Geometry and integrability of
Euler–Poincaré–Suslov equations

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Abstract
We consider nonholonomic geodesic flows of left-invariant metrics and
left-invariant nonintegrable distributions on compact connected Lie groups.
The equations of geodesic flows are reduced to the Euler–Poincaré–Suslov
equations on the corresponding Lie algebras. The Poisson and symplectic
structures give raise to various algebraic constructions of the integrable
Hamiltonian systems. On the other hand, nonholonomic systems are not
Hamiltonian and the integration methods for nonholonomic systems are
much less developed. In this paper, using chains of subalgebras, we give
constructions that lead to a large set of first integrals and to integrable
cases of the Euler–Poincaré–Suslov equations. Further, we give examples
of nonholonomic geodesic flows that can be seen as restrictions of
integrable sub-Riemannian geodesic flows.

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0 Introduction

In this paper we are interested in the geometry and integrability of nonholonomic geodesic flows of left-invariant metrics and left-invariant nonintegrable distributions on compact Lie groups. The equations of geodesic flows are reduced to the Euler–Poincaré–Suslov equations on the corresponding Lie algebras. These systems are natural generalizations to Lie algebras of the Suslov nonholonomic rigid body problem and were introduced independently by Kozlov [12] and Koiller [11]. The known integrable cases are given by Fedorov and Kozlov [8] and author [10].

In the recent years appears many papers concerning geometrical formulation of the nonholonomic mechanics. For instance, see [11, 3, 13] and references therein. However, since nonholonomic systems do not admit a Poisson structure, the integration theory of the constrained mechanical systems is much less developed then for the unconstrained. We mention Chaplygin’s results [7]. By the use of an invariant measure, he had given some of the most interesting examples of the solvable nonholonomic systems and noticed that the phase space could be foliated on invariant tori, placing these systems together with integrable Hamiltonian systems. The methods of integration of systems with an invariant measure, as well as illustrative integrable examples can be found in [1, 2, 3, 4].

Now, we shall briefly describe the results and outline of this paper.

In sections 1 and 2 we shall give basic definitions and notation.

One of the well known ways for studying the integrability of Riemannian geodesic flows is by using certain filtrations of Lie algebras (see [20, 14, 16, 2, 5]). By taking appropriate chains of subalgebras, we can get integrable cases of the Euler–Poincaré–Suslov equations.

In section 3 we shall consider the case when the left-invariant nonintegrable distribution is an invariant subspace of left-invariant metric. Then reduced system has an invariant measure. We shall construct integrable examples by using chains of the form $K \subset H \subset G$, where $(G, H)$ is a symmetric pair. The obtained systems are generalizations of the Fedorov–Kozlov integrable case.

In section 4 we deal with an arbitrary chain $G_0 \subset G_1 \subset \ldots \subset G_n = G$. This gives us opportunity to construct exactly solvable examples without an invariant measure as well.

In some cases the interesting phenomena arises: the nonholonomic geodesic flow could be seen as a restriction of Hamiltonian flow on the whole, unconstrained phase space. We shall present here two families of integrable sub-Riemannian geodesic flows which restrict to the our non-Hamiltonian problem (section 6).

1 Nonholonomic geodesic flows

Let $(Q, (\cdot, \cdot))$ be $n$–dimensional Riemannian manifold with Levi–Civita connection $\nabla$. Let $\mathcal{D}$ be the nonintegrable $\rho$–dimensional distribution distribution of
the tangent bundle. The distribution can be defined by \( n - \rho \) independent one forms \( \alpha_i \) in the following way:

\[
\mathcal{D}_q = \{ \xi \in T_qQ, \alpha_i(\xi) = 0, \ i = 1, \ldots, \rho \}.
\]

The smooth path \( \gamma(t), t \in \Delta \) is called admissible (or allowed by constraints) if velocity \( \dot{\gamma}(t) \) belongs to \( \mathcal{D}_\gamma(t) \) for all \( t \in \Delta \). There are two approaches to define the geodesic lines among admissible paths: by induced connection as a "straightest" lines and by variational principle as a "shortest" lines. We shall deal with the first approach which arises from mechanics. The admissible path \( \gamma(t) \) is called a nonholonomic geodesic line if it satisfied d’Alambert–Lagrange equations:

\[
\pi(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)) = 0,
\]

where \( \pi : T_qQ \to \mathcal{D}_q, q \in Q \) is the orthogonal projection.

Let \( \{\cdot, \cdot\} \) be canonical Poisson brackets on \( T^*Q \). By \( X_f \) we shall denote the Hamiltonian vector field of the function \( f \) (\( dg(X_f) = \{g, f\} \) for all \( g : T^*Q \to \mathbb{R} \)). Let \( \mathcal{M} \) be the constraint submanifold in the phase space \( T^*Q \):

\[
\mathcal{M} = \{ (p, q) \in T^*Q, p \in g_q(D_q) \subset T^*_qQ \},
\]

where we consider scalar product \( (\cdot, \cdot)_q \) as the mapping \( g_q : T_qQ \to T_q^*Q \). Taking for the Hamiltonian \( h(p, q) = \frac{1}{2}p(g_q^{-1}p), p \in T^*_qQ \), we can write (1) in the following form:

\[
\dot{x} = X_h(x) + \sum_{i=1}^{\rho} \lambda_i \text{vert}_{\sigma(x)} \alpha_i(q),
\]

where \( \text{vert}_{\sigma(x)} \alpha_i(q) \in T_x(T^*Q) \) is "vertical" lift of \( \alpha_i|_{\sigma(x)} \in T^*_{\sigma(x)}Q \), \( \sigma : T^*Q \to Q \) is the natural projection and Lagrange multipliers \( \lambda_i \) are chosen such that the phase trajectory \( x(t) \) belongs to \( \mathcal{M} \). In canonical coordinates \( x = (p, q) \) equations (2) are:

\[
\dot{p} = -\frac{\partial h(p, q)}{\partial q} + \sum_{i=1}^{\rho} \lambda_i \alpha_i(q), \quad \dot{q} = \frac{\partial h(p, q)}{\partial p}.
\]

It is important to note that the equations are not Hamiltonian (they are Hamiltonian with respect to the almost-Poisson brackets on \( \mathcal{M} \)) and that the Hamiltonian function is conserved.

Sufficient condition for the integration of the non-Hamiltonian system

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^m,
\]

is existence of \( m - 1 \) independent integrals, or one integral less in the case of the existence of an invariant measure.

Suppose that the system of equations (3) has an invariant measure and \( m - 2 \) first integrals \( F_1, \ldots, F_m \). If \( F_1, \ldots, F_m \) are independent on the invariant set \( M_c = \{ x \in \mathbb{R}^m, F_i(x) = c_i, \ i = 1, \ldots, m - 2 \} \) then the solution of (3)
lying on $M_c$ can be found by quadratures (the Jacobi theorem). Moreover, if $L_c$ is a compact connected component of $M_c$ and $f(x) \neq 0$ on $L_c$ then $L_c$ is diffeomorphic to a two-torus; one can find angular coordinates $\varphi_1, \varphi_2$ on $L_c$ in which equations (4) take the form similar as in the Liouville theorem:

$$\dot{\varphi}_1 = \frac{\omega_1}{\Phi(\varphi_1, \varphi_2)}, \quad \dot{\varphi}_2 = \frac{\omega_2}{\Phi(\varphi_1, \varphi_2)},$$

where $\omega_1, \omega_2$ are constant and $\Phi$ is a smooth positive $2\pi$–periodic function in $\varphi_1, \varphi_2$ (see [1]).

Therefore it is natural to call a non-Hamiltonian system integrable if it can be integrated by the above procedure; or more generally (as it was pointed out in [22]), if the trajectories of the system belong to invariant tori with dynamics of the form

$$\dot{\varphi}_1 = \frac{\omega_1}{\Phi(\varphi_1, \ldots, \varphi_k)}, \ldots, \dot{\varphi}_k = \frac{\omega_k}{\Phi(\varphi_1, \ldots, \varphi_k)},$$

(5)

The flow of (5) is unevenly winding and admit the invariant measure $\mu(D) = \int_D \Phi d\varphi_1 \wedge \ldots \wedge d\varphi_k$. Note that it is shown in [1] that for almost all frequencies $\omega_1, \ldots, \omega_k$, by smooth change of variables

$$T^k \{\varphi_1, \ldots, \varphi_k\} \to T^k \{\theta_1, \ldots, \theta_k\},$$

equations (5) can be reduced to the form

$$\dot{\theta}_i = \Omega_i = \omega_i / \Pi, \quad i = 1, \ldots, k,$$

where $\Pi$ denotes the total measure of $T^k$.

2 Euler–Poincaré–Suslov equations

Now, let $Q$ be a compact connected Lie group $\mathfrak{g}$ with Lie algebra $G = T_e \mathfrak{g}$. In what follows we shall identify $G$ and $G^*$ by $Ad_\mathfrak{g}$ invariant scalar product $\langle \cdot , \cdot \rangle$; $T\mathfrak{g}$ and $T^* \mathfrak{g}$ by bi-invariant metric on $\mathfrak{g}$.

We shall consider left-invariant distributions. Let

$$D = \{ \omega \in G, \langle \omega, a^i \rangle = 0, \ i = 1, \ldots, \rho \} \subset G$$

be the restriction of the left-invariant distribution $D$ to the Lie algebra $G$, for some constant, linearly independent vectors $a^i$ in $G$. The distribution is nonintegrable if and only if $D$ is not a subalgebra. From the invariance, we have that $D_g = g \cdot D$.

Let $I : G \to G$ be a symmetric, positive definite (with respect to $\langle \cdot , \cdot \rangle$) operator that induces left-invariant metric:

$$(\eta_1, \eta_2)_g = \langle I(\omega_1), \omega_2 \rangle, \ \eta_1, \eta_2 \in T_g \mathfrak{g}, \ \omega_1, \omega_2 \in G, \ \eta_1 = g \cdot \omega_1, \ \eta_2 = g \cdot \omega_2.$$  

Let $M$ be the restriction of the constraint submanifold $\mathcal{M}$ to $G$. Then $M = I(D)$ and $\mathcal{M}_g = g \cdot M$. The Hamiltonian of the geodesic flow is function $h : T\mathfrak{g} \to \mathbb{R}$ obtained from reduced Hamiltonian function $H(x) = \frac{1}{2} \langle A(x), x \rangle$ by left-translations (here $A = I^{-1}$).
In such a notation, the equations (2) are reduced to:

\[ \dot{x} = [x, \nabla H(x)] + \sum_{i=1}^{\rho} \lambda_i a^i = [x, A(x)] + \sum_{i=1}^{\rho} \lambda_i a^i, \]

where Lagrange multipliers are chosen such that \( x \) belongs to \( M = I(D) \), i.e. such that \( \omega = A(x) \) belongs to \( D \):

\[ \langle A(x), a^i \rangle = 0, \quad i = 1, \ldots, \rho. \]

According to Fedorov and Kozlov [8] we shall call these equations the Euler–Poincaré–Suslov equations, as a generalization of the Suslov nonholonomic rigid body problem. They have a quite different nature of the corresponding Euler–Poincaré equations \( \dot{x} = [x, A(x)] \). For instance, in general, they do not have a smooth invariant measure (see [12]). The equations (3) could be seen also as a reduced equations from a point of view of a reduction of nonholonomic systems with symmetries given in [3, 13].

Nonholonomic geodesic lines \( g(t) \) are solutions of the kinematic equation \( g^{-1}(t) \cdot \dot{g}(t) = \omega(t) = A(x(t)) \), where \( x(t) \) are solutions of (3). In other words, the following diagram commutes:

\[ \begin{array}{ccc}
   P^t & \rightarrow & P^t \\
   \Lambda & \downarrow & \Lambda \\
   M & \rightarrow & M \\
   \end{array} \]

Here, \( P^t \) and \( P^t \) are phase flows of the nonholonomic geodesic flow and Euler–Poincaré–Suslov equations; \( \Lambda \) maps \( p = g \cdot x \in T_g G \) to \( x \in G \). If Euler–Poincaré–Suslov equations are integrable we shall say that nonholonomic geodesic flow is integrable in the sense of the factorization (7).

The reduced Hamiltonian function \( H(x) = \frac{1}{2} \langle x, A(x) \rangle \) is the first integral of system (3). This follows from the conservation of energy in the natural mechanical nonholonomic systems with linear constraints. One may also prove this fact by a direct computation of the time derivative of \( H(x) \) along the vector field defined by (3). Namely, integral \( F \) of Euler–Poincaré equations \( \dot{x} = [x, A(x)] \) is the integral of Euler–Poincaré–Suslov equations (3) if and only if:

\[ \sum_i \lambda_i \langle \nabla F(x), a^i \rangle |_{x \in M} = 0. \]

3 Symmetric pairs

Let \( L \) be the subspace of \( G \) spanned by \( a^i \), \( i = 1, \ldots, \rho \). In this section we shall consider the case when \( A \) preserve the orthogonal decomposition \( G = L + D \),
i.e., $A = A_L + A_D$, where $A_L : L \to L$, $A_D : D \to D$ are positive definite operators. Then $M = I(D) = D$ and we can write (6) in the following way:

$$
\dot{\xi} = [\xi, A_D(\xi)]_D, \quad \xi \in D
$$

(by $x_K$ we denote the orthogonal projection of $x$ to the linear space $K$).

The equations (6) preserve the standard measure on $D$. Also the constrained reduced Hamiltonian function $H_D = \frac{1}{2} \langle \xi, A_D(\xi) \rangle$ and invariant $F(\xi) = \langle \xi, \xi \rangle$ are always first integrals of the system. Note that by (8), in general, the invariant $F(x) = \langle x, x \rangle$ is not the integral of (6).

**Example 3.1** We have, at least, following integrable cases:

1. If $A_D = s \cdot I d_D$, $s \in \mathbb{R}$ then the solution of (6) are $\xi = \text{const}$. In this case the constraints have no influence to the motion (the Lagrange multipliers in (6) are equal to zero). The nonholonomic geodesic lines are simply given with $g(t) = g_0 \exp(\eta t)$, $\eta \in D$, $g_0 \in \mathfrak{G}$. At the same time these lines are geodesic lines of the left-invariant metric induced by $A = A_L + s \cdot I d_D$. From now on we shall suppose that $A_D \neq s \cdot I d_D$.

2. $\dim D = 2$. Then the solution of (6) are $\xi = \text{const}$.

3. $\dim D = 3, 4$. Then the solutions of (6) belong to the intersections of the spheres and ellipsoids:

$$
M_{c_1, c_2} = \{ \xi \in D, H_D(\xi) = \frac{1}{2} \langle \xi, A_D(\xi) \rangle = c_1, \quad F(\xi) = \langle \xi, \xi \rangle = c_2 \}.
$$

For $\dim D = 3$ general trajectories are periodic. If $\dim D = 4$ then (6) could be integrated by Jacobi theorem as a system with an invariant measure (note that general connected components of invariant submanifolds $M_{c_1, c_2}$ are 2-dimensional spheres).

4. Recall that $(G, H)$ is called a symmetric pair if the following condition is satisfied:

$$
[H, H] \subset H, \quad [H, V] \subset V, \quad [V, V] \subset H,
$$

where $V$ is the orthogonal complement of $H$. If there is a subalgebra $H \subset G$ such that $(G, H)$ is a symmetric pair and that $H \subset L \subset G$, then $[D, D] \subset H$ and $[D, D]_D = 0$. Therefore the solution of (6) are $\xi = \text{const}$.

In the cases 2 and 4 above, nonholonomic geodesic lines are $g(t) = g_0 \exp(\eta t)$, $\eta \in D$, $g_0 \in \mathfrak{G}$, but these lines not need to be geodesic lines of the left-invariant metric induced by $A$.

Motivated with the last example, let us consider the chain of subalgebras:

$$
K \subset H \subset G,
$$

such that $(G, H)$ is a symmetric pair and that $H$ is not a subspace of $L$. Let $G = H + V$ be the orthogonal decomposition. We can consider the adjoint representation of $K$ on the linear space $V$: $\eta \in K \mapsto [\eta, \cdot] \in \text{End}(V)$. Decompose $V$
Theorem 3.1 Suppose that operator $A_D$ preserve the decomposition $D = U + W_0 + W_1 + \ldots + W_n$ and that $A_D|_U = s \cdot \text{Id}_U$, $s \in \mathbb{R}$. Then equations (3), besides functions $H_D(\xi) = \frac{1}{s}(A_D(\xi), \xi)$ and $F(\xi) = (\xi, \xi)$, have a set of the first integrals, the projection of $\xi$ to $W_0$: $F_0(\xi) = \xi_{W_0}$ and functions:

$$F_k(\xi) = \langle B_{W_k}(\xi_{W_k}), \xi_{W_k} \rangle, \quad k = 1, \ldots, n$$

where $B_{W_k} = A_D|_{W_k} - s \cdot \text{Id}_{W_k}$.

Proof. Let $A_W = A_D|_{W}$, $A_{W_k} = A_D|_{W_k}$, $W = W_0 + \ldots + W_n$. The equations (3) have the form:

$$\frac{d}{dt}(\xi_U + \xi_W) = [\xi_U + \xi_W, s\xi_U + A_W(\xi_W)]_D. \quad (10)$$

From $[U, V] \subset V$, $[W, W] \subset H$ and (10) we get:

$$\dot{\xi}_U = [\xi_W, A_W(\xi_W)]_U,$$

$$\dot{\xi}_W = [\xi_U, A_W(\xi_W) - s\xi_W]_W.$$

Since $A_W$ preserve the decomposition $W = W_0 + W_1 + \ldots + W_n$, the second equation is separated on $n + 1$ equations:

$$\dot{\xi}_{W_0} = 0, \quad \dot{\xi}_{W_k} = [\xi_U, B_{W_k}(\xi_{W_k})]_{W_k}, \quad k = 1, \ldots, n. \quad (11)$$

It is clear that $F_k$ are integrals of (11). Note that the invariant $F = (\xi, \xi)$ is dependent of functions $H_D$ and $F_k$, $k = 0, \ldots, n$.

Corollary 3.1 If operators $B_{W_k}$ are positive definite and $c_{n+1}$ satisfied inequality:

$$c_{n+1} > |c_0|^2 + \sum_{k=1}^{n} \frac{c_k}{b_k} \quad b_k = \min_{|\xi_{W_k}| = 1} \langle B_{W_k}(\xi_{W_k}), \xi_{W_k} \rangle,$$

then invariant subspaces:

$$M_c = \{ \xi \in D, \xi_{W_0} = c_0, F_1(\xi) = c_1, \ldots, F_n(\xi) = c_n, F(\xi) = c_{n+1} \}$$

are diffeomorphic to the product of spheres:

$$S^{\dim W_1 - 1} \times \ldots \times S^{\dim W_n - 1} \times S^{\dim U - 1}.$$
Proof. The first part of the corollary follows from the relations:

\[ |\xi_U|^2 = c_{n+1} - |c_0|^2 - \sum_{k=1}^{n} |\xi_{W_k}|^2, \quad |\xi_{W_k}|^2 \leq \frac{c_k}{b_k}, \quad \langle B_{W_k}(\xi_{W_k}), \xi_{W_k} \rangle = c_k. \]

Let \( \dim W_k = 2, k = 1, \ldots, g \), \( \dim W_k = 1, k = g+1, \ldots, n \). Let \( \phi_k \mod 2\pi \) be the angular variables of ellipses \( F_k = c_k, k = 1, \ldots, g \). Let \( \xi_U = \Phi \cdot \eta, \Phi \in \mathbb{R}, \eta \in U, |\eta| = 1 \). Then \( \Phi \) can be expressed in terms of the \( \phi_k \) up to the sign:

\[ \Phi = \Phi(\phi_1, \ldots, \phi_g) = \pm \sqrt{c_{n+1} - |c_0|^2 - \sum_{k=1}^{g} |\xi_{W_k}(\phi_k)|^2 - \sum_{k=g+1}^{n} \frac{c_k}{b_k} \neq 0}. \]

The sign of \( \Phi \) is determined by the choice of the connected component \( T^g \) of \( M_c \).

By substitution of \( \xi_{W_k} = \xi_{W_k}(\phi_k), k = 1, \ldots, g, \xi_U = \Phi(\phi_1, \ldots, \phi_g) \cdot \eta \) to (11), we obtain that angular variables satisfied the following type equations:

\[ \dot{\phi}_1 = \Phi(\phi_1, \ldots, \phi_g) f_1(\phi_1), \ldots, \dot{\phi}_g = \Phi(\phi_1, \ldots, \phi_g) f_g(\phi_g). \quad (12) \]

The system (12) can be integrated in term of the new time \( \tau \) defined by: \( d\tau = \Phi(\phi_1, \ldots, \phi_g) dt \).

If all \( f_k \) are different from zero, then we can introduce angular variables \( \varphi_k \) by averaging:

\[ \varphi_k = \varphi_k(\phi_k) = \omega_k \int_0^{\phi_k} \frac{ds}{\frac{2\pi}{f_k(s)}}, \quad \omega_k = 2\pi \left[ \int_0^{2\pi} \frac{ds}{f_k(s)} \right]^{-1}, \quad k = 1, \ldots, g, \]

in which (12) takes the required form

\[ \dot{\varphi}_1 = \frac{\omega_1}{\Phi - 1}, \ldots, \dot{\varphi}_g = \frac{\omega_g}{\Phi - 1}. \quad (13) \]

Remark 3.1 The frequencies \( \omega_i \) depend only of the metric \( A \). If the trajectories are periodic on one torus, they are periodic on the rest of the tori as well.

Remark 3.2 We have conditions \( \dim W_i = \dim V_i = 2, \dim U = \dim K = 1 \) taking:

\[ G = so(n) = \begin{pmatrix} U & W \\ -W^t & L \end{pmatrix}, \quad U = so(2), L = so(n - 2), D = U + W. \]

So, we can see the corollary as a generalization of the Fedorov–Kozlov integrable case [8].
Remark 3.3 Besides of preserving the standard measure in $D$, we can easily rewrite system (9) in the "Hamiltonian form":

$$\dot{F} = \{F, H_D\}_D, \quad F : D \to \mathbb{R},$$

(14)

where $H_D = \frac{1}{2}\langle \xi, A_D(\xi) \rangle$ and $\{\cdot, \cdot\}_D$ are almost-Poisson brackets defined by:

$$\{F_1, F_2\}_D(\xi) = \langle \xi, [\nabla F_2(\xi), \nabla F_1(\xi)] \rangle, \quad F_1, F_2 : D \to \mathbb{R}.$$  

(15)

These brackets are bi-linear, skew-symmetric and satisfy the Leibniz rule. In the general case, they do not satisfy the Jacobi identity. For $D = G$ these are the usual Lie-Poisson brackets on the Lie algebra $G$.

As for the Poisson brackets, we can define central (or Casimir) functions of the brackets (15). These are the functions $F$ that commute with all functions $F : D \to \mathbb{R}$. Obviously, they are integrals of the system (9). The example of the central function is $F(\xi) = \langle \xi, \xi \rangle$.

More about the almost-Poisson setting for the nonholonomic systems can be found in [13, 14].

4 Chains of subalgebras

Suppose we are given a chain of connected compact subgroups:

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}$$

and the corresponding chain of subalgebras:

$$G_0 \subset G_1 \subset \ldots \subset G_n = G.$$  

Let $G_i = G_{i-1} + V_i$ be the orthogonal decompositions. Then

$$G_i = G_0 + V_1 + \ldots + V_i.$$  

Following [4, 14], consider $A$ of the form:

$$A = A_0 + s_1 \cdot Id_{V_1} + \ldots + s_n \cdot Id_{V_n}, \quad s_i > 0, \quad i = 1, \ldots, n,$$

(16)

where $A_0$ is a symmetric positive operator defined in the subalgebra $G_0$.

Suppose that $D$ has orthogonal decomposition:

$$D = D_0 + D_1 + \ldots + D_n,$$

(17)

$$D_k = \{\omega_k \in V_k, \langle a_k^i, \omega_k \rangle = 0, \quad i = 1, \ldots, \rho_k\}.$$  

Then $D_k, k > 0$ are invariant subspaces of $A$. By $x_k$ denote the orthogonal projection of $x$ to $D_k$, $k > 0$; and by $x_0$ denote the orthogonal projection to $G_0$.  

9
Theorem 4.1 The Euler–Poincaré–Suslov equations (4), with $D$ and operator $A(x)$ of the form (14) and (17), are equivalent to the Euler–Poincaré–Suslov equations on the Lie subalgebra $G_0$:

$$
\dot{x}_0 = [x_0, A_0(x_0)] + \sum_{i=1}^{\rho_0} \mu_i a_0^i,
$$

(18)

$$
\langle A_0(x_0), a_0^i \rangle = 0, \quad i = 1, \ldots, \rho_0,
$$

together with a chain of linear differential equations on the subspaces $D_k$:

$$
\dot{x}_k = [x_k, A_0(x_0) - s_k x_0 + (s_1 - s_k)x_1 + \ldots + (s_{k-1} - s_k)x_{k-1}]_k.
$$

(19)

Proof. Let $x_{V_k}$ be the orthogonal projection of $x$ to $V_k$. The simple computations show:

$$
[x, A(x)] = [x_0 + x_{V_1} + \ldots + x_{V_n}, A_0(x_0) + s_1 x_{V_1} + \ldots + s_n x_{V_n}]
$$

$$
= [x_0, A_0(x_0)] + [x_{V_1}, A_0(x_0) - s_1 x_0 + \ldots +
$$

$$
+ [x_{V_n}, A_0(x_0) - s_n x_0 + (s_1 - s_n)x_{V_1} + \ldots + (s_{n-1} - s_n)x_{V_{n-1}}].
$$

(20)

Taking into account (17), (20) and relations $[G_{k-1}, V_k] \subset V_k$, $k = 1, \ldots, n$, after orthogonal projection of (14) to the linear space $G_0 + D_1 + \ldots + D_n$ the theorem follows.

Remark 4.1 Let $D_k = V_k$ and let $F_k$ be the algebra of polynomials that are constant on the orbits of the adjoint action of the group $G_{k-1}$ on $V_k$. It can be easily seen that they are integrals of (14). The number of functionally independent polynomials in $F_k$ is equal to:

$$
\dim V_k - \dim G_{k-1} + \min_{x_k \in V_k} \dim \{y \in G_{k-1}, [y, x_{k-1}] = 0\}.
$$

If Euler–Poincaré–Suslov equations (18) on $G_0$ are solvable then the integration of original equations (1) reduced to successive integration of the chain of linear dynamical systems (19) for $k > 0$.

The most simplest case is when the solutions of (14) are $x_0 = const$. Then the vector $x_1$ satisfied a linear equation with constant coefficient and it is a elementary functions of the time $t$. This is happened if $A_0 = Id_{G_0}$ or if $G_0$ is a commutative subalgebra (see also example 3.1) In particular if $\dim D_0 = 0$ then we have that $x_0 = 0$. In that case $\dot{x}_1 = 0$ and $x_2$ is a elementary functions of the time $t$. This case will be treated again in the last section (example 6.1).

Example 4.1 In addition, we can obtain exactly solvable cases without an invariant measure. For instance, take the chain:

$$
so(3) \subset so(4) \ldots \subset so(n).
$$

Let $D_0 = \{\omega_0 \in so(3), \langle \omega_0, a_0 \rangle = 0\}$. If $a_0$ is not an eigenvector of the $A_0$ then equations:

$$
\dot{x}_0 = [x_0, A_0(x_0)] + \lambda a_0, \quad \langle A(x_0), a_0 \rangle = 0,
$$

(21)
have no invariant measure \[12\]. After identification \((so(3), [\cdot, \cdot]) = (\mathbb{R}^3, \times)\), the system \[12\] describe the rotation of a rigid body fixed at a point and subject to the constraint: the angular velocity \(\vec{\omega}_0\) is orthogonal to the fixed vector in body coordinates \(\vec{a}_0\). This nonholonomic problem was solved by Suslov \[18\].

The phase space \(M_0 = J_0(D_0)\) has the following property: there is an asymptotic line \(l\) such that \(\lim_{t \to \pm\infty} P_t^0(\Omega_0) \subset l\), where \(P_t^0\) is a phase flow on \(M_0\) and \(\Omega_0\) is any subset of \(M_0\). Then the phase flow \(P^t\) of \((18), (19)\) has asymptotic hyper-plane and also has no invariant measure.

However, for \(\dim D_k \leq 2\) we can easily integrated corresponding systems \((19)\). Suppose that we know \(y_{k-1}(t) = A_0(x_0(t)) - s_k x_0(t) + (s_1 - s_k)x_1(t) + \ldots + (s_{k-1} - s_k)x_{k-1}(t)\) as a function of time. If \(\dim D_k = 1\) then \(x_k = \text{const}\).

For \(\dim D_k = 2\), let \(\phi_k\) be the angular variable of the circle \(I_k = \langle x_k, x_k \rangle = c_k\). Then it can easily be checked that equations \(\dot{x}_k = [x_k, y_{k-1}(t)]_k\) get the form \(\dot{\phi}_k = f_k(t)\), where \(f_k(t)\) is some known function of time. Note that connected components of invariant submanifolds \(H = h, I_1 = c_1, \ldots, I_{n-3} = c_{n-3}\) are diffeomorphic to tori, but the dynamic on tori is quite different from \((13)\).

5 Reconstruction

The integrability of Euler–Poincaré equations \(\dot{x} = [x, A(x)]\) implies non-commutative integrability of unconstrained geodesic flow — the phase space \(T\mathfrak{g}\) is foliated on \(d \leq \dim \mathfrak{g}\) dimensional invariant isotropic tori with quasi-periodic dynamics (see \[14\]). In the nonholonomic case there is no Poisson structure. In order to precisely describe dynamic on the whole phase space \(\mathcal{M}\) we have to solve kinematic equation \(g^{-1}(t) \cdot \dot{g}(t) = \omega(t) = A(x(t))\). This problem, for the Fedorov–Kozlov integrable case, is studied by Zenkov and Bloch \[24, 25\]. Similar analyses can be applied to the considered integrable cases:

(i) The periodic solutions on \(M\) correspond to quasi-periodic motions on the phase space \(\mathcal{M}\).

(ii) Suppose that in the corollary 3.1 we have \(c_k = \epsilon C_k, k = 1, \ldots, g\), where \(\epsilon\) is a small parameter, and that the following Diophantine conditions hold:

\[
|l + i(k_1 \omega_1 + \ldots + k_g \omega_g)| \geq c(|k_1| + \ldots + |k_g|)^{-\gamma},
\]

\[
l = 0, 1, 2, \quad (k_1, \ldots, k_g) \in \mathbb{Z}^g - \{0\},
\]

for some constants \(c > 0\) and \(\gamma > g - 1\). Then the reconstruction of quasi-periodic motion \((13)\) to \(\mathcal{D} = \mathcal{M}\) can be approximated by quasi-periodic dynamics on the time interval of length \(\sim \exp(1/\epsilon)\).

6 Hamiltonian cases

In some cases, the nonholonomic geodesic flow \(\mathcal{D}\) on \(\mathcal{M}\) can be gotten as a restriction of Hamiltonian flow from \(T^*Q\) to submanifold \(\mathcal{M}\). We shall say that nonholonomic geodesic flow \(\mathcal{D}\) has a Hamiltonian restriction property if there
is a function $h^*: T^*Q \to \mathbb{R}$ such that the restriction of Hamiltonian equation $\dot{x} = X_{h^*}$ to $M$ coincide with (3).

The basic examples are when Lagrange multipliers in (3) vanish. Then we can take $h^*(p,q) = h(p,q) = \frac{1}{2}p(g_\alpha^{-1}p)$. However, we shall see that there is another natural choice of the Hamiltonian $h^*$. Note that the cases 2 and 4 of example 3.1 have Hamiltonian restriction property with Lagrange multipliers (in general) different from zero.

From now on, we shall use the notation of section 3.

Let $H(x) = \frac{1}{2}\langle x, A(x) \rangle$, $x \in G$ be the reduced Hamiltonian of the left-invariant geodesic flow such that $A = A_L + A_D$ preserve orthogonal decomposition $G = L + D$. Then $M = I(D) = D$ and Euler–Poincaré–Suslov equations become:

$$\dot{\xi} = [\xi, A_D(\xi)]_D, \quad \xi \in D. \quad (22)$$

Suppose that $L$ is a Lie algebra of some connected Lie subgroup $\mathcal{L}$. Then $[L,L] \subset L$, $[L,D] \subset D$. Further, suppose that $H_D(\xi) = \frac{1}{2}\langle \xi, A_D(\xi) \rangle$ is an invariant of adjoint action of $\mathcal{L}$ on $D$. Recall that function $F : D \to \mathbb{R}$ is $Ad_{\mathcal{L}}$ invariant if it satisfy equation:

$$[\xi, \nabla F(\xi)]_L = 0, \quad \xi \in D. \quad (23)$$

Let $x = \xi + \eta$ be orthogonal decomposition of $x \in G$, such that $\xi \in D$ and $\eta \in L$. Let $A^*(x) = A^*(\xi + \eta) = A_D(\xi)$.

Define the function $H^*: G \to \mathbb{R}$ by:

$$H^*(x) = \frac{1}{2}\langle x, A^*(x) \rangle \quad (24)$$

Using $Ad_{\mathcal{L}}$ invariance of $H_D(\xi)$, we are getting that the Euler–Poincare equations for functions $H(x)$ and $H^*(x)$ leave the plain $D$ invariant and on $D$ coincide with Euler–Poincaré–Suslov equations (22). For example, the system

$$\dot{x} = [x, \nabla H^*(x)] = [x, A^*(x)] \quad (25)$$

after projection to the $L$ and $D$ becomes:

$$\dot{\xi} = [\xi, A_D(\xi)] + [\eta, A_D(\xi)], \quad (26)$$

$$\dot{\eta} = 0.$$

Let $h, h^*: T\mathfrak{g} \to \mathbb{R}$ be the functions obtained by left translations from $H$ and $H^*$. The nonholonomic geodesic flow has a Hamiltonian restriction property with functions $h$ and $h^*$.

The Hamiltonian function $h^*$ has a nice geometrical meaning. Suppose that $D$ generate Lie algebra $G$ by commutations. Then the distribution $\mathcal{D}$ is completely nonholonomic or bracket generating. The Hamiltonian flow $X_{h^*}$ on $T\mathfrak{g}$ is normal sub-Riemannian geodesic flow of the sub-Riemannian metric induced by restriction of the given left-invariant metric to $\mathcal{D}$ (for more details see [4, 13]).

We can summarize previous considerations in the following theorem:
Theorem 6.1 Suppose that constrained reduced Hamiltonian function $H_D(\xi) = \frac{1}{2} \langle \xi, A_D(\xi) \rangle$ is an invariant of adjoint action of the Lie subgroup $L$ on $D$ and that $D$ generate Lie algebra $G$ by commutations. Then on the constrained submanifold $D$ the following three different problems have the same flow: nonholonomic geodesic flow, geodesic flow with Hamiltonian $h$ and sub-Riemannian geodesic flow with Hamiltonian $h^*$. 

Recall that equations (25) are completely integrable if they posses a complete involutive set of integrals. $F = \{ F_1, \ldots, F_k \}$ is complete involutive set of functions on $G$ if $k = \frac{1}{2} (\dim G + \text{rank } G)$, $dF_1 \wedge \ldots \wedge dF_k \neq 0$ on an open dense set $U$ of $G$ and functions in $F$ are in involution with respect to the Lie-Poisson brackets:

$$\{ F_i, F_j \}(x) = \langle x, [\nabla F_j(x), \nabla F_i(x)] \rangle = 0,$$

for all $x \in G$.

If the equations (25) are completely integrable then the set $U$ is foliated on invariant $(\dim G - k)$-dimensional tori with quasi-periodic dynamic (see [1, 21]). Also, the system $\dot{x} = X_h$ is non-commutatively integrable: $T\mathfrak{g}$ is almost everywhere foliated on invariant isotropic tori (see [15]).

Remark 6.1 Integrability of (25) do not implies integrability of the Euler–Poincaré–Suslov equations (22). As an example, it could be happened that $U \cap D = \emptyset$. If $U \cap D$ is an open dense set of $D$, then the Euler–Poincaré–Suslov equations (22) are also integrable. In this case, the reconstruction of the flow on the whole phase space $D = M$ follows from the reconstruction of the flow of Euler–Poincaré equations: the phase space $D$ is foliated on invariant tori with quasi-periodic dynamic.

Example 6.1 Suppose we are given a chain of connected subgroups:

$$L = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}$$

and the corresponding chain of subalgebras:

$$L = G_0 \subset G_1 \subset \ldots \subset G_n = G.$$  \hspace{1cm} (27)

Let $G_i = G_{i-1} + V_i$ be the orthogonal decompositions. Then $D = V_1 + \ldots + V_n$. If we define the left-invariant metric by $A$ of the form:

$$A = A_L + A_D, \quad A_D = s_1 \cdot Id_{V_1} + \ldots + s_n \cdot Id_{V_n}, \quad s_i > 0, \quad i = 1, \ldots, n,$$  \hspace{1cm} (28)

then the function $H_D(\xi)$ will be an invariant of adjoint action of $L$ on $D$. So nonholonomic geodesic flow has a Hamiltonian restriction property.

Suppose that either $(G_i, G_{i-1})$ is a symmetric pair or $V_i$ is a subalgebra of $G$ for all $i = 1, \ldots, n$. Let $F_0$ be arbitrary complete commutative set of functions on the Lie algebra $L$ (see constructions in [21]) lifted to the functions on $G$. Similarly, if $V_i$ is a Lie subalgebra, let $F_i$ be a complete commutative set of
functions on $V_i$ lifted to the functions on $G$. Otherwise (i.e., if $(G_i, G_{i-1})$ is a symmetric pair) we take $\mathcal{F}_i$ to be given by:

$$\mathcal{F}_i = \{ f : G \to \mathbb{R}, f(x) = p(x_{G_{i-1}} + \lambda x_{V_i}), \lambda \in \mathbb{R}, p \in \mathcal{I}(G_i) \},$$

where $\mathcal{I}(G_i)$ is the algebra of $Ad_{\phi_i}$ invariant polynomials on $G_i$.

Then it follows from Mikityuk’s results (see [14]) that Euler–Poincaré equations \([23]\) will be completely integrable. The complete commutative set of integrals is $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 + \ldots + \mathcal{F}_n$.

For example, let us consider the chains:

$$L = so(k) \subset so(k+1) \subset \ldots \subset so(n),$$

$$L = u(k) \subset u(k) + u(1) \subset \ldots \subset u(n-1) + u(1) \subset u(n),$$

$$L = sp(k) \subset sp(k) + sp(1) \subset \ldots \subset sp(n-1) + sp(1) \subset sp(n).$$

Corresponding distributions are completely nonholonomic. Whence we get integrability of sub-Riemannian geodesic flows on Lie groups $SO(n)$, $U(n)$, $Sp(n)$ with left-invariant sub-Riemannian metrics defined above.

**Example 6.2** Let the Lie subgroup $\mathfrak{L}$ be commutative. Then the Lie algebra $L$ is contained in some maximal commutative subalgebra $K \subset G$. Let $D$, $U$ be the orthogonal complements of $L$ in $G$ and $K$ respectively. Further, let $D$ generate $G$ by commutations.

Let $a$ be a regular element of $K$. Then we have that $K = \{ \eta \in G, [\eta, a] = 0 \}$. Let $b$ belongs to $K$ and let $R : K \to K$ be symmetric operator which preserve decomposition $K = L + U$. By $\varphi_{a,b,R}$ denote operator (so called sectional operator \([21]\)) defined with respect to the orthogonal decomposition $G = K + [a, K]$:

$$\varphi_{a,b,R}|_{K} = R, \quad \varphi_{a,b,R}|_{[a,K]} = ad_{a}^{-1}ad_{b}.$$  

For compact groups, among sectional operators there are positive definite. Take such $\varphi_{a,b,R}$. Let $H(x) = \frac{1}{2} \langle x, \varphi_{a,b,R}(x) \rangle$. It can be proved that $H_{D}(\xi) = \frac{1}{2} \langle \xi, \varphi_{a,b,R}|_{D}\rangle(\xi)$ is an invariant of adjoint action of the Lie subgroup $\mathfrak{L}$ on $D$. Thus, nonholonomic geodesic flow has a Hamiltonian restriction property.

The function $H^*$ is of the form: $H^*(x) = \frac{1}{2} \langle x, \varphi_{a,b,R^*}(x) \rangle$, where $R^* : K \to K$ has a kernel equal to $L$. For general $a \in K$, the Euler–Poincaré equations \([23]\) are completely integrable both on $G$ and on $D$. The integrals can be obtained by Mishchenko–Fomenko method, based on the shifting of argument of invariant polynomials (for details see \([21]\)). Thus corresponding nonholonomic and sub-Riemannian geodesic flows are integrable.

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