EQUILIBRIUM AND EQUIVARIANT TRIANGULATIONS OF SOME
SMALL COVERS WITH MINIMUM NUMBER OF VERTICES

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Abstract. Small covers were introduced by Davis and Januszkiewicz in 1991. We introduce
the notion of equilibrium triangulations for small covers. We study equilibrium and ver-
tex minimal $Z^2$-equivariant triangulations of 2-dimensional small covers. We discuss vertex
minimal equilibrium triangulations of $\mathbb{R}P^3 \# \mathbb{R}P^3$, $S^1 \times \mathbb{R}P^2$ and a nontrivial $S^1$ bundle over
$\mathbb{R}P^2$. We construct some nice equilibrium triangulations of the real projective space $\mathbb{R}P^n$ with
$2^n + n + 1$ vertices. The main tool is the theory of small covers.

1. Introduction

Small covers were introduced by Davis and Januszkiewicz in [5], where an $n$-dimensional
small cover is a smooth closed manifold $N^n$ with a locally standard $Z_2^n$-action such that
the orbit space of this action is a simple convex polytope. This gives a nice connection
between topology and combinatorics. A broad class of examples of small covers are all real
projective varieties. Illman’s results [8] on equivariant triangulations on smooth $G$-manifolds
for a finite group $G$ ensure the existence of $Z_2^n$-equivariant triangulations of $n$-dimensional
small covers. Then the following natural question can be asked. Does each $Z_2^n$-equivariant
triangulation of an $n$-dimensional small cover with the orbit space $Q$ induce a triangulation of
$Q$? We give the answer of this question when $n = 2$, see Theorem 3.10. It seems that no
explicit equivariant triangulations are studied for small covers. On the other hand the term
“equilibrium triangulation” first appeared in [1]. They discuss equilibrium triangulations and
simplicial tight embeddings for $\mathbb{R}P^2$ and $\mathbb{C}P^2$. Inspired by the work of [1] and [8], in this
article we construct equilibrium and equivariant triangulations of small covers explicitly with
few vertices. The main results of this article are Theorems 3.10, 3.11, 3.14, 3.18, 3.20, 3.22
and 3.23. Main tool in this article is the theory of small covers.

The arrangement of this article is as follows. We recall the definitions and some basic
properties of small covers following [5] in Subsection 2.1. In Subsection 2.2 we review the def-
initions and some results on triangulation of manifolds following the survey [4]. The definition
of equilibrium triangulations of small covers is introduced in the subsection 3.1 following an
idea of [1]. Some $Z_2^n$-equivariant triangulations of an $n$-dimensional small cover are constructed
in Subsection 3.3. These simple looking triangulations lead to some open problems, see Sub-
section 3.4. We discuss some vertex minimal $Z_2^n$-equivariant triangulations of 2-dimensional
small covers in Subsection 3.3. Subsection 3.4 gives some natural equilibrium triangulations
of 3-dimensional small covers. In this subsection, we study vertex minimal equilibrium tri-
angulations of $\mathbb{R}P^3 \# \mathbb{R}P^3$, a nontrivial $S^1$ bundle over $\mathbb{R}P^2$ and $S^1 \times \mathbb{R}P^2$, see Theorem 3.18,
3.20 and 3.22 respectively. Subsection 3.5 gives some explicit nice equilibrium triangulations
of real projective space $\mathbb{R}P^n$ with $2^n + n + 1$ vertices.

2. Preliminaries
2.1. Small covers. Following [3], we discuss some basic results about small covers. The codimension one faces of a convex polytope are called facets. An n-dimensional simple polytope in \( \mathbb{R}^n \) is a convex polytope if each vertex is the intersection of exactly n facets. We denote the underlying additive group of the vector space \( \mathbb{R}^n \) by \( \mathbb{Z}_2^n \). Let \( \rho : \mathbb{Z}_2^n \times \mathbb{R}^n \to \mathbb{R}^n \) be the standard action. Let \( N \) be an n-dimensional manifold.

**Definition 2.1.** An action \( \eta : \mathbb{Z}_2^n \times N \to N \) is said to be locally standard if the following holds. (1) Every point \( y \in N \) has a \( \mathbb{Z}_2^n \)-stable open neighborhood \( U_y \). (2) There exists a homeomorphism \( \psi : U_y \to V \), where \( V \) is a \( \mathbb{Z}_2^n \)-stable open subset of \( \mathbb{R}^n \). (3) There exists an isomorphism \( \delta_y : \mathbb{Z}_2^n \to \mathbb{Z}_2^n \) such that \( \psi(\eta(t, x)) = \rho(\delta_y(t), \psi(x)) \) for all \( (t, x) \in \mathbb{Z}_2^n \times U_y \).

**Definition 2.2.** A closed n-dimensional manifold \( N \) is said to be a small cover if there is an effective \( \mathbb{Z}_2^n \)-action on \( N \) such that: (1) the action is a locally standard action, (2) the orbit space of the action is a simple polytope (possibly diffeomorphic as manifold with corners to a simple polytope).

Let \( N \) be a small cover and \( \xi : N \to Q \) be the orbit map. We say that \( N \) is a small cover over \( Q \) or \( \xi : N \to Q \) is a small cover over \( Q \).

**Example 2.3.** The natural action of \( \mathbb{Z}_2^n \) defined on the real projective space \( \mathbb{RP}^n \) by

\[
(1) \quad (g_1, \ldots, g_n) \cdot [x_0, x_1, \ldots, x_n] \to [x_0, (-1)^{g_1} x_1, \ldots, (-1)^{g_n} x_n]
\]

is locally standard and the orbit space is diffeomorphic as manifold with corners to the standard n-simplex. Hence \( \mathbb{RP}^n \) is a small cover over the n-simplex \( \Delta^n \).

**Remark 2.4.** The equivariant connected sum of n-dimensional finitely many small covers is also a small cover of same dimension. Details can be found in page-424 of [3].

**Definition 2.5.** Let \( \mathcal{F}(Q) = \{F_1, \ldots, F_m\} \) be the set of facets of a simple n-polytope \( Q \). A function \( \beta : \mathcal{F}(Q) \to \mathbb{F}_2^n \) is called a \( \mathbb{Z}_2 \)-characteristic function on \( Q \) if the span of \( \{\beta(F_j_1), \ldots, \beta(F_j_l)\} \) is an l-dimensional subspace of \( \mathbb{F}_2^n \) whenever the intersection of the facets \( \{F_j_1, \ldots, F_j_l\} \) is nonempty. The vectors \( \beta_j := \beta(F_j) \) are called \( \mathbb{Z}_2 \)-characteristic vectors and the pair \((Q, \beta)\) is called \( \mathbb{Z}_2 \)-characteristic pair.

Note that given a small cover, we can define a \( \mathbb{Z}_2 \)-characteristic pair, see Section 1 in [3]. Let \((Q, \beta)\) be a \( \mathbb{Z}_2 \)-characteristic pair where \( Q \) is a simple n-polytope with facets \( F_1, \ldots, F_m \). Let \( G_F \) be the subgroup of \( \mathbb{Z}_2^n \) generated by \( \{\beta_j_1, \ldots, \beta_j_l\} \), whenever \( F = F_j_1 \cap \ldots \cap F_j_l \).

Define an equivalence relation \( \sim \) on \( \mathbb{Z}_2^n \times Q \) by

\[
(2) \quad (t, p) \sim (s, q) \text{ if } p = q \text{ and } s - t \in G_F,
\]

where \( F \subseteq Q \) is the unique face whose relative interior contains \( p \). We denote the equivalence class of \((t, p)\) by \([((t, p)]^{-}\). The quotient space \( N(Q, \beta) = (\mathbb{Z}_2^n \times Q)/\sim \) is an n-dimensional small cover, see Section 1 in [3] for more details. Following Proposition gives a classification of small covers.

**Proposition 2.6** (Proposition 1.8, [3]). Let \( \xi : N \to Q \) be a small cover over \( Q \) and the function \( \beta : \mathcal{F}(Q) \to \mathbb{F}_2^n \) be its \( \mathbb{Z}_2 \)-characteristic function. Let \( \xi_\beta : N(Q, \beta) \to Q \) be the constructed small cover from the \( \mathbb{Z}_2 \)-characteristic pair \((Q, \beta)\). Then there exists an equivariant homeomorphism from \( N \) to \( N(Q, \beta) \) covering the identity over \( Q \). Hence a small cover \( \xi : N \to Q \) is determined up to equivalence over \( Q \) by its \( \mathbb{Z}_2 \)-characteristic function.

The cubical subdivision of a simple polytope is explicitly discussed in Section 4.2 of [3], a good reference for many interesting developments and applications. We denote a cubical subdivision of the polytope \( Q \) by \( C(Q) \). We consider the vertex in \( C(Q) \) corresponding to \( Q \) is the center of mass of \( Q \).
2.2. Triangulation of manifolds. We recall some basic definitions for triangulations of manifolds following [10]. A compact convex polyhedron which spans a subspace of dimension \( n \) is called an \( n \)-cell. So we can define faces of an \( n \)-cell. A cell complex \( X \) is a finite collection of \( n \)-cells in some \( \mathbb{R}^n \) satisfying, (i) if \( B \) is a face of \( A \) and \( A \in X \) then \( B \in X \), (ii) if \( A, B \in X \) and \( A \cap B \) is nonempty then \( A \cap B \) is a face of both \( A \) and \( B \). Zero dimensional cells of \( X \) are called vertices of \( X \). A cell complex \( X \) is simplicial if each \( A \in X \) is a simplex. We may denote a cell \( \sigma \) with vertices \( v_1, \ldots, v_k \) by \( v_1 v_2 \ldots v_k \). The vertex set of a simplicial complex \( X \) is denoted by \( V(X) \). The union of all cells in a simplicial complex \( X \) is called the geometric carrier of \( X \) which is denoted by \( |X| \). If a Hausdorff topological space \( M \) is homeomorphic to \( |X| \), the geometric carrier of a simplicial complex, then we say that \( X \) is a triangulation of \( M \). If \( |X| \) is a topological \( d \)-ball (respectively, \( d \)-sphere) then \( X \) is called a triangulated \( d \)-ball (resp., triangulated \( d \)-sphere). If \( X, Y \) are two simplicial complexes, then a simplicial map from \( X \) to \( Y \) is a continuous map \( \eta : |X| \to |Y| \) such that for each \( \sigma \in X \), \( \eta|_{\sigma} \) is a cell of \( Y \) and \( \eta|_{\sigma} \) is linear. By a simplicial subdivision of a cell complex \( X \) we mean a simplicial complex \( X' \) together with a homeomorphism from \( |X'| \) onto \( |X| \) which is face-wise linear. From the following proposition we can construct a simplicial complex from a cell complex.

**Proposition 2.7** (2.9 Proposition, [11]). A cell complex can be subdivided to a simplicial complex without introducing any new vertices.

**Definition 2.8.** Let \( G \) be a finite group. A \( G \)-equivariant triangulation of the \( G \)-space \( N \) is a triangulation \( X \) of \( N \) such that if \( \sigma \in X \) then \( f_g(\sigma) \in X \) for all \( g \in G \), where \( f_g \) is the homeomorphism corresponding to the action of \( g \) on \( |X| \).

3. Equilibrium and equivariant triangulations of small covers

3.1. Definition of equilibrium Triangulation. We generalize the definitions of equilibrium set and zones of influence for small covers following the definition of equilibrium triangulation for \( \mathbb{R}P^2 \) and \( \mathbb{C}P^2 \) of [11]. These definitions are also generalized for quasitoric manifolds in [12]. Let \( \xi : N \to Q \) be an \( n \)-dimensional small cover over the \( n \)-dimensional simple polytope \( Q \).

**Definition 3.1.** The equilibrium set of an \( n \)-dimensional small cover \( N \) is defined by the orbit at \( \bar{x} \in N \) such that \( \xi(\bar{x}) \) is the center of mass of \( Q \).

**Definition 3.2.** A zones of influence of an \( n \)-dimensional small cover \( N \) is defined by a \( \mathbb{Z}_2^n \) invariant closed subset such that (1) it contains the equilibrium set, (2) it is \( \mathbb{Z}_2^n \)-equivariantly homeomorphic to a \( \mathbb{Z}_2^n \)-invariant closed ball \( B^n \subset \mathbb{R}^n \).

**Definition 3.3.** A collection \( \{Z_1, \ldots, Z_k\} \) of zones of influences of an \( n \)-dimensional small cover \( N \) is called complete if the collection \( \{\xi(Z_{i_1}) \cap \ldots \cap \xi(Z_{i_l}) : \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}\} \) gives a cubical subdivision of \( Q \).

So, number of vertices in \( Q \) is \( k \). If \( C(Q) \) is a cubical subdivision of \( Q \) with \( n \)-dimensional cells \( I_i \) for \( 1 \leq i \leq k \) then the collection \( \{\xi^{-1}(I_i) : i = 1, \ldots, k\} \) of zones of influences is a complete.

**Definition 3.4.** A triangulation of \( n \)-dimensional small cover \( N \) is said to be an equilibrium triangulation if the equilibrium set and all the zones of influences of a complete collection are triangulated submanifolds of \( N \).

Any equilibrium triangulation of an \( n \)-dimensional small cover \( N \) contains at least \( 2^n \) vertices. We study some equilibrium and equivariant triangulations of small covers in the following subsections.
3.2. Some triangulations of small covers. Let \( \xi : N \to Q \) be a small cover over the \( n \)-dimensional simple polytope \( Q \). Let \( Q' \) be the first barycentric subdivision of \( Q \). Since \( Q \) is a polytope, \( Q' \) is a triangulation of \( Q \). Let \( \alpha \) be an \( l \)-dimensional simplex in \( Q' \) and \( \alpha^0 \) be the relative interior of \( \alpha \). From the equivalence relation ~ in the Equation (2), it is clear that \( \xi^{-1}(\alpha^0) \) is a disjoint union of \( \sigma_1(\alpha)^0, \ldots, \sigma_{2^k}(\alpha)^0 \) in \( N \) where \( \alpha \) belongs to the smallest face \( F^k \) of \( Q \) of dimension \( k \) and \( \sigma_i(\alpha) \) is the closure of \( \sigma_i(\alpha)^0 \) in \( N \). The restriction of \( \xi \) on each \( \sigma_i(\alpha)^0 \) is a diffeomorphism to \( \alpha^0 \). Note that small covers have smooth structure, (cf. [3]). The subset \( \sigma_i(\alpha) \) of \( N \) is diffeomorphic as manifold with corners to an \( l \)-dimensional simplex. Clearly the collection

\[
\Sigma(N) = \{ \sigma_i(\alpha) : \alpha \in Q', \alpha \subset F^k \text{ and } i = 1, \ldots, 2^k \}
\]
gives a triangulation of \( N \). Here \( k \) also depends on \( \alpha \). Note that \( \Sigma(N) \) is an equivariant triangulation of \( N \) with respect to the \( \mathbb{Z}^n_2 \) action. Also this is an equilibrium triangulation of \( N \). With the triangulations \( Q' \) and \( \Sigma(N) \) of \( Q \) and \( N \) respectively the projection map \( \xi \) is simplicial. Hence we get the following.

Lemma 3.5. A small cover \( N \) over \( Q \) where \( f(Q) = (f_0, f_1, \ldots, f_{n-1}) \) has a \( \mathbb{Z}^n_2 \)-equivariant and equilibrium triangulation with \( 2^n + f_02^{n-1} + f_12^{n-2} + \cdots + f_{n-1} \) vertices. Moreover, Euler characteristic of all small covers over \( Q \) are same.

Remark 3.6. Let \( K \) be a contracted pseudotriangulation (crystallization) of an \( n \)-manifold \( M \) with vertex set \( \{v_1, \ldots, v_{n+1}\} \) and with \( f_i(K) \) \( i \)-cells for \( 1 \leq i \leq n \) (for more on Crystallizations and contracted pseudotriangulations of manifolds see [3, 4]). Let \( \sigma^n \) be a simplex with vertex set \( \{1, \ldots, n+1\} \). For any set \( \{v_{i_0}, \ldots, v_{i_j}\} \) of vertices, number of \( j \)-faces in \( K \) whose vertices are \( v_{i_0}, \ldots, v_{i_j} \) is in general more than one and the map \( \varphi \) given by \( v_i \mapsto i \) is not a piece wise linear (pl) map. To make it a pl we have to first subdivide \( K \) to a simplicial complex.

Clearly, we have to add at least \( \Sigma^i_{i=1}(f_i(K) - (i+1)) \) new vertices to make the subdivision a simplicial complex (if \( \alpha_1, \ldots, \alpha_k \) are \( i \)-simplices with same vertex set then we add one vertex in each \( \alpha_2, \ldots, \alpha_k \)). To make the map \( \varphi \) simplicial, we need to add \( \Sigma^i_{i=1}f_i(K) \) vertices. So, the new simplicial complex \( \tilde{K} \) (which is a subdivision of \( K \)) has \( (n+1) + \Sigma^i_{i=1}f_i(K) \) vertices (and \( 2^{n+1} + 2 \) vertices in the subdivision \( \tilde{\sigma} \) of \( \sigma^n \)). This simplicial map \( \varphi : \tilde{K} \to \tilde{\sigma} \) is also a branched covering with quotient polytope a simplex. We have considered such cases in Lemma 3.5 where the quotient polytopes are in general simple polytopes. Our main results (cf. Theorems 3.13, 3.20, 3.22, 3.23) show that we can have branched coverings over simple polytopes with much less number of additional vertices on the manifolds, namely, we add only a vertex in the interior of each \( n \)-simplex.

Observing the above natural triangulations of small covers we may ask the following.

Question 3.1. Let \( \xi : N \to Q \) be a small cover over the \( n \)-dimensional simple polytope \( Q \). What are the vertex minimal equilibrium triangulations of \( N \)?

Question 3.2. Let \( \xi : N \to Q \) be a small cover over the \( n \)-dimensional simple polytope \( Q \). Can we describe all \( \mathbb{Z}^n_2 \)-equivariant triangulations of \( N \) with minimum vertices?

3.3. Triangulations of 2-dimensional small covers. In this subsection, we construct some equilibrium and equivariant triangulations of 2-dimensional small covers with few vertices. We calculate the number of vertices and hence faces of these triangulations. Let \( Q \) be a polygon with vertices \( V_1, \ldots, V_m \) and edges \( F_1, \ldots, F_m \), where the end points of \( F_i \) are \( V_i \) and \( V_{i+1} \) for \( i = 1, \ldots, m - 1 \) and the end points of \( F_m \) are \( V_m \) and \( V_1 \). Let \( \xi : N \to Q \) be a small cover over \( Q \). We denote the fixed point corresponding to the vertex \( V_i \) by the same. Let \( \beta : \{F_1, \ldots, F_m\} \to \mathbb{Z}_2^2 \) be the \( \mathbb{Z}_2 \)-characteristic function of \( N \). Let \( \beta_{k_1} = \cdots = \beta_{k_{k_1}} = (1,0) \), \( \beta_{j_1} = \cdots = \beta_{j_{k_2}} = (0,1) \) and \( \beta_{l_1} = \cdots = \beta_{l_{k_3}} = (1,1) \). So \( k_1 + k_2 + k_3 = m, k_i \geq 0 \) and at most one \( k_1 \) is zero.
Lemma 3.7. There is a $\mathbb{Z}^2$-equivariant and equilibrium triangulation of 2-dimensional small cover $N$ (over $m$-polygon $Q$) with $2m + 4$ vertices. Euler characteristic of $N$ is $4 - m$.

Proof. Let $C(Q)$ be the cubical subdivision of $Q$. Let $C_F$ be the vertex in $C(Q)$ corresponding to the nonempty face $F$ of $Q$. Joining the vertices $C_{F_{i-1}}$ and $C_{F_i}$ in $C(Q)$ for $i = 1, \ldots, m$, we get triangulation $Y$ of $Q$. Note that $N$ is equivariantly diffeomorphic to $N(Q, \beta) := (\mathbb{Z}^2 \times Q)/\sim$ where $\sim$ is defined in the Equation (4). Let $\xi^{-1}(C_Q) = \{C_{ab} \in N : (a, b) \in \mathbb{Z}^2\}$ and $\xi^{-1}(C_{F_i}) = \{C_i', C''_i\}$. Then $\xi^{-1}(C_{F_i}C_{F_{i-1}})$ is the square with vertices $C_i', C_{i-1}', C''_i, C''_{i-1}$ for $i = 1, \ldots, m$. We consider the collection $\Sigma(N) = \{[C_i', C_{i-1}', C''_i, C''_{i-1} : i = 1, \ldots, m] \cup \{[C_{00}, C_i', C''_{i-1}], [C_i, C''_i, C_{i-1}'], [C_{i-1}, C_i', C''_i] : i = 1, \ldots, m\}$. Observe that this gives a $\mathbb{Z}^2$-equivariant triangulation of $N$. Clearly, this is also an equilibrium triangulation of $N$. The face vector of this triangulation is $(2m + 4, 9m, 6m)$. So Euler characteristic of $N$ is $4 - m$. □

Lemma 3.8. If $k_i = 1$ for some $i$ then there is a $\mathbb{Z}^2$-equivariant triangulation of $N$ with $2m$ vertices.

Proof. We may assume $k_1 = 1$. Let $\xi : N \to Q$ be the orbit map, where $Q$ is an $m$-polygon with vertices $V_1, \ldots, V_m$ and edges $F_1, \ldots, F_m$. Let $C_i$ be the middle point of the edge $F_i$ for $i = 1, \ldots, m$. We consider the collection $Y = \{[V_i, C_i, C_{i-1}], [C_1, C_i, C_{i+1}] : i = 1, \ldots, m\}$ of 2-cells in $Q$ with $C_{m+1} = c_0$. Clearly it gives a triangulation of $Q$. Since $k_1 = 1$, with this triangulation of $Q$, similarly as in Lemma 3.7, we can construct a $\mathbb{Z}^2$-equivariant triangulation of $N$ which does not contain any fixed points. Note that this is not an equilibrium triangulation of $N$. □

Example 3.9. We discuss some $\mathbb{Z}^2$-equivariant triangulations of 2-disc $D^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. Let $g_1, g_2$ be the standard generator of $\mathbb{Z}^2$ and $g_i$ acts by reflection along $\{(x_1, x_2) : x_i = 0\}$ for $i = 1, 2$. Following Figure 1 give all possible $\mathbb{Z}^2$-equivariant triangulations of $D^2$ with 4 and 5 vertices. Clearly, any $\mathbb{Z}^2$-equivariant triangulation not containing the fixed point $(0, 0) \in D^2$ contains exactly 4 vertices and the triangulations are given by Figure 1 (1) and (2).

![Figure 1. $\mathbb{Z}^2$-equivariant triangulations of $D^2$ with 4 and 5 vertices.](image)

Theorem 3.10. Let $X$ be a $\mathbb{Z}^2$-equivariant triangulation of the small cover $\xi : N \to Q$ over the polygon $Q$. Then $\{\xi(\sigma) : \sigma \in X\}$ is the set of 2-simplices in a triangulation of $Q$.

Proof. First we show that if $\sigma$ is a 2-cell in $X$ then $\xi(\sigma)$ is a triangle (possibly diffeomorphic to a triangle). This is true if $\sigma \cap \xi^{-1}(F_j)$ is empty for $i = 1, \ldots, m$. From the discussion in example 3.9 it is clear that if $\sigma$ contains a fixed point then $\xi(\sigma)$ is a triangle. Suppose $\sigma \cap \xi^{-1}(F_j) \setminus \{V_{ij}, V_{ij+1}\}$ is nonempty for some $j = 1, \ldots, k$. Observe that $\xi^{-1}(F_j) \setminus \{V_{ij}, V_{ij+1}\}$ is disjoint union of two open arcs, namely $C_{j1}$ and $C_{j2}$, in $N$ for $j = 1, \ldots, k$. Then both $\sigma \cap C_{j1}$ and $\sigma \cap C_{j2}$ cannot be nonempty. Otherwise there will be an edge $e_j$ of $\sigma$ such that $e_j \cap C_{j1}$ and $e_j \cap C_{j2}$ are nonempty. Since $e_j$ does not contain any fixed point and $X$ is $\mathbb{Z}^2$-equivariant, we can show $e_j \cap C_{j1}$ and $e_j \cap C_{j2}$ are the vertices of $e_j$. This makes a 2-vertex triangulation of a circle. Now assume $\sigma \cap C_{j1}$ is nonempty. So either $\sigma \cap C_{j1}$ is a point or a closed interval.
Suppose $\sigma \cap C_{1_1}$ is a point. Then it is a vertex $v_1$ of $\sigma$. Let $v_2$ and $v_3$ be the other two vertices of $\sigma$. Since $\sigma$ does not contain any fixed point, $\sigma - \{v_2, v_3 \cup v_1\}$ is a subset of a connected component of the principal fibration of $N$. So $\xi(\sigma)$ is a triangle in $Q$.

Suppose $\sigma \cap C_{1_1}$ is a closed interval $[a, b]$. Then both the sides of $C_{1_1}$ contain some points of $\sigma$. Since $X$ is $\mathbb{Z}_2^3$-equivariant triangulation, at least one of $a, b$ belongs to $V(\sigma)$. Let both $a, b$ belong to $V(\sigma)$ and $c$ be the third vertex of $\sigma$. Then by equivariance condition on $X$ we get $[a, b] \subset C_{1_1}$ and $\sigma - \{c\}$ belongs to a connected component of the principal fibration. So $\xi(\sigma)$ is a triangle in $Q$.

Let $a$ belongs to $V(\sigma)$ and $c, d$ be the other two vertices of $\sigma$. Then by equivariance condition on $X$ we get $\xi_1(c) = d$ and $\beta_i([a, c, d]) = [a, c, d]$, where $\beta_i$ is the $\mathbb{Z}_2$-characteristic vector associated to the facet $F_i$. Then $\xi([a, c, d]) = \xi[a, b, c]$. By previous arguments on $[a, b, c]$, $\xi(\sigma)$ is a triangle in $Q$. We draw some pictures for the above situations in Figure 3.11.

Since the action is locally standard, for any two 2-cells we have $\xi(\sigma_1) \cap \xi(\sigma_2) = \xi(\sigma_1 \cap \sigma_2)$ and $\xi(\sigma_1 \cap \sigma_2)$ is diffeomorphic to a closed interval. Hence we get the theorem. □

**Theorem 3.11.** Any $\mathbb{Z}_2^3$-equivariant triangulation of a 2-dimensional small cover $N$ over $m$-polygon $Q$ contains at least $2m$ vertices.

**Proof.** Let $X$ be a $\mathbb{Z}_2^3$-equivariant triangulation of the small cover $N$ over $m$-polygon $Q$. Let $V_1, \ldots, V_m$ be the vertices and $F_1, \ldots, F_m$ be the facets of $Q$. Let $[a, b, c]$ be a 2-cell in $X$ such that $V_i \in [a, b, c]$ for some $i \in \{1, \ldots, m\}$. Our claim is that $[a, b, c] \cap \{\cup_{i \notin F_j} \xi^{-1}(F_j)\}$ is empty. From the discussion in Example 3.3, either $V_i$ is a vertex or $V_i$ is an interior point of an edge of $[a, b, c]$. Suppose $V_i = a$. If $[a, b, c] \cap \xi^{-1}(F_j)$ is nonempty with $V_i \notin F_j$ for some $j$, $\xi^{-1}(F_j)$ contains one of $b, c$, which contradicts the equivariant condition on $X$. Now assume $V_i$ belongs to the interior of $[a, b]$. Then $[a, b, c]$ must be similar to one of the 2-cells in the Figure (1) and (2). So $[a, b, c] \subset \xi^{-1}(F_i) \cup \xi^{-1}(F_{i-1})$. This proves the claim. Since $X$ is equivariant, $\{\xi(\sigma) : \sigma \in X \text{ is a 2 cell}\}$ is a set of 2-cells in a triangulation $Y$ of $Q$ (by Theorem 3.11). Now we assign two distinct vertices $x_{i_1}, x_{i_2} \in V(X)$ to the edge $F_i$ for each $i \in \{1, \ldots, m\}$ such that all $\xi_{i_j}$’s are distinct. We do it step by step in following way.

**Step 1:** First consider the edge $F_1$. Let $\tau_1$ be a 2-cell in $Y$ such that $V_1 \in \tau_1$ and $\tau_1 \cap F_1 = [V_1, y_1]$ is an edge of $\tau_1$. Let $\xi(\sigma_1) = \tau_1$ and $y_2$ be the third vertex of $\tau_1$. If $\sigma_1 \cap \xi^{-1}(y_1)$ is nonempty then it is a vertex of $\sigma_1$ and we choose $\{x_{1_1}, x_{1_2}\} = \xi^{-1}(y_2)$ for $F_1$. If $\sigma_1 \cap \xi^{-1}(y_1)$ is empty then $V_1$ is a vertex of $\sigma_1$ and there is an edge, say $[x_{1_1}, x_{1_2}]$ of $\sigma_1$ not containing $V_1$ and $[x_{1_1}, x_{1_2}] \cap \xi^{-1}(y_1)$ is a point and $x_{1_1}, x_{1_2}$ belongs to the principal fibration of $N$. Recall $\xi_i$ is the $\mathbb{Z}_2$-characteristic vector associated to $F_i$. We may assume $x_{1_1} = \{((0, 0), y_1)\}$ and $\xi_1 = (1, 0)$. So $x_{1_2} = \{((1, 0), y_1)\}$. Let $x'_{1_1} = \{((0, 1), y_1)\}$ and $x'_{1_2} = \{((1, 1), y_1)\}$. Then the 2-cells $[V_1, x_{1_1}, x_{1_2}]$ and $[V_1, x'_{1_1}, x'_{1_2}]$ belong to $X$. Then we assign $\{x_{1_1}, x_{1_2}\}$ to $F_1$.  

**Step 2:** Since $\{\xi_1, \xi_2\}$ is a basis of $\mathbb{Z}_2^2$, we may assume $\xi_2 = (0, 1) \in \mathbb{Z}_2^2$. Similarly as step 1, we get either $\xi^{-1}(F_3)$ contains at least 2 vertices, namely $x_{2_1}$ and $x_{2_2}$, of $X$ or there are 2-cells $[V_2, x_{2_1}, x_{2_2}]$ and $[V_2, x'_{2_1}, x'_{2_2}]$ in $X$ with $x_{2_1}$ belongs to principal fibration of $N$. We may assume $x_{2_1} = \{((0, 0), y_1)\}$ for some $y_1 \in Q^0 \setminus y_2$. If $y_2 = y_4$, then $x_{1_1} = x_{2_1}$ and either $x_{1_2} = x'_{2_1}$ or $x_{1_2} = x'_{2_2}$. Suppose $x_{1_2} = x_{2_2}$. So we choose the vertices $x_{2_1}, x_{2_2}$ for $F_2$. If $y_2 \neq y_4$, we get 4 new vertices and we assign two from them to $F_2$.  

**Step 3:** For $F_3$ we have two possibilities, either $\xi_3 = (1, 0)$ or $\xi_3 = (1, 1)$. In either case similarly as step 1, we get either $\xi^{-1}(F_3)$ contains at least 2 vertices, namely $x_{3_1}$ and $x_{3_2}$, of $X$.  

**Figure 2.**
or there are 2-cells \([V_3, x_{31}, x_{32}]\) and \([V_3, x'_{31}, x'_{32}]\) in \(X\) with \(x_{31}\) belongs to principal fibration of \(N\). We may assume \(x_{31} = \left(((0,0), y_6)\right)\) for some \(y_6 \in Q^0\). Suppose \(\xi_3 = (1,0)\). Then \(y_2 \neq y_6\), otherwise we get two 1-cells with same vertices. Assume \(y_4 = y_6\) then similarly as in step 2, we can assign two vertices different from previous steps to the edge \(F_3\). If \(y_4 \neq y_6\), we get 4 new vertices and we assign two from them to \(F_3\). Now suppose \(\xi_3 = (1,1)\). If \(y_6\) is different from one of \(y_2, y_4\), we can proceed similarly as step 2 to assign two new vertices to \(F_4\). Let \(y_2 = y_4 = y_6\). Then we choose \(x_{31} = V_1\) and \(x_{32} = V_2\).

Note if \(\xi_1 = (1,0), \xi_2 = (0,1), \xi_3(1,1)\) and \(y_2 = y_4 = y_6\), then there are edges between any two points of \(\xi^{-1}(y_i)\). So \(y_{2k} \neq y_{2i}\) for any \(3 < k \leq m\). Continuing in this way we can assign two vertices different form the previous vertices to each edge \(F_i\) for \(i > 1\). This proves the theorem.

\[\square\]

**Remark 3.12.** If \(X\) is a \(\mathbb{Z}_2^2\)-equivariant vertex minimal triangulation of a 2-dimensional small cover \(N\) with \(m\) fixed points, then by Lemma 3.7 and Theorem 3.11 \(2m \leq |V(X)| \leq 2m + 4\).

**Example 3.13.** A 12-vertex, 6-vertex and a 7-vertex equivariant triangulations of torus and \(\mathbb{R}P^2\) are given by the following Figure 3 respectively. The minimal triangulation of \(\mathbb{R}P^2\) is given by 6 vertices. So this 6-vertex triangulation is the vertex minimal \(\mathbb{Z}_2^2\)-equivariant of \(\mathbb{R}P^2\). By our construction it is clear that the 7-vertex triangulation is a vertex minimal equilibrium triangulation of \(\mathbb{R}P^2\).

**Figure 3.** Some equivariant triangulations of torus and \(\mathbb{R}P^2\).

### 3.4. Triangulations of 3-dimensional small covers.

Let \(\xi : N \rightarrow Q\) be a small cover over a 3-polytope \(Q\) with \(m\) facets \(F_1, \ldots, F_m\) and with \(k\) vertices \(V_1, \ldots, V_k\). Let \((f_0, f_1, f_2)\) be the \(f\)-vector of \(Q\), where \(f_i\) is the number of codimension \(i + 1\) face of \(Q\). We adhere the notations of Section 3. Let \(I_i\) be the 3-dimensional cube in the cubical subdivision \(C(Q)\) corresponding to the vertex \(V_i\) of \(Q\) for \(i = 1, \ldots, k\). Let \(C_F\) be the vertex of \(C'(Q)\) corresponding to the nonempty face \(F\) of \(Q\). So

\[
\xi^{-1}(C_Q) := \{C_{000}, C_{100}, C_{010}, C_{001}, C_{110}, C_{101}, C_{011}, C_{111}\}.
\]

In this subsection we assume that \(a = C_{000}, b = C_{100}, c = C_{010}, d = C_{001}, e = C_{110}, f = C_{101}, g = C_{011}, h = C_{111}\). Let \(\beta : F(Q) \rightarrow \mathbb{Z}_2^3\) be the \(\mathbb{Z}_2^2\)-characteristic function of \(N\) and \(\beta_i = \beta(F_i)\). Let \(H_i\) be the subgroup generated by \(\beta_i\) of \(\mathbb{Z}_2^3\) and \(\{a_jH_i : j = 1, 2, 3, 4\}\) be the set of distinct cosets of \(H_i\). So \(a_jH_i = \{a_j, a_j + \beta_i\}\) for all \(j = 1, 2, 3, 4\) and \(i = 1, \ldots, m\). Then \(\xi^{-1}(C_QC_{F_i})\) is the disjoint union of the edges \([C_{a_j}, C_{a_j + \beta_i}]\) for \(1 \leq j \leq 4\), where \([C_{a_j}, C_{a_j + \beta_i}]\) is the edge with vertices \(C_{a_j}\) and \(C_{a_j + \beta_i}\) such that \(\xi([C_{a_j}, C_{a_j + \beta_i}])\) contains \(C_{F_i}\). For simplicity, suppose \(V_1 = F_1 \cap F_2 \cap F_3, E_1 = F_1 \cap F_2, E_2 = F_2 \cap F_3\) and \(E_3 = F_3 \cap F_1\). Let \(G_1\) be the subgroup generated by \(\beta_1\) and \(\beta_2\) of \(\mathbb{Z}_2^3\) and \(b_1G_1, b_2G_1\) are distinct cosets of \(G_1\) in \(\mathbb{Z}_2^3\). Then \(\xi^{-1}(C_{F_1}C_{E_1}C_{F_2}C_{Q})\) is the disjoint union of following quadrangles,

\[
C_{b_j}C_{b_j + \beta_1}C_{b_j + \beta_2}C_{b_j + \beta_1 + \beta_2}(E_1)\quad\text{for}\quad j = 1, 2.
\]
Similarly for the sets $\xi^{-1}(C_{F_2}C_{E_2}C_{F_3}C_Q)$ and $\xi^{-1}(C_{F_2}C_{E_3}C_{F_3}C_Q)$. So $\xi^{-1}(I_1)$ is a cube with vertices $\xi^{-1}(C_Q)$ where the boundary is the union of six quadrangles given by

$$\xi^{-1}(C_{F_1}C_{E_1}C_{F_2}C_Q), \xi^{-1}(C_{F_1}C_{E_2}C_{F_3}C_Q) \text{ and } \xi^{-1}(C_{F_1}C_{E_3}C_{F_3}C_Q).$$

For example see Figure 3. Clearly $N = \cup_3 \xi^{-1}(I_j)$ is a cubical subdivision of $N$. Considering a triangulation for $\xi^{-1}(I_j)$, we may triangulate $N$. Even though $N$ is a union of these cubes, we can not consider any triangulation of these cubes. The choice of triangulation for these cubes totally depends on $\mathbb{Z}^3_2$-action on $N$ and hence on the $\mathbb{Z}_2$-characteristic function, see Examples 3.17, 3.19 and 3.21. Since fixed points of $\mathbb{Z}^3_2$-action on $N$ are bijectively correspond to the vertices of $Q$, we also denote them by $V_i$ for $i = 1, \ldots, k$. Each facet of the cube $\partial \xi^{-1}(I_j)$ is a square. So any triangulation of $\partial \xi^{-1}(I_j)$ contains at least 12 triangles. Consider the cone on this triangulation with apex $V_i$. So we get a triangulation of $\xi^{-1}(I_j)$ with at least 12 facets.

Let

$$(3) \quad s(a) = \# \{ F_i : \beta(F_i) = a \}$$

for all $a \in \mathbb{Z}^3_2 - \{0,0,0\}$. Let $F_{i_1}$ and $F_{i_2}$ be two facets such that $\beta_{i_1} = \beta_{i_2}$. So $\xi^{-1}(C_{F_{i_1}}C_QC_{F_{i_2}})$ is a disjoint union of 4 cubes. Then the vertex set $\xi^{-1}(C_{F_{i_1}}C_QC_{F_{i_2}}) \cap \{ \xi^{-1}(C_Q) \cup \{ V_i : i = 1, \ldots, k \} \}$ does not give a triangulation of these cubes. Each of these cubes contains two vertices of $\xi^{-1}(C_Q) \cup \{ V_i : i = 1, \ldots, k \}$. So we need to add one vertex for each facet. We consider the corresponding vertices from the set $\xi^{-1}(C_{F_{i_2}})$. Clearly the minimal triangulation of $\xi^{-1}(C_{F_{i_1}})C_Q \cup \cdots \cup C_{F_{i_2}}C_Q$ needs $\xi^{-1}\{ C_Q, C_{F_{i_1}}, \ldots, C_{F_{i_2}} \}$ vertices. Then the number of vertices in $\xi^{-1}\{ C_{F_{i_1}}, \ldots, C_{F_{i_2}} \}$ is $4s(a) - 4$.

Let $E_j = F_{i_1} \cap F_{i_2}$ be an edge. Let $F_{i_3}$ and $F_{i_4}$ be the facets such that $V_{i_1} = F_{i_3} \cap E_j$ and $V_{i_2} = F_{i_4} \cap E_j$. Let $\beta_{i_3} + \beta_{i_4} \in \{ \beta_{i_1}, \beta_{i_2} \}$. So the minimal triangulations for the boundaries of the cubes $\xi^{-1}(I_{i_1})$ and $\xi^{-1}(I_{i_2})$ do not give a triangulation of the boundary of $\xi^{-1}(I_{i_1} \cap I_{i_2})$. Otherwise we have a 2-edge triangulation of circles belong to the boundary of $\xi^{-1}(I_{i_1} \cup I_{i_2})$. We need to add two vertices corresponding to the point $C_{E_j}$ of the edge $E_j$. Let

$$(4) \quad l = \# \{ E_j : \text{ the condition in the above paragraph holds for the edge } E_j \}.$$ 

So we add $2l$ many vertices corresponding to these edges. With these collections of vertices we triangulate the boundary of each $\xi^{-1}(I_i)$, then make cone at apex $V_i$ by construction, this is an equilibrium triangulation. Thus we can prove the following.

**Theorem 3.14.** Let $N$ be a 3-dimensional small cover with characteristic pair $(Q, \xi)$. There is a natural equilibrium triangulation of $N$ with $4(s(a_1) + \ldots + s(a_7)) + 2l + k - 20$ vertices where $s(a_i)$’s are defined in Equation 3 and $l$ is defined in the previous paragraph.

In the following examples we discuss some irreducible and vertex minimal equilibrium triangulations of some small covers explicitly. We denote the characteristic pairs in Figure 3 (a), (b), (c) and (d) by $(\Delta^3, \beta)$, $(Q, \beta^1)$, $(Q, \beta^2)$ and $(Q, \beta^3)$ respectively, where $Q$ is a 3-dimensional prism with vertices $1, \ldots, 6$. Let $\xi : N \to \Delta^3, \xi_1 : N_1 \to Q, \xi_2 : N_2 \to Q$ and $\xi_3 : N_3 \to Q$ be the respective small covers.

**Example 3.15.** Let $\xi : N \to \Delta^3$ be a small cover over the 3-simplex with vertices $1, \ldots, 4$. So $N$ is $\mathbb{R}P^3$. The $\mathbb{Z}_2$-characteristic function of $\mathbb{R}P^3$ is given in Figure 3 (a).

The triangulations in Figure 3 give a natural 12 vertex triangulation of $\mathbb{R}P^3$ with the vertices $1, 2, 3, 4, a, \ldots, h$. Note that this is an equilibrium but not $\mathbb{Z}_2^2$-equivariant triangulation of $\mathbb{R}P^3$.

**Lemma 3.16.** There is an equilibrium triangulation of $\mathbb{R}P^3$ with 11 vertices.

**Proof.** We delete the vertex 2 from the triangulation of $\mathbb{R}P^3$ in Example 3.13. Then consider the triangulation of $\xi^{-1}(I_2)$ with five 3-simplices. This gives an equilibrium triangulation of $\mathbb{R}P^3$ with 11 vertices. Since any triangulation of $\mathbb{R}P^3$ contains at least 11 vertices, this is a vertex minimal equilibrium triangulation of $\mathbb{R}P^3$. \qed
Example 3.17 (Triangulation for $N_1$). The manifold $N_1$ is $\mathbb{RP}^3 \# \mathbb{RP}^3$. Let $(\xi^1)^{-1}(i) = i$, $i = 1, \ldots, 6$. In this case we have six cubes $(\xi^1)^{-1}(I_i)$ for $i = 1, \ldots, 6$ in the cubical subdivision of $N_1$. For the facets $F_{i_1} = 123$ and $F_{i_2} = 456$ we have $\beta_{i_1}^1 = \beta_{i_2}^1 = (1, 0, 0)$. Since $[d, f], [a, b], [c, e]$ and $[g, h]$ are edges of first 3 cubes, they can not form edges in the remaining 3 cubes. Otherwise we have a 2-edge triangulation of circle. Hence we need to add $p, q, r$ and $s$ in the interior of those edges respectively. Now we can construct a triangulation of the boundary of the cubes $(\xi^1)^{-1}(I_i)$ such that they are compatible on the boundaries, see Figure 4. Then taking cone on the boundary of $(\xi^1)^{-1}(I_i)$ from $i$ for $i = 1, 2, 4, \ldots, 6$ we get an equilibrium triangulation of $N_1$ with 17 vertices. We will show that this is a vertex minimal equilibrium triangulation of $\mathbb{RP}^3 \# \mathbb{RP}^3$. The $f$-vectors of this simplicial complex is $(f_0, f_1, f_2, f_3) = (17, 106, 178, 89)$. Note that the minimal triangulations of $\mathbb{RP}^3 \# \mathbb{RP}^3$ contain 15 vertices.

Theorem 3.18. Let $X$ be an equilibrium triangulation of $\mathbb{RP}^3 \# \mathbb{RP}^3$ with complete zones of influences $Z_1, \ldots, Z_6$. If $V(X) \cap Z_i$ is the equilibrium set for some $i \in \{1, \ldots, 6\}$, then $X$ contains at least 17 vertices.

Proof. We may assume $Z_i = (\xi^1)^{-1}(I_i)$ for $i = 1, \ldots, 6$ and $V(X) \cap Z_3$ is the equilibrium set $\{7, \ldots, 14\}$. So $Z_i \cap X$ is a triangulation of $Z_i$ for $i = 1, \ldots, 6$ and $\{a, \ldots, h\} \subset V(X)$. We know that without extra vertices a 3-cube has 7 distinct triangulations. One with 5 3-dimensional simplices and others with 6 3-dimensional simplices. If we consider the triangulation of $Z_3$ with 6 3-dimensional simplices, there will be a diagonal edge in $Z_3$. So we can not get any triangulations of $Z_1$ and $Z_2$ such that the boundary triangulations of these three cubes $Z_1, Z_2$ and $Z_3$ are compatible. Let us consider the triangulation of $Z_3$ with 5 3-dimensional simplices. Considering this triangulation of $Z_3$, we get a unique triangulation of the boundary of $Z_1$ and $Z_2$, see Figure 5. These boundary triangulations of $Z_1$ and $Z_2$ do not give any triangulations of them without extra vertices. So we need to consider at least one vertex in their interior.
By construction of $N_1$, we get that the intersections $[d, f] \cap Z_4 \cap V(X), [a, b] \cap Z_4 \cap V(X), [c, e] \cap Z_4 \cap V(X)$ and $[g, h] \cap Z_4 \cap V(X)$ are nonempty. Suppose each of these intersections contain $p, q, r$ and $s$ respectively. From the boundary triangulations of $Z_1, Z_2$ and $Z_3$, we get that any one of $[d, h], [g, f], [b, d], [a, f], [a, e], [b, c], [c, h]$ and $[e, g]$ cannot be an edge of the cube $Z_4$. Similarly, any one of $[c, d], [e, f], [b, d], [a, f], [a, g], [b, h], [c, h]$ and $[e, g]$ cannot be an edge of the cube $Z_5$. Also, any one of $[d, h], [f, g], [c, d], [e, f], [d, e], [b, c], [a, g]$ and $[b, h]$ cannot be an edge of the cube $Z_6$. So $\{[p, q], [q, r], [r, s], [p, s]\}$, $\{[p, q], [q, r], [r, s], [p, r]\}$ and $\{[p, s], [q, s], [q, r], [p, r]\}$ are sets of edges in the cubes $Z_4, Z_5$ and $Z_6$ respectively. So, neither $[p, r]$ nor $[g, s]$ can be an edge in the cone on the boundary of $Z_4$, since they are already edge in the cube $Z_5$. For similar reason neither $[q, r]$ nor $[p, s]$ is an edge of the cube $Z_5$ and neither $[p, q]$ nor $[r, s]$ is an edge in the cube $Z_6$.

Also, None of $[d, e], [a, h], [c, f]$ and $[b, g]$ can be an edge in the cube $Z_4$, since they are already edges in the cube $Z_2$. For similar reason, none of $[d, g], [a, c], [b, f], [b, e]$ can be an edge of the cube $Z_5$ and none of $[b, f], [a, d], [c, g], [e, h]$ can be an edge of the cube $Z_6$. So the rectangles with vertices $p, q, r, s$ in $Z_4, Z_5$ and $Z_6$ contain at least one vertex of $X$ other than its vertices. Let $x_1, x_2$ and $x_3$ be the vertices belongs to these rectangles respectively. Since the intersection of these rectangles is $\{p, q, r, s\}$, $x_1, x_2$, and $x_3$ cannot be equal. So $\{x_1, x_2, x_3\} \geq 2$. Suppose $x_1 = x_2$ and $x_1$ belongs to the relative interior of $[p, q]$. Consider the cone from $x_1$ on the boundary of $Z_4$ and $Z_5$. Then the triangle $\{x_1, r, s\}$ will be face of four distinct 3-simplices. So, $x_1, x_2$ and $x_3$ are distinct. Hence $|V(X)| \geq 17$.

Let $1$ and $2$ be points belong to the interior of $Z_1$ and $Z_2$ respectively. Then take cone on the boundary triangulation of $Z_1$ and $Z_2$ from $1$ and $2$ respectively. Let $x_1, x_2$ and $x_3$ be the interior point of the cubes $Z_4, Z_5$ and $Z_6$ respectively. Consider the cone on the boundary of $Z_4, Z_5$ and $Z_6$ from $x_1, x_2$ and $x_3$ respectively. Without loss, we can assume $(x_1, x_2, x_3) = (4, 5, 6)$. Then, we can construct some triangulations of $Z_4, Z_5$ and $Z_6$ such that they are compatible on the boundary. Hence we get some 17 vertex equilibrium triangulations of $\mathbb{RP}^3 \# \mathbb{RP}^2$.

**Example 3.19** (Triangulation for $N_2$). The manifold $N_2$ is a non-trivial $S^1$ bundle over $\mathbb{RP}^2$. Let $(\xi^2)^{-1}(i) = i$ for $i = 1, \ldots, 6$. In this case we have six cubes $(\xi^2)^{-1}(I_i)$ for $i = 1, \ldots, 6$ in the cubical subdivision of $N_2$. Let $E_i = F_{ii} \cap F_{i2}$ be an edge belongs to the triangular
facet whose vertices are \( i_1 = F_i \cap E_i \) and \( i_2 = F_i \cap E_i \). Observe that \( \beta^2_{i_1} + \beta^2_{i_2} \in \{ \beta^2_{i_1}, \beta^2_{i_2} \} \) if \((i_1, i_2) \in \{(1, 3), (2, 3), (2, 5), (4, 6), (5, 6)\}\). So the minimal triangulation for the boundary of the cubes \((\xi^2)^{-1}(I_{i_1})\) and \((\xi^2)^{-1}(I_{i_2})\) may not give a triangulation of the boundary of \((\xi^2)^{-1}(I_{i_1} \cup I_{i_2})\). Otherwise we have a 2-edge triangulation of two circles belongs to the boundary of \((\xi^2)^{-1}(I_{i_1} \cup I_{i_2})\). So we need to add two vertices corresponding to the point \(C_{E_i}\) in \(C(Q)\) of the edge \(E_i\). For any two facets \(F_{i_1}\) and \(F_{i_2}\) we have \(\beta^2_{i_1} \neq \beta^2_{i_2}\). For simplicity of notation, the vertices corresponding to \(C_{E_i}\)’s are denoted by \(q, r, s, t, u, v, w, x, y, z\) accordingly, see Figure \(\textbf{7}\). Now we can construct a triangulation of \(N_2\) by taking cone from \(i\) on the triangulation of the boundary (given in Figure \(\textbf{6}\) of the cube \((\xi^2)^{-1}(I_i)\) for \(i = 1, \ldots, 6\). Clearly, this is an equilibrium triangulation. Observing this triangulation, we can construct some 19-vertex equilibrium triangulation of \(N_2\) using the vertex set \(\{4, a, \ldots, h, q, \ldots, z\}\), see Figure \(\textbf{6}\). The \(f\)-vectors of this simplicial complex is \((f_0, f_1, f_2, f_3) = (19, 111, 184, 92)\).

**Theorem 3.20.** Any equilibrium triangulation of \(N_2\) contains at least 19 vertices. There are some 19 vertex equilibrium triangulations of \(N_2\).

**Proof.** Let \(X\) be an equilibrium triangulation of \(N_2\) with complete zones of influences \(Z_1, \ldots, Z_6\). We may assume \(Z_i = (\xi^2)^{-1}(I_i)\) for \(i = 1, \ldots, 6\). So \(Z_i \cap X\) is a triangulation of \(Z_i\) for \(i = 1, \ldots, 6\) and \(\{a, \ldots, h\} \subset V(X)\) is the equilibrium set.

Observe that neither \([c, d]\) nor \([a, g]\) can be an edge in the cube \(Z_1\), since they are already edge in the cube \(Z_2\). For similar reason neither \([d, g]\) nor \([a, c]\) is an edge of the cube \(Z_2\) and neither \([a, d]\) nor \([c, g]\) is an edge of the cube \(Z_3\). Let \(A_1, A_2\) and \(A_3\) be the facets of \(Z_1, Z_2\) and \(Z_3\) with vertex set \(\{a, c, d, g\}\) respectively. So \(A_i\) contains at least one vertex of \(X\) other than its vertices for \(i = 1, 2, 3\). Suppose \(A_i\) contains only \(x_i \in V(X)\) for \(i = 1, 2, 3\). Since the intersection \(A_1 \cap A_2 \cap A_3 = \{a, c, d, g\}\), \(x_1, x_2, x_3\) can not be equal. Suppose \(x_1 = x_2\) and \(x_1\) belongs to the relative interior of \([a, d]\) in \(Z_1\) and \(Z_2\). Then \(x_3\) can not belong to the boundary of \(A_3\), otherwise we get 2 vertex circle in \(X\). Also for the same reason \(x_1\) can not belong to the boundary of \(A_1\) and \(A_2\). So \(x_i\) belongs to interior of \(A_i\) for \(i = 1, 2, 3\).

Similarly, we can show the interior of the each facets of \(\partial(Z_1), \partial(Z_2)\) and \(\partial(Z_3)\) with vertex set \(\{b, e, f, h\}\) contains at least one vertex of \(X\). Again by similar arguments, we can show the
interior of the each facets of $Z_2$ and $Z_5$ with vertex set $\{c,d,e,f\}$ (resp., $\{a,b,g,h\}$) contain at least one vertex of $X$. Since diagonal edges of $Z_1$ and $Z_4$ can not be edges, any triangulation of $Z_1$ with the vertices as in Figure 3 contain the edge $[g,r]$ and $Z_4$ contains a vertex (say $4 \in V(X)$) in the interior. Hence $|V(X)| \geq 19$. Considering these 19 vertices we can construct some equilibrium triangulations of $N_2$, see Figure 7. Hence we get the theorem. □

**Example 3.21** (Triangulation for $N_3$). The manifold $N_3$ is $S^1 \times \mathbb{RP}^2$. Let $(i^3)^{-1}(i) = i, i = 1, \ldots , 6$. In this case we have six cubes $Z_i = (i^3)^{-1}(I_i)$ for $i = 1, \ldots , 6$. For this $Z_2$-characteristic pair $(Q, \beta^3)$, there are two facets 123 and 456 with characteristic vector $(1,0,0)$. So considering the minimal triangulation of the boundary of these 6 cubes, we may not construct a triangulation of $N_3$. Since $[d,f], [a,b], [c,e]$ and $[g,h]$ are also edges of the cubes $Z_1, Z_2, Z_3$, they can not form edge in the remaining 3 cubes. Otherwise we have a 2-edge triangulation of a circle. Hence we need to add $q, r, s$ and $t$ in the interior of those edges in the last 3 cubes respectively. Since characteristic vectors of 123 and 456 are same, it is enough to consider vertices corresponding to the edges $[1,2], [2,3]$ and $[1,3]$ of $Q_3$. We denote these pair of vertices by $\{u,v\}, \{w,x\}$ and $\{y,z\}$ respectively. Now we can construct a triangulation of the cubes $(i^3)^{-1}(I_i)$ such that their boundaries are compatible and $1$-faces belongs to exactly two $2$-faces of the boundaries, see Figure 8. So we get an equilibrium triangulation of $N_3$. From this triangulation we can construct an equilibrium triangulation of $\mathbb{RP}^2 \times S^1$ with 18 vertices, see Figure 8. The $f$-vectors of this complex is $(18,114,192,96)$. Note that the minimal triangulations of $\mathbb{RP}^2 \times S^1$ contain 14 vertices.

**Theorem 3.22.** Any equilibrium triangulation of $S^1 \times \mathbb{RP}^2$ contains at least 18 vertices. There are 18 vertex equilibrium triangulations of $S^1 \times \mathbb{RP}^2$.

**Proof.** Let $X$ be an equilibrium triangulation of $S^1 \times \mathbb{RP}^2$ with complete zones of influences $Z_1, \ldots , Z_6$. We may assume $Z_i = (i^3)^{-1}(I_i)$ for $i = 1, \ldots , 6$. So $Z_i \cap X$ is a triangulation of $Z_i$ for $i = 1, \ldots , 6$ and $\{a, \ldots , h\} \subset V(X)$ is the equilibrium set. By similar arguments as in Example 3.21 we get $q, r, s, t \in (V(X) \cap Z_i)$ for $i = 4, 5, 6$.

Now, neither $[c,d]$ nor $[a,g]$ can be an edge in the cube $Z_1$, since they are already edge in the cube $Z_2$. For similar reason neither $[d,g]$ nor $[a,c]$ is an edge of the cube $Z_2$ and neither

![Figure 8. Triangulations of zones of influences for $N_3$.](image-url)
suppose $x_i$ contains only $x_i \in V(X)$ for $i = 1, 2, 3$. Since the intersection $A_1 \cap A_2 \cap A_3 = \{a, c, d, g\}$, $x_1, x_2, x_3$ can not be equal. Suppose $x_1 = x_2$ and $x_1$ belongs to the relative interior of $[a, d]$ in $Z_1$ and $Z_2$. Then $x_3$ can not belong to the boundary of $A_3$, otherwise we get 2 vertex circle in $X$. Also for the same reason $x_1$ can not belong to the boundary of $A_1$ and $A_2$. So $x_i$ belongs to interior of $A_i$ for $i = 1, 2, 3$. Hence $A_1 \cap A_2 \cap A_3 \cap V(X)$ contains at least 3 vertices. Similarly, we can show the interiors of the rectangles with vertex set $\{b, e, h, f\}$ in the cubes $Z_1, Z_2$ and $Z_3$ contain at least one vertex of $X$. Hence $|V(X)| \geq 18$. Considering these 18 vertices we can construct some equilibrium triangulations of $N_3$, see Figure 3.

\[\square\]

3.5. Triangulations of real projective spaces. The vertex minimal triangulation of $\mathbb{RP}^n$ is not known for $n > 5$. It is known that $\mathbb{RP}^n$ has triangulation with at least $2^{n+1} - 1$ vertices when $n > 5$, (cf. [4]). A minimal triangulation of $\mathbb{RP}^3$ was constructed by Walkup with 11 vertices, cf. [1]. In 1999 and 2005, F. H. Lutz constructed one 16- and 24-vertex triangulations of $\mathbb{RP}^4$ and $\mathbb{RP}^5$ respectively, see [3]. We construct some nice triangulations of $\mathbb{RP}^n$ in the following.

Theorem 3.23. The real projective space $\mathbb{RP}^n$ has an equilibrium triangulation with $2^n + n + 1$ vertices for all $n$.

Proof. Let $C(\triangle^n)$ be a cubical subdivision of the $n$-dimensional simplex $\triangle^n$. Let $V_1, \ldots , V_n, V_{n+1}$ be the vertices of $\triangle^n$. For $\triangle^n$ and any proper face $F$ of $\triangle^n$, let $O_{\triangle^n}$ and $C_F$ be the corresponding points in $S(\triangle^n)$ respectively. Let $\xi : \mathbb{RP}^n \to \triangle^n$ be the orbit map. Recall the construction of $I^n_i \subset F_2$ from cubical subdivision in page-51 of [4]. Clearly $I^n_i \cup_{\triangle^n} I^n_i$ for all vertex $V_i$ of $\triangle^n$. Let $F$ be a $k$-dimensional face of $\triangle^n$. Then $I^n_i \subset F_2$ is an $(n-k)$-dimensional face of $I^n_i$ for all $V_i \in F$. Since the action of $\mathbb{Z}_2^n$ on $\mathbb{RP}^n$ is locally standard, the subset $\xi^{-1}(I^n_i)$ of $\mathbb{RP}^n$ is diffeomorphic as manifold with corners to an $n$-dimensional cube $A^n_i$ with vertices $\xi^{-1}(O_{\triangle^n})$. The point $\xi^{-1}(V_i)$ belongs to the interior of $A^n_i$. Using the cubical subcomplex of $\triangle^n$ we construct a cell complex of $\mathbb{RP}^n$ in the following way. Let $X_0 = \{\xi^{-1}(V_i)(= V^n_i) : i = 1, \ldots , n + 1 \}$ and $\{\xi^{-1}(O_{\triangle^n})\}$. So the number of points in $X_0$ is $2^n + n + 1$. The cone on $I^n_i \subset F_2$ with apex $V_i$ in $\triangle^n$ is denoted by $V_i \xi^{-1}(I^n_i \subset F_2 \subset \triangle^n)$ for any face $F$ containing $V_i$. Let $X_1 = \{\xi^{-1}(I^n_i \subset F_2 \subset \triangle^n) : F \text{ is an } (n-1)\text{-dimension face of } \triangle^n\} \cup \{\xi^{-1}(V_i O_{\triangle^n}) : i = 1, \ldots , n\}$, where $V_i O_{\triangle^n}$ line segment joining $V_i$ and $O_{\triangle^n}$ in $\triangle^n$. Let $X_2 = \{\xi^{-1}(I^n_i \subset F_2 \subset \triangle^n) : F \text{ is an } (n-2)\text{-dimension face of } \triangle^n\} \cup \{V_i \xi^{-1}(I^n_i \subset F_2 \subset \triangle^n) : i = 1, \ldots , n\}$ : $F$ is an $(n-1)$-dimension face of $\triangle^n$. Continue this process up to $n$th step $X_n$ where $X_n = \{V_i \xi^{-1}(I^n_i \subset F_2 \subset \triangle^n) : i = 1, \ldots , n\} \cup \{\xi^{-1}(I^n_{n+1})\}$. Then $X = \bigcup_{i=0}^n X_i$ gives a cell complex of $\mathbb{RP}^n$. Note that each $X_i$ is a cell complex for $i = 0, \ldots , n$. So by the Proposition 2.7 we can construct nice triangulations of $X$ without adding any new vertices. Clearly, this is an equilibrium triangulation of $\mathbb{RP}^n$ and the number of vertices of the triangulation of $X$ is $2^n + n + 1$.

\[\square\]

Remark 3.24. An equilibrium triangulation of a small cover $N^n$ may not be a $\mathbb{Z}_2^n$-equivariant triangulation and a $\mathbb{Z}_2^n$-equivariant triangulation of a small cover $N^n$ may not be an equilibrium triangulation.

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REFERENCES

[1] T. F. Banchoff and W. Kühnel: Equilibrium triangulations of the complex projective plane, Geom. Dedicata 44 (1992), 413–433.
[2] B. Basak and B. Datta: Crystallizations of 3-manifolds. Available at arXiv:math/13086137v2, 2013.
[3] V. M. Buchstaber and T. E. Panov: Torus actions and their applications in topology and combinatorics, University Lecture Series 24, American Mathematical Society, Providence, RI, 2002.
[4] B. Datta: Minimal triangulations of manifolds. J. Indian Inst. Sci. 87 (2007), no. 4, 429–449.
[5] M. W. Davis and T. Januszkiewicz: Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991), no. 2, 417–451.
[6] B. Datta and S. Sarkar: On equilibrium triangulations of quasitoric 4-manifolds, in preparation.
[7] M. Ferri, C. Gigliardi and L. Grasselli: A graph-theoretic representation of PL-manifolds – a survey on crystallizations, Aequationes Math. 31 (1986), 121–141.
[8] S. Illman: Smooth equivariant triangulations of G-manifolds for a finite group, Math. Ann. 233 (1978), 199–220.
[9] F. H. Lutz: Triangulated manifolds with few vertices: Combinatorial manifolds. Available at arXiv:math/0506372v1, 2005.
[10] C. P. Rourke and B. J. Sanderson: Introduction to piecewise-linear topology. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. Springer-Verlag, New York-Heidelberg, 1972.
[11] D. W. Walkup: The lower bound conjecture for 3- and 4-manifolds, Acta Math. 125 (1970), 75–107.