RIBBON 2-KNOT GROUPS OF COXETER TYPE

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Dedicated to the memory of Stephen Pride

Abstract. Wirtinger presentations of deficiency 1 appear in the context of knots, long virtual knots, and ribbon 2-knots. They are encoded by (word) labeled oriented trees and, for that reason, are also called LOT presentations. These presentations are a well known and important testing ground for the validity (or failure) of Whitehead’s asphericity conjecture. In this paper we define LOTs of Coxeter type and show that for every given \( n \) there exists a (prime) LOT of Coxeter type with group of rank \( n \). We also show that label separated Coxeter LOTs are aspherical.

1. Introduction

Wirtinger presentations of deficiency 1 appear in the context of knots, long virtual knots, and ribbon 2-knots [9]. They are encoded by (word) labeled oriented trees and, for that reason, are also called LOT presentations. Adding a generator to the set of relators in a Wirtinger presentation \( P \) gives a balanced presentation of the trivial group. Thus the associated 2-complex \( K(P) \) is a subcomplex of an aspherical (in fact contractible) 2-complex. Wirtinger presentations are a well known and important testing ground for the validity (or failure) of Whitehead’s asphericity conjecture, which states that a subcomplex of an aspherical 2-complex is aspherical. For more on the Whitehead conjecture see Bogley [3], Berrick-Hillman [1], and Rosebrock [18].

If \( P \) is a Wirtinger presentation and the group \( G(P) \) defined by \( P \) is a 1-relator group, then \( G(P) \) admits a 2-generator 1-relator presentation \( P' \) and the 2-complex \( K(P') \) is aspherical. Since \( K(P') \) and \( K(P) \) have the same Euler characteristic and the same fundamental group, it follows (using Schanuel’s lemma and Kaplansky’s theorem that states that finitely generated free \( \mathbb{Z}G \)-modules are Hopfian) that \( K(P) \) is also aspherical. Thus, when investigating the asphericity of \( K(P) \) for a given Wirtinger presentation \( P \), the first thing to ask is if \( G(P) \) is a 1-relator group.

Many knots have 2-generator 1-relator knot groups. Prime knots whose groups need more than 2 generators were known to Crowell and Fox in 1963. See Bleiler [2] for a good discussion on this topic. As one example, Crowell and Fox consider a certain prime 9 crossing knot, show that its Wirtinger presentation simplifies to

\[
P = \langle x, y, z \mid y^{-1}xy^{-1}y = x^{-1}zx^{-1}zx^{-1}x, x^{-1}zx^{-1}x = y^{-1}yz^{-1}y \rangle,
\]

and that the length of the chain of elementary ideals for this knot group is 2. It follows that the rank (=minimal number of generators) of \( G(P) \) is greater than 2 and therefore equal
to 3. This can also be seen without the use of elementary ideals. We have an epimorphism
\[ G(P) \to \Delta(3,3,3) = \langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (xz)^3, (yz)^3 \rangle \]
sending \( x \to x, y \to y, z \to z \). Since the rank of the Euclidian triangle group \( \Delta(3,3,3) \) is 3
(see Klimento-Sakuma [14]) we have rank \((G(P)) = 3\).

This example is the motivation for this article. It is much easier to construct high rank ribbon 2-knot groups than classical knot groups, because we do not have to verify that a given Wirtinger presentation can be read off a knot projection (a 4-regular planar graph). Below we define (word) labeled oriented trees of Coxeter type and show that given a Coxeter group \( W \), there exists a Coxeter type LOT group that maps onto \( W \). Using this we give examples of LOT groups of arbitrarily high rank.

In the second part of the paper we investigate the question of asphericity of LOTs of Coxeter type. We show that label separated LOTs of Coxeter type are aspherical. It turns out that the study of asphericity can be translated into questions concerning free subgroups of 1-relator LOT groups of dihedral type.

2. Groups defined by graphs

A (word) labeled oriented graph (LOG) is an oriented finite graph \( \Gamma \) on vertices \( x \) and edges \( e \), where each oriented edge is labeled by a word in \( x^\pm 1 \). Associated with a LOG \( \Gamma \) is the presentation
\[ P(\Gamma) = \langle x \mid r = \{ r_e \mid e \in e \} \rangle, \]
where \( r_e = xw(wy)^{-1} \) in case \( e = (x \to y) \) is the edge of \( \Gamma \) starting at \( x \), ending at \( y \), and labeled with the word \( w \) on letters in \( x^\pm 1 \). We denote by \( K(\Gamma) \) and \( G(\Gamma) \) the standard 2-complex and the group defined by \( P(\Gamma) \), respectively. The case where \( \Gamma \) is a tree, now called a (word) labeled oriented tree (LOT), is special. It is known that the groups \( G(\Gamma) \), where \( \Gamma \) is a LOT, are precisely the ribbon 2-knot groups (see Yajima [22], Howie [12], and also Hillman [13], section 1.7), since in that case \( G(\Gamma) \) is a group of weight 1 (normally generated by a single element, in fact by each generator) that has a deficiency 1 presentation \( P(\Gamma) \). The 2-complexes \( K(\Gamma) \), \( \Gamma \) a LOT, are of central importance to Whitehead’s asphericity conjecture, since adding a generator to the set of relators in \( P(\Gamma) \) gives a balanced presentation of the trivial group. So \( K(\Gamma) \) is a subcomplex of a 2-dimensional contractible complex. A question that has been open for a long time asks if \( K(\Gamma) \) is aspherical, i.e. \( \pi_2(K(\Gamma)) = 0 \). See Bogley [3], Berrick-Hillman [1], Rosebrock [18].

Let \( \Upsilon \) be a simplicial graph on vertices \( x \), and suppose edges \( e \) are labeled with integers \( m_e \geq 2 \). Define
\[ P(\Upsilon) = \langle x^2, x \in x, (xy)^{m_e} \text{ if } e = \{x,y\} \text{ is an edge} \rangle. \]
The group \( W = W(\Upsilon) \) defined by this presentation is called a Coxeter group. Let \( K = K(\Upsilon) \) be the 2-complex associated with it. Consider the universal covering \( \tilde{K}(\Upsilon) \). The 1-skeleton of \( \tilde{K}(\Upsilon) \) is the Cayley graph for \((W,x)\). All edges in \( \tilde{K}(\Upsilon) \) are double edges:
For every \( g \in W \) we have an edge \( (g, x) \) connecting \( g \) to \( gx \), and an edge \( (gx, x) \) connecting \( gx \) to \( g \). Note that a double edge pair bounds two 2-cells in \( K(\Upsilon) \), coming from the relator \( x^2 \). We remove one and collapse the other one to an edge. This turns each double edge into a single unoriented edge. Every relator \((xy)^m \) gives rise to \( 2m \) 2-cells with the same boundary. We remove all but one from this set. The 2-complex obtained in this fashion we denote by \( \Sigma(\Upsilon) \). It is the 2-skeleton of the Coxeter complex \( \Sigma(\Upsilon) \). See Proposition 7.3.4 in Davis [5]. Under certain conditions, for example when \( \Upsilon \) is a tree, the Coxeter complex is 2-dimensional: \( \Sigma(\Upsilon) = \Sigma^{(2)}(\Upsilon) \). See [5] Example 7.4.2.

Proposition 2.1. Let \( \Upsilon \) be a tree with associated Coxeter group \( W(\Upsilon) \). Then

1. For every edge \( e = \{x, y\} \) of \( \Upsilon \) we have a 2-cell \( \kappa_e \) in \( \Sigma(\Upsilon) \) attached along a \( 2m_e \)-gon whose edge labels read \( (xy)^{m_e} \).
2. \( \Sigma(\Upsilon) \) is the union of the 2-cells \( w\kappa_e \), \( e \in \{ \text{edges of } \Upsilon \} \), \( w \in W(\Upsilon) \). Furthermore, if \( w_1\kappa_{e_1} \cap w_2\kappa_{e_2} \neq \emptyset \) then \( e_1 \cap e_2 \neq \emptyset \); if \( x = e_1 \cap e_2 \), then the edge \( w_1\kappa_{e_1} \cap w_2\kappa_{e_2} \) carries the label \( x \).
3. \( \Sigma(\Upsilon) \) is a tree of 2-cells: If we connect the barycenters of the 2-cells with the barycenters of their boundary edges we obtain a tree. In particular, if \( M \) is a finite connected union of Coxeter 2-cells \( w_i\kappa_{e_i} \) in \( \Sigma(\Upsilon) \), then there exists a 2-cell \( w\kappa_e \) in \( M \) that intersects with the rest of \( M \) in a single edge.

Proof. The statements (1) and (2) are clear from the construction of \( \Upsilon \). For an edge \( e = \{x, y\} \) let \( P(e) = \langle x, y \mid x^2, y^2, (xy)^{m_e} \rangle \). Let \( D_{m_e} \) be the dihedral group defined by \( P(e) \). Since \( \Upsilon \) is a tree, \( W(\Upsilon) \) is an amalgamated product of the \( D_{m_e} \). The associated Bass-Serre tree can be seen inside the Coxeter complex \( \Sigma(\Upsilon) \). The vertices of that tree are the barycenters of the 2-cells and 1-cells, and the edges connect barycenters of 2-cells to the barycenters of the 1-cells in the boundary of that 2-cell. We can think of \( \Sigma(\Upsilon) \) as a tree of Coxeter 2-cells. An example is shown in Figure 1.

Suppose \( L = \bigcup_{i=0}^k D_i \) is a union of 2-cells. Let \( d_i \) be the barycenter of \( D_i \). Let \( d_p \) be a vertex in the Bass-Serre tree furthest away from \( d_0 \), \( p \in \{0, \ldots, k\} \). Consider a geodesic from \( d_0 \) to \( d_p \) and let \( d_q \) be the barycenter that is encountered just before getting to \( d_p \) when traveling along the geodesic. Then \( \bigcup_{i \neq p} D_i \cap D_p = D_q \cap D_p \), which is a single edge. \( \Box \)

The graph \( \Upsilon \) also defines an Artin presentation. Denote by \( \text{prod}(x, y, k) = x y x y x \ldots \), where the length of the word is \( k \geq 2 \). Note that \( \text{prod}(x, y, k) \) ends with \( x \) in case \( k \) is odd and it ends with \( y \) if \( k \) is even. Define

\[
P_A(\Upsilon) = \langle x \mid \text{prod}(x, y, m_e) = \text{prod}(y, x, m_e), \text{if } e = \{x, y\} \text{ is an edge in } \Upsilon \rangle
\]

and let \( A(\Upsilon) \) be the group defined by \( \Upsilon \).

Definition 2.2. Let \( \Gamma \) be a LOT with vertex set \( x \). We say \( \Gamma \) is of Coxeter type if for every edge \( e = \{x \xrightarrow{w} y\} \) the word \( w \) contains letters \( z \neq x, y \) only with even (positive or negative) exponent.
Figure 1. The Coxeter complex $\Sigma(\Upsilon)$ for $\Upsilon = x^3 - y^3 - z$. It is a tree of Coxeter cells.

Lemma 2.3. Let $\Gamma$ be a LOT of Coxeter type and $e = (x \xrightarrow{w} y)$ an edge. Then the relator $r_e = xw(wy)^{-1}$ reduces (up to cyclic permutation) to $\bar{r}_e = (yx)^{m_e}$, $m_e \geq 1$ and odd, in $\langle x \mid x^2, x \in x \rangle$.

Proof. The word $w$ reduced to an alternating word $\bar{w}$ in the letters $x$ and $y$. There are four cases to consider:

1. $\bar{w}$ starts with $x$ and has even length;
2. $\bar{w}$ starts with $x$ and has odd length;
3. $\bar{w}$ starts with $y$ and has even length;
4. $\bar{w}$ starts with $y$ and has odd length;

In case (1) we have $\bar{w} = xyxy$, say. So $x(xyxy)y(xyxy) = xxyxyxyxyxyxy = xy$. In case (2) we have $\bar{w} = xyxyx$, say. So $x(xyxyx)y(xyxyx) = xxyxyxyxyxyxy = (yx)^5$. In case (3) we have $\bar{w} = yxyx$, say. So $x(yxyx)y(yxyx) = xy$. In case (4) we have $\bar{w} = yxyxy$, say. So $x(yxyxy)y(xyxy) = (xy)^5$. □

Let $\Gamma$ be a LOT of Coxeter type. Define a tree $\Upsilon$ in the following way: Erase orientations in $\Gamma$ and if $e = (x \xrightarrow{w} y)$ is an edge and the LOT relator $r_e = (yx)^{m_e}$ (up to cyclic permutation) in $\langle x \mid x^2, x \in x \rangle$, then label the (unoriented) edge $e$ by $m_e$. We have a map $P(\Gamma) \to P(\Upsilon)$ sending $x$ to $x$ which induces a group epimorphism $G(\Gamma) \to W(\Upsilon)$. This process can be reversed. A LOT is prime if it does not contain a proper subLOT (proper means: not the entire LOT and not a single vertex).

Lemma 2.4. Let $\Upsilon$ be a Coxeter tree where all $m_e \geq 1$ and odd. Then there exists a (prime) LOT of Coxeter type $\Gamma$ such that the process just described produces $\Upsilon$ from $\Gamma$. In particular $G(\Gamma)$ maps onto $W(\Upsilon)$.

Proof. Suppose $e = \{x, y\}$ is an edge in $\Upsilon$. Orient it in from $x$ to $y$. Let $w$ be a word in $x$ that reduces to $(yx)^{m_e-1}$ in $\langle x \mid x^2, x \in x \rangle$. Let $e = (x \xrightarrow{w} y)$ be the corresponding edge in $\Gamma$. □
Remark 1. Let $\Upsilon$ be a Coxeter graph. If $e = \{x, y\}$ is an edge in $\Upsilon$, orient it from $x$ to $y$. If $m_e$ is odd let $e = (x \xrightarrow{w_e} y)$ where $w_e = (yx)^{m_e-1}$. If $m_e$ is even remove the interior of the edge $e$ from $\Upsilon$ and attach a loop $e_x$ at the vertex $x$. Now $e_x = (x \xrightarrow{w_e} x)$ where $w_e = \prod_{e \subseteq \Upsilon} y, x, m_e - 1$. Denote the labeled oriented graph we obtain by $\Gamma$. Observe that $P(\Gamma) = P_A(\Upsilon)$. Thus all Artin groups are LOG groups. This observation arose from a discussion with Gabriel Minian.

Example 1. Let $\Upsilon$ be a triangle with 3 vertices $x, y, z$ and edges $x\xrightarrow{3} y, y\xrightarrow{3} z, x\xrightarrow{2} z$. We get a LOG $\Gamma$ with edges $x \xrightarrow{y} y, y \xrightarrow{z} z, x \xrightarrow{z} x$. Note that $G(\Gamma) = A(2, 3, 3)$ is an Artin group of spherical type with associated Coxeter group $W(2, 3, 3)$, the symmetric group $S_4$. It is known (Mulholland-Rolfsen [16]) that the commutator subgroup of $A(2, 3, 3)$ is finitely generated and perfect. So $G(\Gamma) = A(2, 3, 3)$ is not locally indicable. Whether LOT groups are locally indicable or not is (to our knowledge) an open problem.

3. LOT groups of high rank

Theorem 3.1. (Carette-Weidmann [6]) Let $\Upsilon$ be a graph with $n$ vertices and assume that all the $m_e \geq 6 \cdot 2^n$. Then the rank of $W(\Upsilon)$ is $n$.

Theorem 3.2. Let $W = W(\Upsilon)$ be a Coxeter group such that $W_{ab} = \mathbb{Z}_2$. There exists a (prime) labeled oriented tree $\Gamma$ of Coxeter type so that $G = G(\Gamma)$ maps onto $W$.

Proof. Since $W_{ab} = \mathbb{Z}_2$ the Coxeter graph $\Upsilon$ is connected and contains a maximal tree $\Upsilon_0$ in which all $m_e$ are odd. Then $\Upsilon$ and $\Upsilon_0$ have the same set of vertices and we have an epimorphism $W(\Upsilon_0) \to W(\Upsilon)$. From Lemma 2.4 we know that there is a (prime) LOT $\Gamma$ of Coxeter type so that $G(\Gamma)$ maps onto $W(\Upsilon_0)$. \qed

Corollary 3.3. For any given $n$ there exists a prime labeled oriented tree $\Gamma$ of Coxeter type with $n$ vertices so that $G(\Gamma)$ has rank $n$. In particular if $n \geq 3$ then $G(\Gamma)$ is not a 1-relator group.

Proof. This follows from Theorem 3.2 together with the Carette-Weidmann Theorem 3.1. \qed

Example 2. Let $\Gamma$ be the prime LOT $x \xrightarrow{yz^2x^2} y \xrightarrow{z^2} z$. Note that $G(\Gamma)$ maps onto the amalgamated product $D_3 *_{\mathbb{Z}_2} D_3$ which can not be generated by two elements. Thus the rank of $G(\Gamma)$ is 3 and it follows that this LOT group is not a 1-relator group. If we drop the $z^2$ from the first edge word and $x^2$ from the second edge word we obtain a LOT $\Gamma_0$ that is not prime. In fact $G(\Gamma_0) = A(\Upsilon)$, where $\Upsilon$ is the Coxeter tree $x \xrightarrow{3} y \xrightarrow{3} z$.

Remark 2. Note that if $\Gamma$ is a LOT of Coxeter type and $\Upsilon$ is the associated Coxeter tree, then $W(\Upsilon)$ is an amalgamated product of dihedral groups. A direct way to obtain upper bounds for the rank of $W(\Upsilon)$ without the full force of Theorem 3.1 is via Weidmann [21]. For example the LOT shown in Example 2 does not meet the conditions of the theorem.
Remark 3. A reorientation of a LOT is obtained when changing signs on the exponents of letters that occur in the edge words. Note that reorienting has no effect on the quotient $W(\Upsilon)$. Thus if $rk(G(\Gamma)) = rk(W(\Upsilon))$, then this equation holds also for all reorientations of $\Gamma$.

4. Largeness

A group is large if it has a subgroup of finite index that has a free quotient of rank $\geq 2$. Large groups of deficiency 1 are studied in Button [4]. A list of properties can also be found in that paper. If $G$ is large then

1. $G$ contains free subgroups of rank $\geq 2$;
2. $G$ is SQ-universal (every countable group is the subgroup of some quotient);
3. $G$ has finite index subgroups with arbitrarily large first Betti number;
4. $G$ has uniformly exponential word growth;
5. $G$ has subgroup growth of strict type $n^n$ (which is the largest possible growth for finitely generated groups);
6. the word problem for $G$ is solvable strongly generically in linear time.

Theorem 4.1. Let $\Gamma$ be a LOT of Coxeter type on at least 3 vertices. Let $\Upsilon$ be the corresponding Coxeter tree and assume all $m_e \geq 3$. Then $G(\Gamma)$ is large.

Proof. The conditions imply that $W(\Upsilon)$ is an infinite group that is a finite tree where the vertex groups are dihedral groups of type $m \geq 3$ and the edge groups are $\mathbb{Z}_2$. Thus $W(\Upsilon)$ contains a free subgroup $F$ of rank $\geq 2$ of finite index (see Serre’s book Trees [20], Proposition 11, page 120). Let $H$ be the preimage of $F$ in $G(\Gamma)$. Then $H$ is a subgroup of $G(\Gamma)$ of finite index that maps onto $F$. It follows that $G(\Gamma)$ is large. $\square$

Example 3. As in Example 2 let $\Gamma$ be the prime LOT $x \xrightarrow{y^2} y \xrightarrow{z^2} z$. Then $W(\Upsilon) = D_3 \ast_{\mathbb{Z}_2} D_3$. Let $\Delta(3,3,2)$ be the spherical triangle group (it is the symmetric group $S_4$) defined by $Q = \langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (yz)^3, (xz)^2 \rangle$. We have an epimorphism $W(\Upsilon) \to \Delta(3,3,2)$ and we claim that the kernel $V$ is free of rank $\geq 2$. Indeed, since both $D_3$’s of $W(\Upsilon)$ are also subgroups of $\Delta(3,3,2)$, it follows that $V$ intersects both $D_3$’s trivially and it follows that $V$ acts freely of the Bass-Serre tree $T$ for $W(\Upsilon) = D_3 \ast_{\mathbb{Z}_2} D_3$, and hence is free. Note that the valency of every vertex in $T$ is equal to 3 (since the index of $\mathbb{Z}_2$ in the $D_3$’s is 3), and so $V$ can not be cyclic. Here is why: Note that $V = \pi_1(X)$, where $X = T/V$ is a finite graph where every vertex has valency 3. So $v(X) = \frac{2e(X)}{3}$ and we obtain $\chi(X) = v(X) - e(X) = \frac{2e(X)}{3} - e(X) < 0$. Thus $\dim H_0(X) - \dim H_1(X) = 1 - \dim H_1(X) = \chi(X) < 0$. So $\dim H_1(X) > 1$ and hence $\dim V_{ab} > 1$. One can also check directly that $(xz)^2$ and $x(xz)^2x^{-1} = (zx)^2$ generate a free subgroup of $V$ of rank 2.
5. THE QUESTION OF ASPHERICITY

Let Γ be a labeled oriented tree of Coxeter type and let Υ be the related Coxeter graph. Let \( \hat{K}(Γ) \) be the normal covering space with fundamental group the kernel of the epimorphism \( G(Γ) \to W(Υ) \). We will analyze the structure of \( \hat{K}(Γ) \). We have maps

\[ \hat{K}(Γ) \to \hat{K}(Υ) \to Σ(Υ), \]

and note that \( \hat{K}(Γ) \) and \( \hat{K}(Υ) \) have the same 1-skeleton. Let \( e = (x \to y) \) be an edge in Γ. Let \( P_e = \langle x_e | r_e \rangle \), where \( x_e \subseteq x \) is the subset of the vertices of Γ that occur in \( r_e \). Let \( z = x_e - \{x, y\} \). Then \( P_e = \langle x, y, z | xw = wy \rangle \). The complex \( K(P_e) \) is a subcomplex of \( K(Γ) \). Consider the preimage of \( K(P_e) \) under the covering projection \( \hat{K}(Γ) \to \hat{K}(Υ) \). It is a union of finite subcomplexes \( w\hat{K}_e \), \( w \in W(Υ) \), that we will now describe in detail. The 1-skeleton of \( \hat{K}_e \) is a 2\( m_e \)-gon with double edges labeled in an alternating way by \( x \) and \( y \). At each of the 2\( m_e \) vertices we have a double edge for every \( z \in z \). The situation is depicted in Figure 2. We have 2\( m_e \) 2-cells, attached along the loop with label \( r_e \), starting at every vertex. The dihedral group \( D_{m_e} \), the stabilizer of the Coxeter cell \( κ_e \) in \( Σ(Υ) \), acts freely on \( \hat{L}_e \). It is convenient to replace \( \hat{K}_e \) with a complex with a single \( D_{m_e} \) orbit of vertices. Let \( \hat{L}_e \) be the 2 complex obtained from \( \hat{K}_e \) in the following way: At every vertex collapse one of the \( z \)-edges from the \( z \)-double edge, \( z \in z \). The complex \( \hat{L}_e \) is homotopy equivalent to \( \hat{K}_e \). The 1-skeleton of \( \hat{L}_e \) is an 2\( m_e \)-gon with double edges labeled in an alternating way by \( x \) and \( y \). At each of the 2\( m_e \) vertices we have a loop for every \( z \in z \). Let \( \hat{r}_e \) be the word obtained from \( r_e \) by replacing every \( z^p \), \( z \in z \), by \( z^2 \). Let \( \hat{P}_e = \langle x, y, z | \hat{r}_e \rangle \). Note that the dihedral group \( D_{m_e} \) acts freely on \( \hat{L}_e \) and we have a covering map \( \hat{L}_e \to \hat{L}_e/D_{m_e} = K(\hat{P}_e) \).

**Lemma 5.1.** The 2-complex \( \hat{K}_e \) is aspherical.

**Proof.** The complex \( K(\hat{P}_e) \) is aspherical because \( \hat{P}_e \) is a 1-relator presentation for which the relator is not a proper power. Thus \( \hat{L}_e \) is aspherical, being a covering space of \( K(\hat{P}_e) \). Since \( \hat{K}_e \) is homotopy equivalent to \( \hat{L}_e \), it follows that \( \hat{K}_e \) is aspherical. \( \square \)
An $x$-side of $\bar{K}_e$ consists of a double edge with label $x$ together with all the double edges connected to the two vertices of the $x$-double edge. A $y$-side is defined in the same way. See Figure 2, where the blue part on the left shows a $y$-side. Note that $\bar{K}_e$ has $m_e$ $x$-sides and $m_e$ $y$-sides. We refer to these as the sides of $\bar{K}_e$. We say $\bar{K}_e$ is side injective if the inclusion induced map $\pi_1(S) \to \pi_1(\bar{K}_e)$ is injective for every side $S$. An $x$-side in $\hat{L}_e$ is the image of an $x$-side under $\bar{K}_e \to \hat{L}_e$, etc.

**Lemma 5.2.** Assume $m \geq 3$. Then $\bar{K}_e$ is $x$-side injective if and only
\[ \langle x^2, y^2, z, xy^2x^{-1}, xzx^{-1}, z \in z \rangle \]
is a free subgroup of $G(\hat{P}_e)$ on the given basis.

Proof. An $x$-side $S$ in $\hat{L}_e$ is an $x$-double edge, a $y$-double edge at each of the two vertices, and a loop for every $z \in z$ at each of the two vertices. The image of $\pi_1(S)$ in $G(\hat{P}_e)$ under the covering projection is the group in the statement of the lemma. □

**Remark 4.** Note that in case $m = 1$ the situation can be different. The most extreme case occurs when $m_e = 1$ for all edges $e$. In that case $W(\Upsilon) = Z_2$ and so $\bar{K}(\Gamma)$ has only two vertices. In this situation the 1-skeleton of each $\bar{K}_e$ is all of $\bar{K}(\Gamma)^{(1)}$, and in fact each side of $\bar{K}_e$ is all of $\bar{K}(\Gamma)^{(1)}$. So $\bar{K}_e$ is not side injective.

**Lemma 5.3.** If $T$ is a subgraph of the 1-skeleton of $\bar{K}_e$ that does not involve every letter from $x_e = \{x, y, z\}$, then $\pi_1(T) \to \pi_1(\bar{K}_e)$ is injective.

Proof. We can argue with $\hat{L}_e$ instead of $\bar{K}_e$. A reduced loop $\gamma$ in $T$ gives a reduced word $u$ in the generators of $\hat{P}_e$ that does not involve all letters from $x_e = \{x, y, z\}$. The presentation $\hat{P}_e$ has only one relator $\hat{r}_e$ that does involve all letters from the generating set $x_e = \{x, y, z\}$. The Freiheitssatz for 1-relator group implies that $u$ does not represent the trivial element in $G(\hat{P})$. Thus $\gamma$ is not trivial in $\pi_1(\hat{L}_e)$. □

We continue our analysis. The complex $\bar{K}(\Gamma)$ is a union of the complexes $w\bar{K}_e$, $w \in W(\Upsilon)$, $e \in$ edges of $\Gamma$. The maps
\[ \bar{K}(\Gamma) \to \bar{K}(\Upsilon) \to \Sigma(\Upsilon) \]
give a one-to-one correspondence between the $w\bar{K}_e$ and Coxeter cells $wK_e$. Since $\Upsilon$ is a tree, the Coxeter complex $\Sigma(\Upsilon)$ is a tree of Coxeter cells $wK_e$ and so $\bar{K}(\Gamma)$ is a tree of complexes $w\bar{K}_e$. In complete analogy to Proposition 2.1 we have

**Proposition 5.4.** Consider $\bar{K}(\Gamma) = \bigcup w\bar{K}_e \to \Sigma(\Upsilon) = \bigcup wK_e$.

1. $\bar{K}(\Gamma)$ is the union of the 2-complexes $w\bar{K}_e$, $e \in$ edges of $\Gamma$, $w \in W(\Upsilon)$. Furthermore, if $w_1\bar{K}_{e_1} \cap w_2\bar{K}_{e_2} \neq \emptyset$ then $e_1 \cap e_2 \neq \emptyset$; if $x = e_1 \cap e_2$, then $w_1\bar{K}_{e_1} \cap w_2\bar{K}_{e_2} = T$, where $T$ is the subgraph of an $x$-side $S$ that carries the letters $x_{e_1} \cap x_{e_2}$.

2. $\bar{K}(\Gamma)$ is a tree of 2-complexes. In particular, if $M$ is a finite connected union of 2-complexes $w_1\bar{K}_{e_1}$ in $\bar{K}(\Gamma)$, then there exists a 2-complex $w\bar{K}_e$ in $M$ that intersects with the rest of $M$ in a subgraph of a single side.
Theorem 5.5. Let $\Gamma$ be a LOT of Coxeter type. Then $K(\Gamma)$ is aspherical in case the $K_e$ are side injective for every edge $e$ in $\Gamma$.

Proof. We will show that $\bar{K}(\Gamma)$ is aspherical. It suffices to show that every finite union $\bar{M} = \bigcup_{i=1}^n w_i \bar{K}_{e_i}$ is aspherical. We first claim that the sides of the $w_i \bar{K}_{e_i}$, $\pi_1$-inject into the union $\bar{M}$. We do induction on $n$. If $n = 1$ the result follows from the hypothesis. Assume $n > 1$. Then by Proposition 5.4 part (2) there exists a 2-complex $w\bar{K}_{e}$ in $\bar{M}$ that intersects with the rest of $\bar{M}$ in a subgraph $T$ of a single side $S$ (of course $T$ could be $S$). Now by induction hypothesis the inclusion $S \subseteq \bar{M} - w\bar{K}_{e} = \bar{M}_0$ is $\pi_1$-injective and the inclusion $\bar{S} \subseteq w\bar{K}_{e}$ is $\pi_1$-injective by hypothesis. It follows that $\pi_1(M) = \pi_1(M_0) *_{\pi_1(T)} \pi_1(w\bar{K}_{e})$. And so the inclusion $S \subseteq M$ is $\pi_1$-injective. All other sides that occur in $\bar{M}$ are either contained in $\bar{M}_0$ or in $w\bar{K}_{e}$. $\pi_1$-injectivity follows from the amalgamated product decomposition. Asphericity of $M$ now follows from induction on $n$ and the amalgamated product decomposition $\pi_1(M) = \pi_1(M_0) *_{\pi_1(T)} \pi_1(w\bar{K}_{e})$. \hfill $\Box$

Remark. The above proof shows more than asphericity. Since each $\pi_1(\bar{K}_e)$ is a finite index subgroup of a 1-relator group, we see that $\pi_1(\bar{K})$ is a tree of groups, the vertex groups being finite index subgroups of 1-relator groups, and the edge groups (over which we amalgamate) being finitely generated and free.

Definition 5.6. A labeled oriented tree $\Gamma$ is called \textit{label separated} if for every pair of edges $e_1$ and $e_2$ that have a vertex in common the intersection $x_{e_1} \cap x_{e_2}$ is a proper subset of both $x_{e_1}$ and $x_{e_2}$.

Theorem 5.7. Let $\Gamma$ be a label separated LOT of Coxeter type. Then $K(\Gamma)$ is aspherical.

Proof. The proof is very much the same as the proof of Theorem 5.5. Let $M = \bigcup_{i=1}^n w_i K_{e_i}$ as before. Again it suffices to show that $M$ is aspherical. If $n = 1$ that is clear. It is instructive to look at the case $n = 2$. The intersection $w_1 K_{e_1} \cap w_2 K_{e_2} = T$ is the subgraph of a side that carries the letters $x_{e_1} \cap x_{e_2}$, which is a proper subset of both $x_{e_1}$ and $x_{e_2}$. $\pi_1$-injectivity for the inclusions $T \subseteq w_i K_{e_i}$, $i = 1, 2$, follows from Lemma 5.3. We have $\pi_1(M) = \pi_1(w_1 K_{e_1}) *_{\pi_1(T)} \pi_1(w_2 K_{e_2})$ and $M$ is aspherical. For $n \geq 2$ we argue by induction and obtain (as in the proof of Theorem 5.5) a decomposition $\pi_1(M) = \pi_1(M_0) *_{\pi_1(T)} \pi_1(w\bar{K}_{e})$ which proves asphericity of $M$. \hfill $\Box$

6. Side injectivity

Let $P = \langle a, b, c \mid r \rangle$, be a 1-relator group, where $c$ is a finite set of letters (which could be empty). We assume that $r$ is cyclically reduced and contains all generators. Assume further that $r = (ab)^m$ for some $m \geq 0$ modulo the relations $a^2 = b^2 = c = 1$, $c \in c$, and cyclic permutation. The number $m$ is called the \textit{dihedral type} of $P$.

Let $Q = \langle a, b, c \mid (ab)^m, a^2, b^2, c \in c \rangle$. We have an epimorphism $\phi: G(P) \to G(Q) = D_m$. Let $\bar{K}(P)$ be the covering of $K(P)$ associated with the kernel. Note that $\bar{K}(P)^{(1)} = \bar{K}(Q)^{(1)}$, which is a $2m$-gon, consisting of double edges labeled in an alternating way with
a and b, and at every vertex we have a c loop, for every \( c \in G \). An a-side of \( \bar{K}(P) \) is a connected subgraph of the 1-skeleton that consists of a double edge with label \( a \), together with all the b-double edges and c-loops connected to the two vertices of the a-double edge. A b-side is defined in an analogous way. We say \( P \) is side injective if the inclusion of any side \( S \to \bar{K}(P) \) is \( \pi_1 \)-injective.

**Lemma 6.1.** Assume that \( P \) is of dihedral type \( m \geq 3 \). Then \( P \) is side injective if and only if every cyclically reduced word \( w \) that represents the trivial element in \( G(P) \) contains a reduced subword \( s \) which is a cyclic permutation of

\[
a^{\alpha_1}d_1b^{\beta_1}d_2a^{\alpha_2}d_3b^{\beta_2}
\]

or its inverse. The \( \alpha_i \) and \( \beta_i \) are odd integers and the \( d_i \) are words in the generators containing a and b with even exponents (the \( d_i \) could be trivial).

Proof. Assume first that \( w \) is a cyclically reduced word that represents the trivial element in \( G(P) \) and contains a reduced subword \( s = a^{\alpha_1}d_1b^{\beta_1}d_2a^{\alpha_2}d_3b^{\beta_2} \) (the simplest setting is \( s = abab \), a case the reader should have in mind). This subword does not lift into a side of \( \bar{K}(P) \), and hence \( w \) does not lift into a side. It follows that \( P \) is side injective.

Next assume that \( P \) is not side injective. Assume \( w \) is a cyclically reduced word that represents the trivial element of \( G(P) \) and does lift into an a-side of \( \bar{K}(P) \). Then \( w \) is a reduced word in powers of \( a^2, b^2, ab^2a^{-1}, c, acac^{-1} \), where \( c \in G \). So \( w \) does not contain a subword of the form \( s \).

**Example 4.** \( P = \langle a, b \mid (ab)^m \rangle \), \( m \geq 3 \), is side injective. This is because 1-relator presentations with torsion are Dehn presentations (in particular \( G(P) \) is hyperbolic). See Newman [17]. A word \( w \) that is trivial in the group contains a subword of length more than \( 1/2 \) of a cyclic permutation of the relator or its inverse, hence it contains a a cyclic permutation of \( abab \), or its inverse. The result follows from Lemma 6.1.

**Example 5.** More generally, if \( P = \langle a, b, c \mid r(a, b, c) \rangle \) (\( c \) could be empty) is a Dehn presentation of dihedral type \( m \geq 3 \) so that more than half of a cyclic permutation of the relator or its inverse contains a subword \( s \) as in Lemma 6.1, then \( P \) is side injective. Recall that \( P \) is a Dehn presentation for instance in case it satisfies the small cancellation condition \( C'(1/6) \) or \( C'(1/4) - T(4) \) (see Chapter V, Theorem 4.4 in Lyndon and Schupp [15]). For example if

\[
r(a, b, c) = a^{\alpha_1}d_1b^{\beta_1}d_2a^{\alpha_2}d_3b^{\beta_2}d_4a^{\alpha_3}d_5b^{\beta_3}d_6a^{\alpha_4}d_7b^{\beta_4}d_8
\]

where the \( \alpha_i \) and \( \beta_i \) are odd integers satisfying \( |\alpha_i| = |\alpha_j|, |\beta_i| = |\beta_j|, \forall i, j \leq 4 \) and the \( d_i \) are words of the same length containing a and b with even exponents, and \( P \) satisfies the small cancellation condition \( C'(1/6) \) or \( C'(1/4) - T(4) \), then \( P \) is side injective. Concrete examples are

\[
\langle a, b, c \mid (acbc^{-1}ac^{-1}bc)^2 \rangle
\]
or

\[
\langle a, b, c \mid acbc^{-1}acbc^{-1}ac^{-1}bc \rangle
\]
which are $C'(1/4) = T(4)$ and

$$\langle a, b, c \mid abc^{-1}cba^{-1}a^{-1}c^{-1}bcac^{-1}b^{-1} \rangle$$

which is $C'(1/6)$. These presentations were checked with the help of GAP (see [9]) and the package SMALLCANCELLATION by Ivan Sadofschi Costa (see [17]).

**Example 6.** The Artin presentation $P = \langle a, b \mid \text{prod}(a, b, m) = \text{prod}(b, a, m) \rangle$ is not side injective for $m = 3$, but it is side injective for $m \geq 4$:

1) $m = 3$. We show that $P = \langle a, b \mid aba = bab \rangle$ is not side injective. We have $a^2(aba^2ba)a^{-2} = aba^2ba$ in $G(P)$ because $(aba)^2 = aba^2ba$ is central. So $w = a^2ba^2ba^{-2}b^{-1}a^{-2}b^{-1} = 1$ in $G(P)$. Note that $w$ lifts into a $b$-side of $\bar{K}(P)$.

2) $m = 4$. We show that $P = \langle a, b \mid abab = baba \rangle$ is side injective. Note that $x = abab$ is a central element. The quotient $G(P)/\langle x \rangle$ has a presentation $\langle a, b \mid (ab)^2 \rangle$. Let $y = ba$, then the presentation rewrites to $\langle a, y \mid y^2 \rangle$. In order to show that $P$ is $a$-side injective we have to show that $a^2, b^2, ab^2a^{-1}$ generate a free group of rank 3 in $G(P)$. We will do this by showing that $A = a^2, B = (ya^{-1})^2 = ya^{-1}ya^{-1},$ and $C_0 = a(ya^{-1})^2a^{-1} = aya^{-1}ya^{-1}a^{-1}$ generate a free group in the quotient presented by $Q = \langle a, y \mid y^2 \rangle = \mathbb{Z} \ast \mathbb{Z}_2$. Let $C_1 = BC_0$. We have

$$C_1 = ya^{-1}ya^{-1}aya^{-1}ya^{-1}a^{-1} = ya^{-1}ya^{-1}ya^{-1}ya^{-1}a^{-1} = ya^{-1}a^{-1}ya^{-1}a^{-1} = ya^{-1}a^{-1}ya^{-1}a^{-1} = ya^{-1}a^{-1}ya^{-1}a^{-1} = ya^{-2}ya^{-2}.$$ 

And finally let $C = C_1A = ya^{-2}y$. In summary we have

$$A = a^2, \quad B = ya^{-1}ya^{-1}, \quad C = ya^{-2}y.$$ 

The group $H = \langle A, B, C \rangle$ is a normal free subgroup of $G(Q)$ of rank 3 and index 4. Figure 3 shows a covering space $\pi : \bar{K}(Q) \rightarrow \bar{K}(Q)$ so that $\pi_1(\bar{K}(Q))$ is free of rank 3 and $p_*(\pi_1(\bar{K}(Q))) = \langle A, B, C \rangle \leq \pi_1(K(Q))$. The argument for $b$-side injectivity is analogous.

![Figure 3](image_url)

**Figure 3.** If $Q = \langle a, y \mid y^2 \rangle$ then the universal covering $\bar{K}(Q)$ is a tree with spheres attached. Here we see the intermediate covering $\bar{K}(Q)$ corresponding to the subgroup $H = \langle A, B, C \rangle$. The gray discs with boundary $y^2$ indicate 2-spheres.
3) $m \geq 6$ and even. This case is easy. Let $x = \text{prod}(a, b, m)$. The quotient $G(P)/\langle x \rangle$ is presented by $\langle a, b \mid (ab)^{m/2} \rangle$ which is a Dehn presentation, being a 1-relator presentation with torsion. Since $m \geq 6$, we have $m/2 \geq 3$. Side injectivity follows from Example 3.

4) $m \geq 5$ and odd. Let $x = \text{prod}(a, b, m)$ and $y = ba$. Note that $x = ay^{m-1}$. Using $a = xy^{m+1}/2$ and $b = y^{m+1}/2 x^{-1}$ the presentation $P$ can be rewritten to $\langle x,y \mid x^2 = y^m \rangle$. Thus $G(P)/\langle x^2 \rangle$ is presented by $\langle x,y \mid x^2, y^m \rangle$ which is a Dehn presentation, being a 1-relator presentation with torsion. Since $m \geq 5$, we have $m^2/2 \geq 3$. Side injectivity follows from Example 4.

Let $A = xy^3xy^3, B = y^3xy^3x, C_0 = xyxy^3xy^2x$.

Let $C = C_0A = (xyxy^3xy^2x)(xy^3xy^3) = xyxy$.

$A = xy^3xy^3, B = y^3xy^3x, C = xyxy$.

Note that

$C(y^{-1}Cy) = (xyxy)y^{-1}(xyxy)y = xy^2xy^2 = B^{-1}$

and

$C(y^{-1}Cy)(y^{-2}Cy^2) = xy^2xy^2y^{-2}xyxy^2 = xy^3xy^3 = A$.

So it suffices that to show that

$X = C, Y = y^{-1}Cy, Z = y^{-2}Cy^2$

generate a free subgroup of rank 3. Figure 4 shows a covering space $p: \bar{K}(Q) \to K(Q)$ so that $\pi_1(\bar{K}(Q))$ is free of rank 3 and $p_*(\pi_1(\bar{K}(Q))) = \langle X, Y, Z \rangle \leq \pi_1(K(Q))$. The argument for $b$-side injectivity is analogous.
Example 7. Let $P = \langle a, b, c \mid a(babcaba) = (babcaba)b \rangle$. Then $P$ is side injective by Theorem 6.2.

Theorem 6.2. Suppose $P$ has dihedral type $m \geq 3$ and
- $P = \langle a, b, c \mid a(u_1 c_i u_3) = (u_1 c_i u_3)b \rangle$, or
- $P = \langle a, b, c \mid a(u_1 c_i u_2 c_j u_3) = (u_1 c_i u_2 c_j u_3)b \rangle$, where
  1. $c_i, c_j \in \mathfrak{c}$ ($i = j$ is possible), $\epsilon = \pm 1$;
  2. $u_1$ and $u_3$ do not contain any $c_k \in \mathfrak{c}$ (or $c$ in the first case), and $u_2$ is arbitrary;
  3. both $u_1^{-1}a$ and $u_3b^{-1}$ contain a subword $s$ as in Lemma 6.1.

Then $P$ is side injective.

Proof. We assume we are in the second case and $\epsilon = 1$. The first case is shown in an analogous way. Envision the relator disc placed in the plane as a rectangle, where the $a$ on the very left of the equation and the $b$ on the very right of the equation are horizontal edges, and the word $u_1 c_i u_2 c_j u_3$ is a vertical edge sequence. Connect the midpoints of $c$-edges on the left and right by horizontal red edges. See Figure 5.

Figure 5. The relator disc drawn as a rectangle.
Suppose that $w$ is a cyclically reduced word that represents the trivial element in $G(P)$. Let $D$ be a reduced Van Kampen diagram with boundary $w$. We may assume that $D$ is a topological disc. The red edges in our relator disc will form red circles and red arcs connecting points on the boundary of $D$. See Figure 6.

Consider an innermost red circle. Going around the inside we read off a word that freely reduces to $u_1^{-1}a^ku_1$ or $u_3b^ku_3^{-1}$, for some $k \in \mathbb{Z}$. If $k = 0$, then $D$ is not reduced. If $k \neq 0$, then $G(P)$ has torsion. Both is not the case, hence there are no red circles in $D$. Consider an outermost red arc $\alpha$. Let $E$ be the component of $D - \alpha$ that does not contain anything red. Reading along the part of the boundary of $D$ which belongs to $E$ gives a reduced word (a subword of the reduced word $w$) equal to $u_1^{-1}a^ku_1$ or $u_3b^ku_3^{-1}$. Because $D$ is reduced $k$ cannot be zero. If $k$ is positive then $u_1^{-1}a^ku_1$ contains $u_1^{-1}a$ and hence a word $s$ as in Lemma 6.1. Also, $u_3b^ku_3^{-1}$ contains $bu_3^{-1}$, and since $u_3b^{-1}$ contains a word $s$ as in Lemma 6.1 so does $(u_3b^{-1})^{-1} = bu_3^{-1}$. The case where $k$ is negative goes the same way. It now follows from Lemma 6.1 that $P$ is side injective.

7. Last words about LOT applications

**Theorem 7.1.** Let $\Gamma$ be a LOT of Coxeter type. Suppose that for every edge $e = (a \xrightarrow{w_e} b)$ the word $w_e$ is of the form $u_1c^eu_3$, or $u_1c^eu_2c^eu_3$ for some $c \neq a, b$, as in Theorem 6.2. Then $K(\Gamma)$ is aspherical.

Proof. Each $\tilde{P}_e$ is side injective. This follows from Theorem 6.2. Thus each $\tilde{K}_e$ is side injective. The result follows from Theorem 5.5. \qed
What if side injectivity fails?

**Theorem 7.2.** Suppose $\Gamma$ is a LOT of Coxeter type and there exist two edges $e_1$ and $e_2$ in $\Gamma$ so that

1. $\bar{K}_{e_1} \cap \bar{K}_{e_2} = S$;
2. neither $\bar{K}_{e_1}$ or $\bar{K}_{e_2}$ is side injective, and in fact we have:

   If $N_1 = \ker(\pi_1(S) \to \pi_1(\bar{K}_{e_1}))$ and $N_2 = \ker(\pi_1(S) \to \pi_1(\bar{K}_{e_2}))$ then $\frac{N_1 \cap N_2}{[N_1, N_2]} \neq 1$.

Then Whitehead’s asphericity conjecture is false.

Proof. Suppose Whitehead’s conjecture is true. Then $K(\Gamma)$ and hence $\bar{K}(\Gamma)$ is aspherical. Note that $\bar{K}_{e_1} \cup \bar{K}_{e_2}$ is a subcomplex of $\bar{K}(\Gamma)$. Let $w$ be a reduced edge loop in $S$ that represents a non-trivial element in the quotient $\frac{N_1 \cap N_2}{[N_1, N_2]}$. It is the boundary of a van Kampen diagram $D_1$ for $\bar{K}_{e_1}$ and also the boundary of a van Kampen diagram $D_2$ for $\bar{K}_{e_2}$. The two diagrams can be glued together to form a non-trivial element in $\pi_2(\bar{K}_{e_1} \cup \bar{K}_{e_2})$ (see Gutierrez-Ratcliffe [8]). A contradiction. □

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