Structure Formation in the Early Universe

P. G. Miedema*
Netherlands Defence Academy (NLDA)
Hogeschooilaan 2
NL-4818CR Breda
The Netherlands

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Abstract

The evolution of the perturbations in the energy density and the particle number density in a flat Friedmann-Lemaître-Robertson-Walker universe in the radiation-dominated era and in the epoch after decoupling of matter and radiation is studied. For large-scale perturbations the outcome is in accordance with treatments in the literature. For small-scale perturbations the differences are conspicuous. Firstly, in the radiation-dominated era small-scale perturbations grew proportional to the square root of time. Secondly, perturbations in the Cold Dark Matter particle number density were, due to gravitation, coupled to perturbations in the total energy density. This implies that structure formation could have begun successfully only after decoupling of matter and radiation. Finally, after decoupling density perturbations evolved diabatically, i.e., they exchanged heat with their environment. This heat exchange may have enhanced the growth rate of their mass sufficiently to explain structure formation in the early universe, a phenomenon which cannot be understood from adiabatic density perturbations.

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*pieter.miedema@gmail.com

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1 Introduction

The global properties of our universe are very well described by a $\Lambda$CDM model with a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric within the context of the General Theory of Relativity. To explain structure formation after decoupling of matter and radiation in this model, one has to assume that before decoupling Cold Dark Matter (CDM) has already contracted to form seeds into which the baryons (i.e., ordinary matter) could fall after decoupling. In this article it will be shown that CDM did not contract faster than baryons before decoupling and that structure formation started off successfully only after decoupling.

In the companion article\(^1\) it has been shown that there are two unique gauge-invariant quantities $\varepsilon_{1(i)}$ and $n_{1(i)}$ which are the real, measurable, perturbations to the energy density and the particle number density, respectively. Evolution equations for the corresponding contrast functions $\delta_{\varepsilon}$ and $\delta_{n}$ have been derived for closed, flat and open FLRW universes. In this article these evolution equations will be applied to a flat FLRW universe in its three main phases, namely the radiation-dominated era, the plasma era, and the epoch after decoupling of matter and radiation. In the derivation of the evolution equations, an equation of state for the pressure of the form $p = p(n, \varepsilon)$ has been taken into account, as is required by thermodynamics. As

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\(^1\)Section and equation numbers with a * refer to sections and equations in the companion article [1].
a consequence, in addition to a usual second-order evolution equation (3a) for density perturbations, a first-order evolution equation (3b) for entropy perturbations follows also from the perturbed Einstein equations. This entropy evolution equation is absent in former treatments of the subject. Therefore, the system (3) leads to further reaching conclusions than is possible from treatments in the literature.

Analytic expressions for the fluctuations in the energy density $\delta \varepsilon$ and the particle number density $\delta n$ in the radiation-dominated era and the epoch after decoupling will be determined. It is shown that the evolution equations (3) corroborate the standard perturbation theory in both eras in the limiting case of infinite-scale perturbations. For finite scales, however, the differences are conspicuous. Therefore, only finite-scale perturbations are considered in detail.

A first result is that in the radiation-dominated era oscillating density perturbations with an increasing amplitude proportional to $t^{1/2}$ are found, whereas the standard perturbation equation (61) yields oscillating density perturbations with a constant amplitude. This difference is due to the fact that in the perturbation equations (3) the divergence $\vartheta^{(1)}$ of the spatial part of the fluid four-velocity is taken into account, whereas $\vartheta^{(1)}$ is missing in the standard equation. In the appendix it is made clear why $\vartheta^{(1)}$ is important.

In the radiation-dominated era and the plasma era baryons were tightly coupled to radiation via Thomson scattering until decoupling. A second result is that $\text{CDM}$ was also tightly coupled to radiation, not through Thomson scattering, but through gravitation. This implies that before decoupling perturbations in $\text{CDM}$ have contracted as fast as perturbations in the baryon density. As a consequence, $\text{CDM}$ could not have triggered structure formation after decoupling. This result follows from the entropy evolution equation (3b) since $p_n \leq 0$, (5), throughout the history of the universe as will be shown in Section 3.

From observations [2] of the Cosmic Microwave Background it follows that perturbations were adiabatic at the moment of decoupling, and density fluctuations $\delta \varepsilon$ and $\delta n$ were of the order of $10^{-5}$ or less. Since the growth rate of adiabatic perturbations in the era after decoupling was too small to explain structure in the universe, there must have been, in addition to gravitation, some other mechanism which has enhanced the growth rate sufficiently to form the first stars from small density perturbations. A final result of the present study is that it has been demonstrated that after decoupling such a mechanism did indeed exist in the early universe.

At the moment of decoupling of matter and radiation, photons could not ionize matter any more and the two constituents fell out of thermal equilibrium. As a consequence, the pressure dropped from a very high radiation pressure just before decoupling to a very low gas pressure after decoupling. This fast and chaotic transition from a high pressure epoch to a very low pressure era may have resulted in large relative diabatic pressure perturbations due to very small fluctuations in the kinetic energy density. It is found that the growth of a density perturbation has not only been governed by gravitation, but also by heat exchange of a perturbation with its environment. The growth rate depended strongly on the scale of a perturbation. For perturbations with a scale of 6.5 pc $\approx$ 21 ly (see the peak value in Figure 1) gravity and heat exchange worked perfectly together, resulting in a fast growth rate. Perturbations larger than this scale reached, despite their stronger gravitational field, their non-linear phase at a later time since heat exchange was low due to their larger scales. On the other hand, for perturbations with scales smaller than 6.5 pc gravity was too weak and heat exchange was not sufficient to let perturbations grow. Therefore, density perturbations with scales smaller than 6.5 pc did not
reach the non-linear regime within 13.81 Gyr, the age of the universe. Since there was a sharp decline in growth rate below a scale of 6.5 pc, this scale will be called the relativistic Jeans scale.

The conclusion of the present article is that the ΛCDM model of the universe and its evolution equations for density perturbations (3) explain the so-called (hypothetical) Population III stars and larger structures in the universe, which came into existence several hundreds of million years after the Big Bang [3, 4].

2 Einstein Equations for a Flat FLRW Universe

In this section the equations needed for the study of the evolution of density perturbations in the early universe are written down for an equation of state for the pressure, \( p = p(n, \varepsilon) \).

2.1 Background Equations

The set of zeroth-order Einstein equations and conservation laws for a flat, i.e., \( R(0) = 0 \), FLRW universe filled with a perfect fluid with energy-momentum tensor

\[
T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - pg^{\mu\nu}, \quad p = p(n, \varepsilon),
\]

is given by

\[
\begin{align*}
3H^2 &= \kappa \varepsilon(0), \quad \kappa = 8\pi G N / c^4, \quad \varepsilon = \varepsilon(0), \quad (2a) \\
\dot{\varepsilon}(0) &= -3H \varepsilon(0)(1 + w), \quad w := p(0) / \varepsilon(0), \quad \varepsilon(0), \quad (2b) \\
\dot{n}(0) &= -3H n(0). \quad \varepsilon(0), \quad (2c)
\end{align*}
\]

The evolution of density perturbations took place in the early universe shortly after decoupling, when \( \Lambda \ll \kappa \varepsilon(0) \). Therefore, the cosmological constant \( \Lambda \) has been neglected.

2.2 Evolution Equations for Density Perturbations

The complete set of perturbation equations for the two independent density contrast functions \( \delta_n \) and \( \delta_\varepsilon \) is given by (44∗)

\[
\begin{align*}
\ddot{\delta}_\varepsilon + b_1 \dot{\delta}_\varepsilon + b_2 \delta_\varepsilon &= b_3 \left[ \delta_n - \frac{\delta_\varepsilon}{1 + w} \right], \quad \text{(3a)} \\
\frac{1}{c} \frac{d}{dt} \left[ \delta_n - \frac{\delta_\varepsilon}{1 + w} \right] &= \frac{3H n(0)p_n}{\varepsilon(0)(1 + w)} \left[ \delta_n - \frac{\delta_\varepsilon}{1 + w} \right], \quad \text{(3b)}
\end{align*}
\]

where the coefficients \( b_1, b_2 \) and \( b_3 \), (45∗), are for a flat FLRW universe filled with a perfect fluid described by an equation of state \( p = p(n, \varepsilon) \) given by

\[
\begin{align*}
b_1 &= H(1 - 3w - 3\beta^2) - 2\frac{\beta^2}{\beta}, \quad (4a) \\
b_2 &= \kappa \varepsilon(0) \left[ 2\beta^2(2 + 3w) - \frac{1}{6} (1 + 18w + 9w^2) \right] + 2H \frac{\beta^2}{\beta}(1 + 3w) - \beta^2 \frac{\nabla^2}{a^2}, \quad (4b) \\
b_3 &= \left\{ -\frac{2}{1 + w} \varepsilon(0)p_n(1 + w) + \frac{2p_n \beta}{3H \beta} + p_n(p_\varepsilon - \beta^2) + n(0)n_{nn} \right\} + \frac{n(0)}{\varepsilon(0)} \frac{\nabla^2}{a^2}, \quad (4c)
\end{align*}
\]
where \( p_n(n, \varepsilon) \) and \( p_\varepsilon(n, \varepsilon) \) are the partial derivatives of the equation of state \( p(n, \varepsilon) \):

\[
p_n := \left( \frac{\partial p}{\partial n} \right)_\varepsilon, \quad p_\varepsilon := \left( \frac{\partial p}{\partial \varepsilon} \right)_n.
\]  

(5)

The symbol \( \nabla^2 \) denotes the Laplace operator. The quantity \( \beta(t) \) is defined by

\[
\beta^2 := \frac{\dot{p}(0)}{\dot{\varepsilon}(0)}.
\]

Using that \( \dot{p}(0) = p_n \dot{n}(0) + p_\varepsilon \dot{\varepsilon}(0) \) and the conservation laws (2b) and (2c) one gets

\[
\beta^2 = p_\varepsilon + \frac{n(0)p_n}{\varepsilon(0)(1 + w)}.
\]

(6)

From the definitions \( w := p(0)/\varepsilon(0) \) and \( \beta^2 := \dot{p}(0)/\dot{\varepsilon}(0) \) and the energy conservation law (2b), one finds for the time-derivative of \( w \)

\[
\dot{w} = 3H(1 + w)(w - \beta^2).
\]

(7)

This expression holds true independent of the equation of state.

The pressure perturbation is given by \((49^*)\)

\[
p^g_{(1)} = \beta^2 \varepsilon(0) \delta_\varepsilon + n(0)p_n \left[ \delta_n - \frac{\delta_\varepsilon}{1 + w} \right],
\]

where the first term, \( \beta^2 \varepsilon(0) \delta_\varepsilon \), is the adiabatic part and the second term the diabatic part of the pressure perturbation.

The combined First and Second Law of Thermodynamics reads \((57^*)\)

\[
T(0) s^g_{(1)} = -\frac{\varepsilon(0)(1 + w)}{n(0)} \left[ \delta_n - \frac{\delta_\varepsilon}{1 + w} \right].
\]

(9)

Density perturbations evolve adiabatically if and only if the source term of the evolution equation (3a) vanishes, so that this equation is homogeneous and describes, therefore, a closed system that does not exchange heat with its environment. This can only be achieved for \( p_n \approx 0 \), or, equivalently, \( p \approx p(\varepsilon) \), i.e., if the particle number density does not contribute to the pressure. In this case, the coefficient \( b_3 \), \((4c)\), vanishes.

3 Analytic Solutions

In this section analytic solutions of equations (3) are derived for a flat FLRW universe with a vanishing cosmological constant in its radiation-dominated phase and in the era after decoupling of matter and radiation. It is shown that \( p_n \leq 0 \) throughout the history of the universe. In this case, the entropy evolution equation (3b) implies that fluctuations in the particle number density, \( \delta_n \), are coupled to fluctuations in the total energy density, \( \delta_\varepsilon \), through gravitation, irrespective of the nature of the particles. In particular, this holds true for perturbations in \( cdm \). Consequently, \( cdm \) fluctuations have evolved in the same way as perturbations in ordinary matter. This may rule out \( cdm \) as a means to facilitate the formation of structure in the universe after decoupling. The same conclusion has also been reached by Nieuwenhuizen et al. \([5]\), on different grounds. Therefore, structure formation could start only after decoupling.
3.1 Radiation-dominated Era

At very high temperatures, radiation and ordinary matter are in thermal equilibrium, coupled via Thomson scattering with the photons dominating over the nucleons \( n_\gamma / n_p \approx 10^9 \). Therefore the primordial fluid can be treated as radiation-dominated with equations of state

\[
\varepsilon = a_B T_\gamma^4, \quad p = \frac{1}{3} a_B T_\gamma^4, \tag{10}
\]

where \( a_B \) is the black body constant and \( T_\gamma \) the radiation temperature. The equations of state (10) imply the equation of state for the pressure

\[
p_n = 0, \quad p_\varepsilon = \frac{1}{3} \varepsilon.
\]

Therefore, one has from (6),

\[
\beta^2 = w = \frac{1}{3}.
\]

Using (11) and (12), the perturbation equations (3) reduce to

\[
\ddot{\delta}_\varepsilon - \frac{1}{2} \tau \dot{\delta}_\varepsilon + \left[ \frac{\mu^2}{2} \frac{\tau^2}{\tau} + \frac{1}{2} \tau^2 \right] \delta_\varepsilon = 0, \tag{13a}
\]

\[
\ddot{\delta}_n - \frac{3}{4} \delta_\varepsilon = 0, \tag{13b}
\]

where (58*) has been used. Since \( p_n = 0 \) the right-hand side of (13a) vanishes, implying that density perturbations evolved adiabatically: they did not exchange heat with their environment. Moreover, baryons were tightly coupled to radiation through Thomson scattering, i.e., baryons obey \( \delta_n, \text{baryon} = \frac{3}{4} \delta_\varepsilon \). Thus, for baryons (13b) is identically satisfied. In contrast to baryons, CDM is not coupled to radiation through Thomson scattering. However, equation (13b) follows from the General Theory of Relativity, Section 2.7*. As a consequence, equation (13b) should be obeyed by all kinds of particles that interact through gravitation. In other words, equation (13b) holds true for baryons as well as CDM. Since CDM interacts only via gravity with baryons and radiation, the fluctuations in CDM are coupled through gravitation to fluctuations in the energy density, so that fluctuations in CDM also satisfy equation (13b).

In order to solve equation (13a) it will first be rewritten in a form using dimensionless quantities. The solutions of the background equations (2) are given by

\[
H \propto t^{-1}, \quad \varepsilon(0) \propto t^{-2}, \quad n(0) \propto t^{-3/2}, \quad a \propto t^{1/2},
\]

implying that \( T(0) \propto a^{-1} \). The dimensionless time \( \tau \) is defined by \( \tau := t/t_0 \). Since \( H := \dot{a}/a \), one finds that

\[
\frac{d^k}{d\tau^k} = \left[ \frac{1}{c t_0} \right]^k \frac{d^k}{d\tau^k} = \left[ 2 H(t_0) \right]^k \frac{d^k}{d\tau^k}, \quad k = 1, 2.
\]

Substituting \( \delta_\varepsilon(t, \mathbf{x}) = \delta_\varepsilon(t, \mathbf{q}) \exp(i \mathbf{q} \cdot \mathbf{x}) \) into equation (13a) and using (15) yields

\[
\delta_\varepsilon'' - \frac{1}{2\tau} \delta_\varepsilon' + \left[ \frac{\mu^2}{4\tau} + \frac{1}{2\tau^2} \right] \delta_\varepsilon = 0, \quad \tau \geq 1,
\]

where a prime denotes differentiation with respect to \( \tau \). The parameter \( \mu_\varepsilon \) is given by

\[
\mu_\varepsilon := \frac{2 \pi}{\lambda_\varepsilon} \frac{1}{H(t_0) \sqrt{3}}, \quad \lambda_\varepsilon := \lambda a(t_0),
\]

\[\text{6}\]
with \( \lambda_0 \) the physical scale of a perturbation at time \( t_0 \) (\( \tau = 1 \)), and \( |\mathbf{q}| = 2\pi/\lambda \). To solve equation (16), replace \( \tau \) by \( x := \mu_r \sqrt{\tau} \). After transforming back to \( \tau \), one finds

\[
\delta_\varepsilon(\tau, \mathbf{q}) = \left[ A_1(\mathbf{q}) \sin(\mu_r \sqrt{\tau}) + A_2(\mathbf{q}) \cos(\mu_r \sqrt{\tau}) \right] \sqrt{\tau},
\]

(18)

where the ‘constants’ of integration \( A_1(\mathbf{q}) \) and \( A_2(\mathbf{q}) \) are given by

\[
A_1(\mathbf{q}) = \delta_\varepsilon(t_0, \mathbf{q}) \frac{\mu_r}{H(t_0)} \sin \left[ \mu_r \cos \left( \mu_r \frac{t}{t_0} \right) \right] - \frac{\dot{\delta}_\varepsilon(t_0, \mathbf{q})}{H(t_0)}.
\]

(19)

For large-scale perturbations (\( \lambda \to \infty \)), it follows from (18) and (19) that

\[
\delta_\varepsilon(t) = \left[ \delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \frac{t}{t_0} + \left[ 2\delta_\varepsilon(t_0) - \frac{\dot{\delta}_\varepsilon(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{\frac{1}{2}}.
\]

(20)

The energy density contrast has two contributions to the growth rate, one proportional to \( t \) and one proportional to \( t^{1/2} \). These two solutions have been found, with the exception of the precise factors of proportionality, by a large number of authors [6–11]. Consequently, the evolution equations (13) corroborates for large-scale perturbations the results of the literature.

Small-scale perturbations (\( \lambda \to 0 \)) oscillate with an increasing amplitude according to

\[
\delta_\varepsilon(t, \mathbf{q}) \approx \delta_\varepsilon(t_0, \mathbf{q}) \left( \frac{t}{t_0} \right)^{\frac{1}{2}} \cos \left[ \mu_r \left( \frac{t}{t_0} \right)^{\frac{1}{2}} \sin \left( \mu_r \frac{t}{t_0} \right) \right],
\]

(21)

as follows from (18) and (19). Thus, the evolution equations (13) yield oscillating density perturbations with an increasing amplitude, since in these equations \( \theta^{(1)} \neq 0 \), as follows from their derivation in Section 2.7∗. In contrast, the standard equation (61), which has \( \theta^{(1)} = 0 \), yields oscillating density perturbations with a constant amplitude.

Finally, the plasma era has begun at time \( t_{\text{eq}} \), when the energy density of ordinary matter was equal to the energy density of radiation, (58), and ends at \( t_{\text{dec}} \), the time of decoupling of matter and radiation. In the plasma era the matter-radiation mixture can be characterized by the equations of state (Kodama and Sasaki [12], Chapter V)

\[
\varepsilon(n, T) = nmc^2 + a_B T_\gamma^4, \quad p(n, T) = \frac{1}{3} a_B T_\gamma^4,
\]

(22)

where the contributions to the pressure of ordinary matter and CDM have not been taken into account, since these contributions are negligible with respect to the radiation energy density. Eliminating \( T_\gamma \) from (22), one finds for the equation of state for the pressure, Section 2.1∗,

\[
p(n, \varepsilon) = \frac{1}{3} (\varepsilon - nmc^2),
\]

(23)

so that with (5) one gets

\[
p_n = -\frac{1}{3} nmc^2, \quad p_\varepsilon = \frac{1}{3}.
\]

(24)

Since \( p_n < 0 \), equation (3b) implies that fluctuations in the particle number density, \( \delta_n \), were coupled to fluctuations in the total energy density, \( \delta_\varepsilon \), through gravitation, irrespective of the nature of the particles.
3.2 Era after Decoupling of Matter and Radiation

Once protons and electrons combined to yield hydrogen, the radiation pressure was negligible, and the equations of state have become those of a non-relativistic monatomic perfect gas with three degrees of freedom

\[ \varepsilon(n,T) = nmc^2 + \frac{3}{2}nk_BT, \quad p(n,T) = nk_BT, \]

where \( k_B \) is Boltzmann’s constant, \( m \) the mean particle mass, and \( T \) the temperature of the matter. For the calculations in this subsection it is only needed that the \( cdm \) particle mass is such that for the mean particle mass \( m \) one has \( mc^2 \gg k_BT \), so that \( w := p_{(0)}/\varepsilon_{(0)} \ll 1 \). Therefore, as follows from the background equations (2a) and (2b), one may neglect the pressure \( nk_BT \) and the kinetic energy density \( \frac{3}{2}nk_BT \) with respect to the rest mass energy density \( nmc^2 \) in the unperturbed universe. However, neglecting the pressure in the perturbed universe yields non-evolving density perturbations with a static gravitational field, as is shown in Section 4∗. Consequently, it is important to take the pressure perturbations into account.

Eliminating \( T \) from (25) yields, Section 2.1∗, the equation of state for the pressure

\[ p(n,\varepsilon) = \frac{2}{3}(\varepsilon - nmc^2), \]

so that with (5) one has

\[ p_n = -\frac{2}{3}mc^2, \quad p_\varepsilon = \frac{2}{3}. \]

Substituting \( p_n, p_\varepsilon \) and \( \varepsilon \) (25) into (6) on finds, using that \( mc^2 \gg k_BT \),

\[ \beta \approx \frac{v_s}{c} = \sqrt{\frac{5}{3} \frac{k_BT_{(0)}}{mc^2}}, \]

with \( v_s \) the adiabatic speed of sound and \( T_{(0)} \) the matter temperature. Using that \( \beta^2 \approx \frac{5}{3}w \) and \( w \ll 1 \), expression (7) reduces to

\[ \dot{w} \approx -2Hw, \]

so that with \( H := \dot{a}/a \) one has \( w \propto a^{-2} \). This implies that the matter temperature decays as

\[ T_{(0)} \propto a^{-2}. \]

This, in turn, implies with (28) that \( \dot{\beta}/\beta = -H \). The system (3) can now be rewritten as

\[ \ddot{\delta}_\varepsilon + 3H\dot{\delta}_\varepsilon - \left[ \beta^2 \frac{\nabla^2}{a^2} + \frac{5}{6}k\varepsilon_{(0)} \right] \delta_\varepsilon = -\frac{2}{3} \frac{\nabla^2}{a^2} (\delta_n - \delta_\varepsilon), \]

\[ \frac{1}{c} \frac{d}{dt} (\delta_n - \delta_\varepsilon) = -2H (\delta_n - \delta_\varepsilon), \]

where \( w \ll 1 \) and \( \beta^2 \ll 1 \) have been neglected with respect to constants of order unity. From equation (30b) it follows with \( H := \dot{a}/a \) that

\[ \delta_n - \delta_\varepsilon \propto a^{-2}. \]

Since the system (30) is derived from the General Theory of Relativity, it should be obeyed by all kinds of particles which interact through gravity, in particular baryons and CDM.

It will now be shown that the right-hand side of equation (30a) is proportional to the mean kinetic energy density fluctuation of the particles of a density perturbation. To that end, an
expression for $\varepsilon_{g_1}^{(1)}$ will be derived from (25). Multiplying $\dot{\varepsilon}_{(0)}$ by $\theta_{(1)}/\dot{\theta}_{(0)}$ and subtracting the result from $\varepsilon_{(1)}$, one finds

$$
\varepsilon_{g_1}^{(1)} = n_{g_1}^{(1)}mc^2 + \frac{3}{2}n_{g_1}^{(1)}k_B T_{(0)} + \frac{3}{2}n_{(0)}k_B T_{(1)},
$$

(32)

where also the definitions (40a*) and (52*') have been used. Dividing the result by $\varepsilon_{(0)}$, (25), and using that $k_B T_{(0)} \ll mc^2$, one finds

$$
\delta \varepsilon \approx \delta n + \frac{3}{2}k_B T_{(0)}mc^2 \delta T,
$$

(33)

to a very good approximation. In this expression $\delta \varepsilon$ is the relative perturbation in the total energy density. Since $mc^2 \gg \frac{3}{2}k_B T_{(0)}$, it follows from the derivation of (33) that $\delta n$ can be considered as the relative perturbation in the rest energy density. Consequently, the second term is the fluctuation in the kinetic energy density, i.e., $\delta_{\text{kin}} \approx \delta \varepsilon - \delta n$. The relative kinetic energy density perturbation occurs in the source term of the evolution equation (30a) and is of the same order of magnitude as the term with $\beta^2$ in the left-hand side.

Combining (29) and (31) one finds from (33) that $\delta T$ is constant

$$
\delta T(t, x) \approx \delta T(t_0, x),
$$

(34)

to a very good approximation, so that the kinetic energy density fluctuation is given by

$$
\delta_{\text{kin}}(t, x) \approx \delta \varepsilon(t, x) - \delta n(t, x) \approx \frac{3}{2}k_B T_{(0)}(t)mc^2 \delta T(t_0, x).
$$

(35)

In Section 4 it will be shown that the kinetic energy density fluctuation has played, in addition to gravitation, a role in the evolution of density perturbations. In fact, if a density perturbation was somewhat cooler than its environment, i.e., $\delta T < 0$, its growth rate was, depending on its scale, enhanced.

Using (27) and (33), one finds from (8)

$$
\delta p \approx \frac{5}{3} \delta \varepsilon + \delta T,
$$

(36)

where $\delta p$ is the relative pressure perturbation defined by $\delta p := p_{g_1}^{(1)}/p_{(0)}$, with $p_{(0)}$ given by (25). The term $\frac{5}{3} \delta \varepsilon$ is the adiabatic part and $\delta T$ is the diabatic part of the relative pressure perturbation. The factor $\frac{5}{3}$ is the so-called adiabatic index for a monatomic ideal gas with three degrees of freedom. Thus, relative kinetic energy density perturbations give rise to diabatic pressure fluctuations.

Finally, the perturbed entropy per particle follows from (9) and (33)

$$
\mathcal{S}_{g_1}^{(1)} \approx \frac{3}{2}k_B \delta T.
$$

(37)

In Section 3.2* it has been shown that the background entropy per particle $\mathcal{S}_{(0)}$ is independent of time. In a linear perturbation theory the perturbed entropy per particle is approximately constant, i.e., $\mathcal{S}_{g_1}^{(1)} \approx 0$. Therefore, heat exchange of a perturbation with its environment decays proportional to the temperature, i.e., $T_{(0)}\mathcal{S}_{g_1}^{(1)} \propto a^{-2}$, as follows from (29).

In order to solve equation (30a) it will first be rewritten in a form using dimensionless quantities. The solutions of the background equations (2) are given by

$$
H \propto t^{-1}, \quad \varepsilon_{(0)} \propto t^{-2}, \quad n_{(0)} \propto t^{-2}, \quad a \propto t^{2/3},
$$

(38)
where the kinetic energy density and pressure have been neglected with respect to the rest mass energy density. The dimensionless time \( \tau \) is defined by \( \tau := t/t_0 \). Using that \( H := \dot{a}/a \), one gets

\[
\frac{d^k}{c^k d\tau^k} = \left[ \frac{1}{c^0} \right]^k \frac{d^k}{d\tau^k} = \left[ \frac{2}{\pi} H(t_0) \right]^k \frac{d^k}{d\tau^k}, \quad k = 1, 2.
\]

Substituting \( \delta_\varepsilon(t, \mathbf{x}) = \delta_\varepsilon(t, \mathbf{q}) \exp(i \mathbf{q} \cdot \mathbf{x}) \), \( \delta_n(t, \mathbf{x}) = \delta_n(t, \mathbf{q}) \exp(i \mathbf{q} \cdot \mathbf{x}) \), (28) and (35) into equations (30) and using (29) and (39) one finds that equations (30) can be combined into one equation

\[
\delta''_\varepsilon + \frac{2}{\tau} \delta'_\varepsilon + \left[ \frac{4}{9} \mu_m^2 \tau^2 - \frac{10}{9\tau^2} \right] \delta_\varepsilon = -\frac{4}{15} \mu_m^2 \tau^2 \delta_T(t_0, \mathbf{q}), \quad \tau \geq 1,
\]

where a prime denotes differentiation with respect to \( \tau \). The parameter \( \mu_m \) is given by

\[
\mu_m := \frac{2\pi}{\lambda_0} \frac{1}{H(t_0)} \frac{v_5(t_0)}{c}, \quad \lambda_0 := \lambda a(t_0),
\]

with \( \lambda_0 \) the physical scale of a perturbation at time \( t_0 \) (\( \tau = 1 \)), and \( |\mathbf{q}| = 2\pi/\lambda \). To solve equation (40) replace \( \tau \) by \( x := 2\mu_m \tau^{-1/3} \). After transforming back to \( \tau \), one finds for the general solution of the evolution equation (40)

\[
\delta_\varepsilon(\tau, \mathbf{q}) = \left[ B_1(\mathbf{q}) J_{\pm 7/2}(2\mu_m \tau^{-1/3}) + B_2(\mathbf{q}) J_{-7/2}(2\mu_m \tau^{-1/3}) \right] \tau^{-1/2} - \frac{3}{5} \left[ 1 + \frac{5\tau^2/3}{2\mu_m^2} \right] \delta_T(t_0, \mathbf{q}),
\]

where \( J_{\pm 7/2}(x) \) are Bessel functions of the first kind and \( B_1(\mathbf{q}) \) and \( B_2(\mathbf{q}) \) are the ‘constants’ of integration, calculated with the help of MAXIMA [13]:

\[
B_1(\mathbf{q}) = \frac{3\sqrt{\pi}}{20 \mu_m^{3/2}} \left( 4\mu_m^2 - 5 \right) \frac{\cos 2\mu_m}{\sin 2\mu_m} \pm 10\mu_m \frac{\sin 2\mu_m}{\cos 2\mu_m} \delta_T(t_0, \mathbf{q}) +
\]

\[
\frac{\sqrt{\pi}}{8 \mu_m^{7/2}} \left[ (8\mu_m^4 - 30\mu_m^2 + 15) \frac{\cos 2\mu_m}{\sin 2\mu_m} \pm (20\mu_m^3 - 30\mu_m) \frac{\sin 2\mu_m}{\cos 2\mu_m} \right] \delta_\varepsilon(t_0, \mathbf{q}) +
\]

\[
\frac{\sqrt{\pi}}{8 \mu_m^{7/2}} \left[ (24\mu_m^2 - 15) \frac{\cos 2\mu_m}{\sin 2\mu_m} \pm (8\mu_m^3 - 30\mu_m) \frac{\sin 2\mu_m}{\cos 2\mu_m} \right] \frac{\delta_\varepsilon(t_0, \mathbf{q})}{H(t_0)}.
\]

The particle number density contrast \( \delta_n(t, \mathbf{q}) \) follows from equation (33), (34) and (42). In (42) the first term (i.e., the solution of the homogeneous equation) is the adiabatic part of a density perturbation, whereas the second term (i.e., the particular solution) is the diabatic part.

In the large-scale limit \( \lambda \to \infty \) terms with \( \nabla^2 \) vanish. Therefore, the general solution of equation (40) becomes

\[
\delta_\varepsilon(t) = \frac{1}{7} \left[ 5\delta_\varepsilon(t_0) + \frac{2\delta_\varepsilon(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{3/2} + \frac{2}{7} \left[ \delta_\varepsilon(t_0) - \frac{\delta_\varepsilon(t_0)}{H(t_0)} \right] \left( \frac{t}{t_0} \right)^{-5/4}.
\]

Thus, for large-scale perturbations the diabatic pressure fluctuation \( \delta_T(t_0, \mathbf{q}) \) did not play a role during the evolution: large-scale perturbations were adiabatic and evolved only under the influence of gravity. These perturbations were so large that heat exchange did not play a role during their evolution in the linear phase. For perturbations much larger than the Jeans scale
(i.e., the peak value in Figure 1), gravity alone was insufficient to explain structure formation within 13.81 Gyr, since they grow as $\delta_c \propto t^{2/3}$.

The solution proportional to $t^{2/3}$ is a standard result [6–11]. Since $\delta_c$ is gauge-invariant, the standard non-physical gauge mode proportional to $t^{-1}$ is absent from the solution set of the evolution equations (30). Instead, a physical mode proportional to $t^{-5/3}$ is found. This mode follows also from the standard perturbation equations if one does not neglect the divergence $\vartheta_{(1)}$, as is shown in the appendix. Consequently, only the growing mode of (44) is in agreement with results given in the literature.

In the small-scale limit $\lambda \to 0$, one finds from (42) and (43)

$$\delta_c(t, q) \approx -\frac{2}{5} \delta_T(t_0, q) + \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \left[ \delta_c(t_0, q) + \frac{2}{5} \delta_T(t_0, q) \right] \cos \left[ 2\mu_m - 2\mu_m \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \right],$$

$$\delta_p(t, q) \approx \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \left[ \frac{2}{5} \delta_c(t_0, q) + \delta_T(t_0, q) \right] \cos \left[ 2\mu_m - 2\mu_m \left(\frac{t}{t_0}\right)^{-\frac{1}{3}} \right],$$

where (36) has been used to calculate the fluctuation $\delta_p$ in the pressure. Thus, density perturbations with scales smaller than the Jeans scale oscillated with a decaying amplitude which was smaller than unity: these perturbations were so small that gravity was insufficient to let perturbations grow. Heat exchange alone was not enough for the growth of density perturbations. Consequently, perturbations with scales smaller than the Jeans scale did never reach the non-linear regime.

In the next section it is shown that for density perturbations with scales of the order of the Jeans scale, the action of both gravity and heat exchange together may result in massive structures several hundred million years after decoupling of matter and radiation.

4 Structure Formation after Decoupling of Matter and Radiation

In this section it is demonstrated that the relativistic evolution equations, which include a realistic equation of state for the pressure $p = p(n, \epsilon)$ yields that in the era after decoupling of matter and radiation density perturbations may have grown fast.

Up till now it is only assumed that $mc^2 \gg k_B T$ for baryons and CDM, without specifying the mass of the baryon and CDM particles. From now on it is convenient to assume that the mass of a CDM particle is of the order of magnitude of the proton mass.

4.1 Observable Quantities

The parameter $\mu_m$ (41) will be expressed in observable quantities, namely the present values of the background radiation temperature, $T_{(0)}(t_p)$, the Hubble parameter, $H(t_p)$, and the redshift at decoupling, $z(t_{dec})$. From now on the initial time is taken to be the time at decoupling of matter and radiation: $t_0 = t_{dec}$, so that $\tau := t/t_{dec}$.

The redshift $z(t)$ as a function of the scale factor $a(t)$ is given by

$$z(t) = \frac{a(t_p)}{a(t)} - 1,$$
where $a(t_p)$ is the present value of the scale factor and $z(t_p) = 0$. For a flat FLRW universe one may take $a(t_p) = 1$. Using the background solutions (38), one finds from (46)

\[ H(t) = H(t_p)[z(t) + 1]^{3/2}, \]
\[ t = t_p[z(t) + 1]^{-3/2}, \]
\[ T_{(0)\gamma}(t) = T_{(0)\gamma}(t_p)[z(t) + 1], \]

where it is used that $T_{(0)\gamma} \propto a^{-1}$ after decoupling, as follows from (10) and (14).

The dimensionless time $\tau := t/t_{\text{dec}}$ can be expressed in the redshift

\[ \tau = \left[ \frac{z(t_{\text{dec}}) + 1}{z(t) + 1} \right]^{3/2}, \]

by using that $\tau = (t/t_p)(t_p/t_{\text{dec}})$ and (47b).

Substituting (28) into (41), one gets

\[ \mu_m = \frac{2\pi}{\lambda_{\text{dec}}} \frac{1}{H(t_{\text{dec}})} \sqrt{\frac{5 k_B T_{(0)\gamma}(t_{\text{dec}})}{3 m c^2}}, \quad \lambda_{\text{dec}} := \lambda a(t_{\text{dec}}), \]

where $t_{\text{dec}}$ is the time when matter decouples from radiation and $\lambda_{\text{dec}}$ is the physical scale of a perturbation at time $t_{\text{dec}}$. From (47) one finds

\[ \mu_m = \frac{2\pi}{\lambda_{\text{dec}}} \frac{1}{H(t_p)} [z(t_{\text{dec}}) + 1] \sqrt{\frac{5 k_B T_{(0)\gamma}(t_p)}{3 m c^2}}, \]

where it is used that $T_{(0)\gamma}(t_{\text{dec}}) = T_{(0)\gamma}(t_p)$. With (50) the parameter $\mu_m$ is expressed in observable quantities.

### 4.2 Initial Values from the Planck Satellite

The physical quantities measured by Planck [14] and needed in the parameter $\mu_m$ (50) of the evolution equation (40) are the redshift at decoupling, the present values of the Hubble function and the background radiation temperature, the age of the universe and the fluctuations in the background radiation temperature. The numerical values of these quantities are

\[ z(t_{\text{dec}}) = 1090, \]
\[ cH(t_p) = H(t_p) = 67.31 \text{ km/sec/Mpc}, \]
\[ T_{(0)\gamma}(t_p) = 2.725 \text{ K}, \]
\[ t_p = 13.81 \text{ Gyr}, \]
\[ \delta T_{\gamma}(t_{\text{dec}}) \lesssim 10^{-5}. \]

Substituting the observed values (51a)–(51c) into (50), one finds

\[ \mu_m = \frac{16.57}{\lambda_{\text{dec}}}, \quad \lambda_{\text{dec}} \text{ in pc}, \]

where it is used that the proton mass is $m = m_H = 1.6726 \times 10^{-27} \text{ kg}$, $1 \text{ pc} = 3.0857 \times 10^{16} \text{ m} = 3.2616 \text{ ly}$, the speed of light $c = 2.9979 \times 10^8 \text{ m/s}$ and Boltzmann’s constant $k_B = 1.3806 \times 10^{-23} \text{ JK}^{-1}$. 

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The Planck observations of the fluctuations $\delta_T(t_{\text{dec}})$, (51e), in the background radiation temperature yield for the initial value of the fluctuations in the energy density

$$|\delta_\epsilon(t_{\text{dec}}, q)| \lesssim 10^{-5}.$$  \hfill (53)

In addition, it is assumed that

$$\dot{\delta}_\epsilon(t_{\text{dec}}, q) \approx 0,$$ \hfill (54)

i.e., during the transition from the radiation-dominated era to the era after decoupling, perturbations in the energy density were approximately constant with respect to time.

During the linear phase of the evolution, $\delta_n(t, q)$ follows from (33) so that the initial values $\delta_n(t_{\text{dec}}, q)$ and $\dot{\delta}_n(t_{\text{dec}}, q)$ need not be specified.

### 4.3 Diabatic Pressure Perturbations

At the moment of decoupling of matter and radiation, photons could not ionize matter any more and the two constituents fell out of thermal equilibrium. As a consequence, the high radiation pressure $p = \frac{4}{3} a_B T^4$ just before decoupling did go over into the low gas pressure $p = n k_B T$ after decoupling. In fact, from (47c) and (59) it follows that at decoupling one has

$$n(t_0)(t_{\text{dec}}) \frac{k_B T(t_0)}{3 a_B T_{0}^4_T(t_{\text{dec}})} = \frac{3 k_B T_0(t_p)}{mc^2} \left[z(t_{\text{eq}}) + 1\right] \approx 2.5 \times 10^{-9},$$ \hfill (55)

where it is used that at the moment of decoupling the matter temperature $T(t_0)(t_{\text{dec}})$ was equal to the radiation temperature $T_{0}^4_T(t_{\text{dec}})$. The redshift at matter-radiation equality was $z(t_{\text{eq}}) = 3393$, Planck \[14\]. The fast and chaotic transition from a high pressure epoch to a very low pressure era may have resulted in large relative diabatic pressure perturbations $\delta_T$, (36), due to very small fluctuations $\delta_{\text{kin}}$, (35), in the kinetic energy density. It will be shown in Section 4.4 that density perturbations which were cooler than their environments may have collapsed fast, depending on their scales. In fact, perturbations for which

$$\delta_T(t_{\text{dec}}, q) \lesssim -0.005,$$ \hfill (56)

may have resulted in primordial stars, the so-called (hypothetical) Population III stars, and larger structures, several hundred million years after the Big Bang.

### 4.4 Structure Formation in the Early Universe

In this section the evolution equation (40) is solved numerically \[15, 16\] and the results are summarized in Figure 1, which is constructed as follows. For each choice of $\delta_T(t_{\text{dec}}, q)$ in the range $-0.005, -0.01, -0.02, \ldots, -0.1$ equation (40) is integrated for a large number of values for the initial perturbation scale $\lambda_{\text{dec}}$ using the initial values (53) and (54). The integration starts at $\tau = 1$, i.e., at $z(t_{\text{dec}}) = 1090$ and will be halted if either $z = 0$, i.e., $\tau = [z(t_{\text{dec}}) + 1]^{3/2}$, see (48), or $\delta_\epsilon(t, q) = 1$ for $z > 0$ has been reached. One integration run yields one point on the curve for a particular choice of the scale $\lambda_{\text{dec}}$ if $\delta_\epsilon(t, q) = 1$ has been reached for $z > 0$. If the integration halts at $z = 0$ and still $\delta_\epsilon(t_p, q) < 1$, then the perturbation pertaining to that particular scale $\lambda_{\text{dec}}$ has not yet reached its non-linear phase today, i.e., at $t_p = 13.81$ Gyr. On the other hand, if the integration is stopped at $\delta_\epsilon(t, q) = 1$ and $z > 0$, then the perturbation has
Figure 1: The curves give the redshift and time, as a function of $\lambda_{\text{dec}}$, when a linear perturbation in the energy density with initial values $\delta_\varepsilon(t_{\text{dec}}, \mathbf{q}) \lesssim 10^{-5}$ and $\dot{\delta}_\varepsilon(t_{\text{dec}}, \mathbf{q}) \approx 0$ starting to grow at an initial redshift of $z(t_{\text{dec}}) = 1090$ has become non-linear, i.e., $\delta_\varepsilon(t, \mathbf{q}) = 1$. The curves are labeled with the initial values of the relative perturbations $\delta T(t_{\text{dec}}, \mathbf{q})$ in the diabatic part of the pressure. For each curve, the Jeans scale, i.e., the peak value, is at 6.5 pc.

The growth of a perturbation was governed by gravity as well as heat exchange. From Figure 1 one may infer that the optimal scale for growth was around 6.5 pc $\approx 21$ ly. At this scale, which is independent of the initial value of the diabatic pressure perturbation $\delta T(t_{\text{dec}}, \mathbf{q})$, see (8) and (36), heat exchange and gravity worked together perfectly, resulting in a fast growth. Perturbations with scales smaller than 6.5 pc reached their non-linear phase at a much later time, because their internal gravity was weaker than for large-scale perturbations and heat exchange was insufficient to enhance the growth. On the other hand, perturbations with scales larger than 6.5 pc exchanged heat at a slower rate due to their large scales, resulting also in a smaller growth rate. Perturbations larger than 50 pc grew proportional to $t^{2/3}$, (44), a well-known result. Since the growth rate decreased rapidly for perturbations with scales below 6.5 pc, this scale will be considered as the relativistic counterpart of the classical Jeans scale. The relativistic Jeans scale at decoupling, $\lambda_{\text{J,dec}} \approx 6.5$ pc, was much smaller than the horizon size at decoupling, $d_H(t_{\text{dec}}) = 3ct_{\text{dec}} \approx 3.5 \times 10^5$ pc $\approx 1.1 \times 10^6$ ly.

4.5 Relativistic Jeans Mass

The Jeans mass at decoupling, $M_{\text{J}}(t_{\text{dec}})$, can be estimated by assuming that a density perturbation has a spherical symmetry with diameter the relativistic Jeans scale $\lambda_{\text{J,dec}} := \lambda_{\text{J,0}}(t_{\text{dec}})$. 
The relativistic Jeans mass at decoupling is then given by

\[ M_J(t_{\text{dec}}) = \frac{4\pi}{3} \left[ \frac{1}{2} \lambda_{J,\text{dec}} \right]^3 n_{(0)}(t_{\text{dec}}) m. \]  (57)

The particle number density \( n_{(0)}(t_{\text{dec}}) \) can be calculated from its value \( n_{(0)}(t_{\text{eq}}) \) at the end of the radiation-dominated era. By definition, at the end of the radiation-domination era the matter energy density \( n_{(0)} mc^2 \) was equal to the energy density of the radiation:

\[ n_{(0)}(t_{\text{eq}}) mc^2 = a_B T_{(0)\gamma}(t_{\text{eq}}). \]  (58)

Since \( n_{(0)} \propto a^{-3} \) and \( T_{(0)\gamma} \propto a^{-1} \), one finds, using (46), (47c) and (58), for the particle number density at the time of decoupling \( t_{\text{dec}} \)

\[ n_{(0)}(t_{\text{dec}}) = \frac{a_B T_{(0)\gamma}(t_p)}{mc^2} \left[ z(t_{\text{eq}}) + 1 \right] \left[ z(t_{\text{dec}}) + 1 \right]^3. \]  (59)

Using (51a), the black body constant \( a_B = 7.5657 \times 10^{-16} \text{J/m}^3/\text{K}^4 \), the redshift at matter-radiation equality, \( z(t_{\text{eq}}) = 3393 \), the redshift at decoupling (51a) Planck [14], and the speed of light \( c = 2.9979 \times 10^8 \text{m/s} \), one finds for the Jeans mass (57) at decoupling

\[ M_J(t_{\text{dec}}) \approx 4.4 \times 10^3 M_\odot, \]  (60)

where it is used that one solar mass \( 1 M_\odot = 1.9889 \times 10^{30} \text{kg} \) and the relativistic Jeans scale \( \lambda_{J,\text{dec}} = 6.5 \text{pc} \), the peak value in Figure 1.

A Why the Standard Equation is inadequate to study Density Perturbations

The standard evolution equation for relative density perturbations \( \delta(t, \mathbf{x}) \) in a flat, FRW universe with vanishing cosmological constant, \( \Lambda = 0 \), reads

\[ \ddot{\delta} + 2H \dot{\delta} - \left[ \beta^2 \nabla^2 + \frac{1}{2} \kappa \varepsilon_{(0)}(1 + w)(1 + 3w) \right] \delta = 0. \]  (61)

In the radiation-dominated universe one has \( \beta^2 = w = \frac{1}{3} \) and this equation is identical to the relativistic equation (15.10.57) in the textbook of Weinberg [17]. Since equation (15.10.57) is derived for large-scale perturbations, i.e., \( \nabla^2 \delta \to 0 \), the term with \( \beta^2 \) does not occur. In the epoch after decoupling of matter and radiation \( \beta \) is given by (28), so that \( w \approx \frac{3}{5} \beta^2 \ll 1 \). In this case (61) is identical to equation (15.9.23) of Weinberg which has been derived using the Newtonian Theory of Gravity.

In this appendix it will be shown that the standard equation is inadequate to study the evolution of density perturbations in the universe. To that end, an exact General Relativistic derivation of equation (61) will be compared with the approximate Newtonian derivation.

A.1 General Theory of Relativity

Since the source term of (61) is zero, this equation describes adiabatic perturbations, Section 3.3*, which evolve only under the influence of their own gravitational field. Therefore, the equation of state is given by \( p = p(\varepsilon) \). This implies that \( p_m = 0 \), so that \( \dot{p}_{(0)} = p_\varepsilon \dot{\varepsilon}_{(0)} \) and \( p_{(1)} = p_\varepsilon \dot{\varepsilon}_{(1)} \).
Consequently, the evolution equations for the background particle number density $n_{(0)}$, (2c), and its first-order perturbation $n_{(1)}$, (41b*), need not be considered. From (6) one finds that $p_\epsilon = \beta^2$, so that $p_{(1)} = \beta^2 \varepsilon_{(1)}$. Using the definition $\delta := \varepsilon_{(1)}/\varepsilon_{(0)}$ and $R_{(0)} = 0$, equations (41*) for scalar perturbations can be written in the form

\[
\dot{\delta} + 3H\delta \left[ \beta^2 + \frac{1}{2}(1 - w) \right] + (1 + w) \left[ \dot{\vartheta}_{(1)} + \frac{R_{(1)}}{4H} \right] = 0, \tag{62a}
\]

\[
\dot{\vartheta}_{(1)} + H(2 - 3\beta^2)\vartheta_{(1)} + \frac{\beta^2}{1 + w} \frac{\nabla^2\delta}{a^2} = 0, \tag{62b}
\]

\[
\dot{R}_{(1)} + 2HR_{(1)} - 2\kappa\varepsilon_{(0)}(1 + w)\vartheta_{(1)} = 0, \tag{62c}
\]

where also the background equations (2), have been used. Differentiating (62a) with respect to time and eliminating the time-derivatives of $H$, $\varepsilon_{(0)}$, $\vartheta_{(1)}$ and $R_{(1)}$ with the help of the system of equations (2) and perturbation equations (62b) and (62c), respectively, and, subsequently, eliminating $R_{(1)}$ with the help of (62a), one finds, using MAXIMA [13], that the set of equations (62) can be recast in the form

\[
\dot{\delta} + 2H\delta \left[ 1 + 3\beta^2 - 3w \right] - \left[ \beta^2 \frac{\nabla^2\delta}{a^2} + \frac{1}{2}\kappa\varepsilon_{(0)} \right] \left( 1 + w \right) \left( 1 + 3w \right)
+ 4w - 6w^2 + 12\beta^2 w - 4\beta^2 - 6\beta^4 \right) - 6\beta^3 H \delta = -3H\beta^2(1 + w)\vartheta_{(1)}, \tag{63a}
\]

\[
\dot{\vartheta}_{(1)} + H(2 - 3\beta^2)\vartheta_{(1)} + \frac{\beta^2}{1 + w} \frac{\nabla^2\delta}{a^2} = 0, \tag{63b}
\]

where $\dot{w}$ has been eliminated using (7). The system (63) consists of two relativistic equations for the unknown quantities $\delta$ and $\vartheta_{(1)}$. Thus, the relativistic perturbation equations (41*) for open, flat or closed FLRW universes and a general equation of state for the pressure $p = p(n, \varepsilon)$ reduce for a flat universe and a barotropic equation of state $p = p(\varepsilon)$ to the system (63).

The gauge function $\psi(x)$ is arbitrary and can in no way be determined. Consequently, given the general solution $\{\delta, \vartheta_{(1)}\}$ of the system (63), it is impossible to extract the physical parts $\{\delta_{\mathrm{phys}}, \vartheta_{(1)\mathrm{phys}}\}$ from the general solution. Furthermore, one cannot impose physical initial conditions $\{\delta_{\mathrm{phys}}(t_0, x), \vartheta_{(1)\mathrm{phys}}(t_0, x)\}$ to the system (63) since this would imply that the gauge function $\psi(x)$ could be determined by physical considerations. Therefore, the system (63) is inadequate to study the evolution of density perturbations. This, incidentally, holds true also for the system (41*). Only the quantities $\varepsilon_{(1)}$ and $n_{(1)}^\varrho$, (40a*), combined with the set (41*) have a physical meaning, as has been shown in Section 2.6*.

The fact that the source term of equation (63a) is non-zero does not mean that the density perturbation is diabatic. It only reflects the fact that the homogeneous part of (63a) contains a
physical solution as well as a non-physical gauge mode and \( \vartheta^{(1)\text{phys}} \) yields the particular physical solution for the density fluctuation. As is well-known, evolving density perturbations, whether or not adiabatic, necessarily have \( \vartheta^{(1)\text{phys}} \neq 0 \). In other words, \( \vartheta^{(1)\text{phys}} \) is an intrinsic property of a density perturbation, whether or not it is isolated. For \( \vartheta^{(1)\text{phys}} = 0 \), density perturbations do not evolve, Section 4. Therefore, the correct configuration is that \( \vartheta^{(1)\text{phys}} \) is contained in the left-hand side of an evolution equation, as is the case in equation (3a). A density perturbation which is isolated from its environment evolves only under its own gravitational field and its evolution is, therefore, described by a homogeneous second-order differential equation. The source term of a second-order differential equation describes external influences. In the present case the source term of (3a) describes heat exchange of a density perturbation with its environment.

The relativistic equations (63) are exact for first-order perturbations. This fact has consequences for the standard evolution equation (61), which will be discussed in detail in the next two paragraphs.

**Radiation-dominated Era.** In this era, the pressure is given by a linear barotropic equation of state \( p = w \varepsilon, \) so that \( p_n = 0 \) and \( p_e = w \). Since \( p_e = \beta^2 \), (6), one finds from (7) that \( \beta^2 = w \) is constant. In the case of a radiation-dominated universe this constant is \( w = \beta^2 = \frac{1}{3} \). For a linear barotropic equation of state \( p = w \varepsilon \) equations (63) reduce to

\[
\dot{\delta} + 2H \delta - \left[ w \frac{\nabla^2}{a^2} + \frac{1}{2} \kappa \varepsilon_{(0)} (1 + w)(1 + 3w) \right] \delta = -3Hw(1 + w)\vartheta^{(1)}, \tag{66a}
\]

\[
\dot{\vartheta}^{(1)} + H(2 - 3w)\vartheta^{(1)} + \frac{w}{1 + w} \frac{\nabla^2 \delta}{a^2} = 0. \tag{66b}
\]

The gauge modes (64) are solutions of the system (66) for \( \dot{\varepsilon} = 0 \).

For large-scale perturbations one has \( \nabla^2 \delta_{\text{phys}} \rightarrow 0 \) (this does not necessarily imply that \( \nabla^2 \psi \rightarrow 0 \), since \( \psi(x) \) is an arbitrary function). Using that \( w = \frac{1}{3} \), the solutions (14) of the background equations imply that (66b) yields the physical solution \( \vartheta^{(1)\text{phys}} \propto t^{-1/2} \), so that with (14) one has \( H\vartheta^{(1)\text{phys}} \propto t^{-3/2} \). Therefore, the particular solution of (66a) is \( \delta_{\text{phys}} \propto t^{1/2} \). The solutions of the homogeneous part of (66a) are \( \delta_{\text{phys}} \propto t \) and the gauge mode \( \delta \propto t^{-1} \). This explains the physical modes \( \delta_{\text{phys}} \propto t^{1/2} \) and \( \delta_{\text{phys}} \propto t \) in (20). The standard equation (61) has only one physical mode \( \delta_{\text{phys}} \propto t \) as solution. The physical mode \( \delta_{\text{phys}} \propto t^{1/2} \) cannot be found from the standard equation since \( \vartheta^{(1)\text{phys}} \) is missing in its source term. The fact that equation (13a) yields the solutions \( \delta_{\varepsilon} \propto t^{1/2} \) and \( \delta_{\varepsilon} \propto t \) is a consequence of the fact that \( \vartheta^{(1)\text{phys}} \) forms part of this equation. Hence the differences in appearance between (13a) and (61).

The fact that \( \vartheta^{(1)\text{phys}} \) is absent in the right-hand side of (61) is detrimental to cosmological perturbation theory. Since \( \nabla^2 \delta_{\text{phys}} \) could have been large for small-scale perturbations, the evolution of \( \delta_{\text{phys}} \) may have a large influence on \( \vartheta^{(1)\text{phys}} \) and this may have, in turn, a major impact on the evolution of \( \delta_{\text{phys}} \). This is why (13a) yields oscillating density perturbations with an increasing amplitude — which was, in fact, the real physical behavior — instead of a constant amplitude as follows from (61). Since the standard equation (61) is incomplete, it is inadequate to study small-scale density perturbations in the radiation-dominated era.

**Era after Decoupling of Matter and Radiation.** In this era, the equation of state for the pressure is, according to thermodynamics, given by (26), so that in this case one has \( p \neq p(\varepsilon) \). The case \( p = 0 \) is not considered, since \( p = 0 \) yields the non-relativistic limit, as is shown in
Section 4*. Nonetheless, (61) follows from (63a) since after decoupling $\beta^2$ is given by (28) so that $w \approx \frac{2}{9} \beta^2 \ll 1$. This implies with (29) that $\beta / \beta = -H$. Using that $3H^2 = \kappa \varepsilon \varepsilon (0)$, (2a), one gets $6 \beta H = -2 \kappa \varepsilon \varepsilon (0) \beta^2$. Substituting the latter expression into (63a) and neglecting $w$ and $\beta^2$ with respect to constants of order unity, the system (63) reduces to

$$\dot{\delta} + 2H \delta - \left[ \beta^2 \frac{\nabla^2}{a^2} + \frac{1}{2} \kappa \varepsilon \varepsilon (0) \right] \delta = -3H \beta^2 \dot{\vartheta} (1),$$  \hspace{1cm} (67a)

$$\dot{\vartheta} (1) + 2H \vartheta (1) + \beta^2 \frac{\nabla^2 \delta}{a^2} = 0.$$  \hspace{1cm} (67b)

The gauge modes (64) are solutions of the system (67) for $w \ll 1$ and $\nabla^2 \psi = 0$, as can verified by substitution. Consequently, for the system (67) $\psi$ is an arbitrary infinitesimal constant $C$. This implies that $\vartheta (1) = \vartheta (1) \text{phys}$ is a physical quantity, since its gauge mode $\vartheta (1)$, (64), vanishes identically. However, $\delta$ is still gauge-dependent with gauge mode $\delta = -3H(t)C \propto t^{-1}$, (38) and (64). This fact is in accordance with the residual gauge transformation (64*)

$$x^0 \mapsto x^0 - C, \quad x^i \mapsto x^i - \chi^i (x),$$  \hspace{1cm} (68)

in the non-relativistic limit, since a cosmological fluid for which $w \ll 1$ and $\beta^2 \ll 1$ can be described by non-relativistic equations of state (25). Since the homogeneous part of equation (67a) has the gauge mode $\delta$ as solution, the standard equation (61) yields for all scales gauge-dependent solutions.

Using the solutions (38) of the background equations one finds that for large-scale perturbations, $\nabla^2 \delta \text{phys} \rightarrow 0$, equation (67b) yields the physical solution $\vartheta (1) \text{phys} \propto t^{-4/3}$, so that with $\beta \propto a^{-1}$ one finds that $H \beta^2 \vartheta (1) \text{phys} \propto t^{-11/3}$. Therefore, the particular solution of (67a) is $\delta \text{phys} \propto t^{-5/3}$. The solutions of the homogeneous part of equation (67a) are $\delta \text{phys} \propto t^{2/3}$ and the gauge mode $\delta \propto t^{-1}$. This explains the two physical modes in (44). The physical solution $\delta \text{phys} \propto t^{-5/3}$ cannot be found from (61), since $\vartheta (1) \text{phys}$ is absent in the source term of this equation. The fact that equation (30a) yields the solutions $\delta \propto t^{2/3}$ and $\delta \propto t^{-5/3}$ is a consequence of the fact that $\vartheta (1) \text{phys}$ forms part of the left-hand side of this equation. Hence the differences in appearance between the left-hand side of (30a) and (61).

Just as in the radiation-dominated era, the standard equation (61) lacks the quantity $\vartheta (1) \text{phys}$ in its source term. Although $\nabla^2 \delta \text{phys}$ could have been large for small-scale density perturbations in the early universe the absence of $\vartheta (1) \text{phys}$ is not as harmful as it is in the radiation-dominated phase: due to the smallness of $\beta^2$ and the non-relativistic particle velocities after decoupling, the impact of $\vartheta (1) \text{phys}$ on the evolution of $\delta \text{phys}$ is fairly low. This explains why both (61) and the homogeneous part of (30a) yield oscillating perturbations with a decreasing amplitude as can be inferred from (42) with $\delta_T = 0$.

Finally, in contrast to equations (3), the standard equation (61) is not adapted to a general, realistic, equation of state $\rho = \rho (n, \varepsilon)$. Therefore, equation (61) is incomplete. That is why this equation does not explain structure formation in the universe. It has to be concluded that the standard equation (61) is inadequate to study the evolution of density perturbations in the universe in the era after decoupling of matter and radiation.

A.2 Newtonian Theory of Gravity

It is generally assumed that if the energy density is dominated by non-relativistic particles, so that $w \ll 1$, and if the linear scales involved are small compared with the characteristic scale
$H^{-1}$ of the universe, then one may safely use the Newtonian Theory of Gravity to study the evolution of density perturbations.

The perturbation equations of the (Newtonian) Jeans theory adapted to an expanding universe after decoupling are given by Weinberg [17], equations (15.9.12)–(15.9.16). Substituting $v_1 := a u(1)$, $\rho := \epsilon(0)$ and $\rho_1 := \epsilon(1) = \epsilon(0)\delta$ in these equations and taking the divergence of (15.9.13), one arrives at the Newtonian equations in the notation used in the present article:

\begin{equation}
\dot{\delta} + \vartheta(1) = 0,
\end{equation}

\begin{equation}
\dot{\vartheta}(1) + 2H\vartheta(1) + \beta^2 \frac{\nabla^2 \delta}{a^2} + \frac{\nabla^2 \phi}{a^2} = 0,
\end{equation}

\begin{equation}
\frac{\nabla^2 \phi}{a^2} = \frac{1}{2} \kappa \epsilon(0) \delta,
\end{equation}

where the energy conservation law (2b) with $w \ll 1$ has been used. Differentiating (69a) with respect to time and eliminating $\dot{\vartheta}(1)$ with the help of (69b) and, subsequently, eliminating $\vartheta(1)$ and $\nabla^2 \phi$ with the help of (69a) and (69c), respectively, yields

\begin{equation}
\ddot{\delta} + 2H\dot{\delta} - \left[\beta^2 \frac{\nabla^2}{a^2} + \frac{1}{2} \kappa \epsilon(0)\right] \delta = 0,
\end{equation}

which is precisely the standard equation (61) for $w \ll 1$ and $\beta^2 \ll 1$.

Since the Newtonian Theory of Gravity is invariant under the gauge transformation (68), one has $\psi(x) = C$, just as in the relativistic case (67). Consequently, $\vartheta(1) = \vartheta(1)_{\text{phys}}$ is a physical quantity. From equation (69a) it follows that $\delta$ is also a physical quantity. Therefore, one is tempted to conclude that the standard equation (70) is free from spurious gauge modes, so that $\delta := \epsilon(1)/\epsilon(0)$ would describe the evolution of density perturbations correctly. This would imply that $\dot{\delta} = -3H(t)C \propto t^{-1}$ is a physical solution of the Newtonian equation (70), notwithstanding the occurrence of the gauge constant $C$. This discrepancy will be discussed in the next section.

### A.3 Relativistic versus Newtonian Perturbation Theory

In Section A.1 it has been shown that the left-hand side of the standard equation (61) follows from the General Theory of Relativity. In Section A.2 the Newtonian Theory of Gravity has been used to derive the standard equation for $w \ll 1$ and $\beta$ given by (28).

In the radiation-dominated phase of the universe, the energy density and pressure of the universe is described by relativistic equations of state (10). Consequently, (61) is in this case a relativistic equation which has no Newtonian equivalent. Therefore, its general solution is of the form (65) with $\dot{\delta}$, (64), the non-physical gauge mode.

After decoupling of matter and radiation the matter content of the universe can be described by non-relativistic equations of state (25). In this case, the velocities of the particles are low and the scale of a perturbation is small. According to the literature, it would, therefore, be feasible to derive an evolution equation for density perturbations using the Newtonian Theory of Gravity, if one takes the expansion of the universe into account. The result is equation (70). Again, the general solution of (70) is of the form (65). Since there is no gauge problem in the Newtonian Theory, the solution $\dot{\delta} = -3H(t)C \propto t^{-1}$ would be a physical solution. However, equation (70) can also be derived from the General Theory of Relativity, as follows from the derivation of (67). In this case, $\dot{\delta} = -3H(t)C \propto t^{-1}$ is a gauge mode which has no
physical significance whatsoever. As a consequence, the Newtonian perturbation equation (70) contradicts the relativistic perturbation equation (67a).

However, the General Theory of Relativity is superior to the Newtonian Theory of Gravity. Therefore, $\delta$ is indeed a gauge mode, as will now be explained. In the non-relativistic limit the relativistic gauge transformation $x^\mu \mapsto x^\mu - \xi^\mu(t, x)$ reduces to the Newtonian gauge transformation (68). This implies that the general solution $\delta$ of the homogeneous part of the relativistic equation (67a) is gauge-dependent also in the Newtonian Theory of Gravity. Consequently, the Newtonian equation (70) has a non-physical gauge mode as solution. In other words, the fact that $\delta$ is gauge-dependent in the General Theory of Relativity implies that $\delta$ is also gauge-dependent in the non-relativistic limit. That is why the gauge-invariant quantities (40a*)

$$\varepsilon_{(1)}^{gi} := \varepsilon_{(1)} - \frac{\dot{\varepsilon}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)}, \quad n_{(1)}^{gi} := n_{(1)} - \frac{\dot{n}_{(0)}}{\dot{\theta}_{(0)}} \theta_{(1)},$$

(71)

which are shown [1] to be the unique, real and measurable energy density perturbation and particle number density perturbation, can not become equal to the gauge-dependent quantities $\varepsilon_{(1)}$ and $n_{(1)}$ in the non-relativistic limit, since the latter two quantities are still gauge-dependent in the non-relativistic limit. As has been shown in Section 4*, both $\varepsilon_{(1)}^{gi}$ and $n_{(1)}^{gi}$ become equal to their Newtonian counterparts in the non-relativistic limit. This demonstrates that there is indeed no gauge problem in the Newtonian Theory of Gravity, since $\varepsilon_{(1)}^{gi}$ and $n_{(1)}^{gi}$ are invariant under the gauge transformation (68) of the Newtonian Theory.

Since the Jeans perturbation theory adapted to a non-static universe yields non-physical solutions, it should be concluded that the Newtonian Theory of Gravity is inadequate to study the evolution of density perturbations in a non-static universe.

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References

[1] P. G. Miedema. General Relativistic Evolution Equations for Density Perturbations in Closed, Flat and Open FLRW Universes. http://arxiv.org/abs/1410.0211.

[2] E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright. Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. Astrophys. J. Suppl., 192:18, February 2011. doi: 10.1088/0067-0049/192/2/18. http://arxiv.org/abs/1001.4538.

[3] D. Watson, L. Christensen, K. Kraiberg Knudsen, J. Richard, A. Gallazzi, and M. Jerzy Michałowski. A dusty, normal galaxy in the epoch of reionization. Nature, March 2015. doi: 10.1038/nature14164. http://arXiv.org/abs/1503.00002.
[4] D. Sobral, J. Matthee, B. Darvish, D. Schaerer, B. Mobasher, H. J. A. Röttgering, S. Santos, and S. Hemmati. Evidence for PopIII-like Stellar Populations in the Most Luminous Lyman-α Emitters at the Epoch of Reionization: Spectroscopic Confirmation. *ApJS*, 808:139, August 2015. doi: 10.1088/0004-637X/808/2/139. http://arXiv.org/abs/1504.01734.

[5] T. M. Nieuwenhuizen, C. H. Gibson, and R. E. Schild. Gravitational Hydrodynamics of Large Scale Structure Formation. *Europhysics Letters*, jun 2009. doi: 10.1209/0295-5075/88/49001. http://arXiv.org/abs/0906.5087.

[6] E. M. Lifshitz and I. M. Khalatnikov. Investigations in Relativistic Cosmology. *Adv. Phys.*, 12:185–249, 1963. doi: 10.1080/00018736300101283.

[7] P. J. Adams and V. Canuto. Exact Solution of the Lifshitz Equations Governing the Growth of Fluctuations in Cosmology. *Physical Review D*, 12(12):3793–3799, 1975. doi: 10.1103/PhysRevD.12.3793.

[8] D. W. Olson. Density Perturbations in Cosmological Models. *Physical Review D*, 14(2):327–331, 1976. doi: 10.1103/PhysRevD.14.327.

[9] P. J. E. Peebles. *The Large-Scale Structure of the Universe*. Princeton University Press, New Jersey, 1980. ISBN 978-0691082400.

[10] E. W. Kolb and M. S. Turner. *The Early Universe*. Westview Press, 1994. ISBN 978-0201626742.

[11] W. H. Press and E. T. Vishniac. Tenacious Myths about Cosmological Perturbations larger than the Horizon Size. *The Astrophysical Journal*, 239:1–11, July 1980. doi: 10.1086/158083.

[12] H. Kodama and M. Sasaki. Cosmological Perturbation Theory. *Progress of Theoretical Physics Supplements*, 78:1–166, 1984. doi: 10.1143/PTPS.78.1.

[13] Maxima. Maxima, a Computer Algebra System. Version 5.32.1, 2014. http://maxima.sourceforge.net.

[14] Planck Collaboration, P. A. R. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. J. Banday, R. B. Barreiro, J. G. Bartlett, and et al. Planck 2015 results. XIII. Cosmological parameters. http://arxiv.org/abs/1502.01589.

[15] Karline Soetaert, Thomas Petzoldt, and R. Woodrow Setzer. Solving Differential Equations in R: Package deSolve. *Journal of Statistical Software*, 33(9):1–25, 2010. ISSN 1548-7660. http://www.jstatsoft.org/v33/i09.

[16] R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2015. http://www.R-project.org.

[17] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. John Wiley & Sons, Inc. New York, 1972. ISBN 978-0471925675.
library(deSolve) # load package 'deSolve' to use the solver 'lsodar' at line 109

M_J <- (4/3)*pi*((1/2)*Js*parsec)^3*n_dec*m / m_sun  # (57)

# Calculation of the Jeans mass expressed in sun's mass:
#                Structure Formation in the Early Universe
# Then the "R" file will be send to the reader upon request:
# P.G.Miedema

# Program to calculate Figure 1 in the main text
# 

# dev.off() # close the plotfile

stop()  

box()

# perturbations with scales outside the interval [0.5, 60] parsec do not become
# non-linear within 13.81 Gyr:

result <- lsodar(y, tau.start.end, fun=equation.40, rootfun=stop.conditions, parms)

lambda.nonlin_max <- vector()

# Only the end values, i.e., result[2,..., are needed:

if (round(delta_T, 0)==1.0)
  { i <- i+1
    lambda.nonlin[i] <- lambda_dec
  }

Jeans.scale[k] <- lambda.nonlin_max

# perturbations with scales outside the interval [0.5, 60] parsec do not become
# non-linear within 13.81 Gyr:

# (57)