Vacuum polarization mediated by quark loops is susceptible to external electromagnetic fields as well as to the QCD vacuum structure. Employing the stochastic vacuum model, we calculate the modification of the one-loop Euler-Heisenberg effective action due to stochastic color fields with the Fock-Schwinger technique. Our results indicate nonperturbative light quark contributions of the same order of magnitude as the usual QED terms. Various theoretical and experimental implications are discussed in this progress report.

1 Introduction

The physical vacuum has become a most important research topic with the advent of quantum field theories of the fundamental interactions, of the gauge field theory of the Standard Model in particular. Basic properties of matter, such as masses of elementary particles and ultimately the observed particle spectrum, are induced by the vacuum state and its (broken) symmetries, which in turn are determined by the underlying interactions. Therefore, it is of considerable interest to study in any conceivable way the dynamical vacuum and, especially, its altogether still unknown quantum wave functional.

Our aim here is to perform a first step in studying the interference between the very different color-confining QCD and charge-screening QED vacua. Major efforts have been launched to investigate the ‘melting’ of the QCD vacuum and formation of a quark-gluon plasma in high-energy heavy-ion collisions, see Ref. and earlier references therein. However, relatively little attention has been paid so far to the mutual influence of QCD and QED on the vacuum state and its virtual excitations. Naturally, since quarks carry color as well as electric charges, both contribute to vacuum polarization.

Following earlier theoretical studies of the vacuum induced nonlinear photon-photon interaction, there are ongoing searches for the QED vacuum birefringence effect in high-precision crossed laser/magnetic-field experiments. Most recently, we proposed to evaluate the influence of the QCD vacuum structure on the Euler-Heisenberg (EH) effective action of QED. This well-known one-
loop effective action has been the central quantity for the evaluation of vacuum polarization effects in external electromagnetic fields, most notably Schwinger’s calculation of pair production in a constant electric field.  

Our intention is to calculate the nontrivial correction to the EH effective action arising from QCD interactions and the QCD vacuum, in particular. In lack of a solution of the QCD vacuum problem from first principles, we resort to the stochastic vacuum model (SVM)\(^\text{8}\), which successfully describes color confinement, the quarkonia phenomenology, and produces interesting results in high-energy scattering, see Ref.\(^\text{9}\) and references therein. This model allows us to calculate the one-loop QCD vacuum-to-vacuum amplitude in the presence of external electromagnetic fields.

By means of the SVM, the action-weighted functional integration over gluon (gauge potential) fields, \(B^a_{\mu}, a = 1, \ldots, N_c^2 - 1\) for \(SU(N_c)\), is replaced by an ensemble average over a Gaussian distribution of the color fields, \(G^a_{\mu\nu}\), instead. The latter is characterized by a single nonvanishing vacuum correlation function, which has been computed in nonperturbative lattice gauge theory\(^\text{10}\) and is related to the vacuum correlator appearing in QCD sum rules\(^\text{11}\).

We apply the Fock-Schwinger technique in order to calculate the one-loop effective action of quarks interacting nonperturbatively with an ensemble of time independent homogeneous color fields as well as with an approximately constant electromagnetic background field.

Similarly as the color-magnetic instability of the Mantinyan-Savvidy vacuum ansatz\(^\text{12}\), the SVM points to the importance of inhomogeneous and presumably time dependent color fields, which produce a characteristic correlation length for the QCD vacuum. By the uncertainty principle, a typical vacuum polarization loop has an extent of \(l \approx \Delta \tau \cdot c \approx \hbar c/(2mc^2)\), which amounts to \(l_e \approx 200\) fm for electrons and \(l_u \approx 20\) fm for the lightest up-quarks. Of course, the physically relevant value of \(l_u\) must be expected to differ considerably from this estimate guided by perturbation theory. Results from lattice calculations indicate that important nonperturbative corrections have to be taken into account. One may consider an estimate of a lower bound to be determined by a typical constituent quark mass of \(m_Q \approx 300\) MeV instead, i.e. \(l_Q \approx 0.3\) fm.

Whereas all macroscopic (laboratory) electromagnetic fields are constant on the scale of \(l_e\), the QCD vacuum correlation length obtained from lattice calculations, \(\lambda \approx 0.2\ldots0.4\) fm, is of the same order of magnitude as \(l_Q\) and, therefore, is expected to remain small on the scale of the light quark loops\(^\text{14}\).

Therefore, truly space-time dependent color fields also have to be incorporated in the future. However, their study needs a technically different approach still to be developed.
2 One-loop QCD/QED Effective Action

We briefly rederive the one-loop effective action for fermions in color plus electromagnetic background fields, aiming at the case of a constant electromagnetic field strength with stochastic gluon field fluctuations, in particular. The calculations are performed for Minkowski space with the metric $\mathbb{g}^{\mu\nu} = (1, -1, -1, -1)$, any Lorentz or color indices occurring twice are to be summed over, and we choose units such that $\hbar = c = 1$.

We are interested in the vacuum-to-vacuum amplitude which determines the QCD effective action, $\Gamma_A$, in the presence of an electromagnetic background potential, $A_\mu$:

$$\exp(i\Gamma_A) = \int D\bar{\psi} D\psi D\bar{B} \exp\left\{ i \int d^4x \left( L_{A,B} - \frac{1}{4} F^2 - \frac{1}{4} G^2 \right) \right\},$$

where $L$ denotes the Dirac Lagrangian for the quarks coupled to the electromagnetic and color gauge fields, $A$ and $B$, respectively, $L_{A,B} \equiv \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$, with $D \equiv \partial - ieA - igB$ denoting the covariant derivative in the fundamental representation of the color $SU(N_c)$ group; eventually, one has to sum over the contributions of all fermions, i.e. with different charges and masses. The gauge field actions are written in terms of the field strength tensors, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G^a_{\mu\nu} \equiv \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf_{abc}B^b_\mu B^c_\nu$, respectively.

We did not add gauge fixing terms, since the gluon gauge potentials will be treated in terms of stochastic background field strengths in this work. We remark that the vacuum contribution to $\Gamma_A$ is selected by applying the Feynman boundary condition, which will be made explicit shortly.

In order to proceed, we define the Gaussian stochastic ensemble and its correlators, which will be employed in the following evaluation of the effective action. We assume for simplicity that only approximately constant colormagnetic fields contribute which, however, fluctuate in direction and amplitude.

As previously pointed out in Refs. in the context of QCD background field calculations, the commutator of two covariantly constant color field strength tensors vanishes, $[G_{\mu\nu}, G_{\rho\sigma}] = 0$. Consequently, they can be parametrized as being proportional to the generators of the abelian Cartan (sub-)algebra of the $SU(N_c)$ color group.

We thus take into account $SU(3)$ colormagnetic fields of the form $\vec{B} \equiv \frac{1}{2}(\vec{B}^3\lambda_3 + \vec{B}^8\lambda_8)$, involving the diagonal Gell-Mann matrices $\lambda_{3,8}$. Then, their normalized probability distribution is defined by:

$$P(g\vec{B}^a) d^3(g\vec{B}^a) \equiv \left( \frac{3}{2\pi\langle g^2\vec{B}_a^2\rangle_G} \right)^{3/2} \exp\left( \frac{3\langle g\vec{B}^a\rangle^2}{2\langle g^2\vec{B}_a^2\rangle_G} \right) d^3(g\vec{B}^a),$$

where $\langle \cdots \rangle_G$ denotes the color average.
for \( a = 3, 8 \), with the width determined by the relevant correlator.

In accordance with the SVM model, we replace the functional integral \( \int DB \) in Eq. (3) by the Gaussian ensemble average with the distribution of Eq. (2), which we denote by \( \langle ... \rangle_G \) from now on.

In the absence of a calculation from first principles, the correlator in Minkowski space is obtained by analytical continuation of the Fourier transform of the euclidean one, which is calculated numerically in lattice studies. This procedure is discussed in detail in Ref. 9. Since we consider only stochastic colormagnetic fields, we identify the widths of their distributions with the value of the SVM field correlator:

\[
\langle g^2 B_a^a \mu \nu \rangle_G \equiv \langle g^2 \vec{B}_3^a \cdot \vec{B}_3^a \rangle_G + \langle g^2 \vec{B}_8^a \cdot \vec{B}_8^a \rangle_G = 2 \langle g^2 \vec{B}_3^a \cdot \vec{B}_3^a \rangle_G ,
\]  

(3)

where \( g \) denotes the renormalized QCD coupling. The last equality (isotropy in color space) allows us to relate the respective correlators to the physical gluon condensate:

\[
\langle \frac{\alpha_s}{\pi} G_\mu^a G_\nu^a \rangle_G = \langle \frac{\alpha_s}{\pi} : G_\mu^a G_\nu^a : \rangle = 0.024 \pm 0.011 \text{ GeV}^4 ,
\]  

(4)

where the (running) strong coupling constant is \( \alpha_s = g^2 / 4\pi \), and we cite the empirical value of the condensate (\( \approx (394 \text{ MeV})^4 \)).

For a selfconsistent determination of the stochastic field ensemble, the vacuum condensate in particular, the gluonic contribution to the effective action has to be calculated incorporating gluon fluctuations in the stochastic background together with an appropriate gauge fixing. Here we limit our attention to the fermionic contribution, which generalizes the familiar Euler-Heisenberg effective action of QED.

Obviously, the usual EH effective action \( \Gamma_{EH} \) for leptons, obtained from Eq. (3) by dropping completely the integration over the gluon field \( B \) and setting the corresponding terms of the classical action to zero. In order to evaluate \( \Gamma_A \) instead, employing the SVM model with the stochastic ensemble average defined in Eqs. (2)–(4), we begin by integrating out the fermions:

\[
i \Gamma_A^- = \ln \left( \langle \det \{ [\gamma \cdot D + im][\gamma \cdot \partial + im]^{-1} \} \rangle_G \right) ,
\]  

(5)

where we subtracted the contribution of the noninteracting fermionic vacuum fluctuations for later convenience. From now on, the Feynman boundary condition is implemented by replacing \( m \rightarrow m - i \epsilon \), which we often suppress.

The above result, Eq. (5), is similar to the usual QCD effective action (for \( A_\mu = 0 \)), however, with the vacuum correlator appearing implicitly instead of
the gluon propagator. Next, interchanging the order of taking the logarithm and the ensemble average and using \( \det \{ M \} = \exp \text{Tr} \ln \{ M \} \), the result is:

\[
i(\Gamma_A - A) = 1 - \text{loop} = \langle \text{Tr} \ln \{ [\gamma \cdot D + im][\gamma \cdot \partial + im]^{-1} \} \rangle_G ,
\]

with a trace over space, spin, and color. We remark that interchanging the logarithm with the ensemble average results in the 1-loop approximation here (suppressing this subscript henceforth). Since the dynamics of the Dirac field does not depend on the sign of the mass term, taking the average of both possibilities, the 1-loop action can be expressed conveniently,

\[
i\Gamma_A = \frac{1}{2} \langle \text{Tr} \ln \{ [\gamma \cdot D]^2 + m^2] [\gamma \cdot \partial]^2 + m^2]^{-1} \} \rangle_G ,
\]

where we introduced the kinetic and canonical momentum operators, \( \Pi_\mu \equiv iD_\mu = P_\mu + eA_\mu + gB_\mu \) and \( P_\mu \equiv i\partial_\mu \), respectively; the second equality follows with the help of \( \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \) and \( [\gamma_\mu, \gamma_\nu] \equiv -2i\sigma_{\mu\nu} \), i.e. the (anti-)commutation relations of \( \gamma \)-matrices.

Finally, using the integral representation of the logarithm which presents the starting point of the Fock-Schwinger proper time method, we obtain the familiar looking result:

\[
\Gamma_A = \int_0^\infty \frac{ds}{2is} \langle \text{Tr} \{ \exp(is\Pi^2 - m^2 + \frac{1}{2}\sigma_{\mu\nu}(eF^{\mu\nu} + gG^{\mu\nu})) \} \rangle_G ,
\]

with the mass \((\to m - ie)\) incorporating the Feynman boundary condition.

3 Fock-Schwinger Technique

For arbitrarily varying background fields it is unknown how to evaluate Eq. (8), even for the QED case. Presently, we work with the simplified ensemble of color background fields introduced in the previous section, which have a fluctuating amplitude with respect to space and (color) space direction but are covariantly constant otherwise. We also incorporate the approximately constant electromagnetic field to all orders, expanding the results only in the end.

We proceed by relating the first exponential in Eq. (8) to an unitary evolution operator, \( U(s) \), depending on the proper time variable \( s \):

\[
U(s) \equiv \exp(-isH) , \quad H \equiv \Pi^2 - m^2 + \frac{1}{2}\sigma_{\mu\nu}(eF^{\mu\nu} + gG^{\mu\nu}) ,
\]
where $H$ plays the role of a fictitious Hamiltonian. Following the Fock-Schwinger strategy as described in Ref.\textsuperscript{16}, the aim is to obtain the coordinate space matrix elements of $H$. Using these, the equation of motion for the evolution operator can be converted into an ordinary differential equation. Its solution provides the coordinate space matrix elements of $U(s)$, which are sufficient to calculate the trace and stochastic average in Eq. (8).

To begin with, introducing the Heisenberg operators $\Pi(s) = U(s)^\dagger \Pi U(s)$ and $x(s) = U(s)^\dagger x U(s)$, we obtain the related Ehrenfest equations of motion:

$$\partial_s x_\mu = i[H, x_\mu] = -2\Pi_\mu , \quad (10)$$

$$\partial_s \Pi_\mu = i[H, \Pi_\mu] = 2(eF_{\mu\lambda} + gG_{\mu\lambda})\Pi^\lambda + igJ_\mu + \frac{1}{2}g[D_\mu, \sigma \cdot G] \quad (11)$$

$$\approx 2(eF_{\mu\lambda} + gG_{\mu\lambda})\Pi^\lambda , \quad (12)$$

where the color current, $J_\mu \equiv [D^\lambda, G_{\mu\lambda}]$, is introduced to indicate the physical meaning of this term; similarly, the last term contributes the spin-color coupling here, with $\sigma \cdot G = \sigma_{\mu\nu}G^{\mu\nu}$, while the first term is related to the electromagnetic and color Lorentz forces. In order to arrive at Eqs. (10) and (11), we made use of the coordinate representation of the operators $\Pi, x$ introduced after Eq. (7) and treated $F_{\mu\nu}$ as constant. The last equality, Eq. (12), presents the approximation for covariantly constant fields, i.e. our starting point here.

It is convenient to introduce a combined field strength tensor,

$$\mathcal{F}^{ij}_{\mu\nu} \equiv eF_{\mu\nu}\delta^{ij} + gG_{\mu\nu} t^{ij} , \quad (13)$$

which is a matrix in Lorentz and color indices, with $t^a, a = 1, \ldots, N_c^2 - 1$ denoting the generators of $SU(N_c)$ in the fundamental representation. Then, the solutions of Eqs. (10) and (11) are easily obtained:

$$\Pi(s) = \exp(2\mathcal{F}s)\Pi(0) = \frac{\mathcal{F}}{\exp(-2\mathcal{F}s) - 1} [x(s) - x(0)] , \quad (14)$$

$$x(s) = \frac{1 - \exp(2\mathcal{F}s)}{\mathcal{F}} \Pi(0) + x(0) , \quad (15)$$

where we employed Eq. (13) to eliminate $\Pi(0)$ from Eq. (14).

With the help of Eq. (13) and the basic commutator $[\Pi_\mu(0), x_\nu(0)] = ig_{\mu\nu}$ one derives:

$$[x_\mu(\tau), x_\nu(\tau')] = -2i \left( \exp(\mathcal{F}[\tau - \tau']) \frac{\sinh(\mathcal{F}[\tau - \tau'])}{\mathcal{F}} \right)_{\mu\nu} . \quad (16)$$
Using this, the Hamiltonian of Eqs. (14) can be written in time-ordered form and, then, its coordinate space matrix elements evaluated. We obtain:

\[ H(x', x; s) = \frac{1}{4}[x' - x] \frac{F^2}{\sinh^2(Fs)} [x' - x] - \frac{i}{2} \text{tr}_L [F \coth(Fs)] - m^2 + \frac{1}{2} \sigma \cdot F, \]

(17)

where the trace refers to the Lorentz indices.

Next, we turn to the equation of motion for the proper time evolution operator \( U(s) \), which follows from Eq. (9). Interchanging between the Schrödinger and Heisenberg picture, we obtain in the coordinate representation:

\[
\partial_s U(x', x; s) \equiv \partial_s \langle x'| U(s)|x \rangle = -i \langle x'| H U(s)|x \rangle = -i \langle x'| H(s), x(0); s \rangle \langle x'(s)|x(0) \rangle \equiv -i H(x', x; s) U(x', x; s),
\]

(18)

with \( H(x', x; s) \) from Eq. (17). This presents an ordinary first order differential equation for \( U(x', x; s) \) as a function of \( s \), which can be integrated directly. Since all matrices involved are considered as constants at this point, no ordering prescription for the resulting exponential is needed:

\[ U(x', x; s) = C(x', x) \exp(-i \int_s^0 d\tau H(x', x; \tau)) \] .

(19)

The function \( C(x', x) \) incorporates the usual QED Schwinger string:

\[ C(x', x) \equiv C \exp(i e \int_{x'}^{x} dz^\mu A_\mu(z)) \] , \( z(\xi) \equiv x + \xi(y - x), \) \( 0 \leq \xi \leq 1 \) ,

(20)

as well as a normalization constant \( C \). Furthermore, we calculate:

\[ -i \int_s^0 d\tau H(x', x; \tau) = \frac{i}{4}[x' - x]F \coth(Fs)[x' - x] - \frac{1}{2} \text{tr}_L [\ln(\frac{\sinh(Fs)}{Fs})] \\
+ \ln(s^{-2}) + i m^2 s - \frac{i}{2} \sigma \cdot F, \]

(21)

where we separated the second logarithm for later convenience.

We remark that the Hamiltonian, Eq. (17), is covariant under the global \( SU(N_c) \) gauge transformations admitted here and invariant under arbitrary electromagnetic gauge transformations. Therefore, the evolution operator, Eq. (19), requires the additional electromagnetic string factor, in order to be properly covariant under either gauge transformations.
Finally, the normalization constant $C$ takes the boundary condition (orthogonality and normalization of coordinate eigenstates) into account,

$$\lim_{s \to 0} U(x', x; s) = \lim_{s \to 0} (x'(s)|x(0)) = \delta^4(x' - x) \cdot 1_{S,C},$$

with a unit matrix for spin and color on the right-hand side. The normalization constant is calculated using Eqs. (17)–(22). The result is:

$$C = -\frac{i}{(4\pi)^2}.$$  \hspace{1cm} (23)

Furthermore, Feynman’s $m \to m - i\epsilon$ provides the convergence factor in Eq. (19), thus implementing the asymptotic condition $U(x', x; s \to -\infty) = 0$.

This completes our derivation of the (matrix elements of the) evolution operator $U$. We note in passing that these results immediately yield the propagator in the considered combination of color and electromagnetic background fields, which may be useful for other purposes.

### 4 QCD-modified Euler-Heisenberg Lagrangian

Here we employ our assumptions about the nature of the stochastic color fields, in order to calculate the effective action, Eq. (8), using the results of the previous section, Eqs. (17)–(23) in particular.

We begin by evaluating the traces over spin and space in:

$$\text{tr}_{S,x} U^\dagger(s) = \frac{+i}{(4\pi s)^2} \int d^4x \text{tr}_S \{\exp(-\frac{1}{2} \text{tr}_L \ln(\frac{\sinh(F_s)}{F_s})) - iM^2s + \frac{i}{2} \sigma \cdot F_s\}.$$  \hspace{1cm} (24)

We recall that the covariantly constant color fields are parametrized proportional to the generators of the abelian subgroup of $SU(N_c)$, as discussed in the context of Eqs. (2)–(4), i.e., $F$ is a diagonal color matrix here. Then, the trace over spin is calculated similarly as in the QED case:

$$\text{tr}_S \exp\left(\frac{i}{2} \sigma \cdot F_s\right) = 2(\cos[\frac{s}{2} \sqrt{I_1 + iI_2}] + \cos[\frac{s}{2} \sqrt{I_1 - iI_2}]\right),$$

where two Lorentz invariants of the fields enter,

$$I_1 \equiv 2F_{\mu\nu}F^{\mu\nu}, \quad I_2 \equiv \epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}.$$  \hspace{1cm} (26)

The remaining exponential factor in Eq. (24) can also be simplified:

$$\exp\left(-\frac{1}{2} \text{tr}_L \ln(\frac{\sinh(F_s)}{F_s})\right) = \frac{-is^2}{4} I_2 (\cos[\frac{s}{2} \sqrt{I_1 + iI_2}] - \cos[\frac{s}{2} \sqrt{I_1 - iI_2}])^{-1},$$

$$\hspace{1cm} \text{(27)}$$
Combining Eqs. (24)–(27) and using these in Eq. (8), we obtain the unrenormalized 1-loop effective Lagrangian (Γ
\_\_\_A

= \int d^4x \, \mathcal{L}_A^-):

\[ \mathcal{L}_A^- = \left(-\frac{1}{8\pi^2}\right) \cdot \int_0^\infty \frac{ds}{s^3} e^{-im^2s} \langle \text{tr}_C \{ \frac{g^2}{8} \sqrt{L_1 + iL_2} \cos[\frac{s}{2} \sqrt{L_1 - iL_2}] + \cos[\frac{s}{2} \sqrt{L_1 + iL_2}] - \cos[\frac{s}{2} \sqrt{L_1 - iL_2}] \} - 1 \} \rangle_G, \]

where the color trace and the stochastic average are still left to be done. If we omit these and set all color fields to zero, the usual QED result is recovered.

4.1 Renormalization

Since QCD and QED are both renormalizable, we are guaranteed that the ultraviolet (s → 0) divergences contained in \( \mathcal{L}_A^- \) can be absorbed by renormalization, of the charges and fields in particular. At present we are mostly interested to demonstrate how QCD modifies the usual Euler-Heisenberg effective Lagrangian. Therefore, it is convenient to subtract from \( \mathcal{L}_A^- \) the pure QCD contribution, \( \mathcal{L}_0^- \). It is calculated by setting \( e = 0 \) in Eq. (28). Then, for colormagnetic vacuum fields, we obtain the renormalized result:

\[ \mathcal{L}_0^- = -\frac{1}{8\pi^2} \langle \text{tr}_C \{ g^2 \vec{B} \cdot \vec{B} \int_0^\infty \frac{dx}{x^3} e^{-xz} [x \coth(x) - 1 - \frac{1}{3} x^2] \} \rangle_G, \]

where \( z \equiv m^2/\sqrt{g^2 \vec{B} \cdot \vec{B}} \) is a diagonal color matrix. We remark that here we rotated the integration \( s \rightarrow -ix \) in the complex plane, as compared to Eq. (28); it is obvious now that the integral increases strongly with decreasing \( m \). The corresponding QED integral has been calculated analytically. Using this, we obtain in the limit of strong fields (\( z \ll 1 \)):

\[ \mathcal{L}_0^- = -\frac{1}{24\pi^2} \langle \text{tr}_C \{ g^2 \vec{B} \cdot \vec{B} [\ln(m^2/\sqrt{g^2 \vec{B} \cdot \vec{B}}) + O(z^0)] \} \rangle_G, \]

which is the appropriate limit for the light quarks, given the large value of the gluon condensate, cf. Eq. (4).

After the QCD renormalization, we have to replace \( gG_{\mu\nu} = g_{R}G_{\mu\nu,R} \) also in the remaining terms, i.e. by the renormalized quantities; we drop the subscript \( R \) henceforth. The resulting QCD-subtracted effective Lagrangian is:

\[ \mathcal{L}_A \equiv \mathcal{L}_A^- - \mathcal{L}_0^- \], cf. Eqs. (28)–(30).
In order to proceed, we introduce some useful abbreviations:

\[ \tilde{a} \equiv \frac{1}{4} (2F_{\mu\nu}F^{\mu\nu} - i\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}) , \quad \tilde{d} \equiv \tilde{a}^* , \quad (31) \]

\[ \tilde{b} \equiv \frac{g}{2} (2F_{\mu\nu}G^{\mu\nu} - i\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}G^{\gamma\delta}) , \quad \tilde{e} \equiv \tilde{b}^* , \quad (32) \]

\[ \tilde{c} \equiv \frac{g^2}{4} (2G_{\mu\nu}G^{\mu\nu} - i\epsilon_{\alpha\beta\gamma\delta}G^{\alpha\beta}G^{\gamma\delta}) , \quad \tilde{f} \equiv \tilde{c}^* , \quad (33) \]

where \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are understood as a unit and a diagonal color matrix, respectively. Then, the field dependent factor in the integrand of \( L_{-A} \), Eq. (28), is expressed as:

\[
\frac{\cos[s\sqrt{\tilde{a}e^2 + \tilde{b}e + \tilde{c}}] + \cos[s\sqrt{\tilde{de}^2 + \tilde{ee} + \tilde{f}}]}{\cos[s\sqrt{\tilde{a}e^2 + \tilde{b}e + \tilde{c}}] - \cos[s\sqrt{\tilde{de}^2 + \tilde{ee} + \tilde{f}}]} \left( (\tilde{a} - \tilde{d})e^2 + (\tilde{b} - \tilde{e})e + \tilde{c} - \tilde{f} \right),
\]

where \( \tilde{c} = \tilde{f} \), if we assume only colormagnetic vacuum fields. For the following, we make use of symbolic calculations with Mathematica of Taylor expansions in powers of \( e \) in particular, for which this rewriting helps to organize and cut down the size of the lengthy expressions.

Thus, the ultraviolet \((s \to 0)\) structure of the unrenormalized but QCD-subtracted effective Lagrangian emerges after expanding up to and including terms of \( O(e^3) \):

\[
L_A = \frac{1}{8\pi^2} e^2 \int_0^\infty \frac{ds}{s^3} e^{-im^2s} \cdot \left( \text{tr}_C (\tilde{a} + \tilde{d}) \left[ \frac{1}{6} s^2 + O(\tilde{c}s^4) \right] + O(\tilde{b}\tilde{e}s^4) + O(\tilde{b}^2 + \tilde{e}^2)s^4 \right) \right) G .
\]

Terms which are linear or cubic in \( G_{\mu\nu} \) (and correspondingly in \( e \)) do not contribute here because of the Gaussian ensemble average over vacuum fields; the leading terms \( O(e^0) \) are cancelled by the subtraction of \( L_{-0} \), Eq. (29). Furthermore, using Eqs. (31) – (33), we observe that the term which does not contain color fields, i.e. \( \propto s^{-1}e^2(\tilde{a} + \tilde{d}) \propto s^{-1}e^2F_{\mu\nu}F^{\mu\nu} \) in the integrand of Eq. (34), presents the usual UV divergence which is absorbed by electromagnetic charge and field renormalization. Due to the ensemble average, in particular with \( \langle G^{a\beta}_{\alpha\gamma}G^{b}_{\delta\gamma} \rangle_G \propto \delta^{ab}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \), cf. the correlator (3), the remaining finite terms \( \propto s \) all are proportional to \( e^2F_{\mu\nu}F^{\mu\nu} \) in the end, besides the appropriate power of the gluon condensate (4). Therefore, they are absorbed by an additional finite renormalization.
Following the previous analysis, we obtain the renormalized QCD-modified Euler-Heisenberg Lagrangian:

\[
\mathcal{L}_{EH} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \mathcal{L}_A^0 - \mathcal{L}_0
\]

\[
- \frac{e^2}{1536\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-im^2 s} \left( \text{tr} C \left\{ \frac{1}{(\sqrt{\tilde{c}}\sin[s \sqrt{\tilde{c}}])^3} \left( \cos[s \sqrt{\tilde{c}}](-3[\tilde{b}^2 + \tilde{e}^2] + 12\tilde{c}\tilde{a} + \tilde{d}] + 26s^2\tilde{b}\tilde{c}\tilde{e} + 11s^2\tilde{b}^2 + \tilde{e}^2)) \\
+ \cos[3s \sqrt{\tilde{c}}](3[\tilde{b}^2 + \tilde{e}^2] - 12\tilde{c}\tilde{a} + \tilde{d}] - 2s^2\tilde{b}\tilde{c}\tilde{e} + s^2\tilde{c}\tilde{b}^2 + \tilde{e}^2)) \\
- 24s\sqrt{\tilde{c}}\sin[s \sqrt{\tilde{c}}](2\tilde{c}[\tilde{a} + \tilde{d}] + \tilde{b}\tilde{e}) \right) \right) G \, ,
\]  

(35)

where all charges and fields are the renormalized ones by now, in particular \( \alpha \equiv e^2/4\pi \approx 1/137 \), and where we implemented \( \tilde{c} = \tilde{f} \) for the case of colormagnetic vacuum fields. Naturally, the zeroth order Lagrangian for the electromagnetic field appears here on the right-hand side.

We remark that after renormalization and upon expansion of the effective action \( \mathcal{L}_{EH} \), Eq. (35), the QCD effects enter at \( O(e^4) \) and higher, i.e. affecting the nonlinear effective interaction of electromagnetic fields. In the following section, we calculate the first nontrivial modification of the usual Euler-Heisenberg Lagrangian. Some comments concerning the nonperturbative content of our results will be made shortly.

4.2 Evaluation at \( O(e^4) \)

Here we expand the renormalized QCD-modified Euler-Heisenberg Lagrangian up to \( O(\alpha^2) \), keeping all orders in the strong coupling. We consider as an instructive example the case of the stochastic colormagnetic vacuum fields \( \tilde{B}^a \) together with an (applied) external electric field \( \tilde{E} \). Thus Eqs. (31)–(33) are replaced by:

\[
\tilde{a} = \tilde{d} = \frac{1}{2} F_{\mu \nu} F^{\mu \nu} = -\tilde{E}^2 \, ,
\]

(36)

\[
\tilde{b} = -\tilde{e} = -4i q \tilde{E} \cdot \tilde{B}^a \, ,
\]

(37)

\[
\tilde{c} = \tilde{f} = g^2 \tilde{B} \cdot \tilde{B} \, .
\]

(38)

Using these, we obtain \( \mathcal{L}_{EH} \) at \( O(e^4) \) from Eq. (35),

\[
\mathcal{L}_{EH}^{(4)} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}
\]
\[ L_{EH}^{(4)} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + 2\alpha^2 \int_0^\infty \frac{dx}{x^3} \left\langle \text{tr}_C \left[ \frac{-\tilde{b}_4}{128e^3} + \frac{\tilde{a}_b^2}{16e^3} + \frac{\tilde{a}_d^2}{8e^2} \right] x^2 + \frac{-\tilde{b}_4}{96e^3} + \frac{\tilde{a}_b^2}{24e^3} \right] x^4 \\
+ \left( \frac{-\tilde{b}_4}{128e^3} + \frac{3\tilde{a}_b^2}{16e^3} - \frac{\tilde{a}_d^2}{8e^2} \right) x - \left( \frac{\tilde{b}_4}{192e^3} - \frac{\tilde{a}_b^2}{12e^3} + \frac{\tilde{a}_d^2}{6e^2} \right) x^3 - \frac{\tilde{b}_4}{720e^3} x^5 \right) \coth(x) \\
+ \left( \frac{3\tilde{b}_4}{128e^3} - \frac{\tilde{a}_b^2}{16e^3} - \frac{\tilde{a}_d^2}{8e^2} \right) x^2 - \left( -\frac{\tilde{b}_4}{96e^3} + \frac{\tilde{a}_b^2}{24e^3} \right) x^4 \right) \coth^2(x) \\
+ \left( \frac{\tilde{b}_4}{64e^3} - \frac{\tilde{a}_b^2}{8e^3} + \frac{\tilde{a}_d^2}{4e^2} \right) x^3 \coth^3(x) \right) \right \rangle_G \\
= \frac{1}{2} \tilde{E}^2 - \frac{\alpha^2}{180} \sum_{\text{evs.}} \left\langle \frac{\tilde{b}_4}{e^3} \left[ \frac{1}{2} \zeta(3, z/2) - 2z^{-3} \right] \right \rangle_G \\
= \frac{1}{2} \tilde{E}^2 - \frac{\alpha^2}{180} \sum_{\text{evs.}} \left\langle \frac{\tilde{b}_4}{e^3} \zeta(3, z/2) - 2z^{-3} \right \rangle_G , \tag{40} \]

where \( \zeta \) denotes the Riemann zeta function in two arguments, and we employed formulae 3.551.3 and 9.521.1 of Ref.19 in the second and third equality, respectively. Since the color matrices involved here are diagonal, as discussed, the sum over the eigenvalues of the resulting matrix replaces the color trace.

Then, recalling that the covariantly constant \( SU(3) \) colormagnetic vacuum fields are of the form \( \tilde{B} \equiv \frac{1}{2}(\tilde{B}_3 \lambda_3 + \tilde{B}_8 \lambda_8) \), in terms of the Gell-Mann matrices
\begin{align*}
\lambda_{3,8}, \text{ we obtain:} \\
\mathcal{L}_{EH}^{(4)} &= \frac{1}{2} \vec{E}^2 - \frac{256}{180 \sqrt{3}} \frac{g \alpha^2}{m^6} \left\{ \frac{\langle (\vec{E} \cdot \vec{B})^4 \rangle}{|\vec{B}|^3} + \sum_{\pm} \frac{\langle \vec{E} \cdot (\sqrt{3} \vec{B}^3 \pm \vec{B}^8)/2 \rangle^4}{|(\sqrt{3} \vec{B}^3 \pm \vec{B}^8)/2|^3} \right\}_G \\
&= \frac{1}{2} \vec{E}^2 - \frac{64}{15 \sqrt{3}} \frac{g \alpha^2}{m^6} \left( \frac{\langle \vec{E} \cdot \vec{B}^8 \rangle}{|\vec{B}^8|^3} \right)_G,
\end{align*}

where the last equality follows from the fact that all three contributions are equal. This can be shown by a suitable coordinate transformation in the space of the colormagnetic fields, the ensemble of which is determined by the Gaussian distribution of Eq. (2).

Using the distribution of the stochastic fields, we finally obtain from Eq. (41) the QCD-modified Euler-Heisenberg effective action for the case of external electric fields at O(\(\alpha^2\)):

\begin{align*}
\Gamma_{EH}^{(2)} &= \int d^4 x \left[ \frac{1}{2} \vec{E}^2 - \frac{2\alpha^2}{45} \frac{1}{m^4} \\
&\quad - \frac{2\alpha^2}{45} \langle \vec{E}^2 \rangle^2 \sum_{i=u,d,...} \frac{64\sqrt{2}\pi}{5} \frac{q_i^4}{m_i^2} \left( \frac{4}{m_i^4} \right) \frac{1}{2} \\
&\quad \left( \frac{G_{\mu\nu}^a G_{\mu\nu}^a}{m_i^2} \right) \right],
\end{align*}

where \(m_i\) and \(q_i\) denote the quark masses and charges, respectively, and where we incorporated the usual QED term (\(\propto m^{-4}\)) due to \(e^+ e^-\) vacuum polarization.

For \(m_u \approx 5\text{ MeV} \approx m_d/2\), \(q_u = 2/3 = -2q_d\), and the gluon condensate value of Eq. (4), the sum in the last term in Eq. (42) gives a numerical factor,

\[\sum_{i=u,d,...} \text{constant} \cdot q_i^4 m_i^{-6} \approx 3.86 (1 + 10^{-3} + \ldots) \cdot \frac{1}{m_i^4},\]

where the dots indicate the negligible contribution of the heavier quarks; we factored out \(m_i^{-4}\) for comparison with the QED result.

Clearly, the result of Eq. (43) is nonperturbative in the strong coupling \(\alpha_s\) and strongly dominated by the lightest \(u\)-quark term.

The experimentally interesting situation with crossed electric and magnetic external fields, related to the vacuum birefringence mentioned in Sec. 1, can be studied along the same lines. Presently we presented only the simplest nontrivial case, in order to demonstrate that a sizeable interference between QCD and QED vacuum polarization appears to be possible, based on the present calculation.
A thorough investigation of this effect needs to be performed which also takes the space-time dependent stochastic vacuum fields into account, which are implied by the SVM description of QCD vacuum structure. In particular, we recall from Sec. 1 that they are expected to fluctuate strongly on roughly the same length scale as the light quark loops, which we considered here. Whether this aspect of the nonperturbative stochastic fields tends to reduce the quark vacuum polarization considerably or not, still remains to be seen.

5 Discussion

We derived the modification of the Euler-Heisenberg effective action due to the nonperturbative QCD vacuum structure, which influences the vacuum polarization contribution of electrically and color charged quarks. We employed a simplified version of the stochastic vacuum model (SVM), which has been successful in describing various infrared aspects of QCD including confinement and bound state properties.

In Sec. 4.2 we obtained the light quark contribution to the effective action at $O(\alpha^2)$ for the case of external (macroscopic) electric fields acting simultaneously with stochastic colormagnetic vacuum fields. For simplicity, we assumed the QCD fields to be covariantly constant, such as in earlier background field calculations, however, to be fluctuating in amplitude and (color) space direction. We found a sizeable contribution, of the same order of magnitude as the corresponding electron-positron term.

The extension for the case of crossed electric and magnetic fields, such as employed in the experimental search for the usual QED vacuum birefringence effect, can be obtained in the same way.

However, we discussed and want to stress here once more the fact that in order to obtain quantitatively reliable results, the inhomogeneous SVM fields necessarily have to be incorporated. Their correlation length is expected to interfere in a still unknown way with the length scale of the presumably dominant light quark vacuum polarization loops.

We conclude that a calculational scheme is necessary which allows to handle the important inhomogeneous field configurations. It is well known that applications of the Fock-Schwinger technique are limited to very special field configurations, such as the covariantly constant ones considered here.

Furthermore, our present study indicates that also the properties of the quark condensate, in external electromagnetic fields in particular, may deserve a fresh look when nonperturbative QCD features are implemented with the help of the stochastic vacuum model.

From the definition of the QCD effective action in the presence of external
electromagnetic fields, Eq. (44) in Sec. 2, it follows immediately that the quark condensate could be calculated nonperturbatively indeed:

$$\langle \bar{\psi} \psi \rangle = - \frac{d \Gamma_A}{dm}.$$  

Employing our Eqs. (30) and (42), adding up the contributions, however, the result is quite unsatisfactory. This must be attributed to the fact that the homogeneous stochastic fields employed in a one-loop calculation here still present too crude an approximation for the relevant QCD vacuum properties. It will be very interesting to compare future results of an improved SVM calculation with other phenomenological models. Some related issues have already been discussed previously in a wider context.

Finally, we mention that the fate of nonperturbative particle production poles contained in the effective Lagrangian (35) of Sec. 4.1 after the stochastic averaging over vacuum fields also deserves further study.

Acknowledgments

HTE wishes to thank A. Di Giacomo, H. G. Dosch, E. Ferreira, and T. Kodama for discussions and particularly A. Di Giacomo for his hospitality at the Dipartimento di Fisica (University of Pisa); the invitation by C. A. Bertulani to this stimulating workshop is gratefully acknowledged. This work was supported in part by CNPq-300758/97-9 (Brasil), PRONEX-41.96.0886.00, FAPESP-95/4635-0, and by a grant from the U.S. Department of Energy, DE-FG03-95ER40937.

References

1. S. Weinberg, *The Quantum Theory of Fields*, Vols. I,II (Cambridge Univ. Press, Cambridge, 1995/96).
2. Nucl. Phys. A638, no. 1-2, 1 (1998); see also the earlier proceedings of the series of ‘Quark Matter’ conferences published in Nucl. Phys. A.
3. J. S. Heyl and L. Hernquist, J. Phys. A30, 6485 (1997);
S. L. Adler, Ann. Phys. 87, 599 (1971);
Z. Bialynicki-Birula and I. Bialynicki-Birula, Phys.Rev. D2, 2341 (1970).
4. D. Bakalov et al. (PVLAS Collaboration), *Experimental method to detect the magnetic birefringence of vacuum*, J. Europ. Opt. Soc. B, in press;
G. Cantatore, F. Della Valle, E. Milotti, L. Dabrowski and C. Rizzo, Phys. Lett. B265, 418 (1991);
E. Iacopini and E. Zavattini, Phys. Lett. B85, 151 (1979).
5. W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936);
   W. Heisenberg, Z. Phys. 90, 209 (1935);
   H. Euler and B. Kockel, Naturwissensc., 23, 246 (1935).
6. J. Rafelski and H.-Th. Elze, *Electromagnetic Fields in the QCD Vacuum*,
   in: Proc. Fourth Workshop on Quantumchromodynamics, 1-6 June 1998,
   American Univ. of Paris (France); H. Fried and B. Mueller, eds. (World
   Scientific), in press, [hep-ph/9806389].
7. J. Schwinger, Phys. Rev. 82, 664 (1951).
8. Yu. A. Simonov, Nucl. Phys. B307, 512 (1988);
   H. G. Dosch and Yu. A. Simonov, Phys. Lett. B205, 339 (1988);
   H. G. Dosch, Phys. Lett. B190, 177 (1987).
9. H. G. Dosch, E. Ferreira and A. Krämer, Phys. Rev. D50, 1992 (1994).
10. M. D’Elia, A. Di Giacomo and E. Meggiolaro, Phys. Lett. B408, 315
    (1997); see also earlier references therein.
11. M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147,
    385, 448, 519, (1979);
    M.A. Shifman, *Vacuum Structure and QCD Sum Rules* (North Holland,
    Amsterdam, 1992).
12. S.G. Matinyan and G.K. Savvidy, Nucl. Phys. B134, 539 (1978);
    G.K. Savvidy, Phys. Lett. B71, 133 (1977);
    I.A. Batalin, S.G. Matinyan and G.K. Savvidy, Sov. J. Nucl. Phys. 26,
    214 (1977).
13. A. Yildiz and P. H. Cox, Phys. Rev. D21, 1095 (1980);
    M. Claudson, A. Yildiz and P. H. Cox, Phys. Rev. D22, 2022 (1980).
14. S. Narison, Phys. Lett. B387, 162 (1996); B358, 113 (1995).
15. L. F. Abbott, Nucl. Phys. B185, 189 (1981);
    H.-Th. Elze, Z. Phys. C47 (1990) 647.
16. C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New
    York, 1987).
17. W. Dittrich and M. Reuter, *Effective Lagrangians for Quantum
    Electrodynamics*, Lecture Notes in Physics, Vol. 220 (Springer, Berlin, 1985).
18. S. Wolfram, *Mathematica* (Cambridge Univ. Press, Cambridge, 1996).
19. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals/ Series and Products*
    (Academic Press, New York, 1965).
20. I. A. Shushpanov and A. V. Smilga, Phys. Lett. B402, 351 (1997);
    S. Schramm, B. Müller and A. J. Schramm, Mod. Phys. Lett. A7, 973
    (1992);
    S. P. Klevansky and R. H. Lemmer, Phys. Rev. D39, 3478 (1989).