The effect of Quintessence on the low and high temperature behavior of entanglement entropy for boundary field theory dual to AdS black holes

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The effect of dark energy on the low and high temperature behavior of strip region of the boundary field theory dual to AdS black holes was studied. In this framework, we investigate this behavior for different types of equation state and different regimes of normalization factor \( a \) and \( |\omega| \). We will see that the changes of entanglement entropy is growing up with increasing \( a \) and \( |\omega| \).

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I. INTRODUCTION

In recent decades, quantum entanglement has played an important role in various areas of physics such as quantum field theory, condensed matter systems, statistical mechanics and quantum gravity. The entanglement entropy provides us with a convenient way to measure quantum correlations in a bipartite system. We assume that the system is in pure state \(|\Psi\rangle\) with the density matrix \( \rho = |\Psi\rangle \langle \Psi | \), therefore von-Neuman entropy of this system is defined as \( S = -\text{tr} \rho \log \rho = 0 \). This quantity does not give us any useful information. Thus the total system is divided two subsystems \( A \) that is being studied and \( B \) that is the complement of \( A \). The observer only can access to information from subsystem \( A \) and can not receive any signal from \( B \). This situation is analogous to the case that inside of a black hole \((B)\) is not accessible for an observer in outside of the horizon \((A)\). Mathematically this can be realised by expressing the full Hilbert space \( \mathcal{H} \) as a tensor product of Hilbert spaces \( \mathcal{A} \) and \( \mathcal{B} \), that is, \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \). The entanglement entropy of the system \( A \) is defined as

\[
S_A = -\text{tr} \rho_A \log \rho_A
\]

where \( \rho_A \) is the reduced density matrix of subsystem \( A \) obtained via taking a partial trace \( \rho \) over the substem \( B \), \( \rho_A = \text{tr}_B \rho \). The entanglement entropy of \( B \) can be obtained by the same method too. Entanglement entropy has important properties such as (i) \( S_A = S_B \) and (ii) \( S_A + S_B \geq S_{A\cup B} + S_{A\cap B} \) called subadditivity condition [1].

Direct calculation of von-Neuman entropy in quantum field theory is very complicated. Therefore, a method has been propsed based on replica trick [2–5]. In their approach the entanglement entropy is given by

\[
S_n = \lim_{n \to 1} \frac{1}{1-n} \text{tr} \rho^n_A = -\frac{\partial}{\partial n} \text{tr} \rho^n_A |_{n=1}.
\]

Similar to quantum mechanics, the density matrix is defined in terms of euclidean path integral on an \( n \)-sheeted Riemann surface. This method is used for calculating the entanglement entropy of 1 + 1 CFT in critical [2, 8] and non critical [4, 5] phenamena. Because of the large size of hilbert spaces, replica trick method for calculating entanglement entropy in higher dimensional CFTs encounters problems. Therefore, in spite of noticable successes for 1 + 1 CFT, the direct calculation of entanglement entropy in higher dimensional CFTs is restricted to quasi free fermions and bosons [8, 12].

A very beneficial and applicable approach, at least for quantum systems that have holographic description, is using of holographic entanglement entropy that proposed and developed by the authors in [13, 14]. This method is derived from AdS/CFT duality that relates gravity on an asymptotically local \((d+1)\) dimensional AdS spacetime to \((d)\) dimensional strongly coupled boundary quantum field theory with a UV fixed point [15]. In the next section, this method is described in \((3+1)\) dimensions in details.

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Recently astronomical observations show that the expansion of the universe is accelerated \[22, 23\], that can be explained by the assumption that our universe is filled with a special state of matter with negative pressure. This can be interpreted by a cosmological constant, but the measured vacuum energy density differs largely from zero point energy predicted by quantum field theory. This disagreement is called cosmological constant problem \[24\]. An alternative way may be constructing black hole solutions with quintessence by using dynamical scalar fields \[25–29\]. In this framework, a state equation is obtained as a relation between the pressure and energy density. A static spherically-symmetric exact black hole solutions of Einstein equations with the quintessential matter was proposed by Kiselev in \[30\]. Thermodynamics of the neutral and charged black holes with quintessence was studied in \[31–33\].

The holographic entanglement entropy was used as a nonlocal observable for probing phase structure in quintessence Reissner-Nordström-AdS black hole in \[34\]. From the dual CFT perspective, the effect of quintessence on the formation of superconductor \[35\] and on the non-equilibrium thermalization \[36\] were studied. although at present, the dual field interpretation of the quintessence is not completely known. In this work, we intend to study the behavior of the entanglement entropy of strongly coupled boundary field theory dual to AdS black holes surrounded by quintessence at low and high temperature.

The rest of this paper is organized as follows. In section II we present a review of the holographic entanglement entropy method for a general metric. In the two next sections we calculate the entanglement entropy of AdS black hole surrounded by quintessence at low and high temperature for small and large quintessence charge regimes. Finally in the last section, the results summarized as conclusions.

II. REVIEW OF THE HOLOGRAPHIC ENTANGLEMENT ENTROPY

In the first step we review the holographic entanglement entropy method proposed by Ryu and Takayanagi \[13\] for boundary CFT dual to a static spherically-symmetric \(AdS_{3+1}\) black hole. As mentioned in previous section, a part of the boundary field theory \(A\) is isolated from the rest of system that entangled with \(A\) as shown in Fig 1. This subsystem is entangled with the rest of subsystem. According to this proposal, the entanglement entropy is given by

\[
S_A = \frac{\mathcal{A}(\gamma_A)}{4G^{(3+1)}_N},
\]
where \( G_N^{(3+1)} \) is the Newton’s constant in 3 + 1 dimensional bulk spacetime and \( \gamma_A \) is the (2) dimensional minimal area surface in the bulk whose boundary on the bulk is the boundary of conformal field theory on region \( A \), \( \partial \gamma_A = \partial A \). This formula is motivated from Bekenstein-Hawking entropy due to similarity between entangled quantum systems and black holes. Although, the entanglement entropy is different from Bekenstein-Hawking entropy because \( \text{Area}(\gamma_A) \) is not necessarily the area of event horizon. This represents the surface of region \( A \) lives in the conformal boundary of AdS spacetime. The entanglement entropy of the AdS Schwarzschild black hole \([16–19]\) and Reissner-Nordström-AdS black hole \([20, 21]\) at finite temperature were studied by this method.

The metric for a black hole in poincaré coordinate can be written as

\[
d s_{3+1}^2 = -r^2 f(r) d t^2 + \frac{1}{r^2 f(r)} d r^2 + r^2 (d x^2 + d y^2),
\]

where we assume the AdS length is set to one. The boundary field theory \( A \) is geometrically an infinite strip specified by

\[
x \equiv \in [-\frac{L}{2}, \frac{L}{2}], \quad y \in [-\frac{L}{2}, \frac{L}{2}]
\]

with \( L \to \infty \). The minimal area of the surface enclosed by the boundary of \( A \) is given by

\[
\mathcal{A} = L \int_{r_c}^{\infty} d r \sqrt{r^2 x'^2 + \frac{1}{r^2 f(r)}}.
\]

If we consider the area as an action, we can see that the lagrangian do not depend to \( x \). Therefore the equation of motion leads to a constant of motion

\[
\frac{r^2}{\sqrt{1 + r^4 x'^2 f(r)}} = \text{const.} = c,
\]

thus, we have

\[
\frac{d x}{d r} = \frac{c}{r^4 \sqrt{\left(1 - \frac{c^2}{r^4}\right)f(r)}}.
\]

The constant \( c \) may be found by taking this fact that at turning point \( r_c \), \( x' \) diverges, therefore, we obtain \( c^2 = r_c^4 \). By taking \( x(\infty) = \pm \frac{L}{2} \) one can easily derive the width of strip or extremal length as

\[
\frac{l}{2} = \int_{r_c}^{\infty} d r \sqrt{\frac{r_c^2}{r^4 \sqrt{1 - \left(\frac{c}{r}\right)^4}} f(r)^{-\frac{1}{2}}}.
\]

\[
= \int_{0}^{1} d u \sqrt{\frac{r_c^2}{u^2 \sqrt{1 - u^4} f(u)^{-\frac{1}{2}}}}.
\]

where we used the variable change \( u = \frac{r}{r_c} \). The area of the minimal surface can be expressed as

\[
\mathcal{A} = 2L \int_{r_c}^{\infty} d r \sqrt{\frac{r^2}{1 - \left(\frac{c}{r}\right)^4}} f(r)^{-\frac{1}{2}}.
\]

\[
= 2L \int_{0}^{1} d u \sqrt{\frac{r_c^2}{u^2 \sqrt{1 - u^4} f(u)^{-\frac{1}{2}}}}.
\]

When \( r \to \infty \), the intergrand becomes infinite, therefore, the entanglement entropy has a divergence. Because of too many degrees of freedom, always the entanglement entropy is infinite. For regularization this the entanglement entropy can be divided two parts, a divergene part and a finite part. Divergence part is temperature independent and can be computed easily by introducing an infrared cut off \( r_b \) (dual to UV cut off \( a = \frac{1}{r_b} \) in boundary field). On the
other hand, we use the finite part of entanglement entropy to explore the high and low temperature of entanglement entropy for the boundary field dual to AdS black hole

$$S_A^{\text{finite}} = S_A - S_A^{\text{div}} = \frac{A_A^{\text{finite}}}{4G_N^{3+1}}.$$  \hspace{1cm} (11)

The integrals given in equations (9) and (10) had not any analytical solutions yet. Therefore, a few approximation methods have been proposed for the computation of these integrals, although the most of these methods are valid for low temperature regime. The authors in [18] present a technique based on expansion by which one can calculate the entanglement entropy of a black hole at low and high temperature. In this paper we apply this technique to obtain an analytical expression for entanglement entropy of a boundary CFT dual to AdS black hole surrounded by a quintessence.

The low temperature limit of boundary CFT corresponds to a black hole with small radius. It is shown in [18] that the leading contribution of entanglement entropy is due to pure AdS$^{3+1}$ bulk. In this framework the low temperature limit of entanglement entropy of boundary CFT dual to AdS black hole can be expressed as

$$S_A = S_A^{\text{AdS}} + k(r_h)^3.$$  \hspace{1cm} (12)

On the other hand it was shown that at high temperature, that is when $r_h \to r_c$, the entanglement entropy behaves as the following equation

$$S_A = c_0 L^2 T + T (c_1 + c_2 \epsilon) + O(\epsilon^2),$$

$$\epsilon \approx \epsilon_{\text{ent}} e^{-C T l},$$

where $c_i (i = 0, 1, 2)$ and $\epsilon_{\text{ent}}$ and $C$ are constants. In eq. (13) the first term scales with area of the subsystem (in higher dimension with volume of the subsystem) and corresponds to thermal entropy of the system. The other terms scales with the length of boundary of subsystem (in higher dimension area of the substem) and correspond to the entanglement between the subsystem $A$ and the rest of the system.

### III. THE ENTANGLEMENT ENTROPY OF QUINTESSENCE PLANAR ADS BLACK HOLES

In this section, we firstly present the metric describing AdS planar black holes surrounded by quintessence. Then we use Ryu and Takayanagi approach to compute the entanglement entropy of quintessence AdS black hole dual to a strip region in the boundary field theory. Similar to method used in [30] and considering metric anstaz in eq. (4) one can find the function $f(r)$ as [37] used a technique based on expansion that

$$f(r) = 1 - \frac{2M}{r^3} - \frac{a}{r^3 (\omega + 1)}.$$  \hspace{1cm} (15)

where $M$ is the mass parameter of the black hole, $\rho$ is the quintessence energy density, $\omega$ is state equation parameter and $a$ is the normalization factor that we call it quintessence charge. It relates to density of quintessence as follows

$$\rho = -\frac{3\omega a}{2r^3 (\omega + 1)}.$$  \hspace{1cm} (16)

where for quintessence matter $-1 < \omega < 0$ and $\omega < -1$ is corresponded to phantom dark energy. For $\omega = -1$, the quintessence affects the AdS radius that is due to cosmological constant while for $\omega = -\frac{1}{3}$, dark energy affects the curvature $k$ of the spacetime.

For deriving the relation between $M$ and $a$, it is sufficient to put $f(r_h) = 0$ where $r_h$ is the horizon radius of the black hole, therefore, we have

$$M = \frac{1}{2} \left( r_h^3 - \frac{a}{r_h^{3\omega}} \right).$$  \hspace{1cm} (17)

Here, the positive mass of black hole leads us to have the following condition

$$\frac{a}{r_h^{3(1+\omega)}} < 1.$$  \hspace{1cm} (18)
\[ f(r) = 1 - \left( \frac{r_h}{r} \right)^3 + \frac{a}{r^3} \left( \frac{1}{r_h^2} - \frac{1}{r^{2\omega}} \right) \]  

(19)

The Hawking temperature of the quintessence AdS black holes can be written as

\[ T = \frac{r^2 f'(r)}{4\pi} \bigg|_{r=r_h} = \frac{3r_h}{4\pi} \left( 1 + \frac{a}{r_h^{3(\omega+1)}} \right). \]  

(20)

The allowed range of black hole temperature can be obtained by considering condition (18) as

\[ \frac{3r_h}{4\pi} (1 + \omega) < T < \frac{3r_h}{4\pi} \]  

(21)

By substituting \( f(r) \) from eq. (19) in equations (9) and (10), \( l \) and \( A \) are given by

\[ l = \frac{2}{r_c} \int_0^1 du \frac{u^2}{\sqrt{1-u^3}} \left( 1 - \left( \frac{r_h}{r_c} \right)^3 u^3 + \frac{a}{r_h^{3(\omega+1)}} \left( \frac{r_h^3}{r_c^3} u^3 - \frac{a}{r_h^{3(\omega+1)}} \right) \right)^{-\frac{1}{2}}. \]  

(22)

\[ A = 2L \int_0^1 du \frac{r_c}{u^2\sqrt{1-u^3}} \left( 1 - \left( \frac{r_h}{r_c} \right)^3 u^3 + \frac{a}{r_h^{3(\omega+1)}} \left( \frac{r_h^3}{r_c^3} u^3 - \frac{a}{r_h^{3(\omega+1)}} \right) \right)^{-\frac{1}{2}}. \]  

(23)

For a given \( \omega \), the state space of the boundary field theory dual to quintessence AdS black hole depends on \( T \) and \( a \). Therefore it is needed to work in a specific ensemble. In order to fix \( a \), we consider the system in canonical ensemble. Thus, in our study for finite temperature entanglement entropy, by attention to (21) one can find that the value of \( r_h \) can be considered as a measure of the black hole temperature. Except in the cases \( \omega \) is very close to \(-1\), the large quintessence charge \( a \), \( r_h \) must be large, the larger the radius of black hole, the higher the temperature. We must notice that when \( \omega \rightarrow -1 \), a large value of \( a \), the large value of \( r_h \) is not necessarily related to high temperature. For small values of \( a \), the radius of black hole can be small or large. This can be related to low or high temperature respectively.

In the following subsections we study the behavior of entanglement entropy for the boundary CFT dual to AdS black hole surrounded by quintessence at different quintessence charge regimes.

### IV. BLACK HOLE IN QUINTESSENCE WITH A SMALL CHARGE

In this section we explore the low and high temperature behavior of entanglement entropy for subsystem \( A \) boundary to AdS black hole surrounded by quintessence. We study these behaviors at different region of \( \omega \).

#### A. Small \( a \) regime (low temperature)

At first step we consider a subsystem of boundary field theory dual to AdS quintessence black hole with a small \( a \) parameter at a low temperature. From the mass condition in eq. (18), we can find that when the temperature and charge both are small, the horizon radius is small. In this condition \( r_h \ll r_c \). We can find that when \( \omega \) is not very close to \(-1\), the small temperature and quintessence charge leads to a relation as \( \frac{r_h}{r_c} \sim 1 \). Therefore, in this case the lapse function \( f(u) \) has a form as

\[ f(u) = 1 - \left( \frac{r_h}{r_c} \right)^3 u^3 + \frac{a}{r_h^{3(\omega+1)}} \left( \frac{r_h^3}{r_c^3} u^3 - \left( \frac{r_h^3}{r_c^3} \right)^{3(\omega+1)} u^{3(\omega+1)} \right) \]  

(24)

Defining a new parameter \( \xi = \frac{a}{r_h^{3(\omega+1)}} \), we expand \( f^{-\frac{1}{2}}(u) \) around \( \frac{r_h}{r_c} = 0 \). Therefore by keeping the terms up to \( O\left( \frac{r_h}{r_c} \right)^3 u^3 \), the lapse function has the following form

\[ f^{-\frac{1}{2}}(u) \approx 1 + \frac{1 - \xi}{2} \left( \frac{r_h}{r_c} \right)^3 u^3 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n + 1)} \xi^n \left( \frac{r_h}{r_c} \right)^{3n(\omega+1)} u^{3n(\omega+1)} \]  

(25)
Therefore the subsystem length and minimal surface area are expressed as follows

\[ l \approx \frac{2}{\kappa_c} \int_0^1 \frac{u^2 du}{\sqrt{1 - u^4}} \left( 1 + \frac{1 - \xi}{2} \left( \frac{r_h}{\kappa_c} \right)^3 u^3 + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi n(n + 1)}} \xi^n \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1)} u^{3n(\omega + 1)} \right) \]

(26)

we can solve above integral and derive the relation between \( l \) and \( \kappa_c \) as

\[ \kappa_c \approx \frac{2}{l} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{4 \Gamma(n + 1) \Gamma(n + 1)} \xi^n \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1)} + \frac{\pi (1 - \xi) \left( \frac{r_h}{\kappa_c} \right)^3}{16} \right] \]

(27)

The extremal area surface can be obtained as

\[ A \approx 2 L \kappa_c \int_0^1 \frac{du}{u^2 \sqrt{1 - u^4}} \left( 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} n(n + 1)} \xi^n \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1)} u^{3n(\omega + 1)} + \frac{1 - \xi}{2} \left( \frac{r_h}{\kappa_c} \right)^3 u^3 \right) \]

(28)

the first term in (28) is correspond to pure AdS and is divergent. Therefore, in order to obtain a finite area we introduce an IR cut off \( r_c \) (dual to UV cut off \( c = \frac{1}{r_h} \) of boundary theory) and subtract a counter term from it. Therefore, the first term can be regularized as

\[ A_{1finite} = 2 L \kappa_c \int_0^1 \frac{du}{u^2 \sqrt{1 - u^4}} - 2 L r_b = \frac{\sqrt{\pi} L \kappa_c \Gamma \left( \frac{-1}{4} \right)}{2 \Gamma \left( \frac{1}{4} \right)} \]

(29)

The terms in the sum are divergent when for a special \( n \) in the sum, \( n < \frac{2}{3(1 + \omega)} \). Therefore when in the sum \( n < \frac{2}{3(1 + \omega)} \) and \( 3n(\omega + 1) \neq 1 \) the integral for these term can be solved as follows

\[ A_{2finite} = 2 L \kappa_c \left[ \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} n(n + 1)} \left[ \int_0^1 \frac{du}{u^2 \sqrt{1 - u^4}} \right] \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1) - 2} \right] + \sum_{n=1}^{\infty} \frac{1}{3n(\omega + 1) - 2} \left( \frac{r_c}{r_h} \right)^{3n(\omega + 1) - 2} \left( \frac{r_h}{r_c} \right)^{3n(\omega + 1)} \]

\[ = \frac{L \kappa_c}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1) - 1} \left( \frac{r_h}{r_c} \right)^{3n(\omega + 1)} \]

(30)

Therefore when in the sum \( 3n(\omega + 1) \neq 1 \) the extremal area can be expressed as

\[ A_{finite} = \frac{L \kappa_c}{2} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{-1}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \right] + \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} \left( \frac{3n(\omega + 1) - 1}{4} \right) \xi^n \left( \frac{r_h}{\kappa_c} \right)^{3n(\omega + 1)} \left( \frac{r_h}{r_c} \right)^3 + \frac{\pi}{2} (1 - \xi) \left( \frac{r_h}{r_c} \right)^3 \]

When \( \omega \to -1 \), we have many terms in the lapse function and the calculation is very hard and almost impossible. Fortunately, in this case, \( r_h^{3(\omega + 1)} \to 1 \). Thus \( \xi \ll 1 \) and we can neglect the higher order terms in the lapse function. \( f^{-\frac{3}{4}} \) has a form as follows

\[ f^{-\frac{3}{4}}(u) \approx 1 + \frac{1}{2} \xi \left( \frac{r_h}{r_c} \right)^{3(\omega + 1)} u^{3(\omega + 1)} + \frac{1 - \xi}{2} \left( \frac{r_h}{r_c} \right)^3 u^3 \]

(31)

Therefore the subsystem length and minimal surface area are expressed as follow

\[ r_c \approx \frac{2}{l} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} + \frac{\sqrt{\pi} \Gamma \left( \frac{3(\omega + 6)}{4} \right)}{2 \Gamma \left( \frac{3(\omega + 6)}{4} \right)} \xi (r_h l)^{3(\omega + 1)} - \frac{1}{2 \sqrt{\pi}} (1 - \xi) \left( \frac{r_h}{r_c} \right)^3 \right] \]

(32)

\[ A_{finite} \approx L \left[ - \frac{\pi \Gamma \left( \frac{-3}{4} \right)^2}{4 \Gamma \left( \frac{1}{4} \right)^2} + \frac{1}{2} \Gamma \left( \frac{1}{4} \right)^2 (1 - \xi)(r_l h)^3 + \frac{\pi}{2} \left( \frac{1}{\Gamma \left( \frac{3(\omega + 6)}{4} \right)} - \frac{\Gamma \left( \frac{3(\omega + 2)}{4} \right)}{2 \Gamma \left( \frac{3(\omega + 6)}{4} \right)} \right) \xi (r_l h)^{3(\omega + 1)} \right] \]

(33)
Here we compute the entanglement entropy for several values of $\omega$. For $\omega = -\frac{1}{2}$, $r_c$ and $A$ are expressed as

$$r_c = \frac{2}{l} \sqrt{\pi \Gamma\left(\frac{3}{4}\right)} \left[1 + \frac{\xi}{96\pi \Gamma\left(\frac{3}{4}\right)} (r_h l)^2 + \frac{1 - \xi}{128\pi \Gamma\left(\frac{3}{4}\right)} (r_h l)^3 + \mathcal{O}(r_h l^4) \right]$$

(34)

$$A_{finite} = -\frac{1}{4} \left(\frac{L}{l}\right)^2 \Gamma\left(-\frac{1}{4}\right)^2 \left[1 - \frac{16\xi}{3\pi \Gamma\left(-\frac{1}{4}\right)^4} (r_h l)^2 - \frac{2(1 - \xi)}{\pi} \Gamma\left(-\frac{1}{4}\right)^4 (r_h l)^3 + \mathcal{O}(r_h l^4) \right]$$

(35)

For $\omega = -\frac{1}{2}$, these parameters can be expressed as

$$r_c = \frac{2}{l} \left[ c_0 + c_1 (r_h l)^\frac{3}{2} + c_2 (r_h l)^3 + \mathcal{O}(r_h l)^\frac{3}{2} \right]$$

(36)

$$A_{finite} = \frac{L}{l} \left[ k_0 + k_1 (r_h l)^\frac{3}{2} + k_2 (r_h l)^3 + \mathcal{O}(r_h l)^\frac{3}{2} \right]$$

(37)

where the coefficients $c_i$ and $k_i$ ($i = 0, 1, 2$) are given by the expressions below

$$C_0 = \sqrt{\pi \Gamma\left(\frac{3}{4}\right)} \Gamma\left(\frac{1}{4}\right)$$

(38)

$$C_1 = -\frac{2\pi^\frac{3}{2}}{5} \xi \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(-\frac{1}{4}\right)^3 \Gamma\left(\frac{1}{2}\right)}$$

(39)

$$C_2 = -\frac{3\xi^2}{100\pi \Gamma\left(-\frac{1}{4}\right)^4 \Gamma\left(\frac{1}{2}\right)^2} - \frac{3\xi^2}{8\sqrt{\pi} \Gamma\left(-\frac{1}{4}\right)^3} - \frac{1 - \xi}{2\sqrt{\pi}} \frac{\Gamma\left(-\frac{1}{4}\right)^3}{\Gamma\left(-\frac{1}{2}\right)^3}$$

(40)

$$K_0 = -\frac{\pi \Gamma\left(-\frac{1}{4}\right)^2}{4 \Gamma\left(\frac{1}{2}\right)^2}$$

(41)

$$K_1 = \frac{2\pi^\frac{3}{2}}{10} \xi \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)^3}$$

(42)

$$K_2 = \frac{1}{2} \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(-\frac{1}{2}\right)^2} (1 - \xi) + \frac{1}{50\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(-\frac{1}{4}\right)^3}{\Gamma\left(\frac{1}{2}\right)^3} + \frac{3}{4} \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(-\frac{1}{2}\right)^2} \xi^2$$

(43)

For $\omega = -\frac{1}{2}$ the extremal length can be expressed as

$$r_c = \frac{2}{l} \left[ c_0 + c_1 (r_h l)^\frac{3}{2} + c_2 (r_h l)^3 + \mathcal{O}(r_h l)^\frac{3}{2} \right]$$

(44)

where the coefficients $c_i$ ($i = 0, 1, 2, 3$) can be obtained as

$$c_0 = \sqrt{\pi \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}}$$

(45)

$$c_1 = \frac{1}{50\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(-\frac{1}{4}\right)^3}$$

(46)

$$c_2 = \left( \frac{2}{\pi\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \xi^2 \frac{\Gamma\left(\frac{1}{4}\right)^3}{\Gamma\left(-\frac{1}{4}\right)^3} \right)$$

(47)

$$c_3 = \left[ \left( \frac{4}{\pi^2} - \frac{3}{\pi^2} \right) \frac{\Gamma\left(\frac{1}{4}\right)^5}{\Gamma\left(-\frac{1}{4}\right)^5} + \frac{5}{8} \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(-\frac{1}{4}\right)^2} \right] \xi^3 - \frac{1}{2\sqrt{\pi}} (1 - \xi) \frac{\Gamma\left(\frac{1}{4}\right)^3}{\Gamma\left(-\frac{1}{4}\right)^3}$$

(48)

Therefore the extremal area can be expressed as

$$A_{finite} = \frac{L}{l} \left[ k_0 + k_1 (r_h l)^\frac{3}{2} + k_2 (r_h l)^3 + \mathcal{O}(r_h l)^\frac{3}{2} \right]$$

(49)
The coefficients are given by the following expressions

\[ k_0 = -\frac{\pi \Gamma(-\frac{1}{4})^2}{4 \Gamma(\frac{1}{4})^2} \]  
(50)

\[ k_1 = \xi (\log 2 - \frac{1}{2}) \]  
(51)

\[ k_2 = \frac{\Gamma(\frac{1}{4})^2}{\pi \Gamma(-\frac{1}{4})^2} \xi^2 \]  
(52)

\[ k_3 = \frac{1}{2} \frac{\Gamma(\frac{1}{4})^2}{\Gamma(-\frac{1}{4})^2} (1 - \xi) + \left( \frac{2}{\pi} - 4 \xi^2 \right) \xi^3 \frac{\Gamma(\frac{1}{4})^4}{\Gamma(-\frac{1}{4})^4} \]  
(53)

For \( \omega = -1 \), \( f^{-\frac{1}{2}}(u) \) has the following form

\[ f^{-\frac{1}{2}}(u) = \frac{1}{\sqrt{1-a}} \left( 1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3 \right)^{-\frac{1}{2}} \]  
(54)

The form of the lapse function is the same as that of schwarzschild black holes with the exception that here there is a \( \frac{1}{\sqrt{1-a}} \) factor. For low temperature \( r_c \) and \( \mathcal{A} \) can be expressed as follow

\[ r_c \approx \frac{2}{\sqrt{1-a}} \left[ \sqrt{\frac{\pi \Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})^2}} \right] \frac{\xi}{2} \frac{\Gamma(\frac{1}{4})^3}{\Gamma(-\frac{1}{4})^3} (r_h l)^3 + \mathcal{O}[r_h^6 l^6] \]  
(55)

\[ \mathcal{A}_{\text{finite}} \approx \frac{L}{\sqrt{1-a}} \left[ -\frac{\pi \Gamma(-\frac{1}{4})^2}{4 \Gamma(\frac{1}{4})^2} + \frac{1}{2} \frac{\Gamma(\frac{1}{4})^2}{\Gamma(-\frac{1}{4})^2} (r_h l)^3 + \mathcal{O}[r_h^6 l^6] \right] \]  
(56)

For small quintessence charge regime, we have \( \frac{1}{\sqrt{1-a}} \approx 1 + \frac{1}{2} a \) and in this limit the equations (54) and (55) can be rewritten as

\[ r_c \approx \frac{2(1 + \frac{1}{2} a)}{l} \left[ \sqrt{\frac{\pi \Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})^2}} \right] \frac{\xi}{2} \frac{\Gamma(\frac{1}{4})^3}{\Gamma(-\frac{1}{4})^3} (r_h l)^3 + \mathcal{O}[r_h^6 l^6] \]  
(57)

\[ \mathcal{A}_{\text{finite}} \approx \frac{L(1 + \frac{1}{2} a)}{l} \left[ -\frac{\pi \Gamma(-\frac{1}{4})^2}{4 \Gamma(\frac{1}{4})^2} + \frac{1}{2} \frac{\Gamma(\frac{1}{4})^2}{\Gamma(-\frac{1}{4})^2} (r_h l)^3 + \mathcal{O}[r_h^6 l^6] \right] \]  
(58)

This result is consistent with equation (33) when \( \omega = -1 \).

We can see that in all cases the first term in entanglement entropy is a constant. This term is the entanglement entropy of subsystem \( A \) when the bulk is pure AdS. Also we can find that when the quintessence charge is small, the leading contribution to the entanglement entropy of the subsystem \( A \) dual to AdS black hole surrounded by quintessence arises from the pure spacetime. Besides, the effect of quintessence on entanglement entropy is very small even for \( \omega = -1 \) the dark energy does not affect noticeably the value of entanglement entropy of the subsystem.

## B. High temperature regime

In this subsection we investigate the high temperature behavior of the entanglement entropy of the black hole when the parameter \( a \) is small. For high temperature, the horizon radius is large. Therefore for small quintessence charge we have \( \frac{a}{r_h(\omega+1)} \ll 1 \). We define a new parameter as \( \delta = \frac{a}{r_h(\omega+1)} \), and expand \( f^{-\frac{1}{2}}(u) \) by Taylor series around \( \delta = 0 \) as

\[ f^{-\frac{1}{2}}(u) \approx \frac{1}{\sqrt{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}} - \frac{\delta r_h^3}{2 r_c^3} u^3 \frac{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}{(1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3)^{\frac{3}{2}}} \]  
(59)

Substituting the above approximated forms of lapse function in the integral expressions (22) and (23), we obtain the subsystem length and extremal surface as

\[ l = \frac{2}{r_c} \int_0^1 du \frac{u^2}{\sqrt{1 - u^4}} \left( \frac{1}{\sqrt{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}} - \frac{\delta r_h^3}{2 r_c^3} u^3 \frac{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}{(1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3)^{\frac{3}{2}}} \right), \]  
(60)

\[ \mathcal{A} = 2L \int_0^1 du \frac{r_c}{u^2 \sqrt{1 - u^4}} \left( \frac{1}{\sqrt{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}} - \frac{\delta r_h^3}{2 r_c^3} u^3 \frac{1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3}{(1 - \left( \frac{r_h}{r_c^3} \right)^3 u^3)^{\frac{3}{2}}} \right) \]  
(61)
As showed in [19], the extremal surface can never penetrate the horizon. This implies that \( r_c \) is always larger than \( r_h \). At high temperature \( r_h \) is very large, therefore, it approaches extremal surface, \( r_h \sim r_c \). Assuming \( r_c = r_h(1 + \epsilon) \) and proceeding similar to that used in [20], the form of boundary subsystem length can be obtained as

\[
l r_h = a_1 + \delta a_2 - \frac{1}{\sqrt{3}} \log[3\epsilon] + \mathcal{O}[\epsilon]
\]  

where the coefficients \( a_1 \) and \( a_2 \) are expressed as the following forms

\[
a_1 = \frac{1}{2} \sqrt{\pi} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{4})} + \sum_{n=1}^{\infty} \left( \frac{1}{3n - 1} \frac{\Gamma(n + \frac{1}{4}) \Gamma(\frac{3n + 6}{4})}{\Gamma(n + 1) \Gamma(\frac{3n + 3}{4})} - \frac{2}{3\sqrt{3n^2}} \right)
\]

\[
a_2 = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1) \Gamma(\frac{3n + 3}{4})} - \frac{\Gamma(n + \frac{3}{2}) \Gamma(\frac{3n + 6}{4})}{\Gamma(n + 1) \Gamma(\frac{3n + 3}{4})} - \frac{\omega}{\sqrt{3}} \right)
\]  

We can obtain \( \epsilon \) correction as

\[
\epsilon \approx \epsilon_{ent} e^{-\frac{4\pi}{3} \Gamma(1 - \omega + \frac{3}{4}\delta)}
\]

where \( \epsilon_{ent} \) is a constant has the following form

\[
\epsilon_{ent} \approx \frac{1}{3} \exp(a_1)
\]

The \( \epsilon \) corrections decreases exponentially with the temperature just as they do in the case Schwarzschild black hole in vacuum. The only difference is introduction of two small \( \delta \) terms due to the presence of quintessence. This terms cause \( \epsilon \) to decay in higher temperature rather than Schwarzschild black hole.

The extremal surface area can be written as

\[
\mathcal{A}^{finite} = Llr_h(1 + 2\epsilon) + Lr_h \left[ b_1 + \delta b_2 + b_3 \epsilon + b_4 \log \epsilon + \delta(b_5 \epsilon + b_6 \log[\epsilon] + \mathcal{O}[\epsilon^2]) \right]
\]  

\( b_i(i = 1, 2, 3, 4, 5) \) can be obtained as

\[
b_1 = \sqrt{\pi} \frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + \frac{\pi^2}{9\sqrt{3}} + \sum_{n=1}^{\infty} \left( \frac{1}{3n - 1} \frac{\Gamma(n + \frac{1}{4}) \Gamma(\frac{3n + 3}{4})}{\Gamma(n + 1) \Gamma(\frac{3n + 3}{4})} - \frac{2}{3\sqrt{3n^2}} \right)
\]

\[
b_2 = \frac{\sqrt{\pi}}{2 + 3\omega} \frac{\Gamma(\frac{3n + \omega}{4})}{\Gamma(\frac{3n + 3\omega}{4})} - \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \left( \frac{1}{3n + 2 + 3\omega} \frac{\Gamma(n + \frac{3}{2}) \Gamma(\frac{3n + 3\omega + 6}{4})}{\Gamma(n + 1) \Gamma(\frac{3n + 3\omega + 8}{4})} \right)
\]

\[
b_3 = \sqrt{\pi} \frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})} + \frac{\pi^2}{9\sqrt{3}} + \frac{2 \log 3}{\sqrt{3}} - \sum_{n=1}^{\infty} \left( \frac{\Gamma(n + \frac{1}{4}) \Gamma(\frac{3n + 3}{4})}{\Gamma(n + 1) \Gamma(\frac{3n + 3}{4})} - \frac{6n - 2}{3\sqrt{3n^2}} \right)
\]

\[
b_4 = \frac{2}{\sqrt{3}} \log 3
\]

\[
b_5 = \frac{\pi}{4} \frac{\sqrt{\pi} \Gamma(\frac{6 + 3\omega}{4})}{2 \Gamma(\frac{3 + 3\omega}{4})}
\]

\[
b_6 = \frac{2\omega}{3\sqrt{3}}
\]

where for computing these coefficients we applied the same method used in [20]. Therefore the finite part of the entanglement entropy of boundary field theory dual to quintessence black hole may be written down as follows

\[
S^{finite} = Ll S_{BH}(1 + 2\epsilon) + L l r_h \frac{4G}{b_1 + b_2 + L r_h \epsilon} \left[ b_3 + b_4 \log \epsilon + \delta(b_5 + b_6 \log[\epsilon]) \right] + \mathcal{O}[\epsilon^2]
\]  

This equation is very similar to the equation obtained by authors in ref. [20] for Reissner-Nordström black hole. This is not very unexpected because the charged black holes can be considered as a special form of quintessence black
holes. In the first term, $S_{BH} = \frac{A}{4G}$ corresponds to well-known Bekenstein-Hawking entropy of black hole. We can rewrite (74) in terms of temperature as follows

\[ S^{finite} = \frac{L}{4G} \left( \frac{4\pi T}{3(1+\delta \omega)} \right)^2 (1+2\epsilon) + \frac{L}{4G} \left( \frac{4\pi T}{3(1+\delta \omega)} \right) (b_1 + \delta b_2) + \frac{L\epsilon}{4G} \left( \frac{4\pi T}{3(1+\delta \omega)} \right) \left[ b_3 + b_4 \log \epsilon + \delta (b_5 + b_6 \log |\epsilon|) \right] + \mathcal{O}(\epsilon^2) \] (75)

The first term scales with the area of the subsystem and represents thermal entropy of the region while the subsequent terms are proportional to the length of the boundary separating the subsystem (A) and its complement. and correspond to entanglement between region A and the rest of the system. The quintessence corrections on the entanglement entropy is small and when $\omega \to -1$, these corrections approach to their maximum values. It is trivial that when $\delta \to 0$, the results of Schwatschild black hole is recovered.

V. BLACK HOLE IN QUINTESSENCE WITH A LARGE CHARGE

In this section we explore the low and high temperature behavior of entanglement entropy for subsystem $A$ boundary to AdS black hole surrounded by a large charge quintessence. The low temperature is restricted to $\omega$ close to $-1$, therefore we first study the high temperature of entanglement entropy then investigate the low temperature.

A. high temperature

At high temperature, the horizon radius is very large ($r_h l \gg 1$), therefore, $r_h$ approach $r_c$. Hence, in this region the extremal surface tends to wrap a part of horizon and the leading contributions come from the near horizon part of the surface. In other words, when $\frac{r_h}{r_c} \to 1$, the values of integrand near the horizon, $u \sim u_0 = \frac{r_h}{r_c}$ are great and the leading contributions of the integral correspond to this region. Therefore, by Taylor expansion $f(u)$ around $u_0$, the lapse function has a form as

\[ f(u) = -3 \left( \frac{r_h}{r_c} \right) \left[ 1 + \frac{a\omega}{r_h^{3(\omega+1)}} \right] (u - u_0) + \mathcal{O}((u - u_0)^2) \] (76)

we can rewrite this equation as

\[ f(u) \approx 3 \left[ 1 + \frac{a\omega}{r_h^{3(\omega+1)}} \right] (1 - \frac{r_h}{r_c} u) \] (77)

We denote the prefactor in this equation as $\delta = 3 \left[ 1 + \frac{a\omega}{r_h^{3(\omega+1)}} \right]$. This expression appears in the temperature equation [20]. Therefore, when $\delta \to 0$, the temperature is low and when $\delta \to 3$, the temperature is high. Substituting this approximation form of the lapse function in eq. (22), the subsystem length can be expressed as

\[ l = \frac{2}{r_c \sqrt{\delta}} \int_0^1 du \frac{u^2}{\sqrt{1 - u^2}} \sqrt{1 - \frac{r_h}{r_c} u} \] (78)

This equation is exactly the same derived in [20] for non-extremal black hole. The only difference is the form of $\delta$. For computing entanglement entropy, we use exactly the same method applied in [20]. First by binomial expanding $\frac{1}{\sqrt{1 - \frac{r_h}{r_c} u}}$, we obtain subsystem length as

\[ lr_c = \frac{1}{2 \sqrt{\delta}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + \frac{3}{4})}{\Gamma(n + 1) \Gamma(n + \frac{5}{4})} \left( \frac{r_h}{r_c} \right)^n \] (79)

The series in the above equation goes as $\sim \frac{a^n}{n}$ for large $n$, Thus it is divergent when $r_c \to r_h$. We isolate the divergent term. For large charge regime, the horizon radius is large, therefore, we set $r_c = r_h (1 + \epsilon)$. Substituting this expression in (79) and expanding it in terms of $\epsilon$ we have

\[ lr_h = \frac{1}{\sqrt{\delta}} \left( f - \log |\epsilon| + \mathcal{O}(\epsilon) \right) \] (80)
Therefore the $\epsilon$ correction can be obtained as

$$\epsilon \approx \epsilon_{ent} e^{-\sqrt{2} l r_h} = \epsilon_{ent} e^{-\frac{4\pi^3}{\sqrt{\delta}}}$$  \hspace{1cm} (81)

where $\epsilon = e^f$ and $f$ is given by the following expression

$$f = \frac{\sqrt{\pi} \Gamma\left(\frac{3}{4}\right)}{2 \Gamma\left(\frac{3}{2}\right)} + \sum_{n=1}^{\infty} \left( \frac{\Gamma(n + \frac{3}{2})\Gamma(n + \frac{5}{2})}{2\Gamma(n + 1)\Gamma(n + \frac{3}{2})} - \frac{2}{n^2} \right)$$  \hspace{1cm} (82)

Since $\frac{\delta l c^2}{r_h^{\omega+1}}$ is always negative, $\delta < 3$. Thus $\epsilon$ correction is low for high temperature and is high for high temperature. On the other hand this correction increases with $a$ and $\omega$.

Substituting the form of the lapse function (76) in (23), the extremal surface area can be written as

$$A = \frac{2 L r_c}{\sqrt{\delta}} \int_0^1 \frac{du}{u^2 \sqrt{1 - u^4}} \sqrt{1 - \frac{2a}{r_c} u}$$  \hspace{1cm} (83)

The integral is divergent. Regularization and computation are performed and finite part of extremal surface area is expressed as

$$A^{finite} = L l r_h^2(1 + 2\epsilon) + \frac{2 L r_h}{\sqrt{\delta}} \left[ P_1 + P_2\epsilon + \epsilon \log|\epsilon| + O[\epsilon^2] \right]$$  \hspace{1cm} (84)

where $P_1$ and $P_2$ are given as the following expressions

$$P_1 = -\frac{2\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \log[4] - \frac{10 - 2\pi^2}{8} + \frac{\pi^2}{6} + \sum_{n=2}^{\infty} \left( \frac{1}{n - 1} \frac{\Gamma(n + \frac{3}{2})\Gamma(n + \frac{5}{2})}{\Gamma(n + 1)\Gamma(n + \frac{3}{2})} - \frac{2}{n^2} \right)$$  \hspace{1cm} (85)

$$P_2 = \frac{\pi^2}{6} - \frac{2\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{1}{4} - \sum_{n=2}^{\infty} \left( \frac{\Gamma(n + \frac{3}{2})\Gamma(n + \frac{5}{2})}{\Gamma(n + 1)\Gamma(n + \frac{3}{2})} - \frac{2(n - 1)}{n^2} \right)$$  \hspace{1cm} (86)

Therefore the finite part of entanglement entropy will be expressed as

$$S^{finite} = L l S_{BH}(1 + 2\epsilon) + \frac{L r_h}{2 G \sqrt{\delta}} \left[ P_1 + P_2\epsilon + \epsilon \log|\epsilon| + O[\epsilon^2] \right]$$  \hspace{1cm} (87)

The first term scales with the area of the subsystem and is extensive. This term corresponds to thermal entropy of black hole. The subsequent terms are proportional to length of strip and are measure of the entanglement between subsystem $A$ and the rest of the system. The effect of dark energy appears mainly in these terms as a factor $\frac{\delta}{\sqrt{\delta}}$. For small $|\omega|$ and $a$, this factor is small and the effect of quintessence is also small, but when $|\omega|$ or $a$ goes to larger value, $\delta$ becomes small and consequently $\frac{\delta}{\sqrt{\delta}}$ becomes large. When $\omega \rightarrow -1$, $\frac{\delta}{\sqrt{\delta}} \gg 1$ and the effect of dark energy will be enormous. This can be attributed to the severe entanglement between subsystem and rest of the system at higher $|\omega|$.

### B. low temperature

This case happens only when $\omega < -\frac{2}{3}$ especially when $\omega$ is very close to $-1$. For this case, $\left(\frac{r_h}{l}\right)^{3(\omega+1)} u^{3(\omega+1)} \simeq 1$, therefore, the lapse function has a form as follows

$$f(u) \approx \left[ 1 - \frac{a}{r_h^{3(\omega+1)}} \right] \left[ 1 - \left( \frac{r_h}{l} \right)^3 u^3 \right]$$  \hspace{1cm} (88)

This equation differs from that of pure Schwarzchild black hole in a factor $\left[ 1 - \frac{a}{r_h^{3(\omega+1)}} \right]$ only. For $\omega = -1$ this equation converts to equation (56). The subsystem length and minimal surface area can be obtained as

$$r_c \approx \frac{2}{l} \left[ 1 - \frac{a}{r_h^{3(\omega+1)}} \right]^{-\frac{1}{2}} \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{4}\right)^3}{\Gamma\left(\frac{3}{4}\right)} (r_h l)^3 + O[r_h^{6\gamma}] \right]$$  \hspace{1cm} (89)

$$A^{finite} \approx \frac{L}{l} \left[ 1 - \frac{a}{r_h^{3(\omega+1)}} \right]^{-\frac{1}{2}} \left[ -\frac{\pi}{4} \frac{\Gamma\left(-\frac{1}{2}\right)^2}{\Gamma\left(\frac{3}{4}\right)^2} + \frac{1}{2\Gamma\left(-\frac{1}{2}\right)^2} (r_h l)^3 + O[r_h^{6\gamma}] \right]$$  \hspace{1cm} (90)
The entanglement entropy can be expressed as

$$S_{\text{finite}} \approx \frac{L}{4Gl} \left[ 1 - \frac{a}{r_h^{3(\omega+1)}} \right]^{-\frac{1}{2}} \left[ -\frac{\pi}{4} \frac{\Gamma(-\frac{1}{4})^2}{\Gamma(\frac{1}{4})^2} + \frac{1}{2} \frac{\Gamma(\frac{1}{4})^2}{\Gamma(-\frac{1}{4})^2} (r_h l)^3 + O[r_h^6 l^6] \right]$$

we can see that dark energy affect the entanglement entropy as a constant factor and with increasing \(a\) this effect is increased.

VI. CONCLUSION

In this paper we investigated analytically the entanglement entropy of a strip-like subsystem \((A)\) for the boundary field theory dual to AdS black hole surrounded by a quintessence. We obtained approximated analytical expression for the holographic entanglement entropy of the strip based on the method adopted in [18]. We focused on the dependence of the entanglement entropy on temperature, quintessence type \((\omega)\) and the amount of quintessence \((a)\) that we called the last item as quintessence charge. In this framework we studied the low and high temperature of entanglement entropy for small and high \(a\) regimes and different regions of \(\omega\). We find that at high temperature the entanglement entropy depends on temperature exponentialy and also depends on the factor \(\frac{a}{r_h^{3(\omega+1)}}\) for small and large \(a\) regimes. On the other hand, the changes of the entanglement entropy of quintessence black hole relative to AdS Schwarzschild black is increased when \(\omega\) goes to \(-1\). The behavior of entanglement entropy changes when \(\omega\) approaches to \(-1\) and this behavior has been considered for small and large \(a\) regimes. In general the holographic treatment develops our knowledge about the entanglement and related phenomena in strongly coupled boundary field theory at finite temperature and the charge and type of quintessence.

References

[1] Huzihiro Araki and Elliott H. Lieb, Comm. Math. Phys. 18(2), 160-170 (1970).
[2] C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B 424, 443 (1994).
[3] C. G. Callan, Jr. and F. Wilczek, Phys. Lett. B 333, 55 (1994).
[4] P. Calabrese and J. L. Cardy, J. Stat. Mech. 0406, P06002 (2004).
[5] P. Calabrese and J. Cardy, J. Phys. A 42, 504005 (2009).
[6] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[7] M. Cramer, J. Eisert, M. B. Plenio and J. Dreissig, Phys. Rev. A 73, 012309 (2006).
[8] M. Cramer and J. Eisert, Correlations, New Journal of Physics, 8(5), 71 (2006).
[9] M. Cramer, J. Eisert and M. B. Plenio, Phys. Rev. Lett. 98, 220603 (2007).
[10] M. B. Plenio, J. Eisert, J. Dreissig and M. Cramer, Phys. Rev. Lett. 94, 060503 (2005).
[11] D. Gioev and I. Klich, Phys. Rev. Lett. 96, 100503 (2006).
[12] M. M. Wolf, Phys. Rev. Lett. 96, 010404 (2006).
[13] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006).
[14] V. E. Hubeny, M. Rangamani and T. Takayanagi, JHEP 0707, 062 (2007).
[15] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)].
[16] M. Cadoni and M. Melis, Found. Phys. 40, 638 (2010).
[17] Mariano Cadoni and Maurizio Melis Entropy 12(11), 2244 (2010).
[18] W. Fischler and S. Kundu, JHEP 1305, 098 (2013).
[19] V. E. Hubeny, JHEP 1207, 093 (2012).
[20] P. Chaturvedi, V. Malvimat and G. Sengupta, Phys. Rev. D 94, no. 6, 066004 (2016).
[21] S. Kundu and J. F. Pedraza, JHEP 1608, 177 (2016).
[22] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999).
[23] A. G. Riess et al. [Supernova Search Team], Astron. J. 116, 1009 (1998).
[24] S. Nobbenhuis, Found. Phys. 36, 613 (2006).
[25] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[26] P. J. E. Peebles and B. Ratra, Astrophys. J. 325, L17 (1988).
[27] Wetterich, Nucl. Phys. B 302, 668 (1988).
[28] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).
[29] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82, 896 (1999)
[30] V. V. Kiselev, Class. Quant. Grav. 20, 1187 (2003)
[31] K. Ghaderi and B. Malakolkalami, Nucl. Phys. B 903, 10 (2016).
[32] K. Ghaderi and B. Malakolkalami, Astrophys. Space Sci. 361, no. 5, 161 (2016).
[33] B. B. Thomas, M. Saleh and T. C. Kofane, Gen. Rel. Grav. 44, 2181 (2012)
[34] X. X. Zeng and L. F. Li, Phys. Lett. B 764, 100 (2017)
[35] S. Chen, Q. Pan and J. Jing, Class. Quant. Grav. 30, 145001 (2013)
[36] X. X. Zeng, D. Y. Chen and L. F. Li, Phys. Rev. D 91, no. 4, 046005 (2015)
[37] S. Chen, B. Wang and R. Su, Phys. Rev. D 77, 124011 (2008)
[38] E. Tonni, JHEP 1105, 004 (2011)