Bound States and Heat Kernels for Fractional-Type Schrödinger Operators with Singular Potentials

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Abstract: We consider non-local Schrödinger operators $H = -L - V$ in $L^2(\mathbb{R}^d)$, $d \geq 1$, where the kinetic terms $L$ are pseudo-differential operators which are perturbations of the fractional Laplacian by bounded non-local operators and $V$ is the fractional Hardy potential. We prove pointwise estimates of eigenfunctions corresponding to negative eigenvalues and upper finite-time horizon estimates for heat kernels. We also analyze the relation between the matching lower estimates of the heat kernel and the ground state near the origin. Our results cover the relativistic Schrödinger operator with Coulomb potential.

1. Introduction

Let $d \in \mathbb{N} := \{1, 2, \ldots \}$ and $\alpha \in (0, 2 \wedge d)$. Recall that the fractional Laplacian $L^{(\alpha)} := (-\Delta)^{\alpha/2}$ is a pseudo-differential operator which is defined by

$$
\hat{L^{(\alpha)}} f(\xi) = -|\xi|^\alpha \hat{f}(\xi), \quad f \in \mathcal{D}(L^{(\alpha)}) := \left\{ f \in L^2(\mathbb{R}^d) : |\xi|^\alpha \hat{f}(\xi) \in L^2(\mathbb{R}^d) \right\},
$$

see e.g. Kwaśnicki [29]. Here and in what follows $L^2(\mathbb{R}^d)$ is a complex space with inner product $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$. It is known that

$$
|\xi|^\alpha = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y))\nu^{(\alpha)}(y)dy, \quad \xi \in \mathbb{R}^d,
$$

where

$$
\nu^{(\alpha)}(y) = c_{d,\alpha}|y|^{-d-\alpha} \quad \text{and} \quad c_{d,\alpha} := \frac{\alpha 2^{\alpha-1} \Gamma \left( (d + \alpha)/2 \right)}{\pi^{d/2} \Gamma \left( 1 - \alpha/2 \right)}.
$$

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In this paper we consider the class of Lévy operators $L$ such that
\[ \hat{L}f(\xi) = -\psi(\xi) \hat{f}(\xi), \quad f \in D(L) := \left\{ f \in L^2(\mathbb{R}^d) : \psi \hat{f} \in L^2(\mathbb{R}^d) \right\} \]
(see Jacob [22] and Böttcher, Schilling and Wang [8]), where
\[ \psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y)) \nu(y) \, dy, \quad \xi \in \mathbb{R}^d, \]
and the Lévy density $\nu$ is symmetric (i.e. $\nu(-x) = \nu(x)$) and satisfies the following assumption:

(A1) The density $\nu$ is such that
\[ \sigma := \nu^{(\alpha)} - \nu \geq 0 \]
and $\sigma (dx) = \sigma(x) \, dx$ is a finite measure.

The main goal of this paper is to give pointwise estimates of $L^2$-eigenfunctions corresponding to negative eigenvalues (below we call them bound state eigenfunctions or just bound states) and finite-time horizon heat kernel estimates for non-local Schrödinger operators
\[ H = -L - V, \quad \text{where} \quad V(x) = \frac{\kappa}{|x|^{\alpha}}, \quad 0 < \kappa < \kappa^* = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)^2}{\Gamma\left(\frac{d-\alpha}{2}\right)^2}. \quad (1.1) \]
The operator $H$ is defined in form-sense as a bounded below self-adjoint operator, see Sect. 3.3. An important example is the relativistic Schrödinger operator with Coulomb potential
\[ H = \sqrt{-\Delta + m^2} - m - \frac{\kappa}{|x|}. \]
Here we set $d = 3, \alpha = 1$ and, consequently, $\kappa^* = \frac{2}{\pi}$. The Hamiltonian $H + m$ is known to provide one of possible descriptions (neglecting spin effects) of the energy of a relativistic particle with mass $m$ in the Coulomb field. It has been widely studied as an alternative to the Klein–Gordon and the Dirac theories, see Herbst [21], Weder [41], Daubechies and Lieb [11], and Daubechies [10]. The other examples of operators $L$ (Lévy densities $\nu$) satisfying our assumption (A1) are given in Example 2.3.

The potential $V$ in (1.1) is the Hardy potential corresponding to fractional Laplacian $L^{(\alpha)}$ and $\kappa^*$ is known to be the critical constant in the fractional Hardy inequality
\[ \mathcal{E}^{(\alpha)}[f] \geq \kappa^* \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^\alpha} \, dx, \quad f \in L^2(\mathbb{R}^d); \quad (1.2) \]
$\mathcal{E}^{(\alpha)}$ denotes the quadratic form of the operator $-L^{(\alpha)}$, see Herbst [21], Frank and Seiringer [16], Bogdan, Dyda and Kim [4]. As shown in [4, Section 4] and [5, Section 2.2], for any $0 < \kappa \leq \kappa^*$ there exists a unique number $\delta$ such that
\[ 0 < \delta \leq \frac{d - \alpha}{2} \quad \text{and} \quad \kappa = \frac{2^\alpha \Gamma\left(\frac{\alpha+\delta}{2}\right) \Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right) \Gamma\left(\frac{d-\alpha-\delta}{2}\right)}. \quad (1.3) \]
Our first main result gives pointwise estimates of eigenfunctions corresponding to negative eigenvalues for Schrödinger operators $H$ given by (1.1). It summarizes Proposition 4.1 and Theorem 4.5 which are proven below. Recall that the heat kernel $p(t, x, y) := p_t(y - x)$ and resolvent kernel $g_\lambda(x, y) := g_\lambda(y - x)$ of the operator $-L$ are given by

$$p_t(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\psi(\xi)} e^{-i x \cdot \xi} d\xi, \quad t > 0, \ x \in \mathbb{R}^d,$$

and

$$g_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) dt, \quad \lambda > 0, \ x \in \mathbb{R}^d,$$

see Sect. 2.2 for more details.

**Theorem 1.1.** Let (A1) hold and let $\kappa < \kappa^*$. Let $\varphi \in L^2(\mathbb{R}^d), \|\varphi\|_2 = 1$, be such that

$$H \varphi = E \varphi, \quad \text{for a number } E < 0. \quad (1.4)$$

Then $\varphi$ has a version which is continuous on $\mathbb{R}^d \setminus \{0\}$ and satisfies the following estimates.

(a) For every $R > 0$ there exists $c > 0$ such that

$$|\varphi(x)| \leq c e^{\sigma |x| + E|x|^\delta}, \quad |x| \leq R, \ x \neq 0,$$

where $\delta$ is determined by (1.3); the constant $c$ depends neither on $\varphi, E$ nor $\sigma$.

(b) For every $\varepsilon \in (0, |E| \land 1)$ there is $R = R(\varepsilon) \geq 1$ and $c = c(\varepsilon)$ such that

$$|\varphi(x)| \leq c \sup_{|y| \leq R} g_{|E| - \varepsilon}(x - y), \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (1.5)$$

Furthermore, if $\varphi$ is a ground state (i.e. (1.4) holds with $E = \inf \text{spec}(H) < 0$), then there is $\widetilde{c} > 0$ such that

$$\varphi(x) \geq \widetilde{c} \inf_{|y| \leq 1} g_{|E|}(x - y), \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (1.6)$$

Under the assumption (A1), the operators $-L$ and $-L^{(\alpha)}$ are close to each other in the sense that their Fourier multipliers are asymptotically equivalent at infinity. Indeed, we have

$$\lim_{|\xi| \to \infty} \frac{\psi(\xi)}{|\xi|^\alpha} = 1,$$

which is a direct consequence of

$$\psi(\xi) \leq |\xi|^\alpha = \psi(\xi) + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y))\sigma(y) dy \leq \psi(\xi) + 2|\sigma|, \quad \xi \in \mathbb{R}^d,$$

where $|\sigma| := \sigma(\mathbb{R}^d)$. However, the tail of $\nu(x)$ can be essentially lighter at infinity than that of $\nu^{(\alpha)}(x)$, and therefore some analytic and spectral properties of $H = -L - V$ and $-L^{(\alpha)} - V$ may differ as well. The crucial example is the structure of the spectrum $\text{spec}(H)$ of $H$. First recall that the discrete spectrum $\text{spec}_d(H)$ consists of all isolated eigenvalues of $H$ of finite multiplicity, and the essential spectrum $\text{spec}_e(H)$ can be
defined as $\text{spec}_c(H) := \text{spec}(H) \setminus \text{spec}_d(H)$. It follows from [41, Theorems 3.6, 3.7] (see also the more explicit statement in [24, Proposition 4.1] which is dedicated to self-adjoint Lévy operators with real-valued potentials) that $\text{spec}_c(H) = \text{spec}_c(-L) = [0, \infty)$. This implies that $\text{spec}_d(H) \subset (-\infty, 0)$. Since $L^\alpha$ is a special case of $L$ with $\sigma = 0$, (1.2) gives that $\text{spec}_d(-L^{(\alpha)} - V) = \emptyset$. On the other hand, the discrete spectra $\text{spec}_d(H)$ can be non-empty for Schrödinger operators $H$ as in (1.1) where the fractional Laplacian $L^{(\alpha)}$ is replaced by the proper perturbation $L$. This is exactly the situation in which our Theorem 1.1 applies directly. We also remark that in our settings the ground state eigenfunction is always unique and strictly positive, see e.g. [35, Theorem XIII.44]. This follows from the fact that the integral kernels of the operators $e^{-tH}$, $t > 0$ are strictly positive, see (1.9).

The key example is the relativistic Coulomb model which was mentioned above. It is known to produce infinitely many negative eigenvalues, see [11, 21, 41]. The $L^2$-upper estimates of eigenfunctions for this model have been obtained by Nardini [32] (see also [33]). Our present Theorem 1.1 gives pointwise estimates at infinity and at zero, and together with Theorem 1.3 which is stated below, it also gives two-sided bounds for the ground state. Precise statements and the detailed discussion of this example are postponed to Sect. 6 below. We note that our upper estimate is strongly related to the result of Frank, Merz, Siedentop and Simon [15, Theorem 1.4] which gives an upper bound for the sum of squares of all eigenfunctions of the operator $\sqrt{-\Delta + 1} - \kappa|x|^{-1}$. That result also implies a pointwise upper bound at zero for a single eigenfunction which is similar to that in Theorem 1.1 (a) but with the exponent $\delta + \varepsilon$ (with arbitrarily small $\varepsilon > 0$) for $\kappa$ close to $\kappa^* = 2/\pi$, and $3/2$ for $\kappa$ close to 0 (we note that our $\kappa$ and $\delta$ are equal to $\gamma$ and $\sigma_\gamma$ in the notation of [15]). Moreover, the authors of the quoted paper conjectured that the ground state should behave as $|x|^{-\delta}$ around $x = 0$. Corollary 6.1 below gives a positive answer to this problem.

We want to mention here that the assumption (A1) is motivated by the paper of Ryznar [36], who proposed such an approach in the study of the potential theory of relativistic stable processes. At the level of operators this idea was one of basic tools in the study of the stability of relativistic matter—in that theory, for some estimates, the relativistic operator $\sqrt{-\Delta + m^2} - m$ can be replaced by its bounded perturbation $\sqrt{-\Delta}$, see Fefferman and de la Llave [13], Lieb and Yau [31], Frank, Lieb and Seiringer [14], and the monograph by Lieb and Seiringer [30]. We also refer to the recent paper by Ascione and Lőrinczi [3] for an application to the zero energy inverse problems for relativistic Schrödinger operators.

The decay rates of bound states at infinity for Schrödinger operators involving the Euclidean Laplacian are now a classical topic, see Agmon [1, 2], Reed and Simon [35], and Simon [39]. In the non-local setting this problem has been studied for Schrödinger operators with less singular decaying potentials, see [9, 24] (see also [25] for estimates in the zero energy case). Pointwise estimates as in (1.5) and (1.6) have been first established by Carmona, Masters and Simon in their seminal paper [9]. The authors identified the decay rates at infinity of eigenfunctions corresponding to negative eigenvalues for a large class of Lévy operators $L$ perturbed by Kato-decomposable potentials $V = V_+ - V_-$ (also called Kato-Feller potentials, see [12]) for which the negative part $V_-$ is in the Kato class associated with $L$, and the positive part $V_+$ is locally in that class. Recall that if $V(x) = \kappa|x|^{-\beta}$, for $\kappa, \beta > 0$, then $V$ belongs to the Kato class of $L$ if and only if $\beta < \alpha$, see e.g. [18, Section 3]. Therefore, the singularity of the potential in (1.1) is critical for these operators. Part (b) of Theorem 1.1 states that the estimates of Carmona, Masters and Simon extend to such a case, i.e. the criticality of the potential does not change.
the decay rates of bound states at infinity. The difference is that in our present setting the eigenfunctions can be singular at zero, while for the Kato class potentials they are typically bounded and continuous on $\mathbb{R}^d$. Part (a) of Theorem 1.1 gives the upper estimate of bound states around zero for potentials with critical singularity. The matching lower bound for the ground state is discussed in Theorem 1.3 below. Such pointwise estimates for Lévy operators with the fractional Hardy potential have not been known before. We remark that the estimates of the resolvent kernels for some special examples of Lévy operators can be extracted e.g. from [9, 24].

The methods of the paper [9] are probabilistic—they are based on the Feynman–Kac formula, martingales and the probabilistic potential theory. These tools are not available for more singular potentials as that in (1.1), see the monograph by Demuth and van Casteren [12] for a systematic introduction to the stochastic spectral analysis of Feller operators. It makes the problem we treat in this paper much more difficult. We propose a new, fully analytic approach, which is completely different. The methods we use are based on the technique of perturbations of integral kernels developed by Bogdan, Hansen and Jakubowski [7]. We first construct the kernel $\tilde{p}(t, x, y)$ which is a perturbation of the heat kernel $p(t, x, y)$ by Hardy potential $V$, and we analyze the smoothing properties of the corresponding semigroup of operators $\{\tilde{P}_t\}_{t>0}$ (Sect. 2). In Sect. 3 we study the properties of quadratic forms. We describe in detail the relation between the forms of $L$ and $L^{(\alpha)}$, and show that the form of the Schrödinger operator $H$ can be identified with quadratic form of the semigroup $\{\tilde{P}_t\}_{t>0}$. Consequently, by uniqueness, $e^{-tH} = \tilde{P}_t$, $t > 0$ on $L^2(\mathbb{R}^d)$. In particular, $e^{-tH}$ are integral operators with kernels $\tilde{p}(t, x, y)$, which is crucial for our further investigations. Indeed, it provides direct access to properties of operators $e^{-tH}$. This is given in Theorem 3.3. Finally, all these partial results are applied in Sect. 4 to establish the pointwise estimates for bound states. This part starts with the perturbation formula and combines some direct observations with the self-improving estimate which involves the convolutions of resolvent kernels. Our proofs do not require sharp estimates of the kernel $\tilde{p}(t, x, y)$. On the other hand, we use in an essential way the estimates of the heat kernel for the fractional Laplacian with Hardy potential which have been obtained just recently by Bogdan, Grzywny, Jakubowski and Pilarczyk [5].

The second part of the paper is devoted to estimates of the kernel $\tilde{p}(t, x, y)$. Here, and in what follows, we write $f \asymp g$ on $A$, if $f, g \geq 0$ on $A$ and there is a (comparability) constant $c \geq 1$ such that $c^{-1}g \leq f \leq cg$ holds on $A$ (“$\asymp$” means the comparison with the constant $c$). Also, we use “$\asymp$” to indicate the definition. As usual, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. We write $\nu(\cdot, a) := c(a)$ to indicate that the constant $c$ depends on a parameter $a$. If we want to emphasize that the constant $c$ is independent of a specific parameter, we say this explicitly.

In order to get sharp results, we impose the following additional assumption on the Lévy density $\nu$.

(A2) There exists a non-increasing profile function $f : (0, \infty) \to (0, \infty)$ such that

$$
\nu(x) \asymp f(|x|), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad (1.7)
$$

and the condition

$$
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f_1(|x - y|) f_1(|y|)}{f_1(|x|)} \, dy < \infty, \quad (1.8)
$$

where $f_1 := f \wedge 1$, holds.
As proved by Kaleta and Sztonyk [27], the condition (1.8) provides a minimal regularity of the profile $f$ which is required for comparability $p_t(x) \asymp tv(x)$, for large $x \in \mathbb{R}^d$. Easy-to-check sufficient (and necessary) conditions for it can be found in [26, Section 3.1] and [27, Proposition 2]. In particular, all the examples of Lévy densities $v$ presented in Example 2.3 satisfy (A2).

Our second main result is the following theorem which gives the finite-time horizon upper estimate of $\tilde{p}(t, x, y)$. The construction of this kernel in Sect. 2 was performed for the full range of $\kappa \leq \kappa^*$. Our next theorem also covers the critical case $\kappa = \kappa^*$.

**Theorem 1.2.** Let (A1) and (A2) hold and let $\kappa \leq \kappa^*$. Then for every $T > 0$ there exists a constant $c = c(T)$ such that

$$\tilde{p}(t, x, y) \leq c \left( 1 + \frac{t^{\delta/\alpha}}{|x|^{\delta}} \right) \left( 1 + \frac{t^{\delta/\alpha}}{|y|^{\delta}} \right) p(t, x, y), \quad x, y \in \mathbb{R}^d \setminus \{0\}, \quad t \in (0, T].$$

It is clear that

$$\tilde{p}(t, x, y) \geq p(t, x, y), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$  \hspace{1cm} (1.9)

This means that the estimate in the above theorem is sharp at least for the case when both $|x|$ and $|y|$ are large. One can conjecture that the two-sided estimate as in Theorem 1.2 holds true for the full range of spatial variables. However, a general argument leading to the lower estimate in our present generality seems to be not available at the moment. On the other hand, such a conjecture is supported by our results in [23], where we have proven the two-sided estimate of this form for the case when $L = -(-\Delta + m^{2/\alpha} \alpha^2/2 + m, \alpha \in (0, 2), m > 0$. This is an example of the operator covered by our assumptions (A1)-(A2). Such two-sided bounds, for $t > 0$, have been first obtained for the fractional Laplacian $L^{(\alpha)}$ with Hardy potential [5]. The proof of Theorem 1.2 uses in an essential way the results from [5], the properties of the density $v(x)$ and the sharp bound

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge tv(y - x), \quad t \in (0, T], \quad x, y \in \mathbb{R}^d, \quad \text{ (1.10)}$$

which was established in [27] (see Lemma 5.1 below for details). Observe that (1.10) makes the estimate in Theorem 1.2 explicit. We remark that all of our statements regarding the estimates of the integral kernels in this paper are restricted to finite time horizon. The main reason is that the sharp two-sided large time estimates for the kernels $p(t, x, y)$ of the free kinetic terms are still not available in our present generality. Another reason is the presence of the increasing exponential term in (2.16) which is a consequence of our construction.

Our last result relates the lower estimate of the kernel $\tilde{p}(t, x, y)$ with the lower bound of the ground state at zero for $\kappa < \kappa^*$.

**Theorem 1.3.** Let (A1) and (A2) hold and let $\kappa < \kappa^*$. Let $\delta$ be the number determined by (1.3). Assume that there exists $\varphi \in L^2(\mathbb{R}^d)$ such that $H\varphi = E\varphi$, where $E = \inf \sigma(H) < 0$, i.e. the ground state of $H$ exists. Then the following statements are equivalent.

(a) For every $T > 0$ there exists a constant $c = c(T)$ such that

$$\tilde{p}(t, x, y) \geq c \left( 1 + \frac{t^{\delta/\alpha}}{|x|^{\delta}} \right) \left( 1 + \frac{t^{\delta/\alpha}}{|y|^{\delta}} \right) p(t, x, y),$$

$$|x| \wedge |y| \leq t^{1/\alpha}, \quad x, y \neq 0, \quad t \in (0, T].$$
(b) For every \( R > 0 \) there exists a constant \( c > 0 \) such that
\[
\varphi(x) \geq c|x|^{-\delta}, \quad |x| \leq R, \ x \neq 0.
\]

As mentioned above, for \( |x|, |y| \geq t^{1/\alpha} \) we always have
\[
\tilde{p}(t,x,y) \geq p(t,x,y) \geq \frac{1}{4} \left( 1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) \left( 1 + \frac{t^{\delta/\alpha}}{|y|^\delta} \right) p(t, x, y),
\]
which means that in fact the statement (b) implies the lower bound for the kernel \( \tilde{p}(t,x,y) \) for the full range of time-space variables. In combination with our results in [23], Theorem 1.3 gives the matching lower bound for the ground state of the relativistic Coulomb model. Details are given in Sect. 6 below. Theorem 1.3 is proved in Sect. 5.2. We remark that the sharp two-sided finite-time horizon estimates for heat kernels of fractional Laplacian and more general Lévy operators (covered by our assumptions (A1)–(A2)) with Kato class potentials can be found in Bogdan, Hansen and Jakubowski [7] and Grzywny, Kaleta and Sztonyk [18].

2. Schrödinger Perturbations of Heat Kernels

2.1. General framework. Our approach in this paper uses the perturbation technique of kernels which was developed in [7]. We start with and abstract setting.

Let \((E, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space. Let \( k : (0, \infty) \times E \times E \to [0, \infty) \) be a \(\mathcal{B}((0, \infty)) \times \mathcal{F} \times \mathcal{F}\)-measurable kernel such that
\[
\int_{E} k(s, x, z) k(t, z, y) \mu(\mathrm{d}z) = k(t + s, x, y), \quad x, y \in E, \quad (2.1)
\]
and let
\[
k(t, x, y) = k(t, y, x) \geq 0, \quad t > 0, \ x, y \in E, \quad (2.2)
\]
where
\[
k_0(t, x, y) = k(t, x, y)
\]
\[
k_n(t, x, y) = \int_0^t \int_{E} k(s, x, z) V(z) k_{n-1}(t - s, z, y) \mu(\mathrm{d}z) \mathrm{d}s, \quad n \geq 1.
\]

It is known that \( \tilde{k}(t, x, y) \) is a symmetric transition density, i.e. it satisfies the equalities (2.1), (2.2), see [7]. Moreover, the following perturbation formula
\[
\tilde{k}(t, x, y) = k(t, x, y) + \int_0^t \int_{E} k(s, x, z) V(z) \tilde{k}(t - s, z, y) \mu(\mathrm{d}z) \mathrm{d}s
\]
\[
= k(t, x, y) + \int_0^t \int_{E} \tilde{k}(s, x, z) V(z) k(t - s, z, y) \mu(\mathrm{d}z) \mathrm{d}s, \quad t > 0, \ x, y \in E,
\]
holds. The starting point of our investigations is the following direct observation.
Lemma 2.1. Suppose we are given two kernels $k^{(1)}(t, x, y)$ and $k^{(2)}(t, x, y)$ as above and denote by $\tilde{k}^{(1)}(t, x, y)$ and $\tilde{k}^{(2)}(t, x, y)$ the corresponding perturbed kernels as in (2.3). If
\begin{equation}
k^{(1)}(t, x, y) \leq k^{(2)}(t, x, y), \quad t > 0, \ x, y \in E, \tag{2.4}
\end{equation}
then
\begin{equation}
\tilde{k}^{(1)}(t, x, y) \leq \tilde{k}^{(2)}(t, x, y), \quad t > 0, \ x, y \in E. \tag{2.5}
\end{equation}

Proof. It follows from (2.3), (2.4) and the assumption $V \geq 0$.

This fact will be used below in the following form.

Corollary 2.2. Assume that there is $\lambda > 0$ such that
\begin{equation}
k^{(1)}(t, x, y) \leq e^{\lambda t} k^{(2)}(t, x, y) \quad t > 0, \ x, y \in E. \tag{2.6}
\end{equation}
Then
\begin{equation}
\tilde{k}^{(1)}(t, x, y) \leq e^{\lambda t} \tilde{k}^{(2)}(t, x, y) \quad t > 0, \ x, y \in E.
\end{equation}

Proof. We apply Lemma 2.1 to the kernel $k(t, x, y) := e^{\lambda t} k^{(2)}(t, x, y)$ in place of $k^{(2)}(t, x, y)$. We only have to make sure that $\tilde{k}(t, x, y) = e^{\lambda t} \tilde{k}^{(2)}(t, x, y)$. We have $k_0(t, x, y) = k(t, x, y)$,
\[ k_n(t, x, y) = \int_0^t \int_E k(s, x, z)V(z)k_{n-1}(t-s, x, y)\mu(dz)ds, \quad n \geq 1, \]
and observe that $k_n(t, x, y) = e^{\lambda t} k_n^{(2)}(t, x, y)$, which yields the desired assertion.

2.2. Heat kernels of free operators. Let $d \in \mathbb{N} := \{1, 2, \ldots\}$, $\alpha \in (0, 2)$ and $\alpha < d$. Let
\[ v^{(\alpha)}(y) = c_{d, \alpha} |y|^{-d-\alpha}, \quad y \in \mathbb{R}^d \setminus \{0\}, \]
where
\[ c_{d, \alpha} := \frac{\alpha 2^{\alpha-1} \Gamma((d+\alpha)/2)}{\pi^{d/2} \Gamma(1-\alpha/2)}. \]

This coefficient is chosen so that
\begin{equation}
\int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) v^{(\alpha)}(y) dy = |\xi|^{\alpha}, \quad \xi \in \mathbb{R}^d. \tag{2.7}
\end{equation}

Let $v(dx) = v(x)dx$ be a symmetric Lévy measure such that (A1) holds. Consider
\[ \psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y)) v(dy), \quad \xi \in \mathbb{R}^d. \]

As observed in the introduction, we have
\[ |\xi|^\alpha - 2 |\sigma| \leq \psi(\xi) \leq |\xi|^\alpha, \quad \xi \in \mathbb{R}^d, \]
which gives
\[
\psi(\xi) \asymp |\xi|^\alpha, \quad |\xi| \geq (4|\sigma|)^{1/\alpha}.
\] (2.8)

In particular, \(e^{-t\psi(\cdot)} \in L^1(\mathbb{R}^d)\), for every \(t > 0\). We define
\[
p_t^{(\alpha)}(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{-ix\cdot\xi} \, d\xi, \quad t > 0, \ x \in \mathbb{R}^d.
\] (2.9)

and
\[
p_t(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} e^{-ix\cdot\xi} \, d\xi, \quad t > 0, \ x \in \mathbb{R}^d.
\] (2.10)

For every \(t > 0\), \(p_t^{(\alpha)}\) and \(p_t\) are continuous and bounded probability density functions, see e.g. [28]. Moreover, \(p_t^{(\alpha)}\) is radial and \(p_t\) is symmetric, i.e. \(p_t(-y) = p_t(y), y \in \mathbb{R}^d\). From (2.9) we have
\[
p_t^{(\alpha)}(x) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x).
\] (2.11)

It is also well-known that
\[
p_t^{(\alpha)}(x) \asymp t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \quad t > 0, \ x \in \mathbb{R}^d.
\] (2.12)

We denote
\[
p^{(\alpha)}(t, x, y) = p_t^{(\alpha)}(y - x), \quad p(t, x, y) = p_t(y - x), \quad t > 0, \ x, y \in \mathbb{R}^d.
\]

Clearly, \(p^{(\alpha)}\) and \(p\) are symmetric kernels satisfying the Chapman–Kolmogorov equations:
\[
\int_{\mathbb{R}^d} p^{(\alpha)}(s, x, y)p^{(\alpha)}(t, y, z) \, dy = p^{(\alpha)}(t + s, x, z), \quad x, z \in \mathbb{R}^d, \ s, t > 0,
\] (2.13)
\[
\int_{\mathbb{R}^d} p(s, x, y)p(t, y, z) \, dy = p(t + s, x, z), \quad x, z \in \mathbb{R}^d, \ s, t > 0.
\] (2.14)

We let
\[
P_t^{(\alpha)} f(x) = \int_{\mathbb{R}^d} f(y) p^{(\alpha)}(t, x, y) \, dy, \quad f \in L^2(\mathbb{R}^d), \ t > 0,
\]
\[
P_t f(x) = \int_{\mathbb{R}^d} f(y) p(t, x, y) \, dy, \quad f \in L^2(\mathbb{R}^d), \ t > 0;
\]
\((P_t^{(\alpha)})_{t>0}, (P_t)_{t>0}\) are strongly continuous semigroups of self-adjoint contractions on \(L^2(\mathbb{R}^d)\), and the operators \(L^{(\alpha)}\) and \(L\) are \(L^2\)-generators of these semigroups, see e.g. [22, vol. 1, Example 4.7.28, pp. 407–409]. The kernels \(p^{(\alpha)}(t, x, y), p(t, x, y)\) are heat kernels of the operators \(-L^{(\alpha)}, -L\), respectively.

Since \(v^{(\alpha)} = v + \sigma\), we have
\[
p_t^{(\alpha)}(x) = p_t * \mu_t^\sigma(x) = \int_{\mathbb{R}^d} p_t(x - y) \mu_t^\sigma(\,dy), \quad x \in \mathbb{R}^d, \ t > 0,
\]
where
\[ \mu^\sigma_t(dx) = e^{-|\sigma|t} \sum_{k=0}^{\infty} \frac{t^k \sigma^{k*}(dx)}{k!} = e^{-|\sigma|t} \delta_0(dx) + e^{-|\sigma|t} \sum_{k=1}^{\infty} \frac{t^k \sigma^{k*}(x)}{k!} \]

and \( \sigma^{k*} \) denotes the \( k \)-fold convolution of the measure/density \( \sigma \). Hence,
\[ p^{(\alpha)}_t(x) = e^{-|\sigma|t} p_t(x) + e^{-|\sigma|t} \sum_{k=1}^{\infty} \frac{t^k (p_t \ast \sigma^{k*})(x)}{k!} \]  \hspace{1cm} (2.15)

and, consequently,
\[ p(t, x, y) \leq e^{\|\sigma\|t} p^{(\alpha)} (t, x, y), \quad x, y \in \mathbb{R}^d, \ t > 0. \]  \hspace{1cm} (2.16)

We close this section by collecting some examples of Lévy measures \( \nu \) satisfying (A1) and (A2).

**Example 2.3.** (a) Relativistic stable Lévy density: let \( \alpha \in (0, 2) \) and \( m > 0 \) and let
\[ \nu(x) = \frac{\alpha (4\pi)^{d/2}}{2 \Gamma(1 - \alpha/2)} \int_0^{\infty} \exp \left( -\frac{|x|^2}{4u} - m^{2/\alpha} u \right) u^{-1 - \frac{d-\alpha}{2}} du \]
\[ = \frac{\alpha^{\frac{\alpha-d}{2}} m^{\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}} (m^{\frac{1}{\alpha}} |x|)}{\pi^{\frac{d}{2}} \Gamma(1 - \alpha/2) |x|^{\frac{d+\alpha}{2}}}, \quad x \in \mathbb{R}^d \setminus \{0\}, \]

where
\[ K_{\mu}(r) = \frac{1}{2} \left( \frac{r}{2} \right)^{\mu} \int_0^{\infty} u^{-\mu-1} \exp \left( -u - \frac{r^2}{4u} \right) du, \quad \mu > 0, \ r > 0, \]
is the modified Bessel function of the second kind, see e.g. [34, 10.32.10]. As proved in [36, Lemma 2],
\[ \sigma(x) := \nu^{(\alpha)}(x) - \nu(x) \]
is a positive density of a finite measure such that
\[ |\sigma| = \int_{\mathbb{R}^d} \sigma(x) dx = m \quad \text{and} \quad \sigma(x) \leq \frac{c}{|x|^{d+\alpha-2}}, \quad x \in \mathbb{R}^d \setminus \{0\}, \]
for some constant \( c > 0 \). It is then clear that (A1) holds. In order to verify (A2), we first observe that the density \( \nu(x) \) is radial decreasing function. By using the well-known asymptotics,
\[ \lim_{r \to \infty} K_{\mu}(r) \sqrt{r} e^r = \sqrt{\pi/2}, \]
we can also show that
\[ \nu(x) \asymp e^{-m^{1/\alpha}|x|^{\frac{d+\alpha-1}{2}}}, \quad |x| \geq 1. \]

Consequently, (A2) holds by combination of [27, Proposition 2] and [26, Lemma 3.1].

The Lévy operator associated to \( \nu \) is the relativistic (\( \alpha \)-stable) operator
\[ L = -(-\Delta + m^{2/\alpha})^{\alpha/2} + m. \]
(b) Tempered stable Lévy density: let $\beta > \alpha$ and $\lambda > 0$, and let

$$v(x) = e^{-\lambda |x|^{\beta}} v^{(\alpha)}(x), \quad x \in \mathbb{R}^d \setminus \{0\}. $$

Then

$$\sigma(x) = v^{(\alpha)}(x) - v(x) = c_{d,\alpha}|x|^{\alpha} \left(1 - e^{-\lambda |x|^{\beta}}\right),$$

where

$$c_{d,\alpha} = \frac{\alpha 2^{\frac{\alpha}{2}} \Gamma(\frac{d + \alpha)}{2})}{\pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}.$$

is a positive density of a finite measure. By direct calculations,

$$|\sigma| = \int_{\mathbb{R}^d} |\sigma(x)|dx = c_{d,\alpha} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty (1 - e^{-\lambda u^\beta}) u^{-1 - \frac{\alpha}{\beta}} du$$

$$= \frac{2\alpha \Gamma(\frac{d + \alpha}{2})}{\beta \Gamma(\frac{d}{2}) \Gamma(1 - \frac{\alpha}{2})} \int_0^\infty (1 - e^{-\lambda u^\beta}) u^{-1 - \frac{\alpha}{\beta}} du$$

$$= \frac{2\alpha \Gamma(\frac{d + \alpha}{2})}{\Gamma(\frac{d}{2}) \Gamma(1 - \frac{\alpha}{2})} \lambda^\alpha,$$

where in the last line we have used the standard identity

$$\frac{\gamma}{\Gamma(1 - \gamma)} \int_0^\infty (1 - e^{-\lambda u}) u^{-1 - \gamma} du = \lambda^\gamma, \quad \lambda > 0, \; \gamma \in (0, 1).$$

see e.g. [37, (1)]. In particular, (A1) holds. Moreover, it follows from [27, Proposition 2] that (A2) is satisfied if and only if $\alpha \in (0, 1)$ and $\beta \in (\alpha, 1]$.

(c) Stable Lévy density suppressed on a complement of a neighbourhood of the origin: let $\eta : \mathbb{R}^d \to (0, 1]$ be a function such that there exists a decreasing profile $g : (0, \infty) \to (0, \infty)$ such that $\eta(x) \asymp g(|x|), \; x \in \mathbb{R}^d \setminus \{0\}$. Let $r > 0$ and let

$$v(x) = v^{(\alpha)}(x) 1_{|x| \leq r} + \eta(x) v^{(\alpha)}(x) 1_{|x| > r}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$ 

Then

$$\sigma(x) = v^{(\alpha)}(x) - v(x) = (1 - \eta(x)) v^{(\alpha)}(x) 1_{|x| > r} \geq 0$$

is a density of a finite measure. Indeed,

$$|\sigma| = \int_{\mathbb{R}^d} |\sigma(x)|dx = \int_{|x| > r} (1 - \eta(x)) v^{(\alpha)}(x) dx \leq \int_{|x| > r} v^{(\alpha)}(x) dx < \infty.$$ 

Hence, (A1) holds. If we moreover assume that there exists a constant $c > 0$ such that

$$g(s) g(t) \leq c g(s + t), \quad s, t \geq 1,$$

then (A2) holds with the profile $f(r) = g(r) r^{-d - \alpha}$, cf. [26, Lemma 3.2 b)].

An example of such $v$ is the layered stable Lévy density of the form

$$v(x) = c_{d,\alpha} |x|^{\alpha} (1 \wedge |x|^{-\gamma}), \quad x \in \mathbb{R}^d \setminus \{0\},$$

where $\gamma > 0$. 
2.3. Kernels perturbed by fractional Hardy potential. Let \( d \in \mathbb{N} \) and \( 0 < \alpha < 2 \wedge d \). Following [5, Figure 1 and the discussion in the paragraph above (2.7)], we consider the function \([0, (d - \alpha)/2] \ni \beta \mapsto \kappa_\beta\) given by \(\kappa_0 = 0\) and

\[
\kappa_\beta := \frac{2^\alpha \Gamma \left( \frac{\alpha + \beta}{2} \right) \Gamma \left( \frac{d - \beta}{2} \right)}{\Gamma \left( \frac{\beta}{2} \right) \Gamma \left( \frac{d - \alpha - \beta}{2} \right)}, \quad 0 < \beta < \frac{d - \alpha}{2}.
\]

It is a strictly increasing and continuous function with maximal (critical) value

\[
\kappa^* = \kappa_{(d - \alpha)/2} = \frac{2^\alpha \Gamma \left( \frac{d + \alpha}{4} \right)^2}{\Gamma \left( \frac{d - \alpha}{4} \right)^2}.
\]

It is then clear that it establishes a one-to-one correspondence between numbers in \([0, (d - \alpha)/2]\) and \([0, \kappa^*]\). **Throughout the paper we assume that**

\[
\kappa \in [0, \kappa^*] \text{ is fixed and } \delta \in \left[0, \frac{(d - \alpha)}{2}\right] \text{ is such that } \kappa_\delta = \kappa. \tag{2.17}
\]

Let

\[
V(x) = V_\kappa(x) = \frac{\kappa}{|x|^{\alpha}}.
\]

When \(\kappa < \kappa^*\), then \(V_\kappa\) is said to be sub-critical.

Starting with \(k(t, x, y) = p(t, x, y)\) we can now construct the perturbed kernel \(\tilde{p}(t, x, y)\) according to the procedure described in Sect. 2.1. Recall that \(\tilde{p}(t, x, y)\) is a symmetric transition density such that

\[
\tilde{p}(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p(s, x, z) V(z) \tilde{p}(t - s, z, y) dz ds \tag{2.18}
\]

\[
= p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(s, x, z) V(z) p(t - s, z, y) dz ds,
\]

\(t > 0, \ x, y \in \mathbb{R}^d \setminus \{0\}\).

Such a construction has been recently performed in [5] for the case when \(k(t, x, y) = p^{(\alpha)}(t, x, y)\). It was proved in [5, Theorems 1.1 and 3.1] that

\[
\tilde{p}^{(\alpha)}(t, x, y) \asymp \left(1 + \frac{t^{\delta/\alpha}}{|x|^{\delta}}\right) \left(1 + \frac{t^{\delta/\alpha}}{|y|^{\delta}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|y - x|^{d + \alpha}}\right),
\]

\(x, y \in \mathbb{R}^d \setminus \{0\}, \ t > 0,\) \tag{2.19}

and

\[
\int_{\mathbb{R}^d} \tilde{p}^{(\alpha)}(t, x, y)|y|^{-\delta} dy = |x|^{-\delta}, \quad x \in \mathbb{R}^d \setminus \{0\}, \ t > 0. \tag{2.20}
\]

By (2.16) and Corollary 2.2 applied to

\[
k^{(1)}(t, x, y) = p(t, x, y), \quad k^{(2)}(t, x, y) = p^{(\alpha)}(t, x, y)
\]
and $\lambda = |\sigma|$, we get

$$\int p(t, x, y) \leq e^{\lambda |t| \int p(\alpha)(t, x, y) \lambda x, y \in \mathbb{R}^d \setminus \{0\}, \quad t > 0. \tag{2.21}$$

This domination property will be crucial for our further investigations in this paper. By using (2.21), (2.19) and by following the argument in the proof of [5, Lemmas 4.9–4.10], we can show directly that the function

$$(0, \infty) \times (\mathbb{R}^d \setminus \{0\})^2 \ni (t, x, y) \mapsto \int p(t, x, y)$$

is continuous. \tag{2.22}

2.4. Semigroup of operators defined by the kernel $\int p$. We define

$$\int p(\alpha) f(x) = \int_{\mathbb{R}^d} f(y) \int p(\alpha)(t, x, y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0,$$

$$\int p f(x) = \int_{\mathbb{R}^d} f(y) \int p(t, x, y)dy, \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$ 

It is shown in [5, Proposition 2.4] that $(\int p(\alpha))_{t>0}$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d)$. By similar argument and the domination property (2.21) this extends to $(\int p)_{t>0}$ except the $L^2$-contractivity. Indeed, we only know that $\left\| \int p \right\|_{L^2 \rightarrow L^2} \leq e^{\lambda |t|}, \quad t > 0$. Clearly, each $\int p(\alpha)$ and $\int p$ is a self-adjoint operator.

We will also need the following smoothing properties of the semigroup $(\int p)_{t>0}$.

**Lemma 2.4.** Under (A1), for every $t > 0$, we have

$$\int p \left( L^2(\mathbb{R}^d) \right) \subset C \left( \mathbb{R}^d \setminus \{0\} \right).$$

Furthermore, there exists a constant $c > 0$ such that, for every $x \in \mathbb{R}^d \setminus \{0\}, \quad t > 0$ and $f \in L^2(\mathbb{R}^d)$,

$$|\int p f(x)| \leq ce^{\lambda |t|} t^{-\frac{d-\delta}{\alpha}} \left( 1 + t^{\frac{\delta}{\alpha}} \frac{1}{|x|^{\delta}} \right) \|f\|_2,$$

where $\delta$ is defined by (2.17). The constant $c$ does not depend on $\sigma$.

**Proof.** We first establish the continuity assertion. Fix $f \in L^2(\mathbb{R}^d)$ and $t > 0$. Let $x, z \in \mathbb{R}^d \setminus \{0\}$ be such that $|x - z| < 1 \wedge |x|/2$ and let $R > |x| + 1$. We have

$$|\int p f(x) - \int p f(z)| \leq \int_{|y| \leq 2R} |f(y)| |\int p(t, x, y) - \int p(t, z, y)|dy$$

$$+ \int_{|y| > 2R} |f(y)| |(\int p(t, x, y) + \int p(t, z, y))|dy.$$

By (2.21) and (2.19), we get

$$|\int p(t, x, y) - \int p(t, z, y)| \leq \int p(t, x, y) + \int p(t, z, y)$$

$$\leq c_1 e^{\lambda |t|} \left( 1 + \frac{2t^{\delta/\alpha}}{|x|^{\delta}} \right) \left( 2 \frac{2t^{\delta/\alpha}}{|y - z|^{d+\alpha}} + \frac{t^{\delta/\alpha}}{|y - x|^{d+\alpha}} \right)$$

$$\left( 1 + \frac{t^{\delta/\alpha}}{|y|^{\delta}} \right). \tag{2.23}$$
Recall that $0 \leq \delta \leq (d - \alpha)/2$. The Cauchy–Schwarz inequality implies that
\[
\int_{|y| \leq 2R} |f(y)| \left(1 + \frac{t^{\delta/\alpha}}{|y|^\delta} \right) dy < \infty.
\]
Consequently, the first integral on the right hand side above goes to zero as $z \to x$, by (2.22) and the Lebesgue dominated convergence theorem.

Furthermore, by (2.23) and the Cauchy–Schwarz inequality,
\[
\int_{|y| > 2R} |f(y)| (\tilde{p}(t, x, y) + \tilde{p}(t, z, y)) dy \\
\leq c_1 t^{\sigma|t|/2} \left(1 + \frac{2t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \int_{|y| > 2R} |f(y)| \left(\frac{t}{|y - z|^{d+\alpha}} + \frac{t}{|y - x|^{d+\alpha}} \right) dy \\
\leq 2c_1 t^{\sigma|t|/2} \left(1 + \frac{2t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \|f\|_2 \left(\int_{|y| > R} \frac{1}{|y|^{d+2\alpha}} dy \right)^{1/2} \\
\leq c_2 t^{\sigma|t|/2} \left(1 + \frac{2t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \|f\|_2 R^{-d/2 - \alpha}.
\]

Hence,
\[
\limsup_{z \to x} |\tilde{P}_t f(x) - \tilde{P}_t f(z)| \leq c_2 t^{\sigma|t|/2} \left(1 + \frac{2t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \|f\|_2 R^{-d/2 - \alpha}
\]
and, by letting $R \to \infty$, we get that $\tilde{P}_t f(z) \to \tilde{P}_t f(x)$ as $z \to x$. This shows that $\tilde{P}_t f \in C(\mathbb{R}^d \setminus \{0\})$.

In order to get the second assertion, we proceed in a similar way. Let $f \in L^2(\mathbb{R}^d)$. By (2.21), (2.19) and (2.12), we write
\[
|\tilde{P}_t f(x)| \leq c_3 t^{\sigma|t|/2} \left(1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) \left(\int_{|y| \leq 1} + \int_{|y| > 1} \right) |f(y)| p^{(x)}(t, x, y) \left(1 + \frac{t^{\delta/\alpha}}{|y|^\delta} \right) dy \\
\leq c_3 t^{\sigma|t|/2} \left(1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \int_{\mathbb{R}^d} |f(y)| p^{(x)}(t, x, y) dy + t^{-(d - \delta)/\alpha} \int_{|y| \leq 1} |f(y)| \left(\frac{1}{|y|^\delta} \right) dy
\]
and, by the Cauchy–Schwarz inequality applied to both integrals and (2.12), we get
\[
|\tilde{P}_t f(x)| \leq c_3 t^{\sigma|t|/2} \left(1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) \left(\int_{\mathbb{R}^d} |f(y)|^2 p^{(x)}(t, x, y) dy \right)^{1/2} + c_4 t^{-(d - \delta)/\alpha} \|f\|_2 \\
\leq c_5 t^{\sigma|t|/2} \left(1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) \left(1 + t^{\delta/\alpha} \right) t^{-d/(2\alpha)} + c_5 t^{-(d - \delta)/\alpha} \|f\|_2 \\
\leq c_6 t^{\sigma|t|/2} \left(1 + \frac{t^{\delta/\alpha}}{|x|^\delta} \right) t^{-(d - \delta)/\alpha} (1 + t^{d/(2\alpha)}) \|f\|_2.
\]

This completes the proof. \(\square\)

Observe that the above lemma also applies directly to the semigroup $(\tilde{P}_t^{(\alpha)})_{t > 0}$ as a special case (we just take $\sigma \equiv 0$ in (A1)).
3. Schrödinger Operators with Singular Potentials and Their Semigroups

3.1. Quadratic forms of free operators. We first discuss the relation between quadratic forms of the operators $-L^{(α)}$ and $-L$. Due to (2.15) our analysis will be based on the corresponding operator semigroups. For $f ∈ L^2(\mathbb{R}^d)$, we define

$$\mathcal{E}_t[f] = \frac{1}{t} \langle f - P_t f, f \rangle, \quad \mathcal{E}[f] = \lim_{t \to 0} \mathcal{E}_t[f].$$

Since the operators $P_t$ are self-adjoint contractions on $L^2(\mathbb{R}^d)$, it follows from the spectral theorem that the map

$$(0, \infty) \ni t ↦ \mathcal{E}_t[f]$$

is decreasing for any $f ∈ L^2(\mathbb{R}^d)$, see e.g. [17, Lemma 1.3.4]. In particular, the above limit exists and belongs to $[0, \infty]$. The domain of $\mathcal{E}$ is defined as

$$\mathcal{D}(\mathcal{E}) = \{ f ∈ L^2 : \mathcal{E}[f] < \infty \}.$$

The corresponding sesquilinear form is given by

$$\mathcal{E}[f, g] = \lim_{t \to 0} \frac{1}{t} \langle f - P_t f, g \rangle, \quad f, g ∈ \mathcal{D}(\mathcal{E}),$$

and it satisfies the following polarization identity

$$\mathcal{E}[f, g] = \frac{1}{4} (\mathcal{E}[f + g] - \mathcal{E}[f - g] + i\mathcal{E}[f + ig] - i\mathcal{E}[f - ig]), \quad f, g ∈ \mathcal{D}(\mathcal{E}).$$

The form $(\mathcal{E}^{(α)}, \mathcal{D}(\mathcal{E}^{(α)}))$ of the fractional Laplacian $L^{(α)}$ is defined in the same manner by replacing the operators $P_t$ with $P_t^{(α)}$. Due to (A1), it is a special case of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

For more information on forms we refer the reader to [38, Chapter 10] and [40, Chapter 6].

Below we consider the convolution operators

$$\sigma^{k^*} f(x) = \int_{\mathbb{R}^d} \sigma^{k^*}(x - y) f(y) dy, \quad k ∈ \mathbb{N}, \quad f ∈ L^2(\mathbb{R}^d),$$

where $\sigma$ is defined in (A1). Since $\sigma^{k^*} ∈ L^1(\mathbb{R}^d), k ∈ \mathbb{N}$, it defines a bounded operator on $L^2(\mathbb{R}^d)$. Indeed, by the Cauchy–Schwarz inequality and the Tonelli theorem, for $f ∈ L^2(\mathbb{R}^d)$, we have

$$\left\| \sigma^{k^*} f \right\|^2_2 \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \sigma^{k^*}(x - y) dy \right) \left( \int_{\mathbb{R}^d} |\sigma^{k^*}(x - y)f(y)|^2 dy \right) dx$$

$$= |\sigma|^k \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \sigma^{k^*}(x - y) dx \right) |f(y)|^2 dy = |\sigma|^{2k} \| f \|^2_2. \quad (3.1)$$

**Lemma 3.1.** Under assumption (A1), we have

$$\mathcal{E}[f] + |\sigma| \| f \|^2_2 - \langle f, f \rangle = \mathcal{E}^{(α)}[f], \quad f ∈ L^2(\mathbb{R}^d). \quad (3.2)$$

In particular, $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\mathcal{E}^{(α)})$ and

$$\mathcal{E}[f] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(y)|^2 v(x - y) dx dy, \quad f ∈ L^2(\mathbb{R}^d).$$
Proof. We first prove (3.2). For every $t > 0$, by (2.15) we have
\[
\left\{ f - P_t^{(\alpha)} f, f \right\} = \left\{ f - e^{-|\sigma| t} f, f \right\} + e^{-|\sigma| t} \left\{ f - P_t f, f \right\} - \left\{ e^{-|\sigma| t} tP_t (\sigma f), f \right\} - \left\{ e^{-|\sigma| t} \sum_{k=2}^{\infty} \frac{t^k}{k!} P_t (\sigma^k f), f \right\}.
\]
Hence,
\[
\mathcal{E}^{(\alpha)}_t[f] = \frac{1}{t} \left\{ f - e^{-|\sigma| t} f, f \right\} + e^{-|\sigma| t} \mathcal{E}_t[f] - e^{-|\sigma| t} \left\{ P_t (\sigma f), f \right\} - e^{-|\sigma| t} \left\{ \sum_{k=2}^{\infty} \frac{t^k}{k!} P_t (\sigma^k f), f \right\}.
\]
As shown above, $\sigma f \in L^2(\mathbb{R}^d)$. Thus, by strong continuity, $\langle P_t (\sigma f), f \rangle \to \langle \sigma f, f \rangle$, as $t \to 0$. Moreover, by (3.1),
\[
\left| \sum_{k=2}^{\infty} \frac{t^k}{k!} P_t (\sigma^k f), f \right| \leq \sum_{k=2}^{\infty} \frac{t^k}{k!} \left| P_t (\sigma^k f), f \right| \leq \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} \left\| P_t (\sigma^k f) \right\|_2 \| f \|_2 \leq \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} |\sigma|^k \| f \|_2^2 \leq t|\sigma|^2 \| f \|_2^2 e^{t|\sigma|} \to 0, \text{ as } t \to 0.
\]
Then, by letting $t \to 0$ in (3.3), we get (3.2). In particular, we see that $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\mathcal{E}^{(\alpha)})$.
In order to see the second assertion, we observe that
\[
|\sigma| \| f \|_2^2 - \langle \sigma f, f \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y)) \sigma(x - y) dy f(x) dx = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(y)|^2 \sigma(x - y) dy dx \geq 0, \quad f \in L^2(\mathbb{R}^d),
\]
by symmetrization. By (3.1) this double integral is finite for every $f \in L^2(\mathbb{R}^d)$. Therefore, using (3.2) and a well known fact that
\[
\mathcal{E}^{(\alpha)}[f] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(y)|^2 \nu^{(\alpha)}(x - y) dx dy, \quad f \in L^2(\mathbb{R}^d),
\]
we obtain
\[
\mathcal{E}[f] = \mathcal{E}^{(\alpha)}[f] - \left( |\sigma| \| f \|_2^2 - \langle \sigma f, f \rangle \right) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(y)|^2 \nu^{(\alpha)}(x - y) dx dy - \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x) - f(y)|^2 \sigma(x - y) dy dx.
\]
3.2. Quadratic form of the semigroup $(\tilde{P}_t)_{t>0}$. In this section we identify the form $\tilde{\mathcal{E}}$ of the semigroup $(\tilde{P}_t)_{t>0}$ when acting on $\mathcal{D}(\tilde{\mathcal{E}})$. Similarly as above, we define for $f \in L^2(\mathbb{R}^d)$

$$\tilde{E}_t[f] = \frac{1}{t} (f - \tilde{P}_t f, f), \quad \tilde{E}[f] = \lim_{t \to 0} \tilde{E}_t[f].$$

The limit exists and belongs to $[-|\sigma|, \infty]$, and the domain of $\tilde{E}$ is

$$\mathcal{D}(\tilde{E}) = \{ f \in L^2 : \tilde{E}[f] < \infty \}.$$ 

The corresponding sesquilinear form is then defined as

$$\tilde{E}[f, g] = \lim_{t \to 0} \frac{1}{t} (f - \tilde{P}_t f, g), \quad f, g \in \mathcal{D}(\tilde{E}),$$

and the following polarization identity

$$\tilde{E}[f, g] = \frac{1}{4} (\tilde{E}[f + g] - \tilde{E}[f - g] + i \tilde{E}[f + ig] - i \tilde{E}[f - ig])$$

holds for $f, g \in \mathcal{D}(\tilde{E})$, see [40, Chapter 6] and [38, Chapter 10]. Clearly, the form $(\tilde{\mathcal{E}}^{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)}))$ associated with the semigroup $(\tilde{P}^{(\alpha)}_t)_{t>0}$ is a special case of $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$. The following lemma partly extends [5, Lemma 5.1] to the case $\sigma \neq 0$ (in the quoted lemma an additional form $\tilde{E}$ is considered). Another difference is that we work with the complex $L^2(\mathbb{R}^d)$.

**Lemma 3.2.** Under (A1), for $\kappa \leq \kappa^*$, we have $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\tilde{\mathcal{E}}) \subset \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$ and

$$\tilde{E}[f] = \mathcal{E}[f] - \int_{\mathbb{R}^d} V(x) |f(x)|^2 dx, \quad f \in \mathcal{D}(\mathcal{E}). \quad (3.4)$$

**Proof.** Let first $f \in L^2(\mathbb{R}^d)$ satisfy $f \geq 0$. Then, by (2.18) and Tonelli’s theorem,

$$\tilde{E}_t[f] = \mathcal{E}_t[f] - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \frac{1}{t} \int_0^t \int_{\mathbb{R}^d} p(u, x, z) V(z) \tilde{p}(t-u, z, y) dz du dx dy$$

$$= \mathcal{E}_t[f] - \int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_0^t P_u f(z) \tilde{P}_{t-u} f(z) dz du.$$

Since the integral on the right hand side is nonnegative for every $t > 0$, $f \in \mathcal{D}(\mathcal{E})$ implies $f \in \mathcal{D}(\tilde{\mathcal{E}})$. Furthermore, by (2.21) we have

$$\tilde{E}_t[f] = \frac{1}{t} \left( \langle f, f \rangle - \langle \tilde{P}_t f, f \rangle \right) \geq \frac{1}{t} \left( \langle f, f \rangle - e^{\|\sigma\|} \langle \tilde{P}^{(\alpha)}_t f, f \rangle \right)$$

$$= \frac{1}{t} \left( \langle f, f \rangle - \langle \tilde{P}^{(\alpha)}_t f, f \rangle \right) - \frac{1}{t} \left( e^{\|\sigma\|} - 1 \right) \left\| \tilde{P}^{(\alpha)}_t f \right\|_2^2$$

$$\geq \tilde{E}^{(\alpha)}_t[f] - |\sigma| e^{\|\sigma\|} \|f\|_2^2.$$ 

Thus, $f \in \mathcal{D}(\tilde{\mathcal{E}})$ implies $f \in \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$. The two implications extend to any complex valued $f$, proving $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\tilde{\mathcal{E}}) \subset \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$, as follows.
For any complex \( f \in L^2(\mathbb{R}^d) \) we have
\[
\mathcal{E}_t[f] = \frac{1}{t} \left( (f, f) - \langle Pt_{1/2}f, Pt_{1/2}f \rangle \right) = \mathcal{E}_t[\Re f] + \mathcal{E}_t[\Im f]
\] (3.5)
and for any real \( g \in L^2(\mathbb{R}^d) \)
\[
\mathcal{E}_t[g] = \frac{1}{t} \left( (g, g) - \langle Pt_{1/2}g, Pt_{1/2}g \rangle \right) \geq \frac{1}{t} \left( (|g|, |g|) - \langle Pt_{1/2}|g|, Pt_{1/2}|g| \rangle \right) = \mathcal{E}_t[|g|].
\]
Using the above inequalities, the lower boundedness of the form \( \mathcal{E} \), and the equalities \( g_+ = \frac{|g| + g}{2}, \; g_- = \frac{|g| - g}{2} \), we easily prove the following implication: if \( f \in \mathcal{D}(\mathcal{E}) \), then \( \Re f_+, \Re f_-, \Im f_+, \Im f_- \in \mathcal{D}(\mathcal{E}) \). Since \( \mathcal{D}(\mathcal{E}) \) is a linear space, the inclusion for non-negative functions leads to the full inclusion \( \mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E}) \). Similarly, we infer \( \mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E}(\sigma)) \).

Now we prove the equality. Let \( f \in L^2(\mathbb{R}^d) \) satisfy \( f \geq 0 \). First note that
\[
\frac{1}{t} \int_0^t P_u f \tilde{P}_{t-u} f \, du \xrightarrow{L^1} f^2,
\] (3.6)
as \( t \to 0 \). Indeed,
\[
P_u f \tilde{P}_{t-u} f - f^2 = (P_u f - f) \tilde{P}_{t-u} f + f (\tilde{P}_{t-u} f - f).
\]
Hence, by the Cauchy–Schwarz inequality,
\[
\left\| \frac{1}{t} \int_0^t P_u f \tilde{P}_{t-u} f \, du - f^2 \right\|_1 \leq \frac{1}{t} \int_0^t \left( \| P_u f - f \| \tilde{P}_{t-u} f + f (\tilde{P}_{t-u} f - f) \right) \, du
\]
\[
\leq \frac{1}{t} \int_0^t \left( \| P_u f - f \|_2 \tilde{P}_{t-u} f \right)_2 + \| f \|_2 \left\| \tilde{P}_{t-u} f - f \right\|_2 \, du.
\]
Recall that \( \{ P_t \}_{t>0} \) and \( \{ \tilde{P}_t \}_{t>0} \) are strongly continuous semigroups of bounded operators on \( L^2(\mathbb{R}^d) \). Therefore,
\[
\left\| \frac{1}{t} \int_0^t P_u f \tilde{P}_{t-u} f \, du - f^2 \right\|_1 \leq \| f \|_2 \left( \frac{1}{t} \int_0^t \| P_u f - f \|_2 e^{\sigma |t-u|} + \| \tilde{P}_{t-u} f - f \|_2 \, du \right)
\]
\[
\leq \| f \|_2 e^{\sigma |t|} \sup_{u \in (0, t]} \left( \| P_u f - f \|_2 + \| \tilde{P}_{t-u} f - f \|_2 \right) \xrightarrow{t \to 0} 0.
\]
Now, let \( t_n \to 0 \) be a sequence such that
\[
\liminf_{t \to 0} \int_{\mathbb{R}^d} V(z) \, \frac{1}{t} \int_0^t P_u f (z) \tilde{P}_{t-u} f (z) \, du \, dz
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}^d} V(z) \, \frac{1}{t_n} \int_0^{t_n} P_u f (z) \tilde{P}_{t_n-u} f (z) \, du \, dz
\]
By (3.6), we can choose a subsequence \( t_{n_k} \to 0 \) such that
\[
\frac{1}{t_{n_k}} \int_{t_{n_k}}^{t} P_u f(z) \tilde{P}_{t_{n_k}-u} f(z) \, du k \to \infty \to f^2(z), \quad \text{a.e. } z \in \mathbb{R}^d.
\]
Hence, by Fatou’s lemma, we have
\[
\limsup_{t \to 0} \tilde{E}_t[f] \leq \mathcal{E}[f] - \liminf_{t \to 0} \int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_{0}^{t} P_u f(z) \tilde{P}_{t-u} f(z) \, du \, dz
\]
\[
= \mathcal{E}[f] - \lim_{k \to \infty} \int_{\mathbb{R}^d} V(z) \frac{1}{t_{n_k}} \int_{0}^{t_{n_k}} P_u f(z) \tilde{P}_{t_{n_k}-u} f(z) \, du \, dz
\]
\[
\leq \mathcal{E}[f] - \int_{\mathbb{R}^d} V(z) \lim_{k \to \infty} \frac{1}{t_{n_k}} \int_{0}^{t_{n_k}} P_u f(z) \tilde{P}_{t_{n_k}-u} f(z) \, du \, dz
\]
\[
= \mathcal{E}[f] - \int_{\mathbb{R}^d} V(z) f^2(z) \, dz.
\]
On the other hand, since \( 0 \leq f \in \mathcal{D}(\mathcal{E}) = \mathcal{D}(\mathcal{E}^{(\alpha)}) \), by (2.16) and (2.21),
\[
\liminf_{t \to 0} \tilde{E}_t[f] \geq \mathcal{E}[f] - \limsup_{t \to 0} \int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_{0}^{t} P_u f(z) \tilde{P}_{t-u} f(z) \, du \, dz
\]
\[
\geq \mathcal{E}[f] - \limsup_{t \to 0} e^{\|\sigma\| t} \int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_{0}^{t} P_u^{(\alpha)} f(z) \tilde{P}_{t-u}^{(\alpha)} f(z) \, du \, dz,
\]
and by using (2.18) for \( p^{(\alpha)}(t, x, y) \) and \( \tilde{p}^{(\alpha)}(t, x, y) \), we get
\[
\int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_{0}^{t} P_u^{(\alpha)} f(z) \tilde{P}_{t-u}^{(\alpha)} f(z) \, du \, dz = E_t^{(\alpha)}[f] - \tilde{E}_t^{(\alpha)}[f].
\]
Consequently, by [5, Lemma 5.1],
\[
\lim_{t \to 0} \int_{\mathbb{R}^d} V(z) \frac{1}{t} \int_{0}^{t} P_u^{(\alpha)} f(z) \tilde{P}_{t-u}^{(\alpha)} f(z) \, du \, dz = \int_{\mathbb{R}^d} V(z) f^2(z) \, dz.
\]
Therefore, (3.4) holds for \( 0 \leq f \in \mathcal{D}(\mathcal{E}) \).

Then (3.4) is proved for any real \( f \in \mathcal{D}(\mathcal{E}) \) by standard polarization identities for the sesquilinear forms \( \mathcal{E} \) and \( \tilde{E} \), cf. e.g. the last lines of the proof of [5, Lemma 5.1]. The extension from real functions to complex functions is done via (3.5).

3.3. Schrödinger operators with subcritical singular potential and their semigroups.

Throughout this section we assume that \( \kappa < \kappa^* \), i.e. we consider only the subcritical potential \( V(x) = \kappa |x|^{-\alpha} \). In this case the Schrödinger operator \( H = -L - V \) can be defined as a form-sum. Indeed, by the fractional Hardy inequality (1.2) and the identity in Lemma 3.1, we easily get the inequalities
\[
\kappa \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^\alpha} \, dx \leq \frac{\kappa}{\kappa^*} \mathcal{E}^{(\alpha)}[f] \leq \frac{\kappa}{\kappa^*} \mathcal{E}[f] + \frac{\kappa}{\kappa^*} \|\sigma\| \|f\|_2^2, \quad f \in \mathcal{D}(\mathcal{E}).
\]
This means that the form of the potential \( V(x) = \kappa |x|^{-\alpha} \) is relatively \( \mathcal{E} \)-bounded with relative \( \mathcal{E} \)-bound \( \kappa/\kappa^* < 1 \). Hence, by the KLMN theorem (see e.g. [38, Theorem

\[
\frac{1}{t_{n_k}} \int_{t_{n_k}}^{t} P_u f(z) \tilde{P}_{t_{n_k}-u} f(z) \, du k \to \infty \to f^2(z), \quad \text{a.e. } z \in \mathbb{R}^d.
\]

there exists a unique, bounded below, self-adjoint operator $H$, called the form sum of the operators $-L$ and $-V$ (we simply write $H = -L - V$), such that the form $(\mathcal{E}_H, \mathcal{D}(\mathcal{E}_H))$ of $H$ satisfies

$$\mathcal{D}(\mathcal{E}_H) = \mathcal{D}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}_H[f, g] = \mathcal{E}[f, g] - \int_{\mathbb{R}^d} V(x)f(x)\overline{g(x)}\,dx, \quad f, g \in \mathcal{D}(\mathcal{E}_H).$$

We will now show that the form of the Schrödinger operator $H$ and the form $\tilde{\mathcal{E}}$ discussed in the previous section are equal. Consequently, the strongly continuous semigroup of operators corresponding to $H$ can be identified with the semigroup $\{\tilde{P}_t : t \geq 0\}$ constructed above. In particular, $e^{-tH}$, $t>0$, are integral operators with kernels $\tilde{p}(t,x,y)$. This is crucial for our further investigations.

**Theorem 3.3.** Under (A1), for $\kappa < \kappa^*$, we have

$$\mathcal{D}(\tilde{\mathcal{E}}) = \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)}) = \mathcal{D}(\mathcal{E}^{(\alpha)}) = \mathcal{D}(\mathcal{E}). \quad (3.7)$$

In particular, $(\mathcal{E}_H, \mathcal{D}(\mathcal{E}_H)) = (\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ and the semigroups $\{e^{-tH} : t \geq 0\}$ and $\{\tilde{P}_t : t \geq 0\}$ are equal on $L^2(\mathbb{R}^d)$.

**Proof.** The equality $\mathcal{D}(\mathcal{E}^{(\alpha)}) = \mathcal{D}(\mathcal{E})$ and the inclusions $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\tilde{\mathcal{E}}) \subset \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$ were established in Lemmas 3.1 and 3.2, respectively. In order to complete the proof of (3.7) we show the inclusion $\mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)}) \subset \mathcal{D}(\mathcal{E}^{(\alpha)})$.

First note that by Lemma 3.2 applied to $\sigma = 0$ (see also the original version for real spaces in [5, Lemma 5.1]) and the fractional Hardy inequality (1.2), we have $\mathcal{D}(\mathcal{E}^{(\alpha)}) \subset \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$ and

$$\mathcal{E}^{(\alpha)}[f] = \tilde{\mathcal{E}}^{(\alpha)}[f] + \frac{\kappa}{\kappa^*} \kappa^* \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^\alpha} \,dx \leq \tilde{\mathcal{E}}^{(\alpha)}[f] + \frac{\kappa}{\kappa^*} \mathcal{E}^{(\alpha)}[f], \quad f \in \mathcal{D}(\mathcal{E}^{(\alpha)}).$$

Hence,

$$\mathcal{E}^{(\alpha)}[f] \leq \frac{\kappa^*}{\kappa^* - \kappa} \tilde{\mathcal{E}}^{(\alpha)}[f], \quad f \in \mathcal{D}(\mathcal{E}^{(\alpha)}). \quad (3.8)$$

By [5, Theorem 5.4], $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$ with the norm $\sqrt{\mathcal{E}^{(\alpha)}[\cdot]} + \| \cdot \|_2$. Furthermore, $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{E}^{(\alpha)})$. This was originally proven for a real $L^2$-space, but it easily extends to our present complex case. We will now use these facts to complete the proof.

Let $f \in \mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)})$ and let $f_n \in C_c^\infty(\mathbb{R}^d)$ be a sequence such that $\tilde{\mathcal{E}}^{(\alpha)}[f - f_n] \to 0$ and $\|f - f_n\|_2 \to 0$ as $n \to \infty$. In particular, $\tilde{\mathcal{E}}^{(\alpha)}[f_n - f_m] \to 0$ and $\|f_n - f_m\|_2 \to 0$ as $n, m \to \infty$. By (3.8) we obtain that $\mathcal{E}^{(\alpha)}[f_n - f_m] \to 0$ as $n, m \to \infty$. Since $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$ is a closed form, we obtain that $f \in \mathcal{D}(\mathcal{E}^{(\alpha)})$, showing the inclusion $\mathcal{D}(\tilde{\mathcal{E}}^{(\alpha)}) \subset \mathcal{D}(\mathcal{E}^{(\alpha)})$.

The equality $(\mathcal{E}_H, \mathcal{D}(\mathcal{E}_H)) = (\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ follows directly from (3.7) and Lemma 3.2. Since, by definition, both these forms are symmetric, bounded below and closed forms, they uniquely determine the strongly continuous semigroups of bounded self-adjoint operators on $L^2(\mathbb{R}^d)$, see e.g. [40, Theorem 6.2 (b)]. This completes the proof. \[\square\]
4. Pointwise Estimates of Eigenfunctions

In this section we find pointwise estimates for eigenfunctions of the Schrödinger operators \( H = -L - V \) with subcritical potential \( V(x) = \kappa |x|^{-\alpha} \), i.e. we consider the case \( \kappa < \kappa^* \).

Throughout this section we assume that \( \varphi \in L^2(\mathbb{R}^d) \) is an eigenfunction of the operator \( H \) corresponding to a negative eigenvalue, i.e.

there exists \( E < 0 \) such that \( H \varphi = E \varphi \). \hspace{1cm} (4.1)

It is now crucial that by Theorem 3.3 we have

\[
\varphi(x) = e^{Et} e^{-tH} \varphi(x) = e^{Et} \tilde{p}_t \varphi(x) = e^{Et} \int_{\mathbb{R}^d} \tilde{p}(t, x, y) \varphi(y) dy, \hspace{1cm} t > 0.
\] \hspace{1cm} (4.2)

We first establish the continuity of eigenfunctions and the upper estimate around zero. We always assume that any eigenfunction \( \varphi \) is normalized so that \( \|\varphi\|_2 = 1 \).

**Proposition 4.1.** Let (A1) hold and let \( \kappa < \kappa^* \). Any \( \varphi \in L^2(\mathbb{R}^d) \), \( \|\varphi\|_2 = 1 \), satisfying (4.1) has a version which is continuous on \( \mathbb{R}^d \setminus \{0\} \) and satisfies the following upper estimate: there exists \( c > 0 \) such that

\[
|\varphi(x)| \leq c e^{\sigma |x| + E} \left( 1 + \frac{1}{|x|^\delta} \right), \hspace{1cm} x \in \mathbb{R}^d \setminus \{0\},
\]

where \( \delta \) is determined by (2.17). The constant \( c \) depends neither on \( \varphi \), \( E \) nor \( \sigma \).

**Proof.** It follows directly from (4.2) and Lemma 2.4 by taking \( t = 1 \).

In what follows, we work with the version of the eigenfunction \( \varphi \) which is continuous on \( \mathbb{R}^d \setminus \{0\} \). In particular, the eigenequations (4.2) can always be understood pointwise.

Estimates at infinity will be given in terms of the resolvent kernel of the free operator \( -L \):

\[
g_\lambda(x) = \int_0^\infty e^{-\lambda t} p(t, x) dt, \hspace{1cm} x \in \mathbb{R}^d, \hspace{1cm} \lambda > 0.
\]

It is not difficult to show that for every \( \lambda > 0 \) the map \( x \mapsto g_\lambda(x) \) is continuous on \( \mathbb{R}^d \setminus \{0\} \). For a Borel function \( f \geq 0 \) we also define

\[
G_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt = \int_{\mathbb{R}^d} g_\lambda(x - y) f(y) dy, \hspace{1cm} \lambda > 0.
\]

The last equality is a consequence of the Tonelli theorem. Clearly, the definition easily extends to any (signed) Borel \( f \) for which the integrals are absolutely convergent.

**Lemma 4.2.** Let (A1) hold and let \( \kappa < \kappa^* \). If \( \varphi \in L^2(\mathbb{R}^d) \) is such that (4.1) holds, then

\[
|\varphi(x)| \leq G_{|E|} (V|\varphi|)(x), \hspace{1cm} x \in \mathbb{R}^d \setminus \{0\}.
\] \hspace{1cm} (4.3)

If, in addition, \( \varphi \geq 0 \), then

\[
\varphi(x) = G_{|E|} (V \varphi)(x), \hspace{1cm} x \in \mathbb{R}^d \setminus \{0\}.
\] \hspace{1cm} (4.4)
Proof. Recall that $E < 0$. We let $\lambda := |E| = -E > 0$ to simplify the notation. By the eigenequation (4.2) and the perturbation formula (2.18), we have for $x \in \mathbb{R}^d \setminus \{0\}$

$$\varphi(x) = e^{-\lambda t} \tilde{P}_t \varphi(x) = e^{-\lambda t} P_t \varphi(x) + \int_0^t \int_{\mathbb{R}^d} e^{-\lambda s} p(s, x, z) V(z) e^{-\lambda(t-s)} \tilde{P}_{t-s} \varphi(z) \, dz \, ds$$

$$= e^{-\lambda t} P_t \varphi(x) + \int_0^t \int_{\mathbb{R}^d} e^{-\lambda s} p(s, x, z) V(z) \varphi(z) \, dz \, ds. \quad (4.5)$$

By the Cauchy–Schwarz inequality,

$$P_t |\varphi| (x) \leq \left( \int_{\mathbb{R}^d} p(t, x, y) \, dy \right)^{1/2} \left( \int_{\mathbb{R}^d} p(t, x, y) |\varphi(y)|^2 \, dy \right)^{1/2}$$

$$= \left( \int_{\mathbb{R}^d} p(t, x, y) |\varphi(y)|^2 \, dy \right)^{1/2},$$

and, by (2.10),

$$p(t, x, y) = p_t (y - x) \leq p_t (0) \leq p_1 (0) < \infty, \quad x, y \in \mathbb{R}^d, \quad t \geq 1.$$  

This shows that $e^{-\lambda t} P_t |\varphi| (x) \to 0$ as $t \to \infty$. Therefore, (4.3) and (4.4) follow from (4.5) by letting $t \to \infty$.

Below we use the following identity. It seems to be a standard fact, but we provide here a short proof for reader’s convenience.

**Lemma 4.3.** For $\lambda > 0$ we have

$$g^{*n}_\lambda (x) = \int_0^\infty t^{n-1} e^{-\lambda t} p_t (x) \, dt \quad x \in \mathbb{R}^d, \quad n \in \mathbb{N}. \quad (4.6)$$

**Proof.** We use induction. Clearly (4.6) holds for $n = 1$. Assume (4.6) for some $n \in \mathbb{N}$. Then, by the Tonelli theorem and the Chapman–Kolmogorov identity (2.14),

$$g^{*(n+1)}_\lambda (x) = \int_{\mathbb{R}^d} g^{*n}_\lambda (x - y) g_\lambda (y) \, dy$$

$$= \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty t^{n-1} e^{-\lambda t} p_t (x - y) e^{-\lambda s} p_s (y) \, ds \, dr \, dy$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty t^{n-1} e^{-\lambda (t+s)} p_{t+s} (x) \, ds \, dt$$

$$= \int_0^\infty \int_0^\infty \int_0^s t^{n-1} e^{-\lambda s} p_s (x) \, ds \, dt$$

$$= \int_0^\infty \int_0^s t^{n-1} e^{-\lambda s} p_s (x) \, ds \, dt = \int_0^\infty \frac{s^n}{n!} e^{-\lambda s} p_s (x) \, ds.$$

We are now in a position to make a concluding step in this section.

**Lemma 4.4.** Let (A1) hold and let $\kappa < \kappa^*$. If $\varphi \in L^2 (\mathbb{R}^d)$ is such that (4.1) holds, then for every $\varepsilon \in (0, |E| \wedge 1)$ there is $R = R(\varepsilon) \geq 1$ such that

$$|\varphi(x)| \leq \int_{|y| \leq R} g_{|E| - \varepsilon} (x - y) V(y) |\varphi(y)| \, dy, \quad x \in \mathbb{R}^d \setminus \{0\}. \quad (4.7)$$
Proof. As before, we denote \( \lambda := |E| = -E > 0 \). Let \( \varepsilon \in (0, \lambda \wedge 1) \). We first prove that there are \( R \geq 1 \) and \( M \in (0, 1) \) such that

\[
|\varphi(x)| \leq \sum_{k=1}^{n} \varepsilon^{k-1} \int_{|y| \leq R} g^{*k}_\lambda(x - y) V(y) \varphi(y) dy + M^n \| \varphi_1 \|_{\infty}, \quad x \in \mathbb{R}^d \setminus \{0\}, \quad n \in \mathbb{N},
\]

(4.8)

where \( \varphi_1(y) := \1_{\{|y| > 1\}} \varphi(y) \). Clearly, \( \| \varphi_1 \|_{\infty} < \infty \) because of (4.2) and the estimate in Lemma 2.4.

We will use induction. By (4.3) for every \( R \geq 1 \), we have

\[
|\varphi(x)| \leq \int_{|y| \leq R} g_\lambda(x - y) V(y) |\varphi(y)| dy + \kappa R^{-\alpha} \int_{|y| > R} g_\lambda(x - y) |\varphi(y)| dy \quad (4.9)
\]

Put \( R = 1 \vee (\frac{\varepsilon}{\lambda})^{1/\alpha} \) and \( M = \frac{\varepsilon}{\lambda} < 1 \). Then, using the estimate \( G_\lambda \varphi_1(x) \leq \| \varphi_1 \|_{\infty}/\lambda \) and the inequality \( \frac{\lambda}{R^n} \leq \varepsilon \),

\[
|\varphi(x)| \leq \int_{|y| \leq R} g_\lambda(x - y) V(y) |\varphi(y)| dy + M \| \varphi_1 \|_{\infty}, \quad x \in \mathbb{R}^d \setminus \{0\}.
\]

So (4.8) holds for \( n = 1 \). Now, suppose that (4.8) holds for some \( n \geq 1 \). Then, we can use (4.8) to estimate \( |\varphi| \) under the second integral in (4.9). By proceeding in that way, we get

\[
|\varphi(x)| \leq \int_{|y| \leq R} g_\lambda(x - y) V(y) |\varphi(y)| dy \\
+ \varepsilon \int_{|y| > R} g_\lambda(x - y) \left( \sum_{k=1}^{n} \varepsilon^{k-1} \int_{|z| \leq R} g_\lambda^{*k}(y - z) V(z) |\varphi(z)| dz + M^n \| \varphi_1 \|_{\infty} \right) dy \\
\leq \int_{|y| \leq R} g_\lambda(x - y) V(y) |\varphi(y)| dy \\
+ \sum_{k=1}^{n} \varepsilon^{k} \int_{|z| \leq R} g_\lambda^{*(k+1)}(x - z) V(z) |\varphi(z)| dz + M^{n+1} \| \varphi_1 \|_{\infty} \\
= \sum_{k=1}^{n+1} \varepsilon^{k-1} \int_{|z| \leq R} g_\lambda^{*k}(x - z) V(z) |\varphi(z)| dz + M^{n+1} \| \varphi_1 \|_{\infty}.
\]

Hence, by induction, (4.8) holds for all \( n \geq 1 \) and \( x \in \mathbb{R}^d \setminus \{0\} \).

Lemma 4.3 and Tonelli’s theorem lead us to a concluding estimate

\[
|\varphi(x)| \leq \sum_{k=1}^{n} \varepsilon^{k-1} \int_{|y| \leq R} \int_{0}^{\infty} \frac{t^{k-1}}{(k - 1)!} e^{-\lambda t} p_t(x - y) V(y) |\varphi(y)| dy dt + M^n \| \varphi_1 \|_{\infty} \\
\leq \int_{|y| \leq R} \int_{0}^{\infty} \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k - 1)!} e^{-\lambda t} p_t(x - y) V(y) |\varphi(y)| dy dt + M^n \| \varphi_1 \|_{\infty} \\
= \int_{|y| \leq R} g_{\lambda - \varepsilon}(x - y) V(y) |\varphi(y)| dy + M^n \| \varphi_1 \|_{\infty}.
\]
By letting \( n \to \infty \), we obtain the claimed bound (4.7).

The following theorem summarizes our investigations in this section. It leads to the estimates for eigenfunctions at infinity which are given in terms of the resolvent kernels of the operator \(-L\).

**Theorem 4.5.** Let (A1) hold and let \( \kappa < \kappa^* \). If \( \varphi \in L^2(\mathbb{R}^d) \) is such that (4.1) holds, then for every \( \varepsilon \in (0, |E| \wedge 1) \) there is \( R = R(\varepsilon) \geq 1 \) and \( c = c(\varepsilon) \) such that

\[
|\varphi(x)| \leq c \sup_{|y| \leq R} g|E|_{-\varepsilon}(x - y), \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{4.10}
\]

Furthermore, if \( \varphi \) is a ground state (i.e. \( E = \inf \text{spec}(H) < 0 \)), then there is \( \tilde{c} > 0 \) such that

\[
\varphi(x) \geq \tilde{c} \inf_{|y| \leq 1} g|E|(x - y), \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{4.11}
\]

**Proof.** The upper bound holds by Lemma 4.4 and the lower estimate follows from the second assertion of Lemma 4.2 and strict positivity of \( \varphi \). \( \square \)

5. **Estimates of Heat Kernels**

Throughout this section we assume that \( \kappa \leq \kappa^* \), i.e. we allow for critical potential \( V(x) = \kappa^* |x|^{-\alpha} \). Recall that \( \nu(dx) = \nu(x)dx \) is a symmetric Lévy measure such that the assumption (A1) holds. The heat kernel of the operator \( L \) is given by \( p_t(y-x) \), where \( p_t \) is a probability density function given by (2.10). Hence, for every \( t > 0 \), \( p_t \) is a symmetric, bounded and continuous function such that

\[
\int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t(y) dy = e^{-t\psi(\xi)}, \quad t > 0, \, \xi \in \mathbb{R}^d,
\]

where

\[
\psi(\xi) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y)) \nu(dy), \quad \xi \in \mathbb{R}^d.
\]

5.1. **Upper estimates of the perturbed heat kernel.** In this section we prove the upper estimates for the kernel \( \tilde{p}(t, x, y) \) which was constructed in Sect. 2.3. This will be done for Lévy measures with densities that satisfy our both assumptions (A1) and (A2).

Let

\[
\psi^*(u) := \sup_{|\xi| \leq u} \psi(\xi), \quad u \geq 0,
\]

be a maximal function of \( \psi \). Recall that \( \psi(\xi) \asymp |\xi|^\alpha \), for \( |\xi| \geq (4|\sigma|)^{1/\alpha} \), see (2.8). Since

\[
\psi(\xi) \asymp \psi^*(|\xi|), \quad \xi \in \mathbb{R}^d, \tag{5.1}
\]

(see [27, Lemma 5(a)]), it extends to

\[
\psi(\xi) \asymp |\xi|^\alpha, \quad |\xi| \geq r, \tag{5.2}
\]
for every \( r > 0 \), with comparability constant depending on \( r \). An important consequence of (5.2) is that for every \( r > 0 \) there is a constant \( c > 0 \) such that

\[
v(x) \geq c \psi^\#(1/|x|) / |x|^d, \quad |x| \leq r.
\] (5.3)

This was originally established in [6, Theorem 26] for radial decreasing densities \( v(x) \), but due to (1.7) it easily extends to our setting, see [27, Lemma 5(b)] or [19, Lemat 2.3], [20, Lemat A.3].

We first collect the properties of \( v(x) \) and \( p(t, x, y) \) that are needed below.

**Lemma 5.1.** Under assumptions (A1) and (A2), we have the following statements.

(a) For every \( r > 0 \) there exists a constant \( c > 0 \) such that

\[
c v^{(\alpha)}(x) \leq v(x) \leq v^{(\alpha)}(x), \quad |x| \leq r.
\]

(b) For every \( r > 0 \) there exists a constant \( c > 0 \) such that

\[
v(y) \leq c v(x), \quad |y| \geq 1, \quad |y-x| \leq r.
\] (5.4)

(c) There exists a constant \( c > 0 \) such that

\[
v(y) v(x) \leq c v(y-x), \quad |x|, |y| \geq 1.
\]

(d) For every \( T, K > 0 \) there exist the constants \( c, \tilde{c} \) (depending on \( T \) and \( K \)) such that

\[
p(t, x, y) \asymp t^{-d/\alpha}, \quad t \in (0, T], \quad |y-x| \leq K t^{1/\alpha},
\]

and

\[
p(t, x, y) \asymp t v(y-x), \quad t \in (0, T], \quad |y-x| \geq K t^{1/\alpha}.
\]

In particular,

\[
p(t, x, y) \asymp t^{-d/\alpha} \wedge t v(y-x), \quad t \in (0, T], \quad x, y \in \mathbb{R}^d.
\]

(e) For every \( T, K > 0 \) there exists a constant \( c > 0 \) (depending on \( T \) and \( K \)) such that

\[
p(t, z, y) \geq cp(t, x, y), \quad x, y, z \in \mathbb{R}^d, \quad |x-z| \leq K t^{1/\alpha}, \quad t \in (0, T].
\]

**Proof.** (a) The upper bound is a direct consequence of (A1). The lower bound follows directly from (5.3), (5.1) and (5.2).

(b) Due to (1.7) the assertion is trivial for \( |x| < 1 \). We are left to consider \( |x| \geq 1 \). It follows from a combination of [27, Lemmas 1 and 3], but we provide here a short and direct proof for reader’s convenience. First recall that by Assumption (A2) there exists a decreasing profile \( f \) of the density \( v \) such that

\[
\int_{\mathbb{R}^d} f_1(|x-y|) f_1(|y|) dy \leq c_1 f_1(|x|), \quad x \in \mathbb{R}^d,
\] (5.5)

for a constant \( c_1 \). Here \( f_1 = f \wedge 1 \). Moreover, observe that

\[
c_2 f(|x|) \leq f_1(|x|) \leq f(|x|), \quad |x| \geq 1,
\]
where $c_2 = 1/(1 + f(1))$. Therefore, in order to complete the proof of (5.4), we only need to show that for any $r > 0$ there exists a constant $c_3 = c_3(r) > 0$ such that

$$f_1(u) \leq c_3 f_1(u + r), \quad u \geq 1.$$  

Fix $r > 0$. Let $x = (u + r, 0, \ldots, 0)$, $u \geq 1$. By (5.5) we have

$$c_1 f_1(u + r) = c_1 f_1(|x|) \geq \int_{|y-x|<r} f_1(|x-y|) f_1(|y|) dy \geq f_1(|x|-r) \int_{|y|<r} f_1(|y|) dy.$$

Since $|x| - r = u$ and $\int_{|y|<r} f_1(|y|) dy > 0$, this completes the proof of part (b). (c) This easily follows from the last assertion of [26, Lemma 3.1].

(d) By the first assertion of [26, Lemma 3.1], (5.5) is equivalent to the condition that for every $r > 0$ there is a constant $c_4 = c_4(r) > 0$ such that

$$\int_{|y-x|>r} \nu(x-y) \nu(y) dy \leq c_4 \nu(x), \quad |x| \geq r$$

(this is stated for $r = 1$, but the proof for arbitrary $r$ is the same – this is a consequence of radiality and monotonicity of the profile $f$). This inequality and (5.3) shows that the assumptions (1.1) of [27, Theorem 1] are satisfied for every $r_0 > 0$ (according to the notation in the quoted paper). This theorem gives that for every $T > 0$ there exist constants $c_5, c_6$ (depending on $T$) and $\theta > 0$ such that

$$p(t, x, y) \leq t^{-d/\alpha}, \quad t \in (0, T], \quad |y-x| \leq \theta t^{1/\alpha},$$

and

$$p(t, x, y) \geq t \nu(y-x), \quad t \in (0, T], \quad |y-x| \geq \theta t^{1/\alpha}.$$  

By part (a), for any fixed $r > 0$ (in particular, $r = KT^{1/\alpha}$ for any $K > 0$) we have $\nu(x) \asymp |x|^{-d-\alpha}$, $|x| \leq r$. Hence, we see that

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge t \nu(y-x), \quad t \in (0, T], \quad x, y \in \mathbb{R}^d,$$

and the similar comparabilities hold with $\theta = K$, for any $K > 0$.

We remark that the original statement of [27, Theorem 1] says that we have the estimates for some $T > 0$. However, in our case the assumptions are satisfied for every $r_0 > 0$ and therefore the estimates hold for every $T > 0$. This can be directly checked by inspecting the proof of that theorem.

(e) This bound can be obtained e.g. by iterating the estimate in [18, Corollary 4.2] (with $t_0 = T$). \hfill \Box

For $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$ we denote

$$H(t, x) = 1 + (t^{-1/\alpha} |x|)^{-\delta}.$$  

(5.6)

It follows from [5, Proposition 3.2] and (2.21) that there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} \tilde{p}(t, x, y) dy \leq c e^{\alpha|y|} H(t, x), \quad t > 0, \quad x \in \mathbb{R}^d \setminus \{0\}.$$  

(5.7)
Lemma 5.2. For every $T > 0$ and $R > 0$ there is a constant $c > 0$ such that
\[
\tilde{p}(t, x, y) \leq c \frac{H(t, x)H(t, y)p(t, x, y)}{|x|, |y| \leq R, \ t \in (0, T]}.
\] (5.8)

Proof. Fix $T, R > 0$. By (2.21) and (2.19), for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$, we have
\[
\tilde{p}(t, x, y) \leq c_1 \tilde{p}^{(\alpha)}(t, x, y) \leq c_2 H(t, x)H(t, y) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).
\]
Since $|x - y| \leq |x| + |y| \leq 2R$, Lemma 5.1 (a) implies that there is $c_3 = c_3(R)$ such that
\[
t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \leq c_3 \left( t^{-d/\alpha} \wedge tv(x - y) \right).
\]
Together with Lemma 5.1 (d), this gives the claimed bound. \(\square\)

Observe now that for any $t > 0, x, y \in \mathbb{R}^d$ and $R > 0$, by (2.18), we have
\[
\tilde{p}(t, x, y) \leq p(t, x, y) + \int_0^t \int_{|z| < R} \tilde{p}(t - s, x, z)V(z)p(s, z, y)dzds
\]
\[
+ \int_0^t \int_{|z| > R} \tilde{p}(t - s, x, z)\kappa R^{-\alpha} p(s, z, y)dzds
\]
\[
\leq p(t, x, y) + \int_0^t \int_{|z| < R} \tilde{p}(t - s, x, z)V(z)p(s, z, y)dzds + t\kappa R^{-\alpha} \tilde{p}(t, x, y).
\]
Hence, if $t\kappa R^{-\alpha} < 1$, then
\[
\tilde{p}(t, x, y) \leq \frac{1}{1 - t\kappa R^{-\alpha}} \left( p(t, x, y) + \int_0^t \int_{|z| < R} \tilde{p}(t - s, x, z)V(z)p(s, z, y)dzds \right).
\]
For any fixed $T > 0$ we set
\[
R_0 = R_0(T) := 1 \vee (2T\kappa)^{1/\alpha}.
\] (5.9)
Then,
\[
\frac{1}{1 - t\kappa R_0^{-\alpha}} < 2, \quad t \in (0, T],
\]
and, consequently,
\[
\tilde{p}(t, x, y) \leq 2 \left( p(t, x, y) + \int_0^t \int_{|z| < R_0} \tilde{p}(t - s, x, z)V(z)p(s, z, y)dzds \right). \quad (5.10)
\]

Lemma 5.3. Let $T > 0$ and let $R_0 = R_0(T)$ be a number given by (5.9). Then there is a constant $c = c(T)$ such that for any $|x| \leq R_0 + 1, |y| \geq R_0 + 2$ and $t \in (0, T]$ we have
\[
\tilde{p}(t, x, y) \leq cH(t, x)tv(y - x). \quad (5.11)
\]
Proof. First observe that if $|z| < R_0$, then $|y - z| > 1$ and $|(y - x) - (y - z)| = |x - z| < 2R_0 + 1$. Hence, by using Lemma 5.1 (b), we get $v(y - z) \leq c_1v(y - x)$, for a positive constant $c_1$. Together with the estimate in Lemma 5.1 (d), this implies that

$$p(s, z, y) \leq c_2s v(y - x), \quad |z| < R_0, \quad s \in (0, T],$$

(5.12)

for some $c_2 = c_2(T)$. By (5.10) and (5.12), we get

$$\tilde{p}(t, x, y) \leq 2p(t, x, y) + 2c_2 t v(y - x) \int_0^t \int_{|z| < R_0} \tilde{p}(t - s, x, z)V(z)dzds$$

$$\leq 2p(t, x, y) + 2c_2 t v(y - x) \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t - s, x, z)V(z)dzds.$$

On the other hand, by integrating on both sides of the equality (2.18), we obtain

$$\int_{\mathbb{R}^d} \tilde{p}(t, x, y)dy = 1 + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t - s, x, z)V(z)dzds,$$

and further, by using (5.7), we get

$$\int_0^t \int_{\mathbb{R}^d} \tilde{p}(t - s, x, z)V(z)dzds \leq c_3 H(t, x),$$

for a constant $c_3 = c_3(T)$. This estimate, Lemma 5.1 (d) and the inequality $H(t, x) \geq 1$ lead us to a conclusion

$$\tilde{p}(t, x, y) \leq 2p(t, x, y) + 2c_2 c_3 t v(y - x) H(t, x) \leq c_4 H(t, x) t v(y - x).$$

We are now in a position to make a concluding step in this section.

**Lemma 5.4.** Let $T > 0$ and let $R_0 = R_0(T)$ be a number given by (5.9). Then there is a constant $c = c(T)$ such that for any $x, y \in \mathbb{R}^d$ satisfying the condition $|x| \lor |y| \geq R_0 + 2$ and $t \in (0, T]$ we have

$$\tilde{p}(t, x, y) \leq c H(t, x) H(t, y) t v(y - x).$$

(5.13)

**Proof.** By symmetry of the kernel $\tilde{p}(t, x, y)$ and Lemma 5.3 it suffices to consider only the case $R_0 + 1 < |x| \leq |y|$. By (5.10) and Lemmas 5.1 (d) and 5.3, we have

$$\tilde{p}(t, x, y) \leq 2p(t, x, y) + 2 \int_0^t \int_{|z| < R_0} p(t - s, x, z)V(z)\tilde{p}(s, z, y)dzds$$

$$\leq c_1 t v(y - x) + c_1 \int_0^t \int_{|z| < R_0} (t - s) v(z - x) V(z) s v(y - z) H(s, z)dzds$$

$$\leq c_1 t v(y - x) + c_1 t^2 \int_0^t \int_{|z| < R_0} v(y - z) v(z - x) V(z) H(s, z)dzds.$$

Since for $|z| < R_0$ we have $|y - z|, |z - x| > 1$, Lemma 5.1 (c) implies that

$$v(y - z) v(z - x) \leq c_2 v(y - x).$$
Hence,
\[ \tilde{p}(t, x, y) \leq \left( c_1 + c_1 T \int_0^t \int_{|z| < R_0} V(z) H(s, z) d z d s \right) t v(y - x). \]

By definition of \( H \),
\[ \int_0^t \int_{|z| < R_0} V(z) H(s, z) d z d s \leq \kappa \left( T \int_{|z| < R_0} |z|^{-\alpha} d z + T^{1+\delta/\alpha} \int_{|z| < R_0} |z|^{-\alpha-\delta} d z \right). \]

Recall that \( \alpha < d \), which implies that the first integral on the right hand side is finite. Since \( \alpha \leq \alpha + \delta \leq \alpha + (d - \alpha)/2 = (d + \alpha)/2 < d \), the second integral converges as well. Hence,
\[ \tilde{p}(t, x, y) \leq c_3 t v(y - x) \leq c_3 H(t, x) H(t, y) t v(y - x) \]
with \( c_3 = c_3(T) \), which completes the proof.

**Proof of Theorem 1.2.** Fix \( T > 0 \) and \( R_0 = R_0(T) \) as in (5.9). If \( |x|, |y| \leq R_0 + 2 \), then the claimed estimate follows from Lemma 5.2 with \( R = R_0 + 2 \). If \( |x| \vee |y| \geq R_0 + 2 \), then by (2.21) and (2.19) we have
\[ \tilde{p}(t, x, y) \leq e^{t|\sigma|T} \tilde{p}^{(\alpha)}(t, x, y) \leq c_1 H(t, x) H(t, y) t^{-d/\alpha} \]
and, by Lemma 5.4,
\[ \tilde{p}(t, x, y) \leq c_2 H(t, x) H(t, y) t v(y - x), \]
for every \( t \in (0, T) \). The result follows then from Lemma 5.1 (d).

### 5.2. Lower estimate of the perturbed heat kernel and the ground state.

We will now prove our last main theorem.

**Proof of Theorem 1.3.** (a) \( \Rightarrow \) (b) Fix \( R > 0 \) and take \( T = (2R)^\alpha \). By (4.2) and the estimate in part (a) applied to \( |x| \leq T^{1/\alpha} \), Lemma 5.1 (d), and strict positivity of \( \varphi \),
\[ \varphi(x) = e^{ET} \int_{\mathbb{R}^d} \tilde{p}(T, x, y) \varphi(y) d y \geq c \frac{e^{ET} T^{\delta/\alpha}}{|x|^{\delta}} \int_{|y| < T^{1/\alpha} / 2} p(T, x, y) \varphi(y) d y \geq c_1 \frac{e^{ET} T^{(\delta - d)/\alpha}}{|x|^{\delta}} \int_{|y| < R} \varphi(y) d y = c_1 \frac{e^{E(2R)^\alpha}}{(2R)^{d-\delta}} \left( \int_{|y| < R} \varphi(y) d y \right) \frac{1}{|x|^{\delta}} = c_2 \frac{1}{|x|^{\delta}}, \quad |x| \leq R. \]

This gives the estimate in (b).

(b) \( \Rightarrow \) (a) We adapt the argument from [5, Section 4.2]. First we fix the notation:
\[ \phi_t(x) = 1 + t^{\delta/\alpha} \varphi(x), \quad \mu_t(dz) = \varphi_t^2(z) d z, \quad q(t, x, y) = \frac{\tilde{p}(t, x, y)}{\varphi_t(x) \varphi_t(y)}. \]
Fix $T > 0$. By (4.2), we have for $t \in (0, T)$, $x \in \mathbb{R}^d \setminus \{0\}$,
\[
\int_{\mathbb{R}^d} q(t, x, z) \mu_t(dz) = \int_{\mathbb{R}^d} p(t, x, z)dz + t^{\delta/\alpha} e^{-Et} \phi_t(x) \geq \frac{1 + t^{\delta/\alpha} \phi_t(x)}{\phi_t(x)} = 1.
\]
(5.14)

Recall that $H(t, x) = 1 + t^{\delta/\alpha} |x|^{-\delta}$. Observe that for $|x| \geq T^{1/\alpha}$ we have
\[
\frac{H(t, x)}{\phi_t(x)} \leq 1 + T^{\delta/\alpha} |x|^{-\delta} \leq 2,
\]
while for $|x| \leq T^{1/\alpha}$,
\[
\frac{H(t, x)}{\phi_t(x)} = \frac{1 + t^{\delta/\alpha} |x|^{-\delta}}{1 + t^{\delta/\alpha} \phi_t(x)} \leq c_3,
\]
by the estimate of the ground state in (b). Together with Theorem 1.2 and (2.16) this implies that
\[
q(t, x, z) \leq c_4 p(t, x, z) \leq c_4 e^{\alpha |T|} p^{(\alpha)}(t, x, z), \quad t \in (0, T], \quad x, z \in \mathbb{R}^d \setminus \{0\}.
\]

Hence, by using the scaling property (2.11), (5.14) and by following the argument in [5, Section 4.2, p. 38], we obtain that there are $r \in (0, 1)$ and $R > 1 + 4^{1/\alpha}$ such that
\[
\int_{r \leq \frac{1}{1+r^{1/\alpha}} \leq R} q(t, x, z) \mu_t(dz) \geq \frac{1}{2}, \quad |x| \leq (4t)^{1/\alpha}, \quad x \neq 0, \quad t \in (0, T].
\]
(5.17)

We are now ready to give the proof of (a). By the symmetry of the kernel $\widehat{p}(t, x, y)$, it is enough to establish the estimate for $|x| \leq t^{1/\alpha}$, $t \in (0, T]$, $y \in \mathbb{R}^d$.

Assume first that $|x| \leq (4t)^{1/\alpha}$ and $|y| \geq r(2t)^{1/\alpha}$. By the Chapman–Kolmogorov equation (2.1),
\[
q(2t, x, y) \geq c_5 \int_{r \leq \frac{1}{1+r^{1/\alpha}} \leq R} q(t, x, z)q(t, z, y) \mu_t(dz) \geq c_5 \int_{r \leq \frac{1}{1+r^{1/\alpha}} \leq R} q(t, x, z) \frac{p(t, z, y)}{\phi_t(z)\phi_t(y)} \mu_t(dz).
\]

By Lemma 5.1 (e) (here we use that $|x - z| \leq (4^{1/\alpha} + R)t^{1/\alpha}$) and Proposition 4.1,
\[
\frac{p(t, z, y)}{\phi_t(z)\phi_t(y)} \geq c_6 \frac{p(t, x, y)}{(1 + c_7r^{\delta/\alpha}(1 + |z|^{-\delta}))(1 + c_7r^{\delta/\alpha}(1 + |y|^{-\delta}))} \geq c_6 \frac{p(t, x, y)}{1 + c_7 T^{\delta/\alpha} + r^{-\delta}}^2.
\]

Together with (5.17), this implies that
\[
q(2t, x, y) \geq c_8 \frac{p(t, x, y)}{2(1 + c_7 T^{\delta/\alpha} + r^{-\delta})^2} p(t, x, y),
\]
and, by using the two-sided sharp estimates in Lemma 5.1 (d) (giving $p(t, x, y) \asymp p(2t, x, y)$), we see that
\[
q(2t, x, y) \geq c_9 p(2t, x, y), \quad |x| \leq (4t)^{1/\alpha}, \quad |y| \geq r(2t)^{1/\alpha}, \quad t \in (0, T].
\]
(5.18)
In particular,

\[ q(t, x, y) \geq c_9 p(t, x, y), \quad |x| \leq (2t)^{1/\alpha}, \quad |y| \geq r t^{1/\alpha}, \quad t \in (0, T]. \]  

(5.19)

Consider now the case \( |x| \leq (2t)^{1/\alpha} \) and \( |y| \leq r (2t)^{1/\alpha} \) (in particular, \(|y| < (2t)^{1/\alpha}\)). By the symmetry of the kernel \( q(t, x, y) \) and (5.19), we get

\[
q(2t, x, y) \geq c_5 \int_{r \leq \frac{|y|}{t^{1/\alpha}} \leq R} q(t, x, z)q(t, z, y)\mu_t(\text{dz}) \\
\geq c_{10} \int_{r \leq \frac{|y|}{t^{1/\alpha}} \leq R} q(t, x, z)p(t, z, y)\mu_t(\text{dz}).
\]

One more use of Lemma 5.1 (e), (d) and (5.17) as above leads us to the estimate

\[
q(2t, x, y) \geq c_{11} p(2t, x, y), \quad |x| \leq (2t)^{1/\alpha}, \quad |y| \leq r (2t)^{1/\alpha}, \quad t \in (0, T].
\]

Finally, by combining this with (5.18) and by using (5.15)–(5.16), we obtain the estimate in (a) and complete the proof of the theorem.

6. Applications to Relativistic Coulomb Model

Throughout this section we assume that \( d = 3, \alpha = 1 \) and \( m > 0 \). We are now in a position to apply our results to relativistic Coulomb model. Let

\[
H_m = \sqrt{-\Delta + m^2} - \frac{\kappa}{|x|}, \quad \text{where} \quad 0 < \kappa < \kappa^* := \frac{2}{\pi},
\]

(6.1)

and let \( \delta \) be the unique number such that

\[
0 < \delta < 1 \quad \text{and} \quad \kappa = \frac{2 \Gamma \left( \frac{1+\delta}{2} \right) \Gamma \left( \frac{3-\delta}{2} \right)}{\Gamma \left( \frac{\delta}{2} \right) \Gamma \left( \frac{1}{2} \right)} = (1-\delta) \tan \frac{\pi \delta}{2}.
\]

Herbst [21] and Weder [41] studied the structure of the spectrum of the operator \( H_m \). It is known that \( \text{spec}_c(H_m) = [m, \infty), \)

\[
\text{spec}_d(H_m) \subset \left[ m \sqrt{1 - \left( \frac{\kappa \pi}{2} \right)^2}, m \right),
\]

and \( \text{spec}_d(H_m) \) is infinite. The ground state eigenvalue \( E_{m,0} := \inf \text{spec}(H_m) = \inf \text{spec}_d(H_m) \) is simple and the corresponding ground state eigenfunction \( \varphi_{m,0} \) is strictly positive, see the discussion in Daubechies and Lieb [11], and Daubechies [10].

**Corollary 6.1.** Let \( H_m \) be given by (6.1). Any \( L^2 \)-eigenfunction of \( H_m \) is continuous on \( \mathbb{R}^3 \setminus \{0\} \) and the following pointwise estimates hold.

(a) (General upper estimate) For any eigenfunction \( \varphi_m \) of \( H_m \) corresponding to eigenvalue \( E_m < m \) there exists a constant \( c > 0 \) such that

\[
|\varphi_m(x)| \leq \frac{c}{|x|^\delta}, \quad 0 < |x| \leq 1,
\]

and for every \( \varepsilon > 0 \) there exists a constant \( \tilde{c} \equiv \tilde{c}(\varepsilon) > 0 \) such that

\[
|\varphi_m(x)| \leq \tilde{c} e^{-\left( \sqrt{m^2 - E_m^2} - \varepsilon \right)|x|}, \quad |x| \geq 1.
\]
Remark 6.2. In principle, the constants $c$ and $\tilde{c}$ in the above corollary depend on parameters $m$ and $\kappa$, as well as on $E_m$ and $\varphi_m$. The statements and proofs of our main theorems allow one to track this dependence up to some reasonable level.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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References

1. Agmon, S.: Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators, volume 29 of Mathematical Notes. Princeton University Press, Princeton; University of Tokyo Press, Tokyo (1982)
2. Agmon, S.: Bounds on exponential decay of eigenfunctions of Schrödinger operators. In: Schrödinger Operators (Como, 1984), volume 1159 of Lecture Notes in Math., pp. 1–38. Springer, Berlin (1985)
3. Ascione, G., Lörinczi, J.: Potentials for non-local Schrödinger operators with zero eigenvalues. J. Differ. Equ. 317, 264–364 (2022)
4. Bogdan, K., Dyda, B., Kim, P.: Hardy inequalities and non-explosion results for semigroups. Potent. Anal. 44(2), 229–247 (2016)
5. Bogdan, K., Grzywny, T., Jakubowski, T., Pilarczyk, D.: Fractional Laplacian with Hardy potential. Commun. Part. Differ. Equ. 44(1), 20–50 (2019)
6. Bogdan, K., Grzywny, T., Rydzar, M.: Density and tails of unimodal convolution semigroups. J. Funct. Anal. 266(6), 3543–3571 (2014)
7. Bogdan, K., Hansen, W., Jakubowski, T.: Time-dependent Schrödinger perturbations of transition densities. Studia Math. 189(3), 235–254 (2008)
8. Böttcher, B., Schilling, R., Wang, J., Lévy matters. III, volume 2099 of Lecture Notes in Mathematics. Springer, Cham.: Lévy-type processes: construction, approximation and sample path properties. With a short biography of Paul Lévy by Jean Jacod, Lévy Matters (2013)
9. Carmona, R., Masters, W.C., Simon, B.: Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions. J. Funct. Anal. 91(1), 117–142 (1990)
10. Daubechies, I.: One-electron molecules with relativistic kinetic energy: properties of the discrete spectrum. Commun. Math. Phys. 94(4), 523–535 (1984)
11. Daubechies, I., Lieb, E.H.: One-electron relativistic molecules with Coulomb interaction. Commun. Math. Phys. 90(4), 497–510 (1983)
12. Demuth, M., van Casteren, J.A.: Stochastic spectral theory for self adjoint Feller operators. Probability and Its Applications. Birkhäuser Verlag, Basel (2000). A functional integration approach
13. Fefferman, C., de la Llave, R.: Relativistic stability of matter. I. Rev. Mat. Iberoamericana 2(1–2), 119–213 (1986)
14. Frank, R.L., Lieb, E.H., Seiringer, R.: Stability of relativistic matter with magnetic fields for nuclear charges up to the critical value. Commun. Math. Phys. 275(2), 479–489 (2007)
15. Frank, R.L., Merz, K., Siedentop, H., Simon, B.: Proof of the strong Scott conjecture for Chandrasekhar atoms. Pure Appl. Funct. Anal. 5(6), 1319–1356 (2020)
16. Frank, R.L., Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal. 255(12), 3407–3430 (2008)
17. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet forms and symmetric Markov processes. De Gruyter Studies in Mathematics, vol. 19, extended Walter de Gruyter & Co., Berlin (2011)
18. Grzywny, T., Kaleta, K., Sztonyk, P.: Heat kernels of non-local Schrödinger operators with Kato potentials. J. Differ. Equ. 340, 273–308 (2022)
19. Grzywny, T., Szczypkowski, K.: Lévy processes: concentration function and heat kernel bounds. Bernoulli 26(4), 3191–3223 (2020)
20. Grzywny, T., Szczypkowski, K.: Estimates of heat kernels of non-symmetric Lévy processes. Forum Math. 33(5), 1207–1236 (2021)
21. Herbst, I.W.: Spectral theory of the operator \((p^2+m^2)^{1/2} - Ze^2/r\). Commun. Math. Phys. 53(3), 285–294 (1977)
22. Jacob, N.: Pseudo Differential Operators and Markov Processes, Vol. I, II, III. Imperial College Press, London (2001–2005)
23. Jakubowski, T., Kaleta, K., Szczypkowski, K.: Relativistic stable operators with critical potential (preprint 2022)
24. Kaleta, K., Lőrinczi, J.: Fall-off of eigenfunctions for non-local Schrödinger operators with decaying potentials. Potent. Anal. 46(4), 647–688 (2017)
25. Kaleta, K., Lőrinczi, J.: Zero-energy bound state decay for non-local Schrödinger operators. Commun. Math. Phys. 374(3), 2151–2191 (2020)
26. Kaleta, K., Schilling, R.L.: Progressive intrinsic ultracontractivity and heat kernel estimates for non-local Schrödinger operators. J. Funct. Anal. 279(6), 108606 (2020)
27. Kaleta, K., Sztonyk, P.: Small-time sharp bounds for kernels of convolution semigroups. J. Anal. Math. 132, 355–394 (2017)
28. Knopova, V., Schilling, R.L.: A note on the existence of transition probability densities of Lévy processes. Forum Math. 25(1), 125–149 (2013)
29. Kwaśnicki, M.: Ten equivalent definitions of the fractional Laplace operator. Fract. Calc. Appl. Anal. 20(1), 7–51 (2017)
30. Lieb, E.H., Seiringer, R.: The Stability of Matter in Quantum Mechanics. Cambridge University Press, Cambridge (2010)
31. Lieb, E.H., Yau, H.-T.: The stability and instability of relativistic matter. Commun. Math. Phys. 118(2), 177–213 (1988)
32. Nardini, F.: Exponential decay for the eigenfunctions of the two-body relativistic Hamiltonian. J. Analyse Math. 47, 87–109 (1986)
33. Nardini, F.: On the asymptotic behaviour of the eigenfunctions of the relativistic \(N\)-body Schrödinger operator. Boll. Un. Mat. Ital. A (7) 2(3), 365–369 (1988)
34. Olver, F.W.J., Olde Daalhuis, A.B., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V., Cohl, H.S., McClain, M.A. (Eds.) NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.1.2 of 2021-06-15
35. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. IV. Analysis of Operators. Academic Press (Harcourt Brace Jovanovich, Publishers), New York-London (1978)
36. Ryznar, M.: Estimates of Green function for relativistic \(\alpha\)-stable process. Potential Anal. 17(1), 1–23 (2002)
37. Schilling, R.L., Song, R., Vondraček, Z.: Bernstein Functions, volume 37 of De Gruyter Studies in Mathematics, 2nd edn. Walter de Gruyter & Co., Berlin (2012). Theory and applications
38. Schmüdgen, K.: Unbounded Self-adjoint Operators on Hilbert Space. Graduate Texts in Mathematics, vol. 265. Springer, Dordrecht (2012)
39. Simon, B.: Schrödinger semigroups. Bull. Am. Math. Soc. (N. S.) 7(3), 447–526 (1982)
40. van Casteren, J.A.: Generators of Strongly Continuous Semigroups. Pitman, Boston (1985)
41. Weder, R.A.: Spectral analysis of pseudodifferential operators. J. Funct. Anal. 20(4), 319–337 (1975)

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