Traversable Wormholes in Distorted Gravity

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We consider the effects of Distorted Gravity on the traversability of the wormholes. In particular, we consider configurations which are sustained by their own gravitational quantum fluctuations. The Ultra-Violet divergences appearing to one loop are taken under control with the help of a Noncommutative geometry representation and Gravity’s Rainbow. In this context, it will be shown that for every framework, the self-sustained equation will produce a Wheeler wormhole, namely a wormhole of Planckian size. This means that, from the point of view of traversability, the wormhole will be traversable in principle, but not in practice. To this purpose, in the context of Gravity’s Rainbow we have considered different proposals of rainbow’s functions to see if the smallness of the wormhole is dependent on the chosen form of the rainbow’s function. Unfortunately, we discover that this is not the case and we suggest that the self-sustained equation can be improved to see if the wormhole radius can be enlarged or not. Some consequences on topology change are discussed.

Keywords: Traversable Wormholes; Gravity’s Rainbow; Quantum Gravity.

1. Introduction

In recent years many attempts to modify gravity have been done to explain large and short scale phenomena. This modifications seem to be necessary when one desires to go beyond General Relativity. One proposal comes from adding higher-order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from the effective action of Quantum Gravity. Such corrective terms seem to be unavoidable if we want to obtain the effective action of Quantum Gravity on scales closed to the Planck length. Therefore terms of the form $\mathcal{R}^2$, $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$, $\mathcal{R}^{\mu\alpha\beta\gamma}\mathcal{R}_{\mu\alpha\beta\gamma}$, $\mathcal{R}^2\mathcal{R}$, or $\mathcal{R}^{\square k}\mathcal{R}$ have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. These higher curvature terms can be included in more general forms known as $f(\mathcal{R})$ theories and further generalizations. Another

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*a For a recent review, see Refs. 3–5*
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Proposal comes from modifying the short scale behavior in an attempt to include quantum gravitational effects in the description. To this purpose, Noncommutative geometry, Gravity’s Rainbow and Generalized Uncertainty Principle (GUP) have been widely used to keep under control Ultraviolet (UV) divergences. For example in a Noncommutative spacetime, one introduces a commutator $[x^\mu, x^\nu] = i \theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix which determines the fundamental discretization of spacetime. This feature eliminates point-like structures in favor of smeared objects in flat spacetime. Therefore, one may consider the possibility that noncommutativity could cure the divergences that appear in general relativity. The effect of the smearing is mathematically implemented with a substitution of the Dirac-delta function by a Gaussian distribution of minimal length $\sqrt{\theta}$. In particular, the energy density of a static and spherically symmetric, smeared and particle-like gravitational source has been considered in the following form

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}} \exp \left( -\frac{r^2}{4\theta} \right),$$

where the mass $M$ is diffused throughout a region of linear dimension $\sqrt{\theta}$ due to the intrinsic uncertainty encoded in the coordinate commutator. On the other hand, when Gravity’s Rainbow is considered, spacetime is endowed with two arbitrary functions $g_1 (E/E_P)$ and $g_2 (E/E_P)$ having the following properties

$$\lim_{E/E_P \to 0} g_1 (E/E_P) = 1 \quad \text{and} \quad \lim_{E/E_P \to 0} g_2 (E/E_P) = 1. \quad (2)$$

g_1 (E/E_P) and $g_2 (E/E_P)$ appear into the solutions of the modified Einstein’s Field Equations

$$G_{\mu\nu} (E/E_P) = 8\pi G (E/E_P) T_{\mu\nu} (E/E_P) + g_{\mu\nu} \Lambda (E/E_P), \quad (3)$$

where $G (E/E_P)$ is an energy dependent Newton’s constant, defined so that $G (0)$ is the low-energy Newton’s constant and $\Lambda (E/E_P)$ is an energy dependent cosmological constant. Usually $E$ is the energy associated to the particle deforming the spacetime geometry. Since the scale of deformation involved is the Planck scale, it is likely that spacetime itself begins to fluctuate in such a way to produce a Zero Point Energy (ZPE). In absence of matter fields, the only particle compatible with the deformed Einstein’s gravity is the graviton. As regards the Generalized Uncertainty Principle (GUP) which is a modification of the Heisenberg uncertainty relations, we find that

$$\Delta x \Delta p \geq \hbar + \frac{\lambda^2_p}{\hbar} (\Delta p)^2, \quad (4)$$

where $\hbar$ is the Planck constant and $\lambda_p$ is the Planck length. When the GUP is applied to the Liouville measure, the modified number of quantum states is

$$\frac{d^3x d^3p}{(2\pi\hbar)^3 (1 + \lambda_p^2)^3}. \quad (5)$$
For $\lambda = 0$, the formula reduces to the ordinary counting of quantum states. If Eq. 5 is used for computing black hole entropy, the usual divergence which appear (brick wall) can be removed.\cite{9,10b} To conclude, one could also include, as a further example of a theory which takes under control UV divergences, a very recent and interesting theory known as Hořava-Lifshitz (HL) theory. This is based on a modification of Einstein gravity motivated by the Lifshitz theory in solid state physics.\cite{15,16} This modification allows the theory to be power-counting UV renormalizable and should recover general relativity in the infrared (IR) limit. Nevertheless, HL theory has the unpleasant feature of being noncovariant. Noncommutative geometry, Gravity’s Rainbow, GUP and HL theory are all examples of “Distorted Gravity”. Usually calculations involving quantum fluctuations of the gravitational field manifest UV divergences. To keep under control the UV divergences, one invokes a standard regularization/renormalization process. However in Distorted Gravity this procedure can be avoided\cite{17} especially when a ZPE calculation is considered. What makes a ZPE calculation interesting is that it is strictly related to the Casimir effect. Casimir effect has many applications and it can be considered under different points of view, but it can also be used as a tool to probe another appealing production of the gravitational field theory: a wormhole. A wormhole is often termed Einstein-Rosen bridge because a “bridge” connecting two “sheets” was the result obtained by A. Einstein and N. Rosen in attempting to build a geometrical model of a physical elementary “particle” that was everywhere finite and singularity free.\cite{18} It was J.A. Wheeler who introduced the term wormhole,\cite{19} although his wormholes were at the quantum scale. We have to wait for M. S. Morris and K. S. Thorne\cite{21} to see the subject of wormholes seriously considered by the scientific community. In practice a traversable wormhole is a solution of the Einstein’s Field equations, represented by two asymptotically flat regions joined by a bridge or, in other word, it is a short-cut in space and time. To exist, traversable wormholes must violate the null energy conditions, which means that the matter threading the wormhole’s throat has to be “exotic”. Classical matter satisfies the usual energy conditions. Therefore, it is likely that wormholes must belong to the realm of semi-classical or perhaps a possible quantum theory of the gravitational field. On this ground, the Casimir energy on a fixed background, has the correct properties to substitute the exotic matter: indeed, it is known that, for different physical systems, Casimir energy is negative. Usually one considers some matter or gauge fields which contribute to the Casimir energy necessary to the traversability of the wormholes, nevertheless nothing forbids to use the Casimir energy of the graviton on a background of a traversable wormhole.\cite{22} In this way, one can think that the quantum fluctuations of the gravitational field of a traversable wormhole are the same ones which are responsible to sustain traversability. In this contribution, as an example of Distorted Gravity, we

\footnote{For applications in Quantum Cosmology, see Ref.\cite{14}}

\footnote{Note that in Ref.\cite{22} the Casimir energy was used as an indicator of topology change between wormholes and dark energy stars.
will fix our attention primarily on Gravity’s Rainbow with one extension to Non-commutative geometry. The rest of the paper is structured as follows, in section 2 we define what is a self-sustained traversable wormhole, in section 3 we compute the ZPE graviton energy responsible of the self sustained traversable wormhole in the context of Distorted Gravity, in section 4 we relax the convergence conditions of the integrals involved in the self-sustained calculation. We summarize and conclude in section 5. Units in which $\hbar = c = k = 1$ are used throughout the paper.

2. Self-sustained Traversable Wormholes

In this Section we shall consider the formalism outlined in detail in Refs. \cite{23,24} where the graviton one loop contribution to a classical energy in a wormhole background is used. The spacetime metric representing a spherically symmetric and static wormhole is given by

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (6)$$

where $\Phi(r)$ and $b(r)$ are arbitrary functions of the radial coordinate, $r$, denoted as the redshift function, and the shape function, respectively. The radial coordinate has a range that increases from a minimum value at $r_0$, corresponding to the wormhole throat, to infinity. A fundamental property of a wormhole is that a flaring out condition of the throat, given by $(b - b'/r^2) > 0$, is imposed, and at the throat $b(r_0) = r = r_0$, the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. Another condition that needs to be satisfied is $1 - b(r)/r > 0$. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with $e^{2\Phi} \to 0$, so that $\Phi(r)$ must be finite everywhere. In order to describe a self-sustained traversable wormhole, we need to define the classical energy. A key point is given by the background field super-hamiltonian, $H^{(0)}$. This can be built with the help of the Arnowitt-Deser-Misner (ADM) decomposition\cite{25} of space time based on the following line element

$$ds^2 = g_{\mu
u}(x) dx^\mu dx^\nu = (-N^2 + N_i N^i) dt^2 + 2N_j dt dx^j + g_{ij} dx^i dx^j,$$  \hspace{1cm} (7)$$

where $N$ is the lapse function and $N_i$ the shift function. In terms of the ADM variables, the four dimensional scalar curvature $R$ can be decomposed in the following way

$$R = R + K_{ij} K^{ij} - (K)^2 - 2 \nabla_\mu (K u^\mu + a^\mu),$$  \hspace{1cm} (8)$$

where

$$K_{ij} = -\frac{1}{2N} [\partial_i g_{ij} - N_{i}g_{jj} - N_{j}g_{ii}]$$  \hspace{1cm} (9)$$
is the second fundamental form, $K = g^{ij} K_{ij}$ is its trace, $R$ is the three dimensional scalar curvature and $\sqrt{g}$ is the three dimensional determinant of the metric. The last term in (8) represents the boundary terms contribution where the four-velocity
\( u^\mu \) is the timelike unit vector normal to the spacelike hypersurfaces \((t=\text{constant})\) denoted by \( \Sigma_t \) and \( a^\mu = u^\alpha \nabla_\alpha u^\mu \) is the acceleration of the timelike normal \( u^\mu \). Thus

\[
\mathcal{L}[N, N_i, g_{ij}] = \sqrt{-g} R = \frac{N}{16\pi G} \sqrt{g} \left[ K_{ij} K^{ij} - K^2 + R - 2\nabla_\mu (K u^\mu + a^\mu) \right]
\]

(10)

represents the gravitational Lagrangian density and \( G \) is the Newton’s constant. After a Legendre transformation, we obtain two classical constraints:

\[
\mathcal{H}^{(0)} = (16\pi G) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{16\pi G} R = 0
\]

(11)

and

\[
\mathcal{H}_j = \pi^{ij}_{|j} = 0.
\]

(12)

\( G_{ijkl} \) is the super-metric and the super-momentum \( \pi^{ij} \) is defined as

\[
\pi^{ij} = \frac{\delta \mathcal{L}}{\delta (\partial_t g_{ij})} = (g^{ij} K - K^{ij}) \frac{\sqrt{g}}{16\pi G}.
\]

(13)

Note that \( \mathcal{H} = 0 \) represents the classical constraint which guarantees the invariance under time reparametrization. The other classical constraint represents the invariance by spatial diffeomorphism and the vertical stroke “\(|\) denotes the covariant derivative with respect to the 3D metric \( g_{ij} \). Note that boundary terms become important when one compares different configurations like Wormholes and Dark Stars\(^{22}\) or Wormholes and Gravastars\(^{26}\). When we deal with spherically symmetric line elements, the kinetic term disappears and the hamiltonian constraint (11) reduces to

\[
H^{(0)}_\Sigma = \int_\Sigma d^3 x \mathcal{H}^{(0)} = - \frac{1}{16\pi G} \int_\Sigma d^3 x \sqrt{g} R
\]

\[
= - \frac{1}{2G} \int_{r_0}^{\infty} dr \frac{r^2}{\sqrt{1 - b(r)/r}} \frac{b'(r)}{r^2},
\]

(14)

where we have integrated on a constant time hypersurface \( \mathcal{H}^{(0)} \) and where we have used the explicit expression of the scalar curvature in three dimensions in terms of the shape function. Therefore, from the Hamiltonian point of view, it is not necessary to assume that \( \Phi (r) = \text{const.} \). A traversable wormhole is said to be “self sustained” if

\[
H^{(0)}_\Sigma = -E^{TT},
\]

(15)

where \( E^{TT} \) is the total regularized graviton one loop energy. Basically this is given by

\[
E^{TT} = -\frac{1}{2} \sum_\tau \left[ \sqrt{E_1^2 (\tau)} + \sqrt{E_2^2 (\tau)} \right],
\]

(16)
where $\tau$ denotes a complete set of indices and $E_i^2(\tau) > 0$, $i = 1, 2$ are the eigenvalues of the modified Lichnerowicz operator

$$\left(\Delta^m h^\perp\right)_{ij} = (\Delta_L h^\perp)_{ij} - 4R^k_{ij} h^\perp_{kj} + 3 R h^\perp_{ij},$$

acting on traceless-transverse tensors of the perturbation and where $\Delta_L$ is the Lichnerowicz operator defined by

$$\left(\Delta_L h\right)_{ij} = \Delta h_{ij} - 2R_{ikjl} h^{kl} + R_{jk} h^i_l + R_{jk} h^i_k,$$

with $\Delta = -\nabla^a \nabla_a$. For the background $(6)$, one can define two $r$-dependent radial wave numbers

$$k_i^2(r, l, \omega_{i,nl}) = \omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r) \quad i = 1, 2,$$

where

$$m_1^2(r) = \frac{6}{r^2} \left(1 - \frac{b_0(r)}{r} \right) + \frac{3}{2r^2} b'(r) - \frac{3}{2r^2} b(r)$$

$$m_2^2(r) = \frac{6}{r^2} \left(1 - \frac{b_0(r)}{r} \right) + \frac{1}{2r^2} b'(r) + \frac{3}{2r^2} b(r)$$

are two $r$-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$. When we perform the sum over all modes, $E^{TT}$ is usually divergent. In Refs.[23,24] a standard regularization/renormalization scheme has been adopted to handle the divergences. In this contribution, we will consider the effect of Gravity’s Rainbow and Noncommutative geometry on the graviton to one loop. One advantage in using such a scheme is to avoid the renormalization process and to use only one scale: the Planck scale.

3. Some examples of Distorted Gravity

One of the purposes of Eq.(15) is the possible discovery of a traversable wormhole with the determination of the form of the shape function. However, one could also reverse the strategy: we fix the wormhole shape to see if the self-sustained equation is satisfied. One good candidate is

$$b(r) = \frac{r_0^2}{r},$$

which is the prototype of the traversable wormholes[20,21]. Plugging the shape function (21) into Eq.(15), we find that the left hand side becomes

$$H^{(0)}_\Sigma = \frac{1}{2G} \int_{r_0}^{\infty} dr \frac{r^2}{\sqrt{1 - r_0^2/r^2}} \frac{r_0^2}{r^4},$$

while the right hand side is divergent. To handle with divergences, we have several possibilities. For example, in Ref.[23] a regularization/renormalization scheme has been adopted and the wormhole throat has been fixed at the value $r_0 \simeq 1.16/E_P$. If we adopt a Noncommutative scheme,[23] the distorted Liouville measure

$$dn_i = \frac{d^3x d^3k}{(2\pi)^3} \exp\left(-\frac{\theta}{4} \left(\omega_{i,nl}^2 - m_i^2(r)\right)\right), \quad i = 1, 2,$$
allows the computation of the graviton to one loop. This is possible because the distortion induced by the Noncommutative space time allows the right hand side of Eq. (15) to be finite. Indeed, plugging $dn_i$ into Eq. (15), one finds that the self-sustained equation for the energy density becomes

$$
\frac{3\pi^2}{Gr_0^2} = \int_0^{+\infty} \sqrt{\left(\omega^2 + \frac{3}{r_0^2}\right)} e^{-\frac{4}{\omega^2 + \frac{1}{r_0^2}}} d\omega + \int_{1/r_0}^{+\infty} \sqrt{\left(\omega^2 - \frac{1}{r_0^2}\right)} e^{-\frac{4}{\omega^2 - \frac{1}{r_0^2}}} d\omega,
$$

where we have used the shape function (21) to evaluate the effective masses (20). By defining the dimensionless variable

$$x = \frac{\theta}{4r_0^2},$$

Eq. (24) leads to $(G = l_P^2)$

$$\frac{3\pi^2\theta}{l_P^2} = F(x),$$

where

$$F(x) = \left(1 - x\right) K_1\left(\frac{x}{2}\right) + x K_0\left(\frac{x}{2}\right) \exp\left(\frac{x}{2}\right)$$

$$+ 3 \left(1 + 3x\right) K_1\left(\frac{3x}{2}\right) + 3x K_0\left(\frac{3x}{2}\right) \exp\left(-\frac{3x}{2}\right).$$

$(26)$

$F(x)$ has a maximum for $\bar{x} = 0.24$, where

$$\frac{3\pi^2\theta}{l_P^2} = F(\bar{x}) = 2.20.$$

This fixes $\theta$ to be

$$\theta = \frac{2.20l_P^2}{3\pi^2} = 7.43 \times 10^{-2}l_P^2.\quad (28)$$

and

$$r_0 = 0.28l_P.\quad (29)$$

When we consider Gravity’s Rainbow, spacetime is endowed with two arbitrary functions $g_1(E/E_P)$ and $g_2(E/E_P)$ having the properties shown in (2). As shown in Ref. (30), the self-sustained equation (15) becomes

$$\frac{b'(r)}{2Gg_2(E/E_P) r^2} = \frac{2}{3\pi^2} \left(I_1 + I_2\right),\quad (30)$$

where the r.h.s. of Eq. (30) is represented by

$$I_1 = \int_{E_*}^{\infty} \frac{E g_1(E/E_P)}{g_2^2(E/E_P)} dE \left(\frac{E^2}{g_2^2(E/E_P)} - m_1^2(r)\right)^{\frac{3}{2}} dE\quad (31)$$

and

$$I_2 = \int_{E_*}^{\infty} \frac{E g_1(E/E_P)}{g_2^2(E/E_P)} dE \left(\frac{E^2}{g_2^2(E/E_P)} - m_2^2(r)\right)^{\frac{3}{2}} dE,\quad (32)$$
respectively. $E^*$ is the value which annihilates the argument of the root. Of course, $I_1$ and $I_2$ are finite for appropriate choices of the Rainbow’s functions $g_1(E/E_P)$ and $g_2(E/E_P)$. If we assume that

$$g_1(E/E_P) = \exp\left(-\alpha E^2/E_P^2\right) \quad g_2(E/E_P) = 1,$$

with $\alpha \in \mathbb{R}$, the classical term is not distorted. We find

$$I_1 = 3 \int_0^{\infty} \sqrt{3/r_0} \exp\left(-\alpha E^2/E_P^2\right) E^2 \left(E^2 - \frac{3}{r_0} \right) dE$$

and

$$I_2 = 3 \int_0^{\infty} \exp\left(-\alpha E^2/E_P^2\right) E^2 \left(E^2 + \frac{1}{r_0} \right) dE,$$

where we have fixed the shape function as in Eq. (21). Now, in order to have only one solution with variables $\alpha$ and $r_0$, we demand that

$$\frac{d}{dr_0} \left[ -\frac{1}{2G} \frac{1}{r_0} \right] = \frac{d}{dr_0} \left[ \frac{2}{3\pi^2} \left( I_1 + I_2 \right) \right],$$

which takes the following form after the integration ($G^{-1} = E_P^2$)

$$1 = \frac{1}{2\pi^2 x^2} f(\alpha, x),$$

where $x = r_0 E_P$ and where

$$f(\alpha, x) = \exp\left(\frac{\alpha}{2x^2}\right) K_0\left(\frac{\alpha}{2x^2}\right) - \exp\left(\frac{\alpha}{2x^2}\right) K_1\left(\frac{\alpha}{2x^2}\right) + 9 \exp\left(-\frac{3\alpha}{2x^2}\right) K_0\left(\frac{3\alpha}{2x^2}\right) + \exp\left(-\frac{3\alpha}{2x^2}\right) K_1\left(\frac{3\alpha}{2x^2}\right).$$

$K_0(x)$ and $K_1(x)$ are the modified Bessel function of order 0 and 1, respectively. One finds a root at

$$\bar{x} = r_0 E_P = 2.973786871 \sqrt{\alpha}$$

with $\alpha \simeq 0.242$. This means that $r_0 E_P = 1.46$. It is immediate to understand that the example coming from Noncommutative geometry appears to be “rigid”. With this term we mean that the distorted Liouville measure cannot assume different forms. On the other hand, Gravity’s Rainbow depends on the choice of $g_1(E/E_P)$ and $g_2(E/E_P)$. In this short presentation we have explored only the Gaussian proposal (33) which strongly cuts off the Planckian and trans-Planckian physics. To this purpose, we would like to considerably relax the convergent form of the rainbow’s functions to see if there are some effects on the traversability of the wormhole.
4. Relaxing Gravity’s Rainbow

In this part of the contribution, we would like to explore a relaxed variant of the choice (33). The situation we are going to consider is the following. It is easy to see that if we assume

\[ g_1 \left( \frac{E}{E_P} \right) = \begin{cases} 1 & \text{when } E < E_P \\ \frac{E}{E_P} & \text{when } E > E_P \end{cases} \quad , \tag{40} \]

(31) and (32) reduce to

\[
I_1 = 3 \int_{E_P}^E \sqrt{E^2 - m_1^2(r)} \sqrt{E^2 - m_1^2(r_0)} \, dE , \tag{41}
\]

and

\[
I_2 = 3 \int_{E_P}^E \sqrt{E^2 - m_2^2(r)} \sqrt{E^2 - m_2^2(r_0)} \, dE . \tag{42}
\]

As a specific choice for the shape function we will take under consideration old examples suggested by Morris and Thorne\cite{21}:

\[
b(r) = \frac{4}{3} r_0 - \frac{1}{3} r_0 \left( \frac{r_0}{r} \right)^3 \tag{43}
\]

and

\[
b(r) = \sqrt{r_0 r} . \tag{44}
\]

For both of them, we will also assume that \( \Phi (r) = \text{const} \). Both the examples (43) and (44) satisfy

\[
b(r_0) = r_0, \quad \frac{b(r)}{r} \to 0 \quad \text{when} \quad r \to \infty . \tag{45}
\]

As regards the choice (43), one finds

\[
b'(r) = \frac{2}{3} \left( \frac{r_0}{r} \right)^3 < 1 \tag{46}
\]

and the flare out condition is satisfied on the throat. To see if the wormhole can be self-sustained, we plug the shape function (43) into (31) and (32). Then Eq. (30) simply becomes

\[
\frac{r_0^3}{3Gr^5} = 2 \pi^2 \left( \int_{E_P}^{E_P} \sqrt{E^2 - m_1^2(r_0)} \, dE + \int_{E_0}^{E_P} E^2 \sqrt{E^2 - m_2^2(r_0)} \, dE \right) , \tag{47}
\]

where we have explicitly used the form of the shape function (43) on the classical term. On the throat, the effective masses become

\[
m_1^2(r_0) = \frac{1}{2r_0^2} \quad m_2^2(r_0) = -\frac{11}{6r_0^2} \tag{48}
\]
and Eq. (47) simplifies into
\[
\frac{E_P^2}{3r_0^2} = \frac{2}{\pi^2} \left( \int_{\sqrt{1/2}r_0}^{E_P} E^2 \sqrt{E^2 - \frac{1}{2r_0^2}} dE + \int_{\sqrt{1/6r_0}}^{E_P} E^2 \sqrt{E^2 + \frac{11}{6r_0^2}} dE \right). \tag{49}
\]

The explicit calculation of (49) leads to
\[
1 = \frac{1}{\pi^2} \left[ \frac{3x^2}{2} \left( 1 - \frac{1}{2x^2} \right)^{3/2} + \frac{3}{8} \sqrt{1 - \frac{1}{2x^2}} - \frac{3}{16x^2} \ln \left( \sqrt{2x} + \sqrt{4x^2 - 1} \right) \right] + \frac{3x^2}{2} \left( 1 + \frac{11}{6x^2} \right)^{3/2} - \frac{11}{8} \sqrt{1 + \frac{11}{6x^2}} - \frac{121}{48x^2} \ln \left( \frac{\sqrt{6x^2}}{11} + \frac{\sqrt{4x^2 - 1}}{11} \right) - \frac{7}{4x^2} \ln \left( \frac{\sqrt{4x^2 - 1}}{3} + \frac{\sqrt{4x^2 - 1}}{3} \right) - 1, \tag{50}
\]
where \( x = r_0 E_P \). The solution can be easily computed numerically and we find \( x = r_0 E_P = 1.70 \). As regards the choice (44), one finds
\[
b'(r) = \frac{1}{2} \sqrt{\frac{r_0}{r}} < 1 \tag{51}
\]
and, once again, the flare out condition is satisfied on the throat. In this case, the self-sustained equation reduces to
\[
\frac{r_0}{4Gr^3} = \frac{2}{\pi^2} \left( \int_{\sqrt{m_1^2(r_0)}}^{E_P} E^2 \sqrt{E^2 - m_1^2(r_0)} dE + \int_{\sqrt{m_2^2(r_0)}}^{E_P} E^2 \sqrt{E^2 - m_2^2(r_0)} dE \right). \tag{52}
\]
The effective masses become on the throat
\[
m_1^2(r_0) = \frac{3}{4r_0^2}, \quad m_2^2(r_0) = -\frac{7}{4r_0^2} \tag{53}
\]
and Eq. (52) simplifies into
\[
\frac{E_P^2}{3r_0^2} = \frac{2}{\pi^2} \left( \int_{\sqrt{1/4r_0^2}}^{E_P} E^2 \sqrt{E^2 - \frac{3}{4r_0^2}} dE + \int_{\sqrt{1/4r_0^2}}^{E_P} E^2 \sqrt{E^2 + \frac{7}{4r_0^2}} dE \right). \tag{54}
\]
The explicit calculation of (54) leads to
\[
1 = \frac{1}{\pi^2} \left[ 2x^2 \left( 1 - \frac{3}{4x^2} \right)^{3/2} + \frac{3}{4} \sqrt{1 - \frac{3}{4x^2}} - \frac{9}{16x^2} \ln \left( \sqrt{\frac{4x^2}{3}} + \frac{\sqrt{4x^2 - 1}}{3} \right) \right] + \frac{2x^2}{3} \left( 1 + \frac{7}{4x^2} \right)^{3/2} - \frac{7}{4} \sqrt{1 + \frac{7}{4x^2}} - \frac{49}{16x^2} \ln \left( \frac{\sqrt{4x^2}}{7} + \frac{\sqrt{4x^2 - 1}}{7} \right), \tag{55}
\]
where \( x = r_0 E_P \). Even in this case, the solution can be easily computed numerically and we find \( x = r_0 E_P = 1.51 \). Note that the form in (44) is a special case of a shape function of the form
\[
b(r) = r_0 \left( \frac{r_0}{r} \right)^\omega, \tag{56}
\]
which is obtained imposing an equation of state \( p_r = \omega \rho \).
5. Conclusions

In this contribution we have discussed how some forms of Distorted Gravity can be used to power a traversable wormhole. The distorted Liouville measure (23) has been used as an example of Noncommutative geometry. We have obtained a solution compatible with the procedure located at $r_0 E = 0$. Note that in Ref. [31] a shape form induced by a density profile of the form (1) but with an ordinary regularization/renormalization procedure. It is beyond this contribution to investigate the shape function obtained by (1) combined with the distorted Liouville measure (23) gives a traversable wormhole. As regards Gravity’s Rainbow; the good news is that every shape function analyzed is traversable. The bad news is that the traversability is in principle but not in practice, even if the radii are greater that the radius discovered in Ref. [23] and also larger than the one obtained with the measure (23).

Indeed, all the wormholes discovered survive in the Planckian or Trans-Planckian regime. This is because we are probing a region where the gravitational field develops quantum fluctuations so violent that it is also able to give a topology change. Since the wormhole’s radius is of the Planckian size, we have explored a relaxed version of the rainbow’s functions to see if the smallness of the wormhole radius is a consequence of the strong convergence induced by the choice (33). Once again, the resulting wormhole is traversable in principle but not in practice. This means that we can interpret the self-sustained equation as an ignition equation. To obtain a larger radius, one possibility is to use the self-sustained equation in the following manner

$$\frac{1}{2G} \left( \frac{b'(r)}{r^2 g_2(E)} \right)^{(n)} = \frac{2}{3\pi^2} \left[ I_1 \left( b^{(n-1)}(r) \right) + I_2 \left( b^{(n-1)}(r) \right) \right],$$

where $n$ is the order of the approximation. In this way, if we discover that fixing the radius to some value of a fixed background of the r.h.s., and we discover on the l.h.s. a different radius, we could conclude that if the radius is larger that the original, the wormhole is growing, otherwise is collapsing. Note that in Ref. [32] Eq. (57) has been used to show that a traversable wormhole can be generated with a topology change starting from a Minkowski spacetime.

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