Conserved Quantities
in Perturbed Inflationary Universes

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Abstract

Given that observations seem to favour a $\Omega_0 < 1$, corresponding to an open universe, we consider gauge-invariant perturbations of non-flat Robertson-Walker universes filled with a general imperfect fluid which can also be taken to represent a scalar field. Our aim is to set up the equations that govern the evolution of the density perturbations $\Delta$ so that it can be determined through a first order differential equation with a quantity $K$ which is conserved at any length scale, even in non-flat universe models, acting as a source term. The quantity $K$ generalizes other variables that are conserved in specific cases (for example at large scales in a flat universe) and is useful to connect different epochs in the evolution of density perturbations via a transfer function. We show that the problem of finding a conserved $K$ can be reduced to determining two auxiliary variables $X$ and $Y$, and illustrate the method with two simple examples.

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1 Introduction

Despite much theoretical progress made in the last fifteen years, the problem of large-scale structure formation in the universe remains an unresolved puzzle for cosmologists. In fact, as new and more reliable data becomes available, none of the models proposed during this time seem satisfactory.

In the standard Hot Big Bang model \[1\], the most popular mechanism advocated to explain structure formation is that of the gravitational instability: small perturbations created during an inflationary era eventually grow in the matter dominated era. It is therefore necessary, in order to use observational data to constrain theoretical models, to relate the original amplitude of a perturbation mode at an initial time to that of the same mode today, i.e. to derive a transfer function (cf. \[2, 3\]).

In cosmological perturbation theory the evolution of a perturbation mode is usually given by a second order ordinary differential equation (e.g. see \[4, 5, 6, 2\]). It is therefore very useful to set up the problem in such a way that, at least in the simplest cases, one has a conserved quantity at ones disposal, so that the problem is effectively reduced to solving a first order differential equation with a constant source term which, at least in principle, can always be integrated.

Indeed, such conserved quantities can be found in flat (see \[7, 8, 9\] and \[10, 11, 12, 2\]) and almost flat \[13, 11\]) universe model, for a perturbation mode whose wavelength is much larger than the Hubble radius. This is quite satisfactory in the standard inflationary scenario \[14\], where the present day density parameter \(\Omega_0\) is predicted to be equal to unity with a high degree of accuracy. However, there is no compelling observational evidence for the case of \(\Omega_0 = 1\); rather, a low density universe seems to be favoured by analysis of present day observational data \[15\].

For this reason, and taking into account the appeal of the inflationary paradigms and the fact that this is up to now the only scenario in which perturbations are predicted (by quantum fluctuations within the horizon) rather than assumed as initial data, some authors \[16, 17, 18, 19\] have considered the possibility of a era of “minimal” inflation, in which the accelerated expansion does not last long enough to drive \(\Omega_0\) to unity today. This possibility can be a source of a certain amount of debate \[20\], raising on one hand arguments of fine tuning (which however is in one way or another needed for other reasons in most of the inflationary models proposed up to now), and on the other hand, “how to put a measure on the set of initial data” (lacking this measure, according to this point of view, the fine tuning argument is a weak one).

In this paper we do not enter in such a debate, instead we take the case of a negatively curved universe as an open possibility, and examine perturbations in this context. For the sake of generality, we shall consider an imperfect fluid, which later we shall take to represent a perturbed minimally coupled scalar field.

Our aim is to set up the equations that govern the evolution of the density perturbation mode \(\Delta^{(\nu)}\) so that it can be determined by a first order equation with a conserved quantity \(K^{(\nu)}\) as a source term. In other words, we define a simple algorithm that in principle allows us to find a new quantity \(K^{(\nu)}\) which is conserved at any scale, even in non-flat models. Then \(K\) generalizes known conserved quantities, and allows one to define a transfer function in a more general set of circumstances.
As it will be shown, the problem of finding the quantity $K^{(\nu)}$ in the cases of: i) adiabatic perturbations in a perfect fluid or ii) perturbations in a minimally coupled scalar field, reduces to determine two auxiliary functions $X$ and $Y$.

It is noticeable that a completely consistent theory of large scale Cosmic Microwave Background (CMB) fluctuations in open models seems to be lacking until now [21]. With the cautionary note that even the concept of power spectrum has not yet been generalized to include perturbations over super horizon scales in open universe models, we hope that the work presented here is a step towards a consistent and useful treatment of fluctuations in $\Omega_0 < 1$ universes.

This paper is organized as follows: In section 2 we discuss very briefly the formalism and variables used. In section 2 and 3 we define two basic curvature variables and discuss under what conditions they are conserved both in the case of an imperfect fluid and a minimally coupled scalar field. In section 4 we give a prescription for finding quantities that are conserved at all scales for any given background evolution. In section 5 we apply our algorithm in two simple cases: i) a coasting ($\Omega = \text{const}$) phase and ii) a De-Sitter inflationary phase. In section 6 we briefly summarize how to use our algorithm in order to find a transfer function. Finally in section 7 we end with a discussion.

In this paper we assume standard general relativity, with Einstein equations in the form $R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = \kappa T_{ab}$, $\kappa$ and $\Lambda$ being the gravitational and cosmological constants respectively.

2 Formalism

For the sake of self-consistency, we sketch in this section the covariant approach to density inhomogeneities in a general curved spacetime [22]. This as been applied to almost Friedmann-Lemaître-Robertson-Walker (FLRW) universes dominated by a barotropic fluid [23, 9], as well as by a simple scalar field [13, 12], a general imperfect fluid [10, 6] and a mixture of interacting fluids [24, 25]. Here we just give an outline of the material which will be needed later on, with emphasis on known conserved quantities.

2.1 Covariant cosmology

Let us consider an imperfect fluid flow in a general curved space-time. The 4-velocity of the fluid is the tangent along the fluid flow: $u^a = dx^a/d\tau$ ($u^a u_a = -1$), where $\tau$ is the proper time along the flow lines (the world lines of observers comoving with the fluid). Introducing a projection tensor orthogonal to $u^a$, $h_{ab} = g_{ab} + u_a u_b$ ($h^b_a u_b = 0$) the energy-momentum tensor is decomposed as follows:

\[
T_{ab} = \mu u_a u_b + p h_{ab} + 2q_a u_b + \pi_{ab} ,
\]

where $\mu$ and $p$ are the energy density and pressure of the fluid, and $q_a$ and $\pi_{ab}$ are the energy flux and anisotropic pressure contributions. When the fluid is not perfect (i.e. in it’s perturbed state) the fluid 4-velocity $u^a$ is not uniquely defined [6, 25]; it is however standard to choose $u^a$ either as the particle frame, in which $T_{ab}$ has the general form (1), or as the energy frame, in which $q_a = 0$ (for a discussion of these choices, see [6]). Here we shall take the particle frame choice, as this allows one also to describe more general
cases for example a non-minimally coupled scalar field and higher order gravity theories [27].

The covariant derivative of $u_a$ can be split into four parts:

$$u_{a;b} = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab} - a_a u_b ,$$

(2)

where $\Theta \equiv u^a_{a}$ is volume the expansion, $\sigma_{ab} = \sigma_{(ab)}$ is the shear ($\sigma_{ab} u^b = \sigma^a_{a} = 0$), $\omega_{ab} = \omega_{[ab]}$ is the vorticity ($\omega_{ab} u^b = 0$) and $a_a = \dot{u}_a = u_{a;b} u^b$ is the acceleration (the dot denotes the proper time derivative). It is useful to introduce a length scale factor along flow lines by the relation

$$\frac{1}{3} \Theta = \frac{\dot{\ell}}{\ell} = H ;$$

(3)

when the universe is an exact FLRW space-time $H$ is just the usual Hubble parameter. The evolution equation for the expansion $\Theta$ is the Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + 2(\sigma^2 - \omega^2) - a^a_{a} + \frac{1}{2} \kappa (\mu + 3p) - \Lambda = 0 ,$$

(4)

where $\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab}$ and $\omega^2 = \frac{1}{2} \omega_{ab} \omega^{ab}$ are the shear and vorticity magnitudes.

The matter equations of motion $T_{ab}^{\cdot} = 0$ are equivalent to the energy conservation and momentum conservation equations: retaining only first order contributions in the inhomogeneity variables (the reason for this will be clarified in the next section) we have:

$$\dot{\mu} + 3hH + h (3) \nabla_a \Psi^a = 0 ,$$

(5)

and

$$ha_a + Y_a + h [F_a + \Pi_a] = 0 ,$$

(6)

where for later convenience we define

$$h = (\mu + p) , \quad \Psi_a = \frac{g_a}{h} , \quad F_a = \Psi_a - \left( 3c^2 - 1 \right) H \Psi_a , \quad \Pi_a = \frac{1}{h} (3) \nabla^b \pi_{ab} .$$

(7)

An exact FLRW can be covariantly characterized by the vanishing of the shear and the vorticity of $u^a$ and by the vanishing of the spatial gradients (i.e. orthogonal to $u^a$) of any scalar $f$:

$$\sigma = \omega = 0 , \quad (3) \nabla_a f = 0 ;$$

(8)

in particular the gradients of energy density, pressure and expansion

$$X_a \equiv (3) \nabla_a \mu , \quad Y_a \equiv (3) \nabla_a p \quad Z_a \equiv (3) \nabla_a \Theta ,$$

(9)

vanish, where $Y_a = 0 \Rightarrow a_a = 0$. Then $\mu = \mu(t)$, $p = p(t)$ and $\Theta = \Theta(t) = 3H(t)$ depend only on the cosmic time $t$ defined (up to a constant) by the FLRW fluid flow vector through $u_a = -t_a$. The energy momentum tensor (4) necessarily reduces to the perfect fluid form $T_{ab} = \mu u_a u_b + p h_{ab}$, and it follows that these models are completely determined by an equation of state $p = p(\mu)$, the energy conservation equation (5) (with no imperfect fluid contributions) and the Friedmann equations

$$3 \dot{H} + 3H^2 + \frac{1}{2} \kappa (\mu + 3p) - \Lambda = 0 ,$$

(10)

and

$$H^2 + \frac{\kappa}{3} = \frac{1}{3} \kappa \mu + \frac{1}{3} \Lambda ,$$

(11)

where the latter is a first integral of the former (when $H \neq 0$), which is (4) specialized to a FLRW model.
2.2 Fluid inhomogeneities and gauge invariant variables

The vector $Y_a$ is one of a set of variables we can define \cite{22} to describe the fluid inhomogeneity:

$$Y_a \equiv h_{ab} p_b , \quad D_a \equiv \ell \frac{X_a}{\mu} , \quad Z_a \equiv \ell h_a b \Theta ,$$

(12)

these are the covariantly defined spatial (i.e. orthogonal to $u^a$) gradients of the pressure, energy density and expansion; more precisely the dimensionless comoving fractional spatial gradient of the energy density $D_a$ is the physically relevant variable to characterize the energy density spatial variation. All of these covariantly defined exact variables can be in principle measured as spatial gradients by observers at rest with respect to the fluid, and in general (i.e. in an arbitrary spacetime) they obey exact non-linear and covariant evolution equations which have been derived in \cite{22} for a perfect fluid and in \cite{24} for an imperfect fluid; these equations are coupled with the exact evolution equations for $\sigma_{ab}$ and $\omega_{ab}$ (see e.g. \cite{26}).

The above introduced covariant gradients $Z_a$ and $D_a$, and in general any variable such as those in (8) that vanish in the FLRW background, are gauge - invariant (GI) variables (for more details, see \cite{4, 22, 5} and \cite{28}), so we can proceed to obtain equations for them valid in an almost FLRW universe (a space-time where these variables are small), approaching this universe from a general space-time rather than perturbing an exact FLRW model. Then the linearization procedure we apply consists in dropping terms such as $\sigma^2$ in (4), i.e. terms which are of second order in the GI variables, retaining only terms linear in $D_a$ and $Z_a$ and using zero - order expressions for $\mu$, $p$ and $\Theta = 3H$ [this clarifies the meaning of neglecting higher order terms in (5) and (6)].

At zero - order these variables are functions of the cosmic time only, and satisfy the usual equations valid in an exact FLRW universe. Applying this linearization procedure to a universe where the fluid is barotropic [$p = p(\mu)$], one obtains a closed pair of first - order (in proper time derivatives) equations that couple $D_a$ and $Z_a$ only \cite{22}; this is equivalent to a second - order equation for $D_a$ \cite{23} \cite{3}. It was noted in \cite{3} that in general a source term due to vorticity is present in this equation; it also admits an entropy perturbation source term when a more general non barotropic fluid is considered \cite{10}. Finally, we note that our covariant exact variables can be related to those of Bardeen at first order (see \cite{6, 10}). Using this it is then possible to connect our GI perturbation variables to the standard analysis of the spectrum of perturbations and fluctuations in the cosmic microwave background, which are usually given in some specific gauge.

2.3 Conserved quantities

In order to proceed towards the identification of GI conserved quantities, we introduce the 3 - curvature scalar of the projected metric $h_{ab}$ orthogonal to $u^a$; which in general is (see the appendix in \cite{3})

$$(3)R = 2(-\frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 + \kappa \mu + \Lambda) ,$$

(13)

and when $\omega = 0$ it reduces to the 3-curvature of the surfaces orthogonal to $u^a$. From (13) we can define the GI curvature gradient as

$$C_a \equiv \ell^3 h_a b (3)R_b = -\frac{4}{3} \Theta \ell^2 Z_a + 2\kappa \mu \ell^2 D_a ,$$

(14)
where the second equality holds in the linear approximation; then this expression shows that only two of the three variables \(D_a, Z_a\) and \(C_a\) are linearly independent at first order.

Up to now we have considered vectorial GI variables. However we are interested here in characterizing matter clumping through scalar density perturbations: instead of introducing these through the standard non-local (ADM) splitting [28], we simply locally define GI scalar variables by taking the divergence of the vectorial quantities. In particular we have:

\[
\Delta = \ell h^{ab} D_{ab}, \quad Z = \ell h^{ab} Z_{ab}, \quad C = -\frac{4}{3} \Theta \ell^2 Z + 2 \kappa \mu \ell^2 \Delta, \tag{15}
\]

where \(\Delta\) is the locally defined density perturbation scalar variable describing matter clumping (the factor \(\ell\) in these definitions comes in for dimensional reasons). We also introduce the related quantity

\[
\tilde{C} = C - 4K \gamma^{-1} \Delta, \quad \gamma = 1 + w, \quad w = \frac{p}{\mu}, \tag{16}
\]

which reduces to the GI curvature perturbation \(C\) when \(K = 0\) \([K\) is the curvature constant of FLRW models appearing in \([11]\)]. Once a harmonic analysis is carried out, \(\Delta\) is equivalent to the Bardeen variable \(\epsilon_m\) \([10, 6]\) and \(\tilde{C}\) corresponds to the variable \(\zeta\) defined in \([7]\) (see \([11]\)). Again only two of our variables are independent, therefore the \(\Delta\) evolution can be obtained from a linear system of two equations for \(\Delta\) and \(\tilde{C}\) or \(\Delta\) and \(Z\). For a general fluid with speed of sound \(c_s^2 \equiv dp/d\mu\) we obtain \([6]\):

\[
\dot{\Delta} - \left\{3H w - \left[\frac{sh\ell}{\ell^2} - \frac{K}{\ell^2}\right] H^{-1}\right\} \Delta - \left(\frac{1+w}{4\ell^2 H}\right) \tilde{C}
= 3\ell (1+w) H \left[ F + \Pi \right] - \ell (1+w) \nabla^2 \Psi, \tag{17}
\]

\[
\dot{Z} + 2H Z + \frac{1}{2} \kappa \mu \Delta + \frac{c_s^2}{1+w} \left( \left(3\right) \nabla^2 + \frac{3K}{\ell^2} \right) \Delta \frac{w}{1+w} \left( \left(3\right) \nabla^2 + \frac{3K}{\ell^2} \right) \mathcal{E}
= -\ell \left( \left(3\right) \nabla^2 + \frac{3K}{\ell^2} \right) \left[ F + \Pi \right] + \frac{3}{2} h \left[ F + \Pi \right]. \tag{18}
\]

and

\[
\dot{\tilde{C}} = \frac{4c_s^2 H c_s^2}{(1+w)} \left(3\right) \nabla^2 \Delta + \frac{4c_s^2 H w}{(1+w)} \left[ \left(3\right) \nabla^2 + \frac{3K}{\ell^2} \right] \mathcal{E}
+ 4\ell^3 H \left(3\right) \nabla^2 \left[ F + \Pi \right] + \left[ 4K \ell - 2\ell^3 h \right] \left(3\right) \nabla^2 \Psi, \tag{19}
\]

where

\[
\Pi = \ell \left(3\right) \nabla^2 \Pi_a, \quad F = \ell \left(3\right) \nabla^2 F_a, \quad \Psi = \ell \left(3\right) \nabla^2 \Psi_a. \tag{20}
\]

The system of equations \([17, 19]\) admits \(\dot{\gamma} \neq 0\) (when \(\dot{\gamma} = 0\) one has \(c_s^2 = w = \gamma - 1\)) and is derived for general \(K\) and \(\Lambda\); however in the following we shall assume \(\Lambda = 0\).

It is useful at this point to carry out a harmonic expansion of our variables introducing scalar harmonics \(Q^{(k)}\) which satisfy the Helmholtz equation \(\left(3\right) \nabla^2 Q^{(k)} = -k^2/\ell^2 Q^{(k)}\), where \(k\) is a comoving (i.e. constant) eigenvalue. If \(\nu\) is a non-negative real wavenumber
then in a flat $K = 0$ universe, it is associated with the physical wavelengths $\lambda = 2\pi \ell / \sqrt{H}$ since when $K = 0$, $k = \nu$, however if $K = -1$ then $k^2 = \nu^2 + 1$ [29]. Since observations tend to favor $\Omega_0 \leq 1$ [13], we shall now restrict attention to the cases $K = -1, 0$. We consider the equations for $\Delta$ and $\tilde{C}$, since it is the latter variable (and not the GI curvature perturbation $C$) that will be eventually conserved when $\Omega \neq 1$. Writing equation (19) in terms of harmonic components, the evolution equation for $\tilde{C}^{(\nu)}$ becomes:

$$
\dot{\tilde{C}}^{(\nu)} = -\frac{4\ell^2 H^3 c^2}{(1+w)} \left( \frac{\nu^2}{c^2 H^2} - \frac{K}{c^2 H^2} \right) \Delta^{(\nu)} + \frac{4\ell^2 H^3 w}{(1+w)} \left[ \frac{4K}{c^2 H^2} - \frac{\nu^2}{c^2 H^2} \right] \mathcal{E}^{(\nu)}$

$$
- 4\ell^3 H^3 \left( \frac{\nu^2}{c^2 H^2} - \frac{K}{c^2 H^2} \right) \left[ F^{(\nu)} + \Pi^{(\nu)} \right]
$$

$$
- H^2 \left[ 4K \ell - 2\ell^3 h \right] \left( \frac{\nu^2}{c^2 H^2} - \frac{K}{c^2 H^2} \right) \Psi^{(\nu)}.
$$

\text{(21)}

For large scales, i.e. wavelengths $\lambda \gg H^{-1} \Rightarrow \frac{\nu^2}{c^2 H^2} \ll 1$ this equation reduces to

$$
\dot{\tilde{C}}^{(\nu)} = +\frac{4\ell^2 H^3 c^2}{(1+w)} \left( \frac{K}{c^2 H^2} \right) \Delta^{(\nu)} + \frac{4\ell^2 H^3 w}{(1+w)} \left[ \frac{4K}{c^2 H^2} \right] \mathcal{E}^{(\nu)}
$$

$$
+ 4\ell^3 H^3 \left( \frac{K}{c^2 H^2} \right) \left[ F^{(\nu)} + \Pi^{(\nu)} \right]
$$

$$
+ H^2 \left[ 4K \ell - 2\ell^3 h \right] \left( \frac{K}{c^2 H^2} \right) \Psi^{(\nu)}.
$$

\text{(22)}

Let us now examine the consequences of this equation for a number of different cases.

If the background is flat, $K = 0$, we immediately see that $\tilde{C} = C$ is conserved in the large scale limit even when entropy perturbations and imperfect fluid source terms are present. This is an important result, since we can now use this conserved quantity to write down a general solution of the first order equation for $\Delta^{(\nu)}$, (17), in the long wavelength limit:

$$
\Delta^{(\nu)} = \int_{t_0}^{t} \int_{t_1}^{t} e^{A(t_2)} dt_2 \left[ B^{(\nu)}(t_1) dt_1 \right],
$$

\text{(23)}

where

$$
A(t) = 3H w - \frac{\kappa h}{\sqrt{H}},
$$

\text{(24)}

$$
B^{(\nu)}(t) = \frac{(1+w)}{3\sqrt{H}} \tilde{C}^{(\nu)} + 3\ell (1 + w) H \left[ F^{(\nu)} + \Pi^{(\nu)} \right],
$$

\text{(25)}

and $t_0$ is the epoch at which an initial condition for $\Delta^{(\nu)}$ is specified.

The conservation of $\tilde{C}$ is also very important in the study of perturbations in both standard and non-standard inflationary models (based on generalized gravity theories) since it can be used to directly connect the amplitude of present day large scale structure,

\footnote{A completely consistent theory of CMB fluctuations in open models seems to be lacking until now [21], and even the issue of completeness of the set of eigenfunctions of the Laplacian operator seems unresolved.}

\footnote{A brief summary of the needed harmonic functions is given in Appendix B of [3].}

\footnote{The fluid flow approach to perturbations can also be used to study perturbations in generalized gravity theories. The basic idea is to treat all additional contributions to the field equations except the Einstein tensor part as contributions to an effective energy momentum tensor. Effective fluid quantities are then easy to compute and GI quantities based on spatial gradients can be defined as usual (see [30]).}
which came inside the horizon during the matter dominated era, to the initial conditions just after horizon-crossing during the inflationary era.

We now turn to the case when the background is an open ($K = -1$) model. In this case $\tilde{C}$ is conserved if the following inequalities hold:

\[
\frac{\nu^2}{\ell^2 H^2} \ll 1 \quad \text{and} \quad \frac{1}{\ell^2 H^2} \ll 1,
\]

i.e., if the long wavelength limit is valid up to $\nu = 1$, provided that the entropy perturbations and some of the imperfect fluid terms do not exceed $\Delta^{(\nu)}$. However, in an open universe the cosmological density parameter $\Omega$ is given by

\[
\Omega = 1 - \frac{1}{\ell^2 H^2},
\]

so the second inequality only holds if $\Omega$ is close enough to 1, but then $C$ is also approximately conserved.

### 3 Scalar fields

Let us now extend the analysis carried out up to now to a universe dominated by a scalar field $\phi$, as is the case during an inflationary phase; here we give only a brief outline of the method (details can be found in [12]). We assume for simplicity that $\phi$ is minimally coupled (the non-minimally coupled case will be considered elsewhere [31]): then the Lagrangian density of $\phi$ is

\[
\mathcal{L}_\phi = -\sqrt{-g} \left[ \frac{1}{2} \phi_{,a} \phi^{,a} + V(\phi) \right],
\]

where $V(\phi)$ is a general unspecified potential assumed to be responsible for inflation. We shall also assume that $\phi_{,a}$ is a time-like vector [27], and we make the choice

\[
u^a = -\dot{\phi} \phi^{,a}.
\]

The former assumption will be well justified by a continuity argument in an almost FLRW universe, as it clearly holds in an exact FLRW model; the latter choice of $\nu^a$ is uniquely determined by $\phi^{,a}$ on requiring $\nu^a \nu_a = -1$ (note that $\omega_{ab} = 0$ with this choice). Then the energy-momentum tensor of $\phi$ takes the perfect-fluid form (1), with $p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$ and $\mu = \frac{1}{2} \dot{\phi}^2 + V(\phi)$. With our choice of $\nu^a$ the Klein-Gordon equation $\Box \phi - V'(\phi) = 0$ for $\phi$ (the prime indicates the derivative with respect to $\phi$) takes the form

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0
\]

in any space-time, while the spatial gradient of $\phi$ identically vanishes: $h_{a b} \phi_{,b} = 0$. A gauge-invariant variable associated with $\phi$ is $h_{a b} \phi_{,b}$; from this we can define a dimensionless gauge-invariant scalar variable $\Phi$ simply related to the density perturbation $\Delta$:

\[
\Phi \equiv \ell h^{a c}(h^{-1} h_{a b} \phi_{,b})_{,c} \quad \Rightarrow \quad \Delta = \gamma \Phi.
\]

It is then easy to show that equation (17) for $\Delta$ still holds, whereas the equation for $\tilde{C}$ contains a further term which can be interpreted as due to an entropy perturbation [12, 7].
The equation for the harmonic component $\tilde{C}^{(ν)}$ of $C$ is then
\begin{equation}
\dot{\tilde{C}}^{(ν)} = -\frac{4H^3\ell^2}{\gamma} \left[ \frac{ν^2}{H^2\ell^2} + \frac{6|K|}{H^2\ell^2} \left( \frac{\ddot{ϕ}}{3Hφ} + \frac{7}{6} \right) \right] Δ^{(ν)},
\end{equation}
while (17) still holds for the components $Δ^{(ν)}$ and $\tilde{C}^{(ν)}$. Note that we left an explicit $|K|$ factor in the above equation to show where $K$ appears: when $K = 0$, $\tilde{C}$ (14) reduces to $C$ and the last term in (18) containing $|K|$ can be dropped, i.e. scalar field and barotropic fluid perturbations satisfy formally similar equations [(14) reduces to (19) on putting $c^2 = 1$: however this is only a formal identification: $c^2 = c^2(t) ≠ 1$ for a scalar field]. When $K = -1$ instead, the second term in the square brackets in (18) will not be negligible in general, even for scales $λ ≫ H^{-1}$: $\tilde{C}$ will then decrease if $\dot{V}^2 > 0$, but it will increase in the opposite case. We can ask now in which cases $\tilde{C}^{(ν)}$ will be conserved for scales far outside the effective horizon $H^{-1}$ when $K = -1$. It is clear from the above equation that in general $\tilde{C}$ will not be conserved, even if $ν^2/(ℓH)^2 ≪ 1$, as the other term in the parenthesis will be not negligible in general. However let us consider the horizon crossing condition
\begin{equation}
ν/(ℓH) ≈ 1 ⇔ ν ≈ (1 - Ω_C)^{-\frac{1}{2}} ≫ 1,
\end{equation}
where $Ω_C$ is the value of $Ω$ when the comoving scale fixed by $ν$ crosses the horizon; we see from the above relations that scales with $ν < 1$ never cross the horizon. Scales which are at present inside the horizon, i.e. $λ < H_0^{-1}$, correspond to $ν > ν_0 ≈ (1 - Ω_0)^{-\frac{1}{2}}$ and therefore left the effective horizon when $1 - Ω_C < 1 - Ω_0$. In standard inflationary models that predict $|1 - Ω_0|$ to be exponentially close to zero all scales of major interest left the horizon at the very end of inflation, and the corresponding wavenumbers $ν$ are exponentially large; if instead $|1 - Ω_0| ≈ 1$ (i.e., if we have a minimal inflation), then we can see many of the scales that left the horizon from the very beginning of the inflationary phase, and in this case $ν \geq 1$. In any case these scales, and those outside the present horizon $λ > H_0^{-1}$ that lie in the interval $1 < ν < (1 - Ω_0)^{-\frac{1}{2}}$(practically vanishing for minimal inflationary models), were far outside the horizon only when $Ω$ was already close enough to unity,
\begin{equation}
λ ≫ H^{-1} ⇔ ν(1 - Ω)^{\frac{1}{2}} ≪ 1 ⇔ 1 - Ω ≪ 1 - Ω_C,
\end{equation}
in the last stage of inflation. It follows that, in a universe which is fairly open today, $\tilde{C}^{(ν)}$ cannot be conserved for these scales in epochs when $|1 - Ω| ≈ 1$. For these scales $\tilde{C}^{(ν)}$ will be practically conserved whenever $Ω$ is very close to unity, provided also
\begin{equation}
\frac{\ddot{φ}}{3Hφ} + \frac{7}{6} ≈ 1,
\end{equation}
e.g. during a slow rolling phase implying $ν ≪ 3Hφ$.

Finally, those scales larger than the present horizon $H_0^{-1}$ corresponding to $ν < 1$ have always been outside the horizon, so the condition $ν^2/(ℓH)^2 ≪ 1$ can be satisfied even when $|1 - Ω| ≈ 1$; however at these scales $\tilde{C}^{(ν)}$ will be not conserved in general, unless
\begin{equation}
\left| \frac{1}{6} - \frac{\dot{V}'}{3Hφ} \right| \ll 1.
\end{equation}
One is mainly concerned with the evolution of cosmological perturbations which are at most of the size of the present Hubble radius or slightly larger: if \( |1 - \Omega_0| \sim 1 \) perturbations on such scales cannot be far larger than the effective horizon from the moment they exit the horizon itself up to when \( \Omega \) is close enough to unity. During such an epoch a quantity such as \( \tilde{C} \) cannot be conserved on these scales, and the density perturbation evolution equation (17) is no longer an independent equation with a constant source term responsible for the growth of the density perturbation itself. In the next section we will construct general conserved quantities valid for particular inflationary models.

4 Conserved quantities

In the previous section we discussed the evolution of the scalar curvature variables \( C \) and \( \tilde{C} \) and showed that for a scalar field neither of them are conserved in general when \( K \neq 0 \) for scales much larger than the Hubble radius. It is interesting to ask whether it is possible to construct a quantity \( \mathcal{K} \) from the set of perturbation variables that is conserved at any scale, even for \( K \neq 0 \). It would therefore generalize the two curvature variables already discussed.

Let’s start by defining a general scalar quantity that generalizes \( C \) and \( \tilde{C} \): we found convenient to choose that it take the form

\[
\mathcal{K} = -\frac{4}{3} \epsilon^2 \Theta Z + 2 \mu^2 \left( 1 - \frac{2K}{\ell^2(\mu+p)} \right) \Delta + \epsilon X Z + \epsilon Y \Delta,
\]

where \( X = X(t) \) and \( Y = Y(t) \) are two functions to be determined, and \( \epsilon = 0,1 \). For \( \epsilon = 0 \) \( \mathcal{K} \) reduces to \( \tilde{C} \), and it reduces to \( C \) if in addition we choose \( K = 0 \). As we are now going to show, for \( \epsilon = 1 \), there are cases (clarified below) in which we can determine \( X \) and \( Y \) so that \( \mathcal{K} \) is conserved at any scale (i.e. any mode \( \mathcal{K}^{(\nu)} \) is conserved), both for a flat or for an open universe (the following equations hold only for \( K = 0,-1 \)). To find an evolution equation for \( \mathcal{K} \), we take its time derivative and use the equations for \( Z \) (18) and \( \Delta \) (17). Writing in terms of harmonic components, and keeping the imperfect fluid source terms for generality, we obtain:

\[
\dot{\mathcal{K}}^{(\nu)} = \left\{ \epsilon \left[ \dot{Y} + 3H (\gamma - 1) Y - \frac{1}{5} \kappa \mu X + \frac{H^2 \nu^2}{\gamma} \left( \frac{\nu^2}{\ell^2 H^2} - \frac{4K}{\ell^2 H^2} \right) X \right] - \frac{4 \ell^2 H^2 \nu^2}{\gamma} \left( \frac{\nu^2}{\ell^2 H^2} - \frac{K}{\ell^2 H^2} \right) \right\} \Delta^{(\nu)} + \epsilon \left[ \dot{X} - 2HX - \gamma Y \right] Z^{(\nu)}
+ \frac{H^2 (\gamma - 1)}{\gamma} \left( \frac{\nu^2}{\ell^2 H^2} - \frac{4K}{\ell^2 H^2} \right) \left[ \epsilon X - 4H^2 \ell^2 \right] E^{(\nu)}
+ \left\{ \epsilon \left[ 3\ell \gamma HY + \frac{3}{2} h X + \ell H^2 \left( \frac{\nu^2}{\ell^2 H^2} - \frac{4K}{\ell^2 H^2} \right) X \right] - \frac{4 \ell^2 H^3}{\ell^2 H^2} \right\} \left[ F^{(\nu)} + \Pi^{(\nu)} \right]
+ \left[ \epsilon \ell H^2 Y - 4KL H^2 + 2H^2 \ell^2 h \right] \left( \frac{\nu^2}{\ell^2 H^2} - \frac{K}{\ell^2 H^2} \right) \Psi^{(\nu)}. \tag{38}
\]

When \( \epsilon = 0 \), \( \mathcal{K}^{(\nu)} = \tilde{\mathcal{C}}^{(\nu)} \) and this equation reduces to equation (21) of section 2.3.

It is clear from (38) that \( \mathcal{K}^{(\nu)} \) could be conserved in general only if all the five coefficients of the equation vanish, a condition that clearly cannot be satisfied by only
two arbitrary functions \((X \text{ and } Y)\). However we are primarily interested in the case of perfect fluids, for which \(F = \Pi = \Psi = 0\). Then for such a fluid either we take adiabatic perturbations \((E = 0)\), or we take it to represent a minimally coupled scalar field \(F^{(\nu)} = \Pi^{(\nu)} = \Psi^{(\nu)} = 0\), in which case the entropy perturbation is \(pE^{(\nu)} = (1 - c_s^2) \mu \Delta^{(\nu)} \) \([12]\).

We shall now focus our discussion on the latter case, for which equation \((38)\) becomes:

\[
\dot{K}^{(\nu)} = \left\{ \epsilon \left[ \dot{Y} + 3H (\gamma - 1) Y - \frac{1}{2} \kappa \mu X + \frac{H^2}{\gamma} \left( \frac{\nu^2}{\ell^2 H^2} - \frac{4K}{\ell^2 H^2} \right) X \right] 
- \frac{4H^2 \nu^2}{\ell^2 H^2} + \frac{K}{\ell^2 H^2} (3c_s^2 - 4) \right\} \Delta^{(\nu)} 
+ \epsilon \left[ \dot{X} - \gamma Y - 2HX \right] Z^{(\nu)}. \tag{39}
\]

Then the condition for \(K^{(\nu)}\) to be conserved is that the two coefficients of \(\Delta^{(\nu)}\) and \(Z^{(\nu)}\) vanish, i.e. only the two arbitrary functions \(X\) and \(Y\) are needed; this justifies our choice \((37)\). An equation similar to \((39)\) above is readily obtained from \((38)\) for adiabatic perturbations of a perfect (barotropic) fluid. In a more general case, such as a perfect fluid with isocurvature perturbations \((E \neq 0)\) or an imperfect fluid (as a non-minimally coupled scalar field \([27]\)), \((37)\) should be appropriately generalized with the inclusion of further arbitrary functions.

From \((39)\) with \(\epsilon = 1\), we obtain a pair of first order differential equations for the variables \(X\) and \(Y\):

\[
\dot{Y} + 3H (\gamma - 1) Y - \frac{1}{2} \kappa \mu X + \frac{H^2}{\gamma} \left( \frac{\nu^2}{\ell^2 H^2} - \frac{4K}{\ell^2 H^2} \right) X 
- \frac{4H^2 \nu^2}{\ell^2 H^2} + \frac{K}{\ell^2 H^2} (3c_s^2 - 4) \right\} \Delta^{(\nu)} = 0 \tag{40}
\]

and

\[
\dot{X} - \gamma Y - 2HX = 0. \tag{41}
\]

It must be noted that \(X = X(\nu, t)\) and \(Y = Y(\nu, t)\), i.e. both these functions depend on the wavenumber \(\nu\), as well as \(t\): for each \(\nu\) the two equations above determine a pair \(X, Y\) such that the corresponding harmonic component \(K^{(\nu)}\) of \(K\) will be conserved. These two equations can be combined to give a second order equation for \(X\). Substituting for \(\gamma\) and \(\mu\) in terms of \(\dot{\phi}\) and \(\ddot{\phi}\), we obtain:

\[
\dddot{X} - \left[ 5H + \frac{2\dot{\phi}}{\phi} \right] \ddot{X} + \left[ \frac{1}{2} \dddot{\phi}^2 - 6H^2 + \frac{4H \dddot{\phi}}{\phi} + \frac{1}{6} \nu (\nu^2 - 6K) \right] X = -24H \left[ \frac{K \dddot{\phi}}{3H \phi} - \frac{1}{6} (\nu^2 - 7K) \right]. \tag{42}
\]

We will now apply the above equations to the following two cases commonly used when discussing the evolution of energy-density perturbations during an inflationary period: a coasting era, characterized by a constant value of the density parameter \(\Omega\) and De Sitter exponential inflation (see \([8, 12]\) for a discussion of these models).

\section{5 Applications}
5.1 The coasting solution: $\ell(t) = At$

Using the results given in [18], equation (42) becomes:

$$\ddot{X} - \frac{3}{7} \dot{X} + \frac{1}{7^2} \left[3 + \frac{1}{A^2} (\nu^2 - 5K)\right] X = 4H (\nu^2 - 5K),$$

which we can immediately integrate, giving, as in [12], three classes of solution:

- **a)** $\nu^2 < A^2 + 5K$:
  $$X(t) = X_A t^{2+Q} + X_B t^{2-Q} + 4A^2 t,$$
  where $Q = \sqrt[5]{\frac{5K - \nu^2}{A^2}} + 1$ as before and $X_A$, $X_B$ are two integration constants. The solution for $Y$ then follows from equation (40):
  $$Y(t) = \frac{3}{2} Q \left[X_A t^{1+Q} - X_B t^{1-Q}\right] - 6A^2.$$

- **b)** $\nu^2 = A^2 + 5K$; in this case we obtain:
  $$X(t) = t^2 \left[X^A + X^B \ln(t)\right] + 4A^2 t,$$
  and
  $$Y(t) = \frac{3}{2} X^B t - 6A^2.$$

- **c)** $\nu^2 > A^2 + 5K$; the third case corresponds to damped oscillations:
  $$X(t) = t^2 \left\{X^A \cos [Q \ln(t)] + X^B \sin [Q \ln(t)]\right\} + 4A^2 t,$$
  and
  $$Y(t) = -\frac{3}{2} Q t \left\{X^A \sin [Q \ln(t)] - X^B \cos [Q \ln(t)]\right\} - 6A^2.$$

Substituting these results into equation (37) we can obtain an expression for the conserved quantity $K$.

5.2 De Sitter exponential expansion: $\ell(t) = A \exp(\omega t)$

This time the second order differential equation for $X(t)$ (42) becomes:

$$\ddot{X} - 3w \dot{X} + \left[2w^2 + \frac{1}{12}w (\nu^2 - 5K) \exp(-2wt)\right] X = 4w (\nu^2 - 5K) \omega.$$
\[ a) \nu^2 - 5K < 0 : \]
\[ X(t) = \frac{1}{C^2 p} \left[ X^A \exp(2C\sqrt{p}) + X^B \exp(-2C\sqrt{p}) \right] + \frac{C^2 (\nu^2-5K)}{w} p, \tag{51} \]
where \( C = \sqrt{\frac{\nu^2-5K}{4w^2 A^2}} \), \( p = \exp(-2wt) \) and \( X^A, X^B \) are two integrating constants. The corresponding solution for \( Y \) is:
\[ Y(t) = \frac{1}{\gamma} \left[ \frac{2w}{C^2 p^2} \left\{ X^A (1-p) \exp(2C\sqrt{p}) - \frac{Cp}{\sqrt{p}} \exp(-2C\sqrt{p}) \right\} \right. \]
\[ + X^B \left( (1-p) \exp(-2C\sqrt{p}) + \frac{Cp}{\sqrt{p}} \exp(2C\sqrt{p}) \right) \}
\[ + \frac{2C^2}{p^2} (\nu^2 - 5K)(1-p) \] . \tag{52}

\[ b) \nu^2 - 5K = 0 ; \text{ in this case we obtain:} \]
\[ X(t) = X^A \exp(-\frac{1}{2}p) + X^B \exp(-p), \tag{53} \]
and
\[ Y(t) = \frac{w}{\gamma} \left[ X^A (p-2) \exp(-\frac{1}{2}p) + 2X^B (p-1) \exp(-p) \right] . \tag{54} \]

\[ c) \nu^2 - 5K > 0 : \]
\[ X(t) = \frac{1}{C^2 p} \left[ X^A \cos(2C\sqrt{p}) + X^B \sin(2C\sqrt{p}) \right] + \frac{C^2 (\nu^2-5K)}{w} \frac{1}{p}, \tag{55} \]
and
\[ Y(t) = \frac{1}{\gamma} \left[ \frac{2\omega}{C^2 p^2} \left\{ X^A (1-p) \cos(2C\sqrt{p}) + \frac{Cp}{\sqrt{p}} \sin(2C\sqrt{p}) \right\} \right. \]
\[ + X^B \left( (1-p) \sin(2C\sqrt{p}) - \frac{Cp}{\sqrt{p}} \cos(2C\sqrt{p}) \right) \}
\[ + \frac{2C^2}{p^2} (\nu^2 - 5K)(1-p) \] . \tag{56}

Substituting for \( X(t) \) and \( Y(t) \) into equation \[37\] we can obtain an expression for the corresponding conserved quantity \( K \).

### 6 Transfer functions

In the previous two sections we have illustrated a method to obtain a conserved quantity \( K^{(\nu)} \) via two auxiliary functions \( X(\nu, t) \) and \( Y(\nu, t) \). Without going into further details, in the following we shall sketch how to use \( K^{(\nu)} \) to connect the amplitude of perturbations at different epochs.

Typically, one is interested in connecting an early universe (inflationary) era with the present matter dominated era. For this reason, and for the sake of simplicity, we will consider eras in which we either have a perfect fluid with adiabatic perturbations,
or a minimally coupled scalar field. In both these cases, the evolution of perturbations is given by a system of equations like those of section 2.3, i.e.

\[ \dot{\Delta}^{(\nu)} = a(t)\Delta^{(\nu)} + b(t)Z^{(\nu)}, \]

\[ \dot{Z}^{(\nu)} = c(t)\Delta^{(\nu)} + d(t)Z^{(\nu)}, \]

where at least one of the functions \( a, b, c, d \) depends on \( \nu \) [we remind the reader that the exception is the dust case, for which \( \tilde{C} \) is conserved at all scales, even in an open universe, see (21)], and all depend on the equation of state valid in the given era.

We can write equation (37) in the form

\[ K^{(\nu)} = \kappa_1^{(\nu)}(t)\Delta^{(\nu)} + \kappa_2^{(\nu)}(t)Z^{(\nu)}, \]

where \( \kappa_1 \) and \( \kappa_2 \) depend on \( X(t) \) and \( Y(t) \). Using (59) we then obtain an equation of the same form as (39), and requiring the vanishing of the coefficients of this equation we get (40) and (41), which can then be integrated (at least in some cases) to give us the functions \( X \) and \( Y \). With these results in mind, while at the same time using (59) to substitute for \( Z^{(\nu)} \) in (57), we obtain a first order equation for \( \Delta^{(\nu)} \):

\[ \dot{\Delta}^{(\nu)} = \tilde{a}(t)\Delta^{(\nu)} + \tilde{b}(t)Z^{(\nu)}, \]

and since \( K^{(\nu)} = const \), this equation can always be integrated (at least in principle). Then, neglecting a decaying mode, this will give

\[ \Delta^{(\nu)}(t) = F(\nu, t)K^{(\nu)}, \]

or an integral of the form (23) in the general case. If we want to connect two different eras with different equation of state, we have to repeat the procedure sketched above for the two different cases. Then using \( K^{(\nu)}(t_1) = K^{(\nu)}(t_2) \) one obtains

\[ \Delta^{(\nu)}(t_1) = T(\nu, t_1, t_2)\Delta^{(\nu)}(t_1), \]

where \( T(\nu, t_1, t_2) = F(\nu, t_2)/F(\nu, t_1) \) is the required transfer function.

Thus we have obtained an algorithm through which, provided that the two auxiliary functions \( X \) and \( Y \) can be explicitly determined, the evolution of the density perturbation \( \Delta^{(\nu)} \) is given, for any scale fixed by \( \nu \), by a first order equation with the conserved quantity \( K^{(\nu)} \) as source term. Using this, a transfer function is readily obtained, thus generalizing the procedure illustrate for example in [2, 3] to any \( \nu \) and \( K = 0, -1 \).

7 Discussion

In the present paper we discussed under what conditions the curvature variables \( C \) and \( \tilde{C} \) are conserved both in the case of a general imperfect fluid and in detail for a minimally coupled scalar field. It was shown that in the general case \( \tilde{C} \) is only conserved for large scales when the background is taken to be flat or almost flat (i.e. \( \Omega \approx 1 \)), while for a minimally coupled scalar field an additional condition (35) must be satisfied in the almost flat case. We then explored the possibility of defining a gauge-invariant
curvature variable $K$ which would be conserved in a more general set of circumstances
for all scales of interest. We showed that at least for some models this possibility can
indeed be realized and illustrated this with the example of a minimally coupled scalar
field evolving either in a coasting phase ($\Omega = \text{const}$) or during a De Sitter inflationary
era. We also discussed how to use $K$ to derive a transfer function in order to connect the
amplitude of perturbations at different epochs.

Although we restricted our attention to rather special examples, all the equations
have been derived in general, so this prescription can be applied to a wider variety of
models both inflationary and non-inflationary. In particular, applications to generalized
gravity theories will be of great interest in view of the recent growing activity in such
theories and the generation and evolution of perturbations during a possible inflationary
phase and their final spectrum at second horizon crossing time deserve special attention.
These issues will be discussed in a future paper [31].

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