ON SCATTERING FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION ON WAVEGUIDE \( \mathbb{R}^m \times T \) (WHEN \( m = 2, 3 \))

ZEHUA ZHAO

Abstract. In the article, we prove the large data scattering for two problems, i.e. the defocusing quintic nonlinear Schrödinger equation on \( \mathbb{R}^2 \times T \) and the defocusing cubic nonlinear Schrödinger equation on \( \mathbb{R}^3 \times T \). Both of the two equations are mass supercritical and energy critical. The main ingredients of the proofs contain global Strichartz estimate, profile decomposition and energy induction method. This paper is the second project of our series work (two papers, together with \[38\]) on large data scattering for the defocusing critical NLS with integer index nonlinearity on low dimensional waveguides. At this point, this category of problems are almost solved except for two remaining resonant system conjectures and the quintic NLS problem on \( \mathbb{R} \times T \).

Keywords: NLS, well-posedness, scattering theory, concentration compactness and waveguide manifolds.

1. Introduction

First, we consider the following defocusing nonlinear equation with power-type nonlinearity on the waveguide \( \mathbb{R}^m \times T \) (the product spaces in this form are called waveguide manifold, see \[27, 28\] for information) as follows:

\[
(i\partial_t + \Delta_{\mathbb{R}^m \times T^n})u = F(u) = |u|^{p-1}u,
\]

\[
u(0, x) = u_0 \in H^1(\mathbb{R}^m \times T^n).
\]

where \( \Delta_{\mathbb{R}^m \times T^n} \) is the Laplace-Beltrami operator on \( \mathbb{R}^m \times T^n \) and \( u : \mathbb{R} \times \mathbb{R}^m \times T^n \to \mathbb{C} \) is a complex-valued function. Specifically, when \( m = 3, n = 1, p = 3 \), the equation would become cubic NLS on \( \mathbb{R}^3 \times T \) as follows:

\[
(i\partial_t + \Delta_{\mathbb{R}^3 \times T})u = F(u) = |u|^2u,
\]

\[
u(0, x) = u_0 \in H^1(\mathbb{R}^3 \times T).
\]

In addition, when \( m = 2, n = 1, p = 5 \), the equation would become quintic NLS on \( \mathbb{R}^2 \times T \) as follows:

\[
(i\partial_t + \Delta_{\mathbb{R}^2 \times T})u = F(u) = |u|^4u,
\]

\[
u(0, x) = u_0 \in H^1(\mathbb{R}^2 \times T).
\]

In this paper, we consider the large data scattering for the initial value problem (1.2) and the initial value problem (1.3). There are several reasons why we consider the two problems together and we will explain them shortly. A brief overview of existing related results is as follows:

For Euclidean case, the large data theory of critical and subcritical NLS is much better understood, at least in the defocusing case (see for example \[6, 9, 12, 26, 29\]). For waveguide case, we are also naturally interested in the range of the nonlinearity index \( p \) in (1.1) when the initial value problem is global well-posed and scattering. According to the existing results and theories, we expect that the solution of (1.1) globally exists and scatters in the range \( 1 + \frac{4}{m} \leq p \leq 1 + \frac{4}{m+n-2} \). And for those two problems, the index \( (p = 3) \) in equation (1.2) lies in the range \( \left( \frac{7}{3} \leq p = 3 \leq 3 \right) \); also, the index \( (p = 5) \) in equation (1.3) lies in the range \( (3 \leq p = 5 \leq 5) \). So it is reasonable for us to consider those two problems and expect the solutions of (1.2) and (1.3) are global well-posed and scattering.
There are many related results about NLS on waveguides such as [7] (defocusing cubic $\mathbb{R}^2 \times \mathbb{T}$), [14] (defocusing quintic $\mathbb{R} \times \mathbb{T}^2$), [20] (defocusing cubic $\mathbb{R} \times \mathbb{T}^3$), [36] (defocusing NLS on $\mathbb{R}^d \times \mathbb{T}$) and [38] (defocusing cubic $\mathbb{R}^2 \times \mathbb{T}^2$).

In [7], X. Cheng, Z. Guo, K. Yang and L. Zhao proved the large data scattering for the defocusing cubic NLS on $\mathbb{R}^2 \times \mathbb{T}$. One remarkable point is that they have used resonant system approximation. Additionally, in [37], K. Yang and L. Zhao have proved the large data scattering for the corresponding resonant system.

In [14], Z. Hani and B. Pausader proved the large data scattering for defocusing quintic NLS on $\mathbb{R} \times \mathbb{T}^2$ based on an assumption, i.e. the large data scattering for a corresponding quintic resonant system. A special case of the quintic resonant system coincides with the 1 dimensional mass-critical NLS problem. The large data scattering for 1 dimensional mass-critical NLS problem is proved by B. Dodson ([11]). Additionally, the way to construct global-in-time Stricharz estimate in this paper is very important and useful.

In [20], A. Ionescu and B. Pausader proved the global well-posedness of defocusing cubic NLS on $\mathbb{R} \times \mathbb{T}^3$. We are also largely inspired by the method of this paper.

In [35], N. Tzvetkov and N. Visciglia proved the global well-posedness and scattering for defocusing NLS on $\mathbb{R}^d \times \mathbb{T}$ with nonlinearity index $p$ satisfying $1 + \frac{4}{d} < p < 1 + \frac{4}{d+1}$. In the case, the equation is mass supercritical and energy subcritical. Specifically, when $d = 2$, the corresponding index $p$ satisfies $3 < p < 5$; when $d = 3$, the corresponding index $p$ satisfies $\frac{5}{2} < p < 3$. In this paper, we discuss the critical cases when $d = 3, p = 3$ (equation (1.2)) and $d = 2, p = 5$ (equation (1.3)).

In [38], we proved the large data scattering for defocusing cubic NLS on $\mathbb{R}^2 \times \mathbb{T}^2$ based on an assumption, i.e. the large data scattering for a corresponding cubic resonant system. A special case of the quintic resonant system coincides with the 2 dimensional mass-critical NLS problem. The large data scattering for 2 dimensional mass-critical NLS problem is proved by B. Dodson ([10]). Additionally, one dimensional and higher order dimensional mass-critical NLS problems are also solved by B. Dodson ([12]). We also refer to [24, 33, 34] and they are important results about mass-critical NLS.

Now we consider a series of more specific problems of (1.1), i.e. large data scattering for the defocusing critical NLS with integer index nonlinearity on low dimensional (when $m + n \leq 4$) waveguides. First, noticing the range $1 + \frac{4}{m} \leq p \leq 1 + \frac{4}{m+n-2}$, we have $n = 0, 1, 2$. When $n = 0$, we have $m = 4, p = 3$ (cubic NLS on $\mathbb{R}^4$), $m = 3, p = 5$ (quintic NLS on $\mathbb{R}^3$), $m = 2, p = 3$ (quintic NLS on $\mathbb{R}^2$), $m = 4, p = 2$ (4d mass critical NLS) and $m = 1, p = 5$ (quintic NLS on $\mathbb{R}$). When $n = 1$, we have $m = 1, p = 5$ (quintic NLS on $\mathbb{R} \times \mathbb{T}$), $m = 2, p = 5$ (quintic NLS on $\mathbb{R}^2 \times \mathbb{T}$, discussed in this paper), $m = 3, p = 3$ (cubic NLS on $\mathbb{R}^3 \times \mathbb{T}$, discussed in this paper) and $m = 2, p = 3$ (cubic NLS on $\mathbb{R}^2 \times \mathbb{T}$). When $n = 2$, we have $m = 1, p = 5$ (quintic NLS on $\mathbb{R} \times \mathbb{T}^2$) and $m = 2, p = 3$ (cubic NLS on $\mathbb{R}^2 \times \mathbb{T}^2$). There are totally 11 specific problems. First, critical NLS problems (5 problems) on pure Euclidean domains are well known (see [9, 10, 11, 26]). Additionally, quintic NLS on $\mathbb{R} \times \mathbb{T}$ are expected to be similar to cubic NLS on $\mathbb{R}^2 \times \mathbb{T}$ (see [7]) since both of them are mass critical and energy subcritical. There are 5 problems left, they are discussed in [7, 14, 38] and this paper.

Moreover, [2, 6, 15, 16, 17, 18, 21, 22, 25, 26, 29] are some other important resources and related results. Generally speaking, the difficulty of the critical NLS problems on waveguides $\mathbb{R}^m \times \mathbb{T}^n$ increase if the whole dimension $m + n$ increase or if the $\mathbb{R}$-dimension $m$ decrease. As for the introduction of the related NLS problems on waveguides, we also refer to [7, Introduction], [14, Introduction], [20, Introduction] and [38, Introduction] for more information.

**A word on quintic 3d problems:** While scattering holds for the quintic equation on $\mathbb{R}^3$ (see [9]), $\mathbb{R}^2 \times \mathbb{T}$ (see this paper) and $\mathbb{R} \times \mathbb{T}^2$ (see [14]), it is not expected to hold on $\mathbb{R}^3$. The situation on $\mathbb{R} \times \mathbb{T}^3$ seems to be a borderline case for this question, i.e. defocusing quintic NLS equation on three dimensional waveguides.
A word on cubic 4d problems: Also, while scattering holds for the cubic equation on $\mathbb{R}^{4}$ (see [26]), $\mathbb{R}^{3} \times T$ (see this paper) and $\mathbb{R}^{2} \times T^{2}$, it is not expected to hold on $\mathbb{R} \times T^{3}$ or $T^{4}$. The situation on $\mathbb{R}^{2} \times T^{2}$ seems to be a borderline case for this question, i.e. defocusing cubic NLS equation on four dimensional waveguides.

A word on the remaining problems in this category: Together with [7, 14, 38] (corresponding to cubic $\mathbb{R}^{2} \times T$ problem, quintic $\mathbb{R} \times T^{2}$ problem and cubic $\mathbb{R} \times T^{3}$ problem respectively), large data scattering for defocusing critical NLS with integer-index nonlinearity on low dimensional waveguides are almost solved except for the two resonant system conjectures arising from [14, 38] and the quintic NLS on $\mathbb{R} \times T$. One inspiring thing is, in [37], the corresponding cubic resonant system conjecture (arises from cubic $\mathbb{R}^{2} \times T$ problem) is solved by K. Yang and L. Zhao. We expect one may use [37] and the large data scattering results for defocusing mass critical NLS by B. Dodson ([10, 11, 12]) to prove the two remaining resonant system conjectures. Additionally, we expect one may learn from [7] and use the scattering result for 1d mass critical NLS problem by B. Dodson ([11]) to prove the large data scattering for defocusing mass critical NLS with integer-index nonlinearity on low dimensional waveguides.

Comparison of the two problems: There are some similarities and differences between the two problems. One significant similarity is that both of the two equations are mass supercritical and energy critical which leads that the spirits of the linear profiles and the profile decompositions are same. As for the differences, (1.2) is cubic and (1.3) is quintic thus the nonlinear estimates would be different. Another difference of those two problems would be the whole spatial dimensions (3 and 4). Those two equations are analogues of each other in some sense.

The following two Theorems are the main results of this paper. Theorem 1.1 is for equation (1.2) (cubic $\mathbb{R}^{3} \times T$ problem) and Theorem 1.2 is for equation (1.3) (quintic $\mathbb{R}^{2} \times T$ problem).

**Theorem 1.1.** For any initial data $u_{0} \in H^{1}(\mathbb{R}^{3} \times T)$, there exists a solution $u \in X^{1}_{c}(\mathbb{R})$ to (1.2) that is global and scattering in the sense that there exists $v^{\pm \infty} \in H^{1}(\mathbb{R}^{3} \times T)$ such that

$$||u(t) - e^{i\Delta x^{2} + \tau} v^{\pm \infty}||_{H^{1}(\mathbb{R}^{3} \times T)} \to 0, \ as \ t \to \pm \infty.$$  

The uniqueness space $X^{1}_{c} \subset C_{t}(\mathbb{R} : H^{1}(\mathbb{R}^{3} \times T))$ was first introduced by Herr-Tataru-Tzvetkov ([16]) (see also [14], we define the solution space in a similar way.).

**Theorem 1.2.** For any initial data $u_{0} \in H^{1}(\mathbb{R}^{2} \times T)$, there exists a solution $u \in X^{1}_{c}(\mathbb{R})$ to (1.3) that is global and scattering in the sense that there exists $v^{\pm \infty} \in H^{1}(\mathbb{R}^{2} \times T)$ such that

$$||u(t) - e^{i\Delta x^{2} + \tau} v^{\pm \infty}||_{H^{1}(\mathbb{R}^{2} \times T)} \to 0, \ as \ t \to \pm \infty.$$  

The uniqueness space $X^{1}_{c} \subset C_{t}(\mathbb{R} : H^{1}(\mathbb{R}^{2} \times T))$ was first introduced by Herr-Tataru-Tzvetkov ([16]) (see also [14], we define the solution space in a similar way.).

For those two problems, we will prove small data result first by using global Stricharz estimate and some standard arguments and then we can use profile decomposition and energy induction method to extend our analysis to large data case.

Similar as other related results (see [7, 14, 20, 38]), profile decomposition is a crucial step. In order to understand the appearance of the profiles, specifically for this problem, in view of the scaling-invariant of (1.1) under

$$\mathbb{R}^{3}_{1} \times T_{y} \to M_{3} := \mathbb{R}^{3}_{1} \times (\lambda^{-1}T), \ u \to \tilde{u}(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^{2}t).$$

**Remark.** Similar scaling-invariance analysis also works for the case of quintic NLS on $\mathbb{R}^{2} \times T$ so we omit it.

There are two extreme situations as follows:

When $\lambda \to 0$, the manifolds $M_{3}$ will be similar to $\mathbb{R}^{4}$ and we can use the scattering result for four dimensional energy critical NLS by E. Ryckman and M. Visan ([26]). The appearance is a manifestation of the energy-critical nature of the nonlinearity. This corresponds in $M_{1}$ to solutions
with initial data (This behavior corresponds to Euclidean profiles which we will analyze more precisely in section 5 and section 6.)

\[ u^\lambda(x, y, 0) = \lambda^{-1}\phi(\lambda^{-1}(x, y)) \quad \phi \in C^\infty_0(\mathbb{R}^3), \lambda \to 0. \]

When \( \lambda \to \infty \), the manifolds \( M_\lambda \) become thinner and thinner and resemble \( \mathbb{R}^3 \). The problem will become similar to the following cubic NLS problem on \( \mathbb{R}^3 \):

\[ (i\partial_t + \Delta_x)u = |u|^2u, \quad u(0) \in H^1(\mathbb{R}^3). \]

Those solutions on \( M_\lambda \) correspond to solutions on \( M_1 \) with initial data

\[ u^\lambda(x, y, 0) = \lambda^{-1}\phi(\lambda^{-1}(x, y)) \quad \phi \in C^\infty_0(\mathbb{R}^3 \times T), \lambda \to \infty. \]

Different from [14] (defocusing quintic \( \mathbb{R}^4 \times \mathbb{Z}_2 \) problem) and [38] (defocusing cubic \( \mathbb{R}^2 \times \mathbb{T}^2 \) problem), for those two problems in this paper, large scale profile will not appear in the profile decomposition. Put in another way, extracting orthogonal Euclidean profiles and scale-one profiles are enough for us to control the scattering norm of the linear Schrödinger propagation of the remainder flow and extend our analysis to large data case. The reason is: in (1.6) the \( H^1 \) norm of \( u^\lambda \) will converge to 0 when \( \lambda \to \infty \); additionally, the \( H^1 \) boundedness condition ensures the mass of \( \phi \) is 0, otherwise the mass of \( u^\lambda \) will blow up. Thus \( u^\lambda \) must converge to 0 in \( H^1 \).

Both of the domain and the nonlinearity in the equation play important roles in the asymptotic behavior of the solutions when we consider NLS problems. That is why there are some remarkable differences among those related problems. In [14, 38] (the equation is both mass critical and energy subcritical), there are three types of profiles considered, i.e. Euclidean profiles, scale-one profiles and large-scale profiles. In [20] and this paper (the equations are only energy critical), Euclidean profiles and scale-one profiles are considered. In [7] (the equation is mass critical and energy subcritical), large-scale profiles and scale-one profiles are considered.

The proofs of the Theorem 1.1 and Theorem 1.2 follow from a standard skeleton based on the Kenig-Merle machinery [21, 22]. Mainly there are three important ingredients: global Strichartz estimates, profile decomposition and energy induction method.

The organization of this paper: in Section 2, we introduce some notations and function spaces; in Section 3, we prove the global Strichartz estimates; in Section 4, we prove the local well-posedness and small data scattering of (1.2) and (1.3); in Section 5, we describe Euclidean profile and Euclidean approximation; in Section 6, we obtain the linear profile decomposition that leads us to analyze the large data case; in Section 7, we prove the contradiction argument leading to our large data scattering result, i.e. Theorem 1.1 and Theorem 1.2. Eventually, an important claim is: since the two problems have many similarities, for convenience, when possible, we will mainly discuss cubic \( \mathbb{R}^3 \times \mathbb{T}^2 \) problem and when necessary we will clarify the differences with details.

2. Notation and Preliminaries

About the notation, we write \( A \lesssim B \) to say that there is a constant \( C \) such that \( A \leq C B \). We use \( A \sim B \) when \( A \lesssim B \lesssim A \). Particularly, we write \( A \lesssim_u B \) to express that \( A \leq C(u)B \) for some constant \( C(u) \) depending on \( u \).

We also define the partial Littlewood-Paley projectors \( P_{\leq N}^x \) and \( P_{\geq N}^x \) as follows: fix a real-valued radially symmetrically bump function \( \varphi(\xi) \) satisfying

\[ \varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases} \]

for any dyadic number \( N \in 2^\mathbb{Z} \), let

\[ F_x(P_{\leq N}^x f)(\xi, y) = \varphi(\frac{\xi}{N})(F_x f)(\xi, y), \]

\[ F_x(P_{\geq N}^x f)(\xi, y) = \left( 1 - \varphi(\frac{\xi}{N}) \right)(F_x f)(\xi, y). \]
**Function spaces.** In this paper, we use some function spaces \((X^s, Y^s, N^s)\) based on atomic space and variation space. Those spaces were essentially introduced in [16], see also [17]. Moreover, we construct those spaces as in [14, 38]. We also use \(X^1_\lambda\) to be the basic solution space. Those spaces have some nice properties. For convenience, we give some basic definitions for the case “cubic \(\mathbb{R}^3 \times \mathbb{T}^m\) as follows and we refer to [16, 17] for more description of those spaces.

**Remark.** The setting for quintic \(\mathbb{R}^2 \times \mathbb{T}\) problem is similar and we omit it.

For \(C = [-\frac{1}{2}, \frac{1}{2}]^4 \subset \mathbb{R}^4\) and \(z \in \mathbb{R}^4\), we denote by \(C_z = z + C\) the translate by \(z\) and define the projection operator \(P_{C_z}\) as follows, \((F)\) is the Fourier transform:

\[
F(P_{C_z}f) = \chi_{C_z}(\xi)F(f)(\xi).
\]

As in [16, 17], for \(s \in \mathbb{R}\), we define:

\[
\|u\|_{X^s_\Delta(\mathbb{R})}^2 = \sum_{z \in \mathbb{Z}^4} (z)^{2s}\|P_{C_z}u\|_{L^2_\Delta(\mathbb{R}, L^2)}^2
\]

and similarly we have,

\[
\|u\|_{Y^s_\Delta(\mathbb{R})}^2 = \sum_{z \in \mathbb{Z}^4} (z)^{2s}\|P_{C_z}u\|_{V^2_\Delta(\mathbb{R}, L^2)}^2
\]

where the \(U^p_\Delta\) and \(V^p_\Delta\) are the atomic and variation spaces respectively of functions on \(\mathbb{R}\) taking values in \(L^2(\mathbb{R}^3 \times \mathbb{T})\).

Moreover, for an interval \(I \subset \mathbb{R}\), we can also define the restriction norms \(X^s_\Delta(I)\) and \(Y^s(I)\) in the natural way: \(\|u\|_{X^s_\Delta(I)} = \inf \{ \|v\|_{X^s_\Delta(\mathbb{R})} : v \in X^s_\Delta(\mathbb{R})\text{ satisfying } v|_I = u|_I \}\). And similarly for \(Y^s(I)\).

Additionally, a modification for to \(X^s_\Delta(\mathbb{R})\):

\[
X^s(\mathbb{R}) := \{ u : I_{\Delta} = \lim_{t \to -\infty} e^{-it\Delta}u(t) \text{ exists in } H^s, u(t) - e^{it\Delta}\phi_{\to} \in X^s_\Delta(\mathbb{R}) \} \quad \text{equipped with the norm:}
\]

\[
\|u\|_{X^s(\mathbb{R})}^2 = \|\phi_{\to}\|_{H^s(\mathbb{R}^3 \times \mathbb{T})}^2 + \|u - e^{it\Delta}\phi_{\to}\|_{X^s_\Delta(\mathbb{R})}^2
\]

Our basic space to control solutions is \(X^1_\lambda(I) = X^1(I) \cap C(I : H^1)\). Also we use \(X^1_{\text{loc},\lambda}(I)\) to express the set of all solutions in \(C_{\text{loc}}(I : H^1)\) whose \(X^1(I)\)-norm is finite for any compact subset \(J \subset I\).

At last, in order to control the nonlinearity on interval \(I\), we need to define ‘\(N\) -Norm’ as follows, on an interval \(I = (a, b)\) we have:

\[
\|h\|_{N^s(I)} = \| \int_a^b e^{i(t-s)\Delta}h(s)ds \|_{X^s(I)}
\]

We also need the following theorem which has analogues in [14, 16, 17].

**Theorem 2.1.** If \(f \in L^1_1(I, H^1(\mathbb{R}^3 \times \mathbb{T}))\), then

\[
\|f\|_{N^s(I)} \lesssim \sup_{v \in Y^{-1}(I), \|v\|_{Y^{-1}(I)} \leq 1} \int_{I \times (\mathbb{R}^3 \times \mathbb{T})} f(x, t)\overline{v(x, t)}dxdt.
\]

Also, we have the following estimate holds for any smooth function \(g\) on an interval \(I = [a, b]::

\[
\|g\|_{X^1(I)} \lesssim \|g(0)\|_{H^{1}(\mathbb{R}^3 \times \mathbb{T})} + \sum_{N} \|P_N(i\partial_t + \Delta)g\|_{L^2_1(I, H^{1}(\mathbb{R}^3 \times \mathbb{T}))}^\frac{1}{2}.
\]

**Remark.** We have analogue of the above lemma for quintic \(\mathbb{R}^2 \times \mathbb{T}\) problem.

The following lemma is also useful. It is exactly the analogue of [20, Lemma 2.3].

**Lemma 2.2.** For \(f \in H^1(\mathbb{R}^3 \times \mathbb{T})\), there holds that

\[
\|f\|_{L^2(\mathbb{R}^3 \times \mathbb{T})} \lesssim (\sup_{N} N^{-1}\|P_Nf\|_{L^\infty(\mathbb{R}^3 \times \mathbb{T})})^{\frac{1}{2}} (\|f\|_{H^1(\mathbb{R}^3 \times \mathbb{T})})^{\frac{1}{2}}.
\]
3. Global Strichartz estimate

For cubic $\mathbb{R}^3 \times \mathbb{T}$ problem: we have

**Theorem 3.1.** Now we prove the following Strichartz Estimate:

\[(3.1) \quad \|e^{it\Delta_3 \times T} P_{\leq N} u_0\|_{L_t^p L_x^q_x,\gamma} (\mathbb{R}^3 \times \mathbb{T}^x \times \mathbb{T}^y) \lesssim N^{2-\frac{2}{p}} \|u_0\|_{L^2 (\mathbb{R}^3 \times \mathbb{T})} \]

whenever

\[(3.2) \quad \frac{22}{7} < p < 6 \quad \text{and} \quad \frac{2}{q} + \frac{3}{p} = \frac{3}{2}.

**Proof:** The main idea of the proof is similar to [14, Theorem 3.1] (see also [36, Theorem 3.1]). We use duality argument, $T - T^*$ argument, a partition of unity and then estimate the diagonal part and non-diagonal part separately to obtain the above estimate. First, let us prove a more precise conclusion and we can get the above estimate by duality:

**Lemma 3.2.** For any $h \in C_c^\infty (\mathbb{R}^3 \times T_y \times R_x)$, there holds that

\[(3.3) \quad \| \int_{s \in \mathbb{R}} e^{-is\Delta_3 \times T} P_{\leq N} h(x, y, s) ds \|_{L_t^p L_x^q_x,\gamma} (\mathbb{R}^3 \times \mathbb{T}^x \times \mathbb{T}^y) \leq N^{2-\frac{2}{p}} \|h\|_{L_t^p L_x^q_x,\gamma} (\mathbb{R}^3 \times \mathbb{T}^x \times \mathbb{T}^y) + N^{1-\frac{2}{p}} \|h\|_{L_t^p L_x^q_x,\gamma} (\mathbb{R}^3 \times \mathbb{T}^x \times \mathbb{T}^y)

for any $(p, q)$ satisfies (3.2).

**Remark.** The above threshold $p < 6$ results from the requirement $q' < 2$ noticing the inclusion property of the sequence spaces.

**Corollary 3.3.** Noticing the sequence space inclusion and Lemma 3.2, the following Strichartz estimate also holds:

\[(3.4) \quad \|e^{it\Delta_3 \times T} P_{\leq N} u_0\|_{L_t^p L_x^q_x,\gamma} (\mathbb{R}^3 \times \mathbb{T}^x \times \mathbb{T}^y) \lesssim N^{2-\frac{2}{p}} \|u_0\|_{L^2 (\mathbb{R}^3 \times \mathbb{T})} \]

whenever

\[(3.5) \quad p > \frac{22}{7} \quad \text{and} \quad \frac{2}{q} + \frac{3}{p} = 1.

**Remark.** This estimate corresponds to the scattering norm which we will define in Section 4.

**Proof of Lemma 3.2:** In order to distinguish between the large and small time scales, we choose a smooth partition of unity $1 = \sum_{\gamma \in \mathbb{Z}} \chi(t - 2\pi \gamma)$ with $\chi$ supported in $[-2\pi, 2\pi]$. We also denote by $h_{\alpha}(t) = \chi(t)h(2\pi \alpha + t)$. Using the semigroup property and the unitarity of $e^{it\Delta_3 \times T}$ we can get:

\[
\| \int_{s \in \mathbb{R}} e^{-is\Delta_3 \times T} P_{\leq N} h(x, y, s) ds \|_{L_t^p L_x^q_x} (\mathbb{R}^3 \times \mathbb{T}) = \| \int_{s, t \in \mathbb{T}} \langle \chi(s - 2\pi \alpha), e^{-it\Delta_3 \times T} P_{\leq N} h(t) \rangle_{L_x^p L_y^q} (\mathbb{R}^3 \times \mathbb{T}^y) ds dt \\
= \sum_{\alpha, \beta} \int_{s, t \in \mathbb{R}} \langle \chi(s - 2\pi \alpha), e^{-is\Delta_3 \times T} P_{\leq N} h(s), e^{-it\Delta_3 \times T} P_{\leq N} h(t) \rangle_{L_x^p L_y^q} (\mathbb{R}^3 \times \mathbb{T}^y) ds dt \\
= \sigma_d + \sigma_{nd}.
\]

Here we have,

\[
\sigma_d = \sum_{\alpha \in \mathbb{Z}, |\alpha| \leq 9} \int_{s, t \in \mathbb{R}} \langle e^{-is(2\pi \gamma)} \Delta_3 \times T P_{\leq N} h_{\alpha}(s), e^{-it\Delta_3 \times T} P_{\leq N} h_{\alpha+\gamma}(t) \rangle_{L_x^p L_y^q} ds dt.
\]

\[
\sigma_{nd} = \sum_{\alpha, \gamma \in \mathbb{Z}, |\alpha| > 9} \int_{s, t \in \mathbb{R}} \langle e^{-is(2\pi \gamma)} \Delta_3 \times T P_{\leq N} h_{\alpha}(s), e^{-it\Delta_3 \times T} P_{\leq N} h_{\alpha+\gamma}(t) \rangle_{L_x^p L_y^q} ds dt.
\]
Here, ‘d’ is short for ‘diagonal’ and ‘nd’ is short for ‘non-diagonal’. We will estimate the diagonal part and the non-diagonal part by using different methods.

For the diagonal part: First, we need to use a local-in-time $L^p$ estimate as follows:

**Lemma 3.4.** Let $p_1 = \frac{22}{7}$, then for any $p > p_1$, $N \geq 1$, and $f \in L^2(\mathbb{R}^3 \times T)$,

$$
\|e^{it\Delta} P_N f\|_{L^p(\mathbb{R}^3 \times T \times [0, 2\pi])} \lesssim_p N^{2 - \frac{2}{p}} \|f\|_{L^2(\mathbb{R}^3 \times T)}.
$$

The proof of the local-in-time estimate can also be proved in a similar way as in [20] with small modifications (see also [25] which is a related recent result). We omit it.

According to the estimate (3.6) above, by duality we have

$$
\| \int_{s \in \mathbb{R}} e^{-is\Delta_{\mathbb{R}^3 \times T}} P_{\leq N} h(s) ds \|_{L^2_x y} \lesssim N^{2 - \frac{2}{p}} \|h\|_{L^p_x y, t(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])}
$$

where $h$ is supported in $[-2\pi, 2\pi]$. And consequently,

$$
\sigma_d = \sum_{\alpha \in \mathbb{Z}, |\gamma| \leq 9} \int_{t \in \mathbb{R}} \int_{s \in \mathbb{R}} e^{-is(2\pi\gamma)} \Delta_{\mathbb{R}^3 \times T} P_{\leq N} h_{\alpha}(s), e^{-it\Delta_{\mathbb{R}^3 \times T}} P_{\leq N} h_{\alpha + \gamma}(t) L^2_x y \times L^2_x y \ ds dt
$$

$$
\lesssim N^{2 - \frac{2}{p}} \sum_{\alpha \in \mathbb{Z}, |\gamma| \leq 9} \|h_{\alpha}\|^2_{L^p_x y, t(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])}.
$$

This finishes the estimate for the diagonal part in (3.3).

For the non-diagonal part: we need to use Lemma 3.5 that we will prove soon and we can apply it to estimate the non-diagonal part by using Hölder’s inequality and the discrete Hardy-Sobolev inequality as below:

$$
\sigma_{nd} = \sum_{\alpha, \gamma \in \mathbb{Z}, |\gamma| > 3} \int_{t \in \mathbb{R}} \int_{s \in \mathbb{R}} e^{-is(2\pi\gamma)} \Delta_{\mathbb{R}^3 \times T} P_{\leq N} h_{\alpha}(s), e^{-it\Delta_{\mathbb{R}^3 \times T}} P_{\leq N} h_{\alpha + \gamma}(t) L^2_x y \times L^2_x y \ ds dt
$$

$$
\lesssim N^{2 - \frac{2}{p}} \sum_{\alpha, \gamma \in \mathbb{Z}, |\gamma| > 3} |\gamma| \|h_{\alpha}\|^2_{L^p_x y, t(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])} \|h_{\alpha + \gamma}\|_{L^p_x y, t}.
$$

**Lemma 3.5.** Suppose $\gamma \in \mathbb{Z}$ satisfies $|\gamma| \geq 3$ and that $p > \frac{22}{7}$. For any function $h \in L^\infty_{x,y,t}(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])$, there holds that:

$$
\| \int_{s \in \mathbb{R}} \chi(s)e^{i(t-s+2\pi\gamma)} \Delta_{\mathbb{R}^3 \times T} P_{\leq N} h(s) ds \|_{L^p_{x,y,t}(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])} \lesssim |\gamma| \|h\|_{L^\infty_{x,y,t}(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])} \lesssim 2^{-\frac{2}{p}} N^{2 - \frac{2}{p}} \|h\|_{L^p_{x,y,t}(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])}.
$$

**Proof:** The proof of this lemma is similar to [14, Lemma 3.3] (see also [38]) by using Hardy-Littlewood circle method. The main idea of the proof is to study the Kernel $K_{N, \gamma}$, use a partition and decompose the corresponding index set into three parts and estimate over the three parts separately.

Without loss of generality, we assume that:

$$
h = \chi(s) P_{\leq N} h, \quad \|h\|_{L^p(\mathbb{R}^3 \times T \times [-2\pi, 2\pi])} = 1
$$

and we define:

$$
g(x, y, s) = \int_{s \in \mathbb{R}} e^{i(t-s+2\pi\gamma)} \Delta_{\mathbb{R}^3 \times T} h(x, y, s) ds.
$$
Also the Kernel is defined as follows:
\[ K_N(x, y, t) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^3} [\eta_{\leq N}(\xi_1) \eta_{\leq N}(\xi_2) \eta_{\leq N}(\xi_3)]^2 [\eta_{\leq N}(k)]^2 e^{i(x \cdot \xi + y \cdot k + t|k|^2 + |\xi|^2)] d\xi \]
(3.10)
\[ = \int_{\mathbb{R}^3} [\eta_{\leq N}(\xi_1) \eta_{\leq N}(\xi_2) \eta_{\leq N}(\xi_3)]^2 e^{i(x \cdot \xi + y \cdot k + t|k|^2)} : [\sum_{k \in \mathbb{Z}} [\eta_{\leq N}(k)]^2 e^{i(y \cdot k + t|k|^2)]} \]
\[ = K_N(x, t) \bigotimes K_N(x, t). \]

And we define \( K_{N, \gamma}(x, y, t) := K_N(x, y, 2\pi \gamma + t) \), and so we have \( g(x, y, t) = K_{N, \gamma} * h \). Notice that a remarkable difference is the non-stationary phase estimate because of the dimension, we will have:
\[ ||K_{N, \gamma}||_{L^{\infty}_{x,y,t}} \lesssim |\gamma|^{-\frac{2}{N}} \]
instead of
\[ ||K_N||_{L^{\infty}_{x,y,t}} \lesssim |\gamma|^{-\frac{2}{N^2}}. \]

And
\[ ||F_{x,y,t}K_{N,\gamma}||_{L^{\infty}_{t,k,r}} \lesssim 1 \]
still holds.

For \( \alpha \) a dyadic number, we define \( g^\alpha(x, y, t) = \alpha^{-1} g(x, y, t) 1_{\left\{ \frac{\alpha}{2} \leq |\alpha| \leq \alpha \right\}} \) which has modulus in \( \left[ \frac{1}{2}, 1 \right] \).

We define \( h^\beta \) similarly for \( \beta \in 2^\mathbb{Z} \). And we have the following decomposition:
\[ ||g||_{L^p_{x,y,t}}^p = \langle |g|^{p-2} g, g \rangle = \sum_{\alpha, \beta} \alpha^{p-1} \beta \langle |g|^{p-2} g \alpha, K_{N, \gamma} * h^\beta \rangle 
\]
\[ = \sum_{\alpha, \beta} \alpha^{p-1} \beta \langle |g|^{p-2} g \alpha, K_{N, \gamma} * h^\beta \rangle 
\]
(3.11)
\[ \sum_{\alpha, \beta} \sum_{\alpha} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta} \sum_{\beta}
\]
where \( S_1, S_2, S_3 \) are three index sets. Furthermore, we have the following decomposition:
(1) \( S_1 = \{ (\alpha, \beta) : C|\gamma|^{-\frac{2}{N}} \leq \alpha \beta^{p-1} \}, \)
(2) \( S_2 = \{ (\alpha, \beta) : \alpha \beta^{p-1} \leq CN^\frac{2}{N} |\gamma|^{-\frac{2}{N}} \}, \)
(3) \( S_3 = \{ (\alpha, \beta) : CN^\frac{2}{N} |\gamma|^{-\frac{2}{N}} \leq \alpha \beta^{p-1} \leq C |\gamma|^{-\frac{2}{N}} \} \)

For \( C \) a large constant to be decided later. For fixed \( \alpha, \beta \), we will decompose \( K_{N, \gamma} = K_{N, \gamma, \alpha, \beta} + K_{N, \gamma, \alpha, \beta}^2 \) and estimate them as follows.
\[ \langle |g|^{p-2} g, K_{N, \gamma} \rangle \lesssim ||K_{N, \gamma, \alpha, \beta}||_{L^{\infty}_{x,y,t}} ||g||_{L^1} ||h^\beta||_{L^p} \]
(3.12)
\[ \langle |g|^{p-2} g, K_{N, \gamma} \rangle \lesssim ||F_{x,y,t}K_{N, \gamma, \alpha, \beta}^2||_{L^{\infty}_{t,k,r}} ||g||_{L^p} ||h^\beta||_{L^p} \]
(3.13)
Then we can estimate the three parts as in [14]. The small level and large level are easy to handle, while the estimate for the medium level requires a more delicate decomposition for the kernel as follows:
\[ ||K_{N, \gamma, \alpha, \beta}||_{L^{\infty}_{x,y,t}} \lesssim \alpha \beta^{p-1}, \]
(3.14)
\[ ||F_{x,y,t}K_{N, \gamma, \alpha, \beta}||_{L^{\infty}_{t,k,r}} \lesssim N^\epsilon (\alpha \beta^{p-1})^{-1} |\gamma|^{-\frac{2}{N}}. \]
(3.15)
The above kernel decomposition is similar as in [14] by using Hardy-Littlewood circle method. The rest follows as in [14] so we omit it. Eventually the conclusion estimate is as follows:
\[ ||g||_{L^p_{x,y,t}} \lesssim ||g||_{L^p_{x,y,t}}^\frac{2}{p} \max (N^{-4-i}, N^{\frac{2}{p}+\epsilon}) \lesssim ||g||_{L^p_{x,y,t}}^\frac{2}{p} N^{\frac{2}{p} - \frac{2}{p}} \]
whenever \( p > \frac{2}{\epsilon} \). That finishes the proof of the non-diagonal estimate.
For quintic $\mathbb{R}^2 \times T$ problem: we have the following estimate:

**Theorem 3.6.** Now we prove the following Strichartz Estimate:

\[ \|e^{it\Delta_{\mathbb{R}^2 \times T}} P_{\leq N} u_0\|_{L^p_t L^q_x(\mathbb{R}^2 \times [2\pi \gamma, 2\pi \gamma + 1])} \lesssim N^{\frac{2}{q} - \frac{2}{p}} \|u_0\|_{L^2(\mathbb{R}^2 \times T)} \]

whenever

\[ p > \frac{18}{5} \quad \text{and} \quad \frac{1}{q} + \frac{1}{p} = 1. \]

**Corollary 3.7.** According to the inclusion property of sequence spaces, it is not hard to verify the following estimate also holds:

\[ \|e^{it\Delta_{\mathbb{R}^2 \times T}} P_{\leq N} u_0\|_{L^p_t L^q_x(\mathbb{R}^2 \times [2\pi \gamma, 2\pi \gamma + 1])} \lesssim N^{\frac{2}{q} - \frac{2}{p}} \|u_0\|_{L^2(\mathbb{R}^2 \times T)} \]

whenever

\[ p > \frac{18}{5} \quad \text{and} \quad \frac{1}{q} + \frac{1}{p} = \frac{1}{4}. \]

The following estimate is more precise and in fact it implies Theorem 3.6 by duality.

**Lemma 3.8.** For any $h \in C^\infty_c(\mathbb{R}^2 \times T_y \times \mathbb{T}_t)$, there holds that

\[ \| \int_{s \in \mathbb{R}} e^{-is\Delta_{\mathbb{R}^2 \times T}} P_{\leq N} h(x, y, s) ds \|_{L^p_x(\mathbb{R}^2 \times T_y \times T_t)} \lesssim N^{\frac{2}{q} - \frac{2}{p}} \|h\|_{L^p_x L^q_y(\mathbb{R}^2 \times 2\pi \gamma \times 2\pi \gamma)} + N^{1-\frac{2p}{q}+\frac{2}{p}} \|h\|_{L^p_x L^{q'}_y(\mathbb{R}^2 \times 2\pi \gamma \times 2\pi \gamma)} \]

for any $(p,q)$ satisfies (3.2).

The proof of Lemma 3.8 is similar to [14, Lemma 3.2] and Lemma 3.2 (see also [38, Lemma 3.2]). Since the spirits of the proofs are similar, we omit it.

4. Local theory and small-data result

For cubic $\mathbb{R}^3 \times T$ problem:

We define "$Z$-norm" (scattering norm) as follows

\[ \|u\|_{Z(I)} = \left( \sum_{N \geq 1} N^{6-p_0} \|1_{I}(t) P_N u\|_{L^{p_0}_{t,x,y}(\mathbb{R}^3 \times T \times T)} \right)^{\frac{1}{p_0}}. \]

Here $p_0$ is a constant satisfying $5 < p_0 < \frac{15}{4}$. It is not hard to verify that the scattering norm of the linear solution is control by the $H^1$-norm of the initial data by using Strichartz estimate (Corollary 3.3). For convenience, we define "$Z'$-norm" which is a mixture of $Z$-norm and $X^1$-norm as follows

\[ \|u\|_{Z'(I)} = \|u\|_{Z(I)} \|u\|_{X^1(I)}. \]

Now we are ready to prove the local well-posedness and small-data scattering of (1.2).

**Lemma 4.1 (Bilinear Estimate).** Suppose that $u_i = P_{N_i} u$, for $i = 1, 2$ satisfying $N_1 \geq N_2$. There exists $\delta > 0$ such that the following estimate holds for any interval $I \subseteq \mathbb{R}$:

\[ \|u_1 u_2\|_{L^2_{t,x,y}(\mathbb{R}^3 \times T \times I)} \lesssim \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^\delta \|u_1\|_{Y^0(I)} \|u_2\|_{Z'(I)}. \]

**Proof:** Without loss of generality, we can assume that $I = \mathbb{R}$. On one hand, we need the following estimate which follows as in [17, Proposition 2.8],

\[ \|u_1 u_2\|_{L^2(\mathbb{R}^3 \times T \times \mathbb{R})} \lesssim \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^\delta \|u_1\|_{Y^0(\mathbb{R})} \|u_2\|_{Y^0(\mathbb{R})}. \]

And it suffices to prove the following estimate, if it is hold then we can just combine the two inequalities (noticing the definition of $Z'$-norm) and we will get the lemma completed.

\[ \|u_1 u_2\|_{L^2(\mathbb{R}^3 \times T \times \mathbb{R})} \lesssim \|u_1\|_{Y^0(\mathbb{R})} \|u_2\|_{Z(\mathbb{R})}. \]
We first notice that, by orthogonality considerations, we may replace \( u_1 \) by \( P_C u_1 \) where \( C \) is a cube of dimension \( N_2 \). By using Hölder’s inequality, we have,
\[
\| (P_C u_1) u_2 \|_{L^2_{t,x,y}} \lesssim \| P_C u_1 \|_{L^2_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \| u_2 \|_{L^2_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \\
\lesssim N_2^{-\frac{20}{p_0}} \| P_C u_1 \|_{L^{20}_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \| u_2 \|_{L^{20}_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \\
\lesssim \| P_C u_1 \|_{L^{20}_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \| u_2 \|_{L^{20}_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T \times I_N)} \\
\lesssim \| P_C u_1 \|_{Y^p} \| u_2 \|_{Z(\mathbb{R}).}
\]

**Remark 1.** The index threshold condition in Strichartz estimate \( \frac{2d}{d-2} < \frac{2n}{p_0-n} < 6 \) requires \( 3 < p_0 < \frac{11}{4} \). That is a reason why we have the restriction for \( p_0 \).

**Remark 2.** We have used another version of Strichartz estimate as follows: for \( p > \frac{2d}{d-2} \) and \( q \) as in Theorem 3.1, the following estimate holds for any time interval \( I \subset \mathbb{R} \) and every cube \( Q \subset \mathbb{R}^4 \) of size \( N_1 \):
\[
\| 1_I(t) P_Q u \|_{L^p_{t,x,y}} \lesssim N^{2-d} \| u \|_{L^{20}_{t,x,y} \cap L^{20}_{t,x,y} \cap L^{20}_{t,x,y} (\mathbb{R}^3 \times T)}.
\]

**Remark 3.** By using Strichartz estimate and the embedding properties of the function spaces, it is not hard to verify \( Z \)-norm is weaker than \( X^1 \)-norm.

Based on bilinear estimate, we can prove the following nonlinear estimate which is a crucial step of the local theory. The proof of the following lemmas and theorems follows from standard arguments as in [14, 20, 38].

**Lemma 4.2 (Nonlinear Estimate).** For \( u_i \in X^1(I), i = 1, 2, 3. \) There holds that
\[
\| \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \|_{N(I)} \leq \sum_{(i,j,k)=(1,2,3)} \| u_i \|_{X^1(I)} \| u_j \|_{Z'(I)} \| u_k \|_{Z'(I)}
\]
where \( \tilde{u}_i \) is either \( u_i \) or \( \tilde{u}_i \).

**Proof:** It suffices to prove the following estimate: (Without loss of generality, let \( I = \mathbb{R} \))
\[
\| \sum_{K \geq 1} P_K u_1 \prod_{i=2,3} P_{< CK} \tilde{u}_i \|_{N(\mathbb{R})} \lesssim C \sum_{I=1,2,3} \| u_1 \|_{X^1(\mathbb{R})} \| u_2 \|_{Z'(\mathbb{R})} \| u_3 \|_{Z'(\mathbb{R})}.
\]

Using Theorem 2.1, it suffices to prove for any \( u_0 \in Y^{-1} \) and \( \| u_0 \|_{Y^{-1}} \leq 1 \)
\[
\| \sum_{N_1} \int_{\mathbb{R}^3 \times T \times \mathbb{R}} \tilde{u}_0 P_{N_1} u_1 \prod_{i=2,3} (P_{< CN_1} \tilde{u}_i) dx dy dt \| \| u_0 \|_{Y^{-1}} \| u_1 \|_{X^1(\mathbb{R})} \| u_2 \|_{Z'(\mathbb{R})} \| u_3 \|_{Z'(\mathbb{R})}.
\]

Now we split them as follows, let \( u_i = \sum_{N_i \geq 1} P_{N_1} u_i, i = 0, 1, 2, 3 \), denoting \( u_j^{N_1} = P_{N_1} u_j \) and then the estimate would follow from the following bound:
\[
\sum_{S(N_0,N_1,N_2,N_3)} \| \int u_0^{N_0} u_1^{N_1} u_2^{N_2} u_3^{N_3} dx dy dt \| \| u_0 \|_{Y^{-1}} \| u_1 \|_{X^1(\mathbb{R})} \| u_2 \|_{Z'(\mathbb{R})} \| u_3 \|_{Z'(\mathbb{R})}.
\]

Here we have set of index \( S \) to be \( \{ (N_0,N_1,N_2,N_3) : N_1 \sim \max(N_2,N_0) \geq N_2 \geq N_3 \} \) and we split \( S \) into the disjoint union of \( S_1 \) and \( S_2 \) and \( S_3 \) is for the elements in \( S \) that satisfy \( N_1 \sim N_0 \) and \( S_2 \) if for the elements in \( S \) that satisfy \( N_1 \sim N_2 \). And we will estimate \( S_1 \) and \( S_2 \) separately. We omit the proof for \( S_2 \) part since the estimate is similar.

By using bilinear estimate (4.2) and some basic inequalities and the properties of function spaces, we have, for a term in \( S_1 \):
\[
\| \int u_0^{N_0} u_1^{N_1} u_2^{N_2} u_3^{N_3} dx dy dt \| \leq \| u_0^{N_0} u_2^{N_2} \|_{L^2} \| u_1^{N_1} u_3^{N_3} \|_{L^2} \\
\leq (\frac{N_2}{N_0} + \frac{1}{N_2})^{\gamma} (\frac{N_1}{N_0})^{\delta} \| u_0^{N_0} \|_{Y^p(\mathbb{R})} \| u_1^{N_1} \|_{Y^p(\mathbb{R})} \| u_2^{N_2} \|_{Z'(\mathbb{R})} \| u_3^{N_3} \|_{Z'(\mathbb{R})}.
\]
By using Cauchy-Schwarz inequality, the sum of the terms in $S_1$:

$$S_1 \lesssim \sum_{N_1 \sim N_0} \left( \frac{N_2}{N_0} + \frac{1}{N_2} \delta \frac{N_3}{N_1} + \frac{1}{N_3} \delta \right) \|v_0\|_{Y_0(\mathbb{R})} \|u_1\|_{Y^1(\mathbb{R})} \|u_2\|_{\dot{Z}'(\mathbb{R})} \|u_3\|_{\dot{Z}'(\mathbb{R})}$$

$$\lesssim \left( \sum_{N_1 \sim N_0} \frac{N_0}{N_1} \|v_0\|_{Y^{-1}(\mathbb{R})} \|u_1\|_{Y^1(\mathbb{R})} \|u_2\|_{\dot{Z}'(\mathbb{R})} \|u_3\|_{\dot{Z}'(\mathbb{R})} \right)$$

$$\lesssim \|u_0\|_{Y^{-1}(\mathbb{R})} \|u_1\|_{X^1(\mathbb{R})} \|u_2\|_{\dot{Z}'(\mathbb{R})} \|u_3\|_{\dot{Z}'(\mathbb{R})}.$$ 

This finishes Lemma 4.2.

**Theorem 4.3.** [Local Well-posedness] Let $E > 0$ and $\|u_0\|_{H^1(\mathbb{R}^2 \times [0, T])} < E$, then there exists $\delta_0 = \delta_0(E) > 0$ such that if

$$\|e^{it\Delta}u_0\|_{Z(I)} < \delta$$

for some $\delta \leq \delta_0, 0 \in I$. Then there exists a unique strong solution $u \in X^1(I) \cap X^1([0, T])$ satisfying $u(0) = u_0$ and we can get an estimate,

$$\|u(t) - e^{it\Delta}u_0\|_{X^1(I)} \leq (E\delta)^{\frac{5}{2}}.$$ 

**Remark 1.** Observe that if $u \in X^1(\mathbb{R})$, then $u$ scatters as $t \to \pm \infty$ as in (1.3). Also, if $E$ is small enough, $I$ can be taken to $\mathbb{R}$ which proves the small data scattering of (1.1).

**Proof:** First, we consider a mapping defined as follows,

$$\Phi(u) = e^{it\Delta}u_0 - \int_0^t e^{i(t-s)\Delta}|u(s)|^2u(s)ds.$$ 

And we define a set $B = \{u \in X^1(I) : \|u\|_{X^1(I)} \leq 2E \text{ and } \|u\|_{Z(I)} \leq 2\delta\}$. Now we will verify two properties of $\Phi$: 1. $\Phi$ maps $B$ to $B$. 2. $\Phi$ is a contraction mapping.

1. For $u \in B$, we can use the nonlinear estimate in Lemma 4.2 and let $\delta \leq 1$ small enough to make $E\delta$ small enough, we have:

$$\|\Phi(u)\|_{X^1(I)} \leq \|e^{it\Delta}u_0\|_{X^1(I)} + \|u\|_{N(I)} \leq E + CE\delta \leq 2E,$$

$$\|\Phi(u)\|_{Z(I)} \leq \|e^{it\Delta}u_0\|_{Z(I)} + \|u\|_{N(I)} \leq \delta + CE\delta \leq 2\delta.$$

2. $\|\Phi(u) - \Phi(v)\|_{X^1(I)} \leq \|u - v\|_{X^1(I)} \left( \|u\|_{X^1(I)} + \|v\|_{X^1(I)} \right) \leq C\|u - v\|_{X^1(I)} E\delta \leq C\|u - v\|_{X^1(I)} E\delta \leq C\|u - v\|_{X^1(I)}$.

Thus the result now follows from the Picard’s fixed point argument.

**Theorem 4.4.** [Controlling Norm] Let $u \in X^1_{c, loc}(I)$ be a strong solution on $I \in \mathbb{R}$ satisfying

$$\|u\|_{Z(I)} < \infty.$$ 

Then we have two conclusions,

(1) If $I$ is finite, then $u$ can be extended as a strong solution in $X^1_{c, loc}(I')$ on a strictly larger interval $I' \subset I \subset \mathbb{R}$. In particular, if $u$ blows up in finite time, then the $Z$ norm of $u$ has to blow up.

(2) If $I$ is infinite, then $u \in X^1(I)$.

**Proof:** Without loss of generality, for the finite case we can assume $I = [0, T)$ and we want to extend it to $[0, T + v)$ for some $v > 0$. Denoting $E = \sup_I \|u(t)\|_{H^1(\mathbb{R}^2 \times [0, T])}$ and using the time-divisibility of ‘$Z$-norm’, there exists $T_1$ such that $T - 1 < T_1 < T$ such that

$$\|u\|_{Z([T_1, T])} \leq \epsilon.$$
where $\epsilon$ is to be decided. This allows to conclude:
\[
\|u(t) - e^{i(t-T_1)}\Delta u(T_1)\|_{X^1([T_1,T])} \lesssim \|u\|_{X^1([T_1,T])}^3 \lesssim \|u\|_{Z([T_1, T])}^3 \lesssim C\epsilon^2 \|u\|_{X^1([T_1, T])}^2.
\]
By bootstrap argument, we get,
\[
\|u\|_{X^1([T_1, T])}^2 \lesssim \epsilon.
\]
If $\epsilon$ is small enough and, making $\epsilon$ possibly smaller, we have,
\[
\|e^{i(t-T_1)}\Delta u(T_1)\|_{Z([T_1, T])} \leq \|u\|_{Z([T_1, T])} + \|e^{i(t-T_1)}\Delta u(T_1) - u(t)\|_{Z([T_1, T])}
\leq \epsilon + \|e^{i(t-T_1)}\Delta u(T_1) - u(t)\|_{X^1([T_1, T])}
\leq \epsilon + C\epsilon^2 E^2
\leq \frac{3}{4}\delta_0(E).
\]
Notice that we can let $\epsilon$ small enough s.t. $\epsilon < \frac{1}{4}$ and $\epsilon E < (\frac{1}{2}\delta_0(E))^2$. This allows to find an interval $[T_1, T + \epsilon]$ for which :
\[
\|e^{i(t-T_1)}\Delta u(T_1)\|_{Z([T_1, T + \epsilon])} < \delta_0.
\]
That finishes the proof by using the Theorem 4.3. Moreover, by using the symmetries of the equation, the above argument also covers the case when $I$ is an arbitrary bounded interval.

Now we turn to the infinite case. Without loss of generality, it is enough to consider the case $I = (a, \infty)$. Choosing $T$ to be large enough so that
\[
\|u\|_{Z([T, \infty))} \leq \epsilon,
\]
we get that for any $T' > T$:
\[
\|u(t) - e^{i(t-T)}\Delta u(T)\|_{X^1([T', T])} \lesssim \|u\|_{X^1([T', T])} \|u\|_{Z([T_1, T])} \leq C\epsilon^2 \|u\|_{X^1([T', T])},
\]
which gives that $\|u\|_{X^1([T', T])} \lesssim E$ for any $T' > T$ and we have
\[
\|e^{i(t-T)}\Delta u(T)\|_{Z([T, \infty))} \leq 2\epsilon \leq \delta_0(E)
\]
if $\epsilon$ small enough. The result now follows from by using Theorem 4.3.

**Theorem 4.5.** [Stability Theory] Let $I \in \mathbb{R}$ be an interval, and let $\tilde{u} \in X^1(I)$ solve the approximate solution,
\[
(i\partial_t + \Delta_{\mathbb{R}^3 \times T})\tilde{u} = \rho|\tilde{u}|^2\tilde{u} + e \quad \text{and} \quad \rho \in [0, 1].
\]
Assume that:
\[
\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L^\infty(I, H^1(\mathbb{R}^3 \times T))} \leq M.
\]
There exists $\epsilon_0 = \epsilon_0(M) \in (0, 1]$ such that if for some $t_0 \in I$:
\[
\|\tilde{u}(t_0) - u_0\|_{H^1(\mathbb{R}^3 \times T)} + \|\epsilon\|_{N(I)} \leq \epsilon < \epsilon_0,
\]
then there exists a solution $u(t)$ to the exact equation:
\[
(i\partial_t + \Delta_{\mathbb{R}^3 \times T})u = |u|^2u
\]
with initial data $u_0$ satisfies
\[
\|u\|_{X^1(I)} + \|\tilde{u}\|_{X^1(I)} \leq C(M), \quad \|u - \tilde{u}\|_{X^1(I)} \leq C(M)\epsilon.
\]
**Proof:** The proof is very similar to the proof of [14, Proposition 4.7]. The proof relies tightly on the estimate of Lemma 4.2 (nonlinear estimate) and the trick of division of the intervals.

First, we consider for an interval $J \in I$ s.t. $\|\tilde{u}\|_{Z(J)} \leq \epsilon$ (That is the additional smallness assumption, $\epsilon$ is to be decided). We will prove the theorem under this assumption. By local existing argument for the approximate equation, there exists $\delta_1(M)$ that if
\[
\|e^{i(t-t_*)}\Delta \tilde{u}(t_*)\|_{Z(J)} + \|\epsilon\|_{N(J)} \leq \delta_1
\]
We define the scattering norm for this problem as follows:
\[
\| \tilde{u} - e^{i(t-t_s)\Delta} \tilde{u}(t_s) \|_{X^1(J)} \leq C \| \tilde{u} \|_{X^1(J)}^{\frac{1}{2}} \| \tilde{u} \|_{Z(J)}^{\frac{3}{2}} + \| \epsilon \|_{N(J)}.
\]
We can conclude
\[
\| \tilde{u} \|_{X^1(J)} \lesssim M + 1 \quad \text{and} \quad \| e^{i(t-t_s)\Delta} \tilde{u}(t_s) \|_{Z(J)} \lesssim \epsilon.
\]
if \( \epsilon < \epsilon_1(M) \) is small enough.

Second, let us estimate the difference of the solutions. Consider solution \( u \) with initial data \( u_s \) satisfying \( \| u_s - \tilde{u}(t_s) \|_{H^1} \leq \epsilon \) and living on an interval \( J_{u} \subset J \) containing \( t_s \). And, we want to prove the following estimate for some constant \( C \) independent of \( J_u \) to be specified later:
\[
(4.17) \quad \| u - \tilde{u} \|_{X^1(J_u)} \leq C\epsilon.
\]
Let \( w = u - \tilde{u} \), then we know that \( w \) satisfies:
\[
(i\partial_t + \Delta)w = \rho(|\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2\tilde{u}) - \epsilon.
\]
Adopting the bootstrap hypothesis:
\[
\| w \|_{X^1(I_{\text{new}} \cap [t_*, t_* + t])} \leq 2C\epsilon.
\]
For convenience, we denote \( J_u \cap [t_*, t_* + t] \) by \( J_* \) by using nonlinear estimate, we compute:
\[
\| w \|_{X^1(J_*)} \lesssim \| u(t_*) - \tilde{u}(t_*) \|_{H^1(\mathbb{R}^3 \times \mathbb{T})} + \| w \|_{X^1(J_*)} \| \tilde{u} \|_{X^1(J_*)} \| \tilde{w} \|_{Z(J_*)} + \| \epsilon \|_{N(J_*)} \lesssim \epsilon + C_1 \| w \|_{X^1(J_*)} \| \tilde{u} \|_{X^1(J_*)} \| \tilde{w} \|_{Z(J_*)} \leq C_1 \epsilon + C_1 M \epsilon \| w \|_{X^1(J_*)}.
\]
As a result, if \( \epsilon < \epsilon_1(M) \) with \( \epsilon_1(M) \) small enough in terms of \( M \), we conclude that \( \| u - \tilde{u} \|_{X^1(J_u)} \leq 2C_1 \epsilon \), which close the the bootstrap argument with \( C = 2C_1 \). This finishes the proof under the smallness assumption.

Now, to generalize the argument to the whole interval \( I \), we split \( I \) into \( N = C(M, \epsilon_1(M)) \) intervals \( I_k = [T_k, T_{k+1}] \) such that:
\[
\| u \|_{Z(I_k)} \leq \frac{\epsilon_1(M)}{100} \quad \text{and} \quad \| \epsilon \|_{N(I_k)} \leq \frac{\epsilon_1(M)}{100}.
\]
If \( \epsilon_0(M) \) is chosen sufficiently small in terms of \( N, M \), and \( \epsilon_1(M) \), we can iterate the first part of the proof on each interval \( I_k \) while keeping the condition
\[
\| u(T_k) - \tilde{u}(T_k) \|_{H^1(\mathbb{R}^3 \times \mathbb{T})} + \| \epsilon \|_{N(I_k)} + \| u \|_{Z(I_k)} < \epsilon_1(M)
\]
always satisfied for each \( k \). This finishes the proof of Theorem 4.5.

**For quintic \( \mathbb{R}^2 \times \mathbb{T} \) problem:** we have the following statements for the quintic analogues:

We define the scattering norm for this problem as follows:
\[
(4.18) \quad \| u \|_{Z(I)} = \sum_{p_0 = 8, 10} \left( \sum_{N \geq 1} N^{5 - \frac{p_0}{8}} \| 1_{T}(t) |\nu_{\epsilon_{p_0}} |^{x_p} \right) \| u \|_{L^p_{x_p,t}(\mathbb{R}^2 \times \mathbb{T} \times I_*)}.
\]
For convenience, we define \( Z' \)-norm as a mixture of \( Z \)-norm and \( X^1 \)-norm as follows:
\[
(4.19) \quad \| u \|_{Z'(I)} = \| u \|_{Z(I)}^{\frac{1}{2}} \| u \|_{X^1(I)}^{\frac{1}{2}}.
\]
Now we are ready to prove the local well-posedness and small-data scattering of (1.3). We will prove the following trilinear estimate first:

**Lemma 4.6 (Trilinear estimate).** Suppose that \( u_i = P_N \nu, \) for \( i = 1, 2, 3 \) satisfying \( N_1 \geq N_2 \geq N_3 \). There exists \( \delta > 0 \) such that the following estimate holds for any interval \( I \in \mathbb{R} \):
\[
(4.20) \quad \| u_1 u_2 u_3 \|_{L^2_{x,t}(\mathbb{R}^2 \times \mathbb{T} \times I)} \lesssim \left( \frac{N_3}{N_1} + \frac{N_1}{N_2} \right)^{\delta} \| u_1 \|_{Y_{\epsilon \delta}(I)} \| u_2 \|_{Z'(I)} \| u_3 \|_{Z'(I)}.
\]
Assume that:

\[ \text{Theorem 4.8.} \quad \text{[Local Well-posedness]} \]

Let \( u \) satisfy

\[ \|u_{123}\|_{L^2(\mathbb{R}^2 \times T \times \mathbb{T})} \lesssim N_2 N_3 \left( \frac{N_3}{N_2} + \frac{1}{N_2} \right)^{\delta} \|u_1\|_{Y^0(\mathbb{R})} \|u_2\|_{Y^0(\mathbb{R})} \|u_3\|_{Y^0(\mathbb{R})}. \]

And it suffices to prove the following estimate, if it is hold then we can just combine those two inequalities to get the lemma proved.

\[ \|u_{123}\|_{L^2(\mathbb{R}^2 \times T \times \mathbb{T})} \lesssim \|u_1\|_{Y^0(\mathbb{R})} \|u_2\|_{Z(\mathbb{R})} \|u_3\|_{Z(\mathbb{R})}. \]

By orthogonality considerations, we may replace \( u_1 \) by \( P_C u_1 \) where \( C \) is a cube of dimension \( N_2 \).

By H"older's inequality, we have,

\[
\| (P_C u_1) u_2 u_3 \|_{L^2_{x,t}} \lesssim \| P_C u_1 \|_{L^r_{x,t} L^s_{x,t} L^q_{x,t}} \| u_2 \|_{L^r_{x,t} L^s_{x,t} L^q_{x,t}} \| u_3 \|_{L^r_{x,t} L^s_{x,t} L^q_{x,t}}
\]

\[
\lesssim N_2^\frac{1}{r} \| P_C u_1 \|_{L^r_{x,t}} \| u_2 \|_{L^s_{x,t}} \| u_3 \|_{L^q_{x,t}} \lesssim \| P_C u_1 \|_{L^r_{x,t}} \| u_2 \|_{L^s_{x,t}} \| u_3 \|_{L^q_{x,t}} \lesssim \| P_C u_1 \|_{L^r_{x,t}} \| u_2 \|_{Z(\mathbb{R})} \| u_3 \|_{Z(\mathbb{R})}.
\]

This finishes the proof of (4.20).

Based on the trilinear estimate, we are ready to prove Nonlinear Estimate Lemma, Controlling Norm Theorem and Stability Theorem as follows. We only state those propositions and omit the proofs since the proofs of those propositions are analogues of [14, Lemma 4.3, Proposition 4.5, Proposition 4.6 and Proposition 4.7] once we have the trilinear estimate. (see also [20, 36] and the cubic \( \mathbb{R}^3 \times \mathbb{T} \) case in this paper)

**Lemma 4.7** (Nonlinear Estimate). For \( u_i \in X^1(I), i = 1, 2, 3, 4, 5 \). There holds that

\[
\| \tilde{u} \tilde{v} \tilde{w} \tilde{q} \tilde{r} \|_{N(I)} \leq \sum_{\{i, j, p, m, n\} = \{1, 2, 3, 4, 5\}} \| u_i \|_{X(\mathbb{R})} \| u_j \|_{Y^1(\mathbb{R})} \| u_p \|_{Y^1(\mathbb{R})} \| u_m \|_{Y^1(\mathbb{R})} \| u_n \|_{Y^1(\mathbb{R})}
\]

where \( \tilde{u}_i \) is either \( u_i \) or \( \tilde{u}_i \).

**Theorem 4.8.** [Local Well-posedness] Let \( E > 0 \) and \( \| u_0 \|_{H^s(\mathbb{R}^2 \times T)} < E \), then there exists \( \delta_0 = \delta_0(E) > 0 \) such that if

\[
\| e^{it \Delta} u_0 \|_{Z(\mathbb{R})} < \delta
\]

for some \( \delta \leq \delta_0, \ 0 \in I \). Then there exists a unique strong solution \( u \in X^1(\mathbb{R}) \) satisfying \( u(0) = u_0 \) and we can get an estimate,

\[
\| u(t) - e^{it \Delta} u_0 \|_{X^1(\mathbb{R})} \leq E^2 \delta^3.
\]

**Theorem 4.9.** [Controlling Norm] Let \( u \in X^1_{c, loc}(I) \) be a strong solution on \( I \subseteq \mathbb{R} \) satisfying

\[
\| u \|_{Z(I)} < \infty.
\]

Then we have two conclusions,

(1) If \( I \) is finite, then \( u \) can be extended as a strong solution in \( X^1_{c, loc}(I') \) on a strictly larger interval \( I' \), \( I \subset I' \subset \mathbb{R} \). In particular, if \( u \) blows up in finite time, then the \( Z \)-norm of \( u \) has to blow up.

(2) If \( I \) is infinite, then \( u \in X^1_{c, loc}(\mathbb{R}) \).

**Theorem 4.10.** [Stability Theory] Let \( I \subseteq \mathbb{R} \) be an interval, and let \( \tilde{u} \in X^1(\mathbb{R}) \) solve the approximate solution,

\[
(i \partial_t + \Delta_{\mathbb{R}^2 \times T}) \tilde{u} = \rho |\tilde{u}|^4 \tilde{u} + \varepsilon \quad \text{and} \quad \rho \in [0, 1].
\]

Assume that:

\[
\| \tilde{u} \|_{Z(I)} + \| \tilde{u} \|_{L^\infty_t L^4_x(I; H^1(\mathbb{R}^2 \times T))} \leq M.
\]

There exists \( \epsilon_0 = \epsilon_0(M) \in (0, 1) \) such that if for some \( t_0 \in I \):

\[
\| \tilde{u}(t_0) - u_0 \|_{H^1(\mathbb{R}^2 \times T)} + \| \epsilon \|_{N(I)} \leq \epsilon < \epsilon_0,
\]

\[
\| \tilde{u}(t_0) \|_{Z(I)} + \| \tilde{u} \|_{L^\infty_t L^4_x(I; H^1(\mathbb{R}^2 \times T))} \leq M.
\]

(4.27) There exists \( \epsilon_0 = \epsilon_0(M) \in (0, 1) \) such that if for some \( t_0 \in I \):

(4.28) \( \| \tilde{u}(t_0) - u_0 \|_{H^1(\mathbb{R}^2 \times T)} + \| \epsilon \|_{N(I)} \leq \epsilon < \epsilon_0 \),
then there exists a solution \( u(t) \) to the exact equation:
\[
(i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}})u = |u|^4u
\]
with initial data \( u_0 \) satisfies
\[
||u||_{X^1(I)} + ||u||_{X^1(I)} \leq C(M), \quad ||u - \tilde{u}||_{X^1(I)} \leq C(M)e.
\]

5. Nonlinear Analysis of the Profiles

In this section, we describe and analyze the profiles that appear in the linear and nonlinear profile decomposition.

For cubic \( \mathbb{R}^3 \times \mathbb{T} \) problem:

We recall the motivation discussed in Section 1: In view of the scaling-invariant of the IVP (1.1) under
\[
\mathbb{R}^3 \times T_y \rightarrow M_\lambda := \mathbb{R}^3 \times (\lambda^{-1} \cdot \mathbb{T}) y, \quad u \rightarrow \tilde{u}(x,y,t) = \lambda u(\lambda x, \lambda y, \lambda^2 t).
\]
When \( \lambda \rightarrow 0 \), the manifolds \( M_\lambda \) will be similar to \( \mathbb{R}^4 \). The appearance is a manifestation of the energy-critical nature of the nonlinearity. This extreme behavior corresponds to Euclidean profile. Precise description is as follows.

Remark. We also refer to [14, Section 5], [20, Section 4] and [36, Section 5] for more information. For those problems, Euclidean profiles also appear in the analysis of profile decomposition according to the structures of the corresponding equations.

Euclidean Profiles. The Euclidean profiles define a regime where we can compare solutions of cubic NLS on \( \mathbb{R}^4 \) with those on \( \mathbb{R}^3 \times \mathbb{T} \). We fix a spherically symmetric function \( \eta \in C^\infty_0(\mathbb{R}^4) \) supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Given \( \phi \in H^1(\mathbb{R}^4) \) and a real number \( N \geq 1 \), we define:
\[
Q_N \phi \in H^1(\mathbb{R}^4) \quad (Q_N \phi)(x) = \eta(\frac{x}{N^\frac{3}{2}})\phi(x),
\]
\[
\phi_N \in H^1(\mathbb{R}^4) \quad \phi_N(x) = N(Q_N \phi)(Nx),
\]
\[
f_N \in H^1(\mathbb{R}^3 \times \mathbb{T}) \quad f_N(y) = \phi_N(\Psi^{-1}(y)),
\]
where \( \Psi \) is the identity map from the unit ball of \( \mathbb{R}^4 \) to \( \mathbb{R}^3 \times \mathbb{T} \). Thus \( Q_N \phi \) is a compactly supported modification of the profile \( \phi \), \( \phi_N \) is an \( H^1 \)-invariant rescaling of \( Q_N \phi \), and \( f_N \) is the function obtained by transferring \( \phi_N \) to a neighborhood of 0 in \( \mathbb{R}^3 \times \mathbb{T} \). We notice that
\[
||f_N||_{H^1(\mathbb{R}^3 \times \mathbb{T})} \lesssim ||\phi||_{H^1(\mathbb{R}^4)}.
\]
And we use scattering result for 4d energy critical NLS by E. Ryckman and M. Visan ([26]) in the following form:

Theorem 5.1. Assume \( \psi \in \dot{H}^1(\mathbb{R}^4) \), then there is a unique global solution \( v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4)) \) of the initial-value problem
\[
(i\partial_t + \Delta_{\mathbb{R}^4})v = |v|^4v, \quad v(0) = \psi,
\]
and
\[
||\nabla_{\mathbb{R}^4}v||_{\{L^2_t L^2_y, \cap L^4_t L^4_y\}(\mathbb{R}^4 \times \mathbb{R})} \leq \hat{C}(E_{\mathbb{R}^4}(\psi)).
\]
Moreover this solution scatters in the sense that there exists \( \psi^{\pm \infty} \in \dot{H}^1(\mathbb{R}^4) \) such that
\[
||v(t) - e^{it\Delta} \psi^{\pm \infty}||_{H^1(\mathbb{R}^4)} \rightarrow 0
\]
as \( t \rightarrow \pm \infty \). Besides if \( \psi \in H^5(\mathbb{R}^4) \), then \( v \in C(\mathbb{R} : H^5(\mathbb{R}^4)) \) and
\[
\sup_{t \in \mathbb{R}} ||v(t)||_{H^5(\mathbb{R}^4)} \lesssim ||\psi||_{H^5(\mathbb{R}^4)} 1.
\]
Based on the above result, we have:
Theorem 5.2. Assume \( \phi \in H^1(\mathbb{R}^4) \), \( T_0 \in (0, \infty) \), and \( \rho \in \{0, 1\} \) are given, and we define \( f_N \) as before. Then the following conclusions hold:

1. There is \( N_0 = N_0(\phi, T_0) \) sufficiently large such that for any \( N \geq N_0 \), there is a unique solution \( U_N \in C((-T_0N^{-2}, T_0N^{-2}); H^1(\mathbb{R}^3 \times \mathbb{T})) \) of the initial-value problem
   \[
   (i\partial_t + \Delta)U_N = \rho U_N|U_N|^2, \quad U_N(0) = f_N.
   \]
   Moreover, for any \( N \geq N_0 \),
   \[
   \|U_N\|_{C^1((-T_0N^{-2}, T_0N^{-2}); H^1(\mathbb{R}^3 \times \mathbb{T}))} \lesssim E_{\phi}(\phi).
   \]

2. Assume \( \epsilon_1 \in (0, 1] \) is sufficiently small (depending only on \( E_{\phi}(\phi) \)), \( \phi' \in H^5(\mathbb{R}^4) \), and \( \|\phi - \phi'\|_{H^4(\mathbb{R}^4)} \leq \epsilon_1 \). Let \( v \in C(\mathbb{R} : H^5) \) denote the solution of the initial-value problem
   \[
   (i\partial_t + \Delta)\dot{v} = \rho v|v|^2, \quad \dot{v}(0) = \phi'.
   \]
   For \( R \geq 1 \) and \( N \geq 10R \), we define
   \[
   v_R'(x, t) = e^{\frac{x}{R}}(v'(x, t) - \rho|v'_R|^2 v'_R),
   \]
   \[
   v_{R,N}'(x, t) = N v_R'(Nx, N^2t), \quad V_{R,N}(y, t) = v_{R,N}'(\Psi^{-1}(y), t),
   \]
   \[
   V_{R,N}(y, t) = v_{R,N}'(\Psi^{-1}(y), t) \in \mathbb{R} \times \mathbb{T} \times (-T_0N^{-2}, T_0N^{-2}).
   \]
   Then there is \( R_0 \geq 1 \) such that, for any \( R \geq R_0 \) and \( N \geq 10R \),
   \[
   \lim_{N \to \infty} \sup_{t} \|U_N - V_{R,N}\|_{C^1((-T_0N^{-2}, T_0N^{-2}); H^1(\mathbb{R}^3 \times \mathbb{T}))} \lesssim E_{\phi}(\phi) \epsilon_1.
   \]

Proof: It suffices to prove part (2). All implicit constants are allowed to depend on \( \|\phi\|_{H^1(\mathbb{R}^4)} \). The idea of the proof is to show that with \( R_0 \) chosen large enough, \( V_{R,N} \) is an approximate solution. First, we define:

\[
\dot{e}_R(x, t) := (i\partial_t + \Delta_{R^4})\dot{v}'_R - \rho|v'_R|^2 \dot{v}'_R.
\]

Using the fact that \( \sup_t \|\dot{v}'(t)\|_{H^2} \lesssim \|\phi'\|_{H^5} \), we get:

\[
\|\dot{e}_R(x, t)\|_{L^2} \lesssim \|\phi'\|_{H^5}.
\]

which directly gives that there exists \( R_0 \geq 1 \) such that for all \( R > R_0 \)

\[
\lim_{R \to \infty} \||\dot{e}_R| + |\nabla_{\mathbb{R}^4} \dot{e}_R||_{L^1L^2(\mathbb{R}^4 \times (-T,T))} = 0.
\]

Letting

\[
\dot{e}_{R,N}(x, t) := (i\partial_t + \Delta_{R^4})v_{R,N}' - \rho|v_{R,N}'|^2 v_{R,N},
\]

we have that for any \( R > R_0 \) and \( N \geq 1 \):

\[
\|\dot{e}_{R,N} + |\nabla_{\mathbb{R}^4} \dot{e}_{R,N}||_{L^1L^2(\mathbb{R}^4 \times (-T, T))} \leq 2\epsilon_1
\]

with \( V_{R,N} \) defined on \( \mathbb{R}^3 \times \mathbb{T} \times (-TN^{-2}, TN^{-2}) \). We let

\[
E_{R,N}(y, t) = (i\partial_t + \Delta_{R^4})V_{R,N} - \rho V_{R,N}'^2 V_{R,N} = e_{R,N}(\Psi^{-1}(y), t).
\]

For \( R > R_0 \) and \( N \geq 10R \):

\[
\|E_{R,N} + |\nabla_{\mathbb{R}^4} E_{R,N}||_{L^1L^2(\mathbb{R}^4 \times (-T, T))} \lesssim \epsilon_1
\]

from which it follows (using Theorem 2.1) that:

\[
\|E_{R,N}\|_{L^\infty(\mathbb{R}^3 \times (-T, T))} \lesssim \epsilon_1.
\]

To verify the requirements of Theorem 4.5, we use (5.3) to conclude that:

\[
\|V_{R,N}\|_{L^\infty H^1(\mathbb{R}^3 \times (-T, T))} \lesssim 1.
\]

As for the \( Z \)-norm control, we choose \( N \) to be big enough so that \( TN^{-2} \leq \frac{1}{4} \) which makes all summations in the \( Z \)-norm consist of at most two terms, after which we estimate the \( Z \)-norm by using Littlewood-Paley theory and Sobolev embedding theorem as follows:

\[
\|K_{\phi}^{-1}P_k V_{R,N}\|_{L^p(\mathbb{R}^3 \times (-T, T))} \lesssim \|((1 - \Delta)^{\frac{1}{2}} V_{R,N}\|_{L^p_{\mathbb{R}^4}} \lesssim \|((1 - \Delta)^{\frac{1}{4}} v_{R,N}'\|_{L^p_{\mathbb{R}^4}} \lesssim E(\phi) \|
\]

1.
At last, we know for $R_0$ big enough and $R > R_0$, $N \geq 10R$,
\[
\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{R}^3 \times T)} \lesssim \|Q_N \phi - \phi\|_{H^1(\mathbb{R}^3)} + \|\phi' - \phi\|_{H^1(\mathbb{R}^3)} + \|\phi' - V_R(0)\|_{H^1(\mathbb{R}^3)} \lesssim \epsilon_1.
\]
This completes the verification of the requirements of Theorem 4.5 which concludes the proof.

**Lemma 5.3** (Extinction Lemma). Suppose that $\phi \in \dot{H}^1(\mathbb{R}^4)$, $\epsilon > 0$, and $I \subset \mathbb{R}$ is an interval. Assume that
\[
\|(\phi)\|_{H^1(\mathbb{R}^1)} \leq 1, \quad \|\nabla_x e^{it\Delta} \phi\|_{L^2_T L^2_x(\mathbb{R}^3 \times I)} \leq \epsilon.
\]
For $N \geq 1$, we define as before:
\[Q_N \phi = \eta(N^{-1/2})\phi(x), \quad \phi = N(Q_N \phi)(Nx), \quad f_N(y) = \phi(N(\Psi^{-1}(y))).\]
Then there exists $N_0 = N_0(\phi, \epsilon)$ such that for any $N \geq N_0$,
\[
\|e^{it\Delta} f_N\|_{Z(N^{-2}I)} \lesssim \epsilon.
\]

**Proof**: It suffices to prove that there exists $T_0$ such that for any $N > 1$:
\[
\|e^{it\Delta} f_N\|_{Z(\mathbb{R}\backslash (-N^{-2}T_0,N^{-2}T_0))} \lesssim \epsilon
\]
as the rest follows from Lemma 5.2 (with $\rho = 0$). Without loss of generality, by limiting arguments, we may assume that $\phi \in C_0^\infty(\mathbb{R}^4)$. We have, for any $p$,
\[
f_{N,p}(x) = \frac{1}{2\pi} \int_{\mathbb{T}^3} \phi(x,y)e^{-i(y,p)\cdot dy} = \frac{N}{2\pi} \int_{\mathbb{R}^3} e^{-i(y,p)\cdot \phi(Nx,Ny)}dy.
\]
And using dispersive estimate and unitarity, we have
\[
\|e^{it\Delta} P_M f_N(t)\|_{L^\infty_T L_x^2(\mathbb{R}^3 \times T)} \lesssim \sup_{x \in \mathbb{R}^3} \sum_{|p| \leq M} \|e^{it\Delta} f_{N,p}(x)\| \lesssim \frac{M}{|t|^1} \|f_N\|_{L^2_x} \lesssim \frac{MN^{-3}}{|t|^1}
\]
and
\[
\|e^{it\Delta} P_M f_N(t)\|_{L^\infty_T L_x^2(\mathbb{R}^3 \times T)} \lesssim \|P_M f_N(t)\|_{L^\infty_T L_x^2(\mathbb{R}^3 \times T)} \lesssim M^{-1} \|1 - \Delta\|_{L^2(\mathbb{R}^4)} \lesssim M^{-1} N^{-1}.
\]
Then by interpolation we have (choose $l = 0, 10000$):
\[
\|e^{it\Delta} P_M f_N(t)\|_{L^\infty_T L^2_x(\mathbb{R}^3 \times T)} \lesssim \frac{N^{-2+\frac{4}{l}}}{|t|^{1-\frac{1}{p}}} \sup_{x \in \mathbb{R}^3} \left( \frac{M}{N} \right)^{1-\frac{4}{l} - \frac{1}{2}} \lesssim \frac{N^{-2+\frac{4}{l}}}{|t|^{1-\frac{1}{p}}} \min\left( \frac{M}{N} \right)^{1-\frac{4}{l} - \frac{1}{2}}, \frac{N^{100}}{M^{100}}.
\]
As a result,
\[
\left( \sum_M M^\frac{4}{l} \|e^{it\Delta} P_M f_N\|_{L^\infty_T L_x^2(\mathbb{R}^3 \times (N^2 \Theta \times T))} \right)^{\frac{1}{l}} \lesssim \|\phi_N\|_{H^1} \lesssim 1.
\]
By interpolation, we can obtain (5.11). That finishes the proof of Lemma 5.3.

Now we are ready to describe the nonlinear solutions of (1.1) corresponding to data concentrating at a point. Let $\mathcal{F}_c$ denote the set of renormalized Euclidean frames as follows:
\[
\mathcal{F}_c := \{ (N_k, t_k, x_k)_{k \geq 1} : N_k \in [1, \infty), x_k \in \mathbb{R}^3 \times T, N_k \to \infty, \text{ and either } t_k = 0 \text{ for any } k \geq 1 \text{ or } \lim_{k \to \infty} N_k^2 |t_k| = \infty \}.
\]
Given $f \in L^2(\mathbb{R}^3 \times T)$, $t_0 \in \mathbb{R}$, and $x_0 \in \mathbb{R}^3 \times T$, we define:
\[
\pi_{x_0} f = f(x - x_0), \quad \Pi_{(t_0,x_0)} f = (e^{-it_0 \Delta} f)(x - x_0) = \pi_{x_0} e^{it_0 \Delta} f.
\]
Also for $\phi \in \dot{H}^1(\mathbb{R}^4)$ and $N \geq 1$, we denote the function obtained in (5.1) by:
\[
T^p_{N_k} := N \bar{\phi}(N \Psi^{-1}(x)) \text{ where } \bar{\phi}(y) := \eta\left( \frac{y}{N \sqrt{2}} \right) \phi(y)
\]
and as before observe that $T_N^e : \dot{H}^1(\mathbb{R}^4) \to H^1(\mathbb{R}^3 \times \mathbb{T})$ with \(\|T_N^e \phi\|_{H^1(\mathbb{R}^3 \times \mathbb{T})} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}\).

**Theorem 5.4.** Assume that $\mathcal{O} = (N_k, t_k, x_k) \in \mathcal{F}_e$, $\phi \in \dot{H}^1(\mathbb{R}^4)$, and let $U_k(0) = \Pi_{t_k,x_k}(T_N^e \phi)$:

1. For $k$ large enough, there is a nonlinear solution $U_k \in X^1(\mathbb{R})$ of the equation (1.2) satisfying:

\[
\|U_k\|_{X^1(\mathbb{R})} \lesssim e_{k}(\phi) 1.
\]

2. There exists a Euclidean solution $u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of

\[
(i\partial_t + \Delta_{\mathbb{R}^4})u = |u|^2u
\]

with scattering data $\phi^{\pm \infty}$ defined as in (5.4) such that up to a subsequence: for any $\epsilon > 0$, there exists $T(\phi, \epsilon)$ such that for all $T \geq T(\phi, \epsilon)$ there exists $R(\phi, \epsilon, T)$ such that for all $R \geq R(\phi, \epsilon, T)$, there holds that

\[
\|U_k - \bar{u}_k\|_{X^1(\{t \leq TN^2 \} \times \mathbb{R}^3)} \leq \epsilon,
\]

for $k$ large enough, where

\[
(\pi_{-x_k} \bar{u})(x, t) = N_k \eta(N_k \Psi^{-1}(x)/R)u(N_k \Psi^{-1}(x), N_k^2(t - t_k)).
\]

In addition, up to a subsequence,

\[
\|U_k(t) - \Pi_{(t_k, x_k)} T_N^e \phi^{\pm \infty}\|_{X^1(\{t \leq TN^2 \} \times \mathbb{R}^3)} \leq \epsilon
\]

for $k$ large enough (depending on $(\phi, \epsilon, T, R)$).

**Proof:** This theorem is the analogue of [38, Theorem 5.4] (see also [14, 20]) and the proofs are similar so we omit it.

**For quintic $\mathbb{R}^2 \times \mathbb{T}$ problem:**

We can also consider Euclidean profiles for the 3 dimensional case as follows. Since Theorem 5.6, Lemma 5.7 and Theorem 5.8 are analogues of Theorem 5.2, Lemma 5.3 and Theorem 5.4 respectively and the proofs are similar, we will omit the proofs except for the scattering norm control in Lemma 5.7. (see also [14, section 5], [20, section 5] and [38, section 5])

**Euclidean Profiles.** The Euclidean profiles define a regime where we can compare solutions of cubic NLS on $\mathbb{R}^3$ with those on $\mathbb{R}^2 \times \mathbb{T}$. We fix a spherically symmetric function $\eta \in C_0^\infty(\mathbb{R}^3)$ supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Given $\phi \in H^1(\mathbb{R}^3)$ and a real number $N \geq 1$, we define:

\[
Q_N \phi \in H^1(\mathbb{R}^3) \quad (Q_N \phi)(x) = \eta(\frac{x}{N})\phi(x)
\]

\[
\phi_N \in H^1(\mathbb{R}^3) \quad \phi_N(x) = N^2(Q_N \phi)(Nx)
\]

\[
f_N \in H^1(\mathbb{R}^2 \times \mathbb{T}) \quad f_N(y) = \phi_N(\Psi^{-1}(y)),
\]

where $\Psi$ is the identity map from the unit ball of $\mathbb{R}^3$ to $\mathbb{R}^2 \times \mathbb{T}$. Thus $Q_N \phi$ is a compactly supported modification of the profile $\phi$, $\phi_N$ is an $H^1$-invariant rescaling of $Q_N \phi$, and $f_N$ is the function obtained by transferring $\phi_N$ to a neighborhood of 0 in $\mathbb{R}^2 \times \mathbb{T}$. Notice that

\[
\|f_N\|_{H^1(\mathbb{R}^2 \times \mathbb{T})} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}.
\]

and for this case, we can apply the scattering result for 3d energy critical NLS by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao ([9]) in the following form:

**Theorem 5.5.** Assume $\psi \in \dot{H}^1(\mathbb{R}^3)$, then there is a unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^3))$ of the initial-value problem

\[
(i\partial_t + \Delta_{\mathbb{R}^3})v = v|v|^4, \quad v(0) = \psi,
\]

and

\[
\|\nabla_{\mathbb{R}^3} v\|_{L_T^\infty L_x^2(\mathbb{R}^3 \times \mathbb{R})} \leq \tilde{C}(E_{\mathbb{R}^3}(\psi)).
\]

Moreover this solution scatters in the sense that there exists $\psi^{\pm \infty} \in \dot{H}^1(\mathbb{R}^3)$ such that

\[
\|v(t) - e^{it\Delta} \psi^{\pm \infty}\|_{H^1(\mathbb{R}^3)} \to 0
\]
as \( t \to \pm \infty \). Besides if \( \psi \in H^5(\mathbb{R}^3) \), then \( v \in C(\mathbb{R} : H^5(\mathbb{R}^3)) \) and 
\[
\sup_{t \in \mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^3)} \lesssim \|v\|_{H^5(\mathbb{R}^3)} 1.
\]
Based on the above result, we have:

**Theorem 5.6.** Assume \( \phi \in \dot{H}^1(\mathbb{R}^3), T_0 \in (0, \infty), \) and \( \rho \in \{0,1\} \) are given, and we define \( f_N \) as before. Then the following conclusions hold:

1. There is \( N_0 = N_0(\phi, T_0) \) sufficiently large such that for any \( N \geq N_0 \), there is a unique solution \( U_N \in C((-T_0N^{-2}, T_0N^{-2}); H^1(\mathbb{R}^2 \times T)) \) of the initial-value problem

\[
(i\partial_t + \Delta)U_N = \rho U_N |U_N|^4, \quad U_N(0) = f_N.
\]

Moreover, for any \( N \geq N_0 \),

\[
\|U_N\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim E_{as}(\phi) 1.
\]

2. Assume \( \epsilon_1 \in (0,1] \) is sufficiently small (depending only on \( E_{as}(\phi) \)), \( \phi' \in H^5(\mathbb{R}^3) \), and \( \|\phi - \phi'\|_{H^1(\mathbb{R}^3)} \leq \epsilon_1 \). Let \( v \in C(\mathbb{R} : H^5) \) denote the solution of the initial-value problem

\[
(i\partial_t + \Delta)\nu' = \rho \nu'|\nu'|^4, \quad \nu(0) = \phi'.
\]

For \( R \geq 1 \) and \( N \geq 10R \), we define

\[
v_R(x,t) = \eta(x/R)v(x,t), \quad (x,t) \in \mathbb{R}^3 \times (-T_0, T_0),
\]

\[
v_{R,N}(x,t) = N^2v_R(Nx, N^2t), \quad (x,t) \in \mathbb{R}^3 \times (-T_0N^{-2}, T_0N^{-2}),
\]

\[
V_{R,N}(y,t) = v_{R,N}(\Psi^{-1}(y), t), \quad (y,t) \in \mathbb{R}^2 \times T \times (-T_0N^{-2}, T_0N^{-2}).
\]

Then there is \( R_0 \geq 1 \) (depending on \( T_0 \) and \( \phi' \) and \( \epsilon_1 \)) such that, for any \( R \geq R_0 \) and \( N \geq 10R \),

\[
\limsup_{N \to \infty} \|U_N - V_{R,N}\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim E_{as}(\phi) \epsilon_1.
\]

**Lemma 5.7.** Suppose that \( \phi \in \dot{H}^1(\mathbb{R}^3), \epsilon > 0, \) and \( I \subset \mathbb{R} \) is an interval. Assume that

\[
||\phi||_{H^1(\mathbb{R}^3)} \leq 1, \quad ||\nabla x e^{i\Delta} \phi||_{L^2_T L^6_I(\mathbb{R}^3 \times I)} \leq \epsilon.
\]

For \( N \geq 1 \), we define as before:

\[
Q_N \phi = \eta(N^{-1/2}x)\phi(x), \quad \phi_N = N^2(Q_N \phi)(Nx), \quad f_N(y) = \phi_N(\Psi^{-1}(y)).
\]

Then there exists \( N_0 = N_0(\phi, \epsilon) \) such that for any \( N \geq N_0 \),

\[
||e^{i\Delta} f_N||_{Z(N^{-2}I)} \lesssim \epsilon.
\]

**Proof.** By using similar analysis as in [14, Lemma 5.3] (see also [38, Lemma 5.3] and Lemma 5.3 in this paper), it suffices to prove: there exists \( T_0 \) such that for any \( N > 1 \):

\[
||e^{i\Delta} f_N||_{Z(R \setminus (-T_0N^{-2}, T_0N^{-2}))} \lesssim \epsilon
\]

and we can obtain:

\[
(\sum_M M^{\frac{1}{2}} ||e^{i\Delta} P_M f_N||_{L^2_T L^6_I(\mathbb{R}^3 \times I)}^2) \frac{1}{2} \lesssim T^{-\frac{7}{2}}.
\]

Also, by using Stricharz estimate, for \( p > 10 \), we have

\[
(\sum_M M^{\frac{-1}{2}} ||e^{i\Delta} P_M f_N||_{L^2_T L^6_I(\mathbb{R}^3 \times I)}^2) \frac{1}{2} \lesssim ||\phi_N||_{H^1} \lesssim 1.
\]

By interpolation, we can obtain (5.33). That finishes the proof of Lemma 5.7.
Theorem 5.8. Assume that $\mathcal{O} = (N_k, t_k, x_k) \in \tilde{F}_\varepsilon$, $\phi \in \hat{H}^1(\mathbb{R}^3)$, and let $U_k(0) = \Pi_{t_k,x_k}(T_{N_k}^r \phi)$:

(1) For $k$ large enough, there is a nonlinear solution $U_k \in X^1(\mathbb{R})$ of the equation (1.3) satisfying:

$$\|U_k\|_{X^1(\mathbb{R})} \lesssim E_{\phi, \varepsilon}(\phi)$$

(2) There exists a Euclidean solution $u \in C(\mathbb{R} : \hat{H}^1(\mathbb{R}^3))$ of

$$i\partial_t + \Delta_{\mathbb{R}^3} u = |u|^4 u$$

with scattering data $\phi^{k,\infty}$ defined as in (5.4) such that up to a subsequence: for any $\varepsilon > 0$, there exists $T(\phi, \varepsilon)$ such that for all $T \geq T(\phi, \varepsilon)$ there exists $R(\phi, \varepsilon, T)$ such that for all $R \geq R(\phi, \varepsilon, T)$, there holds that

$$\|U_k - \tilde{u}_k\|_{X^1(\{|t-t_k| \leq TN_k^2\})} \leq \varepsilon,$$

for $k$ large enough, where

$$\pi_{-x_k} \tilde{u}(x, t) = N_k \eta(N_k \Psi^{-1}(x)/R)u(N_k \Psi^{-1}(x), N_k^2 (t - t_k)).$$

In addition, up to a subsequence,

$$\|U_k(t) - \Pi_{(t_k-t, x_k)} T_{N_k}^r \phi^{k,\infty}\|_{X^1(\{|t-t_k| \leq TN_k^2\})} \leq \varepsilon$$

for $k$ large enough (depending on $(\phi, \varepsilon, T, R)$).

6. Profile Decomposition

Definition 6.1 (Frames and Profiles). (1) We define a frame to be sequence $(N_k, t_k, p_k) \in 2^k \times \mathbb{R} \times (\mathbb{R}^3 \times T)$. And we can define some types of profiles as follows.

a) A Euclidean frame is a sequence $(N_k, t_k, p_k)$ with $N_k \geq 1$, $N_k \to \infty, t_k \in \mathbb{R}, p_k \in \mathbb{R}^3 \times T$.

b) A Scale-one frame is a sequence $F_1 = (1, t_k, p_k)$ with $t_k \in \mathbb{R}, p_k \in \mathbb{R}^3 \times T$.

(2) We say that two frames $(N_k, t_k, p_k)$ and $(M_k, s_k, q_k)$ are orthogonal if

$$\lim_{k \to +\infty} \left(\|N_k M_k\| + N_k^2 |t_k - s_k| + N_k |p_k - q_k|\right) = +\infty.$$

(3) We associate a profile defined as:

a) If $\mathcal{O} = (N_k, t_k, p_k)$ is a Euclidean frame and for $\phi \in \hat{H}^1(\mathbb{R}^4)$ we define the Euclidean profile associated to $(\phi, \mathcal{O})$ as the sequence $\tilde{\phi}_{\mathcal{O}, k}$ with

$$\tilde{\phi}_{\mathcal{O}, k} = \Pi_{t_k, p_k} (T_{N_k}^r \phi)(x, y).$$

b) If $\mathcal{O} = (1, t_k, p_k)$ is a scale one frame, if $W \in \hat{H}^1(\mathbb{R}^3 \times T)$, we define the scale one profile associated to $(W, \mathcal{O})$ as $\tilde{W}_{\mathcal{O}, k}$ with

$$\tilde{W}_{\mathcal{O}, k} = \Pi_{t_k, p_k} W.$$

(4) Finally, we say that a sequence of functions $\{f_k\} \subset H^1(\mathbb{R}^3 \times T)$ is absent from a frame $\mathcal{O}$ if, up to a subsequence:

$$\langle f_k, \tilde{\psi}_{\mathcal{O}, k} \rangle_{H^1 \times H^1} \to 0$$

as $k \to \infty$ for any profile $\tilde{\psi}_{\mathcal{O}, k}$ associated with $\mathcal{O}$.

Remark 1. The definition for the case of quintic $\mathbb{R}^2 \times T$ is quite similar. We can replace $\mathbb{R}^3 \times T$ by $\mathbb{R}^2 \times T$ in the Definition 6.1 so we will not repeat it again.

Remark 2. It is very convenient to use the language of frames and profiles to unify Euclidean profiles and scale-one profiles. There are some useful properties about the equivalence of frames. We refer [14, 18, 20, 38] for more information.

For cubic $\mathbb{R}^3 \times T$ problem:

Now, let us state a core lemma which is a key step of the main theorem in this section (profile decomposition). First, for a bounded sequence of functions $\{f_k\} \subset H^1(\mathbb{R}^3 \times T)$, we define the following functional based on a Besov norm:

$$(6.1) \quad \Lambda_{\infty}(\{f_k\}) = \lim_{k \to \infty} \sup_{N \to \infty} \sup_{t \in T, x \in \mathbb{R}^3} \sup_{y \in \mathbb{R}^3} N^{-1} |(e^{it\Delta} P_N f_k)(x, y)|.$$

Given a uniformly bounded sequence in $H^1$, we will extract some profiles whose Besov norms are big to ensure the Besov norm of the linear propagation of the remainder flow is small.
Lemma 6.2. Let $v > 0$. Assume that $\phi_k$ is a sequence satisfying $\|\phi_k\|_{H^1(\mathbb{R}^3)} < E$, then there exists a subsequence of $\phi_k$ (for convenience, we still use $\phi_k$), A Euclidean profiles $\tilde{\varphi}^{\alpha_{\phi}}_{\phi, k}$, and A scale-one profiles $\tilde{W}^{\beta}_{\phi, k}$ such that, for any $k \geq 0$ in the subsequence

$$
\phi_k(x, y) = \phi_k(x, y) - \sum_{1 \leq \alpha \leq A} \tilde{\varphi}^{\alpha}_{\phi, k} - \sum_{1 \leq \beta \leq A} \tilde{W}^{\beta}_{\phi, k}
$$

satisfies

$$
\Lambda_{\infty}(\{\phi_k\}) < v.
$$

Besides, all the frames involved are pairwise orthogonal and $\phi_k'$ is absent from all these frames.

Proof. The proof is similar to [14, Lemma 6.5], [20, Lemma 5.4] and [38, Lemma 6.4]. We omit it.

Theorem 6.3 (Profile decomposition). Assume $\{\phi_k\}_k$ is a sequence of functions satisfying $\|\phi_k\|_{H^1(\mathbb{R}^3)} < E$, up to a subsequence, then there exists a sequence of Euclidean profiles $\tilde{\varphi}^{\alpha_{\phi}}_{\phi, k}$, and scale-one profiles $\tilde{W}^{\beta}_{\phi, k}$ such that, for any $J \geq 0$

$$
\phi_k(x, y) = \sum_{1 \leq \alpha \leq J} \tilde{\varphi}^{\alpha}_{\phi, k} + \sum_{1 \leq \beta \leq J} \tilde{W}^{\beta}_{\phi, k} + R^J_k
$$

where $R^J_k$ is absent from the frames $\mathcal{O}_{\alpha}$ and satisfies

$$
\limsup_{J \to \infty} \Lambda_{\infty}(\{R^J_k\}) = 0
$$

Additionally, we have the following orthogonal relation

$$
\|\phi_k\|_{L^2}^2 = \sum_{\alpha} \|\tilde{\varphi}^{\alpha}_{\phi, k}\|_{L^2}^2 + \sum_{\beta} \|\tilde{W}^{\beta}_{\phi, k}\|_{L^2}^2 + \|R^J_k\|_{L^2}^2 + o_k(1),
$$

$$
\|\nabla \phi_k\|_{L^2}^2 = \sum_{\alpha} \|\nabla \tilde{\varphi}^{\alpha}_{\phi, k}\|_{L^2}^2 + \sum_{\beta} \|\nabla \tilde{W}^{\beta}_{\phi, k}\|_{L^2}^2 + \|\nabla R^J_k\|_{L^2}^2 + o_k(1),
$$

$$
\|\phi_k\|_{L^4}^4 = \sum_{\alpha} \|\tilde{\varphi}^{\alpha}_{\phi, k}\|_{L^4}^4 + \sum_{\beta} \|\tilde{W}^{\beta}_{\phi, k}\|_{L^4}^4 + o_J k(1).
$$

Proof: The proof is similar as in [20, Proposition 5.5]. We omit it. It mainly follows from Lemma 6.1, Lemma 2.2 and some frame equivalence properties.

The following lemma is crucial, which explains how to use the Bosov norm to control the scattering norm of linear Schrödinger propagation.

Lemma 6.4. Assume a sequence $\{f_k\}_k$ satisfying $\sup_k \|f_k\|_{H^1} < E$, we have the following estimate:

$$
\lim_{k \to \infty} \sup_k \|e^{it\Delta} f_k\|_{Z(R)} \lesssim_E (\Lambda_{\infty}(\{f_k\}))^\delta,
$$

where $\delta$ is some positive constant.

Proof. By using interpolation and Stricharz estimate,

$$
\|e^{it\Delta} f_k\|_{Z(R)}^p = \sum_{N \geq 1} N^{d-p} \|1_{I(t)} P_N u\|_{L^{20}_{t,x} L^{\infty}_{x,y,z}}^p
$$

$$
\lesssim \sum_{N} (N^{-1} \|P_N e^{it\Delta} f_k\|_{L^{20}_{t,x}})^{\frac{p}{20}} (N^{-1} \|P_N e^{it\Delta} f_k\|_{L^{20}_{t,x}})^{\frac{p}{20}} (N^{-1} \|f_k\|_{L^2})^{\frac{p}{20}}
$$

$$
\lesssim (\sup_N N^{-1} \|P_N e^{it\Delta} f_k\|_{L^{20}_{t,x}})^{\frac{20}{p}} \|f_k\|_{L^2}^{\frac{p}{20}}
$$

$$
\lesssim_E (\sup_N N^{-1} \|P_N e^{it\Delta} f_k\|_{L^{20}_{t,x}})^{\frac{20}{p}}.
$$

That finishes the proof of Lemma 6.4 noticing that $\|f_k\|_{H^1}$ is uniformly bounded by $E$. 
Remark 1. The range of $p_0$ ensures $2 \frac{2}{\gamma} < \frac{2m}{p} < 6$ so that we can use Strichartz estimate (Theorem 3.1) in the proof.

Remark 2. If we replace $f_k$ by $R^J_k$ which is the remainder term in the profile decomposition, we can obtain the control of the scattering norm of linear Schrödinger propagation of the remainder flow.

Remark 3. We are mainly inspired by [14] and [20]. In those papers, the authors have used similar techniques to estimate the remainder flow.

For quintic $\mathbb{R}^2 \times \mathbb{T}$ problem:

First, we state a core lemma which is a key step of the main theorem in this section (profile decomposition). First, for a bounded sequence of functions $\{f_k\}$ in $H^1(\mathbb{R}^2 \times \mathbb{T})$, we define the following functional based on a Besov norm:

$$(6.8) \quad \Lambda_\infty(\{f_k\}) = \limsup_{k \to \infty} \|e^{it\Delta} f_k\|_{L^p_t L^q_x} = \limsup_{k \to \infty} \sup_{N,t,x,y} N^{-\frac{1}{2}} \|(e^{it\Delta} P_N f_k)(x,y)\|.$$ 

Given a uniformly bounded sequence in $H^1$, we will extract some profiles whose Besov norms are big to ensure the Besov norm of the linear propagation of the remainder flow is small.

Lemma 6.5. Let $\nu > 0$. Assume that $\phi_k$ is a sequence satisfying $\|\phi_k\|_{H^1(\mathbb{R}^2 \times \mathbb{T})} < E$, then there exists a subsequence of $\phi_k$ (for convenience, we still use $\phi_k$), a Euclidean profiles $\varphi^\alpha_{\mathcal{O}^\alpha,k}$, and a scale-one profiles $\tilde{W}^\beta_{\mathcal{O}^\beta,k}$ such that, for any $k \geq 0$ in the subsequence

$$(6.9) \quad \phi_k'(x,y) = \phi_k(x,y) - \sum_{1 \leq \alpha \leq A} \varphi^\alpha_{\mathcal{O}^\alpha,k} + \sum_{1 \leq \beta \leq A} \tilde{W}^\beta_{\mathcal{O}^\beta,k}$$

satisfies

$$(6.10) \quad \Lambda_\infty(\{\phi_k'\}) < \nu.$$ 

Besides, all the frames involved are pairwise orthogonal and $\phi_k'$ is absent from all these frames.

Theorem 6.6 (Profile decomposition). Assume $\{\phi_k\}$ is a sequence of functions satisfying $\|\phi_k\|_{H^1(\mathbb{R}^2 \times \mathbb{T})} < E$, up to a subsequence, then there exists a sequence of Euclidean profiles $\varphi^\alpha_{\mathcal{O}^\alpha,k}$, and scale-one profiles $\tilde{W}^\beta_{\mathcal{O}^\beta,k}$ such that, for any $J \geq 0$

$$(6.11) \quad \phi_k(x,y) = \sum_{1 \leq \alpha \leq J} \varphi^\alpha_{\mathcal{O}^\alpha,k} + \sum_{1 \leq \beta \leq J} \tilde{W}^\beta_{\mathcal{O}^\beta,k} + R^J_k$$

where $R^J_k$ is absent from the frames $\mathcal{O}^\alpha$ and satisfies

$$(6.12) \quad \limsup_{J \to \infty} \Lambda_\infty(\{R^J_k\}) = 0.$$ 

Additionally, we have the following orthogonal relation

$$(6.13) \quad \|\phi_k\|^2_{L^2} = \sum_{\alpha} \|\varphi^\alpha_{\mathcal{O}^\alpha,k}\|^2_{L^2} + \sum_{\beta} \|\tilde{W}^\beta_{\mathcal{O}^\beta,k}\|^2_{L^2} + \|R^J_k\|^2_{L^2} + o_k(1),$$

$$(6.14) \quad \|\nabla \phi_k\|^2_{L^2} = \sum_{\alpha} \|\nabla \varphi^\alpha_{\mathcal{O}^\alpha,k}\|^2_{L^2} + \sum_{\beta} \|\nabla \tilde{W}^\beta_{\mathcal{O}^\beta,k}\|^2_{L^2} + \|\nabla R^J_k\|^2_{L^2} + o_k(1),$$

The following estimate is crucial and it is the analogue of Lemma 6.3.

Lemma 6.7 (Control of the remainder term). Assume a sequence $\{f_k\}$ satisfying $\sup_k \|f_k\|_{H^1} < E$, we have the following estimate:

$$(6.15) \quad \limsup_{k \to \infty} \|e^{it\Delta} f_k\|_{Z(\mathbb{R})} \lesssim E (\Lambda_\infty(\{f_k\}))^\delta,$$

where $\delta$ is some positive constant.
Proof. By using interpolation and Strichartz estimate,

\[ ||e^{it\Delta}f_k||_{L^2_\omega} = \sum_{p_0=8,10} \left( \sum_{N \geq 1} N^{5-\frac{1}{p_0}}||1_{(t)}P_N u||_{L^p_{t,x}(\mathbb{R}^2 \times \mathbb{T} \times I_x)}^{p_0} \right)^{\frac{1}{p_0}} \]

\[ \lesssim \sum_{p_0=8,10} \sum_{N} (N^{-\frac{1}{2}}||P_N e^{it\Delta}f_k||_{L^2_{\omega}^\omega}) \frac{1}{2} (N^\frac{p_0-1}{2}||P_N e^{it\Delta}f_k||_{L^2_{\omega}^\omega}) \frac{1}{2} \]

\[ \lesssim (\sup_N N^{-\frac{1}{2}}||P_N e^{it\Delta}f_k||_{L^2_{\omega}^\omega})^2 \sum_N (N||f_k||_{L^2})^2 \]

\[ \lesssim (\sup_N N^{-\frac{1}{2}}||P_N e^{it\Delta}f_k||_{L^2_{\omega}^\omega})^2 ||f_k||_{H^1} \]

\[ \lesssim E (\sup_N N^{-\frac{1}{2}}||P_N e^{it\Delta}f_k||_{L^2_{\omega}^\omega})^2. \]

That finishes the proof of Lemma 6.7 noticing that \( ||f_k||_{H^1} \) is uniformly bounded by \( E \).

7. Induction on Energy

We are now ready to prove the main theorem. We follow an induction on energy method formalized in [21, 22]. Similar as related results ([7, 14, 20, 38]), we also consider the full energy since it is a \( H^1 \) data problem. We define the following functional

\[ \Lambda(L) = \sup\{||u||_{Z(t)} : u \in X^1_{\text{loc}}(I), E(u) + M(u) \leq L \} \]

where the supremum is taken over all strong solutions of full energy less than \( L \). According to the local theory, this is sublinear in \( L \) and finite for \( L \) sufficiently small. We also define

\[ L_{\text{max}} = \sup\{L : \Lambda(L) < +\infty\}. \]

In order to prove the large data scattering of (1.2) and (1.3), it suffices to show that \( L_{\text{max}} = +\infty \) according to Theorem 4.4 and Theorem 4.9. That is our goal. The key proposition is as follows (Theorem 7.1 and Theorem 7.4):

**For cubic \( \mathbb{R}^3 \times \mathbb{T} \) problem:**

**Theorem 7.1.** Assume that \( L_{\text{max}} < +\infty \). Let \( \{t_k\}, \{a_k\}, \{b_k\} \) be arbitrary sequences of real numbers and \( \{u_k\} \) be a sequence of solutions to (1.1) such that \( u_k \in X^1_{\text{loc}}(t_k-a_k, t_k+b_k) \) and satisfying

\[ L(u_k) \to L_{\text{max}}, \quad ||u_k||_{Z(t_k-a_k, t_k)} \to +\infty, \quad ||u_k||_{Z(t_k, t_k+b_k)} \to +\infty. \]

Then passing to a subsequence, there exists a sequence \( x_k \in \mathbb{R}^3 \) and \( \omega \in H^1(\mathbb{R}^3 \times \mathbb{T}) \) such that

\[ \omega_k(x,y) = u_k(x-x_k, y, t_k) \to \omega \]

strongly in \( H^1(\mathbb{R}^3 \times \mathbb{T}) \).

The proof mainly follows from profile decomposition and perturbation theory. We will give the proof at the end of Section 7. Now let us show how to use Theorem 7.1 to close the contradiction argument and finish the proof of the main theorem. The following analysis (Corollary 7.2 and Theorem 7.3) is similar to [14] (see also [38]).

**Corollary 7.2.** Assume that \( L_{\text{max}} < +\infty \). Then there exists \( u \in X^1_{\text{c,loc}}(\mathbb{R}) \) solving (1.1) and a Lipschitz function \( \bar{x} : \mathbb{R} \to \mathbb{R}^3 \) such that \( L(u) = L_{\text{max}} \) and

\[ \sup_{t \in \mathbb{R}} |\bar{x}'(t)| \lesssim 1, \]

\[ (u(x-\bar{x}(t), y, t) : t \in \mathbb{R}) \]

is precompact in \( H^1(\mathbb{R}^3 \times \mathbb{T}) \).

**Remark.** The proof is similar to [14, Corollary 7.2] by using perturbation theory. We omit it.

**Theorem 7.3.** Assume that \( u \) satisfies the conclusion of Corollary 7.2, then \( u = 0 \).
Proof: Assume $u \neq 0$. Then, from the compactness property, we see that there exists $\rho > 0$ such that
\[ (7.4) \quad \inf_{t \in \mathbb{R}} \min(||u(t)||_{L^4_{x,y}((\mathbb{R}^3 \times \mathbb{T})}, ||u(t)||_{L^2_{x,y}((\mathbb{R}^3 \times \mathbb{T})})} \geq \rho. \]
Now let us consider the conserved momentum
\[ P(u) = Im \int_{\mathbb{R}^3 \times \mathbb{T}} \bar{u}(x, y, t) \partial_x u(x, y, t) dx dy. \]
Considering the Galilean transform
\[ v(z, t) = e^{-i|\xi_0|^2 t + i(z, \xi_0)} u(z - 2\xi_0 t, t), \]
and letting
\[ \xi_0 = -\frac{P(u)}{M(u)}, \]
without loss of generality, we can assume that
\[ (7.5) \quad P(u) = 0. \]
Then we define the Virial action by
\[ A_R(t) = \int_{\mathbb{R}^3 \times \mathbb{T}} \chi_R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t))Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dx dy \]
for $\chi_R(x) = \chi(x/R)$ and $\chi$ satisfies $\chi(x) = 1$ when $|x| \leq 1$ and $\chi(x) = 0$ when $|x| \geq 2$ ($x \in \mathbb{R}$).
On one hand, clearly
\[ (7.6) \quad \sup_t |A_R(t)| \lesssim R. \]
On the other hand, we compute that
\[ \frac{d}{dt} A_R = -z_1'(t) Im \int_{\mathbb{R}^3 \times \mathbb{T}} \bar{u}(x, y, t) \partial_x u(x, y, t) dx dy \]
\[ + \frac{z_1'(t)}{R} \int_{\mathbb{R}^3 \times \mathbb{T}} \{(\chi')R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t)) - (1 - \chi_R(x_1 - \bar{z}_1(t)))] Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dx dy \]
\[ + \int_{\mathbb{R}^3 \times \mathbb{T}} \chi_R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t)) \partial_t Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] dx dy. \]
The first term will vanish automatically based on the assumption (7.5) and the second term can be bounded by
\[ \int_{\{x - \bar{z}(t) \geq R\}} \int_T \frac{||u(x, y, t)||^2}{R} + ||\nabla u(x, y, t)||^2 dx dy = O_R(t), \]
\[ \sup_t O_R(t) \rightarrow 0 \quad \text{as} \quad R \rightarrow +\infty. \]
Notice that
\[ \partial_t Im[\bar{u}(x, y, t) \partial_x u(x, y, t)] = \partial_x \Delta \frac{|u|^2}{2} - 2div[Re[\partial_x u \nabla u]] - \frac{1}{4} \partial_{x_1} |u|^4. \]
For the last term, we have
\[ \frac{d}{dt} A_R = \int_{\mathbb{R}^3 \times \mathbb{T}} \chi_R(x_1 - \bar{z}_1(t)) \left[ \frac{1}{4} ||u(x, y, t)||^4 + \frac{1}{2} ||\partial_x u(x, y, t)||^2 \right] dx dy \]
\[ + \int_{\mathbb{R}^3 \times \mathbb{T}} \chi_R(x_1 - \bar{z}_1(t)) \frac{x_1 - \bar{z}_1(t)}{R} \left[ \frac{1}{4} ||u(x, y, t)||^4 + \frac{1}{2} ||\partial_x u(x, y, t)||^2 \right] dx dy \]
\[ - \int_{\mathbb{R}^3 \times \mathbb{T}} \frac{|u(x, y, t)|^2}{2} \partial_{x_1}^2 \left[ \chi_R(x_1 - \bar{z}_1(t))(x_1 - \bar{z}_1(t)) \right] dx dy + O_R(t) \]
\[ = \int_{\mathbb{R}^3 \times \mathbb{T}} \left[ \frac{1}{4} ||u(x, y, t)||^4 + \frac{1}{2} ||\partial_x u(x, y, t)||^2 \right] dx dy + \tilde{O}_R(t). \]
Integrating this equality, we obtain
\[ |A_R(t) - A_R(0)| \geq C \rho t - t \sup_t \tilde{O}_R(t). \]
Taking \( R \) sufficiently large enough, when \( t \) is sufficiently large, we see there is a contradiction. This finishes the proof of Theorem 7.3.

**Proof of Theorem 7.1:** The proof is similar to [20, Theorem 6.1] (see also [38, Theorem 7.1]).

First, we apply profile decomposition to the bounded \( H^1 \) sequence \( u_k(0) \) and then consider three cases, i.e. no profile, only one profile and multiple profiles. The first two cases can be easily handled by using Lemma 6.3 and approximation result and the last case can be handled by constructing approximate solution and perturbation theory.

Without loss of generality, we assume \( t_k = 0 \), and we apply Theorem 6.2 (profile decomposition) to \( \{u_k(0)\}_k \) which is a bounded sequence in \( H^1(\mathbb{R}^3 \times \mathbb{T}) \). For all \( J \), we have

\[
(7.7) \quad u_k(0) = \sum_{1 \leq \alpha \leq J} \tilde{\varphi}_{\tilde{O},k}^\alpha + \sum_{1 \leq \beta \leq J} \tilde{W}_{\tilde{O}_{\alpha},k}^\beta + R^J.
\]

**Case 1:** There are no profiles. By using Lemma 6.3 (scattering norm estimate), we have, if we take \( J \) sufficiently large, we will have:

\[
\|e^{it\Delta}u_k(0)\|_{L^1(\mathbb{R})} = \|e^{it\Delta}R^J_k\|_{L^1(\mathbb{R})} \leq \delta_0/2
\]

for \( k \) sufficiently large, where \( \delta_0 \) is given in Theorem 4.3. Then we know that \( u_k \) can be extended on \( \mathbb{R} \) and that

\[
\lim_{k \to +\infty} \|u_k\|_{L^1(\mathbb{R})} \leq \delta_0.
\]

It is a contradiction. Hence there are at least one profile. There are two other cases left: only one profile (Case 2) and multiple profiles (Case 3). Furthermore, only one profile contains two cases, i.e. only one Euclidean profile (Case 2a) and only scale-one profile (Case 2b). Except for (Case 1), we will also rule out Case 2a and Case 3. Actually the conclusion statement in Theorem 7.1 is the only case that will happen. That is our goal for Theorem 7.1 which is an important preparation for the compactness argument (Corollary 7.2).

Moreover, for every linear profile, we define the associated nonlinear profile as the maximal solution of (1.2) with the corresponding initial data as in [14].

For any profile, we consider operator \( L \) such that

\[
L(\alpha) := \lim_{k \to \infty} (E(\tilde{\varphi}_{\tilde{O},k}^\alpha) + M(\tilde{\varphi}_{\tilde{O},k}^\alpha)) \in (0, L_{\text{max}}].
\]

According to the orthogonal properties in profile decomposition, we have:

\[
\lim_{J \to +\infty} \left[ \sum_{1 \leq \alpha, \beta \leq J} [L(\alpha) + L(\beta)] + \lim_{k \to +\infty} L(R^J_k) \right] \leq L_{\text{max}}.
\]

**Case 2a:** There are only one Euclidean profile in the profile decomposition, that is

\[
u_k(0) = \tilde{\varphi}_{\tilde{O},k} + o_k(1)
\]

in \( H^1(\mathbb{R}^3 \times \mathbb{T}) \), where \( \varepsilon \) is a Euclidean frame. In this case, since the corresponding nonlinear profile \( U_k \) satisfies \( \|U_k\|_{L^1(\mathbb{R})} \lesssim E_{\tilde{O},k} \) and \( \lim_{k \to +\infty} \|U_k(0) - u_k(0)\|_{H^1(\mathbb{R}^3 \times \mathbb{T})} \to 0 \). We can use Theorem 4.5 to deduce that

\[
\|u_k\|_{L^1(\mathbb{R})} \lesssim \|U_k\|_{L^1(\mathbb{R})} \lesssim L_{\text{max}},
\]

which contradicts (7.1).

**Case 2b:** There are only one scale-one profile in the profile decomposition, we have that

\[
u_k(0) = \tilde{\omega}_{\tilde{O},k} + o_k(1)
\]

in \( H^1(\mathbb{R}^3 \times \mathbb{T}) \), where \( \tilde{O} = \{1, t_k, x_k\} \) is a scale-one frame. If \( t_k \equiv 0 \), this is precisely the conclusion (7.2).

If \( t_k \to +\infty \), then

\[
\|e^{it\Delta}u_k(0)\|_{L^1(\mathbb{R},0)} \leq \|e^{it\Delta}u_k(0)\|_{L^1(-\infty,0)} = \|e^{it\Delta}u_k(0)\|_{L^1(-\infty,-t_k)}
\]

which goes to 0 as \( t_k \to +\infty \). Using Theorem 4.3, we see that, for \( k \) large enough,

\[
\|u_k\|_{L^1(-\infty,0)} \leq \delta_0.
\]
It contradicts (7.1). The case $t_k \to -\infty$ is similar.

Case 3: There are multiple profiles in the profile decomposition. We can construct approximate equation and use perturbation theory to rule out this case. It is similar as to [14, Proposition 7.1, case 3] and [20, Proposition 6.1, case 3] (see also [38, Theorem 7.1]). We omit it.

For quintic $\mathbb{R}^2 \times T$ problem: we have,

**Theorem 7.4.** Assume that $L_{max} < +\infty$. Let $\{t_k\}_k, \{a_k\}_k, \{b_k\}_k$ be arbitrary sequences of real numbers and $\{u_k\}$ be a sequence of solutions to (1.1) such that $u_k \in X_{c,loc}(t_k - a_k, t_k + b_k)$ and satisfying

$$L(u_k) \to L_{max}, \quad ||u_k||_{L^1(t_k - a_k, t_k)} \to +\infty, \quad ||u_k||_{L^1(t_k, t_k + b_k)} \to +\infty.$$  

Then passing to a subsequence, there exists a sequence $x_k \in \mathbb{R}^2$ and $\omega \in H^1(\mathbb{R}^2 \times T)$ such that

$$\omega(x, y) = u_k(x - x_k, y, t_k) \to \omega$$

strongly in $H^1(\mathbb{R}^2 \times T)$.

The proof of Theorem 7.4 mainly follows from profile decomposition and perturbation theory. Based on this, we can prove:

**Corollary 7.5.** Assume that $L_{max} < +\infty$. Then there exists $u \in X_{c,loc}(\mathbb{R})$ solving (1.1) and a Lipschitz function $\underline{u} : \mathbb{R} \to \mathbb{R}^2$ such that $L(u) = L_{max}$ and

$$\sup_{t \in \mathbb{R}} |\underline{u}'(t)| \lesssim 1,$$

$$(u(x - \underline{u}(t), y, t) : t \in \mathbb{R}) \text{ is precompact in } H^1(\mathbb{R}^2 \times T).$$

**Theorem 7.6.** Assume that $u$ satisfies the conclusion of Corollary 7.5, then $u \equiv 0$.

The proofs of the above three theorems are similar to those for the cubic case and we omit them. We also refer to [14, 19, 38].

Acknowledgments. I would like to express my deep thanks to my thesis advisor Professor Benjamin Dodson for many useful discussions, suggestions and comments. Also, I really appreciate Professor Zaher Hani, Professor Benoit Pausader, Qingtang Su and Professor Lifeng Zhao for insightful suggestions and valuable comments. Moreover, part of this work was progressed when the author attended the PDE conference (Waves, Spectral Theory and Applications Part 2) at UNC and I would like to thank the organizers for hosting.

**References**

[1] P. Bégout and A. Vargas, Mass concentration phoneme for the $L^2$-critical nonlinear Schrödinger equations, Trans. Amer. Math. Soc. 359(11), (2007), 5257-5282.

[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3(1993), 107-156.

[3] J. Bourgain, Exponential sums and nonlinear Schrödinger equations, Geom. Funct. Anal. 3 (1993), 157178.

[4] N. Burq, P. Gerard, and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math. 126 (2004), 569605.

[5] R. Carles, and S. Keraani, On the role of quadratic oscillations in the nonlinear Schrödinger equation II, the $L^2$-critical case, Trans. Amer. Math. Soc. 359 (2007), 3362.

[6] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[7] X. Cheng, Z. Guo, K. Yang and L. Zhao, On scattering for the cubic defocusing nonlinear Schrödinger equation on wave guide $\mathbb{R}^2 \times T$, arXiv: 1705.00954v1.

[8] X. Cheng, C. Miao, and L. Zhao, Global well-posedness and scattering for nonlinear equations with combined nonlinearities in the radial case, J. Differential Equations 261 (2016), no. 6, 2881-2914.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Ann. of Math. 167 (2008), 767865.

[10] B. Dodson, Global Well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equations when $d = 2$, Duke Math J. Volume 165, Number 18 (2016), 3435-3516.

[11] B. Dodson, Global Well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equations when $d = 1$, arXiv: 1100.0040.

[12] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equations when $d \geq 3$, Journals of the American Mathematical Society, 25 no. 2 (2012) 429-463.
13. B. Dodson, C. Miao, J. Murphy and J. Zheng, The defocusing quintic NLS in four space dimensions, arXiv:1508.07298
14. Z. Hani and B. Pausader, On scattering for the quintic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$, Comm. Pure Appl. Math. 67 (2014), no. 9, 1466-1542.
15. Z. Hani, B. Pausader, N. Tzvetkov, and N. Visciglia, Modified scattering for the cubic equation on product spaces and applications, Forum of Mathematics, Pi. (2015), Vol. 3, 1-63.
16. S. Herr, D. Tataru, and N. Tzvetkov, Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(T^3)$, Duke Math. J. 159 (2011), no. 2, 329-349.
17. S. Herr, D. Tataru, and N. Tzvetkov, Strichartz estimates for partially periodic solutions to Schrödinger equations in 4d and applications, J. Ang. Math. 690 (2014), 65-78
18. A. D. Ionescu, B. Pausader, and G. Staffilani, On the global well-posedness of energy-critical Schrödinger equations in curved spaces, Analysis and PDE, arXiv: 1008.1237.
19. A. D. Ionescu and B. Pausader, The energy-critical defocusing NLS on $\mathbb{T}^3$, Duke Math. J. 161 (2012), no. 8, 1581-1612.
20. A. D. Ionescu and B. Pausader, Global well-posedness of the energy critical defocusing NLS on $\mathbb{R} \times \mathbb{T}^3$ Comm. Math. Phys. 312 (2012), no. 3, 781-831.
21. C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case, Invent. Math. 166 (2006), no. 3, 645-675.
22. C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, nonlinear wave equation, Acta. Math. 201 (2008), 147-212.
23. R. Killip and M. Visan, Nonlinear Schrödinger Equations at critical regularity, Proceedings for the Clay summer school “Evolution Equations”, Edigerössische technische, Zürich, 2008.
24. R. Killip and M. Visan, The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher, Analysis and PDE 1, no. 2 (2008) 229-266.
25. R. Killip and M. Visan, Scale invariant Strichartz estimates on tori and applications, arXiv:1409.3603v1 [math.AP] .
26. E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R} \times \mathbb{R}^4$, arXiv: math/0501462
27. T. Schneider, Nonlinear optics in telecommunications, Springer, Berlin, 2004.
28. A. W. Snyder and J. Love, Optical waveguide theory, Springer, US, 1983.
29. T. Tao, Nonlinear Dispersive Equations.Local and Global Analysis, CBMS regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, R.I., 2006.
30. T. Tao, A sharp bilinear restriction estimate for paraboloids, Geom. Funct. Anal. 13 (2003), 1359-1384.
31. T. Tao, A pseudoconformal compactication of the nonlinear Schrödinger equation and applications, New York J. Math. 15 (2009), 265282.
32. T. Tao, M. Visan and X. Zhang, Minimal-mass blowup solutions of the mass-critical NLS, Forum Math. 20 (2008), 881-919.
33. T. Tao, M. Visan and X. Zhang, Global well-posedness and scattering for the defocusing mass - critical nonlinear Schrödinger equation for radial data in high dimensions, Duke Mathematical Journal, 140 no. 1 (2007) 165 - 202.
34. M. Tarulli, Well-posedness and scattering for the mass-energy NLS on $\mathbb{R}^N \times \mathbb{R}$ arxiv: 1510.01710.
35. N. Tzvetkov and N. Visciglia, Small data scattering for the nonlinear Schrödinger equation on product spaces, Comm. Partial Differential Equations. 37 (2012), no. 1, 125-135
36. N. Tzvetkov and N. Visciglia, Well-posedness and scattering for NLS on $\mathbb{R}^d \times \mathbb{T}$ in the energy space, arXiv: 1409.3938.
37. K. Yang and L. Zhao, Global well-posed and scattering for NLS mass-critical, defocusing, infinite dimensional vector-valued resonant nonlinear Schrödinger system, arXiv: 1704.08976.
38. Z. Zhao, Global well-posedness and scattering for the defocusing cubic Schrödinger equation on waveguide $\mathbb{R}^2 \times \mathbb{T}^2$ arXiv:1710.00702