Curvature, Expansion, Kolmogorov’s Diameter, Hilbert’s Rational Designs and Overtwisted Immersions I

Misha Gromov

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Abstract

We prove the existence of locally distance increasing maps with controllably small curvatures between Riemannian manifolds, where our main construction depends on the presence of particular spherical and almost spherical sections of the unit balls in the $l_{p-4}$ spaces.

In the part II [Gr 2022] we prove similar results for $C^\infty$-smooth isometric immersions $X^m \to Y^N$, where our approach allows an improvement of the present-day bounds on the dimension $N$ of the ambient manifold $Y$ in certain cases.

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1 Introduction

Immersions are $C^1$-maps $f: X \to Y$ between smooth manifolds, such that their differentials $df: T(X) \to T(Y)$ nowhere vanish:

$$df(\tau) = 0 \implies \tau = 0, \tau \in T(X).$$

The (maximal normal bundle) curvature of an immersed $X$ in a Riemannian $Y$,

$$f: X \hookrightarrow Y,$$

is the supremum of the $Y$-curvatures of geodesics $\gamma \subset X$, for the induced Riemannian metric in $X$,

$$\text{curv}^i(X) = \text{curv}^i(f(X)) = \text{curv}^i(f) = \text{curv}^i(X \hookrightarrow Y) = \text{curv}^i(X \hookrightarrow Y),$$

Minimal Curvature Problem. What is the infimum of curvatures of immersions $f: X \hookrightarrow Y$,

$$\mincurv^i(X,Y) = \mincurv^i(X \hookrightarrow Y)?$$

Product Example. If $X$ is a product of spheres,

$$X = \bigtimes_{i=1}^{l} S^{m_i},$$

and $Y$ is the unit ball $B^N(1) \subset \mathbb{R}^N$ then (apart from the trivial case of $l = 1$) we know the exact value of $\mincurv^i(X \hookrightarrow Y, B^N(1))$ only where all $m_i = 1$, i.e. for the torus $T^l$, and where $N$ is large:

$$[\sqrt{3}]_T \text{mincurv}^i(T^l, B^N(1)) = \sqrt{3 \frac{l}{l+2}}, \quad N >> l^2.$$  

(See sections 3, 5 and [Pet2023].)

1Immersions are locally one-to-one but globally they may have self intersections. Immersions without self intersections are called embeddings, where, if $X$ is non-compact, one may require the induced topology in $X$ to be equal the original one.
But if all \( m_i = 2 \), for instance, i.e. \( X = (S^2)^l \) we neither can show that

\[
\text{min.curv}^i((S^2)^l, B^{2l+1}) \to \infty \text{ for } l \to \infty
\]

nor that

\[
\frac{\text{min.curv}^i((S^2)^l, B^{10l})}{\sqrt{l}} \to 0 \text{ for } l \to \infty.
\]

**Hand-Made Immersions**

**Clifford Embeddings.** The product \( X \) of spheres \( S^{m_i}(r_i) \subset R^{m_i+1}, i = 1, \ldots, l \), for \( \sum_{i=1}^{l} r_i^2 = 1 \) naturally isometrically imbeds to the boundary of the unit \( N \)-ball for \( N = k + \sum_{i} m_i \):

\[
\text{Cl} : X = S^{m_1}(r_1) \times \ldots \times S^{m_l}(r_l) \to S^{N-1}(1) \subset B^{N}(1) \subset R^{m_i+1} \times \ldots \times R^{m_i+1}
\]

where, clearly,

\[
\text{curv}^i(X \subset B^{N}) = \max_i 1/r_i.
\]

This, for \( r_1 = r_2 = \ldots = r_l \), delivers a codimension \( l \)-embedding with curvature \( \sqrt{l} \). Thus,

\[
\text{min.curv}^i \left( \bigtimes_{i=1}^{l} S^{m_i}(B^{N}(1)) \right) \leq \sqrt{l}, \; N = l + \sum_{i} m_i.
\]

If \( l = 1 \), then this is optimal. In fact, it is obvious that

\[
\text{curv} \left( X \rightarrow B^{m}(1) \times R^{N} \right) \geq 1, \; \text{for } n \geq 2.
\]

for all smoothly immersed closed \( m \)-manifolds \( X \) in the "unit band" \( B^m(1) \times R^N \).

But, for instance, the equality

\[
\text{min.curv}^i(T^{m} \rightarrow B^{2m}) = \sqrt{m}
\]

is problematic for all \( m \geq 2 \).

**Round \( m \)-Tori in the Unit \( (m+1) \)-Balls.**

\[
\text{min.curv}^i(T^2 \rightarrow B^3) \leq 3:
\]

the boundary of the \( \frac{1}{3} \)-neighbourhood of the circle of radius \( \frac{2}{3} \) in the space has \( \text{curv}^i(T^2 \subset R^3) = 3 \).

Similarly (see section 4.1)

\[
\text{min.curv}^i(T^3 \rightarrow B^4) \leq 2\sqrt{2} + 1 < 4
\]

\[
\text{min.curv}^i(T^7 = T^3 \times T^3 \times T^1 \rightarrow B^8) \leq 8 + 2\sqrt{2} + 1 < 12
\]

.............

\[
\text{min.curv}^i(T^m, B^{m+1}) < \frac{m}{2}, \; m = 2^k - 1.
\]

**Veronese embedding**\(^2\) of the real projective spaces satisfy (see 5.1),

\[
\text{curv} \left( \mathbb{R}P^m \rightarrow B^{m(m+1)} \right) = \sqrt{\frac{2m}{m+1}}, \; \text{e.g.}
\]

\(^2\)These are flashes from a superior world.
Conjecture.

\[ \min_cirv(X^m, B^N) < \sqrt{\frac{2m}{m+1}} \implies X = \text{diff geo } S^m. \]

1.1 Immersions with Small Curvature and \( D(m, N) \)-Approximation

Expansion. A map between metric spaces,

\( f : X \to Y, \)

is \( \lambda \)-expanding, \( \lambda > 0, \) if it increases the the length of curves \( \xi : [0 : 1] \to X \) by a factor \( \geq \lambda, \)

\[ \text{length}(f \circ \xi) \geq \lambda \cdot \text{length}(\xi) \]

for all continuous maps \( \xi : [0 : 1] \to X. \)

Continuous maps.

Expanding is an abbreviation for "1-expanding".

Riemannian Example. A \( C^1 \)-smooth map \( f \) between Riemannian manifolds, e.g. open subsets in Euclidean spaces, is \( \lambda \)-expanding if and only if

\[ ||df(\tau)|| \geq ||\lambda \tau|| \]

for all tangent vectors \( \tau \in T(X). \)

Thus, smooth expanding maps are immersion and every immersion \( f \) expands with respect to some Riemannian metrics \( g = g(f) \) in \( X \) and \( h = h(f) \) in \( Y. \)

Equidimensional example. If \( \dim(X) = \dim(Y) \) then smooth immersions \( X \hookrightarrow Y \) are local diffeomorphisms and \( C^1 \)-smooth expanding maps are locally distance increasing.

The relative (maximal) curvature of an immersion between Riemannian manifolds,

\( (X, g) \hookrightarrow (Y, h) \)

is the supremum of \( h \)-curvatures in \( Y, \) of \( g \)-geodesics \( \gamma \subset X, \)

\[ \text{curv}(f) = \text{curv}^X(f) = \text{curv}^Y(f) = \sup_{\gamma \subset X} \text{curv}_h(f(\gamma)). \]

If \( g = f^*(h) \) is the induced Riemannian metric in \( X, \) this is our curvature of \( X \) in \( Y, \)

\[ \text{curv}^X_{\lambda}(f) = \text{curv}^1(X \hookrightarrow Y). \]

(This \( \text{curv}^1(X) \) unlike \( \text{curv}(f) \) is defined for immersions of smooth manifolds with no metrics on them.)

Equidimensional example. If \( \dim(X) = \dim(Y) \), then \( \text{curv}^1(X \hookrightarrow Y) = 0, \) while \( \text{curv}^X(f) \) measures by how much \( f \) deviates from a projective map.

Normal Immersions, where \( \text{curv}^X_P(X) = \text{curv}^X(f). \) Call an immersion between Riemannian manifolds \( f : X(g) \hookrightarrow Y(h) \) normal if for all normal vectors to \( X \) in \( Y, \)

\[ \nu \in T^X_\nu(X) = T_f(x)(Y) \cap df(T_x(X)) \]

\[ \text{Expanding} \] locally homeomorphic maps are also locally distance increasing, but the absolute value map \( x \mapsto |x|, \) for example, is 1-expanding but not locally homeomorphic.
the second quadratic form $\Pi_2$ of the immersed $X \hookrightarrow \mathbb{R}^n$ is simultaneously diagonalizable with the quadratic forms $g(x)$ and $f^*(h)$ on the tangent space $T_x(X)$. For instance, isometric immersions are normal.

Clearly, $\text{curv}^l(X) = \text{curv}^N(f)$ for isometric immersions $f$.

**Curvature in Spheres.** If an immersion $X \hookrightarrow S^{N-1}(1)$ is normal then so is the corresponding immersion to $\mathbb{R}^N \supset S^{N-1}(1)$, where the spherical curvature of $X$ is related to the Euclidean one by the Pythagorean theorem:

$$(\text{curv}^i(X \hookrightarrow S^{N-1}(1))^2 = (\text{curv}^i(X \hookrightarrow \mathbb{R}^N))^2 - 1.$$  

Notice that the Clifford embeddings to the unit sphere are known to be optimal for $l = 2$,

$$\min \text{curv}^i(S^{m_1} \times S^{m_2}, S^{m_1+m_2+1}(1)) = 1, \ m_1, m_2 \geq 1,$$

but the corresponding Euclidean equality

$$\min \text{curv}^i(S^{m_1} \times S^{m_2}, B^{m_1+m_2+2}(1)) = \sqrt{2},$$

remains conjectural for all $m_1, m_2 \geq 1$, except for $m_1 = m_2 = 1$ [Pet].

**Curvature in Codimension 1.** This curvature of $X^m \to Y^{m+1}$ is the supremum of the principal curvatures of $X$ in $Y$ over all points $x \in X$.

Here normality means that the induced quadratic form $f^*(g)(x)$ on the tangent space $T_x(X)$ is, at all $x \in X$, diagonalizable in the same basis as the second fundamental form $\Pi$ of $X$.

**Example.** the immersion $S^m(r) \times S^1 \to \mathbb{R}^{m+2}$ obtained by rotating $S^m(r) \to \mathbb{R}^{m+1}$ around a line in $\mathbb{R}^{m+1}$ within distance $R > r$ from the origin is normal with curvature max$(\frac{1}{R}, \frac{1}{r})$.

**Expanding Immersions and Regular Homotopies.**

The minimal curvature problem can be refined in two ways as follows.

What is the minimal curvature of expanding immersions between given Riemannian manifolds? What is the minimal curvature in a given homotopy or regular homotopy class of immersions?

Below are partial answers to these questions.

**$\mathcal{D}(m, N)$: Curvature of Euclidean Expanding Maps.** Let $\mathcal{D}(m, N)$ be the infimum of the relative curvatures of the smooth expanding maps $f$ from the Euclidean $m$-space to the unit $N$-ball,

$$\mathcal{D}(m, N) = \inf_{f} \text{curv}_{e^m_N}(f),$$

where $e_m$ and $e_N$ denote the Euclidean metrics in $\mathbb{R}^m$ and $\mathbb{R}^N \supset B^N(1)$.

**Example.** The composition of the toral Clifford embedding $\mathbb{T}^m \to B^{2m}(1)$ with the universal covering $\mathbb{R}^m \to \mathbb{T}^m$ followed the Euclidean homothety $x \mapsto (\sqrt{m})x$ is an isometric immersion $\mathbb{R}^m \to B^{2m}(1)$ with curvature $\sqrt{m}$. Hence,

$$\mathcal{D}(m, 2m) \leq \sqrt{m}$$

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4See [Ge2021], section 3.7.3 in [Gr2022] and section 5.5 in the present paper.

5A $C^1$-continuous homotopy $f_t$ of smooth maps is regular if the maps $f_t$ are immersions for all $t$. 

5
Question. Is $D(m, 2m)$ equal to $\sqrt{m}$?

1.1.A. Euclidean $D(m, N)$-Theorem.

• $\leq 2m$ If $N \geq 2m$, then
  \[ D(m, N) \leq \sqrt{\frac{3m}{m+2}} + C_0 \frac{m}{\sqrt{N}}. \]
  where $C_0$ is a universal constant (see section 3). Moreover, if $N \geq 100m^2$, then
  \[ D(m, N) = \sqrt{\frac{3m}{m+2}}. \]

• $\geq 2m$ If $m + 1 \leq N < 2m$, then
  \[ D(m, N) \leq 6 \frac{m^2}{N - m}. \]

About the Proof. The upper bound on $D(m, N)$ is proven in section 3 for $N \geq 2m$ and in section 4 for $N \leq 2m$.

The lower bound on $D(m, N)$ and the issuing equality $D(m, N) = \sqrt{\frac{4m}{m+2}}$ is proven in section 5 by reproducing Petrunin’s argument from [Pet2023].

1.1.B. $\delta$-Approximation Corollary. Let $X = X^m$ be a smooth manifold and $f : X \rightarrow \mathbb{R}^N$ a continuous map.

• $\leq$ If $N \geq 2m - 1$ then $f$ can be $\delta$-approximated by smooth immersions $f_\delta : X \rightarrow \mathbb{R}^N, \delta > 0,$ with curvatures
  \[ \text{curv}^+ f_\delta (X) \leq \frac{1}{\delta} \left( \sqrt{\frac{6m - 2}{2m + 1}} + C_0 \frac{m}{\sqrt{N}} \right) + o(1), \quad \delta \rightarrow 0, \]
  where "$\delta$-approximated" means that
  \[ \text{dist}_{\mathbb{R}^N} (f_\delta (x), f_0 (x)) \leq \delta, \quad x \in X. \]

• $\geq$ If $X$ admits an immersion to $\mathbb{R}^n$, $n < N$, and $N \leq 2m$, then $f$ can be $\delta$-approximated by smooth immersions $f_\delta : X \rightarrow \mathbb{R}^N, \delta > 0,$ with curvatures
  \[ \text{curv}^+ f_\delta (X) \leq \frac{1}{\delta} \frac{6n}{N - n} + o \left( \frac{1}{\delta} \right). \]

Proof. Let $\phi : X = X^m \rightarrow \mathbb{R}^n$ be a smooth immersion and observe the following.

1.1.C. Stretching Lemma. If $n \geq m + 1$, then, for all Riemannian metrics $g$ on $X$ and all positive functions $\varepsilon (x)$, there exists an a $g$-expanding immersion

\[ 6 \]

\[ ^6 \text{All } X^m \text{ immerse to } \mathbb{R}^{2m - 1}, \text{ if } m \geq 2, \text{ by the Whitney theorem}. \]
\( \psi : X \to \mathbb{R}^n \) regularly homotopic to \( \phi \), i.e. it can be joined with \( \phi \) by a \( C^1 \)-continuous homotopy of smooth immersion, and such that \( \text{curv}_\psi(X, x) \leq \varepsilon(x) \).

**Proof.** If \( X \) is compact, scale \( \phi \to \psi = \lambda \phi \) and send \( \lambda \to \infty \).

If \( X \) is non-compact and \( n < m \) regularly homotop \( \phi \) to it a proper (infinity goes to infinity) immersion with a use of Hirsch’ immersion theorem and let \( \psi_\lambda : X \to \mathbb{R}^n \) be the composition of \( \psi \) with a \( \lambda(y) \)-expanding map \( : \mathbb{R}^n \to \mathbb{R}^m \), \( y \in \mathbb{R}^n \), for a large and fast growing function \( \lambda(y) \).

Now, \( \varepsilon \)-approximate \( f \) by a smooth map \( f' \) and add to it the composed map of \( \delta^{-1} \psi_\lambda = \psi_\delta^{-1} \lambda \) with an expanding map \( f_\delta : \mathbb{R}^n \to \mathbb{R}^N \) times \( \delta \). It is clear that if the function \( \lambda(x) = \lambda_f(x) \) is sufficiently large, depending on the norms of the fist and the second differentials \( \|d_f(x)\| \) and \( \|d^2 f'(x)\| \), then the curvature of this sum
\[
f_{\delta, \lambda}(x) = f'(x) + \delta \cdot f^\circ \psi_\delta^{-1} \lambda(\delta^{-1} x)
\]
is bounded by
\[
\frac{\text{curv}(f^\circ)}{\delta} + o\left(\frac{1}{\delta}\right)
\]
and the proof follows with \( \varepsilon \to 0 \).

**Remark 1.** If \( f = 0 \), and \( X \) immerses to \( \mathbb{R}^n \), then the above delivers an immersion \( f_1 \) of \( X \) to the unit ball \( B^{m+1} = B^{n+1}(1) \) with a bound on the curvature of \( f_1 \) depending only on the dimension \( m \) of \( X \), e.g.,
\[
\text{min.curv}^i(X, B^N) \leq \sqrt{\frac{3(2m-1)}{2m+1}} = \sqrt{3 - \frac{6}{2m+1}} \quad \text{for } N \geq 100m^2.
\]

Moreover, we show in section 3) the following.

**1.1.D.** As \( N \) becomes very large depending on the topology of \( X \), then \( \text{min.curv}^i(X, B^N) < \sqrt{\frac{3(2m-1)}{2m+1}} \). In fact,
\[
\lim_{N \to \infty} \text{min.curv}^i(X, B^N) \leq \sqrt{\frac{3m}{m+2}} = \sqrt{3 - \frac{6}{m+1}}.
\]

**1.1.E. Conjecture.** If \( N \geq 100m^2 \) then all \( m \)-manifolds \( X \) admit immersions to the unit sphere \( S^N(1) \) with curvatures
\[
\text{curv}^i(X \to S^N(1)) \leq \sqrt{\frac{3m}{m+2}} - 1 = \sqrt{\frac{2m-1}{m+2}}.
\]

The bound [\( N >> \)], albeit unlikely, may be optimal\footnote{Anton Petrunin [Pet2014] proved it is optimal, see section 5.} but our bounds on on \( \text{curv}^i_{\lambda_f}(X) \) for small \( N \) are far from optimal. For instance, Clifford embeddings of products of \( l \) spheres to the unit balls have curvatures \( l^+ \ll l^2 \).

But the Clifford embeddings are not optimal either: there are products of \( l \) spheres, which admit codimension 1 (not \( l! \)) immersions with curvatures bounded by a universal constant, where the best available – we don’t know if this is optimal – such a constant is \( 1 + 2\sqrt{\frac{d-3}{d+1}} \) according to the following.
1.1.F. Codim 1 Theorem/Example. (See section 4.2) Let
\[ X = S^k \times S^1 \times \ldots \times S^1. \]
If \( k \geq l \) then there exists an immersion
\[ F : X \hookrightarrow B^{k+l}(1) \]
with
\[ \text{curv}_F^c(X) \leq 1 + 2 \sqrt{\frac{3l - 3}{l + 1}} < 4.5. \]

Remark II. The proof of the remark I doesn’t apply to immersions to \( \mathbb{R}^n \) without passing to \( \mathbb{R}^{n+1} \) but this is taken care of by the following (see section 4.3).

1.1.G. Regular Homotopy/Approximation Theorem. Let \( f : X = X^m \rightarrow \mathbb{R}^n \) be an immersion. If \( n > m \), then \( f \) can be \( \delta \)-approximated by immersions \( f_\delta : X \rightarrow \mathbb{R}^n \) which are regularly homotopic to \( f \) and such that
\[ \text{curv}_{f_\delta}^c(X) \leq \frac{500}{\delta} m^{\frac{3}{2}} + o\left(\frac{1}{\delta}\right). \]

1.1.H. Remarks/Questions. We don’t know how close this inequality to the minimal values of the curvatures of codim1 immersions of products of spheres is.

(a) For instance let \( P^{l-1} \) be an \((l-1)\)-dimensional manifold diffeomorphic to a product of spheres where some of these have dimensions \( \geq 2 \). Then, if \( k >> l \), there exist immersions
\[ F_\varepsilon : S^k \times P^{l-1} \hookrightarrow B^{k+l}(1) \]
with
\[ \text{curv}_{F_\varepsilon}^c(S^k \times P^{l-1}) \leq 1 + 2 \sqrt{\frac{3l - 3}{l + 1}} + \varepsilon \]
for all \( \varepsilon > 0 \).

But this is unclear for \( \varepsilon = 0 \), even for the product \( S^1 \times S^k \), which embeds to the ball \( B^{k+2}(1) \) with curvature 3 for all \( k \) and where we don’t know if there are immersions of \( S^1 \times S^{k+2} \) (or other closed non-spherical manifolds of dimension \( k + 1 \)) to the unit ball \( B^{k+2}(1) \) with curvatures < 3.

(b) It is not impossible according to what we know, that \( m \)-dimensional products of spheres of dimensions \( \geq 2 \) admit immersions to \( B^{m+1}(1) \) with curvature < 100.

But the best we can do (see section 4.1) are immersions with curvatures \( \leq m^{\frac{3}{2}}. \)

\[ ^8 \text{The hugeness of this number is the product of my perfunctory interpretation of Hilbert’s argument in [H1909].} \]
1.2 Equidimensional Expanding Maps

Affine Expanding Maps. The product of $r_i$-balls admits an affine equidimensional expanding map to the $R$-ball

$$f : \bigtimes_{i=1}^k B^n(r_i) \to B^N(R), ~ N = \sum_i r_i,$$

if and only if

$$[\Sigma r_i^2] \quad \sum_i r_i^2 \leq R^2,$$

where – all this is, of course, obvious – in the case of equality $\Sigma r_i^2 = R^2$, such an $f$ is an isometric embedding.

But – this was pointed out to me by Roman Karasev – it is unlikely that there is a simple criterion for the existence of such embeddings to cubes, not even for rectangular solids,

$$B^n(r) = \bigtimes_{i=1}^n [-r_i, r_i] \to [-r, r]^n.$$

1.2.A. Rolled Band Example. What is more interesting from our perspective is a $(1-\varepsilon)$-expanding map, for a given $\varepsilon > 0$, from the infinite cylinder $X = B^{n-1}(r) \times \mathbb{R}^1$ to the ball $B^n(2r)$,

$$f_\varepsilon : B^{n-1}(r) \times \mathbb{R}^1 \to B^n(2r),$$

where this $f_\varepsilon$ comes as the composition of two maps.

1. The first map is the universal covering map from the cylinder $B^{n-1}(r-\varepsilon) \times \mathbb{R}^1$ to the round solid torus embedded to the ball,

$$f_1 : B^{n-1}(r) \times \mathbb{R}^1 \to \mathbb{T}_{\text{std}}(r, r-\varepsilon) \subset B^n(2r),$$

where this torus is equal to the $(r-\varepsilon)$-neighbourhood of a planar circle

$$S^1(r) \subset B^n(2r)$$

of radius $r$, where the center of $S^1(r+\varepsilon)$ is positioned at the center of the ball $B^n(2r)$.

Observe that the map $f_1$ is isometric on the $(n-1)$-balls

$$B^{n-1}(r-\varepsilon) \times t \subset B^{n-1}(r-\varepsilon) \times \mathbb{R}^1, \quad t \in \mathbb{R}^1.$$

2. The second map $f_2$ is the linear (scaling) diffeomorphism

$$f_2 : B^{n-1}(r) \times \mathbb{R}^1 \to B^{n-1}(r-\varepsilon) \times \mathbb{R}^1$$

for $f_2 : (s, t) \mapsto \left( \frac{s}{1-\varepsilon}, \varepsilon^{-1}t \right)$,

where, clearly, the composition

$$B^{n-1}(r) \times \mathbb{R}^1 \xrightarrow{f_2} B^{n-1}(r-\varepsilon) \times \mathbb{R}^1 \xrightarrow{f_1} \mathbb{T}_{\text{std}}(r, r-\varepsilon) \subset B^n(2r)$$

is the required $(1-\varepsilon)$-expanding map $B^{n-1}(r) \times \mathbb{R}^1 \xrightarrow{f_\varepsilon} B^n(2r)$. 

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1.2.B $\left[ f \times f \right]$-Corollary. The Cartesian powers of

$$f_r: [-r, +r] \times \mathbb{R}^1 \to B^2(2r) \subset \mathbb{R}^2$$

deliver expanding maps

$$B^m(r) \times \mathbb{R}^m \subset [-r, +r]^m \times \mathbb{R}^m \to B^{2m} \left( 1 + \frac{1}{\sqrt{m}} \right)$$

for all $m = 1, 2, \ldots$ and $r < \frac{1}{\sqrt{m}}$.

1.2.C. $\frac{1}{2}$-Exercise. Show that if $\xi \leq 2r$, then the cylinder $B^{n-1}(r) \times \mathbb{R}^1$ admits no expanding map $f$ to the ball $B^n(\xi)$.

Hint. (i) The axes – the central line $0 \times \mathbb{R}^1$ of the cylinder – must go by $f$ to the concentric ball $B^{n-1}(\xi - r) \subset B^n(\xi)$.

(ii) The longest straight segment with respect to the $f$-induced flat metric between pairs of points on this axes must have length $> 2\xi - r$.

The above 1.2.B is generalized in section 4.2 as follows.

1.2.D. Rolled Band into Ball Theorem. If $M \geq 100m^2$, and

$$r < \frac{\sqrt{m} + 2}{\sqrt{3m} + \sqrt{m} + 2} \left( > \frac{1}{3} \right),$$

then the product $B^M(r) \times \mathbb{R}^m$ admits an equidimensional expanding map to the unit ball,

$$F_r: B^M(r) \times \mathbb{R}^m \to B^{m+M}(1).$$

Remark/Question. If $m = 1$, then, by the above $\frac{1}{2}$-exercise, the bound $r < 1/2$ is optimal, but it is not clear for $m = 2$.

Here the above inequality for $m = 2$, which allows expanding maps from $B^2(r) \times \mathbb{R}^2$ to the unit ball $B^{m+M}(1)$, where the supremum of the possible $r$ is

$$\sup r = \frac{2}{\sqrt{6} + 2} - \varepsilon (\approx 0.45),$$

is implemented with $M = 4$ by means of the normal exponential map for the 2-subtorus in Clifford torus $T^3 \subset B^6(1)$, which is is normal to the principal diagonal in $T^3$.

Similarly the normal exponential map for the Clifford torus $T^2 \subset B^4(1)$ leads to such maps $B^2(r) \times \mathbb{R}^2 \subset B^4(1)$ with

$$\sup = \frac{1}{1 + \sqrt{2}} \approx 0.41 < 0.45,$$

while the best $B^1(r) \times \mathbb{R}^2 \subset B^3$, where

$$\sup r = \frac{1}{3} < 0.41,$$

is obtained with the normal exponential map for the standard round torus in $\mathbb{R}^3$. 

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And the only known upper bound on $r$ is for $M = 1$:

$$r \leq \frac{\pi}{2\sqrt{\lambda_1(B^3(1))}} = \frac{\pi}{2j_{1/2}} = \frac{1}{2} \sqrt{\frac{2}{\sqrt{6} + 2}} \approx 0.45,$$

where this $\lambda_1$ is the first Dirichlet eigenvalue of the Laplacian in the unit 3-ball, and $j_{1/2} = \pi$ is the first Bessel function zero (see section 5.1).

None of these four inequalities is known to be (or not to be) optimal.

1.3 Remarks, Acknowledgements and the Plan of the Paper

The lower bounds on curvatures of tori (see section 1.3) in concert with the "natural symmetry" of Clifford's manifolds may lead one to believe that such bounds persist in all codimensions. But when I mentioned this to Fedor Bogomolov, "everything is possible in large dimensions" – he responded.

Then my attempts to prove lower bounds on the curvatures of $m$-tori in $n$-dimensional balls for $n \sim 2m$ were arrested by what Gilles Pisier explained to me about norms of generic linear families of selfadjoint operators.

Also Gilles pointed out to me on the criticality of dimensions $N \sim m^2$ (example 3.1 in [FLM1977]) and the present state of art with Dvoretzky-Milman inequalities for the $l_p$-spaces was explained to me by Grigoris Paouris who also suggested to me the relevance [K1995] for evaluation of the Kolmogorov diameter $D$.

Then Bo'az Klartag and Noga Alon patiently explained me the essential properties on the spherical designs and construction of these based on binary codes, allowing sharp bound on $D$ in moderately high dimensions. We present all this in section 2.

In section 3, we show how bounds on the Kolmogorov $m$-diameter of the space $l_4^N$ translate to corresponding inequalities for curvatures $\text{curv}^\perp(X \hookrightarrow \mathbb{R}^{2N})$ for submanifolds $X$ in the Clifford tori $\mathbb{T}^N \subset \mathbb{R}^{2N}$.

In section 4.1 we elaborate on the round torus construction from section 1 needed for immersions below $4m - 2$.

In section 4.2 we exhibit codim 1 immersions with small curvatures as boundaries of "tubular neighbourhoods" of immersion with high codimension constructed in the previous sections and similarly construct expanding maps in the cases indicated in section 1.2.

In section 4.3 we describe a twisting procedure of immersed manifolds by regular homotopies with controlled curvature and in section 4.4. we outline a similar procedure based on Poenaru-Eliashberg’s folding idea.

In section 5 we collect (mostly) known bounds on expansion and on the curvature of immersions, including the recent sharp $\sqrt{3}$-inequality by Petrunin.

In section 6 we discuss curvature problems similar to but different from the ones we address in the main body of the paper.
2 KOLMOGOROV’S $D = D(m, N, p)$, HILBERT’S THEOREM AND SPHERICAL DESIGNS

**K-Diameter $\sqrt[p]{D(m, N, p)}$.** Let $\|y\|_{L_p} = (\frac{1}{N} \sum_{i=1}^{N}|y_i|^p)^{\frac{1}{p}}$

Let $D(m, N, p)$ denotes the infimum of the numbers $D > 0$ such that $\mathbb{R}^N$ contains an $m$-dimensional linear subspace $X$, such that

$$\|x\|_{L_p}^p \leq D\|x\|_{L_2}^p, \text{ for all } x \in X.$$  

Observe that $D(1, N, p) = 1$, $D(m, m, p) = m^{\frac{p}{2} - 1}$, that $D(m, N, p)$ is monotone increasing in $m$ and decreasing in $N$ and let

$$D(m, p) = D(m, \infty, p) = \lim_{N \to \infty} D(m, N, p).$$

**2.1.A. Gamma Function Design Formula.** If $p = 4, 6, 8, \ldots$, then a simple $O(m)$-averaging argument, shows that

$$[\Gamma/T] \quad D(m, p) = \frac{\int_{S^{m-1}} |l(s)|^p ds}{\left(\int_{S^{m-1}} |l(s)|^2 ds\right)^{\frac{p}{2}}} = \frac{m^{\frac{p}{2} - 1} \cdot 3 \cdot 5 \cdots (p - 1)}{(m + 2) \cdot (m + 4) \cdots (m + p - 2)},$$

where $l(s)$ is a non-zero linear function on the sphere.

**2.1.B. Hilbert Connection.** In his proof of the Waring problem, Hilbert shows the existence of $M = \left(\frac{m + p - 1}{m - 1}\right) + 1$ rational points $s_i \in S^{m-1}$ and of positive rational weight $w_i > 0$, $\sum_i^M w_i = 1$, such that $\sum_i w_i l^d(s_i) = \int_{S^{m-1}} l^d(s) ds$ for all linear functions on the sphere.

This, after partitioning each $s_i$ into $\Delta$ atoms for $\Delta$ being the smallest common denominator $\Delta$ of $w_i$, becomes what is no-a-days called spherical design of cardinality $N = \Delta M$ of $w_i$, which yields (this is nearly obvious, see 2.1.C below) the following.

**D(m, N) - Stabilization:** $D(m, N, p) = D(m, \infty, p)$ for all sufficiently large $N \geq N_{Hilb}(m, p) \leq N_{M}$, where $- \to$ be safe let it be rough– $N_{Hilb} \leq m^m$.

**Design Rationality:** If $N \geq N_{Hilb}$ then the space $l^N$ contains a rational linear subspace $X$ of dimension $m$, such that

$$\|x\|_{L_p}^p = D(m, p)\|x\|_{L_2}^p \text{ for all } x \in X.$$  

**2.1.C. Spherical Designs and the Equality $D(m, N) = D(m, \infty)$**

A design of even degree $p = 2, 4, \ldots$ and cardinality $N$ on the sphere $S^{m-1}$ is a map from a set $\Sigma$ of cardinality $N$ to the sphere, written as $\sigma \mapsto s(\sigma)$, such that the linear functions $l(\sigma)$ on the sphere $S^{m-1} \subset \mathbb{R}^m$ satisfy

$$\frac{1}{N} \sum_{\sigma \in \Sigma} l^d(s(\sigma)) = \int_{S^{m-1}} l^d(s) ds, \quad d = 2, \ldots, p,$$

where $ds$ is the $O(m)$ invariant probability measure on the sphere.
Hence, the linear map from the space $\mathbb{R}^{m^2}(=\mathbb{R}^m)$ of linear functions on the sphere $S^{m-1} \subset \mathbb{R}^m$ to $\mathbb{R}^N = \mathbb{R}^\Sigma$ preserves both, the $L_2$ and the $L_p$-norms and, by the above $[\Gamma/\Gamma]$,

the existence a design of cardinality $N$ implies that $D(m,N,p) = D(m,p)$\(^9\)

Non-rational designs, at least for $p = 4$, are known to exit for $N \ll N_{Hilb}$.

2.1.D $2m^2$-Design Construction. If $p = 4$, and if $m$ is a power of 2, then there exists a spherical designs of cardinality $N = 2m^2 + 4m$.\(^{10}\)

This, now for all $m$, shows that

(i) \[ D(m,N,4) = \frac{3m}{m+2} \text{ for } N \geq 8(m^2 + m). \]

$[\mathbb{R}^2$ in $l^3]$-Example. \(D(2,N,4) = \frac{3}{2}\) for $N \geq 3$, with four (rational) planes $X \subset \mathbb{R}^2 = l^2_1$, where $\|x\|_{l_4}^4 = \frac{3}{4}\|x\|_{l_2}^4$; these are the normals to the vectors $(1,1,-1), (1,-1,1), (1,1,1), (1,-1,-1)$.

2.1.E. $D(m,N)$-Inequalities. If $N \leq m^2$, then upper bounds on $D^4(m,N,4)$ follow from the corresponding estimates in the randomization proofs of the Dvoretzky theorem for the $l^p$-spaces, where the following inequality follow from (the argument in) \[PVZ2017\] as it was spelled out in details in a mesage by Grigoris Paouris to me.

(ii) \(D(m,N,4) \leq 3 + const_{(ii)} \frac{m^2}{N} \text{ for } N \geq m^2\)\(^{11}\)

(iii) \(D(m,N,4) \leq const_{(iii)} \frac{m^2}{N} \text{ for } 2m \leq N \leq m^2\)\(^{12}\)

2.1.F. $D(m,N)$ Concentration Property. The existence of $m$-subspaces $X \in l^N_1$ in \[FLM1977\] and \[PVZ2017\], such that

\[ [D] \]

\[ \|x\|_{l_4} \leq D\|x\|_{l_2}, \text{ } x \in X, \]

is derived from a lower bound the measure of those $m$-subspaces $X \subset \mathbb{R}^N$, where this inequality fails for some $x \in X$.

In particular, the argument used in \[FLM1977\] implies that the measure $\mu_D$ of those $X \subset \mathbb{R}^N$ with respect to the $O(N)$-invariant probability measure in the Grassmanian $Gr_m(\mathbb{R}^N)$ where $\|x\|_{l_4}^4 \geq D\|x\|_{l_2}^4$, for some $x \in X$ satisfies:

If

\[ D > \frac{3m}{m+2} \]

then

\[ \mu_D \rightarrow 0, \text{ for } N \rightarrow \infty. \]

Nash Connection. Besides applications to lower bounds on curvatures of immersions (see next section), Hilbert’s argument, combined with a Nash-like twist, leads to $C^2$-smooth isometric Riemannian immersions with (large)
prescribed curvatures and also to a solution of the differential geometric Warning problem:

collection of isometric $C^1$-immersions of manifolds with symmetric differential forms of degrees $d > 2$, (see 2.4 (B)(4) on p. 205 in [Gr1986] and [Gr2017]).

3 Equivariant Immersions $\mathbb{R}^m \to S^{2N-1}$ and Euclidean $\mathcal{D}(m,N)$-Theorem for $N \geq 4n$

3.A. Curvatures of the Clifford Tori. Let

$$T^N \subset S^{2N-1} \subset B^{2N}(1) \subset (B^2(1))^N \subset \mathbb{R}^{2N}\) be the Clifford torus and observe that the second quadratic form of this torus in the ambient Euclidean space $\mathbb{R}^{2N} \supset S^{2N-1} \subset T^N$, regarded as a quadratic form with values in the normal bundle, is

$$\Pi = \sqrt{N} \sum_{i=1}^N \nu_i dt_i^2,$$

where $t_i$ are the cyclic coordinates on the torus and $\{\nu_i \in T^i(T^N \subset \mathbb{R}^{2N})\}$ is the corresponding orthonormal frame of normal vectors to $T^N$.

This, in terms of the orthonormal tangent frame $\{e_i = \frac{d}{dt_i} \in T(T^N)\}$, means that

$$\Pi: e_i \otimes e_i \mapsto \sqrt{N} \nu_i \text{ and } \Pi: e_i \otimes e_j \mapsto 0 \text{ for } i \neq j.$$

Thus, the curvature of $T^N$ in $B^N$ along a unit tangent vector $\bar{x} \in T(T^N)$,

$$\bar{x} = \sum_i x_i e_i, \text{ where } \sum_i x_i^2 = 1,$$

is

$$\text{curv}^i(T^N, \bar{x}) = ||\Pi(\bar{x} \otimes \bar{x})|| = ||\Pi(\sum_i x_i e_i \otimes \sum_i x_i e_i)|| =$$

$$||\Pi(\sum_i x_i e_i \otimes e_i)|| = \sqrt{N} \sum_i x_i^2 \nu_i = \sqrt{N} \sqrt{\sum_i x_i^2} = \sqrt{N} \sqrt{\sum_i x_i^4} =$$

where $||\bar{x}||^2 = ||\bar{x}||_{l_2}^2 = \sum_{i=1}^N x_i^2$.

Hence,

$$\text{curv}^i(T^N, \bar{x}) = \left(\sqrt{N} \frac{||\bar{x}||_{l_4}}{||\bar{x}||_{l_2}}\right)^2 \left(\frac{||\bar{x}||_{l_4}}{||\bar{x}||_{l_2}}\right)^2,$$

where, recall, the $L_p$-norms refer to the finite probability spaces with $N$ equal atoms,

$$||\bar{x}||_{L_p} = \frac{||\bar{x}||_{l_p}}{\sqrt{N}}.$$

3.B. Proof of the Euclidean $\mathcal{D}(m,N)$-Theorem 1.1.A for $N \geq 2m$.

The above (★) implies the existence of an equivariant isometric immersion from the Euclidean $m$-space to the Clifford $N$-torus,

$$f^\circ : \mathbb{R}^m \to T^N \subset S^{2N} \subset \mathbb{R}^{2N}$$. 

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with the relative curvature \(\text{curv}_E^\phi(f^\circ)\) (for the Euclidean metrics \(e\) in \(\mathbb{R}^m\) and \(E\) in \(\mathbb{R}^{2N}\)) equal to \(\sqrt{D(m,N)} = \sqrt{D(m,N,4)}\).

Hence,
\[
\mathcal{D}(m,N) \leq \sqrt{D(m,M)}
\]
for all \(m\) and \(N \geq 2M\); thus the above \(D(m,N)\)-inequalities (i),(ii),(iii) yield the corresponding \(\mathcal{D}(m,N)\) inequalities in 1.1.

In addition to that, if the \(l^2_m\)-space contains a rational \(m\)-subspace \(X\) with \(\frac{\|x\|_{l^2_m}}{\|y\|_{l^2_m}} = D\), then \(\mathbb{T}^N\) contains an \(m\)-subtorus with the ambient Euclidean curvature \(\sqrt{D}\).

3.C. Proof of \([N >>]\) from 1.1.D. Embed \(X^m \to \mathbb{R}^{2m} \subset \mathbb{R}^N, N > 2m,\) and apply orthogonal transformations \(o \in O(M)\) to \(X\). Since \(X\) is compact (non-compact manifolds are irrelevant here) the \(D(m,N)\)-concentration 2.1.F implies that there exist an \(o \in O(n)\), such that all tangent vectors \(\tau \in \sigma(T(X)) \subset \mathbb{R}^N\) satisfy
\[
||\tau||_{l^2_m} \leq \left(\frac{3m}{m + 2} + \varepsilon N\right) ||\tau||_{l^2_4}, \text{ where } N \varepsilon \to \infty \text{ for } N \to \infty.
\]

Thus, arguing as earlier, the \(\lambda\)-scaled manifold \(X\) imbeds to the Clifford torus \(\mathbb{T}^N \subset S^{2N-1} \subset \mathbb{R}^{2N}\) with
\[
\text{curv}_E^X(X \to \mathbb{R}^{2N}) \leq \left(\sqrt{\frac{3m}{m + 2} + \varepsilon N + \varepsilon \lambda}\right), \text{ where } \varepsilon \lambda \to 0 \text{ for } \lambda \to \infty
\]
and the proof follows.

3.D. \(\delta\)-Approximation in Non-Euclidean Riemannian Manifolds.

The derivation of the \(\delta\)-approximation from expanding Euclidean maps in section 1.1 easily generalizes, albeit with limitations, to Riemannian manifolds as follows.

Theorem. Let \(Y\) be a complete Riemannian manifold with sectional curvature \(\text{sect, curv}_E^X(Y) \leq \kappa^2\) and let \(f : X \to Y\) be a continuous map.

If the induced bundle \(f^\ast(T(Y)) \to X\) contains a subbundle isomorphic to \(X \times \mathbb{R}^N\), (i.e. a trivial one) and if \(X\) admits an immersion to \(\mathbb{R}^N\), e.g. \(2m - 1 \leq N \leq \text{dim}Y - \text{dim}(Y) - 1\), then, for all positive \(\delta \leq \frac{1}{2m}\), the map \(f\) can be \(\delta\)-approximated by immersions \(f_\delta : X \to Y\), such that
\[
\text{curv}_{f_\delta}(X) \leq 1 + \frac{2\kappa}{\delta} \sqrt{\mathcal{D}(m,N)},
\]
where
\[
\mathcal{D}(m,N) \leq \frac{3m}{m + 2} + \text{const}\frac{m}{\sqrt{N}} \text{ for } N \geq 2m
\]
and
\[
\mathcal{D}(m,N) \leq \frac{6m^2}{N - m} \text{ for } N \leq 2m.
\]

Proof. Proceed as in the proof of 1.1.B, where instead of adding \(\delta \cdot f^\circ \circ \psi_{\varepsilon,3,\lambda}\) to \(f'_\varepsilon\) we the compose exponential map with a (fiberwise injective) bundle homomorphism from the trivial bundle \(X \times \mathbb{R}^N\) to \(X\) over the smooth map \(f'_\varepsilon\); (this map \(\varepsilon\)-approximates \(f\)).

\(^{13}\)One may allow a boundary, but this is a minor problem.
3.1 Subtori in Non-Equilateral Clifford Tori

All invariant $N$-tori in the sphere $S^{2N-1} \subset \mathbb{R}^n$ are (equal, up to isometries of $S^{2N-1}$, to) the orbits of the product action of $N$-copies of the standard action of $\mathbb{T}^1$ in the plane, where these orbits are equal to the non-equilateral Clifford tori

$$T^N(\tilde{r}) \times S^1(r_i), \quad \text{for} \quad \tilde{r} = (r_1, ..., r_N), \quad \text{where} \quad \|\tilde{r}\|^2 = \sum_i r_i^2 = 1$$

Then, similarly to the above (\star), the values of the curvature operator of this torus at the unit tangent vectors $\tilde{x} = (x_1, ..., x_N) \in T(T^N(\tilde{r}))$ are

\[\text{curv}^i(T^N(\tilde{r}), \tilde{x}) = \left\| \sum_i \frac{x_i^2}{r_i} \right\| = \sqrt{\sum_i x_i^4} \]

where, if all $r_i = \frac{1}{\sqrt{N}}$, this reduces to (\star) for

$$\sqrt{\sum_i x_i^4} = \sqrt{\frac{\sum_i |x_i|^4}{N}}$$

and where we denote

$$\|x\|_{L_4(\tilde{r})} = \sqrt[4]{\sum_i x_i^4}$$

3.1.A. Conclusion. There is a one-to-one correspondence between

equivariant $\mathbb{R}^m \subset S^{2N-1}$ with $\text{curv}^i(\mathbb{R}^m) < \alpha$

and pairs $(\tilde{r}, X)$, where $\tilde{r} = (r_1, ..., r_N)$ is a unit vector with positive entries,

$$\sum_{i=1}^N r_i^2 = 1, \quad r_i > 0,$$

and subspaces $X \subset Y = \mathbb{R}^N = l_2^N$ is a

such that all $x \in X$ satisfy

$$\|x\|_{L_4(\tilde{r})} < \sqrt{\alpha} \cdot \|y\|_{L_2},$$

where, recall, the $L_2$-norm of $y \in Y$, including $y \in X \subset Y$, is

$$\|y\|_{L_2} = \sqrt{\frac{\sum_{i=1}^N y_i^2}{N}} = \frac{\|y\|}{\sqrt{N}}$$

Conceivably, $m$-torical orbits not contained in $T^N_{\mathbb{C}^1}$, e.g. those maximizing the $m$-volumes of the respective $m$-tori actions, may have slightly smaller curvatures than Kolmogorov’s $D(m, N)$, that is, as we know, is equal to the infimum of the curvatures of $m$-subtori in $T^M_{\mathbb{C}^1}$.

This can be stated with the $\tilde{r}$-counterpart of Kolmogorov’s $D(m, N)$, denoted $\check{D}(m, N) (\leq D(m, N))$ that is the infimum of the suprema of the ratios of the two norms:

$$\check{D}(m, N) = \inf_{Y, \tilde{r}, 0 \neq y \in Y} \sup_{\|y\|_{L_4(\tilde{r})}} \frac{\|y\|_{L_4(\tilde{r})}}{\|y\|_{L_2}}$$

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where the infimum is taken over all \( m \)-dimensional linear subspaces \( Y \subset \mathbb{R}^N \) and all positive unit vectors \( r \).

**Question.** Is, ever, \( \hat{\mathcal{Q}}(m, N) < D(m, N) \)?

The space \( \mathcal{I}_\alpha = \mathcal{I}(m, N, \alpha) \) of isometric equivariant immersions \( \mathbb{R}^m \to S^{2N-1} \) with curvatures \( \leq \alpha \) is a semi algebraic subset in the (Euclidean) space \( J_N(m, N) \) of \( N \)-jets at 0 \( \in \mathbb{R}^m \) of smooth maps \( \mathbb{R}^m \to \mathbb{R}^N \) which is invariant under the action of the orthogonal group \( O(2N) \), and where the \( O(2N) \)-orbit of an \( I \in \mathcal{I} \) in \( S^{2N-1} \) is equal to

\[ W_I / O(2N) / T^N, \text{ where } W_I \text{ is the subgroup of the Weyl group of } O(2N), \text{ which preserves } I, \text{ (this is empty for generic } I \).\]

There can be something geometrically interesting in the \( O(N) \)-topology of \( \mathcal{I}_\alpha \) depending on \( \alpha \), but all one can say off hand is the Petrovsky-Thom-Milnor bound on the homology of \( \mathcal{I}_\alpha \) by the algebraic degree of this set.

## 4 Normal Immersions in Small Codimensions

### 4.1 Proof of Euclidean \( D(m, N) \)-Theorem for \( N \leq 2m \)

**\( \triangleright \)-Construction.** Let \( \phi_1 : X_1 = X_1^{m_1} \to \mathbb{R}^{m_1+n_1} \), be an immersion with a trivial normal bundle, where this "triviality" is implemented by a smooth map

\[ \Phi_1 : X_1 \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1+n_1} \]

and let \( \phi_2 : X_2 = X_2^{m_2} \to \mathbb{R}^{n_2} \) be another immersion. If \( \phi_2 \) lands in the \( r \)-ball in \( \mathbb{R}^{n_1} \) for some \( r_1 > 0 \),

\[ \phi_2(X_2) \subset B_{\phi_1}(r) \subset \mathbb{R}^{n_1} \]

and

\[ \text{curv}^r_{\phi_1}(X_1) \leq \alpha_1 < 1/r, \]

then the composed map \( (x_1, x_2) \mapsto \Phi_1(x_1, \phi_2(x_2)) \) is an immersion, say

\[ \phi_1 \times \phi_2 : X_1 \times X_2 \to \mathbb{R}^{m_1+n_1}. \]

Recall that the normal connection \( \nabla^I \) in the (trivial) normal bundle

\[ X_1 \times \mathbb{R}^{n_1} = T^I(X_1) = T(\mathbb{R}^{m_1+n_1}) \oplus T(X^1) \to X_1 \]

is defined by the field \( r^I \) of tangent \( m_1 \)-planes in \( X_1 \times \mathbb{R}^{n_1} \), which are normal to the Euclidean fibers \( x_1 \times \mathbb{R}^{n_1} \) with respect to the (flat) Riemannian metric induced by the map \( \Phi_1 : X_1 \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_1+n_1} \).

**Flat Split Bundles and \( \nabla^I \)-Trivial Immersions** The connection \( \nabla^I \) is called flat split if the map \( \Phi_1 \) is \( \nabla^I \)-parallel that is the field \( \nabla^I \) is normal to the

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14 Grigoris Paouris has sent to me a message with an evaluation of \( \hat{\mathcal{Q}} = \inf_Y \sup_{\phi \in Y} \inf_{x_1 \in X_1} \frac{||\phi_x||_{L^2}}{||x_1||_{L^2}} \), for several classes of \( \mathcal{I} \).

15 This can’t happen for large \( N >> m^2 \) by Petrunin’s inequality.

16 The space \( J_k(m, N) \) is isomorphic to the space of polynomial maps \( \mathbb{R}^m \to \mathbb{R}^N \) of degrees \( \leq k \).

17 The corresponding space \( X(m, N, \sqrt{\mathcal{I}}) \) of \( m \)-subspaces \( X \in L^N_4 \) with \( \frac{||x||_{L^2}}{||x||_{L^2}} = \sqrt{\mathcal{I}} \), which, albeit being also semi algebraic, has more combinatorial flavour than \( \mathcal{I} \).

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fibers $x_1 \times \mathbb{R}^{n_1}$ with respect the product metric in $X^1 \times \mathbb{R}^{n_1}$ and the immersion $\phi_1$ is called $\nabla^1$-trivial in this case.

4.1.A. List of $\nabla^1$-Trivial Examples. (a) Immersions $\mathbb{R}^1 \to \mathbb{R}^n$ are $\nabla^1$-trivial.
(b) Codimension 1 immersion of orientable manifolds, $X^m \to \mathbb{R}^{m+1}$, are $\nabla^1$-trivial.
(c) Equivariant immersions of tori, $T^m \to \mathbb{R}^n$, are $\nabla^1$-trivial.
(d) Direct products of $\nabla^1$-trivial-immersions $\phi_i : X_i \to \mathbb{R}^{n_i}$

\[ \prod_i \phi_i : \prod_i X_i \to \mathbb{R}^{\sum_i n_i} \]

are $\nabla^1$-trivial.
(e) The above "semidirect products" $\phi_1 \times \phi_2 : X_1 \times X_2 \to \mathbb{R}^{m_1+n_1}$ of $\nabla^1$-trivial $\phi_1 : X_1 \to \mathbb{R}^{m_1+n_1}$ and $\phi_2 : X_2 \to \mathbb{R}^{n_1}$ are $\nabla^1$-trivial.

4.1.B. (Obvious) $\nabla^1$-Normality Lemma. Let $\phi_1 : X_1 \to \mathbb{R}^{n_1}$ and $\phi_2 : X_2 \to \mathbb{R}^{n_1}$ be $\nabla^1$-trivial immersions. Then:
- $\nabla^n$ If $\phi_1$ and $\phi_2$ are normal (see 1.1) then $\phi_1 \times \phi_2$ is also normal.
- $\nabla^n$ If $\phi_2(X_2) \subset B^{n_1}(r) \subset \mathbb{R}^{n_1}$, then

\[ \text{foc.rad}_{\phi_1} \times \phi_2(X_1 \times X_2) \geq \min(\text{foc.rad}_{\phi_2}(X_2), \text{foc.rad}_{\phi_1}(X_1) - r) \]

and in the normal case the relative curvature of $\phi_1 \times \phi_2$ (as well as the curvature $\text{curv}(X) = \text{foc.rad}(X)^{-1}$ itself), satisfies the corresponding inequality.

\[ \text{curv}^i(\phi_1 \times \phi_2) \leq (\min(\text{curv}^i(\phi_2)^{-1}, \text{curv}^i(\phi_1)^{-1} - r))^{-1}. \]

4.1.C. Torus-by-Torus Construction. Let

\[ [-1,1] \times T^1 \to [-2,2]^2 \times B^2(2) \]

be the map obtained by rotation of the segment $[0,2]$ around the origin in the plane (which is an immersion away from the "interior" boundary circle) and let

\[ f_1 = f_{1/k}^2 : [-1,1]^k \times T^k = \left([-1,1]^k \times T^1\right)^k \to \left([-2,2]^2\right)^k = [-2,2]^{2k} \]
\[ f_2 : [-1,1]^k \times T^{2k} = [-1,1]^k \times T^k \times T^{2k} \to [-2,2]^{2k} \times T^{2k} = \left([-2,2]^k \times T^k\right)^2 \to [-4,4]^{4k} \]

\[ f_i : [-1,1]^k \times T^{2^{i-1}-k} \to [-2^i,2^i]^{2^i}. \]

It follows by the construction, that this map is normal and that the normal exponential map of the central torus

\[ T^{2^{i-1}} = 0 \times T^{2^{i-1}} \]

(immersed actually embedded) to the cube $[-2^i,2^i]^{2^i}$ is injective in the interior of $[-1,1]^k \times T^{2^{i-1}-k}$. Hence, the curvature of this torus and the (relative) curvature of the immersion $f_i$ are bounded by 1 and the corresponding scaled map $f : T^{2^{i-1}-k} \to B^{2^{i}k^2}$ satisfies

\[ \text{curv}^i_k(T^{2^{i-1}-k}) = \text{curv}^i_k(T^{2^{i-1}-k})(f) \leq 2^i \cdot \sqrt{k^{2i}}. \]
or, in terms of \( m = k2^i - k \),

\[
\text{curv}_E^k(T^m) \leq \left( \frac{m}{k} + 1 \right) \sqrt{m + k},
\]

which implies for all \( m \) and \( k \leq m \):

\[
\text{curv}_E^k(T^m) = \text{curv}^m(f) < \frac{m^2}{k}.
\]

The proof of theorem 1.1.B is concluded.

### 4.2 Proofs of the Codim 1 and the Rolled Band Theorems

Let \( f : X^m \to Y \) be an immersion with \( \text{foc.rad}(X) = R \) and \( S^i(r)(X) \to X \) be the bundle of normal \( r \)-spheres \( S^i_{X^m}(r) \subset T^i_{X^m}(X) \), \( T_{f(x)}(Y) \oplus T_x(X) = \mathbb{R}^{N-m} \).

If \( r \leq R \) then the normal exponential map \( E : S^i(r)(X) \to Y \) is an immersion, where \( \text{foc.rad}_E(S^i(r)(X)) = \min(r, R - r) \).

For instance, if \( X \to B^N(1) \) is an immersion with trivial normal bundle and \( \text{curv}_E^1(X) \leq \), then the immersion

\[
E_f : \left( 1 + \frac{1}{2c} \right)^{-1} E : X \times S^{N-m-1} = S^i \left( \frac{1}{2c} \right)(X) \to B^N(1)
\]

has

\[
\text{curv}_E^{E_f} \left( X \times S^{N-m-1} \to B^{N-m-1} \right) \leq 2c \left( 1 + \frac{1}{2c} \right) = (2c + 1).
\]

#### 4.2.A. Codim1 Conclusion

This, applied to immersions of tori \( T^{l-1} \to B^N(1) \) with large \( N \) curvature \( \frac{N-1}{l+1} \), yields codimension codimension one immersions with small curvature as stated in 1.1.G.

#### 4.2.B. Generalization from 1-Tori to l-Polyhedra

Given a compact polyhedral (or cellular) space \( P \) of dimension \( l \), there exists a compact \( N \)-manifold \( X \), for all \( N \geq 2l - 1 \), such that:

- \( \bullet_P \) there is a continuous map \( K \to X \), which is a homotopy equivalence in dimensions \( < N/2 \), i.e. this map induces isomorphisms of the homotopy groups, \( \pi_i(P) \to \pi(X) \) for \( i < N/2 \);
- \( \bullet_200 \) if \( N \geq 200l^2 \) then, for all \( \varepsilon > 0 \), \( X \) admits an immersion to \( B^{N+1}(1) \) with

\[
\text{curv}^i(X \to B^{N+1}(1)) \leq 1 + \sqrt{\frac{3l}{l+2} \varepsilon}.
\]

In fact, the boundary of the regular neighbourhood of \( P \) embedded to \( \mathbb{R}^{N+1} \) can be taken for \( X \).

**Embedding Remark.** This, \( X \), by its very construction, embeds to \( \mathbb{R}^{N+1} \), but one can show (section 5.3) that there is no universal bound on the curvature of embeddings of \( X \) to the unit ball in \( \mathbb{R}^{N+1} \).

For instance if \( P \) is a connected sum of different lens spaces, e.g.

\[
P_k = \#_{i=1}^k S^3_{P_{k_i}},
\]

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where \( p_1 < \ldots < p_i < \ldots < p_k \) are prime numbers, then the curvatures of all smooth embeddings \( F : X \to B^{N+1}(1) \) satisfy:

\[
\text{curv}_F(X) \geq \log \log (k)/N^N.
\]

**Question.** What, roughly, is the minimum of the curvatures of embeddings \( T^i \times S^N \to B^{N+i+1}(1) \)? (See section 6.3 for more about it.)

### 4.2.C. The proof of the "rolled band theorem"

The proof proceeds similarly to the above. Let \( f : \mathbb{R}^m \to B^{m+M}(1) \) be an immersion with curvature bounded by \( D = D(m, m + M) \) as in 1.1. Let

\[
e = e_f : \mathbb{R}^m \times B^M(r) \to \mathbb{R}^m \to B^{m+M}(1 + r), \quad r < \frac{1}{D},
\]

be the normal exponential map for \( \mathbb{R}^m \) immersed to \( \mathbb{R}^m \to B^{m+M} \) and let

\[
E_\lambda : \mathbb{R}^m \times B^M(r) \to \mathbb{R}^m \to B^{m+M}(1) \text{ for } (x, b) \mapsto (1 + r)^{-1}e(\lambda x, b).
\]

If \( \lambda \) is sufficiently large, then the map \( E_\lambda \) is expanding in the \( \mathbb{R}^m \) directions, i.e. it expands \( \mathbb{R}^m \times b \) for all \( b \in B \) and since it is isometric in the \( B^M \)-directions it is expanding on \( \mathbb{R}^m \times B^M(r) \) except for one problem:

- the normal \( M \)-ball bundle \( B^i(r) \to \mathbb{R}^m \) of the immersed \( \mathbb{R}^m \to \mathbb{R}^{m+M} \) is trivial, it is indeed, isomorphic to the product \( \mathbb{R}^m \times B^M(r) \) but the map \( (x, b) \mapsto \lambda(x, b) \) is not necessarily expanding with respect to the (Euclidean) metric induced by the exponential map. (Look at the planar map \( (x, y) \mapsto (0, 10x + y) \))

Fortunately, the normal bundles of our immersions constructed in sections and 3. are flat split, (see 4.1) the map \( E_\lambda \) is expanding and it can be taken for the required \( F_r \).

### 4.2.D. Expanding Maps \( F_r \) for all \( m \) and \( M \).

The above argument delivers expanding maps \( F_r : \mathbb{R}^m \times B^M(r) \to B^{M+m}(1) \) provided \( r \leq (1 + \Delta)^{-1}, \)

where \( \Delta \) is taken according to the \( D(m, N) \) inequalities (see section 1.1 and 3).

\[
\Delta = \sqrt{\frac{3m}{m + 2} + C_0 \frac{m}{\sqrt{M}}}, \quad \text{for } M \geq m,
\]

and

\[
\Delta = 6 \frac{m^\frac{3}{2}}{M}, \quad \text{for } M < m.
\]

### 4.3 Proof of the Regular Homotopy/Approximation Theorem.

**Step 1. Slicing.** Given an immersed manifold

\[
X = X^m \overset{\phi}{\to} \mathbb{R}^n, \quad n > m,
\]

and \( (\text{small}) \) positive numbers \( \varepsilon, \delta > 0 \) there exists an immersion

\[
X \overset{\psi}{\to} \mathbb{R}^n
\]
regularly homotopic tp $\phi$, such that

- $\bullet_{\text{curv}_\varphi}(X) \leq \varepsilon$.

- $\bullet_\varphi$ the first coordinate function $y_1(x) = y_1(\varphi(x))$ of $y = \varphi(x) \in \mathbb{R}^n = \{y_1, \ldots, y_n\}$ is proper Morse, where there are no critical points of $y_1$ on the $\delta i$ levels of $y_1$ for integer $i = \ldots - 2, -1, 0, 1, 2 \ldots$, i.e. the hyperplanes where $y_1 = \delta i$ in $\mathbb{R}^n$ are transversal to $\varphi(X) \subset \mathbb{R}^n$ and

- $\bullet_\varepsilon$ the curvatures of these $\delta i$ levels are bounded by $\varepsilon$.

**Proof.** If $X$ is compact, then $\bullet_{\text{curv}_\varphi}$ achieved achieved by scaling: $x \mapsto \lambda \varphi(x)$ for a large $\lambda$ and then one gets $\bullet_\varphi$ by a preliminary generic rotation of $\varphi(X) \in \mathbb{R}^n$, where then the critical values of $y_1(x)$ moved to the centers of the segments $[\delta i, \delta (i+1)]$, let $\frac{1}{\lambda} = o(\lambda)$ and conclude the proof with the following obvious (but essential)

**4.3.A. Levels Curvature Sublemma.** Let $y(x)$ be a Morse function on a compact Riemannian manifold $X$ and $x_0$ be a critical point, where $y(x_0) = 0$. Then the curvatures of the $\delta$-levels $f^{-1}(\delta) \subset X$ satisfy

$$\text{curv}^i(f^{-1}(\delta)) = o\left(\frac{1}{\delta}\right).$$

**Step 2. Zigzag Folding and Compression.** Reflect the $X$-bands $y_i^{-1}[\delta i, \delta (i+1)] \subset X$ in the hyperplanes $y_1 = \delta i$, $i \in \mathbb{Z}$, and thus "compress" $\varphi(X)$ to a zigzag map $\zeta$ from $X$ to the Euclidean $\delta$-band between a pair of such hyperplane, say between $y_1 = 0$ and $y_1 = \delta$.

**Step 3. Twisted Regularization with Controlled Curvature.** There exists a smooth 10$\delta$-approximation of $\zeta$ by a smooth immersion $\zeta_\varepsilon : X \to \mathbb{R}^n$, such that

- $\bullet_\varepsilon$ the immersion $\zeta_\varepsilon$ is equal to $\zeta$ outside the $\varepsilon$-neighbourhood of the corners of $\zeta$, that is the subset $y_i^{-1}(\delta Z) \subset X$, where $\varepsilon > 0$ en is a given number which may be taken much smaller than $\delta$;

- $\bullet_{\text{reg}}$ the immersion $\zeta_\varepsilon$ is regularly homotopic to $\varphi$,

- $\bullet_{\text{curv}/\delta}$ the curvature $\zeta_\varepsilon$ is bounded by $\frac{1}{\delta}$.

**Proof.** To see how it works, let $\theta_\varepsilon$ and $\theta_{\varphi}$ be two immersions of the circle to the plane, each having a single corner point, both with the same corner angle. If we align these corners properly and attach the immersions one to another at the corner points, we obtain a composed smooth immersion $\theta_\varepsilon$ where, if $\theta_{\varphi}$ is $\varphi$-shaped, this $f_\varepsilon$ is regularly homotopic to $f_{\varphi}$.

Now, in he case of a corner along a hypersurface $X_i = \varphi^{-1}\delta$) attach the product $X_i \times \varphi$ to $\zeta(X)$ along this corner and by doing it to all $X_i$ we obtain a smooth immersion regularly homotopic to $\varphi$ where the conditions $\bullet_\varepsilon$ and $\bullet_{\text{curv}/\delta}$ are easily achievable 10$\delta$ close to $\zeta$. Details are left to the reader.

**Step 4. Rolling Bands into Balls.** The band $\mathbb{R}^{n-1} \times [-10\delta, 11\delta] \supset \zeta_\varepsilon(X)$ is mapped to $B^n(1)$ by "rolled band" immersion $F_r : \mathbb{R}^n \times [-r, r] \to B^n(r)$ for $r$ from 1.2.D, where $F_r$ is restricted to the sub-band $\mathbb{R}^n \times [-r/2, r/2] \mathbb{R}^n \times [-r, r]$ and where we then let $\delta = \frac{1}{2r}$.

In order estimate the curvature of the composed map $\Phi = F_r \circ \zeta_\varepsilon$,

$$X \xrightarrow{\zeta_\varepsilon} \mathbb{R}^n \times [-r/2, r/2] \xrightarrow{F_r} B^n(1),$$

by $\text{curv}_{\zeta_\varepsilon}(X) \leq c = \frac{1}{\delta}$ we recall the construction of the underlying normal immersion

$$f = F_r|_{\mathbb{R}^{n-1} \times \{0\}} : \mathbb{R}^{n-1} \to B^n(1-r),$$
where $\text{curv}(f) \leq 6(1-r)^{-1}n^{2}$ and where also (the differential of) this map has controllably bounded anisotropy,

$$\frac{||d\tau_1||}{||d\tau_2||} \leq 2n$$

for all unit tangent vectors $\tau_1, \tau_2 \in T(X)$. It follows that the curvature $\text{curv}_\delta(X)$ is bounded essentially in the same way as that of $F_r$,

$$\text{curv}_\delta(X) \leq 420n^2,$$

and the corresponding approximation inequality follows as in the proof in the genera case of the $\delta$-approximation theorem. (This $\delta$ and that in $[-10\delta, 11\delta]$, albeit similar, are not the same.)

### 4.3.B. Immersions to non-Euclidean $Y$

The above argument, unlike the proof of the the $\delta$-approximation theorem as explained in 3.D doesn’t generalize to immersions from $X$ to general Riemannian manifolds $Y$.

Yet, a combination of the above "twisted regularization" on the top of a routine induction by skeleta delivers the following.

### 4.3.C. Rough Exponential Bound on Curvature

Let $Y$ be a complete Riemannian manifold with $|\text{sect.curv}| \leq \kappa^2$ and let $f : X = X^m \hookrightarrow Y$ be a smooth immersion. If $\text{dim}(Y) > m$ then, for all positive $\delta \leq \frac{1}{\kappa}$, the map $f$ can be $\delta$-approximated by immersions $f_\delta : X \to Y$, which are regularly homotopic to $f$ and such that

$$\text{curv}^\perp_\delta(X) \leq \frac{(1 + \kappa)100^m}{\delta}.$$

### 4.4 Unfolding Folds and other Singularities

Below is another proof of the regular homotopy/approximation theorem for orientable hypersurfaces, which leads to a better, possibly sharp in some cases, bounds on the curvature.

**Unfolding Lemma.** Let $X = X^m$ be an orientable manifold and $f : X \to \mathbb{R}^{m+1}$ be an immersion. Then, for all $\varepsilon > 0$, there is an immersion,

$$\zeta : X \to \mathbb{R}^m \times [-1, 1],$$

which is regularly homotopic to $f$ and such that

$$\text{curv}_\delta^\perp(X) \leq 1 + \varepsilon.$$

**Proof.** Apply Poenaru’s $h$-principle for pleated maps (see (C) on p.56 in [Gr1986]), and obtain a smooth map $f_1 : X \to \mathbb{R}^{m+1}$ regularly homotopic to $f$, such that the only singularity of the normal projection $\zeta : X \to \mathbb{R}^m \subset \mathbb{R}^{m+1}$ is a folding along a smooth hypersurface $\Sigma = \Sigma^{m-1} \subset X$.

Make the curvature of the immersion $\zeta : \Sigma \to \mathbb{R}^m$ as small as you wish by $\lambda$-scaling as we did earlier and thus also separate different part of $\Sigma$ far one from another, such that, on the balls of large radii $R \sim \lambda$ in $X$, the scaled map is $\varepsilon$-close to the standard fold $(x_1, ..., x_m) \mapsto (x_1, ..., x_m^2)$.
"Unfold" $\zeta \sim \zeta_0 = (\lambda \zeta, y) \in \mathbb{R}^{m+1}$, where $y : X \to \mathbb{R}$ is a smooth function on $X$, which, in the obvious normal coordinates, depends only on the last coordinate $x = x_m$, where it is $\varepsilon$-close to a lift $\eta_0 : \mathbb{R} \to \mathbb{R}_+ \times [-1, 1]$ of the standard fold $\mathbb{R} \to \mathbb{R}_+, x \mapsto y = x^2$, where $\eta_0(x) = (x, y(x))$ and where

the $x$-segment $[-1, 1]$ is sent by $\eta_0$ to the semicircle in the half plane $(x, y)_{y \geq 0}$ and $\eta_0(x) = -1$ for $x < -1$ and $\eta_0(x) = 1$ for $x > 1$.

Conclude the proof by rolling the band $\mathbb{R}^m \times [-1, 1]$ into the ball as in the above step 4.

Remarks. (a) Our unfolding with controlled curvature quantifies a single step in removal of the singularities argument (see [GE1971] and section 2.1 in [Gr1986].)

To do the same for all step and thus unfold more general Thom-Boardman singularities with controlled curvature start by observing that our image curve $\eta_0(\mathbb{R}) \subset \mathbb{R}_+ \times [-1, 1]$, (which is only $C^1$-smooth), is equal to the boundary of the 1-neighbourhood of the ray $[1, \infty) \subset \mathbb{R} \times [-1, 1]$.

Then, to unfold $\Sigma^{1,...,1}$, of depth $k$, where $1, ..., 1 = 1, ..., 1$, the natural model to use is the boundary of the 1-neighbourhood of the positive quadrant $\mathbb{R}_+^k \subset \mathbb{R}^k \times [-1, 1]$, which has $\text{curv} \leq 1$ as well. But I haven’t checked if this actually works.\(^{18}\)

(b) It could be interesting to quantify the approximation procedure of smooth maps by immersion in Sobolev spaces from [GE1971] and also a similar approximation in [Be1991].

(c) It is unclear how to "controllably unfold" in $\mathbb{R}^{m+1}$ more general singularities of smooth maps $X^m \to \mathbb{R}^m \subset \mathbb{R}^{m+l}$.

This leaves the following question open.

Do smooth immersions $f : X^m \to \mathbb{R}^{m+l}$ are regularly homotopic to immersions $f_0$, the curvatures of which are bounded up to a multiplicative constant by the minimal relative curvatures of $\nabla^1$-trivial immersions of flat tori $T^m \to \mathbb{R}^{m+l}$.

For instance it remains problematic if

all $m$-manifolds $X$ admit immersions $f : X \to B^{2m}(1)$ with curvatures $\text{curv}^1_p(X) \leq \text{cost}\sqrt{m}$, say for $\text{const} = 100$.

5 Lower Bounds on Curvature and upper Bounds on Expansion

5.1 Briefly on Scalar Curvature: $\exists \text{PSC}$, $\exists \text{Sc}^*$, $\exists \text{pss}$, $\text{Ros.ind}$, $\text{SYS}$, etc

All known lower bounds on the curvatures of immersions $X \to Y$ (except for hypersurfaces in spheres) depend on obstruction to positivity of the scalar curvature of Riemannian metrics on $X$ and/or on submanifolds in $X$.

Below is a (non-complete) summary of what is known in this respect.

$\text{Sc}^*(X)$ and $\exists \text{PSC}$. Let $X$ be a compact Riemannian manifold (possibly) with a boundary, let $\text{Sc}(X, x)$ denote the scalar curvature at $x \in X$, that is

\(^{18}\)Beware of non-coorientable folds, such as of the Möbius strip along he central line.

\(^{19}\)If $\text{curv}^1(X^m \to S^{m+1}) < 1$, then $X^m$ is homeomorphic to $S^m$. See [Ge2021] and section 3.7.3 in [Gr2021].

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the sum of the values of the sectional curvatures $\kappa$ at the $m(m-1)$ (ordered) orthonormal bivectors in $T_x(X)$.

For instance, $Sc(S^m(R)) = m(m-1)/R^2$.

Let $\lambda[\beta]$ be the lowest eigenvalue of the operator

$$-\Delta_X + \beta Sc(X)$$
on X with the Dirichlet boundary condition.

Recall that

$$\lambda[\beta](X) = \sup_{\Theta} \inf_{x \in X} (Sc(X,x) - \beta^{-1} (\Delta \Theta(x) + \|\nabla \Theta\|^2)),$$

where the supremum is taken over all smooth functions $\Theta(x)$ on $X$.

that $\lambda[1/4](X)$, denoted $Sc^*(X)$,

serves as a worthwhile substitute for $\inf_{x \in X} Sc(X,x)$ (see [Gr2023]), for compact as well as noncompact ones where in the latter case $Sc^*(X)$ is defined as the limit of $Sc^*(X_i)$ for compact manifolds $X_1 \subset X_2 \subset \ldots \subset X$, which exhaust $X$.

Examples. If $Y$ has constant scalar curvature $\sigma$, then

$$Sc^*(X) = 4\lambda_1(X) + \sigma$$

For instance, the rectangular solids satisfy

$$Sc^* \left( \mathbb{R}^n \right) = 4 \sum_{i=1}^{n} \lambda_i[a_i,b_i] = \sum_{i=1}^{n} \frac{4\pi^2}{(b_i - a_i)^2},$$

the unit hemispheres satisfy:

$$Sc^*(S^n) = n(n-1) + 4n = n(n+3)$$

and

$$Sc^*(B^n) = 4f^2_\nu,$$

for the first zero of the Bessel function $J_\nu$, $\nu = \frac{\pi}{2} - 1$, where $j_{-1/2} = \frac{\pi}{2}$, $j_0 = 2.4042\ldots$, $j_{1/2} = \pi$ and if $\nu > 1/2$, then

$$\nu + a \frac{\nu^{1/2}}{2^{1/2}} < j_\nu < \nu + a \frac{\nu^{1/2}}{2^{1/2}} + \frac{3}{20} \frac{2^{2/3} a^2}{\nu^{1/3}},$$

where $a = \left( \frac{2\pi}{n} \right)^{2/3} (1 + \varepsilon) = 2.32$ with $\varepsilon < 0.13 \left( \frac{8}{24 \pi^2} \varepsilon \right)^2 < 0.1$ [QW1999].

Question. Is there a non-trivial bound

$$\lambda_1(X_1 \times [-r,r]) \geq \lambda_1(B^{m+1}(1)) + \varepsilon,$$

where $X = X^m$ is immersed to the unit ball $B^{m+1}(1)$ with $foc.rad(X) > r$, where $X_1 \times [-r,r]$ is endowed with the (flat) metric induced by the normal exponential map $X_1 \times [-r,r] \to B^{m+1}(1)$ and where $\varepsilon = \varepsilon(X) > 0$ for non-spherical $X$, e.g. $\varepsilon(X) \geq 1/10^{m+1}$ for $X = homeo \mathbb{T}^m$, $m \geq 2$?

$x\text{PSC and Enlargeability}$. A smooth manifold $X$ is $\exists\text{PSC}$ if admits a metric with $Sc > 0$, otherwise it is called $\exists\text{PSC}$. 

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For instance, if a Riemannian manifold \( X = (X, g) \) has \( Sc^n > 0 \) then it is \( \exists PSC \). In fact, there exists a conformal metric \( Sc(\phi g) > 0 \) for some function \( \phi(x) > 0 \) by Kazdan-Warner theorem.

A basic class of \( \exists PSC \) is constituted by enlargeable manifolds\(^2\), where a closed Riemannian \( m \)-manifold \( X = (X, g) \) is enlargeable if it admits a sequence of orientable covering \( \tilde{X}_i \rightarrow X \) and \( \lambda_i \)-Lipschitz maps \( f_i : (\tilde{X}_i, \tilde{g}_i) \rightarrow S^m(1) \), such that

- the maps \( f_i \) are locally constant at infinity and have non-zero degrees,
- \( \lambda_i \rightarrow 0 \) for \( i \rightarrow \infty \).

Clearly, enlargeability is a homotopy invariant of \( X \); moreover, if \( X_1 \rightarrow X_2 \) is a map with non zero degree and \( X_2 \) is enlargeable, then \( X_1 \) is also enlargeable.

**Examples.** Tori, as well as all manifolds \( X \) with \( sect\text{-}curv^t(X) \leq 0 \) are, obviously, enlargeable and (slightly less obviously) aspherical locally homogeneous Riemannian manifolds are also enlargeable.

Aspherical 3-manifolds are enlargeable and there is no example at the present moment of a non-enlargeable aspherical m-manifold for \( m \geq 4 \).

**Rosenberg Index.** This is an invariant of closed smooth manifolds, which takes values in some (algebraic K-theory) group, where non-vanishing, \( Ros.ind(X) \neq 0 \) for spin manifolds \( X \) is, essentially, a shorthand for: "\( X \) is \( \exists PSC \), where this property is provable by a Dirac-theoretic argument".

**Examples.** The following manifolds \( X \), if spin, have \( Ros.ind(X) \neq 0 \).

- \( 4k \)-Manifolds, where a certain Pontryagin number, called \( \tilde{A} \)-genus, doesn't vanish.
- Hitchin’s spheres: manifolds homeomorphic (but not diffeomorphic) to the spheres \( S^m \), \( m = 8k + 1; 8k + 2 \), which don’t bound spin manifolds; these do exist for all \( k = 1, 2, 3, \ldots \).
- Enlargeable manifolds and their products by those in \( \bullet 1 \& \bullet 2 \).
  Moreover, if \( X^l \) is enlargeable, and \( X^{l+m} \) admits a smooth map \( X^{l+m} \rightarrow X^l \), such that the pullback \( Y = Y_x \) of a generic point \( x \in X^l \) has \( Ros.ind(Y) \neq 0 \), then \( Ros.ind(X^{l+m}) \neq 0 \) (see [WXY2021], [Ku 2021] and references therein.)

**spin-Remark.** If \( Ros.ind(X) \) is non-torsion, then, in many (all?) cases (e.g. for enlargeable manifolds and their products by those with \( \tilde{A} \neq 0 \)) the spin condition \( X \) in the proof of \( \exists PSC \) can be replaced by spin of the universal covering \( \tilde{X} \) of \( X \), where the latter is satisfied, for instance, if \( \pi_2(X) = 0 \).

**SYS-Manifolds, SYS-Enlargeability and \( \exists pss \).** SYS is a condition on the integer homology of \( X \) introduced by Schoen and Yau who proved in [SY1979] using minimal hypersurfaces that \( SYS \implies \exists PSC \) (here \( X \) doesn’t need to be spin) for \( m = dim(X) \leq 7 \), where this inequality is due to a possible existence of perturbation stable singularities of minimizing hypersurfaces.

Conjecturally \( \exists pss \) holds for minimizing hypersurfaces of all dimensions (this means that the set of metric \( g \) on \( X \), such that all \( g \)-volumes minimizing hypersurfaces \( \Sigma \subset X \) are smooth, is \( C^2 \)-dense in the space of all Riemannian metrics on \( X \). Also, \( \exists pss \) is expected of stable \( \mu \)-bubbles.

This was confirmed for \( m = 8 \) in [Sm1993] and in [CMS 2023] for \( m \leq 10 \) in the volume minimizing case.

Besides a 2d-partial \( \exists pss \) is presented in [SY2017], where it is used for the proof of \( \exists PSC \) for SYS-manifolds of all dimensions \( m \).

\(^2\)The implication enlargeable \( \implies \exists PSC \) and related \( \exists PSC \) results and problems are extensively discussed in [Gr 2021], where the reader finds further references.
Experience shows, the types of arguments used in these papers for minimizing hypersurfaces equally apply to the stable \( \mu \)-bubbles, but I checked this only for \( \dim(X) \leq 8 \).

SYS-enlargeability generalizes enlargeability, for instance

products of SYS-manifolds by enlargeable ones are SYS-enlargeable,
circle bundles over enlargeable manifolds (albeit not necessarily enlargeable)
are SYS-enlargeable

and

2d-partial \( \mathfrak{pss} \) suffices for the implication SYS-enlargeable \( \implies \) \( \mathfrak{PSC} \) [Gr2018]

\( \mathfrak{PSC} \times \text{Enlargeable} \). If \( X \) is \( \mathfrak{PSC} \) manifolds and \( Z \) is a closed enlargeable manifold of dimension \( m \neq 4 \), then, granted \( \mathfrak{pss} \) for the stable \( \mu \)-bubbles, the product \( X \times Z \) is \( \mathfrak{PSC} \).

Indeed, \( \mathfrak{pss} \) allows, for all \( \varepsilon > 0 \), a representation of the homology class \( [X] \in H_m(X \times Z) \) by a submanifold \( X_\varepsilon \subset X \times Z \), which is (normally) frame bordant in \( X \times Z \circ X^{22} \) and such that

\[
\mathfrak{Sc}^n(X_\varepsilon) \geq \mathfrak{Sc}(X \times Z) - \varepsilon
\]

(see section 2 in [Gr2023]).

Then a codimension \( \geq 2 \) surgery applied to \( X_\varepsilon \) brings you back to \( X \), now endowed with a metric with \( \mathfrak{Sc}(X) \geq \mathfrak{Sc}(X_\varepsilon - \varepsilon^{22}) \).

Remark. The simplest case of this, where \( Z = S^1 \) and where \( \mathfrak{pss} \) is needed only for minimal hypersurfaces; thus the above is unconditionally true for \( \dim(X \times Z) \leq 10^{23} \).

All applications of \( \mathfrak{pss} \) in the present paper are derived from the following.

5.1.A. \( \mathbb{T}^m \)-Stabilized Band Inequality. Let \( X \) be a Riemannian manifold without boundary and let \( f : X \to [-r, r] \) be a 1-Lipschitz, i.e. distance non-increasing function. Then, granted \( \mathfrak{pss} \) for stable \( \mu \)-bubbles of dimensions \( \leq \dim(X) - 1 \), there exists a smooth properly embedded hypersurface \( \Sigma \), which separates \( f^{-1}(-r) \subset X \) from \( f^{-1}(r) \subset X^{24} \) and such that

\[
\mathfrak{Sc}^n(\Sigma) \geq \mathfrak{Sc}^n(X) - \mathfrak{Sc}^n([-r, r]) = \mathfrak{Sc}^n(X) - \pi^2 j r^2.
\]

(See [Gr 2023] and references therein.)

intersect \( \Sigma \). (This condition is non-vacuous only if both sets \( f^{-1}(-r) \) and \( f^{-1}(r) \) are non-empty.) and such that

\[
\mathfrak{Sc}^n(\Sigma) \geq \mathfrak{Sc}^n(X) - \mathfrak{Sc}^n([-r, r]) = \mathfrak{Sc}^n(X) - \pi^2 j r^2.
\]

(See [Gr 2023] and references therein.)

\(^{21}\)This, strictly speaking, makes sense only for orientable \( X \), and it should be phrased more carefully if \( X \) is non-orientable, where one must be aware that double covers of \( \mathfrak{PSC} \) manifolds can be \( \mathfrak{PSC} \).

\(^{22}\)If \( X \) is spin this follows directly from theorem 1.8 in [St2002].

\(^{23}\)In fact, the \( \times S^1 \)-stability of \( \mathfrak{PSC} \times \) is formulated as conjecture 1.24 in [R 2006].

But since the above argument for \( m = 2, 3, 5, 6 \) is missing in this paper, I am worried of myself making a silly mistake.

\(^{24}\)This means that all curve-segments in \( X \) with the ends in \( f^{-1}(-r) \) and \( f^{-1}(r) \) intersect \( \Sigma \). (This condition is non-vacuous only if both sets \( f^{-1}(-r) \) and \( f^{-1}(r) \) are non-empty.)
5.2 Norms on Curvature, \( m \)-th Scalar Curvature, Gauss Formula and Petrovčkā's Inequality

Besides the normal curvature of an immersion \( f : X \to Y \) at a point \( x \in X \), that is
\[
\text{curv}^f_j(X, x) = \sup_{\tau, \nu} \| \Pi_\nu(\tau, \tau) \|,
\]
where \( \Pi = \Pi_\nu(\tau_1, \tau_2) \) is the second fundamental form the supremum is taken over the unit tangent vectors \( \tau \in T_x(X) \) and unit normal vectors \( \nu \in T^*_f(Y) \) and \( f(T(x)) = T^*_f(Y) \) define the \( L^2 \) norm of the second fundamental form \( \Pi = \Pi_f(X, x) \) as follows.

\[
\| \Pi \|^2_{L^2} = \sum_{j=1}^{m} \sum_{i_1, i_2=1}^{m} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2,
\]
where \( \{\tau_i\}, \ i = 1, ..., m = \dim(X) \), is a frame of orthonormal vectors in the tangent space \( T_x(X) \subset T^*_f(Y) \) and \( \{\nu_j\}, \ j = 1, ..., k = \codim(X \to Y) \) is such a frame in the normal space \( T^*_f(Y) \subset T_x(X) \).

Observe that
\[
\| \Pi \|^2_{L^2} = \sum_{i_1, i_2=1}^{m} \sum_{j=1}^{m} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2 \leq k \sum_{i_1, i_2} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2 \leq km \cdot \text{curv}^f(X)^2
\]
and that
\[
\| \Pi \|^2_{L^2} = \sum_{i_1, i_2=1}^{m} \sum_{j=1}^{m} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2 \leq m^2 \sum_{j} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2 \leq m^2 \text{curv}^f(X)^2,
\]
because
\[
\text{curv}^f(X)^2 = \sup_{\nu, \tau} \Pi_{\nu}(\tau, \tau) \geq \sup_{j, i_1, i_2} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2,
\]
for principal curvatures \( \alpha_{j,l} \) of \( X \) with respect to \( \nu_j \) and
\[
\sum_{j} \Pi_{\nu_j}(\tau_{i_1}, \tau_{i_2})^2 = \sup_{\nu} \Pi_{\nu}(\tau_{i_1}, \tau_{i_2})^2,
\]
since \( \Pi_{\nu_j}(\tau_1, \tau_2) \) is a linear function in \( \nu \in T^*_f(Y) \) for all pairs \( \tau_1, \tau_2 \in T_x(X) \).

Next, following [Pet2023], define \( \Pi = \Pi_f(X, x) \) as the average of \( \sum_j \| \Pi_{\nu_j}(\tau, \tau) \|^2 \) over the unit vectors \( \tau \in S^{m-1}_x \subset T_x(X) \).

Clearly,
\[
\Pi_f(X, x) \leq (\text{curv}^f(X, x))^2,
\]
where the equality holds if and only if \( \| \Pi \|^2_{L^2} = \| \text{mean.curv}^f(X, x) \|^2 \). (If \( \codim(X) = 1 \), this means that all principal curvatures \( X \) at \( x \) are mutually equal.)

Furthermore, integration of \( \| \Pi(\tau, \tau) \|^2 \), which is a 4th-degree polynomial in \( \tau \), over the unit sphere \( S^{m-1}_x \) shows (as in section 2 and in [Pet2023]) that
\[
\Pi \leq \frac{2}{m(m+2)} \left( \| \Pi \|^2_{L^2} + \frac{1}{2} \| \text{mean.curv}^f \|^2 \right)
\]

\footnote{Gilles Pisier explained to me that random (in a suitable sense) \( \Pi \) have \( \| \Pi \|^2 \geq \text{const} \cdot \sup_{\nu, \tau} \Pi_{\nu}(\tau, \tau) \), that is the inequality \( \| \Pi_f(X) \|^2 \leq km \cdot \text{curv}^f(X)^2 \) is optimal up to a multiplicative constant.}

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or
\[ ||\Pi||_2^2 = \frac{m(m+2)}{2} \Pi - \frac{1}{2} ||\text{mean.curv}^+||^2 \]

For instance, if \( m = \text{dim}(X) = 1 \) and \( n = m + k = \text{dim}(Y) = 3 \) and the principal curvatures of \( X \) at \( x \) are \( \alpha_1 \) and \( \alpha_2 \), then
\[
\text{curv}^+(X, x) = \max(|\alpha_1|, |\alpha_2|),
\]
\[
||\Pi||_2^2 = \alpha_1^2 + \alpha_2^2,
\]
\[
||\text{mean.curv}^+|| = |\alpha_1 + \alpha_2|
\]
and
\[
\Pi = \frac{1}{4} (\alpha_1^2 + \alpha_2^2) + \frac{1}{8} (\alpha_1 + \alpha_2)^2 = \frac{3}{8} (\alpha_1^2 + \alpha_2^2) + \frac{1}{4} \alpha_1 \alpha_2;
\]
if \( X = S^2 \subset Y = \mathbb{R}^3 \), where \( \alpha_1 = \alpha_2 = 1 \), this makes \( \Pi = 1 \) as well.

Now let us turn to the curvature the ambient Riemannian manifold \( Y \) of dimension \( n \geq m \) and define the function \( Sc_{\partial m}(Y) \) on the tangent \( m \)-planes \( T_y^m \subset T(Y) \) in \( Y \), as the sum of the sectional curvatures \( \kappa \) of \( Y \) on the bivectors in \( T_y^m \) at \( y \), that is the scalar curvature of submanifold \( y \ni Y^m \subset Y \) tangent \( T_y^m \), i.e. \( T_y(Y^m) = T_y^m \subset T_y(Y) \) and having zero relative curvature in \( Y \) at \( y \),
\[
Sc_{\partial m}(Y, T_y^m) = Sc(T_y^m, y) = \sum_{i<j=1,...,n} \kappa(e_i \wedge e_j)
\]
for a frame of orthonormal vectors \( e_i \in T_y^m \).

In this terms, the Gauss formula for the scalar curvature of \( X \rightarrow Y \) reads:
\[
Sc(X, x) = Sc_{\partial m}(Y, T_x(X)) + ||\text{mean.curv}^+(X, x)||^2 - ||\Pi||_2^2,
\]
where by Petrunin’s formula
\[
||\text{mean.curv}^+(X, x)||^2 - ||\Pi(X, x)||_2^2 = ||\frac{1}{2} \text{mean.curv}^+(X, x)||^2 - \frac{m(m+2)}{2} \Pi.
\]

Hence, the inequality \( Sc_{\partial m}(Y) \geq \sigma_m \) implies that
\[
Sc(X) \geq \sigma_m - ||\Pi(X, x)||^2.
\]
Therefore
\[
[km] \quad Sc(X) \geq \sigma_m - km \cdot \text{curv}^+(X)^2
\]
for \( k \leq m \) and
\[
Sc(X) \geq \sigma_m - m^2 \text{curv}^+(X)^2.
\]
for all \( k \), where Petrunin’s formula yields the better inequality
\[
Sc(X) \geq \sigma_m - \frac{m(m+2)}{2} \Pi \geq \sigma_m - \frac{m(m+2)}{2} \text{curv}^+(X)^2.
\]

It follows that if the manifold \( X \) is \( \not\exists \text{PSC}, \) i.e. it admis no metric with \( Sc > 0 \), then
\[
\text{curv}^+(X) \geq \sqrt{\Pi} \geq \sqrt{\frac{2\sigma_m}{m(m+2)}} \text{ for all } k \text{ and } n = m + k = \text{dim}(Y), \ Y \leftrightarrow X,
\]
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and
\[ \text{curv}^i(X) \geq \sqrt{\frac{\sigma_m}{km}} \text{ for } k < mn/2. \]

**Examples and Corollaries.**
Let \( X \) be an \( m \)-dimensional \( \mathbb{HP} \) PSC manifold, e.g. the \( m \)-torus \( \mathbb{T}^m \) or Hitchin’s exotic sphere and let us indicate several examples of lower bounds on curvatures of immersions from \( X \) to "small" manifolds \( Y \).

\( (*_{S^n}) \) Immersions \( f \) from \( X \) to the unit spheres \( S^n \), \( n = m + k \), where \( SC_{\text{mil}}(S^n) = SC(S^m) = m(m - 1) \), satisfy
\[ \text{curv}^i_f(X) \geq \sqrt{\frac{m - 1}{k}} \]
and
\[ \text{curv}^c_f(X) \geq \sqrt{\frac{2m - 2}{m + 2}} \text{ for all } k, \]
where the latter inequality (1.2.(c) in [Pet 2023]) is sharp as it is seen with \( \mathbb{T}^m \subset S^{2N_{\text{mil}}^2}(1) \) in section 2.1, where \( \sqrt{\Pi_{\text{Hilb}}} = \text{curv}(\mathbb{T}^m) = \sqrt{\frac{2m - 2}{m + 2}} \).

\( (c) \) Let \( Y \) be the product, \( Y = Y_1^{n_1} \times Y_2^{n_2} \), where the sectional curvatures of the factors are bounded from below by
\[ \text{sect.curv}^i(Y_1) \geq 1 \text{ and } \text{sect.curv}^c(Y_2) \geq -k. \]

For instance \( Y_1 = S^{n_1} \) and \( Y_2 = H^{n_2} \).
Let \( f \) be an immersion from an \( m \)-dimensional \( \mathbb{HP} \) PSC manifold to \( Y \) and let
\[ \frac{(m - n_2)(m - n_2 - 1)}{n_2(n_2 - 1)} \geq k. \]
Then the curvature of \( X \) is bounded from below by
\[ \text{curv}^c_f(X) \geq \sqrt{\frac{(m - n_2)(m - n_2 - 1) - kn_2(n_2 - 1)}{m(n_1 + n_2 - m)}}. \]
For instance, if \( m = 2n_2 < 2n_1 \) and \( k = \frac{1}{2} \), then
\[ \text{curv}^c_f(X) \geq \sqrt{\frac{n_2 - 1}{4(n_1 - n_2)}}. \]
Indeed, the \( m \)-th scalar curvature of \( Y \) is bounded from below by
\[ SC(Y) \geq m - n_2 m - n_2 - 1 - kn_2(n_2 - 1). \]

\( (a) \) On Extremality of Veronese. (a) The smallest curvature of immersions of closed connected non-spherical manifolds to \( S^N \) is, prbably, that of the Veronese embeddings (see 6.1) \( \mathbb{R}P^m \rightarrow S^{m(m+3)/2}(1) \), where \( \sqrt{\Pi_{\text{Ver}}} = \text{curv}(\mathbb{R}P^m) = \sqrt{\frac{m - 1}{m + 1}}. \)

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Let us show in this regard that closed connected non-spherical manifolds $X^m$ admit no immersions to $S^N(1)$ with curvatures

$$\text{curv}^i(X \to S^N(1)) < \sqrt{\frac{3}{10}} \left( = \sqrt{\frac{1}{3} - \frac{1}{30}} \right).$$

Indeed, let such an $X = X^m$ be immersed to the unit $N$-sphere, such that

$$\text{curv}^i(X^m \to S^N(1)) \leq \delta, \ m \geq 2, \ \delta < 1/\sqrt{2}.$$

Then the sectional curvature of the induced metric in $X$ satisfies

$$1 - 2\delta^2 \leq \text{sect.curv}^\perp(X) \leq 1 + 2\delta^2.$$

(●) If $\delta = \text{curv}^i(X^m \to S^N) \leq 1/\sqrt{2}$, then

$$\text{sect.curv}^i(X^m) \geq 0 \implies \text{rank}(H_*(X;\mathbb{F})) \leq \text{const}_m \text{ for all fields } \mathbb{F}.$$

and the universal covering $\tilde{X}$ of $X$ satisfies

$$\text{diam}(\tilde{X}) \leq \pi/\sqrt{1 - 2\delta^2}.$$

(●●) If $\delta \leq \sqrt{\frac{3}{10}}$, then

$$\frac{1 + 2\delta^2}{1 - 2\delta^2} \leq 4$$

and the universal covering of $X$ is diffeomorphic to $S^m$.

(●●●) If $\text{curv}^i((X^m) \to S^N(1)) \leq \delta$ then

$$\text{inj.rad}(X) \geq \pi\sqrt{\frac{1}{1 + \delta^2}},$$

and if $X$ is not simply connected, then the universal covering of $X$ satisfies

$$\text{diam}(\tilde{X}) \geq \frac{2\pi}{\sqrt{1 + \delta^2}}.$$

Since the inequality

$$\frac{\pi}{\sqrt{1 - 2\delta^2}} \geq \frac{2\pi}{\sqrt{1 + \delta^2}}$$

implies that $\delta \geq \frac{1}{\sqrt{3}} > \sqrt{\frac{3}{10}}$, the proof follows.

(b) **Rigidity Remark.** Looking closer at the above argument reveals that the inequality

$$\text{curv}^i(X^2 \to S^N(1)) \leq \frac{1}{\sqrt{3}}$$

for closed connected non-simply connected surfaces $X^2$ implies that these are congruent to the Veronese surface.

(c) **Small Volume Remark.** If

$$\text{curv}^i(X^m \to S^N(1)) \leq \delta < 1/\sqrt{2}, \ m \geq 2,$$
then the volume of $X$ is, on one hand, bounded by

$$\text{vol}(X) \leq \left( \frac{1}{\sqrt{1 - 2\delta^2}} \right)^m \text{vol}(S^m(1))$$

and on the other hand

$$\text{vol}(X) \geq \left( \frac{1}{\sqrt{1 + 2\delta^2}} \right)^m \text{vol}(S^m(1)).$$

It follows that the order of the fundamental group of $X$ is bounded by the ratio of these two numbers.

$$\text{card}(\pi_1(X)) \leq \left( \frac{\sqrt{1 + 2\delta^2}}{\sqrt{1 - 2\delta^2}} \right)^m.$$

(d) **On Almost Flat** $X \hookrightarrow S^N$. If $\text{curv}_\perp(X \hookrightarrow S^N(1)) \leq \delta$ for a small $\delta$ - I roughly checked this for $\delta \leq 1/4$ - the manifold $X$ lies $\delta$-close to an equatorial $S^m \subset S^N$ and the normal projection $X^m \rightarrow S^m$ is a diffeomorphism.

(e) **On Non-spherical** $Y \leftarrow X$. The inequalities ($\ast_{S^N}$) implies similar inequalities for Riemannian manifolds $Y^N$ with boundaries (e.g. Euclidean and hyperbolic balls) by means of suitable (e.g. projective as in the proof of Burago’s inequality in 3.2.3 in [Gr2086]) diffeomorphisms $F$ from $Y^N$ into $S^N(1)$, where these $F$ controllably increase curvatures of curves in $Y^N$.

For instance

let $Y$ be a complete Riemannian $n$-manifold with the sectional curvature bounded in the absolute value by $|\text{sect.curv}_\perp(Y)| \leq 1$ and with $\text{inj.rad} \geq 1$ and let $f$ be an immersion from an $m$-dimensional $\mathbb{H}P^c$ manifold to $Y$ such that the diameter of the $f$-image of $X$ in $Y$ is at most $0.1$.

Then the curvature of $X$ in $Y$ is bounded from below by

$$\text{curv}_\perp^f(X) \geq 0.1 \sqrt{\frac{m}{n - m} - 1}.$$ 

**Sketch of the Proof.** Conformally modify the Riemannian metric $g$ of $Y$ in the vicinity of $f(X)$ with a conformal factor $\phi(x) = \psi(\text{dist}_g(x, x_0))$, where $\psi(d)$ is defined by the following condition.

if $Y$ is isometric to the hyperbolic $n$-space $\mathbb{H}^n$ with curvature $-1$, then the unit ball $B_{x_0} \subset \mathbb{H}^n$, when endowed with the new metric $\psi(\text{dist}_{\text{hyp}}(x, x_0))g_{\text{hyp}}$ becomes isometric to the hemisphere $S^m_+ \subset \mathbb{S}^n$.

Then one can show that the curvature of $X$ with respect to the metric $\phi g$ is not much greater than that that with respect to $g$ and he proof follows.

**Remark.** The constant 0.1 is very crude; the proof of a similar nearly optimal inequality is presented in the next section following Petrunin’s argument from §4 in [Pet2023].

### 5.2.1 On Meanings of $S_{c,m}$

The above notwithstanding, the role of $S_{c,m}$ and especially of the bound $S_{c,m} \geq \sigma$ played in shaping the global (geo)metric and/or topological properties of
Riemannian n-manifolds $Y$ with $Sc_{|m}(Y) \geq \sigma$ for $n > m \geq 3$ essentially remain 100% problematic.

One expects that positivity of $[Sc_{|m}](Y)$ for $m < n = dim(Y)$ has greater significance than positivity of $Sc(Y) = [Sc_{|m}](Y)$. Below is an, albeit weak, confirmation to this.

Let $Y$ be a Riemannian manifold, the boundary $\partial Y$ of which is divided into two disjoint parts, $\partial Y = \partial_1 Y \cup \partial_2 Y$, where $\partial_1 Y$ and $\partial_2 Y$ are unions of connected components of $\partial Y$.

Let

$$\text{dist}(\partial_1 Y, \partial_2 Y) = 2r,$$

let the sectional curvature of $Y$ be bounded from below,

$$\kappa(Y) \geq \kappa_-$$

and let

$$Sc_{(n-1)} \geq \sigma.$$

Then

$Y$ contains a smooth hypersurface $X \subset Y$, which separates $\partial Y$ from $\partial_2 Y$ (recall that $\partial Y = \partial_1 Y \cup \partial_2 Y$) and such that the scalar curvature of the induced Riemannian metric in $X$ satisfies:

$$[\sigma|\alpha]$$

$$Sc(X) \geq \sigma - (n-1)\alpha_{\kappa_-}(r)^2,$$

where $\alpha_{\kappa_-}(r)$ denotes the curvature of the circle of radius $r$ in the standart surface with constant curvature $\kappa_-$, e.g.

- $\alpha_1(r) = \frac{\cos r}{\sin r}$
- $\alpha_0(r) = \frac{1}{r}$
- $\alpha_{\kappa_-}(r) = \frac{\kappa_- + e^{-r}}{e^{-r} - e^{-r}}$

Proof. Let $X_{[2r]} \subset Y$ be the $2r$-equidistance hypersurface to $\partial Y$ and $X_{[2r-\varepsilon]} \subset Y$ be the $r$-equidistant to $X_{2r}$ on the side of $\partial Y$. Then clearly ($\sigma_\varepsilon$) the hypersurface $X_{[2r-\varepsilon]}$ is $C^{1,1}$-smooth with the curvature, i.e. with the norm of the second fundamental form, bounded by $\alpha_{\kappa_-}(r)$.

Hence, $X_{[2r-\varepsilon]}$ can be approximated by $C^\infty$-smooth hypersurfaces $X_\varepsilon \subset Y$ with curvatures bounded by $\alpha_{\kappa_-}(r) = \varepsilon$ for all $\varepsilon > 0$. QED.

Remark. If $n \leq 8$ (and $\mathbb{H}^3$ is known for the $mu$-bubbles), then $\partial Y$ and $X_2 \subset Y$ can be separated by a smooth stable $\mu$-bubble $X_\varepsilon \subset Y$ such that the scalar curvature of a warped product metric $g^\mu = g^\mu(x, t) = dx^2 + \varphi(x)^2 dt^2$ on $X \times T^1$ is bounded from below in terms of $\sigma = \inf_y Sc(Y,y)$ and $r$ as follows (see section 3.7 in [Gr2021]),

$$Sc(X) \geq \sigma - \frac{(n-1)\pi^2}{n r^2}.$$

Although this is not formally stronger than $[\sigma|\alpha]$, it is by far more general and informative.

Questions. (a) Does ($\sigma_\varepsilon$) generalize to submanifolds $X \subset Y$ of codimensions $k > 1$, where $Y$ is, in some way, "wide in k-directions"?

(b) Much stronger results for another "intermediate scalar curvature" are obtained in [BHJ 2022].
For instance, let $Y$ be a Riemannian manifold homeomorphic to $X_0 \times B^k(1)$, where $X_0$ is a closed manifold of dimension $n - k$, let the sectional curvature of $Y$ be bounded by $|\kappa(Y)| \leq 1$ and the injectivity radius by $\text{inj.rad}(Y) \geq 1$ (compare with [Gr2022]).

What else do you know about $Y$ to effectively evaluate the minimal $\alpha$, such that $Y$ contains a submanifold $X \subset Y$ homologous to $X_0 = X_0 \times \{0\} \subset X_0 \times B^k(1) = X$, such that the curvature of $X$ in $Y$ is bounded by $\alpha$?

What is the best bound on $\alpha$ in a presence of a proper (boundary-to-boundary) $\lambda$-Lipschitz map $X \to B^k(1)$?

The known (unless I am missing some) quantitative transversality theorems applied to maps $X \to B^k$ deliver submanifolds $X \subset Y$ with $\alpha \leq \text{const}_n$, but we need $X$ with $\alpha \leq \text{const}_k$ for our purposes.

Alternatively, an inductive use of $(\mathcal{O}_r)$ leads to a bound with

$$\text{const} \sim 100^k(1 + \text{diam}(Y))$$

but this is not satisfactory either.

(b) How much (if at all) do (essential) global (geo)metric and/or topological properties of Riemannian $n$-manifolds $Y$ with $\text{Sc}^{\mathbb{c}}(Y) \geq m(m - 1)$ for $m \geq 3$ differ from those with $\text{Sc}(Y) \geq n(n - 1)$?

For instance, does the product $\mathbb{T}^{n-2} \times S^2$, $n \geq 4$, admit a metric with $\text{Sc}^{\mathbb{c}} > 0$?

### 5.3 Mean$_{(\mathcal{O}_r)}(\partial Y)$-Curvature, $Sc^*$-Curvature and Immersions to Riemannian Manifolds with Boundaries

Let $\Phi = \Phi(y)$ be a smooth function on a Riemannian manifold $Y$, let $\nabla \Phi$ be the gradient field of $\phi$ and let $\Delta_{\text{in}} \Phi(y)$ be the maximum of the $V$-derivatives of the $m$-volumes on the tangent $m$-planes $\tau \subset T_y(Y)$.

\textbf{Example.} Let $Y = \mathbb{R}^N = \{y_1, \ldots, y_N\}$ and and $\Phi(y) = -(y_1^2 + \ldots + y_{N-l})^2$. If $l \geq m$, then $\Delta_{\text{in}} \Phi(y) = 0$ and if $l < m$ then $\Delta_{\text{in}} \Phi(y) = -2(m - l)$. In particular if $m = N$, then $\Delta_{\text{in}} \Phi(y) = \Delta \Phi(y)$ where $\Delta \Phi$ is the ordinary Laplacian.

****************************************************

Given an $m$-submanifold $X^m \subset Y$, decompose $V$ on $X$ into a tangent and normal fields to $X$, $V|_X = \tau|_X + \nu|_X$, where $\tau|_X = \text{grad}_X \Phi(x)$ and recall that the $V$-derivative of the volume (element) on $X$ is

$$\Delta_X \Phi(x) + \langle M(x), \nabla \Phi(x) \rangle,$$

where $\Delta_X$ is the Laplacian on $X$ with respect to the induced metric on $X$ and where $M(x)$ is the mean curvature vector of $X$, such that $\tau$-direction of $M$ is chosen such that the volume of $X$ increases under the $V$-flow.

It follows that

$$-\Delta_X \Phi \geq -\Delta_{\text{in}} \Phi + \langle M, \nabla \Phi \rangle.$$

This yields the following lower bound on the first eigenvalue $\lambda^{[2]}$ of the operator $-\Delta_X + \beta \text{Sc}(X)$ (see section 5.1) on $X$ with the induced Riemannian metric and the Dirichlet boundary condition.

$$\lambda^{[2]} \geq \inf_{x \in X} \left( M(x)^2 - \|\nabla \Phi(x)\|_2^2 - \beta^{-1} \left( \Delta_{\text{in}} \Phi + \|\nabla \Phi(x)\|^2 \right) + \langle M(x), \nabla \Phi(x) \rangle \right)$$

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\[
\geq \inf_{x \in X} \left( -\|\nabla\|^2_{L^2}(x) - \beta^{-1}\Delta_{\text{int}} \Phi - \left( \beta^{-1} + \frac{1}{2} \right) \|\nabla \Phi(x)\|^2 \right)
\]
for all smooth functions \( \Phi \) on \( Y \).

**Mean\( m \)curv \( \geq m \).** Define the mean curvature counterpart of \( SC_{m+1} \) as

\[
\text{mean}_{m} \text{curv}^+(\partial Y) = c_1 + c_2 + \ldots + c_m,
\]
where \( c_i \) are the first \( m \) smallest principal curvatures of the boundary of \( Y \).

**Example.** If \( Y = B^m(R) \times \mathbb{R}^{l} \subset \mathbb{R}^{k+l} \), then \( \text{mean}_{m} \text{curv}^+(\partial Y) = \max((m-l)R^{1}, 0) \).

(Our understanding of \( \text{mean}_{m} \text{curv} \) is as meager as that of \( SC_{m} \). For instance, does the product \( T^{n-2} \times B^{2} \), \( n \geq 4 \), admit an immersion to \( \mathbb{R}^{n} \) with \( \text{mean}_{3} \text{curv}^+ \text{ of } \partial T^{n-2} \times B^{2} > 0 \)?

This may be related to the above question about \( SC_{m+1} \) by the doubling construction as it is done for convex \( (m = 1) \) and mean convex \( (m = n-1) \) hypersurfaces.

Let \( Y \) be a complete connected Riemannian manifold with non-negative sectional curvature and with non-empty boundary, and let

\[
\text{mean}_{m} \text{curv} \geq m.
\]

Let \( \Phi(y) = -\gamma(1 - \frac{1}{2} \text{dist}(y, \partial Y))^2 \) and observe that \(-\Delta_{\text{int}} \Phi \geq m \) and \( \|\nabla \Phi\| \leq 1 \).

Let \( \text{codim}(X) = \text{dim}(Y) - m \leq k \) and \( \text{curv}^+(X) \leq \sqrt{C} \) then

\[
\lambda^m \geq C_{km} + \beta^{-1} \gamma m - \gamma^2 \left( \beta^{-1} + \frac{1}{2} \right) \text{ for all } \gamma > 0.
\]

The relevant for us \( \beta = \frac{1}{4} \), where

\[
\lambda^{[1/4]} = Sc^*(X) \geq \max_{\gamma > 0} \left( 4\gamma m - C_{km} - \frac{9}{2} \gamma^2 \right) = \frac{16m}{9} - C_{km} - \frac{8m}{9} = \left( \frac{8m}{9} - C_{k} \right)m.
\]

For instance if \( Sc^*(X) \leq 0 \), then

\[
\text{curv}^+(X) \geq \sqrt{\frac{8m}{9k}}.
\]

**5.3.A. Corollary.** Let \( f: \mathbb{R}^{m} \rightarrow B^{m+k}(1) \subset \mathbb{R}^{m+k} \) be a smooth distance increasing immersion. Then

\[
\text{curv}^j_\|\text{R}^{m}\| \geq \sqrt{\frac{8m}{9k}}.
\]

**Remarks.** (a) The above is a (minor) modification of (a part of) Petrunin’s argument from [Pet2023] used for the proof of the inequality

\[
\text{curv}^j_\|\mathbb{R}^{m}\| \rightarrow B^{N}(1) \geq \sqrt{\frac{3m}{m+2}}
\]
for immersions of \( \mathbb{R} \)PSC manifolds \( X \).
(b) If the curvature of an immersion of a closed manifold to the unit ball satisfies
\[ \text{curv}^1(X^m \rightarrow B^N(1)) \leq \delta \]
then \( X^m \) is contained in the \( 2\delta \)-neighbourhood of an equatorial \( S_{eq}^m \subset S^N \) and if \( \delta \leq 0.05 \), then the normal projection \( X^m \rightarrow S_{eq}^m \) is a diffeomorphism.

(This can be derived from 5.2(d) by radially projecting \( X^m \) to the boundary sphere \( S^{N-1} = \partial B^N(1) \).)

5.3.B. **Problem.** Identify \( m \)-dimensional manifolds \( X, m \geq 3 \), which admit metrics with \( Sc > 0 \), yet satisfy Petrunin’s inequality \( \text{curv}^1(X^m \rightarrow B^N(1)) \geq \sqrt{\frac{3m}{m+2}} \) for all \( N \) and all immersions \( f \).

Moreover, classify \( m \)-manifolds \( X \) which admit immersions \( f : X \rightarrow B^N \) with \( \text{curv}^1(X^m \rightarrow B^N(1)) \leq (1 - \delta)\sqrt{\frac{3m}{m+2}} \) for a given \( \delta > 0 \).

5.4. **Low Bounds on Expansion of Equidimensional Immersions.**

Let \( f : X \rightarrow Y \) be an expanding, i.e. locally distance nondecreasing map between compact manifolds with boundaries

The simplest invariant which is monotone increasing under such an \( f \) is the inradius of \( X \),
\[ \text{inrad}(Y) \geq \text{inrad}(X), \]
where \( \text{inrad}(X) = \sup_{x \in X} \text{dist}(x, \partial X) \).

In fact,
\[ \text{dist}(x, \partial X) \leq \text{dist}(f(x), \partial Y) \text{ for all } x \in X, \]
since (some connected component of) the \( f \) pullback of a curve from \( f(x) \) to \( \partial Y \) in \( Y \) connects \( x \) with partial \( X \).

This also shows that if
\[ \text{dist}(x_1, x_2) \leq \text{dist}(x_1, \partial X), \]
then
\[ \text{dist}(f(x_1), f(x_2)) \geq \text{dist}(x_1, x_2), \]
that is \( f \) is distance increasing on all balls in the interior of \( X \).

**Example of a Corollary.** Let \( X \subset \mathbb{R}^n \) be the convex hull of two balls of radii \( r_1 \) and \( r_2 \leq r_1 \) in \( \mathbb{R}^n \), such that distance \( d \) between their centers is \( \geq r_1 \) and let \( f : X \rightarrow \mathbb{R}^n \) be an expanding map. Then
\[ \text{diam}(f(X)) \geq 2r_1 + \frac{r_1(d + r_2)}{r_1 + d + r_2} + \frac{r_2r_1}{r_1 + d + r_2} = \frac{r_1(d + 2r_2)}{r_1 + d + r_2} \]

In fact, the image of \( f \) contains the union of an \( r_1 \)-ball \( B_1 = B(r_1) \subset \mathbb{R}^n \) and an \( r \)-ball \( B_2 = B(r) \) for \( r = \frac{r_1(d + 2r_2)}{r_1 + d + r_2} \), with the center of \( B_2 \) in the boundary \( \partial B_1 \).

5.4.A. **Question.** Do expanding self-maps \( X \rightarrow X \) of compact manifolds with boundaries send \( \partial X \rightarrow \partial X \)?

More interestingly, let \( X \) and \( Y \) be closed connected domains in the Euclidean space \( \mathbb{R}^n \) and let \( \lambda_1(X) \) and \( \lambda_1(Y) \) be the first eigenvalues of the Laplace operators in these domains with the Dirichlet boundary conditions.
5.4.B. Large Rectangle Theorem. If $X$ is a rectangular solid, $X = \times_{i=1}^{n}[0,d_i]$ and if $X$ admits an expanding map $f : X \to Y$, then
\[ \lambda_1(X) \geq \lambda_1(Y). \]

In fact, let $(g_f)$ be the flat metric induced by $f$ in $X$ and observe that

\begin{align*}
\bullet \quad & \lambda_1(X,g_f) \geq \lambda_1(Y) \\
\bullet\bullet\quad & \text{there exists a distance decreasing map } (X,g_f) \to X\text{ of positive degree } (\text{this is } f^{-1}, \text{of course}).
\end{align*}

This yields the proof by the following theorem (see [Gr2023] and references herein).

Sc”-Extremality of Rectangular Solids. Let $Z$ be a compact orientable Riemannian manifold with a boundary and
\[ \Phi = \{\phi_1, \ldots, \phi_n\} \to \times_{i=1}^{n}[0,d_i] = [0,d_1] \times \ldots \times [0,d_n] \]
be a continuous map, where the functions $\phi_i : Z \to [0,d_i]$ are 1-Lipschitz (distance non-increasing) and where $\Phi$ sends $\partial Z \to \partial \times_{i=1}^{n}[0,d_i]$.

If the map $\Phi$ has non-zero degree, then
\[ Sc^n(Z) \leq Sc^n\left(\times_{i=1}^{n}[0,d_i]\right). \]

5.4.C. Question. Does 5.4.B holds true for all convex $X$?

5.5 Bounds on Focal Radii of Immersions

The focal radius of an immersed manifold $X \overset{f}{\hookrightarrow} Y$,
\[ \text{foc.rad}(X) = \text{foc.rad}(X \hookrightarrow Y) = \text{foc.rad}_f(X) \]
is the supremum of those $R$, for which the differential of the normal exponential map, denoted
\[ \exp^\perp : T^\perp(X) \to Y, \]
is injective along all normal segments of length $< R$, where, in the case of a non-complete $Y$ or a presence of a boundary $\partial Y$, one has to say "defined and injective...".

If $Y$ has constant sectional curvature, then the focal radii of submanifolds are intimately related to their curvatures in $Y$.

For instance,
\[ \text{foc.rad}(X \hookrightarrow \mathbb{R}^N) = \frac{1}{\text{curv}^\perp(X \hookrightarrow \mathbb{R}^N)}. \]

and
\[ \text{foc.rad}(X \hookrightarrow B^N(1)) = \min\left(\frac{1}{\text{curv}^\perp(X)}, \text{dist}(X,\partial Y)\right). \]

More generally, if $\text{curv}^\perp(X) \leq \alpha$, then $\text{foc.rad}(X)$ is bounded by the radii of circles in $S^2$ with curvatures $\alpha$ and if $\text{sect.curv}^\perp(Y) \leq \kappa$, then $\text{foc.rad}(X)$ is
bounded from below by the radii of circles in surfaces with constant curvature $\kappa$.

**Codimension 1.** If $X^m \to Y^{m+1}$ is a coorientable immersion and $r < foc.rad(X)$ then the Riemannian metric $g_r$ induced in $X \times [-r, r]$ by the exponential map satisfies

$$Sc^*(X \times [-r, r]) > Sc^*(Y)$$

and $dist_{g_r}(X \times \{-r\}, X \times \{r\}) = 2r,$

where the $T^*$-stabilized band inequality implies (see section 3.6 [Gr 2021] and references therein):

$$\left[ \frac{\pi}{\sqrt{Sc^*}} \right], \quad r \leq \frac{\pi}{\sqrt{Sc^*(X \times [-r, r])}} < \frac{\pi}{\sqrt{Sc^*(Y)}}$$

provided one of the following two conditions is satisfied.

(i) $X$ has non-zero Rosenberg index.

(ii) No smooth hypersurface $X' \subset X \times [-1, 1]$, which is homologous to $X \times \{0\}$ admits a metric with $Sc > 0$ and $m = dim(X) \leq 7$ (see section 5 in [Gr 2021] and section 2 in [Gr 2022]).

**Comparison with the Gaussian Curvature Inequalities in Sections 5.2, 5.3.** If $Y = S^{m+1}(1)$, then the Gauss formula implies that $\text{curv}^*(X) \geq \sqrt{m-1}$ which shows that

$$\text{foc.rad}(X) < \frac{1}{\sqrt{m-1}},$$

while the above inequality

$$\text{foc.rad}(X) < \frac{\pi}{\sqrt{m(m-1)}}$$

serves better for $m \geq 10 \approx \pi^2$.

Apparently, there must be an inequality better than both of the two.

**Expanding Quantification of $\left[ \frac{\pi}{\sqrt{Sc^*}} \right]$.** Let $X$ be a compact orientable $m$-dimensional manifold with a boundary, let $Y$ be a Riemannian manifold of dimension $m + 1$ and let $X \to Y$ be a smooth immersion.

Let $\phi : X \to [-1, 1]^m$ be a continuous map, which sends the boundary of $X$ to the boundary of the cube $[-1, 1]^m$ and let $deg(f) \neq 0$.

Let $d_i \geq r = 2r$, $i = 1, ..., m$ be the distances between the pullbacks of the opposite faces of the cube with respect the Riemannian metric induced in $X$ by the immersion $X \to Y$.

Then, under the above assumptions (i) or (ii), e.g. if $X$ is spin or if $dim(X) \leq 7$,

$$\frac{\pi^2}{4r^2} + \sum_{i=1}^{m} \frac{\pi^2}{(d_i - 2r)^2} \geq \frac{1}{4} Sc^*(Y),$$

that is

$$\left[ \frac{d_i - 2r}{\pi^2} \right] \leq \left( \frac{Sc^*(Y)}{\pi^2} - \sum_{i=1}^{m} \frac{4}{(d_i - 2r)^2} \right)^{-\frac{1}{2}}.$$

\(^{27}\)Coorientability can be achieved by taking a double cover of $X$.

\(^{28}\)See section 5.1 for examples of manifolds $X$ with $Ros.ind(X) \neq 0$.

\(^{29}\)If $m = dim(X) \neq 4$ this is equivalent to $3PS\text{-property}$ of $X$.

\(^{30}\)If $m > 7$ one needs, $2\text{pss}$ for the stable bubbles in $(m+1)$-dimensional manifold, which, most likely, follows for $m \leq 9$ by the argument from [CMS2023].

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**Codimension 2.** Let be a coorientable immersion with $foc.rad(X) = r$. Then the exponential map from the the normal $\frac{r}{2}$-sphere bundle $S$ to $Y$ is an immersion with $foc.rad = \frac{r}{2}$.

**Case 1:** $\text{Euler}^i = 0$. If $X$ is a $\exists PCS$ manifold and $\dim(Y) = m + 2 = \dim(X) + 2$, and the normal Euler class of the immersion vanishes, (e.g. $X \to Y$ is an embedding homologous to 0), then $S \to X$ is a trivial circle bundle, $S = X \times S^1$. Hence, $S$ is a $\exists PCS$ with a few (some still unsettled) exceptions (see section 5.1) and $[\frac{2\pi}{\sqrt{Sc^*}}]$ applied to $S$ then shows that

$$\frac{2\pi}{\sqrt{Sc^*(X \times [-r, r])}} < \frac{2\pi}{\sqrt{Sc^*(Y)}}$$

and all smooth hypersurfaces $X'' \subset X \times \mathbb{T}^1 \times [-1, 1]$, which are homologous to $X \times \mathbb{T}^1 \times \{0\}$ satisfy either the above (i) or (ii).

**Case 2:** $\text{Euler}^i \neq 0$. Probably, if the Rosenberg index is non-torsion, then the same is true for (non-trivial) circle bundles $S \to X$ in most (all?) cases and $[\frac{2\pi}{\sqrt{Sc^*}}]$ holds true in such a case for spin manifolds.

Also, if $X$ is enlargeable, then, granted $\exists pss$ for $m + 2 = \dim(Y)$ for $Y \hookrightarrow X$, (unconditionally for $m = \dim(X) \leq 6$), then 5.1.A applies and shows that the relative homology class of the fiber of the normal $r$-disc bundle $D(r) \to X$ is realizable by a surface $\Delta \subset D$ with $Sc^*(\Delta) \geq Sc^*(Y)$

and, according to the $\mathbb{T}^*$-stable Bonnet-Myers diameter inequality 2.8(b) in [Gr2021],

$$r \leq \frac{2\pi}{\sqrt{Sc^*(Y)}}$$

for enlargeable $X$ as well.

(In fact, one needs here only a 2d-partial $\exists pss$ for the stable $\mu$-bubbles, while 2d-partial $\exists pss$ for minimal hypersurfaces yields the bound $r \leq 8\pi/\sqrt{Sc^*(Y)}$.)

**Codimension $\geq 3$.** Given a closed orientable manifold $\Sigma$ of dimension $l \geq 2$, define the minimal parametric (cospherical) area $\text{PAR}(\Sigma)$ as the infimum of the numbers $A$, such that there exist a compact smooth orientable $M$-dimensional manifold $P$ (possibly) with a boundary and a smooth $P$-family $\Phi_p$ of smooth maps from $\Sigma$ to the unit $l + M$-sphere, that is a smooth map

$$\Phi: \Sigma \to P \to S^{l+M}(1)$$

such that the maps

$$\Phi_p: \Sigma \to S^{l+M}(1)$$

are area $A$-contracting i.e. the areas of all smooth surfaces $S \subset \Sigma$

$$\text{area}(F_p(S)) \leq A^{-1} \text{area}(S)$$

for all $p \in P$

and such that the map $\Phi$ is constant on $\sigma \times \partial P$ and the degree of $\Phi$ is non-zero.
One knows (see [Gr 20/1] and references therein) that if $\Sigma$ is spin$^{31}$, then

$$\text{Par} < ...$$.

$$\text{Par}(\Sigma) \cdot \text{Sc}^n(\Sigma) \leq \text{Par}(S^l(1))\text{Sc}(S^l) = 4\pi l(l-1).$$

Now, let $X$ be closed enlargeable $m$-manifold, $Y$ be a Riemannian $(m+k)$-manifold, let $f : X \to Y$ be a smooth immersion, such that

$$\text{foc.rad}_f(X) > r$$

map from $Y$, and let $B = B(r) \to X$ be the normal $r$-ball bundle endowed with the Riemannian metric induced by the normal exponential map $\exp^B : B \to Y$.

Then, granted $2\text{pss}$ for $\mu$-bubbles of dimensions $\leq m + k - 1^{32}$ there exists a $(k-1)$-dimensional submanifold $\Sigma = \Sigma^{k-1}$ in the $1/3$-annulus in the normal $r$-ball bundle $B = B(r) \to X$

$$\Sigma \subset B(2r/3) \setminus B(r/3)$$

with

$$\text{Sc}^n(\Sigma) \geq \sigma_r = \text{Sc}^n(Y) - \text{Sc}^n[0, r/3] = \text{Sc}^n(Y) - 36\pi^2/r^2$$

and such that the homology class $[\Sigma] \in H_{k-1}(B(2r/3) \setminus B(r/3))$ is equal to a non-zero multiple of the class of the fiber $S^{k-1}_x = \partial B_x(1/3) \subset B(2r/3) \setminus B(r/3)$. (see [Gr2022] and references therein).

To simplify, let the normal bundle of $X$ in $Y$ be trivial$^{33}$ let $P = X \times [r/3, 2r/3] \to B(2r/3) \setminus B(r/3)$ where the embedding is naturally naturally associated with a section of the $r$-sphere bundle $S = S(r) = \partial B(r) \to X$.

Let

$$\tilde{B}^*_p(\rho) \subset B, \ p \in P, \rho \leq r/3,$$

be the one point compactifications of the (open) $\rho$-balls in $B$.

Let

$$\text{inj.rad}_{\exp^B} \rho(Y) \geq r/3^{34}$$

choose a normal frame over $X$ and (radially diffeomorphically) map the balls $B^*_p(\rho) \subset Y$ to $B^{m+k}(\rho) \subset \mathbb{R}^{m+k}$ by means of the inverse exponential maps in $Y$ at the points $y = \exp^B \rho \in Y$.

Finally, scale $B^{m+k}(\rho)$ to the parameter $\pi$-ball $\tilde{B}^{m+k}(\pi)$ and radially map it to the unit sphere $S^{m+k}(1)$.

Thus we obtain a map $F : \Sigma \times P \to S^{m+k}(1)$ such that

- if $\rho < r/3$ then $\deg(F) = 1$,
- if $\text{sect.curv}^*(Y) \leq 1/\rho$, then $F$ is $A$-contracting for $A \geq \rho^2/\pi^2$.

Therefore,

- if $\text{inj.rad}_X(Y) \geq r/3$ and $\text{sect.curv}^*(Y) \leq 1/3r$, where $r = \text{foc.rad}(X)$, then

$$\left(\frac{r^2}{\pi^2}\right) \cdot (\text{Sc}^n(Y) - 36\pi^2/r^2) \leq 4\pi(k-1)(k-2),$$

that is,

$$\text{foc.rad}(X) \leq \pi \sqrt{\frac{4\pi(k-1)(k-2) + 36}{\text{Sc}^n(Y)}}.$$

$^{31}$The spin condition can be relaxed to that for the universal covering of $\Sigma$, but dropping spin all together remains problematic.

$^{32}$If $k = 3$, then 2d-partial $2\text{pss}$ suffices.

$^{33}$The vanishing of he Euler class suffices for our argument.

$^{34}$The inequality $\text{inj.rad}_Y(Y) \geq R$ signifies that $\text{dist}(y, \partial Y) \geq R$ and that the exponential map $\exp_y$ in $Y$ at $y$ smoothly embed the tangent $R$-ball from $T_y(Y)$ to $Y$. 

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Remarks. (a) A similar result holds for immersions of product manifolds $X = X_0^l \times Z^m \to Y^{l+m+k}$, where $Z = Z^m$ is enlargeable,

$$foc.rad(X) \leq \pi \sqrt{\frac{4\pi(k+l-1)(k+l-2) + 36}{Sc^*(Y)}} (k-1),$$

provided $Euler(T^i(X)) = 0$ and granted $\not\exists$ for $\mu$-bubbles of dimensions $\leq m + k + l - 1$.

(See [Gr///] for sharper and more general inequalities of this type.)

5.6 Conjectures and Problems

**Codimension $k$ Conjecture.** The inequality $curv^i(X^m \to B^m(1)) \geq \frac{2\nu}{(n-m)\pi} - 1$ holds for all compact enlargeable $m$-manifolds, all $n > m$ and the first zero of the Bessel function $J_\nu$, $\nu = \frac{n}{2} - 1$.

(Overoptimistic?) Conjecture. If the cohomology of a closed $m$-manifold $X^m$ with coefficients in some field $K$ contains $l$ elements with non-zero product,

$$h_i \lhd ... \lhd h_i \neq 0, \; h_i \in H^*(X;K),$$

(e.g. $X^m = S^{m_1} \times ... \times S^{m_l}$, $m_1 + ... + m_l = m$),

then the curvatures of immersion $f : X^m \to B^{m+k}(1)$ bounded from below as follows,

$$curv^i_f(X) \geq 0.1 \frac{l^2}{mk}.$$  

**Clifford Tori Extremality Problem.** Does the $m$-torus admit an immersion to the unit $2m$-ball with curvature $< \sqrt{m}$?

For all we know, all flat $m$-tori admit smooth isometric immersions to $B^{2m}(1)$ with curvatures $< 10$.

**$m^2\beta$-Problem.** What is the minimal $\beta$, such that the tori of all dimensions $m$ admit immersion to the unit $(m + 1)$-balls,

$$f : T^m \to B^{m+1}(1),$$

with curvatures $curv^i_f(T^m) \leq 100m^2\beta$? (We know that $\beta \leq \frac{3}{2}$.)

**Simply Connected Codim 1 Curvature Problem.** Do all compact smoothly imbedded simply connected hypersurfaces $X^m \subset \mathbb{R}^{m+1}$, e.g. products of spheres of dimensions $\geq 2$, admit immersion to the unit ball,

$$f : X^m \to B^{m+1}(1)$$

with curvature $curv^i_f(X) \leq 100$?

A. How large can be the ratio

$$\frac{\min.curv^i(X^m \to B^M(1))}{\min.curv^i(X^m \to B^{M+1}(1))}$$

\[\text{If } Z = T^m, \text{ then } \not\exists \text{ is needed for minimizing hypersurfaces of dimensions } \leq m + k + l - 1, \text{ while } \not\exists \text{ for the stable } \mu\text{-bubbles is needed only for dimensions } \leq k - 1. \text{ And if } k = 3, \text{ then } 2d\text{-partial } \not\exists \text{ suffices.}\]
provided $X$ immerses to the Euclidean space $\mathbb{R}^M$, e.g. for $M \geq 2m-1$?

Is this ratio bounded by a universal constant, say by $\text{const} \leq 100$?

**B.** What is the homological Morse spectrum of the function $M : f \mapsto \text{curv}_X f(X)$ on the space of immersions $f : X \to Y$?

(An $r \in \mathbb{R}_+$ is in the Morse spectrum of a function $M : F \to \mathbb{R}_+$ if there exists a homology class $h \in H_*(F;A)$ with some coefficient group $A$, such that the $r$-sublevel of $M$ contains $h$, i.e. $h$ is contained in the image of the inclusion homomorphism

$$H_*(M^{-1}[0,r];A) \to H_*(X;A)$$

while the lower sublevels $M^{-1}[0,p] \subset M^{-1}[0,r]$, $p < r$, don’t contain $h$. See [Gr1988], [Gr2017] for more about it.)

**C.** What are bounds on the averages of powers of the curvatures of immersions, $\frac{1}{\text{vol}_X} \int_X (\text{curv}_X(X \to Y))^p dx$ for $p \geq 1$?

An (See [Pet 2023] for such an inequality with the mean curvature and consult [LB2021] for the bounds on the Yamabe invariant for 4-manifolds, which may (?) apply here.

**Product of Balls Problem.** Given positive numbers $r_i$, $R_i$ and positive integers $m_i$, $n_i$, $i = 1,...k$, such that $\sum_i m_i = \sum_i n_i$, evaluate, let it be only roughly, the maximal $\lambda > 0$ such that the product of $m_i$-dimensional $r_i$-balls $B^{m_i}(r_i) \mathbb{R}^{m_i}$ admit a $\lambda$-expanding map to the product of $n_i$-dimensional $R_i$-balls,

$$\prod_{i=1}^k B^{m_i}(r_i) \to \prod_{i=1}^k B^{n_i}(R_i).$$

**Cube Extremality Problem.** Does, the unit $n$-cube $[-1,1]^n$ admits an expanding map to the $n$-ball of radius $< \sqrt{n}$?

**Expansion on Mesoscale.** Find unified generalizations of the above results to classes of continuous maps $f : X \to Y$ stable under $C^0$-perturbations.

**B. Example 1.** Given a function $\delta(d)$, study continuous maps $f : X \to Y$, such that

$$\text{dist}_Y(f(x_1)f(x_2)) \geq \delta(\text{dist}_X(x_{1,2})).$$

(A possible $\delta$ may be supported in the segment $[c,100c]$ if $d$, where $\delta(d) \geq d$ for $c < d < 100d$ and where eventually $c \to 0$.

**C. Example 2.** Study embeddings $\phi : X^m \to V^n$ where $V \supset X$ retracts to $X$ and where $\text{dist}(X,\partial V) \geq r$; then study composed maps $f = \psi \circ \phi : X \to Y^n$,

$$X^m \overset{\psi}{\to} V \overset{\psi}{\to} Y^n,$$

where $\psi$ is an expanding map.

**D. Example 3.** Specialize the above to (piecewise linear) maps including non-locally trivial $p.l.$ immersions and to more general piecewise smooth maps.
6 Miscellaneous

6.1 Veronese Maps.

Besides invariant tori, there are other submanifolds in the unit sphere $S^{N-1}$, which have small curvatures and which are transitively acted upon by subgroups in the orthogonal group $O(N)$.

The generalized Veronese maps are a minimal equivariant isometric immersions of spheres to spheres, with respect to certain homomorphisms (representations) between the orthogonal groups $O(m+1) \rightarrow O(m+1)$,

$$\text{ver} = \text{ver}_s = \text{ver}_s^m : S^m(R_s) \rightarrow S^m = S^{m_s} = S^{m_s}(1),$$

where

$$m_s = (2s + m - 1) \frac{s + m - 2}{s!} < 2^{s+m}$$

and $R_s = R_s(m) = \sqrt{\frac{s(s + m - 1)}{m}}$,

for example,

$$m_2 = \frac{m(m+3)}{2} - 1, R_2(m) = \frac{2(m+1)}{m}$$

(see [DW1971]If $s = 2$ these, called classical Veronese maps, are defined by taking squares of linear functions (forms) $l = l(x) = \sum_i l_i x_i$ on $\mathbb{R}^{m+1}$,

$$\text{Ver} : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{M_m}, M_m = \frac{(m+1)(m+2)}{2},$$

where this $\mathbb{R}^{M_m}$ is represented by the space $Q = Q(\mathbb{R}^{m+1})$ of quadratic functions (forms) on $\mathbb{R}^{m+1}$,

$$Q = \sum_{i=1,j=1}^{m+1,m+1} q_{ij} x_i x_j.$$

The Veronese map, which is (obviously) equivariant for the natural action of the orthogonal group $O(n+1)$ on $Q$, where, observe, this action fixes the line $Q_o$ spanned by the form $Q_o = \sum_i x_i^2$ as well as the complementary subspace $Q_o$ of the traceless forms $Q$, where the action of $O(n+1)$ is irreducible and, thus, it has a unique, up to scaling Euclidean/Hilbertian structure.

Then the normal projection $\text{normal}$ defines an equivariant map to the sphere in $Q_o$

$$\text{ver} : S^m \rightarrow S^{M_m-2}(r) \subset Q_o,$$

where the radius of this sphere, a priori, depends on the normalization of the $O(m+1)$-invariant metric in $Q_o$.

Since we want the map to be isometric, we either take $r = \frac{1}{R_2(m)} = \sqrt{\frac{m}{2(m+1)}}$ and keep $S^m = S^{m}(1)$ or if we let $r = 1$ and $S^m = S^m(R_2(m))$ for $R_2(m) =$ \sqrt{\frac{2(m+1)}{m}}.

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[36] The splitting $Q = Q_o \oplus Q_o$ is necessarily normal for all $O(m+1)$-invariant Euclidean metrics in $Q$. 

Also observe that the Veronese maps, which are not embeddings themselves, factor via embeddings of projective spaces to spheres

$$S^m \to \mathbb{P}^m \subset S^{m-2} \subset \mathbb{R}^{M_m-1} = Q_m, \quad M_m = \frac{(m+1)(m+2)}{2}.$$  

**Curvature of Veronese.** Let is show that

$$\text{curv}^v(S^m(R_2(m)) \to S^{M_m-2}(1)) = \sqrt{\frac{R_2(1)}{R_2(m)}} - 1 = \sqrt{\frac{m-1}{m+1}}.$$  

Indeed, the Veronese map sends equatorial circles from $S^m(R_2(m))$ to planar circles of radii $R_2(m)/R_2(1)$, the curvatures of which in the ball $B^{M_m-1}$ is $R_2(1)/R_2(m) = 2\sqrt{\frac{m}{m+1}}$ and the curvatures of these in the sphere,

$$\text{curv}^v(S^1 \subset S^{M_m-2}(1)) = \sqrt{\text{curv}(S^1 \subset B^{M_m-1}(1))^2 - 1} = \sqrt{\frac{4m}{m+1} - 1} = \sqrt{\frac{3m-1}{m+1}}$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \to S^{M_m-2}(1)$ itself

$$\sqrt{R_2(1)/R_2(m)} = \sqrt{\frac{2m}{m+1}}; \quad \text{and the curvatures of these in the sphere,}$$

$$\text{curv}^v(S^1 \subset S^{M_m-2}(1)) = \sqrt{\text{curv}(S^1 \subset B^{M_m-1}(1))^2 - 1},$$

is equal to the curvature of the Veronese $S^m(R_2(m)) \to S^{M_m-2}(1)$ itself. QED.

It may be hard to prove (conjecture in section 1) that Veronese manifolds have the smallest possible curvatures among non-spherical $m$-manifold in the unit ball: if a smooth compact $m$-manifold $X$ admits a smooth immersion to the unit ball $B^N = B^N(1)$ with curvature $\text{curv}^v(X \to B^N) < \sqrt{\frac{2m}{m+1}}$, then $X$ is diffeomorphic to $S^m$.

It is more realistic to show that the Veronese have smallest curvatures among submanifolds $X \subset B^N$ invariant under subgroups in $O(N)$, which transitively act on $X$.

Remark. Manifolds $X^m$ immersed to $S^{m+1}$ with curvatures $< 1$ are diffeomorphic to $S^m$, see 5.5, but, apart from Veronese’s, we can’t rule out such $X$ in $S^N$ for $N \geq m + 2^{37}$ and, even less so, non-spherical $X$ immersible with curvatures $< \sqrt{2}$ to $B^N(1)$, even for $N = m + 1$.

It seems hard to decide this way or another, but it may be realistic to try to prove sphericity of simply connected manifolds immersed with curvatures $< 1$ to $S^N(1)$ for all $N$.

The curvatures of Veronese maps can be also evaluated with the Gauss formula, (teorema egregium), which also gives the following formula for curvatures of all $\text{ver}_s$:

$$m = 2, \quad 1 - 2c^2 = 1/3, \quad 2c^2 = 2/3 \ c\sqrt{1/3}$$

$$C = \sqrt{1 + 1/3} = 2/\sqrt{3}$$

**From Veronese to Tori.** The restriction of the map $\text{ver}_s : S^{2m-1}(R_s) \to S^{N_s}$ to the Clifford torus $\mathbb{T}^m \subset S^{2m-1}(R_s)$ obviously satisfies

$$\text{curv}_{\text{ver}}(\mathbb{T}^m) \leq \text{A}_{2m-1,s} + \sqrt{\frac{m}{R_s}} = \sqrt{3 - \frac{5}{2}m + \varepsilon(m,s)}$$

$$^{37}\text{Hermitian Veronese maps from the complex projective spaces } \mathbb{C}P^m \text{ to the spaces } \mathcal{H}_n \text{ of Hermitian forms on } \mathbb{C}^{m+1} \text{ are among the prime suspects in this regard.}$$
for
\[ \varepsilon(m, s) = \frac{2}{4m^2} - \frac{4m - 2}{s(s + 2m - 2)} + \frac{5(2m - 1)}{2ms(s + 2m - 2)} - \frac{2m - 1}{(ms(s + 2m - 2))^2}. \]

This, for \( s \gg m^2 \), makes \( \varepsilon(m, s) = O\left(\frac{1}{m}\right) \).

Since \( N_s < 2^{s+2m} \),

starting from \( N = 2^{10m_3} \)

\[ \text{curv}_{\text{ver}_s}(\mathbb{T}^m) < \sqrt{3 - \frac{5}{2}m}. \]

where it should be noted that

the Veronese maps restricted to the Clifford tori are \( \mathbb{T}^m \)-equivariant

and that

this bound is weaker than the optimal one \( \|\|_1^2 \geq \sqrt{3 - \frac{1}{m+2} + \varepsilon} \) from the previous section.

Remarks. (a) It is not hard to go to the (ultra)limit for \( s \to \infty \) and thus obtain an

equivariant isometric immersion \( \text{ver}_\infty \) of the Euclidean space \( \mathbb{R}^m \) to the unit sphere in the Hilbert space, such that

\[ \text{curv}_{\text{ver}_\infty}(\mathbb{R}^m \to S^\infty) = \sqrt{\frac{(m-1)(2m+1)}{(m+1)^2}} = \sqrt{2 - \frac{5}{m+1} + \frac{2}{(m+1)^2}}, \]

where equivariance is understood with respect to a certain unitary representation of the isometry group of \( \mathbb{R}^m \).

Probably, one can show that this \( \text{ver}_\infty \) realizes the minimum of the curvatures among all equivariant maps \( \mathbb{R}^m \to S^\infty \).

(b) Instead of \( \text{ver}_s \), one could achieve (essentially) the same result with a use of compositions of the classical Veronese maps, \( \text{ver} : S^{m_1} \to S^{m_{i+1}}, \ i+1 = \frac{(m+1)(m+2)}{2} - 2 \),

starting with \( m_1 = 2m - 1 \) and going up to \( i = m \). (Actually, \( i \sim \log m \) will do.)

6.2 Product Manifolds, Connected Sums and Related Constructions

Let \( f_i : X_i^{m_i} \to B^{m_i+1}(1), \ i = 1, \ldots, l \), be immersions with focal radii \( r \) and let \( f_0 : X_0^{m_0} \to B^l(1) \) be an immersion with \( \text{foc.rad}_f(X_0^{m_0}) = r_0 \).

Then the \( \times \)-construction (see 4.1) delivers an immersion

\[ f : X = \bigtimes_0^l X_i \to B^N(1), \ N = l + \sum_1^l m_i, \]

such that

\[ \text{foc.rad}_f(X) \geq \max_{0 \leq i \leq 1} \frac{\min(r - \lambda, \lambda r_0)}{\sqrt{1 + \lambda r_0}}. \]
Similarly, if $X_0^{m_0}$ admits a $\nabla^j$-trivial (see 4.1) immersion to $B^M(1)$ with focal radius $r_0$, then $X$ admits an immersion to $B^{M+k}(1)$ for all $k \geq 1 - M + \sum_i m_i$, such that

$$foc.rad_f(X) \geq \max_{0<\lambda \leq 1} \frac{\min(r_0 - \lambda, \lambda r/\sqrt{I})}{\sqrt{I} + \lambda r/\sqrt{I}}$$

6.2.A. Example: Product of Spheres. Let

$$X = X^m = \times_i S^m_i, \sum_i m_i = m,$$

and let $\mu = \min_i m_i$. Then there exists an immersion $f : X \to B^{m+1}(1)$, such that

$$\text{curv}_f(X) \leq \text{const}_\mu m + \frac{\text{const}}{\delta^2}.$$

**Proof.** Adopt the torus-by-torus construction 4.1.C to product of spheres, where instead of squaring maps at each step, use (Cartesian) product of at least $\mu$ of maps, where then the above inequality for $foc.rad$ translated to curvature apply.

**Embedding Remark.** Observe that the resulting maps $X^m \to B^{m+1}(1)$ are embeddings.

6.2.B. Connected Sums. If $m$-manifolds $X_i, i = 1, 2, \ldots, l$, admit immersions to the unit ball $B^n = B^n(1), n > m$, with the curvatures bounded by a constant $C$, then the connected sum $X_1 \# \ldots \# X_l$ can immersed to $B^n$ with curvature bounded by $5C$.

**Proof.** Make geometric connected sums of all $X_i$ with the unit equatorial sphere $S^m \subset S^{n} = \partial B^n$, where this is done with each $X_i$ individually with a copy of $S^m \subset B^n$ by connecting $X_i$ with $S^m_{i} \# S^m_i$ with a tube with curvature $< 5C$. Then the connected sum between $X_i$ is implemented by making similar tubes between $S^m_i$.

**Example.** Since there are 2-Tori in the unit 3-ball with $\text{curv} = 3$, the minimal possible curvatures of orientable surfaces $X$ satisfy

$$\text{min.curv}^i(X^2 \# \to B^3(1)) < 15,$$

while non-orientable ones have

$$\text{min.curv}^i(X^2 \to B^3(1)) \leq 5\text{min.curv}^i(\mathbb{R}P^2 \to B^3(1)) < 50,$$

the Boy surface seem to have curvature about 10, Probably, all surfaces have $\text{min.curv}^i < 10$, but it is unclear, not even for the 2-torus, what actually minimal curvatures of surfaces in $B^3(1)$ are.

**Attaching $k$-Handles for $k \geq 2$.** To attach a handle to a sphere $S^{k-1} \subset X$ with a controlled the curvature, with a controllable increase of the curvature, one needs a regular $\delta$-neighbourhood of this sphere in $X$ with $\delta$ controllably bounded from below: this which would allow attaching a $k$ handle with the curvature increase roughly by $1/\delta$.

For instance, if $k = 2$ an $S^1 \subset X$ is the shortest non-contractible curve in $X$, then it does admits such a neighbourhood in $X$ with $\delta$ controllably bounded from below by the curvature of $X$; thus attaching with certain normal frames 2-handles to it is possible with curvature increase by a definite multiplicative constant.
In general one can show the following.

6.2.C. Handles Stretch Proposition. (Compare with 4.3.C.) Let an immersed manifold $X_m \xrightarrow{\phi} B^N(1)$ be obtained from $X^m \xrightarrow{f} B^n(1)$ by attaching $l$-handles for $l \leq k$ where, all steps surgery keep in the class of immersed manifolds.

Then $\phi$ is regularly homotopic to an immersion $\phi_1 : X \xrightarrow{} B^n(1)$, such that

$$\text{curv}^1_{\phi_1}(X) \leq C^{2k}\text{curv}^1_f(X)$$

for $C \leq 10,000$.

Sketch of the Proof. Regularly homotop $f$ in $B^n(1)$ to an immersion $f_1$ with $\text{curv}^1_{f_1}(X) \leq 100^{2k}\text{curv}^1_f(X)$ and such that that the $f_1$-induced Riemannian metric in a (small) neighbourhood $U$ of the $2k$-skeleton of a smooth triangulation of $X$ is by an arbitrarily large (independently of $U$) factor $\lambda$ greater than the $f$-induced metric.

Assume without loss of generality that all spheres $S^i$, at which the surgery performed are located and in $U$ don’t intersect there (this is possible for $m \geq 2k$, which we may assume with no problem) and choose $\lambda$ so large that the union of these spheres has a nice thick regular neighbourhood, where the surgery can be made with at most $100^{2k}$ increase in the curvature.

Remark. It is not hard to visualise an actual proof along these lines but I don’t see how to write it down in a readable form.

6.3 Embeddings with Small Curvatures

Connected Sums of Embedded Manifolds. If $X = X^m$ admits an embedding (i.e. a immersion with no self-intersection) to $B^{m+1}(1)$ with curvature $\leq c$, then the connected sums of $2l$-copies of $X$ embed to $B^{m+1}(1)$ with curvatures $< 100c$.

Proof. Let $X_l \subset B^{m+1}(1)$ be obtained from $X$ by attaching a single 1-handle $S^{m-1} \times [0, 1]$, such that $\text{curv}X_l \subset B^{m+1}(1) < 10c$.

Let $\tilde{X}_l$ be the natural cyclic covering of $X_l$ of order $l$ and let $\tilde{X}_l$ be obtained by cutting $\tilde{X}_l$ along the sphere $S^{m-1} \subset \tilde{X}_l$ from the handle.

Observe that this $\tilde{X}_l$ is a manifold with two spherical boundary components and that it (almost) naturally embeds to $B^{m+1}(1)$ with curvature $< 10c$.

Let $X' \subset B^{m+1}(1) \setminus X_l$ be obtained by a slight normal displacement of $\tilde{X}_l$ and let us attach $X'_{2l}$ to $\tilde{X}_l$ along a pair of nearby $(m-1)$-spheres and also fill in the remaining two boundary spheres with $m$-balls. Clearly, the resulting manifold, call it $X_{2l}$, is diffeomorphic to the connected sum of $2l$ copies of $X$ and it is not hard to arrange an embedding of $X_{2l}$ to the unit ball with curvature $< 100$.

Exercises. (a) Let $X = X^m$ be a connected sum of an arbitrary number of manifolds diffeomorphic to product of spheres. Show that $X$ embeds to the unit $(m + 1)$-ball with curvature $< 500 \cdot 2^{\frac{m+1}{2}}$.

Hint. Embed mutually non-diffeomorphic products of spheres into $2^m$ disjoint $r$-balls in $B^{m+1}(1)$ of radii $r = 2^{-\frac{m+1}{2}}$.

(b) Let $X = X^m$ be disconnected closed manifold, which contains $l$ mutually non-diffeomorphic components. Show that

$$\text{curv}^1_f(X \xrightarrow{} B^{m+1}) \geq \text{const}_m l, \text{const}_m \geq \frac{1}{(10m)^m},$$

for all embeddings $f : X \xrightarrow{} B^{m+1}(1)$.  

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(c) Construct closed $m$-dimensional manifolds $X_i$, $i = 1, 2, \ldots$ for all $m \geq 6$, such that all of them embed to $B^7(1)$ and such that embedding of connected sums of $l$ among these manifolds have curvatures $\geq const.$

**Question.** Can one have these $X_i$ embeddable to $\mathbb{R}^{m+1}$ with curvatures $< 1000000$?

### 6.4 Cycles with Small Curvature

Our equidimensional expanding maps are effective in delivering immersed submanifolds with controllably bounded curvatures, because these maps themselves, besides being expanding, have controllably bounded second derivatives.

In general, it is hard to construct an immersed $m$-dimensional submanifolds $X \hookrightarrow Y$ with small curvature and with non-zero homology classes $[X] \in H_m(Y)$.

Apparently, all known results of this kind badly depend on the dimension and/or codimension of $X$, see [CDM2016].

A happy exception is the codimension one case, $m = n - 1$, where there is no topological obstructions for the existence of $X$ and where an equidistant smoothing delivers hypersurfaces with controllably small curvatures as follows.

Let $Y$ be a proper Riemannian band of dimension $n$, that is a Riemannian manifold, the boundary $\partial Y$ of which is divided into two disjoint parts, $\partial Y = \partial_1 Y \cup \partial_2 Y$, where $\partial_2 Y$ are unions of connected components of $\partial Y$, and denote by $d$ the width of $Y$,

$$d = \text{width}(Y) = \text{dist}(\partial_1 Y, \partial_2 Y).$$

Let us $d_1$-equidistantly push $\partial_1 Y$ inside $Y$ for $d_1 < d$ and then $d_2$-equidistantly move the resulting hypersurface, denoted $\partial_{d_1, d_2}$, back toward $\partial_2 Y$ with $d_2 < d_1$.

That is, $\partial_{d, d_1}$ is equal to the (topological) boundary of the $d_1$-neighbourhood $U_{d_1}(\partial_1 Y) \subset Y$ and the result of the second move, call it $X_\circ = \partial_{-d_1, d_2} \subset U_{d_2}(\partial_2 Y)$, is the boundary of $U_{d_2}(\partial_{-d_1}) \subset U_{d_2}(\partial_2 Y)$.

Let us evaluate the curvature of $X_\circ$ in terms of the sectional curvatures of $Y$, where we observe the following.

1. If $Y$ has constant sectional curvature $\pm \kappa^2$, then $X_\circ$ is $C^{1,1}$-smooth and

$$\text{foc.rad}(X_\circ) \geq (\text{min}(d_2, d_1 - d - 2));$$

accordingly $\text{curv}^+(X) \leq \alpha^+(\text{min}(d_2, d_1 - d - 2))$ for the function $\alpha^+$ from 1.B.

2. If more generally, the sectional curvatures of $Y$ is pinched between two values, that are the curvatures of two standard surfaces $S_\pm$ with constant curvatures,

$$\text{sect.curv}^+(S_+) \leq \text{sect.curv}^+(Y) \leq \text{sect.curv}^+(S_-),$$

then the curvature of $X_\circ$ is bounded by the maximum the two numbers:

- the first number is the curvature of the circle of the radius $d_2$ in $S_-$;
- the second number is the curvature of the circle $S^1(r) \subset S_+$, such that the curvature of the concentric circle $S^1(r + d_2)$ is equal to the curvature of the $d_1$-circle in $S_+$.

It follows, for instance, that

$$(\circ_d) \quad \text{if} \quad -1 \leq \text{sect.curv}^+(Y) \leq 1$$
and \( d = \text{width}(Y) \leq 1 \), then

\[
Y \text{ contains a smooth hypersurface, which separates } \partial Y \text{ from } \partial Y \text{ and such that }
\]

\[
\text{curv}^\perp(X) \leq \frac{4}{d}.
\]

**Corollary.** Let \( Y \) be a complete Riemannian \( n \)-manifold with \(|\text{sect}\text{curv}^\perp(Y)| \leq \kappa^2\) and with \( \text{inj.rad}(Y) \geq r \).

Then

\((\mathcal{O}_{\kappa,r})\) all integer \((n-1)\)-dimensional homology classes \( h \in H_{n-1}(Y) \) are realizable by smoothly immersed oriented hypersurfaces \( X \hookrightarrow Y \) with \( \text{curv}^\perp(X) \leq 10\kappa + \frac{100}{r}\).

Indeed, given a homology class \( h \in H_1(Y) \), apply \((\mathcal{O}_d)\) to the infinite cyclic covering of \( Y \), which is defined by this class.

**Questions.** (a) Do \((\mathcal{O}_{\kappa,\kappa})\) and \((\mathcal{O}_r)\) meaningfully generalize to submanifolds \( X \subset Y \) of codimensions \( k > 1 \), where \( Y \) is, in some way, "wide in \( k \)-directions"?

For instance, Let \( Y \) be a Riemannian manifold homeomorphic to \( X_0 \times B^k(1) \), where \( X_0 \) is a closed connected orientable manifold of dimension \( n-k \), let the sectional curvature of \( Y \) be bounded by \(|\kappa(Y)| \leq 1 \) and the injectivity radius by \( \text{inj.rad}(Y) \geq 1 \).

What else need you know about \( Y \) to effectively bound the minimal possible curvature of a submanifold \( X \subset Y \) homologous to \( X_0 = X_0 \times \{0\} \subset X_0 \times B^k(1) = X \)?

What is the best bound on this curvature in a presence of a proper \((\text{boundary-to-boundary})\) \( \lambda \)-Lipschitz map \( X \rightarrow B^k(1) \)?

Are, similarly to \((\mathcal{O}_{\kappa,\kappa})\), non-zero multiples of the homology classes \( h \in H_m(Y) \), for all \( m \leq \dim(Y) \), realizable by immersed \( m \)-dimensional submanifolds \( X \hookrightarrow Y \) with \( \text{curv}^\perp(X) \leq 100m^{100}(\kappa^{100}) \)?

**From Focal Radius to Expansion.** Let us turn to the opposite problem: In what cases does the \( r \)-neighbourhood \( U_r(X) \subset X \) of an embedded manifold \( X \subset Y \) with "large" universal covering, e. g. for \( X \) homeomorphic to \( \mathbb{T}^m \), and with large \( \text{foc.rad}(X) \), receive an expanding map from a "large manifold" e.g. from \( B^m(R) \times B^{n-m}(\frac{1}{100}) \) with large \( R \)?

Here the answer is positive for \( m = n-1 \) and \( m = n-2 \):

if \( X \) receives expanding maps from the balls \( B^m(R) \) for all \( R \) (as e.g. the \( m \)-torus does), then, in the case \( m = n-1 \), the neighbourhood \( U_r(X) \) receives expanding maps from \( B^m(R) \times B^1(\frac{\sqrt{2}}{2} R - \varepsilon) \) for all \( R \rightarrow \infty \) and positive \( \varepsilon \rightarrow 0 \).

And if \( m = n-2 \), then \( U_r(X) \) receives such maps from \( B^{n+1}(R) \times B^1(\frac{\sqrt{2}}{2} R - \varepsilon) \).

**Proof.** The required map for \( m = n-1 \) and coorientable \( X \subset Y \) is obtained with the obvious splitting \( U_r(X) = X \times B^1(r) \) and the case \( m = n-2 \) follows by applying this to the hypersurface \( Z = \partial U_{r/2}(X) \subset U_r(X) \), where, clearly, \( \text{foc.rad}(Z) = \frac{1}{2} \text{foc.rad}(X) \geq \frac{1}{2} \), and where the case of a non-trivial normal bundle of \( X \subset Y \) needs a little thinking about.

But when it comes to \( m \leq n-3 \) nothing of the kind seems to be true, where the apparent difficulty stems from the following phenomenon.

\[38\] If \( Y \) is, Riemannian flat, then the term \( 10/r \) is unneeded and if \( Y \) is almost flat one can do without it for multiples of \( h \) and I am not certain about examples where the term \( 10/r \) is truly needed.
If \( m, k \geq 2 \), then the topologically trivial sphere bundle \( V = \mathbb{R}^m \times S^k \to \mathbb{R}^m \) admits an orthogonal connection \( \nabla \) with an arbitrary small curvature such that all smooth sections \( \phi : \mathbb{R}^n \to V \) satisfy

\[
\sup_{x \in \mathbb{R}^m} ||\nabla \phi(x)|| = \infty.
\]

Despite this, our \( U_r(X) \), still looks large for all \( m \) and large \( r = \text{foc.rad}(X) \), but I don’t know, how to make precise sense of largeness for these \( U_r \).

Here is a specific question.

Let us regard \( U = B^k(r) \times B^m(R) \) as (the total space of) a \( B^k(r) \)-bundle over the ball \( B^m(R) \), let \( \nabla \) be a Euclidean connection in this bundle and \( g_\nabla \) the corresponding Riemannian metric on \( U \), that is the sum of the differential quadratic form induced by the map \( U = B^k(t) \times B^m(R) \to B^m(R) \) with the Euclidean metrics in the fibers \( B^k_0(t) \subset U \), \( x \in B^m(R) \) extended to \( T(U) \) by zero on the \( \nabla \)-horizontal vectors.

For which \( r, R \) and \( R \) the manifolds \((U, g_\nabla)\) admit no expanding maps \((U, g_\nabla) \to B^{m+k}(R)\) for all connections \( \nabla \)?

Conversely, from what kind of manifolds do \((U, g_\nabla)\) receive expanding maps?

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