A light scalar mode from holographic deformations

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We consider different ways of deforming the mass spectrum of a strongly-coupled gauge theory with confinement using the AdS/CFT correspondence. Single - and multi - trace deformations are introduced such that the resulting theory has a mode lighter than the confinement scale. It is shown that the particular type of deformation determines the elementary/composite admixture of the light mode.

I. INTRODUCTION

Based on our experience with QCD, we know that the composite states of a strongly-coupled gauge theory have masses around the confinement scale. Usually exceptions to this occur in the presence of an underlying symmetry as in the case of pions. Spontaneously broken global symmetries might be introduced to have light states when the symmetry is also broken explicitly. In this paper, we explore other ways of producing scalars lighter than the confinement scale of the theory.

One can modify the theory in different ways such as coupling an elementary scalar to a composite operator or introducing effective interactions in the strong sector. These changes modify the mass spectrum of the theory, however, it is not straightforward to compute the effect due to the strong dynamics. Therefore, inspired by the AdS/CFT correspondence \[1\], we can construct the dual model in the extra dimensional space where the modifications of the mass spectrum are more tractable. In Sec II, we present the five-dimensional model dual to a composite states of a strongly-coupled gauge theory having confinement using the AdS/CFT correspondence. Single - and multi - trace deformations are introduced in this specific form just to be consistent with the solutions which will be presented later. It behaves near the AdS boundary, \( y \rightarrow -\infty \), as

\[
\Phi(p, y) \rightarrow - C_1(p) C_2(p) \frac{2\pi \Gamma(\alpha)}{\pi} \left( \frac{ip}{k} \right)^{-\alpha} e^{(2-\alpha)ky} + C_1(p) \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \left( \frac{ip}{k} \right)^{\alpha} e^{(2+\alpha)ky},
\]

which is consistent with \[2\]. Now let us consider a slice of AdS\(_5\) where the extra coordinate is restricted by UV and IR branes that exist at \( y_0 \) and \( y_1 \) respectively \[3\]. It is customary to define a variable for the UV boundary value of the bulk field as \( \hat{\Phi}(p) \equiv \Phi(p, y_0) \). The solution then becomes

\[
\Phi(p, y) = \hat{\Phi}(p) e^{2k(y-y_0)} \frac{J_\alpha(ip/k_y) + C_2(p)Y_\alpha(ip/k_y)}{J_\alpha(ip/k_y) + C_2(p)Y_\alpha(ip/k_y)},
\]

Note that defining a variable for the UV boundary value does not specify the UV boundary condition yet. Minimizing the action on the branes as well determines the boundary conditions and the result depends on whether the boundary value is fixed, \( \delta \hat{\Phi}|_{y=y_0} = 0 \), or not. Since we do not fix the IR boundary value of the bulk field, the IR boundary condition is determined using \( \delta \hat{\Phi}|_{y=y_1} \neq 0 \). Therefore, \( C_2(p) \) can be computed using the IR boundary condition. The existence of an IR brane implies that the conformal symmetry is broken in the dual theory. While there was no mass scale in the CFT before introducing the UV and IR branes, now the particle bound states appear at the IR scale. The mass spectrum of the theory can be computed from the AdS/CFT correspondence.

II. HOLOGRAPHIC BASIS WITH A MASSLESS MODE

Consider the action for a real scalar field, \( \Phi(x, y) \), propagating on the AdS\(_5\) background,

\[
\mathcal{S} = \int d^5x \sqrt{-g} \left[ -\frac{1}{2} \left( \partial_M \Phi \right)^2 - \frac{1}{2} a k^2 \Phi^2 \right],
\]

where \( a \geq -4 \) parameterizes the bulk mass and \( y \) is the coordinate of the fifth dimension. The metric for this background is

\[
ds^2 = e^{-2ky} \eta_{\mu\nu}dx^\mu dx^\nu + dy^2,
\]

where \( k \) is the AdS curvature scale. Using the AdS/CFT dictionary, we know that this five-dimensional field corresponds to an operator of dimension: \( \text{dim}[\mathcal{O}] \equiv \Delta = 2 + \sqrt{4 + a} \) in the four-dimensional theory \[2\]. The equation of motion derived from the variation of the action is

\[
\Box \Phi + e^{2ky} \partial_y \left( e^{-4ky} \partial_y \Phi \right) - a k^2 e^{-2ky} \Phi = 0,
\]

where \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu \). The solution for \( \Phi \) can be written in momentum space as

\[
\Phi(p, y) = C_1(p) e^{2ky} \left[ J_\alpha \left( \frac{ip}{k_y} \right) + C_2(p) Y_\alpha \left( \frac{ip}{k_y} \right) \right],
\]

where \( k_y \equiv k e^{-ky} \) and \( \alpha \equiv \sqrt{4 + a} \). \( C_1(p) \) and \( C_2(p) \) are unknown functions that can be calculated once the boundary conditions are imposed. The solution is presented in this specific form just to be consistent with the solutions which will be presented later. It behaves near the AdS boundary, \( y \rightarrow -\infty \), as

\[
\Phi(p, y) \rightarrow - C_1(p) C_2(p) \frac{2\pi \Gamma(\alpha)}{\pi} \left( \frac{ip}{k} \right)^{-\alpha} e^{(2-\alpha)ky} + C_1(p) \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \left( \frac{ip}{k} \right)^{\alpha} e^{(2+\alpha)ky},
\]
is known that there is no massless mode for the broken CFT described by Eq. [14]. The massless mode requires us to add brane masses in the following way,

\[ S = \int d^5x \sqrt{-g} \left[ -\frac{1}{2} \left( \partial_M \Phi \right)^2 - \frac{1}{2} \phi^2 \Phi^2 - bk\Phi^2 \left[ \delta(y - y_0) - \delta(y - y_1) \right] \right], \tag{7} \]

where \( b \equiv 2 \pm \alpha \). The parameter range, \( b < 2 \) (\( b > 2 \)), is called \(-\) (\(+\) ) branch respectively. Supersymmetry forces both brane masses to be related to each other in this form, so it can be generalized to different brane masses if the theory is not supersymmetric [5]. These terms on the branes are necessary to have a light mode but not sufficient as we will see later. The term on the IR brane simply modifies the IR boundary condition and hence \( C_2(p) \). As mentioned earlier, since we do not fix the IR boundary value, the IR boundary condition becomes \((\partial_y - bk) \Phi|_{y=y_1} = 0\). Plugging the solution, Eq. [6], back into the action, Eq. [7], the on-shell bulk action can be written as

\[ I[\Phi] = -\frac{e^{-3ky_0}}{2} \int d^4p \, P(p) F(p,y_0, \Phi(p,y_0) \Phi(-p)) - \frac{1}{2} \int d^4p \, \Phi(p) \Sigma(p) \Phi(-p), \tag{8} \]

where

\[ G = J_o \left( \frac{ip}{k_y} \right) Y_{\alpha \pm 1} \left( \frac{ip}{k_y} \right) - Y_o \left( \frac{ip}{k_y} \right) J_{\alpha \pm 1} \left( \frac{ip}{k_y} \right) \]

\[ F = J_{\alpha \pm 1} \left( \frac{ip}{k_y} \right) Y_{\alpha \pm 1} \left( \frac{ip}{k_y} \right) - Y_{\alpha \pm 1} \left( \frac{ip}{k_y} \right) J_{\alpha \pm 1} \left( \frac{ip}{k_y} \right). \tag{9} \]

Another useful quantity that can be computed in the five-dimensional theory is the conjugate variable,

\[ \Phi = -\frac{\delta I[\Phi]}{\delta \Phi}, \tag{10} \]

which is dual to \( \langle \mathcal{O} \rangle \). The original AdS/CFT correspondence recipe lets us determine the \( n \)-point functions, \( \langle \mathcal{O}...\mathcal{O} \rangle \), for the strongly-coupled CFT from the five-dimensional theory [8]. However, for general deformations of the bulk action, \( \int d^4p \, (W[\Phi] + \phi_0 \Phi/g_5) \), where \( g_5 \) is an expansion parameter with \( \text{dim}[g_5] = -1/2 \), \( W[\Phi] \) is an arbitrary function of \( \Phi \), and \( \phi_0 \) is the four-dimensional source, we follow [7] to compute the improved correspondence formula. The Legendre transform of \( I[\Phi] \),

\[ J[\Phi] = I - \int d^4p \, \Phi \frac{\delta I}{\delta \Phi}, \tag{11} \]

can then be used to construct a generating functional, \( S_{\text{holo}} \), from which one can compute the mass spectrum. Let us construct the generating functional,

\[ S_{\text{holo}} = J[\Phi] + \int d^4p \, \left( W[\Phi] + \frac{1}{g_5} \phi_0 \Phi \right), \tag{12} \]

whose minimization, \( \delta S_{\text{holo}}/\delta \Phi = 0 \), determines the relationship between \( \Phi \) and the source, \( \phi_0 \). Plugging the solution for \( \Phi \) back into Eq. [12] results in a functional, \( S_{\text{holo}}[\phi_0] \). The AdS/CFT correspondence can therefore be expressed as the following relation between the four- and five-dimensional theories,

\[ e^{-\langle S_{\text{holo}}[\phi_0] - S_{\text{holo}}[\phi_0=0] \rangle} = \left( e^{-\int d^4p \, \frac{\Lambda_{UV}^{2\Delta-6}}{5\Lambda_{UV}} \phi_0 \mathcal{O}} \right) W[\mathcal{O}], \tag{13} \]

where \( \Lambda_{UV} \equiv 2k_{y_0} \) is the cut-off scale. This definition can be considered as choosing an origin for the location of the UV brane in the fifth dimension. With this definition, following the literature and picking \( y_0 = 0 \), the curvature would obey the inequality, \( k/\Lambda_{UV} \lesssim 2 \), in order for the classical metric solution to be valid [8].

Let us start with no deformation of the CFT other than the addition of the source term, \( W = 0 \). For such linear deformations of the CFT, i.e. single-trace perturbations, the result of minimizing \( S_{\text{holo}} \), \( \Phi = \phi_0/g_5 \), is compatible with the original AdS/CFT recipe. This results in a trivial generating functional, \( S_{\text{holo}}[\phi_0] = I[\phi_0/g_5] \). Using the generating functional from the five-dimensional theory, and noting that \( S_{\text{holo}}[\phi_0 = 0] = 0 \) for \( W = 0 \), we can compute the two-point functions,

\[ \Lambda_{UV}^{2\Delta-6} \frac{\delta^2 S_{\text{holo}}[\phi_0]}{\delta \phi_0 \delta \phi_0} (-p) = \langle O(p) O(-p) \rangle 
- \Lambda_{UV}^{2\Delta-6} \Sigma(p) \sim \left\{ \begin{array}{l} p^{2\alpha} + \text{c. t.} \quad y_1 \to \infty \text{ finite \ y}_1 \end{array} \right. \tag{14} \]

where \( a_n = \langle 0 | O | \phi^n \rangle \) is the matrix element for \( O \) to create the \( n \)th meson, \( \phi^n \), with mass \( m_n \) from the vacuum [9]. The dimension of the operator, \( \mathcal{O} \), can be read by considering the limit, \( y_1 \to \infty \). The mass spectrum of the particle bound states after confinement can then be found by calculating the poles of \( pF(p,y_0)/G(p,y_0) \) for a finite \( y_1 \), which are given by the solutions of the following equation,

\[ J_o \left( \frac{m_n}{k_y} \right) \] \[ Y_o \left( \frac{m_n}{k_y} \right) \]

\[ \frac{J_{\alpha \pm 1} \left( \frac{m_n}{k_y} \right) Y_{\alpha \pm 1} \left( \frac{m_n}{k_y} \right)}{Y_o \left( \frac{m_n}{k_y} \right) J_o \left( \frac{m_n}{k_y} \right)} \]

\[ J_{\alpha \pm 1} \left( \frac{m_n}{k_y} \right) Y_{\alpha \pm 1} \left( \frac{m_n}{k_y} \right) \]

\[ \frac{J_o \left( \frac{m_n}{k_y} \right) }{Y_o \left( \frac{m_n}{k_y} \right) } \]

\[ \text{This means that the bulk field can decomposed as a tower of the particle bound states,} \]

\[ \Phi(x,y) = \sum_{n=1}^{\infty} \phi^n(x) g^n(y). \tag{16} \]

The profiles, \( g^n(y) \), that satisfy the bulk equation of motion, Eq. [3], with \( \Box \phi^n = m_n^2 \phi^n \) can be written as

\[ g^n(y) = N_n e^{2ky} \left[ J_o \left( \frac{m_n}{k_y} \right) + \kappa(m_n) Y_o \left( \frac{m_n}{k_y} \right) \right]. \tag{17} \]

where \( N_n \) is the normalization constant and \( \kappa(m_n) \) is determined by imposing the boundary conditions. Consider the lightest mode in the mass spectrum coming from
Eq. (15),
\[ m_1 \sim \begin{cases} k_{y_1} e^{\alpha k(y_0 - y_1)} - \text{branch} \\ k_{y_1} e^{\alpha k(y_0 - y_1)} + \text{branch}. \end{cases} \] (18)

Note that only the + branch has a mode whose mass is lighter than the confinement scale, \( \Lambda_{IR} \equiv k_{y_1} \). To compute the mass spectrum for the pure bound states, one needs to remove the UV brane, \( y_0 \to -\infty \). This tells us that the + branch has a massless mode but it is modified by the finite UV cut-off effects. On the other hand, the \( - \) branch never has a mode lighter than the confinement scale. Eq. (15) implies the following boundary conditions for the profiles of the composite states:
\[ g^n(y_0) = 0 \quad \text{and} \quad (\partial_y - bk) g^n|_{y=y_1} = 0. \] (19)

Note that this implies a fixed UV boundary value of the bulk field, \( \delta \Phi|_{y=y_0} = 0 \), even though the bound states are dynamical fields with \( \delta \varphi^n \neq 0 \). Since \( \varphi_0 \) is not a dynamical field, one can set it to zero at the end of the calculations. Therefore, we need to deform the action to have a massless or a light mode in the theory especially for the - branch. Now we consider different ways of deforming the theory for that purpose.

### A. A Linear Mixing with an Elementary Scalar

We introduce an elementary scalar field, \( \varphi^s \), that linearly mixes with the operator, \( \mathcal{O} \). This is also known as the single - trace deformation of the CFT, \( \int d^4 p \varphi^s \mathcal{O}/\Lambda_{UV}^2 \), which means \( W[\mathcal{O}] = 0 \). Then the five-dimensional theory is perturbed by \( \int d^4 p \varphi^s \Phi / g_5 \), and observe that this is very similar to the term that we added to compute the two-point functions. The difference is that now \( \varphi^s \) is a dynamical field with \( \delta \varphi^s \neq 0 \) unlike the fixed source, \( \varphi_0 \). \( \varphi^s \) mixes with the composite states and the mass spectrum of the theory is modified. The modified mass spectrum can be computed by minimizing the effective action, \( \delta S_{\text{holog}}[\varphi^n] \). Minimizing the effective action, \( \delta S_{\text{holog}}[\varphi^n] = \frac{\delta S_{\text{holog}}[\varphi^n]}{\varphi^n} \delta \varphi^n = 0 \), requires \( \Sigma(p) \) to be zero since \( \varphi^n(-p) \neq 0 \), which is satisfied only for certain momentum values. Therefore, the modified mass spectrum, \( M_n \), is given by the zeros of \( pF(p, y_0)/G(p, y_0) \) instead of its poles. The zeros are given by
\[ \frac{J_{a+1}(M_n/k_{y_0})}{\bar{Y}_{a+1}(M_n/k_{y_0})} = \frac{J_{a+1}(M_n/k_{y_0})}{\bar{Y}_{a+1}(M_n/k_{y_0})}. \] (20)

In this case, there is a massless mode, \( M_0 = 0 \). We conclude that these modified modes are admixtures of \( \varphi^s \) and \( \varphi^n \). Then the decomposition of the bulk field should be supplemented by a new four-dimensional field, \( \varphi^s(x) \), with a profile, \( g^s(y) \), that has a nonzero value on the UV brane,
\[ \Phi(x, y) = \varphi^s(x)g^s(y) + \sum_{n=1}^{\infty} \varphi^n(x)g^n(y), \] (21)

where \( g^s(y) \) also satisfies the bulk equation of motion, Eq. (3), with \( \Box \varphi^s = m_s^2 \varphi^s \) and it is normalized so that the kinetic terms in the resulting four-dimensional theory are canonical. This is called the holographic basis.

The new profile, \( g^s(y) \), can also be written as
\[ g^s(y) = N_s e^{2ky} \left[ J_\nu \left( \frac{m_s k}{y} e^{ky} \right) + \kappa(m_s) Y_\nu \left( \frac{m_s k}{y} e^{ky} \right) \right], \] (22)

where \( N_s \) is the normalization constant and \( m_s \) is the mass term for \( \varphi^s \). The constant, \( \kappa(m_s) \), is determined by imposing the boundary conditions. Since \( \varphi^s \) is proportional to \( \Phi(p, y_0) \), we expect its profile to satisfy the boundary condition, \( [\partial_y - (2 - \alpha)k] g^s(y)|_{y=y_0} = 0 \), from Eq. (5). This boundary condition makes sure that the profile, \( g^s(y) \), looks like the dominant term in Eq. (3) near the UV boundary. On the other hand, different boundary conditions for \( g^s(y) \) on the IR brane can be imposed, which in turn determines the nature of the mixing between the elementary and composite scalars. For example, \( \Box \) implicitly picks \( [\partial_y - (2 - \alpha)k] g^s(y)|_{y=y_0} = 0 \), which brings the following equation for the \( - \) branch, \( b < 2 \),
\[ \frac{J_{a-1}(m_s/k_{y_0})}{\bar{Y}_{a-1}(m_s/k_{y_0})} = \frac{J_{a-1}(m_s/k_{y_0})}{\bar{Y}_{a-1}(m_s/k_{y_0})}, \] (23)

which is the same with Eq. (20). Therefore, one of the solutions to this equation is \( m_s = 0 \). In this case there is only a kinetic mixing between \( \varphi^s \) and \( \varphi^n \) in the resulting four-dimensional theory. However, for the + branch, \( b > 2 \), \( \Box \) finds both kinetic and mass mixing. The main conclusion for this way of achieving a massless mode is that the massless scalar is an admixture of elementary and composite scalars.

### B. A Multi - Trace Deformation

We introduce a new interaction, \( W[\mathcal{O}] = -\xi \mathcal{O}^2 / \Lambda_{UV}^{2\Delta - 4} \), into the theory, where \( \xi \) is a dimensionless constant. This is also known as the multi - trace deformation of the CFT. Since we did not introduce a dynamical field like \( \varphi^s \) in this case, we need to add an additional deformation, \( \varphi_0 \mathcal{O}/\Lambda_{UV}^{2\Delta - 4} \), where \( \delta \varphi_0 = 0 \) as before to be able to probe the theory and calculate the mass spectrum. To determine the five-dimensional counterpart of this deformation, we need to know the relationship between \( \Phi \) and \( \langle \mathcal{O} \rangle \). This can be achieved by comparing the linear deformations with \( \varphi_0 \) from both sides,
\[ \frac{\Phi}{g_5} \leftrightarrow \langle \mathcal{O} \rangle / \Lambda_{UV}^{\Delta - 3}. \] (24)

Then the five-dimensional theory is deformed by
\[ \int d^4 p \left( -\xi \Phi^2 / (g_5^2 \Lambda_{UV}^2) + \varphi_0 \Phi / g_5 \right). \] For such multi-trace deformations of the CFT, minimizing \( S_{\text{holog}} \) results in a
more complicated generating functional,
\[ S_{\text{holo}}[\varphi_0] = -\frac{1}{2} \int d^4p \; \varphi_0(p) \frac{\Sigma(p)/g_5^2}{1 - \xi \Sigma(p)/(g_5^2 \Lambda_{UV}^2)} \varphi_0(-p). \]  
(25)

Using the generating functional from the five-dimensional theory and noting that \( S_{\text{holo}}[\varphi_0 = 0] = 0 \), we can compute the two-point functions,
\[
\frac{\Lambda_{UV}^{2\Delta - 6}}{2} \frac{\delta^2 S_{\text{holo}}[\varphi_0]}{\delta \varphi_0(p) \delta \varphi_0(-p)} = \langle O(p)O(-p) \rangle
\]
\[
-\frac{\Lambda_{UV}^{2\Delta - 4}}{2} \frac{\Sigma(p)}{g_5^2 \Lambda_{UV}^2 - \xi \Sigma(p)} \sim \sum_n \frac{a_n^2}{p^2 + m_n^2} \quad \text{finite } y_1.
\]  
(26)

Note that the expression for \( y_1 \to \infty \) would not be as trivial as the one in Eq.(14). This means that the dimension of the operator is not trivially related to the bulk mass parameter anymore. The mass spectrum of the particle bound states after confinement are then given by the solutions of the following equation,
\[
1 = \frac{\Sigma(p)}{g_5^2 \Lambda_{UV}^2}.
\]  
(27)

Note that the mass spectrum of the original broken CFT, \( \xi = 0 \), was given by the poles of \( \Sigma(p) \). If we consider the limit, \( \xi \to \infty \), the solutions, \( m_n \), are given by the zeros of \( \Sigma(p) \). The mass spectrum of the maximally deformed CFT, \( \xi \to \infty \), is then computed by using the following equation,
\[
\frac{J_{\alpha \pm 1}(m_n/k_{y0})}{Y_{\alpha \pm 1}(m_n/k_{y0})} = \frac{J_{\alpha \pm 1}(m_n/k_{y1})}{Y_{\alpha \pm 1}(m_n/k_{y1})},
\]  
(28)

which is the same with the equation, Eq.(26). Therefore, this particular type of maximal CFT deformation modifies the mass spectrum such that \( m_n = M_n \) and there is a massless scalar without introducing a new elementary field like \( \varphi^s \). How the mass eigenvalues change from the poles of \( \Sigma(p) \) to the zeros of \( \Sigma(p) \) can be seen in Fig.1. This phenomenon is also observed in a string theory setup, [11], where a light composite fermion is achieved in a similar limit. The bulk field can then be decomposed as a tower of the particle bound states,
\[
\Phi(x, y) = \sum_{n=0}^{\infty} \varphi^n(x) f^n(y),
\]  
(29)

where \( f^n(y) \) are the new profiles. These profiles, \( f^n(y) \), that satisfy the bulk equation of motion, Eq.(3), with \( \Box \varphi^n = M_n^2 \varphi^n \) can be written as
\[
f^n(y) = N_n e^{2ky} \left[ J_{\alpha} \left( \frac{M_n}{k_y} \right) + \kappa(M_n) Y_{\alpha} \left( \frac{M_n}{k_y} \right) \right],
\]  
(30)

where \( N_n \) is the normalization constant and \( \kappa(M_n) \) is determined by imposing the boundary conditions,
\[
(\partial_y - bk) f^n \bigg|_{y = y_0} \quad \text{and} \quad (\partial_y - bk) f^n \bigg|_{y = y_1}.
\]  
(31)

This process describes only how the on-shell five-dimensional action needs to be modified for such deformations, \( W[O] \). How \( \xi \) can take large values is not obvious but we can understand its physical implications. Expanding Eq.(26) in this limit,
\[
\lim_{\xi \to \infty} \frac{\Lambda_{UV}^{2\Delta - 4}}{2} \frac{\Sigma(p)}{g_5^2 \Lambda_{UV}^2 + \xi \Sigma(p)} \sim -\frac{\Lambda_{UV}^{2\Delta - 4}}{2\xi} + \frac{g_5^2 \Lambda_{UV}^{2\Delta - 2}}{2 \xi^2} \frac{1}{\Sigma(p)},
\]  
(32)

and looking at the leading non-analytic term with \( p \) for \( y_1 \to \infty \) we can compute the dimension, \( \Delta' \), of the modified operator, \( O' \). Note that this result, \( \Delta' = 2 - \alpha \), is very similar to the one in [12], where the Legendre transform of the on-shell bulk action inverts the expression for the two-point function, \( \langle O(p)O(-p) \rangle \). Of course, this interpretation should be restricted to the following parameter values, \( 0 \leq \alpha \leq 1 \), so that \( \Delta' \geq 1 \). Considering the AdS/CFT formula, Eq.(13), this result can be summarized as
\[
\lim_{\xi \to \infty} \langle e^{-\int d^4p \; \frac{\Lambda_{UV}}{45} \varphi_0 O} \rangle_{\xi \Lambda_{UV}^2} \to \langle e^{-\int d^4p \; \frac{\Lambda_{UV}}{45} \varphi_0 O'} \rangle_0.
\]  
(33)

The modified generating functional for the same limit in the five-dimensional theory should thus be constructed differently,
\[
S'_{\text{holo}} = I[\Phi] + \int d^4p \; \frac{\Lambda_{UV}}{g_5} \varphi_0 \Phi,
\]  
(34)

whose minimization, \( \delta S'_{\text{holo}}/\delta \Phi = 0 \), determines the relationship between \( \Phi \) and the source, \( \varphi_0 \). This is consistent with the interpretation of [12] that says that the roles of two solutions near the AdS boundary, \( \hat{\Phi} \) and \( \Phi \), are interchanged. Plugging the solution for \( \Phi \) back into Eq.(34)
results in a functional,
\[ S'_{\text{holo}}[\phi_0] = -\int d^4p \, \frac{\Lambda_{\text{UV}}^2}{2g_5^2} \phi_0(p) \frac{1}{\Sigma(p)} \phi_0(-p), \] (35)
which is equivalent to Eq. (32) in terms of the dimension of the composite operator that it describes and the mass spectrum when there is confinement.

III. AN ALTERNATIVE HOLOGRAPHIC BASIS WITH A MASSLESS MODE

In Sec. II A we stated that different boundary conditions for the profile, \( g^s(y) \), can be imposed on the IR brane. Let us pick \( g^s(y_1) = 0 \) instead of the modified Neumann condition that we used before. These boundary conditions lead to the following equation for \( b < 2 \),
\[ J_{\alpha - 1}(m_s/k_{y_0}) = J_b(m_s/k_{y_0}), \]
that can be solved for new nonzero \( m_s \) values. The smallest solution is given by
\[ m_s \sim \begin{cases} k_{y_1} e^{(\alpha - 1)/k(y_0-y_1)} & \alpha > 1 \\ k_{y_1} & 0 \leq \alpha < 1. \end{cases} \] (37)

We showed that there is no \( m_s = 0 \) for \( b < 2 \) unlike the case in Sec. II A but we need to check whether this new solution also allows the system to have a massless mode. Inserting the holographic basis into the action, we find
\[ S = \int d^5x \left[ S(\phi^s) + S(\phi^n) + S_{\text{mix}} \right], \] (38)
where \( S_{\text{mix}} \) includes the kinetic, \( z_n \), and mass mixings, \( \mu_n^2 \),
\[ S_{\text{mix}} = \int d^4x \sum_{n=1}^{\infty} \left[ -\frac{1}{2} z_n \partial_\mu \phi^s \partial^\mu \phi^n - \frac{1}{2} \mu_n^2 \phi^s \phi^n \right]. \] (39)

Then we have a mass matrix of the following form,
\[ m^2 = \begin{pmatrix} \mu_2^2 & \mu_1^2 & \mu_2^2 & \ldots \\ \mu_1^2 & m_1^2 & 0 & \ldots \\ \mu_2^2 & 0 & m_2^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \] (40)
If we can show that the determinant of the mass matrix is zero, we are done with proving that there is a massless mode. For this purpose, we need to know what the mass mixing terms, \( \mu_n^2 \), are. Using the profiles, they are given by
\[ \mu_n^2 = \int_{y_0}^{y_1} dy \, e^{-4ky} \left[ \partial_y g^s \partial_y g^n + 2bkg^s g^n \right. \\
+ \left. 2bkg^s g^n \left[ \delta(y - y_0) - \delta(y - y_1) \right] \right], \] (41)

There are two ways of differentiating by parts,
\[ \mu_n^{2A} = \int_{y_0}^{y_1} dy \, m_n^2 g^s \left[ e^{-2ky} g^n - y_1^n(y_1) e^{-bky} e^{(b-2)ky} \right] \]
\[ \mu_n^{2B} = \int_{y_0}^{y_1} dy \, m_n^2 g^n \left[ e^{-2ky} g^s - y_0^n y_0 e^{-bky} e^{(b-2)ky} \right], \] (42)
where \( \mu_n^2 = \mu_n^{2A} = \mu_n^{2B} \). The mass matrix has a massless mode if \( \sum_{n=1}^{\infty} \mu_n^2 = m_s^2 \). This equation can be proven by computing \( \sum_{n=1}^{\infty} \mu_n^{2A} \mu_n^{2B} \) with the help of the completeness relation.

IV. CONCLUSION

We showed how an effective interaction term in the strong sector can produce a composite scalar lighter than the confinement scale. How much lighter this state is depends on the interaction strength. The physical origin of this interaction strength is how it can take large values are subjects that need further investigation. However, when one does not need a big suppression from the confinement scale, a reasonable interaction strength is useful enough. A light composite scalar as we discussed in this paper might be used for dark matter models with light scalar mediators. Moreover, the strong dynamics that produce a composite Higgs boson could also give rise to such light composite scalars, which would affect the Higgs decay channels. Or more drastically, Higgs boson itself could be imagined as the light composite state of a strongly-coupled gauge theory with a confinement scale around TeV. These kind of models would allow for heavier composite scalars to be above the several TeV range that is consistent with LHC results.

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