Hidden Markov Model Where Higher Noise Makes Smaller Errors

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Abstract

We consider the problem of parameter estimation in a partially observed linear Gaussian system with small noises in the state and observation equations. We describe asymptotic properties of the MLE and Bayes estimators in the setting with state and observation noises of possibly unequal intensities. It is shown that both estimators are consistent, asymptotically normal with convergent moments and asymptotically efficient. This model has an unusual feature: larger noise in the state equation yields smaller estimation error. The proofs are based on asymptotic analysis of the Kalman-Bucy filter and the associated Riccati equation in particular.

MSC 2000 Classification: 62M02, 62G10, 62G20.

Key words: Partially observed linear system, parameter estimation, small noise asymptotic, asymptotic properties.

1 Introduction

We consider partially observed stochastic linear system

\[\begin{align*}
dX_t &= f(\vartheta, t) Y_t dt + \varepsilon \sigma(t) dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
dY_t &= a(\vartheta, t) Y_t dt + \psi \varepsilon b(\vartheta, t) dV_t, \quad Y_0 = y_0, \quad 0 \leq t \leq T,
\end{align*}\]

where \(W_t, 0 \leq t \leq T\) and \(V_t, 0 \leq t \leq T\) are two independent Wiener processes, \(f(\cdot, \cdot), a(\cdot, \cdot), b(\cdot, \cdot)\) and \(\sigma(\cdot)\) are known functions, \(\varepsilon \in (0, 1]\) and
\( \psi \in (0, 1] \) are noise intensities. The initial value \( y_0 \) is deterministic. The parameter \( \vartheta \in \Theta = (\alpha, \beta) \) is unknown and has to be estimated using the observations \( X^T = (X_t, 0 \leq t \leq T) \). The Gaussian process \( Y^T = (Y_t, 0 \leq t \leq T) \) is unobservable (hidden). Construction of the maximum likelihood estimator (MLE) \( \hat{\vartheta}_e \) and the Bayesian estimator (BE) \( \tilde{\vartheta}_e \) is based on the likelihood function \( \text{(3)} \)

\[
L(\vartheta, X^T) = \exp \left\{ \int_0^T \frac{M(\vartheta, t)}{\varepsilon^2 \sigma(t)^2} \mathrm{d}X_t - \int_0^T \frac{M(\vartheta, t)^2}{2 \varepsilon^2 \sigma(t)^2} \mathrm{d}t \right\}, \quad \vartheta \in \Theta.
\]

Here \( M(\vartheta, t) = f(\vartheta, t) m(\vartheta, t) \) and the conditional expectation \( m(\vartheta, t) = E_\vartheta (Y_t|X_s, 0 \leq s \leq t) \) is the solution of the Kalman-Bucy filtering equations \( \text{(15)} \).

The MLE is solution of the following equation

\[
L(\hat{\vartheta}_e, X^T) = \sup_{\vartheta \in \Theta} L(\vartheta, X^T).
\]

If this equation has more than one solution, then any one of them can be taken as MLE.

To introduce BE we assume that the unknown parameter \( \vartheta \) is a random variable with known density \( p(\vartheta), \vartheta \in \Theta \). Then for quadratic loss function the BE is the conditional expectation

\[
\tilde{\vartheta}_e = \int_{\Theta} \vartheta p(\vartheta|X^T) \, \mathrm{d}\vartheta, \quad p(\vartheta|X^T) = \frac{p(\vartheta) L(\vartheta, X^T)}{\int_{\Theta} p(\vartheta) L(\vartheta, X^T) \, \mathrm{d}\vartheta}.
\]

Systems as \( \text{(1)} \) and \( \text{(2)} \), either in continuous or discrete time, are the typical models to which the Kalman-Bucy method is applicable. Nowadays it is widely used in various branches of sciences and technology: industrial production \( \text{(2)} \), GPS localization \( \text{(11), (26), (9)} \), chemistry and biochemistry \( \text{(5), (6), (31)} \), physics \( \text{(17)} \), finance \( \text{(32)} \). This is only a short list, which can be easily extended. Note that similar models were studied in non linear filtration with small noise in observations (see \( \text{(10), (30)} \) and references there in).

The engineering literature on adaptive Kalman-Bucy filtering is vast, but mathematical study of statistical problems for such systems is not yet sufficiently developed. Statistical problems for discrete time models were studied more extensively, see e.g. monographs \( \text{(3), (7) and (8)} \). The continuous time hidden Markov processes with discrete state space and white Gaussian noise observations were studied in \( \text{(3), (16)} \).
For continuous time linear systems such as (1)-(2) with constant functions $f(\varphi, t) = f(\varphi)$, $a(\varphi, t) = a(\varphi)$, $b(\varphi, t) = b(\varphi)$, $\sigma(t) = \sigma$, ($\varepsilon = 1$, $\psi_\varepsilon = 1$), asymptotic analysis with respect to $T \to \infty$ appeared in [18], [13], [24]. The survey [25] reviews some results on parameter estimation in the model (1)-(2) in both large time and small noise asymptotics.

The partially observed system (1)-(2) with $\psi_\varepsilon = \varepsilon \to 0$ as well as some of its generalizations were studied in [19], Chapter 6. Let us briefly recall some of the results in [19] and compare them with those obtained recently. The processes $X_T^\varepsilon, Y_T^\varepsilon$ as $\varepsilon \to 0$ converge to the deterministic solutions of the ordinary differential equations

$$
\frac{\partial x_t(\varphi)}{\partial t} = f(\varphi, t) y_t(\varphi), \quad \frac{\partial y_t(\varphi)}{\partial t} = -a(\varphi, t) y_t(\varphi),
$$

with initial values $x_0(\varphi) = 0$ and $y_0(\varphi) = y_0$ respectively. It is shown that under appropriate regularity conditions the MLE $\hat{\varphi}_\varepsilon$ and BE $\tilde{\varphi}_\varepsilon$ are consistent, asymptotically normal

$$
\frac{\hat{\varphi}_\varepsilon - \varphi_0}{\varepsilon} \Rightarrow \tilde{\zeta} \sim N(0, \hat{I}(\varphi_0)^{-1}), \quad \frac{\tilde{\varphi}_\varepsilon - \varphi_0}{\varepsilon} \Rightarrow \tilde{\zeta},
$$

the moments converge and both estimators are asymptotically efficient. Here $\varphi_0$ is the true parameter value and $\hat{I}(\varphi_0)$ is the Fisher information,

$$
\int_0^T \frac{\dot{M}(\varphi_0, t)^2}{\sigma(t)^2} dt \to \int_0^T \frac{\left[ \ddot{f}(\varphi_0, t) y_t(\varphi_0) + f(\varphi_0, t) \dot{y}(\varphi_0, t) \right]^2}{\sigma(t)^2} dt = \hat{I}(\varphi_0).
$$

Here and below the derivative in $\varphi$ are denoted by dots and

$$
\dot{M}(\varphi_0, t) = \dot{f}(\varphi_0, t) m(\varphi_0, t) + f(\varphi_0, t) \dot{m}(\varphi_0, t).
$$

The function $\dot{y}(\varphi_0, t)$ is distinct from $\dot{y}_t(\varphi)$ and is the solution of a certain auxiliary linear equation. It is important to note that if $y_0 = 0$, then $\hat{I}(\varphi) = 0$. Therefore $y_0 \neq 0$ is a necessary condition for (7).

Another instance of the model (1)-(2) with $\psi_\varepsilon = 1$ was studied in [23]. In this case the limit system is

$$
\ddot{x}_t = f(\varphi, t) Y_t(\varphi), \quad \ddot{x}_0 = 0,
$$

$$
dY_t = a(\varphi, t) Y_t dt + b(\varphi, t) dV_t, \quad Y_0 = 0, \quad 0 \leq t \leq T,
$$

where we denoted $\ddot{x}_t = \frac{\partial X_t(\varphi)}{\partial \varphi}$. Roughly speaking the question of consistency of estimators is reduced to the following one: is it possible to determine the
parameter \( \vartheta \) exactly, using the observations \( \tilde{x}^T = (\tilde{x}_t, 0 \leq t \leq T) \)? In [23] it was shown that if the identifiability condition

\[
\inf_{|\vartheta - \vartheta_0| > \nu} \int_0^T \frac{[S(\vartheta, t) - S(\vartheta_0, t)]^2}{S(\vartheta, t) \sigma(t)} dt > 0, \quad \forall \nu > 0, \quad (10)
\]

where \( S(\vartheta, t) = f(\vartheta, t) b(\vartheta, t) \), is satisfied, then the answer to the above question is in positive.

Moreover, under mild regularity conditions the MLE and BE are asymptotically normal

\[
\frac{\hat{\vartheta}_x - \vartheta_0}{\sqrt{\varepsilon}} \Rightarrow \zeta \sim N(0, I(\vartheta_0)^{-1}), \quad \frac{\hat{\vartheta}_x - \vartheta_0}{\sqrt{\varepsilon}} \Rightarrow \zeta, \quad (11)
\]

the moments converge and these estimators are asymptotically efficient. Here \( \hat{I}(\vartheta) \) the Fisher information is given by a different expression.

This model has an interesting feature. If \( f(\vartheta, t) = f(t), b(\vartheta, t) = b(t) \) or \( f(\vartheta, t) = \vartheta f(t), b(\vartheta, t) = \vartheta^{-1} b(t) \) then the identifiability condition \( (10) \) fails and convergence \( (11) \) does not hold. Note also that condition \( (10) \) does not depend on \( a(\vartheta, t) \) and \( y_0 = 0 \). Remark that the condition \( y_0 = 0 \) was omitted in [23], but the limit of estimators in the case \( y_0 \neq 0 \) is different.

It was shown that

\[
\int_0^T \frac{\dot{M} (\vartheta_0, t)^2}{\sigma(t)^2} dt \xrightarrow{\varepsilon \to 0} 0. \quad (12)
\]

and

\[
\frac{\dot{f}(\vartheta_0, t) m(\vartheta_0, t) + f(\vartheta_0, t) \dot{m}(\vartheta_0, t)}{\sqrt{\varepsilon \sigma(t)}} \Rightarrow h(t) \xi_t, \quad t \in (0, T],
\]

where \( h(\cdot) \) is a bounded function and \( \xi_t, t \in (0, T] \) is a family of independent Gaussian \( N(0, 1) \) random variables. However, the convergence

\[
\int_0^T \frac{\dot{M} (\vartheta_0, t)^2}{\varepsilon \sigma(t)^2} dt \Rightarrow \int_0^T h(t)^2 \xi_t^2 dt,
\]

does not hold for a number of reasons. In particular, the latter integral does not exist in any reasonable sense because \( \xi_t, t \in (0, T] \) is not a separable process. Instead, it is shown in [23] that

\[
\int_0^T \frac{\dot{M} (\vartheta_0, t)^2}{\varepsilon \sigma(t)^2} dt \to \int_0^T h(t)^2 dt = \int_0^T \frac{\dot{S}(\vartheta_0, t)^2}{2S(\vartheta_0, t) \sigma(t)} dt = I(\vartheta). \quad (13)
\]
The present work is concerned the setting, intermediate between these two. We consider the model (1)-(2) with \( \varepsilon \to 0 \), \( \psi_\varepsilon \to 0 \) and
\[
\frac{\varepsilon}{\psi_\varepsilon^3} \to 0.
\] (14)

For \( \psi_\varepsilon = \varepsilon^\delta \), (14) corresponds to \( \delta \in \left( 0, \frac{1}{3} \right) \). The limit system in this case coincides with (6), we have the convergence (12) and the convergence as in (13) but with different normalization
\[
\int_0^T \dot{M}(\vartheta_0, t)^2 \varepsilon \psi_\varepsilon \sigma(t)^2 dt \longrightarrow I(\vartheta_0)
\]
where Fisher information is given by (13). It will be shown that under suitable regularity conditions, the MLE and BE are consistent, asymptotically normal
\[
\sqrt{\psi_\varepsilon} \left( \hat{\vartheta}_\varepsilon - \vartheta_0 \right) \Longrightarrow \zeta \sim N(0, I(\vartheta_0)^{-1}), \quad \sqrt{\psi_\varepsilon} \left( \tilde{\vartheta}_\varepsilon - \vartheta_0 \right) \Longrightarrow \zeta,
\] (15)
the moments converge and the both estimators are asymptotically efficient.

The error asymptotics implied by (15) is somewhat surprising,
\[
E_{\vartheta_0} \left( \hat{\vartheta}_\varepsilon - \vartheta_0 \right)^2 = \frac{\varepsilon (1 + o(1))}{\psi_\varepsilon I(\vartheta_0)} , \quad E_{\vartheta_0} \left( \tilde{\vartheta}_\varepsilon - \vartheta_0 \right)^2 = \frac{\varepsilon (1 + o(1))}{\psi_\varepsilon I(\vartheta_0)}.
\]
This means that larger noise in the state equation causes smaller estimation errors (Theorem 1 below). The best case corresponds to the situation, where the noise in the state equation does not tend to zero (\( \psi_\varepsilon = 1 \)). We say surprising because for the values \( \psi_\varepsilon = \varepsilon^\delta, \delta \in \left( \frac{1}{3}, 1 \right) \) the situation changes essentially, \( \psi_\varepsilon \) became a true noise and the normalization of estimators is different (see section 3).

Convergence of MLE and BE for the model (1)-(2) with \( \psi_\varepsilon = \varepsilon \) has a different rate in the case of change-point in the observation equation, i.e., if
\[
f(\vartheta, t) = q(t) \mathbb{I}_{\{t<\vartheta\}} + r(t) \mathbb{I}_{\{t\geq\vartheta\}}.
\]
Then it can be shown that
\[
\frac{\hat{\vartheta}_\varepsilon - \vartheta_0}{\varepsilon^2} \Longrightarrow \zeta^*, \quad \frac{\tilde{\vartheta}_\varepsilon - \vartheta_0}{\varepsilon^2} \Longrightarrow \zeta^*,
\]
where \( \zeta^* \) and \( \zeta^* \) are two different random variables. Only the BE is asymptotically efficient in this case. When the change-point is introduced into the state equation \( a(\vartheta, t) = q(t) \mathbb{I}_{\{t<\vartheta\}} + r(t) \mathbb{I}_{\{t\geq\vartheta\}} \), the estimators are asymptotically normal with regular rate \( \varepsilon \) as in (7), see [19].
2 Main result

Consider the observation model

\[ \begin{align*}
    dX_t &= f(\vartheta, t) Y_t \, dt + \varepsilon \sigma(t) \, dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
    dY_t &= a(\vartheta, t) Y_t \, dt + \psi \varepsilon b(\vartheta, t) \, dV_t, \quad Y_0 = 0, \quad 0 \leq t \leq T.
\end{align*} \]

Our goal is to estimate \( \vartheta \) using the observations \( X^T \). To this end, we will study the asymptotic behavior of the MLE and BE defined in (4) and (5) respectively. In the case of BE we assume that the density \( p(\cdot) \) is a continuous positive function on \( \Theta = (\alpha, \beta) \).

These estimators are based on the family of stochastic processes \( (m(\vartheta, t), 0 \leq t \leq T), \vartheta \in \Theta, \) where the conditional expectation \( m(\vartheta, t), 0 \leq t \leq T \) satisfies the Kalman-Bucy filtering equations [15] (see details in [29, Theorem 10.1])

\[ \begin{align*}
    dm(\vartheta, t) &= -a(\vartheta, t) m(\vartheta, t) \, dt \\
                   &+ \frac{\gamma(\vartheta, t) f(\vartheta, t)}{\varepsilon^2 \sigma(t)^2} [dX_t - f(\vartheta, t) \, m(\vartheta, t) \, dt], \quad (16)
\end{align*} \]

subject to \( m(\vartheta, 0) = 0 \), where the function \( \gamma(\vartheta, t) = E_\vartheta (m(\vartheta, t) - Y_t)^2 \) solves the Riccati equation

\[ \begin{align*}
    \frac{\partial \gamma(\vartheta, t)}{\partial t} &= -2a(\vartheta, t) \gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} + \psi^2 b(\vartheta, t)^2, \quad (17)
\end{align*} \]

subject to initial condition \( \gamma(\vartheta, 0) = 0 \).

The true value will be denoted by \( \vartheta_0 \). The equations (16) and (17) generate the conditional expectation if \( \vartheta = \vartheta_0 \). For other values of \( \vartheta \) these equations define stochastic processes, which do not coincide in general with the conditional expectation of \( Y_t \). It will be convenient to introduce the function \( \gamma_*(\vartheta, t) = \gamma(\vartheta, t) / (\varepsilon \psi) \). Equations (16) and (17) can be written as

\[ \begin{align*}
    dm(\vartheta, t) &= -q_\varepsilon(\vartheta, t) m(\vartheta, t) \, dt \\
                   &+ \frac{\psi \varepsilon \gamma_*(\vartheta, t) f(\vartheta, t)}{\sigma(t)^2} \, dX_t, \quad (18)
\end{align*} \]

\[ \begin{align*}
    \frac{\varepsilon}{\psi \varepsilon} \frac{\partial \gamma_*(\vartheta, t)}{\partial t} &= -2 \frac{\varepsilon}{\psi \varepsilon} a(\vartheta, t) \gamma_*(\vartheta, t) - \frac{\gamma_*(\vartheta, t)^2 f(\vartheta, t)^2}{\sigma(t)^2} + b(\vartheta, t)^2, \quad (19)
\end{align*} \]

subject to \( m(\vartheta, 0) = 0, \gamma_*(\vartheta, 0) = 0 \) and where

\[ q_\varepsilon(\vartheta, t) = a(\vartheta, t) + \frac{\psi \varepsilon \gamma_*(\vartheta, t) f(\vartheta, t)^2}{\sigma(t)^2}. \]
In the case $\vartheta = \vartheta_0$ we have

$$
\text{d}m(\vartheta_0, t) = -a(\vartheta_0, t)m(\vartheta_0, t)\text{d}t + \psi_\varepsilon \frac{\gamma_\varepsilon(\vartheta_0, t)f(\vartheta_0, t)}{\sigma(t)} \text{d}W_t, \quad m(\vartheta_0, 0) = 0.
$$

Here $\bar{W}_t, 0 \leq t \leq T$ is the innovation Wiener process (see Theorem 7.12 in [29]).

Finding the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ by means of (4),(5) requires solving equations (18),(19) for all $\vartheta \in \Theta$ and is therefore computationally inefficient. In Section 3 below we discuss the possibility of estimating $\vartheta$ using a much more simple algorithm.

The properties of estimators will be derived under the following regularity conditions.

**Conditions $\mathcal{A}$.**

$\mathcal{A}_1$. The functions $f(\vartheta, t), a(\vartheta, t), b(\vartheta, t), t \in [0, T], \vartheta \in \Theta$ and $\sigma(t), t \in [0, T]$ have continuous derivatives in $t$.

$\mathcal{A}_2$. The functions $f(\vartheta, t), b(\vartheta, t), t \in [0, T], \vartheta \in \Theta$ and $\sigma(t), 0 \leq t \leq T$ are separated from zero.

$\mathcal{A}_3$. The functions $f(\vartheta, t), a(\vartheta, t), b(\vartheta, t), t \in [0, T], \vartheta \in \Theta$ are two times continuously differentiable in $\vartheta$ and the derivatives $\dot{f}(\vartheta, t), \dot{b}(\vartheta, t)$ have continuous derivatives in $t$.

$\mathcal{A}_4$. The function $\psi_\varepsilon = \varepsilon^\delta \to 0$, where $0 < \delta < \frac{1}{3}$.

For the sake of simplicity and without loss of generality, we assume that the functions $f(\cdot), b(\cdot)$ and $\sigma(\cdot)$ are positive.

The Fisher information in this problem is

$$
I(\vartheta) = \int_0^T \frac{\hat{S}(\vartheta, t)^2}{2S(\vartheta, t)\sigma(t)} \text{d}t.
$$

Define the function

$$
G(\vartheta, \vartheta_0) = \int_0^T \frac{[S(\vartheta, t) - S(\vartheta_0, t)]^2}{4S(\vartheta, t)\sigma(t)} \text{d}t.
$$

**Conditions $\mathcal{B}$.**

$\mathcal{B}_1$. The Fisher information is positive

$$
\inf_{\vartheta \in \Theta} I(\vartheta) > 0.
$$
\textbf{B2.} For any $\vartheta_0 \in \Theta$ and $\nu > 0$

$$\inf_{|\vartheta - \vartheta_0| > \nu} G(\vartheta, \vartheta_0) > 0.$$ 

The family of measures \( \left\{ P^{(e)}_{\vartheta}, \vartheta \in \Theta \right\} \) induced by the observation process \( X_T = (X_t, 0 \leq t \leq T) \) on the space \( (C[0, T], \mathcal{B}) \) of continuous functions on \( [0, T] \) is locally asymptotically normal (LAN) (see Lemma 4 below). Therefore the mean squared estimation error satisfies the Hajek-Le Cam’s minimax lower bound: for any $\vartheta_0 \in \Theta$ and any estimator $\hat{\vartheta}_e$

$$\lim_{\nu \to 0} \lim_{\epsilon \to 0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \frac{\psi_{\vartheta}}{\epsilon} E_\vartheta |\hat{\vartheta}_e - \vartheta|^2 \geq I(\vartheta_0)^{-1}. \quad (20)$$

The estimator $\hat{\vartheta}_e$ is called asymptotically efficient if for any $\vartheta_0 \in \Theta$

$$\lim_{\nu \to 0} \lim_{\epsilon \to 0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \frac{\psi_{\vartheta}}{\epsilon} E_\vartheta |\hat{\vartheta}_e^* - \vartheta|^2 = I(\vartheta_0)^{-1}.$$

For the proof of a more general result see, e.g., [12].

The main result of this paper is the following theorem.

\textbf{Theorem 1.} Assume that conditions \( A \) and \( B \) are satisfied. Then the MLE $\hat{\vartheta}_e$ and BE $\tilde{\vartheta}_e$ are consistent, asymptotically normal

$$\sqrt{\frac{\psi_{\vartheta}}{\epsilon}} (\hat{\vartheta}_e - \vartheta_0) \Rightarrow \zeta \sim N(0, I(\vartheta_0)^{-1}) \quad \text{and} \quad \sqrt{\frac{\psi_{\vartheta}}{\epsilon}} (\tilde{\vartheta}_e - \vartheta_0) \Rightarrow \zeta,$$

the convergence of moments holds,

$$\left( \frac{\psi_{\vartheta}}{\epsilon} \right)^{\frac{p}{2}} E_{\vartheta_0} |\hat{\vartheta}_e - \vartheta_0|^p \Rightarrow E_{\vartheta_0} |\zeta|^p, \quad \left( \frac{\psi_{\vartheta}}{\epsilon} \right)^{\frac{p}{2}} E_{\vartheta_0} |\tilde{\vartheta}_e - \vartheta_0|^p \Rightarrow E_{\vartheta_0} |\zeta|^p,$$

for any $p > 0$, and both estimators are asymptotically efficient.

\textbf{Proof.} The proof of this theorem is based on the general results of Ibragimov and Khasminskii [12].

Let us denote $M(\vartheta, t) = f(\vartheta, t) m(\vartheta, t)$, where $m(\vartheta, t)$ is the solution of equation (18). The method is based on asymptotics of the likelihood ratios

$$L(\vartheta, X_T) = \exp \left\{ \int_0^T \frac{M(\vartheta, t)}{\epsilon^2 \sigma(t)^2} dX_t - \int_0^T \frac{M(\vartheta, t)^2}{2\epsilon^2 \sigma(t)^2} dt \right\}, \quad \vartheta \in \Theta = (\alpha, \beta).
Define the normalized likelihood ratio process

$$Z_\varepsilon(u) = \frac{L(\vartheta_0 + \varphi_\varepsilon u, X^T)}{L(\vartheta_0, X^T)}, \quad u \in U_\varepsilon = \left(\frac{\alpha - \vartheta_0}{\varphi_\varepsilon}, \frac{\beta - \vartheta_0}{\varphi_\varepsilon}\right),$$

where \(\varphi_\varepsilon = \sqrt{\varepsilon/\psi_\varepsilon}\). Let us first sketch the main idea of the proof of asymptotic normality of the MLE \(\hat{\vartheta}_\varepsilon\) (Theorem 3.1.1 in [12]). Suppose that we already proved the weak convergence of the random process \(Z_\varepsilon(\cdot)\) to the limit process \(Z(\cdot)\) as \(\varepsilon \to 0\), where

$$Z(u) = \exp\left\{u \Delta (\vartheta_0) - \frac{u^2}{2} I(\vartheta_0)\right\}, \quad u \in \mathcal{R},$$

and \(\Delta(\vartheta_0) \sim \mathcal{N}(0, I(\vartheta_0))\). Then for any \(x \in \mathcal{R}\)

$$\mathbf{P}_{\vartheta_0}\left(\sqrt{\frac{\varepsilon}{\varphi_\varepsilon}} (\hat{\vartheta}_\varepsilon - \vartheta_0) < x\right) = \mathbf{P}_{\vartheta_0}\left(\hat{\vartheta}_\varepsilon < \vartheta_0 + \varphi_\varepsilon x\right)$$

$$= \mathbf{P}_{\vartheta_0}\left(\sup_{\vartheta < \vartheta_0 + \varphi_\varepsilon x} L(\vartheta, X^T) > \sup_{\vartheta_0 + \varphi_\varepsilon x} L(\vartheta_0, X^T)\right)$$

$$= \mathbf{P}_{\vartheta_0}\left(\sup_{\vartheta < \vartheta_0 + \varphi_\varepsilon x} \frac{L(\vartheta, X^T)}{L(\vartheta_0, X^T)} > \sup_{\vartheta_0 + \varphi_\varepsilon x} \frac{L(\vartheta_0, X^T)}{L(\vartheta_0, X^T)}\right)$$

$$= \mathbf{P}_{\vartheta_0}\left(\sup_{u < x} Z_\varepsilon(u) > \sup_{u \geq x} Z_\varepsilon(u)\right)$$

$$\longrightarrow \mathbf{P}_{\vartheta_0}\left(\sup_{u < x} Z(u) > \sup_{u \geq x} Z(u)\right) = \mathbf{P}_{\vartheta_0}\left(\frac{\Delta(\vartheta_0)}{I(\vartheta_0)} < x\right).$$

Here we changed the variable so that \(\vartheta = \vartheta_0 + \varphi_\varepsilon u\). Note that the random function \(Z(\cdot)\) has the unique maximum at the point \(\zeta = \Delta(\vartheta_0) I(\vartheta_0)^{-1} \sim \mathcal{N}(0, I(\vartheta_0)^{-1})\).

Similar calculations show that the BE \(\hat{\vartheta}_\varepsilon\) is asymptotically normal with the same limit variance (see details in [12]).

For the model under consideration

$$\ln Z_\varepsilon(u) = \int_0^T \frac{M(\vartheta_u, t) - M(\vartheta_0, t)}{\varepsilon^2 \sigma(t)^2} dX_t - \int_0^T \frac{[M(\vartheta_u, t) - M(\vartheta_0, t)]^2}{2\varepsilon^2 \sigma(t)^2} dt,$$

where \(\vartheta_u = \vartheta_0 + \varphi_\varepsilon u\). The formal Taylor formula yields

$$M(\vartheta_0 + \varphi_\varepsilon u, t) - M(\vartheta_0, t) = \varphi_\varepsilon u \tilde{M}(\vartheta_0, t) + \mathcal{O}(\varphi_\varepsilon).$$
Recall that $\dot{M}(\vartheta, t)$ is the derivative in $\vartheta$ of the random process $M(\vartheta, t)$,

$$
\dot{M}(\vartheta, t) = \dot{f}(\vartheta, t) m(\vartheta, t) + f(\vartheta, t) \dot{m}(\vartheta, t).
$$

Thus we need to find the asymptotics of the derivative $\dot{m}(\vartheta, t)$ of $m(\vartheta, t)$ and the related derivative $\dot{\gamma}_*(\vartheta, t)$ of $\gamma_*(\vartheta, t)$, since the equation (18) for $m(\vartheta, t)$ depends on $\gamma_*(\vartheta, t)$.

Let us denote

$$
\gamma_0(\vartheta, t) = b(\vartheta, t) \sigma(t), \quad A(\vartheta, t) = \gamma_*(\vartheta, t) f(\vartheta, t) \sigma(t)^2,
$$

$$
q_\varepsilon(\vartheta, t) = a(\vartheta, t) + \frac{\psi_\varepsilon}{\varepsilon} A(\vartheta, t) f(\vartheta, t).
$$

Then equations (18), (19) take the form

$$
dm(\vartheta, t) = -q_\varepsilon(\vartheta, t) m(\vartheta, t) \, dt + \frac{\psi_\varepsilon}{\varepsilon} A(\vartheta, t) \, dX_t,
$$

$$
\frac{\partial \gamma_*(\vartheta, t)}{\partial t} = -2a(\vartheta, t) \gamma_*(\vartheta, t) - \frac{\psi_\varepsilon}{\varepsilon} A(\vartheta, t)^2 \sigma(t)^2 + \frac{\psi_\varepsilon}{\varepsilon} b(\vartheta, t)^2,
$$

with the initial values $m(\vartheta, 0) = 0, \gamma_*(\vartheta, 0) = 0$.

**Lemma 1.** Let the conditions $A_1, A_2$ be satisfied and $\varepsilon \psi_\varepsilon \to 0$. Then, for any $t_0 \in (0, T]$,

$$
\sup_{t_0 \leq t \leq T} |\gamma_*(\vartheta, t) - \gamma_0(\vartheta, t)| = O\left(\frac{\varepsilon}{\psi_\varepsilon}\right).
$$

**Proof.** Introduce the additional Riccati equation

$$
\frac{\partial \hat{\gamma}(t)}{\partial t} = -2a_m \hat{\gamma}(t) - \frac{\psi_\varepsilon}{\varepsilon} \hat{\gamma}(t)^2 \frac{f_m^2}{S^2} + \frac{\psi_\varepsilon}{\varepsilon} B^2, \quad \hat{\gamma}(0) = 0,
$$

where we denoted

$$
a_m = \inf_{0 \leq t \leq T} \inf_{\vartheta \in \Theta} |a(\vartheta, t)|, \quad f_m = \inf_{0 \leq t \leq T} \inf_{\vartheta \in \Theta} f(\vartheta, t), \quad \sigma_M = \sup_{0 \leq t \leq T} \sigma(t).
$$

Note that by condition $A_2$ we have $f_m > 0$. Due to the comparison theorem for ordinary differential equations, $\gamma_*(\vartheta, t) \leq \hat{\gamma}(t)$ holds for all $t \in [0, T]$. This bound is intuitive, i.e., as replacing the “noise coefficient” $\sigma(t)$ by its maximal value should increase the estimation error. Further, as we take smaller coefficient $f_m$ in the drift $f(\vartheta, t)Y_t$ the error increases as well. The
solution of Riccati equation (25) with constant coefficients can be written explicitly (see [1])

\[ \hat{\gamma} (t) = e^{-2R_{\varepsilon}t} \left[ \frac{\psi_{m} f_{m}^{2} (1 - e^{-2R_{\varepsilon}t})}{2\varepsilon R_{\varepsilon} \sigma_{M}^{2}} - \hat{\gamma}^{-1} \right]^{-1} + \hat{\gamma} \]

Here

\[ R_{\varepsilon} = \left( a_{m}^{2} + \frac{\psi_{\varepsilon}^{2} B^{2} f_{m}^{2}}{\varepsilon^{2} \sigma_{M}^{2}} \right)^{1/2} = \frac{\psi_{m} B f_{m}}{\varepsilon \sigma_{M}} \left( 1 + O \left( \frac{\varepsilon^{2}}{\psi_{\varepsilon}^{2}} \right) \right) \]

and

\[ \hat{\gamma} = \frac{\varepsilon}{\psi_{\varepsilon}} \frac{a_{m} \sigma_{M}^{2}}{f_{m}^{2}} \left[ \left( 1 + \frac{\psi_{\varepsilon}^{2} B^{2} f_{m}^{2}}{\varepsilon^{2} a_{m}^{2} \sigma_{M}^{2}} \right)^{1/2} - 1 \right] = \frac{B \sigma_{M}}{f_{m}} \left( 1 + O \left( \frac{\varepsilon}{\psi_{\varepsilon}} \right) \right). \]

Therefore for any \( t_{0} \in (0, T] \)

\[ \sup_{t_{0} \leq t \leq T} \left| \hat{\gamma} (t) - \frac{B \sigma_{M}}{f_{m}} \right| = O \left( \frac{\varepsilon}{\psi_{\varepsilon}} \right). \] (26)

Let us write the Riccati equation in integral form

\[ \frac{\psi_{\varepsilon}}{\varepsilon} \left[ \gamma_{*} (\vartheta, t) - \gamma_{*} (\vartheta, t_{0}) \right] + 2 \int_{t_{0}}^{t} a (\vartheta, s) \gamma_{*} (\vartheta, s) \, ds \]

\[ = - \int_{t_{0}}^{t} \frac{\gamma_{*} (\vartheta, s)^{2} f (\vartheta, s)^{2}}{\sigma (s)^{2}} \, ds + \int_{t_{0}}^{t} b (\vartheta, s)^{2} \, ds. \]

As \( \gamma_{*} (\vartheta, t) \leq \hat{\gamma} (t) \) and \( \hat{\gamma} (t) \) according to (26) is bounded, the left hand side of the integral equation tends to 0. Hence

\[ \int_{t_{0}}^{t} \frac{\gamma_{*} (\vartheta, s)^{2} f (\vartheta, s)^{2}}{\sigma (s)^{2}} \, ds = \int_{t_{0}}^{t} b (\vartheta, s)^{2} \, ds + O \left( \frac{\varepsilon}{\psi_{\varepsilon}} \right) \]

and

\[ \gamma_{*} (\vartheta, t)^{2} = \frac{b (\vartheta, t)^{2} \sigma (t)^{2}}{f (\vartheta, t)^{2}} + O \left( \frac{\varepsilon}{\psi_{\varepsilon}} \right) = \gamma_{0} (\vartheta, t)^{2} + O \left( \frac{\varepsilon}{\psi_{\varepsilon}} \right). \]
It can be checked that all terms of order $O \left( \frac{\varepsilon}{\psi} \right)$ above satisfy
\[
\left| O \left( \frac{\varepsilon}{\psi} \right) \right| \leq C \frac{\varepsilon}{\psi},
\]
where the constant $C > 0$ can be chosen independent on $\vartheta$.

Let us denote
\[
\dot{\gamma}_0 (\vartheta, t) = \frac{b(\vartheta, t) \sigma (t)}{f(\vartheta, t)} \frac{\partial}{\partial \vartheta} \left[ \ln \frac{b(\vartheta, t)}{f(\vartheta, t)} \right].
\]

**Lemma 2.** Let the conditions $A_1 - A_3$ be satisfied and $\frac{\varepsilon}{\psi} \to 0$, then, for any $t_0 \in (0, T]$,
\[
\sup_{t_0 \leq t \leq T} |\dot{\gamma}_* (\vartheta, t) - \dot{\gamma}_0 (\vartheta, t)| = O \left( \frac{\varepsilon}{\psi} \right) \quad (27)
\]

**Proof.** The derivative $\dot{\gamma}_* (\vartheta, t)$ solves the equation
\[
\frac{\partial \dot{\gamma}_* (\vartheta, t)}{\partial t} = -2q_\varepsilon (\vartheta, t) \dot{\gamma}_* (\vartheta, t) + 2\frac{\psi_\varepsilon}{\varepsilon} b(\vartheta, t) \dot{b}(\vartheta, t)
\]
\[
- 2 \left[ \dot{a} (\vartheta, t) + \frac{\psi_\varepsilon}{\varepsilon} A (\vartheta, t) \dot{f} (\vartheta, t) \right] \gamma_* (\vartheta, t)
\]
subject to $\dot{\gamma}_* (\vartheta, 0) = 0$, which can be justified by the usual argument, using the Gronwall-Bellman lemma. This is linear equation and its solution can be written as follows
\[
\dot{\gamma}_* (\vartheta, t) = -2 \int_0^t e^{-2 \int_s^t q_\varepsilon (\vartheta, \nu) d\nu} \dot{a} (\vartheta, s) \gamma_* (\vartheta, s) ds
\]
\[
+ \frac{2\psi_\varepsilon}{\varepsilon} \int_0^t e^{-2 \int_s^t q_\varepsilon (\vartheta, \nu) d\nu} \left[ b(\vartheta, s) \dot{b}(\vartheta, s) - \frac{\gamma_* (\vartheta, s)^2 f(\vartheta, s) \dot{f}(\vartheta, s)}{\sigma (s)^2} \right] ds.
\]

By Lemma 1 in [23] and (24) we have the representation
\[
\dot{\gamma}_* (\vartheta, t) = \frac{b(\vartheta, t) \dot{b}(\vartheta, t) \sigma (t)}{f(\vartheta, t)} \frac{\gamma_* (\vartheta, t) f(\vartheta, t)^2}{f(\vartheta, t)} + O \left( \frac{\varepsilon}{\psi} \right)
\]
\[
= \frac{b(\vartheta, t) \sigma (t)}{f(\vartheta, t)} \left[ \frac{\dot{b}(\vartheta, t)}{b(\vartheta, t)} - \frac{\dot{f}(\vartheta, t)}{f(\vartheta, t)} \right] + O \left( \frac{\varepsilon}{\psi} \right)
\]
\[
= \frac{b(\vartheta, t) \sigma (t)}{f(\vartheta, t)} \frac{\partial}{\partial \vartheta} \left[ \ln \frac{b(\vartheta, t)}{f(\vartheta, t)} \right] + O \left( \frac{\varepsilon}{\psi} \right).
\]
\[
\blacksquare
\]
As before, the terms \( O \left( \frac{\varepsilon}{\psi} \right) \) can be shown to satisfy

\[
\left| O \left( \frac{\varepsilon}{\psi} \right) \right| \leq C \frac{\varepsilon}{\psi},
\]

with a constant \( C > 0 \) independent of \( \vartheta \).

Below \( O_p \left( \frac{\varepsilon}{\psi} \right) \) and \( o_p \left( \frac{\varepsilon}{\psi} \right) \) means that for any \( p > 1 \)

\[
E_{\vartheta_0} \left| O_p \left( \frac{\varepsilon}{\psi} \right) \right|^p \leq C \left( \frac{\varepsilon}{\psi} \right)^p, \quad \mathbb{E}_{\vartheta_0} \left| o_p \left( \frac{\varepsilon}{\psi} \right) \right|^p \leq c_\varepsilon \left( \frac{\varepsilon}{\psi} \right)^p, \quad c_\varepsilon \to 0,
\]

where the constants \( C > 0 \) and \( c_\varepsilon > 0 \) do not depend on \( \vartheta \). Recall that \( S(\vartheta, t) = f(\vartheta, t) b(\vartheta, t) \).

**Lemma 3.** Let conditions \( \mathcal{A} \) be satisfied, then

\[
\dot{M}(\vartheta_0, t) = \sqrt{\varepsilon \psi} \sqrt{\frac{\sigma(t)}{2S(\vartheta_0, t)}} \dot{S}(\vartheta_0, t) \xi_{t,\varepsilon} (1 + o(1)) + O_p \left( \frac{\varepsilon}{\psi} \right),
\]

(29)

with \( \xi_{t,\varepsilon}(\vartheta) \rightarrow \xi_t \sim \mathcal{N}(0, 1) \). Here \( \xi_t, t \in (0, T] \) are mutually independent random variables.

**Proof.** The formal derivative of equation (22) with respect to \( \vartheta_0 \) gives the equation for derivative \( \dot{m}(\vartheta_0, t) \),

\[
d\dot{m}(\vartheta_0, t) = -q_\varepsilon(\vartheta_0, t) \dot{m}(\vartheta_0, t) dt + \frac{\psi_\varepsilon}{\varepsilon} \dot{A}(\vartheta_0, t) dX_t
\]

\[
- \left[ \dot{a}(\vartheta_0, t) + \frac{\psi_\varepsilon}{\varepsilon} F(\vartheta_0, t) \right] m(\vartheta_0, t) dt, \quad \dot{m}(\vartheta_0, 0) = 0,
\]

(30)

where we denoted

\[
F(\vartheta_0, t) = \dot{A}(\vartheta_0, t) f(\vartheta_0, t) + A(\vartheta_0, t) \dot{f}(\vartheta_0, t).
\]

To verify that \( m(\vartheta_0, t) \) has derivative in the mean square, we can write the equation for \( m(\vartheta_0 + h, t) \) and then consider the equation for the difference \( m(\vartheta_0 + h, t) - m(\vartheta_0, t) - h \dot{m}(\vartheta_0, t) \). Using Gronwall-Bellman lemma we obtain the estimate

\[
\mathbb{E}_{\vartheta_0} \left[ m(\vartheta_0 + h, t) - m(\vartheta_0, t) - h \dot{m}(\vartheta_0, t) \right]^2 = o(h^2).
\]
Due to the *Innovation Theorem*, the observation process can be written as

\[ dX_t = f(\vartheta_0, t) m(\vartheta_0, t) \, dt + \varepsilon \sigma(t) \, d\tilde{W}_t, \quad 0 \leq t \leq T. \]

Substitution of this differential in (30) gives the equation

\[
\begin{align*}
\dot{m}(\vartheta_0, t) &= -q_\varepsilon(\vartheta_0, t) \dot{m}(\vartheta_0, t) \, dt - \frac{\psi_\varepsilon}{\varepsilon} A(\vartheta_0, t) \dot{f}(\vartheta_0, t) m(\vartheta_0, t) \, dt \\
&\quad - \dot{a}(\vartheta_0, t) m(\vartheta_0, t) \, dt + \psi_\varepsilon \dot{A}(\vartheta_0, t) \sigma(t) \, d\tilde{W}_t, \quad \dot{m}(\vartheta_0, 0) = 0.
\end{align*}
\]

Hence we have

\[
\dot{m}(\vartheta_0, t) = \frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} A(\vartheta_0, s) \sigma(s) \, d\tilde{W}_s \\
- \frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} A(\vartheta_0, s) \dot{f}(\vartheta_0, s) m(\vartheta_0, s) \, ds \\
- \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} \dot{a}(\vartheta_0, s) m(\vartheta_0, s) \, ds.
\]

(31)

The main contribution in these integrals is due to the values at the vicinity of the point \( t \). Therefore the functions \( A(\vartheta_0, s), f(\vartheta_0, s), \sigma(s) \) in these integrals we can replaced by the values \( A(\vartheta_0, t), f(\vartheta_0, t), \sigma(t) \). Here we use the Taylor expansion such as \( A(\vartheta_0, s) = A(\vartheta_0, t) + (s - t) A'(\vartheta_0, \tilde{s}). \)

By Lemma 1 in [23], these integrals satisfy

\[
\frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} A(\vartheta_0, s) \dot{f}(\vartheta_0, s) m(\vartheta_0, s) \, ds \\
= \frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} A(\vartheta_0, s) \dot{f}(\vartheta_0, s) [m(\vartheta_0, s) - m(\vartheta_0, t)] \, ds \\
+ \frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} A(\vartheta_0, s) \dot{f}(\vartheta_0, s) m(\vartheta_0, t) \\
= I_\varepsilon + \frac{\dot{f}(\vartheta_0, t) m(\vartheta_0, t)}{f(\vartheta_0, t)} + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right).
\]

where \( I_\varepsilon \) denotes the integral with \( m_{t,s} = m(\vartheta_0, t) - m(\vartheta_0, s) \).

Similarly

\[
\begin{align*}
\int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} \dot{a}(\vartheta_0, s)m(\vartheta_0, s) \, ds \\
= \int_0^t e^{-\int_s^t q_\varepsilon(\vartheta_0, v) \, dv} \dot{a}(\vartheta_0, s) [m(\vartheta_0, s) - m(\vartheta_0, t)] \, ds + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right).
\end{align*}
\]
Changing integration variable to \( s = t - \frac{r\varepsilon}{\psi_s} \) gives

\[
m_{t,s} = - \int_s^t a(\vartheta_0, v) m(\vartheta_0, v) \, dv + \psi_\varepsilon \int_s^t A(\vartheta_0, v) \sigma(v) \, d\tilde{W}_v
\]

\[
= \frac{r\varepsilon}{\psi_s} a(\vartheta_0, t) m(\vartheta_0, t) (1 + o_p(1))
\]

\[
+ \psi_\varepsilon A(\vartheta_0, t) \sigma(t) \left[ \tilde{W}_t - \tilde{W}_{t - \frac{r\varepsilon}{\psi_s}} \right] (1 + o(1)).
\]

(32)

Changing the integration variables and letting \( k(t) = A(\vartheta_0, t) f(\vartheta_0, t) \) we also get

\[
\psi_\varepsilon \int_0^t e^{-\int_s^t q(\vartheta_0, v) \, dv} A(\vartheta_0, s) \dot{f}(\vartheta_0, s) \left[ m(\vartheta_0, s) - m(\vartheta_0, t) \right] \, ds
\]

\[
= A(\vartheta_0, t) \dot{f}(\vartheta_0, t) \int_0^{\frac{r\varepsilon}{\psi_s}} e^{-rk(t)} \left[ -\frac{r\varepsilon}{\psi_\varepsilon} a(\vartheta_0, t) m(\vartheta_0, t) \right.
\]

\[
+ \sqrt{\varepsilon \psi_s} A(\vartheta_0, t) \sigma(t) w_\varepsilon(r) \big] \, dr \left[ 1 + o(1) \right)
\]

\[
= A(\vartheta_0, t) \dot{f}(\vartheta_0, t) \sigma(t) \sqrt{\varepsilon \psi_s} \int_0^{\frac{r\varepsilon}{\psi_s}} e^{-rk(t)} w_\varepsilon(r) \, dr \left[ 1 + o(1) \right),
\]

where \( w_\varepsilon(r) = \sqrt{\frac{t}{\psi_s}} \left[ \bar{W}_t - \bar{W}_{t - \frac{r\varepsilon}{\psi_s}} \right] \) is a Wiener process. Further,

\[
\int_0^{\frac{r\varepsilon}{\psi_s}} e^{-rk(t)} w_\varepsilon(r) \, dr = \frac{1}{k(t)} \int_0^{\frac{r\varepsilon}{\psi_s}} \left[ e^{-k(t)z} - e^{-tk(t)\bar{W}_z} \right] \, dw_z(z)
\]

\[
= \frac{1}{k(t)^{3/2}} \int_0^{\frac{r\varepsilon}{\psi_s}} e^{-y} \, d\tilde{w}_\varepsilon(y) \left[ 1 + o_p(1) \right)
\]

\[
= \frac{1}{A(\vartheta_0, t)^{3/2} f(\vartheta_0, t)^{3/2} \sqrt{2} \xi_{t,\varepsilon}} \left[ 1 + o_p(1) \right)
\]

Here \( \tilde{w}_\varepsilon(y) = k(t)^{1/2} w_\varepsilon(y/k(t)) \) and

\[
\xi_{t,\varepsilon} = \sqrt{2} \int_0^{\frac{r\varepsilon}{\psi_s}} e^{-y} \, d\tilde{w}_\varepsilon(y) \implies \xi_t \sim \mathcal{N}(0, 1),
\]

where \( \xi_t, t \in (0, T] \) are mutually independent random variables.
For the stochastic integral similar argument implies

\[
\sqrt{\frac{\psi}{\varepsilon}} \int_0^t e^{-\int_0^s q(\vartheta, \nu) \, d\nu} \dot{A}(\vartheta_0, s) \sigma(s) \, d\tilde{W}_s
\]

\[
= \dot{A}(\vartheta_0, t) \sigma(t) \sqrt{\frac{\psi}{\varepsilon}} \int_0^t e^{-\int_0^s \frac{1}{k(t)} (t-s) \, d\tilde{W}_s} (1 + o_p(1))
\]

\[
= \frac{\dot{A}(\vartheta_0, t) \sigma(t)}{\sqrt{k(t)}} \int_0^t e^{-y} \, d\tilde{w}_\varepsilon(y) (1 + o_p(1))
\]

\[
= \frac{\dot{A}(\vartheta_0, t) \sigma(t)}{\sqrt{2A_0(\vartheta_0, t) f(\vartheta_0, t)}} \xi_{t, \varepsilon} (1 + o_p(1)).
\]

Define the limit functions, \( \varepsilon \to 0 \),

\[
A_0(\vartheta_0, t) = \frac{\gamma_0(\vartheta_0, t) f(\vartheta_0, t)}{\sigma(t)^2} = \frac{b(\vartheta_0, t)}{\sigma(t)},
\]

\[
\dot{A}_0(\vartheta_0, t) = \frac{\gamma_0(\vartheta_0, t) f(\vartheta_0, t) + \gamma_0(\vartheta_0, t) \dot{f}(\vartheta_0, t)}{\sigma(t)^2} = \frac{\dot{b}(\vartheta_0, t)}{\sigma(t)}.
\]

Then

\[
\frac{A_0(\vartheta_0, t)^2 \dot{f}(\vartheta_0, t) \sigma(t)}{A_0(\vartheta_0, t)^{3/2} f(\vartheta_0, t)^{3/2}} + \frac{\dot{A}_0(\vartheta_0, t) \sigma(t)}{A_0(\vartheta_0, t) f(\vartheta_0, t)}
\]

\[
= \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{f(\vartheta_0, t)}} \left( \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} + \frac{\dot{b}(\vartheta_0, t)}{b(\vartheta_0, t)} \right).
\]

All this allows us to write the expansion for the derivative

\[
\dot{m}(\vartheta_0, t) = -\frac{\dot{f}(\vartheta_0, t) m(\vartheta_0, t)}{f(\vartheta_0, t)} + O \left( \frac{\varepsilon}{\psi_\varepsilon} \right)
\]

\[
+ \sqrt{\varepsilon \psi_\varepsilon} \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{2f(\vartheta_0, t)}} \left( \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} + \frac{\dot{b}(\vartheta_0, t)}{b(\vartheta_0, t)} \right) \xi_{t, \varepsilon} (1 + o(1)).
\]

Now \( \text{(29)} \) follows from this representation and \( \text{(21)} \).

\[ \blacksquare \]

Let us now return to the normalized likelihood ratio \( Z_\varepsilon(u) \).
Lemma 4. Let the conditions $\mathcal{A}$ and $\mathcal{B}_1$ be satisfied, then the family of measures $\{P^{(\vartheta)}_{\vartheta}, \vartheta \in \Theta\}$ is locally asymptotically normal in $\Theta$, i.e., the normalized likelihood ratio $Z_{\vartheta}(u)$ for all $\vartheta_0 \in \Theta$ admits the representation

$$Z_{\vartheta}(u) = \exp \left\{ u \Delta_{\vartheta}(\vartheta_0, X^T) - \frac{u^2}{2} I(\vartheta_0) + r_{\vartheta} \right\},$$

where

$$\Delta_{\vartheta}(\vartheta_0, X^T) = \int_0^T \frac{\dot{S}(\vartheta_0, t) \xi_{t,\vartheta}}{\sqrt{2S(\vartheta_0, t) \sigma(t)}} \, d\bar{W}_t \to \Delta(\vartheta_0) \sim \mathcal{N}(0, I(\vartheta_0)).$$

Here $r_{\vartheta} \to 0$ and $\xi_{t,\vartheta} \Rightarrow \xi_t$ for the random variables $\xi_t, t \in (0, T]$.

Proof. The process $Z_{\vartheta}(u)$ admits of the representation

$$\ln Z_{\vartheta}(u) = \psi_{\vartheta} \frac{\xi_{t,\vartheta}^2}{\varepsilon} \int_0^T \dot{M}(\vartheta_0, t) \, d\bar{W}_t - \frac{u^2}{2} \frac{\varepsilon^2}{\psi_{\vartheta}^2} \int_0^T \dot{M}(\vartheta_0, t)^2 \, dt + o_p(1)$$

$$= u \int_0^T \frac{\dot{S}(\vartheta_0, t) \xi_{t,\vartheta}}{\sqrt{2S(\vartheta_0, t) \sigma(t)}} \, d\bar{W}_t - \frac{u^2}{2} \int_0^T \frac{\dot{S}(\vartheta_0, t)^2 \xi_{t,\vartheta}^2}{2S(\vartheta_0, t) \sigma(t)} \, dt + o_p(1).$$

Let us denote

$$N_{\vartheta} = \sqrt{\psi_{\vartheta} \int_0^T R(t) (\xi_{t,\vartheta}^2 - 1) \, dt}$$

where $R(\cdot)$ is some bounded function. As in the proof of Lemma 4 in [23], for any $n > 0$,

$$E_{\vartheta_0} |N_{\vartheta}|^{2n} \leq C.$$  (33)

Therefore we can write

$$E_{\vartheta_0} \left( \int_0^T \frac{\dot{S}(\vartheta_0, t)^2 \xi_{t,\vartheta}^2}{2S(\vartheta_0, t) \sigma(t)} \, dt - \int_0^T \frac{\dot{S}(\vartheta_0, t)^2}{2S(\vartheta_0, t) \sigma(t)} \, dt \right)^2 \leq C \frac{\varepsilon}{\psi_{\vartheta}}$$

and obtain the convergence in probability

$$\int_0^T \frac{\dot{S}(\vartheta_0, t)^2 \xi_{t,\vartheta}^2}{2S(\vartheta_0, t) \sigma(t)} \, dt (1 + o(1)) \to \int_0^T \frac{\dot{S}(\vartheta_0, t)^2}{2S(\vartheta_0, t) \sigma(t)} \, dt = I(\vartheta_0).$$

This limit in probability allows us to apply the central limit theorem for the stochastic integral and to obtain the convergence

$$\int_0^T \frac{\dot{S}(\vartheta_0, t) \xi_{t,\vartheta}}{\sqrt{2S(\vartheta_0, t) \sigma(t)}} \, d\bar{W}_t \to \mathcal{N}(0, I(\vartheta_0)).$$
Let us denote
\[ G_\varepsilon (\vartheta, \vartheta_0) = -\varepsilon \psi_\varepsilon \ln \frac{L(\vartheta, X^T)}{L(\vartheta_0, X^T)}. \]

**Lemma 5.** Assume that the conditions \( A_1, A_2, A_4 \) are satisfied, then
\[
G_\varepsilon (\vartheta, \vartheta_0) = \int_0^T \left[ \frac{\varepsilon T_t^2}{4S(\vartheta, t) \sigma(t)} \right] \varepsilon \sigma(t) dt (1 + o(1)) + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right).
\]

**Proof.** We can write
\[
G_\varepsilon (\vartheta, \vartheta_0) = \int_0^T \frac{M(\vartheta, t) - M(\vartheta_0, t)}{\psi_\varepsilon \sigma(t)} d\bar{W}_t + \int_0^T \left[ \frac{M(\vartheta, t) - M(\vartheta_0, t)}{2\varepsilon \psi_\varepsilon \sigma(t)^2} \right] dt,
\]
where
\[
M(\vartheta, t) - M(\vartheta_0, t) = f(\vartheta, t) [m(\vartheta, t) - m(\vartheta_0, t)] + [f(\vartheta, t) - f(\vartheta_0, t)] m(\vartheta_0, t).
\]

To study the difference \( m_t(\vartheta, \vartheta_0) = m(\vartheta, t) - m(\vartheta_0, t) \), subtract the equations
\[
\begin{align*}
dm(\vartheta, t) &= -q_\varepsilon(\vartheta, t) m(\vartheta, t) dt + \psi_\varepsilon A(\vartheta, t) d\bar{W}_t, \\
dm(\vartheta_0, t) &= -a(\vartheta_0, t) m(\vartheta_0, t) dt + \psi_\varepsilon A(\vartheta_0, t) \sigma(t) d\bar{W}_t
\end{align*}
\]
to obtain
\[
\begin{align*}
dm_t(\vartheta, \vartheta_0) &= -q_\varepsilon(\vartheta, t) m_t(\vartheta, \vartheta_0) dt + \psi_\varepsilon [A(\vartheta, t) - A(\vartheta_0, t)] \sigma(t) d\bar{W}_t \\
&\quad - a(\vartheta, t) m(\vartheta_0, t) dt - \frac{\psi_\varepsilon}{\varepsilon} A(\vartheta, t) [f(\vartheta, t) - f(\vartheta_0, t)] m(\vartheta_0, t) dt.
\end{align*}
\]
The solution of this equation is
\[
\begin{align*}
m_t(\vartheta, \vartheta_0) &= \psi_\varepsilon \int_0^t e^{-\int_0^s q_\varepsilon(\vartheta, \vartheta) ds} [A(\vartheta, s) - A(\vartheta_0, s)] \sigma(s) d\bar{W}_s \\
&\quad - \frac{\psi_\varepsilon}{\varepsilon} \int_0^t e^{-\int_0^s q_\varepsilon(\vartheta, \vartheta) ds} A(\vartheta, s) [f(\vartheta, s) - f(\vartheta_0, s)] m(\vartheta_0, s) ds \\
&\quad - \int_0^t e^{-\int_0^s q_\varepsilon(\vartheta, \vartheta) ds} a(\vartheta, s) m(\vartheta_0, s) ds.
\end{align*}
\]
By the same arguments as above, the following expansions of the integrals holds,

\[
\psi_\varepsilon \int_0^t e^{-\int_0^t q_\vartheta (\vartheta, \varphi) d\varphi} \left[ A (\vartheta, s) - A (\vartheta_0, s) \right] \sigma (s) \, d\tilde{W}_s \\
= \sqrt{\varepsilon \psi_\varepsilon} \left[ A (\vartheta, t) - A (\vartheta_0, t) \right] \frac{\sigma (t)}{\sqrt{2A (\vartheta, t) f (\vartheta, t)}} \xi_{t,\varepsilon} (1 + o (1)),
\]

\[
\int_0^t e^{-\int_0^t q_\vartheta (\vartheta, \varphi) d\varphi} a (\vartheta, s) m (\vartheta_0, s) \, ds \\
= \frac{\varepsilon}{\psi_\varepsilon} \frac{a (\vartheta, t)}{A (\vartheta, t) f (\vartheta, t)} m (\vartheta_0, t) (1 + o_p (1)),
\]

\[
\psi_\varepsilon \int_0^t e^{-\int_0^t q_\vartheta (\vartheta, \varphi) d\varphi} A (\vartheta, s) \left[ f (\vartheta, s) - f (\vartheta_0, s) \right] m (\vartheta_0, s) \, ds \\
= A (\vartheta, t) \frac{\left[ f (\vartheta, s) - f (\vartheta_0, s) \right] m (\vartheta_0, t)}{A (\vartheta, t) f (\vartheta, t)} + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right)
\]

\[+ \psi_\varepsilon H_t (\vartheta, \vartheta_0) \int_0^t e^{-\int_0^t q_\vartheta (\vartheta, \varphi) d\varphi} \left[ m (\vartheta_0, s) - m (\vartheta_0, t) \right] ds.
\]

Here we denoted \(H_t (\vartheta, \vartheta_0) = A (\vartheta, t) \left[ f (\vartheta, t) - f (\vartheta_0, t) \right].\) The difference \(m (\vartheta_0, s) - m (\vartheta_0, t)\) was already evaluated in (32). Therefore we can write

\[
m_t (\vartheta, \vartheta_0) = \sqrt{\varepsilon \psi_\varepsilon} \frac{[A (\vartheta, t) - A (\vartheta_0, t)] \sigma (t)}{\sqrt{2A (\vartheta, t) f (\vartheta, t)}} \xi_{t,\varepsilon} (1 + o_p (1))
\]

\[+ \frac{A (\vartheta, t) \left[ f (\vartheta, t) - f (\vartheta_0, t) \right] m (\vartheta_0, t)}{A (\vartheta, t) f (\vartheta, t)} + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right)
\]

\[+ \sqrt{\varepsilon \psi_\varepsilon} \frac{A (\vartheta, t) \left[ f (\vartheta, t) - f (\vartheta_0, t) \right] A (\vartheta_0, t) \sigma (t)}{\sqrt{2A (\vartheta, t) f (\vartheta, t)}} \xi_{t,\varepsilon} (1 + o_p (1))
\]

\[= \frac{A (\vartheta_0, t) - f (\vartheta_0, t)}{f (\vartheta, t)} m (\vartheta_0, t) + O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right)
\]

\[+ \sqrt{\varepsilon \psi_\varepsilon} \frac{[A (\vartheta_0, t) f (\vartheta, t) - A (\vartheta_0, t) f (\vartheta_0, t)] \sigma (t)}{\sqrt{2A (\vartheta, t) f (\vartheta, t)}} \xi_{t,\varepsilon} (1 + o_p (1)).
\]

Hence

\[
M (\vartheta, t) - M (\vartheta_0, t) = O_p \left( \frac{\varepsilon}{\psi_\varepsilon} \right)
\]

\[+ \sqrt{\varepsilon \psi_\varepsilon} \frac{[b (\vartheta, t) f (\vartheta, t) - b (\vartheta_0, t) f (\vartheta_0, t)] \sqrt{\sigma (t)}}{\sqrt{2b (\vartheta, t) f (\vartheta, t)}} \xi_{t,\varepsilon} (1 + o_p (1)).
\]
For the stochastic integral we obtain the relation
\[
\int_0^T \frac{M(\vartheta, t) - M(\vartheta_0, t)}{\psi \sigma(t)} d\bar{W}_t
= \sqrt{\frac{\varepsilon}{\psi \varepsilon}} \int_0^T \frac{S(\vartheta, t) - S(\vartheta_0, t)}{\sqrt{2S(\vartheta, t) \sigma(t)}} \xi_{t, \varepsilon} d\bar{W}_t (1 + o_p(1)) \rightarrow 0.
\]

For the ordinary integral this gives us the limit
\[
\int_0^T \left[ M(\vartheta, t) - M(\vartheta_0, t) \right]^2 \frac{2 \varepsilon \psi \sigma(t)^2}{dt}
= \int_0^T \left[ S(\vartheta, t) - S(\vartheta_0, t) \right]^2 \frac{\varepsilon^2}{4S(\vartheta, t) \sigma(t)} \xi_{t, \varepsilon} dt (1 + o(1)) + O_p\left( \frac{\varepsilon}{\psi \varepsilon} \right)
\rightarrow \int_0^T \left[ S(\vartheta, t) - S(\vartheta_0, t) \right]^2 \frac{\varepsilon}{4S(\vartheta, t) \sigma(t)} d\vartheta = G(\vartheta, \vartheta_0).
\] (34)

\[\square\]

**Lemma 6.** Assume that conditions A, B are satisfied, then for some constant \( \kappa > 0 \) and any \( N > 0 \) there exists a constant \( C_N > 0 \) such that
\[
P_{\vartheta_0} \left\{ Z_\varepsilon(u) \geq e^{-\kappa u^2} \right\} \leq \frac{C_N}{|u|^N}.
\] (35)

**Proof.** Take a sufficiently small \( \delta \) such that for \( |h| \leq \delta \)
\[
G(\vartheta_0 + h, \vartheta_0) = \frac{1}{2} h^2 I(\vartheta_0) (1 + o(1)) \geq \frac{1}{4} \frac{I(\vartheta_0)}{h^2}.
\]

Let us denote \( g(\vartheta_0, \delta) = \inf_{|\vartheta - \vartheta_0| > \delta} G(\vartheta, \vartheta_0) \) and recall that by condition B2 \( g(\vartheta_0, \delta) > 0 \). Then for \( |h| > \delta \) we have
\[
G(\vartheta_0 + h, \vartheta_0) \geq g(\vartheta_0, \delta) \geq \frac{g(\vartheta_0, \delta)}{(\beta - \alpha)^2} h^2.
\]

Combining these two estimates we obtain
\[
G(\vartheta_0 + h, \vartheta_0) \geq \kappa_* h^2, \quad \kappa_* = \left( \frac{g(\vartheta_0, \delta)}{(\beta - \alpha)^2} \wedge \frac{I(\vartheta_0)}{4} \right).
\] (36)
Denote $\Delta M_t = M(\vartheta_u, t) - M(\vartheta_0, t)$ and write

$$
P_{\vartheta_0}\{ \ln Z_{\varepsilon}(u) \geq -\kappa u^2 \} = P_{\vartheta_0}\left\{ \int_0^T \frac{\Delta M_t}{2 \varepsilon \sigma(t)} \, d\bar{W}_t - \int_0^T \frac{\Delta M_t^2}{4 \varepsilon^2 \sigma(t)^2} \, dt \geq -\kappa u^2 \right\}
$$

$$
\leq P_{\vartheta_0}\left\{ \int_0^T \frac{\Delta M_t}{2 \varepsilon \sigma(t)} \, d\bar{W}_t - \int_0^T \frac{\Delta M_t^2}{8 \varepsilon^2 \sigma(t)^2} \, dt \geq \frac{\kappa}{2} u^2 \right\} + P_{\vartheta_0}\left\{ - \int_0^T \frac{\Delta M_t^2}{8 \varepsilon^2 \sigma(t)^2} \, dt \geq -\kappa u^2 \right\}
$$

$$
\leq e^{-\frac{\kappa}{2} u^2} + P_{\vartheta_0}\left\{ \int_0^T \frac{\Delta M_t^2}{8 \varepsilon^2 \sigma(t)^2} \, dt \leq 8 \kappa u^2 \right\}. \quad (37)
$$

Here we used the equality

$$
E_{\vartheta_0} \exp \left\{ \int_0^T \frac{\Delta M_t}{2 \varepsilon \sigma(t)} \, d\bar{W}_t - \int_0^T \frac{\Delta M_t^2}{8 \varepsilon^2 \sigma(t)^2} \, dt \right\} = 1.
$$

To estimate the latter probability in (37) we consider separately two cases $A = (u : |u| \leq \varphi_{\varepsilon}^{1/2})$ (local) and $A^c = (u : |u| > \varphi_{\varepsilon}^{1/2})$ (global). Locality is with respect to the values of $u$ for which $|\vartheta_u - \vartheta_0| \leq \varphi_{\varepsilon}^{1/2}$.

Let $u \in A$. The proof of (29) shows that this representation is valid for the values $u \in A$. Moreover the residuals $o(1)$ and $O\left(\frac{\varepsilon}{\psi(\varepsilon)}\right)$ have bounded polynomial moments of all orders. Hence

$$
\frac{M(\vartheta_0 + \varphi_{\varepsilon} u, t) - M(\vartheta_0, t)}{\varepsilon \sigma(t)} = \frac{\varphi_{\varepsilon} u}{\varepsilon} \frac{\dot{M}(\vartheta_0, t)}{\sigma(t)} (1 + o_p(1))
$$

$$
= \frac{u}{\sqrt{\varphi_{\varepsilon}}} \frac{\dot{M}(\vartheta_0, t)}{\sigma(t)} (1 + o_p(1))
$$

$$
= \frac{u}{\sqrt{\varphi_{\varepsilon}}} \frac{\dot{S}(\vartheta_0, t)}{\sigma(t)} \xi_{t, \varepsilon} (1 + o_p(1)) + uO_p\left(\frac{\varepsilon}{\psi_{\varepsilon}^3}\right).
$$

Therefore we can write

$$
\int_0^T \frac{\Delta M_t^2}{\varepsilon^2 \sigma(t)^2} \, dt = u^2 \int_0^T \frac{\dot{S}(\vartheta_0, t)^2}{2S(\vartheta_0, t) \sigma(t)} \xi_{t, \varepsilon}^2 \, dt (1 + o_p(1)) + u^2 O_p\left(\frac{\varepsilon}{\psi_{\varepsilon}^3}\right)
$$

$$
= u^2 I(\vartheta_0) + u^2 \int_0^T \frac{\dot{S}(\vartheta_0, t)^2}{2S(\vartheta_0, t) \sigma(t)} \left(\xi_{t, \varepsilon}^2 - 1\right) \, dt (1 + o(1)) + u^2 O_p\left(\frac{\varepsilon}{\psi_{\varepsilon}^3}\right)
$$

$$
= u^2 I(\vartheta_0) + u^2 \sqrt{\frac{\varepsilon}{\psi_{\varepsilon}}} N_{\varepsilon} (1 + o(1)) + u^2 \frac{\varepsilon}{\psi_{\varepsilon}^2} Q_{\varepsilon}
$$

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with the obvious notations. Further, let us denote \( \hat{\kappa} = \inf_{\vartheta \in \Theta} I(\vartheta) > 0 \) and introduce the sets

\[
N_\varepsilon = \left\{ \sqrt{\frac{\varepsilon}{\psi(\varepsilon)}} |N_\varepsilon|(1 + o(1)) \leq \frac{\hat{\kappa}}{4} \right\}, \quad Q_\varepsilon = \left\{ \frac{\varepsilon}{\psi_\varepsilon^3} |Q_\varepsilon| \leq \frac{\hat{\kappa}}{4} \right\}.
\]

Then we can write

\[
P_{\vartheta_0} \left\{ \int_0^T \frac{\Delta M^2_t}{\varepsilon^2 \sigma(t)^2} dt \leq 8\kappa u^2 \right\} \leq N_\varepsilon, Q_\varepsilon \right\} + P_{\vartheta_0} (N_\varepsilon^c) + P_{\vartheta_0} (Q_\varepsilon^c).
\]

If we let \( \kappa = \hat{\kappa}/32 \) then for \( |u| > 0 \) the first probability satisfies

\[
P_{\vartheta_0} \left\{ \int_0^T \frac{\Delta M^2_t}{\varepsilon^2 \sigma(t)^2} dt \leq 8\kappa u^2, N_\varepsilon, Q_\varepsilon \right\} \leq P_{\vartheta_0} \left\{ u^2 I(\vartheta_0) - \frac{\hat{\kappa}}{2} u^2 \leq 8\kappa u^2, N_\varepsilon, Q_\varepsilon \right\} \leq P_{\vartheta_0} \left\{ \frac{\hat{\kappa}}{2} u^2 \leq 8\kappa u^2, N_\varepsilon, Q_\varepsilon \right\} = 0.
\]

As the moments of \( N_\varepsilon \) and \( Q_\varepsilon \) are bounded we can write, for any \( n > 0 \),

\[
P_{\vartheta_0} \{ N_\varepsilon^c \} \leq C \left( \frac{\varepsilon}{\psi_\varepsilon} \right)^n = C \varepsilon^{n(1-\delta)},
\]

\[
P_{\vartheta_0} \{ Q_\varepsilon^c \} \leq C \left( \frac{\varepsilon}{\psi_\varepsilon^3} \right)^n = C \varepsilon^{n(1-3\delta)}.
\]

Let \( \delta_\ast = \frac{(1-\delta)}{2} \wedge (1 - 3\delta) \), then

\[
P_{\vartheta_0} \left\{ \int_0^T \frac{\Delta M^2_t}{\varepsilon^2 \sigma(t)^2} dt \leq 8\kappa u^2 \right\} \leq P_{\vartheta_0} \{ N_\varepsilon^c \} + P_{\vartheta_0} \{ Q_\varepsilon^c \} \leq C \varepsilon^{n\delta_\ast}.
\]

Recall that

\[
\vartheta_0 + \varphi_\varepsilon u \in (\alpha, \beta), \quad |u| \leq \frac{\beta - \alpha}{\varphi_\varepsilon} = \varepsilon^{\frac{1-\delta}{2}} (\beta - \alpha), \quad \varepsilon < \frac{(\beta - \alpha)^{2-s}}{|u|^{1-s}}.
\]

Hence if for any \( N > 0 \) we take \( n = \frac{N(1-\delta)}{2\delta_\ast} \) then

\[
P_{\vartheta_0} \left\{ \int_0^T \frac{\Delta M^2_t}{\varepsilon^2 \sigma(t)^2} dt \leq 8\kappa u^2 \right\} \leq \frac{C}{|u|^{\frac{2-s}{1-s}}} \leq \frac{C}{|u|^N}.
\]
Thus we obtained the estimate (35) for $u \in A$.

Suppose that $u \in A^c$. The relations (34) and (36) allow us to write

$$
\int_0^T \frac{\Delta M^2_s}{2 \varepsilon^2 \sigma(t)^2} \, dt = \frac{\psi_{\varepsilon}}{\varepsilon} \frac{1}{\varepsilon \psi_{\varepsilon}} \int_0^T \frac{\Delta M^2}{2 \sigma(t)^2} \, dt
$$

$$
= \frac{\psi_{\varepsilon}}{\varepsilon} \int_0^T \frac{[S(\vartheta, t) - S(\vartheta_0, t)]^2}{4 S(\vartheta, t) \sigma(t)} \varepsilon^2 \, dt (1 + o_p(1)) + O_p(1)
$$

$$
= \frac{\psi_{\varepsilon}}{\varepsilon} G(\vartheta, \vartheta_0) + \sqrt{\frac{\psi_{\varepsilon}}{\varepsilon}} R_{\varepsilon} + O(1) \geq \kappa^* u^2 + \sqrt{\frac{\psi_{\varepsilon}}{\varepsilon}} R_{\varepsilon} + O(1),
$$

where we denoted

$$
R_{\varepsilon} = \sqrt{\frac{\psi_{\varepsilon}}{\varepsilon}} \int_0^T \frac{[S(\vartheta, t) - S(\vartheta_0, t)]^2}{4 S(\vartheta, t) \sigma(t)} \left[ \xi^2_{t, \varepsilon} - 1 \right] \, dt (1 + o_p(1)).
$$

Therefore if we take $\kappa = \kappa^*/16$ and recall that for $u \in A^c$ we have $|u| > \sqrt{\psi_{\varepsilon}}$, then we can write

$$
P_{\vartheta_0} \left\{ \int_0^T \frac{\Delta M^2}{\varepsilon^2 \sigma(t)^2} \, dt \leq 8 \kappa u^2 \right\}
$$

$$
\leq P_{\vartheta_0} \left\{ - \frac{\psi_{\varepsilon}}{\varepsilon} |R_{\varepsilon}| + O_p(1) \leq -(\kappa^* - 8 \kappa) u^2 \right\}
$$

$$
\leq P_{\vartheta_0} \left\{ |u| |R_{\varepsilon}| + O_p(1) \geq \frac{\kappa^*}{2} u^2 \right\} \leq \frac{C}{|u|^N}.
$$

Lemma 7. Assume that conditions $A$ and $B_1$ are satisfied, then

$$
E_{\vartheta_0} \left[ Z_{\varepsilon}(u_1)^{1/2} - Z_{\varepsilon}(u_2)^{1/2} \right]^2 \leq C \left| u_2 - u_1 \right|^2
$$

(38)

Proof. As usually in such situations (see, e.g., [19]) we write

$$
E_{\vartheta_0} \left[ Z_{\varepsilon}(u_1)^{1/2} - Z_{\varepsilon}(u_2)^{1/2} \right]^2 = 2 - 2E_{\vartheta_{u_1}} \left( \frac{Z_{\varepsilon}(u_2)}{Z_{\varepsilon}(u_1)} \right)^{1/2} = 2 - 2E_{\vartheta_{u_1}} V_T,
$$

where we denoted $\vartheta_{u_1} = \vartheta_0 + \varphi_{\varepsilon} u_1$, and introduce the process

$$
V_t = \exp \left\{ \int_0^t \frac{\Delta M_s}{2 \varepsilon^2 \sigma(s)^2} \, dX_s - \int_0^t \frac{M(\vartheta_{u_2}, s)^2 - M(\vartheta_{u_1}, s)^2}{4 \varepsilon^2 \sigma(s)^2} \, ds \right\}.
$$
Here \( 0 \leq t \leq T \) and \( \Delta M_s = M (\varphi_0 + \varphi u_2, s) - M (\varphi_0 + \varphi u_1, s) \). This process with \( P_{\vartheta_1} \)-probability 1 has stochastic differential

\[
dV_t = -\frac{(\Delta M_t)^2}{8\varepsilon^2\sigma(t)^2} V_t dt + \frac{\Delta M_t}{2\varepsilon\sigma(t)} V_t d\tilde{W}_t, \quad V_0 = 1.
\]

Therefore

\[
2 - 2\mathbb{E}_{\vartheta_1} V_T = \int_0^T \mathbb{E}_{\vartheta_1} \frac{(\Delta M_t)^2}{8\varepsilon^2\sigma(t)^2} dt \leq \frac{1}{4\varepsilon^2} \int_0^T \mathbb{E}_{\vartheta_2} (\Delta M_t)^2 \sigma(t)^2 dt + \frac{1}{4\varepsilon^2} \int_0^T \mathbb{E}_{\vartheta_2} (\Delta M_t)^2 \sigma(t)^2 dt.
\]

Here we used the inequality \( V_t (\Delta M_t)^2 \leq 2V_t^2 (\Delta M_t)^2 + 2 (\Delta M_t)^2 \) and changed the measure \( \mathbb{E}_{\vartheta_1} V_t^2 = \mathbb{E}_{\vartheta_2} \). Now the bound (38) follows from the representation (29), where \( \vartheta_0 \) is replaced with \( \vartheta_1 \) and \( \vartheta_2 \) respectively.

It can be shown that the convergence in Lemma 4 is uniform on the compacts of \( \Theta \) and the constants in the Lemmas 6 and 7 can be chosen independent on \( \vartheta_0 \).

The properties of the normalized likelihood ratio \( Z_{\varepsilon} (\cdot) \) established in Lemmas 4, 6 and 7 verify the sufficient conditions \( N_1 - N_4 \) of Theorems 3.1.1 and 3.2.1 in [12], which, in turn, imply the properties of the MLE and BE claimed in Theorem 1.

Since the convergence of moments is uniform on compacts of \( \Theta \), we can prove the asymptotic efficiency of estimators as follows. The uniform convergence of moments for the MLE imply

\[
\lim_{\varepsilon \to 0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \frac{\psi}{\varepsilon} \mathbb{E}_{\vartheta} \left[ \hat{\vartheta}_\varepsilon - \vartheta \right]^2 = \sup_{|\vartheta - \vartheta_0| \leq \nu} I (\vartheta)^{-1} \frac{\nu \to 0}{\nu} I (\vartheta_0)^{-1}.
\]

The same convergence holds for BE \( \hat{\vartheta}_\varepsilon \). For more general loss functions see Theorem 3.1.3 in [12].

3 Discussions

Suppose that \( f (\vartheta, t) = \vartheta f (t), b (\vartheta, t) = \vartheta^{-1} b (t) \) and \( a (\vartheta, t) \) depend on \( \vartheta \). Then the function \( S (\vartheta, t) = f (t) b (t) \) does not depend on \( \vartheta \) and the conditions \( B \) fail. Now the existence of the consistent estimator depends on the
value of $y_0$. If $y_0 \neq 0$ then the rate of convergence of the estimators is different. Let us construct a consistent and asymptotically normal estimator in this situation. Introduce the notations

$$H(\vartheta, t) = y_0 \int_0^t \exp \left( \int_0^s a(\vartheta, v) \, dv \right) f(s) \, ds, \quad \eta(t) = \int_0^t \sigma(s) \, dW_s,$$

$$\pi(\vartheta, t) = \int_0^t \int_0^s \exp \left( \int_r^s a(\vartheta, v) \, dv \right) b(r) \, dV_r f(s) \, ds.$$

Then the observed process can be written as follows

$$X_t = H(\vartheta_0, t) + \psi \varepsilon \pi(\vartheta_0, t) + \varepsilon \eta(t), \quad 0 \leq t \leq T,$$

where $\pi(\cdot)$ and $\eta(\cdot)$ are independent Gaussian processes. The minimum distance estimator (MDE) $\vartheta^*_\varepsilon$ is the solution of equation

$$\int_0^T [X_t - H(\vartheta^*_\varepsilon, t)]^2 \, dt = \inf_{\vartheta \in \Theta} \int_0^T [X_t - H(\vartheta, t)]^2 \, dt.$$

The identifiability condition is

$$\inf_{|\vartheta - \vartheta_0| > \nu} \int_0^T [H(\vartheta, t) - H(\vartheta_0, t)]^2 \, dt > 0, \quad \forall \nu > 0, \quad (39)$$

If this condition is satisfied then the MDE is consistent (see [19]). Moreover, it can be shown that

$$\vartheta^*_\varepsilon = \vartheta_0 + \psi \varepsilon \int_0^T \pi(\vartheta_0, t) D(\vartheta_0, t) \, dt (1 + o(1))$$

$$+ \varepsilon \int_0^T \eta(t) D(\vartheta_0, t) \, dt (1 + o(1)), \quad D(\vartheta_0, t) = \frac{\dot{H}(\vartheta_0, t)}{\int_0^t \dot{H}(\vartheta_0, t)^2 \, dt}.$$

Therefore if we denote $\phi_\varepsilon = \max(\varepsilon, \psi \varepsilon)$, then for any $\varepsilon \to 0$ and $\psi \varepsilon \to 0$, the asymptotic normality of $\vartheta^*_\varepsilon$ holds,

$$\frac{\vartheta^*_\varepsilon - \vartheta_0}{\phi_\varepsilon} \Rightarrow \mathcal{N}(0, d(\vartheta_0)^2) \quad (40)$$

with the corresponding limit variance $d(\vartheta_0)^2$.

Let us consider the possibility of the adaptive filtration for this model of observations. As mentioned in the Introduction, finding the MLE and the BE for the partially observed linear system (1), (2) is computationally inefficient, since it requires solving the filtering equations (18), (19) for all $\vartheta \in \Theta$. The
for the MLE we have to solve the maximization problem (41). Instead we can use a much simpler algorithm, based on the multi-step approach, recently developed in [22], [16], [28]. Let us consider such construction of preliminary estimator in the case of observations (1) omitting the technical details. Fix a small \( \tau \in (0, T) \) and let \( \hat{\vartheta}_{\tau, \epsilon} \) be the MLE \((y_0 = 0)\) and \( \vartheta^*_{\tau, \epsilon} \) be the MDE \((y_0 \neq 0)\) described above based on the observations \( X^\tau = (X_t, t \in [0, \tau]) \). Assume that the corresponding identifiability and regularity conditions are fulfilled then the both estimators are consistent and asymptotically normal. Below we use notation \( \vartheta^*_{\tau, \epsilon} \) for preliminary estimator assuming that it can be the MLE too. Let \( \psi_{\epsilon} = \epsilon^\delta, \, \delta \in \left( \frac{1}{5}, \frac{1}{3} \right) \) and define the one-step MLE-process \( \vartheta^*_{t, \epsilon}, \tau < t \leq T \), where

\[
\vartheta^*_{t, \epsilon} = \vartheta^*_{\tau, \epsilon} + I^t_\tau \left( \vartheta^*_{\tau, \epsilon} \right)^{-1} \int_\tau^t \frac{M \left( \vartheta^*_{\tau, \epsilon}, s \right)}{\epsilon \psi_{\epsilon} \sigma (s)^2} \left[ dX_s - M \left( \vartheta^*_{\tau, \epsilon}, s \right) ds \right].
\]  

(41)

The Fisher information here

\[
I^t_\tau (\vartheta) = \int_\tau^t \frac{\dot{S} (\vartheta, s)^2}{2S (\vartheta, s) \sigma (s)} ds
\]

is supposed to be positive for all \( t \in (\tau, T] \). We have to explain how calculate \( \dot{M} \left( \vartheta^*_{\tau, \epsilon}, s \right) \) and \( M \left( \vartheta^*_{\tau, \epsilon}, s \right) \). Recall that by (22) \((y_0 = 0)\) we have

\[
m (\vartheta, t) = \frac{\psi_{\epsilon} e^{-\int_0^t q_v (\vartheta, v) \, dv}}{\epsilon} \int_0^t e^{\int_0^s q_v (\vartheta, v) \, dv} A (\vartheta, s) \, dX_s
\]

\[
= h (\vartheta, t) \int_0^t H (\vartheta, s) \, dX_s
\]

with obvious notation. The stochastic integral can be replaced using the following relation

\[
\int_0^t H (\vartheta, s) \, dX_s = H (\vartheta, t) \, X_t - \int_0^t H' (\vartheta, s) \, X_s \, ds.
\]

The right hand side of this equality we denote as \( Z (\vartheta, t, X^t) \) and put

\[
m (\vartheta^*_{\tau, \epsilon}, s) = h \left( \vartheta^*_{\tau, \epsilon}, s \right) Z \left( \vartheta^*_{\tau, \epsilon}, s, X^s \right).
\]

For \( \dot{M} \left( \vartheta^*_{\tau, \epsilon}, s \right) \) can be obtained the similar expression.

Following the usual calculations in such situation (see, e.g., [27], [22]) we
obtain the relations

\[
\sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) = \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) + \frac{1}{\Pi'_\tau (\vartheta^*_\tau,\varepsilon)} \int_\tau^t \hat{M} (\vartheta^*_\tau,\varepsilon, s) \varepsilon \sqrt{\psi_\varepsilon} \sigma (s) d\hat{W}_s \\
+ \sqrt{\psi_\varepsilon} \frac{1}{\Pi'_\tau (\vartheta^*_\tau,\varepsilon)} \int_\tau^t \frac{\hat{M} (\vartheta^*_\tau,\varepsilon, s)}{\varepsilon \sqrt{\psi_\varepsilon} \sigma (s)^2} [M (\vartheta_0, s) - M (\vartheta^*_\tau,\varepsilon, s)] ds \\
= \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) + \frac{1}{\Pi'_\tau (\vartheta_0)} \int_\tau^t \frac{\dot{S} (\vartheta_0, s)}{\sqrt{2S (\vartheta_0, s) \sigma (s)}} \xi_{s,\varepsilon} d\hat{W}_s (1 + o (1)) \\
- \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) \int_\tau^t \frac{\dot{S} (\vartheta_0, s)^2}{2S (\vartheta_0, s) \sigma (s)} \xi_{s,\varepsilon}^2 ds (1 + O (\vartheta^*_\tau,\varepsilon - \vartheta_0)) \\
= \frac{1}{\Pi'_\tau (\vartheta_0)} \int_\tau^t \frac{\dot{S} (\vartheta_0, s)}{\sqrt{2S (\vartheta_0, s) \sigma (s)}} \xi_{s,\varepsilon} d\hat{W}_s (1 + o_p (1)) \\
+ \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0)^2 O_p (1) + o_p (1).
\]

Since \( \delta \in \left( \frac{1}{5}, \frac{1}{3} \right) \),

\[
\sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0)^2 = \sqrt{\psi_\varepsilon} \psi_\varepsilon^2 O_p (1) = \varepsilon^{\frac{2}{5}} (\delta - \frac{1}{3}) O_p (1) \rightarrow 0
\]

and hence

\[
\int_\tau^t \frac{\dot{S} (\vartheta_0, s)}{\sqrt{2S (\vartheta_0, s) \sigma (s)}} \xi_{s,\varepsilon} d\hat{W}_s \implies \mathcal{N} (0, \Pi'_\tau (\vartheta_0))
\]

and, consequently,

\[
\sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) \implies \mathcal{N} (0, \Pi'_\tau (\vartheta_0)^{-1}).
\]

Thus we constructed estimator which is consistent and asymptotically normal with good rate. It requires solving the Riccati equation just for one value \( \vartheta = \vartheta^*_\tau,\varepsilon \) and the random functions \( m(\vartheta^*_\tau,\varepsilon, t) \) and \( \hat{m}(\vartheta^*_\tau,\varepsilon, t) \). Though the presentation here was formal, all calculations can be made precise using the technique developed in [27], [16], [21], [22], [28].

The adaptive filtration can be realized with the help of the equations (41) and

\[
d\hat{m}_t = -a (\vartheta^*_\tau,\varepsilon, t) \hat{m}_t dt + \frac{\psi_\varepsilon b (\vartheta^*_\tau,\varepsilon, t)}{\sigma (t)} [dX_t - f (\vartheta^*_\tau,\varepsilon, t) \hat{m}_t dt], \quad \tau < t \leq T,
\]

and

\[
\sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) = \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) + \frac{1}{\Pi'_\tau (\vartheta^*_\tau,\varepsilon)} \int_\tau^t \hat{M} (\vartheta^*_\tau,\varepsilon, s) \varepsilon \sqrt{\psi_\varepsilon} \sigma (s) d\hat{W}_s \\
+ \sqrt{\psi_\varepsilon} \frac{1}{\Pi'_\tau (\vartheta^*_\tau,\varepsilon)} \int_\tau^t \frac{\hat{M} (\vartheta^*_\tau,\varepsilon, s)}{\varepsilon \sqrt{\psi_\varepsilon} \sigma (s)^2} [M (\vartheta_0, s) - M (\vartheta^*_\tau,\varepsilon, s)] ds \\
= \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) + \frac{1}{\Pi'_\tau (\vartheta_0)} \int_\tau^t \frac{\dot{S} (\vartheta_0, s)}{\sqrt{2S (\vartheta_0, s) \sigma (s)}} \xi_{s,\varepsilon} d\hat{W}_s (1 + o (1)) \\
- \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0) \int_\tau^t \frac{\dot{S} (\vartheta_0, s)^2}{2S (\vartheta_0, s) \sigma (s)} \xi_{s,\varepsilon}^2 ds (1 + O (\vartheta^*_\tau,\varepsilon - \vartheta_0)) \\
= \frac{1}{\Pi'_\tau (\vartheta_0)} \int_\tau^t \frac{\dot{S} (\vartheta_0, s)}{\sqrt{2S (\vartheta_0, s) \sigma (s)}} \xi_{s,\varepsilon} d\hat{W}_s (1 + o_p (1)) \\
+ \sqrt{\psi_\varepsilon} (\vartheta^*_\tau,\varepsilon - \vartheta_0)^2 O_p (1) + o_p (1).
\]
subject to initial value \( \hat{m}_\tau = m(\vartheta^*_{\tau,\tau}, \tau) \). Moreover, as it was shown in the mentioned above works in similar situations this estimation of \( m(\vartheta_0, t) \) can have some properties of optimality.

Consider the observations model (11), (12) and assume that conditions \( A, B \) hold. As \( y_0 = 0 \) the processes \( X^T, Y^T \) converge with probability 1 to 0,

\[
\sup_{0 \leq t \leq T} |X_t| \rightarrow 0, \quad \sup_{0 \leq t \leq T} |Y_t| \rightarrow 0.
\]

This means that the limit observations are \( X_t \equiv 0 \), but nevertheless the MLE and BE still have all the properties, claimed in Theorem 11. On the other hand, this shows the essential difference between the observation model (11), (12) with \( \psi_{\varepsilon} = \varepsilon \) studied in [19] and the present one. Unlike in our case, there we have a deterministic dynamical system (6) (limit model, \( y_0 \neq 0 \)) perturbed by small noise.

The result of this paper is in a sense surprising. We see that the error of estimation decreases if noise intensity \( \psi_{\varepsilon} \) in the state equation increases in some region (\( \psi_{\varepsilon} = \varepsilon^\delta, 0 < \delta < \frac{1}{2} \)). This can be explained heuristically as follows. Suppose that the conditions \( A, B \) hold and \( y_0 = 0 \). Then

\[
Y_t = \psi_{\varepsilon} \int_0^t e^{-\int_0^s a(\vartheta_0, v) dv} b(\vartheta_0, s) dV_s = \psi_{\varepsilon} \hat{Y}_t(\vartheta_0),
\]

where \( \hat{Y}_t(\vartheta_0) \) is defined by the latter equality. The observed process can be rewritten as

\[
dX_t = \psi_{\varepsilon} f(\vartheta_0, t) \hat{Y}_t(\vartheta_0) \, dt + \varepsilon \sigma(t) \, dW_t.
\]

Here \( \varepsilon \to 0 \) much faster than \( \psi_{\varepsilon} \to 0 \). Suppose that \( \varepsilon = 0 \) holds already but \( \psi_{\varepsilon} > 0 \). Then the observations \( x_t = \frac{dX_t}{dt}, t \in [0, T] \) are \( x_t = \psi_{\varepsilon} f(\vartheta_0, t) \hat{Y}_t(\vartheta_0) \), \( t \in [0, T] \) and the process \( \hat{x}_t = \psi_{\varepsilon}^{-1} x_t = f(\vartheta_0, t) \hat{Y}_t(\vartheta_0) \) has the differential

\[
d\hat{x}_t = [f'(\vartheta_0, t) - f(\vartheta_0, t) a(\vartheta_0, t)] \hat{Y}_t(\vartheta_0) \, dt + S(\vartheta_0, t) \, dV_t, \quad \hat{x}_0 = 0,
\]

where as before \( S(\vartheta_0, t) = f(\vartheta_0, t) b(\vartheta_0, t) \). The Itô formula for \( \hat{x}_t^2 \) gives the equation

\[
d\hat{x}_t^2 = 2\hat{x}_t d\hat{x}_t + S(\vartheta_0, t)^2 \, dt, \quad \hat{x}_0 = 0.
\]

Therefore we can write

\[
\int_0^T S(\vartheta_0, t)^2 \, dt = \hat{x}_T^2 - 2 \int_0^T \hat{x}_t d\hat{x}_t.
\]
This equation (under identifiability condition) allows us to find \( \vartheta_0 \) precisely. For example, let \( f(\vartheta, t) = f(t) \) and \( b(\vartheta, t) = \sqrt{b(t)} \), then

\[
\vartheta_0 = \left( \int_0^T f(t)^2 b(t)^2 \, dt \right)^{-1} \left[ \frac{x_T^2}{2} - 2 \int_0^T \tilde{x}_t \, dt \right].
\]

We see that if \( \varepsilon = 0 \), then the value \( \vartheta_0 \) by observations \( x_t, t \in [0, T] \) can be calculated without error.

Note that in observations (42) the function \( \psi_\varepsilon \) plays the role of amplitude of the signal \( \psi_\varepsilon f(\vartheta_0, t) \hat{Y}_t(\vartheta_0) \) containing unknown parameter. The increase in \( \psi_\varepsilon \) leads to increase in the signal-to-noise ratio, \( \text{SNR} \approx \frac{\psi_\varepsilon^2}{\varepsilon^2} \). Of course, all these explications are only heuristics and for the other range of \( \psi_\varepsilon \) they may not apply. In particular, if \( \psi_\varepsilon = \varepsilon \), the rate of convergence of the estimators is essentially better than \( \sqrt{\frac{\varepsilon}{\psi_\varepsilon}} \) (see (7)).

The normalization \( \varphi_\varepsilon \) is determined by the key representation (29). We obtained

\[
\varphi_\varepsilon = \sqrt{\frac{\psi_\varepsilon}{\psi_\varepsilon}} = \varepsilon^{\frac{1}{2}(1-\delta)} = \varepsilon^\gamma, \frac{1}{3} < \gamma < \frac{1}{2}
\]

because we assumed that \( \varepsilon/\psi_\varepsilon \to 0 \) or let \( \psi_\varepsilon = \varepsilon^\delta, 0 < \delta < \frac{1}{3} \). If we assume that \( \varepsilon/\psi_\varepsilon \to \infty \) then the main term in (29) will have order \( O \left( \frac{\varepsilon^\delta}{\psi_\varepsilon} \right) \) and this would probably lead to the normalization \( \varphi_\varepsilon = \psi_\varepsilon = \varepsilon^\delta, \frac{1}{3} < \delta < 1 \).

Remark. The case \( y_0 \neq 0 \) merits a special study because there is an “atom” at the point \( t = 0 \). Note that the ordinary integral in the expression (31) converges to the similar limit \( (t = \frac{\psi_\varepsilon}{\psi_\varepsilon} \ln \varepsilon^{-1} \to 0) \)

\[
\psi_\varepsilon \int_0^t e^{-\int_0^s \psi_\varepsilon} \psi_\varepsilon A(\vartheta_0, s) \dot{f}(\vartheta_0, s) \, ds \to \frac{\dot{f}(\vartheta_0, 0)}{f(\vartheta_0, 0)} y_0.
\]

To verify it we use the expansion \( \gamma_*(\vartheta_0, t) = \frac{\psi_\varepsilon}{\psi_\varepsilon^2} b(\vartheta_0, 0)^2 t (1 + o(1)) \) for small \( t \) and change the variables \( v = \frac{\psi_\varepsilon}{\psi_\varepsilon}, s = \frac{\psi_\varepsilon}{\psi_\varepsilon} \). The other two integrals in (31) converge to zero. Therefore, once more we have

\[
\dot{f}(\vartheta_0, t) m(\vartheta_0, t) + f(\vartheta_0, t) \dot{m}(\vartheta_0, t) \to 0.
\]

The rate of convergence of estimators has to be at least \( \sqrt{\frac{\varepsilon}{\psi_\varepsilon}} \). For example, if we suppose that \( f(\vartheta, t) = \vartheta f(t) \), \( f(0) \neq 0 \) and introduce the estimator

\[
\tilde{\vartheta}_{\tau_\varepsilon} = \frac{X_{\tau_\varepsilon}}{y_0 f(0) \tau_\varepsilon},
\]

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then for $\tau_\varepsilon = \varepsilon/\psi_\varepsilon$ using elementary calculations we obtain

$$
\sqrt{\frac{\psi_\varepsilon}{\varepsilon}} (\bar{\vartheta}_\varepsilon - \vartheta_0) \Rightarrow N \left( 0, d(\vartheta_0)^2 \right)
$$

with some $d(\vartheta_0)^2 > 0$. Of course, this estimator can be used as preliminary in the construction (11) of one-step MLE-process $\vartheta^*_\varepsilon$, $\tau_\varepsilon < t \leq T$ for this model. Note that such ($\tau_\varepsilon \to 0$) preliminary estimators in one-step estimation were used many times (see, e.g. [27], [25] and references there in), but the rate of convergence of them were always slower than the rate of convergence of estimators constructed by observations on a fixed interval.

Acknowledgment. I am grateful to P. Chigansky for useful comments and especially for attracting my attention to the situation with $y_0 \neq 0$ (see Remark above). This research was supported by RSF project no 20-61-47043.

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