STUDYING VELOCITY TURBULENCE FROM DOPPLER-BROADENED ABSORPTION LINES: STATISTICS OF OPTICAL DEPTH FLUCTUATIONS

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ABSTRACT

We continue our work on developing techniques for studying turbulence with spectroscopic data. We show that Doppler-broadened absorption spectral lines, in particular, saturated absorption lines, can be used within the framework of the previously introduced technique termed the velocity coordinate spectrum (VCS). The VCS relates the statistics of fluctuations along the velocity coordinate to the statistics of turbulence; thus, it does not require spatial coverage by sampling directions in the plane of the sky. We consider lines with different degree of absorption and show that for lines of optical depth less than one, our earlier treatment of the VCS developed for spectral emission lines is applicable, if the optical depth is used instead of intensity. This amounts to correlating the logarithms of absorbed intensities. For larger optical depths and saturated absorption lines, we show that only wings of the line are available for the analysis. In terms of the VCS formalism, this results in introducing an additional window, whose size decreases with the increase of the optical depth. As a result, strongly saturated absorption lines only carry the information about the small-scale turbulence. Nevertheless, the contrast of the fluctuations corresponding to the small-scale turbulence increases with the increase of the optical depth, which provides advantages for studying turbulence by combining lines with different optical depths. By combining different absorption lines one can develop a tomography of the turbulence in the interstellar gas in all its complexity.

Subject headings: ISM: general — ISM: structure — MHD — radio lines: ISM — turbulence

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1. INTRODUCTION

Turbulence is a key element of the dynamics of astrophysical fluids, including those of the interstellar medium (ISM), clusters of galaxies, and circumstellar regions. The realization of the importance of turbulence induces sweeping changes, for instance, in the paradigm of the ISM. It became clear, for instance, that turbulence substantially affects star formation, mixing of gas, and transfer of heat. observationally, it is known that the ISM is turbulent on scales ranging from AU to kpc (see Armstrong et al. 1995; Elmegreen & Scalo 2004), with an embedded magnetic field that influences almost all of its properties.

The issue of quantitative descriptors that can characterize turbulence is not a trivial one (see discussion in Lazarian 1999 and references therein). One of the most widely used measures is the turbulence spectrum, which describes the distribution of turbulent fluctuations over scales. For instance, the famous Kolmogorov model of incompressible turbulence predicts that the difference in velocities at different points in turbulent fluid increases on average with the separation between points as a cubic root of the separation, i.e., \[ \frac{\Delta v}{\Delta r} \sim \frac{1}{\sqrt[3]{r}}. \] In terms of a direction-averaged energy spectrum, this gives the famous Kolmogorov scaling \( E(k) \sim 4\pi k^2 P(k) \sim k^{5/3} \), where \( P(k) \) is a three dimensional (3D) energy spectrum defined as the Fourier transform of the correlation function of velocity fluctuations \( \xi_i(r) = \langle \delta v(x) \delta v(x + r) \rangle \). Note that in this paper we use angle brackets to denote the averaging procedure.

Quantitative measures of turbulence, in particular, the turbulence spectrum, became important recently also due to advances in the theory of MHD turbulence. As we know, astrophysical fluids are magnetized, which makes one believe that the correspondence should exist between astrophysical turbulence and MHD models of the phenomenon (see Vazquez-Semadeni et al. 2000; Mac Low & Klessen 2004; Ballesteros-Paredes et al. 2007; McKee & Ostriker 2007 and references therein).

In fact, without observational testing, the application of the theory of MHD turbulence to astrophysics could always be suspect. Indeed, from the point of view of fluid mechanics astrophysical turbulence is characterized by huge Reynolds numbers, \( Re \), which is the inverse ratio of the eddy turnover time of a parcel of gas to the time required for viscous forces to slow it appreciably. For \( Re \gg 100 \) we expect gas to be turbulent, and this is exactly what we observe in HI (for HI Re \( \sim 10^5 \)). In fact, very high astrophysical Re and its magnetic counterpart, the magnetic Reynolds number Rm (that can be as high as Rm \( \sim 10^{16} \)), present a big problem for numerical simulations that cannot possibly get even close to the astrophysically motivated numbers. The currently available 3D simulations can have Re and Rm up to \( \sim 10^4 \). Both scale as the size of the box to the first power, while the computational effort increases as the fourth power (three coordinates + time), so the brute-force approach cannot begin to resolve the controversies related, for example, to ISM turbulence.

We expect that observational studies of turbulence velocity spectra will provide important insights into ISM physics. Even in the case of much simpler oceanic (essentially incompressible) turbulence, studies of spectra allow one to identify meaningful energy injection scales. In the interstellar and intracluster media, in addition to that, we expect to see variations of the spectral index arising from the variations of the degree of compressibility, magnetization, interaction of different interstellar phases, etc.
How to get the turbulence spectra from observations is a problem with a long standing. While density fluctuations are readily available through both interstellar scintillations and studies of column density maps, the more coveted velocity spectra have been difficult to obtain reliably until very recently.

Turbulence is associated with fluctuating velocities that cause fluctuations in the Doppler shifts of emission and absorption lines. Observations provide integrals of either emissivities or opacities, both proportional to the local densities, at each velocity along the line of sight. It is far from trivial to determine the properties of the underlying turbulence from the observed spectral line.

Centroids of velocity (Munch 1958) have been an accepted way of studying turbulence, although it was not clear when and to what extend the measure really represents the velocity. Recent studies (Lazarian & Esquivel 2003; Esquivel & Lazarian 2005; Ossenkopf et al. 2006; Esquivel et al. 2007) have showed that the centroids are not a good measure for supersonic turbulence, which means that while the results obtained for H ii regions (O’Dell & Castaneda 1987) are probably OK, those for molecular clouds are unreliable.

An important development in the analytical description of the relation between the spectra of turbulent velocities and the observable spectra of fluctuations of spectral intensity was obtained in Lazarian & Pogosyan (2000, hereafter LP00). This description paved the way for two new techniques, which were later termed velocity channel analysis (VCA) and velocity coordinate spectrum (VCS).

The techniques provide different ways of treating observational data in position-position-velocity (PPV) data cubes. While VCA is based on the analysis of channel maps, which are the velocity slices of PPV cubes, the VCS analyzes fluctuations along the velocity direction. While the slices have been used earlier for turbulence studies, although the relation between the spectrum of intensity fluctuations in the channel maps and the underlying turbulence spectrum was unknown, the analysis of the fluctuations along the velocity coordinate was initiated by the advent of the VCS theory.

With the VCA and the VCS one can relate both observations and simulations to turbulence theory. For instance, the aforementioned turbulence indices are very informative, e.g., velocity indices steeper than the Kolmogorov value of $-5/3$ are likely to reflect the formation of shocks, while shallower indices may reflect scale-dependent suppression of cascading (see Beresnyak & Lazarian 2006 and references therein). By associating the variations of the index with different regions of the ISM, e.g., with high or low star formation, one can get an important insight into the fundamental properties of ISM turbulence, its origin, evolution, and dissipation.

The absorption of the emitted radiation was a concern of the observational studies of turbulence from the very start of work in the field (see discussion in Munch 1999). A quantitative study of the effects of the absorption was performed for the VCA in Lazarian & Pogosyan (2004, hereafter LP04) and for the VCS in Lazarian & Pogosyan (2006, hereafter LP06). In LP06 it was stressed that absorption lines themselves can be used to study turbulence. Indeed, the VCS is a unique technique that does not require a spatial coverage to study fluctuations. Therefore, individual point sources sampling the turbulent absorbing medium can be used to get the underlying turbulent spectra.

However, LP06 discusses only the linear regime of absorption, i.e., when the absorption lines are not saturated. This substantially limits the applicability of the technique. For instance, for many optical and UV absorption lines, e.g., Mg ii, S ii, Si ii, the measured spectra show saturation. This means that a part of the wealth of the unique data obtained, e.g., by HST and other instruments, cannot be handled with the LP06 technique.

The goal of this paper is to improve this situation. In particular, in what follows, we develop a theoretical description that allows one to relate the fluctuations of the absorption line profiles and the underlying velocity spectra in the saturated regime.

Below, in § 2 we describe the setting of the problem we address, while our main derivations are in § 3. The discussion of the new technique of turbulence study is provided in § 4, while the summary is in § 5.

2. ABSORPTION LINES AND MATHEMATICAL SETTING

While our earlier publications (LP00, LP04, LP06) concentrated on emission lines, in particular radio emission lines, e.g., H i and CO, absorption lines present the researchers with well-defined advantages. For instance, they allow one to test turbulence with a pencil beam and suffer less from uncertainties in path length. In fact, studies of absorption features in the spectra of stars have proven useful in outlining the gross features of gas kinematics in the Milky Way. Recent advances in the sensitivity and spectral resolution of spectrographs allow studies of turbulent motions.

Among the available techniques, VCS is the leading candidate to be used with absorption lines. Indeed, it is only with extended sources that either the centroid or VCA studies are possible. At the same time, VCS makes use not of the spatial but of the frequency resolution. Thus, potentially, turbulence studies are possible if absorption along a single line is available. In reality, information along a few lines of sight, as is shown in Figure 1, is required to improve the statistical accuracy of the measured spectrum. Using simulated data sets, Chepurnov & Lazarian (2006a, 2006b) experimentally established that the acceptable number of lines ranges from 5 to 10.

For weak absorption, the absorption and emission lines can be analyzed in the same way, namely, the way suggested in LP06. For this case, the statistic to analyze is the squared Fourier transform of the Doppler-shifted spectral line, irrespective of whether this is an emission or an absorption spectral line. Such a “spectrum of spectrum” is not applicable for saturated spectral lines, whose width is still determined by the Doppler broadening. It is known (see Spitzer 1978) that this regime corresponds to the optical depth $\tau$ ranging from 10 to $10^3$. The present paper will concentrate on this regime.1

Consider the problem in a more formal way. Intensity of the absorption line at frequency $\nu$ is given as

$$I(\nu, \nu_0) = I_0 e^{-\tau(\nu, \nu_0)}$$  \hspace{1cm} (1)

where $\tau(\nu)$ is the optical depth.

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1 It is known that for $\tau$ larger than $10^3$ the line width is determined by atomic constants, and therefore, it does not carry information about turbulence.
In the limit of vanishing intrinsic width of the line $\phi_i(v - \nu_0) = \alpha \delta(v - 3\nu_0)$, the frequency spread of $\tau(\nu, \nu_0)$ is determined solely by the Doppler shift of the absorption frequency from moving atoms. The column density of absorbers of mass $m_a$ that are moving with the line-of-sight velocity $v \approx (c/\nu_0)(\nu - \nu_0)$ is

$$\rho_\nu(v) = \int_0^S dz n(z) \phi(v - u(z) - v_{\text{reg}}(z)), \quad (2)$$

where $n(z)$ is the number density of atoms at the position $z$ along the line of sight and $\phi$ is the Maxwellian thermal distribution

$$\phi(v - u(z) - v_{\text{reg}}(z)) = \frac{1}{\sqrt{2\pi}\beta} \exp\left\{ - \frac{[v - u(z) - v_{\text{reg}}(z)]^2}{2\beta} \right\}, \quad \beta = \frac{k_B T}{m_a},$$

centered at each $z$ at the local mean velocity given by the sum of the turbulent $u(z)$ and the regular $v_{\text{reg}}(z)$ flow at that point. The $\rho_\nu$ is the density in PPV coordinates that we introduced in LP00. The optical depth as a function of velocity is $\tau(v) = \alpha(\nu_0)\rho_\nu(v)$, where $\alpha$ is the absorption coefficient. The intrinsic line width is accounted for by the convolution

$$\tau(v) = \alpha(\nu_0) \int dw \rho_\nu(w) \phi_i(w - v), \quad (3)$$

or in an expanded form,

$$\tau(v) = \alpha(\nu_0) \int_0^S dz n(z) \int dw \phi(w - u(z) - v_{\text{reg}}(z)) \phi_i(w - v). \quad (4)$$

With the intrinsic profile given by the Lorentz form $\phi_i(w - v) = (a/\pi)/[(w - v)^2 + a^2]$, the inner integral gives the shifted Voigt profile

$$H(w - u(z) - v_{\text{reg}}(z)) = \frac{1}{\pi \sqrt{2\pi}\beta} \int dw \frac{a}{(w - v)^2 + a^2} \exp\left\{ - \frac{[w - u(z) - v_{\text{reg}}(z)]^2}{2\beta} \right\}, \quad (5)$$

so we have another representation

$$\tau(v) = \alpha(\nu_0) \int_0^S dz n(z) H(v - u(z) - v_{\text{reg}}(z)). \quad (6)$$

We clearly see from equation (6) that the line is affected by both Doppler shifts and atomic constants. In what follows we assume that the regular motions are absent.

### 3. FLUCTUATIONS OF OPTICAL DEPTH

#### 3.1. Statistics of Optical Depth Fluctuations

The optical depth as a function of frequency contains a fluctuating component arising from turbulent motions and the associated density inhomogeneities of the absorbers. Statistics of optical depth fluctuations along the line of sight therefore carries information about turbulence in the ISM.

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2 Formally, one can always make use of the known atomic constants and get insight into turbulence. This is not practical, however, for an actual spectrum in the presence of noise.
The optical depth is determined by the density of the absorbers in the PPV space, $\rho_s$. In our previous work we have studied statistical properties of $\rho_s$ in the context of emission lines, using both structure function and power spectrum formalisms. Absorption lines demonstrate several important differences that warrant separate study.

First, our ability to recover the optical depth from the observed intensity

$$I(\nu) = I_0e^{-\tau(\nu)} + N$$

depends on the magnitude of the absorption as well as the sensitivity of the instrument and the level of measurement noise $N$. For lines with optical depth low enough that $\tau(0) < -\ln(N/I_0)$, we can in principle measure the optical depth throughout the whole line. At higher optical depths, the central part of the line is saturated below the noise level, and the useful information is restricted to the wings of the line. This is the new regime that is the subject of this paper.

In this regime the data is available over a window of frequencies limited to velocities high enough so that $\tau(\nu) < -\ln(N/I_0)$ but not so high as to have a Lorentz tail define the line. The higher the overall optical depth, the narrower the wings are (following Spitzer, at $\tau(0) > 10^2$ the wings are totally dominated by the Lorentz factor). We shall denote this window by $W(\nu - v_0, \Delta)$, where $v_0$ is the velocity that the window is centered on (describing the frequency position of the wing) and $\Delta$ is the wing width. It acts as a mask on the “underlying” data,

$$\tau(\nu) \rightarrow \tau(\nu)W(\nu - v_0, \Delta).$$

Second, fluctuations in the wings of a line are superimposed on the frequency-dependent wing profile. In other words, the statistical properties of the optical depth are inhomogeneous in this frequency range, with a frequency-dependent statistical mean value. While fluctuations of the optical depth $\delta\tau(\nu)$ that have their origin in the turbulence can still be assumed to be statistically homogeneous, the mean profile of a wing must be accounted for.

What statistical descriptors should one choose in the case of line-of-sight velocity data given over a limited window? Primary descriptors of a random field, here $\tau(\nu)$, are the ensemble average product of the values of the field at separated points—the two-point correlation function

$$\xi(\nu_1, \nu_2) = \langle \tau(\nu_1)\tau(\nu_2) \rangle$$

and, reciprocally, the average square of the amplitudes of its (Fourier) harmonics decomposition—and the power spectrum

$$P_{\tau}(k_1, k_2) = \langle \tau(k_1) \tau^*(k_2) \rangle.$$  

In practice, these quantities are measurable if one can replace ensemble averages by averaging over different positions, which relies on some homogeneity properties of the stochastic process. We assume that underlying turbulence is homogeneous and isotropic. This does not make the optical depth statistically homogeneous in the wings of the line, but it allows one to introduce the fluctuations of $\tau$ on the background of the mean profile $\bar{\tau}(\nu)$, $\Delta\tau(\nu) = \tau(\nu) - \bar{\tau}(\nu)$, which are (LP04)

$$\xi_{\Delta\tau}(\nu) = \langle \Delta\tau(\nu)\Delta\tau(\nu_2) \rangle, \quad \nu = \nu_1 - \nu_2,$$

$$P_{\Delta\tau}(k_1, k_2) = \langle \Delta\tau(k_1)\Delta\tau^*(k_2) \rangle \delta(k_1 - k_2).$$

The homogeneous correlation function depends only on a point separation, and amplitudes of distinct Fourier harmonics are independent. The obvious relations are

$$\xi_{\tau}(\nu_1, \nu_2) = \xi_{\Delta\tau}(\nu) + \bar{\tau}(\nu_1)\bar{\tau}(\nu_2),$$

$$P_{\tau}(k_1, k_2) = P_{\Delta\tau}(k) + \bar{\tau}(k_1)\bar{\tau}(k_2).$$

Although mathematically the power spectrum is just a Fourier transform of the correlation function,

$$P_{\Delta\tau}(k) = \int e^{ik\cdot\nu} \xi_{\Delta\tau}(\nu) d\nu,$$
Conversely, the correlation function is localized in configuration space and can be measured for nonuniformly sampled data. However, at each velocity separation it contains contributions from all scales and may mix together the physical effects from different scales. In particular, $\xi(v)$ even diverges everywhere for pure power-law spectra $P(k_r) \sim k_r^{-m}$, where $m$ is the index of the line-of-sight component of the velocity structure function. For Kolmogorov turbulence $m = 2/3$, and for turbulent motions dominated by shocks $m = 1/2$. These spectra are steep, $\sim k_r^{-3}$ and $\sim k_r^{-4}$, which makes the direct measurement of the structure functions impractical (although for $m > 2/3$ the structure function can be defined). At the same time, in our present studies we deal with a limited range of data in the wings of the absorption lines, which complicates the direct measurements of the power spectrum. Below, we first describe the properties of the power spectrum $P_{\Delta r}(k_r)$ in this case and, next, develop the formalism of higher order structure functions.

3.2. Power Spectrum of Optical Depth Fluctuations

Let us derive the power spectrum of the optical depth fluctuations in the absorption line, $P_r(k_r) \equiv \langle \tau(k_r)\tau^*(k_r) \rangle$, in the medium with stochastically distributed absorbers that are subject to turbulent motions. Here, $k_r$ is a wavenumber reciprocal to the velocity (frequency) separation between two points on the line of sight, and angle brackets denote an ensemble averaging over stochastic properties of the absorbing medium.\(^3\)

The Fourier transform of equation (6) with respect to velocity is

$$
\tau(k_r) = \alpha(v_0) \int_0^S dz n(x) \int dk_r' e^{-|k_r'|u} e^{-k_r'^2/2} e^{-ik_r'u(x)} W(k_r - k_r', v_0, \Delta),
$$

and the power spectrum is given by

$$
P_r(k_r) = \left\langle \alpha(v_0)^2 \int_0^S dz_1 \int_0^S dz_2 n(z_1) n(z_2) \int dk_r' \int dk_r'' e^{-|k_r'|u} e^{-|k_r''|u} e^{-\left(k_r'^2+k_r''^2\right)/2} e^{-i(k_r'u(z_1) - k_r''u(z_2))} W(k_r - k_r') W^* (k_r - k_r'') \right\rangle,
$$

which is useful to express using the average velocity $u_+ = [u(z_1) + u(z_2)]/2$ and the velocity difference $u = u(z_1) - u(z_2)$, as well as corresponding variables for the wavenumbers $k_r^+ = (k_r' + k_r'')/2$ and $k_r^- = k_r' - k_r''$, as

$$
P_r(k_r) = \left\langle \alpha(v_0)^2 \int_0^S dz_1 \int_0^S dz_2 n(z_1) n(z_2) \int_{-\infty}^{\infty} dk_r^- e^{-k_r^-2/2} e^{ik_r^-u} \right.$$  

$$
\times \int_{-\infty}^{\infty} dk_r^+ e^{-\left(1/4\right)k_r^+2/2} e^{ik_r^+u} e^{-\left(|k_r^-|+|k_r^+|\right)u} \left\langle W(k_r - k_r^+ - k_r^-/2) W^* \left(k_r - k_r^+ + k_r^-/2\right) \right\rangle.
$$

The fluctuating, random quantities over which the averaging is performed are the density $n(z)$ and the line-of-sight component of the velocity of the absorbers $u(z)$, varying along the line of sight. For statistically isotropic turbulence we describe their properties (see Monin & Yaglom 1975) by the density correlation function

$$
\xi(r) = \langle n(x)n(x + r) \rangle
$$

and the velocity structure tensor $\langle \Delta u \otimes \Delta u \rangle$, $\Delta u = u(x + r) - u(x)$, for which we are only interested in the z-projection that can be expressed through standard longitudinal $D_{\Delta L}$ and transverse $D_{\Delta N}$ structure functions as

$$
D_r(r) = \hat{z} \cdot \langle \Delta u \otimes \Delta u \rangle \cdot \hat{z} = D_{\Delta L}(r) + |D_{\Delta N}(r)| \hat{r} \cdot \hat{z}.
$$

For separations $r$ along the line of sight, $\hat{r} \cdot \hat{z} = 1$, $r = z \equiv z_1 - z_2$, and $D_z(z) = D_{\Delta L}(z)$. At small scales, the velocity structure function is frequently power law, $D_z(z) = D_z(S)(r/S)^m$, in particular for Kolmogorov turbulence $m = 2/3$, while at scales approaching the size of the absorbing cloud $S$, it is expected to saturate at the $D_z(S)$ value. We consider the velocity $u$, but not necessarily the density $n$, to be a Gaussian random field.

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3. In physically realistic situations, the power law will not, of course, extend infinitely to large scales. Mathematical divergence of the correlation function in practice means that for steep spectra the largest present scale determines the correlation for all, even small, separations $r$.

4. Note that due to having measurements in the finite window, the amplitudes at different waves numbers are, in general, correlated, $\langle \tau(k_r)\tau^*(k_r') \rangle \neq 0, k_r \neq k_r'$. Here, we restrict ourselves to diagonal terms only.
In our earlier papers (see LP00, LP04) we argued that in many important cases density and velocity can be considered as uncorrelated between themselves, so that

$$\langle n(z_1)n(z_2)e^{ik_yn}e^{ik_zu}\rangle = \langle n(z_1)n(z_2)\rangle e^{ik_iu}e^{ik_ju}e^{-(1/2)|k|^2D_z(z)}e^{-(1/4)|k|^2[\bar{u}(z)-\langle u_1(z)\rangle]^2},$$

(23)

where we have noted $\langle u^2 \rangle = D_z(z)$, $\langle u_1^2 \rangle = \frac{1}{2}[D_z(S) - \frac{1}{2}D_z(z)]$, and $\langle u_1, u \rangle = 0$. The dependence of $\xi(z)$ and $D_z(z)$ only on spatial separation between a pair of absorbers reflects the assumed statistical homogeneity of the turbulence model. Introducing $z = z_1 - z_2$ and performing integration over $z_+ = (z_1 + z_2)/2$ in equation (20), one obtains (see the Appendix)

$$P_r(k_r) = \alpha^2 S \int_0^S dz(S - |z|)\xi(z) \int_0^\infty dk_x^+ e^{-2k_x^+a}e^{-(1/2)k_x^+D^-} \int_0^\infty dk_x^- e^{-(1/4)k_x^-D^+} \tilde{W}(k_x, k_x^+ , \frac{k_x^-}{2}) + \text{correction terms},$$

(24)

where $D^- = D_z(z) + 2\beta$, $D^+ = D_z(S) - D_z(z)/2 + \beta$, and the symmetrized window $\tilde{W}$ is defined in the Appendix. This is a general expression for the power spectrum of the optical depth measured in the masked absorption line. The statistics of the underlying turbulence is imprinted mainly through $\xi(z)$ and $D^+(z)$, while $D^-(z)$ primarily reflects the amplitude of the velocities at the scale of the cloud and is insensitive to small-scale turbulent motions [at $z \ll SD_z \approx D(S) + \beta = \text{const.}$].

If the whole line is available for analysis, the masking window will be flat with a $\delta(k)$-function Fourier transform. The combination of the windows in the power spectrum will translate to $\delta(k_x - k_x^+)\delta(k_x^-)$ and

$$P_r(k_r) \sim 2\alpha \langle \nu_0 \rangle^2 S\varepsilon^{-2k_x^+a}e^{-k_x^+D^-} \int_0^S dz(S - |z|)\xi(z)e^{-(1/2)k_x^-D_z(z)}.$$  

(25)

This expression is equivalent to the previously obtained results for 1D velocity power spectra of emissivity (LP00, LP06) which demonstrates that the optical depth of the absorption lines contains similar information about the turbulence. A detailed investigation of the integral is given in LP06.

In Figure 2 we summarize different regimes for studies of turbulence using absorption lines from point sources for which angular resolution can be considered as perfect. For shallow density, i.e., for $\xi(\varphi) \approx \bar{\varphi}^2[1 + (\rho_0/\rho)^2]$, $\gamma > 0$, the spectrum of optical depth fluctuations that arise from inhomogeneities in density projected onto the velocity coordinate scales as $P_r(k_\|) \sim k_{\|}^{-2(3-\gamma)m}$. For sufficiently large $k_\|$, its contribution dominates the contribution $P_r(k_\perp) \sim k_{\perp}^{-2-\gamma}$ coming from the purely velocity moving uniformly distributed, $\xi(\varphi)$, gas. If we measure the point $\Delta V_{\|} \approx [D_z(S)]^{1/2}(\rho_0/\rho)^m/L_\|$, at which $P_r$ becomes dominant

The following averaging procedure over a Gaussian variable with variance $D$ is used

$$\langle e^{ik_n} \rangle = \frac{1}{\sqrt{2\pi D}} \int du e^{-u^2/(2D)} e^{ik_n} = e^{-(1/2)k^2D}.$$  

Absolute values in the Fourier image of the Lorentz transform require separate consideration of each quadrant of wavenumbers. This is carried out in the Appendix.
over \(P_m\), we can estimate the density correlation radius \(r_0\) which has the physical meaning of the scale at which the dispersion of the fluctuations of density equal the mean density (see LP06). However, whether we can observe both regimes or just one depends on the mask that is imposed by the saturation of the absorption line, as well as the simultaneous estimate of the density correlation radius \(r_0\), which has the physical meaning of the scale at which the small-scale filtering arises from thermal broadening. For \(\gamma < 0\), \(P_\nu\) always dominates.

Masking the data has the effect of aliasing modes of the large scales that exceed the available data range, to shorter wavelength. This is represented by the convolution with the Fourier image of the mask. A secondary effect is the contribution of the modes with nonzero wavenumbers \(k^- \neq 0\) to the diagonal part of the power spectrum. This again reflects the situation that different Fourier components are correlated in the presence of the mask.

To illustrate the effects of the mask, let us assume that we select the line wing with the help of a Gaussian mask of width \(\Delta\), centered in the middle of the wing at \(v_1\),

\[
W(v - v_1, \Delta) = e^{-(v - v_1)^2/(2\Delta^2)},
\]

\[
W(k_v) = \sqrt{2\pi} \Delta e^{-ik_v v_1} e^{-(1/2)k_v^2 \Delta^2}.
\]

This gives

\[
\mathcal{W} = 4\pi \Delta^2 \cos \left( \frac{k_v}{2} v_1 \right) e^{-k_v^2 \Delta^2/4} \left( e^{-(k_v - k_0)^2 \Delta^2} + e^{-(k_v + k_0)^2 \Delta^2} \right),
\]

and all the integrals can now be carried out to obtain

\[
P_v(k_v) \sim \alpha^2 S \int_0^S dz(S - z) \xi(z) \Delta^2 \exp \left\{ -(1/4) \frac{v_1^2}{(D^+ + \Delta^2)} \right\} \frac{\exp \left( -\frac{\Delta^2 D^-}{D^- + 2\Delta^2} k_v^2 \right)}{\sqrt{(D^+ + 2\Delta^2)(D^- + 2\Delta^2)}} \times \exp \left( -\frac{2a^2}{D^- + 2\Delta^2} \right) \left\{ \exp \left( \frac{-4a^2 k_0}{D^- + 2\Delta^2} \right) \text{Erfc} \left( \frac{\sqrt{2} (a - \Delta^2 k_0)}{\sqrt{D^- + 2\Delta^2}} \right) + (k_v \rightarrow -k_v) \right\}.
\]

This formula includes both the masking effects and the contribution from the intrinsic line width.

Eliminating the mask by taking \(\Delta \rightarrow \infty\), one obviously recovers equation (25). This, however, is not a realistic regime when one is restricted to the wings of the line, in which case the window width is necessarily limited by \(\Delta^2 < D(S) + 2\beta\). The effect of the finite mask is particularly clearly demonstrated when the intrinsic width can be ignored, \(a \rightarrow 0\),

\[
P_v(k_v) \sim \alpha (v_0)^2 S \int_0^S dz(S - z) \xi(z) \Delta^2 \exp \left\{ -(1/4) \frac{v_1^2}{(D^+ + \Delta^2)} \right\} \frac{\exp \left( -\frac{\Delta^2 D^-}{D^- + 2\Delta^2} k_v^2 \right)}{\sqrt{(D^+ + 2\Delta^2)(D^- + 2\Delta^2)}}.
\]

The power spectrum is corrupted at scales \(k_v < \Delta^{-1}\), but still maintains information about turbulence statistics for \(k_v \gg \Delta^{-1}\). Indeed, in our integral representation the power spectrum at \(k_v\) is determined by the linear scales such that \(\Delta^2 D^-/(D^- + 2\Delta^2) k_v^2 < 1\), which
translates into $D^- < 2\Delta^2/(k^2 \Delta^2 - 1)$. Thus, if $k_\Delta \Delta > 1$ over all scales defining power at $k_\Delta$, one has $\Delta \gg D^-$ and there is no significant power aliasing.\footnote{In this regime the first factor is essentially constant, since $D^+ (z) = D_4 (S) + 2/3 - D^- (z)/2$ varies little, $D^+ (z) \approx D^- (0)$ in the interval $2/3 < D^- < 2k_\Delta^2 < 2\Delta^2 < D_4 (S) + 2/3$.} For intermediate scales there is a power aliasing as numerical results demonstrate in Figure 3.

Doppler broadening described by $D^-$ incorporates both turbulent and thermal effects. Thermal effects are especially important in the case of narrow-line wings, since the range of the wavenumbers is relatively unaffected by both thermal motions, and the mask is limited, $1/\sqrt{\beta} > k_\Delta > \Delta^{-1}$, and exists only for relatively wide wings $\Delta > \sqrt{\beta}$. For narrower wings the combined turbulent and thermal profile must be fitted to the data, possibly determining the temperature of the absorbers at the same time. This recipe is limited by the assumption that the temperature of the gas is relatively constant for the absorbers of a given type.

We should note that the Gaussian window provides one of the ideal cases, limiting the extent of power aliasing, since the window Fourier image falls off quickly. One of the worst scenarios is represented by a sharp top-hat mask, whose Fourier image falls off only as $k^{-2}$, spreading the power from large scales further into short scales. For the steep spectra that we have in VCS studies, all scales may experience some aliasing. This argues for extra care while treating the line wings through the power spectrum or for the use of alternative approaches.

3.3 Second-Order Structure Function

As an alternative to power spectrum measurement in the case of steep spectra with the data limited to the section of the lines, we introduce the second-order structure function of the optical depth defined as

$$dD_{\Delta^2} (v) = \frac{1}{2} \left[ \Delta \tau (v_1 + v) + \Delta \tau (v_1 - v) - 2 \Delta \tau (v_1) \right]^2. \quad (31)$$

Note that this function still, as the ordinary structure function, determines the quadratic statistics, but on the triplets of data points with equally spaced frequencies. This is different from “higher order” correlation functions that deal with cubic and higher products of the data values and, thus, with higher moments of the distribution. The masking of the data enters the definition only implicitly, through averaging over the triplets drawn from frequency intervals where the data exists.

Using this second-order structure function represents additional regularization of the correlation function beyond what the ordinary structure function provides. We can formally write

$$dD_{\Delta^2} (v) = 3 \xi_{\Delta^2} (0) - 4 \xi_{\Delta^2} (v) + \xi_{\Delta^2} (2v)$$

$$= 2D_{\Delta^2} (v) - \frac{1}{2} D_{\Delta^2} (2v). \quad (33)$$

For steep spectra, each of the correlation terms in the first expression, and for spectra as steep as we encounter in VCS, even the structure functions in the second expression, are divergent, but their combination is well behaved. Let us look into this in more detail.

The $D_{\Delta^2} (v)$ is proportional to the 3D velocity-space density structure function at zero angular separations $d_\rho (0, v)$, discussed in LP06. Using the results of LP06 for $d_\rho (0, v)$, we obtain

$$dD_{\Delta^2} (v) \sim \left( \frac{(r_0)}{S} \right)^\gamma \int_1 d \tilde{z} \frac{1}{|z|^{1+m/2}} \left[ 3 - 4 \exp \left( - \frac{\tilde{z}^2}{2|z|^2} \right) + \exp \left( - 2\tilde{z}^2 \right) \right]$$

$$\times \frac{\tilde{z}^2 S^2}{D_2 (S) m} \left( \frac{(r_0)}{S} \right)^\gamma \left( \frac{2^p \Gamma (- p) (2p - 1) \Gamma (- p) \tilde{z}^{2p}}{p-2} \right), \quad (34)$$

where $p = (1 - \gamma)/m - 1/2 > 0$ and $\gamma$ is the correlation index that describes spatial inhomogeneities of the absorbers. To shorten intermediate formulae, the dimensionless quantities $\tilde{z} = vD_1^1 (S), \tilde{z} = z/S, \tilde{z} = r/S$ are introduced. The first term in the expansion contains information about the underlying field, while the power-law series represent the effect of boundary conditions at the cloud scale. In contrast to the ordinary correlation function, they are not dominant until $p \geq 2$, i.e., for $(1 - \gamma)/m < 5/2$ the second-order structure function is well defined. When turbulent motions provide the dominant contribution to optical depth fluctuations, $\gamma = 0$, we see that by measuring the $dD_{\Delta^2}$, one can recover the turbulence scaling index if $m > 2/5$, which includes both interesting cases of Kolmogorov turbulence and shock-dominated motions. This condition is replaced by $m > 2/5 (1 - \gamma)$ if the density fluctuations, described by correlation index $k_{\gamma}$, are dominant. Thus, at sufficiently small scales the second-order structure function has the same scaling as the first-order one

$$dD_{\Delta^2} (v) \approx \frac{\tilde{z}^2 S^2}{D_2 (S) m} \left( \frac{(r_0)}{S} \right)^\gamma (2p-1) \Gamma (- p) \tilde{z}^{2p}, \quad (35)$$

but the range of its applicability is extended to steeper spectra $m > 2/5 (1 - \gamma)$.

A practical issue of measuring the structure functions directly in the wing of the line is to take into account the line profile. The directly accessible

$$dD_{\Delta^2} (v_1, v) = \frac{1}{2} \left[ \tau (v_1 + v) + \tau (v_1 - v) - 2 \tau (v_1) \right]^2$$

$$\text{for } v_1, v \text{ close to } 0.$$
is related to the structure function of the fluctuations as

\[ dD_{\Delta_v}(v) = dD_r(v_1, v) - \frac{1}{2} [\tilde{\tau}(v_1 + v) + \tilde{\tau}(v_1 - v) - 2\tilde{\tau}(v_1)]^2, \]  

(37)

where the mean profile of the optical depth \( \tilde{\tau} \) is related to the mean profile of the PPV density \( \tilde{\rho} \) given in Appendix B of LP06. At small separations \( \epsilon \), the correction to the structure function due to the mean profile behaves as \( \epsilon^4 \) and is subdominant. This is a reflection of the fact that the second-order structure function is insensitive to the gradients in the data.

The price one pays when utilizing higher level structure functions is their higher sensitivity to the noise in the data. While the correlation function itself is not biased by the noise except at zero separations (assuming noise is uncorrelated),

\[ \langle [\tau(v_1) + N(v_1)][\tau(v_1 + v) + N(v_1 + v)] \rangle = \xi_r(v) + \langle N^2 \rangle \delta(v), \]

(38)

already the structure function is biased by the noise which contributes to all separations,

\[ \langle [\tau(v_1) + N(v_1) - \tau(v_1 + v) - N(v_1 + v)]^2 \rangle = D_r(v) + 2\langle N^2 \rangle. \]

(39)

This effect is further amplified for the second-order structure function,

\[ \frac{1}{2} \langle [\tau(v_1 + v) + \tau(v_1 - v) - 2\tau(v_1) + N(v_1 + v) + N(v_1 - v) - 2N(v_1)]^2 \rangle = dD_r(v) + 3\langle N^2 \rangle. \]

(40)

The error in the determination of the structure functions of higher order due to noise also increases.

3.4. Comparison of the Approaches

Structure functions and power spectra are used interchangeably in the theory of turbulence (see Monin & Yaglom 1975). However, complications arise when spectra are “extremely steep,” for the 1D case \( P \propto k^{-n} \), \( n > 3 \). For such random fields, one cannot use ordinary structure functions.

As a rule, one does not have to deal with such steep spectra in the theory of turbulence (see, however, Cho et al. 2003; Cho & Lazarian 2004). However, within the VCS, such “extremely steep” spectra emerge naturally, even when the turbulence is close to being Kolmogorov. This was noted in LP06, where the spectral approach was presented as the correct one to studying turbulence using fluctuations of intensity along the \( v \)-coordinate.

The disadvantage of the spectral approach is when the data is being limited by a non-Gaussian window function. Then, the contributions from the scales determined by the window function may interfere in the obtained spectrum at large \( k \). An introduction of an additional narrower Gaussian window function may mitigate the effect, but limits the range of \( k \), for which turbulence can be studied. Thus, higher order structure functions (see § 3.3) are advantageous for the practical data handling.

In terms of the VCA theory, we used a mostly spectral description in LP00, while in LP04, dealing with absorption, we found it advantageous to deal with real rather than Fourier space. In doing so, however, we faced the steepness of the spectrum along the \( v \)-coordinate and provided a transition to the Fourier description to avoid the problems with the “extremely steep” spectrum. Naturally, our approach of higher order structure functions is applicable to dealing with the absorption within the VCA technique.

4. DISCUSSION

4.1. Simplifying Assumptions

In the paper above we have discussed the application of VCS to strong absorption lines. The following assumptions were used. First of all, considering the radiative transfer we neglected the effects of stimulated emission. This assumption is well satisfied for optical or UV absorption lines (see Spitzer 1978). Then, we assumed that the radiation is coming from a point source, which is an excellent approximation for the absorption of the light of a star or a quasar. Moreover, we disregarded the variations of temperature in the medium.

Within our approach the last assumption may be the most questionable. Indeed, it is known that the variations of temperature do affect absorption lines. Nevertheless, our present study, as well as our earlier studies, prove that the effects of the variations of density are limited. It is easy to see that the temperature variations can be combined together with the density ones to get effective renormalized “density” whose effects we have already quantified.

Our formalism can also be generalized to include a more sophisticated radiative transfer and the spatial extend of the radiation source. In the latter case we shall have to consider the case of both a narrow and a broad telescope beam, the way it has been done in LP06. Naturally, the expressions in LP06 for broad-beam observations can be straightforwardly applied to the absorption lines, substituting the optical depth variations instead of intensities. The advantage of the extended source is that not only VCS, but also VCA can be used (see Deshpande et al. 2000). A disadvantage of an extended source is the steepening of the observed \( v \)-coordinate spectrum for studies of unresolved turbulence. This, for instance, may require employing even higher order structure functions, if one has to deal with windows arising from saturation of the absorption line.

In LP06 we have studied the VCS technique in the presence of absorption and formulated the criterion for the fluctuations of intensity to reliably reflect the fluctuations in turbulent velocities. In this paper, however, we used the logarithms of intensities and showed that this allows turbulence studies beyond the regime at which fluctuations of intensity would be useful.
A similar approach, potentially, is applicable to the VCS for studies of emitted radiation in the presence of significant absorption. Indeed, the equation of the radiative transfer provides us in this case with

$$I_X(\nu) = \frac{c}{\alpha} \left(1 - e^{-\alpha \rho(X,\nu)}\right), \quad (41)$$

where the first term is a constant that can be subtracted. If this is done, a logarithm can be taken of the intensity. Potentially, this should allow VCS studies of emission lines where, otherwise, absorption distorts the statistics.

The difficulty of such an approach is the uncertainty of the base level of the signal. Taking a logarithm is a nonlinear operation that may distort the result, if the base level of the signal is not accounted for properly. However, the advantage of the approach is that it potentially allows studies of velocity turbulence when the traditional VCA and VCS fail. Further research should clarify the utility of this approach.

4.2. VCS Cookbook for Absorption Lines

In LP06 we presented a cookbook for emission lines. A simplifying element that is introduced by absorption lines is that they provide measurements in the “high-resolution regime,” provided that the absorption against a point source is used. To get proper averaging, one requires more than one point source. Numerical experiments in Chepurnov & Lazarian (2006a, 2006b, 2008) showed that measuring spectra along 5–10 lines of sight is enough to recover the underlying spectrum properly.

In the simplest case, Figure 2 may illustrate how the information on velocity and density can be obtained from observations for the case of shallow density. In the case of steep density the measured spectrum is always $\sim k_e^{-2/m}$, which allows one to find the spectral index of velocity $m$.

Similar to the case of emitting gas, the VCS study of absorption gas is dominated by the colder component of the gas along the line of sight. In LP06 we were discussing the possibility of correcting for the exponential factors $\exp(-\beta k_e^2)$ reflecting the effect of thermal broadening. However, a practical data handling using the VCS technique in Chepurnov et al. (2006) showed that obtaining a fit of observational data using integral expressions is a more reliable way of obtaining both parameters of the underlying turbulence and temperature of the emitting gas.

The differences in data handling for absorption compared to that in Chepurnov et al. (2006) are twofold. First of all, one has to take into account masking of the data as discussed in § 3.2. Second, in Chepurnov et al. (2006) the fitting was done while varying the spatial resolution of the data. As the effects of particular factors, e.g., thermal broadening, are different at different spatial scales, one can provide additional testing of the accuracy of the fit by varying the angular resolution. Varying the resolution is not an option for the absorption data from point sources, but is possible when spatially extended sources are used to sample the medium.\(^8\)

4.3. Prospects of Studying ISM Turbulence with Absorption Lines

The study of turbulence using the modified VCS technique above should be reliable for optical depths $\tau$ up to $10^3$. For this range of optical depths, the line width is determined by Doppler shifts rather than the atomic constants. While formally the entire line profile provides information about the turbulence, in reality, the flat saturated part of the profile will contain only noise and will not be useful for any statistical study. Thus, the wings of the lines will contain signal.

As several absorption lines can be available along the same line of sight, this allows one to extend the reliability of measurements by combining them together. We believe that piecewise analyses of the wings belonging to different absorption lines is advantageous. The actual data analysis may employ fitting the data with models that, apart from the spectral index, specify the turbulence injection scale and velocity dispersion, as this is done in Chepurnov et al. (2006).

Note that measurements of turbulence in the same volume using different absorption lines can provide complementary information. Formally, if lines with weak absorption, i.e., $\tau(0) < 1$ are available, there is no need for other measurements. However, in the presence of inevitable noise, the situation may be far from trivial. Naturally, noise of a constant level, e.g., instrumental noise, will affect more weak absorption lines. The strong absorption lines, in terms of VCS, sample turbulence only for sufficiently large $k_e$. This limits the range of turbulent scales that can be sampled with the technique. However, the contrast that is obtained with the strong absorption lines is higher, which provides an opportunity for increasing the signal-to-noise ratio for the range of $k_e$ that is sampled by the absorption lines. If, however, a single strong absorption line is used, an analogy with a two-dish radio interferometer is appropriate. Every dish of the radio interferometer samples spatial frequencies in the range approximately $[1/\ell, 1/d]$, where $\ell$ is the operational wavelength and $d$ is the diameter of the dish. In addition, the radio interferometer samples the spatial frequency $1/D$, where $D$ is the distance between the dishes. Similarly, a strong absorption line provides information on turbulent velocity at the largest spatial scale of the emitting objects, as well as the fluctuation corresponding to the scales $k_{\text{win}}$ and $k_{\text{abs}}$.

In LP06 we concentrated on obtaining asymptotic regimes for studying turbulence. At the same time, in Chepurnov et al. (2006) fitting models of turbulence to the data was attempted. In the latter approach, non-power-law observed spectra can be used, which is advantageous for actual data for which the range of scales in $k_e$ is rather limited. Indeed, for H\textsc{i} with the injection velocities of $10$ km s$^{-1}$ and the thermal velocities of $1$ km s$^{-1}$ it provides an order of magnitude of effective “inertial range.” Correcting for thermal velocities, one can increase this range by a factor which depends on the signal-to-noise ratio of the data. Using heavier species rather than hydrogen, one can increase the range by a factor $(m_{\text{heavy}}/m_{\text{H}})^{1/2}$. This may or may not be enough for observing good asymptotics.

We have seen in § 3.2 that for absorption lines the introduction of windows determined by the width of the line wings introduces additional distortions of the power spectrum. However, this is not a problem if, instead of asymptotics, fitting of the model is used. Compared to the models used in Chepurnov et al. (2006), the models for absorption lines should also have to model the window induced by the absorption. The advantage is, however, that absorption lines provide a pure pencil-beam observations.

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\(^8\) In the latter case, the VCA-type studies of the absorption line wings are also possible.
4.4. Comparison and Synergy with Other Techniques

Formally, there exists an extensive list of different tools to study turbulence that predated our studies (see Lazarian 1999 and references therein). However, a closer examination shows that this list is not as impressive as it looks. Moreover, our research showed that some techniques may provide confusing, if not erroneous, output, unless theoretical understanding of what they measure is achieved. For instance, we mentioned in §1 an example of the erroneous application of velocity centroids to supersonic molecular cloud data. Note that clump- and shell-finding algorithms would find a hierarchy of clumps/shells for synthetic observations obtained with incompressible simulations. This calls for a more cautious approach to the interpretation of the results of some of the accepted techniques.

For instance, the use of different wavelets for the analysis of data is frequently treated in the literature as different statistical techniques of turbulence studies (Gill & Henriksen 1990; Stutzki 2001; Cambresy 1999; Khalil et al. 2006), which creates an illusion of an excessive wealth of tools and approaches. In reality, while Fourier transforms use harmonics of $e^{ikr}$, wavelets use more sophisticated basis functions, which may be more appropriate for the problems at hand. In our studies we also use wavelets both to analyze the results of computations (see Kowal & Lazarian 2006) and synthetic maps (Ossenkopf et al. 2006; Esquivel et al. 2007), along with or instead of Fourier transforms or correlation functions. Wavelets may reduce the noise arising from the inhomogeneity of the data, but we found in the situations when correlation functions of centroids that we studied were failing as the Mach number was increasing, a popular wavelet ($\Delta$-variance) was also failing (cf. Esquivel & Lazarian 2005; Ossenkopf et al. 2006; Esquivel et al. 2007).

While in wavelets the basis functions are fixed, a more sophisticated technique, principal component analysis (PCA), chooses basis functions that are, in some sense, the most descriptive. Nevertheless, while our research in LP00 shows that for density spectra $E_\rho \sim k^{-\alpha}$, for $\alpha < 1$ both velocity and density fluctuations influence the statistics of PPV cubes, no dependencies of PPV statistics on density have been reported so far in PCA studies. This also may reflect the problem of finding the underlying relations empirically with data cubes of limited resolution. The latter provides a special kind of shot noise, which is discussed in a number of papers (Lazarian et al. 2001; Esquivel et al. 2003; Chepurnov & Lazarian 2006).

The spectral correlation function (SCF; see Rosolowsky et al. 1999 for its original form) is another way to study turbulence. Further development of the SCF technique in Padoan et al. (2001) removed the adjustable parameters from the original expression for the SCF and made the technique rather similar to VCA in terms of the observational data analysis. Indeed, both SCF and VCA measure correlations of intensity in PPV “slices” (channel maps with a given velocity window $\Delta v$), but if SCF treats the outcome empirically, the analytical relations in LP00 relate the VCA measures to the underlying velocity and density statistics. Mathematically, SCF contains additional square roots and normalizations compared to the VCA expressions. Those make the analytical treatment, which is possible for simpler VCA expressions, prohibitive. One might speculate that, similar to the case of conventional centroids and unnormalized centroids introduced in Lazarian & Esquivel (2003), the actual difference between the statistics measured by the VCA and SCF is not significant.

In fact, we predicted several physically motivated regimes for VCA studies. For instance, slices are “thick” for eddies with velocity ranges less than $\Delta v$ and “thin” otherwise. VCA relates the spectral index of intensity fluctuations within channel maps to the thickness of the velocity channel and to the underlying velocity and density in the emitting turbulent volume. In the VCA these variations of indices with the thickness of a PPV “slice” are used to disentangle velocity and density contributions. We suspect that similar “thick” and “thin” slice regimes should be present in the SCF analysis of data, but they have not been reported yet. While the VCA can be used for all the purposes the SCF is used for (e.g., for an empirical comparisons of simulations and observations), the opposite is not true. In fact, Padoan et al. (2004) stressed that VCA eliminates errors inevitable for empirical attempts to calibrate PPV fluctuations in terms of the underlying 3D velocity spectrum.

VCS is a statistical tool that uses the information about fluctuations along the velocity axis of the PPV. Among all the tools that use spectral data, including the VCA, it is unique, as it does not require spatial resolution. This is why, dealing with the absorption lines, where good spatial coverage is problematic, we employed the VCS. Potentially, having many sources sampling the object one can create PPV cubes and also apply the VCA technique. However, this requires very extended data sets, while for the VCS, sampling with 5 or 10 sources can be sufficient for obtaining good statistics (Chepurnov & Lazarian 2006a).

We feel that when dealing with the ISM turbulence, it is synergetic to combine different approaches. For the wavelets used, their relation with the underlying Fourier spectrum is usually well defined. Therefore, the formulation of the theory (presented in this work as well as in our earlier papers in terms of the Fourier transforms) in terms of wavelets is straightforward. At the same time, the analysis of data with the wavelets may be advantageous, especially in the situations when one has to deal with window functions.

4.5. Information Available

Fitting of the observed spectra with the model as in Chepurnov et al. (2006) allows one to determine the injection scale of turbulence, the injection velocity, and the velocity spectrum. Applying this to different regions of interstellar gas, one can determine sources and sinks of interstellar turbulence as well as the variations of spectral index. One may also study whether the turbulent energy is injected or drained at intermediate scales.

In terms of velocity spectrum, numerical simulations provide us with the choice of Kolmogorov-type turbulence with $E(k) \sim k^{-5/3}$, which also corresponds to the Goldreich & Sridhar (1995) magnetized turbulence, and turbulence of shocks with $E(k) \sim k^{-2}$. There are other contested spectra, e.g., a slightly shallower spectrum of magnetic turbulence, arising when the effects of dynamical alignment, polarization intermittency, or nonlocality are important (see Boldyrev 2005; Beresnyak & Lazarian 2006; Gogoberidze 2007), spectra of MHD fast modes (Goldreich & Sridhar 1995; Cho & Lazarian 2002; Chandran 2005; Suzuki et al. 2007), and spectra of imbalanced turbulence, arising when the flow of Alfvén waves in one direction is larger than the flow of waves in the opposite direction.

9 We showed that much of the earlier confusion stemmed from different observational groups having used velocity channels of different thicknesses (compare, e.g., Green 1993 and Stanimirovic et al. 1999).
Lithwick et al. 2007; Beresnyak & Lazarian 2008; Chandran 2008). Observational data can clarify what is happening in the actual astrophysical conditions.

5. SUMMARY

In the paper above we have shown that:

1. Studies of turbulence with absorption lines are possible with the VCS technique if, instead of intensity $I(\nu)$, one uses the logarithm of the absorbed intensity $\log I_{\text{abs}}(\nu)$, which is equivalent to the optical depth $\tau(\nu)$.

2. In the weak-absorption regime, i.e., when the optical depth at the middle of the absorption line is less than unity, the analysis of $\tau(\nu)$ coincides with the analysis of the intensities of emission for ideal resolution that we discussed in LP06.

3. In the intermediate-absorption regime, i.e., when the optical depth at the middle of the absorption line is larger than unity, but less than $10^3$, the wings of the absorption line can be used for the analysis. The saturated part of the line is expected to be noise dominated.

4. The higher the absorption, the less the portion of the spectrum corresponds to the wings available for the analysis. In terms of the mathematical setting, this introduces an additional window in the expressions for the VCS analysis. However, the contrast of the small-scale fluctuations increases with the decrease of the window.

5. For the strong-absorption regime, the broadening is determined by Lorentzian wings of the line, and therefore, no information on turbulence is available.

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APPENDIX

DERIVATION OF THE POWER SPECTRUM $P_\tau$

Following equations (20) and (23), the power spectrum of the optical depth is

$$P_\tau(k_v) = \alpha (\nu_0)^2 \int_0^{S_\text{z1}} dz_1 \int_0^{S_\text{z2}} dz_2 \xi(|z_1 - z_2|) \int_{-\infty}^{\infty} dk'_v \int_{-\infty}^{\infty} dk''_v$$

$$\times e^{-\frac{1}{2}k_v^2D^+} e^{-\frac{1}{2}k_v^2D^-} e^{-\frac{1}{2}(k'_v + k''_v)^2} W \left( k_v - k'_v + \frac{k''_v}{2} \right) W \left( k_v - k''_v + \frac{k'_v}{2} \right),$$

(A1)

where $D^+ = D_2(|z_1 - z_2|) + 2\beta$ and $D^- = D_2(S) - D_2(|z_1 - z_2|)/2 + \beta$, while $k'_v = (k'_v + k''_v)/2$ and $k''_v = k'_v - k''_v$. Since the mask is real, $W\left(k_v\right) = W\left(-k_v\right)$.

To deal with absolute values in the Lorentz transform, we split integration regions into quadrants: I—$(k'_v > 0, k''_v > 0)$; II—$(k'_v < 0, k''_v > 0)$; III—$(k'_v < 0, k''_v < 0)$; and IV—$(k'_v > 0, k''_v < 0)$. Integration over quadrants III and IV can be folded into integration over regions I and II, respectively, by substitution, $k'_v \rightarrow -k'_v$, $k''_v \rightarrow -k''_v$, $k'_v \rightarrow -k'_v$, and $k''_v \rightarrow -k''_v$. Writing out only integration over $k''_v$,

$$\text{I + III : } \int_{-\infty}^{\infty} dk'_v \int_{-\infty}^{\infty} dk''_v e^{-\frac{1}{2}k'_v^2D^+} e^{-\frac{1}{2}k''_v^2D^-} e^{-2ak'_v} \times \left[ W \left( k_v - k'_v + \frac{k''_v}{2} \right) W \left( k'_v - k_v - \frac{k''_v}{2} \right) + W \left( k_v + k'_v + \frac{k''_v}{2} \right) W \left( -k_v - k'_v + \frac{k''_v}{2} \right) \right],$$

(A2)

$$\text{II + IV : } \int_{-\infty}^{\infty} dk'_v \int_{-\infty}^{\infty} dk''_v e^{-\frac{1}{2}k'_v^2D^+} e^{-\frac{1}{2}k''_v^2D^-} e^{-2ak'_v} \times \left[ W \left( k_v - k'_v + \frac{k''_v}{2} \right) W \left( k'_v - k_v - \frac{k''_v}{2} \right) + W \left( k_v + k'_v + \frac{k''_v}{2} \right) W \left( -k_v - k'_v + \frac{k''_v}{2} \right) \right].$$

Changing variables of integration to $k'_v$ and $k''_v$,

$$\text{I + III : } \int_{-\infty}^{\infty} dk'_v e^{-\frac{1}{2}k'_v^2D^-} e^{-2ak'_v} \int_{-2k'_v}^{2k'_v} dk''_v e^{-\frac{1}{2}(k'_v + k''_v)^2} \times \left[ W \left( k_v - k'_v + \frac{k''_v}{2} \right) W \left( k'_v - k_v - \frac{k''_v}{2} \right) + W \left( k_v + k'_v + \frac{k''_v}{2} \right) W \left( -k_v - k'_v + \frac{k''_v}{2} \right) \right],$$

(A4)

$$\text{II + IV : } \int_{-\infty}^{\infty} dk'_v e^{-\frac{1}{2}k'_v^2D^-} \int_{2k'_v}^{\infty} dk''_v e^{-\frac{1}{2}(k'_v + k''_v)^2} e^{-2ak'_v} \times \left[ W \left( k_v - k'_v + \frac{k''_v}{2} \right) W \left( k'_v - k_v - \frac{k''_v}{2} \right) + W \left( k_v + k'_v + \frac{k''_v}{2} \right) W \left( -k_v - k'_v + \frac{k''_v}{2} \right) \right].$$

(A5)
At the end, the integrals can be combined into the main contribution and the correction that manifests itself only when Lorentz broadening is significant,

\[
\text{main: } \int_0^\infty dk_+^* e^{-(1/2)\Delta^2 k_+^*} e^{-2ak_+^*} \int_0^\infty dk_-^* e^{-(1/4)\Delta^2 k_-^*} W(k_+, k_+^*, k_-),
\]

\[
\text{correction: } \int_0^\infty dk_+^* e^{-(1/2)\Delta^2 k_+^*} \int_0^\infty dk_-^* e^{-(1/4)\Delta^2 k_-^*} \left( e^{-ak_-^*} - e^{-2ak_-^*} \right) W(k_+, k_+^*, k_-),
\]

where the symmetrized window is

\[
\tilde{W}(k_+, k_+^*, k_-^*) \equiv W(k_+ - k_-^* - \frac{k_+^*}{2}) W(k_+^* - k_+ - \frac{k_+}{2}) + W(k_+^* + k_+ - \frac{k_+}{2}) W(-k_- - k_+^* + \frac{k_+}{2}) + W(k_+ - k_-^* - \frac{k_+^*}{2}) W(k_+^* - k_+ - \frac{k_+}{2}) + W(k_+^* + k_+ - \frac{k_+}{2}) W(-k_- - k_+^* + \frac{k_+}{2}).
\]

The final expression for \( P_+^*(k_+) \) is then

\[
P_+^*(k_+) = \alpha^2 S \int_0^S dz(S - |z|) \tilde{\xi}(z) \int_0^\infty dk_+^* e^{-2k_+^* a} e^{-(1/2)\Delta^2 k_+^*} \int_0^\infty dk_-^* e^{-(1/4)\Delta^2 k_-^*} W(k_+, k_+^*, k_-^*) \int_0^\infty dk_-^* e^{-(1/4)\Delta^2 k_-^*} \left( e^{-ak_-^*} - e^{-2ak_-^*} \right) \tilde{W}(k_+, k_+^*, k_-).
\]

REFERENCES

Armstrong, J. W., Rickett, B. J., & Spangler, S. R. 1995, ApJ, 443, 209

Ballesteros-Paredes, J., Klessen, R., Mac Low, M., & Vasquez-Semadeni, E. 2007, in Protostars and Planets V, ed. B. Reipurth, D. Jewitt, & K. Keil (Tucson: Univ. Arizona Press), 63

Berensyak, A., & Lazarian, A. 2006, ApJ, 640, L175

———. 2008, ApJ, 682, 1070

Boldyrev, S. 2005, ApJ, 626, L37

Cambresy, L. 1999, A&A, 345, 965

Chandran, B. 2005, Phys. Rev. Lett., 95, 265004

———. 2008, ApJ, in press (arXiv: 0801.4903)

Chepurnov, A., & Lazarian, A. 2006a, preprint (astro-ph/0611463)

———. 2006b, preprint (astro-ph/0611465)

———. 2008, ApJ, submitted

Chepurnov, A., Lazarian, A., Stanimirovic, S., Peek, J., & Heiles, C. 2006, preprint (astro-ph/0611462)

Cho, J., & Lazarian, A. 2002, Phys. Rev. Lett., 88, 245001

———. 2004, ApJ, 615, L41

Cho, J., Lazarian, A., Hoeink, A., Kneepkens, B., Kassinos, S., & Moin, P. 2003, ApJ, 589, L77

Deshpande, A. A., Dwarakanath, K. S., & Goss, W. M. 2000, ApJ, 543, 227

Elmegreen, B., & Scalo, J. 2004, ARA&A, 42, 211

Esquivel, A., & Lazarian, A. 2005, ApJ, 631, 320

Esquivel, A., Lazarian, A., Horibe, S., Ossenkopf, V., Stutzki, J., & Cho, J. 2007, MNRAS, 381, 1733

Esquivel, A., Lazarian, A., Pogosyan, D., & Cho, J. 2003, MNRAS, 342, 325

Gill, A., & Henriksen, R. N. 1990, ApJ, 365, L27

Gogoberidze, G. 2007, Phys. Plasmas, 14, 022304

Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763

Green, D. A. 1993, MNRAS, 262, 327

Khail, A., Joncas, G., Necka, F., Kestener, P., & Arneodo, A. 2006, ApJS, 165, 512

Kowal, G., & Lazarian, A. 2006, preprint (astro-ph/0611396)

Lazarian, A. 1999, in Plasma Turbulence and Energetic Particles in Astrophysics, ed. M. Ostrowski & R. Schlickeiser (Kraw: Obs. Astron., Univ. Jagiellonski), 28

Lazarian, A., & Esquivel, E. 2003, ApJ, 592, L37

Lazarian, A., & Pogosyan, D. 2000, ApJ, 537, 720 (LP00)

———. 2004, ApJ, 616, 943 (LP04)

———. 2006, ApJ, 652, 1348 (LP06)

Lazarian, A., Pogosyan, D., Vázquez-Semadeni, E., & Pichardo, B. 2001, ApJ, 555, 130

Lithwick, Y., Goldreich, P., & Sridhar, S. 2007, ApJ, 655, 269

Mac Low, M.-M., & Klessen, R. S. 2004, Rev. Mod. Phys., 76, 125

McKee, C., & Ostriker, E. 2007, ARA&A, 45, 565

Monin, A. S., & Yaglom, A. M. 1975, Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2 (Cambridge: MIT Press)

Munch, G. 1958, Rev. Mod. Phys., 30, 1035

———. 1999, in Interstellar Turbulence, ed. J. Franco & A. Carraminana (Cambridge: Cambridge Univ. Press), 1

O’Dell, C. R., & Casteneda, H. 1987, ApJ, 317, 686

Ossenkopf, V., Esquivel, A., Lazarian, A., & Stutzki, J. 2006, A&A, 452, 223

Padoan, P., Jimenez, R., Juvela, M., & Nordlund, Å. 2004, ApJ, 604, L49

Padoan, P., Rosolowsky, E. W., & Goodman, A. A. 2001, ApJ, 547, 862

Rosolowsky, E. W., Goodman, A. A., Wilner, D. J., & Williams, J. P. 1999, ApJ, 524, 887

Spitzer, L. 1978, Physical Processes in the Interstellar Medium (New York: Wiley)

Stanimirovic, S., Staveley-Smith, L., Dickey, J. M., Sault, R. J., & Snowden, S. L. 1999, MNRAS, 302, 417

Stutzki, J. 2001, Ap&SS Suppl., 277, 39

Suzuki, T., Lazarian, A., & Berensyak, A. 2007, ApJ, 662, 1033

Vázquez-Semadeni, E., Ostriker, E. C., Passot, T., Gammie, C. F., & Stone, J. M. 2000, in Protostars and Planets IV, ed. V. Mannings, A. P. Boss, & S. S. Russell (Tucson: Univ. Arizona Press), 3