Precise relativistic orbits in Kerr and Kerr-(anti) de Sitter spacetimes.

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September 21, 2018

Abstract

The timelike geodesic equations resulting from the Kerr gravitational metric element are derived and solved exactly including the contribution from the cosmological constant. The geodesic equations are derived, by solving the Hamilton-Jacobi partial differential equation by separation of variables. The solutions can be applied in the investigation of the motion of a test particle in the Kerr and Kerr-(anti) de Sitter gravitational fields. In particular, we apply the exact solutions of the timelike geodesics i) to the precise calculation of dragging (Lense-Thirring effect) of a satellite’s spherical polar orbit in the gravitational field of Earth assuming Kerr geometry, ii) assuming the galactic centre is a rotating black hole we calculate the precise dragging of a stellar polar orbit around the galactic centre for various values of the Kerr parameter including those supported by recent observations. The exact solution of non-spherical geodesics in Kerr geometry is obtained by using the transformation theory of elliptic functions. The exact solution of spherical polar geodesics with a nonzero cosmological constant can be expressed in terms of Abelian modular theta functions that solve the corresponding Jacobi’s inversion problem.
1 Introduction

1.1 Motivation

Most of the celestial bodies deviate very little from spherical symmetry, and
the Schwarzschild spacetime is an appropriate approximation for their gravita-
tional field [2]. However, for some astrophysical bodies the rotation of the mass
distribution cannot be neglected. A more general spacetime solution of the grav-
itational field equations should take this property into account. In this respect,
the Kerr solution [7] represents, the curved spacetime geometry surrounding a
rotating mass [8]. Moreover, the above solution is also important for probing
the strong field regime of general relativity [9]. This is significant, since general
relativity has triumphed in large scale cosmology [5, 4, 51, 10], and in predicting
solar system effects like the perihelion precession of Mercury with a very high
precision [1, 3]. In a recent paper [3] Kraniotis and Whitehouse, provided the
exact solution in closed analytic form, of the time-like geodesics that describe
the motion of a test particle in the Schwarzschild gravitational field including
the contribution from the cosmological constant. The solution for the orbit was
expressed in terms of Abelian hyperelliptic genus 2 modular functions using the
solution of Jacobi’s inversion problem for hyperelliptic integrals. For zero cos-

omological constant the solution was provided by the Weierstraß Jacobi modular
form. Subsequently, the authors applied the solution to the compact calculation
of the orbit and the perihelion precession of Mercury around the Sun and com-
pared with current experimental data. The impressive agreement of the precise
theory developed in [3] for the perihelion precession and the eccentricity of the
orbit, based on the exact solution of the geodesic equations, with experiment,
the precise determination of the cosmological constant effect in conjuction with
the dedicated efforts of the experimentalists to measure the orbit of Mercury
with higher accuracy in a series of experiments (BepiColombo ESA [37], Mes-
senger NASA [38]) motivates the development of the theory beyond the static
nature of the Schwarzschild spacetime. In this way, one can determine all the
relativistic effects, and compare the theory with further measurements of the
cosmological and orbital physical parameters.

Furthermore, as was discussed in [3], the investigation of spacetime struc-
tures near strong gravitational sources, like neutron stars or candidate black
hole (BH) systems is of paramount importance for testing the predictions of the
theory in the strong field regime. The study of geodesics are crucial in this re-
spect, in providing information of the structure of spacetime in the strong-field
limit. The generalization of the precise theory, to the gravitational field of a
rotating mass, will have important applications also in this domain.

The study of the geodesics from the Kerr metric are additionally motivated
by recent observational evidence of stellar orbits around the galactic centre,
which indicates that the spacetime surrounding the Sgr A* radio source, which is
believed to be a supermassive black hole of 3.6 million solar masses, is described
by the Kerr solution rather than the Schwarzschild solution, with the Kerr
parameter \[22\]

\[
\frac{J}{GM_{BH}/c} = 0.52 \pm 0.1, \pm 0.08, \pm 0.08
\]  

(1)

where the reported high-resolution infrared observations of Sgr A∗ revealed ‘quiescent’ emission and several flares. This is half the maximum value for a Kerr black hole \[11\]. In the above equation \(J\) \(^1\) denotes the angular momentum of the black hole (The error estimates here the uncertainties in the period, black hole mass \((M_{BH})\) and distance to the galactic centre, respectively; \(G\) is the gravitational constant and \(c\) the velocity of light.)

Taking into account the cosmological constant \(\Lambda\) contribution, the generalization of the Kerr solution is described by the Kerr –de Sitter metric element which in Boyer-Lindquist (BL) coordinates \(^2\) is given by \[15, 16\]:

\[
ds^2 = \frac{\Delta_r}{\Xi^2 \rho^2} (c dt - a \sin^2 \theta d\phi)^2 - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2
\]

\[
- \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} (ac dt - (r^2 + a^2) d\phi)^2
\]

(2)

where

\[
\Delta_r := (1 - \frac{\Lambda}{3} r^2)(r^2 + a^2) - \frac{2GMr}{c^2}
\]

\[
\Delta_\theta := 1 + \frac{a^2 \Lambda}{3} \cos^2 \theta
\]

\[
\Xi := 1 + \frac{a^2 \Lambda}{3}, \quad \rho^2 := r^2 + a^2 \cos^2 \theta
\]

(3)

The above solution is stationary, but not static.

It is the purpose of this paper, to derive the geodesic equations that describe the motion for a test particle in the Kerr and Kerr –de Sitter gravitational fields, and obtain exact solutions of the corresponding equations in an interesting class of possible types of motion, generalizing the results of \[3\].

We also apply the exact solutions of the geodesic equations to the following situations:

**Frame dragging from rotating gravitational mass:** An essential property of the geodesics in Schwarzschild spacetime is that although the orbit precesses relativistically it remains in the same plane; the Kerr rotation adds longitudinal dragging to this precession. For instance, in the spherical polar orbits we will discuss in the main text, (where the particle traverses all latitudes, passes through

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\(^1\)\(J = ca\) where \(a\) is the Kerr parameter. The interpretation of \(ca\) as the angular momentum per unit mass was first given by Boyer and Price \[13\]. In fact, by comparing with the Lense-Thirring calculations \[14\] they determined the Kerr parameter to be: \(a = -\frac{2\Omega l^2}{5c}\), where \(\Omega \) and \(l\) denote the angular velocity and radius of the rotating sphere.

\(^2\)These coordinates have the advantage that reduce to the Schwarzschild solution with a cosmological constant in the limit \(a \to 0\), see \[12\].
the symmetry axis $z$, infinitely many times) the angle of longitude increases after a complete oscillation in latitude. This phenomenon, is in accordance with Mach’s principle.

We shall calculate the dragging of inertial frames in the following situations. a) Dragging of a satellite’s spherical polar orbit in the gravitational field of Earth assuming Kerr geometry. b) Dragging of a stellar, spherical polar orbit, in the gravitational field of a rotating galactic black hole.

The material of this paper is organized as follows. In section 2, starting with the Kerr metric we derive the geodesic equations by integrating the Hamilton-Jacobi partial differential equation by separation of variables. In section 3, we solve exactly the timelike spherical polar geodesics. The solution for the orbit is expressed in terms of the Weierstraß elliptic function. We then apply the exact solution in determining the frame-dragging (Lense-Thirring effect) of a satellite’s spherical polar orbit in the gravitational field of Earth as well as the dragging of a stellar spherical polar orbit by a galactic black hole. The exact expression of dragging is proportional to the real half-period of the Weierstraß Jacobi modular form. An alternative expression in terms of a hypergeometric function is also provided. In section 4, we solve exactly spherical non-polar geodesics. In section 5, we describe the general solution of (non-spherical) timelike geodesics in Kerr metric, using the transformation theory of elliptic functions, an approach developed by Abel in [17]. In section 6, we derive the timelike geodesic equations in Kerr metric that includes the contribution from the cosmological constant. The derivation is produced, by integrating the Hamilton-Jacobi partial differential equation again by a separation of variables. We subsequently discuss the exact solution, for spherical polar geodesics with a cosmological constant. The integration can be achieved, by solving the corresponding Jacobi’s inversion problem for a system of Abelian integrals that we have formulated. Afterwards, the equatorial geodesis with a cosmological constant are presented together with the exact solution of the corresponding circular equatorial orbits. The application of the exact solutions with the $\Lambda$ term present will be a subject of a future publication. Finally, in section 7 we present our conclusions and discuss future research directions. In the appendices, we collect some of our formal calculations as well as some useful formulas for the solution of cubic and quartic equations and definitions of genus-2 theta functions.

2 Geodesics and the separability of the Hamilton-Jacobi differential equation in Kerr metric

2.1 Determination of first geodesic integrals using the Hamilton-Jacobi approach assuming vanishing cosmological constant

In this section we are going to review the method of integration of the Hamilton-Jacobi differential equation, when separation of variables is taking place and
apply it in deriving the timelike geodesic equations that describe the motion of a test particle in a Kerr spacetime. The more general case of Kerr-de Sitter spacetime is treated in section 6. The method for deriving the geodesics, assuming vanishing cosmological constant in the Kerr spacetime solution, was first applied from Carter [23].

The Hamilton-Jacobi differential equation is given by:

$$\frac{\partial W}{\partial t} + H\left(t; q_1, q_2, \cdots, q_k; \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \cdots, \frac{\partial W}{\partial q_k}\right) = 0$$  \hspace{1cm} (4)

where $H$, the characteristic function, depends on the degrees of freedom $q_1, q_2, \cdots, q_k, t$ and the momenta $p_i := \frac{\partial W}{\partial q_i}, i = 1, \cdots k$. Also $H$, is a function of second degree in the partial derivatives of $W$. Assuming, that an integral $W(t; q_1, q_2, \cdots, q_k; \alpha_1, \cdots, \alpha_k)$ has been found, where $\alpha_1, \cdots, \alpha_k$ are integration constants, then the general integral of the differential equations of mechanics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$  \hspace{1cm} (5)

is provided by

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \quad p_1 = \frac{\partial W}{\partial q_1}$$

$$\cdots$$

$$\frac{\partial W}{\partial \alpha_k} = \beta_k, \quad p_k = \frac{\partial W}{\partial q_k}$$  \hspace{1cm} (6)

where $\beta_1, \cdots, \beta_k$ denote new arbitrary constants.

An interesting case, which was first studied in great detail by Stäckel [25] and Levi-Civita [26], is when one can integrate the Hamilton-Jacobi differential equation through separation of variables. In this case, the characteristic function $W$ takes the form:

$$W = \sum_{i=1}^{n} W_i(q_i)$$  \hspace{1cm} (7)

where every term on the right hand side of eq. (7) $W_i(q_i)$ depends only on one of the variables $q_i$.

Assuming now, that the characteristic function separates in the Kerr metric

$$W = W_t(t) + W_\phi(\phi) + W_\theta + W_r$$  \hspace{1cm} (8)

where $W_\theta, W_r$ are functions of $\theta$ and $r$ coordinates respectively, and then plug this expression in the relativistic Hamiltonian in BL coordinates

$$H(x^\alpha, p_\beta) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$$  \hspace{1cm} (9)
and the relativistic form of eq.(4),

\[ g^{\mu \nu} \frac{\partial W}{\partial x^\mu} \frac{\partial W}{\partial x^\nu} + \mu^2 = 0, \]

one gets

\[ W = -Ect + \int \frac{\sqrt{R}}{\Delta} dr + \int \sqrt{\Theta} d\theta + L\phi \]  

(10)

where

\[ \Theta := Q - \left[ a^2(\mu^2 - E^2) + \frac{L^2}{\sin^2 \theta} \right] \cos^2 \theta \]  

(11)

and

\[ R := \left[ (r^2 + a^2)E - aL \right]^2 - \Delta \left[ \mu^2 r^2 + (L - aE)^2 + Q \right] \]  

(12)

with \( \Delta := r^2 + a^2 - \frac{2GM}{r} \). Also \( E, L \) are constants of integration associated with the isometries of the Kerr metric. Carter’s constant of integration is denoted by \( Q \). Differentiation of (10), with respect to the integration constants \( E, L, Q, \mu \) leads to the following set of first-order equations of motion [23]:

\[
\begin{align*}
\rho^2 \frac{cdt}{d\lambda} &= \frac{r^2 + a^2}{\Delta} P - a \left( a E \sin^2 \theta - L \right) \\
\rho^2 \frac{dr}{d\lambda} &= \pm \sqrt{R} \\
\rho^2 \frac{d\theta}{d\lambda} &= \pm \sqrt{\Theta} \\
\rho^2 \frac{d\phi}{d\lambda} &= \frac{a}{\Delta} P - aE + \frac{L}{\sin^2 \theta}
\end{align*}
\]  

(13)

where

\[ P := E(r^2 + a^2) - aL \]  

(14)

We proceed now to discuss the exact solution of timelike geodesic equations (13) in closed analytic form, and apply them to the precise determination of Lense-Thirring precession of a satellite’s spherical polar orbit in the gravitational field of Earth, as well as in stellar orbits around the galactic centre. We should mention at this point, the extreme black hole solutions \( a = 1 \) of spherical non-polar geodesics obtained in [24] expressed in terms of formal integrals. The more general case in the presence of the cosmological constant, will be discussed in section 6.

### 3 Spherical Polar Geodesics

Depending on whether or not the coordinate radius \( r \) is constant along a given timelike geodesic, the corresponding particle orbit is characterized as spherical or non-spherical respectively. In this section, we will concentrate on spherical polar orbits.

Assuming zero cosmological constant, \( r = r_f \), where \( r_f \) is a constant value and using the last two equations of (13) we obtain:

\[ \frac{d\phi}{d\theta} = \frac{\frac{aP}{\Delta} - aE + L/\sin^2 \theta}{\sqrt{\Theta}} \]  

(15)
Equation (15) can be rewritten

\[
d\phi = \frac{d\theta \sin \theta \left\{ \frac{|a|E(r^2 + a^2 - aL)}{r^2 + a^2 - a^2 \cos^2 \theta} - aE + \frac{L}{\sin^2 \theta} \right\}}{\sqrt{Q(1 - \cos^2 \theta) - [a^2(\mu^2 - E^2)(1 - \cos^2 \theta) + L^2] \cos^2 \theta}}
\]  

(16)

Now by defining \( z := \cos^2 \theta \), the previous equation can be written as follows:

\[
d\phi = -\frac{1}{2} \frac{dz}{\sqrt{z^3\alpha - z^2(\alpha + \beta) + Qz}} \times \left\{ \frac{AP'}{\Delta} - aE + \frac{L}{1 - z} \right\}
\]

(17)

where \( \alpha := a^2(\mu^2 - E^2), \beta := Q + L^2 \)

(18)

It has been shown [27] that a necessary condition for an orbit to be polar (meaning to intersect the symmetry axis of the Kerr gravitational field) is the vanishing of the parameter \( L \), i.e., \( L = 0 \). Assuming \( L = 0 \), in equation (17), we can transform it into the Weierstraß form of an elliptic curve by the following substitution

\[
z := -\frac{\xi + \frac{\alpha + \beta}{12}}{-\alpha/4}
\]

(19)

Thus, we obtain the integral equation

\[
\int d\phi = \int \frac{1}{2} \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} \times \left\{ \frac{AP'}{\Delta} - aE \right\}
\]

(20)

and this orbit integral can be inverted by the Weierstraß modular Jacob form

\[
\xi = \wp(\phi/A)
\]

(21)

where \( A := -\frac{1}{2} \left( \frac{AP'}{\Delta} - aE \right), P' = E(r^2 + a^2) \).

The Weierstraß invariants are given by

\[
\begin{align*}
g_2 &= \frac{\alpha^2}{12} + \frac{\beta^2}{12} + \frac{\alpha\beta}{6} - \frac{Q\alpha}{4} = \frac{1}{12}(\alpha + \beta)^2 - \frac{Q\alpha}{4} \\
g_3 &= \frac{\alpha^3}{216} + \frac{\alpha^2\beta}{72} + \frac{\alpha\beta^2}{72} + \frac{\beta^3}{216} - \frac{\alpha^2Q}{48} - \frac{Q\alpha\beta}{48} = \frac{1}{216}(\alpha + \beta)^3 - \frac{Q\alpha^2}{48} - \frac{Q\alpha\beta}{48}
\end{align*}
\]

(22)

We note that the discriminant \( \Delta^c := g_2^3 - 27g_3^2 \) of the elliptic curve, vanishes for \( a = 0 \), \(^4\) and this is consistent with the fact that circular orbits in the Schwarzschild spacetime do not have modular properties [3].

Now

\[
\frac{AP'}{\Delta} - aE = a \left\{ \frac{E(r^2 + \frac{1}{r^2}) + 1}{r^2 + \frac{1}{r^2} - 2GMr/c^2a^2} - E \right\}
\]

(23)

\(^3\)For more information on the properties of the Weierstraß function, the reader is referred to the monographs [34, 40], and the appendix of [10].

\(^4\)in this case \( a = 0 \), \( \beta = Q + L^2 (= Q \) for polar orbits).
Then
\[ d\phi = \frac{dz}{\sqrt{z^3\alpha' - z^2(\alpha' + \beta')} + Q'z} A' \]  

where

\[ A' := \left\{ \frac{E(\frac{r^2}{a^2} + 1)}{c \frac{r^2}{a^2} + 1 - \frac{2GMr}{c^2a^2}} - E \right\} \times \left( -\frac{1}{2} \right) \]

\[ = \frac{-EGMr}{c^2a^2} \]  

and \( \alpha' := \frac{\alpha}{a^2}, \beta' := Q/a^2 = Q' \). Using Eq.(19) with \( \alpha \to \alpha', \beta \to \beta' \), and integrating Eq.(24) we get

\[ \xi = \varphi(\phi/A') \]  

with the Weierstraß invariants \( g_2', g_3' \) given by the expressions

\[ g_2' = \frac{\alpha'^2}{12} + \frac{\beta'^2}{12} + \frac{\alpha'\beta'}{6} - \frac{Q'\alpha'}{4} \]

\[ g_3' = \frac{\alpha'^3}{216} + \frac{\alpha'^2\beta'}{72} + \frac{\alpha'\beta'^2}{72} + \frac{\beta'^3}{216} - \frac{\alpha'^2Q'}{48} - \frac{\alpha'\beta'Q'}{48} \]  

Equation (26) represents the first exact solution of a spherical polar orbit assuming zero cosmological constant, in closed analytic form, in terms of the Weierstraß Jacobi modular form.

Equation (26) can be rewritten in terms of the original variables as

\[ \varphi(\phi + \epsilon) = \frac{\alpha''}{4}\cos^2\theta - \frac{1}{12}(\alpha'' + \beta'') \]  

where \( \alpha'' := \alpha'/A'^2, \beta'' := \beta'/A'^2, Q'' := Q'/A'^2 \) with corresponding Weierstraß invariants \( g_2'', g_3'' \). Also \( \epsilon \) is a constant of integration.

### 3.1 Frame dragging in spherical polar geodesics assuming vanishing cosmological constant

We are now going to calculate the Lense-Thirring effect which as we discussed in the introduction is the imprint of rotating black holes or rotating mass distributions, i.e. the dragging of inertial frames. For this we shall need to calculate the periods of the Weierstraß modular form.

In order to calculate the periods of the Weierstraß function we need to determine the roots of the cubic. The Weierstraß invariants are given in terms of

\[ 8 \]
the Kerr parameter $a$ and the initial conditions $E, Q$ as follows:\footnote{We have set $\mu = 1$.}:

\[
g''_2 = \frac{1}{12} \left( \frac{a^2(1 - E^2) + Q}{a^4 A^4} \right)^2 - \frac{1}{4} \frac{Q(1 - E^2)}{a^2 A^4}
\]

\[
g''_3 = \frac{1}{432a^6 A^6} \left[ 2a^6(1 - E^2)^3 - 3a^4(1 - E^2)^2 Q - 3a^2(1 - E^2) Q^2 + 2Q^3 \right]
\]

while the discriminant $\Delta^c$ of the cubic equation is given by the expression

\[
\Delta^c = \frac{(1 - E^2)^2 Q^2 (a^2(-1 + E^2) + Q)^2}{256a^8 A^{12}}
\]

(29)

The sign of the discriminant $\Delta^c$ determines the roots of the elliptic curve: $\Delta^c > 0$, corresponds to three real roots while for $\Delta^c < 0$ two roots are complex conjugates and the third is real. In the degenerate case $\Delta^c = 0$, (where at least two roots coincide) the elliptic curve becomes singular and the solution is not given by modular functions. The analytic expressions for the three roots of the cubic, which can be obtained by applying the algorithm of Tartaglia and Cardano [54], are given by

\[
\begin{align*}
e_1 &= \frac{(a^2(-1 + E^2) + 2Q) \Delta^2}{12a^2E^2 r^2 (GM/c^2)^2} \\
e_2 &= -\frac{(2a^2(-1 + E^2) + Q) \Delta^2}{12a^2E^2 r^2 (GM/c^2)^2} \\
e_3 &= \frac{(a^2(-1 + E^2) - Q) \Delta^2}{12a^2E^2 r^2 (GM/c^2)^2}
\end{align*}
\]

(31)

Since we are assuming spherical orbits, there are two conditions from the vanishing of the quartic polynomial $R$ and its first derivative. Implementing, these two conditions expressions for the parameter $E$ and Carter’s constant $Q$ are obtained,

\[
\begin{align*}
E^2 &= \frac{r \Delta^2}{(r^2 + a^2)Z} \\
Q &= \frac{r \left( r^3 GM/c^2 + a^2 r^2 - 3a^2 GM/c^2 + a^4 \right)}{Z} - a^2 E^2
\end{align*}
\]

(32)

where $Z := r^3 + a^2 r - 3 \frac{GM}{c^2} r^2 + a^2 \frac{GM}{c^2}$ [27].
Using the first and fourth line of Eq.(13) with \( L = 0 \), we obtain

\[
\frac{cdt}{d\phi} = \frac{r^2 + a^2 P' - a^2 E \sin^2 \theta}{\Delta} - \frac{a}{\Delta} - aE + \frac{a^2 E \cos^2 \theta}{\Delta} - \frac{aE}{\Delta}
\]

or

\[
ct + E = a\phi + \frac{r^2 P'/\Delta}{(-2aA')} \phi - \frac{4a^2 E/\alpha''}{(-2aA')} \left(\zeta(\phi) - \frac{1}{12} (\alpha'' + \beta'') \phi\right)
\]

where we used Eq.(28) and the fact that, \( \int \varphi(\phi) d\phi = -\zeta(\phi) \), where \( \zeta(z) \) denotes the Weierstraß zeta function. Also \( E \) denotes a constant of integration.

Our free parameters are \( a \) and \( r \). For a given choice of our free parameters, \( E \) and Carter’s constant \( Q \) are fixed with the aid of (32).

The two half-periods \( \omega \) and \( \omega' \) are given by the following Abelian integrals (for \( \Delta^c > 0 \)) [53]:

\[
\omega = \int_{\epsilon_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \quad \omega' = i \int_{-\infty}^{-\epsilon_3} \frac{dt}{\sqrt{-4t^3 + g_2 t + g_3}}
\]

The values of the Weierstraß function at the half-periods \(^6\) are the three roots of the cubic. For positive discriminant \( \Delta^c \) one half-period is real while the second is imaginary \(^7\). The period ratio is defined as \( \tau = \frac{\omega'}{\omega} \).

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical polar orbit in the General Theory of Relativity (GTR), increases by

\[
\Delta \phi^{GTR} = 4\omega
\]

### 3.1.1 Dragging of satellite polar orbit in the gravitational field of rotating Earth

In this subsection, assuming the Earth’s gravitational field is described by a Kerr space-time, we apply the exact solution of the Kerr spherical polar geodesics obtained in the previous section, in determining the effect of Earth’s rotation on the motion of its orbiting satellites. Thus we perform a precise calculation of the corresponding Lense-Thirring effect.

Experimentally, a polar orbital geometry has the advantage that classical nodal precessions due to the even zonal harmonics of the geopotential, are vanishing. We note, that the Gravity Probe B (GP-B) mission, launched in April

\(^6\)An alternative expression for the real half-period \( \omega \) of the Weierstraß function is: \( \omega = \sqrt{1 - \frac{4a^2 E}{\alpha'' + \beta''}} \), where \( F(a, \beta, \gamma, x) \) is the hypergeometric function \( 1 + \frac{\alpha \beta}{\gamma + 1} x + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{(\gamma + 1)(\gamma + 2)} x^2 + \cdots \).

\(^7\)We organize the roots as: \( \epsilon_1 > \epsilon_2 > \epsilon_3 \).
The Kerr parameter that corresponds to the angular momentum of the Earth is equal to \( a_\oplus = 329.432 \text{ cm} = 371.398 (2GM_\oplus/c^2) \). For the radius of our spherical orbit we use the semi-major axis of the GP-B mission \( r = r_{\text{GP-B}} = 7027.4 \text{ km} \). Then Carter’s constant \( Q \) and the parameter \( E \) are determined by eq.(32). Then, the half-period \( \omega \) has the value \( \omega = 3.699746 \times 10^{-11} \), which leads to the precise frame-dragging effect of \( \Delta \phi^{\text{GTR}} = 0.164 \text{ arcs yr}^{-1} \). We repeated the analysis, for slight different values of radius \( r \), and for fixed Kerr parameter. In particular we used as a value for \( r \) the semi-major axis of the Polares satellite, i.e. \( r = r_{\text{Polares}} = 8378 \text{ km} \). We list our results in table 1.

We also compare our precise results with the Lense-Thirring (LT) formula [14]:

\[
\Delta \phi^{\text{LT}} = \frac{2GJ}{c^2r_e^3(1-e^2)^{3/2}}
\]

in which \( J \) is the magnitude of the central body’s angular momentum, \( r_e \) the semi-major axis of the orbit and \( e \) the eccentricity of the orbit. For zero eccentricity, formula (37) predicts \( 0.164 \text{ arcs yr}^{-1} \) and \( 0.096 \text{ arcs yr}^{-1} \) for the radii of GP-B mission and Polares satellites, respectively.

Table 1: Predictions for frame dragging of a satellite’s polar spherical orbit in the gravitational field of Earth. The period ratio is \( \tau = 6.18642t \).

| parameters | half-period | predicted dragging |
|------------|-------------|---------------------|
| \( a_\oplus = 329.432 \text{ cm}, r = r_{\text{GP-B}} = 7027.4 \text{ km} \) | \( \omega = 3.699746 \times 10^{-11} \) | \( \Delta \phi^{\text{GTR}} = 0.164 \text{ arcs yr}^{-1} \) |
| \( a_\oplus = 329.432 \text{ cm}, r = r_{\text{POLARES}} = 8378 \text{ km} \) | \( \omega = 2.8421923 \times 10^{-11} \) | \( \Delta \phi^{\text{GTR}} = 0.096 \text{ arcs yr}^{-1} \) |

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8http://einstein.stanford.edu/
9Indeed if the orbit is exactly polar the precession due to Earth’s quadrupole moment \( J_2 \), vanishes. Practically, it had originally been suggested by Van Patten and Everitt a two satellite experiment, with the two satellites in similar nearly polar orbits. The quadrupole moment of the Earth, then produces precessions of opposite directions for the two satellites while the Lense–Thirring precession produces the same effect for both. To separate these two effects, the small angle between their orbits needs to be measured in addition to the precession rates of the two satellites. The distance of the two satellites at each polar crossing is measured by Doppler ranging. Substracting, the contribution of the quadrupole moment from the actual measurement determines the Lense-Thirring effect. Ciufolini [29] alternatively suggested, to make use of a satellite already in orbit LAGEOS and require only one new satellite LAGEOS II. Although the orbit of LAGEOS, is far from polar, if the orbit of LAGEOS II is inclined at an exactly opposite angle with respect to the axis of the Earth, the Lense-Thirring effect can directly be measured, since the contributions from the quadrupole moment cancel exactly. See also [33] for a different approach.
10We note the small eccentricity (0.014) of the GP-B satellite.
Table 2: Predictions for frame dragging from galactic black hole, with Kerr parameter $a_{\text{Galactic}} = 0.52\frac{GM_{BH}}{c^2}$, for two different values of orbital radius. The values of the radii are in units of $GM_{BH}/c^2$. The period ratios, $\tau$, are $2.92558i$ and $3.16583i$ respectively.

| parameters          | half-period | predicted dragging         |
|---------------------|-------------|----------------------------|
| $a_{\text{Galactic}} = 0.52, r = 10$ | $\omega = 0.0515693$ | $\Delta \phi_{\text{GTR}} = 11.8188^\circ$ per revolution = $42531.4$ arcs/revolution |
| $a_{\text{Galactic}} = 0.52, r = 50$ | $\omega = 0.00462023$ | $\Delta \phi_{\text{GTR}} = 1.0589^\circ$ per revolution = $3810.9$ arcs/revolution |

Table 3: Predictions for frame dragging from galactic black hole, with Kerr parameter $a_{\text{Galactic}} = 0.9939\frac{GM_{BH}}{c^2}$, for two different values of orbital radius. The values of the radii are in units of $GM_{BH}/c^2$. The period ratios, $\tau$, are $2.5086i$ and $3.40363i$ respectively.

| parameters          | half-period | predicted dragging         |
|---------------------|-------------|----------------------------|
| $a_{\text{Galactic}} = 0.9939, r = 10$ | $\omega = 0.0981121$ | $\Delta \phi_{\text{GTR}} = 22.485^\circ$ per revolution = $80917.2$ arcs/revolution |
| $a_{\text{Galactic}} = 0.9939, r = 50$ | $\omega = 0.00882902$ | $\Delta \phi_{\text{GTR}} = 2.023^\circ$ per revolution = $7281.66$ arcs/revolution |

3.1.2 Precise calculation of dragging of stellar polar orbits from a galactic black hole

Assuming, that the centre of the Milky Way is a black hole and that the structure of spacetime near the region Sgr A*, is described by the Kerr geometry as is indicated by observations Eq.(1), we determined the precise frame dragging (Lense-Thirring effect) of a stellar orbit with a spherical polar geometry. The results are displayed in table 2.

We repeated the analysis for a value of the Kerr parameter as high as $a_{\text{Galactic}} = 0.9939$. Such high values for the angular momentum of the black hole, have been recently reported from X-ray flare analysis of the galactic centre [30]. The results are given in table 3.

4 Spherical geodesics with $L \neq 0$

We now integrate Eq.(17) including the contribution from the parameter $L$. Let us define:

$$\Pi := \int \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}}$$

thus $\xi = \varphi(\Pi+\epsilon)$, and $A'' \int \frac{dz}{\sqrt{z^3(\alpha + \beta) + Qz}} = A'' \int \frac{dz}{\sqrt{4\xi^3 - g_2\xi - g_3}} = A''\Pi$, $A'' := -\frac{1}{2}(\frac{dP}{d\lambda} - aE)$ and $\epsilon$ is a constant of integration. Now

$$-\frac{1}{2}L \int \frac{dz}{(1-z)\sqrt{z^3(\alpha + \beta) + Qz}}$$
under the substitution (19) becomes

\[- \frac{L\alpha}{8} \int \frac{d\xi}{\left( \frac{4}{9}(1 - \frac{\alpha + \beta}{3\alpha}) - \xi \right) \sqrt{4\xi^4 - g_2\xi - g_3}} = - \frac{L\alpha}{8} \int \frac{d\xi}{(w - \xi) \sqrt{4\xi^4 - g_2\xi - g_3}}
\]

\[- \frac{L\alpha}{8} \int \frac{d\Pi}{\psi'(\Pi)\Theta(\Pi)} = - \frac{L\alpha}{8} \int \frac{d\Pi}{(w - \psi(\Pi + \epsilon)) \sqrt{4\psi^3(\Pi) - g_2\psi(\Pi) - g_3}}
\]

\[- \frac{L\alpha}{8} \int \frac{d\Pi}{\psi'(\Pi)}
\]

\[- \frac{L\alpha}{8} \left[ \log \frac{\sigma(\Pi + \epsilon - v_0)}{\sigma(\Pi + \epsilon + v_0)} + 2\Pi \zeta(v_0) \right] \times \frac{1}{\psi'(v_0)} \]  

(40)

where \( w := \frac{\alpha}{4} \left( 1 - \frac{\alpha + \beta}{3\alpha} \right) = \psi(v_0) \). Also \( \sigma(z) \) denotes the Weierstraß sigma function. Thus the equation for the orbit is given by

\[ \phi = \int d\phi = A''\Pi - \frac{L\alpha}{8} \left[ \log \frac{\sigma(\Pi + \epsilon - v_0)}{\sigma(\Pi + \epsilon + v_0)} + 2\Pi \zeta(v_0) \right] \times \frac{1}{\psi'(v_0)} \]  

(41)

and \( \psi'(v_0) = 4\psi^3(v_0) - g_2\psi(v_0) - g_3 = 4w^3 - g_2w - g_3 \). In terms of the integration constants, \( w \) is given by the expression:

\[ w = \frac{a^2(1 - E^2)}{4} - \frac{a^2(1 - E^2) + Q + L^2}{12} \]  

(42)

Similarly using the first and third line of Eq.(13), we obtain for \( t \) the expression

\[ c t = \frac{\sqrt{\Delta}}{P} \frac{\Pi}{-2} + \frac{a\Pi}{2}(aE - L) + \frac{a^2E \Pi}{2} \left( -\frac{1}{3} \frac{\alpha + \beta}{\alpha} \right) + \frac{a^2E \Pi}{2} \frac{4}{\alpha} \zeta(\Pi) \]  

(43)

5 General solution for the time-like geodesics in Kerr metric

In the general case, of non-spherical orbits one has to solve the differential equations

\[ \int^\theta \frac{d\theta}{\sqrt{\Theta}} = \int^r \frac{dr}{\sqrt{R}} \]  

(44)

where \( R(r) \) is a quartic polynomial given by

\[ R = \left[ (r^2 + a^2)r - aL \right]^2 - \Delta \left[ r^2 + (L - aE)^2 + Q \right] \]  

(45)

and \( \Theta(\theta) \) is given by equation (11). Note that this equation is an equation between two elliptic integrals, in the general case when all roots are distinct.
The left hand side can be transformed into an elliptic integral with variable \( z \) or \( \xi \) in Weierstraß normal form, see Eq.(19). In order to solve this differential equation and determine \( r \) as a function of \( \theta \), we will employ the theory of Abel for the transformation of elliptic functions [17].

The general case of non-spherical orbits is of importance for determining the precession of perihelion, periapsis, and perinigricon (point of closest approach to the black hole).

5.1 Exact solution of the general Kerr geodesic using the transformation theory of elliptic functions

Abel in [17] deals with the following question:

"Find all possible cases in which the differential equation

\[
\frac{dy}{\sqrt{(1 - c^2y^2)(1 - e^2x^2)}} = \pm C \frac{dx}{\sqrt{(1 - c^2x^2)(1 - e^2x^2)}},
\]

is satisfied by putting \( y \) an algebraic function of \( x \), rational or irrational."

He explains that the problem may be reduced to the case in which \( y \) is a rational function of \( x \). Below we shall describe his approach, which uses the idea of inversion of elliptic integrals and then we shall apply it to the solution of eq.(44). Abel's notation is as follows: From the elliptic integral

\[
\vartheta = \int_0^x \frac{dx}{\sqrt{(1 - c^2x^2)(1 - e^2x^2)}},
\]

he defines the inverse function

\[
x := \lambda \vartheta
\]

while

\[
\frac{\omega}{2} = \int_0^{1/c} \frac{dx}{\sqrt{(1 - c^2x^2)(1 - e^2x^2)}}, \quad \frac{\omega'}{2} = \int_0^{1/e} \frac{dx}{\sqrt{(1 - c^2x^2)(1 - e^2x^2)}},
\]

and \( \Delta \vartheta = \sqrt{(1 - c^2x^2)(1 - e^2x^2)} \). Also the solution of the equation \( \lambda \vartheta' = \lambda \vartheta \) is \( \vartheta' = (-1)^{m+m'} \vartheta + m\omega + m'\omega' \) where \( m, m' \) are arbitrary integral numbers.

Let \( y = \psi(x) \) be the rational functions we are looking for and \( x = \lambda \vartheta, x_1 = \lambda \vartheta_1 \) two roots of the equation \( y = \psi(x), (y = \psi(x) = \psi(x_1)) \). Let the radical on the left hand side of Eq.(46), be denoted by \( \sqrt{R} \), then

\[
\frac{dy}{\sqrt{R}} = \pm C d\vartheta
\]

Then by changing \( x \) with \( x_1 \), we have \( \pm \frac{dy}{\sqrt{R}} = \pm C d\vartheta \), which gives \( d\vartheta_1 = \pm d\vartheta \) or after integration \( \vartheta_1 = \epsilon \pm \vartheta \) where \( \epsilon \) is a constant. The quantities \( \lambda \vartheta \) and \( \lambda (\vartheta + \epsilon) \) are roots, thus we have

\[
y = \psi(\lambda \vartheta) = \psi(\lambda (\vartheta + \epsilon))
\]
Now

\[ y = \psi(\lambda \vartheta) = \psi(\lambda(\vartheta + \epsilon)) = \psi(\lambda(\vartheta + 2\epsilon)) = \cdots = \psi(\lambda(\vartheta + k\epsilon)) \]  

(52)

where \( k \) denotes an arbitrary integral number. As the equation \( y = \psi(x) \) has only a finite number of roots, there exist \( k \) and \( k' \) distinct such that \( \lambda(\vartheta + k\epsilon) = \lambda(\vartheta + k'\epsilon) \). Assuming \( n := k - k' > 0 \), and replacing \( \vartheta \) by \( \vartheta - k'\epsilon \), we get

\[ \lambda(\vartheta + n\epsilon) = \lambda \vartheta \]  

(53)

Thus we get \( \vartheta + n\epsilon = (-1)^{m+m'}\vartheta + m\omega + m'\omega' \), and since \( \vartheta \) is a variable we get \((-1)^{m+m'} = 1\) and

\[ \epsilon = \frac{m}{n}\omega + \frac{m'}{n}\omega' \]  

(54)

where \( \mu := m/n, \mu' := m'/n \in Q, m + m' \) is even number. Assuming the degree of the equation \( y = \psi(x) \) is such that it has roots other then \( \lambda(\vartheta + k\epsilon) \), anyone of them has the form \( \lambda(\vartheta + \epsilon_1) \) where \( \epsilon_1 = \mu_1\omega + \mu'_1\omega' \), \( (\mu_1, \mu'_1) \in Q \) and all the \( \lambda(\vartheta + k\epsilon + k_1\epsilon_1) \) are roots of the equation. Continuing in this fashion the roots of \( y = \psi(x) \) are of the form

\[ x = \lambda(\vartheta + k_1\epsilon_1 + k_2\epsilon_2 + \cdots + k_{\nu}\epsilon_{\nu}) \]  

(55)

where \( k_1, k_2, \cdots k_{\nu} \) are integers and the quantities \( \epsilon_1, \epsilon_2, \cdots \epsilon_{\nu} \) are of the form \( \mu\omega + \mu'\omega' \). By denoting the values of the expression \( \lambda(\vartheta + k_1\epsilon_1 + k_2\epsilon_2 + \cdots k_{\nu}\epsilon_{\nu}) \) by \( \lambda_\vartheta, \lambda(\vartheta + \epsilon_1), \lambda(\vartheta + \epsilon_2), \cdots \lambda(\vartheta + \epsilon_{m-1}) \), and writing \( \psi(x) = \frac{\varphi}{\eta}, \) where \( p, q \) are polynomials of degree \( m \) in \( x \), with no common divisor, the equation \( y = \psi(x) \) is written

\[ p - qy = (f - gy)(x - \lambda\vartheta)(x - \lambda(\vartheta + \epsilon_1)) \cdots (x - \lambda(\vartheta + \epsilon_{m-1})) \]  

(56)

where the constants \( f, g \) are the dominant coefficients of \( p, q \) respectively. If \( f' \) and \( g' \) denote the coefficients of \( x^{m-1} \) we have that

\[ f' - g'y = -(f - gy)[\lambda\vartheta + \lambda(\vartheta + \epsilon_1) + \lambda(\vartheta + \epsilon_2) + \cdots \lambda(\vartheta + \epsilon_{m-1})] \]  

(57)

and if we define

\[ \varphi\vartheta := \lambda\vartheta + \lambda(\vartheta + \epsilon_1) + \lambda(\vartheta + \epsilon_2) + \cdots \lambda(\vartheta + \epsilon_{m-1}) \]  

(58)

we get for \( y \) the expression

\[ y = \frac{f' + f\varphi\vartheta}{g' + g\varphi\vartheta} \]  

(59)

By hypothesis, \( y \) is a rational function in \( x \) and so \( \varphi \) must be the same. The goal is to express \( \varphi \) rationally in \( x \) and to determine \( f, f', g, g', \epsilon_1, \epsilon_1 \) and \( C \) in order that (46) is satisfied.

First, we search for values of \( \epsilon \) which satisfy \( \lambda(\vartheta - \epsilon) = \lambda(\vartheta + \epsilon) \), that is to say \( \lambda(\vartheta + 2\epsilon) = \lambda\vartheta \). According to (54),

\[ \epsilon = \frac{m}{2}\omega + \frac{m'}{2}\omega' \]  

(60)
where \(m + m' \in 2\mathbb{Z}\). In this case \(\lambda(\vartheta + \epsilon)\) takes the following distinct values \([17]\)

\[
\begin{align*}
\lambda \vartheta &= x, \\
\lambda(\vartheta + \omega) &= -\lambda \vartheta = -x, \\
\lambda(\vartheta + \frac{\omega + \omega'}{2}) &= -\frac{1}{1 - ec x}, \\
\lambda(\vartheta + \frac{3\omega + \omega'}{2}) &= -\frac{1}{1 - ec \lambda(\vartheta + \omega)} = \frac{1}{1 - ec x}
\end{align*}
\]

(61)

All the other roots \(\lambda(\vartheta + \epsilon)\) come with a term \(\lambda(\vartheta - \epsilon)\) and therefore we can write an expression for \(\varphi \vartheta\) as follows \([17]\):

\[
\varphi \vartheta = \lambda \vartheta + k\lambda(\vartheta + \omega) + k'\lambda(\vartheta + \frac{\omega + \omega'}{2}) + k''\lambda(\vartheta + \frac{3\omega + \omega'}{2}) + \lambda(\vartheta + \epsilon_1) + \lambda(\vartheta - \epsilon_1) + \lambda(\vartheta + \epsilon_2) + \lambda(\vartheta - \epsilon_2) + \cdots + \lambda(\vartheta + \epsilon_n) + \lambda(\vartheta - \epsilon_n)
\]

(62)

where \(k, k', k''\) are equal to 0 or 1.

### 5.1.1 Rational solution for \(y = \psi(x)\), when \(k = k' = k'' = 0\)

In this particular case, we have

\[
\begin{align*}
\varphi \vartheta &= \lambda \vartheta + \lambda(\vartheta + \epsilon_1) + \lambda(\vartheta - \epsilon_1) + \lambda(\vartheta + \epsilon_2) + \lambda(\vartheta - \epsilon_2) + \cdots + \lambda(\vartheta + \epsilon_n) + \lambda(\vartheta - \epsilon_n) \\
\end{align*}
\]

(63)

or written in terms of \(x\)

\[
\varphi \vartheta = x + 2x \sum_i \frac{\Delta \epsilon_i}{1 - e^2 \epsilon_i^2 \lambda^2 x^2}
\]

(64)

Thus, the prime condition that \(y\) is rational in \(x\) is fulfilled. Let \(\delta, \delta', \eta, \eta'\) be the values of \(\vartheta\) that correspond to \(y = 1/c_1, -1/c_1, 1/c_1, -1/c_1\). Thus, for \(y = 1/c_1\),

\[
1 - c_1 y = \frac{g' - c_1 f' + (g - c_1 f) \varphi \delta}{g' + g \varphi \delta}
\]

or

\[
1 - c_1 y = \frac{g' - c_1 f'}{\Sigma} \left[ \frac{1 - \varphi \vartheta}{\varphi \delta} \right]
\]

(65)

where \(\Sigma := g' + g \varphi \vartheta\). Similarly, we get \([17]\)

\[
\begin{align*}
1 + c_1 y &= \frac{g' + c_1 f'}{\Sigma} \left[ \frac{1 - \varphi \vartheta}{\varphi \delta'} \right] \\
1 - e_1 y &= \frac{g' - e_1 f'}{\Sigma} \left[ \frac{1 - \varphi \vartheta}{\varphi \eta} \right] \\
1 + e_1 y &= \frac{g' + e_1 f'}{\Sigma} \left[ \frac{1 - \varphi \vartheta}{\varphi \eta'} \right]
\end{align*}
\]

(66)
From the expression of σ, one gets

\[
1 - \frac{\varphi \partial}{\varphi \delta} = \frac{1 + A_1 x + A_2 x^2 + \cdots A_{2n+1} x^{2n+1}}{(1 - e^{2c^2 \lambda^2 \epsilon_1 x^2})(1 - e^{2c^2 \lambda^2 \epsilon_2 x^2}) \cdots (1 - e^{2c^2 \lambda^2 \epsilon_n x^2})} \tag{67}
\]

For \( \vartheta = \delta \) the right hand side of (67) must vanish and similarly for \( \delta \pm \epsilon_1, \cdots \delta \pm \epsilon_n \) with arbitrary \( \delta \). Thus we have that

\[
1 + A_1 x + \cdots + A_{2n+1} x^{2n+1} = \left(1 - \frac{x}{\lambda \delta}ight) \left(1 - \frac{x}{\lambda (\delta + \epsilon_1)} \right) \left(1 - \frac{x}{\lambda (\delta - \epsilon_1)} \right) \times \cdots \left(1 - \frac{x}{\lambda (\delta + \epsilon_n)} \right) \left(1 - \frac{x}{\lambda (\delta - \epsilon_n)} \right) \tag{68}
\]

Eq.(46) can be written as

\[
\sqrt{(1 - c_1^2 y^2)(1 - c_1^2 y'^2)} = \frac{1}{C} \frac{dy}{dx} \sqrt{(1 - c^2 x^2)(1 - e^2 x^2)} \tag{69}
\]

and we can see that the four functions \( 1 \pm c_1 y, 1 \pm \epsilon_1 y \) must vanish when \( x = \pm \frac{1}{c}, \pm \frac{\epsilon_1}{c} \) that is when \( \vartheta = \pm \frac{\varphi_1}{\lambda}, \pm \frac{\epsilon_1 \varphi_1}{\lambda} \). Thus for \( \delta = \frac{\varphi_1}{\lambda}, \delta' = -\frac{\varphi_1}{\lambda}, \eta = \frac{\varphi_1}{\lambda}, \eta' = -\frac{\varphi_1}{\lambda} \)
we get \( g' = c_1 f \varphi \left( \frac{\varphi_1}{\lambda} \right) = c_1 f \varphi \left( \frac{\varphi_1}{\lambda} \right) \) and \( f' = \frac{\kappa}{c_1} \varphi \left( \frac{\varphi_1}{\lambda} \right) = \frac{\kappa}{c_1} \varphi \left( \frac{\varphi_1}{\lambda} \right) \). A solution of this system is

\[
g = f' = 0, \quad f' = 0, \quad f' = 0, \quad g = 0 \tag{70}
\]

and \( \kappa \) is arbitrary. Then

\[
y = \frac{1}{\kappa} \varphi \partial \tag{71}
\]

and \( ^{12} \)

\[
1 - \frac{\varphi \partial}{\varphi \delta} = \frac{1}{i \left(1 - c x \right)} \left(1 - \frac{x}{\lambda \left( \frac{\varphi_1}{\lambda} - \epsilon_1 \right)} \right)^2 \left(1 - \frac{x}{\lambda \left( \frac{\varphi_1}{\lambda} - \epsilon_2 \right)} \right)^2 \cdots \left(1 - \frac{x}{\lambda \left( \frac{\varphi_1}{\lambda} - \epsilon_n \right)} \right)^2 \tag{72}
\]

where \( \iota := (1 - e^{2c^2 \lambda^2 \epsilon_1 x^2})(1 - e^{2c^2 \lambda^2 \epsilon_2 x^2}) \cdots (1 - e^{2c^2 \lambda^2 \epsilon_n x^2}) \). Similar expressions are obtained for \( 1 + \frac{\varphi \partial}{\varphi \delta} \), \( 1 \pm \frac{\varphi \partial}{\varphi \delta} \). Then \( 1 - c_1^2 y^2 = (1 - \frac{\varphi \partial}{\varphi \delta})(1 + \frac{\varphi \partial}{\varphi \delta}) = (1 - c^2 x^2) \frac{d^2}{dx^2}, 1 - c_1^2 y'^2 = (1 - c^2 x^2) \frac{d^2}{dx^2} \) or

\[
\sqrt{(1 - c_1^2 y^2)(1 - c_1^2 y'^2)} = \pm \frac{d d'}{\iota^2} \sqrt{(1 - c^2 x^2)(1 - e^2 x^2)} \tag{73}
\]

\(^{11}\) We have used the fact that Abel’s function satisfies: \( \varphi \left( \frac{\varphi_1}{\lambda} \right) = -\varphi \left( \frac{\varphi_1}{\lambda} \right), \varphi \left( \frac{\varphi_1}{\lambda} \right) = -\varphi \left( \frac{\varphi_1}{\lambda} \right) \)

\(^{12}\) \( \lambda \frac{\varphi_1}{\lambda} = 1/c. \)
where \( d := \left(1 - \frac{e^2}{\sqrt{1 - e^2}}\right) \cdots \left(1 - \frac{e^2}{\sqrt{1 - e_n^2}}\right) \) and a similar expression for \( d' \) by substituting \( \omega/2 \rightarrow \omega'/2 \). Then it is shown [17] that \( \frac{d'}{dd'} \) is a constant \( C \) and Eq.(46) is satisfied.

Two other expressions for \( y \) are also provided [17]

\[
y = C \frac{x \left(1 - \frac{e^2}{\sqrt{1 - e_1^2}}\right) \left(1 - \frac{e^2}{\sqrt{1 - e_2^2}}\right) \cdots \left(1 - \frac{e^2}{\sqrt{1 - e_n^2}}\right)}{(1 - e^2 \lambda^2 e_1 x^2)(1 - e^2 \lambda^2 e_2 x^2) \cdots (1 - e^2 \lambda^2 e_n x^2)}
\]

(74)

and

\[
y = \frac{1}{\kappa} (ec)^{2n} b \lambda^2 \lambda(\epsilon_1 + \vartheta) \lambda(\epsilon_1 - \vartheta) \cdots \lambda(\epsilon_n + \vartheta) \lambda(\epsilon_n - \vartheta)
\]

(75)

with \( b := \lambda^2 \epsilon_1 \lambda^2 \epsilon_2 \lambda^2 \epsilon_3 \cdots \lambda^2 \epsilon_n \). For \( \vartheta = \omega/2, \vartheta = \omega'/2 \), with corresponding values for \( y = 1/\epsilon_1 \) and \( 1/\epsilon_1 \) we obtain

\[
\frac{1}{\epsilon_1} = (-1)^n b \kappa^{-2n} \epsilon^{2n} \left[ \lambda\left(\frac{\omega}{2} - \epsilon_1\right) \lambda\left(\frac{\omega}{2} - \epsilon_2\right) \cdots \lambda\left(\frac{\omega}{2} - \epsilon_n\right) \right]^2
\]

(76)

As is discussed in [17] (see also [18]) the transformation found by Jacobi in [20], corresponds to \( \epsilon_1 = \frac{2\omega}{2n+1}, c = c_1 = 1 \), and then \( \epsilon_2 = \frac{4\omega}{2n+1} = 2\epsilon_1, \epsilon_n = n\epsilon_1 \)

\[
y = \frac{\lambda \vartheta \lambda \left(\frac{2\omega}{2n+1} + \vartheta\right) \lambda \left(\frac{2\omega}{2n+1} - \vartheta\right) \cdots \lambda \left(\frac{2n\omega}{2n+1} + \vartheta\right) \lambda \left(\frac{2n\omega}{2n+1} - \vartheta\right)}{\left[ \lambda \left(\frac{1}{2n+1} \frac{\omega}{2}\right) \lambda \left(\frac{3}{2n+1} \frac{\omega}{2}\right) \cdots \lambda \left(\frac{2n-1}{2n+1} \frac{\omega}{2}\right) \right]^2}
\]

\[
C = \left\{ \lambda \left(\frac{1}{2n+1} \frac{\omega}{2}\right) \lambda \left(\frac{3}{2n+1} \frac{\omega}{2}\right) \cdots \lambda \left(\frac{2n-1}{2n+1} \frac{\omega}{2}\right) \right\}^2
\]

\[
\epsilon_1 = e^{2n+1} \lambda \left(\frac{1}{2n+1} \frac{\omega}{2}\right) \lambda \left(\frac{3}{2n+1} \frac{\omega}{2}\right) \cdots \lambda \left(\frac{2n-1}{2n+1} \frac{\omega}{2}\right) \lambda \left(\frac{2n}{2n+1} \frac{\omega}{2}\right)
\]

(77)

In addition in [17], two more cases are discussed i) those in which \( k = 0 \) and \( k' \) or \( k'' \) is equal to 1 and ii) \( k = 1 \). The general theorem for the first real transformation of arbitrary order \( n \) is also stated [17].

In Jacobi’s language the indefinite elliptic integral \( x = \int \frac{dy}{\sqrt{(1-y^2)(1-e^2 y^2)}} \) defines the elliptic function \( y = \text{sn}(x, k) \). Thus, in Eq.(77), \( x = \lambda \vartheta = \text{sn}(\vartheta, e) \) \footnote{One can also switch to Legendre’s notation by defining \( y = \sin \psi, x = \sin \varphi' \). Then the last line of (77) can be written as \( \frac{dy}{\sqrt{1-e^2 \sin^2 \psi}} = C \frac{d\varphi'}{\sqrt{1-e^2 \sin^2 \varphi'}} \). For large \( n \) the modulus \( \epsilon_1 \) becomes negligible and then \( \psi = \sin^{-1} y = C \int \frac{d\varphi'}{\sqrt{1-e^2 \sin^2 \varphi'}} \) for \( \varphi = \varphi' \).}.\footnote{One can also switch to Legendre’s notation by defining \( y = \sin \psi, x = \sin \varphi' \). Then the last line of (77) can be written as \( \frac{dy}{\sqrt{1-e^2 \sin^2 \psi}} = C \frac{d\varphi'}{\sqrt{1-e^2 \sin^2 \varphi'}} \). For large \( n \) the modulus \( \epsilon_1 \) becomes negligible and then \( \psi = \sin^{-1} y = C \int \frac{d\varphi'}{\sqrt{1-e^2 \sin^2 \varphi'}} \) for \( \varphi = \varphi' \).}
5.2 Transforming the geodesic elliptic integrals into Abel’s form

The transformation

\[ x \to e_3 + \frac{(e_1 - e_3)}{x^2} \quad (78) \]

transforms the elliptic integral in Weierstraß form into Abel’s and Jacobi’s form:

\[
\int \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} \to -\frac{1}{\sqrt{e_1-e_3}} \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad (79)
\]

with \(k^2 = \frac{e_2 - e_3}{e_1 - e_3}\).

Similarly, the quartic can be brought to Jacobi’s form \(y^2 = (1-x^2)(1-k_4^2x^2)\).

Indeed, as we saw in section 3 we have

\[
\frac{d\theta}{\sqrt{\Theta}} = -\frac{1}{2} \frac{dz}{\sqrt{z^3\alpha - z^2(\alpha + \beta) + Qz}} = -\frac{1}{2} \frac{d\xi}{\sqrt{4\xi^3 - g_2\xi - g_3}} \quad (80)
\]

where

\[
g_2 = \frac{1}{12} (a^2(1 - E^2) + Q + L^2)^2 - \frac{Q}{4} (a^2(1 - E^2))
\]

\[
g_3 = \frac{1}{216} (a^2(1 - E^2) + Q + L^2)^3 - \frac{Q}{48} a^4 (1 - E^2)^2 - \frac{Q}{48} (a^2(1 - E^2)(Q + L^2))
\]

(81)

In this case the three roots of the cubic \(4\xi^3 - g_2\xi - g_3\) are given by the expressions

\[
e_1 = \frac{1}{24} \left( a^2(1 - E^2) + L^2 + Q + 3\sqrt{(a^2(-1 + E^2) + Q)^2 + 2a^2L^2 - 2a^2E^2L^2 + L^4 + 2L^2Q} \right)
\]

\[
e_2 = \frac{1}{24} \left( a^2(1 - E^2) + L^2 + Q - 3\sqrt{(a^2(-1 + E^2) + Q)^2 + 2a^2L^2 - 2a^2E^2L^2 + L^4 + 2L^2Q} \right)
\]

\[
e_3 = \frac{1}{12} (a^2(E^2 - 1) - L^2 - Q)
\]

(82)

Thus the Jacobi modulus \(k^2 = \frac{e_2 - e_3}{e_1 - e_3}\) is given by

\[
k^2 = \frac{-a^2(-1 + E^2) + L^2 + Q - \sqrt{a^4(-1 + E^2)^2 - 2a^2(-1 + E^2)(L^2 - Q) + (L^2 + Q)^2}}{-a^2(-1 + E^2) + L^2 + Q + \sqrt{a^4(-1 + E^2)^2 - 2a^2(-1 + E^2)(L^2 - Q) + (L^2 + Q)^2}} \quad (83)
\]

It has the correct limit for polar geodesics with \(L = 0\)

\[ k^2(L = 0) = \frac{a^2(1 - E^2)}{Q} \quad (84) \]

Also

\[
\frac{1}{\sqrt{e_1 - e_3}} = \frac{2\sqrt{2}}{\sqrt{-a^2(-1 + E^2) + L^2 + Q + \sqrt{a^4(-1 + E^2)^2 - 2a^2(-1 + E^2)(L^2 - Q) + (L^2 + Q)^2}}} \quad (85)
\]

19
and it also has the correct limit for polar geodesics

\[
\frac{1}{\sqrt{e_1 - e_3}} = \frac{2}{\sqrt{Q}}
\] (86)

Thus, we get

\[
\int \frac{d\vartheta}{\sqrt{\Theta}} = -\frac{1}{2} \frac{1}{\sqrt{e_1 - e_3}} \int \frac{d\zeta'}{\sqrt{(1 - \zeta'^2)(1 - k^2\zeta'^2)}}
\] (87)

Now we shall discuss the transformation of the right hand side of Eq.(44). Inspired by the work of Jacobi and Abel, Luchterhand obtained the following transformation formula [21]:

\[
\frac{\partial x}{M \sqrt{(1 - x^2)(1 - k_1^2 x^2)}} = \frac{\partial y}{\sqrt{(y - \alpha)(y - \beta)(y - \gamma)(y - \delta)}}
\] (88)

where the Jacobi modulus \(k_1\) and the coefficient \(M\), are given in terms of the roots \(\alpha, \beta, \gamma, \delta\) of the quartic, by the following expressions

\[
k_1 = \sqrt{\frac{(\alpha - \delta)(\beta - \gamma)}{(\alpha - \gamma)(\beta - \delta)}}
\]

\[
M = \sqrt{\frac{(\alpha - \gamma)(\beta - \delta)}{(\alpha - \beta)(\gamma - \delta)}}
\] (89)

and the integration variables are related by

\[
\frac{1 - x}{1 + x} = \frac{(\gamma - \delta)(y - \alpha)(y - \beta)}{(\alpha - \beta)(y - \gamma)(y - \delta)}
\] (90)

Also we assume that the roots \(\alpha, \beta, \gamma, \delta\) of the quartic are real and are organized in the following ascending order of magnitude: \(\alpha > \beta > \gamma > \delta\).

At this stage, we have succeeded in transforming both sides of Eq.(44), into Abel’s form. We can also provide a nice formula for \(r\) in terms of the Jacobi’s sinus amplitudinus function \(^{14}\)

\[
\frac{(\gamma - \delta)(r - \alpha)(r - \beta)}{(\alpha - \beta)(r - \gamma)(r - \delta)} = \frac{1 - \text{sn}\left(M \int \frac{d\vartheta}{\sqrt{\Theta}}, k_1\right)}{1 + \text{sn}\left(M \int \frac{d\vartheta}{\sqrt{\Theta}}, k_1\right)}
\] (91)

A systematic investigation of the phenomenological aspects of the above exact solution like the periapsis precession will be reported elsewhere.

### 6 Separability of Hamilton-Jacobi’s differential equation in Kerr-(anti) de Sitter metric and derivation of geodesics.

In the presence of the cosmological constant we proved (see Appendix A for details) the important result that the Hamilton-Jacobi differential equation can be

\(^{14}\)We write \(\int \frac{dr}{\sqrt{r}} = \int \frac{d\vartheta}{\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} = \int \frac{d\vartheta}{M \sqrt{(1 - x^2)(1 - k_1^2 x^2)}} = \int \frac{d\vartheta}{\sqrt{\Theta}}.

20
solved by separation of variables. Thus in this case, the characteristic function separates and takes the form

\[
W = -Ect + L\phi + \int \frac{\sqrt{[Q + (L - aE)^2 - \mu^2a^2 \cos^2 \theta] \Delta_\theta - \frac{\Xi^2(aE \sin^2 \theta - L)^2}{\sin^2 \theta}}}{\Delta_\theta} d\theta
\]

\[
+ \int \frac{\sqrt{\Xi^2[(r^2 + a^2)E - aL]^2 - \Delta_r (\mu^2r^2 + Q + \Xi^2(L - aE)^2)}}{\Delta_r} dr
\]

By differentiating now with respect to constants of integration, \(Q, L, E, \mu\), we obtain the following set of geodesic differential equations

\[
\int \frac{dr}{\sqrt{R'}} = \int \frac{d\theta}{\sqrt{\Theta'}}
\]

\[
\rho^2 \frac{d\phi}{d\lambda} = -\frac{\Xi^2}{\Delta_\theta \sin^2 \theta} (aE \sin^2 \theta - L) + \frac{a\Xi^2}{\Delta_r} [(r^2 + a^2)E - aL]
\]

\[
cp^2 \frac{dt}{d\lambda} = \frac{\Xi^2(r^2 + a^2) [(r^2 + a^2)E - aL] - a\Xi^2(aE \sin^2 \theta - L)}{\Delta_r}
\]

\[
\rho^2 \frac{\Delta_r}{d\lambda} = \pm \sqrt{R'}
\]

\[
\rho^2 \frac{\Delta_\theta}{d\lambda} = \pm \sqrt{\Theta'}
\]

where

\[
R' := \Xi^2 [(r^2 + a^2)E - aL]^2 - \Delta_r (\mu^2r^2 + Q + \Xi^2(L - aE)^2)
\]

\[
\Theta' := [Q + (L - aE)^2 - \mu^2a^2 \cos^2 \theta] \Delta_\theta - \frac{\Xi^2(aE \sin^2 \theta - L)^2}{\sin^2 \theta}
\]

The first line of Eq. (92) is a differential equation that relates a hyperelliptic Abelian integral to an elliptic integral which is the generalisation of the theory of transformation of elliptic functions discussed in section 5, in the case of non-zero cosmological constant. The mathematical treatment of such a relationship was first discussed by Abel in [19].

### 6.1 Exact solution of spherical polar orbits with a cosmological constant

Using the second and the fifth line of Eq. (92), for \(L = 0\) and assuming a constant value for \(r\), we obtain

\[
\frac{d\phi}{d\theta} = \frac{-\Xi^2 aE + a\Xi^2(r^2 + a^2)E}{\Delta_r \sqrt{\Theta'}} = \frac{-\Xi^2 aE B \Delta_\theta}{\Delta_\theta \sqrt{\Theta'}}
\]
where \( B := \frac{a \Xi^2 (r^2 + a^2) E}{\Delta_r} \).

Similarly, using the third and fifth line we obtain

\[
\frac{cdt}{d\theta} = \frac{\Xi^2 (r^2 + a^2) E}{\Delta_r \sqrt{\Theta'}} - a^2 \Xi^2 E \sin^2 \theta \Delta_r \sqrt{\Theta'}
\]

\[= \frac{\Gamma \Delta_\theta - a^2 \Xi^2 E \sin^2 \theta \Delta_r \sqrt{\Theta'}}{\Delta_r \sqrt{\Theta'}} \tag{95}\]

and \( \Gamma := \frac{\Xi^2 (r^2 + a^2) E}{\Delta_r} \).

Now using the variable \( z = \cos^2 \theta \), we obtain the following system of integral equations:

\[
\phi = \int \frac{\Xi^2 a E/2}{\sqrt{f(z)}} dz + \int \frac{B(1 + \frac{a^2}{3} z)/(-2)}{\sqrt{f(z)}} dz
\]

\[= \int \frac{\Gamma (1 + \frac{a^2}{3} z)}{\sqrt{f(z)}} dz - \int \frac{a^2 \Xi^2 E (1 - z) dz}{\sqrt{f(z)}} \tag{96}\]

or

\[
\phi = \int^z \frac{(\alpha_1 + \beta_1 z) dz}{\sqrt{f(z)}}
\]

\[= \int^z \frac{(\gamma_1 + \delta_1 z) dz}{\sqrt{f(z)}} \tag{97}\]

where \( f(z) = z(1 - z)(Q + z(Qa^2 \Delta + \Xi^3 a^2 E^2 - \mu^2 a^2) + z^2(-\mu^2 a^4 \Delta))(1 + a^2 \Delta z)^2 \).

Also we have defined

\[
\alpha_1 = \Xi^2 \frac{a E}{2} - \frac{1}{2} \frac{1}{\Delta_r} a \Xi^2 (r^2 + a^2) E
\]

\[
\beta_1 = -\frac{1}{2} \frac{1}{\Delta_r} a \Xi^2 (r^2 + a^2) E a^2 \Lambda
\]

\[
\gamma_1 = -\frac{1}{2} \frac{1}{\Delta_r} \Xi^2 (r^2 + a^2) E + \frac{a^2 \Xi^2 E}{2}
\]

\[
\delta_1 = -\frac{a^2 \Lambda \Xi^2 (r^2 + a^2)^2 E}{6} - \frac{a^2 \Xi^2 E}{2} \tag{98}\]

Equation (96) is a system of equations of Abelian integrals, whose inversion in principle, involves genus-2 Abelian-Siegelsche modular functions. Indeed, this system is a particular case of Jacobi’s inversion problem of hyperelliptic Abelian integrals of genus 2 [44]-[50] (see Appendix B for details). Then, one can express \( z \) as a single valued genus two Abelian theta function with arguments \( t, \phi \).

However, since the polynomial \( f(z) \) of sixth degree possesses a double root it may well be that the Abelian genus-2 theta function degenerates and the final result can be expressed in terms of genus-1 modular functions. This issue will be investigated in detail elsewhere.
6.2 Equatorial geodesics including the contribution of the cosmological constant

The equatorial geodesics (i.e. \( \theta = \pi/2, Q = 0 \)), with a nonzero cosmological constant, may be obtained by Eq.(92) for the particular values of \( Q, \theta \). The characteristic function in this case, has the form

\[
W = -Ect + \int \frac{\sqrt{R'}}{\Delta_r} dr + L\phi \tag{99}
\]

and the geodesics are given by the expressions:

\[
\frac{dr}{\sqrt{R'}} = \frac{d\lambda}{r^2}
\]

\[
r^2 \frac{d\phi}{d\lambda} = \frac{a((1 + \frac{4a^2 \Lambda}{3})(E(r^2 + a^2) - La) + (L - aE)(1 + \frac{1}{3}a^2\Lambda)^2}{(1 - \frac{\Lambda}{3}r^2)(r^2 + a^2) - \frac{2GMr}{c^2}}
\]

\[
cr^2 \frac{dt}{d\lambda} = \frac{(1 + \frac{4a^2 \Lambda}{3})(r^2 + a^2)[(r^2 + a^2)E - aL]}{\Delta_r} + (1 + \frac{1}{3}a^2\Lambda)^2a(L - aE)
\]

where

\[
R' = (1 + \frac{1}{3}a^2\Lambda)^2\left[[((r^2 + a^2)E - aL)^2 - \Delta_r((L - aE)^2)]\right) \tag{100}
\]

for null-geodesics and

\[
R' = (1 + \frac{1}{3}a^2\Lambda)^2\left[((r^2 + a^2)E - aL)^2 - \Delta_r((L - aE)^2)] - \Delta_r(\mu^2 r^2)\right) \tag{102}
\]

for time-like geodesics.

6.3 Various limits of equatorial geodesic equations.

For \( \Lambda = 0, a \neq 0 \) (Kerr limit) Eqs(100) reduces to

\[
\frac{dr}{\sqrt{R'}} = \frac{d\lambda}{r^2}
\]

\[
r^2 \frac{d\phi}{d\lambda} = \frac{a(E(r^2 + a^2) - La)}{r^2 + a^2 - \frac{2GM}{c^2}} + (L - aE)
\]

\[
\frac{cr^2}{d\lambda} \frac{dt}{d\lambda} = \frac{(r^2 + a^2)[(r^2 + a^2)E - aL]}{\Delta} + a(L - aE) \tag{103}
\]

\[\text{See appendix A for details.}\]
Similarly for $a = 0, \Lambda \neq 0$ (Schwarzschild-de Sitter limit), Eqs(100) reduce to

\[
\sqrt{r^4 E^2 - (1 - \frac{\Lambda r^2}{3}) r^2 - \frac{2GM}{c^2} r^2 \left[ \mu^2 r^2 + L^2 \right]} = \frac{d\lambda}{r^2}
\]

\[
r^2 \frac{d\phi}{d\lambda} = L
\]

\[
cdt = \frac{E}{1 - \frac{\Lambda r^2}{3} - \frac{2GM}{rc^2}} d\lambda
\]

(104)

The exact solution of (104) have been given in [3].

### 6.4 Circular geodesics with a cosmological constant

In this section we shall study the circular equatorial geodesics and we shall prove the result:

\[
\frac{dt}{d\phi} = \frac{-\Gamma_{03}^1 \pm \left[ (\Gamma_{03}^1)^2 - \Gamma_{33}^1 \Gamma_{00}^1 \right]^{1/2}}{\Gamma_{00}^1}
\]

\[
= \frac{a}{c} \pm \frac{1}{c} \sqrt{\frac{\sqrt{r}}{(\frac{2GM}{c^2} - \frac{\Lambda r}{c^2})}}
\]

(105)

where $\Gamma_{jk}^i$ denotes the Christoffel symbols.

The way to arrive to this result is to demand $\Gamma_{jk}^i u^j u^k = 0$, together with the conditions:

\[
u^1 = \frac{dr}{ds} = 0
\]

\[
u^2 = \frac{d\theta}{ds} = 0
\]

(106)

Alternatively since we have derived the general form for equatorial geodesics by requiring $\nu^3 = 0$ or equivalently that $R'(r) = 0$ (and its first derivative also vanishes) will lead us to a condition for eliminating the constants of integration and we should arrive in (105).

We also derive the following expression for $\nu^3 = \frac{d\phi}{ds}$.

\[
\nu^3 = \frac{d\phi}{ds} = \left[ g_{00} \left( \frac{dt}{d\phi} \right)^2 + 2g_{03} \frac{dt}{d\phi} + g_{33} \right]^{-1/2}
\]

(107)
This is equivalent to the following equation

\[
\frac{d\phi}{ds} = \frac{1}{\sqrt{c^2(1 - \frac{\Lambda}{3}(r^2 + a^2) - \frac{2GM}{c^2r}) \left( \frac{dt}{d\phi} \right)^2 + 2 \left( \frac{2GMc}{c^2r} + \frac{c\alpha(r^2+a^2)\Lambda}{(1+\frac{a^2}{c^2})^2} \right) \frac{dt}{d\phi} - \left[ \frac{2GMr}{c^2} + \frac{(r^2+a^2)}{1+\frac{a^2}{c^2}} \right]}}
\]

(108)

The above expression may be obtained from the metric

\[
ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 + 2g_{03}d\phi dt
\]

(109)

Some aspects of the equatorial circular geodesics have been investigated in [36].

7 Conclusions

In this work, we have investigated the motion of a test particle in the gravitational field of Kerr space-time with and without the cosmological constant. The geodesic equations were derived by integrating the relativistic Hamilton-Jacobi partial differential equation by separation of variables. As we saw in the main body of the paper, the structure of the exact solutions of the corresponding geodesic equations is quite rich and mathematically very interesting. The integration of the geodesic equations involves the inversion problem of Abel and Jacobi for hyperelliptic and elliptic Abelian integrals, as well as the theory of transformations of elliptic functions. As we saw, in the case of spherical polar orbits with a cosmological constant the solution involves directly the solution of Jacobi’s inversion problem for a system of Abelian integrals.

Physically, the new types of relativistic motions lead to the interesting phenomenon of frame dragging of inertial frames. In this respect, we have applied the solution of spherical polar geodesics in Kerr space-time in two situations. First, by modelling the gravitational field of Earth by the Kerr geometry, we determined the longitudinal dragging of a satellite’s polar spherical orbit around the Earth using as radius the semi-major axis of the polar orbit of GP-B mission launched in April 2004. Secondly, by assuming the galactic centre is a rotating black hole, according to the interpretations of recent observations of the galactic centre [22], whose Kerr parameter is determined by experiment as for instance in Eq.(1), we determined the longitudinal dragging of a star in spherical polar orbit around and close to the galactic centre. In principle, these predictions can be tested by observations.

A systematic investigation of the physical implications of the exact solutions of the geodesic equations for the various possible types of motion for the test particle, in the presence of the cosmological constant, will be reported elsewhere [52]. For instance, one can investigate the effect of the rotation of the Sun in the perihelion precession of Mercury and a comparison can be made with the cosmological constant contribution that has been calculated in [3].

It can also be applied in determining the cosmological constant effect in rotating galaxies, and in particular in the velocity rotation curves. Assuming
the galactic centre is a rotating black hole with the Kerr parameter measured as in eq.(1), what is the effect of the cosmological constant in the motion of stars as a function of the distance from the galactic centre?

Probing general relativity in the strong-field regime will certainly be one of the most exciting endeavours.

Acknowledgements

This work is supported by a Max Planck research fellowship at the Max Planck Institute for Physics in Munich. The author also acknowledges the support from a research stipendium, during the early stages of this work, from the Max Planck Institute for Gravitational Physics (Albert Einstein Institute) at Golm. We are grateful to D. Lüst for useful discussions.

A Separability of Hamilton-Jacobi equation in the presence of a cosmological constant

In this appendix, we will prove that the Hamilton-Jacobi differential equation preserves the important property of separability in the presence of the Cosmological constant, which has been used in deriving the geodesic equations in section 6 from the characteristic function $W$.

The non-zero elements of the inverse metric are

$$
g_{00} = \frac{\Xi^2 [(r^2 + a^2)\Xi - a^2 \sin^2 \theta (r^2 + a^2)\Xi - \frac{2GMr}{c^2}]}{c^2 \Delta_r \Delta_\theta \rho^2}$$

$$
g_{03} = \frac{a\Xi^2 (6GMr + c^2 \Lambda (a^2 + r^2)\rho^2)}{3c^2 \Delta_r \rho^2 \Delta_\theta c}$$

$$
g_{11} = -\frac{\Delta_r}{\rho^2}$$

$$
g_{22} = -\frac{\Delta_\theta}{\rho^2}$$

$$
g_{33} = \Xi^2 \left(\frac{a^2}{\Delta_r \rho^2} - \frac{1}{\rho^2 \sin^2 \theta \Delta_\theta}\right)$$

(110)

The Hamilton-Jacobi differential equation takes the form:

$$
\Xi^2 \times \left[\frac{(r^2 + a^2)^2 \Xi - a^2 \sin^2 \theta (r^2 + a^2)\Xi - \frac{2GMr}{c^2}}{\rho^2 \Delta_r \Delta_\theta \sin^2 \theta}\right] E^2 \sin^2 \theta + \Xi^2 (\sin^2 \theta \Delta_\theta a^2 - \Delta_r) L^2 \frac{\rho^2 \Delta_r \Delta_\theta \sin^2 \theta}{\rho^2 \Delta_r \Delta_\theta \sin^2 \theta} + 2a\Xi^2 \left(\frac{2GMr}{c^2} + \frac{a^2}{2} \rho^2 (a^2 + r^2)(-EL) \sin^2 \theta\right) \frac{\rho^2 \Delta_r \Delta_\theta \sin^2 \theta}{\rho^2 \Delta_r \Delta_\theta \sin^2 \theta} + \frac{-\Delta_r}{\rho^2} \left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{-\Delta_\theta}{\rho^2} \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \mu^2 = 0
$$

(111)
or

\[
\Xi^2 \left[ (r^2 + a^2)E - aL \right]^2 - \Xi^2 (L - aE)^2 \frac{\Delta_r}{\Delta_r} + \Xi^2 (L - aE)^2 \frac{\Delta_\theta}{\Delta_\theta} - \frac{\Xi^2 (aE \sin^2 \theta - L)^2}{\sin^2 \theta} + \\
+ \mu^2 \rho^2 + (-\Delta_r) \left( \frac{\partial W_r}{\partial r} \right)^2 + (-\Delta_\theta) \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 = 0
\]

(112)

For equatorial geodesics the Hamilton-Jacobi differential equation takes the form

\[
g^{00}(-Ec)^2 + g^{33}L^2 + g^{11} \left( \frac{\partial W_r}{\partial r} \right)^2 + 2g^{03}(-EcL) + \mu^2 = 0
\]

(113)

Using eq. (110) with \( \theta = \pi/2 \) in (113) we obtain

\[
\frac{\partial W_r}{\partial r} = \frac{\sqrt{\Xi^2 \left[ (r^2 + a^2)E - aL \right]^2 - \Delta_r(L - aE)^2} - \Delta_r \mu^2 r^2}{\Delta_r}
\]

(114)

**B Definitions of genus-2 theta functions that solve Jacobi’s inversion problem**

Riemann’s theta function [49] for genus \( g \) is defined as follows:

\[
\Theta(u) := \sum_{n_1,\ldots,n_g} e^{2\pi i un + i\Omega n^2}
\]

(115)

where \( \Omega n^2 := \Omega_{11} n_1^2 + \cdots + 2\Omega_{12} n_1 n_2 + \cdots \) and \( un := u_1 n_1 + \cdots + u_g n_g \). The symmetric \( g \times g \) complex matrix \( \Omega \) whose imaginary part is positive definite is a member of the set called Siegel upper-half-space denoted as \( \mathbb{H}_g \). It is clearly the generalization of the ratio of half-periods \( \tau \) in the genus \( g = 1 \) case. For genus \( g = 2 \) the Riemann theta function can be written in matrix form:

\[
\Theta(u, \Omega) = \sum_{n \in \mathbb{Z}^2} e^{\pi i n \Omega n + 2\pi i nu}
\]

\[
= \sum_{n_1, n_2} e^{\pi i \begin{pmatrix} n_1 & n_2 \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + 2\pi i \begin{pmatrix} n_1 & n_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}
\]

(116)

Riemann’s theta function with characteristics is defined by:

\[
\Theta(u; q, q') := \sum_{n_1,\ldots,n_g} e^{2\pi i u(n + q') + i\Omega(n + q')^2 + 2\pi i q(n + q')}
\]

(117)
the summation extends to all positive and negative integer values of \(q\). Eq.(117) can be rewritten in a suggestive matrix form:

\[
\Theta \left[ \begin{array}{c} q' \\ q \end{array} \right] (u, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (n+q)\Omega (n+q') + 2\pi i (n+q') (u+q)}, \quad q, q' \in \mathbb{Q}^g
\] (118)

The theta functions whose quotients provide a solution to Abel-Jacobi’s inversion problem are defined as follows [50]:

\[
\theta(u; q, q') := \sum_{n \in \mathbb{Z}^g} e^{au^2 + 2hu(n+q') + b(n+q')^2 + 2i\pi q(n+q')}
\] (119)

where the summation extends to all positive and negative integer values of the \(q\) integers \(n_1, \cdots, n_g\), \(a\) is any symmetrical matrix whatever of \(g\) rows and columns, \(h\) is any matrix whatever of \(g\) rows and columns, in general not symmetrical, \(b\) is any symmetrical matrix whatever of \(g\) rows and columns, such that the real part of the quadratic form \(bh\) is necessarily negative for all real values of the quantities \(m_1, \cdots, m_g\), other than zero, and \(q, q'\) constitute the characteristics of the function. The matrix \(b\) depends on \(\frac{1}{2}g(g+1)\) independent constants; if we put \(i\pi \Omega = b\) and denote the \(g\)-quantities \(hu\) by \(i\pi U\), we obtain the relation with Riemann’s theta function:

\[
\theta(u; q, q') = e^{au^2} \Theta(U; q, q')
\] (120)

The dependence of genus-2 theta functions on two complex variables is denoted by: \(\theta(u; q, q') = \theta(u_1, u_2; q, q')\), the dependence on the Siegel moduli matrix \(\Omega\) by: \(\theta(u_1, u_2; \Omega; q, q')\). To every half-period one can associate a set of characteristics. For instance, the period \(u^{a, a_1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) while \(\theta(u)\) is a theta function of two variables with zero characteristics, i.e. \(\theta(u) = \theta(u; 0, 0) = \theta \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] (u, \Omega)\). Also, Weierstraß had associated a symbol for each of the six odd theta functions with characteristics and the ten even theta functions of genus two. For example, \(\theta(u)\) is associated with the Weierstraß symbol 5 or occasionally the number appears as a subscript, i.e. \(\theta(u)_5\).

Let the genus \(g\) Riemann hyperelliptic surface be described by the equation:

\[
y^2 = 4(x - a_1) \cdots (x - a_g)(x - c_1)(x - c_2) \cdots (x - c_g)
\] (121)

For \(g = 2\) the above hyperelliptic Riemann algebraic equation reduces to:

\[
y^2 = 4(x - a_1)(x - a_2)(x - c_1)(x - c_2)
\] (122)

where \(a_1, a_2, c_1, c_2\) denote the finite branch points of the surface.

The Jacobi’s inversion problem involves finding the solutions, for \(x_i\) in terms of \(u_i\), for the following system of equations of Abelian integrals [50]:

\[
\begin{align*}
u_1^{x_1, a_1} + \cdots + u_1^{x_g, a_g} & \equiv u_1 \\
+ \cdots + : & \vdots \\
u_g^{x_1, a_1} + \cdots + u_g^{x_g, a_g} & \equiv u_g
\end{align*}
\] (123)
where \( u_1^{x\mu} = \int_a^x \frac{dx}{y} \), \( u_2^{x\mu} = \int_a^x \frac{x dx}{y} \cdot \cdot \cdot , u_g^{x\mu} = \int_a^x \frac{x^{g-1} dx}{y} \).

For \( g = 2 \) the above system of equations takes the form:

\[
\begin{align*}
&\int_{a_1}^{x_1} \frac{dx}{y} + \int_{a_2}^{x_2} \frac{dx}{y} = u_1 \\
&\int_{a_1}^{x_1} \frac{x dx}{y} + \int_{a_2}^{x_2} \frac{x dx}{y} = u_2
\end{align*}
\]  
\quad(124)

where \( u_1, u_2 \) are arbitrary. The solution is given by the five equations [50]

\[
\frac{\theta^2(u|u^{b,a})}{\theta^2(u)} = A(b-x_1)(b-x_2) \cdot \cdot \cdot A(b-x_g)
\]

\[
= A(b-x_1)(b-x_2)
\]

\[
= \pm \frac{\theta^2(u|u^{b,a})}{\sqrt{\epsilon^{\pi i P \bar{P}'} f(b)}};
\]  
\quad(125)

where \( f(x) = (x-a_1)(x-a_2)(x-c_1)(x-c_2) \), and \( \epsilon^{\pi i P \bar{P}'} = \pm 1 \) according as \( u^{b,a} \) is an odd or even half-period. Also \( b \) denotes a finite branch point and the branch place \( a \) being at infinity [50]. The symbol \( \theta(u|u^{b,a}) \) denotes a genus 2 theta function with characteristics: \( \theta(u; q, q') \) [50], where \( u = (u_1, u_2) \), denotes two independent variables, see appendix A for further details. From any 2 of these equations, eq.(125), the upper integration bounds \( x_1, x_2 \) of the system of differential equations eq.(124) can be expressed as single valued functions of the arbitrary arguments \( u_1, u_2 \). For instance,

\[
x_1 = a_1 + \frac{1}{A_1(x_2-a_1)} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)}
\]  
\quad(126)

and

\[
x_2 = -\frac{[(a_2-a_1)(a_2+a_1) + \frac{1}{A_1} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} - \frac{1}{A_2} \frac{\theta^2(u|u^{a_2,a})}{\theta^2(u)}]}{2(a_1-a_2)}
\]

\[
\pm \frac{\sqrt{[(a_2-a_1)(a_2+a_1) + \frac{1}{A_1} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} - \frac{1}{A_2} \frac{\theta^2(u|u^{a_2,a})}{\theta^2(u)}]^2 - 4(a_1-a_2)\eta}}{2(a_1-a_2)}
\]  
\quad(127)

where

\[
\eta := \frac{a_2}{a_1} (a_1-a_2) - \frac{a_2}{A_1} \frac{\theta^2(u|u^{a_1,a})}{\theta^2(u)} + \frac{a_1}{A_2} \frac{\theta^2(u|u^{a_2,a})}{\theta^2(u)}
\]  
\quad(128)

Also, \( A_i = \frac{1}{\sqrt{\epsilon^{\pi i P \bar{P}'} f(a_i)}} \).

The solution can be reexpressed in terms of generalized Weierstraß functions:

\[
x_k^{(1,2)} = \frac{\wp_{2,2}(u) \pm \sqrt{\wp_{2,2}^2(u) + 4 \wp_{2,2}(u)}}{2}, \quad k = 1, 2
\]  
\quad(129)

29
while \( \Omega \) and described below \([17, 18, 20]\).

One of the applications supplied by the transformation theory of Elliptic functions is of great importance in Number theory, are the modular equations which can be transformed to Jacobi’s form by the substitution \( z = \frac{1}{2} \frac{\phi_1}{\phi_2} \), for \( \Lambda = 0 \), \( \beta_1 = 0 \) and

\[
\phi_2,2(u) = \frac{(a_1 - a_2)(a_2 + a_1) - \frac{1}{A_1} \frac{\theta_2 (u | u_1 | a_1)}{\theta_2 (u)} + \frac{1}{A_2} \frac{\theta_2 (u | u_2 | -a_1)}{\theta_2 (u)}}{a_1 - a_2} (130)
\]

and

\[
\phi_2,1(u) = \frac{-a_1 a_2 (a_1 - a_2) - \frac{a_1}{A_2} \frac{\theta_2 (u | u_2 | -a_1)}{\theta_2 (u)} + \frac{a_2}{A_1} \frac{\theta_2 (u | u_1 | a_1)}{\theta_2 (u)}}{a_1 - a_2} (131)
\]

Thus, \( x_1, x_2 \), that solve Jacobi’s inversion problem Eq.(124), are solutions of a quadratic equation \([45, 50]\)

\[
Ux^2 - U'x + U'' = 0 (132)
\]

where \( U, U', U'' \) are functions of \( u_1, u_2 \). In the particular case that the coefficient of \( x^2 \) in the quintic polynomial is equal to 4, \( U = 1, U' = \phi_2,2(u), U'' = \phi_2,1(u) \).

The matrix elements \( h_{ij}, \Omega_{ij} \) can be explicitly written in terms of the half-periods \( U_2^{e_i}, e_j \). For clarity, \( U_2^{e_1}, e_2 = \int_{e_1}^{e_2} x dx/y, U_2^{e_1}, e_3 = \int_{e_2}^{e_3} dx/y \) etc. The roots have been arranged in ascending order of magnitude and are denoted by \( e_{2g}, e_{2g-1}, \cdots, e_0, g = 2 \), so that \( e_{2i}, e_{2i-1} \) are respectively, \( e_0, e_{-1}, \cdots \) and \( e_0 = c \). For instance, the matrix element \( h_{ij} = \frac{U_2^{e_i}, e_j}{2(U_2^{e_i}, e_{i+1} - U_2^{e_{i-1}}, e_2)} \times \pi i \), while \( \Omega_{11} = \frac{U_2^{e_1}, e_1 - U_2^{e_1}, e_2}{U_2^{e_1}, e_2 U_2^{e_1}, e_1} \).

Eq.(96) has the correct limit for vanishing cosmological constant. Indeed, for \( \Lambda = 0 \), \( \beta_1 = 0 \) and

\[
\phi = \int \frac{\alpha_1 dz}{\sqrt{z(1-z)(Q + za^2(E^2 - 1))}} (133)
\]

which can be transformed to Jacobi’s form by the substitution \( z = \frac{1}{2} \frac{\phi_1}{\phi_2} \). Then the solution is given by the Weierstrass function of section 3.

C Transformation Theory of Elliptic functions and Modular equations

One of the applications supplied by the transformation theory of Elliptic functions, which is of great importance in Number theory, are the modular equations described below \([17, 18, 20]\).

For a rational solution of the differential equation

\[
\frac{dy}{\sqrt{(1-y^2)(1-\epsilon_1^2 y^2)}} = C \frac{dx}{\sqrt{(1-x^2)(1-\epsilon_1^2 x^2)}} (134)
\]

the necessary conditions among the periods

\[
K(e_1) = a_0 CK(e) + a_1 CiK'(e) \quad \quad \quad iK(e_1) = b_0 CK(e) + b_1 CiK'(e) (135)
\]
with the period ratios (moduli) of the associated modular theta functions being given by

$$\tau = \frac{b_0 + b_1 \tau'}{a_0 + a_1 \tau'}$$  \hspace{1cm} (136)$$

are also sufficient, when

$$a_0 b_1 - a_1 b_0 = n$$  \hspace{1cm} (137)$$

is a positive integer number. The integer $n$ is called the degree of transformation.

Equation (137) for $a_0, b_1, a_1, b_0 \in \mathbb{Z}$ when viewed as the determinant of a matrix $\in GL(2, \mathbb{Z})$, sometimes is called a modular correspondence of level $n$.

It can be shown that the *inequivalent reduced forms of modular correspondences*

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix},$$

are of the form

$$\begin{pmatrix} q & 0 \\ 16\xi & q' \end{pmatrix}$$

where $q$ a positive part of $n$ represents, $q' := \frac{n}{q}$, and $0 \leq \xi \leq q' - 1$. For instance for $n = p$ a prime number, there are $p + 1$ inequivalent reduced forms of the form $^{16}$

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 16 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 16.2 & p \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 16(p-1) & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

Also the multiplication factor $C$ in Eq.(134) is given by

$$C = \frac{1}{q} \frac{K(e_1)}{K(e)}$$  \hspace{1cm} (138)$$

which for a degree of transformation that is a prime number ($n = p$) is equal to $\frac{K(e_1)}{K(e)}$ or $(1/n) \frac{K(e_1)}{K(e)}$. Eq.(138) can be reexpressed in terms of Jacobi theta functions as follows

$$C = \frac{1}{q} \vartheta_2^2(0, \tau) \vartheta_3^2(0, \frac{\tau - 16\xi}{q})$$  \hspace{1cm} (139)$$

The modular equations are equations relating the Jacobi modulus $e(p\tau)$ to $e(\tau)$ which are of the form

$$F_p \left[ \left( \frac{2}{p} \right) \sqrt{e(\tau)}, \sqrt[4]{e \left( \frac{\tau - 16\xi}{p} \right)} \right] = 0$$  \hspace{1cm} (140)$$

$^{16}$In a more familiar notation these classes of inequivalent reduced forms are

$$a_i = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, ad = n, (a, b, d) = 1, 0 \leq b < d \text{ and } a, b, d \in \mathbb{Z}.$$
where \((\frac{x}{y})\) denotes the Legendre symbol \(^{17}\). Equivalently, modular equations can be written in terms of the absolute modular invariant function \(j(\tau)\), and relate the reduced absolute modular invariant \(j^*\) to \(j\) by polynomial equations of the form

\[
\Phi_p(j^*, j) = 0
\]

where \(j^* := j \alpha_i = j \left(\frac{\omega i}{2(2n + 1)}\right)\). The explicit form of \(\Phi_2(j^*, j) = 0\), has been given in [41].

An additional example considered in [18] is that of the differential equation

\[
\frac{dy}{\sqrt{(1 - y^2)(1 + e^2 y^2)}} = C \frac{dx}{\sqrt{(1 - x^2)(1 + e^2 x^2)}}
\]

(142)

where \(y = \pm ie^n x \left(\frac{\phi^2(\frac{\omega}{2(2n + 1)}) \cdots \phi^2(\frac{(n - 1)\omega}{2(2n + 1)})}{\phi^2(\frac{\omega}{2(2n + 1)}) \cdots \phi^2(\frac{(n - 1)\omega}{2(2n + 1)})}\right)^2 \frac{1}{e}\), while the modulus \(e\) is given by \(1 = e^{n+1} \left(\phi^2(\frac{\omega}{2(2n + 1)}) \cdots \phi^2(\frac{(n - 1)\omega}{2(2n + 1)})\right)^2\) and \(C = \left(\frac{\phi^2(\frac{\omega}{2(2n + 1)}) \cdots \phi^2(\frac{(n - 1)\omega}{2(2n + 1)})}{\phi^2(\frac{\omega}{2(2n + 1)}) \cdots \phi^2(\frac{(n - 1)\omega}{2(2n + 1)})}\right)^2 \frac{1}{e}\).

For \(n = 1\) it gives \(e = \sqrt{3} + 2, \phi^2(\frac{\omega}{2}) = 2\sqrt{3} - 3\). Also here \(\varphi \vartheta = x\), where \(\vartheta = \int \frac{dx}{\sqrt{(1 - x^2)(1 + e^2 x^2)}}\).

\[\text{D Quartic and cubic for spherical orbits assuming } \Lambda = 0\]

In the case of spherical orbits (polar with \(L = 0\), or non-polar) and with the assumption of a vanishing cosmological constant, the test particle’s radial coordinate will be stable at some value \(r_f\) if \(R(r)\) vanishes at \(r = r_f\) and goes negative nearby. This will be the case if

\[
R(r_f) = 0, \quad \frac{dR}{dr}|_{r = r_f} = 0
\]

(143)

and the second derivative of the polynomial \(R(r)\) negative.

These conditions result in the solution of a quartic and a cubic algebraic equation. Let us describe first how we can solve the quartic equation:

For a quartic polynomial Ferrara (1545) solved the general equation of degree 4:

\[
P_4(x) = x^4 - c_1 x^3 + c_2 x^2 - c_3 x + c_4 = 0
\]

(144)

as follows: using the substitution \(x \to x + c_1 / 4\), we can eliminate the cubic term \((c_1 = 0)\). The reduced equation is equivalent to

\[
(x^2 + c_0)^2 = y x^2 + c_3 x + c_0^2 - c_4
\]

(145)

\(^{17}(\frac{x}{y}) = e^{\frac{x^2 - 1}{y} i \pi}\).
where \( c_0 = \frac{1}{2} (c_2 + y) \). The discriminant \( \Delta \) of the right hand side (RHS) is:

\[
\Delta = c_3^2 - 4y(c_0^2 - c_4) = c_3^2 - 4y \left( \frac{1}{4} (c_2 + y)^2 - c_4 \right)
\]  

(146)

Its vanishing results in a cubic in \( y \) to be solved. Then the RHS has a double root \( x_0 = -\frac{c_3}{2y} \), and \( x^2 + c_0 = \pm \sqrt{y} (x - x_0) \) maybe solved by the extraction of one more square root.

As we described, the vanishing of the discriminant \( \Delta = 0 \) results in a cubic equation:

\[
-y^3 - 2c_2y^2 + y(4c_4 - c_2^2) + c_3^2 = 0
\]  

(147)

or \( y^3 + 2c_2y^2 + y(c_2^2 - 4c_4) - c_3^2 = 0 \). Under the substitution

\[
y = -4\xi - \frac{2c_2}{3}
\]

we get the cubic

\[
-64\xi^3 + \xi \left( \frac{4}{3} c_2^2 + 16c_4 \right) - \frac{2}{27} c_2^2 - c_3^2 + \frac{8c_2c_4}{3}
\]  

(149)

or \( \xi^3 + \xi m - n = 0 \) with

\[
m := -\frac{m'}{64}, \quad m' = \frac{4}{3} c_2^2 + 16c_4,
\]

\[
n := \frac{n'}{64}, \quad n' = -\frac{2}{27} c_2^2 - c_3^2 + \frac{8c_2c_4}{3}
\]  

(150)

Then the roots are:

For polar orbits the coefficients \( c_2, c_3, c_4 \) in terms of the constants of integration \( E, Q \), \( (L = 0) \) and the Kerr parameter \( a \) are given by

\[
c_3 = -\frac{2GM}{c^2}(Q + a^2E^2)\left(\frac{E^2}{E^2 - 1}\right),
\]

\[
c_2 = a^2 - \frac{Q}{E^2 - 1},
\]

\[
c_4 = -\frac{Qa^2}{E^2 - 1}
\]  

(151)

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