FULL COUNTING STATISTICS

An elementary derivation of Levitov’s formula

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The field of quantum noise has been rapidly developing in recent years, with the growing possibilities in precision measurements [1], and interest in mesoscopic systems as well as in technological applications of physical effects at the micrometer and nanometer scales.

Of particular interest is the study of the statistics of charge transport between materials coupled through a contact or through a time dependent scatterer. The full statistics of charge transport was studied in a series of works by Levitov et al. [2, 3]. This approach yielded interesting results, and in particular, they where able to express the full counting statistics in terms of a determinant of a single particle operator. Several aspects of Levitov’s formula (4) where discussed in following papers [4, 5].

Our aim in this paper is to present a novel derivation of the original Levitov formula. This is done by proving a trace formula (8), which relates certain traces in Fock space to single particle determinants. Using the present approach we find in addition several generalizations, such as a corresponding formula for Bosons.

1. The full counting statistics

The typical setting is the following: consider particle reservoirs, with given temperatures and chemical potentials, which are separated at time zero, and are evolving by a second quantized hamiltonian $H_0$. At some time the reservoirs are coupled, through a scattering region, and evolve by a new, time dependent hamiltonian. After a time $T$ they are decoupled again and one is interested in the statistics of charge transported from side to side, i.e. to compute $<Q>$, $<Q^2>$ and higher moments. The role of the third moment was recently discussed in [6].

The full statistics of charge transfer may be conveniently represented by the characteristic function of the (charge transport) probability distribution
function, defined by:
\[
\chi(\lambda_1, \ldots; T) = \sum_{\alpha, \beta} P(\alpha(t = 0), \beta(t = T)) e^{iq\sum_i \lambda_i(\beta_i - \alpha_i)}
\]  
(1)

Here the summation is over all states \(\alpha = (\alpha_1, \ldots), \beta = (\beta_1, \ldots)\) labelling the Fock space in the occupation number representation: for fermions these are vectors of zeros and ones and for bosons vectors with integer coefficients, where \(\alpha_i\) is the number of particles occupying the single particle state \(i\) in \(\alpha\). \(P(\alpha(t = 0), \beta(t = T))\) is the probability that we started in state \(\alpha\) at time 0 and finished in a state \(\beta\) at time \(T\). Thus terms \((\alpha_i - \beta_i)\) appearing in the exponent are just the change in the number of particles occupying the single particle state \(i\). And the parameters \(\lambda_i\) are introduced in the standard manner to calculate different moments. By taking derivatives of \(\chi\) with respect to \(\lambda_i\), one can calculate arbitrary moments of the charge accumulation in state \(i\). For example,
\[
< Q_i > = -i\partial_{\lambda_i} \log \chi|_{\lambda_1, \ldots = 0}
\]  
(2)

And
\[
(\Delta Q_i)^2 = < Q_i^2 > - < Q_i >^2 = -\partial_{\lambda_i}^2 \log \chi|_{\lambda_1, \ldots = 0}.
\]  
(3)

It is also possible to compute the moments of the charge which is transferred to a particular reservoir by taking the derivative with respect to \(\lambda\) after setting \(\lambda_i = \lambda\) for all states \(i\) belonging to the desired reservoir. \(\chi(\lambda)\) may be interpreted as the reaction of the system to coupling with a classical field \(\lambda\) which measures the number of electrons on each side.

For adiabatic change, and short scattering time Levitov et al [2], obtained the following expression for \(\chi\):
\[
\chi(\lambda) = \det(1 + n(S^e^{i\lambda}S e^{-i\lambda} - 1))
\]  
(4)

Where \(n\) is the occupation number operator and \(S\) is the scattering matrix. As was remarked [2], this expression requires careful understanding and regularization. In the following we derive this formula in a new manner which, we hope will allow a convenient way to address these issues.

In order to proceed we first write \(\chi\) as a trace in Fock space:
\[
\chi(\lambda, T) = \sum_{\alpha, \beta} \langle \alpha | \rho_0 | \alpha \rangle | \beta \rangle | \beta \rangle \langle \alpha | U^{\dagger} e^{iq\sum_i \lambda_i(\beta_i - \alpha_i)}
\]
\[= \text{Tr}(\rho_0 U^{\dagger} e^{iq\sum_i \lambda_i a_i^\dagger a_i} U e^{-iq\sum_i \lambda_i a_i a_i})
\]  
(5)

Where \(\rho_0\) is the density matrix at the initial time \((t = 0)\), \(U\) is the evolution (in Fock space) from time \(t = 0\) to time \(t = T\) and \(a_i^\dagger, a_i\) are the creation and
annihilation operators for a given one particle state $i$. Here it is assumed that the occupation number basis is chosen such that the initial time density matrix $\rho_0$ is diagonal in it, which implies that the states $\alpha$ are eigenstates of the initial Hamiltonian, and measurement of charge in a specific state is meaningful.

Next, we define the second quantized version of a single particle operator $A$ (i.e. an operator on the single particle Hilbert space) to be the Fock space operator:

$$\Gamma(A) = \sum_i <i|A|j> a_i^\dagger a_j.$$ (6)

Then (5) can be written as:

$$\chi(\lambda, T) = \text{Tr}(\rho_0 e^{i[q\Gamma(\lambda) U e^{-iq\Gamma(\lambda)}]}).$$ (7)

Here $\lambda$ is the matrix $\text{diag}(\lambda_1, \lambda_2, ...)$. To handle this kind of expressions (and to obtain Levitov’s formula) we prove in the following section a trace formula.

2. A trace formula

In this section we prove the following:

$$\text{Tr}(e^{\Gamma(A)} e^{\Gamma(B)}) = \det(1 - \xi e^A e^B)^{-\xi}$$ (8)

Where $\xi = 1$ for bosons and $\xi = -1$ for fermions (i.e. the creation and annihilation operators satisfy $a_j a_i^\dagger - \xi a_i^\dagger a_j = \delta_{ij}$).

We prove this result for the finite dimensional Hilbert space case, and avoid at this point questions regarding the limit of infinite number of states, to be addressed elsewhere [7].

Proof:

For an $N$ dimensional single particle Hilbert space $\Gamma$ is a representation of the usual Lie algebra of matrices $gl(N)$. Indeed, substituting the definition (6), together with the relations obeyed by the creation and annihilation operators it is straightforward to check that

$$[\Gamma(A), \Gamma(B)] = \Gamma([A, B])$$ (9)

is true for bosons and for fermions. By Baker Campbell Hausdorff there exists a matrix $C$ such that $e^A e^B = e^C$. $C$ is an element of $gl(N)$ and is given by a series of commutators, since $\Gamma$ is a representation, it holds that

$$e^A e^B = e^C \rightarrow e^{\Gamma(A)} e^{\Gamma(B)} = e^{\Gamma(C)}.$$ (10)
Now let us evaluate $\text{Tr}(e^{\Gamma(C)})$. Any matrix $C$ can be written in a basis in which it is of the form $\text{diag}(\mu_1, \ldots, \mu_n) + K$ where $K$ is an upper triangular, thus we have
\begin{equation}
\text{Tr}(e^{\Gamma(C)}) = \text{Tr}(e^{\Gamma(\text{diag}(\mu_1, \ldots, \mu_n))} + \Gamma(K)) = \text{Tr}(e^{\Gamma(\text{diag}(\mu_1, \ldots, \mu_n))}) = \prod_i e^{\mu_i a_i^\dagger a_i} = \prod_i (1 - \xi e^{\mu_i})^{-\xi} = \det(1 - \xi e^{C})^{-\xi}
\end{equation}
(One may also think of $\text{Tr}(e^{\Gamma(C)})$ as the partition function of a system with Hamiltonian $-C$ at temperature $k_B T = 1$). From this equation (8) follows:
\begin{equation}
\text{Tr}(e^{\Gamma(A)} e^{\Gamma(B)}) = \text{Tr}(e^{\Gamma(C)}) = \det(1 - \xi e^{C})^{-\xi} = \det(1 - \xi e^{A} e^{B})^{-\xi}
\end{equation}
We remark at this point that this relation can immediately be generalized in the same way to products of more than two operators.

- Let us illuminate our identity with a trivial example: Let $H$ be an $N$-dimensional Hilbert space, and choose $A = B = 0$. Then the dimension of the appropriate Fock space is given by
\begin{equation}
\text{Tr}(I) = \text{Tr}(e^{\Gamma(0)} e^{\Gamma(0)}) = \det(1 - \xi)^{-\xi} = \begin{cases} 2^N & \text{Femions} \\ \infty & \text{Bosons} \end{cases}
\end{equation}
as it should be.

3. **Levitov's formula**

We now turn to give a novel derivation of Levitov's result for the full counting statistics. In the framework of non interacting fermions the evolution $U$ in the expression (5) for $\chi$ is just the Fock space implementation of the single particle evolution $U$. That means that $U^\dagger \Gamma(\lambda) U = \Gamma(U^\dagger \lambda U)$ so that by the trace formula (8) for 3 operators, we immediately have
\begin{equation}
\chi(\lambda, T) = \text{Tr}\left(\frac{e^{-\beta G_0}}{Z} e^{iq \Gamma(U^\dagger \lambda U)} e^{-iq \Gamma(\lambda)}\right)
\end{equation}
\begin{equation}
\frac{1}{Z} \det(1 + e^{-\beta H_0} (U^\dagger e^{iq \lambda} U e^{-iq \lambda})) = \det(1 + n(U^\dagger e^{iq \lambda} U e^{-iq \lambda} - 1))
\end{equation}
Where $Z = \det(1 + e^{-\beta H_0})$ and $n$ is the occupation number operator $\frac{e^{-\beta H_0}}{1 + e^{-\beta H_0}}$ at the initial time. We note that the result (13) should be viewed as the general expression for the counting statistics of noninteracting fermions, at any given time, and without any approximation. And may be a good start for studying different limits of the problem, as well as regularization difficulties.

Finally, if the scattering time is very small compared with the entire evolution, then one may describe the problem in terms of dynamical scattering operators, $S = \lim_{t \to \infty} e^{i H_0 t} U(t, -t) e^{i H_0 t}$ where $H_0$ is the initial free
evolution. Using the fact that $\lambda$ commutes with $H_0$, one obtains in the limit of $T \to \infty$:

$$
\chi(\lambda) = \det(1 + n(S^t e^{i\lambda S} e^{-i\lambda} - 1))
$$

(14)

Which is Levitov's result (4), as promised.

We now add a few remarks:

1. Convergence and regularization: First we note that as long as we assume that $\rho$ exists and has trace 1, then trace of $\rho$ times a bounded operator is also finite, so that $\chi$ is well defined. However, problems might arise when taking the thermodynamic limit. Taking the infinite volume limit may cause the Fock space density matrix to be ill defined (i.e. it cannot be normalized to trace 1), however, expectation values obtained using it may still have meaning.

If one uses the scattering matrix as in (4), regularization of the determinant is needed, since in this case one uses the static scattering matrix as an approximation for the true evolution. Indeed, in the limit $T \to \infty$ arbitrarily large charges can pass from side to side \(^1\), so that the information about the length of the time interval has to be put in by hand \([2]\). The equation (13), however, can be shown to be well defined even in the thermodynamic limit, for a finite time interval \([7]\).

2. Bosons: It is now straightforward to derive an analogous formula for the full counting statistics of bosons. The result is simply:

$$
\chi_B(\lambda, T) = \frac{1}{\det(1 - n_B(U^\dagger e^{i\lambda U} e^{-i\lambda} - 1))}.
$$

(15)

Where $n_B$ is the occupation number operator for bosons.

3. Rate of charge accumulation: Here we give an example of how one may compute the rate of charge accumulation in a box $A$. We choose a box in space, which can be described by a projection $P_A$ in the single particle Hilbert space (with matrix elements $\langle x | P_A | x' \rangle = \delta(x - x')$ if $x$ is in the box, and zero otherwise). By setting $\lambda = 0$ for all states that are outside the box $A$, one finds

$$
\dot{Q}_A = -i\partial_t \partial_\lambda \log \chi(\lambda, t)|_{\lambda=0} =
$$

$$
q \partial_t \text{Tr}(n(U^\dagger P_A U - P_A)) = q \text{Tr}(U^\dagger P_A U + U^\dagger P_A U) =
$$

$$
q \text{Re} < U \dot{U}^\dagger P_A >_t
$$

The angular brackets describe averaging over the distribution at the time of measurement $t$. Charge accumulation is equivalent to current if the box $A$ is

\(^1\)To see this we note that the first moment of (4), which is the transported charge, diverges as time goes to infinity if there is a bias between the reservoirs.
connected via just one contact to the other reservoirs. This equation should be compared to the formula for the current in terms of scattering matrices [8, 9, 10, 11] which is of fundamental interest in the field of quantum pumps.

4. summary

To conclude, we presented a novel derivation of Levitov’s determinant formula for the full counting statistics of charge transfer. This was done by introducing a trace formula (8) which is suitable for translating problems of non-interacting particles from Fock space to the single particle Hilbert space. The derivation is general enough to allow consideration of new problems of counting statistics, in particular, further problems involving bosons, or measurement of other operators then charge. We hope that some properties of the determinant (13) under various limits, such as adiabatic and thermodynamic limits will now be easier to address.

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