Abstract
Quantum field planes furnish a noncommutative differential algebra $\Omega$ which substitutes for the commutative algebra of functions and forms on a contractible manifold. The data required in their construction come from a quantum field theory. The basic idea is to replace the ground field $\mathbb{C}$ of quantum planes by the noncommutative algebra $A$ of observables of the quantum field theory.

1 A picture of quantum space time
We wish to compound a global space time, which is quantum, from local Lorentz frames. The physics in the local Lorentz frames shall be described by a special relativistic local quantum field theory on a classical space time of some dimension.

To accomplish such a construction, we need to generalize the classical setup of general relativity which involves a classical space time manifold and a differential calculus and geometry on it [3].

Quantum field planes furnish a noncommutative differential algebra $\Omega$ which substitutes for the commutative algebra of functions and forms on a contractible manifold. The basic data to construct it come from a quantum field theory in the local Lorentz frame.

$$\Omega = \mathcal{F} \otimes \Phi$$

as a linear space, where $\mathcal{F}$ plays the role of an “algebra of scalar functions on quantum space time”, and the fibre $\Phi$ is the algebra of field operators of the quantum field theory in the local Lorentz frame.

Quantum field planes are generalizations of the quantum planes which were studied by Manin [22], Wess and Zumino [26], and others, and were generalized to the quasi-associative

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case by the authors \[21\]. Basically the construction of quantum field planes replaces the
ground field \( \mathbb{C} \) of quantum planes by the noncommutative algebra \( \mathcal{A} \) of observables of the
quantum field theory in the local Lorentz frame. Its elements will be called \( a \)-numbers.

There is one extra piece of structure which has no analogue in quantum planes. It
comes from the operator product expansions in the quantum field theory \( \Phi \).

We begin with the general description of the construction of the differential calculus,
without imposing operator product expansions at first. Then examples will be presented,
including the classical calculus, quantum planes, and quantum field planes. Finally, the
extra consistency conditions from operator product expansions will be handled.

The Doplicher Haag Roberts construction of superselection sectors in local quantum
field theories is basic \[6, 7\]. It starts from localizable morphisms of the algebra of observ-
able. The reconstruction theorem of Doplicher and Roberts for 4-dimensional theories \[8\]
was recently generalized by one of us to situations with braid statistics \[24\]. Here we will use
the language of amplimorphisms which was introduced by Vecsernyes and Szlachányi \[23\].
It simplifies our proofs considerably and permits a unified point of view. Moreover, quasi-
associative possibilities appear very natural in this setting.

2 Differential calculus on generalized quantum planes

In the first subsection we describe an algebraic structure, which is often met in physics.
We will show later that quasi-Hopf algebras as well as observable algebras in quantum
field theory provide examples. This algebraic structure is sufficient for many constructions
known in the theory of quantum groups. As an example we will discuss how quantum
planes and differential calculi thereon can be obtained within the general framework.

2.1 Amplimorphisms and their intertwiners

Throughout this paper, let
\( \mathcal{K} \) - \(*\)-algebra with unit \( e \).
\( \mathcal{R}ep \) - a set of \( * \)-representations \( \pi \) of \( \mathcal{K} \).
\( \pi_0 \) - a fixed representation in \( \mathcal{R}ep \). It will act in a Hilbert space \( \mathcal{H}_0 \)

Let \( W \) be some finite dimensional Hilbert space.

**Definition 1** \[23, 25\] (Amplimorphisms) An amplimorphism
\[ \mu : \mathcal{K} \mapsto \text{End}(W) \otimes \mathcal{K} \]
is a not necessarily unit-preserving \( * \)-homomorphism of algebras.

Given the representation \( \pi_0 \) of \( \mathcal{K} \) and an amplimorphism \( \mu : \mathcal{K} \mapsto \text{End}(W) \otimes \mathcal{K} \), a new
representation
\[ \pi = \pi_0 \circ \mu \equiv (id \otimes \pi_0) \circ \mu. \] \hspace{1cm} (2.1)
can be constructed. Its representation space is \( W \otimes \mathcal{H}_0 \).
This construction principle is very powerful. It generalizes the use of morphisms in the theory of superselection sectors of algebraic field theory. Using the fact that amplimorphisms can be composed in a natural way,

\[ \mu_2 \circ \mu_1 : \mathcal{K} \mapsto \text{End}(W_1) \otimes \text{End}(W_2) \otimes \mathcal{K}, \]

they lead to a definition of a product of representations of \( \mathcal{K} \).

**Definition 2** (product of representations of \( \mathcal{K} \)) Suppose that \( \pi_i = \pi_0 \circ \mu_i \). Then

\[ (\pi_2 \boxtimes \pi_1)(\xi) \equiv \pi_0((\mu_1 \circ \mu_2)(\xi)). \]

Now let us consider a family of Hilbert spaces, \( W_\pi, \pi \in \text{Rep} \), \( W_{\pi_0} = \mathbb{C} \) and a family of amplimorphisms \( \mu_\pi, \pi \in \text{Rep} \) such that

1. \( \mu_{\pi_0} = \text{id}_\mathcal{K} \),
2. \( \pi_0 \circ \mu_{\pi_1} \circ \ldots \circ \mu_{\pi_n} \in \text{Rep} \),
3. \( \pi(\xi) = \pi_0 \circ \mu_\pi(\xi) \).

**Definition 3** A family of amplimorphisms with the properties 1., 2., 3. is said to generate all the representations \( \pi \in \text{Rep} \) of \( \mathcal{K} \) from \( \pi_0 \).

Besides the amplimorphisms, one considers their intertwiners.

**Definition 4** (intertwiners) Given amplimorphisms \( \mu_i : \mathcal{K} \mapsto \text{End}(W_i) \otimes \mathcal{K} \), an element \( T \in \text{End}(W_1, W_2) \otimes \mathcal{K} \) is called an intertwiner between \( \mu_1 \) and \( \mu_2 \) if

\[ T \mu_1(\xi) = \mu_2(\xi) T \]

Of particular interest are intertwiners \( \varphi(\mu_\pi, \mu_\sigma) \in \text{End}(W_\sigma \boxtimes \pi, W_\sigma \otimes W_\pi) \otimes \mathcal{K} \) which intertwine \( \mu_\pi \circ \mu_\sigma \) and \( \mu_\sigma \boxtimes \pi \), viz.

\[ \varphi(\mu_\pi, \mu_\sigma)(\mu_\sigma \boxtimes \pi(\xi)) = \mu_\pi \circ \mu_\sigma(\xi) \varphi(\mu_\pi, \mu_\sigma) \] (2.2)

Note the contravariant nature of this relation: The order of \( \sigma \) and \( \pi \) gets interchanged.

An assignment \( \varphi : (\mu_\pi, \mu_\sigma) \mapsto \varphi(\mu_\pi, \mu_\sigma) \) with this intertwining property will be called a reassociator. The name comes from the fact that the product \( \circ \) of amplimorphisms is always associative, while the product \( \boxtimes \) of representations of \( \mathcal{K} \) is not necessarily associative, but only quasiassociative in the sense that a relation of the form (2.2) holds.

Statistics elements are another important class of intertwiners. They permute factors in \( \mu_\pi \circ \mu_\sigma \). To state their properties we will use the usual permutation \( P \),

\[ P : W_\mu \otimes W_\nu \mapsto W_\nu \otimes W_\mu \]
Definition 5 (statistics elements) A family $\mathcal{R}(\mu, \nu)$ which assigns a unitary element in $\text{End}(W_\nu) \otimes \text{End}(W_\mu) \otimes \mathcal{K}$ to every pair $\mu, \nu$ of amplimorphisms is called a family of statistics elements if $\mathcal{R}(\mu, \nu) = P \mathcal{R}(\mu, \nu)$ has the following properties

1. $\hat{\mathcal{R}}(\mu_1, \mu_2)$ furnishes an intertwiner between $\mu_1 \circ \mu_2$ and $\mu_2 \circ \mu_2$, i.e.
   \[ \hat{\mathcal{R}}(\mu_1, \mu_2) \circ \mu_2(\xi) = \mu_2 \circ \mu_1(\xi) \hat{\mathcal{R}}(\mu_1, \mu_2) \quad , \]  
   (2.3)

2. if $T_i, i = 1, 2$ are two intertwiners with the property $T_i \mu_i = \mu_i T_i$ then
   \[ \hat{\mathcal{R}}(\mu_1, \mu_2) T_1 T_2 = T_2 T_1 \hat{\mathcal{R}}(\nu_1, \nu_2) \]  
   (2.4)

3. elements $\hat{\mathcal{R}}(\mu, \nu)$ satisfy the following generalized quasitriangularity relations
   \[ \hat{\mathcal{R}}(\mu_1, \mu_2, \nu) = \hat{\mathcal{R}}(\mu_1, \nu \mu_2) \hat{\mathcal{R}}(\mu_2, \nu) \]  
   (2.5)
   \[ \hat{\mathcal{R}}(\nu, \mu_1 \circ \mu_2) = \mu_1(\hat{\mathcal{R}}(\nu, \mu_2)) \hat{\mathcal{R}}(\nu, \mu_1) \]  
   (2.6)

4. $\hat{\mathcal{R}}(\mu, \nu)$ is trivial, if one of its arguments is trivial, i.e.
   \[ \hat{\mathcal{R}}(\text{id}_\mathcal{K}, \mu) = \hat{\mathcal{R}}(\mu, \text{id}_\mathcal{K}) = 1_{W_\mu} \otimes e. \]  
   (2.7)

Remark: 3. are basically Drinfelds quasitriangularity-relations [8] for R-matrices which imply the Yang Baxter equations. They can be restated with the help of reassociators. From

\[ \hat{\mathcal{R}}(\mu_\pi \circ \mu_\sigma, \nu_\pi) \varphi(\mu_\pi, \mu_\sigma) = \nu(\varphi(\mu_\pi, \mu_\sigma)) \hat{\mathcal{R}}(\mu_\sigma \otimes \pi, \nu_\pi) \]  
   (2.8)
\[ \hat{\mathcal{R}}(\nu, \mu_\pi \circ \mu_\sigma) \nu(\varphi(\mu_\pi, \mu_\sigma)) = \varphi(\mu_\pi, \mu_\sigma) \hat{\mathcal{R}}(\nu, \mu_\sigma \otimes \pi) \]  
   (2.9)

it follows that
\[ \nu(\varphi(\mu_\pi, \mu_\sigma)) \hat{\mathcal{R}}(\mu_\sigma \otimes \pi, \nu_\sigma) \varphi^{-1}(\mu_\pi, \mu_\sigma) = \hat{\mathcal{R}}(\mu_\pi, \nu) \mu_\pi(\hat{\mathcal{R}}(\mu_\sigma, \nu)) \]  
   (2.10)
\[ \varphi(\mu_\pi, \mu_\sigma) \hat{\mathcal{R}}(\nu, \mu_\sigma \otimes \pi) \nu(\varphi^{-1}(\mu_\pi, \mu_\sigma)) = \mu_\pi(\hat{\mathcal{R}}(\nu, \mu_\sigma)) \hat{\mathcal{R}}(\nu, \mu_\sigma). \]  
   (2.11)

In representation theory, the notion of conjugate representation is important. This translates into the language of amplimorphisms as follows

Definition 6 (conjugation) The family $\{\mu_\pi\}$ of amplimorphisms admits a conjugation if there is for every $\mu$ an amplimorphism $\bar{\mu}$ and an intertwiner $g(\mu, \bar{\mu}) \in \text{End}(W_{\bar{\mu}} \otimes W_\mu, \mathcal{C}) \otimes \mathcal{K}$ such that
\[ g(\mu, \bar{\mu}) \mu \circ \bar{\mu}(\xi) = \xi g(\mu, \bar{\mu}) \]

We introduce the notation $\chi_\mu \equiv g(\mu, \bar{\mu}) \mu(g^*(\bar{\mu}, \mu))$. It has the property
\[ \mu(\xi) \chi_\mu = \chi_\mu \mu(\xi) \]

Sometimes it is convenient to admit the possibility that the conjugate amplimorphism $\bar{\mu}$ and the intertwiner $g(\bar{\mu}, \mu)$ exist, but $\bar{\mu}$ does not belong to the family which is associated with representations $\pi \in \mathcal{R}_{\text{rep}}$. 

4
2.2 Differential calculus

We assume that a family of amplimorphisms is given which generates all the representations \( \pi \in \mathcal{R}ep \) of \( K \) in the sense of definition 3. We assume in addition that statistics elements are defined, that a reassociator exists, and that the family of amplimorphisms admits conjugation.

We construct a differential algebra. Here we consider only a special case. We assume that there exists a special amplimorphism \( \mu \), such that all the representations in \( \mathcal{R}ep \) can be constructed as subrepresentations of \( \pi_0 \circ \mu \circ \ldots \circ \mu \), i.e. as subrepresentations of products of representations

\[
\pi_f = \pi_0 \circ \mu^f.
\]

The construction can be generalizes to situations with several fundamental representations. From now on we will write \( \mu \) in place of \( \mu_f \).

By assumption, \( \mathcal{R}(\mu, \mu) \) is unitary and \( P^2 = id \). Therefore \( \hat{\mathcal{R}} \) can be diagonalized and admits a spectral decomposition

\[
\hat{\mathcal{R}}(\mu, \mu) = \sum_{\alpha=0}^{M} \gamma_{\alpha} E_{\alpha},
\]

where \( \gamma_{\alpha} \) are the eigenvalues of \( \hat{\mathcal{R}}(\mu, \mu) \). We define

\[
\Pi_\gamma = \prod_{\gamma_{\alpha} \neq \gamma} (\hat{\mathcal{R}}(\mu, \mu) - \gamma_{\alpha})(\gamma - \gamma_{\alpha})^{-1}, \quad \bar{\Pi}_\gamma = \prod_{\gamma_{\alpha} \neq \gamma} (\hat{\mathcal{R}}(\bar{\mu}, \bar{\mu}) - \gamma_{\alpha})(\gamma - \gamma_{\alpha})^{-1}.
\]

In the following we assume that one of the eigenvalues \( \gamma \in \{\gamma_{\alpha}\} \) has been singled out.

For the sake of clarity, let us introduce a basis \( \{e^a\}, (a = 1 \ldots n_f) \) in \( W_{\pi f} \). This is not necessary, but it will facilitate comparison with the classical calculus.

We construct a \( \mathbb{Z} \)-graded algebra.

\[
\Omega = \sum_{n \geq 0} \Omega_n
\]

\( \Omega_0 \) is a substitute for the space of functions on a manifold, and \( \Omega_n \) substitutes for the space of \( n \)-forms. Elements of \( \Omega_n \) are said to be of degree \( n \).

**Definition 7** (Differential Algebra) The algebra \( \Omega \) is generated by \( K \), generators \( Z_a \) of degree \( \deg(Z_a) = 0 \) and generators \( \Theta_a, a = 1 \ldots n_f \) of degree 1, subject to the following relations for \( Z = \sum_a Z_a e^a \), \( \Theta = \sum_a \Theta_a e^a \).

\[
\xi Z = Z \mu(\xi), \quad \xi \Theta = \Theta \mu(\xi) \quad \text{for all } \xi \in K
\]

\[
ZZ \Pi_\gamma = 0, \quad \Theta \Theta \hat{\mathcal{R}}(\mu, \mu) = \gamma \Theta \Theta
\]

\[
Z \Theta \hat{\mathcal{R}}(\mu, \mu) = -\gamma \Theta Z
\]
\( \Omega \) is made into a differential algebra by adjoining to it an operator \( d \) subject to the relations

\[\begin{align*}
\xi d &= d\xi \quad \text{for all } \xi \in \mathcal{K} \quad \text{(invariance)} \\
\xi^2 &= 0 \\
dZ &= \Theta + Zd \quad \text{(Leibniz rule)} \\
d\Theta &= -\Theta d
\end{align*}\] (2.16) (2.17) (2.18) (2.19)

\( Z_a \) substitute for coordinate functions, and \( \Theta_a \) substitute for differentials.

Proof that the definition is meaningful:

Consistency of the braid relations (2.14, 2.15) follows from the Drinfeld relations, definition 5 in the standard way.

Consistency of \( d \): It must be verified that the relation \( dZ = \Theta + Zd \) is consistent with \( ZZ\Pi = 0 \). We move \( d \) through \( ZZ\Pi \).

\[d(ZZ\Pi) = \Theta Z\Pi + ZdZ\Pi = \Theta Z\Pi + Z\Theta\Pi + ZZd\Pi \]

(2.20)

\[\Theta = 1 - \frac{\gamma}{\gamma} R(\mu, \mu) \Pi \gamma = 0 \]

(2.21)

Consistency of the other relations with \( d \) is obvious q.e.d.

In the classical calculus, \( d \) can be constructed from partial differential operators \( \partial_a \) and differentials \( \theta_a = dz_a \). This is also possible here. Our \( \Theta_a \) substitutes for the operator of multiplication with \( \theta_a \), and similarly for \( Z_a \).

We introduce a basis \( \{e_b\} \) in \( W_{\pi^f} \), \( \pi^f = \pi_0 \circ \mu \).

**Definition 8** (partial derivatives) We adjoin partial derivatives \( \partial = \sum \partial \partial e_b \) to \( \Omega \). They are \( \mathcal{K} \)-covariant in the sense that

\[\xi \partial = \partial \mu(\xi),\]

(2.22)

and are subject to the relations

\[\begin{align*}
\partial \Pi \gamma &= 0 \\
\partial \partial \Pi \gamma &= 0 \\
\Theta \partial &= -\gamma \partial \Theta R(\mu, \mu) \\
d Z &= g(\mu, \mu) - \gamma Z\partial R(\mu, \mu)
\end{align*}\]

(2.23) (2.24) (2.25)

**Theorem 9** (Consistency of the differential calculus) The relations (2.24) for the partial derivatives are consistent with the relations in \( \Omega \). If we define \( d \) in terms of differentials \( \Theta \) and partial derivatives \( \partial \) by

\[d = \Theta \partial g(\mu, \mu) \gamma^{-1},\]

then \( d \) has the properties stated in definition 8; in particular it satisfies \( d^2 = 0 \) and the Leibniz rule.

The proof requires a number of consistency checks. It will be relegated to Appendix A.
3 Examples

This section is devoted entirely to examples of the structures introduced above. We start with the simplest commutative case. Then the above construction will be applied to (quasi-) Hopf-algebras. In this case we recover the standard calculus on (quasi-) quantum planes \[26, 21\]. Algebraic quantum field theory provides examples where the amplimorphisms are not obtained from a co-product. This finally leads to quantum field planes as new examples of non-commutative differential geometries.

3.1 Calculus on \( \mathbb{C}^N \)

The algebra of polynomial (or holomorphic) functions of \( N \) complex variables is generated by coordinate functions \( z_a, a = 1...N \). We denote by \( Z_a \) the operator of multiplication by \( z_a \). The algebra of forms is obtained by adjoining \( \theta_a = dz_a, (a = 1...N) \). We denote by \( \Theta_a \) the operator of multiplication by \( \theta_a \).

The group \( GL(N, \mathbb{C}) \) can act. It possesses a trivial 1-dimensional representation \( \pi_0 \).

\[ \pi_0(\xi) = 1 \] for \( \xi \in GL(N, \mathbb{C}) \). It has also a defining \( N \)-dimensional representation

\[ \pi^f(\xi) = \xi. \]

We denote by \( Rep \) the set of all those irreducible representations of \( GL(N, \mathbb{C}) \) by matrices \( \pi(\xi) \) which appear as subrepresentations of products of defining representations. They act in representation spaces

\[ W_{\pi} = \mathbb{C}^{n_\pi}. \]

All these representations extend to representations of the group algebra \( \mathcal{K} = CGL(N, \mathbb{C}) \).

The conjugate \( \bar{\pi} \) of the fundamental representation is defined by

\[ \bar{\pi}(\xi) = t \xi^{-1} \]

It does not belong to \( Rep \), but we consider it anyway.

With every representation \( \pi \in Rep \) we associate an amplimorphism

\[ \mu_\pi : \mathcal{K} \rightarrow End(W_{\pi}) \otimes \mathcal{K} = Mat_{n_\pi}(\mathcal{K}) \]

\[ \mu_\pi(\xi) = \pi(\xi) \otimes \xi, \quad (\xi \in GL(N, \mathbb{C})) \] (3.1)

(3.2)

It follows that

\[ \pi_0 \circ \mu_\pi(\xi) = \pi(\xi)\pi_0(\xi) \]

(3.3)

\[ = \pi(\xi) \text{ for } \xi \in GL(N, \mathbb{C}) . \] (3.4)

The amplimorphism \( \mu = \mu^f \) is associated with the \( N \)-dimensional fundamental representation, \( \bar{\mu} \) with its conjugate.

The intertwiner between \( \mu \circ \bar{\mu} \) and the identity morphism has the form

\[ g(\mu, \bar{\mu}) = \bar{\gamma}(\mu, \bar{\mu})e \]

\[ \bar{\gamma}_{ab}(\mu, \bar{\mu}) = \delta_{ab} \]
It follows that \( \tilde{g}_{ab}(\mu, \bar{\mu}) \xi^{-1}_{ac} \xi_{bd} = \tilde{g}_{cd}(\mu, \bar{\mu}) \) as required.
Finally we define \( R(\mu, \mu) = id \otimes id \), so that \( \hat{R} \) is the permutation operator.

\[
\hat{R}(\mu, \mu) = P, \tag{3.5}
\]
i.e. \( \hat{R}(\mu, \mu)_{ab,cd} = \delta_{ad} \delta_{bc} \). It has two eigenvalues

\[
\gamma_0 = +1, \quad \gamma_1 = -1 \tag{3.6}
\]
The reassociator is trivial, \( \varphi(\mu_\pi, \mu_\sigma) = id \otimes e \).

It is now readily verified that the formulae of the differential calculus as set out in definitions \ref{eq:339} reproduce the classical calculus.

The algebra \( \Omega \) of section \ref{sec:32} is generated by \( K = C GL(N, \mathbb{C}) \) and generators \( Z_a, \Theta_a \). Here it is the algebra of polynomial or holomorphic \( C GL(N, \mathbb{C}) \)-valued functions and forms of \( N \) complex variables \( z_1...z_N \). For convenience we use multiplication operators, but this makes no difference in cases with a trivial reassociator. The braid relations \ref{eq:214} - \ref{eq:215} become

\[
Z_a Z_b = + Z_b Z_a, \quad \Theta_a \Theta_b = - \Theta_b \Theta_a, \tag{3.7}
\]
\[
Z_a \Theta_b = \Theta_b Z_a, \tag{3.8}
\]
\[
d Z_a = \Theta_a + Z_a d. \tag{3.9}
\]

Both generators \( Z_a \) and \( \Theta_a \) transform according to the defining representation of \( GL(n, \mathbb{C}) \), viz.

\[
\xi Z_a = Z_b \mu_{ba}(\xi) = Z_b \pi_{ba}(\xi), \tag{3.10}
\]
\[
\xi \Theta_a = \Theta_b \mu_{ba}(\xi) = \Theta_b \pi_{ba}(\xi), \tag{3.11}
\]

Therefore exterior differentiation \( d \) commutes with the action of \( GL(N, \mathbb{C}) \).

The partial derivatives commute among themselves and with \( \Theta_a \). They transform according to the conjugate of the defining representation.

\[
\partial_a \partial_b = \partial_b \partial_a, \quad \xi \partial_a = \partial_b \mu_{ba}(\xi) = \partial_b (t^{-1})_{ba} \xi, \tag{3.12}
\]
\[
\Theta_a \partial_b = \partial_b \Theta_a, \tag{3.13}
\]
\[
\partial_a Z_b = \delta_{ab} e + Z_b \partial_a, \tag{3.14}
\]
\[
d = \Theta_a \partial_b \delta_{ab}. \tag{3.15}
\]

### 3.2 Quantum planes

In the quantum planes and their generalizations, the symmetry group \( GL(N, \mathbb{C}) \) or \( SU(N) \) of the classical calculus is replaced by a quantum symmetry. In the most general case it can be a weak quasitriangular quasi Hopf *-algebra \( \mathcal{G} \) \ref{eq:19}.

Throughout this subsection, let \( \mathcal{K} = \mathcal{G} \) - semisimple *-algebra with unit \( e \).
\(\mathcal{R}ep\) - set of all unitary representations \(\tau\) of \(\mathcal{G}\)

\(\pi_0 = \epsilon\) - the co-unit of \(\mathcal{G}\)

\(W_\tau = V_\tau\) - carrier space of the representation \(\tau\)

The counit \(\epsilon\) defines a 1-dimensional representation of \(\mathcal{G}\), i.e. a homomorphism of \(\mathcal{G}\) into \(\mathbb{C}\). \(\mathcal{G}\) comes equipped with a comultiplication

\[
\Delta : \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}.
\]

(3.16)

\(\Delta\) is a not necessarily unit-preserving *-homomorphism of algebras. If \(\Delta(e) \neq e \otimes e\) we speak of truncation. It is required that

\[
(id \otimes \epsilon)\Delta = id = (\epsilon \otimes id)\Delta
\]

(3.17)

\(\Delta\) is required to be quasiassociative in the following sense. There exists an element \(\varphi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}\) which possesses a quasi-inverse \(\varphi^{-1}\) such that

\[
\varphi(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\varphi \quad (\xi \in \mathcal{G}).
\]

(3.18)

The quasi-inverse property is

\[
\varphi\varphi^{-1} = (id \otimes \Delta)\Delta(e) \quad \varphi^{-1}\varphi = (\Delta \otimes id)\Delta(e).
\]

(3.19)

The coproduct is required to be braid commutative in the following sense.

Let \(\Delta'(\xi)\) differ from \(\Delta(\xi)\) by interchange of factors. There exists an element \(R \in \mathcal{G} \otimes \mathcal{G}\) with quasi-inverse \(R^{-1}\) such that

\[
R\Delta(\xi) = \Delta'(\xi).
\]

(3.20)

The quasi-inverse property states that

\[
RR^{-1} = \Delta'(e) \quad R^{-1}R = \Delta(e).
\]

(3.21)

\(R\) and \(\varphi\) must satisfy a number of consistency relations - Drinfelds hexagon and pentagon identities. There is also an antipode \(S\) with certain properties. It is an anti-automorphism of \(\mathcal{G}\).

The representation theory of the quantum symmetry \(\mathcal{G}\) can be rephrased in the language of amplimorphisms by defining amplimorphisms and their intertwiners as follows [25].

\[
\mu_\tau(\xi) = (\tau \otimes id)\Delta(\xi) \quad \text{for all} \quad \xi \in \mathcal{G}
\]

(3.22)

\[
\varphi(\mu_\tau, \mu_\sigma) = (\sigma \otimes \tau \otimes id)(\varphi),
\]

\[
\mathcal{R}(\mu_\tau, \mu_\sigma) = (\sigma \otimes \tau \otimes id)(\varphi_{213}R_{12}\varphi^{-1})
\]

We use the standard notations: If \(\varphi_{123} = \sum_{\sigma} \varphi_{\sigma}^1 \otimes \varphi_{\sigma}^2 \otimes \varphi_{\sigma}^3\) then

\[
\varphi_{s(1)s(2)s(3)} = \sum_{\sigma} \varphi_{\sigma}^{s(1)} \otimes \varphi_{\sigma}^{s(2)} \otimes \varphi_{\sigma}^{s(3)}
\]
etc. The properties of $\varphi$, $R$ can be checked. The equations in the remark after the definition of statistics elements are equivalent to the following equations for $R$, $\varphi$.

$$
(id \otimes id \otimes \Delta)(\varphi)_{2314} = \varphi_{134}(id \otimes id \otimes \Delta)R
$$

$$
(id \otimes id \otimes \Delta)(R)(id \otimes id \otimes \Delta)(\varphi^{-1}) = [(id \otimes id \otimes \Delta)(R)]_{1324}R_{234}
$$

These relations are in turn equivalent to Drinfelds hexagon and pentagon equations.

The conjugate amplimorphism is defined with the help of the antipode, $\bar{\mu}_\tau = \mu_{\bar{\tau}}$ with $\bar{\tau}(\xi) = \tau(S^{-1}(\xi))$. The intertwiner $g(\bar{\mu}, \mu)$ exists [25, 19].

The definition of the product of representations with the help of amplimorphisms is equivalent to the usual definition of tensor products of representations of quantum symmetries $(\tau_2 \otimes \tau_1)(\xi) = (\tau_2 \otimes \tau_1)(\Delta(\xi))$. (3.23)

We assume that all the irreducible representations of $G$ can be obtained by reducing products of one fundamental representation $\tau_f = \tau$ of $G$ of dimension $n_f$. The associated amplimorphism will be denoted by $\mu_f = \mu$.

The algebra $\Omega$ of section 3 will be generated by $G$ together with $n_f$ generators $Z_a$ of degree 0 and $n_f$ generators $\Theta_a$ of degree 1 as before. This algebra substitutes for an algebra of $G$-valued functions and forms of $n_f$ noncommuting variables $z_a$. The algebra is only quasi-associative if $G$ is not coassociative, i.e. if $\varphi \neq e \otimes e \otimes e$. In the quasiassociative case it is essential to have $G$ as a part of the algebra to start with. $G$ can be factored out also in this case, but this involves a nontrivial construction, see [21].

If $\tau^f \otimes \tau^f(R)$ possesses only two different eigenvalues $\gamma_0$ and $\gamma$, then the construction of section 2 reproduces the well known quantum planes and their quasi-associative generalizations [21]. The ZZ braid relations reduce in this case to

$$
ZZ \tilde{R}(\mu, \mu) = \gamma_0 ZZ .
$$

In the coassociative case, $R(\mu, \mu) = (\tau \otimes \tau)(R) \otimes e$, i.e. it is a numerical matrix. So we get the standard quantum planes.

### 3.3 Algebraic quantum field theory

In special relativistic local quantum field theory there is a *-algebra $A$ of observables. For convenience, one considers complex linear combinations of bounded self adjoint observables as elements of $A$. There are subalgebras $A(\mathcal{O})$ which contain observables that can be measured in bounded contractible domains $\mathcal{O}$ of space time. Locality says that observables in relatively spacelike domains commute. There is also a Hamiltonian $H$ affiliated with $A$. In a positive energy representation it has nonnegative spectrum.

Typically, the algebra $A$ has a number of inequivalent irreducible positive energy representations $\pi$. They are called superselection sectors [28].

Among these representations is the vacuum representation $\pi_0$. It acts on a Hilbert space $\mathcal{H}_0$ which contains the ground state of $H$. This sector is called the vacuum sector.
One starts from general assumptions of algebraic field theory. These assumptions are fulfilled in conformal field theories, provided one completes $\mathcal{A}(\mathcal{O})$ to von Neumann algebras (acting in $\mathcal{H}^0$), and the global algebra $\mathcal{A}$ is constructed from the local algebras $\mathcal{A}(\mathcal{O})$ in an appropriate way \[1, 12\]. It follows that there exist localizable unital $*$-endomorphisms

$$\rho^J : \mathcal{A} \mapsto \mathcal{A}$$

such that all positive energy representations $\pi^J$ can be obtained from the vacuum representation by composition with these morphisms

$$\pi^J = \pi^0 \circ \rho^J \text{ i.e. } \pi^J(A) = \pi^0(\rho^J(A)).$$

By definition, unital $*$-endomorphisms of $\mathcal{A}$ are $\mathbb{C}$-linear maps subject to the conditions

$$\rho(AB) = \rho(A)\rho(B),$$
$$\rho(A^*) = \rho(A)^*,$$
$$\rho(1) = 1.$$  \tag{3.27}

Localizability means the following. To every bounded space time domain $\mathcal{O}$ there exists a unitary $u \in \mathcal{A}$ such that the morphism

$$\rho^J_u(A) = u^* \rho^J(A) u,$$  \tag{3.28}

is localized in $\mathcal{O}$. This means that

$$\rho^J_u(A) = A \text{ if } A \in \mathcal{A}(\mathcal{O}')$$  \tag{3.29}

and if $\mathcal{O}'$ is relatively spacelike to $\mathcal{O}$.

Using these morphisms, a product of irreducible positive energy representations of $\mathcal{A}$ can be defined as

$$\pi^I \boxtimes \pi^J = \pi^0 \circ \rho^I \circ \rho^J.$$  \tag{3.30}

These products will decompose into irreducibles according to

$$\pi^I \boxtimes \pi^J = \sum_K \pi^K N^K_{IJ},$$  \tag{3.31}

with multiplicities $N^K_{IJ}$. It follows from the standard assumptions of algebraic field theory that the observable algebra determines statistics operators $\epsilon(\rho^I, \rho^K) \in \mathcal{A}$ which enjoy standard properties \[13\].

One defines spaces of intertwiners between morphisms,

$$T \in T(\rho_1, \rho_2) \subset \mathcal{A} \iff T \rho_1(A) = \rho_2(A)T \text{ for all } A \in \mathcal{A}.$$  

For every pair of localizable endomorphisms $\rho_i, i = 1, 2$ there is a unitary local intertwiner, the statistics operator $\epsilon(\rho_1, \rho_2) \in T(\rho_1 \circ \rho_2, \rho_2 \circ \rho_1)$.
Explicitly, its intertwining property is

\[ \epsilon(\rho_1, \rho_2) \rho_1 \circ \rho_2(A) = \rho_2 \circ \rho_1(A) \epsilon(\rho_1, \rho_2) \]

The collection of statistics operators in uniquely determined by the following equations (a detailed proof can be found in [15])

\[ \epsilon(\rho_1, \rho_2) \rho_1(T_2)T_1 = T_2 \sigma_2(T_1) \epsilon(\sigma_1, \sigma_2) \quad \text{for all} \quad T_i \in \mathcal{T}(\sigma_i, \rho_i) , \]

\[ \epsilon(\rho_1 \circ \rho_2, \sigma) = \epsilon(\rho_1, \sigma) \rho_1(\epsilon(\rho_2, \sigma)) \quad \epsilon(\sigma, \rho_1 \circ \rho_2) = \rho_1(\epsilon(\sigma, \rho_2)) \epsilon(\sigma, \rho_1) \quad (3.32) \]

\[ \epsilon(\rho_1, \rho_2) = 1 \quad \text{whenever} \quad \rho_1 > \rho_2. \]

In the last row, \( \rho_1 > \rho_2 \) if \( \rho_1 \) and \( \rho_2 \) are localized in relatively spacelike domains, and if these domains are ordered, when they can be ordered. This last property will not be used here. In chiral conformal field theories on the circle, ordering is left or right, after a reference point on the circle has been chosen. Trivialization for \( \rho_1 < \rho_2 \) would give rise to the opposite statistics operator \( \epsilon(\rho_2, \rho_1)^* \).

Here we restrict our attention to sectors with finite statistics. For such sectors, the sum in the decomposition (3.31) of the product of representations is finite. This assumption is used in the construction of field operators in the next section.

All this fits in with the setup described at the beginning of section 2 if we make the following identifications

\[ \mathcal{K} = \mathcal{A} \quad \text{observable algebra of a QFT satisfying standard assumptions} \]

\[ \mathcal{R}ep \quad \text{set of positive energy representations with finite statistics} \]

\[ \pi_0 \quad \text{vacuum representation of} \mathcal{A} \]

\[ W_\pi = \mathbb{C}, \]

\[ \mathcal{R}(\rho, \sigma) = \epsilon^*(\sigma, \rho) \quad (\text{one could also take} \, \epsilon(\rho, \sigma)) \]

The amplimorphisms \( \mu \) are the endomorphisms \( \rho^J \)

Every sector with finite statistics has a unique conjugate. We denote by \( \bar{\rho} \) the morphism which generates it. It can be shown that the intertwiner \( g(\bar{\rho}, \rho) \) exists for sectors of finite statistics and has the form

\[ g(\bar{\rho}, \rho) = \sqrt{d_\rho} R^*_\rho. \]

where \( R_\mu \) is an isometry with property \( \bar{\rho} \circ \rho(\xi)R_\rho = R_\rho \xi. \)

\( d_\rho > 0 \) is called the statistics of the sector generated by \( \rho. \)

**Remark:** phases in \( R_\rho \) can be adjusted such that \( \chi_\rho = \pm 1, \) where the sign depends on the sector but + for all sectors which are not selfconjugate [11].

The reassociator is trivial.

This setup is not suitable yet to construct an interesting differential calculus because the spaces \( W_\pi \) are trivial. But this changes when the gauge symmetry of the theory is taken into account.
3.4 Quantum field planes

A gauge symmetry in quantum mechanics is a symmetry which leaves the observables invariant. Here we are interested in gauge symmetries of first kind (i.e. global ones), to start with. According to the principles of gauge theories and of general relativity, these gauge symmetries should become local symmetries (gauge symmetries of second kind), when the local frames are glued together in a covariant fashion.

Let us start from an algebra of observables in a local quantum field theory and its superselection sectors as described in the last section.

We assume that there exists a quantum symmetry \( G \) which encodes the selection rules \((3.31)\). In detail this means that there should exist a semisimple bi-*-algebra \( G \) such that the irreducible representations \( \tau^I \) of \( G \) are in bijective correspondence with the superselection sectors \( \pi_J \), and the selection rules match. That is

\[
\tau^I \otimes \tau^J = \sum_K \tau^K N_{IK}^J
\]  

(3.33)

with the same multiplicities \( N_{IK}^J \) as in \((3.31)\).

This requirement makes sense. By definition, a bi-*-algebra \( G \) is a *-algebra with unit which is equipped with a coproduct \( \Delta \) and a counit \( \epsilon \). They are *-homomorphisms of \( G \) into \( G \otimes G \) resp. \( \mathbb{C} \), and are subject to the relation \((3.17)\). The product \( \otimes \) of representations \( \tau \) of \( G \) is therefore defined by eq.\((3.23)\) and admits a decomposition into irreducibles by semisimplicity.

It was shown by one of us [24] that one can find the necessary intertwiners to make \( G \) into a weak quasitriangular quasi Hopf algebra as described in subsection 3.2. The input for this construction is furnished by the algebra of observables, its morphisms and their intertwiners as described in subsection 3.3. Using this, the amplimorphisms for \( G \) and their intertwiners can be constructed as in subsection 3.2. We write \( \mu_J^G \) for the amplimorphism of \( G \) which is associated with its representation \( \tau^J \), etc.

We take the tensor product of both structures. In the following, letters \( a \) will stand for elements of \( A \) and \( \xi \) for elements of \( G \). By definition, \( \rho^J(a) \in A \), and \( \epsilon(\xi) \in \mathbb{C} \). Let

\[ K = G \otimes A , \]

\[ \text{Rep} - \text{representations } \pi^J(a\xi) = \pi^J(a) \otimes \tau^J(\xi), \]

\[ \pi_0(a\xi) = \epsilon(\xi)\pi_0(a) \]. It acts in \( H_0 \).

\[ W_{\tau^J} \equiv W_J = W_{\tau^J} - \text{carrier space of the representation } \tau^J \text{ of } G \]

The amplimorphisms and their intertwiners become

\[ \mu^J(a\xi) = \mu^J_G(\xi)\rho^J(a) \in \text{End}(W_J) \otimes K, \]

\[ \varphi(\mu^J, \mu^K) = \varphi_G(\mu^J_G, \mu^K_G), \]

\[ \mathcal{R}(\mu^J, \mu^K) = \mathcal{R}_G(\mu^J_G, \mu^K_G)e^*(\rho^K, \rho^J), \]

and similarly for \( g(\mu^J, \mu^J) \).

These quantities satisfy the requirements laid down at the beginning of section 3, because the factors from \( G \) and from \( A \) do.
Now we can introduce the differential calculus. We assume for simplicity that all the representations $\tau^J$ of $\mathcal{G}$ are contained in products of a single fundamental representation $\tau^f = \tau$ of dimension $n_f$. The corresponding endomorphism of $\mathcal{A}$ is denoted by $\rho^f = \rho$.

The algebra $\Omega$ is generated by generators $Z_\alpha$ of degree 0 and $\Theta_\alpha$ of degree 1, $\alpha = 1\ldots n_f$. The multiplication operators $\Theta_\alpha$ will be called \textit{protofields} for reasons that will become clear in the next section.

Their relations are as stated in section 2. Let us spell out some of them. Covariance under $G \otimes A$ yields

$$a \Theta_\alpha = \Theta_\alpha \rho^f(a) \quad (a \in \mathcal{A})$$

$$\xi \Theta_\alpha = \sum \Theta_\beta \tau_{\beta \alpha}(\xi^1) \xi^2 \quad (\xi \in \mathcal{G}),$$

if $\Delta(\xi) = \sum \xi^1 \otimes \xi^2$. The same relations hold with $Z$ substituted for $\Theta$.

Similarly the braid relations (2.14) for $\Theta$ read

$$\Theta_\alpha \Theta_\beta \mathcal{R}^G_{\beta \alpha, \gamma \delta} \epsilon^\gamma(\rho^f, \rho^f) = \gamma \Theta_\alpha \Theta_\delta$$

$$\mathcal{R}^G = (\tau^f \otimes \tau^f \otimes \text{id})(\varphi_{213} R_{12} \varphi^{-1})$$

$\mathcal{R}^G$ is constructed from the $R$-element of $\mathcal{G} \otimes \mathcal{G}$ and the reassociator for $\mathcal{G}$, as in subsection 3.2. If $\mathcal{G}$ is coassociative then $\mathcal{R}^G_{\beta \alpha, \gamma \delta}$ is a number, otherwise it is an element of $\mathcal{G}$. $\epsilon^\gamma(\ldots)$ is an $a$-number. $\gamma$ is a complex number (of modulus 1).

If $\mathcal{R}^G$ has only two distinct eigenvalues then the braid relations for $Z$ look the same as for $\Theta$, with another complex factor $\gamma_0$. The $Z\Theta$-braid relations look the same as the $\Theta\Theta$ braid relations except for an overall $-\text{sign}$.

Let us note that all these relations are very much the same as in a quantum plane, except for the appearance of $a$-numbers in place of $c$-numbers. Note however that $a$-numbers do not commute with generators $\Theta_\alpha$ and $Z_\alpha$. Instead one has the commutation relations (3.34), and the same with $Z$ substituting for $\Theta$.

It is appropriate to call the number $n_f$ of generators $Z_\alpha$ the \textit{dimension} of the quantum space time. It is determined by the symmetry $\mathcal{G}$ and is independent of the dimension of the classical space time in the local Lorentz frames.

4 Differential geometry

Let $A \in \Omega_1$. It substitutes for a $\mathcal{G} \otimes \mathcal{A}$-valued 1-form. The covariant exterior derivative

$$D = d + A$$

transforms covariantly under invertible elements $\Xi \in \mathcal{F} = \Omega_0$. $\Xi$ substitutes for a $\mathcal{G} \otimes \mathcal{A}$-valued function.

$$D \Xi = \Xi D'$$

$$D' = d + A'$$

$$A' = \Xi^{-1} A \Xi + \Xi^{-1} [d, \Xi]$$
The field strength $F \in \Omega_2$ is defined in the customary way,

$$F = D^2 = \{d, A\} + AA.$$  \hfill (4.42)

## 5 Operator product expansions

It is possible to impose additional relations in the algebra $\Omega$. They are the operator product expansions of quantum field theory. They are more restrictive than the braid relations alone.

The algebra generated by $\mathcal{G} \otimes \mathcal{A}$ and the protofields $\Theta_\alpha$ becomes the algebra of field operators of the local field theory with observable algebra $\mathcal{A}$ and gauge symmetry of first kind $\mathcal{G}$, upon imposing operator product expansions for the $\Theta$'s. More precisely $\Phi$ includes besides the quantum fields also the representation operators of the elements $\xi$ of the symmetry $\mathcal{G}$; we use the same symbol $\xi$ for them.

The operator product expansions look as follows. The algebra $\Phi$ is the $\mathcal{A}$-linear span of basis vectors $\Theta^{J}_\alpha$, $\alpha = 1...n_J =$ dimension of representation $\tau^J$ of $\mathcal{G}$. They are $K$-covariant in the sense that

$$a\xi \Theta^J = \Theta^J \mu^J(a\xi).$$  \hfill (5.1)

There exist intertwiners: $C(K^J_1) \in End(W_J \otimes W_I, W_K) \otimes K$ with

$$C(K^J_1) \mu^I \circ \mu^J(\xi) = \mu^K(\xi) C(K^J_1).$$

One may impose orthonormality conditions on them. The operator product expansions are the multiplication law for the basis vectors. They read

$$\Theta^J \Theta^I = \sum_K \Theta^K C(K^J_1).$$  \hfill (5.2)

In general $C(K^J_1)$ are matrices with entries in $\mathcal{G} \otimes \mathcal{A}$. In case $\mathcal{G}$ is coassociative, the entries are $a$-numbers.

These operator product expansions look very much like those in topological field theory\cite{5}. The difference is that the coefficients are $a$-numbers in place of $c$-numbers. In fact they are convergent and partially summed up forms of the standard operator product expansions, as are familiar from conformal field theory. They are valid in any quantum field theory which satisfies the standard assumption of algebraic field theory. They were proven in \cite{24}.

One can use the operator product expansions to construct all the protofields $\Theta^J$ from products of the fundamental $\Theta^J$ from products of the fundamental $\Theta$'s.

The operator product expansions will in general not respect the $\mathbb{Z}$-grading. We assume, however, that they admit a $\mathbb{Z}_2$ symmetry. This leads to a surviving $\mathbb{Z}_2$-grading.

**Theorem 10** (Consistency of operator product expansions) The operator product expansions (5.2) are consistent with the other relations in the algebra $\Omega$, except for the $\mathbb{Z}$-grading.\footnote{It was shown long ago that operator product expansions in conformal field theory are convergent.}
**Proof:** The consistency of the $\Theta \Theta$ braid relations with the operator product expansions is an intrinsic property of the field algebra $\Phi$ which is well known [24].

The consistency of the $Z \Theta$ braid relations with the operator product expansions must be checked. Multiply the difference of the right hand side and left hand side of (5.2) by $Z$ and push the $Z$’s through the $\Theta$’s. The result must be zero. This is indeed true as a result of properties of $\hat{R}$, definition 5.

$$
\mu^L(C(\kappa^J)) \hat{R}(\mu^L, \mu^L) \mu^L(\hat{R}(\mu^J, \mu^L))
= \mu^L(C(\kappa^J)) \hat{R}(\mu^L \circ \mu^J, \mu^L)
= \hat{R}(\mu^K, \mu^L) C(\kappa^J)
$$

(5.3)

**Comments on local fields**

To explain the relation with the standard lore, let us first explain what the local field operators are.

One defines the extended algebras $\Phi(O)$ of field operators which are localized in space time regions $O$. $\Phi(O)$ consists of those elements of $\Phi$ which commute with all observables $a \in A(O')$. Herein $O'$ is the spacelike complement of $O$ as before. The qualification ”extended” refers to the fact that $G \subset \Phi(O)$ for all $O$.

Let us suppose that $\Psi^J = \Theta^J u$ with $u$ a unitary element of $A$. Then one can use eq. (5.1) to compute

$$
a \Psi^J = \Psi^J \rho_u(a),
$$

(5.4)

$$
\rho_u(a) = u^{-1} \rho^*(a) u.
$$

(5.5)

Thus, $\Psi^J \in F(O)$ if $\rho_u$ is localized in $O$, cp. section 3.3.

From (5.2) operator product expansions for localized fields $\Psi^J$ are obtained. One uses the $a$-covariance relation 5.1 to push factors $u$ through $\Theta$. For further details see [24].

Fields $\Psi^J(x)$ at a point can in principle be obtained as limits of fields in $\Phi(O)$ when $O$ shrinks to a point. A simple example is found in [2] where this construction is carried through explicitly.

Let us briefly recall how the Clebsch Gordan kernels $C(\kappa^J)$ are constructed. This will make it clear why one needs the symmetry $G$ to construct field operators which obey operator product expansions on the whole Hilbert space of physical states.

There exist intertwining operators $T^i(\kappa^J)$ for $A$ [10, 18, 24]. They are used to reduce the representation $\pi^I \boxtimes \pi^J$ of $A$. Because the representation $\tau^K$ appears in $\pi^I \boxtimes \pi^J$ with multiplicity $N^I_K$, these intertwiners are labelled by $i = 1...N^I_K$.

Similarly there are Clebsch-Gordan kernels $C^i_G(\kappa^J)$ for $G$. By assumption, the representation $\tau^K$ appears in $\tau^I \boxtimes \tau^J$ with multiplicity $N^I_K$. Therefore these Clebsch Gordan kernels are also labelled by $i = 1...N^I_K$, with the same multiplicities $N^I_K$. One may contract over the index $i$ to obtain

$$
C(\kappa^J) = \sum_i C^i_G(\kappa^J) T^i(\kappa^J)
$$

(5.6)
6 Divorce of the Lorentz group

When one starts from a 4-dimensional field theory, the symmetry group is a compact group. In other cases it substitutes for a compact internal symmetry group. Here we wish to use it as a substitute for the Lorentz group. This requires a comment.

Classical general relativity may be regarded as a gauge theory with gauge group $SO(3, 1)$. Note that the local symmetry in global space time is a global symmetry in local Lorentz frames. Of course, $SO(3, 1)$ is noncompact. But its finite dimensional representations are tensor products of the "left handed" and "right handed" fundamental representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. These representations are both unitary irreducible representations of $SU(2)$, extended to $SL(2, \mathbb{C})$ by analyticity.

In their geometric interpretation of the standard model, Connes and Lott proposed that space time is double sheeted. It is then natural to postulate that left handed matter fields live on one sheet, and right handed matter fields on the other. In this way the Lorentz group gets divorced into two SU(2) groups. At the level of the fundamental theory there is no need for a group which accommodates representations which are tensor products of representations for left handed and for right handed matter. That comes only at the level of effective theories at length scales longer than the distance between the two sheets of space time.

This suggests that we may replace the Lorentz group by a compact group like SU(2) or possibly a deformation of it, i.e. by some weak quasi Hopf algebra.

Since the algebra $A$ is to become part of the local gauge symmetry, there can be two algebras $A^{left}$ and $A^{right}$ attached to the two sheets of space time.

7 Appendix A: Proof of Theorem 9

Consistency of the differential calculus

(1) check that the constant term in the $\partial - Z$ equations gives no new relations.

$$\partial Z \Pi_{\gamma} = Z_{\mu}(g(\mu, \bar{\mu}))\Pi_{\gamma} - \gamma Z\partial Z \mu(\hat{R}(\mu, \bar{\mu}))\Pi_{\gamma} = Z_{\mu}(g(\mu, \bar{\mu})) - \gamma g(\mu, \bar{\mu})\mu(\hat{R}(\mu, \bar{\mu}))\Pi_{\gamma} + \gamma^2 ZZ \hat{R}(\mu, \bar{\mu})\mu(\hat{R}(\mu, \bar{\mu}))\Pi_{\gamma}$$

We derive two formulas for the intertwiners

i) Since $\mu \circ \mu(\xi)\Pi_{\gamma} = \Pi_{\gamma} \mu \circ \mu(\xi)$ it follows that

$$\hat{R}(\mu, \bar{\mu})\mu(\hat{R}(\mu, \bar{\mu}))\Pi_{\gamma} = \hat{R}(\mu \circ \mu, \bar{\mu})\Pi_{\gamma} = \bar{\mu}(\Pi_{\gamma})\hat{R}(\mu \circ \mu, \bar{\mu})$$

ii)

$$\mu(g(\mu, \bar{\mu})) - \gamma g(\mu, \bar{\mu})\mu(\hat{R}(\mu, \bar{\mu})) = \mu(g(\mu, \bar{\mu})) - \gamma g(\mu, \bar{\mu})\mu(\hat{R}(\mu, \bar{\mu}))\hat{R}(\mu, \mu)\hat{R}^{-1}(\mu, \mu) = \mu(g(\mu, \bar{\mu})) - \gamma g(\mu, \bar{\mu})\hat{R}(\mu, \mu)\hat{R}^{-1}(\mu, \mu) = \mu(g(\mu, \bar{\mu}))\hat{R}^{-1}(\mu, \mu)[\hat{R}(\mu, \mu) - \gamma]$$
It follows that $\partial Z \Pi_\gamma = 0$.

(2) The relation $dZ = \Theta + Zd$.

$$dZ = \Theta \partial g^*(\bar{\mu}, \mu) \chi_{\mu}^{-1} Z$$
$$= \Theta g(\mu, \bar{\mu}) \mu(g^*(\bar{\mu}, \mu)) \chi_{\mu}^{-1} - \gamma \Theta Z \partial \hat{\mathcal{R}}(\mu, \bar{\mu}) \mu(g^*(\bar{\mu}, \mu)) \chi_{\mu}^{-1}$$
$$= \Theta + Z \Theta \partial \bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}(\mu, \bar{\mu}) \mu(g^*(\bar{\mu}, \mu)) \chi_{\mu}^{-1}$$

For the intertwiner we obtain

$$\bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}(\mu, \bar{\mu}) (g^*(\bar{\mu}, \mu)) = \hat{\mathcal{R}}(\mu, \bar{\mu} \circ \mu)(g^*(\bar{\mu}, \mu)) = g^*(\bar{\mu}, \mu)$$

so that the Leibniz rule follows.

(3) To prove $d^2 = 0$ we use the following lemma

**Lemma:** $\Pi_\gamma \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) = \bar{\mu} \circ \mu(\Pi_\gamma) \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu)$

**Proof:**

$$\bar{\mu}(\hat{\mathcal{R}}^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) = \bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}^*(\bar{\mu}, \mu) \mu(g^*(\bar{\mu}, \mu))$$
$$= \bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}^*(\bar{\mu}, \mu) \circ \mu(g^*(\bar{\mu}, \mu))$$
$$= \hat{\mathcal{R}}(\mu, \bar{\mu}) \mu(g^*(\bar{\mu}, \mu))$$

$$\bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) g^*(\bar{\mu}, \mu) = \bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}^*(\bar{\mu}, \mu) \mu(g^*(\bar{\mu}, \mu))$$
$$= \bar{\mu}(\hat{\mathcal{R}}(\mu, \bar{\mu})) \hat{\mathcal{R}}^*(\bar{\mu}, \mu) \circ \mu(g^*(\bar{\mu}, \mu))$$
$$= \hat{\mathcal{R}}^*(\bar{\mu}, \mu) \mu(g^*(\bar{\mu}, \mu))$$

From these two formulas we infer

$$\hat{\mathcal{R}}(\bar{\mu}, \bar{\mu}) \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) = \bar{\mu}(\hat{\mathcal{R}}^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \mu(g^*(\bar{\mu}, \mu)) \mu(g^*(\bar{\mu}, \mu))$$

$$= \bar{\mu}(\hat{\mathcal{R}}^*(\bar{\mu}, \mu)) \mu(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu)$$

This proves the lemma.

Now we are prepared to calculate $d^2$.

$$d^2 = \Theta \Theta \partial \bar{\mu} \circ \mu(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \chi_{\mu}^2$$
$$= -\Theta \Theta \partial \bar{\mu}(\hat{\mathcal{R}}^*(\bar{\mu}, \mu)) \mu(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \chi_{\mu}^2$$
$$= -\Theta \Theta \partial \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \chi_{\mu}^2$$
$$= -\Theta \Theta \partial \bar{\mu}(1 - \Pi_\gamma) \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \chi_{\mu}^2$$
$$= -\Theta \Theta \partial \bar{\mu}(1 - \Pi_\gamma) \bar{\mu}(g^*(\bar{\mu}, \mu)) g^*(\bar{\mu}, \mu) \chi_{\mu}^2 = 0$$

because $\Theta \Theta (1 - \Pi_\gamma) = 0$. 

18
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