A long route to consensus: Two stage coarsening in binary choice voting model

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Formation of consensus, in binary yes/no type of voting, is a well defined process. However, even in presence of clear incentives, the dynamics involved can be incredibly complex. Specifically, formations of large groups of similarly opinionated individuals could create a condition of ‘support-bubbles’ or spontaneous polarization that renders consensus virtually unattainable (e.g. the question of the UK exiting the EU). There have been earlier attempts in capturing the dynamics of consensus formation in societies through simple \(Z_2\)-symmetric models hoping to capture the essential dynamics of average behavior of a large number of individuals in a statistical sense. However, in absence of external noise, they tend to reach a frozen state with fragmented and polarized states i.e. two or more groups of similarly opinionated groups with frozen dynamics. Here we show, using a kinetic exchange opinion model (KEM), that while such frozen states could be avoided, formation of consensus in those situations takes a much longer time scale. Specifically, the system could either reach consensus in a time that scales as \(L^2\) or a long lived metastable state (termed a domain-wall state) for which formation of consensus takes a time scaling as \(L^{3.6}\), much slower than that in similar familiar dynamical models. The late-time anomaly in the time scale is reflected in the persistence probability of the model. Finally, the interval of zero-crossing of the average opinion i.e. the time interval over which the average opinion does not change sign is shown to follow a scale free distribution, which is compared with that seen in the opinion surveys regarding Brexit and associated issues in the last 40 years. The issue of minority spreading is also addressed by calculating the exit probability.

I. INTRODUCTION

The dynamical evolution of opinion formation in a society is a complex process involving myriads of socio-economic, to mention psychological, issues. Even the quantification of opinions by numbers, therefore, could be difficult to define, let alone their evolution [1–4]. However, when the choice is binary, e.g. in a yes/no referendum, a two party voting etc., the said quantification is straight-forward. It is often done with \(\pm 1\) values, i.e. like spins with Ising symmetry, for example in the voter model [5–7]. Additionally, it could be crucial to have neutral opinions - with opinion value zero - reflecting the population that does not belong to either of these groups [8].

In spite of the complexity involved, the literature over the last three decades or so indicates, at least in the statistical physics community, that the essence of the opinion evolution in a society could be captured by models that assume that the change in the individual opinion is activated by interaction between two or few agents [9]. The diversity can come through various factors. The choice of the partner for interaction can be crucial. Most of the models consider only nearest neighbors interactions. But there can be additional restrictions, e.g., for the bounded-confidence models, where the interaction is conditional upon a limited range of the absolute difference of opinions of the two individuals [10–11]. One can consider kinetic exchange models (KEM), where the relative weights of one’s own opinion and that of their peers [11–12] are both decisive factors in the evolution of the opinions. Unlike the one-to-one interaction of the KEM, to take peer-pressure into account, in some models, a group of individuals having identical opinion can together affect one individual opinion, as in [13–14]. The topology and/or the hierarchy of the neighborhood of the agents (social-networking) also play a key role by limiting the interaction range. Of course, many other factors can be incorporated which add to the intricacy of the models but are beyond the scope of discussion in the context of the present paper (see for a recent review [2]).

Each of these variations has its own value in representing certain global or average feature of the opinion formation in a society of a large number of interacting agents. However, in terms of formation of an overall consensus [21], these models, especially in dimensions two or higher, are often inadequate. Many of these models reach a frozen or a fragmented state in the long time. The dynamics after that either cease to continue or evolve in a way that no further movement towards consensus is possible. Examples range from the two state models like the Ising model (which can be regarded as a simple opinion dynamics model) in two dimensions [22] and both the Ising and voter models on well-studied complex networks [23–24] to the bounded confidence models with multistate or continuous opinions (see e.g. [2–3, 10–14], 5–6).

Here we show that in a kinetic exchange model of opin-
ion formation on square ($L \times L$ lattices), which allows neutral opinions in an otherwise binary choice situation, a consensus is always reached, but it is reached in a two-stage process. In the first stage, starting from the random initial conditions, we note the usual Ising-like coarsening dynamics, i.e., the order parameter scales as $t^{1/2}$. Beyond a time scale $\tau_1 \propto L^2$, the order parameter shows a different behavior with time as, with a finite probability, the system reaches a so-called domain-wall state. In the two-dimensional Ising coarsening, all initially random states do not reach consensus but some get stuck in a locally stable state characterized by rectangular domains of $\pm 1$ state. Here, states which consist of domains with nearly straight edges are reached somewhat reminiscent of the frozen states in the Ising model. Although these states are long-lived, the domain walls are eventually unstable. The system always evolves to a consensus state, but in a much longer time-scale, $\tau_2 \sim L^{3.6}$. It is in this long meta-stable state that the two competing, relatively similar-sized domains of opinion values, continue to evolve, showing a scale-free scaling of the size of the intervals within which the society as a whole switches the sign of majority (zero-crossing probability of the average opinion).

The model is described in detail in the next section. It belongs to the Ising universality class as far as equilibrium properties are concerned [34,39]. So one of the concerns is whether/how the dynamics are different from the kinetic Ising model. Apart from the time to reach consensus, the persistence probability \[ P(t) \] as a function of time $t$ is also studied. $P(t)$ is the probability that a field has not changed sign till time $t$, in opinion dynamics (spin) models, it is the fraction of opinions (spins) that remain unchanged till $t$. In the zero-temperature Ising Glauber model in two dimensions, it is known to behave as $P(t) = t^{-\theta}$ with $\theta$ very close to 0.20 [11]. Although a power law regime exists in the present model, we find a different value for $\theta$ and finite-size scaling behavior.

The three states of opinions are represented by 1, 0, -1 in the kinetic exchange model; however only the ‘all 1’ / ‘all -1’ states are the fixed points of the dynamics whenever the initial state has at least one agent with nonzero opinion value. In general we study the dynamics starting with an uniform distribution of opinions with average opinion equal to zero. One can also choose a biased initial configuration by taking 1/3 of the opinions equal to zero, \( \frac{1}{3} + \frac{2}{3} \) fraction equal to 1 and the rest equal to -1. Then the exit probability $E(\Delta)$ measures the probability that the system reaches a ‘all 1’ state eventually. It is interesting to know that when $\Delta < 0$, whether the system can reverse altogether, i.e., give a nonzero value of $E(\Delta)$, which will correspond to a case of minority spreading [12].

Finally, for comparison with a similar situation in society, we turn to the opinion surveys for the last 40 years in the UK regarding its issue of exiting the European Union (recently came to be referred to as Brexit). We note that this is indeed a situation where similarly sized domains continue to evolve. We compare the interval distributions of the zero-crossing time of the average opinions (intervals to switch between “leave” to “remain” majority) with that computed in our model.

II. MODEL AND SALIENT DYNAMICAL FEATURES

The kinetic exchange model for opinion formation [16] is considered on a square lattices where each lattice site is occupied by an agent. For the agent at site $i$ (called the $i$th agent), interacting with the $j$th agent, the opinion value $o_i$ changes according to

$$o_i(t+1) = o_i(t) + \mu_{ij} o_j,$$

(1)

No sum over the index $j$ is implied, as the model is binary-exchange. The $j$-th agent must be one of the four nearest neighbors of the randomly selected $i$-th agent. A non-linearity enters the model from the imposed bounds in the opinion values for the extreme ends at $\pm 1$, i.e., $|o_i| \leq 1$, signifying the limit to an extreme opinion. The values of $\mu_{ij}$ are -1 with probability $p$ and +1 otherwise.
Clearly, if the initial condition of the model is restricted to the agents having opinion values $\pm 1$ and 0, for all subsequent times, the opinion values are confined within these three values. For a binary choice case, the two given choices, say ‘leave’ and ‘remain’ (e.g., in Brexit) are represented by opinion states $\pm 1$ and the agents who remain neutral/indifferent are assigned opinion “0”. This steady state properties of this model have been studied in mean field and other topologies to some extent (see e.g. 36–39). A trivial fixed point of the model is when all $\alpha_i = 0$, but this is unstable with respect to any single agent with a nonzero opinion being present. Given that our initial conditions never start with all neutral agents, this state is never reached. For the purpose of this work, where we only look for paths to consensus, we keep $p = 0$ i.e. $\mu_{ij} = 1$ here, implying a conducive environment for positive interaction, as $\mu_{ij}$ reflects the likelihood of positive interaction between the $i$-th and $j$-th agents.

With the condition of $p = 0$, after a transient state, the neutral opinions are exclusively confined at the domain boundaries of opposite signs. This is because neutral agents necessarily interacting with agents having extreme opinion values, are unstable. It is worthwhile to note at this point that as a consequence of the above observation, in one dimension, the dynamics of this model are equivalent to that of the zero temperature coarsening of the Ising model using Glauber dynamics. The only difference is the presence of domains of neutral opinions in the kinetic exchange model. However, such domains are limited in their life-spans that are proportional to their initial lengths which eventually makes the two models behave in the same manner. As long as the lower dimensions are concerned, the first occurrence of a non-trivial difference in the dynamics of the model is for dimension $d = 2$. The reason for this is straightforward. As mentioned before, the neutral agents appear only along the domain boundaries of the two extreme values. Given the dynamical evolution rules of the model, those neutral agents keep the dynamics alive along the domain boundaries, as long as domain boundaries exist. This means that the system is dynamically active until a consensus is reached. However, the time needed for reaching a consensus as a result of this ‘active-boundaries’ is much higher than the usual coarsening time.

### III. NUMERICAL RESULTS

A qualitative idea of the dynamics can be gained by looking at the opinion configurations of the KEM at various stages of the dynamics (see Fig. 1). The snapshots clearly indicate the presence of two time scales. At the initial times ($t < \tau_1$), the spin domain configurations are random. At a later stage ($\tau_1 < t < \tau_2$), if the system has not already entered a fully consensus state, it is in a state with two (or more, in principle) large domains (taking the periodic boundary conditions into account).

#### A. System size scaling of the two consensus time scales

We first study the behavior of the global order parameter given by $O(t) = \frac{1}{N} \sum_i \alpha_i(t)$. The simulations are done on square lattices, with the sizes $(N = L \times L)$ indicated in the figures and following periodic boundary conditions. Unless otherwise mentioned, the initial conditions are completely random i.e. equal probability for the three opinion values $\pm 1$ and 0. The realization averages are between 10000 to 100, depending on system size. In Fig. 2 we show how the order parameter evolves in time for various system sizes. It is clear from Fig. 2(a) that the evolution shows the effect of the two time scales mentioned earlier. For the initial part of the dynamics, the order parameter increases in a power-law manner with time, with the exponent value about 0.5, which is the same as in Ising model coarsening. The growth, however, drastically slows down when the second time scale manifests itself. The subsequent growth is much slower, which continues until complete consensus is reached. The inset of Fig. 2(a) confirms that $\tau_1 \propto L^{z_1}$ where $z_1 = 2$ is a dynamic exponent of growth. This value of $z_1$ indicates a dynamical evolution which is curvature driven and valid for the zero temperature coarsening of Ising model in all dimensions.

For $t > \tau_1$, the order parameter can be fitted to a functional form

$$O(t) \sim 1 - a \exp(-bt), \tag{2}$$

indicating that $1/b$ represents another time scale which we identify as $\tau_2$. $1/b$ is shown in the inset of Fig. 2(b) and we find

$$1/b \sim \tau_2 \propto L^{z_2} \tag{3}$$

where $z_2 = 3.60 \pm 0.01$. In Fig. 2(b) we obtain a partial collapse of the data for the late time regime using using Eqs 2 and 3.

We also obtain the distribution of the consensus times as shown in the inset of Fig. 3. The peak in the distribution function corresponds to the shorter consensus time or domain-wall formation time scale $\tau_1$, where the configurations either reach consensus avoiding the domain wall formation or reach the domain-wall states. However, the tail of the distribution, which represents the time scales for the domain walls to merge, i.e. formation of the overall consensus, gives the second, much larger, time scale $\tau_2$. One can obtain a collapse of the data for the tail by scaling $t$ by $L^{\alpha}$. It is also observed that the scaling function has an exponential behavior.

#### B. Persistence and exit probabilities

The persistence of the opinion value of a given agent is an interesting measure of the stability of dynamics in a
model. In zero temperature quench of Ising model, this is well studied. The persistence probability $P(t)$ at any time is the fraction of agents who did not change their opinion values at all up to that time. Fig. 4 shows the decay of the persistence probability for the KEM. Although it does not show a clear power law behavior for small system sizes, for the largest size, a power law behavior can be detected over a considerable time range. From this regime, one obtains

$$P(t) \sim t^{-\theta},$$

with $\theta \approx 0.27$, clearly distinct from that found in the two dimensional Ising model. Furthermore, due to the fact that the dynamics may not be frozen after time $\tau_1 \sim L^2$, the persistence probability does not become time independent beyond $L^2$ (before reaching a saturation value) as in the case of Ising model. However, the subsequent decay of the persistence probability is, like the growth of the order parameter discussed before, very slow. Hence, the effect of the two time scales described above is also visible in this measure. We also note that the saturation value of $P(t)$ varies with $L$ as $L^{-0.76}$ shown in the inset of Fig. 4.

The exit probability is another important measure that quantifies the probability to reach a particular consensus state with an initial bias. In particular, if an initial bias is ineffective (e.g., an initial configuration with more $-1$ ultimately ends up in an all $+1$ state) it can be called a case of minority spreading. This is again a very well studied quantity in Ising model and other opinion dynamics models. The general scaling form of the
The finite size scaling exponent \( (\nu = 1.05 \pm 0.01) \) is distinct from what is seen in the case of Ising model \((1.25)\) [25]. The collapsed curve fits with a function of the form \( f(x) = (1 + \tanh(cx))/2 \) with \( c = 0.547 \).

The exit probability \( E(\Delta, L) \) reads
\[
E(\Delta, L) = F \left( \left( \Delta - \Delta_c \right) L^{1/\nu} \right) \tag{5}
\]
where \( \Delta \) is the bias in the initial opinion fractions (initial densities are \( \frac{1}{2}, \frac{1}{3} \) and \( \frac{1}{2} - \frac{1}{3} \)) for opinion values 0, \( \pm 1 \) respectively; \( -2/3 \leq \Delta \leq 2/3 \), with \( \Delta_c = 0 \) and \( \nu \) being interpreted as a correlation length exponent (to be distinguished from the critical correlation length). The exit probabilities and their finite size scaling are shown in Fig. 5. The exponent value of \( \nu \) for the collapse is \( 1.05 \pm 0.05 \) which is again distinct from what is seen in the Ising model in two dimensions \((\approx 1.3)\) and [49].

IV. COMPARISON WITH OTHER DYNAMICAL MODELS

The present model, although a three state model, has only two stable fixed points, all +1/ all -1 and hence can be compared with other \( Z_2 \) models studied extensively in the literature. We have in mind precisely the Ising model or voter like models which are special cases of the generalized voter model [51]. We have already mentioned some differences compared to the Ising model, the most important of which is the absence of frozen states at extremely large times. This is attributed to the fact that the zero states primarily found at the domain boundaries, ultimately “melt”, albeit very slowly, to lead to consensus in KEM in a finite fraction of initial states (about 30%). Such metastability or freezing is possible only in dimensions greater than one, hence in one dimension, dynamics of both Ising model and KEM coincide. It is already known that the voter model dynamics are identical to that of Ising in one dimension, so that all three models show the same dynamical behavior in one dimension.

The most striking result for the KEM is the presence of the two time scales, the second much longer than the first. It is true that in the two dimensional Ising model also, one gets a two time scale scenario due to the slow relaxation of the so called diagonal states [22] where the scaling of the second time scale is quite similar to what we get. However, such cases were much more rare (nearly 5% of the consensus states) to affect the bulk dynamical behavior compared to KEM where the slow dynamics occur in about 30% cases. In the Ising model, the dynamics are always curvature driven. In the KEM, the initial evolution is perhaps the same indicated by the value of \( z_2 = 2 \). On the other hand, in the voter model, the dynamics are interfacial noise driven so that it shows a different dynamical behavior in two dimensions. In the KEM, the dynamics are like the voter model only for the zero state which simply copies the state of the interacting agent and in that sense partial interfacial noise driven. Since at a later stage, it is basically the zeroes which undergo a change, the dynamics become more and more interfacial noise driven. In the generalized voter model, the interplay of the interfacial noise and curvature driven dynamics was studied recently [48] which showed crossover from a faster to a slower dynamics due to the presence of metastable states with similar nearly straight-edged domains. It was the interfacial noise which was found to lead the system to consensus in the later stages and one can argue that the same is happening in KEM. In the generalized voter model also, two time scales were detected, however, such a large value of \( z_2 \) was not indicated in that case.

We also noted that the persistence exponent in KEM
is larger compared to the Ising model, which is obviously because the dynamics continue for a longer time. On the other hand, the persistence probability decays slower in comparison to the voter model where $d = 2$ is a critical dimension for which the dynamics are known to be slow for all configurations. Consistently, the saturation value of the persistence, which signifies the end of the dynamical evolution, comes at a much later stage in KEM. This in turn, affects the scaling form of the persistence probability. In Ising model the saturation value varies as $L^{-\alpha}$, with $\alpha = \theta z$ [52], where $z = z_1 = 2$ is the unique dynamical exponent for the Ising model. But in KEM, the saturation value scales as $L^{-0.76}$, which does not satisfy the scaling relation mentioned above for either $z_1$ or $z_2$, with $\theta \approx 0.27$.

The exit probability in the voter model is known to vary linearly with the bias $\Delta$ in all dimensions. This follows from the global conservation of the opinions over all ensemble, also true for the one dimensional Ising model. In two dimensions, the Ising model and the Sznajd model show a behavior similar to what we obtain for the KEM, however, the values of the parameters in the scaled functions for the Ising model are quite different ($\nu \approx 1.3$ and $c \approx 1.2$ [59]). Although in the thermodynamic limit, the exit probability assumes a step functional form, for finite sizes, there is evidence of minority spreading as indicated by the presence of the universal scaling function, a feature shared by the Ising model, though in the latter it happens to a lesser extent as the value of $c$ is higher.

V. COMPARISONS WITH DATA

There are examples of issues that eluded consensus in the society for decades and formed a spontaneous polarization like the domain wall state discussed here. One recent example is the question of the United Kingdom leaving the European Union - commonly referred to as Brexit. This issue has certainly made the UK and EU as a whole, more polarized. From opinion surveys dated back to 1970s [50], there seems to be a fluctuating tendency in the opinion values, while no side got overwhelming support. These are qualitative features we also find in the KEM considered here. However, while in the model we can quantify these features with various measures, with the data of opinion surveys on Brexit (or related questions), such quantification are difficult.

Nevertheless, we can look at the intervals of the so called ‘zero-crossing’ of the overall opinion values, i.e. the intervals in which the majority opinion shifted between “leave” and “remain”. Fig. 6 shows a rank plot of the intervals, which suggests an approximate exponent value of $-0.6$ for the cumulative distribution, which in turn suggests $-1.6$ as the exponent value for the probability distribution function [54] of the interval of switching of opinions.

For comparison, we considered the time series of the global order parameter of KEM only for the configurations that did not reach consensus in $\tau_1$ i.e. those configurations which reached a domain-wall state. The distribution $D(I)$ of the zero-crossing intervals for those configurations only, gives an exponent value $-1.5$ (see Fig. 6), which closely resembles the value suggested from data. However, as noted before, the data available are rather sparse in measurement intervals, which can affect the distribution especially in the lower-cutoff region (not shown). Nevertheless, a manifestation of an active exchange of opinions between two polarized groups is seen in the data that was described in the model.

VI. DISCUSSIONS AND CONCLUSION

Formation of consensus in a multi-agent society has been long viewed as a collective emergent phenomenon. In various models that considers the phenomenon as a critical behavior, in presence of sufficiently weak noise, a majority opinion builds up [2, 4, 9]. Here we have shown, however, that even in the absence of external noise, formation of opinions can be hindered due to spatial constraint in interactions. Specifically, in the kinetic exchange model of opinion dynamics studied here, we find that for about 30% of realizations, the configuration of opinions in a two-dimensional lattice go to a segregated or domain-wall state, where there are two large groups of opposing opinion values. The groups are, however, not frozen in time. The domain walls move, due to interactions mediated by neutral agents, and eventually merge to form a consensus state. The time-scale to reach the consensus, however, is significantly higher than the time scale for reaching consensus for the non domain-wall states. Indeed, the two time-scales scale differently with system size as well. The scaling of the persistence probability and the overall order parameter quantify these two time scales in the model. This behavior is distinct from other coarsening dynamics, say in two dimensional Ising model, where the domain wall state is frozen in time in absence of external noise.

There are real world scenarios where formation of consensus takes exceptionally long time. It is long seen that in presence of external noise, bounded confidence or zealots, inertial agents etc., formation of consensus could be hindered. But even without such effect, consensus formation could be difficult. We have considered one such case, which is the question of the UK leaving the EU or Brexit. This has seen fragmentation of two groups, none of which could achieve overwhelming support i.e. consensus was not formed. There is a long history of opinion surveys on this question, which shows the fluctuating nature of the two opinion groups. Indeed, in this case there could be external noise, bounded confidence and even zealots, but a simple model could also give the qualitative agreement of the size distribution of the intervals of zero crossing seen in the data and in the model.

In conclusion, we have presented a simple model of
kinetic exchange for opinion formation, where two time scales of consensus formation are clearly demonstrated. In about 30% of cases, the model reaches a segregated state, where consensus formation takes a much longer time ($\tau_2 \sim L^{2.6}$) than the Ising-like coarsening time-scale ($\tau_1 \sim L^2$). We have compared the fluctuating nature of the two opinion groups with opinion survey data for Brexit to find a rough agreement.

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