Correspondence between many-particle excitations and the entanglement spectrum of disordered ballistic one-dimensional systems

Shaul Leiman, Ariel Eisenbach and Richard Berkovits

Department of Physics, Jack and Pearl Resnick Institute, Bar-Ilan University - Ramat-Gan 52900, Israel

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Abstract – Using exact diagonalization for non-interacting systems and density matrix renormalization group for interacting systems we show that Li and Haldane’s conjecture on the correspondence between the low-lying many-particle excitation spectrum and the entanglement spectrum holds for disordered ballistic one-dimensional many-particle systems. In order to demonstrate the correspondence we develop a computationally efficient way to calculate the entanglement spectrum of low-lying excitation of non-interacting systems. We observe and explain the presence of an unexpected shell structure in the excitation spectrum. The low-lying shells are robust and survive even for strong electron-electron interactions.

Introduction. – Concepts developed in the field of quantum information are gaining a foothold in condensed matter physics [1,2]. One of the most influential concepts is related to quantifying the entanglement between two parts of a system. Specifically, a many-particle system in a pure state may be divided into two regions A and B. The entanglement between the regions A and B can be quantified by different measures such as the entanglement entropy, Rényi entropy and entanglement spectrum (ES), connected to the reduced density matrix (RDM) of area A, ρ_A or B, ρ_B.

The canonical method for utilizing the information embedded in the eigenvalues \( \lambda_i^A \) of \( \rho_A \) is the entanglement entropy: \( S_A = -\sum_i \lambda_i^A \ln \lambda_i^A \). The ES is constructed out of the set of eigenvalues \( \{\lambda_i^A\} \) by the transformation \( \{\varepsilon_i^A = -\ln \lambda_i^A\} \). Several years ago Li and Haldane [3] came up with an intriguing conjecture regarding the ES of a partitioned fractional quantum Hall \( \nu = 5/2 \) state resembled the minimal edge excitation spectrum, and suggest a connection between the properties of the ES and the topological order of this state. They show that the ES of a partitioned fractional quantum Hall \( \nu = 5/2 \) state resembled the minimal edge excitation spectrum, and suggest a connection between the properties of the ES and the topological order of this state.

This idea led to many additional studies [4–21], mainly focused on systems with topological behavior, suggesting that the low-energy ES distribution shows some correspondence to the true many-body excitations (MBE) of the partitioned segment (region A). Since the reduced density matrix of a region is adiabatically connected to the MBE of the disconnected region, one may expect that the exact ground-state eigenfunction of the whole system encodes information on the sub-system’s low-lying excitations.

In this letter we use Li and Haldane’s conjecture in order to investigate the properties of the excitation spectrum of fermionic disordered systems. The effect of disorder on the ES of edge states in topological insulators has been previously studied [22,23]. Properties of several low-lying ES levels have been used to identify a metallic phase in one-dimensional Bose-Hubbard models [24]. Nevertheless, a direct comparison between the properties of many-particle excitations in a weakly disordered system and the ES of a corresponding segment is still lacking.

The field of MBE statistics has a long history, and continues to draw interest in diverse areas such as cold atoms in the presence of quasi-periodic potentials [25–28], and the many-body localization (MBL) transition [29–31]. For disordered systems it is natural to seek knowledge on the statistical properties of the spectrum. Specifically, we are interested in answering the question: does the ES exhibit statistical properties corresponding to the MBE spectrum? For the disordered single-particle spectrum there is an extensive literature on the statistical properties of the energy spectrum in the localized, critical, diffusive
and chaotic regimes [32]. Furthermore, disordered systems show different energy spectra and wave function statistics, depending on their symmetry. If the disordered system has time reversal symmetry it follows the Gaussian orthogonal ensemble (GOE) statistics, while if time reversal symmetry is broken it will follow Gaussian unitary statistics (GUE). In the cases where spin-orbit interactions are present a Gaussian symplectic statistics (GSE) will be observed [33–35]. The energy spectrum statistics can be used to identify the Anderson localization transition [36].

The statistics of MBE in disordered interacting systems have an interesting twist. For non-interacting many-particle systems the level spacing MBE distribution is expected to follow the Poisson distribution for excitation energies above the second spacing, without depending on the single-level distribution [37]. On the other hand, once repulsive interactions between the particles are considered, a transition to the Wigner distribution for higher excitations is observed [38–42]. The main difficulty in studying this transition is that exact diagonalization needed to study excited states is limited to very small systems. Here, the Li and Haldanes’ conjecture can come to the rescue, since as we shall demonstrate below, one may extract the ES up to a few hundreds of states. Thus, if indeed there is a connection between the ES and the MBE, the low-level excitations of rather large many-body systems are numerically available.

In this letter we shall explicitly demonstrate the correspondence between the excitation spectrum and the ES, for non-interacting one-dimensional disordered systems in the chaotic regime. In order to facilitate the calculation of the ES for large systems, we present a numerical method based on the correlation matrix eigenvalues, which could be used for non-interacting systems in any dimension. We show that the average entanglement spectrum level spacings (ESLS) show a shell structure, with a large average spacing appearing according to the combinatorical distribution for the large spacings. Adding interactions removes the shell structure for strong interactions at higher levels, but the shell structure is rather robust for the low-lying portion of the ES, corresponding to the MBE which are close to the Fermi energy.

**Model.** – As an example of a disordered many-particle system, we consider a spinless 1D electron system of size $L$ with repulsive nearest-neighbor interactions and on-site disordered potential. The Anderson tight-binding model Hamiltonian is given by [43]:

$$H = \sum_{j=1}^{L} \epsilon_j \hat{c}_j^\dagger \hat{c}_j - t \sum_{j=1}^{L-1} (\hat{c}_j^\dagger \hat{c}_{j+1} + \text{h.c.}) + U \sum_{j=1}^{L-1} \left( \hat{c}_j^\dagger \hat{c}_j - \frac{1}{2} \right) \left( \hat{c}_{j+1}^\dagger \hat{c}_{j+1} - \frac{1}{2} \right)$$

(1)

where $\epsilon_j$ is the on-site energy drawn from a uniform distribution $[-W/2,W/2]$, $\hat{c}_j^\dagger$ is the creation operator of an electron at site $j$, and $t=1$ is the hopping matrix element. The repulsive interaction strength is depicted by $U \geq 0$, and a positive background is considered.

**Non-interacting electrons.** – All single-electron states of such a 1D system are localized [44], with states at the middle of the band having a localization length $\xi \approx 105/W^2$ [45]. We consider a case where the disorder was chosen as $W = 0.3$ and $L = 350$, i.e., $\xi \approx 1100 > L$. For this case the single-electron energies $\epsilon_i$ and eigenstates $\psi_i$ are readily available via exact diagonalization. Unless otherwise specified, we perform our analysis over 10000 realizations of disorder. First, we consider the single-electron level spacing distribution, for the $i$-th spacing $\delta_i = \epsilon_{i+1} - \epsilon_i$. The unfolded spacing is defined as $s_i = \delta_i / (\delta_1)$, where $\langle \ldots \rangle$ denotes averaging over the different disorder realization. The single-electron level spacing will follow Poisson statistics ($P_{\text{Poisson}}(s_i) = \exp(-s_i)$) as long as $L \gg \xi$, while in the metallic (diffusive) regime it should follow the Wigner (GOE) distribution ($P_{\text{GOE}}(s_i) = (\pi s_i^2/4) \exp(-\pi s_i^2/4)$) [36]. For a one-dimensional disordered system there is no genuine metallic regime since $L < \xi \approx \ell$ (where $\ell$ is the mean free path) and therefore the system crosses from a localized to a disordered ballistic regime, resulting in a narrower distribution concentrated around the mean ($s_i = 1$). This can be clearly seen in fig. 1, where the distribution of the single-electron level spacings close to the middle of the band of a ballistic system is presented.

The MBE of the system cannot be obtained by exact diagonalization since the size of the many-body Hilbert space grows as $(\frac{L}{\xi})^2$, which for half-filling ($N = L/2 = 175$), where $N$ is the number of particles, is of order $10^{104}$. Thus, another tack is needed. For the non-interacting case one may calculate the low-lying excitations by considering the

![Fig. 1: (Colour online) The single-electron level spacing distribution $P(s_i)$ for states $i = 173, \ldots, 179$ around the middle of the band for a $L = 350$, $W = 0.3$ wire. For this strength of disorder the system is chaotic, $L < \xi$, and the distribution is peaked around the mean (where the level at the middle of the band $i = 176$ is more peaked than the others). The Poisson and GOE distributions are provided for comparison.](image-url)
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Fig. 2: (Colour online) The properties of the many-body excitation spectrum for 10000 realizations of the non-interacting system ($U = 0$) at half-filling whose single-electron level spacings are depicted in fig. 1. Top panel: level spacing distributions for the eight lowest many-particle levels. Lower panel: average spacings as a function of the level number. The vertical lines depict the positions of the peaks predicted by the partition function, i.e., the $j$-th peak will appear at $\sum_{m=0}^{j-1} p(m) = 1, 2, 4, 7, 12, 19, 30, 45, 67 \ldots$

different permutations of the single-electron occupations, $n_i = 0, 1$, of the single-electron states. Here, each permutation which obeys $N = \sum_i n_i$ is a many-body state, with the total many-body energy $E = \sum_i n_i \epsilon_i$. These many-body states may be enumerated by arranging the energies, such that $E_1 < E_2 < E_3 < \ldots$. Of course there are still an astronomical number of these states and covering them all is impossible. Nevertheless, if one constrains the search to low-lying excitations, one can limit the permutations to $p$ (of order $p \sim E/\delta$) single-electron states around the Fermi energy. Thus, only $\binom{N}{p}$ MBE are considered. Similar to the single-particle case the MBE level spacing $\Delta_i = E_{i+1} - E_i$ can be extracted, as well as the average $(\Delta_i)$ and distribution $P(S_i)$, where $S_i = \Delta_i/(\Delta)$.

The results for the average MBE spacings and their distribution are plotted in fig. 2. It is immediately apparent that the average MBE spacing exhibits a shell-like (also known as magic numbers in the context of nuclear physics) structure. Spacings $1, 2, 4, 7, 12, 19, 30, 45, 67 \ldots$ are significantly larger than their neighboring spacings. The reason for this behavior is combinatorical. In the non-interacting case a many-particle state is defined by the occupation of the single-particle states. Thus a many-particle state with $N$ fermions is given by a vector of $n_1, n_2, \ldots, n_i, \ldots$, where $n_i = 0$ or 1 is the occupation of the single-particle state, $\sum n_i = N$, and the many-particle energy $E = \sum_\epsilon n_i \epsilon_i$ is the single-particle energy. For the ground state $n_{i\leq N} = 1$, while $n_{i>N} = 0$. By moving particles from single-particle states below the Fermi energy ($i \leq N$) to states above ($i > N$) all MBE may be obtained. For equal single-level spacings (i.e., $\delta_i = \delta$ or $\epsilon_n = n\delta$), the MBE can acquire energies of $n\delta$ above the ground state, $\mathcal{E}_\text{g}$, where $m = 1, 2, \ldots$. For a specific excitation energy $\mathcal{E}_\text{g} + m\delta$ there are different corresponding many-particle states. The most obvious is to move the highest electron from the Fermi energy $i = N$ to the $i = N + m$ state and not move any other electron. One could move two electrons, the first an electron at the Fermi energy to $i = N + m - 1$ and the second electron from $i = N - 1$ to $N$. This is of course equivalent to moving one electron from $i = N - 1$ to $i = N + m - 1$, but since electrons are identical it is the same excited state. Finally, one can move all $m$ electrons below the Fermi energy $(N - m < i \leq N)$ one state above, which is equivalent to moving the electron at $i = N - m + 1$ to $i = N + 1$. Thus, counting the number of different many-particle states that have the same excitation energy $\mathcal{E}_\text{g} + m\delta$ is equivalent to counting the number of ways of writing the integer $m$ as a sum of positive integers, where the order of addends is not significant. This corresponds to a problem in combinatorics known as the partition problem, and the answer is given by the partition function $p(m)$ (which is not the thermodynamical partition function), which has a recurrence relation $p(m) = \sum_{k=1}^{m} (-1)^{k+1} [p(m - k(3k - 1)/2) + p(m - k(3k + 1)/2)]$ [46]. Thus, $p(m)$ for $m = 0, 1, 2, \ldots$ are $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 44 \ldots$ respectively. This discrete series has implementations to the fields of symmetric functions, prime numbers and graph theory, and can be visualized by Young tableau diagrams [46–48]. Returning to the MBE energy spacing, this degeneracy implies that for each excitation energy $\mathcal{E}_\text{g} + m\delta$, there are $p(m)$ states for which the spacing $\Delta$ is zero, followed by a spacing $\Delta = \delta$ as the next possible excitation energy $\mathcal{E}_\text{g} + (m + 1)\delta$ is reached. Thus, large spacings will appear at the $j$-th level corresponding to $\sum_{m=1}^{j} p(m) = 1, 2, 4, 7, 12, 19, 30, 45, 67 \ldots$. In the case where the single-particle spacing is not constant the degeneracy is lifted, but as long as the variation is not too large, the shell structure remains and the MBE level spacing between a closed shell and the next open one is relatively large (of order $\delta$), as can be seen in fig. 2. Thus, the number of MBE states between the $j$-th MBE shell (i.e., large level spacing) and the $j + 1$ shell is $p(j)$, in agreement with the numerical results. For high excitations the shell structure will be wiped out. Nevertheless, $p(m)$ can still be used to evaluate the MBE level spacing which should be proportional to $\delta/p(m)$. Interestingly, Hardy and Ramanujan have devised an asymptotic form $p(m) \sim (4m\sqrt{3})^{-1} \exp(\pi \sqrt{2m/3})$ [47], which corresponds well to independent estimations of the many-particle density of states [49].

The MBE spacing distributions $P(S_i)$ also retain the shell structure, where the distributions corresponding to the large spacings are similar to the single-electron spacings (fig. 1), i.e., narrow and peaked around the mean, while for the other spacings follow the Poisson distribution, expected when the non-interacting MBE differ by more than a single-electron occupation [37,38] (see fig. 2).

Does the ES of a finite segment of a disordered system exhibit a similar behavior as might be expected from Li and Haldane’s conjecture [3]? First we have to calculate
the ES, i.e., the eigenvalues of the RDM. The density matrix renormalization group (DMRG) [50,51] is a natural candidate for calculating the ground state of disordered interacting 1D system and the corresponding eigenvalues of the reduced density matrix. We will use it for the interacting case. For non-interacting systems, one would expect that it is possible to extract the eigenvalues without resort to DMRG. A direct calculation of the RDM is impossible since its size is of order $2^{L_A}$ (where $L_A$ is the size of region $A$), so a different approach is needed.

Reduced density matrix eigenvalues using the correlation matrix. – In order to calculate the ESLS for the non-interacting case we use the connection between the correlation matrix (CM) and the RDM [52]. The CM between two sites in region $A$ is a unitary matrix given by

$$C_{mn}^t = \left[ \text{Tr} \{ \rho_A c_m^\dagger c_n \} \right] = \text{Tr} \{ c_m^\dagger c_n \rho_A \} C_{nm}. \tag{2}$$

One may diagonalize the CM and write the trace in terms of the eigenvectors. Thus,

$$\text{Tr} \{ \rho_A a_q^\dagger a_p \} = \nu_q \delta_{qp}, \tag{3}$$

where $a_q^\dagger$ is the creation operator of the $q$-th eigenvector and $\nu_q$ is the corresponding eigenvalue. The RDM for non-interacting electrons can be written as a product state of local density matrices [52]:

$$\rho_A = \rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_{L_A}, \tag{4}$$

where $\rho_q$ is the density matrix associated with the $q$-th eigenvalue of the CM. Moreover, due to the fermionic nature of the particles, $\rho_q$ has to be diagonal.

The relation between $\nu_q$ and $\rho_q$ is easily extracted [52]:

$$A_{qq} = \text{Tr} \{ \rho_A a_q^\dagger a_q \} = \text{Tr} \{ \rho_q a_q^\dagger a_q \} = \nu_q.$$ \tag{5}

Since $\rho_q$ are independent any eigenvalue of the RDM can be constructed by multiplying a permutation of either $(1 - \nu_q)$ (no particle) or $\nu_q$ (one particle) occupying the $q$-th state. For the case where no particles occupy region $A$ ($N_A = 0$), the single eigenvalue of the RDM is $\lambda_1^{N_A=0} = \prod_{q=1}^{L_A} (1 - \nu_q)$. For the $N_A = 1$ the RDM has $L_A$ eigenvalues, where the $i$-th eigenvalue, $\lambda_i^{N_A=1} = (1 - \nu_1)(1 - \nu_2) \ldots (1 - \nu_{L_A})$. Generalizing to any $N_A = p$ particle block in the RDM the first eigenvalue is constructed by $\lambda_1^p = \nu_1 \ldots \nu_p (1 - \nu_{p+1}) \ldots (1 - \nu_{L_A})$, while the other $(L_A^p)$ permutations define the rest of the eigenvalues. Since usually many of the CM eigenvalues are either extremely small or very close to one, it is possible to reach an accurate estimation of $\lambda$, with significantly fewer permutations.

The ES obtained for a wire of length $L = 700$ and $L_A = 350$ occupied by $N_A$ particles and averaged over 10000 realizations of disorder are presented in fig. 3. This size was chosen in order to correspond to the sizes of the system for which the MBE were calculated. We chose the block (of the RDM) which corresponds to occupation of $N_A = L_A/2$ or averaged over $L_A/2 - 6 < N_A < L_A/2 + 6$. For the ES we perform the transformation $\epsilon_i^{N_A} = -\ln(\lambda_i^{N_A})$ and calculate the ESLS $\Delta_i^{N_A} = \epsilon_{i+1} - \epsilon_i^{N_A}$, the average spacing $\langle \Delta_i \rangle$ and distribution $P(\Delta_i)$, where $\Delta_i = \Delta_i / \langle \Delta_i \rangle$. Indeed, the low-lying ESLS average as well as the spacing distributions show the same general features shown by the MBE. The shell structure is reproduced for spacings 1, 2, 4, 7, 12, small deviations in the peak positions appear for the higher spacings at 19 and 30, and is almost wiped up at peaks higher than 45. The general behavior of the ESLS follows the distribution exhibited by the excitation spacings with narrow distributions for spacings 1, 2, 4 and 7, while the other spacings follow the Poisson distribution. Nevertheless, details such as the exact width of the distribution vary between the excitation and entanglement spacings.

The influence of interactions. – Once interactions are added, our previous arguments do not hold anymore. Nevertheless, following the ideas leading to the Fermi liquid picture and quasi-particles, one expects that for not too strong interactions and close to the Fermi energy the non-interacting shell picture will continue to provide a good description of the system.

As we cannot use the combinatoric approach to calculate the interacting MBE energy levels, and neither exact diagonalization nor DMRG can provide more than a few excited states, we will use the correspondence between the MBE and the ES of a finite section shown for the non-interacting case to study the evolution of the spectrum as a function of the electron-electron interaction strength. Thus we turn on the interaction $U$ in eq. (1) and use...
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The average ELS $\hat{\Delta}_n$ and distribution $P(\hat{S}_i)$, are calculated and presented in fig. 4. For weak interactions $U = 0.3$, the shell structure of the average spacing is hardly affected. As interactions increase, the higher shells are washed out, until for $U = 2.4$ only spacings 1, 2 and 4 remain larger than their neighbors. The entanglement spacing distribution is also transformed as a function of the interaction strength. For weak interactions ($U = 0.3$) the distribution $P(\hat{S}_i)$ is close to Poisson, while the distribution for the 4th (large) spacings, $P(\hat{S}_i)$, is peaked around the mean, as for the non-interacting case (see the inset of fig. 4). This distinction is blurred as the interactions become stronger and $P(\hat{S}_i)$ for any $i$ approaches the GOE distribution. This is in agreement with the observations for the distribution of interacting MBE [37–42] which show a transition to GOE statistics as interactions become stronger. Thus, it seems that Li and Haldane’s conjecture holds even for the ES of interacting systems.

**Discussion.** – Thus, Li and Haldane’s conjecture on the correspondence between the low-lying MBE spectrum, and its ES is confirmed for non-interacting disordered systems. Turning on interactions gradually change the ES of the system in a manner that fits the expectations for the behavior of the MBE. We conclude that ES is a promising route to study MBE properties beyond the first few excitations, which are impossible to study by any other way for large systems. This could be a promising way to study phenomena occurring only for the MBE, such as the many-body localization transition [29]. Although still limited to low-lying excitations, the fact that higher many-body excitations may be probed improves the possibility to glean useful information on the transition. Indeed in the behavior of the shell structure as a function of the interaction strength presented in fig. 4, where as interaction increases lower shell peaks disappear, may show a glimpse of this phenomena.

For non-interacting systems we have presented a computational effective method to extract the low-lying ES from the CM. This method could be used for the study of the ES for higher dimensions, which is otherwise quite daunting.

We have also shown a new shell (magic number) structure appearing for disordered ballistic one-dimensional systems and explained its origin. The low-lying shells are robust and are not wiped out by interactions. It would be interesting to understand whether this behavior survives for stronger disorder where the localization length is smaller than the length of region $A$. On the one hand, since the disorder is larger it is expected to wipe out the shell structure, on the other hand, the area in region $A$ entangled with the rest of the system remains of order $\xi$ and thus shrinks with the disorder, *i.e.*, one effectively samples a smaller part of the system. Also the study of higher dimensions may prove interesting.

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