Lafforgue’s variety and irreducibility of induced representations

Kostas I. Psaromiligkos

Abstract

We construct the Lafforgue variety, an affine variety parametrizing the simple modules of a non-commutative algebra $R$ for which the center $Z(R)$ is finitely generated and $R$ is finite as a $Z(R)$-module. Using our construction in the case of Hecke algebras, we provide a characterization for irreducibility of induced representations via the vanishing of a generalized discriminant. We explicitly compute this discriminant in the case of an Iwahori-Hecke algebra. We construct well-behaved maps from the Lafforgue variety to Solleveld’s extended quotient and in the case $R$ is a complex finite type algebra to the primitive ideal spectrum.

Contents

0 Introduction ................................................................. 2
  0.1 Summary ............................................................... 2
  0.2 Outline ................................................................. 4
  0.3 Future work ........................................................... 4
  0.4 Acknowledgements .................................................. 5

1 The Lafforgue variety .................................................... 5
  1.1 Non-commutative Hilbert scheme .................................. 6
  1.2 Trace map .............................................................. 7
  1.3 Determinant map ..................................................... 9
  1.4 Dependence on the central subalgebra .............................. 10

2 Jacobson stratification and irreducibility of induced representations ........................................ 11
  2.1 Jacobson stratification ................................................. 11
  2.2 Definition of the generalized discriminant ........................ 13
  2.3 Irreducibility of induced representations .......................... 14
0 Introduction

0.1 Summary

In [Laf06], L. Lafforgue asserted the existence of an algebraic variety classifying smooth irreducible representations of a $p$-adic reductive group. As this very basic assertion does not seem to be known, and since Lafforgue did not provide us with an argument, we provide a proof for the existence of this variety.

Smooth representations of reductive $p$-adic groups correspond to non-degenerate modules of Hecke algebras. We work in the following more general setting. Let $R$ be a possibly non-commutative $k$-algebra over a field $k$, such that the center $Z(R)$ is finitely generated and $R$ is finite as a $Z(R)$-module. Hecke algebras satisfy the above condition. Let $A$ be any subalgebra of $Z(R)$ such that $R$ is a finite $A$-module.

Lafforgue’s construction in the characteristic 0 case works as follows. Let $\text{Irr}(R)$ be the class of simple modules of $R$. For each $r \in R$, we define a function $f_r : \text{Irr}(R) \to k$ by $f_r(V) = tr_r(V)$. The trace is well-defined due to the finiteness property. Let $T_R$ be the ring of functions on $\text{Irr}(R)$ generated by all $f_r, r \in R$, which we call the ring of traces. $\text{Irr}(R)$ can be considered as a subset of $\text{Spec}(T_R)(k)$, since any $V \in \text{Irr}(R)$ gives a geometric point $\text{Hom}(T_R, k)$ via the evaluation homomorphism, and this assignment is injective. We define $\text{Laf}_{R/A} := \text{Spec}(T_R)$ to be the Lafforgue variety.

For an algebraically closed $k$, the subalgebra $A \subseteq Z(R)$, by virtue of Schur’s lemma, and thus $\text{Laf}_{R/A}$ can be thought as a scheme over $\text{Spec}(A)$.

If the characteristic of $k$ is nonzero, we construct $\text{Laf}_{R/A}$ in a similar way, using the determinant instead of the trace. For the details, see subsection 1.3.

We can now state our main theorem.

**Theorem 1.** $\text{Irr}(R)$ forms the set of $k$-points of a dense Zariski open subscheme $\text{iLaf}_{R/A} \subseteq \text{Laf}_{R/A}$. The projection $p : \text{Laf}_{R/A} \to \text{Spec}(A)$ is finite.

The space $\text{Laf}_{R/A}$ turns out to be independent of the choice of $A$. We can stratify $\text{Spec}(A)$ according to the cardinality of the fibers of $p$.

If $A$ is regular and $R$ is a locally free $A$-module, we have a concrete description of this stratification. Fixing a central character $\chi : A \to k$, a simple $R$-module with central character $\chi$ corresponds to an $R_{\chi} = R \otimes_{A, \chi} k$-module. We can then describe the stratification by studying the rank of the Jacobson radical of $R_{\chi}$.
We mainly apply this result to the open dense stratum $X_0$. We define a notion of a generalized discriminant $d_{R/A}$, which is a principal ideal of $A$, with the property that the complement of its zero set in $\text{Spec} A$ is $X_0$.

As a main application of our results, we consider the case of a Hecke algebra $H_K$ of a reductive $p$-adic group $G$ and a compact open subgroup $K$. Each smooth irreducible representation $\rho$ of $G$ with a $K$-fixed vector corresponds to a cuspidal datum $(M, \sigma)$, where $M$ is a Levi subgroup of $G$ and $\sigma$ a cuspidal representation of $M$, such that $\rho$ is a subquotient of the induced representation $i^G_M(\sigma)$. Cuspidal data for smooth irreducible representations of compact level $K$ are parametrized up to association by $\text{Spec}(Z_K)$ for $Z_K$ the center of $H_K$. $\text{Spec}(Z_K)$ is a subvariety of the Bernstein variety defined in [BD84], parametrizing all cuspidal data up to association. Smooth irreducible representations are thus in a finite-to-one correspondence with the points of the Bernstein variety. For a great introduction to representations of $p$-adic groups that includes all results we use, we recommend [Ber92].

Applying Theorem 1 to $H_K$, the Lafforgue variety $\text{Laf}_{H_K/Z_K}$ parametrizes smooth irreducible representations of $G$ with a $K$-fixed vector, and $\rho$ sends an irreducible representation $\rho$ to its cuspidal datum $(M, \sigma)$. Let $X_0$ be the open dense stratum for the stratification of $\text{Spec}(Z_K)$ by the cardinality of the fiber.

We can choose a regular central subalgebra $A \subseteq Z_K$ with $f : \text{Spec}(Z_K) \rightarrow \text{Spec}(A)$ finite. Using the Jacobson stratification, we prove the following.

**Theorem 2.** Let $(M, \sigma)$ be a cuspidal datum. Then, $i^G_M(\sigma)$ is irreducible if and only if $(M, \sigma) \in X_0$. Outside of the singular locus $Z(f)$, this is equivalent to

$$d_{H_K/A}(f(M, \sigma)) \neq 0.$$  

We explicitly calculate the discriminant for the unramified component of the Bernstein variety to recover results about irreducibility of principal series appearing in [Kat82]. Lafforgue’s variety is related to other parametrizing spaces used in the literature. In particular, the primitive ideal spectrum $\text{Prim}(R)$ of a complex finite type algebra $R$ is a topological space that also parametrizes $\text{Irr}(R)$, used in [KNS98] to study the Hochschild homology of $R$. We prove that there is a unique continuous bijection

$$i_{\text{Laf}}_{R/A}(\mathbb{C}) \rightarrow \text{Prim}(R)$$

respecting the bijections with the set $\text{Irr}(R)$.

Furthermore, in the case of an Iwahori-Hecke algebra $H$ of a split reductive $p$-adic group $G$, Solleveld used a space called the extended quotient to non-canonically parametrize the irreducible representations. If $\hat{T}$ is the dual torus of $G$, and $W$ its Weyl group, let $W_0$ be a choice of representatives from the conjugacy classes of $W$. The extended quotient was defined in [Sol10] to be

$$\hat{T} \sslash W := \bigsqcup_{w \in W_0} T^w / Z_W(w).$$

Solleveld also used the extended quotient to prove results about Hochschild homology. We show for each choice of bijection between the closed points of $\hat{T} \sslash W$ and $\text{Irr}(H)$ the corresponding bijection

$$i_{\text{Laf}}_{H/A}(\mathbb{C}) \rightarrow \hat{T} \sslash W$$

is algebraic.
0.2 Outline

In Section 1, we recall some basic concepts related to $p$-adic reductive groups, and show that our Theorem 1 implies Corollary 4, which was stated by Lafforgue [Laf16]. We then proceed to carry out the proof of Theorem 1. In doing so, we define a non-commutative version of the classical Hilbert-Chow morphism, that may be of independent interest. We then show that the Lafforgue variety is independent of the choice of $A$.

In Section 2, we construct the Jacobson stratification. We define generalized discriminants and study the case of a Hecke algebra to prove Theorem 2.

In Section 3 we prove properties of discriminants to make them more amenable to calculation. Said properties include a generalization of the classical behavior of the discriminant in a tower of extensions of number rings to general commutative algebras, which doesn't seem to appear in the literature for our case. In particular, if $N_{B/A}$ is the norm function (Definition 14), using the generalized Riemann-Hurwitz formula, we show the following.

**Lemma 3.** For a tower of extensions $C/B/A$ such that each algebra is a finite free module over the next one, $A, B$ are commutative and regular, and $C$ is commutative we have that

$$d_{C/A} = (d_{B/A})^n \cdot N_{B/A}(d_{C/B}),$$

where $n = [C : B]$.

We then compute the discriminant for the case of an Iwahori-Hecke algebra of a split reductive $p$-adic group.

In Section 4, we work out the relation of Lafforgue’s variety to the primitive ideal spectrum and Solleveld’s extended quotient. We also show how we can recover the trace Paley-Weiner theorem [BDK86] by Theorem 1.

0.3 Future work

The Bernstein decomposition theorem gives a precise decomposition of the category of smooth representations of a $p$-adic reductive group as

$$\mathcal{M} \cong \prod_{\Omega} \mathcal{M}(\Omega)$$

where $\Omega$ ranges over connected components of the Bernstein variety, and $\mathcal{M}(\Omega)$ is the subcategory of smooth representations whose Jordan-Hölder components’ cuspidal data are in $\Omega$. $\mathcal{M}(\Omega)$ has a finitely generated projective generator $\Pi(\Omega)$, and thus $\mathcal{M}(\Omega) \cong \mathcal{M}(\text{End}(\Pi(\Omega)))$, where the latter is category of modules of the algebra of endomorphisms of $\Pi(\Omega)$. Recently, Maarten Solleveld proved that $\text{End}(\Pi(\Omega))$ is Morita equivalent to a twisted affine Hecke algebra $H_\Omega$, and the center of $H_\Omega$ admits a concrete description [Sol20]. We wish to use the tools in section 3 to describe the discriminant of any twisted affine Hecke algebra, and by virtue of Theorem 2 derive explicit criteria on the cuspidal data whose corresponding induced representation is irreducible.

Another natural problem would be to provide a map from the Lafforgue variety to the moduli stack of $L$-parameters of [DHKM20], [Zhu20], thereby giving a description of $L$-packets as fibers of said map.
0.4 Acknowledgements

I warmly thank Anne-Marie Aubert, Aristides Kontogeorgis, Zhilin Luo, Benedict Morrissey, Bao Châu Ngô, Minh-Tâm Quang Trinh, Yiannis Sakellaridis, Maarten Solleveld and Griffin Wang for their interest, valuable comments and many helpful discussions.

I am indebted to Anne-Marie Aubert, Zhilin Luo, Benedict Morrissey, Bao Châu Ngô, and Yiannis Sakellaridis for many constructive comments on this manuscript.

This work is part of my PhD thesis at the University of Chicago, advised by Bao Châu Ngô, to whom I am greatly thankful for his mentorship and guidance throughout the process, as well as suggesting this problem.

My work was supported by a Graduate Research Fellowship from the Onassis foundation.

1 The Lafforgue variety

The main goal of this section is to give a proof of Theorem \[\square\] First, we describe the case of $p$-adic reductive groups.

To state Lafforgue’s assertion with some precision, we need to recall some standard concepts and notations. Let $G$ be an algebraic group over a nonarchimedean local field $F$. The group $G(F)$, equipped with the adic topology is totally disconnected in the sense that there exists a system of neighborhoods of the identity which are compact open subgroups. A smooth representation of $G(F)$ is a linear action of $G(F)$ on a complex vector space $V$ such that the stabilizer of every vector $v \in V$ is an open subgroup of $G(F)$. A smooth representation $V$ is said to be admissible if, for every compact open subgroup $K$ of $G(F)$, the space $V^K$ of $K$-fixed vectors is finite dimensional. A theorem of Bernstein asserted that under the assumption that $G$ is reductive, every smooth irreducible representation of $G(F)$ is admissible. From now on, we assume that $G$ is reductive.

We will denote $\mathcal{H}$ the algebra of locally constant compactly supported measures on the group $G(F)$, whose multiplicative structure is given by the convolution product. This algebra does not have a unit but a system of idempotent elements $e_K$ associated with compact open subgroups $K$ of $G(F)$. If, for every compact open subgroup $K$ of $G(F)$, we denote $\mathcal{H}_K$ the unital subalgebra $e_K \star \mathcal{H} \star e_K$, we have $\mathcal{H} = \bigcup_K \mathcal{H}_K$. A smooth representation of $G(F)$ is a linear action of $G(F)$ on a vector space $V$ such that the stabilizer of every vector $v \in V$ is an open subgroup of $G(F)$. There is an equivalence of categories between smooth representations of $G(F)$ and non-degenerate modules of $\mathcal{H}$ i.e $\mathcal{H}$-modules $V$ such that $V = \bigcup_K e_K V$. For an admissible representation $V$ of $G(F)$, the subspace $e_K V = V^K$ is finite dimensional for every compact open subgroup $K$ of $G$. Let $\Pi$ denote the set of equivalence classes of smooth irreducible admissible representations of $G(F)$, and $\Pi_K$ the subset of equivalence classes of smooth irreducible representations with a non-zero $K$-fixed vectors. There is a canonical bijection between $\Pi_K$ and the set of isomorphism classes of simple $\mathcal{H}_K$-modules that are all finite dimensional as a consequence of Bernstein’s admissibility theorem.

Hecke algebras satisfy our finiteness property. Let $T_K$ be the trace ring of $\mathcal{H}_K$. Our Theorem \[\square\] implies Lafforgue’s original assertion which is the following.

**Corollary 4.** For each compact open subgroup $K$ of a $p$-adic reductive group $G(F)$, the trace algebra $T_K$ is finitely generated as algebra over $\mathbb{C}$. Moreover, the set $\Pi_K$ of isomorphism classes of irreducible
representations of $G(F)$ with non-zero $K$-fixed vectors form the set of $\mathbb{C}$-points of a Zariski open subscheme of $\text{Spec}(T_K)$.

The main difficulty is that Lafforgue’s construction of the trace algebra $T_K$ is constructed through some sort of Gelfand duality which does not provide much information on its structure. Returning to the general case of a non-commutative algebra $R$, what we need is thus a framework in algebraic geometry which gives rise to a more workable definition of the trace ring $T_R$. This framework turns out to be a non-commutative generalization of the classical Hilbert-Chow morphism.

If $M$ is a $\bar{k}$-finite dimensional simple $R$-module, then $A$ must act on $M$ through a character $\alpha: A \rightarrow \bar{k}$. Lafforgue’s variety can thus be thought of as a scheme over $\text{Spec}(A)$. Our strategy is as follows: we will first construct a non-commutative Hilbert scheme for the finite $A$-algebra $R$, which is a proper $A$-scheme $Q$. Next using the trace (or the determinant in positive characteristic case), we construct a morphism from the Hilbert scheme to an affine $A$-scheme $V$. The morphism $Q \rightarrow V$ can then be shown to factor through a closed subscheme of $V$ which is finite over $\text{Spec}(A)$, of which the coordinate ring can be identified with $T_R$.

1.1 Non-commutative Hilbert scheme

Let $A$ be a commutative ring which is contained in the center of a possibly non-commutative ring $R$. We consider the functor $\text{Hilb}_{R/A}$ which associates to every commutative $A$-algebra $B$ the set of isomorphism classes of $R \otimes_A B$-modules $M$, which are flat as $B$-modules, equipped with a surjective $R \otimes_A B$-linear map $R \otimes_A B \rightarrow M$. If we consider $R$ just as a finite $A$-module, the functor $\mathcal{Q}_{R/A}$ associating to every commutative $A$-algebra $B$ the set of isomorphism classes of flat $B$-modules $M$ equipped with a surjective $R \otimes_A B$-linear map $m : R \otimes_A B \rightarrow M$ is representable by a projective scheme over $\text{Spec}(A)$. Since the functor $\text{Hilb}_{R/A}$ is a closed subfunctor of $\mathcal{Q}_{R/A}$, it is also representable by projective scheme over $\text{Spec}(A)$.

There is a decomposition of $\text{Hilb}_{R/A}$ into open and closed subschemes

$$\text{Hilb}_{R/A} = \bigsqcup_{d \in \mathbb{N}} \text{Hilb}_{R/A}^d$$

where $\text{Hilb}_{R/A}^d$ classifies $R \otimes_A B$-linear maps $m : R \otimes_A B \rightarrow M$ with $M$ being a locally free $B$-module of rank $d$.

Since we are mainly interested in irreducible modules, it will also be useful to consider the nested Hilbert scheme $n\text{Hilb}_{R/A}$ which associates to every commutative $A$-algebra $B$ the set of isomorphism classes of pairs of $R \otimes_A B$-modules $M, N$, which are flat as $B$-modules, equipped with surjective $R \otimes_A B$-linear maps $R \otimes_A B \rightarrow M \rightarrow N$. We also assume that the kernel of $M \rightarrow N$ is non-zero, or in other words, the rank of $M$ is strictly greater then the rank of $N$ as locally free $B$-modules. Again, $n\text{Hilb}_{R/A}$ is representable by a proper scheme over $\text{Spec}(A)$, which is a closed subscheme of a relative flag variety.

The map $n\text{Hilb}_{R/A} \rightarrow \text{Hilb}_{R/A}$ defined by $(M, N) \mapsto M$ is a proper morphism as $n\text{Hilb}_{R/A}$ is proper over $\text{Spec}(A)$ and $\text{Hilb}_{R/A}$ is separated over $\text{Spec}(A)$, by [Sta22] Tag 01W0, Lemma 29.41.7. Its image is a closed subscheme of $\text{Hilb}_{R/A}$ of which we will denote by $i\text{Hilb}_{R/A}$ the open complement. A geometric point $x \in i\text{Hilb}_{R/A}(\bar{k})$ over a point $a : A \rightarrow \bar{k}$ consists of a quotient $M_x$ of
the algebra \( R_a = R \otimes_A \bar{k} \) by a maximal left ideal, or in other words, \( M_x \) is a simple \( R_a \)-module equipped with a generator.

We consider the group scheme \( G_{R/A} \) over \( \text{Spec}(A) \) which associates to every commutative \( A \)-algebra the group \( (R \otimes_A B)^\times \) of invertible elements of the possibly non-commutative algebra \( R \otimes_A B \). This group scheme is smooth over \( \text{Spec}(A) \) if \( R \) is a finite locally free \( A \)-module.

The group scheme \( G_{R/A} \) acts on \( \text{Hilb}_{R/A} \) relative to \( \text{Spec}(A) \). For a \( B \)-point of \( (M, m) \in \text{Hilb}_{R/A}(B) \) we will denote the action of \( R \otimes_A B \) on \( M \) by \( (r, m) \mapsto e(r)m \). If \( g \in (R \otimes_A B)^\times \) we define the action of \( g \) on \( (M, m) \) to be \( g(M, m) = (M', m') \) where \( M' = M \) as a \( B \)-module equipped with the structure of \( R \otimes_A B \)-module given by \( e'(r)m = e(g^{-1}rg)m \), and \( m' = mg \).

The group scheme \( G_{R/A} \) also acts on the nested Hilbert scheme \( n \text{Hilb}_{R/A} \) in the way that makes the morphism \( n \text{Hilb}_{R/A} \to \text{Hilb}_{R/A} \) equivariant. Its image is thus a \( G_{R/A} \)-equivariant closed subscheme of \( \text{Hilb}_{R/A} \). It follows that its complement, the open subscheme \( \text{iHilb}_{R/A} \) of \( \text{Hilb}_{R/A} \), is also stable under the action of \( G_{R/A} \).

### 1.2 Trace map

Following Grothendieck, we define a generalized vector bundle \( V_{R/A} \) over a commutative ring attached to an \( A \)-module \( R \). As a functor, \( V_{R/A} \) attaches to each \( A \)-algebra \( B \) the abelian group \( \text{Hom}_A(R, B) \) of all \( A \)-linear maps \( R \to B \). This functor is represented by the symmetric algebra \( \text{Sym}_A(R) \): it is the \( \mathbb{N} \)-graded \( A \)-algebra with \( \text{Sym}_A^0(R) = A \), \( \text{Sym}_A^1(R) = R \), and for every \( d \in \mathbb{N} \), the \( d \)th symmetric power \( \text{Sym}_A^d(R) \) is the largest quotient of the \( d \)th fold tensor power \( R \otimes_A^d \) of \( R \) over \( A \) on which the symmetric group \( S_d \) acts trivially. We claim that the morphism of functors on \( A \)-algebras:

\[
\text{Hom}_A-\text{Alg}(\text{Sym}_A(R), B) \to \text{Hom}_A(R, B),
\]

defined as the restriction an \( A \)-algebra homomorphism \( x : \text{Sym}_A(R) \to B \) to the degree 1 component \( \text{Sym}_A^1(R) = R \), is an isomorphism of functors. Indeed, every \( A \)-linear map \( y : R \to B \), induces an \( A \)-linear map \( R \otimes_A^d \to B \otimes_A^d \to B \) which factors through an \( A \)-linear map \( y^d : \text{Sym}_A^d(R) \to B \). It’s not hard to check that the \( A \)-linear map \( x : \bigoplus_{d \in \mathbb{N}} \text{Sym}_A^d(R) \to B \) given by \( x = \bigoplus_{d \in \mathbb{N}} y^d \) is a homomorphism of \( A \)-algebras. It is also clear that the map \( y \mapsto x \) thus defined gives rise to an inverse of the functor \( x \mapsto y \). We conclude that the functor \( V_{R/A} \) is representable by the affine scheme \( \text{Spec}(\text{Sym}_A(R)) \) which is a generalized vector bundle in the sense of Grothendieck.

Let us assume that \( R \) is a finite \( A \)-module which is equipped with a structure of a possibly non-commutative algebra containing \( A \) in its center. We can construct the trace map

\[
\text{tr}_{R/A} : \text{Hilb}_{R/A} \to V_{R/A}
\]

as follows. For every point \( (M, m) \in \text{Hilb}_{R/A}(B) \), where \( M \) is an \( R \otimes_A B \)-module that is locally free and finite as a \( B \)-module. Every \( r \in R \) defines a \( B \)-linear operator of \( M \) given by the structure of an \( R \otimes_A B \)-module. Since \( M \) is a finitely generated locally free \( B \)-module, the trace \( \text{tr}_B(r) \in B \) is well defined. This gives rise to an \( A \)-linear map \( \text{tr}_M : R \to B \) and thus to a \( B \)-point of \( V_{R/A} \).

Since \( \text{Hilb}_{R/A} \) is proper over \( \text{Spec}(A) \), whereas \( V_{R/A} \) is affine by construction, we define \( \text{LaR}_{R/A} \) to be the scheme-theoretic image of \( \text{tr} \). Then \( \text{tr} \) factors through a proper surjective map \( \text{tr}_L : \text{Hilb}_{R/A} \to \text{LaR}_{R/A} \), and \( \text{LaR}_{R/A} \) is proper over \( \text{Spec}(A) \) by \textbf{Sta22}, Tag 01W0, Lemma.
29.41.10. Since \( p : \text{La}f_{R/A} \to \text{Spec}(A) \) is proper and affine, it is finite. Let \( i\text{La}f_{R/A} \) denote the open subscheme of \( \text{La}f_{R/A} \) which is the complement of the closed subset that is defined as the image of the proper map \( \text{tr}_L : n\text{Hilb}_{R/A} \to \text{La}f_{R/A} \).

**Proposition 5.** Assume that \( A \) is a finitely generated \( k \)-algebra where \( k \) is a field of characteristic zero. Then the preimage \( \text{tr}_L^{-1}(i\text{La}f_{R/A}) \) is \( i\text{Hilb}_{R/A} \). Moreover for every geometric point \( l \in i\text{La}f_{R/A}(\bar{k}) \) over \( a : A \rightarrow \bar{k} \), the group \( \mathcal{G}_{Ra} \) acts transitively on the fiber \( \text{tr}_L^{-1}(l) \).

This assertion is nothing but a reformulation of well known facts about modules over a finite-dimensional algebra, improperly referred to as Brauer-Nesbitt’s theorem. We refer to Lang’s book [Lan02, chapter XVII Cor 3.8] for more information.

**Proposition 6** (Bourbaki). Assume that \( k \) is a field of characteristic zero and \( R \) is a finite-dimensional \( k \)-algebra possibly non-commutative. Let \( M \) and \( N \) be \( k \)-finite dimensional \( R \)-modules such that for all \( x \in R \), we have \( \text{tr}_x(M) = \text{tr}_x(N) \), then \( M \) and \( N \) have the same semi-simplification. In particular if \( \text{tr}_L(x_1) = \text{tr}_L(x_2) \), and if \( M_1 \) is a simple \( R_a \)-module then \( M_2 \) is also simple and \( M_2 \cong M_1 \).

The last statement of Proposition 6 implies that \( \text{tr}_L^{-1}(i\text{La}f_{R/A}) = i\text{Hilb}_{R/A} \). It also implies that if \( l \in i\text{La}f_{R/A}(\bar{k}) \) over \( a : A \rightarrow \bar{k} \), and if \( x_1, x_2 \in \text{tr}_L^{-1}(l) \) are represented by quotients \( M_1 \) and \( M_2 \) of \( R_a \) then \( M_1 \) and \( M_2 \) are isomorphic simple \( R_a \)-modules. It follows that there exists \( g \in M_a^x \) such that \( gx_1 = x_2 \). In other words, the fiber of \( \mathcal{G}_{R_a} \) over \( a \) acts transitively on the fiber of \( \text{tr} \) over \( l \).

As we want to extend the construction of Lafforgue’s variety to the case of positive characteristic, we sketch the proof of Proposition 6 which was given in full details in Lang’s book.

**Proof.** The assertion is obvious in one direction. If the quotients \( M_1 \) and \( M_2 \) of \( R_a \) have the same semi-simplification as \( R_a \)-modules then the induced linear forms \( \text{tr}_{M_1}, \text{tr}_{M_2} : R_a \rightarrow \bar{k} \) are equal because traces only depend of semi-simplification. Conversely, Jacobson’s density theorem implies the existence of projectors: if \( V_0, V_1, \ldots, V_n \) are non-isomorphic simple \( R_a \)-modules, there exists an element \( e_0 \in R \) which acts as the identity on \( V_0 \) and \( 0 \) on \( V_1, \ldots, V_n \). Note that Jacobson’s density theorem is valid in any characteristic. Now let \( V_0, \ldots, V_n \) be simple \( R_a \)-modules occurring as a simple subquotient of \( M_1 \) or \( M_2 \) and write decompositions of the semi-simplifications of \( M_1 \) and \( M_2 \) as \( M_1^{ss} = V_0^{m_1} \oplus U_1 \) and \( M_2^{ss} = V_0^{m_2} \oplus U_2 \) where \( U_1 \) and \( U_2 \) are semi-simple modules with no occurrences of \( V_0 \). To prove that \( M_1 \) and \( M_2 \) have the same semi-simplification, it is enough to prove \( m_1 = m_2 \). This derives from the equalities

\[
m_1 \dim(V_0) = \text{tr}_{M_1}(e_0) = \text{tr}_{M_2}(e_0) = m_2 \dim(V_0)
\]
as elements of \( \bar{k} \). The characteristic zero assumption is only used to guarantee that in \( \bar{k} \) we have \( \dim(V_0) \neq 0 \).

Without hypothesis on characteristic, we have to replace the trace by the determinant. Let us formalize the construction of the determinant map as an analogue of the trace map previously defined.
1.3 Determinant map

Let $A$ be a commutative ring. An $A$-module $R$ gives rise to a functor $R : B \mapsto R \otimes_A B$ from the category of $A$-algebras to the category of sets. Following Roby [Rob63], we call a polynomial law on $R$ a morphism of functors $f : R \rightarrow A$ where $A$ is the functor $B \mapsto A \otimes_A B = B$. Thus, a polynomial law $f$ on $R$ consists of a family of set theoretical maps $f_B : R \otimes_A B \rightarrow B$ depending on $B$ in a functorial way. We denote by $\text{Pol}_A(R)$ the set of all polynomial laws on the $A$-module $R$.

If $r_1, \ldots, r_n \in R \otimes_A B$ form a finite sequence $r$ of elements of $R \otimes_A B$, then $f$ gives rise to a polynomial $f_r \in B[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are free variables, such that for every $x_1, \ldots, x_n \in B$, we have $f_r(x_1, \ldots, x_n) = f(x_1 r_1 + \cdots + x_n r_n)$. Indeed, if we take

$$X_r = r_1 \otimes X_1 + \cdots + r_n \otimes X_n \in R \otimes_A B[X_1, \ldots, X_r],$$

then we set

$$f_r := f_B(X_r) \in B[X_1, \ldots, X_r],$$

see [Rob63, Thm 1.1]. The main point in Roby’s concept of polynomial law is that the polynomial $f_r$ is a part of the data of $f$.

We say that the polynomial law $f : R \rightarrow A$ is homogeneous of degree $d \in \mathbb{N}$ if for every $A$-algebra $B$, $x \in B$ and $r \in R \otimes_A B$ we have $f_B(xr) = x^d f_B(r)$. It’s not hard to check that the polynomial law $f$ is homogeneous of degree $d$ if and only if for every finite sequence $r = (r_1, \ldots, r_n)$ of elements of $R \otimes_A B$ for any $A$-algebra $B$, $f_r$ is a homogeneous polynomial of degree $d$ with coefficients in $B$, [Rob63, Prop. 11, p. 226]. We denote by $\text{Pol}_A^d(R)$ the set of all homogeneous polynomial laws of degree $d$ on the $A$-module $R$. A homogeneous polynomial law of degree 1 on $R$ consists thus in a family of linear forms $f_B : M \otimes_A B \rightarrow B$ depending functorially on $B$ which is equivalent to the initial linear form $f_A : M \rightarrow A$.

Now we will generalize Grothendieck’s construction of generalized vector bundle associated to an $A$-module, by replacing linear forms on $M$ by homogeneous polynomial laws of degree $d$. Let $A$ be a commutative ring and $R$ an $A$-module. We consider the functor $S^dV_{R/A}$ which attaches to every $A$-algebra $B$ the set $\text{Pol}_B^d(R \otimes_A B)$ of polynomial laws on the $B$-module $R \otimes B$ which are homogeneous of degree $d$. For $d = 1$, $\text{Pol}_B(R \otimes_A B) = \text{Hom}_B(R, B)$ and we have an isomorphism of functors $S^dV_{R/A} = V_{R/A}$ which are represented by the affine scheme $\text{Spec}(\text{Sym}_A(R))$. For every $d \in \mathbb{N}$, Roby constructed a canonical isomorphism of functors $\text{Pol}_B^d(R \otimes_A B) = \text{Hom}_B(\Gamma_d^a R, B)$, see [Rob63, Thm II.3 p. 262, IV.1 p. 266] where $\Gamma_d^a R$ is the $d$th divided power of the $A$-module $M$ [Rob63, ch. III p. 249]. It follows that the functor $B \mapsto S^dV_{R/A}(B) = \text{Pol}_B^d(R \otimes_A B)$ is representable by the affine scheme $\text{Spec}(\text{Sym}_A(\Gamma_d^a R))$.

Let $R$ be a possibly non-commutative algebra containing a commutative ring $A$ in its center such that $R$ is finitely generated locally free $A$-module. For every $d \in \mathbb{N}$, the determinant now defines a morphism

$$\det_{R/A} : \text{Hilb}^d_{R/A} \rightarrow S^dV_{R/A} \tag{3}$$

Indeed, for every point $x \in \text{Hilb}^d_{R/A}(B)$ represented by an $R \otimes_A B$-quotient module $M$ of $R \otimes_A B$ which, as a $B$-module, is locally free of rank $d$. This gives rise to a map $R \otimes_A B \rightarrow B$ given by $r \mapsto \det_{M}(r)$ which is homogenous of degree $d$. By choosing local generators of $M$ as locally free $B$-module, we see that $r \mapsto \det_{M}(r)$ gives rise to a morphism $V_{R/A} \otimes_A B \rightarrow \mathbb{G}_{a,B}$ which is homogenous of degree $d$ and therefore a point $\det(x) \in S^dV_{R/A}(B)$. 

9
Again, since \( \text{Hilb}^d_{R/A} \) is a proper scheme over \( A \), and \( S^dV_{R/A} \) is affine, the morphism \( \det_{R/A} \) factors through a closed subscheme \( \text{Laf}^d_{R/A} \) of \( S^dV_{R/A} \) which is finite over \( A \). We thus get a proper surjective map

\[
\det^d : \text{Hilb}^d_{R/A} \twoheadrightarrow \text{Laf}^d_{R/A}.
\]

Using the nested Hilbert scheme \( n\text{Hilb}^d_{R/A} \) as before, we can define open subschemes \( i\text{Hilb}^d_{R/A} \) and \( i\text{Laf}^d_{R/A} \) as the complements of the images of \( n\text{Hilb}^d_{R/A} \). Geometric points \( x \in i\text{Hilb}^d_{R/A}(\overline{k}) \) over \( a : A \to \overline{k} \) correspond to \( R_a \)-quotient modules of \( R_a \) that are simple.

**Proposition 7.** We have \( \det^{-1}_L(i\text{Laf}^d_{R/A}) = i\text{Hilb}^d_{R/A} \). For every geometric point \( l \in i\text{Hilb}^d_{R/A}(\overline{k}) \) over \( a : A \to \overline{k} \), the group \( \mathcal{G}_{R_a} \) acts transitively on the fiber \( (\det^d_L)^{-1}(l) \).

Again, this assertion is nothing but a reformulation of well known facts about modules over a finite-dimensional algebra.

**Proposition 8.** Let \( R \) be a possibly non-commutative finite-dimensional algebra over a field \( k \) and \( M, N \) be \( R \)-modules which are \( d \)-dimensional \( k \)-vector spaces. Assume that \( \det_M = \det_N \) as homogeneous polynomial of degree \( d \) on \( R \), then \( M \) and \( N \) have isomorphic semi-simplifications. In particular, if \( M \) is a simple \( R \)-module then \( N \) is also simple and \( N \cong M \).

**Proof:** The assertion is obvious in one direction. If the factors \( M \) and \( N \) of \( R \) have the same semi-simplification then the induced homogenous forms \( \det_M, \det_N : R \to k \) are equal because the determinant only depends on the semi-simplification. Conversely, Jacobson’s density theorem implies the existence of projectors: if \( V_1, \ldots, V_r \) are non-isomorphic simple \( R \)-modules, then there exists an element \( e_i \in R \) which acts as identity on \( V_i \) and 0 on \( V_j \) for \( j \neq i \). Now let \( V_1, \ldots, V_r \) be the simple \( R \)-modules occurring as a simple subfactors of \( M \) or \( N \) and decompose the semi-simplifications of \( M \) and \( N \) as

\[
M^{ss} = \bigoplus_{i=1}^{m_1} V_1^{m_1} \oplus \cdots \oplus \bigoplus_{i=1}^{m_r} V_r^{m_r}
\]

\[
N^{ss} = \bigoplus_{i=1}^{n_1} V_1^{n_1} \oplus \cdots \oplus \bigoplus_{i=1}^{n_r} V_r^{n_r}
\]

(4)

If \( X_1, \ldots, X_r \) are free variables then we a the formula

\[
\det_M(X_1e_1 + \cdots + X_re_r) = X_1^{\dim(V_1)} \cdots X_r^{\dim(V_r)}
\]

for the determinant of \( x_1e_1 + \cdots + x_re_r \) on \( M \) and similarly for \( N \). The equality

\[
\det_M(X_1e_1 + \cdots + X_re_r) = \det_N(X_1e_1 + \cdots + X_re_r)
\]

of polynomials of variables \( X_1, \ldots, X_r \) implies that \( m_i = n_i \) for all \( i \). It follows that \( M \) and \( N \) have isomorphic semi-simplifications.

\[\Box\]

### 1.4 Dependence on the central subalgebra

If \( A \) is a commutative \( k \)-algebra contained in the center of a possibly non-commutative \( k \)-algebra \( R \), assuming \( k \) to be algebraically closed, then Schur’s lemma guarantees that \( A \) acts on every finite-dimensional simple \( R \)-module through a character \( a : A \to k \). This implies that the set of \( k \)-points of the Lafforgue variety \( i\text{Laf} \) doesn’t depend on the choice of \( A \). In this section, we will prove that the Lafforgue variety itself is independent of the choice of \( A \). This will follow from a relative version of Schur’s lemma.
Proposition 9. Let \( R \) be a possibly non-commutative ring containing commutative rings \( A \subset A' \) in its center. The natural morphism \( \iota \mathrm{Laf}_{R/A'} \to \iota \mathrm{Laf}_{R/A} \) is an isomorphism.

Proof: It is enough to prove that the morphism \( \iota \mathrm{Hilb}_{R/A'} \to \iota \mathrm{Hilb}_{R/A} \) is an isomorphism. It is enough to prove that for any \( A \)-algebra \( B \), every morphism \( \mathrm{Spec}(B) \to \iota \mathrm{Laf}_{R/A} \) can be canonically lifted to a morphism \( \mathrm{Spec}(B) \to \iota \mathrm{Laf}_{R/A'} \), which is the content of the following assertion.

Proposition 10. Let \( R \) be a possibly non-commutative ring containing commutative rings \( A \subset A' \) in its center. Assume that that \( R \) is finite as an \( A \)-module. Let \( B \) be an \( A \)-algebra and \( M \) a finite locally free \( A \)-module equipped with a structure of an \( R \otimes_A B \)-module such that over every geometric point \( b \in \mathrm{Spec}(B) \) over \( a \in \mathrm{Spec}(A) \), \( M_b \) is a simple \( R_a \)-module. Then the ring homomorphism \( A' \to \mathrm{End}_B(M) \) factors through \( B \).

Proof. The homomorphism \( R \to \mathrm{End}_B(M) \) is surjective as it is surjective fiberwise over \( \mathrm{Spec}(B) \) by the Jacobson density theorem. It follows that the image of the central subalgebra \( A' \) is contained in \( B \).

2 Jacobson stratification and irreducibility of induced representations

As before, let \( R \) be a possibly non-commutative \( k \)-algebra over a field \( k \) such that \( A \) is a finitely generated subalgebra \( A \) of the center with \( R \) being finite as an \( A \)-module. Then by Theorem 1, the projection \( \iota \mathrm{Laf}_{R/A} \to \mathrm{Spec}(A) \) is finite which implies we can stratify \( \mathrm{Spec}(A) \) according to the number of points in the fiber. In the case where \( A \) is regular and \( R \) is also a locally free \( A \)-module, we can explicitly describe this stratification using the rank of the Jacobson ideal. In this section, we construct this stratification.

We then apply it to irreducibility of induced representations of \( p \)-adic reductive groups. We recall that, for a reductive group \( G \) over a non-archimedean local field \( F \), the Bernstein variety parametrizes cuspidal data \( (M, \sigma) \) up to association, and therefore parametrizes smooth irreducible representations in a finite-to-one manner. Fixing a choice of a compact open subgroup \( K \subset G(F) \), we consider the Hecke algebra \( \mathcal{H}_K \) of compactly supported measures on \( G(F) \) which are left and right invariant under \( K \). Then simple \( \mathcal{H}_K \)-modules are the smooth irreducible \( G(F) \) representations that have a \( K \)-fixed vector, and \( \mathrm{Spec}(Z_K) \) for \( Z_K \) being the center of \( \mathcal{H}_K \) parametrizes their cuspidal data. The projection \( \iota \mathrm{Laf}_{\mathcal{H}_K/Z_K} \to \mathrm{Spec}(Z_K) \) sends a smooth irreducible representation \( \rho \) to its cuspidal datum \( (M, \sigma) \), defined by \( \rho \to \iota_M^G(\sigma) \). For a generic \( (M, \sigma) \), the induced representation \( \iota_M^G(\sigma) \) is irreducible. As a main application of this section, we prove Theorem 2 providing a computational criterion for the irreducibility of \( \iota_M^G(\sigma) \) outside a singular locus.

2.1 Jacobson stratification

Let \( A \) be a commutative ring contained in the center of a possibly non-commutative ring \( R \). From now on we will assume that \( R \) is a finite locally free \( A \)-module. We will also assume that \( A \) contains a field \( k \) of characteristic zero.
For every point \( a : A \to k(a) \) of \( \text{Spec}(A) \), \( k(a) \) being a field, the fibre \( R_a = R \otimes_A k(a) \) is a finite-dimensional \( k(a) \)-algebra. The Jacobson radical \( J_a = \text{rad}(R_a) \), defined as the intersection of all maximal left ideals of \( R_a \), is a 2-sided ideal which can be characterized in multiple ways, namely it is the intersection of the annihilators of simple left \( R_a \)-modules, or the maximal left (or right) nilpotent ideals, see [Lam91, 4.2,4.12]. The quotient \( R_a/J_a \) is a semi-simple \( k \)-algebra which, by the Artin-Weddenburn theorem, is isomorphic to a product of matrix algebras \( R_a/J_a = \prod_{i=1}^r M_{n_i}(D_i) \) where \( M_{n_i}(D_i) \) is a matrix algebra over a skew field \( D_i \) containing \( k(a) \) in its center.

**Proposition 11.** The function \( r_{\text{Jac}} : \text{Spec}(A) \) given by \( a \mapsto \dim_{k(a)} J_a \) is upper semi-continuous.

The assertion will follow from yet other interpretation of the Jacobson radical as the kernel of a trace form. We recall that as \( R \) is a finite locally free \( A \)-module, for every element \( r \in R \), the \( A \)-linear operator on \( R \) given by \( x \mapsto rx \) has a well defined trace \( \text{tr}_R/A(r) \). It follows that we have a symmetric \( A \)-bilinear form on \( R \) given by \( \text{tr}_R/A(x,y) = \text{tr}_R/A(xy) \), or equivalently a \( A \)-linear map \( \text{Tr}_{R/A} : R \to R^\vee \). The construction of the trace form and the bilinear form \( \text{Tr}_{R/A} \) commute in the obvious way with base change and for every geometric point \( a : A \to k(a) \), we have a trace form \( \text{tr}_a : R_a \to k(a) \) and a symmetric bilinear form \( \text{Tr}_{R/A,a} \) on \( R_a \), or equivalently a linear form \( \text{Tr}_{R/A,a} : R_a \to R_a^\vee \).

**Proposition 12.** For every point \( a : A \to k(a) \) of \( \text{Spec}(A) \), the Jacobson radical \( J_a \) is the kernel the bilinear form \( \text{Tr}_{R/A,a} : R_a \to R_a^\vee \).

**Proof.** Since \( J_a \) is a nilpotent ideal, for every \( x \in J_a \) and \( y \in R_a \), we have \( \text{tr}_a(xy) = 0 \). It follows that \( J_a \) is contained in the kernel of \( \text{Tr}_{R/A,a} \). Moreover, the Artin-Weddenburn theorem implies that \( \text{Tr}_{R/A,a} \) induces a non-degenerate bilinear form on \( R_a/J_a \) and therefore \( J_a \) is exactly equal to the kernel of \( \text{Tr}_{R/A,a} \).

We will now construct the stratification of \( \text{Spec}(A) \) by the rank of the Jacobson ideal using the concept of determinantal ideals. Assume that \( R \) is a locally free \( A \)-module of rank \( n \). Locally for the Zariski topology we may assume that \( R \) is a free \( A \)-module of rank \( n \), and the trace form \( \text{Tr} : R \to R^\vee \) is given by a \( n \times n \)-matrix. For every positive integer \( i \), we define \( I_i \) to be the ideal of \( A \) such that locally for the Zariski topology, \( I_i \) is generated by the minors to the order \( n - i + 1 \) of the local matrix of \( \text{Tr} \). We know a chain of inclusions of ideals \( 0 = I_0 \subset I_1 \subset \cdots \) which induces a chain of inclusion of closed subsets \( X_0 \supset X_1 \supset \cdots \) where \( X_i = \text{Spec}(A/I_i) \). Over \( X_i \) the complement of \( X_{i+1} \) in \( X_i \), the rank of the Jacobson radical is constant of value \( i \).

In fact, over \( X_i \) the trace form \( \text{Tr}_{X_i} : R \otimes_A O_{X_i} \to R^\vee \otimes_A O_{X_i} \) has kernel a locally free \( O_{X_i} \)-module \( J_i \) of rank \( i \), and image a locally free \( O_{X_i} \)-module \( \overline{R}_{X_i} \) of rank \( n - i \). The trace form \( \text{Tr}_{X_i} \) induces a non-degenerate symmetric bilinear form \( \overline{\text{Tr}}_{X_i} \) on \( \overline{R}_{X_i} \). In particular, for every point \( a : A \to k(a) \) of \( \text{Spec}(A) \) belonging to the stratum \( X_i \), \( \overline{R}_{X_i} \otimes_{O_{X_i}} k(a) \) is a semi-simple algebra over \( k(a) \). Let \( \overline{a} : A \to k(a) \) a geometric point over \( a \). Then \( \overline{R}_{X_i} \otimes_{O_{X_i}} k(a) \) is isomorphic to a product of matrix algebras

\[
\overline{R}_{X_i} \otimes_{O_{X_i}} k(a) = \prod_{i=1}^r M_{n_i}(k(\overline{a}))
\]

where \( \overline{n}(a) = (n_1, \ldots, n_r) \) is unordered sequence of positive integers depending only on \( a \).
Proposition 13. The function \( a \mapsto n(a) \) is locally constant on \( X_i \).

Proof. Since \( \overline{R}_{X_i} \) is a locally free \( O_{X_i} \)-module equipped with a structure of associative algebra which is fiberwise semisimple over \( X_i \), its invertible elements define a smooth group scheme \( G_{\overline{R}_{X_i}} \) over \( X_i \). Its geometric fiber over a geometric point \( \overline{a} \) is isomorphic to \( GL_{n_1} \times \cdots \times GL_{n_r} \). Thus \( G_{\overline{R}_{X_i}} \) is a smooth reductive group scheme whose geometric fiber over \( \overline{a} \) is isomorphic to \( GL_{n_1} \times \cdots \times GL_{n_r} \). A general theorem in SGA 3 on smooth reductive group schemes implies that the function \( a \mapsto n(a) \) is locally constant [ABD+66][Exposé XIX, Corollaire 2.6].

2.2 Definition of the generalized discriminant

We mainly use the Jacobson stratification over the open dense stratum \( X_0 \). We need the following definition.

Definition 14. The norm function \( N_{R/A} : \text{End}_A(R) \to A \) is the map sending an endomorphism \( f \in \text{End}_A(R) \) to its determinant. If \( r \in R \), by considering \( r \) as an endomorphism via left multiplication, we also define \( N_{R/A}(r) \).

Now we can define the discriminant. For simplicity, as all constructions are local, from now on we assume \( R \) is free as an \( A \)-module.

Definition 15. Let \( \text{Tr}_{R/A} : R \otimes_A R \to A \) be the trace form defined by \( \text{Tr}_{R/A}(r_1, r_2) = tr_A(r_1 r_2) \). We consider it as a function \( \text{Tr}_{R/A} : R \to R^\vee = \text{Hom}_A(R, A) \). Up to a non-canonical identification \( R^\vee \cong R \), we can consider \( \text{Tr}_{R/A} \) as an element of \( \text{End}_A(R) \), thus we can take its norm \( N_{R/A}(\text{Tr}_{R/A}) \). This is an element of \( A \) well-defined up to \( A^\times \), since any identification is related to every other by an invertible change of basis.

Thus, we get a principal ideal of \( A \) generated by all such choices that we call the discriminant \( d_{R/A} \) of \( R \) over \( A \).

Remark 16. In the case of number rings, our definition agrees with the classical discriminant of algebraic number theory.

Immediately by the definition, we get

Lemma 17. The open stratum \( X_0 \) in the Jacobson stratification of \( \text{Spec}A \) is the complement of the zero set \( V(d_{R/A}) \).

Proof: Notice that the zero set is well-defined since any two elements of the discriminant are related by an invertible element, thus the zero set does not change. Now by the definition of the discriminant, the zero set is exactly the locus where the trace form is an isomorphism, thus the Jacobson radical is zero by proposition 12.

Any choice of generator of \( d_{R/A} \) gives a regular function on \( \text{Spec}(A) \) with the same zero set, so we often treat \( d_{R/A} \) as a function.
2.3 Irreducibility of induced representations

In the case of the Hecke algebra $\mathcal{H}_K$, we know after [BBK18] that $\mathcal{H}_K$ is a finite Cohen-Macaulay module over its center $Z_K$ which is itself a Cohen-Macaulay algebra. We can apply the Lafforgue variety construction for $R = \mathcal{H}_K$ and $A = Z_K$, but we can also apply it for $A$ being a regular algebra contained in $Z_K$ such that $Z_K$ is a finite $A$-module. If $A$ is regular, then both $\mathcal{H}_K$ and $Z_K$ are finite locally free $A$-modules. In this case, the group scheme $\mathcal{G}_{R/A}$ is smooth acting on the Hilbert scheme $\text{Hilb}_{R/A}$ which is a closed subscheme of $\mathcal{A}_{R/A}$ the familiar relative Grassmannian scheme attached to a vector bundle.

If $f : \text{Spec}(Z_K) \to \text{Spec}(A)$ is the projection corresponding to the inclusion $A \subseteq Z_K$, we define $Z(f)$ to be the closed subset of $\text{Spec}(Z_K)$ where $f$ is not smooth. Let $X_0$ be the open dense stratum of $\text{Spec}(Z_K)$ given by the cardinality of the fiber of the projection from the Lafforgue variety. We identify a cuspidal datum $(M, \sigma)$ with the corresponding point in $\text{Spec}(Z_K)$.

We recall and prove Theorem 2.

**Theorem 2.** Let $(M, \sigma)$ be a cuspidal datum. Then, $i_M^G(\sigma)$ is irreducible if and only if $(M, \sigma) \in X_0$. Outside of the singular locus $Z(f)$, this is equivalent to

$$d_{\mathcal{H}_K/A}(f(M, \sigma)) \neq 0.$$  

**Proof.** By Theorem 1 we get finite projections

$$\text{Laf}_{\mathcal{H}_K/Z_K} \cong \text{Laf}_{\mathcal{H}_K/A} p \rightarrow \text{Spec}(Z_K) f \rightarrow \text{Spec}(A)$$

By the definition of $p$, $|JH(i_M^G(\sigma))| = |p^{-1}(M, \sigma)|$. Since generically an induced representation is irreducible, over $X_0$ the cardinality of the fiber is 1 which proves the first assertion.

Let $Y_0$ be the open dense stratum of the Jacobson stratification for $\text{Spec}(A)$, and $n = \text{deg}(f)$. Then, for a generic point $a \in \text{Spec}(A)$, the cardinality of $f \circ p$ is $n$, and thus the cardinality of a point $a \in \text{Spec}(A)$ is $\geq n$ with equality if and only if it is a point of $Y_0$.

Since the fibers of $f$ outside the singular locus have cardinality $n$, Lemma 7 implies the second assertion. 

**Example 18.** Let $H$ be the Iwahori-Hecke algebra of $GL_2$. It is generated over the group algebra of the cocharacter lattice $R \cong \mathbb{C}[x_1^+, x_2^+]$ by elements $T_e, T_s$ satisfying a quadratic and an intertwining relation, see [HP02]. If $W = S_2$ is the Weyl group, the center is $R^W = \mathbb{C}[x_1^+, x_2^+] S_2$. A cuspidal datum in this case corresponds to a choice of an unordered pair of complex numbers defining an unramified character of a split maximal torus $M$.

We can use the presentation of $H$ to determine that the trace ring is $T_H = \mathbb{C}[x_1^+, x_2^+] S_2 \oplus \mathbb{C}[x_1^+] \oplus \mathbb{C}[x_1^+]$. The Lafforgue variety and the projection are therefore roughly given by the following picture.

In this case, the center $R^W$ is already regular, and the discriminant can be computed, directly or using the results of section 3, to be

$$d_{H/R^W} = (x_2 - qx_1)^2(x_1 - qx_2)^2,$$

which retrieves that the induction $i_M^G(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1 \chi_2^{-1} = q^\pm$. When this is not the case, the Jordan-Holder constituents of the induction are an irreducible character and a Steinberg representation corresponding to the two other connected components shown in the figure.
3 Generalized discriminants

Let $R$ be a possibly non-commutative algebra which is a finite locally free module over a finitely generated regular commutative $k$-algebra $A$. We assume $k$ to be an algebraically closed field of characteristic zero. In this section prove some properties for the discriminant and use them to compute the discriminant for the Iwahori-Hecke algebra of a split reductive $p$-adic group, first for the case of an adjoint group where the center is already regular, and then for the general case, where we need to choose a regular subalgebra.

3.1 Basic properties

First, we recall some elementary properties of the norm.

**Lemma 19.** For the norm function $N_{R/A}$ we have

- $N_{R/A}(fg) = N_{R/A}(f)N_{R/A}(g)$,
- If $a \in A$, $N_{R/A}(a) = a^n$ where $n$ is the rank of $R$ over $A$.

**Proof:** The first part is just multiplicativity of the determinant. For the second, $a$ can be identified with a scalar matrix. \[ \square \]

The norm is also transitive, ie.

**Lemma 20.** If $B$ is a commutative $A$-algebra that is free as an $A$ module and $C$ is a $B$-algebra such that $C$ is locally free over $B$, we have

$$N_{C/A} = N_{B/A} \circ N_{C/B}.$$  

In particular, if $n$ is the rank of $C$ over $B$,

$$N_{C/A}(b) = (N_{B/A}(b))^n$$  

15
This is actually not a trivial result, for a proof, see [Cas86, Appendix B, lemma 4].

In algebraic number theory, there is a useful formula allowing us to compute the discriminant of a tower of extensions in terms of the discriminants of the intermediate steps. As stated in the introduction, it turns out it can be generalized to our case. We recall Lemma 3.

**Lemma 3.** For a tower of extensions $C/B/A$ such that each algebra is a finite free module over the next one, $A, B$ are commutative and regular, and $C$ is commutative we have that

$$d_{C/A} = (d_{B/A})^n \cdot N_{B/A}(d_{C/B}),$$

where $n = [C : B]$.

**Proof.** Let $X = \text{Spec} A, Y = \text{Spec} B, Z = \text{Spec} C$ and $g : Z \to Y$ and $f : Y \to X$ the maps corresponding to inclusion.

We know $f_* R_{Y/X} = \text{div}(d_{B/A})$. We have the relative short exact sequence of Kahler differentials

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_C/A \to \Omega_C/B \to 0,$$

where the first map is injective due to smoothness.

We take determinants in the sense of [Har77, exercise II.6.11], to get

$$\det(\Omega_C/A) \cong \det(\Omega_{B/A} \otimes_B C) \otimes \det(\Omega_{C/B}).$$

Now by the smoothness of the maps $\det(\Omega_C/A) = \omega_{C/A}, \det(\Omega_{B/A}) = \omega_{C/B}$ and thus $\det(\Omega_{B/A} \otimes_B C) = g^* \omega_{B/A}$. Therefore,

$$\omega_{C/A} \cong \omega_{C/B} \otimes g^* \omega_{B/A}.$$

We know that $\omega_{C/A} = \mathcal{L}(R_{Z/X})$ is the invertible sheaf corresponding to the ramification divisor $R_{Z/X}$. Thus, taking associated divisors,

$$R_{Z/X} \cong R_{Z/Y} + g^* R_{Y/X}$$

We take pushforward by $f \circ g$ to get the divisor corresponding to the discriminant. Since $g_* g^*$ for divisors is multiplication by the degree, and $f_* \text{div}(b) = \text{div}(N_{B/A}(b))$, we get

$$\text{div}(d_{C/A}) \cong f_* \text{div}(d_{C/B}) + f_*(nR_{Y/X})$$

$$\cong \text{div}(N_{B/A}(d_{C/B})) + \text{div}((d_{B/A})^n)$$

$$\cong \text{div}((d_{B/A})^n N_{B/A}(d_{C/B}))$$

\[\square\]

**Remark 21.** We can also deduce Lemma 3 by repeated application of the generalized Riemann-Hurwitz formula.
Indeed, we have

\[ R_{Z/X} \equiv K_Z - (f \circ g)^* K_X \]
\[ \equiv K_Z - g^* f^* K_X \]
\[ \equiv K_Z - g^* (K_Y - R_{Y/X}) \]
\[ \equiv R_{Z/Y} + g^* R_{Y/X} \]

and we conclude as before.

### 3.2 Discriminant of adjoint reductive groups

The center \( R^W \) of the Iwahori-Hecke algebra is also the coordinate ring of \( \hat{T}/\hat{W} \) where \( \hat{T} \) is the dual torus. In this subsection we assume \( G \) is an adjoint group, thus \( \hat{T} \) is the torus of a simply-connected group, and it is well-known that \( \hat{T}/\hat{W} \). Thus, for an adjoint group \( R^W \) is already regular. In this case we can retrieve Kato’s result by computing \( d_{H/R^W} \), which makes sense in this case.

The computation essentially will be performed in two steps, from \( H \) to \( R \) and from \( R \) to \( R^W \), in a similar fashion to lemma 3, which cannot be used directly since \( H \) is non-commutative. It will turn out that the discriminant behaves in a similar fashion nonetheless.

Let \( W = \{ w_1, \ldots, w_n \} \) and \( I_{w_i}, K_{w_i} \) be the intertwiners/normalized intertwiners as we defined them in chapter 2. Then \( d_{H/R^W} \) is the discriminant of the lattice \( \{ I_{w_i} \pi^{\mu_j} \}_{i,j} \) for proper \( \mu_j \).

Thus, we need to compute the determinant \( \{ tr(I_{w_i} \pi^{\mu_j} I_{w_k} \pi^{\mu_j}) \}_{i,j,k,l[n]} = \{ tr(I_{w_i} I_{w_k} \pi^{w_k(\mu_j) \pi^{\mu_j}}) \}_{i,j,k,l[n]} \).

We notice that for \( w_i \neq w_k^{-1} \) the trace is zero, because elements \( I_{w_i} \) for \( w \neq e \) permute the generalized eigenvectors, so we have \( n \times n \times n \) blocks. Also, we recall that setting \( e_a = 1 - q^{-1} a^r, d_a = 1 - a^r \), gives

\[ I_w I_{w^{-1}} = \prod_{a \in R_w} \frac{e_a e_{-a}}{d_a d_{-a}} \]

by the Gindikin-Karpelevich formula [HP09][Corollary 1.13.2].

Thus, we can simplify the calculation using the following.

**Lemma 22.** Let \( R \) be a commutative algebra over the commutative algebra \( A \). Let \( p, r_1, \ldots, r_n \in R. \) Then we have that

\[ \det \{ tr(pr_ir_j) \} = N_{R/A}(p) \cdot \det \{ tr(r_ir_j) \}. \]

**Proof.** Consider a basis of generalized eigenvectors \( v_i \) and let \( \kappa_i \) be the eigenvalues of \( p \) and \( \lambda_i^j \) be the eigenvalues of \( r_j \). Then \( tr(pr_ir_j) = \sum \kappa_k \lambda_k^j \lambda_l^k \). We therefore have

\[
\begin{pmatrix}
tr(p r_1^2) & \cdots & tr(p r_1 r_n) \\
\vdots & \ddots & \vdots \\
tr(p r_n r_1) & \cdots & tr(p r_n^2)
\end{pmatrix} =
\begin{pmatrix}
\kappa_1 \lambda_1^1 & \cdots & \kappa_1 \lambda_1^n \\
\vdots & \ddots & \vdots \\
\kappa_n \lambda_n^1 & \cdots & \kappa_n \lambda_n^n
\end{pmatrix}
\begin{pmatrix}
\lambda_1^1 & \cdots & \lambda_1^n \\
\vdots & \ddots & \vdots \\
\lambda_n^1 & \cdots & \lambda_n^n
\end{pmatrix}
\]

And this is equal to

\[
\kappa_1 \cdots \kappa_n \cdot
\begin{pmatrix}
\lambda_1^1 & \cdots & \lambda_1^n \\
\vdots & \ddots & \vdots \\
\lambda_n^1 & \cdots & \lambda_n^n
\end{pmatrix}
\cdot
\begin{pmatrix}
\lambda_1^1 & \cdots & \lambda_1^n \\
\vdots & \ddots & \vdots \\
\lambda_n^1 & \cdots & \lambda_n^n
\end{pmatrix}
= \det(p) \cdot
\begin{pmatrix}
tr(r_1^2) & \cdots & tr(r_1 r_n) \\
\vdots & \ddots & \vdots \\
tr(r_n r_1) & \cdots & tr(r_n^2)
\end{pmatrix}
\]
By Lemma 22, we get

\[ d_{H/R^W} = \left( \prod_{i=1}^{n} \prod_{a \in R_{\omega_i}} \frac{e_a e_{-a}}{d_a d_{-a}} \right) \cdot d_{R/R^W}^n \]

Notice that we also have

\[ \prod_{i=1}^{n} \prod_{a \in R_{\omega_i}} \frac{e_a e_{-a}}{d_a d_{-a}} = \prod_{a \in \Phi} \prod_{w \in W, a \in R_w} \frac{e_a e_{-a}}{d_a d_{-a}} = \left( \prod_{a \in \Phi} \frac{e_a e_{-a}}{d_a d_{-a}} \right)^{n/2}, \]

so, since this element is \( W \)-invariant and thus by Lemma 19 its norm is itself to the \( n \)-th power, we get the general formula

\[ d_{H/R^W} = \left( \prod_{a \in \Phi} \frac{e_a e_{-a}}{d_a d_{-a}} \right)^{n/2} \cdot d_{R/R^W}^n. \] (5)

We compute the discriminant for \( R/R^W \). Indeed, it is enough to calculate the ramification divisor of the map \( \text{Spec} R \to \text{Spec} R^W \). Ramification happens when \( d_a = 0 \) for some \( a \). Indeed, in that case, the corresponding homomorphism \( R \to k \) was \( s_a \)-invariant, and in that case two sheets degenerate in one in every ramified point. Thus, \( d_a \) appears with an exponent of 1 in the ramification divisor. Pushed forward, this gives that the discriminant of the extension \( R/R^W \) is \( (\prod_{a \in \Phi} d_a)^n \). This computation is also carried out algebraically by Steinberg [Ste74][pp. 125-127].

Combined with the fact that \( d_a = d_{-a} \) up to an invertible element, equation (5) gives that for an adjoint group \( G \) we have

**Proposition 23.** If \( G \) is adjoint, we have

\[ d_{H/R^W} = \left( \prod_{a \in \Phi} e_a e_{-a} \right)^{n/2} \cdot d_{R/R^W}^n. \]

**Remark 24.** Since the zero locus of \( d_{H/R^W} \) is exactly the locus where the induced representation is irreducible by Theorem 2, for the case of an adjoint group we recover Kato’s result [Kat82][Theorem 2.2]. Notice that the second condition in Kato’s theorem is always satisfied for adjoint groups.

### 3.3 Discriminant in the non-adjoint case

If \( G \) is not adjoint, \( R^W \) is not regular anymore, so we need to restrict to some subalgebra \( A \) that is regular. We will make a canonical such choice.

**Definition 25.** We identify the fundamental weights \( \omega_1, \ldots, \omega_n \) with the trace function of the corresponding fundamental representations. Then there are integers \( d_1, \ldots, d_n \) such that \( \omega_i^{d_i} \in R^W \). We define \( A = \mathbb{C}[\omega_1^{d_1}, \ldots, \omega_n^{d_n}] \subseteq R^W \) to be the algebra of fundamental weights.
A is obviously regular as it is a polynomial algebra.

By exactly the same procedure as in the previous subsection, equation 5 is still true upon replacing \( d_{R/W} \) by \( d_{R/A} \), so now we want to compute \( d_{R/A} \). Recall that \( R \) is the group algebra of the cocharacter lattice, so alternatively, it is the function ring of the dual torus \( \hat{T} \). The fact that \( R/W \) is regular when \( G \) is adjoint comes from the fact that the dual torus would be simply connected: indeed, in that case it is known that \( \hat{T} \parallel W \) is an affine space of dimension \( \text{rank}(G) \), and its coordinate ring are polynomials over the trace functions corresponding to the fundamental weights.

For the general case, we consider a simply connected cover \( \hat{T} \) of \( \hat{T} \) such that \( \hat{T} = \hat{T} \parallel Z \). We define \( R^+ = k[\hat{T}] \). Then \( R^+ \) is regular and also \( (R^+)^W \) is regular.

Consider the following diagram

\[
\begin{array}{ccc}
R & \longrightarrow & R^+ \\
\downarrow & & \downarrow \\
R^W & \longrightarrow & (R^+)^W \\
A & \rightarrow & \end{array}
\]

We want to compute \( d_{R/A} \). By the transitivity lemma, it is enough to compute \( d_{R^+/R}, d_{R^+/(R^+)^W}, d_{(R^+)^W} \). We already know \( d_{R^+/(R^+)^W} \). Since \( \hat{T} \parallel W \cong \mathbb{C}[\omega_1, \ldots, \omega_n] \), we have \( d_{(R^+)^W} = \omega_1^{d_1(d_1-1)} \cdots \omega_n^{d_d(d_d-1)} \).

We also have that \( N_{R/A}(d_{R+/R}) \) is an invertible element. Let \( n = |W| \) as before, and \( [R^+:R] = r \). Then if we set \( [R^+:A] = nd \) we have \( [(R^+):A] = rd \). By using Lemma 3, we get the following.

**Proposition 26.** In the general case, and for \( A \) being the algebra of fundamental weights, we have

\[
d_{H/A} = \left( \prod_{a \in \Phi} e_a e_{-a} \right)^{d_n^2/2} \cdot \left( \omega_1^{d_1(d_1-1)} \cdots \omega_n^{d_n(d_n-1)} \right)^{n/r}.
\]

**Proof.** By the same method as in the adjoint case,

\[
d_{H/A} = \left( \prod_{a \in \Phi} \frac{e_a e_{-a}}{d_a d_{-a}} \right)^{d_n^2/2} \cdot d_{R/A}^n.
\]

By Lemma 3, writing the discriminant \( d_{R+/A} \) in two different ways and ignoring the invertible factor \( N_{R/A}(d_{R+/R}) \) we have

\[
d_{R+/A} = (d_{R+/R})^r d_{(R^+)^W}^n.
\]

Combining the two equations with \( d_{R^+/R} = (\prod_{a \in \Phi} e_a^2)^n \) gives the result. \( \square \)

**Example 27** (\( SL_2 \) case). For \( SL_2 \) the dual group is \( PGL_2 \), and the simply connected cover would be again \( SL_2 \). This gives \( R^+ = \mathbb{C}[x^\pm] \), while \( R^+ \parallel \mathbb{Z}_2 = \mathbb{C}[x^2] \). Then \( (R^+)^W \cong \mathbb{C}[x + x^{-1}] \) and \( R^W = \mathbb{C}[x^2 + x^{-2}] = A \). This is the only simply connected group for which that is correct, since \( \omega_1 = x + x^{-1} \) so \( \omega_1^2 \) generates \( R^W \).
It is easy now to compute directly $d_{R^+/R} = x^2, d_{R^+/R^+} = (1 - x^{-2})^2, d_{(R^+)w/A} = (1 + x^{-2})^2,$ which gives (one can also do this directly to get the same result) $d_{R^+/A} = (1 - x^{-4})^2$ since $x^2$ is invertible.

Therefore, either by Proposition 26 or by direct computation,

$$d_{H/R^+} = (1 - q^{-1}a^\vee)^2 \cdot (1 - q^{-1}a^{-2})^2 \cdot (1 - a^\vee)^4 (1 + a^\vee)^4.$$  

As a corollary, the induced representation is irreducible if and only if one of the three factors is zero. This agrees with Kato’s result and the example in [Sol21, pg. 1020].

4 Relation to other parametrizing spaces

4.1 The primitive ideal spectrum

The primitive ideal spectrum is a topological space attached to an algebra, that in the particular case of complex finite type algebras parametrizes their irreducible representations, see [KNS98]. Most notably, it was used in [KNS98] to derive relations between its topology and the periodic cyclic and Hochschild homologies of a complex finite type algebra. In the particular case of the Hecke algebra of a $p$-adic reductive group $G$, there was already an algebraic description of its periodic cyclic homology in terms of the action on the Bruhat-Tits building of $G$, derived independently in [HN96] and [Sch96]. Using the topology of the primitive ideal spectrum, Kazhdan, Nistor and Schneider, the latter two being authors of the aforementioned two works, derived in [KNS98] an alternate description of the periodic cyclic homology groups of $\mathcal{H}(G)$ in terms of the representation theory of $G$. Here, we recall the definition of the primitive ideal spectrum and we prove that the Lafforgue variety admits a continuous bijection to it.

Let $R$ be a complex finite type algebra.

**Definition 28.** A primitive ideal $J$ of $R$ is a two-sided ideal that arises as the kernel of a non-trivial complex representation of $R$.

We will endow the set of primitive ideals $\text{Prim}(R)$ with a natural topology. For an arbitrary subset $S \subseteq R$, we define

$$V(S) = \{ J \in \text{Prim}(R) \mid S \subseteq J \}$$

We have the following easy lemma.

**Lemma 29.** The sets $V(S)$ as $S$ ranges over subsets of $R$ form the class of closed sets of a topology on $\text{Prim}(R)$.

**Proof.** It is easy to verify all properties needed. Indeed,

- $V(\emptyset) = \text{Prim}R$ trivially.
- $V(R) = \emptyset$, since primitive ideals come from non-trivial representations.
- $\bigcap_i V(S_i) = V(\bigcup_i S_i)$.
- $V(S_1) \cup V(S_2) = V(S_1 \cap S_2)$.

$\Box$
Definition 30. The topology on Prim\(R\) that has \(V(S), S \subseteq R\) as closed sets is called the Jacobson topology on Prim\(R\).

Prim\(R\) equipped with the Jacobson topology is called the primitive ideal spectrum of \(R\).

The following lemma is obvious.

Lemma 31. There is a bijection on the level of points between Irr\((R)\) and Prim\((R)\).

Remark 32. Without the finiteness properties, the previous lemma does not hold. In particular, it does not hold for the algebra of endomorphisms of an infinite dimensional vector space.

We have the following easy lemma relating the Lafforgue variety with the primitive ideal spectrum.

Lemma 33. Let \(R\) be a complex algebra that is a finite module over a finitely generated commutative subalgebra \(A\) of its center. Then the map \(iLa_f_{R/A}(\mathbb{C}) \to \text{Prim}(R)\) defined by the corresponding bijections with Irr\((R)\) is a continuous bijection.

Proof. It is enough to show that for \(r \in R\), the subset \(V_r \subseteq iLa_f_{R/A}\) of modules annihilated by \(r\) is closed. For any \(s \in R\), let \(U_s \subseteq iLa_f_{R/A}\) be the subset defined by the trace of \(s\) being zero. \(U_s\) is closed by the definition of the trace ring.

It is enough to show

\[ V_r = \bigcap_{s \in R} U_{sr}. \]

Indeed, if \(r\) annihilates a module \(M\), we obviously have \(tr_M(sr) = 0\) for any \(s \in R\). If \(r\) does not annihilate a simple module \(M\), then there is an \(m \in M\) such that \(rm = m' \neq 0\). By simplicity of \(M\), there exists an \(s \in R\) such that \(sm' = m\). Then the element \(sr\) has at least one nontrivial eigenvalue, since \(srm = sm' = m\).

Since \(R\) is complex, there must be a nonzero \(tr_M((sr)^n)\) for some \(n \in \mathbb{N}\).

4.2 The extended quotient

Let \(G\) be a split reductive \(p\)-adic group with maximal torus \(T\), Weyl group \(W\), dual torus \(\hat{T}\) and \(H_I\) the Iwahori-Hecke algebra. In [Sol10], the extended quotient was used to parametrize the irreducible representations.

Definition 34. Let \(W_0 \subseteq W\) be a choice of one representative for each conjugacy class of \(W\).

The extended quotient of \(\hat{T}\) by \(W\) is the space

\[ \hat{T} \sslash W := \bigsqcup_{w \in W_0} T^w / Z_W(w). \]

This is naturally an algebraic variety.

In [Sol10], it is shown that \(\hat{T} \sslash W\) admits non-canonical bijections to Irr\((H_I)\). Notice that the extended quotient also admits a finite projection to the Bernstein variety.

Up to a natural choice of such a bijection, the following is true.
Proposition 35. The map \( f : iLaf_{H_{1}/A}(\mathbb{C}) \to \hat{T} \parallel W \) preserving the bijections with \( \text{Irr}(H_{1}) \) is algebraic.

Proof. We will show algebraicity by looking at each connected component \( L \) of \( iLaf_{H_{1}/A} \) separately. It is enough to show that the restriction of \( p \) to \( L \) is injective, as then both \( L \) and its image in \( \hat{T} \parallel W \) will be isomorphic to the same subset of the Bernstein variety.

We notice that \( p(L) \) must be a subset of the closure of some stratum \( X_{n} \) of the stratification by rank of the fiber. Fix the maximal split torus \( T \) and let \( \chi \) be the universal character with values in the coordinate ring of \( p(L) \) whose specializations give the cuspidal data of \( p(L) \). We get a universal module \( i^{G}_{T}(\chi) \) for the induced representations of \( p(L) \), and looking at its composition series we derive that representations in the same component of the Lafforgue variety differ by an unramified twist. \( \square \)

Remark 36. In a similar fashion, considering \( \Omega \) to be a connected component of the Bernstein variety, we see that \( p^{-1}(\Omega) \subseteq \text{Laf}_{H_{1}/Z_{K}} \) has a component of maximal dimension \( L \) whose points correspond to the representations \( i^{G}_{M}(\sigma\chi) \) for \( (M, \sigma) \) a choice of cuspidal datum in \( \Omega \) with \( i^{G}_{M}(\sigma) \) irreducible.

Restricting \( p \) to \( L \), we get an isomorphism with \( \Omega \). By the definition of the trace ring, the fact that \( p \) is an isomorphism recovers the trace Paley-Wiener Theorem \([BDK86]\).

References

[ABD+66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre. Schémas en groupes. Séminaire de Géométrie Algébrique de l’Institut des Hautes Études Scientifiques. Institut des Hautes Études Scientifiques, Paris, 1963/1966.

[BBK18] Joseph Bernstein, Roman Bezrukavnikov, and David Kazhdan. Deligne-Lusztig duality and wonderful compactification. Selecta Mathematica, 24(1):7–20, 2018.

[BD84] J. Bernstein and P. Deligne. Le centre de Bernstein, 1984.

[BDK86] J. Bernstein, P. Deligne, and D. Kazhdan. Trace Paley-Wiener theorem for reductive \( p \)-adic groups. Journal d’Analyse Mathematique, 47(1):180–192, December 1986.

[Ber92] J. Bernstein. Representations of \( p \)-adic groups, 1992.

[Cas86] J. W. S. Cassels. Local Fields. London Mathematical Society Student Texts. Cambridge University Press, 1986.

[DHKM20] Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss. Moduli of Langlands Parameters, 2020.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HN96] Nigel Higson and Victor Nistor. Cyclic homology of totally disconnected groups acting on buildings. Journal of Functional Analysis, 141(2):466–495, November 1996.
[HP09] Kottwitz Haines and Prasad. Iwahori-Hecke Algebras, 2009.

[Kat82] Shinichi Kato. Irreducibility of principal series representations for Hecke algebras of affine type, 1982.

[KNS98] D. Kazhdan, Victor Nistor, and P. Schneider. Hochschild and cyclic homology of finite type algebras. Selecta Mathematica, New Series, 4:321–359, 06 1998.

[Laf16] Laurent Lafforgue. Le principe de fonctorialité de Langlands comme un problème de généralisation de la loi d’addition, 2016.

[Lam91] T Y Lam. A first course in noncommutative rings, volume 131 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.

[Lan02] Serge Lang. Algebra, volume 211 of Graduate Texts in Mathematics. Springer-Verlag, New York, New York, NY, third edition, 2002.

[Rob63] Norbert Roby. Lois polynomes et lois formelles en théorie des modules. Annales Scientifiques de l’Ecole Normale Supérieure. Quatrième Série, 80(3):213–348, 1963.

[Sch96] P. Schneider. The cyclic homology of p-adic reductive groups. Journal für die reine und angewandte Mathematik, 1996.

[Sol10] Maarten Solleveld. On the classification of irreducible representations of affine Hecke algebras with unequal parameters. 2010.

[Sol20] Maarten Solleveld. Endomorphism algebras and Hecke algebras for reductive p-adic groups. 2020.

[Sol21] Maarten Solleveld. Affine Hecke algebras and their representations. Indagationes Mathematicae, 32(5):1005–1082, sep 2021.

[Sta22] Stacks project authors. Stacks Project, 2022.

[Ste74] Robert Steinberg. Reductive and semisimple algebraic groups, regular and subregular elements, pages 76–155. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974.

[Zhu20] Xinwen Zhu. Coherent sheaves on the stack of langlands parameters, 2020.