Periodic Points of Asymmetric Bernoulli Shifts

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1. Introduction

In 1964, Sharkovskii [1] firstly introduced a special ordering on the set of positive integers. This ordering implies that if \( p < q \) and a continuous self-map of a closed bounded interval has a point of period \( p \); then it has a point of period \( q \). The least number with respect to this ordering is 3. Thus, if a map has a point of period 3, then it has points of any periods. In 1975, the latter result was rediscovered by Li and Yorke [2].

Then numerous papers are devoted to the study of interval maps (see e.g., [3–5] and references therein).

It is well-known that Sharkovskii’s theorem gives a complete structure of periodic order for a continuous self-map on a closed bounded interval. As a further study, a natural problem is how to determine the location and number of periodic points for a specific map. This paper considers the periodic points of asymmetric Bernoulli shift, which is a piecewise linear chaotic map.

Given a positive integer \( n \), one interesting question is how to find all \( n \)-periodic points of \( F \). The other is how many \( n \)-periodic points of \( F \).

In this paper, we study periodic orbits of \( F \). In the next section, we present dynamics of jumps of \( F^n \). Section 3 recalls the real number representation, i.e., \( F \)-expansion. In Section 4, we use the \( F \)-expansion to give explicit formulas of \( F^n(x) \) for \( n \in \mathbb{N} \), explicit formulas of jumps of \( F^n(x) \), explicit formulae of fixed points of \( F^n(x) \), and explicit formulas of all \( n \)-periodic points of \( F(x) \). The last section gives the number \( h(n) \) of periodic orbits of a given period \( n \) for \( F \) and the limit behavior of \( h(n) \).

2. Dynamics of Jumps of \( F^n \)

For \( n \in \mathbb{N} \), let \( F^n(x) \) denote the \( n \)-th iterate of \( F \), which is recursively defined by \( F^0(x) = x \) and \( F^n = F(F^{n-1}(x)) \) for \( x \in [0, 1] \).

A point \( c \in (0, 1) \) is called a jump of \( F \) if the one-sided limits, \( F(c−) \) and \( F(c+) \), exist and are finite, but are not equal. The set of jumps of \( F \) is denoted by \( \mathcal{J}(F) \). One can see that

\[
\mathcal{J}(F) \subseteq \mathcal{J}(F^2) \subseteq \cdots \subseteq \mathcal{J}(F^n) \subseteq \mathcal{J}(F^{n+1}) \subseteq \cdots. \tag{2}
\]

Each element of \( \mathcal{J}(F^{n+1})/\mathcal{J}(F^n) \) must be a preimage under \( F \) of a point from \( \mathcal{J}(F^n) \). More precisely,
\[ \mathcal{J}(F^{m+1}) \setminus \mathcal{J}(F^m) = F^{-1}(\mathcal{J}(F^m)) \setminus \mathcal{J}(F^m). \quad (3) \]

The map \( F \) has the unique jump \( a \). Put \( x_{1, 0} = 0, x_{1, 1} = a, \) and \( x_{1, 2} = 1 \). Let \( I \) denote the unit interval \([0, 1]\), \( I_{1, 1} = (x_{1, 0}, x_{1, 1}) \), and \( I_{1, 2} = (x_{1, 1}, x_{1, 2}) \). One can see that \( F^m \) has \( 2^m - 1 \) jumps for \( m \geq 2 \) by induction. For \( i, j \in \mathbb{N} \), let \( x_{i, 0} = 0, x_{i, 2} = 1, \) and \( x_{i, j} \) denote the \( j \)-th jumps of \( F^i \) in the following order:

\[ 0 = x_{i, 0} < x_{i, 1} < x_{i, 2} < \cdots < x_{i, j} < \cdots < x_{i, 2^i-1} < x_{i, 2^i} = 1. \quad (4) \]

Put \( I_{j, i} = (x_{i-j, i}, x_{i, j}) \) for every \( j \in \{1, 2, 3, \ldots, 2^i\} \). It is clear that \( I_{j, i} \) is the \( j \)-th monotonic interval of \( F^i \).

**Lemma 1.** For \( n \geq 1 \), the jumps of \( F^n \) and \( F^{n-1} \) have the following relationship:

(i) \( F(x_{n,k}) = F(x_{n,2^{n-1}+k}) = x_{n-1,k} \) for \( 1 \leq k \leq 2^{n-1} - 1 \)

(ii) \( x_{n,2^{n-1}} = x_{n-1,2^n-2} = a \)

(iii) \( F^j(I_{n,k}) = F^j(I_{n,2^{n-1}+k}) = I_{n-j,k} \) for \( 1 \leq k \leq 2^{n-1} \) and \( 1 \leq j \leq n-1 \)

**Proof.** We first claim that \( a \) is a jump of \( F^m \) for every \( n \geq 1 \). In fact, since \( a \) is a jump of \( F(x) \), \( a \) is also a jump of \( F^m(x) \) for \( n \geq 2 \). Moreover, it is easy to check that \( F^n(a) = 1 \) for \( n \geq 2 \).

Next, we prove (i) and (ii) by induction. It is clear that these results hold for \( n = 2 \).

Assume that these results hold for \( n = m \geq 2 \), i.e.,

(i) \( F(x_{m,k}) = F(x_{m,2^{m-1}+k}) = x_{m-1,k} \) for \( 1 \leq k \leq 2^{m-1} - 1 \)

(ii) \( x_{m,2^{m-1}} = x_{m-1,2^m-2} = a \)

Now we shall prove these results hold for \( n = m + 1 \). Denote \( 2^m - 1 \) jumps of \( F^m \) by

\[ 0 < x_{m,1} < x_{m,2} < \cdots < x_{m,k} < \cdots < x_{m,2^{m-1}} < 1. \quad (5) \]

Since \( F \) is strictly increasing on the subinterval \( I_{1,1} \) and \( F(I_{1,1}) = (0, 1) \), for each \( k \in \{1, 2, \ldots, 2^m - 1\} \), there exists the unique point, denoted by \( x_{m+1,k} \), in \( I_{1,1} \) such that \( F(x_{m+1,k}) = x_{m,k} \). Since \( F \) is strictly increasing on \( I_{1,1} \), one can see that

\[ 0 < x_{m+1,1} < x_{m+1,2} < \cdots < x_{m+1,k} < \cdots < x_{m+1,2^{m-1}-1} < a. \quad (6) \]

Further, by the definition of jump, \( x_{m+1,k} \) is a jump of \( F^{m+1} = F^m \circ F \) for each \( k \in \{1, 2, \ldots, 2^m - 1\} \).

Similarly, since \( F \) is strictly increasing on the subinterval \( I_{1,2} \) and \( F(I_{1,2}) = (0, 1) \), for each \( k \in \{1, 2, \ldots, 2^m - 1\} \), there exists the unique point, denoted by \( x_{m+1,2^{m-1}+k} \), in \( I_{1,2} \) such that \( F(x_{m+1,2^{m-1}+k}) = x_{m,k} \). Since \( F \) is strictly increasing on \( I_{1,2} \), one can see that

\[ a < x_{m+1,2^{m-1}+1} < x_{m+1,2^{m-1}+2} < \cdots < x_{m+1,2^m+k} < \cdots < x_{m+1,2^{m-1}+1} < 1. \quad (7) \]

Further, by the definition of jump, \( x_{m+1,2^m+k} \) is a jump of \( F^{m+1} = F^m \circ F \) for each \( k \in \{1, 2, \ldots, 2^m - 1\} \). Let \( x_{m+1,2^m+k} \) denote \( a \). Therefore,

(i) \( F(x_{m+1,k}) = F(x_{m,2^m+k}) = x_{m,k} \) for \( 1 \leq k \leq 2^m - 1 \)

(ii) \( x_{m+1,2^m+k} = x_{m,2^m+k} = a \)

It follows from (i) that for \( 1 \leq k \leq 2^n - 1 \),

\[ F(I_{n,k}) = F(I_{n,2^{n-1}+k}) = I_{n-k}. \quad (8) \]

Then for \( 1 \leq k \leq 2^n - 1 \) and \( 1 \leq i \leq n - 1 \),

\[ F^i(I_{n,k}) = F^i(I_{n,2^{n-1}+k}) = I_{n-i,k}. \quad (9) \]

This completes the proof. \( \square \)

### 3. \( F \)-Expansion

In this section, we will introduce a new real number representation.

**Definition 1.** A sequence \( \{\epsilon_k\}_{k \in \mathbb{N}} \) of 0 and 1 is called the itinerary of \( x \in [0, 1] \) with respect to the asymmetric Bernoulli shift \( F: [0, 1] \rightarrow [0, 1] \) and \( a \in (0, 1) \), if, for \( k \geq 1 \),

\[ \epsilon_k = \begin{cases} 0, & F^{k-1}(x) \leq a, \\ 1, & F^{k-1}(x) > a. \end{cases} \quad (10) \]

In fact, the itinerary of \( x \in [0, 1] \) with respect to \( F \) and \( a \in (0, 1) \) is just the \( F \)-expansion of a real \( x \in [0, 1] \). According to [10], or these two classic papers [11, 12], we have an expansion for \( x \) in powers of the numbers \( a \) and \( 1 - a \):

\[ x = \sum_{k=1}^{\infty} \epsilon_k a^{k-1} (1 - a)^{\epsilon_k} = \sum_{k=1}^{\infty} \epsilon_k a^{k-1} a^{1-a} \]

(11)

where \( s_0 = 0 \) and \( s_k = \sum_{j=1}^{k} \epsilon_j \) for \( k \geq 1 \).

Thus, every \( x \in [0, 1] \) can be represented through its digit sequence \( \{\epsilon_k\}_{k \in \mathbb{N}} \). In this situation, write \( x = [\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots] \) for short. One can see that every infinite \( F \)-expansion is unique, whereas each \( x \in (0, 1) \) with a finite \( F \)-expansion can be expanded in exactly two ways, namely, one immediately verifies that

\[ x = [\epsilon_1, \ldots, \epsilon_{k-1}, 1] = [\epsilon_1, \ldots, \epsilon_{k-1}, 0, 1, 1, \ldots]. \quad (12) \]

In the following, we employ a convention in which finite fractions such as

\[ [1, 1, 0, 0, 0, 0, \ldots] = [1, 0, 1, 1, 1, 1, \ldots], \]

\[ [1, 0, 1, 0, 0, 0, 0, \ldots] = [1, 0, 1, 1, 1, 1, \ldots], \quad (13) \]

are represented as finite fractions with infinite zeros, as \([1, 1, 0, 0, 0, \ldots]\) or \([1, 0, 1, 0, 0, 0, \ldots]\), unless otherwise stated.
Lemma 2. If \( x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots] \), then \( F(x) = [\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots] \).

Proof. It follows from a property of the asymmetric Bernoulli shift \( F(x) \) that \( x \preceq a \) provided that \( \varepsilon_1 = 0 \) in \( x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots] \). One then finds

\[
F(x) = \frac{x}{a} = \sum_{k=1}^{\infty} \varepsilon_k a^{k-1} \left(1 - \frac{a}{x}\right)^{s_k-1}
\]

\[
= \sum_{k=1}^{\infty} \varepsilon_k a^{k-1} \left(1 - \frac{a}{x}\right)^{s_k-1} \leq \frac{x}{a}
\]

\[
= [\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots].
\]

4. The Explicit Formula of \( F^n \)

Since \( F^n \) is a piecewise linear map, and \( F^n \) is strictly increasing on each subinterval \( I_{n,k} \). One can obtain the explicit formula of \( F^n \).

Theorem 1. If \( x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots] \), then

\[
F^n(x) = \frac{x}{a^{m-n} (1-a)^n} - \sum_{j=1}^{n} \varepsilon_j a^{-j-s_{m-n}^{s_{m-n}+j} (1-a)^{s_{m-n}^{s_{m-n}+j}+1}}
\]

for \( n \in \mathbb{N}^+ \).

Proof. We prove this result by mathematical induction.

Firstly consider the trivial case \( n = 1 \). If \( x \in I_{1,1} \), then \( \varepsilon_1 = 0 \) and \( s_1 = 0 \). Thus,

\[
F(x) = \frac{x}{a^{1-s_1} (1-a)^{s_1}} = \frac{x}{1-a} - \frac{a}{1-a}
\]

If \( x \in I_{1,2} \), then \( \varepsilon_1 = 1 \) and \( s_1 = 1 \). Thus,

\[
F(x) = \frac{x}{a^{1-s_1} (1-a)^{s_1}} - \sum_{j=1}^{1} \varepsilon_j a^{-j-s_{m-n}^{s_{m-n}+j} (1-a)^{s_{m-n}^{s_{m-n}+j}+1}}
\]

Therefore, the result holds for \( n = 1 \).

Assume that the result holds for \( n = m \geq 1 \), i.e.,

\[
F^m(x) = \frac{x}{a^{m-n} (1-a)^n} - \sum_{j=1}^{m} \varepsilon_j a^{-j-s_{m-n}^{s_{m-n}+j} (1-a)^{s_{m-n}^{s_{m-n}+j}+1}}
\]

Now we shall prove that the result holds for \( n = m + 1 \). If \( F^m(x) \leq a \); then \( \varepsilon_{m+1} = 0 \) and \( s_{m+1} = s_m \). Thus,

\[
F^{m+1}(x) = \frac{1}{a} \left( \frac{x}{a^{m-n} (1-a)^n} - \sum_{j=1}^{m} \varepsilon_j a^{-j-s_{m-n}^{s_{m-n}+j} (1-a)^{s_{m-n}^{s_{m-n}+j}+1}} \right)
\]

\[
= \frac{x}{a^{m+1-n-s_{m+1}} (1-a)^{s_{m+1}}} - \sum_{j=1}^{m+1} \varepsilon_j a^{-j-s_{m+1}^{s_{m+1}+j} (1-a)^{s_{m+1}^{s_{m+1}+j}+1}}
\]

If \( F^m(x) > a \); then \( \varepsilon_{m+1} = 1 \) and \( s_{m+1} = s_m + 1 \). Thus,

\[
F^{m+1}(x) = \frac{1}{a} \left( \frac{x}{a^{m-n} (1-a)^n} - \sum_{j=1}^{m} \varepsilon_j a^{-j-s_{m-n}^{s_{m-n}+j} (1-a)^{s_{m-n}^{s_{m-n}+j}+1}} \right)
\]

\[
= \frac{x}{a^{m+1-n-s_{m+1}} (1-a)^{s_{m+1}}} - \sum_{j=1}^{m+1} \varepsilon_j a^{-j-s_{m+1}^{s_{m+1}+j} (1-a)^{s_{m+1}^{s_{m+1}+j}+1}}
\]
\[ F^{m+1}(x) = \frac{1}{1-a} \cdot \frac{x}{a_{m-s_m} (1-a)^n} - \sum_{j=1}^{m} \frac{\epsilon_j}{a_{m-j-s_m+s_j-1} (1-a)^{s_j+1}} \cdot \frac{a}{1-a}. \]

Therefore, the result holds for \( n = m + 1 \). The proof is completed. \( \square \)

As a corollary, we present the exact formulas of these jumps of \( F^n \).

**Corollary 1.** All jumps of \( F^n \) are given by

\[ \sum_{j=1}^{a} \epsilon_j a^{j+1-s_j} (1-a)^{s_j-1}, \quad (23) \]

where \( \epsilon_j \) are 0 or 1, and not all are \( \epsilon_j \) equal to 0.

**Proof.** If all \( \epsilon_j \) are zero, then \( x_{1,0} = 0 \), and it is not a jump. From Theorem 2, solving \( F^n(x) = 0 \), we can obtain all these jumps of \( F^n \). \( \square \)

**Definition 2.** A point \( x \) in \( X \) is called a periodic point of a self-mapping \( f: X \to X \) if there exists an integer \( n \) such that

\[ f^n(x) = x. \quad (24) \]

The smallest positive integer \( n \) satisfying the above is called the prime period or least period of the point \( x \), the point \( x \) is called an \( n \)-periodic point of \( f \), and the sequence \( \{ x, f(x), \ldots, f^{n-1}(x) \} \) is called an \( n \)-periodic orbit.

In particular, an 1-periodic point is a fixed point.

The following corollary presents the exact formulas of all fixed points of \( F^n \).

**Corollary 2.** All fixed points of \( F^n \) are given by

\[ \frac{1}{1-a} \cdot \frac{x}{a_{m-s_m} (1-a)^n} - \sum_{j=1}^{m} \frac{\epsilon_j}{a_{m-j-s_m+s_j-1} (1-a)^{s_j+1}} \cdot \frac{a}{1-a}. \quad (25) \]

where \( \epsilon_j \) are 0 or 1.

**Proof.** Since the curve of \( y = F^n(x) \) intersects the line of \( y = x \) at \( 2^n \) points, \( F^n(x) \) has \( 2^n \) fixed points. Solving \( F^n(x) = x \), we have

\[ x = \frac{1}{1-a} \cdot \frac{x}{a_{m-s_m} (1-a)^n} - \sum_{j=1}^{m} \frac{\epsilon_j}{a_{m-j-s_m+s_j-1} (1-a)^{s_j+1}}. \quad (26) \]

\[ \square \]

5. The Number of \( n \)-Periodic Points

The fixed points of \( F \) are the intersections of \( y = F(x) \) and \( y = x \), namely, two points \( x = 0, 1 \). The intersections of \( y = F^n(x) \) and \( y = x \) have four points where there are two 2-periodic points, namely, two points

\[ \frac{a^2}{1-a+a^2}, \quad \frac{a}{1-a+a^2}. \quad (27) \]

The other two intersections \( x = 0 \) and 1 are the fixed points. The intersections of \( y = F^n(x) \) and \( y = x \) have \( 2^n \) periodic points. If \( x \) is a \( p \)-periodic point of \( F \), then \( p | n \). Let \( h(p) \) denote the number of the \( p \)-periodic points. Then,

\[ \sum_{p|n} h(p) = 2^n, \quad \text{for every integer } n \geq 1, \quad (28) \]

where the sum extends over all positive divisors \( p \) of \( n \).

In order to obtain the exact number \( h(n) \) of \( n \)-periodic points of \( F \), we need to introduce the Möbius function and Möbius inversion formula (see, for example, [13, 14]).

Define the Möbius function \( \mu: \mathbb{N} \to \{0, 1, \} \) by

\[ \mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = q_1 q_2 \cdots q_r, q_1 < q_2 < \cdots < q_r, \\ 0, & \text{otherwise}. \end{cases} \quad (29) \]

Thus, if \( (n_1, n_2) = 1 \), then \( \mu(n_1 n_2) = \mu(n_1) \mu(n_2) \), and for any \( n \in \mathbb{N} \), there holds

\[ \sum_{k|n} \mu(k) = \left[ \frac{1}{n} \right]. \quad (30) \]

**Lemma 3** (Möbius inversion formula). If \( h \) and \( g \) are arithmetic functions, i.e., from \( \mathbb{N} \) to \( \mathbb{C} \), satisfying
\[ g(n) = \sum_{d|n} h(d), \quad \text{for every integer } n \geq 1. \quad (31) \]

Then,
\[ h(n) = \sum_{d|n} \mu(d) g\left( \frac{n}{d} \right), \quad \text{for every integer } n \geq 1, \quad (32) \]

where \( \mu \) is the M"obius function and the sums extend over all positive divisors \( d \) of \( n \).

In effect, the original \( h(n) \) can be determined given \( g(n) \) by using the inversion formula.

**Corollary 3.** The number of \( n \)-periodic points of \( F \) is given by,
\[ h(n) = \sum_{d|n} \mu\left( \frac{n}{d} \right) 2^d, \quad \text{for every integer } n \geq 1. \quad (33) \]

Let \( N(n) \) denote the number of \( n \)-periodic orbits of \( F \). Then,
\[ N(n) = \frac{h(n)}{n} = \frac{1}{n} \sum_{d|n} \mu\left( \frac{n}{d} \right) 2^d, \quad \text{for every integer } n \geq 1. \quad (34) \]

The first several \( N(n) \) are
- \( N(1) = 2, \)
- \( N(2) = 1, \)
- \( N(3) = 2, \)
- \( N(4) = 3, \)
- \( N(5) = 6, \)
- \( N(6) = 9, \)
- \( N(7) = 18, \)
- \( N(8) = 30, \quad (35) \)

while \( N(n) \) for larger \( n \) are
- \( N(16) = 4080, \)
- \( N(20) = 52377, \)
- \( N(32) = 134215680, \)
- \( N(64) = 288230376084602880. \quad (36) \)

If the values just above are compared to
\[ \frac{2^6}{6} = 4096, \]
\[ \frac{2^{20}}{20} = 52428.8, \quad (37) \]
\[ \frac{2^{32}}{32} = 134217728, \]
\[ \frac{2^{64}}{64} = 288230376151711744, \]
one finds that the ratio of \( N(n) \) and \( (2^n/n) \) approaches 1 as \( n \to +\infty \).

Now we shall prove that the ratio of \( h(n) \) and \( 2^n \) approaches 1 as \( n \to +\infty \).

**Theorem 2.** Let \( h(p) \) be the number of the \( p \)-periodic points of \( F \). Then,
\[ \lim_{n \to +\infty} \frac{h(n)}{2^n} = \lim_{n \to +\infty} \frac{\sum_{d|n} \mu(n/d) 2^d}{2^n} = 1, \quad (38) \]

where \( \mu \) is the M"obius function.

**Proof.** On one hand,
\[ \sum_{d|n} \mu\left( \frac{n}{d} \right) 2^d \leq \sum_{d|n} 2^d \]
\[ = 2^n + \sum_{d|n, d \neq n} 2^d \]
\[ \leq 2^n + \sum_{1 \leq d \leq [n/2]} 2^d \]
\[ \leq 2^n + 2^{(n/2)+1} - 2. \quad (39) \]

On the other hand,
\[ \sum_{d|n} \mu\left( \frac{n}{d} \right) 2^d = \sum_{d|n, d \neq n} \mu\left( \frac{n}{d} \right) 2^d \]
\[ \geq 2^n - \sum_{d|n, d \neq n} 2^d \]
\[ \geq 2^n - \sum_{1 \leq d \leq [n/2]} 2^d \]
\[ \geq 2^n - 2^{(n/2)+1} + 2. \quad (40) \]
Consequently,

\[
1 = \lim_{n \to +\infty} \frac{2^n - 2^{(n/2)+1} + 2}{2^n} \leq \lim_{n \to +\infty} \frac{\sum_{d \mid n} \mu(n/d) 2^d}{2^n} \leq \lim_{n \to +\infty} \frac{2^n + 2^{(n/2)+1} - 2}{2^n} = 1.
\]  

By the squeeze theorem,

\[
\lim_{n \to +\infty} \frac{h(n)}{2^n} = \lim_{n \to +\infty} \frac{\sum_{d \mid n} \mu(n/d) 2^d}{2^n} = 1.
\]  

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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