Lectures on perverse sheaves on instanton moduli spaces

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1. Introduction

Let $G$ be an almost simple simply-connected algebraic group over $\mathbb{C}$ with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We assume $G$ is of type ADE, as there arise technical issues for type BCFG. (We will remark them at relevant places. See footnotes 5, 12.) At some points, particularly in this introduction, we want to include the case $G = GL(r)$. We will not make clear distinction between the case $G = SL(r)$ and $GL(r)$ in the main text.

Let $G_c$ denote a maximal compact subgroup of $G$. Our main player is

$$U^d_G = \text{the Uhlenbeck partial compactification}$$

of the moduli spaces of framed $G_c$-instantons on $S^4$ with instanton number $d$. The framing means a trivialization of the fiber of the $G_c$-bundle at $\infty \in S^4$. Framed instantons on $S^4$ are also called instantons on $\mathbb{R}^4$, as they extend across $\infty$ if their curvature is in $L^2(\mathbb{R}^4)$. We follow this convention. These spaces were first considered in a differential geometric context by Uhlenbeck, Donaldson and others, for more general 4-manifolds and usually without framing.

The Uhlenbeck compactification has been used to define differential topological invariants of 4-manifolds, i.e., Donaldson invariants, as integral of cohomology classes over moduli spaces of instantons: Moduli spaces are noncompact,
therefore the integral may diverge. Thus compactification is necessary to make the integral well-defined. (See e.g., [16].)

Our point of view here is different. We consider Uhlenbeck partial compactification of instanton moduli spaces on $\mathbb{R}^4$ as objects in geometric representation theory. We study their intersection cohomology groups and perverse sheaves in view of representation theory of the affine Lie algebra of $\mathfrak{g}$ or the closely related $\mathcal{W}$-algebra.\footnote{If $G$ is not of type ADE, we need to replace the affine Lie algebra of $\mathfrak{g}$ by its Langlands dual $\mathfrak{g}_{aff}$. It is a twisted affine Lie algebra, and should not be confused with the untwisted affine Lie algebra of the Langlands dual of $\mathfrak{g}$.} We will be concerned only with a very special 4-manifold, i.e., $\mathbb{R}^4$ (or $\mathbb{C}^2$ as we will use an algebro-geometric framework). On the other hand, we will study instantons for any group $G$, while $G_c = SU(2)$ is usually enough for topological applications.

We will drop ‘Uhlenbeck partial compactification’ hereafter unless it is really necessary, and simply say instanton moduli spaces or moduli spaces of framed instantons, as we will keep ‘U’ in the notation.

We will study equivariant intersection cohomology groups of instanton moduli spaces

$$IH^*_T \times \mathbb{C}^\times \times \mathbb{C}^\times (U_{G_d}),$$

where $T$ is a maximal torus of $G$ acting by the change of the framing, and $\mathbb{C}^\times \times \mathbb{C}^\times$ is a maximal torus of $GL(2)$ acting on $\mathbb{R}^4 = \mathbb{C}^2$. The $T$-action has been studied in topological context: it is important to understand singularities of instanton moduli spaces around reducible instantons. The $\mathbb{C}^\times \times \mathbb{C}^\times$-action is specific for $\mathbb{R}^4$, nevertheless it makes the tangent bundle of $\mathbb{R}^4$ nontrivial, and yields a meaningful counter part of Donaldson invariants, as Nekrasov partition functions. (See below.)

More specifically, we will explain the author’s joint work with Braverman and Finkelberg [12] with an emphasis on its geometric part in these lectures. The stable envelop introduced by Maulik-Okounkov [34] and its reformulation in [45] via Braden’s hyperbolic restriction functors are key technical tools. They also appear in other situations in geometric representation theory. Therefore those will be explained in a general framework. In a sense, a purpose of lectures is to explain these important techniques and their applications.

It is not our intention to reproduce the proof of [12] in full here. As lectures move on, we will often leave details of arguments to [12]. I hope that a reader is comfortable to read [12] at that stage after learning preliminary materials assumed in [12].

Prerequisite

- The theory of perverse sheaves is introduced in de Cataldo’s lectures in the same volume. We will also use materials in [15, §8.6], in particular the isomorphism between convolution algebras and Ext algebras.
• We assume readers are familiar with [41], at least for Chapters 2, 3, 6. (Chapter 6 presents Hilbert-Chow morphisms as examples of semi-small morphisms. They also appear in de Cataldo’s lectures.) Results explained there will be briefly recalled, but proofs are omitted.

• We will use equivariant cohomology and Borel-Moore homology groups. A brief introduction can be found in [46]. We also use derived categories of equivariant sheaves. See [5] (and/or [33, §1] for summary).

• We will not review the theory of $W$-algebras, such as [19, Ch. 15]. It is not strictly necessary, but better to have some basic knowledge in order to appreciate the final result.

**History** Let us explain history of study of instanton moduli spaces in geometric representation theory.

Historically relation between instanton moduli spaces and representation theory of affine Lie algebras was first found by the author in the context of quiver varieties [38]. The relation is different from one we shall study in this paper. Quiver varieties are (partial compactifications of) instanton moduli spaces on $\mathbb{R}^4/\Gamma$ with gauge group $G = \text{GL}(r)$. Here $\Gamma$ is a finite subgroup of $\text{SL}(2)$, and the corresponding affine Lie algebra is not for $G$: it corresponds to $\Gamma$ via the McKay correspondence. The argument in [38] works only for $\Gamma \neq \{1\}$. The case $\Gamma = \{1\}$ corresponds to the Heisenberg algebra, that is the affine Lie algebra for $\mathfrak{gl}(1)$. The result for $\Gamma = \{1\}$ was obtained later by Grojnowski and the author independently [24, 39] for $r = 1$, Baranovsky [4] for general $r$. This result will be recalled in §3, basically for the purpose to explain why they were not enough to draw a full picture.

In the context of quiver varieties, a $\mathbb{C}^\times$-action naturally appears from an action on $\mathbb{R}^4/\Gamma$. (If $\Gamma$ is of type $A$, we have $\mathbb{C}^\times \times \mathbb{C}^\times$-action as in these lectures.) The equivariant K-theory of quiver varieties are related to representation theory of quantum toroidal algebras, where $\mathbb{C}^\times$ appears as a quantum parameter $q$. More precisely the representation ring of $\mathbb{C}^\times$ is the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$. See [42]. The corresponding result for equivariant homology/affine Yangian version, which is closer to results explained in this lecture series was obtained by Varagnolo [57]. But these works covered only the case $\Gamma \neq \{1\}$. It is basically because the construction relies on a particular presentation of quantum toroidal algebras and affine Yangian, which was available only for $\Gamma \neq \{1\}$ when the paper [42] was written.

In physics side, Nekrasov [51] introduced ‘partition functions’ which are roughly considered as generating functions of equivariant Donaldson invariants on $\mathbb{R}^4$ with respect to the $\mathbb{C}^\times \times \mathbb{C}^\times$-action. The ordinary Donaldson invariants do not make sense for $\mathbb{R}^4$ (or $S^4$) as there is no interesting topology on $\mathbb{R}^4$. But equivariant Donaldson invariants are nontrivial, and contain interesting information. Nekrasov partition functions have applications to ordinary Donaldson invariants, see e.g., [22, 23]. Roughly equivariant variables for $\mathbb{C}^\times \times \mathbb{C}^\times$ play the role of Chern
roots of the tangent bundle of a complex surface. Important parts of Donaldson invariants (namely wall-crossing terms and coefficients relating Donaldson and Seiberg-Witten invariants) are universal in Chern classes, hence it is enough to compute for \( \mathbb{R}^4 \).

Another conjectural connection between affine Lie algebras and cohomology groups of instanton moduli spaces on \( \mathbb{R}^4/\mathbb{Z}_\ell \), called the geometric Satake correspondence for affine Kac-Moody groups, was found by [10]. Recall the affine Grassmannian \( \text{Gr}_G = G((z))/G[[z]] \) for a finite dimensional complex reductive group has a \( G[[z]] \)-action whose orbits are parametrized by dominant coweights \( \lambda \). The geometric Satake correspondence roughly says IC of the \( G[[z]] \)-orbit through \( \lambda \) knows the irreducible representation \( V(\lambda) \) with highest weight \( \lambda \) of the Langlands dual group \( G^\vee \). The affine Grassmannian of an affine Kac-Moody group is difficult to make sense as a geometric object, but instanton moduli spaces on \( \mathbb{R}^4/\mathbb{Z}_\ell \) play the role of a slice to an orbit in the closure of another larger orbit. In this connection, the \( \mathbb{C}^\times \times \mathbb{C}^\times \)-action is discarded, but the intersection cohomology is considered. The main conjecture says the graded dimension of the intersection cohomology groups can be computed in terms of the \( q \)-analog of weight multiplicities of integrable representations of the affine Lie algebra of \( g \) with level \( \ell \). (See [55] also for an important correction of the conjectural definition of the filtration.) We will visit a simpler variant of the conjecture in §7.5.

When an affine Kac-Moody group is of affine type \( A \), the Uhlenbeck partial compactification is a quiver variety, and has the Gieseker partial compactification, as a symplectic resolution of singularities. Then one can work on ordinary homology/K-theory of the resolution. There is a close relation between ordinary homology of the symplectic resolution and intersection cohomology. (See (3.7.1).) In particular, the above conjectural dimension formula was checked for type \( A \) (after discarding the grading) [44]. A key is that the quiver variety is already linked with an affine Lie algebra, as we have recalled above. More precisely, for \( G = \text{SL}(r) \), the quiver variety construction relates the IH of an instanton moduli space to \( (\mathfrak{sl}_\ell)_{\text{aff}} \) of level \( r \), while the conjectural geometric Satake correspondence relates it to \( (\mathfrak{sl}_\ell)_{\text{aff}} \) of level \( \ell \). They are related by I. Frenkel’s level-rank duality, and the weight multiplicities are replaced by tensor product multiplicities.

We still lack a precise (even conjectural) understanding of IH of instanton moduli spaces for general \( G \) and general \( \mathbb{R}^4/\Gamma \), not necessarily of type \( A \), say both \( G \) and \( \Gamma \) are of type \( E_8 \). It is a good direction to pursue in future.

Despite its relevance for the study of Nekrasov partition function, the equivariant K-theory and homology group for the case \( \Gamma = \{1\} \) were not understood for several years. They were more difficult than the case \( \Gamma \neq \{1\} \), because of a technical issue mentioned above. In the context of the geometric Satake correspondence, \( \Gamma = \{1\} \) corresponds to level 1 representations, which have explicit constructions.
from the Fock space (Frenkel Kac construction), and the conjectural dimension formula was checked \[10,55\]. But the role of $C^{\times} \times C^{\times}$-action was not clear.

In 2009, Alday-Gaiotto-Tachikawa \[1\] has connected Nekrasov partition for $G = SL(2)$ with the representation theory of the Virasoro algebra via a hypothetical 6-dimensional quantum field theory. This AGT correspondence is hard to justify in a mathematically rigorous way, but yet gives a very good viewpoint. In particular, it predicts that the equivariant intersection cohomology of instanton moduli space is a representation of the Virasoro algebra for $G = SL(2)$, and of the $W$-algebra associated with $g$ in general.

On the other hand, in mathematics side, equivariant $K$ and homology groups for the case $\Gamma = \{1\}$ had been understood gradually around the same time. Before AGT,\(^3\) Feigin-Tsymbaliuk and Schiffmann-Vasserot \[18,54\] studied the equivariant $K$-theory for the $G = GL(1)$-case. Then research was continued under the influence of the AGT correspondence, and the equivariant homology for the $G = GL(r)$-case was studied by Schiffmann-Vasserot, Maulik-Okounkov \[34,53\] for $G = GL(r)$ and the homology group.

In particular, the approach taken in \[34\] is considerably different from previous ones. It does not use a particular presentation of an algebra. Rather it constructs the algebra action from the $R$-matrix, which naturally arises on equivariant homology group. This sounds close to a familiar RTT construction of Yangians and quantum groups, but it is more general: the $R$-matrix is constructed in a purely geometric way, and has infinite size in general. Also for $\Gamma \neq \{1\}$, it defines a coproduct on the affine Yangian. It was explicitly constructed for the usual Yangian for a finite dimensional complex simple Lie algebra long time ago by Drinfeld \[17\], but the case of the affine Yangian was new.\(^4\)

In \[45\], the author reformulated the stable envelop, a geometric device to produce the $R$-matrix in \[34\], in a sheaf theoretic language, in particular using Braden’s hyperbolic restriction functors \[9\]. This reformulation is necessary in order to generalize the construction of \[34\] for $G = GL(r)$ to other $G$. It is because the original formulation of the stable envelop required a symplectic resolution. We have a symplectic resolution of $U_G^{\text{\textit{d}}} \text{for } G = GL(r)$, as a quiver variety, but not for general $G$.

Then the author together with Braverman, Finkelberg \[12\] studies the equivariant intersection cohomology of the instanton moduli space, and constructs the $W$-algebra representation on it. Here the geometric Satake correspondence for the affine Lie algebra of $g$ gives a philosophical background: the reformulated stable envelop is used to realize the restriction to the affine Lie algebra of a Levi subalgebra of $g$. It nicely fits with Feigin-Frenkel description of the $W$-algebra \[19, \text{Ch. 15}\].

\(^3\)Preprints of those papers were posted to arXiv, slightly before \[1\] was appeared on arXiv.

\(^4\)It motivated the author to define the coproduct in terms of standard generators in his joint work in progress with Guay.
Lectures on perverse sheaves on instanton moduli spaces

Convention

(1) A partition $\lambda$ is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ of nonnegative integers with $\lambda_N = 0$ for sufficiently large $N$. We set $|\lambda| = \sum \lambda_i$, $l(\lambda) = \#\{i \mid \lambda_i \neq 0\}$. We also write $\lambda = (\alpha_1^2 \alpha_2 \cdots)$ with $\alpha_k = \#\{i \mid \lambda_i = k\}$.

(2) For a variety $X$, let $D^b(X)$ denote the bounded derived category of complexes of constructible $\mathbb{C}$-sheaves on $X$. Let $IC(X_0, \mathcal{L})$ denote the intersection cohomology complex associated with a local system $\mathcal{L}$ over a Zariski open subvariety $X_0$ in the smooth locus of $X$. We denote it also by $IC(X)$ if $\mathcal{L}$ is trivial. When $X$ is smooth and irreducible, $\mathcal{E}_X$ denotes the constant sheaf on $X$ shifted by $\dim X$. If $X$ is a disjoint union of irreducible smooth varieties $X_\alpha$, we understand $\mathcal{E}_X$ as the direct sum of $\mathcal{E}_{X_\alpha}$.

(3) We make a preferred degree shift for Borel-Moore homology groups, and denote them by $H_{[\ast]}(X)$, where $H_{[\ast]}(X) = H_{[\ast] + \dim X}(X)$ for a smooth variety $X$. More generally, if $L$ is a closed subvariety in a smooth variety $X$, we consider $H_{[\ast]}(L) = H_{[\ast] + \dim X}(L)$.

Post-requisite  Further questions and open problems are listed in [12, 1(xi)]. In order to do research in those directions, the followings would be necessary besides what are explained in this lecture series.

- The AGT correspondence predicts a duality between $4d$ $N = 2$ SUSY (supersymmetric) quantum field theories and $2d$ conformal field theories. In order to understand it in mathematically rigorous way, one certainly needs to know the theory of vertex algebras (e.g., [19]). In fact, we still lack a fundamental understanding why equivariant intersection cohomology groups of instanton moduli spaces have structures of vertex algebras. We do want to have an intrinsic explanation without any computation, like checking Heisenberg commutation relations.

- The AGT correspondence was originally formulated in terms of Nekrasov partition functions. Their mathematical background is given for example in [49].

- In view of the geometric Satake correspondence for affine Kac-Moody groups [10], the equivariant intersection cohomology group $IH^*_G \times \mathbb{C} \times \mathbb{C} \times$ of instanton moduli spaces of $\mathbb{R}^4/\mathbb{Z}_\ell$ should be understood in terms of representations of the affine Lie algebra of $\mathfrak{g}$ and the corresponding generalized $W$-algebra. We believe that necessary technical tools are more or less established in [12], but it still needed to be worked out in detail. Anyhow, one certainly needs knowledge of $W$-algebras in order to study their generalization.

Coulomb branches  About a few month before I delivered lectures, I found a mathematical approach to the so-called Coulomb branches of $3$-dimensional SUSY gauge theories [47]. Such a gauge theory is associated with a pair $(H_c, M)$ of a compact Lie group and its quaternionic representation. Coulomb branches are
hyper-Kähler manifolds with SU(2)-action possibly with singularities. They have been studied in physics for many years. But physical definition contains quantum corrections, which were difficult to justify in a mathematically rigorous way. On the other hand, the so-called Higgs branches of gauge theories are hyper-Kähler quotients of $M$ by $H_c$. This is a mathematically rigorous definition, and quiver varieties mentioned above are examples of Higgs branches of particular $(H_c,M)$, called (framed) quiver gauge theories.

At the time these notes are written, I together with Braverman and Finkelberg have established a rigorous definition of Coulomb branches as affine algebraic varieties with holomorphic symplectic structures under the condition $M = N \oplus N^*$ for some $N$ [13]. The spaces $U^d_G$ discussed in these notes, and even more generally moduli spaces of framed $G_c$-instantons on $R^4/Z_\ell$ conjecturally appear as Coulomb branches of $(H_c,M)$. The gauge theories are framed quiver gauge theories of affine type ADE, whose Higgs branches are quiver varieties of affine types, i.e., moduli spaces of $GL(r)$-instantons on $R^4/\Gamma$. A proof of this conjecture for type A will be given in [48].

Thus Coulomb branches open a new way to approach to instanton moduli spaces. For examples, their quantization, i.e., noncommutative deformation of the coordinate rings naturally arise from the construction. We also obtain variants of $U^d_G$, arising from the same type of framed quiver gauge theory, for each weight $\mu \leq \Lambda_0$ where $\Lambda_0$ is the 0th fundamental weight of the affine Lie algebra for $G$. Here $U^d_G$ corresponds to $\mu = \Lambda_0 - d\delta$, where $\delta$ is the primitive positive imaginary root. In fact, the Coulomb branch of this category is conjecturally an instanton moduli space of Taub-NUT space. The Taub-NUT space is a hyper-Kähler manifold which is isomorphic to $C^2$ as a holomorphic symplectic manifold. The Riemannian metric is different from the Euclidean metric: The size of fibers of the Hopf fibration at infinity remains bounded. There could be nontrivial monodromy on fibers at infinity, which is the extra data cannot be seen in $U^d_G$.

Moreover Higgs and Coulomb branches are tightly connected in my proposed approach: The construction of Coulomb branches uses moduli spaces of twisted maps from $P^1$ to Higgs branches. These links should fit with symplectic duality proposed in [7]. The intersection cohomology groups of $U^d_G$ studied in this paper should be looked at from this view point. See Remark 5.6.2 for an example.

Thus it is clear that the whole stories talked here must be redone for instanton moduli spaces on Taub-NUT space. I would optimistically hope that they are similar, and our lectures still serve as basics.

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I am grateful to A. Braverman and M. Finkelberg for the collaboration [12], on which the lecture series is based. I still vividly remember when Sasha said that symplectic resolution exists only in rare cases, hence we need to study singular
spaces. It was one of motivations for me to start instanton moduli spaces for general groups, eventually it led me to the collaboration.

Prior to the PCMI, lectures were delivered at University of Tokyo, September 2014. I thank A. Tsuchiya for invitation.

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2. Uhlenbeck partial compactification – in brief

We do not give a precise definition of Uhlenbeck partial compactification of an instanton moduli space on \( \mathbb{R}^4 \), which was introduced in [11]. For type \( A \), there is an earlier alternative construction as an example of a quiver variety, as reviewed in §3.1.

For our purpose, we will only use properties explained in this section. These are easy to check in the alternative construction, hence the reader should accept them at the first reading.

2.1. Moduli space of bundles

First of all, \( \mathcal{U}_d^d \) is a partial compactification of \( \text{Bun}_d^d \), the moduli space of framed holomorphic (or equivalently algebraic) \( G \)-bundles \( \mathcal{F} \) over \( \mathbb{P}^2 \) of instanton number \( d \), where the framing is the trivialization \( \varphi \) of the restriction of \( \mathcal{F} \) at the line \( \ell_\infty \) at infinity. It will be denoted by \( \mathcal{U}^d \) throughout these lectures. It is an affine algebraic variety.

Here the instanton number is the integration over \( \mathbb{P}^2 \) of the characteristic class of \( G \)-bundles corresponding to the invariant inner product on \( g \), normalized as \( \langle \theta, \theta \rangle = 2 \) for the highest root \( \theta \).

Remark 2.1.1. By an analytic result due to Bando [3], we can replace instanton moduli spaces on \( \mathbb{R}^4 \) by \( \text{Bun}_d^d \). We use algebro-geometric approaches to instanton moduli spaces hereafter.

We also use that \( \text{Bun}_d^d \) is a smooth locus of \( \mathcal{U}_d^d \), and its tangent space at \((\mathcal{F}, \varphi)\) is

\[ H^1(\mathbb{P}^2, g_{\mathcal{F}}(-\ell_\infty)), \]

where \( g_{\mathcal{F}} \) is the associated vector bundle \( \mathcal{F} \times_G g \).

In the alternative construction, \( \text{Bun}_d^d \) is defined as the space of linear maps satisfying a quadratic equation modulo the action of \( \text{GL}(d) \). The tangent space is the quotient of the derivative the defining equation modulo the differential of the action. Properties of tangent spaces which we will use can be equally checked by using such a description.

\[^5\text{When an embedding } \text{SL}(2) \rightarrow G \text{ corresponding to a coroot } \alpha^\vee \text{ is given, we can induction a } G\text{-bundle } \mathcal{F} \text{ from a SL}(2)\text{-bundle } \mathcal{F}_{\text{SL}(2)} \text{. Then we have } d(\mathcal{F}) = d(\mathcal{F}_{\text{SL}(2)}) \times (\alpha^\vee, \alpha^\vee)/2. \text{ Instanton numbers are preserved if } G \text{ is type } \text{ADE}, \text{ but not in general.}\]
2.2. Stratification We will use its stratification:

\[(2.2.1) \quad \mathcal{U}_G^d = \bigsqcup_{0 \leq d' \leq d} \text{Bun}_G^{d'} \times S^{d-d'}(\mathbb{C}^2),\]

where \( \text{Bun}_G^{d'} \) is the moduli space with smaller instanton number \( d' \).

Let us refine the stratification in the symmetric product \( S^{d-d'}(\mathbb{C}^2) \) part:

\[(2.2.2) \quad \mathcal{U}_G^d = \bigsqcup_{d = |\lambda| + d'} \text{Bun}_G^{d'} \times S_\lambda(\mathbb{C}^2),\]

where \( S_\lambda(\mathbb{C}^2) \) consists of configurations of points in \( \mathbb{C}^2 \) whose multiplicities are given by the partition \( \lambda \).

2.3. Factorization morphism We will also use the factorization morphism

\[\pi^d: \mathcal{U}_G^d \to S^d \mathbb{C}^1.\]

The definition is given in [11, §6.4]. It depends on a choice of the projection \( \mathbb{C}^2 \to \mathbb{C}^1 \). We do not recall the definition, but its crucial properties are

1. On the factor \( S^{d-d'}(\mathbb{C}^2) \), it is given by the projection \( \mathbb{C}^2 \to \mathbb{C}^1 \).
2. Consider \( C_1 + C_2 \in S^d \mathbb{C}^1 \) such that \( C_1 \in S^{d_1} \mathbb{C}^1 \), \( C_2 \in S^{d_2} \mathbb{C}^1 \) are disjoint. Then \( (\pi^d)^{-1}(C_1 + C_2) \) is isomorphic to \( (\pi^{d_1})^{-1}(C_1) \times (\pi^{d_2})^{-1}(C_2) \).

Intuitively \( \pi^d \) is given as follows. By (1), it is enough to consider the case of genuine framed \( G \)-bundles \( (\mathcal{F}, \varphi) \). Let \( x \in \mathbb{C}^1 \) and consider the line \( \mathbb{P}_x = \{ z_1 = xz_0 \} \) in \( \mathbb{P}^2 \). If \( x = \infty \), we can regard \( \mathbb{P}_x^1 \) as the line \( \ell_{\infty} \) at infinity. If we restrict the \( G \)-bundle \( \mathcal{F} \) to \( \ell_{\infty} \), it is trivial. Since the triviality is an open condition, the restriction \( \mathcal{F}_{|\ell} \) is also trivial except for finitely many \( x \in \mathbb{C}^1 \), say \( x_1, x_2, \ldots, x_k \). Then \( \pi^d(\mathcal{F}, \varphi) \) is the sum \( x_1 + x_2 + \cdots + x_k \) if we assign the multiplicity appropriately.

2.4. Symplectic resolution As we will see in the next section, \( \mathcal{U}_G^d \) for \( G = \text{SL}(r) \) has a symplectic resolution. On the other hand, \( \mathcal{U}_G^d \) for \( G \neq \text{SL}(r) \) does not have a symplectic resolution. This can be checked as follows. First consider the case \( d = 1 \). It is well known that \( \mathcal{U}_G^1 \) is the product of \( \mathbb{C}^2 \) and the closure of the minimal nilpotent orbit in \( \varphi \).\footnote{I do not know who first noticed this fact, and even where a proof is written. In [2] it was shown that any \( G_c \) 1-instanton is reduced to an \( \text{SU}(2) \) 1-instanton, and the instanton moduli space (but not framed one) for \( \text{SU}(2) \) is determined. But this statement itself is not stated.} Fu proved that the minimal nilpotent orbit has no symplectic resolution except \( G = \text{SL}(r) \) [20]. Next consider \( d > 1 \). Take the second largest stratum \( \text{Bun}_G^{d-1} \times \mathbb{C}^2 \). By the factorization, \( \mathcal{U}_G^d \) is isomorphic to a product of a smooth variety and \( \mathcal{U}_G^d \) in a neighborhood of a point in the stratum. Fu’s argument is local (the minimal nilpotent orbit has an isolated singular point), hence \( \mathcal{U}_G^d \) does not have a symplectic resolution except \( G = \text{SL}(r) \).

2.5. Group action We have an action of a group \( G \) on \( \mathcal{U}_G^d \) by the change of framing. We also have an action of \( \text{GL}(2) \) on \( \mathcal{U}_G^d \) induced from the \( \text{GL}(2) \) action on \( \mathbb{C}^2 \). Let \( T \) be a maximal torus of \( G \). In these notes, we only consider the
action of the subgroup $T \times C^\times \times C^\times$ in $G \times GL(2)$. Let us introduce the following notation: $T = T \times C^\times \times C^\times$.

Our main player will be the equivariant intersection cohomology group $IH_T^*(\mathcal{U}^d_G)$. It is a module over

$$H^*_T(pt) \cong \mathbb{C}[\text{Lie } T] \cong \mathbb{C}[\varepsilon_1, \varepsilon_2, \bar{a}],$$

where $\varepsilon_1, \varepsilon_2$ (resp. $\bar{a}$) are coordinates (called equivariant variables) on $\text{Lie}(C^\times \times C^\times)$ (resp. $\text{Lie } T$).

We also use the notation

$$A_T = \mathbb{C}[\text{Lie } T] = \mathbb{C}[\bar{a}, \varepsilon_1, \varepsilon_2], \quad A = \mathbb{C}[\varepsilon_1, \varepsilon_2].$$

Their quotient fields are denoted by

$$F_T = \mathbb{C}(\text{Lie } T), \quad F = \mathbb{C}(\varepsilon_1, \varepsilon_2).$$

3. Heisenberg algebra action on Gieseker partial compactification

This lecture is an introduction to the actual content of subsequent lectures. We consider instanton moduli spaces when the gauge group $G$ is $SL(r)$. We will explain results about (intersection) cohomology groups of instanton moduli spaces known before the AGT correspondence was found. Then it will be clear what were lacking, and readers are motivated to learn more recent works.

3.1. Gieseker partial compactification

When the gauge group $G$ is $SL(r)$, we denote the corresponding Uhlenbeck partial compactification $\mathcal{U}^d_G$ by $\tilde{\mathcal{U}}^d_T$.

For $SL(r)$, we can consider a modification $\mathcal{U}^d_T$ of $\tilde{\mathcal{U}}^d_T$, called the Gieseker partial compactification. It is a moduli space of framed torsion free sheaves $(E, \varphi)$ on $\mathbb{P}^2$, where the framing $\varphi$ is a trivialization of the restriction of $E$ to the line at infinity $\ell_\infty$. It is known that $\tilde{\mathcal{U}}^d_T$ is a smooth (holomorphic) symplectic manifold. It is also known that there is a projective morphism $\pi: \tilde{\mathcal{U}}^d_T \to \mathcal{U}^d_T$, which is a resolution of singularities.

When $r = 1$, the group $SL(1)$ is trivial. But the Giesker space is nontrivial: $\tilde{\mathcal{U}}^d_T$ is the Hilbert scheme $(\mathbb{C}^2)^d$ of $d$ points on the plane $\mathbb{C}^2$, and the Uhlenbeck partial compactification\footnote{Since $\text{Bun}^d_{GL(1)} = \emptyset$ unless $d = 0$, this is a confusing name.} $\mathcal{U}^d_T$ is the $d^{th}$ symmetric product $S^d(\mathbb{C}^2)$ of $\mathbb{C}^2$. The former parametrizes ideals $I$ in the polynomial ring $\mathbb{C}[x, y]$ of two variables with colength $d$, i.e., $\dim \mathbb{C}[x, y]/I = d$. The latter is the quotient of the Cartesian product $(\mathbb{C}^2)^d$ by the symmetric group $S_d$ of $d$ letters. It parametrized $d$ unordered points in $\mathbb{C}^2$, possibly with multiplicities. We will use the summation notation like $p_1 + p_2 + \cdots + p_d$ or $d \cdot p$ to express a point in $S^d(\mathbb{C}^2)$.

For general $r$, these spaces can be understood as quiver varieties associated with Jordan quiver. It is not my intention to explain the theory of quiver varieties, but here is the definition in this case: Take two complex vector spaces of dimension $d, r$ respectively. Consider linear maps $B_1, B_2, I, J$ as in Figure 3.1.1. We
impose the equation
\[ \mu(B_1, B_2, I, J) \text{ def. } [B_1, B_2] + IJ = 0 \]
Then we take two types of quotient of \( \mu^{-1}(0) \) by \( GL(d) \). The first one corresponds to \( U_d \), and is the affine algebro-geometric quotient \( \mu^{-1}(0) // GL(d) \). It is defined as the spectrum of \( C[\mu^{-1}(0)]^{GL(d)} \), the ring of \( GL(d) \)-invariant polynomials on \( \mu^{-1}(0) \). Set-theoretically it is the space of closed \( GL(d) \)-orbits in \( \mu^{-1}(0) \). The second quotient corresponds to \( \tilde{U}_d \), and is the geometric invariant theory quotient with respect to the polarization given by the determinant of \( GL(d) \). It is \( \text{Proj} \left( \bigoplus_{n \geq 0} C[\mu^{-1}(0)]^{GL(d), \text{det}^n} \right) \), the ring of \( GL(d) \)-semi-invariant polynomials. Set-theoretically it is the quotient of stable points in \( \mu^{-1}(0) \) by \( GL(d) \). Here \((B_1, B_2, I, J)\) is stable if there is no proper subspace \( T \) of \( C^d \) which is invariant under \( B_1, B_2 \) and is containing the image of \( I \).

From this description, we can check the stratification \((2.2.1)\). If \((B_1, B_2, I, J)\) has a closed \( GL(d) \)-orbit, it is semisimple, i.e., a ‘submodule’ (in appropriate sense) has a complementary submodule. Thus \((B_1, B_2, I, J)\) decomposes into a direct sum of simple modules, which do not have nontrivial submodules. There is exactly one simple summand with nontrivial \( I, J \), and all others have \( I = J = 0 \). The former gives a point in \( \text{Bun}_d \). The latter is a pair of commuting matrices \([B_1, B_2] = 0\), and the simplicity means that the size of matrices is 1. Therefore the simultaneous eigenvalues give a point in \( C^2 \).

Let us briefly recall how those linear maps determine points in \( \tilde{U}_d \) and \( U_d \). The detail was given in [41, Ch. 2]. Given \((B_1, B_2, I, J)\), we consider the following complex
\[
\begin{align*}
0 & \oplus d \\
0 \oplus d (-1) & \xrightarrow{\alpha} 0 \oplus d & \xrightarrow{\beta} 0 \oplus d (1), \\
& \oplus & \oplus
\end{align*}
\]
where
\[
\begin{pmatrix}
z_0 B_1 - z_1 \\
z_0 B_2 - z_2 \\
z_0 I
\end{pmatrix}, \quad \begin{pmatrix}
- (z_0 B_2 - z_2) & z_0 B_1 - z_1 & z_0 I
\end{pmatrix}.
\]
Here \([z_0 : z_1 : z_2]\) is the homogeneous coordinate system of \(\mathbb{P}^2\) such that \(\ell_\infty = \{z_0 = 0\}\). The equation \(\mu = 0\) guarantees that this is a complex, i.e., \(ba = 0\). One sees easily that \(a\) is injective on each fiber over \([z_0 : z_1 : z_2]\) except for finitely many. The stability condition implies that \(b\) is surjective on each fiber. It implies that \(E \overset{\text{def}}{=} \ker b / \im a\) is a torsion free sheaf of rank \(r\) with \(c_2 = d\). Considering the restriction to \(z_0 = 0\), one sees that \(E\) has a canonical trivialization \(\varphi\) there. Thus we obtain a framed sheaf \((E, \varphi)\) on \(\mathbb{P}^2\). If \(a\) is injective on any fiber, \(E\) is a locally free sheaf, i.e., a vector bundle.

**Exercise 3.1.2.** (a) We define the factorization morphism \(\pi^d\) for \(G = \text{SL}(r)\) in terms of \((B_1, B_2, I, J)\). Let \(\pi^d(\mathcal{B}_1, \mathcal{B}_2, I, J) \in S^d C\) be the spectrum of \(B_1\) counted with multiplicities. Check that \(\pi^d\) satisfies the properties (1),(2) above.

(b) Check that \(\mathcal{U}_I\) is trivial if \(B_1 - x\) is invertible.

More generally one can define the projection as the spectrum of \(a_1B_1 + a_2B_2\) for \((a_1, a_2) \in \mathbb{C}^2 \setminus \{0\}\), but it is enough to check this case after a rotation by the \(\text{GL}(2)\)-action.

**3.2. Tautological bundle** Let \(V\) be the tautological bundle over \(\mathbb{A}^d_r\). It is a rank \(d\) vector bundle whose fiber at a framed torsion free sheaf \((E, \varphi)\) is \(H^1(\mathbb{P}^2, E(-\ell_\infty))\). In the quiver variety description, it is the vector bundle associated with the principal \(\text{GL}(d)\)-bundle \(\mu^{-1}(0)_{\text{stable}} \to \mathbb{A}^d_r\). For Hilbert schemes of points, parametrizing ideals \(I\) in \(\mathbb{C}[x, y]\), the fiber at \(I\) is \(\mathbb{C}[x, y]/I\).

**3.3. Gieseker-Uhlenbeck morphism**

**Proposition 3.3.1** ([41, Exercise 5.15]). \(\pi_! : \mathbb{A}^d_r \to \mathcal{U}_I^d\) is semi-small with respect to the stratification (2.2.2). Moreover the fiber \(\pi_!(x)\) is irreducible at any point \(x \in \mathcal{U}_I^d\).

Recall that a (surjective) projective morphism \(\pi : M \to X\) from a nonsingular variety \(M\) is semi-small if \(X\) has a stratification \(X = \bigsqcup X_\alpha\) such that \(\pi^{-1}(x_\alpha)\) is a topological fibration, and \(\dim \pi^{-1}(x_\alpha) \leq \frac{1}{2}\ \text{codim} \ X_\alpha\) for \(x_\alpha \in X_\alpha\).

This semi-smallness result is proved for general symplectic resolutions by Kaledin [26].

[41, Exercise 5.15] asks the dimension of the central fiber \(\pi^{-1}(d \cdot 0)\). Let us explain why the estimate for the central fiber is enough. Let us take \(x \in \mathcal{U}_I^d\) and write it as \((E, \varphi, \sum \lambda_i x_i)\), where \((E, \varphi) \in \text{Bun}^d_{\text{SL}(r)}, x_i \neq x_j\). The morphism \(\pi^{-1}(x)\) is assigning \((E^{\vee \vee}, \varphi, \text{Supp}(E^{\vee \vee}/E))\) to a framed torsion free sheaf \((E, \varphi) \in \mathbb{A}^d_r\). See [41, Exercise 3.53]. Then \(\pi^{-1}(x)\) parametrizes quotients of \(E^{\vee \vee}\) with given multiplicities \(\lambda_i\) at \(x_i\). Then it is clear that \(\pi^{-1}(x)\) is isomorphic to the product of quotients of \(E^{\otimes r}\) with multiplicities \(\lambda_i\) at \(0\), i.e., \(\prod \pi^{-1}(\lambda_i \cdot 0)\). If one knows each \(\pi^{-1}(\lambda_i \cdot 0)\) has dimension \(r\lambda_i - 1\), we have \(\dim \pi^{-1}(x) = \sum (r\lambda_i - 1) = \frac{1}{2}\ \text{codim} \ \text{Bun}^d_{\text{SL}(r)} \times \mathcal{S}(\mathbb{C}^2)\). Thus it is enough to check that \(\pi^{-1}(d \cdot 0) = rd - 1\).

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\*The solution of exercise only shows there is only one irreducible component of \(\pi^{-1}(x_\alpha)\) with dimension \(\frac{1}{2}\ \text{codim} \ X_\alpha\). The irreducibility was proved by Baranovsky and Ellingsrud-Lehn.*
3.4. Equivariant cohomology groups of Giesker partial compactification

We have a group action of $G \times \text{GL}(2)$ on $\tilde{U}_r^d$, $U_r^d$ as in §2.5. The Gieseker-Uhlenbeck morphism $\pi: \tilde{U}_r^d \to U_r^d$ is equivariant. Let $T = T \times C^\times \times C^\times$ as in §2.5. We will study the equivariant cohomology groups of Giesker spaces

$$H^s_T(\tilde{U}_r^d), \quad H^s_{T,c}(\tilde{U}_r^d)$$

with arbitrary and compact support respectively.

We denote the equivariant variables by $\bar{a} = (a_1, \ldots, a_r)$ with $a_1 + \cdots + a_r = 0$ for $T$, and $\varepsilon_1, \varepsilon_2$ for $C^\times \times C^\times$. (See (2.5.1).)

We have the intersection pairing

$$H^s_T(\tilde{U}_r^d) \otimes H^s_{T,c}(\tilde{U}_r^d) \to H^s_T(\text{pt}); \ c \otimes c' \mapsto (-1)^{dr} \int_{\tilde{U}_r^d} c \cup c'.$$

This is of degree 0. The sign $(-1)^{dr}$ is introduced to save $(-1)^r$ in the later formula. The factor $dr$ should be understood as the half of the dimension of $\tilde{U}_r^d$. Similarly the intersection form on $H^s_{C^\times \times C^\times}(C^2)$ has the sign factor $(-1)^1 = (-1)^{\dim C^2/2}$.

3.5. Heisenberg algebra via correspondences

Let $n > 0$. Let us consider a correspondence in the triple product $\tilde{U}_r^{d+n} \times U_r^d \times C^2$:

$$P_n \overset{\text{def.}}{=} \left\{ (E_1, \varphi_1, E_2, \varphi_2, x) \in \tilde{U}_r^{d+n} \times \tilde{U}_r^d \times C^2 | E_1 \subset E_2, \text{Supp}(E_2/E_1) = \{x\} \right\}.$$

Here the condition $\text{Supp}(E_2/E_1) = \{x\}$ means the quotient sheaf $E_2/E_1$ is 0 outside $x$. In the left hand side the index $d$ is omitted: we understand either $P_n$ is the disjoint union for various $d$, or $d$ is implicit from the situation. It is known that $P_n$ is a lagrangian subvariety in $\tilde{U}_r^{d+n} \times \tilde{U}_r^d \times C^2$.

We have two projections $q_1: P_n \to \tilde{U}_r^{d+n}$, $q_2: P_n \to \tilde{U}_r^d \times C^2$, which are proper. The convolution product gives an operator

$$p_\Delta_n(\alpha): H^s_T(\tilde{U}_r^d) \to H^{s+\deg \alpha}_T(\tilde{U}_r^{d+n}); c \mapsto q_1^*(q_2^*(c \otimes \alpha) \cap (P_n))$$

for $\alpha \in H^s_{C^\times \times C^\times}(C^2)$. The meaning of the superscript $'\Delta'$ will be explained later. We also consider the adjoint operator

$$P^n_\Delta(\alpha) = (p_\Delta_n(\alpha))^*: H^{s+\deg \alpha}_{T,c}(\tilde{U}_r^{d+n}) \to H^s_{T,c}(\tilde{U}_r^d).$$

A class $\beta \in H^s_{C^\times \times C^\times}(C^2)$ with compact support gives operators $p^n_\Delta_n(\beta): H^s_{T,c}(\tilde{U}_r^d) \to H^{s+\deg \beta}_T(\tilde{U}_r^{d+n})$, and $P^n_\Delta_n(\beta) = (p^n_\Delta_n(\beta))^*: H^s_{T,c}(\tilde{U}_r^d) \to H^{s+\deg \beta}_T(\tilde{U}_r^{d+n})$.

**Theorem 3.5.1** ([24, 39] for $r = 1$, [4] for $r \geq 2$). As operators on $\bigoplus_d H^s_{T,c}(\tilde{U}_r^d)$ or $\bigoplus_d H^s_T(\tilde{U}_r^d)$, we have the Heisenberg commutator relations

$$(3.5.2) \quad \left[ p^n_\Delta_m(\alpha), p^n_\Delta_n(\beta) \right] = r m \delta_{m,n} \langle \alpha, \beta \rangle \text{id}.$$

Here $\alpha, \beta$ are equivariant cohomology classes on $C^2$ with arbitrary or compact support. When the right hand side is nonzero, $m$ and $n$ have different sign, hence one of $\alpha, \beta$ is compact support and the other is arbitrary support. Then $\langle \alpha, \beta \rangle$ is well-defined.
Historical Comment 3.5.3. As mentioned in Introduction, the author [38] found relation between representation theory of affine Lie algebras and moduli spaces of instantons on $\mathbb{C}^2/\Gamma$, where the affine Lie algebra is given by $\Gamma$ by the McKay correspondence. It was motivated by works by Ringel [52] and Lusztig [30], constructing upper triangular subalgebras of quantum enveloping algebras by representations of quivers.

The above theorem can be regarded as the case $\Gamma = \{1\}$, but the Heisenberg algebra is not a Kac-Moody Lie algebra, and hence it was not covered in [38], and dealt with later [4, 24, 39]. Note that a Kac-Moody Lie algebra only has finitely many generators and relations, while the Heisenberg algebra has infinitely many.

A particular presentation of an algebra should not be fundamental, so it was desirable to have more intrinsic construction of those representations. More precisely, a definition of an algebra by convolution products is natural, but we would like to understand why we get a particular algebra, namely the affine Lie algebra in our case. We do not have a satisfactory explanation yet. The same applies to Ringel, Lusztig’s constructions.

Exercise 3.5.4. [41, Remark 8.19] Define operators $P_{\pm 1}^{\alpha} \left( \alpha \right)$ acting on $\bigoplus_n H^a(S^n X)$ for a (compact) manifold $X$ in a similar way, and check the commutation relation (3.5.2) with $r = 1$.

3.6. Dimensions of cohomology groups When $r = 1$, it is known that the generating function of dimension of $H_T^{a}((\mathbb{C}^2)^{|d|})$ over $A_T = H_T^{a}(pt)$ for $d \geq 0$ is

$$\sum_{d=0}^{\infty} \dim H_T^{a}((\mathbb{C}^2)^{|d|}) q^d = \prod_{d=1}^{\infty} \frac{1}{1-q^d}.$$ (See [41, Chap. 5].) This is also equal to the character of the Fock space of the Heisenberg algebra. Therefore the Heisenberg algebra action produces all cohomology classes from the vacuum vector $|\text{vac}\rangle = 1_{(\mathbb{C}^2)^{[0]}} \in H_T^{a}((\mathbb{C}^2)^{[0]})$

On the other hand we have

$$\sum_{d=0}^{\infty} \dim H_T^{a}(\tilde{U}_r^d) q^d = \prod_{d=1}^{\infty} \frac{1}{(1-q^d)^r}.$$ (3.6.1)

for general $r$. Therefore the Heisenberg algebra is smaller than the actual symmetry of the cohomology groups.

Let us explain how to see the formula (3.6.1). Consider the torus $T$ action on $\tilde{U}_r^d$. A framed torsion free sheaf $(E, \varphi)$ is fixed by $T$ if and only if it is a direct sum $(E_1, \varphi_1) \oplus \cdots \oplus (E_r, \varphi_r)$ of rank 1 framed torsion free sheaves. Rank 1 framed torsion free sheaves are nothing but ideal sheaves on $\mathbb{C}^2$, hence points in Hilbert

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The Fock space is the polynomial ring of infinitely many variables $x_1, x_2, \ldots$. The operators $P_{\pm 1}^{\alpha} \left( \alpha \right)$ act by either multiplication of $x_n$ or differentiation with respect to $x_n$ with appropriate constant multiplication. It has the highest weight vector (or the vacuum vector) $1$, which is killed by $P_{\pm 1}^{\alpha} \left( \alpha \right)$ with $n > 0$. The Fock space is spanned by vectors given by operators $P_{\pm 1}^{\alpha} \left( \alpha \right)$ successively to the highest weight vector.
schemes. Thus
\[(\tilde{U}_r^d)^T = \bigcup_{d_1+\ldots+d_r=d} (C^2)^{[d_1]} \times \ldots \times (C^2)^{[d_r]}.
\]
Hence
\[
\bigoplus_d H_T^{[s]}((\tilde{U}_r^d)^T) = \bigoplus_{d_1,\ldots,d_r} H_T^{[s]}((C^2)^{[d_1]} \times \ldots \times (C^2)^{[d_r]})
\]
\[(3.6.3)
= \bigotimes_{i=1}^\infty \bigoplus_{d_i=0}^r H_T^{[s]}((C^2)^{[d_i]}).
\]

To compute the dimension of $H_T^{[s]}(\tilde{U}_r^d)$ over $A_T$, we restrict equivariant cohomology groups to generic points, that is to consider tensor products with the fractional field $\mathbb{F}$ of $A_T$. Then the localization theorem for equivariant cohomology groups gives an isomorphism between $H_T^{[s]}(\tilde{U}_r^d)$ and $H_T^{[s]}((\tilde{U}_r^d)^T)$ over $\mathbb{F}$. Therefore the above observation gives the formula (3.6.1).

In view of (3.6.1), we have the action of $r$ copies of Heisenberg algebra on $\bigoplus_d H_T^{[s]}((\tilde{U}_r^d)^T)$ and hence on $\bigoplus_d H_T^{[s]}(\tilde{U}_r^d)$ by the localization theorem. It is isomorphic to the tensor product of $r$ copies of the Fock module, so all cohomology classes are produced by the action. This is a good starting point to understand $\bigoplus_d H_T^{[s]}(\tilde{U}_r^d)$. However this action cannot be defined over non localized equivariant cohomology groups. In fact, $P_n^A(\alpha)$ is the ‘diagonal’ Heisenberg in the product, and other non diagonal generators have no description like convolution by $P_n$.

The correct algebra acting on $\bigoplus_d H_T^{[s]}(\tilde{U}_r^d)$ is the $W$-algebra $W(gl(r))$ associated with $gl(r)$. It is the tensor product of the $W$-algebra $W(sl(r))$ and the Heisenberg algebra (as the vertex algebra). Its Verma module has the same size as the tensor product of $r$ copies of the Fock module.

This result is due to Schiffmann-Vasserot [53] and Maulik-Okounkov [34] independently.

3.7. Intersection cohomology group  Recall that the decomposition theorem has a nice form for a semi-small resolution $\pi: M \to X$:
\[(3.7.1) \quad \pi_* (\mathcal{E}_M) = \bigoplus_{\alpha,\chi} IC(X_{\alpha},\chi) \otimes H_{[0]}(\pi^{-1}(x_{\alpha}))_{\chi},
\]
where we have used the following notation:

- $\mathcal{E}_M$ denotes the shifted constant sheaf $\mathcal{C}_M[\dim M]$.
- $IC(X_{\alpha},\chi)$ denotes the intersection cohomology complex associated with a simple local system $\chi$ on $X_{\alpha}$. (We will simply write $IC(X_{\alpha})$ when $\chi$ is trivial. We may also write it $IC(X)$ if $X_{\alpha}$ is the open dense in $X$.)
- $H_{[0]}(\pi^{-1}(x_{\alpha}))$ is the homology group of the shifted degree 0, which is the usual degree codim $X_{\alpha}$. When $x_{\alpha}$ moves in $X_{\alpha}$, it forms a local system. $H_{[0]}(\pi^{-1}(x_{\alpha}))_{\chi}$ denotes its $\chi$-isotropic component.
Exercice 3.7.2. Let $Gr(d,r)$ be the Grassmannian of $d$-dimensional subspaces in $\mathbb{C}^r$, where $0 \leq d \leq r$. Let $M = T^*Gr(d,r)$. Determine $X = \text{Spec}(\mathbb{C}[M])$. Study fibers of the affinization morphism $\pi: M \to X$ and show that $\pi$ is semi-small. Compute graded dimensions of $H^*$ of strata, using the well-known computation of Betti numbers of $T^*Gr(d,r)$.

Consider the Gieseker-Uhlenbeck morphism $\pi: \tilde{U}^d \to U^d$. By Proposition 3.3.1, any fiber $\pi^{-1}(x_{\alpha})$ is irreducible. Therefore all the local systems are trivial, and

$$\pi_*(e_{U^d}) = \bigoplus_{d = |\lambda| + d'} IC(Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2)) \otimes \mathbb{C}[\pi^{-1}(x_{\lambda}')]$$

where $x_{\lambda}'$ denotes a point in the stratum $Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2)$, and $[\pi^{-1}(x_{\lambda}')]$ denotes the fundamental class of $\pi^{-1}(x_{\lambda}')$, regarded as an element of $H_0((\mathbb{C}^2) \otimes \mathbb{C}[\pi^{-1}(x_{\lambda}')]$.

The main summand is $IC(Bun_{SL(r)}^d)$, and other smaller summands could be understood recursively as follows. Let us write the partition $\lambda$ as $(1^{a_1}2^{a_2}\ldots)$, when $1$ appears $a_1$ times, $2$ appears $a_2$ times, and so on. We set $l(\lambda) = a_1 + a_2 + \cdots$ and $\text{Stab}(\lambda) = S_{a_1} \times S_{a_2} \times \cdots$. Note that we have total $l(\lambda)$ distinct points in $S_{\lambda}(\mathbb{C}^2)$. The group $\text{Stab}(\lambda)$ is the group of symmetries of a configuration in $S_{\lambda}(\mathbb{C}^2)$. We have a finite morphism

$$\xi: \tilde{U}^d \to Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2)$$

extending the identity on $Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2)$. Then $IC(Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2))$ is the direct image of IC of the domain. We have the Künneth decomposition for the domain, and the factor $(\mathbb{C}^2)^{l(\lambda)} / \text{Stab}(\lambda)$ is a quotient of a smooth space by a finite group. Therefore the IC of the second factor is the (shifted) constant sheaf. We thus have

$$IC(Bun_{SL(r)}^d \times S_{\lambda}(\mathbb{C}^2)) \equiv \xi_* \left( IC(Bun_{SL(r)}^d) \boxtimes e_{(\mathbb{C}^2)^{l(\lambda)} / \text{Stab}(\lambda)) \right).$$

Thus

$$H^{|\lambda|}(\tilde{U}^d) = \bigoplus_{d = |\lambda| + d'} H^{|\lambda|}_{\mathbb{T}}(Bun_{SL(r)}^d) \otimes H^{|\lambda|}_{\mathbb{T}}((\mathbb{C}^2)^{l(\lambda)} / \text{Stab}(\lambda)) \otimes \mathbb{C}[\pi^{-1}(x_{\lambda}')] \text{.}$$

This decomposition nicely fits with the Heisenberg algebra action. Note that the second and third factors are both 1-dimensional. Thus we have 1-dimensional space for each partition $\lambda$. If we take the sum over $d$, it has the size of the Fock module, and it is indeed the submodule generated by the vacuum vector $|\text{vac}\rangle = 1_{\tilde{U}^0} \in H^{|\lambda|}(\tilde{U}^d)$. This statement can be proved by the analysis of the convolution algebra in [15, Chap. 8], but it is intuitively clear as the 1-dimensional space corresponding to $\lambda$ is the span of $P_{-1}(1)^{a_1}P_{-2}(1)^{a_2}\cdots|\text{vac}\rangle$.

The Heisenberg algebra acts trivially on the remaining factor

$$\bigoplus_d H^{|\lambda|}_{\mathbb{T}}(\tilde{U}_{SL(r)}^d).$$
(Here we write $\IH^s_T(U\mathfrak{l}^d_{\mathfrak{sl}(r)})$ instead of $\IH^s_T(Bun^d_{\mathfrak{sl}(r)})$, as $Bun^d_{\mathfrak{sl}(r)} = U\mathfrak{l}_{\mathfrak{sl}(r)}$.) The goal of these notes is to see that it is a module of $W(sl(r))$, and the same is true for ADE groups $G$, not only for $SL(r)$.

**Exercise 3.7.4.** Show the above assertion that the Heisenberg algebra acts trivially on the first factor $\bigoplus_d \IH^s_T(U\mathfrak{l}^d_{\mathfrak{sl}(r)})$.

### 4. Stable envelops

The purpose of this lecture is to explain the stable envelop introduced in [34]. It will nicely explain a relation between $H^s_T(U\mathfrak{l}^d)$ and $H^s_T(U\mathfrak{l}^d_T)$. This is what we need to clarify, as we have explained in the previous lecture. The stable envelop also arises in many other situations in geometric representation theory. Therefore we explain it in a wider context, as in the original paper [34].

#### 4.1. Setting – symplectic resolution

Let $\pi: M \to X$ be a resolution of singularities of an affine algebraic variety $X$. We assume $M$ is symplectic. We suppose that a torus $T$ acts on both $M$, $X$ so that $\pi$ is $T$-equivariant. We suppose $T$-action on $X$ is linear. We also assume a technical condition that $T$ contains $\mathbb{C}^\times$ such that $X$ is a cone with respect to $\mathbb{C}^\times$ and the weight of the symplectic form is positive except in §4.7. (This assumption will be used when we quote a result of Namikawa later.) Let $T$ be a subtorus of $T$ which preserves the symplectic form of $M$.

**Example 4.1.1.** Our basic example is $M = U\mathfrak{u}^d$, $X = \mathfrak{u}^d$ and $\pi$ is the Gieseker-Uhlenbeck morphism with the same $T$, $T$ as above. In fact, we can also take a larger torus $T \times C^\times_{hyp}$ in $T$, where $C^\times_{hyp} \subset C^\times \times C^\times$ is given by $t \mapsto (t, t^{-1})$.

**Example 4.1.2.** Another example is $M = T^\ast(\text{flag variety}) = T^\ast(G/B)$, $X = (\text{nilpotent variety})$ and $\pi$ is the Grothendieck-Springer resolution. Here $T$ is a maximal torus of $G$ contained in $B$, and $T = T \times C^\times$, where $C^\times$ acts on $X$ by scaling on fibers.

We can also consider the same $\pi: M \to X$ as above with smaller $T$, $T$. Let $M^T$ be the $T$-fixed point locus in $M$. It decomposes $M^T = \bigsqcup F_\alpha$ to connected components, and each $F_\alpha$ is a smooth symplectic submanifold of $M$. Let $i: M^T \to M$ be the inclusion. We have the pull-back homomorphism

$$i^*: H^s_T|M^T| \to H^s_T|U\mathfrak{l}^d + \text{codim } X^T|(M^T) = \bigoplus_\alpha H^s_T|U\mathfrak{l}^d + \text{codim } F_\alpha|(F_\alpha).$$

Here we take the degree convention as before. Our degree 0 is the usual degree $\dim_\mathbb{C} M$ for $H^0_T(M)$, and $\dim_\mathbb{C} F_\alpha$ for $H^0_T(F_\alpha)$. Since $i^*$ preserves the usual degree, it shifts our degree by $\text{codim } F_\alpha$. Each $F_\alpha$ has its own codimension, but we denote the direct sum as $H^s_T|U\mathfrak{l}^d + \text{codim } X^T|(M^T)$ for brevity.

The stable envelop we are going to construct goes in the opposite direction $H^s_T(M^T) \to H^s_T(M)$ and preserves (our) degree.
In the above example $M = \tilde{U}_r^d$, the $T$ and $T \times C_{hyp}^\times$-fixed point loci are

$$\{\tilde{U}_r^d\}^T = \bigsqcup_{d_1+\ldots+d_r=d} \tilde{U}_1^{d_1} \times \cdots \times \tilde{U}_r^{d_r},$$

$$\{\tilde{U}_r^d\}^{T \times C_{hyp}^\times} = \bigsqcup_{|\lambda_1|+\ldots+|\lambda_r|} \{I_{\lambda_1} \oplus \cdots \oplus I_{\lambda_r}\},$$

where $\lambda_i$ is a partition, and $I_{\lambda_i}$ is the corresponding monomial ideal sheaf (with the induced framing).

Also

$$T^*(G/B)^T = W,$$

where $W$ is the Weyl group.

**4.2. Chamber structure** Let us consider the space $\text{Hom}_{grp}(C \times, T)$ of one parameter subgroups in $T$, and its real form $\text{Hom}_{grp}(C \times, T) \otimes_\mathbb{R} \mathbb{R}$. A generic one parameter subgroup $\rho$ satisfies $M^\rho(C \times) = M^T$. But if $\rho$ is special (the most extreme case is $\rho$ is the trivial), the fixed point set $M^\rho(C \times)$ could be larger. This gives us a 'chamber' structure on $\text{Hom}_{grp}(C \times, T) \otimes_\mathbb{R} \mathbb{R}$, where a chamber is a connected component of the complement of the union of hyperplanes given by $\rho$ such that $M^\rho(C \times) \neq M^T$.

**Exercise 4.2.1.** (1) In terms of $T$-weights on tangent spaces $T_p M$ at various fixed points $p \in M^T$, describe the hyperplanes.

(2) Show that the chamber structure for $M = T^*(\text{flag variety})$ is identified with usual Weyl chambers.

(3) Show that the chamber structure for $M = \tilde{U}_r^d$ is identified with the usual Weyl chambers for $\text{SL}(r)$.

(4) Compute the chamber structure for $M = \tilde{U}_r^d$, but with the larger torus $T \times C_{hyp}^\times$.

For a chamber $\mathcal{C}$, we have the **opposite chamber** $-\mathcal{C}$ consisting of one parameter subgroups $t \mapsto \rho(t^{-1})$ for $\rho \in \mathcal{C}$.

The stable envelop depends on a choice of a chamber $\mathcal{C}$.

**4.3. Attracting set** Let $\mathcal{C}$ be a chamber and $\rho \in \mathcal{C}$. We define the **attracting set** $A_X$ by

$$A_X = \left\{ x \in X \mid \exists \lim_{t \to 0} \rho(t)x \right\}.$$ 

We similarly define the attracting set $A_M$ in $M$ in the same way. As $\pi$ is proper, we have $A_M = \pi^{-1}(A_X)$. We put the scheme structure on $A_M$ as $\pi^{-1}(A_X)$ in these notes.

**Example 4.3.1.** Let $X = \mathcal{U}_2^d$. In the quiver description, $A_X$ consists of closed $\text{GL}(d)$-orbits $\text{GL}(d)(B_1, B_2, I, J)$ such that $Jf(B_1, B_2)I$ is upper triangular for any noncommutative monomial $f \in C(x, y)$. It is the **tensor product variety** introduced in [43], denoted by $\pi(3)$ therein.
As framed sheaves, \( A_M \) consists of \((E, \varphi)\) which are written as an extension \(0 \to E_1 \to E \to E_2 \to 0\) (compatible with the framing) for some \(E_1 \in \mathcal{U}_1^{d_1}, E_2 \in \mathcal{U}_1^{d_2}\) with \(d = d_1 + d_2\).

We have the following diagram
\[
X^{\rho(C^\times)} = X^T_p \xrightarrow{\rho} A_X \xrightarrow{i} X,
\]
where \(i, j\) are natural inclusion, and \(p\) is given by \(A_x \ni x \mapsto \lim_{t \to 0} \rho(t)x\).

### 4.4. Leaves

Since \(M^T = \bigsqcup F_{\alpha}\), we have the corresponding decomposition of \(A_M = \bigsqcup p^{-1}(F_{\alpha})\). Let \(\text{Leaf}_{\alpha} = p^{-1}(F_{\alpha})\). By the Bialynicki-Birula theorem ([6], see also [14]), \(p: \text{Leaf}_{\alpha} \to F_{\alpha}\) is a \(T\)-equivariant affine bundle. Similarly \(\text{Leaf}_{\alpha}^\perp \to F_{\alpha}\) denote the corresponding affine bundle for the opposite chamber \(\perp\). (For quiver varieties, they are in fact isomorphic to vector bundles \(L_{\alpha}^\pm\) below. See [43, Prop. 3.14].)

Let us consider the restriction of the tangent bundle \(TM\) to a fixed point component \(F_{\alpha}\). It decomposes into weight spaces with respect to \(\rho\):
\[
TM|_{F_{\alpha}} = \bigoplus T(m), \quad T(m) = \{ v | \rho(t)v = t^m v \}.
\]
Then \(T \text{Leaf}_{\alpha} = \bigoplus_{m \geq 0} T(m)\). Note also \(T(0) = TF_{\alpha}\). Since \(T\) preserves the symplectic form, \(T(m)\) and \(T(-m)\) are dual to each other. From these, one can also check that \(\text{Leaf}_{\alpha}^\perp \times F_{\alpha} \to M \times F_{\alpha}\) is a lagrangian embedding.

For a later purpose let
\[
L_{\alpha}^\pm \overset{\text{def.}}{=} \bigoplus_{m > 0} T(m).
\]
The normal bundle of \(F_{\alpha}\) in \(\text{Leaf}_{\alpha}\) (resp. \(\text{Leaf}_{\alpha}^\perp\)) is \(L_{\alpha}^\pm\) (resp. \(L_{\alpha}^\perp\)).

**Example 4.4.3.** Let \(\pi: M = T^*\mathbb{P}^1 \to X = \mathbb{C}^2/\pm\). Let \(T = \mathbb{C}^\times\) act on \(X\) and \(\mathbb{C}^2/\pm 1\) so that it is given by \(t(z_1, z_2) \mod \pm = (tz_1, tz_2^{1-1}) \mod \pm\). Then \(X^T\) consists of two points \((0, 0)\) in the zero section \(\mathbb{P}^1\) of \(T^*\mathbb{P}^1\). If we take the ‘standard’ chamber containing the identity operator, \(\text{Leaf}_0\) is the zero section \(\mathbb{P}^1\) minus \(\infty\). On the other hand \(\text{Leaf}_\infty\) is the (strict transform of) the axis \(z_2 = 0\). See Figure 4.4.4.

For the opposite chamber, \(\text{Leaf}_0\) is the axis \(z_1 = 0\), and \(\text{Leaf}_\infty\) is the zero section minus \(0\).

**Definition 4.4.5.** We define a partial order \(\succeq\) on the index set \((\alpha)\) for the fixed point components so that
\[
\overline{\text{Leaf}_{\beta} \cap F_{\alpha}} \neq \emptyset \implies \alpha \succeq \beta.
\]
We have \(\infty \leq 0\) in Example 4.4.3.

Let
\[
A_{M, \leq \alpha} = \bigcup_{\beta : \beta \leq \alpha} \text{Leaf}_{\beta}.
\]
Then \(A_{M, \leq \alpha}\) is a closed subvariety. We define \(A_{M, < \alpha}\) in the same way.
Proposition 4.4.6. (1) $H^T_{[s]}(A_{M, \leq \alpha})$ vanishes in the odd degree.
(2) We have an exact sequence

$$0 \to H^T_{[s]}(A_{M, < \alpha}) \to H^T_{[s]}(A_{M, \leq \alpha}) \to H^T_{[s]}(\text{Leaf}_\alpha) \to 0.$$  

Proof. Consider the usual long exact sequence

$$H^T_{[s]}(A_{M, < \alpha}) \to H^T_{[s]}(A_{M, \leq \alpha}) \to H^T_{[s]}(\text{Leaf}_\alpha) \to H^T_{[s-1]}(A_{M, < \alpha}).$$

Recall that Leaf$_\alpha$ is an affine bundle over $F_\alpha$. Hence the pullback $H^T_{[s]}(F_\alpha) \to H^T_{[s]}(\text{Leaf}_\alpha)$ is an isomorphism. It is known that $H^T_{[s]}(F_\alpha)$ vanishes in odd degrees. (It follows from [41, Exercise 5.15] for $\tilde{U}^d$, and is a result of Kaledin [26] in general.)

Let us show that $H^T_{[s]}(A_{M, \leq \alpha})$ vanishes in odd degrees by the descending induction on $\alpha$. In particular, the assertion for $H^T_{[s]}(A_{M, < \alpha})$ implies $\delta = 0$, i.e., (2).

If $\alpha$ is larger than the maximal element, $A_{M, \leq \alpha} = \emptyset$, hence the assertion is true. Suppose that the assertion is true for $H^T_{[s]}(A_{M, < \alpha})$. Then the above exact sequence and the odd vanishing of $H^T_{[s]}(\text{Leaf}_\alpha)$ implies the odd vanishing of $H^T_{[s]}(A_{M, \leq \alpha})$. □

Exercise 4.4.7. (1) Determine $A_M$ for $M = T^*(\text{flag variety})$.
(2) Determine $A_M$ for $M = \tilde{U}^d$ with respect to $T \times C^\infty_{\text{hyp}}$.

4.5. Steinberg type variety  Recall that Steinberg variety is the fiber product of $T^*(\text{flag variety})$ with itself over nilpotent variety. Its equivariant K-group realizes the affine Hecke algebra (see [15, Ch. 7]), and it plays an important role in the geometric representation theory.

Let us recall the definition of the product in our situation. We define $Z = M \times_X M$, the fiber product of $M$ itself over $X$. Its equivariant Borel-Moore homology group has the convolution product:

$$H^T_{[s]}(Z) \otimes H^T_{[s]}(Z) \ni c \otimes c' \mapsto p_{13*}(p_{12}^*c \cap p_{23}^*c') \in H^T_{[s]}(Z),$$
where \( p_{ij} : M \times M \times M \to M \times M \) is the projection to the product of \( i^{th} \) and \( j^{th} \) factors.\(^{10}\) When \( M \to X \) is semi-small, one can check that the multiplication preserves the shifted degree \([*]\).

We introduce a variant, mixing the fixed point set \( M^T \) and the whole variety \( M \): Let \( Z_A \) be the fiber product of \( A_M \) and \( M^T \) over \( X^T \), considered as a closed subvariety in \( M \times M^T \):

\[
Z_A = A_M \times_{X^T} M^T \subset M \times M^T,
\]

where \( A_M \to X^T \) is the composite of \( p : A_M \to M^T \) and \( \pi : M^T \to M_0 \), or alternatively \( \pi |_{A_M} : A_M \to A_{M_0} \) and \( p : A_{M_0} \to X^T \). Here we denote projections \( A_M \to M^T \) and \( A_X \to X^T \) both by \( p \) for brevity. As a subvariety of \( M \times M^T \), \( Z_A \) consists of pairs \((x, x')\) such that \( \lim_{t \to 0} p(t) \pi(x) = \pi(x') \).

The convolution product as above defines a \((H^T_{[s]}(Z), H^T_{[s]}(Z_A))\)-bimodule structure on \( H^T_{[s]}(Z_A) \). Here \( Z^T \) is the \( T \)-fixed point set in \( Z \), or equivalently the fiber product of \( M^T \) with itself over \( X^T \).

In our application, we use \( H^T_{[s]}(Z) \) as follows: we shall construct an operator \( H^T_{[s]}(M^T) \to H^T_{[s]}(M) \) by

\[
H^T_{[s]}(M^T) \ni \xi, \quad L \overset{\xi}{\to} \bigl( p_1^*(p_2^* \xi \cap L) \bigr) \in H^T_{[s]}(M)
\]

for a suitably chosen (degree 0) equivariant class \( L \) in \( H^T_{[0]}(Z_A) \). Note that the projection \( Z_A \to M \) is proper, hence the operator in this direction is well-defined. On the other hand, \( Z_A \to M^T \) is not proper. See §4.9 below.

Recall \( A_M \) and \( M^T \) decompose as \( \bigsqcup \text{Leaf}_\beta \), \( \bigsqcup \text{Leaf}_\alpha \) respectively. Therefore

\[
Z_A = \bigsqcup \text{Leaf}_{\alpha, \beta} \times_{X^T} \text{F}_\alpha.
\]

We have the projection \( p : \text{Leaf}_\beta \times_{X^T} \text{F}_\alpha \to \text{F}_\beta \times_{X^T} \text{F}_\alpha \).

**Proposition 4.5.2.** \( Z_A \) is a lagrangian subvariety in \( M \times M^T \). If \( Z^F_1, Z^F_2, \ldots \) denote the irreducible components of \( \bigcap \text{Leaf}_\beta \times_{X^T} \text{F}_\alpha \), then closures of their inverse images

\[
Z_1 \overset{\text{def.}}{=} \overline{p^{-1}(Z^F_1)}, Z_2 \overset{\text{def.}}{=} \overline{p^{-1}(Z^F_2)}, \ldots
\]

are the irreducible components of \( Z_A \).

**Proof.** Consider an irreducible component \( Z^F_\nu \) of \( \text{F}_\beta \times_{X^T} \text{F}_\alpha \). Using the semi-smallness of \( \pi : M^T \to X^T \), one can check that \( Z^F_\nu \) is half-dimensional. Then the dimension of \( Z_\nu \) is

\[
\frac{1}{2} (\dim \text{F}_\beta + \dim \text{F}_\alpha) + \frac{1}{2} \dim \text{F}_\beta = \frac{1}{2} (\dim X + \dim \text{F}_\alpha),
\]

as the dimension of fibers of \( \text{Leaf}_\beta \) is the half of codimension of \( \text{F}_\beta \). Therefore \( Z_\nu \) is half-dimensional in \( \text{F}_\beta \times M \).

We omit the proof that \( Z_\nu \) is lagrangian. \( \square \)

### 4.6. Polarization

Consider the normal bundle \( N_{\text{F}_\alpha / M} \) of \( \text{F}_\alpha \) in \( M \). It is the direct sum \( L^\perp_\alpha \oplus L_{\overline{\alpha}}^\perp \), where \( L^\perp_\alpha \) is as in (4.4.2). Since \( L^\perp_\alpha, L_{\overline{\alpha}}^\perp \) are dual to each other with

\(^{10}\)We omit explanation of pull-back with supports \( p_{12*}, p_{23*} \), etc. See [15] for more detail.
We will understand a polarization as a choice of ±. Choose and fix a chamber.

Theorem 4.8.1. Let \( e(\mathcal{L}) \) be the restriction of the diagram \((\mathcal{L})\) to \(\text{Leaf}_\alpha\). Then for \(\gamma_\alpha \in H_{T^*_\alpha}^*(\mathcal{F}_\alpha)\) we have

\[
[Z_{\mathcal{A}}] \ast \gamma_\alpha = j_\alpha \ast p_\alpha^*(\gamma_\alpha).
\]

In particular, the following properties are clearly satisfied.

\[
i_\alpha^*([Z_{\mathcal{A}}] \ast \gamma_\alpha) = e(\mathcal{L}) \cup \gamma_\alpha,
\]

\[
i_\beta^*([Z_{\mathcal{A}}] \ast \gamma_\alpha) = 0 \quad \text{for } \beta \neq \alpha.
\]

4.8. Definition of the stable envelop. Now we return back to a general situation. Let \( i_{\beta,\alpha} : \mathcal{F}_\beta \times \mathcal{F}_\alpha \to M \times M^T \) denote the inclusion.

Theorem 4.8.1. Choose and fix a chamber \(\mathcal{C}\) and a polarization ±. There exists a unique homology class \(\mathcal{L} \in H_{T^*_0}^*(Z_{\mathcal{A}})\) with the following three properties:
(1) \( \mathcal{L} \big|_{M \times F_\infty} \) is supported on \( \bigcup_{t \leq \alpha} \text{Leaf}_t \times X_t F_\alpha \).

(2) Let \( e(L_\alpha) \) denote the equivariant Euler class of the bundle \( L_\alpha \) over \( F_\alpha \). We have

\[
\iota_{\alpha, \alpha}^* \mathcal{L} = \pm e(L_\alpha) \cap [\Delta_{\alpha}]_{F_\alpha}.
\]

(3) For \( \beta < \alpha \), we have \( \iota_{\beta, \alpha}^* \mathcal{L} = 0 \).

**Remark 4.8.2.** Since \( Z_\alpha \) is lagrangian in \( M \times M^T \), \( H^T_0(Z_\alpha) \) has a base by irreducible components of \( Z_\alpha \). The same is true for the \( T \)-equivariant Borel-Moore homology, and even for the non-equivariant homology. In particular, though \( \mathcal{L} \) is constructed in \( H^T_0(Z_\alpha) \), it automatically gives a well-defined class in \( H^T_0(Z_\alpha) \).

This is important as we eventually want to understand \( H^T_0(M) \).

Note that the property (2) still holds in \( T \)-equivariant cohomology group by the construction. However \( \iota_{\beta, \alpha}^* \mathcal{L} \neq 0 \) in general.

**Proof of the existence.** We first prove the existence.

We apply the construction in §4.7 after a deformation as follows.

We use the following fact due to Namikawa [50]: \( M \) and \( X \) have a 1-parameter deformation \( M, X \) together with \( \Pi: M \to X \) over \( \mathbb{C} \) such that the original \( M, X \) are fibers over \( 0 \in \mathbb{C} \), and other fibers \( M^t, X^t (t \neq 0) \) are isomorphic, both are smooth and affine. Moreover the \( T \)-action extends to \( M^t, X^t \).

For \( \mathbb{L}_d \to \mathbb{L}_d^T \), such a deformation can be defined by using quiver description: we perturb the moment map equation as \( [B_1, B_2] + IJ = t \text{id} \). The \( T \)-action does not preserve this equation, as it scales \( t \text{id} \). But the \( T \)-action preserves the equation. For \( T^\vee (\text{flag variety}) \to \{\text{nilpotent variety}\} \), we use deformation to semisimple adjoint orbits. (See [15, §3.4].)

We define \( Z^T_\alpha \) for \( M^t \) in the same way. We consider its fundamental class \( [Z^T_\alpha] \) as in §4.7. We now define

\[
\mathcal{L} = \sum_{\alpha} \mathcal{L}_\alpha; \quad \mathcal{L}_\alpha \overset{\text{def.}}{=} \pm \lim_{t \to 0} [Z^T_\alpha],[
\]

where \( \lim_{t \to 0} \) is the specialization in Borel-Moore homology groups (see [15, §2.6.30]).

The conditions (1), (2), (3) are satisfied for \( [Z^T_\alpha] \). Taking the limit \( t \to 0 \), we see that three conditions (1), (2), (3) are satisfied also for \( \mathcal{L} \).

The uniqueness can be proved under a weaker condition. Let us explain it before starting the proof.

Note that \( T \) acts on \( M^T \) trivially. Therefore the equivariant cohomology \( H^T_0(M^T) \) is isomorphic to \( H^T_0(M) \otimes H^T_+(\text{pt}) \). The second factor \( H^T_+(\text{pt}) \) is the polynomial ring \( \mathbb{C}[\text{Lie} T] \).

We define the degree \( \text{deg}_T \) on \( H^T_0(M^T) \) as the degree of the component of \( H^T_+(\text{pt}) = \mathbb{C}[\text{Lie} T] \). Now (3) is replaced by

(3)’ \quad \text{deg}_T \iota_{\beta, \alpha}^* \mathcal{L} < \frac{1}{2} \text{codim}_M F_\beta.
Proof of the uniqueness. We decompose \( L = \sum L_\alpha \) according to \( M^T = \bigsqcup F_\alpha \) as above. Let \( Z_1, Z_2, \ldots \) be irreducible components of \( Z \) as in Proposition 4.5.2. Each \( Z_k \) is coming from an irreducible component of \( F_\beta \times_X T \) \( F_\alpha \). So let us indicate it as \( Z_k^{(\beta, \alpha)} \). Since \( L \) is the top 1-degree class, we have \( L_\alpha = \sum a_k[Z_k^{(\beta, \alpha)}] \) with some \( a_k \in C \). Moreover by (1), \( Z_k^{(\beta, \alpha)} \) with nonzero \( a_k \) satisfies \( \beta \leq \alpha \).

Consider \( Z_k^{(\alpha, \alpha)} \). By (2), we must have \( Z_k^{(\alpha, \alpha)} = \text{Leaf}_\alpha, \ a_k = \pm 1, \) where \( \text{Leaf}_\alpha \) is mapped to \( Z_{\alpha} = M \times_X T \) by (inclusion) \( \times p \).

Suppose that \( L_{1,\alpha}^1, L_{2,\alpha}^2 \) satisfy conditions (1),(2),(3)' Then the above discussion says \( L_{1,\alpha}^1 - L_{2,\alpha}^2 = \sum a_k[Z_k^{(\beta, \alpha)}] \), where \( Z_k^{(\beta, \alpha)} \) with nonzero \( a_k \) satisfies \( \beta \leq \alpha \).

Suppose that \( L_{1,\alpha}^1 \neq L_{2,\alpha}^2 \) and take a maximum \( \beta_0 \) among \( \beta \) such that there exists \( Z_k^{(\beta_0, \alpha)} \) with \( a_k \neq 0 \).

Consider the restriction \( t_{p_0,\alpha}^k(L_{1,\alpha}^1 - L_{2,\alpha}^2) \). By the maximality, only those \( Z_k^{(\beta_0, \alpha)} \)'s with \( \beta = \beta_0 \) contribute and hence

\[
t_{p_0,\alpha}^k(L_{1,\alpha}^1 - L_{2,\alpha}^2) = \sum a_k e(L_{p_0}) \cap [Z_k^T],
\]

where \( Z_k^T \) is an irreducible component of \( F_{p_0} \times_X T \) \( F_\alpha \) corresponding to \( Z_k^{(\beta_0, \alpha)} \).

Since \( e(L_{p_0}) \cap [Z_k^T] \) is the product of weights \( \lambda \) such that \( \langle p, \lambda \rangle < 0 \) (cf. (4.4.1)), it has degree exactly equal to \( \text{codim} F_\beta / 2 \). But it contradicts with the condition (3)'. Therefore \( L_{1,\alpha}^1 = L_{2,\alpha}^2 \).

**Definition 4.8.3.** The operator defined by the formula (4.5.1) given by the class \( L \) constructed in Theorem 4.8.1 is called the **stable envelop**. It is denoted by \( \text{Stab}_c \). (The dependence on the polarization is usually suppressed.)

Let us list several properties of stable envelops.

4.9. **Adjoint** By a general property of the convolution product, the adjoint operator

\[
\text{Stab}_c^*: H_{T_c}(M) \to H_{T_c}(M^T)
\]

is given by changing the first and second factors in (4.5.1):

\[
H_{T_c}(M) \ni c \mapsto (-1)^{\text{codim } X^T} p_{2*}(p_1^* c \cap L) \in H_{T_c}(M^T),
\]

where the sign comes from our convention on the intersection form.

4.10. **Image of the stable envelop** From the definition, the stable envelop defines a homomorphism

\[
H_{T_s}^s(M^T) \to H_{T_{[-s]}(A_M)},
\]

so that the original \( \text{Stab}_c \) is the composition of the above together with the pushforward \( H_{T_s}(A_M) \to H_{T_{[-s]}(M)} \) of the inclusion \( A_M \to M \) and the Poincaré duality \( H_{T_{[-s]}(M)} \cong H_{T_s}^s(\text{Leaf}_\alpha) \).

**Proposition 4.10.2.** (4.10.1) is an isomorphism.

By Proposition 4.4.6, \( H_{T_s}^s(A_M) \) has a natural filtration \( \bigcup H_{T_s}^s(A_{M, \leq s}) \) whose associated graded is \( \bigoplus H_{T_s}^s(\text{Leaf}_\alpha) \). From the construction, the stable envelop
is compatible with the filtration, where the filtration on \( H^{[s]}_{\mathbb{T}}(M^T) \) is one induced from the decomposition to \( \bigoplus H^{[s]}_{\mathbb{T}}(F_\alpha) \). Moreover the induced homomorphism \( H^{[s]}_{\mathbb{T}}(F_\alpha) \to H^{[s]}_{\mathbb{T}}(\operatorname{Leaf}_\alpha) \) is the pull-back, and hence it is an isomorphism. Therefore the original stable envelop is also an isomorphism.

4.11. **Subtorus** Let \( T' \) be a subtorus of \( T \) such that \( M^{T'} \) is strictly larger than \( M^T \). We have a natural inclusion \( \operatorname{Hom}_{\operatorname{grp}}(C^\times, T') \otimes_{\mathbb{Z}} \mathbb{R} \subset \operatorname{Hom}_{\operatorname{grp}}(C^\times, T) \otimes_{\mathbb{Z}} \mathbb{R} \). Then a one parameter subgroup in \( T' \) is contained in the union of hyperplanes, which we removed when we have considered chambers. We take a chamber \( \mathcal{C}' \) for \( T' \) and a chamber \( \mathcal{C} \) for \( T \) such that

\[ \mathcal{C}' \subset \mathcal{C}. \]

A typical example will appear in Figure 6.2.1, where \( T \) is 2-dimensional torus \( \{(t_1, t_2, t_3) \mid t_1 t_2 t_3 = 1\} \), and \( T' \) is 1-dimensional given by, say, \( t_1 = t_2 \). For example, we take \( \mathcal{C} \) as in the figure, and \( \mathcal{C}' \) as a half line in \( a_1 = a_2 \) from the origin to the right.

We consider \( \operatorname{Stab}_{\mathcal{C}'}: H^{[s]}_{\mathbb{T}}(M^{T'}) \to H^{[s]}_{\mathbb{T}}(M) \) defined by \( \mathcal{C}' \). We can also consider \( M^T \) as the fixed point locus in \( M^{T'} \) with respect to the action of the quotient group \( T/T' \). Then the chamber \( \mathcal{C} \) projects a chamber for \( T/T' \subset M^{T'} \). Let us denote it by \( \mathcal{C}/\mathcal{C}' \). It also induces \( \operatorname{Stab}_{\mathcal{C}/\mathcal{C}'}: H^{[s]}_{\mathbb{T}}(M^T) \to H^{[s]}_{\mathbb{T}}(M^{T'}) \). Here note that a polarization for \( \mathcal{C} \) induces ones for \( \mathcal{C}' \) and \( \mathcal{C}/\mathcal{C}' \) by considering of restriction of weights from \( \text{Lie } T \) to \( \text{Lie } T' \). The uniqueness implies

**Proposition 4.11.1.**

\[ \operatorname{Stab}_{\mathcal{C}'} \circ \operatorname{Stab}_{\mathcal{C}/\mathcal{C}'} = \operatorname{Stab}_{\mathcal{C}}: H^{[s]}_{\mathbb{T}}(M^T) \to H^{[s]}_{\mathbb{T}}(M). \]

5. **Sheaf theoretic analysis of stable envelopes**

In this lecture, we will study stable envelopes in sheaf theoretic terms. In particular, we will connect them to hyperbolic restriction introduced in [9].

5.1. **Nearby cycle functor** This subsection is a short detour, or a warm up, proving \( \pi_e(\mathcal{C}_M) \) is perversive for a symplectic resolution \( \pi: M \to X \) without quoting Kaledin’s semi-smallness result, mentioned in a paragraph after Proposition 3.3.1. It uses the nearby cycle functor,\(^{11}\) which is important itself, and useful for sheaf theoretic understanding of stable envelopes in §5.4. Moreover we will use the same idea to show that hyperbolic restriction functors appearing in symplectic resolution preserve perversity. See §5.4.

Let us recall the definition of the nearby cycle functor in [27, §8.6].

Let \( X \) be a complex manifold and \( f: X \to C \) a holomorphic function. Let \( X = f^{-1}(0) \) and \( r: X \to \mathcal{X} \) be the inclusion. We take the universal covering \( \tilde{C}^* \to C^* \) of \( C^* = C \setminus \{0\} \). Let \( c: \tilde{C}^* \to C \) be the composition of the projection and inclusion

\(^{11}\)I learned usage of the nearby cycle functor for symplectic resolutions from Victor Ginzburg many years ago. He attributed it to Gaitsgory. See [21].
We take the fiber product $\hat{X}^*$ of $X$ and $\hat{C}^*$ over $C$. We consider the diagram

$\xymatrix{ \hat{X}^* \ar[r] \ar[d]^c & \hat{C}^* \ar[d]^c \\
X \ar[r]^r & X \ar[r]^f & C.}$

Then the nearby cycle functor from $D^b(X)$ to $D^b(X)$ is defined by

\[
(5.1.1) \quad \psi_f(\circ) = r^*\hat{c}^*\hat{c}^*(\circ).
\]

Note that it depends only on the restriction of objects in $D^b(X)$ to $X \setminus X$.

We now take a deformation $X, M$ used in the proof of Theorem 4.8.1. Take the projection $X, M \to C$ as $f$. Let us denote them by $f_X, f_M$ respectively. (Singularities of $X$ causes no trouble in the following discussion: we replace $X$ by an affine space to which $X$ is embedded.) We consider the nearby cycle functors $\psi_{f_X}, \psi_{f_M}$.

Since $\Pi: M \to X$ is proper, one can check

\[
(5.1.2) \quad \psi_{f_X}\Pi_* = \pi_*\psi_{f_M}
\]

from the base change.

Now we apply the both sides to the shifted constant sheaf $\mathcal{E}_M$. Since $M \to C$ is a smooth fibration, i.e., there is a homeomorphism $X \to M \times C$ such that $f_M$ is sent to the projection $M \times C \to C$. Then we have

\[
(5.1.3) \quad \psi_{f_M}\mathcal{E}_M = \mathcal{E}_M[1].
\]

On the other hand, since we only need the restriction of $\Pi_*\mathcal{E}_M$ to $X \setminus X$, we may replace $\pi_*\mathcal{E}_M$ by $\mathcal{E}_X$ as $M \setminus M \to X \setminus X$ is an isomorphism. (In fact, $\Pi_*\mathcal{E}_M = IC(X)$ as it is known that $\Pi$ is small.) We thus have

\[
(5.1.4) \quad \psi_{f_X}\mathcal{E}_X = \pi_*[\mathcal{E}_M][1].
\]

We have a fundamental property of the nearby cycle functor: $\psi_f[-1]$ sends perverse sheaves on $X$ to perverse sheaves on $X$. (See [27, Cor. 10.3.13].) Hence $\pi_*\mathcal{E}_M$ is perverse.

**Exercise 5.1.5.**

(1) Check (5.1.2).

(2) Check (5.1.3).

**5.2. Homology group of the Steinberg type variety**

Recall that the variety $Z_A$ is defined as a fiber product $A_M \times_X T \times T$, analog of Steinberg variety. The homology group of Steinberg variety, or more generally the fiber product $M \times X M$ nicely fits with framework of perverse sheaves by Ginzburg’s theory [15, §8.6]. A starting point of the relationship is an algebra isomorphism (not necessarily grading preserving)

\[
(5.2.1) \quad H_*(Z) \cong \text{Ext}^*_D(\mathcal{E}_M, \mathcal{E}_M).
\]
Moreover, if $M \to X$ is semi-small, $\pi_*(\mathcal{E}_M)$ is a semisimple perverse sheaf, and (5.2.2) $H_{[0]}(Z) \cong \Hom_{D^b(X)}(\pi_*\mathcal{E}_M, \pi_*\mathcal{E}_M)$.

See [15, Prop. 8.9.6].

We have the following analog for $Z_A$:

**Proposition 5.2.3** (cf. [45, Lemma 4]). *We have a natural isomorphism*

\[ H_*(Z_A) \cong \Ext^*_A(X_j)(\pi_*(\mathcal{E}_M^r), p_*j^!\pi_*(\mathcal{E}_M)). \]  

Recall that $H_*(Z_A)$ is an $(H^*_j(Z), H^*_j(Z^T))$-bimodule. Similarly the right hand side of the above isomorphism is a bimodule over

\[ \langle \Ext^*_A(X_j)(\pi_*(\mathcal{E}_M^r), \pi_*(\mathcal{E}_M)), \Ext^*_A(X_j)(\pi_*(\mathcal{E}_M), \pi_*(\mathcal{E}_M)) \rangle. \]

Under (5.2.1) and its analog for $M^T$, the above isomorphism respects the bimodule structure.

In view of (5.2.1, 5.2.2, 5.2.4), it is natural to ask what happens if we replace the left hand side of (5.2.4) by the shifted degree 0 part $H_{[0]}(Z_A)$, where the cycle $\text{Stab}_C = L$ in Theorem 4.8.1 lives. Since $\pi^T : M^T \to X^T$ is semi-small, $\pi^T_*(\mathcal{E}_M^r)$ is a semisimple perverse sheaf. As we shall see in the next two subsections, the same is true for $p_*j^!\pi_*(\mathcal{E}_M)$. Then we have

\[ H_{[0]}(Z_A) \cong \Hom_{D^b(X_j)}(\pi^T_*(\mathcal{E}_M^r), p_*j^!\pi_*(\mathcal{E}_M)). \]

Now $\text{Stab}_C$ can be regarded as the *canonical* homomorphism from $\pi^T_*(\mathcal{E}_M^r)$ to $p_*j^!\pi_*(\mathcal{E}_M)$. Since it has upper triangular and invertible diagonal entries, it is an isomorphism. (In fact, we will see it directly in §5.4.) Therefore $\text{Stab}_C$ is the *canonical* isomorphism. In particular, we have an algebra homomorphism

(5.2.5) \[ H^T_{[a]}(Z) = \Ext^*(\pi_*(\mathcal{E}_M), \pi_*(\mathcal{E}_M)) \xrightarrow{p_*j^!} \Ext^*(\pi_*(\mathcal{E}_M), \pi_*(\mathcal{E}_M)) \xrightarrow{\cong} \Ext^*(\pi^T_*(\mathcal{E}_M^r), \pi^T_*(\mathcal{E}_M^r)) = H^T_{[a]}(Z^T). \]

This homomorphism can be regarded as a ‘coproduct’, as it will be clear in §6.

**Exercise 5.2.6.** Give a proof of Proposition 5.2.3.

**5.3. Hyperbolic restriction** The claim that $p_*j^!\pi_*(\mathcal{E}_M)$ is a semisimple perverse sheaf is a consequence of two results:

(a) Braden’s result [9] on preservation of purity.
(b) Dimension estimate of fibers, following an idea of Mirkovic-Vilonen [36].

One may formally compare these results for proper pushforward homomorphisms:

(a) the decomposition theorem, which is a consequence of preservation of purity.
(b) Semi-smallness implies the preservation of perversity.

In fact, the actual dimension estimate (b) required for $p_*j^!\pi_*(\mathcal{E}_M)$ is rather easy to check, once we use the nearby cycle functor. The argument can be compared with one in §5.1.
4.3.2 We have the diagram

\[ X^T \xrightarrow{\pi} R_X \xrightarrow{j} X, \]

as for the attracting set.

**Theorem 5.3.1** ([9]). We have a natural isomorphism \( p_*j^! \cong p_*j^* \) on \( T \)-equivariant complexes \( D^b_T(X) \).

This theorem implies the preservation of the purity for \( p_*j^! = p_*j^* \) as \( p_*, j^! \) increase weights while \( p_!, j^* \) decrease weights. In particular, semisimple complexes are sent to semisimple ones. Thus \( p_*j^! \pi_*(\mathcal{E}_M) \) is semisimple.

**Definition 5.3.2.** The functor \( p_*j^! = p_*j^* \) is called the hyperbolic restriction.

Note that we have homomorphisms

\[
\begin{align*}
H^*_T(X^T, p_*j^! \mathcal{F}) &\rightarrow H^*_T(X, \mathcal{F}), \\
H^*_T(X, p_*j^* \mathcal{F}) &\rightarrow H^*_T(X^T, p_*j^! \mathcal{F})
\end{align*}
\]

for \( \mathcal{F} \in D^b_T(X) \) by adjunction. These become isomorphisms when we take \( \otimes_{H^*_T(pt)} \text{Frac}(H^*_T(pt)) \) by the localization theorem in equivariant cohomology.

5.4. **Exactness by nearby cycle functors** Consider one parameter deformation \( f_M, f_X : M, X \rightarrow \mathbb{C} \) as in §5.1. We have the diagram

\[ X^T \xrightarrow{\pi} A_X \xrightarrow{j} X, \]

as in (4.3.2). We have the hyperbolic restriction functor \( p_{X^T,j}^! \).

We also have a family \( f_{X^T} : X^T \rightarrow \mathbb{C} \).

The purpose of this subsection is to prove the following.

**Proposition 5.4.1.** (1) The restriction of \( p_{X^T,j}^! \mathcal{E}_X \) to \( X^T \setminus X^T \) is canonically isomorphic to the constant sheaf \( \mathcal{E}_{X^T \setminus X^T} \).

(2) The nearby cycle functors commute with the hyperbolic restriction:

\[ p_*j^! \psi_{f_X} = \psi_{f_{X^T}} p_{X^T,j}^! \mathcal{E}_X. \]

**Corollary 5.4.2.** (1) \( p_*j^! \pi_*(\mathcal{E}_M) \) is perverse.

(2) The isomorphism \( \mathcal{E}_{X^T \setminus X^T} \xrightarrow{\cong} p_{X^T,j}^! \mathcal{E}_{X^T \setminus X^T} \) induces an isomorphism \( \pi_*(\mathcal{E}_{M^T}) \xrightarrow{\cong} p_*j^! \pi_*(\mathcal{E}_M) \).

In fact, \( \pi_*(\mathcal{E}_M) \xrightarrow{\psi_{f_X}} \mathcal{E}_{X}[\neg 1] \) by (5.1.4). Therefore Proposition 5.4.1(2) implies \( p_*j^! \pi_*(\mathcal{E}_M) \xrightarrow{\psi_{f_{X^T}}} p_{X^T,j}^! \mathcal{E}_{X}[\neg 1] \). Now we can replace \( p_{X^T,j}^! \mathcal{E}_{X}[\neg 1] \) by \( \mathcal{E}_{X^T}[\neg 1] \) by (1) over \( X^T \setminus X^T \). Since \( \psi_{f_{X^T}}[\neg 1] \) sends a perverse sheaf to a perverse sheaf, we get the assertion (1).

Applying \( \psi_{f_{X^T}} \) to \( \mathcal{E}_{X^T \setminus X^T} \xrightarrow{\cong} p_{X^T,j}^! \mathcal{E}_{X^T \setminus X^T} \), we get an isomorphism

\[ \pi_*(\mathcal{E}_{M^T}) \xrightarrow{\psi_{f_{X^T}}} \mathcal{E}_{X^T} \xrightarrow{\cong} \psi_{f_{X^T}} p_{X^T,j}^! \mathcal{E}_X = p_*j^! \pi_*(\mathcal{E}_M). \]
This is (2).

Moreover, the isomorphism (2) coincides with one given by the class \( \mathcal{L} \in H^T_{\{0\}}(Z_A) \cong \text{Hom}_{D^b(X^T)}(\pi^T_*(\mathcal{E}_{M_T}), p^* j^! \pi_*(\mathcal{E}_M)) \). This is clear from the definition, and the fact that the nearby cycle functor coincides with the specialization of Borel-Moore homology groups.

We have canonical homomorphisms \( \text{IC}(X^T) \to \pi^T_*(\mathcal{E}_{M_T}) \) and \( \pi_*(\mathcal{E}_M) \to \text{IC}(X) \) as the inclusion and projection of direct summands. Therefore we have a canonical homomorphism

\[
\text{IC}(X^T) \to p^* j^! \text{IC}(X).
\]

This is a nice reformulation of a stable envelop, which makes sense even when we do not have a symplectic resolution. See §5.6 and §7.

**Historical Comment 5.4.3.** A different but similar proof of Corollary 5.4.2(1) was given by Varagnolo-Vasserot [56] for hyperbolic restrictions for quiver varieties, and it works for symplectic resolutions. It uses hyperbolic semi-smallness for the case of symplectic resolution on one hand, and arguments in [32, 3.7] and [31, 4.7] on the other hand. The latter two originally gave decomposition of restrictions of character sheaves, restrictions of perverse sheaves corresponding to canonical bases respectively. (Semisimplicity follows from a general theorem 5.3.1. But decomposition must be studied in a different way.)

**Proof of Proposition 5.4.1.** (1) The key is that \( M \setminus M \) is isomorphic to \( X \setminus X \). We have the decomposition \( M^T = \bigsqcup \mathcal{F}_\alpha \) corresponding to \( M^T = \bigsqcup F_\alpha \) to connected components. Then we have the induced decomposition

\[
A_X \setminus A_X = \bigsqcup p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha)
\]

to connected components, such that each \( p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \) is an affine bundle over \( \mathcal{F}_\alpha \setminus F_\alpha \). The same holds for \( R_X \setminus R_X = \bigsqcup p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \).

Moreover \( p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \) is a smooth closed subvariety of \( X \setminus X \). Its codimension is equal to the dimension of fibers of \( p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \).

Thus the hyperbolic restriction \( p_X j^! X \mathcal{E}_X \) is the constant sheaf \( e_{X \setminus X^T} \) up to shifts. (Shifts could possibly different on components.)

Now normal bundles of \( \mathcal{F}_\alpha \setminus F_\alpha \) to \( p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \) and \( p^{-1}_X(\mathcal{F}_\alpha \setminus F_\alpha) \) are dual vector bundles with respect to the symplectic structure. In particular the dimension of fibers are equal. This observation implies that shifts are unnecessary, \( p_X j^! X \mathcal{E}_X \) is equal to \( e_{X \setminus X^T} \). Moreover the isomorphism is given by the restriction of the constant sheaves and the Thom isomorphism for the cohomology group of an affine bundle. Therefore it is canonical.

(2) Recall that the nearby cycle functor \( \psi_{f_X} \) is given by \( r^*_X \mathcal{E}_X^* \) as in (5.1.1), where we put subscripts \( X \) to names of maps to indicate we are considering the family \( X \rightarrow C \).
We replace $p_\ast j^\ast$ by $p_\ast j^\ast$ by Theorem 5.3.1. We have the commutative diagram

$$
\begin{array}{c}
X^T \leftarrow & \mathcal{R}_X \rightarrow & X \\
\downarrow & \downarrow & \downarrow \\
\mathcal{X}^T \leftarrow & \mathcal{R}_\mathcal{X} \rightarrow & \mathcal{X},
\end{array}
$$

where both squares are cartesian. Thus $p_\ast j^\ast \mathcal{R}_X = p_\ast p_\ast j^\ast \mathcal{R}_\mathcal{X} = p_\ast j^\ast \mathcal{X}^T p_\ast$. Namely the hyperbolic restriction commutes with the restrictions $r_\ast$, $r_\ast^\circ$ to 0-fibers.

Next we replace back $p_\ast j^\ast$ to $p_\ast j^\ast$ and consider the diagram

$$
\begin{array}{c}
\mathcal{X}^T \leftarrow & \mathcal{A}_X \rightarrow & \mathcal{X} \\
\uparrow & \uparrow & \uparrow \\
\mathcal{X}^T \leftarrow & \mathcal{A}_X^\circ \rightarrow & \mathcal{X}^\circ.
\end{array}
$$

The bottom row is pull back to the universal cover $\tilde{\mathcal{X}}^\circ$ of the upper row. This is commutative and both squares are cartesian. Thus we have $p_\ast j^\ast \mathcal{A}_X = p_\ast j^\ast \mathcal{A}_X^\circ = p_\ast j^\ast \mathcal{X}^T p_\ast$. Thus the hyperbolic restriction commutes with the pushforward for the coverings $\tilde{\mathcal{X}}$, $\tilde{\mathcal{X}}^T$.

Finally we commute the hyperbolic restriction with pullbacks for $\mathcal{X}$, $\mathcal{X}^T$. We do not need to use Theorem 5.3.1 as we saw $j^\ast$, $p_\ast$ are (union of) embedding of smooth closed subvarieties and projections of vector bundles. Therefore $\ast$ and $!$ are the same up to shift. (Also Theorem 5.3.1 is proved for algebraic varieties, and is not clear whether the proof works for $\mathcal{X}^\circ$ in general. This problem disappears if we consider the nearby cycle functor in algebraic context.) This finishes the proof of (2). □

5.5. Hyperbolic semi-smallness  Looking back the proof of Proposition 5.4.1, we see that a key observation is the equalities $\text{rank } \mathcal{A}_X = \text{rank } \mathcal{R}_X = \text{codim}_X X^T/2$. (More precisely restriction to each component of $X^T$.)

In order to prove the exactness of the hyperbolic restriction functor in more general situation, in particular, when we do not have symplectic resolution, we will introduce the notion of hyperbolic semi-smallness in this subsection.

The terminology is introduced in [12], but the concept itself has appeared in [36] in the context of the geometric Satake correspondence.

Let $X = \bigsqcup X_\alpha$ be a stratification of $X$ such that $i^!_\alpha \text{IC}(X)$, $i^!_\alpha \text{IC}(X)$ are shifts of locally constant sheaves. Here $i_\alpha$ denotes the inclusion $X_\alpha \rightarrow X$. We suppose that $X_0$ is the smooth locus of $X$ as a convention.

We also suppose that the fixed point set $X^T$ has a stratification $X^T = \bigsqcup Y_\beta$ such that the restriction of $p$ to $p^{-1}(Y_\beta) \cap X_\alpha$ is a topologically locally trivial fibration over $Y_\beta$ for any $\alpha$, $\beta$ (if it is nonempty). We assume the same is true for $p_\ast$. 

Theorem 5.5.3. Suppose \((Stab, \rho)\) and it is isomorphic to the Langlands dual of \(G\) and \(H\). \(G\) sheaves on \(Gr\), \(\chi\) simple local system have bases given by \((5.5.2)\)

\[
\begin{align*}
\dim p^{-1}(y_\beta) \cap X_\alpha &\leq \frac{1}{2}(\dim X_\alpha - \dim Y_\beta), \\
\dim p^{-1}(y_\beta) \cap X_\alpha &\leq \frac{1}{2}(\dim X_\alpha - \dim Y_\beta).
\end{align*}
\]

These conditions are for \(p\) and \(j\). Nevertheless we will often say the functor \(p_\ast j_!\) is hyperbolic semi-small for brevity. There should be no confusion if we use \(p_\ast j_!\) only for \(\text{IC}(X)\).

Suppose \(X\) is smooth (and \(X = X_0\)). Then \(X^T\) is also smooth. We decompose \(X^T = \bigsqcup Y_\beta\) into connected components as usual. Then the above inequalities must be equalities, i.e., \(\text{rank} A_{\chi} = \text{rank} R_{\chi} = \text{codim}_X Y_\beta / 2\). They are the condition which we have mentioned in the beginning of this subsection.

Note that \(p^{-1}(y_\beta) \cap X_0\) and \(p^{-1}(y_\beta) \cap X_0\) are at most \((\dim X - \dim Y_\beta) / 2\)-dimensional if \(\Phi\) is hyperbolic semi-small. In this case, cohomology have bases given by \((\dim X - \dim Y_\beta) / 2\)-dimensional irreducible components of \(p^{-1}(y_\beta) \cap X_0\) and \(p^{-1}(y_\beta) \cap X_0\) respectively. Let \(H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_X\) and \(H_c^{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_X\) denote the components corresponding to a simple local system \(\chi\) on \(Y_\beta\).

Theorem 5.5.3. Suppose \((p, j)\) is hyperbolic semi-small. Then \(p_\ast j_!(\text{IC}(X))\) is perverse and it is isomorphic to

\[
\bigoplus_{\beta, X} \text{IC}(Y_\beta, \chi) \otimes H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_X.
\]

Moreover, we have an isomorphism

\[
H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_X \cong H_c^{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_X.
\]

The proof is similar to one in [36, Theorem 3.5], hence the detail is left as an exercise for the reader. In fact, we only use the case when \(X^T\) is a point, and the argument in detail for that case was given in [12, Th. A.5].

Exercise 5.5.4. Consider \(X = \mathbb{C}^2\) with the hyperbolic action \((x, y) \mapsto (tx, t^{-1}y)\) for \(t \in \mathbb{C}^\times\). We choose a chamber \(R_{>0}\). Check that \((S^d X, C^\times_{hyp})\) is hyperbolic semi-small. For a partition \(\lambda\) of \(d\), take an irreducible representation \(\rho\) of \(\text{Stab}(\lambda)\). \((\text{Stab}(\lambda)\) was introduced in §3.7.) We consider it as a local system on \(S_\lambda(\mathbb{C}^2)\), and take the associated IC sheaf \(\text{IC}(S_\lambda(\mathbb{C}^2), \rho)\). Compute the hyperbolic restriction of \(\text{IC}(S_\lambda(\mathbb{C}^2), \rho)\).

Exercise 5.5.5. Give a proof of Theorem 5.5.3. Assume \(X^T\) is a single point for brevity.

5.6. Hyperbolic restriction for affine Grassmannian. Recall the affine Grassmannian \(Gr_G = G([z])/G([z])\) is defined for a complex reductive group \(G\). Perverse sheaves on \(Gr_G\) are related to finite dimensional representations of \(G^\vee\), the Langlands dual of \(G\) by the geometric Satake correspondence. Let us review how
the hyperbolic restriction functor appears in Mirkovic-Vilonen’s work \[36\] on the geometric Satake correspondence. This topic as well as further topics on the geometric Satake can be found in Zhu’s lectures in the same volume. So we will omit details.

Let $\text{Perv}_G(\text{Gr}_G)$ be the abelian category of $G[[z]]$-equivariant perverse sheaves on $\text{Gr}_G$. Let $\nu$ be a coweight and $z^\nu$ be the corresponding $T$-fixed point in $\text{Gr}_G$. Let $S^\nu$ be the corresponding attracting set $S^\nu = \{ x \in \text{Gr}_G | \lim_{t \to 0} 2\rho^\vee(t)x = z^\nu \}$.

We consider the diagram

\[
\begin{array}{ccc}
\{z^\nu\} & \overset{p^\nu}{\underset{i^\nu}{\leftrightarrow}} & S^\nu \cup \text{Gr}_G \\
\end{array}
\]

as in (4.3.2), and the corresponding hyperbolic restriction $p^\nu_* j^\nu!$. Here we need to treat fixed points separately. In fact, we do not have the projection $p$ on the union of $S^\nu$, considering it as a closed subscheme of $\text{Gr}_G$. This problem has not occur, as we have only considered hyperbolic restriction functors for affine varieties so far.

Let $\lambda$ be a dominant weight and $\text{Gr}^\lambda_G$ be the $G[[z]]$-orbit through $z^\lambda$. Let $\overline{\text{Gr}^\lambda_G}$ be its closure. It is the union of $\text{Gr}^\mu_G$ over dominant weights $\mu \leq \lambda$. Then it was proved in \[36\], Th. 3.2 that the restriction of (5.6.1) to $\overline{\text{Gr}^\lambda_G}$ is hyperbolic semismall. Therefore $p^\nu_* j^\nu! IC(\overline{\text{Gr}^\lambda_G})$ is perverse, in other words it is concentrated in degree 0, as $z^\nu$ is a point. In Mirkovic-Vilonen’s formulation of geometric Satake correspondence, the equivalence $\text{Perv}_G(\text{Gr}_G) \cong \text{Rep} G^\vee$ is designed so that we have an isomorphism

\[ H^0(p^\nu_* j^\nu! IC(\overline{\text{Gr}^\lambda_G})) \cong V^\nu(\lambda), \]

where $V(\lambda)$ is the irreducible representation of $G^\vee$ with highest weight $\lambda$, and $V^\nu(\lambda)$ is its weight $\nu$ subspace. (See \[36\], Cor. 7.4.) In this case, the construction also gives a basis of $H^0(p^\nu_* j^\nu! IC(\overline{\text{Gr}^\lambda_G}))$, and hence of $V^\nu(\lambda)$ given by irreducible components of $S^\nu \cap \text{Gr}^\lambda_G$.

**Remark 5.6.2.** A resemblance between pushforward of (partial) resolution morphisms and hyperbolic restrictions is formal at this stage. But the following observation, given in \[44\] in the context of geometric Satake correspondence for loop groups \[10\], suggests a deeper relation is hidden. Also it is natural to conjecture that a similar relation exists in more general framework of the symplectic duality \[7\] and Higgs/Coulomb branches of gauge theories \[47\].

It is known that the representation theory of $G$ is related to geometry in two ways. One is through $\text{Gr}_G$ as we have just reviewed. Another is via quiver
varieties of finite types. In these connections, partial resolution and hyperbolic restrictions appear in different places:

| affine Grassmannian | hyperbolic restrictions | partial resolution |
|---------------------|-------------------------|--------------------|
| quiver varieties    | weight spaces           | tensor products    |
|                     | tensor products         | weight spaces      |

For $\text{Gr}_G$, a hyperbolic restriction is used to realize weight spaces as we have reviewed just above. For quiver varieties, weight spaces are top homology groups of central fibers of affinization morphisms $\pi: M \to X$ ([40]). It means that weights spaces are isotypical components of $\mathcal{C}_0$ in $\pi_* (\mathcal{C})$. Here $\mathcal{C}_0$ is the constant sheaf at the point 0.

On the other hand, a hyperbolic restriction appears for realization of tensor products for quiver varieties. (We will see it for Gieseker spaces in §6.3.) For $\text{Gr}_G$, tensor products are realized by the convolution diagram (see [36, §4]). We will not review the construction, but we just mention that tensor product multiplicities are understood as dimension of isotypical components of $\mathcal{IC}(\text{Gr}_G^\lambda)$ in the pushforward of a certain perverse sheaf under a semi-small morphism.

Affine Grassmannian (more precisely slices of $\text{Gr}_G^\lambda$ in the closure of another $G[[z]]$-orbit) and quiver varieties of finite types are examples of Coulomb and Higgs branches of common gauge theories (see [47]). It is natural to expect that similar exchanges of two functors appear for pairs of dual symplectic varieties and Coulomb/Higgs branches. Toric hyper-Kähler manifolds provide us other examples [8].

Let us consider a 3-dimensional SUSY gauge theory associated with $(H_c, M)$ in general. A maximal torus of the normalizer of $H_c$ in $\text{Sp}(M)$ (flavor symmetry group) gives a torus action on the Higgs branch, and (partial) resolution on the Coulomb branch [47, §5]. On the other hand, the Pontryagin dual of the fundamental group of the gauge group $H_c$ gives a torus action on the Coulomb branch and (partial) resolution on the Higgs branch [47, §4(iv)]. Therefore the exchange of partial resolution and hyperbolic restriction are natural.

### 6. R-matrix for Gieseker partial compactification

In this lecture we introduce $R$-matrices of stable envelops for symplectic resolutions. Then we study them for Gieseker partial compactification, in particular relate them to the Virasoro algebra.

#### 6.1. Definition of $R$-matrix

We consider the setting in §4.1.

Let $\iota: M^T \to M$ be the inclusion. Then $\iota^* \circ \text{Stab}_C \in \text{End}(H^*_T(M^T))$ is upper triangular, and the diagonal entries are multiplication by $e(L^-_{\alpha})$. Since $e(L^-_{\alpha})|_{H^*_T(\text{pt})}$ is nonzero, $\iota^* \circ \text{Stab}_C$ is invertible over $F_T = C(\text{Lie } T)$. By the localization theorem for equivariant cohomology groups, $\iota^*$ is also invertible. Therefore $\text{Stab}_C$ is invertible over $F_T$. 

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Definition 6.1.1. Let \( \mathcal{C}_1, \mathcal{C}_2 \) be two chambers. We define the R-matrix by

\[
R_{\mathcal{C}_1, \mathcal{C}_2} = \text{Stab}_{\mathcal{C}_1} \circ \text{Stab}_{\mathcal{C}_2} \in \text{End}(H_T^{[\mathcal{C}_1]}(M^T)) \otimes F_T.
\]

Example 6.1.2. Consider \( M = T^* \mathbb{P}^1 \) with \( T = \mathbb{C}^X, T = \mathbb{C}^X \times \mathbb{C}^X \) as before. We denote the corresponding equivariant variables by \( u, h \) respectively. (So \( H_T^*(pt) = C[\text{Lie } T] = C[u], H_T^*(pt) = C[\text{Lie } T] = C[u, h] \)) Choose a chamber \( \mathcal{C} = \{ u > 0 \} \). Then \( R_{\mathcal{C}, \mathcal{C}} \) is the middle block of Yang’s R-matrix

\[
R = 1 - \frac{h}{u}, \quad P = \sum_{i,j=1}^2 e_{ij} \otimes e_{ji},
\]

where \( e_{ij} \) is the matrix element acting on \( C^2 \) up to normalization.

Exercise 6.1.3. Check this example.

It is customary to write the R-matrix as \( R(u) \) to emphasize its dependence on \( u \). This variable \( u \) is called a spectral parameter in the context of representation theory of Yangian.

The reason why it should be called the R-matrix is clear if we consider the ‘coproduct’ in (5.2.5). Let us denote it \( \Delta_{\mathcal{C}} \) as it depends on a choice of a chamber \( \mathcal{C} \). If we take two chambers \( \mathcal{C}_1, \mathcal{C}_2 \) as above, we have two coproducts \( \Delta_{\mathcal{C}_1}, \Delta_{\mathcal{C}_2} \). Then we can regard \( H_T^{[\mathcal{C}_1]}(M^T) \) as an \( H_T^T(Z) \)-module in two ways, through \( \Delta_{\mathcal{C}_1} \) or \( \Delta_{\mathcal{C}_2} \). Let us distinguish two modules as \( H_T^{[\mathcal{C}_1]}(M^T)_{\mathcal{C}_1}, H_T^{[\mathcal{C}_2]}(M^T)_{\mathcal{C}_2} \). Then the R-matrix

\[
R_{\mathcal{C}_1, \mathcal{C}_2} : H_T^{[\mathcal{C}_1]}(M^T)_{\mathcal{C}_2} \otimes F_T \rightarrow H_T^{[\mathcal{C}_2]}(M^T)_{\mathcal{C}_1} \otimes F_T
\]

is an intertwiner.

Yang’s R-matrix above originally appeared in a quantum many-body problem. Subsequently it is understood that \( R(u) \) is an intertwiner between \( V(a_1) \otimes V(a_2) \) and \( V(a_2) \otimes V(a_1) \), where \( V(a_1) \) is a 2-dimensional evaluation representation of the Yangian \( Y(sl_2) \) associated with \( sl_2 \). Conversely the Yangian \( Y(sl_2) \) can be constructed from the R-matrix by the so-called RTT relation. See [37]. Maulik-Okounkov [34] apply this construction to the R-matrix for \( \tilde{U}_\lambda \), and more generally for quiver varieties. For quiver varieties of type ADE, this construction essentially recovers the usual Yangian \( Y(g) \) associated with \( g \) [35]. But it is a new Hopf algebra for quiver varieties of other types.

We will not review the RTT construction, as the W-algebra associated with \( g \) (not of type A) is something different.

For Yangian, we have two coproducts \( \Delta, \Delta^{op} \) where the latter is given by exchanging two factors in tensor products. The R-matrix is an intertwiner of two coproducts.

6.2. Yang-Baxter equation Suppose that \( T \) is two dimensional such that \( \text{Lie } T = \{ a_1 + a_2 + a_3 = 0 \} \). We suppose that there are six chambers given by hyperplanes \( a_1 = a_2, a_2 = a_3, a_3 = a_1 \) as for Weyl chambers for \( sl(3) \). The cotangent bundle of
the flag variety of $\text{SL}(3)$ and $\tilde{\mathcal{U}}_d^3$ are such examples by Exercise 4.2.1. We factorize the $R$-matrix from $\mathcal{C}$ to $-\mathcal{C}$ in two ways to get the Yang-Baxter equation

$$R_{12}(a_1-a_2)R_{13}(a_1-a_3)R_{23}(a_2-a_3) = R_{23}(a_2-a_3)R_{13}(a_1-a_3)R_{12}(a_1-a_2).$$

See Figure 6.2.1.

![Figure 6.2.1. Yang-Baxter equation](image)

Let $\mathcal{C}$, $\mathcal{C}_2$ be chambers as in Figure 6.2.1. Let $\mathcal{C}'$ be a half line separating $\mathcal{C}$, $\mathcal{C}_2$, considered as a chamber for a subtorus $T'$ corresponding to $a_1 = a_2$. Then by Proposition 4.11.1 we have

$$R_{12} = R_{\mathcal{C}_2, \mathcal{C}_1} = \text{Stab}_{\mathcal{C}_2/\mathcal{C}_1}^{-1} \circ \text{Stab}_{\mathcal{C}/\mathcal{C}_1}. \tag{6.2.2}$$

Here $\text{Stab}_{\mathcal{C}_2/\mathcal{C}_1}$, $\text{Stab}_{\mathcal{C}/\mathcal{C}_1}$ are $H[\ast]T(M\tilde{\mathcal{U}}_d^r)$ to $H[\ast]T(MT')$. Therefore the right hand side is the $R$-matrix for the action of $T/T'$ on $MT'$. For the cotangent bundle of the flag variety of $\text{SL}(3)$ and $\tilde{\mathcal{U}}_d^3$, it is the $R$-matrix for $\text{SL}(2)$ and $\tilde{\mathcal{U}}_d^2$ respectively.

### 6.3. Heisenberg operators

In the remainder of this lecture, we study the $R$-matrix for the case $X = \tilde{\mathcal{U}}_d^r$. But we first need to consider behavior of Heisenberg operators under the stable envelop.

Let us observe that the Heisenberg operators $P^{D_m}(\alpha)$ sends $H_{[\ast]}(A_{\tilde{\mathcal{U}}_d^r})$ to $H_{[\ast]}(A_{\tilde{\mathcal{U}}_d^{r+m}})$. This is because $q_1(q_2^{-1}(A_{\tilde{\mathcal{U}}_d^r} \times C^2)) \subset A_{\tilde{\mathcal{U}}_d^{r+m}}$. Therefore the direct sum $\bigoplus_d H_{[\ast]}(A_{\tilde{\mathcal{U}}_d^r})$ is a module over the Heisenberg algebra. Recall we have isomorphisms

$$\bigoplus_d H_{[\ast]}(A_{\tilde{\mathcal{U}}_d^r}) \cong \bigoplus_d H_{[\ast]}((\tilde{\mathcal{U}}_d^d)^{\wedge}) \cong \bigotimes_{i=1}^{\infty} H_{[\ast]}((C^2)_{d_i}^{\wedge}). \tag{3.6.3}$$

Note that the rightmost space is the tensor product of $r$ copies of the Fock space. Therefore it is a representation of the product of $r$ copies of the Heisenberg algebra.

We compare two Heisenberg algebra representations.
Proposition 6.3.1 ([34, Th. 12.2.1]). The operator $P^\Delta_m(\alpha)$ is the diagonal Heisenberg operator

$$\sum_{i=1}^{r} \text{id} \otimes \cdots \otimes \text{id} \otimes P_{-m}(\alpha) \otimes \text{id} \otimes \cdots \otimes \text{id}.$$  

Thus $P^\Delta_m(\alpha)$ is a primitive element with respect to the coproduct $\Delta$.

Exercise 6.3.2. Give a proof of the above proposition.

6.4. $R$-matrix as a Virasoro intertwiner  Consider the $R$-matrix for $\tilde{U}^d_T$. By (6.2.2), it is enough to consider the $r = 2$ case.

We mostly consider the localization

$$H^a_T((\tilde{U}^d_T)^T) \otimes \text{id} F,$$

from now. It means that we consider variables $\epsilon_1, \epsilon_2, \bar{a}$ take generic value. By (3.6.3) $\bigoplus H^a_T((\tilde{U}^d_T)^T)$ is isomorphic to the tensor product of two copies of Fock space. Let us denote the Heisenberg generator for the first and second factors by $P^{(1)}_m, P^{(2)}_m$ respectively. Here we take $1 \in H^a_T(C^2)$ for the cohomology class $\alpha$ in §3.5, and omit (1) from the notation. Note $P^{(1)}_n, P^{(2)}_n$ are well-defined operators on $H^a_T((\tilde{U}^d_T)^T)$ only for $n < 0$ (a creation operator). To make sense also for $n > 0$, we need to consider the localized equivariant cohomology group as above. Note also that we have $(1, 1) = -1/\epsilon_1 \epsilon_2$ in the commutation relation. This does not make sense unless $\epsilon_1, \epsilon_2$ are invertible, i.e., in $C(\epsilon_1, \epsilon_2)$. We will discuss the integral form in §8.

We have $P^\Delta_m = P^{(1)}_m + P^{(2)}_m$ by Proposition 6.3.1. Since $P^\Delta_m$ is defined a correspondence which makes sense without going to fixed points $(\tilde{U}^d_T)^T$, it commutes the $R$-matrix. Therefore the $R$-matrix should be described by anti-diagonal Heisenberg generators $P^{(1)}_n - P^{(2)}_n$. Let us denote them by $P_m$.

Let us denote the corresponding Fock spaces by $F^\Delta$ and $F^-$. Therefore we have $\bigoplus H^a_T((\tilde{U}^d_T)^T) \cong F^\Delta \otimes F^-$. The above observation means that the $R$-matrix is of a form $\text{id}_{F^\Delta} \otimes R'$ for some operator $R'$ on $F^-$. We now characterize $R'$ in terms of the Virasoro algebra, acting on $F^-$ by the well-known Feigin-Fuchs construction.

Let us recall the Feigin-Fuchs construction. We put

$$P^-_0 \overset{\text{def.}}{=} \frac{1}{\epsilon_1 \epsilon_2} (a_1 - a_2 - (\epsilon_1 + \epsilon_2)).$$

This element is central, i.e., it commutes with all other $P^-_m$. In particular, the Heisenberg relation (3.5.2) remains true.

We then define

$$L_n \overset{\text{def.}}{=} \frac{\epsilon_1 \epsilon_2}{4} \sum_m :P^-_m P^-_{-m}: - \frac{n + 1}{2} (\epsilon_1 + \epsilon_2) P^-_n.$$

These $L_n$ satisfy the Virasoro relation

$$[L_m, L_n] = (m - n) L_{m+n} + \left(1 + \frac{6(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \right) \delta_{m,-n} \frac{m^3 - m}{12}.$$  

with the central charge $1 + 6(\varepsilon_1 + \varepsilon_2)^2 / \varepsilon_1 \varepsilon_2$.

The vacuum vector $|\text{vac}\rangle = 1_{\tilde{U}_2} \in H^{|x|}(\tilde{U}_2)$ is a highest weight vector, it is killed by $L_n$ ($n > 0$) and satisfies

$$L_0 |\text{vac}\rangle = -\frac{1}{4} \left( \frac{(a_1 - a_2)^2}{\varepsilon_1 \varepsilon_2} - \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1 \varepsilon_2} \right) |\text{vac}\rangle.$$  

Here we have used the normal ordering $:\cdot: ;$, which is defined by moving all annihilation operators to the right. See [41, Def. 9.34] for more detail.

It is known that the Fock space, as a representation of the Virasoro algebra, irreducible if its highest weight is generic. Moreover its isomorphism class is determined by its highest weight (and the central charge).

Looking at the above formula for the highest weight, we see that it is unchanged under the exchange $a_1 \leftrightarrow a_2$. Therefore there exists the unique automorphism on $F_{-}(\text{over } F_T)$ sending $|\text{vac}\rangle$ to itself, and intertwining $L_n$ and $L_n$ with $a_1 \leftrightarrow a_2$. It is called the reflection operator.

Now a fundamental observation due to Maulik-Okounkov is the following result:

**Theorem 6.4.4 ([34, Th. 14.3.1]).** The $R$-matrix is $\text{id}_{\Delta F} \otimes (\text{reflection operator})$.

We will give a sketch of proof of Theorem 6.4.4 in §6.6 after recalling the relation between Virasoro algebra and cohomology groups of Hilbert schemes in the next subsection.

**6.5. Virasoro algebra and Hilbert schemes** Historically the first link between the Virasoro algebra and cohomology groups of instanton moduli spaces was found for the rank 1 case by Lehn [29]. Lehn’s result holds for an arbitrary nonsingular complex quasiprojective surface, but let us specialize to the case $\mathbb{C}^2$.

Recall the tautological bundle $V$ over the Hilbert scheme $(\mathbb{C}^2)^d$ (§3.2). We consider its first Chern class $c_1(V)$.

Let $P_n$ denote the Heisenberg operator $P_n^\Delta(1)$ for the $r = 1$ case.

**Theorem 6.5.1 ([29]).** We have

$$c_1(V) \cup \bullet = -\frac{(\varepsilon_1 \varepsilon_2)^2}{3!} \sum_{m_1 + m_2 + m_3 = 0} :P_{m_1} P_{m_2} P_{m_3}: - \frac{\varepsilon_1 \varepsilon_2 (\varepsilon_1 + \varepsilon_2)}{4} \sum_m (|m| - 1) :P_{-m} P_{m}:.$$  

Taking the commutator with $P_n$, we have

$$[c_1(V) \cup \bullet, P_n] = \frac{n \varepsilon_1 \varepsilon_2}{2} \sum_{l + m = n} :P_l P_m: - \frac{n(|n| - 1)}{2} (\varepsilon_1 + \varepsilon_2) P_n.$$  

If we compare this with (6.4.1), we find that this looks very similar to $n L_n$, except a mysterious expression $|n|$.
A different proof, which works only for $C^2$, was given in [46]. Relation with Jack polynomials is explained there.

6.6. R-matrix at the minimal element  
Let us give a sketch of the proof of Theorem 6.4.4.

The first step is to determine the classical r-matrix, which is the second coefficient of $R$:

$$R = 1 + \frac{(\epsilon_1 + \epsilon_2)}{a_1 - a_2} r + O((a_1 - a_2)^{-2}).$$

Recall we have $\bigoplus H_T^{|\sigma|}(\tilde{\mathcal{U}}_2^d) \supseteq F^\Lambda \otimes F^-$ such that the R-matrix is of a form $id_{F^\Lambda} \otimes R'$ for some operator $R'$ on $F^- \otimes_{A_T} F_T$. In order to determine an operator of this form, it is enough to determine the matrix element for the component $(C^2)^{0} \times (C^2)^{|d|}$ in $(\tilde{\mathcal{U}}_2^d)^T$. (See [34, Lem. 12.4.2].) This is the minimum component with respect to the partial order in Definition 4.4.5. By the triangularity of the stable envelop, the matrix element is explicitly written in terms of $e(L_\alpha)$. The vector bundles $L^+_{\alpha}$, $L^-_{\alpha}$ are the tautological vector bundle $\mathcal{V}$, tensored with appropriate characters. Here $(C^2)^{0}$ is a point, so $(C^2)^{0} \times (C^2)^{|d|} = (C^2)^{|d|}$. Since the rank of $\mathcal{V}$ is $d$, which is $-\epsilon_1 \epsilon_2 \sum_{n>0} P_{-n} P_n$ (the leading part of $L_0$), we determine the classical r-matrix as

$$r = -\epsilon_1 \epsilon_2 \sum_{n>0} P_{-n} P_n.$$

(See [34, Th. 12.4.4].)

Next we consider the first Chern class $c_1(\mathcal{V})$ of the tautological bundle over $\tilde{\mathcal{U}}_2^d$. Recall the formula of the coproduct $\Lambda$ on Heisenberg operator $P^\Lambda_m(\alpha)$ in Proposition 6.3.1. We have $\Lambda P^\Lambda_m(\alpha) = P_{-m}(\alpha) \otimes 1 + 1 \otimes P_{-m}(\alpha)$ for rank 2. The underlying geometric reason of this formula is that $P^\Lambda_m(\alpha)$ is given by the lagrangian correspondence. Note $c_1(\mathcal{V})$ is the fundamental class of the diagonal cut out by $c_1(\mathcal{V})$ as a correspondence. It is not lagrangian, as it is cut. But one can show that $\Lambda c_1(\mathcal{V}) - c_1(\mathcal{V}) \otimes 1 - 1 \otimes c_1(\mathcal{V})$ involves only the classical r-matrix. (See [34, §10.1.2].) From this together with the above formula of the classical r-matrix, one calculate $\Lambda c_1(\mathcal{V}) - c_1(\mathcal{V}) \otimes 1 - 1 \otimes c_1(\mathcal{V})$. Then we combine it with the formula of $c_1(\mathcal{V}) \otimes 1 - 1 \otimes c_1(\mathcal{V})$ in Theorem 6.5.1. The formula is given in [34, Th. 14.2.3], but only its restriction on $F^-$ is necessary for us.

Let us consider the commutator $[\Lambda c_1(\mathcal{V}), P^A_n]$ acting on $\bigoplus H_T^{|\sigma|}(\tilde{\mathcal{U}}_2^d)^T \otimes_{A_T} F_T \supseteq F^A_{\text{loc}} \otimes F^-_{\text{loc}}$, where the subscript ‘loc’ means $\otimes_{A_T} F_T$. We take its restriction to $1 \otimes F^-_{\text{loc}} \equiv F^-_{\text{loc}}$, composed with the projection $F^A_{\text{loc}} \otimes F^-_{\text{loc}} \rightarrow 1 \otimes F^-_{\text{loc}}$. It is an operator on $F^-_{\text{loc}}$. Let us denote it by $[\Lambda c_1(\mathcal{V}), P^A_n]_{F^-_{\text{loc}}}$. Then the formula [34, Th. 14.2.3] gives us

**Theorem 6.6.1.** We have $[\Lambda c_1(\mathcal{V}), P^A_n]_{F^-_{\text{loc}}} = n L_n$, where $L_n$ is given by (6.4.1).

Since $R$ intertwines $\Lambda c_1(\mathcal{V})$ and $\Lambda^op c_1(\mathcal{V})$ by its definition, the above implies that it intertwines Virasoro operators.
Note that $F_{\text{loc}}$ is characterized in $F_{\text{loc}}^x \otimes F_{\text{loc}}^\mathbb{C}$ as the intersection of kernels of $p_{-m}$ for $m > 0$. This subspace is nothing but $\bigoplus_d H^1_{T}(\mathbb{C}_{\text{SL}(2)} \otimes A_1 \otimes T)$ by Exercise 3.7.4. Thus $\bigoplus_d H^1_{T}(\mathbb{C}_{\text{SL}(2)} \otimes A_1 \otimes T)$ is a module of the Virasoro algebra, which is the W-algebra associated with $g = \mathfrak{sl}(2)$. This is the first case of the AGT correspondence mentioned in Introduction.

Moreover the operator $[\Delta_{c_1}(V), p_{-n}]$ is well-defined on non-localized equivariant cohomology $H^1_{T}(\mathbb{C}_{\text{SL}(2)} \otimes A_1 \otimes T)$ if $p_{-n} = p_{-n}^\mathbb{C}$ is replaced by $\epsilon_1 \epsilon_2 p_{-n} = p_{-n}^\mathbb{C}$ for $n > 0$. This is the starting point of our discussion on integral forms. See §8.2.

Remark 6.6.2. For quiver varieties, we have tautological vector bundles $V_i$ associated with vertexes $i$. The formula of $\Delta_{c_1}(V_i)$ is given in the same way (see [34, Th. 10.1.1]). On the other hand, the coproduct of the Yangian $Y(g)$ associated with a finite dimensional simple Lie algebra $g$ has an explicit formula for the first Fourier mode of fields corresponding to Chevalley generators of $g$. It is given in terms of the classical $r$-matrix as in the geometric construction, where $r$ is the invariant bilinear form. The constant Fourier modes are primitive, and there are no known explicit formula for second or higher Fourier modes.

When $c_1(V_i)$ is identified with $h_{i,1}$ for type ADE quiver varieties, the first Fourier mode of the field corresponding to a Cartan element $h_i$, geometric and algebraic coproducts coincide. Since $h_{i,1}$ together with constant modes generates $Y(g)$, two coproducts are equal [35].

7. Perverse sheaves on instanton moduli spaces

We now turn to $\mathbb{L}_G^d$ for general $G$.

7.1. Hyperbolic restriction on instanton moduli spaces

Let $\rho: \mathbb{C}^\times \to T$ be a one parameter subgroup. We have associated Levi and parabolic subgroups

$$L = G^{\rho}(\mathbb{C}^\times), \quad P = \left\{ g \in G \mid \exists \lim_{t \to 0} \rho(t) g \rho(t)^{-1} \right\}.$$  

Unlike before, here we allow nongeneric $\rho$ so that $G^{\rho}(\mathbb{C}^\times)$ could be different from $T$. This is not an actual generalization. We can replace $T$ by $Z(L)^0$, the connected center of $L$. Then $\rho$ above can be considered as a generic one parameter subgroup in $Z(L)^0$.

We consider the induced $\mathbb{C}^\times$-action on $\mathbb{L}_G^d$. Let us introduce the following notation for the diagram (4.3.2):

$$\mathbb{L}_L^d \overset{\text{def}}{=} (\mathbb{L}_G^d)^{\rho(\mathbb{C}^\times)}, \quad \mathbb{L}_T^d \overset{\text{def}}{=} \mathbb{L}_G^d \otimes A_1^d \overset{1_T}{\to} \mathbb{L}_G^d.$$  

Let us explain how these notation can be justified. If we restrict our concern to the open subscheme $\text{Bun}_G^d$, a framed $G$-bundle $(\mathcal{F}, \varphi)$ is fixed by $\rho(\mathbb{C}^\times)$ if and only if we have an $L$-reduction $\mathcal{F}_L$ of $\mathcal{F}$ (i.e., $\mathcal{F} = \mathcal{F}_L \times_L G$) so that $\mathcal{F}_L|_{\ell_\infty}$ is sent to $\ell_\infty \times L$ by the trivialization $\rho$. Thus $(\text{Bun}_G^d)^{\rho(\mathbb{C}^\times)}$ is the moduli space
of framed L-bundles, which we could write \( \text{Bun}_L^d \). The definition of Uhlenbeck partial compactification is a little delicate, and is defined for almost simple groups. Nevertheless it is still known that \( \mathcal{U}_L^d \) is homeomorphic to the Uhlenbeck partial compactification for \([L, L]\) when it has only one simple factor ([12, 4.7]), though we do not know they are the same as schemes or not. We will actually use this fact later, therefore the same notation for fixed point subschemes and genuine Uhlenbeck partial compactifications are natural for us.

On the notation \( \mathcal{U}_P^d \): If we have a framed \( P \)-bundle \( (\mathcal{F}_P, \varphi) \), the associated framed \( G \)-bundle \( (\mathcal{F}_P \times_P G, \varphi \times_P G) \) is actually point in the attracting set \( \mathcal{A}_{\mathcal{U}_G^d} \). Thus the moduli space of framed \( P \)-bundle \( \text{Bun}_L^d \) is an open subset in \( \mathcal{A}_{\mathcal{U}_G^d} \). This is the reason why we use the notation \( \mathcal{U}_P^d \). However a point in \( \mathcal{U}_L^d \cap \text{Bun}_G^d \) is not necessarily coming from a framed \( P \)-bundle like this. See Exercise 7.1.3 below. Nevertheless we believe that it is safe to use the notation \( \mathcal{U}_P^d \), as we never consider genuine Uhlenbeck partial compactifications for the parabolic subgroup \( P \).

**Example 7.1.2.** If \( G = \text{SL}(r) \) and \( L = S(\text{GL}(r_1) \times \text{GL}(r_2)) \), \( \text{Bun}_L^d \) is the moduli space of pairs of framed vector bundles \((E_1, \varphi_1), (E_2, \varphi_2)\). On the other hand, \( \text{Bun}_G^d \) is the moduli space of vector bundles \( E \) which arise as extension \( 0 \to E_1 \to E \to E_2 \to 0 \). (In this situation, one can show that \( E \) determines \( E_1, E_2 \).)

**Exercise 7.1.3** (cf. an example in [12, §4(iv)]). Consider the case \( G = \text{SL}(r) \). Suppose \((E, \varphi)\) is a framed vector bundle which fits in an exact sequence \( 0 \to E_1 \to E \to E_2 \to 0 \) compatible with the framing. Here we merely assume \( E_1, E_2 \) are torsion-free sheaves. Are \( E_1, E_2 \) locally free? Give a counter-example.

**Definition 7.1.4.** Now we consider the hyperbolic restriction functor \( p^* j_* : \text{D}^b_T(\mathcal{U}_G^d) \to \text{D}^b_T(\mathcal{U}_L^d) \) and denote it by \( \Phi_{L,G}^P \). If groups are clear from the context, we simply denote it by \( \Phi \).

We have the following associativity of hyperbolic restrictions.

**Proposition 7.1.5.** Let \( Q \) be another parabolic subgroup of \( G \), contained in \( P \) and let \( M \) denote its Levi subgroup. Let \( Q_L \) be the image of \( Q \) in \( L \) and we identify \( M \) with the corresponding Levi group. Then we have a natural isomorphism of functors

\[
\Phi_{M,L}^Q \circ \Phi_{L,G}^P \cong \Phi_{M,G}^Q.
\]

**Proof.** It is enough to show that

\[
\mathcal{U}_L^d \times \mathcal{U}_Q^d = \mathcal{U}_Q^d.
\]

This is easy to check. See [12, 4.16].

---

12Here we use the assumption \( G \) is of type ADE. The instanton number is defined via an invariant bilinear form on \( g \). For almost simple groups, we normalize it so that the square length of the highest root \( \theta \) is 2. If \( G \) is of type ADE, instanton numbers are preserved for fixed point sets, but not in general. See [12, 2.1].
7.2. Exactness For a partition \( \lambda \), let \( S_\lambda \mathbb{C}^2 \) be a stratum of a symmetric product as before. Let \( \text{Stab}(\lambda) = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \) if we write \( \lambda = (1^{\alpha_1}2^{\alpha_2} \cdots) \). We consider an associated covering
\[
(\mathbb{C}^2)^{\alpha_1} \times (\mathbb{C}^2)^{\alpha_2} \times \cdots \setminus \text{diagonal} \to S_\lambda(\mathbb{C}^2).
\]
Let \( \rho \) be a simple local system over \( S_\lambda \mathbb{C}^2 \) corresponding to an irreducible representation of \( \text{Stab}(\lambda) \).

We consider the following class of perverse sheaves:

**Definition 7.2.1.** Let \( \text{Perv}(\mathcal{U}_\Lambda^d) \) be the additive subcategory of the abelian category of semisimple perverse sheaves on \( \mathcal{U}_\Lambda^d \), consisting of finite direct sum of \( \text{IC}(\text{Bun}_{\mathcal{U}_\Lambda^d}^{\rho}, S_\lambda(\mathbb{C}^2), 1 \boxtimes \rho) \) for various \( d', \lambda, \rho \).

Here we consider the stratification of \( \mathcal{U}_\Lambda^d \) as in (2.2.2):
\[
\mathcal{U}_\Lambda^d = \bigsqcup_{d = |\lambda| + d'} \text{Bun}_{\mathcal{U}_\Lambda^{d'}} \times S_\lambda(\mathbb{C}^2).
\]
It is the restriction of the stratification (2.2.2) to \( \mathcal{U}_\Lambda^d \).

Let us explain why we need to consider nontrivial local systems, even though our primary interest will be on \( \text{IC}(\mathcal{U}_\Lambda^d) \). When we analyze \( \text{IC}(\mathcal{U}_\Lambda^d) \) through hyperbolic restriction functor, IC sheaves for nontrivial local systems occur. This phenomenon can be seen for type \( A \) as follows.

Let us take the Gieseker space \( \mathcal{U}_r^d \) and consider the hyperbolic restriction for a chamber \( \mathcal{C} \) for the \( T \)-action. By Corollary 5.4.2(2), we have \( p_j^* \pi_* (\mathcal{C} \circ \mathcal{U}_r^d) \cong \pi^T (\mathcal{C} \circ \mathcal{U}_r^d)^T \), where \( (\mathcal{U}_r^d)^T \) is the fixed point set and \( \pi^T : (\mathcal{U}_r^d)^T \to (\mathcal{U}_r^d)^T \) is the restriction of \( \pi \). The fixed point sets are given by Hilbert schemes and symmetric products, and \( \pi^T \) factors as
\[
(\mathcal{U}_r^d)^T = \bigsqcup_{d_1 + \cdots + d_r = d} (\mathbb{C}^2)^{[d_1]} \times \cdots \times (\mathbb{C}^2)^{[d_r]},
\]
where \( \kappa \) is the ‘sum map’, defined by \( \kappa(C_{1}, \cdots, C_r) = C_1 + \cdots + C_r \). If we use the ‘sum notation’ for points in symmetric products, like \( x_1 + x_2 + \cdots + x_d \). The pushforward for the first factor \( \pi \times \cdots \times \pi \) is
\[
\bigoplus_{|\lambda_1| + \cdots + |\lambda_r| = d} \mathcal{C}_{\lambda_1}(\mathbb{C}^2) \boxtimes \cdots \boxtimes \mathcal{C}_{\lambda_r}(\mathbb{C}^2)
\]
by discussion in §3.7, where \( \lambda_1, \ldots, \lambda_r \) are partitions. Therefore we do not have nontrivial local systems. But \( \kappa \) produces nontrivial local systems. Since \( \kappa \) is a finite morphism, in order to calculate its pushforward, we need to study how \( \kappa \) restricts to covering on strata. For example, for \( \lambda = (1^T) \), \( d_1 = d_2 = \cdots = d_r = 1 \), it is a standard \( S_r \)-covering \( (\mathbb{C}^2)^r \setminus \text{diagonal} \to S_{(1^r)}(\mathbb{C}^2) \).

We have the following

**Theorem 7.2.2.** \( \Phi_{L,G}^\rho \) sends \( \text{Perv}(\mathcal{U}_\Lambda^d) \) to \( \text{Perv}(\mathcal{U}_\Lambda^d) \).
By the remark after Theorem 5.3.1 we know that $\Phi_{L,G}^P$ send $\text{Perv}(\mathcal{U}_G^d)$ to semisimple complexes. From the factorization, it is more or less clear that they must be direct sum of shifts of simple perverse sheaves in $\text{Perv}(\mathcal{U}_G^d)$. Therefore the actual content of this theorem is the $t$-exactness, that is shifts are unnecessary. For type $A$, it is a consequence of Corollary 5.4.2.

For general $G$, we use hyperbolic semi-smallness (Theorem 5.5.3). The detail is given in [12, Appendix A], and is a little complicated to be reproduced here. Let us mention only several key points: By a recursive nature and the factorization property of instanton moduli spaces, it is enough to estimate dimension of the extreme fibers, i.e., $p^{-1}(d \cdot 0)$, $p^{-1}(d \cdot 0)$ in (5.5.2). Furthermore using the associativity of hyperbolic restriction (Proposition 7.1.5), it is enough to prove the case $L = T$. In fact, we can further reduce to the case of the hyperbolic restriction for the larger torus $T \times \mathbb{C}_\text{hyp}$. The associativity (Proposition 7.1.5) remains true for the larger torus. Finally, we consider affine Zastava spaces, i.e., Uhlenbeck partial compactifications corresponding to moduli spaces of framed parabolic bundles. In some sense, these spaces behave well, and we have required dimension estimate there.

7.3. Calculation of the hyperbolic restriction  Our next task is to compute $\Phi_{L,G}^P(\text{IC}(\mathcal{U}_G^d))$.

We have two most extreme simple direct summands in it:

(a) $\text{IC}(\mathcal{U}_L^d)$,
(b) $C_{S(\mathfrak{a})}(\mathbb{C}^2)$.

Other direct summands are basically products of type (a) and (b). Let us first consider (a). Let us restrict the diagram (7.1.1) to $\text{Bun}_L^d$. Note that a point in $p^{-1}(\text{Bun}_L^d)$ is a genuine bundle, cannot have singularities, as singularities are equal or increased under $p$. Thus the diagram sit in moduli spaces of genuine bundles as

$$\text{Bun}_L^d \xleftarrow{p} \text{Bun}_G^d \xrightarrow{j} \text{Bun}_G^d.$$ 

Then $\text{Bun}_L^d$ is a vector bundle over $\text{Bun}_G^d$. This can be seen as follows. For $\mathcal{F} \in \text{Bun}_G^d$, the tangent space $T_{\mathcal{F}} \text{Bun}_G^d$ is $H^1(\mathbb{P}^2, g_{\mathcal{F}}(-\ell_\infty))$, where $g_{\mathcal{F}}$ is the associated bundle $\mathcal{F} \times_G \mathfrak{g}$. If $\mathcal{F} \in \text{Bun}_L^d$, we have the decomposition

$$H^1(\mathbb{P}^2, g_{\mathcal{F}}(-\ell_\infty)) \cong H^1(\mathbb{P}^2, \mathfrak{i}_{\mathcal{F}}(-\ell_\infty)) \oplus H^1(\mathbb{P}^2, n_{\mathcal{F}}^+(\mathfrak{i}(-\ell_\infty))) \oplus H^1(\mathbb{P}^2, n_{\mathcal{F}}^-(\mathfrak{i}(-\ell_\infty))),$$

according to the decomposition $\mathfrak{g} = \mathfrak{i} \oplus n^+ \oplus n^-$. Here $n^+$ is the nilradical of $p$, and $n^-$ is the nilradical of the opposite parabolic. The first summand gives the tangent bundle of $\text{Bun}_L^d$. Thus the normal bundle is given by sum of the second and third summands. Moreover, the second and third summands are Leaf and Leaf$^-$ respectively.

Therefore we have the Thom isomorphism $p_* j^! C_{\text{Bun}_G^d} \cong C_{\text{Bun}_L^d}$. It extends to a canonical isomorphism

$$\text{Hom}(\text{IC}(\text{Bun}_L^d), p_* j^! \text{IC}(\mathcal{U}_G^d)) \cong \mathbb{C}.$$
In other words, we have \( p^{-1}(y_\beta) \cap X_0 \) in Theorem 5.5.3 is the fiber of the vector bundle \( \text{Bun}_d^\text{p} \to \text{Bun}_d \), hence we have the canonical isomorphism \( H_{\dim X - \dim Y_P} (p^{-1}(y_\beta) \cap X_0)|_X \cong C \), given by its fundamental class.

On the other hand, the multiplicity of the direct summand \( \mathcal{E}_{S_{(d)}(C^2)} \) of \( \text{IC} \)-sheaves of simple constitutes. W e understand the associated IC-sheaf as the direct sum of \( \lambda \) where the lower isomorphism is the special case of the upper one as \( \text{Bun}_d \).

Thus \( \mathcal{U}^d = \mathcal{U}^d_{L,G} \) defined by \( \text{Hom}(\mathcal{E}_{S_{(d)}(C^2)}, \Phi_{L,G}^P [\text{IC}(\mathcal{U}_G^d)]) \).

From the factorization, we have the canonical isomorphism \( \Phi_{L,G}^P [\text{IC}(\mathcal{U}_G^d)] \cong \bigoplus_{d=|\lambda| + d'} \text{IC}(\text{Bun}_d^G \times S_\lambda(C^2), (\mathcal{U}^1)^{\otimes \alpha_1} \otimes (\mathcal{U}^2)^{\otimes \alpha_2} \otimes \cdots) \),

where \( \lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots) \) and \( (\mathcal{U}^1)^{\otimes \alpha_1} \otimes (\mathcal{U}^2)^{\otimes \alpha_2} \otimes \cdots \) is a representation of \( \text{Stab}(\lambda) = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \). Though \( (\mathcal{U}^1)^{\otimes \alpha_1} \otimes (\mathcal{U}^2)^{\otimes \alpha_2} \otimes \cdots \) may not be irreducible, it is always semisimple. We understand the associated IC-sheaf as the direct sum of IC-sheaves of simple constitutes.

If we take the global cohomology group, contributions of nontrivial local systems vanish. (See [12, Lemma 4.35].) Therefore (after taking an isomorphism \( H^*(S_\lambda C^2) \cong A_T \)) we get

\[
\bigoplus_{d} H^*_T(\Phi_{L,G}^P [\text{IC}(\mathcal{U}_G^d)]) \cong \bigoplus_{d} H^*_T(\text{IC}(\text{Bun}_d^G)) \otimes_C \text{Sym} (\mathcal{U}^1 \oplus \mathcal{U}^2 \oplus \cdots),
\]

\[
\bigoplus_{d} H^*_T(\Phi_{L,G}^B [\text{IC}(\mathcal{U}_G^d)]) \cong A_T \otimes_C \text{Sym} (\mathcal{U}^1 \oplus \mathcal{U}^2 \oplus \cdots),
\]

where the lower isomorphism is the special case of the upper one as \( \text{Bun}_d^G = \emptyset \) for \( d \neq 0 \) and is a point for \( d = 0 \).

The first result on \( \mathcal{U}^d \) is

**Lemma 7.3.2** ([12, 4.35]).

\[
\dim \mathcal{U}^d = \text{rank } G - \text{rank } [L, L],
\]

in particular \( \dim \mathcal{U}^d = \text{rank } G \) if \( L = T \).

This result was proved as

1. a reduction to the case \( L = T \) via the associativity (Proposition 7.1.5),
2. a reduction to a computation of the ordinary restriction to \( S_{(d)} A^2 \) thanks to a theorem of Laumon [28], and
3. the computation of the ordinary restriction to \( S_{(d)} A^2 \) in [11, Theorem 7.10].

Instead of explaining the detail of this argument, let us give two explanations of \( \dim \mathcal{U}^d = \text{rank } G \) for \( L = T \) in the next two subsections. In fact, we will also see that there is an isomorphism \( \mathcal{U}^d \cong h \), the Cartan subalgebra of \( G \). (This will be actually one of our goal.)
7.4. Hyperbolic restriction for Gieseker partial compactification  Let us suppose \( G = \text{SL}(r) \) and consider the hyperbolic restriction of \( \pi_*(\mathcal{E}_{U_E}) \). Let us suppose \( L = T \) for brevity. We introduce

\[
V^d \overset{\text{def}}{=} \text{Hom}(\mathcal{E}_{S(d)}(C^2), \Phi^B_{T,G}(\pi_*(\mathcal{E}_{U_E})))
\]
as an analog of \( U^d \). As in (7.3.1), we have

\[
\bigoplus_d H^*_T(\Phi^B_{T,G}(\pi_*(\mathcal{E}_{U_E}))) \cong A_T \otimes \text{Sym}(V^1 \oplus V^2 \oplus \cdots).
\]

By the stable envelop, we have an isomorphism \( \Phi^B_{T,G}(\pi_*(\mathcal{E}_{U_E})) \cong \pi_T^*(\mathcal{E}_{(\mathcal{U}_E)^T}) \) (see Corollary 5.4.2). We decompose \( \pi_T^*(\mathcal{E}_{(\mathcal{U}_E)^T}) \) according to the the description (3.6.2) of \( (\mathcal{U}_E^d)^T \). One can easily check that a direct summand has nonzero contribution to \( V^d \) only when \( d_1, \ldots, d_r \) are zero except one. That is \( (d_1, \ldots, d_r) = (1,0,\ldots,0), (0,1,0,\ldots,0), \text{or } (0,\ldots,0,1) \). Moreover \( \text{Hom}(\mathcal{E}_{S(d)}(C^2), \pi_*(\mathcal{E}_{C^2}(d))) \) is naturally isomorphic to the fundamental class \( [\pi^{-1}(d \cdot 0)] \) of the fiber of \( \pi \) at \( d \cdot 0 \in S(d)(C^2) \). In particular, we have a base \( \{e_1, \ldots, e_r\} \) corresponding to summands \( (d_1, \ldots, d_r) = (1,0,\ldots,0), (0,1,0,\ldots,0), \text{or } (0,\ldots,0,1) \), and \( V^d \) is \( r \)-dimensional.

Moreover in the decomposition of \( \pi_*(\mathcal{E}_{U_E}) \) in (3.7.3), a direct summand contributes to \( V^d \) only if \( d' = d \) (and \( \lambda = 0 \)) or \( \lambda = (d) \) (and \( d' = 0 \)). Therefore we have a natural direct sum decomposition

\[
V^d = U^d \oplus C[\pi^{-1}(d \cdot 0)],
\]

where \( d \cdot 0 \) is considered as a point in the stratum \( \text{Bun}_{SL(r)}^0 \times S(d)(C^2) \). In particular, we obtain \( \dim U^d = r - 1 = \text{rank SL}(r) \) as expected.

Note also that the class \( [\pi^{-1}(d \cdot 0)] \) is \( P^\Delta_d([0])|_{\text{vac}} = \epsilon_1 \epsilon_2 P^\Delta_d|_{\text{vac}} \). By Proposition 6.3.1 it is equal to \( e_1 + \cdots + e_r \). Since \( U^d \) corresponds to the subspace killed by the Heisenberg operators, we also see

\[
U^d \cong \left\{ a_1 e_1 + \cdots + a_1 e_r \mid \sum a_i = 0 \right\}.
\]

Thus \( U^d \) is isomorphic to the Cartan subalgebra of \( sl(r) \). Furthermore for \( r = 2 \), we have \( F_{\text{loc}} = F_T \otimes \text{Sym}(U^1 \oplus U^2 \oplus \cdots) \), where \( F_{\text{loc}} \) is the Fock space of the Heisenberg operator \( P_{\pi}^\Delta = P_{(1)} - P_{(2)} \) considered in the previous lecture. In fact, it will be naturally considered as the Fock space for \( e_1 \epsilon_2 P_{\pi}^\Delta \), as \( [\pi^{-1}(d \cdot 0)] \) is a cohomology class with compact support, when we will consider integral forms in §8.

7.5. Affine Grassmannian for an affine Kac-Moody group  In this subsection, we drop the assumption that \( G \) is of type ADE.

Recall that it is proposed in [10] that G-instanton moduli spaces on \( \mathbb{R}^4/\mathbb{Z}_\ell \) plays a role of affine Grassmannian for an affine Kac-Moody group, as we mentioned in Introduction. Let us give an interpretation of \( \dim U^d = \text{rank } G \) in this framework.
We restrict ourselves to the case $\ell = 1$. Then there are no differences on instanton moduli spaces for various $G$ with the common Lie algebra. Therefore we expect that corresponding representations are of the Langlands dual affine Lie algebra. Let us denote it by $g^{\vee}_{\text{aff}}$. If $G$ is of type $\text{ADE}$, then it is the untwisted affine Lie algebra of Lie $G$. If $G$ is of type $\text{BCFG}$, it is the Langlands dual of the untwisted affine Lie algebra of Lie $G$, hence the notation is reasonable. Here the Langlands dual is just given by reversing arrows in the Dynkin diagram. Concretely it is a twisted affine Lie algebra given by the following table:

| $G$          | the affine Lie algebra $g^{\vee}_{\text{aff}}$ |
|-------------|-----------------------------------------------|
| $X_\ell$ ($X = \text{ADE}$) | $X^{(1)}_\ell$ |
| $B_\ell$    | $A^{(2)}_{2\ell}$ |
| $C_\ell$    | $D^{(2)}_{\ell+1}$ |
| $F_4$       | $E^{(2)}_6$ |
| $G_2$       | $D^{(3)}_4$ |

Here we follow [25] for the notation of Lie algebras.

Since we are considering $\ell = 1$, our $U^d_G$ should correspond to a level $1$ representation of $g^{\vee}_{\text{aff}}$. It is known that all level $1$ representations are related by Dynkin diagram automorphisms for our choice of $g^{\vee}_{\text{aff}}$ (e.g., rotations for $A^{(1)}_\ell$ [25, (12.4.5)]. Therefore we could just take the 0th fundamental weight $\Lambda_0$ without loss of generality. In fact, we do not have a choice of a highest weight in $U^d_G$, so the geometric Satake could not make sense otherwise.

Let us consider $L(\Lambda_0)$, the irreducible representation of $g^{\vee}_{\text{aff}}$ with highest weight $\Lambda_0$. The weight corresponding to $U^d_G$ is $\Lambda_0 - d\delta$. We only have discrete parameter $d$ in the geometric side. Fortunately all dominant weights of $L(\Lambda_0)$ are of this form, and the geometric Satake for affine Kac-Moody group is formulated only for dominant weights when [10] was written.

By [25, Prop. 12.13] we have

$$\text{mult}_{L(\Lambda_0)}(\Lambda_0 - d\delta) = p^\Lambda(d),$$

where

$$\sum_{d \geq 0} p^\Lambda(d)q^d = \prod_{n \geq 1} (1 - q^n)^{-\text{mult}n\delta}.$$ 

Moreover $\text{mult}n\delta = \ell$ if $g^{\vee}_{\text{aff}} = X^{(1)}_\ell$ for type ADE. Next suppose $g^{\vee}_{\text{aff}}$ is the Langlands dual of $X^{(1)}_\ell$ where $X$ is of type BCFG. Let $r^{\vee}$ be the lacing number of $g^{\vee}_{\text{aff}}$, i.e., the maximum number of edges connecting two vertexes of the Dynkin diagram of $g^{\vee}_{\text{aff}}$ (and $X^{(1)}_\ell$). Then $\text{mult}n\delta = \ell$ if $n$ is a multiple of $r$, and equals to the number of long simple roots in the finite dimensional Lie algebra $X_\ell$ otherwise. Explicitly $\ell - 1$ for $B_\ell$, $1$ for $C_\ell$ and $G_2$, and $2$ for $F_4$. See [25, Cor. 8.3].

Let us turn to the geometric side. As we have explained in §5.6, the hyperbolic restriction with respect to a maximal torus corresponds to a weight space. Our $T$ is a maximal torus of $G$, but not of the affine Kac-Moody group. So let us
consider a larger torus \( \tilde{T} = T \times C_\text{hyp}^\times \), where the second \( C_\text{hyp}^\times \) is a subgroup \((t, t^{-1})\) of \( C^\times \times C^\times \) acting on \( C^2 \) preserving the symplectic form. There is only one fixed point (for each \( U^d \)) with respect to \( \tilde{T} \), as \( (C^2)^{C^\times} = \{0\} \). Let us denote the corresponding hyperbolic restriction by \( \tilde{\Phi} \). It is hyperbolic semi-small. (In fact, we proved the property first for \( \tilde{\Phi} \), as we mentioned before.) Therefore we expect

\[
\tilde{\Phi}(\text{IC}(U^d))
\]

is naturally isomorphic to a weight space of a level 1 representation of \( g^{\text{aff}} \). In our case, we have already computed this space implicitly in §7.3 for type ADE. It is equal to the degree \( d \) part of the right hand side of (7.3.1). Otherwise solve Exercise 5.5.4. When \( G \) is of type BCF, we have a different behavior because of the mismatch of instanton numbers under the restriction, explained in footnote 5. One can check that it matches with the above character formula.

For type ADE, the representation \( L(\Lambda_0) \) has an explicit realization by vertex operators (Frenkel-Kac construction) [25, §14.8]. The underlying vector space is given by

\[
\text{Sym}(\bigoplus_{d>0} z^{-d} \otimes h) \otimes \mathbb{C}[Q],
\]

where \( h \) (resp. \( Q \)) is the Cartan subalgebra (the root lattice) of Lie \( G \). Therefore it is natural to identify \( U^d \) with \( z^{-d} \otimes h \).

Note however that this is just an explanation, as the geometric Satake correspondence for affine Kac-Moody groups is not established.

### 7.6. Heisenberg algebra representation on localized equivariant cohomology

Let us fix a Borel \( B \), and consider the parabolic subgroups \( P_i \supset B \) for each vertex \( i \) of the Dynkin diagram of \( G \). Let \( L_i \) be the Levi factor. By the associativity of the hyperbolic restriction (Proposition 7.1.5), we have a natural decomposition

\[
U^d_{T,G} \cong U^d_{T,L_i} \oplus U^d_{L_i,G}.
\]

Since \( [L_i, L_i] \cong \text{SL}(2) \), we apply the construction in §7.4 to get a Fock space representation on \( F_T \otimes \text{Sym}(U^1_{T,L_i} \oplus U^2_{T,L_i} \oplus \ldots) \). Let us denote the Heisenberg operator by \( P^i_m \). We extend it to \( F_T \otimes \text{Sym}(U^1_{T,G} \oplus U^2_{T,G} \oplus \ldots) \) by letting act trivially on \( U^d_{L_i,G} \). So we have rank \( G \) copies of Heisenberg generators action on \( F_T \otimes \text{Sym}(U^1_{T,G} \oplus U^2_{T,G} \oplus \ldots) \).

**Proposition 7.6.1.** We have

\[
[P^i_m, P^j_n] = -m\delta_{m,-n}(\alpha_i, \alpha_j) \frac{1}{\epsilon_1 \epsilon_2}
\]

where \( \alpha_i \) is the simple root of \( G \) corresponding to \( P_i \). Therefore

\[
\bigoplus_d H^T_i(0^{\text{B}_{T,G}}(\text{IC}(U^d_G)) \otimes_A F_T \cong F_T \otimes \text{Sym}
\]

is naturally the Fock representation of the Heisenberg algebra associated with the Cartan subalgebra \( h \) of \( g = \text{Lie} G \).
Thus highest weights and level are identified with equivariant variables. This will be supposed to be discussed in future.

For two distinct vertexes \(i, j\), we consider the corresponding parabolic subgroup \(P_{i,j}\) with Levi factor \(L_{i,j}\). We have \(U^d_{i,G} \cong U^d_{i,L_{i,j}} \oplus U^d_{L_{i,j},G}\). Since \([L_{i,j}, L_{i,j}]\) is either \(SL(3)\) or \(SL(2) \times SL(2)\). For \(SL(2) \times SL(2)\), the commutator is clearly 0. For \(SL(3)\), we consider \(\bigoplus H^*_T((\bar{U}^d_3)^T)\) as the tensor product of three copies of Fock space as in §6.4. Then we have three Heisenberg operators \(P_{m_i}^{(1)}, P_{m_i}^{(2)}, P_{m_i}^{(3)}\) and \(P_{m_i}^\Delta = P_{m_i}^{(1)} + P_{m_i}^{(2)} + P_{m_i}^{(3)}\). It is also clear that \(P_{m_i}^1 = P_{m_i}^{(1)} - P_{m_i}^{(2)}\) by a consideration of the associativity of the hyperbolic restriction.

Hence we get (7.6.2). In fact, there is a delicate point here. We need to choose a polarization for instanton moduli spaces so that it is compatible with a polarization for \(\bar{U}^d_3\) for each \(L_{i,j}\). See [12, §6(ii)] for detail. \(\square\)

8. \(\mathcal{W}\)-algebra representation on equivariant intersection cohomology groups

The goal of this final lecture is to explain a construction of \(\mathcal{W}\)-algebra representation on \(\bigoplus_d H^*_T(\bar{U}^d_G)\). In fact, if we assume level is generic, or take \(\bigotimes A_T F_T\) in equivariant cohomology, we have already achieved it. This is because the \(\mathcal{W}\)-algebra is known to be a (vertex) subalgebra of the Heisenberg (vertex) algebra at generic level by Feigin-Frenkel (see [19, Ch. 15]). So we just restrict the Heisenberg algebra representation on \(\bigoplus_d H^*_T(\Phi^B_T, G(\text{IC}(\bar{U}^d_G)))\) to \(\mathcal{W}\), and use the localization theorem \(H^*_T(\Phi^B_T, G(\text{IC}(\bar{U}^d_G))) \bigotimes A_T F_T \cong H^*_T(\bar{U}^d_G) \bigotimes A_T F_T\). But this is not an interesting assertion, as it does not explain any geometric meaning of the \(\mathcal{W}\)-algebra.

One way to explain such a meaning is to prove the Whittaker conditions are satisfied for fundamental classes \(\{\bar{U}^d_G\} \in H^*_0(\bar{U}^d_G)\) as in [12, §8]. We will go a half way towards this goal, namely we will explain a construction of a representation of an integral form of \(\mathcal{W}\)-algebra on \(\bigoplus d H^*_T(\bar{U}^d_G)\). The remaining half requires an introduction of generating fields (denoted by \(W_i\) in [19] and by \(\hat{W}^{(k)}\) in [12]), which is purely algebraic, and hence is different from the theme of lectures.

We think that the integral form itself is an important object, as we can recover arbitrary level (and highest weight) representation by a specialization \(A_T \rightarrow \mathbb{C}\). Thus highest weights and level are identified with equivariant variables. This will be supposed to be discussed in future.

8.1. Four types of nonlocalized equivariant cohomology groups

We consider cohomology groups with compact and arbitrary support with coefficients in \(\text{IC}(\bar{U}^d_G)\) and its hyperbolic restriction \(\Phi_{T,G}(\text{IC}(\bar{U}^d_G))\). So we consider four types of nonlocalized equivariant cohomology groups. They are related by natural adjunction
homomorphisms as
\[
\bigoplus_d H^*_{T,c}(\mathcal{U}_G^d) \to \bigoplus_d H^*_T(\Phi^{B\,\mathcal{U}_G^d}_{T,G})(IC_{\mathcal{U}_G^d})
\]
\[
\to \bigoplus_d H^*_T(\Phi^{B\,\mathcal{U}_G^d}_{T,G})(IC_{\mathcal{U}_G^d}) \to \bigoplus_d H^*_T(\mathcal{U}_G^d).
\]
(8.1.1)

(See (5.3.3).) One can show that they are free $A_T$-modules, and homomorphisms are inclusion, which become isomorphisms over $F_T$ ([12, §6]).

Moreover, we have a natural Poincaré pairing between the first and fourth cohomology groups. When we compare it with Kac-Shapovalov form so that it is a pairing between the Verma module with highest weight $\alpha$ and the dual of the Verma module with highest weight $-\alpha$, we need to make a certain twist of this pairing. In particular, the pairing is sesquilinear in equivariant variables. See [12, §6(viii)]. Let us ignore this point, as we will not discuss details of highest weights. The pairing between the second and third can be defined if we replace one of the hyperbolic restriction by the opposite one $\Phi^{B\,\mathcal{U}_G^d}_{T,G}$ as $D_{pi,j} = p_{ij}^*D = p_{-i}^*j$ by Theorem 5.3.1. This is compensated by the twist above, hence we have a pairing between the second and third.

Let $P^i_n = \epsilon_1 \epsilon_2 P^i_n = P^i_n(\{0\})$ be the Heisenberg operator associated with the fundamental class of the origin $0 \in \mathbb{C}^2$. This operator is always well-defined on the middle two cohomology groups in (8.1.1). It satisfies
\[
[P^i_m, P^j_n] = -m \delta_{m,-n}(\alpha_i, \alpha_j) \epsilon_1 \epsilon_2.
\]
Note that the right hand side is a polynomial in $\epsilon_1, \epsilon_2$ contrary to (7.6.2). The pairing is invariant.

Let $\mathfrak{Heis}_A(h)$ denote the algebra over $A \overset{\text{def}}{=} C[\epsilon_1, \epsilon_2]$ generated by $P^i_n$ with the above relations. It is an $A$-form of the Heisenberg algebra. Two middle cohomology groups in (8.1.1) are $\mathfrak{Heis}_A(h)$-modules dual to each other.

Let us look at (7.3.1). The isomorphism uses the identification $H^*_{T}(\mathfrak{S}_\mathbb{C}^2) \cong \mathfrak{A}_T$, which is given by the fundamental class $[\mathfrak{S}_\mathbb{C}^2]$. It is not compatible with $P^i_n$, but $H^*_T(\mathfrak{S}_\mathbb{C}^2) \cong \mathfrak{A}_T$ does as it is given by the fundamental class $[0]$. Therefore we should consider the cohomology group with compact support. We get

**Proposition 8.1.2.** $\bigoplus_d H^*_T(\Phi^{B\,\mathcal{U}_G^d}_{T,G})(IC_{\mathcal{U}_G^d})$ is a highest weight module of $\mathfrak{Heis}_A(h)$ with the highest weight vector $|\text{vac}\rangle$, i.e.,
\[
\bigoplus_d H^*_T(\Phi^{B\,\mathcal{U}_G^d}_{T,G})(IC_{\mathcal{U}_G^d}) \cong \mathfrak{Heis}_A(h)|\text{vac}\rangle.
\]

**8.2. Integral form of Virasoro algebra** Consider the case $G = \text{SL}(2)$. Recall that we can normalize Virasoro operators so that they act on non-localized equivariant cohomology as we have discussed after Theorem 6.6.1. More precisely, as we set $\tilde{P}_n^i = \epsilon_1 \epsilon_2 P^i_n$, it is natural to introduce $\tilde{L}_n = \epsilon_1 \epsilon_2 L_n$. Then it is a well-defined operator on $IH^*_T(\mathcal{U}_{\text{SL}(2)}^d)$ and $IH^*_T(\mathcal{U}_{\text{SL}(2)}^d)$ by Theorem 6.6.1.
The relation (6.4.2) is modified to
\[
[\tilde{L}_m, \tilde{L}_n] = \varepsilon_1 \varepsilon_2 \left\{ (m-n)\tilde{L}_{m+n} + \left( \varepsilon_1 \varepsilon_2 + 6(\varepsilon_1 + \varepsilon_2)^2 \right) \delta_{m,-n} \frac{m^3 - m}{12} \right\}.
\]

The relation is defined over \(\mathbb{C}[\varepsilon_1, \varepsilon_2]\). Therefore we have the integral form of the Virasoro algebra defined over \(\mathbb{C}[\varepsilon_1, \varepsilon_2]\). Let us denote it by \(\mathfrak{Vir}_A\). We have an embedding \(\mathfrak{Vir}_A \subset \mathfrak{sl}_2\), where \(\mathfrak{sl}_2\) is the Cartan subalgebra of \(\mathfrak{sl}_2\).

**Proposition 8.2.1.** \(\mathbb{I}^*_A(\mathfrak{sl}_2)\) is a highest weight module with the highest weight vector \(|\text{vac}\rangle\).

We already know that \(|\text{vac}\rangle\) is killed by \(\tilde{L}_n\). The highest weight is computed in (6.4.3). Hence we need to check \(\mathbb{I}^*_A(\mathfrak{sl}_2) = \mathfrak{Vir}_A|\text{vac}\rangle\). This is done by comparing graded dimensions of both sides ([12, Prop. 8.11]). (This argument works for general \(G\).)

We call \(\mathbb{I}^*_A(\mathfrak{sl}_2)\) the *universal Verma module*, as it is specialized to the Verma module in the usual sense by a specialization \(A_T \rightarrow \mathbb{C}\). Then \(\mathbb{I}^*_A(\mathfrak{sl}_2)\) is the dual universal Verma module.

### 8.3. Integral form of \(\mathcal{W}\)-algebra

Let us take the parabolic subgroup \(P_i\) and its Levi factor \(L_i\) as in §7.6. Then we have the corresponding integral Virasoro algebra as a subalgebra in \(\mathfrak{Vir}_i\). Let us denote it by \(\mathfrak{Vir}_i\). Let \(\mathfrak{Heis}_A(\alpha_i)\) denote the integral Heisenberg algebra for the root hyperplane \(\alpha_i = 0\) corresponding to \(L_i\). Then \(\mathfrak{Vir}_i \otimes_A \mathfrak{Heis}_A(\alpha_i)\) is an \(A\)-subalgebra of \(\mathfrak{Heis}_A(h)\). More precisely we need to consider them as *vertex algebras*, but we ignore this point. Then an integral version of Feigin-Frenkel’s result is

**Theorem 8.3.1** ([12, Th. B.49]). Let \(\mathcal{W}_A(g)\) be an \(A\)-form of the \(\mathcal{W}\)-algebra defined by an \(A\)-form of the BRST complex. Then

\[
\mathcal{W}_A(g) \cong \bigcap_i \mathfrak{Vir}_{i,A} \otimes_A \mathfrak{Heis}_A(\alpha_i^+) ,
\]

where the intersection is taken in \(\mathfrak{Heis}_A(h)\).

The original version states the same result is true for *generic* level, and can be deduced from above by taking \(\otimes_A \mathfrak{C}(\varepsilon_1, \varepsilon_2)\). The level \(k\) in the usual approach is related to \(\varepsilon_1, \varepsilon_2\) by the formula

\[
k + h^\vee = \frac{\varepsilon_2}{\varepsilon_1},
\]

where \(h^\vee\) is the dual Coxeter number of \(g = \text{Lie } G\).

**Remark 8.3.2.** Two equivariant variables \(\varepsilon_1, \varepsilon_2\) are symmetric in the geometric context. On the other hand, \(\varepsilon_1\) and \(\varepsilon_2\) play very different role in the \(A\)-form on the BRST complex above. On the other hand, \(\mathfrak{Vir}_A\) and \(\mathfrak{Heis}_A\) have symmetry \(\varepsilon_1 \leftrightarrow \varepsilon_2\). Therefore the theorem implies the symmetry on \(\mathcal{W}_A(g)\). This symmetry is nontrivial, and is called the Langlands duality of the \(\mathcal{W}\)-algebra. (If \(g\) is not of type
Theorem 8.4.1 (§8(i)). $\bigoplus_d H^*_\text{I}_A^+ (\mathcal{U}^d_T^\text{I}_A^+) \subset \mathcal{W}_A (g)$-module. It is a highest weight module with the highest weight vector $|\text{vac}\rangle$.

We do not review the highest weight, but it is given by an explicit formula in equivariant variables as in (6.4.3). We call $\bigoplus_d H^*_\text{I}_A^+ (\mathcal{U}^d_T^\text{I}_A^+)$ the universal Verma module as for the Virasoro algebra.

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