A space-time finite element method for fractional wave problems

Binjie Li · Hao Luo · Xiaoping Xie

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Abstract

In this paper, we analyze a space-time finite element method for fractional wave problems involving the time fractional derivative of order $\gamma$ ($1 < \gamma < 2$). We first establish the stability of the proposed method and then derive the optimal convergence rate in $H^1(0, T; L^2(\Omega))$-norm and suboptimal rate in discrete $L^\infty(0, T; H^1_0(\Omega))$-norm. Furthermore, we discuss the performance of this method when the true solution has singularity at $t = 0$ and show that optimal convergence rate with respect to $H^1(0, T; L^2(\Omega))$-norm can still be achieved by using graded temporal grids. Finally, numerical experiments are performed to verify the theoretical results.

Keywords Fractional wave problem · Space-time finite element · Convergence · Singularity · Graded grid

1 Introduction

This paper considers the following fractional wave problem:

\[
\begin{align*}
D_{0+}^\gamma (u - u_0 - tu_1) - \Delta u &= f & \text{in } \Omega \times (0, T), \\
u &= 0 & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0 & \text{in } \Omega, \\
\frac{\partial u}{\partial t}(\cdot, 0) &= u_1 & \text{in } \Omega,
\end{align*}
\]

(1)
where \( 1 < \gamma < 2 \), \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) is a polygon/polyhedron, and \( u_0, u_1, \) and \( f \) are given functions. Here \( u_t \) is the derivative of \( u \) with respect to the time variable \( t \), and \( D^\gamma_{0+} \) is a Riemann–Liouville fractional differential operator of order \( \gamma \).

Over the last two decades, the numerical treatment to time fractional partial differential equations has been an active research area. The main difference of these numerical methods is how to discretize the fractional derivatives. So far, there are three types of approaches to discretize the fractional derivatives: finite difference methods, spectral methods, and finite element methods. For the first type of algorithms that use the finite difference methods to discretize the fractional derivatives, we refer the reader to \([3, 7, 9–12, 19, 21, 30, 33, 36–38]\) and the references therein. These algorithms are easy to implement, but are generally of low temporal accuracy. For the second type of algorithms based on the spectral methods, we refer the reader to \([16, 20, 39, 41–44]\). These algorithms have high-order accuracy if the solution is sufficiently regular. Since singularity is an important feature of time fractional problems, the high-order accuracy of these algorithms is limited. Besides, these algorithms often lead to large scale dense systems to solve. For the third type of algorithms that use the finite element method to discretize fractional derivatives, we refer the reader to \([13–15, 17, 18, 22–28]\). Similar to the first type of algorithms, the discrete systems arising from these algorithms are solved successively in the time direction. Furthermore, these algorithms possess high-order accuracy, and if the solution has singularity, these algorithms can also have high-order accuracy by using graded grids in the time discretization.

Due to the nonlocal property of fractional derivatives, the history information has to be stored to compute the solution at each stage \([6, 40]\). Hence, the storage and computing cost to solve a time fractional wave problem is significantly more expensive than that to solve a standard wave problem. A natural idea is to develop high-order temporal accuracy algorithms. However, it is well known that time fractional wave problems generally have singularity at \( t = 0 \), despite how regular the initial and boundary data are. This makes developing high-order accuracy algorithms more challenging. As mentioned earlier, the finite element methods are generally flexible in meshing and easy to attain high-order accuracy. This motivates us to develop high-order accuracy finite element methods that can also tackle the singularity at \( t = 0 \).

In this paper, we propose a space-time finite element method for the fractional wave problem (1). This method employs a Petrov-Galerkin type time-stepping scheme to discretize the fractional derivative, which uses continuous piecewise polynomials (of degree \( \leq m \)) as trial functions and totally discontinuous piecewise polynomials (of degree \( \leq m - 1 \)) as test functions. We establish the stability of this method and derive two a priori error estimates under a reasonable regularity assumption on the solution (cf. Theorem 4.3). The estimates show that the proposed method possesses temporal accuracy order \( m \) in the \( H^1(0,T;L^2(\Omega)) \)-norm and temporal accuracy order \( m - 1/2 \) \((m \geq 2)\) in the discrete \( L^\infty(0,T;H^1_0(\Omega)) \)-norm (cf. Remark 4.5), provided that the exact solution is sufficiently regular. Furthermore, we analyze the case that the solution has singularity at \( t = 0 \) and show that the presented method can still achieve temporal accuracy of order \( m \) in the \( H^1(0,T;L^2(\Omega)) \)-norm by using suitable graded grids in the time discretization.
The rest of this paper is organized as follows. Section 2 introduces some vector-valued spaces, the Riemann–Liouville fractional calculus operators, and the weak form to problem Section 1. Section 3 describes a space-time finite element method, and Section 4 investigates its stability and convergence. Section 5 performs some numerical experiments to verify the theoretical results.

2 Preliminaries

We first introduce some vector-valued spaces. Let \( X \) be a separable Hilbert space with an inner product \((\cdot, \cdot)_X\) and an orthonormal basis \( \{e_j : j \in \mathbb{N}\} \), and let \(-\infty \leq a < b \leq \infty\). For \( 0 < \alpha < \infty \), define

\[
H^\alpha(a, b; X) := \left\{ v \in L^2(a, b; X) : \sum_{j=0}^{\infty} \left\| (v, e_j)_X \right\|_{H^\alpha(a, b)}^2 < \infty \right\}
\]

and endow this space with the norm

\[
\| \cdot \|_{H^\alpha(a, b; X)} := \left( \sum_{j=0}^{\infty} \left\| (\cdot, e_j)_X \right\|_{H^\alpha(a, b)}^2 \right)^{1/2},
\]

where \( L^2(a, b; X) \) is an \( X \)-valued Bochner \( L^2 \) space. If \( 0 < \alpha < 1/2 \), we also introduce the following two norms:

\[
|v|_{H^\alpha(a, b; X)} := \left( \sum_{j=0}^{\infty} \left\| (v, e_j)_X \right\|_{H^\alpha(a, b)}^2 \right)^{1/2},
\]

\[
\|v\|_{H^\alpha(a, b; X)} := \inf_{\tilde{v}\in H^\alpha((-\infty, b); X)} \left| \tilde{v} \right|_{H^\alpha((-\infty, b); X)},
\]

for all \( v \in H^\alpha(a, b; X) \). Here, \( H^\alpha(a, b) \) is a standard Sobolev space (see [35]), and

\[
\left| v \right|_{H^\alpha(a, b)} := \left( \int |\xi|^{2\alpha} \left| \mathcal{F}(v \cdot \chi_{(a,b)})(\xi) \right|^2 \, d\xi \right)^{1/2}
\]

for each \( v \in H^\alpha(a, b) \) with \( 0 < \alpha < 1/2 \), where \( \mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is the Fourier transform operator and \( \chi_{(a,b)} \) is the indicator function of \((a, b)\). Moreover, for \( 0 < \alpha < 1/2 \), we use \( H^{-\alpha}(\mathbb{O}; X^*) \) to denote the dual space of \( H^\alpha(a, b; X) \), where \( X^* \) is the dual space of \( X \) and \( H^\alpha(a, b; X) \) is endowed with the norm \( |\cdot|_{H^\alpha(a, b; X)} \). For \( v \in H^i(a, b; X) \) with \( i \in \mathbb{N}_{>0} \), we use \( v^{(i)} \) to denote its \( i \)th weak derivative, and \( v^{(1)} \) and \( v^{(2)} \) are abbreviated to \( v' \) and \( v'' \), respectively.

Additionally, for \( 0 \leq \delta < 1 \), define

\[
L^2_\delta(a, b; X) := \left\{ v \in L^1(a, b; X) : \|v\|_{L^2_\delta(a, b; X)} < \infty \right\},
\]

where

\[
\|v\|_{L^2_\delta(a, b; X)} := \left( \int_a^b \|t\|^\delta \|v(t)\|_X^2 \, dt \right)^{1/2}.
\]
Conventionally, $C([a, b]; X)$ is the set of all $X$-valued continuous functions defined on $[a, b]$, and $P_j ([a, b]; X)$ is the set of all $X$-valued polynomials (of degree $\leq j$) defined on $[a, b]$. For convenience, $\| \cdot \|_{L^2_\delta (a, b; \mathbb{R})}$ and $P_j ([a, b]; \mathbb{R})$ are abbreviated to $\| \cdot \|_{L^2_\delta (a, b)}$ and $P_j ([a, b])$, respectively.

Now we introduce the Riemann–Liouville fractional calculus operators.

**Definition 2.1** For $0 < \alpha < \infty$, define

$$ (I_{a+}^\alpha v)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} v(s) \, ds, \quad t \in (a, b), $$

$$ (I_{b-}^\alpha v)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha - 1} v(s) \, ds, \quad t \in (a, b), $$

for all $v \in L^1 (a, b; X)$, where $L^1 (a, b; X)$ denotes a standard $X$-valued Bochner $L^1$ space and $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2** For $j - 1 < \alpha < j$ with $j \in \mathbb{N}_{>0}$, define

$$ D_{a+}^\alpha := D^j I_{a+}^{j-\alpha}, $$

$$ D_{b-}^\alpha := (-D)^j I_{b-}^{j-\alpha}, $$

where $D$ is the first-order differential operator in the distribution sense.

Finally, let us define the weak solution to problem (1). Throughout this paper, we assume that $u_0 \in H^1_0 (\Omega)$, $u_1 \in L^2 (\Omega)$, and $f \in H^{-\gamma_0} (0, T; L^2 (\Omega))$, where $\gamma_0 = (\gamma - 1)/2$. We assume that problem (1) admits a weak solution

$$ u \in H^{1+\gamma_0} (0, T; L^2 (\Omega)) \cap L^2 (0, T; H^1_0 (\Omega)) $$

such that $u(0) = u_0$ and

$$ \langle D_{0+}^\gamma (u - u_0 - tu_1), v \rangle_{H^{\gamma_0} (0, T; L^2 (\Omega))} + \langle \nabla u, \nabla v \rangle_{\Omega \times (0, T)} = \langle f, v \rangle_{H^{-\gamma_0} (0, T; L^2 (\Omega))} $$

for all $v \in H^{\gamma_0} (0, T; L^2 (\Omega)) \cap L^2 (0, T; H^1_0 (\Omega))$. Above and throughout, if $D$ is a Lebesgue measurable set of $\mathbb{R}^l$ ($l = 1, 2, 3, 4$), then the symbol $\langle p, q \rangle_D$ means $\int_D pq$ whenever $pq \in L^1 (D)$, and if $D$ is a Banach space, then $\langle \cdot, \cdot \rangle_D$ means the duality pairing between $D^*$ and $D$.

**Remark 2.1** By means of the technique developed in [16, Appendix A] and [18], we can establish the well posedness of the weak solution to problem (1). However, this is already beyond the scope of this work.

### 3 Discretization

For $\sigma \geq 1$ and $J \in \mathbb{N}_{>0}$, define

$$ t_j := (j/J)^\sigma T \quad \text{for all } 0 \leq j \leq J. $$
For each $1 \leq j \leq J$, we set $I_j := (t_{j-1}, t_j)$ and $\tau_j := t_j - t_{j-1}$. Let $\mathcal{K}_h$ be a triangulation of $\Omega$ consisting of $d$-simplexes, and we use $h$ to denote the maximum diameter of the elements in $\mathcal{K}_h$. For $m, n \in \mathbb{N}_{>0}$, we define

$$S_h := \left\{ v_h \in H^1_0(\Omega) : v_h|_K \in P_n(K) \quad \forall \, K \in \mathcal{K}_h \right\},$$

$$M_{h, \tau} := \left\{ V \in H^1(0, T; S_h) : V|_{I_j} \in P_m(I_j; S_h) \quad \forall \, 1 \leq j \leq J \right\},$$

$$W_{h, \tau} := \left\{ V \in L^2(0, T; S_h) : V|_{I_j} \in P_{m-1}(I_j; S_h) \quad \forall \, 1 \leq j \leq J \right\}.$$

Now, inspired by the weak formulation (2), we construct a space-time finite element method as follows: seek $U \in M_{h, \tau}$ such that $U(0) = R_h u_0$ and

$$\left\langle D_{0+}^\gamma (U - U(0) - t \Pi_h u_1), V \right\rangle_{H^\gamma(0, T; L^2(\Omega))} + \langle \nabla U, \nabla V \rangle_{\Omega \times (0, T)} = \left\langle f, V \right\rangle_{H^\gamma(0, T; L^2(\Omega))}$$

(3)

for all $V \in W_{h, \tau}$, where $\Pi_h$ is the $L^2(\Omega)$-orthogonal projection operator onto $S_h$, and $R_h : H^1_0(\Omega) \to S_h$ is defined by

$$\langle \nabla (v - R_h v), \nabla v_h \rangle_{\Omega} = 0 \quad \forall \, v_h \in S_h,$$

for all $v \in H^1_0(\Omega)$.

**Remark 3.1** Given $V \in M_{h, \tau}$, a straightforward calculation yields that

$$\left\langle D_{0+}^\gamma (V - V(0)), \varphi \right\rangle = \left\langle \mathcal{L}_{0+}^\gamma (V - V(0)), \varphi'' \right\rangle_{(0, T)} = \left\langle V - V(0), \mathcal{L}_{T^-}^\gamma \varphi'' \right\rangle_{(0, T)}$$

$$= \left\langle V - V(0), \left( \mathcal{I}_{T^-}^\gamma \varphi \right)' \right\rangle_{(0, T)} = -\left\langle V', \left( \mathcal{I}_{T^-}^\gamma \varphi \right)' \right\rangle_{(0, T)}$$

$$= \sum_{j=1}^{J} \left\langle \varphi'' \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{I_j} + \sum_{j=1}^{J-1} \left\langle \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{(t_j)}$$

for any $\varphi \in C_0^\infty(0, T)$, where $\left\langle \mathcal{I}_{T^-}^\gamma \varphi \right\rangle := \lim_{t \to t_j-} V'(t) - \lim_{t \to t_j+} V'(t)$. Setting $Z \in L^1((0, T); S_h)$ by

$$Z|_{I_j} := (V|_{I_j})'' \quad \text{for all } 1 \leq j \leq J,$$

we obtain

$$\sum_{j=1}^{J} \left\langle \varphi'' \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{I_j} = \left\langle Z, \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{(0, T)} = \left\langle \mathcal{I}_{0+}^\gamma Z, \varphi \right\rangle_{(0, T)}.$$

Additionally, it is easy to see

$$\sum_{j=1}^{J-1} \left\langle \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{(t_j)} = \sum_{j=1}^{J-1} \left\langle \mathcal{I}_{T^-}^\gamma \varphi \right\rangle_{(0, T)}$$

where

$$\omega_j := \begin{cases} 0 & \text{if } 0 < t < t_j, \\ \frac{(t_j)^{1-\gamma}}{\Gamma(2-\gamma)} & \text{if } t_j < t < T. \end{cases}$$
Consequently,

\[
\{ D_{0+}^\gamma (V - V(0)) , \varphi \} = \left( I_{0+}^{2-\gamma} Z + \sum_{j=1}^{J-1} \| V_j' \| \cdot \omega_j , \varphi \right)_{(0,T)}
\]

for all \( \varphi \in C_0^\infty (0,T) \), which indicates

\[
D_{0+}^\gamma (V - V(0)) = I_{0+}^{2-\gamma} Z + \sum_{j=1}^{J-1} \| V_j' \| \cdot \omega_j.
\]

\textbf{Remark 3.2} Since \( W_{h,\tau} \) is totally discontinuous, we can solve \( U|_{I_j} \) successively from \( j = 1 \) to \( j = J \).

\section{4 Stability and convergence}

For convenience, \( a \lesssim b \) means that there exists a positive constant \( C \), depending only on \( \gamma, T, m, n \), or the shape regularity of \( K_h \), such that \( a \leq Cb \), and \( a \sim b \) means \( a \lesssim b \lesssim a \). Moreover, if the symbol \( C \) has subscript(s), then it means a positive constant depends only on its subscript(s) unless otherwise stated, and its value may differ at each of its occurrence(s). For example, \( C_{\gamma,T} \) is a positive constant that depends only on \( \gamma \) and \( T \), and its value may differ at different places.

\subsection*{4.1 Two interpolation operators}

Let \( X \) be a separable Hilbert space. For each \( v \in C((0,T]; X) \), define \( P^X_{\tau} v \) by

\[
\begin{aligned}
(P^X_{\tau} v)|_{I_j} &\in P_{m-1}(I_j; X), \\
\lim_{t \to t^-_j} (P^X_{\tau} v)(t) &= v(t_j), \\
\langle v - P^X_{\tau} v, q \rangle_{I_j} &= 0 \quad \text{for all } q \in P_{m-2}(I_j),
\end{aligned}
\]

for all \( 1 \leq j \leq J \), where \( P_{-1}(I_j) := \emptyset \) in the case of \( m = 1 \). For any \( v \in H^{1+\gamma_0}(0,T; X) \), define \( Q^X_{\tau} v \in C([0,T]; X) \) by

\[
\begin{aligned}
(Q^X_{\tau} v)(0) &= v(0), \\
(Q^X_{\tau} v)|_{I_j} &\in P_m(I_j; X) \quad \text{for all } 1 \leq j \leq J, \\
\langle D_{0+}^{\gamma_0} (v - Q^X_{\tau} v)' , w_{\tau} \rangle_{H^{\gamma_0}(0,T)} &= 0 \quad \text{for all } w_{\tau} \in W_{\tau},
\end{aligned}
\]

where

\[
W_{\tau} := \left\{ w_{\tau} \in L^2(0,T) : w_{\tau}|_{I_j} \in P_{m-1}(I_j) \quad \forall 1 \leq j \leq J \right\}.
\]

For simplicity, we shall suppress the superscript \( X \) of \( Q^X_{\tau} \) and \( P^X_{\tau} \) when no confusion will arise.

\textbf{Remark 4.1} Clearly, Lemma A.1 implies that \( Q^X_{\tau} \) is well defined.

\textbf{Lemma 4.1} \([32, 34] \) If \( 1 \leq j \leq J \) and \( v \in H^m(I_j) \), then

\[
\| v - P_{\tau} v \|_{L^2(I_j)} + \tau_j \| v - P_{\tau} v \|_{H^1(I_j)} \leq C_{\gamma,T_j}^m \| v \|_{H^m(I_j)}.
\]
Lemma 4.2 [35] If $0 < \alpha < 1/2$, then

$$|v|_{H^\alpha(0,1)} \sim \|v\|_{H^\alpha(0,1)}$$

for all $v \in H^\alpha(0,1)$, and

$$|v|_{H^\alpha(\mathbb{R})} \sim \left(\int_\mathbb{R} \int_\mathbb{R} \frac{|v(s) - v(t)|^2}{|s - t|^{1+2\alpha}} \ ds \ dt\right)^{\frac{1}{2}}$$

for all $v \in H^\alpha(\mathbb{R})$.

Lemma 4.3 If $0 < \alpha < 1/2$ and $v \in H^\alpha(0, T)$ with $v' \in L^1(0,T)$, then

$$|v - P_\tau v|_{H^\alpha(0,t_j)} \leq C_\alpha \left(\sum_{i=1}^j \tau_i^{2-2\alpha} \inf_{0 \leq \delta < 1} \frac{t_i - \delta}{1 - \delta} \left\| (v - P_\tau v)' \right\|^2_{L^2_\delta(I_i)}\right)^{\frac{1}{2}}$$

for each $1 \leq j \leq J$.

Proof Setting $g := (I - P_\tau)v$, by Lemma 4.2 we only need to prove

$$E_1 + E_2 + E_3 \leq C_\alpha \sum_{i=1}^j \tau_i^{2-2\alpha} \inf_{0 \leq \delta < 1} \frac{t_i - \delta}{1 - \delta} \left\| (v - P_\tau v)' \right\|^2_{L^2_\delta(I_i)}, \tag{4}$$

where

$$E_1 = \sum_{i=1}^j \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} |g(t) - g(s)|^2 |t - s|^{-1-2\alpha} \ ds \ dt,$$

$$E_2 = \sum_{i=1}^j \sum_{l=i+1}^j \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t_l} |g(t) - g(s)|^2 |t - s|^{-1-2\alpha} \ ds \ dt,$$

$$E_3 = \int_0^{t_j} |g(t)|^2 \left(\int_{t_j}^{\infty} (s - t)^{-1-2\alpha} \ ds + \int_{-\infty}^{0} (t - s)^{-1-2\alpha} \ ds\right) \ dt.$$

A straightforward calculation yields

$$\sum_{i=1}^j \sum_{l=i+1}^j \int_{t_{l-1}}^{t_l} \int_{t_{l-1}}^{t_l} g^2(t) |t - s|^{-1-2\alpha} \ ds \ dt$$

$$= \frac{1}{2\alpha} \sum_{i=1}^j \sum_{l=i+1}^j \int_{t_{l-1}}^{t_l} g^2(t) \left((t_{l-1} - t)^{-2\alpha} - (t_l - t)^{-2\alpha}\right) \ dt$$

$$\leq \frac{1}{2\alpha} \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} g^2(t) (t_i - t)^{-2\alpha} \ dt.$$
and
\[
\sum_{i=1}^{j} \sum_{l=i+1}^{j} \int_{t_{l-1}}^{t_{l}} dt \int_{t_{l-1}}^{t_{l}} g^2(s) |t-s|^{-1-2\alpha} ds
\]
\[
= \frac{1}{2\alpha} \sum_{i=1}^{j} \sum_{l=i+1}^{j} \int_{t_{l-1}}^{t_{l}} g^2(s) \left((s-t_{l})^{-2\alpha} - (s-t_{l-1})^{-2\alpha}\right) ds
\]
\[
\leq \frac{1}{2\alpha} \sum_{l=2}^{j} \int_{t_{l-1}}^{t_{l}} g^2(s)(s-t_{l-1})^{-2\alpha} ds.
\]

It follows that
\[
E_2 \leq \frac{1}{\alpha} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_{i}} g^2(t) \left((t_{i}-t)^{-2\alpha} + (t-t_{i-1})^{-2\alpha}\right) dt.
\]

In addition, it is evident that
\[
E_3 \leq \frac{1}{2\alpha} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_{i}} g^2(t) \left((t_{i}-t)^{-2\alpha} + (t-t_{i-1})^{-2\alpha}\right) dt.
\]

Therefore, using Lemma B.1 yields
\[
E_2 + E_3 \leq \frac{3}{2\alpha} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_{i}} g^2(t) \left((t_{i}-t)^{-2\alpha} + (t-t_{i-1})^{-2\alpha}\right) dt
\]
\[
\leq C_\alpha \sum_{i=1}^{j} \tau_i^{-2\alpha} \inf_{0 \leq \delta < 1} \frac{t_{i-\delta}}{1-\delta} \|g'\|_{L_2^2(I_i)}^2.
\]

As Lemma B.1 also implies
\[
E_1 \leq C_\alpha \sum_{i=1}^{j} \tau_i^{-2\alpha} \inf_{0 \leq \delta < 1} \frac{t_{i-\delta}}{1-\delta} \|g'\|_{L_2^2(I_i)}^2,
\]
we readily obtain (4) and thus conclude the proof. \(\square\)

**Lemma 4.4** Define
\[
v(t) := t^r, \quad 0 < t < T,
\]
where \(1 < r \leq m + 1/2\). Let \(1 \leq j \leq J\) and set
\[
\sigma^* := \frac{2m + 1 - \gamma}{2r - \gamma} \geq 1.
\]

If \(1 \leq \sigma < \sigma^*\), then
\[
\sum_{i=1}^{j} \tau_i^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_{i-\delta}}{1-\delta} \left\|\left(v' - P_{\tau} v'\right)\right\|_{L_2^2(I_i)}^2 \leq C_{r,m,\gamma,\sigma,T} J^{\sigma(\gamma - 2r)}.
\]
If $\sigma = \sigma^*$, then
\[
\sum_{i=1}^{j} \tau_i^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \left\| (v' - P_\tau v')' \right\|_{L^2(I_i)}^2 \leq C_{r,m,\gamma,\sigma,T} (1 + \ln j) J^{-(2m+1-\gamma)}. \tag{6}
\]
Moreover, if $\sigma > \sigma^*$, then
\[
\sum_{i=1}^{j} \tau_i^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \left\| (v' - P_\tau v')' \right\|_{L^2(I_i)}^2 \leq C_{r,m,\gamma,\sigma,T} J^{-(2m+1-\gamma)}. \tag{7}
\]

**Proof** Since the proofs of (5) and (6) are similar to that of (7), we only prove the latter. Set
\[
\delta_0 := \begin{cases} 
1/2 & \text{if } r \geq \frac{3}{2}, \\
2 - r & \text{if } 1 < r < \frac{3}{2},
\end{cases}
\]
then we have
\[
\left\| v'' \right\|_{L^2_0(I_1)} = \frac{r(r-1)\tau_1^{r+\delta_0/2-3/2}}{\sqrt{\delta_0 + 2(r-2) + 1}}.
\]
Applying a standard scaling argument yields the estimate
\[
\left\| (P_\tau v')' \right\|_{L^2_0(I_1)} \leq C_{r,m} \left\| v'' \right\|_{L^2_0(I_1)} \leq C_{r,m} \tau_1^{2r-\gamma}.
\]
and it follows that
\[
\tau_1^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \left\| (v' - P_\tau v')' \right\|_{L^2(I_i)}^2 \leq C_{r,m} \tau_1^{2r-\gamma}.
\]
Therefore, by the fact $\sigma > \sigma^*$ and the evident estimate
\[
\tau_1^{2r-\gamma} = T^{2r-\gamma} J^{-\sigma(2r-\gamma)} < T^{2r-\gamma} J^{-(2m+1-\gamma)},
\]
we obtain
\[
\tau_1^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \left\| (v' - P_\tau v')' \right\|_{L^2(I_i)}^2 \leq C_{r,m,\gamma,T} J^{-(2m+1-\gamma)}. \tag{8}
\]
In addition, Lemma 4.1 implies
\[
\sum_{i=1}^{j} \tau_i^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \left\| (v' - P_\tau v')' \right\|_{L^2(I_i)}^2 \leq C_m \sum_{i=2}^{j} \tau_i^{3-\gamma + 2(m-1)} \int_{t_{i-1}}^{t_i} t^{2(r-m-1)} dt.
\]
Then, by the inequality
\[
\tau_i < \sigma^{2\sigma-1} J^{-1/\sigma} t_1^{1-1/\sigma}, \quad 2 \leq i \leq j,
\]
we obtain
\[
\sum_{i=2}^{j} \tau_i^{3-\gamma} \inf_{0 \leq \delta < 1} \frac{I_i^{\delta}}{1-\delta} \left\| (v' - P_\tau v')' \right\|_{L_2^d(I_i)}^2 \leq C_{m,\gamma,T} J^{-(2m+1-\gamma)} \int_{t_1}^{t_j} t^{2(r-m-1) + (1-1/\sigma)(2m+1-\gamma)} \, dt \\
\leq C_{r,m,\gamma,T} J^{-(2m+1-\gamma)}.
\] (9)

Finally, combining (8) and (9) yields (7) and thus proves the lemma.

Lemma 4.5 If \(0 \leq \beta < \infty\) and \(v \in H^\alpha(0, T)\) with \(0 \leq \alpha < 1/2\), then
\[
\inf_{w_\tau \in W_{\tau}} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(0,t_j)} \leq C_{\alpha,\beta,m,\gamma,T} \tau_j \inf_{w_\tau \in W_{\tau}} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} \leq C_{\alpha,\beta,m,T} \left\| v \right\|_{H^{\varrho}(0,t_j)}
\] for each \(1 \leq j \leq J\), where
\[\varrho := \min\{\alpha, \max\{0, m - \beta\}\} .\]

Proof Let \(J^*\) be the smallest integer such that \(T/J^* \leq \tau_1\), and define
\[W_{\tau}^* := \left\{ w_{\tau} \in L^2(-T,T) : w_{\tau}|_{I_i} \in P_{m-1}(I_i), -(J^* - 1) \leq i \leq J \right\},\]

where
\[I_i := \left( \frac{i-1}{J^*}, \frac{i}{J^*} \right) \quad \text{for all } -(J^* - 1) \leq i \leq 0.\]

Extending \(v\) to \((-T, 0)\) by zero, by the definition of the norm \(\left\| \cdot \right\|_{H^{\gamma_0}(0,t_j)}\), we obtain
\[
\inf_{w_\tau \in W_{\tau}} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(0,t_j)} \leq \inf_{w_\tau \in W_{\tau}^*} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} \leq \inf_{w_\tau \in W_{\tau}^*} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} ,
\]
so that by Lemma 4.2
\[
\inf_{w_\tau \in W_{\tau}} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(0,t_j)} \leq C_{\gamma,T} \inf_{w_\tau \in W_{\tau}^*} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} .
\]

Additionally, a standard scaling argument yields
\[
\left\| I_{t_j}^\beta v \right\|_{H^{\min\{\alpha+\beta,m\}(-T,t_j)}} \leq C_{\alpha,\beta,m,T} \left| v \right|_{H^\varrho(-T,t_j)} = C_{\alpha,\beta,m,T} \left| v \right|_{H^{\varrho}(0,t_j)} ,
\]
by Lemmas 4.2 and A.4. Therefore, (10) follows from the standard approximation estimate (see [1, Chapter 14])
\[
\inf_{w_\tau \in W_{\tau}^*} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} \leq C_{\alpha,\beta,m,T} \tau_j \inf_{w_\tau \in W_{\tau}^*} \left\| I_{t_j}^\beta v - w_\tau \right\|_{H^{\gamma_0}(-T,t_j)} .
\]

This concludes the proof of the lemma. □
Remark 4.2 Observe that the constant in (10) is independent of \( t_j \), which is crucial in our analysis.

Lemma 4.6 If \( v \in H^{1+\gamma_0}(0, T) \) and \( v' \in C(0, T) \), then
\[
\left\| (v - Q_{\tau}v)' \right\|_{H^0(0, t_j)} \lesssim \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)},
\]
(11)
\[
\left\| (v - Q_{\tau}v)' \right\|_{L^2(0, t_j)} \lesssim \tau_j \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)},
\]
(12)
\[
\left\| v - Q_{\tau}v \right\|_{H^{-\gamma_0}(0, t_j)} \lesssim \tau_j \min\{\gamma, m - \gamma_0\} \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)},
\]
(13)
\[
\left\| v - Q_{\tau}v \right\|_{L^2(0, t_j)} \lesssim \tau_j \min\{1 + \gamma_0, m - \gamma_0\} \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)},
\]
(14)
for each \( 1 \leq j \leq J \). Moreover, if \( m \geq 2 \), then
\[
\left\| (v - Q_{\tau}v)\right\|_{H^0(0, t_j)} \lesssim \tau_j^{\gamma_0 + 1/2} \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)}.
\]
(15)

Proof Set \( g := v - Q_{\tau}v \) and let us first prove (11). Observing that the definition of \( Q_{\tau} \) implies
\[
\left\{ D_{0+}^{2\gamma_0} (v - Q_{\tau}v)', (Q_{\tau}v)' - P_{\tau}v' \right\}_{H^0(0, t_j)} = 0,
\]
we obtain
\[
\left\{ D_{0+}^{2\gamma_0} ((Q_{\tau}v)' - P_{\tau}v'), (Q_{\tau}v)' - P_{\tau}v' \right\}_{H^0(0, t_j)} = \left\{ D_{0+}^{2\gamma_0} (v' - P_{\tau}v'), (Q_{\tau}v)' - P_{\tau}v' \right\}_{H^0(0, t_j)}.
\]
Therefore, using Lemma A.2 yields
\[
\left\| (Q_{\tau}v)' - P_{\tau}v' \right\|_{H^0(0, t_j)} \lesssim \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)},
\]
and so (11) follows from the triangle inequality
\[
\left\| g' \right\|_{H^0(0, t_j)} \leq \left\| (Q_{\tau}v)' - P_{\tau}v' \right\|_{H^0(0, t_j)} + \left\| (I - P_{\tau})v' \right\|_{H^0(0, t_j)}.
\]

Then let us prove (12). Since Lemma A.2 implies
\[
\left\| g' \right\|_{L^2(0, t_j)}^2 = \left\{ D_{0+}^{2\gamma_0} g', D_{1-j}^{2\gamma_0} g' \right\}_{H^0(0, t_j)} = \left\{ D_{0+}^{2\gamma_0} g', I_{1-j}^{2\gamma_0} g' \right\}_{H^0(0, t_j)},
\]
the definition of \( Q_{\tau} \) implies that
\[
\left\| g' \right\|_{L^2(0, t_j)}^2 = \left\{ D_{0+}^{2\gamma_0} g', I_{1-j}^{2\gamma_0} g' - w_{\tau} \right\}_{H^0(0, t_j)}
\]
for all \( w_{\tau} \in W_{\tau} \). Therefore, by Lemmas A.3 and 4.5, we obtain
\[
\left\| g' \right\|_{L^2(0, t_j)}^2 \lesssim \left\| g' \right\|_{H^0(0, t_j)} \inf_{w_{\tau} \in W_{\tau}} \left\| I_{1-j}^{2\gamma_0} g' - w_{\tau} \right\|_{H^0(0, t_j)} \lesssim \left\| g' \right\|_{H^0(0, t_j)} \tau_j \left\| g' \right\|_{L^2(0, t_j)}.
\]
It follows that
\[
\left\| g' \right\|_{L^2(0, t_j)} \lesssim \tau_j \left\| g' \right\|_{H^0(0, t_j)},
\]
which, together with (11), proves estimate (12).
Analogously, we can obtain (14) and (13). Since \( g(0) = 0 \), using integration by parts gives
\[
\left| g(t_j) \right|^2 = \int_0^{t_j} 2g(t)g'(t) \, dt \leq 2 \| g \|_{L^2(0,t_j)} \| g' \|_{L^2(0,t_j)}.
\]
Therefore, combining (12) and (14) proves (15). This completes the proof. 

**Lemma 4.7** If \( m = 1 \) and \( v \in H^{1+\gamma_0}(0, T) \), then for \( 1 \leq j \leq J \),
\[
\left\| (v - Q_\tau v)(t_j) \right\|_{H^{\gamma_0}(0,t_j)} \lesssim \begin{cases} 
\tau_j^{\gamma_0 + 1/2} & \text{if } \gamma_0 < 1/4, \\
\sqrt{1 + \ln(t_j/\tau_j)} \tau_j^{1-\gamma_0} & \text{if } \gamma_0 = 1/4, \\
\tau_j^{1-\gamma_0} & \text{if } \gamma_0 > 1/4.
\end{cases}
\]

**Proof** Putting \( G(t) := \frac{(t_j - t)^{2\gamma_0}}{\Gamma(1 - 2\gamma_0)} \), \( 0 < t < t_j \), by a direct computing, we obtain
\[
(v - Q_\tau v)(t_j) = \left( D_{0+}^{2\gamma_0} (v - Q_\tau v)', G \right)_{H^{\gamma_0}(0,t_j)}.
\]
From the definition of \( Q_\tau \), it follows that
\[
(v - Q_\tau v)(t_j) = \left( D_{0+}^{2\gamma_0} (v - Q_\tau v)', G - w_\tau \right)_{H^{\gamma_0}(0,t_j)}
\]
for all \( w_\tau \in W_\tau \). Hence, Lemma A.2 implies
\[
\left\| (v - Q_\tau v)(t_j) \right\|_{H^{\gamma_0}(0,t_j)} \lesssim \left\| (v - Q_\tau v)' \right\|_{H^{\gamma_0}(0,t_j)} \inf_{w_\tau \in W_\tau} |G - w_\tau|_{H^{\gamma_0}(0,t_j)}.
\]
By the similar techniques as that used in Lemma 4.3 and 4.4, a tedious but straightforward calculation yields
\[
\inf_{w_\tau \in W_\tau} |G - w_\tau|_{H^{\gamma_0}(0,t_j)} \lesssim \begin{cases} 
\tau_j^{\gamma_0 + 1/2} & \text{if } \gamma_0 < 1/4, \\
\sqrt{1 + \ln(t_j/\tau_j)} \tau_j^{1-\gamma_0} & \text{if } \gamma_0 = 1/4, \\
\tau_j^{1-\gamma_0} & \text{if } \gamma_0 > 1/4.
\end{cases}
\]
Combining the above two estimates gives (16) and thus concludes the proof. 

**4.2 Main results**

In the rest of this paper, for each \( 1 \leq j \leq J \), we define \( \epsilon_j \) as follows: if \( m = 1 \), then set
\[
\epsilon_j := \begin{cases} 
\tau_j^{1/2+\gamma_0} & \text{if } \gamma_0 < 1/4, \\
\sqrt{1 + \ln(t_j/\tau_j)} \tau_j^{1-\gamma_0} & \text{if } \gamma_0 = 1/4, \\
\tau_j^{1-\gamma_0} & \text{if } \gamma_0 > 1/4,
\end{cases}
\]
and if \( m \geq 2 \), then set
\[
\epsilon_j := \tau_j^{1/2+\gamma_0}.
\]
Theorem 4.1 It holds that
\[
\left| U' \right|_{H^{\gamma_0}(0, t_j; L^2(\Omega))} + \| U(t_j) \|_{H^1_0(\Omega)} \\
\lesssim \| u_0 \|_{H^1_0(\Omega)} + t_j^{1/2 - \gamma_0} \| u_1 \|_{L^2(\Omega)} + \| f \|_{H^{-\gamma_0}(0, t_j; L^2(\Omega))}
\] (17)
for each \( 1 \leq j \leq J \).

Remark 4.3 Note that our space-time method (3) leads to a linear square system. Hence, the above theorem also implies the unique existence of \( U \).

Remark 4.4 We recall that \( H^{-\gamma_0}(0, t_j; L^2(\Omega)) \) is the dual space of \( H^{\gamma_0}(0, t_j; L^2(\Omega)) \), where \( H^{\gamma_0}(0, t_j; L^2(\Omega)) \) is endowed with the norm \( | \cdot |_{H^{\gamma_0}(0, t_j; L^2(\Omega))} \). Using the same technique as that used in the proof of Lemma 4.9, we easily derive that
\[
L^2_{2\gamma_0}(0, t_j; L^2(\Omega)) \subset H^{-\gamma_0}(0, t_j; L^2(\Omega)).
\]
This indicates that even if \( f \) has singularity at \( t = 0 \), problem (3) may also be stable. Moreover, since
\[
\| v \|_{H^{\gamma_0}(0, t_j; L^2(\Omega))} = \| v \|_{H^{\gamma_0}(0, T; L^2(\Omega))}
\]
for all \( v \in H^{\gamma_0}(0, T; L^2(\Omega)) \) such that \( v|_{(t_j, T)} = 0 \), we obtain
\[
\| f \|_{H^{-\gamma_0}(0, T; L^2(\Omega))} \leq \| f \|_{H^{-\gamma_0}(0, T; L^2(\Omega))}.
\]

Theorem 4.2 If \( u \in H^{1+\gamma_0}(0, T; H^1_0(\Omega) \cap H^2(\Omega)) \) and \( u'' \in L^1(0, T; H^2(\Omega)) \), then
\[
\| (u - U)(t_j) \|_{H^1_0(\Omega)} \lesssim \eta_{j,1} + \eta_{j,2} + \eta_{j,3} + \eta_{j,5},
\]
\[
\| (u - U)' \|_{L^2(0, t_j; L^2(\Omega))} \lesssim \eta_{j,4} + t_j^{\gamma_0}(\eta_{j,2} + \eta_{j,5}),
\]
for each \( 1 \leq j \leq J \), where
\[
\eta_{j,1} := \| (I - R_h)u(t_j) \|_{H^1_0(\Omega)},
\]
\[
\eta_{j,2} := \| (I - R_h)u' \|_{H^{\gamma_0}(0, t_j; L^2(\Omega))},
\]
\[
\eta_{j,3} := \epsilon_j \left( \sum_{i=1}^j \tau_i^{2-2\gamma_0} \inf_{0 \leq \delta < 1} \frac{t_i^{1-\delta}}{1 - \delta} \left\| (I - P_{\tau})R_h u' \right\|_{L^2_0(0, t_j; H^1_0(\Omega))} \right)^{1/2},
\]
\[
\eta_{j,4} := \tau_j^{\gamma_0} \left( \sum_{i=1}^j \tau_i^{2-2\gamma_0} \inf_{0 \leq \delta < 1} \frac{t_i^{1-\delta}}{1 - \delta} \left\| (I - P_{\tau})R_h u' \right\|_{L^2_0(I_i, L^2(\Omega))} \right)^{1/2},
\]
\[
\eta_{j,5} := \tau_j^{\min\{\gamma, m - \gamma_0\}} \left( \sum_{i=1}^j \tau_i^{2-2\gamma_0} \inf_{0 \leq \delta < 1} \frac{t_i^{1-\delta}}{1 - \delta} \left\| (I - P_{\tau})\Delta u' \right\|_{L^2_0(I_i, L^2(\Omega))} \right)^{1/2}.
\]
4.2.1 High regularity case

By Theorem 4.2, Lemma 4.1 and the standard estimate that [2]
\[ \| (I - R_h)v \|_{L^2(\Omega)} + h \| (I - R_h)v \|_{H_0^1(\Omega)} \lesssim h^{n+1} \| v \|_{H^{n+1}(\Omega)} \]
for all \( v \in H_0^1(\Omega) \cap H^{n+1}(\Omega) \), we readily conclude the following convergence estimates.

**Theorem 4.3** If
\[ u \in H^{m+1}(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^{1+\gamma_0}(0, T; H_0^1(\Omega) \cap H^{n+1}(\Omega)), \]
then
\[ \| (u - U)(t_j) \|_{H_0^1(\Omega)} \lesssim v_{j,1} + v_{j,3} + v_{j,4} + v_{j,5} \]
\[ \| (u - U)' \|_{L^2(0,t_j;L^2(\Omega))} \lesssim v_{j,2} + t_{j,0} (v_{j,3} + v_{j,5}) \]
for each \( 1 \leq j \leq J \), where
\[ v_{j,1} := h^n \| u(t_j) \|_{H^{n+1}(\Omega)}, \]
\[ v_{j,2} := \tau_j^m \| u \|_{H^{m+1}(0,t_j;H_0^1(\Omega))}, \]
\[ v_{j,3} := h^{n+1} \| u' \|_{H^{\gamma_0}(0,t_j;H^{n+1}(\Omega))}, \]
\[ v_{j,4} := \epsilon_j \tau_j^{m-\gamma_0} \| u \|_{H^{m+1}(0,t_j;H_0^1(\Omega))}, \]
\[ v_{j,5} := \tau_j^{\min\{\gamma,m-\gamma_0\}+m-\gamma_0} \| u \|_{H^{m+1}(0,t_j;H^2(\Omega))}. \]

**Remark 4.5** Under the assumption of the above theorem, we have the following results. The accuracy in \( H^1(0, T; L^2(\Omega)) \)-norm is optimal
\[ \| (u - U)' \|_{L^2(0,T;L^2(\Omega))} = O\left(h^{n+1}\right) + O\left(J^{-m}\right), \]
which is verified by numerical examples. If \( m = 1 \), then
\[ \max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H_0^1(\Omega)} = O(h^n) + \begin{cases} O\left(J^{-3/2}\right) & \text{if } 1 < \gamma < 3/2, \\ O\left(J^{\gamma-3}\right) & \text{if } 3/2 < \gamma < 2, \\ O\left(\sqrt{\ln J} J^{-3/2}\right) & \text{if } \gamma = 3/2, \end{cases} \]
and if \( m \geq 2 \), then
\[ \max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H_0^1(\Omega)} = O\left(h^n\right) + O\left(J^{-m-1/2}\right). \]
Hence, the spatial error in discrete \( L^\infty(0, T; H_0^1(\Omega)) \)-norm is still optimal, and this is validated by our numerical tests. However, from numerical examples, we see that
\[ \max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H_0^1(\Omega)} = O(h^n) + \begin{cases} O\left(J^{-m-1}\right) & \text{if } m \geq 2, \\ O\left(J^{-2}\right) & \text{if } m = 1, 1 < \gamma < 3/2, \\ O\left(J^{\gamma-3}\right) & \text{if } m = 1, 3/2 \leq \gamma < 2. \end{cases} \]
Therefore, ignoring the logarithm factor, our theoretical rate \(3 - \gamma\) is optimal when \(m = 1\) and \(3/2 \leq \gamma < 2\). In the rest cases, the theoretical rate \(m + 1/2\) is suboptimal.

**Remark 4.6** Note that when \(m = 1\), the proposed space-time finite element method coincides with the well-known L1 scheme, which was recently analyzed in [12, 18] for fractional wave problems (1) with nonsmooth data. Particularly, the accuracy \(3 - \gamma\) in discrete \(L^\infty(0, T; L^2(\Omega))\)-norm was derived in [12], under uniform temporal grid and the assumption \(\tau^\alpha/h_{\min}^2 < \infty\), where \(h_{\min}\) is the minimum diameter of the elements in the spatial triangulation. In view of this and our numerical results, we expect that the suboptimal rate \(3/2\) for \(m = 1\) and \(1 < \gamma < 3/2\) implied by Theorem 4.3 may be improved to \(3 - \gamma\) or even the optimal rate 2.

### 4.2.2 Singularity case

Let us first consider the following fractional ordinary problem:

\[
\begin{align*}
\frac{D_{0+}^\gamma}{t} (y - c_0 - tc_1) + \lambda y &= g \quad \text{in} \ (0, T), \\
y(0) &= c_0, \quad y'(0) = c_1,
\end{align*}
\]

where \(c_0, c_1 \in \mathbb{R}, \lambda \in \mathbb{R}_{>0}\), and \(g\) are given functions. It is well known that we can turn the above problem into the following integral form:

\[
y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} (g(s) - \lambda y(s)) \, ds, \quad 0 < t < T.
\]

Suppose that \(g\) is sufficiently smooth on \([0, T]\). It is clear that if \(g \neq \lambda c_0\), then \(y\) is dominated by

\[
g(0) - \lambda c_0 \frac{t^\gamma}{\Gamma(1 + \gamma)}
\]

near \(t = 0\). This motivates us to investigate the accuracy of \(U\) in the case that \(u\) is of the form

\[u(x, t) = t^r \phi(x), \quad (x, t) \in \Omega \times (0, T),\]

where \(\phi \in H^1_0(\Omega) \cap H^{n+1}(\Omega)\) and \(1 < r \leq m + 1/2\) with \(r \notin \mathbb{N}\).

To this end, let us introduce \(\varepsilon_\gamma\) as follows:

\[
\begin{align*}
\text{if } m &= 1, \quad \text{then } \varepsilon_\gamma := \begin{cases} 
J^{-\gamma/2} & \text{if } 1 < \gamma < 3/2, \\
J^{(\gamma-3)/2} & \text{if } 3/2 < \gamma < 2, \\
\sqrt{\ln J} J^{(\gamma-3)/2} & \text{if } \gamma = 3/2,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{if } m &\geq 2, \quad \text{then } \varepsilon_\gamma := J^{-\gamma/2}.
\end{align*}
\]

We also set

\[
\sigma^* := \frac{2m + 1 - \gamma}{2r - \gamma} \geq 1.
\]

By Theorem 4.2 and Lemma 4.4, we easily obtain the following error estimates.
Theorem 4.4 If $1 \leq \sigma < \sigma^*$, then
\[
\| (u - U) \|_{L^2(0,T;L^2(\Omega))} \leq C_1 \left( h^{n+1} + J^{\sigma(\gamma/2-r)-\gamma_0} \right),
\]
\[
\max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^1_0(\Omega)} \leq C_2 \left( h^n + \varepsilon \gamma J^{\sigma(\gamma/2-r)} \right).
\]
If $\sigma = \sigma^*$, then
\[
\| (u - U) \|_{L^2(0,T;L^2(\Omega))} \leq C_3 \left( h^{n+1} + \sqrt{\ln J} J^{-m} \right),
\]
\[
\max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^1_0(\Omega)} \leq C_4 \left( h^n + \varepsilon \gamma J^{\gamma_0-m} \right). \tag{20}
\]
Furthermore, if $\sigma > \sigma^*$, then
\[
\| (u - U) \|_{L^2(0,T;L^2(\Omega))} \leq C_5 \left( h^{n+1} + J^{-m} \right),
\]
\[
\max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^1_0(\Omega)} \leq C_6 \left( h^n + \varepsilon \gamma J^{\gamma_0-m} \right). \tag{21}
\]
Above $C_i \ (1 \leq i \leq 6)$ are positive constants that depend only on $m$, $n$, $\gamma$, $\sigma$, $r$, $\phi$, $T$, and the regularity of $K_r$.

Remark 4.7 Except the rates in (20) and (21) for $m \geq 2$, all the temporal accuracies given by Theorem 4.4 are verified by our numerical tests. Ignoring the logarithm factor, when $m \geq 2$ and $\sigma \geq \sigma^*$, we conclude from Theorem 4.4 that
\[
\max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^1_0(\Omega)} \leq C \left( h^n + J^{-m-1/2} \right).
\]
However, numerical results show the optimal rate
\[
\max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^1_0(\Omega)} \leq C \left( h^n + J^{-m-1} \right).
\]

4.3 Proofs of Theorem 4.1 and 4.2

Proof of Theorem 4.1 Inserting $V = U' \chi((0,t_j))$ into (3) yields
\[
\begin{align*}
& \left\{ D_{0+}^{\gamma'} \left( U - U(0) - t \Pi h u_1 \right), U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\{ \nabla U, \nabla U' \right\}_{\Omega \times (0,t_j)} \\
= & \left\{ f, U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\{ D_{0+}^{\gamma'} \Pi h u_1, U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))}.
\end{align*}
\]
Since
\[
D_{0+}^{\gamma'} (U - U(0) - t \Pi h u_1) = D_{0+}^{\gamma_0} (U' - \Pi h u_1),
\]
it follows that
\[
\begin{align*}
& \left\{ D_{0+}^{\gamma_0} \left( U' - \Pi h u_1 \right), U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\{ \nabla U, \nabla U' \right\}_{\Omega \times (0,t_j)} \\
= & \left\{ f, U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\{ D_{0+}^{\gamma'} \Pi h u_1, U' \right\}_{H^{\gamma_0}(0,t_j;L^2(\Omega))}.
\end{align*}
\]
In addition, using integration by parts gives
\[
2 \left\{ \nabla U, \nabla U' \right\}_{\Omega \times (0,t_j)} = \| U(t_j) \|_{H^1_0(\Omega)}^2 - \| U(0) \|_{H^1_0(\Omega)}^2.
\]
and Lemma A.2 implies
\[
\begin{align*}
\left\langle D_{0+}^{2\gamma_0} U', U^\prime \right\rangle_{H^{\gamma_0}(0,t_j;L^2(\Omega))} & \sim |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2, \\
\left\langle D_{0+}^{2\gamma_0} \Pi_h u_1, U^\prime \right\rangle_{H^{\gamma_0}(0,t_j;L^2(\Omega))} & \lesssim |\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}.
\end{align*}
\]
Consequently,
\[
\begin{align*}
|U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 & \lesssim \|U(0)\|_{H^0_0(\Omega)}^2 + \|f\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 + |\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} \\
& \lesssim \|U(0)\|_{H^0_0(\Omega)}^2 + \|f\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 + |\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}.
\end{align*}
\]
By Young’s inequality with \( \epsilon \), it follows that
\[
\begin{align*}
|U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 & \le \|U(0)\|_{H^0_0(\Omega)}^2 + \|f\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 + |\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} \\
& \lesssim \|U(0)\|_{H^0_0(\Omega)}^2 + \|f\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 + |\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} |U'|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}.
\end{align*}
\]
Therefore, (17) follows from the following evident estimates:
\[
\begin{align*}
\|U(0)\|_{H^0_0(\Omega)}^2 & = \|R_h u_0\|_{H^0_0(\Omega)}^2 \lesssim \|u_0\|_{H^0_0(\Omega)}^2, \\
|\Pi_h u_1|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} & \lesssim t_j^{1/2-\gamma_0} \|u_1\|_{L^2(\Omega)}.
\end{align*}
\]
This completes the proof.

To prove Theorem 4.2, let us first prove the following two lemmas.

**Lemma 4.8** If \( u \in H^{1+\gamma_0}(0,T;H^1_0(\Omega) \cap H^2(\Omega)) \) and \( u'' \in L^1(0,T;H^2(\Omega)) \), then
\[
\left| (U - Q_{\tau} R_h u)' \right|_{H^{\gamma_0}(0,t_j;L^2(\Omega))}^2 + \left\| (U - Q_{\tau} R_h u) (t_j) \right\|_{H^0_0(\Omega)}^2 \lesssim \eta_{j,2} + \eta_{j,5}
\]
for each \( 1 \le j \le J \), where \( \eta_{j,2} \) and \( \eta_{j,5} \) are defined as that in Theorem 4.2.

**Proof** Since
\[
D_{0+}^\gamma (u - U - (u - U)(0) - t(I - \Pi_h)u_1) = D_{0+}^{2\gamma_0} ((u - U)' - (I - \Pi_h)u_1),
\]
combining (2) and (3) yields
\[
\left\langle D_{0+}^{2\gamma_0} ((u - U)' - (I - \Pi_h)u_1), \theta \right\rangle_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\langle \nabla (u - U), \nabla \theta \right\rangle_{\Omega \times (0,t_j)} = 0,
\]
where \( \theta := U - Q_{\tau} R_h u. \) Then, as the definition of \( \Pi_h \) implies
\[
\left\langle D_{0+}^{2\gamma_0} (I - \Pi_h)u_1, \theta \right\rangle_{H^{\gamma_0}(0,t_j;L^2(\Omega))} = 0,
\]
we obtain
\[
\left\langle D_{0+}^{2\gamma_0} (u - U)', \theta \right\rangle_{H^{\gamma_0}(0,t_j;L^2(\Omega))} + \left\langle \nabla (u - U), \nabla \theta \right\rangle_{\Omega \times (0,t_j)} = 0.
\]
Therefore, a simple calculation gives
\[
\left\{ D_{0+}^{2γ_0}θ', 0' \right\}_{H^{γ_0}(0,t_j;L^2(Ω))} + \left\{ θ', θ' \right\}_{H^{γ}(0,t_j;L^2(Ω))} = E_1 + E_2,
\]
where
\[
E_1 := \left\{ D_{0+}^{2γ_0}(u - Q_τ R_h)u', θ' \right\}_{H^{γ_0}(0,t_j;L^2(Ω))},
\]
\[
E_2 := \left\{ θ', θ' \right\}_{H^{γ}(0,t_j;L^2(Ω))}.
\]
As the fact θ(0) = 0 implies
\[
2\left\{ θ', θ' \right\}_{Ω×(0,t_j)} = \left\| θ(t_j) \right\|_{H^1_0(Ω)},
\]
by Lemma A.2, we obtain
\[
\left| θ' \right|_{H^{γ_0}(0,t_j;L^2(Ω))} + \left\| θ(t_j) \right\|_{H^1_0(Ω)}^2 \lesssim E_1 + E_2. \tag{22}
\]
Next, let us estimate E_1 and E_2. As the definition of Q_τ indicates
\[
E_1 = \left\{ D_{0+}^{2γ_0}(u - R_h)u', θ' \right\}_{H^{γ_0}(0,t_j;L^2(Ω))},
\]
using Lemma A.2 yields
\[
E_1 \lesssim \left| (I - R_h)u' \right|_{H^{γ_0}(0,t_j;L^2(Ω))} \left| θ' \right|_{H^{γ_0}(0,t_j;L^2(Ω))}. \tag{23}
\]
By the definitions of R_h and Q_τ, a straightforward computing gives
\[
E_2 = \left\{ θ', θ' \right\}_{Ω×(0,t_j)} = -\left\{ Δu - Δ(Q_τ u), θ' \right\}_{Ω×(0,t_j)}
= -\left\{ (I - Q_τ)Δu, θ' \right\}_{Ω×(0,t_j)},
\]
so that Lemma 4.6 implies
\[
E_2 \lesssim \left\| (I - Q_τ)Δu \right\|_{H^{-γ_0}(0,t_j;L^2(Ω))} \left| θ' \right|_{H^{γ_0}(0,t_j;L^2(Ω))}.
\]
Finally, by Young’s inequality with ε, combining (22), (23), and (24) yields
\[
\left| θ' \right|_{H^{γ_0}(0,t_j;L^2(Ω))} + \left\| θ(t_j) \right\|_{H^1_0(Ω)} \lesssim \left| (I - R_h)u' \right|_{H^{γ_0}(0,t_j;L^2(Ω))} + \tau_j^{\min\{γ,m-γ_0\}} \left| (I - P_τ)Δu' \right|_{H^{γ_0}(0,t_j;L^2(Ω))}. \tag{24}
\]
Therefore, using Lemma 4.3 proves (4.8).

Lemma 4.9 If v ∈ H^{γ_0}(0,b) with 0 < b < ∞, then
\[
\| v \|_{L^2(0,b)} \leq C_γ b^{γ_0} | v |_{H^{γ_0}(0,b)}. \tag{25}
\]

Proof Extending v to ℝ \ (0,b) by zero, by Lemma 4.2, we have
\[
\int_{ℝ} \int_{ℝ} \frac{|v(s) - v(t)|^2}{|s - t|^{1 + 2γ_0}} ds dt \leq C_γ | v |_{H^{γ_0}(ℝ)}^2 = C_γ | v |_{H^{γ_0}(0,b)}^2.
\]
Since a simple calculation yields
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|v(s) - v(t)|^2}{|s - \tau|^{1+2\gamma_0}} \, ds \, dt = \frac{1}{\gamma_0} \int_{0}^{b} s^{-2\gamma_0} v^2(s) \, ds + \frac{1}{\gamma_0} \int_{0}^{b} (t - s)^{-2\gamma_0} v^2(s) \, ds + \int_{0}^{b} \int_{0}^{b} \frac{|v(s) - v(t)|^2}{|s - t|^{1+2\gamma_0}} \, ds \, dt,
\]
we obtain
\[
\int_{0}^{b} v^2(s) \, ds \leq b^{2\gamma_0} \int_{0}^{b} s^{-2\gamma_0} v^2(s) \, ds \leq C_{\gamma} b^{2\gamma_0} |v|_{H^{\gamma_0}(0,b)}^2.
\]
This proves (25) and thus completes the proof.

Proof of Theorem 4.2 As the proof of (18) is trivial by Lemmas 4.6, 4.7, and 4.8, we only prove (19). To do so, we set
\[
E_1 := \|(I - R_h)u'\|_{L^2(0,t_j;L^2(\Omega))},
\]
\[
E_2 := \|(U - Q_\tau R_hu)'\|_{L^2(0,t_j;L^2(\Omega))},
\]
\[
E_3 := \|((I - Q_\tau)R_hu)'\|_{L^2(0,t_j;L^2(\Omega))}.
\]
Since Lemma 4.9 implies
\[
E_1 \lesssim t_j^{\gamma_0} \|(I - R_h)u'\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} = t_j^{\gamma_0} \eta_{j,2},
\]
\[
E_2 \lesssim t_j^{\gamma_0} \|(U - Q_\tau R_hu)'\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))},
\]
by Lemma 4.8, we obtain
\[
E_1 + E_2 \lesssim t_j^{\gamma_0} (\eta_{j,2} + \eta_{j,s}).
\]
Also, by Lemmas 4.1 and 4.6,
\[
E_3 \lesssim t_j^{\gamma_0} \|((I - P_\tau)R_hu)'\|_{H^{\gamma_0}(0,t_j;L^2(\Omega))} \lesssim \eta_{j,4}.
\]
As a consequence,
\[
E_1 + E_2 + E_3 \lesssim \eta_{j,4} + t_j^{\gamma_0} (\eta_{j,2} + \eta_{j,s}).
\]
Therefore, (19) follows from the estimate
\[
\|(u - U)'\|_{L^2(0,t_j;L^2(\Omega))} \leq E_1 + E_2 + E_3.
\]
This completes the proof.

5 Numerical results

This section performs some numerical experiments in two-dimensional space to verify the theoretical results. We set \(\Omega := (0, 1)^2\), \(T := 1\), and
\[
u(x, t) := r^xy(1 - x)(1 - y), \quad (x, t) \in \Omega \times (0, T),
\]
Table 1 $\gamma = 1.5, r = 2, m = 2, \sigma = 1, J = 64$

| $1/h$ | $n = 1$ | Order | $\mathcal{E}_2(U)$ | Order | $n = 2$ | Order | $\mathcal{E}_2(U)$ | Order |
|-------|---------|--------|---------------------|--------|---------|--------|---------------------|--------|
| 8     | 1.59e−3 | –      | 3.02e−2             | –      | 3.68e−5 | –      | 2.11e−3             | –      |
| 16    | 4.03e−4 | 1.98   | 1.52e−2             | 0.99   | 4.59e−6 | 3.00   | 5.31e−4             | 1.99   |
| 32    | 1.01e−4 | 2.00   | 7.60e−3             | 1.00   | 5.73e−7 | 3.00   | 1.33e−4             | 2.00   |
| 64    | 2.53e−5 | 2.00   | 3.80e−3             | 1.00   | 7.17e−8 | 3.00   | 3.32e−5             | 2.00   |

where $r > 1$. In addition, we introduce the following notations:

$$\mathcal{E}_1(U) := \| (u - U)^\prime \|_{L^2(0,T;L^2(\Omega))}, \quad \mathcal{E}_2(U) := \max_{1 \leq j \leq J} \| (u - U)(t_j) \|_{H^0_0(\Omega)}.$$  

Experiment 1 This experiment verifies the spatial accuracy of $U$ in the case of $\gamma = 1.5$. To ensure that the spatial discretization is dominating, we set $r = 2$, $m = 2$, and $J = 64$. The numerical results in Table 1 illustrate $\mathcal{E}_1(U) = O(h^n + 1)$ and $\mathcal{E}_2(U) = O(h^n)$, which agrees well with Theorem 4.3.

Experiment 2 This experiment verifies the temporal accuracy indicated by Theorem 4.3. We set $n = 4$ and $h = 1/16$ to ensure that the spatial error is negligible. The numerical results displayed in Table 2 are summarized as follows.

- The accuracy $\mathcal{E}_1(U) = O(J^{-m})$ is well verified.

Table 2 $r = 3, n = 4, h = 1/16, \sigma = 1$

| $\gamma$ | $m = 1$ | $m = 2$ |
|----------|---------|---------|
| $J$      | $\mathcal{E}_1(U)$ | Order | $\mathcal{E}_2(U)$ | Order | $J$      | $\mathcal{E}_1(U)$ | Order | $\mathcal{E}_2(U)$ | Order |
| 1.2      | 16      | 2.08e−3 | –      | 2.51e−4 | –      | 16      | 2.97e−5 | –      | 6.50e−7 | –     |
| 32       | 1.04e−3 | 1.00   | 6.12e−5 | 2.03   | 32      | 7.48e−6 | 1.99   | 8.02e−8 | 3.02   |
| 64       | 5.21e−4 | 1.00   | 1.49e−5 | 2.04   | 64      | 1.88e−6 | 1.99   | 1.00e−8 | 3.00   |
| 128      | 2.60e−4 | 1.00   | 3.60e−6 | 2.05   | 128     | 4.70e−7 | 2.00   | 1.25e−9 | 3.01   |
| 1.5      | 128     | 2.60e−4 | –      | 7.72e−6 | –      | 16      | 3.42e−5 | –      | 1.71e−6 | –     |
| 256      | 1.30e−4 | 1.00   | 2.91e−6 | 1.41   | 32      | 8.59e−6 | 1.99   | 2.07e−7 | 3.05   |
| 512      | 6.51e−5 | 1.00   | 1.07e−6 | 1.44   | 64      | 2.15e−6 | 2.00   | 2.52e−8 | 3.04   |
| 1024     | 3.26e−5 | 1.00   | 3.90e−7 | 1.46   | 128     | 5.38e−7 | 2.00   | 3.08e−9 | 3.03   |
| 1.8      | 128     | 2.62e−4 | –      | 8.13e−5 | –      | 16      | 4.13e−5 | –      | 3.36e−6 | –     |
| 256      | 1.31e−4 | 1.00   | 3.60e−5 | 1.18   | 32      | 1.03e−5 | 2.00   | 4.05e−7 | 3.05   |
| 512      | 6.54e−5 | 1.00   | 1.58e−5 | 1.19   | 64      | 2.59e−6 | 2.00   | 4.89e−8 | 3.05   |
| 1024     | 3.27e−5 | 1.00   | 6.91e−6 | 1.19   | 128     | 6.47e−7 | 2.00   | 5.93e−9 | 3.05   |
\[ \gamma = 1.4, n = 4, h = 1/16 \]

| \( \sigma \) | \( m = 1, r = 1.1 \) | \( m = 2, r = 1.6 \) |
|---|---|---|
| 1 | | |
| 2(\( \sigma^* \)) | | |
| 2.1 | | |

- For \( m = 1 \), the accuracy \( \mathcal{E}_2(U) = \mathcal{O}(J^{-1.2}) \) in the case of \( \gamma = 1.8 \) is verified, the numerical results about \( \mathcal{E}_2(U) \) in the case of \( \gamma = 1.5 \) also agree with the theoretical accuracy \( \mathcal{O}(\sqrt{\ln JJ^{-3/2}}) \), but in the case of \( \gamma = 1.2 \), the numerical results illustrate \( \mathcal{E}_2(U) = \mathcal{O}(J^{-2}) \), which exceeds the theoretical accuracy \( \mathcal{O}(J^{-1.5}) \).
- For \( m = 2 \), the numerical results indicate that \( \mathcal{E}_2(U) = \mathcal{O}(J^{-3}) \), which exceeds the theoretical accuracy \( \mathcal{O}(J^{-2.5}) \).

\[ \gamma = 1.6, n = 4, h = 1/16 \]

| \( \sigma \) | \( m = 1, r = 1.15 \) | \( m = 2, r = 1.65 \) |
|---|---|---|
| 1 | | |
| 2(\( \sigma^* \)) | | |
| 2.1 | | |
Experiment 3 This experiment verifies the temporal accuracy implied by Theorem 4.4. Here we set \( n = 4, h = 1/16 \) so that the spatial error is sufficiently small. The numerical results are presented in Tables 3 and 4. Obviously, the numerical results verifies well that \( \mathcal{E}_1(U) = \mathcal{O}(J^{1/2-r}) \) for \( \sigma = 1 \) and that \( \mathcal{E}_1(U) = \mathcal{O}(J^{-m}) \) for \( \sigma \geq \sigma^* \). For \( m = 2 \), the accuracy \( \mathcal{E}_2(U) = \mathcal{O}(J^{-r}) \) in the case of \( \sigma = 1 \) is verified, but \( \mathcal{E}_2(U) = \mathcal{O}(J^{-3}) \) in the case of \( \sigma \geq \sigma^* \) is also observed, which exceeds the theoretical accuracy \( \mathcal{O}(J^{-2.5}) \).

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Appendix A: Properties of fractional calculus operators

**Lemma A.1** [4, 29, 31] Let \(-\infty < a < b < \infty\). If \( 0 < \alpha, \beta < \infty \), then

\[
I_{a}^{\alpha} I_{a}^{\beta} = I_{a}^{\alpha + \beta}, \quad I_{b}^{\alpha} I_{b}^{\beta} = I_{b}^{\alpha + \beta}.
\]

If \( 0 < \alpha < \beta < \infty \), then

\[
D_{a}^{\beta} I_{a}^{\alpha} = D_{a}^{\beta-\alpha}, \quad D_{b}^{\beta} I_{b}^{\alpha} = D_{b}^{\beta-\alpha}.
\]

**Lemma A.2** [5] Assume that \(-\infty < a < b < \infty \) and \( 0 < \alpha < 1/2 \). If \( v \in H^\alpha(a,b) \), then

\[
\| D_{a+}^\alpha v \|_{L^2(a,b)} \leq |v|_{H^\alpha(a,b)}, \quad \| D_{b-}^\alpha v \|_{L^2(a,b)} \leq |v|_{H^\alpha(a,b)},
\]

\[
\{ D_{a+}^\alpha v, D_{b-}^\alpha v \}_{(a,b)} = \cos(\alpha \pi) |v|_{H^\alpha(a,b)}^2,
\]

\[
\{ D_{a+}^\alpha v, D_{b-}^\alpha w \}_{(a,b)} \leq |v|_{H^\alpha(a,b)} |w|_{H^\alpha(a,b)}, \quad \forall w \in H^\alpha(a,b),
\]

\[
\left \{ D_{a+}^{2\alpha} v, w \right \}_{H^\alpha(a,b)} = \left \{ D_{a+}^\alpha v, D_{b-}^\alpha w \right \}_{(a,b)} = \left \{ D_{b-}^{2\alpha} w, v \right \}_{H^\alpha(a,b)}, \quad \forall w \in H^\alpha(a,b).
\]

**Lemma A.3** Suppose that \(-\infty < a < b < \infty \) and \( 0 < \alpha < 1/2 \). If \( v, w \in H^\alpha(a,b) \), then

\[
\{ D_{a+}^\alpha v, D_{b-}^\alpha w \}_{(a,b)} \leq |v|_{H^\alpha(a,b)} \|w\|_{H^\alpha(a,b)}.
\]

**Proof** By the definition of \( \|\| \|_{H^\alpha(a,b)} \), this lemma is a direct consequence of Lemma A.2. 

\( \square \)
Lemma A.4 If $\alpha \in [0, 1) \setminus \{0.5\}$ and $0 < \beta < \infty$, then
\[
\left\| I_{1-}^\beta v \right\|_{H^{\alpha+\beta}(0,1)} \leq C_{\alpha,\beta} \| v \|_{H^\alpha(0,1)} \tag{26}
\]
for all $v \in H_0^\alpha(0,1)$.

Proof The proof is a simple modification of that of [16, Lemma 5.7]. Let us first prove that
\[
\left\| I_{1-}^\beta w \right\|_{H^{\beta}(0,1)} \leq C_{\beta} \| w \|_{L^2(0,1)} \tag{27}
\]
for all $w \in L^2(0,1)$ and $0 < \beta < 1$. Extending $w$ to $\mathbb{R} \setminus (0,1)$ by zero, we define
\[
G(t) := \frac{1}{\Gamma(\beta/2)} \int_{t}^{\infty} (s-t)^{\beta/2-1} w(s) \, ds, \quad -\infty < t < \infty.
\]
Since $0 < \beta/2 < 1/2$, a routine calculation yields $G \in L^2(\mathbb{R})$, and then [31, Theorem 7.1] implies
\[
\|G\|_{H^{\beta/2}(\mathbb{R})} \leq C_{\beta} \| w \|_{L^2(0,1)},
\]
and hence
\[
\left\| I_{1-}^{\beta/2} w \right\|_{H^{\beta/2}(0,1)} \leq C_{\beta} \| w \|_{L^2(0,1)}. \tag{28}
\]
In addition, if $w \in H_0^1(0,1)$, then, since
\[
\text{Di}_{1-}^{\beta/2} w = -\text{Di}_{1-}^{\beta/2} I_{1-} w' = I_{1-}^{\beta/2} w',
\]
the estimate (28) implies
\[
\left\| I_{1-}^{\beta/2} w \right\|_{H^{1+\beta/2}(0,1)} \leq C_{\beta} \| w \|_{H_0^1(0,1)}.
\]
Consequently, [35, Lemma 22.3] yields
\[
\left\| I_{1-}^{\beta/2} w \right\|_{H^{\beta}(0,1)} \leq C_{\beta} \| w \|_{H_0^{\beta/2}(0,1)} \quad \text{for all } w \in H_0^{\beta/2}(0,1). \tag{29}
\]
Therefore, since $I_{1-}^{\beta} w = I_{1-}^{\beta/2} I_{1-}^{\beta/2} w$, combining (28) and (29) indicates that (27) holds for all $w \in L^2(0,1)$ and $0 < \beta < 1$.

Next, let us proceed to prove (26). Since the case of $\beta \in \mathbb{N}$ is trivial, we assume that $k < \beta < k+1$ with $k \in \mathbb{N}$, and so it suffices to prove
\[
\left\| I_{1-}^{\beta-k} v \right\|_{H^{\alpha+\beta-k}(0,1)} \leq C_{\alpha,\beta} \| v \|_{H^\alpha(0,1)}. \tag{30}
\]
Since we have already proved that (27) holds for all $w \in L^2(0,1)$ and $0 < \beta < 1$, we obtain
\[
\left\| I_{1-}^{\beta-k} w \right\|_{H^{\beta-k}(0,1)} \leq C_{\beta} \| w \|_{L^2(0,1)} \quad \text{for all } w \in L^2(0,1),
\]
\[
\left\| I_{1-}^{\beta-k} w \right\|_{H^{1+\beta-k}(0,1)} \leq C_{\beta} \| w \|_{H_0^1(0,1)} \quad \text{for all } w \in H_0^1(0,1).
Therefore, using \([35, \text{Lemma 22.3}]\) again proves (30) and thus concludes the proof of this lemma.

\[ \int_a^b \int_a^b |v(s) - v(t)|^2 |s - t|^{-1 - \alpha} \, ds \leq \frac{8b^\delta}{1 - \delta} (b - a)^{2 - \alpha} \|v'\|_{L^2_\delta(a, b)}^2. \]  

Moreover, if \(v(b) = 0\), then

\[ \int_a^b v^2(t)(t - a)^{-\alpha} \, dt \leq \frac{b^{\delta}(b - a)^{2 - \alpha}}{(1 - \delta)(1 - \alpha)} \|v'\|_{L^2_\delta(a, b)}^2, \]  

\[ \int_a^b v^2(t)(b - t)^{-\alpha} \, dt \leq \frac{b^{\delta}(b - a)^{2 - \alpha}}{(1 - \delta)(1 - \alpha)} \|v'\|_{L^2_\delta(a, b)}^2. \]

**Proof** Let us first establish (32). For \(a < t < b\), a simple computing gives

\[
|v(t)| = \left| \int_t^b v'(s) \, ds \right| \leq \int_t^b |v'(s)| \, ds \\
\leq \left( \int_t^b s^{-\delta} \, ds \right)^{\frac{1}{2}} \left( \int_t^b |v'(s)|^2 \, ds \right)^{\frac{1}{2}} \\
= \sqrt{\frac{b^{1 - \delta} - t^{1 - \delta}}{1 - \delta}} \|v'\|_{L^2_\delta(a, t)} \leq \sqrt{\frac{b^{\delta}(b - a)}{1 - \delta}} \|v'\|_{L^2_\delta(a, b)},
\]

so that we obtain

\[
\int_a^b v^2(t)(t - a)^{-\alpha} \, dt \leq \frac{b^{\delta}(b - a)}{1 - \delta} \int_a^b (t - a)^{-\alpha} \, dt \|v'\|_{L^2_\delta(a, b)}^2 \\
= \frac{b^{\delta}(b - a)^{2 - \alpha}}{(1 - \delta)(1 - \alpha)} \|v'\|_{L^2_\delta(a, b)}^2,
\]

namely, estimate (32). Similarly, we can derive (33) by that

\[
\int_a^b v^2(t)(b - t)^{-\alpha} \, dt \leq \frac{b^{\delta}(b - a)}{1 - \delta} \int_a^b (b - t)^{-\alpha} \, dt \|v'\|_{L^2_\delta(a, b)}^2 \\
= \frac{b^{\delta}(b - a)^{2 - \alpha}}{(1 - \delta)(1 - \alpha)} \|v'\|_{L^2_\delta(a, b)}^2.
\]
Then, let us prove (31). Since
\[
\int_a^b dt \int_t^b \left| v(s) - v(t) \right|^2 \left| s - t \right|^{-1-\alpha} ds
\]
\[
= 2 \int_a^b dt \int_t^b \left( \int_0^1 v'(t + \theta(s-t)) d\theta \right)^2 (s-t)^{-1-\alpha} ds
\]
\[
\leq 2(b-a)^{1-\alpha} \int_a^b dt \int_t^b \left( \int_0^1 v'(t + \theta(s-t)) d\theta \right)^2 ds,
\]
applying Minkowski’s integral inequality (cf. [8, Eq. 6.13.9]) yields that
\[
\int_a^b dt \int_t^b \left| v(s) - v(t) \right|^2 \left| s - t \right|^{-1-\alpha} ds
\]
\[
\leq 2(b-a)^{1-\alpha} \int_a^b dt \left( \int_0^1 \theta^{-1/2} \sqrt{\int_t^{t+\theta(b-t)} |v'(\eta)|^2 d\eta} d\theta \right)^2
g \leq 2(b-a)^{1-\alpha} \left( \int_0^1 \theta^{-1/2} \sqrt{\int_t^{t+\theta(b-t)} |v'(\eta)|^2 d\eta} d\theta \right)^2 dr.
\]
Finally, the inequality (31) is a direct consequence of
\[
\int_a^b \left( \int_0^1 \theta^{-1/2} \sqrt{\int_t^{t+\theta(b-t)} |v'(\eta)|^2 d\eta} d\theta \right)^2 dr
\]
\[
\leq \int_a^b \left( \int_0^1 \theta^{-1/2} \sqrt{\int_t^{t+\theta(b-t)} (\eta/\delta |v'(\eta)|^2 d\eta} d\theta \right)^2 dr
\]
\[
= \int_a^b t^{-\delta} \left( \int_0^1 \theta^{-1/2} \sqrt{\int_t^{t+\theta(b-t)} \eta^\delta |v'(\eta)|^2 d\eta} d\theta \right)^2 dr
\]
\[
\leq \int_a^b t^{-\delta} \left( \int_0^1 \theta^{-1/2} d\theta \right)^2 dr \left\| v' \right\|_{L^2_0(a,b)}^2
\]
\[
\leq \frac{4b^{-\delta}(b-a)}{1-\delta} \left\| v' \right\|_{L^2_0(a,b)}^2.
\]
This lemma is thus proved. \(\square\)
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