Energy-Based In-Domain Control and
Observer Design for Infinite-Dimensional
Port-Hamiltonian Systems

Tobias Malzer ∗ Jesús Toledo ∗∗ Yann Le Gorrec ∗∗ Markus Schöberl ∗

Abstract: In this paper, we consider infinite-dimensional port-Hamiltonian systems with in-
domain actuation by means of an approach based on Stokes-Dirac structures as well as in
a framework that exploits an underlying jet-bundle structure. In both frameworks, a dynamic
controller based on the energy-Casimir method is derived in order to stabilise certain equilibrias.
Moreover, we propose distributed-parameter observers deduced by exploiting damping injection
for the observer error. Finally, we compare the approaches by means of an in-domain actuated
vibrating string and show the equivalence of the control schemes derived in both frameworks.

Keywords: infinite-dimensional systems, partial differential equations, in-domain actuation,
port-Hamiltonian systems, structural invariants

1. INTRODUCTION

Energy-based approaches are well established for the con-
trol design of dynamical systems since they allow to exploit
the physical properties of the underlying system, see e.g.
vander Schaft (2000); Ortega et al. (2001). In this context,
especially the port-Hamiltonian (pH) system representa-
tion, which originally was designed for systems described
by ordinary differential equations (ODEs), has proven to
be an appropriate tool. From a control-engineering point
of view, a major advantage of the pH-framework is the fact
that so-called power ports can be introduced, which can
be used to incorporate the externally supplied energy and
therefore enable the interconnection with other systems.
Recently, a lot of research effort has been invested in the
adaptation of the pH-framework and energy-based control
strategies to systems governed by partial differential equa-
tions (PDEs).

However, it is worth stressing that the pH-system represen-
tation is not unique. With respect to control-engineering
purposes, in particular two different frameworks have turned
out to be especially suitable. The main difference
between the so-called Stokes-Dirac scenario, see van der
Schaft and Maschke (2002); Le Gorrec et al. (2005), and
the jet-bundle approach, see Ennsbrunner and Schlacher
(2005); Schöberl and Siuka (2011, 2014), is the choice of the
variables (energy variables vs. derivative variables), and
consequently, the generation of the power ports strongly
depends on the chosen approach; see e.g. Schöberl and
Siuka (2013) for a comparison of these frameworks by

This work has been supported by the Austrian Science Fund
(FWF) under grant number P 29964-N32.

means of the well-known Mindlin plate. In this contribu-
tion, we intend to compare not only the system representa-
tions based on these frameworks, but also the controller
as well as the observer design and draw some interesting
conclusions.

Similar to Macchelli and Melchiorri (2004); Macchelli
et al. (2017) and Schöberl and Siuka (2011); Rams and
Schöberl (2017), where dynamic control schemes based
on the well-known energy-Casimir method for so-called
boundary-control systems are addressed within the Stokes-
Dirac scenario and the jet-bundle framework, respectively,
in this paper the aim is to derive dynamic control laws,
where we focus on infinite-dimensional systems with in-
domain actuation. While the energy-Casimir method has
been extended to this system class in Malzer et al. (2019)
for the jet-bundle approach, in this paper we discuss the
controller design for in-domain actuated systems within
both frameworks and aim at comparing the results.

The energy-Casimir method is working appropriately if
the initial conditions of the plant are known precisely; how-
ever it yields unsatisfactory results for uncertain initial
conditions, see e.g. Rams et al. (2017), where this problem
is briefly discussed for a boundary-control system. To
overcome this obstacle, in this paper the energy-based
control law shall rely on a distributed-parameter observer.
In the literature, a lot of different observer-design schemes
for the infinite-dimensional scenario are available, see e.g.
Smyshlyaev and Kristic (2005) for an approach that relies
on the so-called backstepping methodology or Schaum
et al. (2016) for a dissipativity-based observer design.
Motivated by the fact that we focus on pH-systems, in this
contribution we intend to exploit energy considerations with respect to the observer design.

Thus, the main contributions are as follows: i) within both considered frameworks, we derive a Casimir-based control law for in-domain actuated systems, see Section 4. ii) moreover, in Section 5, a distributed-parameter observer is deduced within both considered frameworks by exploiting energy considerations for the observer error, iii) we demonstrate the capability of the proposed approach by means of an in-domain actuated vibrating string, where the equivalence of both control laws deduced in the different frameworks is shown, see Section 6.

2. NOTATION AND PRELIMINARIES

In this paper, we investigate distributed-parameter systems with a 1-dimensional spatial domain that shall be equipped with the coordinate \( z \in [0, L] \). The standard inner product on \( L^2([0, L]; \mathbb{R}^n) \) is denoted by \( \langle \cdot, \cdot \rangle_{L^2} \), while a Sobolev space of order \( p \) is indicated by \( H^p([0, L]; \mathbb{R}^n) \). In the following, we discuss some differential-geometric preliminaries. Formulas are kept short and readable by applying tensor notation and especially Einstein’s convention on sums, where we will not indicate the range of the indices when they are clear from the context. Furthermore, we use the standard symbols \( d \), \( \wedge \) and \( [ \) for the exterior derivative, the exterior (wedge) product and the natural contraction between tensor fields, respectively. Moreover, we avoid the use of pull-back bundles. The set of all smooth functions on a manifold \( \mathcal{M} \) is denoted by \( C^\infty(\mathcal{M}) \).

The total manifold \( \mathcal{E} \) of a bundle \( \pi : \mathcal{E} \to \mathcal{B} \) is equipped with the coordinates \( (z, x^\alpha) \), where \( x^\alpha \), with \( \alpha = 1, \ldots, n \), denotes the dependent variables, while the 1-dimensional base manifold possesses the independent coordinate \( z \), and therefore, a bundle – often denoted by \( \pi : (z, x^\alpha) \to (z) \) – allows to easily distinguish between dependent and independent variables. Note that a mathematical expression restricted to the boundary of \( \mathcal{E} \) is indicated by \( \mid_{\mathcal{E}} \).

To be able to introduce so-called derivative variables, we consider jet manifolds, where for instance the 2nd jet manifold \( \mathcal{J}^2(\mathcal{E}) \) possesses the coordinates \( (z, x^\alpha, x^\beta_\alpha, x^\beta_\alpha_{\alpha}) \), with \( x^\beta_\alpha \) denoting the 2nd-order derivative variable or jet coordinate, i.e. the 2nd derivative of \( z^\alpha \) with respect to \( z \).

Next, we introduce the tangent bundle \( \tau_{\mathcal{E}} = T(\mathcal{E}) \to \mathcal{E} \), which is equipped with the coordinates \( (z, x^\alpha, z, x^\alpha) \) together with the fibre bases \( \partial_2 = \partial / \partial z \) and \( \partial_\alpha = \partial / \partial x^\alpha \). Furthermore, the vertical tangent bundle \( \nu_{\mathcal{E}} : \nu(\mathcal{E}) \to \mathcal{E} \) possessing the coordinates \( (z, x^\alpha, \dot{x}^\alpha) \) is a submodule of \( \tau_{\mathcal{E}} \), and hence, a vertical vector field \( v = v^\alpha \partial_\alpha \in \nu(\mathcal{E}) \). Consequently, by means of the total derivative \( v^\alpha \partial_\alpha + x^\alpha \partial_\alpha + \partial_\alpha \dot{x}^\alpha + \ldots \), with the abbreviation \( \partial_\alpha = \partial / \partial x^\alpha \), the 1st prolongation of a vertical vector field is given as \( j^1(v) = v^\alpha \partial_\alpha + d_z (v^\alpha) \partial_\alpha \).

The so-called co-tangent bundle \( \tau^*_\mathcal{E} : T^*(\mathcal{E}) \to \mathcal{E} \), which possesses the coordinates \( (z, x^\alpha, \dot{z}, \dot{x}^\alpha) \) and the bases \( dz \) and \( dx^\alpha \), is a further important differential geometric object and allows to define a one-form \( w : \mathcal{E} \to T^*(\mathcal{E}) \) as a section given in local coordinates as \( w = \delta z w^z + w^\alpha dx^\alpha \) with \( w, w^\alpha \in C^\infty(\mathcal{E}) \). In this contribution, we are interested in densities \( \mathbf{F} = F dz \) where the coefficients may depend on 1st-order jet variables, i.e. \( \mathcal{F} \in C^\infty(\mathcal{J}^1(\mathcal{E})) \). In particular, we focus on the formal change of the corresponding integrated quantity \( \mathcal{F} = \int_0^L \mathcal{F} dz \) along solutions of a generalised vertical vector field, where for the calculation we exploit the so-called Lie derivative reading as \( L_v(\mathcal{F}) \) for a differential form \( v \). Hence, for 1st-order densities the formal change can be decomposed according to

\[
\mathcal{F} = \left. \int_0^L L^j_v(\mathcal{F} \mathcal{W}) \right|_{z = 0} = \left. \int_0^L v^\alpha \delta \mathcal{W} + (v^\alpha \mathcal{J}^\alpha_\beta \mathcal{J}_{\beta \alpha}) \right|_{z = 0}
\]

by using integration by parts and Stoke’s theorem. In (1), the map \( \delta \mathcal{W} = \delta_v(\mathcal{F}^\alpha \delta_\alpha \mathcal{W}) \) is called variational derivative and is given as \( \delta_v(\mathcal{F}) = \delta_\alpha(\mathcal{F} - d_z (\delta_\alpha \mathcal{F})) \) in local coordinates, while the boundary operator locally reads as \( \delta_z^j \mathcal{F} = \partial_z^j \mathcal{F} \).

3. PORT-HAMILTONIAN FRAMEWORK

As already mentioned, the pH-system representation in the infinite-dimensional scenario is not unique. In the following, two different approaches that have proven to be adequate frameworks in particular with respect to control-engineering purposes are presented, where both approaches rely on energy considerations. Although the structures of the considered system representations are quite different – stemming from the fact that different state variables are used – it should be stressed that the governing physical, i.e. the underlying system of PDEs, is the same.

3.1 Geometric Approach based on Jet-Bundle Structure

First, a pH-system representation that is particularly suitable for systems that allow for a variational characterisation is discussed. The approach exploits an underlying jet-bundle structure and makes heavy use of a certain power-balance relation, see Ennsbruner and Schlacher (2005); Schöberl et al. (2008); Schöberl (2014) for instance. To this end, we consider the bundle \( \pi : (z, x^\alpha) \to (z) \). Then, a pH-system with 1st-order Hamiltonian \( \mathcal{H} = \mathcal{H} dz \), i.e. \( \mathcal{H} \in C^\infty(\mathcal{J}^1(\mathcal{E})) \), including in- and outputs on the domain can be given as

\[
\dot{x} = (J - \mathcal{R})(\delta \mathcal{H}) + v \mathcal{G}
\]

\[
y = \mathcal{G}^\top (\delta \mathcal{H})
\]

together with appropriate boundary conditions. It should be stressed that the linear operators \( J, \mathcal{R} : T^*(\mathcal{E}) \wedge T^*(\mathcal{B}) \to V(\mathcal{E}) \), describing the internal power flow and the dissipation effects of the system, respectively, as well as the input operator \( \mathcal{G} : \mathcal{U} \to \nu(\mathcal{E}) \) in general can be differential operators. However, in this contribution it is sufficient to use bounded linear mappings, where the coefficients of the skew-symmetric interconnection tensor \( J \) satisfy \( J^\alpha_\beta = -J^\beta_\alpha \in C^\infty(\mathcal{J}^2(\mathcal{E})) \), while \( \mathcal{R}^\alpha_\beta = \mathcal{R}^\beta_\alpha \in C^\infty(\mathcal{J}^2(\mathcal{E})) \) and \( |\mathcal{R}^\alpha_\beta| \geq 0 \) is valid for the coefficient matrix of the symmetric and positive semidefinite dissipation map \( \mathcal{R} \). The input map \( \mathcal{G} \), where the components \( \mathcal{G}^\alpha_\beta \) may depend (amongst others) on the spatial coordinate \( z \), enables to incorporate external inputs located within the spatial domain. While we focus on systems with lumped inputs \( u^z \in \mathcal{U} \), the output components \( y^z \in \mathcal{Y} \) can be interpreted as distributed output densities because \( \mathcal{G}^\alpha_\beta \) are the components of the adjoint output map \( \mathcal{G}^\top : T^*(\mathcal{E}) \wedge T^*(\mathcal{B}) \to \mathcal{Y} \) as well. Since the input bundle \( \rho : \mathcal{U} \to \mathcal{E} \)}
\( \mathcal{J}(\mathcal{E}) \) is dual to the output bundle \( \phi : \mathcal{Y} \to \mathcal{J}(\mathcal{E}) \), see (Ennsbrunner and Schlacher, 2005, Section 4) or (Schöberl et al., 2008, Section 3), one can deduce the important relation

\[
(u|\mathcal{G})|\delta \mathbf{y} = u|\mathcal{G}^{*}|\delta \mathbf{y} = u|\mathbf{y}.
\]

Thus, by replacing \( \mathcal{G} \) by \( \mathcal{G} \) in (1) and substituting (2a) for \( \mathcal{G} \), for the system class under consideration the formal change of the Hamiltonian functional \( \mathcal{H} = \int_{\mathcal{L}} H dz \) follows to

\[
\mathcal{H} = -\int_{0}^{L} \mathcal{R}(\delta \mathbb{S})|\delta \mathbf{y} \mathcal{S} + \int_{0}^{L} u|\mathcal{G}^{*}y + (v|\delta \mathbf{y})|L
\]

where the first part describes the energy that is dissipated and the remaining parts correspond to collocation on the domain as well as on the boundary. Moreover, if we introduce a local representation for (2) as

\[
\dot{x}^0 = (\mathcal{J}^0 - \mathcal{R}^0)\delta \mathbb{S} + \mathcal{G}^0 u^\xi
\]

(5a)

\[
y^e = \mathcal{G}^e \delta \mathbb{H}.
\]

(5b)

with \( \alpha, \beta = 1, \ldots, n \) and \( \xi = 1, \ldots, m \), the power-balance relation (4) can be stated as

\[
\mathcal{H} = -\int_{0}^{L} \delta \mathbf{y}(\mathcal{H})|\delta \mathbb{S} \mathcal{H} + \int_{0}^{L} u^e \delta \mathbb{y} \mathcal{S} + (\dot{x}^0 \delta \mathbf{y} \mathcal{H})|L.
\]

At this point it should be mentioned that in this contribution we consider systems with trivial boundary conditions, and therefore, the boundary ports \((\dot{x}^0 \delta \mathbf{y} \mathcal{H})|L\) vanish for the considered systems. However, the boundary terms could easily be determined by applying the boundary operator \( \delta \mathbf{y} \), which will also play an important role for the determination of so-called Casimir conditions in Subsection 4.1.

### 3.2 Approach based on Stokes-Dirac Structure

In this subsection, we present the pH-approach that relies on Stokes-Dirac structures and is closely related to functional analysis, see e.g. Le Gorrec et al. (2005); Jacob and Zwart (2012) for a detailed analysis regarding the well-posedness and stability of boundary-controlled pH-systems. Here, we consider pH-systems according to

\[
\dot{\mathbf{y}}(z, t) = \mathbf{P}_1 \partial \mathbf{y}(\mathbf{Q}(\mathbf{z})\chi(z, t)) + \ldots + \mathbf{P}_0 \mathbf{G}(\mathbf{z}) + \mathbf{B}(\mathbf{z}) u(t)
\]

(6a)

together with the collocated output densities

\[
y = \mathbf{B}^T(\mathbf{Q}(\mathbf{z})\chi(z, t))
\]

(6b)

and appropriate boundary conditions given as

\[
\mathbf{W}_e \left[ j^0_e(t) \mathbf{e}_0(t) \right]^T = 0.
\]

(6c)

In (6a), the input \( u \in \mathbb{R}^m \) depends on the time \( t \) solely, i.e. we restrict ourselves to lumped inputs, whereas the output \( \mathcal{H}(z) \) depends on the spatial coordinate \( z \), implying that the output densities (6b) might be distributed over the spatial domain \( z \in [0, L] \). It is worth stressing that we intentionally use the same notation for the input \( u \) and the output \( y \) as for (2), since we assume that the inputs and outputs are the same in both frameworks. However, as already mentioned, the two approaches mainly differ in the choice of the variables which shall be highlighted by denoting the system state \( \chi(z, t) \in \mathbb{R}^n \). For the matrices in (6a) we have that \( \mathbf{P}_1 \in \mathbb{R}^{n \times n} \) and is invertible, \( \mathbf{P}_0 = -\mathbf{P}_0 \in \mathbb{R}^{n \times n} \), \( \mathbf{P}_n \in \mathbb{R}^{n \times n} \) and is invertible, and \( \mathbf{Q}(\mathbf{z}) \in \mathbb{L}^2([0, L]; \mathbb{R}^{n \times n}) \) denotes a bounded and continuously differentiable matrix-valued function, where \( \mathbf{Q}(\mathbf{z}) = \mathbf{Q}^T(\mathbf{z}) \)

and \( m I \leq \mathbf{Q}(\mathbf{z}) \leq MI \), with the constants \( m, M > 0 \), is valid for all \( z \in [0, L] \). Note that \( \mathbf{Q} \) and \( \mathbf{Q}^T \) are often used instead of \( \chi(z, t) \) and \( \mathbf{Q}(\mathbf{z}) \) for the sake of simplicity. Moreover, the state space is \( \mathbf{X} = \mathbb{L}^2([0, L]; \mathbb{R}^n) \), which is equipped with the inner product \( \langle \chi_1, \chi_2 \rangle_\mathbb{Q} = \langle \chi_1, \chi_2 \rangle_{\mathbb{L}^2} \) and the norm \( \| \chi \|_\mathbb{Q} = \| \chi \|_{\mathbb{L}^2} \). Moreover, the Hamiltonian can be given as \( H(t) = \frac{1}{2} \| \chi \|_\mathbb{Q}^2 \), emphasising that the norm is related to the stored energy of a system. Hence, \( \chi(z, t) \) are called energy variables, while \( \mathbf{Q}(\mathbf{z}) \chi(z, t) \) represents the co-energy variables. To be able to reformulate the boundary conditions of a system according to (6c), the boundary port variables (Macchelli et al., 2017, Eq. (2))

\[
\left[ f_0(t) \mathbf{e}_0(t) \right] = \frac{1}{\sqrt{2}} \left[ \mathbf{P}_1 - \mathbf{P}_1 \mathbf{Q}(\mathbf{L}) \mathbf{Q}(\mathbf{0}) \mathbf{Q}(\mathbf{L}) \right],
\]

(7)

which correspond to a linear combination of the co-energy variables restricted to the boundary, are introduced, where \( \mathbf{I} \) denotes the identity matrix of appropriate dimension. Hence, the time derivative of \( H(t) \) can be deduced to

\[
\dot{H} = -\int_{0}^{L} \dot{\mathbf{y}}^T \mathbf{G} \mathbf{y} dz + \int_{0}^{L} u^T \mathbf{y} dz + B,
\]

(8)

with \( B = \frac{1}{2} \| \mathbf{Q}(\mathbf{L}) \mathbf{P}_1 \mathbf{Q}(\mathbf{0}) \| \), where the derivation given in (Jacob and Zwart, 2012, Section 7.2) for boundary-control systems without dissipation can be adopted in a straightforward manner. Like (4), the balance equation (8) decomposes into dissipation (first term) and into collocation on the domain as well as on the boundary.

### 4. ENERGY-BASED IN-DOMAIN CONTROL

The following section deals with the energy-based design of control laws for infinite-dimensional systems with in-domain actuation within both discussed approaches. In particular, we focus on systems with lumped inputs, where the in-domain actuators exhibit a non-vanishing spatial distribution, naturally requiring the use of finite-dimensional controllers. Furthermore, it is of interest to compare the results obtained within both approaches.

The main idea of the energy-Casimir method is to couple the plant to a dynamic controller – which will beneficially be given in a pH-formulation – in order to shape the total energy of the closed loop and to inject damping into the system. The latter can be accomplished either by means of controller states that are not related to the plant, where the pH-structure of the controller shall be exploited, see e.g. Rams and Schöberl (2017), or by using an additional input \( u' \) in the interconnection of plant and controller given as

\[
u = -\gamma_e + u', \quad u_e = \gamma.
\]

(9)

In (9), \( u_e \) and \( u_e \) denote the in- and the output of the dynamic controller that will be declared subsequently, while \( \gamma \) corresponds to the integrated output density of the plant.
behaviour. To this end, we consider a finite-dimensional dynamic controller given in the pH-formulation

\[ \dot{z}^\alpha_c = \left( \Pi^\alpha_c \beta - R^\alpha_c \xi_c \right) \delta^\beta_c \dot{H}_c + C^\alpha_c \xi_c^\alpha_c, \]

(10a)

\[ y_c = C^\alpha_c \xi_c \delta^\alpha_c \dot{H}_c, \]

(10b)

with \( \alpha_c, \beta_c = 1, \ldots, n_c \) and \( \xi = 1, \ldots, m \). If we use the interconnection (9), locally given as

\[ u^\xi = -\delta^\xi y_c - u^\xi, \quad u^\xi = \delta^\xi \int_0^L y_c dz, \]

(11)

with the Kronecker-Delta symbol meeting \( \delta^\xi = 1 \) for \( \xi = \eta \) and \( \delta^\xi = 0 \) for \( \xi \neq \eta \), we obtain a closed-loop system with

\[ \mathcal{H}_d(x, x_c) = \int_0^L \tilde{H} dz + H_c(x_c). \]

(12)

Consequently, by using the interconnection (11) and taking the fact that we consider systems with in-domain actuation solely – i.e. no power flow takes place through the boundary ports – into account, the formal change of \( \mathcal{H}_d \) follows to

\[ \tilde{\mathcal{H}}_d = -\int_0^L \delta^\alpha_c (H) R^\alpha_c \delta^\beta_c (H) dz + \ldots \]

\[ - \partial^\alpha_c (H) R^\alpha_c \delta^\beta_c (H) + u^\xi \int_0^L y_c dz. \]

Remark 1. It should be noted that a comprehensive proof of stability for systems governed by PDEs requires functional analysis, whereas the focus of this contribution is on structural/geometric considerations. Thus, no detailed stability investigations are presented here. Nevertheless, in Section 6 we exemplarily sketch the necessary procedure by means of the observer error of an in-domain actuated vibrating string.

To be able to use the closed-loop Hamiltonian as Lyapunov candidate, it must be ensured that \( \mathcal{H}_d \) exhibits a minimum at the desired equilibrium point, entailing that the controller-Hamiltonian has to be designed properly. Therefore, we exploit Casimir functionals of the form

\[ \mathcal{C}^\alpha = x^\alpha_c + \int_0^L C^\lambda dz, \]

(13)

where it should be stressed that they in general may depend on 1st-order jet variables, i.e. \( \mathcal{C}^\alpha \in \mathcal{C}^{\alpha \beta} (\mathcal{I}^1(\mathcal{E})) \). Thus, (13) has to fulfil \( \dot{\mathcal{C}}^\alpha = 0 \) independently of the system-energy function in order to serve as conserved quantity. Consequently, due to \( \dot{\mathcal{C}}^\alpha = \dot{\mathcal{C}}^\alpha \lambda = \lambda c, \) where the constants \( \lambda = \mathcal{C}^{\alpha \beta} \lambda^\beta_{\alpha^\beta} \) depend on the initial states of the plant and the controller, we can express the controller states that are related to the plant as \( x^\alpha_c = \lambda^\alpha - \int_0^L \mathcal{C}^\lambda dz \). Hence, if each controller state is related to the plant, we are able to write \( \mathcal{H}_d = \mathcal{H}_d(x) + H_c(x) \) indicating that we are able to shape the minimum of the closed loop. However, it should be mentioned that \( \lambda \) cannot be determined exactly when the initial conditions of the plant are not known precisely, which would yield an offset in the resulting control law, implying a deviation regarding the desired equilibrium. To overcome this drawback, we design a distributed parameter observer in Section 5, as the fact that the estimated state will converge to the real one implies that these offset vanishes.

Proposition 2. Consider the closed loop that results due to the interconnection of the plant (5) and the controller (10) by means of (11) with \( u^\xi = 0 \). Then, if the functionals (13) fulfil the conditions

\[ \delta^\alpha_c (J^\alpha - R^\alpha_c) = 0 \]

(14a)

\[ \delta^\alpha_c (\mathcal{F}^\alpha - R^\alpha_c) + G^\alpha_c \mathcal{K}^\alpha_c G^\alpha_c = 0 \]

(14b)

\[ \delta^\alpha_c \mathcal{G}^\alpha_c K^\alpha_c G^\alpha_c = 0 \]

(14c)

\[ (\mathcal{F}^\alpha - R^\alpha_c)^2) L_0 = 0 \]

(14d)

for \( \lambda = 1, \ldots, n \), they serve as conserved quantities.

Proof. For the proof we refer to Malzer et al. (2019).

4.2 Stokes-Dirac Approach

Next, we want to deduce a control law based on the energy-Casimir method within the Stokes-Dirac framework. Thus, we consider a dynamic controller given in the pH-form

\[ \dot{v}_c = (A_c - S_c)Q_c v_c + B_c u_c \]

(15a)

\[ \dot{y}_c = B_c^T Q_c v_c \]

(15b)

with \( v_c \in \mathbb{R}^{n_c} \), where it should be noted that the output \( u_c \in \mathbb{R}^m \) is the same as for (10), but the output \( y_c \in \mathbb{R}^m \) might be different as (10b). In (15a), we have the Hamiltonian \( H_c = \frac{1}{2} \| \chi \|_{Q_c}^2 + \frac{1}{2} \| v_c \|_{Q_c}^2 \). To be able to properly shape \( H_d(x, v_c) \), we introduce Casimir functions of the form

\[ C = \Gamma T v_c + \int_0^L \chi (\Psi (T)) dz, \]

(16)

representing a special case of (13) if \( \Gamma \) is a unit vector.

Proposition 3. The functionals (16) serve as structural invariants of the closed loop, stemming from the interconnection of the plant (6) and the controller (15) via (9) with \( u' = 0 \), if they meet the conditions

\[ (A_c + S_c)\Gamma = 0 \]

(17a)

\[ P_1 \partial \Psi + (P_0 + G_0) \Psi - B_1 B_1^T \Gamma = 0 \]

(17b)

\[ B_1 B_1^T \Psi = 0 \]

(17c)

\[ [\epsilon_0 f_0] R \Psi (L) \Psi (0) = 0 \]

(17d)

Proof. Next, we show the proof of Prop. 3, which is a trivial adaptation of the boundary-control case, and hence, we deduce the formal change of (16). Therefore, if we substitute (6), (15) and the interconnection (9) in

\[ \dot{\mathcal{C}} = \Gamma^T v_c + \int_0^L \Psi T \partial \mathcal{C} dz, \]

an integration by parts yields

\[ \dot{\mathcal{C}} = \Gamma^T (A_c - S_c)Q_c v_c + \ldots \]

\[ - \gamma_0 (\Gamma^T B_c B_1^T - \partial \Psi) T P_1 + \Psi T (P_0 - G_0) Q_c v_c + \ldots \]

\[ - \int_0^L \Psi T B_1 B_1^T Q_c v_c dz + (\Psi^T P_1 Q_c)_{L_0}^0. \]

Thus, the conditions (17a), (17b) and (17c) follows immediately by considering the properties of \( A_c, S_c, P_1, P_0, G_0 \). If we rewrite the expression restricted to the boundary as

\[ (\Psi^T P_1 Q_c)_{L_0}^0 = \left[ \begin{array}{c} \Psi (L) \Psi (0) T P_1 \end{array} \right] \]

and use the relation \( R^T \Sigma R = \left[ \begin{array}{c} P_1 \end{array} \right] \), by means of (7) we are able to deduce condition (17d).
Although the considered approaches are quite different, in fact the conditions (17) yield a similar result as Prop. 2. In particular, the conditions (17b) and (14b) allows to relate the plant within the domain to the controller, which is different compared to the Casimir conditions for boundary-control systems given in (Macchelli et al., 2017, Prop. 3.1). Like (14c), condition (17c) implies that we cannot find Casimir functions depending on system states where an input appears in the corresponding system equation.

5. OBSERVER DESIGN

As already mentioned, an uncertain initial configuration of the plant would cause some problems regarding the Casimir-based control law. To be able to apply the control scheme anyway, we propose a distributed-parameter observer in both considered frameworks. The idea is to design an observer based on energy considerations such that the observer-error system exhibits a desired behaviour.

5.1 Jet-Bundle Approach

Now, the objective is to design an observer system by exploiting an underlying jet-bundle structure such that the observer error tends to zero. To this end, we introduce the bundle $\tilde{\pi} : \tilde{E} \to \tilde{B}$ with coordinates $(\tilde{x}, \tilde{x}^\alpha)$ for $\tilde{E}$ and $(\tilde{z})$ for $\tilde{B}$. Next, we extend the copy of the plant (5) by an error-injection term exploiting the additional input 

$$u^0 = \delta^{\alpha}(\tilde{y}_\alpha - \tilde{y}_\alpha^0),$$

and consequently, the observer system can be written as

$$\dot{\tilde{x}}^\alpha = ((\mathcal{F}^\alpha_\beta - \mathcal{R}^\alpha_\beta)\delta_\beta \mathcal{H} + \mathcal{G}^\alpha_\eta \delta^\eta + \mathcal{K}^\alpha_\eta u^0_\eta),$$

with $\alpha, \beta = 1, \ldots, n$ and $\xi, \eta = 1, \ldots, m$, by means of the observer-energy density $\mathcal{H}$. In (18), $\tilde{y}_\xi$, which corresponds to the integrated output density of the plant according to $\tilde{y}_\xi = \int_0^L \tilde{y}_\xi dz$, is assumed to be available as measurement quantity, while $\tilde{y}_\xi$ represents the copy of the integrated plant output according to $\tilde{y}_\xi = \int_0^L \tilde{y}_\xi dz$ with

$$\tilde{y}_\xi = \mathcal{G}^\xi_\delta \delta_\delta \mathcal{H}.$$  

The aim is to design the observer gain $\mathcal{K}^\alpha_\xi$ such that the observer error $\tilde{x} = x - \tilde{x}$ tends to $0$. If we substitute (5a) and (19a) in $\tilde{x} = x - \tilde{x}$, we have

$$\dot{\tilde{x}}^\alpha = (\mathcal{F}^\alpha_\beta - \mathcal{R}^\alpha_\beta)\delta_\beta \tilde{H} + \mathcal{K}^\alpha_\xi \delta^\xi \tilde{y}_\xi,$$

for the dynamics of the observer error. Thus, if we consider

$$\dot{\tilde{y}}_\xi = -\mathcal{K}^\xi_\delta \delta_\delta \tilde{H},$$

as collocated output densities, the observer error (20) can be interpreted as a pH-system with the error-Hamiltonian $\mathcal{H} = \int_0^L \tilde{H} dz$, where it should be mentioned that $\mathcal{H} \neq \mathcal{H} + \mathcal{H}$. Thus, the formal change of $\mathcal{H}$ follows to

$$\dot{\mathcal{H}} = -\int_0^L \left(\delta_\delta (\tilde{H}) \mathcal{R}^\delta_\beta \delta_\beta \tilde{H} + \delta_\delta (\tilde{H}) \mathcal{K}^\delta_\xi \delta^\xi (\tilde{y}_\xi - \tilde{y}_\xi^0)\right) dz$$

by means of (18). Therefore, by choosing the components $\mathcal{K}^\alpha_\xi$ properly, it is possible to achieve an error system with $\mathcal{H} \leq 0$, i.e. the observer error is non-increasing. Of course, with regard to the observer error it is necessary to show that it converges to $0$, which is discussed in Section 6 for a concrete example.

5.2 Stokes-Dirac Approach

Next, we intend to exploit the framework proposed in Subsection 3.2 in order to deduce a distributed-parameter observer for systems with in-domain actuation and collocated measurement. Again, the main idea is to extend the copy of the plant by an error-injection term – where like in Subsection 5.1 it is assumed that $\tilde{y}$ is available as measurement quantity – and determine the observer gain such that the observer error converges to $0$. In general, the infinite-dimensional observer system can be introduced as

$$\partial_t \tilde{x}(z,t) = P_1 \partial_z (Q(z)\tilde{x}(z,t)) + (P_0 - G_0) (Q(z)\tilde{x}(z,t)) + B(z)u(t) + \mathcal{L}(z)(\tilde{y} - \tilde{y}) + \tilde{x},$$

where the corresponding boundary port variables can be deduced according to (7) with the observer state $\tilde{x}$. In (21a), we have the same $P_1, P_0, G_0, Q$ and $B$ as defined for (6a), and $\mathcal{L}(z) \in L^2([0, L]; \mathbb{R}^{\infty \times \infty})$ denotes the observer gain. If we define the observer error $\tilde{x} = \chi - \tilde{x}$, the dynamics of the observer error can be formulated as pH-system

$$\partial_t \tilde{x}(z,t) = P_1 \partial_z (Q(z)\tilde{x}(z,t)) + \ldots + (P_0 - G_0) (Q(z)\tilde{x}(z,t)) + \mathcal{L}(z)(\tilde{y} - \tilde{y}),$$

together with the collocated output density

$$\tilde{y} = -\mathcal{L}^T(z)Q\tilde{x},$$

and the boundary conditions

$$W_B \left[ f_0^T(t) \tilde{y}_0^T(t) \right]^T = 0, \quad (22b)$$

To properly design $\mathcal{L}(z)$, we consider the formal change of the energy $\tilde{H} = \frac{1}{2} \|\tilde{x}\|^2$, which follows to

$$\dot{\tilde{H}} = -\int_0^L (Q\tilde{x})^T G_0 Q\tilde{x} dz - \int_0^L (\tilde{y} - \tilde{y})^T \mathcal{L}^T Q\tilde{x} dz$$

by taking the boundary conditions (22c) into account. The objective is to determine $\mathcal{L}(z)$ such that the observer error $\tilde{x}$ tends to $0$, which is demonstrated for an in-domain actuated vibrating string in the following example.

6. EXAMPLE: VIBRATING STRING

To illustrate the proposed approaches, we consider a vibrating string actuated within the spatial domain that can be modelled according to

$$\rho(z) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial z}(T(z) \frac{\partial w}{\partial z}) + f(z,t),$$

where $w$ describes the vertical deflection of the string, $\rho(z) \equiv \text{const}$ the mass density and $T(z) \equiv \text{const}$ Young’s modulus. We assume that the string is clamped at $z = 0$ and free at $z = L$, i.e. the boundary conditions

$$w(0,t) = 0, \quad T \frac{\partial w}{\partial z}(L,t) = 0$$

are valid. The distributed force $f(z,t) = g(z)u(t)$ shall be generated by an actuator behaving like a piezoelectric patch, which is located between $z = z_{p1}$ and $z = z_{p2}$, mathematically described by $g(z) = h(z - z_{p1}) - h(z - z_{p2})$, with $h(\cdot)$ denoting the Heaviside function. In fact, the force-distribution on the domain $z_{p1} \leq z \leq z_{p2}$ is supposed to be constant and is scaled by $u(t)$, which can
be interpreted as an input voltage applied to the actuator. The objective is to stabilise the desired equilibrium
\[ u^d = \begin{cases} 
  az & \text{for } 0 \leq z < z_{p1} \\
 -b(z - z_{p2})^2 + c & \text{for } z_{p1} \leq z < z_{p2} \\
  c & \text{for } z_{p2} \leq z \leq L
\end{cases}, \quad (24) \]
with \( a, b > 0 \) and \( c = b(z_{p2} - z_{p1})^2 + az_{p1}. \)

6.1 Jet-Bundle Approach

First, we consider the bundle \( \pi : (z,w,p) \rightarrow (z) \), where we have introduced the generalised momenta \( p = \rho \bar{v} \).

Consequently, the governing equation of motion reads as
\[ \dot{p} = Tw_{zz} + g(z)u. \quad (25) \]

Hence, by means of the Hamiltonian density \( H = \frac{1}{2}p^2 + \frac{1}{2}T(w)^2 \), we are able to reformulate (25) according to
\[ \begin{bmatrix} \dot{w} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_wH \\ \delta_pH \end{bmatrix} + \begin{bmatrix} 0 \\ g(z) \end{bmatrix}u, \quad y = g(z)\frac{p}{\rho}. \]

The integrated plant-output \( \tilde{y} = \int_0^L g(z)\frac{p}{\rho}dz \) corresponds to the current through the actuator and is available as measurement quantity. Furthermore, the formal change of the Hamiltonian functional \( \mathcal{H} \) reduces to \( \mathcal{H} = \int_0^L g(z)\frac{p}{\rho}dz \) because of the boundary conditions (23b).

Control Design

In the following, we are interested in designing a dynamic controller by exploiting the energy-Casimir method proposed in Subsection 4.1. As we only have one output of the plant, one controller state shall be related to the plant. Moreover, due to the fact that we intend to inject damping by an additional input \( u' \), we do not further extend the dimension of the controller, i.e. \( n_c = 1 \). To be able to shape the closed-loop energy, we choose the ansatz \( C_1 = -g(z)w \), which fulfills the conditions (14) for \( G_1^1 = 1 \) and allows for a relation between the plant and the controller state according to
\[ x_c^1 = \int_0^L g(z)wdz \quad (26) \]
by choosing the initial controller states properly.

However, the conditions (14) restrict the controller dynamics to \( x_c^1 = u_c \). If we set the controller-Hamiltonian to \( H_c = c_1(x_c^1 - x_c^1,d - \bar{x}_w) \), where we use the constant \( c_1 > 0 \) and
\[ x_c^1,d = \int_0^L g(z)wdz, \quad (27) \]
the equilibrium (24) becomes a part of the minimum of \( \mathcal{H}_c \). Worth stressing is the fact that we have included the term with \( u_c \) in \( H_c \), because the equilibrium (24) requires non-zero power. Furthermore, the output of the dynamic controller follows to \( y_c = c_1(\bar{x}_w - x_c^1,d - \bar{x}_w) \). If we use the interconnection \( u = -y_c + u' \) and \( u_c = \int_0^L ydz \) together with the dissipative output feedback \( u' = -c_2\bar{y} \), where \( c_2 > 0 \), the formal change of \( \mathcal{H}_c \) can be deduced to \( \mathcal{H}_c = -c_2\bar{y}^2 \leq 0 \). To allow for a comparison with the controller that is derived within the Stokes-Dirac scenario later, we state the total-control law, which can be given as
\[ u = -c_1(\int_0^L g(z)wdz - \int_0^L g(z)wdz) - c_2\bar{y} + u_s \quad (28) \]
by substituting (26) and (27).

Observer Design

Next, an observer for the in-domain actuated vibrating string can be introduced as
\[ \begin{bmatrix} \dot{\bar{w}} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_wH \\ \delta_pH \end{bmatrix} + \begin{bmatrix} 0 \\ g(z) \end{bmatrix}u + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}(\bar{y} - \bar{y}), \quad \text{by means of the observer density } H = \frac{1}{2}\bar{p}^2 + \frac{1}{2}T(\bar{w})^2, \]
where \( \bar{y} = \int_0^L g(z)\frac{p}{\rho}dz \) corresponds to the copy of the plant output. Next, we introduce the error coordinates \( \bar{w} = w - \bar{w}, \bar{p} = p - \bar{p} \), implying that the error dynamics follows to
\[ \begin{bmatrix} \dot{\bar{w}} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_wH \\ \delta_pH \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}(\bar{y} - \bar{y}). \quad (29a) \]

Consequently, by means of the energy density of the observer error \( \tilde{H} = \frac{1}{2}\bar{p}^2 + \frac{1}{2}T(\bar{w})^2 \), the dynamics of the observer error can be given in the pH-formulation
\[ \begin{bmatrix} \dot{\bar{w}} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta_wH \\ \delta_pH \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}(\bar{y} - \bar{y}), \quad (30a) \]
\[ \bar{y} = -\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} \delta_wH \\ \delta_pH \end{bmatrix} = k_1T\bar{w}_{zz} - k_2\bar{p}. \quad (30b) \]
A straightforward calculation of the formal change of the error-Hamiltonian functional \( \mathcal{H}_c \) yields the relation
\[ \dot{\mathcal{H}} = \int_0^L (T\bar{w}_{zz}k_1(\bar{y} - \bar{y}) - \bar{p}k_2(\bar{y} - \bar{y}))dz. \]

If we choose \( k_1 = 0 \) and \( k_2 = kg(z) \) with \( k > 0 \), by keeping in mind that \( (\bar{y} - \bar{y})^2 = \int_0^L (g(z)\frac{p}{\rho})dz \) holds, one can conclude that \( \dot{\mathcal{H}}(\bar{w},\bar{p}) = -k(\bar{y} - \bar{y})^2 \leq 0 \), and therefore, the observer error is non-increasing. Hence, by initialising the controller with \( \bar{x}_c^1(0) = \int_0^L g(z)\bar{w}(z,0)dz \) and using \( \bar{x}_c^1 = u_c = \bar{y} \) for the controller dynamics, the control law (28) can be interpreted as \( u = -c_1(\int_0^L g(z)wdz - \int_0^L g(z)wdz - c_2\bar{y} + u_s \). If \( \bar{x} \) converges to \( x \), the combination of controller and observer stabilises the desired equilibrium exactly, and therefore, in the following we sketch the procedure for proving well-posedness and asymptotic stability of the observer error.

Remarks on the Observer Convergence and Simulation Results

To analyse the well-posedness and stability of the observer-error system, we introduce the state vector \( \tilde{w} = [w^1, w^2]^T = [\bar{w}, \bar{p}]^T \) together with the state space \( W = H^1_c(0,L) \times L^2(0,L), \) where \( H^1_c(0,L) = \{ w^1 \in H^1(0,L)|w^1(0) = 0 \} \). Furthermore, we define the (energy) norm \( ||w||^2_W = T(\bar{w}_z, \bar{w}_z)_L + \frac{1}{2}(|\bar{p}, \bar{p}|_L^2), \) where it can be shown that it is equivalent to the standard norm. Next, we reformulate the observer-error dynamics as an abstract Cauchy problem \( \dot{\tilde{w}} = \mathcal{A}\tilde{w}, \) with the operator \( \mathcal{A} : D(\mathcal{A}) \subset W \rightarrow W \) given as
\[ \mathcal{A} : \begin{bmatrix} \bar{w} \\ \bar{p} \end{bmatrix} \mapsto \begin{bmatrix} \bar{p} \\ T\bar{w}_{zz} - kg(z)\int_0^L g(z)\rho^{-1}pdz \end{bmatrix}, \]
where it can be shown that the inverse operator \( \mathcal{A}^{-1} \) exists. Due to the relations \( \mathcal{A}^* = \frac{1}{2}||w||_W^2 \) and \( \mathcal{A}^* \leq 0 \) it follows that \( \mathcal{A} \) is dissipative, and hence, we are able to apply a variant of the Lumer-Phillips theorem (Liu and Zheng, 1999, Thm. 1.2.4) in order to show that \( \mathcal{A} \) generates a \( C_0 \)-contraction semigroup. Furthermore, since \( \mathcal{A}^{-1} \) is closed – which can be shown similarly to the proof
it is shown that the proposed controller in combination with the observer stabilizes the desired equilibrium (24).

As already mentioned, this framework relies on the use of observer quantities $\hat{w}$, and consequently, we are interested in one Casimir function, i.e., we choose $\Gamma = 1$. Next, (17c) implies that $\Psi_{\hat{2}} = 0$ must be valid. Furthermore, if we set $B_c = 1$, an evaluation of the condition (17b) yields the restrictions $\partial_2 \Psi_1 = g(z)$, where $\Psi_1$ has to be chosen such that (17d) together with the boundary conditions (23b) is satisfied, i.e., $(\Psi_1 \hat{z})_{z=0} = 0$ must be valid. Thus, a relation between the plant and the controller is given by

$$v_c = - \int_0^L \Psi_1 q dz,$$  
(33)

and the dynamic of the controller is restricted to $\dot{v}_c = u_c$.

To achieve that the desired equilibrium (24) becomes a part of the minimum of $H_c = H + H_c$, we set $\dot{H}_c = -d_1(v_c - \dot{w}_c - \frac{q_c}{\Psi_1})^2$, with $d_1 > 0$ and (33) with $q = \partial_z w|_L$ instead of $q$ for $v^d_c$. If we use $u' = -d_2 \dot{y}_c$ for the damping injection, we obtain $\dot{H}_d = -d_2 \dot{y}_c^2$, and the control law follows to

$$u = -d_1(- \int_0^L \Psi_1 q dx + \int_0^L \Psi_1 q^d dx) - d_2 \dot{y}_c + u_s.$$  
(34)

Next, it is of particular interest to compare the control laws (28) and (34). In fact, by substituting $q = \partial_z w$ and using integration by parts, we find that

$$- \int_0^L \Psi_1 \partial_z w dx = \left[ \Psi_1 w \right]_0^L + \int_0^L \partial_z (\Psi_1) w dx .$$

Taking the boundary conditions and the restriction $\partial_2 \Psi_1 = g(z)$ into account, the condition (33) can be rewritten as

$$v_c = \int_0^L g(z) w dx.$$  

The same can be done for $v^d_c = - \int_0^L \Psi_1 q^d dx$, and consequently, we are able to give (34) as

$$u = -d_1(- \int_0^L g(z) w dx + \int_0^L g(z) w^d dx) - d_2 \dot{y}_c + u_s,$$  
(35)

which is exactly the same control law as (28) if we set $c_1 = d_1$ and $c_2 = d_2$.

**Observer Design** To be able to exploit the control scheme presented in the previous subsection, in the following we design an observer according to the strategy proposed in Subsection 5.2. Therefore, for the in-domain actuated vibrating string, the observer can be introduced as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \partial_z \left( \begin{bmatrix} T & 0 \\ 0 & \rho^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \right) + \begin{bmatrix} 0 \\ g(z) \end{bmatrix} u .$$  
(31)

Consequently, by means of (7) we are able to determine the boundary-port variables

$$f_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} \rho^{\gamma}(L) - \rho^{-\gamma}(0) \\ \rho^{\gamma}(L) + \rho^{-\gamma}(0) \end{bmatrix} \int_0^L \begin{bmatrix} \rho^{\gamma}(z) \\ \rho^{-\gamma}(z) \end{bmatrix} dz ,$$

which further can be used to reformulate the boundary conditions (23b) according to (6c) with

$$W_\partial = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$  
(32)

**6.2 Stokes-Dirac Approach**

As already mentioned, this framework relies on the use of energy variables, and therefore, the strain $q = \partial_z w$ is introduced instead of the deflection $w$, i.e., we use the state $\chi = [q, p]^T$ to describe the underlying system. Hence, an alternative pH-system representation for the in-domain actuated vibrating string (23a) reads as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \partial_z \left( \begin{bmatrix} T & 0 \\ 0 & \rho^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \right) + \begin{bmatrix} 0 \\ g(z) \end{bmatrix} u .$$  
(31)
\[
\begin{bmatrix}
\dot{q} \\
\dot{\rho}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\partial_z \left( \begin{bmatrix}
T & 0 \\
0 & \rho^{-1}
\end{bmatrix}
\begin{bmatrix}
\dot{q} \\
\dot{\rho}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
g
\end{bmatrix} u + \begin{bmatrix}
l_1 \\
l_2
\end{bmatrix} (y - \bar{y})
\right),
\]

\[\text{together with } W_B \begin{bmatrix} f_0 \\ e_0 \end{bmatrix} = \begin{bmatrix} \rho(0)^{-1} \dot{\rho}(0) \\ T(L) \dot{q}(L) \end{bmatrix} = 0 \text{ by means of (32). Thus, by using } \bar{q} = q - \bar{q} \text{ and } \bar{\rho} = \rho - \bar{\rho}, \text{ we have}
\]

\[\begin{bmatrix}
\dot{q} \\
\dot{\rho}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\partial_z \left( \begin{bmatrix}
T & 0 \\
0 & \rho^{-1}
\end{bmatrix}
\begin{bmatrix}
\dot{q} \\
\dot{\rho}
\end{bmatrix}
- \begin{bmatrix}
l_1 \\
l_2
\end{bmatrix} (y - \bar{y})
\right)
\]

for the error dynamics and the collocated output reads as

\[\bar{y} = - \begin{bmatrix} l_1 \\
l_2 \end{bmatrix} \begin{bmatrix} T \dot{q} \\
\rho^{-1} \dot{\rho}
\end{bmatrix} \].

Next, a careful investigation of the formal change of \( \dot{H} = \frac{1}{2} \int_0^L (\dot{q}^2 + \dot{\rho}^2)dz \) which can be deduced to

\[\dot{H} = - \int_0^L (l_1 (\bar{y} - \dot{y}) T \dot{q} + l_2 (\bar{y} - \dot{y}) \frac{1}{\rho} \dot{\rho}) dz \quad \text{(36)}\]

by means of integration by parts and taking the boundary conditions \( W_B \begin{bmatrix} f_0 \\ e_0 \end{bmatrix} = \begin{bmatrix} \rho(0)^{-1} \dot{\rho}(0) \\ T(L) \dot{q}(L) \end{bmatrix} = 0 \) into account, allows to determine the observer gains \( l_1 \) and \( l_2 \) properly. In fact, the choice \( l_1 = 0 \) and \( l_2 = k g(z) \) – i.e. we have exactly the same components for the observer gain as in Subsection 6.1.2 – renders (36) to \( \dot{H} = - k (\bar{y} - \dot{y})^2 \leq 0 \).

Let us finally mention again that the main difference of the proposed approaches is the choice of the variables, cf. \( x = [w, y, p]^T \) for the jet-bundle approach and \( \chi = [q, p]^T \) for the Stokes-Dirac approach, which implies that the Hamiltonian function is different within the considered frameworks, although the energy of course is the same. The choice of the variables also affect the ansatz of the Casimir functions which might have a further impact on the controller states; however, for the system under consideration we have shown that we obtain the same control law, cf. (28) and (35). Furthermore, although we considered two completely different observer systems – i.e. we have different observer states – we obtain the same dissipation rate for the observer errors.

7. CONCLUSION AND OUTLOOK

In this paper, we have considered the controller and observer design for infinite-dimensional \( p \)-systems with in-domain actuation based on a jet-bundle as well as on a Stokes-Dirac approach. In particular, we have shown that – although the underlying system descriptions and therefore the controller and observer design are quite different – the proposed approaches yield the same results. As we only sketched the proof of stability for the observer error, stability investigations for the controller as well as for the combination with the observer remain to be done.

REFERENCES

Ennsbrunner, H. and Schlacher, K. (2005). On the Geometrical Representation and Interconnection of Infinite Dimensional Port Controlled Hamiltonian Systems. Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conf., (5263–5268).

Guo, W. and Guo, B.Z. (2011). Parameter estimation and stabilisation for a one-dimensional wave equation with boundary output constant disturbance and non-collocated control. International Journal of Control, 84(2), 381–395.

Jacob, B. and Zwart, H.J. (2012). Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces. Birkhäuser.

Liu, Z. and Zheng, S. (1999). Semigroups Associated with Dissipative Systems. Research Notes in Mathematics Series. Chapman and Hall/CRC.

Luo, Z.H., Guo, B.Z., and Morgul, O. (1998). Stability and Stabilization of Infinite Dimensional Systems with Applications. Springer.

Macchelli, A., Le Gorrec, Y., Ramirez, H., and Zwart, H. (2017). On the Synthesis of Boundary Control Laws for Distributed Port-Hamiltonian Systems. IEEE Trans. Autom. Control, 62(4), 1700–1713.

Macchelli, A. and Melechiorri, C. (2004). Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach. SIAM J. Control Optim., 43(2), 743–767.

Malzer, T., Rams, H., and Schöberl, M. (2019). Energy-Based In-Domain Control of a Piezo-Actuated Euler-Bernoulli Beam. In Proceedings of the 3rd IFAC Workshop on Control of Systems Governed by Partial Differential Equations (CPDE), IFAC-PapersOnLine, 147–152.

Le Gorrec, Y., Zwart, H.J., and Maschke, B. (2005). Dirac structures and boundary control systems associated with skew-symmetric differential operators. SIAM J. Control Optim., 44(5), 1864–1892.

Ortega, R., van der Schaft, A.J., Mareels, I., and Maschke, B. (2001). Putting energy back in control. IEEE Control Syst. Mag., 21(2), 18–33.

Rams, H. and Schöberl, M. (2017). On Structural Invariants in the Energy Based Control of Port-Hamiltonian Systems with Second-Order Hamiltonian. In Proceedings of the American Control Conference (ACC), 1139–1144.

Rams, H., Schöberl, M., and Schlacher, K. (2017). Optimal Motion Planning and Energy-Based Control of a Single Mast Stacker Crane. IEEE Transactions on Control Systems Technology, to appear.

Schaum, A., Moreno, J.A., and Meurer, T. (2016). Dissipativity-based observer design for a class of coupled 1-D semi-linear parabolic PDE systems. In Proceedings of the 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations (CPDE), volume 49, issue 8 of IFAC-PapersOnLine, 98–103.

Schöberl, M. (2014). Contributions to the Analysis of Structural Properties of Dynamical Systems in Control and Systems Theory, volume 21 of Modellierung und Regelung komplexer dynamischer Systeme. Shaker Verlag.

Schöberl, M., Ennsbrunner, H., and Schlacher, K. (2008). Modelling of Piezoelectric structures - a Hamiltonian Approach. Mathematical and Computer Modelling of Dynamical Systems, 14(3), 179–193.

Schöberl, M. and Siuka, A. (2011). On Casimir Functionals for Field Theories in Port-Hamiltonian Description of Control Purposes. In Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 7759–7764.

Schöberl, M. and Siuka, A. (2013). Analysis and Comparison of Port-Hamiltonian Formulations for Field Theories
- demonstrated by means of the Mindlin plate. 548–553.
Schöberl, M. and Siuka, A. (2014). Jet bundle formulation of infinite-dimensional port-Hamiltonian systems using differential operators. *Automatica*, 50(2), 607–613.
Smyshlyaev, A. and Kristic, M. (2005). Backstepping observers for a class of parabolic PDEs. *Systems and Control Letters*, 54, 613–625.
Stürzer, D., Arnold, A., and Kugi, A. (2016). Closed-loop Stability Analysis of a Gantry Crane with Heavy Chain.
van der Schaft, A.J. (2000). *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer.
van der Schaft, A.J. and Maschke, B. (2002). Hamiltonian formulations of distributed parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42(1-2), 166–194.