The probability that planar loop-erased random walk uses a given edge

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Abstract

We give a new proof of a result of Rick Kenyon that the probability that an edge in the middle of an $n \times n$ square is used in a loop-erased walk connecting opposite sides is of order $n^{-3/4}$. We, in fact, improve the result by showing that this estimate is correct up to multiplicative constants.

1 Introduction

Loop-erased random walk is a process obtained by erasing loops from simple random walk. Although the process can be defined for arbitrary Markov chains, we will discuss the process only on the planar integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$. We start this paper by stating our main result.

Let

$$A_n = \{j + ik \in \mathbb{Z} + i\mathbb{Z} : -n + 1 < j < n, -n < k < n\},$$

$$\partial A_n = \{z \in \mathbb{Z}^2 : \text{dist}(z, A_n) = 1\}.$$ 

Let $K_n$ denote the set of nearest neighbor paths $\omega = [\omega_0, \ldots, \omega_k]$ with $\text{Re}[\omega_0] = -n, \text{Re}[\omega_k] = n + 1$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n$. We write $|\omega| = k$ for the number of steps, and let $p(\omega) = 4^{|\omega|}$ be the simple random walk measure. Let

$$f(n) = \sum_{\omega \in K_n} p(\omega).$$

It is known that $\lim_{n \to \infty} f(n) = c_1 \in (0, \infty)$ (see, e.g., [4] Proposition 8.1.3]), where the constant $c_1$ can be given in terms of the Green’s function of Brownian motion on a domain bounded by a square.

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Let $\mathcal{W}_n$ denote the set of self-avoiding walks (SAWs) $\eta = [\eta_0, \ldots, \eta_k] \in \mathcal{K}_n$. For each $\omega \in \mathcal{K}_n$ there is a unique path $L(\omega) \in \mathcal{W}_n$ obtained by chronological loop-erasing (see \cite{4} Chapter 9 for appropriate definitions). The loop-erased measure $\hat{p}_n(\eta)$ is defined by

$$\hat{p}_n(\eta) = \sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} p(\omega).$$

Note that

$$\sum_{\eta \in \mathcal{W}_n} \hat{p}_n(\eta) = f(n).$$

Let $\mathcal{W}_n^+$ denote the set of $\eta \in \mathcal{W}_n$ that contain the directed edge $[0, 1]$ and $\mathcal{W}_n^-$ those that contain $[1, 0]$. Let $\mathcal{W}_n^* = \mathcal{W}_n^- \cup \mathcal{W}_n^+$ be the set of $\eta \in \mathcal{W}_n$ that contain the edge $[0, 1]$ in either direction. We write $a_n \asymp b_n$ to mean that $a_n/b_n$ and $b_n/a_n$ are uniformly bounded. The goal of this paper is to prove the following theorem.

**Theorem 1.1.** As $n \to \infty$,

$$\sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta) \asymp n^{-3/4}. \quad (1)$$

With a little more work, we could establish the existence of the limit

$$\lim_{n \to \infty} n^{3/4} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta),$$

but we will not do it here. (Our argument would not give the value of the limit. While we believe we might be able to some of the relevant asymptotic constants, we definitely do not know how to compute the value of the limit in (16) below.) Our result can be considered a strengthening of a result of Kenyon \cite{1} who proved that

$$\sum_{\eta \in \mathcal{W}_n} \hat{p}_n(\eta) \approx n^{-3/4}, \quad (2)$$

where $\approx$ indicates that the logarithms of both sides are asymptotic. His proof used the relationship between loop-erased walks and two other models, dimers and uniform spanning trees. Another proof of (2) was given by Masson \cite{10} using the relationship between loop-erased walk and the Schramm-Loewner evolution (SLE). We do not need to make reference to any of these models in our proof of (1). There are two main steps.

- A combinatorial identity is proved which writes the left-hand side of (1) in terms of simple random walk quantities.
- The simple random walk quantities are estimated.
Our computation to obtain the exponent $3/4$ uses the Brownian loop measure to estimate the random walk loop measure. This is in the spirit of Kenyon’s calculations [1] since the loop measure is closely related to the determinant of the Laplacian.

Although the proof is self-contained (other than some estimates for simple random walk) it does use a key idea from Kenyon’s work as discussed in [2, Section 5.7]. For each random walk path $\omega$, we let $J(\omega)$ be the number of times that the path crosses an edge of the form $[-ki, -ki+1]$ or $[-ki+1, -ki]$ where $k$ is a positive integer. Let $q(\omega) = (-1)^{J(\omega)} p(\omega)$. Let $Y_+(\omega)$ denote the number of times that $\omega$ uses the directed edge $[0, 1]$, $Y_-(\omega)$ the number of times that $\omega$ uses the directed edge $[1, 0]$, and $Y(\omega) = Y_+(\omega) - Y_-(\omega)$. The combinatorial identity is obtained by writing the quantity

$$\Lambda_n = \sum_{\omega \in \mathcal{K}_n} q(\omega) Y(\omega) = \sum_{\omega \in \mathcal{K}_n} p(\omega) (-1)^{J(\omega)} Y(\omega).$$

in two different ways.

The paper is written using the perspective of loop-erased walk in terms of the random walk loop measure as in [4, Chapter 9]. We start by reviewing this perspective in Section 2 and then we prove the identity in Section 3. Section 4 discusses the random walk estimates.

After the main result is proved, we generalize the combinatorial identity to random walk starting at any two points on $\partial A_n$. The argument is essentially the same, and one can compute the dependence of the probability on the starting points. A somewhat analogous calculation was done in [1], where the boundary was fixed but the interior point was allowed to vary. In the last section, this dependence on the boundary point is explained in terms of the scaling limit of loop-erased random walk, the Schramm-Loewner evolution with parameter $SLE_2$. Here we do not give all details.

One of the main motivations for doing the estimates in this paper is to show that the loop-erased random walk converges to $SLE_2$ in the natural parametrization [5, 7]. Up-to-constant estimates for the loop-erased walk probability can be viewed as a step in the program to establish this result.

## 2 Random walk loop measure

The random walk loop measure is a measure on unrooted random walk loops. A rooted loop is a nearest neighbor path

$$l = [l_0, l_1, \ldots, l_{2k}]$$

with $k \geq 0$ and $l_0 = l_{2k}$. We call $l_0$ the root of the loop. An unrooted loop $\bar{l}$ is an equivalence class of rooted loops with $k > 0$ under the equivalence

$$[l_j, l_{j+1}, \ldots, l_{2k}, l_1, l_2, \ldots, l_j] \sim [l_0, l_1, \ldots, l_{2k}]$$

for all $j$. Note that the orientation of the loops is maintained. The random walk loop measure $m$ is defined by

$$m(\bar{l}) = 4^{-|l|} \frac{d(\bar{l})}{|l|},$$
where \(d(\bar{l})\) is the number of rooted loops in the equivalence class of the unrooted loop \(\bar{l}\). Note that \(d(\bar{l})\) is always an integer dividing \(|\bar{l}|\). In a slight abuse of notation, if \(l\) is a loop and \(A \subset \mathbb{Z}^2\), we write \(l \subset A\) to mean that the vertices of \(l\) are contained in \(A\) and \(l \cap A\) for the set of vertices in \(A\) that \(l\) visits.

There is an equivalent way of defining this measure that is sometimes useful. Enumerate \(\mathbb{Z}^2 = \{v_1, v_2, \ldots, v_n\}\) and let \(V_n = \{v_1, \ldots, v_n\}\). We define a different measure on rooted loops by assigning to each (rooted) loop as in (4) with \(k > 0\), \(l \subset V_n\), and \(l_0 = v_n\) measure \(s^{-1} 4^{-2k}\) where \(s = \#\{j; 1 \leq j \leq 2k, l_j = v_n\}\). This induces a measure on unrooted loops by summing over rooted loops that generate an unrooted loop. (The factor \(s^{-1}\) compensates for the fact that several rooted loops give the same unrooted loop.) One can check that the induced measure on unrooted loops is the same as the loop measure above regardless of which enumeration is chosen. For computations it is often convenient to choose an enumeration in which \(|v_j|\) is nondecreasing.

If \(V = \{v_1, \ldots, v_k\} \subset A \not\subset \mathbb{Z}^2\), we define

\[
F_V(A) = \exp \left\{ \sum_{l \subset A, l \cap V \neq \emptyset} m(\bar{l}) \right\} = \prod_{j=1}^{k} G_{U_j}(v_j, v_j).
\]

Here \(U_j = A \setminus \{v_1, \ldots, v_{j-1}\}\) and \(G_U\) denotes the usual random walk Green’s function in the set \(U\). The loop-erased measure satisfies \([4\) Proposition 9.5.1]\)

\[
\hat{p}_n(\eta) = p(\eta) F_\eta(A_n).
\] (5)

We can also define a loop measure using the signed weight \(q(\omega) = (-1)^{J(l)} p(\omega)\). The quantities \(J(l), Y(l)\) as defined in the introduction are functions of the unrooted loop \(\bar{l}\). (Note that \(Y(l)\) does depend on the orientation of \(l\), so it is important that we are considering oriented, unrooted loops.) Let \(\mathcal{J}_A\) denote the set of unrooted loops \(\bar{l} \subset A\) such that \(J(\bar{l})\) is odd. If \(V \subset A\), let \(\mathcal{J}_{A,V}\) denote the set of unrooted loops \(\bar{l} \in \mathcal{J}_A\) that intersect \(V\). Let

\[
Q_V(A) = \exp \left\{ \sum_{l \subset A, l \cap V \neq \emptyset} m(\bar{l}) (-1)^{J(l)} \right\} = F_V(A) \exp \{ -2m(\mathcal{J}_{A,V}) \}.
\]

As in the case for \(F\), if \(V = \{v_1, \ldots, v_k\} \subset A\), then

\[
Q_V(A) = \prod_{j=1}^{k} g_{U_j}(v_j, v_j),
\]

where \(U_j = A \setminus \{v_1, \ldots, v_{j-1}\}\) and

\[
g_{U_j}(v_j, v_j) = \sum_{l} q(l) = \sum_{l} (-1)^{J(l)} p(l)
\]
where the sum is over all (rooted) loops \( l \) from \( v_j \) to \( v_j \) staying in \( U \). In particular, if \( \eta \in \mathcal{W}_n \), then when the algebraic computation which gives (5) is applied to \( q \), we get

\[
\sum_{\omega \in K_n, L(\omega) = \eta} q(\omega) = q(\eta) Q_\eta(A_n).
\]

This implies that

\[
\sum_{\omega \in K_n, L(\omega) = \eta} (-1)^{J(\omega) - J(\eta)} p(\omega) = p(\eta) Q_\eta(A_n).
\]

If \( V = \{0\} \) is a singleton set, then

\[
\lim_{n \to \infty} Q_V(A_n) = Q_V(\mathbb{Z}^2) = \sum_{k=0}^{\infty} s^k = (1 - s)^{-1} > 0.
\]

Here \( s = \mathbb{E}[J'] \) where \( J' = (-1)^{J(S[0,T_0])} \), \( S \) is a simple random walk starting at the origin, and \( T_0 = \min\{n \geq 1 : S_n > 0\} \). Since \( \mathbb{P}\{J' = 1\} > 0 \) and \( \mathbb{P}\{J' = -1\} > 0 \), we see that \( |s| < 1 \) and hence the limit exists and is positive. A similar argument shows that if \( \mathbb{Z}^2 \setminus U \) is finite and non-empty, and \( v \) is in the unbounded component of \( U \), then \( g_U(v,v) \) is finite and strictly positive. Given this and (6), it is straightforward to show that if \( V \) is finite, then

\[
Q_V = Q_V(\mathbb{Z}^2) = \lim_{n \to \infty} Q_V(A_n)
\]
exists and is strictly positive. We will write \( Q_{01}(A_n) \) for \( Q_{\{0,1\}}(A_n) \).

For the important computation of the random walk loop measure, we will use the Brownian loop measure as introduced in [9]. There are several equivalent definitions. We give one here that is convenient for computational purposes and is analogous to the second definition we gave for the random walk loop measure. We start with the Brownian (boundary) bubble measure in the upper half plane \( \mathbb{H} \) started at the origin. It is the limit as \( \epsilon \downarrow 0 \) of a measure on paths from \( i\epsilon \) to 0 in \( \mathbb{H} \) which we now describe. For each \( \epsilon \) consider the measure of total mass \( \epsilon^{-1} \) on paths whose normalized probability measure is that of a Brownian \( h \)-process to 0. (An \( h \)-process can be viewed roughly as a Brownian motion conditioned to leave \( \mathbb{H} \) at 0.) As \( \epsilon \downarrow 0 \), the limit measure is a \( \sigma \)-finite measure \( \nu_{\mathbb{H}}(0) \) on loops from 0 to 0 otherwise in \( \mathbb{H} \). The normalization is such that the measure of bubbles that hit the unit circle equals one. This definition can be extended to simply connected domains with smooth boundaries either by the analogous definition or by the following conformal covariance rule: if \( f : \mathbb{H} \to D \) is a conformal transformation, then

\[
f \circ \nu_{\mathbb{H}}(0) = |f'(0)|^2 \nu_D(f(0)).
\]

(In the definition of \( f \circ \nu_{\mathbb{H}}(0) \), we need to modify the parametrization of the curve using Brownian scaling, but the parametrization is not important in this paper.)

Given the Brownian bubble measure, the Brownian loop measure restricted to curves in the unit disk \( \mathbb{D} \) can be written as

\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \nu_{\mathbb{D}}(re^{i\theta}) r \, d\theta \, dr,
\]

(8)
To be more precise, the loop measure is the measure on unrooted loops induced by the above measure on rooted loops. (This representation of the measure on unrooted loops focuses on the rooted representative with root as far from the origin as possible.) The Brownian loop measure is the scaling limit of the random walk loop measure in a sense made precise in \([8]\). We discuss this more in Section 4.

3 A combinatorial identity

Let \(K'_n\) denote the set of nearest neighbor paths \(\omega = [\omega_0, \omega_1, \ldots, \omega_k] \) with \(\text{Re}[\omega_0] = -n, \omega_k = 0\) and \(\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus [0, \infty)\). Let \(K''_n\) denote the set of nearest neighbor paths \(\omega = [\omega_0, \omega_1, \ldots, \omega_k] \) with \(\text{Re}[\omega_0] = n + 1, \omega_k = 1\) and \(\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus (-\infty, 1]\). There is a natural bijection between \(K'_n\) and \(K''_n\) obtained by reflection about the line \(\{\text{Re}(z) = 1/2\}\). Let

\[
R_n = \sum_{\omega \in K'_n} p(\omega) = \sum_{\omega \in K''_n} p(\omega).
\]

Note that \(R_n\) equals \(\mathbb{P}\{\text{Re}(S_\tau) = -n\}\) where \(S\) is a simple random walk starting at the origin and \(\tau = \min\{j > 0 : S_j \in \partial A_n \cup [0, \infty)\}\). It is known (e.g., \([4]\) Proposition 5.3.2]) that

\[
R_n \asymp n^{-1/2}, \quad n \to \infty.
\]

The goal of this section is to prove the following combinatorial identity which relates the probability that loop-erased walk uses the undirected edge \(\{0, 1\}\) to some simple random walk quantities.

**Theorem 3.1.**

\[
4 \sum_{\eta \in \mathcal{W}^*_n} \hat{p}_n(\eta) = Q_{01}(A_n) R_n^2 \exp\{2m(J_{A_n})\}.
\]

**Proof.** We claim the following:

\[
(-1)^{J(\eta)} Y(\eta) = 1 \quad \text{if} \quad \eta \in \mathcal{W}^*_n.
\]

To see this consider the path \(\eta\) as a continuous path from \(\{\text{Re}(z) = -n\}\) to \(\{\text{Re}(z) = n + 1\}\) in the domain \(D = \{x + iy \in \mathbb{C} : -n < x < n + 1, -n < y < n\}\). Then \(\eta\) is a crosscut of \(D\) such that \(D \setminus \eta\) consists of two components, the “top” component \(D^+\) and the “bottom” component \(D^-\). Each ordered edge \([w, w']\) in \(\eta\) can be considered as subsets of \(\partial D^+\) and \(\partial D^-\). As we traverse from \(w\) to \(w'\), the left-hand side of \([w, w']\) (considered as a prime end) is in \(\partial D^+\) and the right-hand side is in \(\partial D^-\). Let \(N_+\) be the set of integers \(k\) such that the ordered edge \([ki, ki + 1]\) is contained in \(\eta\), \(N_-\) be the set of integers \(k\) such that the ordered edge \([ki + 1, ki]\) is contained in \(\eta\), and \(N = N_+ \cup N_-\). We claim that if \(j \in N_+\) and \(k\) is the largest integer less than \(j\) with \(k \in N\), then \(k \in N_-\). For otherwise, the open line segment from \(ji + (1/2)\) to \(ki + (1/2)\) would be in both \(D^+\) and \(D^-\). We now consider the smallest \(k\) such that \(k \in N\). The line segment from \(-ni + (1/2)\) to \(ki + (1/2)\) is contained in \(D^-\)
and hence \( k \in N_+ \). As we continue up the line \( \{ \text{Re}(z) = 1/2 \} \) we see that when we intersect edges in \( \eta \), they alternate being in \( N_+ \) or \( N_- \), the first in \( N_+ \), the second in \( N_- \), the third in \( N_+ \), etc. When we reach the unordered edge \( \{ 0, 1 \} \), we see that if \( 0 \in N_+ \), then there have been an even number of edges before \( \{ 0, 1 \} \) and if \( 0 \in N_- \), there have been an odd number of edges. In other words, \((-1)^{J(\eta)} = 1 \) if \( \eta \in W_n^+ \) and \((-1)^{J(\eta)} = -1 \) if \( \eta \in W_n^- \). This gives (10).

Let \( \Lambda_n \) be defined as in (3). We claim that

\[
\Lambda_n = \sum_{\omega \in K_n} q(\omega) Y(L(\omega)) = \sum_{\eta \in W_n^+} \sum_{L(\omega) = \eta} p(\omega) (-1)^{J(\omega) - J(\eta)}.
\]  

To see this, suppose that \( L(\omega) = \eta = [\eta_0, \ldots, \eta_k] \). We can write

\[
\omega = [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \oplus \cdots \oplus [\eta_{k-2}, \eta_{k-1}] \oplus l_{k-1} \oplus [\eta_{k-1}, \eta_k],
\]

where \( l_j \) is a loop rooted at \( \eta_j \) that does not enter \( \{ \eta_1, \ldots, \eta_{j-1} \} \). We write

\[
J(\omega) = J(\eta) + J_L(\omega), \quad Y(\omega) = Y(\eta) + Y_L(\omega),
\]

where \( J_L, Y_L \) denote the contributions from the loops. Then

\[
Y(\omega) = Y(\eta) + \sum_{j=1}^{k-1} Y(l_j).
\]

For each loop \( l_j \) there is the corresponding reversed loop \( l_j^R \) for which \( Y(l_j^R) = -Y(l_j) \). Since \( J(l_j^R) = J(l_j) \) and \( Y(l_j^R) = -Y(l_j) \), we get cancellation. Doing this for all the loops, we see that

\[
\sum_{\omega \in K_n, L(\omega) = \eta} q(\omega) [Y(\omega) - Y(\eta)] = 0.
\]

This gives the first equality in (11). The second equality uses (10) and the fact that \( Y(\eta) = 0 \) if \( \eta \not\in W_n^+ \).

If \( \eta \in W_n^+ \), then

\[
\sum_{L(\omega) = \eta} p(\omega) (-1)^{J(\omega) - J(\eta)} = p(\eta) \sum_{L(\omega) = \eta} \frac{p(\omega)}{p(\eta)} (-1)^{J_L(\omega)}
\]

\[
= p(\eta) Q_{\eta}(A_n)
\]

\[
= p(\eta) \exp \left\{ \sum_{\ell \subseteq A_n, \ell \cap \eta \neq \emptyset} (-1)^{J(\ell)} m(\ell) \right\}
\]

\[
= p(\eta) F_{\eta}(A_n) \exp \left\{ -2 \sum_{\ell \subseteq A_n, \ell \cap \eta \neq \emptyset, J(\ell) \text{ odd}} m(\ell) \right\}.
\]
If \( J(\bar{l}) \) is odd, then \( \bar{l} \) must include at least one unordered edge \( \{k_i, k_i + 1\} \) with \( k \geq 0 \) and at least one unordered edge \( \{k_i, k_i + 1\} \) with \( k < 0 \). Therefore, topological considerations imply that if \( \eta \in \mathcal{W}_n^* \), then \( \eta \cap \bar{l} \neq \emptyset \). Hence

\[
\sum_{\bar{l} \in A_n, \bar{l} \cap \eta \neq \emptyset, J(\bar{l}) \text{ odd}} m(\bar{l}) = \sum_{\bar{l} \in A_n, J(\bar{l}) \text{ odd}} m(\bar{l}) = m(J_{A_n}).
\]

Combining this with (11), we see that

\[
\Lambda_n = \sum_{\eta \in \mathcal{W}_n^*} p(\eta) Q_{\eta}(A_n) = e^{-2m(J_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} p(\eta) F_{\eta}(A_n) = e^{-2m(J_{A_n})} \sum_{\eta \in \mathcal{W}_n^*} \hat{p}_n(\eta). \tag{12}
\]

We will now compute \( \Lambda_n \) as defined in (3) a different way. Let \( \omega = [\omega_0, \ldots, \omega_r] \in \mathcal{K}_n \). If \( \omega \) does not visit 0 or 1, then \( Y(\omega) = 0 \). Hence, we need only consider the sum over \( \omega \in \mathcal{K}_n \) that visit both 0 and 1.

Suppose that \( T_0 = \min\{j : \omega_j = 0\} \) and define \( T_0' = \max\{j < \tau : \omega_j = 0\} \) and define \( T_1, T_1' \) similarly.

Suppose that \( T_0 < T_1, T_0' > T_1' \). In this case we write

\[
\omega = \omega^- \oplus l \oplus \omega^+ \tag{13},
\]

where \( l \) is the loop \([\omega_{T_0}, \ldots, \omega_{T_0'}] \). Note that \( Y(\omega) = Y(l) \). For any such loop, there is the corresponding reversed loop \( l^R = [\omega_{T_0'}, \omega_{T_0-1}, \ldots, \omega_{T_0}] \) for which \( Y(l^R) = -Y(l) \). These terms cancel and hence the sum in (3) over \( \omega \) with \( T_0 < T_1, T_0' > T_1' \) is zero. Similarly, the sum over \( \omega \) with \( T_1 < T_0 \leq T_0' < T_1' \) is zero.

Suppose that \( T_0 > T_1, T_0' > T_1' \). Then we can write \( \omega \) as

\[
\omega = \omega^- \oplus l_1 \oplus \omega' \oplus l_0 \oplus \omega^+ \tag{14}.
\]

Here \( l_0 \) is a loop rooted at 0, \( l_1 \) is a loop rooted at 1, \( \omega' \) is a path from 1 to 0, \( \omega^- \) is a path from \( \{\text{Re}(z) = -n\} \) to 1 avoiding 0, \( \omega^+ \) is a path from 0 to \( \{\text{Re}(z) = n + 1\} \) avoiding 1. Let \( \tilde{\omega}^- \) be the reflection of \( \omega^- \) about the real axis. Then \( J(\omega^-) + J(\tilde{\omega}^-) \) is odd and these terms will cancel. Hence the sum over all \( \omega \) with \( T_0 > T_1, T_0' > T_1' \) is zero.

Let \( \mathcal{K}_n^1 \) be the set of paths in \( \mathcal{K}_n \) that visit both 0 and 1 and satisfy \( T_0 < T_1, T_0' < T_1' \). We have shown that

\[
\Lambda_n = \sum_{\omega \in \mathcal{K}_n^1} q(\omega) Y(\omega).
\]

If \( \omega \in \mathcal{K}_n^1 \), let \( \rho = \min\{j > T_0' : \omega_j = 1\} \). Then we can write \( \omega \) as

\[
\omega = \omega^- \oplus l_0 \oplus \omega' \oplus l_1 \oplus \omega^+. \tag{14}
\]

Here \( l_0 \) is a loop rooted at 0, \( l_1 \) is a loop rooted at 1, \( \omega' = [\omega_{T_0'}, \ldots, \omega_{\rho}] \) is a path from 0 to 1, \( \omega^- \) is a path from \( \{\text{Re}(z) = -n\} \) to 0, \( \omega^+ \) is a path from 1 to \( \{\text{Re}(z) = n + 1\} \). The paths \( \omega', \omega^-, \omega^+ \) do not enter \( \{0, 1\} \) except at their endpoints. The loop \( l_1 \) does not visit 0. All points other than the endpoints must lie in \( A_n \). These are all the restrictions on the paths.
Note that $Y(\omega) = Y(l_0) + Y(\omega')$. As in the previous arguments, we can replace $l_0$ with the reversed loop $l_0^R$, to see that

$$
\sum_{\omega \in \mathcal{K}_n^1} (-1)^{J(\omega)} Y(l_0) p(\omega) = 0.
$$

Also $Y(\omega') \in \{0, 1\}$ with $Y(\omega') = 1$ if and only if $T_0' + 1 = \rho$, that is, if $\omega' = [0, 1]$. Therefore, if $\mathcal{K}_n^2$ denotes the set of paths in $\mathcal{K}_n^1$ with $\omega' = [0, 1]$, then

$$
\Lambda_n = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega)} p(\omega) = \sum_{\omega \in \mathcal{K}_n^2} (-1)^{J(\omega^-) + J(l_0) + J(l_1) + J(\omega^+)} p(\omega). \quad (15)
$$

If $\omega \in \mathcal{K}_n^2$, let $\xi$ be the smallest $j$ such that $\omega_j$ is on the positive real axis. Suppose for the moment that $\xi < T_0$. Then we can write $\omega^--\omega^- = \omega^- \oplus \omega^-$, by splitting the path at time $\xi$. The path $\omega^- \oplus \omega^-$ is a path from the positive real axis to 0 that does not go through the point 1. Hence, $J(\omega^-) + J(\omega^-)$ is odd, where $\omega^- \oplus \omega^-$ is the reflection of $\omega^- \oplus \omega^-$ about the real axis. These terms will cancel in the sum (15), and hence it suffices to sum over $\omega^-$ such that $\omega^- \cap [1, \infty) = \emptyset$. For these $\omega^-$, we can see by topological reasons that $(-1)^{J(\omega^-)} = 1$. By a similar argument, it suffices to sum over $\omega^+$ satisfying $\omega^+ \cap (-\infty, 0] = \emptyset$, and for these walks $(-1)^{J(\omega^+)} = 1$. Therefore, if $\mathcal{K}_n^3$ denote the set of paths in $\mathcal{K}_n^2$ satisfying

$$
\omega^- \cap [1, \infty) = \emptyset, \quad \omega^+ \cap (-\infty, 0] = \emptyset,
$$

we see that

$$
\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} p(\omega).
$$

Let us write any $\omega \in \mathcal{K}_n^3$ as in (14). We must choose $\omega^- \in \mathcal{K}_n', (\omega^+)^R \in \mathcal{K}_n''$ and $\omega' = [0, 1]$. Summing over all of these possibilities, gives a factor of $R_n^2/4$. The choices of $l_0, l_1$ are independent of the choices of $\omega^-$ and $\omega^+$. The only restriction is that the loops lie in $A_n$ and $l_1$ does not contain the origin. By our definition,

$$
\sum_{l_0, l_1} (-1)^{J(l_0) + J(l_1)} p(l_0) p(l_1) = g_{A_n}(0, 0) g_{A_n \setminus \{0\}}(1, 1) = Q_{01}(A_n).
$$

Therefore,

$$
\Lambda_n = \sum_{\omega \in \mathcal{K}_n^3} (-1)^{J(l_0) + J(l_1)} p(\omega) = \frac{1}{4} R_n^2 Q_{01}(A_n).
$$

Comparing this with (12) gives the theorem.
4 Estimate on the random walk loop measure

Using Theorem 3.1 and the estimates (7) and (9), we see that
\[ \sum_{\eta \in W_n^*} \hat{p}_n(\eta) \asymp n^{-1} \exp\{2m(J_{A_n})\}. \]

The proof of (1) is finished with the following proposition.

**Proposition 4.1.** There exists \( c < \infty \) such that for all \( n \),
\[ \left| m(J_{A_n}) - \frac{1}{8} \log n \right| \leq c. \]

**Proof.** Let \( C_n = \{ z \in \mathbb{Z}^2 : |z| < e^n \} \). We will prove the stronger fact that the limit
\[ \lim_{n \to \infty} \left[ m(J_{C_n}) - \frac{n}{8} \right] \]
exists by showing that
\[ \sum_{n=1}^{\infty} \left| m(J_{C_{n+1}} \setminus J_{C_n}) - \frac{1}{8} \right| < \infty. \]  (17)

Let \( \mu \) denote the Brownian loop measure, and let \( \tilde{J} \) denote the set of unrooted Brownian loops \( \gamma \) in the unit disk that intersect \( \{ |z| \geq e^{-1} \} \) and such that the winding number of \( \gamma \) about the origin is odd. We will establish (17) by showing that \( \mu(\tilde{J}) = 1/8 \) and
\[ \left| m(J_{C_{n+1}} \setminus J_{C_n}) - \mu(\tilde{J}) \right| = O(n^{-2}). \]  (18)

For the Brownian loop measure, we do a computation similar to that in [3, Proposition 3.9]. Using (8), we write
\[ \mu(\tilde{J}) = \frac{1}{\pi} \int_{e^{-1}}^{1} \int_0^{2\pi} \phi(r, \theta) d\theta \, r \, dr, \]
where \( \phi(r, \theta) \) denotes the Brownian bubble measure of loops in \( r\mathbb{D} \) rooted at \( re^{i\theta} \) with odd winding number about the origin. Rotational symmetry implies that \( \phi(r, \theta) = \phi(r, 0) \) and conformal covariance implies that \( \phi(r, 0) = r^{-2} \phi \) where \( \phi = \phi(1, 0) \). Hence,
\[ \mu(\tilde{J}) = \frac{\phi}{\pi} \int_{e^{-1}}^{1} \int_0^{2\pi} r^{-2} \, d\theta \, r \, dr = 2\phi. \]  (19)

By considering the (multi-valued) covering map \( f(z) = i \log z \) which satisfies \( |f'(1)| = 1 \), we see that
\[ \phi = \sum_{k \text{ odd}} H_{\partial \mathbb{D}}(0, 2\pi k), \]
where \( H_{\partial \mathbb{H}} \) denotes the boundary Poisson kernel in the upper half-plane \( \mathbb{H} \) normalized so that \( H_{\partial \mathbb{H}}(0, x) = x^{-2} \). Therefore,

\[
2\phi = 2 \sum_{k=-\infty}^{\infty} \frac{1}{[2\pi(2k+1)]^2} = \frac{1}{\pi^2} \left[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right] = \frac{1}{8}.
\]

If \( s > 2 \), let \( \tilde{\mathcal{J}}_s^* \) denote the set of Brownian loops in \( \mathbb{D} \) that intersect both \( \{ |z| \geq e^{-1} \} \) and \( \{ |z| \leq e^{-s} \} \). We claim that as \( s \to \infty \),

\[
\mu(\tilde{\mathcal{J}}^*_s) = s^{-1} + O(s^{-2}),
\]

(20)

\[
\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}^*_s] = (2s)^{-1} + O(s^{-2}).
\]

(21)

To see this, we first consider the boundary bubble measure \( \lambda_s \) of loops in \( \mathbb{D} \) rooted at 1 that enter \( \{ |z| \leq e^{-s} \} \). An exact expression is given as follows. Let \( B_t \) be a Brownian motion and \( \sigma_s = \inf \{ t : |B_t| = e^{-s} \} \). Then,

\[
\lambda_s = \lim_{\epsilon \downarrow 0} \mathbb{E}^\epsilon [H_{\mathbb{D}}(B_{\sigma_s}, 1); \sigma_s < \sigma_0].
\]

The Poisson kernel in the disk is well known; for our purpose we need only know that

\[
H_{\mathbb{D}}(z, 1) = \frac{1}{2} + O(|z|),
\]

and a standard estimate for Brownian motion gives

\[
\mathbb{P}^{1-\epsilon} \{ \sigma_s < \sigma_0 \} = \frac{\log(1-\epsilon)}{-s} \sim \frac{\epsilon}{s}.
\]

Therefore, \( \lambda_s = (2s)^{-1} + O(e^{-s}) \). Using rotational invariance, and conformal covariance, if \( r \geq e^{-1} \) and \( \lambda(r, \theta, s) \) denotes the bubble measure of bubbles in \( r \mathbb{D} \) rooted at \( re^{i\theta} \) that enter \( \{ |z| \leq e^{-s} \} \), then

\[
\lambda(r, \theta, s) = r^{-2} (2s)^{-1} [1 + O(s^{-1})].
\]

If we compute as in (19), we get (20). The relation (21) is done similarly except that we have to worry about the winding number of the loop. This gives a factor of \( 1/2 \). Note that we have

\[
\mu[\tilde{\mathcal{J}} \cap \tilde{\mathcal{J}}^*_s] = \frac{1}{2} \mu[\tilde{\mathcal{J}}^*_s] [1 + O(s^{-1})].
\]

For each unrooted random walk loop \( \bar{l} \in \mathcal{J}_{\mathbb{C}} \setminus \mathcal{J}_{\mathbb{C}_{\alpha-1}} \), there is a corresponding continuous unrooted loop \( \bar{l}^{(n)} \) in \( \mathbb{D} \) obtained from linear interpolation and Brownian scaling. We will write \( d(\bar{l}, \gamma) \leq \delta \), if we can parametrize and root the loops \( \bar{l}^{(n)} \) and \( \gamma \) such that the loops are within \( \delta \) in the supremum norm. In [3] it was shown that there exists \( \alpha > 0 \) and a coupling of the random walk and Brownian loop measures in \( D \), restricted to loops of diameter at least \( 1/e \), so that the total masses agree up to \( O(e^{-\alpha n}) \) and such that in the coupling, except for a set of paths of size \( O(e^{-\alpha n}) \), we have \( d(\bar{l}, \gamma) < e^{-\alpha n} \). We would like to say that in the
coupling, the Brownian loop has odd winding number if and only \(J(\bar{l})\) is odd. If the loops stay away from the origin, this holds. However, if the loops are near the origin, it is possible for the winding numbers of the continuous and the discrete walks to be different. However, and this is why we can prove what we need, it is also true that if a macroscopic loop (either continuous or discrete) gets close to the origin, then it is just about equally likely to have an odd as an even winding number. Let us be more precise.

Let \(\beta < \alpha\) and let \(\mathcal{J}^n\) denote the set of loops in \(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}\) that intersect \(|z| \leq e^{-\beta n} e^{n+1}\). Using the coupling and (21), we see that

\[
m(\mathcal{J}^n) = \mu(\mathcal{J}_{\beta n}) + O(n^{-2}) = (\beta n)^{-1} + O(n^{-2}).
\]

Let us split these paths into two sets: those for which \(\text{dist}(0, \gamma) \leq 2e^{-n\alpha}\) and those for which \(\text{dist}(0, \gamma) > 2e^{-n\alpha}\). If \(\text{dist}(0, \gamma) > 2e^{-n\alpha}\) and \(d(\bar{l}, \gamma) \leq e^{-n\alpha}\), then \(J(\bar{l})\) is odd if and only if the winding number of \(\gamma\) is odd. Therefore

\[
m((\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) \setminus \mathcal{J}^n) = \mu(\mathcal{J} \setminus \mathcal{J}_{\beta n}) + O(n^{-2}).
\]

(The error term \(O(n^{-2})\) is comparable to the measure of loops \(\gamma\) such that \(e^{-n\beta} \leq \text{dist}(0, \gamma) \leq 2e^{-n\beta}\).)

A coupling argument can be used to give a random walk analogue of (22),

\[
m(\mathcal{J}^n \cap (\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n})) = \frac{1}{2} m(\mathcal{J}^n)[1 + o(n^{-1})].
\]

We sketch the proof. We use the definition of the loop measure using an enumeration of \(\mathbb{Z}^2 = \{z_1, z_2, \ldots\}\) such that \(|z_j|\) increases. Then an unrooted loop in \(\mathcal{J}^n\) is obtained from a loop rooted in \(C_{n+1} \setminus C_n\). Let us call the root \(z_k\) and so the loops lies in \(V_k = \{z_1, \ldots, z_k\}\). Let us stop the walk at the first time it reaches a point, say \(z'\), in \(|z| \leq e^{-\beta n} e^{n+1}\). The remainder of the loop acts like a random walk started at \(z'\) conditioned to reach \(z_k\) without before leaving \(V_k\). Let \(J'\) denote the number of times such a walk crosses the half line \((1/2) + iy : y < 0\). We claim that the probability that \(J'\) is odd equals \(\frac{1}{2} + O(e^{-un})\) for some \(u > 0\). Indeed, we can couple two walks starting at the point so that each walk has the distribution of random walk conditioned to reach \(z_k\) before leaving \(V_k\) and that, except for an event of probability \(O(e^{-\delta})\), the parity of \(J'\) is different for the two walks. This uses a standard technique. The key estimate is the following. There exists \(c > 0\) such that if \(S\) is a simple random walk starting at \(z \in C_{j-1}\) and \(T = \min\{j : S_y \in C_j\}\), then for all \(w \in \partial C_j\) with \(\text{Im}(w) > 0\),

\[
\mathbb{P}\{S(T) = w, J' \text{ odd}\} \geq c e^{-j},
\]

\[
\mathbb{P}\{S(T) = w, J' \text{ even}\} \geq c e^{-j}.
\]

Without the restriction of the parity of \(J'\), see, for example, [4, Lemma 6.3.7]. To get the result with the restriction, we just note that there is a positive probability of making a loop in the annulus \(C_j \setminus C_{j-1}\), and this increases \(J'\) by one. Hence, we can find a coupling and a \(\rho > 0\) such that at each annulus there is a probability \(\rho\) of a successful coupling given that
the walks have not yet been coupled. Since there are of order $\beta n$ annuli, we can couple the processes so that the probability of not being coupled is $(1 - \rho)^{\beta n} = O(e^{-un})$ for some $u$.

From the last two estimates and (21), we see that

$$\left| \mu(\mathcal{J}) - m(\mathcal{J}_{C_{n+1}} \setminus \mathcal{J}_{C_n}) \right| \leq cn^{-2}.$$ 

This gives (18).

5 Different boundary conditions

In this section, we generalize Theorem 3.1 to more general boundary conditions. Let $\bar{\zeta} = (\zeta_1, \zeta_2)$ be an ordered pair of distinct points in $\partial A_n$, and let $K_n(\bar{\zeta})$ to be the set of nearest neighbor paths $\omega = [\omega_0, \ldots, \omega_k]$ with $\omega_0 = \zeta_1, \omega_k \in \zeta_2$ and $\{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus \{0, 1\}$. Let $W^+_{n}(\bar{\zeta}), W^-_{n}(\bar{\zeta}), W_{n}^*(\bar{\zeta})$ be defined similarly with the new boundary condition. As before, if $\eta \in W_{n}^*(\bar{\zeta})$, then

$$\hat{p}_n(\eta) = \sum_{\omega \in K_n(\bar{\zeta}), L(\omega) = \eta} \hat{p}(\omega).$$

Define $R_n(\zeta_j)$ as follows. Let $S_n$ be a simple random walk starting at the origin and let $T_n = T_n = \min\{j \geq 1 : S_j \in [0, \infty) \cup \partial A_n\}$.

Then

$$R_n(\zeta_j) = I(\zeta_j) \mathbb{P}\{S_T = \zeta_j\},$$

where $I(\zeta_j) = -1$ if Re$(\zeta_j) > 0, \text{Im}(\zeta_j) < 0$ and $I(\zeta_j) = 1$ otherwise. Note that if $\omega \in K_n(\zeta_1)$ with $\omega \cap [1, \infty) = \emptyset$, then $I(\zeta_1) = (-1)^{|J(\zeta_1)|}$. Define

$$\Phi_n(\bar{\zeta}) = |R_n(\zeta_1) R_n(1-\zeta_2) - R_n(1-\zeta_1) R_n(\zeta_2)|.$$

The combinatorial identity takes the following form.

**Theorem 5.1.** If $\bar{\zeta} = (\zeta_1, \zeta_2) \in \partial A_n \times \partial A_n$, then

$$\sum_{\eta \in W_{n}^*(\bar{\zeta})} \hat{p}(\eta) = \frac{1}{4} Q_{01}(A_n) \exp\{2m(J_{A_n})\} \Phi_n(\bar{\zeta}).$$

(23)

Note that the factor $Q_{01}(A_n) \exp\{2m(J_{A_n})\}/4$ does not depend on $\bar{\zeta}$. Hence the theorem implies that if $\bar{\zeta}, \bar{\zeta}' \in \partial A_n \times \partial A_n$,

$$\frac{\sum_{\eta \in W_{n}^*(\bar{\zeta})} \hat{p}(\eta)}{\sum_{\eta \in W_{n}^*(\bar{\zeta}')} \hat{p}(\eta)} = \frac{\Phi_n(\bar{\zeta})}{\Phi_n(\bar{\zeta}')}.$$
Proof. Let $J(\omega), Y(\omega)$ be as in the introduction. Both sides of (23) vanish if $\zeta_1 = \zeta_2$ so we assume that $\zeta_1 \neq \zeta_2$. Both sides of (23) are invariant under the transformation $(\zeta_1, \zeta_2) \mapsto (\zeta_2, \zeta_1)$. Let

$$D_n = \{x + iy \in \mathbb{C} : -n < x < n + 1, -n < y < n\}. \quad (24)$$

The boundary $\partial D_n$ is a rectangle. Without loss of generality we will assume that the order $(\zeta_1, \zeta_2)$ is chosen so that the point $\frac{1}{2} - ni$ is on the arc of $\partial D_n$ from $\zeta_1$ to $\zeta_2$ in the counterclockwise direction. If $\eta \in W_n^*(\zeta)$, considered as a simple curve by linear interpolation, then $D_n \setminus \eta$ consists of two components, $D^+, D^-$ where $D^-$ denotes the component containing $\frac{1}{2} - ni$ on its boundary. By our choice of order of $\zeta_1, \zeta_2$, as we traverse $\eta$ from $\zeta_1$ to $\zeta_2$, the right-hand side of $\eta$ is on $\partial D^-$ and the left-hand side is $\partial D^+$. Using this, we can use the topological argument as in (10) to conclude that

$$(-1)^{J(\eta)} Y(\eta) = 1 \quad \text{if} \quad \eta \in W_n^*(\bar{\zeta}). \quad (25)$$

Let

$$\Lambda = \Lambda_n(\bar{\zeta}) = \sum_{\omega \in \mathcal{K}_n(\bar{\zeta})} (-1)^{J(\omega)} p(\omega) Y(\omega). \quad (26)$$

We claim that

$$\Lambda = \exp \{-2m(\mathcal{J}_{A_n})\} \sum_{\eta \in W_n^*(\bar{\zeta})} \hat{p}_n(\eta).$$

Indeed, given (25), the proof is identical to that of (12). Hence to prove the theorem, it suffices to prove that

$$\Lambda = \frac{Q_{01}(A_n)}{4} \Phi_n(\bar{\zeta}). \quad (27)$$

As in the proof of Theorem 3.1 we note that if $\omega$ does not contain the unordered edge $\{0, 1\}$, then $Y(\omega) = 0$. Hence the sum in (26) is over $\omega$ such that $T_0, T_0', T_1, T_1'$, as defined in the proof of Theorem 3.1, are well defined. As in that proof, if $T_0 < T_1$ and $T_0' > T_1$, we can write $\omega$ as in (13). By considering the path with the reversed loop, we see that the sum over $\omega$ with $T_0 < T_1$ and $T_0' > T_1$ equals zero. Similarly, the sum over $\omega$ with $T_0 > T_1$, $T_0' < T_1'$ is zero. Hence we need only consider the sum over paths with $T_0 < T_1$, $T_0' < T_1'$ or $T_0 > T_1$, $T_0' > T_1'$. However, unlike in the proof of Theorem 3.1 the sum over $\omega$ with $T_0 > T_1, T_0' > T_1'$ does not vanish.

Let $\mathcal{K}^5 = \mathcal{K}_n^5(\bar{\zeta})$ be the set of $\omega \in \mathcal{K}_n(\bar{\zeta})$ such that $T_0 < T_1$ and $S_{T_0 + 1} = 1$. Let $\mathcal{K}^6 = \mathcal{K}_n^6(\bar{\zeta})$ be the set of $\omega \in \mathcal{K}_n(\bar{\zeta})$ such that $T_1 < T_0$ and $S_{T_0 - 1} = 1$. Then the argument leading to (15) in this case gives

$$\Lambda = \sum_{\omega \in \mathcal{K}^5} (-1)^{J(\omega)} p(\omega) - \sum_{\omega \in \mathcal{K}^6} (-1)^{J(\omega)} p(\omega).$$

If $\omega = [\omega_0, \ldots, \omega_r] \in \mathcal{K}^5$, we write

$$\omega = \omega_- \oplus l_0 \oplus [0, 1] \oplus l_1 \oplus \omega_+$$
where
\[ \omega_- = [\omega_0, \ldots, \omega_T_0], \quad l_0 = [\omega_{T_0}, \ldots, \omega_{T_0}] , \]
\[ l_1 = [\omega_{T_0+1}, \ldots, \omega_{T_1}], \quad \omega_+ = [\omega_{T_1}, \ldots, \omega_T] . \]

Note that \( \omega_- \) is a path from \( \zeta_1 \) to 0; \( l_0 \) is a loop rooted at 0; \( l_1 \) is a loop rooted at 1 that avoids the origin; and \( \omega_+ \) is a path from 1 to \( \zeta_2 \). The points in \( l_0, l_1 \) must lie in \( A_n \). The points in \( \omega_-, \omega_+ \) other than the endpoints must lie in \( A_n \setminus \{0, 1\} \). Let \( K^6 = K^6_n(\zeta_1) \) be the set of nearest neighbor paths \( \omega = [\omega_0, \ldots, \omega_k] \) with \( \omega_0 = \zeta_1, \omega_k = 0, \{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus \{0, 1\} \). Let \( K^7 = K^7_n(\zeta_2) \) be the set of nearest neighbor paths \( \omega = [\omega_0, \ldots, \omega_k] \) with \( \omega_0 = 1, \omega_k = \zeta_2, \{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \setminus \{0, 1\} \). Then
\[
\sum_{\omega \in K^6} (-1)^{J(\omega)} p(\omega) = \frac{Q_{0,1}(A_n)}{4} \left[ \sum_{\omega \in K^6} (-1)^{J(\omega)} p(\omega) \right] \left[ \sum_{\omega \in K^7} (-1)^{J(\omega)} p(\omega) \right] .
\]

If \( \omega = [\omega_0, \ldots, \omega_k] \in K^6 \), let \( j \) be the first index (if it exists) such that \( \omega_j \in [1, \infty) \). If \( j \) exists, we can write
\[
\omega = [\omega_0, \ldots, \omega_j] \oplus [\omega_j, \ldots, \omega_k] .
\]
The path \( \omega' = [\omega_j, \ldots, \omega_k] \) is a path from the positive real line to the origin. If \( \tilde{\omega}' \) denotes the reflection of \( \omega \) about the real axis, then \( J(\omega') + J(\tilde{\omega}') \) is odd. Hence, these terms will cancel in the sum. Therefore,
\[
\sum_{\omega \in K^6} (-1)^{J(\omega)} p(\omega) = \sum_{\omega \in K^8} (-1)^{J(\omega)} p(\omega),
\]
where \( K^8 \) denotes the set of paths in \( K^6 \) that do not intersect \([1, \infty) \). If \( \omega \in K^8 \), then \( \omega \) cannot hit the positive real line, so we can see that \( J(\omega) \) is even unless \( \omega \) lies in the quadrant \( \{\text{Re}(\zeta_1) > 0, \text{Im}(\zeta_1) < 0\} \) in which case \( J(\omega) \) is odd. Therefore,
\[
\sum_{\omega \in K^8} (-1)^{J(\omega)} p(\omega) = R_n(\zeta_1) .
\]

Using symmetry about the line \( \{\text{Re}(z) = 1/2\} \), we can see that
\[
\sum_{\omega \in K^7} (-1)^{J(\omega)} p(\omega) = R_n(1 - \zeta_2) .
\]

Similarly, we see that
\[
\sum_{\omega \in K^6} (-1)^{J(\omega)} p(\omega) = \frac{Q_{01}(A_n)}{4} R_n(\zeta_2) R_n(1 - \zeta_1) .
\]
6 Scaling limit

The scaling limit of the loop-erased walk is known in some sense to be the Schramm-Loewner evolution with parameter 2 (SLE$_2$), see [6]. Here we would like to give a stronger conjecture about this convergence that has not been proved and then show how the estimates here could be used to prove one case of this conjecture. Although we could probably given the details of the proof, we choose not to bother since it is only a start to the main result we hope to obtain later.

We start by reviewing SLE$_\kappa$ in the simple curve case. We take a “partition function” view with curves parametrized by natural parametrization as outlined in [3]. Suppose $0 < \kappa \leq 4$ and let $a = 2/\kappa \in [1/2, \infty)$. We will consider bounded simply connected domains $D \subset \mathbb{C}$ containing the origin with piecewise analytic boundaries. We will consider what is sometimes called two-sided radial SLE$_\kappa$, or, more convenient for use, as chordal SLE$_\kappa$ from $z$ to $w$ conditioned to go through the origin. This is a finite measure on simple curves $\gamma: [0, T_\gamma] \rightarrow D$ that go through the origin with $\gamma(0) = z, \gamma(T_\gamma) = w, \gamma(0, T_z) \subset D$. We denote this measure by $\mu_D(0; z, w)$ and write it as

$$
\mu_D(0; z, w) = \Psi_D(0; z, w) \mu^\#_D(0, z, w)
$$

where $\Psi_D(0; z, w)$ denotes the total mass of the measure and $\mu^\#_D(0, z, w)$ is a probability measure. The measure is supported on curves of Hausdorff dimension $d = d_\kappa = 1 + \frac{8}{\kappa}$. The family of measures satisfies a conformal covariance property that we now describe. Suppose $F: D \rightarrow F(D)$ is a conformal transformation with $F(0) = 0$ and such that $\partial F(D)$ is locally analytic near $F(z), F(w)$. If $\gamma$ is a curve, define the curve $F \circ \gamma$ to be the image of the curve reparametrized so that the time to traverse $F(\gamma[r, s])$ is

$$
\int_r^s |F'(t)|^d dt.
$$

If $\nu$ is a measure on curves on $D$, we define $F \circ \nu$ to be the measure on curves on $F(D)$,

$$
F \circ \nu(V) = \nu\{\gamma: F \circ \gamma \in V\}.
$$

Then the conformal covariance rule is

$$
F \circ \mu_D(0; z, w) = |F'(z)|^b |F'(w)|^b |F'(0)|^{2-d} \mu_{F(D)}(F(0); F(z), F(w)),
$$

where $b = 3a - 1 = \frac{6 - \kappa}{2\kappa}$. This rule can be considered as a combination of two rules,

$$
\Psi_D(0; z, w) = |F'(z)|^b |F'(w)|^b |F'(0)|^{2-d} \Psi_{F(D)}(0; F(z), F(w)),
$$

$$
F \circ \mu^\#_D(0; z, w) = \mu^\#_{F(D)}(F(0); F(z), F(w)).
$$

For each $\kappa$ there is a unique such family of measures up to two arbitrary multiplicative constants. (If $\mu_D$ satisfies the conditions above, then so does $c \mu_D$, and also the measure
obtained by changing the unit of time, that is, replacing \( \gamma(s), 0 \leq s \leq t \) with \( \tilde{\gamma}(s) = \gamma(sr), 0 \leq s \leq t/r \). For simply connected \( D \), the total mass is given (up to multiplicative constant) by

\[
\Psi_D(0; z, w) = H_{\partial D}(z, w)^b G_D(0; z, w)
\]

where \( G_D(0; z, w) \) denotes the \( SLE_\kappa \) Green's function giving the normalized probability that a chordal \( SLE_\kappa \) path from \( z \) to \( w \) goes through the origin. Up to multiplicative constant,

\[
G_D(0; 1, e^{2i\theta}) = |\sin \theta|^{4a-1}.
\]

and for other domains can be determined by

\[
G_D(0; z, w) = |F'(0)|^{2-d} G_F(D)(F(z), F(w)).
\]

Loop-erased walk corresponds to \( \kappa = 2 \) \((a = 1, b = 1, d = 5/4)\), and for the remainder of this section we fix \( \kappa = 2 \). If \( U, V \) are closed subarcs of \( \partial D \), we define

\[
\Psi_D(0; U, V) = \int_U \int_V \Psi_D(0; z, w) |dz| |dw|,
\]

\[
\mu_D(U, V) = \int_U \int_V \mu_D(0; z, w) |dz| |dw|.
\]

Since \( b = 1 \), the scaling rule \((28)\) can be used to see that

\[
\Psi_D(0; U, V) = |F'(0)|^{3/4} \Psi_F(D)(0; F(U), F(V)).
\]

Given this, we can extend the definition to domains with rough boundaries.

We will now state a conjecture about the convergence of loop-erased walk to \( SLE_2 \). We will state it for simply connected domains, but we expect it to be true for finitely connected domains as well.

- Suppose \( D \) is a bounded, simply connected domain containing the origin whose boundary \( \partial D \) is a Jordan curve.
- Let \( U, V \) be disjoint, closed subarcs of \( \partial D \).
- Let \( S = \{x + iy \in \mathbb{C} : |x|, |y| \leq 1/2 \} \) and if \( z \in \mathbb{Z} \times i\mathbb{Z} \), let \( S_z = z + S \).
- Let \( A_n = A_n(D) \) be the connected component containing the origin of the set of \( z \in \mathbb{Z} \times i\mathbb{Z} \) such that \( S_z \subset nD \).
- Let \( U_n \) be the set of \( z \in \partial A_n \) such that \( S_z \cap nU \neq \emptyset \). Define \( V_n \) similarly.
- Let \( \mathcal{K}_n = \mathcal{K}_n(D, U, V) \) denote the set of nearest neighbor paths \( \omega = [\omega_0, \ldots, \omega_k] \) with \( \omega_0 \in U_n, \omega_k \in V_n \) and \( \{\omega_1, \ldots, \omega_{k-1}\} \subset A_n \). Let \( \mathcal{W}_n^* = \mathcal{W}_n^*(D, U, V) \) denote the set of self-avoiding paths in \( \mathcal{K}_n \) that contain the unordered edge \( \{0, 1\} \). As before, let

\[
\hat{p}_n(\eta) = \sum_{\omega \in \mathcal{K}_n, L(\omega) = \eta} p(\omega).
\]
• If \( \eta = [\eta_0, \ldots, \eta_k] \in \mathcal{W}_n^* \), define the scaled path \( \eta^n(t) \) by

\[
\eta^{(n)}(jn^{-5/4}) = \eta_j/n, \quad j = 0, 1, \ldots, n
\]

and linear interpolation in between. Let \( \mu^{(n)} = \mu_D^{(n)}(0; U, V) \) denote the measure of paths that gives measure \( \hat{p}_n(\eta) \) to \( \eta^{(n)} \) for each \( \eta \in \mathcal{W}_n^*(D, U, V) \).

**Conjecture 6.1.** We can choose the two arbitrary constants in the definition of the measure \( \mu_D \) such for every \( D, U, V \) as above,

\[
\lim_{n \to \infty} n^{3/4} \mu^{(n)} = \mu_D(0; U, V).
\]

In particular,

\[
\lim_{n \to \infty} \sum_{\eta \in \mathcal{W}_n} \hat{p}_n(\eta) = \Psi_D(0; U, V). \tag{29}
\]

We will show how Theorem 5.1 can be interpreted as one case of the conjecture (29). Let \( \mathbb{D} \) denote the unit disk in \( \mathbb{C} \) and \( D \) the square

\[
D = \{x + iy : |x|, |y| < 1\}.
\]

If \( z \in \partial D \), let \( z_n \) be the corresponding point in \( \partial A_n \). (Approximately \( z_n = nz \), but we need to round to the nearest integer and compensate for the fact that \( A_n \) is not exactly a square centered at the origin. These are very small errors.) Then, we would expect from (23) that

\[
n \Phi_n(z_n, w_n) \sim c \Psi_D(0; z, w) \sim c' H_{\partial D}(z, w) G_D(0; z, w). \tag{30}
\]

We will show why this holds. We will not calculate the right-hand side explicitly (we could by computing the map \( F \) below, but it will not be necessary). However, we do know that

\[
H_{\partial D}(e^{2i\theta_1}, e^{2i\theta_2}) G_D(0; e^{2i\theta_1}, e^{2i\theta_2}) \sim c |\sin(\theta_1 - \theta_2)|^{-2} |\sin(\theta_1 - \theta_2)|^3 = c |\sin(\theta_1 - \theta_2)|.
\]

There is a unique conformal transformation

\[
F : \mathbb{D} \to D,
\]

with \( F(-1) = -1, F(0) = 0, F(1) = 1 \). One way to show it exists is to define \( F : \mathbb{D} \cap \mathbb{H} \to D \cap \mathbb{H} \) to be the unique conformal transformation that fixes the boundary points \(-1, 0, 1\), and then to extend \( F \) to \( \mathbb{D} \) by Schwarz reflection. Let \( \mathbb{D}^+ = \mathbb{D} \setminus (-1, 0], \mathbb{D}^- = \mathbb{D} \setminus [0, 1), D^+ = D \setminus (-1, 0], D^- = D \setminus [0, 1) \), and note that \( F \) conformally maps \( \mathbb{D}^+ \) onto \( D^+ \) and \( \mathbb{D}^- \) onto \( D^- \). Using conformal invariance, we see that

\[
G_D(0; F(e^{2i\theta_1}), F(e^{2i\theta_2})) = |F'(0)|^{-3/4} G_D(0; e^{2i\theta_1}, e^{2i\theta_2}) = c_3 |\sin(\theta_2 - \theta_1)|^3,
\]

\[
H_{\partial D}(F(e^{2i\theta_1}), F(e^{2i\theta_2})) G_D(0; F(e^{2i\theta_1}), F(e^{2i\theta_2})) = c_4 |F'(e^{2i\theta_1})|^{-1} |F'(e^{2i\theta_2})|^{-1} |\sin(\theta_2 - \theta_1)|. \tag{31}
\]
Let \( g_+ \) denote the Poisson kernel in the slit domain \( \mathbb{D}^+ \) defined as follows. Start a Brownian motion near the origin in \( \mathbb{D}^+ \) and condition the Brownian motion to exit \( \mathbb{D}^+ \) on the unit circle \( \partial \mathbb{D} \). Then \( g_+ \) is the conditional density of the exit distribution (taken in the limit as the starting point approaches the origin). It is not difficult to compute \( g_+ \) using conformal invariance

\[
g_+ \left( e^{2i\theta} \right) = c_1 \sin \theta.
\]

(In this section, we will use \( c_1, c_2, \ldots \) for absolute constants whose values could be computed, but we will not bother to.) If \( g_- \) is the corresponding density in \( \mathbb{D}^- \),

\[
g_- \left( e^{2i\theta} \right) = c_1 \cos \theta.
\]

Let \( \hat{g}_+, \hat{g}_- \) denote the corresponding densities for the slit domains \( D^+, D^- \). Conformal covariance implies that

\[
\hat{g}_+(F(e^{i\theta})) = c_2 |F'(e^{2i\theta})|^{-1} \sin \theta, \quad \hat{g}_-(F(e^{i\theta})) = c_2 |F'(e^{2i\theta})|^{-1} \cos \theta.
\] (32)

Suppose that \( z = F(e^{2i\theta_1}), w = F(e^{2i\theta_2}) \). The relation (32) suggests (and, in fact, it can be proved) that

\[
|R_n(z_n)| \sim c_6 n^{-1/2} |F'(e^{2i\theta_1})|^{-1} \sin \theta_1, \quad |R_n(1 - z_n)| \sim c_6 n^{-1/2} |F'(e^{2i\theta_1})|^{-1} \cos \theta_1.
\]

(Here we use the fact that \( |F'(x + iy)| = |F'(-x + iy)| \).) Let us consider two cases. First, suppose that \( z, w \) are both in the upper half plane, that is, \( 0 \leq \theta_2 < \theta_1 \leq \pi/2 \). Then,

\[
n \Phi_n(z_n, w_n) = n \left[ R_n(z_n) R_n(1 - w_n) - R_n(1 - z_n) R_n(w_n) \right]
\]

\[
\sim c_6^2 |F'(e^{2i\theta_1})|^{-1} |F'(e^{2i\theta_1})|^{-1} \left[ \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2 \right]
\]

\[
= c_6^2 |F'(e^{2i\theta_1})|^{-1} |F'(e^{2i\theta_1})|^{-1} \sin(\theta_1 - \theta_2)
\]

As a second case, suppose that \( z = F(e^{2i\theta_1}), w = F(e^{-2i\theta_2}) \), where we also assume that \( \theta_2 \leq \pi/4 \) so that \( w_n \) is in the southeast quadrant. Then \( R_n(w_n) < 0, R_n(1 - w_n) > 0 \), and

\[
n \Phi_n(z_n, w_n) = n \left[ R_n(z_n) R_n(1 - w_n) - R_n(1 - z_n) R_n(w_n) \right]
\]

\[
\sim c_6^2 |F'(e^{2i\theta_1})|^{-1} |F'(e^{2i\theta_1})|^{-1} \left[ \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \right]
\]

\[
= c_6^2 |F'(e^{2i\theta_1})|^{-1} |F'(e^{2i\theta_1})|^{-1} \sin(\theta_1 + \theta_2)
\]

If we compare this with (31), we see that we get the prediction (30).

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