Two-loop graviton scattering relation
and IR behavior in $\mathcal{N} = 8$ supergravity

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Abstract

We derive an ABDK-like relation between the one- and two-loop four-graviton amplitudes in $\mathcal{N} = 8$ supergravity. Specifically we show that the infrared divergent part of the two-loop amplitude is one-half the square of the one-loop amplitude, suggesting an exponential structure for IR divergences. The difference between the two-loop amplitude and one-half the square of the full one-loop amplitude is therefore finite, and expressible in a relatively simple form. We give arguments for generalizations to higher loops and $n$-point functions, suggesting that the exponential of the full one-loop amplitude may be corrected, to low orders, by only simple finite terms.
1 Introduction

Many advances have been made recently in understanding the structure of the loop expansions of $\mathcal{N} = 4$ super Yang-Mills (in particular, in completely computing its gluon scattering amplitudes) and $\mathcal{N} = 8$ supergravity scattering amplitudes.

The realization that the loop expansion of $\mathcal{N} = 4$ SYM amplitudes has an iterative structure began with the result of Anastasiou, Bern, Dixon, and Kosower (ABDK) relating the two-loop planar four-point gluon scattering amplitude to the one-loop amplitude [1]

$$ M_4^{(2)}(\epsilon) = \frac{1}{2} \left[ M_4^{(1)}(\epsilon) \right]^2 + (-\zeta_2 - \zeta_3 \epsilon + \cdots) M_4^{(1)}(2\epsilon) + \text{const} + \mathcal{O}(\epsilon) . \quad (1.1) $$

Although $\mathcal{N} = 4$ SYM theories are UV finite, scattering amplitudes contain infrared divergences, which are controlled by dimensional regularization in $D = 4 - 2\epsilon$ dimensions. Subsequently, Bern, Dixon, and Smirnov (BDS) [2], building on the work of refs. [3, 4], realized that the IR divergent factors of planar $n$-point amplitudes in $\mathcal{N} = 4$ SYM have an exponential form, and are completely governed by two functions of the coupling $\lambda_{SYM}$: the cusp anomalous dimension $f(\lambda_{SYM})$ and the collinear anomalous dimension $g(\lambda_{SYM})$. They also conjectured a complete nonperturbative exponential ansatz for planar, MHV $n$-point scattering amplitudes in ref. [2]. In their ansatz for the four-point function, the finite part is completely determined by the cusp anomalous dimension $f(\lambda_{SYM})$. The form of their full four-point ansatz was subsequently confirmed in the large coupling limit using the AdS-CFT correspondence [5]. The nonperturbative form of $f(\lambda_{SYM})$ was computed in ref. [6].

The BDS ansatz for the four and five-point function was also proved using dual conformal symmetry [7, 8], while it was found that for six-point functions and above, there are finite corrections to the BDS ansatz [9, 10, 11, 12, 13]. The form of the IR divergent factor for any $n$ was confirmed in ref. [14].

In a parallel development, $\mathcal{N} = 8$ supergravity amplitudes have been found to be much better behaved in the UV than previously thought, and generally to be much simpler than a field theory of quantum gravity is a priori expected to be. In particular, an explicit three-loop four-graviton scattering calculation found no UV divergences [15] and various arguments have been given that $\mathcal{N} = 8$ supergravity is UV finite in four dimensions up to eight loops [16] or even to all orders in perturbation theory [17, 18, 16, 15]. There may be, however, nonperturbative obstructions to UV finiteness [19].

Despite the fact that $\mathcal{N} = 4$ SYM theory is a (finite, superconformal) gauge theory and $\mathcal{N} = 8$ supergravity a (potentially non-renormalizable) theory of quantum gravity, there are deep connections between their perturbative scattering amplitudes. Their tree-level amplitudes are closely related by the string theory relations of Kawai, Lewellen, and Tye (KLT) [20]. These tree-level relations were employed for loop calculations of $\mathcal{N} = 8$ supergravity amplitudes using unitarity methods [21].

In this paper, we begin addressing the question: is it possible, in virtue of the KLT relations, that the exponential structure of both the infrared divergent and the finite parts of the $\mathcal{N} = 4$ SYM amplitudes extends to $\mathcal{N} = 8$ supergravity amplitudes at least up to the order to which $\mathcal{N} = 8$ supergravity is UV finite? In fact, we prove a relation for the

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3Since we work in dimensional regularization, the UV and IR divergences will mix up afterwards, making any structure harder to disentangle.
four-graviton scattering amplitude analogous to the ABDK relation (1.1)

\[ M_4^{(2)}(\epsilon) = \frac{1}{2} \left[ M_4^{(1)}(\epsilon) \right]^2 + \text{finite} + \mathcal{O}(\epsilon) \]  

(1.2)

where the explicit form of the finite part is specified in eq. (2.27). The relation (1.2) for \( \mathcal{N} = 8 \) supergravity is not as strong as the ABDK relation (1.1), in which the finite part is actually a constant rather than a function of the kinematic variables.

We make several observations about this result. First, whereas the ABDK result (1.1) only holds in the large-\( N \) limit, and therefore only involves planar diagrams, the analogous result (1.2) for supergravity requires collusion between planar and non-planar diagrams.

Second, eq. (1.2) implies that the IR-divergent part of the scattering amplitude through two loops is given exactly by the exponential of the one-loop amplitude (and as a result depends not only on the divergent but also on the finite part of the one-loop amplitude). This relation is actually simpler than that for \( \mathcal{N} = 4 \) SYM, where the two-loop divergences are modified by terms proportional to the \( \mathcal{O}(\lambda_{\text{SYM}}^2) \) coefficients of \( f(\lambda_{\text{SYM}}) \) and \( g(\lambda_{\text{SYM}}) \). The absence of such corrections in \( \mathcal{N} = 8 \) supergravity may be explained by the dimensionality of the gravitational coupling \( \kappa \), which dictates that a term like \( M_4^{(1)}(2\epsilon) \) would need to be multiplied by a function of \( s, t, \) and \( u \) of degree one. Cyclic symmetry, however, allows only \( s + t + u \), which vanishes for massless gravitons.

Third, the finite remainder in eq. (1.2), while apparently not expressible in terms of the one-loop amplitude, is much simpler than the complete finite piece of the two-loop amplitude itself, as we will see in section 2. Hence, a large part of the finite two-loop amplitude is determined by the square of the one-loop amplitude. It is therefore probably similar to the case of the six-point gluon amplitude in \( \mathcal{N} = 4 \) SYM [11].

The paper is organized as follows: in section 2, we perform the main calculation of this paper, obtaining the ABDK-like relation (1.2) for the two-loop amplitude. In section 3, we analyze more generally the IR behavior of the four-graviton amplitude, and make some conjectures for higher \( n \)-point functions, as well as for higher loop contributions. Section 4 contains our conclusions.

## 2 Two-loop relation for the four-graviton amplitude

The full all-loop-orders graviton four-point amplitude of \( \mathcal{N} = 8 \) supergravity is proportional to the tree-level four-point amplitude [21]

\[ \mathcal{M}_4 = \mathcal{M}_4^{\text{tree}} \left[ 1 + M_4^{(1)} + M_4^{(2)} + \cdots \right] \]  

(2.1)

where \( \mathcal{M}_4^{\text{tree}} \) contains all the helicity information of the external gravitons, and \( M_4^{(L)} \) is a scalar (momentum-dependent) factor appearing at \( L \) loops. In this section, we will prove the ABDK-like relation (1.2) between \( M_4^{(1)} \) and \( M_4^{(2)} \), suggestive of an exponential form for the full amplitude.

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4 After the work described in this paper was completed, one of the authors learned from Lance Dixon that he was previously aware of the relation (1.2).
Due to the KLT relations [20], \( \mathcal{N} = 8 \) supergravity graviton amplitudes are closely related to \( \mathcal{N} = 4 \) SYM gluon amplitudes, and so loop amplitudes in \( \mathcal{N} = 8 \) supergravity can be expressed in terms of the same scalar integrals that appear in \( \mathcal{N} = 4 \) SYM theory. The one-loop four-graviton amplitude is given by [21]

\[
M_{4}^{(1)} = -i \left( \frac{K}{2} \right)^2 stu \left[ \mathcal{I}_{4}^{(1)}(s, t) + \mathcal{I}_{4}^{(1)}(s, u) + \mathcal{I}_{4}^{(1)}(t, u) \right]
\]

where \( s = (k_1 + k_2)^2 \), \( t = (k_1 + k_4)^2 \), and \( u = (k_1 + k_3)^2 \) are the usual Mandelstam variables, obeying \( s + t + u = 0 \) for massless external gravitons, and \( \mathcal{I}_{4}^{(1)}(s, t) \) corresponds to the scalar box integral

\[
\mathcal{I}_{4}^{(1)}(s, t) = \mathcal{I}_{4}^{(1)}(t, s) = \mu^{4-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}.
\]

We regularize loop integrals by evaluating them in \( D = 4 - 2\epsilon \) dimensions. In the region where \( s, t < 0 \), the scalar box integral \([23]\) is given by [2]

\[
\mathcal{I}_{4}^{(1)}(s, t) = \frac{i\mu^2 e^{-\epsilon}(4\pi)^{-D/2}}{(-s)^{1+\epsilon}(-t)} \left\{ \frac{4}{\epsilon^2} + \frac{2 \log \left( -s \right)}{\epsilon} - \frac{2 \log \left( -t \right)}{\epsilon} + \epsilon \left( 2 \text{Li}_3 (x) + 2 \text{Li}_2 (x) \right) \right\}
\]

where \( x = -t/s \), \( L = -\log(-x) = \log(s/t) \), and \( U = \log(1-x) = \log(-u/s) \), and we have explicitly written \( \mathcal{O}(\epsilon) \) terms that will be needed later. For now we drop the \( \mathcal{O}(\epsilon) \) terms to write (again for \( s, t < 0 \))

\[
\mathcal{I}_{4}^{(1)}(s, t) = \frac{i\epsilon e^{-\epsilon}(4\pi)^{-D/2}}{st} \left\{ \frac{2}{\epsilon^2} \left( \frac{\mu^2}{-s} \right) - \frac{2 \log \left( -s / \mu^2 \right)}{\epsilon} - \frac{4 \pi^2}{3} + \mathcal{O}(\epsilon) \right\}
\]

If we wish to evaluate this in the region \( t > 0 \) and \( s < 0 \), we continue \( t \) from the negative to the positive real axis in the upper half plane to obtain

\[
\mathcal{I}_{4}^{(1)}(s, t) = \frac{i\epsilon e^{-\epsilon}(4\pi)^{-D/2}}{st} \left\{ \frac{4}{\epsilon^2} - \frac{2 \log \left( -s / \mu^2 \right)}{\epsilon} + \epsilon \left[ \log \left( -t / \mu^2 \right) + i\pi \right] \right. \\
\left. + 2 \log \left( -s / \mu^2 \right) \left[ \log \left( -t / \mu^2 \right) - i\pi \right] - \frac{4 \pi^2}{3} + \mathcal{O}(\epsilon) \right\}.
\]

Therefore, using the expression \([2.5]\) or \([2.6]\) as appropriate for each term, we write the full one-loop scattering amplitude \((2.2)\) in the physical region \( t > 0 \) and \( s, u < 0 \),

\[
M_{4}^{(1)} = \frac{\lambda}{8\pi^2} \left\{ \frac{1}{\epsilon} \left( s \log \left( -s / \mu^2 \right) + t \left[ \log \left( t / \mu^2 \right) - i\pi \right] + u \log \left( -u / \mu^2 \right) \right) \right.
\]

\[
+ s \log \left( -u / \mu^2 \right) \left[ \log \left( t / \mu^2 \right) - i\pi \right] + t \log \left( -u / \mu^2 \right) \log \left( -s / \mu^2 \right) \]

\[
+ u \log \left( -s / \mu^2 \right) \left[ \log \left( t / \mu^2 \right) - i\pi \right] + \mathcal{O}(\epsilon) \right\}
\]
where

\[
\lambda = \left( \frac{\kappa}{2} \right)^2 \left( 4\pi e^{-\gamma} \right)^\epsilon. \tag{2.8}
\]

The one-loop scattering amplitude in the region \( s > 0 \) and \( t, u < 0 \) may be obtained by simply exchanging \( s \leftrightarrow t \) in eq. (2.7). Note that, despite the \( 1/\epsilon^2 \) divergence of the scalar loop integral \( \mathcal{I}_4^{(1)}(s, t) \), the full one-loop four-graviton amplitude only has a \( 1/\epsilon \) IR divergence. This is as expected for gravity, as discussed in sec. 3 of this paper.

The one-loop expression (2.7) may be written in a completely permutation symmetric way as

\[
M^{(1)}_4 = \frac{\lambda}{8\pi^2} \left\{ \frac{1}{\epsilon} \left[ s \log \left( -\frac{s}{\mu^2} \right) + t \log \left( -\frac{t}{\mu^2} \right) + u \log \left( -\frac{u}{\mu^2} \right) \right] + s \log \left( -\frac{t}{\mu^2} \right) \log \left( -\frac{u}{\mu^2} \right) + t \log \left( -\frac{u}{\mu^2} \right) \log \left( -\frac{s}{\mu^2} \right) + u \log \left( -\frac{s}{\mu^2} \right) \log \left( -\frac{t}{\mu^2} \right) + \mathcal{O}(\epsilon) \right\} \tag{2.9}
\]

an expression which is manifestly real in the Euclidean region \( s, t, u < 0 \).

Now we turn to the two-loop four-point graviton amplitude [21]

\[
M^{(2)}_4 = \left( \frac{\kappa}{2} \right)^4 s^3tu \left[ \mathcal{I}^{(2)P}_4(s, t) + \mathcal{I}^{(2)P}_4(s, u) + \mathcal{I}^{(2)NP}_4(s, t) + \mathcal{I}^{(2)NP}_4(s, u) \right] + \left( \text{cyclic perms of } s, t, u \right) \tag{2.10}
\]

which receives contributions both from the scalar double-box integral

\[
\mathcal{I}^{(2)P}_4(s, t) = \mu^{8-2D} \int \frac{d^Dp}{(2\pi)^D} \frac{d^Dq}{(2\pi)^D} \frac{1}{p^2 (p + q)^2 q^2 (p - k_1)^2 (p - k_1 - k_2)^2 (q - k_3)^2 (q - k_3 - k_4)^2} \tag{2.11}
\]

as well as from the two-loop non-planar integral

\[
\mathcal{I}^{(2)NP}_4(s, t) = \mu^{8-2D} \int \frac{d^Dp}{(2\pi)^D} \frac{d^Dq}{(2\pi)^D} \frac{1}{p^2 (p + q)^2 q^2 (p - k_2)^2 (p + q + k_1)^2 (q - k_3)^2 (q - k_3 - k_4)^2} \tag{2.12}
\]

The non-planar integral has been evaluated by Tausk [22], who writes it as

\[
\mathcal{I}^{(2)NP}_4(s, t) = \mathcal{I}^{(2)NP}_4(s, u) = -\left( 4\pi \right)^{-D} \Gamma(1 + \epsilon)^2 \left\{ \frac{F_t}{s^2 t} + (t \leftrightarrow u) \right\} \tag{2.13}
\]

where the expression for \( F_t \) takes different forms in different regions depending on the signs of \( s, t, \) and \( u \). We use eq. (2.13) to re-express the two-loop amplitude (2.10) as

\[
M^{(2)}_4 = \left( \frac{\kappa}{2} \right)^4 s^3tu \left[ \mathcal{I}^{(2)P}_4(s, t) - 2\left( 4\pi \right)^{-D} \Gamma(1 + \epsilon)^2 \frac{F_t}{s^2 t} \right] + \left( \text{all perms of } s, t, u \right). \tag{2.14}
\]

We begin with an expression [23, 2] for the scalar double-box integral in the region \( s, t < 0 \) (hence \( u > 0 \)):

\[
\mathcal{I}^{(2)P}_4(s, t) = \left( i\mu^2 e^{-\gamma} (4\pi)^{-D/2} \right)^2 \frac{1}{(-s)^{2+2\epsilon}(-t)} \left\{ -\frac{4}{\epsilon^4} - \frac{5L}{\epsilon^3} + \frac{1}{\epsilon^2} \left( -2L^2 + \frac{5\pi^2}{2} \right) \right\}
\]
\[ + \frac{1}{\epsilon} \left\{ -4 \text{Li}_3(x) - 4 L \text{Li}_2(x) + 2 U L^2 + \frac{2}{3} L^3 + 2 \pi^2 U + \frac{11 \pi^2}{2} L + \frac{65}{3} \zeta_3 \right\} 
+ 44 \text{Li}_4(x) - 4 S_{2,2}(x) + \left( 24 L - 4 U \right) \text{Li}_3(x) - 4 L S_{1,2}(x) 
+ \left( 2 L^2 - 4 U L + \frac{20 \pi^2}{3} \right) \text{Li}_2(x) + U^2 L^2 + \frac{8}{3} U L^3 + \frac{4}{3} L^4 
+ \left( U^2 + \frac{10}{3} U L + 6 L^2 \right) \pi^2 + \left( 4 U + \frac{88}{3} L \right) \zeta_3 + \frac{29 \pi^4}{30} + \mathcal{O}(\epsilon) \right\} . \] (2.15)

It will be convenient to evaluate \( M^{(2)}_4 \) in a region where \( t > 0 \) and \( s, u < 0 \), and therefore we must analytically continue eq. (2.15) into this region. To do so, first we re-express the generalized polylogarithms \( \text{Li}_n(x) \) and \( S_{n,p}(x) \) appearing in eq. (2.15) as functions of \( y \equiv 1/x \) using identities (A.4) given in the appendix. Next, we analytically continue \( t \) from the negative to the positive real axis through the upper half plane (holding \( s \) fixed), which takes \( L \to -T + \pi i \) and \( U \to V + T - \pi i \), where \( T = \log(x) = -\log(y) = -\log(-s/t) \) and \( V = \log(1 - y) = \log(-u/t) \). After the continuation, we have \( 0 < y < 1 \), so that polylogarithms with argument \( y \) do not pick up additional contributions from the analytic continuation (since the branch cut for polylogarithms along the positive real axis starts to the right of unity). Finally, we write \( (-s)^{-2\epsilon} = t^{-2\epsilon} \exp(2\epsilon T) \) to obtain

\[ T^{(2)P}_4(s,t) = \frac{\mu^{4\epsilon} e^{-2\epsilon(4\pi)^{-D}}}{s^{2\epsilon} t^{1+2\epsilon}} \left\{ -\frac{4}{\epsilon^4} + \frac{1}{\epsilon^3} \left( -3 T - 5 \pi i \right) + \frac{1}{\epsilon^2} \left( \frac{9 \pi^2}{2} - 6 \pi i T \right) 
+ \frac{1}{\epsilon} \left( -4 \text{Li}_3(y) - 4 T \text{Li}_2(y) + \frac{2}{3} T^3 + 2 V T^2 + \frac{11 \pi^2}{2} T + \frac{65}{3} \zeta_3 \right) 
+ \frac{i \pi}{\epsilon} \left( 4 \text{Li}_2(y) - 2 T^2 - 4 V T + \frac{7 \pi^2}{2} \right) - 36 \text{Li}_4(y) - 4 S_{2,2}(y) 
- 4 T S_{1,2}(y) + \left( -28 T - 4 V \right) \text{Li}_3(y) + \left( -10 T^2 - 4 V T - \frac{14 \pi^2}{3} \right) \text{Li}_2(y) 
+ 2 V T^3 + V^2 T^2 + \left( \frac{14}{3} V T + \frac{7 \pi^2}{3} \right) \pi^2 + \left( 18 T + 4 V \right) \zeta_3 - \frac{113 \pi^4}{90} \right\} 
+ \frac{i \pi}{\epsilon} \left[ 20 \text{Li}_3(y) + 4 S_{1,2}(y) + \left( 4 V + 12 T \right) \text{Li}_2(y) 
+ \frac{4}{3} T^3 - 2 V T^2 - 2 V^2 T + \left( \frac{19}{3} T - \frac{2}{3} V \right) \pi^2 + \frac{76}{3} \zeta_3 \right] + \mathcal{O}(\epsilon) \right\} \] (2.16)

valid in the region \( t > 0 \) and \( u, s < 0 \). In the same region, the non-planar integral is given by \cite{22}.

\[ F_t = \left( \frac{\mu^2}{t} \right)^{2\epsilon} \left\{ -\frac{2}{\epsilon^4} + \frac{1}{\epsilon^3} \left( 2 T + \frac{7}{2} V - \frac{5 \pi i}{2} \right) 
+ \frac{1}{\epsilon^2} \left( 2 T^2 + T V - V^2 + 6 T + 6 V + \frac{31 \pi^2}{12} + i \pi [T + 4 V] \right) 
+ \frac{1}{\epsilon} \left( 2 S_{1,2}(y) - \frac{2}{3} T^3 - 2 T^2 V - 2 T V^2 - V^3 - 24 T - 24 V + \left( \frac{23}{6} T - \frac{41}{6} V \right) \pi^2 \right\} . \]
Inserting eqs. (2.16) and (2.17) into eq. (2.14), and using

\[ M_4^{(2)} = \frac{\lambda^2 s u}{(4\pi)^4} \left( \frac{\mu^2}{t} \right)^2 \left\{ - \frac{7}{\epsilon^3} (T + V) + \frac{1}{\epsilon^2} \left( 2 V^2 - 2 V T - 4 T^2 - 12 T - 12 V - 8i \pi [T + V] \right) \right. \]
\[ + \frac{1}{\epsilon} \left( -4 \mathrm{Li}_3 (y) - 4 S_{1,2}(y) - 4 T \mathrm{Li}_2 (y) + 2 T^3 + 6 V T^2 + 4 V^2 T + 2 V^3 \right) \]
\[ + \left( \frac{25 \pi^2}{2} + 48 \right) (T + V) + 4 \zeta_3 + i \pi \left[ 8 \mathrm{Li}_2 (y) - 10 T^2 - 8 V T + 6 V^2 - 24(T + V) - \frac{2 \pi^2}{3} \right] \]
\[ - 60 \mathrm{Li}_4 (y) + 120 S_{2,2}(y) - 52 S_{1,3}(y) + \left( -52 T + 32 V - 48 \right) \mathrm{Li}_3 (y) \]
\[ + \left( 84 T - 12 V - 48 \right) S_{1,2}(y) + \left( -22 T^2 + 32 V T - 48 T - \frac{92 \pi^2}{3} \right) \mathrm{Li}_2 (y) + \frac{11}{3} T^4 \]
\[ + \frac{32}{3} V T^3 - V^2 T^2 - 4 V^3 T - \frac{8}{3} V^4 + 8 T^3 + 24 V T^2 + 8 V^3 + \left( \frac{49}{3} T^2 + \frac{59}{3} V T - 12 V^2 \right) \pi^2 \]
\[ + (26 \pi^2 - 192) (T + V) + \left( \frac{332}{3} T + \frac{260}{3} V + 48 \right) \zeta_3 + \frac{283 \pi^4}{90} + i \pi \left[ -44 \mathrm{Li}_3 (y) - 52 S_{1,2}(y) \right] \]
\[ + \left( -20 V - 16 T + 96 \right) \mathrm{Li}_2 (y) - 6 T^3 - 6 V T^2 - 2 V^2 T - \frac{8}{3} V^3 - 24 T^2 - 48 V T + 24 V^2 \]
\[ + \left( 6 T + 10 V \right) \pi^2 + 96(T + V) + 52 \zeta_3 - 8 \pi^2 \right\} + \mathcal{O}(\epsilon) \right\} + \left( \text{all perms of } s, t, u \right). \]  

Note that the leading $1/\epsilon^4$ pole of the planar integral (2.15), which is present in the two-loop $\mathcal{N} = 4$ SYM amplitude, is cancelled in the two-loop $\mathcal{N} = 8$ supergravity amplitude by the
1/\epsilon^4 pole of the two-loop non-planar integral. Additional cancellations of poles will occur when we add the other permutations of s, t, and u.

First we consider the permutation that exchanges s and u, which can be obtained by simply letting \( y \to 1 - y \), \( T \to -V \), and \( V \to -T \) in the expression above (without any analytic continuation required — this is the reason we chose to evaluate the amplitude in the region \( t > 0 \) and \( s, u < 0 \)). Adding eq. (2.19) and the \( (s \leftrightarrow u) \) permutation, and using identities \( \text{(A.6)} \) relating polylogarithms with argument \( 1 - y \) to those with argument \( y \), we obtain

\[
M_4^{(2)} = \frac{\lambda^2 su}{(4\pi)^4} \left( \frac{\mu^2}{t} \right)^{2\epsilon} \left\{ -\frac{2}{\epsilon^2} (T + V)^2 - \frac{4\pi i}{\epsilon} (T + V)^2 - 8 \text{Li}_4(y) + 8 S_{1,3}(y) \\
+ (-4V - 12T) \text{Li}_3(y) - 4(T + V) S_{1,2}(y) + (-8T^2 - 4VT) \text{Li}_2(y) \\
+ T^4 + 4VT^3 + 2V^2T^2 + \frac{2}{3}V^3T + V^4 + \frac{13\pi^2}{3}(T + V)^2 + 4(T + V)\zeta_3 + \frac{4\pi^4}{15} \\
i\pi \left[ 8 \text{Li}_3(y) - 8 S_{1,2}(y) + (16T + 8V) \text{Li}_2(y) - \frac{10}{3}T^3 - 2V^2T - 10VT^2 + \frac{10}{3}V^3 \\
- \frac{2\pi^2}{3}(T + V) + 8\zeta_3 \right] + \mathcal{O}(\epsilon) \right\} + \left( \text{cyclic perms of } s, t, u \right). \tag{2.20}
\]

At this point, the 1/\epsilon^3 pole has also cancelled, leaving an expression whose leading divergence is 1/\epsilon^2. This is as expected for a two-loop gravity amplitude, as discussed in sec. 3.

Observe that one can define a degree of transcendentality for each term in an expression, with \( \log^k z \), \( \text{Li}_k(z) \), \( S_{n,k-n}(z) \), \( \zeta_k \), and \( \pi^k \) (since \( \zeta_{2m} \sim \pi^{2m} \)) all having degree \( k \), where \( z \) is any ratio of momentum invariants (e.g., \( x \) or \( y \)). The degree of transcendentality is preserved by all (generalized) polylogarithm identities, and therefore well-defined. Both the one-loop (2.7) and two-loop (2.20) results satisfy a simple rule: all terms proportional to \( (\lambda/\epsilon^2)^L \cdot \epsilon^k \) have degree of transcendentality \( k \), where \( L \) is the loop order. Note, however, that while the one- and two-loop planar integrals (2.4) and (2.15) also satisfy this rule, the two-loop nonplanar integral (2.17) does not, as it contains terms of subleading transcendentality. The terms of subleading transcendentality only cancel out when we add the \( u \leftrightarrow s \) permutation in eq. \( \text{(2.20))} \]. We used this cancellation of terms of subleading transcendentality as a useful check on our intermediate calculations. It remains an interesting question whether this “conservation law for transcendentality” persists to higher orders in the loop expansion.

Note also that the coefficients of the 1/\epsilon^2 and 1/\epsilon poles of the amplitude are considerably simpler than those of the original planar and non-planar integrals that contributed to it. Its form suggests that it may be related to the square of the one-loop amplitude (2.7), as we will now show.

Using \( s + t + u = 0 \), we may express the square of the one-loop amplitude (2.22) as

\[
\left[ M_4^{(1)} \right]^2 = \left( \frac{k}{2} \right)^4 su \left[ st T_4^{(1)}(s,t) - ut T_4^{(1)}(u,t) \right]^2 + \left( \text{cyclic perms of } s, t, u \right). \tag{2.21}
\]

Continuing the expression (2.4) to the region \( t > 0 \) and \( s, u < 0 \) as we did before for the

\[\text{footnote:We would like to thank Lance Dixon for pointing out to us that this fact may not be widely known.}\]
For $u < 0$ and $t > 0$, we may find $\mathcal{I}_4^{(1)}(u, t)$ by simply letting $y \to 1 - y$, $T \to -V$, and $V \to -T$ in eq. (2.22). Using the identities (A.6), we then obtain

\begin{equation}
\begin{split}
st \mathcal{I}_4^{(1)}(s, t) - ut \mathcal{I}_4^{(1)}(u, t) &= \frac{i\mu^2 e^{-\epsilon\gamma}(4\pi)^{-D/2}}{t^{\epsilon}} \left\{ \frac{2}{\epsilon} (T + V) + 2\pi i (T + V) \\
&+ \epsilon \left( 2 \text{Li}_3(y) + 2 S_{1,2}(y) + 2 T \text{Li}_2(y) - \frac{1}{3} T^3 - T^2 V - \frac{1}{3} V^3 - \frac{7\pi^2}{6} (T + V) \\
&- 2\zeta_3 + i\pi \left[ -4 \text{Li}_2(y) + T^2 + 2VT - V^2 + \frac{\pi^2}{3} \right] \right) + \mathcal{O}(\epsilon^2) \right\}. 
\end{split}
\end{equation}

Inserting this result in eq. (2.21), we find that the difference between the two-loop amplitude and half of the square of the one-loop amplitude is finite, expressible in the rather compact form

\begin{equation}
M_4^{(2)} - \frac{1}{2} [M_4^{(1)}]^2 = \left( \frac{\kappa}{8\pi} \right)^4 \left[ 8 S_{1,3}(y) + \frac{1}{3} \log^4 y + 8\zeta_4 + i\pi \left[ -8 S_{1,2}(y) + \frac{4}{3} \log^3 y + 8\zeta_3 \right] \\
+ (y \to 1 - y) \right] + \left( \text{cyclic perms of } s, t, u \right). \tag{2.24}
\end{equation}

This relatively simple expression suggests that a large portion of the rather complicated finite piece of the two-loop amplitude (2.20) comes from the square of the one-loop amplitude, in particular involving nontrivially both the finite term and the $\mathcal{O}(\epsilon)$ term. The difference (2.24) may be rewritten as

\begin{equation}
M_4^{(2)} - \frac{1}{2} [M_4^{(1)}]^2 = \left( \frac{\kappa}{8\pi} \right)^4 \left\{ su \left[ h(t, s, u) + h(t, u, s) \right] + tu \left[ h(s, t, u) + h(s, u, t) \right] \\
+ st \left[ h(u, s, t) + h(u, t, s) \right] \right\}. \tag{2.25}
\end{equation}

where $h(t, s, u)$ is given in the region $t > 0$ and $s, u < 0$ by the expression

\begin{equation}
h(t, s, u) = 8 S_{1,3}(-s/t) + \frac{1}{3} \log^4(-s/t) + 8\zeta_4 + i\pi \left[ -8 S_{1,2}(-s/t) + \frac{4}{3} \log^3(-s/t) + 8\zeta_3 \right]. \tag{2.26}
\end{equation}

To obtain an explicit expression for (2.25) we must analytically continue eq. (2.26) into several other regions. To obtain $h(s, t, u)$, we must first analytically continue $h(t, s, u)$ to
the region where \( s > 0 \) and \( t, u < 0 \) (the explicit expression is in the appendix), and then exchange \( s \) and \( t \) in the resulting expression. To obtain \( h(s, u, t) \), we must first analytically continue \( h(t, s, u) \) to the region where \( u > 0 \) and \( s, t < 0 \) (also in the appendix), and then permute \( s \to u \to t \to s \) in the result. Using several additional identities, we may combine these pieces to obtain our final result

\[
M_4^{(2)} - \frac{1}{2} \left[ M_4^{(1)} \right]^2
= \left( \frac{\kappa}{8\pi} \right)^4 \left\{ su \left( 8 S_{1,3}(y) + \frac{1}{3} \log^4 y + 8 \zeta_4 + i\pi \left[ -8 S_{1,2}(y) + \frac{4}{3} \log^3 y + 8 \zeta_3 \right] \right) + tu \left( 8 S_{2,2}(y) - 8 S_{1,3}(y) - 8 \log y S_{1,2}(y) - 4\pi^2 \text{Li}_2(y) + \frac{1}{3} \log^4(1-y) \right. \\
- \frac{4}{3} \log y \log^3(1-y) + 2 \log^2 y \log^2(1-y) + 2\pi^2 \log^2(1-y) - 4\pi^2 \log y \log(1-y) \\
\left. + i\pi \left[ 8 S_{1,2}(1-y) + \frac{8\pi^2}{3} \log(1-y) - 8 \zeta_3 \right] \right\} + (s \leftrightarrow u, y \to 1-y) \tag{2.27}
\]

valid in the physical region \( t > 0 \) and \( s, u < 0 \). If we wish to obtain the result in the region \( s > 0 \) and \( t, u < 0 \), we simply exchange \( s \leftrightarrow t \) which means that \( y = -s/t \) is replaced by \( x = -t/s \) throughout the expression above.

The function (2.26) can be written rather elegantly using eq. (A.3) as

\[
h(t, s, u) = 16 \zeta_4 \\
+ \int_1^y \frac{dy}{y} \left\{ \frac{4}{3} \left( \log^3(e^{i\pi}y) - \log^3(e^{i\pi}(1-y)) \right) + 4\pi^2 \left( \log(e^{i\pi}y) - \log(e^{i\pi}(1-y)) \right) \right\} \tag{2.28}
\]

The expression (2.27) is not manifestly permutation symmetric in \( s, t, \) and \( u \) since in the physical region in which we are working, \( t > 0 \) whereas \( u, s < 0 \). However, if we analytically continue this expression to the Euclidean domain \( s, t, u < 0 \), then eq. (2.27) becomes

\[
M_4^{(2)} - \frac{1}{2} \left( M_4^{(1)} \right)^2 = \left( \frac{\kappa}{8\pi} \right)^4 \left\{ s \zeta_4 + \int_1^{\sqrt{s}} \frac{d\bar{y}}{\bar{y}} \left[ \log \left( \frac{s}{\bar{y}} \right) \right] \left( \frac{\log^3(\frac{\bar{y}}{\sqrt{s}}) - \log^3(\frac{s}{\bar{y}})}{3} + \pi^2 \log \frac{s}{\bar{y}} \right) + (u \leftrightarrow s) \right\} + \text{(cyclic perms of } s, t, u) \tag{2.29}
\]

This expression is now explicitly symmetric in \( s, t, u \) (since \( s, t, u < 0 \) there is nothing to break the symmetry). In order to go back to the polylogarithm form we must choose a Euclidean region constraint (thus breaking the symmetry). Choosing \(-|t| = u + s\), we obtain

\[
M_4^{(2)} - \frac{1}{2} \left( M_4^{(1)} \right)^2 = \left( \frac{\kappa}{8\pi} \right)^4 \left\{ u s \left( 8 S_{1,3}(\frac{u - s}{|t|}) + \zeta_4 + \frac{\log^4(\frac{u - s}{|t|})}{3} + 4\pi^2 \left( \text{Li}_2(\frac{u - s}{|t|}) - \zeta_2 + \frac{\log^2(\frac{u - s}{|t|})}{2} \right) \right) \right\} \tag{2.30}
\]

\[\text{Note that the analytical continuation between various regions can be taken as } -s = se^{-i\pi}, \text{ and similarly for } t \text{ and } u, \text{ i.e. } -s = |s|e^{-i\pi\theta(s)}, \ s = |s|e^{i\pi\theta(-s)} \text{ and similarly for } t \text{ and } u. \text{ We have checked that this gives the correct continuation of the planar and nonplanar integrals in refs. [22].}\]
\[ +tu \left[ 8 \left( S_{1,3}(-\frac{u}{s}) + 4\zeta_4 - S_{1,3}(-1) - \text{Li}_2(-\frac{u}{s}) + \log(-\frac{u}{s}) \text{Li}_2(-\frac{u}{s}) - \frac{1}{2} \log^2(-\frac{u}{s}) \text{Li}_2(-\frac{u}{s}) \right) \right. \\
+ \frac{1}{3} \left( \log^4\left(\frac{|t|}{s^2}\right) + \log^4\left(\frac{u}{s}\right) - 4 \log^3\left(\frac{u}{s}\right) \log\left(\frac{|t|}{s}\right) \right) + 2\pi^2 \log^2\left(\frac{|t|}{s}\right) + \frac{\pi^4}{3} \right] \} + (u \leftrightarrow s). \]

3 Infrared behavior and generalizations

In the previous section, we calculated the two-loop four-point function in \( \mathcal{N} = 8 \) supergravity, and noted the particularly simple structure of its infrared divergences in eq. (2.20). In this section, we will derive the form of the leading-power divergence more heuristically, in a way that can be generalized to higher-\( n \)-point functions. Before discussing \( \mathcal{N} = 8 \) supergravity, we briefly review IR divergences for \( \mathcal{N} = 4 \) SYM theories [2] (for a review, see ref. [24]).

When we dimensionally regularize a theory in \( D = 4 - 2\epsilon \) dimensions, both UV and IR divergences appear as poles in \( \epsilon \). In a UV finite theory, such as \( \mathcal{N} = 4 \) SYM, the poles in \( \epsilon \) are solely due to IR divergences. In gluon-gluon scattering in \( \mathcal{N} = 4 \) SYM, IR divergences arise both from soft gluons and from collinear gluons (which can exchange a virtual gluon with soft transverse momentum), each of which gives rise to a \( 1/\epsilon \) pole at 1-loop, leading to a \( 1/\epsilon^2 \) pole at that order. At \( L \) loops, the leading IR divergence is therefore \( \mathcal{O}(1/\epsilon^{2L}) \), arising from multiple soft gluon exchanges. In the large-\( N \) (planar) limit, these IR divergences can be characterized by the Sudakov factor \( A_{\text{div,SYM}}(s) \), with one such factor for each pair of adjacent (external) gluons in the \( n \)-gluon amplitude,

\[
\prod_{i=1}^{n} A_{\text{div,SYM}}(s_{i,i+1})
\]

where \( s_{i,i+1} = (k_i + k_{i+1})^2 \). For \( n = 4 \), this becomes \( A_{\text{div,SYM}}^2(s)A_{\text{div,SYM}}^2(t) \), where \( s = s_{1,2} = s_{3,4} \) and \( t = s_{2,3} = s_{4,1} \). The Sudakov factor in the one-loop approximation is

\[
A_{\text{div,SYM}}(s) = \exp \left[ -\frac{\lambda_{\text{SYM}}}{4\pi\epsilon^2} \left( \frac{\mu^2}{-s} \right)^\epsilon + \mathcal{O}(\lambda_{\text{SYM}}^2) \right]
\]

where the SYM coupling \( \lambda_{\text{SYM}} \) is the dimensionless 't Hooft coupling \( g^2 N \). The exponential Sudakov factor is modified at higher-loop order, but can be completely characterized by two functions of \( \lambda_{\text{SYM}} \): the cusp anomalous dimension \( f(\lambda_{\text{SYM}}) \) and the collinear anomalous dimension \( g(\lambda_{\text{SYM}}) \).

Now consider \( \mathcal{N} = 8 \) supergravity, which is UV finite at least to third (maybe eighth) order in perturbation theory, and possibly to all orders [17, 18, 16, 15]. Therefore, to at least third (maybe eighth) order, the poles in the \( \mathcal{N} = 8 \) scattering amplitudes are due only to IR divergences. It has been known for some time that gravity theories have infrared divergences due to soft gravitons, but that collinear divergences are absent [25, 26]. Hence at \( L \) loops, IR divergences are expected to give rise to a leading \( 1/\epsilon^L \) divergence. This is borne out at two loops by the calculations of the last section, where the \( 1/\epsilon^4 \) and \( 1/\epsilon^3 \) poles cancel out of the final result (2.20). In ref. [27], Dunbar and Norridge showed that the one-loop amplitude has a \( 1/\epsilon \) divergence.

A deep relationship exists between the perturbative amplitudes of \( \mathcal{N} = 8 \) supergravity and \( \mathcal{N} = 4 \) SYM theory, going back to the work of Kawai, Lewellen, and Tye [20].
expressed in terms of the same scalar integral $\mathcal{I}^{(1)}_4(s,t)$, and the IR divergences are described by the same product of Sudakov factors at one loop, with two differences. The first difference is that in gravity theories, there is no large-$N$ limit, so we must consider planar and non-planar graphs on the same footing. As a result, there is a factor of $\mathcal{A}_{\text{div}}(s)$ for every pair of external gravitons, not just adjacent gravitons
\[
\prod_{i<j} \mathcal{A}_{\text{div}}(s_{i,j}).
\] (3.3)

For the four-point function, this becomes $\mathcal{A}_{\text{div}}^2(s)\mathcal{A}_{\text{div}}^2(t)\mathcal{A}_{\text{div}}^2(u)$ where $s = s_{1,2} = s_{3,4}$, $t = s_{2,3} = s_{1,4}$, and $u = s_{1,3} = s_{2,4}$. The second difference is that the supergravity coupling
\[
\lambda = \left(\frac{\kappa}{2}\right)^2 \left(4\pi e^{-\gamma}\right)^\epsilon
\] (3.4)
is dimensionful, so the factor of $\lambda_{\text{SYM}}$ in the Sudakov factor (3.2) must be replaced by the dimensionless effective coupling $\lambda \cdot s$. Hence, the IR divergent part of the four-graviton amplitude at one loop is expected to be
\[
\mathcal{A}_{\text{div}}^2(s)\mathcal{A}_{\text{div}}^2(t)\mathcal{A}_{\text{div}}^2(u)
\]
\[
= \exp\left\{ -\frac{2\lambda s}{(4\pi\epsilon)^2} \left(\frac{\mu^2}{s}\right)^\epsilon - \frac{2\lambda t}{(4\pi\epsilon)^2} \left(\frac{\mu^2}{t}\right)^\epsilon - \frac{2\lambda u}{(4\pi\epsilon)^2} \left(\frac{\mu^2}{u}\right)^\epsilon \right\}_{\text{divergent}} + O(\lambda^2) \}
\]
\[
= \exp\left\{ \frac{\lambda}{8\pi^2\epsilon} \left[ s \log \left(\frac{-s}{\mu^2}\right) + t \log \left(\frac{-t}{\mu^2}\right) + u \log \left(\frac{-u}{\mu^2}\right) \right] + O(\lambda^2) \}
\] (3.5)
where the $1/\epsilon^2$ term vanishes because it is multiplied by $s + t + u = 0$. Thus our heuristic argument reproduces the IR divergence of the one-loop amplitude (2.9). The one-loop IR divergence (3.5) was obtained over a decade ago by Dunbar and Norridge [27].

Analogously, the IR divergent part of the $n$-graviton amplitude should depend on the product of all distinct factors of $\mathcal{A}_{\text{div}}(s_{i,j})$, since again planar and nonplanar graphs are on equal footing and since the same divergent function as for SYM appears in the scalar diagrams (due to the KLT relations). Therefore at one-loop, the IR divergent factor for the $n$-graviton amplitude is
\[
\prod_{i<j} \mathcal{A}_{\text{div}}(s_{i,j}) = \exp\left\{ -\frac{\lambda}{(4\pi\epsilon)^2} \sum_{i<j} s_{i,j} \left(\frac{\mu^2}{-s_{i,j}}\right)^\epsilon \right\}_{\text{divergent}} + O(\lambda^2) \}
\]
\[
= \exp\left\{ -\frac{\lambda}{8\pi^2\epsilon} \sum_{i<j} k_i \cdot k_j + \frac{\lambda}{16\pi^2\epsilon} \sum_{i<j} s_{i,j} \log \left(\frac{-s_{i,j}}{\mu^2}\right) + O(\lambda^2) \}
\]
\[
= \exp\left\{ \frac{\lambda}{16\pi^2\epsilon} \sum_{i<j} s_{i,j} \log \left(\frac{-s_{i,j}}{\mu^2}\right) + O(\lambda^2) \}
\] (3.6)
where $s_{i,j} = (k_i + k_j)^2 = 2k_i \cdot k_j$ because external states are massless, and the coefficient of $1/\epsilon^2$, namely $\sum_{i<j} k_i \cdot k_j$, vanishes for massless gravitons due to momentum conservation.
\[ \sum_{i=1}^{n} k_i = 0. \] The IR divergence of the one-loop \( n \)-graviton amplitude was also obtained in ref. [27].

The exponent of the SYM Sudakov factor (3.2) gets a correction at \( O(\lambda_{SYM}^2) \) due to the cusp anomalous dimension \( f(\lambda_{SYM}) \). In principle, the analogous factors \( A_{\text{div}}(s) \) in eqs. (3.5) and (3.6) could get an \( O(\lambda^2) \) IR divergent correction, but the two-loop calculation of the previous section revealed the absence of such a correction. Because there is no analog of the function \( f(\lambda_{SYM}) \) for supergravity, \( A_{\text{div}}(s) \) differs from \( A_{\text{div,SYM}}(s) \) at higher orders.

The calculation of the previous section showed that the leading \( 1/\epsilon^2 \) pole of the two-loop four-point amplitude is indeed correctly given by eq. (3.5), with no \( O(\lambda_{SYM}^2) \) modification. One could reasonably conjecture that the leading \( 1/\epsilon^2 \) divergence for the \( L \)-loop \( n \)-point amplitude is also given by eq. (3.5), namely
\[
\frac{1}{L!} \left( \frac{\lambda}{8\pi^2\epsilon} \right)^L \left[ s \log \left( \frac{-s}{\mu^2} \right) + t \log \left( \frac{-t}{\mu^2} \right) + u \log \left( \frac{-u}{\mu^2} \right) \right]^L + O(1/\epsilon^{L-1}) \tag{3.7}
\]
and similarly that the leading divergence for the \( L \)-loop \( n \)-point function is given by
\[
\frac{1}{L!} \left( \frac{\lambda}{16\pi^2\epsilon} \right)^L \left[ \sum_{i<j} s_{i,j} \log \left( \frac{-s_{i,j}}{\mu^2} \right) \right]^L + O(1/\epsilon^{L-1}) \tag{3.8}
\]
These are consistent with general expectations for the order of the leading divergence, but we have not attempted to verify them beyond two loops.

We found in fact a stronger result for the four-point function at two loops; namely, that both the leading \( 1/\epsilon^2 \) and the subleading \( 1/\epsilon \) IR divergence are given by the exponential of the one-loop amplitude (2.7)
\[
A_{\text{div}}^2(s) A_{\text{div}}^2(t) A_{\text{div}}^2(u) = \exp \left[ M^{(1)}_4(\epsilon) + O(\lambda^3) \right] \bigg|_{\text{divergent}}. \tag{3.9}
\]
This implies that the total two-loop divergence involves the finite as well as the divergent part of the exponent. Equation (3.9) differs from \( N = 4 \) SYM theory, in which the two-loop divergences are given by
\[
\exp \left[ a M^{(1)}_4(\epsilon) - a^2 (\zeta_2 + \epsilon \zeta_3) M^{(1)}_4(2\epsilon) + O(a^3) \right] \bigg|_{\text{divergent}}, \quad a = \left( \frac{\lambda_{SYM}}{8\pi^2} \right) \left( 4\pi e^{-\gamma} \right)^\epsilon \tag{3.10}
\]
where the second term, which contributes to the \( 1/\epsilon^2 \) and \( 1/\epsilon \) divergences, comes from the \( O(\lambda_{SYM}^2) \) coefficients of the anomalous dimensions \( f(\lambda_{SYM}) \) and \( g(\lambda_{SYM}) \). An argument for the absence of an \( O(\lambda^2) \) correction to the exponent in eq. (3.9) would go as follows: due to the dimensionality of the coupling \( \lambda \), the second term would have to be multiplied not by \( \lambda \), but by some function of \( \lambda s, \lambda t, \) and \( \lambda u \), but the only symmetric term at first order, \( \lambda (s + t + u) \), vanishes. At three loops, of course, a nonvanishing term \( \lambda^2 (s^2 + t^2 + u^2) \)

\[\text{As we can see from ref. [27], there is no fundamental difference between } n = 4 \text{ and } n > 4 \text{ amplitudes as far as IR divergences are concerned, so we can extend the } n > 4 \text{ results to the same loop order as the } n = 4 \text{ result.}\]
could in principle come in. The only allowed possibility that would mimic the $\mathcal{N} = 4$ SYM result, an $f(\lambda^2(s^2 + t^2 + u^2)) M^{(1)}$ term, implies a factorization of the momentum dependence which seems unlikely. One cannot exclude, however, the possibility that higher-order corrections do not organize into a single function, but give, e.g., an infinite series $\sum_{n \geq 2} c_n \lambda^n (s^n + t^n + u^n) M^{(1)}$. After all, $\mathcal{N} = 8$ supergravity is potentially nonrenormalizable, being a field theory of quantum gravity.

On the more optimistic side, it remains possible that the simple behavior in eq. (3.9) continues at higher loops (at least up to the order to which the theory is UV finite), and that the IR divergences (both leading and subleading) of the four-point function are exactly given by

$$A^2_{\text{div}}(s) A^2_{\text{div}}(t) A^2_{\text{div}}(u) = \exp \left[ M^{(1)}_4(\epsilon) \right] \bigg|_{\text{divergent}}$$

(3.11)

to all orders in the coupling $\lambda$. An even more daring conjecture is that the complete IR divergences of the $n$-point amplitudes are given by the exponential of the 1-loop amplitude

$$\prod_{i < j} A_{\text{div}}(s_{i,j}) = \exp \left[ M^{(1)}_n(\epsilon) \right] \bigg|_{\text{divergent}}$$

(3.12)

to all orders in the coupling $\lambda$. In principle, the expressions for the IR-divergent contributions (3.11) and (3.12) could be modified by functions $E_n(\epsilon)$ that vanish as $\epsilon \to 0$, as in the case of $\mathcal{N} = 4$ SYM theory [2]. On the other hand, such functions $E_n(\epsilon)$ could be absorbed into the $\epsilon$-expansions of $f^{(l)}(\epsilon)$ [2] and thus related to the anomalous dimensions. As we pointed out in the previous paragraph, such “anomalous dimension-like” terms may well be absent in supergravity.

In summary, we have found similarities between the IR divergences of $\mathcal{N} = 8$ supergravity and those of planar $\mathcal{N} = 4$ SYM, as well as significant differences, due to the absence of collinear divergences, and due to the presence of a dimensionful coupling constant for supergravity.

## 4 Conclusions

In this paper, the one- and two-loop graviton four-point amplitudes in $\mathcal{N} = 8$ supergravity were explicitly computed. A number of regularities appeared, most importantly an ABDK-like relation (2.27) between the one- and two-loop amplitudes.

Specifically, we found that the IR divergent part of the two-loop amplitude is the divergent part of one-half the square of the full one-loop amplitude, suggesting an exponentiation of the IR divergences. We gave a heuristic argument for the IR divergences of graviton scattering amplitudes which allows a generalization to one-loop $n$-point amplitudes, and a conjectured generalization to $L$-loop $n$-point amplitudes.

Moreover, most of the finite part of the two-loop amplitude also comes from the square of the full one-loop amplitude (i.e., including the order $\epsilon$ part), with a very simple remainder. This is reminiscent of $\mathcal{N} = 4$ SYM, where the presence of the dual conformal symmetry...
of the dual Wilson loop restricts the form of the \( n = 4 \) and \( n = 5 \) amplitudes \([7, 8]\) to the BDS exponential form (essentially the exponential of the one-loop amplitude, together with the extra information contained in the functions \( f(\lambda_{\text{SYM}}) \) and \( g(\lambda_{\text{SYM}}) \)). But for the \( n = 6 \) amplitude, dual conformal symmetry does not fix the result, and it was found that at two-loops, besides the BDS exponential form, there is a small remainder function \([10, 11, 12]\) (small means, \( e.g. \), that it does not affect Regge behavior \([28]\) and it arises as a correction \([30]\)). One could expect that something similar is at work here (for supergravity there is no dual conformal symmetry to fix the amplitude): since there are no analogs of \( f(\lambda_{\text{SYM}}) \) and \( g(\lambda_{\text{SYM}}) \) due to the dimensionality of the coupling, the amplitude is given by the exponential of the one-loop amplitude, with a simple finite remainder (at least to the order to which \( \mathcal{N} = 8 \) supergravity is finite).

The discussion in this paper does not assume that \( \mathcal{N} = 8 \) supergravity is or is not perturbatively UV finite. If UV finiteness breaks down at \( L \)-loops, then our conjectures could nonetheless be valid up to that loop level.

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**Appendix**

The generalized polylogarithms of Nielsen are defined by \([31]\)

\[
S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 dt \log^{n-1}(t) \log^p(1-xt) \frac{1}{t}, \quad n, p \geq 1, \quad x \leq 1
\]

(A.1)

which in the case of \( p = 1 \) reduce to the usual polylogarithms

\[
S_{n-1,1}(x) \equiv \text{Li}_n(x)
\]

(A.2)

For \( n = 1 \), eq. (A.1) may be rewritten as

\[
S_{1,p}(x) = \int_0^x \frac{dz}{z} \frac{(-\log(1-z))^p}{p!}
\]

(A.3)

The following identities for generalized polylogarithms are valid for \( x < 0 \), with \( y = 1/x \) and \( L = -\log(-x) = \log(-y) \):

\[
\text{Li}_2(x) = -\text{Li}_2(y) - \frac{1}{2} L^2 - \frac{\pi^2}{6}
\]

(A.4)

\(* But see ref. \([29]\).*
\[ \text{Li}_3(x) = \text{Li}_3(y) + \frac{1}{6}L^3 + \frac{\pi^2}{6}L \]

\[ S_{1,2}(x) = -S_{1,2}(y) + \text{Li}_3(y) - L \text{Li}_2(y) - \frac{1}{6}L^3 + \zeta_3 \]  
(A.4)

\[ \text{Li}_4(x) = -\text{Li}_4(y) - \frac{1}{24}L^4 - \frac{\pi^2}{12}L^2 - \frac{7\pi^4}{360} \]

\[ S_{2,2}(x) = S_{2,2}(y) - 2\text{Li}_4(y) + L \text{Li}_3(y) + \frac{1}{24}L^4 - \zeta_3L - \frac{7\pi^4}{360} \]

\[ S_{1,3}(x) = -S_{1,3}(y) + S_{2,2}(y) - \text{Li}_4(y) - L S_{1,2}(y) + L \text{Li}_3(y) - \frac{1}{2}L^2 \text{Li}_2(y) - \frac{1}{24}L^4 - \frac{\pi^4}{90} \]

and are used in sec. 2 to convert eq. (2.15) to (2.16) and eq. (2.4) to (2.22). The polylogarithms also obey the following identity when \( 0 < y < 1 \)

\[ S_{n,p}(1-y) = \sum_{s=0}^{n-1} \frac{\log^s(1-y)}{s!} \left[ S_{n-s,p}(1) - \sum_{r=0}^{p-1} \frac{(-\log y)^r}{r!} S_{p-r,n-s}(y) \right] + \frac{(-1)^p}{n! p!} \log^n(1-y) \log^p y \]  
(A.5)

which becomes, where \( T = -\log y \) and \( V = \log(1-y) \),

\[ \text{Li}_2(1-y) = -\text{Li}_2(y) + VT + \frac{\pi^2}{6} \]

\[ \text{Li}_3(1-y) = -S_{1,2}(y) - V \text{Li}_2(y) + \frac{1}{2}TV^2 + \frac{\pi^2}{6}V + \zeta_3 \]

\[ S_{1,2}(1-y) = -\text{Li}_3(y) - T \text{Li}_2(y) + \frac{1}{2}VT^2 + \zeta_3 \]  
(A.6)

\[ \text{Li}_4(1-y) = -S_{1,3}(y) - V S_{1,2}(y) - \frac{1}{2}V^2 \text{Li}_2(y) + \frac{1}{6}TV^3 + \frac{\pi^2}{12}V^2 + \zeta_3V + \frac{\pi^4}{90} \]

\[ S_{2,2}(1-y) = -S_{2,2}(y) - T S_{1,2}(y) - V \text{Li}_3(y) - VT \text{Li}_2(y) + \frac{1}{4}V^2T^2 + \zeta_3V + \frac{\pi^4}{360} \]

\[ S_{1,3}(1-y) = -\text{Li}_4(y) - T \text{Li}_3(y) - \frac{1}{2}T^2 \text{Li}_2(y) + \frac{1}{6}VT^3 + \frac{\pi^4}{90} \]

which are used in sec. 2 to obtain eqs. (2.20) and (2.23).

In eq. (2.23), we obtained the expression

\[ h(t, s, u) = 8S_{1,3}(y) + \frac{1}{3} \log^4 y + \frac{4\pi^4}{45} + i\pi \left[ -8S_{1,2}(y) + \frac{4}{3} \log^3 y + 8\zeta_3 \right], \quad y = -\frac{s}{t} \]  
(A.7)

valid in the region \( t > 0 \) and \( s, u < 0 \), for the function that appears in eq. (2.25), the difference between the two-loop amplitude and one-half the square of the one-loop amplitude. To obtain the full result for the difference, we need to analytically continue \( h(t, s, u) \) to other regions.

To analytically continue \( h(t, s, u) \) to the region \( s > 0 \) and \( t, u < 0 \), we let \( s \) and \( t \) traverse the upper half plane (in opposite directions), which causes \( y = -s/t \) to go from a point between 0 and 1 on the real axis clockwise through an angle \( 2\pi \) around the origin, ending up at a point to the right of 1 on the real axis. Hence, \( \log y \to \log y - 2\pi i \) and
$S_{n,p}(y) \to S_{n,p}(y+i0)$. Next, we use eqs. (A.12) and (A.13) from ref. [32] to re-express this as

$$h(t, s, u) = -8 \text{Li}_4(x) + 8 S_{2,2}(x) - 8 S_{1,3}(x) + 8 \log x \left[ \text{Li}_3(x) - S_{1,2}(x) \right] - 2\pi^2 \log^2 x$$

$$-4 \left( \log^2 x + \pi^2 \right) \text{Li}_2(x) - \frac{13\pi^4}{3} + i\pi \left[ \frac{4}{3} \log^3 x + \frac{8\pi^2}{3} \log x \right], \quad x = -\frac{t}{s} \quad (A.8)$$

valid for $s > 0$ and $t, u < 0$ (that is, for $0 < x < 1$). Finally, we simply let $x \to y$ to obtain

$$h(s, t, u) = -8 \text{Li}_4(y) + 8 S_{2,2}(y) - 8 S_{1,3}(y) + 8 \log y \left[ \text{Li}_3(y) - S_{1,2}(y) \right] - 2\pi^2 \log^2 y$$

$$-4 \left( \log^2 y + \pi^2 \right) \text{Li}_2(y) - \frac{13\pi^4}{3} + i\pi \left[ \frac{4}{3} \log^3 y + \frac{8\pi^2}{3} \log y \right], \quad y = -\frac{s}{t} \quad (A.9)$$

valid for the region $t > 0$ and $s, u < 0$.

To analytically continue $h(t, s, u)$ to the region $u > 0$ and $s, t < 0$, we let $u$ and $t$ traverse the upper half plane (in opposite directions), which causes $y = -s/t$ to go from a point between 0 and 1 on the real axis clockwise through an angle $2\pi$ around the point $y = 1$, ending up at a point on the negative real axis. As a result

$$\log y \to \log(-y) - i\pi$$

$$S_{1,2}(y) \to S_{1,2}(y) - 2\pi^2 \log(-y) + i\pi \left[ 2 \text{Li}_2(y) + \frac{5\pi^2}{3} \right]$$

$$S_{1,3}(y) \to S_{1,3}(y) - 2\pi^2 \text{Li}_2(y) - \pi^4 + i\pi \left[ 2 S_{1,2}(y) - \frac{4\pi^2}{3} \log(-y) - 2\zeta_3 \right]. \quad (A.10)$$

Inserting these into eq. (A.7), we obtain

$$h(t, s, u) = 8 S_{1,3}(y) + \frac{1}{3} \log^4(-y) + 2\pi^2 \log^2(-y) + \frac{199\pi^4}{45} + i\pi \left[ 8 S_{1,2}(y) + \frac{8}{3} \pi^2 \log(-y) - 8\zeta_3 \right] \quad (A.11)$$

valid for $u > 0$ and $s, t < 0$ (that is, for $y < 0$). Then, to obtain $h(s, u, t)$ for $t > 0$ and $s, u < 0$, we permute $s \to u \to t \to s$, which takes $y \to (y-1)/y$. Polylogarithms with argument $(y-1)/y$ can be expressed as polylogarithms with argument $y/(y-1)$ by using eqs. (A.4), and the latter can be expressed as polylogarithms with argument $y$ by using eqs. (A.15) through (A.20) of ref. [32], resulting in

$$h(s, u, t) = 8 \text{Li}_4(y) - 8 \log y \text{Li}_3(y) + 4 \log^2 y \text{Li}_2(y) + \frac{1}{3} \log^4(1-y) - \frac{4}{3} \log y \log^3(1-y)$$

$$+ 2 \log^2 y \log^2(1-y) + 2\pi^2 \left[ \log(1-y) - \log y \right]^2 + \frac{13\pi^4}{3} + i\pi \left[ -8 \text{Li}_3(y) \right]$$

$$+ 8 \log y \text{Li}_2(y) + 4 \log^2 y \log(1-y) - \frac{4}{3} \log^3 y + \frac{8\pi^2}{3} \log(1-y) - \frac{8\pi^2}{3} \log y \right] \quad (A.12)$$

valid for $t > 0$ and $s, u < 0$. Finally, we add eqs. (A.9) and (A.12) to obtain the coefficient of $tu$ in eq. (2.27).
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