TWO NEW LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE

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Abstract. In this paper, we obtain two new lower bounds for the smallest singular value of non-singular matrices which is better than the bound presented by Zou [1], Lin and Xie [2] under certain circumstances.

1. Introduction

Let $M_n$ ($n \geq 2$) be the space of $n \times n$ complex matrices. Let $\sigma_i$ ($i = 1, \ldots, n$) be the singular values of $A \in M_n$ which is nonsingular and suppose that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-1} \geq \sigma_n > 0$. For $A = [a_{ij}] \in M_n$, the Frobenius norm of $A$ is defined by

$$\|A\|_F = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \text{tr}(A^H A)^{1/2}$$

where $A^H$ is the conjugate transpose of $A$. The relationship between the Frobenius norm and singular values is

$$\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.$$ 

It is well known that lower bounds for the smallest singular value $\sigma_n$ of a nonsingular matrix $A \in M_n$ have many potential theoretical and practical applications [3, 4]. Yu and Gu [5] obtained a lower bound for $\sigma_n$ as follows:

$$\sigma_n \geq |\det A| \cdot \left( \frac{n-1}{\|A\|_F^2} \right)^{(n-1)/2} = \ell > 0.$$ 

The above inequality is also shown in [6]. In [1], Zou improved the above inequality by showing that

$$\sigma_n \geq |\det A| \left( \frac{n-1}{\|A\|_F^2 - \ell^2} \right)^{(n-1)/2} = \ell_0.$$ 

In [2], Lin, Minghua and Xie, Mengyan improve a lower bound for smallest singular value of matrices by showing that $a$ is the smallest positive solution to the equation

$$x^2 (\|A\|_F^2 - x^2)^{n-1} = |\det A|^2 (n-1)^{n-1}$$ 

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and \( \sigma \geq a > l_0 \).

In this paper, we obtain two new lower bounds for the smallest singular value of nonsingular matrices. We give some numerical examples which will show that our result is better than \( l_0 \) and \( a \) under certain circumstances.

2. Main results

**Lemma 1.** Let
\[
l_0 = |\det A| \left( \frac{n - 1}{\|A\|_F^2 - l_0^2} \right)^{(n-1)/2}
\]
then \( \sigma_n > l_0 \).

*Proof.* In [1], we have
\[
\sigma_n \geq |\det A| \left( \frac{n - 1}{\|A\|_F^2 - \sigma^2_n} \right)^{(n-1)/2}
\]
since \( \sigma_n \geq l_0 > l \), thus
\[
\sigma \geq |\det A| \left( \frac{n - 1}{\|A\|_F^2 - \sigma^2_n} \right)^{(n-1)/2}
\geq |\det A| \left( \frac{n - 1}{\|A\|_F^2 - l_0^2} \right)^{(n-1)/2}
\]
\[
> |\det A| \left( \frac{n - 1}{\|A\|_F^2 - l_0^2} \right)^{(n-1)/2} = l_0
\]
so \( \sigma_n > l_0 \). \( \square \)

**Theorem 1.** Let \( A \in M_n \) be nonsingular. Then
\[
\left( l_0^n + |\det(l_0^2 I_n - A^H A)| \left( \frac{n - 1}{\|A\|_F^2 - nl_0^2} \right)^{n-1} \right)^{1/2} = l_1
\]
then \( \sigma_n \geq l_1 \), where
\[
l = |\det A| \left( \frac{n - 1}{\|A\|_F^2} \right)^{\frac{n-1}{2}}, \quad l_0 = |\det A| \left( \frac{n - 1}{\|A\|_F^2 - l_0^2} \right)^{\frac{n-1}{2}}.
\]

*Proof.* Let \( 0 < \lambda < \sigma^2_n \), then
\[
| (\lambda - \sigma^2_1) (\lambda - \sigma^2_2) \cdots (\lambda - \sigma^2_{n-1}) | \leq \left( \frac{\sigma^2_1 + \cdots + \sigma^2_{n-1} - (n-1)\lambda}{n-1} \right)^{n-1}.
\]
Since
\[ |(\lambda - \sigma_1^2)(\lambda - \sigma_2^2)\cdots(\lambda - \sigma_{n-1}^2)| = \frac{|(\lambda - \sigma_1^2)(\lambda - \sigma_2^2)\cdots(\lambda - \sigma_n^2)|}{\sigma_n^2 - \lambda} \]
then
\[ \frac{|\det(\lambda I_n - A^H A)|}{\sigma_n^2 - \lambda} \leq \left( \frac{\sigma_1^2 + \cdots + \sigma_{n-1}^2 - (n-1)\lambda}{n-1} \right)^{n-1} \]
\[ \sigma_n^2 \geq \lambda + |\det(\lambda I_n - A^H A)| \left( \frac{n-1}{\sigma_1^2 + \cdots + \sigma_{n-1}^2 - (n-1)\lambda} \right)^{n-1} \]
\[ \sigma_n \geq \left( \lambda + |\det(\lambda I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)\lambda} \right) \right)^{1/2} \]
By Lemma 1, \( l_0 < \sigma_n \), \( l_0^2 < \sigma_n^2 \), let \( \lambda = l_0^2 \), then
\[ \sigma_n \geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2} \right) \right)^{1/2} \]  \hspace{1cm} (1)
Therefore
\[ \sigma_n \geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - nl_0^2} \right) \right)^{1/2} \] \hspace{1cm} \square

**Theorem 2.** Let \( A \in M_n \) be nonsingular. Let
\[ b_{k+1} = \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - (n-1)l_0^2 - b_k^2} \right) \right)^{1/2}, \quad k = 1, 2, \ldots \]
with \( l = |\det A| \left( \frac{n-1}{\|A\|_F^2 - l^2} \right)^{n-1}, \ l_0 = |\det A| \left( \frac{n-1}{\|A\|_F^2 - l^2} \right)^{n-1} \)
\[ b_1 = \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - (n-1)l_0^2} \right) \right)^{1/2} \]
then \( 0 < b_k < b_{k+1} \leq \sigma_n, \ k = 1, 2, \ldots, \lim_{k \to \infty} b_k \) exists.

**Proof.** We show by induction on \( k \) that
\[ \sigma_n \geq b_{k+1} > b_k > 0. \]
By (1), we have
\[
\sigma_n \geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2}
\]
so \(\sigma_n \geq b_1\), then
\[
\sigma_n \geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - (n-1)l_0^2 - b_1^2} \right)^{n-1} \right)^{1/2} = b_2
\]
\[
> \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2} = b_1 > 0.
\]
When \(k = 1\), we have
\[
\sigma_n \geq b_2 > b_1 > 0.
\]
Assume that our claim is true for \(k = m\), that is \(\sigma_n \geq b_{m+1} > b_m > 0\). Now we consider the case when \(k = m + 1\). By (1), we have
\[
\sigma_n \geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2}
\]
so
\[
\geq \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - b_{m+1}^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2} = b_{m+2}
\]
\[
> \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - b_m^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2} = b_{m+1} > 0.
\]
Hence \(\sigma_n \geq b_{m+2} > b_{m+1} > 0\). This proves \(\sigma_n \geq b_{k+1} > b_k > 0, k = 1, 2, \ldots\) By the well known monotone convergence theorem, \(\lim_{k \to \infty} b_k\) exists. \(\square\)

**Theorem 3.** Let \(b = \lim_{k \to \infty} b_k\),
\[
f(x) = \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - x^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2}
\]
then \(b\) is the smallest positive solution to the equation \(x = f(x)\), and \(\sigma_n \geq b\).
Proof. Let \( x_0 \) is the smallest positive solution to the equation \( x = f(x) \), we show by induction on \( k \) that \( x_0 > b_k, k = 1, 2, \cdots \). When \( k = 1 \)

\[
x_0 = \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2}
\]

\[
> \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2} = b_1.
\]

Assume that our claim is true for \( k = m \), that is \( \sigma_n > b_m \). Now we consider the case when \( k = m + 1 \).

\[
x_0 = \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2}
\]

\[
> \left( l_0^2 + |\det(l_0^2 I_n - A^H A)| \left( \frac{n-1}{\|A\|_F^2 - (n-1)l_0^2} \right)^{n-1} \right)^{1/2} = b_{m+1}.
\]

Hence \( x_0 > b_{m+1} \). This proves \( x_0 > b_k, k = 1, 2, \cdots \). Since \( b \) is a positive solution to the equation \( x = f(x) \) and \( x_0 > b_k, k = 1, 2, \cdots \), then \( b = x_0 \). Therefore \( b \) is the smallest positive solution to the equation \( x = f(x) \) and \( \sigma_n > b \). \( \square \)

Therefore we obtain two new lower bounds \( l_1 \) and \( b \) for the smallest singular value of nonsingular matrices.

3. Numerical examples

We use Examples 1 and Example 2 to compare the values of \( l, l_0, l_1 \).

**EXAMPLE 1.** Let

\[
A = \begin{bmatrix}
4 & -4 & -3 \\
3 & 4 & 2 \\
4 & 1 & 0
\end{bmatrix}.
\]

Then \( \sigma_{\min} = 0.0231 \), and

\[
l = 0.0229885 \\
l_0 = 0.0229886.
\]

Our result:

\[
l_1 = 0.0230691.
\]

**EXAMPLE 2.** Let

\[
A = \begin{bmatrix}
4 & 0 & 0 \\
-1 & 5 & 0 \\
0 & 5 & 4
\end{bmatrix}.
\]
Then
\[ l = 1.92771 \]
\[ l_0 = 2.01806. \]

Our result:
\[ l_1 = 2.31515. \]

Next we use the following example to compare the values of \( a, b, l_1 \).

**Example 3.** Let
\[
A = \begin{bmatrix}
3 & 2 & 0 \\
1 & 9 & 5 \\
0 & 5 & 7
\end{bmatrix}.
\]

Then
\[ a = 1.0367. \]

Our result:
\[ l_1 = 1.3434 \]
\[ b = 1.3455. \]

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