Statistical-mechanical approaches to the problem of phase retrieval in adaptive optics in astronomy

Yohei Saika and Hidetoshi Nishimori

1 Department of Electrical and Computer Engineering, Wakayama National College of Technology, 77 Noshima, Nada, Gobo, Wakayama 644-0023, Japan
2 Department of Physics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro, Tokyo 152-8551, Japan

E-mail: saika@wakayama-nct.ac.jp

Abstract. We construct the statistical-mechanical formulation for the problem of phase retrieval in adaptive optics using the Q-Ising model on the square lattice. This method is based on the maximizer of the posterior marginal (MPM) estimate which is classified into two kinds selectively used according to the sensitivity of measurement. Next, using the Markov-chain Monte Carlo simulation for a typical wave-front in adaptive optics, we obtain the result that our method works well for phase retrieval if we appropriately select the model of the true prior respective of the sensitivity of measurement. Further, these results are qualitatively confirmed by the replica theory for the infinite-range model.

1. Statistical-mechanical formulation for phase retrieval in adaptive optics

In recent years, a lot of researchers have been working on the problems of information science, related to image analysis, statistics for spatial data and Markov-random fields [1,2]. On the basis of the analogy between probabilistic information processing and statistical mechanics, various statistical-mechanical approaches [3,4], such as the mean-field theory and the replica theory, have been applied to the problems of image restoration and error-correcting codes. Recently, these approaches have been used for various problems, such as quantum computation and mobile communication.

In the field of adaptive optics in astronomy, wave-fronts generated by a reference source carry information to a receiver at the ground-based telescope through the transmission channel with atmospheric turbulence and measurement errors. Therefore, in order to construct reliable wave-front compensation system, it is important to reconstruct wave-fronts using the wave-front slopes observed by the optical system with the wave-front sensor. For this problem, the least-squares estimation [5] and its variants, the MAP estimate [6] using various cost function have been attempted.

In this study, on the basis of statistical mechanics of the Q-Ising model on the square lattice, we formulate the problem of phase retrieval in adaptive optics for astronomy using the maximizer of the posterior marginal (MPM) estimate, which is divided into two kinds according to the sensitivity of measurement.

If the Nyquist condition holds at every sampling point, we retrieve wave-fronts so as to maximize the posterior marginal probability using a set of observed wave-front slopes due to the shearing interferometer. Here the posterior probability is estimated using the models of true prior and noise probability using the Bayes formula. The model prior is the Boltzmann factor of the Q-Ising model under the uniform field enhancing smooth structures in the $z>0$ region. Then, noise probability is the
Boltzmann factor of random couplings using the observed wave-front slopes restricted to the principal interval \([-\pi, \pi]\). In this study, we generate a typical wave-front in adaptive optics for astronomy. Then, we consider that the wave-front is corrupted by the Gaussian noise with a variance corresponding to atmospheric turbulence and that the wave-front slopes are corrupted by the complex Gaussian noise corresponding to measurement errors, when they are observed by the shearing interferometer. On the other hand, if “aliasing” occurs due to under-sampling, we retrieve wave-fronts by the MPM estimate using an initial wave-front constructed so as to minimize the differences between neighboring wave-front slopes.

2. Performance estimation

Using the Markov-chain Monte Carlo simulation for a typical wave-front in adaptive optics, we estimate the performance of the MPM estimate through the static and dynamic properties.

When Nyquist condition holds at every sampling point, as is shown in figure 1, the Monte Carlo simulation indicates that the MPM estimate realizes phase retrieval more accurately than the conventional least-squares estimation even if we use the model prior with uniform distribution, and that the performance of the MPM estimate is improved by the model prior expressed by the Boltzmann factor of the uniform field, if we appropriately set parameters. Then, we note here that these results are qualitatively confirmed by the replica theory. Then, as is shown in figure 2, the Monte Carlo simulation shows that the mean square error smoothly diverges if \( h \) is less than the optimal value \( h_{\text{opt}}=0.02 \), and that the mean square error shows the non-monotonic behavior, \( h \) is larger than \( h_{\text{opt}} \). That is, On the other hand, if “aliasing” occurs due to under-sampling, the Monte Carlo simulation derives the result that the wave-front compensation succeeds if we start from an initial wave-front constructed due to the smoothness condition for the wave-front slopes between neighboring sampling points.

From the above results, our method based on the MPM estimate works well for the problem of phase retrieval, if we selectively choose our methods according to the sensitivity of measurement.

![Image](image1.png)

**Figure 1.** The mean square error as a function of \( h \) due to the MPM estimate

![Image](image2.png)

**Figure 2.** Time evolution of the mean square error due to the MPM estimate

References

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