On Odd Torsion in Even Khovanov Homology

Sujoy Mukherjee
Department of Mathematics, The Ohio State University, Columbus, OH, USA

ABSTRACT
This short note resolves the most important part of the PS braid conjecture while introducing the first known examples of knots and links with odd torsion of order 9, 27, 81, and 25 in their even Khovanov homology.

What does torsion in KH signify?
For completeness, a very brief description of KH is first provided below. For more details, the reader is directed to the articles [2, 15, 16]. This is followed by the discussion on the PS braid conjecture. The final part of the paper introduces knots and links with odd torsion of order 9, 27, 81, and 25 in their even KH.

1. Introduction
For $D$, an unoriented diagram of a link, let $cro(D)$ denote its set of crossings. $s : cro(D) \rightarrow \{A, B\}$ is called a Kauffman state of $D$. Let $KSD(D)$ denote the set of all Kauffman states. Further, let $\sigma = \sigma(s) = |s^{-1}(A)| - |s^{-1}(B)|$ for all $s \in KSD(D)$.

Denote by $cir(s)$, the set of circles obtained from a Kauffman state $s$ by smoothing the crossings of $D$ according to the convention described in Figure 1 and let $t_s : cir(s) \rightarrow \{-1, 1\}$. $t_s$ is called an enhanced Kauffman state of $D$. Denote the set of enhanced Kauffman states of a link diagram $D$ by $EKSD(D)$. For each enhanced Kauffman state $t_s$, let $\tau = \tau(t_s) = \sum_{c \in cir(s)} t_s(c)$.

For an enhanced Kauffman state $t_s$ obtained from a Kauffman state $s$, let $(a(t_s), b(t_s)) = (\sigma(s), \sigma(s) + 2\tau(t_s))$. Now, let $C_{a,b}(D)$ be the free Abelian group with generators: $\{t_s | t_s \in EKSD(D) \text{ with } a(t_s) = a \text{ and } b(t_s) = b\}$.

Let $t_s$ and $t_s'$ be enhanced Kauffman states. Then, $t_s'$ is said to be adjacent to $t_s$, if $a(t_s) = a(t_s') + 2$, $b(t_s) = b(t_s')$, and the Kauffman states corresponding to $t_s$ and $t_s'$ differ at exactly one crossing with $s$ having an $A$ value and $s'$ having a $B$ value assigned to the crossing.

Finally, order the crossings of $D$ and for a fixed integer $b$, consider the descendant complex:

$$\cdots \rightarrow C_{a+2,b}(D) \xrightarrow{\partial,b} C_{a,b}(D) \xrightarrow{\partial,b} C_{a-2,b}(D) \rightarrow \cdots$$

CONTACT Sujoy Mukherjee sujoymukherjee.math@gmail.com Department of Mathematics, The Ohio State University, Columbus, OH, USA.

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with differential $\partial_{a,b}(t_i) = \sum_{i' \in \mathcal{X}_S} (t_i : t_{i'}) t_{i'}$, where $(t_i : t_{i'}) \in \{-1, 0, 1\}$. If $t_{i'}$ is not adjacent to $t_i$, $(t_i : t_{i'}) = 0$ and otherwise $(t_i : t_{i'}) = (-1)^\omega$, with $\omega$ the number of $B$-labeled crossings in $t_i$ coming after the crossing at which the Kauffman states corresponding to $t_i$ and $t_{i'}$ differ.

With the notation above, the groups $H^{a,b}(D) = \ker(\partial_{a,b}) / \text{im}(\partial_{a,b})$ are invariant under Reidemeister moves II and III. These groups categorify the unreduced Kauffman bracket polynomial [5] and are called the framed Khovanov homology groups.

Let $\mathcal{D}$ be the diagram obtained after fixing an orientation for $D$ and let $w = w(\mathcal{D})$ be its writhe. Then, the classical Khovanov (co)homology of $\mathcal{D}$ can be obtained from the framed Khovanov homology in the following way:

$$H^{a,b}(\mathcal{D}) = H_{w-2,3w-2}(D).$$

To avoid cumbersome notation, in the remaining part of this article, $D$ (instead of $\mathcal{D}$) is used to denote an oriented link diagram.

Most of the previously known examples of knots and links with interesting torsion are derived from torus links. In fact, it is conjectured that torus knots of type $(p, p+1)$, where $p$ is a prime, contain $\mathbb{Z}_p$ torsion. The trend of relying on derivatives of torus links for torsion in KH continues in this note too, however, with modifications to torus links in a different way. In fact, the author hopes that the KH data of the links in these families will encourage the readers to compute their other invariants.

Let $B_n$ denote the braid group with $(n-1)$ generators. Further, let $0 \leq i \neq j < n$. For $i < j$, denote by $w_{i,j}$ the braid word: $\sigma_i \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j$, and similarly, by $w_{i,j}^{-1}$, the braid word: $\sigma_j^{-1} \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1}$. Figure 2 shows $w_{1,4} \in B_5$. Analogously, for $i > j$, denote by $w_{j,i}$ the braid word: $\sigma_{i} \sigma_{i+1} \cdots \sigma_{j-1} \sigma_j$, and by $w_{j,i}^{-1}$, the braid word: $\sigma_j^{-1} \sigma_{j+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_i^{-1}$.

### 2. The PS braid conjecture

In 2012, the following conjecture addressing the potential connection between the torsion subgroups that appear in the KH of a closed braid and its braid index was posed by Przytycki and Szadanović [11].

**Conjecture 1.**

1. KH of a closed 3-braid can have only $\mathbb{Z}_2$ torsion.
2. KH of a closed 4-braid cannot have odd torsion.
3. KH of a closed 4-braid can have only $\mathbb{Z}_2$ and $\mathbb{Z}_4$ torsion.
4. KH of a closed n-braid cannot have $\mathbb{Z}_p$ torsion for $p > n$, where $p$ is prime.

It is believed that the first part of the PS braid conjecture is true. In fact, it has been proven to be true for four out of the seven families of 3-braids [3, 10]. In particular, these infinite families contain the torus links $T(3, n)$. Parts two and three were disproved by the closure of the braid: $w_{1,3}^3 \in B_4$ [8]. The last part was disproved by the family of links with large torsion subgroups of even order in their KH. A prominent member of this family is the flat two cabling of the smallest non-alternating knot $8_{19}$. The final part of the PS braid conjecture is resolved by the following theorem [9].

**Theorem 2.** The part of the PS braid conjecture stating that the order of torsion subgroups, when prime, appearing in the KH of a closed braid is bounded above by its braid index, is false.

**Proof.** Consider the braid: $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^5 \cdot w_{1,5} \in B_8$. The closure of this braid is a knot and its KH contains $\mathbb{Z}_7$ torsion in bigradings $(23, 71)$ and $(24, 75)$ as seen in Figure 3.

**Remark 3.** There are braids with smaller crossing numbers and braid indexes whose closures are counterexamples to the last part of the PS braid conjecture. However, all such known examples are links. One of them is $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^5 \cdot w_{1,4}^6 \in B_5$. The closure of this braid contains $\mathbb{Z}_7$ in its KH.

### 3. More knots and links with odd torsion in KH

After [8], the state of affairs regarding odd torsion subgroups in KH was that infinitely many examples of odd torsion subgroups up to $\mathbb{Z}_7$ were known and the next higher one, meaning $\mathbb{Z}_9$, was expected to show up in the KH of the torus knot of type $(9, 10)$. However, the KH of this knot is beyond reach even at present due to limited computational abilities. The challenge, therefore, was to find a knot
or link within computational reach with \( \mathbb{Z}_9 \) torsion in its KH. Surprisingly, not only was it possible to find knots and links with \( \mathbb{Z}_9 \) torsion in their KH, but also ones with \( \mathbb{Z}_{27}, \mathbb{Z}_{81}, \) and \( \mathbb{Z}_{25} \) torsion in their KH.

**Example 4.**

1. The closure of \( (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^5 \cdot w_{1,5} \) contains \( \mathbb{Z}_9 \) torsion in its KH. Also, the connected sum of the torus knot of type \((5, 6)\) with itself contains \( \mathbb{Z}_9 \) torsion in its KH.

2. The closure of the braid: \( (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^6(\sigma_4 \sigma_3 \sigma_6 \sigma_7)^6 \), the overlapping connected sum of the torus knot of type \((5, 6)\) with itself twice, contains \( \mathbb{Z}_{27} \) torsion in its KH.

3. The connected sum of the closure of the braid: \( (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^6 \) with itself contains \( \mathbb{Z}_{25} \) torsion in its KH. See Figures 5–7 in the Appendix.

The article [8] proposed a family of links with torsion subgroups of order \( 2^s \) in their KH. The following conjecture, verified up to \( m = 4 \), proposes a similar family for torsion subgroups of order \( 3^m \).

**Conjecture 5.** The KH of the link \( T(2, 3)_m(\sigma_1 \sigma_2 \sigma_3)^4 \sigma_1 \sigma_2 \), for \( m \in \mathbb{Z}^+ \), contains the torsion subgroups \( \mathbb{Z}_{23}, \mathbb{Z}_{29}, \ldots, \mathbb{Z}_{3^m} \), where denotes the connected sum operation.

**Figure 3.** KH of the closure of \( (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^7 \cdot w_{1,5} \).

**Figure 4.** The braid whose closure is the 3-component link \( T^{-1}(4, 5)T^{-1}(4, 5)T(2, 3) \) having \( \mathbb{Z}_9 \) torsion in its KH. Here, \( T(2, 3) \) and \( T(3, 4) \) represent the braids whose closures give the corresponding torus knots \( T(2, 3) \) and \( T(3, 4) \), respectively.
one component is involved. Observe that $T^{-1}(4, 5)$ is a link of two components, one of which is the torus knot $T(3, 4)$. In the above conjecture, the $T^{-1}(4, 5)$ factors are added by taking the connected sum of the non-$T(3, 4)$ component of the previous link with the $T(3, 4)$ component of the new $T^{-1}(4, 5)$ factor. At the very end, $T(2, 3)$ is added by taking the connected sum with the non-$T(3, 4)$ component of the previous $T^{-1}(4, 5)$ factor. Figure 4 shows the example of the procedure when the final link has two factors of $T^{-1}(4, 5)$ along with the Trefoil knot. At this point, an interesting
The question to ask is: Does the KH of a non-prime link depend on the choice of base points when a connected sum operation was performed? The answer is yes! In fact, in the above case, if the $T(3, 4)$ component is used for all the connected sum operations, the torsion subgroup $\mathbb{Z}_3 \times \mathbb{Z}_2$ does not appear in the KH of the resulting link.

In general, the computational complexity of KH is exponential. However, from some experience it is clear that for the KH of certain families of links, the computational complexity seems to be smaller. It may be useful to study this under a general framework when looking to predict the presence of larger torsion subgroups in KH in a consistent way.

**Figure 6.** The KH table of the connected sum of the closure of the braid: $(\sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1)^6 \sigma_1 \sigma_2 \sigma_3 \sigma_4$ with itself for $i$-grading between 19 and 28.
Moreover, it is still unknown how non-$\mathbb{Z}_2$ torsion sub-
groups appear in KH. The note is concluded with the fol-
lowing question in relation to [17].

**Question 6.** Are there families of knots and links having the
same free part but different torsion subgroups in their KH?

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**Appendix**

Let $m$ and $n$ be positive integers. In the KH tables, an (broken into
three pieces), an entry $m$ denotes $\mathbb{Z}^m$ while an entry $m_n$ denotes $\mathbb{Z}^m_n$. Th

Thus, the entry at bigrading $(36, 109)$ in Figure 7

denotes $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$. 

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**Figure 7.** The KH table of the connected sum of the closure of the braid: $(\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_3)^6 \sigma_1 \sigma_2 \sigma_3 \sigma_4$ with itself for $i$-grading between 29 and 40.
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Declaration of Interest

No potential conflict of interest was reported by the author.

ORCID

Sujoy Mukherjee http://orcid.org/0000-0002-3319-6703

References

[1] Asaeda, M. M, Przytycki, J. H. (2004). Khovanov homology: torsion and thickness Advances in Topological Quantum Field Theory, 135–166. NATO Sci. Ser. II Math. Phys. Chem., 179, Dordrecht: Kluwer Acad. Publ.

[2] Bar-Natan, D. (2002). On Khovanov’s categorification of the Jones polynomial. Algebr. Geom. Topol. 2(1): 337–370. doi:10.2140/agt.2002.2.337

[3] Chandler, A., Lowrance, A., Sazdanović, R, Summers, V. Torsion in thin regions of Khovanov homology. Preprint https://arxiv.org/abs/1903.05760.

[4] Jones, V. F. R. (1985). A polynomial invariant for knots via von Neumann algebras. Bull. Amer. Math. Soc. 12(1): 103–111. doi:10.1090/S0273-0979-1985-15304-2

[5] Kauffman, L. H. (1987). State models and the Jones polynomial. Topology 26(3): 395–407. doi:10.1016/0040-9383(87)90009-7

[6] Khovanov, M. (2000). A categorification of the Jones polynomial. Duke Math. J. 101(3): 359–426. doi:10.1215/S0012-7094-00-10131-7

[7] Kronheimer, P. B, Mrowka, T. S. (2011). Khovanov homology is an unknot-detector. Publmathihies. 113(1): 97–208. doi:10.1007/s10240-010-0030-y

[8] Mukherjee, S., Przytycki, J. H., Silvero, M., Wang, X, Yang, S. Y. (2018). Search for Torsion in Khovanov Homology. Exp. Math 27(4): 488–497. doi:10.1080/10586458.2017.1320242

[9] Mukherjee, S. (2019). On Skein Modules and homology theories related to knot theory. Ph.D. thesis. The George Washington University. 124pp. Available at: https://pqdtopen.proquest.com/pubnum/13810465.html.

[10] Murasugi, K. (1974). On closed 3-braids. Memoirs of the American Mathematical Society. No. 151. Providence, RI: American Mathematical Society.

[11] Przytycki, J. H, Sazdanović, R. (2014). Torsion in Khovanov homology of semi-adequate links. Fund. Math. 225(1): 277–304. doi:10.4064/fm225-1-13

[12] Rasmussen, J. (2005). Knot polynomials and knot homologies. In Geometry and topology of manifolds 261–280. Fields Inst. Commun., 47. Providence, RI: Amer. Math. Soc. https://bookstore.ams.org/fic-47/.

[13] Schütz, D. Torsion calculations in Khovanov cohomology (in preparation). http://maths.dur.ac.uk/~dma0ds/

[14] Shumakovitch, A. N. (2014). Torsion of Khovanov homology. Fund. Math. 225(1): 343–364. doi:10.4064/fm225-1-16

[15] Turner, P. (2017). Five lectures on Khovanov homology. J. Knot Theory Ramifications 26(3): 41pp., 1741009. doi:10.1142/S0218216517410097

[16] Viro, O. (2004). Khovanov homology, its definitions and ramifications. Fund. Math. 184: 317–342. doi:10.4064/fm184-0-18

[17] Watson, L. (2007). Knots with identical Khovanov homology. Algebr. Geom. Topol. 7(3): 1389–1407. doi:10.2140/agt.2007.7.1389