HOMOLOGY OF EVEN ARTIN KERNELS

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ABSTRACT. We explicitly compute the homology groups with coefficients in a field of characteristic zero of cocyclic subgroups or even Artin groups of FC-type. We also give some partial results in the case when the coefficients are taken in a field of prime characteristic.

1. Introduction

The family of Artin-Tits groups has received increasing attention in the past years due to their intrinsic geometrical nature. They are closely related to Coxeter groups, that is, groups generated by reflections. Like Coxeter groups, Artin-Tits groups are defined in a combinatorial manner starting from a labeled graph which describes a presentation. Several questions arise from this fact, trying to determine to what extend properties of the groups can be described or characterized combinatorially, that is, in terms of a defining graph.

Several questions regarding this connection between group and defining graph remain open in general, but are solved for particular subfamilies of Artin-Tits groups, such as the family of right-angled Artin groups. Right-angled Artin groups are defined only by commutation relations among some of their generators. Properties such as polyfreeness or residually finiteness are satisfied for these groups, at least those associated with finite graphs (see [15, 14, 13]). Other important properties are also described combinatorially, such as, rigidity, the $K(\pi, 1)$-conjecture, quasi-projectivity or the main focus of this paper: the homology of Artin kernels.

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In short, an Artin kernel is a cocyclic subgroup of an Artin-Tits group. The homology with trivial coefficients of such a subgroup can be seen as a module over the ring of deck transformations. Precise definitions will be provided in §2.

The main purpose of this paper is to give an explicit combinatorial description of Artin kernels for a family of Artin-Tits groups that generalizes right-angled Artin groups, namely, even Artin groups. Our results generalize those in [16], [2].

The systematic study of even Artin groups was initiated by the first author in his Ph.D. thesis [4]. Some of the results in this thesis were published separately, for example in [5] there is a characterization of the even Artin groups which are quasi-projective in terms of the graph and in [6] it is shown that even Artin groups of FC-type are poly-free. The $K(\pi, 1)$-conjecture is also known to be true for even Artin groups of FC-type. As precursors of this thesis, one can find some results in the literature about even Coxeter groups (see for instance [11]) and a brief reference to even Artin groups in [2].

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These groups receive different names in the literature, note that we will refer to them as Artin-Tits groups since they are attributed to both E. Artin and J. Tits in their full generality. However we use right-angled Artin groups and even Artin groups for the subfamilies and Artin kernels for their cocyclic subgroups.

The paper is organized as follows. In §2 the main definitions of even Artin groups and Artin kernels are given together with the construction of the Salvetti complex. The Salvetti complex provides a $K(\pi, 1)$-model for the group whenever the defining graph is of FC-type. In §3 we use this complex to construct, for a given Artin kernel, a chain complex whose homology is the homology of that Artin kernel. Using this complex, we prove some partial results about the homology groups of Artin kernels with coefficients in an arbitrary field $\mathbb{K}$. These homology groups are in a natural way modules over a polynomial ring $\mathbb{K}[t^{\pm 1}]$ and thus decompose as a direct sum of a free part and a torsion part as follows

$$H_{k+1}(A^\chi_k; \mathbb{K}) = \mathbb{K}[t^{\pm 1}]^{r_k} \oplus \left( \mathbb{K}[t^{\pm 1}] \frac{\dim_k \ker \partial_{k+1}}{(t - 1)} \right)^{\infty} \bigoplus_{d \in \mathbb{T}_\gamma} \bigoplus_{j=1}^{\infty} \left( \mathbb{K}[t^{\pm 1}] \Phi_d(t) \right)^{n_{k,j}(d)},$$

where $r_k := \dim_k \tilde{H}_k(\mathcal{F}^f(\Gamma); \mathbb{K})$, $\mathcal{F}^f(\Gamma)$ is the finite type flag complex associated with $\Gamma$, $n_{k,j}(d) \in \mathbb{Z}_{\geq 0}$, $\Phi_d(t)$ is the $d$-th cyclotomic polynomial in $\mathbb{K}[t]$, and $\mathbb{T}_\gamma$ is a finite set. This is the main result of Theorem 3.9. The last two sections are devoted to calculating both the free and torsion part in terms of $\Gamma, \chi, \gamma$, and the characteristic of the base field $\mathbb{K}$. In particular, we dedicate §4 to determining the rank of the free part of such modules in the most general case, that is, any Artin kernel and any characteristic for the field $\mathbb{K}$ in Theorems 4.1 and 4.2. This gives an insight into the problem of finiteness properties of Artin kernels in terms of $\Gamma$ and $\chi$ as discussed in Example 3.3. Finally, in §5 the main results are proved, namely, a combinatorial description of the $k$-th homology with coefficients in a field of characteristic zero, of Artin kernels of even Artin groups of FC-type in terms of their defining graph. A non-resonance condition on the morphism $\chi$ is required for the techniques to work, namely, $\chi(g_v) \neq 0$ for all the standard generators $g_v$ of the Artin-Tits group $A_\Gamma$. A first discussion on the torsion part of $H_{1}(A^\chi_1; \mathbb{K})$ in terms of spanning trees is provided in Theorem 5.2. This approach is classical for right-angled Artin kernels and can be applied to even Artin kernel as well. The second part of the section deals with the combinatorial description of the torsion part of $H_{k+1}(A^\chi_k; \mathbb{K})$ by introducing a multiplicity spectral sequence $\{E^s_d\}$ of the finite type flag complex associated with the graph and with $d \in \mathbb{T}_\gamma$. The main result is provided in Theorem 5.7 where the sequence of $k$-th relative Euler characteristics of the multiplicity spectral sequence at the different pages of $\{E^s_d\}$, completely determines the invariant factors of the torsion part of $H_{k+1}(A^\chi_k; \mathbb{K})$ by a set of linear equations of the following type

$$\sum_{j \geq s} n_{k,j}(d) = \chi^{rel}_k(E^s_d),$$

where $n_{k,j}(d)$ is defined in §1 for any $d \in \mathbb{T}_\gamma$. Moreover, the Jordan blocks associated with the torsion of $H_{k+1}(A^\chi_k; \mathbb{K})$ have size at most $k + 2$. We end this paper with an example that shows that the bound provided for the Jordan blocks is sharp.

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2. Preliminaries

For the sake of completeness and to fix notation, we will give the explicit definition of Artin groups. To do that we will use the graph notation but note that our graphs are not the Coxeter-Dynkin diagrams of Artin groups. Instead, the way in which we define an Artin group associated to a graph generalizes the usual graph notation for right-angled Artin groups.

2.1. Even Artin groups. Let $\Gamma = (V, E, \ell)$ be a labeled finite simplicial graph. By a label we mean a map $\ell : E \to \mathbb{Z}_{>1}$. The Artin-Tits group associated with $\Gamma$ has the following finite presentation

\[ A_\Gamma := \langle g_v, v \in V : R(e, \ell(e)), \ e \in E \rangle, \]

where $R(e, \ell(e))$, $e = \{v, w\}$ represents the relation $(g_v g_w)^k = (g_w g_v)^k$ if $\ell(e) = 2k$ and $(g_v g_w)^k g_v = (g_w g_v)^k g_w$ if $\ell(e) = 2k + 1$.

In this paper we consider positive even labels, that is, $\ell(e) = 2 \ell(e)$ with $\ell(e) \geq 1$. An Artin-Tits group is called even (EAG for short) if its associated graph $\Gamma$ has only even labels. In the case of EAGs note that the relations $R(e, \ell(e))$ are commutator relations $[a, b]^k = (ab)^k (ba)^{-k}$ generalizing $[a, b] = [a, b]_1$.

An Artin-Tits group $A_\Gamma$ is called of spherical type if $W_\Gamma$ is a finite group. Another interesting class of Artin-Tits groups is given as follows. Consider $X \subseteq \Gamma$ a labeled subgraph of $\Gamma$, $A_X$ the Artin-Tits group associated with $X$ and $V_X \subseteq V$ the set of vertices in $X$. The subgroup of $W_\Gamma$ generated by $\{g_v, v \in V_X\}$ is in fact the Coxeter group $W_X$ (see [1]) called the standard parabolic subgroup of $W_\Gamma$ associated with $X$. An Artin-Tits group $A_\Gamma$ is said to be of FC-type if all the standard parabolic groups $W_X$ associated with complete subgraphs $X \subseteq \Gamma$ are finite.

A standard operation to obtain new Artin-Tits groups from old ones comes from taking the 2-join of two graphs, namely, the join of the graphs in which all the new edges are labeled by two. Formally, consider $\Gamma_1 = (V_1, E_1, \ell_1)$ and $\Gamma_2 = (V_2, E_2, \ell_2)$ two labeled graphs and define $\Gamma_1 \ast \Gamma_2 = (V, E, \ell)$ as $V := V_1 \cup V_2$, $E := E_1 \cup E_2 \cup V_1 \times S V_2$ (where $V_1 \times S V_2$ denotes the symmetric product of $V_1$ and $V_2$) and

\[ \ell(e) = \begin{cases} 
\ell_i(e) & \text{if } e \in E_i \\
2 & \text{if } e \in V_1 \times S V_2.
\end{cases} \]

Note that $A_{\Gamma_1 \ast \Gamma_2} = A_{\Gamma_1} \times A_{\Gamma_2}$. We refer to a labeled graph or to its associated Artin-Tits group as irreducible if it is not the 2-join of two labeled graphs.

Example 2.1. Artin-Tits complete graphs with two vertices are always spherical and their Coxeter groups are dihedral groups of order $2\ell$. For $\ell \geq 3$ they correspond to the Dynkin diagrams of types $A_2$, $B_2$, resp. $I_2(\ell)$ for $\ell = 3, 4$, resp. $\ell \geq 5$. And their defining graphs in our sense are of the form: (see [11][12]).

\[ \bullet \ell \bullet \]
The only irreducible spherical graphs with three vertices are those whose Dynkin diagram is of type $A_3$, $B_3$, and $H_3$. Note that none of them yields an even Artin group. In particular, the only irreducible EAGs are the cyclic kind $A_1$, the dihedral kind $B_2$, and $H_2(2k)$, $k > 2$. In other words, any spherical EAG must be a 2-join of these.

Example 2.2. As a consequence of the discussion above, RAAGs are of FC-type and all their subgroups associated to complete subgraphs of $\Gamma$ are free abelian.

2.2. Finite type flag complex. Artin-Tits groups of FC-type satisfy the $K(\pi, 1)$ conjecture (see [10]), which in particular means that there is a nice combinatorial description of a natural Eilenberg-MacLane space. In this section we will briefly describe such spaces for EAGs. This space can be described via a CW-complex called the Salvetti complex (see [9, 19]).

We first define the finite type flag complex $F^f(\Gamma)$ of $\Gamma = (V, E, \ell)$ as follows. Consider $S_k^f := \{X \subset \Gamma \mid W_X \text{ is finite}\}$. To construct $F^f(\Gamma)$, a simplex of dimension $k$ of $F^f(\Gamma)$ is given by any subgraph $X \subset \Gamma$, $V_X = \{v_0, \ldots, v_k\} \subset V$ such that $X \in S_k^f$. We will fix an order in the set of vertices of $\Gamma$. This order yields an orientation in each $X \in S_k^f$. Let $K$ be a field (of arbitrary characteristic at this point). The $K$-chain complex $C^f_k(\Gamma)$ can be constructed as follows.

$$C^f_k(\Gamma) = \sum_{|V_X| = k + 1} c_X K$$

with differential

$$\partial_k(c_X) = \sum_{v \in X} \langle X_v | X \rangle c_{X_v},$$

where $X_v$ results from $X$ after deleting the vertex $v \in X$ and $\langle X_v | X \rangle$ is the incidence of $X_v$ in $X$ and it is 1 or $-1$ according to the orientation given in $X$, namely, if $X = (v_0, \ldots, v_k)$, then $\langle X_v | X \rangle = (-1)^i$.

This simplicial complex will be called the finite type flag complex $F^f(\Gamma)$ associated with $\Gamma$.

2.3. The Salvetti complex of an FC-type graph. The Salvetti complex $S\operatorname{al}(\Gamma)$ of an FC-type graph $\Gamma$ can be briefly defined as the 2-presentation complex associated to the presentation $(2)$ of the Artin-Tits group $A_\Gamma$ after attaching higher dimensional cells for each complete subgraph of $\Gamma$. As a first approximation its 0-skeleton is given by a unique cell $K[\Gamma] = K[\chi(\Gamma)] = K[\ell_\Gamma]$, $v \in V$, and its 2-skeleton is given by 2 cells $\sigma_e, e \in E$. The differential of the Salvetti complex is zero in the case when $A_\Gamma$ is even.

2.4. Artin kernels and the equivariant $\partial^\chi$-complex. Consider a non-trivial morphism $\chi : A_\Gamma \to \mathbb{Z}$ for an even Artin group $A_\Gamma$. The kernel of this homomorphism is called the Artin kernel of $A_\Gamma$ associated with $\chi$ and will be denoted by $A_\Gamma^\chi$. We will say $\chi$ is resonant if $\chi(g_v) = 0$ for some $v \in V_\Gamma$, otherwise $\chi$ will be called non-resonant. Let us denote $m_v := \chi(g_v)$ so that an Artin kernel is represented by a tuple $(m_v)_{v \in V}$.

Note that the abelianization of $A_\Gamma$ is a free abelian group $H_\Gamma := A_\Gamma/A_\Gamma^\chi$ of rank $|V|$ and hence the universal abelian cover of $S\operatorname{al}(\Gamma)$ is given by a cell decomposition $C^f_k(S\operatorname{al}(\Gamma)) = C^f_k(S\operatorname{al}(\Gamma)) \times \mathbb{K}[H_\Gamma]$ where $\mathbb{K}[H_\Gamma] = \mathbb{K}[t_v^{\pm 1}, v \in V]$ is the group algebra of $H_\Gamma$ over $\mathbb{K}$. The action of $t_v \in H_\Gamma$ on $g_v$ is given by conjugation and it is represented as $t_v \ast g_v = g_v g_v^{-1}$ and one can check that it does not depend on the choice of representative in $H_\Gamma$. 


If $\chi$ is surjective, then it determines an infinite cyclic cover of $\overline{\text{Sal}}(\Gamma)$ which will be denoted as $\overline{\text{Sal}}^\chi(\Gamma)$ and whose chain complex $C_k(\overline{\text{Sal}}^\chi(\Gamma))$ has
\[
C_k(\overline{\text{Sal}}^\chi(\Gamma)) = \overline{C}_k(\overline{\text{Sal}}(\Gamma)) \oplus \mathbb{K}[t^{\pm 1}].
\]
Here, $\mathbb{K}[t^{\pm 1}]$ is a $\mathbb{K}[H_\Gamma]$-module by the action $t_v \ast 1 = t_v m_v$, where $t$ geometrically represents the action on the cyclic cover associated with the choice of a generator of $\text{im} \chi$.

This complex is called the equivariant $\partial^\chi$-complex associated with $\Gamma$ and $\chi$.

**Remark 2.3.** From the previous discussion note that, without loss of generality, one can assume that $\chi$ is an epimorphism, that is $\gcd\{m_v \mid v \in V\} = 1$. Otherwise, $\text{im} \chi = d\mathbb{Z}$ for $d = \gcd\{m_v \mid v \in V\}$, the action will be given by $t^d$ and $C_k(\overline{\text{Sal}}^\chi(\Gamma)) \cong C_k(\overline{\text{Sal}}^\chi(\Gamma))$ where $\frac{1}{d}\chi$ is now an epimorphism.

It is obvious from this description that the universal abelian cover could have been avoided altogether, however, it is sometimes more convenient from a conceptual point of view to present the cyclic covers this way. The universal abelian cover notation will be used throughout the paper to simplify some formulas.

Since $\text{Sal}(\Gamma)$ is an Eilenberg-MacLane space, $\overline{\text{Sal}}^\chi(\Gamma)$ is an Eilenberg-MacLane space as well and thus $H_k(\overline{\text{Sal}}^\chi(\Gamma)) = H_k(A^\chi)\overline{\text{Sal}}$ is a $\mathbb{K}[t^{\pm 1}]$-module.

Let us look at the structure of $\overline{\text{Sal}}^\chi(\Gamma)$ in more detail. First, the 0-skeleton is given by the orbit of a 0-cell $\sigma_0\chi$ by the cyclic group $\Delta r / \ker \chi$, that is, $t^n \sigma_0\chi$, where $\sigma_0\chi$ is a choice of a preimage of $\sigma_\theta$ by the cyclic cover. Hence $C_0(\overline{\text{Sal}}^\chi(\Gamma)) = \mathbb{K}[t^{\pm 1}]\sigma_0\chi$.

The 1-skeleton of $\overline{\text{Sal}}^\chi(\Gamma)$ is given by the 1-cells $t^n \sigma_1\chi$, where $\sigma_1\chi$ is a choice of 1-cell in the preimage of $\sigma_v$ by the cyclic cover. Note that $\sigma_1\chi$ is not a closed cell anymore and its boundary map is given by $\partial_0^\chi \sigma_1\chi = (t_v - 1)\sigma_1\chi = (t_v m_v - 1)\sigma_1\chi$.

Analogously, the 2-skeleton of $\overline{\text{Sal}}^\chi(\Gamma)$ is given by the 2-cells $t^n \sigma_2\chi$, $e = \langle v, w \rangle \in E$ and
\[
\partial_2^\chi \sigma_2\chi = \begin{cases} \langle t_v - 1 \rangle \sigma_2\chi - 1 \sigma_2\chi q_{\ell}(c)(t_v t_w), & \text{if } v \neq w \\ \langle t^n - 1 \rangle \sigma_2\chi - 1 \sigma_2\chi q_{\ell}(c)(t^n v), & \text{otherwise} \end{cases}
\]
where $q_{\ell}(x) = x^\chi - 1$, $m_v = m_v + m_w$ and $\ell = 2 \ell^X$. Note that this map is sensitive to the orientation given to $e = \langle v, w \rangle$.

Finally, the $k$-skeleton of $\overline{\text{Sal}}^\chi(\Gamma)$ is given by the $k$-cells $t^n \sigma_k\chi$, $X \in S^k$ and
\[
\partial_k^\chi \sigma_k\chi = \sum_{v \in X} \langle X_v \mid X \rangle (t_v - 1) \left[ \prod_{\ell \in c(v, w)} q_{\ell}(c)(t_v t_w) \right] \sigma_k^X.
\]

3. On the homology of the equivariant $\partial^\chi$-complex with coefficients in an arbitrary field

Recall that $\chi$ is an epimorphism (see Remark 2.3) and consider the equivariant $\partial^\chi$-complex $(C^\chi_k(\Gamma), \partial^\chi_k) = (C_{k+1}(\overline{\text{Sal}}^\chi(\Gamma)), \partial^\chi_{k+1})$ associated with $\Gamma$ and $\chi$ as described in §2.1.

\[
\text{...} \rightarrow C^\chi_k(\Gamma) \xrightarrow{\partial^\chi_k} C^\chi_{k-1}(\Gamma) \rightarrow \ldots
\]

**Proposition 3.1.** The following isomorphism holds as $\mathbb{K}[t^{\pm 1}]$-modules
\[
H_k(C^\chi_k(\Gamma), \partial^\chi_k) = H_{k+1}(A^\chi_k; \mathbb{K}).
\]
As a consequence, the homology groups \( H_{k+1}(A^\chi; \mathbb{K}) \) can be seen as \( \mathbb{K}[t^{\pm 1}] \)-modules. Since \( \mathbb{K}[t^{\pm 1}] \) is a principal ideal domain, these modules decompose as a torsion part and a free part. In the rest of this section we will see how to determine the free part at least in some cases and will prove some useful results about the torsion part.

Let us use the following notation. Recall that we are denoting \( m_v = \chi(v) \) and \( t_v = t^{m_v} \) for \( v \in V \) and \( t_e = t_v t_w \) for \( e = \{v, w\} \) an edge in \( \Gamma \).

Define the following resonance set of simplices \( \mathcal{R}(\Gamma, \chi, \mathbb{K}) = V_R \cup E_R \), where
\[
V_R := \{ v \in V \mid m_v = 0 \} \quad \text{and} \quad E_R := \{ e \in E \mid m_e = 0 \} \text{ and } \ell(e) \cdot 1_\mathbb{K} = 0 \}.
\]
Note that \( t_v - 1 \neq 0 \) if and only if \( v \notin V_R \) and \( q_{\ell(e)}(t_e) \neq 0 \) if and only if \( e \notin E_R \).

**Definition 3.2.** For \( X \in S^I \),
\[
p_{X} := \prod_{v \in V_X \setminus V_R} (t_v - 1) \quad \text{and} \quad q_{X} := \prod_{e \in E_X \setminus E_R} q_{\ell(e)}(t_e).
\]
If \( X = \{X_1, \ldots, X_r\} \) is a set of elements in \( S^I \), we also use the following notation
\[
p_{\bar{X}} := \prod_{i=1}^{r} p_{X_i} \quad \text{and} \quad q_{\bar{X}} := \prod_{i=1}^{r} q_{X_i}.
\]

Then \( p_X q_X \neq 0 \) for any \( X \in S^I \). Therefore one can formally rewrite (6) as
\[
\frac{1}{p_X q_X} \partial_X^\chi \sigma_X^\chi = \sum_{\substack{Y \subset X \backslash \text{non-resonant} \atop |Y| = k}} (Y|X) \frac{1}{p_Y q_Y} \sigma_Y^\chi,
\]
where the sum is taken over the non-resonant \( Y \subset X \), that is, \( v \notin V_R \) for any \( v \in X \setminus Y \), and \( e = \{v, w\} \notin E_R \) for any \( v \in X \setminus Y \) and \( w \in Y \).

**Definition 3.3.** We say \( \chi \) is \( \mathbb{K} \) non-resonant if \( \mathcal{R}(\Gamma, \chi, \mathbb{K}) = \emptyset \).

Note that if \( \chi \) is \( \mathbb{K} \) non-resonant, for \( X \in S^I \), one obtains
\[
p_{X} := \prod_{v \in V_X} (t_v - 1) \quad \text{and} \quad q_{X} := \prod_{e \in E_X} q_{\ell(e)}(t_e).
\]

Note that if \( \mathbb{K} \) has characteristic zero, then a character is \( \mathbb{K} \) non-resonant if and only if it is non-resonant.

For the rest of this section we will fix a field \( \mathbb{K} \) of arbitrary characteristic and assume that the character \( \chi \) is \( \mathbb{K} \) non-resonant.

### 3.1. The free part in the \( \mathbb{K} \) non-resonant case.

**Theorem 3.4.** \( \mathbb{K} \) Let \( \chi : A_{\Gamma} \to \mathbb{Z} \) be a \( \mathbb{K} \) non-resonant epimorphism. If \( A^\chi_{\Gamma} := \ker \chi \), then the free part of \( H_{k+1}(A^\chi_{\Gamma}; \mathbb{K}) \) as a \( \mathbb{K}[t^{\pm 1}] \)-module has rank \( r_k := \dim_{\mathbb{K}} \tilde{H}_k(F^I(\Gamma); \mathbb{K}) \).

**Proof.** For simplicity let us denote by \( F = F^I(\Gamma) \) the finite type flag complex of \( \Gamma \), by \( F_k = F_k^I(\Gamma) \) its set of \( k \)-simplices, and by \( C_k = C_k^I(\Gamma) \) the free abelian group generated by \( F_k \). Note that \( \left( \frac{1}{p_X q_X} \sigma_X^\chi \right)_{X \in F_k} \) is a basis of the vector space \( C_k \otimes \mathbb{K}(t) \). Analogously, \( (\sigma_X)_{X \in F_k} \) a basis of \( C_k \otimes \mathbb{K} \). Note that both spaces have the same dimension over their respective fields, moreover by (7) and since \( \chi \) is \( \mathbb{K} \) non-resonant, both boundary maps are given by the incidence matrix \((Y|X)_{X,Y} \), where \((Y|X)\) is defined for \( X \in C_k \), \( Y \in C_{k-1} \) and is given as in (6) if \( Y \subset X \) and as 0 otherwise. Hence the result follows. \( \square \)
3.2. A resolution matrix. We will use the notation $M^X_k(t)$ to denote the matrix of the homomorphism $\partial_k$ of $\mathbb{K}[t^{\pm 1}]$-modules with respect to the natural bases $C^X_k(\Gamma)$ and $C^X_{k-1}(\Gamma)$. Also, $M_k$ will represent the matrix with respect to the analogous basis over $\mathbb{K}$ of the homomorphism $\partial_k$ of the complex $(C^X(\Gamma), \partial^X)$ defined in §2.2. In order to give formulas for the torsion we will study the set of invariants of the matrices $M^X_k(t)$. This is a consequence of the following well-known result.

**Lemma 3.5.** The torsion part of $H_{k+1}(A^X_1; \mathbb{K})$ coincides with the torsion part of $\ker \partial^X_k$. In particular, the non-trivial invariant factors of $M^X_k(t)$ determine the torsion part of $H_{k+1}(A^X_1; \mathbb{K})$.

**Proof.** The short exact sequence

$$0 \rightarrow H_{k+1}(A^X_1; \mathbb{K}) = H_k(C^X_k(\Gamma), \partial^X_k) \rightarrow C^X_k(\Gamma) \xrightarrow{\im \partial^X_k} \ker \partial^X_k \cong \im \partial^X_k \rightarrow 0$$

follows from Proposition 3.1. The right-most term is free since it is a submodule of $C^X_k(\Gamma)$, which is a free module over a PID, hence the first part follows. The second part is a consequence of the structure theorem for modules over a PID and the fact that $M^X_k(t)$ is the free presentation matrix of $\ker \partial^X_k$.



3.3. The Fitting ideals of $M^X_{k+1}(t)$. By Lemma 3.5 it is enough to calculate the invariant factors of $M^X_{k+1}(t)$, or equivalently, its Fitting ideals. Recall that the $s$-th Fitting ideal $I_s$ associated with an $R$-module $U$ is given as the ideal generated by the minors of size $r \times r$ for $r = m - s$ of any free presentation matrix $M$ of $U$, that is, $R^m \xrightarrow{M} R^m \rightarrow U \rightarrow 0$. To see this, recall that

$$I_s \subseteq I_{s+1}.$$ 

If $R$ is a PID and we write $I_s = f_s R$, then we have $f_{s+1} | f_s$ and the invariant factors of $M$ are the elements $g_s := \frac{f_s}{f_{s+1}}$. These ideals yield the usual a decomposition of $U$ as a sum of a free $R$-module and modules of the form $R/g_s R$.

Note that a square submatrix of $M^X_{k+1}(t)$ of size $r \times r$ is determined by the choice of $r$ $(k+1)$-simplices and $r$ $(k)$-simplices. We will denote such a submatrix by $S(\bar{X}, \bar{Y})$, where $\bar{X} = \{X_1, \ldots, X_r\}$ (resp. $\bar{Y} = \{Y_1, \ldots, Y_r\}$) is a list of $(k+1)$-simplices (resp. $(k)$-simplices). We define by $m^X_{(\bar{X}, \bar{Y})}$ (resp. $m_{(\bar{X}, \bar{Y})}$) the minors $\det(S(\bar{X}, \bar{Y}))$ of the matrix $M^X_{k+1}(t)$ (resp. $M_{k+1}$). One has the following immediate properties.

**Proposition 3.6.**

a) If $Y_i \not\subset X_j$ for some $j$ and any $i = 1, \ldots, r$, then $m^X_{(\bar{X}, \bar{Y})} = m_{(\bar{X}, \bar{Y})} = 0$. Analogously, if $Y_i \not\subset X_j$ for some $i$ and any $j = 1, \ldots, r$, then $m^X_{(\bar{X}, \bar{Y})} = m_{(\bar{X}, \bar{Y})} = 0$.

b) If $\bar{X}$ contains a $(k+1)$-cycle, that is, $\sigma := \sum_{i=1}^{\lambda_1} \lambda_i \sigma_{X_i}$, for some non-trivial choice $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ and $\partial^X_{k+1}(\sigma) = 0$, then $m^X_{(\bar{X}, \bar{Y})} = m_{(\bar{X}, \bar{Y})} = 0$.

In order to characterize the choices of $(\bar{X}, \bar{Y})$ whose associated minor $m^X_{(\bar{X}, \bar{Y})}$ is non zero we need the following.

**Definition 3.7.** Let $\bar{X}$ (resp. $\bar{Y}$) be a list of $(k+1)$-simplices (resp. $(k)$-simplices) in the finite type flag complex $\mathcal{F}^f$. Consider $N(\bar{X}) := \mathcal{F}^f_k \cup \cup_{\sigma_{k+1} \in \bar{X}} \sigma_{k+1}$ and $N(\bar{Y}^-) := \mathcal{F}^f_{(k-1)} \cup \cup_{\sigma_{k} \not\subset \bar{Y}} \sigma_{k}$. We say $(\bar{X}, \bar{Y})$ is $(k+1)$-acyclic of order $r$ (or simply acyclic of order $r$) if $(N(\bar{X}), N(\bar{Y}^-))$ is acyclic and $|\bar{X}| = |\bar{Y}| = r$. 


Using the notation above one has the following result on the minors of $M_{k+1}^X(t)$ in $\mathbb{K}[t]$.

**Proposition 3.8.** Let $\chi : A \rightarrow \mathbb{Z}$ be a $\mathbb{K}$-non-resonant morphism and let $m_{(X,Y)}^\chi$ be a minor of size $r \times r$ of the matrix $M_{k+1}^X(t)$ associated with the pair $(X,Y)$. Then

(i) $m_{(X,Y)}^\chi = \frac{p_{X,q_X}m_{(X,Y)}}{p_{r,q_Y}}$.

(ii) $m_{(X,Y)}^\chi$ is non-zero if and only if $(X,Y)$ is acyclic of order $r$.

(iii) The biggest possible size $r$ such that $m_{(X,Y)}^\chi \neq 0$ is $r = \dim_\mathbb{K} \im \partial_{k+1}$.

**Proof.** Part (i) is an immediate consequence of (7) since $\chi$ is $\mathbb{K}$-non-resonant. To prove part (ii) note that

$$C_i(N(X), N(Y^c)) = \begin{cases} C_{k+1}(X) & \text{if } i = k+1, \\ C_k(X, Y^c) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

since $X$ (resp. $Y^c$) has dimension $k+1$ (resp. $k$). Hence $H_{k+1}(N(X), N(Y^c); \mathbb{K}) = 0$ is equivalent to asking $C_{k+1}(X; \mathbb{K}) \hookrightarrow C_k(X, Y^c; \mathbb{K})$. However, since they both have the same dimension and $\det(S_{(X,Y)}) = m_{(X,Y)}$ this is in fact an isomorphism whose associated matrix is $S_{(X,Y)}$ and the result follows. Finally, the rank of $M_{k+1}^X$ is given by the dimension of $\im \partial_{k+1}$ and thus part (iii) follows.

### 3.4. Torsion in $H_{k+1}(A_t^X; \mathbb{K})$.

Note that the discussion about the Fitting ideals above together with Proposition 3.8[i] imply that the torsion part $H_{k+1}(A_t^X; \mathbb{K})$ can be described in terms of the $d$-th cyclotomic polynomials $\Phi_d(t)$ for $\mathbb{K}$ for $d$ dividing either $m_\varphi$ or $\ell(e)m_\epsilon$. Moreover, Proposition 3.8[ii] is also true even if the character is $\mathbb{K}$-resonant. However, we consider the $\mathbb{K}$-non-resonant case only to give a more detailed description in our next result.

**Theorem 3.9.** Consider $H_{k+1}(A_t^X; \mathbb{K})$ as a $\mathbb{K}[t^{\pm 1}]$-module where $\chi : A_t \rightarrow \mathbb{Z}$ is $\mathbb{K}$-non-resonant. Then

$$H_{k+1}(A_t^X; \mathbb{K}) = \mathbb{K}[t^{\pm 1}]^{r_k} \oplus \bigoplus_{d \in T_E} \left( \mathbb{K}[t^{\pm 1}] / (t-1)^{\dim_\mathbb{K} \im \partial_{k+1}} \right)^{n_{k,j}(d)} \oplus \bigoplus_{d \in T_V} \left( \mathbb{K}[t^{\pm 1}] / \Phi_d(t)^j \right)^{n_{k,j}(d)},$$

for some $n_{k,j}(d) \in \mathbb{Z}_{\geq 0}$, where $\Phi_d(t)$ is the $d$-th cyclotomic polynomial in $\mathbb{K}[t]$, $r_k := \dim_\mathbb{K} \tilde{H}_k(F^I(\Gamma); \mathbb{K})$, and

$$T_E = T_{V_r} \cup T_{E_r},$$

(8)

$$T_{V_r} = \bigcup_{e \in V_r} \{ d \in \mathbb{Z}_{>1} \mid m_\varphi = 0 \mod d \}, \text{ and}$$

$$T_{E_r} = \bigcup_{e \in E_r} \{ d \in \mathbb{Z}_{>1} \mid \ell(e)m_\epsilon = 0 \mod d, \text{ but } m_\epsilon \neq 0 \mod d \}.$$

**Proof.** The free part was given in Theorem 3.8. By Proposition 3.8[i] the only possible torsion appears as a root of polynomials of type either $p_X$ or $q_X$, that is, as roots of either $t^{m_\varphi} - 1$ or $q_{\ell(e)}(t^{m_\varphi})$. The union of the first type of roots is given by $T_{V_r}$ whereas the second type of roots is given by $T_{E_r}$. To end the proof, note that the hypothesis that $\chi$ is $\mathbb{K}$-non-resonant together with Proposition 3.8[ii] imply that the polynomial $(t-1)$ is a factor with multiplicity precisely $r$ of each $r \times r$ non-zero minor $m_{(X,Y)}^\chi$. Together with the discussion above this implies that the $(t-1)$-part of the torsion module is semisimple. Moreover, according to Proposition 3.8[iii], the biggest possible such $r$ is $\dim_\mathbb{K} \im \partial_{k+1}$, hence the result follows. \(\square\)
4. ON THE FREE PART OF $H_{k+1}(A_1^g;\mathbb{K})$ IN THE RESONANT CASE

In this section we will see how to adapt $\Gamma$ to apply Theorem 3.4 in the $\mathbb{K}$ resonant case for $H_i(A_1^g;\mathbb{K})$, $i = 1, 2$. The case of higher homology groups will be treated in a forthcoming paper. Let $\chi$ be a character and $R(\Gamma, \chi)$ its resonant set. In the following particular cases of edges $e = \{v, w\}$ one has

$$\partial^X(\sigma^\chi_e) = \begin{cases} (e|w)p_uq_v\sigma^\chi_e & \text{if } v \in V_R, w \notin V_R \\ 0 & \text{if either } v, w \in V_R \text{ or } e \in E_R. \end{cases}$$

Consider the graph $\Gamma_1$ obtained from $\Gamma$ after deleting the (open) edges $e = \{v, w\} \in E$ for which either $v, w \in V_R$ or $e \in E_R$ and the vertices $v \in V_R$ whose link intersects $V \setminus V_R$. There is a morphism between the (flag) chain complexes associated to $\Gamma$ and $\Gamma_1$ which is a quasi-isomorphism up to $H_0$. Moreover, $\chi$ induces a character $\chi_1 : A_{\Gamma_1} \to \mathbb{Z}$ which produces another Artin kernel $A_1^g$ such that $H_1(A_1^g;\mathbb{K}) \cong H_1(A_1^g;\mathbb{K})$. The character $\chi_1$ might still be K resonant for $\Gamma_1$, but the vertices in the kernel of $\chi_1$ are isolated in $\Gamma_1$. Hence the proof in Theorem 3.4 can be applied to $\Gamma_1$ to obtain the following result.

**Theorem 4.1.** Let $A_1^g$ be the Artin kernel of a general character $0 \neq \chi : A_{\Gamma} \to \mathbb{Z}$ of an even Artin-Tits group. Then the rank of the free part of $H_1(A_1^g;\mathbb{K})$ as a $\mathbb{K}[t^\pm 1]$-module is $\dim H_0(\Gamma;\mathbb{K})$. In particular, $H_1(A_1^g;\mathbb{K})$ is a torsion module if and only if $\Gamma_1$ is connected.

The previous theorem, when applied to right-angled Artin groups, recovers well-known results [8, 17], characterizing the group $A_1^g$ to be finitely generated (which implies $H_1(A_1^g;\mathbb{K})$ is torsion) if and only if $\Gamma \setminus V_R$ is connected and $\chi$ is dominant, that is, for any $v \in V_R$ there exists a $w \in V_{\Gamma} \setminus V_R$ such that $e = \{v, w\} \in E_{\Gamma}$, which is equivalent to asking $\Gamma_1$ to be connected.

Analogously, for the following particular cases of sets $X = \{u, v, w\} \in S^I$ note that

$$\partial^X(\sigma^\chi_e) = \begin{cases} (X_u|X)p_uq_v\sigma^\chi_e & \text{if } v, w \in V_R, u \notin V_R \\ (X_u|X)p_uq_v\sigma^\chi_e & \text{if } e = \{v, w\} \in E_R, u \notin V_R \\ 0 & \text{otherwise}. \end{cases}$$

The simplicial subcomplex $F_2$ obtained from the 2-skeleton of the finite type flag complex $F^I(\Gamma)$ after removing the 2-cells $X = \{u, v, w\} \in F^I(\Gamma)$ such that either $\{u, v\} \in E_R$ and $w \in V_R$ or $u, v, w \in V_R$, removing the 1-cells $e = \{v, w\}$ in $E_R$ or $v, w \in V_R$ whose link intersects $V_{\Gamma} \setminus V_R$ and then identifying the ends of the remaining 1-cells $e = \{u, v\}$ such that $u, v \in V_R$ or $e \in E_R$. As before one obtains a morphism of complexes which induces $H_2(A_1^g;\mathbb{K}) \cong H_1(C^I(F_2);\mathbb{K})$. In that case, $F_2$ might not be the finite type flag complex of an Artin-Tits group. However, the proof of Theorem 3.3 still applies to obtain the following result.

**Theorem 4.2.** Let $A_1^g$ be the Artin kernel of a general character $0 \neq \chi : A_{\Gamma} \to \mathbb{Z}$ of an even Artin-Tits group. Then the rank of the free part of $H_2(A_1^g;\mathbb{K})$ as a $\mathbb{K}[t^\pm 1]$-module is $\dim H_2(F_2;\mathbb{K})$. In particular, $H_2(A_1^g;\mathbb{K})$ is a torsion module if and only if $F_2$ is 1-acyclic.

**Example 4.3.** To illustrate the different behavior of the homology groups according to whether the character is or not $\mathbb{K}$ resonant, we can consider just the example of a dihedral Artin-Tits group. Let $\Gamma$ be the complete graph with two vertices as in Example 2.1 with label 4. Put
\[ V_\Gamma = \{ u, v \}, \quad E_\Gamma = \{ e = \{ u, v \} \}, \] and let \( \chi : A_\Gamma \to \mathbb{Z} \) be the character defined by \( \chi(g_u) = 1, \) \( \chi(g_v) = -1. \) Then \( \chi \) is \( \mathbb{K} \) non-resonant if and only if \( \text{char}(\mathbb{K}) \neq 2. \) Note that \( A_{\Gamma} = \langle g_u, g_v \mid (g_u g_v)^2 = (g_v g_u)^2 \rangle \) and \( A_{\chi}^{\Gamma} = \langle w_n = g_u^{n+1} g_v g_u^{-n} \mid w_n^2 = w_n^0, n \in \mathbb{Z} \rangle \) and hence

\[
H_1(A_{\chi}^{\Gamma}; \mathbb{K}) = \begin{cases} 
\mathbb{K}[t^{\pm 1}] & \text{if char}(\mathbb{K}) = 2, \\
\mathbb{K}[t^{\pm 1}] & \text{if char}(\mathbb{K}) \neq 2.
\end{cases}
\]

Theorems 3.4 and 3.9 hold for \( \text{char}(\mathbb{K}) \neq 2, \) since \( \tilde{H}_k(\mathcal{F}^{f}(\Gamma); \mathbb{K}) = 0 \) (\( \Gamma \) is contractible) and \( \dim_{\mathbb{K}} \partial_1 = 1 \) (\( \partial_1 \) is injective and \( C_1^{\chi}(\Gamma) = c_\mathbb{K} \)).

Note that if \( \text{char}(\mathbb{K}) = 2, \) then Theorems 3.4 and 3.9 do not hold, but one can apply the discussion in §4. In this case, the free part of \( H_1(A_{\chi}^{\Gamma}; \mathbb{K}) \) comes from the fact that \( \tilde{\Gamma}_1 \) is not connected, since it results from \( \Gamma \) after removing the edge.

In addition, the non-finiteness presentation of \( A_{\chi}^{\Gamma} \) is a consequence of the non-triviality of the free part of \( H_1(A_{\chi}^{\Gamma}; \mathbb{K}) \).

Also observe that the same argument applies to all dihedral Artin-Tits groups with edge labeled by \( 2\ell \) for the character \( \chi \) defined as above, i.e., that \( H_1(A_{\chi}^{\Gamma}; \mathbb{K}) \) is not a torsion module whenever \( \text{char}(\mathbb{K}) | \ell, \) thus showing that \( A_{\chi}^{\Gamma} \) does not admit a finite presentation.

**Example 4.4.** Consider the Artin-Tits group associated with the graph \( \Gamma \) shown in Figure 1 and the character \( \chi : A_\Gamma \to \mathbb{Z} \) defined as \( \chi(v_1) = \chi(v_2) = \chi(v_3) = 1, \) and \( \chi(v_0) = -1. \)

![Figure 1](image.png)

Note that \( \chi \) is \( \mathbb{F}_2 \) resonant with resonance set \( R(\Gamma, \chi, \mathbb{F}_2) = E_R = \{ \sigma_{02} \}. \) Its equivariant complex can be described as

\[
\begin{array}{cccccc}
0 & \to & C_3^\chi(\Gamma) & \to & C_2^\chi(\Gamma) & \to & C_1^\chi(\Gamma) & \to & C_0^\chi(\Gamma) & \to & 0 \\
\sigma_{012} & \mapsto & (t+1)\sigma_{02} & \mapsto & \sigma_0 & \mapsto & (t^{-1}+1)\sigma_0 \\
\sigma_{023} & \mapsto & (t+1)\sigma_{02} & \mapsto & \sigma_i, i = 1, 2, 3 & \mapsto & (t+1)\sigma_0 \\
\sigma_{02} & \mapsto & 0 \\
\sigma_{12} & \mapsto & (t+1)(\sigma_1 + \sigma_2) \\
\sigma_{23} & \mapsto & (t+1)(\sigma_2 + \sigma_3) \\
\sigma_{01} & \mapsto & (t+1)(\sigma_0 + t^{-1}\sigma_1) \\
\sigma_{03} & \mapsto & (t+1)(\sigma_0 + t^{-1}\sigma_3)
\end{array}
\]
Hence

\[ H_1(A_1^2; F_2) = \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} \sigma_1 + \sigma_2 \right) \oplus \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} (\sigma_2 + \sigma_3) \right) \oplus \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} (\sigma_2 + t \sigma_0) \right) \]

\[ H_2(A_1^2; F_2) = \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} \sigma_{02} \right) \oplus F_2[t^{\pm 1}] ((\sigma_{12} + \sigma_{23}) + t(\sigma_{01} + \sigma_{03})). \]

Note that \( \text{dim}_{F_2} H_1(A_1^2; F_2) < \infty \) but \( \text{dim}_{F_2} H_2(A_1^2; F_2) = \infty \) is consistent with the discussion above since \( F_2 = \Gamma_1 = \Gamma \setminus \{ \sigma_{02} \} \) is connected but not simply connected.

For the same graph but a different character \( \chi' : A_{\Gamma} \to \mathbb{Z} \) defined as \( \chi'(v_1) = \chi'(v_3) = 0 \), \( \chi'(v_2) = 1 \), and \( \chi'(v_0) = -1 \) note that \( \Gamma_1 = \Gamma \setminus \{ \sigma_{02} \} \) as before, however \( \sigma_{02} \) remains in the construction of \( F_2 \) as described above. In fact, \( F_2 = S^1 \vee S^1 \vee S^1 \), which implies,

\[ H_1(A_1^2; F_2) = \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} \sigma_1 \right) \oplus \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} \sigma_3 \right) \oplus \left( \frac{F_2[t^{\pm 1}]}{(t + 1)} (\sigma_2 + t \sigma_0) \right) \]

\[ H_2(A_1^2; F_2) = F_2[t^{\pm 1}] \sigma_{02} \oplus F_2[t^{\pm 1}] (\sigma_{12} + t \sigma_{01}) \oplus F_2[t^{\pm 1}] (\sigma_{23} + t \sigma_{03}). \]

5. Torsion in \( H_{k+1}(A_1^2; K) \) for \( \text{char}(K) = 0 \) and \( \chi \) non-resonant

The purpose of this section is to give more specific formulas for the invariants \( n_{k,j}(d) \) as introduced in Theorem 3.9 in the particular case \( \text{char}(K) = 0 \). Note that in this case, the notion of non-resonant and \( K \) non-resonant are equivalent.

We will assume in this section that \( \Gamma \) is connected. Otherwise \( A_{\Gamma} = A_{\Gamma_1} \ast A_{\Gamma_2} \) is a free product of groups and one can check that the corresponding equivariant complexes fit in a short exact sequence

\[ 0 \to C_\chi^X(\Gamma_1) \to \bar{C}_\chi^X(\Gamma) \to C_\chi^X(\Gamma_2) \to 0 \]

where \( \chi_i \) are the corresponding restrictions of \( \chi \) to \( \Gamma_i \) and \( \bar{C}_\chi^X(\Gamma) \) is a variation of \( C_\chi^X(\Gamma) \) where we replace \( C_\chi^X(\Gamma_1) \) by \( C_\chi^X(\Gamma_1) = C_\chi^X(\Gamma_1) \oplus C_\chi^X(\Gamma_2) \). This implies that as abelian groups

\[ H_{k+1}(A_1^2; K) = H_{k+1}(A_1^2; K) \oplus H_{k+1}(A_1^2; K) \]

for \( k > 0 \). However, as a word of caution, their submodule structure depends on the fact that \( \chi_i \) is a restriction of \( \chi \), for instance, in case \( \chi_i \) is not an epimorphism. In the case of \( k = 0 \), by Lemma 3.5 we get the same decomposition for the \( K[t^{\pm 1}] \)-torsion submodules.

First, a discussion for \( H_1(A_1^2; K) \) is included based on the structure of spanning trees of \( \Gamma \) following the ideas in [2]. A second discussion for the general \( H_{k+1}(A_1^2; K) \) requires the introduction of the multiplicity spectral sequence.

5.1. The \( H_1(A_1^2; K) \) case. In order to calculate \( n_{0,j}(d) \) we will obtain the invariants of the matrix \( M_1^X(t) \) which defines the boundary map of the \( \partial^X \)-complex with respect to the natural basis, as described in the previous section.

Note that \( \text{dim}_{F_2} \text{im} \partial_2 = |V_\Gamma| - 1 \) and hence maximal 1-acyclic pairs \((\bar{X}, Y)\) are given by \( \bar{X} \) the set of edges of a spanning tree \( T \) and \( Y^c \) the choice of a vertex \( v_1 \in V_\Gamma = V_T \). (observe that the number of vertices in any tree is exactly one plus the number of edges) We will call \( (T, v_1) \) a rooted spanning tree. All 1-acyclic pairs \((\bar{X}, Y)\) of size \( r = |V_\Gamma| - s \) can be obtained as follows. Consider \( F_s = T_1 \cup \ldots \cup T_s \) an s-forest, that is, a disjoint union of s trees. By convention, a tree with zero edges is just a vertex. An s-forest \( F_s \) is called a spanning s-forest if the union of its vertices is \( V_\Gamma \), that is, \( V_\Gamma = \cup V_{T_i} \). A pair \((F_s, \bar{v})\) for \( \bar{v} = (v_1, \ldots, v_s) \), where \( v_i \in V_{T_i} \) is called a rooted spanning s-forest of \( \Gamma \). The following result is immediate.
Lemma 5.1. Any 1-acyclic pair \((\bar{X}, Y)\) of order \(r = |V_T| - s\) is given as \(\bar{X} = E_F\) and \(Y^c = \bar{v}\) for a rooted spanning s-forest \(F_s\) of \(\Gamma\). Moreover, if \((F_{s+1}, \bar{v}')\) is obtained from \((F_s, \bar{v})\) after eliminating an edge, \((\bar{X}, Y)\) (resp. \((\bar{X}', Y')\)) denotes their corresponding 1-acyclic pairs, and \(m\) (resp. \(m'\)) denotes the multiplicity of a root \(\zeta_d\) of \(\Phi_d(t)\) in the polynomial \(m_{(\bar{X}, Y)}\) (resp. \(m_{(\bar{X}', Y')}\)), then \(m \leq m' + 2\).

Proof. The first part is immediate. For the moreover part, let us denote by \(\bar{v}' = (v_1, ..., v_s, v_{s+1})\), where \(\bar{v} = (v_1, ..., v_s)\) and by \(e \in E_T\), the edge removed to obtain the \((s+1)\)-forest \(F_{s+1}\). Note that from Proposition 3.8(iii) one obtains

\[
m_{(\bar{X}, Y)} = \frac{\text{gcd}(\bar{p}_e, q_e)}{p_{v_{s+1}}}.
\]

Since the polynomials \(p_e, q_e\) only have simple roots, the claim follows. \(\square\)

We can use this to obtain a more detailed description of \(H_1(A^X_{\bar{v}}; \mathbb{K})\) as follows.

Theorem 5.2. The invariants \(n_{0,j}(d)\) of Theorem 5.1 vanish for \(j > 2\) so

\[
H_1(A^X_{\bar{v}}; \mathbb{K}) = \mathbb{K}[t^{\pm 1}]^r \bigoplus_{d \in T_T} \left[ \frac{\mathbb{K}[t^{\pm 1}]}{\Phi_d(t)} \right]^{n_{0,1}(d)} \bigoplus \left[ \frac{\mathbb{K}[t^{\pm 1}]}{\Phi_d(t)} \right]^{n_{0,2}(d)}.
\]

Proof. Define for each \(1 \leq s \leq |V_T|\),

\[
f_s = \text{gcd} \left( \frac{\bar{p}_F, q_F, p_e}{p_v} : (F_s, \bar{v}) \text{ is a rooted spanning s-forest} \right)
\]

where by \(p_F, q_F\) we denote \(p_X, q_X\) where \(X\) is the set of edges of the forest \(F_s\).

By Proposition 3.8(ii) and Lemma 5.1, the ideals \(I_s = (f_s(t))\) are the Fitting ideals rank \(s\). Recall that the invariant factors of \(H_1(A^X_{\bar{v}}; \mathbb{K})\) are obtained as \(f_s/f_{s+1}\). The result follows by observing that Lemma 5.1 implies that the difference between the multiplicity of a root \(\zeta_d\) of \(\Phi_d(t)\) in the polynomials \(f_s\) ans \(f_{s+1}\) is at most 2. \(\square\)

The rest of the section will be devoted to calculating the invariants \(n_{k,j}(d)\) in terms of the graph \(\Gamma\) and a non-resonant epimorphism \(\chi\).

5.2. A weight map. For a polynomial \(f \in \mathbb{K}[t^{\pm 1}]\), let \(\text{mult}_d(f)\) denote the biggest integer \(m\) such that \(\Phi_d(t)^m \ | \ f\), or equivalently the multiplicity of a primitive \(d\)-th root of unity \(\zeta_d\) as a root of \(f\). From Theorem 5.2 we see that the invariants \(n_{0,1}(d)\) and \(n_{0,2}(d)\) can be computed by computing \(\text{mult}_d(f_s)\) for each \(1 \leq s \leq |V_T|\). For example, in the case of \(f_1\) this multiplicity is

\[
\text{mult}_d(f_s) = \min \left\{ \text{mult}_d \left( \frac{p_v(t)p_F(t)q_F(t)}{p_v(t)} \right) \ | \ (T, v) \text{ is a rooted spanning tree of } \Gamma \right\}.
\]

Assume \(d \in T_T\) (in particular \(d \neq 1\)), then

\[
\text{mult}_d(p_v) \geq 1 \iff d \mid m_v
\]

\[
\text{mult}_d(q_e) \geq 1 \iff d \notin l_e m_e
\]

\[
\text{mult}_d(q_e) \geq 1 \iff d \notin l_e m_e
\]

The following result describes the multiplicity \(\text{mult}_d(p_e, q_e)\).

Lemma 5.3. Under the conditions above, \(\text{mult}_d(p_e, q_e) \leq 2\).

Proof. Assuming \(\text{mult}_d(p_e, q_e) \geq 1\), these are the possibilities for \(d\):

\[
\text{mult}_d(p_e) \geq 1 \iff d \mid m_e
\]

\[
\text{mult}_d(q_e) \geq 1 \iff d \notin l_e m_e
\]

\[
\text{mult}_d(q_e) \geq 1 \iff d \notin l_e m_e
\]
a) If \( d \mid m_e \) and \( d \mid m_v \), then \( \text{mult}_d(p_e) = 2 \) and \( \text{mult}_d(q_e) = 0 \), since \( d \mid m_e \).

b) Otherwise, if say \( d \mid m_v \) and \( d \nmid m_e \), then \( \text{mult}_d(p_e, q_e) = \text{mult}_d(p_e) + \text{mult}_d(q_e) = 1 + 1 = 2 \).

c) Otherwise, if say \( d \mid m_e \) but \( d \nmid m_v, m_e \), then \( \text{mult}_d(p_e, q_e) = \text{mult}_d(p_e) = 1 \).

d) Finally, if \( d \nmid m_v, d \mid m_e, d \mid \tilde{\ell}_e m_e \), and \( d \nmid m_e \), then \( \text{mult}_d(p_e, q_e) = \text{mult}_d(q_e) = 1 \).

This proves the claim.

The multiplicity maps on the vertices \( \Gamma \rightarrow \mathbb{Z}_{\geq 0} \), \( v \mapsto \text{mult}_d(p_e) \) and edges \( E \rightarrow \mathbb{Z}_{\geq 0} \), \( e \mapsto \text{mult}_d(p_e, q_e) \) determine the invariant factors of \( H_1(A^2; K) \). We generalize this to a weight map \( w : F^j(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} \) on the finite type flag complex \( F^j(\Gamma) \) defined as \( w(X) := \text{mult}_d(p_X q_X) \), which will play an important role in the general case.

5.3. The multiplicity spectral sequence. Given \( d \in \mathbb{T}_\Gamma \) consider \( w : F^j(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} \), \( w(X) := \text{mult}_d(p_X q_X) \) as defined above. Associated with this weight map one can construct the standard increasing weight filtration of simplicial complexes \( F_{d,*} \), as follows:

\[
F_{d,p} C_q := \langle X \in F^j_q(\Gamma) \mid w(X) \leq p \rangle \subset C^j_q(\Gamma),
\]

that is, generated by the \( q \)-simplices \( X \in F^j_q(\Gamma) \) for which \( \Phi_d \) has multiplicity at most \( p \) in the polynomial \( p_X q_X \). For convenience, if \( p < 0 \), then \( F_{d,p} C_q = \{0\} \). Note that \( F_{d,p} C_q \subset F_{d,p+1} C_q \) and \( \partial F_{d,p} C_{q+1} \subset F_{d,p} C_q \). The spectral sequence associated with this filtration starts with the term \( E^0_{d,(p,q)} := F_{d,p} C_{p+q}/F_{d,p-1} C_{p+q} \) and \( \partial^0 : E^0_{d,(p,q)} \rightarrow E^0_{d,(p,q-1)} \) is well defined by \( \partial \), since \( \partial \circ F_{d,p} C_{p+q} \subset F_{d,p} C_{p+q-1} \) and \( \partial \circ F_{d,p} C_{p+q-1} \subset F_{d,p-1} C_{p+q-1} \). Note that \( E^1_{d,(p,q)} = H_{p+q}(F_{d,p} C_q) \) and the spectral sequence \( \{ (E^k_{d,(p,q)}, \partial^k) \} \) is well defined where \( \partial^k \) is a morphism of type \((-k,k-1)\). This weight map in more generality can be found in [20].

Proposition 5.4. The multiplicity spectral sequence \( (E^p_{d,(p,q)}, \partial^k) \) associated with the flag complex \( F^j(\Gamma) \) of FC-type is bounded and satisfies the following properties:

(i) \( E^0_{d,(p,q)} = \{0\} \) if \( p + q < -1 \)

(ii) \( E^0_{d,(p,-p-1)} = \)

\[
\begin{cases}
\langle \sigma_0 \rangle_X & \text{if } p = 0 \\
\{0\} & \text{otherwise}
\end{cases}
\]

(iii) \( E^0_{d,(p,q-p)} = \)

\[
\begin{cases}
\langle \sigma_X \mid X \in F^j_q(\Gamma), w(X) = p \rangle_X & \text{if } q \geq 0, p \leq q + 1 \\
\{0\} & \text{otherwise.}
\end{cases}
\]

Proof. Part [1] (resp. [ii]) is an immediate consequence of \( C^j_k(\Gamma) = \{0\} \) if \( k < -1 \) \( (C^j_{-1}(\Gamma) = \langle \sigma_0 \rangle) \). Part [iii] can be proved by induction. The first step, \( q = 1 \), is Lemma [5,3]. Let \( m = w(X') \), and let us consider \( X = X' \cup \{v\} \) a \((q+1)-1\)-simplex. If \( w(v) = 1 \), then the only new edges \( e = \{v, v'\} \) such that \( \text{mult}_d(q_e) = 1 \) are those for which \( \text{mult}_d(p_{v'}) = 0 \). Moreover, by the FC-type condition, if \( \text{mult}_d(q_e) = 1 \) with \( e = \{v, v'\} \), then for all \( e' = \{v', v''\} \) one has \( \text{mult}_d(q_{e'}) = 0 \). Hence there are at most \( q+1-m \) of such edges. Thus \( w(X) = w(v)+\sum_{v' \in X} \text{mult}_d(q_{v'}) + w(X') \leq 1 + (q + 1 - m) + m = q + 2 \). As a consequence of [1] and [iii] \( E^0_{d,(p,q)} = \{0\} \) if \( p < 0, p + q < -1 \), and \( p + q > w(\Gamma) \) the clique number of \( \Gamma \), that is, the dimension of \( F^j(\Gamma) \). Hence the multiplicity spectral sequence is bounded.
5.4. The \((\Phi_d)\)-adic filtration. In order to study the primary part of \(H_{k+1}(A^\chi_k; \mathbb{K})\), in the decomposition given in Theorem 5.3 we will fix \(d \in \mathbb{N}^r\), denote by \(\Phi_d(t)\) the cyclotomic polynomial of order \(d\), and consider \(\hat{\Lambda}\) the completion of \(\Lambda = \mathbb{K}[t^\pm 1]\) with respect to the \((\Phi_d)\)-adic filtration as defined in [18, §4.2]. If \(\mathbb{K}_d = \Lambda/(\Phi_d)\) denotes the residue field and \(\iota : \Lambda \hookrightarrow \hat{\Lambda}\) the natural inclusion, then \(\text{gr}(\hat{\Lambda}) \cong \mathbb{K}_d[t]\) and if \(f \in \Lambda\) is a polynomial, then \(\iota(f)\) is a unit in \(\hat{\Lambda}\) if and only if its class in \(\mathbb{K}_d\) is non-trivial, that is, \(\Phi_d(f)\).

Alternatively, one can work with \(\hat{\Lambda} := \mathbb{K}_d[t]_p\) where the subindex \(p\) means localization at the ideal \(P\) which is the ideal generated by \(\tau = t - \zeta_d\) with \(\zeta_d \in \mathbb{K}_d\) root of \(\Phi_d(t)\).

**Proposition 5.5.** Under the above conditions

\[
\dim_{\mathbb{K}_d} \left( H_k(C^\chi_k(\Gamma); \hat{\Lambda}) \otimes_{\hat{\Lambda}} H^\Lambda(\tau^s) \right) = sr_k + \sum_{j=1}^{s-1} jn_{k,j}(d) + \sum_{j \geq s} sn_{k,j}(d)
\]

\[
\dim_{\mathbb{K}_d} \text{Tor}_1^\hat{\Lambda} \left( H_k(C^\chi_k(\Gamma); \hat{\Lambda}) \otimes_{\hat{\Lambda}} H^\Lambda(\tau^s) \right) = \sum_{j=1}^{s-1} jn_{k,j}(d) + \sum_{j \geq s} sn_{k,j}(d).
\]

**Proof.** It follows immediately from

\[
\text{Tor}_1^\hat{\Lambda} \left( \hat{\Lambda}(\tau^{s_1}), \hat{\Lambda}(\tau^{s_2}) \right) = \left( \frac{\tau^M}{\tau^{s_1+s_2}} \right)
\]

and

\[
\hat{\Lambda}(\tau^{s_1}) \otimes_{\hat{\Lambda}} \hat{\Lambda}(\tau^{s_2}) = \hat{\Lambda}(\tau^m),
\]

where \(M := \max\{s_1, s_2\}, m := \min\{s_1, s_2\}\). \(\square\)

In order to describe the homology of the Artin kernels we need to introduce some notation associated with invariants of the multiplicity-spectral sequence. Let us denote \(h^*_{d.(p,q)} := \dim_{\mathbb{K}} E^*_{d,(p,q)}\), \(h^*_{d,q} := \sum_{p \geq 0} h^*_{d,(p,q-p)}\), and \(\chi_k^{rel}(E^*_{d,q}) := \sum_{q=0}^{k} (-1)^{k-q}(h^*_{d,q} - h^*_{d,q+1})\), this is the \(k\)-th relative Euler characteristic of \(E^*_{d,q}\). Note that these are combinatorial invariants of the flag complex \(\mathcal{F}(\Gamma)\) and the weight map.

**Proposition 5.6.**

\[
\dim_{\mathbb{K}_d} H_k(C^\chi_k(\Gamma); \hat{\Lambda}/(\tau^s)) = \sum_{j=1}^{s} h^*_{d,j,k}.
\]

Moreover, \((E^*_{d,(p,q)} \otimes_{\hat{\Lambda}} \hat{\Lambda}(\tau^q), \partial^\bullet)\) degenerates at the \(s\)-th page.

**Proof.** The spectral sequence \((E^*_{d,(p,q)} \otimes_{\hat{\Lambda}} \hat{\Lambda}(\tau^q), \partial^\bullet)\) associated with the complex \(C^\chi_k(\Gamma) \otimes_{\hat{\Lambda}} \hat{\Lambda}(\tau^q)\), where

\[
\partial(\sigma^X_1 \otimes 1) = \sum_{X_s} \langle X_1 \mid X \rangle \tau^{w(X) - w(X_s)} \sigma^X_s \otimes 1
\]

and the decreasing filtration \(F^q = (\tau^q)\hat{\Lambda} \cdot C^\chi_p(\Gamma)\) is bounded and such that \(d^q = 0\) if \(q \geq s\). By an argument generalizing [18, Corollary 5.5] the abutment of this spectral sequence is \(H_k(C^\chi_k(\Gamma); \hat{\Lambda}/(\tau^s))\). Moreover, if \(\sigma^X_1 \in F_{d,p}C^\chi_k(\Gamma)\), then

\[
\partial(\sigma^X_1 \otimes 1) = \sum_{i=0}^{s-1} \partial_i(\sigma^X_1 \otimes 1)\tau^i,
\]

where \(\partial_i(\sigma^X_1 \otimes 1) \in F_{d,p-i}C^\chi_{k-1}(\Gamma)\).
For $s = 1$, the first page of this spectral sequence coincides with that of $(E^s_{d,(p,q)}, d)$ and it degenerates at this page, hence $\dim_{K_d} H_k(C^2_d(\Gamma); \hat{\Lambda}/(s)) = \sum_{p+q=k} h^2_{d,(p,q)} = h^1_{d,k}$. Similarly, $\dim_{K_d} H_k(C^2_d(\Gamma); \text{gr}^2(\Lambda)) = \sum_{p+q=k} h^1_{d,(p,q)} = h^1_{d,k}$. The result follows by induction and the Künneth formula.

5.5. The general $H_{k+1}(A^1; \mathbb{K})$ case. The previous discussion provides the following formula for the invariants of $H_{k+1}(A^1; \mathbb{K})$.

Theorem 5.7. Under the previous notation,

$$r_k + \sum_{j \geq s} n_{k,j}(d) = \sum_{p \geq 0} h^2_{d,(p,k-p)} - \sum_{j \geq s} n_{k-1,j}(d).$$

Equivalently,

$$\sum_{j \geq s} n_{k,j}(d) = \chi^\text{rel}_k(E^s_d).$$

This determines completely the $\Phi_d$-primary part of $H_{k+1}(A^1; \mathbb{K})$.

Moreover, the Jordan blocks associated with the torsion of $H_{k+1}(A^1; \mathbb{K})$ have size at most $k+2$.

Proof. By the universal coefficient theorem $H_k(C^2_d(\Gamma); \hat{\Lambda}) = H_k(C^2_d(\Gamma); \mathbb{K}) \otimes_{\mathbb{K}} \hat{\Lambda}$ and

$$0 \to H_k(C^2_d(\Gamma); \hat{\Lambda}) \otimes_{\hat{\Lambda}} \hat{\Lambda}/(\tau^s) \to H_k(C^2_d(\Gamma); \hat{\Lambda}/(\tau^s)) \to \text{Tor}_1^\hat{\Lambda} \left( H_{k-1}(C^2_d(\Gamma); \hat{\Lambda}), \hat{\Lambda}/(\tau^s) \right) \to 0.$$ The result follows from comparing (12) for $s$ and $s - 1$ and using Propositions 5.5 and 5.6. By Proposition 5.6,

$$\dim_{K_d} \left( H_k(C^2_d(\Gamma); \hat{\Lambda}) \otimes_{\hat{\Lambda}} \hat{\Lambda}/(\tau^s) \right) - \dim_{K_d} \left( H_k(C^2_d(\Gamma); \hat{\Lambda}) \otimes_{\hat{\Lambda}} \hat{\Lambda}/(\tau^{s-1}) \right) = r_k + \sum_{j \geq s} n_{k,j}(d).$$

By Proposition 5.6,

$$\dim_{K_d} \left( H_k(C^2_d(\Gamma); \hat{\Lambda}/(\tau^s)) \right) - \dim_{K_d} \left( H_k(C^2_d(\Gamma); \hat{\Lambda}/(\tau^{s-1})) \right) = \sum_{i \geq 0} h^2_{d,(i,k-i)}.$$

By Proposition 5.5,

$$\dim_{K_d} \text{Tor}_1^\hat{\Lambda} \left( H_{k-1}(C^2_d(\Gamma); \hat{\Lambda}), \hat{\Lambda}/(\tau^s) \right) - \dim_{K_d} \text{Tor}_1^\hat{\Lambda} \left( H_{k-1}(C^2_d(\Gamma); \hat{\Lambda}), \hat{\Lambda}/(\tau^{s-1}) \right) = \sum_{j \geq s} n_{k-1,j}(d).$$

By (12) one has

$$r_k + \sum_{j \geq s} n_{k,j}(d) = \sum_{i \geq 0} h^2_{d,(i,k-i)} - \sum_{j \geq s} n_{k-1,j}(d).$$

To obtain (11) it is enough to use $r_k = \sum_{i \geq 0} h^\infty_{d,(i,k-i)}$ and induction. The moreover part is a consequence of Proposition 5.2(iii).

Example 5.8. Consider Figure 4 as a graph whose edges are labeled by the even numbers outside of the brackets on the edges. The even Artin group $A^1$ associated with this labeled graph has a presentation

$$A^1 = \langle g_1, g_2, g_3, g_4 : [g_1, g_4] = 1, (g_1g_2)^2 = (g_2g_3)^2 = (g_3g_4)^2 = (g_4g_3)^2 \rangle$$

Consider the character $\chi$ is given in Figure 4 by sending each generator at $v_i$ to the corresponding number in parenthesis. The first homology of the Artin kernel $A^1$ associated with $\chi$ can be
studied using the spectral sequence associated with the multiplicity filtration. For instance, note that \( \mathbb{T}_\Gamma = \{2, 3, 6\} \).

The filtration given by \( F_{6,*} \) (resp. \( F_{2,*} \)) can be summarized by labeling vertices and edges of \( \Gamma \) with their corresponding weight. This is done in Figure 2 using the first (resp. second) number in brackets. Therefore, for \( F_{6,*} \) note that \( E^1_{6,(0,0)} \cong \mathbb{K}^2 \) is generated by the cycles \( \langle v_2 - v_1, v_3 - v_1 \rangle \) whereas \( v_1 - v_1 \) is the image of the edge \( e_{1,4} = \{v_1, v_4\} \in F_{0,0}C_1 \). This spectral sequence degenerates at \( E^\infty_{6,(p,q)} = E^2_{6,(p,q)} \). Moreover,

\[
E^2_{6,(p,q)} = \begin{cases} 
\mathbb{K} & \text{if } p = 1, q = 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The only other non-zero term is \( E^1_{6,(1,0)} \cong \mathbb{K}^3 \) generated by the edges \( \langle e_{1,2}, e_{2,3}, e_{3,4} \rangle \). Hence the only non-trivial dimension in the spectral sequence \( E^k_{6,(p,q)} \), \( k \geq 1 \) is \( h^1_{6,(0,0)} = 2 \). By Theorem 5.7 one obtains

\[
n_{0,1}(6) + n_{0,2}(6) = \chi_0^{rel}(E^1_6) = h^1_{6,0} + h^\infty_{6,0} = 2,
n_{0,2}(6) = \chi_0^{rel}(E^2_6) = h^2_{6,0} - h^\infty_{6,0} = 0,
n_{1,1}(6) + n_{1,2}(6) + n_{1,3}(6) = \chi_1^{rel}(E^1_6) = (h^1_{6,1} - h^\infty_{6,1}) - (h^1_{6,0} - h^\infty_{6,0}) = (3 - 1) - 2 = 0.
\]

Thus \( n_{0,2}(6) = 0, n_{0,1}(6) = 2, \) and \( n_{1,1}(6) = n_{1,2}(6) = n_{1,3}(6) = 0. \)

Analogously, for \( F_{2,*} \) note that \( E^1_{2,(0,0)} \cong \mathbb{K}^2 \)

\[
0 \to E^0_{2,(2,-1)} = \langle e_{1,2}, e_{2,3}, e_{3,4} \rangle \to E^0_{2,(2,-2)} = \{0\} \to 0 \to 0 \\
0 \to E^0_{2,(1,0)} = \langle e_{1,4} \rangle \to E^0_{2,(1,-1)} = \langle v_2, v_4 \rangle \to 0 \to 0 \\
0 \to E^0_{2,(0,1)} = \{0\} \to E^0_{2,(0,0)} = \langle v_1, v_3 \rangle \to E^0_{2,(0,-1)} = \langle \sigma_9 \rangle \to 0
\]

which gives

\[
E^1_{2,(p,q)} = \begin{cases} 
\langle v_4 - v_1 \rangle & \text{if } p = q = 0 \\
\langle v_2 \rangle & \text{if } p = 1, q = -1 \\
\langle e_{1,2}, e_{2,3}, e_{3,4} \rangle & \text{if } p = 2, q = -1 \\
\{0\} & \text{otherwise.}
\end{cases}
\]
In particular, the only non-trivial dimensions in the spectral sequence $E^{k}_{2,(p,q)}$, $k \geq 1$ are $h^{1}_{2,(0,0)} = h^{2}_{2,(1,-1)} = 1$, $h^{1}_{2,(2,-1)} = 3$, $h^{2}_{2,(2,-1)} = 2$, and $h^{3}_{2,(2,-1)} = 1$. Hence according to Theorem 5.7:

\[
n_{0,1}(2) + n_{0,2}(2) = \chi_{0}^{rel}(E^{1}_{2}) = h^{1}_{2,0} - h^{\infty}_{2,0} = 2,
\]
\[
n_{0,2}(2) = \chi_{0}^{rel}(E^{2}_{2}) = h^{2}_{2,0} - h^{\infty}_{2,0} = 1
\]

\[
n_{1,1}(2) + n_{1,2}(2) + n_{1,3}(2) = \chi_{1}^{rel}(E^{1}_{2}) = (h^{1}_{2,1} - h^{\infty}_{2,1}) - (h^{1}_{2,0} - h^{\infty}_{2,0}) = (3 - 1) - 2 = 0,
\]

that is, $n_{0,1}(2) = n_{0,2}(2) = 1$ and $n_{1,1}(2) = n_{1,2}(2) = n_{1,3}(2) = 0$. Moreover, $\text{im} \partial_1 = 3$, $\text{im} \partial_2 = 0$, and $r_0 = 0$, $r_1 = 1$. By Theorem [7,9]  

\[
H_{1}(A_{1}^{\partial}; K) = \left( \frac{K[t^{\pm 1}]}{(t - 1)} \right)^{3} \oplus \frac{K[t^{\pm 1}]}{(t + 1)} \oplus \frac{K[t^{\pm 1}]}{(t^{2} + 1)}
\]

and

\[
H_{2}(A_{1}^{\partial}; K) = K[t^{\pm 1}].
\]

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