AN EXTENSION OF HARMONIC FUNCTIONS ALONG FIXED DIRECTION

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Abstract. Let a function \( u(x, y) \) be harmonic in the domain
\[
D \times V_r = D \times \{ y \in \mathbb{R}^m : |y| < r \} \subset \mathbb{R}^n \times \mathbb{R}^m
\]
and for each fixed point \( x^0 \) from some a set \( E \subset D \), the function \( u(x^0, y) \), as a function of variable \( y \), can be extended to a harmonic function on the whole \( \mathbb{R}^m \). Then \( u(x, y) \) harmonically extends to the domain \( D \times \mathbb{R}^m \) as a function of variables \( x \) and \( y \).

1. Introduction

The well known Hartogs lemma concerns the extension of holomorphic functions along fixed direction (see [1]) and states that if a function \( f(z, w) \) is holomorphic in the domain
\[
D \times \{ w \in \mathbb{C} : |w| < R \} \subset \mathbb{C}^n \times \mathbb{C}
\]
and for each fixed \( z^0 \in D \) the function \( f(z^0, w) \) of variable \( w \) holomorphically extends to the disk \( \{ w \in \mathbb{C} : |w| < R \} \), then \( f(z, w) \) can be holomorphically extended to the domain \( D \times \{ w \in \mathbb{C} : |w| < R \} \) as a function of variables \( z \) and \( w \).

Note that the Hartogs lemma remains to hold also for pluriharmonic and harmonic functions (see [3]) and can be formulated as follows. Let a function \( u(x, y) \) be harmonic in the domain
\[
D \times V_r = D \times \{ y \in \mathbb{R}^m : |y| < r \} \subset \mathbb{R}^n \times \mathbb{R}^m,
\]
and for each fixed \( x^0 \in D \) the function \( u(x^0, y) \) of variable \( y \) harmonically extends to the ball \( V_R = \{ y \in \mathbb{R}^m : |y| < R \} \), \( R > r \). Then \( u(x, y) \) harmonically extends to \( D \times V_R \) as a function of variables \( x \) and \( y \).

One of the main methods used in the investigation of harmonic extensions is to convert this problem to holomorphic extensions. For this purpose, the following lemma is proved in [3].

Lemma 1.1. Consider the space \( \mathbb{R}^n(x) \) embedded into \( \mathbb{C}^n(z) = \mathbb{R}^n(x) + i\mathbb{R}^n(y) \), where \( z = (z_1, z_2, ..., z_n) \), \( z_j = x_j + iy_j \), \( j = 1, 2, ..., n \). Then for each domain \( D \subset \mathbb{R}^n(x) \) there exists some holomorphy domain \( \tilde{D} \subset \mathbb{C}^n(z) \), such that \( D \subset \tilde{D} \) and each harmonic function in \( D \) can be holomorphically extended to the domain \( \tilde{D} \), i.e., there exists a holomorphic function \( f_u(z) \in \tilde{D} \) such that \( f_u|_D = u \).

The main result of our paper reads as follows. The concept of \( N \)-sets of \( Lh_0(D) \) which appears there is defined in the next section.

Theorem 1.2. Let a function \( u(x, y) \) be harmonic in the domain
\[
D \times V_r = D \times \{ y \in \mathbb{R}^m : |y| < r \} \subset \mathbb{R}^n \times \mathbb{R}^m
\]

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and for each fixed point \(x^0\) from some set \(E \subset D\), which is not embedded into a countable union of \(N\)-sets of \(Lh_0(D)\), the function \(u(x^0, y)\) of variable \(y\) can be extended to a harmonic function on the whole \(\mathbb{R}^m\). Then \(u(x, y)\) harmonically extends to the domain \(D \times \mathbb{R}^m\) as a function of variables \(x\) and \(y\).

2. The class of functions \(Lh_0(D)\)

Let \(D \subset \mathbb{R}^n\) and \(h(D)\) be the space of harmonic functions in \(D\). By \(Lh_\varepsilon(D)\) we denote the minimal class of functions which contains functions of the form \(\alpha \ln |u(x)|, \ u(x) \in h(D), \ 0 < \alpha < \varepsilon\), and it is closed with respect to the operation of “upper regularization”, i.e., for any set of functions \(u_\lambda(x) \in Lh_\varepsilon(D), \ \lambda \in \Lambda\), where \(\Lambda\) is an index set, the function

\[
\lim_{y \to x} \sup\{u_\lambda(y) : \lambda \in \Lambda\}
\]

also belongs to the class \(Lh_\varepsilon(D)\)([4]).

The union of function classes \(Lh_0(D) = \bigcup_{\varepsilon > 0} Lh_\varepsilon(D)\) we call the set of \(Lh_0-\)functions.

Now, we will define \(N\)-set of the class \(Lh_0(D)\). Let \(\vartheta_k(x) \in Lh_0(D)\) be a monotonically increasing sequence of locally uniformly from above bounded functions. We denote

\[
\vartheta(x) = \lim_{y \to x} \lim_{k \to \infty} \vartheta_k(y) = x \in D.
\]

Then everywhere in \(D\) holds the inequality

\[
\lim_{k \to \infty} \vartheta_k(y) \leq \vartheta(x).
\]

**Definition 2.1.** The subsets of the set

\[
\{x \in D : \lim_{k \to \infty} \vartheta_k(y) < \vartheta(x)\}
\]

are called \(N\)-sets of the class \(Lh_0(D)\).

Note that if \(\vartheta_k(x) \in Lh_0(D)\) is a sequence of locally uniformly from above bounded functions and

\[
\lim_{y \to x} \lim_{k \to \infty} \vartheta_k(y) = \vartheta(x), \ x \in D,
\]

then the set

\[
E = \{x \in D : \lim_{k \to \infty} \vartheta_k(y) < \vartheta(x)\}
\]

consists of countable union of \(N\)-sets of the class \(Lh_0(D)\). Indeed, consider the sequence of functions

\[
w_{l,j}(x) = \max_{l \leq k \leq j} \vartheta_k(x).
\]

Clearly

\[
\lim_{k \to \infty} \vartheta_k(y) = \lim_{l \to \infty} \lim_{j \to \infty} w_{l,j}(x).
\]

Since the sequence is increasing in \(j\), we have

\[
\lim_{j \to \infty} w_{l,j}(x) \leq \lim_{l \to \infty} \lim_{j \to \infty} w_{l,j}(y), \ x \in D,
\]

and the sets

\[
E_l = \{x \in D : \lim_{j \to \infty} w_{l,j}(x) < \lim_{l \to \infty} \lim_{j \to \infty} w_{l,j}(y)\}, \ l = 1, 2, \ldots,
\]
are $N$-set of class the $Lh_0(D)$. On the other hand, the sequences $\lim_{j \to \infty} w_{1,j}(x)$ and $\overline{\lim}_{l \to \infty} \lim_{j \to \infty} w_{1,j}(y)$, $j = 1, 2, \ldots$, are monotonically decreasing, and

\[
\overline{\lim}_{k \to \infty} \vartheta_k(x) = \lim_{l \to \infty} \lim_{j \to \infty} w_{1,j}(x) = \vartheta(x) = \lim_{l \to \infty} \lim_{y \to x} \lim_{j \to \infty} w_{1,j}(y), \quad x \in D \setminus \bigcup_{l=1}^{\infty} E_l.
\]

It follows that $E \subset \bigcup_{l=1}^{\infty} E_l$, i.e. $E = \bigcup_{l=1}^{\infty} (E_l \cap E)$.

**Definition 2.2.** The set $E \subset D$ is called $Lh_0$-polar with respect to $D$, if there exists a function $\vartheta(x) \in Lh_0(D)$ such that $\vartheta(x) \not\equiv -\infty$ and $\vartheta(x)|_E = -\infty$.

Note that if $u(x) \in h(D)$, $u(x) \not\equiv 0$ and $E \subset \{u(x) = 0\}$, then $E$ is $Lh_0$-polar with respect to $D$.

**Proposition 2.3.** Each $Lh_0$-polar set with respect to $D$ is contained in a countable union of $N$-sets of the class $Lh_0(D)$.

Indeed, let $E$ be a $Lh_0$-polar set with respect to $D$. Then by Definition 2.2 there exists a function $\vartheta(x) \in Lh_0(D)$ such that $\vartheta(x) \not\equiv -\infty$ and $\vartheta(x)|_E = -\infty$. Consider the sequence of functions $\vartheta_k(x) = \frac{1}{k} \vartheta(x)$. Obviously, $\vartheta_k(x) \in Lh_0(D)$ and $\vartheta_k(x) \not\equiv -\infty$ and $\vartheta_k(x)|_E = -\infty$. Moreover, $\lim_{k \to \infty} \vartheta_k(x) = 0$ for almost all $x \in D$ and $\lim_{k \to \infty} \vartheta_k(x) = -\infty$ for all $x \in E$. It follows that

\[
E \subset \left\{ x \in D : \lim_{k \to \infty} \vartheta_k(x) < \overline{\lim}_{y \to x} \lim_{k \to \infty} \vartheta_k(x) \right\}.
\]

On the other hand, as we have showed above, the set

\[
\left\{ x \in D : \lim_{k \to \infty} \vartheta_k(x) < \overline{\lim}_{y \to x} \lim_{k \to \infty} \vartheta_k(x) \right\}
\]

consists of a countable union of $N$-sets of the class $Lh_0(D)$.

3. **Proof of the theorem**

Suppose that the number $r$ is sufficiently large such that $\hat{V} \subset \mathbb{C}^m$ contains the closure of the unit polydisc

\[
U = \left\{ w = (w_1, w_2, \ldots, w_m) \in \mathbb{C}^m : |w_1| < 1, |w_2| < 1, \ldots, |w_m| < 1 \right\},
\]

since the general case easily may be reduced to this case by linear changing of $y$.

By the lemma of [3], for each fixed $x \in D$ the function $u(x, y)$ of variable $y$ can be extended holomorphically to $\hat{V}_r \subset \mathbb{C}^m$ and can be expanded into a Hartogs’ series

\[
u(x, w) = \sum_{|k|=0}^{\infty} c_k(x) w^k,
\]

where $k = (k_1, k_2, \ldots, k_m)$ is a multiindex. Clearly, the function $u(x, y)$ is harmonic in each variable and coefficients $c_k \in h(D)$. Also, according to the Cauchy’s inequalities (see [1], [2]), for each set $D_0 : E \subset D_0 \subset \subset D$ the following estimation holds

\[
|c_k(x)| \leq M, \quad \forall k, \quad \forall x \in D_0,
\]

where $M = \sup\{|u(x, w)| : (x, w) \in \overline{D_0 \times U}\}$, i.e., the sequence of functions $c_k(x)$ is locally uniformly bounded in the domain $D$. On the other hand, by the same lemma
from $[3]$, for each fixed $x \in E$ the function $u(x, w)$ of the variable $w$ is holomorphic everywhere in $\mathbb{C}^m$, which means that for all $x \in E$
\[ \lim_{|k| \to \infty} \frac{|k|}{|k|} \sqrt{|c_k(x)|} = 0. \]
It follows that the sequence $\frac{1}{|k|} \ln |c_k(x)|$ is locally uniformly bounded from above and
\[ \lim_{|k| \to \infty} \frac{1}{|k|} \ln |c_k(x)| = -\infty, \ \forall x \in E. \]
Put $\lim_{y \to x} \lim_{|k| \to \infty} \frac{1}{|k|} \ln |c_k(y)| = \vartheta(x)$. Since the set
\[ F = \left\{ x \in D : \lim_{|k| \to \infty} \frac{1}{|k|} \ln |c_k(x)| < \vartheta(x) \right\} \]
consists of the countable union of the $N$-sets of $Lh_0(D)$ and $E$ is not contained in a countable union of the $N$-sets of $Lh_0(D)$, the set $E \setminus F = \{ x \in E : \vartheta(x) = -\infty \}$ is not contained in a countable association of the $N$-sets of $Lh_0(D)$ as well. By Proposition $2.3$ the set $E \setminus F$ is not $Lh_0$-polar with respect to $D$. Consequently, $\vartheta(x) \equiv -\infty$, i.e.,
\[ \lim_{|k| \to \infty} \frac{1}{|k|} \ln |c_k(x)| = -\infty, \ \forall x \in D. \]
Thus, we get
\[ \lim_{|k| \to \infty} \frac{|k|}{|k|} \sqrt{|c_k(x)|} = 0, \ \forall x \in D. \]
Therefore, for all $x \in D$, the function $u(x, y)$ of variable $y$ extends to a harmonic function on the whole $\mathbb{R}^m$. Hence, $u(x, y)$ harmonically extends in $D \times \mathbb{R}^m$. The theorem is proved.

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