Ehrhart series of fractional stable set polytopes of finite graphs

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Abstract. The fractional stable set polytope $\text{FRAC}(G)$ of a simple graph $G$ with $d$ vertices is a rational polytope that is the set of nonnegative vectors $(x_1, \ldots, x_d)$ satisfying $x_i + x_j \leq 1$ for every edge $(i, j)$ of $G$. In this paper we show that (i) The $\delta$-vector of a lattice polytope $2\text{FRAC}(G)$ is alternatingly increasing, (ii) The Ehrhart ring of $\text{FRAC}(G)$ is Gorenstein, (iii) The coefficients of the numerator of the Ehrhart series of $\text{FRAC}(G)$ are symmetric, unimodal and computed by the $\delta$-vector of $2\text{FRAC}(G)$.

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Introduction

The Ehrhart series of a rational convex polytope is one of the most important topics in combinatorics. Let $P$ be a $d$-dimensional rational convex polytope in $\mathbb{R}^N$. For each $n \in \mathbb{N}$, let $nP = \{n\alpha \mid \alpha \in P\}$ and define the function $i(P, n) := \sharp(nP \cap \mathbb{Z}^N)$. Thus $i(P, n)$ is the number of lattice points contained in $nP$, called the Ehrhart quasi-polynomial of $P$. It is known that $i(P, n)$ is indeed a quasi-polynomial of degree $d$. In particular, if $P$ is a lattice polytope, i.e., all vertices of $P$ are lattice points, then $i(P, n)$ is a polynomial and called the Ehrhart polynomial of $P$. The generating function of the Ehrhart quasi-polynomial is defined by $E(P, t) := 1 + \sum_{n=1}^{\infty} i(P, n)t^n$ and called the Ehrhart series of $P$. Let $m$ be the smallest natural number $k$ for which $kP$ is a lattice polytope and let $v$ be the smallest natural number $k$ for which $kP$ has a lattice point in its interior. It is known that $E(P, t)$ is a rational function of degree $-v$ and has an expression $E(P, t) := g(P, t)/(1 - t^m)^{d+1}$ where $g(P, t)$ is a polynomial of degree $m(d+1) - v$ with nonnegative integer coefficients. In particular, if $P$ is a lattice polytope, then $m = 1$ and hence
\[ E(P, t) = \delta(P, t)/(1 - t)^{d+1} \]

where \(\delta(P, t) = \delta_0 + \delta_1 t + \cdots + \delta_{d+1-v} t^{d+1-v} \) is a polynomial of degree \(d + 1 - v\), called the \(\delta\)-polynomial of \(P\). The coefficients \((\delta_0, \ldots, \delta_{d+1-v})\) of \(\delta(P, t)\) is called the \(\delta\)-vector (or \(h\)-vector, \(h^*\)-vector) of \(P\). Next, we define the Ehrhart ring of a \(d\)-dimensional rational polytope \(P \subset \mathbb{R}^N\). Let \([A_K(P)]_n\) be the linear space over a field \(K\) whose basis is the set of Laurent monomials \(x_1^{\alpha_1} \cdots x_N^{\alpha_N} t^n\) with \((\alpha_1, \ldots, \alpha_N) \in n \mathbb{P} \cap \mathbb{Z}^N\). Then \(A_K(P) := \bigoplus_{n \geq 0}[A_K(P)]_n\) is called the Ehrhart ring of \(P\). If \(A_K(P)\) is Gorenstein, then the coefficients of the numerator of \(E(P, t)\) are symmetric. The dual polytope of \(P\) is defined by \(P^\vee = \{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P\}\), where \(\langle x, y \rangle\) is the usual inner product of \(\mathbb{R}^d\). The notion of dual polytopes appears in a criterion for \(A_K(P)\) to be Gorenstein. Let \(P \subset \mathbb{R}^d\) be a lattice polytope of dimension \(d\). We say that \(P\) is a Fano polytope if the origin of \(\mathbb{R}^d\) is the unique lattice point belonging to the interior of \(P\). A Fano polytope is called Gorenstein if its dual polytope is a lattice polytope. (A Gorenstein Fano polytope is often called a reflexive polytope in the literature.)

Let \(G\) be a finite simple graph on the vertex set \([d] = \{1, 2, \ldots, d\}\) and let \(E(G)\) be the edge set of \(G\). Throughout this paper, we always assume that \(G\) has no isolated vertices. Given a subset \(W \subset [d]\), we associate the \((0, 1)\)-vector \(\rho(W) = \sum_{j \in W} e_j \in \mathbb{R}^d\). Here, \(e_i\) is the \(i\)-th unit coordinate vector of \(\mathbb{R}^d\). In particular, \(\rho(\emptyset)\) is the origin of \(\mathbb{R}^d\). A subset \(W\) is called stable if \(\{i, j\} \not\in E(G)\) for all \(i, j \in W\) with \(i \neq j\). Note that the empty set and each single-element subset of \([d]\) are stable. Let \(S(G)\) denote the set of all stable sets of \(G\). The stable set polytope (independent set polytope) \(\text{STAB}(G) \subset \mathbb{R}^d\) of a simple graph \(G\) is the \((0, 1)\)-polytope which is the convex full of \(\{\rho(W) \mid W \in S(G)\}\). Stable set polytopes are very important in many areas, e.g., optimization theory. The \(\delta\)-vector of the stable set polytope of a perfect graph is studied in [1, 7]. On the other hand, the fractional stable set polytope \(\text{FRAC}(G)\) of \(G\) is the \(d\)-polytope in \(\mathbb{R}^d\) defined by

\[
\text{FRAC}(G) := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \left| \begin{array}{c}
  x_i \geq 0 \\
  x_i + x_j \leq 1 \\
  (1 \leq i \leq d) \\
  ((i, j) \in E(G))
\end{array} \right. \right\}.
\]

In general, we have \(\text{STAB}(G) \subset \text{FRAC}(G)\). Each vertex of \(\text{FRAC}(G)\) belongs to \(\{0, 1/2, 1\}^d\) (see, e.g., [5]). It is known that \(\text{FRAC}(G) = \text{STAB}(G)\) if and only if \(G\) is bipartite. If \(G\) is bipartite, then \(\text{STAB}(G)\) has a unimodular triangulation, and the \(\delta\)-vector of \(\text{STAB}(G)\) is symmetric and unimodal (see [1, 3, 7]). Note that, if \(G\) is bipartite, then \(\text{STAB}(G)\) is the chain polytope of a poset \(P\) of rank 1 whose comparability graph is \(G\), and affinely equivalent to the order polytope of the poset \(P\) (see [4]). The purpose of this paper is to study the Ehrhart series of \(\text{FRAC}(G)\). The following two polytopes will play important roles:

\[
P(G) = 2 \cdot \text{FRAC}(G),
\]

\[
Q(G) = 3 \cdot \text{FRAC}(G) - (1, \ldots, 1)
\]

\[
= \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \left| \begin{array}{c}
  x_i \geq -1 \\
  x_i + x_j \leq 1 \\
  (1 \leq i \leq d) \\
  ((i, j) \in E(G))
\end{array} \right. \right\}.
\]
In [13], Steingrímsson called the lattice polytope \( P(G) \) the extended 2-weak vertex-packing polytope of \( G \) and studied the structure of \( P(G) \). In particular, he constructed a unimodular triangulation of \( P(G) \) and showed that the \( \delta \)-vector of \( P(G) \) is obtained by a descent statistic on a subset of the hyperoctahedral group determined by \( G \).

This paper is organized as follows. In Section 1, we show that the \( \delta \)-vector \( (\delta_0, \ldots, \delta_{d-1}) \) of \( P(G) \) is alternatingly increasing ([10, Definition 2.9]), i.e.,

\[
\delta_0 \leq \delta_{d-1} \leq \delta_1 \leq \cdots \leq \delta_{\lfloor d/2 \rfloor - 1} \leq \delta_{d-\lfloor d/2 \rfloor}.
\]

In Section 2, we study the structure of \( Q(G) \) in order to show that the Ehrhart ring of \( \text{FRAC}(G) \) is Gorenstein. By using this result, in Section 3, we give a formula for the numerator of the Ehrhart series \( E(\text{FRAC}(G), t) := g(\text{FRAC}(G), t) / (1 - t^2)^{d+1} \) via the \( \delta \)-vector of \( P(G) \). Since the Ehrhart ring of \( \text{FRAC}(G) \) is Gorenstein and since the \( \delta \)-vector of \( P(G) \) is alternatingly increasing, it follows that the coefficients of \( g(\text{FRAC}(G), t) \) is symmetric and unimodal. Finally, in Section 4, we discuss the dual polytope \( Q(G)^\lor \) of \( Q(G) \).

1. The \( \delta \)-vector of \( P(G) \)

First, we review the results in [13]. Let \( B_d \) denote the all signed permutation words on \([d] = \{1, 2, \ldots, d\}\). For example, if \( d = 2 \),

\[
B_2 = \{1 \ 2, \ 2 \ 1, \ 1 \ \bar{2}, \ 2 \ \bar{1}, \ 1 \ \bar{1}, \ 2 \ \bar{2}, \ \bar{1} \ \bar{2}, \ \bar{2} \ \bar{1}\},
\]

where \( \bar{1} = -1 \) and \( \bar{2} = -2 \). We order the letters in signed permutations as integers, i.e., \( \cdot \cdot \cdot < 3 < 2 < \bar{1} < 0 < 1 < 2 < 3 < \cdot \cdot \cdot \). An element \( i \in [d] \) is called a descent in \( \pi = a_1 \cdots a_d \in B_d \) if one of the following holds ([13, Definition 5]):

(i) \( i < d \) and \( a_i > a_{i+1} \);
(ii) \( i = d \) and \( a_i > 0 \).

Let \( \text{des}(\pi) \) denote the number of descents in \( \pi \in B_d \). For example, for \( \pi = 2 \ \bar{3} \ \bar{4} \ 1 \in B_4 \), \( \text{des}(\pi) = 3 \) since the descents of \( \pi \) are 1, 2 and 4. For any subset \( S \) of \( B_d \), the descent polynomial of \( S \) is \( D(S, t) := \sum_{\pi \in S} t^{\text{des}(\pi)} \). Let \( G \) be a simple graph on the vertex set \([d]\) and the edge set \( E(G) \). We define a subset \( \Pi(G) \) of \( B_d \) as follows ([13, Definition 11 and Theorem 12]):

\[
\Pi(G) = \left\{ \pi \in B_d \biggm| \begin{array}{l}
\text{if } (i, j) \in E(G) \text{ and } +i \text{ appears in } \pi, \\
\text{then } -j \text{ must precede } +i \text{ in } \pi
\end{array} \right\}.
\]

**Proposition 1.1 ([13]).** Let \( G \) be a finite simple graph. Then the \( \delta \)-polynomial of \( P(G) \) equals the descent polynomial \( D(\Pi(G), t) \).

By using this fact, we will show Theorem 1.2 below. Note that a similar decomposition technique (i.e., \( a(t) + tb(t) \)) was used in [12] to establish Ehrhart inequalities originally due to Stanley and Hibi.
Theorem 1.2. Let $G$ be a simple graph with $d$ vertices. Then there exist symmetric and unimodal polynomials $a(t)$ of degree $d - 1$ and $b(t)$ of degree $d - 2$ such that $\delta(P(G), t) = a(t) + tb(t)$. In particular, the $\delta$-vector $(\delta_0, \delta_1, \ldots, \delta_{d-1})$ of $P(G)$ is alternatingly increasing, i.e.,

$$\delta_0 \leq \delta_{d-1} \leq \delta_1 \leq \delta_{d-2} \leq \cdots \leq \delta_{\lfloor d/2 \rfloor - 1} \leq \delta_{d - \lfloor d/2 \rfloor} \leq \delta_{\lfloor d/2 \rfloor}.$$ 

Proof. Let $\Pi_+$ (resp. $\Pi_-$) denote the set of all $\pi \in \Pi(G)$ such that the last number of $\pi$ is positive (resp. negative). Note that the first number of $\pi \in \Pi(G)$ is always negative since $G$ has no isolated vertices.

Let $\pi \in \Pi_+$. Then $\pi$ has a representation

$$\pi = m_1^{(1)} \cdots m_{\alpha_1}^{(1)} p_1^{(1)} \cdots p_{\beta_1}^{(1)} m_1^{(2)} \cdots m_{\alpha_2}^{(2)} p_1^{(2)} \cdots p_{\beta_2}^{(2)} \cdots m_1^{(\gamma)} \cdots m_{\alpha_\gamma}^{(\gamma)} p_1^{(\gamma)} \cdots p_{\beta_\gamma}^{(\gamma)}$$

where $p_1^{(j)} > 0$ and $m_1^{(j)} < 0$. Let $S(\pi)$ denote the set of all signed permutation words on $[d]$ of the form

$$m_{\sigma_1(1)}^{(1)} \cdots m_{\sigma_1(\alpha_1)}^{(1)} p_{\tau_1(1)}^{(1)} \cdots p_{\tau_1(\beta_1)}^{(1)} \cdots m_{\sigma_\gamma(1)}^{(\gamma)} \cdots m_{\sigma_\gamma(\alpha_\gamma)}^{(\gamma)} p_{\tau_\gamma(1)}^{(\gamma)} \cdots p_{\tau_\gamma(\beta_\gamma)}^{(\gamma)}$$

where $\sigma_k \in S_{\alpha_k}$ and $\tau_k \in S_{\beta_k}$ are permutations. It is easy to see that $S(\pi) \subset \Pi_+$. Let $A_k(t) = \sum_{i=0}^{k-1} A(k, i)t^i$ denote the Eulerian polynomial whose coefficients $A(k, i)$ is Eulerian number. It is known that $A_k(t) = \sum_{\pi \in S_k} t^{d(\pi)}$, where $d(\pi)$ is the number of usual descent of $\pi$ (i.e., no signs involved and never a descent at $k$). See, e.g., [3]. Thus

$$D(S(\pi), t) = t^\gamma \prod_{j=1}^{\gamma} A_{\alpha_j}(t) A_{\beta_j}(t).$$

It is known that $(A(k, 0), A(k, 1), \ldots, A(k, k-1))$ is symmetric and unimodal, i.e., $A(k, i) \leq A(k, i + 1)$ for $0 \leq i \leq \lfloor 2/k \rfloor$. The degree of $D(S(\pi), t)$ is $\gamma + \sum_{j=1}^{\gamma} (\alpha_j + \beta_j - 2) = d - \gamma$. Since $A_k(t)$ is symmetric and unimodal, so is $A_{\alpha_j}(t) A_{\beta_j}(t)$. Hence

$$D(S(\pi), t) = s_\gamma t^\gamma + \cdots + s_{d-\gamma} t^{d-\gamma}$$

satisfies that $(s_\gamma, \ldots, s_{d-\gamma})$ is symmetric and unimodal. Since

$$D(\Pi_+, t) = u_1 t + \cdots + u_{d-1} t^{d-1}$$

is a sum of such $D(S(\pi), t)$’s, $(u_1, \ldots, u_{d-1})$ is symmetric and unimodal.

Let $\pi \in \Pi_-$. Then $\pi$ has a representation

$$\pi = m_1^{(1)} \cdots m_{\alpha_1}^{(1)} p_1^{(1)} \cdots p_{\beta_1}^{(1)} m_1^{(2)} \cdots m_{\alpha_2}^{(2)} p_1^{(2)} \cdots p_{\beta_2}^{(2)} \cdots m_1^{(\gamma)} \cdots m_{\alpha_\gamma}^{(\gamma)},$$

where $p_1^{(j)} > 0$ and $m_1^{(j)} < 0$. Define $S(\pi)$ as before. Then we have

$$D(S(\pi), t) = t^{\gamma-1} A_{\alpha_1}(t) \prod_{j=1}^{\gamma-1} A_{\alpha_j}(t) A_{\beta_j}(t).$$

The degree of $D(S(\pi), t)$ is $\gamma - 1 + \alpha_\gamma - 1 + \sum_{j=1}^{\gamma-1} (\alpha_j + \beta_j - 2) = d - \gamma$. Since

$$D(\Pi_-, t) = v_0 + v_1 t + \cdots + v_{d-1} t^{d-1}$$
is a sum of such $D(S(\pi, t)')s, (v_0, v_1, \ldots, v_{d-1})$ is symmetric and unimodal.

We now show that the $\delta$-vector $(\delta_0, \ldots, \delta_{d-1}) = (v_0, u_1 + v_1, \ldots, u_{d-1} + v_{d-1})$ of $P(G)$ is alternatingly increasing. First, $\delta_{d-1} - \delta_0 = u_{d-1} + v_{d-1} - v_0 = u_{d-1} \geq 0$. Moreover, for $i = 1, 2, \ldots, [d/2]$, we have $\delta_i - \delta_{d-i} = u_i + v_i - u_{d-i} - v_{d-i} = v_i - v_{i-1} \geq 0$, and for $i = 1, 2, \ldots, [d/2] - 1$, we have $\delta_{d-i-1} - \delta_i = v_{d-i-1} + v_{d-i} - u_i - v_i = u_{i+1} - u_i \geq 0$. Thus the $\delta$-vector of $P(G)$ is alternatingly increasing. \hfill \square

2. The Ehrhart ring of $\text{FRAC}(G)$

In this section, we will show that the Ehrhart ring of $\text{FRAC}(G)$ is Gorenstein. In order to show that the Ehrhart ring of $\text{FRAC}(G)$ is Gorenstein, we will use the following criterion [5, Theorem 1.1]:

**Proposition 2.1.** Let $P \subset \mathbb{R}^d$ be a rational convex polytope of dimension $d$ and let $\delta \geq 1$ denote the smallest integer for which $\delta(P \setminus \partial P) \cap \mathbb{Z}^d \neq \emptyset$. Fix $\alpha \in \delta(P \setminus \partial P) \cap \mathbb{Z}^d$ and let $Q = \delta P - \alpha \subset \mathbb{R}^d$. Then the Ehrhart ring $A_K(P)$ of $P$ is Gorenstein if and only if the following conditions are satisfied:

(i) The dual polytope $Q^\vee$ of $Q$ is a lattice polytope;
(ii) Let $\tilde{P} \subset \mathbb{R}^{d+1}$ denote the rational convex polytope which is the convex hull of the subset $\{(\beta, 0) \in \mathbb{R}^{d+1} \mid \beta \in P\} \cup \{(0, \ldots, 0, 1/\delta)\}$ in $\mathbb{R}^{d+1}$. Then $\tilde{P}$ is facet-rectangular, that is to say, if $H$ is a hyperplane in $\mathbb{R}^{d+1}$ and if $H \cap \tilde{P}$ is a facet of $\tilde{P}$, then $H \cap \mathbb{Z}^{d+1} \neq \emptyset$.

It is clear that there exists no lattice points in the interior of $P(G) = 2\text{FRAC}(G)$, and that the lattice point $(1, \ldots, 1)$ belongs to the interior of $3\text{FRAC}(G)$. Thus it is enough to show that conditions (i) and (ii) in Proposition 2.1 are satisfied when $P = \text{FRAC}(G)$, $\delta = 3$, $\alpha = (1, \ldots, 1)$ and $Q = Q(G)$. A criterion for a vector to be a vertex of $\text{FRAC}(G)$ is given in [13, Theorem 15]:

**Lemma 2.2.** Let $G$ be a finite simple graph with $d$ vertices. Suppose that $v = (v_1, \ldots, v_d) \in \{0, 1/2, 1\}^d$ belongs to $\text{FRAC}(G)$. Let $G_S$ be the subgraph of $G$ induced by $S = \{i \in [d] \mid v_i = 1/2\}$. Then $v$ is a vertex of $\text{FRAC}(G)$ if and only if either $S = \emptyset$ or each connected component of $G_S$ contains an odd cycle.

Using Lemma 2.2, we determine when $Q(G)$ is a lattice polytope.

**Proposition 2.3.** Let $G$ be a finite simple graph without isolated vertices. Then the following conditions are equivalent.

(i) The graph $G$ is a bipartite graph;
(ii) The polytope $\text{FRAC}(G)$ is a lattice polytope;
(iii) The polytope $Q(G)$ is a lattice polytope.

**Proof.** If $G$ is bipartite, then $\text{FRAC}(G) = \text{STAB}(G)$ is a lattice polytope. Hence (i) $\Rightarrow$ (ii) holds. Moreover, (ii) $\Rightarrow$ (iii) is trivial. We now show that (iii) $\Rightarrow$ (i). Suppose $G$ contains an odd cycle $C$. Let $H$ be a connected component
of $G$ that contains $C$ and let $V(H)$ be the set of vertices of $H$. Here, we define $v = (v_1, \ldots, v_d)$ by $v_i = 1/2$ if $i \in V(H)$ and $v_i = 0$ if $i \notin V(H)$. Then $v$ is a $(0, 1/2)$-vector in $\text{FRAC}(G)$. Moreover, since $v$ satisfies the condition in Lemma \ref{lem:1}, $v$ is a vertex of $\text{FRAC}(G)$. Then $3v - (1, \ldots, 1) \in \{-1, 1/2\}^d$ is a vertex of $Q(G)$ that is not a lattice point. Hence $Q(G)$ is not a lattice polytope.

Next we show that $Q(G)^\vee$ is a lattice polytope.

**Proposition 2.4.** Suppose $G$ is a finite simple graph without isolated vertices. Then the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $Q(G)$ and

$$\{e_i + e_j \mid (i, j) \in E(G)\} \cup \{-e_i \mid 1 \leq i \leq d\}$$

is the vertex set of $Q(G)^\vee$. In particular, if $G$ is a bipartite graph, then $Q(G)$ is a Gorenstein Fano polytope.

**Proof.** It is known that the inequalities $x_i \geq 0$ $(1 \leq i \leq d)$ and $x_i + x_j \leq 1 ((i, j) \in E(G))$ define the facets of $\text{FRAC}(G)$. Hence the inequalities $x_i \geq -1$ $(1 \leq i \leq d)$ and $x_i + x_j \leq 1 ((i, j) \in E(G))$ define the facets of $Q(G)$. Thus a vector $(v_1, \ldots, v_d) \in \mathbb{R}^d$ belongs to the interior of $Q(G)$ if and only if $v_i > -1$ $(1 \leq i \leq d)$ and $v_i + v_j < 1 ((i, j) \in E(G))$. It is clear that the origin of $\mathbb{R}^d$ belongs to the interior of $Q(G)$. Suppose that $(v_1, \ldots, v_d) \in \mathbb{Z}^d$ belongs to the interior of $Q(G)$. Since $v_i$ and $v_i + v_j$ are integers, we have $v_i \geq 0$ $(1 \leq i \leq d)$ and $v_i + v_j \leq 0 ((i, j) \in E(G))$. Hence $v_i = 0$ for all $i$, i.e., $(v_1, \ldots, v_d) = 0$. It is known that there is a one-to-one correspondence between the facets of $Q(G)$ and the vertices of $Q(G)^\vee$. The set $\{e_i + e_j \mid (i, j) \in E(G)\} \cup \{-e_i \mid 1 \leq i \leq d\}$ of coefficient vectors of inequalities that define facets is the set of vertices of $Q(G)^\vee$. Thus, in particular, $Q(G)^\vee$ is a lattice polytope. By Proposition \ref{prop:3}, if $G$ is a bipartite graph, then $Q(G)$ is a lattice polytope, and hence a Gorenstein Fano polytope. \hfill $\Box$

We are now in the position to show that the Ehrhart ring of $\text{FRAC}(G)$ is Gorenstein.

**Theorem 2.5.** Let $G$ be a finite simple graph without isolated vertices. Then the Ehrhart ring of $\text{FRAC}(G)$ is Gorenstein.

**Proof.** It is enough to show that conditions (i) and (ii) in Proposition \ref{prop:4} are satisfied when $P = \text{FRAC}(G)$, $\delta = 3$, $\alpha = (1, \ldots, 1)$ and $Q = Q(G)$. First, Proposition \ref{prop:4} guarantees that $Q^\vee$ is a lattice polytope. Let $F = H \cap \tilde{P}$ be a facet of $\tilde{P}$, where $H$ is a hyperplane in $\mathbb{R}^{d+1}$. We may assume that $H \neq \{x_{d+1} = 0\}$. Then $F' = F \cap \{x_{d+1} = 0\}$ is a facet of $\{((\beta, 0) \in \mathbb{R}^{d+1} \mid \beta \in P\}$ whose supporting hyperplane is $H$. Therefore, $H' = H \cap \{x_{d+1} = 0\}$ is defined by $\{x_{d+1} = 0\}$ and either $x_i + x_j = 1 ((i, j) \in E(G))$ or $x_i = 0 (1 \leq i \leq d)$. Hence it is clear that there exists a lattice point in $H' \subset H$. Thus condition (ii) in Proposition \ref{prop:4} holds. Therefore, the Ehrhart ring of $P$ is Gorenstein by Proposition \ref{prop:4}. \hfill $\Box$
3. The Ehrhart series of FRAC(G)

In this section, we show that we can calculate the Ehrhart series and the Ehrhart quasi-polynomial of FRAC(G) from that of P(G). Let G be a simple graph on the vertex set [d] without isolated vertices. Since the interior of P(G) possesses no lattice points, and the interior of 2P(G) has a lattice point, it follows that deg δ(P(G), t) = d + 1 − 2 = d − 1. On the other hand, the degree of E(FRAC(G), t) is −3 as a rational function. Given a rational convex polytope P, the period of i(P, n) is a divisor of the smallest positive integer α for which αP is a lattice polytope. See [9, Theorem 4.6.25]. Hence i(FRAC(G), n) is a quasi-polynomial of period at most 2. Thus there exist polynomials i^{odd}(FRAC(G), n) and i^{even}(FRAC(G), n) of degree d such that

\[
i(FRAC(G), n) = \begin{cases} i^{odd}(FRAC(G), n) & \text{if } n \text{ is odd,} \\ i^{even}(FRAC(G), n) & \text{if } n \text{ is even.}
\end{cases}
\]

In particular, if G is bipartite, then i^{odd}(FRAC(G), n) = i^{even}(FRAC(G), n).

Theorem 3.1. Let G be a simple graph on the vertex set [d] without isolated vertices and let δ(P(G), t) = δ₀ + δ₁t + ⋯ + δ_{d−1}t^{d−1}. Then we have

\[
E(FRAC(G), t) = \frac{δ(P(G), t^2) + t^{2d−1}δ(P(G), 1/t^2)}{(1 − t^2)^{d+1}}
= \frac{δ₀ + δ_{d−1}t + δ₁t^2 + δ_{d−2}t^3 + ⋯ + δ_{d−1}t^{2d−2} + δ₀t^{2d−1}}{(1 − t^2)^{d+1}},
\]

where (δ₀, δ_{d−1}, δ₁, δ_{d−2}, ⋯, δ_{d−1}, δ₀) is symmetric and unimodal. In addition,

\[
i^{odd}(FRAC(G), 2k + 1) = (-1)^d i^{even}(FRAC(G), −2k − 4) = (-1)^d i(P(G), −k − 2).
\]

Proof. Let W = FRAC(G) and P = P(G). Then

\[
E(W, t) = \sum_{k \geq 0} i^{even}(W, 2k)t^{2k} + \sum_{k \geq 0} i^{odd}(W, 2k + 1)t^{2k+1}.
\]

Since i^{even}(W, 2k) = i(2W, k) = i(P, k), we have

\[
\sum_{k \geq 0} i^{even}(W, 2k)t^{2k} = \sum_{k \geq 0} i(P, k)(t^2)^k = \frac{δ(P, t^2)}{(1 − t^2)^{d+1}}.
\]

Since the degree of i^{odd}(W, 2k + 1) is d, by [8, Corollary 4.3.1], we have

\[
\sum_{k \geq 0} i^{odd}(W, 2k + 1)t^{2k+1} = t \sum_{k \geq 0} i^{odd}(W, 2k + 1)(t^2)^k = t \frac{a(t^2)}{(1 − t^2)^{d+1}}
\]

where a(t) is a polynomial of degree \leq d. Thus

\[
E(W, t) = \frac{δ(P, t^2)}{(1 − t^2)^{d+1}} + t \cdot \frac{a(t^2)}{(1 − t^2)^{d+1}} = \frac{δ(P, t^2) + ta(t^2)}{(1 − t^2)^{d+1}}.
\]
Since the degree of \( E(W, t) \) is \(-3\) as a rational function, the degree of \( \delta(P, t^2) + ta(t^2) \) is \( 2d - 1 \). Hence \( \deg a(t) = d - 1 = \deg \delta(P, t) \). Moreover, since the Ehrhart ring of \( W \) is Gorenstein, the coefficients of \( \delta(P, t^2) + ta(t^2) \) are symmetric. Thus \( a(t) = t^{d-1} \delta(P, 1/t) \) and \( \delta(P, t^2) + ta(t^2) = \delta(P, t^2) + t^{2d-1} \delta(P, 1/t^2) \). It is known that \( E(P, 1/t) = -\sum_{k \geq 1} i(P, -k)t^k \) (see [9, Chapter 4]). Hence

\[
\sum_{k \geq 0} i^{\text{odd}}(W, 2k + 1)t^{2k+1} = \frac{t^{2d-1}\delta(P, 1/t^2)}{(1 - t^2)^{d+1}}
\]

\[
= \frac{(-1)^{d+1} \delta(P, 1/t^2)}{t^3 (1 - t^2)^{d+1}}
\]

\[
= \frac{(-1)^{d+1}}{t^3} E(P, 1/t^2)
\]

\[
= \frac{(-1)^d}{t^3} \sum_{k \geq 1} i(P, -k)t^{2k}.
\]

Thus \( i^{\text{odd}}(W, 2k + 1) = (-1)^d i(P, -k - 2) = (-1)^d i^{\text{even}}(W, -2k - 4) \), as desired. \( \square \)

**Example 3.2.** Let \( W = \text{FRAC}(K_d) \) and \( P = \mathcal{P}(K_d) \) where \( K_d \) is a complete graph with \( d \) vertices. It is known [13, Example 27] that \( \delta(P, t) = A_d(t) + dtA_{d-1}(t) \). Let

\[
E(W, t) = \frac{b_0 + b_1t + \cdots + b_{2d-1}t^{2d-1}}{(1 - t^2)^{d+1}}.
\]

Since

\[
\delta(P, t) = A_d(t) + dtA_{d-1}(t)
\]

\[
= \sum_{i=0}^{d-1} A(d, i)t^i + dt \sum_{i=0}^{d-2} A(d - 1, i)t^i
\]

\[
= \sum_{i=0}^{d-1} A(d, i)t^i + d \sum_{i=0}^{d-1} A(d - 1, i - 1)t^i
\]

\[
= 1 + \sum_{i=1}^{d-1} (A(d, i) + dA(d - 1, i - 1))t^i
\]

hold, the \( \delta \)-vector \( (\delta_0, \ldots, \delta_{d-1}) \) of \( P \) satisfies \( \delta_0 = 1 \) and \( \delta_i = A(d, i) + dA(d - 1, i - 1) \) for \( i = 1, 2, \ldots, d - 1 \). By Theorem [31] we can obtain the formula \( b_0 = 1 \) and \( b_i = A(d, [i/2]) + dA(d - 1, [(i - 1)/2]) \) for \( i = 1, 2, \ldots, 2d - 1 \).

**Example 3.3.** Let \( W_d = \text{FRAC}(C_d) \) where \( C_d \) is an odd cycle of length \( d \). We computed the numerator \( g(W_d, t) \) of \( E(W_d, t) = g(W_d, t)/(1 - t^2)^{d+1} \) for \( d = 3, 5, 7, 9 \) by using software Normaliz [2].

\[
g(W_3, t) = 1 + 4t + 7t^2 + 7t^3 + 4t^4 + t^5.
\]
The edge incidence matrix of any bipartite graph is a lattice polytope if and only if the corresponding maximal minor of the configuration matrix of the graph is an identity matrix. Then \( Q \) is normal if and only if \( Q \) is Fano, and since \( (Q(G))\) is Gorenstein Fano if and only if \((Q(G))\) is normal.

In this section, we will discuss the dual polytope \( Q(G) \) of \( Q(G) \). Recall that

\[
Q(G) = \text{Conv}(\{ e_i + e_j \mid (i, j) \in E(G) \} \cup \{-e_i \mid 1 \leq i \leq d \})
\]

if \( G \) has no isolated vertices. It is easy to see that \( Q(G) \) is Fano. A lattice polytope \( P \subset \mathbb{R}^d \) is called normal if \( \mathbb{Z}_{\geq} A = \mathbb{Q}_{\geq} A \cap \mathbb{Z} A \), where

\[
A = \begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \end{pmatrix}
\]

such that \( \{a_1, \ldots, a_n\} = P \cap \mathbb{Z}^d \). Here \( \mathbb{Z} A = \{ \sum_{i=1}^n z_i (a_i, 1) \mid z_i \in \mathbb{Z} \} \), for example. A triangulation \( \Delta \) of \( P \) is called unimodular if the normalized volume of each maximal simplex of \( \Delta \) is one. If \( \mathbb{Z} A = \mathbb{Z}^{d+1} \), then the normalized volume of each maximal simplex is equal to the absolute value of the corresponding maximal minor of \( A \). See, [4 Section 5.5]. It is known that a lattice polytope \( P \) is normal if \( P \) has a unimodular triangulation ([4, Theorem 5.6.7]).

**Theorem 4.1.** Let \( G \) be a finite simple graph without isolated vertices. Then the following conditions are equivalent.

1. The graph \( G \) is a bipartite graph;
2. The dual polytope \( Q(G)^\vee \) has a unimodular triangulation;
3. The dual polytope \( Q(G)^\vee \) is normal;
4. The dual polytope \( Q(G)^\vee \) is a Gorenstein Fano polytope.

**Proof.** Since \( Q(G)^\vee \) is Fano, and since \((Q(G)^\vee)^\vee = Q(G)\), \( Q(G)^\vee \) is Gorenstein Fano if and only if \( Q(G) \) is a lattice polytope. By Proposition [2,3], \( Q(G) \) is a lattice polytope if and only if \( G \) is bipartite. Hence we have (i) \( \iff \) (iv).

Moreover, (ii) \( \Rightarrow \) (iii) holds in general. Let \( \mathcal{A}_G \) be the vertex-edge incidence matrix of \( G \) and let \( \mathcal{A}_G' \) be the configuration matrix of \( Q(G)^\vee \), namely,

\[
\mathcal{A}_G' = \begin{pmatrix} 0 & \mathcal{A}_G & -E_d \\ 1 & 1 \cdots & 1 \end{pmatrix},
\]

where \( E_d \) is an identity matrix. Then \( \mathbb{Z}\mathcal{A}_G' = \mathbb{Z}^{d+1} \). Hence \( Q(G)^\vee \) is normal if and only if \( \mathbb{Z}_{\geq} \mathcal{A}_G' = \mathbb{Q}_{\geq} \mathcal{A}_G' \cap \mathbb{Z}^{d+1} \).

(i) \( \Rightarrow \) (ii): Suppose that \( G \) is bipartite. It is known that the vertex-edge incidence matrix of any bipartite graph is totally unimodular; i.e., the

\[
g(W_5, t) = 1 + 11t + 51t^2 + 131t^3 + 206t^4 + 206t^5 + 131t^6 + 51t^7 + 11t^8 + t^9.
\]

\[
g(W_7, t) = 1 + 29t + 281t^2 + 1408t^3 + 4320t^4 + 8814t^5 + 12475t^6 + 12475t^7 + 8814t^8 + 4320t^9 + 1408t^{10} + 281t^{11} + 29t^{12} + t^{13}.
\]

\[
g(W_9, t) = 1 + 76t + 1450t^2 + 12844t^3 + 67000t^4 + 230986t^5 + 561004t^6 + 996310t^7 + 1321369t^8 + 1321369t^9 + 996310t^{10} + 561004t^{11} + 230986t^{12} + 67000t^{13} + 12844t^{14} + 1450t^{15} + 76t^{16} + t^{17}.
\]
determinant of every square non-singular submatrix is ±1. Hence it follows that the submatrix \( B = (A_G - E_d) \) of \( A'_G \) is totally unimodular. Let \( \Delta \) be a pulling triangulation (see [1, 4 Proposition 5.6.5]) of \( Q(G)^\vee \) such that the origin is a vertex of every maximal simplex in \( \Delta \). Such a triangulation is obtained by a Gröbner basis of the toric ideal of \( A'_G \) with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin. Then the normalized volume of each maximal simplex in \( \Delta \) is equal to the absolute value of the corresponding maximal minor of \( B \). Since \( B \) is totally unimodular, each maximal minor of \( B \) is ±1, and hence the triangulation \( \Delta \) is unimodular.

(iii) ⇒ (i): Suppose that the graph \( G \) contains an odd cycle \( C \). Now, we will show that \( Q(G)^\vee \) is not normal, that is, \( \mathbb{Z}_{\geq 0}A'_G \neq \mathbb{Q}_{\geq 0}A'_G \cap \mathbb{Z}^d \). We may assume that \( C = (1, 2, \ldots, 2k+1) \). Let

\[
\mathbf{u} = \frac{1}{2} \left( e_{d+1} + (e_1 + e_{2k+1} + e_{d+1}) + \sum_{i=1}^{2k} (e_i + e_{i+1} + e_{d+1}) \right)
\]

\[
= (k+1)e_{d+1} + \sum_{i=1}^{2k+1} e_i.
\]

Then \( \mathbf{u} \) belongs to \( \mathbb{Q}_{\geq 0}A'_G \cap \mathbb{Z}^d \). It is enough to show that \( \mathbf{u} \notin \mathbb{Z}_{\geq 0}A'_G \).

Suppose

\[
\mathbf{u} = \gamma e_{d+1} + \sum_{(i,j) \in E(G)} \alpha_{ij} (e_i + e_j + e_{d+1}) + \sum_{i=1}^d \beta_i (-e_i + e_{d+1}) \tag{1}
\]

for some \( \alpha_{ij}, \beta_i \in \mathbb{Z}_{\geq 0} \). Then the coefficient of \( e_i \) (\( 1 \leq i \leq 2k+1 \)) in (1) is \( 1 = \sum_{(i,j) \in E(G)} \alpha_{ij} - \beta_i \), and that of \( e_i \) (\( 2k+2 \leq i \leq d \)) in (1) is \( 0 = \sum_{(i,j) \in E(G)} \alpha_{ij} - \beta_i \). By summing up the equations for \( 1 \leq i \leq d \), we obtain

\[
2k+1 = \sum_{(i,j) \in E(G)} \alpha_{ij} - \sum_{i=1}^d \beta_i. \tag{2}
\]

On the other hand, the coefficient of \( e_{d+1} \) in (1) is

\[
k + 1 = \gamma + \sum_{(i,j) \in E(G)} \alpha_{ij} + \sum_{i=1}^d \beta_i. \tag{3}
\]

Since \( \gamma \) and \( \sum_{i=1}^d \beta_i \) are nonnegative, by equations (2) and (3), we obtain

\[
k + \frac{1}{2} \leq \sum_{(i,j) \in E(G)} \alpha_{ij} \leq k + 1.
\]

Since \( \sum_{(i,j) \in E(G)} \alpha_{ij} \in \mathbb{Z} \), it follows that \( \sum_{(i,j) \in E(G)} \alpha_{ij} = k + 1 \). Hence, by equation (2), we have \( \sum_{i=1}^d \beta_i = 1 \). Thus, by equation (3), we have \( \gamma + 1 = 0 \), which is a contradiction. \( \Box \)
References

[1] C. A. Athanasiadis, $h^*$-vectors, Eulerian polynomials and stable polytopes of graphs, *Electron. J. Combin.* 11 (2004/06), no. 2, Research Paper 6, 13 pp. (electronic).

[2] W. Bruns, B. Ichim, T. Römer, R. Sieg and C. Söger, Normaliz. Algorithms for rational cones and affine monoids. Available at [https://www.normaliz.uni-osnabrueck.de](https://www.normaliz.uni-osnabrueck.de).

[3] W. Bruns and T. Römer, $h$-Vectors of Gorenstein polytopes, *J. Combin. Theory Ser. A* 114 (2007), 65–76.

[4] T. Hibi, Ed., “Gröbner Bases: Statistics and Software Systems,” Springer, 2013.

[5] E. De Negri and T. Hibi, Gorenstein algebras of Veronese type, *J. Algebra* 193 (1997), 629–639.

[6] G. L. Nemhauser and L. E. Trotter, Jr., Properties of vertex packing and independence system polyhedra, *Math. Programming* 6 (1974), 48–61.

[7] H. Ohsugi and T. Hibi, Special simplices and Gorenstein toric rings, *J. Combin. Theory Ser. A* 113 (2006), 718–725.

[8] R. P. Stanley, “Enumerative Combinatorics” Volume 1 second edition, Wadsworth & Brook, Monterey, Wadsworth & Brooks/Cole Math Series, 1986.

[9] R. P. Stanley, Two poset polytopes, *Discrete Comput. Geom.* 1 (1986), 9–23.

[10] J. Schepers and L. Van Langenhoven, Unimodality questions for integrally closed lattice polytopes, *Ann. Comb.* 17 (2013), 571–589.

[11] A. Schrijver, “Theory of Linear and Integer Programming,” John Wiley & Sons, Ltd., Chichester, 1986.

[12] A. Stapledon, Inequalities and Ehrhart $\delta$-vectors, *Trans. Amer. Math. Soc.* 361 (2009), 5615–5626.

[13] E. Steingrímsson, A decomposition of 2-weak vertex-packing polytopes, *Discrete Comput. Geom.* 12 (1994), 465–479.

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