Contact process on weighted planar stochastic lattice

Sidney G Alves* and Marcelo M de Oliveira
Departamento de Estatística, Física e Matemática, Universidade Federal de São João Del-Rei, 36490-972 Ouro Branco, MG, Brazil
E-mail: sidiney@ufsj.edu.br and mmdeoliveira@ufsj.edu.br

Received 14 March 2022
Accepted for publication 16 May 2022
Published 9 June 2022

Online at stacks.iop.org/JSTAT/2022/063201
https://doi.org/10.1088/1742-5468/ac70dc

Abstract. We study the absorbing state phase transition in the contact process on the weighted planar stochastic (WPS) lattice. The WPS lattice is multifractal. Its dual network has a power-law degree distribution function and is also embedded in a bidimensional space. Moreover, it represents a novel way to introduce coordination disorder in lattice models. We investigated the critical behavior of the disordered system using extensive simulations. Our results show the critical behavior is distinct from that on a regular lattice, suggesting it belongs to a different universality class. We evaluate the exponent governing the bond fluctuations and our results agree with the Harris–Barghathi–Vojta criterium for relevant fluctuations.

Keywords: phase transitions into absorbing states, absorbing states

*Author to whom any correspondence should be addressed.
1. Introduction

In nonequilibrium systems, an absorbing-state phase transition occurs when a control parameter is varied, and the system experiences a phase transition from an active (fluctuating) state to an absorbing (with no fluctuations) state [1, 2]. Absorbing-state phase transitions have attracted considerable interest in recent years, as they appear in a broad class of phenomena, such as epidemic spreading [3], chemical reactions [4], population dynamics [5], and others [6].

As in equilibrium phase transitions, it is expected that critical phase transitions into absorbing states can be classified in a finite number of universality classes, depending on a few characteristics of the model, such as symmetries, conserved quantities, and dimensionality [7]. Models such as the contact process (CP), with short-range interactions and without a conserved quantity or symmetry beyond translational invariance, belong to the directed percolation (DP) universality class, the most prominent class of absorbing-state transitions [8, 9].

The disorder is an unavoidable ingredient in real systems, so it is of primary interest to understand how spatially quenched (i.e. that does not change in time) disorder affects the critical behavior of absorbing-state phase transitions. On a regular lattice, the quenched disorder is usually introduced by dilution (random deletion of sites or bonds) [10–13] or by random spatial fluctuations of the control parameter [14–16]. In these cases, where the disorder is uncorrelated, rare regions that are locally supercritical (even when the whole system is sub-critical) emerge. The lifetime of these active rare regions grows exponentially with the domain size, leading to slow dynamics, with nonuniversal exponents, for some interval of the control parameter $\lambda_c(0) < \lambda < \lambda_c$ with $\lambda_c(0)$ and $\lambda_c$ being the critical points of the clean and disordered systems, respectively. This kind of behavior marks a Griffiths phase (GP), and it appears in DP models with uncorrelated disorder, irrespective of the disorder strength [13, 17].

https://doi.org/10.1088/1742-5468/ac70dc
Strong disorder renormalization group methods and numerical studies showed that the activated-disorder behavior corresponds to the universality class of the random transverse Ising model \cite{18, 19}. These findings are in agreement with the heuristic Harris’ criterion \cite{20}, which states that uncorrelated quenched disorder is a relevant perturbation if $d\nu_\perp < 2$, where $d$ is the dimensionality, and $\nu_\perp$ is the correlation length exponent of the clean model. Here, it is noteworthy to mention that in DP class, this inequality is satisfied for all dimensions $d < 4$, since $\nu_\perp = 1.096854(4), 0.7333(75)$ and $0.584(5)$, for $d = 1, 2$ and $3$, respectively \cite{2}.

A further step is to understand the effects of topological disorder, which appears in nonperiodic, random structures. One example is the random lattice generated by the Voronoi–Delaunay (VD) triangulation \cite{21}, which is a bidimensional connected graph with a Poissonian distribution of connectivity with average degree $7 = 6$ (where we use the term degree to designate the node number of edges). In such a case, it was found that the disorder does not alter the critical behavior exhibited by the clean CP \cite{22, 23}. These results are in clear contrast with the above-mentioned results for uncorrelated disorder, which lead to an infinite-randomness critical point and strong GPs.

In order to determine the relevance of the disorder in these cases, extending early works by Luck \cite{24}, Barghathi and Vojta \cite{25} proposed what is known as the Harris–Barghathi–Vojta (HBV) criterion. It is based on the idea that the stability of the critical point is governed by the decay of spatial fluctuations of the local coordination numbers. Averaging the coordination numbers $q$ of the lattice nodes over coarse-grained blocks, the corresponding variance $\sigma_q$ decays algebraically with the block size, $\sigma_q(L_b) \sim L_b^{-a}$. Therefore, the HBV criterion states that quenched disorder is an irrelevant perturbation if the inequality $a\nu_\perp > 1$ holds (the exponent $a$ describes the decay of the fluctuations). Note that, in the case of independent dilution, one has $a = d/2$, and the HBV criterion reduces to the original Harris criterion. Besides, the HBV criterion has successfully explained the critical behavior in the VD triangulation, and in other examples with a topological disorder, very recently examples of a class of systems that violate even the HBV criterion were found \cite{26–28}.

In this work, we study the critical behavior of the CP in a distinct disordered structure, proposed by Hassan et al \cite{29}, the weighted planar stochastic (WPS) lattice. In comparison with the VD lattice, the WPS lattice neither has a fixed cell size nor a fixed coordination number. On the other hand, the WPS lattice introduces topological disorder emerging from a multifractal structure \cite{30}, and its coordination number follows a power-law. Also, the WPS lattice resembles a crack tessellation \cite{31}, which is applied in city modeling \cite{32}.

Hassan and Rahman \cite{33} found that the isotropic percolation (bond and site) universality class on the WPS lattice is distinct from the one for all of the known planar lattices \cite{34}. It is suggested that the power-law distribution of the coordination number is the ingredient that changes the universality class. Liu et al \cite{35} also found that an opinion dynamic model exhibits distinct critical behavior on the WPS lattice, in contrast with the critical exponents of VD, which are the same as two-dimensional Ising model \cite{36}.

So, due to the importance of the DP universality class in the context of nonequilibrium phase transitions, and to give a further step in the understanding of the effects of
distinct kinds of topological disorder in critical phenomena, in the present contribution, we aim to investigate whether disorder in the form of a coordination disorder exhibited by the WPS lattice alters the critical behavior of the CP.

The remainder of this paper is organized as follows. In section 2, we present the model and methods employed in the work. In section 3, we show and discuss our simulation results; section 4 is to summarize our conclusions.

2. Models and methods

The WPS lattice is built following the recipe in [29]. The planar lattice construction begins by taking a square and dividing it randomly into four smaller blocks. Then, these blocks are labeled by their respective areas in a clockwise fashion, starting from the upper left block. A block is chosen with a probability proportional to its area in the next steps. This step is repeated considering the block area weighting the probability, until the desired total iteration number (the process conserves the area of the starting square during the fragmentation process). After a number \( N_{it} \) of iterations following such a process, the resulting planar lattice has \( 3N_{it} + 1 \) blocks, and its dual lattice contains \( N = 3N_{it} + 1 \) nodes. Each block is considered as a node and two blocks are connected if they share any part of their borders. The total number of blocks sharing any part with a given block defines its neighbors number or degree. We show a snapshot of a block diagram obtained after \( N_{it} = 200 \) steps of the WPS construction in figure 1(a), and in (b), its dual network.

In comparison with the VD lattice, the WPS lattice neither has a fixed cell size nor a fixed degree. However, while in the VD lattice, the degree probability distribution function (PDF) is peaked around its mean, in the dual of a WPS lattice, the degree PDF
follows a power-law, $P(q) \sim q^{-5.6}$, as shown in figure 2. The WPS lattice also presents size disorder as the size of its blocks exhibits a multifractal structure [30]. In addition, the partitioning process can be used as a way to grow networks. So, we note the dual WPS network grows by a partitioning process, where four new nodes replace one old node at each iteration. In the present work, all the simulations were performed using the dual WPS network (see figure 1(b)) as the substrate for the standard CP.

The CP is a stochastic interacting particle system defined on a lattice, where each site $i$ either can be occupied ($\sigma_i(t) = 1$), or vacant ($\sigma_i(t) = 0$). Particles are created at a site $i$ if at least one of its neighbors is occupied: the transition from $\sigma_i = 0$ to $\sigma_i = 1$ occurs at rate $\lambda r$, where $r$ is the fraction of nearest neighbors of site $i$ that are occupied. Particle annihilation, i.e. transitions from $\sigma_i = 1$ to $\sigma_i = 0$ occur at a unity rate, independent of the neighboring sites. Therefore, the state $\sigma_i = 0$ for all $i$ is absorbing. For a critical value of the control parameter $\lambda = \lambda_c$, one observes a continuous phase transition between an active phase (with a positive density $\rho$ of occupied sites) to the absorbing state (with $\rho = 0$).

In the computational scheme, we employ the usual simulation procedure: an occupied site is chosen at random. Then, we choose between annihilation events with probability $1/(1 + \lambda)$ and creation events with probability $\lambda/(1 + \lambda)$. In the case of annihilation, the chosen site is vacated, while, for creation events, one of its $q$ nearest-neighbor sites is selected at random and, if it is currently vacant, the chosen site becomes occupied. The time increment associated with each such event is $\Delta t = 1/N_{occ}$, where $N_{occ}$ is the number of occupied sites just before the attempted transition.

To find the critical creation rate $\lambda_c$, we carry out spreading analysis using WPS lattices with $N_{it} = 10^6$ iterations starting the simulation with one node occupied. After, we take averages restricted to the samples that did not visit an absorbing state. This was performed employing the quasi-stationary simulation method [37, 38], which is based on maintaining, and gradually updating, a set of configurations visited during the evolution. The size of the set is 2000 different configurations, and this set is updated with
Figure 3. Spreading of activity from a single seed. The straight line is a power law function with exponent 0.18.

3. Results and discussion

The first step in analyzing our results is to determine the critical creation rate $\lambda_c$. For this purpose, we study the number of active particles, $n(t)$ via spreading analysis using WPS lattices with $10^6$ iterations and a mean over $5 \times 10^3$ networks with 10 runs in each one. The critical value $\lambda_c$ is then defined as the smallest $\lambda$ supporting asymptotic growth. In figure 3, we show the time evolution of the number of particles $n(t)$. From the data in the figure, we obtain $\lambda_c = 1.5525(5)$. We also observe that the asymptotic evolution follows a power law,

$$n(t) \propto t^\eta,$$

where $\eta = 0.185(5)$. This value is distinct from the value $\eta = 0.2295(10)$ for the DP class.

Now, we perform extensive simulations of the CP on WPS random lattices with iteration numbers $N_{\text{it}} = 128, 256, \ldots, 8192$, using the QS simulation method. Each realization of the process is initialized with all sites occupied, and runs for at least $10^6$ time steps. Averages are taken in the QS regime, after discarding an initial transient that depends on the system size. This procedure is repeated for each realization of disorder (for each size studied, we performed averages over 100 different lattices).
In figure 4 we show the quasistationary density $\rho$ as a function of the control parameter $\lambda$ for several values of $N_{it}$. We see, as expected, a continuous phase transition from an active to an absorbing state at $\lambda_c = 1.5525(5)$. Note that the threshold is slightly higher than those obtained for regular triangular lattices $\lambda_c = 1.54780(5)$ and for the Voronoi triangulation $\lambda_c = 1.54266(4)$, which also has $q = 6$.

Analyzing the results shown in figure 5, we see that at criticality, the quasistationary density of active sites, $\rho$, follows a power-law

$$\rho \propto L^{-\beta/\nu_\perp},$$

where $L \propto \sqrt{N}$, is the linear system size. (Here we employed system of sizes generated after up to $N_{it} = 2^{16}$ interactions.) Also, $\beta$ in the above equation is the critical exponent associated with the scaling of the density of active sites, $\rho \sim |\lambda - \lambda_c|^\beta$. Finally, $\nu_\perp$ is the exponent associated with the divergence of the spatial correlation length $\varepsilon_\perp$ at criticality, $\varepsilon_\perp \sim |\lambda - \lambda_c|^{\nu_\perp}$. From the data in figure 5, we obtain the exponent ratio $\beta/\nu_\perp = 0.84(1)$, which is distinct from the value $0.797(3)$ exhibited by models in the DP universality class.

Now we turn to the dynamical exponent $z = \nu_\parallel/\nu_\perp$. We see that the lifetime of the QS state, $\tau$ (which we take as the mean time between two attempts to visit the absorbing state in the QS simulation), follows

$$\tau \propto L^z$$

at criticality. From the data in figure 6, we obtain the exponent ratio $z = 1.59(1)$, which is distinct from the value $1.7674(6)$ exhibited by models in the DP universality class.

We also search for a GP, which would be a remark of activated dynamics. A GP is a region inside the subcritical phase where the long-time decay of $\rho$ towards the extinction is algebraic (with non-universal exponents), rather than exponential, that emerges in the presence of relevant disorder in the activated dynamics scenario [12]. So, in figure 7, we present results from initial decay simulations, where the system starts its dynamics
from a fully occupied lattice. We only observe exponential decay in the subcritical phase, without any sign of a GP. At criticality, we note

$$\rho \propto t^{-\delta},$$

and we obtain $\delta = 0.57(3)$, which is far from the DP value, $\delta = 0.4505(1)$.

In summary, our results reveal that the absorbing phase transition of the CP defined on the WPS lattice does not belong to the DP universality class. It is important to mention that, while the exponents differ from those obtained for the DP class, they still obey the scaling relation $4\delta + 2\eta = 2d/z$ [1]. We also did not find any hallmark of a GP, however, we cannot rule out its existence if larger systems sizes are considered.
Now, to check the agreement of our simulation results with the predictions given by the HBV criterion, we evaluate the wandering exponent of the WPS lattice. We investigate the degree fluctuation, the system is divided into blocks of size $\epsilon$ and we calculate the block-averaged degree and its standard deviation $\sigma_Q$ [25]. In figure 8, we plot the fluctuation of the mean connection $\sigma_Q$ as a function of the grid size $\epsilon$. We obtain the exponent $a = 1.2$, which yields $a\nu_\perp = 0.88$ for $d = 2$. Therefore, according to the HBV criterion, the disorder presented by the WPS lattice should be relevant, which is in agreement with our findings.
4. Conclusions

We performed extensive large-scale simulations of the CP on the WPS lattice, which presents the multifractality and coordination number power-law distribution. We evaluated several critical exponents and found that they are distinct from those of the DP class. So, our results reveal that the topological disorder exhibited by the WPS lattice is a relevant perturbation to the DP universality class. By numerically evaluating the exponent governing the fluctuations due to the lattice topology, we notice that our findings agree with the predictions of the HBV criterion.

As a final remark, despite the quenched disorder present in the WPS dual network, it is important to mention that for infinite dimensional power-law networks, it is known that the critical behavior depends on the degree distribution exponent [39]. Therefore, it would be interesting to investigate if some kind of modification in the algorithm of the partitioning process can generate dual networks with distinct exponents. This would be useful in clarifying the role of long-range connections and the quenched disorder in changing the critical behavior of the CP on WPS networks.

Acknowledgments

We acknowledge the anonymous referees for several useful suggestions. This work was supported by CNPq and FAPEMIG, Brazilian agencies.

References

[1] Marro J and Dickman R 1999 Nonequilibrium Phase Transitions in Lattice Models (Cambridge: Cambridge University Press)
[2] Henkel M, Hinrichsen H, Lü S and Pleimling M 2008 Non-Equilibrium Phase Transitions vol 1 (Berlin: Springer)
[3] Pastor-Satorras R, Castellano C, Van Mieghem P and Vespignani A 2015 Rev. Mod. Phys. 87 925
[4] de Oliveira M M and Dickman R 2004 Physica A 343 525
[5] de Oliveira M M, Dos Santos R V and Dickman R 2012 Phys. Rev. E 86 011121
[6] Hinrichsen H 2000 Adv. Phys. 49 815
[7] Ódor G 2004 Rev. Mod. Phys. 76 663
[8] Janssen H 1981 Z. Phys. B 42 151
[9] Grassberger P 1982 Z. Phys. B 47 365
[10] Noest A J 1988 Phys. Rev. B 38 2715
[11] Dickman R and Moreira A G 1998 Phys. Rev. E 57 1263
[12] Vojta T and Lee M Y 2006 Phys. Rev. Lett. 96 035701
[13] de Oliveira M M and Ferreira S C 2008 J. Stat. Mech. P11001
[14] Bramson M, Durrett R and Schonmann R H 1991 Ann. Probab. 19 960
[15] Faria M S, Ribeiro D J and Salinas J S A 2008 J. Stat. Mech. P11001
[16] Amaral L A and de Oliveira M M 2021 Phys. Rev. E 104 064102
[17] Vojta T, Farquhar A and Mast J 2009 Phys. Rev. E 79 011111
[18] Hooyberghs J, Iglói F and Vanderzande C 2003 Phys. Rev. Lett. 90 100601
[19] Hooyberghs J, Iglói F and Vanderzande C 2004 Phys. Rev. E 69 066140
[20] Harris A B 1974 J. Phys. C: Solid State Phys. 7 1671
[21] Okabe A, Boots B, Sugihara K and Chiu S N 2000 Spatial Tessellations: Concepts and Applications of Voronoi Diagrams (New York: Wiley)
[22] de Oliveira M M, Alves S G, Ferreira S C and Dickman R 2008 Phys. Rev. E 78 031133
[23] de Oliveira M M, Alves S and Ferreira S 2016 Phys. Rev. E 93 012110

https://doi.org/10.1088/1742-5468/ac70dc
Contact process on weighted planar stochastic lattice

[24] Luck J M 1993 Europhys. Lett. 24 359
[25] Barghathi H and Vojta T 2014 Phys. Rev. Lett. 113 120602
[26] Schrauth M, Portela J S E and Goth F 2018 Phys. Rev. Lett. 121 100601
[27] Schrauth M and Portela J S E 2019 Phys. Rev. E 100 062118
[28] Schrauth M and Portela J S E 2019 Phys. Rev. Res. 1 033061
[29] Hassan M K, Hassan M Z and Pavel N I 2010 New J. Phys. 12 093045
[30] Dayeen F R and Hassan M K 2016 Chaos Solitons Fractals 91 228
[31] Nagel W and Weiss V 2005 Adv. Appl. Probab. 37 859–83
[32] Courtat T, Gloaguen C and Douady S 2011 Phys. Rev. E 83 036106
[33] Hassan M K and Rahman M M 2016 Phys. Rev. E 94 042109
[34] Hsu H-P and Huang M-C 1999 Phys. Rev. E 60 6361
[35] Liu X-S, Wu Z-X and Guan J-Y 2018 Eur. Phys. J. B 91 220
[36] Lima F, Moreira J, Andrade J and Costa U 2000 Physica A 283 100
[37] de Oliveira M M and Dickman R 2005 Phys. Rev. E 71 016129
[38] de Oliveira M M and Dickman R 2006 Braz. J. Phys. 36 685
[39] Ferreira S C, Ferreira R S, Castellano C and Pastor-Satorras R 2011 Phys. Rev. E 84 066102

https://doi.org/10.1088/1742-5468/ac70dc