An investigation on existence and uniqueness of solution for Integro differential equation with fractional order

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Abstract. Present article deals with some theoretical results regarding the solution of integro differential equation with integral boundary condition. Main idea is to convert the integro differential equation into corresponding integral equation. Then we adopt a suitable transformation to implement Banach contraction and Schaefer-Krasnoselskii fixed point Theorem. Finally by properties of fractional calculus and fixed point theory we derive the main results. An example is considered for the verification of theoretical work.

1. Introduction
In this paper our object is to derive some qualitative analysis of the following problem

\[ D^\xi_c \phi(t) = H(t, \phi(t)) + \int_0^t Q(t, p, \phi(p)) dp, \]

\[ \phi(0) = \phi_0 - \Psi \phi(t) \]

where \( t \in \mathcal{I} = [0, T] \), \( \phi_0 \) is some prescribed constant, \( D^\xi_c \) is standard Caputo fractional derivative with order \( \xi \in (0, 1) \), \( H : \mathcal{I} \times \mathbb{R} \to \mathbb{R} \). \( \Psi : U(\mathcal{I}, \mathbb{R}) \to \mathbb{R} \) is considered as

\[ \Psi \phi(t) = \lambda(t) \int_0^T \phi(p) dp, \]

with \( \lambda : \mathcal{I} \to \mathbb{I}, Q : \mathcal{I} \times \mathcal{I} \times \mathbb{R} \to \mathbb{R} \) and \( U = U(\mathcal{I}, \mathbb{R}) \) is the space of all bounded continuous functions from \( \mathcal{I} \) to \( \mathbb{R} \) induced by \( ||.||_\infty \).

With the gradual development of calculus in fractional order systems, integro differential equations of non integer order appear frequently in various physical situations; such as biology, continuum mechanics, nonlinear oscillations, dynamics in Hamiltonian systems, electromagnetics, etc. Many authors have contributed in this particular field of integro differential
equation by their extensive theoretical findings. Over a period of time qualitative research on integro differential equation have been carried out by Ahmad and Nieto [1], Lakshmikantham and Leela et al [2, 3], Bhaskar [4], Ahmad and Sivasundaram [5], Abbas [6], Abdo and Wahash et al [7] and several other authors, see [8-12] and references therein. In this testimony our primary object is to implement Banach contraction and Schaefer-Krasnoselskii fixed point Theorem on transformed form of corresponding integro differential equation.

The arrangement of this article is as follows, basic concepts are covered in section 2, main results are described in section 3, an example is illustrated in section 4 in support of the theoretical results and finally we conclude some closing remarks in section 5.

2. Basic concepts

**Definition 2.1.** [11] The Riemann-Liouville fractional integral operator with order $\xi$, of a function $\phi \in U$ is defined by

$$I_{RL}^\xi \phi(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t-\tau)^{\xi-1} \phi(\tau)d\tau, \text{ for } t \in \mathbb{R}^+$$

and

$$I_{RL}^0 \phi(t) = \phi(t).$$

**Definition 2.2.** [11] The fractional derivative of $\phi(t) \in U$ in Caputo sense is defined by

$$D_c^\xi \phi(t) = I_{RL}^{n-\xi} D_c^n \phi(t) = \frac{1}{\Gamma(n-\xi)} \int_0^t (t-\xi)^{n-\xi-1} \frac{d^n \phi(\xi)}{d\xi^n} d\xi,$$

for $y \in \mathbb{R}^+$ and $n-1 < \xi \leq n$, $n \in \mathbb{N}$.

**Theorem 2.3.** [13] Every contraction mapping on a Banach space admits a fixed point uniquely.

**Theorem 2.4.** [14] Consider a Banach space $B$. $\tilde{B}$ be a closed bounded and convex subset of $B$ with a completely continuous operator $S : \tilde{B} \rightarrow \tilde{B}$. Then $S$ posses a fixed point.

3. Main results

Our basic aim is to investigate existence and uniqueness theory for the solution of (1). Initially we introduce Riemann-Liouville fractional integral operator and transform (1) – (2) into corresponding integral equation

$$\phi(t) = \phi_0 - \lambda(t) \int_0^T \phi(p)dp + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} \left\{ H(p, \phi(p)) + \int_0^p Q(p, s', \phi(s'))ds' \right\} dp$$

where $t \in [0, T]$. Now we introduce $T : U(\mathcal{I}, \mathbb{R}) \rightarrow U(\mathcal{I}, \mathbb{R})$ defined by

$$(T \phi)(t) = \phi_0 - \lambda(t) \int_0^T \phi(p)dp + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} \left\{ H(p, \phi(p)) + \int_0^p Q(p, s', \phi(s'))ds' \right\} dp. \quad (3)$$

It is evident that, $\phi$ is solution of (1) – (2) only when $\phi \in U$ becomes fixed point of $T$. With this motivation and under some relevant assumptions we proceed further.

(A1) For $p, s' \in \mathcal{I}$ and $\phi_1, \phi_2 \in U$, there exist $L_F, L_K \in \mathbb{R}^+$ so that

$$|H(p, \phi_1) - H(p, \phi_2)| \leq L_F |\phi_1 - \phi_2|, \quad |Q(p, s', \phi_1) - Q(p, s', \phi_2)| \leq L_K |\phi_1 - \phi_2|.$$

(A2) There exist constant $\mathcal{M}, \mathcal{M}^* > 0$ such that

$$\sup |\lambda(t)| \leq \mathcal{M} \text{ and } |\phi(t)| \leq \mathcal{M}^*, \text{ for } t \in \mathcal{I}.$$
(A3) The function \( \phi : \mathcal{I} \rightarrow \mathbb{R} \) is continuous.

**Theorem 3.1.** If problem (1) – (2) satisfy (A1)-(A2), then it admits unique solution if
\[
\mathcal{M} T + \frac{L_F T^\xi}{\Gamma(\xi+1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi+2)} < 1
\]
holds.

Proof. Consider \( \phi_1, \phi_2 \in \mathcal{U} \) such that
\[
(\mathcal{T} \phi_1)(t) = \phi_0 - \lambda(t) \int_0^t \phi_1(p) \, dp + \frac{1}{\Gamma(\xi)} \int_0^t (t - p)^{\xi-1} \left\{ H(p, \phi_1(p)) + \int_0^p Q(p, s', \phi_1(s')) \, ds' \right\} \, dp,
\]
and
\[
(\mathcal{T} \phi_2)(t) = \phi_0 - \lambda(t) \int_0^t \phi_2(p) \, dp + \frac{1}{\Gamma(\xi)} \int_0^t (t - p)^{\xi-1} \left\{ H(p, \phi_2(p)) + \int_0^p Q(p, s', \phi_2(s')) \, ds' \right\} \, dp.
\]
Consequently, we obtain
\[
|\mathcal{T} \phi_1(t) - \mathcal{T} \phi_2(t)| \leq |\mathcal{T} \phi_1(t) - \mathcal{T} \phi_2(t)| \leq |\mathcal{T} \phi_1(t) - \mathcal{T} \phi_2(t)| =
\]
\[
\leq \left( \mathcal{M} T + \frac{L_F T^\xi}{\Gamma(\xi+1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi+2)} \right) |\phi_1(t) - \phi_2(t)|.
\]
This implies
\[
||\mathcal{T} \phi_1(t) - \mathcal{T} \phi_2(t)||_\infty \leq W(\xi, \mathcal{M}) ||\phi_1(t) - \phi_2(t)||_\infty,
\]
where \( W(\xi, \mathcal{M}) = \mathcal{M} T + \frac{L_F T^\xi}{\Gamma(\xi+1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi+2)} \).

For \( \mathcal{M} T + \frac{L_F T^\xi}{\Gamma(\xi+1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi+2)} < 1 \), \( \mathcal{T} \) turns out to be a contraction mapping. Banach fixed point theory immediately establishes that, \( \mathcal{T} \) posses a fixed point. That concludes the uniqueness of solution for the problem (1) – (2).

In the next theorem we shall study the existence of solution.

**Theorem 3.2.** If the conditions (A1)-(A3) are satisfied, then (1) – (2) posses atleast one solution.

Proof. To investigate existence of solution we approach with the following three steps

**Step 1:** Consider the sequence \( \{ \phi_j \}_{j \in \mathbb{N}} \), which converges to \( \phi \) in \( \mathcal{U} \). Consequently for \( t \in \mathcal{I} \) we obtain
\[
|\mathcal{T} \phi_j(t) - \mathcal{T} \phi(t)| \leq |\lambda(t)| \int_0^t |\phi_j(p) - \phi(p)| \, dp + \frac{1}{\Gamma(\xi)} \int_0^t (t - p)^{\xi-1} |H(p, \phi_j(p)) - H(p, \phi(p))| \, dp + \frac{1}{\Gamma(\xi)} \int_0^t (t - p)^{\xi-1} \left( \int_0^p |Q(p, s', \phi_j(s')) - Q(p, s', \phi(s'))| \, ds' \right) \, dp
\]
\[
\leq \left( \mathcal{M} T + \frac{L_F T^\xi}{\Gamma(\xi+1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi+2)} \right) |\phi_j(t) - \phi(t)|.
\]
Assumption (A3) provides
\[ \|T\phi_j(t) - T\phi(t)\|_\infty \to 0, \text{ as } j \to \infty. \]
This implies, \( T \) is continuous on \( U \).

**Step 2:** Here, our goal is to show that \( T \) transforms bounded sets into bounded sets in \( U \).
We consider \( \mathcal{B}_r = \{ \phi \in U : \|\phi\|_\infty \leq r, r > 0 \} \), where \( \|\phi\|_\infty = sup\{\|\phi(t)\|, t \in \mathcal{I}\} \). This \( \mathcal{B}_r \) is a closed bounded convex subset of \( U \). Now, for all \( \phi \in \mathcal{B}_r \) we get
\[
|(T\phi)(t)| \leq |\phi_0| + |\lambda(t)| \int_0^T |\phi(p)|ds + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} |H(p, \phi(p)) - H(p, 0)|dp + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} \int_0^p |Q(p, s', \phi(s')) - Q(p, 0)|ds' dp + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} \int_0^p |Q(p, s, 0)|ds' dp + \frac{1}{\Gamma(\xi)} \int_0^t (t-p)^{\xi-1} |H(p, 0)|dp.
\]
Thus
\[
\|T\phi(t)\|_\infty \leq |\phi_0| + M\mathcal{M}^*T + \frac{(L_F\mathcal{M}^* + F^*)T_\xi}{\Gamma(\xi+1)} + \frac{(L_K\mathcal{M}^* + K^*)T_{\xi+1}}{\Gamma(\xi+2)}.
\]
So, \( \|T\phi(t)\|_\infty < \tilde{r} \), for all \( t \in \mathcal{I} \)
We denote, \( |\phi_0| + M\mathcal{M}^*T + \frac{(L_F\mathcal{M}^* + F^*)T_\xi}{\Gamma(\xi+1)} + \frac{(L_K\mathcal{M}^* + K^*)T_{\xi+1}}{\Gamma(\xi+2)} = \tilde{r} \). That leads to, \( \|T\phi(t)\|_\infty \leq \tilde{r} \)
for all \( \phi \in \mathcal{B}_r \). Thus \( T\mathcal{B}_r \subset \mathcal{B}_{\tilde{r}} \).

**Step 3:** We claim \( T \) is completely continuous on \( U \).
For \( \phi \in \mathcal{B}_r \) and \( \eta_1, \eta_2, \eta \in \mathcal{I} \) with \( \eta_1 < \eta < \eta_2 \)
\[
|(T\phi)(\eta_1) - (T\phi)(\eta_2)| \leq (\eta_1 - \eta_2) \lambda(\eta)\mathcal{M}^*T \frac{1}{\Gamma(\xi)} \int_0^\eta \left\{ (\eta_1 - p)^{\xi-1} \right\} |H(p, \phi(p))|dp + \frac{1}{\Gamma(\xi)} \int_0^\eta \left\{ (\eta_1 - p)^{\xi-1} - (\eta_2 - p)^{\xi-1} \right\} \int_0^p |Q(p, s, \phi(s))|ds' dp + \frac{1}{\Gamma(\xi)} \int_0^\eta \left\{ (\eta_1 - p)^{\xi-1} \right\} \int_0^p |Q(p, s', \phi(s'))|ds' dp \\
\leq (\eta_1 - \eta_2) \lambda(\eta)\mathcal{M}^*T \frac{F_\xi}{\Gamma(\xi+1)} \left(2(\eta_2 - \eta_1)\xi + \eta_1^\xi - \eta_2^\xi\right) + \frac{K_\xi}{\Gamma(\xi+2)} \left(2(\eta_2 - \eta_1)\xi + \eta_1^\xi - \eta_2^\xi\right).
\]
Consequently we obtain \( \|T\phi(\eta_1) - (T\phi)(\eta_2)\|_\infty \to 0 \) as \( \eta_1 \to \eta_2 \) and \( \{T\mathcal{B}_r\} \) immediately becomes equicontinuous. By Arzela-Ascoli theorem, we conclude that \( T \) is completely continuous.

From Schaefer’s fixed point theorem (Theorem 2.4.) it is confirmed that, the problem (1) - (2) admit at least one solution on \( U \).

### 4. Numerical example
We study an example to verify the theoretical work.

**Example 4.1.**
\[
D_c^{1/3} \phi(t) = \frac{1+\sin(\phi(t))}{3} + \frac{1}{2} t^t e^{-\frac{1}{2}}(\phi(p))dp, \ t \in (0, 1],
\]
with initial condition \( \phi(0) = \frac{1}{2} - \frac{(t^2 - 2t)}{6} \int_0^1 \phi(p)dp \).

Here \( \xi = \frac{1}{3}, \lambda(t) = \frac{(t^2 - 2t)}{6}, Q(t, p, \phi(p)) = e^{-\frac{1}{2}}(\phi(p)) \) and
\[ H(t, \phi(t)) = \frac{1+\sin(\phi(t))}{3}. \]

This leads to, \( \mathcal{M} T + \frac{L_K T^\xi}{\Gamma(\xi + 1)} + \frac{L_K T^{\xi+1}}{\Gamma(\xi + 2)} = 0.7499 < 1 \).

By Theorem 3.1., we conclude that example 4.1. has a unique solution.

5. Conclusion

In this study, our focus is to develop some theoretical results for qualitative study of fractional order integro differential equation with integral type condition. By virtue of fractional calculus and fixed point theory we derive the proposal. An example is also studied as an application and numerical verification purpose.

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