Cops and robber on grids and tori

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Abstract

This paper is a contribution to the classical cops and robber problem on a graph, directed to two-dimensional grids and toroidal grids. These studies are generally aimed at determining the minimum number of cops needed to capture the robber and proposing algorithms for the capture. We apply some new concepts to propose a new solution to the problem on grids that was already solved under a different approach, and apply these concepts to give efficient algorithms for the capture on toroidal grids. As for grids, we show that two cops suffice even in a semi-torus (i.e. a grid with toroidal closure in one dimension) and three cops are necessary and sufficient in a torus. Then we treat the problem in function of any number \( k \) of cops, giving efficient algorithms for grids and tori and computing lower and upper bounds on the capture time. Conversely we determine the minimum value of \( k \) needed for any given capture time and study a possible speed-up phenomenon.

Keywords: Cops, Robber, Capture time, Grid, Tori, Speed-up.

1 Introduction

Cops and robber is a well known game on a graph. Cops and robber stay on the vertices and, at each round, can move to adjacent vertices or stay still. A round is composed of a parallel move of the cops followed by a move of the robber, and this is captured when a cop reaches its vertex. Initially the cops select their positions, then the robber chooses its own accordingly.

Such a simple game, initially defined for one cop by Nowakowski and Winkler [20] and Quillot [21], has generated a complex theory, and the problem has been studied extensively by several points of view. The first aim was to find graphs where the game can be won by a single cop. A
The large part of the literature has then been devoted to solve the problem on a specific class of graphs with a fixed number of cops. A general survey can be found in the comprehensive book by Bonato and Nowakowski [3] which brings together the main structural and algorithmic results on the field.

In [11] it was originally shown that determining if \( k \) cops can catch a robber is EXPTIME-complete if the initial positions are given. Then in [9], beside other results, it was proved that the finding the minimal number of cops able to catch the robber is NP-hard.

Many variants of the basic problem exist, for particular classes of graphs, such as considering more than one robber; or cops and robber that move at different speeds; or cops that can be invisible for some rounds; or their initial positions are imposed. In a recent evolution of the game [5] the robber must try to escape surveillance, that is being distant from any cop more than a fixed value \( d \). Other important related problems, born in the field of distributed computing to model moving agents, are graph search, intruder capture, network decontamination, see for instance [17] for graph search, [14, 15] for network decontamination and [3, 10] for intruder capture.

The main goals studied in the literature are to determine the cop number needed for the capture, or to attain the minimum capture time once the team is given, where these parameters depend on the network topology. In [4] it is shown that determining the cop number \( k \) needed for the capture in no more than a given capture time is NP-hard for general graphs.

Results on two-dimensional grids can be found in [6, 7, 19, 22], where the visibility power of each cop is limited to edges and vertices of the column (row) where the cop is located. In [6, 19] the cops win when they can see the robber, while in the other papers the capture is as usual. In [22] it is shown that the formulation of the problem with limited visibility has application in motion planning of multiple robots and, if the robber moves as fast as the cops, two cops are necessary and sufficient for the capture. The study of [22] has been revisited in [7] and algorithms for the capture using one, two or three cops having constant maximal speed are defined. In [2] the number of cops needed to the capture a robber who can move at arbitrary speed along the edge grid is determined. A more recent work [8] considers a problem where the initial positions of cops and robber are chosen randomly. In [1] the study is extended to multi dimensional grids, showing that \( n \) cops are necessary if one wants to catch the robber starting from all possible initial configurations. Finally in [18] the capture time is studied for a two-dimensional grid.

There is a wealth of possibilities, however, we limit our treatment to the standard game on 2-dimensional grids starting from the results of [18], and
then extend it to toroidal grids. Our new results are the following. First we introduce some new concepts on the capture valid for general graphs, showing how the results of [18] can be found with a new different approach. This approach is then applied to semi-tori (i.e. grids with toroidal closure in one dimension) and to tori. We give efficient algorithms for the capture that use two or three cops respectively, and show that these numbers are the minimal possible. We then treat the capture problem in function of any (hence not necessarily minimum) number of cops, giving efficient algorithms also for these cases.

For any given capture time \( \bar{t} \) we also determine the minimum number of cops needed for the capture in at most \( \bar{t} \) rounds. In this context we adopt the concept of work \( w_k \) of an algorithm run by \( k \) cops in time \( t_k \) inherited from parallel processing, where \( w_k = k \cdot t_k \) [12, 13]. Using two teams of \( i \) or \( j \) cops for the capture, the speed-up between the action of \( j \) over \( i \) cops, with \( j > i \), is then defined as \( \frac{w_i}{w_j} \). If the algorithms run by the two teams are provably optimal, the speed-up is an important measure of the advantage of using more cops. A related study has been carried out in [16] for butterfly decontamination.

2 Basic model and properties

We adopt the basic model of the cops and robber problem on an undirected and connected graph \( G = (V, E) \). One or more cops and one robber, collectively called agents, are placed on the vertices of \( G \). The game develops in consecutive rounds, each composed of a cops turn followed by a robber turn. In the cops turn each cop may move to an adjacent vertex or stay still. In the robber turn, the robber may move to an adjacent vertex or stay still. The game is over when a cop reaches the vertex of the robber.

The initial positions of the cops are arbitrarily chosen, then the initial position of the robber is chosen accordingly. The aim of the cops is capturing the robber in a number of rounds as small as possible, called capture time \( t \); while the robber tries to escape the capture as long as possible. If needed two or more cops can stay on the same vertex and move along the same edge. All agents are aware all the time of the locations of the other agents. \( k \), the cop number, denotes the smallest number of cops needed to capture the robber.

As discussed in the following sections we will direct our study to 2-dimensional grids, whose bounding edges may also be connected in the form of a semi-torus or a torus. However, first we pose some preliminary prop-
erties valid for all undirected and connected graphs \( G = (V,E) \), partly extending known facts. For a vertex \( v \in V \), let \( \mu(v) \) denote the set of neighbors of \( v \), and let \( \nu(v) = \mu(v) \cup \{v\} \) denote the closed set of neighbors. We pose:

**Definition 1.** A siege \( S(v) \) of a vertex \( v \) is a minimum set of vertices containing cops, such that at least one vertex \( u \in S(v) \) is in \( \mu(v) \), and \( \bigcup_{w \in S(v)} \nu(w) \supseteq \mu(v) \). Among all the sieges of \( v \), \( \bar{S}(v) \) denotes the one of minimal cardinality.

Definition 2 depicts the situation shown in figure 1, where black and white circles on the graph denote vertices occupied by the cops, or by the robber, respectively. Let the robber be in \( v \), and assume that the cops have just been moved into the vertices of \( S(v) \). Now the robber has to complete the current round, but whether it moves or stands still it will be captured in the next round. In fact the condition \( \bigcup_{w \in S(v)} \nu(w) \supseteq \mu(v) \) indicates that all the escape routes for the robber have been cut. We immediately have:

**Lemma 1.** The robber is captured in round \( i \) if and only if at round \( i - 1 \) the robber is in a vertex \( v \) and there is a siege \( S(v) \).

![Figure 1: A minimal siege \( \bar{S}(v) \) with the robber (white circle) in \( v \) and three cops (black circles) in \( \bar{S}(v) \).](image)

A lower bound on the number \( k \) of cops needed to capture the robber immediately follows from Lemma 1, namely:

**Lemma 2.** Let \( v \) be a vertex for which \( \bar{S}(v) \) has minimal cardinality among all the vertices of the graph. Then \( k \geq |\bar{S}(v)| \).

Based on the definition of siege we can also establish a lower bound on the capture time \( t \). In fact once the initial positions of the cops have been chosen, we shall determine an initial position and a moving strategy of the robber that forces the cops to make at least a certain number of moves for establishing a siege. For this purpose we pose:
Definition 2. For a graph $G$ and an integer $e \geq 4$, an $e$-loop $E$ is a chordless cycle of $e$ vertices where each vertex of $G \setminus E$ is adjacent to at most one vertex of $E$.

Note that a single cop would chase forever a robber that moves inside an $e$-loop. We have:

Lemma 3. Let the initial positions of the $k$ cops $c_1, c_2, \ldots, c_k$ be established; let $v$ be the initial position of the robber; let $d_1 \leq d_2 \leq \cdots \leq d_k$ be the distances (number of edges in the shortest paths) of $c_1, c_2, \ldots, c_k$ from $v$; and let $h$ be the cardinality of a minimal siege for $G$, $2 \leq h \leq k$. We have:

(i) $t \geq d_1$;
(ii) if $v$ belongs to an $e$-loop, $t \geq d_h - \left\lfloor \frac{e}{2} \right\rfloor - 2$.

Proof. (i) Since the robber must be reached by one cop, and may stand still until a cop becomes adjacent to it, we have $t \geq d_1$.
(ii) At least $d_1 - 1$ rounds are needed to bring $c_1$ to a vertex adjacent to $v$, while the other cops may also move towards the robber. At this point either the robber is surrounded by a siege and the game ends in the next round with $t = d_1$; or the robber completes the current round moving away, in particular to another vertex of the $e$-loop that may be one step closer to the other cops. For the capture, $c_1$ must wait the arrival of other $h - 1$ cops to set up a siege and may chase the robber along the $e$-loop to force it to move further towards the other cops, for a maximum of $\left\lfloor \frac{e}{2} \right\rfloor - 1$ positions. Since in a siege $c_h$ must be at a distance 2 from the robber, the total number of moves of $c_h$ must be at least $d_h - (\left\lfloor \frac{e}{2} \right\rfloor - 1) - 2$, to which the final capture move in the siege must be added. \qed

Clearly between the two bounds of Lemma 3 the greater will be applied. In particular bound (ii) is greater than (i) for $d_h > d_1 + \left\lfloor \frac{e}{2} \right\rfloor + 2$.

3 Capture on grids

An elegant approach to studying the capture on an $m \times n$ grid has been presented in [18], where the grid is treated as the Cartesian product of two paths. This leads to prove that two cops are needed and the capture time is $t = \left\lceil \frac{m+n}{2} \right\rceil - 1$. We examine this problem under a different viewpoint, as a basis for studying robber capture on toroidal grids.

Formally an $m \times n$ grid $G_{m,n}$ is a graph whose vertices are arranged in $m$ rows and $n$ columns, where each vertex $v_{i,j}$, $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$ is connected to the four vertices $v_{i-1,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}$, whenever these
indices stay inside the closed intervals \([0, m-1]\) and \([0, n-1]\) respectively. Then the vertices can be divided into three sets, namely: corner vertices, where both the subscripts \(i\) and \(j\) have the values 0 or \(m-1\), and 0 or \(n-1\), respectively; border vertices, where one of the subscripts \(i\) and \(j\) has the value 0 or \(m-1\), or 0 or \(n-1\), respectively; internal vertices, i.e. all the others. Corner, border, and internal vertices have two, three, and four neighbors each.

If two vertices \(u, w\) of a grid are adjacent, the set \(\mu(u) \cap \mu(w)\) is empty; if \(w\) is at a distance two from \(u\), the set \(\mu(u) \cap \mu(w)\) contains one or two vertices. This implies that the siege \(S(u)\) has cardinality three if \(u\) is an internal vertex, or cardinality two if \(u\) is a border or corner vertex, see figure 2. This bears some initial consequences.

Figure 2: Examples of a siege \(S(u)\) in a grid, if \(u\) is an internal vertex, a border vertex, or a corner vertex.

Since the minimal siege for a grid has cardinality two, the number of cops needed to capture the robber is \(k \geq 2\) by Lemma 2. In fact it has been proved in [18] that two cops suffice. Moreover Lemma 1 shows that any algorithm using two cops must push the robber to a border or to a corner vertex to establish a siege around it, as three cops would be needed for a siege around an internal vertex. This is what our algorithm will do. Finally it can be easily shown that, wherever the cops are initially placed, there is a vertex \(v\) at distance \(d = \left\lfloor \frac{m+n}{2} \right\rfloor - 1\) where the robber can be placed, hence \(t \geq d\) by Lemma 3 case (i).

The basic concept is the one of the shadow cone of a cop \(c\), namely a zone (set of vertices) of the grid ending on the border, from where the robber is impeded by \(c\) to escape. To this end let \(c\) be in vertex \(u = v_{i,j}\) and consider two straight lines at \(\pm 45^\circ\) through \(u\) that divide the grid into four zones whose borders contain vertices placed on the two lines, called edges of the zone, and vertices placed on the border of the grid, see figure 3. The shadow cone of \(c\) is one of the four zones, chosen by \(c\). In particular the robber is said to stay within the cone if it stays in the cone but not on one of its edges.

Without loss of generality, let the shadow cone of \(c\) lay ”below” the cop as shown in figure 3. Two cases may occur, to which the following Cone
Figure 3: A cop $c$ divides the grid in four zones limited by two straight lines at $\pm 45^\circ$ passing through the vertex of $c$, and by the grid border. Two positions $x,y$ of the robber are shown, inducing different cop movements.

Rule applies:

**CONE RULE**

The cop $c$ is in vertex $v_{i,j}$, and the robber is in the shadow cone of $c$.

1. Let be the cop’s turn to move. (i) If the robber is within the cone (vertex $x$ in figure 3) the cop moves “down” to vertex $v_{i+1,j}$ thereby reducing the size of the cone while keeping the robber in it. (ii) If the robber is on an edge of the cone (e.g. in vertex $y$) the cop does not move.

2. Let be the robber’s turn to move. (iii) If the robber remains in the cone the subsequent cop’s move takes place as specified in points 1.i or 1.ii whichever applies. (iv) If the robber moves out of the cone from one of its edges (e.g. from vertex $y$, moving “up” or “to the right”), the cop moves across to vertex $v_{i,j+1}$ thereby shifting its shadow cone by one positions to keep the robber in the cone.

Once the shadow cone is established at the beginning of the operations as the one containing the robber, the role of the Cone Rule is to keep the robber in the cone, possibly moving the cone by one position to compensate the robber’s movement (point 2.iv of the rule). An important consequence is the following:

**Lemma 4.** By applying the Cone Rule, if the robber reaches an edge of the cone it will never be able to reach the opposite edge.

**Proof.** For reaching the opposite edge of the cone the robber should traverse the cone, and at each round the cop would become closer to it (point 2.iii of the rule), to become adjacent to the robber in the center of the cone and capture it. \[ \square \]

Note that if at each round the robber stands on an edge of the cone, the cop would not be able to reach the robber. In fact two cops $c_1, c_2$ are
needed for the capture. We now give the following algorithm GRID that runs in \( t = \lceil \frac{m+n}{2} \rceil - 1 \) rounds as for the algorithm of [18], but is useful for the discussion that follows on toroidal grids. W.l.o.g. we let \( m \leq n \), \( m_1 = \lceil \frac{m-1}{2} \rceil \), \( m_2 = \lceil \frac{m-1}{2} \rceil \), \( n_1 = \lceil \frac{n-1}{2} \rceil \), \( n_2 = \lceil \frac{n-1}{2} \rceil \).

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algorithm GRID(m,n)
1. initial positions of the cops \( c_1, c_2 \):
   for \( m \) even and \( n \) odd \((e|o)\), or for \( e|e\), place \( c_1 \) in \( v_{m_1,n_1} \) and \( c_2 \)
   in \( v_{m_2,n_1} \); for \( o|o\), place \( c_1 \) is in \( e_{m_1-1,n_1} \) and \( c_2 \) is in \( e_{m_1,n_1} \), see figure 4.a; for \( o|e\), place \( c_1 \) in \( e_{m_1,n_1} \) and \( c_2 \) is in \( v_{m_1,n_2} \);
   // assume to work on an \( o|o \) grid (the others are treated similarly)
   // the shadow cones are chosen so that at least one of them will contain
   the robber; assume that they lay below the cops as in figure 4
2. initial position of the robber:
   place the robber in any vertex not adjacent to a cop;
3. repeat
4. cops’ turn to move:
4.1 if (the robber is in the two cones, as vertices \( x,y \) in figure 4.b)
   move both cops one step down
4.2 else (the robber is on the edge of a cone but outside the other
   cone, vertex \( y \) in the figure) or (the robber is outside the two
   cones, e.g. after moving from vertex \( z \) of figure 4.b)
   move both cops horizontally in the direction of the robber
   (to the right in the figure);
5. robber’s turn to move:
   move the robber in any way to try to escape from the cops
6. until the robber makes its last move inside a siege;
   // the siege is established with the robber in a grid corner (figure 4.c)
7. capture the robber;
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To understand how the algorithm works some observations are in order. The cops are initially adjacent, then their cones have a large portion in common but their edges are disjoint, see figure 4. Then, up to the round in which the siege is established, they move in parallel so their mutual positions do not change. At the beginning the robber is in at least one of the shadow cones and is kept in this condition after each cops’ move. Note that to delay the capture as much as possible the robber must eventually escape from one side of a cone. By Lemma 4, however, it is forced to escape always from the same side until it ends up in a siege, in a grid corner. We can state:
Figure 4: (a) Initial placement of the two cops for grids with $m$ odd and $n$ odd, denoted as $o|o$. (b) The cops push the robber towards the border of the grid. A similar situation occurs for grids $e|o$ and $o|e$, and for grids $e|e$ exchanging rows with columns. $x, y, z$ indicate particular positions of the robber (steps 4.1 and 4.2 of algorithm GRID). (c) The final siege.

**Theorem 1.** In a grid $G_{m,n}$ two cops can capture the robber in $t = \left\lfloor \frac{m+n}{2} \right\rfloor - 1$ rounds.

**Proof.** Use algorithm GRID.

(i) *Capture.* After the execution of one round (steps 4 and 5), either the shadow cones become tighter around the robber (step 4.1), or the robber exits from the edge of one cone and moves towards the border of the grid. By Lemma 4 this exit may be repeated in further rounds only from the same edge. Then the robber will be eventually pushed by the cops to the border of the grid, and from there to a corner, surrounded by a siege. Step 7 completes the capture.

(ii) *Evaluation of $t$.* Each movement of the cops up to establishing a siege reduces by one their distance from a grid corner where the robber is eventually pushed. With the chosen initial displacement, $d = \left\lfloor \frac{m+n}{2} \right\rfloor - 1$ is the maximum distance between a grid corner and the closest cop. Then the siege is reached in $d - 1$ rounds and the capture is done in $d$ rounds.

4 Capture on toroidal grids

We now extend our study to 2-dimensional grids in the form of semi-tori $S_{m,n}$ and tori $T_{m,n}$. In $S_{m,n}$ a toroidal closure occurs in the first dimension, that is each vertex $v_{i,0}$ is connected with $v_{i,n-1}$ by an edge, $0 \leq i \leq m-1$. In $T_{m,n}$ the toroidal closure occurs in both dimensions, that is also each vertex $v_{0,j}$ is connected with $v_{m-1,j}$, $0 \leq j \leq n-1$. Then semi-tori have a top and a bottom border and tori have no borders.
4.1 Capture on semi-tori

Consider a $S_{m,n}$ with the toroidal closures on the rows, and let $m \geq 3, n \geq 4$ to avoid trivial cases. At least two cops are needed since the vertices on the border have a siege of minimum cardinality 2. In fact the following algorithm SGRID proves that two cops suffice.

\textbf{algorithm SGRID}(m,n)

1. initial positions of the cops $c_1, c_2$:
   \hspace{1em} place $c_1$ in $v_{\lfloor \frac{m-1}{2} \rfloor, 0}$; place $c_2$ in $v_{\lfloor \frac{m-1}{2} \rfloor, \lceil \frac{n}{2} \rceil}$;
   \hspace{1em} let $\gamma_1, \gamma_2$ be the shadow cones of $c_1, c_2$;
   \hspace{1em} // assume that $\gamma_1, \gamma_2$ are chosen below the cops as in figure 5a

2. initial position of the robber $r$:
   \hspace{1em} place the robber in any vertex not adjacent to a cop;

3. repeat
   4. cops’ turn to move:
      \hspace{1em} if ($r$ is outside $\gamma_1$ and $\gamma_2$) move $c_1$ and $c_2$ horizontally towards $r$
      \hspace{1em} else if ($r$ is within $\gamma_1$ and/or within $\gamma_2$) move $c_1$ and $c_2$ down
      \hspace{1em} else if ($r$ is on an edge of $\gamma_1$ (resp. $\gamma_2$) and outside $\gamma_2$ (resp. $\gamma_1$))
         \hspace{2em} move $c_2$ (resp. $c_1$) horizontally towards that edge
      \hspace{1em} else if ($r$ is on an edge of $\gamma_1$ and on an edge of $\gamma_2$)
         \hspace{2em} \{ if (the cops are in different rows)
         \hspace{3em} move down the cop in the highest row
         \hspace{2em} else move down one of the cops; \}

4. robber’s turn to move:
   \hspace{1em} move the robber in any way to try to escape from the cones

5. until the robber makes its last move inside a siege;
   \hspace{1em} // the siege is established with the robber on the lower border of the grid

6. capture the robber;

The cops $c_1, c_2$ are initially placed in row $\lfloor \frac{m-1}{2} \rfloor$, and columns 0 and $\lceil \frac{n}{2} \rceil$ respectively, so that there are two cop-free gaps of $\lfloor \frac{n}{2} \rfloor - 1$ columns between $c_1$ and $c_2$, and of $\lfloor \frac{n}{2} \rfloor - 1$ columns between $c_2$ and $c_1$ (around the semi-torus), see figure 5a. A crucial configuration is the one of a \textit{pre-siege} where one cop is on the same row of the robber and adjacent to it, and the other cop is one row above the robber and at a diagonal distance 2 from it, see figure
b, so the robber can move only one vertex down and will eventually be pushed to a siege in the bottom row (if the robber is already in the bottom row the pre-siege is in fact a siege).

Figure 5: (a) Initial positions of $c_1, c_2$ in $S_{6,9}$, with $\lfloor \frac{m-1}{2} \rfloor = 2, \lceil \frac{n}{2} \rceil = 5$. Note the cop-free gaps of 4 and 3 columns between the cops. (b) Cop paths to a pre-siege.

To understand how algorithm SGRID works observe the following.

- If the robber $r$ is initially in a gap between the cops $c_1, c_2$ and outside both cones it will always remain in that gap. In fact the cops move towards $r$ (step 4.1) until $r$ is on the edge of a cone and will not be allowed to reach the opposite edge by the Cone Rule that is enforced in steps 4.3 and 4.4. Therefore in the longest chase the robber will be captured on the bottom border, in a column of the larger gap.

- If $r$ is initially within a cone of a cop $c$ it will be kept in this cone by $c$ that moves down reducing the size of the cone around $r$ at each round (step 4.2). If $r$ reaches an edge of the cone, steps 4.3 and 4.4 maintain $r$ in the gap where the capture will take place.

- If $r$ is on an edge of the cone of $c$ and not on an edge of the other cone, and $r$ escapes from the cone in its turn to move, then $c$ moves the cone to recapture $r$ in its cone (step 4.1 or 4.3) and the gap between the cops becomes smaller. If $r$ is on an edge of both cones and escapes from one or both of them, it is recaptured in one or both cones (step 4.1 or 4.3).

- The cops are kept in the same row in steps 4.1, 4.2, and 4.3, and may occupy two adjacent rows in step 4.4. However they will never will be at a larger vertical distance, and will regain the same row with a new application of step 4.4.

- The cops always move towards $r$, but only one cop moves in a round in steps 4.3 and 4.4. This has an impact on the capture time that is
maximized if the robber forces the cops to repeat these two steps as many times as possible.

**Theorem 2.** In a semi-torus $S_{m,n}$ two cops can capture the robber in time:

(i) $t = \lceil \frac{n}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor - 2$, for $\lfloor \frac{m}{2} \rfloor \leq \lceil \frac{n-2}{4} \rceil$;

(ii) $t = \lceil \frac{n}{2} \rceil + \lceil \frac{n-2}{4} \rceil + \lfloor \frac{m}{2} \rfloor - 2$, for $\lfloor \frac{m}{2} \rfloor > \lceil \frac{n-2}{4} \rceil$.

**Proof.** Use algorithm SGRID.

*Capture.* if the robber is within or outside both cones, the two cops become closer to it with one move, and repeat the round until the robber ends up on the edge of one or both cones. Now one cop moves closer to the robber and the other remains still. This inevitably bring cops and robber in a siege or in a pre-siege condition, with the robber pushed down to a siege on the border. Then the capture takes place in the next round.

*Evaluation of $t$.* The cops always move in the direction of the robber, either horizontally or vertically, until the robber is captured in a border vertex $v_{m-1,j}$, with $1 \leq j \leq \lceil \frac{n}{2} \rceil - 1$. Let $h = \lceil \frac{n}{2} \rceil$ and $k = 2\lfloor \frac{m}{2} \rfloor$ respectively denote the sum of the horizontal and of the vertical distances between the initial positions of the two cops $c_1, c_2$ and $v_{m-1,j}$. Note that $h$ and $k$ are independent of $j$. Letting $r_1, r_2, r_3, r_4$ be the number of rounds respectively executed in steps 4.1, 4.2, 4.3, 4.4 of the algorithm, the siege is reached in $t = r_1 + r_2 + r_3 + r_4$ rounds subject to the conditions $2r_1 + r_3 = h - 2$ and $2r_2 + r_4 = k - 1$, because when the siege is reached the values of $h$ and $k$ are respectively reduced to 2 and 1. The highest value of $t$ then occurs when $r_1$ and $r_2$ are minimized. For this purpose the robber must stay on the edge of a cone and not within the other cone, for as many rounds as possible.

(i) For $\lfloor \frac{m}{2} \rfloor \leq \lceil \frac{n-2}{4} \rceil$ the robber can force the cops to always apply steps 4.3 and 4.4 if it starts in the lowest vertex of the edge of one of the cones, e.g. in vertex $v_{m-1,\lfloor \frac{n}{2} \rfloor}$, and stays still until the siege is built. The total number of rounds will be then $t = r_3 + r_4 + 1$ including the last round following the siege, subject to the conditions $r_3 = h - 2 = \lceil \frac{n}{2} \rceil - 2$ and $r_4 = k - 1 = 2\lfloor \frac{m}{2} \rfloor - 1$, for a total of $t = \lceil \frac{n}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor - 2$.

(ii) For $\lfloor \frac{m}{2} \rfloor > \lceil \frac{n-2}{4} \rceil$ the two cones $\gamma_1$ and $\gamma_2$ intersect and the cops will always be able to apply steps 4.1 and/or 4.2 for some times. To reduce $r_1$ and/or $r_2$ as much as possible the robber must start on the lowest vertex on the edge of a cone (say $\gamma_1$) and not within the other cone, and stay there until a pre-siege is built. For such a vertex $v_{i,j}$ we have $i = \lceil \frac{m-1}{2} \rceil + \lfloor \frac{n-2}{4} \rfloor$, $j = \lceil \frac{n-2}{4} \rceil$, with $1 \leq \lfloor \frac{n-2}{4} \rfloor \leq \lfloor \frac{m}{2} \rfloor - 1$. To build a pre-siege steps 4.3 and 4.4 are applied in $r_3 + r_4$ rounds, subject to the conditions $r_3 = h - 2 = \lceil \frac{n}{2} \rceil - 2$, and $r_4 = k - 1 = 2\lfloor \frac{n-2}{4} \rfloor - 1$ since $v_{i,j}$ is at a vertical distance $\lceil \frac{n-2}{4} \rceil$ from
both cops. We then have \( r_3 + r_4 = \left\lfloor \frac{n}{2} \right\rfloor + 2\left\lceil \frac{n-2}{4} \right\rceil - 3 \). Once a pre-siege is built, the robber must move down and step 4.2 is applied until \( c_1 \) reaches the siege in border in row \( m-1 \), in \( r_2 = m-1 - \left( \left\lfloor \frac{m-1}{2} \right\rfloor + \left\lceil \frac{n-2}{4} \right\rceil \right) = \left\lfloor \frac{m}{2} \right\rfloor - \left\lceil \frac{n-2}{4} \right\rceil \) rounds. In total \( t = r_2 + r_3 + r_4 + 1 = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n-2}{4} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor - 2 \).

For \( S_{6,9} \) of figure 5 we have \( \left\lfloor \frac{m}{2} \right\rfloor = 3 \) and \( \left\lceil \frac{n-2}{4} \right\rceil = 2 \), so case (ii) of Theorem 2 applies and the capture takes \( t_2 = 5 + 2 + 3 - 2 = 8 \) rounds.

Since all the vertices of a grid belong to an \( e \)-loop consisting of square cycles of \( e = 4 \) vertices, a lower bound can be established by Lemma 3 on the capture time on \( S_{m,n} \). We have:

**Lemma 5.** The capture time in a semi-torus \( S_{m,n} \) admits a lower bound \( t_{LS} = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 4 \).

**Proof.** For any vertex \( u \) of the semi-torus there is a vertex \( w \) whose distance from \( u \) is at least \( \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \) (e.g. this occurs between vertices \( v_{0,0} \) and \( v_{\left\lfloor \frac{m}{2} \right\rfloor,\left\lfloor \frac{n}{2} \right\rfloor} \)). If \( c_2 \) is initially placed in \( u \), then \( r \) can be placed in \( w \) and we have from Lemma 3:

\[
t_{LS} = d_2 - \left\lfloor \frac{e}{2} \right\rfloor - 2 = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor - 4.
\]

Letting \( t_{US} \) be the upper bound to \( t \) given in Theorem 2 we have:

**Corollary 1.** In a semi-torus \( S_{m,n} \) the ratio \( t_{US}/t_{LS} \to 1 \) for \( n/m \to \infty \) and for \( n/m \to 0 \).

**Proof.** For \( n/m \to \infty \) case (i) of Theorem 2 applies and we have:

\[
t_{US}/t_{LS} = \left( \left\lfloor \frac{n}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{4} \right\rceil - 2 \right)/\left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n-2}{4} \right\rceil - 4 \right) \to 1.
\]

For \( n/m \to 0 \) case (ii) of Theorem 2 applies and we have:

\[
t_{US}/t_{LS} = \left( \left\lceil \frac{n-2}{4} \right\rceil + \left\lceil \frac{m}{2} \right\rceil - 2 \right)/\left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil - 4 \right) \to 1.
\]

Corollary 1 shows that if \( n \) is much greater or is much smaller than \( m \), algorithm SGRID tends to be optimal with regard to the capture time.

### 4.2 Capture on tori

The capture on tori \( T_{m,n} \) is more difficult as there are no borders where to push the robber. All the vertices now admit a siege of cardinality 3, then at least three cops are needed, see figure 2. The following algorithm TGRID proves that three cops suffice.

Without loss of generality we define the algorithm for \( n \geq m \) (simply exchange rows with columns if \( m > n \)), and let \( m \geq 6 \) and \( n \geq 6 \) to avoid trivial cases. Place all the cops \( c_1, c_2, c_3 \) in row 0, and in columns \( 0, \left\lfloor \frac{m}{2} \right\rfloor \), and \( \left\lceil \frac{n}{3} \right\rceil \), respectively (see figure 6). Note that initially there is a cop-free gap of \( \left\lfloor \frac{n-3}{3} \right\rfloor \) columns between \( c_1 \) and \( c_3 \), and a cop-free gap of \( \left\lfloor \frac{n-3}{3} \right\rceil \) or
Figure 6: Chase with three cops in \( T_{7,15} \) up to a pre-siege. The first two moves of \( c_3, c_2, \) and \( r \) take place in the guard phase.

\[ \left\lfloor \frac{n-3}{3} \right\rfloor \] columns between \( c_3 \) and \( c_2 \) and between \( c_2 \) and \( c_1 \) around the torus. Starting with the cops in any row will be the same as we work on a torus. The strategy is to bring a cop to guard the robber \( r \), that is the cop will reach the column of \( r \) and then follow \( r \) if it moves horizontally, so to build a virtual border along the row of the guard that prevents \( r \) from traversing it. When the guard is established, the other cops start chasing \( r \) with an immediate extension of algorithm SGRID. Without loss of generality we assume that the initial position of the robber is such that \( c_3 \) becomes the guard. To understand how algorithm TGRID works observe the following.

- The algorithm is divided in a phase for establishing the guard with \( c_3 \), followed by a phase of chasing. The first phase, however, is repeated until \( c_2 \) reaches the column \( \left\lceil \frac{n}{2} \right\rceil \) to start the chase with \( c_1, c_2 \) even if the guard has been established in a previous round.

- For the chase phase all the considerations made for SGRID apply. As before the robber \( r \) must start on the edge of a cone to delay the capture as much as possible, but now the best position for it is not below row \( \left\lfloor \frac{m}{2} \right\rfloor \) (figure 6), otherwise \( c_1 \) and \( c_2 \) would chase it “from the bottom” of the torus.

- When \( c_1 \) and \( c_2 \) have established a pre-siege, \( r \) must move down. The novelty here is that \( c_3 \) moves towards \( r \) in step 8.6, reducing its distance from \( r \) hence the number of rounds for the capture.

Computing a lower and an upper bound to the capture time using algorithm TGRID implies using several floor and ceilings approximations depending on the parity of \( m \) and \( n \), and on the divisibility of \( n \) by 3. We have:
algorithm TGRID(m,n)
1. initial positions of the cops c₁, c₂, c₃:
   place c₁ in v₀,0; place c₂ in v₀,⌈2n/3⌉; place c₃ in v₀,⌈n/3⌉;
   let γ₁, γ₂ be the shadow cones of c₁, c₂;
   // as in algorithm SGRID assume that γ₁, γ₂ are chosen below the cops
2. initial position of the robber r:
   place r in any vertex vᵢ,j not adjacent to a cop;
   // w.l.o.g let ⌈n/3⌉ ≤ j ≤ ⌈2n/3⌉, i.e. r is in a column between c₃ and c₂
3. repeat  // ESTABLISHING THE GUARD
   let x, y be the columns of c₂, c₃;
4.  cops' turn to move:
4.1 move c₂ to the left (x = x − 1);
4.2 if (y < j) move c₃ to the right (y = y + 1)
4.3 else if (y > j) move c₃ to the left (y = y − 1); // else do not move c₃
5.  robber's turn to move:
   move the robber in any way to try to escape the guard
6.  until x = ⌈n/2⌉;
7. repeat  // CHASING WITH AN EXTENSION OF SGRID
8.  cops' turn to move:
8.1 if (r is outside γ₁ and γ₂) move c₁ and c₂ horizontally towards r
8.2 else if (r is within γ₁ and/or within γ₂) move c₁ and c₂ down
8.3 else if (r is on an edge of γ₁ (resp. γ₂)
    and outside γ₂ (resp. γ₁))
    move c₂ (resp. c₁) horizontally towards that edge
8.4 else if (r is on an edge of γ₁ and on an edge of γ₂)
    {if (the cops are in different rows)
    move down the cop in the highest row
    else move down one of the cops};
8.5 if (c₃ and r are in different columns) move c₃ to the column of r
8.6 else if (c₁, c₂ build a pre-siege)
    move c₃ from its row z to row (z − 1) mod m;
9.  robber's turn to move:
   move the robber in any way to try to escape from the cones
10. until the robber makes its last move inside a siege;
    // the siege is established with the robber adjacent to c₃
11. capture the robber;

15
Theorem 3. In a torus $T_{m,n}$ three cops can capture the robber in time $t$ such that:

(i) $\frac{2n}{3} + \frac{5m}{4} - \frac{9}{2} \leq t \leq \frac{2n}{3} + \frac{5m}{4} - \frac{25}{12}$, for $m \leq \left\lceil \frac{n}{2} \right\rceil$;
(ii) $\frac{25n}{24} + \frac{m}{2} - \frac{9}{2} \leq t \leq \frac{25n}{24} + \frac{m}{2} - \frac{17}{8}$, for $\left\lceil \frac{n}{2} \right\rceil < m \leq n$.

Proof. Use algorithm TGRID.

Capture. In the first phase of the algorithm the guard is established (say by $c_3$) with an obvious procedure, and $c_2$ is brought to column $\left\lceil \frac{n}{2} \right\rceil$ to start the chase with $c_1$. Depending on the position of the robber, either one or both cops move closer to it in each round until a pre-siege is inevitably built around $r$, which is then pushed towards $c_3$ to end in a siege, and then is captured.

Evaluation of $t$. The algorithm requires three consecutive times $t_1, t_2, t_3$, respectively needed for the guard phase, the construction of a pre-siege, and the construction of a siege, plus an additional round for final capture. $t_1$ is the number of rounds to bring $c_2$ from column $\left\lceil \frac{2n}{3} \right\rceil$ to column $\left\lceil \frac{n}{2} \right\rceil$, that is $t_1 = \left\lceil \frac{2n}{3} \right\rceil - \left\lceil \frac{n}{2} \right\rceil$. The values of $t_2$ and $t_3$ depend on the value of $m$.

(i) Let $m \leq \left\lceil \frac{n}{2} \right\rceil$. Starting in row $\left\lceil \frac{n}{2} \right\rceil$, in the lowest vertex on the edge of a cone and not inside the other cone, the robber forces the cops to apply steps 8.3 and 8.4 as many times as possible until a pre-siege is established in that row while $c_3$ is moved to row $m-1$, and the robber makes a step down to row $\left\lceil \frac{n}{2} \right\rceil+1$. As shown in the proof of Theorem 2 this requires $\left\lceil \frac{n}{2} \right\rceil - 2$ steps 8.3 plus $2 \left\lceil \frac{n}{2} \right\rceil - 1$ steps 8.4, then we have $t_2 = \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \frac{n}{2} \right\rceil - 3$. At this point there is a gap of $\lambda$ cop-free rows between the robber $r$ and $c_3$, with $\lambda = m - 1 - \left(\left\lceil \frac{m}{2} \right\rceil + 1\right) - 1 = \left\lceil \frac{m}{2} \right\rceil - 3$. From now on $c_1$ and $c_2$ move down pushing $r$ down, and $c_3$ moves up, until $\lambda$ becomes equal to zero or to one and a siege is established. Since at each round the value of $\lambda$ decreases by two we have $t_3 = \left\lceil \left((\left\lceil \frac{m}{2} \right\rceil - 3)/2\right) \right\rceil = \left\lceil \frac{m-6}{4} \right\rceil$. We have $t = t_1 + t_2 + t_3 + 1$. With easy approximations to substitute floor and ceiling operators we obtain the bounds specified in the theorem.

(ii) Let $\left\lceil \frac{n}{2} \right\rceil < m \leq n$. Even in this case the robber must start on the lowest possible row $j$ on the edge of a cone and not inside the other cone to force the cops to apply steps 8.3 and 8.4 as many times as possible, but the value of $j$ is smaller than in case (i) because the cones have a larger intersection. We have $j = \left\lceil \frac{n}{2} \right\rceil$ for $\left\lceil \frac{n}{2} \right\rceil$ even, or $j = \left\lceil \frac{n}{2} \right\rceil$ for $\left\lceil \frac{n}{2} \right\rceil$ odd (note that now $j$ does not depend on $m$). So $\left\lceil \frac{n}{2} \right\rceil - 2$ steps 8.3 plus $\left\lceil \frac{n}{2} \right\rceil - 1$ steps 8.4 are required in the first case, that is $t_2 = 2 \left\lceil \frac{n}{2} \right\rceil - 3$ for $\left\lceil \frac{n}{2} \right\rceil$ even; or $\left\lceil \frac{n}{2} \right\rceil - 2$ steps 8.3 plus $\left\lceil \frac{n}{2} \right\rceil - 2$ steps 8.4 are required in the second case, that is $t_2 = 2 \left\lceil \frac{n}{2} \right\rceil - 4$ for $\left\lceil \frac{n}{2} \right\rceil$ odd. The robber is now pushed down along a gap
of \( \lambda \) cop-free rows, with \( \lambda = m - 1 - (\lceil \frac{n}{4} \rceil + 1) - 1 = m - \lceil \frac{n}{2} \rceil - 3 \) for \( \lceil \frac{n}{2} \rceil \) even, or \( \lambda = m - \lceil \frac{n}{4} \rceil - 3 \) for \( \lceil \frac{n}{2} \rceil \) odd, and the siege is reached in \( t_3 = \lceil \frac{m}{2} \rceil \) rounds. Also here we have \( t = t_1 + t_2 + t_3 + 1 \), and with easy approximations we obtain the lower and upper bounds specified in the theorem, respectively computed for \( \lceil \frac{n}{2} \rceil \) odd and \( \lceil \frac{n}{2} \rceil \) even.

For \( T_{7,15} \) of figure 6 case (i) of Theorem 3 applies and we have \( 11.75 < t < 16.67 \), that is \( 12 < t < 16 \) since \( t \) must be an integer. Computing \( t \) without approximation, using the exact values shown in the proof of the theorem, we have \( t_1 = 2, t_2 = 11, t_3 = 1 \) hence \( t = 15 \).

We can establish a lower bound on the capture time in a torus identical to the one of Lemma 5, namely:

**Lemma 6.** The capture time in a torus \( T_{m,n} \) admits a lower bound \( t_{LT} = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor - 4 \).

**Proof.** For any vertex \( u \) of a torus there is a vertex \( w \) in a 4-loop whose distance from \( u \) is exactly \( \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \). If cop \( c_h \) is initially placed in \( u \), then \( r \) can be placed in \( w \), and we have from Lemma 3: \( t_{LT} = d_h - \lfloor \frac{n}{2} \rfloor - 2 = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor - 4 \). □

Letting \( t_{UT} \) be the value of \( t \) given in Theorem 3 we immediately have:

**Corollary 2.** In a torus \( T_{m,n} \) the ratio \( t_{UT}/t_{LT} \to \frac{4}{3} \) for \( n/m \to \infty \) and \( t_{UT}/t_{LT} \to \sim \frac{37}{24} \) for \( n/m \to 1 \).

It is worth noting that letting \( n < m \), new values for \( t_{UT} \) are simply built from the ones of Theorem 3 exchanging \( n \) with \( m \), while the lower bound \( l_{LT} \) of Lemma 6 holds unchanged. So the first statement of Corollary 2 is rephrased as: \( t_{UT}/t_{LT} \to 4/3 \) for \( m/n \to \infty \). Comparing all these results with the ones found in Theorem 2 and Corollary 1 for semi-tori we see that both algorithms improve performance for increasing difference of the grid dimensions. The reason why algorithm SGRID performs comparatively better than TGRID depends in the latter on guard phase and on the necessity of pushing the robber towards the guard that is at a larger distance than the border of a semi-torus.

### 5 Using larger teams of cops

The cops and robber problem is traditionally focused on studying the minimum number of cops needed for capturing a robber in a given family of graphs, and on the algorithms to successfully attain the capture. Let us
now take a new approach, discussing how the capture time decreases using an increasing number of cops, and conversely which is the minimum number of cops needed to attain the capture within a given time.

This approach has a twofold purpose. On one hand, the possibility of employing the cops immediately in a new chase when they have completed their previous job; on the other hand, completing a job within a required time when a smaller team of cops cannot meet that deadline. This inherits the concept of speed-up introduced in parallel processing, where the work $w_k$ of a process carried out by $k$ agents in time $t_k$ is defined as $w_k = k \cdot t_k$, and the speed-up between the actions of $j$ over $i < j$ agents to catch the robber is defined as $w_i/w_j$. If the algorithms run by the two teams of $i$ and $j$ agents are provably optimal, the speed-up is an important measure of the efficiency of parallelism. Referring to the cops and robber problem, the speed-up is a measure of the gain obtained using an increasing number of cops with the best available algorithms. In this paper we obviously direct our investigation to two-dimensional grids, semi-tori, and tori.

### 5.1 $k$ cops on a grid

Let us consider the case of $k > 2$ cops on a grid $G_{m,n}$, with $m \geq 4, n \geq 4$ to avoid trivial cases. W.l.o.g let $m \leq n$. A new algorithm GRID-K can be designed as an extension of algorithm GRID, taking $k$ even. The structure of GRID-K is given below, limited to its main lines for brevity. Still this formulation is sufficient for computing the capture time.

The cops $c_1, \ldots, c_k$ are placed in $h$ pairs of adjacent cops, $k = 2h$ with $h > 1$. The cops of each pair are placed in rows $\lfloor m/2 \rfloor - 1$ and $\lfloor m/2 \rfloor$, and the pairs are almost equally spaced, with $\lceil n/h \rceil$ and $\lfloor n/h \rfloor$ cop-free columns between them, except for the leftmost and the rightmost groups of columns of almost equal sizes whose sum is again $\lceil n/h \rceil$ or $\lfloor n/h \rfloor$, for example see figure 7 for $k = 4$.

![Figure 7: Two pairs of cops in $G_{4,13}$](image)

In algorithm GRID-K the robber may be captured on a left or on a right corner of the grid by the leftmost or by the rightmost pair of cops; or it may
be captured on the top or on the bottom border by two cops, one from each pair, in a vertex between the two pairs. We have:

**Theorem 4.** In a grid $G_{m,n}$, $k = 2h$ cops, with $h > 1$, can capture the robber in $t_k = \lceil \frac{n-h}{2h} \rceil + \lceil \frac{m-2}{2} \rceil$ rounds.

**Proof.** Use algorithm GRID-K. If the robber is chased by repetitions of steps 1 and 2 it is captured in a corner in $\lceil \frac{n-h}{2h} \rceil + \lceil \frac{m-2}{2} \rceil$ rounds as in algorithm GRID. If the robber is chased by repetitions of steps 1 and 3, it is pushed to the border in an almost central vertex between the pairs and is captured there, again in $\lceil \frac{n-h}{2h} \rceil + \lceil \frac{m-2}{2} \rceil$ rounds. \(\square\)

**ALGORITHM GRID-K (SCHEMATIC)**

Let the cones lay below the cops.

1. If the robber $r$ is in both cones of a pair (vertex $u$ of figure 7), all the cops move vertically towards $r$.
2. If $r$ is in a column at the right (resp. left) of the rightmost (resp. leftmost) pair of cops and is not within a cone of the pair (vertices $x,w$ of the figure), $r$ is captured in a corner as in algorithm GRID by repetitions of steps 1 and 2.
3. If $r$ is in a column between two pairs of cops and not within a cone (vertices $y,z$ of the figure), both pairs of cops move horizontally towards $r$ until it ends in a pair of cones. Then steps 1 and 3 are repeated until $r$ is pushed in a siege on the border with the concurrence of both pairs of cops.

For example in $G_{4,13}$ of figure 7 we have $t_4 = \lceil \frac{13-2}{4} \rceil + \lceil \frac{4-2}{2} \rceil = 4$. The longest capture takes place in the rightmost right corner, or in the border between the two pairs of cops. Note that, if computed with $h = 1$, the result of Theorem 4 does not coincide with the one of Theorem 1 for $m$ odd and $n$ even.

We now compute the minimum number $k$ of cops needed to attain the capture within a given time $\bar{t}$. From Theorem 4 we have $t_k \leq \frac{n-h}{2h} + \frac{m-2}{2}$ and we easily derive:

$$k \geq \frac{2n}{2\bar{t} - m + 3}; \quad \text{valid for } G_{m,n}. \tag{1}$$

In the example of figure 7 we have seen that 4 cops capture the robber in 4 rounds. If we wish to attain the capture in $\bar{t} = 3$ rounds we must employ $k \geq \frac{24}{6} = 5.2$ cops, that is 3 pairs of cops are needed.

The speed-up for $k = 2h$ cops versus 2 cops is given by:
\[
\frac{w_2}{w_k} = 2\left(\left\lfloor \frac{m+n}{2} \right\rfloor - 1\right)/2h\left(\left\lceil \frac{n-k}{2h} \right\rceil + \left\lfloor \frac{m-2}{2} \right\rfloor\right).
\]

For example, for a grid \(G_{4,18}\) we have \(t = 10\) with \(k = 2\), hence \(w_2 = 20\).
Applying algorithm \(\text{GRID-K}\) with \(k = 4\) we have \(t_4 = 5\) and \(w_4 = 20\), so the speed-up is one in this case.

### 5.2 \(k\) cops on a semi-torus

Let us now consider the case of \(k > 2\) cops on a semi-torus \(S_{m,n}\), with \(m \geq 3, n \geq 2k\) to avoid trivial cases. A new algorithm \(\text{SGRID-K}\), whose main lines are given below, can be built as an immediate extension of algorithm \(\text{SGRID}\). This simplified formulation is however sufficient for computing the capture time.

As for algorithm \(\text{SGRID}\), the cops \(c_1, \ldots, c_k\) are placed in row \(\left\lfloor \frac{m-1}{2} \right\rfloor\), with \(c_1\) in column 0 and the others almost equally spaced along the row, with a gap between two consecutive cops of \(\left\lceil \frac{n-k}{k} \right\rceil\) or \(\left\lfloor \frac{n-k}{k} \right\rfloor\) cop-free columns according to the value of \(n\). In the longest chase of algorithm \(\text{SGRID-K}\) the robber is captured by two cops separated by a larger gap.

**ALGORITHM SGRID-K (SCHEMATIC)**

Let the chase take place in the cones below the cops.

1. Until the robber \(r\) is within one or more cones, all the cops move down vertically. This eventually brings \(r\) on the edge of a cone.
2. Until \(r\) is in the gap between two consecutive cops and outside of their cones, the two cops move horizontally towards \(r\). This eventually brings \(r\) on the edge of a cone.
3. If \(r\) is on the edge of a cone, and therefore between two consecutive cops, it is captured by these two cops with algorithm \(\text{SGRID}\).

Using algorithm \(\text{SGRID-K}\), and considering the column gap \(\left\lceil \frac{n-k}{k} \right\rceil\) instead of \(\left\lceil \frac{n-2}{4} \right\rceil\) between the leftmost cops in the proof of Theorem \(2\), we have with straightforward computation:

**Theorem 5.** In a semi-torus \(S_{m,n}\) \(k \geq 2\) cops can capture the robber in time:

(i) \(t_k = \left\lceil \frac{n}{k} \right\rceil + 2\left\lfloor \frac{m}{2} \right\rfloor - 2\), for \(\left\lfloor \frac{m}{2} \right\rfloor \leq \left\lceil \frac{n-k}{2k} \right\rceil\);

(ii) \(t_k = \left\lceil \frac{n}{k} \right\rceil + \left\lceil \frac{n-k}{2k} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor - 2\), for \(\left\lfloor \frac{m}{2} \right\rfloor > \left\lceil \frac{n-k}{2k} \right\rceil\).
Theorem 5 is an immediate extension of Theorem 2 and coincides with it for \( k = 2 \).

We now compute the minimum number \( k \) of cops needed to attain the capture within a given time \( \bar{t} \). From Theorem 5 we have:

Case (i) \( t_k \geq \frac{n}{k} + m - 2 \), hence \( k \geq \frac{n}{m-2} \), for \( m \) even;

\[
t_k \geq \frac{n}{k} + m - 3, \quad \text{hence } k \geq \frac{n}{m+3}, \quad \text{for } m \text{ odd.} \tag{2.1}
\]

Case (ii) \( t_k \geq \frac{n}{k} + \frac{n-k}{2k} + \frac{m}{2} - 2 \), hence \( k \geq \frac{3n}{2k-m+5} \), for \( m \) even;

\[
t_k \geq \frac{n}{k} + \frac{n-k}{2k} + \frac{m}{2} - 3, \quad \text{hence } k \geq \frac{3n}{2k-m+7}, \quad \text{for } m \text{ odd.} \tag{2.2}
\]

As an example of speed-up consider the semi-torus \( S_6,9 \) in figure 8. For \( k = 2 \) (figure 5) we have already found \( t = 8 \) hence \( w_2 = 16 \). For \( k = 3 \), case (ii) of Theorem 5 applies and we have \( t_3 = \left[ \frac{n}{k} \right] + \left[ \frac{n-k}{2k} \right] + \left[ \frac{m}{2} \right] - 2 = 5 \), hence \( w_3 = 15 \) and \( \frac{w_3}{w_2} > 1 \). This is a case of super-linear speed-up computed with the best available algorithms for semi-tori. Recall that the speed-up may be different and clearly more significant using provably optimal algorithms if they were known.

Figure 8: Movements of three cops and the robber in \( S_{6,9} \), up to a siege: the last two moves of \( c_1 \) are done concurrently with the moves of \( c_2 \).

5.3 \( k \) cops on a torus

Let us now consider \( k \) cops working on a torus \( T_{m,n} \) with \( k \geq 4 \). W.l.o.g. let \( n \geq m \), and let \( m \geq 6, n \geq 2k \) to avoid trivial cases. As before a schematic formulation of TGRID-K, given as an immediate extension of TGRID, is sufficient for computing the capture time.

The \( k + 1 \) cops are placed in row 0 in the order \( c_1, c_k, c_2, c_3, \ldots, c_{k-1} \), with the first in column 0 and the others almost equally spaced along the row, with a gap between two consecutive cops of \( \left[ \frac{n-k}{k} \right] \) or \( \left[ \frac{n-k}{2} \right] \) cop-free columns according to the value of \( n \). Assume that the larger gaps occur between the cops at the beginning of the sequence, so \( c_k \) and \( c_2 \) respectively
start in columns $\left\lceil \frac{n}{k} \right\rceil$ and $2\left\lceil \frac{n}{k} \right\rceil$. W.l.o.g. assume that cop $c_k$ will be the guard and the longest chase will be done by $c_1$ and $c_2$.

In the guard phase of algorithm TGRID-K the guard is established by $c_k$ and the cops $c_1, \ldots, c_{k-1}$ are brought to almost equally spaced positions in row 0 (the new gaps will be $\left\lceil \frac{n-(k-1)}{k-1} \right\rceil$ or $\left\lfloor \frac{n-(k-1)}{k-1} \right\rfloor$) to be prepared for chasing the robber (which, in the longest chase, will be captured by $c_1$ and $c_2$). For this purpose $c_1, \ldots, c_{k-1}$ move together rightwards for the needed number of steps, depending on the sizes of the gaps between the cops. In any case $c_2$ is placed in column $\left\lfloor \frac{n}{k-1} \right\rfloor$ with $2\left\lceil \frac{n}{k} \right\rceil - \left\lfloor \frac{n}{k-1} \right\rfloor$ moves, and no other cop makes more moves in this phase of the algorithm. In the chase phase of the algorithm, first the robber is confined in a set of columns between two cops (say $c_1$ and $c_2$), then is chased as in TGRID in this narrower section of the torus.

**ALGORITHM TGRID-K (SCHEMATIC)**

Let the robber start in the gap between $c_k$ and $c_2$.

1. **GUARD PHASE.**
   
   $c_k$ moves rightwards and $c_2, \ldots, c_{k-1}$ move leftwards, concurrently in row 0, until they reach their proper positions for the chase. $c_k$ eventually becomes the guard and the phase ends when $c_2$ reaches column $\left\lceil \frac{n}{k-1} \right\rceil$.

2. **CHASE PHASE.**

   2.1 While the robber $r$ is within one or more cones, all the cops except $c_k$ move down vertically. This eventually brings $r$ on the edge of a cone. If needed, $c_k$ moves horizontally to stay in the same column of $r$.

   2.2 While $r$ is in the gap between two consecutive cops (assume that they are $c_1$ and $c_2$ for the longest chase), or on the edge of one or both cones, it is captured by these two cops with algorithm TGRID run by them together with $c_3$.

For torus $T_{7,15}$ with $k = 4$, the initial positions of the cops and the robber, and their evolution according to algorithm TGRID-K, are indicated in figure 9. The analysis of TGRID-K is an extension of the one of TGRID. We have:

**Theorem 6.** In a torus $T_{m,n}$, $k > 3$ cops can capture the robber in time $t_k$ such that:

(i) $\frac{2n}{k} + \frac{5m}{4} - \frac{9}{2} \leq t_k \leq \frac{2n}{k} + \frac{5m}{4} + \frac{k-1}{4} - \frac{11}{4}$, for $m \leq \left\lfloor \frac{n}{k-1} \right\rfloor$;

(ii) $\frac{2n}{k} + \frac{3m}{4(k-1)} + \frac{m}{2} - \frac{9}{2} \leq t_k < \frac{2n}{k} + \frac{3m}{4(k-1)} + \frac{m}{2} - \frac{1}{2}$, for $\left\lceil \frac{n}{k-1} \right\rceil < m \leq n$.  

22
Figure 9: Chase with four cops in $T_{7,15}$ up to a pre-siege. The first three moves of $c_1, c_2,$ and $r$ take place in the guard phase. Compare the moves with the ones for $k = 3$ reported in figure 6.

**Proof.** Use algorithm TGRID-K, and refer to the proof of Theorem 3 for comparison. Attaining the capture is obvious. The guard time is $t_1 = \lceil \frac{2n}{k} \rceil - \lceil \frac{n}{k-1} \rceil$.

In case (i), the time to establish a pre-siege is $t_2 = \lceil \frac{n}{k-1} \rceil + 2\lceil \frac{m}{2} \rceil - 3$, and the following time to establish a siege is $t_3 = \lceil \frac{m-6}{4} \rceil$. With proper approximations of the ceiling and floor functions the capture time $t_k = t_1 + t_2 + t_3 + 1$ can be bounded as in the statement of the theorem.

In case (ii) the time $t_2$ to establish a pre-siege depends on the parity of $\lceil \frac{n}{k-1} \rceil$. The robber is placed in row $\lceil \frac{n}{2(k-1)} \rceil$ for $\lceil \frac{n}{k-1} \rceil$ even, or in row $\lfloor \frac{n}{2(k-1)} \rfloor$ for $\lceil \frac{n}{k-1} \rceil$ odd. Refer to algorithm TGRID used inside TGRID-K.

In the first case $\lceil \frac{n}{k-1} \rceil - 2$ steps 8.3 plus $\lceil \frac{n}{k-1} \rceil - 1$ steps 8.4 are required, that is $t_2 = 2\lceil \frac{n}{k-1} \rceil - 3$ for $\lceil \frac{n}{k-1} \rceil$ even. In the second case $\lceil \frac{n}{k-1} \rceil - 2$ steps 8.3 plus $\lceil \frac{n}{k-1} \rceil - 2$ steps 8.4 are required, that is $t_2 = 2\lceil \frac{n}{k-1} \rceil - 4$ for $\lceil \frac{n}{k-1} \rceil$ odd. The following time to establish a siege is $t_3 = \lceil \lambda \rceil$, with $\lambda = m - \lceil \frac{n}{2(k-1)} \rceil - 3$ for $\lceil \frac{n}{k-1} \rceil$ even, and $\lambda = m - \lfloor \frac{n}{2(k-1)} \rfloor - 3$ for $\lceil \frac{n}{k-1} \rceil$ odd. With proper approximations of the ceiling and floor functions the capture time $t_k = t_1 + t_2 + t_3 + 1$ can be bounded as in the statement of the theorem.

Note that the bounds for $t_k$ given in Theorem 6 coincide with the ones of Theorem 3 for $k = 3$, except for the upper bound of case (ii) that has now been evaluated with a stronger approximation (thus yielding a $<$ sign instead of the stricter $\leq$ sign) to avoid a complicated formula. In the torus $T_{7,15}$ of figure 9 using $k = 4$ cops and applying the exact values of the numbers of steps reported in the proof of Theorem 6 case (ii) with $\lceil \frac{n}{k-1} \rceil = 5$ odd we
have: \( t_4 = 3 + 6 + 1 + 1 = 11 \).

The minimum number \( k \) of cops needed to attain the capture within a given time \( \bar{t} \) is derived in the two cases of Theorem 6 with some further approximations. We have:

**Case (i)** 
\[
\frac{8n}{4\bar{t} - 6m + 18} \leq k < \frac{8n}{4\bar{t} - 6m + 7}, \quad \text{for } m \leq \left\lceil \frac{n}{k-1} \right\rceil;
\]

**Case (ii)** 
\[
\frac{11n}{4\bar{t} - 2m + 18} < k < \frac{11n}{4\bar{t} - 2m + 2} + 1, \quad \text{for } \left\lceil \frac{n}{k-1} \right\rceil < m \leq n.
\]

For torus \( T_{7,15} \) of figure 9 case (ii) applies for any \( k > 3 \). Imposing a capture time of at most \( \bar{t} = 12 \), from relation (3.2) we have \( 3.17 < k < 5.58 \), that is the required number of cops is between 4 and 5. In fact we have already seen that 3 cops require 15 rounds and 4 cops require 11 rounds. This also implies that \( w_3 = 45 \) and \( w_4 = 44 \), hence a slightly super-linear speed-up occurs.

### 6 Concluding remarks

In this work we have extended the cops and robber problem in two-dimensional grids to semi toroidal and fully toroidal grids. We have introduced the concepts of siege around the robber and of shadow-cones of the cops to reconstruct the already known result on grids, as basic tools for studying the new chase on toroidal grids. We have given matching lower and upper bounds on the number of cops and polynomial time algorithms for solving the different instances of the problem. Although we have not been able to prove that our algorithms are optimal for toroidal grids in relation to the capture time, we have shown that their behavior improves if the ratio between the numbers of rows and columns becomes unbalanced.

We have then introduced a discussion on the effect of using an arbitrary number of cops, with a twofold purpose of studying the minimum number of cops needed if the capture time is fixed, and computing the speed-up obtained if that number increases. To justify this task assume that a capture can be done by 2 cops in 8 rounds, and by 4 cops in 3 rounds. If four cops are available, two robbers can be captured in 8 rounds with two parallel chases with 2 cops each, or in 6 rounds with two sequential chases with 4 cops each. Depending on the requests of the problem the latter approach may be preferred. The concept of “work” inherited from parallel processing, that is the product between the number of cops and the capture time, has been adopted for proving the possible interest of using a non minimum number of cops.

A immediate extension of our work should be studying the capture on
multi-dimensional grids and tori, and possibly on different families of graphs. More ambitiously our approach could be applied to other classes of problems solved with mobile agents, studying the impact of using teams with different numbers of agents. The present work is to be seen as a first step in this direction.

References

[1] S. Bhattacharya, G. Paul and S. Sanyal. A cops and robber game in multidimensional grids. *Discrete Applied Mathematics* 158, 1745-1751, 2010.

[2] S. Bhattacharya, A. Banerjee and S. Badyopadhy. CORBA-based analysis of multi-agent behavior. *Journal of Computer Science and Technology* 20 (1), 118-124, 2005.

[3] A. Bonato and R. Nowakovski. *The Game of Cops and Robbers on Graphs*. American Mathematical Society, 2011.

[4] L. Blin, P. Fraignaud, N. Nisse, and S. Vial. Distributed chasing of network intruders. *Theoretical Computer Science* 399, 12-37, 2008.

[5] N. Cohen, M. Hilaire, N.A. Martins, N. Nisse, and S. Perennes. Spy-game on graphs. *Proc. 8-th International Conference FUN 2016* DOI 10.4230/LIPIcs.FUN.2016.10

[6] R. Dawes. Some pursuit-evasion problems on grids. *Information Processing Letters* 43, 241-247, 1992.

[7] A. Dumitrescu, H. Kok, I. Suzuki and P. Zylinski. Vision based pursuit-evasion on a grid. *Proc. 11-th Scandinavian Workshop on Algorithm Theory, SWAT 2008* LNCS 5124, 45-64, 2008.

[8] J. Ellis and R. Warren. Lower bounds on the pathwidth of some grid-like graphs. *Discrete Applied Mathematics* 156, 545-555, 2008.

[9] F. Fomin, P. Golovach, J. Kratochvil, N. Nisse and K. Suchan. Pursuing a fast robber on a graph. *Theoretical Computer Science* 411, 1167-1181, 2010

[10] D. Ilcinkas, N. Nisse and D. Soguet. The cost of monotonicity in distributed graph searching. *Distributed Computing* 22(2) 117-127, 2009
[11] F. Goldstein and E. Reingold. The complexity of pursuing a graph. *Theoretical Computer Science* 143, 93-112, 1995

[12] R.M. Karp and V. Ramachandran. Parallel algorithms for shared memory machines. In: J. van Leeuwen (ed) *Handbook of Theoretical Computer Science, Vol. A*. North Holland, New York, 869-941, 1990.

[13] F. Luccio, L. Pagli, and G. Pucci. Three non Conventional Paradigms of Parallel Computation. In: *Parallel Architectures and Their Efficient Use*. LNCS 678, 166-175, 1992.

[14] F. Luccio, L. Pagli, and N. Santoro. Network decontamination in presence of local immunity. *International Journal of Foundation of Computer Science* 18(3), 457–474, 2007.

[15] F. Luccio and L. Pagli. A general approach to toroidal mesh decontamination with local immunity. *Proceedings of the 23rd IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, 1-8, 2009.

[16] F. Luccio and L. Pagli. More agents may decrease global work: A case in butterfly decontamination. *Theoretical Computer Science* 655, 41-57, 2016.

[17] N. Megiddo, S. Hakimi, M. Garey, D. Johnson and C. Papadimitriou. The complexity of searching a graph. *Journal of the ACM* 35(1), 18–44, 1988.

[18] A. Mehrabian. The capture time of grids. *Discrete Mathematics* 311, 102-105, 2011.

[19] S. Neufeld. A pursuit-evasion problem on a grid *Information Processing Letters* 58, 5-9, 1996.

[20] R. Nowakowski and P. Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Mathematics* 43, 253-259, 1983.

[21] P. Quillot. A short note about pursuit games payed on a graph of given genus. *Journal on Combinatoric Theory* Ser. B 38, 89-92, 1985.

[22] K. Sugihara and I. Suzuki Optimal Algorithm for a pursuit-evasion problem. *SIAM Journal of Discrete Mathematics* 2, 126-143, 1989.