On the uniform one-dimensional fragment

Antti Kuusisto
Tampere University and University of Helsinki

Abstract

The uniform one-dimensional fragment of first-order logic, $U_1$, is a formalism that extends two-variable logic in a natural way to contexts with relations of all arities. We survey properties of $U_1$ and investigate its relationship to description logics designed to accommodate higher arity relations, with particular attention given to $\mathcal{DLR}_{reg}$. We also define a description logic version of a variant of $U_1$ and prove a range of new results concerning the expressivity of $U_1$ and related logics.

1 Introduction

Two-variable logic [10, 26] and the guarded fragment [1] are currently perhaps the most widely studied subsystems of first-order logic. Two-variable logic $\text{FO}^2$ was proved decidable in [21], and the satisfiability problem of $\text{FO}^2$ was shown to be NEXPTIME-complete in [6]. The extension of two-variable logic with counting quantifiers, $\text{FOC}^2$, was proved decidable in [8, 22] and subsequently shown to be NEXPTIME-complete in [23]. Research on extensions and variants of two-variable logic is currently very active. Recent research has mainly concerned decidability and complexity issues in restriction to particular classes of structures and also questions related to different built-in features and operators that increase the expressivity of the base language. Recent articles in the field include for example [3, 4, 12, 27] and several others.

The guarded fragment was shown 2EXPTIME-complete in [7] and in fact EXPTIME-complete over bounded arity vocabularies in the same article. The guarded fragment has since then generated a vast literature. The fragment has recently been significantly generalized in the article [2] which introduces the guarded negation first-order logic $\text{GNFO}$. Intuitively, $\text{GNFO}$ only allows negations of formulae that are guarded in the sense of the guarded fragment. The guarded negation fragment has been shown complete for 2EXPTIME in [2].

The article [9] introduced the uniform one-dimensional fragment, $U_1$, which is a natural generalization of $\text{FO}^2$ to contexts with relations of arbitrary arities. Intuitively, $U_1$ is a fragment of first-order logic obtained by
restricting quantification to blocks of existential (universal) quantifiers that
leave at most one free variable in the resulting formula. Additionally, a uni-
formity condition applies to the use of atomic formulae: if \( n, k \geq 2 \), then a
Boolean combination of atoms \( R(x_1, ..., x_k) \) and \( S(y_1, ..., y_n) \) is allowed only if the sets \( \{x_1, ..., x_k\} \) and \( \{y_1, ..., y_n\} \) of variables are equal. Boolean com-
binations of formulae with at most one free variable can be formed freely,
and the use of equality is unrestricted. Several variants of \( U_1 \) have also been
investigated in [9] and the two subsequent papers [13, 14].

Perhaps the easiest way to gain intuitive insight on \( U_1 \) is to consider the
fully uniform fragment, \( FU_1 \), which is a slight restriction of \( U_1 \) introduced
in the current article. It turns out that \( FU_1 \) can be represented roughly as
the standard polyadic modal logic where novel accessibility relations can be
formed by the Boolean combination and permutation of atomic accessibility
relations. Recall that polyadic modal logic is the extension of modal logic
with formulae \( \chi := \langle R \rangle (\varphi_1, ..., \varphi_k) \) interpreted such that \( M, w \models \chi \) iff there
exist points \( u_1, ..., u_k \) such that \( (w, u_1, ..., u_k) \in R \) and \( M, u_i \models \varphi_i \) for each \( i \).
It also turns out, as we shall see, that over vocabularies with at most binary
relations, \( FU_1 \) is in fact equi-expressive with \( FO^2 \). This result extends a
similar observation from [20] concerning Boolean modal logic with the inverse
operator and a built-in identity modality. It was proved in [20] that this logic
is expressively complete for \( FO^2 \). The fact that \( FU_1 \) collapses to \( FO^2 \) over
binary vocabularies can be taken to indicate that \( FU_1 \) is a natural and in
some sense minimal generalization of \( FO^2 \) to higher arity contexts.

The uniform one-dimensional fragment \( U_1 \) was shown to have the finite
model property and a NEXPTIME-complete decision problem in [13], thereby
establishing that the transition from \( FO^2 \) to \( U_1 \) comes without a cost in
complexity. It was also shown in [13] that \( U_1 \) is incomparable in expressivity
with \( FO^2 \); we will prove in the current article that \( U_1 \) is incomparable with
GNFO, too. We note, however, that the article [9] already established a sim-
ilar incomparability result concerning GNFO and the equality-free
fragment of \( U_1 \). The article [13] also showed that the extension of \( U_1 \) with counting
quantifiers is undecidable. The article [9], in turn, established that relaxing
either of the two principal constraints of the syntax of \( U_1 \)-formulae—leaving
two free variables after quantification or violating the uniformity condition—
leads to undecidability. Building on [9] and [13], the article [14] investigated
variants of \( U_1 \) in the presence of built-in equivalence relations. It was shown,
e.g., that while \( U_1 \) becomes 2NEXPTIME-complete when a built-in equivalence
is added, a certain natural restriction of \( U_1 \) (which still contains \( FO^2 \))
remains NEXPTIME-complete. In the current article we briefly discuss the
above collection of results on \( U_1 \) and its variants and list a number of related
open problems.

Unlike the guarded fragment and GNFO, two-variable logic does not
cope well with relations of arities greater than two, and the same applies to
\( FO^2 \). In database theory contexts, for example, this can be a major draw-
back. Therefore the scope of research on two-variable logics is significantly restricted. The uniform one-dimensional fragment $U_1$ extends two-variable logics in a way that leads to the possibility of investigating systems with relations of all arities.

Another possible advantage of $U_1$ is its one-dimensionality, i.e., the fact that its formulae are essentially of the type $\varphi(x)$, where $x$ is a free variable. This links $U_1$ to description logics in a natural way, as formulae of $U_1$ can be regarded as concepts in the description logic sense. Below we make use of this issue and define a description logic $\mathcal{DL}_{FU_1}$, which we prove to be expressively equivalent to the fully uniform one-dimensional fragment $FU_1$. The logic $\mathcal{DL}_{FU_1}$ makes explicit the link between $FU_1$ and polyadic modal logic we mentioned above. It can be seen as the canonical extension of the description logic $\mathcal{ALBO}^{id}$ [21] to higher arity contexts. While $\mathcal{ALBO}^{id}$ is $\mathcal{ALC}$ extended with Boolean and inverse operators on roles, an identity role and singleton concepts, $\mathcal{DL}_{FU_1}$ is essentially the same system with roles of all arities. The relational inverse operator is generalized to an operator that slightly generalizes the relational permutation operator.

Higher arity relations arise naturally in contexts relevant to description logics. Consider for example the ternary role $R$ such that $R(a, b, c)$ if $a$ has contracted a virus $b$ in country $c$, or the quaternary role $S$ such that $S(c, d, e, f)$ if $c$ and $d$ have sold $e$ to $f$. It is easy to see by a counting argument that a $k$-ary relation cannot be encoded by a finite number of relations of lower arity without changing the domain, and therefore—in addition to aesthetic considerations—a direct access to higher arity roles can be advantageous.

Higher arity roles have of course been investigated before in the description logic literature, for example in [5, 19, 25]. Below we compare $\mathcal{DL}_{FU_1}$ and the system $\mathcal{DLR}_{reg}$ from [5], which includes, e.g., the union, composition and transitive reflexive closure operators for binary roles as well as operators that enable the creation of binary relations from higher arity roles. We show that $\mathcal{DL}_{FU_1}$ and $\mathcal{DLR}_{reg}$ are incomparable in expressivity. While this result itself is not at all surprising, it is still worth proving since the related arguments directly demonstrate the relative expressivities of $\mathcal{DLR}_{reg}$ and $\mathcal{DL}_{FU_1}$. We end the article by identifying a fragment of $\mathcal{DLR}_{reg}$ which is in a certain sense maximal with the property that it embeds into $\mathcal{DL}_{FU_1}$. In the context of this investigation we discuss the curious fact that while $U_1$ can count, it cannot count well enough to express the number restriction operators of $\mathcal{DLR}_{reg}$. In the investigations below concerning expressivity issues, we make occasional use of the novel Ehrenfeucht-Fraïssé (EF) game for $U_1$ from [14]. The related concrete arguments shed light on the expressivity properties of $U_1$.

Finally, it is worth pointing out here that a rather nice and potentially fruitful feature of $\mathcal{DL}_{FU_1}$ is that it is based on the syntactically and semantically same approach as standard polyadic modal logic. Thereby $\mathcal{DL}_{FU_1}$
extends the celebrated and fruitful link between modal and description logics to higher arity contexts in a way that preserves the close relationship between the two fields.

\section{Preliminaries}

We let $\text{VAR}$ denote a countably infinite set of variable symbols. Let $X = \{x_1, ..., x_k\}$ be a finite set of variable symbols and let $R$ be an $n$-ary relation symbol; $R$ is not allowed to be the identity symbol here. An atomic formula $R(x_{i_1}, ..., x_{i_n})$ is called an $X$-atom if $\{x_{i_1}, ..., x_{i_n}\} = X$. For example, assuming $x, y, z$ to be distinct variables, both $S(x, y)$ and $T(x, x, y, y, x)$ are $\{x, y\}$-atoms while $P(x)$ and $R(x, y, z)$ are not.

Let $\mathbb{Z}_+$ be the set of positive integers. We let $V$ denote the infinite relational vocabulary $V := \bigcup_{k \in \mathbb{Z}_+} \tau_k$, where $\tau_k$ is a countably infinite set of $k$-ary relation symbols; the equality symbol is not in $V$. A unary $V$-atom is an atomic formula of the form $P(x)$ or $R(x, ..., x)$, where $P, R \in V$. Here $(x, ..., x)$ denotes the tuple that repeats $x$ exactly $n$ times, $n$ being the arity of $R$.

The set of formulae of the equality-free uniform one-dimensional fragment $U_1(\text{wo }=)$ of first-order logic is the smallest set $\mathcal{F}$ satisfying the following conditions (cf. \cite{9}).

1. Every unary $V$-atom is in $\mathcal{F}$. Also $\bot, \top \in \mathcal{F}$.
2. If $\varphi \in \mathcal{F}$, then $\neg \varphi \in \mathcal{F}$.
3. If $\varphi, \psi \in \mathcal{F}$, then $(\varphi \land \psi) \in \mathcal{F}$.
4. Let $Y := \{x_0, ..., x_k\} \subseteq \text{VAR}$ and $X \subseteq Y$. Let $\varphi$ be a Boolean combination of $X$-atoms over $V$ and formulae in $\mathcal{F}$ whose free variables (if any) are in $Y$. Then $\exists x_0 ... \exists x_k \varphi \in \mathcal{F}$ and $\exists x_0 ... \exists x_k \varphi \in \mathcal{F}$.

For example

$$\exists y \exists z((\neg R(x, y, z) \lor T(z, y, x, x)) \land P(z))$$

is a $U_1(\text{wo }=)$-formula, while

$$\exists y \exists z(S(x, y) \land S(y, z) \land P(z))$$

is not because $\{x, y\} \neq \{y, z\}$. This latter formula is said to violate the uniformity condition of $U_1$. The formula $\exists y R(x, y, z)$ is also illegitimate because it violates one-dimensionality, leaving two variables free instead of one. However, the sentence $\exists x \exists z \exists y R(x, y, z)$ is legitimate, and so is

$$\forall x \exists z \exists y (R(x, y, z) \land \exists u \neg U(y, u)),$$

while the sentence $\forall x \forall z \exists y R(x, y, z)$ is not.
The fully uniform one-dimensional fragment \( \text{FU}_1 \) is the logic whose formulae are obtained from formulae of \( \text{U}_1(\text{wo} =) \) by allowing the free substitution of any collection of binary relation symbols by the equality symbol \( = \). The uniform one-dimensional fragment \( \text{U}_1 \) is obtained by adding to the above four clauses that define the set \( \mathcal{F} \) of formulae of \( \text{U}_1(\text{wo} =) \) the additional clause \( x = y \in \mathcal{F} \).

For example

\[
\exists y \exists z (R(y, z, x) \land x \neq y \land \exists z S(y, z))
\]

is a formula of \( \text{U}_1 \) but not of \( \text{FU}_1 \). Clearly \( \text{FU}_1 \) is a fragment of \( \text{U}_1 \). The following proposition, where \( \text{FO}^2 \) denotes two-variable logic with equality, is easy to prove using disjunctive normal form representations of formulae.

**Proposition 1.** \( \text{FU}_1 \) and \( \text{FO}^2 \) are equi-expressive over models with at most binary relations. That is, in restriction to models with relations of arity at most two, each formula of \( \text{FU}_1 \) with at most two free variables has an equivalent \( \text{FO}^2 \)-formula, and each \( \text{FO}^2 \)-formula has an equivalent \( \text{FU}_1 \)-formula.

However, \( \text{U}_1 \) is strictly more expressive than two-variable logic \( \text{FO}^2 \) even over the empty vocabulary, because \( \text{U}_1 \) can count better than \( \text{FO}^2 \): we observe that for example the sentence

\[
\exists x \exists y \exists z (x \neq y \land x \neq z \land y \neq z)
\]

is a \( \text{U}_1 \)-formula. It is well known and easy to show by a two-pebble-game argument (see [18] for pebble games) that this sentence is not expressible in \( \text{FO}^2 \).

It is easy to see that \( \text{FO}^2 \) and therefore \( \text{FU}_1 \) can define the property that \( |P| = 1 \) for a unary predicate \( P \). Thus nominals can be simulated in those logics. The logic \( \text{U}_1 \) can define even the properties \( |P| \leq k \), \( |P| \geq k \) and \( |P| = k \) for any finite \( k \). However, the counting capacity of \( \text{U}_1 \) is restricted in an interesting way, as we will see later on: \( \text{U}_1 \) cannot make counting statements about the in-degrees and out-degrees of binary relations.

Finally, the \( \text{U}_1 \)-sentence

\[
\exists x \forall y \forall z (R(y, z) \rightarrow (x = y \lor x = z))
\]

provides a possibly more interesting example of what is definable in \( \text{U}_1 \) but not in \( \text{FO}^2 \). This sentence states that there is an element that belongs to every edge of \( R \). It is easy to see by a two-pebble-game argument that this property is not expressible in \( \text{FO}^2 \): the Duplicator wins the two-pebble-game played on \( K_2 \) and \( K_3 \), where \( K_n \) is the \( n \)-clique. Recall that the \( n \)-clique is the structure with \( n \) elements where \( R \) is the total binary relation with the reflexive loops removed.
3 Complexity of $U_1$ and its variants

The complexity of $U_1$ was identified in [13] by showing that the logic has the exponential model property.

**Theorem 1 ([13]).** Every satisfiable $U_1$-formula $\varphi$ has a model whose size is bounded exponentially in $|\varphi|$.

**Theorem 2 ([13]).** The satisfiability problem (= finite satisfiability problem) for $U_1$ is $\text{NEXPTIME}$-complete.

The argument in [13] leading to the above results bears at least some degree of resemblance to the NEXPTIME upper bound proof of $\text{FO}^2$ by Grädel, Kolaitis and Vardi in [6]. It turns out that $U_1$-formulae can be transferred into equisatisfiable formulae in a generalized version of the Scott normal form specially designed for $U_1$, and the exponential model property can then be established by appropriately modifying and extending the arguments applied in [6].

The complexity results of the article [13] were extended in [14]. If $L$ denotes a fragment of first-order logic and $R_1, \ldots, R_k$ are binary relation symbols, then we let $L(R_1, \ldots, R_k)$ denote the language obtained by allowing for the free substitution of identity symbols in $L$-formulae by the special symbols $R_i$. The article [14] investigated $U_1$ and its variants over models with a built-in equivalence relation $\sim$. It was shown that the satisfiability (SAT) and finite satisfiability (FINSAT) problems for $U_1(\sim)$ are complete for $\text{2NEXPTIME}$. The article [14] also identified a natural restriction $SU_1$ of $U_1$ that still extends $\text{FO}^2$ and showed that the SAT and FINSAT problems for $SU_1(\sim)$ are only $\text{NEXPTIME}$-complete; see [14] for the formal definition of $SU_1$. Furthermore, the article [14] established that the SAT and FINSAT-problems of $SU_1(\sim, \sim_2)$, i.e., $U_1$ with two built-in equivalences, is undecidable. This contrasts with the case for $\text{FO}^2$ which remains decidable with two equivalences (SAT [15] and FINSAT [16]).

Several immediately interesting open problems remain, for example the decidability issue for $U_1(\leq)$, where $\leq$ denotes a built-in linear order. Also, while $U_1(tr)$ (i.e., $U_1$ with a built-in transitive relation $tr$) was shown undecidable in [14], it was left open whether $U_1(tr(\text{uniform}))$ is decidable; here $U_1(tr(\text{uniform}))$ denotes the language obtained from $U_1$ by allowing the free substitution of any instances of a binary relation (rather than the equality symbol) by the built-in transitive relation $tr$.

4 Expressivity issues

In this section we provide an overview on the expressivity of $U_1$ and its variants. The following theorem from [13] relates the expressivities of $U_1$ and $\text{FO}^2$. 


Theorem 3 (13). $U_1$ and FOC$^2$ are incomparable in expressivity.

Proof. It is easy to show that the $U_1$-sentence $\exists x \exists y \exists z R(x, y, z)$ cannot be expressed in FOC$^2$, and therefore $U_1 \not\leq$ FOC$^2$. To prove that FOC$^2 \not\leq U_1$, let $S$ be a binary relation symbol. We will show that $U_1$ cannot express the FOC$^2$-definable condition that the in-degree (with respect to the relation $S$) at every node is at most one. Assume $\varphi(S)$ is a $U_1$-formula that defines the property. Consider the formula

$$\varphi(S) \land \forall x \exists y S(x, y) \land \exists x \forall y \neg S(y, x).$$

It is clear that this formula has only infinite models, and thereby the assumption that $U_1$ can express $\varphi(S)$ is false by the finite model property of $U_1$ (Theorem 1).

We next consider $U_1$ over vocabularies with at most binary relations.

Theorem 4 (13). Consider models over a relational vocabulary $\tau$ with the arity bound two. Suppose that $\tau$ indeed contains at least one binary relation symbol. Then FOC$^2 < U_1 <$ FOC$^2$.

Proof. We already discussed the strict inclusion FOC$^2 < U_1$ above in the preliminaries section. A lengthy proof of the inclusion $U_1 \leq$ FOC$^2$ is given in [13]. The strictness of this inclusion follows from the proof of Theorem 3 where we showed that $U_1$ cannot express that the in-degree of a binary relation is at most one.

We then compare the expressivities of $U_1$ and the guarded negation fragment GNFO [2]. The first non-inclusion ($U_1 \not\leq$ GNFO) of the following theorem has been proved in [9], where only the equality-free fragment of $U_1$ was investigated. The second non-inclusion ($\text{GNFO} \not\leq U_1$) is new.

Theorem 5. $U_1$ and GNFO are incomparable in expressivity.

Proof. Define the two structures $\langle \{a\}, \{(a,a)\} \rangle$ and $\langle \{a,b\}, \{(a,a),(b,b)\} \rangle$. It is straightforward to establish by using the bisimulation for GNFO, provided in [2], that these two structures are bisimilar in the sense of GNFO. Thus the $U_1$-sentence $\exists x \exists y \neg R(x, y)$ is not expressible in GNFO. Hence $U_1 \not\leq$ GNFO.

Consider then the GNFO-sentence

$$\varphi := \exists x \exists y \exists z (Rxy \land Ryz \land Rzx).$$

Let $\mathcal{A}$ denote the model consisting of four disjoint copies of the directed cycle with three elements. Let $\mathcal{B}$ be the model with three disjoint copies of the directed cycle with four elements. It follows rather directly from the Ehrenfeucht-Fraïssé game for $U_1$ (which is defined in [14]) that $\mathcal{A}$ and $\mathcal{B}$
satisfy the same U₁-sentences. For the game-based argument to work, it is essential that the two models A and B have the same cardinality, because bijections between subsets of the domains of A and B are used in the game. (See [14] for a detailed discussion of the game.) With A and B defined in this way, the rest of the game-based argument is straightforward. We can therefore now conclude that U₁ cannot express the GNFO-sentence \( \varphi \) we fixed above, and hence GNFO \( \not\leq \) U₁.

Before we close the current section, we observe that all the above results concerning expressivity hold even if attention is limited to finite models only. The same proofs apply without modification, as the reader can check. This is especially interesting in the case of Theorem 3 whose proof makes use of the finite model property of U₁.

5 Undecidability of U₁ with counting quantifiers

Since FO\(^2\) and U₁ are both NEXPTIME-complete, it is natural to ask whether the extension of U₁ by counting quantifiers (UC₁) remains decidable. Formally, UC₁ is obtained from U₁ by allowing the free substitution of quantifiers \( \exists \) by quantifiers \( \exists \geq k \), \( \exists \leq k \), \( \exists = k \).

While the transition from FO\(^2\) to FOC\(^2\) preserves NEXPTIME-completeness, the analogous step from U₁ to UC₁ crosses the undecidability barrier.

**Theorem 6 ([13]).** The satisfiability and finite satisfiability problems of UC₁ are \( \Pi^0_1 \)-complete and \( \Sigma^0_1 \)-complete, respectively.

Thereby UC₁ has the same complexity as full first-order logic. It is an interesting open problem to identify natural logics that extend FOC\(^2\) into higher arity contexts in a way that preserves decidability. Possible research directions here could involve for example investigating restrictions of UC₁ based on somewhat more limited ways of using the quantifiers \( \exists \geq k \), \( \exists \leq k \), \( \exists = k \).

6 U₁ and description logics

In this section we define a novel logic \( \mathcal{DL}_{FU_1} \) which is a description logic version of FU₁ and compare it to \( \mathcal{DL}_{R_{reg}} \), which is a well-known description logic that accommodates higher arity relations.

We first generalize the relational inverse operation to contexts with higher arity relations. When \( n \) is a positive integer, we let \([n]\) denote the set \( \{1, \ldots, n\} \). We let SRJ denote the set of all surjections \( \sigma : [k] \rightarrow [m] \), such that \( 2 \leq m \leq k \). When \( m = k \), then \( \sigma \) is a permutation; permutations are natural generalizations of the relational inverse operator into higher arity contexts, and surjections generalize permutations an inch further. When we use SRJ in constructing the syntax of \( \mathcal{DL}_{FU_1} \) below, we assume each
function $\sigma \in \text{SRJ}$ to be a suitable string listing the ordered pairs $(n, k)$ such that $\sigma(n) = k$ in binary.

The set $\mathcal{R}$ of roles of $\mathcal{DL}_{FU_1}$ is defined by the grammar

$$\mathcal{R} ::= R | \varepsilon | -\mathcal{R} | (\mathcal{R}_1 \cap \mathcal{R}_2) | \sigma \mathcal{R}$$

where $R$ denotes an atomic role, $\varepsilon$ the binary identity role and $\sigma \in \text{SRJ}$. Here $R$ can have any arity greater or equal to two, and the arity of $\varepsilon$ is two. The intersection of relations of different arity will produce the empty relation, so we may as well allow such terms. (We fix the arity of the empty relation in such cases to be two.) The set of concepts of $\mathcal{DL}_{FU_1}$ is given by the grammar

$$C ::= A | -C | (C_1 \cap C_2) | \exists R.(C_1, ..., C_n)$$

where $A$ is an atomic concept and the arity of the relation term $R$ is $n + 1$. An interpretation $\mathcal{I}$ is a pair $(\Delta, \cdot)$, where $\Delta$ is a nonempty set and $\cdot$ a function such that $R^\mathcal{I} \subseteq \Delta^k$ and $A^\mathcal{I} \subseteq \Delta$ for atomic roles $R$ and atomic concepts $A$; here $k$ is the arity of $R$.

In the pathological case where $\sigma : [n] \to [m]$ acts on a relation $\mathcal{R}$ whose arity is not equal to $n$, the empty binary relation is produced. We need the surjection operators (rather than simply permutations) in order to express in $\mathcal{DL}_{FU_1}$ conditions such as the one given by the FU$_1$-formula $\exists y(R(x, y) \wedge S(x, y, x) \wedge P(y))$. In the following theorem, equivalence means equivalence in the standard sense in which formulae of modal and predicate logic are compared.

**Theorem 7.** $\mathcal{DL}_{FU_1}$ and FU$_1$ are equi-expressive: each FU$_1$-formula $\varphi(x)$ has an equivalent $\mathcal{DL}_{FU_1}$-concept, and vice versa.

**Proof.** We only provide a rough sketch the proof. The most involved issue here is the translation of FU$_1$-formulae of the type $\exists x_1...\exists x_k \varphi$ into $\mathcal{DL}_{FU_1}$, where $\varphi$ is a Boolean combination of higher arity atoms and at most unary FU$_1$-formulae. Here we put $\varphi$ into disjunctive normal form and distribute
the quantifier prefix over the disjunctions in order to obtain a disjunction of
formulae of the type
\[ \exists x_1 \ldots \exists x_k (T(y_1, \ldots, y_n) \land \chi_1(u_1) \land \cdots \land \chi_m(u_m)) \]
such that the following three conditions hold.

- \{y_1, \ldots, y_n\} \subseteq \{x_0, x_1, \ldots, x_k\}
- \{u_1, \ldots, u_m\} \subseteq \{x_0, x_1, \ldots, x_k\}
- The term \( T(y_1, \ldots, y_n) \) is a conjunction of higher arity literals (atoms and negated atoms) such that each literal has exactly the same set \{y_1, \ldots, y_n\} of variables.

Such formulae can easily be translated into \( \mathcal{DL}_{FU_1} \), assuming inductively that we already know how to translate the unary \( \mathcal{FU}_1 \)-formulae \( \chi_i(u_i) \).

We then define the description logic \( \mathcal{DLR}_{reg} \) from [5] and compare it to \( \mathcal{DL}_{FU_1} \). \( \mathcal{DLR}_{reg} \) is defined by the grammar

\[
\begin{align*}
R &::= \top_n \mid R \mid (\$i/n : C) \mid \neg R \mid (R_1 \cap R_2) \\
E &::= \varepsilon \mid R_{[\delta_i, \delta_j]} \mid (E_1 \circ E_2) \mid (E_1 \cup E_2) \mid E^* \\
C &::= \top_1 \mid A \mid \neg C \mid (C_1 \cap C_2) \mid \exists E.C \mid \exists[\delta_i]R \mid (\leq k[\delta_i]R)
\end{align*}
\]

where \( R \) is an atomic role and \( A \) an atomic concept from a finite set \( V \) of atomic role and concept symbols. The indices \( i \) and \( j \) denote integers between 1 and \( n_{max} \) (where \( n_{max} \) is the maximum arity of the symbols in \( V \)), \( n \) denotes an integer between 2 and \( n_{max} \) and \( k \) denotes a non-negative integer. All these numbers are encoded in binary.

An interpretation \( I = (\Delta, \cdot^I) \) for \( \mathcal{DLR}_{reg} \) over \( V \) is any structure such that the following conditions are met (cf. [5]).

1. For each atomic concept \( A \in V \) and atomic role \( R \in V \), we have \( A \subseteq \Delta \) and \( R \subseteq \Delta^n \), where \( n \) is the arity of \( R \).
2. For each \( n > 1, (\top_n)^I \) is a subset of \( \Delta^n \) that covers the relations of arity \( n \).
3. \((\$i/n : C)^I \) is the set of tuples \((u_1, \ldots, u_n) \in (\top_n)^I \) such that \( u_i \in C^I \).
4. \((\neg R)^I = (\top_n)^I \setminus R^I \) when \( R \) is an \( n \)-ary term and \((R_1 \cap R_2)^I = R_1^I \cap R_2^I \).
5. \( \varepsilon^I = \{ (u, u) \mid u \in \Delta \} \) and \((R_{[\delta_i, \delta_j]})^I \) is the relation
   \[ \{ (u, v) \mid u = w_i \text{ and } v = w_j \text{ for some tuple } (w_1, \ldots, w_n) \in R^I \} \].
6. The operators \( \circ \), \( \cup \) and \( \cdot \) in the terms \((E_1 \circ E_2), \ (E_1 \cup E_2)\) and \(E^*\) are interpreted in the usual way, i.e., \( \circ \) is the relational composition operator, \( \cup \) the union and \( \cdot \) the transitive reflexive closure operator.

7. \((\top_1)^I = \Delta, \ (-C)^I = (\top_1)^I \setminus C^I\) and \((C \cap D)^I = C^I \cap D^I\).

8. \((\exists \mathcal{E}. C)^I = \{ u \mid \text{exists } (u, v) \in \mathcal{E}^I \text{ such that } v \in C^I \}\)

9. \((\exists \{i\} \mathcal{R})^I = \{ u \mid \text{exists } (v_1, \ldots, v_n) \in \mathcal{R}^I \text{ such that } u = v_1 \}\)

10. \((\leq k \{i\} \mathcal{R})^I = \{ u \mid |\{ u \mid \text{exists } (v_1, \ldots, v_n) \in \mathcal{R}^I \text{ s.t. } u = v_1 \}| \leq k \}\).

\(\mathcal{DLR}_{\text{reg}}\) interpretations are associated with the atomic built-in relations \(\top_n\). When comparing the expressivity of \(\mathcal{DLR}_{\text{reg}}\) with \(\mathcal{DL}_{\text{FU}_1}\) below, we consider interpretations \(I\) where the relations \(\top_n\) are appropriate atomic built-in roles and thus directly available also in \(\mathcal{DL}_{\text{FU}_1}\).

**Proposition 2.** \(\mathcal{DLR}_{\text{reg}}\) and \(\mathcal{DL}_{\text{FU}_1}\) are incomparable in expressivity.

**Proof.** It is easy to see that \(\mathcal{DLR}_{\text{reg}}\) is closed under disjoint copies such that if \(C^I = U\) for some \(\mathcal{DLR}_{\text{reg}}\)-concept \(C\), then \(C^{|I_1+I_2|} = U_1 \cup U_2\), where \(I_1+I_2\) consists of two disjoint copies of \(I\) and obviously \(U_1\) and \(U_2\) are the related copies of \(U\). Because of the free use of role negation in \(\mathcal{DL}_{\text{FU}_1}\), the same does not hold in that logic. For example the \(\mathcal{DL}_{\text{FU}_1}\)-concept \(\neg \exists (\neg R)A\), where \(R\) is a binary role, is satisfied in an interpretation consisting of a single element \(u\) that satisfies \(A\) and connects to itself via \(R\). This interpretation together with a disjoint copy of itself does not satisfy \(\neg \exists (\neg R)A\). Thus \(\mathcal{DL}_{\text{FU}_1}\) is not contained in \(\mathcal{DLR}_{\text{reg}}\).

For the converse, it suffices to observe that \(\mathcal{DL}_{\text{FU}_1}\) cannot define the concept \(\exists (R^*)A\). It is well known that this property is not first-order expressible, and thus it is not definable in \(\mathcal{DL}_{\text{FU}_1}\).

We finish up the current section by identifying a maximal fragment of \(\mathcal{DLR}_{\text{reg}}\) that embeds into \(\mathcal{DL}_{\text{FU}_1}\). What exactly we mean by maximality in this context will become clear below.

Let \(\mathcal{DLR}_{\text{reg}}^0\) denote the fragment of \(\mathcal{DLR}_{\text{reg}}\) without Kleene star and counting, i.e., \(\mathcal{DLR}_{\text{reg}}^0\) is obtained by the grammar that drops the terms \(E^*\) and \((\leq k \{i\} \mathcal{R})\) from the grammar of \(\mathcal{DLR}_{\text{reg}}\). For each positive integer \(k\), we let \(\mathcal{DLR}_{\text{reg}}^0[\leq k]\) denote the system we obtain if we add the terms \((\leq k \{i\} \mathcal{R})\) (with each arity for \(\mathcal{R}\) and each related \(i\) included) to the grammar of \(\mathcal{DLR}_{\text{reg}}^0\). (Note that \((\leq 0 \{i\} \mathcal{R})\) is equivalent to \(\neg \exists \{i\} \mathcal{R}\).) Similarly, we let \(\mathcal{DLR}_{\text{reg}}^0[\ast]\) be the logic we obtain by adding the term \(E^*\) to the grammar of \(\mathcal{DLR}_{\text{reg}}^0\).

We will show that while \(\mathcal{DLR}_{\text{reg}}^0\) embeds into \(\mathcal{DL}_{\text{FU}_1}\) (Theorem 3), neither \(\mathcal{DLR}_{\text{reg}}^0[\ast]\) nor any of the logics \(\mathcal{DLR}_{\text{reg}}^0[\leq k]\) does (Theorem 4). We already observed above that the operator \(\ast\) of \(\mathcal{DLR}_{\text{reg}}\) is inexpressible in
The fact that the number restriction operators ($\leq k[\$i]R$) are definable neither in $DL_{FU_1}$ nor in $U_1$, as we shall prove, is somewhat more surprising since $U_1$ can do some counting. However, as we already discussed earlier, the counting ability of $U_1$ is limited.

Finally, it is not entirely trivial that we can indeed keep the composition operator in $DLR_0^{reg}$ and still embed this logic into $DL_{FU_1}$. This is because the use of the composition operator often requires the three-variable fragment of first-order logic, and $DL_{FU_1}$ collapses to $FO^2$ on binary vocabularies.

**Theorem 8.** $DLR_0^{reg}$ embeds into $DL_{FU_1}$.

**Proof.** We begin by showing that we can eliminate the composition operator $\circ$ from $DLR_0^{reg}$ altogether. Consider a concept $D$ of $DLR_0^{reg}$. We first observe that we can use the standard identity $R(S\cup T) = (R\circ S)\cup(R\circ T)$ of relation algebra to obtain from $D$ an expression where the composition operators are on the “atomic” level, with the relational terms $\varepsilon$ and $R_{\$i,\$j}$ of the grammar of $DLR_{reg}$ regarded as atoms. We then use the equivalence

$$\exists(\varepsilon_1 \cup ... \cup \varepsilon_m).C \equiv (\exists\varepsilon_1.C) \cup ... \cup (\exists\varepsilon_m.C)$$

to obtain a disjunction of formulae $\exists\varepsilon_i.C$ where $\varepsilon_i$ is a composition of “atomic” terms $S$. To eliminate the composition operators from the terms $\varepsilon_i = S_1 \circ ... \circ S_n$, we use the equivalence

$$\exists(S_1 \circ ... \circ S_n).C \equiv \exists S_1.\exists S_2.\exists S_3 ... \exists S_n.C.$$  

Thus we can eliminate instances of $\circ$ from $DLR_0^{reg}$.

Next we note that all the remaining union operators are also eliminable, using the equivalence

$$\exists(\varepsilon_1 \cup ... \cup \varepsilon_m).C \equiv (\exists\varepsilon_1.C) \cup ... \cup (\exists\varepsilon_m.C).$$

We then show how to translate the obtained formula (which is free of union and composition operators) into $DL_{FU_1}$. For presentational reasons, we will translate the formula into the first-order fragment $FU_1$. The syntax of $DLR_{reg}$ without composition and union is given by the grammar

$$\mathcal{R} ::= \top_n \mid R \mid (\$i/n:C) \mid \neg\mathcal{R} \mid (\mathcal{R}_1 \cap \mathcal{R}_2)$$

$$\mathcal{E} ::= \varepsilon \mid \mathcal{R}_{\$i,\$j}$$

$$\mathcal{C} ::= \top_1 \mid A \mid \neg A \mid (C_1 \cap C_2) \mid \exists\mathcal{E}.C \mid \exists[\$i]\mathcal{R}$$

where $\mathcal{R}_{\$i,\$j}$ with $i = j$ is not allowed; these are easy to eliminate. Our translation will be defined with three translation operators $s,t,T$ that correspond to, respectively, the terms for $\mathcal{R}, \mathcal{E}, \mathcal{C}$ above. Each of these operators is parameterized by an appropriate tuple of variables. We first define $T$ as follows.
1. $T[x](\top) := \top$ and $T[x](A) := A(x)$. 
2. $T[x](-C) := -T[x](C)$ and $T[x](C_1 \cap C_2) := T[x](C_1) \land T[x](C_2)$. 
3. $T[x](\exists \mathcal{E} C) := \exists y(t[x,y](\mathcal{E}) \land T[y]C)$, where $t$ is the translation for terms $\mathcal{E}$ to be defined below. 

We then define the operator $t$.

1. $t[x,y](\varepsilon) := x = y$. 
2. $t[x,y](\mathcal{R}_{i \leq j}) := \exists \pi(s[\pi](\mathcal{R}))$, where $\exists \pi$ quantifies existentially each of the variables $x_1, \ldots, x_n$ except for $x_i$ and $x_j$, and where $\pi$ is obtained from the tuple $(x_1, \ldots, x_n)$ by replacing $x_i$ by $x$ and $x_j$ by $y$. Here $n$ is the arity of the relation $\mathcal{R}$ and $s$ is the translation for $\mathcal{R}$. 

We finally define the operator $s$ as follows.

1. $s[x_1, \ldots, x_n](\top) := \top_n(x_1, \ldots, x_n)$ and $s[x_1, \ldots, x_n](R) := R(x_1, \ldots, x_n)$ for atomic roles $R$ and the built-in relation $\top_n$. 
2. $s[x_1, \ldots, x_n](\langle \mathcal{R}/[i : l] \rangle) := T[x_i](C) \land \top_n(x_1, \ldots, x_n)$, where $T$ is the translation for $C$. 
3. $s[x_1, \ldots, x_n](\neg R) := \top_n(x_1, \ldots, x_n) \land \neg s[x_1, \ldots, x_n](\mathcal{R})$. 
4. $s[x_1, \ldots, x_n](\mathcal{R}_1 \cap \mathcal{R}_2) := s[x_1, \ldots, x_n](\mathcal{R}_1) \land s[x_1, \ldots, x_n](\mathcal{R}_2)$. 

The translated formula is now easily modified to a formula of FU1. This involves shifting the quantifiers introduced in clause 2 of the translation $t[x,y]$. 

We then show that none of the operators of $\mathcal{DLR}_{reg}$ missing from $\mathcal{DLR}_{reg}^0$ could be added to $\mathcal{DLR}_{reg}^0$ without losing the embedding into $\mathcal{DL}_{FU_1}$. By an operator we here mean a term $[\leq k][\mathcal{R}/i]$ with $k \in \mathbb{Z}_+$. Note that for a fixed $k$, the term $[\leq k][\mathcal{R}/i]$ strictly speaking denotes a collection of operators, because we could vary $i$ and the arity of $\mathcal{R}$. Thus a more fine-grained analysis than the one below could be given. We ignore this issue for the sake of simplicity.

**Theorem 9.** $\mathcal{DLR}_{reg}^0[\ast]$ and $\mathcal{DLR}_{reg}^0[\leq k]$ for each $k \in \mathbb{Z}_+$ are all incomparable with $\mathcal{DL}_{FU_1}$.
Proof. We already observed in the proof of Proposition\(^2\) that \(\mathcal{DL}_{FU_1}\) cannot define the concept \(\exists(R^*).A\) and that \(\mathcal{DLR}_{reg}\) cannot define \(\neg\exists(\neg R).A\), where \(\neg\) is the full negation of \(\mathcal{DL}_{FU_1}\). Thus it now suffices to show that for each \(k \in \mathbb{Z}_+\), the concept \((\leq k[\$2]R)\) is not expressible in \(\mathcal{DL}_{FU_1}\). Here \(R\) is a binary relation.

In the proof of Theorem\(^3\) we already dealt with the special case where \(k = 1\): if \(\varphi(x)\) was an FU\(_1\)-formula defining the concept \((\leq 1[\$2]R)\), then the FU\(_1\)-sentence \(\forall x\varphi(x)\) would define that the in-degree of \(R\) is at most one. Thus we can now fix a \(k \geq 2\) and define two interpretations, one consisting of \(k + 1\) copies of the clique of size \(k\) and the other one of \(k\) copies of the clique of size \(k + 1\). (Recall that a clique is a structure where the binary relation \(R\) is the total relation with the reflexive loops removed).

We have prepared the setting in such a way that it is now easy to show, using once again the EF-game for U\(_1\) (defined in [14]), that the two structures satisfy exactly the same U\(_1\)-sentences. However, the concept \((\leq k - 1[\$2]R)\) is satisfied by every element in the first structure and by none of the elements of the second one. Thus no U\(_1\)-formula is equivalent to \((\leq k - 1[\$2]R)\), because if \(\varphi(x)\) was equivalent to \((\leq k - 1[\$2]R)\), the U\(_1\)-sentence \(\exists x\varphi(x)\) would be satisfied by the first structure but not the second one.

Logics building on the two-variable logic, U\(_1\) and \(\mathcal{DL}_{FU_1}\) soon become highly undecidable if extended with fixed points or even transitive closure. Notable exceptions to this pattern can be obtained via the game-theoretic recursion (see, e.g., the article [11] for the definitions, and note that here we are mainly—but not exclusively—interested in the unbounded semantics). The related logics have many nice properties. For example validity for the so extended two-variable logic is complete for co-NEXPTIME [11], and we get recursive enumerability of validities even for the extension of FO with game-theoretic recursion.

The logics with game-theoretic recursion constitute an interesting collection of systems with self-referential statements. Note that the formula \(C \neg C\) gives a formulation of the liar paradox. Within the standard semantics (cf. [11]), this is an indeterminate statement. To consider the paradox generally, let LS stand for the informal claim “this sentence is not true”. Note that LS can be interpreted as the claim “this sentence is not \(T_1\)” where \(T_1\) refers to some reasonably ordinary notion of truth, let us call it “first-level truth”. This could be, for example, the “well-founded truth” [17], but various notions—formal and informal—are fine here. With this interpretation, we can associate LS with a second-level truth value \(U_2\). Here \(U_2\) can mean, for example, “second-level true” or “second-level false”, or perhaps “not second-level true” or “not second-level false”, to give a few options out of many. In each case, we avoid the usual paradox associated with the claim “this sentence is not true”. This is because \(U_2\) operates on a different level than \(T_1\), that is, gives a different sense of truth. So “second-level true”
means true on some other level (or in a different sense) than “first-level true” \( T_1 \), and similarly for falsity and other truth notions. Altogether, the above discussion gives one suggestion for a resolution of the paradox associated with “this sentence is not true”. In general, there is no paradox in stating, to give one example, that it is not second-level true that “this sentence is not first-level true”. It is not so central what second-level truth value we give “this sentence is not true”, the point is that we can give it any second-level truth value, as it does not interact with a first-level truth values. The same analysis gives—for similar reasons—a similar suggestion for resolving the paradox with “this sentence is false” (where we talk about falsity instead of not being true).

Let us get back to the game-theoretic recursion and give a related example. Considering the standard semantics for \( C \neg C \), we can take “first-level true” be the well-founded truth mentioned above, meaning Eloise having a winning strategy. We can also take “first-level false” to mean a similar, well-founded notion meaning that Abelard has a winning strategy. Since \( C \neg C \) is indeterminate, the sentence \( C \neg C \) is neither “first-level true” nor “first-level false”. Thus it is not “first-level true”, and it is not “first-level false”. However, to give one possibility, \( C \neg C \) can, without problems, be considered “second-level false”, with intuition would be that Eloise does not have a winning strategy, i.e., “second-level false” is equated with not being “first-level true”. On the other hand, giving a somewhat different interpretation to second-level truth values, \( C \neg C \) can also be considered to not be “second-level false” since Abelard has no winning strategy (so here we equate “second-level false” with “first-level false” on the formal level). And so on. The general point is simply that we resolve the paradox by suggesting to give new truth values that operate on different levels (or in different senses). This is often natural, as for example “well-founded truth” (a possible interpretation for “first-level true”) and similarly “well-founded falsity” are naturally different from many second-level truth values that talk about indeterminacy due to neither player having a winning strategy. And, in the general case, since the truth values on different levels do not interact (in any forced way), there is no paradox. There is indeed no problem stating (for example) that the sentence “this sentence is not first-level true” is not second-level true. Indeed, we can read \( C \neg C \) as stating that “this sentence is not first-level true” and then give it the semantic judgement that it is not “second-level true”, where “second-level true” means Eloise having a winning strategy. And so on, with many interesting different options. The general resolution simply says that we can give a second-level truth value in more or less any way we like. Some options can be more natural than others, but there is no inconsistency. Indeed, defining what positive real numbers are extends the definition of positive integers, and that generalization is natural but not something that the properties of integers force. We could extend the definition in other, alternative ways without direct inconsistency.
There are many suggested resolutions of the general paradox in the literature. In particular, we note that there is a significant difference between Tarski’s suggested resolution and the suggestion discussed above. In Tarski’s suggestion, the liar sentence is denied the role of an admissible sentence, as the truth predicate should not apply on any level \( n \) to sentences on the same level \( n \). This kind of a constraint is not used in the suggestion we discussed above. Note also that our suggestion indeed bears a nice link of the syntax and semantics of the \( C \)-operator as used in [11] and the articles leading to that paper (see the introduction of [11] for more information). In general, the framework with \( C \) relates nicely to many issues on self-reference. However, it of course does not touch all issues. For example, in the related logics (so far), \( \varphi \) is equivalent to \( \varphi \land \varphi \). This breaks with sentences such as “this sentence has less than fifty letters” which is true but its conjunction with itself is—at least under one natural interpretation—not true.

Concerning the syntax of logics with the game-theoretic looping operator \( C \), it is sometimes convenient to write \( L_C \) for the non-atomic occurrences of \( C \) while atomic operators are kept as they are. Now \( C \neg C \) becomes \( L_C \neg C \). This formula fits both the predicate logic as well as the description logic context. Concerning the latter family of logics, for example the formula \( C \exists R.(A \cup C) \) becomes \( L_C \exists R.(A \cup C) \). Eloise wins the game for this formula precisely from those points from where we can reach a point satisfying the atomic concept \( A \). Note that \( C \) is here of course an iterator variable, not a metasymbol denoting a concept. To avoid confusion, it may indeed be convenient to use some other letter for iterator variables than \( C \) in the context of description logics.

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