BETTI NUMBERS ARE TESTABLE*

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We prove that the Betti numbers of simplicial complexes of bounded vertex degrees are testable in constant time.

1. INTRODUCTION

Property testing in bounded degree graphs was introduced in the paper of Goldreich and Ron [4]. In this paper we study property testing for bounded degree simplicial complexes in higher dimensions. Let $d \geq 2$ be a natural number and consider finite simplicial complexes where each vertex (zero dimensional simplex) is contained in at most $d$ edges (1-dimensional simplex). Of course, such a complex can be at most $d$-dimensional. What does it mean to test the $p$-th Betti number of such a simplicial complex? First fix a positive real number $\varepsilon > 0$. A tester takes a simplicial complex $K$ as an input and pick $C(\varepsilon)$ random vertices. Then it looks at the $C(\varepsilon)$-neighborhoods of the chosen vertices. Based on this information the tester gives us a guess $\hat{b}^p(K)$ for the $p$-th Betti number $b^p(K)$ of the simplicial complex such a way that:

$$\text{Prob} \left( \left| \frac{\hat{b}^p(K) - b^p(K)}{|V(K)|} \right| > \varepsilon \right) < \varepsilon,$$

where $V(K)$ is the set of vertices in $K$. In other words, we can estimate the $p$-th Betti number very effectively with high probability knowing only

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a small (random) part of the simplicial complex. The goal of this paper is to show the existence of such a tester for any $\varepsilon > 0$. That is to prove the following theorem:

**Theorem 1.** Betti-numbers are testable for bounded degree simplicial complexes.

For graphs the 0-th Betti number is just the number of components and the first Betti number can be computed via the 0-th Betti number and the Euler-characteristic, hence it is not hard to see that such tester exists. For connected surfaces one can also calculate the first Betti number using just the number of vertices, edges and triangles. However in higher dimensions there is no such formula even for triangulated manifolds. Note that this paper was not solely motivated by the paper of Goldreich and Ron, but also by the solution of the Kazhdan-Gromov Conjecture by Wolfgang Lück [5]. The workhorse lemma of our paper is basically extracted from his paper using a slightly different language. It is very important to note that our proof works only for Betti numbers of real coefficients and we do not claim anything for the Betti numbers of mod-$p$ coefficients.

### 2. The Convergence of Simplicial Complexes

Let $\Sigma^d$ be the set of finite simplicial complexes $K$ of vertex degree bound $d$ that is any 0-dimensional simplex is contained in at most $d$ 1-dimensional simplices. We denote by $K_i$ the set of $i$-simplices in $K$ and by $G_K$ the 1-skeleton of $K$, that is $V(G_K) = K_0$, $E(G_K) = K_1$. A rooted $r$-ball of degree bound $d$ is a simplex $L \in \Sigma^d$ with a distinguished vertex $x$ such that for any $y \in V(G_L)$, $d(x, y) \leq r$, where $d(x, y)$ is the shortest path distance of $x$ and $y$ in the graph $G_L$. We denote by $Z_{r,d}$ the rooted isomorphism classes of rooted $r$-balls. If $K \in \Sigma^d$ and $p \in V(K)$ then let $G_r(p)$ be the rooted $r$-ball in the 1-skeleton $G_K$ and $B_r(p)$ is the set of simplices $\sigma$ such that all vertices of $\sigma$ are in $G_r(p)$. Then $B_r(p)$ is a rooted $r$-ball of vertex degree bound $d$.

For $\alpha \in Z_{r,d}$ we denote by $T(K, \alpha)$ the set of vertices $p$ such that $B_r(p) \cong \alpha$. We set

$$p_K(\alpha) := \frac{|T(K, \alpha)|}{|K_0|}.$$

We say that $\{K^n\}_{n=1}^{\infty} \subset \Sigma^d$ is convergent (see [1] for the graph case) if