Gluon propagator and confinement scenario in Coulomb gauge

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We present numerical results in SU(2) lattice gauge theory for the instantaneous part of the gluon propagator in Coulomb gauge $D_{44,\text{inst}} = V_{\text{coul}}(R) \delta(t)$. Data are taken on lattice volumes $24^4$ and $28^4$ for 7 values of $\beta$ in the interval $2.2 \leq \beta \leq 2.8$. The data are confronted with the confinement scenario in Coulomb gauge. They are consistent with a linearly rising color-Coulomb potential $V_{\text{coul}}(R)$.

1. CONFINEMENT SCENARIO IN COULOMB GAUGE

A particularly simple confinement scenario \cite{1} is available in the minimal Coulomb gauge. It attributes confinement of color to the long range of the color-Coulomb potential $V_{\text{coul}}(R)$. This quantity is the instantaneous part of the 4-4 component of the gluon propagator, $D_{\mu\nu}(x) \equiv \langle gA_{\mu}(x) gA_{\nu}(0) \rangle$, namely $D_{44}(x,t) = V_{\text{coul}}(|x|) \delta(t) + P(x,t)$. The vacuum polarization term $P(x,t)$ is less singular than $\delta(t)$ at $t = 0$. Since $A_4$ couples universally to color charge, the long range of $V_{\text{coul}}(R)$ suffices to confine all color charge. It was conjectured that it is linearly rising at large $R$, $V_{\text{coul}}(R) \sim -\sigma_{\text{coul}} R$. If an external quark-antiquark pair is present, the physical potential $V_W(R)$ between them may be extracted from a Wilson loop. The color-Coulomb potential contributes the term $-CV_{\text{coul}}(R)$ directly to the Wilson loop, where $C = (N^2 - 1)/(2N)$ in SU(N) gauge theory with external quarks in the fundamental representation. The minus sign occurs because the antiquark has opposite charge to the quark. The vacuum-polarization term is screening, and one expects that $V_W(R)$ is bounded above by this term asymptotically at large $R$, $V_W(R) \leq -CV_{\text{coul}}(R)$. If $V_W(R)$ is also linearly rising, $V_W(R) \sim \sigma R$, where $\sigma$ is the conventional string tension, we get $\sigma \leq C\sigma_{\text{coul}}$. If dynamical quarks are present, the string “breaks” at some radius $R_b$, and the conventional asymptotic string tension vanishes, $\sigma = 0$. String-breaking is easily explained in the Coulomb-gauge confinement scenario if $V_{\text{coul}}(R)$ is linearly rising even in the presence of dynamical quarks, as was also conjectured \cite{1}. For, if so, it is energetically preferable to polarize a pair of sea quarks from the vacuum.

Here we report the confrontation of this scenario with the numerical data for $V_{\text{coul}}(R)$ for the case of pure gluodynamics.

1.1. Relation of $V_{\text{coul}}$ to $\alpha_s$

We may identify $V_{\text{coul}}(R)$ with the phenomenological potential that is the starting point for QCD bound state calculations \cite{2}. The identification of a phenomenological potential with the instantaneous part of the gluon propagator, a fundamental quantity in the gauge theory, is possible because, remarkably, $V_{\text{coul}}(R)$ is a renormalization-group invariant, and thus scheme-independent, so it is independent of the cut-off $\Lambda$ and of the renormalization mass $\mu$.

This follows from the non-renormalization of $gA_4$, as expressed by the identity $g_{(0)} A_4^{(0)} = g_{(r)} A_4^{(r)}$, where 0 and $r$ refer to unrenormalized and renormalized quantities in the Coulomb gauge \cite{3}. This identity has no direct analog in a Lorentz-covariant gauge. Because of the scheme-independence of $V_{\text{coul}}(R)$, its Fourier transform $\tilde{V}_{\text{coul}}(k)$ provides a scheme-independent definition for the running coupling constant of QCD,
Table 1  
Values of fitting parameters from formula (1).

| $\beta$ | $A$         | $B$       | $A$ | $W^2$ | $\chi^2$/dof |
|---------|-------------|-----------|-----|-------|-------------|
| 2.2     | 7.38 ± 0.35 | 8.15 ± 15.58 | 8.83 ± 19.34 | 9.91 ± 19.01 | 0.81        |
| 2.3     | 5.81 ± 0.30 | 24.57 ± 83.41 | 32.57 ± 52.59 | 17.26 ± 27.18 | 0.72        |
| 2.4     | 6.60 ± 0.64 | 6.72 ± 10.15 | 7.70 ± 15.90 | 3.99 ± 6.03  | 0.20        |
| 2.5     | 7.45 ± 0.47 | 6.00 ± 2.65 | 6.16 ± 2.58 | 3.06 ± 1.26  | 0.05        |
| 2.6     | 8.23 ± 1.48 | 8.61 ± 1.68 | 4.53 ± 1.93 | 3.84 ± 0.34  | 0.07        |
| 2.7     | 11.30 ± 0.81 | 7.73 ± 0.47 | 6.87 ± 0.67 | 3.64 ± 0.15  | 0.07        |
| 2.8     | 8.18 ± 2.85 | 10.38 ± 0.46 | 5.58 ± 0.08 | 0.06 ± 0.05  | 0.19        |

$k^2\tilde{V}(k) = x_0 g_{\text{coul}}^2(|k|)$, and of $\alpha_s \equiv \frac{g^2(k/\Lambda_{\text{coul}})}{4\pi}$. Here $x_0 = \frac{12N_c}{12N_c - 2n_f}$ and $\Lambda_{\text{coul}}$ is a finite QCD mass scale.

2. NUMERICAL STUDY OF $V_{\text{coul}}$

2.1. Method

We have previously studied 4 both space- and time-components of the gluon propagator, $D_{ij}(k)$ and $D_{44}(k) = \tilde{V}_{\text{coul}}(k)$ at equal time, at $\beta = 2.2$, on various lattice volumes, $14^4$ to $30^4$, in the minimal Coulomb gauge. This gauge is achieved numerically by (i) maximizing $\sum_{x, i=1}^{N} \text{Tr} U_{x, i}$ with respect to all local gauge transformations $g(x)$, where the sum is on all horizontal or spatial links, and then (ii) maximizing $\sum_x \text{Tr} U_{x, 4}$ with respect to all $x$-independent but $x_4$-dependent gauge transformations $g(x_4)$ where the sum is on all vertical or time-like links. This makes the 3-vector potential $A_i$, for $i = 1, 2, 3$ transverse, $\partial_t A_i = 0$, so $A_i = A_i^\tau$.

In this gauge, the horizontal link variables $U_{x, i}$, for $i = 1, 2, 3$ are as close to the identity as possible, but the vertical variables $U_{x, 4}$ are much further from the identity. Not surprisingly, we found that, whereas $D_{ij}^0(k)$ gave values that could be reasonably extrapolated to the continuum, this was not true for $D_{44}(k)$. In order to remedy this, in the present study we extended our investigation of $D_{44}(k)$ to $\beta = 2.2, 2.3, \ldots, 2.8$, on lattice volumes $24^4$ and $28^4$. We have also determined $V_{\text{coul}}(R)$ numerically by 2 quite different methods. Method I relies on the standard formula $D_{44}(x, t) = V_{\text{coul}}(|x|) \delta(t) + P(x, t)$. Method II relies on the lattice analog of the continuum formula that is obtained 3 by integrating out $A_4$, namely, $V(|x - y|) = \langle (M^{-1}(-\nabla^2)M^{-1})|_{x, y} \rangle$. Here $M(A) = -\nabla \cdot D(A)$ is the 3-dimensional Faddeev-Popov operator, and $D(A) = \nabla + \nabla \times \Lambda$ is the gauge-covariant derivative. Method II requires numerically inverting the lattice Faddeev-Popov matrix to obtain $M^{-1}(A)_{x, y}$, but it has the advantage that $D_{44}(k)$ is expressed entirely in terms of the spatial link variables $U_{x, i}$ for $i = 1, 2, 3$, that are close to the identity. Moreover, method II involves only the horizontal link variables that lie within a single time slice, so it is independent of the gauge fixing (ii) on vertical links. Thus it measures a truly instantaneous quantity. We found that method I did not exhibit scaling in the above range of $\beta$, and we report here only the result of method II.

2.2. Results

For each $\beta$ we have 100 configurations on volume $V = 24^4$ and 50 configurations on volume $V = 28^4$. Runs have been done on the PC clusters at the Physics Department of New York University and at the IFSC of São Paulo University.

In figure 1 we plotted the results for $\beta = 2.2, 2.5,$ and 2.8 respectively. The horizontal axis measures $k^2 = 4a^{-2} \sin^2(n\pi/L)$, for $L = 24, 28$, rescaled to physical units by setting the physical string tension equal to $\sigma = (0.44\text{GeV})^2$ and using Table 3 of 3, so the lattice spacing at $\beta = 2.2, \ldots, 2.8$ is (in $\text{GeV}^{-1}$) $a = 1.066, 0.839, 0.605, 0.433, 0.309, 0.231, 0.165$, respectively. The vertical axis measures $|k|^2\tilde{V}_{\text{coul}}(k)$ in phys-
ical units, so string tension may be read off from the vertical intercept (see below). Finite-volume artifacts are clearly visible at low momentum. To control these, in our fits we have dropped those low-momentum points for which appreciably different values are obtained at volumes $24^4$ and $28^4$ and, for the fit, we used only data points obtained at $V = 28^4$.

A simple parametrization of $\tilde{V}_{\text{coul}}(R)$ would be $-\tilde{V}_{\text{coul}}(R) = \sigma_{\text{coul}} R - c/R$, which has the Fourier transform $\tilde{V}_{\text{coul}}(k) = 8\pi \sigma_{\text{coul}} / |k|^4 + 4\pi c / |k|^2$. We have used the fitting formulas

$$|k|^4 \tilde{V}_{\text{coul}}(k) = A + \frac{Bk^2}{W^2 + \ln(1 + k^2/A^2)}$$  

(1)

$$|k|^4 \tilde{V}_{\text{coul}}(k) = A + Bk^2$$  

(2)

$$|k|^6 \tilde{V}_{\text{coul}}(k) = A|k|^6 + Bk^2.$$  

(3)

Fit (1) has the asymptotic behavior at large $k$, consistent with the 1-loop $\beta$-function. We report in Table 1 the values of the parameters from fit (1).

The most striking feature of the data is the finite intercepts $A$. This is consistent with a finite string tension $\sigma_{\text{coul}} = A/(8\pi)$. It scales rather nicely, varying from $A = 7.38 \pm 0.35$ GeV$^2$ at $\beta = 2.2$ to $A = 8.18 \pm 2.85$ GeV$^2$ at $\beta = 2.8$, but with considerable variation in between. From the lowest and highest values, $A = 5.8$ and $A = 11.3$, we get respectively $\sigma_{\text{coul}} = (0.48\text{GeV})^2$ and $\sigma_{\text{coul}} = (0.67\text{GeV})^2$. The inequality $\sigma \leq (3/4)\sigma_{\text{coul}}$ for SU(2), with $\sigma = (0.44\text{GeV})^2$, reads for these values: $0.44 \leq 0.42(2)$ and $0.44 \leq 0.58(3)$. We appear to be at or near saturation.

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Figure 1. Fit of $|k|^4 \tilde{V}_{\text{coul}}(k)$ using eq. (1). Data are for lattice volumes $V = 24^4$ (×) and $V = 28^4$ (∗) and $\beta = 2.2, 2.5$ and 2.8 from top to bottom.