Nu\textsubscript{k}-CP: An Improved Local Search Algorithm for Maximum \textit{k}-Club Problem

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Abstract

The maximum \textit{k}-club problem (M\textit{k}CP) is an important clique relaxation problem with wide applications. Previous M\textit{k}CP algorithms only work on small-scale instances and are not applicable for large-scale instances. For solving instances with different scales, this paper develops an efficient local search algorithm named Nu\textsubscript{k}CP for the M\textit{k}CP which mainly includes two novel ideas. First, we propose a dynamic reduction strategy, which makes a good balance between the time efficiency and the precision effectiveness of the upper bound calculation. Second, a stratified threshold configuration checking strategy is designed by giving different priorities for the neighborhood in the different levels. Experiments on a broad range of different scale instances show that Nu\textsubscript{k}CP significantly outperforms the state-of-the-art M\textit{k}CP algorithms on most instances in terms of solution quality.

Introduction

A clique is a subset of vertices of an undirected graph in which each pair of vertices are adjacent. Clique is one of the basic concepts of graph theory and has been widely studied (Ouyang et al. 1997; Batenko and Wilhelm 2006). However, the constraint condition of clique model is too strict for many real-world applications since the aim of these applications is to find some dense structures rather than a complete subgraph. These structures can usually be seen as clique relaxation models and are mainly divided into two categories: density-based models such as \textit{k}-plex (Gao et al. 2018) and quasi-clique (Chen et al. 2021) as well as diameter-based models such as \textit{k}-clique (Cavique, Mendes, and Santos 2009) and \textit{k}-club (Shahinpour and Batenko 2013b).

In this paper, we focus on studying the maximum \textit{k}-club problem (M\textit{k}CP) which has been used in various domains. For example, the M\textit{k}CP can help to facilitate the search on the internet since it can cluster topically related information (Pattillo, Youssef, and Butenko 2013). In social network, the internet since it can cluster topically related information. For example, the M\textit{k}CP is also used to analyze the roles of functional modules by mining important substructures in biological network (Balasundaram, Butenko, and Trukhanov 2005; Jia et al. 2018).

Given a graph \(G = (V, E)\) and a fixed integer \(k\), a \textit{k}-club \(S\) is a subset of vertices inducing a subgraph of diameter at most \(k\). It is easy to see that 1-club is a clique. The M\textit{k}CP aims to identify the \textit{k}-club with the maximum size in a graph. It is NP-hard, even for any fixed \(k > 1\) (Bourjolly, Laporte, and Pesant 2002). Up to now, there are mainly two types of algorithms for the M\textit{k}CP, i.e., exact algorithms and heuristic algorithms.

Several exact algorithms have been designed to solve the M\textit{k}CP. Bourjolly et al. (2002) proposed a classic branch-and-bound algorithm for the M\textit{k}CP and could solve the instances involving up to 200 vertices. The branch-and-bound algorithms for the M\textit{k}CP were further improved by considering the \textit{k}-coloring number as an upper bound (Pajouh and Balasundaram 2012) and designing a dynamic data structure (Chang et al. 2013). Recently, another paradigm solved the M\textit{k}CP by using different integer linear programming formulations (Almeida and Carvalho 2012, 2014a; Veremyev et al. 2021). Although exact algorithms can guarantee the optimality of their solutions, they may fail to solve the problem within acceptable time, especially for large-scale instances.

In practice, for solving the large-scale M\textit{k}CP instances, researchers resort to designing heuristic algorithms for obtaining good solutions. Bourjolly et al. (2000) proposed three simple and effective heuristic algorithms for the M\textit{k}CP, including Constellation, Drop and \textit{k}-Clique & Drop. Shahinpour and Batenko (2013a) developed a variable neighborhood search called VNS for the M\textit{k}CP and firstly tested the performance of VNS on the DIMACS benchmark. Afterwards, two hybrid algorithms including mS\textsubscript{IP} specialized for the 2-club problem and mB\textsubscript{IP} specialized for the 3-club problem were introduced by combining heuristic algorithms and integer linear programming formulations (Almeida and Carvalho 2014b). In the same work, Almeida and Carvalho (2014b) also proposed a heuristic algorithm called Backbone and proved its superiority over the previous algorithm Constellation on the 3-club problem. Moradi and Balasundara (2018) proposed a heuristic algorithm for the M\textit{k}CP called ITDBC which used a combination of graph decomposition and model decomposition techniques. In the ITDBC, a reduction method called TRIM played an impor-
tant role in cutting down the search space. Results showed that ITDBC achieved the best results on most instances.

In this paper, we develop an efficient local search algorithm named NukCP which can apply to different scale instances and performs better on almost all benchmarks. There are two main novel ideas in our algorithm.

The first idea is called dynamic reduction method (DRM) designed for quickly reducing the size of graphs. The previous reduction method TRIM (2018) used a kind of static upper bound calculation, which is time-consuming and thus is not applicable for large-scale instances. Compared to TRIM, the DRM allows to delete vertices during the calculation of upper bounds. Additionally, the DRM introduces a novel method to dynamically update the upper bound values instead of recalculating them as the TRIM does. These two techniques make our reduction method much faster than TRIM. Experiments show that the DRM can achieve good practical results within short time.

The second idea is an improved version of the configuration checking (CC) strategy called stratified threshold configuration checking (STCC). CC firstly proposed by Cai et al. (2011) was used to overcome the cycling problem during local search. Recently, different variants of CC have been designed for solving the clique related problems such as SCC for the maximum weight clique problem (Wang, Cai, and Yin 2016), DCC for the maximum k-plex problem (Chen et al. 2020) and BoundedCC for the maximum quasi-clique problem (Chen et al. 2021). Different from the above problems, the M$k$CP needs to consider the multi-level neighborhood of vertices due to its characteristics. In our work, we take the characteristics into account and maintain the multi-level neighborhood information. Although previous CC version named CC$^2$V3 (Wang et al. 2018) considers the two-level neighborhood, for the M$k$CP we need to take more than two-level neighborhood into account, which leads to a serious problem, i.e., how to maintain the neighborhood information. To address this, the spanning tree is used to dynamically maintain the multi-level neighborhood of vertices in the candidate solution.

We carry out extensive experiments to evaluate the NukCP on the benchmarks used in the literature for the M$k$CP as well as a suite of massive graphs (Rossi and Ahmed 2015). Compared with three state-of-the-art heuristic algorithms, NukCP obtains the best results for almost all benchmarks. Besides, our experimental analyses verify the effectiveness of the proposed strategies.

The remainder of the paper is organized as follows. The next section introduces some basic definitions. Section 3 presents a new reduction strategy. Section 4 presents a variant of CC designed for the M$k$CP. Section 5 describes our NukCP algorithm. Experimental results are shown in Section 6 and Section 7 gives concluding remarks.

Preliminaries

An undirected graph $G = (V, E)$ consists of a vertex set $V$ and an edge set $E$. Two vertices are neighbors if they belong to one edge. We denote $N_G(v) = \{ u \in V \mid \{ u, v \} \in E \}$ as the set of neighbors of a vertex $v$ and its degree is $\deg_G(v) = |N_G(v)|$. Given a pair of vertices $u, v \in V$, the distance $\text{dist}_G(u, v)$ is the number of edges in a shortest path connecting them and $\text{dist}_G(v, v) = 0$ particularly. The diameter of $G$ denoted as $\text{diam}(G)$ is the maximum distance between any pair of vertices. We define the $i$-th level neighborhood of $v$ in $G$ as $N_i(G)(v) = \{ u \in V \mid \text{dist}_G(u, v) = i \}$, $N_i(G)[v] = N_i(G)(v) \cup \{ v \}$ and $N_k^S[v] = \bigcup_{i=1}^{k} N_i(G)[v]$. Thus, $N_i(G)(v) = N(G)(v)$. For $S \subseteq V$, $G[S] = (V_S, E_S)$ is a subgraph in $G$ induced by $S$ whose vertex set is $S$ and whose edge set consists of all of the edges in $E$ that have both endpoints in $S$.

Given a graph $G$ and a fixed integer $k$, a $k$-club $S$ is a subset of $V$ such that $\text{diam}(G[S]) \leq k$. The maximum $k$-club problem (M$k$CP) is to find a $k$-club with the most vertices.

A Dynamic Reduction Strategy

Although reduction methods have been widely used in the clique related problems (Cai and Lin 2016; Jiang, Li, and Manyà 2016; Wang et al. 2020a; Zhou et al. 2021), we are aware of only one reduction method for the M$k$CP, which is named TRIM (Moradi and Balasundaram 2018). In this section, we propose a novel reduction method for the M$k$CP, which is built upon the rules of TRIM but significantly faster than TRIM.

Previous Static Reduction Strategy

For the M$k$CP, the general principle for reduction is as follows. For a graph $G$, an upper bound function calculates for each vertex a value $ub_G(v)$ such that the size of any $k$-club that $v$ belongs to is at most $ub_G(v)$. A lower bound function calculates a value $lb(G)$ such that the size of the largest $k$-club in $G$ is not smaller than $lb(G)$. All vertices with $ub_G(v) \leq lb(G)$, along with their incident edges, can be safely deleted since they cannot be part of any optimal solution.

The TRIM reduction method employs simple but effective lower bound and upper bound.

- $lb(G) = \max \{ |N_{G}^{(k/2)}[v]| \mid v \in V \}$
- $ub_G(v) = |N_G^k[v]|$

When applying reduction rules, the graph $G$ always refers to the current graph in process and thus can be omitted. For convenience, $lb(G)$ and $ub_G(v)$ are denoted as $lb$ and $ub(v)$. TRIM works in an iterative way and each iteration is consisted of two steps as below.

- Step 1. Calculate $X = \{ v \in V \mid ub(v) \leq lb \}$;
- Step 2. If $X$ is not empty, then $V = V \setminus X$ and TRIM goes to step 1. Otherwise, TRIM breaks the iteration.

In practice, TRIM is time-consuming and not suitable for the massive graphs due to the following two reasons. (1) The deletion of a vertex $v$ would change the $ub$ values of those vertices in $N_G^k[v]$. TRIM recalculates $ub$ values for such vertices. What is worse, the $ub$ values for some vertices are recalculated multiple times. (2) In each iteration, TRIM first calculates $ub$ values for all vertices, and then removes those vertices satisfying the reduction condition. In this way, TRIM does not update the $ub$ values in real time based on the current simplified graph. This leads to more iterations than necessary for the reduction process to converge.
Dynamic Reduction Method

In order to address the drawback of TRIM, we develop a new reduction method for the MkCP called dynamic reduction method (DRM). There are two main ideas in the DRM, with the aim of resolving the two issues of TRIM. Firstly, DRM only calculates the $ub$ values once for all vertices and then dynamically updates the $ub$ values instead of recalculating them whenever a vertex is deleted. Secondly, DRM allows a vertex to be deleted during the process of calculating its $ub$ value — this strategy not only decreases the iterations of reduction, but also accelerates the subsequent calculation or updating of $ub$ values as the computation is executed on the simplified graph.

The proposed DRM is consisted of two phases: the dynamic calculation phase and the iterative deletion phase. During the DRM, the calculation of $ub$ is only executed once at the beginning of the algorithm and the iterative deletion phase is used for further reduction. Before presenting our DRM, we first introduce a novel dynamic maintenance rule named dynamic update rule (DUR) for updating the neighborhood of a deleted vertex.

**Dynamic Update Rule.** When a vertex $v \in V$ is deleted from $G$, $ub(u) = ub(u) - 1$ for all $u \in N_G^2[v]$. The above rule has a low time complexity of $O(|V|)$. Based on the DUR, we present the dynamic calculation phase whose implementation efficiency is closely related to the sequence of vertices and the iterative deletion phase which is implemented by an auxiliary queue.

**Dynamic Calculation Phase.** Initially, the positions of vertices are arranged in an ascending order of their degrees, $V = \{v_1, v_2, \ldots, v_n\}$, s.t. $deg_G(v_1) \leq deg_G(v_2) \leq \cdots \leq deg_G(v_n)$. If $ub(v_i) \leq lb$, then $v_i$ is deleted from $G$ in advance, and at the same time the $ub$ values for its neighborhood $v_j \in N_G^2[v_i]$ with $j < i$ are updated by the DUR.

**Iterative Deletion Phase.** An auxiliary queue is used to store all vertices $v \in V$ with $ub(v) \leq lb$. In each iteration, a vertex in the queue is deleted from $G$ and the $ub$ values of each vertex $u \in N_G^2[v]$ are updated by the DUR. If $ub(u) \leq lb$ and $u$ is not in the queue, then $u$ is added into the queue. The loop continues until the queue is empty. Notice that during the first phase, when deleting $v_i$ from $G$, we do not update the $ub$ value of the vertex $v_j$ whose position is behind $v_i$ (i.e., $j > i$) since $ub(v_j)$ has not been initialized yet. Additionally, the deletion of $v_i$ reduces the size of $G$, which makes the subsequent process of calculating $ub$ values faster. Thus, before calculating $ub$ values, we sort all vertices in an ascending order according to their degrees, trying to delete vertices as early as possible, which accelerates the overall deletion process.

As explained above, the DUR avoids the recalculating of $ub$ values. On the other hand, this comes with a price that the $ub$ values calculated in this way may be actually larger than the $ub$ of TRIM whose value is always equal to $|N_G^2|$. It is mainly caused by the following situation: when a vertex $v$ is deleted from $G$, the $|N_G^2[u]|$ values for $u \in N_G^2[v]$ may decrease by more than one while the DUR just subtracts one for $ub(u)$.

We use an example to illustrate this situation in Figure 1. Assume that an original graph is $G = (V, E)$ and $V = \{v_1, v_2, \ldots, v_8\}$ whose positions have been already arranged. In order to show the changes in the size of the graph during the first phase, we use $G_1, G_2, \ldots, G_8$ to denote the corresponding graphs where $V_{i+1} = V_i \setminus \{v_i\}$ if $v_i$ is deleted and $V_{i+1} = V_i$ otherwise. We can easily get $lb = |N_G^2[v_6]| = 5$. At the beginning, there are no vertices being deleted inducing that $G_1 = G$. Based on the $G_1$, $ub_G(v_1) = 4 < lb$, $v_1$ should be deleted and $V_2 = V_1 \setminus \{v_1\}$. Afterwards, we cannot delete $v_2$ since $ub_G(v_2) = 7$, and thus $G_2 = G_2$. $ub_G(v_3) = 5$ induces that $v_3$ should be deleted. When deleting $v_3$, we update its neighborhood according to the DUR which decreases $ub(v_2)$ by one, i.e., $ub_G(v_2) = 6$. However, if we recalculate $ub(v_2)$ on $G_3$ as what TRIM does, the value of $|N_G^2[v_2]|$ should be 5. This is because the deletion of $v_3$ makes $v_4$ no longer be in the neighborhood of $v_2$. Thus, in some cases, our upper bound value is larger than the one calculated by TRIM.

Even though our method may sacrifice the precision of the $ub$ values, it has greatly improved the efficiency of updating process and can be used for massive graphs. In fact, our experimental results show that the reduction ability of our dynamic reduction method is close to that of TRIM, whilst it greatly improves the efficiency.

**The DRM Function** The pseudo code of DRM is presented in Algorithm 1, which includes the dynamic calculation phase (lines 2–13) and the iterative deletion phase (lines 14–18). Input parameter $mode = 1$ means the algorithm calls the DRM function on the initialization phase, while $mode = 2$ means that the DRM function is used when the algorithm obtains a better solution during the local search phase. At first, the algorithm initializes a deletion queue named $Q_d$ as an empty set (line 1). When entering the first phase, the algorithm first sorts all vertices in an ascending order according to their degrees (line 3). After that, for each vertex $v_i$, its $ub(v_i)$ value will be calculated. If $ub(v_i) \leq lb$, then $v_i$ will be deleted from $V$ in advance (line 7). For $v_i$ in the $v_i$’s neighborhood and $j < i$, $ub(v_j)$ will be updated according to the DUR (lines 8–9). If $ub(v_j) \leq lb$, then $v_j$ will be added into $Q_d$ (line 10). If NuCP calls the DRM during the local search phase (i.e., $mode = 2$), then the vertices $v_i \in V$ with $ub(v_i) \leq lb$ will be added into $Q_d$ (lines 11–13).

In the iterative deletion phase, during each iteration (lines 14–18), a vertex $v \in Q_d$ will be deleted from $G$ and its neighborhood should be updated according to the DUR. Af-
while pop a vertex
foreach u of its neighbors. For a vertex which refers to the states (i.e., in the configuration) tar 2011) and has been successfully used in several NP-hard problems for the multi-level neighborhood of the removal vertex. Different from DCC and BoundedCC which only preserve the first k-level neighborhood of each vertex in the candidate solution S as the configuration information. Moreover, the first k-level neighborhood of vertices for the Mi\&CP will be dynamically changed with respect to S.

In order to facilitate the dynamic maintenance of the neighborhood, for each v ∈ S, we build a spanning tree for it, denoted as T^v = (V^v, E^v), where V^v = {u ∈ V | dist_G[S∪(u)](u, v) ≤ k}. dist_T^v(u, v) is used to denote the depth value of vertex u in the T^v. During the search process, we maintain |S| spanning trees. Three corresponding updating rules of spanning tree are described as follows.

**Constructing Rule.** For v ∈ S, a spanning tree T^v = (V^v, E^v) is constructed via using breadth-first search. During this process, three expansion ways are used. 1) For u_1 \notin S and dist_G[S∪(u_1)](u_1, v) ≤ k, we do not expand it and directly mark it as a leaf vertex. 2) For u_2 ∈ S and dist_G[S](u_2, v) < k, u_2 will be expanded. If u_2 cannot expand any vertices, then we mark u_2 as a leaf vertex. 3) For u_3 ∈ S and dist_G[S](u_3, v) = k, u_3 needs to be marked as a leaf vertex.

**Adding Rule.** When vertex v is added into S, T^v will be generated according to the constructing rule. For ∀u ∈ (T^v ∩ S), T^v needs to be updated in the following method. For v and w ∈ V^v with dist_T^v(u, w) > dist_T^v(u, v), these vertices should be re-expanded by using the expansion ways of the constructing rule.

**Removing Rule.** When vertex v is removed from S, we delete its spanning tree T^v and update T^u for ∀u ∈ (T^v ∩ S). For ∀w ∈ V^u with dist_T^v(u, w) ≥ dist_T^v(u, v), these vertices should be re-expanded by using the expansion ways of the constructing rule.

The vertices inside S constitute the trunk or leaf vertices of the spanning tree, and some vertices outside S constitute the remaining leaf vertices. The complexity of constructing or updating a spanning tree T\^v is O(|V_{S\cup{v}}| + |E_{S\cup{v}}|). To make the updating rules more comprehensive, we show an example in Figure 2 with regard to maintaining T\^v when adding v_3 and then removing v_5.

### Data Structure of STCC

Different from DCC and BoundedCC which only preserve the first level neighborhood, we need to maintain the first k-level neighborhood of each vertex in the candidate solution S as the configuration information. Moreover, the first k-level neighborhood of vertices for the Mi\&CP will be dynamically changed with respect to S.

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Update Rule for STCC

For $\forall v \in V$, the proposed STCC strategy is implemented with two arrays $conf[v]$ and $thread[v]$ to denote the configuration and threshold of vertices, respectively. Only when $conf[v] \geq thread[v]$, $v$ is allowed to be added into $S$. The novel STCC strategy is specified by the following rules.

**STCC-InitialRule.** For $\forall v \in V$, $conf[v]$ and $thread[v]$ are both initialized to $k$.

**STCC-AddRule.** When $v$ is added into the candidate solution, for $\forall u \in V_{k}^{2}$, $conf[u]$ is increased by $k + 1 - dist_{TV}(u, v)$ and $thread[v]$ is increased by $1$.

**STCC-RemoveRule.** When $v$ is removed from the candidate solution, $conf[v]$ is reset to $0$.

In the above rules, we increase $conf[u]$ by different values according to their $dist_{TV}$ to reflect the impact of changes in the states of neighborhood at different levels.

### The Nu/kCP Algorithm

According to the above strategies, we propose a local search algorithm for the M\&CP named Nu/kCP. Before introducing the Nu/kCP, we present the scoring function used in our work, which has been used in (Bourjolly, Laporte, and Pesant 2000; Almeida and Carvalho 2014b) for the M\&CP. We denote our scoring function as $score(v)$ for $\forall v \in V$.

$$score(v) = |\{ u \in S | dist_{G[S \cup \{v\}]}(u, v) \leq k \}|$$

Based on the above scoring function, our vertex selection rules are given as below.

**Selection Adding Rule.** Select a vertex $v \in V \setminus S$ with the highest $score(v)$ value, breaking ties by the oldest one.

**Selection Removing Rule.** Select a vertex $v \in S$ with the lowest $score(v)$ value, breaking ties by the oldest one.

Note that only when the $score$ values of all vertices in $S$ are equal to $|S|$, then we say $S$ is a feasible solution for the M\&CP. In addition, for $k = 3$, we use a tighter lower bound $lb = \max\{|N_1(v) \cup N_2(v)| | \{u, v\} \in E\}$ proposed by Almeida and Carvalho (2014b).

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**Algorithm 2: the Nu/kCP algorithm**

**Input:** graph $G$, the $cutoff$ time, parameter $k$

**Output:** the best $k$-club $S^*$ found

1. $S^* := \emptyset$
2. if $k \neq 3$ then $lb := \max\{|N_1^{(k/2)}[v]| | v \in V\}$
3. else $lb := \max\{|N_1(v) \cup N_2(v)| | \{u, v\} \in E\}$
4. $G := DRM(G, lb, k, 1)$
5. while elapsed time $<$ cutoff do
6. $S := InitConstruct(G)$
7. $S_{best} := ClubSearch(G, S)$
8. if $|S_{best}| > |S^*|$ then
9. $S^* := S_{best}$
10. $G := DRM(G, |S^*|, k, 2)$
11. if $|V_G| \leq |S^*|$ then return $S^*$
12. return $S^*$

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**The Main Framework of Nu/kCP**

The pseudo code of Nu/kCP is outlined in Algorithm 2. At first, the algorithm initializes $S^*$ (line 1) and calculates the lower bound $lb$ according to $k$ (lines 2-3). The algorithm reduces the original graph by calling the DRM (line 4).

In each loop (lines 5–11), the algorithm first constructs an initial candidate solution $S$ by calling our $InitConstruct$ process (line 6). Specifically, $InitConstruct$ first selects a random vertex $v \in V$, and then sets $S = N_1^k[v]$ as an initial solution. $InitConstruct$ iteratively removes vertices according to the selection removing rule until $S$ becomes a feasible solution, and $S$ will be returned. Afterwards, the algorithm calls the $ClubSearch$ process to improve the current solution (line 7). If the local best solution $S_{best}$ in this search trajectory is better than $S^*$, then $S^*$ is updated by $S_{best}$ and the algorithm tries to reduce the graph $G$ again (lines 8–10). If the size of remaining vertices in $G$ is smaller than $|S^*|$ which means the optimal solution is found, then we can return $S^*$ in advance (line 11). Otherwise, we return $S^*$ when the time limit is reached (line 12).

**The ClubSearch Function**

After getting the initial solution, the algorithm calls the $ClubSearch$ function to improve this solution. We formalize the $ClubSearch$ function in Algorithm 3 as below. At the beginning, $step$ and $S_{best}$ are set to $0$ and $S$, respectively (line 1). The spanning tree of each vertex in $S$ is initialized according to the constructing rule (line 2). The algorithm sets the $conf$ and $thread$ of each vertex to $k$ (line 3). After that, the algorithm searches for a local optimal solution denoted by $S_{best}$ (lines 4–18). Finally, the algorithm returns $S_{best}$ when $step$ reaches parameter $stepMax$ (line 19).

During each step, if $S$ is a feasible solution, $S_{best}$ is updated by $S$ and $step$ is reset to $0$ (line 6). The algorithm selects a vertex by using the selection adding rule and the STCC strategy, and then adds it into $S$ (line 7). Otherwise, if $S$ is not a feasible solution, the algorithm uses the above same adding strategy to add at most two vertices into $S$ (lines 9–11). This is because for the M\&CP, adding two vertices may make the solution become feasible while adding only one vertex fails to obtain a feasible solution in some
Algorithm 3: ClubSearch(G, S)

Input: Graph G, a feasible solution S
Output: the best local solution S_{best} found

1. $step := 0$, $S_{best} := S$;
2. build spanning tree $T^w$ for $\forall v \in S$ according to Constructing Rule;
3. $conf[v] := \text{thr}[v] := k$ for $\forall v \in V$;
4. while $step < stepMax$ do
5. if $S$ is a $k$-club then
6. $S_{best} := S$, $step := 0$;
7. select $v_1$ with $conf[v_1] \geq \text{thr}[v_1]$ according to Selection Adding Rule;
8. $S := S \cup \{v_1\}$ and update the spanning trees, $conf$ and $\text{thr}$;
9. while $S$ is infeasible and $|S| \leq |S_{best}| + 2$ do
10. select $v_2$ with $conf[v_2] \geq \text{thr}[v_2]$ according to Selection Adding Rule;
11. $S := S \cup \{v_2\}$ and update the spanning trees, $conf$ and $\text{thr}$;
12. if $S$ is infeasible then
13. select $v_1$ according to Selection Removing Rule;
14. $S := S \setminus \{v_1\}$ and update the spanning trees and $conf$;
15. if $S$ is infeasible and with probability $\alpha$ then
16. select $v_2$ according to Selection Removing Rule;
17. $S := S \setminus \{v_2\}$ and update the spanning trees and $conf$;
18. $step := step + 1$;
19. return $S_{best}$.

Experimental Evaluation

We carry out experiments to evaluate Nu\textsc{c}P on a broad range of random and DIMACS benchmarks as well as massive graphs for $k = 2, 3, 4$. We compare Nu\textsc{c}P with three previous algorithms, including VNS (2013a), mS/B (2014b)\(^2\) and ITDBC (2018). Note that mS/B is only designed for $k = 2$ and 3.

All algorithms were implemented in C++ and compiled by g++ with ‘-O3’ option. CPLEX 12.63\(^3\) and Gurobi\(^4\) are used in the mS/B and ITDBC, respectively. We set the same parameters as described in the corresponding literature and optimize these parameters for the new added instances. The parameters step\textsc{Max} and $\alpha$ in Nu\textsc{c}P are set to 100 and 0.6 according to our preliminary experiment. All experiments are run on Intel Xeon E5-2640 v4 @ 2.40GHz CPU with 128GB RAM under CentOS 7.5.

We use the same generator method as (Moradi and Balasundaram 2018) to randomly generate 90 instances for $k = 2, 3, 4$. These random graphs are divided into 9 families, each of which has 10 instances. As for the DIMACS benchmark\(^5\), we collect all DIMACS instances used in the previous literature for the MinCP. Thus, we select a total of 22 DIMACS instances. We consider massive real-world graphs from the Network Data Repository (Rossi and Ahmed 2015) and among them we choose 65 graphs with more than $10^5$ vertices and more than $10^6$ edges in this work, which has been widely used into testing different graph problems (Cai et al. 2020; Wang et al. 2020a,b).

For each instance, VNS and Nu\textsc{c}P are executed 10 times with random seeds from 1 to 10 while mS/B and ITDBC only execute one time since the random seeds do not affect them. The time limit of all algorithms is set to 1000 seconds for the random and DIMACS benchmarks, while the time limit is 3600 seconds for the massive graphs. For each instance, $\text{max}$ denotes the best size found and $\text{avg}$ denotes the average size obtained over 10 runs. When $\text{max} = \text{avg}$, we do not report $\text{avg}$. For random graphs, we report for each family the average value of $\text{max}$, denoted as $\overline{\text{max}}$. If an algorithm fails to provide a solution for an instance, then the corresponding column is marked as “N/A”. If an algorithm proves the optimal solution, the corresponding column is marked with a “*”. The bold values indicate the best solution value among the different algorithms. Due to space

\(^2\)mS/IP and mB/IP (2014b) are specialized for 2-club and 3-club, respectively. We use mS/B to denote these two algorithms.

\(^3\)https://www.ibm.com/products/software

\(^4\)http://www.gurobi.com

\(^5\)https://www.cc.gatech.edu/dimacs10
The results on the massive graphs are presented in Tables 3 and 4, where we only present the results for \( k = 2,3, \) and the results for \( k = 4 \) can be found in the supplementary material. Nu\( k \)CP performs better on almost all instances except two instances where mS/B finds better solutions. Among these instances, our Nu\( k \)CP can prove the optimal solution for 44 out of 195 instances, and most of them are concentrated in the case of \( k = 2 \). Also, the results show that the performance of our Nu\( k \)CP algorithm becomes a bit worse as the value of \( k \) increases. This is because when \( k \) has a large value, the size of solution is also large and thus Nu\( k \)CP costs a lot of time to calculate the upper bound as well as maintain the solution.

We compare the average run time of these four algorithms.

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| Instance                        | \( k = 2 \) | \( k = 3 \) |
|---------------------------------|-------------|-------------|
|                                | Nu\( k \)CP | VNS         | ITDBC       | mS/IP       | Nu\( k \)CP | VNS         | ITDBC       | mB/IP       |
|                                | max (avg)   | max (avg)   | max (avg)   | max (avg)   | max (avg)   | max (avg)   | max (avg)   | max (avg)   |
|--------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| bn-human-BNU1_0*5_s*n1-1-bg    | 8255        | 8081        | 8081        | 8227        | 15101       | N/A         | 8081        | 15101       |
| bn-human-BNU1_0*5_s*n2-bg      | 7494        | 7432        | 7432        | 7441        | 12117       | (12034)     | N/A         | 7432        | 12056       |
| co-authors-dblp                | 3300*       | 3300        | 3300        | 3300        | 7098        | 4369        | 3300        | 6652        |
| ca-dblp-2012                   | 344*        | 344         | 344         | 344         | 2136        | 608         | 2136        | 2136        |
| ca-hollywood-2009              | 11468       | 11468       | N/A         | 11468       | 17777       | N/A         | N/A         | 17777       |
| channel-500x100x100-b050       | 19          | 19          | 19          | 19          | 44          | 30          | 44          | 44          |
| dbpedia-link                   | 293749      | N/A         | N/A         | 293749      | 369531      | N/A         | N/A         | 369531      |
| delaunay_n22                   | 24          | 24          | 24          | 24          | 52          | 38          | 52          | 52          |
| delaunay_n23                   | 29          | 29          | 29          | 29          | 62          | 48          | 62          | 62          |
| delaunay_n24                   | 27          | 27          | 27          | 27          | 58          | 46          | 58          | 58          |
| friendster                      | 3005        | N/A         | N/A         | 3005        | 5932        | N/A         | N/A         | 5932        |
| hugebubbles-00020              | 5           | 4           | 5           | 4           | 7           | 6           | 7           | 7           |
| hugetrace-00010                | 5           | 4           | 5           | 4           | 7           | 6           | 7           | 7           |
| hugetrace-00020                | 5           | 4           | 5           | 4           | 7           | 6           | 7           | 7           |
| inf-europe_osm                 | 14          | 14          | 14          | 14          | 23          | (20.5)      | 15          | 14          | 18          |
| inf-germany_osm                | 14          | 14          | 14          | 14          | 23          | 15          | 14          | 18          |
| inf-roadNet-CA                 | 13*         | 13          | 13          | 13          | 17          | 16          | 17          | 17          |
| inf-roadNet-PA                 | 10*         | 10          | 10          | 10          | 15*         | 14          | 15          | 15          |
| inf-road-usa                   | 10*         | 10          | 10          | 10          | 16          | 12          | 10          | 16          |
| rec-dating                     | 33412       | N/A         | N/A         | 33412       | 54051       | N/A         | N/A         | 54051       |
| rec-epinions                   | 158933      | N/A         | N/A         | 158933      | 191728      | N/A         | N/A         | 191728      |
| rec-libimseti-dir              | 33390       | N/A         | N/A         | 33390       | 56801       | N/A         | N/A         | 56801       |
| rgg_n2_23_s0                   | 41          | 41          | 41          | 41          | 65          | 55          | 65          | 65          |
| rgg_n2_24_s0                   | 42          | 41          | 42          | 41          | 75          | 58          | 75          | 75          |
| rt-retweet-crawl               | 5071*       | 5071        | 5071        | 5071        | 6499        | 5732        | 6338        |               |
| sc-ldoor                       | 77          | 77          | 77          | 77          | 133         | 112         | 133         | 130         |
| sc-msdoor                      | 77          | 77          | 77          | 77          | 126         | 126         | 126         | 126         |
| sc-pwtk                        | 180*        | 180         | 180         | 180         | 214         | 214         | 214         | 214         |
| sc-re9                         | 168         | 168         | 168         | 168         | 266         | N/A         | 266         | N/A         |
| sc-shipsec1                    | 71          | 71          | 71          | 71          | 162         | 160         | 162         | 162         |
| sc-shipsec5                    | 90          | 90          | 90          | 90          | 187         | 187         | 187         | 187         |
| soc-buzznet                    | 64290*      | 64290       | 64290       | 64290       | 72280       | (72279)     | N/A         | 72276       |
| soc-delicious                  | 3217*       | 3217        | 3217        | 3217        | 6369        | (6317.1)    | 4244        | 3217        | 5465        |
| soc-digg                       | 17644*      | 17644       | 17644       | 17644       | 27771       | (27680)     | N/A         | 27653       |
| soc-dogster                    | 46504       | N/A         | N/A         | 46504       | 70507       | N/A         | N/A         | 70507       |
| socfb-B-anon                   | 4916*       | 4916        | 4916        | 4916        | 8903        | N/A         | N/A         | 8903        |
| socfb-B-unon                   | 4357        | 4357        | 4357        | 4357        | 7402        | (7153)      | N/A         | 6946        |
| socfb-uci-uni                  | 4961*       | N/A         | N/A         | 4961        | 11088       | (7378.2)    | N/A         | 6928        |
| soc-flickr                     | 4370        | 4370        | 4370        | 4370        | 9976        | (9883)      | N/A         | 9923        |

Table 3: Experiment results on the massive graphs I.

limitations, the detailed results as well as the source code of our Nu\( k \)CP can be found in the supplementary material\(^6\).

Tables 1 and 2 summarize the results of the random and DIMACS benchmarks, respectively. Most instances are so easy that all algorithms find the same quality values. We do not report the detailed results of these instances in Tables 1 and 2. For three instance families in the random graphs, Nu\( k \)CP finds better solutions. As for the DIMACS benchmark, only for email with \( k = 3 \), Nu\( k \)CP fails to find the same solution as VNS. Moreover, for 75 out of 270 random graphs and 30 out of 66 DIMACS instances, Nu\( k \)CP can prove the optimal solution.

\(^6\)https://github.com/yiyuanwang1988/NuCP.git
Table 4: Experiment results on the massive graphs II.

| Instance                  | k=2      |   | k=3      |   |
|---------------------------|---------|---|---------|---|
|                           | Nu\k CP | VNS | ITDBC   | mS/IP | Nu\k CP | VNS | ITDBC   | mB/IP |
|                           | max     | max | max     | max   | max     | max | max     | max   |
|                           | (avg)   | (avg)| (avg)   | (avg) | (avg)   | (avg)| (avg)   | (avg) |
| soc-flickr-und            | 27237   | N/A | N/A     | 27237 | 40654   | N/A | N/A     | 40654 |
| soc-flixster              | 1475    | 1475| 1475    | 1475  | 3891(3800.1)| N/A | N/A     | 3709  |
| soc-FourSquare            | 106229  | N/A | N/A     | 106229| 106513  | N/A | N/A     | 106513|
| soc-lastfm                | 5151*   | 5151| 5151    | 5151  | 8227(8196.5)| N/A | N/A     | 8105  |
| soc-livejournal           | 2652    | 2652| 2652    | 2652  | 3517     | N/A | N/A     | 3683  |
| soc-livejournal-user-groups| 1053721 | N/A | N/A     | 1053721| 1300141 | N/A | N/A     | 1300141|
| soc-LiveMocha             | 2981    | 2981| 2981    | 2981  | 8182(8133)| N/A | N/A     | 8242  |
| soc-ljournal-2008         | 19433   | N/A | N/A     | 19433 | 25624   | N/A | N/A     | 25624 |
| soc-orcut                 | 27467   | N/A | N/A     | 27467 | 50600   | N/A | N/A     | 50600 |
| soc-orcut-dir             | 33314   | N/A | N/A     | 33314 | 59375   | N/A | N/A     | 59375 |
| soc-pokec                 | 14855*  | 14855| 14855  | 14855| 16415(16414.7)| N/A | N/A     | 16290 |
| soc-sinaweibo             | 278490  | N/A | N/A     | 278490| 382513  | N/A | N/A     | 382513|
| soc-twitter-higgs         | 51387   | N/A | N/A     | 51387 | 78697   | N/A | N/A     | 78697 |
| soc-youtube               | 25410*  | 25410| 25410  | 25410| 33636(33538)| N/A | N/A     | 33413 |
| soc-youtube-snap          | 28755*  | 28755| 28755  | 28755| 41215(40864)| N/A | N/A     | 40605 |
| tech-as-skitter           | 35456   | 35456| 35456  | 35456| 57395(57369)| N/A | N/A     | 57394 |
| tech-ip                   | 1833162 | N/A | N/A     | 1833162| 1855649| N/A | N/A     | 1855649|
| twitter_mpi               | 532053  | N/A | N/A     | 532053| 765315  | N/A | N/A     | 765315|
| web-arabic-2005           | 1103*   | 1103| 1103    | 1103  | 1137*   | 1137| 1137    | 1137  |
| web-baidu-baike           | 97849   | N/A | N/A     | 97849 | 97849   | N/A | N/A     | 97849 |
| web-it-2004               | 470*    | 470 | 470     | 470   | 1064494| 1064494| N/A     | 1064494|
| web-uk-2005               | 851*    | 851 | 851     | 851   | 302564  | 302564| N/A     | 302564|
| web-wikipedia_link        | 825148  | N/A | N/A     | 825148| 302564  | 302564| N/A     | 302564|
| web-wikipedia2009         | 2625*   | 2625| 2625    | 2625  | 3183(2858.1)| N/A | N/A     | 2685  |
| web-wikipedia-growth      | 226074* | N/A | N/A     | 226074| 28755   | N/A | N/A     | 28755 |
| wikipedia_link_en         | 68873   | N/A | N/A     | 68873 | 68873   | N/A | N/A     | 68873 |

Table 5: Average run time for all benchmarks

| benchmark                     | k | Nu\k CP | VNS | ITDBC | mS/B |
|-------------------------------|---|---------|-----|-------|------|
| random graphs                 | 2 | 0.01    | 0.01| 0.01  | 0.01 |
|                               | 3 | 0.01    | 0.06| 0.02  | 0.01 |
|                               | 4 | 0.08    | 2.4 | 1.3   |      |
| DIMACS                         | 2 | 0.01    | 49.98| 6.28  | 0.01 |
|                               | 3 | 9.09    | 46.77| 40.52 | 29.43|
|                               | 4 | 6.77    | 18.72| 9.59  |      |
| massive graphs                | 2 | 33.82   | 1288.89| 1230.22| 71.33|
|                               | 3 | 912.61  | 3054.07| 2375.97| 1229.91|
|                               | 4 | 2071.4  | 3465.55| 2436.12|      |

Table 6: Reduction efficiency of TRIM and DRM.

The Effectiveness of the Proposed Strategies

Table 4 reports the reduction ratio ($r\%$) and the time consumption ($time(s)$) of TRIM (2018) and DRM for the initial reduction on all benchmarks with the same bound functions. The difference of the reduction ration between DRM and TRIM is obvious on the massive graphs. This is because TRIM costs too much time on calculating upper bound values and fails to delete any vertices.

To further verify the effectiveness of the proposed Nu\k CP, we also evaluate the performance of Nu\k CP on two popular benchmarks, including Stanford Large Network Dataset Collection\footnote{http://snap.stanford.edu/data} and DIMACS10\footnote{https://www.cc.gatech.edu/dimacs10/}. Due to space limitations, we present the detailed results of Nu\k CP and all competitors in the supplementary material.
As we know, it is the first work for solving the M$k$C problem. The proposed STCC plays a key role in the Nu$k$CP, and thus the proposed Nu$k$CP can help establish such a standard for the M$k$C problem. As for future work, STCC can be considered as a general idea to solve many other optimization problems with connectivity constraints.

## Conclusion

In this paper, we propose an efficient reduction strategy and a variant of configuration checking strategy for the M$k$C problem. Based on the above strategies, we develop a local search algorithm called Nu$k$CP. Experiments show that Nu$k$CP significantly outperforms the state-of-the-art heuristic algorithms. As we know, it is the first work for solving the M$k$C problem on the massive graphs, and thus the proposed Nu$k$CP can help establish such a standard for the M$k$C problem. As for future work, STCC can be considered as a general idea to solve many other optimization problems with connectivity constraints.

## Acknowledgements

This work was supported by National Key R&D Program of China (2020AAA0105200), NSFC under Grant No. 61806050, 61972063, 61976050, the Fundamental Research Funds for the Central Universities (2412020FZ030, 2412018QD022), and Jilin Science and Technology Association (QT202005).

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| Benchmark         | vs. Nu$k$CP1 | vs. Nu$k$CP2 |
|-------------------|--------------|--------------|
|                   | #better | #worse | #better | #worse |
| random graphs     | 1     | 1      | 1      | 1      |
| DIMACS            | 1     | 0      | 1      | 0      |
| massive graphs    | 15    | 4      | 18     | 3      |

Table 7: Compare Nu$k$CP with two modified versions on all benchmarks. #better and #worse denote the number of instances where Nu$k$CP finds better and worse results, respectively.

Figure 3: Average run time of Nu$k$CP and competitors.

![Graph showing average run time of Nu$k$CP and competitors]
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