GLOBAL SOBOLEV INEQUALITIES AND DEGENERATE P-LAPLACIAN EQUATIONS

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ABSTRACT. We prove that a local, weak Sobolev inequality implies a global Sobolev
estimate using existence and regularity results for a family of $p$-Laplacian equations.

Given $\Omega \subset \mathbb{R}^n$, let $\rho$ be a quasi-metric on $\Omega$, and let $Q$ be an $n \times n$ semi-definite
matrix function defined on $\Omega$. For an open set $\Theta \subset \Omega$, we give sufficient conditions
to show that if the local weak Sobolev inequality
$$\left( \int_B |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left[ r(B) \int_B |\sqrt{Q} \nabla f|^{p} \, dx + \int_B |f|^p \, dx \right]^{\frac{1}{p}}$$
holds for some $\sigma > 1$, all balls $B \subset \Theta$, and functions $f \in \text{Lip}_0(\Theta)$, then the global
Sobolev inequality
$$\left( \int_\Theta |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left( \int_\Theta |\sqrt{Q} \nabla f(x)|^p \, dx \right)^{\frac{1}{p}}$$
also holds. Central to our proof is showing the existence and boundedness of solutions of the Dirichlet problem
$$\begin{cases}
X_{p,\tau} u = \varphi & \text{in } \Theta \\
u = 0 & \text{in } \partial \Theta,
\end{cases}$$
where $X_{p,\tau}$ is a degenerate $p$-Laplacian operator with a zero order term:
$$X_{p,\tau} u = \text{div} \left( |\sqrt{Q} \nabla u|^{p-2} \nabla u \right) - \tau |u|^{p-2} u.$$

1. Introduction

Given an open set $\Theta \subset \mathbb{R}^n$, the classical Sobolev inequality,
$$\left( \int_\Theta |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left( \int_\Theta |\nabla f|^p \, dx \right)^{\frac{1}{p}},$$
holds for $1 \leq p < n$, $\sigma = \frac{n}{n-p} > 1$, and all functions $f \in Lip_0(\Theta)$ (that is, Lipschitz function such that $\text{supp}(f) \subseteq \Omega$). For this result and extensive generalizations, see [E, GT, HK].

We are interested in determining sufficient conditions for a degenerate Sobolev inequality,

$$\left( \int_{\Theta} |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C \left( \int_{\Theta} |\sqrt{Q} \nabla f|^p \, dx \right)^{\frac{1}{p}},$$

(1.1)

to hold, where $1 < p < \infty$, $\sigma > 1$, and $Q$ is an $n \times n$ matrix of measurable functions defined on $\Theta$ such that for almost every $x \in \Theta$, $Q(x)$ is semi-definite. Such inequalities arise naturally in the study of degenerate elliptic PDEs: a global Sobolev inequality is necessary to prove the existence of weak solutions (see, for instance [CMN, MR]) and to prove compact embeddings of (degenerate) Sobolev spaces (see [CRW]).

Our goal is to show that such global estimates can be derived from weaker, local Sobolev inequalities.

**Definition 1.1.** Given $1 \leq p < \infty$ and $\sigma > 1$, a local Sobolev property of order $p$ with gain $\sigma$ holds in $\Omega$ if there is a constant $C_0 > 0$ and a positive, continuous function $r_1: \Omega \to (0, \infty)$ such that for any $y \in \Omega$, $0 < r < r_1(y)$, and $f \in Lip_0(B(y,r))$,

$$\left( \int_{B(y,r)} |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C_0 r \left( \int_{B(y,r)} |\sqrt{Q} \nabla f|^p \, dx \right)^{\frac{1}{p}} + C_0 \left( \int_{B(y,r)} |f|^p \, dx \right)^{\frac{1}{p}},$$

(1.2)

Local Sobolev estimates arise naturally in the study of regularity of degenerate elliptic equations (see [SW1, SW2]), but they are not sufficient for proving the existence of solutions. So it is natural to ask if local inequalities imply global ones. The obvious approach is to use a partition of unity argument, but this does not work. If the local Sobolev property of order $p$ with gain $\sigma$ holds on $\Omega$, then given any open set $\Theta \subseteq \Omega$ a partition of unity argument shows that there is a constant $C(\Theta)$ such that

$$\left( \int_{\Theta} |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C(\Theta) \left[ \left( \int_{\Theta} |\sqrt{Q} \nabla f|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Theta} |f|^p \, dx \right)^{\frac{1}{p}} \right],$$

(1.3)

holds for every $f \in Lip_0(\Theta)$. However, we cannot remove the second term on the right of (1.3) even when the second term on the right of (1.2) is not present.

Nevertheless, with some additional assumptions we are able to pass from a local to a global Sobolev inequality. To set the stage for our main result, we first state an important special case.

**Theorem 1.2.** Fix $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $Q$ is a semi-definite matrix function in $L^\infty(\Omega)$. Suppose that the local Sobolev property
of order \( p \) with gain \( \sigma > 1 \) (1.2) holds, and suppose further that a local Poincaré inequality

\[
\left( \int_{B(y,r)} |f - f_{B(y,r)}|^p \, dx \right)^{\frac{1}{p}} \leq Cr \left( \int_{B(y,\beta r)} |\sqrt{Q \nabla f}|^p \, dx \right)^{\frac{1}{p}}
\]

holds for \( y \in \Omega, \beta \geq 1, \, 0 < \beta r < r_1(y), \) and \( f \in \text{Lip}_0(\Omega) \). Then given any open set \( \Theta \Subset \Omega \), there exists a constant \( C(\Theta) \) such that (1.1) holds.

Our main result, Theorem 2.7, generalizes Theorem 1.2 in several ways. First, we remove the assumption that \( Q \) is bounded, allowing it to be singular as well as degenerate. Second, we can change the underlying geometry by replacing the Euclidean metric with a quasi-metric, and defining balls with respect to this metric. The statement is rather technical and requires some additional hypotheses, which is why we have deferred the statement until below.

The remainder of the paper is organized as follows. In Section 2 we give the necessary assumptions and definitions and then state Theorem 2.7. In Section 3 we give an application of Theorem 2.7 to a family of Lipschitz vector fields. Such vector fields are a natural example of where degenerate \( p \)-Laplacians arise.

A central and somewhat surprising part of our proof of Theorem 2.7 is to prove the existence and boundedness of solutions of the Dirichlet problem

\[
\begin{cases}
X_{p,\tau} u = \varphi \text{ in } \Theta \\
u = 0 \text{ in } \partial \Theta,
\end{cases}
\]

where \( X_{p,\tau} \) is a degenerate \( p \)-Laplacian operator with a zero order term:

\[
X_{p,\tau} u = \text{div} \left( |\sqrt{Q \nabla u}|^{p-2} Q \nabla u \right) - \tau |u|^{p-2} u.
\]

We prove the existence of solutions using Minty’s theorem in Section 4 and we prove boundedness using ideas from [CRW, MRW1] in Section 5. Finally, in Section 6 we prove Theorem 2.7.

Throughout this paper, \( \Omega \) will be a fixed open, connected subset of \( \mathbb{R}^n \). We say an open set \( \Theta \) is compactly contained in \( \Omega \) and write \( \Theta \Subset \Omega \) if \( \Theta \) is bounded and \( \overline{\Theta} \subset \Omega \). The set \( \text{Lip}_0(\Omega) \) consists of all Lipschitz functions \( f \) such that \( \text{supp}(f) \Subset \Omega \). A constant \( C \) may vary from line to line; if necessary we will denote the dependence of the constant on various parameters by writing, for instance, \( C(p) \).

2. The main result

In order to state Theorem 2.7 we need to make some technical assumptions and give some additional definitions. We begin with the topological framework. Fix \( \Omega \subset \mathbb{R}^n \) and let \( \rho : \Omega \times \Omega \to \mathbb{R} \) be a symmetric quasimetric on \( \Omega \): that is, there is a constant \( \kappa \geq 1 \) so that for all \( x, y, z \in \Omega \):
Given $x \in \Omega$ and $r > 0$ we will always denote the $\rho$-ball of radius $r$ centered at $x$ by $B(x, r)$; that is $B(x, r) = \{ y \in \Omega : \rho(x, y) < r \}$. We will assume that the balls $B(x, r)$ are Lebesgue measurable. We will also use $C(x, r)$ to denote the corresponding Euclidean ball $\{ x \in \Omega : |x - y| < r \}$. We will always assume that the quasi-metric $\rho$ and the Euclidean distance satisfy the following:

\begin{equation}
(2.1) \quad \text{given } x, y \in \Omega, \ |x - y| \to 0 \text{ if and only if } \rho(x, y) \to 0.\end{equation}

Equivalently, we may assume that given $x \in \Omega$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $D(x, \delta) \subset B(x, \epsilon)$, and that given any $\delta > 0$ there exists $\gamma > 0$ such that $B(x, \gamma) \subset D(x, \delta)$. Either of these conditions hold if we assume that the topology generated by the balls $B(x, r)$ is equivalent to the Euclidean topology on $\Omega$.

**Remark 2.1.** This assumption on the topology of $(\Omega, \rho)$ is taken from [MRW1]; it is closely related to a condition first assumed by C. Fefferman and Phong [FP] (see also [SW1]).
holds for every \( f \in \text{Lip}(B(y, \beta r)) \) such that \( \sqrt{Q} \nabla f \in L^p(B(y, \beta r)) \), and where \( f_{B(y,r)} = \int_{B(y,r)} f(x) \, dx \).

**Remark 2.3.** Inequality (2.2) will be used to establish that the embedding of the degenerate Sobolev space \( \hat{H}^{1,p}_Q \) (that we will define below) into \( L^p \) is compact. We will also use it to prove a product rule for functions in this Sobolev space. The parameter \( t' \) is determined by the regularity of \( Q \): the more regular \( Q \) is, the smaller \( t' \) is permitted to be. In fact, if \( Q \) is locally bounded in \( \Omega \), (2.2) is not required to establish the product rule: see the proof of Lemma 6.1.

**Remark 2.4.** The problem of determining sufficient conditions on the matrix \( Q \) for the Poincaré inequality (2.2) to hold has been considered in a somewhat different form in [CIM, CRR]. It is interesting to note that in this case the condition involves the solution of a Neumann problem for a degenerate \( p \)-Laplacian operator.

Our final definition is a technical assumption on the geometry of \((\Omega, \rho)\). This condition, which we refer to as the “cutoff” condition, ensures the existence of accumulating sequences of Lipschitz cutoff functions on \( \rho \)-balls. Again, the function \( r_1 \) is assumed to be the same as in Definition (1.2).

**Definition 2.5.** Given \((\Omega, \rho)\) and a matrix \( Q \), a cutoff condition of order \( 1 \leq s \leq \infty \) holds if there exist constants \( C_3, N > 0 \) and \( 0 < \alpha < 1 \) such that given \( x \in \Omega \) and \( 0 < r < r_1(x) \) there exists a sequence \( \{\psi_j\}_{j=1}^{\infty} \subset \text{Lip}_0(B(x,r)) \) such that for all \( j \in \mathbb{N} \),

\[
\begin{align*}
0 &\leq \psi_j \leq 1, \\
\text{supp } \psi_j &\subset B(x,r), \\
B(x, \alpha r) &\subset \{ y \in B(x,r) : \psi_j(y) = 1 \}, \\
\text{supp } \psi_{j+1} &\subset \{ y \in B(x,r) : \psi_j(y) = 1 \}, \\
\left( \int_{B(x,r)} |\sqrt{Q(y)} \nabla \psi_j(y)|^s dy \right)^{\frac{1}{s}} &\leq C_3 \frac{N^j}{r}.
\end{align*}
\]

**Definition 2.5** first appeared in [SW1], though it is a generalization of a concept that has appeared previously in the literature; see [SW1] for further references. If \( \rho \) is the Euclidean metric and \( Q \) is bounded, then this sequence of cutoff functions can be taken to be the standard Lipschitz cutoff functions on \( \rho \)-balls. More generally, it was shown in [SW1] that with our assumptions on \( \rho \), if \( Q \) is continuous, then such a sequence exists with \( s = \infty \). This cutoff condition holds for a wide variety of geometries, including \( \rho \) which produce highly degenerate balls. See, for example, [M].

**Remark 2.6.** There is a close connection between the cutoff condition and doubling. In [KMR] it was shown that if the local Sobolev property of order \( p \) and gain \( \sigma \) and the cutoff condition (2.3) both hold, then Lebesgue measure is locally doubling for the
collection of $\rho$-balls $\{B(x,r)\}_{x \in \Omega, r > 0}$. That is, there exists a positive constant $C$ so that given any $x \in \Omega$ and $0 < r < r_1(x)$ then $|B(x,2r)| \leq C|B(x,r)|$. Consequently, for any $0 < r \leq s < r_1(x)$,
\begin{equation}
|B(x,s)| \leq \tilde{C}(\frac{s}{r})^{d_0}|B(x,r)|,
\end{equation}
where $d_0 = \log_2(C)$. We will use this fact to prove Proposition 5.3 below.

We can now state our main result.

**Theorem 2.7.** Given a set $\Omega \subset \mathbb{R}^n$, let $\rho$ be a quasi-metric on $\Omega$. Fix $1 < p < \infty$ and $1 < t \leq \infty$, and suppose $Q$ is a semi-definite matrix function such that $Q \in L^{\infty}_{\text{loc}}(\Omega)$. Suppose further that the cutoff condition of order $s > p\sigma'$, the local Poincaré property of order $p$ with gain $t' = \frac{s}{s-p}$, and the local Sobolev property of order $p$ with gain $\sigma \geq 1$ hold. Then, given any open set $\Theta \subset \Omega$ there is a positive constant $C(\Theta)$ such that the global Sobolev inequality
\begin{equation}
\left( \int_\Theta |f|^{ps}dx \right)^{\frac{1}{ps}} \leq C(\Theta) \left( \int_\Theta \sqrt{Q} \nabla f|^{p}dx \right)^{\frac{1}{p}}
\end{equation}
holds for all $f \in \text{Lip}_0(\Theta)$.

**Remark 2.8.** If $Q \in L^{\infty}_{\text{loc}}(\Omega)$, then we can take $t = \infty$ and $t' = 1$, so that $s = \infty$. Thus, we only need to assume a local Poincaré inequality of order $p$ without gain. If we assume that $\rho$ is the Euclidean metric, then as we noted above, the cutoff condition holds with $s = \infty$. Thus, Theorem 1.2 follows immediately from Theorem 2.7.

### 3. Example: diagonal Lipschitz vector fields

In this section we give an illustrative example of the application of Theorem 2.7. Let $\Omega$ be any bounded domain in $\mathbb{R}^n$ and let $1 < p < \infty$. Fix a vector function $a = (a_1, \ldots, a_n)$ where $a_1 = 1$ and $a_2, \ldots, a_n : \Omega \to \mathbb{R}$ are such that for $2 \leq j \leq n$, $a_j$ is bounded, nonnegative and Lipschitz continuous. Further, assume that the $a_j$ satisfy the $RH_\infty$ condition in the first variable $x_1$ uniformly in $x_2, \ldots, x_n$; there exists a constant $C$ for each interval $I$ and $x = (x_1, \ldots, x_n) \in \Omega$,
\begin{equation}
a_j(x_1, \ldots, x_n) \leq C \int_I a(z_1, x_2, \ldots, x_n)dz_1.
\end{equation}
For instance, we can take $a_j(x_1, \ldots, x_n) = |x_1|^{\alpha_j} b_j(x_2, \ldots, x_n)$, where $\alpha_j \geq 0$ and $b_j$ is a non-negative Lipschitz function. (For more on the $RH_\infty$ condition, see [CN].)

Now let $X_j = a_j \frac{\partial}{\partial x_j}$ and $\nabla_a = (X_1, \ldots, X_n)$, and define the associated $p$-Laplacian
\begin{equation}
L_{p,a}u = \text{div}_a \left(|\nabla_a u|^{p-2} \nabla_a u \right).
\end{equation}
If we let \( Q(x) = \text{diag}(1, a_2^2, \ldots, a_n^2) \), then we have that \( L_{p,a} = \text{div}\left(|\sqrt{Q}\nabla u|^{p-2} Q\nabla u\right) \).

It is shown in [MRW2] that each of Definitions 1.1, 2.2, and 2.5 hold with respect the family of non-interference balls \( A(x,r) \) defined as in [SW1], and there exists a quasi-metric \( \rho \) such that the non-interference balls are equivalent to the \( \rho \)-balls. In fact, setting \( B(x,r) = A(x,r) \) and \( r_1(x) = \delta' \text{dist}(x, \partial \Omega) \) for \( \delta' > 0 \) sufficiently small depending on \( \|Q\|_{\infty} \), condition (2.3) holds with \( s = \infty \), (2.2) holds with \( t' = 1 \) and (1.2) holds with \( \sigma = d_0 d_0^{-p} \) where \( d_0 \) is the doubling exponent associated to Lebesgue measure and the collection of balls \( A(x,r) \), as in (2.4). As a result, both [MRW1, (1.15) and (1.16)] hold with \( t = \infty \) and \( t' = 1 \). Therefore, we can apply Theorem 2.7 to get that for any open subdomain \( \Theta \subset \subset \Omega \), there exists a constant \( C(\Theta) \) such that the Sobolev inequality

\[
(3.2) \quad \left( \int_{\Theta} |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C(\Theta) \left( \int_{\Theta} |\sqrt{Q} \nabla f|^{p} \, dx \right)^{\frac{1}{p}}
\]

holds for all \( f \in \text{Lip}_0(\Theta) \). Moreover, by the doubling property (2.4), if we let \( \Theta = A(x,r) \) for \( 0 < r < \delta \text{dist}(x, \partial \Omega) \), then we get

\[
\left( \int_{A(x,r)} |f|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} \leq C r \left( \int_{A(x,r)} |\sqrt{Q} \nabla f|^{p} \, dx \right)^{\frac{1}{p}}
\]

for any \( f \in \text{Lip}_0(A(x,r)) \).

As a consequence, when \( p = 2 \) inequality (3.2) is sufficient to use [R, Theorem 3.10] to prove the existence of a unique weak solution of the linear Dirichlet problem

\[
\begin{cases}
\text{div } (Q\nabla u) = f & \text{in } \Theta \\
u = 0 & \text{on } \partial \Theta.
\end{cases}
\]

4. Weak solutions of degenerate \( p \)-Laplacians

A key step in the proof of Theorem 2.7 is to prove the existence and boundedness of solutions of the Dirichlet problem

\[
(4.1) \quad \begin{cases}
X_{p,\tau} u = \varphi & \text{in } \Theta \\
u = 0 & \text{in } \partial \Theta,
\end{cases}
\]

where \( X_{p,\tau} \) is a degenerate \( p \)-Laplacian operator with a zero order term:

\[
(4.2) \quad X_{p,\tau} u = \text{div}\left(|\sqrt{Q}\nabla u|^{p-2} Q\nabla u\right) - \tau |u|^{p-2} u.
\]

In this section we will define weak solutions to this equation and prove that they exist.

As the first step we define the degenerate Sobolev spaces related to \( Q \). Detailed discussions of these spaces can be found in [CMN, CRR, CRW, MRW1, MRW2, SW2];
here we will sketch the key ideas and refer the reader to these references for further information. Fix \( 1 \leq p < \infty \) and a matrix function \( Q \) such that \( \sqrt{Q} \in L^p_{\text{loc}}(\Omega) \). Fix an open set \( \Theta \Subset \Omega \), and for \( 1 \leq p < \infty \) define \( \mathcal{L}_Q^p(\Theta) \) to be the collection of all measurable \( \mathbb{R}^n \) valued functions \( f = (f_1, \ldots, f_n) \) that satisfy

\[
\|f\|_{\mathcal{L}_Q^p(\Theta)} = \left( \int_\Theta |\sqrt{Q}f|^p \, dx \right)^{1/p} < \infty.
\]

More properly we define \( \mathcal{L}_Q^p(\Theta) \) to be the normed vector space of equivalence classes under the equivalence relation \( f \equiv g \) if \( \|f - g\|_{\mathcal{L}_Q^p(\Theta)} = 0 \). Note that if \( f(x) = g(x) \) a.e., then \( f \equiv g \), but the converse need not be true, depending on the degeneracy of \( Q \).

Let \( \text{Lip}_Q(\Theta) \) be the collection of all functions \( f \in \text{Lip}_{\text{loc}}(\Theta) \) such that \( f \in L^p(\Theta) \) and \( \nabla f \in \mathcal{L}_Q^p(\Theta) \). We now define the corresponding degenerate Sobolev space \( \hat{H}_Q^{1,p}(\Theta) \) to be the formal closure of \( \text{Lip}_Q(\Theta) \) with respect to the norm

\[
\|f\|_{\hat{H}_Q^{1,p}(\Theta)} = \left[ \int_\Theta |f|^p \, dx + \int_\Theta Q(x, \nabla f) \frac{\|f\|_{\mathcal{L}_Q^p(\Theta)}}{\sqrt{Q}} \, dx \right]^{1/p} = \left[ \int_\Theta |f|^p \, dx + \int_\Theta |\sqrt{Q} \nabla f|^p \, dx \right]^{1/p}.
\]

Similarly, define \( \hat{H}_{Q,0}^{1,p}(\Theta) \subset \hat{H}_Q^{1,p}(\Theta) \) to be the formal closure of \( \text{Lip}_{0}(\Theta) \) with respect to this norm.

Because of the degeneracy of \( Q \), we cannot represent either \( \hat{H}_{Q,0}^{1,p}(\Theta) \) or \( \hat{H}_Q^{1,p}(\Theta) \) as spaces of functions except in special situations. But, since \( L^p(\Theta) \) and \( \mathcal{L}_Q^p(\Theta) \) are complete, given an equivalence class of \( \hat{H}_Q^{1,p}(\Theta) \) there exists a unique pair \( \tilde{f} = (f, g) \in L^p(\Theta) \times \mathcal{L}_Q^p(\Theta) \) that we can use to represent it. Such pairs are unique and so we refer to elements of \( \hat{H}_Q^{1,p}(\Theta) \) using their representative pair. However, because of the classical example in [FKS], \( g \) need not be uniquely determined by the first component \( f \) of the pair: if we think of \( g \) as the “gradient” of \( f \), then there exist non-constant functions \( f \) whose gradient is 0.

On the other hand, since \( \sqrt{Q} \in L^p_{\text{loc}}(\Omega) \) and since constant sequences are Cauchy, if \( f \in \text{Lip}_Q(\Theta) \), then \( (f, \nabla f) \in \hat{H}_Q^{1,p}(\Theta) \) where \( \nabla f \) is the classical gradient of \( f \) in \( \Theta \); see [GT].

We need one structural result about these Sobolev spaces. The following result is proved in [CRR] for the space \( H_Q^{1,p}(\Theta) \), which is the closure of \( C^1(\Theta) \) with respect to the \( \hat{H}_Q^{1,p}(\Theta) \) norm, but the proof is identical in our case.

**Lemma 4.1.** Given \( 1 \leq p < \infty \), \( \Theta \subset \Omega \), and a matrix \( Q \), \( \hat{H}_Q^{1,p}(\Theta) \) and \( \hat{H}_{Q,0}^{1,p}(\Theta) \) are separable Banach spaces. If \( p > 1 \), they are reflexive.

We now define the weak solution of the Dirichlet problem (4.1) for equation (4.2). We will assume that \( 1 < p < \infty, \tau \geq 0, \varphi \in L^{p'}_{\text{loc}}(\Omega) \), and \( \sqrt{Q} \in L^p_{\text{loc}}(\Omega) \). Associated
to the Dirichlet problem is the non-linear form

\[ A_{p,\tau}(\vec{u}) : \tilde{H}^{1,p}_{Q}(\Theta) \times \tilde{H}^{1,p}_{Q}(\Theta) \rightarrow \mathbb{R}, \]

defined for \( \vec{u} = (u, \vec{g}) \) and \( \vec{v} = (v, \vec{h}) \) by

\[ (4.4) \quad A_{p,\tau}(\vec{u})(\vec{v}) = \int_{\Theta} \langle Q\vec{g}, \vec{g} \rangle^{\frac{p-2}{2}} \langle Q\vec{g}, \vec{h} \rangle \, dx + \tau \int_{\Theta} |u|^{p-2}uv \, dx; \]

we use the convention that \( A_{p,\tau}(\vec{u}) = 0 \) if \( 1 < p < 2 \). The notation used on the left-hand side of (4.4) is meant to suggest that for each fixed \( \vec{u} = (u, \vec{g}) \in \tilde{H}^{1,p}_{Q}(\Theta) \), the operator \( A_{p,\tau}(\vec{u})(\cdot) \in (\tilde{H}^{1,p}_{Q}(\Theta))' \); see Lemma 4.7 below.

We use this form to define a weak solution.

**Definition 4.2.** A weak solution to the Dirichlet problem (4.1) is an element \( \vec{u} = (u, \vec{g}) \in \tilde{H}^{1,\infty}_{Q,0}(\Theta) \) such that the equality

\[ (4.5) \quad A_{p,\tau}(\vec{u})(\vec{v}) = \int_{\Theta} \langle Q\vec{g}, \vec{g} \rangle^{\frac{p-2}{2}} \langle Q\vec{g}, \nabla \vec{v} \rangle \, dx + \tau \int_{\Theta} |u|^{p-2}uv \, dx = -\int_{\Theta} \varphi \, v \, dx \]

holds for every \( v \in \text{Lip}_{0}(\Theta) \).

**Remark 4.3.** If \( \vec{u} \in \tilde{H}^{1,\infty}_{Q,0}(\Theta) \) is a weak solution, then by a standard density argument we have that (4.5) holds if we replace \( \langle v, \nabla v \rangle \) with any \( \vec{v} = (v, \vec{h}) \in \tilde{H}^{1,\infty}_{Q,0}(\Theta) \).

We can now state and prove our existence result.

**Proposition 4.4.** Let \( \Omega \subset \mathbb{R}^n \) be open. Given \( 1 < p < \infty \) and \( \tau > 0 \), suppose \( Q \in L_{\text{loc}}^{\frac{2}{p}}(\Omega) \). Then, for any open set \( \Theta \subset \Omega \) the Dirichlet problem (4.1) with \( \varphi \in L^{p'}(\Theta) \) has a weak solution \( \vec{u} = (u, \vec{g}) \in \tilde{H}^{1,\infty}_{Q,0}(\Theta) \).

To prove Proposition 4.4 we will use Minty’s theorem as found in [Sh]; this result is a generalization of the Lax-Milgram theorem to general Banach spaces. To state it we fix some notation. Let \( X \) be a separable, reflexive Banach space with norm \( \| \cdot \|_X \), and let \( X^* \) denote its dual space. Given a map \( T : X \rightarrow X^* \) and \( u, v \in X \), we will write \( T(u)(v) = \langle T(u), v \rangle \).

**Theorem 4.5.** (Minty) Let \( X \) be a separable, reflexive Banach space and fix \( \Gamma \in X^* \). Let \( T : X \rightarrow X^* \) be an operator that is:

- bounded: \( T \) maps bounded subsets of \( X \) to bounded subsets of \( X' \);
- monotone: \( \langle T(u) - T(v), u - v \rangle \geq 0 \) for all \( u, v \in X \);
- hemicontinuous: for \( z \in \mathbb{R} \), the mapping \( z \mapsto T[u + zv](v) \) is continuous for all \( u, v \in X \);
- almost coercive: there exists \( \beta > 0 \) such that \( \langle Tv, v \rangle > \langle \Gamma, v \rangle \) for all \( v \in X \) such that \( \|v\|_X > \beta \).
Then the set \( u \in X \) such that \( T(u) = \Gamma \) is non-empty.

To apply Theorem 4.5 to solve the Dirichlet problem (4.1), let \( X = \hat{H}_{Q,0}^{1,p}(\Theta) \); then by Lemma 4.1, \( X \) is a separable, reflexive Banach space. Fix \( \varphi \in L^p(\Theta) \); given \( \vec{v} = (v, \vec{h}) \), define \( \Gamma \in X^* \) by

\[
\Gamma(\vec{v}) = -\int_\Theta \varphi v \, dx.
\]

Let \( T = A_{p,\tau} \); then \( \vec{u} \in \hat{H}_{Q,0}^{1,p}(\Theta) \) is a weak solution if \( A_{p,\tau}(\vec{u})(\vec{v}) = \Gamma(\vec{v}) \) for every \( \vec{v} = (v, \vec{h}) \in \hat{H}_{Q,0}^{1,p}(\Theta) \). By Minty’s theorem, such a \( \vec{u} \) exists if \( A_{p,\tau} \) is bounded, monotone, hemicontinuous, and almost coercive. To complete the proof of Proposition 4.4, we will prove each of these properties in turn.

We begin with three useful inequalities which we record as a lemma. For their proof, see [PL, Ch. 10].

**Lemma 4.6.** For all \( s, r \in \mathbb{R}^n \),

\[
\langle |s|^{p-2} s - |r|^{p-2} r, s - r \rangle \geq 0;
\]

if \( p \geq 2 \),

\[
|s|^{p-2} s - |r|^{p-2} r \leq c(p) (|s|^{p-2} + |r|^{p-2}) |s - r|;
\]

if \( 1 < p \leq 2 \),

\[
|s|^{p-2} s - |r|^{p-2} r \leq c(p) |s - r|^{p-1}.
\]

**Lemma 4.7.** \( A_{p,\tau} \) is bounded on \( \hat{H}_{Q,0}^{1,p}(\Theta) \) for all \( 1 < p < \infty \) and \( \tau \in \mathbb{R} \).

**Proof.** Fix \( 1 < p < \infty \) and \( \tau \in \mathbb{R} \). Let \( \vec{u} = (u, \vec{g}), \vec{v} = (v, \vec{h}) \in \hat{H}_{Q,0}^{1,p}(\Theta) \). If we apply Hölder’s inequality twice, then we have that

\[
|A_{p,\tau}(\vec{u})(\vec{v})| \leq \left| \int_{\Theta} \langle Q\vec{g}, \vec{g} \rangle^\frac{p-2}{2} \langle Q\vec{g}, \vec{h} \rangle \, dx \right| + |\tau| \left| \int_{\Theta} |u|^{p-2} uv \, dx \right|
\]

\[
\leq \int_{\Theta} \langle Q\vec{g}, \vec{g} \rangle^\frac{p-1}{2} \langle Q\vec{h}, \vec{h} \rangle^\frac{1}{2} \, dx + |\tau| \|u\|_{L^p(\Theta)}^{p-1} \|v\|_{L^p(\Theta)}
\]

\[
\leq \|Q\sqrt{\vec{g}}\|_{L^p(\Theta)}^{p-1} \|Q\sqrt{\vec{h}}\|_{L^p(\Theta)} \|v\|_{L^p(\Theta)} + |\tau| \|u\|_{L^p(\Theta)}^{p-1} \|v\|_{L^p(\Theta)}
\]

\[
\leq (1 + |\tau|) \|\vec{u}\|_{\hat{H}_{Q,0}^{1,p}(\Theta)}^{p-1} \|\vec{v}\|_{\hat{H}_{Q,0}^{1,p}(\Theta)}.
\]

It follows at once from this inequality that \( A_{p,\tau} \) is bounded. \( \square \)

**Lemma 4.8.** \( A_{p,\tau} \) is monotone for all \( 1 < p < \infty \) and \( \tau \geq 0 \).
**Proof.** Fix $p$ and $\tau$, and let $\overline{u}, \overline{v} \in \hat{H}^{1,p}(\Theta)$ be as in Lemma 4.7. Then,

$$\langle A_{p,\tau} \overline{u} - A_{p,\tau} \overline{v}, \overline{u} - \overline{v} \rangle = \int_{\Theta} \langle Q \overline{g}, \overline{g} \rangle \frac{p-2}{2} \langle Q \overline{g}, \overline{g} - \overline{h} \rangle \, dx - \int_{\Theta} \langle Q \overline{h}, \overline{h} \rangle \frac{p-2}{2} \langle Q \nabla \overline{h}, \overline{g} - \overline{h} \rangle \, dx + \tau \int_{\Theta} \left( |u|^{p-2}u - |v|^{p-2}v \right) (u - v) \, dx \equiv I_1 + \tau I_2.$$

We estimate $I_1$ and $I_2$ separately, beginning with $I_2$. By inequality (4.7),

$$I_2 = \int_{\Theta} \langle Q \overline{g}, \overline{g} \rangle \frac{p-2}{2} \langle Q \overline{g}, \overline{g} - \overline{h} \rangle \, dx \geq 0.$$

To estimate $I_1$ note that by the symmetry of $Q$ we have that $\langle Q \overline{g}, \overline{g} \rangle = |\sqrt{Q} \overline{g}|$. Hence,

$$I_1 = \int_{\Theta} |\sqrt{Q} \overline{g}|^{p-2} \langle Q \overline{g}, \overline{g} - \overline{h} \rangle \, dx - \int_{\Theta} |\sqrt{Q} \overline{h}|^{p-2} \langle Q \overline{h}, \overline{g} - \overline{h} \rangle \, dx = \int_{\Theta} \langle |\sqrt{Q} \overline{g}|^{p-2} \sqrt{Q} \overline{g} - |\sqrt{Q} \overline{h}|^{p-2} \sqrt{Q} \overline{h}, \sqrt{Q} \overline{g} - \sqrt{Q} \overline{h} \rangle \, dx.$$

With $s = \sqrt{Q} \overline{g}$ and $r = \sqrt{Q} \overline{h}$, the integrand is again of the form (4.7) and so non-negative. Thus $I_1 \geq 0$ and our proof is complete. \qed

**Lemma 4.9.** $A_{p,\tau}$ is hemicontinuous for all $1 < p < \infty$ and $\tau \in \mathbb{R}$.

**Proof.** Fix $\overline{u}, \overline{v} \in \hat{H}^{1,p}_{Q,0}(\Theta)$ as in the previous lemmas. For $z \in \mathbb{R}$, let $z \overline{v} = z(v, \overline{h}) = (zv, z\overline{h}) \in \hat{H}^{1,p}_{Q,0}(\Theta)$; we will show that the function $z \mapsto A_{p,\tau}(\overline{u} + z \overline{v})(\overline{v})$ is continuous. By the definition of $A_{p,\tau}$ we can split this mapping into the sum of two parts:

$$z \mapsto G_p(\overline{u} + z \overline{v})(\overline{v}) = \int_\Theta \langle Q(\overline{g} + z \overline{h}), (\overline{g} + z \overline{h}) \rangle \frac{p-2}{2} \langle Q(\overline{g} + z \overline{h}), \overline{h} \rangle \, dx,$$

$$z \mapsto H_{p,\tau}(\overline{u} + z \overline{v})(\overline{v}) = \int_\Theta \tau |u + zv|^{p-2}(u + zv) \, dx.$$

We will show each part is continuous in turn.

To show that the mapping $z \mapsto G_p(\overline{u} + z \overline{v})(\overline{v})$ is continuous, we modify an argument from the proof of [CMN, Proposition 3.15]. Fix $z, w \in \mathbb{R}$; then (since $Q = \sqrt{Q} \sqrt{Q}$
is symmetric),
\[
\left| G_p(\bar{u} + z\bar{v})(\bar{v}) - G_p(\bar{u} + w\bar{v})(\bar{v}) \right| \\
= \left| \int_\Theta |\sqrt{Q}(\bar{g} + z\bar{h})|^{p-2}\langle \sqrt{Q}(\bar{g} + z\bar{h}), \sqrt{Q}\bar{h} \rangle \, dx \right. \\
- \left. \int_\Theta |\sqrt{Q}(\bar{g} + w\bar{h})|^{p-2}\langle \sqrt{Q}(\bar{g} + w\bar{h}), \sqrt{Q}\bar{h} \rangle \, dx \right| \\
\leq \int_\Theta ||\sqrt{Q}(\bar{g} + z\bar{h})|^{p-2}\sqrt{Q}(\bar{g} + z\bar{h}) - |\sqrt{Q}(\bar{g} + w\bar{h})|^{p-2}\sqrt{Q}(\bar{g} + w\bar{h})| \, |\sqrt{Q}\bar{h}| \, dx;
\]
if \( p \geq 2 \), then by (4.8) and Hölder’s inequality with exponent \( \frac{p}{p-2} \),
\[
\leq C(p) \int_\Theta \left( (|\sqrt{Q}(\bar{g} + z\bar{h})|^{p-2} + |\sqrt{Q}(\bar{g} + w\bar{h})|^{p-2})|z - w||\sqrt{Q}\bar{h}|^2 \, dx \right) \\
\leq C(p) \left( \int_\Theta (|\sqrt{Q}(\bar{g} + z\bar{h})|^{p-2} + |\sqrt{Q}(\bar{g} + w\bar{h})|^{p-2}) \frac{p}{p-2} \, dx \right)^{\frac{p-2}{p}} \\
\times |z - w| \left( \int_\Theta |\sqrt{Q}\bar{h}|^p \, dx \right)^{\frac{2}{p}} \\
\leq C(p)|z - w| (||\bar{g}||_{L^p(\Theta)} + (|z| + |w|)||\bar{h}||_{L^p(\Theta)})^{p-2} ||\bar{h}||_{L^p(\Theta)}^2.
\]
Since the norms in the final term are all finite, we see that this term tends to 0 as \( w \to z \); thus the mapping \( z \mapsto G_p(\bar{u} + z\bar{v})(\bar{v}) \) is continuous, when \( p \geq 2 \).

When \( 1 < p < 2 \), we can essentially repeat the above argument but instead apply (4.9) to get that
\[
\left| G_p(\bar{u} + z\bar{v})(\bar{v}) - G_p(\bar{u} + w\bar{v})(\bar{v}) \right| \leq C(p) \int_\Theta |z - w|^{p-1}|\sqrt{Q}\bar{h}|^p \, dx,
\]
and the desired continuity again follows.

To show that the mapping \( z \mapsto H_{p,\tau}(\bar{u} + z\bar{v})(\bar{v}) \) is continuous, again fix \( z, w \in \mathbb{R} \).

Then
\[
\left| H_{p,\tau}(\bar{u} + z\bar{v})(\bar{v}) - H_{p,\tau}(\bar{u} + w\bar{v})(\bar{v}) \right| \\
= \left| \int_\Theta \tau|u + zv|^{p-2}(u + zv) - \tau|u + wv|^{p-2}(u + wv) \, v \, dx \right| \\
\leq |\tau| \int_\Theta \left( |u + zv|^{p-2}(u + zv) - |u + wv|^{p-2}(u + wv) \right) |v| \, dx.
\]
The integrand in the final term tends to 0 pointwise as \( w \to z \), so the desired continuity will follow by the dominated convergence theorem if we can prove that the
integrand is dominated by an integrable function. But we have that
\[
|u + zv|^{p-2}(u + zv) - |u + wv|^{p-2}(u + wv)|v| \\
\leq |u + zv|^{p-1}|v| + |u + wv|^{p-1}|v| \\
\leq C(p)(|u|^{p-1}|v| + (|z|^{p-1} + |w|^{p-1})|v|^{p}) \\
\leq C(p, |z|)(|u|^{p-1}|v| + |v|^{p}).
\]

By Hölder’s inequality, the final term is in \( L^1(\Theta) \). Hence, we have that the mapping \( z \mapsto H_{p, \tau}(\hat{u} + z\hat{v})(\vec{v}) \) is continuous and this completes the proof.

\[ \square \]

**Lemma 4.10.** Given \( 1 < p < \infty \) and \( \varphi \in L^{p^\prime}(\Theta) \), define \( \Gamma \) by (4.6). Then for all \( \tau > 0 \), \( A_{p, \tau} \) is almost coercive.

**Proof.** Let \( \bar{u} = (u, \bar{g}) \in \tilde{H}^{1,0}_{Q,0}(\Theta) \) and \( \varphi \in L^{p^\prime}(\Theta) \). Then we have that
\[
A_{p, \tau}(\bar{u}) = \int_{\Theta} \langle Q\bar{g}, \bar{g} \rangle \frac{u^2}{|u|^{p-2}} dx + \tau \int_{\Theta} |u|^{p-2} u^2 dx \\
= \int_{\Theta} \langle Q\bar{g}, \bar{g} \rangle \frac{\bar{u}^2}{|\bar{u}|^{p-2}} dx + \tau \int_{\Theta} |\bar{u}|^{p-2} \bar{u}^2 dx \geq \eta \left( \int_{\Theta} \langle Q\bar{g}, \bar{g} \rangle \frac{\bar{u}^2}{|\bar{u}|^{p-2}} dx + \int_{\Theta} |\bar{u}|^{p} dx \right) = \eta \||\bar{u}||_{{p^\prime}, 1}^{p}.
\]
where \( \eta = \min\{1, \tau\} > 0 \). Therefore, if we let \( \beta = (\eta^{-1}\||\varphi||_{p^\prime}^{p^\prime-1}) \), then by Hölder’s inequality we have that for all \( \||\bar{u}||_{{p^\prime}, 1}^{p} > \beta \),
\[
|\Gamma(v)| = \left| \int_{\Theta} \varphi u dx \right| \leq \||\varphi||_{p^\prime}||u||_p \leq ||\varphi||_{p^\prime}||\bar{u}||_{{p^\prime}, 1}^{1,0}(\Theta) < \eta \||\bar{u}||_{{p^\prime}, 1}^{p} \leq A_{p, \tau}(\bar{u}).
\]
Thus, \( A_{p, \tau} \) is almost coercive. \[ \square \]

5. **Boundedness of solutions to degenerate p-Laplacians**

In this section we will prove that solutions of the Dirichlet problem (4.1) are bounded. The proof is quite technical, as it relies on a very general result from [MRW1] and much of the work in the proof is checking the hypotheses.

**Proposition 5.1.** Given a set \( \Omega \subset \mathbb{R}^n \), let \( \rho \) be a quasi-metric on \( \Omega \). Fix \( 1 < p < \infty \) and \( 1 < t \leq \infty \), and suppose \( Q \) is a semi-definite matrix function such that \( Q \in L^1_{loc}(\Omega) \). Suppose further that that the cutoff condition of order \( s > p\rho^t \), the local Poincaré property of order \( p \) with gain \( t' = \frac{t}{s-p} \), and the local Sobolev property of order \( p \) with gain \( \sigma \geq 1 \) hold. Given any \( \Theta \subset \Omega \) and \( q \in [p', \infty) \cap (p\sigma', \infty) \), if \( \varphi \in L^q(\Theta) \), then there exists a positive constant \( C \) such that for all \( \tau \in (0, 1) \), the corresponding weak solution \( \bar{u}_\tau = (u_\tau, \bar{g}_\tau) \in \tilde{H}^{1,0}_{Q,0}(\Theta) \) of (4.1) satisfies
\[
\text{ess sup}_{x \in \Theta} |u_\tau(x)| \leq C\||\varphi||_{L^q(\Theta)}^{\frac{p}{p^t}}.
\]
The constant $C$ is independent of $\varphi$, $\bar{u}_r$, and $\tau$.

**Remark 5.2.** The hypotheses of Proposition 5.1 are the same as those of Theorem 2.7 except that we do not require higher integrability on $Q$: $Q \in L^2_{\text{loc}}(\Omega)$ is sufficient for this result.

The proof of Proposition 5.1 requires that the mapping $I : \tilde{H}^{1,p}_{Q,0}(\Theta) \rightarrow L^p(\Theta)$, $I((u, \vec{g})) = u$, is compact. This is a consequence of the following result.

**Proposition 5.3.** Given a set $\Omega \subset \mathbb{R}^n$, let $\rho$ be a quasi-metric on $\Omega$. Fix $1 < p < \infty$ and $1 < t \leq \infty$, and suppose $Q$ is a semi-definite matrix function such that $Q \in L^2_{\text{loc}}(\Omega)$. Suppose further that the cutoff condition of order $s > p\sigma'$, the local Poincaré property of order $p$ with gain $t' = \frac{s}{s-p}$, and the local Sobolev property of order $p$ with gain $\sigma \geq 1$ hold. Fix $\Theta \Subset \Omega$; then the mapping $I : \tilde{H}^{1,p}_{Q,0}(\Theta) \rightarrow L^p(\Theta)$, $I((u, \vec{g})) = u$, is compact.

**Proof.** Proposition 5.3 is a particular case of a general imbedding result from [CRW, Theorem 3.20]. So to prove it we only need to show that the hypotheses of this result are satisfied. We will go through these in turn but for brevity we have omitted restating the precise form of each hypothesis as given there and instead refer to them as they are stated in the theorem and the preliminaries in [CRW, Section 3]. We refer the reader to this paper for complete details.

Since $(\Omega, \rho)$ is a quasi-metric space, $\rho$ satisfies (2.1), and by Remark 2.6 Lebesgue measure satisfies a local doubling property for metric balls, the topological assumptions of Section 3A and condition (3-12) in [CRW] hold. In particular, since we assume that the function $r_1$ in Remark 2.6 is continuous, the local geometric doubling condition in [CRW, Definition 3.3] holds.

In the definition of the underlying Sobolev spaces, and in the Poincaré and Sobolev inequalities [CRW, Definitions 3.5, 3.16], we let the measures $\mu, \nu, \omega$ all be the Lebesgue measure. Since the local Poincaré inequality of order $p$, Definition 2.2, holds, and since $r_1$ is assumed to be continuous, [CRW, Definition 3.5] holds. (See also [CRW, Remark 3.6].) Similarly, since we assume that the local Sobolev inequality of order $p$ with gain $\sigma$, Definition (1.2), holds, [CRW, Definition 3.16] holds.

The existence of an accumulating sequence of cut-off functions, Definition 2.5, lets us prove the cut-off property of order $s \geq p\sigma'$ in [CRW, Definition 3.18]. Fix a compact subset $K$ of $\Omega$ and let $R > 0$ be the minimum of $r_1$ on $K$. Given $x \in K$ and $0 < r < R$, since $r < r_1(x)$, the cutoff condition of order $s \geq p\sigma'$ gives $\psi_1 \in \text{Lip}_0(B(x, r))$ such that

1. $0 \leq \psi_1(x) \leq 1$,
2. $\psi_1(x) = 1$ on $B(y, \alpha r)$,
3. $\nabla \psi_1 \in L^s_Q(\Omega)$. 

This yields the desired function in [CRW, Definition 3.18].

Finally, we show that the weak Sobolev inequality, [CRW, Inequality (3.33)], holds with $t' = \frac{s}{s-p}$. Since $s \geq p\sigma'$, $t = \frac{s}{p} \geq \sigma'$, and so $1 < t' \leq \sigma$. Again let $K$ be a compact subset of $\Omega$ and $R > 0$ the minimum value of $r_1$ on $K$. If $x \in K$ and $0 < r < R$, then by the local Sobolev property with $B = B(x, r)$ we have that for any $u \in Lip_0(B)$,

$$
\left( \int_B |u|^{p\sigma'} dy \right)^{\frac{1}{p\sigma'}} \leq C(B) \left( \int_B |u|^{p\sigma} dy \right)^{\frac{1}{p\sigma}} \leq C(B) \| (u, \nabla u) \|_{\dot{H}^{1,p}_0(\Omega)},
$$

which gives us inequality (3.33).

Thus, we have shown that we satisfy the necessary hypotheses, and so Proposition 5.3 follows from [CRW, Theorem 3.20]. \hfill \Box

**Proof of Proposition 5.1.** Let $\Theta \Subset \Omega$ and $q \in [p', \infty) \cap (p\sigma', \infty)$ with $\varphi \in L^q(\Theta)$. Note that since $q \geq p'$ and $\Theta$ is bounded, it follows that $\varphi \in L^{p'}(\Theta)$. Fix $0 < \tau < 1$; then by Proposition 4.4 there exists a weak solution $\tilde{u}_r = (u_r, g_r) \in \dot{H}^{1,p}_0(\Theta)$ of the Dirichlet problem (4.1). To complete the proof, we will first use [MRW1, Theorem 1.2] to show that $\tilde{u}_r$ satisfies (5.1). We will then show using Proposition 5.3 that the constant is independent of $\tau$. (By [MRW1, Theorem 1.2] we have that it is independent of $\varphi$ and $\tilde{u}_r$.)

To apply [MRW1, Theorem 1.2], first note that $(\Omega, \rho)$ is a quasi-metric space and (2.1) holds, we satisfy the topological assumptions of this paper, including [MRW1, (1.9)]. As a result, if $0 < \beta r < r_1(y)$, the local Sobolev condition, Definition 1.1, the Poincaré inequality, Definition 2.2, and the cutoff condition, Definition 2.5, hold. This shows that assumptions [MRW1, (1.13), (1.14), (1.15), (1.16)] hold with $t' = \frac{s}{s-p}$.

(For condition (1.16), see also [MRW1, Remark 1.1].)

We now show that $\tilde{u}_r$ is the solution of an equation with the appropriate properties.

Define $A, \tilde{A} : \Theta \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $B : \Theta \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

1. $A(x, z, \xi) = \langle Q(x) \xi, \xi \rangle^\frac{p-2}{2} Q(x) \xi$,
2. $\tilde{A}(x, z, \xi) = \langle Q(x) \xi, \xi \rangle^\frac{p-2}{2} \sqrt{Q(x)} \xi$,
3. $B(x, z, \xi) = \varphi(x) + \tau|z|^{p-2}z$.

Given these functions, the differential equation (4.2) can be rewritten as

$$\text{div}(A(x, u, \nabla u)) = B(x, u, \nabla u),$$

which is [MRW1, (1.1)]. Furthermore, we have that $A(x, z, \xi) = \sqrt{Q(x)} \tilde{A}(x, z, \xi)$, and for all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and a.e. $x \in \Theta$,

1. $\xi \cdot A(x, z, \xi) = \langle Q(x) \xi, \xi \rangle^\frac{p-2}{2} \langle Q(x) \xi, \xi \rangle = |\sqrt{Q} \xi|^p$,
2. $|\tilde{A}(x, z, \xi)| = |\sqrt{Q(x)} \xi|^{p-1},$
3. $|B(x, z, \xi)| \leq |\varphi(x)| + \tau|z|^{p-1} < |\varphi(x)| + |z|^{p-1}.$

Therefore, the structural conditions [MRW1, (1.3)] are satisfied with the exponents $\delta = \gamma = \psi = p$ and the coefficients $a = 1$, $b = 0$, $c = 0$, $d = 1$, $e = 0$, $f = |\varphi|$, $g = 0$, and $h = 0$.

The above shows that we satisfy the hypotheses of [MRW1, Theorem 1.2]. Therefore, fix $\epsilon \in (0, 1]$ such that $p - \epsilon > 1$ and for $k > 0$ (to be fixed below) set $\tau_r = |u_r| + k$. Then for each $y \in \Theta$ and $0 < \beta r < r_1(y)$, we have the $L^\infty$-estimate

$$
\text{(5.2)} \quad \text{ess sup}_{x \in B(y, \alpha r)} |u_r(x)| \leq C \frac{1}{|B(y, r)|} \int_{B(y, r)} |u_r|^{s^*_p} \, dx \frac{1}{s^*_p},
$$

where $s^*$ is the dual exponent of $s_1 > \sigma'$, define by $s = s_1 p$. (In (5.2), $\alpha$ is the constant from (2.3); note that in [MRW1] it is called $\tau$.) The term $Z$ on the right-hand side is defined by

$$
Z = \left[ 1 + \left( r^p |B(y, r)| - \frac{1}{p - \epsilon} \|1 + k^{1-p}|\varphi||\hat{F}_{\tau_r}^{p-1}(B(y, r))\| \right)^{\frac{1}{p - \epsilon}} \right]^{s^*_p}.
$$

Since $\frac{p}{p - \epsilon} < q$ and $\varphi \in L^q(\Theta)$, $Z$ is bounded with a bound independent of $\tau$ for any $k > 0$ but depending on the ball $B(y, r)$.

If $\varphi \neq 0$, let $k = \|\varphi\|_{L^q(\Theta)} > 0$; then by Minkowski’s inequality and the local Sobolev inequality (1.2), (5.2) becomes

$$
\text{ess sup}_{x \in B(y, \alpha r)} |u_r(x)| \leq C \left[ \left( \frac{1}{|B(y, r)|} \int_{B(y, r)} |u_r|^{p\sigma} \, dx \right)^{\frac{1}{p\sigma}} + \|\varphi\|_{L^{p\sigma}(\Theta)} \right]^{\frac{1}{p\sigma}}
$$

$$
\leq C \left[ \frac{1}{|B(y, r)|^{\frac{1}{p}}} \|\tilde{u}_r\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)} + \|\varphi\|_{L^{\frac{1}{p-1}}(\Theta)} \right],
$$

The constant $C$ depends on $B(y, r)$ but not on $\tau$ or $\varphi$. The case when $\varphi = 0$ is similar and left to the reader.

We now extend this estimate to all of $\Theta$ using the fact that $\overline{\Theta}$ is compact. By assumption (2.1), the balls $B(y, r)$ are open, so we can find a finite cover $B_j = \tilde{B}(y_j, r_j)$, $1 \leq j \leq N$, with $0 < \beta r_j < r_1(y_j)$. Hence, we have that

$$
\text{ess sup}_{y \in \Theta} |u_r(x)| \leq C_0 \left[ \|\tilde{u}_r\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)} + \|\varphi\|_{L^{\frac{1}{p-1}}(\Theta)} \right],
$$

where the constant $C_0$ depends on $\min\{r_j : 1 \leq j \leq N\} > 0$ but not on $\tau$ or $\varphi$.

To complete the proof we will show that

$$
\|\tilde{u}_r\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)} \leq C \|\varphi\|_{L^{\frac{1}{p-1}}(\Theta)}
$$

$\text{(5.4)}$
with a constant $C$ independent of $\tau$. To do so we will use Proposition 5.3. Suppose to the contrary that (5.4) is false. Then there exists a sequence $\{\tau_k\} \subset (0,1)$ and corresponding sequence of weak solutions $\{\tilde{u}_{\tau_k}\} = \{u_{\tau_k}, \tilde{g}_{\tau_k}\} \subset \tilde{H}^{1,p}_{Q,0}(\Theta)$ of (4.1) such that

$$\|\tilde{u}_{\tau_k}\|_{\tilde{H}^{1,p}(\Theta)} \rightarrow \infty$$

as $k \rightarrow \infty$. We must have that $\tau_k \rightarrow 0$ as $k \rightarrow \infty$. To see this, note that since $\tilde{u}_{\tau_k}$ is a valid test function in the definition of a weak solution, we have that

$$\|\tau_k\|_{\tilde{H}^{1,p}(\Theta)} = \frac{\tau_k \left[ \int_{\Theta} |u_{\tau_k}|^{p-2} u_{\tau_k} u_{\tau_k} \, dx + \int_{\Theta} \langle Q\tilde{g}_{\tau_k}, \tilde{g}_{\tau_k} \rangle \frac{p-2}{2} \langle Qg_{\tau_k}, g_{\tau_k} \rangle \, dx \right]}{\|u_{\tau_k}\|_{\tilde{H}^{1,p}(\Theta)}} \leq \left\|A_{p,\tau_k} \tilde{u}_{\tau_k}(\tilde{u}_{\tau_k}) \right\| = \left\|u_{\tau_k}\|_{\tilde{H}^{1,p}(\Theta)} \|\varphi\|_{L^{p'}(\Theta)} \right\|,$$

Since $\|\tilde{u}_{\tau_k}\|_{\tilde{H}^{1,p}(\Theta)} \not= 0$ this inequality implies that

$$\|\tilde{u}_{\tau_k}\|_{\tilde{H}^{1,p}(\Theta)} \leq \left( \frac{1}{\tau_k} \right)^{\frac{1}{p-1}}$$

which in turn implies that $\tau_k \rightarrow 0$.

For each $k \in \mathbb{N}$ define $\tilde{v}_{\tau_k} = \|\tilde{u}_{\tau_k}\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)}^{-1} \tilde{u}_{\tau_k}$. Then, $\tilde{v}_{\tau_k} = (v_{\tau_k}, \tilde{h}_{\tau_k}) \in \tilde{H}^{1,p}_{Q,0}(\Theta)$,

$$\|\tilde{v}_{\tau_k}\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)} = 1,$$

and $\tilde{v}_{\tau_k}$ is a weak solution of the Dirichlet problem

$$\begin{cases}
\text{div} \left( \langle Q\nabla w, \nabla w \rangle^{\frac{p-2}{2}} Q\nabla w \right) - \tau_k |w|^{p-2} w = \varphi_k \text{ in } \Theta \\
\quad w = 0 \text{ on } \partial \Theta
\end{cases}$$

where $\varphi_k = \|\tilde{u}_{\tau_k}\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)}^{-1} \varphi$. By Proposition 5.3, $\tilde{H}^{1,p}_{Q,0}(\Theta)$ is compactly embedded in $L^p(\Theta)$. Therefore, since $\{\tilde{v}_{\tau_k}\}$ is a bounded sequence in $\tilde{H}^{1,p}_{Q,0}(\Theta)$, by passing to a subsequence (renumbered for simplicity of notation) we have that there exists $v \in L^p(\Theta)$ such that $v_{\tau_k} \rightarrow v$ in $L^p(\Theta)$. Furthermore, arguing as we did above to prove $\tau_k \rightarrow 0$, we have a Caccioppoli-type estimate:

$$\int_{\Theta} \langle Q\tilde{h}_{\tau_k}, \tilde{h}_{\tau_k} \rangle \frac{q}{2} \, dx = \int_{\Theta} \langle Q\tilde{h}_{\tau_k}, \tilde{h}_{\tau_k} \rangle \frac{p-2}{2} \langle Q\tilde{g}_{\tau_k}, \tilde{g}_{\tau_k} \rangle \, dx$$

$$\leq A_{p,\tau_k} \tilde{v}_{\tau_k}(\tilde{v}_{\tau_k}) \leq \left\|v_{\tau_k}\|_{\tilde{H}^{1,p}_{Q,0}(\Theta)} \right\| \left\|\varphi \right\|_{L^{p'}(\Theta)}.$$
By the Poincaré inequality (2.2) we have that for any \( y \in \Theta \) and \( r > 0 \) sufficiently small,
\[
\int_{B(y,r)} |v - v_{B(y,r)}|^p dx = 0.
\]
Hence \( v \) is constant a.e. (or, more properly, constant on each connected component of \( \Theta \)). We claim that \( v \) is constant a.e., which would contradict the fact that \( \|v\|_{L^p(\Theta)} = 1 \).
To show this, extend \( v \) to all of \( \Omega \) as follows. Let \( \{\psi_j\} \subset Lip_0(\Theta) \) be such that \( (\psi_j, \nabla \psi_j) \to \bar{v}_0 \) in \( \hat{H}^{1,p}_{Q,0}(\Theta) \) norm. Define \( \eta_j \in Lip_0(\Omega) \) so that \( \eta_j = \psi_j \) in \( \Theta \) and \( \eta_j = 0 \) on \( \Omega \setminus \Theta \). Then \( \{(\eta_j, \nabla \eta_j)\} \) is Cauchy in the \( \hat{H}^{1,p}_{Q,0}(\Omega) \) norm and so converges to some \( \bar{w}_0 = (w,0) \in \hat{H}^{1,p}_{Q,0}(\Omega) \). If we again apply Poincaré’s inequality, we see that \( \bar{v} \) is constant in \( \Omega \) (since \( \Omega \) is connected). However, \( w = v \) in \( \Theta \) and \( w = 0 \) in \( \Omega \setminus \Theta \), and so we must have that \( v = 0 \) a.e.

From this contradiction we have that our assumption is false and so (5.4) holds with a constant independent of \( \tau \). This completes our proof.

\[ \square \]

6. Proof of Theorem 2.7

Before proving our main result, we give one more lemma, a product rule for degenerate Sobolev spaces. The proof is adapted from the proof of [MRW1, Proposition 2.2].

**Lemma 6.1.** Given \( 1 \leq p < \infty \) and \( 1 < t \leq \infty \), suppose \( \sqrt{Q} \in L^t_{loc}(\Omega) \) and that the local Poincaré property of order \( p \) with gain \( t' \) holds. Let \( \Theta \subset \Omega \). If \( \tilde{u} = (u,\tilde{g}) \in \hat{H}^1_{Q,0}(\Theta) \) and \( v \in Lip_0(\Theta) \), then \( uv, v\tilde{g} + u\nabla v \in \hat{H}^1_{Q,0}(\Theta) \).

**Proof.** By assumption, there exists a sequence \( \{\psi_j\} \subset Lip_0(\Theta) \) such that \( \psi_j \to u \) in \( L^p(\Theta) \) and \( \nabla \psi_j \to \tilde{g} \) in \( L^p_Q(\Theta) \) as \( j \to \infty \). For each \( j \in \mathbb{N} \), define \( \phi_j = \psi_j v \). Then \( \{\phi_j\} \subset Lip_0(\Theta) \) and
\[
\|uv - \phi_j\|_{L^p(\Theta)} \leq \|v\|_{L^\infty(\Theta)} \|u - \psi_j\|_{L^p(\Theta)};
\]
hence, \( \phi_j \to uv \) in \( L^p(\Theta) \) as \( j \to \infty \).

To complete the proof, we will show that \( \nabla \phi_j \to v\tilde{g} + u\nabla v \) in \( L^p_{Q,0}(\Theta) \). Since \( \nabla \phi_j = v\nabla \psi_j + \psi_j \nabla v \), we have that
\[
\int_\Theta |\sqrt{Q}(v\tilde{g} + u\nabla v - v\nabla \psi_j - \psi_j \nabla v)|^p dx
\]
\[
\leq C \left[ \int_\Theta |\sqrt{Q}(\tilde{g} - \nabla \psi_j)|^p |v|^p dx + \int_\Theta |\sqrt{Q} \nabla v|^p |u - \psi_j|^p dx \right]
\]
\[
\leq C \left[ \|v\|_{L^\infty(\Theta)} \int_\Theta |\sqrt{Q}(\tilde{g} - \nabla \psi_j)|^p dx + \|\sqrt{Q} \nabla v\|_{L^{p'(\Theta)}} \|u - \psi_j\|_{L^{p'(\Theta)}} \right].
\]
The first term on in the last line goes to 0 by our choice of \( \psi_j \) and if \( t' = 1 \), then the second term does as well. If \( t' > 1 \), then to estimate the second term, note first
it follows from the local Poincaré inequality (2.2) that for all \( y \) and \( r > 0 \) sufficiently small and \( f \in \text{Lip}_0(\Omega) \),
\[
\left( \int_{B(y,r)} |f|^{p'} dx \right)^{\frac{1}{p'}} \leq C r \left( \int_{B(y,br)} |\sqrt{Q} \nabla f|^p dx \right)^{\frac{1}{p}} + \left( \int_{B(y,r)} |f|^p dx \right)^{\frac{1}{p}}.
\]
Therefore, by a partition of unity argument like that used to prove the weak global Sobolev inequality (1.3) from the weak local Sobolev inequality (1.2), we have that
\[
(6.1) \quad \left( \int_{\Theta} |f|^{p'} dx \right)^{\frac{1}{p'}} \leq C \left( \int_{\Theta} |\sqrt{Q} \nabla f|^p dx \right)^{\frac{1}{p}} + \left( \int_{\Theta} |f|^p dx \right)^{\frac{1}{p}}.
\]
But then, in the second term we have that \( \| \sqrt{Q} \nabla v \|_{L^p(\Theta)} < \infty \) since \( \nabla v \in L^\infty(\Theta) \) and \( Q \in L^{p,\mu}_{\text{loc}}(\Omega) \). Moreover, by (6.1) we have that
\[
\| u - \psi_j \|_{L^{p'}(\Theta)} \leq \| u - \psi_j \|_{\widehat{H}^1_{Q,\theta}(\Theta)};
\]
and the right-hand term goes to 0 as \( j \to \infty \). Therefore, we have shown that \((uv, v\overline{g} + u\nabla v) \in \widehat{H}^1_{Q,0}(\Theta)\).

\(\square\)

**Proof of Theorem 2.7.** Fix \( v \in \text{Lip}_0(\Theta) \). Our goal is to show that the global Sobolev estimate (2.5) holds. It will be enough to show that for some \( \eta, 1 < \eta < \sigma \),
\[
(6.2) \quad \left( \int_{\Theta} |v|^{p\eta} dx \right)^{\frac{1}{p\eta}} \leq C \left( \int_{\Theta} |\sqrt{Q} \nabla v|^p dx \right)^{\frac{1}{p}}.
\]
For given this, by the weak global Sobolev inequality (1.3), we have that
\[
\left( \int_{\Theta} |v|^{p\sigma} dx \right)^{\frac{1}{p\sigma}} \leq C \left[ \left( \int_{\Theta} |\sqrt{Q} \nabla v|^p dx \right)^{\frac{1}{p}} + \left( \int_{\Theta} |v|^p dx \right)^{\frac{1}{p}} \right] \leq C \left( \int_{\Theta} |\sqrt{Q} \nabla v|^p dx \right)^{\frac{1}{p}},
\]
and this is the desired inequality.

To prove (6.2), fix \( q \in [p', \infty) \cap (p\sigma', \infty) \) and let \( \eta = q' > 1 \); note that \( 1 < \eta < \sigma \) since \( q > p\sigma' > \sigma' \). Then by duality we have that
\[
(6.3) \quad \left( \int_{\Theta} |v|^{pq} dx \right)^{\frac{1}{pq}} = \left[ \left( \int_{\Theta} (|v|^p)^q dx \right)^{\frac{1}{q}} \right]^{\frac{1}{p}} = \sup \left[ \int_{\Theta} \varphi |v|^p dx \right]^{\frac{1}{p}},
\]
where the supremum is taken over all non-negative \( \varphi \in L^q(\Theta) \), \( \| \varphi \|_{L^q(\Theta)} = 1 \).

Fix a non-negative function \( \varphi \in L^q(\Theta) \), \( \| \varphi \|_{L^q(\Theta)} = 1 \); we estimate the last integral. Fix \( \tau \in (0, 1) \); the exact value of \( \tau \) will be determined below. Since \( q \geq p' \) and \( \Theta \) is bounded, \( \varphi \in L^{p'}(\Theta) \), and so by Proposition 4.4, there exists \( \bar{u}_\tau = (u_\tau, \bar{g}_\tau) \in \widehat{H}^1_{Q,0}(\Theta) \)
that is a weak solution of the Dirichlet problem (4.1). Since $|v|^p \in Lip_0(\Theta)$, we can use it as a test function in the definition of weak solution. This yields

$$
\int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} \langle Q\bar{g}_\tau, \nabla(|v|^p) \rangle \, dx + \tau \int_\Theta |u_\tau|^{p-2} u_\tau |v|^p \, dx = -\int_\Theta \varphi |v|^p \, dx.
$$

If we take absolute values, rearrange terms and apply Hölder's inequality, we get

$$
\int_\Theta \varphi |v|^p \, dx \leq p \int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} \langle Q\bar{g}_\tau, \nabla v \rangle |v|^{p-1} \text{sgn}(v) \, dx
$$

$$
+ \tau \int_\Theta |u_\tau|^{p-2} u_\tau |v|^p \, dx
$$

$$
\leq p \int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} \langle Q\bar{g}_\tau, \nabla v \rangle |v|^{p-1} \, dx
$$

$$
+ \tau \|u_\tau\|_{L^\infty(\Theta)}^{p-1} \|v\|^p_{L^p(\Theta)}
$$

$$
\leq p \left( \int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle |v|^p \, dx \right)^{\frac{1}{p}} \left( \int_\Theta \langle Q\nabla v, \nabla v \rangle \frac{x}{2} |v|^{p-1} \, dx \right)^{\frac{1}{p}}
$$

$$
+ \tau \|u_\tau\|_{L^\infty(\Theta)}^{p-1} \|v\|^p_{L^p(\Theta)}.
$$

(6.4)

To estimate (6.4), define

$$
A = \int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} |v|^p \, dx.
$$

Let $\bar{h} = pu_\tau |v|^{p-1} \text{sgn}(v) \nabla v + |v|^p \bar{g}_\tau$; then by Lemma 6.1 we have $(u_\tau |v|^p, \bar{h}) \in \tilde{H}^{1,p}_{Q,0}(\Theta)$. Moreover, we have that

$$
\int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} \langle Q\bar{g}_\tau, \bar{h} \rangle \, dx = A + p \int_\Theta \langle Q\bar{g}_\tau, \bar{g}_\tau \rangle \frac{p^2}{2} \langle Q\bar{g}_\tau, \nabla v \rangle u_\tau \text{sgn}(v) |v|^{p-1} \, dx.
$$
Since $\overrightarrow{u}_\tau$ is a weak solution of (4.1) and $(u_\tau |v|^p, \overrightarrow{h})$ can be used as a test function, we have that

$$A \leq \left| \int_\Theta \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \overrightarrow{g}_{\tau} \rangle \frac{p^2}{2} \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \nabla v \rangle u_\tau \text{sgn}(v) |v|^{p-1} dx \right| + p \int_\Theta \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \overrightarrow{g}_{\tau} \rangle \frac{p^2}{2} \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \nabla v \rangle u_\tau dx$$

$$\leq \left| \int_\Theta \varphi u_\tau |v|^p dx \right| + \tau \int_\Theta |u_\tau| |v|^p dx$$

$$+ p \int_\Theta \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \overrightarrow{g}_{\tau} \rangle \frac{p^2}{2} \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \nabla v \rangle \frac{1}{2} |u_\tau||v|^{p-1} dx$$

$$\leq \left| \int_\Theta \varphi u_\tau |v|^p dx \right| + \tau \int_\Theta |u_\tau| |v|^p dx + C(p) \int_\Theta \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \nabla v \rangle \frac{1}{2} |u_\tau|^p dx$$

$$+ \frac{1}{2} \int_\Theta \langle \overrightarrow{Q}\overrightarrow{g}_{\tau}, \overrightarrow{g}_{\tau} \rangle \frac{1}{2} |v|^p dx;$$

the last inequality follows from Young’s inequality. The last term equals $\frac{1}{2}A$. Moreover, since $\|\varphi\|_{L^2(\Theta)} = 1$, by Proposition 5.1 there exists a constant $K$, independent of $\tau$, such that $\|u_\tau\|_{L^\infty(\Theta)} \leq K$. Therefore, the above inequality yields

$$(6.5) \quad A \leq C(p) \left[ K \int_\Theta \varphi |v|^p dx + \tau K^p \|v\|_{L^p(\Theta)}^p + K^p \int_\Theta \langle \overrightarrow{Q}\overrightarrow{v}, \overrightarrow{v} \rangle \frac{1}{2} dx \right].$$

Irrespective of which term on the right-hand side of (6.5) is the maximum, if we combine (6.4) with (6.5) we get

$$(6.6) \quad \int_\Theta \varphi |v|^p dx \leq C(p, K) \left[ \int_\Theta \langle \overrightarrow{Q}\overrightarrow{v}, \overrightarrow{v} \rangle \frac{1}{2} dx + \tau \|v\|_{L^p(\Theta)}^p \right].$$

To see that this is true, suppose first that the largest term is $\tau K^p \|v\|_{L^p(\Theta)}^p$. Then by Young’s inequality we have that

$$\int_\Theta \varphi |v|^p dx \leq C \tau^{\frac{p-1}{p}} K^{p-1} \|v\|_{L^p(\Theta)}^{p-1} \left( \int_\Theta \langle \overrightarrow{Q}\overrightarrow{v}, \overrightarrow{v} \rangle \frac{1}{2} dx \right)^{\frac{1}{p}} + \tau K^{p-1} \|v\|_{L^p(\Theta)}^p$$

$$= C K^{p-1} \left[ \tau^{\frac{p-1}{p}} \|v\|_{L^p(\Theta)}^{p-1} \left( \int_\Theta \langle \overrightarrow{Q}\overrightarrow{v}, \overrightarrow{v} \rangle \frac{1}{2} dx \right)^{\frac{1}{p}} + \|v\|_{L^p(\Theta)}^p \right]$$

$$\leq C K^{p-1} \left[ \int_\Theta \langle \overrightarrow{Q}\overrightarrow{v}, \overrightarrow{v} \rangle \frac{1}{2} dx + \tau \|v\|_{L^p(\Theta)}^p \right].$$

The other estimates are proved similarly.
Given inequality (6.6) it is now straightforward to prove the desired estimate:

\[
\left( \int_{\Omega} \varphi |v|^p \, dx \right)^{\frac{1}{p}} \leq C(p, K)^{p} \left[ \int_{\Omega} \langle Q \nabla v, \nabla v \rangle^{\frac{p}{2}} \, dx + \tau \|v\|_{L^p(\Omega)}^p \right]^{\frac{1}{p}}
\]

\[
\leq C(p, K)^{p} \left[ \left( \int_{\Omega} \langle Q \nabla v, \nabla v \rangle^{\frac{p}{2}} \, dx \right)^{p} + \tau \left( \int_{\Omega} |v|^{pm} \, dx \right)^{p} \right].
\]

Fix \( \tau < 1 \) such that \( \tau C(p, K) \leq \frac{1}{2} \). Since this constant is independent of \( \varphi \), if we combine this inequality with the duality estimate (6.3), we get that

\[
\left( \int_{\Omega} |v|^{pm} \, dx \right)^{\frac{1}{pm}} \leq C(p, K) \left( \int_{\Omega} \langle Q \nabla v, \nabla v \rangle^{\frac{p}{2}} \, dx \right)^{p} + \frac{1}{2} \left( \int_{\Omega} |v|^{pm} \, dx \right)^{\frac{1}{pm}}.
\]

If we re-arrange terms we get (6.2) and our proof is complete. \( \Box \)

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