An Ergodic Dilation of Completely Positive Maps

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Abstract

We shall prove the following Stinespring-type theorem: there exists a triple \((\pi, \mathcal{H}, \mathbf{V})\) associated with an unital completely positive map \(\Phi : \mathfrak{A} \to \mathfrak{A}\) on C*-algebra \(\mathfrak{A}\) with unit, where \(\mathcal{H}\) is a Hilbert space, \(\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})\) is a faithful representation and \(\mathbf{V}\) is a linear isometry on \(\mathcal{H}\) such that \(\pi(\Phi(a)) = \mathbf{V}^* \pi(a) \mathbf{V}\) for all \(a\) belong to \(\mathfrak{A}\). The Nagy dilation theorem, applied to isometry \(\mathbf{V}\), allows to construct a dilation of ucp-map, \(\Phi\), in the sense of Arveson, that satisfies ergodic properties of a \(\Phi\)-invariant state \(\varphi\) on \(\mathfrak{A}\), if \(\Phi\) admit a \(\varphi\)-adjoint.

1 Introduction

A discrete quantum process is a pair \((\mathfrak{M}, \Phi)\) consisting of a von Neumann algebra \(\mathfrak{M}\) and a normal unital completely positive map \(\Phi\) on \(\mathfrak{M}\). In this work we shall prove that any quantum process is possible dilate to quantum process where the dynamic \(\Phi\) is a \(\ast\)-endomorphism of a larger von Neumann algebra.

In dynamical systems, the process of dilation has taken different meanings. Here we adopt the following definition (See Ref. Muhly-Solel [6]):

Suppose \(\mathfrak{M}\) acts on Hilbert space \(\mathcal{H}\), a dilation of a quantum process \((\mathfrak{M}, \Phi)\) is a quadruple \((\mathfrak{M}, \Theta, \mathcal{K}, z)\) where \((\mathfrak{M}, \Theta)\) is a quantum process with \(\mathfrak{M}\) acts on Hilbert space \(\mathcal{K}\) and \(\Theta\) is a homomorphism (i.e. \(\ast\)-endomorphism on von Neumann algebra \(\mathfrak{M}\)) with \(z : \mathcal{K} \to \mathcal{K}\) isometric embedding such that:

- \(z\mathfrak{M}z^* \subset \mathfrak{M}\) and \(z^* \mathfrak{M}z \subset \mathfrak{M}\);
- \(\Phi^n(a) = z^* \Theta^n(zaz^*)z\) for all \(a \in \mathfrak{M}\) and \(n \in \mathbb{N}\);
- \(z^* \Theta^n(X)z = \Phi^n(z^* X z)\) for all \(X \in \mathfrak{M}\) and \(n \in \mathbb{N}\).

Many authors in the past have been applied to problems very similar to the one we described above. We remember the work of Arveson [2] on the Eo-semigroups, of Bhat-Parthasarathy on the dilations of nonconservative dynamical semigroups [3] and finally, the most recent work of Muhly-Solel [6].

We shall prove the existence of dilation using the Nagy theorem for linear contraction (See Fojas-Nagy Ref. [7]) and of a particular covariant representation obtained through the Stinespring’s theorem for completely positive maps (See Stinespring Ref. [10]).

We recall that a covariant representation of discrete quantum process \((\mathfrak{M}, \Phi)\) is a triple \((\pi, \mathcal{H}, \mathbf{V})\) where \(\pi : \mathfrak{M} \to \mathfrak{B}(\mathcal{H})\) is a normal faithful representation on the Hilbert space \(\mathcal{H}\) and \(\mathbf{V}\) is an isometry on \(\mathcal{H}\) such that for \(a \in \mathfrak{M}\) and \(n \in \mathbb{N}\),

\[
\pi(\Phi^n(a)) = \mathbf{V}^n \pi(a) \mathbf{V}^n.
\]

Since the covariant representation is faithful and normal, we identify the von neuman algebra \(\mathfrak{M}\) with \(\pi(\mathfrak{M})\) and in sec. 3 we construct a dilation of the quantum process \((\pi(\mathfrak{M}), \Psi)\) where \(\Psi\) is the following completely positive map \(\Psi(\pi(x)) = \pi(\Phi(x))\) for all \(n \in \mathbb{N}\).

In fact, if the triple \((\mathbf{V}, \mathcal{H}, z)\) is the minimal unitary dilation of isometry \(\mathbf{V}\), we can construct a von Neumann algebras \(\mathfrak{M} \subset \mathfrak{B}(\mathcal{H})\) with following properties: \(\mathbf{V}^* \mathfrak{M} \mathbf{V} \subset \mathfrak{M}\) and \(z^* \mathfrak{M} z = \mathfrak{M}\).

Of fundamental importance to quantum process theory, is the \(\varphi\)-adjointness properties. The dynamic \(\Phi\) admit a \(\varphi\)-adjoint (See Kunmerer Ref. [11]) relative to the normal \(\Phi\)-invariant state \(\varphi\) on \(\mathfrak{M}\), if there is a normal unital completely positive map \(\Phi \varphi : \mathfrak{M} \to \mathfrak{M}\) such that for \(a, b \in \mathfrak{M}\),

\[
\varphi(\Phi(a)b) = \varphi(a \Phi \varphi(b)).
\]
The relationship between reversible process, modular operator and \( \varphi \)-adjointness has been studied by Accardi-Cecchini in [1] and Majewski in [5].

In sec. 4 we shall prove that our dilation satisfies ergodic properties of a \( \Phi \)-invarante state \( \varphi \) on \( \mathcal{M} \) if the dynamic \( \Phi \) admit a \( \varphi \)-adjoint.

More precisely, let \((\mathcal{M}, \Theta)\) be our dilation of quantum process \((\mathcal{M}, \Phi)\), we shall prove that if

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)| = 0,
\]

for all \( a, b \in \mathcal{M} \), we have

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |\varphi(z^* X \Theta^k(Y)z) - \varphi(z^* X z)\varphi(z^* Y z)| = 0,
\]

for all \( X, Y \in \mathcal{R} \).

For generality, we will work with concrete unital C*-algebras \( \mathfrak{A} \) and unital completely positive map \( \Phi \) (briefly ucp-map). The results obtained are easily extended to the quantum process \((\mathcal{M}, \Phi)\).

Before introducing the proof about existence of dilation of discrete quantum process, it is necessary to recall the fundamental Nagy dilation theorem, subject of the next section.

2 Nagy dilation theorem

If \( V \) is a linear isometry on Hilbert space \( \mathcal{H} \), there is a triple \((\hat{V}, \hat{\mathcal{H}}, Z)\) where \( \hat{\mathcal{H}} \) is a Hilbert space, \( Z : \mathcal{H} \to \hat{\mathcal{H}} \) is a lineary isometry, while \( \hat{V} \) is an unitary operator on \( \hat{\mathcal{H}} \) such that for \( n \in \mathbb{N} \),

\[
\hat{V}^n Z = Z V^n,
\]

with the following minimal properties:

\[
\hat{\mathcal{H}} = \bigvee_{k \in \mathbb{Z}} \hat{V}^k Z \mathcal{H}.
\]

For our purposes it is useful to recall here the structure of the unitary minimal dilation of a contraction (See Fojas-Nagy Ref. [7]).

Let \( \mathcal{K} \) be a Hilbert space, by \( l^2(\mathcal{K}) \) we denote the Hilbert space \( \{ \xi : \mathbb{N} \to \mathcal{K} : \sum_{n \geq 0} ||\xi(n)||^2 < \infty \} \).

We now get the orthogonal projection \( F = I - VV^* \) and the following Hilbert space \( \hat{\mathcal{H}} = \mathcal{H} \oplus l^2(F\mathcal{H}) \) and define the following unitary operator on the Hilbert space \( \hat{\mathcal{H}} \):

\[
\hat{V} = \begin{pmatrix} V & F \Pi_0 \end{pmatrix},
\]

where for each \( j \in \mathbb{N} \) we have set with \( \Pi_j : l^2(F\mathcal{H}) \to \mathcal{H} \) the canonical projections:

\[\Pi_j(\xi_0, \xi_1...\xi_n...) = \xi_j,\]

while \( W : l^2(F\mathcal{H}) \to l^2(F\mathcal{H}) \) is the linear operator

\[W(\xi_0, \xi_1...\xi_n...) = (\xi_1, \xi_2...),\]

for all \( (\xi_0, \xi_1...\xi_n...) \in l^2(F\mathcal{H}) \).

If \( Z : \mathcal{H} \to \hat{\mathcal{H}} \) is the isometry defined by \( Z\Psi = \Psi \oplus 0 \) for all \( \Psi \in \mathcal{H} \), it’s simple to prove that the relationships [1] and [2] are given.

We observe that for each \( n \in \mathbb{N} \) we have

\[
\hat{V}^n = \begin{pmatrix} V^n & C(n) \end{pmatrix},
\]

(3)
where \( C(n) : L^2(\mathcal{H}) \to \mathcal{K} \) are the following operators:

\[
C(n) = \sum_{j=1}^{n} V^{n-j} F \Pi_{j-1}, \quad n \geq 1.
\]

Furthermore, for each \( n, m \in \mathbb{N} \) we obtain:

\[
\Pi_n W^m = \Pi_{n+m} \quad \text{and} \quad \Pi_n W^m = \begin{cases} 
\Pi_{n-m} & n \geq m \\
0 & n < m
\end{cases},
\]

since

\[
W^m (\xi_0, \xi_1, ..., \xi_n) = (0, 0, ..., 0, \xi_0, \xi_1, ..., \xi_n),
\]

while for each \( k \) and \( p \) natural number, we obtain:

\[
\Pi_p C(k)^* = \begin{cases} 
FV^{(k-p-1)} & k > p \\
0 & \text{elsewhere}
\end{cases}
\]

since

\[
C(k)^* \Psi = (FV^{(k-1)} \Psi, ..., FV \Psi, F \Psi, 0, 0, ...)
\]

for all \( \Psi \in \mathcal{H} \).

### 3 Invariant algebra

Let be \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) a C*-algebras with unit and \( V \) an isometry on Hilbert space \( \mathcal{H} \) such that

\[
V^* \mathfrak{A} V \subset \mathfrak{A}.
\]

If \((\hat{V}, \hat{\mathcal{H}}, Z)\) denotes the minimal unitary dilation of the isometry \( V \) we shall prove the following proposition:

**Proposition 1.** There exists a C*-algebra with unit \( \hat{\mathfrak{A}} \subset \mathcal{B}(\hat{\mathcal{H}}) \) such that:

1. \( \mathfrak{A} \subset \hat{\mathfrak{A}} \subset \mathfrak{A} \) and \( Z^* \mathfrak{A} Z \subset \mathfrak{A} \),
2. \( V^* \mathfrak{A} V \subset \mathfrak{A} \),
3. \( Z^* V^* X V Z = V^* Z X V \), \quad for all \( X \in \mathfrak{A} \),
4. \( Z^* V^* (Z \mathfrak{A} Z^*) V = V^* A V \), \quad for all \( A \in \mathfrak{A} \).

first of all we want to consider some special operators on Hilbert space \( \mathcal{H} \).

#### 3.1 The gamma operators associated to pair \((\mathfrak{A}, V)\)

The sequences of elements of type \( \alpha = (n_1, n_2, ..., n_r, A_1, A_2, ..., A_r) \), with \( n_j \in \mathbb{N} \) and \( A_j \in \mathfrak{A} \) for all \( j = 1, 2, ..., r \), are called strings of \( \mathfrak{A} \) of length \( r \) and weight \( \sum_{i=1}^{r} n_i \).

For each \( \alpha \) string of \( \mathfrak{A} \), we associate the following operators of \( \mathcal{B}(\mathcal{H}) \):

\[
|\alpha\rangle = A_1 V^{n_1} \cdots A_r V \quad \text{and} \quad (\alpha) = V^{n_r} A_r \cdots V^{n_1} A_1,
\]

furthermore \( \hat{\alpha} = \sum_{i=1}^{r} n_i \) and \( l(\alpha) = r \), while \( |n\rangle \) denote the set operators \( |\alpha\rangle \) with \( \hat{\alpha} = n \) and usually

\[
|n\rangle \mathfrak{A} = \left\{|\alpha\rangle A : A \in \mathfrak{A} \right\} \quad \text{with} \quad \hat{\alpha} = n.
\]

The symbols \( |n\rangle \) and \( \mathfrak{A} |n\rangle \) have the same obvious meaning of above.
Proposition 2. Let $\alpha$ and $\beta$ be strings of $\mathfrak{A}$ for each $R \in \mathfrak{A}$ we have:

$$\langle \alpha \mid R \mid \beta \rangle \in \begin{cases} \mathfrak{A} \left( \alpha - \beta \right) & \text{if } \alpha \geq \beta \\ \left| \beta - \alpha \right| \mathfrak{A} & \text{if } \alpha < \beta \end{cases},$$

(7)

and with a simple calculation

$$|\alpha \rangle R |\beta \rangle \in |\alpha + \beta \rangle.$$  

(8)

Proof. For each $m, n \in \mathbb{N}$ and $R \in \mathfrak{A}$ we have:

$$V^m R V^n \in \begin{cases} V^{(m-n)} \mathfrak{A} & m \geq n \\ \mathfrak{A} V^{(n-m)} & m < n \end{cases}$$

(9)

Let $\alpha = (m_1, m_2, \ldots, m_r, A_1, A_2, \ldots, A_r)$ and $\beta = (n_1, n_2, \ldots, n_s, B_1, B_2, \ldots, B_s)$ strings of $\mathfrak{A}$, we obtain:

$$\langle \alpha \mid R \mid \beta \rangle = V^{m_r} A_r \cdots V^{m_1} A_1 R B_1 V^{n_s} \cdots B_s V^{n_1} = (\tilde{\alpha} | I | \tilde{\beta})$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are strings of $\mathfrak{A}$ with $l(\tilde{\alpha}) + l(\tilde{\beta}) = l(\alpha) + l(\beta) - 1$. Moreover if $\alpha \geq \beta$ we have $\tilde{\alpha} \geq \tilde{\beta}$ while if $\alpha < \beta$ it follows that $\tilde{\alpha} < \tilde{\beta}$.

In fact if $m_1 \geq n_1$ we obtain:

$$\langle \alpha \mid R \mid \beta \rangle = V^{m_r} A_r \cdots V^{m_2} A_2 R_1 V^{n_s} \cdots V^{n_2} B_2 \cdots B_s V^{n_1} = (\tilde{\alpha} | I | \tilde{\beta})$$

where $R_1 = V^{n_1} A_1 R B_1 V^{n_1}$, $\tilde{\alpha} = (m_1 - n_1, m_2, \ldots, m_r, R_1, A_2, \ldots, A_r)$ and $\tilde{\beta} = (n_2, \ldots, n_s, B_2, \ldots, B_s)$.

If $m_1 < n_1$ we can write:

$$\langle \alpha \mid R \mid \beta \rangle = V^{m_r} A_r \cdots V^{m_2} A_2 R_1 V^{n_1} A_1 R B_1 V^{n_1} \cdots B_s V^{n_1} = (\tilde{\alpha} | I | \tilde{\beta})$$

where $R_1 = V^{n_1} A_1 R B_1 V^{n_1}$, $\tilde{\alpha} = (m_2, \ldots, m_r, A_2, \ldots, A_r)$ and $\tilde{\beta} = (n_1 - m_1, n_2, \ldots, n_s, R_1, B_2, \ldots, B_s)$. Then by induction on number $\nu = l(\alpha) + l(\beta)$ we have the relationship

$$\blacksquare$$

For each string $\alpha$ of $\mathfrak{A}$ with $\alpha \geq 1$, we define the linear operators:

$$\Gamma(\alpha) = (\alpha | \mathbf{F} \Pi_{\alpha-1}^+)$$

that will be the gamma associated operators to the pair $(\mathfrak{A}, V)$.

Proposition 3. For each $\alpha$ and $\beta$ strings of $\mathfrak{A}$ with $\alpha, \beta \geq 1$, the gamma operators associated to $(\mathfrak{A}, V)$ satisfy the following relationship:

$$\Gamma(\alpha) \cdot \Gamma(\beta)^* \in \mathfrak{A}.$$  

Proof. We obtain:

$$\Gamma(\alpha) \cdot \Gamma(\beta)^* = (\alpha | \mathbf{F} \Pi_{\alpha-1}^+ \mathbf{F}^* \beta) = \begin{cases} (\alpha | F | \beta) & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases},$$

in fact

$$\langle \alpha \mid F \mid \beta \rangle = \langle \alpha \mid (I - VV^*) \mid \alpha \rangle = \langle \alpha \mid I \mid \alpha \rangle - \langle \alpha \mid VV^* \mid \alpha \rangle \in \mathfrak{A},$$

since we have $\langle \alpha \mid V \in |\alpha - 1\rangle$ while $V^* \mid \alpha \rangle \in |\alpha - 1\rangle$ and by relationship it follows that:

$$\left(\alpha - 1 \mid I \mid \alpha - 1\right) \in \mathfrak{A}.$$  

$$\blacksquare$$
We have an operator system \( \Sigma \) of \( \mathcal{B}(l^2(F^H)) \) this is:

\[
\Sigma = \{ T \in \mathcal{B}(l^2(F^H)) : \Gamma_1 T \Gamma_2 \in \mathfrak{A} \text{ for all gamma operators } \Gamma, \text{ associated to } (\mathfrak{A}, V) \}. \quad (10)
\]

We observe that \( I \in \Sigma \) and \( \Gamma_1^* \Gamma_2 \in \Sigma \) for all gamma operators \( \Gamma_i \). Moreover \( \Sigma \) is a norm closed, while it is a weakly closed if \( \mathfrak{A} \) is a W*-algebra.

**3.2 The napla operators**

For each \( \alpha, \beta \) strings of \( \mathfrak{A} \), \( A, \alpha, \beta \) and \( k \in \mathbb{N} \) we define the napla operators of \( \mathcal{B}(l^2(F^H)) \):

\[
\Delta_k(A, \alpha, \beta) = \Pi_{\alpha+k}^* F(\alpha) A(\beta) \Pi_{\beta+k}^*.
\]

For each \( h, k \geq 0 \) we obtain the following results:

\[
\Delta_k(A, \alpha, \beta)^* = \Delta_k(A^*, \beta, \alpha),
\]

and

\[
\Delta_k(A, \alpha, \beta) \Delta_h(B, \gamma, \delta) = \begin{cases} 0 & k + \beta \neq h + \gamma, \\
\Delta_k(R, \alpha, \vartheta) & k + \beta = h + \gamma, \quad h - k \geq 0, \text{ with } \vartheta = \delta + h - k \text{ and } R \in \mathfrak{A} \\
\Delta_h(R, \vartheta, \delta) & k + \beta = h + \gamma, \quad k - h > 0, \text{ with } \vartheta = \delta + k - h \text{ and } R \in \mathfrak{A}
\end{cases}
\]

In fact we have:

\[
\Delta_k(A, \alpha, \beta) \cdot \Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^* F(\alpha) A(\beta) \Pi_{\beta+k}^* \Pi_{\gamma+h}^* F(\gamma) B(\delta) \Pi_{\delta+h}^*
\]

and if \( k + \beta \neq h + \gamma \) follows that \( \Pi_{\beta+k}^* \Pi_{\gamma+h}^* = 0 \), while if \( k + \beta = h + \gamma \), without losing generality we can get \( h \geq k \), and we obtain \( \beta = \gamma + h - k \geq \gamma \). Moreover by relationship \( \mathbb{I} \)

\[
(\beta \mid F \mid \gamma) \in \mathfrak{A} \left( \beta - \gamma \right)
\]

then

\[
A(\beta \mid F \mid \gamma) B(\delta) \in \mathfrak{A} \left( \delta + \beta - \gamma \right).
\]

there exists \( \vartheta \) string of \( \mathfrak{A} \) with \( \vartheta = \hat{\vartheta} + \hat{\beta} - \hat{\gamma} \) and a \( R \in \mathfrak{A} \) such that:

\[
A(\beta \mid F \mid \gamma) B(\delta) = R(\vartheta).
\]

Since \( \hat{\vartheta} = \delta + h - k \) we have:

\[
\Delta_k(A, \alpha, \beta) \cdot \Delta_h(B, \gamma, \delta) = \Pi_{\alpha+k}^* F(\alpha) R(\vartheta) \Pi_{\delta+h}^* = \Pi_{\alpha+k}^* F(\alpha) R(\vartheta) \Pi_{\delta+h}^* = \Delta_k(R, \alpha, \vartheta).
\]

**Proposition 4.** The linear space \( \mathfrak{X}_\alpha \) generated by napla operators, is a *-subalgebra of \( \mathcal{B}(l^2(F^H)) \) included in the operator systems \( \Sigma \) defined in \( \mathbb{I} \).

**Proof.** From relationship \( \mathbb{I} \) the linear space \( \mathfrak{X}_\alpha \) is a *-algebra. Moreover for each gamma operators \( \Gamma(\alpha) \) and \( \Gamma(\beta) \) we obtain:

\[
\Gamma(\alpha) \Delta_k(A, \gamma, \delta) \Gamma(\beta)^* = (\alpha \mid F \Pi_{\alpha-1} \Pi_{\gamma+k}^* F \mid \gamma) A(\beta) \Pi_{\delta+k}^* \Pi_{\beta-1} F \mid \beta) \in \mathfrak{A},
\]

since by the relationships \( \mathbb{I} \) and \( \mathbb{S} \) we have:

\[
(\alpha \mid F \Pi_{\alpha-1} \Pi_{\gamma+k}^* F \mid \gamma) A(\beta) \Pi_{\delta+k}^* \Pi_{\beta-1} F \mid \beta) \in \begin{cases} (k + 1) \mathfrak{A} \mid k + 1 & \hat{\alpha} - 1 = \gamma + k, \hat{\beta} - 1 = \delta + k \\
0 & \text{elsewhere}
\end{cases}
\]
In fact if \( \hat{\alpha} = \hat{\gamma} + k + 1 \) we can write:

\[
(\alpha|\text{FI}_{\alpha_1}\cdots\text{FI}_{\alpha_k}|\gamma) = (\alpha|F|\gamma) = (\alpha|I|\gamma) - (\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1)
\]

since

\[
(\alpha|I|\gamma) \in \mathfrak{A}(k+1) \quad \text{and} \quad (\alpha|VV^*|\gamma) \in \mathfrak{A}(k+1)
\]

while if \( \hat{\beta} = \hat{\delta} + k + 1 \) we obtain

\[
(\delta|\text{FI}_{\delta+k}\cdots\text{FI}_{\delta-1}|\beta) \in (k+1)\mathfrak{A}.
\]

**Corollary 1.** The \(*\)-algebra \( \hat{\mathfrak{X}} \) and the operator systems \( \Sigma \) are \( \mathbb{W} \)-invariant:

\[
\mathbb{W}^*\mathfrak{X}_o\mathbb{W} \subset \hat{\mathfrak{X}} \quad \text{and} \quad \mathbb{W}^*\Sigma\mathbb{W} \subset \Sigma.
\]

**Proof.** Let be \( T \) belong to \( \Sigma \), for each gamma operators \( \Gamma(\alpha) \) and \( \Gamma(\beta) \) we have:

\[
\Gamma(\alpha)(\mathbb{W}^*TW)\Gamma(\beta)^* = (\alpha|\text{FI}_{\alpha_1}\cdots\text{FI}_{\alpha_{m-1}}\text{FI}_{\alpha_k}|\beta) = (\alpha|\text{FI}_{\alpha_1}\cdots\text{FI}_{\alpha_k}|\beta) \in \mathbb{A}V^*\Gamma_1(\alpha_o)T\Gamma_2(\beta_o)V\mathbb{A} \subset \mathbb{W}^*\mathfrak{A}\mathbb{V} \subset \mathfrak{A}.
\]

where \( \alpha_o \) and \( \beta_o \) are strings of \( \mathfrak{A} \) with \( \alpha_o = \hat{\alpha} - 1 \) and \( \beta_o = \hat{\beta} - 1 \).

In fact let \( \alpha = (m_1,m_2,...,m_r,A_1,A_2,...,A_r) \) by definition of gamma operator, there is \( i \leq r \) with \( m_i \geq 1 \) such that

\[
(\alpha|\text{FI}_{\alpha_{m_1}}\cdots\text{FI}_{\alpha_{m_i-1}}\text{FI}_{\alpha_k}|\beta) = (\alpha_o|\text{FI}_{\alpha_{m_1}}\cdots\text{FI}_{\alpha_{m_i-1}}\text{FI}_{\alpha_k}|\beta) \in \mathbb{A}V^*\Gamma_1(\alpha_o)T\Gamma_2(\beta_o)V\mathbb{A} \subset \mathbb{W}^*\mathfrak{A}\mathbb{V} \subset \mathfrak{A}.
\]

where \( \alpha_o = (0,0,m_1-1,m_{i+1},...,m_r,A_1,A_2,...,A_r) \) with \( \alpha_o = \hat{\alpha} - 1 \).

Let \( \hat{\mathfrak{X}} \) be the closure in norm of the \(*\)-algebra \( \mathfrak{X}_o \). Since \( \Sigma \) is a norm closed set, we have \( \mathfrak{X} \subset \Sigma \) while if \( \mathfrak{A} \) is a von Neumann algebra of \( \mathfrak{B}(\mathfrak{H}) \) then \( \Sigma \) is weakly closed and \( \mathfrak{X}_o^\prime \subset \Sigma \).

**Proposition 5.** The set

\[
\mathfrak{S} = \left\{ \begin{array}{c} A \\ \Gamma_1 \\ T \end{array} \right| A \in \mathfrak{A}, \; T \in \mathfrak{X} \; \text{and} \; \Gamma_i \; \text{are gamma op. of } (\mathfrak{A}, \mathfrak{V}) \right\}, \tag{12}
\]

is an operator system of \( \mathfrak{B}(\mathfrak{H}) \) such that:

\[
\hat{\mathbb{V}}^*\mathfrak{S}\hat{\mathbb{V}} \subset \mathfrak{S}.
\]

Furthermore

\[
\hat{\mathbb{V}}^*C^*(\mathfrak{S})\hat{\mathbb{V}} \subset C^*(\mathfrak{S}),
\]

where \( C^*(\mathfrak{S}) \) is the \( C^* \)-algebra generated by the set \( \mathfrak{S} \).

**Proof.** We obtain:

\[
\hat{\mathbb{V}}^*\mathfrak{S}\hat{\mathbb{V}} = \begin{vmatrix} V^*AV & V^*AC(1) + V^*\Gamma_1W \\ C(1)^*AV + W^*\Gamma_2V & C(1)^*AC(1) + W^*\Gamma_2C(1) + C(1)^*\Gamma_1W + W^*TW \end{vmatrix},
\]

where the operators \( V^*\Gamma(\alpha)W \) and \( V^*AC(1) \) are gamma operators associated to pair \( (\mathfrak{A}, \mathfrak{V}) \), while \( C(1)^*AC(1), C(1)^*\Gamma(\alpha)W, \) and \( W^*TW \) are operators belonging to \( \mathfrak{X} \).

In fact we have the following relationships:

\[
V^*AC(1) = V^*AFI_0 = \Gamma(\theta) \quad \text{with} \quad \theta = (1,A).
\]
while if α = (m₁, m₂,..,m_r, A₁, A₂,..A_r) we obtain:

\[ V^* Γ'(a) W = V^* (α| F\Pi_{α-1} W = Γ'(b), \]

with θ = (m₁ + 1, m₂,..,m_r, A₁, A₂,..A_r) since \( Π_{α-1} W = Π_{α} \).

Furthermore

\[ C(1)^* A C(1) = Π_{α}^0 F Π_{α-1} W = Π_{α}^0 F |γ⟩ (a| F \Pi_{α+0} = Δ_0 (I,γ,α) \]

with α = β = (0, I).

We observe that the *-algebra \( A^*(S) \) generated by the operator system \( S \) is given by

\[ A^*(S) = \begin{vmatrix} A & \mathcal{A}X \\ \mathcal{X}^* \mathcal{A} & X \end{vmatrix}. \]  

(13)

Now we can easily prove proposition II

**Proof.** We get \( C^*(S) \), the \( C^* \)-algebra generated by \( S \) defined in (12) by the definition \( \mathbb{Z}\mathbb{A}z^* \subset S \) then

\[ \mathbb{Z} C^*(S) \mathbb{Z} \subset \mathbb{A}. \]

Moreover for \( X \in C^*(S) \) we have:

\[ \mathbb{Z}^* \mathbb{V}^* X \mathbb{V} \mathbb{Z} = V^* X Z V, \]

since \( \mathbb{V} \mathbb{Z} = \mathbb{Z} \mathbb{V} \).

Let be \( \mathbb{S} \) the family of \( C^* \)-subalgebras \( \mathbb{B} \) with unit of \( C^*(S) \) such that \( \mathbb{Z}\mathbb{A}z^* \subset \mathbb{B} \) and \( \mathbb{V}^* \mathbb{B} \mathbb{V} \subset \mathbb{S} \). The family \( \mathbb{S} \) with inclusion is partially ordered set, then for Zorn lemma’s exists a minimal element that we shall denote with \( \mathbb{A}z \).

\[ \square \]

### 4 Stinespring’s theorem and dilations

We examine a concrete \( C^* \)-algebra \( \mathbb{A} \) of \( \mathcal{B} \)(\( H \)) with unit and an ucp-map \( \Phi : \mathbb{A} \to \mathbb{A} \). By the Stinespring theorem for the ucp-map \( \Phi \), we can deduce a triple \( (\mathbb{V}_\Phi, \sigma_\Phi, \mathcal{L}_\Phi) \) constituted by a Hilbert space \( \mathcal{L}_\Phi \), a representation \( \sigma_\Phi : \mathbb{A} \to \mathcal{B}(\mathcal{L}_\Phi) \) and a linear contraction \( \mathbb{V}_\Phi : \mathcal{K} \to \mathcal{L}_\Phi \) such that for \( \mathbb{a} \),

\[ \Phi(\mathbb{a}) = \mathbb{V}_\Phi^* \sigma_\Phi(\mathbb{a}) \mathbb{V}_\Phi. \] 

(14)

We recall that on the algebraic tensor \( \mathbb{A} \otimes \mathcal{K} \) we can define a semi-inner product by

\[ \langle a_1 \otimes \Psi_1, a_2 \otimes \Psi_2 \rangle_\mathcal{K} = \langle \Psi_1, \Phi(a_1^* a_2) \Psi_2 \rangle_{\mathcal{K}}, \]

for all \( a_1, a_2 \in \mathbb{A} \) and \( \Psi_1, \Psi_2 \in \mathcal{K} \) furthermore the Hilbert space \( \mathcal{L}_\Phi \) is the completion of the quotient space \( \mathbb{A}\otimes_{\Phi} \mathcal{K} \) of \( \mathbb{A} \otimes \mathcal{K} \) by the linear subspace

\[ \{ X \in \mathbb{A} \otimes \mathcal{K} : \langle X, X \rangle_{\mathcal{K}} = 0 \} \]

with inner product induced by \( \langle , \rangle_{\mathcal{K}} \). We shall denote the image at \( a \otimes \Psi \in \mathbb{A} \otimes \mathcal{K} \) in \( \mathbb{A}\otimes_{\Phi} \mathcal{K} \) by \( a\otimes_{\Phi} \Psi \), so that we have

\[ \langle a_1 \otimes_{\Phi} \Psi_2, a_2 \otimes_{\Phi} \Psi_2 \rangle_{\mathcal{L}_\Phi} = \langle \Psi_1, \Phi(a_1^* a_2) \Psi_2 \rangle_{\mathcal{K}}, \]

for all \( a_1, a_2 \in \mathbb{A} \) and \( \Psi_1, \Psi_2 \in \mathcal{K} \).

Moreover \( \sigma_{\Phi}(a) (x\otimes_{\Phi} \Psi) = ax \otimes_{\Phi} \Psi \), for each \( x\otimes_{\Phi} \Psi \in \mathcal{L}_\Phi \) and \( \mathbb{V}_\Phi \Psi = \mathbb{1}\otimes_{\Phi} \Psi \) for each \( \Psi \in \mathcal{K} \).

Since \( \Phi \) is unital map, the linear operator \( \mathbb{V}_\Phi \) is an isometry with adjoint \( \mathbb{V}_\Phi^* \) defined by

\[ \mathbb{V}_\Phi^* a\otimes_{\Phi} \Psi = \Phi(a) \Psi, \]

for all \( a \in \mathbb{A} \) and \( \Psi \in \mathcal{K} \).
for all \( a \in \mathfrak{A} \) and \( \Psi \in \mathcal{K} \).

We recall that the multiplicative domain of the ucp-map \( \Phi : \mathfrak{A} \to \mathfrak{A} \) is the C*-subalgebra of \( \mathfrak{A} \) such defined:

\[
\mathcal{D}_\Phi = \{ a \in \mathfrak{A} : \Phi(a^*)\Phi(a) = \Phi(a^*a) \text{ and } \Phi(a^*)\Phi(a) = \Phi(a^*a) \},
\]

we have the following implications (See Paulsen Ref.\([3]\)):

\( a \in \mathcal{D}_\Phi \) if and only if \( \Phi(a^*)\Phi(x) = \Phi(ax) \) and \( \Phi(x)\Phi(a) = \Phi(xa) \) for all \( x \in \mathfrak{A} \).

**Proposition 6.** The ucp-map \( \Phi \) is a multiplicative if and only if \( \mathbf{V}_\Phi \) is an unitary. Moreover if \( x \in \mathcal{D}(\Phi) \) we have:

\[
\sigma_\Phi(x) \mathbf{V}_\Phi \mathbf{V}_\Phi^* = \mathbf{V}_\Phi \mathbf{V}_\Phi^* \sigma_\Phi(x).
\]

**Proof.** For each \( \Psi \in \mathcal{K} \) we obtain the following implications:

\[
a^* \mathbf{V}_\Phi \Phi(a) \Psi = \mathbf{1} \mathbf{V}_\Phi \Phi(a) \Psi \iff \Phi(a^*) = \Phi(a^*) \Phi(a),
\]

since

\[
\|a^* \mathbf{V}_\Phi \Phi(a) \Psi - \mathbf{1} \mathbf{V}_\Phi \Phi(a) \Psi\| = \langle \Psi, \Phi(a^*) \Psi \rangle - \langle \Psi, \Phi(a^*) \Phi(a) \Psi \rangle.
\]

Furthermore, for each \( a \in \mathfrak{A} \) and \( \Psi \in \mathcal{K} \) we have \( \mathbf{V}_\Phi \mathbf{V}_\Phi^* a^* \mathbf{V}_\Phi \Phi(a) \Psi = \mathbf{1} \mathbf{V}_\Phi \Phi(a) \Psi. \quad \square \)

Now we prove the following Stinespring-type theorem (See Zsido Ref.\([11]\)):

**Proposition 7.** Let \( \mathfrak{A} \) be a concrete C*-subalgebra with unit of \( \mathfrak{B}(\mathcal{K}) \) and \( \Phi : \mathfrak{A} \to \mathfrak{A} \) an ucp-map, then there exists a faithful representation \((\pi_\infty, \mathcal{K}_\infty)\) of \( \mathfrak{A} \) and an isometry \( \mathbf{V}_\infty \) on Hilbert Space \( \mathcal{K}_\infty \) such that for each \( a \in \mathfrak{A} \),

\[
\mathbf{V}_\infty^* \pi_\infty(a) \mathbf{V}_\infty = \pi_\infty(\Phi(a)),
\]

where

\[
\sigma_0 = \text{id}, \quad \Phi_n = \sigma_n \circ \Phi
\]

and \((\mathbf{V}_n, \sigma_n, \mathcal{K}_{n+1})\) is the Stinespring dilation of \( \Phi_n \) for every \( n \geq 0 \),

\[
\mathcal{K}_\infty = \bigoplus_{j=0}^\infty \mathcal{K}_j, \quad \mathcal{K}_j = \mathfrak{A} \mathbf{V}_\Phi \mathcal{K}_{j-1}, \quad \text{for } j \geq 1 \text{ and } \mathcal{K}_0 = \mathcal{K};
\]

and

\[
\mathbf{V}_\infty(\Psi_0, \Psi_1, \Psi_2, \ldots) = (0, \mathbf{V}_0 \Psi_0, \mathbf{V}_1 \Psi_1, \ldots)
\]

for each \((\Psi_0, \Psi_1, \Psi_2, \ldots) \in \mathcal{K}_\infty\).

Furthermore the map \( \Phi \) is a homomorphism if and only if \( \mathbf{V}_\infty \mathbf{V}_\infty^* \in \pi_\infty(\mathfrak{A})' \).

**Proof.** By the Stinespring theorem there is triple \((\mathbf{V}_0, \sigma_1, \mathcal{K}_1)\) such that for each \( a \in \mathfrak{A} \) we have \( \Phi(a) = \mathbf{V}_0^* \sigma_1(a) \mathbf{V}_0 \). The application \( a \in \mathfrak{A} \to \sigma_1(\Phi(a)) \in \mathfrak{B}(\mathcal{K}_1) \) is a composition of cp-maps therefore it is also a cp map. Set \( \Phi_1(a) = \sigma_1(\Phi(a)) \). By applying the Stinespring's theorem to \( \Phi_1 \), we have a new triple \((\mathbf{V}_1, \sigma_2, \mathcal{K}_2)\) such that \( \Phi_1(a) = \mathbf{V}_1^* \sigma_2(a) \mathbf{V}_1 \).

By induction for \( n \geq 1 \) we define \( \Phi_n(a) = \sigma_n(\Phi(a)) \) and we have a triple \((\mathbf{V}_n, \sigma_{n+1}, \mathcal{K}_{n+1})\) such that \( \mathbf{V}_n : \mathcal{K}_n \to \mathcal{K}_{n+1} \) and \( \Phi_n(a) = \mathbf{V}_n^* \sigma_{n+1}(a) \mathbf{V}_n \).

We get the Hilbert space \( \mathcal{K}_\infty \) defined in \([11]\) and the injective representation of the C*-algebra \( \mathfrak{A} \) on \( \mathcal{K}_\infty \):

\[
\pi_\infty(a) = \bigoplus_{n \geq 0} \sigma_n(a)
\]

with \( \sigma_0(a) = a \), for each \( a \in \mathfrak{A} \).

Let \( \mathbf{V}_\infty : \mathcal{K}_\infty \to \mathcal{K}_\infty \) be the isometry defined by

\[
\mathbf{V}_\infty(\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots) = (0, \mathbf{V}_0 \Psi_0, \mathbf{V}_1 \Psi_1, \ldots, \mathbf{V}_n \Psi_n, \ldots),
\]
for all $\Psi_i \in \mathcal{H}_i$ with $i \in \mathbb{N}$.

The adjoint operator of $V_\infty$ is

$$V_\infty^*(\Psi_0, \Psi_1, ..., \Psi_n...) = (V_0^*\Psi_1, V_1^*\Psi_2, ..., V_{n-1}^*\Psi_n...) \quad (19)$$

for all $\Psi_i \in \mathcal{H}_i$ with $i \in \mathbb{N}$, therefore

$$V_\infty^*\pi_\infty (a) V_\infty \bigoplus \Psi_n = \bigoplus V_n^*\sigma_{n+1} (a) V_n \Psi_n = \bigoplus \Phi_n (a) \Psi_n = \bigoplus \sigma_n (\Phi (a)) \Psi_n = \pi_\infty (\Phi (a)) \bigoplus \Psi_n.$$

We notice that $E_n = V_n V_n^*$ be the orthogonal projection of $B(\mathcal{H}_{n-1})$, we have:

$$E (\Psi_0, \Psi_1...\Psi_n...) = (0, E_0\Psi_1, E_1\Psi_2,...E_n\Psi_{n+1}...).$$

Finally for the proof of the last statement we only need to note that $x$ belong to multiplicative domains $\mathbb{D} (\Phi)$ if and only if we have:

$$\pi_\infty (x) V_\infty V_\infty^* = V_\infty V_\infty^* \pi_\infty (x).$$

\[ \square \]

Remark 1. Let $(\mathfrak{M}, \Phi)$ be a quantum process, the representation $\pi_\infty (a) : \mathfrak{M} \rightarrow B(\mathcal{H}_\infty)$ defined in proposition \ref{prop:stinespring} is normal, since the Stinespring representation $\sigma_\Phi : \mathfrak{A} \rightarrow B(L_\Phi)$ is a normal map. Then $(\pi_\infty, \mathcal{H}_\infty, V_\infty)$ is a covariant representation of quantum process.

### 4.1 Dilations of ucp-Maps

If $(\mathcal{H}_\infty, \pi_\infty, V_\infty)$ is the Stinespring representation of proposition \ref{prop:stinespring} we have that $V_\infty^*\pi_\infty (\mathfrak{A}) V_\infty \subset \pi_\infty (\mathfrak{A})$ and by proposition \ref{prop:closedness} there exists a $C^*$-algebra with unit of $B (\mathcal{H})$ such that:

1. $\mathfrak{A} \subset \mathcal{A};$
2. $\mathfrak{A}^* \hat{\mathfrak{A}} = \pi_\infty (\mathfrak{A});$
3. $\hat{\mathfrak{A}}^* X \hat{\mathfrak{A}} = V \pi_\infty (Z^* X Z) V,$ for all $X \in \hat{\mathfrak{A}}.$

Furthermore, we have a homomorphism $\hat{\Phi} : \hat{\mathfrak{A}} \rightarrow \hat{\mathfrak{A}}$ thus defined

$$\hat{\Phi}(X) = \hat{\Phi}^* X \hat{\Phi}$$

for all $X \in \hat{\mathfrak{A}},$ such that for $A \in \mathfrak{A},$ $X \in \hat{\mathfrak{A}}$ and $n \in \mathbb{N}$ we have:

$$\Phi^n(A) = Z^* \hat{\Phi}^n(Z A Z^*)Z$$

and

$$Z^* \hat{\Phi}^n(X) Z = \Phi^n(Z^* X Z).$$

The quadruple $(\hat{\Phi}, \hat{\mathfrak{A}}, \mathcal{H}, Z)$ with the above properties, is said to be a multiplicative dilation of ucp-map $\Phi : \mathfrak{A} \rightarrow \mathfrak{A}.$

Remark 2. It is clear that these results are easily extended to the von Neumann algebras $\mathfrak{M}$ with $\Phi$ normal ucp-map. In this way we obtain a dilation of discrete quantum process $(\mathfrak{M}, \Phi)$.

### 5 Ergodic properties

Let $\mathfrak{A}$ be a concrete $C^*$-algebra of $B (\mathcal{H})$ with unit, $\Phi : \mathfrak{A} \rightarrow \mathfrak{A}$ an ucp-map and $\varphi$ a state on $\mathfrak{A}$ such that $\varphi \circ \Phi = \varphi.$ We recall (See N.S.Z. Ref.\cite{Ref}) that the state $\varphi$ is a ergodic state, relative to the ucp-map $\Phi$, if

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^{n} [\varphi(a\Phi^k(b)) - \varphi(a)\varphi(b)] = 0.$$
for all \( a, b \in \mathfrak{A} \), while is weakly mixing if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} |\varphi(a \Phi^k(b)) - \varphi(a)\varphi(b)| = 0,
\]

for all \( a, b \in \mathfrak{A} \).

We observe that by the Stinepring-type theorem \[2\] we can assume, without losing generality, that \( \mathfrak{A} \) is a concrete C*-algebra of \( \mathfrak{B} (\mathcal{H}) \), and that there is a linear isometry \( V \) on \( \mathcal{H} \) such that:

\[
\Phi (A) = V^* AV \text{ for all } A \in \mathfrak{A}.
\]

Then \( \left( \hat{V}, \hat{\mathcal{H}}, \hat{Z} \right) \) is the minimal unitary dilation of \( (V, \mathcal{H}) \) and the C*-algebra \( \hat{\mathfrak{A}} \) defined in proposition \[1\] is included in \( \mathfrak{B} (\hat{\mathcal{H}}) \).

We want to prove the following ergodic theorem, for dilation ucp-map \( (\hat{\Phi}, \hat{\mathfrak{A}}, \hat{\mathcal{H}}, \hat{Z}) \) previously defined:

**Proposition 8.** If the ucp-map \( \Phi \) admits a \( \varphi \)-adjoint and \( \varphi \) is a ergodic state, we obtain:

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi(Z^* X \hat{V}^k Y \hat{V}^k Z) = \varphi(Z^* X Z \varphi(Z^* Y Z)) = 0,
\]

while if \( \varphi \) is weakly mixing:

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} |\varphi(Z^* X \hat{V}^k Y \hat{V}^k Z) - \varphi(Z^* X Z)\varphi(Z^* Y Z)| = 0,
\]

for all \( X, Y \in \hat{\mathfrak{A}} \).

If we write every element \( X \) of \( \mathfrak{B} (\hat{\mathcal{H}}) \) in matrix form \( X = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix} \) with \( \hat{\mathcal{H}} = \mathcal{H} \oplus l^2 (\mathcal{H}) \) we obtain:

\[
\varphi(Z^* X \hat{V}^k Y \hat{V}^k Z) = \varphi(X_{1,1} V^k Y_{1,1} V^k) + \varphi(X_{1,2} C (k)^* Y_{1,1} V^k) + \varphi(X_{1,2} W^k Y_{2,1} V^k)
\]

and the proof of previous proposition is an easy consequence of the following lemma:

**Lemma 1.** Let \( X \in \mathcal{A}^*(\mathcal{S}) \), the *-algebra generated by operator system \( \mathcal{S} \) defined in \[2\] and \( Y \in \hat{\mathfrak{A}} \),

a) if \( \varphi \) is an ergodic state we have:

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C (k)^* Y_{1,1} V^k + X_{1,2} W^k Y_{2,1} V^k \right) = 0, \tag{20}
\]

b) if \( \varphi \) is weakly mixing we have:

\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \left| \varphi \left( X_{1,2} C (k)^* Y_{1,1} V^k + X_{1,2} W^k Y_{2,1} V^k \right) \right| = 0. \tag{21}
\]

**Proof.** Since \( X \in \mathcal{A}^*(\mathcal{S}) \) we can assume that \( X_{1,2} = A \Gamma (\gamma) \Delta_m (B, \alpha, \beta) \) with \( A, B \in \mathfrak{A} \) and \( \gamma \) string of \( \mathfrak{A} \). Then:

\[
X_{1,2} = A (\gamma | F)_{\alpha_m}^* \Pi^{\gamma}_{\alpha_m} F | \alpha \beta (\beta | F)_{\beta+m} = \begin{cases} A (\gamma | F) | \alpha \beta (\beta | F)_{\beta+m} & \text{if } \gamma = 1 = \alpha + m \\ 0 & \text{elsewhere} \end{cases} \tag{22}
\]

Now we observe that there is a natural number \( \kappa \), such that for each \( k > \kappa \) we obtain:

\[
X_{1,2} W^k Y_{2,1} V^k = 0
\]
In fact we have that
\[ W^{k^*} (\xi_0, \xi_1, \ldots) = \left( \frac{k}{k \text{-time}} \frac{1}{N+1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k + X_{1,2} W^{k^*} Y_{2,1} V^k \right) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k \right), \]
for all \((\xi_0, \xi_1, \ldots) \in l^2 (F\mathcal{H})\) then \(\Pi_{\beta+m} W^{k^*} = 0\) for all \(k > \beta + m\).
It follows that:
\[ \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k + X_{1,2} W^{k^*} Y_{2,1} V^k \right) = \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k \right), \]
Then we compute only the term \(\varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k \right)\) and by relationship \(22\) we can write that:
\[ X_{1,2} C \left( k^* \right) Y_{1,1} V^k = A (\gamma | F | \alpha) B (\beta | F)) \Pi_{\beta+m} C \left( k^* \right) Y_{1,1} V^k \]
moreover by relationship \(6\) for \(k > \beta + m\) we have:
\[ \Pi_{\beta+m} C \left( k^* \right) = F V^{(k-\beta-m-1)^*}, \]
it follows that
\[ X_{1,2} C \left( k^* \right) Y_{1,1} V^k = A (\gamma | F | \alpha) B (\beta | F)) F V^{(k-\beta-m-1)^*} Y_{1,1} V^k = A (\gamma | F | \alpha) B (\beta | F) \Phi^{(k-\beta-1)} (Y_{1,1}) V^{\beta+m+1}. \]
Since \(\gamma = \alpha + m + 1\), by relationship \(7\) we obtain:
\[ A (\gamma | F | \alpha) B (\beta | F) \in \mathfrak{A} \left( \beta + m + 1 \right), \]
it follows that there exists a \(\vartheta\) string of \(\mathfrak{A}\) with \(\vartheta = \beta + m + 1\) and an operator \(R \in \mathfrak{A}\), such that
\[ A (\gamma | F | \alpha) B (\beta | F) = R (\vartheta). \]
Then
\[ X_{1,2} C \left( k^* \right) Y_{1,1} V^k = R (\vartheta | F) \Phi^{(k-\beta-1)} (Y_{1,1}) V^{\beta+m+1}. \]
If we set \(\vartheta = (n_1, n_2, \ldots, n_r, A_1, A_2, \ldots, A_r)\). we have \(n_1 + n_2 + \ldots + n_r = \beta + m + 1\) and
\[ R (\vartheta | F) \Phi^{(k-\beta-1)} (Y_{1,1}) V^{\beta+m+1} = R V^{n_r} A_r V^{n_{r-1}} A_{r-1} \cdots A_2 V^{n_2} A_2 \Phi^{(k-\beta-1)} (Y_{1,1}) V^{\beta+m+1} = \]
\[ = R \Phi^{n_r} (A_r \Phi^{n_{r-1}} (A_{r-1} \cdots \Phi^{n_2} (A_2 R_k)))), \]
where
\[ R_k = \Phi^{n_r} (A_r) \Phi^{(k-\beta-1)} (Y_{1,1}) - \Phi^{n_{r-1}} (\Phi (A_r) \Phi^{(k-\beta)} (Y_{1,1})). \]
We have:
\[ \varphi \left( X_{1,2} C \left( k^* \right) Y_{1,1} V^k \right) = \varphi \left( R \Phi^{n_r} \left( A_r \Phi^{n_{r-1}} (A_{r-1} \cdots \Phi^{n_2} (A_2 R_k)))) \right) = \]
\[ = \varphi \left( \Phi_{k}^{n_r} (R) A_{k} \Phi^{n_{r-1}} (A_{r-1} \cdots \Phi^{n_2} (A_2 R_k)))) = \]
\[ = \varphi \left( \Phi_{k}^{n_{r-1}} \left( \Phi_{k}^{n_r} (R) A_{r-1} (A_{r-2} \cdots \Phi^{n_2} (A_2 R_k)) \right) = \]
\[ = \varphi \left( \Phi_{k}^{n_2} \left( \Phi_{k}^{n_3} \cdots \Phi_{k}^{n_{r-1}} \left( \Phi_{k}^{n_r} (R) A_{r-1} (A_{r-2} \cdots \Phi^{n_2} (A_2 R_k)) \right) \right) \right) \]
and replacing \(R_k\), we obtain:
\[ \Phi_{k}^{n_2} \left( \Phi_{k}^{n_3} \cdots \Phi_{k}^{n_{r-1}} \left( \Phi_{k}^{n_r} (R) A_{r-1} (A_{r-2} \cdots \Phi^{n_2} (A_2 R_k)) \right) \right) A_2 R_k = \]
11
If the state $N$

Adding and subtracting the element $\phi$

It follows that:

\[
\frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C (k)^* Y_{1,1} V^k \right) =
\]

\[
= \frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1} (A_1) \Phi^{(k-\beta-1)} (Y_{1,1})
\]

\[
- \frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1-1} (\Phi (A_1) \Phi^{(k-\beta)} (Y_{1,1}))
\].

If the state $\varphi$ is ergodic we have:

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1-1} (\Phi (A_1) \Phi^{(k-\beta)} (Y_{1,1})) =
\]

\[
= \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1} (A_1) \varphi (Y_{1,1})
\]

while

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1} (A_1) \varphi (Y_{1,1}) =
\]

\[
= \varphi \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_3 \right) A_2 \Phi^{n_1} (A_1) \varphi (Y_{1,1})
\]

then we obtain

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \varphi \left( X_{1,2} C (k)^* Y_{1,1} V^k \right) = 0.
\]

In weakly mixing case, using the previous results, we obtain:

\[
|\varphi \left( X_{1,2} C \ast Y_{1,1} V^k \right)| = \left| \varphi \left( B \Phi^{n_1} (A_1) \Phi^{(k-\beta-1)} (Y_{1,1}) \right) - \varphi \left( B \Phi^{n_1-1} \left( \Phi (A_1) \Phi^{(k-\beta)} (Y_{1,1}) \right) \right) \right|
\]

where $B = \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n \left( \Phi^{\hat{Y}}^n (R) A_r \right) \right) \cdots \right) A_2$.

Adding and subtracting the element $\varphi (B \Phi^{n_1} (A_1)) \varphi (Y_{1,1})$ we can write:

\[
\left| \varphi \left( B \Phi^{n_1} (A_1) \Phi^{(k-\beta-1)} (Y_{1,1}) \right) - \varphi \left( B \Phi^{n_1-1} \left( \Phi (A_1) \Phi^{(k-\beta)} (Y_{1,1}) \right) \right) \right| \leq
\]

\[
\left| \varphi \left( B \Phi^{n_1} (A_1) \Phi^{(k-\beta-1)} (Y_{1,1}) \right) - \varphi (B \Phi^{n_1} (A_1)) \varphi (Y_{1,1}) \right| +
\]

\[
\left| \varphi \left( B \Phi^{n_1-1} \left( \Phi (A_1) \Phi^{(k-\beta)} (Y_{1,1}) \right) \right) - \varphi (B \Phi^{n_1} (A_1)) \varphi (Y_{1,1}) \right|.
\]
Moreover
\[
\left| \varphi \left( B \Phi^{n_1-1} \left( \Phi \left( A_1 \right) \Phi \left( k-\beta \right) \left( Y_{1,1} \right) \right) \right) - \varphi \left( B \Phi^{n_1} \left( A_1 \right) \right) \varphi \left( Y_{1,1} \right) \right| =
\]
\[
= \left| \varphi \left( \Phi^{n_1-1} \left( B \right) \Phi \left( A_1 \right) \Phi \left( k-\beta \right) \left( Y_{1,1} \right) \right) - \varphi \left( \Phi^{n_1-1} \left( B \right) \Phi \left( A_1 \right) \right) \varphi \left( Y_{1,1} \right) \right|
\]
and by the weakly mixing properties we obtain:
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \left| \varphi \left( B \Phi^{n_1} \left( A_1 \right) \Phi \left( k-\beta-1 \right) \left( Y_{1,1} \right) \right) - \varphi \left( B \Phi^{n_1} \left( A_1 \right) \right) \varphi \left( Y_{1,1} \right) \right| = 0,
\]
and
\[
\lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \left| \varphi \left( \Phi^{n_1-1} \left( B \right) \Phi \left( A_1 \right) \Phi \left( k-\beta \right) \left( Y_{1,1} \right) \right) - \varphi \left( \Phi^{n_1-1} \left( B \right) \Phi \left( A_1 \right) \right) \varphi \left( Y_{1,1} \right) \right| = 0.
\]

Finally, the proof of proposition is a simple result of the previous lemma.

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