Fluctuations and photon statistics in quantum metamaterial near the superradiant transition

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The analysis of single-mode photons fluctuations and their counting statistics at the superradiant phase transition is presented. The study concerns the equilibrium Dicke model in a regime where the Rabi frequency, related to a coupling of the photon mode with a finite-number qubit environment, plays a role of the transition’s control parameter. We use the effective Matsubara action formalism based on the representation of Pauli operators as bilinear forms with complex and Majorana fermions. Then, we address to the average photon number, define the fluctuational Ginzburg-Levanyuk region of the phase transition and analyze fluctuations of quasiparticles and that of the superradiant order parameter. We determine the cumulant generating function which describes a full counting statistics of equilibrium photon numbers.

I. INTRODUCTION

The dynamics of quantum metamaterials [1–9], which are the hybrid systems where cavity photons interact with multi-qubit environment, attracts a great interest. The behavior of such systems is captured by the Dicke model [10–12]. The interactions can be characterized by a collective Rabi energy proportional to a product of the individual qubit-cavity coupling and square root of the qubit number. If the Rabi energy is larger than the photon mode frequency then the superradiant transition, characterized by an emergence of a large photon number and order parameter, occurs for temperatures lower than a critical value. The rigorous study of such transition was proposed by Fedotov and Popov in the framework of Matsubara effective action for the photon field [13]. In that work the chemical potential was assumed to be zero and, consequently, the excitations’ number was not constrained. In Refs. [14, 15] an opposite case of finite chemical potential in the Dicke model was addressed and it was shown that the Bose condensation of polaritons is emerged [13, 15] The Keldysh diagrammatic approach for finite-N corrections, as well as effects of dissipation and external driving, were studied in Refs. [16, 17].

Alternatively to the temperature driven transition discussed in Ref. [13], the superradiance can be turned on by an increase of the interaction. It takes place if the Rabi frequency is above a critical value. A realization of a control parameter as the interaction strength, in principle, is possible for quantum metamaterials such as superconducting qubits arrays [1, 2, 18, 19] integrated with a GHZ transmission line via tunable couplers [20–23]. Also, this may be done in hybrid systems with a controllable amount of nitrogen-vacancy (NV) centers in a diamond sample which interact with an electromagnetic field in a single-mode cavity [24–26].

In the present paper we address to the situation where the Rabi frequency in a quantum metamaterial is varied from weak to ultra-strong coupling domains whether the temperature is kept constant. We also preserve a constant number of qubits N. We assume that N is large but finite. The finiteness of N in our consideration means that the superradiant transition is smoothed by the fluctuations of the order parameter and, beside of that, thermal fluctuations of polariton quasiparticles. The aim of this work is (i) to describe fluctuations of the above two types and (ii) formulate a full counting statistics for the photon numbers in this regime.

The main results consist of analytical expressions for the average photons number, its fluctuations and full counting statistics as functions of the collective Rabi frequency. Our formalism provides a solution for low temperature T and large N if they satisfy the inequalities \(\hbar \omega \gg k_B T \gg \hbar \omega / N\) (in this case all qubits are assumed to be in a resonance with the photon mode of the frequency \(\omega\)). The generalizations for the high-temperature limit, \(k_B T \gg \hbar \omega\), and dispersive regime, where a spectral density of qubits energies is strongly broadened, are also discussed.

The paper is organized as follows. In the Sec. II we present Matsubara action for the Dicke model, where the qubits degrees of freedom are formulated through the
Majorana and complex fermion variables. This is one of possible representations of Pauli operators acting in a Hilbert space of a two-level system. In Sec. III we derive the photon mode’s effective action which was obtained in previous works [13, 15] by means of alternative techniques. In the Sec. IV we present the general expressions for the average photon number and their fluctuations in a region near the superradiant transition. In the Secs. V, VI we discuss statistical properties at the critical point and generalize the results for high temperatures and inhomogeneous broadening in qubit ensemble. In Sec. VII the cumulant generating function for the photon number is derived. In the Sec. VIII we conclude. In the Appendix X we derive the conditions, where our solution based on the Gaussian approximation for thermal fluctuations is strict.

II. PATH INTEGRAL FORMULATION

The Dicke Hamiltonian of N qubits reads (we set $\hbar = 1$ and $k_B = 1$ throughout the text):

$$
\hat{H} = \omega \hat{\psi}^+ \hat{\psi} + \sum_{j=1}^{N} \frac{\epsilon_j}{2} \hat{\sigma}^+_j + \sum_{j=1}^{N} g_j (\hat{\psi} \hat{\sigma}^-_j + \hat{\psi}^+ \sigma^-_j). \tag{1}
$$

Here $\epsilon_j$ are the qubits energies, $g_j$ are the individual coupling strengths between $j$-th qubit and the photon field in a single-mode cavity. The fundamental frequency of the photon mode is $\omega$. The coupling term is introduced in the standard rotating wave approximation.

In a path integral formulation the photon mode is described by a conventional complex bosonic fields $\hat{\psi}, \hat{\psi}^+$. The Pauli operators, $\hat{\sigma}^+_j, \hat{\sigma}^-_j$, acting on the $j$-th qubit degrees of freedom, may be represented in path integrals in different ways. It can be bosonic Holstein-Primakoff representation [27] or bilinear forms of fermions. What concerns the fermion representations for the Dicke model in previous works, a techniques based on imaginary chemical potential [13] or auxiliary boson field [14] were used. Additional constraints are necessary to eliminate the emergent unphysical states and to reduce a phase space to that of a spin-1/2. We use another one fermion representation in our approach. It reads as the product of a complex $\hat{c}_j \neq \hat{c}_j^+$ and Majorana $\hat{d}_j = \hat{d}^+_j$ fermion operators [28]:

$$
\hat{\sigma}^+_j = \hat{c}^+_j \hat{d}_j, \quad \hat{\sigma}^-_j = \hat{d}_j \hat{c}_j. \tag{2}
$$

They correspond to three Grassmann fields $\bar{c}, c$ and $d$. The use of Majorana fermion allows to avoid the use of auxiliary constraints in the action. This fermion representation has been recently applied to spin-boson model [29, 30] and to a description of spin-spin interaction via helical Luttinger liquid [31]. In our consideration of the Dicke model (1), the use of the representation (2) seems to be useful for a study of fluctuation-dominated regimes.

The effective action, which describes superradiant transition, has been derived in Refs. [13, 15]. Below we demonstrate how one can obtain it with the use of the fermionization (2). The starting point of such consideration is the path integral formulation of the partition function $Z$ in terms of the boson complex fields $\Psi_\tau = [\hat{\psi}_\tau, \psi_\tau]$ and fermion fields $\bar{c}, c, d$ [32]:

$$
Z = \int \mathcal{D}[\Psi, \bar{c}, c, d] \exp(-S[\Psi, \bar{c}, c, d]) \tag{3}
$$

with the action is

$$
S[\Psi, \bar{c}, c, d] = S_{ph}[\Psi] + S_q[\bar{c}, c, d] + S_{int}[\bar{c}, c, d] + \ln Z_{ph} Z_q. \tag{4}
$$

Here $S_{ph}[\Psi], S_q[\bar{c}, c, d]$ and $S_{int}[\bar{c}, c, d]$ are the Matsubara actions of the photon mode, qubit environment and their interaction, respectively. The last term is due to a normalization of $Z$ to unity at the decoupled limit of $g_j = 0$.

Below we define the terms in (4) in more details. Both of the qubit and photon subsystems are assumed to be in thermal equilibrium at the temperature $T$. The photon mode action, defined on the imaginary time interval $\tau \in [0, \beta]$, where $\beta = 1/T$, is

$$
S_{ph}[\Psi] = \int_{0}^{\beta} \bar{\psi}_\tau (-G_{ph;\tau'\tau}) \psi_{\tau'} d\tau, \tag{5}
$$

where the inverse Green function of free photon mode

$$
G_{ph;\tau'\tau}^{-1} = \delta_{\tau'\tau}(-\partial_{\tau'} - \omega). \tag{6}
$$

The Fourier transformations from $\tau$ to Matsubara frequencies $2\pi n T$ are defined for the fields and for the Green functions as

$$
\psi_n = T \int_{0}^{\beta} \bar{\psi}_\tau e^{i 2\pi n T \tau} d\tau, \quad \bar{\psi}_n = T \int_{0}^{\beta} \psi_{\tau'} e^{-i 2\pi n T \tau'} d\tau' \tag{7}
$$

and

$$
G_{ph;n}^{-1} = \int_{0}^{\beta} G_{ph;\tau'\tau}^{-1} e^{i 2\pi n T \tau'} d\tau' = i 2\pi n T - \omega. \tag{8}
$$

In this representation the photon mode action (5) is transformed into

$$
S_{ph}[\Psi] = \beta \sum_{n} \bar{\psi}_n (-G_{ph;n}) \psi_n. \tag{9}
$$

The qubit ensemble action is

$$
S_q[\bar{c}, c, d] = \frac{1}{2} \sum_{j=1}^{N} \int_{0}^{\beta} \left[ \bar{c}_j c_j + d_j \right] (-G_{j;\tau'\tau}) \left[ d_j \bar{c}_j \right] d\tau d\tau'. \tag{10}
$$
The matrix \( G^{-1}_{j;\tau-\tau'} \) describes the \( j \)-th qubit. It contains the inverse Green functions for the \( j \)th complex fermion and its conjugate with the energies \( \pm \epsilon_j \), respectively, and the Majorana fermion of zero energy:

\[
- G^{-1}_{j;\tau-\tau'} = \begin{bmatrix}
\partial_{\tau'} + \epsilon_j & 0 & 0 \\
0 & \partial_{\tau'} - \epsilon_j & 0 \\
0 & 0 & \partial_{\tau'} \end{bmatrix}.
\] (11)

Bilinear forms \( c_j d_j \) and \( \bar{c}_j d_j \) appear in \( S[\Psi, \bar{c}, c, d] \) due to the qubit-cavity coupling encoded by the matrix \( V_j[\Psi_\tau] \):

\[
S_{\text{int}}[\bar{c}, c, d] = \frac{1}{2} \sum_{j=1}^{N} \int_0^\beta [\bar{c}_j c_j d_j] \delta_{\tau-\tau'} V_j[\Psi_\tau] \begin{bmatrix} c_j \\ \bar{c}_j \\ d_j \end{bmatrix} d\tau d\tau'.
\] (12)

This is the matrix which involve the complex boson fields \( \psi, \bar{\psi} \) as follows:

\[
V_j[\Psi_\tau] = g_j \begin{bmatrix} 0 & 0 & -\psi_\tau \\ 0 & 0 & \bar{\psi}_\tau \\ -\bar{\psi}_\tau & \psi_\tau & 0 \end{bmatrix}.
\] (13)

The normalization term in (4) is the product of partition functions of non-interacting photon mode and \( N \) qubits. The logarithms of their partition functions \( Z_{\text{ph}} = \int D[\Psi] \exp(-S_{\text{ph}}[\Psi]) \) and \( Z_q = \int D[\bar{c}, c, d] \exp(-S_q[\bar{c}, c, d]) \) are the following:

\[
\ln Z_{\text{ph}} = -\text{Tr} \ln(-G^{-1}_{\text{ph};\tau-\tau'})
\] (14)

and

\[
\ln Z_q = \frac{1}{2} \sum_{j=1}^{N} \text{Tr} \ln(-G^{-1}_{j;\tau-\tau'}).
\] (15)

Expanding the logarithm in the last term of (16) we obtain that odd order terms are equal to zero. It follows from the diagonal and non-diagonal structures of \( G_j \) and \( V_j \), respectively. The resummation back of the non-zero terms of even orders gives the identity:

\[
\text{Tr} \ln(-G^{-1}_{j;\tau-\tau'} + \delta_{\tau-\tau'} V_j[\Psi_\tau]) = \ln Z_q + \frac{1}{2} \text{Tr} \ln(-G^{-1}_{j;\tau-\tau'} + V_j[\Psi_\tau] G_{j;\tau-\tau'} V_j[\Psi_\tau]).
\] (17)

A direct first order expansion of the logarithm in the second line of (17) by \( V[\Psi_\tau] G_{j;\tau-\tau'} \) provides Gaussian action for all Matsubara modes \( \psi_n \). As it will be shown in Sec. IV, this expansion results in divergent number of photons at the critical Rabi energy near the transition into superradiant phase (see Eq. 40). This follows from an infinite occupation of zero-frequency component of the field \( \psi_0 \equiv \int_0^\beta \psi d\tau \). In order to regularize this approximation, one should leave \( \psi_0 \) in zero order term of (17) and expand the logarithm by the fluctuations \( \delta \psi_{\tau} \equiv \psi_{\tau} - \psi_0 \). Note, that Fourier transformation \( \delta \psi_{\tau} \) gives the non-zero Matsubara components \( \psi_n \neq 0 \). The field \( \psi_0 \) is related to superradiant order parameter while the components \( \psi_n \neq 0 \) are related to thermal fluctuations of polaritonic quasiparticles.

The regularization assumes a redefinection of the Green function, \( G_j \rightarrow G_j[\Psi_0] \) with \( \Psi_0 = [\psi_0, \bar{\psi}_0] \), as follows:

\[
G^{-1}_{j;\tau-\tau'}[\Psi_0] \equiv G^{-1}_{j;\tau-\tau'} - V_j[\Psi_0] G_{j;\tau-\tau'} V_j[\Psi_0].
\] (18)

Here we introduce the matrix with zero-mode components

\[
V_j[\Psi_0] = \frac{1}{\beta} \int_0^\beta V_j[\Psi_\tau] d\tau.
\]

Below we limit our consideration of the fluctuations by their quadratic combinations in the action \( \delta \psi_{\tau} \delta \bar{\psi}_{\tau}, \delta \bar{\psi}_{\tau} \delta \psi_{\tau}, \) and \( \delta \psi_{\tau} \delta \bar{\psi}_{\tau} \). For certain values of large enough \( N \) and not arbitrary small \( T \) this approach is exact in that sense that corrections of higher orders are small enough. Namely, for a situation where qubits and photon mode are close to a resonance, \( \epsilon_j \approx \omega \), the second order expansion by fluctuations is valid if the condition

\[
T \gg \frac{\omega}{N}
\] (19)

holds (see Appendix X for a detailed derivation). This condition provides the range of parameters where one can go beyond the thermodynamic limit. In the further consideration the ratio \( \frac{\omega}{NT} \) defines the small parameter in our calculations.

Under the assumption of (19), we expand (17) up to a second order by the matrix

\[
V_j[\delta \Psi_\tau] \equiv V_j[\Psi_\tau] - V[\Psi_0],
\] (20)
which involves the fluctuating parts in $\delta \Psi_T = [\delta \bar{\psi}_r, \delta \psi_r]$. We note that the first order contribution by $V_\Psi \delta \Psi_T$ equals zero in this method. As a result, we obtain

$$S_{\text{eff}}[\Psi] = S_{\text{ph}}[\Psi] + S_{\text{zm}}[\Psi_0] + S_{\text{fl}}[\Psi] + \ln Z_{\text{ph}}.$$  \hspace{1cm} (21)

The first term in not changed. The second term $S_{\text{zm}}[\Psi_0] = -\frac{1}{2} \sum_n \text{Tr} \ln \left( \mathbf{G}_n \mathbf{G}_n^{-1} [\Psi_0] \right)$ involves the zero-frequency mode $\Psi_0$ only. Note, that in the Dicke model (1) the interaction is limited by the rotating wave approximation which conserve the excitations number. In this case $S_{\text{zm}}$ depends on the zero mode’s magnitude squared, $\Phi \equiv \bar{\psi}_0 \psi_0$, and is independent on its complex phase $\varphi \equiv \arg \psi_0$. Thus, $S_{\text{zm}}[\Psi_0] = S_{\text{zm}}[\Phi]$ and its explicit expression is

$$S_{\text{zm}}[\Phi] = -\sum_{j=1}^N \ln \frac{\cosh \frac{\sqrt{z_j^2 + 2T^2}}{2T}}{\cosh \frac{z_j}{2T}}.$$ \hspace{1cm} (22)

This result follows from a representation of the Green functions $\mathbf{G}$ and $\mathbf{G}$ in Matsubara frequencies $\omega_n$. $S_{\text{zm}}$ is reduced to a calculation of infinite product by $n$. For the values of $N$ and $T$ restricted by the condition (19) the quadratic expansion by $\Phi$ in $S_{\text{zm}}[\Phi]$ allows to capture the superradiant transition (this corresponds to taking into account the non-Gaussian $|\psi_0|^4$).

The third term in (21) quadratic by quasiparticle fluctuations reads

$$S_{\text{fl}}[\Psi] =$$

$$\frac{\beta}{2} \sum_{n \neq 0} \left[ \bar{\psi}_n \psi_{-n} \right] \left[ \begin{array}{c} \Sigma_n[\Psi_0] \\ \Sigma_n[\Phi] \end{array} \right] \left[ \begin{array}{c} \psi_n \\ \bar{\psi}_{-n} \end{array} \right].$$ \hspace{1cm} (23)

This is dissipative part of the action, it correspond to effective photon-photon interaction via qubits degrees of freedom. The self-energy operators $\Sigma_n[\Psi_0]$ and $\Sigma_n[\Phi]$ provide normal and anomalous channels of the photon-photon interactions, respectively. They results from a summation over the fermionic Matsubara frequencies. From calculations it follows that normal self-energy depends on $\Phi$ only, $\Sigma_n[\Phi] = \Sigma_n[\Phi]$, while the anomalous one depends also on the phase, i.e., $\Sigma_n[\Psi_0] = \Sigma_n[\Phi, \varphi]$. The above results for $S_{\text{zm}}$ and $S_{\text{fl}}$ are in full correspondence with that derived in Refs. [13, 15].

The action $S_{\text{eff}}$ allows to calculate the thermodynamical average value $\langle \Phi \rangle$ which is known to be the superradiant order parameter. A non-zero dispersion of this variable $\langle (\Phi - \langle \Phi \rangle)^2 \rangle$ corresponds to the order parameter fluctuations near its average value. Due to the finite $N$ in our consideration this transition is smoothed by fluctuations. As long as $S_{\text{eff}}$ indicates the superradiance as a second order transition, we can take $\Phi = 0$ in the self-energies and suppose that the dependence on $\Phi$ is a perturbation. Expansion up to the first order by $\Phi$ gives for the normal part

$$\Sigma_n[\Phi] \approx \Sigma_n[0] + \Phi \Sigma'_n[0].$$ \hspace{1cm} (24)

The linear in $\Phi$ perturbation results in the small correction to average photon number if the condition (19) is satisfied (see Appendix X for details). The anomalous terms can be neglected as well because they are quadratic by $\Phi$ in the leading order, i.e., $\Sigma_n[\Phi, \varphi] \propto \Phi^2$. In other words, the consideration $S_{\text{fl}}[\Phi] = S_{\text{fl}}[\Phi=0, \delta \Psi] = 0$. Note, that $S_{\text{fl}}$ is purely Gaussian in this case because the terms proportional to $\bar{\psi}_n \psi_n$ are neglected.

To summarize the above, in the limit of $T \gg \omega/N$ we apply quadratic expansion by $\Phi$ in $S_{\text{zm}}$ and set $\Sigma_n[\Phi] = \Sigma_n[0] = 0$ and $\Sigma_n[\Phi, \varphi] = 0 = 0$. We arrive at $S_{\text{eff},0}$ which is used in the further studies:

$$S_{\text{eff},0}[\Phi, \bar{\psi}_n, \psi_n] = A \Phi + \Gamma \Phi^2 +$$

$$+ \beta \sum_{n \neq 0} \left[ -i2\pi nT + \omega + \Sigma_n[0] \right] \bar{\psi}_n \psi_n.$$ \hspace{1cm} (25)

The parameters are:

$$A = \beta \omega - \beta \sum_{j=1}^N \frac{g_j^2}{\epsilon_j} \tanh \frac{\beta \epsilon_j}{2},$$ \hspace{1cm} (26)

$$\Gamma = \frac{\beta}{4} \sum_{j=1}^N \frac{g_j^2}{\epsilon_j^3} \left[ (\cosh \beta \epsilon_j + 1) \right],$$ \hspace{1cm} (27)

and

$$\Sigma_n[0] = \sum_{j=1}^N \frac{g_j^2}{2(2\pi nT - \epsilon_j)}.$$ \hspace{1cm} (28)

In the above formulation, the superradiant transition does occur if the parameter $A$ is in its critical point $A_c = 0$. For $A < 0$ the system is in the superradiant phase with large amount of photons. In other words, if $A < 0$ then $S_{\text{eff},0}$ has a minimum at $\Phi = \Phi^*$ with

$$\Phi^* = -\frac{A}{2T}.$$ \hspace{1cm} (29)

In terms of the initial photon field this corresponds to a saddle line which is a circle in the complex $\psi_0$-plane.

The control parameter of the phase transition is the collective Rabi frequency. We define it here as

$$\Omega = \sqrt{\frac{N}{2} \langle g^2 \rangle_j}$$ \hspace{1cm} (30)

with the averaging over the qubit ensemble. The superradiance condition $A < 0$ corresponds to the Rabi frequency exceeding a certain critical value $\Omega > \Omega_c$. For the homogeneous limit where all qubits have the same energy, $\epsilon_j = \bar{\epsilon}$, the saddle point (29) is given by

$$\Phi^* = N \frac{\epsilon^2 (\Omega^2 - \bar{\epsilon} \omega)}{\Omega^2} \frac{1 + \cosh \bar{\epsilon} \tanh \frac{\beta \bar{\epsilon}}{2}}{\sinh \beta \bar{\epsilon} \tanh \frac{\beta \bar{\epsilon}}{2}}.$$ \hspace{1cm} (31)

Here we introduced $\Omega_T$ which is the Rabi frequency renormalized by a finite temperature, $\Omega_T = \Omega \sqrt{\tanh \frac{\beta \bar{\epsilon}}{2T}}$. 


For the resonant limit of the superradiance with the use of $S$ where

$$
\langle \Phi \rangle = \frac{1}{\sqrt{2\pi\alpha T}} e^{-\frac{\alpha T}{2}}.
$$

The disorder in $g_j$, in its turn, is taken into account. The parameters (26, 27) are reduced to

$$
\alpha = A_{\epsilon_j=\omega} = \beta \omega \left(1 - \frac{\Omega_c^2}{\omega^2}\right),
$$

$$
\gamma = \Gamma_{\epsilon_j=\omega} = q f(\beta \omega) \frac{\beta \Omega_c^4}{\omega^3}.
$$

We introduced here the function $f(x) = \frac{\sinh x - x}{1 + \cosh x}$; the parameter $q$ is a ratio between fourth and second moments for coupling parameters, $q = \langle g^4 \rangle / \langle g^2 \rangle^2$ and $\langle g^k \rangle = N^{-1} \sum_{j=1}^N g_j^k$. The absence of the disorder in $g_j$ corresponds to $q = 1$; in disordered case $q > 1$; $q = 9/5$ for a flat distribution ranging from $g_{\min}$ to $g_{\max}$ with $g_{\max} \gg g_{\min}$.

In further consideration the photon number

$$
\langle N_{ph} \rangle = \beta^{-1} \int_0^\beta \langle \bar{\psi}_\tau \psi_\tau \rangle d\tau
$$

is analyzed. Alternatively, it is given by the following identity

$$
\langle N_{ph} \rangle = T \sum_n (-G_n) - \frac{1}{2},
$$

$$
G_n = -\beta \langle \bar{\psi}_n \psi_n \rangle
$$

where $G_n$ is $n$-th component of Matsubara Green function. If the Gaussian approximation is applied, i.e. $\gamma = 0$ in the action $S_{eff,0}[\Phi, \bar{\psi}_n, \psi_n]$, then the following expression for the Green function is obtained for arbitrary $\epsilon_j$ and $\omega$:

$$
G_n = \frac{1}{[2\pi n T - \omega - \Sigma_n[0]]}.
$$

For the resonant limit $\epsilon_j = \omega$ we have:

$$
G_n(\epsilon_j=\omega) = \frac{\omega - 2i\pi n T}{(2\pi n T + i\omega)^2 + \Omega_T^2}.
$$

It is used in the calculations below. This expression holds for any $n$ in the Gaussian approach ($\gamma = 0$). After the summation one yields the average photon number:

$$
\langle N_{ph} \rangle_{Gauss} = \frac{1}{4} \left[ \coth \frac{\omega - \Omega_T}{2T} + \coth \frac{\omega + \Omega_T}{2T} \right] - \frac{1}{2}.
$$

One can see that $\langle N_{ph} \rangle_{Gauss}$ is divergent at the critical value of the renormalized Rabi energy $\Omega_{T,c} = \omega$ and is negative for $\Omega_T > \omega$. This follows from the Green function at $n = 0$, which is $G_0_{Gauss} = -\frac{1}{\alpha T}$. It is divergent at the critical point where $\alpha = 0$, see (33).

After the regularization with $\gamma \neq 0$ the zero mode’s Green function is changed to $G_z = -\langle \Phi \rangle_0$ which is not divergent. For non-zero modes the expression $G_{n \neq 0}$ is the same as in (39). In this non-Gaussian by $\psi_n$ expansion we rewrite the the average $\langle N_{ph} \rangle$ in terms of the action $S_{eff}[\Phi, \bar{\psi}_n, \psi_n]$ variables as

$$
\langle N_{ph} \rangle = \langle \Phi \rangle + \sum_{n \neq 0} \langle \bar{\psi}_n \psi_n \rangle - \frac{1}{2}.
$$

Let us calculate both of the contributions originating from the superradiant mode, $\langle \Phi \rangle$, and from the thermal excitations $\langle \bar{\psi}_n \psi_n \rangle$. As long as there is no explicit dependence on $\varphi$, one has $\int \int d\varphi d\varphi$ $\langle \bar{\psi}_n \psi_n \rangle = \pi \int_0^\infty d\Phi$. For $\langle \Phi \rangle$ we find

$$
\langle \Phi \rangle = \frac{\int \Phi e^{-S_0[\Phi]} d\Phi}{\int e^{-S_0[\Phi]} d\Phi} = -\frac{\alpha}{2\gamma} + \frac{e^{-\frac{\alpha \beta}{2}}}{\sqrt{\pi} \gamma} \text{erfc} \frac{\alpha}{\sqrt{\pi} \gamma},
$$

with the complementary error function is $\text{erfc} = 1 - \text{erf}$. Summation over $n \neq 0$ gives the quasiparticle contribution

$$
\sum_{n \neq 0} \langle \bar{\psi}_n \psi_n \rangle = \langle N_{ph} \rangle_{Gauss} - \frac{1}{\alpha}.
$$

Finally, for the average photon number (41) we obtain

$$
\langle N_{ph} \rangle = -\frac{\alpha}{2\gamma} + \frac{e^{-\frac{\alpha \beta}{2}}}{\sqrt{\pi} \gamma} \text{erfc} \frac{\alpha}{\sqrt{\pi} \gamma} + \langle N_{ph} \rangle_{Gauss} - \frac{1}{\alpha}.
$$

The fluctuations of the photon number are given by the second cumulant $\langle N_{ph}^2 \rangle = \langle N_{ph}^2 \rangle - \langle N_{ph} \rangle^2$. With use of the above notations it is reduced to the following:

$$
\langle N_{ph}^2 \rangle = \langle \Phi \rangle^2 + T^2 \sum_{n \neq 0} G_n^2.
$$

Calculation of the integrals by $\Phi$ and summation over $n$ provides

$$
\langle N_{ph}^2 \rangle = \frac{1}{2\gamma} + \frac{\sqrt{\frac{\alpha \beta}{2}}}{2\sqrt{\pi}} \text{erfc} \frac{\alpha}{\sqrt{2\pi} \gamma} + e^{-\frac{\alpha^2}{2}} + \langle N_{ph}^2 \rangle_{Gauss} - \frac{1}{\alpha^2}.
$$
We introduced here the second cumulant in Gaussian approximation \( \langle N_{ph}^2 \rangle = T^2 \sum_n G_n^2 \). It reads

\[
\langle N_{ph}^2 \rangle_{\text{Gauss}} = \cosh \frac{\beta \Omega}{2} \left( \cosh \frac{\Omega}{T} + \frac{T \Omega}{2 \Omega_T} \sinh \frac{\Omega}{T} \right) - 1 - \frac{T \Omega}{2 \Omega_T} \sinh \frac{2 \Omega}{T} \frac{4 \left( \cosh \frac{\beta \Omega}{2} - \cosh \frac{\Omega}{T} \right)}{4 \left( \cosh \frac{\beta \Omega}{2} - \cosh \frac{\Omega}{T} \right)}^2. \tag{47}
\]

Similar to (44), the divergent zero frequency term in the sum is canceled by \( 1/\alpha^2 \) in (46). In the Section V the properties of the photon number and its cumulant are analyzed in details.

V. PHASE TRANSITION AT LOW AND HIGH TEMPERATURES

A. Average photon number

In the following consideration at low temperatures \( T \ll \omega \), we should emphasize that there is also a limitation (19) which means that \( T \) can not be arbitrary small. Namely, it belongs to the domain

\[
\omega \gg T \gg \frac{\omega}{N}. \tag{48}
\]

In such limit we set \( f(\beta \omega) = 1 \) and \( \Omega_T = \Omega \) with an exponential accuracy.

Let us start from the average photon number as the function of \( \Omega \). The typical dependence of \( \langle N_{ph} \rangle \) is shown as bold curve in Fig. 1 in logarithmic scale. The parameters here are: \( \omega = 10^7 \), the critical Rabi frequency \( \Omega_c = \omega = 10^7 \) and qubits number \( N = 100 \). In the vicinity of the phase transition \( \langle N_{ph} \rangle \) demonstrates the rapid growth.

We obtain an analytical expansion of photon number \( \langle N_{ph} \rangle \) (44) around the critical point \( \Omega_c = \omega \). The expansion in series by the dimensionless detuning \( \Omega - \omega / \omega \) for \( \langle N_{ph} \rangle \) is obtained below:

\[
\langle N_{ph} \rangle \approx \left[ \sqrt{\frac{NT}{\pi q \omega}} + \delta n_0 \right] - \frac{1}{2} + \left[ \frac{N(\pi - 2)}{\pi q} + \delta n_1 \right] \frac{\Omega - \omega}{\omega} + O \left[ \left( \frac{\Omega - \omega}{\omega} \right)^2 \right]. \tag{49}
\]

The main contribution to \( \langle N_{ph} \rangle \) follows from the \( \psi_0 \) mode as powers of \( \sqrt{N T / \omega} \). The prefactors contain the leading term given by zero mode, and small corrections \( \delta n_1 \), which follows from the fluctuations of the modes \( \psi_n \neq 0 \). Their expressions might be obtained from the expansion of (43) as

\[
\delta n_0 = \frac{1}{4} - \frac{T}{4 \omega}. \tag{50}
\]

Let us start from the average photon number as the function of \( \Omega \). The typical dependence of \( \langle N_{ph} \rangle \) is shown as bold curve in Fig. 1 in logarithmic scale. The parameters here are: \( \omega = 10^7 \), the critical Rabi frequency \( \Omega_c = \omega = 10^7 \) and qubits number \( N = 100 \). In the vicinity of the phase transition \( \langle N_{ph} \rangle \) demonstrates the rapid growth.

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\[
\langle N_{ph} \rangle \approx \left[ \sqrt{\frac{NT}{\pi q \omega}} + \delta n_0 \right] - \frac{1}{2} + \left[ \frac{N(\pi - 2)}{\pi q} + \delta n_1 \right] \frac{\Omega - \omega}{\omega} + O \left[ \left( \frac{\Omega - \omega}{\omega} \right)^2 \right]. \tag{49}
\]

The zero order term in (49) gives large but finite photon number at the critical point

\[
\langle N_{ph} \rangle_c = \sqrt{\frac{NT}{\pi q \omega}} - \frac{1}{4}. \tag{52}
\]

The leading term is much higher than unity under the condition (48). If one take a formal limit of \( T \to 0 \) in (52) then obtain a unphysical negative value. It demonstrates that \( S_{\text{eff}, 0} \) can not be applied for low temperatures and non-Gaussian fluctuations of \( \psi_n \neq 0 \) or the dependence on \( \Phi \) and \( \varphi \) in the self-energy operators should be taken into account.

In the limit \( T \to 0 \) and above the critical point one has \( \langle N_{ph} \rangle = 1/2 \). It follows from the ground state wave function which is changed in this case from \( |0 \rangle \) to a superposition with equal weights of qubit and single photon excitations. In Appendix X we demonstrate how the constant value in \( \langle N_{ph} \rangle_c \) (52) is changed if one adds the correction \( \propto \psi_0 \psi_n \) to \( S_{\text{eff}, 0} \). This correction is due to the dependence of the normal part of the self-energy \( \Sigma_n \) on \( \Phi \). Integration over all non-zero modes provides the perturbation \( \delta S[\Phi] = -\delta \alpha \Phi \) to \( S_{\text{eff}} \). The parameter \( \delta \alpha = \frac{1}{2g \sqrt{T \omega}} \) is small. Calculation at the critical point \( \alpha = 0 \) with the perturbed action \( S_{\text{eff}}[\Phi] + \delta S[\Phi] \) provides the change of the constant term from \(-1/4 \) to \( 1/8 \) in \( \langle N_{ph} \rangle_c \). In other words, if the linear by \( \Phi \) term in \( \Sigma_n[\Phi] \) is taken into account in the action then going out of the applicability range to very low \( T \) does not give the unphysical negative result for \( \langle N_{ph} \rangle_c \).
B. Fluctuations and Fano factor at the superradiant critical point

At the critical point the fluctuations are contributed by the zero-mode in the low temperature limit:

\[ \langle N_{ph}^2 \rangle_c = \frac{(\pi - 2)NT}{2\omega}. \]  

(53)

We neglected here by a small contribution of the order \( T/\omega \) due to the \( \psi_r \)-fluctuational corrections. The relative value of fluctuations

\[ r = \frac{\langle N_{ph}^2 \rangle}{\langle N_{ph} \rangle^2}, \]

(54)

is large as \( e^{\omega/T} \) in the decoupling limit \( \Omega = 0 \) and decays monotonously due to the decreasing of the second cumulant. It is less than unity above the phase transition. Using the expressions for \( \langle N_{ph} \rangle \) and \( \langle N_{ph}^2 \rangle \), Eqs. (44) and (46), one obtains the expansion near the phase transition up to the first order by the dimensionless detuning:

\[ r \approx \frac{\pi - 2}{2} - (\pi - 3)\sqrt{\frac{\pi N \omega}{qT}} \frac{\Omega - \omega}{\omega}. \]  

(55)

At the critical point \( (\Omega = \omega) \) the main contribution is due to the zero mode and, consequently, \( r_c = \frac{\langle \Phi^4 \rangle}{\langle \Phi \rangle^2} \). For large enough detuning with \( \Omega > \Omega_c \), where the saddle point for \( \Phi \) appears, \( \sqrt{\langle N_{ph}^2 \rangle} \) is small compared to the average \( \langle N_{ph} \rangle \) (see dashed and bold curves in Fig. 1). The universal value of the relative fluctuations \( r \) at the transition point is

\[ r_c = \frac{\pi}{2} - 1. \]  

(56)

It follows from the \( \Phi \)-integrals (42) at \( \alpha = 0 \) and is exact up to the small correction \( \sim N^{-1/2} \).

From the expansion (55) the width of the fluctuational Ginzburg-Levanyuk region, \( \Omega_{GL} \), near the critical Rabi frequency can be defined. This is the zone where fluctuations and average value are of the same order. This consideration can be applied straightforward to the superradiant phase where \( \Omega > \Omega_c \). There is \( r \ll 1 \) beyond the fluctuational region, where \( \Omega - \Omega_c \gg \Omega_{GL} \). The parameter \( \Omega_{GL} \) is obtained from the matching conditions

\[ r(\Omega) \sim 1, \ \Omega - \Omega_c \sim \Omega_{GL}, \]

(57)

which give

\[ \Omega_{GL} \sim \sqrt{\frac{\omega T}{N}}. \]  

(58)

Approaching the critical point from the normal phase, i.e., \( \Omega < \Omega_c \), fluctuations are always greater that average values and the definition (57) is not valid. Instead of

(57) we introduce the width \( \Omega' \) where the superradiant order parameter fluctuations start to grow and become relevant. In this region the contribution to \( \langle N_{ph} \rangle \) due to the non-Gaussian fluctuations of \( |\psi_0|^4 \) is comparable with the quasiparticle’s part related to \( \psi_{r\neq0} \). We define \( \Omega' \) through the value of \( \Omega = \Omega_c - \Omega' \) which provides the matching between the average values obtained in the Gaussian and non-Gaussian approaches:

\[ \langle N_{ph} \rangle_{Gauss} \sim \langle N_{ph} \rangle, \ \Omega_c - \Omega \sim \Omega'. \]  

(59)

From (40) and (44) it follows that \( \langle N_{ph} \rangle_{Gauss} \sim T/\Omega' \) and \( \langle N_{ph} \rangle \sim \sqrt{NT/\omega} \). The width \( \Omega' \) of fluctuation-dominated region in the normal phase is obtained of the same order as in the superradiant phase, i.e.,

\[ \Omega' \sim \Omega_{GL} \sim \sqrt{\frac{\omega T}{N}}. \]  

(60)

It is rather narrow and is much less than the temperature due to the condition (48).

The Fano factor, defined as

\[ F = \frac{\langle N_{ph}^2 \rangle}{\langle N_{ph} \rangle^2}, \]

reveals a non monotonous dependence on \( \Omega \) as shown in the Fig. 1 (dashed curve). It starts to decrease from

\[ F = \frac{1}{1 - e^{-\beta \omega}}, \]  

(61)

being a value close to unity in the decoupled limit (see Section VII C for details). The decreasing to lower values means a negative correlation between photons which interact through the qubit environment. There is the minimum in \( F \) at a particular \( \Omega_{min} < \Omega_c \) and the maximum at the transition to the superradiant phase \( \Omega = \Omega_c \):

\[ F_c = \frac{1}{2} (\pi - 2) \sqrt{\frac{NT}{\pi q \omega}}. \]  

(62)

This value exceeds unity and, consequently, corresponds to significant positive correlations at the critical point.

VI. SOME GENERALIZATIONS

A. High temperatures

Below we discuss results obtained at the critical point for the high temperature regime \( T \gg \omega \). Note that the phase transition at \( \alpha = 0 \) (see Eq. (26)) is given by the increased collective coupling:

\[ \Omega_c = \sqrt{T/\omega}. \]  

(63)

We use (44) and (46) to obtain the leading order expansions for \( \langle N_{ph} \rangle \) and \( \langle N_{ph}^2 \rangle \) by the large parameter \( T/\omega \).
In the Appendix X we show that the Gaussian approximation for quasiparticle fluctuations, where the corrections due to cross terms $\Phi \bar{\psi}_n \psi_n$ are small, is valid for $N \gg 1$. This requirement is more soft than that of $N \gg \omega/T \gg 1$ in the low-temperature limit addressed above.

For the high temperature regime, the average photon number at very large $N \gg \omega/T \gg 1$ is contributed by the Matsubara zero mode $\Phi$ only. This regime is related to the thermodynamical limit. We obtain that the photon number at the critical point grows faster than (52) with the temperature as:

$$\langle N_{ph}\rangle_c = \sqrt{\frac{3N T}{\pi \omega}} \quad (64)$$

The fluctuations of photons,

$$\langle N_{ph}^2\rangle_c = \frac{3(\pi - 2)N T^2}{2\pi \omega^2} + \frac{T^{5/2}}{8\sqrt{2} \omega^{5/2}} \quad (65)$$

oppositely to (53), contain not only the contributions from $\Phi$ (first term), but from the non-zero modes $\psi_n$ as well (second term). Thus, the high temperature limit is distinct in that sense that there are two domains of $N$ where fluctuations have different contributions. The first domain for $N$ is related to the thermodynamical limit of very large qubit number. It is given by (65) as

$$N \gg \sqrt{\frac{T}{\omega}} \quad (66)$$

where the superradiant zero mode contributes only. The second one is the intermediate region,

$$\sqrt{\frac{T}{\omega}} \gtrsim N \gg 1 \quad (67)$$

where the thermal fluctuations of quasiparticles are more relevant than that of the order parameter. The relative value at the transition for this intermediate domain,

$$r_c = \frac{\pi - 2}{2} + \frac{\pi \sqrt{T}}{24 \sqrt{2} N \sqrt{\omega}} \quad (68)$$

shows a deviation from the universal value $\pi/2 - 1$ due to the second term. Thus, $N \sim \sqrt{T/\omega}$ defines a border between the domains where the superradiant transition behavior is related to the thermodynamic limit or to the regime of strong fluctuations.

B. Inhomogeneous broadening

In the above results for the resonant limit a spread of coupling energies $g_j$ yields the prefactor $g^{-1}$ for the qubit number. The inhomogeneous broadening of qubit energies modifies the expressions in a more significant way described below.

We assume that qubit frequencies are distributed in a certain interval, temperatures are low enough, $T \ll \epsilon_1$, and couplings are homogeneous, $g_j \equiv g$. We assume that the system is in the critical point, $\alpha = 0$, and photons number (44) is contributed by the zero mode only, i.e., $N_{ph} = \frac{1}{\sqrt{N}}$, and quasiparticles contributions are neglected. In the definition for $\gamma$ (27) the sum over qubit index is replaced by the integral over energies, $\sum_j \rightarrow N \int \rho(E)dE$ with the density of states $\rho(E)$ is normalized to unity ($E$ is a qubit’s energy). We discuss two cases which correspond to flat distributions with finite and very broad widths.

In the first case we consider the distribution with a median energy at $\epsilon$ and width $\Delta$, hence, the density of states is

$$\rho(E) = \frac{1}{\Delta} \theta(\Delta/2 - |E - \epsilon|) \quad (69)$$

The photon number is obtained as

$$\langle N_{ph}\rangle = z(\Delta/\epsilon)\sqrt{NT/\epsilon} \quad (70)$$

where dimensionless prefactor $z$ is

$$z(x) = \left(\frac{1}{x} - \frac{x}{4}\right) \ln \frac{1 + x/2}{1 - x/2} \quad (71)$$

In the homogeneous limit, $\Delta = 0$, this prefactor is unity.

Note, that the expression (70) provides the photon number at the critical point for the off-resonant regime, where $\epsilon \neq \omega$.

In the second case of the very broad distribution, qubits energies belong to the interval from $\epsilon$ up to large $\epsilon_c$ which is spectrum cut-off. This case is considered as a thermodynamic limit where the average level spacing can be introduced, $\delta \epsilon \equiv \epsilon_c/N$. Under the assumption

$$T \ll \{\epsilon, \omega\} \ll \epsilon_c \quad (72)$$

we find that

$$\langle N_{ph}\rangle = \sqrt{\frac{2T}{\pi \delta \epsilon \omega}} \ln \frac{\epsilon_c}{\epsilon} \quad (73)$$

In a physically relevant situation the lower edge of the qubits spectrum $\epsilon$ may be of the order of the resonator mode frequency, hence, their fraction is order of unity. The logarithm is also not very large number. Interestingly, that in this case we obtain that the photon number is affected mainly by the ratio between the smallest energy scales – the temperature and level spacing.

VII. COUNTING STATISTICS

A. Generating action

The effective action for quantum fluctuations (25) allows to we derive the full counting statistics for photon
numbers. These are cumulant and moment generating functions (CGF and MGF). These are functions of real counting variable $\xi$. In our consideration the generating action is introduced on the imaginary time.

The CGF and MGF are defined as follows through the partition function $Z(\xi)$

$$\text{CGF}(\xi) = \ln \text{MGF}(\xi), \quad \text{MGF}(\xi) = \frac{Z(\xi)}{Z(0)},$$

(74)

$$Z(\xi) = \int D[\Psi] \exp \left[ - S_{\text{eff},0}[\Phi, \bar{\psi}_n, \psi_n] - i \xi \left( \Phi + \sum_{n\neq 0} \bar{\psi}_n \psi_n - 1/2 \right) \right].$$

(75)

$T$-ordering in the imaginary time representation of the path integrals assumes that the photon number, introduced in (35), is defined as

$$N_{\text{ph}} = T \int_0^\beta \bar{\psi}_t \psi_{t+\beta} d\tau$$

(76)

in a generating term. Alternatively, the generating term can be also represented as a half sum of (76) with $+\theta$ and $-\theta$, which is symmetric under $T$- and anti-$T$-ordering. Due to the commutation of photon operators, we must involve $-1/2$ in (75). In the Matsubara representation we obtain the generating action in the form of (75) after such a symmetrization.

The photon number moments $\langle N_{\text{ph}}^n \rangle \equiv \langle (\hat{\psi}^+ \hat{\psi})^n \rangle$ are given by the derivatives

$$\langle N_{\text{ph}}^n \rangle = \left( i^n \frac{\partial^n}{\partial \xi^n} \text{MGF}(\xi) \right)_{\xi=0},$$

(77)

while the cumulants are defined as

$$\langle N_{\text{ph}}^n \rangle = \left( i^n \frac{\partial^n}{\partial \xi^n} \text{CGF}(\xi) \right)_{\xi=0}.$$

(78)

Path integration in (75) is reduced to the infinite product of Matsubara Green functions involving the counting variable

$$\text{MGF}(\xi) = e^{i\xi/2} \int_0^\infty e^{-(\alpha+i\xi)\Phi - \gamma \Phi^2} d\Phi \prod_{n\neq 0} G_n(\xi) G_n(0),$$

(79)

The Green function with the counting variable reads as

$$G_n(\xi) = \frac{1}{2\pi in - (\omega + i\xi T) - \Sigma_n[0]}, \quad n \neq 0.$$

(80)

Calculation of the integrals and product in (79) yields for the resonant case ($\epsilon_j = \bar{\epsilon} = \omega$):

$$\text{MGF}(\xi) = \text{MGF}_0(\xi) \text{MGF}_\beta(\xi),$$

(81)

where the zero mode’s and quasiparticles’ parts are

$$\text{MGF}_0(\xi) = \exp \left[ \frac{2i\alpha \xi - \xi^2}{4\gamma} \right] \frac{\text{erfc} \frac{\alpha + i\xi}{\sqrt{\pi\gamma}}}{\text{erfc} \frac{\xi}{2\sqrt{\gamma}}},$$

(82)

and

$$\text{MGF}_\beta(\xi) =$$

$$\left[ 1 + \frac{i\xi T \omega}{\omega^2 - \Omega^2} \right] \frac{\cosh \frac{\xi T}{\omega} - \cosh \frac{\Omega T}{\omega}}{\cosh \frac{\xi + i\xi T}{\omega} - \cosh \frac{\Omega + i\xi T}{\omega} - \xi^2 - \Omega^2},$$

(83)

With the use of this result for MGF one can obtain the above expressions for the photon number and its fluctuations (44) and (46).

### B. Full counting statistics at the phase transition

In the thermodynamic limit of large enough $N$, the leading contribution to cumulants is described by that of the zero mode $\text{MGF}_0(\xi)$. Thus, the CGF for the critical point is

$$\text{CGF}_0(\xi) = \frac{-\xi^2}{4\gamma} + \ln \left[ \frac{\text{erfc} \frac{\xi}{2\sqrt{\gamma}}}{\text{erfc} \frac{\xi}{\sqrt{\gamma}}} \right].$$

(84)

The first six cumulants, which follows from $\text{CGF}_0(\xi)$, are:

$$\langle N_{\text{ph}} \rangle = \frac{1}{\sqrt{\pi\gamma}},$$

(85)

$$\langle N_{\text{ph}}^2 \rangle = \frac{\pi - 2}{2\pi\gamma},$$

(86)

$$\langle N_{\text{ph}}^3 \rangle = \frac{4 - \pi}{2(\pi\gamma)^{3/2}},$$

(87)

$$\langle N_{\text{ph}}^4 \rangle = \frac{2(\pi - 3)}{(\pi\gamma)^2},$$

(88)

$$\langle N_{\text{ph}}^5 \rangle = \frac{96 - 40\pi + 3\pi^2}{4(\pi\gamma)^{5/2}},$$

(89)

$$\langle N_{\text{ph}}^6 \rangle = \frac{60(\pi - 2) - 7\pi^2}{(\pi\gamma)^3}. $$

(90)

From a numerical calculation it follows that higher cumulants alter their signs, for instance, as it seen from the negativity of the 5th and 6th ones. The non-zero cumulants for $n > 2$ is the consequence of that fact that photons’ probability distribution function is half of a Gaussian because of the positively defined variable of integration $\Phi$ in (79).

The Fourier transformation of the MGF provides the probability density to measure $N_{\text{ph}}$ photons on average

$$P(N_{\text{ph}}) = \int_{-\infty}^{\infty} \text{MGF}(\xi) e^{i\xi N_{\text{ph}}} d\xi.$$

(91)
Note, that $\mathcal{P}$ is a non-zero function of the continuous variable $N_{\text{ph}}$. This is due to that $N_{\text{ph}}$ is not an eigenvalue of the Hamiltonian (1). Hence, non-integer values $N_{\text{ph}}$ are assumed to be observed as the thermodynamical averages.

As long as the $\psi_n$-fluctuations are frozen out in a vicinity of the critical point and at the thermodynamic limit, one finds from (75) and (91) that the probability density is identical to the exponent in $Z(75)$ as

$$
\mathcal{P}_0(N_{\text{ph}}) = 2\pi\theta(N_{\text{ph}}) \frac{\exp[-\alpha N_{\text{ph}} - \gamma N^2_{\text{ph}}]}{Z(0)}.
$$

In particular, at the critical point $\text{MGF}_0(\xi)$ from (81) the distribution is

$$
\mathcal{P}_c(N_{\text{ph}}) = \left\{ \begin{array}{ll}
4\sqrt{\pi}\gamma \exp[-\gamma N^2_{\text{ph}}], & \text{if } N_{\text{ph}} \geq 0, \\
0, & \text{if } N_{\text{ph}} < 0.
\end{array} \right.
$$

This is the half of the Gaussian for $N_{\text{ph}} > 0$, while for unphysical $N_{\text{ph}} < 0$ it is zero. At the critical point (we assume below that $\Omega_c = \omega$), the distribution’s maximum is located at $N_{\text{ph}} = 0$. In the superradiant phase, the maximum of $\mathcal{P}(N_{\text{ph}})$ is shifted to a non-zero value reinstating a Gaussian tail. In other words, for higher values $\Omega \gg \omega$ one obtains from $\ln[\text{MGF}_0(\xi)]$ that in the leading order $\langle N_{\text{ph}} \rangle = \frac{\omega^2}{2\Omega}$ and $\langle N^2_{\text{ph}} \rangle = \frac{\gamma\omega^2}{2\Omega^2}$. The higher cumulants are strongly suppressed by the exponent: for instance, the third one is $\langle N^3_{\text{ph}} \rangle \sim e^{-N/\lambda}$.

### C. Statistics for weak interaction and normal phase

In this part we discuss MGF at the normal phase and weak coupling limit. It is assumed that the system is far away from the fluctuational region, i.e., $\Omega_T \ll \omega$ (see Eq. 60). Taking the limit $\gamma \rightarrow 0$ in (81) one obtains the MGF for the normal phase of the Dicke model:

$$
\text{MGF}(\xi) = \frac{(\cosh \frac{\Omega}{T} - \cosh \Omega_T) e^{\xi/2}}{\cosh \left[\frac{T}{2} + i\frac{\Omega}{2T}\right] - \cosh \left[\frac{\Omega}{T} - i\frac{\xi}{4}\right]}.
$$

In the decoupled limit, where the Rabi frequency is the smallest scale $\Omega_T \ll \{T, \omega\}$, one arrives at the MGF of the free photon mode of the frequency $\omega$

$$
\text{MGF}(\xi) = \frac{1 - e^{-\beta\omega}}{1 - e^{-i\xi - \beta\omega}}.
$$

Note, that it is $2\pi$-periodic function of the counting variable. The discrete Fourier transformation of (94) at the finite interval $[0; 2\pi]$ of the single period yields the standard Hibs distribution probabilities

$$
P_n = (1 - e^{-\beta\omega}) e^{-n\beta\omega}, \quad n \geq 0.
$$

Obviously, the infinite integral definition (91) one would obtains delta-peaks in the probability distribution density located at $N_{\text{ph}} = n \geq 0$, being the eigenvalues of the free photon mode Hamiltonian, as

$$
\mathcal{P}(N_{\text{ph}}) = \frac{1}{2\pi} \sum_{n \geq 0} P_n \delta(N_{\text{ph}} - n).
$$

Note that the cumulant generating function for the free mode is

$$
\text{CGF}(\xi) = \frac{i\xi}{2} - \ln \frac{\sinh \frac{\omega + i\xi}{2T}}{\sinh \frac{\omega}{2T}}.
$$

The cumulants itself are

$$
\langle \langle N^n_{\text{ph}} \rangle \rangle = \left\{ \begin{array}{ll}
\frac{i}{2} \coth \frac{\omega}{2T} - \frac{1}{2}, & n = 1; \\
(1 - 1)^{n-1} \frac{\rho^{\pi-1}}{\rho^{\pi-1}} \coth \xi |_{\xi = \frac{\omega}{2T}}, & n \geq 2.
\end{array} \right.
$$

One arrives at the mentioned above Fano factor $F = (1 - e^{-\beta\omega})^{-1}$ in (61) and the relative fluctuations parameter $\tau = e^{\beta\omega}$.

### VIII. CONCLUSIONS

In this work we addressed to fluctuations near superradiant transition which is driven by an interaction between a single-mode photons and multi-qubit environment. In such consideration the collective Rabi frequency is varied (it can be close to the critical value of superradiant transition), while the temperature $T$ is kept unchanged. We did not assume the thermodynamic limit of infinite qubits number $N$ and consider it as large enough but finite value. Our analysis was focused on two types of competing fluctuations – the thermal one and that of the superradiant order parameter. This regime is opposite to the transition by the temperature studied in Ref. [13].

We used Majorana fermion representation of qubits’ Pauli operators in order to formulate a path integral approach. Having started from the Dicke Hamiltonian, we demonstrate how one can derive the effective action for the photon mode, obtained by alternative fermionization techniques in Refs. [13, 15]. After that we calculated the average photons number and equilibrium fluctuations in terms of the effective action formalism. As a generalization, the full counting statistics, providing higher order cumulants of the photon numbers, was formulated.

The central result of this paper concerns a low temperature regime and a resonance between qubits and photon mode frequency $\omega$. It was shown that the Gaussian approximation for thermal fluctuations is exact and analytical solution can be found, if $\hbar\omega \gg k_BT \gg \hbar\omega/N$. In this limit the critical value of the collective Rabi frequency is $\Omega_c = \omega$ and the average photon number at this point is $\langle N_{\text{ph}} \rangle = \sqrt{N k_B T / (\pi \hbar \omega)}$. The relative fluctuations parameter $r_c = \langle \langle N_{\text{ph}}^2 \rangle \rangle / \langle N_{\text{ph}} \rangle^2$, where the second cumulant is $\langle \langle N^2_{\text{ph}} \rangle \rangle = \langle N^2_{\text{ph}} \rangle - \langle N_{\text{ph}} \rangle^2$, is universal at the critical point $r_c = \pi/2 - 1$. A zone near $\Omega_c$ in the superradiant phase, where $r$ is not suppressed, corresponds to the
fluctuational Ginzburg-Levanyuk region. The width of such frequency range is proportional to $\sqrt{\omega k_B T/(\hbar N)}$; this is much smaller than $k_B T$ and shrinks at thermodynamic limit. The another characteristic, Fano factor $F \equiv \langle N_{ph}^2 \rangle / \langle N_{ph} \rangle$, decreases from the unity in decoupled limit $\Omega \ll \Omega_c$ to a minimum $F < 1$ at $\Omega \lesssim \Omega_c$. The latter indicates a negative correlation between photons. The further increase of $\Omega$ up to the critical value $\Omega_c$ shows a non-universal enhancement of $r_{ph}$, which reveals a two-level nature of qubits environment. We believe that the above results can be of an interest in a context of state-of-the-art hybrid systems and quantum metamaterials based on ensembles of NV-centers or Josephson qubits tunable into strong coupling regimes.

For high temperatures, $k_B T \gg \hbar \omega$, the solution is obtained for $N \gg 1$. This condition is less strict compared to that in the low-temperature regime. The finiteness of the qubit number can change a behavior of fluctuations at the critical point. Namely, for $\sqrt{k_B T/(\hbar \omega)} \gtrsim N \gtrsim 1$ the quasiparticle fluctuations become greater than that of superradiant order parameter. This intermediate region shows a non-universal enhancement of $r_{ph}$, which reveals a two-level nature of qubits environment. We believe that the above results can be of an interest in a context of state-of-the-art hybrid systems and quantum metamaterials based on ensembles of NV-centers or Josephson qubits tunable into strong coupling regimes.

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X. APPENDIX

In this Appendix we calculate the non-Gaussian correction $\propto \Phi \psi_n \psi_n$ for the effective action $S_{\text{eff}, 0}$ and obtain a condition on the low temperatures where this correction is small and can be neglected. Namely, we consider the correction for $S_{\text{eff}, 0}$ which follows from the first order expansion by $\Phi$ in the normal component of the self-energy $\Sigma_n(\Phi)$ (24). We perform Gaussian integration over fluctuations $\psi_n, \psi_n$ with $n \neq 0$ in $S_{\text{eff}}[\Phi]$ (21) where the self-energies include $\Psi_0$. The integration gives the effective action for zero-mode field

$$S_0[\Phi, \varphi] = \beta \omega \Phi + S_{\text{zm}}[\Phi, \varphi] + \delta S[\Phi, \varphi].$$

The correction $\delta S[\Phi, \varphi]$ here is the result of the integration over $\psi_n \neq 0$. It reads as

$$\delta S[\Phi, \varphi] = \frac{1}{2} \sum_{n \neq 0} \text{tr} \ln \beta \left[ -i2\pi n T + \omega + \Sigma_n(\Phi) \quad \tilde{\Sigma}_n(\Phi, \varphi) \right]$$

$$\left[ -i2\pi n T + \omega + \tilde{\Sigma}_n(\Phi, \varphi) \right]$$

The expansion of the logarithm up to linear order term $\Phi$ provides

$$\delta S[\Phi] = \delta \alpha \Phi$$

where

$$\delta \alpha = \sum_{n \neq 0} \frac{\Sigma'_n[0]}{-i2\pi n + \beta \omega + \beta \Sigma_n[0]}.$$ (101)

We expand $S_{\text{zm}}[\Phi]$ up to $\Phi^2$ which yields the approximation for (98):

$$S_0[\Phi] = \alpha \Phi + \gamma \Phi^2 + \delta S[\Phi].$$ (102)

Note, that a quadratic term $\Phi^2$ from the logarithm gives a non-relevant small correction to a non-vanishing $\gamma$. Oppositely, $\alpha$ can be arbitrary small at the transition point. In this case only the linear by $\Phi$ part form $\delta S[\Phi, \varphi]$ is important.

Below we estimate $\delta S[\Phi]$ for the case of full resonance $\epsilon_j = \bar{\epsilon} = \omega$ and absence of disorder in coupling terms, i.e. $g_j = \tilde{g}$. In this resonant case it reads as

$$\delta \alpha = \frac{\Phi \bar{g}^4 N}{16 T^2 \omega^4} \left( 6 T^2 + \frac{\omega^2}{1 + \cosh \beta \omega} \tanh \beta \omega \frac{2}{3 T \omega} \right).$$ (103)

For low temperatures $T \ll \omega$, the critical Rabi frequency is $\Omega_c \equiv g \sqrt{N/2} = \omega$ and

$$\delta \alpha_c = \frac{3 \omega}{4 N T}.$$ (104)

Let us estimate a character value of $\Phi'$ where the integral over $\exp[-(\alpha + \delta \alpha)\Phi - \gamma \Phi^2]$ in the partition function does converge. In the vicinity of the phase transition ($\alpha = 0$) it is given by the Gaussian integrand’s width, i.e., $\Phi' \sim \gamma^{-1/2} \sim \sqrt{N T}/\omega$. The self-consistency of the expansion (102) means that the correction $\delta S[\Phi]$ must be much less than the unity at the convergence region, namely, $\delta S[\Phi'] = \delta \alpha \Phi' \ll 1$. The latter results in the following condition:

$$T \gg \frac{\omega}{N}.$$ (105)

It provides the limitation for low temperatures in our approach based on the Gaussian approximation for $S_0$.

Note, that for much lower temperatures $T \ll \omega/N$ the non-Gaussian contributions by $\psi_n$ as well as $\Psi_0$-dependencies in $\Sigma_n$ and $\tilde{\Sigma}_n$ can not be neglected in the action for quasiparticle fluctuations $S_0$. Technically, it means that the higher order terms in the expansion of (17) by $V[\Psi', G_{\tau-r}\Psi, V[\Psi, \tau]]$ should be taken into account.

In the high temperature limit, $T \gg \omega$, the critical coupling is enhanced as $\Omega_c = \sqrt{NT}$ and we obtain $\delta S[\Phi'] \sim 1/\sqrt{N}$. The self-consistency of the Gaussian approximation in high temperature limit also requires $\delta S[\Phi'] \ll 1$. It provides

$$N \gg 1.$$ (106)

Such condition on the qubits number is less strict than that in the low temperature regime (105), which assumes that $N \gg \omega/T \gg 1$. 

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