THE CRITICAL COHA OF A SELF DUAL QUIVER WITH POTENTIAL

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Abstract. In this paper we provide an explanation for the many beautiful infinite product formulas for generating functions of refined DT invariants for symmetric quivers with potential: they are characteristic functions of free supercommutative algebras. We prove this by first showing that, for a quiver with potential that satisfies a notion of self duality, introduced below, the critical cohomological Hall algebra is supercommutative, and admits a kind of localised coproduct, which is enough to guarantee freeness. The primitive graded pieces of the space of generators of the cohomological Hall algebra are a putative definition of the space of BPS states in String Theory, and we discuss the issue of their finite-dimensionality. We consider the conjecture that in the presence of the above-mentioned self duality these spaces are always finite-dimensional, and finish with some examples, in which, amongst other things, this conjecture can be seen to hold. Some of these examples use a cohomological dimensional reduction of the sort used by Behrend, Bryan and Szendrői to calculate the motivic DT invariants of $\mathbb{C}^3$. We prove the required isomorphism holds in the appendix. The final example pulls together all of the technology in this paper to describe the link between critical CoHAs and character varieties: we build a free supercommutative critical CoHA out of the cohomology of untwisted character varieties, and present some evidence that the spaces of primitive generators in this case are given by the cohomology of their twisted counterparts, as studied in [14]. In a separate paper we discuss our other main example, which yields a new proof of the Kac conjecture.

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1. Introduction

The purpose of this paper is to study the cohomological Hall algebra (henceforth CoHA) of a quiver with potential \((Q,W)\), as introduced in [19]. In fact there are two different versions of this algebra – we study the one arising from taking cohomology of vanishing cycles, or ‘critical cohomology’, which we will denote \(\mathcal{H}_{Q,W}\) and call the critical CoHA. Its underlying \(\mathbb{Z}^{Q_0}\)-graded vector space is defined by

\[
\mathcal{H}_{Q,W} = \bigoplus_{\gamma \in \mathbb{Z}^{Q_0}} H_c,\text{GL}_C(\gamma)(M_\gamma, \varphi_{1r}(W))[-\chi(\gamma, \gamma)]^*
\]

which is the dual of the compactly supported cohomology of the moduli stack of representations of the Jacobi algebra for \((Q,W)\) with coefficients in the vanishing cycle complex. The cohomological degree of an element in this space takes values in \(\mathbb{Z}_{\text{sc}} := \mathbb{Z}\) - the \text{sc} stands for shifted cohomological degree. So the unshifted, \(\mathbb{Z}_{\text{co}}\)-degree of an element of \(\mathbb{Z}_{\text{sc}}\)-degree \(k\), is \(k + \chi(\gamma, \gamma)\).

Many of the quivers with potential that arise ‘in nature’ admit a kind of involution \(\sigma\), which in particular acts on dimension vectors. Let \(\mathcal{H}_{Q,W}^\sigma\) be the restricted CoHA of representations
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with dimension vector invariant under \(\sigma\). Let

\[ C := \left(\mathbb{Z}/2\mathbb{Z}\right)^{Q_0} \oplus \left(\mathbb{Z}/2\mathbb{Z}\right) \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^{n+1}, \]

where \(n = |Q_0|\). Then \(H^\sigma_{Q,W}\) is given a \(C\)-grading by setting the \(C\)-degree of a \(\mathbb{Z}\)co-degree \(k\) element of \(H_c, GL_C(\gamma)\) \((M_{\gamma}, \varphi_{tt(W)})[-\chi(\gamma, \gamma)]^*\) to be \((\kappa(\gamma), k)\), for the linear map \(\kappa : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}\) defined by

\[ \kappa(\gamma) = (b_{i1}\gamma(1), \ldots, b_{in}\gamma(n)), \]

with \(b_{ii}\) the number of loops at vertex \(i\), minus one. Then we show that \(H^\sigma_{Q,W}\) is a \(\mathbb{Z}Q_0\) graded \(C\)-supercommutative algebra, i.e. the multiplication respects the \(C\oplus \mathbb{Z}Q_0\)-grading and

\[ \zeta \cdot \zeta' = (-1)^{c'c}\zeta' \cdot \zeta \]

for \(\zeta\) and \(\zeta'\) of \(C\)-degree \(c\) and \(c'\), and \(c \cdot c'\) the usual dot product of vectors. Furthermore the main structural result of this paper, which applies to all quivers with potential and a self-duality \(\sigma\), is the following.

**Theorem 1.1.** The critical CoHA \(H^\sigma_{Q,W}\) is a free \(C\)-supercommutative algebra.

In fact in the presence of some splitting results, we can often do better than this and prove that \(H^\sigma_{Q,W}\) is a free algebra object in some richer category, for example the category of \(C\)-graded monodromic mixed Hodge structures, to be discussed later – see Theorem 6.9 for an exact statement. At a first level of refinement (see Theorem 6.10) we can prove that \(Gr_{wt}(H^\sigma_{Q,W})\), the associated graded of \(H^\sigma_{Q,W}\) with respect to the weight filtration of 19, is a free supercommutative algebra in the category of \(\mathbb{Z}^{Q_0} \oplus \mathbb{Z}_{sc} \oplus \mathbb{Z}_{wt} \oplus \mathbb{C}\)-graded vector spaces, where the supercommutativity again refers to the Koszul sign rule being applied, but solely with respect to the \(C\)-grading – this is enough to make sense of (2) below. Here and from now on, we identify \(\mathbb{Z}_{wt} := \mathbb{Z} =: \mathbb{Z}_{sc}\), the subscripts help us to remember which copy of \(\mathbb{Z}\) is keeping track of the (shifted) cohomological degree, and which copy of \(\mathbb{Z}\) is keeping track of the weight degree.

The approach to proving Theorem 1.1 is to look for extra structures, in particular, some variation of a comultiplication \(\Delta : H^\sigma_{Q,W} \to H^\sigma_{Q,W} \otimes H^\sigma_{Q,W}\) such that \(\Delta\) is an algebra homomorphism (i.e. making \(H^\sigma_{Q,W}\) into a bialgebra), which force the theorem to be true, without identifying any particular generators for \(H^\sigma_{Q,W}\). At the end of this section we consider a little how this works for the non critical cohomological Hall algebra, which is isomorphic to the special case \(W = 0\) (see Section 7.1).

It is worth noting the shortcoming of this approach: at no point do we identify canonical generators of \(H^\sigma_{Q,W}\). The question of whether these exist remains completely open and is surely worthy of closer study.

The subgroup \(C^* \to GL_C(\gamma)\) given by \(z \mapsto z \cdot id\) acts trivially on all the spaces \(M_{\gamma}\), and so \(H_{Q,W,\gamma}\) is free as a \(\mathbb{C}[y]\) module, for \(\gamma \neq 0\), where \(y\) is the generator of the equivariant cohomology \(H^*_C(pt, \mathbb{Q})\) (it is probably not obvious how or why the dual of compactly supported equivariant cohomology of equivariant vanishing cycles should form a module over the ordinary equivariant
cohomology of a point – this will be explained in Section 2). Since the multiplication and
comultiplication respect this action, it follows that the generators of \( \text{Gr}_{wt}(H_{Q,W}^\sigma) \) can themselves
be given a \( \mathbb{C}[y] \) action which is again free, so we may describe the generating space as \( V_{\text{prim}}[y] \subset \text{Gr}_{wt}(H_{Q,W}^\sigma) \). Considering the algebra \( H_{Q,W}^\sigma \) as an algebra in the category of \( \mathbb{Z}_{sc} \oplus \mathbb{Z}^{Q_0} \)-graded monodromic mixed Hodge structures, we take the weight polynomial, and Theorem 1.1 implies
\[
\sum_{\gamma \in \mathbb{Z}^{Q_0}} \chi_q(H_{Q,W,\gamma}^\sigma,q^{1/2})x^\gamma := \prod_{\gamma \in \mathbb{Z}^{Q_0} \setminus 0} (1 - x^\gamma)^{\Omega_{\gamma}(q^{1/2})(q-1)^{-1}}.
\]
assuming we can make sense of (3), i.e., assuming that each \( V_{\text{prim},\gamma} \) is \( \mathbb{Z}_{wt} \)-graded finite dimen-
sional. Now in the event that we have an equality
\[
\sum_{\gamma \in \mathbb{Z}^{Q_0}} \chi_q(H_{Q,W,\gamma}^\sigma,q^{1/2})x^\gamma := \prod_{\gamma \in \mathbb{Z}^{Q_0} \setminus 0} (1 - x^\gamma)^{\Omega_{\gamma}(q^{1/2})(q-1)^{-1}}.
\]
for \( \Omega_{\gamma}(q) \) Laurent polynomials in \( q^{1/2} \), one says that the left hand side of (4) is admissible,
or alternatively that the refined Donaldson-Thomas invariants, which are defined to be the polynomials \( \Omega_{\gamma}(q) \), are well-defined. So the well-definedness of the DT invariants is implied by finite-dimensionality of the \( V_{\text{prim},\gamma} \).

The following is the obvious extension of Conjecture 1 of [19], proved for the non critical CoHA in [12].

Conjecture 1.2. The space \( V_{\text{prim},\gamma} \) is finite dimensional for each \( \gamma \).

There are certain instances in which the well-definedness of the refined DT invariants itself
implies finite-dimensionality of the \( V_{\text{prim},\gamma} \). This occurs, for example, when \( V_{\text{prim},\gamma} \) vanishes in
even or odd degree, or when the monodromic mixed Hodge structure on \( V_{\text{prim},\gamma} \) is pure. In
fact one can show that at least one of these two conditions holds in a wide range of examples
in which the refined DT invariants have been shown to exist and have been calculated, and so
deduce finite-dimensionality of the \( V_{\text{prim},\gamma} \).

We finish the paper by describing some of these examples. In this introduction let us merely
point towards an interesting application that we don’t have space to develop in this paper,
the details are in [5]. It can be shown that out of a quiver \( Q \) (without a potential) one may
canonically build a new quiver \( \tilde{Q} \) with potential \( W \) such that the refined DT invariants are the
Kac polynomials for the original quiver, and for which the associated spaces \( V_{\text{prim},\gamma} \), of which
the Kac polynomials are weight polynomials, have cohomology only in even degree and are
pure. This provides a new proof of the Kac conjecture, recently proved by Hausel, Letellier and
Rodriguez-Villegas in [14].

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1.1. Plan of the paper. The primary purpose of this paper is to prove that certain critical CoHAs are free $C$-supercommutative, and up until we start into the examples of Section 7 this is where everything is leading to. We finish this introductory section with a brief discussion of the CoHA associated to a quiver without a potential. It turns out that while everything is much simpler here, many of the main ideas that will follow are already hinted at in this special case; in Section 7.1 it is fully explained why the CoHA of a quiver $Q$ is a special case of the critical CoHA of a quiver with potential.

In Section 2 we collect together everything we will use regarding sheaves, Hodge modules, monodromic mixed Hodge structures, and vanishing cycles. The cohomology theory underlying the critical CoHA is the dual of compactly supported equivariant cohomology with coefficients in vanishing cycles. So in Section 2 we recall the definition of equivariant compactly supported cohomology of vanishing cycles, and also explain why its dual is a module over ordinary equivariant cohomology.

The Hall algebra product can be informally described as follows: there is a diagram of stacks

$$
\mathcal{X}_{\gamma_1} \times \mathcal{X}_{\gamma_2} \xrightarrow{\pi_1 \times \pi_3} \mathcal{X}_{\gamma_1, \gamma_2} \xrightarrow{\pi_2} \mathcal{X}_{\gamma_1 + \gamma_2}
$$

where $\mathcal{X}_{\gamma}$ is the stack of representations of $Q$ of dimension vector $\gamma$, $\mathcal{X}_{\gamma_1, \gamma_2}$ is the stack of flags $F_1 \subset F_2$ where $\dim(F_1) = \gamma_1$ and $\dim(F_2/F_1) = \gamma_2$. The CoHA multiplication is given by pulling back dual compactly supported cohomology, always with coefficients in the vanishing cycle complex, along $\pi_1 \times \pi_3$, and then pushing forward along $\pi_2$. An important observation in constructing the localised coproduct is that these maps in dual compactly supported cohomology all admit umkehr maps, so that out of exactly the same diagram of stacks we can construct a putative comultiplication. The background for this is explained in Section 2.7 where push-forwards and pullbacks of dual compactly supported equivariant cohomology are studied, and localisation of dual compactly supported cohomology appears as a part of the definition of these umkehr maps.

In Section 3 we recall the definitions of spaces of quiver representations and superpotentials on them, and recall also the definition of the critical CoHA multiplication. This multiplication operation is in fact a composition of six different morphisms, all of which are discussed in the background section on sheaves and Hodge structures.

Section 4 is rather abstract, and introduces the notion of a $Q$-localised bialgebra object in a Tannakian category. First let’s motivate this slightly higher level of abstraction.

While we can be totally unambiguous throughout the paper over the issue of the underlying vector space of the critical CoHA $\mathcal{H}_{Q,W}$, it is advantageous to be somewhat flexible over the issue of what extra structure we enrich $\mathcal{H}_{Q,W}$ with. An enrichment of $\mathcal{H}_{Q,W}$ in all instances means
finding an algebra object in a Tannakian category such that the result of applying the fibre functor is the original CoHA, considered as an algebra object in graded vector spaces. A good reason for this flexibility is as follows. We would like to use the fact that some suitably enriched version of $\mathcal{H}_{Q,W}$ is a symmetric algebra object in some sufficiently rich category to explain infinite product formulae that appear when we take characteristic functions of it. But only if a Tannakian category is semisimple can we deduce that an algebra object $\mathcal{A}$ is the free symmetric algebra object on some generating object $V$ from the analogous statement for the image of $\mathcal{A}$ under the fibre functor. The critical CoHA $\mathcal{H}_{Q,W}$ is an algebra object in the category $\text{MMHS}$ of mixed monodromic Hodge structures, and this is the highest refinement we will encounter, in the sense that there are forgetful functors to all the others. This is not a semisimple category, but there are forgetful functors (e.g. the functor taking a mixed monodromic Hodge structure to the associated graded for the weight filtration) from $\text{MMHS}$ that do land in semisimple categories, and from which the data of refined partition functions of $\mathcal{H}_{Q,W}$ can still be calculated. This helps to explain why infinite product expansions of these partition functions are so prevalent.

The second reason for working at the level of abstraction of Section 4 takes us beyond the scope of this paper, and is discussed only briefly, in Section 6.2. From the data of a Tannakian category $\mathcal{C}$ and a quiver $Q$ we build a new category that is not Tannakian, it is only braided monoidal, and not symmetric - its analogue of the Koszul sign rule is built out of the Ringel form, which needn’t be symmetric. This category is graded by dimension vectors and the part $D$ that is concentrated in self dual dimension vectors is a tensor category, and becomes the subject of interest for the main theorem of this paper. But it is worth noting that along the way we prove that the critical CoHA $\mathcal{H}_{Q,W}$, considered as an object of the braided monoidal category $\mathcal{C}_Q$ still admits a localised bialgebra structure. We conjecture that in fact $\mathcal{H}_{Q,W}$ can be considered as a symmetric localised bialgebra object in a suitable tensor category containing $D_{\text{wt}}$, the category obtained from $D$ by applying the fibre functor landing in $\mathbb{Z}^{Q_0} \oplus \mathbb{Z}_{\text{wt}}$-graded vector spaces.

We finish Section 4 by recalling the link between supercommutative localised bialgebras and free supercommutative algebras. If we remove the word ‘localised’ this link is classical, and can be found e.g. in Borel’s thesis. During the course of this recollection we meet the extra ingredient that is required in the localised case to deduce free supercommutativity from a commutative bialgebra structure: the localisation map must itself be an injection.

In Section 5 we recast the critical CoHA in terms of the fixed part of the Weyl group action on a torus equivariant critical CoHA. This is a standard move, and is introduced here to make what follows easier. In particular, it turns out to be easier to prove supercommutativity of $\mathcal{H}_{Q,W}^\tau$, the part of the critical CoHA concentrated in self-dual dimension vectors, using the torus equivariant version. We also prove in this section the fact mentioned above, that the localisation maps in the localisations that we will use are indeed injective – this turns out to be a slightly modified version of the Atiyah-Bott lemma.

In Section 6 we prove our main result. The bulk of the section is devoted to defining the localised comultiplication on $\mathcal{H}_{Q,W}$ and proving that it endows $\mathcal{H}_{Q,W}$ with the structure of a localised bialgebra. Up to a constant factor (denoted $\tilde{E}_{\gamma_1,\gamma_2}$ below), the definition of the comultiplication is essentially given by running some of the six maps that appear in the multiplication
the ‘wrong way’ (i.e. taking their umkehr maps), and using the fact that the remaining map, the Thom–Sebastiani isomorphism, is an isomorphism, and taking its inverse. We prove that this gives us a localised bialgebra structure, and putting everything else in the paper together, that $H_{Q,W}^\sigma$ is indeed a free $\mathbb{C}$-supercommutative algebra.

In the final section we work through some examples. Examples of the application of the work of this paper come in Representation Theory, Topology and Algebraic Geometry. The Algebraic Geometry examples are perhaps quite well known, but we still deduce some surprising results from our main theorem – that the CoHAs of zero-dimensional coherent sheaves on $\mathbb{C}^3$ supported on the hyperplane defined by $x = 0$, of coherent sheaves on the noncommutative resolved conifold supported on the exceptional locus, and coherent sheaves on the noncommutative resolved $(-2)$ curve supported on the exceptional locus are free supercommutative, with explicitly calculated dimensions for their generating sets. We can remove the support constraint in all of these examples and we obtain the same result, that the critical CoHA is free supercommutative, but then calculating the dimensions of the generating sets comes down to a purity conjecture contained in [5].

In Representation Theory the one example that is worked out gives a new proof of Kac’s conjecture, already proved in [14]. The passage from the results of this paper to this result is actually rather straightforward, and rests on a purity conjecture, related to the one above, which we can in fact prove. This example probably deserves its own paper, as this one is already looking rather long, and is dealt with in [5].

Finally in Topology there turns out to be a rather intriguing way to import the study of character varieties and their twisted counterparts into the world of CoHAs, in such a way that the results of this paper have a direct bearing on the cohomology of these varieties. In short, it turns out that the cohomologies of the untwisted character varieties form a critical CoHA, and we conjecture that the generators are given by the cohomologies of the twisted character varieties. If this conjecture is true it implies a fascinating interplay between these two different types of cohomology, which we hope has some bearing on the conjectures of [15], as well as the P=W conjecture of [11].

In all three types of example we end up substituting critical cohomology ($:=\text{dual compactly supported cohomology with coefficients in the sheaf of vanishing cycles}$) with ordinary dual compactly supported cohomology. This is achieved via a process known as dimensional reduction, which is explained in the appendix, where we prove the theorem that allows us to make this substitution.

1.2. The main result for quivers without potential. Let $Q$ be an arbitrary quiver. Define $Q^T$ to be the quiver obtained from $Q$ by reversing all the arrows, so that for each arrow $a \in Q_1$ we have an arrow $a^* \in (Q_1)^T$ with $h(a^*) = t(a)$ and $t(a^*) = h(a)$. A self-duality $\sigma$ for $Q$ is an isomorphism $Q \to Q^T$, and we say a dimension vector is self dual if it is invariant under $\sigma$.

Example 1.3. If $Q$ is symmetric in the sense that for every two vertices $i, j \in Q_0$ there are as many arrows from $i$ to $j$ as from $j$ to $i$, $Q$ carries a self-duality $\sigma$ fixing all vertices, and all dimension vectors are self-dual for $\sigma$. 
Example 1.4. Let $Q$ carry an action of the cyclic group $\mu_n := \mathbb{Z}/n\mathbb{Z}$, which acts by cyclic permutation on the vertices $\{1, \ldots, n\}$. Then $Q$ also carries a self-duality $\sigma$, and the group of self dual dimension vectors is of rank $\lfloor n/2 \rfloor + 1$.

Example 1.5. In general, let $Q$ be a quiver with $B$ the matrix defined by

$$b_{ij} = \delta_{ij} - \#\{a \in Q_1 | s(a) = i, t(a) = j\}.$$ 

Then a permutation matrix $M$ of order two is given by the underlying action on $Q_0$ of a self-duality of $Q$ if and only if $MBM = B^T$.

Now we fix a quiver $Q$ and a self duality structure $\sigma : Q \to Q^T$. Consider the non critical cohomological Hall algebra concentrated at self-dual dimension vectors $\mathcal{H}_{Q}^{\sigma} := \bigoplus_{\text{self-dual } \gamma} \mathcal{H}_{Q, \gamma}$, where $\mathcal{H}_{Q, \gamma}$ is defined to be the subspace of $\bigotimes_{i \in Q_0} \mathbb{C}[x_{i,1}, \ldots, x_{i, \gamma(i)}]$ invariant under the action of $\text{Sym}_{\gamma} := \prod_{i \in Q_0} \text{Sym}_{\gamma(i)}$. The spaces $\mathcal{H}_{Q, \gamma}$ are given a $\mathbb{Z}_{sc}$-grading by putting monomials of polynomial degree $d$ in $\mathbb{Z}_{sc}$-degree $\chi_Q(\gamma, \gamma) + 2d$. They are given a $\mathbb{C}$-grading by putting the whole of $\mathcal{H}_{Q, \gamma}$ in degree $(\kappa(\gamma), 0)$, where $\kappa$ is as in [11]. The $\mathbb{Z}_{sc}$-graded multiplication operation is defined by identifying these spaces of symmetric polynomials with equivariant cohomology spaces and then using correspondences in cohomology, as in [19]. As noted in [19] there is, however, a direct formula for the multiplication, given as follows:

$$(5) \quad f_1 \cdot f_2(x_{1,1}, \ldots, x_{1,\gamma_1(1)+\gamma_2(1)}, \ldots, x_{n,1}, \ldots, x_{n,\gamma_1(n)+\gamma_2(n)}) =$$

$$\sum_{\pi \in \mathcal{P}(\gamma_1, \gamma_2)} f_1(x_{1,\pi_1(1)}; \ldots, x_{1,\pi_1(\gamma_1(1))}; \ldots, x_{n,\pi_n(1)}; \ldots, x_{n,\pi_n(\gamma_1(n))}),$$

$$f_2(x_{1,\pi_1(\gamma_1(1)+1)}; \ldots, x_{1,\pi_1(\gamma_1(1)+\gamma_2(1))}; \ldots, x_{n,\pi_n(\gamma_1(n)+1)}; \ldots, x_{n,\pi_n(\gamma_1(n)+\gamma_2(n)))},$$

$$\prod_{i,j \in Q_0} \prod_{\alpha=1}^{\gamma_1(i)} \prod_{\beta=\gamma_1(j)+1}^{\gamma_1(j)+\gamma_2(j)} (x_{j,\pi(j)} - x_{i,\pi(i)})^{-b_{ij}}$$

where $\mathcal{P}(\gamma_1, \gamma_2)$ is the subset of $\pi \in \text{Sym}_{\gamma_1+\gamma_2}$ such that $\pi_1$ preserves the ordering of $\{1, \ldots, \gamma_1(i)\}$ and $\{\gamma_1(i)+1, \ldots, \gamma_1(i)+\gamma_2(i)\}$ for each $i \in Q_0$ (the set of shuffles of $(\gamma_1, \gamma_2)$ into $\gamma$).

Although it is possible for $-b_{ij}$ to be negative, it turns out that [5] always defines a polynomial function, so gives a well defined multiplication. To the extent that a tradition exists, it is traditional to make this algebra supercommutative by twisting the multiplication by a sign, as described in Section [5,11] but for now we continue with the approach described in the introduction whereby we twist what we mean by supercommutative – it is easy to check that $\mathcal{H}_{Q}^{\sigma}$ is $\mathbb{C}$-supercommutative (note that the entire algebra lies in even $\mathbb{Z}_{sc}$-degree). The question is: is there a reasonable coproduct on $\mathcal{H}_{Q}^{\sigma}$ that we can cook up straight from formula [5]?

For example if $Q$ is symmetric, as in Example [1,3] we know the answer must be yes, by work of Efimov [12]. He shows (again, directly from the formulas for multiplication) that in this instance $\mathcal{H}_{Q}^{\sigma}$ is supercommutative and free, from which it follows that there is a comultiplication for $\mathcal{H}_{Q}^{\sigma}$ (the shuffle coproduct for free supercommutative algebras). But the direction this paper takes is in some sense the other way around: we are going to use the existence of a coproduct (or something like it) to demonstrate supercommutative freeness in the first place.
We examine instead a different way to see that at least one of the $H_\sigma^Q$ carries a coproduct. Let $Q$ be the quiver with one vertex and one loop, equipped with the unique self duality $\sigma : Q \to Q^T$. Then from (5), $H_\sigma^Q = H_Q$ is a shuffle algebra on countably many variables. This has an obvious coproduct, which we now describe. For $\gamma_1 + \gamma_2 = \gamma$, there is a natural inclusion

$$i_{\gamma_1, \gamma_2} : H_{Q, \gamma} \to H_{Q, \gamma_1} \otimes H_{Q, \gamma_2}$$

given by considering a $\text{Sym}_\gamma$ invariant polynomial as a $\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}$ invariant one. We define

$$\Delta = \sum_{\gamma_1 + \gamma_2 = \gamma} i_{\gamma_1, \gamma_2}.$$

It is a worthwhile check to see that this really does define a compatible coassociative coproduct $\Delta$, where by compatible we mean that $\Delta$ is an algebra homomorphism, i.e. the following diagram commutes

$$\xymatrix{ H_Q \otimes H_Q \ar[r]^-{\Delta \otimes \Delta} \ar[d]^-{(a,b)\mapsto a \cdot b} & H_Q \otimes H_Q \otimes H_Q \otimes H_Q \ar[d]^-{(a,b,c,d)\mapsto \gamma(x_{i,j} - x_{i,a}) b_{ij}} \\
H_Q \ar[r]^-{\Delta} & H_Q \otimes H_Q,}$$

where $| \bullet |_c$ denotes the $c$-degree. It’s worth noting also that $\Delta$ is the same as the coproduct $\Delta'$ one gets by considering $H_Q$ as a free supercommutative algebra in the first place (it is freely generated by the space of polynomials in one variable, i.e. $H_{Q,1}$), as for each $i$ we have

$$\Delta'(x^i) = 1 \otimes x^i + x^i \otimes 1 = \Delta(x^i).$$

When we try to generalise $\Delta$, for other quivers $Q$, we need to adjust for the product terms in the last line of (5). In fact it’s not hard to guess what the correct approach to this should be, and the guess turns out to be almost right:

$$\Delta = \sum_{\gamma_1 + \gamma_2 = \gamma} i_{\gamma_1, \gamma_2} \prod_{i,j \in Q_0} \prod_{\alpha = 1}^{\gamma_1(i)} \prod_{\beta = \gamma_1(j)+1}^{\gamma_2(j)} (x_{j,\beta} - x_{i,\alpha}) b_{ij}.$$

The ‘almost’ here is not referring to noncommutativity of (5), but to the fact that $\Delta$ no longer really defines a morphism between the constituent terms of (5). The problem is that in inverting the signs of the $b_{ij}$, we lose the guarantee we had in the case of the multiplication that the result of feeding in polynomials is a new polynomial, as opposed to a rational function. But if we just treat multiplication and $\Delta$ as operations on rational functions in variables $\{x_{1,1}, \ldots, x_{1,\gamma_1(1)}, \ldots, x_{n,1}, \ldots, x_{n,\gamma(n)}\}$, the morphisms in (5) do commute. It turns out this is enough (see Section 6) to recover the first part of the following theorem, proved in [12] in the case that $Q$ is symmetric.

**Theorem 1.6.** The non critical CoHA $H_\sigma^Q$ is a free $c$-supercommutative algebra. We can choose a primitive generating set $V \subset H_{Q,W}$ such that $V_\gamma = V_\gamma^{\text{prim}} \otimes \mathbb{C}[y]$, where $y = x_{1,1} + \ldots + x_{n,\gamma(n)}$, and the operation of $y$ on $V$ is via the usual multiplication operation between polynomials, and $V_\gamma^{\text{prim}} \to H_{Q,\gamma}/y \cdot H_{Q,\gamma}$ is an injection.
The second part, which deals with the specific properties of a generating set, will be proved later, in the more general context of critical cohomological Hall algebras. The appearance of rational functions in the naive definition of the comultiplication generalising the natural comultiplication on the shuffle algebra explains the appearance of localisations of compactly supported equivariant cohomology in this paper.

2. Constructible sheaves and vanishing cycles

2.1. Verdier duality. Let \( f : X \to Y \) be a morphism of manifolds. Then the proper push forward functor \( f_! \) defines a functor \( D^b(X) \to D^b(Y) \) on the category of Abelian sheaves with constructible cohomology, with right adjoint \( f^! \). This adjoint functor is in general not the derived functor of a functor from the Abelian category of sheaves on \( Y \) to the Abelian category of sheaves on \( X \), though if \( f \) is an affine fibration or a locally closed inclusion it is, at least up to a cohomological shift.

The “six functors”, namely \( f_!, f^!, f_* , f^* , \mathcal{H}om \), and \( \otimes \) themselves descend from functors on the categories \( D^b(\text{MHM}(X)) \) and \( D^b(\text{MHM}(Y)) \), and the adjunction between \( f_! \) and \( f^! \) lifts to the level of mixed Hodge modules too. For an introduction to the theory of mixed Hodge modules see [25]. For the reader that prefers never to think about mixed Hodge modules, the paper can still be read, with the rule that every time one meets the symbols \( \bullet \{d\} \) one should read \( \bullet \otimes \mathbb{Q}_{2d,2d} \), the operation of tensoring with the rational vector space concentrated in \( \mathbb{Z}_{\text{sc}} \oplus \mathbb{Z}_{\text{wt}}\)-degree \( (2d,2d) \).

Let \( p : Y \to \text{pt} \) be the projection to a point. There is an isomorphism
\[
(7) \quad p_! \mathbb{Q} \to \mathbb{Q}_Y \otimes p^* \mathbb{Q}(\dim(Y))[-2\dim(Y)]
\]
in the category of mixed Hodge modules on \( Y \). Here \( \mathbb{Q}(\dim(Y)) \) is the pure one-dimensional weight \( 2\dim(Y) \) Hodge structure. A choice of isomorphism \( (7) \) corresponds via Poincaré duality to a choice of a dual class to \( H_0(Y,\mathbb{Q}) \). There is, then, a canonical isomorphism \( (7) \), given by the map sending the class of a point in \( H_0(Y) \) to 1. For notational convenience we define
\[
\mathbb{Q}\{d\} := \mathbb{Q}(d)[-2d]
\]
and
\[
\mathbb{Q}_Y\{d\} := \mathbb{Q}_Y \otimes p^* \mathbb{Q}\{d\},
\]
and we define the derived functor \( \mathcal{F} \to \mathcal{F}\{d\} \) via
\[
(8) \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathbb{Q}_Y\{d\}.
\]

Recall that the Verdier dual of \( \mathcal{F} \) is defined to be \( R\mathcal{H}om(\mathcal{F},p'_! \mathbb{Q}) \), and so there is a canonical isomorphism of functors \( D\mathcal{F} \to \mathcal{F}^\vee \otimes \mathbb{Q}_Y\{\dim(Y)\} \), where \( -^\vee \) is the duality functor \( R\mathcal{H}om(-,\mathbb{Q}_Y) \). The multiplication map \( \mathbb{Q}_Y \otimes \mathbb{Q}_Y \to \mathbb{Q}_Y \) induces an isomorphism \( \mathbb{Q}_Y \to \mathbb{Q}_Y \), and so we deduce that there is a canonical isomorphism
\[
(9) \quad \mathbb{Q}_Y\{\dim(Y)\} \to D\mathbb{Q}_Y.
\]
2.2. Vanishing cycles of sheaves. We first discuss vanishing cycles of sheaves, without the added worry of Hodge structures – that will come later. Let $Y$ be a connected complex manifold, and let $Z \subset Y$ be a closed subspace. Then for $\mathcal{F}$ an Abelian sheaf on $Y$, we define

$$\Gamma_Z \mathcal{F}(U) = \ker \left( \mathcal{F}(U) \to \mathcal{F}(U \setminus Z) \right).$$

The associated derived functor commutes with Verdier duality, and the following diagram commutes too, for $c$ the map induced by the canonical inclusion

$$
\begin{array}{ccc}
\Gamma_Z \{ \dim(X) \} & \xrightarrow{c} & \id \{ \dim(X) \} \\
\downarrow \Gamma_Z \nu & & \downarrow \nu \\
\text{DR} \Gamma_Z D & \xleftarrow{Dc} & D
\end{array}
$$

where we have identified $R\Gamma_Z D$ with $\text{DR} \Gamma_Z$.

Let $f : Y \to \mathbb{C}$ be a holomorphic function. We define

$$\varphi_f \mathcal{F}[-1] := (R\Gamma_{\{\text{Re}(f)\leq 0\}} \mathcal{F})|_{f^{-1}(0)},$$

the vanishing cycles functor for $f$. We can consider this as a functor $\varphi_f : D^b(Y) \to D^b(Y)$ between the derived categories of Abelian sheaves with constructible cohomology. It is perhaps more standard to consider $\varphi_f$ as a functor $D^b(Y) \to D^b(f^{-1}(0))$, but we prefer to keep all our sheaves as sheaves on smooth manifolds, as we use Verdier duality a great deal and always take Verdier duals in categories of sheaves on smooth manifolds. Here we are using a nonstandard definition for $\varphi_f$ that is equivalent to the usual one in the complex case (see Exercise VIII.13 of [16]). Often we will abbreviate $\varphi_f \mathbb{Q}_{Y}[-1]$ to just $\varphi_f$. The isomorphism (9) induces isomorphisms in $D^b(Y)$

$$\varphi_f \mathbb{Q}[-2 \dim(Y)] \to \varphi_f D \mathbb{Q}$$

which will be used heavily in the sequel in order to construct ‘umkehr’ maps. For instance these maps are required for the definition of compactly supported equivariant cohomology with coefficients given by vanishing cycles.

**Remark 2.1.** From now on all functors will be functors between derived categories of sheaves, and we will abbreviate $R\Gamma_-$ to $\Gamma_-$, $Rj_*$ to $j_*$, etc.

If $Y' \xrightarrow{j} Y \xrightarrow{f} X$ is a composition of maps of manifolds, and $Z \subset X$ is a closed subspace, there is a natural transformation of functors $D^b(Y) \to D^b(Y)$

$$\Gamma_{f^{-1}(Z)} \to j_* \Gamma_{(j; f)^{-1}(Z)} j^*$$

which induces a natural transformation

$$\varphi_f \to j_* \varphi_{j; f} j^*$$

in the case that $f$ is a regular function. The natural transformation (13) is generally not an isomorphism. On the other hand, if $j$ is a closed embedding then the composition

$$\Gamma_{f^{-1}(Z)} j_* \to j_* \Gamma_{(j; f)^{-1}} j^* j_* \xrightarrow{\cong} j_* \Gamma_{(j; f)^{-1}}$$

is a natural isomorphism of functors $D^b(Y') \to D^b(Y)$, and

\[(15) \quad \varphi_f j_* \to j_! \varphi_f j^*\]

is a natural isomorphism of functors $D^b(Y') \to D^b(Y)$. Similarly, if $j$ is an affine fibration, \[\text{[13]}\] is a natural equivalence, which when evaluated on $\mathbb{Q}_Y$ gives a quasi isomorphism of complexes of sheaves

$$\varphi_f \to j_* \varphi_f j^*.$$  

It is easy to find examples showing that if $i : X \to Y$ is the inclusion of a closed subspace, and $f : Y \to \mathbb{C}$ is a holomorphic function, then tensoring with $i_* \mathbb{Q}_X$ doesn't commute with taking vanishing cycles. For example, one can show that the sheaf of vanishing cycles $\varphi_f$ is supported on the critical locus of $f$, while the complex $\varphi_f i_* \mathbb{Q}_X$ has cohomology supported on the critical locus of $f|_X$ by \[\text{[14]}\]. However we do have the following useful fact:

**Proposition 2.2.** Let $D^b_{lf}(X)$ be the full subcategory of bounded derived category of sheaves on $X$ consisting of objects with locally free cohomology. There is a natural equivalence of bifunctors $D^b_{lf}(X) \times D^b(X) \to D^b(X)$

$$\nu : (\mathcal{L} \times \mathcal{F} \mapsto \varphi_f(\mathcal{F} \otimes \mathcal{L})) \to ((\mathcal{L} \times \mathcal{F} \mapsto \varphi_f(\mathcal{F}) \otimes \mathcal{L}).$$

If $Y' \to Y$ is an affine fibration, there is a natural equivalence $j_! j^* \mathcal{F} \to \mathcal{F} \otimes j_! \mathbb{Q}_{Y'}$, and so we deduce the following corollary.

**Corollary 2.3.** Let $Y' \to Y$ be an affine fibration. Then there is a natural equivalence $\varphi_f j_! j^* \to j_! \varphi_f j^*$.  

### 2.3. Monodromic mixed Hodge structures

The critical CoHA $\mathcal{H}_{Q,W}^{sp}$ defined below will be an algebra object in the category of $\mathbb{Z}_{sc} \oplus \mathbb{Z}^{Q_0}$-graded vector spaces, where $\mathbb{Z}_{sc}$ is the shifted cohomological grading, as in the introduction. For some applications it is advantageous to consider a more refined algebra object, in the category of *monodromic mixed Hodge structures*. For example, one cannot recover the theory of refined DT invariants without at least considering the extra $\mathbb{Z}^{wt}$-grading on $Gr_{wt}(\mathcal{H}_{Q,W}^{sp})$ coming from the weight filtration associated to monodromic mixed Hodge structures.

We will give an outline of the relevant definitions and propositions here. For many applications, the following remark will suffice: there is a full and faithful exact tensor functor $j_* s_* : 

\text{MHS} \to \text{MMHS}$, and in the event that all our vanishing cycles $\varphi_f$ lie in the image of this functor, we may as well consider $\mathcal{H}_{Q,W}^{sp}$ as an algebra object in the category of $\mathbb{Z}_{sc} \oplus \mathbb{Z}^{Q_0}$-graded mixed Hodge structures.

The category $\text{MHM}(X)$ of mixed Hodge modules on a manifold $M$ is a full sub-tensor category of the category $F_W \text{MF}_{rh}(X)$, which consists of filtered objects of $\text{MF}_{rh}(X)$, which itself consists of triples

1. A perverse sheaf $L$ on $X$
2. A regular holonomic $D_X$-module $M$ with an isomorphism $DR(M) \cong L$.
3. A good filtration $F$ on $M$. 

Actually describing which objects of $F_W \text{MF}_{ch}(X)$ belong to $\text{MHM}(X)$ is a rather complicated matter, to which we refer the reader to [25]. There is a forgetful functor $\text{rat} : D^b(\text{MHM}(X)) \to D^b(X)$, given by remembering only the perverse sheaf, and this functor is exact and faithful.

The category $\text{MMHS}$ is a full subcategory of $\text{EMHS}$, the category of exponential mixed Hodge structures. The category $\text{EMHS}$ is in turn defined as the full subcategory of $\text{MHM}(\mathbb{A}^1)$ such that the corresponding perverse sheaf $L$ satisfies $R\Gamma(L) = 0$. Let

$$\text{sum} : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

be the addition morphism, then the tensor product in $\text{EMHS}$ is defined as

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \text{sum}_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Let

$$j : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$$

be the inclusion. There is an obvious inclusion of categories $j_* : \text{EMHS} \to \text{MHM}(\mathbb{A}^1)$, with $j^* := \text{sum}_*(-,j_!\mathbb{A}^1\setminus\{0\}[1])$ its left adjoint. The functor $j^*$ is a symmetric monoidal functor. There is a tensor functor $s_* : \text{MHS} \to \text{MHM}(\mathbb{A}^1)$ given by pushforward along the inclusion

$$s : \{0\} \to \mathbb{A}^1$$

and so a tensor functor

$$(16) \quad j_*s_* : \text{MHS} \to \text{EMHS}.$$ 

In this way we realise the category of mixed Hodge structures as a subcategory of the category of exponential mixed Hodge structures. The weight filtration for $\mathcal{F} \in \text{EMHS}$ is defined by

$$W^\text{EMHS}_m \mathcal{F} := j^*W^\text{MHM}(\mathbb{A}^1)_m j_*\mathcal{F}.$$ 

**Definition 2.4.** We say an object of $D^b(\text{EMHS})$ is pure if its $m$th cohomology is pure of weight $m$.

Since all our exponential mixed Hodge structures will in fact belong to the subcategory of monodromic mixed Hodge structures, the fibre functor to vector spaces admits an easy description: let $i : \{1\} \to \mathbb{A}^1$ be the inclusion of the point, then

$$\text{rat} i^* : \text{MMHS} \to \text{Vect}$$

provides a fibre functor (i.e. this functor commutes with tensor products and their symmetry isomorphisms, and is exact and faithful).

**Definition 2.5.** If $V$ is a vector space, an exponential mixed Hodge structure on $V$ consists of an element $\bar{V} \in \text{EMHS}$ and an isomorphism $\text{rat} i^*\bar{V} \cong V$.

If $f : X \to \mathbb{C}$ is a holomorphic function on a smooth manifold $X$, we define the element $\varphi_f \in \text{MHM}(X)$ as in [21].

Let $Z \subset Y$ be a submanifold, and let $f$ be a function on $Y$. The cohomology $H_\ast(Z, \varphi_f \mathbb{Q}_Y)$ is given an exponential mixed Hodge structure as follows. Let $u$ be the coordinate on $\mathbb{C}^*$, and consider the mixed Hodge module

$$(\mathbb{C}^* \to \mathbb{A}^1)_!(Z \times \mathbb{C}^* \to \mathbb{C}^*)!(Z \times \mathbb{C}^* \to Y \times \mathbb{C}^*)^*\varphi_f/\partial u \mathbb{Q}_{Y \times \mathbb{C}^*}.$$
This is an element of $L \in \text{MMHS} \subset \text{EMHS}$, and we have a natural isomorphism

$$\text{rat} \ i^*L \cong H_c(Z, \varphi_f \mathbb{Q}_Y)$$

as required. Note here that $f$ is considered throughout as a function on $Y$, not $Z$.

**Proposition 2.6.** Let $Y' \to Y$ be a closed embedding of manifolds, or a smooth map, and let $Y \to \mathbb{C}$ be a holomorphic function. The natural transformation (13), applied to $\mathbb{Q}_Y$, lifts to a morphism of mixed Hodge structures.

By Proposition 2.6 and faithfulness of the fibre functor, all the isomorphisms of the previous section can (and will) be lifted to isomorphisms of exponential mixed Hodge structures.

2.4. **Compactly supported equivariant cohomology.** Now assume that $Y$ carries a $G$ action, where $G$ is an algebraic group, and we are given an inclusion $G \subset \text{GL}_C(n)$ for some $n$. For $N \geq n$ we define $\text{fr}(n, N)$ to be the space of $n$-tuples of linearly independent vectors in $\mathbb{C}^N$, and we define $(Y, G)_N := Y \times \text{fr}(n, N)$ and $(Y, G)_{N+1} := Y \times G \text{fr}(n, N)$. If we assume furthermore that $f : Y \to X$ is $G$-invariant, we obtain functions

$$f_N : (Y, G)_N \to X$$

and objects $\Gamma_{f_N^{-1}(Z)}(\mathbb{Q}(Y, G)_N)$ on each of the spaces $(Y, G)_N$. There is a natural inclusion $\mathbb{C}^N \to \mathbb{C}^{N+1}$ sending $(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_N, 0)$ inducing maps $\text{fr}(n, N) \to \text{fr}(n, N+1)$ and $i_N : (Y, G)_N \to (Y, G)_{N+1}$. Combining (11) and (15) there are maps

$$i_N! \varphi_{f_N} \mathbb{Q}(Y, G)_N \{\dim((Y, G)_N)\} \to (i_N)! \varphi_{f_N} D\mathbb{Q}(Y, G)_N \varphi_{f_N+1} \mathbb{Q}(Y, G)_{N+1} \{\dim((Y, G)_{N+1})\}$$

where $e$ is defined via applying Verdier duality to the natural map

$$\mathbb{Q}(Y, G)_{N+1} \to (i_N)! (i_N)^* \mathbb{Q}(Y, G)_{N+1}.$$

For $X$ a sub $G$-equivariant subvariety of $Y$, we define

$$H_{c,G}(X, \varphi_f) := \lim\{H_c(X, \varphi_{f_N}) \{\dim(\text{fr}(n, N))\}\}$$

where $X_N \subset (Y, G)_N$ is the subspace of points projecting to $X$.

**Remark 2.7.** We have picked the unique normalizing $\mathbb{Q}\{-\}$ twist in (13) such that if $Y$ is acted on freely by $G$ in a $G$-equivariant open neighbourhood $U$ of $X$, we recover the usual definition of the compactly supported cohomology of $\varphi_g$ along $Y/G$, where $g$ is the induced function on $U/G$.

2.5. **Thom–Sebastiani isomorphism.** Now let $Y_1$ and $Y_2$ be a pair of complex manifolds, acted on by $G_1$ and $G_2$ respectively, where again we have embeddings $G_i \subset \text{GL}_C(n_i)$, and let $f_1 : Y_1 \to \mathbb{C}$ and $f_2 : Y_2 \to \mathbb{C}$ be $G_1$ and $G_2$ invariant functions, respectively. The inclusion of closed subsets

$$\{\text{Re}(f_1)_N \leq 0\} \times \{\text{Re}(f_2)_N \leq 0\} \subset \{\text{Re}(f_1)_N + (f_2)_N \leq 0\} \subset (Y_1, G_1)_N \times (Y_2, G_2)_N$$
induces a morphism of objects in \(\text{D}^b((Y_1,G_1)_N \times (Y_2,G_2)_N)\)
\[
R\Gamma_{\{\text{Re}(f_1)_N \leq 0\} Q_{(Y_1,G_1)_N}} \boxtimes R\Gamma_{\{\text{Re}(f_2)_N \leq 0\} Q_{(Y_2,G_2)_N}} \rightarrow R\Gamma_{\{\text{Re}(f_1)_N \boxplus (f_2)_N \leq 0\} Q_{(Y_1,G_1)_N \times (Y_2,G_2)_N}}
\]
inducing a map
\[
H_c((f_1)_N^{-1}(0),\varphi_{(f_1)_N}) \otimes H_c((f_2)_N^{-1}(0),\varphi_{(f_2)_N}) \rightarrow H_c(((f_1)_N^{-1}(0) \times (f_2)_N^{-1}(0)),\varphi_{(f_1 \boxplus f_2)_N})
\]
and by the main result of \([21]\) this is an isomorphism, inducing an isomorphism of cohomologically graded vector spaces
\[
(19) \quad H_{c,G_1}((Y_1)_0,\varphi_{f_1}) \otimes H_{c,G_2}((Y_2)_0,\varphi_{f_2}) \rightarrow H_{c,G_1 \times G_2}((Y_1)_0 \times (Y_2)_0,\varphi_{f_1 \boxplus f_2})
\]
in the limit. The following observation will be important in dealing with some delicate sign issues later:

**Proposition 2.8.** The morphism \([19]\) intertwines the symmetric monoidal structures on \(\mathbb{Z}_{co}^-\) graded spaces (and tensor product) and topological spaces (with Cartesian product).

**Remark 2.9.** In order for the cohomological Hall algebra, defined in Section 3, to be an algebra object in the derived category \(\text{D}^b(\text{MMHS})\) of monodromic mixed Hodge structures, as discussed in Section 2.3 By unpublished work of Saito, \([19]\) becomes an isomorphism in this category.

Working with unpublished results is not ideal; there are three solutions to this situation.

1. One can go ahead and consider the CoHA as an algebra object in the category of monodromic mixed Hodge structures.
2. One can forget the Hodge structure on both sides of \([19]\) and consider it as an isomorphism in \(\text{D}^b(\text{pt})\).
3. One can assume that both sides of \([19]\) are genuine Hodge structures, in the sense that they lie in the image of the map \([16]\). This is indeed a safe assumption in a wide range of cases, most notably those coming from ‘dimensional reduction’ – see \([6]\). These cases include, for example, the non commutative conifold, and enough examples to reprove the Kac conjecture, see \([5]\), and analyse Hodge structures on twisted and untwisted character varieties using CoHAs.

### 2.6. Module structure.

In general, for \(X\) a \(G\) equivariant space, the space \(H_G(X,Q)\) is an algebra via the usual cohomology operations, and \(H_G(X,Q)\) is a module over \(H_G(\text{pt},Q)\). For example, each \(G_\gamma\) graded piece \(H_{Q,\gamma}\) of the non critical cohomological Hall algebra \(H_Q\) carries a \(H_{G_\gamma}(\text{pt},Q)\) action, since in fact there is an isomorphism in cohomology \(H_{Q,\gamma} \cong H_{G_\gamma}(\text{pt},Q)\). Sticking with quivers without potential, consider the spaces
\[
H_{c,G_\gamma}(M_Q,Q_{M_Q,\gamma})^* := Da_i Q_{M_Q,\gamma}
\]
where \( a : M_{Q, \gamma} \to \text{pt} \) is the projection to a point. Via the isomorphisms \( \mathbb{Q}_{M_{Q, \gamma}} \{ \dim(M_{Q, \gamma}) \} \to D_{\mathbb{Q}_{M_{Q, \gamma}}} \) and \( D(a)! \cong a_* \) we obtain an isomorphism

\[
H_{c, G, \gamma}(M_{Q, \gamma}, \gamma(Q) \{ \dim(M_{Q, \gamma}) \}) \to H_{G, G, \gamma}(Q_{M_{Q, \gamma}} \{ \dim(M_{Q, \gamma}) \}),
\]

and so we deduce that the dual of the compactly supported cohomology with trivial coefficients is the space that most naturally inherits the \( H_{G, \gamma}(\text{pt}, Q) \) action; given a \( G \) equivariant manifold \( X \) and a \( G \) invariant function \( f \) we construct a \( H_{G, \gamma}(X, \varphi_f)^* \) that becomes the above action in the special case \( G = G_\gamma, X = M_{Q, \gamma} \) and \( f = 0 \).

Define

\[
\Delta_N : (X \times_G \text{fr}(n, N)) \to (X \times_G \text{fr}(n, N)) \times (\text{pt} \times_G \text{fr}(n, N))
\]

\[
(x, z) \mapsto ((x, z), (\text{pt}, z)).
\]

Let \( B_N \) be the target of \( \Delta_N \) and \( A_N \) be the preimage. Let \( f_N \) be the function induced by \( f \) on \( A_N \), and let \( g_N \) be the function induced by \( f \) on \( B_N \). Then applying Verdier duality to

\[
\mathbb{Q}_{B_N} \to (\Delta_N)_! \mathbb{Q}_{A_N}
\]

and applying the vanishing cycle functor we obtain a morphism

\[
\varphi_{g_N}((\Delta_N)_! \mathbb{Q}_{A_N} \to \mathbb{Q}_{B_N} \{ \dim(\text{fr}(n, N)/G) \}).
\]

Via Corollary 2.3 we obtain morphisms

\[
(\Delta_N)_! \varphi_{g_N} \to \varphi_{g_N} \{ \dim(\text{fr}(n, N)/G) \},
\]

and taking duals of compactly supported cohomology, morphisms

\[
H_c((X, G)_N \times \text{fr}(G)_N, \varphi_{g_N}) \{ \dim(\text{fr}(n, N)/G) \}^* \to H_c((X, G)_N, \varphi_{\Delta_N})^*.
\]

Applying the Thom-Sebastiani isomorphism, observing that \( g_N = g_N \pi_1 \), where \( \pi_1 \) is projection onto the first factor of \( \mathbb{Q}_N \), we may rewrite the left hand side of \( 21 \) as

\[
H_c((X, G)_N, \varphi_{\Delta_N})^* \otimes H_c(\text{fr}(n, N)/G, Q) \{ \dim(\text{fr}(n, N)/G) \}^*
\]

and so by \( 20 \) we obtain a morphism

\[
H_c((X, G)_N, \varphi_{\Delta_N})^* \otimes H(\text{fr}(n, N)/G, Q) \to H_c((X, G)_N, \varphi_{\Delta_N})^*
\]

which gives an action of \( H_G(\text{pt}, Q) \) on \( H_{c, G}(X, \varphi_f)^* \).

Now let

\[
\Sigma_N : (X \times_G \text{fr}(n, N)) \to (X \times_G \text{fr}(n, N)) \times (X \times_G \text{fr}(n, N))
\]

be the diagonal embedding. Denote the target by \( \mathbb{Q}_N \). We define a function \( \mathbb{N}_N = \mathbb{N}_N \pi_1 \), where \( \pi_1 \) is projection onto the first factor of \( \mathbb{Q}_N \). Then again, \( \Sigma_N \mathbb{N}_N = \mathbb{N}_N \), and we build in the same way an extended action

\[
H_{G}(X, Q) \otimes H_{c, G}(X, \varphi_f)^* \to H_{c, G}(X, \varphi_f)^*.
\]
In all our applications, \( X \) will be contractible, and so there will be no difference between the two actions.

### 2.7. Umkehr maps in localised compactly supported cohomology

Associated to maps \( g : X \to Y \) of \( G \)-equivariant manifolds, with \( G \subset GL_C(n) \) an algebraic group and \( f \) a \( G \)-invariant function on \( Y \), we will often want to associate maps going \textit{both} ways between \( H_{c,G}(X,\varphi_f)^* \) and \( H_{c,G}(Y,\varphi_f)^* \). The maps \( g \) for which we wish to do this fall into essentially two different types.

Firstly, let \( \pi : X \to Y \) be an affine fibration of \( G \)-equivariant vector bundles. Then there is an isomorphism
\[
\pi^* : H_{G,c}(Y,\varphi_f)\{\dim(\pi)\}^* \to H_{G,c}(X,\varphi_f)^*
\]
which we call the pullback map associated to \( \pi \). This is constructed as follows. We denote by \( \pi_N \) the natural projection \( (X,G)_N \to (Y,G)_N \). Firstly, there is a natural isomorphism
\[
\varphi_f^N : (\pi_N)_!DQ_{(X,G)_N} \to DQ_{(Y,G)_N}.
\]
Applying \( \varphi_f^N \) to the Verdier dual of (23) we obtain maps
\[
\varphi_f^N : (\pi_N)_!Q_{(X,G)_N} \to Q_{(Y,G)_N}\{\dim(\pi)\}.
\]
By (11) this gives us an isomorphism
\[
(\pi_N)_!\varphi_f^N \to \varphi_f^N\{\dim(\pi)\}.
\]
From Corollary 2.3 we obtain an isomorphism
\[
(\pi_N)_!\varphi_f^N \to \varphi_f^N\{\dim(\pi)\}
\]
and taking compactly supported cohomology, passing to the limit, and taking duals, the isomorphism (22).

We make the further assumption that the Euler characteristic of \( \pi \) is not a zero divisor in \( H_{c,G}(Y,\varphi_f)^* \) for the extended action introduced in the previous section, and define the pushforward map associated to \( \pi \) to be
\[
\pi_* := (\pi^*)^{-1} \cdot \mathbf{cu}(\pi)^{-1} : H_{G,c}(X,\varphi_f)^* \to H_{G,c}(Y,\varphi_f)^*[\mathbf{cu}(\pi)^{-1}].
\]

If instead \( p : X \to Y \) is a proper map of manifolds, then later in (38) we use a pushforward
\[
p_* : H_c(X,\varphi_f)^* \to H_c(Y,\varphi_f)^*
\]
defined via the maps
\[
\varphi_f(Q_Y \to p_*Q_X)
\]
and the isomorphism \( \varphi_f(p_*Q_X) \cong p_*\varphi_{fp}Q_Y \) of Corollary 2.3 using that \( p_* \cong p_l \) since \( p \) is proper.

Taking Verdier duals we obtain maps
\[
\varphi_f(p_*Q_X) \to Q_Y\{\dim(Y) - \dim(X)\}.
\]
inducing what we will refer to as the pullback map
\[
p^* : H_c(Y,\varphi_f)\{\dim(Y) - \dim(X)\}^* \to H_c(X,\varphi_{fp})^*.
\]
Proposition 2.10. Let

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow j & & \downarrow j' \\
Y & \xrightarrow{h} & Y'
\end{array}
\]

be a Cartesian diagram of complex manifolds, in which all maps are affine fibrations, or surjective and proper, or inclusions of closed submanifolds. If \( j' \) and \( h \) are inclusions, we furthermore assume that \( Y \) and \( X' \) intersect transversally in \( Y' \). Let \( f \) be a holomorphic function on \( Y' \). Then the following diagram commutes

\[
\begin{array}{ccc}
H^c_c(X, \varphi_{fj'g})^* & \xrightarrow{g^*} & H^c_c(X', \varphi_{fj'})^* \\
\downarrow j^* & & \downarrow j'^* \\
H^c_c(Y, \varphi_{fh})^* & \xrightarrow{h^*} & H^c_c(Y', \varphi_f)^*.
\end{array}
\]

Proof. If \( h \) is an affine fibration, then since (25) is Cartesian, we have \( j^* \mathfrak{e}u(h) = \mathfrak{e}u(g) \) and so it is enough to prove that the diagram

\[
\begin{array}{ccc}
H^c_c(X, \varphi_{fj'g})^* & \xleftarrow{g^*} & H^c_c(X', \varphi_{fj'})^* \\
\downarrow j^* & & \downarrow j'^* \\
H^c_c(Y, \varphi_{fh})^* & \xleftarrow{h^*} & H^c_c(Y', \varphi_f)^*.
\end{array}
\]

commutes, which follows from the commutativity of (25). Similarly, if \( j' \) is an affine fibration, it is sufficient to prove that

\[
\begin{array}{ccc}
H^c_c(X, \varphi_{fj'g})^* & \xrightarrow{g^*} & H^c_c(X', \varphi_{fj'})^* \\
\downarrow j^* & & \downarrow j'^* \\
H^c_c(Y, \varphi_{fh})^* & \xrightarrow{h^*} & H^c_c(Y', \varphi_f)^*.
\end{array}
\]

commutes, which again follows directly from the commutativity of (25). So now assume that \( h \) and \( j' \) are proper.

First we claim that

\[
\begin{array}{ccccc}
j'_* \mathbb{Q}_X^Y & \longrightarrow & \mathbb{Q}_Y^Y & \longrightarrow & \mathbb{Q}_Y^Y \\
\downarrow & & \downarrow & & \downarrow \\
j'_* \mathbb{Q}_X & \longrightarrow & j'_* \mathfrak{e}u \mathbb{Q}_X & \longrightarrow & \mathfrak{e}u h_* \mathbb{Q}_Y
\end{array}
\]

commutes. By base change we may assume that \( X' \) is a point, and \( j' \) is surjective, in which case the vertical maps in (26) are isomorphisms and commutativity is clear.
Since (25) is Cartesian, and if it is the inclusion of two closed submanifolds they intersect transversally, we have the natural isomorphism
\[ j'_*(\omega_X) \otimes \omega_{Y'}^{-1} \cong h_*(j_*\omega_X) \otimes \omega_Y^{-1} \]
and so tensoring (26) by \( v := j'_*(\omega_X \otimes \omega_Y^{-1}) \) we obtain the commuting diagram
\[ \begin{array}{ccc}
Q_X & \rightarrow & h_*Q_Y \\
\downarrow & & \downarrow \\
Q_X' \otimes v & \rightarrow & Q_X' \otimes v \\
\end{array} \]
It follows that
\[ \phi(f)Q_X' \rightarrow \phi(h)_*Q_Y \\
\phi(f)(Q_X' \otimes v) \rightarrow \phi(f)(Q_X' \otimes v) \\
\phi(f)(Q_X' \otimes v) \rightarrow \phi(f)(Q_X' \otimes v) \]
commutes, where here we have used Proposition 2.2 for commutativity of the bottom square. Taking dual compactly supported cohomology of the outer diagram implies commutativity of (25). □

In extending to the equivariant case we have to take care of \( G \) actions on the relative Verdier dual \( p^!Q_Y \). In [24] we have used the trivialization of the dualizing sheaves to identify
\[ \tau^!Q \otimes p_*p^!\tau^!Q^{-1} \cong \mathbb{Q}\{\dim(Y) - \dim(X)\}, \]
where \( \tau \) is the structure morphism for \( Y \). However the trivializations are not in general liftable to the equivariant category, and we have in the \( G \)-equivariant case
\[ \tau^!Q \otimes p_*p^!\tau^!Q^{-1} \cong \mathfrak{o}(p)\{\dim(Y) - \dim(X)\} \]
where \( \mathfrak{o}(p) \) is the sheaf of relative orientations of \( p \).

**Proposition 2.11.** Let \( g : X \rightarrow Y \) be a \( G \)-equivariant map of manifolds. Then
\begin{itemize}
  \item If \( g \) is a closed embedding, the map \( g^*g_* : H_{c,G}(X,\varphi_{fg})^* \rightarrow H_{c,G}(X,\varphi_{fg})^* \) is given by multiplication by the Euler class of the normal bundle \( N_{X/Y} \)
  \item If \( g \) is a smooth proper map, and the Euler class of \( \mathfrak{o}(g) \) is a non zero divisor in \( H_G(pt) \), then the induced map
    \[ g^*g_* : H_{c,G}(X,\varphi(f))^* \rightarrow H_{c,G}(X,\varphi(f))^* \otimes_{H_G(X,Q)} H_G(X,Q)[\mathfrak{o}(\varphi(g))^{-1}] \]
    is given by division by \( [\mathfrak{o}(g)] \).
\end{itemize}
3. Cohomological Hall algebra

3.1. Spaces of quiver representations. Let $Q$ be a quiver with potential $W \in \mathbb{C} Q/[\mathbb{C} Q, \mathbb{C} Q]$. We call such a pair $(Q, W)$ a QP from now on. For $\gamma \in \mathbb{N} Q_0$ denote by $M_{Q, \gamma}$ the affine space 
\[ \bigoplus_{a \in \mathbb{Q}_1} \text{Hom}(\mathbb{C}^{\gamma_s(s(a))}, \mathbb{C}^{\gamma_t(t(a))}). \]
This is considered in a natural way as a space of representations of $Q$, or a space of left $\mathbb{C} Q$-modules. If $\gamma_1, \gamma_2 \in \mathbb{N} Q_0$ are a pair of dimension vectors, denote by $M_{Q, \gamma_1, \gamma_2}$ the affine space 
\[ \bigoplus_{a \in \mathbb{Q}_1} \{ f_a \in \text{Hom}(\mathbb{C}^{\gamma_{1s}(s(a))} \oplus \mathbb{C}^{\gamma_{2s}(s(a))}, \mathbb{C}^{\gamma_1(t(a))} \oplus \mathbb{C}^{\gamma_2(t(a))}) \mid f_a(\mathbb{C}^{\gamma_{2s}(s(a))}) \subset \mathbb{C}^{\gamma_2(t(a))} \}. \]
In what follows, if the QP $(Q, W)$ is fixed, we will abbreviate $M_{Q, \gamma}$ to $M_{\gamma}$ and abbreviate $M_{Q, \gamma_1, \gamma_2}$ to $M_{\gamma_1, \gamma_2}$.

Define $G_{\gamma} := \prod_{i \in \mathbb{Q}_0} \text{GL}_C(\gamma(i))$ and $G_{\gamma_1, \gamma_2} := \prod_{i \in \mathbb{Q}_0} \text{GL}_C(\gamma_1(i), \gamma_2(i))$, where $\text{GL}_C(m, n)$ is the subgroup of $\text{GL}_C(m + n)$ preserving the flag $0 \subset \mathbb{C}^n \subset \mathbb{C}^{m+n}$. On each $M_{\gamma}$ there is a function $\text{tr}(W)_{\gamma}$, which is invariant with respect to the action of $G_{\gamma}$.

We define in the same way the function $\text{tr}(W)_{\gamma_1, \gamma_2}$ on $M_{\gamma_1, \gamma_2}$, which is again invariant with respect to the $G_{\gamma_1, \gamma_2}$ action. There is a natural projection $p : M_{\gamma_1, \gamma_2} \to M_{\gamma_1} \times M_{\gamma_2}$ and an inclusion $\eta : M_{\gamma_1, \gamma_2} \to M_{\gamma_1+\gamma_2}$ and we have 
\[ \text{tr}(W)_{\gamma_1, \gamma_2} = p^*(\text{tr}(W)_{\gamma_1} \oplus \text{tr}(W)_{\gamma_2}) \]
\[ = \eta^* \text{tr}(W)_{\gamma_1+\gamma_2}. \]
Throughout this section we assume that we are given subspaces $M^\text{sp}_{\gamma} \subset M_{\gamma}$.

Assumption 3.1. These subspaces are required to satisfy the property that there is an equality 
\[ M^\text{sp, ext} \gamma_1, \gamma_2 := p^{-1}(M^\text{sp}_{\gamma_1} \cap \text{crit}(\text{tr}(W)_{\gamma_1}) \times (M^\text{sp}_{\gamma_2} \cap \text{crit}(\text{tr}(W)_{\gamma_2}))) = \]
\[ M^\text{sp} \gamma_1, \gamma_2 : = \eta^{-1}(M^\text{sp}_{\gamma_1+\gamma_2} \cap \text{crit}(\text{tr}(W)_{\gamma_1+\gamma_2})). \]
We will also assume that $0 \in M^\text{sp}_{\gamma_0}$, which is equivalent to not all of the $M^\text{sp}_{\gamma}$ being empty.

For convenience we assume that $M^\text{sp}_{\gamma} \cap \text{crit}(\text{tr}(W)_{\gamma}) \subset \text{tr}(W)^{-1}(0)$.

Remark 3.2. This last requirement can be relaxed at the expense of some extra minor complications, but none of the applications we’re interested in will require this level of generality.

Now define 
\[ H^\text{sp}_{Q, W, \gamma} := H_{c, G_{\gamma}}(M^\text{sp}_{\gamma}, \varphi_{\text{tr}(W)_{\gamma}}) \{ \chi(\gamma, \gamma)/2 \}^*, \]
where the twist is as defined in \[ \mathbb{S} \] and 
\[ H^\text{sp}_{Q, W} := \bigoplus_{\gamma \in \mathbb{Q}_0} H^\text{sp}_{Q, W, \gamma}. \]
In the case in which all $M^\text{sp}_{Q, W, \gamma} = M_{Q, W, \gamma}$ we denote $H^\text{sp}_{Q, W}$ by $H_{Q, W}$. The space $H^\text{sp}_{Q, W, \gamma}$ carries the shifted cohomological grading, i.e. it is a $\mathbb{Z}_{ac} := \mathbb{Z}$-graded object, where the grading is the
usual cohomological grading on \( H_{c,G}(M_{\gamma}^{\text{sp}}, \varphi_{\tr(W)\gamma})^* \) shifted by \( \chi(\gamma, \gamma) \). As noted, we use the subscript \( \text{sc} \) when we take account of the \( \{\chi(\gamma, \gamma)/2\} \) twist.

**Remark 3.3.** As an aside we explain the representation-theoretic origin of \( H_{c,G}(M_{\gamma}^{\text{sp}}, \varphi_{\tr(W)\gamma}) \).

Firstly, given a quiver \( Q \), and a potential \( W = \sum_{m \in S} \lambda_m a_m \) for it, where each of the \( a_m \) is a scalar, the noncommutative derivative \( \partial W/\partial a \) is defined by setting

\[
\partial W/\partial a := \sum_{m \in S} \lambda_m \sum_{a_m = uav | u,v \text{ words in } Q} vu.
\]

Then the Jacobi algebra for the QP \((Q, W)\) is defined by

\[
J(Q, W) := \mathbb{C}Q/\langle \partial W/\partial a | a \in Q_1 \rangle.
\]

Representations of the Jacobi algebra form a Zariski closed subscheme \( V_{\gamma} \) of \( M_{Q,\gamma} \) in a natural way, and we have the equality of subschemes

\[
V_{\gamma} = \text{crit} (\text{tr}(W)\gamma).
\]

In addition, the vanishing cycle complex \( \varphi_{\tr(W)\gamma} \) is supported on the critical locus of \( \text{tr}(W)\gamma \).

So the compactly supported equivariant cohomology \( H_{c,G}(M_{\gamma}^{\text{sp}}, \varphi_{\tr(W)\gamma}) \) can be thought of as the compactly supported equivariant cohomology of the space of \( \gamma \)-dimensional representations of \( J(Q, W) \), with coefficients in the vanishing cycle complex.

### 3.2. Multiplication operations.

As above, all functors will be considered as derived functors.

Consider the affine fibrations

\[
p_{\gamma_1,\gamma_2,N} : (M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N \to (M_{\gamma_1} \times M_{\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N
\]

these induce isomorphisms

\[
\varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2} (\mathcal{Q}_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N}) \to (p_{\gamma_1,\gamma_2,N})_*(\mathcal{Q}_{(M_{\gamma_1} \times M_{\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N})
\]

and via Verdier duality

\[
\varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2} (\mathcal{Q}_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N}) \to (p_{\gamma_1,\gamma_2,N})_*(\mathcal{Q}_{(M_{\gamma_1} \times M_{\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N})
\]

or

\[
\varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2} (\mathcal{Q}_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N}) \to (p_{\gamma_1,\gamma_2,N})_*(\mathcal{Q}_{(M_{\gamma_1} \times M_{\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N})
\]

where \( l = \sum_{a \in Q_1} \gamma_1(t(a))\gamma_2(s(a)) \), using (11). By Corollary 2.3 we obtain isomorphisms

\[
(p_{\gamma_1,\gamma_2,N})!\varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2} (\mathcal{Q}_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N}) \to (p_{\gamma_1,\gamma_2,N})_*(\mathcal{Q}_{(M_{\gamma_1} \times M_{\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N})\{l\}
\]

Passing to the limit and taking compactly supported cohomology, we arrive at an isomorphism

\[
\alpha^* : H_{c,G_{\gamma_1} \times G_{\gamma_2}} (M_{\gamma_1,\gamma_2}^{\text{sp}}, \varphi_{\tr(W)\gamma_1 \gamma_2}) \to H_{c,G_{\gamma_1} \times G_{\gamma_2}} (M_{\gamma_1}^{\text{sp}} \times M_{\gamma_2}^{\text{sp}}, \varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2})\{l\},
\]

though in the end we work with

\[
\alpha : H_{c,G_{\gamma_1} \times G_{\gamma_2}} (M_{\gamma_1}^{\text{sp}} \times M_{\gamma_2}^{\text{sp}}, \varphi_{\tr(W)\gamma_1 \boxplus \tr(W)\gamma_2})\{l\} \to H_{c,G_{\gamma_1} \times G_{\gamma_2}} (M_{\gamma_1,\gamma_2}^{\text{sp}}, \varphi_{\tr(W)\gamma_1 \gamma_2})^*,
\]
the dual map, as \(H_{Q,W}^{sp}\) is defined in terms of dual spaces of compactly supported equivariant cohomology. In the terminology of Section 2.7, these maps are induced by the pullback maps associated to 28.

Similarly, the affine fibrations

\[
q_{\gamma_1,\gamma_2,N} : (M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N \to (M_{\gamma_1,\gamma_2}, G_{\gamma_1,\gamma_2})_N
\]

induce maps

\[
(q_{\gamma_1,\gamma_2,N})! \varphi_{tr(W)_{\gamma_1,\gamma_2,N}} : \left(Q_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1} \times G_{\gamma_2})_N}\right) \to \varphi_{tr(W)_{\gamma_1,\gamma_2,N}}(Q_{(M_{\gamma_1,\gamma_2}, G_{\gamma_1,\gamma_2})_N}) \{l'\}
\]

where

\[
l' = \sum_{a \in Q_1} \gamma_1(s(a)) \gamma_2(t(a)).
\]

Taking compactly supported cohomology and taking duals we obtain pullback isomorphisms

\[
\beta : H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1,\gamma_2}}) \{l'\}^* \to H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1,\gamma_2}})^*.
\]

Consider the projections

\[
pr_{\gamma_1,\gamma_2,N} : (M_{\gamma_1,\gamma_2})_N \to (M_{\gamma_1}, G_{\gamma_2})_N.
\]

The natural transformation of functors

\[
\varphi_{tr(W)_{\gamma,N}} \to (pr_{\gamma_1,\gamma_2,N})! \varphi_{tr(W)_{\gamma,N}}(pr_{\gamma_1,\gamma_2,N})^*
\]

applied to \(Q_{(M_{\gamma_1}, G_{\gamma_2})_N}\) induces a map

\[
\delta^* : H_{c,G_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}}) \to H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}})
\]

or, taking duals

\[
\delta : H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}})^* \to H_{c,G_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}})^*.
\]

the pushforward map associated to the maps (36).

Next, consider the inclusions \(i_{\gamma_1,\gamma_2,N} : (M_{\gamma_1,\gamma_2}^{sp}, G_{\gamma_1,\gamma_2})_N \to (M_{\gamma_1}^{sp}, G_{\gamma_1,\gamma_2})_N\). These induce maps

\[
\varphi_{tr(W)_{\gamma,N}} \to (i_{\gamma_1,\gamma_2,N})_*(i_{\gamma_1,\gamma_2,N})^* \varphi_{tr(W)_{\gamma,N}}
\]

and maps

\[
\zeta^* : H^*_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}}) \to H^*_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma}})
\]

or, again taking duals,

\[
\zeta : H^*_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma}})^* \to H^*_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma}^{sp}, \varphi_{tr(W)_{\gamma}})^*.
\]

Finally, the natural transformation

\[
\varphi_{tr(W)_{\gamma,N}} \to (i_{\gamma_1,\gamma_2,N})_*(\varphi_{tr(W)_{\gamma_1,\gamma_2,N}}(i_{\gamma_1,\gamma_2,N})^*)
\]

induces maps

\[
e : H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1,\gamma_2}})^* \to H_{c,G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma}})^*
\]

as in the definition of \(\delta\).
Throughout we have used that \( pr_{\gamma_1, \gamma_2, N} \) and \( i_{\gamma_1, \gamma_2, N} \) are proper in order to identify \( (pr_{\gamma_1, \gamma_2, N})_* \) with \( (pr_{\gamma_1, \gamma_2, N})! \) and \( (i_{\gamma_1, \gamma_2, N})_* \) with \( (i_{\gamma_1, \gamma_2, N})! \).

We define a product
\[
m : H_{Q,W}^{sp}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \otimes H_{Q,W}^{sp}(M_{\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_2}}) \to H_{Q,W}^{sp}(M_{\gamma_1}^{sp} \times M_{\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}} \oplus tr(W)_{\gamma_2})\]
to be the map which, when restricted to \( H_{Q,W, \gamma_1}^{sp} \otimes H_{Q,W, \gamma_2}^{sp} \), is given by the composition of maps \( \delta \zeta \epsilon \beta^{-1} \alpha TS \), where \( TS \) is the Thom–Sebastiani isomorphism
\[
H_{c,G, \gamma_1}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ \chi(\gamma_1, \gamma_1) \}^* \otimes H_{c,G, \gamma_2}(M_{\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_2}}) \{ \chi(\gamma_2, \gamma_2) \}^* \to
H_{c,G \times \gamma_2}(M_{\gamma_1}^{sp} \times M_{\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}} \oplus tr(W)_{\gamma_2}) \{ \chi(\gamma_1, \gamma_1) + \chi(\gamma_2, \gamma_2) \}^*.
\]

**Proposition 3.4** ([19]). *The map \( \epsilon \zeta \) is an isomorphism.*

This is the Hodge theoretic version of the “integral identity” from [18], proved as Theorem 13 of [19].

**Remark 3.5.** The map \( \zeta \epsilon \) is the pushforward
\[
\zeta : H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \to H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}})\]
associated to \( i_{\gamma_1, \gamma_2} \). We have split it into two maps as in [19]. In subsequent sections we will just use \( \zeta \).

**Remark 3.6.** All the maps in the definition of \( H_{Q,W}^{sp} \), apart from the Thom–Sebastiani isomorphism also are maps of monodromic mixed Hodge structures. So depending on the reader's preference regarding Remark 2.4, the CoHA \( H_{Q,W}^{sp} \) can be considered as an algebra object in the category MMHS.

For the benefit of the reader we represent the multiplication in a different way:

\[
\begin{array}{c}
H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ \chi_Q(\gamma_2, \gamma_1) \}^* \xrightarrow{\zeta} H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ - \chi_Q(\gamma_2, \gamma_1) \}^* \\
H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ \chi_Q(\gamma_2, \gamma_1) \}^* \xrightarrow{\beta^{-1}} H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ - \chi_Q(\gamma_2, \gamma_1) \}^* \\
H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ \chi_Q(\gamma_2, \gamma_1) \}^* \xrightarrow{\alpha} H_{c,G, \gamma_1, \gamma_2}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}}) \{ - \chi_Q(\gamma_2, \gamma_1) \}^* \\
\end{array}
\]

where we have abbreviated the notation in the obvious ways, and we use \( \chi_Q(\gamma_2, \gamma_1) = \gamma' - \gamma \).

For each \( \gamma \in \mathbb{Z}Q_0 \) we have an inclusion \( \mathbb{C}^* \to G_{\gamma} \) given by \( z \to (z \cdot id_{(i) \times \gamma(i)})_{i \in Q_0} \). It follows that \( M_{\gamma_1}^{sp}, M_{\gamma_1}^{sp} \times M_{\gamma_2}^{sp} \) and \( M_{\gamma_1, \gamma_2}^{sp} \) carries a \( \mathbb{C}^* \)-action, which can be seen to be trivial, and so each of the constituent terms of the above diagram are free \( \mathbb{C}[y] \)-modules. Since the underlying maps of spaces are \( \mathbb{C}^* \)-equivariant, we deduce
Proposition 3.7. The multiplication $m$ is a $\mathbb{C}[y]$-module homomorphism between free $\mathbb{C}[y]$-modules.

Note that for $\tau \otimes \tau' \in H_{c,G_{\gamma_1}}(M_{\gamma_1}^{sp}, \varphi_{U(W)_{\gamma_1}})^* \otimes H_{c,G_{\gamma_2}}(M_{\gamma_2}^{sp}, \varphi_{U(W)_{\gamma_2}})^*$ we have

$$y(\tau \otimes \tau') = ((y\tau) \otimes \tau') + (\tau \otimes (y\tau')).$$

4. Generalities on localised comultiplications

4.1. Self-dual dimension vectors. Let $(Q, W)$ be a QP. We form a new QP $(Q, W)^T := (Q^T, W^T)$ by reversing all the arrows in $Q$ and reversing the order of the cyclic words in $W$. A self-duality structure for $(Q, W)$ is an isomorphism of quivers $\sigma : Q \to Q^T$ such that

$$\sigma^* W^T = \lambda W$$

for some scalar $\lambda$.

Proposition 4.1. If $\sigma$ is a self-duality structure for the QP $(Q, W)$, then it is a self-duality structure for $(Q, W')$, where $\sigma^*(W - W') = \lambda(W - W')$ for $\lambda$ as in (41). In particular, $\sigma$ is a self-duality structure for $(Q, 0)$.

We say a dimension vector $\gamma$ is self-dual if $\sigma^* \gamma = \gamma$. The set of self-dual vectors is closed under addition, and self-dual vectors satisfy the relation

$$\chi(\gamma, \gamma') = \chi(\gamma', \gamma).$$

Example 4.2. Consider the quiver $Q_2$

with superpotential

$$W_{2} = z_1y_2x_3 + z_2y_3x_1 + z_3y_1x_2 - z_1y_3x_2 - z_2y_1x_3 - z_3y_2x_1.$$

It is easy to construct an isomorphism $\sigma : Q_{2} \to Q_{2}^T$ swapping vertices 2 and 3, with $\sigma^* W_{2}^T = W_2$. The group of self dual dimension vectors is the set of $\gamma \in \mathbb{Z}^{Q_0}$ such that $\gamma(2) = \gamma(3)$.

The CoHA of self dual vectors $\mathcal{H}_{Q,W}^{sp,\sigma}$ is formed by setting

$$M_{Q,W,\gamma}^{sp,\sigma} := \begin{cases} M_{Q,W,\gamma}^{sp} & \text{if } \sigma^* \gamma = \gamma \\ \emptyset & \text{if } \sigma^* \gamma \neq \gamma. \end{cases}$$

Note that 0 is always a self-dual vector, so $\mathcal{H}_{Q,W,0}^{sp,\sigma} = \mathbb{C}$ and $\mathcal{H}_{Q,W}^{sp,\sigma}$ is a unital associative algebra.

The following proposition is a result of [12] and the Tate twist [27] in the definition of the critical cohomological Hall algebra. The associativity is proved in [19, Sec.2.3].
Proposition 4.3. The critical CoHA $\mathcal{H}_{Q,W}^{ps}$ is a $\mathbb{Z}_{Q_0} \oplus \mathbb{Z}_{ac}$-graded associative algebra object in the category of monodromic mixed Hodge structures.

4.2. $Q$-localised comultiplications in Tannakian categories. In this section we state a very general supercommutative freeness result. We state it in this level of generality because it gives us enough flexibility to deal with many different types of CoHAs at once (critical CoHAs, noncritical CoHAs, associated graded CoHAs, etc), and also points the way to the extension of the results of this paper to the non-self-dual case.

Recall that a rigid tensor category $\mathcal{C}$ is a symmetric monoidal category with a monoidal unit $1$, such that all objects admit duals, i.e. for all $X \in \mathcal{C}$ there is an internal Hom object $\text{Hom}(X, 1)$. Let $\mathcal{C}$ be a rigid Abelian $\mathbb{Q}$-linear tensor category with an exact faithful $\mathbb{Q}$-linear tensor functor $\text{fib}: \mathcal{C} \to \text{Vect}$ to the tensor category of $\mathbb{Q}$-vector spaces, i.e. a Tannakian category. Out of the category $\mathcal{C}$ we form a new $\mathbb{Z}_{Q_0}$-graded braided monoidal category $\mathcal{C}_Q$ of $\mathbb{Z}_{Q_0}$-graded objects in $\mathcal{C}$, for which the monoidal structure is twisted by setting

$$a \otimes b = (-1)^{\chi(\gamma_1, \gamma_2)} b \otimes a$$

for $a$ and $b$ of $\mathbb{Z}_{Q_0}$-degree $\gamma_1$ and $\gamma_2$ respectively.

Let $n = Q_0$ and set $A_\gamma = \mathbb{C}[x_{1,1}, \ldots, x_{1,\gamma(1)}, \ldots, x_{n,1}, \ldots, x_{n,\gamma(n)}]^{\text{Sym}_\gamma}$ and

$$A_{\gamma_1, \gamma_2} = \left(\mathbb{C}[x_{1,1}, \ldots, x_{1,\gamma(1)}, \ldots, x_{n,1}, \ldots, x_{n,\gamma(n)}] \left[ \prod_{i,j \in \mathbb{Q}_0} \prod_{a=1}^{\gamma_1(i)} \prod_{b=1}^{\gamma_2(i)} (x_{i,a} - x_{j,\gamma_1(j) + b})^{-1} \right]^{\text{Sym}_\gamma} \right)_{\gamma_1, \gamma_2}.$$

Definition 4.4. A $Q$-localised bialgebra object in $\mathcal{C}$ is the data of an algebra object $(\tilde{B}, \tilde{m}, \nu: 1 \to B)$ in $\mathcal{C}_Q$ and an $A_\gamma$ module structure for each $\tilde{B}_\gamma$, along with morphisms

$$\tilde{B}_\gamma \rightarrow \tilde{B}_{\gamma_1} \otimes \tilde{B}_{\gamma_2} \otimes A_{\gamma_1, \gamma_2}$$

for all decompositions $\gamma = \gamma_1 + \gamma_2$. We require also that the morphisms

$$(\tilde{B}_{\gamma_1} \otimes \tilde{B}_{\gamma_2}) \rightarrow (\tilde{B}_{\gamma_1} \otimes \tilde{B}_{\gamma_2}) \otimes A_{\gamma_1, \gamma_2}$$

are injective for all $\gamma = \gamma_1 + \gamma_2$, and that we are given an algebra structure $\tilde{m} \otimes \tilde{m}$ on $\bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Q}_0} \tilde{B}_{\gamma_1} \otimes \tilde{B}_{\gamma_2} \otimes A_{\gamma_1, \gamma_2}$ extending the algebra $(\tilde{B} \otimes \tilde{B}, (m \otimes m) \circ (\text{id} \otimes \text{sw}_{C_0} \otimes \text{id}))$, such that the following diagram commutes

$$\begin{align*}
\tilde{B}_{\gamma_1} \otimes \tilde{B}_{\gamma_2} &\xrightarrow{\Delta \otimes \Delta} (\tilde{B}_{\gamma_1'} \otimes \tilde{B}_{\gamma_2'} \otimes \tilde{B}_{\gamma_3} \otimes \tilde{B}_{\gamma_4}) \otimes A_{\gamma_1, \gamma_2} (A_{\gamma_1', \gamma_2'} \otimes A_{\gamma_3, \gamma_4'}) \\
\tilde{B}_{\gamma} &\xrightarrow{\Delta} (\tilde{B}_{\gamma_1^0} \otimes \tilde{B}_{\gamma_2^0}) \otimes A_{\gamma_1^0, \gamma_2^0} (\tilde{m} \otimes \tilde{m} \circ (\text{id} \otimes \text{sw}_{\gamma_1', \gamma_3'} \otimes \text{id}))
\end{align*}$$

where

$$\gamma_1 = \gamma_1' + \gamma_2'$$

$$\gamma_2 = \gamma_3' + \gamma_4'$$

$$\gamma_1^0 = \gamma_1' + \gamma_3'$$
\[ \gamma_0^0 = \gamma_2' + \gamma_4', \]
and \( \cdot \bar{\mathcal{E}}_{\gamma_2', \gamma_3'} \) is multiplication by
\[
\prod_{i,j \in Q_0} \left( \prod_{m=1}^{\gamma_j'} \prod_{m'=0}^{\gamma_2'} (x_i, \gamma_j'+m - x_j, \gamma_2'+m')^{b_{ij}} (x_j, \gamma_2'+m' - x_j, \gamma_j'+m')^{b_{ji}} \right) (-1)^{\chi(\gamma_2', \gamma_3')}.
\]

If \( \mathcal{E}_{\gamma_2', \gamma_3'} \) is not a polynomial in the variables \( x_{i,t} \), we make the further assumption that \( \bar{\gamma} \bar{\gamma} \) factors through the injective localisation
\[
(\bar{B}_{i_1} \otimes \bar{B}_{i_2} \otimes \bar{B}_{i_3} \otimes \bar{B}_{i_4}) \otimes A, \quad (A_{i_1, \gamma_{i_2}'} \otimes A_{i_3, \gamma_4'}) \to (\bar{B}_{i_1} \otimes \bar{B}_{i_2'} \otimes \bar{B}_{i_3} \otimes \bar{B}_{i_4'}) \otimes A, \quad (A_{i_1, \gamma_{i_2}'})
\]
in order to make sense of the rightmost vertical map in (45).

**Proposition 4.5.** Let \( \sigma : Q \to Q^T \) be a self-duality structure for the QP \((Q,0)\), and let \( \tilde{B} \) be a Q-localised bialgebra object in the category \( \mathcal{C} \) with commutative underlying algebra. Assume
- \( B := \text{fib}(\tilde{B}) \) is connected and positively graded with respect to the \( \mathbb{Z}^{Q_0} \)-grading, i.e. \( B_{\gamma} = 0 \) for all \( \gamma \) such that \( \gamma(i) < 0 \) for some \( i \in Q_0 \), and \( B_0 = Q \).
- Let \( B = B_0 \oplus B^+ \) be the graded decomposition. For all \( a \in B^+, \Delta(a) = a \otimes 1 + 1 \otimes a + b \), where \( b \) is strictly positively graded with respect to the \( \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{Q_0} \)-grading, i.e. its \((0,\gamma)\) and \((\gamma,0)\)-graded pieces are zero for all \( \gamma \in \mathbb{Z}^{Q_0} \).
- \( B_{\gamma} = 0 \) for all \( \gamma \) such that \( \sigma(\gamma) \neq \gamma \).

Then \( \text{fib}(B) \) is a free \( \mathcal{C} \)-supercommutative algebra. Assume that Image(\( \tilde{B}^+ \oplus \tilde{B}^+ \to \tilde{B} \)) \( \to \tilde{B} \) splits in \( \mathcal{C} \). Then \( \tilde{B} \) is a free commutative algebra in \( \mathcal{C}_Q \).

Note that the subcategory of \( \mathcal{C}_Q \) satisfying the third condition of Proposition 4.5 is automatically a symmetric monoidal \( \mathcal{C} \)-graded category, by the relation \( \chi(\gamma_1, \gamma_2) = \chi(\gamma_2, \gamma_1) \) for self dual \( \gamma_1 \) and \( \gamma_2 \), and the sign change rule (43). Commutativity for an algebra in the subcategory of \( \mathcal{C}_Q \) satisfying this condition just means that the multiplication commutes with the symmetrizing morphism.

**Proof.** Under the assumption that \( B_{\gamma} = 0 \) unless \( \gamma \) is self dual, we have
\[ \chi(\gamma_1, \gamma_2) = |\gamma_1|:|\gamma_2| \]
and so \( \mathcal{C} \)-supercommutativity of \( \text{fib}(\tilde{B}) \) follows. Under the same assumption, we may rewrite (46) as
\[ \prod_{i,j \in Q_0} \left( \prod_{m=1}^{\gamma_j'} \prod_{m'=0}^{\gamma_2'} (x_i, \gamma_j'+m - x_j, \gamma_2'+m')^{b_{ij}} (x_j, \gamma_2'+m' - x_j, \gamma_j'+m')^{b_{ji}} \right) (-1)^{\chi(\gamma_2', \gamma_3')} = 1. \]

The proof of the first part of the proposition is then an easy variation of the standard proof of a theorem of Borel, which we recall in this adjusted context.
First, we take the filtration
\[(48) \quad B \supset F_1(B) := B^+ \supset F_2(B) := (B^+)^2 \supset \ldots\]
and let \(\text{Gr}(B)\) be the associated graded algebra object - note that by our assumptions, the filtration \((48)\) is exhaustive. Taking the associated graded \(\text{Gr}(B)\) introduces a new grading with values in \(\mathbb{Z}_{\text{pd}} := \mathbb{Z}\) (\(\text{pd}\) stands for polynomial degree, a convention justified by the fact, which we will prove, that \(\text{Gr}(B)\) is freely generated in \(\text{pd}\) degree one).

We write \((B \otimes B)_{\text{loc}} := \bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Z}^Q_0} B_{\gamma_1} \otimes B_{\gamma_2} \otimes A_{\gamma_1, \gamma_2}.\)

Now, define \(I_1\) to be the two-sided ideal generated by
\[(B^+ \otimes 1 + 1 \otimes B^+) \subset (B \otimes B)_{\text{loc}}\]
and \(I_2\) to be the two-sided ideal generated by
\[\bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Z}^Q_0 | \gamma_1, \gamma_2 \neq 0} B_{\gamma_1} \otimes B_{\gamma_2} \otimes A_{\gamma_1, \gamma_2} \subset (B \otimes B)_{\text{loc}}.\]

Define
\[F_i((B \otimes B)_{\text{loc}}) := \sum_{a_1 \ldots a_t \in \{1, 2\}^{s}} I_{a_1} \cdot \ldots \cdot I_{a_t} \subset (B \otimes B)_{\text{loc}}.\]

where the multiplication is with respect to the product \(m \otimes m.\) Let \(\text{Gr}((B \otimes B)_{\text{loc}})\) be the associated \(\mathbb{Z}_{\text{pd}}\)-graded algebra. By the second assumption of the proposition, \(\Delta(B^+) \subset (B \otimes B)^+\), and so we obtain a commutative diagram of \(\mathbb{Z}_{\text{pd}}\)-graded objects
\[(49) \quad \text{Gr}(B) \otimes \text{Gr}(B) \xrightarrow{\text{Gr} \otimes \text{Gr}} \text{Gr}((B \otimes B)_{\text{loc}}) \otimes \text{Gr}((B \otimes B)_{\text{loc}}) \xrightarrow{m \otimes m_{\text{Gr}}} \text{Gr}((B \otimes B)_{\text{loc}}).\]

Now let \(S\) be a \(C\)-graded basis for \(\text{Gr}^1(B)\). Such a basis provides a \(\mathbb{Z}_{\text{pd}}\)-graded map
\[C := \text{Sym}(x_s | s \in S) \xrightarrow{f} \text{Gr}(B),\]
where each variable \(x_s\) is placed in the same \(C\)-degree as \(s\), and in \(\mathbb{Z}_{\text{pd}}\)-degree 1. The morphism \(f\) is clearly surjective, since \(\text{Gr}(B)\) is generated in \(\mathbb{Z}_{\text{pd}}\)-degree 1.

Recall the definition of \(\mathcal{P}(\gamma_1, \gamma_2)\) from Section 1.2; it is the set \(\pi\) of \(Q_0\)-tuples of permutations \(\pi_i\) of \(\{1, \ldots, \gamma_1(i) + \gamma_2(i)\}\) preserving the ordering of \(\{1, \ldots, \gamma_1(i)\}\) and \(\{\gamma_1(i) + 1, \ldots, \gamma_1(i) + \gamma_2(i)\}\).

We will prove the following two claims by induction. For all \(t\)

1. The morphism \(f\) is an isomorphism in \(\mathbb{Z}_{\text{pd}}\)-degree \(\leq t.\)
2. In \(\mathbb{Z}_{\text{pd}}\)-degree \(\leq t\) the map \(f\) commutes with \(Q\)-localised coproducts, where for monomials \(x_{i_1} \ldots x_{i_t}\) we have
\[\Delta_{\text{Gr}}(x_{i_1} \ldots x_{i_t}) = \sum_{t_1 + t_2 = t} \sum_{\pi \in \mathcal{P}(t_1, t_2)} (-1)^{\pi} x_{\pi(1)} \ldots x_{\pi(t_1)} \otimes x_{\pi(t_1+1)} \ldots x_{\pi(t)} ,\]
with $\psi = \sum_{i<j\leq t}\xi_{\pi(i)\pi(j)}|x_i|\xi|_\psi|x_j|\xi$.

In the case $t = 1$, the first claim is a tautology, while the second follows from our assumptions on $B$. Now assume they are both true for $t' - 1$—we will show that they are also true for $t'$. It suffices to check the second claim on monomials of $\mathbb{Z}_{pd}$-degree $t'$, for which it follows from the commutativity of (49), the claim for $t = t' - 1$ and also $t = 1$. Now let $p \in C$ be a nonzero polynomial of polynomial degree $t'$. Pick one of the variables $x_s$ contained in $p$. We may rewrite $p = ux_s + v$

where $v$ does not contain $x_s$. By the second claim, the map $\Delta_{Gr}f$ factors through the inclusion $Gr(B \otimes B) \to Gr((B \otimes B)_{loc})$, since the coproduct for $C$ involves only polynomial terms. Furthermore, the $\bullet \otimes s$ coefficient of $\Delta_{Gr}(f(p))$ is given by $f((\partial u x_s/\partial x_s))$, which is nonzero as $\partial u x_s/\partial x_s \neq 0$, and by the inductive hypothesis, $f$ is injective on elements of $\mathbb{Z}_{pd}$-degree $\deg(\partial u x_s/\partial x_s) = t' - 1$. Since the localisation maps (44) are injective, it follows that $\Delta_{Gr}(f(p)) \neq 0$, and so $f(p) \neq 0$. We deduce both the claims, and in particular we deduce that $Gr(B)$ is isomorphic to a free supercommutative algebra. It follows that the map $C \to B$ is an isomorphism, since it is an isomorphism after passing to the associated graded algebras ($C$ is isomorphic to its own associated graded algebra), and the grading on $B$ is exhaustive.

Now let $F_2(\tilde{B}) \to F_1(\tilde{B}) \to V$

be the canonical short exact sequence in $C$, and assume it is split by the morphism $V \xrightarrow{\varrho} F_1(\tilde{B})$. There is a unital algebra morphism $\text{Sym}(V) \xrightarrow{\text{Sym}(\varrho)} \tilde{B}$ in $C$, and by the first part it projects to an isomorphism under $\text{fib}$. It follows from the faithfulness of $\text{fib}$ that $\text{Sym}(V) \cong \tilde{B}$ as algebra objects in $\mathcal{C}$.

4.3. The space of refined DT invariants. We retain the setup from the previous subsection, i.e. assume that $\tilde{B}$ is an algebra with a $Q$-localised comultiplication in a Tannakian category $\mathcal{C}$. By assumption, each $\tilde{B}_t$ carries an endomorphism which we denote

$$y := \sum_{i \leq t_0} \sum_{m=1}^{\gamma(i)} x_{i,m}.$$  

Each $B_{\gamma_1} \otimes B_{\gamma_2}$ carries a $y$ action, given by setting $(\cdot y)((\tau \otimes \tau') = ((\cdot y)\tau \otimes \tau') + (\tau \otimes (\cdot y)\tau')$. We abuse notation and denote by $\cdot y$ the induced endomorphism on $\text{fib}(B)$. In the sequel, we will be interested in generators for the CoHA $\mathcal{H}^{sp,\sigma}_{Q,W}$ which do not lie in the image of this endomorphism. To be precise, a choice of complement to the object $\cdot y(\tilde{V})$ in $\tilde{V}$ will give a choice of realisation of the space of $\mathcal{C}$-refined DT invariants.

**Proposition 4.6.** Let $\tilde{B}$ be a $Q$-localised bialgebra object in $\mathcal{C}$ satisfying (1), (2) and (3) from Proposition 4.5, and assume that $\text{fib}(B^+)$ is a free $\mathbb{C}[y]$-module, and that the multiplication and comultiplication are $\mathbb{C}[y]$-module maps. Then the short exact sequence

(50) \quad \text{Image}(B^+ \otimes B^+ \to B^+) \to B^+ \to V

splits in the category of $\mathbb{C}[y]$-modules, and so we have $V \cong V^{\text{prim}} \otimes \mathbb{C}[y]$ for some $\mathbb{C}$-graded subspace $V^{\text{prim}} \subset \mathcal{H}^{sp,\sigma}_{Q,W}$. If, in addition, we can lift the split short exact sequence (50) to the
category of objects of $\mathcal{C}$ equipped with an action by $\cdot y$, and the inclusion $\cdot y(\bar{V}) \to \bar{V}$ splits, then we may write $\bar{V} = \bar{V}^{\text{prim}}[y] \subset \mathcal{H}^{sp}_{Q,W}$, and $\mathcal{H}^{sp}_{Q,W} \cong \text{Sym}(\bar{V}^{\text{prim}} \otimes \mathbb{C}[y])$ as commutative algebra objects in $\mathcal{C}_Q$, and we say that $\bar{V}^{\text{prim}}$ is the space of $\mathcal{C}$-refined DT invariants.

Proof. Recall that we have an isomorphism of algebras $\text{Gr}(B) \to B$, and by our assumptions this is an isomorphism of $\mathbb{C}[y]$-modules. Elements $\tau$ in the $\mathbb{Z}_{pd}$-degree one piece of $\text{Gr}(B)$ are characterised by the equation

$$\Delta_{\text{Gr}}(\tau) = \tau \otimes 1 + 1 \otimes \tau.$$ 

Now say $y\tau$ satisfies this equation, then it is easy to see that $\tau$ must too, and so we deduce that $V$ can be given a free $\mathbb{C}[y]$-module structure, and the first statement follows. As in the proof of Proposition 1.5 the second statement is again a trivial consequence of the faithfulness of $\text{fib}$. \hfill $\square$

5. The $T$-equivariant critical CoHA

5.1. Twisted multiplications. Consider again the bilinear form $\chi : Z^{Q_0} \otimes Z^{Q_0} \to Z$ given by

$$\chi(\gamma_1, \gamma_2) = \sum_{i \in Q_0} \gamma_1(i)\gamma_2(i) - \sum_{a \in Q_1} \gamma_1(s(a))\gamma_2(t(a)).$$

We define a bilinear form on $Z^{Q_0} \oplus Z_{co}$ via

$$(51) \quad e'(((\gamma_1, k_1), (\gamma_2, k_2)) := \chi(\gamma_1, \gamma_2) + \chi(\gamma_1, \gamma_1)\chi(\gamma_2, \gamma_2) + k_1\chi(\gamma_2, \gamma_2) + k_2\chi(\gamma_1, \gamma_1).$$

This descends to a symmetric bilinear form $e'_2$ on $(\mathbb{Z}/2\mathbb{Z})^{Q_0} \oplus (\mathbb{Z}_{co}/2\mathbb{Z}_{co})$ such that

$$(52) \quad e'_2(((\gamma_1, k_1), (\gamma_2, k_2)) = \tau'((\gamma_1, k_1), (\gamma_2, k_2)) + \tau'((\gamma_2, k_2), (\gamma_1, k_1)).$$

We may twist the multiplication by setting $a \oplus b = (-1)^{\tau((\gamma_1, k_1), (\gamma_2, k_2))}a \cdot b$ for $a \in \mathcal{H}_{Q,W,\gamma_1}$ of $Z_{co}$-degree $k_1$ and $b \in \mathcal{H}_{Q,W,\gamma_2}$ of $Z_{co}$-degree $k_2$.

Remark 5.1. In [19] it is proved that $(a, b) \mapsto a \cdot b$ defines an associative product on $\mathcal{H}^{sp}_{Q,W}$. In general, if we have a graded associative product $\cdot$ on a $\mathbb{Z}^t$-graded vector space $A$, for some $t \in \mathbb{N}$, and $\tau$ is a bilinear form on $(\mathbb{Z}/2\mathbb{Z})^t$, the bilinearity of $\tau$ implies that the product $\ast$, that sends $(a, b) \mapsto (-1)^{\tau((a), (b))}a \cdot b$ for $a$ of homogeneous $\mathbb{Z}^t$-degree $[a]$ and $b$ of homogeneous $\mathbb{Z}^t$-degree $[b]$, is also associative.

Now we describe the standard twisting of the product on $\mathcal{H}^{sp}_{Q,W}$. Let $\epsilon$ now denote the bilinear form on $Z^{Q_0}$ defined by

$$\epsilon(\gamma_1, \gamma_2) := \chi(\gamma_1, \gamma_2) + \chi(\gamma_2, \gamma_1)$$

and define $\epsilon_2$ as the induced bilinear form on $(\mathbb{Z}/2\mathbb{Z})^{Q_0}$. Then again there is a $\tau$ such that

$$\tau(\gamma_1, \gamma_2) + \tau(\gamma_2, \gamma_1) = \epsilon_2(\gamma_1, \gamma_2)$$

and we may instead twist the multiplication on $\mathcal{H}_{Q,W}$ by setting $a \ast b = (-1)^{\tau(\gamma_1, \gamma_2)}a \cdot b$ for $a \in \mathcal{H}^{sp}_{Q,W,\gamma_1}$ and $b \in \mathcal{H}^{sp}_{Q,W,\gamma_2}$. 


In what follows we will focus mainly on the untwisted multiplication, we mention the twisted versions in order to make further contact with the existing literature. We finish this section by restating our main results in terms of each twist.

(1) For the untwisted multiplication $\cdot$, the algebra $\mathcal{H}_{Q,W}^{sp,\sigma}$ is free $C$-supercommutative, and admits a $Q$-localised comultiplication. The morphism $sw$ in $[45]$ is the twisted monoidal structure given by the Koszul sign rule $a \otimes b \mapsto (-1)^{|a||b|} e_b \otimes a$.

(2) We give $\mathcal{H}_{Q,W}^{sp,\sigma}$ a $C' := (\mathbb{Z}/2\mathbb{Z})^2$-grading by setting the degree of $a \in H^k_{c,G,\gamma}(M^{sp}_{Q,W,\gamma_1} \times \ldots \times M^{sp}_{Q,W,\gamma_m}, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu}$ to be $(k, \chi(\gamma, \gamma))$. Then $(\mathcal{H}_{Q,W}^{sp,\sigma}, \star)$ is free $C'$-supercommutative, and admits a $Q$-localised comultiplication.

(3) For the multiplication $\otimes$, the algebra $\mathcal{H}_{Q,W}^{sp,\sigma}$ is free supercommutative with respect to the $Z_{gc}$-grading, but does not in general admit a $Q$-localised comultiplication.

5.2. Localisation. Let $\mathcal{H}_{Q,W;\gamma_1}^{sp} \otimes \ldots \otimes \mathcal{H}_{Q,W;\gamma_m}^{sp}$ be an arbitrary finite tensor product of $\mathbb{N}Q_0$ graded pieces of the CoHA $\mathcal{H}_{Q,W}^{sp}$. Via the Thom Sebastiani isomorphism we may identify the vector spaces

$$\mathcal{H}_{Q,W;\gamma_1}^{sp} \otimes \ldots \otimes \mathcal{H}_{Q,W;\gamma_m}^{sp} \cong H_{c,G,\gamma_1 \times \ldots \times \gamma_m}(M^{sp}_{\gamma_1} \times \ldots \times M^{sp}_{\gamma_m}, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu},$$

where we have left out the shift in cohomology. Let $M := M^{sp}_{\gamma_1} \times \ldots \times M^{sp}_{\gamma_m}$ and $G := G_{\gamma_1} \times \ldots \times G_{\gamma_m}$. Let $c, d \in \{1, \ldots, m\}$ be a pair of distinct numbers, and let $i, i' \in Q_0$. Let

$$T = M \times \text{Hom}(\mathbb{C}^{\gamma_1}(i), \mathbb{C}^{\gamma_d}(i')).$$

Then $T$ is naturally a $G$-equivariant vector bundle over $M$, with projection $\pi : T \to M$, and Euler class

$$\text{cu}(i, i', c, d) := \pi_1(1).$$

Proposition 5.2. Multiplication by $\text{cu}(i, i', c, d)$ is an injective morphism on $\mathcal{H}_{Q,W;\gamma_1}^{sp} \otimes \ldots \otimes \mathcal{H}_{Q,W;\gamma_m}^{sp}$.

Proof. This is a small variation of the Atiyah Bott lemma [2]. In detail, let $S^1 \to G_{\gamma_i}$ be defined by $e^{i\theta} \mapsto e^{i\theta}$. Via the inclusion $G_{\gamma_i} \to G_{\gamma_1} \times \ldots \times G_{\gamma_m}$ this defines an action on the vector bundle $T$ such that the fixed point set is exactly $M$. Let $G' = G/S^1$. Then $H_{c,G}(M, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu}$ is filtered by

$$F_p(H_{c,G}(M, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu}) := \bigoplus_{G'} H_{c,G'}(M, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu} \otimes H_{S^1}(pt, \mathbb{Q}),$$

and we denote by $N$ the associated graded object, which is acted on freely by $H_{S^1}(pt, \mathbb{Q})$. Let

$$\tilde{\mu} \in H_{c,G}(M, \varphi_{tr(W)}{\gamma_1 \ldots \otimes \text{tr}(W)_{\gamma_m}})_{\mu},$$

and let $\mu \in N$ be the associated homogeneous element. Let $s$ equal the degree of $\mu$ with respect to the grading induced by $F$. Then projecting $\text{cu}(a, c, d)\mu$ onto its degree $s$ part, also with respect to the grading induced by $F$, it is given by $\text{cu}_{S^1}(a, c, d)\mu$, where now $\text{cu}_{S^1}(a, c, d)$ is the $S^1$-equivariant Euler characteristic of $T$, which is nonzero since $M$ is the fixed locus of the $S^1$ action on $T$. \qed
Now we define
\[
\mathcal{E}(Q_1, \gamma_1, \gamma_2) = \prod_{a \in Q_1} \prod_{m=1}^{\gamma_1(s(a))} \prod_{m'=1}^{\gamma_2(t(a))} (x_{t(a), m'} + \gamma_1(t(a)) - x_{s(a), m})
\]
and
\[
\mathcal{E}(Q_0, \gamma_1, \gamma_2) = \prod_{a \in Q_0} \prod_{m=1}^{\gamma_1(i)} \prod_{m'=1}^{\gamma_2(i)} (x_{i, m'} + \gamma_1(i) - x_{i, m}).
\]
Each of these classes is a product of classes of the form (53), and so we deduce the following corollary.

**Corollary 5.3.** For every pair \(\gamma_1, \gamma_2 \in \mathbb{Z}^{Q_0}\), the localisation maps
\[
\mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}} \to \left(\mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}}\right) \otimes_{\mathcal{H}_{G,\gamma_1} \times \mathcal{H}_{G,\gamma_2}} (\mathcal{H}_{G,\gamma_1} \times \mathcal{H}_{G,\gamma_2})(\text{pt}, \mathbb{Q}) \mathcal{E}(Q_0, \gamma_1, \gamma_2)^{-1}
\]
for \(s = 0, 1\) are injective. More generally, the maps
\[
\mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}} \to \mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}} \otimes \mathcal{H}_{A_1+\gamma_2}^{\text{sp}} A_{\gamma_1,\gamma_2}
\]
and
\[
\mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_3}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_4}^{\text{sp}} \to \mathcal{H}_{Q,\gamma_1}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_2}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_3}^{\text{sp}} \otimes \mathcal{H}_{Q,\gamma_4}^{\text{sp}} \otimes \mathcal{H}_{A_1+\gamma_2}^{\text{sp}} A_{\gamma_1,\gamma_2}
\]
are too, where here we set
\[
\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4
\]
\[
\gamma_0 = \gamma_1 + \gamma_2
\]
\[
\gamma_2 = \gamma_2 + \gamma_4.
\]
This establishes the injectivity of the localisation maps required for the definition of a localised bialgebra structure (see Definition 4.4).

### 5.3. Definition of the \(T\)-equivariant CoHA.

Define \(T_\gamma := \prod_{i \in Q_0} (\mathbb{C}^*)^{\gamma(i)}\). After choosing a basis for \(\bigoplus C^{\gamma(i)}\) there is a natural inclusion \(T_\gamma \subset G_\gamma\). Let \(N(T_\gamma)\) be the normalizer of \(T_\gamma\) inside \(G_\gamma\). For every natural number \(N\) there are morphisms
\[
s_N : (M_{Q,\gamma}, N(T_\gamma))_N \to (M_{Q,\gamma}, G_\gamma)_N
\]
inducing maps
\[
\varphi_{(\text{tr}(W)_\gamma)_N} \to (s_N)_* \varphi_{(\text{tr}(W)_\gamma)_N}^{s_N}
\]
and
\[
(s_N)_! \varphi_{(\text{tr}(W)_\gamma)_N}^{s_N} \{c_0\} \to \varphi_{(\text{tr}(W)_\gamma)_N}^{s_N}
\]
where \(c_0 = \sum_{i \in Q_0} (\gamma_i^2 - \gamma_i)\).

**Proposition 5.4.** There are natural maps
\[
H_{c,T_\gamma}(M_{\gamma}^{\text{sp}}, \varphi_{\text{tr}(W)_\gamma})^{\text{sym}} \{c_0\} \to H_{c,G_\gamma}(M_{\gamma}^{\text{sp}}, \varphi_{\text{tr}(W)_\gamma})
\]
which are isomorphisms.
Proof. For each $N$, there is a map

$$w : (\mathcal{M}^{sp}_{\gamma}, T_{\gamma})_{N} \to (\mathcal{M}^{sp}_{\gamma}, N(T_{\gamma}))_{N}$$

with fibre $\text{Sym}_{\gamma}$, from which we deduce that there is an isomorphism in compactly supported cohomology

$$H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{\gamma}, w^{*}\varphi_{\text{tr}(W)}\gamma) \otimes \text{Sym}_{\gamma} \to H_{c,N(T_{\gamma})}(\mathcal{M}^{sp}_{\gamma}, \varphi_{\text{tr}(W)}\gamma),$$

and so it suffices to prove that

$$H_{c,N(T_{\gamma})}(\mathcal{M}^{sp}_{\gamma}, \varphi_{\text{tr}(W)}\gamma) \{ -c_0 \} \to H_{c,G_{\gamma}}(\mathcal{M}^{sp}_{\gamma}, \varphi_{\text{tr}(W)}\gamma)$$

is an isomorphism. This will follow from the claim that (54) is an isomorphism in $D^b((\mathcal{M}^{sp}_{\gamma}, G_{\gamma})_N)$. Since $s_N$ is smooth, it follows that

$$\varphi_{\text{tr}(W)}\gamma_N s_N \cong (s_N)^{*}\varphi_{\text{tr}(W)}\gamma_N,$$

and so the proof follows from the claim that if $F$ is a sheaf on $D^b((\mathcal{M}^{sp}_{\gamma}, G_{\gamma})_N)$, then the natural map $F \to (s_N)^{*}(s_N)^{*}F$ is an isomorphism. But this follows from the fact that the fibres of $s_N$ have cohomology equal to $\mathbb{Q}$ in degree zero, and zero in other degrees. □

We now describe new cohomological Hall algebra operations on the underlying vector space $T_{Q,W} := \bigoplus T_{Q,w,\gamma}$

where

$$T_{Q,w,\gamma} := \left( H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{Q,\gamma}, \varphi_{\text{tr}(W)}\gamma)^{*}\text{Sym}_{\gamma} \right).$$

Firstly, define

$$\overline{T}_{Q,w,\gamma} := \left( H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{Q,\gamma}, \varphi_{\text{tr}(W)}\gamma)^{*} \right)^{*}.$$

- Define

$$\delta_T : H_{c,T_{\gamma}}(\mathcal{M}_{\gamma}, \varphi_{\text{tr}(W)}\gamma)^{*} \to H_{c,T_{\gamma}}(\mathcal{M}_{\gamma}, \varphi_{\text{tr}(W)}\gamma)^{*}[\mathcal{E}(Q_0, \gamma_1, \gamma_2)^{-1}]$$

to be division by $\mathcal{E}(Q_0, \gamma_1, \gamma_2)$.

- Define $\overline{\delta}_T : H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{\gamma_1,\gamma_2}, \varphi_{\text{tr}(W)}\gamma_1, \gamma_2)^{*} \to H_{c,T_{\gamma}}(\mathcal{M}_{\gamma_1,\gamma_2}, \varphi_{\text{tr}(W)}\gamma_1)^{*}$ as the pushforward induced by the inclusion $M_{\gamma_1,\gamma_2} \to M_{\gamma}$.  

- Define $\overline{\sigma}_T : H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{\gamma_1} \times M^{sp}_{\gamma_2}, \varphi_{\text{tr}(W)}\gamma_1, \varphi_{\text{tr}(W)}\gamma_2)^{*} \to H_{c,T_{\gamma}}(\mathcal{M}^{sp}_{\gamma_1,\gamma_2}, \varphi_{\text{tr}(W)}\gamma_1, \gamma_2)^{*}$ as the pullback induced by the affine fibration $M_{\gamma_1,\gamma_2} \to M_{\gamma_1} \times M_{\gamma_2}$.

Then define

$$\overline{m} : \overline{T}_{Q,w,\gamma_1} \otimes \overline{T}_{Q,w,\gamma_2} \to \overline{T}_{Q,w,\gamma}[\mathcal{E}(Q_0, \gamma_1, \gamma_2)^{-1}]$$
by \( \overline{m} = \overline{\delta_T} \overline{\zeta} \overline{T} \). The space \( \mathcal{T}_{Q,W,\gamma} \) carries a \( \text{Sym}_\gamma \)-action, and by definition we have \( \mathcal{T}_{Q,W,\gamma} := \mathcal{T}_{Q,W,\gamma}^{\text{Sym}_\gamma} \). We define

\[
\begin{align*}
\mathcal{T}_T := & \mathcal{T}_T^{\text{Sym}_1 \times \text{Sym}_2}, \\
\alpha_T := & \alpha_T^{\text{Sym}_1 \times \text{Sym}_2}, \\
\zeta_T := & \zeta_T^{\text{Sym}_1 \times \text{Sym}_2}, \\
\delta_T := & (\sum_{c \in C} c) \delta_T^{\text{Sym}_1 \times \text{Sym}_2},
\end{align*}
\]

where \( C \) is a set of representatives of the cosets of \( \text{Sym}_1 \times \text{Sym}_2 \) inside \( \text{Sym}_\gamma \). Composing these maps we build a map

\[
\delta_T \zeta_T \alpha_T \mathcal{T}_T : \mathcal{T}_{Q,W,\gamma_1} \otimes \mathcal{T}_{Q,W,\gamma_2} \rightarrow (H_{c,T_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_\gamma}
\]

where the subscript \( L \) means we formally invert \( \pi_* \mathcal{E}(Q_0; \gamma_1, \gamma_2) \) for every \( \pi \in \mathcal{P}(\gamma_1, \gamma_2) \).

**Proposition 5.5.** The following diagram commutes:

(56)

\[
\begin{array}{ccc}
(H_{c,T_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_\gamma)(-c_1) & \xrightarrow{\xi_1} & H_{c,G_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_\gamma}
\end{array}
\]

\[
\begin{array}{ccc}
(H_{c,T_{\gamma}}(M_{\gamma}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_1 \times \text{Sym}_2)(-l' - c_1) & \xrightarrow{\xi_2} & H_{c,G_{\gamma_{1,2}}}(M_{\gamma}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_1 \times \text{Sym}_2)(-l')
\end{array}
\]

\[
\begin{array}{ccc}
(H_{c,T_{\gamma}}(M_{\gamma_{1,2}}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_1 \times \text{Sym}_2)(-l' - c_3) & \xrightarrow{\xi_3} & H_{c,G_{\gamma_{1,2}}}(M_{\gamma_{1,2}}^{sp}, \varphi_{tr(W)})\}^{\text{Sym}_1 \times \text{Sym}_2)(-l')
\end{array}
\]

\[
\begin{array}{ccc}
(H_{c,T_{\gamma}}(M_{\gamma}^{sp} \times M_{\gamma}^{sp}, \varphi)(-c_4 + \chi(\gamma_2, \gamma_1)) & \xrightarrow{\xi_4} & H_{c,G_{\gamma_{1,2}}}(M_{\gamma_{1,2}}^{sp} \times M_{\gamma_{1,2}}^{sp}, \varphi)(\chi(\gamma_2, \gamma_1))
\end{array}
\]

where \( l' \) is defined by \( (5E) \), in the last line \( \varphi = \varphi_{tr(W)}\gamma_1 \otimes \varphi_{tr(W)}\gamma_2 \), and all of the \( \xi_t \) for \( t \geq 2 \) are isomorphisms. The \( c_s \) are defined by

\[
c_1 = c_3 = \sum_{i \in Q_0} \binom{(\gamma(i) + 1)}{2} + \gamma_1(i)\gamma_2(i)
\]

\[
c_4 = \sum_{i \in Q_0} \binom{(\gamma_1(i) + 1)}{2} + \binom{(\gamma_2(i) + 1)}{2}.
\]

**Proof.** All of the \( \xi_t \) are defined as follows. Firstly, we define, for \( V \) an arbitrary \( \text{Sym}_\gamma \)-equivariant vector space, the natural restriction map

\[
(V^*)^{\text{Sym}_\gamma} \rightarrow (V^{\text{Sym}_\gamma})^*
\]

defining a natural isomorphism of contravariant functors, since we always work over a field of characteristic zero. So we may interchange the operations of taking invariants and taking vector duals in the left hand column of (57). Then, we take the duals of the maps of Proposition 5.4

We deal with the commutativity of the constituent squares, working from top to bottom.

**Square 1** For \( V \) a \( \text{Sym}_\gamma \) equivariant vector space, the following diagram commutes

\[
\begin{array}{ccc}
(V^*)_{\text{Sym}_\gamma} & \cong & (V^*)_{\text{Sym}_\gamma}^* \\
\downarrow \sum c_i & & \downarrow i^* \\
(V^*)_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}} & \cong & (V^*)_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}}^*
\end{array}
\]

where \( i^* : (V^*)_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}} \rightarrow (V^*)_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}}^* \) is the inclusion. It follows that we have a commutative diagram

\[
\begin{array}{ccc}
(H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{\text{Sym}_\gamma})^* & \cong & (H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{\text{Sym}_\gamma})^* \\
\downarrow \delta_T & & \downarrow \delta_T' \\
(H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}}^* & \cong & (H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}})^*
\end{array}
\]

where \( \delta_T' \) is given by composing division by \( \mathcal{E}(Q_0, \gamma_1, \gamma_2) \) with the dual of the inclusion

\[
H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{L}^* \rightarrow H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}\{c_0\}_{L}^* \rightarrow \text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}.
\]

Consider the commutative diagram of groups

\[
\begin{array}{ccc}
N(T_\gamma) & \rightarrow & G_\gamma \\
\downarrow & & \downarrow \\
N(T_{\gamma_1}) \times N(T_{\gamma_2}) & \rightarrow & G_{\gamma_1, \gamma_2}
\end{array}
\]

inducing the commutative diagram of spaces

\[
\begin{array}{ccc}
(M^\sp_{\gamma}, N(T_{\gamma}))_N & \eta_{1,N} & (M^\sp_{\gamma}, G_\gamma)_N \\
\downarrow \eta_{2,N} & & \downarrow \eta_{3,N} \\
(M^\sp_{\gamma}, N(T_{\gamma_1}) \times N(T_{\gamma_2}))_N & \eta_{4,N} & (M^\sp_{\gamma, \gamma_1, \gamma_2}, G_{\gamma_1, \gamma_2})_N
\end{array}
\]

The map \( \eta_{2,N} \) is a finite cover, and the pushforward

\[
(\eta_2)_* : H_{c,N(T_{\gamma_1}) \times N(T_{\gamma_2})}(M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma})^* \rightarrow H_{c,N(T_{\gamma})}(M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma})^*
\]

is given by the dual of the inclusion of the \( \text{Sym}_\gamma \)-invariant part of \( H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\} \) into the \( \text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2} \)-invariant part. Since (57) commutes, it follows that the pushforward

\[
(\eta_3)_*(\eta_4)_* : (H_{c,T}\{M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma}\}_{\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}})^* \rightarrow H_{c,G_\gamma}(M^\sp_{\gamma}, \varphi_{\text{tr}(W)_\gamma})^*
\]
factors through \((\eta_2)_*\), and the map \((\eta_3)^* (\eta_1)_*\) in dual compactly supported equivariant cohomology is induced by the map \((\eta_3)^* (\eta_3)_* (\eta_4)_*\). The commutativity then follows from Proposition 2.10 and the equation 

\[\mathcal{E}(Q_0, \gamma_1, \gamma_2).\]

(Square 2) The following diagram of spaces is a commutative Cartesian diagram, in which the vertical maps are closed inclusions and the horizontal maps are smooth projections:

\[
\begin{array}{ccc}
(M^\text{sp}_{\gamma_1, \gamma_2}, N(T_\gamma))_N & \rightarrow & (M^\text{sp}_{\gamma_1, \gamma_2}, G_{\gamma_1, \gamma_2})_N \\
\downarrow & & \downarrow \\
(M^\text{sp}_{\gamma_1, \gamma_2}, N(T_\gamma))_N & \rightarrow & (M^\text{sp}_{\gamma_1, \gamma_2}, G_{\gamma_1, \gamma_2})_N
\end{array}
\]

and \(\zeta_T, \zeta\) are obtained by pushforward along the vertical maps, while \(\epsilon_2\) and \(\epsilon_3\) are obtained by pullback. Commutativity then follows by Proposition 2.10.

(Square 3) The proof of the commutativity of the bottom square is as in the proof of the commutativity of the second square, using Proposition 2.10 in the affine fibration case.

Since the map \(\delta\) lands in \(H_{c, G, (M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)})^*}\) we deduce the following corollary.

**Corollary 5.6.** The map

\[m_T : \mathcal{T}_{Q, W, \gamma_1} \otimes \mathcal{T}_{Q, W, \gamma_2} \rightarrow \left(H_{c, T, (M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g}\]

factors through \(\mathcal{T}_{Q, W, \gamma}\), and induces an associative multiplication on \(\mathcal{T}_{Q, W}\).

**Corollary 5.7.** There is an isomorphism of algebras \(\mathcal{T}_{Q, W} \cong H_{Q, W}\).

### 5.4. The critical CoHA as a shuffle algebra.

Using the proposition we may recast the \(T\)-equivariant CoHA as follows:

- The underlying vector space of \(\mathcal{T}_{Q, W} := \bigoplus \mathcal{T}_{Q, W, \gamma}\) where
  \[\mathcal{T}_{Q, W, \gamma} := \left(H_{c, T, (M^\text{sp}_{Q, \gamma}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\}

- For each \(\pi \in \mathcal{P}(\gamma_1, \gamma_2)\), let \(S_\pi \in G_\pi\) be the associated \(Q_0\)-tuple of permutation matrices. Consider \(M^\text{sp}_{Q, \gamma_1} \times M^\text{sp}_{Q, \gamma_2}\) as a subspace of \(M^\text{sp}_{Q, \gamma}\) consisting of block diagonal matrices. Then via the Thom-Sebastiani isomorphism, there is an isomorphism
  \[\mathcal{T}_\pi : \left(H_{c, T, (M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\} \rightarrow \left(H_{c, T, (M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\}

The \(T_\gamma\) action on \((S_\pi)_*(M^\text{sp}_{Q, \gamma_1} \times M^\text{sp}_{Q, \gamma_2})\) is defined by \(z \cdot S_\pi(x) = S_\pi(z \cdot x)\) – so we are considering \((S_\pi)_*\) as a \(T_\gamma\)-equivariant map, but for a twisted action.

- The maps
  \[\pi_{\gamma} : \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\} \rightarrow \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\}
  \[\delta_{\pi} : \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\} \rightarrow \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\}
  \[\pi_{\gamma} : \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\} \rightarrow \left(H_{c, T, (S_\pi)_*(M^\text{sp}_{Q, \gamma_1}, \varphi_{\text{tr}(W)_\gamma})_L} \right)^{\text{Sym}_g} \cdot \chi(\gamma, \gamma)/2\}

are defined in the same way as $\pi_T$, $\overline{\Sigma}_T$, and $\overline{\Omega}_T$.

- Define $\Psi_\pi = \delta_\pi \sum_{\Sigma_\pi} T_{\Sigma_\pi}$ and $\overline{\Psi}_\pi$, then $\overline{\Psi} := \sum_{\pi \in \mathcal{P}(\gamma_1, \gamma_2)} \overline{\Psi}_\pi$ factors through a map $\mathcal{T}_{Q,W, \gamma_1} \otimes \mathcal{T}_{Q,W, \gamma_2} \to \mathcal{T}_{Q,W, \gamma}$, recovering the multiplication of the previous subsection.

The benefit of this construction is that it makes it clearer how to define a multiplication $m \otimes m$ on the localised object

$$H_{Q,W}^{sp,2sp} := \bigoplus_{\gamma_1, \gamma_2 \in \mathcal{Q}_0} \left( (H_{Q,W, \gamma_1}^{sp} \otimes H_{Q,W, \gamma_2}^{sp}) \otimes A_{\gamma_1 + \gamma_2} A_{\gamma_1, \gamma_2} \right).$$

In brief, let $J_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} \in \text{Sym}_{\gamma_1 + \ldots + \gamma_4}$ be the $Q_0$-tuple of permutations sending

$$(1, \ldots, \gamma_1(i), \gamma_1(i) + 1, \ldots, \gamma_1(i) + \gamma_2(i), \ldots, \gamma_1(i) + \ldots + \gamma_4(i)) \mapsto$

$$(1, \ldots, \gamma_1(i), \gamma_1(i) + \gamma_2(i) + 1, \ldots, \gamma_1(i) + \gamma_2(i) + \gamma_3(i), \gamma_1(i) + \ldots + \gamma_4(i))$$

Let $\mathcal{P}'(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \subset \text{Sym}_{\gamma_1 + \ldots + \gamma_4}$ be the set of all permutations that can be written as $(\pi_1 \otimes \pi_2)J_{\gamma_1, \gamma_2, \gamma_3, \gamma_4}$ for some $\pi_1 \in \mathcal{P}(\gamma_1, \gamma_3)$ and $\pi_2 \in \mathcal{P}(\gamma_2, \gamma_4)$. Then just as above we define maps

$$\mathcal{T}_\pi : H_{c,T_{\gamma_1}}(M_{Q, \gamma_1}^{sp}, \varphi_{\text{tr}(W)_{\gamma_1}}) \otimes \ldots \otimes H_{c,T_{\gamma_4}}(M_{Q, \gamma_4}^{sp}, \varphi_{\text{tr}(W)_{\gamma_4}}) \to$$

$$H_{c,T_{\gamma_1 + \ldots + \gamma_4}}\left( (S_\pi)_* (M_{Q, \gamma_1}^{sp} \times \ldots \times M_{Q, \gamma_4}^{sp}), \varphi(S_\pi)_* \text{tr}(W)_{\gamma_1 + \ldots + \gamma_4} \right)$$

and maps $\overline{\pi}_\pi, \ldots, \overline{\delta}_\pi$ that compose to form a map

$$H_{c,T_{\gamma_1}}(M_{Q, \gamma_1}^{sp}, \varphi_{\text{tr}(W)_{\gamma_1}}) \otimes \ldots \otimes H_{c,T_{\gamma_4}}(M_{Q, \gamma_4}^{sp}, \varphi_{\text{tr}(W)_{\gamma_4}}) \to$$

$$H_{c,T_{\gamma_1 + \ldots + \gamma_4}}\left( (S_{\pi_1})_* (M_{\gamma_1 + \gamma_3}, \varphi(S_{\pi_1})_* \text{tr}(W)_{\gamma_1 + \gamma_3}) \otimes H_{c,T_{\gamma_2 + \gamma_4}}\left( (S_{\pi_2})_* (M_{\gamma_2 + \gamma_4}, \varphi(S_{\pi_2})_* \text{tr}(W)_{\gamma_2 + \gamma_4}) \right) \right),$$

which is a twisted $\text{H}_{T_{\gamma_1 + \ldots + \gamma_4}}(pt, Q)$ module map, and so extends to a map between localisations. Summing over all such $\pi$ gives the required map $m \otimes m$.

5.5. **Supercommutativity of $H_{Q,W}^{sp,\sigma}$.** A representation $\psi$ of $Q$ of dimension vector $\gamma$ induces a representation $\psi^T$ of $Q^T$ of dimension vector $\gamma$, where the space we associate to each $i \in Q_0$ is $(\mathbb{C}^{\gamma(i)})^*$ and $\psi(a^*)^T = \psi(a)^*$. Consider the canonical Hermitian inner product on $\mathbb{C}^{\gamma(i)}$, for each $i \in Q_0$. This gives isomorphisms $(\mathbb{C}^{\gamma(i)})^* \cong \mathbb{C}^{\gamma(i)}$, and so we may consider $\psi^T$ as a representation of $Q^T$ with $\psi^T(a) = \psi(a)^\dagger$. Finally, via the isomorphism $\sigma$ we obtain a $Q$ representation $\sigma(\psi)$ sending $a$ to $\psi^T(\sigma(a))$. In short, $\sigma$ induces an automorphism of $M_{Q,W, \gamma}$. We label this automorphism $\Gamma_\gamma$, and we make the following additional assumption.

**Assumption 5.8.** The automorphism $\Gamma_\gamma$ also takes $M_{Q,W, \gamma}^{sp,\sigma}$ to $M_{Q,W, \gamma}^{sp,\sigma}$.

Note that by our assumption on $W$, we have the identity $\text{tr}(W)_\gamma \Gamma_\gamma = \lambda \text{tr}(W)_\gamma$ for some nonzero scalar $\lambda$.

Consider the automorphism $\Lambda : G_\gamma \to G_\gamma$ sending $(g_i)_{i \in Q_0} \mapsto ((g_{\sigma(i)}^\dagger)^{-1})_{i \in Q_0}$. Then if we twist the action on the target of $\Gamma_\gamma$ by $\Lambda$, $\Gamma_\gamma$ is a $G_\gamma$-equivariant map, inducing a map $H_{c,G_\gamma}(\Gamma_\gamma) :$
H_{c,G_{\gamma}}(M_{Q,W,\gamma}^{sp,\sigma}, \varphi_{tr(W)_{\gamma}}) \rightarrow H_{c,G_{\gamma}}(M_{Q,W,\gamma}^{sp,\sigma}, \varphi_{tr(W)_{\gamma}}), \text{ given by taking the Sym}_{\gamma}-\text{invariant part of the map}

\begin{equation}
H_{c,T_{\gamma}}(\Gamma_{\gamma}) : H_{c,T_{\gamma}}(M_{Q,W,\gamma}^{sp,\sigma}, \varphi_{tr(W)_{\gamma}}) \rightarrow H_{c,T_{\gamma}}(M_{Q,W,\gamma}^{sp,\sigma}, \varphi_{tr(W)_{\gamma}}),
\end{equation}

defined below.

**Proposition 5.9.** The underlying map on vector spaces (58) is given by multiplication by 

\((−1)^{\chi(\gamma,\gamma)}\).

**Proof.** Let \(r_n\) be the automorphism of \(C^n \setminus 0\) given by \(z \rightarrow z/|z|^2\). Then \(H_{c,T_{\gamma}}(\Gamma_{\gamma})\) is induced by the automorphisms \(\Gamma_{\gamma,n}\) of \(M_{Q,W,\gamma}^{sp,\sigma} \times T \prod (C^n \setminus 0)^{\sum \gamma_i}\) sending \((\psi, x) \rightarrow (\Gamma_{\gamma}(\psi), r_n \otimes \sum \gamma_i(x))\). Finally, the claim follows by considering the action of \(\Gamma_{\gamma,n}\) on the orientation sheaf. Firstly, \(\Gamma_{\gamma}\) reverses exactly \(\sum_{\sigma \in Q_1} \gamma(s(\sigma))\gamma(t(\sigma))\) directions, as it is given by performing complex conjugation on this many variables. Also, \(r_n\) is a degree \(-1\) map, since in polar coordinates it reverses only one direction (the radius). We deduce that the action of \(\Gamma_{\gamma,n}\) on the compactly supported cohomology of \(M_{Q,W,\gamma}^{sp,\sigma} \times T \prod (C^n \setminus 0)^{\sum \gamma_i}\) is given by the sign \(−1)^{\sum_{\sigma \in Q_1} \gamma(s(\sigma))\gamma(t(\sigma))}\), and also that the action of \(\Gamma_{\gamma}\) on the relative dualizing complex of the morphism

\[ M_{Q,W,\gamma}^{sp,\sigma} \times T \prod (C^n \setminus 0)^{\sum \gamma_i} \rightarrow M_{Q,W,\gamma}^{sp,\sigma} \times T \prod (C^{n+1} \setminus 0)^{\sum \gamma_i} \]

is trivial. Putting these two facts together proves the claim. □

Using Proposition 2.8 we deduce the following.

**Proposition 5.10.** Let

\[sw_{co} : H_{Q,W,\gamma}^{sp,\sigma} \otimes H_{Q,W,\gamma'}^{sp,\sigma} \rightarrow H_{Q,W,\gamma}^{sp,\sigma} \otimes H_{Q,W,\gamma'}^{sp,\sigma}\]

be the involution defined by \((\zeta, \zeta') \rightarrow (-1)^{kk'}(\zeta', \zeta)\), for \(\zeta\) and \(\zeta'\) of \(\mathbb{Z}_{co}\)-degree \(k\) and \(k'\) respectively. Then we have that

\[m sw_{co} = H_{c,G_{\gamma}+\gamma'}(\Gamma_{\gamma+\gamma'})\tilde{m}(H_{c,G_{\gamma}}(\Gamma_{\gamma}) \otimes H_{c,G_{\gamma'}}(\Gamma_{\gamma'})),\]

where \(\tilde{m}\) is defined in the same way as \(m\), but with the modification that the division defining \(\tilde{\sigma}_T\) is with respect to the twisted \(H_{T_{\gamma}}(pt, \mathbb{Q})\)-module structure, and the trivialization of the dualizing sheaf defining \(\tilde{\pi}_T\) is twisted by \(H_{c,T_{\gamma}}(\Gamma_{\gamma})\) as well.

By the same reasoning as that contained in the proof of Proposition 5.9, the above modification to \(\tilde{\sigma}_T\) introduces a sign

\[(-1)^{\sum_{\sigma \in Q_0} \gamma'(i)\gamma(i)},\]

while the modification to \(\tilde{\pi}_T\) introduces a sign

\[(-1)^{\sum_{\sigma \in Q_1} \gamma'(t(\sigma))\gamma(s(\sigma))},\]

and so we deduce that

\[\tilde{m} = (-1)^{\chi(\gamma',\gamma)}m.\]
Proposition 5.11. Let \( \zeta_1 \in H^{k_1}_{c,T_{\gamma_1}}(M_{\gamma_1}^{sp,\sigma}, \varphi_{tr(W)_{\gamma_1}}) \) and \( \zeta_2 \in H^{k_2}_{c,T_{\gamma_2}}(M_{\gamma_2}^{sp,\sigma}, \varphi_{tr(W)_{\gamma_2}}) \) be of homogeneous \( \mathcal{C} \)-degree \( c_1 \) and \( c_2 \), with \( \gamma_1 \) and \( \gamma_2 \) self-dual dimension vectors. Then
\[
\zeta \cdot \zeta' = (-1)^{c_1 \cdot c_2} \zeta' \cdot \zeta
\]
Equivalently \( \mathcal{H}_{Q,W}^{sp,\delta} \cong \mathcal{T}_{Q,W}^{sp,\sigma} \) is supercommutative with respect to the \( \mathcal{C} \)-grading.

Proof. Let \( k \) and \( k' \) be equal to the \( \mathbb{Z}_{\geq 0} \) degrees of \( \zeta \) and \( \zeta' \). By Propositions 5.9, 5.10 and equation (59), we have that
\[
(-1)^{kk'} \zeta_1 \cdot \zeta_2 = (-1)^{\chi(\gamma_1, \gamma_2) + \chi(\gamma_2, \gamma_1) + \chi(\gamma_1 + \gamma_2, \gamma_1 + \gamma_2)} (-1)^{\chi(\gamma_1, \gamma_2)} \zeta_2 \cdot \zeta_1.
\]
Since \( \gamma_1 \) and \( \gamma_2 \) are self dual it follows that
\[
\chi(\gamma_1, \gamma_2) = \chi(\gamma_2, \gamma_1)
\]
and
\[
\chi(\gamma_1, \gamma_2) \equiv \sum_{i \in \mathbb{Q}_0} b_i \gamma_1(i) \gamma_2(i) \pmod{2}.
\]
We deduce that
\[
\zeta \cdot \zeta' = (-1)^{c_1 \cdot c_2} \zeta' \cdot \zeta.
\]
\[\square\]

6. Proof of the main result

6.1. Comultiplication operations. Let \( \gamma = \gamma_1 + \gamma_2 \). We now define a map
\[
(60) \quad \mathcal{H}_{Q,W,\gamma}^{sp} \to \left( \mathcal{H}_{Q,W,\gamma_1}^{sp} \otimes \mathcal{H}_{Q,W,\gamma_2}^{sp} \right) \otimes A_{\gamma_1 + \gamma_2} A_{\gamma_1, \gamma_2}
\]
that we will then go on to show defines the structure of a \( Q \)-localised comultiplication for \( \mathcal{H}_{Q,W}^{sp} \) in the category of \( \mathbb{Z}_{Q_0} \otimes \mathbb{Z}_{\mathcal{C}} \)-graded monodromic mixed Hodge structures, or the category of \( \mathbb{Z}_{Q_0} \oplus \mathbb{Z}_{ac} \)-graded ordinary mixed Hodge structures in the event that all of the monodromic mixed Hodge structures on \( \mathcal{H}_{Q,W}^{sp} \) come from ordinary mixed Hodge structures, or in the category of \( \mathbb{Z}_{Q_0} \oplus \mathcal{C} \) graded vector spaces, if one prefers to ignore Hodge structures.

Firstly, it will be more convenient to work with the \( T_{\gamma} \)-equivariant critical cohomological Hall algebra \( \mathcal{T}_{Q,W}^{sp} \cong \mathcal{H}_{Q,W}^{sp} \). Firstly we define
\[
\Delta : H_{c,T_1}(M_{Q,\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}})^* \to H_{c,T_{\gamma_1}}(M_{Q,\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}})^* \otimes H_{c,T_{\gamma_2}}(M_{Q,\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_2}})^* \left[ \mathcal{E}(Q_T^1, \gamma_1, \gamma_2)^{-1} \right]
\]
as the composition of the following maps:
- Define \( \Delta_T : H_{c,T_{\gamma}}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1, \gamma_2}})^* \to H_{c,T_{\gamma_1}}(M_{\gamma_1}^{sp} \times M_{\gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1} \oplus tr(W)_{\gamma_2}})^*[\mathcal{E}(Q_T^1, \gamma_1, \gamma_2)^{-1}] \) as the pushforward associated to the affine fibration \( M_{\gamma_1, \gamma_2} \xrightarrow{\pi} M_{\gamma_1} \times M_{\gamma_2} \). Here we are using that \( \mathcal{E}(Q_T^1, \gamma_1, \gamma_2) \) is the Euler class of \( \pi \).
- Define \( \Delta_T : H_{c,T_{\gamma}}(M_{\gamma_1}^{sp}, \varphi_{tr(W)_{\gamma_1}})^* \to H_{c,T_{\gamma_1}}(M_{\gamma_1, \gamma_2}^{sp}, \varphi_{tr(W)_{\gamma_1, \gamma_2}})^* \) as the pullback induced by the inclusion \( M_{\gamma_1, \gamma_2}^{sp} \to M_{\gamma_1}^{sp} \).
• Define $\delta_T : H_{c,T}(M_{\gamma}, (\varphi_{\text{tr}}(W))_{\gamma})^* \to H_{c,T}(M_{\gamma}, (\varphi_{\text{tr}}(W))_{\gamma})^*$ to be multiplication by

$$\prod_{a \in Q_0} \prod_{m=1}^{\gamma_1(i)} \prod_{m'=1}^{\gamma_2(i)} (x_i,m - x_{i,m'+\gamma_1(i)}) = (-1)^{\sum_{i \in Q_0} \gamma_1(i)\gamma_2(i)} \mathcal{E}(Q_0, \gamma_1, \gamma_2).$$

**Definition 6.1.** The map $\nabla \gamma \to (\gamma_1, \gamma_2)$:

$$\nabla \gamma \to (\gamma_1, \gamma_2) : \mathcal{T}_{Q,W,\gamma}^p \to \mathcal{T}_{Q,W,\gamma_1}^p \otimes \mathcal{T}_{Q,W,\gamma_2}^p [\mathcal{E}(Q_1, \gamma_1, \gamma_2)^{-1}]$$

is defined by $\nabla \gamma \to (\gamma_1, \gamma_2) = T^{-1} \alpha T \delta T \delta T \cdot \mathcal{E}(\gamma_1, \gamma_2)(-1)^{\chi(\gamma_1, \gamma_2)}$. The map

$$\Delta : \mathcal{T}_{Q,W}^p \to \bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Z}^Q_0} (\mathcal{T}_{Q,W,\gamma_1}^p \otimes \mathcal{T}_{Q,W,\gamma_2}^p) [\mathcal{E}(Q_1, \gamma_1, \gamma_2)^{-1}]$$

is defined to be the sum

$$(61) \Delta = \sum_{\gamma_1 + \gamma_2 = \gamma} \nabla \gamma \to (\gamma_1, \gamma_2),$$

and

$$\Delta : \mathcal{T}_{Q,W}^p \to \bigoplus_{\gamma_1, \gamma_2 \in \mathbb{Z}^Q_0} (\mathcal{T}_{Q,W,\gamma_1}^p \otimes \mathcal{T}_{Q,W,\gamma_2}^p) [\mathcal{E}(Q_1, \gamma_1, \gamma_2)^{-1}]$$

is defined by restricting to the $\text{Sym}_{\gamma_1} \times \text{Sym}_{\gamma_2}$ invariant part.

Let $\sigma : Q \to Q^T$ be a self-duality for the QP $(Q, W)$. Restricting $\Delta$ to $\mathcal{H}_{Q,W}^{p,\sigma}$, the factor $\mathcal{E}(\gamma_1, \gamma_2)$ disappears by (77), and $\chi$ becomes symmetric. Then the following is proved in the same way as Proposition 5.11

**Proposition 6.2.** The restriction of $\Delta$ to $\mathcal{H}_{Q,W}^{p,\sigma}$ is $\mathcal{E}$-supercocommutative.

The following proposition follows in the same way as Proposition 3.7 from the observation that all the maps constituting $\Delta$ commute with the endomorphism $\cdot y$

**Proposition 6.3.** The map $\Delta$ commutes with the endomorphism $\cdot y$.

The following is proved in just the same way as the associativity of $m$, see [19], and also Comment 5.1

**Proposition 6.4.** The following diagram commutes for all decompositions $\gamma = \gamma_1 + \gamma_2 + \gamma_3$

$$\mathcal{H}_{Q,W,\gamma}^{p,\sigma} \xrightarrow{\Delta_{\gamma \to (\gamma_1 + \gamma_2, \gamma_3)}} \mathcal{H}_{Q,W,\gamma_1}^{p,\sigma} \otimes \mathcal{H}_{Q,W,\gamma_2 + \gamma_3}^{p,\sigma} [\mathcal{E}(Q_1, \gamma_1, \gamma_2 + \gamma_3)^{-1}]$$

with

$$P = \mathcal{H}_{Q,W,\gamma_1}^{p,\sigma} \otimes \mathcal{H}_{Q,W,\gamma_2}^{p,\sigma} \otimes \mathcal{H}_{Q,W,\gamma_3}^{p,\sigma} [\mathcal{E}(Q_1, \gamma_1, \gamma_2 + \gamma_3)^{-1}] [\mathcal{E}(Q_1, \gamma_1 + \gamma_2, \gamma_3)^{-1}].$$

Now we come to the main theorem regarding the operation $\Delta$. 
Theorem 6.5. The following diagram commutes:

$$
\begin{array}{c}
\mathcal{H}_{Q,W,\gamma}^{sp} \otimes \mathcal{H}_{Q,W,\gamma}^{sp} \\
\sum_{\gamma_1+\gamma_2=\gamma_0, \gamma_3+\gamma_4=\gamma_0} (m) \otimes (\Delta_{Q,W,\gamma_1} \otimes \Delta_{Q,W,\gamma_2}) \\
\sum_{\gamma_1+\gamma_2=\gamma_0, \gamma_3+\gamma_4=\gamma_0} (m) \otimes (\Delta_{Q,W,\gamma_1} \otimes \Delta_{Q,W,\gamma_2})
\end{array}
$$

where $\text{sw}_Q$ is defined by the sign rule for $\mathbb{Z}^{Q_0} \oplus \mathbb{Z}_{co}$-graded objects $a \otimes b = (-1)^{\chi(\gamma,\gamma') + kk'} b \otimes a$ if $a$ and $b$ are of degree $(\gamma, k)$ and $(\gamma', k')$. In other words, $\mathcal{H}_{Q,W}^{sp}$ carries a $Q$-localised comultiplication in the category $\text{MMHS}_{co}$.

The reader might be wondering what has happened to the $\mathbb{Z}_{ac}$-grading. To clarify, we consider objects in $\text{MMHS}_{co}$ to have two gradings, the unshifted $\mathbb{Z}_{co}$-grading, which appears in the Koszul sign rule, but which is not preserved by the multiplication, and the shifted $\mathbb{Z}_{ac}$-grading, which is respected by the multiplication and comultiplication, and which does not feature in the symmetric monoidal structure. If this appears confusing it’s because it is...

Proof of Theorem 6.5. The proof of Theorem 6.5 will occupy us for the rest of this section.

By Section 5.4 the composition $\Delta m$ is given by a sum over shuffles $\pi$ of $(\gamma_0, \gamma_0)$ into $\gamma$. Given such a shuffle, let

$$
S_1(i) := \pi_i^{-1}(\{1, \ldots, \gamma_1(i)\}) \cap \{1, \ldots, \gamma_0(i)\}
$$

$$
S_2(i) := \pi_i^{-1}(\{\gamma_1(i) + 1, \ldots, \gamma(i)\}) \cap \{1, \ldots, \gamma_0(i)\}
$$

$$
S_3(i) := \pi_i^{-1}(\{1, \ldots, \gamma_1(i)\}) \cap \{\gamma_0(i) + 1, \ldots, \gamma(i)\}
$$

$$
S_4(i) := \pi_i^{-1}(\{\gamma_1(i) + 1, \ldots, \gamma(i)\}) \cap \{\gamma_0(i) + 1, \ldots, \gamma(i)\}
$$

And let $\gamma'_e(i) := |S_e(i)|$ for $e = 1, 2, 3, 4$. Then

$$
\gamma'_1 + \gamma'_2 = \gamma'_0
$$

and

$$
\gamma'_3 + \gamma'_4 = \gamma'_0
$$
and π determines a shuffle π₁ of (γ₁, γ₂) into γ₁ and a shuffle π₂ of (γ₂, γ₄) into γ₂. Let L ∈ G_γ be the n-tuple of permutation matrices given by the set of permutations sending

\[ (1, \ldots, γ₁^{′}(i), γ₁(i) + 1, \ldots, γ(i)) \mapsto (1, \ldots, γ₁^{′}(i), γ₂(i) + 1, \ldots, γ₁^{′}(i) + γ₂(i) + γ₃(i), γ₁^{′}(i) + 1, \ldots, γ₁^{′}(i) + γ₂(i) + γ₃(i) + 1, \ldots, γ(i)). \]

There is a unique shuffle of (γ₀, γ₂) into γ associated to the data of a pair of decompositions (63) and (64) and a pair of shuffles π₁ of (γ₁, γ₃) into γ₁ and (γ₂, γ₄) into γ₂. This correspondence can be expressed as the following map from pairs of n-tuples of permutation matrices in G_γ₁ and G_γ₂ respectively:

\[ (\pi₁, \pi₂) \mapsto (S_{\pi₁} \otimes S_{\pi₂})L \]

So we fix this data and instead consider the diagram

\[ \begin{array}{ccc}
\mathcal{H}_{\gamma₁} \otimes \mathcal{H}_{\gamma₂} & \rightarrow^\pi & \mathcal{H}_\gamma[\mathcal{E}(Q₀, γ₂, γ₁)⁻¹] \\
\mathcal{H}_{\gamma₁} \otimes \mathcal{H}_{\gamma₃} \otimes \mathcal{H}_{\gamma₄} & \rightarrow^\pi & \mathcal{H}_\gamma[\mathcal{E}(Q₁, γ₁, γ₃)⁻¹][\mathcal{E}(Q₀, γ₂, γ₁)⁻¹]
\end{array} \]

where \( \pi \) is defined as in the definition of \( \pi_\pi \). This incorporates the sign rule for id \( \otimes \text{sw}_{\text{co}} \otimes \text{id} \) by Proposition 2.8 and the decomposition (65) of \( S_\pi \).

Define

\[ M_{\pi} \]

to be \( S_\pi \) applied to the subspace of representations \( \rho ∈ M^{sp} \) satisfying \( \rho(C_γ^\prime) ⊂ C_γ^{\prime₁} ⊕ C_γ^{\prime₂} ⊕ C_γ^{\prime₄} \).

We extend this notation in the obvious way, so for example

\[ M_{\pi} \]

in the diagram below is the subspace of \( M_{Q,\gamma}^{sp} \) of representations \( \rho \) such that

\[ \begin{aligned}
\rho(a)(S^{-1}_\pi C_γ^{\prime₁}) &⊂ S^{-1}_\pi C_γ^{\prime₁} \\
\rho(a)(S^{-1}_\pi C_γ^{\prime₂}) &⊂ S^{-1}_\pi C_γ^{\prime₁} ⊕ S^{-1}_\pi C_γ^{\prime₂} \\
\rho(a)(S^{-1}_\pi C_γ^{\prime₄}) &⊂ S^{-1}_\pi C_γ^{\prime₁} ⊕ S^{-1}_\pi C_γ^{\prime₄}.
\end{aligned} \]

For \( D \) a selection of dots in a 4 by 4 grid, we denote by \( W_D \) the potential induced by \( W \) on the space \( M_D \). So as long as \( D_1 \) contains \( D_2 \), we may define

\[ H_{c,T_γ}(M_{D₂}, W_{D₁}) := H_{c,T_γ}(M_{D₂}, ϕ_{W_{D₁}}). \]
We also abbreviate $D(-) := H_{c,T\gamma}(-)^*$. Finally, for $D$ a collection of dots in a 4 by 4 grid, and an understood decomposition $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ we set

$$\mathcal{E}_D := \prod_{D \text{ contains a dot in place } (b,c), \text{ and } b \neq c} \left( \prod_{a \in Q_1} \prod_{m=1}^{\gamma_a'(s(a))} \prod_{m'=1}^{\gamma_a'(t(a))} (x_{i(t(a),\gamma_a'(t(a)) + m'} - x_{s(a),\gamma_a'(s(a)) + m}) \right)$$

$$\mathcal{W}_D := \prod_{D \text{ contains a dot in place } (b,c), \text{ and } b \neq c} \left( \prod_{i \in Q_0} \prod_{m=1}^{\gamma_c'(i)} \prod_{m'=1}^{\gamma_c'(i)} (x_{i,\gamma_c'(i) + m'} - x_{i,\gamma_c'(i) + m}) \right)$$

For arbitrary $\gamma_1, \ldots, \gamma_4$ we have

$$(68) \quad \mathcal{W}_{DT} = (-1)^{\sum_{b < c} e(D,b,c) \sum_{i \in Q_0} \gamma_a'(i) \gamma_c'(i)} \mathcal{W}_D$$

where

$$e(D, b, c) := \begin{cases} 1 & \text{if } D \text{ has a dot in position } (b, c) \\ 0 & \text{otherwise.} \end{cases}$$

In analogy with (46), we define

$$\bar{\mathcal{E}}_D := \mathcal{E}_D^{-1} \mathcal{E}_{DT} \mathcal{W}_D \mathcal{W}_{DT}^{-1} (-1)^{\sum_{b < c} e(D,b,c) \chi(\gamma_a',\gamma_c')}.$$ 

There are morphisms

$$i_* : H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^* \rightarrow H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^*$$

and

$$i^* : H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^* \rightarrow H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^*$$

and Proposition 2.11 implies that we have

$$(69) \quad i^* i_* = \cdot \mathcal{E}_D.$$ 

Similarly, there are morphisms

$$v_* : H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^* \rightarrow H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^*$$

and

$$v^* : H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^* \rightarrow H_{c,T\gamma}(M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}, W_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}})^*$$

induced by the affine fibration

$$M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}} \rightarrow M_{\boxed{\begin{array}{c} \bullet \bullet \bullet \bullet \\end{array}}}$$

and the definitions of the pushforward and pullback map imply

$$(70) \quad v^* v_* = \cdot \mathcal{E}_D^{-1}.$$
**Lemma 6.6.** Using the notation just introduced, consider the following diagram:

\[
\begin{array}{c}
\text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_1} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_2} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \\
\text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_3} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_4} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \\
\text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_5} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_6} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \\
\text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_7} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_8} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \\
\text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_9} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \xrightarrow{j_{10}} \text{D}(M_{\bullet \bullet}, W_{\bullet \bullet}) \\
\end{array}
\]

We define the constituent maps as follows. The maps \(j_2, j_7, \) and \(j_{12}\) are pushforwards associated to inclusions, \(j_3, j_4, \) and \(j_5\) are pullbacks associated to inclusions, \(j_1, j_6, \) and \(j_{11}\) are pullbacks associated to affine fibrations, and \(j_8, j_9, \) and \(j_{10}\) are pushforwards associated to affine fibrations. Then we have

\[j \triangleq j_{10} j_5 j_2 j_1 = \mathcal{E}^{-1} \mathcal{E} j_{12} j_{11} j_8 j_3 = : \mathcal{E}^{-1} \mathcal{E} j \]

\[(72)\]

In other words, up to a prescribed twist, the two maps from the top left corner of (71) to the bottom right corner given by passing along only external edges of (71) commute.

**Proof.** We have commutativity of the top left and bottom right squares, as they are composed entirely of pullbacks or entirely of pushforwards, and the squares

\[
\begin{array}{c}
M_1 \xleftarrow{j_1} M_2 \\
\downarrow \quad \downarrow \\
M_3 \xleftarrow{j_2} M_4
\end{array}
\]

and

\[
\begin{array}{c}
M_1 \xrightarrow{j_3} M_2 \\
\downarrow \quad \downarrow \\
M_3 \xrightarrow{j_4} M_4
\end{array}
\]

commute. So we have established that every small square of (71) commutes, with the possible exceptions of the top right and bottom left squares. In fact these final squares do not commute, and this is the source of the twist in (72).
The following squares are Cartesian:

\[
\begin{array}{ccc}
M & \overset{h_1}{\rightarrow} & M \\
\downarrow{l_1} & & \downarrow{h_2} \\
M & \overset{h_2}{\rightarrow} & M \\
\end{array}
\]

and

\[
\begin{array}{ccc}
M & \overset{h_3}{\rightarrow} & M \\
\downarrow{l_3} & & \downarrow{h_4} \\
M & \overset{h_4}{\rightarrow} & M \\
\end{array}
\]

with the first composed entirely of affine fibrations, and the second the inclusion of two transversally intersecting manifolds in an ambient manifold. We have

\[
\begin{align*}
j_6 &= v^* h_1^* \\
j_9 &= (h_2)_* v_* \\
j_2 &= i_*(h_3)_* \\
j_5 &= h_4^* i_* \\
j_8 &= (l_1)_* \\
j_{11} &= (l_2)^* \\
j_4 &= (l_3)^* \\
j_7 &= (l_4)_*
\end{align*}
\]

from which we deduce from Proposition 2.10 and Lemma 6.6 follows.

The morphism \( j \) differs from \( \Delta \) in the following respects

1. We have left out the sign \((-1)^{\chi(\gamma'_1 + \gamma'_3 - \gamma'_4)}\) appearing in the definition of \( \Delta \).
2. We have left out the multiplication by \( \gamma_1 \) contained in the definition of \( \delta_T \).
3. We have left out the division by \( \gamma_2 \) contained in the definition of \( \delta_T \).
4. We have left out the multiplication by \( \gamma_3 \) contained in the definition of \( \Delta \).
Similarly, the morphism \( j \) differs from \( \Psi \circ \Delta \) in the following ways:

1. We have left out the sign \((-1)^{\chi(\gamma_1, \gamma_2)}(-1)^{\chi(\gamma_3, \gamma_4)}\) appearing in the definition of the co-multiplication.
2. We have left out the division by \( \Psi \) contained in the definition of \( \delta_{\pi_1} \circ \delta_{\pi_2} \).
3. Since \( j \) accounts for the sign appearing in \((\text{id} \otimes \text{sw}_Q \otimes \text{id})\), and not the sign appearing in \((\text{id} \otimes \text{sw}_Q \otimes \text{id})\), there is a difference in sign given by the parity of \( \chi(\gamma_2, \gamma_3) \).
4. We have left out multiplication by \( \Psi \) contained in the definition of \( \delta T \otimes \delta T \).
5. We have left out the multiplication by \( \Psi \) contained in the definition of \( \Delta \otimes \Delta \).

We have that

\[
\Psi \Psi^{-1} = \Psi^{-1} (\Psi \Psi^{-1})
\]

Putting these factors together we deduce that

\[
\Delta \Psi = (m_{\pi_1} \otimes m_{\pi_2}) (\text{id} \otimes \text{sw}_Q \otimes \text{id}) \Delta \Psi_0 \otimes \Delta \Psi_0 (\Psi \Psi^{-1} \Psi^{-1} \Psi).
\]

Tracking through the definitions of the constituent terms, using the identity

\[
\Psi^{-1} = (\Psi^{-1}) (\Psi^{-1})
\]

we find that

\[
\Psi \Psi^{-1} \Psi^{-1} \Psi^{-1} (-1)^{\chi(\gamma_2, \gamma_3)} + \chi(\gamma_4) + \chi(\gamma_3, \gamma_2) + \chi(\gamma_1, \gamma_2) + \chi(\gamma_1, \gamma_3) = 0 \pmod 2,
\]

which is a trivial consequence of the fact that \( \chi \) is a symmetric bilinear form on the lattice of self-dual dimension vectors.

**Remark 6.7.** The result is simplified somewhat by restricting to \( \mathcal{H}_{Q,W}^{sp,\sigma} \). Then all \( \Psi \) factors become 1, while \( \Psi \Psi^{-1} \Psi^{-1} \Psi^{-1} \) becomes \((-1)^{\chi(\gamma_1, \gamma_3)}\), and so \( \Psi \Psi^{-1} \Psi^{-1} \Psi^{-1} \) reduces to checking that

\[
\chi(\gamma_1, \gamma_3) + \chi(\gamma_2, \gamma_3) + \chi(\gamma_1, \gamma_3, \gamma_2) + \chi(\gamma_1, \gamma_3, \gamma_2) + \chi(\gamma_1, \gamma_3, \gamma_2) + \chi(\gamma_1, \gamma_3, \gamma_2) \equiv 0 \pmod 2,
\]

which is a trivial consequence of the fact that \( \chi \) is a symmetric bilinear form on the lattice of self-dual dimension vectors.

### 6.2. Generalisation to non self dual dimension vectors.

This subsection is an aside, and can be ignored unless the reader is interested in a preview of future directions.

Theorem 6.5 shows that \( \mathcal{H}_{Q,W}^{sp} \) is always a \( Q \)-localised bialgebra in a suitable braided monoidal category. It would be nice to extend the supercommutativity theorem below to the statement that \( \mathcal{H}_{Q,W}^{sp} \) is a commutative \( Q \)-localised bialgebra to the whole of \( \mathcal{H}_{Q,W}^{sp} \), but the lack of symmetry in the braided monoidal structure we have chosen makes this tricky. A way of interpreting
the term \((\text{id} \otimes \tilde{E}_{\gamma_1, \gamma_2} \otimes \text{id})\) appearing in the definition of a \(Q\)-localised bialgebra is as a compensation term for the fact that we are working in the wrong monoidal category: the symmetrising morphism \(\text{sw}_{CQ}\) should really be \(\text{sw}'\), which we define by

\[
z_1 \otimes z_2 \mapsto \tilde{E}_{\gamma_1, \gamma_2} (-1)^{(\gamma_1, \gamma_2)} z_2 \otimes z_1,
\]

for \(z_i\) of \(Z^{Q_0}\)-degree \(\gamma_i\), but we have chosen a symmetrising morphism \(\text{sw}_{CQ}\) that agrees with this on the subcategory of objects concentrated on self-dual dimension vectors, since \(\text{sw}'\) only makes sense in general after localising.

We define in the natural way a localised commutative bialgebra for the localised tensor category \(\mathcal{D}_Q := (\text{MMHS}_{Z^{Q_0}, \text{sw}'})\), and make the following:

**Conjecture 6.8.** The critical CoHA \(\mathcal{H}_{Q,W}^{sp}\) is a localised commutative bialgebra in the category \(\mathcal{D}_Q\).

One may quickly check that the conjecture is true in case \(W = 0\), using the identification of Section 7.1 between the critical CoHA with zero potential and the ordinary CoHA for the quiver \(Q\), and the explicit presentation of the multiplication given in [19], which we have recalled in Section 1.2.

### 6.3. Free supercommutativity of \(\mathcal{H}_{Q,W}^{sp,\sigma}\)

We can finally put everything together and prove our main theorem. We first recall the setup. We are given the data \((Q, W)\) of a quiver with potential, and also a self-duality \(\sigma\) as defined in Section 4.1, which also contains the definition of \(\mathcal{H}_{Q,W}^{sp,\sigma}\). By Proposition 4.3, \((\mathcal{H}_{Q,W}^{sp,\sigma}, \cdot)\) is a \(Z^{Q_0} \oplus Z_{sc}\)-graded associative algebra object in the category of monodromic mixed Hodge structures. By Proposition 5.11, the algebra \((\mathcal{H}_{Q,W}^{sp,\sigma}, \cdot)\) is, in addition, \(C\)-supercommutative.

Next, by Theorem 6.5, \(\Delta\), as defined in (61), endows \(\mathcal{H}_{Q,W}^{sp}\) with the structure of a \(Q\)-localised comultiplication in the category of \(Z_{sc}\)-graded monodromic mixed Hodge structures, as defined in Section 4.2. By Propositions 3.7 and 6.3 we see that both the multiplication and the localised comultiplication commute with the endomorphism \(\cdot y\) defined on \(\mathcal{H}_{Q,W}^{sp,\sigma}\) and \(\mathcal{H}_{Q,W}^{sp,\sigma} \otimes \mathcal{H}_{Q,W}^{sp,\sigma}\), and in addition, the underlying graded vector space \(\text{fib}(\mathcal{H}_{Q,W}^{sp,\sigma})\) of \(\mathcal{H}_{Q,W}^{sp,\sigma}\) is a free \(C[y]\)-module.

Now Propositions 4.5 and 4.6 amount to the statement of our main theorem:

**Theorem 6.9.** Let \((Q, W)\) be a QP admitting a self-duality structure \(\sigma\), and assume that the spaces \(M_{Q,\gamma}^{\text{sp}}\) satisfy Assumptions 5.8 and 6.7. Considered as an object in the category of \(Z^{Q_0} \oplus Z_{sc}\)-graded vector spaces, \(\mathcal{H}_{Q,W}^{sp,\sigma}\) is a free \(C\)-supercommutative algebra, freely generated by a graded subspace of the form \(V_{\text{prim}}[y]\), where each \(V_{\text{prim}}[y] \subset \mathcal{H}_{Q,W,\gamma}^{sp,\sigma}\) is a free \(C[y]\)-submodule. If the inclusion

\[
\text{Image} \left( m : (\mathcal{H}_{Q,W}^{sp,\sigma})^+ \otimes (\mathcal{H}_{Q,W}^{sp,\sigma})^+ \to \mathcal{H}_{Q,W}^{sp,\sigma} \right) \to \mathcal{H}_{Q,W}^{sp,\sigma}
\]

splits in the category \(\text{MMHS}_{Z^{Q_0} \oplus Z_{sc}}\), then \(\mathcal{H}_{Q,W}^{sp,\sigma}\) is a free supercommutative algebra in the category \(\text{MMHS}_{Z^{Q_0} \oplus Z_{sc}}\). If (73) is split by an inclusion \(V \to \mathcal{H}_{Q,W}^{sp,\sigma}\), where \(V\) is stable under the action of \(\cdot y\), and if the inclusion

\[
\cdot y(V) \to V
\]
splits in the category $\text{MMHS}_{\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac}}$, then $\mathcal{H}^{sp,\sigma}_{Q,W}$ is freely generated in the category $\text{MMHS}_{\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac}}$ by an object $V_{\text{prim}}[y]$.

Let $\text{Gr}_{wt}(\mathcal{H}^{sp,\sigma}_{Q,W})$ be the associated graded of $\mathcal{H}^{sp,\sigma}_{Q,W}$ with respect to the weight filtration defined in [19]. Then $\text{Gr}_{wt}(\mathcal{H}^{sp,\sigma}_{Q,W})$ is a $\mathcal{C}$-supercommutative $\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac} \oplus \mathbb{Z}_{wt}$-graded algebra by [19, Prop.4], and $\text{Gr}_{wt}(\Delta)$ endows $\text{Gr}_{wt}(\mathcal{H}^{sp,\sigma}_{Q,W})$ with the structure of a $Q$-localised comultiplication in the category of $\mathbb{Z}_{ac} \oplus \mathbb{Z}_{wt}$-graded vector spaces. This category is semisimple, so all short exact sequences split. We deduce the following.

**Theorem 6.10.** The vector space $\text{Gr}_{wt}(\mathcal{H}^{sp,\sigma}_{Q,W})$ is a free supercommutative algebra in the category of $\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac} \oplus \mathbb{Z}_{wt}$-graded vector spaces, freely generated by a $\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac} \oplus \mathbb{Z}_{wt}$-graded subspace $V_{\text{prim},wt}[y]$.

**Remark 6.11.** Let $\mathcal{C}$ be the category of pairs $(V,T)$, where $V$ is a $\mathbb{Z}_{ac} \oplus \mathbb{Z}_{Hdg} \oplus \mathbb{Z}_{wt}$-graded vector space, and $T$ is a semisimple quasi-unipotent endomorphism of $V$. Then $\mathcal{C}$ has a convolution tensor product, mimicking the tensor product in $\text{MMHS}_{\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac}}$. Furthermore, this category is also semisimple, and $\mathcal{H}^{sp}_{Q,W}$ may be considered as a free $\mathcal{C}$-supercommutative algebra in the category $\mathcal{C}_Q$, by applying $\text{rat} \otimes \mathcal{C}$ to the Hodge theoretic version of $\mathcal{H}^{sp}_{Q,W}$, and taking the associated graded with respect to the Hodge and weight filtrations. Since $\mathcal{C}$ is again semisimple, the analogue of Theorem 6.10 still holds. In terms of refined DT invariants, this implies that if the spaces $V_{\text{prim},wt,\gamma}$ are finite-dimensional, the extra-refined DT invariants, given by taking Hodge polynomials instead of weight polynomials, are well-defined. Here we define the Hodge polynomial of a filtered object in the category of vector spaces equipped with a quasi-unipotent endomorphism as in [18] - in particular, it is a polynomial in two variables $u$ and $v$ with fractional powers. Explicitly, if $V$ is a complex of mixed Hodge structures with a monodromy action $N$, then consider the class $[\text{Gr}_{Hdg} \text{Gr}_{wt} \mathcal{H}^i(V)]$ in the Grothendieck group $K_0(\mathcal{E})$ of the category $\mathcal{E}$ of bigraded complex vector spaces with a $\mu_n$ action, where $n$ is the least common multiple of the generalised eigenvalues of $N$. Then apply the ring homomorphism $K_0(\mathcal{E}) \rightarrow \mathbb{Z}[u^a,v^b|a,b \in \mathbb{Q}]$ sending

$$[U] \mapsto \sum_{a,b \in \mathbb{Z}} \dim(\text{Gr}_{Hdg}^a(\text{Gr}_{wt}^b(U^0)))u^av^{b-a} + \sum_{a,b \in \mathbb{Z}} \dim(\text{Gr}_{Hdg}^a(\text{Gr}_{wt}^b(U^c)))u^{a+c/n}v^{b-a+1-c/n},$$

where $U^s$ is the part of $U$ with character $s$ under the $\mu_n$ action.

### 7. Some examples

#### 7.1. The case with zero potential

Let $Q$ be as always a quiver, and let $\sigma : Q \rightarrow Q^T$ be an isomorphism: we will think of $\sigma$ as a self duality structure for the QP $(Q,0)$. The CoHA $H^{sp,\sigma}_{Q,0}$ is an algebra object in the category of $\mathbb{Z}Q_0 \oplus \mathbb{Z}_{ac}$-graded mixed Hodge structures. For each $\gamma \in \mathbb{Z}Q_0$, and for each $m \in \mathbb{Z}$, the $m$th graded piece (with respect to the $\mathbb{Z}_{ac}$-grading) of $H^{sp,\sigma}_{Q,0,\gamma}$ is pure of weight $m$, and admits a finite filtration with quotients given by $Q(m)$. Since $\text{Ext}_{\text{MHMS}}^i(Q(m),Q(m)) = 0$, it follows that

$$\text{Image} \left( (H^{sp,\sigma}_{Q,0})^+ \otimes (H^{sp,\sigma}_{Q,0})^+ \rightarrow (H^{sp,\sigma}_{Q,0})^+ \right) \rightarrow (H^{sp,\sigma}_{Q,0})^+$$
splits in $\text{MHS}_{\mathbb{Z}^Q_0 \oplus \mathbb{Z}^\text{sc}}$ and we deduce from Theorem 6.3 that $\mathcal{H}^{sp,\sigma}_Q$ is a free supercommutative algebra in the category of mixed Hodge structures, and is generated by $V_{\text{prim}}[y]$ for $V_{\text{prim},\gamma}$ a shifted pure Hodge structure, in the sense of Definition 2.4.

Recall the cohomological Hall algebra $\mathcal{H}^\sigma_Q$, considered in Section 1.2. We define the underlying $\mathbb{Z}^Q_0 \oplus \mathbb{Z}^\text{sc}$-graded mixed Hodge structure by setting

$$\mathcal{H}^\sigma_{Q, \gamma} := \begin{cases} \emptyset & \text{if } \sigma(\gamma) \neq \gamma \\ H_{G,\gamma}(M_{\gamma}, \mathbb{Q})\{\chi(\gamma, \gamma)/2\} & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$\mathcal{H}^{\sigma}_{Q,0,\gamma} := \begin{cases} \emptyset & \text{if } \sigma(\gamma) \neq \gamma \\ H_{c, G, \gamma}(M_{\gamma}, \mathbb{Q})\{\chi(\gamma, \gamma)/2\}^* & \text{otherwise.} \end{cases}$$

Using the isomorphism $\varphi_0 \cong \mathbb{Q}_X$ for general $X$, with 0 considered as the constant function $X \to \{0\}$.

For a smooth complex manifold $X$, with structure morphism $\tau : X \to \text{pt}$, we have isomorphisms

$$\tau_+ \mathbb{Q}_X \to D_\tau D\mathbb{Q}_X \to D_\tau \mathbb{Q}_X \{\dim X\}$$

which induce isomorphisms

$$\mathcal{H}^\sigma_{Q, \gamma} \cong \mathcal{H}^{\sigma}_{Q,0,\gamma}$$

using $\chi(\gamma, \gamma) = -\dim([M_{Q, \gamma}/G_{\gamma}])$ and the convention of Remark 2.7. One may quickly check that the Verdier duals of the morphisms found in [19] that define the cohomological Hall algebra structure on $\mathcal{H}^\sigma_Q$ are the same as the operations defining the critical cohomological Hall algebra $\mathcal{H}^\sigma_{Q,0}$, and so there is an isomorphism of $\mathbb{Z}^Q_0 \oplus \mathbb{Z}^\text{sc}$-graded algebra objects in the category of mixed Hodge structures

$$\mathcal{H}^\sigma_{Q,0} \cong \mathcal{H}^\sigma_Q,$$

and so we may deduce Theorem 1.6 from Theorem 6.3.

7.2. Examples coming from dimensional reduction. The following examples rely upon the technique of dimensional reduction – for more details see the appendix.

Example 7.1. Let $Q_{\mathbb{C}^3}$ be the quiver with one vertex and three loops, and let $W_{\mathbb{C}^3} = xyz - xzy$. Then $(Q_{\mathbb{C}^3}, W_{\mathbb{C}^3})$ has a self-duality $\sigma$ sending $x$, $y$ and $z$ to $x^T$, $y^T$ and $z^T$ respectively. Since $\sigma$ is the identity on the unique vertex of $Q_{\mathbb{C}^3}$, every dimension vector is self dual.

In addition, this QP admits a cut in the sense of Appendix A.2. Explicitly, setting $S = \{x\}$, every word of $W_{\mathbb{C}^3}$ contains exactly one instance of a letter in $S$. So by Theorem A.10 there is an isomorphism

$$(76) \quad H_{c,G,m}(M_{Q_{\mathbb{C}^3},m}, \mathbb{P}W_{\mathbb{C}^3}) \cong H_{c,G,m}(\mathbb{Z}_m, \mathbb{Q})$$

where $\mathbb{Z}_m$ is the space of representations of $\mathbb{C}Q_{\mathbb{C}^3}$ satisfying $[y, z] = 0$. Note that the Jacobi algebra $J(Q_{\mathbb{C}^3}, W_{\mathbb{C}^3})$ is isomorphic to $\mathbb{C}[x, y, z]$, the free commuting polynomial algebra in three generators. The critical locus of $\text{tr}(W_{\mathbb{C}^3})_m$ on $M_{Q_{\mathbb{C}^3},m}$ is exactly the space of representations of the Jacobi algebra (this is a general fact for quivers with superpotential), and $\mathcal{H}_{Q_{\mathbb{C}^3}, W_{\mathbb{C}^3}}$ is the critical CoHA for the category of zero-dimensional coherent sheaves on $\mathbb{C}^3$. 
In [5] we conjecture that the Hodge structure (76) is pure: evidence for this claim is presented in that paper. Note that purity here is meant in the usual sense for mixed (not mixed monodromic) Hodge structures, as this is the category that the right hand side of (76) belongs to.

From the explicit calculation of the refined DT invariants found in [3], and the above conjecture, we deduce that the odd cohomology vanishes. Since \( b_3 \equiv 0 \) (mod 2) we deduce that the whole of \( \mathcal{H}_{Q_{C^3}, W_{C^3}} \) lives in \( C \)-degree 0, and so by Proposition 5.11 \( \mathcal{H}_{Q_{C^3}, W_{C^3}} \) is a free commutative algebra, with

\[
\dim(V_{prim,m}^k) = \begin{cases} 
1 \quad &\text{if } k = -2, m > 0 \\
0 \quad &\text{otherwise.}
\end{cases}
\]

Here the superscript keeps track of the \( Z_{sc} \)-grading.

Now define \( M_{Q_{C^3},m}^{sp} \) to be the subspace of \( M_{Q_{C^3},m} \) defined by the condition that \( z \) is sent to a nilpotent matrix. One may check that this choice of \( M_{Q_{C^3},m}^{sp} \) satisfies Assumption A.9 since the condition is independent from the assignment of a matrix to \( x \), and so we have an isomorphism

\[(77) \quad H_{c,G_m}(M_{Q_{C^3},m}^{sp}, \varphi_{W_{C^3}}) \cong H_{c,G_m}(Z_{nilp,m}, \mathbb{Q}),\]

where \( Z_{nilp,m} \subset Z_m \) is defined by the extra condition that \( z \) is sent to a nilpotent matrix. The Assumption 3.1 is also satisfied: a matrix of the form

\[
\begin{pmatrix} 
M_1 & * \\
0 & M_2 
\end{pmatrix},
\]

for \( M_1 \) and \( M_2 \) square matrices, is nilpotent if and only if \( M_1 \) and \( M_2 \) are. Similarly, Assumption 5.8 is satisfied since the conjugate transpose of a matrix \( M \) is nilpotent if and only if \( M \) is. So we deduce from Theorem 6.9 that \( \mathcal{H}_{Q_{C^3}, W_{C^3}}^{sp} \) is a free supercommutative algebra.

It is proved in [5] that the right hand side of (77) is a pure Hodge structure. Also, by the proof of the main result of [3], the partition function

\[
\chi_q(H_{Q_{C^3}, W_{C^3}}^{sp}) := \sum_{\gamma \in \mathbb{Z}^\mathbb{Q}_0} \chi_q(H_{Q_{C^3}, W_{C^3}, m}^{sp}, q^{1/2}) x^m \\
= \prod_{m>0} (1 - x^{-m})(1-q)^{-1}
\]

and so we deduce

\[
\dim(V_{prim,m}^{sp,k}) = \begin{cases} 
1 \quad &\text{if } k = 0, m > 0 \\
0 \quad &\text{otherwise.}
\end{cases}
\]

**Example 7.2.** Let \( Q_{con} \) be the quiver pictured below
Let $W_{\text{con}} = ACBD - ADHC$. Then there is a duality $\sigma : Q \to Q^T$ sending $C$, $D$ $A$ and $B$ to $A^T$, $B^T$, $C^T$ and $D^T$ respectively, with $\sigma^{-1}W_{\text{con}}^T = -W_{\text{con}}$. Since $\sigma$ is the identity on vertices, all dimension vectors are self-dual.

The QP $(Q_{\text{con}}, W_{\text{con}})$ admits a cut in the sense of Section A.2. Explicitly, we give edge $D$ weight one, and all other edges weight zero. Then by Theorem A.10 there is an isomorphism

$$H_{c,G}(M_{Q_{\text{con}},\gamma}, \varphi_{tr(W)_\gamma}) \cong H_{c,G}(Z_\gamma, \mathbb{Q})$$

where

$$Z_\gamma \subset M_{Q_{\text{con}},\gamma}$$

is the subspace cut out by the relation $R = ACB - BCA$. Let

$$Z_\gamma \subset \bigoplus_{e \in \{a,b,c\}} \text{Hom}(\mathbb{C}^{\gamma(s(e))}, \mathbb{C}^{\gamma(t(e))})$$

be the subspace cut out by the same relation. Then there is an affine fibration of Artin stacks

$$[Z_\gamma/G_\gamma] \to [Z/G_\gamma].$$

Let $Q'$ be the quiver obtained from $Q_{\text{con}}$ by deleting the edge $D$. One can check that there is a derived equivalence $\Psi$ from the bounded derived category of modules for the quiver algebra $\mathbb{C}Q'/\langle R \rangle$ with finite dimensional cohomology modules, to the derived coherent sheaves on $X := \text{Tot}(O_{\mathbb{P}^1}(-1))$ with compactly supported cohomology sheaves. In a little more detail, $\mathbb{C}Q'/\langle R \rangle$ is the Yoneda algebra of $O_X \oplus \pi^*O_{\mathbb{P}^1}(1)$, where $\pi : X \to \mathbb{P}^1$ is the natural projection.

Let $\theta : Z_{\geq 0}^{(Q_{\text{con}})} \to \mathbb{H}^+ := \{re^{\sqrt{-1}t} | r \in \mathbb{R}_{>0}, t \in (0, \pi)\}$ be a semigroup morphism, i.e. a stability condition for the category of $J(Q_{\text{con}}, W_{\text{con}})$-modules, and let us assume furthermore that it is generic, in the sense that the image doesn’t lie on a single ray. The map $\theta$ defines a stability condition on $\mathbb{C}Q'/\langle R \rangle$-mod in the sense of [17], and the stable objects are given by

- A stable rigid object $L_+^n$ of dimension vector $(n,n+1)$ for all $n \in \mathbb{Z}_{>0}$. Under the derived equivalence $\Psi$ these are Serre twists of the structure sheaf of the zero section of $X := \text{Tot}(O_{\mathbb{P}^1}(-1))$.
- A stable rigid object $L_-^{n+1}$ of dimension vector $(n+1,n)$ for all $n \in \mathbb{Z}_{>0}$. Under $\Psi$ these correspond to sheaves $F[1]$, where the $F$ are as above Serre twists of the structure sheaf of $\mathbb{P}^1$.
- Representations of dimension vector $(1,1)$, which correspond under $\Psi$ to skyscraper sheaves over points of $X$.

Every $\mathbb{C}Q'/\langle R \rangle$ representation is a direct sum of representations of the first two types, and a representation obtained by repeated extensions of representations of the third type. The moduli stack of representations $[Z_\gamma/G_\gamma]$ admits a filtration $\emptyset =: N_0 \subset \ldots \subset N_t := M_{Q_{\text{con}},\gamma}$, where the difference $N_t \setminus N_{t-1}$ is defined by the property that the representations of the first two types appearing in this direct sum decomposition are fixed. Now fix the multiset

$$U = \{L_+^a, L_-^a, L_0^a, L_0^b, L_0^c, L_0^d, L_0^e, L_0^f, L_0^g, \ldots\}$$

represented by

- $L_+^a$ times $a_1$ times
- $L_-^a$ times $a_0$ times
- $L_0^b$ times $b_0$ times
- $L_0^c$ times $b_1$ times
- $L_0^d$ times $b_2$ times
- $L_0^e$ times $b_3$ times
- $L_0^f$ times $b_4$ times
- $L_0^g$ times $b_5$ times
- $\ldots$
of the first two types appearing in a $\gamma$-dimensional representation $L$, then we fix the dimension vector $\gamma' = (m, m)$ of the summand of the third type, and since $\gamma$ is finite, the above set $U$ is too. In the above ordering, nonzero homomorphisms can only pass between nonisomorphic objects $L$ and $L'$ if $L'$ is to the left of $L$. In addition, there are no nonzero homomorphisms from degree zero modules to elements $L_i^+$ or from elements $L_i^-$ to degree zero modules. It follows that there is an affine fibration

$$N_s \to N_s$$

where

$$N_s := \left( \prod_i N^+_{a_i} \right) \times N^\text{deg 0} \times \left( \prod_i N^-_{b_i} \right),$$

$N^+_{a_i}$ and $N^-_{b_i}$ are the stacks of representations isomorphic to $(L^+)^{\oplus a_i}$, and $(L^-)^{\oplus b_i}$ respectively, and $N^\text{deg 0} \times (m,m)$ is the stack of length $m$ zero-dimensional sheaves on $X$.

It follows from the existence of the affine fibration (80) and from the decomposition (81) that the compactly supported equivariant cohomology of $N_s$ is pure if and only if the compactly supported equivariant cohomology of each of the stacks in the decomposition of the right hand side of (81) are. There is an isomorphism of stacks

$$N^+_s \cong BGL_n$$

for which purity is well-known. So in the end, purity of the compactly supported cohomology of $N_s$ is equivalent to purity for $N^\text{deg 0} \times (m,m)$. From the long exact sequence in compactly supported cohomology, purity for the compactly supported equivariant cohomology of $[Z_\gamma/G_\gamma]$ will follow from the purity for each of the $N_s$. Since (79) is another affine fibration, purity for $[Z_\gamma/G_\gamma]$ follows also from purity of the $N_s$, and so finally, by (78), purity for the CoHA $\mathcal{H}_{Q, W \text{con}}$ is implied by the purity of the compactly supported equivariant cohomology of the $N^\text{deg 0} \times (m,m)$.

Next pick a point $p \in \mathbb{P}^1$. Then $N^\text{deg 0} \times (m,m)$ is itself filtered by stacks $P_0 \subset P_1, \ldots, P_m$, where $P_i$ is the substack of sheaves $\mathcal{F}$ in $N^\text{deg 0} \times (m,m)$ satisfying the condition that the length of the sheaf $\mathcal{F} \otimes \pi^* \mathcal{O}_p$ is of length at most $i$. Then there is an isomorphism

$$P_i \setminus P_{i-1} \cong S^\text{nilp}_i \times S_{m-i}$$

where $S_i$ is the stack of length $i$ zero-dimensional sheaves on $\mathbb{C}^2$, and $S^\text{nilp}_i$ is the stack of length $i$ zero-dimensional sheaves on $\mathbb{C}^2$ supported topologically on the $x$-axis.

By [24 Thm.3.4] applied to the one loop quiver $Q_{\text{Jor}}$ defined below, the mixed Hodge structure on the compactly supported equivariant cohomology of $S^\text{nilp}_i$ is pure. By the conjecture of the previous section, the mixed Hodge structure on the compactly supported equivariant cohomology of $S_i$ is pure also. We deduce, again using the long exact sequence in compactly supported cohomology, that purity of the CoHA $\mathcal{H}_{Q, W \text{con}}$ follows from the conjecture.

Now let us assume that $\mathcal{H}_{Q, W \text{con}}$ is indeed pure. In [22] it is calculated that

$$\chi_q(\mathcal{H}_{Q, W \text{con}}) = \prod_{i \geq 0} \left( 1 - x^{(i,i+1)}q^{-1/2}(1-q)^{-1} \left( 1 - x^{(i+1,i)}q^{-1/2}(1-q)^{-1} \left( 1 - x^{(i,i)}(-1-q)(1-q)^{-1} \right) \right) \right),$$
From the main theorem, still using the conjecture, we deduce that $H_{Q_{\text{con}},W_{\text{con}}}$ is a free $\mathfrak{C}$-supercommutative algebra with generating set satisfying

$$\dim(V^k_{\text{prim},\gamma}) = \begin{cases} 1 & \text{if } \gamma(1) = \gamma(2) \text{ and } k = 0, 2 \\ 1 & \text{if } \gamma(1) = \gamma(2) \pm 1, \gamma(1) \geq 0 \leq \gamma(2), \text{ and } k = -1. \end{cases}$$

We finish by translating this statement into a statement involving usual $\mathbb{Z}_2$-graded supercommutativity. By our calculations (as ever, assuming the purity conjecture regarding zero-dimensional sheaves on $\mathbb{C}^2$), all (unshifted) cohomology is even. In addition, $b_{11} \equiv b_{22} \equiv 1 \pmod{2}$. So the nonzero $V_{\text{prim},\gamma}$ are concentrated in $\mathfrak{C}$-degree $(0,0)$ for $\gamma(1) = \gamma(2)$, and in $\mathfrak{C}$-degree $(0,1)$ for $\gamma(1) = \gamma(2) \pm 1$. We deduce that $H_{Q_{\text{con}},W_{\text{con}}}$ is the tensor product of a free symmetric algebra, generated by the $V_{\text{prim},\gamma}[y]$ for which $\gamma(1) = \gamma(2)$, and a free exterior algebra, generated by the $V_{\text{prim},\gamma}[y]$ for which $\gamma(1) = \gamma(2) \pm 1$.

### 7.3. Examples with monodromy.

Each of the examples considered below has a monodromic mixed Hodge structure that properly belongs to the category $\text{MMHS}$, and is not in the essential image of the inclusion $\text{MHS} \to \text{MMHS}$.

**Example 7.3.** Let $Q_{\text{Jor}}$ be the quiver with one vertex and one loop, which we will label $X$ ($Q_{\text{Jor}}$ is known as the Jordan quiver). Let $W_{\text{Jor}} = X^{d+1}$ for some $d \geq 1$. Then one may easily show that the mixed Hodge structure on $H_{Q_{\text{Jor}},W_{\text{Jor}},1}$ is pure (for example this is a special case of [8, Thm.3.1], or [27]). According to [26], the map

$$(82) \quad \text{Sym}(H_{Q_{\text{Jor}},W_{\text{Jor}},1}) \to H_{Q_{\text{Jor}},W_{\text{Jor}}}$$

is an isomorphism. This is in fact equivalent to purity of $H_{Q_{\text{Jor}},W_{\text{Jor}}}$. One way to see this is to argue as follows. In the first instance, (82) is an injection, by our main result. In the second instance, the weight polynomial of the right hand side is calculated in [9] and agrees with the left hand side. So purity implies that (82) is an isomorphism. On the other hand, symmetric powers of a pure exponential Hodge structure are themselves pure, from which we deduce the opposite implication.

So the statement that (82) is an isomorphism provides a new verification of the calculation of the refined DT invariants of $(Q_{\text{Jor}},W_{\text{Jor}})$ found in [9]. By Remark 6.11 we may go further and calculate the Hodge polynomial of $H_{Q_{\text{Jor}},W_{\text{Jor}}}$ to be

$$\text{Sym} \left( (1 - x)(uv)^{-1/2}(u^{1/d_1}v^{(d_1-1)/d_1} + \ldots + u^{(d_1-1)/d_1}v^{1/d_1}) \right),$$

as obtained by applying the Hodge polynomial to the calculation of the motivic partition function found in [9]. The space $M_1 \cong \mathbb{C}$ carries the trivial $\mathbb{C}^*$ action, and so

$$(83) \quad H_{Q_{\text{Jor}},W_{\text{Jor}},1} \cong H_c(\mathbb{C}, \varphi X^{d+1})[y].$$

The cohomology $H_c(\mathbb{C}, \varphi X^{d+1})$ lives entirely in degree one, while for the one vertex $i$, we have $b_{ii} \equiv 0 \pmod{2}$. It follows that all the generators of $H_{Q_{\text{Jor}},W_{\text{Jor}},1}$ live in $\mathfrak{C}$-degree $(0,1)$, and from the Koszul sign rule for $\mathfrak{C}$-graded vector spaces that the algebra $H_{Q_{\text{Jor}},W_{\text{Jor}}}$ is the free exterior algebra generated by (83).
Example 7.4. Consider the example studied in [10]. We take a quiver $Q_2$ given as follows

\[
\begin{array}{c}
1 \\
A \\
B \\
C \\
2 \\
3
\end{array}
\]

We set

\[
W = \frac{1}{d+1} X_1^{d+1} - \frac{1}{d+1} X_2^{d+1} - X_1 CA + X_1 DB + X_2 AC - X_2 BD.
\]

The resulting cohomology $H_{c,G,\gamma}(M_{Q_2,\gamma}, \varphi_{tr(W,\gamma)})$ is supported on the space of representations of the Jacobi algebra

\[
A_2 := \mathbb{C}Q_2/(\partial W_2/\partial a|a \in Q_1),
\]

and we have

\[
D^b(A_2) \cong D^b(Y_d),
\]

where $Y_d$ is a small resolution of the singularity

\[
X_d := \text{Spec} \left( \mathbb{C}[x, y, z, w]/(x^2 + y^2 + (z + w^2)(z - w^d) \right),
\]

see [10] for more details of all this. We define $M_{Q_2,W_2,\gamma}^{sp}$ to be the subspace of $A_2$ representations that live on the exceptional locus of this resolution. Then one can show that the CoHA multiplication induces a surjection

\[
\cdots \otimes \mathcal{H}_{Q_2,W_2,(m+1,m)}^{sp,st} \otimes \mathcal{H}_{Q_2,W_2,(m,m-1)}^{sp,st} \otimes \cdots \otimes \mathcal{H}_{Q_2,W_2,(1,1)}^{sp,st} \otimes \cdots \otimes \mathcal{H}_{Q_2,W_2,(m-1,m)}^{sp,st} \otimes \cdots \rightarrow \mathcal{H}_{Q_2,W_2}^{sp}
\]

where $\mathcal{H}_{Q_2,W_2,(m\pm 1,m)}^{sp,st}$ is the CoHA of extensions of a fixed $A_2$ representation of dimension $(m \pm 1, m)$, and $\mathcal{H}_{Q_2,W_2,(1,1)}^{sp,st}$ is the CoHA of representations of $A_2$ that correspond to zero dimensional sheaves under the equivalence (84). By Example 8.3

\[
\mathcal{H}_{Q_2,W_2,(m\pm 1,m)}^{sp,st} \cong \mathcal{H}_{Q_{ser},W_{ser}}
\]

carries a pure Hodge structure, and by results of [9], so does $\mathcal{H}_{Q_2,W_2,(1,1)}^{sp,st}$. We deduce from the surjectivity of (84) that $\mathcal{H}_{Q_2,W_2}^{sp}$ is pure. By the main result of [10] there is an equality of generating functions

\[
\chi_q(\mathcal{H}_{Q_2,W_2,\gamma}^{sp}) = \left( \prod_{m \geq 1} (1 - x^{(m,m+1)})^{d_1} (1-q)^{-1} \right) (1 - x^{(m+1,m)})^{d_2} (1-q)^{-1} (1-x^{(m,m)})^{d_3} (1-q)^{-1}
\]

and so from purity we can calculate the dimensions of the spaces $V_{prim,\gamma}$: for $\gamma = (m, m \pm 1)$ they are concentrated in odd degree and are of dimension $d_1$; for $\gamma = (m, m)$ they are of dimension one and in even degree. In particular, $\mathcal{H}_{Q_2,W_2}^{sp}$ is a tensor product of an exterior algebra on countably many generators and a symmetric algebra on countably many generators.
7.4. **Length 2 rational curves.** Examples [7.2] and [7.4] concern noncommutative crepant resolutions (in the sense of [28]) of 3-dimensional isolated singularities admitting crepant resolutions for which the exceptional locus is $\mathbb{P}^1$. In such a situation, the normal bundle to the $\mathbb{P}^1$ appearing in the resolution is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. In the first instance we are considering the conifold, which is the most well-known example, and in the second, which corresponds to Example [7.4] we are considering a ramified cover of the conifold, whose geometry is perhaps only slightly more complicated (and for which the classification of such curves is not too complicated, being given by the parameter $d$). The third class of examples is something of a zoo, and not even a complete classification is at hand. They are partly parameterised by a measure called “length”, and we are able to say a little about some length 2 examples.

**Example 7.5.** Firstly we have the Morrisson-Pinkham example, for which we follow [1]. This is the singularity

$$x^2 + y^3 + wz^2 + w^3y - \lambda wy^2 - \lambda w^4 = 0$$

where $\lambda$ is a fixed parameter. The associated noncommutative crepant resolution has quiver

```
1 \rightarrow 2 \rightarrow 3
```

and superpotential

$$W_\lambda = b^2 dc + \frac{1}{2} dcdc + a^2 b + \frac{1}{3} \lambda b^3 + \frac{1}{4} b^4$$

and the QP $(Q, W_\lambda)$ is obviously self dual for every $\lambda$, with self-duality $\sigma$ fixing the vertices $Q_0$. It follows from Theorem [6.9] that the critical CoHA $\mathcal{H}_{Q, W}$ is free $\mathbb{C}$-supercommutative.

**Example 7.6.** Next consider the equation

$$x^2 + y^3 + wz^2 + w^{2n+1}y = 0.$$ 

This is a class of examples considered by Laufer [20], which again gives a class of length two singularities. The noncommutative crepant resolution is written down in [1], it is given by setting $Q_{\text{Lau}}$ to be the quiver

```
C \rightarrow 1 \rightarrow 2
```

and setting

$$W_{\text{Lau}} := dwc + dcb^2 + a^2 b + \frac{w^{n+1}}{n+1} + \frac{(-1)^n b^{2n+2}}{2n+2}.$$ 

Again, our main theorem implies that $\mathcal{H}_{Q_{\text{Lau}}, W_{\text{Lau}}}$ is a free $\mathbb{C}$-supercommutative algebra.
7.5. Twisted and untwisted character varieties. Let $\Sigma_g$ denote the genus $g$ topological Riemann surface. We define
\[
\text{Rep}_m(\Sigma_g) := \{ A_1, \ldots, A_g, B_1, \ldots, B_g \in \text{GL}_C(m) \mid \prod (A_i, B_i) = \text{id}_{m \times m} \},
\]
where $(A_i, B_i)$ is the group theoretic commutator. This variety is a space of homomorphisms $\pi_1(\Sigma_g) \to \text{GL}_C(m)$, or representations of $\pi_1(\Sigma_g)$, and the stack theoretic quotient $[\text{Rep}_m(\Sigma_g)/\text{GL}_C(m)]$ is the stack of $n$ dimensional $\pi_1(\Sigma_g)$ representations. Let $\zeta_m$ be a primitive $m$th root of unity.

We will also consider the twisted counterpart of this variety
\[
\text{Rep}^{\zeta_m}_m(\Sigma_g) := \{ A_1, \ldots, A_g, B_1, \ldots, B_g \mid \prod (A_i, B_i) = \zeta_m \text{id}_{m \times m} \}.
\]

By [14, Cor.2.2.4], up to isomorphisms in cohomology lifting to isomorphisms in Hodge structure, it doesn’t matter which primitive $m$th root of unity we pick. We will build a free supercommutative CoHA $H^\mathbb{C}_{\Sigma_g}^{\text{sp}}$ such that there are natural isomorphisms
\[
H^\mathbb{C}_{\Sigma_g}^{\text{sp}}(\text{Rep}_m(\Sigma_g), \mathbb{Q})\{(1 - g)m^2\}^* \cong H^\mathbb{C}_{\text{GL}_C(m)}(\text{Rep}_m(\Sigma_g), \mathbb{Q})\{(1 - g)m^2\}^*,
\]
and conjectural isomorphisms
\[
V_{(m,m,m,m)} \cong H^\mathbb{C}_{\text{GL}_C(m)}(\text{Rep}^{\zeta_m}_m(\Sigma_g), \mathbb{Q})\{(1 - g)m^2\}^*\]

between the space of generators of degree $m$ and the compactly supported equivariant cohomology of $\text{Rep}^{\zeta_m}_m(\Sigma_g)$. By [14, Cor.2.2.7] $\text{Rep}^{\zeta_m}_m(\Sigma_g)$ is acted on freely by $\text{PGL}_C(m)$, so there is a natural isomorphism
\[
H^\mathbb{C}_{\text{GL}_C(m)}(\text{Rep}^{\zeta_m}_m(\Sigma_g), \mathbb{Q})\{(1 - g)m^2\}^* \cong H^\mathbb{C}_{\text{PGL}_C(m)}(\text{Rep}^{\zeta_m}_m(\Sigma_g)/\text{PGL}_C(m), \mathbb{Q})\{(1 - g)m^2\}^*[y]
\]

where $y$ is as ever the degree 2 generator of $H^\mathbb{C}_*(\text{pt}, \mathbb{Q})$. In other words, we conjecture that there are isomorphisms
\[
V_{\text{prim.}(m,m,m,m)} \cong H^\mathbb{C}_{\text{PGL}_C(m)}(\text{Rep}^{\zeta_m}_m(\Sigma_g)/\text{PGL}_C(m), \mathbb{Q})\{(1 - g)m^2\}^*
\]

between the space of generators of $H_{Q_{\Sigma_g},W_{\Sigma_g}}$ and the dual compactly supported cohomology of the twisted character varieties for $\Sigma_g$. Since by [14, Thm.2.2.5] the twisted character varieties are smooth, we may alternatively restate this as an isomorphism between the primitive generators of $H_{Q_{\Sigma_g},W_{\Sigma_g}}$ and the cohomology of the twisted character varieties.

Example 7.7. For ease of exposition we consider only the Riemann surface of genus 2, but everything generalises in a way that is hopefully obvious. We break the surface $\Sigma_2$ into 4 tiles. The front two tiles are as drawn in black in Figure 1, they are glued along the dashed line. The back two tiles are as drawn in Figure 2, they are again glued along the dashed line, and the back two tiles are glued to the front two by gluing along the solid black lines. We have drawn, in red and blue, the dual quiver to this tiling, this will be our quiver $Q_{\Sigma_2}$, which we reproduce
Now define
\[ W_{\Sigma_2} := lgfa - jgda + jhdb - kheb + kiec - lifc. \]
The recipe for this quiver with potential is as follows - in the literature it is called the QP associated to a brane tiling of a surface, see e.g. [4] for a detailed reference, [13] for the Physics.
To a tiling $\Delta$ of a Riemann surface, the 1-skeleton of which is given the structure of a bipartite graph, the quiver is just the dual quiver, as above, oriented so that the arrows go clockwise around the black vertices. The potential is given by taking the alternating sum

$$W_\Delta := \sum_{v \in \Delta_0 | v \text{ is white}} l_v - \sum_{v \in \Delta_0 | v \text{ is black}} l_v$$

where $l_v$ is the shortest cycle going around the vertex $v$ in the dual quiver.

Returning to our special case, we define $M_{sp\gamma} \subset M_{\gamma}$ by the condition that every red arrow is sent to an isomorphism, and since an upper block triangular matrix is invertible if and only if its diagonal blocks are, we deduce that these $M_{sp\gamma}$ satisfy Assumption 3.1. Note that $M_{sp\gamma} = \emptyset$ unless $\gamma = (m,m,m,m)$ for some $m$. The QP $(W_{\Sigma_2}, W_{\Sigma_2})$ admits a self-duality $\sigma$ defined by

$$\sigma(a), \sigma(b), \sigma(c) = a^T, b^T, c^T$$

respectively

$$\sigma(d), \sigma(e), \sigma(f) = j^T, k^T, l^T$$

respectively

$$\sigma(g), \sigma(h), \sigma(i) = g^T, h^T, i^T$$

respectively

$$\sigma(j), \sigma(k), \sigma(l) = g^T, h^T, i^T$$

respectively,

and one may verify that $\sigma^* W^T = W$, and that the spaces $M_{sp\gamma}^{Q_{\Sigma_2}}$ satisfy Assumption 5.8.

The QP $(W_{\Sigma_2}, W_{\Sigma_2})$ admits a cut, given by setting $S = \{a, b, c\}$, and the moduli spaces $M_{sp\gamma}^{Q_{\Sigma_2}}$ satisfy Assumption 5.8 so that we have an isomorphism in cohomology

$$H_{c,G\gamma}(M_{sp\gamma}^{Q_{\Sigma_2}}, \varphi_{tr(W_{\Sigma_2})}\gamma) \cong H_{c,G\gamma}(Z_{\gamma}, \mathbb{Q})$$

where $Z_{\gamma}$ is the space of representations of $Q_{\Sigma_2}$ such that all red arrows are sent to isomorphisms, and the relations

$$\partial W_{\Sigma_2}/\partial a = lgf - jgd$$

$$\partial W_{\Sigma_2}/\partial b = jhd - khe$$

$$\partial W_{\Sigma_2}/\partial c = kie - lif$$

are satisfied. Let $Z_{\gamma}$ be the space of representations of the quiver $Q'$, obtained by deleting arrows $a, b$ and $c$, still satisfying the relations 86. Up to gauge transformation we may assume that $d', g$ and $j$ are all the identity matrix, and consider $[Z_{\gamma}/G_{\gamma}]$ as the stack of representations of the 6 loop quiver with loops labelled by $e, f, h, i, k$ and $l$, satisfying the relations

$$fl = 1$$

$$h = khe$$

$$lie = kie$$

Substituting $l = f^{-1}$ and $k = he^{-1}h^{-1}$, we deduce that $Z_{\gamma}$ is the stack of $m$-dimensional representations of the 4 loop quiver, with loops labelled $f, h, e, i$, satisfying the one relation

$$he^{-1}h^{-1}ie = f^{-1}if,$$

which becomes

$$he^{-1}h^{-1}e = i^{-1}f^{-1}if$$
after the substitution $h \mapsto ih$. In other words, $[Z_\gamma/G_\gamma]$ is isomorphic to the stack $[\text{Rep}_m(\Sigma_2)/\text{GL}_C(m)]$ of $m$-dimensional representations of $\pi_1(\Sigma_g)$.

Via the affine fibration $\overline{Z}_\gamma \to Z_\gamma$ we obtain an isomorphism of mixed Hodge structures

$$H_{c,G}(M_{Q_{\Sigma_2},W_{\Sigma_2}}^{sp},\varphi_{tr}(W_{\Sigma_2}),\gamma) \simeq H_{c,\text{GL}_C(m)}(\text{Rep}_m(\Sigma_2),\Q)\{3m^2\}.$$ 

Since $\chi((m,m,m,m),(m,m,m,m)) = -8m^2$ we deduce that

$$\mathcal{H}_{Q_{\Sigma_2},W_{\Sigma_2}}^{sp}(m,m,m,m) \simeq H_{c,\text{GL}_C(m)}(\text{Rep}_m(\Sigma_2),\Q)\{m^2\}^*.$$ 

Generalising the above construction we have

$$\mathcal{H}_{Q_{\Sigma_g},W_{\Sigma_g}}^{sp}(m,m,m,m) \simeq H_{c,\text{GL}_C(m)}(\text{Rep}_m(\Sigma_g),\Q)\{(1-g)m^2\}^*.$$ 

We finish by returning to the conjectural form for the generators of $\mathcal{H}_{Q_{\Sigma_g},W_{\Sigma_g}}^{sp}$.

\textbf{Conjecture 7.8.} There are isomorphisms in MHS

$$V_{\text{prim},(m,m,m,m)} \simeq H_c(\text{Rep}_m^{\text{can}}(\Sigma_g)/\text{PGL}_m,\Q)\{(1-g)m^2\}^*.$$ 

The evidence for this conjecture comes from taking weight polynomials (E polynomials, in the terminology of [13]). In [13] the following calculation is made:

$$\chi_q(\mathcal{H}_{Q_{\Sigma_g},W_{\Sigma_g}}^{sp}) = \prod (1-x_m^g)\chi_q(H_{c,\text{GL}_C(m)}(\text{Rep}_m^{\text{can}}(\Sigma_g),\Q)\{(1-g)m^2\}^*),$$

which means that, assuming $\mathcal{H}_{Q_{\Sigma_g},W_{\Sigma_g}}^{sp}$ really does have finite dimensional spaces of primitive generators, these spaces have the same weight polynomials as the twisted character varieties $\text{Rep}_m^{\text{can}}(\Sigma_g)$, up to a Tate twist.

\textbf{Appendix A. Dimensional reduction for quivers with potential and a cut}

\textbf{A.1. Relating critical cohomology to ordinary cohomology.} Let $Y := X \times \mathbb{A}^n$ be the total space of the trivial vector bundle, carrying the $\mathbb{C}^*$ action that acts trivially on $X$ and with weight one on $\mathbb{A}^n$ (the rescaling action). Let $f : Y \to \mathbb{A}^1$ be $\mathbb{C}^*$ equivariant, where $\mathbb{C}^*$ acts with weight one on the target. We may express $f$ as follows

$$f = \sum_{a=1}^{a=n} f_a x_a$$

where $x_a$ is a linear coordinate system on $\mathbb{A}^n$, i.e. each $x_a$ is acted on with weight one by the $\mathbb{C}^*$ action, and the $f_a$ are functions on $X$. We define $Z \subset X$ to be the reduced scheme theoretic vanishing locus of all the $f_a$. Clearly $Z$ doesn’t depend on the linear coordinate system we pick for $\mathbb{A}^n$ - it can be defined without reference to it, as the space of points $z \in X$ such that $\pi^{-1}_X(z) \subset f^{-1}(0)$.

\textbf{Theorem A.1.} In the above situation, let $i : Z \to X$ be the closed inclusion. There is a natural isomorphism of functors $D^b(X) \to D^b(X)$

$$\pi_!\varphi f\pi^*[-1] \cong \pi_!\pi^*i_*i^*$$ (87)
So that in particular:

\[(88) \quad H_c^*(Y, \varphi_f) \cong H_c^*(Z \times \mathbb{A}^n, \mathbb{Q}) \cong H_c^{*+2n}(Z, \mathbb{Q})\]

The proof of Theorem A.1 is a little complicated – we first prove the special case \(n = 1\) and then reduce the general case to this.

**Lemma A.2.** Let \(X, Y\) and \(f\) be as above, and assume \(n = 1\). Then \(\psi_f\) has locally constant cohomology along the fibres of the projection \(X \times (\mathbb{A} \setminus \{0\}) \to X\).

**Proof.** We have \(f = f_1 x\) for \(x\) a coordinate on \(\mathbb{A}^1\). The lemma, restated in terms of \(f_1\), is the statement that \((f_1^{-1}(x \Re(z) \leq 0) \to X)_*\mathcal{Q}_{f_1^{-1}(x \Re(z) \leq 0)}\) gives a family of objects in the derived category of constructible sheaves on \(\mathbb{A}^1 \setminus \{0\} \times X\) which is locally trivial in the \((\mathbb{A}^1 \setminus \{0\})\) direction. This follows from the original definition of the vanishing cycles functor (the monodromy around 0 \(\in \mathbb{A}^1\) is precisely the monodromy operator on vanishing cycles). \(\square\)

**Lemma A.3.** Let \(\mathcal{L} \in D^b(X \times (\mathbb{A} \setminus \{0\}))\) have locally constant cohomology sheaves in the \(A\) direction, i.e. for each point \(x \in X\), if we let \(\overline{p}\) be the inclusion of the fibre over \(x\) into \(X \times (\mathbb{A} \setminus \{0\})\), \(\overline{p}^* \mathcal{L}\) has locally constant cohomology sheaves. Let \(r : X \times (\mathbb{A} \setminus \{0\}) \to Y\) be the inclusion. Then \(\pi r_* = 0\).

**Lemma A.4.** Theorem A.1 is true in the case \(n = 1\).

**Proof.** Let \(\tilde{i} : Z \times \mathbb{A}^1 \to Y\) be the natural inclusion. Let \(j\) be the inclusion of the open complement to \(\tilde{i}\). Then we have a distinguished triangle in the derived category of constructible sheaves on \(Y\)

\[
j_* j^* \varphi_f \pi^* \to \varphi_f \pi^* \to \tilde{i}_* \tilde{i}^* \varphi_f \pi^*
\]

and so an isomorphism \(\varphi_f \pi^* \cong i_* i^* \varphi_f \pi^*\), since \(Z \times \mathbb{A}^1\) contains the critical locus of \(f\), the support of \(\varphi_f\). Now by definition we have a distinguished triangle

\[
\pi \tilde{i}_* \tilde{i}^* \varphi_f \pi^*[-1] \to \pi \tilde{i}_* \tilde{i}^* \pi^* \to \pi \tilde{i}_* \tilde{i}^* \psi f \pi^*
\]

and the proposition will follow from the claim that \(\pi \tilde{i}_* \tilde{i}^* \psi f \pi^* = 0\). In other words we must show that

\[
\pi \tilde{i}_* \tilde{i}^*(Y_+ \to Y)_*(Y_+ \to Y)^* \pi^* = 0.
\]

There is an inclusion \(Y_+ \subset X \times (\mathbb{A}^1 \setminus \{0\})\) and so we must prove that

\[
\pi \tilde{i}_* \tilde{i}^*(X \times (\mathbb{A}^1 \setminus \{0\}) \to Y)_*(Y_+ \to X \times (\mathbb{A}^1 \setminus \{0\}))_*(Y_+ \to Y)^* \pi^* = 0,
\]

or

\[
i_* i^* \pi_1(X \times (\mathbb{A}^1 \setminus \{0\}) \to Y)_*(Y_+ \to X \times (\mathbb{A}^1 \setminus \{0\}))_*(Y_+ \to Y)^* \pi^* = 0,
\]

which is the same by base change. This follows from Lemmas A.2 and A.3. \(\square\)

**Proof of Theorem A.1.** Let \(g : Y \to Y\) be the blowup of \(Y\) along \(X \times \{0\}\), with exceptional divisor \(E\), and with \(\tilde{Z} := (Z \times \mathbb{A}^n) \times_Y Y\). As in the proof of Lemma A.4 we need to show that
The inclusion $Y_+ \to Y$ factors through $\bar{Y} \to Y$, and so we may write

$$R\pi_1 i^* \psi_f \pi^* = 0.$$ The inclusion $Y_+ \to Y$ factors through $\bar{Y} \to Y$, and so we may write

$$\pi_1 i^* \psi_f \pi^* \cong (Y \to X)_!(Z \times \mathbb{A}^n \to Y)_*(Z \times \mathbb{A}^n \to Y)^*(Y_+ \to Y)_*(Y_+ \to Y)^*(Y \to X)^*$$

$$\cong (Y \to X)_!(Z \times \mathbb{A}^n \to Y)_*(Z \times \mathbb{A}^n \to Y)^*(\bar{Y} \to Y)_*(Y_+ \to \bar{Y})_*$$

$$\cong (Y \to X)_!(\bar{Y} \to Y)_*(\bar{Y} \to Y)_*(Y_+ \to \bar{Y})^*(\bar{Y} \to E)^*(E \to X)^*$$

$$\cong (E \to X)_!(\bar{Y} \to E)_!(\bar{Y} \to E)_*(Y_+ \to \bar{Y})^*(Y_+ \to \bar{Y})^*(\bar{Y} \to E)^*(E \to X)^*$$

and the result will follow from the claim that

$$(89) \quad (\bar{Y} \to E)_!(\bar{Y} \to E)_!(\bar{Y} \to Y)_*(Y_+ \to \bar{Y})^*(\bar{Y} \to E)^*(E \to X)^* = 0.$$ If we let $U \subset \bar{Y}$ be defined by the condition $x_j \neq 0$ for some $j \leq n$, we may write

$$fg = \mathcal{L} \sum f_i x_i / x_j$$

from which we deduce that we are in the situation of Lemma A.4, with $L$ our coordinate on $\mathbb{A}^1$ and $f'_i = \sum f_i x_i / x_j$. The space $\bar{Z} \cap U$ is contained in the vanishing locus of the function $f'_i$ since on an open subset of $\bar{Z} \cap U$ all the $f_i$ vanish, from which we deduce (89) from Lemma A.4. □

Let $Y = Y_1 \times Y_2$, and let $f = f_1 \oplus f_2$. Assume furthermore that both $Y_1$ and $Y_2$ admit product decompositions $Y_i \cong X_i \times \mathbb{A}^{n_i}$ such that the $f_i$ are $\mathbb{C}^*$ equivariant, where $\mathbb{C}^*$ acts with weight one on $\mathbb{A}^{n_i}$, trivially on $X_i$, and with weight one on the target $\mathbb{C}$. From the commutativity of the square of derived functors

$$\Gamma_{\{\Re f_1 \leq 0\}}(\bullet) \oplus \Gamma_{\{\Re f_2 \leq 0\}}(\bullet) \longrightarrow \Gamma_{\{\Re (f_1 \oplus f_2) \leq 0\}}(\bullet \oplus \bullet)$$

$$\text{id}(\bullet) \oplus \text{id}(\bullet) \longrightarrow \text{id}(\bullet \oplus \bullet)$$

we deduce the following proposition.

**Proposition A.5.** The following diagram of isomorphisms commutes

$$\begin{array}{ccc}
H^*_c(f_1^{-1}(0), \varphi_{f_1}) \otimes H^*_c(f_2^{-1}(0), \varphi_{f_2}) & \longrightarrow & H^*_c(f_1^{-1}(0) \times f_2^{-1}(0), \varphi_{f_1 \oplus f_2}) \\
\downarrow & & \downarrow \\
H^*_c(Z_1 \times \mathbb{A}^{n_1}, \mathbb{Q}) \otimes H^*_c(Z_2 \times \mathbb{A}^{n_2}, \mathbb{Q}) & \longrightarrow & H^*_c(Z_1 \times Z_2 \times \mathbb{A}^{n_1+n_2}, \mathbb{Q})
\end{array}$$

where TS is the Thom-Sebastiani isomorphism, Ku is the Kunneth isomorphism, and the vertical isomorphisms are as in Theorem A.1.

Theorem A.1 can be expressed by saying that the natural transformation of sheaves $\pi_1(Z \times \mathbb{A}^n \to Y)_*(Z \times \mathbb{A}^n \to Y)^*(\Gamma_{\{\Re f \leq 0\}} \to \text{id})$ is an isomorphism. Since this lifts to a natural transformation of mixed Hodge modules, and the forgetful functor from mixed Hodge modules to constructible sheaves is faithful, we deduce the following corollary.

**Corollary A.6.** There is a natural isomorphism $\pi_! \varphi_f \pi^*[-1] \to \pi_! \pi^* i_* i^*$ in the category $D^b(MHM(X))$. 
Corollary A.7. Let $X^{sp} \subset X$ be a subvariety of $X$, and define $Y^{sp} = X^{sp} \times \mathbb{A}^n$. There is a natural isomorphism in $\text{MMHS}$

$$H_c(Y^{sp}, \varphi_f) \cong H_c((Z \cap X^{sp}) \times \mathbb{A}^n, \mathbb{Q}).$$

Proof. Let $u$ be the coordinate on $\mathbb{C}^*$. By Theorem A.1 there is an isomorphism

$$(Y^{sp} \times \mathbb{C}^* \to Y \times \mathbb{C}^*)^*(\pi!\varphi_f/\pi^*Q_{X \times \mathbb{C}^*}[-1] \to \pi!\pi^*i_*i^*Q_{X \times \mathbb{C}^*})$$

which induces the desired isomorphism after taking proper pushforward along the projection to $\mathbb{C}^*$. □

Corollary A.8. With $Y$ as in the statement of Theorem A.1, assume also that $Y$ is a $G$-equivariant vector bundle over $X$, and $Y^{sp} = X^{sp} \times \mathbb{A}^n$ is the total space of a sub $G$-bundle. Then there is an isomorphism in $\text{MMHS}$

$$H_{c,G}(Y^{sp}, \varphi_f) \to H_{c,G}(X^{sp} \times \mathbb{A}^n, \mathbb{Q}).$$

Proof. Let $h : (Y,G)_N \to (Y,G)_{N+1}$ be the inclusion. The diagram commutes. Taking the Verdier dual, we deduce that the middle square of the following diagram commutes

$$\begin{array}{ccc}
\Gamma_{\{\text{Re}(f_{N+1}) \leq 0\}} Q_{(Y,G)_{N+1}} & \longrightarrow & h_*\Gamma_{\{\text{Re}(f_N) \leq 0\}} Q_{(Y,G)_N} \\
\downarrow & & \downarrow \\
Q_{(Y,G)_{N+1}} & \longrightarrow & h_*Q_{(Y,G)_N}
\end{array}$$

while the other two squares commute by commutativity of (10). Applying $(Y^{sp},G)_{N+1} \to (Y,G)_{N+1}^*$ and taking compactly supported cohomology, (90) becomes a diagram of isomorphisms by Theorem A.1 and so the entire diagram commutes. Commutativity of the outer
A.9. There is a canonical isomorphism

\[ H_c((Y^{sp}, G)_{N+1}, \mathbb{Q}) \{t\} \xleftarrow{\sim} H_c((Y^{sp}, G)_{N+1}, \varphi_T) \{t\} \]

and the corollary follows. \( \square \)

A.2. The critical cohomology of a quiver with potential and a cut. Let \( Q \) be a quiver with algebraic potential \( W \in \mathbb{C}Q/\mathbb{C}Q, \mathbb{C}Q \). We call such a pair \((Q, W)\) a QP from now on. We further assume that the QP \((Q, W)\) admits a cut. That is, there is a grading \( \nu : Q_1 \to \mathbb{N} \) of the edges of \( Q \) with zeroes and ones such that \( W \) is homogeneous of degree one. Equivalently, there is a subset \((\nu^{-1}(1) = \gamma)\) such that every cyclic word in \( W \) contains exactly one instance of exactly one arrow of \( S \). For \( \gamma \in \mathbb{N}_{Q_0} \) denote by \( M_{\gamma} \), as before, the affine space

\[ \bigoplus_{a \in \mathbb{N}_{Q_0}} \text{Hom}(\mathbb{C}^{(s(a))}, \mathbb{C}^{(\ell(a))}). \]

The space \( M_{\gamma} \) admits a decomposition \( M_{\gamma} = (M_{\gamma})_0 \oplus (M_{\gamma})_1 \), where

\[ (M_{\gamma})_i := \bigoplus_{a \in \nu^{-1}(i)} \text{Hom}(\mathbb{C}^{(s(a))}, \mathbb{C}^{(\ell(a))}). \]

This is the weight space decomposition of the natural \( \mathbb{C}^* \) action on \( M_{\gamma} \), defined via \( \nu \).

Up to cyclic permutation of words in \( Q \), we may write \( W = \sum_{a \in \nu^{-1}(1)} a p_a \), where \( p_a \) are linear combinations of paths in \( Q \) containing only arrows in \( \nu^{-1}(0) \). We define

\[ \tilde{\iota}_\gamma : Z_\gamma \times \mathbb{A}^{\sum_{a \in \nu^{-1}(1)} \gamma(s(a)) \gamma(\ell(a))} \to M_{\gamma} \]

to be the inclusion of subscheme cut out by the matrix valued equations \( \{p_a = 0 | a \in \nu^{-1}(1)\} \).

As in the main document we assume that we are given subspaces \( M_{\gamma}^{sp} \subset M_{\gamma} \) satisfying the following additional assumption.

Assumption A.9. The spaces \( M_{\gamma}^{sp} \) are in turn given by pullbacks of subspaces \( \pi_{\gamma,0}^{-1}(M_{\gamma})^{sp} \) for \( (M_{\gamma})_0^{sp} \subset (M_{\gamma})_0 \) and \( \pi_{\gamma,0} : M_{\gamma} \to (M_{\gamma})_0 \) the natural projection.

The following is then a direct application of Corollary A.8

**Theorem A.10.** Let \((Q, W)\) admit a cut, and assume that the spaces \( M_{\gamma}^{sp} \) satisfy Assumption A.9. There is a canonical isomorphism

\[ H_c(G_{\gamma}(M_{\gamma}^{sp}, \varphi_{tr(W)}_{\gamma})) \xrightarrow{\cong} H_c(G_{\gamma}(\pi_{\gamma,0}^{-1}(Z_{\gamma}^{sp}), \mathbb{Q})). \]
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