \cos \theta_{2n-1},
\]
\[
y_n = \rho \sin^{\frac{n-k}{2}} \theta \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{2n-2} \sin \theta_{2n-1},
\]
\[
l = \rho^{2k} \cos \theta,
\]

where \( R_1 < \rho < R_2 \), \( 0 < \theta < \pi \), \( 0 < \theta_j < \pi \), \( j = 1, \ldots, 2n-2 \) and \( 0 < \theta_{2n-1} < 2\pi \). Let \( J(\Phi) \) be the Jacobian of \( \Phi \). A direct computation shows that

\[
|J(\Phi)| = \rho^{Q-1} \sin^{\frac{n-k}{2}} \theta \sin^{2n-2} \theta_1 \cdots \sin^{2} \theta_{2n-2} \sin \theta_{2n-1}.
\]

Therefore, the volume element satisfies the following relation

\[
dw = \rho^{Q-1} d\rho \sin^{\frac{n-k}{2}} \theta d\theta \sin^{2n-2} \theta_1 \cdots \sin^{2} \theta_{2n-3} \sin \theta_{2n-2} d\theta_{2n-1}
\]

and

\[
|z|^p = \rho^p \sin^{\frac{p}{2}} \theta, \quad p > 1.
\]

Notice that, if \( u \) has the form

\[
u(w) = \frac{|z|^{p(2k-1)}}{\rho^{p(2k-1)}} v(\rho) = |\nabla_k \rho|^p v(\rho),
\]

then we readily have

\[
\int_D u(w) dw = \int_D |\nabla_k \rho|^p v(\rho) dw
\]
\[
= \gamma_n \left( \int_0^\pi (\sin \theta)^{\frac{n-k}{2} + \frac{p(2k-1)}{2}} d\theta \right) \left( \int_{R_1}^{R_2} \rho^{Q-1} v(\rho) d\rho \right)
\]
\[
= s_n \int_{R_1}^{R_2} \rho^{Q-1} v(\rho) d\rho.
\]
Here,
\[ \gamma_n := \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \sin^{2n-2} \theta_1 \cdots \sin \theta_{2n-1} d\theta_1 \cdots d\theta_{2n-2} d\theta_{2n-1} \]
\[ = 2\pi \left( \int_0^\pi \sin^{2n-2} \theta_1 d\theta_1 \right) \cdots \left( \int_0^\pi \sin \theta_{2n-2} d\theta_{2n-2} \right) \]
is the $2n$–Lebesgue measure of the unitary Euclidean sphere in $\mathbb{R}^{2n}$ and
\[ s_n := \gamma_n \left( \int_0^\pi (\sin \theta)^{n-k+\frac{2(2-k)}{2n-k}} d\theta \right). \]

**3. General weighted Rellich type inequalities.** The classical Picone identity states that
\[ |\nabla u|^2 + \frac{u^2}{\vartheta^2} |\nabla \vartheta|^2 - 2 \frac{u}{\vartheta} \nabla u \cdot \nabla \vartheta = |\nabla u|^2 - \nabla \left( \frac{u^2}{\vartheta} \right) \cdot \nabla \vartheta \geq 0, \quad (12) \]
where $u \geq 0$ and $\vartheta > 0$ are smooth functions in $\mathbb{R}^n$. Allegretto and Huang [5] generalized (12) to the $p$–Laplacian case. With a similar approach as in the [5], Niu, Zhang and Wang [25] obtained a Picone type identity for the $p$–sub-Laplacian and the $p$–biharmonic operators in the Heisenberg group.

Before proving main result of this section, we first give the following Picone type identity for the $p$–biharmonic operators in the sub-Riemannian space $\mathbb{R}^{2n+1}$ defined by the Greiner vector fields (7) that will be used in the sequel.

**Lemma 3.1.** Let $u$ be a nonnegative smooth function and $\vartheta$ be a positive supersolution of the Greiner operator $\Delta_k$ in $\mathbb{R}^{2n+1}$. If
\[ R(u, \vartheta) = |\Delta_k u|^p - \Delta_k \left( \frac{u^p}{\vartheta^{p-1}} \right) |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \]
and
\[ L(u, \vartheta) = |\Delta_k u|^p - p \frac{u^{p-1}}{\vartheta^{p-1}} \Delta_k u |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta + (p-1) \frac{u^p}{\vartheta^p} |\Delta_k \vartheta|^p \]
\[ - p (p-1) \frac{u^{p-2}}{\vartheta^{p-1}} |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \left( \nabla_k u - \frac{u}{\vartheta} \nabla_k \vartheta \right)^2 \]
with $p > 1$, then one has
\[ L(u, \vartheta) = R(u, \vartheta) \geq 0. \]

**Proof.** The proof follows from direct calculations. It contains Lemma 3.1 in [25]. \qed

We now prove main theorem of this section and then show several applications leading to various Rellich inequalities with known and also new weights.

**Theorem 3.2.** Let $a \in C^2(\mathbb{R}^{2n+1})$ and $b \in L^1_{loc}(\mathbb{R}^{2n+1})$ be nonnegative functions and $p > 1$. If $\vartheta \in C^\infty(\mathbb{R}^{2n+1})$ is a positive supersolution of the Greiner operator $\Delta_k$ such that
\[ \Delta_k \left( a |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \right) \geq b \vartheta^{p-1} \quad a.e. \text{ in } \mathbb{R}^{2n+1}, \]
then for any $u \in C^\infty_0(\mathbb{R}^{2n+1})$ one has
\[ \int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dw \geq \int_{\mathbb{R}^{2n+1}} b |u|^p \, dw. \]
Proof. We first consider $0 \leq u \in C^{\infty}_0(\mathbb{R}^{2n+1})$. Together with the Lemma 3.1 and the integration by parts formula we obtain

\[
0 \leq \int_{\mathbb{R}^{2n+1}} aL(u, \vartheta) \, dw = \int_{\mathbb{R}^{2n+1}} aR(u, \vartheta) \, dw
\]

\[
= \int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dw - \int_{\mathbb{R}^{2n+1}} a \Delta_k \left( \frac{u^p}{p-1} \right) |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \, dw
\]

\[
= \int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dw - \int_{\mathbb{R}^{2n+1}} \frac{u^p}{p-1} \Delta_k \left( a |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \right) \, dw
\]

\[
\leq \int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dw - \int_{\mathbb{R}^{2n+1}} b u^p \, dw,
\]

where we have used the given differential inequality

\[
\Delta_k \left( a |\Delta_k \vartheta|^{p-2} \Delta_k \vartheta \right) \geq b \vartheta^{p-1}.
\]

Therefore, for any nonnegative function $u \in C^{\infty}_0(\mathbb{R}^{2n+1})$ one gets

\[
\int_{\mathbb{R}^{2n+1}} a |\Delta_k u|^p \, dw \geq \int_{\mathbb{R}^{2n+1}} b |u|^p \, dw.
\]

For general $u$, by letting $u = u^+ - u^-$, we complete the proof.

3.1. Applications of Theorem 3.2. This subsection is devoted to examples of various weighted Rellich type inequalities, which are the direct consequences of Theorem 3.2. Let $\epsilon > 0$ be given. To make the following arguments rigorous, we should replace the function $\rho$ with its regularization

\[
\rho_\epsilon := \left( |z|^{4k} + t^2 \right)^{1/4},
\]

where $|z| := \left( \epsilon^2 + \sum_{j=1}^n (x_j^2 + y_j^2) \right)^{1/2}$ and after the computation take the limit as $\epsilon \to 0$. In order to avoid the tedious presentation we shall, however, proceed formally.

Cautiously choosing $a$ and $\vartheta$ in Theorem 3.2, a variety of weighted Rellich type inequality can be obtained immediately. We start by considering the model functions

\[
a = \frac{\rho^\alpha}{|\nabla_k \rho|^2} \quad \text{and} \quad \vartheta = \rho^{-\left( \frac{Q+\alpha-4}{2} \right)}
\]

in Theorem 3.2. This yields subsequent result due to Ahmetolan and Kombe [4].

**Corollary 1.** Let $\alpha \in \mathbb{R}$, $Q+\alpha > 4$ and $Q > \alpha$. Then for any $u \in C^{\infty}_0(\mathbb{R}^{2n+1}\setminus\{0\})$, one has

\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha}{|\nabla_k \rho|^2} |\Delta_k u|^2 \, dw \geq \frac{(Q + \alpha - 4)^2 (Q - \alpha)^2}{16} \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\nabla_k \rho|^2 u^2 \, dw.
\]

Moreover, the constant $\frac{(Q + \alpha - 4)^2 (Q - \alpha)^2}{16}$ is sharp (for the proof of sharpness see [4]).

We now set the pair as

\[
a = \frac{\rho^\alpha}{|\nabla_k \rho|^{2p-2}} \quad \text{and} \quad \vartheta = \rho^{-\left( \frac{Q+a-2}{p} \right)}.
\]

Then via Theorem 3.2, we get the weighted $L^p$ Rellich inequality proved in [19].
Corollary 2. Let $1 < p < \frac{Q + \alpha}{2}$ and $\alpha \in \mathbb{R}$. Then for any $u \in C_0^\infty (\mathbb{R}^{2n+1} \setminus \{0\})$, we have

$$\int_{\mathbb{R}^{2n+1}} \rho^\alpha |\Delta_k u|^p \, dw \geq C_{Q,p,\alpha}^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\nabla_k \rho|^{2p} |u|^p \, dw,$$

where $C_{Q,p,\alpha} = (Q-p-Q(\alpha+2p)).$ Moreover, the positive constant $C_{Q,p,\alpha}^p$ is sharp (for the proof of sharpness see [19]).

If we apply Theorem 3.2 together with the couple

$$a = \rho^\alpha \quad \text{and} \quad \vartheta = \rho^{-(\frac{Q+\alpha-2p}{p})},$$

we have the following weighted Rellich type inequality which is again from [19].

Corollary 3. Let $1 < p < \frac{Q + \alpha}{2}$ and $\alpha \in \mathbb{R}$. Then for any $u \in C_0^\infty (\mathbb{R}^{2n+1} \setminus \{0\})$, we have

$$\int_{\mathbb{R}^{2n+1}} \rho^\alpha |\Delta_k u|^p \, dw$$

$$+ 2(p-1)(2k-1)(Q+4pk-6k-2p)C^{p-1} \int_{\mathbb{R}^{2n+1}} \rho^\alpha \left(\frac{|z|}{\rho}\right)^{4k(p-1)} |u|^p \, dw$$

$$\geq \left[C^p + 2(p-1)(2k-1)(Q+4pk-4k-2p)C^{p-1}\right] \int_{\mathbb{R}^{2n+1}} \rho^\alpha \left(\frac{|z|}{\rho}\right)^{4kp} |u|^p \, dw,$$

where $C = C_{Q,p,\alpha} = (Q-p-Q(\alpha+2p)).$}

Remark 1. Lian [19] also showed that the constant

$$C^p + 2(p-1)(2k-1)(Q+4pk-4k-2p)C^{p-1}$$

appeared on the right hand side of the above inequality is sharp.

We should emphasize here that one can apply Theorem 3.2 to obtain not only known weighted Rellich type inequalities but also other new ones on some special domains in $\mathbb{R}^{2n+1}$. For instance, specializing the functions

$$a = \frac{\rho^\alpha}{|\nabla_k \rho|^{2p-2}} \quad \text{and} \quad \vartheta = \log \frac{R}{\rho}, \quad R > \sup_{w \in \Omega} \rho$$

on bounded domains $\Omega$ in $\mathbb{R}^{2n+1}$, we get the following inequality.

Corollary 4. Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^{2n+1}$. Let $\alpha \geq 0, \frac{Q+\alpha}{2} > p > \frac{\alpha+2}{2}$ and $R > \sup_{w \in \Omega} \rho$. Then for every $u \in C_0^\infty (\Omega)$, we have

$$\int_{\Omega} \frac{\rho^\alpha}{|\nabla_k \rho|^{2p-2}} |\Delta_k u|^p \, dw \geq \tilde{C}_{Q,\alpha,p} \int_{\Omega} \left(\frac{\rho^\alpha}{\log \frac{R}{\rho}}\right)^{p-1} |\nabla_k \rho|^{2p} |u|^p \, dw,$$

where $\tilde{C}_{Q,\alpha,p} := (2p - \alpha - 2)(Q + \alpha - 2p)(Q-2)^{p-1}$.

A direct consequence of Theorem 3.2 with the pair

$$a = \frac{\rho^\alpha}{|\nabla_k \rho|^{2p-2}} \quad \text{and} \quad \vartheta = R - \rho$$

on the $\rho-$ball $B_R$ in $\mathbb{R}^{2n+1}$ is the following result.
Corollary 5. Let \( Q + \alpha - 1 > p > \alpha + 1 \) and \( \alpha \geq 0 \). Then for every \( u \in C_0^\infty (B_R) \), one has
\[
\int_{B_R} \frac{\rho^\alpha}{|\nabla_k \rho|^{2p-2}} |\Delta_k u|^p dw \geq \tilde{C}_{Q,\alpha,p} \int_{B_R} \frac{\rho^\alpha}{(R - \rho)^{p+1}} |\nabla \rho|^2 |u|^p dw,
\]
where \( \tilde{C}_{Q,\alpha,p} := (p - \alpha - 1)(Q + \alpha - p - 1)(Q - 1)^{p-1} \) and \( B_R \) is the \( \rho \)-ball.

3.1.1. Various Rellich type inequalities with non-radial weights. Most of the inequalities discussed to this point involve the weights of the form \( \rho^\alpha |z|^2 \) for some real numbers \( \alpha \) and \( \beta \). However, we now consider versions of \( L^p \) Rellich type inequalities with non-radial weights related to the Greiner operator \( \Delta_k \) on some special domains \( \Omega \) in \( \mathbb{R}^{2n+1} \) via Theorem 3.2. As an example, the following choice of non-symmetric functions
\[
a = y_1^{2p-2} \log x_1 \quad \text{and} \quad \vartheta = \log y_1
\]
on \( \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : x_1 > 1, y_1 > 1 \} \) produces the first result in this direction.

Corollary 6. For any \( u \in C_0^\infty (\Omega) \) and \( p > 1 \), one has
\[
\int_{\Omega} y_1^{2p-2} \log x_1 |\Delta_k u|^p dw \geq \int_{\Omega} \frac{|u|^p}{x_1^{2p-1} y_1} dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : x_1 > 1, y_1 > 1 \} \).

On the other hand, considering the non-radial functions
\[
a = \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \log y_1 \quad \text{and} \quad \vartheta = \log l
\]
on \( \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1, y_1 > 1 \} \) gives the following \( L^p \) Rellich type inequality with non-radial weights.

Corollary 7. For any \( u \in C_0^\infty (\Omega) \) and \( p > 1 \), one has
\[
\int_{\Omega} \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \log y_1 |\Delta_k u|^p dw \geq (2k)^{2p-2} \int_{\Omega} \frac{|u|^p}{y_1^2 \log^{p-1} l} dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1, y_1 > 1 \} \) and \( k \geq 1 \).

Another direct consequence of Theorem 3.2 with the following non-radial model functions
\[
a = \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \log l \quad \text{and} \quad \vartheta = \log l
\]
on \( \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \} \) leads us to the subsequent inequality with non-radial weights again.

Corollary 8. For any \( u \in C_0^\infty (\Omega) \) and \( p > 1 \), one has
\[
\int_{\Omega} \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \log l |\Delta_k u|^p dw \geq (2k)^{2p} \int_{\Omega} \frac{|z|^{4k-2}}{l^2 \log^{p-1} l} |u|^p dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \} \) and \( k \geq 1 \).
We now set the non-radial functions
\[ a = \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \sqrt{\theta} \quad \text{and} \quad \theta = \log l \]
on \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \}. Hence, Theorem 3.2 immediately yields another \( L^p \) Rellich type inequality with non-radial weights.

**Corollary 9.** For any \( u \in C_0^\infty(\Omega) \) and \( p > 1 \), one has
\[
\int_\Omega \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \sqrt{\theta} |\Delta_k u|^p \, dw \geq \frac{(2k)^{2p} - 2}{4} \int_\Omega \frac{|z|^{4k-2}}{\sqrt{\theta}^p \log^{p-1} l} |u|^p \, dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \} \) and \( k \geq 1 \).

When one takes the pair
\[ a = \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \sqrt{x_1} \quad \text{and} \quad \theta = \log l \]
on \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1, x_1 > 0 \} in Theorem 3.2, one easily obtains the following \( L^p \) Rellich type inequality with different non-radial weights.

**Corollary 10.** For any \( u \in C_0^\infty(\Omega) \) and \( p > 1 \), one has
\[
\int_\Omega \left( \frac{|z|^{2k-1}}{l} \right)^{2-2p} \sqrt{x_1} |\Delta_k u|^p \, dw \geq \frac{(2k)^{2p} - 2}{4} \int_\Omega \frac{|u|^p}{\sqrt{x_1} \log^{p-1} l} \, dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1, x_1 > 0 \} \) and \( k \geq 1 \).

Finally, if we consider the couple
\[ a = \left( \frac{\sqrt[4k-2]{l}}{|z|^{2k-1}} \right)^{p-1} \log l \quad \text{and} \quad \theta = \sqrt{\theta} \]
on \Omega := \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \}, then we can deduce the following result via Theorem 3.2.

**Corollary 11.** For any \( u \in C_0^\infty(\Omega) \) and \( p > 1 \), one has
\[
\int_\Omega \left( \frac{\sqrt[4k-2]{l}}{|z|^{2k-1}} \right)^{p-1} \log l |\Delta_k u|^p \, dw \geq \frac{4k^{2p}}{\sqrt{\theta}^{p+3}} \int_\Omega |u|^p \, dw,
\]
where \( \Omega = \{ w = (x, y, l) \in \mathbb{R}^{2n+1} : l > 1 \} \) and \( k \geq 1 \).

4. **Weighted Rellich type inequality II.** This section is concerned with the sharp weighted \( L^p \) Rellich type inequality that connects first to second order derivatives associated to the Greiner operator \( \Delta_k \). Here is the result in this direction.

**Theorem 4.1.** Let \( Qp > Q + \alpha > p > 1 \) and \( \alpha \in \mathbb{R} \). Then for any function \( u \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\}) \), one has
\[
\int_{\mathbb{R}^{2n+1}} \rho^\alpha |\Delta_k u|^p \, dw \geq \frac{(Qp - Q - \alpha)}{p} \int_{\mathbb{R}^{2n+1}} \rho^{\alpha} |\nabla_k u|^p \, dw.
\]
Furthermore, the constant \( \left( \frac{Qp - Q - \alpha}{p} \right)^p \) is sharp.
Proof. For any \( u \in C_0^\infty(\mathbb{R}^{2n+1}\setminus\{0\}) \) and \( \beta \in \mathbb{R} \), we set
\[
v = \rho^\beta \nabla_k \rho \cdot \nabla_k u.
\] (13)

In view of (10), an evident computation gives that
\[
\nabla_k v = \rho^\beta \Delta_k u \nabla_k \rho + (Q + \beta - 1) \rho^{\beta - 1} |\nabla_k \rho|^2 \nabla_k u.
\]

We now employ the following convexity inequality
\[
|\xi + \eta|^p \geq |\xi|^p + p|\xi|^{p-2}\eta \cdot \eta,
\]
where \( \xi, \eta \in \mathbb{R}^n \) and \( p > 1 \) (see [20]) to arrive at
\[
|\rho^\beta \Delta_k u \nabla_k \rho|^p = \left| -(Q + \beta - 1) \rho^{\beta - 1} |\nabla_k \rho|^2 \nabla_k u + \nabla_k v \right|^p
\geq (Q + \beta - 1)^p \rho^{(\beta - 1)p} |\nabla_k \rho|^{2p} |\nabla_k u|^p
- p (Q + \beta - 1)^{p-1} \rho^{(\beta - 1)(p-1)} |\nabla_k \rho|^{2(p-1)} |\nabla_k u|^{p-2} \nabla_k u \cdot \nabla_k v.
\] (14)

One can deduce from (13) that \( v \nabla_k \rho \cdot \nabla_k u = \rho^\beta |\nabla_k \rho|^2 |\nabla_k u|^2 \) and hence
\[
|\nabla_k u|^{p-2} \nabla_k u \cdot \nabla_k v = \frac{\rho^{(1-p)\beta} \nabla_k \rho \cdot \nabla_k |v|^p}{p |\nabla_k \rho|^p}.
\] (15)

Combining (14) and (15) we obtain
\[
\rho^\beta |\nabla_k \rho|^p |\Delta_k u|^p \geq (Q + \beta - 1)^p \rho^{(\beta - 1)p} |\nabla_k \rho|^{2p} |\nabla_k u|^p
- (Q + \beta - 1)^{p-1} \rho^{1-p} |\nabla_k \rho|^{p-2} \nabla_k \rho \cdot \nabla_k |v|^p.
\] (16)

Choosing \( \beta = \frac{p-Q-\alpha}{p} \) and then multiplying both sides of (16) by \( \frac{\rho^{Q-2p+2\alpha + 1}}{|\nabla_k \rho|^{p+2}} \) leads to
\[
\frac{\rho^\alpha}{|\nabla_k \rho|^p} |\Delta_k u|^p \geq \left( \frac{Qp - Q - \alpha}{p} \right)^p \rho^\alpha |\nabla_k u|^p
- \left( \frac{Qp - Q - \alpha}{p} \right)^{p-1} \rho^{Q-2p+2\alpha + 1} |\nabla_k \rho|^{2p+2} \nabla_k \rho \cdot \nabla_k |v|^p.
\]

As an immediate consequence of integration by parts we get
\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha}{|\nabla_k \rho|^p} |\Delta_k u|^p \, dw \geq \left( \frac{Qp - Q - \alpha}{p} \right)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\nabla_k u|^p \, dw
+ \left( \frac{Qp - Q - \alpha}{p} \right)^{p-1} \int_{\mathbb{R}^{2n+1}} \nabla_k \cdot \left( \frac{\rho^{Q-2p+2\alpha + 1}}{|\nabla_k \rho|^{p+2}} \nabla_k \rho \right) |v|^p \, dw.
\]

Together with the identities (10) and (11), we compute
\[

abla_k \cdot \left( \frac{\rho^{Q-2p+2\alpha + 1}}{|\nabla_k \rho|^{p+2}} \nabla_k \rho \right) = 2 (Q + \alpha - p) \rho^{Q+2\alpha-2p} |\nabla_k \rho|^p.
\]

Since \( Qp > Q + \alpha > p > 1 \), we obtain the desired inequality:
\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha}{|\nabla_k \rho|^p} |\Delta_k u|^p \, dw \geq \left( \frac{Qp - Q - \alpha}{p} \right)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |\nabla_k u|^p \, dw.
\] (17)
It only remains to show that the constant \( \left( \frac{Qp - Q - \alpha}{p} \right)^p \) is sharp and for this we use the following family of radial functions

\[
  u_\epsilon (\rho) = \begin{cases} 
    - \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right) (\rho - 1) + 1 & \text{if } \rho \leq 1, \\
    \rho^{-\left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)} & \text{if } \rho > 1,
  \end{cases}
\]

where \( \epsilon > 0 \). Note that \( u_\epsilon \) can be well approximated by smooth functions with compact support in \( \mathbb{R}^{2n+1} \). A direct calculation yields that

\[
|\nabla_k u_\epsilon|^p = \begin{cases} 
  \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p |\nabla_k \rho|^p & \text{if } \rho \leq 1, \\
  \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \rho^{-Q - \alpha + ep} |\nabla_k \rho|^p & \text{if } \rho > 1
  \end{cases}
\]

and

\[
|\Delta_k u_\epsilon|^p = \begin{cases} 
  (Q - 1)^p \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \rho^{-p} |\nabla_k \rho|^{2p} & \text{if } \rho \leq 1, \\
  \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \left( \frac{Qp - Q - \alpha}{p} + \epsilon \right)^p \rho^{-Q - \alpha + ep} |\nabla_k \rho|^{2p} & \text{if } \rho > 1.
  \end{cases}
\]

We decompose the integral on the left-hand side of (17) as follows:

\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha}{|\nabla_k \rho|^p} |\Delta_k u_\epsilon|^p \, dw = A_{Q,p,\alpha,\epsilon} \int_{\rho \leq 1} \rho^{Q + \alpha - p - 1} |\nabla_k \rho|^p \, dw + B_{Q,p,\alpha,\epsilon} \int_{\rho > 1} \rho^{-Q - ep} |\nabla_k \rho|^p \, dw,
\]

where

\[
A_{Q,p,\alpha,\epsilon} := (Q - 1)^p \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p
\]

and

\[
B_{Q,p,\alpha,\epsilon} := \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \left( \frac{Qp - Q - \alpha}{p} + \epsilon \right)^p.
\]

By the spherical coordinates in Section 2, we have

\[
\int_{\rho \leq 1} \rho^{Q + \alpha - p - 1} \, d\rho = s_n \int_0^1 \rho^{Q + \alpha - p - 1} \, d\rho = \frac{s_n}{Q + \alpha - p},
\]

and

\[
\int_{\rho > 1} \rho^{-Q - ep} \, d\rho = s_n \int_1^{\infty} \rho^{-ep - 1} \, d\rho = \frac{s_n}{ep}.
\]

It therefore follows from (18) and (19) that

\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha}{|\nabla_k \rho|^p} |\Delta_k u_\epsilon|^p \, dw = s_n \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \left( \frac{(Q - 1)^p}{Q + \alpha - p} + \frac{\left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p}{ep} \right).
\]

Similarly, the integral on the right-hand side of (17) can be computed as

\[
\int_{\mathbb{R}^{2n+1}} \frac{\rho^\alpha |\nabla_k u_\epsilon|^p}{\rho^p} \, dw = s_n \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \left( \int_0^1 \rho^{Q + \alpha - p - 1} \, d\rho + \int_1^{\infty} \rho^{-ep - 1} \, d\rho \right)
\]

\[
= s_n \left( \frac{Q + \alpha - 2p}{p} + \epsilon \right)^p \left( \frac{1}{Q + \alpha - p} + \frac{1}{ep} \right).
\]
Hence, the Rayleigh quotient is given by
\[
\frac{\int_{\mathbb{R}^{2n+1}} \frac{a}{|\nabla_k \rho|^2} |\Delta_k u|^2 \, dw}{\int_{\mathbb{R}^{2n+1}} \frac{a}{|\nabla_k \rho|^2} |\nabla_k u|^2 \, dw} = \frac{(Q - 1)^p}{1 + \frac{Q + \alpha - p}{\epsilon p}} + \left(\frac{Qp - Q - \alpha}{p} + \epsilon\right)^p.
\]
(20)

Letting \(\epsilon \to 0\) in (20), we obtain the sharp constant \(\left(\frac{Qp - Q - \alpha}{p}\right)^p\). Therefore, this completes the proof. \(\square\)

5. Improved two-weight Rellich type inequalities. In this section, we first introduce an improved two-weight \(L^p\) Rellich type inequality related to the Greiner operator \(\Delta_k\) on the basis of a particular partial differential inequality and then give explicit examples to illustrate our result for different weights.

**Theorem 5.1.** Let \(\Omega\) be a bounded domain with smooth boundary \(\partial\Omega\) in \(\mathbb{R}^{2n+1}\). Let \(a \geq 0\) and \(\vartheta > 0\) be smooth functions satisfying \(-\nabla_k \cdot (a \rho^2 - Q \nabla_k \vartheta) \geq 0\) a.e. in \(\Omega\). Then for any \(u \in C_0^\infty(\Omega)\), one has
\[
\int_{\Omega} \frac{a \rho^\alpha}{|\nabla_k \rho|^{2p - 2}} |\Delta_k u|^p \, dw \geq C_p \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 |u|^p \frac{\rho^2}{\rho^{2p}} \, dw
\]
+ \frac{(p - 1) C^{p - 1}}{p} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 |u|^p \frac{\rho^2}{\rho^{2p}} \, dw
\]
+ \frac{2 (p - 1) (Q + \alpha - 2p)}{p} \int_{\Omega} \rho^\alpha \nabla_k \rho \cdot \nabla_k a |u|^p \frac{\rho^2}{\rho^{2p - 2}} \, dw
\]
+ \frac{2p C^{p - 1}}{p} \int_{\Omega} \rho^\alpha \nabla_k \rho \cdot \nabla_k a |u|^{p - 1} \frac{\rho^2}{\rho^{2p - 2}} \, dw
\]
+ \frac{C^{p - 1}}{p} \int_{\Omega} \rho^\alpha \Delta_k a |u|^p \frac{\rho^2}{\rho^{2p - 2}} \, dw,
\]
where \(C = C_{Q, p, \alpha} = \frac{(Qp - Q - \alpha)(Q + \alpha - 2p)}{\rho^2}\) and \(Q + \alpha > 2p\).

**Proof.** Together with the identity (10) direct calculation implies that
\[
\Delta_k \rho^{-2p + 2} = (Q + \alpha - 2p) (\alpha - 2p + 2) \rho^{-2p} |\nabla_k \rho|^2.
\]
(21)

Multiplying both sides of (21) by \(a |u|^p\) and applying integration by parts two times over \(\Omega\) yield
\[
\int_{\Omega} \rho^{-2p + 2} \Delta_k (a |u|^p) \, dw = (Q + \alpha - 2p) (\alpha - 2p + 2) \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 |u|^p \frac{\rho^2}{\rho^{2p}} \, dw.
\]
(22)

Since
\[
\Delta_k (a |u|^p) = \Delta_k a |u|^p + 2p \nabla_k u \cdot \nabla_k a |u|^{p - 1}
\]
+ \(p (p - 1) a |\nabla_k u|^2 |u|^{p - 2} + pa |u|^{p - 1} \Delta_k u,
\]

we obtain
\[
\int_{\Omega} \rho^{-2p + 2} \Delta_k (a |u|^p) \, dw \geq C_p \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 |u|^p \frac{\rho^2}{\rho^{2p}} \, dw
\]
+ \frac{(p - 1) C^{p - 1}}{p} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 |u|^p \frac{\rho^2}{\rho^{2p}} \, dw
\]
+ \frac{2 (p - 1) (Q + \alpha - 2p)}{p} \int_{\Omega} \rho^\alpha \nabla_k \rho \cdot \nabla_k a |u|^p \frac{\rho^2}{\rho^{2p - 2}} \, dw
\]
+ \frac{2p C^{p - 1}}{p} \int_{\Omega} \rho^\alpha \nabla_k \rho \cdot \nabla_k a |u|^{p - 1} \frac{\rho^2}{\rho^{2p - 2}} \, dw
\]
+ \frac{C^{p - 1}}{p} \int_{\Omega} \rho^\alpha \Delta_k a |u|^p \frac{\rho^2}{\rho^{2p - 2}} \, dw.
\]
the equality (22) can be written as
\[
\int_{\Omega} a \rho^\alpha |\nabla_k u|^2 \frac{|u|^{p-2}}{\rho^{2p-2}} \, dw - \frac{(\alpha - 2p + 2) (Q + \alpha - 2p)}{p (p-1)} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]
\[
= - \frac{1}{p-1} \int_{\Omega} a \rho^\alpha \Delta_k u \frac{|u|^{p-1}}{\rho^{2p-2}} \, dw - \frac{2}{p-1} \int_{\Omega} \rho^\alpha \nabla_k \cdot \nabla_k u \frac{|u|^{p-1}}{\rho^{2p-2}} \, dw
\]
\[
- \frac{1}{p (p-1)} \int_{\Omega} \rho^\alpha \Delta_k \rho \frac{|u|^p}{\rho^{2p-2}} \, dw.
\]  (23)

Applying successively Hölder and Young inequalities to the integral expression

\[
I_1 := - \frac{1}{p-1} \int_{\Omega} a \rho^\alpha \Delta_k u \frac{|u|^{p-1}}{\rho^{2p-2}} \, dw
\]
on the right hand side of (23), we estimate

\[
I_1 \leq \frac{1}{p-1} \left( \int_{\Omega} \frac{a \rho^\alpha}{|\nabla_k \rho|^{2p-2}} |\Delta_k u|^p \, dw \right)^{\frac{1}{p}} \left( \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw \right)^{\frac{p-1}{p}}
\]
\[
\leq \frac{e^p}{p (p-1)} \int_{\Omega} \frac{a \rho^\alpha}{|\nabla_k \rho|^{2p-2}} |\Delta_k u|^p \, dw + \frac{e^{\frac{p}{2}}}{p} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]  (24)

for any \( \epsilon > 0 \). As a next step, we first use the identity \( |\nabla_k u|^2 |u|^{p-2} = \frac{1}{p} |\nabla_k (u^{p/2})|^2 \) and then employ the following improved two-weight Hardy type inequality from [28]

\[
\int_{\Omega} a \rho^\alpha |\nabla_k u|^2 \, dw \geq \frac{(Q + \alpha - 2)^2}{4} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 \frac{u^2}{\rho^2} \, dw
\]
\[
+ \frac{(Q + \alpha - 2)}{2} \int_{\Omega} a \rho^\alpha u^2 \rho \, dw
\]
\[
+ \frac{1}{4} \int_{\Omega} a \rho^\alpha \frac{|\nabla_k \theta|^2}{\rho^2} u^2 \, dw
\]
to the integral expression

\[
I_2 := \int_{\Omega} a \rho^\alpha |\nabla_k u|^2 \frac{|u|^{p-2}}{\rho^{2p-2}} \, dw
\]
on the left-hand side of the equality (23). Then we get

\[
I_2 = \frac{4}{p^2} \int_{\Omega} a \rho^{\alpha - 2p + 2} |\nabla_k (u^{p/2})|^2 \, dw \geq \frac{(Q + \alpha - 2p)^2}{p^2} \int_{\Omega} a \rho^\alpha |\nabla_k \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]
\[
+ \frac{2 (Q + \alpha - 2p)}{p^2} \int_{\Omega} \rho^\alpha \nabla_k \cdot \nabla_k \rho \frac{|u|^p}{\rho^{2p-1}} \, dw
\]
\[
+ \frac{1}{p^2} \int_{\Omega} a \rho^\alpha \frac{|\nabla_k \theta|^2}{\rho^2} \frac{|u|^p}{\rho^{2p-2}} \, dw.
\]  (25)
Combining (24) and (25) with (23) we deduce that
\[
\int_{\Omega} \frac{a\rho^{\alpha}}{|\nabla k\rho|^{2p-2}} |\Delta_k u|^p \, dw \geq f(\epsilon) \int_{\Omega} \frac{a\rho^{\alpha}}{|\nabla k\rho|^{2p-2}} |u|^p \, dw \\
+ \frac{(p-1)}{p} \int_{\Omega} \rho^\alpha |\nabla k\vartheta|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw \\
+ \frac{2p}{\epsilon p} \int_{\Omega} \rho^\alpha \nabla k\alpha \cdot \nabla k\rho \frac{|u|^p}{\rho^{2p-2}} \, dw \\
+ \frac{2(p-1)(Q + \alpha - 2p)}{p} \int_{\Omega} \rho^\alpha |\nabla k\alpha \cdot \nabla k\rho| \frac{|u|^p}{\rho^{2p-1}} \, dw \\
+ \frac{1}{\epsilon p} \int_{\Omega} \rho^\alpha \Delta_k \rho |u|^p \rho^{2p-2} \, dw,
\]
where \( \epsilon > 0 \) and
\[
f(\epsilon) := \epsilon^{-p} \left[ \frac{(p-1)(Q + \alpha - 2p)^2}{p} - (Q + \alpha - 2p)(\alpha - 2p + 2) - (p-1) \epsilon^{p-1} \right].
\]
Observe that \( f(\epsilon) \) attains the maximum for \( \epsilon_0 = C \frac{1-p}{Q - p} \) and this maximum value is equal to \( f(\epsilon_0) = C^p \), where \( C = C_{Q,p,\alpha} = \frac{(Qp - Q - \alpha)(Q + \alpha - 2p)}{p^2} \). The theorem is therefore proved.

5.1. **Applications of Theorem 5.1.** We should mention here that one can obtain as many weighted improved \( L^p \) Rellich type inequalities associated to the Greiner operator \( \Delta_k \) as one can construct functions \( a \) and \( \vartheta \) satisfying the differential inequality \( -\nabla k \cdot (a\rho^{2-Q}\nabla k\vartheta) \geq 0 \). In what follows we apply Theorem 5.1 to concrete cases. For instance, a direct consequence of Theorem 5.1 together with the following choice
\[
a \equiv 1 \quad \text{and} \quad \vartheta = \log \left( \frac{R}{\rho} \right), \quad R > \sup_{w \in \Omega} \rho
\]
on bounded domains \( \Omega \) in \( \mathbb{R}^{2n+1} \) leads to the weighted \( L^p \) Rellich type inequality including a logarithmic remainder.

**Corollary 12.** Let \( \Omega \) be a bounded domain with smooth boundary \( \partial \Omega \) in \( \mathbb{R}^{2n+1} \) and \( R > \sup_{w \in \Omega} \rho \). Then for any \( u \in C^\infty_0(\Omega) \), we have
\[
\int_{\Omega} \frac{\rho^\alpha}{|\nabla k\rho|^{2p-2}} |\Delta_k u|^p \, dw \geq C \int_{\Omega} \rho^\alpha |\nabla k\rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw \\
+ \frac{(p-1)}{p} \int_{\Omega} \frac{\rho^\alpha}{|\nabla k\rho|^2} \frac{|u|^p}{\rho^{2p}} \, dw,
\]
where \( C = C_{Q,p,\alpha} = \frac{(Qp - Q - \alpha)(Q + \alpha - 2p)}{p^2} \) and \( Q + \alpha > p \geq 2 \).

By considering the pair as
\[
a \equiv 1 \quad \text{and} \quad \vartheta = \log \left( \frac{\log R}{\rho} \right), \quad R > e \sup_{w \in \Omega} \rho
\]
in Theorem 5.1, we acquire another result having a different logarithmic remainder.
Corollary 13. Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^{2n+1}$ and $R > e \sup_{x \in \Omega} \rho$. Then for any $u \in C_0^\infty(\Omega)$, we have
\[
\int_{\Omega} \frac{\rho^\alpha}{|\nabla \rho|^{2p-2}} |\Delta_k u|^p \, dw \geq C_p \int_{\Omega} \rho^\alpha |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]
\[
+ \frac{(p-1)}{p} \int_{\Omega} \rho^{\alpha} e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw
\]
\[
+ \frac{2(p-1)}{p} \left(\int_{\Omega} \rho^\alpha e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw\right) \Omega
\]
where $C = C_{Q,p,\alpha} = \frac{(Qp-Q-\alpha)(Q+\alpha-2p)}{p^2}$ and $\frac{Q+\alpha}{2} > p \geq 2$.

If we apply Theorem 5.1 with the couple
\[
a = e^\rho \quad \text{and} \quad \vartheta = e^{-\rho},
\]
then we get the subsequent two-weight $L^p$ Rellich type inequality containing more nonnegative remainders.

Corollary 14. Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^{2n+1}$. Then for any $u \in C_0^\infty(\Omega)$, we have
\[
\int_{\Omega} \frac{\rho^\alpha e^\rho}{|\nabla \rho|^{2p-2}} |\Delta_k u|^p \, dw \geq C_p \int_{\Omega} \rho^\alpha e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]
\[
+ \frac{(2p-1)}{p} \int_{\Omega} \rho^\alpha e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw
\]
\[
+ \frac{(2(p-1)(Q+\alpha-2p)+Qp-p)}{p} \int_{\Omega} \rho^\alpha e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw
\]
\[
+ 2pC^{p-1} \int_{\Omega} \rho^\alpha e^\rho |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw,
\]
where $C = C_{Q,p,\alpha} = \frac{(Qp-Q-\alpha)(Q+\alpha-2p)}{p^2}$ and $\frac{Q+\alpha}{2} > p \geq 2$.

Another consequence of Theorem 5.1 with the special functions
\[
a \equiv 1 \quad \text{and} \quad \vartheta = R - \rho
\]
on the $\rho$-ball $B_R$ centered at the origin with radius $R$ is the following result.

Corollary 15. Let $B_R$ be the $\rho$-ball centered at the origin with radius $R$ in $\mathbb{R}^{2n+1}$. Then for any $u \in C_0^\infty(B_R)$, we have
\[
\int_{B_R} \frac{\rho^\alpha}{|\nabla \rho|^{2p-2}} |\Delta_k u|^p \, dw \geq C_p \int_{B_R} \rho^\alpha |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p}} \, dw
\]
\[
+ \frac{(p-1)}{p} \int_{B_R} \rho^\alpha (R-\rho)^2 |\nabla \rho|^2 \frac{|u|^p}{\rho^{2p-2}} \, dw,
\]
where $C = C_{Q,p,\alpha} = \frac{(Qp-Q-\alpha)(Q+\alpha-2p)}{p^2}$ and $\frac{Q+\alpha}{2} > p \geq 2$.

Remark 2. The lack of regularities on the above choices can be readily handled by replacing the function $\rho$ with $\rho_\epsilon$ and then passing to the limit as $\epsilon \to 0$. 
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