A Level-1 Limit Order Book with Time Dependent Arrival Rates

Jonathan A. Chávez-Casillas¹ · Robert J. Elliott²,³ · Bruno Rémillard⁴ · Anatoliy V. Swishchuk⁵

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Abstract
We propose a simple stochastic model for the dynamics of a limit order book, extending the recent work of Cont and de Larrard (SIAM J Financial Math 4(1), 1–25 2013), where the price dynamics are endogenous, resulting from market transactions. We also show that the conditional diffusion limit of the price process is the so-called Brownian meander.

Keywords Limit order book · Inhomogeneous Poisson process · Brownian motion · Brownian meander

Mathematics Subject Classification (2010) Primary 60G50 · 60G52; Secondary 60F05 · 91G80

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Bruno Rémillard
bruno.remillard@hec.ca

Jonathan A. Chávez-Casillas
jchavezc@uri.edu

Robert J. Elliott
elliott@ucalgary.ca

Anatoliy V. Swishchuk
aswish@ucalgary.ca

¹ Department of Mathematics, University of Rhode Island, Kingston, RI 02881, USA
² Haskayne School of Business, University of Calgary, Calgary, Canada
³ Centre for Applied Financial Studies, University of South Australia, Adelaide, Australia
⁴ GERAD, CRM, and Department of Decision Sciences, HEC Montréal, Montreal, Canada
⁵ Department of Mathematics and Statistics, University of Calgary, Calgary, Canada
1 Introduction

The limit order book gives the list of possible bid/ask prices together with the size (number of shares available) at each price. It changes rapidly over time, many orders possibly arriving within a millisecond. Either for testing high frequency trading strategies or deciding on an optimal way to buy or sell a large number of shares, it is important to try to model the behavior of limit order books. Several authors suggested interesting models for limit order books. For example, in Smith et al. (2003), the authors assumed that the markets orders (bid/ask) arrive independently at rate $\mu$ in chunks of $m$ shares; since these orders reduce the number of shares at the best bid or best ask price, they are usually combined with order cancellations. In their model, the limit orders (bid/ask) also arrive independently at rate $\lambda$ in chunks of $m$ shares; the associated price is said to be selected “uniformly” amongst the possible bid prices or ask prices, whatever that means. Basically, they examined some properties of the resulting limit order book, trying to use techniques used in physics to characterize some macro quantities of their model.

More recently, Cont and de Larrard (2013) proposed a similar model for the time arrivals of the limit orders, but they only considered the level-1 order book, meaning that the best bid and best ask prices are taken into account. They also assumed that markets orders for the best bid/ask prices arrive independently at rate $\mu$, in chunks of $m$ shares, and limit orders for the best bid/ask prices arrive independently at rate $\lambda$, also in chunks of $m$ shares. When the size (number of shares) of the best bid price attains 0, the bid price decreases by $\delta$ and so does the ask price; the sizes of the best bid/ask prices are then chosen at random from a distribution $\tilde{f}$. When the size of the best ask price attains 0, the ask price increases by $\delta$ and so does the bid price; the sizes of the best bid/ask prices are then chosen at random from a distribution $f$. With this simple but tractable model, they were able to determine the asymptotic behavior of the price process, instead of assuming it. They also found the asymptotic behavior of the price.

According to some participants in the high frequency trading world, the hypothesis of constant arrivals of orders is not justified. Therefore, one should assume that the arrival rates are time-dependent. This is the model proposed here. We extend the Cont and de Larrard (2013) setting by assuming that the rates for market orders and limit orders depend on time and that they are also different if they are bid or ask orders. As in Cont and de Larrard (2013), under some simple assumptions, we are also able to find the limiting behavior of the price process, and we show how to estimate the main parameters of the model. The main ingredients are the random times at which the price changes, the associated counting process, and the distribution of the price changes.

More precisely, in Section 2, we present the construction of the model we consider. Under some simplifying assumptions, we derive in Section 3 the distribution of the random times at which the price changes. The asymptotic distribution of the price process is examined in Section 4, while the estimation of the parameters is discussed in Section 5, together with an example of implementation. The proofs of the main results are given in Appendix B.

2 Description of the Model

We discuss a level-1 limit order Book model using as a framework the model proposed in Cont and de Larrard (2013). However, the point processes describing the arrivals of Limit orders have time-dependent periodic rates proportional to the rate describing the arrival of market orders plus cancellations.
Recalling the Cont-de Larrard model we will define the level-1 limit order book model as follows:

- There is just one level on each side of the order book, i.e., one knows only the best bid and the best ask prices, together with their sizes (number of available shares at these prices).
- The spread is constant and always equals the tick size $\delta$.
- Order volume is assumed to be constant (set as one unit).
- Limit orders at the bid and ask sides of the book arrive independently according to inhomogeneous Poisson processes $\mathcal{L}_t^b$ and $\mathcal{L}_t^a$, with intensities $\lambda_t^b$ and $\lambda_t^a$ respectively.
- Market orders plus cancellations at the bid and ask sides of the book arrive independently according to inhomogeneous Poisson processes $\mathcal{M}_t^b$ and $\mathcal{M}_t^a$, with intensities $\mu_t^b$ and $\mu_t^a$ respectively.
- The processes $\mathcal{L}_t^a$, $\mathcal{L}_t^b$, $\mathcal{M}_t^a$ and $\mathcal{M}_t^b$ are all independent.
- Every time there is a depletion at the ask side of the book, both the bid and the ask prices increase by one tick, and the size of both queues gets redrawn from some distribution $f \in \mathbb{N}^2$.
- Every time there is a depletion at the bid side of the book, both the bid and the ask prices decrease by one tick, and the size of both queues gets redrawn from some distribution $f \in \mathbb{N}^2$.

2.1 Construction of the Processes

First, consider the following infinitesimal generators of birth and death processes:

\[
(L_t^a)_{ij} = \begin{cases} 
0, & i = 0, j \geq 0, \\
\mu_t^a, & 1 \leq i, j = i - 1, \\
\lambda_t^a, & 1 \leq i, j = i + 1, \\
-(\mu_t^a + \lambda_t^a), & 1 \leq i, j = i, \\
0, & \text{otherwise.}
\end{cases}
\]

(1)

\[
(L_t^b)_{ij} = \begin{cases} 
0, & i = 0, j \geq 0, \\
\mu_t^b, & 1 \leq i, j = i - 1, \\
\lambda_t^b, & 1 \leq i, j = i + 1, \\
-(\mu_t^b + \lambda_t^b), & 1 \leq i, j = i, \\
0, & \text{otherwise.}
\end{cases}
\]

(2)

Note that 0 is an absorbing state for any Markov chain with generators $L^a$ or $L^b$. When a chain reaches the absorbing point 0, one calls it extinction.

To describe precisely the behavior of the price process $S_t$ and the queues sizes process $q_t = (q^b_t, q^a_t)^T$, one needs to define the following sequence of random times. Let $\sigma_{x_0,1}$ and $\sigma_{y_0,1}$ be the extinction times of independent Markov chains $X^{(b,1)}$ and $X^{(a,1)}$ with generators $L^{(b,1)}$ and $L^{(a,1)}$, starting from $x_0$ and $y_0$ respectively, where $L_t^{(a,1)} = L_t^a$ and $L_t^{(b,1)} = L_t^b$. Further set $\tau_0 = 0$ and $\tau_1 = \min\left(\sigma_{x_0,1}, \sigma_{y_0,1}\right)$.

Having defined $\tau_1, \ldots, \tau_{n-1}$, set $V_{n-1} = \sum_{k=0}^{n-1} \tau_k$, and let $\sigma_{x_{n-1}}$ and $\sigma_{y_{n-1}}$ be the extinction times of independent Markov chains $X^{(b,n)}$ and $X^{(a,n)}$ with generators $L^{(b,n)}$ and $L^{(a,n)}$, starting respectively from $x_{n-1}$ and $y_{n-1}$, where $L_t^{(a,n)} = L_t^a V_{n-1+t}$ and $L_t^{(b,n)} = L_t^b V_{n-1+t}$, $t \geq 0$; then set $\tau_n = \min\left(\sigma_{x_{n-1}}, \sigma_{y_{n-1}}\right)$. Here the random variables $(x_k, y_k)$ are $\mathcal{F}_{\tau_k}$-measurable, for any $k \geq 0$. In fact, $(x_0, y_0)$ is chosen at random from distribution $f_0$,.
while \((x_n, y_n)\) is chosen at random from distribution \(f_n\) if \(\sigma_{x_{n-1}}^{(a,n)} < \sigma_{y_{n-1}}^{(b,n)}\) and chosen at random from distribution \(\tilde{f}_n\) if \(\sigma_{x_{n-1}}^{(a,n)} > \sigma_{y_{n-1}}^{(b,n)}\). Now for \(t \in [V_{n-1}, V_n)\), \(q^b_t = X_t^{(b,n)}\) and \(q^a_t = X_t^{(a,n)}\) starting respectively from \(x_{n-1}\) and \(y_{n-1}\) at time \(V_{n-1}\). Finally, the price process \(S\), representing either the price or the log-price, is defined the following way: for \(t \in [V_{n-1}, V_n)\),

\[
S_t = SV_{n-1} - \delta \quad \text{if} \quad \sigma_{x_{n-1}}^{(a,n)} < \sigma_{y_{n-1}}^{(b,n)}
\]

and

\[
S_t = SV_{n-1} + \delta \quad \text{if} \quad \sigma_{x_{n-1}}^{(b,n)} < \sigma_{y_{n-1}}^{(a,n)}.
\]

In Cont and de Larrard (2013), the authors assumed that the arrivals were time homogeneous, meaning that \(L^a_t \equiv Q^a\) and \(L^b_t \equiv Q^b\). In fact, most of their results were stated for the case \(Q^a = Q^b = Q\), where

\[
Q^a_{ij} = \begin{cases} 
0 & \text{if } i = 0, \ j \geq 0, \\
\mu^a & \text{if } 1 \leq i, \ j = i - 1, \\
\lambda^a & \text{if } 1 \leq i, \ j = i + 1, \\
-(\lambda^a + \mu^a) & \text{if } 1 \leq i, \ j = i, \\
0 & \text{if } |i - j| > 1. 
\end{cases} \tag{3}
\]

and

\[
Q^b_{ij} = \begin{cases} 
0 & \text{if } i = 0, \ j \geq 0, \\
\mu^b & \text{if } 1 \leq i, \ j = i - 1, \\
\lambda^b & \text{if } 1 \leq i, \ j = i + 1, \\
-(\lambda^b + \mu^b) & \text{if } 1 \leq i, \ j = i, \\
0 & \text{if } |i - j| > 1. 
\end{cases} \tag{4}
\]

3 Distributional Properties

Because of the independence between the ask and the bid side of the book before the first price change, to analyze the distribution of \(\tau_1\), it is enough to study one side of the order book, say the ask. In this case, an explicit formula for \(P[\sigma^{(a,1)} > t]\) is given in the next section.

3.1 Distribution of the Inter-Arrival Time Between Price Changes

Let \(L_t\) be the infinitesimal generator of a non homogeneous birth and death process \(X\) given by

\[
(L_t)_{ij} = \begin{cases} 
0 & \text{if } i = 0, \ j \geq 0, \\
\mu_t & \text{if } 1 \leq i, \ j = i - 1, \\
\lambda_t & \text{if } 1 \leq i, \ j = i + 1, \\
-(\lambda_t + \mu_t) & \text{if } 1 \leq i, \ j = i, \\
0 & \text{if } |i - j| > 1. 
\end{cases} \tag{6}
\]

Notice that 0 is an absorbing state. Also, let \(\sigma_X\) be the first hitting times of 0 for this process, i.e.,

\[
\sigma_X := \inf\{t > 0 \mid X_t = 0\}. \tag{7}
\]

Then since 0 is an absorbing state, one has \(P_x[\sigma_X \leq t] = P_x[X_t = 0]\).
It is hopeless to expect solving the problem for general generators so as a first approach, some assumptions on the infinitesimal generators $L^a$ and $L^b$ will be made.

**Assumption 1** There exists a measurable function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that $A_t = \int_0^t \alpha_s ds < \infty$ for any $t \geq 0$, with $L^a_t = \alpha_t Q^a$ and $L^b_t = \alpha_t Q^b$.

**Remark 3.1** Under the assumption that $L_t = \alpha_t Q$, a process $X$ with infinitesimal generator $L_t$ can be seen as a time change of a process $Y$ with infinitesimal generator $Q$, viz. $X_t = Y_{\alpha_t}$. In particular, if $\sigma_X$ and $\sigma_Y$ are respectively the first hitting time of 0 for $X$ and $Y$, then for any $t \geq 0$,

$$F_L(t; x) := \mathbb{P}[\sigma_X \leq t \mid X_0 = x] = \mathbb{P}[\sigma_Y \leq t \mid Y_0 = x] := F_Q(A_t; x).$$

This result is essential in what follows since it implies that the distribution of the time between price changes in the present model is comparable to the distribution of the inter-arrival time between price changes for the model considered by Cont and de Larrard (2013).

The following lemma gives the distribution of the extinction time $\sigma_Y$ of a birth and death process $Y$ with generator $Q$.

**Lemma 3.2** Let $Y$ be a birth and death process with generator $Q$ given by Eq. 5. If $\lambda \leq \mu$, then $1 - F_Q(t; x) = \mathbb{P}_x[\sigma_Y > t] = u_{\lambda, \mu}(t, x)$, where

$$u_{\lambda, \mu}(t, x) = x \left(\frac{\mu}{\lambda}\right)^{x/2} \int_t^\infty \frac{1}{s} I_x \left(2s\sqrt{\lambda \mu}\right) e^{-s(\lambda + \mu)} ds,$$

and where $I_0(\cdot)$ is the modified Bessel function of the first kind.

If $\lambda > \mu$, then

$$u_{\lambda, \mu}(t, x) = 1 - \left(\frac{\mu}{\lambda}\right)^{x} + x \left(\frac{\mu}{\lambda}\right)^{x/2} \int_t^\infty \frac{1}{s} I_x \left(2s\sqrt{\lambda \mu}\right) e^{-s(\lambda + \mu)} ds.$$

In particular, $\mathbb{P}_x[\sigma_Y = +\infty] = 1 - \left(\frac{\mu}{\lambda}\right)^{x} > 0$.

**Remark 3.3** The case $\lambda \leq \mu$ is proven in Cont and de Larrard (2013). For the case $\lambda > \mu$, note that $\mathbb{E}_x[e^{-s\sigma_Y}] = \left(\frac{\lambda + s - \sqrt{(\lambda + s)^2 - 4\lambda \mu}}{2\lambda}\right)^x$, so letting $s \downarrow 0$ yields $\mathbb{P}_x(\sigma_Y < \infty) = \left(\frac{\mu}{\lambda}\right)^x$. It then follows that $\mathbb{P}_x[\sigma_Y > t \mid \sigma_Y < \infty] = u_{\lambda, \mu}(t, x)$. Then $\mathbb{P}_x[\sigma_Y > t] = 1 - \left(\frac{\mu}{\lambda}\right)^x u_{\lambda, \mu}(t, x)$. Hence the result.

It is important to analyze the tail behavior of the survival distribution for $\sigma_Y$. The following lemma, whose proof is deferred to Appendix B, establishes such behavior. Recall that $\Gamma(s, x) = \int_x^\infty u^{s-1}e^{-u} du$ is the incomplete gamma function.

**Lemma 3.4** Let $Y$ be a birth and death process with generator $Q$ given by Eq. 5, and assume that $\lambda \leq \mu$. Set $C = (\sqrt{\mu} - \sqrt{\lambda})^2$. Then, for a sufficiently large $T$,

$$\mathbb{P}[\sigma_Y > T \mid Y_0 = x] \sim \left\{\begin{array}{ll}
\left(\frac{\mu}{\lambda}\right)^{x/2} \frac{x}{\sqrt{\pi} \sqrt{\lambda \mu}} \left[\frac{e^{-TC}}{T^{3/2}} - \sqrt{C}\Gamma\left(\frac{1}{2}, TC\right)\right] & \text{if } \lambda < \mu;
\frac{x}{\lambda \sqrt{\pi} \sqrt{T}} & \text{if } \lambda = \mu.
\end{array}\right.$$

Consequently, as expected, if $\lambda = \mu$, $\mathbb{E}_x[\sigma_Y] = \infty$, whereas if $\lambda < \mu$, $\mathbb{E}_x[e^{\theta \sigma_Y}] < \infty$ for $\theta < C$. In particular, $\mathbb{E}[\sigma_Y^k] < \infty$ for every $k \in \mathbb{N}$.
Remark 3.5 Note that if $\lambda = \mu$, the results in Lemma 3.4 agree with the results obtained in Eq. 6 in Cont and de Larrard (2013). However, if $\lambda < \mu$, Eq. 5 in Cont and de Larrard (2013) says that $\mathbb{P}[\sigma_Y > T \mid Y_0 = x] \sim \frac{\lambda + \mu}{2(\mu - \lambda)} \frac{1}{T}$, which is incorrect, since for a birth and death process with death rate larger than its birth rate, the extinction time $\sigma_Y$ has moments of all orders. An easy way to see this is to use the moment generating function (mgf) computed in Proposition 1 of Cont and de Larrard (2013) and observe that if $\lambda < \mu$, then the mgf is defined on an open interval around 0; see, e.g., (Billingsley 1995, Section 21).

Lemma 3.2 allows a closed formula to be obtained for the distribution of $\sigma_X$, when the rates are proportional to each other, as in Assumption 1. Such a formula is described in the following proposition, whose proof is deferred to Appendix B.

**Proposition 3.6** Let $X$ be a birth and death process with generator $L$ satisfying $L_t = \alpha_t Q$. If $\lambda \leq \mu$, then the distribution of $\sigma_X$ is given by

$$
\mathbb{P}_x[\sigma_X > T] = x \left(\frac{\mu}{\lambda}\right)^{x/2} \int_0^\infty \frac{1}{s} I_x \left(2s\sqrt{\lambda\mu}\right) e^{-s(\lambda + \mu)} ds.
$$

**Corollary 3.7** Under Assumption 1, for $A_t = \int_0^t \alpha_s ds$, the distribution of $\tau_1$ is given by

$$
\mathbb{P}_Q[\tau_1 > T \mid q_0 = (x, y)] = \mathbb{P}_Q[\sigma_y^{(a,1)} > A_T \mid \sigma_x^{(a,1)} > A_T] = \mathbb{P}_Q[\tau_1 > A_T \mid q_0 = (x, y)].
$$

**Proof** The result follows from the fact that $\tau_1 = \sigma_y^{(a,1)} \land \sigma_x^{(b,1)}$, Proposition 3.6 and the independence between $\sigma_y^{(a,1)}$ and $\sigma_x^{(b,1)}$. $\square$

Now, we present the asymptotic behavior of the survival distribution function of $\tau_1$ under $\mathcal{L}$. It follows directly from Lemma 3.4 and Corollary 3.7.

**Lemma 3.8** Let $C_a = (\sqrt{\mu_a} - \sqrt{\lambda_a})^2$, $C_b = (\sqrt{\mu_b} - \sqrt{\lambda_b})^2$, and set $F_{\mathcal{L}}(t : x, y) = \mathbb{P}_\mathcal{L}[\tau_1 \leq t \mid q_0 = x, q_0^\prime = y], t \geq 0$. Assume that $\lambda_a \leq \mu_a$ and $\lambda_b \leq \mu_b$. Then, as $T \to \infty$, $1 - F_{\mathcal{L}}(T : x, y)$ is asymptotic to

$$
\left(\frac{\mu_a}{\lambda_a}\right)^{x/2} \left(\frac{\mu_b}{\lambda_b}\right)^{y/2} \frac{xy}{\pi (\lambda_a \lambda_b \mu_a \mu_b)^{1/4}} \left[\exp(-A_T C_a) e^{-\sqrt{C_a} \Gamma \left(\frac{1}{2}, A_T C_a\right)}\right]^x \times \left[\exp(-A_T C_b) e^{-\sqrt{C_b} \Gamma \left(\frac{1}{2}, A_T C_b\right)}\right]^y.
$$

In particular, if $\lambda_a = \mu_a$ and $\lambda_b = \mu_b$, then

$$
A_T \mathbb{P}_\mathcal{L}[(\tau_1 > T \mid q_0 = (x, y)) \to \infty \frac{xy}{\pi \sqrt{\lambda_a \lambda_b}}.
$$

**Remark 3.9** It might happen that either $\lambda_a > \mu_a$ or $\lambda_b > \mu_b$. If both these conditions hold, there is a positive probability that the queues will never deplete, so this case must be excluded. There are basically two cases left. The following result follows directly from the proof of Lemma 3.8.
Suppose that $\lambda^b > \mu^b$ and $\lambda^a \leq \mu^a$. Then, as $T \to \infty$, $1 - F_L(T : x, y)$ is asymptotic to
\[
\left[ 1 - \left( \frac{\lambda^b}{\lambda^b} \right)^x \right] \frac{\lambda^a}{\lambda^a} \frac{y}{\pi(\lambda^a \mu^a)^{1/4}} \left[ \frac{\exp(-A_T \Sigma)}{\sqrt{A_T}} - \sqrt{\Sigma} \Gamma \left( \frac{1}{2}, A_T \Sigma \right) \right].
\]

Suppose that $\lambda^a > \mu^a$ and $\lambda^b \leq \mu^b$. Then, as $T \to \infty$, $1 - F_L(T : x, y)$ is asymptotic to
\[
\left[ 1 - \left( \frac{\lambda^a}{\lambda^a} \right)^y \right] \frac{\lambda^b}{\lambda^b} \frac{x}{\pi(\lambda^b \mu^b)^{1/4}} \left[ \frac{\exp(-A_T \Sigma)}{\sqrt{A_T}} - \sqrt{\Sigma} \Gamma \left( \frac{1}{2}, A_T \Sigma \right) \right].
\]
In particular, if $\lambda^a > \mu^a$ and $\lambda^b = \mu^b$, then
\[
\sqrt{A_T} \mathbb{P}[T | q_0 = (x, y)] \to \frac{x}{\pi \sqrt{\lambda^b}} \left[ 1 - \left( \frac{\mu^a}{\lambda^a} \right)^y \right].
\]

### 3.2 Probability of a Price Increase

In Cont and de Larrard (2013, Proposition 3), the authors considered an asymmetric order flow as given here by the processes $Y^a$ and $Y^b$ for computing the probability of a price increase. This was not used elsewhere in their paper. They obtained the following result, which we cite without much changes. However there are some typos that are corrected here. The proof of the result is given in Van Leeuwaarden et al. (2013).

**Proposition 3.10** Suppose that $\lambda^a \leq \mu^a$ and $\lambda^b \leq \mu^b$. Given $(q^b, q^a) = (x, y)$, the probability $p^{up}(x, y)$ that the next price change is an increase is
\[
p^{up}(x, y) = 1 - \frac{1}{\pi} \left( \frac{\lambda^a}{\lambda^a} \right)^y \left( \frac{2 \sqrt{\Sigma \mu^a}}{\mu^a + \lambda^a} \right) \int_0^\pi H_i \sin(yt) \sin(t) \times \left\{ \frac{2 \lambda^b H_i - G_i}{2 \sqrt{\Sigma \mu^a} \cos(t) - 1} \right\} \frac{1}{\sqrt{G_i^2 - 4 \lambda^b \mu^b}} dt,
\]
where $\Sigma = \mu^a + \mu^b + \lambda^a + \lambda^b$, $G_i = \Sigma - 2 \sqrt{\lambda^a \mu^a} \cos(t)$, and $H_i = G_i - \sqrt{G_i^2 - 4 \lambda^b \mu^b} / 2 \lambda^b$.

Under Assumption 1, the same result applies for our model since $X^a_t = Y^a_{A_t}$ and $X^b_t = Y^b_{A_t}$.

**Remark 3.11** One can also use Lemma 3.2 and Proposition 3.6 to obtain the previous result by integration.

### 4 Diffusion Limit of the Price Process

Let $V_n$ be the time of the $n$-th jump in the price, as defined in Section 2.1. We are interested in analyzing the asymptotic behavior of the number of price changes up to time $t$, that is, in describing the counting process
\[
N_t := \max\{n \geq 0 \mid V_n \leq t\}, \quad t \geq 0.
\]

\[
(11)
\]
4.1 Asymptotic Behavior of the Counting Process \( N \)

The next proposition, whose proof is deferred to Appendix B, provides an expression which relates the distribution of the partial sums for the waiting times between price changes for the models with the generators \( L \) and \( Q \). This result is based on a new assumption, stated below.

**Assumption 2**

\[
\sum_{(x,y) \in \mathbb{N}^2} \tilde{f}(x,y)P^{\text{Q}}[\tau_1 \leq t | q_0^b = x, q_0^a = y] = \sum_{(x,y) \in \mathbb{N}^2} f(x,y)P^{\text{Q}}[\tau_1 \leq t | q_0^b = x, q_0^a = y] = F_1, Q(t). \]

This is true for example, when (i) \( \tilde{f}(x,y) = f(y,x) \) and \( Q^a = Q^b \), or (ii) \( \tilde{f} = f \). Properties (i) and (ii) are used for example in Cont and de Larrard (2013).

**Proposition 4.1**

Recall that \( A_t = \int_0^t \alpha_s ds \). Then, under Assumptions 1–2,

\[
P_L[V_n \leq t | q_0^b = x, q_0^a = y] = P_Q[V_n \leq A_t | q_0^b = x, q_0^a = y].
\]

**Remark 4.2**

Under generator \( Q \), \( \tau_1, \tau_2, \ldots, \tau_n \) are independent and \( \tau_2, \ldots, \tau_n \) are i.i.d.

In order to deal with the counting process \( N \), we need another assumption.

**Assumption 3**

There exists a positive constant \( \nu \) such that \( A_t/t \rightarrow \nu \) as \( t \rightarrow \infty \).

**Remark 4.3**

Assumption 3 is true for example if \( \alpha \) is periodic. Such an assumption makes sense. One can easily imagine that \( \alpha \) repeats itself everyday. Of course, it must be validated empirically. One can also suppose that \( \alpha \) is random but independent of the other processes. In this case, \( \alpha \) would act as a random environment and if we assume that \( \alpha \) is stationary and ergodic, then Assumption 3 holds almost surely. However, in this case, all computations are conditional on the environment.

In order to obtain the asymptotic behavior of the prices, there are two cases to be taken into account: \( C_a + C_b > 0 \) and \( C_a + C_b = 0 \).

### 4.1.1 Case \( C_a + C_b > 0 \)

First, assume that

\[
\gamma_1 = \sum_{(x,y) \in \mathbb{N}^2} xy \left( \frac{\mu^b}{\lambda^b} \right)^{x/2} \left( \frac{\mu^a}{\lambda^a} \right)^{y/2} f(x,y) < \infty. \tag{12}
\]

Now, from Abramowitz and Stegun (1972, p. 376), \( I_n(z) = \frac{1}{\pi} \int_0^\pi e^{\cos \theta} \cos(n\theta) d\theta \), so for any \( x \in \mathbb{N} \), \( I_n(z) \leq e^z \). In this case, it follows from Lemma 3.2 and Lemma 3.4 that

\[
E_Q(\tau_1) = \sum_{(x,y) \in \mathbb{N}^2} xy \left( \frac{\mu^b}{\lambda^b} \right)^{x/2} \left( \frac{\mu^a}{\lambda^a} \right)^{y/2} f(x,y) \int_0^\infty \int_0^\infty t \wedge s g_{x,b}(t)g_{y,a}(s) dt ds
\]

\[
\leq \frac{\gamma_1}{\max(C_a, C_b)} < \infty,
\]

where \( g_{y,a}(s) = \frac{1}{\lambda} I_s \left( 2s \sqrt{\lambda^a \mu^a} \right) e^{-s(\lambda^a + \mu^a)} \) and \( g_{x,b}(s) = \frac{1}{\lambda} I_s \left( 2s \sqrt{\lambda^b \mu^b} \right) e^{-s(\lambda^b + \mu^b)} \).

Then, under Assumptions 1–2 and under model \( Q \), \( V_n/n \rightarrow E_Q(\tau_1) < \infty \) a.s.
Assumption 3 and Lemma 3.8, one then finds that under model $L$, $V_n/n$ converges in probability to $c_1 = E_Q(\tau_1)/\nu$. Finally, using Propositions A.1–A.2, one finds that under $L$, $N_t/t$ converges in probability to $\frac{1}{\sigma_1} = \nu/E_Q(\tau_1)$. In addition, $\frac{N_{[nt]} - nt/\sigma_1}{\sqrt{n}} \overset{\text{in probability}}{\to} \frac{1}{\sigma_1} \mathbb{W}(t)$, where $\mathbb{W}$ is a Brownian motion. This follows from the convergence of $V_n$, under $Q$, to a Brownian motion. It also holds under $L$, using Assumption 3.

### 4.2.1 Case $C_a = C_b = 0$

Assume that

$$\gamma_0 = \sum_{(x,y) \in \mathbb{N}^2} xyf(x, y) < \infty. \quad (13)$$

Then it follows from Lemma 3.8 and Proposition A.4 that

$$T^\mathbb{P}_L[\tau_1 > T] \overset{T \to \infty}{\to} c_0 = \frac{\gamma_0}{\nu \pi \sqrt{\lambda_\alpha \lambda_b}}.
$$

As a result, using Propositions A.1–A.2 with $f(n) = n \log n$, one finds that under $L$, $N_t/(t/\log t)$ converges in probability to $\frac{1}{c_0} = \frac{\nu \pi \sqrt{\lambda_\alpha \lambda_b}}{\gamma_0}$. In particular, if $a_n = n \log n$, then $N(a_n)/n$ converges in probability to $\frac{1}{c_0}$. Also, $\frac{V_m - c_0 \log n}{n/\log n} \overset{\text{in probability}}{\to} \frac{1}{\nu} \mathcal{V}_1$, where $\mathcal{V}_1$ is a stable process of index 1. It then follows that $\frac{N(a_n \log _n)}{n/\log n} \overset{\text{in probability}}{\to} -\frac{1}{c_0 \nu} \mathcal{V}_1$. Note that $\mathcal{V}_1$ is the weak limit of $\frac{\nu n}{n} - c_0 \nu \log n$ under $Q$, and $\mathcal{V}_1 = \hat{\mathcal{V}}_1 + d_0$, where $d_0$ is the limit of $nb_n - c_0 \nu \log n$, where $b_n = E_Q\{\sin(\tau_1/n)\}$. Next, it follows from Feller (1971) that the characteristic function of $\mathcal{V}_1$ is $e^{\psi(\zeta)}$, where

$$\psi(\zeta) = -|\zeta| c_0 \nu \left\{ \frac{\pi}{2} + i \text{sgn}(\zeta) \log |\zeta| \right\}.$$

### 4.2 Asymptotic Behavior of the Price Process

Under no other additional hypothesis on $f$ and $\tilde{f}$ than Assumption 2, the sequence $(\xi_t)$ of price changes is an ergodic Markov chain with transition matrix $\Pi$; the sequence is also independent from $N_t$. Note that $P(\xi_2 = \delta|\xi_1 = \delta) = \sum_{(i, j) \in \mathbb{N}^2} f(i, j)P^{up}(i, j)$ and $P(\xi_2 = \delta|\xi_1 = -\delta) = \sum_{i, j} f(i, j)P^{up}(i, j)$, so the associated transition matrix $\Pi$ is given by

$$\Pi = \begin{bmatrix} P(\xi_2 = -\delta|\xi_1 = -\delta) & P(\xi_2 = \delta|\xi_1 = -\delta) \\ P(\xi_2 = -\delta|\xi_1 = \delta) & P(\xi_2 = \delta|\xi_1 = \delta) \end{bmatrix},$$

with stationary distribution $(\nu, 1 - \nu)$ satisfying

$$\nu = P(\xi_1 = -\delta) = \frac{P(\xi_2 = -\delta|\xi_1 = \delta)}{P(\xi_2 = -\delta|\xi_1 = \delta) + P(\xi_2 = \delta|\xi_1 = -\delta)}.$$

If $[c]$ stands for the largest integer smaller or equal to $c$, then the sequence $\mathcal{W}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} (\xi_i - E(\xi_i))$ converges in law to $\sigma \mathcal{W}(t)$, where $\mathcal{W}$ is a Brownian motion, and the variance $\sigma^2$ is given by

$$\sigma^2 = 4\delta^2 \left[ \nu(1 - \nu) + \nu \sum_{k=1}^{\infty} \left( (\Pi^k)_{11} - \nu \right) - (1 - \nu) \sum_{k=1}^{\infty} \left( (\Pi^k)_{21} - \nu \right) \right], \quad (14)$$

with $(\Pi^k)_{ij}$ being the element $(i, j)$ of $\Pi^k$. 

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Remark 4.4 If \( \tilde{f} = f \), then the variables \( \xi_j, j \geq 1 \), are i.i.d., which is the case considered by Cont and de Larrard (2013). This is why our formula (14) is different. In fact,

\[
P(\xi_2 = \delta | \xi_1 = \delta) = \sum_{(i,j) \in \mathbb{N}^2} f(i, j) P_{up}(i, j)
\]

and

\[
P(\xi_2 = \delta | \xi_1 = -\delta) = \sum_{i,j} \tilde{f}(i, j) P_{up}(i, j) = \sum_{i,j} f(i, j) P_{up}(i, j)
\]

\[= P(\xi_2 = \delta | \xi_1 = \delta).\]

Note also that the variables \( \xi_j, j \geq 1 \), are independent from \( \tau_1, \ldots, \tau_n \). However, unless \( Q^a = Q^b \) and \( f \) is symmetric, one cannot conclude that \( P(\xi_i = \delta) = 1/2 \).

Finally, the price process \( S \) can be expressed as

\[
S_t = S_0 + \sum_{i=1}^{N_{ant}} \xi_i, \quad t \geq 0.
\]

To state the final results, set \( a_n = n \log n \) or \( n \), according as \( C_a + C_b = 0 \) or not. Then, using the results of Section 4.1, \( N_{ant}/n \) converges in probability to \( t/c \), where \( c = c_0 \) or \( c = c_1 \) according as \( C_a + C_b = 0 \) or not. It is then easy to show that \( n^{-1/2} \sum_{i=1}^{N_{ant}} \{ \xi_i - E(\xi_1) \} \sim \frac{\sigma}{\sqrt{c}} \tilde{W} \), where \( \tilde{W} \) is a Brownian motion. In fact, for any \( t \geq 0 \), \( \tilde{W}_t = \sqrt{c} W_{t/c} \). Next,

\[
S_{ant} - nt E(\xi_1)/c = \sum_{i=1}^{N_{ant}} \{ \xi_i - E(\xi_1) \} + E(\xi_1)(N_{ant} - nt/c).
\]

This expression shows that there are really two sources of randomness involved in the asymptotic behavior of \( S_{ant} - nt E(\xi_1)/c \). As before, one must consider the cases \( C_a + C_b > 0 \) and \( C_a + C_b = 0 \).

4.2.1 \( C_a + C_b > 0 \)

In this case, setting \( W_n(t) = \{ S_{nt} - nt/c_1 E(\xi_1) \} / \sqrt{n} \), then \( W_n \sim \tilde{\sigma} W \), where \( W \) is a Brownian motion and

\[
\tilde{\sigma} = \left[ \frac{\sigma^2}{c_1} + \frac{(E(\xi_1))^2}{c_1^3} \right]^{1/2}.
\]

In fact, \( \tilde{\sigma} W_t = \frac{\sigma}{\sqrt{c_1}} \tilde{W}_t + \frac{E(\xi_1)}{c_1^{3/2}} W_{t/c_1} \), where \( \tilde{W} \) and \( W \) are the two independent Brownian motions appearing respectively in the asymptotic behaviour of the Markov chain and the counting process. Note that the volatility \( \tilde{\sigma} \) could be estimated by taking the standard deviation of the price increments every 10 minutes, as proposed in Cont and de Larrard (2013); see also Swishchuk et al. (2016). More generally, if \( \Delta \) is the time in seconds between successive prices and \( s_{\Delta} \) is the corresponding standard deviation of the price increments over interval of size \( \Delta \), then \( \tilde{\sigma} = s_{\Delta}/\sqrt{\Delta} \).

4.2.2 \( C_a + C_b = 0 \)

In this case, if \( E(\xi_1) = 0 \), then using Eq. 15, one obtains that \( S_{n \log nt} / \sqrt{n} \sim \frac{1}{\sqrt{c_0}} W_t \), where \( W \) is the Brownian motion resulting from the convergence of the Markov chain.
Table 1  Spread distribution in cents for Facebook, from November 3rd, 2014 to November 7th, 2014

| Day | Spread 1 | Spread 2 | Spread 3 | Spread 4 | Spread 5 | Ave.  |
|-----|----------|----------|----------|----------|----------|-------|
| 1   | 91.6%    | 91.8%    | 89.7%    | 88.4%    | 93.6%    | 91.0% |
| 2   | 7.6%     | 8.0%     | 10.1%    | 11.1%    | 5.9%     | 8.5%  |
| > 2 | 0.8%     | 0.2%     | 0.2%     | 0.5%     | 0.5%     | 0.5%  |

However, if $E(\xi_1) \neq 0$, then $(S_{nt} - nt/c_0 E(\xi_1)) / (n/ \log n) \rightsquigarrow -\frac{E(\xi_1)}{c_0 V} V_t$, where $V$ is the stable process defined in Section 4.1.2.

Remark 4.5 Note that in Cont and de Larrard (2013), $E(\xi_1) = 0$, so the limiting process is a Brownian motion whether $C_a + C_b = 0$ or $C_a + C_b > 0$.

4.3 Conditioned Limit of the Price Process

In general, what one wants to achieve in rescaling the price process $S$ is to replace a discontinuous process by a more amenable process if possible, over a given time interval. However, on this time interval, the price is known to be positive, so the limiting distribution should be positive as well.

Fig. 1  Graphs of $M^a_{it}/t$ and $M^b_{it}/t$ for each of the five days

Fig. 1  Graphs of $M^a_{it}/t$ and $M^b_{it}/t$ for each of the five days
If the unconditioned limit is a Brownian motion, then the conditioned limit, i.e., conditioning on the fact that the Brownian motion is positive, is called a Brownian meander (Durrett et al. 1977; Revuz and Yor 1999). If the unconditioned limit is a stable process, then the conditioned limit could be called a stable meander. See, e.g., Caravenna and Chaumont (2008) for more details. Note that according to Durrett et al. (1977), a Brownian meander $W_t^+$ over $(0, 1)$ has conditional density

$$P(W_t^+ \in dy|W_s^+ = x) = \{\phi_{t-s}(y-x) - \phi_{t-s}(y+x)\} \left(\frac{\Phi_{1-t}(y) - 1/2}{\Phi_{1-s}(x) - 1/2}\right),$$

$0 < s < t < 1, x, y > 0$, where $\Phi_t$ is the distribution function of a centered Gaussian variable with variance $t$ and associated density $\phi_t$. It then follows that the infinitesimal generator $H_t$ of $W_t^+$ is given by

$$H_t f(x) = f'(x)\{1 + \phi_{1-t}(x)\} + \frac{f''(x)}{2}, \quad x > 0.$$

### 5 Estimation of Parameters

In order to have identifiable parameters, one has to answer the following question about $\alpha$: What happens if $\alpha$ is multiplied by a positive factor $h$? Then, the value $v$ in Assumption 3

![Graphs of $M^a_t/t$ and $M^b_t/t$ for five days](image-url)
is multiplied by $h$. Thus the parameters $\lambda^a$, $\lambda^b$, $\mu^a$, and $\mu^b$ are all divided by $h$, since for example, $\lambda^a_t = \lambda^a_0 \alpha t$. As a result, $E_Q(\tau_1)$ is then multiplied by $h$ and so is $\gamma_0$. It then follows that $c_0$ and $c_1$ are invariant by any scaling. So, one could normalize $\alpha$ so that $v = 1$. This is what we will assume from now on. The estimation of the parameters will then be easier.

Next, one of the assumptions of the model is that the size of the orders are constant, which is not the case in practice. So in view of applications, and depending on the statistics of sizes for level-1 orders, if the chosen size is 100 say, then an order of size 324 would count for 3.24 orders.

Assume that data are collected over a period of $n$ days. Recall that time 0 corresponds to the opening of the market at 9:30:00 ET. Let $\Lambda^b_{it}$ and $\Lambda^a_{it}$ be the number of limit orders for bid and ask respectively up to time $t$ (measured in seconds) for day $i$. Further let $t_d$ be the number of seconds considered in a day. Typically, $t_d = 23400$. Finally, let $M^b_{it}$ and $M^a_{it}$ be the number of market orders and cancellations for bid and ask respectively up to time $t$ (measured in seconds) for day $i$. For any $i \geq 1$, set $v_i = \{A_{itd} - A_{(i-1)t_d}\}/t_d$, and set $\hat{v} = \bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i$. Then for any $i \geq 1$, one should have approximately

$$
\hat{\mu}^a v_i = \frac{M^a_{itd}}{t_d}, \quad \hat{\mu}^b v_i = \frac{M^b_{itd}}{t_d},
$$

$$
\hat{\lambda}^a v_i = \frac{\Lambda^a_{itd}}{t_d}, \quad \hat{\lambda}^b v_i = \frac{\Lambda^b_{itd}}{t_d}.
$$

Fig. 3 Graphs of $\Lambda^a_{it}/t$ and $\Lambda^b_{it}/t$ for each of the five days
Having assumed that \( v = 1 \), one can set

\[
\hat{\mu}^a = \frac{1}{ntd} \sum_{i=1}^{n} M_{itd}^a, \quad \hat{\mu}^b = \frac{1}{ntd} \sum_{i=1}^{n} M_{itd}^b,
\]

\[
\hat{\lambda}^a = \frac{1}{ntd} \sum_{i=1}^{n} \Lambda_{itd}^a, \quad \hat{\lambda}^b = \frac{1}{ntd} \sum_{i=1}^{n} \Lambda_{itd}^b.
\]

Finally, note that the transition matrix \( \Pi \) can be estimated directly from the data, as is \( 1/c_1 \) from \( N_{t}/t \).

### 5.1 Example of Implementation

For this example, we use the Facebook data provided in Cartea et al. (2015), from November 3rd, 2014 to November 7th, 2014. At the moment, since there is no test of independence available in the literature, we can only assume that the buyers and sellers act independently of each others.

First, the results for the spread are given in Table 1, from which we can see that most of the time, the spread \( \delta \) is \( .01 \$ \).

One can see from Figs. 1, 2, 3 and 4 that as time increases, the ratios become more and more stable, enabling us to estimate the parameters \( \lambda^a, \mu^a, \lambda^b, \mu^b \) according to the formulas given in the beginning of Section 5. These estimations are reported in Table 2. It then follows that \( \hat{\lambda}^a < \hat{\mu}^a \) and \( \hat{\lambda}^b < \hat{\mu}^b \), see also Fig. 5. So with these data, we are in the case where

![Fig. 4 Graphs of \( \Lambda^a_{it}/t \) and \( \Lambda^b_{it}/t \) for five days](image-url)
\( C_a + C_b > 0 \), meaning that the unconditioned limiting price process is a Brownian motion with volatility satisfying (16).

**Remark 5.1** According to Fig. 6, on November 3rd, the ratio \( \Lambda^{a}_{1t}/M^{a}_{1t} \) is bigger than one, while the ratio \( \Lambda^{b}_{1t}/M^{b}_{1t} \) is smaller than one, meaning that most of the time, the bid queue will be depleted before the ask queue, so the price has a negative trend throughout that day. This is well illustrated in Fig. 7, where it is seen that the price indeed goes down on that day.

![Graphs of \( \Lambda^{a}_{1t}/M^{a}_{1t} \) and \( \Lambda^{b}_{1t}/M^{b}_{1t} \) for five days](image_url)
5.1.1 Estimations of $\tilde{\sigma}$

There are basically two ways of estimating $\tilde{\sigma}$. One can use the standard deviation of high-frequency data, as exemplified in Table 3, or we could use the analytic expression given by Eq. 16, as proposed in Swishchuk and Vadori (2017); Swishchuk et al. (2016).
Table 3  Estimation of \( \tilde{\sigma} = s/\sqrt{\Delta} \) using high-frequency standard deviations

| Day   | 10-minute | 5-minute | 1-minute |
|-------|-----------|----------|----------|
| 1     | 0.0040    | 0.0052   | 0.0057   |
| 2     | 0.0079    | 0.0073   | 0.0075   |
| 3     | 0.0069    | 0.0070   | 0.0082   |
| 4     | 0.0071    | 0.0062   | 0.0059   |
| 5     | 0.0038    | 0.0040   | 0.0051   |
| pooled| 0.0062    | 0.0060   | 0.0066   |

First, the estimations of \( \tilde{\sigma} \) are presented in Table 3 for each of the five days and for three frequencies: 1-minute, 5-minute and 10-minute. One can see from Table 3 that although the daily estimations differ for the three frequencies considered, pooling the data over five days reduces a lot the differences between the three frequencies. In fact, they are quite similar.

Next, to estimate \( \tilde{\sigma} \) analytically, one needs the estimation of the transition matrix \( \Pi \). With the data set, we get \( \hat{\Pi} = \begin{bmatrix} 0.4731177 & 0.5268512 \\ 0.5241391 & 0.475891 \end{bmatrix} \). It then follows that \( \hat{\nu} = 0.4987 \), so \( E(\xi_1) = 0.0026 \), and using formula (14), one obtains \( \sigma = 0.0066 \). Next, \( 1/\hat{c}_1 = 0.6194786 \), so the analytical estimation of \( \tilde{\sigma} \) is 0.0053, which is quite close to the pooled values in Table 3.

Appendix A: Auxiliary Results

**Proposition A.1** Suppose that \( V_n = X_1 + \cdots + X_n \), where the variables \( X_i \) are i.i.d. with \( x P(X_i > x) \xrightarrow{x \to \infty} c \in (0, \infty) \). Then \( \frac{V_n}{n \log n} \xrightarrow{P} c \), as \( n \to \infty \).

**Proof** First, for any \( s > 0 \) and \( T > 0 \),

\[ s \int_T^\infty \frac{e^{-sx}}{x} dx = s \int_T^\infty \frac{e^{-y}}{y} dy = -s \log(Ts)e^{-Ts} + s \int_T^\infty \log(y)e^{-y} dy, \]

so as \( s \to 0 \),

\[ s \int_T^\infty \frac{e^{-sx}}{x} dx \sim -s \log s. \]

Next, for any non negative random variable \( X \) and any \( s \geq 0 \),

\[ \mathbb{E}[e^{-sX}] = 1 - s \int_0^\infty P(X > x)e^{-sx} dx. \]

As a result, if \( P(X > x) \sim c/x \), as \( x \to \infty \), then, as \( s \to 0 \),

\[ \mathbb{E}[e^{-sX}] = 1 + cs \log s + o(s \log s). \]

Therefore, setting \( a_n = n \log n \), one obtains, for a fixed \( s > 0 \),

\[ \mathbb{E}[e^{-sV_n/a_n}] = \left[ \mathbb{E}[e^{-sX_1/a_n}] \right]^n \]

\[ \xrightarrow{n \to \infty} e^{-cs} \]

since \( \frac{a_n}{n \log (sa_n)} \to s \) as \( n \to \infty \). Hence, \( V_n/a_n \xrightarrow{P} c \), as \( n \to \infty \).
**Proposition A.2** Suppose that $V_n/f(n) \xrightarrow{P_r} c$, as $n \to \infty$, where $f(n) \to \infty$ is regularly varying of order $\alpha$. Define $N_t = \max[n \geq 0; V_n \leq t]$ and suppose that for some function $g$ on $(0, \infty)$, $f \circ g(t) \sim g \circ f(t) \sim t$, as $t \to \infty$. Then $N_t/g(t) \xrightarrow{P_r} c^{-1/\alpha}$.

**Proof** The proof is similar to the proof of the renewal theorem in Durrett (1996)[Theorem 7.3]. By definition, $V_{N_t} \leq t < V_{N_t+1}$. As a result,

$$\frac{V_{N_t}}{f(N_t)} \leq \frac{t}{f(N_t)} < \frac{V_{N_t}}{f(N_t + 1)} \cdot$$

By hypothesis, $V_n/f(n)$ converges in probability to $c \in (0, \infty)$, as $n \to \infty$. Also, since $V_n$ is finite for any $n \in \mathbb{N}$, it follows that $N_t$ converges in probability to $\infty$ as $t \to \infty$. Next, since $f(n + 1)/f(n) \to 1$ as $n \to \infty$, it follows that as $t \to \infty$, $f(N_t)/t$ converges in probability to $c$. Also, $g$ is regularly varying of order $1/\alpha$, so one may conclude that $N_t/g(t) \xrightarrow{P_r} c^{-1/\alpha}$.

**Remark A.3** If $f(t) = t \log t$, then $\alpha = 1$ and one can take $g(t) = t/\log t$.

**Proposition A.4** Set $\psi\lambda(t, x) = \int_t^\infty \frac{1}{u} I_x(2u\lambda)e^{-2u^2}du$, for any $t$, $x$, $\lambda > 0$. Then there exists a constant $C$ so that for any $x$, $\lambda > 0$, and any $t \geq \frac{1}{2\lambda}$, $\psi\lambda(t, x) \leq \frac{C}{\sqrt{2\lambda}}$.

**Proof** First, note that $\psi\lambda(t, x) = \psi_{1/2}(2\lambda t, x)$. It is well-known that

$$I_x(z) = \frac{1}{\pi} \int_0^\pi e^z \cos \theta \cos(x\theta) d\theta \leq \frac{1}{\pi} \int_0^\pi e^z \cos \theta d\theta \leq \frac{1}{2} + \frac{1}{\pi} \int_0^1 \frac{e^z}{\sqrt{1 - s^2}} ds.$$

Next, set $E_1(u) := \int_u^\infty \frac{e^{-w}}{w} dw$, $u > 0$. Then

$$\psi_{1/2}(t, x) \leq \int_t^\infty \frac{e^{-u}}{u} \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^1 \frac{e^{us}}{\sqrt{1 - s^2}} ds \right\} du = \frac{1}{2} E_1(t) + \frac{1}{\pi} \int_t^\infty \int_0^1 \frac{e^{-su}}{u \sqrt{2 - s}} ds du = \frac{1}{2} E_1(t) + \frac{1}{\pi} \int_0^1 \frac{E_1(st)}{\sqrt{2 - s}} ds.$$

According to Olver et al. (2010, Section 6.8.1), $E_1(u) \leq e^{-u} \ln (1 + 1/u)$ for any $u > 0$. Furthermore, $\ln(1 + x) \leq x$ and $\ln(1 + x) \leq x^{2/5}$ for any $x \geq 0$. As a result,

$$\psi_{1/2}(t, x) \leq \frac{e^{-t}}{2t} + \frac{t^{-1/2}}{\pi} \int_0^t s^{-9/10} e^{-s} ds \leq \frac{e^{-t}}{2t} + \frac{\Gamma\left(\frac{1}{10}\right)}{\pi t^{1/2}} \leq Ct^{-1/2}$$

for any $t \geq 1$, where $C = \frac{e^{-1}}{2} + \frac{\Gamma\left(\frac{1}{10}\right)}{\pi}$.
Appendix B: Proofs

Proof of Lemma 3.4 From Olver et al. (2010)[Formula 10.30.4], for fixed \(v\), \(I_v(z) \sim \frac{e^z}{\sqrt{2\pi z}}\) as \(z \to \infty\). Also, from Abramowitz and Stegun (1972, p. 376), \(I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z\cos \theta} \cos(n\theta) d\theta\), so for any \(x \in \mathbb{N}\), \(I_n(z) \leq e^z\). Thus, as \(T \to \infty\),

\[
\mathbb{P}_x[\sigma_Y > T] = \left(\frac{\mu}{\lambda}\right)^{x/2} \int_T^\infty x \frac{I_x(s)}{\sqrt{2\pi s\lambda}} e^{-s(\lambda+\mu)} ds
\]

\[
\sim \left(\frac{\mu}{\lambda}\right)^{x/2} \int_T^\infty \frac{x}{\sqrt{2\pi s\lambda}} e^{-s(\lambda+\mu)} ds
\]

\[
\sim \left(\frac{\mu}{\lambda}\right)^{x/2} \frac{x}{\sqrt{2\pi \lambda}} \int_T^\infty s^{-3/2} e^{-sc} ds.
\]

Also, for any \(x \in \mathbb{N}\),

\[
\mathbb{P}_x[\sigma_Y > T] \lesssim x \left(\frac{\mu}{\lambda}\right)^{x/2} \int_T^\infty s^{-1} e^{-sc} ds.
\]

(17)

Consequently, if \(\lambda = \mu, C = 0\) and

\[
\mathbb{P}_x[\sigma_Y > T] \sim \frac{x}{2\lambda\sqrt{\pi}} \int_T^\infty s^{-3/2} ds \sim \frac{x}{2\lambda\sqrt{\pi}} \frac{2}{\sqrt{T}} \sim \frac{x}{\lambda\sqrt{\pi}}.
\]

This agrees with the result proved in Cont and de Larrard (2013). However, if \(\lambda < \mu\), using the change of variable \(u = sC\), one gets

\[
\mathbb{P}_x[\sigma_Y > T] \sim C^{1/2} \left(\frac{\mu}{\lambda}\right)^{x/2} \frac{x}{2\sqrt{\pi} \sqrt{\lambda}} \int_{TC}^\infty u^{-3/2} e^{-u} du
\]

\[
\sim \left(\frac{\mu}{\lambda}\right)^{x/2} \frac{x}{\sqrt{2\pi} \sqrt{\lambda}} \left[ \frac{e^{-TC}}{\sqrt{T}} - \sqrt{C} \Gamma \left(\frac{1}{2}, TC\right) \right].
\]

To compute the expectation in the case where \(\lambda = \mu\), note that for large enough \(T\), \(\mathbb{E}_x[\sigma_Y] = \int_0^\infty \mathbb{P}_x[\sigma_Y > t] dt \geq \frac{x}{2\lambda\sqrt{\pi}} \int_T^\infty \frac{1}{\sqrt{T}} dt = \infty\), whereas if \(\lambda < \mu\), for a sufficiently large \(T\), there are finite constants \(C_1\) and \(C_2\) such that for any \(0 \leq \theta < C\),

\[
\mathbb{E}_x[e^{\theta \sigma_Y}] = 1 + \theta \int_0^\infty e^{\theta t} \mathbb{P}_x[\sigma_Y > t] dt \leq C_1 + \theta C_2 \int_T^\infty e^{-t(C-\theta)} dt
\]

\[
= C_1 + C_2 \frac{e^{-T(C-\theta)}}{(C-\theta)} < \infty.
\]

\[\square\]

Proof of Proposition 4.1 Let \(F_{n,Q}(t; x, y)\) and \(F_{n,L}(t; x, y)\) denote the cdf of \(S^n_0\) and \(S^n_L\), respectively, starting from \(z_0 = (x, y)\), with densities \(f_{n,Q}(t; z_0)\) and \(f_{n,L}(t; z_0)\), where \(F_{n,Q}(\cdot; z_0)\) is the convolution of \(F_{1,Q}(n-1)\) times with \(F_{1,Q}(\cdot; z-0)\). The result will be proven by induction. The base case \(n = 1\) is given in Corollary 3.7. Assume the result is true for any \(m \leq n \in \mathbb{N}\). Then by Corollary 3.7 and the induction hypothesis,

\[
F_L(t; x, y) = F_Q(A_t; x, y) \quad \text{and} \quad f_{n,L}(t; x, y) = f_{n,Q}(A_t; x, y)\alpha_t.
\]

(18)
Also, by the definition of $\tau_n$ and $V_n$, under Assumption 2, if $z_0 = (x, y)$, then

$$F_{n,L}(t; z_0) = P_{L}[V_{n+1} \leq t \mid q_0 = z_0] = P_{L}[V_n \leq t, \tau_{n+1} \leq t - V_n \mid q_0 = z_0]$$

$$= \sum_z f(z) \int_0^t P_{L}[\tau_{n+1} \leq t - u \mid q_u = z] f_{n,L}(u; z_0) du$$

$$= \sum_z f(z) \int_0^t P_{Q}[\tau_{n+1} \leq A(n+1) t - u \mid q_u = z] f_{n,Q}(A u; z_0) \alpha_u du$$

$$= \int_0^t F_{1,Q}(A t - u) f_{n,Q}(A u; z_0) \alpha_u du = \int_0^A F_{1,Q}(A t - u) f_{n,Q}(u; z_0) du$$

$$= \int_0^A F_{1,Q}(A t - u) dF_{n,Q}(u; z_0) = F_{1,Q}(A t - u)$$

where we used the fact that for any $s \geq 0$, $\alpha^{(n+1)}(s) = \alpha(s + u)$ given $V_n = u$, so $A^{(n+1)}(t) = \int_0^t \alpha(s + u) ds = A_{t+u} - A_u$. Furthermore, in the last equality we used the fact that for $X$ and $Y$, non-negative independent random variables,

$$FX + Y(t) = P[X + Y \leq t] = FX * FY(t) = \int_0^t FX(t - x) dFY(x),$$

with $F_X$ and $F_Y$ denoting the cdfs of $X$ and $Y$. Furthermore, starting $q_0$ from distribution $f$, one obtains that $P_{L}[V_n \leq t] = P_{Q}[V_n \leq A_t]$.

### References

Abramowitz M, Stegun IE (1972) Handbook of mathematical functions with formulas, graphs, and mathematical tables, volume 55 of applied mathematics series. National bureau of standards, tenth edition

Billingsley P (1995) Probability and measure. Wiley series in probability and mathematical statistics, 3rd edn. Wiley, New York. A Wiley-Interscience Publication

Caravenna F, Chaumont L (2008) Invariance principles for random walks conditioned to stay positive. Ann Inst Henri Poincaré Probab Stat 44(1):170–190

Cartea A, Jaimungal S, Penalva J (2015) Algorithmic and high-frequency trading. Cambridge University Press, Cambridge

Cont R, de Larrard A (2013) Price dynamics in a Markovian limit order market. SIAM J Financial Math 4(1):1–25

Durrett R (1996) Probability: Theory and examples, 2nd edn. Duxbury Press, Belmont

Durrett RT, Iglehart DL, Miller DR (1977) Weak convergence to Brownian meander and Brownian excursion. Ann Probab 5(1):117–129

Feller W (1971) An introduction to probability theory and its applications, volume II of Wiley series in probability and mathematical statistics, 2nd edn. Wiley, New York

Olver FW, Lozier DW, Boisvert RF, Clark CW (2010) NIST Handbook of mathematical functions. Cambridge University Press, New York

Revuz D, Yor M (1999) Continuous martingales and Brownian motion, volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental principles of mathematical sciences], 3rd edn. Springer, Berlin

Smith E, Farmer JD, Gillemot L, Krishnamurthy S (2003) Statistical theory of the continuous double auction. Quantitative Finance 3(6):481–514
Swishchuk A, Cera K, Schmidt J, Hofmeister T (2016) General semi-Markov model for limit order books: theory, implementation and numerics. arXiv:1608.05060
Swishchuk AV, Vadori N (2017) A semi-Markovian modeling of limit order markets. SIAM Journal on Financial Mathematics. (in press)
Van Leeuwaarden JS, Raschel K et al (2013) Random walks reaching against all odds the other side of the quarter plane. J Appl Probab 50(1):85–102

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