Uniqueness properties of functionals with Lipschitzian derivative

BIAGIO RICCERI

1. Introduction

Let $X$ be a real Hilbert space and $J$ a $C^1$ functional on $X$. For $x_0 \in X$, $r > 0$, set $S(x_0, r) = \{x \in X : \|x - x_0\| = r\}$.

Also on the basis of the beautiful theory developed and applied by Schechter and Tintarev in [2], [3], [4] and [5], it is of particular interest to know when the restriction of $J$ to $S(0, r)$ has a unique maximum.

The aim of the present paper is to offer a contribution along this direction.

We show that such a uniqueness property holds (for suitable $r$) provided that $J'$ is Lipschitzian and $J'(0) \neq 0$. At the same time, we also show that (for suitable $s$) the set $J^{-1}(s)$ has a unique element of minimal norm.

After proving the general result (Theorem 1), we present an application to a semilinear Dirichlet problem involving a Lipschitzian nonlinearity (Theorem 2).

2. The main result

With the usual convention $\inf \emptyset = +\infty$, our main result reads as follows:

THEOREM 1. - Let $X$ be a real Hilbert space and let $J : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous $C^1$ functional, with Lipschitzian derivative. Let $L$ be the Lipschitz constant of $J'$.

Then, for each $x_0 \in X$ with $J'(x_0) \neq 0$, if we set

$$\alpha_0 = \inf_{x \in M_{\frac{1}{L}}} J(x)$$

and

$$\beta_0 = \text{dist}(x_0, M_{\frac{1}{L}}),$$

where $M_{\frac{1}{L}}$ is the set of all global minima of the functional $x \to \frac{1}{2}\|x - x_0\|^2 - \frac{1}{L}J(x)$, we have $\alpha_0 > J(x_0)$, $\beta_0 > 0$, and the following properties hold:

(i) for every $r \in ]J(x_0), \alpha_0[$ there exists a unique $y_r \in J^{-1}(r)$ such that

$$\|x_0 - y_r\| = \text{dist}(x_0, J^{-1}(r)).$$

(ii) for every $r \in ]0, \beta_0[$ the restriction of the functional $J$ to the set $S(x_0, r)$ has a unique global maximum.
The main tool used to get Theorem 1 is the following particular case of Theorem 3 of [1].

**THEOREM A.** - Let \( X \) be a reflexive real Banach space, \( I \subseteq \mathbb{R} \) an interval and \( \Psi : X \times I \to \mathbb{R} \) a function such that \( \Psi(x, \cdot) \) is concave and continuous for all \( x \in X \), while \( \Psi(\cdot, \lambda) \) is sequentially weakly lower semicontinuous and coercive, with a unique local minimum for all \( \lambda \in \text{int}(I) \).

Then, one has
\[
\sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda) .
\]

We will also use the two propositions below.

**PROPOSITION 1.** - Let \( Y \) be a nonempty set, \( f, g : Y \to \mathbb{R} \) two functions, and \( a, b \) two real numbers, with \( a < b \). Let \( y_a \) be a global minimum of the function \( f - ag \) and \( y_b \) a global minimum of the function \( f - bg \).

Then, one has \( g(y_a) \leq g(y_b) \). If either \( y_a \) or \( y_b \) is strict and \( y_a \neq y_b \), then \( g(y_a) < g(y_b) \).

**PROOF.** We have
\[
f(y_a) - ag(y_a) \leq f(y_b) - ag(y_b)
\]
as well as
\[
f(y_b) - bg(y_b) \leq f(y_a) - bg(y_a) .
\]
Summing, we get
\[
-ag(y_a) - bg(y_b) \leq -ag(y_b) - bg(y_a)
\]
and so
\[
(b - a)g(y_a) \leq (b - a)g(y_b)
\]
from which the first conclusion follows. If either \( y_a \) or \( y_b \) is strict and \( y_a \neq y_b \), then one of the first two inequalities is strict and hence so is the third one. \( \triangle \)

**PROPOSITION 2.** - Let \( Y \) be a real Hilbert space and let \( \varphi : Y \to \mathbb{R} \) be a sequentially weakly upper semicontinuous \( C^1 \) functional whose derivative is a contraction.

Then, for every \( y_0 \in Y \), the functional \( y \to \frac{1}{2}\|y - y_0\|^2 - \varphi(y) \) is coercive and has a unique local minimum.

**PROOF.** Let \( \nu \) be the Lipschitz constant of \( \varphi' \). So, \( \nu < 1 \), by assumption. For each \( y \in Y \), we have
\[
\varphi(y) = \varphi(0) + \int_0^1 \langle \varphi'(ty), y \rangle dt
\]
and so
\[
|\varphi(y)| \leq |\varphi(0)| + \int_0^1 |\langle \varphi'(ty), y \rangle| dt \leq |\varphi(0)| + \|y\| \int_0^1 \|\varphi'(ty)\| dt \leq |\varphi(0)| + \|y\| \left( \frac{1}{2}\|\varphi'(ty) - \varphi'(0)\| dt + \|\varphi'(0)\| \right) \leq |\varphi(0)| + \frac{\nu}{2}\|y\|^2 + \|\varphi'(0)\|\|y\| .
\]
From this, we then get
\[ \frac{1}{2}||y - y_0||^2 - \varphi(y) \geq \frac{1 - \nu}{2}||y||^2 - (||\varphi'(0)|| + ||y_0||)||y|| + \frac{1}{2}||y_0||^2 - |\varphi(0)| \]
and hence
\[ \lim_{||y|| \to +\infty} \frac{1}{2}||y - y_0||^2 - \varphi(y) = +\infty \]
which yields our first claim. Then, the functional \( y \to \frac{1}{2}||y - y_0||^2 - \varphi(y) \) has a global minimum, since it is sequentially weakly lower semicontinuous. But the critical points of this functional are exactly the fixed points of the operator \( \varphi' + y_0 \) that is a contraction. So, the functional has a unique local minimum (that is its global minimum). \( \triangle \)

**Proof of Theorem 1.** First, note that, for each \( \gamma > L \), the operator \( \frac{1}{\gamma}J' \) is a contraction, and so, by Proposition 2, the functional \( x \to \frac{1}{2}||x - x_0||^2 - \frac{1}{\gamma}J'(x) \) has a unique global minimum, say \( x_{\perp} \). Fix \( \gamma > L \). By Proposition 1, we have \( J(x_0) \leq J(x_{\perp}) \). We claim that \( J(x_0) < J(x_{\perp}) \). Arguing by contradiction, assume that \( J(x_0) = J(x_{\perp}) \). Then, by Proposition 1 again, we would have \( x_{\perp} = x_0 \). Consequently, the derivative of the functional \( x \to \frac{1}{2}||x - x_0||^2 - \frac{1}{\gamma}J'(x) \) would vanish at \( x_0 \), that is \( -\frac{1}{\gamma}J'(x_0) = 0 \), against one of the hypotheses. Then, we have
\[ J(x_0) < J(x_{\perp}) \leq J(x) \]
for all \( x \in M_{\perp} \), and so \( J(x_0) < \alpha_0 \). Clearly, \( x_{\perp} \) is the global minimum of the functional \( x \to \frac{1}{2}||x - x_0||^2 - J(x) \), while any \( z \in M_{\perp} \) is a global minimum of the functional \( x \to \frac{L}{2}||x - x_0||^2 - J(x) \). Consequently, if we apply Proposition 1 again (with \( f(x) = -J(x) \), \( g(x) = -||x - x_0||^2 \), \( a = \frac{L}{2}, b = \frac{L}{2} \)), for any \( z \in M_{\perp} \), we get
\[ -||z - x_0||^2 \leq -||x_{\perp} - x_0||^2, \]
and so
\[ \beta_0 \geq ||x_{\perp} - x_0|| > 0. \]
Now, to prove (i), fix \( r \in ]J(x_0), \alpha_0[ \) and consider the function \( \Psi : X \times [0, \frac{1}{\gamma}] \to \mathbb{R} \) defined by
\[ \Psi(x, \lambda) = \frac{1}{2}||x - x_0||^2 + \lambda(r - J(x)) \]
for all \( (x, \lambda) \in X \times [0, \frac{1}{\gamma}] \). Taken Proposition 2 into account, it is clear that the function \( \Psi \) satisfies all the assumptions of Theorem A. Consequently, we have
\[ \sup_{\lambda \in [0, \frac{1}{\gamma}]} \inf_{x \in X} \Psi(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in [0, \frac{1}{\gamma}]} \Psi(x, \lambda). \]
The functional \( \sup_{\lambda \in [0, \frac{1}{\gamma}]} \Psi(\cdot, \lambda) \) is sequentially weakly lower semicontinuous and coercive, and so there exists \( x^* \in X \) such that
\[ \sup_{\lambda \in [0, \frac{1}{\gamma}]} \Psi(x^*, \lambda) = \inf_{x \in X} \sup_{\lambda \in [0, \frac{1}{\gamma}]} \Psi(x, \lambda). \]
Also, the function \( \inf_{x \in X} \Psi(x, \cdot) \) is upper semicontinuous, and so there exists \( \lambda^* \in [0, \frac{1}{L}] \) such that
\[
\inf_{x \in X} \Psi(x, \lambda^*) = \sup_{\lambda \in [0, \frac{1}{L}]} \inf_{x \in X} \Psi(x, \lambda) .
\]
Hence, from this it follows that
\[
\frac{1}{2} \|x^* - x_0\|^2 + \lambda^*(r - J(x^*)) = \inf_{x \in X} \frac{1}{2} \|x - x_0\|^2 + \lambda^*(r - J(x)) = \sup_{\lambda \in [0, \frac{1}{L}]} \frac{1}{2} \|x^* - x_0\|^2 + \lambda(r - J(x^*)) .
\]

We claim that \( J(x^*) = r \). Indeed, if it were \( J(x^*) < r \), then we would have \( \lambda^* = \frac{1}{L} \), and so \( x^* \in M_{\frac{1}{L}} \), against the fact that \( r < \alpha_0 \). If it were \( J(x^*) > r \), then we would have \( \lambda^* = 0 \), and so \( x^* = x_0 \), against the fact that \( J(x_0) < r \). We then have
\[
\frac{1}{2} \|x^* - x_0\|^2 = \inf_{x \in X} \frac{1}{2} \|x - x_0\|^2 + \lambda^*(r - J(x)) .
\]

This implies, on one hand, that \( \lambda^* < \frac{1}{L} \) (since \( r < \alpha_0 \)) and, on the other hand, that each global minimum (and \( x^* \) is so) of the restriction to \( J^{-1}(r) \) of the functional \( x \to \frac{1}{2} \|x - x_0\|^2 \) is a global minimum in \( X \), of the functional \( x \to \frac{1}{2} \|x - x_0\|^2 - \lambda^* J(x) \). But this functional (just because \( \lambda^* < \frac{1}{L} \)) has a unique global minimum, and so \( (i) \) follows. Let us now prove \( (ii) \). To this end, fix \( r \in [0, \beta_0] \) and consider the function \( \Phi : X \times [L, +\infty[ \to \mathbb{R} \) defined by
\[
\Phi(x, \lambda) = \frac{\lambda}{2}(\|x - x_0\|^2 - r^2) - J(x)
\]
for all \( (x, \lambda) \in X \times [L, +\infty[ \). Applying Theorem A, we get
\[
\sup_{\lambda \in [L, +\infty[} \inf_{x \in X} \Phi(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in [L, +\infty[} \Phi(x, \lambda) .
\]
Arguing as before (note, in particular, that \( \lim_{\lambda \to +\infty} \inf_{x \in X} \Phi(x, \lambda) = -\infty \)), we get \( \hat{x} \in X \) and \( \hat{\lambda} \in [L, +\infty[ \) such that
\[
\sup_{\lambda \in [L, +\infty[} \Phi(\hat{x}, \lambda) = \inf_{x \in X} \sup_{\lambda \in [L, +\infty[} \Phi(x, \lambda)
\]
and
\[
\inf_{x \in X} \Phi(x, \hat{\lambda}) = \sup_{\lambda \in [L, +\infty[} \inf_{x \in X} \Phi(x, \lambda) .
\]
So that
\[
\frac{\hat{\lambda}}{2}(\|\hat{x} - x_0\|^2 - r^2) - J(\hat{x}) = \inf_{x \in X} \frac{\lambda}{2}(\|x - x_0\|^2 - r^2) - J(x) = \sup_{\lambda \in [L, +\infty[} \frac{\lambda}{2}(\|\hat{x} - x_0\|^2 - r^2) - J(\hat{x}) .
\]
From this it follows at once that \( \|\hat{x} - x_0\|^2 \leq r^2 \). But, if it were \( \|\hat{x} - x_0\|^2 < r^2 \) we would have \( \hat{\lambda} = L \). This, in turn, would imply that \( \hat{x} \in M_{\frac{1}{L}} \), against the fact that \( r < \beta_0 \). Hence, we have \( \|\hat{x} - x_0\|^2 = r^2 \). Consequently
\[
-\frac{1}{\lambda} J(\hat{x}) = \inf_{x \in X} \frac{1}{2}(\|x - x_0\|^2 - r^2) - \frac{1}{\lambda} J(x) .
\]
This implies, on one hand, that $\hat{\lambda} > L$ (since $r < \beta_0$) and, on the other hand, that each global maximum (and $\hat{x}$ is so) of the restriction of the functional $J$ to the set $S(x_0, r)$ is a global minimum in $X$ of the functional $x \rightarrow \frac{1}{2}\|x - x_0\|^2 - \frac{1}{\lambda}J(x)$. Since $\hat{\lambda} > L$, this functional has a unique global minimum, and so (ii) follows.

REMARK 1. - It is clear from the proof that the assumption $J'(x_0) \neq 0$ has been used to prove $\alpha_0 > J(x_0)$ and $\beta_0 > 0$, while it has no role in showing (i) and (ii). However, when $J'(x_0) = 0$, it can happen that $\alpha_0 = J(x_0)$, $\beta_0 = 0$, with (i) (resp. (ii)) holding for no $r > \alpha_0$ (resp. for no $r > 0$). To see this, take, for instance, $X = \mathbb{R}$, $J(x) = \frac{1}{2}x^2$, $x_0 = 0$.

3. An application

From now on, $\Omega$ is an open, bounded and connected subset of $\mathbb{R}^n$ with sufficiently smooth boundary, and $X$ denotes the space $W^{1,2}_0(\Omega)$, with the usual norm

$$\|u\| = \left(\int_{\Omega} |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}}.$$

Moreover, $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitzian function, with Lipschitz constant $\mu$.

Let $\lambda \in \mathbb{R}$. As usual, a classical solution of the problem

$$\begin{cases}
-\Delta u = \lambda f(u) & \text{in } \Omega \\
u|_{\partial\Omega} = 0
\end{cases} \quad (P_\lambda)$$

is any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, zero on $\partial\Omega$, which satisfies the equation pointwise in $\Omega$.

For each $u \in X$, put

$$J(u) = \int_{\Omega} \left(\int_0^{u(x)} f(\xi) \, d\xi\right) \, dx.$$

By classical results, the functional $J$ is continuously Gâteaux differentiable and sequentially weakly continuous in $X$, and one has

$$J'(u)(v) = \int_{\Omega} f(u(x))v(x) \, dx$$

for all $u, v \in X$. Moreover, by a standard regularity result, the critical points in $X$ of the functional $u \rightarrow \frac{1}{2}\|u\|^2 - \lambda J(u)$ are exactly the classical solutions of problem $(P_\lambda)$.

Denote by $\lambda_1$ the first eigenvalue of the problem

$$\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \\
u|_{\partial\Omega} = 0
\end{cases}.$$
Recall that $\|u\|_{L^2(\Omega)} \leq \lambda_1^{-\frac{1}{2}} \|u\|$ for all $u \in X$.

We are now in a position to state the following

**THEOREM 2.** Assume that $f(0) \neq 0$. For each $r > 0$, put

$$\gamma(r) = \sup_{\|u\|^2 = r} J(u).$$

Further, put

$$\delta_0 = \inf_{u \in M} \|u\|^2$$

where $M$ is the set of all global minima in $X$ of the functional $u \rightarrow \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{\mu} J(u)$.

Then, $\delta_0 > 0$, the function $\gamma$ is $C^1$ and $\gamma'$ is positive in $]0, \delta_0[$ and there exists a continuous function $\varphi : ]0, \delta_0[ \rightarrow X$ such that, for each $r \in ]0, \delta_0[$, $\varphi(r)$ is a classical solution of the problem

$$\begin{cases}
-\Delta u = \frac{1}{2\gamma(r)} f(u) & \text{in } \Omega \\
u|_{\partial \Omega} = 0
\end{cases}$$

satisfying $\|\varphi(r)\|^2 = r$ and $J(\varphi(r)) = \gamma(r)$.

**PROOF.** Fix $u, v, w \in X$, with $\|w\| = 1$. We have

$$|J'(u)(w) - J'(v)(w)| \leq \int_{\Omega} |f(u(x)) - f(v(x))||w(x)| dx \leq \mu \|u-v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \leq \frac{\mu}{\lambda_1} \|u-v\|,$$

and hence

$$\|J'(u) - J'(v)\| \leq \frac{\mu}{\lambda_1} \|u-v\|.$$ 

That is, $J'$ is Lipschitzian in $X$, with Lipschitz constant $\frac{\mu}{\lambda_1}$. Moreover, since $f(0) \neq 0$, we have $J'(u) \neq 0$ for all $u \in X$. Then, thanks to Theorem 1, for each $r \in ]0, \delta_0[$, the restriction of the functional $J$ to the sphere $S(0, \sqrt{r})$ has a unique maximum. At this point, taken into account that $\gamma(r) > 0$ for all $r > 0$, the conclusion follows directly from Lemma 2.1 and Corollary 2.13 of [2].

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Department of Mathematics  
University of Catania  
Viale A. Doria 6  
95125 Catania  
Italy  
*e-mail address*: ricceri@dmi.unict.it