Anisotropy, Cracking and the Stability of Relativistic Fluid Spheres

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January 2019

Abstract. The stability of static, uncharged, anisotropic fluid sphere in general relativity has been studied by considering static spherically symmetric spacetime metric. It has been noticed that anisotropy plays the important role for stability of stellar structure. It is shown that nature of anisotropy can be considered to decide potentially stable/unstable regions. Finally potentially stable/unstable regions of two known models of relativistic star are checked.

Keywords: Anisotropy, Stability, Cracking

1. Introduction

Chandrasekhar[1] studied radial perturbation for isotropic fluid spheres in general relativity and derived the pulsation equation. This method was generalized by Dev and Gleiser[4] for anisotropic fluid distribution. Herrera[5] and DiPrisco et. al.[6] introduced concept of cracking or overturning which describes the alternative way to determine the stability of anisotropic fluid spheres in general relativity. Abreu et. al.[7] and González et. al.[8] proved that the regions for which \(-1 \leq V_{s \perp}^2 - V_{sr}^2 \leq 0\) are potentially stable and the regions for which \(0 < V_{s \perp}^2 - V_{sr}^2 \leq 1\) are potentially unstable, where \(V_{s \perp}^2 = \frac{dp}{d\rho}\) and \(V_{sr}^2 = \frac{dp}{dr}\).

In this work the role of anisotropy in describing potentially stable/unstable regions has been studied. With simple calculations it has been shown that potentially stable/unstable regions can be recognized from gradient of anisotropy with respect to radial variable \(r\).
2. Static, Uncharged, Anisotropic Matter Distribution

The interior spacetime metric for static spherically symmetric fluid distribution is considered as

\[ ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

and the energy-momentum tensor for anisotropic fluid distribution is taken as

\[ T_{ij} = (\rho + p) u_i u_j - p g_{ij} + \pi_{ij}, \]  

where \( \rho \) and \( p \) represent energy-density and isotropic pressure respectively. \( u^i \) is 4-velocity of fluid distribution and \( \pi_{ij} \) represent anisotropic stress tensor which has the form

\[ \pi_{ij} = \sqrt{3}S \left[ C_i C_j - \frac{1}{3} (u_i u_j - g_{ij}) \right], \]  

where \( C^i = (0, -e^{-\lambda/2}, 0, 0) \) is radially directed vector and \( S \) which is function of radial variable \( r \) denotes magnitude of anisotropy. The nonvanishing components of energy-momentum tensor are computed as

\[ T^0_0 = \rho, \quad T^1_1 = - \left( p + 2S \sqrt{3} \right), \quad T^2_2 = T^3_3 = - \left( p - \frac{S}{\sqrt{3}} \right). \]  

The radial pressure \( (p_r) \) and tangential pressure \( (p_\perp) \) can be defined from equation (4) as

\[ p_r = p + \frac{2S}{\sqrt{3}}, \]  

\[ p_\perp = p - \frac{S}{\sqrt{3}}, \]  

and the magnitude of anisotropy takes the form

\[ \sqrt{3}S = p_r - p_\perp. \]  

The Einstein’s field equations corresponding to spacetime metric (1) with energy-momentum tensor (4) by considering \( G = c = 1 \) can be written as

\[ e^{-\lambda} = 1 - \frac{2m}{r}, \]  

\[ r (r - 2m) \nu' = 8\pi p_r r^3 + 2m, \]  

\[ (8\pi \rho + 8\pi p_r) \nu' + 2(8\pi p'_r) = - \frac{4}{r} \left( 8\pi \sqrt{3}S \right). \]  

Combining equations (9) and (10) we get,

\[ (8\pi p'_r) + (8\pi \rho + 8\pi p_r) \left( \frac{4\pi r^3 p_r + m}{r (r - 2m)} \right) + \frac{4}{r} \left( 8\pi \sqrt{3}S \right) = 0, \]  

which is relativistic hydrostatic equilibrium equation. When \( p_r = p_\perp \) i.e. in isotropic case equation (11) takes the form of TOV equation.

For the physically plausible matter configuration \( 0 \leq V^2_{sr} \leq 1 \) and \( 0 \leq V^2_{s\perp} \leq 1 \).

\[ \therefore \quad Min \left( V^2_{s\perp} - V^2_{sr} \right) = -1, \quad Max \left( V^2_{s\perp} - V^2_{sr} \right) = 1. \]
Theorem: For static spherically symmetric spacetime metric (1) with energy-momentum tensor (4) inside the stellar configuration and having decreasing density with respect to radial variable $r$, the potentially stable regions are those for which gradient of anisotropy with respect to radial variable $r$ is less than or equal to zero and potentially unstable regions are those for which gradient of anisotropy with respect to radial variable $r$ is positive.

Proof: Case-I Gradient of anisotropy with respect to radial variable $r$ is less than or equal to zero

\[ \therefore \frac{d}{dr} \left( 8\pi \sqrt{3} S \right) \leq 0 \]

\[ \therefore \frac{d}{d\rho} \left( \frac{8\pi \sqrt{3} S}{d\rho} \right) \geq 0 \]

\[ \therefore \frac{d}{d\rho} \left( 8\pi \rho_r - 8\pi \rho_\perp \right) \geq 0 \]

\[ \therefore V_{sr}^2 - V_{s\perp}^2 \geq 0 \]

\[ \therefore V_{s\perp}^2 - V_{sr}^2 \leq 0 \]

(13)

Hence from (12) and (13) \(-1 \leq V_{s\perp}^2 - V_{sr}^2 \leq 0\), which represent potentially stable regions as described by Abreu et. al. [7].

Case-II Gradient of anisotropy with respect to radial variable $r$ is positive

\[ \therefore \frac{d}{dr} \left( 8\pi \sqrt{3} S \right) > 0 \]

\[ \therefore \frac{d}{d\rho} \left( \frac{8\pi \sqrt{3} S}{d\rho} \right) < 0 \]

\[ \therefore \frac{d}{d\rho} \left( 8\pi \rho_r - 8\pi \rho_\perp \right) < 0 \]

\[ \therefore V_{sr}^2 - V_{s\perp}^2 < 0 \]

\[ \therefore V_{s\perp}^2 - V_{sr}^2 > 0 \]

(14)

Hence from (12) and (14) \(0 \leq V_{s\perp}^2 - V_{sr}^2 \leq 1\), which represent potentially unstable regions as described by Abreu et. al. [7].

3. Analysis of Two Known Models

1. Sharma and Ratanpal[9] model: In their work the graph of anisotropy is shown as figure (7), which clearly indicates \( \frac{d}{dr} \left( 8\pi \sqrt{3} S \right) \leq 0 \). Hence their model is potentially stable as per theorem described in section 2. However there is typographic error in figure 6 of Sharma and Ratanpal[9] model, instead of \( V_{sr}^2 - V_{s\perp}^2 \) it should be \( V_{s\perp}^2 - V_{sr}^2 \).
2. Ratanpal et. al.[10] model: In their work the graph of anisotropy is shown as figure (3). Here $8\pi \sqrt{3} S$ first increases, reaches its maximum value at $r = 2.54$ and then decreases, crosses the x-axis (for variable $r$) at $r = 4.98861$ and further decreases. Hence from $0 \leq r \leq 2.54$ region is potentially unstable and from $2.54 < r \leq 9.69$ region is potentially stable. The adiabatic index $\Gamma > \frac{4}{3}$ throughout the star. Hence further investigation is needed for absolute stable regions.

4. Conclusion

The results presented here suggest that by looking at the graph of anisotropy potentially stable/unstable regions can be identified. However it should be noted that absolute stable regions can not by identified from nature of anisotropy. It can be seen in [10] adiabatic index $\Gamma > \frac{4}{3}$ throughout the matter distribution but the matter distribution has two parts one potentiall unstable and another potentially stable. Hence further analysis is required connecting cracking and adiabatic index $\Gamma$.

Acknowledgement

BSR would like to thank IUCAA, Pune for the facilities and hospitality provided to them where the part of work was carried out.

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