Residual Tangent Kernels

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Abstract
A recent body of work has focused on the theoretical study of neural networks at the regime of large width. Specifically, it was shown that training infinitely-wide and properly scaled vanilla ReLU networks using the L2 loss, is equivalent to kernel regression using the Neural Tangent Kernel (NTK), which is deterministic, and remains constant during training. In this work, we derive the form of the limiting kernel for architectures incorporating bypass connections, namely residual networks (ResNets), as well as to densely connected networks (DenseNets). In addition, we derive finite width and depth corrections for both cases. Our analysis reveals that deep practical residual architectures might operate much closer to the “kernel regime” than their vanilla counterparts: while in networks that do not use skip connections, convergence to the NTK requires one to fix depth, while increasing the layers’ width. Our findings show that in ResNets, convergence to the NTK may occur when depth and width simultaneously tend to infinity, provided proper initialization. In DenseNets, however, convergence to the NTK as the width tend to infinity is guaranteed, at a rate that is independent of both depth and scale of the weights.

1. Introduction
Understanding the effect of different architectures on the ability to train deep networks has long been a major topic of research. A popular playing ground for studying the forward and backward propagation of signals at the point of initialization, is the “infinite width” regime (Arora et al., 2020; Sirignano & Spiliopoulos, 2019; Schoenholz et al., 2017; Yang & Schoenholz, 2017). In this regime and for some architectures, the central limit theorem applies when the weights are sampled i.i.d, guaranteeing the convergence of pre-activations to Gaussian Processes (GP), simplifying the analysis considerably (Lee et al., 2017). By viewing neural networks at their initialized state through the lens of GP’s, a link to kernel methods is revealed. Indeed, it has been shown that under certain conditions, shallow training (where only the top layer is trained), is equivalent to kernel regression, where the kernel is defined by a GP (Arora et al., 2019).

While the analysis of initialized networks has traditionally been thought of as a first step towards understanding of the full training dynamics of neural networks, recent findings suggest a much stronger link between initialized networks, and their final convergent state post-training, especially for large width networks. While characterizing the full training dynamics of practically-sized networks remains an open problem, a certain limiting behavior was shown to emerge in large width networks, where perfect fitting of the training data using gradient descent is achieved with a minimal alteration of the weights compared to their initialized state. This notion is made precise by a recent paper on the Neural Tangent Kernel (NTK) (Jacot et al., 2018), in which it is shown that the training dynamics of fully connected networks trained with gradient descent can be characterized by a kernel, when the width of the network approaches infinity. In other words, the evolution through time of the function computed by the network follows the dynamics of kernel gradient descent.

Given a dataset \( \{x_i\}_{i=1}^{N} \), let \( f(x, W) \in \mathbb{R} \) denote the output of a fully connected feed forward network, with weights \( W \) for input \( x \). The NTK is given by \( K \in \mathbb{R}^{N \times N} \) such that:

\[
K(x, x') = \mathbb{E}_W \left[ \frac{\partial f(x)}{\partial W} \frac{\partial f(x')}{\partial W}^\top \right] = \mathbb{E}_W [G(x, x')] \tag{1}
\]

As the width of each layer approaches infinity, provided proper scaling and initialization of the weights, the expectation in Eq. 1 can be removed, and it holds that \( G_{i,j} \) converges in probability to the kernel function:

\[
\lim_{\text{width} \to \infty} G(x, x') \to K(x, x') \tag{2}
\]

In the limit, minimizing the squared loss \( \mathcal{L}(W) \) by gradient descent with an infinitely small learning rate is equivalent to kernel regression, with the evolution of the cost obeying the following dynamics:

\[
\frac{\partial \mathcal{L}(W)}{\partial t} = \frac{\partial \mathcal{L}(W)}{\partial \mathbf{f}} K \frac{\partial \mathcal{L}(W)}{\partial \mathbf{f}}^\top \tag{3}
\]
where \( f = [f(x_1, W) \ldots f(x_N, W)]^\top \in \mathbb{R}^{N \times 1} \) denotes the vectorized form of the output on the entire training dataset.

From the positive definiteness of the limiting kernel \( K \), convergence to zero training error when using gradient descent is then guaranteed.

While this view of neural network optimization does shed light on the empirical observations regarding the ease of optimization, the question of generalization remains elusive. Recently, empirical support has demonstrated the power of CNTK (convolutional neural tangent kernel) on practical datasets, demonstrating new state of the art results for kernel methods, surpassing other known kernels by a large margin (Arora et al., 2019; Yu et al., 2020). Although the performance gap compared with state of the art deep learning architectures remains non trivial, understanding the source of such a leap in performance compared with other kernel methods might shed new light on the inductive bias of neural networks, as explored in (Bietti & Mairal, 2019). It is, therefore, interesting to understand how far the training dynamics of practically sized architectures deviate from the “infinite width” regime. To that end, an important subtlety worth considering is the rate of convergence in Eq. 2 to the limit, and its dependence on other hyper parameters, such as depth and scale. This question has recently been addressed in the case of vanilla feed forward fully connected networks (Hanin & Nica, 2020), where it is shown that the variance of the diagonal entries of \( G \) is exponential in the ratio between width and depth, and so convergence to the limiting kernel cannot happen when both are taken to infinity at the same rate. This important observation suggests that deep vanilla networks operate far from the “infinite width” regime. We would like to emphasize a few attractive outcomes of such an analysis. Although, from a learning perspective, the view of neural network training through the lenses of Kernel methods might seem as a step backwards, the theoretical benefits are clear. Deep neural networks displaying “close to” kernel dynamics, are amenable to theoretical analysis by means of investigating the properties of the corresponding kernel (Yang & Salman, 2019). Moreover, even when the training dynamics drift away from the kernel regime, after the onset of gradient descent, the conditions at the point of initialization, when the kernel regime approximation is valid, might shed light on the inductive bias of the model.

In this work, we derive the corresponding neural tangent kernel of residual architectures, namely ResNets (He et al., 2015) and DenseNets (Huang et al., 2016). Furthermore, we rigorously derive finite width and depth corrections for both architectures, revealing a fundamentally different relationship between width, depth and the NTK. Unlike vanilla architectures, when properly scaled, convergence to the limiting kernel is achieved when taking both the width and the depth of the architecture to infinity simultaneously. Our main contributions are:

- Using the GP approximation obtained in the infinite-width limit, we derive the corresponding GP and NTK for residual architectures, namely ResNets and DenseNets.
- We prove a forward-backward norm propagation duality for a wide family of ReLU architectures, and use it to derive the rate of convergence of \( G \) to the NTK, for finite sized networks. Specifically, we show that for ResNets, there exists a constant \( C \) such that:

\[
\frac{\mathbb{E}[G(x, x)^2]}{K(x, x)^2} \sim \exp \left[ \frac{C}{n} \sum_{l=1}^L \frac{\alpha_l}{1 + \alpha_l} \right] \left( 1 + \mathcal{O}(\frac{1}{n}) \right)
\]

where \( m \) is the depth of each residual branch, \( L \) is the number of residual branches, \( n \) is the width, and \( \{\alpha_l\}_{l=1}^L \) are initialization constants multiplying normally distributed i.i.d weights. For DenseNets, our results show:

\[
\frac{\mathbb{E}[G(x, x)^2]}{K(x, x)^2} \sim \exp \left[ \frac{C}{n} \right] \left( 1 + \mathcal{O}(\frac{1}{n}) \right)
\]

Our results prove that ResNet models exhibit convergence to the diagonal elements of the NTK, when both \( n, L \to \infty \) simultaneously, provided that \( \frac{1}{n} \sum_{l=1}^L \alpha_l \to 0 \). For DenseNets, surprisingly, the same convergence property is independent of both depth and initialization scale. These results are in stark contrast to the vanilla fully connected case, where the same quantity is exponential in \( \frac{L}{n} \), and convergence to the NTK is only feasible when the depth is fixed. (Hanin & Nica, 2020).

2. Related Work

The study of infinitely wide neural networks has been in the forefront of theoretical deep learning research in the last few years. A major line of work has been focused on the mean field approach, where the propagation of signals in random networks are analyzed in the regime of infinite width (Yang & Schoenholz, 2017; Schoenholz et al., 2017; Xiao et al., 2018). In that case, the pre-activations in any layer are approximated by a Gaussian distribution with moments depending on the specific layer and initialization parameters. Depth scales that limit the maximum depth for which information can propagate can then be derived for different architectures and initialization. For standard fully connected feed forward models, for example, exponential collapsing of the input geometry is observed in (Schoenholz et al., 2017). Mean field analysis of residual networks is presented in (Yang & Schoenholz, 2017), where it is shown that ResNets exhibit sub-exponential, or even polynomial dynamics. A more refined analysis presented in (Pennington et al., 2017),
considers the full spectrum of the input output Jacobian of infinitely wide networks. exploding or vanishing gradients are then prevented, by analyzing conditions that give rise to dynamical isometry, a state in which the squared eigenvalues of $J_{IO}$ are all concentrated around 1. in (Pennington et al., 2017) it is shown that fully connected ReLU networks are incapable of reaching dynamical isometry, as opposed to sigmoidal networks, while in (Pennington et al., 2018) it is shown that dynamical isometry is achieved in a universal manner for a variety of activation functions. More recently, the dynamics of training infinitely wide networks were precisely described with the introduction of the Neural Tangent Kernel (NTK) (Jacot et al., 2018), and its convolutional counterpart (Yu et al., 2020). A notable generalization of the GP and NTK results are given in (Yang, 2019a) and (Yang, 2019b), where it is shown that neural nets of any architecture (including weight-tied resnet, densenet, or RNNs) converge to GPs in the infinite width limit, and prove the existence of the infinite width NTKs. Finite width corrections to the NTK in the vanilla case were introduced in (Hanin & Nica, 2020). in general, finite width corrections to various statistical quantities have been presented in numerous previous articles (Hanin & Rolnick, 2018),(Hanin, 2018).

(Hanin & Rolnick, 2018) tackle two failure modes that are caused in finite size networks by exponential explosion or decay of the norm of intermediate layers. it is shown that for random fully connected vanilla ReLU networks, the variance of the squared norm of the activations exponentially increases, even when initializing with the $\frac{2}{\sqrt{nm}}$ initialization. For ResNets, this failure mode can be overcome by correctly rescaling the residual branches. However, it is not clear how such a rescaling affects the back propagation of gradients.  

(Hanin, 2018) explores the conditions at initialization that give rise to the exploding or vanishing gradient problem, by analyzing the individual entries of the input-output Jacobian. as far as we can ascertain, finite width and depth correction analysis of parameter gradients in residual architectures, as required to observe $G$ and its relationship with the NTK, have yet to be rigorously explored.

3. Preliminaries And Notations

we make use of the following notations: $f(x, W)\in\mathbb{R}$ denotes the output of a neural network $N$ on input $x$, where we assume without loss of generality that $\|x\|^2=1$. The ReLU non-linearity is denoted by $\phi()$, and intermediate outputs are denoted $\{y^l(x)\}_{l=0}^L$, for a fixed input $x\in\mathbb{R}^d$.

For simplicity, the dependence of the outputs on $x$ is often made implicit $\{y^l\}_{l=0}^L$ when the specific input used to calculate the outputs can be inferred from context. $y^l_i$ denotes the $i$th component of the vector $y^l$, and $n_1 \ldots n_L$ denote the width of the corresponding layers, with $d$ the length of the input vector. $\|\cdot\|^2$ is the squared Euclidean norm. We denote the weight matrix associated with layer $l$ by $W^l$, with lower case letters $w^l_{i,j}$ denoting the individual components of $W^l$. Additional superscripts $W^{l,k}$ are used, when several weight matrices are associated with layer $l$. Weights appearing without any superscript $W$ denote all the weights concatenated to a vector. Each model analyzed begins and ends with an affine layer (no non-linearity), such that $y^0=W^s x$, and $f(x)=\frac{1}{\sqrt{n_L}} W^f y^L$, where $W^s \in \mathbb{R}^{d \times n_0}$, $W^f \in \mathbb{R}^{n_L \times 1}$. throughout the paper, we assume that the weights are sampled i.i.d from a normal distribution.

4. Signal Propagation in Multi-pathway Networks

The purpose of this section is to derive the form of the NTK for residual architectures using the GP approximation of infinitely wide neural networks, namely ResNets and DenseNets. as observed in (Lee et al., 2017), the pre-activations of a feed forward neural network tend to i.i.d Gaussian processes as the width of the hidden layers tend to infinity. the proof of this claim usually involves taking the width of individual layers to infinity in a sequential manner, ensuring that the previous layer is governed by its limiting distribution before considering the current layer. this technical limitation was recently made redundant in (Yang, 2019b), which proposes a general framework for viewing neural network computations, elucidating the GP correspondence for general architectures, including the ones discussed in the present paper. in light of this, we may safely assume the GP approximation holds when taking the width of every layer to infinity, without resorting to the specific order in which the limits were taken.

4.1. ResNets

Residual networks have reintroduced the concept of bypass connections, allowing the training of deep and narrow models with relative ease. a generic, residual architecture, with residual branches of depth $m$, takes the form:

$$\forall 0<\ell \leq L, y^\ell = [y^{\ell-1}]_{n_\ell} + \sqrt{\alpha_{\ell}} y^{\ell-1,m}$$

where $\{\alpha_{\ell}\}_{\ell=1}^L$ are scaling coefficients, $[y^{\ell-1}]_{n_\ell}$ denotes the first $n_\ell$ components of $y^{\ell-1}$ $(n_0 \geq n_1 \ldots \geq n_L)$, and:

$$y^{\ell-1,h} = \begin{cases} \frac{1}{\sqrt{n_{\ell-1,h}}} W^{l,h} q^{\ell-1,h-1} & 1 < h \leq m \\ \frac{1}{\sqrt{n_{\ell-1}}} W^{l,h} y^{\ell-1} & h = 1 \end{cases}$$

where $\forall 1 \leq h < m$, $W^{l,h} \in \mathbb{R}^{n_{\ell-1,h-1} \times n_{\ell-1,h}}$, $W^{l,1} \in \mathbb{R}^{n_{\ell-1} \times n_{\ell-1,h}}$, $W^{l,m} \in \mathbb{R}^{n_{\ell-1,m-1} \times n_{\ell-1}}$, $q^{l,h} = \sqrt{2}\phi(y^{l,h})$ (see Fig. 2 for an illustration).

we may now take the width of each layer to infinity. Recall that the first layer in any model considered is a simple affine
layer with i.i.d Gaussian weights. Since the inputs are fixed, the first layer \( y^0 \) is, therefore, trivially a zero mean Gaussian process with covariance:

\[
\forall 0<j\leq n_0 \quad \mathbb{E}[y^0(x)_j y^0(x')_j] = \Lambda^0(x,x') = x^\top x'
\]  

(8)

When \( n_0 \to \infty \), the pre-activations \( y^{0,1} \) are given by an infinite sum of i.i.d random variables, and hence by the multivariate central limit theorem, tend to a GP with covariance:

\[
\forall 0<j\leq n_{0,1} \quad \mathbb{E}[y^{0,1}(x)_j y^{0,1}(x')_j] = \frac{1}{n_0} \mathbb{E}\left[\left(\sum_{i=1}^{n_0} y^0(x)_i u^1_{i,j}\right)\left(\sum_{i=1}^{n_0} y^0(x')_i u^1_{i,j}\right)\right] = \Lambda^0(x,x')
\]  

(9)

In addition, the components of \( q^{0,1} = \sqrt{2}\phi(y^{0,1}) \) are then also distributed i.i.d , with covariance

\[
\mathbb{E}[q^{0,1}(x)_j q^{0,1}(x')_j] = \Sigma^{0,1}(x,x')
\]

(10)

such that:

\[
\Sigma^{0,1}(x,x') = 2 \mathbb{E}_{u,v \sim \mathcal{N}(0,\Lambda^0)}[\phi(u)\phi(v)]
\]

(10)

When \( n_0, n_{0,1}, n_{1,1}, \ldots, n_L \to \infty \), the GP parameters of all subsequent layers in any residual branch can easily be derived in a similar fashion. We arrive at the following inductive formula for \( \Sigma^{l,h}(x,x') \):

\[
\Sigma^{l,h}(x,x') = \mathbb{E}[q^{l,0,1}(x)_j q^{l,0,1}(x')_j] =
\begin{cases} 
2 \mathbb{E}_{u,v \sim \mathcal{N}(0,\Sigma^{l,h-1}(x,x'))}[\phi(u)\phi(v)] & 1 < h \\
2 \mathbb{E}_{u,v \sim \mathcal{N}(0,\Lambda^l(x,x'))}[\phi(u)\phi(v)] & h = 1
\end{cases}
\]

(11)

Note that since both \( y^{l-1} \) and \( y^{l-1,m} \) are independent GPs, \( y' = [y^{l-1}]_{n_l} + y^{l-1,m} \) is a GP with covariance given by the sum:

\[
\Lambda^l(x,x') = \Lambda^{l-1}(x,x') + \alpha_l \Sigma^{l-1,m-1}(x,x')
\]

(12)

4.2. DenseNets

DenseNets were recently introduced, demonstrating faster training, as well as improved performance on several popular datasets. The main architectural features introduced by DenseNets include the connection of each layer output to all subsequent layers, using concatenation operations instead of summation, such that the weights of layer \( l \) multiply the concatenation of the outputs \( y^0 \ldots y^{l-1} \). The output of layer \( l \) is given by:

\[
\forall 0<i\leq L, \quad y^l = \sqrt{\frac{\alpha}{n_{l-1}^t}} \sum_{h=0}^{l-1} W^{l,h\top}[q^h]_{n_{l-1}^t}
\]

(13)

where \( \alpha \) is a scaling coefficient, \([q^h]_{n_{l-1}^t}\) denotes the first \( n_{l-1}^t \) components of \( q^h \) (\( \forall i \), \( n_{l-1}^t \leq n_h \)), and \( W^{l,h} \in \mathbb{R}^{n_{l-1}^t \times n_l} \) (see Fig. 2 for an illustration).

. Similar to the ResNet case, the pre-activations of the first layer \( y^0 \), given a fixed input, are centered i.i.d GP, with covariance \( \Lambda^0(x,x') = x^\top x' \). When \( n_0, n_0', n_1, n_1', n_2, \ldots n_L \to \infty \), the GP parameters of each layer are derived by the following inductive formula:

\[
\forall 0<i\leq L, \quad \Lambda^l(x,x') = \mathbb{E}[y^l(x)_j y^l(x')_j] =
\begin{cases} 
\frac{\alpha}{n_{l-1}^t} \left( \sum_{h=0}^{l-1} W^{l,h\top}[q^h(x)]_{n_{l-1}^t} \right) \left( \sum_{h=0}^{l-1} W^{l,h\top}[q^h(x')]_{n_{l-1}^t} \right) & 1 < h \\
\frac{\alpha}{n_{l-1}^t} \sum_{h=0}^{l-1} \Sigma^h(x,x') & h = 1
\end{cases}
\]

(14)

where \( \Sigma^l(x,x') = \mathbb{E}[q^l(x)_j q^l(x')_j] \) is given by:

\[
\Sigma^l(x,x') = 2 \mathbb{E}_{u,v \sim \mathcal{N}(0,\Lambda^l(x,x'))}[\phi(u)\phi(v)]
\]

(15)
Given a depth $L$ ResNet, with positive initialization constants $\{\alpha_1\}_{i=1}^L$, it holds at initialization that $G(x, x')$ converges (in law) to $K_{L}^R(x, x')$ as $n_0, n_1, \ldots, n_L \to \infty$, such that:

$$K_{L}^R = K_{L-1}^R(\alpha_L \prod_{h=1}^{m} \Sigma_{h-1}^{L-1} + 1)$$

$$+ \alpha_L \sum_{h=1}^{m} \left( \Sigma_{h-1}^{L-1} \prod_{h'=h}^{m} \tilde{\Sigma}_{h'}^{L-1} \right)$$

where $K_{1}^R = \alpha_1 \sum_{h=1}^{m} \left( \Sigma_{h-1}^{0} \prod_{h'=h}^{m} \tilde{\Sigma}_{h'}^{0} \right)$, $\{\Sigma_{l}^{h}\}$ are defined in Eq. 11, and:

$$\tilde{\Sigma}_{l}^{h}(x, x') =
\begin{cases}
2 \mathbb{E}_{u, v \sim \mathcal{N}(0, \Sigma_{l-1}^{h-1}(x, x'))} [\hat{\phi}(u) \hat{\phi}(v)] & 1 < h \\
2 \mathbb{E}_{u, v \sim \mathcal{N}(0, \Lambda(x, x'))} [\hat{\phi}(u) \hat{\phi}(v)] & h = 1
\end{cases}$$

where $\hat{\phi}(u)$ denotes the derivative of $\phi$ by $u$.

The following theorem states our result for DenseNets.

**Theorem 2.** Given a depth $L$ DenseNet, with positive initialization constant $\alpha$, it holds at initialization that $G(x, x')$ converges (in law) to $K_{L}^{D}(x, x')$ as $n_0, n_1, \ldots, n_L \to \infty$, such that:

$$K_{L}^{D} = K_{L-1}^{D} \left( \frac{\alpha \Sigma_{L-1}^{1} + L - 1}{L} \right) + \frac{\alpha \Sigma_{L-1}^{1}}{L}$$

where $\{\Sigma_{l}^{l}\}$ are defined in Eq. 15, and:

$$\Sigma_{l}^{l}(x, x') = 2 \mathbb{E}_{u, v \sim \mathcal{N}(0, \Lambda(x, x'))} [\hat{\phi}(u) \hat{\phi}(v)]$$

We employed Monte Carlo simulations in order to verify the results of Theorems 1 and 2, as illustrated in Fig. 1. There is an excellent match between the simulations and our derivations.

**5. Finite Width and depth Corrections**

The NTK provides an elegant glimpse into the training dynamics of infinitely wide networks. In this regime, provided that the loss is well-behaved, gradient descent becomes kernel gradient descent in functional space with the fixed neural tangent kernel, where no features are learned.

In practice, zero training error is also achieved also for deep and narrow residual architectures. It is, therefore, worth investigating how close the actual training dynamics of modern architectures (which usually contain residual connections) to the training dynamics of infinitely wide networks.

Specifically, we relax the fixed kernel assumption with a smoothness assumption. A differentiable function $f(W) : \mathbb{R}^d \to \mathbb{R}$ is $\beta$-smooth, if its gradient is $\beta$-Lipschitz. That is, for any $u, v$, it holds $\|\nabla f(u) - f(v)\| \leq \beta \|u - v\|$.

The following lemma gives a sufficient condition for gradient descent to converge to zero training error for $\beta$-smooth functions.

**Lemma 1.** Given a $\beta$-smooth function $f(u) \in [0, 1]$, a learning rate $0 \leq \alpha < \frac{1}{\beta}$ and a constant $0 < c < 1$. Let $\{f(u_t)\}_{t \geq 0}$ be the series of outputs given by applying gradient descent, such that $f(u_{t+1}) = f(u_t) - \alpha \nabla f(u_t)$. Assuming the following holds for any point $u$:

$$\|\nabla f(u)\| \geq \sqrt{2c\beta f(u)}$$

Figure 2. An illustration of (a) ResNet and (b) DenseNet as given in Eq. 6 and Eq. 13 (with constant width and absent scaling coefficients).
it then holds that \( f(u_\infty) = 0 \).

When \( f \) is given by a neural network equipped with a strongly convex loss function, the condition on the gradient given in Eq. 21 can be guaranteed by the positive definiteness of \( \mathcal{G} \). Similarly to Eq. 3 in the infinite case, the batch gradient squared norm in the finite case is given by:

\[
\| \nabla L(W) \|^2 = \frac{\partial L}{\partial \mathcal{G}} \mathcal{G}^T \frac{\partial L}{\partial \mathcal{G}}
\]  

(22)

Considering a regression task, with input label pairs \( \{x_i, y_i\} \) such that \( f, y_i \in [0, 1] \), the functional gradient is lower bounded:

\[
\| \frac{\partial L}{\partial \mathcal{G}}(W) \| = 2|f(W) - y| \geq 2L(W)
\]  

(23)

Plugging into Eq. 22, the condition on the gradient in Lemma 1 is, therefore, guaranteed when the minimum eigenvalue \( \lambda_{\text{min}} \) of \( \mathcal{G} \) is lower bounded by:

\[
\lambda_{\text{min}}(\mathcal{G}) \geq \frac{c\beta}{2}.
\]  

(24)

For infinitely wide networks, there exists a constant \( c < 1 \) such that the condition in Eq. 24 is met, due to the positive definiteness of the NTK. For finite width networks, \( \mathcal{G} \) is random at initialization, and changes during training. Moreover, the rate in which \( \mathcal{G} \) converges to its limit as width increases, depends on the depth of the network.

For vanilla feed forward models, the variance of the entries of \( \mathcal{G} \) is exponential in the ratio between depth to width (Hanin & Nica, 2020). Therefore, taking both parameters to infinity simultaneously, and at the same rate, maintains this variance, and \( \mathcal{G} \) does not converge to the NTK. However, as we show next, models employing residual connections may convergence to the NTK at rates that are independent of depth.

5.1. Forward-Backward Norm Propagation Duality

We aim to derive an expression for the first and second moments of the diagonal entries of \( \mathcal{G} \) at the point of initialization, given by the Jacobian squared norm evaluated on \( x_i \):

\[
\mathcal{G}(x, x) = \| J(x) \|^2 = \sum_k \| J^k(x) \|^2
\]  

(25)

where \( J^k(x) = \frac{\partial f(x)}{\partial \mathcal{G}^k} \) denotes the per-weight Jacobian (here we use a single superscript \( k \) to identify weight matrices), and \( \sum_k \| J^k \|^2 \) denotes summation over every weight matrix in the network. In the following analysis, We assume the output \( f \) is computed using a single fixed sample \( x \). To facilitate our derivation, we introduce a link between the propagation of the norm of the activations, and the norm of the per-layer Jacobian in random ReLU networks of finite width and depth. This link will then allow us to study the statistical properties of the full Jacobian in general architectures incorporating residual connections and concatenations with relative ease. Specifically, we would like to establish a connection between the first and second moments of the squared norm of the output \( (f)^2 \), and those of the per layer Jacobian norm \( \| J^k \|^2 \). Using a path-based notation, for any weight matrix \( W^k \), the output \( f \) can be decomposed to paths that go through \( W^k \) (i.e., paths that include weights from \( W^k \)), denoted by \( f^k \), and paths that skip \( W^k \), denoted by the complement \( (f^k)^c \). Namely:

\[
f = f^k + (f^k)^c
\]  

\[
= \sum_{\gamma \in \{\gamma_k\}} c_\gamma z_\gamma \prod_{l=1}^{|\gamma|} w_{\gamma,l} + \sum_{\gamma \notin \{\gamma_k\}} c_\gamma z_\gamma \prod_{l=1}^{|\gamma|} w_{\gamma,l}
\]  

(26)

where the summation is over paths from input to output, indexed by \( \gamma \), with \( |\gamma| \) denoting the length of the path indexed by \( \gamma \), and \( c_\gamma \) a scaling factor. In non-residual networks, we have \( |\gamma| = L + 2 \) (when considering the initial and final projections \( W^0, W^L \)) and \( \sum_\gamma \) sums over \( d \prod_{l=0}^L n_l \) paths.

The term \( \sum_{\gamma \in \{\gamma_k\}} c_\gamma z_\gamma \prod_{l=1}^{|\gamma|} w_{\gamma,l} \) denotes the product of weights along path \( \gamma \), multiplied by a binary variable \( z_\gamma \in \{0, 1\} \), indicating whether path \( \gamma \) is active (i.e all relevant activations along \( \gamma \) have a scaling factor). In non-residual networks, we have \( |\gamma| = L + 2 \) (when considering the initial and final projections \( W^0, W^L \)) and \( \sum_\gamma \) sums over \( d \prod_{l=0}^L n_l \) paths.

The form \( \sum_{\gamma \notin \{\gamma_k\}} c_\gamma z_\gamma \prod_{l=1}^{|\gamma|} w_{\gamma,l} \) indicates summation over all possible paths that include weights from both \( W^k \), while \( \sum_{\gamma \notin \{\gamma_k\}} \) indicate summation over paths that do not include weights from \( W^k \). In all relevant architectures, the squared norm of the per weight Jacobian \( \| J^k \|^2 \) is given by:

\[
\| J^k \|^2 = \| \frac{\partial f^k}{\partial W^k} + \frac{\partial (f^k)^c}{\partial W^k} \|^2 = \| \frac{\partial f^k}{\partial W^k} \|^2
\]  

(27)

where the last transition stems from the fact that \( z_\gamma \) binary, and \( \frac{\partial (f^k)^c}{\partial W^k} = 0 \) everywhere except for a set of zero measure, where the derivative is ill-defined. The derivative of output with respect to weight \( w_{ij} \) is given by the sum of all paths \( \gamma \) that contain \( w_{ij} \) divided by \( w_{ij} \). We make the following definition:

**Definition 1.** Reduced network \( \mathcal{N}^{(k)} \): the outputs of a reduced network \( f^{(k)}(W) \) given by \( y^{(k)}_0 \ldots y^{(k)}_L \) given input \( x \) are obtained by removing all connections bypassing weights \( W^k \) from the network \( \mathcal{N} \).

Note that for vanilla networks, it holds that \( f^{(k)} = f^k = f \), and \( y^{(k)}_i = y^i \). The following Theorem links between the moments of \( f^{(k)} \) and those of \( f^k \):

**Theorem 3.** For any architecture \( \mathcal{N} \) described in Sec 4 (Eq. 6 and Eq. 13), the following holds at initialization for any non-negative even integer \( m \):

\[
\forall W^k \in \mathcal{N}, \ E \left[ (f^{(k)})^m \right] = E \left[ (f^k)^m \right]
\]  

(28)
In the general case, the equality \( f(k) = f^k \) does not hold, since \( f(k) \) contains different activation patterns, induced by the removal of residual connections. However, Theorem 3 states that the moments of both are equal in the family of considered ReLU networks (see Fig. 3 for an illustration). The following theorem relates the moments of \( \|J^k\|^2 \) with those of \( f(k) \):

**Theorem 4.** For any architecture \( \mathcal{N} \) described in Sec 4 (Eq. 6 and Eq. 13), the following holds at initialization:

1. \( \forall W_k \in \mathcal{N}: \mathbb{E} \left[ \|J^k\|^2 \right] = \mathbb{E} \left[ (f(k))^2 \right] \)
2. \( \forall W_k \in \mathcal{N}: \frac{\mathbb{E} \left[ (f(k))^4 \right]}{c_4} \leq \mathbb{E} \left[ \|J^k\|^4 \right] \leq \mathbb{E} \left[ (f(k))^4 \right] \)

where \( c_4 \) is the fourth moment of the weights. From Eq. 25 and Theorem 4, we can derive bounds on the second moment of \( G(x, x) \), by observing the moments of \( f(k) \). In addition, Theorem 4 also allows us to derive bounds on the convergence rate of \( G(x, x) \) to \( K_L(x, x) \), given by the ratio:

\[
\eta = \frac{\mathbb{E}[G(x, x)^2]}{K_L(x, x)^2}
\]

(29)

An important distinction must be made between the convergence of \( \eta \) as a function of width and depth, and the convergence of \( K_L(x, x) \) as a function of depth. Recall that \( K_L(x, x) \) is a result of taking the infinite width limit of \( G(x, x) \) while keeping the depth fixed. And so, in the general case, the infinite depth limit of \( K_L(x, x) \) may not exist. This fact, however, should not prevent us from considering \( \eta \), even for diverging NTKs, given that a divergent NTK can easily be made convergent by simply scaling the output layer accordingly. In Theorems 5,6, we derive bounds on the asymptotic behavior of \( \eta \), with respect to both width and depth.

**Theorem 5.** For a constant width ResNet (Eq. 6 with \( n_0, n_{0,1}...n_{1,1}...n_L = n \)), with positive initialization constants \( \{\alpha_l\}_{l=1}^L \), there exists a constant \( C > 0 \) such that:

\[
\max \left[ 1, \frac{\sum u \alpha_u^2}{\sum_{u,v} \alpha_u \alpha_v} \right] \leq \eta(n, L) \leq \xi
\]

(30)

where:

\[
\xi = \exp \left[ \frac{C}{n} \sum_{l=1}^L \alpha_l \right] \left( 1 + O\left( \frac{1}{n} \right) \right)
\]

(31)

From the result of Theorem 5, the convergence rate is exponential in \( \frac{1}{n} \sum_{l=1}^L \alpha_l \). Setting \( \{\alpha_l\}_{l=1}^L \) such that \( \frac{1}{n} \sum_{l=1}^L \alpha_l \) vanishes as \( n \) tends to infinity, insures the convergence of \( \eta \) to 1, regardless of depth. Note that the \( \sum_{l=1}^L \alpha_l \sim O(1) \) initialization achieves (though not required) this requirement, and was also suggested in (Zhang et al., 2019) as a way to train ResNets without batchnorm. Our results, however, reveal a much stronger implication of this initialization, as it also bounds the fluctuations of the squared Jacobian norm, implying a closer relationship with the “kernel regime” when training deep ResNets. From Theorem 5, we conclude that a proper initialization plays a crucial role in determining the asymptotic behavior of \( \eta \) in deep ResNets. Surprisingly, this relationship between initialization and \( \eta \) breaks down when considering DenseNets, as illustrated in the following Theorem.

**Theorem 6.** For a constant width DenseNet (Eq. 13 with \( n_0, n_0'...n_L = n \)), with initialization constant \( \alpha > 0 \), there exists constants \( C_1, C_2 > 0 \) such that:

\[
\max \left[ 1, \frac{C_1}{L \log(L)^2} \xi \right] \leq \eta(n, L) \leq \xi
\]

(32)

where:

\[
\xi = \exp \left[ \frac{C_2}{n} \right] \left( 1 + O\left( \frac{1}{n} \right) \right)
\]

(33)

Surprisingly, the depth parameter \( L \), as well as the initialization scale \( \alpha \), are absent in the upper bound of Eq. 32, revealing a depth and scale-invariant property unique to DenseNets. In other words, the convergence rate of \( \eta \) to 1 is exponential in \( \frac{C_2}{n} \), and does not depend on depth, or the scaling coefficient of the weights. This property represents a fundamental differentiating aspect of DenseNets, which might explain practical advantages observed in models incorporating dense residual connections. Finally, it is worth
Figure 4. The (a) second (b) fourth moments, in log scale, of the per layer Jacobian norm $\|J^k\|$ and the squared norm of the output of the corresponding reduced architecture $\|f(k)\|$, for a ResNet with $m = 2, \forall l \alpha_l = 0.3$ and varying depth. The results were obtained from the simulated results of 200 independent runs per depth, where the value for $k$ is random for each depth. All networks were initialized using normal distributions. As can be seen, the mean of both $\|J^k\|^2$ and $\|f(k)\|^2$ closely match, while the fourth moment $\mathbb{E}[\|J^k\|^4]$ is upper and lower bounded by the corresponding moments of the output, as predicted in Theorem 4.

Empirical support We employed Monte Carlo simulations in order to verify our theoretical findings in Theorems 4,5 and 6. This is shown in Figure 4 and 5.

6. Conclusions

The Neural Tangent Kernel has provided new insights into the training dynamics of wide neural networks, as well as their generalization properties, by linking them to kernel methods. In this work, we have derived the corresponding neural tangent kernel for the ResNet and DenseNet architectures. Using a duality principle between forward and backward norm propagation, we have derived finite width and depth corrections for both architectures, and have shown convergence properties of deep residual models that are absent in the vanilla fully connected architectures. Our results shed new light on the effect of residual connections on the training dynamics of practically sized networks, suggesting that models incorporating residual connections may operate much closer to the so called “kernel regime” approximation, than vanilla architectures, even at large depths.

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A. Proofs

A.1. Proof of Theorem 1

Proof. Recall that we need to compute
\[ K_L(x, x')^R = \lim_{n_0, \ldots, n_L \to \infty} \sum_{l=1}^{L} \sum_{h=1}^{m} \left( \frac{\partial f(x, W)}{\partial W^{l,h}} \cdot \frac{\partial f(x', W)}{\partial W^{l,h}} \right) . \]

The derivative \( \frac{\partial f(x, W)}{\partial W^{l,h}} \) can be written in a compact matrix form:
\[
\forall 0 < t \leq L, \quad \frac{\partial f(x, W)}{\partial W^{l,h}} = \begin{cases} 
\sqrt{\prod_{h'=h}^{m-1} q^{l-1,h-1}(x) \left( C_h^{l}(x) A^{l+1}(x) \right)^\top} & 1 < h \leq m \\
\sqrt{\prod_{h'=h}^{m-1} q^{l-1,h-1}(x) \left( C_h^{l}(x) A^{l+1}(x) \right)^\top} & h = 1
\end{cases}
\]

where:
\[
C_h^l(x) = \begin{cases} 
\prod_{l'=h}^{m-1} \left( \hat{Z}^{l,h'}(x) W^{l,h'+1} \right) & 1 < h \leq m \\
I & \text{else}
\end{cases}
\]

and:
\[
A^l(x) = \begin{cases} 
\left( \sqrt{\prod_{h'=h}^{m-1} W^{l,1} C_h^l(x)} + I^l \right) A^{l+1}(x) & 0 < l \leq L \\
1 & l = L
\end{cases}
\]

\( Z^{l,h} \) is a diagonal matrix holding the binary activation variables of layer \( h \) in residual branch \( l \) in its diagonal, and
\( I^l = [I_{n_1}, \theta_{n_l-1-n_l}]^\top \in \mathbb{R}^{n_l-1 \times n_l} \) is a concatenation of an \( n_l \) dimensional identity matrix, and a zero matrix \( \theta_{n_l-1-n_l} \in \mathbb{R}^{n_l \times n_l} \).

It follows:
\[
\forall 1 < h \leq L, \quad \left( \frac{\partial f(x, W)}{\partial W^{l,h}} \cdot \frac{\partial f(x', W)}{\partial W^{l,h}} \right)
\]
\[
= \frac{a_l 2^{m-h}}{\prod_{h'=h}^{m-1} n_{l-1,h'}} \begin{pmatrix} q^{l-1,h-1}(x) \left( C_h^l(x) A^{l+1}(x) \right)^\top, q^{l-1,h-1}(x') \left( C_h^l(x') A^{l+1}(x') \right)^\top \end{pmatrix}
\]
\[
= \frac{a_l 2^{m-h}}{\prod_{h'=h}^{m-1} n_{l-1,h'}} \begin{pmatrix} q^{l-1,h-1}(x), q^{l-1,h-1}(x') \end{pmatrix} A^{l+1}(x)^\top C_h^l(x)^\top C_h^l(x') A^{l+1}(x')
\]

In the following, we take the limits \( n_0, n_0, \ldots, n_{l-1,1+} \to \infty \) consecutively. After taking the limits \( n_0, \ldots, n_{l-1,1+} \to \infty \), it holds that
\[
\frac{1}{n_{l-1,1-}} \begin{pmatrix} q^{l-1,h-1}(x), q^{l-1,h-1}(x') \end{pmatrix} \to \Sigma^{l-1,h-1}(x, x').
\]

We are left with:
\[
\lim_{n_0, \ldots, n_{l-1,1-}} \left( \frac{\partial f(x, W)}{\partial W^{l,h}} \cdot \frac{\partial f(x', W)}{\partial W^{l,h}} \right)
\]
\[
= \frac{a_l 2^{m-h}}{\prod_{h'=h}^{m-1} n_{l-1,h'}} \begin{pmatrix} \Sigma^{l-1,h-1}(x, x') A^{l+1}(x)^\top C_h^l(x)^\top C_h^l(x') A^{l+1}(x') \end{pmatrix}
\]

Taking \( n_{l-1,1-} \to \infty \), it holds that
\[
\frac{2}{n_{l-1,1-}} W^{l,h+1} \hat{Z}^{l,h}(x) \hat{Z}^{l,h}(x') W^{l,h+1} \to \Sigma^{l-1,h}(x, x') I.
\]

And so:
\[
\lim_{n_0, \ldots, n_{l-1,1-}} \left( \frac{\partial f(x, W)}{\partial W^{l,h}} \cdot \frac{\partial f(x', W)}{\partial W^{l,h}} \right)
\]
\[
= \frac{a_l 2^{m-h}}{\prod_{h'=h}^{m-1} n_{l-1,h'}} \begin{pmatrix} \Sigma^{l-1,h-1}(x, x') A^{l+1}(x)^\top C_h^l+1(x) C_h^l(x') A^{l+1}(x') \end{pmatrix}
\]
\[
= a_l \Sigma^{l-1,h-1}(x, x') \prod_{h'=h}^{m-1} \Sigma^{l-1,h'}(x, x') A^{l+1}(x)^\top A^{l+1}(x')
\]
Expanding $A^{l+1}(x)^\top A^{l+1}(x')$:

$$
\langle A^{l+1}(x), A^{l+1}(x') \rangle = A^{l+2}(x)^\top \left( \sqrt{\frac{a_{l+2}^m}{\prod_{l'=0}^{m-1} n_{l,l'}}} W^{l+1,1} C_1^{l+1}(x) + I^{l+1} \right)^\top \ldots 
$$

(46)

$$
\left( \frac{a_{l+2}^m}{\prod_{l'=0}^{m-1} n_{l,l'}} \right) W^{l+1,1} C_1^{l+1}(x) + I^{l+1} A^{l+2}(x')
$$

(47)

$$
= \frac{a_{l+2}^m}{\prod_{l'=0}^{m-1} n_{l,l'}} W^{l+2}(x)^\top C_1^{l+1}(x) W^{l+1,1} I^{l+1} A^{l+2}(x)
$$

(48)

$$
+ \frac{a_{l+2}^m}{\prod_{l'=0}^{m-1} n_{l,l'}} W^{l+2}(x')^\top C_1^{l+1}(x') W^{l+1,1} I^{l+1} A^{l+2}(x)
$$

(49)

$$
+ \frac{a_{l+2}^m}{\prod_{l'=0}^{m-1} n_{l,l'}} A^{l+2}(x)^\top C_1^{l+1}(x') W^{l+1,1} A^{l+2}(x')
$$

(50)

$$
+ \left( \langle A^{l+2}(x), A^{l+2}(x') \rangle \right)
$$

(51)

$$
= T_1 + T_2 + T_3 + T_4
$$

(52)

Note that $\sqrt{\frac{1}{n_l} \| W^{l+1,1} I^{l+1} \|} \rightarrow 0$ as $n_l$ tends to infinity, and so $T_1, T_2 \rightarrow 0$. Taking $n_{l+1} \rightarrow \infty$, we have $\frac{1}{n_l} W^{l+1,1} I^{l+1} \rightarrow I$. Taking $n_0 \ldots n_{l-1} \rightarrow \infty$, we have that $T_3 \rightarrow \prod_{l'=1}^{m-1} \Sigma^{l,h'}(x, x') \langle A^{l+2}(x), A^{l+2}(x') \rangle$.

And so it follows:

$$
\lim_{n_0 \ldots n_L \rightarrow \infty} \langle A^{l+1}(x), A^{l+1}(x') \rangle = \lim_{n_0 \ldots n_L \rightarrow \infty} \langle A^{l+2}(x), A^{l+2}(x') \rangle \left( \alpha_{l+1} \prod_{l'=1}^{m-1} \Sigma^{l,h'}(x, x') + I \right)
$$

(53)

$$
= \prod_{l=1}^{L-1} \left( \alpha_{l+1} \prod_{h'=1}^{m-1} \Sigma^{l,h'}(x, x') + I \right)
$$

(54)

And finally:

$$
\lim_{n_0 \ldots n_L \rightarrow \infty} \left\langle \frac{\partial f(x, W)}{\partial W^{l,h}}, \frac{\partial f(x', W)}{\partial W^{l,h}} \right\rangle = \alpha_l \Sigma^{l-1,h-1}(x, x') \prod_{h'=h}^{m-1} \Sigma^{l-1,h'}(x, x') \prod_{l'=1}^{L-1} \left( \alpha_{l'+1} \prod_{h'=1}^{m-1} \Sigma^{l',h'}(x, x') + I \right)
$$

(55)

(56)

The derivation for the case of $h = 1$ gives an identical result, and so summing over all the weights:

$$
K^{R}_L(x, x') = \lim_{n_0 \ldots n_L \rightarrow \infty} \sum_{l=1}^{L} \sum_{h=1}^{m} \left\langle \frac{\partial f(x, W)}{\partial W^{l,h}}, \frac{\partial f(x, W)}{\partial W^{l,h}} \right\rangle
$$

(57)

$$
= \sum_{l=1}^{L} \sum_{h=1}^{m} \alpha_l \left( \sum_{l'=1}^{L-1} \sum_{h'=h}^{m-1} \Sigma^{l-1,h-1}(x, x') \prod_{l'=1}^{L-1} \left( \alpha_{l'+1} \prod_{h'=1}^{m-1} \Sigma^{l',h'}(x, x') + 1 \right) \right)
$$

(58)

$$
= K^{R}_{L-1}(x, x') \left( \alpha_L \prod_{h=1}^{m-1} \Sigma^{L-1,h}(x, x') + 1 \right) + \alpha_L \sum_{h=1}^{m} \left( \sum_{h'=h}^{m-1} \Sigma^{L-1,h'}(x, x') \prod_{h'=1}^{m-1} \Sigma^{L,h'}(x, x') \right)
$$

(59)

\[\square\]

### A.2. Proof of Theorem 2

**Proof.** Similarly to the ResNet case, the derivative $\frac{\partial f(x, W)}{\partial W^{l,h}}$ can be written in a compact matrix form:

$$
\forall 0 \leq h < L, \quad \frac{\partial f(x, W)}{\partial W^{l,h}} = \sqrt{\frac{\alpha}{\ln_{l-1}^{m} (g^l(x))_{n_{l-1}^{l-1}}} \left( A^{l+1,l}(x) \right)^\top}
$$

(60)
where:

\[
A^{l,h}(x) = \begin{cases} 
\sqrt{\frac{2\alpha}{ln_{l-1}}} Z^h(x) I^{h,l-1} W^{l,h} A^{l+1,l}(x) + A^{l+1,h}(x) & 0 \leq h < l < L \\
\sqrt{\frac{2\alpha}{ln_{l-1}}} Z^h(x) I^{h,l-1} W^{l,h} A^{l+1,l}(x) & l = L \\
\frac{1}{\sqrt{n}} W^f & l = L + 1
\end{cases}
\] (61)

where \( Z^l \) is a diagonal matrix holding the binary activation variables of layer \( l \) in its diagonal, and \( I^{h,l} = [I_n^l; 0_{n_h-n_i}^l]^T \in \mathbb{R}^{n_h \times n_i^l} \) is a concatenation of an \( n_h \) dimensional identity matrix, and a zero matrix \( 0_{n_h-n_i^l} \in \mathbb{R}^{n_i^l \times n_h - n_i^l} \). We then have:

\[
\forall 0 \leq h < l \leq L, \quad \left\langle \frac{\partial f(x, W)}{\partial W^h}, \frac{\partial f(x', W)}{\partial W^h} \right\rangle = \frac{\alpha}{ln_{l-1}} \left\langle [q^h(x)]_{n_l-1'}, [q^h(x')]_{n_l-1} \right\rangle \left\langle A^{l+1,l}(x), A^{l+1,l}(x') \right\rangle
\] (62)

In the following we take the limits \( n_0, n_0'...n_L \to \infty \) consecutively. After taking the limit \( n_0, n_0'...n_{l-1}' \to \infty \), it holds that \( \frac{\alpha}{ln_{l-1}} \left\langle [q^h(x)]_{n_l-1'}, [q^h(x')]_{n_l-1} \right\rangle \to \frac{\alpha}{l} \sum (h, x') \). We are left with computing the limit of \( \left\langle A^{l+1,l}(x), A^{l+1,l}(x') \right\rangle \). It follows that:

\[
\forall 0 \leq h < l \leq L, \quad \left\langle A^{l,h}(x), A^{l,h}(x') \right\rangle = \sqrt{\frac{2\alpha}{ln_{l-1}}} A^{l+1,h}Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x')
\] (63)

\[
\left\langle A^{l,h}(x), A^{l,h}(x') \right\rangle = \sqrt{\frac{2\alpha}{ln_{l-1}}} A^{l+1,h}Z^h(x) I^{h,l-1} W^{l,h} A^{l+1,l}(x)
\] (64)

\[
\sqrt{\frac{2\alpha}{ln_{l-1}}} A^{l+1,l}Z^h(x') W^{l,h} I^{h,l-1} Z^h(x) Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x') + A^{l+1,h}Z^h(x) A^{l+1,l}(x')
\] (65)

\[
T_1 = T_2 + T_3 + T_4
\] (66)

Expanding \( T_1 \):

\[
T_1 = \sqrt{\frac{2\alpha}{ln_{l-1}}} A^{l+1,h}Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x')
\] (67)

\[
= \left( \sqrt{\frac{2\alpha}{(l+1)n_l}} Z^h(x) I^{h,l} W^{l+1,h} A^{l+2,l+1}(x) + A^{l+2,h}(x) \right) \left( \sqrt{\frac{2\alpha}{ln_{l-1}}} Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x') \right)
\] (68)

\[
= \frac{2\alpha}{\sqrt{(l+1)n_l} n_{l-1}'n_i^l} A^{l+2,l+1}Z^h(x) W^{l+1,h} I^{h,l+1}Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x')
\] (69)

\[
+ A^{l+2,h}(x) \left( \sqrt{\frac{2\alpha}{ln_{l-1}}} Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x') \right)
\] (70)

Looking at the first term of the expansion, notice that after taking the limit \( n_0...n_{l-1}' \to \infty \), it holds that \( \frac{1}{\sqrt{n_{l-1}'}} ||I^{h,l}Z^h(x)Z^h(x') I^{h,l-1} W^{l,h}|| \to 0 \), and so we are left with the second term:

\[
\lim_{n_0...n_{l-1}' \to \infty} T_1 = \lim_{n_0...n_{l-1}' \to \infty} A^{l+2,h}(x) \left( \sqrt{\frac{2\alpha}{ln_{l-1}'}} Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x') \right)
\] (71)

Recursively expanding \( A^{l+2,h}(x) \), we get similar terms that vanish in the limit, therefore:

\[
\lim_{n_0...n_{l-1}' \to \infty} T_1 = \lim_{n_0...n_{l-1}' \to \infty} A^{L,h}(x) \left( \sqrt{\frac{2\alpha}{ln_{l-1}'}} Z^h(x') I^{h,l-1} W^{l,h} A^{l+1,l}(x') \right) \to 0
\] (72)
And so \( \lim_{n_0 \rightarrow \infty} T_1, T_2 \rightarrow 0 \). We are left with evaluating the limit of \( T_3, T_4 \). Expanding \( T_3 \):

\[
\lim_{n_0 \rightarrow \infty} T_3 = \lim_{n_0 \rightarrow \infty} \frac{2\alpha}{m_{n_0}} A^{l+1,l \uparrow}(x) W^{l,h \uparrow} I^{h,l \uparrow} Z^{h}(x) Z^{h}(x') I^{h,l \downarrow} W^{l,h} A^{l+1,l}(x')
\]

(73)

Note that it holds that \( \lim_{n_0 \rightarrow \infty} \frac{2\alpha}{m_{n_0}} W^{l,h \uparrow} I^{h,l \uparrow} Z^{h}(x) Z^{h}(x') I^{h,l \downarrow} W^{l,h} \rightarrow \alpha \hat{\Sigma}^h(x,x')I \), and so:

\[
\lim_{n_0 \rightarrow \infty} T_3 = \frac{\alpha \hat{\Sigma}^h(x,x')}{l} A^{l+1,l \uparrow}(x) A^{l+1,l}(x')
\]

(74)

Plugging back into Eq. 63:

\[
\lim_{n_0 \rightarrow \infty} \left\langle A^{l,h}(x), A^{l,h}(x') \right\rangle = \lim_{n_0 \rightarrow \infty} T_3 + T_4
\]

(75)

\[
= \frac{\alpha \hat{\Sigma}^h(x,x')}{l} \left\langle A^{l+1,l}(x), A^{l+1,l}(x') \right\rangle + \left\langle A^{l+1,h}(x), A^{l+1,h}(x') \right\rangle
\]

(76)

and:

\[
\lim_{n_0 \rightarrow \infty} \left\langle A^{L,h}(x), A^{L,h}(x') \right\rangle = \lim_{n_0 \rightarrow \infty} T_3
\]

(77)

\[
= \lim_{n_0 \rightarrow \infty} \frac{\alpha \hat{\Sigma}^h(x,x')}{L} \left\langle A^{L+1,l}(x), A^{L+1,l}(x') \right\rangle = \frac{\alpha \hat{\Sigma}^h(x,x')}{L}
\]

(78)

Denoting by \( m^{l,h}(x,x') = \lim_{n_0 \rightarrow \infty} \left\langle A^{l,h}(x), A^{l,h}(x') \right\rangle \), it holds that:

\[
\lim_{n_0 \rightarrow \infty} m_{L+h} = \frac{1}{n_L} Wf^\uparrow Wf = 1
\]

(79)

Plugging into Eq. 62, we arrive at:

\[
\lim_{n_0 \rightarrow \infty} \left\langle \frac{\partial f(x,W)}{\partial W^{l,h}}, \frac{\partial f(x',W)}{\partial W^{l,h}} \right\rangle = \frac{\alpha \Sigma^h(x,x')}{l} m^{l+1,l}(x,x')
\]

(80)

with the following recursion:

\[
m_{l,h}(x,x') = \begin{cases} 
\frac{\alpha \hat{\Sigma}^h(x,x')}{l} m^{l+1,l}(x,x') + m^{l+1,h}(x,x') & 0 \leq h < l < L \\
\frac{\alpha \hat{\Sigma}^h(x,x')}{L} & 0 \leq h < L, l = L \\
1 & \text{else}
\end{cases}
\]

(81)

Summing over all the weights, we have:

\[
K^D_L(x,x') = \sum_{l=1}^{L} \sum_{h=0}^{l-1} \frac{\alpha \hat{\Sigma}^h(x,x')}{l} m^{l+1,l}(x,x') = K^D_{L-1}(x,x') \left( \frac{\alpha \Sigma^{L-1}(x,x')}{L} + \frac{L-1}{L} \right) + \frac{\alpha \Sigma^{L-1}(x,x')}{L}
\]

(82)

\[
\Box
\]

A.3. Proof of lemma 1

Proof. Smoothness of \( f \) implies that for any \( u, v \) it holds \( \| \nabla f(u) - \nabla f(v) \| \leq \frac{\beta}{2} \| u - v \| \), and the following hold:

\[
f(v) \leq f(u) + (\nabla f(u), v - u) + \frac{\beta}{2} \| v - u \|^2
\]

(83)

Setting \( v = u - \frac{1}{\beta} \nabla f(u) \), we have (using the fact that \( f \geq 0 \)):

\[
\| \nabla f(u) \| \leq \sqrt{2\beta f(u)}
\]

(84)
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together with the assumption that \( \|\nabla f(u)\| \geq \sqrt{2c\beta f(u)} \), we have:

\[
\sqrt{2c\beta f(u)} \leq \|\nabla f(u)\| \leq \sqrt{2\beta f(u)}
\]  

(85)

Setting \( v = u - \alpha \nabla f(u) \) in Eq. 83:

\[
f(v) \leq f(u) - \alpha \|\nabla f(u)\|^2 \left( 1 - \frac{\alpha \beta}{2} \right) \leq f(u) - 2c\alpha \beta f(u) \left( 1 - \frac{\alpha \beta}{2} \right)
\]

\[
= f(u) \left( 1 - 2c\alpha \beta \left( 1 - \frac{\alpha \beta}{2} \right) \right)
\]  

(87)

For \( 0 \leq \alpha < \frac{1}{\beta} \), it holds that \( 0 \leq \left( 1 - 2c\alpha \beta \left( 1 - \frac{\alpha \beta}{2} \right) \right) < 1 \), and so it follows:

\[
f(u_{\infty}) \leq f(u_0) \left( 1 - 2c\alpha \beta \left( 1 - \frac{\alpha \beta}{2} \right) \right)^{\infty} = 0
\]  

(88)

We make use of the following propositions and definitions to aid in the proofs of Theorems 3 and 4.

**Proposition 1.** Given a random vector \( w = [w_1, \ldots, w_n] \) such that each component is identically and symmetrically distributed i.i.d random variable with moments \( E[w_i^m] = c_m \) (\( c_0 = 1, c_1 = 0 \)), a set of non negative integers \( m_1, \ldots, m_l \) such that \( \sum_{i=1}^l m_i \) is even, and a random binary variable \( z \in \{0, 1\} \) such that \( (z|w) = 1 - (z|-w) \), then it holds that:

\[
E[\prod_{i=1}^l w_i^{m_i} z] = \prod_{i=1}^l \frac{c_{m_i}}{2}
\]

(89)

**Proof.** We have:

\[
\prod_{i=1}^l c_{m_i} = \int \prod_{i=1}^l w_i^{m_i} p(w)dw = \int_{w|z=1} \prod_{i=1}^l w_i^{m_i} p(w)dw + \int_{w|z=0} \prod_{i=1}^l w_i^{m_i} p(w)dw
\]

(90)

\[
= \int_{w|z=1} \prod_{i=1}^l w_i^{m_i} p(w)dw + \int_{w|z=1} \prod_{i=1}^l (-w_i)^{m_i} p(w)dw
\]

(91)

\[
= \int_{w|z=1} \prod_{i=1}^l w_i^{m_i} zp(w)dw + \int_{w|z=1} \prod_{i=1}^l (-w_i)^{m_i} zp(w)dw
\]

(92)

For even \( \sum_{i=1}^l m_i \), it follows that:

\[
\int_{w|z=1} \prod_{i=1}^l (-w_i)^{m_i} zp(w)dw = \int_{w} \prod_{i=1}^l w_i^{m_i} zp(w)dw
\]

(93)

\[
\prod_{i=1}^l c_{m_i} = 2 \int_{w} \prod_{i=1}^l w_i^{m_i} zp(w)dw
\]

(94)

yielding:

\[
\prod_{i=1}^l \frac{c_{m_i}}{2} = \int_{w} \prod_{i=1}^l w_i^{m_i} zp(w)dw = E[\prod_{i=1}^l w_i^{m_i} z]
\]

(95)
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**Proposition 2.** Given a random vector \( w = [w_1...w_n] \) such that each component is identically and symmetrically distributed i.i.d random variable with moments \( E(w_i^m) = c_m \) (\( c_0 = 1, c_1 = 0 \)), two sets of non negative integers \( m_1...m_l, n_1...n_l \), such that \( \sum_{i=1}^l m_i, \sum_{i=1}^l n_i \) are even, \( \forall i, m_i \geq n_i \), and a random binary variable \( z \in \{0, 1\} \), such that \( (z|w) = 1 - (z|\bar{w}) \), then it holds that:

\[
E\left[ \frac{1}{w_i} \prod_{i=1}^l w_i^{m_i} z \right] = \prod_{i=1}^l c_{m_i-n_i} \frac{1}{2}
\]

**Proof.** This is trivially true from proposition 1 since \( \sum_{i}(m_i - n_i) \) is even. \( \square \)

**Definition 2.** ResNet path parametrization: A path from input to output \( \gamma \) in a ResNet, with two layer residual branches \( (m=2) \), defines a product of weights along the path denoted by:

\[
P_{\gamma} = \prod_{l=0}^{L+1} p_{\gamma,l}
\]

where:

\[
p_{\gamma,l} = \begin{cases} 
1 & l \notin \gamma \\
w_{\gamma,l}^1 z_{\gamma,l} w_{\gamma,l}^2 & l \in \gamma, 0 < l \leq L \\
w_{\gamma,l} & l = \{0, L+1\}
\end{cases}
\]

Here, \( w_{\gamma,l}^1, w_{\gamma,l}^2 \) are weights associated with residual branch \( l \), \( w_{\gamma,0}, w_{\gamma,L+1} \) belong to the first and last linear projection matrices \( W^s, W^f \), and \( z_{\gamma,l} \) is the binary activation variable relevant for weight \( w_{\gamma,l}^1 \). (Note that \( z_{\gamma,l} \) depends on \( w_{\gamma,l}^1 \) but not on \( w_{\gamma,l}^2 \)). \( l \notin \gamma \) indicates if layer \( l \) is skipped.

**Definition 3.** DenseNet path parametrization: Similarly to the ResNet case, a path \( \gamma \) from input in to output in a DenseNet, defines a product of weights along the path denoted by:

\[
P_{\gamma} = \prod_{l=0}^{L+1} p_{\gamma,l}
\]

where:

\[
p_{\gamma,l} = \begin{cases} 
1 & l \notin \gamma \\
w_{\gamma,l} z_{\gamma,l} & l \in \gamma, 0 < l \leq L \\
w_{\gamma,l} & l = \{0, L+1\}
\end{cases}
\]

Here, \( w_{\gamma,l} \) is a weight associated with layer \( l \), \( w_{\gamma,0}, w_{\gamma,L+1} \) belong to the first and last linear projection matrices \( W^s, W^f \), and \( z_{\gamma,l} \) is the binary activation variable relevant for weight \( w_{\gamma,l} \). \( l \notin \gamma \) indicates if layer \( l \) is skipped.

A.4. Proof of Theorem 3

**Proof.** We present the proof using the DenseNet path parameterization. Extending to ResNet parameterization is trivial and requires no additional arguments. We aim to show that for an even \( m \), and \( \forall W^s, W^f \in N \):

\[
E \left[ \| f(k) \|^m \right] = E \left[ \| f(k) \|^m \right]
\]

The output \( f^k \) can be expressed as follows:

\[
f^k = \sum_{\gamma \in \{\gamma_k\}} c_{\gamma} \prod_{l=0}^{L+1} p_{\gamma,l}
\]
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Since the output \( f(x) \) is composed of products of weights and activations along the same paths \( \gamma \in \{ \gamma_k \} \) as \( f^k \) (with different activation variables), we only need to prove the following:

For any weight matrix \( W_k \in \mathcal{N} \), and a set of \( m \) paths \( \gamma^1, ..., \gamma^m \in \{ \gamma_k \} \), it holds that:

\[
\mathbb{E}_\mathcal{N} \left[ \prod_{i=1}^{m} P_{\gamma^i} \right] = \mathbb{E}_{\mathcal{N}_{(k)}} \left[ \prod_{i=1}^{m} P_{\gamma^i} \right] \tag{103}
\]

where \( \mathbb{E}_\mathcal{N} \) stands for expectation using the full architecture, and \( \mathbb{E}_{\mathcal{N}_{(k)}} \) stands for expectation using the reduced architecture.

We have that:

\[
\mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i} \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} \left( \prod_{i=1}^{m} P_{\gamma^i, l} \right) \right] \tag{104}
\]

From the linearity of the last layer, it follows that:

\[
\mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i} \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} \left( \prod_{i=1}^{m} P_{\gamma^i, l} \right) \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} \left( \prod_{i=1}^{m} P_{\gamma^i, l} \right) \right] \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} w_{\gamma^i, l+1} \right] \tag{105}
\]

We denote by \( \{ w_{u, l+1} \}_{u=1}^{s} \), \( s \leq m \) the set of \( s \) unique weights in \( \{ w_{\gamma^i, l+1} \}_{i=1}^{m} \), with corresponding multiplicity \( \{ m_{u}^{l+1} \} \), such that \( \sum_{u} m_{u}^{l+1} = m \). It follows that:

\[
\mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i} \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} \left( \prod_{i=1}^{m} P_{\gamma^i, l} \right) \right] \mathbb{E}_{\mathcal{N}} \left[ \prod_{u} \left( w_{u}^{l+1} \right)^{m_{u}^{l+1}} \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{l=0}^{L} \left( \prod_{i=1}^{m} P_{\gamma^i, l} \right) \right] \prod_{u} c_{m_{u}^{l+1}} \tag{106}
\]

where \( c_{m_{u}^{l+1}} \) is the \( m_{u}^{l+1} \)th moment of a normal distribution.

Since the computations done by all considered architectures form a markov chain, such that the output of any layer depends only on up-stream weights, denoted by \( R^{l-1} \), we have that:

\[
\mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i, L} \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i, L} \left| R^{L-1} \right. \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i, L} \left| q^0 \ldots q^{L-1} \right. \right] \tag{107}
\]

Conditioned on \( q^0 \ldots q^{L-1} \), the pre-activations \( y^L \) are zero mean iid Gaussian variables. In addition, the activations \( q^L = 2 \phi(y^L) \) are iid distributed. We denote by \( \{ z_{u} \}_{u=1}^{s} \) the set of unique activation variables in the set \( \{ z_{\gamma^i, L} \}_{i=1}^{m} \). For each \( z_{u} \), we denote by \( \{ w_{u,v}^{L} \} \) the set of unique weights in \( \{ w_{\gamma^i} \}_{i=1}^{m} \) multiplying \( z_{u} \), with corresponding multiplicity \( m_{u,v}^{L} \), such that \( \sum_{v} m_{u,v}^{L} = m \), and \( \sum_{u} m_{u,v}^{L} = m_{u}^{L+1} \). Note that, from the symmetry of the normal distribution, it holds that odd moments vanish, and so we only need to consider even \( m_{u}^{L+1} \) for all \( u \). From the independence of the set \( \{ z_{u} \} \), the expectation takes a factorized form:

\[
\mathbb{E}_{\mathcal{N}} \left[ \mathbb{E}_{\mathcal{N}} \left[ \prod_{i=1}^{m} P_{\gamma^i, L} \left| q^0 \ldots q^{L-1} \right. \right] \right] = \mathbb{E}_{\mathcal{N}} \left[ \prod_{u=1}^{s} \mathbb{E}_{\mathcal{N}} \left[ z_{u} \prod_{v} (w_{u,v}^{L})^{m_{u,v}^{L}} \left| q^0 \ldots q^{L-1} \right. \right] \right] \tag{108}
\]

\[
= \mathbb{E}_{\mathcal{N}} \left[ \prod_{u=1}^{s} \mathbb{E}_{\mathcal{N}} \left[ z_{u} \prod_{v} (w_{u,v}^{L})^{m_{u,v}^{L}} \sum_{h=0}^{L-1} || q^{h} || > 0 \right] \right] \tag{109}
\]

Note that for both the full and reduced architecture, flipping the sign of all weights in layer \( k \), will flip the activation variables (except for a set of zero measure defined by \( \sum_{h=0}^{L-1} W^k h q^h = 0 \), which does not affect the expectation). So, using Proposition 1:

\[
\mathbb{E}_{\mathcal{N}} \left[ \prod_{u=1}^{s} \mathbb{E}_{\mathcal{N}} \left[ z_{u} \prod_{v} (w_{u,v}^{L})^{m_{u,v}^{L}} \sum_{h=0}^{L-1} || q^{h} || > 0 \right] \right] = \mathbb{E} \left[ \prod_{u=1}^{s} \frac{c_{m_{u,v}^{L}}}{2} \right] \tag{110}
\]
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It then follows:

\[
E_N \left[ \prod_{l=0}^{L-1} \prod_{i=1}^{m} p_{\gamma,i,l} \right] = E_N \left[ \prod_{l=0}^{L-1} \prod_{i=1}^{m} p_{\gamma,i,l} \right] \prod_{u=1}^{s} \left( \frac{\prod_{v} e_{m_{w,v}}^k}{2} \right)
\]

Similarly, it holds for the reduced architecture:

\[
E_{N(k)} \left[ \prod_{l=0}^{L-1} \prod_{i=1}^{m} p_{\gamma,i,l} \right] = E_{N(k)} \left[ \prod_{l=0}^{L-1} \prod_{i=1}^{m} p_{\gamma,i,l} \right] \prod_{u=1}^{s} \left( \frac{\prod_{v} e_{m_{w,v}}^k}{2} \right)
\]

Recursively going through \( l = L - 1 \ldots 0 \) completes the proof for DenseNet.

\[\square\]

A.5. proof of Theorem 4

Proof. We present the proof using the DenseNet path parameterization. Extending to ResNet parameterization is trivial and requires no additional arguments. Neglecting scaling coefficients for notational simplicity, for any weight matrix \( W^k \in N \), the Jacobian \( J^k \) can be expressed as follows:

\[
E \left[ \left\| J^k \right\|^2 \right] = E \left[ \left\| \frac{\partial f^k}{\partial W^k} \right\|^2 \right] = \sum_{i,j} E \left[ \left( \sum_{\gamma \mid w_{i,j}^k \in \gamma} \frac{1}{w_{i,j}^k} P_{\gamma} \right)^2 \right]
\]

where \( \sum_{\gamma \mid w_{i,j}^k \in \gamma} \) represents sum over paths that include weight \( w_{i,j}^k \). From applying Eq. 111 recursively, the expectation is factorized as follows:

\[
E \left[ \left\| \frac{\partial f^k}{\partial W^k} \right\|^2 \right] = \sum_{i,j} \sum_{\gamma \mid w_{i,j}^k \in \gamma} E \left[ \left( \frac{1}{w_{i,j}^k} P_{\gamma, k} \right)^2 \right] \prod_{h=0}^{k-1} \left( \sum_{l \neq k} \left( P_{\gamma, l} \right)^2 \right) \prod_{h=0}^{k-1} \left( \left\| q_{\gamma, l} \right\|^2 > 0 \right)
\]

Using Propositions 1 and 2, we have that

\[
\forall \gamma \mid w_{i,j}^k \in \gamma, \left( \frac{1}{w_{i,j}^k} P_{\gamma, k} \right)^2 \sum_{h=0}^{k-1} \left\| q_{\gamma, l} \right\|^2 > 0 = E \left[ \left( \frac{1}{w_{i,j}^k} P_{\gamma, l} \right)^2 \sum_{h=0}^{k-1} \left\| q_{\gamma, l} \right\|^2 > 0 \right] = \frac{1}{\sum_{h=0}^{k-1} \left\| q_{\gamma, l} \right\|^2}
\]

Inserting into Eq. 116, and using Theorem 3 proves the first claim.

For the second claim, it follows:

\[
E \left[ \left\| J^k \right\|^4 \right] = E \left[ \left\| \frac{\partial f^k}{\partial W^k} \right\|^2 \right] \left\| \frac{\partial f^k}{\partial W^k} \right\|^2
\]

\[
= \sum_{i,j} \sum_{i',j'} E \left[ \left( \sum_{\gamma \mid w_{i,j}^k \in \gamma} \frac{1}{w_{i,j}^k} P_{\gamma} \right)^2 \left( \sum_{\gamma \mid w_{i',j'}^k \in \gamma} \frac{1}{w_{i',j'}^k} P_{\gamma} \right)^2 \right]
\]

\[
= \sum_{i,j} \sum_{i',j'} E \left[ \left( \frac{1}{w_{i,j}^k w_{i',j'}^k} \right)^2 \sum_{\gamma, \gamma' \mid w_{i,j}^k \in \gamma, \gamma' \sum_{\gamma, \gamma' \mid w_{i',j'}^k \in \gamma', \gamma''} P_{\gamma'} P_{\gamma''} P_{\gamma} \right)^2 \right]
\]
We use the following proposition to aid in the proofs of Theorems 5 and 6.

Inserting into Eq. 122, using Theorem 3, and the fact that \( c \) from Proposition 1, it follows that:

Conditioning on \( R \), the expectation is factorized as follows:

For vanilla fully connected network, with intermediate outputs given by:

Absorbing the scale \( W \), we denote by \( Z \) the diagonal matrix holding in its diagonal the activation variables \( z_j \) for unit \( j \) in layer \( l \), and so we have:

We use the following proposition to aid in the proofs of Theorems 5 and 6.

**Proposition 3.** For vanilla fully connected network, with intermediate outputs given by:

where the weight matrices \( W^l \in \mathbb{R}^{n_l-1 \times n_l} \) are normally distributed, the following holds at initialization:

**Proof.** Absorbing the scale \( \sqrt{\frac{2}{n_l-1}} \) into the weights, we denote by \( Z \) the diagonal matrix holding in its diagonal the activation variables \( z_j \) for unit \( j \) in layer \( l \), and so we have:

Conditioning on \( R^l-1 = \{W^1...W^{l-1}\} \) and taking expectation:

From Proposition 1, it follows that:

\[
E[\|y\|^2] = E \left[ E[\|y\|^2|R^{l-1}] \right] = \frac{n_L}{n_{L-1}} E[\|y^{l-1}\|^2]
\]
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Similarly:

$$\mathbb{E} \left[ \| \mathbf{y}' \|_4^4 \left| R^{l-1} \right. \right] = \mathbb{E} \left[ (y^{l-1}_1)^4 \right] + \frac{6n_l}{n^2_{l-1}} \sum_{i_1 \neq i_2} \mathbb{E} \left[ (y^{l-1}_{i_1})^2 (y^{l-1}_{i_2})^2 \right]$$

From Proposition 1, and the independence of the activation variables conditioned on \( R^{l-1} \):

$$\mathbb{E} \left[ w_{i_1,j_1}^l w_{i_2,j_1}^l w_{i_3,j_2}^l w_{i_4,j_2}^l z_{j_1}^l z_{j_2}^l \left| R^{l-1} \right. \right]$$

and so:

$$\mathbb{E} \left[ \| \mathbf{y}' \|_4^4 \right] = \frac{n_l c_4}{2} \sum_{i=1} \mathbb{E} \left[ (y^{l-1}_i)^4 \right] + \frac{6n_l}{n^2_{l-1}} \sum_{i_1 \neq i_2} \mathbb{E} \left[ (y^{l-1}_{i_1})^2 (y^{l-1}_{i_2})^2 \right]$$

$$+ \frac{n_l(n_l - 1)}{n^2_{l-1}} \sum_{i_1,i_2} \mathbb{E} \left[ (y^{l-1}_{i_1})^2 (y^{l-1}_{i_2})^2 \right]$$

For Gaussian distributions, \( \Delta = 0 \), proving the claim.

**Proposition 4.** For a vanilla fully connected linear network, with intermediate outputs given by:

$$\forall 0 \leq l \leq L, \quad y^l = \frac{1}{\sqrt{n_{l-1}}} W^l y^{l-1}$$

, where the weight matrices \( W^l \in \mathbb{R}^{n_l \times n_l} \) are normally distributed, the following holds at initialization:

$$\mathbb{E} \left[ \| \mathbf{y}' \|_2^2 \right] = \frac{n_l}{n_{l-1}} \mathbb{E} \left[ \| \mathbf{y}' \|_2^2 \right]$$

$$\mathbb{E} \left[ \| \mathbf{y}' \|_4^4 \right] = \frac{n_l(n_l + 2)}{n^2_{l-1}} \mathbb{E} \left[ \| \mathbf{y}' \|_4^4 \right]$$

**Proof.** The proof follows the derivation of Proposition 3 exactly, and will be omitted for brevity.

**A.6. Proof of Theorem 5**

Using the result of Theorem 4, and using Cauchy–Schwarz inequality, an upper bound to \( \eta \) can be derived:

$$\eta = \frac{\mathbb{E} \left[ \mathcal{G}(x, x) \right]}{K_L(x, x)^2} = \frac{\sum_{u,v} \mathbb{E} \left[ \| J_u \|_2^2 \| J_v \|_2^2 \right]}{K_L(x, x)^2} \leq \frac{\sum_{u,v} \sqrt{\mathbb{E} \left[ \| J_u \|_4^4 \right] \mathbb{E} \left[ \| J_v \|_4^4 \right]}}{K_L(x, x)^2} \leq \frac{\sum_{u,v} \sqrt{\mathbb{E} \left[ \| f(u) \|_4^4 \right] \mathbb{E} \left[ \| f(v) \|_4^4 \right]}}{K_L(x, x)^2}$$

The lower bound is similarly derived using Theorem 4:

$$\eta \geq \frac{\sum_{k} \mathbb{E} \left[ \| J_k \|_4^4 \right]}{K_L(x, x)^2} \geq \frac{1}{c_4} \frac{\sum_{k} \mathbb{E} \left[ \| f(k) \|_4^4 \right]}{K_L(x, x)^2}$$
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The asymptotic behaviour of $\eta$ is therefore governed by the propagation of the fourth moment $\mathbb{E}[\|y_{(k)}^L\|^4]$ through the model.

In the following proof, for the sake of notation simplicity, we omit the notation $k$ in $y_{(k)}^l$, and assume that $y^l$ stands for the reduced network $y_{(k)}^l$. The recursive formula for the intermediate outputs of the reduced network are given by:

$$y^l = \begin{cases} 
    y^{l-1} + \sqrt{\alpha_l} y^{l-1,m} & 0 < l \leq L, l \neq k \\
    \sqrt{\alpha_l} y^{l-1,m} & l = k 
\end{cases} \quad (149)$$

where:

$$y^{l-1,h} = \begin{cases} 
    \sqrt{\frac{1}{n}} W^{l,h} q^{l-1,h-1} & 1 < h \leq m \\
    \sqrt{\frac{1}{n}} W^{l,h} y^{l-1} & h = 1 
\end{cases} \quad (150)$$

with $q^{l-1,h} = \sqrt{2} \phi(y^{l-1,h})$.

We have for layer $L$, using the results of Propositions 3 and 4:

$$\mathbb{E} \left[ \|y^L\|^2 \right] = \mathbb{E} \left[ \|y^{L-1}\|^2 \right] + \frac{\alpha_L}{n} \mathbb{E} \left[ y^{L-1,m-1} W^L W^{L-1} y^{L-1,m-1} \right] = \mathbb{E} \left[ \|y^{L-1}\|^2 \right] (1 + \alpha_L) \quad (151)$$

$$\mathbb{E} \left[ \|y^L\|^2 \right] = \mathbb{E} \left[ \|y^{L-1}\|^2 \right] + \alpha_L \mathbb{E} \left[ \|y^{L-1,m-1}\|^2 \right] = \mathbb{E} \left[ \|y^{L-1}\|^2 \right] (1 + \alpha_L) \quad (152)$$

$$\mathbb{E} \left[ \|y^L\|^2 \right] = \mathbb{E} \left[ \|y^{L-1}\|^2 \right] \prod_{i=k}^L (1 + \alpha_i) \quad (153)$$

For the fourth moment, using the results of proposition 3 and 4 (taking into account that odd powers will vanish in expectation), it holds:

$$\mathbb{E} \left[ \|y^L\|^4 \right] = \mathbb{E} \left[ \|y^{L-1}\|^4 \right] + \alpha_L^2 \mathbb{E} \left[ \|y^{L-1,m}\|^4 \right] + 4 \alpha_L \mathbb{E} \left[ (y^{L-1,m-1} y^{L-1})^2 \right] + 2 \alpha_L \mathbb{E} \left[ \|y^{L-1,m}\|^2 \|y^{L-1}\|^2 \right] \quad (155)$$

We now handle each term separately:

$$\mathbb{E} \left[ \|y^{L-1,m}\|^4 \right] = \mathbb{E} \left[ \mathbb{E}[\|y^{L-1,m}\|^4 | R^{L-1} ] \right] \quad (156)$$

Using the results of Propositions 3 and 4:

$$\mathbb{E} \left[ \|y^{L-1,m}\|^4 | R^{L-1} \right] = (1 + \frac{2}{n}) (1 + \frac{5}{n})^m \|y^{L-1}\|^4 \sim (1 + \frac{5}{n})^m \|y^{L-1}\|^4 \quad (157)$$

$$\mathbb{E} \left[ (y^{L-1,m-1} y^{L-1})^2 \right] = \frac{1}{n} \sum_{j_1,j_2} \mathbb{E} \left[ y_{i_1}^{L-1,m-1} y_{j_1}^{L-1,m-1} y_{i_2}^{L-1} y_{j_2}^{L-1} w_{i_1,j_1} L m w_{i_2,j_2} \right] \quad (158)$$

$$\mathbb{E} \left[ \|y^{L-1,m-1}\|^2 \|y^{L-1}\|^2 \right] = \frac{1}{n} \mathbb{E} \left[ \|y^{L-1}\|^4 \right] \quad (159)$$

$$\mathbb{E} \left[ \|y^{L-1,m}\|^2 \|y^{L-1}\|^2 \right] = \mathbb{E} \left[ \|y^{L-1}\|^4 \right] \quad (160)$$

Plugging it all into Eq. 155, and denoting:

$$\beta_L = 1 + 2 \alpha_L (1 + \frac{2}{n}) + \alpha_L^2 (1 + \frac{5}{n})^m \quad (161)$$

$$\beta_L = 1 + 2 \alpha_L (1 + \frac{2}{n}) + \alpha_L^2 (1 + \frac{5}{n})^m \quad (162)$$

we have after recursing through $l = L - 1 \ldots k + 1$:

$$\mathbb{E} \left[ \|y^L\|^4 \right] \sim \mathbb{E} \left[ \|y^k\|^4 \right] \prod_{l=k+1}^L \beta_l \quad (163)$$
In the reduced architecture, the transformation from layer $k - 1$ to layer $k$ is given by an $m$ layer fully connected network, with a linear layer on top, we can use the results from the vanilla case, and assigning $\|y_j^0\|^4 = 1$:

$$
\mathbb{E} \left[ \|y^L\|^4 \right] = \alpha_k^2 (1 + \frac{2}{n}) (1 + \frac{5}{n})^{m-1} \prod_{l \neq k}^L \beta_l \sim \alpha_k^2 (1 + \frac{5}{n})^m \prod_{l \neq k}^L \beta_l
$$

(164)

Denoting $\rho = (1 + \frac{5}{n})^{\frac{m}{2}}$, and using the following inequality:

$$
\beta_l \sim (1 + \alpha_l \rho)^2
$$

(165)

it follows that:

$$
\mathbb{E}[G(x, x)^2] \lesssim \sum_{u, v} \sqrt{\mathbb{E}[\|y_{(u)}^L\|^4] \mathbb{E}[\|y_{(v)}^L\|^2]}
$$

(166)

$$
\sim (1 + \frac{5}{n})^m \sum_{u, v} \alpha_u \alpha_v \sqrt{\prod_{l \neq u}^L \beta_l (\prod_{l \neq v}^L \beta_l)}
$$

(167)

$$
= (1 + \frac{5}{n})^m \sum_{u, v} \alpha_u \alpha_v \left( \prod_{l \neq u}^L (1 + \rho \alpha_l) \right) \left( \prod_{l \neq v}^L (1 + \rho \alpha_l) \right)
$$

(168)

Similarly, we have:

$$
\mathbb{E}[G(x, x)^2] \lesssim \sum_k \mathbb{E}[\|J_k\|^4] \sim (1 + \frac{5}{n})^m \sum_{u} \alpha_u^2 \prod_{l \neq u}^L \beta_l = (1 + \frac{5}{n})^m \sum_{u} \alpha_u^2 \prod_{l \neq u}^L (1 + \rho \alpha_l)^2
$$

(169)

Using Eq. 151, we have that:

$$
K_L^R(x, x)^2 = \sum_{u, v} \alpha_u \alpha_v \left( \prod_{l \neq u}^L (1 + \alpha_l) \right) \left( \prod_{l \neq v}^L (1 + \alpha_l) \right)
$$

(170)

yielding:

$$
\frac{\mathbb{E}[G(x, x)^2]}{K_L^R(x, x)^2} \lesssim (1 + \frac{5}{n})^m \sum_{u, v} \alpha_u \alpha_v \left( \prod_{l \neq u}^L (1 + \rho \alpha_l) \right) \left( \prod_{l \neq v}^L (1 + \rho \alpha_l) \right)
$$

(171)

$$
\sim (1 + \frac{5}{n})^m \sum_{u, v} \alpha_u \alpha_v \left( \prod_{l \neq u}^L (1 + \alpha_l) \right) \left( \prod_{l \neq v}^L (1 + \alpha_l) \right)
$$

(172)

$$
= (1 + \frac{5}{n})^m \left( \prod_{l=1}^L (1 + \rho \alpha_l) \right)^2 = (1 + \frac{5}{n})^m \left( \prod_{l=1}^L (1 + \alpha_l(\rho - 1)) \right)^2
$$

(173)

$$
\sim \exp \left[ C \frac{\alpha_l}{n} \sum_{l=1}^L \frac{\alpha_l}{1 + \alpha_l} \right] \left( 1 + O\left( \frac{1}{n} \right) \right)
$$

(174)

for the lower bound, we have:

$$
\frac{\mathbb{E}[G(x, x)^2]}{K_L^R(x, x)^2} \lesssim (1 + \frac{5}{n})^m \sum_{u, v} \alpha_u \alpha_v \left( \prod_{l \neq u}^L (1 + \rho \alpha_l) \right)^2
$$

(175)

$$
\sim \frac{\sum_{u} \alpha_u^2}{\sum_{u, v} \alpha_u \alpha_v} \exp \left[ C \frac{L}{n} \sum_{l=1}^L \frac{\alpha_l}{1 + \alpha_l} \right] \left( 1 + O\left( \frac{1}{n} \right) \right)
$$

(176)
Since \( \mathbb{E}[g(x, x)^2] > K_L^2(x, x)^2 \), the lower bound is given by:

\[
\frac{\mathbb{E}[g(x, x)^2]}{K_L^2(x, x)^2} \geq \max \left[ 1, \frac{\sum_n \alpha_n^2}{\sum_{u, v} \alpha_u \alpha_v} \exp \left( \frac{C}{n} \sum_{l=1}^L \frac{\alpha_l}{1 + \alpha_l} \right) (1 + \mathcal{O}(\frac{1}{n})) \right]
\]

(177)

### A.7. Proof of Theorem 6

In the following proof, for the sake of notation simplicity, we omit the notation \( k \) in \( y_{(k)}^l \), and assume that \( y^l \) stands for the reduced network \( y_{(k)}^l \). The recursive formula for the intermediate outputs of the reduced network are given by:

\[
y^l = \begin{cases} 
\sqrt{\frac{2\alpha}{nL}} \sum_{h=k}^{l-1} W^{l, h^T} q^h & k < l \leq L \\
\sqrt{\frac{2\alpha}{nL}} \sum_{h=0}^{l-1} W^{l, h^T} q^h & 1 \leq l < k \\
\sqrt{\frac{2\alpha}{nL}} W^{k, k-1^T} q^{k-1} & l = k
\end{cases}
\]

(178)

with \( q^h = \sqrt{2\phi(y^h)} \). We have:

\[
\mu_L = \mathbb{E} \left[ \|q^L\|^2 \right] = \frac{2\alpha}{nL} \mathbb{E} \left[ \left( \sum_{l=k}^{L-1} q^{l^T} W^{L_l} \right) Z_L^s \left( \sum_{l=k}^{L-1} q^{l^T} W^{L_l} \right) \right] = \frac{\alpha}{L} \sum_{l=k}^{L-1} \mu_l
\]

(179)

Telescoping the mean:

\[
\mu_L = \frac{\alpha}{L} \sum_{l=k}^{L-1} \mu_l = \frac{\alpha \mu_{L-1}}{L} + \frac{L-1}{L} \mu_{L-1} = \mu_{L-1} (1 + \frac{\alpha - 1}{L}) = \mu_{k+1} \prod_{l=k+2}^{L} \left( 1 + \frac{\alpha - 1}{l} \right)
\]

(180)

and so:

\[
K_L^2(x, x)^2 = \left( \sum_{k=1}^{L} \mu_L \right)^2 = \left( \sum_{k=1}^{L} \frac{\alpha}{k+1} \right)^2 \prod_{l=k+2}^{L} \left( 1 + \frac{\alpha - 1}{l} \right)^2 \sim \alpha^2 \log(L)^2 \prod_{l=1}^{L} \left( 1 + \frac{\alpha - 1}{l} \right)^2
\]

(181)

(182)

For the fourth moment:

\[
\mathbb{E} \left[ \|q^L\|^4 \right] = \frac{4\alpha^2}{n^2 L^2} \mathbb{E} \left[ \left( \sum_{l=k}^{L-1} (q^{l^T} W^{L_l}) Z_L^s \sum_{l=k}^{L-1} (q^{l^T} W^{L_l}) \right)^2 \right]
\]

(183)

\[
= \frac{4\alpha^2}{n^2 L^2} \mathbb{E} \left[ \left( \sum_{l_1=k}^{L-1} (q^{l_1^T} W^{L_{l_1}}) Z_L^s \sum_{l_2=k}^{L-1} (q^{l_2^T} W^{L_{l_2}}) \sum_{l_3=k}^{L-1} (q^{l_3^T} W^{L_{l_3}}) Z_L^s \sum_{l_4=k}^{L-1} (q^{l_4^T} W^{L_{l_4}}) \right)^2 \right]
\]

(184)

Using the results from the vanilla architecture, and denoting \( C_{l', l''} = \mathbb{E} \left[ \|q^{l'}\|^2 \|q^{l''}\|^2 \right] \), it then follows:

\[
C_{L,L} = \frac{\alpha^2(n + 5)}{nL^2} \sum_{l_1, l_2=k}^{L-1} C_{l_1, l_2}
\]

(185)

From Eq. 185, it also holds that:

\[
\sum_{l_1, l_2=k}^{L-2} C_{l_1, l_2} = \frac{n(L - 1)^2}{(n + 5)\alpha^2} C_{L-1, L-1}
\]

(186)
It then follows:
\[
\mathbb{E}(\|q_L\|^4) = C_{L,L}
\]
\[
= \frac{\alpha^2}{L^2} (1 + \frac{5}{n}) \sum_{t_1, t_2 = k}^{L-1} C_{t_1 t_2}
\]
\[
= \frac{\alpha^2}{L^2} (1 + \frac{5}{n}) \left( C_{L-1,L-1} + \sum_{t_1, t_2 = k}^{L-2} C_{t_1 t_2} + 2 \sum_{l=k}^{L-2} C_{L-1,l} \right)
\]
\[
= \frac{\alpha^2}{L^2} (1 + \frac{5}{n}) \left( C_{L-1,L-1} + \frac{(L-1)^2 n}{\alpha^2 (n+5)} C_{L-1,L-1} + 2 \sum_{l=k}^{L-2} C_{L-1,l} \right)
\]

The following also holds:
\[
\forall t_1 > t_2 \geq k, \quad C_{t_1, t_2} = \frac{\alpha}{nL} \mathbb{E} \left[ \sum_{l=k}^{t_1-1} q_l^T W_l W_l^T ||y||^2 \right] = \frac{\alpha}{L} \sum_{l=k}^{t_1-1} C_{l, l_2}
\]
and so:
\[
C_{L,L} = \frac{\alpha^2 (n+5)}{nL^2} \left( C_{L-1,L-1} + \frac{(L-1)^2 n}{\alpha^2 (n+5)} C_{L-1,L-1} + \frac{2\alpha}{L-1} \sum_{l=k}^{L-2} \sum_{l_2 = k}^{L-2} C_{l_1, t_2} \right)
\]
\[
= \frac{\alpha^2 (n+5)}{nL^2} \left( C_{L-1,L-1} + \frac{(L-1)^2 n}{\alpha^2 (n+5)} C_{L-1,L-1} + \frac{2n(L-1)}{\alpha(n+5)} C_{L-1,L-1} \right)
\]
\[
= \frac{\alpha^2 (n+5)}{nL^2} C_{L-1,L-1} \left( 1 + \frac{(L-1)^2 n}{\alpha^2 (n+5)} + \frac{2n(L-1)}{\alpha(n+5)} \right)
\]
\[
= C_{L-1,L-1} \left( 1 + \frac{\alpha-1}{L} \right)^2 + \frac{5\alpha^2}{nL^2}
\]

Telescoping through \(l = L-1 \ldots k+1\):
\[
C_{L,L} = C_{k+1,k+1} \prod_{l=k+2}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]

For the reduced architecture, the transition from \(q^k\) to \(q^{k+1}\) is a vanilla ReLU block, and so using the result from the vanilla architecture:
\[
C_{L,L} = C_{k,k} \frac{\alpha^2 (n+5)}{n(k+1)^2} \prod_{l=k+2}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]
\[
= \frac{\alpha^2 (n+5)}{n(k+1)^2} \prod_{l=k+1}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]
\[
\sim \frac{\alpha^2 (n+5)}{n(k+1)^2} \prod_{l=1}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]

where we assumed \(C_{0,0} = 1\). It follows:
\[
\mathbb{E}[G(x, x)] \lesssim \sum_{u,v} \sqrt{\mathbb{E}[||y^u_{(x)}||^4] \mathbb{E}[||y^v_{(x)}||^2]}
\]
\[
\sim \left( \sum_{k=1}^{L} \frac{1}{k+1} \right)^2 \frac{\alpha^2 (n+5)}{n} \prod_{l=1}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]
\[
\sim \log(L) \frac{\alpha^2 (n+5)}{n} \prod_{l=1}^{L} \left( 1 + \frac{\alpha-1}{l} \right)^2 + \frac{5\alpha^2}{nL^2}
\]
Similarly, we have:

\[
\mathbb{E}[\mathcal{G}(x,x)^2] \geq \sum_k \mathbb{E}[\|J^k\|^4] = \sum_{k=1}^{L} \frac{\alpha^2(n+5)}{n(k+1)^2} \prod_{l=1}^{L} \left( \left(1 + \frac{\alpha - 1}{l}\right)^2 + \frac{5\alpha^2}{nl^2} \right) \tag{203}
\]

\[
\sim \frac{\alpha^2(n+5)}{nL} \prod_{l=1}^{L} \left( \left(1 + \frac{\alpha - 1}{l}\right)^2 + \frac{5\alpha^2}{nl^2} \right) \tag{204}
\]

yielding:

\[
\frac{\mathbb{E}[\mathcal{G}(x,x)^2]}{K^D_L(x,x)^2} \geq \frac{\sum_{k=1}^{K_K^D} \left( \left(1 + \frac{\alpha - 1}{l}\right)^2 + \frac{5\alpha^2}{nl^2} \right)}{\prod_{l=1}^{L} \left(1 + \frac{\alpha - 1}{l}\right)^2} \tag{205}
\]

\[
= \frac{(n+5)}{n} \prod_{l=1}^{L} \left(1 + \frac{5\alpha^2}{n(l+\alpha-1)^2} \right) \tag{206}
\]

\[
\sim \exp \left[ \sum_{l=1}^{L} \frac{5\alpha^2}{n(l+\alpha-1)^2} \right] \left(1 + O\left( \frac{1}{n} \right) \right) \tag{207}
\]

\[
\sim \exp \left[ \frac{C}{n} \right] \left(1 + O\left( \frac{1}{n} \right) \right) \tag{208}
\]

For the lower bound, we have:

\[
\frac{\mathbb{E}[\mathcal{G}(x,x)^2]}{K^D_L(x,x)^2} \leq \frac{\sum_{k=1}^{K_K^D} \left( \left(1 + \frac{\alpha - 1}{l}\right)^2 + \frac{5\alpha^2}{nl^2} \right)}{\prod_{l=1}^{L} \left(1 + \frac{\alpha - 1}{l}\right)^2} \tag{209}
\]

\[
\sim \frac{1}{L \log(L)^2} \exp \left[ \frac{C}{n} \right] \left(1 + O\left( \frac{1}{n} \right) \right) \tag{210}
\]

Since \(\mathbb{E}[\mathcal{G}(x,x)^2] > K^D_L(x,x)^2\), the lower bound is given by:

\[
\frac{\mathbb{E}[\mathcal{G}(x,x)^2]}{K^D_L(x,x)^2} \geq \max \left[ 1, \frac{1}{L \log(L)^2} \exp \left[ \frac{C}{n} \right] \left(1 + O\left( \frac{1}{n} \right) \right) \right] \tag{211}
\]