$\mathcal{N} = 2$, conformal gauge theories at large R-charge: the $SU(N)$ case

Based on hep-th/2001.06645 (JHEP03(2020)160)
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Introduction and Motivation
Double Scaling Limits

- 't Hooft realized that $SU(N)$ gauge theory simplifies in the limit $g \to 0, \ N \to \infty$, with $g^2 N$ a constant.
- This is the prototypical example of a double scaling limit.
- Another class of examples comes from considering a QFT with some coupling $g$ and studying the operators with large charge $n$ under a global symmetry. [Hellerman et al. -2015; Arias-Tamargo et al - 2019]
- $\mathcal{N} = 2$ superconformal theories with gauge group $SU(N)$ are an attractive setup. We will study correlation functions of Coulomb branch operators with large $U(1)_R$-charge.
- The goal is to exhibit the simplicity that emerges in the double scaling limit.
- As we will see this limit enables us to probe some massive BPS states in the theory.
Two Point Functions in Conformal Field Theories

- For isolated conformal field theories, two point functions of primary operators are trivial: they are fixed by conformal symmetry up to normalization.
- For conformal field theories that allow exactly marginal deformations, the normalization is not global: the two point functions have a non-trivial dependence on exactly marginally couplings.
- The complexified gauge coupling $\tau$ is always exactly marginal for a superconformal $\mathcal{N} = 2$, $SU(N)$ theory.
- For a superconformal primary $\mathcal{O}$, the two point function is:

$$\langle \mathcal{O}(x), \bar{\mathcal{O}}(y) \rangle = \frac{G_{\mathcal{O}\bar{\mathcal{O}}}(\tau, \bar{\tau})}{(x - y)^{2\Delta(\mathcal{O})}}$$
The Coulomb branch operators of an $\mathcal{N} = 2$, $SU(N)$ theory are generated by $\text{tr} \phi^k$ with $1 < k < N$.

Their VEVs parameterize the Coulomb branch of vacua.

Using supersymmetric localization the partition function of any superconformal $\mathcal{N} = 2$ theory on 4-sphere can be reduced to finite dimensional integral over the Coulomb branch. [Pestun - 2007].

For an $SU(N)$ gauge theory, this is a one matrix model i.e an integral over a matrix $M$ that depends only on traces of $M$.

$$Z_{S^4} = \int [da] \exp(-4\pi \text{Im} \tau \text{tr} a^2) Z_{1\text{-loop}}(\text{tr} a^2, \text{tr} a^3, \cdots)$$

With $[da] = \prod_{\mu=1}^N da_\mu \prod_{\nu<\mu}(a_\mu - a_\nu)^2 \delta\left(\sum_\mu a_\mu\right)$. 

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Coulomb Branch Operators and Localization

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Correlation functions from Localization [Gerchkovitz et al. 2017]

- We will consider the simplest infinite sequence of Coulomb branch operators with increasing $R$-charge $O_n = (\text{tr}\phi^2)^n$.
- On $S^4$, the correlation function can be evaluated using localization.

$$\langle O_n(N)\bar{O}_m(S) \rangle_{S^4} = \partial^n_{\tau} \partial^m_{\bar{\tau}} Z_{S^4}$$

- This is not diagonal! Metric on sphere and flat space are conformally equivalent but due to conformal anomaly the map between flat space operators and those on $S^4$ is not trivial.
- To get flat space operator : $O_n$ : we need to perform Gram-Schmidt orthogonalization on $1, O_1, O_2, \cdots, O_n$.

[Bourget et al - 2018]
A Double Scaling Limit?

- Let’s consider the double scaling limit:

\[ F(\kappa) = \lim_{n \to \infty} \frac{\langle O_n(x), \bar{O}_n(y) \rangle_{\mathcal{N}=2}}{\langle O_n(x), \bar{O}_n(y) \rangle_{\mathcal{N}=4}} \]

With \( \kappa \) the finite coupling \( \frac{2\pi n}{\text{Im} \tau} \)

- Does this limit even exist? Maybe it is trivial?
- Localization seems to provide a path to answer this question but it is complicated by conformal anomaly.
- Progress can be made by exploiting the integrable structures in \( \mathcal{N} = 2 \) theories. For SQCD see: [Bourget et al - 2018, Beccaria - 2018]
- Grassi, Komargodski and Tizzano realized that for \( SU(2) \) the Gram-Schmidt process is hiding another “dual” matrix model.
- This observation in fact generalizes to higher rank case.
Large $n$ Correlators and Positive Matrices
Correlators from Determinants

- Define the the $n \times n$ matrix $M^{(n)}$ by $M_{kl}^{(n)} = \partial^k_\tau \partial^l_{\bar{\tau}} Z_{S^4}$.
- Then the flat space correlator can be written as a ratio of determinants.

\[ G_{2n} = \frac{\det M^{(n+1)}}{\det M^{(n)}} \]

- Using the localization result $M^{(n)}$ is a matrix with each element a finite dimensional integral. We can exchange $\det M^{(n)}$ for an integral over determinants.

\[ \det M^{(n)} = \frac{1}{n!} \int [da_i] e^{-4\pi \text{Im } \tau \text{ tr } a^2_i} Z_{1\text{-loop}}(a_i) \prod_{j<i} (\text{tr } a^2_i - \text{tr } a^2_j)^2 \]

- We have an integral over a matrix $W$ whose eigenvalues are $\text{tr } a^2_i$!
The Dual Matrix Model

- The result is that we are dealing with a matrix integral
  \[ \text{det } M^{(n)} = \frac{1}{n!} \int [dW] \exp(-V(W)) \]

- Eigenvalues of \( W \) are \( \text{tr } a_i^2 \): \( W \) is a positive matrix.
- The large \( n \)-limit of potential \( V \) can be determined from the interacting action of the \( \mathcal{N} = 2 \) theory.
- It turns out that if rank of gauge group is greater than 1, \( V \) contains higher traces of all orders!
- The higher trace operators are suppressed just right to contribute at the same order as single trace operators.
- So the large \( n \) limit exists but it is not planar.
Planarity and Diagrams

- This non-planarity has a very interesting analog in the super-diagram analysis.
- In the ’t Hooft limit only the planar diagrams contribute to leading order in $N$.
- In contrast the large $n$ limit is dominated by diagrams that maximize genus at a given order in gauge coupling.
- Concretely the relevant diagrams are all possible completions of the skeleton.
- The 1-loop correction is planar but the 2-loop correction has genus 1 due to an insertion of the box diagram.
Perturbative results

• In summary, we have an efficient algorithm for perturbative calculations able to quickly produce long series expansion to very high order. For example for $\mathcal{N} = 2$ Superconformal QCD we obtain:

$$\log F(\kappa) = -\frac{9 \zeta(3)}{2} \kappa^2 + \frac{25(2N^2 - 1) \zeta(5)}{N(N^2 + 3)} \kappa^3 - \frac{1225(8N^6 + 4N^4 - 3N^2 + 3) \zeta(7)}{16N^2(N^2 + 1)(N^2 + 3)(N^2 + 5)} \kappa^4 + \ldots$$

• The algorithm is completely generic and doesn’t require any assumptions beyond a simple gauge group and the input of partition function on $S^4$ as an integral over Coulomb branch.

• But non-planarity makes it hard to resum the perturbative results in a way amenable to probing the large $\kappa$ regime, in contrast to $SU(2)$ where it is possible [Beccaria 2019, Grassi et al. 2019].
One Point Functions in the Presence of Wilson Loop

Figure credits: M. Billo, F. Galvagno, P. Gregori and A. Lerda
Wilson Loops

- For a more striking simplification we turn to one point functions of chiral operators in the presence of Wilson loops.
- These can also be computed using localization,
  \[ \langle : \mathcal{O}_n : \mathcal{W} \rangle \propto \int [da] : \mathcal{O}_n : \text{tr} \exp(2\pi a) \exp(-4\pi \text{Im} \tau \text{tr} a^2) Z_{1\text{-loop}}(a) \]
- It turns out that the large \( n \) limit is the same as that of two point functions, \( \langle : \mathcal{O}_n : \mathcal{W} \rangle \to \langle : (\text{tr} a^2)^n : \text{tr} a^{2n} \rangle \).
- The large \( n \) limit of this two point function is captured by an \("SU(2)"\) like matrix model!
  \[ Z_{\text{eff}} = \int dr \ r^{N^2-2} \exp(-4\pi \text{Im} \tau r^2) Z_{1\text{-loop}}(ra_0). \]
- \( a_0 = \left( \frac{1}{\sqrt{N(N-1)}}, \frac{1}{\sqrt{N(N-1)}}, \cdots, \frac{1}{\sqrt{N(N-1)}}, -\sqrt{\frac{N-1}{N}} \right) \) is the point on \( S^{N-1} \) that maximizes \( \text{tr} a^{2n} \).
A Simple Final Result

- As a result the large $n$-limit is planar. This allows us to conjecture all order resummations that reveals a strikingly simple structure.

$$\lim_{n \to \infty} \log \frac{\langle \mathcal{O}_n : \mathcal{W} \rangle^{\mathcal{N}=2}}{\langle \mathcal{O}_n : \mathcal{W} \rangle^{\mathcal{N}=4}} = \int_0^\infty \frac{dt \, e^t}{t(e^t - 1)^2} \mathcal{J}(t)$$

- The "SU(2) like" $Z_{\text{eff}}$ is an integral over the line $r a_0$ in Coulomb branch. On this line, the VEVs of $\phi$ break $SU(N) \to U(N - 1)$.

- The supermultiplets split as representations of this $U(N - 1)$. The VEVs of $\phi$ also give mass to some of resulting fields.

- Each such massive representation $r$ of $U(N - 1)$ contributes a term to $\mathcal{J}(t)$ which is $\pm 2 \dim r [J_0(\sqrt{2}m_r t) - 1]$.

- $m_r$ is the mass of $r$ at the point $\kappa a_0$ of the moduli space.
An Example: $\mathcal{N} = 2$ SQCD

- $2N$ hypermultiplets in the fundamental of $U(N)$.
- Each fundamental hypermultiplet splits into a fundamental and a singlet of $U(N - 1)$. At $\kappa a_0$,
  - $U(N - 1)$ fundamental has mass $\sqrt{\frac{\kappa}{N(N-1)}}$.
  - $U(N - 1)$ singlet has mass $\sqrt{\frac{\kappa(N-1)}{N}}$.
- The vector multiplet splits as
  - Adjoint of $U(N - 1)$ which is massless as expected from unbroken $U(N - 1)$ gauge symmetry.
  - 2 massive $W$-bosons in the fundamental of $U(N - 1)$ with mass $\sqrt{\frac{\kappa N}{N-1}}$.

The large $n$ limit we are after is
\[
4 \int_0^\infty \frac{dt e^t}{(e^t - 1)^2} \left[ N \tilde{J}_0\left( t \sqrt{\frac{2(N-1)\kappa}{N}} \right) + N(N - 1) \tilde{J}_0\left( t \sqrt{\frac{2\kappa}{N(N-1)}} \right) - (N - 1) \tilde{J}_0\left( t \sqrt{\frac{2N\kappa}{N-1}} \right) \right]
\]
This contains both perturbative and exponentially suppressed non-perturbative terms.
Conclusions and Outlook

• Leveraging localization and random matrix theory we can learn about the large $R$-charge limit of $\mathcal{N} = 2$, $SU(N)$ theories.
• This requires choosing the observables (or rather the sequence of observables) carefully.
• Is there a similar story for generic sequence of operators with increasing $R$-charge?
• The large $n$ limit of one point functions of chiral operators in the presence of Wilson loop remains planar for $SU(N)$ gauge group.
• It also admits a simple and interesting interpretation in terms of the mass spectrum at the relevant point in moduli space.
• Maybe it contains a hint of an EFT description generalizing the $SU(2)$ case?