Combining Kasparov’s theorem of Voiculescu and Cuntz’s description of $KK$-theory in terms of quasihomomorphisms, we give a simple construction of the Kasparov product. This will be used in a more general context of locally convex algebras in order to treat products of certain universal cycles.

1 Introduction

The goal of this note is to establish existence of the Kasparov product based on Kasparov’s theorem of Voiculescu ([Kas80a]), and to examine how this construction is related to the one used by Kasparov.

In the first section, we interpret the connection condition and the existence of Kasparov product ([Kas80b]) as the existence of a certain extension of a quasihomomorphism ([Cun87]). Such extensions always exist, as can be seen by applying split exactness of $KK$ to a certain algebra $D_\alpha$ that is a semidirect product of the domain and target of a quasihomomorphism. The resulting description of the Kasparov product already yields a useful way to construct the Kasparov product; it is particularly well adapted to generalisations of the bimodule-formalism to locally convex algebras, where it may be used to calculate products of certain "smooth" submodules, and is used in [Gre] in a crucial manner.

In the second section, it is shown that, without making use of split exactness of $KK$, one can, in case that Kasparov’s theorem of Voiculescu is available, construct the product by using this interpretation. First we show how to reduce quasihomomorphisms to a single morphism and a unitary; and if an absorbing morphism is chosen, all classes of quasihomomorphisms are obtained from it by conjugation by a unitary. Applying this to a pair of composable quasihomomorphisms, we see that it suffices to extend quasihomomorphisms to just...
one 'universal' algebra; if further the domain or target of the first quasihomomorphism is nuclear, there is a canonical way to extend quasihomomorphisms.

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## 2 The Kasparov product revisited

Quasihomomorphisms were introduced by Cuntz in [Cun83] and further developed in [Cun87].

**Definition 1.** Let $B$ be stable, $\hat{B}$ a $C^*$-algebra containing $B$ as an ideal; then a quasihomomorphism from $A$ to $B$ is a pair of homomorphisms from $A$ to $\hat{B}$ such that $\alpha(a) - \bar{\alpha}(a) \in B$ for all $a \in A$.

For nonstable $B$, a quasihomomorphism from $A$ to $B$ is by definition a quasihomomorphism from $A$ into the stabilisation $\mathbb{K} \otimes B$ of $B$.

Let $(E, \varphi, F)$ be a Kasparov $(A, B)$-module with $A$ and $B$ trivially graded. If $F$ is selfadjoint and invertible, then with respect to the grading:

$$\varphi = \begin{pmatrix} \varphi^{(0)} \\ \varphi^{(1)} \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} T & T^{-1} \end{pmatrix}$$

where the $\varphi^{(i)}$ are homomorphisms $A \to \mathbb{B}(E^{(i)})$ and $T$ is by hypothesis a unitary in $\mathbb{B}_B(E^{(0)}, E^{(1)})$. Hence we obtain a quasihomomorphism $(\alpha, \bar{\alpha}) := (\varphi^{(0)}, T^{-1}\varphi^{(1)}T)$ from $A$ to $\mathbb{K}_B(E^{(0)})$ simply by identifying $E^{(0)}$ and $E^{(1)}$ via $T$, and where we view the latter as a subalgebra of $\mathbb{K} \otimes B$ via the stabilization-theorem.

We may always reduce to this case by using the standard simplifications in $KK$-theory, and therefore we can define an associated quasihomomorphism $Qh(x)$ to every Kasparov module.

The original construction of the Kasparov product from [Kas80b] was quite technical. We will use the version based on the notion of connection introduced by Connes and Skandalis. We fix the following setting: Let $E_1$ be a graded Hilbert $B$-module, $E_2$ a graded Hilbert $C$-module, $\varphi : B \to \mathbb{B}_C(E_2)$ a $*$-homomorphism and $F$ an odd selfadjoint operator on $E_2$. We set $E_{12} := E_1 \otimes_B E_2$, and define for every $x \in E_1$ an operator $T_x : E_2 \to E_1 \otimes_B E_2$, $y \mapsto x \otimes y$.

Note that the adjoint of $T_x$ is given by the mapping $E_{12} \to E_2$, $y \otimes z \mapsto \varphi((x|y))z$, and $T_xT_x^* = \theta_{x,x} \otimes \text{id}_{E_2}$. 
Definition 2. An $E_1$-connection for an odd operator $F$ is an odd selfadjoint operator $G$ such that for all homogeneous $x \in E_1$

\[ T_x F - (-1)^{\partial_x}GT_x \in K_C(E_2, E_{12}) \text{ and } FT_x^* - (-1)^{\partial_x}T_x^*G \in K_C(E_{12}, E_2). \]

As a consequence of the stabilisation theorem, such connections exist in case we deal with Kasparov modules; more precisely:

Proposition 3. If $E_1$ is countably generated and $[F, b]$ is compact for all $b \in B$, then there exists an odd $E_1$-connection for $F$.

If $(E_1, \varphi_1, F_1)$ is a Kasparov $(A, B)$-module, $(E_2, \varphi_2, F_2)$ a Kasparov $(B, C)$-module, $G$ an $F_2$-connection for $E_1$, then $(E_{12}, \text{id}_{K_B(E_1)} \otimes 1, G)$ is a Kasparov $(K_B(E_1), C)$-module.

The existence statement stems from [CS84]; the second fact was stated in [Ska84], Proposition 9.

The composition product is given in terms of the representatives of the cycles involved: If $(\varphi_1, E_1, F_1)$ is a Kasparov $(A, B)$-module and $(\varphi_2, E_2, F_2)$ a Kasparov $(B, C)$-module, then a $F_2$-connection for $E_1$, then $(E_{12}, \text{id}_{K_B(E_1)} \otimes 1, G)$ is called a product of $(E_1, \varphi_1, F_1)$ and $(E_2, \varphi_2, F_2)$ if

(i) $F_{12}$ is an $E_1$-connection for $F_2$ (connection condition)

(ii) For all $a \in A$, $\varphi_1(a) \otimes 1[F_1 \otimes 1, F_{12}]\varphi_1(a)^* \otimes 1$ is positive in the quotient $B_C(E_{12})/K_C(E_{12})$ (positivity condition).

The set of operators $F_{12}$ satisfying the above conditions will be denoted $F_1 \# F_2$.

Using Kasparov’s technical theorem, one can show that a product as above always exists if $A$ is separable, is unique up to operator homotopy, and passes to homotopy classes (cf. [Ska84]).

Recall also that a Hilbert $B$-module $E$ is called full if the linear span of $\langle E | E \rangle$ is dense in $B$.

Definition 4. Let $A$ and $B$ be graded $C^*$-algebras. A graded Morita(-Rieffel) equivalence between $A$ and $B$ is given by a graded full Hilbert $B$-module $E$, called the equivalence bimodule, and a graded isomorphism $\varphi : A \rightarrow K_B(E)$.

We identify $A$ with $K(E)$ and drop the isomorphism $\varphi$. If $E$ is a graded Morita equivalence bimodule from $A = K_B(E)$ to $B$, then we define the $(B, K(E))$-module $E^*: = K(E, B)$. The $K(E)$-valued scalar product is simply $\langle T | S \rangle := R^*S$, and this makes $E^*$ into a graded Hilbert $K(E)$-module.

Let $A$ and $B$ be separable. Then the class $[(E, \text{id}_A, 0)]$ of the equivalence bimodule yields a $KK$ equivalence from $A$ to $B$ with inverse $[(E^*, \text{id}_B, 0)]$. 

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*A note on Kasparov products*
Conversely, any given full Hilbert $B$-module $E$ may be viewed as a graded Morita equivalence from $\mathbb{K}_B(E)$ to $B$.

If $y = (E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, $E_1$ is a Hilbert $B$-module, and $w$ denotes the Kasparov module defined by the Morita equivalence determined by $E_1$, then the operator $G$ in a product $w \cap x$ is exactly an $E_1$-connection for $F_2$, as the positivity condition is trivially satisfied.

If $x = (E_1, \varphi_1, F_1)$ and $v = (E_1^*, \text{id}_B, 0)$ is the inverse of $w$, then the product $x \cap v$ is represented by $(\mathbb{K}_B(E_1), \varphi_1, F_1)$, where the bounded operators on $E$ are considered to act on the Hilbert $\mathbb{K}_B(E_1)$-module by multiplication. This is easily seen by using the explicit form of the isomorphism $U : E_1 \hat{\otimes}_BE_1^* \to \mathbb{K}_B(E_1)$ given above, as $U^T \theta_{E_1}(|\xi\rangle|\eta\rangle) = |T\xi\rangle|\eta\rangle$ for all $T \in \mathbb{B}(E)$. Hence compact operators on $E$ act again by compact operators on $\mathbb{K}_B(E)$, and therefore $(\mathbb{K}_B(E_1), \varphi, F_1)$ does indeed define a cycle, the connection condition is obvious, and positivity follows from $a[F_1, F_1]a^* = a(2F_1^2)a^* = aa^*$ modulo compacts.

We fix two Kasparov bimodules $(E_1, \varphi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \varphi_2, F_2) \in \mathbb{E}(B, C)$, and denote their classes in $KK$ by $x$ and $y$. The module $E_1$ is seen as a Morita equivalence from $\mathbb{K}_B(E_1)$ to $B$, whose class in $KK$ we denote by $w$, and its inverse by $v$. Let $(\alpha, \bar{\alpha}) : A \to D \boxtimes \mathbb{K}_B(E_1)$ be the quasihomomorphism associated to $x' := x \cap v$, and recall that $y' := w \cap y$ may be viewed as the class of the Kasparov module defined via an $E_1$ connection for $F_2$. If we define $D_\alpha$ as the sub-$C^*$-algebra of $A \oplus D$ generated by $(\alpha, \alpha(a))$ and $0 \oplus B$, $a \in A$, we obtain the double split short exact sequence

$$0 \to \mathbb{K}_B(E_1) \xrightarrow{\iota} D_\alpha \xrightarrow{\text{id}_A \oplus \bar{\alpha}} A \to 0$$

which in turn, by split exactness of $KK$, yields a long exact sequence

$$0 \to KK(A, C) \to KK(D_\alpha, C) \to KK(\mathbb{K}_B(E_1), C) \to 0.$$ 

We may thus assume that $y' = \iota^* z$ for some $z \in KK(D_\alpha, C)$. We claim that $\alpha^*(z) - \bar{\alpha}^*(z) = y \cap x$. This follows as $KK((\alpha, \bar{\alpha}), C)$ is multiplication by $x'$ on the left, and therefore

$$x \cap y = x' \cap y' = KK((\alpha, \bar{\alpha}), C)(y') = (\alpha^* - \bar{\alpha}^*)(\iota^*)^{-1}\iota^*(z) = (\alpha^* - \bar{\alpha}^*)(z).$$

Calculating a representative for the last expression, we have thus proved:

**Theorem 5.** Let $x \in KK(A, B)$, $y = [(E_2, \varphi_2, F_2)] \in KK(B, C)$. Then the Kasparov product of $x$ and $y$ may be defined by
(i) representing \( x \) as a quasihomomorphism \((\alpha, \bar{\alpha}) : A \rightrightarrows B \Rightarrow K_B(E_1)\).

(ii) choosing an \( E_1 \) connection \( G \) for \( F_2 \)

(iii) lifting the Kasparov \((K_B(E_1), C)\)-module \((E_1 \otimes_B E_2, \text{id}_{K_B(E_1)} \otimes 1, G)\) along the canonical inclusion of \( K_B(E_1) \to D_\alpha \) to a Kasparov \((D_\alpha, C)\)-module \((\hat{\varphi}, \hat{E}, \hat{G})\),

(iv) and setting

\[
x \cap y := \left[ \left( \left( \varphi \circ \alpha \circ \bar{\alpha} \circ \varepsilon \right), \hat{E} \oplus \hat{E}^{op}, \left( \hat{G} \right) \right) \right] \in KK(A, C)
\]

where \( E^{op} \) denotes the Hilbert \( B \)-module \( E \) with inversed grading, and \( \varepsilon \) the grading operator on \( E \).

Here (iii) means exactly that the quasihomomorphism

\[ Qh(E_1 \otimes_B E_2, \text{id}_{K_B(E_1)} \otimes 1, G) \]

extends to a quasihomomorphism on the larger algebra \( D_\alpha \); note that the class of the cycle \( x \cap y \) as defined above is independent of the choice of the extension.

3 Reduction of quasihomomorphisms and a construction of the Kasparov product

For a given linear map \( \varphi : A \to B \) of \( E \), where \( E \) is a Hilbert \( B \)-module, we define \( E^\infty := \bigoplus_{n=1}^{\infty} E \), and \( \varphi^\infty : A \to B \) as the diagonal action of \( \varphi \).

**Proposition 6.** The class of every quasihomomorphism is represented by a quasihomomorphism of the form \((\alpha, U \circ \alpha)\), where \( U \) is a unitary.

**Proof.** Let \((\alpha, \bar{\alpha}) : A \rightrightarrows B \gtrless B \) be a quasihomomorphism. We may assume that \( B = B(E) \) and \( B = K_B(E) \) for some Hilbert \( B \)-module \( E \). We may replace \((\alpha, \bar{\alpha})\) by

\[(\alpha \oplus a^\infty \oplus \bar{a}^\infty, \alpha \oplus a^\infty \oplus \bar{a}^\infty) : A \to B \oplus E^\infty \oplus E^\infty \supset K_B(E \oplus E^\infty \oplus E^\infty)\]

because \((a^\infty \oplus \bar{a}^\infty, a^\infty \oplus \bar{a}^\infty)\) is degenerate.

Now let \( U \) be the unitary on \( E \oplus E^\infty \oplus E^\infty \) that maps

\[
(\xi_0, (\xi_1, \xi_2, \ldots), (\eta_1, \eta_2, \ldots)) \to (\xi_1, (\xi_2, \xi_3, \ldots), (\xi_0, \eta_1, \eta_2, \ldots)).
\]

Then

\[(a(a) \oplus a^\infty(a) \oplus \bar{a}^\infty(a))U = U(a(a) \oplus a^\infty(a) \oplus \bar{a}^\infty(a)).\]
Definition 7. Let $A$ and $B$ be $C^*$-algebras, $\beta : A \to \mathbb{B}(\mathcal{H}_B)$ a $*$-homomorphism such that for every $*$-homomorphism $\alpha : A \to \mathbb{B}(\mathcal{H}_B)$ there exists a unitary $U$ with $\alpha \oplus \beta = U^*\beta U$ modulo compact operators. Then $\beta$ will be called absorbing.

The following theorem was proved in [Kas80a]:

**Theorem 8 (Kasparov-Voiculescu).** Let $A$ and $B$ be separable $C^*$-algebras and $\beta_0 : A \to \mathbb{B}(\mathcal{H})$ a faithful representation of $A$ such that $(\tilde{\beta}_0)^{-1}(\mathbb{K}(\mathcal{H})) = \{0\}$. We denote by $\beta$ the inclusion of $A$ into $\mathbb{B}_B(\mathcal{H}_B)$ obtained from $\beta_0$ by viewing $\mathbb{B}(\mathcal{H})$ as a subalgebra of $\mathbb{B}_B(\mathcal{H}_B)$. If either $A$ or $B$ is nuclear, then $\beta$ is absorbing.

In general, there is a result of Thomsen from [Tho01], Theorem 2.7, which shows that for $A$ and $B$ separable, there is an absorbing homomorphism from $A$ into the stable multiplier algebra $M(B \otimes \mathbb{K})$ of $B$.

**Lemma 9.** Let $(\alpha, \alpha^U) : A \rightrightarrows \mathcal{B} \ni B$ be a quasihomomorphism, and $\beta : A \to \mathcal{B}$ a homomorphism such that $\alpha(a) - \beta(a) \in B$ for all $a$. Then $(\beta, \beta^U)$ is a quasihomomorphism equivalent to $(\alpha, \alpha^U)$.

**Proof.** Using the usual rotation matrices, we obtain a path of unitaries

$$U_t := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$ 

Reparametrizing, we get a homotopy

$$(\alpha \oplus \beta, \text{Ad}_{U_t} \circ \alpha \oplus \beta)$$

of the quasihomomorphisms $(\alpha, \text{Ad}_U \circ \alpha) \oplus (\beta, \beta)$ and $(\alpha, \alpha) \oplus (\beta, \text{Ad}_U \circ \beta)$.

**Proposition 10.** Let $\beta : A \to \mathbb{B}(\mathcal{H}_B)$ be absorbing. Then every element of $KK(A, B)$ is represented by a quasihomomorphism of the form

$$(\beta, \text{Ad}_U \circ \beta) : A \to \mathbb{B}_B(\mathcal{H}_B) \ni \mathbb{K} \otimes B,$$

where $U \in \mathbb{B}_B(\mathcal{H}_B)$ is a unitary.

**Proof.** By Proposition we may assume that we are given a quasihomomorphism $(\alpha, \alpha^U) : A \rightrightarrows \mathbb{B}_B(\mathcal{H}_B) \ni \mathbb{K}_B(\mathcal{H}_B)$, where $U$ is a unitary in $\mathbb{B}_B(\mathcal{H}_B)$. Let $V$ be a unitary such that $\alpha \oplus \beta = V^*\beta V$. Then we get

$$(\alpha, \alpha^U) \sim (\alpha \oplus \beta, \alpha^U \oplus \beta) \sim (\beta^V, \beta^V(\mathbb{U} \oplus 1) V^*) \sim (\beta, \beta^V(U \oplus 1)V^*)$$

by the above Lemma.
A note on Kasparov products

Corollary 11. Let $A$, $B$, $C$ be separable $C^*$-algebras, $\beta$ as in the above Proposition absorbing, $(\gamma, \bar{\gamma})$ a quasihomomorphism from $B$ to $C$. Then it suffices to find an extension of $(\gamma, \bar{\gamma})$ to the one algebra $D_{\beta}$, in order to calculate explicitly all products of $(\gamma, \bar{\gamma})$ with elements from $KK(A, B)$ (as in [2]).

One can use these ideas to construct the Kasparov product in good cases:

Let $(\alpha, \bar{\alpha}) : A \to B \otimes K(H)$ and $(\beta, \bar{\beta}) : B \to \hat{C} \supseteq C$ be another quasihomomorphism. We may extend $(\beta, \bar{\beta})$ to a quasihomomorphism $(\beta', \bar{\beta}') : 1 \otimes B(H) + B \otimes K(H) \to \mathcal{M}(\hat{C} \otimes K(H)) \supseteq C \otimes K(H)$ by first stabilizing and then setting $\beta'(1 \otimes T + x) := 1 \otimes T + \beta \otimes \text{id}_K(x)$. Because $D_{\bar{\alpha}} \subseteq 1 \otimes B(H) + B \otimes K(H)$, we have constructed a product. Note further that because $(\beta', \bar{\beta}')$ represents zero on the image of $\bar{\alpha}$, the product has a very simple form:

$[\alpha, \bar{\alpha}] [\beta', \bar{\beta}'] = [\beta' \circ \alpha, \bar{\beta}' \circ \alpha].$

In particular, if we have any two quasihomomorphisms $(\alpha, \bar{\alpha})$ from $A$ to $B$ and $(\beta, \bar{\beta})$ from $B$ to $C$ and either $A$ or $B$ is nuclear, then by Proposition 10 we may assume that $\bar{\alpha}$ is obtained from a faithful representation $A$ whose image is disjoint from the compacts, and then apply the construction as above. More generally, one may construct on this way the Kasparov product for the functor $KK_{nuc}$ from [Ska88].

This construction of the product coincides with the one by Kasparov by the preceding section.

References

[CS84] A. Connes and G. Skandalis. The longitudinal index theorem for foliations. Publ. Res. Inst. Math. Sci., 20(6):1139–1183, 1984.

[Cun83] J. Cuntz. Generalized homomorphisms between $C^*$-algebras and $KK$-theory. In Dynamics and processes (Bielefeld, 1981), volume 1031 of Lecture Notes in Math., pages 31–45. Springer, Berlin, 1983.

[Cun87] J. Cuntz. A new look at $KK$-theory. K-Theory, 1(1):31–51, 1987.

[Gre] M. Grensing. Universal cycles and homological invariants of locally convex algebras. preprint.
[Kas80a] G. G. Kasparov. Hilbert $C^*$-modules: theorems of Stinespring and Voiculescu. J. Operator Theory, 4(1):133–150, 1980.

[Kas80b] G. G. Kasparov. The operator $K$-functor and extensions of $C^*$-algebras. Izv. Akad. Nauk SSSR Ser. Mat., 44(3):571–636, 719, 1980.

[Ska84] G. Skandalis. Some remarks on Kasparov theory. J. Funct. Anal., 56(3):337–347, 1984.

[Ska88] Georges Skandalis. Une notion de nucléarité en $K$-théorie (d’après J. Cuntz). K-Theory, 1(6):549–573, 1988.

[Tho01] Klaus Thomsen. On absorbing extensions. Proc. Amer. Math. Soc., 129(5):1409–1417 (electronic), 2001.