ON THE ALGEBRAIC STRUCTURES IN $A_\Phi(G)$

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Abstract. Let $G$ be a locally compact group and $(\Phi, \Psi)$ be a complementary pair of $N$-functions. In this paper, using the powerful tool of porosity, it is proved that when $G$ is an amenable group, then the Figà-Talamanca-Herz-Orlicz algebra $A_\Phi(G)$ is a Banach algebra under convolution product if and only if $G$ is compact. Then it is shown that $A_\Phi(G)$ is a Segal algebra, and as a consequence, the amenability of $A_\Phi(G)$ and the existence of a bounded approximate identity for $A_\Phi(G)$ under the convolution product is discussed. Furthermore, it is shown that for a compact abelian group $G$, the character space of $A_\Phi(G)$ under convolution product can be identified with $\hat{G}$, the dual of $G$.

1. Introduction

The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced and studied by Eymard in [4]. Then Herz in [7, 8] generalized Fourier algebras into Figà-Talamanca-Herz algebras, denoted by $A_p(G)$ for $1 \leq p \leq \infty$. Indeed, it was shown that $A_2(G) = A(G)$. The space $A_p(G)$ is determined by a homomorphic image of the projective tensor product of the classical Lebesgue spaces $L^p(G)$ and $L^q(G)$, where $q$ is the conjugate exponent to $p$. Fourier and Figà-Talamanca-Herz algebras are well-known concepts in harmonic analysis, have been investigated thoroughly and there is a rich literature on them. For further background, we refer the reader to the valuable books by Pier [9] and Derighetti [3].

Orlicz spaces are a type of function spaces generalizing the Lebesgue spaces. A great deal of linear aspects of Orlicz spaces have been completely characterized in the last few decades. Rao and Ren’s books [13, 14] contain comprehensive information on Orlicz spaces and so in this paper, we use the definitions and basic facts from these two books.

In [1], the first and second authors introduced the Figà-Talamanca-Herz-Orlicz algebra $A_\Phi(G)$ for an $N$-function $\Phi$ and a locally compact group $G$. This is a natural generalization of Figà-Talamanca-Herz algebras $A_p(G)$ to Orlicz spaces and it is shown that $A_\Phi(G)$ is a Banach algebra under pointwise product of functions.

In this article, exploring this space more from the algebraic structure point of view, using the concept of porosity, we address the following question: when is $A_\Phi(G)$ a Banach algebra under the convolution product? Indeed, in Section 3, assuming $G$ is amenable, we show

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that $A_{\Phi}(G)$ is closed under the convolution product if and only if $G$ is compact. As a consequence, we show that $A_{\Phi}(G)$ is a Segal algebra. The latter statement, in turn, helps us to characterize amenability of $A_{\Phi}(G)$ and the existence of a bounded approximate identity in $A_{\Phi}(G)$. Lastly, we characterize the character space of $A_{\Phi}(G)$ under the convolution product when $G$ is compact and abelian.

2. Preliminaries

In this paper, $G$ denotes a locally compact group with a fixed left Haar measure $\lambda$. As usual, by $L^0(G)$ we show the set of all $\lambda$-measurable complex-valued functions on $G$ when $\lambda$-almost everywhere equal functions are identified. The convolution product of $f$ and $g$ in $L^0(G)$ is defined on $G$ by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,d\lambda(y),$$

whenever the function $y \mapsto f(y)g(y^{-1}x)$ is Haar integrable for $\lambda$-almost all $x \in G$.

A locally compact group $G$ is called amenable if it admits a left invariant mean \[9, 17\]. It turns out that the amenability of $G$ can be characterized by Leptin’s condition, that is, for any $\varepsilon > 0$ and compact subset $K \subseteq G$, there exists a compact subset $U \subseteq G$ such that $0 < \lambda(U) < \infty$ and $\lambda(KU) < (1 + \varepsilon)\lambda(U)$ (see \[9, Theorem 7.9, Proposition 7.11\]). For an overview concerning amenability of a locally compact group see \[9, 17\].

A convex even function $\Phi : \mathbb{R} \to [0, \infty)$ is called an $N$-function if it satisfies $\Phi(0) = 0$, $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$ and $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$. The convexity of $\Phi$ and $\Phi(0) = 0$ implies that $\Phi$ is strictly increasing and so invertible. Indeed, $N$-functions are Young functions with a reasonable behavior \[13, 14\].

For each $N$-function one can associate another $N$-function $\Psi : \mathbb{R} \to [0, \infty)$, termed the complementary $N$-function to $\Phi$, defined by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}.$$ 

The pair $(\Phi, \Psi)$ is then called the complementary pair of $N$-functions and satisfies a useful inequality called Young’s inequality:

$$xy \leq \Phi(x) + \Psi(y) \quad (x, y \in \mathbb{R}). \quad (2.1)$$

A number of various $N$-functions, borrowed from \[14\], are listed below.

1. For each $1 < p < \infty$, let $\Phi(x) = x^p/p$. Then $\Psi(y) = y^q/q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

2. If $\Phi(x) = (1 + |x|)\log(1 + |x|) - |x|$, then $\Psi(y) = e^{|y|} - |y| - 1$.

3. Let $\Phi(x) = \cosh(x) - 1$. Then $\Psi(x) = |x|\ln(|x| + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1$.

According to \[13, Proposition 2.1.1(ii)\], the inverse functions of a complementary pair of $N$-functions $(\Phi, \Psi)$ satisfy the following inequality:

$$x < \Phi^{-1}(x)\Psi^{-1}(x) \leq 2x. \quad (2.2)$$
For each $f \in L^0(G)$ we define

$$\rho_{\Phi}(f) = \int_G \Phi(|f(x)|) \, d\lambda(x).$$

Given an $N$-function $\Phi$, the **Orlicz space** $L^\Phi(G)$ is defined by

$$L^\Phi(G) = \{ f \in L^0(G) : \rho_{\Phi}(af) < \infty \text{ for some } a > 0 \}. $$

Similarly, the **Morse-Transue space** on $G$ is

$$M^\Phi(G) = \{ f \in L^0(G) : \rho_{\Phi}(af) < \infty \text{ for all } a > 0 \}. $$

The Orlicz space $L^\Phi(G)$ is a Banach space under the Luxemburg-Nakano norm $N_{\Phi}(-)$ defined for any $f \in L^\Phi(G)$ by

$$N_{\Phi}(f) = \inf \{ k > 0 : \rho_{\Phi}(f/k) \leq 1 \}. $$

Also, if $\chi_F$ denotes the characteristic function of a subset $F \subseteq G$ with $0 < \lambda(F) < \infty$, by [13, corollary 3.4.7] we have

$$N_{\Phi}(\chi_F) = \left[ \Phi^{-1}\left( \frac{1}{\lambda(F)} \right) \right]^{-1}. \tag{2.3}$$

Moreover, since $\Psi(x) > 0$ for each $x \neq 0$, then

$$\|f\|_\Phi = \sup \left\{ \int_G |fg| \, d\lambda : g \text{ is measurable and } N_{\Psi}(g) \leq 1 \right\},$$

is another norm called the **Orlicz norm** on $L^\Phi(G)$. It follows from [13, Proposition 3.3.4] that these two norms are equivalent and for each $f \in L^\Phi(G)$,

$$N_{\Phi}(f) \leq \|f\|_\Psi \leq 2N_{\Phi}(f). \tag{2.4}$$

The space $A_{\Phi}(G)$ introduced in [1], consists of those $u \in C_0(G)$, the space of all complex-valued continuous functions on $G$ vanishing at infinity, such that there are sequences $(f_n)_{n=1}^\infty \subseteq M^\Phi(G)$ and $(g_n)_{n=1}^\infty \subseteq M^\Psi(G)$ with $\sum_{n=1}^\infty N_{\Phi}(f_n)\|g_n\|_\Psi < \infty$ and $u = \sum_{n=1}^\infty f_n * \tilde{g}_n$, where $\tilde{g}_n(x) = g(x^{-1})$ for $x \in G$. The norm of $u \in A_{\Phi}(G)$ is defined by

$$\|u\|_{A_{\Phi}} = \inf \left\{ \sum_{n=1}^\infty N_{\Phi}(f_n)\|g_n\|_\Psi : u = \sum_{n=1}^\infty f_n * \tilde{g}_n \right\}. $$

It can be readily seen that $\|u\|_\infty \leq \|u\|_{A_{\Phi}}$ (see the proof of [6, Proposition A.4.5] for example).
3. Convolution product on $A_{\Phi}(G)$

This section is devoted to the clarification of the fact that when $A_{\Phi}(G)$ is closed under convolution product. To put in a nutshell, in the case when $G$ is amenable, we will show that for an $N$-function $\Phi$, the space $A_{\Phi}(G)$ is a Banach algebra under convolution product if and only if $G$ is compact. This manifests the reason for choosing pointwise product for $A_{\Phi}(G)$.

Let us recall the notion of porosity. For a metric space $X$ the open ball around $x \in X$ of radius $r > 0$ is denoted by $B(x, r)$. Given a real number $0 < c \leq 1$, a subset $M$ of $X$ is called $c$-lower porous if

$$\liminf_{R \to 0^+} \frac{\gamma(x, M, R)}{R} \geq \frac{c}{2},$$

for all $x \in M$, where

$$\gamma(x, M, R) = \sup \{r \geq 0 : \exists z \in X, B(z, r) \subseteq B(x, R) \setminus M\}.$$

Clearly $M$ is $c$-lower porous if and only if

$$\forall x \in M \ \forall \alpha \in (0, c/2) \ \exists R_0 > 0 \ \forall R \in (0, R_0) \ \exists z \in X, \ B(z, \alpha R) \subseteq B(x, R) \setminus M.$$

A set is called $\sigma$-c-lower porous if it is a countable union of $c$-lower porous sets with the same constant $c > 0$.

It is not hard to observe that a $\sigma$-c-lower porous set is meager and the notion of $\sigma$-porosity is stronger than meagerness. For more details see [18].

Let us also remark that for two normed spaces $A$ and $B$, we can endow the spaces $A \times B$ and $A \cap B$ with the norms

$$\| (f, g) \| = \max \{\| f \|_A, \| g \|_B \} \quad \text{and} \quad \| f \| = \| f \|_A + \| f \|_B,$$

respectively, for every $f \in A$ and $g \in B$. Notice that we consider the fact that $A \cap B$ is defined for two appropriate normed spaces $A$ and $B$.

The following, taken from [1], is a vital lemma in proving the main result of this section. However, we include its proof here for convenience and the fact that we will use some of its notation in next theorems.

**Lemma 3.1.** [1] Suppose that $G$ is an amenable locally compact group. Let $E$ be a compact subset of $G$ and $\Phi$ be an $N$-function. Then for every $\epsilon > 0$ there exists a function $u \in A_{\Phi}(G) \cap A_{\Psi}(G)$ such that $\| u \|_{A_{\Phi}} < 2(1 + \epsilon)$, $\| u \|_{A_{\Psi}} = \| \check{u} \|_{A_{\Phi}} < 2(1 + \epsilon)$ and $u = 1$ on $E$.

**Proof.** Consider a compact subset $F$ of $G$ with $\lambda(F) > 0$, and define

$$v(x) = \frac{1}{\lambda(F)}(\chi_{EF} \ast \check{\chi}_F)(x) = \frac{\lambda(xF \cap EF)}{\lambda(F)}.$$

Then $v \in A_{\Phi}(G)$ and $0 \leq v \leq 1$. If $x \in E$, then $\lambda(xF \cap EF) = \lambda(xF) = \lambda(F)$, so that $v(x) = 1$, while if $x \notin E_{FF^{-1}}$, then $xF \cap EF = \emptyset$ and therefore $v(x) = 0$. It follows that $\text{supp}(v) \subseteq E_{FF^{-1}}$ is compact. Now, since $G$ is amenable, by Leptin’s condition we may
select a compact set \( V \subseteq G \) of positive measure provided the inequality \( \lambda(EV) < (1+\epsilon)\lambda(V) \) holds. Now let \( f = \chi_{EV} \), \( g = \frac{1}{\lambda(V)}\chi_V \) and \( u = f \ast \tilde{g} \). Then, as \( u \) is a special case of \( v \), we have \( u \in A_\Phi(G) \cap A_\Psi(G) \) and \( u = 1 \) on \( E \). Moreover, using (2.4) we obtain

\[
\|u\|_{A_\Phi} \leq \frac{1}{\lambda(V)}\|\chi_V\|_{\Psi} N_{\Phi}(\chi_{EV}) \leq \frac{2}{\lambda(V)} \left[ \Phi^{-1}\left(\frac{1}{\lambda(\chi_{EV})}\right)\right]^{-1} \left[ \Psi^{-1}\left(\frac{1}{\lambda(V)}\right)\right]^{-1} \leq \frac{2}{\lambda(V)} \left[ \Phi^{-1}\left(\frac{1}{1+\epsilon}\lambda(V)\right)\right]^{-1} \left[ \Psi^{-1}\left(\frac{1}{1+\epsilon}\lambda(V)\right)\right]^{-1} < 2(1+\epsilon).
\]

Similarly, we can get \( \|u\|_{A_\Psi} < 2(1+\epsilon) \).

As an immediate consequence of Lemma 3.1 we can determine when \( A_\Phi(G) \) is unital.

**Corollary 3.2.** Let \( G \) be a locally compact group and \( \Phi \) be an \( N \)-function. Then \( A_\Phi(G) \) is unital if and only if \( G \) is compact.

**Proof.** If \( G \) is compact, taking \( E = G \) in Lemma 3.1 implies that the constant function 1 belongs to \( A_\Phi(G) \) and so it is unital. On the other hand, if \( A_\Phi(G) \) is unital, considering the fact that \( A_\Phi(G) \) is equipped with pointwise product, we conclude that its unit is the constant function 1. Since \( A_\Phi(G) \subseteq C_0(G) \), we earn \( 1 \in C_0(G) \). Therefore, \( G \) is compact. \( \square \)

**Theorem 3.3.** Let \( G \) be an amenable locally compact group and \( \Phi \) be an \( N \)-function. If \( G \) is not compact and \( V \) is a symmetric compact neighborhood of the identity, then the set

\[
E = \{ (f, g) \in A_\Phi(G) \times (A_\Phi(G) \cap A_\Psi(G)) : |f \ast \tilde{g}(x)| < \infty, \forall x \in V \},
\]

is a \( \sigma \)-c-lower porous set for some \( c > 0 \).

**Proof.** Let \( V \) be a symmetric compact neighborhood of \( e \), the identity of \( G \). Since \( G \) is not compact, there exists a sequence \( (a_m)_{m \in \mathbb{N}} \subseteq G \) such that \( a_i V \cap a_j V = \emptyset \) for distinct natural numbers \( i \) and \( j \). To be quite explicit, let \( a_1 = e \). Suppose \( a_1, a_2, \ldots, a_{m-1} \) are chosen. Since \( G \) is not compact, there exists \( a_m \in G \) so that \( a_m \notin \bigcup_{i=1}^{m-1} a_i V^2 \). Then the collection \( \{a_m V\}_{m \in \mathbb{N}} \) contains pairwise disjoint subsets of \( G \). Now for each \( n \in \mathbb{N} \) set

\[
E_n = \left\{ (f, g) \in A_\Phi(G) \times (A_\Phi(G) \cap A_\Psi(G)) : \int_G |f(y)||g(x^{-1}y)|d\lambda(y) \leq n, \forall x \in V \right\}.
\]

Clearly, \( E = \bigcup_{n \in \mathbb{N}} E_n \). Hence, we only need to show that for each \( n \in \mathbb{N} \), \( E_n \) is \( c \)-lower porous for some \( 0 < c < 1/32 \).

Fix \( n \in \mathbb{N} \) and \( R > 0 \) and suppose \( (f, g) \in E_n \). Then set

\[
A_f^1 := \{ x \in G : \text{Ref}(x) \geq 0 \} \quad \text{and} \quad A_f^{-1} := \{ x \in G : \text{Ref}(x) < 0 \}.
\]

In the same way we define \( A_g^1 \) and \( A_g^{-1} \). It follows that

\[
G = \bigcup_{i,j \in \{1,-1\}} \left( A_f^i \cap A_g^j \right).
\]
Then for some function \( s : \{1, 2\} \to \{1, -1\} \), we can get
\[
\lambda \left( \bigcup_{m=1}^{\infty} \left( a_m V \cap A_f^{s(1)} \cap A_g^{s(2)} \right) \right) = \infty.
\]
Assume, without loss of generality, that \( s(1) = s(2) = 1 \). Hence there exist \( m_0 \in \mathbb{N} \) such that
\[
\lambda \left( \bigcup_{m=1}^{m_0} \left( a_m V \cap A_f^{1} \cap A_g^{1} \right) \right) > \frac{512n}{R^2}.
\]
Let \( K = \bigcup_{m=1}^{m_0} \left( a_m V \cap A_f^{1} \cap A_g^{1} \right) \). Applying Lemma 3.1 with \( \varepsilon = 1 \), there exists \( u \in A_\Phi(G) \cap A_\Psi(G) \) with \( 0 \leq u \leq 1 \), \( u = 1 \) on \( K \) and \( \|u\|_{A_\Phi} + \|u\|_{A_\Psi} \leq 8 \). Define functions \( \tilde{f} \) and \( \tilde{g} \) on \( G \) by setting
\[
\tilde{f}(x) := f(x) + \frac{R}{8} u(x) \quad \text{and} \quad \tilde{g}(x) := g(x) + \frac{R}{16} u(x).
\]
Now, it can be easily verified that \( B((\tilde{f}, \tilde{g}), R/32) \subset B((f, g), R) \). Therefore, it suffices to show \( B((\tilde{f}, \tilde{g}), R/32) \cap E_n = \emptyset \). Fix \( (h, k) \in B((\tilde{f}, \tilde{g}), R/32) \) and let \( K_1 = \{ x \in K : |h(x)| \geq R/16 \} \). Then we have
\[
R/32 > \| \tilde{f} - h \|_{A_\Phi} \geq \| \tilde{f} - h \|_\infty \geq \sup_{x \in K \setminus K_1} \left\{ |\tilde{f}(x)| - |h(x)| \right\} \geq R/16.
\]
However, this is impossible and thus \( K_1 = K \). Similarly,
\[
R/32 > \| \tilde{g} - k \|_{A_\Phi \cap A_\Psi} \geq \sup_{x \in G} \left\{ |\tilde{g}(x)| - |k(x)| \right\} \geq \sup_{x \in K} \{ R/16 - |k(x)| \},
\]
from which for every \( x \in K \) we obtain \( |k(x)| > R/32 \). On the other hand, since \( k \) is a continuous function, we can find a symmetric compact neighborhood \( U \) contained in \( V \) such that \( |k(x)| > R/32 \) for every \( x \in UK \). Now consider an arbitrary element \( z \in U \). Then one may conclude that
\[
\int_K |h(y)||k(z^{-1}y)|d\lambda(y) \geq \frac{R^2}{512} \lambda(K) > n.
\]
Thus \( (h, k) \notin E_n \), as required.

**Corollary 3.4.** Let \( G \) be an amenable locally compact group and \( \Phi \) be an \( N \)-function. Then the convolution of each two functions in \( A_\Phi(G) \) exists if and only if \( G \) is compact. In particular, \( A_\Phi(G) \) is a Banach algebra under convolution product if and only if \( G \) is compact.

**Proof.** Since every Banach space is of second category, the Banach space \( A_\Phi(G) \times (A_\Phi(G) \cap A_\Psi(G)) \) cannot coincide with the set \( E \) in Theorem 3.3 provided \( G \) is not compact.

Conversely, let \( G \) be compact and \( C(G) \) denote the space of continuous functions on \( G \). Then \( C(G) = C_0(G) \) and
\[
A_\Phi(G) \ast A_\Phi(G) \subseteq C(G) \ast C(G) \subseteq A_\Phi(G)
\]
whence $A_{\Phi}(G)$ is closed under convolution product. Moreover, since $C(G)$ is included in $M^\Phi(G)$, it is a Banach space with respect to the norm $\|f\| := \|f\|_\infty + N_\Phi(f)$, for $f \in C(G)$.

Let $I : (C(G), \| \cdot \|_0) \rightarrow (C(G), \| \cdot \|_\infty)$ be the identity map, which is continuous and one-to-one. By a consequence of the open mapping theorem there exists some $\alpha > 0$ such that $N_\Phi(f) \leq \alpha \|f\|_\infty$ for all $f \in C(G)$. The same argument can be used to derive some $\beta > 0$ in order to have $\|f\|_\Psi \leq \beta \|f\|_\infty$. Now by the definition of $\| \cdot \|_{A_{\Phi}}$ for $u, v \in A_{\Phi}(G)$ we have

$$\|u * v\|_{A_{\Phi}} \leq N_\Phi(u)\|v\|_{\Psi} \leq \alpha \beta \|u\|\|v\|_{\infty} \leq \alpha \beta \|u\|_{A_{\Phi}} \|v\|_{A_{\Phi}}.$$ 

Therefore $(A_{\Phi}(G), \| \cdot \|_{A_{\Phi}})$ is a Banach algebra. \hfill \Box

We remark that, unfortunately, we were unable to prove Corollary 3.4 without taking the advantage of amenability property of $G$. However, we conjecture that this results holds without the amenability constraint.

We recall that a subspace $S(G)$ of $L^1(G)$ is called a Segal algebra if it satisfies the following four conditions:

1. $S(G)$ is dense in $L^1(G)$;
2. $(S(G), \| \cdot \|_S)$ is a Banach space such that $\|f\|_1 \leq \|f\|_S$ for all $f \in S(G)$;
3. $L_t f \in S(G)$ for all $f \in S(G)$ and $t \in G$, where $L_t f(x) = f(t^{-1}x)$ for $x \in G$;
4. For all $f \in S(G)$ the map $\Gamma_f : G \rightarrow S(G)$ by $\Gamma_f(t) = L_t f$ is continuous.

A Segal algebra is called symmetric if

1. $\mathcal{R}_t f \in S(G)$ for all $f \in S(G)$ and $t \in G$, where $\mathcal{R}_t f(x) = f(tx)$ for $x \in G$;
2. For all $f \in S(G)$ the map $\Lambda_f : G \rightarrow S(G)$ by $\Lambda_f(t) = \mathcal{R}_t f$ is continuous.

According to Proposition 1 of [10], every Segal algebra is an abstract Segal algebra (see [10, 2] for its definition) with respect to $L^1(G)$.

As an immediate consequence of Corollary 3.4, we obtain the following.

**Theorem 3.5.** Let $G$ be a compact group with normalized left Haar measure and $\Phi$ be an $N$-function. Then $(A_{\Phi}(G), \| \cdot \|_{A_{\Phi}})$ is a symmetric Segal algebra with respect to $L^1(G)$. In particular, $(A_{\Phi}(G), \| \cdot \|_{A_{\Phi}})$ is an abstract Segal algebra with respect to $L^1(G)$.

**Proof.** By Lemma 3.1, $A_{\Phi}(G)$ strongly separates the points of $L^1(G)$. Therefore, $A_{\Phi}(G)$ is dense in $L^1(G)$ by the Stone-Weierstrass theorem. Also, since $\lambda(G) = 1$, for any $f \in A_{\Phi}(G)$ we have $\|f\|_1 \leq \|f\|_\infty \leq \|f\|_{A_{\Phi}}$.

To complete the proof we note that for any Young function $\Phi$, $M^\Phi(G)$ is left translation invariant and $\|L_t h\|_{\Phi} = \|h\|_{\Phi}$ whenever $h \in M^\Phi(G)$. Now take $u \in A_{\Phi}(G)$. For simplicity assume $u = f * \tilde{g}$, where $f \in M^\Phi(G)$ and $g \in M^\Phi(G)$. Then for any $t \in G$, it can be readily observed that $L_t u = (L_t f) * \tilde{g}$ and $R_t u = f * (L_t \tilde{g})$. This yield that $L_t u, R_t u \in A_{\Phi}(G)$ and $\|L_t u\|_{A_{\Phi}} = \|R_t u\|_{A_{\Phi}} = \|u\|_{A_{\Phi}}$. Finally, by Proposition 5.3 in [1] the mappings $t \rightarrow L_t f$ and $t \rightarrow R_t f$ from $G$ to $A_{\Phi}(G)$ are continuous. \hfill \Box
It turns out that a symmetric Segal algebra is indeed a two-sided ideal in $L^1(G)$ and has a $L^1$-bounded approximate identity (see [15, 16]). The followings are some consequences of the preceding theorem. For the notion of amenability of a Banach algebra see [17].

**Corollary 3.6.** Let $G$ be a compact group, $\Phi$ be an $N$-function and consider $A_\Phi(G)$ as a Banach algebra under convolution product. Then the following are equivalent:

- (i) $A_\Phi(G)$ has a bounded approximate identity.
- (ii) $A_\Phi(G)$ is unital.
- (iii) $A_\Phi(G)$ is amenable.
- (iv) $G$ is finite.

**Proof.** As $A_\Phi(G)$ is a Segal algebra by Theorem 3.5, according to [2, Theorem 1.2], it cannot have a $\| \cdot \|_{A_\Phi}$-bounded approximate identity unless $A_\Phi(G) = L^1(G)$. Since $A_\Phi(G) \subseteq C(G) \subseteq L^1(G)$, we conclude that $C(G) = L^1(G)$ and so $G$ is finite. This also is equivalent to the existence of a unit for $A_\Phi(G)$. As amenability implies the existence of a bounded approximate identity by [17, Proposition 2.2.1], (iv) is equivalent to others too. 

We remark that Corollaries 3.2 and 3.6 show that how algebraic structures can make really different spaces.

The character space of a Banach algebra $A$ is defined to be the set of all bounded multiplicative linear functionals on $A$ and is denoted by $\Delta(A)$. As another application of Theorem 3.5, along with [2, Theorem 2.1], we present the following result about compact abelian groups.

**Corollary 3.7.** Let $G$ be a compact abelian group and $\Phi$ be an $N$-function. Then $\Delta(A_\Phi(G))$ is homeomorphic to $\hat{G}$, the dual of $G$, where $A_\Phi(G)$ is considered with convolution product.

**Remark.** In the first draft of the paper it was shown that the character space of $A_\Phi(G)$ under pointwise product coincides with $G$, for an arbitrary locally compact group $G$ and an $N$-function $\Phi$. It is, however, pointed out by the reviewer that it was already given in [11] (see Theorems 3.6(i), 3.7 and Corollary 3.8). Therefore, we preferred to do not include them in the final version of the paper. However, it is worth mentioning that our approach was taken from [6] (specially Theorem 2.9.4), which was a bit different from the approach given in [11].

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ON THE ALGEBRAIC STRUCTURES IN $A_{\Phi}(G)$

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