SOME PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY MODULES WITH RESPECT TO A PAIR OF IDEALS

TRAN TUAN NAM AND NGUYEN MINH TRI

Abstract. We introduce a notion of generalized local cohomology modules with respect to a pair of ideals \((I, J)\) which is a generalization of the concept of local cohomology modules with respect to \((I, J)\). We show that generalized local cohomology modules \(H^i_{I,J}(M, N)\) can be computed by the Čech cohomology modules. We also study the artinianness of generalized local cohomology modules \(H^i_{I,J}(M, N)\).

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1. Introduction

Throughout this paper, \(R\) is a noetherian commutative (non-zero identity) ring. In [10], Takahashi, Yoshino and Yoshizawa introduced the local cohomology modules with respect to a pair of ideals \((I, J)\). For an \(R\)-module \(M\), the \((I, J)\)-torsion submodule of \(M\) is \(\Gamma_{I,J}(M) = \{x \in M | I^n x \subseteq Jx \text{ for some positive integer } n\}\). \(\Gamma_{I,J}\) is a covariant functor from the category of \(R\)-modules to itself. The \(i\)-th local cohomology functor \(H^i_{I,J}\) with respect to \((I, J)\) is defined to be the \(i\)-th right derived functor of \(\Gamma_{I,J}\). When \(J = 0\), the \(H^i_{I,J}\) coincides with the usual local cohomology functor \(H^i_I\).

For two \(R\)-modules \(M\) and \(N\), we define \(\Gamma_{I,J}(M, N)\) to be the \((I, J)\)-torsion submodule of \(\text{Hom}_R(M, N)\). For each \(R\)-module \(M\), there is a covariant functor \(\Gamma_{I,J}(M, -)\) from the category of \(R\)-modules to itself. The \(i\)-th generalized local cohomology functor \(H^i_{I,J}(M, -)\) with the respect to pair of ideals \((I, J)\) is the \(i\)-th right derived functor of \(\Gamma_{I,J}(M, -)\). This definition is really a generalization of the local cohomology functors \(H^i_{I,J}\) with respect to \((I, J)\).

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The organization of the paper is as follows. In the next section, we study some elementary properties of generalized local cohomology modules with respect to a pair of ideals \((I, J)\). We also show that generalized local cohomology modules \(H^i_{I,J}(M, N)\) can be computed by Čech cohomology modules (Theorem 2.8).

The last section is devoted to study the artinianness of local cohomology modules \(H^i_{I,J}(M, N)\). In Theorem 3.1 we prove that if \(M, N\) are two finitely generated \(R\)-modules with \(p = \text{pd}(M)\) and \(d = \dim(N)\), then \(H^{p+d}_{I,J}(M, N) \cong \text{Ext}^p_R(M, H^d_{I,J}(N))\) and \(H^{p+d}_{I,J}(M, N)\) is an artinian \(R\)-module. Theorem 3.2 shows that if \(M\) is a finitely generated \(R\)-module and \(H^i_{I,J}(N)\) is artinian for all \(i < t\), then \(H^i_{I,J}(M, N)\) is artinian for all \(i < t\). On the other hand, \(\text{Ext}^i_R(R/\mathfrak{a}, N)\) is also artinian for all \(i < t\) and for all \(\mathfrak{a} \in \tilde{W}(I, J)\).

Let \(I, J\) be two ideals of the local ring \((R, \mathfrak{m})\) such that \(\sqrt{I+J} = \mathfrak{m}\) and \(M, N\) are two finitely generated \(R\)-modules with \(\dim(N) < \infty\). If \(H^i_{I,J}(M, N)\) is an artinian \(R\)-module for all \(i > t\), then \(H^i_{I,J}(M, N)/JH^i_{I,J}(M, N)\) is also an artinian \(R\)-module (Theorem 3.4). This section is closed by Theorem 3.7 which says that \(H^i_{I,J}(M, N)\) is artinian for all \(i \geq 0\) provided \(M\) is a finitely generated \(R\)-module and \(N\) is an artinian \(R\)-module.

2. Some basic properties of generalized local cohomology modules with respect to a pair of ideals

Let \(I, J\) be two ideals of \(R\). For an \(R\)-module \(M\), the \((I, J)\)-torsion submodule of \(M\) is

\[ \Gamma_{I,J}(M) = \{ x \in M | I^n x \subset Jx \text{ for some positive integer } n \} \]  

We introduce the following definition.

**Definition 2.1.** For two \(R\)-modules \(M, N\) we denote by \(\Gamma_{I,J}(M, N)\) the following module

\[ \Gamma_{I,J}(M, N) = \Gamma_{I,J}(\text{Hom}_R(M, N)). \]

In the special case \(M = R\), \(\Gamma_{I,J}(R, N) = \Gamma_{I,J}(N)\) the \((I, J)\)-torsion submodule of \(N\). Note that an element \(f \in \Gamma_{I,J}(M, N)\) if and only if there is an integer \(n > 0\) such that \(I^n f(x) \subset Jf(x)\) for all \(x \in M\).

For each \(R\)-module \(M\), \(\Gamma_{I,J}(M, -)\) is a left exact covariant functor from the category of \(R\)-modules to itself.

Let us denote by \(H^i_{I,J}(M, -)\) the \(i\)-th right derived functor of \(\Gamma_{I,J}(M, -)\) and call the \(i\)-th generalized local cohomology functor with the respect to pair of ideals \((I, J)\).
Theorem 2.2. Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

$$\Gamma_{I,J}(M,N) = \text{Hom}_R(M, \Gamma_{I,J}(N)).$$

Proof. If $f \in \Gamma_{I,J}(M,N)$, there exists an integer $n > 0$ such that $I^n f(x) \subset Jf(x)$ for all $x \in M$. Since $f(x) \in N$, we get $f(x) \in \Gamma_{I,J}(N)$ for all $x \in M$ and then $f \in \text{Hom}_R(M, \Gamma_{I,J}(N))$.

Let $f \in \text{Hom}_R(M, \Gamma_{I,J}(N))$. Assume that $x_1, x_2, \ldots, x_m$ are generators of $M$.

Since $f(x_i) \in \Gamma_{I,J}(N)$, there exist an integer $n_i$ such that $I^{n_i} f(x_i) \subset Jf(x_i)$ for $i = 1, 2, \ldots, m$.

Set $n = n_1n_2 \ldots n_m$, then $I^n f(x_i) \subset Jf(x_i)$ for all $i = 1, 2, \ldots, m$.

It follows $I^n f(x) \subset Jf(x)$ for all $x \in M$. So $I^n f \subset Jf$ and then $f \in \Gamma_{I,J}(\text{Hom}_R(M,N)) = \Gamma_{I,J}(M,N)$. \qed

Note that in [11] Zamani introduced another definition of local cohomology functors $H_{I,J}$ as follow

$$H_{I,J}^i(M,N) = H^i(\text{Hom}_R(M, \Gamma_{I,J}(E^\bullet)))$$

for all $i \geq 0$, where $E^\bullet$ is an injective resolution of $R$-module $N$. Thus from 2.2 we see that our definition is coincident with Zamani’s one.

We have a property of the set of associated primes of $\Gamma_{I,J}(M,N)$.

Corollary 2.3. Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Then

$$\text{Ass}(\Gamma_{I,J}(M,N)) = \text{Supp}(M) \cap \text{Ass}(N) \cap W(I, J).$$

Proof. Since $M$ is a finitely generated $R$-module, $\text{Ass}(\text{Hom}_R(M,K)) = \text{Supp}(M) \cap \text{Ass}(K)$ for all $R$-modules $K$. By [2.2] we have

$$\text{Ass}(\Gamma_{I,J}(M,N)) = \text{Ass}(\Gamma_{I,J}(\text{Hom}_R(M,N)))$$

$$= \text{Ass}(\text{Hom}_R(M, \Gamma_{I,J}(N)))$$

$$= \text{Supp}(M) \cap \text{Ass}(\Gamma_{I,J}(N))$$

$$= \text{Supp}(M) \cap \text{Ass}(N) \cap W(I, J)$$

as required. \qed

The following proposition is an extension of [10, 1.4].

Proposition 2.4. Let $M$ be a finitely generated $R$-module and $N$ an $R$-module. Let $I, I', J, J'$ be ideals of $R$. Then

(i) $\Gamma_{I,J}(\Gamma_{I',J'}(M,N)) = \Gamma_{I',J'}(\Gamma_{I,J}(M,N))$.

(ii) If $I \subseteq I'$, then $\Gamma_{I,J}(M,N) \supseteq \Gamma_{I',J}(M,N)$. 

(iii) If \( J \subseteq J' \), then \( \Gamma_{I,J}(M, N) \subseteq \Gamma_{I,J'}(M, N) \).

(iv) \( \Gamma_{I,J} (\Gamma_{I',J'}(M, N)) = \Gamma_{I+I',J'}(M, N) \).

(v) \( \Gamma_{I,J} (\Gamma_{I,J'}(M, N)) = \Gamma_{I,J\cap J'}(M, N) \). Moreover, \( H^i_{I,J'}(M, N) = H^i_{I,J}(M, N) \) for all \( i \).

(vi) If \( J' \subseteq J \), then \( \Gamma_{I+J',J}(M, N) = \Gamma_{I,J}(M, N) \). Moreover, \( \Gamma_{I+J,J}(M, N) = \Gamma_{I,J}(M, N) \) and \( H^i_{I+J,J}(M, N) = H^i_{I,J}(M, N) \) for all \( i \).

(vii) If \( \sqrt{I} = \sqrt{I'} \), then \( H^i_{I,J}(M, N) = H^i_{I,J'}(M, N) \) for all \( i \). In particular, \( H^i_{I,J}(M, N) = H^i_{\sqrt{I},J}(M, N) \) for all \( i \).

(viii) If \( \sqrt{J} = \sqrt{J'} \), then \( H^i_{I,J}(M, N) = H^i_{I,J'}(M, N) \) for all \( i \).

Proof. We only prove (i), the others are similar.

Combining [10, 1.4] and [22] we have

\[
\Gamma_{I,J} (\Gamma_{I',J'}(M, N)) = \Gamma_{I,J} (\text{Hom}_R(M, \Gamma_{I',J'}(N)))
= \text{Hom}_R(M, \Gamma_{I,J}(\Gamma_{I',J'}(N)))
= \text{Hom}_R(M, \Gamma_{I,J'}(\Gamma_{I,J}(N)))
= \Gamma_{I',J'}(\Gamma_{I,J}(M, N))
\]

as required. \( \square \)

Lemma 2.5. If \( E \) is an injective \( R \)-module, then \( \Gamma_{I,J}(E) \) is also injective.

Proof. From [10, 3.2] we have

\[
\Gamma_{I,J}(E) \cong \lim_{a \in \tilde{W}(I,J)} \Gamma_a(E),
\]

where \( \tilde{W}(I,J) \) is the set of ideals \( a \) of \( R \) such that \( I^n \subseteq a + J \) for some integer \( n \).

Since \( E \) is an injective \( R \)-module, \( \Gamma_a(E) \) is also injective. Moreover, \( R \) is a Noetherian ring, then the direct limit of injective modules is injective. Therefore we have the conclusion. \( \square \)

It is well-known that \( H^i_I(M, N) \cong \text{Ext}^i_R(M, N) \), where \( N \) is an \( I \)-torsion \( R \)-module. The following proposition gives a similar result when \( N \) is \((I,J)\)-torsion.

Proposition 2.6. Let \( N \) be an \((I,J)\)-torsion \( R \)-module. Then

\[
H^i_{I,J}(M, N) \cong \text{Ext}_R^i(M, N)
\]

for all \( i \geq 0 \).
Proof. From \cite[1.12]{10} there exists an injective resolution $E^\bullet$ of $N$ such that each term is an $(I, J)$-torsion $R$-module. Then we have by \ref{7.2}

$$H^i_{I, J}(M, N) \cong H^i(\text{Hom}_R(M, \Gamma_{I, J}(E^\bullet)))$$

$$\cong H^i(\text{Hom}_R(M, E^\bullet))$$

$$= \text{Ext}^i_R(M, N)$$

for all $i \geq 0$. \hfill $\square$

When $N$ is a $J$-torsion $R$-module, we have the following proposition.

**Proposition 2.7.** If $N$ is a $J$-torsion $R$-module, then

$$H^i_{I, J}(M, N) \cong H^i_I(M, N)$$

for all $i \geq 0$.

**Proof.** It is obvious that $\Gamma_I(N) \subset \Gamma_{I, J}(N)$. Let $x \in \Gamma_{I, J}(N)$, there exist integers $n, k$ such that $I^n x \subset Jx$ and $J^k x = 0$. Hence $I^{nk} x = 0$ and then $x \in \Gamma_I(N)$. Thus $\Gamma_{I, J}(N) = \Gamma_I(N)$.

It remains to prove that $\Gamma_{I, J}(M, N) \cong \Gamma_I(M, N)$. From \ref{2.2} we have

$$\Gamma_{I, J}(M, N) = \text{Hom}_R(M, \Gamma_{I, J}(N))$$

$$= \text{Hom}_R(M, \Gamma_I(N))$$

$$\cong \Gamma_I(M, N).$$

By the property of derived functors, we obtain $H^i_{I, J}(M, N) \cong H^i_I(M, N)$ for all $i \geq 0$. \hfill $\square$

Let $J$ be an ideal of $R$. For an element $a \in R$ the set

$$S_{a,J} = \{a^n + j \mid n \in \mathbb{N}, j \in J\}$$

is a multiplicatively closed subset of $R$ \cite[2.1]{10}. Let $M$ be a finitely generated $R$-module. Denote by $M_{a,J}$ the module of fractions of the $R$-module $M$ with respect to $S_{a,J}$. The complex $C_{a,J}^\bullet$ was given by

$$C_{a,J}^\bullet : 0 \to R \to R_{a,J} \to 0.$$

For a sequence $a = \{a_1, a_2, \ldots, a_r\}$ of elements of $R$, the Čech complex $C_{a,J}^\bullet$ was defined as

$$C_{a,J}^\bullet = \bigotimes_{i=1}^r C_{a_i,J}^\bullet$$

$$= \left(0 \to R \to \prod_{i=1}^r R_{a_i,J} \to \prod_{i<j} (R_{a_i,J})_{a_j,J} \to \cdots \to (\cdots (R_{a_2,J}) \cdots)_{a_r,J} \to 0\right).$$
In [10, 2.4], there is a natural isomorphism $H^i_{I,J}(M) \cong H^i(C^\bullet_{a,J} \otimes_R M)$, where $a = \{a_1, a_2, \ldots, a_r\}$ is a sequence of elements of $R$ that generates $I$.

Let

$$F_\bullet : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a free resolution of $M$ with the finitely generated free modules.

Apply the functor $\text{Hom}_R(-, N)$ to above resolution, we have a complex

$$\text{Hom}_R(F_\bullet, N) : 0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(F_0, N) \rightarrow \text{Hom}_R(F_1, N) \rightarrow \cdots$$

Then there is a bicomplex $C^\bullet_{a,J} \otimes_R \text{Hom}_R(F_\bullet, N) = \{C^p_{a,J} \otimes_R \text{Hom}_R(F_q, N)\}$, where $C^p_{a,J}$ is the $p$-th position in the Čech complex $C^\bullet_{a,J}$. Thus we get a total complex $\text{Tot}(M, N)$ of bicomplex $C^\bullet_{a,J} \otimes_R \text{Hom}_R(F_\bullet, N)$ where

$$\text{Tot}(M, N)^n = \bigoplus_{p+q=n} C^p_{a,J} \otimes_R \text{Hom}_R(F_q, N).$$

**Theorem 2.8.** Let $M$ be a finitely generated $R$-module. Then for all $R$-modules $N$ and $n \geq 0$,

$$H^n_{I,J}(M, N) \cong H^n(\text{Tot}(M, N)).$$

**Proof.** It is clear that $\{H^n(\text{Tot}(M, -))\}_n$ and $\{H^n_{I,J}(M, -)\}_n$ are exact connected right sequences of functors.

We next prove that $H^0(\text{Tot}(M, N)) \cong H^0_{I,J}(M, N)$. Consider the homomorphism $d^0 : \text{Tot}(M, N)^0 \rightarrow \text{Tot}(M, N)^1$. As $C^0_{a,J} = R$, we get

$$H^0(\text{Tot}(M, N)) \cong \text{Ker}(\text{Hom}_R(F_0, N) \xrightarrow{d^0} \text{Hom}_R(F_1, N) \oplus (C^1_{a,J} \otimes_R \text{Hom}_R(F_0, N))).$$

Thus

$$H^0(\text{Tot}(M, N)) = \text{Hom}_R(M, N) \bigcap \Gamma_{I,J}(\text{Hom}_R(F_0, N))$$

$$= \Gamma_{I,J}(\text{Hom}_R(M, N)) = H^0_{I,J}(M, N)$$

by [10, 2.3(5)].

The proof is completed by showing that $H^n(\text{Tot}(M, E)) = 0$ for all $n > 0$ and for any injective $R$-module $E$. It follows from [8, 10.18] a spectral sequence

$$E^{p,q}_2 = H^p H^q(C^\bullet_{a,J} \otimes_R \text{Hom}_R(F_\bullet, E)) \Rightarrow H^n(\text{Tot}(M, E)).$$

Note that

$$E^{p,q}_1 = H^q(C^\bullet_{a,J} \otimes \text{Hom}_R(F_p, E)).$$
From the proof of [10, 2.4], $H^i(C^a_{\alpha J} \otimes_R E) = 0$ for all $i > 0$ and for any injective $R$–module $E$. Note that $\text{Hom}_R(F_q, E)$ is also an injective $R$-module for all $q \geq 0$. Hence

$$E_{p,q}^1 = \begin{cases} 
0, & q > 0 \\
\text{Ker}(\text{Hom}_R(F_p, E) \rightarrow \prod_{i=1}^r \text{Hom}_R(F_p, E)_{a_i,j}), & q = 0.
\end{cases}$$

Combining [10, 2.3(5)] with 2.2 yields

$$\text{Ker}(\text{Hom}_R(F_p, E) \rightarrow \prod_{i=1}^r \text{Hom}_R(F_p, E)_{a_i,j}) \cong \Gamma_{I,J}(\text{Hom}_R(F_p, E)) \cong \text{Hom}_R(F_p, \Gamma_{I,J}(E)).$$

It follows

$$E_{p,q}^2 = \begin{cases} 
0, & q > 0 \\
H^n(\text{Hom}_R(F_\bullet, \Gamma_{I,J}(E))), & q = 0.
\end{cases}$$

As $\Gamma_{I,J}(E)$ is an injective $R$–module, the following sequence is exact

$$\text{Hom}_R(F_\bullet, \Gamma_{I,J}(E)) : 0 \rightarrow \text{Hom}_R(M, \Gamma_{I,J}(E)) \rightarrow \text{Hom}_R(F_0, \Gamma_{I,J}(E)) \rightarrow \text{Hom}_R(F_1, \Gamma_{I,J}(E)) \rightarrow \ldots.$$

Thus $E_{p,0}^2 = 0$ for all $p > 0$. From [8, 10.21 (ii)] we get

$$H^n(\text{Tot}(M, E)) \cong E_{2,n,0}^n = 0$$

for all $n > 0$. The proof is complete. \qed

3. On artinianness of generalized local cohomology modules with respect to a pair of ideals

We have the following theorem.

**Theorem 3.1.** Assume that $(R, m)$ is a local ring. Let $M, N$ be two finitely generated $R$–modules with $r = \text{pd}(M)$ and $d = \text{dim}(N)$. Then

$$H^{r+d}_{I,J}(M, N) \cong \text{Ext}^r_R(M, H^d_{I,J}(N)).$$

Moreover $H^{r+d}_{I,J}(M, N)$ is an artinian $R$–module.
Proof. Let $G(-) = \Gamma_{I,J}(-)$ and $F(-) = \text{Hom}_R(M,-)$ be functors from category of $R$-modules to itself. Then $FG = \Gamma_{I,J}(M,-)$ and $F$ is left exact. For any injective module $E$

$$R^i F(G(E)) = R^i \text{Hom}_R(M, \Gamma_{I,J}(E)) = 0$$

for all $i > 0$, as $\Gamma_{I,J}(E)$ is an injective $R$-module. By [8, 10.47] there is a Grothendieck spectral sequence

$$E^{pq}_2 = \text{Ext}^p_R(M, H^q_{I,J}(N)) \Rightarrow H^{p+q}_{I,J}(M, N).$$

We now consider the homomorphisms of the spectral

$$E^r_k = E^{r-k,d+k-1}_k \rightarrow E^r_d \rightarrow E^{r+k,d+1-k}_k.$$ 

We have $H^q_{I,J}(N) = 0$ for all $q > d$ by [10, 4.7]. Then $E^{pq}_2 = 0$ for all $p > r$ or $q > d$. Thus $E^{r-k,d+k-1}_k = E^{r+k,d+1-k}_k = 0$ for all $k \geq 2$, so

$$E^{r_2}_2 = E^{r_3}_3 = \ldots = E^{r_\infty}_\infty.$$ 

It remains to prove that $E^{r_\infty}_\infty \cong H^{r+d}_{I,J}(M, N)$. Indeed, there is a filtration $\Phi$ of $H^{r+d} = H^{r+d}_{I,J}(M, N)$ such that

$$0 = \Phi^{r+d+1} H^{r+d} \subseteq \Phi^{r+d} H^{r+d} \subseteq \ldots \subseteq \Phi^1 H^{r+d} \subseteq \Phi^0 H^{r+d} = H^{r+d}_{I,J}(M, N)$$

and

$$E^{i,r+d-i}_\infty = \Phi^i H^{r+d}/\Phi^{i+1} H^{r+d}, \quad 0 \leq i \leq r + d.$$ 

From the above proof we have $E^{i,r+d-i}_2 = \text{Ext}^i_R(M, H^{r+d-i}_{I,J}(N)) = 0$ for all $i \neq r$. Hence

$$\Phi^{r+1} H^{r+d} = \Phi^{r+2} H^{r+d} = \ldots = \Phi^{r+d+1} H^{r+d} = 0$$

and

$$\Phi^r H^{r+d} = \Phi^{r-1} H^{r+d} = \ldots = \Phi^0 H^{r+d} = H^{r+d}_{I,J}(M, N).$$

This gives

$$E^{r}_\infty \cong \Phi^r H^{r+d}/\Phi^{r+1} H^{r+d} \cong H^{r+d}_{I,J}(M, N).$$

Thus $\text{Ext}^r_R(M, H^{r}_{I,J}(N)) \cong H^{r+d}_{I,J}(M, N)$. It follows from [2, 2.1] that $H^{r}_{I,J}(N)$ is an artinian $R$-module. Therefore $H^{r+d}_{I,J}(M, N)$ is also an artinian $R$-module. \hfill \square

Next theorem, we show the connection between the artinnianness of $H^{r}_{I,J}(N)$ and $H^{r+d}_{I,J}(M, N)$.

**Theorem 3.2.** Let $M$ be a finitely generated $R$-modules and $N$ an $R$-module. Let $t$ be a positive integer. If $H^{r}_{I,J}(N)$ is artinian for all $i < t$, then
Some properties of generalized local cohomology modules...

(i) $H^i_{I,J}(M, N)$ is artinian for all $i < t$.

(ii) $\text{Ext}_R^i(R/\mathfrak{a}, N)$ is artinian for all $i < t$ and for all $\mathfrak{a} \in \tilde{W}(I, J)$.

Proof. (i) We use induction on $t$. When $t = 1$, by 2.2 we have

$$\Gamma_{I,J}(M, N) = \text{Hom}_R(M, \Gamma_{I,J}(N)).$$

Since $\Gamma_{I,J}(N)$ is artinian, the statement is true in this case.

Let $t > 1$ and we assume that the statement is true for $t - 1$ and for any $R$-module $N$. Denote by $E(N)$ the injective hull of $N$. Applying the functors $\Gamma_{I,J}(\cdot)$ and $\Gamma_{I,J}(M, \cdot)$ to the following short exact sequence

$$0 \to N \to E(N) \to E(N)/N \to 0$$

we get isomorphisms

$$H^i_{I,J}(E(N)/N) \cong H^{i+1}_{I,J}(N)$$

and

$$H^i_{I,J}(M, E(N)/N) \cong H^{i+1}_{I,J}(M, N)$$

for all $i > 0$. From the hypothesis, $H^i_{I,J}(N)$ is artinian for all $i < t$. It follows that $H^i_{I,J}(E(N)/N)$ is also artinian for all $i < t - 1$. By the inductive hypothesis on $E(N)/N$, $H^i_{I,J}(M, E(N)/N)$ is artinian for all $i < t - 1$. We conclude from the second isomorphism that $H^i_{I,J}(M, N)$ is artinian for all $i < t$.

(ii) The proof is by induction on $t$. When $t = 1$, the short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0.$$ 

deduces an exact sequence

$$0 \to \text{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(N)) \to \text{Hom}_R(R/\mathfrak{a}, N) \to \text{Hom}_R(R/\mathfrak{a}, N/\Gamma_{\mathfrak{a}}(N)) \to \cdots$$

As $N/\Gamma_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-torsion-free, we have $\text{Hom}_R(R/\mathfrak{a}, N/\Gamma_{\mathfrak{a}}(N)) = 0$ and then $\text{Hom}_R(R/\mathfrak{a}, N) \cong \text{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(N))$. Note that $\Gamma_{\mathfrak{a}}(N) \subset \Gamma_{I,J}(N)$. By the hypothesis, $\Gamma_{\mathfrak{a}}(N)$ is an artinian $R$-module and then $\text{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(N))$ is also an artinian $R$-module.

The proof for $t > 1$ is similar to (i). \qed

In [2, 2.4], when $(R, \mathfrak{m})$ is a local ring and $N$ is a finitely generated $R$-module, there is an equality

$$\inf\{i \mid H^i_{I,J}(N) \text{ is not artinian}\} = \inf\{\text{depth}\mathfrak{p} \mid \mathfrak{p} \in W(I, J) \setminus \{\mathfrak{m}\}\}.$$ 

We have the following consequence.
Corollary 3.3. Let \((R, \mathfrak{m})\) be a local ring. If \(M\) and \(N\) are two finitely generated \(R\)-modules, then
\[
\inf \{\text{depth}_p N | p \in W(I, J) \backslash \{\mathfrak{m}\}\} \leq \inf \{i | H^i_{I, J}(M, N) \text{ is not artinian}\}.
\]
Proof. From 3.2, we have the following inequality
\[
\inf \{i | H^i_{I, J}(N) \text{ is not artinian}\} \leq \inf \{i | H^i_{I, J}(M, N) \text{ is not artinian}\}.
\]
Thus the conclusion follows from [2, 2.4]. □

Theorem 3.4. Let \(I, J\) be two ideals of the local ring \((R, \mathfrak{m})\) such that \(\sqrt{I + J} = \mathfrak{m}\). Assume that \(M, N\) are two finitely generated \(R\)-modules with \(\dim(N) < \infty\) and \(t\) is a non-negative integer. If \(H^i_{I, J}(M, N)\) is an artinian \(R\)-module for all \(i > t\), then \(H^t_{I, J}(M, N)/JH^t_{I, J}(M, N)\) is also an artinian \(R\)-module.

Proof. Combining 2.4(vi) with 2.4(vii) we conclude that \(H^i_{I, J}(M, N) = H^i_{\mathfrak{m}, J}(M, N)\) for all \(i \geq 0\), as \(\sqrt{I + J} = \mathfrak{m}\). Thus, without loss of generality we can assume that \(I = \mathfrak{m}\).

We now use induction on \(\dim(N) = d\). When \(d = 0\), \(N\) is \(\mathfrak{m}\)-torsion and then \(N\) is \((\mathfrak{m}, J)\)-torsion. From 2.6, there is an isomorphism \(H^i_{\mathfrak{m}, J}(M, N) \cong \text{Ext}^i_R(M, N)\) for all \(i \geq 0\). Since \(N\) is artinian, it follows that \(H^i_{\mathfrak{m}, J}(M, N)\) is an artinian \(R\)-module for all \(i \geq 0\). Therefore the statement is true in this case.

Let \(d > 0\). The short exact sequence
\[
0 \to \Gamma_J(N) \to N \to N/\Gamma_J(N) \to 0.
\]
induces a long exact sequence
\[
H^t_{\mathfrak{m}, J}(M, \Gamma_J(N)) \to H^t_{\mathfrak{m}, J}(M, N) \to H^t_{\mathfrak{m}, J}(M, N/\Gamma_J(N)) \to \cdots
\]
Since \(\Gamma_J(N)\) is a \(J\)-torsion \(R\)-module, there is an isomorphism
\[
H^i_{\mathfrak{m}, J}(M, \Gamma_J(N)) \cong H^i_{\mathfrak{m}}(M, \Gamma_J(N))
\]
by 2.7. From [4, 2.2] \(H^i_{\mathfrak{m}, J}(M, \Gamma_J(N))\) is artinian for all \(i\).

From the long exact sequence, we get two short exact sequences
\[
0 \to \text{Im} \alpha \to H^t_{\mathfrak{m}, J}(M, N) \to \text{Im} \beta \to 0
\]
and
\[
0 \to \text{Im} \beta \to H^t_{\mathfrak{m}, J}(M, N/\Gamma_J(N)) \to \text{Im} \gamma \to 0.
\]
Two above exact sequences deduce long exact sequences
\[
\cdots \to \text{Im} \alpha/J \text{Im} \alpha \to H^t_{\mathfrak{m}, J}(M, N)/JH^t_{\mathfrak{m}, J}(M, N) \to \text{Im} \beta/J \text{Im} \beta \to 0
\]
and
\[
\cdots \to \text{Tor}^R_1(R/J, \text{Im} \gamma) \to \text{Im} \beta/J \text{Im} \beta \to \cdots
\]
Some properties of generalized local cohomology modules...

\[ \rightarrow H_{m,J}^t(M, N/\Gamma_J(N))/JH_{m,J}^t(M, N/\Gamma_J(N)) \rightarrow \text{Im}\gamma/J\text{Im}\gamma \rightarrow 0. \]

Note that Im\(\alpha\) and Im\(\gamma\) are artinian \(R\)-modules. The proof is completed by showing that \(H^t_{m,J}(M, N/\Gamma_J(N))/JH^t_{m,J}(M, N/\Gamma_J(N))\) is an artinian \(R\)-module.

Let \(\overline{N} = N/\Gamma_J(N)\), then \(\overline{N}\) is \(J\)-torsion-free. Thus there exists an element \(x \in J\) that is a non-zerodivisor on \(\overline{N}\).

The short exact sequence
\[ 0 \rightarrow \overline{N} \xrightarrow{\cdot x} N \rightarrow N/\overline{N} \rightarrow 0 \]
implies the following long exact sequence
\[ \cdots \rightarrow H^t_{m,J}(M, \overline{N}) \xrightarrow{f} H^t_{m,J}(M, N/\overline{N}) \xrightarrow{g} H^{t+1}_{m,J}(M, \overline{N}) \rightarrow \cdots \]
From the hypothesis, we get that \(H^i_{m,J}(M, \overline{N}/x\overline{N})\) is artinian for all \(i > t\). As \(\dim(\overline{N}/x\overline{N}) \leq d - 1\), \(H^t_{m,J}(M, \overline{N}/x\overline{N})/JH^t_{m,J}(M, \overline{N}/x\overline{N})\) is artinian by the inductive hypothesis.

We now consider two exact sequences
\[ 0 \rightarrow \text{Im}\ f \rightarrow H^t_{m,J}(M, \overline{N}/x\overline{N}) \rightarrow \text{Im}\ g \rightarrow 0 \]
and
\[ H^t_{m,J}(M, \overline{N}) \xrightarrow{\cdot x} H^t_{m,J}(M, \overline{N}) \rightarrow \text{Im}\ f \rightarrow 0. \]
They give two long exact sequences
\[ \text{Tor}^R_1(R/J, \text{Im}\ g) \rightarrow \text{Im}\ f/J\text{Im}\ f \rightarrow H^t_{m,J}(M, \overline{N}/x\overline{N})/JH^t_{m,J}(M, \overline{N}/x\overline{N}) \rightarrow \text{Im}\ g/J\text{Im}\ g \rightarrow 0 \]
and
\[ H^t_{m,J}(M, \overline{N})/JH^t_{m,J}(M, \overline{N}) \xrightarrow{\cdot x} H^t_{m,J}(M, \overline{N})/JH^t_{m,J}(M, \overline{N}) \rightarrow \text{Im}\ f/J\text{Im}\ f \rightarrow 0. \]
Since \(x \in J\), we obtain from the exact sequence that
\[ H^t_{m,J}(M, \overline{N})/JH^t_{m,J}(M, \overline{N}) \cong \text{Im}\ f/J\text{Im}\ f. \]

On other hand, \(\text{Tor}^R_1(R/J, \text{Im}\ g)\) is artinian, as \(\text{Im}\ g \subset H^{t+1}_{m,J}(M, \overline{N})\) an artinian \(R\)-module. Hence \(\text{Im}\ f/J\text{Im}\ f\) is an artinian \(R\)-module and the proof is complete. \(\square\)

**Theorem 3.5.** Let \(M, N\) be two finitely generated \(R\)-modules and \(t\) a positive integer such that \(H^t_{I,J}(M, R/p)\) is artinian for all \(p \in \text{Supp}(N)\). Then \(H^t_{I,J}(M, N)\) is also artinian.
Proof. As $N$ is finitely generated, there is a chain of submodules of $N$

$$0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_k = N$$

such that $N_i/N_{i-1} \cong R/p_i$ for some $p_i \in \text{Supp}(N)$.

For each $1 \leq i \leq k$, the short exact sequence

$$0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow R/p_i \rightarrow 0$$

duces a long exact sequence

$$\cdots \rightarrow H^i_{I,J}(M, N_{i-1}) \rightarrow H^i_{I,J}(M, N_i) \rightarrow H^i_{I,J}(M, R/p_i) \rightarrow \cdots$$

In particular, $H^i_{I,J}(M, N_1) \cong H^i_{I,J}(M, R/p_1)$. From the exact sequence, it follows that $H^i_{I,J}(M, N_i)$ is artinian for all $1 \leq i \leq k$. This finishes the proof. \(\square\)

From Theorem 3.5 we have the following immediate consequence.

**Corollary 3.6.** Let $M, N$ be finitely generated $R$-modules and $t$ a positive integer. Assume that $H^i_{I,J}(M, R/p)$ is artinian for all $p \in \text{Supp}(N)$.

(i) If $L$ is a finitely generated $R$-module such that $\text{Supp}(L) \subset \text{Supp}(N)$, then $H^i_{I,J}(M, L)$ is artinian.

(ii) If $a$ is an ideal of $R$ such that $V(a) \subset \text{Supp}(N)$, then $H^i_{I,J}(M, R/a)$ is artinian.

In the following theorem, we study the artinianness of $H^i_{I,J}(M, N)$ when $N$ is artinian.

**Theorem 3.7.** Let $M$ be a finitely generated $R$-module and $N$ an artinian $R$-module. Then $H^i_{I,J}(M, N)$ is artinian for all $i \geq 0$.

Proof. We use induction on $i$. When $i = 0$, we have $\Gamma_{I,J}(M, N) = \text{Hom}_R(M, \Gamma_{I,J}(N))$ is artinian, as $\Gamma_{I,J}(N) \subset N$.

Let $i > 0$, denote by $E(N)$ the injective hull of $N$. Note that, if $N \subset K$ is an essential submodule, then $N$ is artinian if and only if $K$ is artinian. Hence $E(N)$ is also artinian.

Now the short exact sequence

$$0 \rightarrow N \rightarrow E(N) \rightarrow E(N)/N \rightarrow 0$$

duces a long exact sequence

$$\cdots \rightarrow H^{i-1}_{I,J}(M, E(N)/N) \rightarrow H^i_{I,J}(M, N) \rightarrow H^i_{I,J}(M, E(N)) \rightarrow \cdots$$

Since $H^i_{I,J}(M, E(N)) = 0$ for all $i > 0$, there are isomorphims

$$H^{i-1}_{I,J}(M, E(N)/N) \cong H^i_{I,J}(M, N)$$
for all $i > 1$.

$H_{I,J}^{i-1}(M, E(N)/N)$ is artinian by inductive hypothesis. Therefore $H_{I,J}^{i}(M, N)$ is also artinian. □

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DEPARTMENT OF MATHEMATICS-INFORMATICS, HO CHI MINH UNIVERSITY OF PEDAGOGY, HO CHI MINH CITY, VIET NAM.

E-mail address: namtuantran@gmail.com

DEPARTMENT OF NATURAL SCIENCE EDUCATION, DONG NAI UNIVERSITY, DONG NAI, VIET NAM.

E-mail address: triminhng@gmail.com