The bremsstrahlung spectrum for the electric dipole which nonradially falls into a black hole

Alexander Shatskiy, 1 I.D. Novikov, 1, 2 and Alexandr Malinovsky

1P.N. Lebedev Physical Institute, Astro Space Center,
84/32 Profsoyuznaya st., Moscow, GSP-7, 117997, Russia

2The Niels Bohr International Academy, The Niels Bohr Institute,
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

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The electromagnetic bremsstrahlung spectrum for the dipole which falling by a spiral orbit into the Schwarzschild black hole was found. The characteristic features in this electromagnetic spectrum can be used for determine of the black hole mass by the new way. This new way (if implemented) provides higher accuracy in determining of the black hole mass. Also these features in the spectrum can be used for determine of the certain characteristics in the black hole magnetosphere or in the accretion disk characteristics around the black hole. It is also shown that the asymptotic behavior of this spectrum (at high frequencies) is practically independent from the impact parameter of the falling dipole.

I. INTRODUCTION

As is well known (see eg [1, 2]) black hole have no ”hair”. Therefore, all electromagnetic fields from multipole moments will disappear if the system of charges is approaching to black hole horizon. For the static point charge the problem of computing the field was solved by Linet [3]. In this problem, it has been shown, in particular, that the field of a point charge close to the field of the charged black hole (with the same charge) as the charge approaches to the horizon. Hence it follows that all fields from all electric and magnetic multipole moments should be radiated by approaching the charge (or a system of charges or currents) to the black hole horizon.

For accelerated motion of monopole (single charge) the energy loss is determined mainly
by bremsstrahlung. This radiation is a dipolar, since the radiated field components are inversely proportional to the $c^2$ (square of the speed of light).

The existence of dipole radiation it is not obvious for a massive dipole falls into a black hole (due to the fact that both of the charges of the dipole are moving and accelerating in the same direction, and the signs of the charges are opposite). However, the increasing of the space-time curvature leads to the emission of the dipole type radiation (see section IV).

Earlier, a similar radiation was investigated in many papers – see for example [4–10]; We used to do this a slightly different method, which is much more convenient for numerical calculations – see [11, 12].

In the paper [13] was found the bremsstrahlung radiation for a charge passing (with a constant velocity) through the wormhole and its spectrum.

In this paper we will solve our problem in general case of non-radial dipole which free falling into a Schwarzschild black hole.

II. LAW OF MOTION OF FREE-FALLING PARTICLE

Schwarzschild metric for a nonrotating and uncharged black hole has the form:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2 dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$  (1)

Here: $r_g = 2GM/c^2$ – the radius of the black hole horizon, $G$ – gravitational constant, $M$ – the mass of the black hole.

Let us remind the law of motion for the test particles in the Schwarzschild gravitational field. We suppose that the particle motion occurs in the equatorial plane ($\theta = \pi/2$). Let us write the geodesic equation for the particle (see. [14] [15], §87):

$$\frac{du_i}{ds} = \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} u^k u^l$$  (2)

Hence for the metric $[1]$ and $i = 0$ (corresponding to the time coordinate $ct$) we have the integral of motion$^1$: $u_0 := \epsilon = \text{const}$ (the specific energy the particles); and for $i$, which corresponding coordinates $\varphi$ we have the integral of motion: $u_\varphi = L/(mc) := h\epsilon = \text{const}$ (here $L$ – angular momentum, $m$ – mass and $h$ – the impact parameter of the particle).

$^1$ In our work, the Latin indices run a series of spatial coordinates and time coordinates, and Greek indices run a series of spatial coordinates.
We assume that fall of particles happens with the initial radius \( r_0 \) and the result of this fall is to capture of the particles by a black hole (ie initial conditions for the particle motion does not allow to remain untrapped). We also assume that the mass of the particle is much smaller than the mass of the black hole: \( m \ll M \). From equations [2] and taking into account the identity \( u_i u^i \equiv 1 \), we have:

\[
\frac{cdt}{ds} \equiv u^0 = \epsilon/(1 - r_g/r) \tag{3}
\]

\[
\frac{dr}{ds} \equiv u^r = -\sqrt{\epsilon^2 - (1 - r_g/r)(1 + h^2\epsilon^2/r^2)} \tag{4}
\]

\[
\frac{d\varphi}{ds} \equiv u^\varphi = -h\epsilon/r^2 \tag{5}
\]

Here \( r(t) \) – the current radial coordinate of the particle.

To find the highest possible value\(^2\) of the impact parameter of the particle \( h_{max} \) (in which it will be captured by a black hole), we proceed similarly the work [16].

The value of \( h_{max} \) is defined as the root of the equation \( [u^r]^2_{(h,r)} = 0 \), where this root at the same time should be a point of minimum for function \( [u^r]^2_{(r)} \). Thus the point of minimum \( r_m \) determined by solution of the equation:

\[
(\epsilon^2 - 1)r_m^2 + (2 - 1.5\epsilon^2)r_mr_g - r_g^2 = 0 \tag{6}
\]

From the two roots of this equation we need to choose a smaller (which corresponding to the plus sign and the minimum distance to the black hole). Then the value of \( h_{max} \) is determined by the expression:

\[
h_{max}^2 = \frac{(\epsilon^2 r_m - r_m + r_g)r_m^2}{\epsilon^2(r_m - r_g)} \tag{7}
\]

At \( \epsilon = 1 \) (particle is at rest at infinity) we have:

\[
h_{max}^{(\epsilon=1)} = 2r_g \tag{8}
\]

In the limit \( \epsilon \to \infty \) (for photons), we have:

\[
h_{max}^{(\epsilon\to\infty)} = \frac{3\sqrt{3}r_g}{2} \tag{9}
\]

The minimum valid value for \( \epsilon \) is determined by the inequality: \( \epsilon^2 > 8/9 \).

\(^2\) This corresponds to the minimum-possible value of the impact parameter of the particle, in which it still will not be captured by the by a black hole.
III. RADIATION OF A CHARGE

For covariant 4-vector potential $A_i$ (for electromagnetic field), we have two invariants: $\text{inv}_1 = A_i u^i$ and $\text{inv}_2 = A_i A^i$. In accordance with the gauge invariance of the 4-vector potential we choose the calibration $\tilde{A}_i$ in the comoving (free-falling) reference system so that $\tilde{A}_\gamma := 0$. Thus, in the comoving reference frame we have: $\tilde{A}_0 = q/R$, where $q$ – particle charge, $R = |\vec{r}_{ob} - \vec{r}|$, $\vec{r}_{ob}$ and $\vec{r}$ – the radius-vectors to the observer and to the charge (respectively). Hence, according to the Lorentz transformations, we obtain for the covariant spatial components of potential in Schwarzschild reference frame near the particle:

$$A_0 = \frac{q}{R} u_0, \quad A_\gamma = \frac{q}{R} u_\gamma. \quad (10)$$

To get from here the covariant components of the electric field $F_{\gamma} \equiv \partial_\gamma A_0 - \partial_0 A_\gamma$ is necessary to differentiate expressions (10) by the time coordinate $ct_{ob}$ (at the point where the observer) and by the spatial coordinates of the radius vector $\vec{r}_{ob}$ (at the observation point). Because values of $u_0$ and $u_\phi$ are constants (see section II) and $u_\theta = 0$, we have:

$$F_{0r} = \frac{q}{R} \partial_0 u_r \quad (11)$$

In the expression (11) we have discarded a member with the asymptotic $\propto 1/R^2$ and allowed only a member with the required asymptotic behavior: $\propto 1/R$ in the field of the electromagnetic wave (emw):

$$F_{\gamma}^{\text{emw}} F_{\gamma}^{\text{emw}} \propto \frac{1}{R^2}, \quad F_{\alpha\gamma}^{\text{emw}} F_{\alpha\gamma}^{\text{emw}} \propto \frac{1}{R^2}. \quad (12)$$

The asymptotic behavior $\propto 1/R$ in the electromagnetic field corresponds to the field of the wave propagating along the vector at infinity $\vec{R}$. In this case, the field asymptotic behavior $\propto 1/R^2$ does not correspond to electromagnetic waves, as does not satisfy the conservation of energy flux through a sphere with radius $r >> r_g$. At the same time, at distances which much larger than the size of the radiating system (ie maximum radial coordinates of the particle), wave approximation works and we can talk about radiation of photons by the

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3 In general, the gauge transformation $A_i \rightarrow A_i + \partial_i f(x^k)$ can not be made to vanish by all three spatial components $A_\gamma$, but in the comoving (for incident particle) reference system it is possible to do – see for example [15], §65.

4 Therefore, to find the radiated field components we need to differentiate only $u_i$. 
particle. But we are interested not only processes at large distances in the wave zone, and (mostly) processes near the black hole and near the particle. When you change locations of the charge (or a system of charges) in Schwarzschild coordinates the corresponding change of electromagnetic field propagates at the speed of light. We are interested in the change in electromagnetic field, which corresponds to the asymptotic behavior of the wave field \( \propto \frac{1}{R} \) and then we will talk about such a field – as an the radiation field of of the electromagnetic wave with wave vector \( k^i \). At the same time, we remember that we are considering processes occur not only in the wave zone.

To find the Fourier transform of the radiation field we proceed similarly to\(^5\) our work [11]:

\[
F_{0\gamma}^{(w)} = \frac{1}{2} \int_0^\infty [\sin(wt_{ob}) + \cos(wt_{ob})] \ F_{0\gamma} \ dt_{ob}
\]  

(13)

Here, the value \( w \equiv ck_0 \) has the physical meaning of covariance, time component of the wave 4-vector\(^6\) \( k_0 \) – integral of the motion for the photon (analogue \( u_0 \) from [2]).

Because the observer with the radial coordinate \( r_{ob} \) and 4-velocity \( U^i \) (relative to predetermined reference system) registers the frequency of the electromagnetic field \( \hat{w} := ck_i U^i \), then in general the frequency of which registers the fixed observer is equal to \( \hat{w} = w/\sqrt{1 - r_g/r_{ob}} \).

The formula (13) from the point of view of the stationary observer (at radius \( r_{ob} \)) can be rewritten as:

\[
F_{0\gamma}^{(\hat{w})} = \frac{1}{\sqrt{2}} \int_0^\infty \cos \left( \hat{w}t_{ob} \sqrt{1 - r_g/r_{ob}} - \frac{\pi}{4} \right) \ F_{0\gamma} \ dt_{ob}
\]  

(14)

At \( r_{ob} >> r_g \) we have \( \hat{w} \approx w \).

In formulas (13) and (14) the point of time \( t_{ob} \) is the time of arrival of the wave to the observer. At the same time \( t_{ob} \) should be carried differentiation (to obtain the field \( F_{0\gamma} \) from

\(^{5}\) Here we take into account that up to time \( t_{ob} = 0 \) the particle at rest (for a stationary observer), therefore the emitted fields are absent. Since any function can be represented as a superposition of symmetric and antisymmetric parts, then we extrapolating the function \( F_{1\gamma} \) to the negative half-time even and odd way – in the half-sum we get zero, and each of the parts (even and odd) we use for the Fourier transform.

\(^{6}\) The values of the contravariant components of the wave 4-vector \( k^i \) (corresponding to the electromagnetic wave which propagating in the plane of the equator) are obtained from expressions (3-5) by the replacement \( u^i \rightarrow \epsilon k^i \) in the limit \( \epsilon \rightarrow \infty \).
potential $A_\gamma$), Therefore, in expression $[14]$ can use integration by parts$^7$:

$$F_{0\gamma}^{(\hat{w})} = -\frac{A_\gamma(t_{ob} = 0)}{2} + \frac{\hat{w}\sqrt{1 - r_g/r_{ob}}}{\sqrt{2}} \int_0^\infty \sin\left(\hat{w}t_{ob}\sqrt{1 - \frac{r_g}{r_{ob}} - \frac{\pi}{4}}\right) A_\gamma(t_{ob}) \ dt_{ob} \quad (15)$$

In order to determine the time $t_{ob}$ we consider two consecutive and near events in the Schwarzschild reference system:

1) field measurement by observer at the time of arrival of the particle at the point with the radial coordinate $r_1$;
2) field measurement by observer at the time of arrival of the particle at the point with the radial coordinate $r_2$. And $r_1 - r_2 := dr > 0$.

The corresponding total changing the time $dt_{ob}$ consists of two components:

1) Period of time$^8$ $\Delta t_f$, which corresponds to the difference in length of paths $\delta l_1$ and $\delta l_2$ for distribution (with speed of light) a retarded potential of the radiated field. Moreover, the retarded potential extends from the points with radial coordinates $r_1$ and $r_2$ (respectively) to the observer (with the radial coordinate $r_{ob}$). If you know the impact parameter for the photon $h_f$, then from expressions $[3]$ and $[4]$ at $\epsilon \rightarrow \infty$ (for distribution of retarded potential) we get by clock a stationary observer at infinity $\Delta t_f = \sqrt{g_{00}^{ob}(u^0/u^r)}dr/c$:

$$\Delta t_f = \frac{-\sqrt{1 - r_g/r_{ob}} \ dr}{c(1 - r_g/r)\sqrt{1 - (1 - r_g/r)h_f^2/r^2}} \quad (16)$$

2) the time required to move the particle from the point $r_1$ to the point $r_2$ (by clock a stationary observer) – see $[3]$ and $[5]$).

Thus, the sum of these two components gives:

$$dt_{ob} = \frac{-\sqrt{1 - r_g/r_{ob}} \ dr}{c(1 - r_g/r)\sqrt{1 - (1 - r_g/r)h_f^2/r^2}} + \frac{-\sqrt{1 - r_g/r_{ob}} \ dr}{c(1 - r_g/r)\sqrt{1 - (1 - r_g/r)(1/e^2 + h_f^2/r^2)}} \quad (17)$$

### IV. THE RADIATION OF THE DIPOLE

The field of retarded potentials of the dipole is determined by us in the standard way: as a superposition of fields from each of the charges of the dipole. The dipole moment $d_0$ is

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$^7$ In formula $[15]$ was taken into account that on the horizon for the dipole we have $A_\gamma = 0$ – see. $[23]$.

$^8$ Hereafter the subscript "\textit{f}" corresponds to the retarded potential of the field.
also defined in the standard way:

\[ d_0 := q \cdot \delta l = \text{const}, \]  

(18)

here \( \delta l \) – constant length of the rigid dipole which defined in the framework of general relativity. To determine the \( \delta l \) we assume that in the reference frame which associated with the dipole one of its charges emits a gamma quantum, and the other charge through the proper time \( \delta \tau \) this gamma quantum are absorbed. Then the square of the interval \( ds^2 \) between these two events get (in the comoving and Schwarzschild reference systems, respectively):

\[ ds^2 = c^2 \delta \tau - \delta l^2 = 0, \quad ds^2 = g_{ij} \delta x^i \delta x^j = 0. \]  

(19)

To determine from here the magnitude \( \delta l \) we use another invariant – see (2):

\[ c \delta \tau = \delta x^i u_i = \text{inv} \]  

(20)

We consider three possible variants for the dipole orientation in space: along \( \delta r \), along \( \delta \theta \) and along \( \delta \phi \). For the dipole orientation along the increment \( \delta x^\alpha \) (in spatial coordinates) we introduce the notation: \( \delta x^\alpha := \delta l \cdot f^\alpha \). From equations (19) and (20) we obtain for these three possible orientations of the dipole:

\[ f^r = \frac{(1 - r_g/r)}{\epsilon - u^r}, \quad f^\theta = \frac{\sqrt{1 - r_g/r}}{\epsilon r}, \quad f^\phi = \frac{\sqrt{1 - r_g/r}}{\epsilon \left( r + h \sqrt{1 - r_g/r} \right)}. \]  

(21)

Then, in the formula (10), which written for the dipole, will be a corresponding increment \( \delta u_\gamma \) for 4-velocity under parallel translation \( u_\gamma \) along \( \delta x^\alpha \) in a curved space-time:

\[ A_\gamma = \frac{q}{R} \delta u_\gamma \equiv \frac{q}{R} \left( \partial_\alpha u_\gamma - \Gamma^k_{\gamma \alpha} u_k \right) \delta x^\alpha = \frac{d_0 f^\alpha}{R} \left( \partial_\alpha u_\gamma - \Gamma^k_{\gamma \alpha} u_k \right) \]  

(22)

Here \( \Gamma^i_{kl} := \frac{1}{2} g^{im} \left( \partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl} \right) \) – Christoffel symbols.

The Fourier transform of the electric dipole field at infinity we obtain by substituting (22) in the formula (15).

The retarded potentials (10) and (22) in the leading order are proportional to \( 1/c \) (inversely proportional the speed of light), because in the nonrelativistic limit \( (\epsilon \rightarrow 1, \ r_g/r \rightarrow 0) \). The spatial components for 4-velocity inversely proportional to the speed of light \( (u^\gamma \propto v/c) \), and zero (time) component of the particle 4-velocity in the main approximation by \( c \) does not depend. Therefore the emitted fields of the charge and dipole in the main approximation are correspond to the emission of the dipole type radiation (which inversely proportional to the square of the speed of light).
V. THE CALCULATION OF THE SPECTRAL DENSITY OF DIPOLE RADIATION

Then we pass to dimensionless units: \( r_g = 1 \) and \( c = 1 \). Let us consider the simplest (in terms of computations) dipole orientation\(^9\): \( \delta x^\alpha = \delta_\theta^\alpha \cdot \delta l \cdot f^\theta \). In this case, the orientation of the dipole in spherical coordinates stored over time. Let the observer has the coordinates: \( r_{ob} >> r_g, \theta_{ob} = \pi/2, \varphi_{ob} = 0 \) (at the equator plane). Then from (22) we obtain:

\[
A_\gamma = -\delta_\gamma^\theta \cdot \frac{d_0 f^\theta ru}{R} = \frac{\delta_\theta^\theta d_0}{R} \sqrt{\left(1 - \frac{r_g}{r}\right) - \left(1 - \frac{r_g}{r}\right)^2 \left(\frac{1}{\epsilon^2} + \frac{h_f^2}{r^2}\right)}
\]

(23)

Note that the form of the expression (23) ensures the convergence of the integral (15) throughout the range of the radial coordinate \( r \) and time \( t_{ob} \) (despite the divergence of expressions for \( dt_{ob} \) near the horizon of the black hole).

For brevity, we will talk about the photons (instead of the above-mentioned fields of retarded potential of electromagnetic wave), thus all characteristics of the photon determined by us away from the black hole (in the wave zone). Accordingly, for a dipole we introduce the quantities subscripts ’’\( d \)’’, and for the field of retarded potential – ’’\( f \)’’.

Denoting: \( \xi := 1/r \), from (3-5) we obtain for the total deflection angle of the photon:

\[
\varphi_f = \int_{\xi_{ob}}^{\xi_d} \frac{h_f d\xi}{\sqrt{1 - h_f^2\xi^2(1 - \xi)}} + \varphi_d
\]

(24)

During the movement the particle (dipole) emits photons in opposite directions, but we are interested in only those photons which are reach the observer, ie at position \( \varphi_f = \varphi_{ob} := 0 \) and \( \xi_{ob} = 1/r_{ob} \). In this case, the initial coordinates for photons are: \( r_d, \varphi_d(r_d) \) and time \( t_d(r_d) \) (for each of the photon – its value \( r_d \)). To calculate the trajectory of these photons, we also need to know their impact parameters \( h_f(r_d) \). Therefore, to calculate the change of the impact parameter \( dh_f \) (between different photons – during the motion of a particle) we obtain from (24) the differential relation:

\[
d\varphi_f(h_f, \xi_d) = \frac{\partial \varphi_f}{\partial \xi_d} d\xi_d + \frac{\partial \varphi_f}{\partial h_f} dh_f + d\varphi_d(\xi_d) = 0,
\]

(25)

\(^9\) Here we have designated: \( \delta_\theta^\theta \) – Kronecker symbol.
Figure 1: The spectral density of the dipole radiation $\left| F^{(\hat{w})}_{\hat{\phi}}(\hat{w}) \right|^2$. The parameters $h_d$, $\theta_0$, $\epsilon$, total radiated energy $E_{\text{tot}}$ (in units $d_0^2/(r_g R^2)$) and the amplitudes $\left| F^{(\hat{w})}_{\hat{\phi}} \right|$ (in units $d_0/R$) for all these cases displayed on the panels. The left-hand sides of panels correspond $\hat{w} = 0$, right-hand sides – $\hat{w} = c/r_g$.

where

$$\frac{\partial \varphi_f}{\partial \xi_d} = \frac{h_f}{\sqrt{1 - h_d^2 \xi_d^2(1 - \xi_d)}}$$

(26)

$$\frac{\partial \varphi_f}{\partial h_f} = \int_{\xi_{\text{in}}}^{\xi_d} \frac{d\xi}{\left[ 1 - h_f^2 \xi^2(1 - \xi) \right]^{3/2}}$$

(27)

and taking into account (4) and (5) we have:

$$d\varphi_d = \frac{-h_d d\xi_d}{\sqrt{1 - (\epsilon^{-2} + h_d^2 \xi_d^2)(1 - \xi_d)}}$$

(28)

At $\varphi_d = 0$ we have $h_f = 0$ (the initial moment $t_d = 0$). Thus, from the relation (25) we can calculate $h_f$. 
Knowing the value of \( h_f \) for the photon, we can find the time of photon propagation \( \Delta t_f \) from the current position of the particle to the observer – see (16):

\[
\Delta t_f = \int_{\xi_{ob}}^{\xi_d} \frac{\sqrt{1 - \xi} \, d\xi}{\xi^2(1 - \xi)\sqrt{1 - h_f^2\xi^2(1 - \xi)}}
\]

(29)

At \( \xi_{ob} \to 0 \) we have: \( \Delta t_f \to \infty \) – since the time of propagation for the photon to an infinitely for the distant observer tends to infinity. Therefore, the value \( \xi_{ob} \) must be greater than zero.

VI. CHARACTERISTIC FEATURES OF THE SPECTRUM

Try to explain the characteristic features of the spectrum for falling into a black hole dipole and shown in Fig 1 (for low frequencies \( w \leq c/r_g \)). For this purpose, we according to the most general considerations:

1. At the initial moment a distant observer can detect the electric dipole field, as dipole is sufficiently far away from the black hole.

2. At the final moment (in the limit \( t_{ob} \to \infty \)) the dipole disappears below the horizon of the black hole, all electromagnetic fields associated with the dipole must be radiating to infinity, so a distant observer can not register a dipole field.

The law of decrease of the field for a distant observer was calculated in the previous section. In general, this law can be approximated by any decaying function. In the simplest case, we can use this step function: \( F_{0\gamma} = F_{\text{step}} \cdot \Theta(t_1 - t_{ob}) \), according to the formula (13) we have:

\[
F_{0\gamma}(w) = F_{\text{step}} \cdot \frac{1 - \cos(wt_1) + \sin(wt_1)}{2w}
\]

(30)

If we use for this exponent: \( F_{0\gamma} = F_{\exp} \cdot \exp(-\alpha t_{ob}) \cdot \Theta(t_{ob} - t_1), \alpha > 0 \), we have:

\[
F_{0\gamma}(w) = F_{\exp} \cdot \exp(-\alpha t_1) \cdot \frac{\alpha \sin(wt_1) + w \cos(wt_1) + \alpha \cos(wt_1) - w \sin(wt_1)}{2(\alpha^2 + w^2)}
\]

(31)

From the last two formulas becomes clear why the oscillations in the spectra on Fig. 1 and their asymptotic behavior: \( \propto 1/w \).
Figure 2: The characteristic shape of the spectrum asymptotics at $w \to \infty$ (this expression is (35), which squared and multiplied by $w^2$). Panels A and B correspond to the different possible relationships between the coefficients $F_{0\alpha}(0)$ and $\sqrt{2} \sum_{n=0}^{\infty} \frac{F_{0\alpha}^{(n)}(0)}{n!} t^n$ in (35).

VII. ASYMPTOTICS OF THE SPECTRUM OF THE RADIATED FIELD AT $\omega \to \infty$

For calculating the required asymptotics (at $w \to \infty$ – in dimensionless units) we proceed similarly to our work [12]. The integral in (15) splits into two parts: the first part – from zero to the observer’s time $t_1$ and the second – from $t_1$ to infinity (which corresponds to achievement by the particle the horizon by the hour stationary observer). Moreover, the time $t_1$ is chosen so that the corresponding radial coordinate of the particle $r_1$ differs from $r_g$ by a small amount $\delta_1 := r_1 - r_g << r_g$.

The potential $A_\gamma$ in the second part of the integral, according to (23), is equal to $\sim (d_0/R) \sqrt{\delta_1/r_g}$. In addition, according to (17), the element of time $dt_{ab}$ is equal to $\sim r_g dr/\delta_1$. Then in the main approximation by small value $\delta_1$ the second part of the integral in (15) will be small compared with the first part ($\propto \sqrt{\delta_1}$).

Since the main contribution to the integrals (13), (14) and (15) gives the first part (from
0 to $t_1$), then the second part we neglect and consider the first integral in (13) in more detail. We decompose the integrand function $F_{0\alpha}(t_{\text{ob}})$, in a Taylor series with respect to dimensionless time $t$

$$F_{0\alpha}(t) = \sum_{n=0}^{\infty} \frac{F_{0\alpha}^{(n)}(0)}{n!} \cdot t^n$$  \hspace{1cm} (32)$$

Use the recurrence relation:

$$\int t^n [\sin(wt) + \cos(wt)] \, dt = \frac{t^n}{w} [\sin(wt) - \cos(wt)] - \frac{n}{w} \int t^{n-1} [\sin(wt) - \cos(wt)] \, dt =$$

$$= \frac{t^n}{w} [\sin(wt) - \cos(wt)] - \frac{nt^{n-1}}{w^2} [\sin(wt) + \cos(wt)] + \frac{n(n-1)}{w^2} \int t^{n-2} [\sin(wt) + \cos(wt)] \, dt = ...$$

And we use its mathematical limit (the leading order) at $w \to \infty$:

$$\int_{0}^{t_1} t^n [\sin(wt) + \cos(wt)] \, dt \approx \frac{t_1^n [\sin(wt_1) - \cos(wt_1) + \delta^n]}{w}$$  \hspace{1cm} (33)$$

Using this expression in the decomposition (32), for (13) we obtain the asymptotic behavior of the spectrum of the radiated field:

$$\lim_{w \to \infty} F_{0\alpha}^{(w)} \approx \frac{\sin(wt_1) - \cos(wt_1)}{2w} \left( \sum_{n=0}^{\infty} \frac{F_{0\alpha}^{(n)}(0) t^n_1}{n!} \right) + \frac{F_{0\alpha}(0)}{2w}$$  \hspace{1cm} (34)$$

This shows that the spectral energy density attractor $E_w$ of radiated field (which proportional to the square of this expression) is inversely proportional to the square of the frequency:

$$E_w \propto \frac{1}{w^2}.$$ 

Expression (34) can be rewritten as:

$$\lim_{w \to \infty} F_{0\alpha}^{(w)} \approx \frac{F_{0\alpha}(0)}{2w} - \frac{\sqrt{2}}{2w} \left( \sum_{n=0}^{\infty} \frac{F_{0\alpha}^{(n)}(0) t^n_1}{n!} \right) \cos \left( wt_1 + \frac{\pi}{4} \right)$$  \hspace{1cm} (35)$$

At $w \to \infty$ the shape of this spectrum is almost not depending on magnitude of the impact parameter $h_d$ – see Fig. 2.

VIII. THE DEPENDENCE OF THE POYNTING VECTOR ON TIME

Poynting vector $P$ is given by\textsuperscript{10}:

$$P := \frac{F_{\theta\theta}^2}{4\pi r^2}, \quad F_{\theta\theta} = \frac{dA_\theta}{dt_{\text{ob}}}.$$  \hspace{1cm} (36)$$

\textsuperscript{10} Strictly speaking the value of $P$ is not the Poynting vector, but only one of its components (at infinity).
Figure 3: Dependencies of the Poynting vectors $P := F_{0\theta}^2/(4\pi r^2)$ from $t_{ob}$. The maxima of the curves are placed in the center panel, the width of the panels – $60r_g/c$, the distance between the vertical lines is equal to $15r_g/c$, therefore the width of the "bells" at half height $\sim 10r_g/c$.

The amplitude maximum (at different $h_d$) indicated in units $d_0^2/(r^2R^2)$ (at $r_g = 1$ and $c = 1$).

For each panel, the red curve (with highest amplitude) corresponds to $h_d = 0$, the green curve – $h_d = 0.5h_{max}$, and the blue curve (with more than one "bell") – $h_d = 0.99h_{max}$.

And, according to[17], we replace the differentiation with respect $dt_{ob}$ to the differentiation with respect $r$ and obtain:

$$|F_{0\theta}| = \frac{d_0 \sqrt{(1 - r_g/r) \left[ r^3 - (r - r_g)h_f^2 \right] [r_g r^3 - 2r_g(r - r_g)(r^2/\epsilon^2 + h_f^2)] + 2(r - r_g)^2h_f^2]}{2Rr^5 \sqrt{1 - r_g/R_{ob}}} \left[ \sqrt{r^3 - (r - r_g)h_f^2} + \sqrt{r^3 - (r - r_g)(r^2/\epsilon^2 + h_f^2)} \right]$$

(37)

As was shown in [12], and as shown in Fig. 3 profile shape depending on the vector Pointing on time $t_{ob}$ is practically independent from the dipole orbital parameters (the initial radius of falling). And the profile width at half maximum $\Delta t_{ob}$ directly related with the mass of the black hole: $\Delta t_{ob} \sim 10r_g/c$. In general case, non-radial dipole falling into Schwarzschild black hole the profile form $P(t_{ob})$ becomes dependent on $\epsilon$ and $r_0$ at $h_d > 0$ (see Fig. 3). At $h_d \sim 1$
in the function $P(t_{ob})$ appears several local maxima, although the maximum amplitude is reduced and at the same time slightly reduced the total radiated energy – see Fig. [1]

IX. DISCUSSION AND CONCLUSIONS

As can be seen from the results of the numerical calculations (see. Fig. [1]), the main features of the spectrum are also saved for $h_d > 0$. (asymptotic behavior and an increase in the frequency of oscillations with increasing $r_0$). In this case, all the spectra dependence (on the value of $h_d$) is detectable only at small $w < \sim 1$, and at the asymptotic behavior ($w \to \infty$) (as shown in Section VII) the behavior of spectra is identical.

The dependence of the amplitude and the total radiated energy on the value $h_d$ is weak, therefore almost all of the results and conclusions obtained by us for the case $h_d = 0$, are also applicable to the general case $h_d > 0$. We list here all these results (see [12]):

1. In the case of supervisory detection of such spectra (or rather their asymptotics) will have a real possibility by new and independent way to determine the main characteristics of a black hole - its mass. In addition, probably also will be able to determine (by circumstantial evidence) some properties of magnetized matter, which accreting to this black hole.

2. The main problem, which in this case will be facing the observer, would be too weak energy flow (for measuring on the available radio frequencies) for observation of such process (dipole falling into a black hole).

3. In addition, the falling of the strongly magnetized compact bodies onto black holes are apparently very rare events, which also represents a serious obstacle for observations.

4. It is also likely that scan the entire range of radio waves on the subject the search of such spectra lead to the discovery of new black holes in our galaxy and studying of their properties.

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[1] *Black Holes: The Membrane Paradigm* (Kip S. Thorne, 1986), Douglas A. MacDonald, Richard H. Price ed.

[2] V. P. Frolov and I. D. Novikov, *Black Hole Physics. Basic Concepts and New Developments* (Kluver AP, 1998).

[3] B. Linet, J. Phys. A 9, 1081 (1976).

[4] F.J. Zerilli, Phys. Rev. D 2, 2141 (1970).

[5] M. Davis, R. Ruffini, W.H. Press and R.H. Price, Phys. Rev. Lett. 27, 1466 (1971).

[6] D.K. Ross, Astron. Soc. Pacific 83, 633 (1971).

[7] S.A. Teukolsky, Phys. Rev. Lett. 29, 1114 (1972).

[8] D.G. Yakovlev, JETP 41, 179 (1975).

[9] I.G. Dymnikova, Astrophysics and Space Science 51, 229 (1977).

[10] Karl Martel and Eric Poisson (2008), ArXiv: gr-qc/0107104

[11] Shatskiy A.A., Novikov I.D., Lipatova L.N., Journal of Experimental and Theoretical Physics 116, 904 (2013).

[12] Shatskii A.A., Novikov I.D., Lipatova L.N., Astronomy Reports 58, 39 (2014).

[13] Nail Khusnutdinov, PHYSICAL REVIEW D 89, 024012 (2014).

[14] G.C. Graves, D.R. Brill, Phys. Rev. 120, 1507 (1960).

[15] L.D. Landau and E.M. Lifshitz, *Course of Theoretical Physics, Volume 2: The Classical Theory of Fields* (Nauka, Moscow, 1988; Butterworth–Heinemann, Oxford, 1990, 1988).

[16] Ya.B. Zel’dovich and I.D. Novikov, *Relativistic Astrophysics* (Nauka, Moscow [in Russian], 1967).