Sharp Adams inequalities with exact growth conditions on metric measure spaces and applications

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Abstract

Adams inequalities with exact growth conditions are derived for Riesz-like potentials on metric measure spaces. The results extend and improve those obtained recently on $\mathbb{R}^n$ by the second author, for Riesz-like convolution operators. As a consequence, we will obtain new sharp Moser-Trudinger inequalities with exact growth conditions on $\mathbb{R}^n$, the Heisenberg group, and Hadamard manifolds. On $\mathbb{R}^n$ such inequalities will be used to prove the existence of radial ground states solutions for a class of quasilinear elliptic equations, extending results due to Masmoudi and Sani.

1 Introduction

There are several extensions of the classical Moser-Trudinger inequality (MTI) for functions in $W^{\alpha,n/\alpha}_0(\Omega)$, with $\Omega \subset \mathbb{R}^n$ and $\alpha$ an integer with $0 < \alpha < n$. The most important types can be categorized using the single inequality

$$\int_{\{|u| \geq 1 \}} \frac{\exp \left[ \gamma |u|^{n-\alpha} \right]}{|u|^\lambda} \, dx \leq C(1 - \epsilon + \epsilon \|u\|^{n/\alpha}_{n/\alpha}), \quad \kappa \|u\|^{n/\alpha}_{n/\alpha} + \|\nabla^\alpha u\|^{n/\alpha}_{n/\alpha} \leq 1 \quad (1.1)$$

where $\nabla^\alpha = (-\Delta)^{\alpha/2}$ if $\alpha$ is even, and $\nabla^\alpha = \nabla(-\Delta)^{\alpha-1/2}$ if $\alpha$ is odd.

The classical MTI corresponds to the case $\lambda = \epsilon = \kappa = 0$, and it holds for $\gamma$ up to and including an explicit constant $\gamma_{n,\alpha}$ (see (8.3)) and not for a larger $\gamma$, provided that $\Omega$ is an open set with finite measure. This is the classical result by Adams [A1]. More generally, in [FM3] it was shown that the classical MTI holds is $\Omega$ is Riesz-subcritical, i.e., roughly speaking, if it misses enough dimensions at infinity.

The Ruf MTI corresponds to $\lambda = \epsilon = 0$ and $\kappa = 1$, and it holds on $W^{\alpha,n/\alpha}_0(\mathbb{R}^n)$ for $\gamma \leq \gamma_{n,\alpha}$ and it fails for $\gamma > \gamma_{n,\alpha}$. The first such inequality was derived by Ruf in [Ruf] for the gradient in $\mathbb{R}^2$, later extended to all dimensions in [LR], the Laplacian in [LL2], arbitrary $\alpha$ in [FM2]. The inequality fails at $\gamma = \gamma_{n,\alpha}$ (and it’s true for $\gamma < \gamma_{n,\alpha}$) if the Ruf norm condition is replaced by the weaker condition $\max\{\|u\|^{n/\alpha}_{n/\alpha}, \|\nabla^\alpha u\|^{n/\alpha}_{n/\alpha}\} \leq 1$.

The Adachi-Tanaka MTI is the case $\lambda = \kappa = 0$ and $\epsilon = 1$, and it is true only for $\gamma < \gamma_{n,\alpha}$ and not true for $\gamma = \gamma_{n,\alpha}$ regardless of whether or not $\Omega$ has infinite measure [AT], [FM2]. In fact, such an inequality with $\gamma = \gamma_{n,\alpha}$ fails even if $0 < \lambda < \frac{n}{n-\alpha}$. The Adachi-Tanaka inequality is implied by the Ruf MTI.

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The MTI with exact growth condition, which we will also call Masmoudi-Sani inequality (MSI), corresponds to \( \lambda = \frac{n}{n-\alpha} \), \( \epsilon = 1 \), \( \kappa = 0 \), and it holds on any \( \Omega \subseteq \mathbb{R}^n \) for \( \gamma \leq \gamma_{n,\alpha} \) while it fails for \( \gamma > \gamma_{n,\alpha} \). This result was derived Masmoudi, Sani and other authors in [MS1], [MS2], [MS3], [IMN], [LTZ]. When \( \Omega = \mathbb{R}^n \) the inequality yields, in particular, a uniform bound of the exponential integral with \( \gamma = \gamma_{n,\alpha} \) under the condition \( \max\{\|u\|_{n/\alpha}, \|\nabla^\alpha u\|_{n/\alpha}\} \leq 1 \) (which is not possible if \( \lambda < \frac{n}{n-\alpha} \)), which is weaker than the Ruf norm condition. It is in fact possible to derive the Ruf MTI from the MSI, even though both inequalities are in effect giving optimal growth conditions under different Sobolev norms.

When \( \Omega \) has infinite measure, or when \( \epsilon = 1 \), all of the above inequalities can be stated in equivalent forms with the integrals taken over the entire \( \Omega \), provided the exponential is suitably regularized and the denominator is replaced by \( 1 + |u|^{\lambda} \).

As a side note, we mention that in \( \mathbb{R}^2 \) the limiting case \( \alpha = 2 \) in \( L^1 \) has been studied in [CRT] and [FM4], in the context of the so-called “reduced Sobolev spaces”, where the Moser type inequalities are often called “Brezis-Merle inequalities”. The case where no boundary condition is assumed, i.e. \( W^{\alpha,n/\alpha}(\Omega) \), has been settled completely by Cianchi in [Ci1] for \( \alpha = 1 \), whereas when \( \alpha = 2 \) sharp results have been derived in [FM5] when \( \Omega \) is a ball. We also mention [Ci2], where Cianchi derives the several sharp Moser-Trudinger inequalities with respect to Frostman (or upper Ahlfors-regular) measures, on a wide class of bounded domains of \( \mathbb{R}^n \), with and without zero boundary conditions.

In this paper we shall call “Adams inequality” (AI) any of the versions of (1.1) applied to potentials rather than Sobolev functions, namely

\[
\int_{\{|Tf| \geq 1\}} \frac{\exp\left[\gamma |Tf|^{\frac{n}{n-\alpha}}\right]}{|Tf|^{\lambda}} \, dx \leq C(1 - \epsilon + \epsilon \|Tf\|_{n/\alpha}^{n/\alpha}), \quad \kappa \|Tf\|_{n/\alpha}^{n/\alpha} + \|f\|_{n/\alpha}^{n/\alpha} \leq 1 \tag{1.2}
\]

where \( f \) is a compactly supported function in \( L^{n/\alpha}(\Omega) \), and \( Tf \) is an integral operator with a kernel \( k \) which behaves like a Riesz potential, in a suitable sense. The classical AI was first derived by Adams for the Riesz potential in [A1]. In a series of papers ([FM1], [FM2], [FM3]) the first author and Fontana refined the Adams machinery and used it derive Adams inequalities in several settings. In particular, they proved that if \( Tf = K \ast f \), where \( K(x) \sim g(x^*)|x|^{\alpha-n} \) for small \( x \), where \( x^* = x/|x| \), then the classical AI holds for \( \gamma \leq \gamma_g \), where \( \gamma_g \) is given explicitly in terms of \( g \), provided \( \Omega \) has finite measure, or more generally if \( K \) satisfies a critical integrability condition at infinity ([FM1], [FM3]). In [FM2] it was shown that for such \( K \) the Ruf AI holds for \( \gamma \leq \gamma_g \) when \( \Omega = \mathbb{R}^n \). In all these cases the constant \( \gamma_g \) is sharp (largest) provided \( K \) is regular enough. These results applied to the Riesz potential imply the corresponding MT inequalities mentioned above.

The Masmoudi-Sani Adams inequality was recently settled by the second author in [Qin2], for convolution operators \( Tf = K \ast f \) where \( K(x) \sim g(x^*)|x|^{\alpha-n} \) for small \( x \) and for large \( x \), with the same sharp exponential constant \( \gamma_g \). As a consequence Qin derived sharp MTI with exact growth condition for arbitrary fractional powers of the Laplacian and for general homogeneous differential operators with constant coefficients.
In this paper we push the techniques used in [Qin2] to general metric measure spaces, and derive sharp and improved Masmoudi-Sani and Ruf’s Adams inequalities for Riesz-like potentials in that setting. So far only the classical Adams inequalities are known in the context of measure spaces [FM1], [FM3]; the metric structure was not needed there, but we could not avoid it in this paper.

As a consequence, we will obtain Moser-Trudinger inequalities with exact growth conditions on the Heisenberg group and on Hadamard manifolds. We will also give an application concerning the existence of a ground state solution of a nonlinear PDE in the same spirit as in [MS2], but for a slightly more general equation. Other applications in the context of Riemannian manifolds with nonnegative curvature will appear in a forthcoming paper.

The outline of the paper is as follows.

In Section 2 we will first introduce the assumptions we need on our metric measure spaces. We will only require that the measure of balls is a finite continuous function of the radius, and we will not require the doubling property.

Next, we will introduce the Riesz-like kernels. There are several papers dealing with Riesz kernels on a metric measure space $X$, with metric $d$ and Borel measure $\mu$. Normally such kernels $k(x, y)$ are comparable to $\mu(B(x, d(x, y)))^{-\beta}$ ($\beta > 0$), where $B(x, r)$ is the ball centered at $x$ and with radius $r$, or in other cases they are defined as $d(x, y)^\alpha \mu(B(x, d(x, y)))^{-1}$. If $\mu$ is Ahlfors-regular of order $d$ (i.e. balls of radius $r$ have measures comparable to $r^d$) these definitions amount to assuming that $k$ is essentially comparable to $d(x, y)^\alpha - d$ (see for example [ARSW], [HK]). The key step in the derivation of sharp MTI using Adams inequalities, is to represent a Sobolev function in terms of a (pseudo-)differential operator of order $\alpha$, and to have precise information on the behavior of its fundamental solution (Green’s function) near the diagonal. In general we cannot expect the Green function behaves like a constant multiple of $d(x, y)^{\alpha - d}$, or $\mu(B(x, d(x, y)))^{-\frac{d}{d - \alpha}}$, when $d(x, y)$ is small. On the Heisenberg group $\mathbb{H}^n$, for example, powers higher than 2 of the subLaplacian have a Green function of type $g((xy^{-1})^*|xy^{-1}|^{a - Q}$, where $Q = 2n + 2$, and $x^* = x/|x|$, for some nonconstant function $g$ on the Heisenberg sphere.

For our purpose we will need sharp asymptotic Riesz-like behavior of the kernel both for small and large distances, which will be captured by integral conditions on the kernel, rather than pointwise.

The main inequalities of the paper are stated in Theorem 1, the sharpness statements are in Theorem 2. Besides being in the context of measure spaces, our MSI result contains two other substantial improvements from the known versions. The first improvement is the explicit dependence of sharp exponential constant from either the asymptotic behavior of the kernel along the diagonal, or the one at infinity. It has been already observed by Qin [Qin2, Remark 3] that the MSI for the Riesz kernel $|x - y|^{a-n}$ in $\mathbb{R}^n$ modified to be $2|x - y|^{a-n}$ for $|x - y| \geq 2$ cannot hold with the same sharp exponential constant for the unmodified Riesz kernel.

Secondly, we will allow the power of $|Tf|$ in the denominator to vary, as well as the norm on the right-hand side, accordingly. As consequence of this result, for example we obtain the following improvement of the original MSI on $\mathbb{R}^n$: 

3
\[
\int_{\{|u|\geq 1\}} \exp\left[\frac{\gamma_{n,\alpha} \cdot |u|^\frac{n}{n-\alpha}}{|u|^\frac{p}{p-\alpha}}\right] dx \leq C\|u\|_p^p, \quad \|\nabla^\alpha u\|^\frac{n}{n-\alpha} \leq 1 \tag{1.3}
\]

valid for all \(p \geq 1\), whereas the known results in the literature have \(p = n/\alpha\). The natural space of functions associated to such an inequality is a Sobolev space with mixed norms.

In Sections 3, 4, 5 we will prove Theorem 1. The proof follows the same lines as the one in [Qin2], however there are many changes and improvements, due to the different background space, the more general assumptions on the kernels, and the different conclusions of the theorem. The key ideas used in [Qin2] was to use an improved O’Neil’s Lemma combined with a suitable potential version of the “optimal descending growth condition”. Such condition was introduced in [IMN], [MS2] for radial functions in \(W^{1,n}(\mathbb{R}^n)\), and states that for \(u \in W^{1,n}(\mathbb{R}^n)\), radially decreasing and such that \(u(r) \geq 1\) and \(\|\nabla u\chi_{\{|x|\geq r\}}\|_n \leq 1\) we have

\[
\exp\left[\frac{\gamma_{n,1} \cdot u(r)^\frac{n}{n-1}}{u(r)^\frac{1}{n-1}}\right] r^n \leq C\|u\chi_{\{|x|\geq r\}}\|_n^n \tag{1.4}
\]

(for simplicity we let \(u(x) = u(|x|)\)), where \(\gamma_{n,1} = n\omega_{n-1}^{\frac{1}{n-1}}\) is the sharp constant in the MTI for the gradient. This estimate quantifies optimally the growth of radially decreasing Sobolev functions under the sole condition on their gradients, from which the authors in [IMN], [MS2] were able to derive the MSI for the case \(\alpha = 1\). A version of (1.4) for the case \(\alpha = 2\), using Lorentz norms of the Laplacian, was obtained in [MS1], [MS3], [LTZ].

Following [Qin2] a version of (1.4) will be obtained for general potentials, see (3.30) in Remark 6 where the function \(u(r)\) above is replaced by

\[
(Tf)^\circ(\tau) = \esssup_{x \in E_\tau} \left|T\left(f\chi_{F_\tau \cap B_\tau(x)^c}\right)(x)\right| \tag{1.5}
\]

where \(E_\tau, F_\tau\) are level sets of \(Tf\) and \(f\) resp., with volume \(\tau\), and where \(B_\tau(x)\) is a ball centered at \(x\), with volume \(\tau\). When \(T\) is the Riesz potential in \(\mathbb{R}^n\) and \(f\) is radially decreasing, then \((Tf)^\circ\) can be compared from above or from below by the decreasing rearrangement \((Tf)^*\) in some instances, and it actually coincides with it when \(Tf = \nabla|y|^{2-n} * f\), if \(|f|\) is radially decreasing (see Appendix, Prop. 4).

The potential version of (1.4) is stated in Proposition 2 in a slightly improved form, together with a version under the Ruf condition. Both of these results are key tools from which the main inequalities will be obtained. Although the Ruf inequality can be deduced from the MSI, we prefer to derive it directly from the optimal descending growth condition under Ruf norm. The proof of Proposition 2 is the most challenging one, and will be effected through an improved version of a discretization procedure introduced in [Qin2], which was itself adapted from the original papers by Masmoudi et al.

In Section 7 we will prove Theorem 2, which gives sufficient conditions for the exponential constants in the inequalities of Theorem 1 to be sharp.
In Section 8 we will apply inequality (1.3) in the case $\alpha = 1$ to show the existence of a radial ground state solution of the nonlinear equation

$$-\Delta_n u + V_0 |u|^{p-2} u = f(u)$$

where $\Delta_n$ is the $n-$Laplacian, $V_0$ is a positive constant, $p > 1$, $\|u\|_p < \infty$, $\|\nabla u\|_n < \infty$, and where $f$ satisfies certain critical exponential growth conditions. This result extends those in [MS2], which were proved for $p = n$.

Section 9 is devoted to the Heisenberg group, where we obtain, as an application of Theorems 1 and 2, sharp Moser-Trudinger inequalities with exact growth for the horizontal gradient and for the powers of the subLaplacian. In the latter case we also obtain sharp Ruf inequalities, which were previously known in the gradient case [LL1].

Finally, in Section 10 we apply Theorem 1 and obtain sharp MSI on complete, simply connected Riemannian manifolds with sectional curvature bounded above and below by two negative constants. The case $\alpha = 1$ in the hyperbolic space was settled in [LT]. Recently Bertrand and Sandeep [BS] proved that the classical MTI in such manifold holds, extending previous results on the Hyperbolic space by Fontana and Morpurgo in [FM3]. Both papers exploit the fact that the Green kernel for $\nabla^\alpha$ on simply connected manifolds with constant and negative curvatures have a nice exponential decay, and satisfy a critical integrability condition at infinity.

Some results in this paper are included in Qin’s PhD thesis [Qin1], specifically: 1) the Adams MSI with varying powers in the denominator, for Riesz-like convolution potentials on $\mathbb{R}^n$; 2) the application of (1.3) to the quasilinear equation (1.6); 3) the MSI inequality on $\mathbb{H}^n$, in particular (9.2) of Theorem 5, and Theorem 6 below (but without varying powers in the denominator). The latter results were obtained by adapting the proof in $\mathbb{R}^n$ in the context of the Heisenberg group.

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2 Adams inequalities on metric measure spaces

Let $(M, d, \mu)$ be a metric measure space, that is a set $M$ endowed with a distance function $d$ and a Borel measure $\mu$. Define

$$B(x, r) = \{y : d(x, y) \leq r\}, \quad V_x(r) = \mu(B(x, r)).$$

We assume that for all $x \in M$

i) $\forall r > 0, V_x(r) < \infty$, 
ii) $r \to V_x(r)$ continuous.

Such hypothesis easily implies that the boundary of any ball has zero measure, and that for any $x \in M$ and any $E$ measurable the function $r \to \mu(\overline{B(x, r) \cap E})$ is continuous on $[0, \infty)$. In particular, the measure $\mu$ is nonatomic.
A measurable function \( k : M \times M \to \mathbb{R} \) will be called a \textit{Riesz-like kernel of order} \( \beta > 1 \) and \textit{normalization constants} \( A_0 > 0 \) and \( A_\infty \geq 0 \) if there exists \( B \geq 0 \) such that for all \( x \in M \)

\[
\int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) \leq A_0 \log \frac{V_x(r_2)}{V_x(r_1)} + B, \quad 0 < V_x(r_1) < V_x(r_2) \leq 1, \tag{K1}
\]

\[
\int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) \leq A_\infty \log \frac{V_x(r_2)}{V_x(r_1)} + B, \quad 1 \leq V_x(r_1) < V_x(r_2) \tag{K2}
\]

for all \( x, y \in M \)

\[
|k(x,y)| \leq BV_x(d(x,y))^{-1/\beta} \quad |k(x,y)| \leq BV_y(d(x,y))^{-1/\beta} \tag{K3}
\]

for each \( \delta > 0 \) there is \( B_\delta > 0 \) such that for all \( x \in M \)

\[
\int_{d(x,y) \geq R} |k(x',y) - k(x,y)|^\beta d\mu(y) \leq B_\delta, \quad V_x(R) \geq (1 + \delta)V_x(r), \quad \forall x' \in B(x,r). \tag{K4}
\]

The classical Riesz kernel on \( \mathbb{R}^n \), i.e. \( |x - y|^{\alpha - n} \), satisfies (K1)-(K4) with \( \beta = \frac{n}{n-\alpha} \) and \( A_0 = A_\infty = |B_1| \), the volume of the unit ball. More generally, the convolution kernels treated in [FM2] or [Qin2] are of the above type. When \( \beta = 1 \) kernels satisfying conditions similar to (K3) and (K4) are known as “standard kernels” in the literature involving singular integrals (see e.g. [Ch]).

It’s easy to check that the continuity hypothesis on \( V_x \) implies that

\[
\left( \frac{1}{V_x(d(x,\cdot))} \right)^* (t) = \frac{1}{t} \quad x \in M, \ t > 0 \tag{2.1}
\]

where the function on the left is the decreasing rearrangement of \( V_x(d(x,\cdot)) \)^{-1} for each fixed \( x \) (for the definition of decreasing rearrangement see (3.1) below). Using (2.1) we get that (K3) implies

\[
\int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) \leq B^\beta \log \frac{V_x(r_2)}{V_x(r_1)}, \quad 0 < V(r_1) < V(r_2) < \infty. \tag{2.2}
\]

In essence, conditions (K1) and (K2) quantify the asymptotic behavior of \( k(x,y) \) when \( V_x(d(x,y)) \) is either small or large, in terms of specific constants \( A_0, A_\infty \).

The regularity condition (K4) is a consequence of the following pointwise regularity estimate

\[
|k(x',y) - k(x,y)| \leq C_\delta V_x(d(x,x'))^\eta V_x(d(x,y))^{-\eta - 1/\beta} \tag{2.3}
\]

valid for some \( \eta > 0 \), and for all \( x \in M \), all \( x' \in B(x,r) \), and all \( y \notin B(x,R) \), whenever \( V_x(R) \geq (1 + \delta)V_x(r) \).
We also note that conditions (K1), (K2) can be expressed in terms of the decreasing rearrangement of \(k\). For \(t > 0\), let

\[
\begin{align*}
k_1^*(t) &= \sup_{x \in M} (k(x, \cdot))^*(t), \quad k_2^*(t) = \sup_{y \in M} (k(\cdot, y))^*(t)
\end{align*}
\]

where \((k(x, \cdot))^*(t)\) and \((k(\cdot, y))^*(t)\) are the decreasing rearrangements of \(k(x, y)\) for fixed \(x\) and fixed \(y\), respectively.

The first observation is that condition (K3) implies

\[
k_1^*(t) \leq C t^{-1/\beta}, \quad k_2^*(t) \leq C t^{-1/\beta}, \quad t > 0
\]

(with \(C = B^{1/\beta}\)). It is also true (and it will be used later) that if \(A_\infty = 0\) and \(k\) is symmetric, then the validity of (K3) for \(V_x(d(x, y)) \leq 1\) is enough to imply (K3') (see Remark 13 in Appendix A).

Secondly, one can see that assuming (K3) conditions (K1), (K2) are equivalent to

\[
\begin{align*}
\int_{t_1}^{t_2} (k^*(x, u))^\beta du \leq A_0 \log \frac{t_2}{t_1} + B, & \quad 0 < t_1 < t_2 \leq 1, \forall x \in M \quad (K1') \\
\int_{t_1}^{t_2} (k^*(x, u))^\beta du \leq A_\infty \log \frac{t_2}{t_1} + B, & \quad 1 \leq t_1 < t_2, \forall x \in M \quad (K2')
\end{align*}
\]

where for simplicity we let

\[
k^*(x, t) = (k(x, \cdot))^*(t)
\]

(see Appendix A for a proof).

In turn, the integral conditions above are implied by following pointwise asymptotic conditions on \(k_1^*\):

\[
k_1^*(t) \leq A_0^{1/\beta} t^{-1/\beta} + C(t^{-1/\beta + \delta_1}) \quad 0 < t \leq \gamma \quad (K1'')
\]

\[
k_1^*(t) \leq A_\infty^{1/\beta} t^{-1/\beta} + C(t^{-1/\beta - \delta_2}) \quad t > \gamma \quad (K2'')
\]

for some \(\delta_1, \delta_2, \gamma > 0\). Condition (K1''), together with the second condition in (K3'), was first introduced in [FM1], as a main hypothesis in order to derive Adams inequalities on measure spaces of finite measure, for potentials with kernel \(k\). We note here, and this will be also used later, that the main result in [FM1], Theorem 1, is true under the more general conditions (K1) and (K3), or (K1') and (K3').

The special case \(A_\infty = 0\) in (K2') corresponds to what in [FM3] was labeled as a “critical integrability” condition, namely

\[
\text{ess sup}_{x \in M} \int_1^{\infty} (k^*(x, t))^\beta dt < \infty.
\]

It was proven in [FM3] that the above condition together with (K1'') and (K3') guarantee a classical Adams inequality with exponential constant \(1/A_0\), on general measure spaces. Even in this case the proof goes though if one only assumes (K1) and (K3').

7
A Riesz-like potential is an integral operator

\[ Tf(x) = \int_M k(x, y) f(y) d\mu(y). \]  

where \( k(x, y) \) is Riesz-like. Such operator is well defined for \( f \in L^{\beta'} \) and with compact support. Here and from now on \( \beta' \) will denote the exponent conjugate to \( \beta \):

\[ \frac{1}{\beta} + \frac{1}{\beta'} = 1. \]

**Theorem 1.** If \( k(x, y) \) is a Riesz-like kernel of order \( \beta > 1 \) and normalization constants \( A_0, A_\infty \), and if \( p \geq 1 \), then there is a constant \( C \) such that for any measurable function \( f \) supported in a ball with

\[ \|f\|_{\beta'} \leq 1 \]  

we have

\[ \int_M \frac{\exp[p/\beta - 1]}{1 + |Tf|^{p/\beta'}} \left[ \frac{1}{\max\{A_0, A_\infty\}} |Tf|^\beta \right] d\mu(x) \leq C \|Tf\|_p^p. \]  

Moreover, if \( A_\infty > 0 \) and given any \( \kappa > 0 \) there exists \( C \) (depending on \( \kappa \)) such that for any measurable \( f \) supported in a ball with

\[ \|f\|_{\beta'}^\prime + \kappa \|Tf\|_{\beta'}^\prime \leq 1 \]  

we have

\[ \int_M \frac{\exp[p/\beta - 1]}{1 + |Tf|^{p/\beta'}} \left[ \frac{1}{A_0} |Tf|^\beta \right] d\mu(x) \leq C \|Tf\|_p^p, \]  

and

\[ \int_M \exp[\beta' - 2] \left[ \frac{1}{A_0} |Tf|^\beta \right] d\mu(x) \leq C. \]

If \( A_\infty = 0 \) (critical integrability) then \((2.11), (2.12)\) hold under \((2.8)\), and the same is true if the first estimate in (K3) and (K4) hold only for \( V_x(d(x, y)) \leq 1 \), and if (K3) holds.

**Remark 1.** If \( M = \mathbb{R}^n \) = Lebesgue measure and \( k(x, y) = |x - y|^\alpha - n \), the Riesz kernel, then it is easy to see by dilation that the constant \( C \) in (2.11) is independent of \( \kappa \), whereas in (2.12) it is of type \( C/\kappa \) for any \( \kappa > 0 \). In general, it is possible to see that the current proof only yields this behavior for large \( \kappa \).

**Remark 2.** The conclusions of Theorem 1 hold if \( k \) and \( f \) are vector-valued. Specifically, if \( k = (k_1, ..., k_m) \) is defined on \( M \times M \) and measurable, and if \( f = (f_1, ..., f_m) \) is measurable on \( M \), then let \( Tf(x) = \int_M k(x, y) \cdot f(y) d\mu \) where the “\( \cdot \)” denotes the standard Euclidean inner product on \( \mathbb{R}^m \). If \( k \) satisfies (K1)-(K4), and if \( |k| = (k \cdot k)^{1/2}, |f| = (f \cdot f)^{1/2}, \)

\[ \|f\|_{\beta'} = \left( \int_M |f|^{\beta'} \right)^{1/\beta'} \]

then the conclusions of Theorem 1 holds (just apply the inequality \( |k \cdot f| \leq |k||f| \) in the proof of Theorem 1).
Remark 3. We will see in Section 10 an instance where \(A_\infty = 0\) but the estimates in [K3] is only available for balls with volumes less than or equal 1.

Remark 4. If \(\mu(M) < \infty\) then condition (K3) implies condition (K2) with \(A_\infty = 0\). Hence, if \(k\) is a Riesz-like kernel satisfying (K1), (K3) and (K4) on a space with finite measure, we can deduce (2.11), (2.12) under \(\|f\|_{\mathcal{O}} \leq 1\). Note that the validity of (2.12) under critical integrability is guaranteed by results in [FM3].

The next theorem deals with the sharpness of the exponential constants in Theorem 1. The essence is that if [K1] or [K2] are sharp at some \(x\), then under further regularity conditions on \(k\) the exponential constants in Theorem 1 are sharp.

We say that a Riesz-like kernel with normalization constants \(A_0, A_\infty\) is **proper on the diagonal** if there exists \(x_0 \in M\) such that

\[
\int_{r_1 \leq d(x_0, y) \leq r_2} |k(x_0, y)|^\beta d\mu(y) \geq A_0 \log \frac{V_{x_0}(r_2)}{V_{x_0}(r_1)} - C, \quad 0 < V_{x_0}(r_1) < V_{x_0}(r_2) \leq 1, \tag{2.13}
\]

and that is **proper at infinity** if \(\mu(M) = +\infty\), \(A_\infty > 0\), and there is \(x_0 \in M\) such that

\[
\int_{r_1 \leq d(x_0, y) \leq r_2} |k(x_0, y)|^\beta d\mu(y) \geq A_\infty \log \frac{V_{x_0}(r_2)}{V_{x_0}(r_1)} - C, \quad 1 \leq V_{x_0}(r_1) < V_{x_0}(r_2). \tag{2.14}
\]

It is easy to check that, due to the \(x\)-regularity in the form [K4], for any given \(r_1, r_2 > 0\) if (2.13) holds, then there is \(r' < r_1\) (depending on \(r_1, r_2\)) so that (2.13) holds with \(x_0\) replaced by any \(x \in B(x_0, r')\), and the same is true for (2.14).

We will also require a slightly stronger \(x\)-regularity condition than [K4], and an even higher regularity of \(k(x, y)\) in the \(y\) variable. For simplicity we will assume pointwise regularity in the same spirit as (2.3).

We will say that a Riesz-like kernel has a **Taylor formula with volume remainder of order \(\eta \geq 0\) in the \(y\) variable at \(x_0\)** if there is an integer \(m \geq 1\) and measurable functions \(k_j(x, x_0), p_j(y, x_0), j = 0, 1, \ldots, m - 1\), where the \(\{p_j\}_{j=0}^{m-1}\) are bounded and linearly independent on \(B(x_0, r)\) for any \(r > 0\), and such that

\[
\left| k(x, y) - \sum_{j=0}^{m-1} k_j(x, x_0)p_j(y, x_0) \right| \leq C_\delta V_{x_0}(d(y, x_0))^{\eta} V_{x_0}(d(x, x_0))^{-\eta - 1/\beta} \tag{2.15}
\]

whenever \(V_x(R) \geq (1 + \delta)V_x(r), \ d(y, x_0) \leq r, \ d(x, x_0) \geq R\). We will also require that if \(\{v_j(y, x_0)\}_{j=0}^{m-1}\) is an orthonormal basis of the space spanned by the \(p_j\) restricted to the ball \(B(x_0, r)\), then

\[
|v_j(y, x_0)| \leq \frac{C}{\sqrt{V_{x_0}(r)}}, \quad y \in B(x_0, r), \quad j = 0, 1, \ldots, m - 1. \tag{2.16}
\]

In concrete cases where \(k(x, y)\) is a differentiable function one can typically choose the \(p_j\) so that for \(j = 0, 1, \ldots, m - 1\)

\[
|k_j(x, x_0)| \leq C_\delta V_{x_0}(d(x, x_0))^{-\eta_j - 1/\beta}, \quad |p_j(y, x_0)| \leq C_\delta V_{x_0}(d(y, x_0))^{\eta_j} \tag{2.17}
\]
with \( \eta_0 = 0, k_0 = p_0 = 0 \) (which is (K3)), and \( k_1(x, x_0) = k(x, x_0), p_1(y, y_0) = 1 \), which gives the same regularity condition as in (2.3) but in the \( y \) variable.

The Riesz kernel in \( \mathbb{R}^n \) and the \( m \)-regular Riesz-like kernels defined in [FM2], and [Qin2] are obvious examples of kernels of this type, being approximated by their classical Taylor polynomials of order \( m - 1 \), with volume remainders of order \( m/n \) (see for example eq. (95) in [FM2]).

**Theorem 2.** Under the same assumptions of Theorem 1, assume further that there is \( x_0 \) such that \( k(x, y) \) satisfies (2.3) with \( x = x_0 \), for some \( \eta > 0 \), and has a Taylor formula with volume remainder of order \( \eta \geq 0 \) in the \( y \) variable at \( x_0 \). If \( p > (1 + p/\beta')(1 + \eta)^{-1} \) then the following hold:

(a) If \( k(x, y) \) is proper on the diagonal, then the exponential constant \( A_0^{-1} \) in (2.11), (2.12) is sharp, and it is also sharp in (2.9) when \( A_0 \geq A_\infty \). The power of the denominators \( p\beta/\beta' \) in (2.9) and (2.11) is also sharp.

(b) If \( k(x, y) \) is proper at infinity, then the exponential constant \( A_\infty^{-1} \) in (2.9) is sharp when \( A_0 \leq A_\infty \). The power of the denominator \( p\beta/\beta' \) in (2.9) is also sharp.

Here the meaning of “sharp” is to be intended as follows: the ratios of the left-hand sides and right-hand sides of (2.9), (2.11), (2.12) can be made arbitrarily large if either the exponential constants are larger or the powers of the denominators are smaller. There is only one case where we can say more, namely in case (a) the left-hand sides of (2.9), (2.11) can be arbitrarily large while the right-hand sides stay small, if the exponential constants are larger (see Proof of Theorem 2).

**Notes.** 1. The conclusions of the above theorem remain valid under slightly weaker regularity conditions, namely (7.3), (7.12).

2. If \( p > 1 + p/\beta' \) then we can take \( m = 0, \eta_0 = 0 \) and just use (K3) to derive sharpness. If one can choose \( m \) high enough with \( \eta \geq p/\beta' \), then sharpness holds for all \( \beta, p > 1 \). For example for the Riesz kernel in \( \mathbb{R}^n \), and \( p = \beta' \) then \( \eta = 1 \) is obtained by taking the Taylor expansion of order \( n - 1 \) of \( |x - y|^{n-n} \).

### 3 Proof of Theorem 1

We will prove Theorem 1 under the hypothesis that (K1)-(K4) hold and \( A_\infty > 0 \). In Section 6 we will outline the changes of the proof in order to deal with the case \( A_\infty = 0 \), under the weaker conditions stated at the end of Theorem 1. We will also assume throughout the proof, WLOG, that \( B \geq 2^{1/\beta'} \).

Let us recall the definition of decreasing rearrangement. Given a measurable function \( f : M \to [-\infty, \infty] \) its distribution function is defined as

\[
m_f(s) = \mu(\{x \in M : |f(x)| > s\}), \quad s \geq 0.
\]

Assuming that the distribution function of \( f \) is finite for all \( s > 0 \), the decreasing rearrangement of \( f \) is defined as

\[
f^*(t) = \inf\{s \geq 0 : m(f, s) \leq t\}, \quad t > 0,
\] (3.1)
and we also define

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(u)du, \quad t > 0. \]

From now on let us assume that

\[ \|f\|_{\beta'} \leq 1, \quad \text{supp } f := \{x : f(x) \neq 0\} \subseteq B(x_0, R), \quad (3.2) \]

for some \( x_0, \in M, R > 0 \) (depending on \( f \)) and after possibly redefining \( f \) to be 0 on a set of 0 measure outside \( B(0, R) \).

Under these circumstances \( Tf(x) \) is well-defined and finite for a.e. \( x \) in \( M \). This follows from the weak type estimate

\[ (m_{T'f}^s(s))^{1/\beta} \leq \frac{\beta' B \|f\|_1}{s} \]

where \( T' \) is the operator with kernel \( |k(x, y)| \) (see [Qin2], estimate (A.1)).

By the “Exponential Regularization Lemma” (see [FM2], Lemma 9, [Qin2], Lemma A) (2.9) and (2.11) are equivalent to

\[ \int_{\|Tf\| \geq 1} \exp \left[ \frac{1}{A} |Tf|^\beta \right] \frac{1}{1 + |Tf|^{p\beta/\beta'}} d\mu \leq C \|Tf\|_p \]

and (2.12) is equivalent to

\[ \int_{\{\|Tf\| \geq 1\}} \exp \left[ \frac{1}{A_0} |Tf|^\beta \right] d\mu \leq C, \]

where

\[ A = \begin{cases} \max\{A_0, A_\infty\} & \text{given (2.8)} \\ A_0 & \text{given (2.10)}. \end{cases} \]

Let

\[ t_0 = \mu \{x \in M : |Tf(x)| \geq 1\} \]

so that (3.4) is equivalent to

\[ \int_0^{t_0} \exp \left[ \frac{1}{A_0} ((Tf)^*(t))^{\beta} \right] \frac{1}{1 + ((Tf)^*(t))^{p\beta/\beta'}} dt \leq C \|Tf\|_p \]

and (3.5) is equivalent to

\[ \int_0^{t_0} \exp \left[ \frac{1}{A_0} ((Tf)^*(t))^{\beta} \right] dt \leq C. \]

A first estimate for \( (Tf)^*(t) \) is given via the O’Neil functional, defined as

\[ Uf(t) = Ct^{-\frac{1}{p}} \int_0^t f^*(u)du + \int_t^\infty k_1^*(u)f^*(u)du. \]
The O’Neil Lemma on measure spaces (see [FM1], [FM3]) gives \((Tf)^*(t) \leq (Tf)^*(t) \leq U_f(t)\) for any \(t > 0\).

The basic Adams inequality on measure spaces with finite measure can be stated in terms of \(U_f\) as follows:

\[
\int_0^\tau \exp \left[ \frac{1}{A_0} (U_f(t))^\beta \right] dt \leq C (\tau + \mu(\supp f)), \quad \tau > 0
\]

(3.10)

under conditions (K1), (K3), where \(C\) is independent of \(f, T\). Such an inequality has been derived in [FM2], under more restrictive pointwise conditions such as (K1'), (K3'), and it is the key to the proof of the main results in [FM2]. The proof under the more general conditions (K1), (K3) follows by modifying the function \(g(x, \xi, \eta)\) in [FM2, p. 10], letting it equal to \(k_\tau(x, e^{-\xi})e^{-\frac{\beta\tau}{\beta+1}}\), when \(\xi \in (\infty, \eta)\), and by applying the integral condition in the form (K1) to obtain the estimate on p. 29, line 3.

The key starting point in [Qin2], in the case of convolution Riesz-like kernels in \(\mathbb{R}^n\), was to split \(f\) on a suitable level set of given measure \(\tau\).

To this end, for each \(\tau > 0\) we consider a measurable set \(F\) satisfying

\[
\left\{ x : |f(x)| > f^*(\tau) \right\} \subseteq F \subseteq \left\{ x : |f(x)| \geq f^*(\tau) \right\}
\]

(3.11)

the existence of which is guaranteed by the continuity assumption of \(\mu(B(x, r))\) in \(r\), for all \(x \in M\). Such a set may not be unique, for example if \(f^*\) is constant around 0, however a prescription can be given to identify it uniquely, once a preferred point \(x_0 \in M\) is fixed. Namely (see [Qin2]) if \(V_1 = \{ x : |f(x)| > f^*(\tau) \}\) and \(V_2 = \{ x : |f(x)| \geq f^*(\tau) \}\), then \(F\) can be found of type \(V_1 \cup (B(x_0, r) \cap (V_2 \setminus V_1))\), for some \(r \in [0, \infty]\), uniquely up to a set of zero measure. Note that in \(\mathbb{R}^n\) if \(f\) is radially decreasing and \(\tau = |B(0, r)|\) then \(F\), either the open or the closed ball of center 0 and radius \(r\).

The set \(F\) with \(\mu(F) = \tau\) satisfies the identity

\[
\int_{F_\tau} \Phi(|f(x)|)d\mu(x) = \int_0^\tau \Phi(f^*(u))du
\]

(3.12)

where \(\Phi\) is any non-negative, measurable function on \([0, \infty]\). See [BeSh, Lemma 2.5], for the existence of such set on finite nonatomic measure spaces, and where the above identity is given for \(\Phi(u) = u\), and which can be easily extended to arbitrary \(\Phi\) (see also [Qin2, (3.10)]).

Likewise, we define a corresponding level set \(E_\tau\) relative to \(Tf\):

\[
\left\{ x : |Tf(x)| > (Tf)^*(\tau) \right\} \subseteq E_\tau \subseteq \left\{ x : |Tf(x)| \geq (Tf)^*(\tau) \right\}
\]

(3.13)

\[
\mu(E_\tau) = \tau.
\]

Let

\[
f_\tau = f_{X_{F_\tau}}, \quad f_\tau' = f_{X_{F_\tau'}} = f - f_\tau
\]

and let

\[
r_x(\tau) = \min\{r > 0 : V_x(r) = \tau\}.
\]

(3.14)
We will use for simplicity the notation
\[ B_\tau(x) = B(x, r_x(\tau)) \] (3.15)
to denote the smallest ball centered at \( x \) and with volume \( \tau \).

As in [Qin2], the crucial quantity we will use to control \( f \) outside \( F_\tau \) is defined as follows:
\[
(Tf)^\alpha(\tau) := \operatorname{ess sup}_{x \in E_\tau} |T(f^\prime \chi_{F_\tau(x)})(x)| = \operatorname{ess sup}_{x \in E_\tau} \left| \int_{B_\tau(x) \cap F_\tau^c} k(x, y) f(y) d\mu(y) \right|. 
\] (3.16)
which is finite for every \( \tau \) from Hölder’s inequality, (K3), and (3.2), which imply
\[
(Tf)^\alpha(\tau) \leq \left( \frac{BV_x(R)}{\tau} \right)^{1/\beta}.
\] (3.17)

**Lemma 1.** There is \( z \in E_\tau \) such that for \( 0 < t \leq \tau \) we have
\[
(Tf)^*(t) \leq (Tf)^*(t) \leq U_f(t) + B\|f^\prime \chi_{F_\tau(z)}\|_{\beta'} + (Tf)^\alpha(\tau). 
\] (3.18)

**Proof.** Arguing exactly as in [Qin2], using the improved O’Neil Lemma ([Qin2, Lemma 1]) we obtain
\[
(Tf)^*(t) \leq U_f(t) + \operatorname{ess sup}_{x \in E_\tau} \int_{\tau}^{2\tau} k_1^*(u)(f^\prime \chi_{F_\tau(z)})(u - \tau) du + (Tf)^\alpha(\tau). 
\] (3.19)
Note that by [K3] we have, for each \( x \in E_\tau \),
\[
\int_{\tau}^{2\tau} k_1^*(u)(f^\prime \chi_{F_\tau(z)})(u - \tau) du \leq B^{1/\beta} \int_{\tau}^{2\tau} u^{-\frac{1}{\beta}}(f^\prime \chi_{F_\tau(z)})(u - \tau) du \leq B^{1/\beta}\|f^\prime \chi_{F_\tau(z)}\|_{\beta'} \leq B^{1/\beta} \operatorname{ess sup}_{x \in E_\tau} \|f^\prime \chi_{F_\tau(z)}\|_{\beta'} \leq 2B^{1/\beta}\|f \chi_{F_\tau(z)}\|_{\beta'} \leq B\|f^\prime \chi_{F_\tau(z)}\|_{\beta'}
\] (3.20)
for some \( z \in E_\tau \) (note that \( \|f^\prime \chi_{F_\tau(z)}\|_{\beta'} \) is continuous in \( x \), and that we also assumed \( B \geq 2^{1/\beta'} \)).

Using the above lemma we can estimate the integrand in (3.8). For every \( \epsilon \in (0, 1) \), using \((a + b)\beta \leq \epsilon^{1-\beta}a\beta + (1 - \epsilon)^{1-\beta}b\beta \) we have
\[
\exp \left[ \frac{1}{A} (\frac{(Tf)^*(t)}{\beta}) \right] \leq \exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A} \left( (Tf)^\alpha(\tau) + B\|f^\prime \chi_{F_\tau(z)}\|_{\beta'} \right)^{\beta} \right] 
\] (3.21)
and
\[
\exp \left[ \frac{1}{A_0} ((T f)^*(t))^\beta \right] \leq C \exp \left[ \frac{\epsilon^{1-\beta}}{A_0} (U f_\tau(t))^\beta \right] \cdot \exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A_0} ((T f)^\circ \tau + B \| f_\tau' \chi_{B_r(z)} \|_{\beta'})^\beta \right]
\] (3.22)
for some \( C > 0 \) independent of \( f, \tau, \epsilon \).

The following is an easy consequence of (3.10):

**Proposition 1.** There is \( C > 0 \) such that for any \( \epsilon > 0 \)
\[
\int_0^\tau \exp \left[ \frac{\epsilon^{1-\beta}}{A_0} (U f_\tau(t))^\beta \right] \leq C \tau, \quad \tau > 0 \quad (3.23)
\]
provided that\( \| f_\tau \|_{\beta'} \leq \epsilon \). \( (3.24) \)

**Proof of Proposition 1.** Let
\[
\tilde{f} := \frac{f_\tau}{\epsilon^{1/\beta'}},
\]
then we have that \( \tilde{f} \) has measure of support \( \mu(\text{supp} \tilde{f}) \leq \tau \) with
\[
\| \tilde{f} \|_{\beta'} \leq 1.
\]
Therefore (3.23) follows from the (improved) Adams inequality in the form (3.10). \( \square \)

**Proposition 2.** There exists a constant \( C^* > 0 \) such that for \( \kappa \geq 0, \epsilon \in [0, 1) \) and for any measurable \( f \) supported in a ball satisfying
\[
\| f \|_{\beta'} + \kappa \| Tf \|_{\beta'} \leq 1 \quad (3.25)
\]
\[
\| f_\tau' \|_{\beta'} + \kappa \| (T f) \chi_{E^c} \|_{\beta'} \leq 1 - \epsilon \quad (3.26)
\]
\[
(T f)^\circ (\tau) > C^* \quad (3.27)
\]
we have
\[
\exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A} ((T f)^\circ (\tau) + B \| f_\tau' \chi_{B_r(z)} \|_{\beta'})^\beta \right] \leq C \tau^{\kappa/\beta'} \quad (3.28)
\]
where \( A \) is the same as in (3.6), and
\[
\exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A_0} ((T f)^\circ (\tau) + B \| f_\tau' \chi_{B_r(z)} \|_{\beta'})^\beta \right] \leq \frac{C}{\tau} \quad (3.29)
\]
where \( C \) is a constant independent of \( f, \epsilon \). 14
Remark 5. The set $E_\tau$ in (3.26) and (3.28) can also be replaced by any measurable set $E$ with measure less than or equal to $\tau$.

Remark 6. An immediate consequence of (3.28) is that under $\|f\|_{\beta'} \leq 1$, $\|f\|_{\beta'} \leq 1$ and $(Tf)^*(\tau) > C^*$ we have the following form of the “optimal descending growth condition”:

$$\exp \left[ \frac{1}{\max\{A_0, A_\infty\}} \left( \frac{(Tf)^*(\tau) + B\|f\|_{\beta'} X_{B_\tau(z)}\|_{\beta'}}{(Tf)^*(\tau) + B\|f\|_{\beta'} X_{B_\tau(z)}\|_{\beta'} + \rho\beta / \beta'} \right)^\beta \right] \leq C \frac{\|f\|_{\beta'}^\beta}{\|f\|_{\beta'}^\beta + \kappa\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta},$$

(3.30)

which includes the corresponding condition (1.4) derived in [IMN] and [MS2] in $\mathbb{R}^n$, when $k(x, y) = c_2|x - y|^{2-n}$ and $f = \nabla u$, where $u$ positive and radially decreasing, $-\partial_\rho u$ decreasing, and $u(R) > 1$. In this case we indeed have $(Tf)^*(\tau) = u^*(R)$, if $\tau = |B(0, R)|$ (see Appendix E).

The proof of Proposition 2 is quite involved and will be given in Section 4. Assuming Proposition 2 we will now derive the main inequalities of Theorem 1.

Proof of (2.9) and (2.11). For notational convenience let

$$I_1(\tau, \epsilon) = \exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A} \left( \frac{(Tf)^*(\tau) + B\|f\|_{\beta'} X_{B_\tau(z)}\|_{\beta'}}{(Tf)^*(\tau) + B\|f\|_{\beta'} X_{B_\tau(z)}\|_{\beta'} + \rho\beta / \beta'} \right)^\beta \right]$$

(3.31)

and

$$I_2(\tau, \epsilon, t) = \exp \left[ \frac{\epsilon^{1-\beta}}{A_0} \left( \frac{U f(t)}{(Tf)^*(\tau)} \right)^\beta \right]$$

(3.32)

First note that if $(Tf)^*(\tau) \leq C^*$, then by direct computation and the fact that $(Tf)^*(t) \geq 1$ for $t < t_0$ we have

$$I_1(\tau, \epsilon) \leq e^{(1-\epsilon)^{1-\beta} A^{-1}(C^*+B)\beta -1 \|Tf\|_{\beta'}^\beta}.$$  

(3.33)

Let

$$\epsilon_\tau = \min \left\{ \frac{\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta}{\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta}, \frac{1}{4} \right\},$$

(3.34)

then

$$3/4 \leq 1 - \epsilon_\tau < 1.$$  

(3.35)

For any $\kappa \geq 0$, using condition (3.25) and (3.34) we get

$$\|f^\beta_{\beta'} + \kappa(Tf)X_{E_\tau}^\beta_{\beta'} \leq \|f\|_{\beta'}^\beta - \|f^\beta_{\beta'} + \kappa(Tf)^\beta_{\beta'}.$$

$$= (\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta) \left( 1 - \frac{\|f^\beta_{\beta'} + \kappa(Tf)^\beta_{\beta'}}{\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta} \right) \leq (\|f\|_{\beta'}^\beta + \kappa\|Tf\|_{\beta'}^\beta) (1 - \epsilon_\tau) \leq 1 - \epsilon_\tau.$$  

(3.36)
Taking $\epsilon = \epsilon_\tau$ in (3.24) and (3.26), the proof of (2.9), (2.11) follows in the same manner as in [Qin2, proof of (3.34)]. Let us define

$$\tau_0 = \max \left\{ \tau \in [0, t_0] : \frac{\|f_\tau\|_{\beta'}^\gamma}{\|f\|_{\beta'}^\gamma + \kappa\|Tf\|_{\beta'}^\gamma} \leq \frac{1}{4} \right\}.$$ 

It is clear that by the definition of $\epsilon_\tau$ that $\epsilon_{\tau_0} \geq \|f_\tau\|_{\beta'}$, hence from Propositions 1, 2 and (3.33) we get

$$I_1(\tau_0, \epsilon_{\tau_0}) \leq C \|Tf\|_p^p \quad \text{and} \quad \int_0^{\tau_0} I_2(\tau_0, t, \epsilon_{\tau_0}) dt \leq C\tau_0. \quad (3.37)$$

Therefore, using (3.21) it is immediate that

$$\int_0^{\tau_0} \exp \left[ \frac{1}{A} \frac{(Tf)^\beta}{1 + ((Tf)^\beta)^{\rho\beta'/\beta'}} \right] dt \leq \int_0^{\tau_0} I_1(\tau_0, \epsilon_{\tau_0}) I_2(\tau_0, t, \epsilon_{\tau_0}) dt \leq C\|Tf\|_p^p. \quad (3.38)$$

Next, we take $\tau = t$ for $\tau_0 \leq t \leq t_0$, and $\epsilon = 1/8$ in $I_2$. Then by definition of O’Neil’s operator and the fact that the support $f_t$ has measure less than or equal to $t$, an using (3.12)

$$Uf_t(t) = C_0 t^{-\frac{\theta}{2}} \int_0^t f_t^*(u) du \leq C \left( \int_0^t (f_t^*)_{\beta'} \right)^{1/\beta'} = C\|f_t\|_{\beta'} \leq C.$$

So we have

$$I_2\left( t, t, \frac{1}{8} \right) \leq C. \quad (3.39)$$

Since $\tau_0 \leq t \leq t_0$, by definition we have $\epsilon_\tau = 1/4$. Take $\theta = (\frac{1-\frac{1}{4}}{1-1/8})^{\beta-1} < 1$ we get

$$I_1\left( t, \frac{1}{8} \right) = \exp \left[ \frac{(7/8)^{1-\beta}}{A} \frac{((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^\beta}{1 + ((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^{\rho\beta'/\beta'}} \right] \leq \left( \frac{\exp \left[ \frac{(3/4)^{1-\beta}}{A} \frac{((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^\beta}{1 + ((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^{\rho\beta'/\beta'}} \right]}{\theta} \right)^\theta \leq \left( \frac{\exp \left[ \frac{(1-\epsilon_t)^{1-\beta}}{A} \frac{((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^\beta}{1 + ((Tf)^{\beta}(t) + B\|f_t X_{B_t(z)}\|_{\beta'})^{\rho\beta'/\beta'}} \right]}{\theta} \right) = \frac{C}{t^\theta} \|Tf\|_p^{\theta p}. \quad (3.40)$$

Using (3.39), (3.40) and that $t_0 \leq \|Tf\|_p^p$ (see (3.7)),

$$\int_{\tau_0}^{t_0} I_1\left( t, \frac{1}{8} \right) I_2\left( t, t, \frac{1}{8} \right) dt \leq C \int_{\tau_0}^{t_0} \frac{1}{t^\theta} \|Tf\|_p^{\theta p} dt \leq C t_0^{1-\theta} \|Tf\|_p^{\theta p} \leq C\|Tf\|_p^{(1-\theta)p} = C\|Tf\|_p^p. \quad (3.41)$$
Combining (3.38) and (3.41) we have inequality (2.11).

**Proof of (2.12).** The proof is almost the same as above. Let \( k > 0 \) and define
\[
I_3(\tau, \epsilon) = \exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A_0} \left( (T f)^{\circ}(\tau) + B \| f^{\prime}_\tau \chi_{B_r(z)} \|_{\beta'} \right)^\beta \right] \leq \frac{C}{\tau},
\]
(3.42) from (3.29).

We take \( \epsilon_\tau \) as in (3.34) and estimate similarly as in (3.38), (3.41). By using Lemmas 1, 2 we get
\[
\int_0^{\tau_0} \exp \left[ \frac{1}{A_0} \left( (T f)^{(f)^*}(t) \right)^\beta \right] dt \leq \int_0^{\tau_0} I_3(\tau_0, \epsilon_\tau_0) I_2(\tau_0, t, \epsilon_\tau_0) dt \leq C,
\]
(3.43) and
\[
\int_0^{\tau_0} I_3\left( t, \frac{1}{8} \right) I_2\left( t, t, \frac{1}{8} \right) dt \leq C \int_0^{\tau_0} \frac{1}{t^\theta} dt \leq C t_0^{1-\theta} \leq C \| T f_\|_{\beta'}^{1-\theta} \leq C.
\]
(3.44)

\[\blacksquare\]

## 4 Proof of Proposition 2

Let us assume throughout this section that \((3.25), (3.26)\) hold for some \( \kappa \geq 0 \) i.e.
\[
\| f \|_{\beta'} + \kappa \| T f \|_{\beta'} \leq 1, \quad \| f^{\prime}_\tau \|_{\beta'} + \kappa \| (T f) \chi_{E_{\tau}} \|_{\beta'} \leq 1 - \epsilon
\]

**Proof of (3.28).**

For \( 0 < \tau < \mu(M) \) and \( x \in E_{\tau} \) define
\[
(T f)^{\circ}(\tau, x) = |T(f^{\prime}_\tau \chi_{B^c(x, r_\tau(\tau))})(x)|,
\]
(4.1) so that \((T f)^{\circ}(\tau) = \text{ess sup}_{x \in E_{\tau}} (T f)^{(f)^*}(\tau, x)\). Recall that \( r_\tau(\tau) \) is the smallest \( r \) such that \( \mu(B(x, r)) = \tau \).

To prove \((3.28)\) we will show that there are constants \( C, C^* > 0 \) such that for \( 0 < \tau < t_0 \) and for all \( x \in E_{\tau} \) with \((T f)^{\circ}(\tau, x) > C^*\)
\[
\exp \left[ \frac{(1 - \epsilon)^{1-\beta}}{A} \left( (T f)^{\circ}(\tau, x) + B \| f^{\prime}_\tau \chi_{B_r(z)} \|_{\beta'} \right)^\beta \right] \leq C \frac{\| (T f) \chi_{E_{\tau}} \|_{\beta'}}{\tau(1 - \epsilon)^{p\beta/\beta'}}.
\]
(4.2)

where \( z \in E_{\tau} \) is as in Lemma 1.

Now let us state a more general version of the exact growth condition for sequences given in [IMN], [MS1]-[MS3], [LT], [LTZ], the proof of which is postponed to the Appendix.

Given any sequence \( a = \{a_k\}_{k \geq 0} \) we will let
\[
\| a \|_q = \left( \sum_{k=0}^{\infty} |a_k|^q \right)^{1/q}
\]
(4.3) and given a sequence \( \lambda = \{\lambda_k\}_{k \geq 0} \) we will let \( \lambda a = \{\lambda_k a_k\}_{k \geq 0} \).
Lemma 2. For any $\beta > 1$, $p > 0$ and given a sequence $\lambda = \{\lambda_k\}$ such that $\lambda_k \in (0, 1]$ and $\lambda_k < 1$ for at most finitely many $k$, define, for $h > 0$,

$$\mu_d(h) = \inf \left\{ \sum_{k=0}^{\infty} |a_k|^p e^{\beta' k} : \|a\|_1 = h, \|\lambda a\|_{\beta'} \leq 1 \right\}. \quad (4.4)$$

Then, for each $h_0 > 0$ there exist positive constants $C = C(p, \beta, h_0), C' = C'(p, \beta, h_0)$ such that

$$Ce^{-L_{\beta'}} \exp \left[ \frac{\beta' h^\beta}{h^{p\beta'/\beta'}} \right] \leq \mu_d(h) \leq C' \frac{\exp \left[ \frac{\beta' h^\beta}{h^{p\beta'/\beta'}} \right]}{h^{p\beta'/\beta'}} \quad \forall h \geq h_0. \quad (4.5)$$

where

$$L_{\lambda} := \sum_{k=0}^{\infty} \left( \lceil \lambda_k^{-\beta} \rceil - 1 \right). \quad (4.6)$$

Remark 7. The original lemma in [IMN],[MS1]-[MS3],[LT],[LTZ] (and which was also used in [Qin2]) had $\lambda_k = 1$ for all $k$, and $p = \beta'$. The original result for $\beta' = 2$ was given in [IMN] and [MS1], and was obtained for $\beta' > 2$ independently in [MS2] and [LT], while in [LTZ] the authors settled the case $1 < \beta' < 2$. Our proof of Lemma 2, given in the Appendix, is a bit shorter than the existing ones, and works for all $\beta' > 1$.

From the above lemma we have that given a sequence $\lambda$ satisfying $L_{\lambda} \leq C$, and given any $\beta' > 1$, $\mu > 0$, $h > 1$, there is $C$ such that for any sequence $\{a_k\}$ satisfying

$$\sum_{k=0}^{\infty} |a_k| = h \quad \sum_{k=0}^{\infty} \lambda_k^{\beta'} |a_k|^{\beta'} \leq \mu \quad (4.7)$$

we have

$$\exp \left[ \frac{\beta' \mu^{1-\beta} h^\beta}{h^{p\beta'/\beta'}} \right] \leq Ce^{L_{\lambda} \beta'} \mu^{-p\beta'/\beta'} \sum_{k=0}^{\infty} |a_k|^{p e^{\beta' k}}. \quad (4.8)$$

We will apply this version of Lemma 3 to prove Proposition 2, by finding a number $h_1$, a sequence $\{a_k\}$, both depending on $f$ and $x$, and a sequence $\{\lambda_k\}$ also depending on $f$ and $x$ but satisfying $L_{\lambda} \leq C$ independent of $f, x$, such that

$$\beta' - \frac{1}{p} A^{-\frac{1}{p}} \left( (Tf)^{\circ} (\tau, x) + B \|f_{x'} X_{B_{r(z)}}\|_{p'} \right) \leq h_1 \quad (4.9)$$

$$\sum_{k=0}^{\infty} |a_k| = h_1, \quad \sum_{k=0}^{\infty} \lambda_k^{\beta'} |a_k|^{\beta'} \leq 1 - \epsilon, \quad (4.10)$$

and

$$\sum_{k=0}^{\infty} |a_k|^{p e^{\beta' k}} \leq \frac{C}{\tau (1 - \epsilon)^{p\beta'/\beta'}} \|Tf\|_{E_{q}^p}^{p}. \quad (4.11)$$

Fix $x \in M, 0 < \tau < t_0$. Recall that $t_0 = \mu\{x : |Tf| \geq 1\} \leq \mu(M)$. Fix $R > 0$ such that $\text{supp } f \subseteq B(x, R)$. 18
Let
\[
N = \begin{cases} 
0 & \text{if } V_x(R) \leq \tau \\
1 & \text{if } \tau < V_x(R) \leq \tau e^{2\beta'} \\
\left[\frac{1}{\beta'} \log \frac{V_x(R)}{\tau}\right] - 1 & \text{if } V_x(R) > \tau e^{2\beta'}
\end{cases}
\] (4.12)
\[
R_j = r_x(\tau e^{\beta' j}), \quad j = 0, 1, 2, \ldots
\] (4.13)
so that
\[
e^{-\beta'} V_x(R) \leq V_x(R_N) = \tau e^{\beta' N} < V_x(R) \leq \tau e^{\beta'(N+1)}.
\] (4.14)

Define
\[
\begin{cases} 
r_0 = R_0 = r_x(\tau) \\
r_N = R & \text{if } N \geq 1
\end{cases}
\] (4.15)
and if \(N \geq 2\) let for \(j \leq N - 1\)
\[
r_j = \begin{cases} 
\sup \left\{ r \leq R_j : \int_{r_{j-1} \leq d(x,y) \leq r} |k(x,y)|^\beta d\mu(y) \leq \beta' A_0 \right\}, & \text{if } r_{j-1} < r_x(1) \\
\sup \left\{ r \leq R_j : \int_{r_{j-1} \leq d(x,y) \leq r} |k(x,y)|^\beta d\mu(y) \leq \beta' A_\infty \right\}, & \text{if } r_{j-1} \geq r_x(1)
\end{cases}
\] (4.16)
and let
\[
\hat{j}_1 = \begin{cases} 
\min \left\{ j : V_x(r_j) \geq 1 \right\} & \text{if } \exists j \leq N : V_x(r_j) \geq 1 \\
N & \text{if } \forall j \leq N, V_x(r_j) < 1.
\end{cases}
\] (4.17)

Note that with this notation we have, for \(N \geq 1\),
\[
r_j \leq R_j, \quad j = 0, 1, \ldots, N - 1, \quad r_N = R > R_N
\] (4.18)
where the last inequality is from (4.14) and the fact that \(V_x(r)\) is increasing.

**Remark 8.** From the continuity in \(r\) of the integral inside (4.16) we have that the sup is actually a max, and when \(r_j < R_j\) there is equality in the integral condition inside (4.16).

It turns out that \(B(x, r_j)\) and \(B(x, R_j)\) have comparable volumes: If \(N \geq 2\) then we have
\[
e^{-\frac{\beta}{\beta_0} - \frac{\beta}{\beta_\alpha}} V(R_j) \leq V_x(r_j) \leq e^{\beta'} V_x(R_j), \quad 0 \leq j \leq N.
\] (4.19)
\[
Q_1 V(R_j) \leq V_x(r_{j+1}) - V_x(r_j) \leq e^{2\beta'} V_x(R_j), \quad 0 \leq j \leq N - 1,
\] (4.20)
for some constant \(Q_1\) depending only on \(\beta', B, A_\infty, A_0\). The proof of the above estimates will be given in the Appendix.

Later we will need the following estimate, valid for all \(x \in M:\)
\[
\int_{r_{j-1} \leq d(x,y) < r_j} |k(x,y)|^\beta d\mu(y) \leq Q_2, \quad j = 1, \ldots, N,
\] (4.21)
which follows easily from (K1), (K2), (4.14), and (4.19).
By (4.13), (1.19), (4.20), there exists an integer $m$ depending only on $k, \beta$ such that if $N \geq m + 1$ then

$$\begin{cases} V_x(r_j) - V_x(r_0) \geq 2\tau & m \leq j \leq N - 1. \\
V_x(r_{j+1}) - V_x(r_j) \geq 2\tau & m \leq j \leq N - 1.
\end{cases}$$

(4.22)

On the other hand, if $N \leq m$, by (4.21) we get

$$(T_f)^\circ(\tau, x) = |T(f'_{\tau}X_{B^e(x, r_\tau)}) (x)| = \left| \int_{r_\tau \leq d(x, y) < R} k(x, y) f'_{\tau}(y) d\mu(y) \right| \leq \left( \int_{r_0 \leq d(x, y) \leq \tau N} |k(x, y)|^\beta d\mu(y) \right)^{1/\beta} \leq (NQ_2)^{1/\beta} \leq (mQ_2)^{1/\beta} \leq Q_2,$$

(4.23)

where we assumed WLOG that $Q_2 > m^{\frac{1}{\beta - 1}}$.

From now on we assume $(T_f)^\circ(\tau, x) > Q_2$ which then implies that $N \geq m + 1 \geq 2$ and that (4.22) holds.

For $x \in M$ we will let

$$\begin{align*}
D_j &= B(x, r_j) \quad \text{for } j = 0, 1, \ldots, N \\
D'_0 &= B(z, r_\tau)
\end{align*}$$

(4.24)

where the $r_j$ are defined as above (depending on $x, f, \tau$) and where $z$ is defined as in (5.18).

In what follows we will assume, for notational simplicity, that $m = 1$, hence $N \geq 2$. For general $m$, there are a few modifications to make. One is to replace, in the proof below, $D_j$ by $D_{j+m-1}$ for $1 \leq j \leq N - m + 1$, and $D_1 \setminus D_0$ by $D_m \setminus D_0$. The range of $j$, $J$ should also be changed to $0, \ldots, N - m + 1$ accordingly. Note that by definition $\mu(D_{j+m} \setminus D_{j+m-1})$ is comparable to $V_x(r_{j+1}) - V_x(r_j)$ for $1 \leq j \leq N - m$, and $\mu(D_m \setminus D_0)$ is comparable to $V_x(r_1) - V_x(r_0)$. This fact is used later in (5.15), (D.2) and (D.8). The other modification is to define $j_1$ as follows

$$j_1 = \begin{cases} 0 & \text{if } V_x(r_0) \geq 1 \\
\min\{j : V_x(r_{j+m-1}) \geq 1\} & \text{if } V_x(r_0) < 1 \text{ and } \exists j \leq N - m + 1 : V_x(r_{j+m-1}) \geq 1 \\
N - m + 1 & \text{if } \forall j \leq N - m + 1 : V_x(r_{j+m-1}) < 1
\end{cases}$$

(4.25)

It is clear that this definition coincides with the previous definition of $j_1$ for $m = 1$ in (4.17). Note that if $j_1 \neq N - m + 1$, we have $\tau e^{\beta' j_1} = e^{-\beta'(m-1)} \tau e^{\beta'(j_1+m-1)} \geq e^{-\beta'(m-1)}$, which is needed to get (4.42) and (4.43).

Let us split $f'_{\tau} = fX_{F^\circ}$ as follows

$$f'_{\tau} = \sum_{j=0}^{N-1} f'_{\tau}X_{(D_{j+1} \setminus D_j) \setminus D'_0} + f'_{\tau}X_{D_0 \setminus D'_0} + f'_{\tau}X_{D'_0},$$

(4.26)

and define

$$\alpha_j = \|f'_{\tau}X_{(D_{j+1} \setminus D_j) \setminus D'_0}\|_{\beta'}; \quad \alpha_{-1} = \|f'_{\tau}X_{D_0 \setminus D'_0}\|_{\beta'}; \quad \alpha' = \|f'_{\tau}X_{D'_0}\|_{\beta'},$$

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\[
\overline{\alpha}_j = \max \{\alpha_0, \ldots, \alpha_j\}, \quad \beta_j = \|f'_\tau X_{D_j\setminus D_0}\|_{\beta'}, \quad \beta_{-1} = \|f'_\tau\|_{\beta'}.
\]

Notice that for any \( j \)
\[
\alpha_j \leq \beta_j \leq 1, \quad \beta_{j-1} \leq \beta_j + \alpha_{j-1}. \quad (4.27)
\]

Clearly \( \beta_j \) is decreasing, and it vanishes when \( j = N \), since \( \text{supp} \, f \subseteq D_N = B(x, r_N) \).

**Proposition 3.** There exist constants \( C_1, C_2 > 0 \) and \( C^* > Q_2 \) independent of \( f \) such that, such that for any \( f \) supported in a ball with \( \|f\|_{\beta'} \leq 1 \) and such that \( (Tf)\circ(\tau, x) > C^* \) there is an integer \( J \leq N - 1 \) (depending on \( f, x \)) with

\[
A_{0}^{-\frac{1}{p}}(\beta')^{-\frac{1}{p}}\left( (Tf)\circ(\tau, x) + B\|f'_\tau X_{B_r(z)}\|_{\beta'} \right) \leq \begin{cases} 
\sum_{j=0}^{J} \alpha_j + C_1 \overline{\alpha}_J + C_1 \beta_J, & \text{if } J \leq j_1 - 1, \\
\sum_{j=0}^{j_1-1} \alpha_j + \frac{A_{0}^{-\frac{1}{p}}}{A_{0}^{-\frac{1}{p}}} \sum_{j=j_1}^{J} \alpha_j + C_1 \overline{\alpha}_J + C_1 \beta_J, & \text{if } J \geq j_1.
\end{cases}
\]

and

\[
\sum_{j=0}^{J} \alpha_j^{p} e^{\beta' j} + C_1 \overline{\alpha}_J^{p} + C_1 \beta_J^{p} \leq \frac{C_2}{r} \|(Tf)\chi_{E_{r}^{c}}\|_{p}. \quad (4.28)
\]

The proof of Proposition 3 will be given later. Assuming the proposition, our goal is to find a number \( h_1 \) and a sequence \( a = \{a_k\} \) that satisfies (4.9), (4.10) and (4.11).

**Proof of (4.2) in the case \( \kappa = 0. \)**

Assume \( \|f\|_{\beta'} \leq 1, \|f'_\tau\|_{\beta'} \leq 1 - \epsilon, \) and \( (Tf)\circ(\tau, x) > C^* \), where \( C^* \) is as in Proposition 3. Let

\[
h_1 = \sum_{j=0}^{J} \alpha_j + C_1 \overline{\alpha}_J + C_1 \beta_J. \quad (4.30)
\]

If \( A = \max\{A_0, A_\infty\} \), then by Proposition 3, we have

\[
A_{0}^{-\frac{1}{p}}(\beta')^{-\frac{1}{p}}\left( (Tf)\circ(\tau, x) + B\|f'_\tau X_{B_r(z)}\|_{\beta'} \right) \leq h_1. \quad (4.31)
\]

Let

\[
a_k = \begin{cases} 
C_1 \overline{\alpha}_J & \text{if } k = 0 \\
C_1 \beta_J & \text{if } k = 1 \\
\alpha_{k-2} & \text{if } k = 2, \ldots, J + 2.
\end{cases} \quad (4.32)
\]

Then it is clear that we have

\[
\sum_{k=0}^{J+2} |a_k| = \sum_{k=0}^{J+2} a_k = h_1,
\]

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and
\[
\sum_{k=0}^{J+2} |a_k| e^{\beta' k} = C \sum_{j=0}^{J} \alpha_j e^{\beta_j} + C e^{\beta_p} \beta_j + e^{2\beta'} \sum_{j=0}^{J} \alpha_j e^{\beta_j} \leq \frac{C}{\kappa} \| (T f) \chi_{E^*_\kappa} \|_p, \tag{4.33}
\]
where in the last inequality we used \((4.29)\) in Proposition \([3]\).

Next, we show that the second inequality in \((4.10)\) holds for such sequence \(\{a_k\}\). If there exists \(j^* \in \{0, ..., J - 1\}\) such that \(\alpha_{j^*} = \alpha_{j^*}^\ast\), we take
\[
\lambda_k = \begin{cases} 
(1 + C^{\beta'_p})^{-1/\beta'} & \text{if } k = 0, 1, j^* + 2, J + 2 \\
1 & \text{otherwise}
\end{cases}
\]
so that \(L_\lambda = \sum_{k=0}^{\infty} \left( \lambda_k^{\beta} - \beta \right) \leq C\) independent of \(f, x, z\) and
\[
\sum_{k=0}^{\infty} \lambda_k^{\beta} |a_k|^{\beta} = \sum_{k=0, j \neq j^*}^{J-1} \alpha_j^{\beta'} + \frac{C^{\beta'} + 1}{(1 + C^{\beta'})} \alpha_j^{\beta'} + \frac{1}{1 + C^{\beta'}} \alpha_j^{\beta'} + \frac{C^{\beta'}}{1 + C^{\beta'}} \beta_j^{\beta'}
\leq \sum_{j=0}^{J-1} \alpha_j^{\beta'} + \beta_j^{\beta'} = \| f^\prime \chi_{D^*_0} \|_{\beta'} \leq 1 - \epsilon. \tag{4.34}
\]

In the case where \(\alpha_j = \alpha_j^\ast\) and \(\alpha_{j^*} = \alpha_{j^*}^\ast\), it is obvious that by taking slightly different \(\lambda_k\) we still get \((4.10)\), so we omit it here. Therefore we find a sequence that satisfies \((4.9)\) to \((4.11)\) and hence \((3.28)\) in Proposition \([2]\) follows.

Proof of \((4.2)\) in the case \(\kappa > 0\).
Recall that in this case \(A = A_0\). Also, it is enough to assume \(A_\infty > A_0\), otherwise we can apply the previous case, since \((3.25)\) implies \((2.8)\). Let
\[
h_1 = \begin{cases} 
\sum_{j=0}^{J} \alpha_j + C_1 \alpha_{j^*} + C_1 \beta_{j^*}, & \text{if } J \leq j_1 - 1, \\
\sum_{j=0}^{J} \alpha_j + \frac{A_0^{1/\beta}}{A_0^{1/\beta}} \sum_{j=j_1}^{J} \alpha_j + C_1 \alpha_{j^*} + C_1 \beta_{j^*}, & \text{if } J \geq j_1.
\end{cases} \tag{4.35}
\]
Then clearly
\[
A_0^{-\frac{1}{\beta}} \beta^{\prime - \frac{1}{\beta'}} \left( (T f)^\circ (\tau, x) + B \| f^\prime \chi_{B_{r}(x)} \|_{\beta'} \right) \leq h_1. \tag{4.36}
\]
If \(J \leq j_1 - 1\), then the sequence defined in \((4.32)\) will work by the same argument as in the proof of \((4.2)\) under condition \((2.8)\). Now suppose that \(J \geq j_1\) and define
\[
a_0 = C_1 \alpha_{j^*}, \quad a_1 = C_1 \beta_{j^*}, \quad \text{and} \quad a_k = \begin{cases} 
\frac{\alpha_{k-2}}{A_0^{1/\beta}} & \text{for } k = 2, ..., j_1 + 1, \\
\frac{A_0^{1/\beta}}{A_0^{1/\beta}} \frac{\alpha_{k-2}}{A_0^{1/\beta}} & \text{for } k = j_1 + 2, ..., J + 2. \tag{4.37}
\end{cases}
\]
Then the \(\{a_k\}\) satisfies the first identity in \((4.10)\), and the estimate \((4.11)\), using the same argument as the one under condition \((2.8)\).
To deal with $\|\lambda a\|_{\beta'}$, fix any $n_0 \in \mathbb{N}$ and suppose that
\begin{equation}
0 \leq J - j_1 \leq n_0.
\tag{4.38}
\end{equation}

In view of (4.37), that we can find suitable $\{\lambda_k\}$, where at most $n_0 + 4$ of the $\lambda_k$ are less than 1, and the value of each $\lambda_k$ only depends on $C_1, A_\infty, A_0, \beta$, so that $L_\lambda \leq C n_0$. Arguing as in the previous case we see that the second inequality in (4.10) holds. For example, in the case that $x_j = \alpha_j$ with $j^* \in \{0, ..., j_1 - 1\}$, we can take
\begin{equation}
\lambda_k = \begin{cases}
(1 + C_1^{\beta'})^{-1/\beta'} & \text{if } k = 0, j^* + 2 \\
\left(\frac{A_0}{A_0} + C_1^{\beta'}\right)^{-1/\beta'} & \text{if } k = 1, j_1 + 2, ..., J + 2 \\
1 & \text{otherwise}
\end{cases}
\tag{4.39}
\end{equation}

Therefore, arguing as in (4.34), we have the following
\begin{equation}
\sum_{k=0}^{\infty} \lambda_k^{\beta'} |a_k|_{\beta'}^{\beta'} \leq \|f_{\tau e}^{(Tf)X_{E_{\hat{f}}}}\|_{\beta'}^{\beta'} + \kappa \|(Tf)X_{E_{\hat{f}}}/\beta'\|^{\beta'}_{\beta'} \leq 1 - \epsilon.
\end{equation}

On the other hand, assume that (4.38) does not hold, that is
\begin{equation}
J - j_1 > n_0.
\tag{4.40}
\end{equation}

Note that by (4.29) with $p = \beta'$, we have that for any $j \leq J$,
\begin{equation}
\tau e^{\beta' j} \sum_{k=j}^{J} \alpha_k^{\beta'} \leq C_2 \|X_{E_{\hat{f}}}/\beta'\|_{\beta'}^{\beta'}.
\tag{4.41}
\end{equation}

Since $\tau e^{\beta' j_1} = V_x(R_{j_1}) \geq V_x(r_{j_1}) \geq 1$, from (4.37), (4.40), and (4.41) (with $j = j_1 + n_0$) we have
\begin{equation}
\sum_{k=j_1+n_0+2}^{J+2} |a_k|_{\beta'}^{\beta'} = \frac{A_\infty}{A_0} \sum_{j=j_1+n_0}^{J} \alpha_j^{\beta'} \leq \frac{C_2 A_\infty}{A_0 e^{\beta' n_0}} \|(Tf)X_{E_{\hat{f}}}/\beta'\|^{\beta'}_{\beta'} \leq \kappa \|(Tf)X_{E_{\hat{f}}}/\beta'\|^{\beta'}_{\beta'},
\tag{4.42}
\end{equation}

provided $n_0$ is chosen sufficiently large, independently of $x, z, f$. Once again, we can choose suitable $\lambda_k$ similar to (4.39), with at most $n_0 + 2$ of the $\lambda_k$ less than 1. Indeed, we can write
\begin{equation}
h_1 = \sum_{k=0}^{J+2} a_k = C_1 x_j + C_1 \beta_j + \sum_{j=0}^{j_1-1} \alpha_j + \sum_{j=j_1}^{j_1+n_0-1} \alpha_j + \sum_{j=j_1+n_0}^{J} \alpha_j
\tag{4.43}
\end{equation}
so that we choose $\lambda_k = 1$ for $2 \leq k \leq j_1 + 1$ and $j_1 + n_0 + 2 \leq k \leq J + 2$, and the other $n_0 + 2$ values of $\lambda_k$ chosen as in (4.39). With this choice we have $L_\lambda \leq C n_0 \leq C$ and
\begin{equation}
\sum_{k=0}^{\infty} \lambda_k^{\beta'} |a_k|_{\beta'}^{\beta'} = \sum_{k=0}^{j_1+n_0+1} + \sum_{k=j_1+n_0+2}^{J+2} \leq \|f_{\tau e}^{(Tf)X_{E_{\hat{f}}}}\|_{\beta'}^{\beta'} + \kappa \|(Tf)X_{E_{\hat{f}}}/\beta'\|^{\beta'}_{\beta'} \leq 1 - \epsilon.
\tag{4.44}
\end{equation}
Therefore (4.10) holds and (3.28) in Proposition 2 follows.

**Remark 3.** In the above proof the reader can appreciate the reason why under \( \|f\|_{\beta} \leq 1 \) one cannot use the constant \( A_0 \) in the main inequality of Proposition 2, and hence of Theorem 1. Indeed, if \( A_{\infty} > A_0 \) one can choose \( h_1 \) as (4.33) so that (4.36) holds. The corresponding sequence \( \lambda = \{\lambda_k\} \) defined in (4.39) will satisfy \( \|\lambda a\|_{\beta'} \leq \|f\|_{\beta'}(1 - \epsilon) \), however the number of \( \lambda_k \) strictly less than 1 can grow arbitrarily, as the support of \( f \) gets larger, so that the inequality \( L_{\lambda} \leq C \) can fail.

**Proof of (3.29).**

In this case we are assuming that (3.25) and (3.26) hold with \( \kappa > 0 \). It is enough to show that for all \( x \in E_\tau \) with \( (Tf)^\circ(\tau, x) > C^* \) (with \( C^* \) as in Prop. 3) we have

\[
\exp\left[\frac{(1 - \epsilon)^{1 - \beta}}{A_0}(Tf)^\circ(\tau, x) + B\|f^t\chi_{B_{\tau}(z)}\|_{\beta'}\right] \leq \frac{C}{\tau}.
\]  

(4.45)

We first consider the case where \( \overline{\alpha}_J = \alpha' \) and \( J - j_1 \geq 2 \). By (3.26) and (4.29) we have

\[
\sum_{j=0}^{J-1} \alpha_j^{\beta'} + (\alpha')^{\beta'} + \beta_j + C_2^{-1}\kappa\tau \sum_{j=0}^{J-1} \alpha_j^{\beta'}e^{\beta j}
\]

(4.46)

\[
\leq \|f^t\|_{\beta'} + \kappa\|Tf\chi_{E_\tau^*}\| \leq 1 - \epsilon
\]

By (4.28) in proposition 3 we get

\[
A_0^{-1/\beta}(Tf)^\circ(\tau, x) + B\|f^t\chi_{B_{\tau}(z)}\|_{\beta'} \leq (\beta')^{1/\beta}\left(\sum_{j=0}^{J-1} \alpha_j + \left(\frac{A_{\infty}}{A_0}\right)^{1/\beta}\sum_{j=j_1}^{J-1} \alpha_j + C_1\alpha' + (1 + C_1)\beta_j\right)
\]

\[
\leq (\beta')^{1/\beta}\left(\sum_{j=0}^{J-1} (1 + C_2^{-1}\kappa\tau e^{\beta j})\alpha_j^{\beta'} + (\alpha')^{\beta'} + \beta_j^{\beta'}\right)^{1/\beta'}
\]

\[
\times \left(\sum_{j=0}^{J-1} (1 + C_2^{-1}\kappa\tau e^{\beta j})^{-\beta/\beta'} + \frac{A_{\infty}}{A_0}\sum_{j=j_1}^{J-1} (1 + C_2^{-1}\kappa\tau e^{\beta j})^{-\beta/\beta'} + (2\beta)(C_1^3 + 1)\right)^{1/\beta}.
\]

(4.47)

By definition of \( j_1 \) in (4.17) and (4.19), (4.20) we have that

\[
\tau e^{\beta j_1} \geq 1, \quad j_1 \leq \frac{1}{\beta'} \log \frac{1}{\tau} + C.
\]

Hence by (4.46) we get

\[
A_0^{-1}(Tf)^\circ(\tau, x) + B\|f^t\chi_{B_{\tau}(z)}\|_{\beta'} \leq \beta'(1 - \epsilon)^{\beta - 1}(j_1 + C\sum_{j=j_1}^{J-1} e^{-\beta(j-j_1)} + C)
\]

(4.48)

\[
\leq (1 - \epsilon)^{\beta - 1}(\log \frac{1}{\tau} + C).
\]

For the other cases such as \( \overline{\alpha}_J = \alpha_j \) for some \( j \leq J \), or \( J \leq j_1 - 1 \), the proof is completely similar, so we omit it here.
5 Proof of Proposition 3

Recall that \( R_j = r_x(\tau e^{\beta j}) \), \( V_\alpha(R_j) = \mu(B(x, R_j)) = \tau e^{\beta j} \), and \( r_j \) is defined in (4.16). The integer \( N \) was defined in (4.12), and we are assuming \( (Tf)^\circ(\tau, x) > Q_2 \), where \( Q_2 \) is as in (4.21), which implies \( N \geq 2 \) and (4.22).

For the rest of this section we will set for any measurable function \( \phi : M \to \mathbb{R}^m \)
\[
S_j \phi = \phi \chi_{D_j \setminus D_0'};
\]
where we defined \( D_j \), \( D_0' \) in (4.24). With this notation we then have
\[
(S_j - S_{j+1})f_\tau' = f_\tau' \chi_{(D_{j+1} \setminus D_j) \setminus D_0'}, \quad \alpha_j = \|(S_j - S_{j+1})f_\tau'\|_{\beta'}, \quad \beta_j = \|S_j f_\tau'\|_{\beta'}, \quad j \geq 0.
\]

We first give some preliminary estimates on \((Tf)^\circ(\tau, x)\). Recall that
\[
(Tf)^\circ(\tau, x) = |T(f_\tau' \chi_{D_0'})| \leq |T f_\tau' \chi_{D_0' \setminus D_0}(x)| + |T f_\tau' \chi_{D_0' \setminus D_0}(x)|.
\]
By Hölder and (K3) we have
\[
|T f_\tau' \chi_{D_0' \setminus D_0}(x)| \leq \left( \int_{\{d(x, y) \geq r_0\} \cap D_0'} |k(x, y)|^\beta d\mu(y) \right)^{1/\beta} \left( \int_{D_0'} |f_\tau'(y)|^\beta d\mu(y) \right)^{1/\beta'} \leq B^{1/\beta} V_\alpha(r_0)^{-1/\beta} \mu(D_0')^{1/\beta} \|f_\tau' \chi_{D_0'}\|_{\beta'} = B^{1/\beta} \alpha' \leq B\alpha'.
\]
By (5.1) we can write, for any \( J \in \{0, 1, ..., N - 1\} \)
\[
(Tf)^\circ(\tau, x) + B \|f_\tau' \chi_{B_r(x)}\|_{\beta'} \leq |TS_0 f_\tau'(x)| + 2B\alpha' = \left| \sum_{j=0}^{J} T(S_j - S_{j+1})f_\tau' + TS_{j+1}f_\tau' \right| + 2B\alpha'
\]
\[
\leq \sum_{j=0}^{J} |T(S_j f_\tau' - S_{j+1}f_\tau')(x)| + |TS_{j+1}f_\tau'(x)| + 2B\alpha'.
\]

For any integer \( j \), by (4.16), (4.21) we have the estimate
\[
|T(S_j f_\tau' - S_{j+1}f_\tau')(x)| \leq \left( \int_{D_{j+1} \setminus D_j} |k(x, y)|^\beta d\mu(y) \right)^{1/\beta} \|S_j f_\tau' - S_{j+1}f_\tau'\|_{\beta'} \leq \begin{cases} (\beta' A_0)^{1/\beta} \alpha_j, & \text{if } 1 \leq j \leq j_1 - 1, \ j \neq N - 1 \\ (\beta' A_\infty)^{1/\beta} \alpha_j, & \text{if } j_1 \leq j \leq N - 1, \ j \neq 0 \\ (Q_2)^{1/\beta} \alpha_j, & \text{if } j = 0, \ j = N - 1. \end{cases}
\]

We then get that there is \( Q_3 \) such that for any \( J = 0, 1, ..., N - 1 \)
\[
(Tf)^\circ(\tau, x) + B \|f_\tau' \chi_{B_r(x)}\|_{\beta'} \leq \begin{cases} (\beta' A_0)^{1/\beta} \sum_{j=0}^{J} \alpha_j + Q_3 \alpha_j + |TS_{j+1}f_\tau'(x)|, & \text{if } J \leq j_1 - 1, \\ (\beta' A_0)^{1/\beta} \sum_{j=0}^{j_1 - 1} \alpha_j + (\beta' A_\infty)^{1/\beta} \sum_{j=j_1}^{J} \alpha_j + Q_3 \alpha_j + |TS_{j+1}f_\tau'(x)|, & \text{if } J \geq j_1. \end{cases}
\]
What remains to be proved is that for \( (T f)^n(\tau, x) > C^* \) large enough, there is \( J \in \{0, 1, ..., N - 1\} \) (depending on \( x \)) and \( Q^* \) (depending on \( k, \beta \)) such that

\[
|TS_{J+1}f^*_\tau(x)| \leq Q^*\alpha_J + Q^*\beta_J, \tag{5.6}
\]

(so that \(4.28\) holds for some \( C_1 \)) and \(4.29\) holds i.e.

\[
\sum_{j=0}^{J} \alpha_j\beta_j + C_1\alpha_J + C_1\beta_J \leq \frac{C_2}{\tau} \| (T f)\chi_{E_c \tau} \|_p, \tag{5.7}
\]

for some \( C_2 > 0 \).

For each \( j = 0, 1, ..., N - 1 \) consider the estimates

\[
\beta_{j-1} \leq \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} |T f| d\mu. \tag{5.8}
\]

\[
|TS_jf^*_\tau(x)| \leq \frac{2e^{\beta_j - 1}}{e^{\beta_j - 1} + 1} |TS_{j+1}f^*_\tau(x)|. \tag{5.9}
\]

If \(5.8\) holds for \( j = 0 \) let

\[
J_1 = \max \{ k \in \{1, ..., N\} : \text{ (5.8) holds for } 0 \leq j \leq k - 1 \} \tag{5.10}
\]

whereas if \(5.8\) does not hold for \( j = 0 \) let \( J_1 = 0 \).

Similarly, if \(5.9\) holds for \( j = 0 \) let

\[
J_2 = \max \{ k \in \{1, ..., N\} : \text{ (5.9) holds for } 0 \leq j \leq k - 1 \} \tag{5.11}
\]

whereas if \(5.9\) does not hold for \( j = 0 \) let \( J_2 = 0 \).

The integers \( J_1, J_2 \) could be considered as stopping times: they represent the first times after 0 that \(5.8, 5.9\) do not occur, if they do occur at 0.

**Proof of \(5.6, 5.7\) in case \( 0 \leq J_2 \leq J_1 \leq N \).**

First assume that \( J_2 \neq N \). From the definition of \( J_2 \) we have

\[
|TS_{J_2+1}f^*_\tau(x)| < \frac{e^{\beta_j - 1} + 1}{2e^{\beta_j - 1}} |TS_{J_2}f^*_\tau(x)| \leq \frac{e^{\beta_j - 1} + 1}{2e^{\beta_j - 1}} \left( |TS_{J_2+1}f^*_\tau(x)| + |T(S_{J_2} - S_{J_2+1})f^*_\tau(x)| \right) \tag{5.12}
\]

\[
\leq \frac{e^{\beta_j - 1} + 1}{2e^{\beta_j - 1}} \left( |TS_{J_2+1}f^*_\tau(x)| + Q_4\alpha_{J_2} \right)
\]

where the last inequality is by \(5.4\). This implies

\[
|TS_{J_2+1}f^*_\tau(x)| < Q_5\alpha_{J_2} \leq Q_5\alpha_{J_2}, \tag{5.13}
\]

which is \(5.6\) with \( J = J_2 \).
To show (5.7), note that if \( J_1 = 0 \), then \( J_2 = 0 \) and from (5.5) there is \( Q_6 \) such that

\[
(T f)^o(\tau, x) \leq Q_6 \overline{\tau}_0 \leq Q_6. \tag{5.14}
\]

Therefore, assuming \((T f)^o(\tau, x) > \max\{Q_6, Q_2\}\) we have \( J_1 \neq 0 \) and (5.8) is true for all \( j \leq J - 1 \), so

\[
\sum_{j=0}^{J} \alpha_j^p e^{\beta_j} + C_1 \overline{\tau}_j^p + C_1 \beta_j^p \leq Q_7 \sum_{j=0}^{J-1} \beta_{j-1}^p e^{\beta_j} \leq Q_7 \sum_{j=0}^{J-1} \left( \int_{(D_{j+1}\setminus D_j)} |T f|^p d\mu \right)^p e^{\beta_j}
\]

\[
\leq Q_7 \sum_{j=0}^{J-1} e^{\beta_j} \int_{(D_{j+1}\setminus D_j)} |T f|^p d\mu \leq 2Q_7 \sum_{j=0}^{J-1} \frac{V_x(r_j)}{\tau} \int_{(D_{j+1}\setminus D_j)} |T f|^p d\mu \leq \frac{C_2}{\tau} \sum_{j=0}^{J-1} \int_{(D_{j+1}\setminus D_j)} |T f|^p d\mu \leq \frac{C_2}{\tau} \|\|(T f)\chi_{\bigcap_j E_r}\|_p
\]

(5.15)

where the first inequality is due to the fact that \( \beta_j \) is decreasing, the third last inequality is by that \( \mu(E_r) = \tau \) and (4.22), and whereas the second last inequality is by (4.20).

If \( J_2 = N \), then \( J_1 = N \), and if we take \( J = N - 1 \) it is clear that (5.9) holds, since \( TS_N f_t(x) = 0 \). Note that (5.9) is true for \( j = 0, \ldots, N - 1 \), therefore (5.7) still follows by the same calculations as in (5.15).

**Proof of (5.6), (5.7) in case \( N \geq J_2 \geq J_1 + 1 \).**

If \( J_1 = N - 1 \) then the proof is the same as in the case \( J_1 = J_2 = N \) given above, so we can assume \( J_1 < N - 1 \).

We will need the following lemma to handle this case. Let us state it here, and its proof will be postponed to the Appendix.

**Lemma 3.** There exist constants \( C_3, C > 0 \), depending only on \( k, \beta \), such that for any \( j \leq N - 2 \)

\[
\int_{(D_{j+1}\setminus D_j)} |T f|^p d\mu \leq C_3 e^{-\beta' - 1} j, \tag{5.16}
\]

\[
|TS_{j+1} f_t(x) - \int_{(D_{j+1}\setminus D_j)} TS_{j+2} f_t d\mu| \leq C \beta_{j+1}, \tag{5.17}
\]

\[
\left| \int_{(D_{j+1}\setminus D_j)} T(S_0 - S_{j+2}) f_t d\mu \right| \leq C \overline{\tau}_{j+1}, \tag{5.18}
\]

\[
\left| \int_{(D_{j+1}\setminus D_j)} T(f_t^r \chi_{D_0\setminus D_0'}) d\mu \right| \leq C \alpha_{j-1}, \tag{5.19}
\]

and

\[
\left| \int_{(D_{j+1}\setminus D_j)} T(f_t^r \chi_{D_0'}) d\mu \right| \leq C \alpha'. \tag{5.20}
\]
By (5.17) in Lemma 3 and noticing that
\[ f = f_\tau + f_\tau' = f_\tau + f_\tau'X_{D_0} \setminus D_0' + f_\tau'X_{(D_{J_1+2}\setminus D_0)} + f_\tau'X_{D_0' \setminus D_0} + f_\tau'X_{D_0'} , \]
we have, using (5.17)-(5.20),
\[
|TS_{J_1+1}f_\tau'(x)| \leq \left| \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} TS_{J_1+2}f_\tau'd\mu \right| + C\beta_{J_1+1}
= \left| \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} (Tf - Tf_\tau - T(f_\tau'X_{D_0} \setminus D_0') - T(S_0 - S_{J_1+2})f_\tau' - T(f_\tau'X_{D_0'})\right)d\mu \right| + C\beta_{J_1+1}
\leq \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} |Tf|d\mu + \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} |Tf_\tau|d\mu + C\alpha_{-1} + C\alpha_{J_1+1} + C\beta_{J_1+1}.
\]

To estimate the second integral, note first that by (5.1), (5.2)
\[ (Tf) = (Tf_\tau) \geq \max\{B + 4C_3, Q_2\}, \]
where \( C_3 \) is the constant in (5.16), then \( C_3 \leq \frac{1}{2}|TS_0 f_\tau'(x)| \). Using (5.16) in Lemma 3 and condition (5.9) applied \( J_1 + 1 \) times (which is possible since \( J_1 + 1 \leq J_2 \), we get
\[
\int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} |Tf_\tau|d\mu \leq C_3 e^{-(\beta'-1)J_1} \leq \frac{1}{4} e^{-(\beta'-1)J_1} |TS_0 f_\tau'(x)|
\leq \frac{1}{4} e^{-(\beta'-1)J_1} \left( \frac{2 e^{\beta'-1}}{e^{\beta'-1} + 1} \right)^{J_1+1} |TS_{J_1+1} f_\tau'(x)|
= \frac{1}{4} \left( \frac{2}{e^{\beta'-1} + 1} \right)^{J_1} \left( \frac{2 e^{\beta'-1}}{e^{\beta'-1} + 1} \right)^{J_1} |TS_{J_1+1} f_\tau'(x)| \leq \frac{1}{2} |TS_{J_1+1} f_\tau'(x)|,
\]
and
\[
|TS_{J_1+1} f_\tau'(x)| \leq \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} |Tf|d\mu + \frac{1}{2} |TS_{J_1+1} f_\tau'(x)| + C\alpha_{J_1+1} + C\beta_{J_1}.
\]
So the above inequality, along with the fact that (5.8) is false for \( j = J_1 \), give us
\[
|TS_{J_1+1} f_\tau'(x)| \leq 2 \int_{(D_{J_1+1} \setminus D_{J_1}) \setminus E_r} |Tf|d\mu + 2C\alpha_{J_1+1} + 2C\beta_{J_1+1}
\leq 2\beta_{J_1} + 2C\alpha_{J_1+1} + 2C\beta_{J_1+1} \leq Q_3 \alpha_{J_1} + Q_3 \beta_{J_1},
\]
which is (5.6), with \( J = J_1 + 1 \) (in the last inequality above we used from (4.27) that for any \( j \) we have \( \beta_{j-1} \leq \beta_j + \alpha_{j-1} \leq \beta_j + \alpha_j \) and that \( \alpha_{j+1} \leq \max\{\alpha_j, \alpha_{j+1}\} \leq \alpha_j + \beta_{j+1} \leq \alpha_j + \beta_j \).)

By the same argument as in the previous case, if \( J_1 = 0 \) then we have, from (5.5) that \( (Tf) = (Tf_\tau) < Q_9 \) for some \( Q_9 > 0 \), so that if \( (Tf) \geq \max\{Q_9, Q_2, B + 4C_3\} \) then \( J_1 \neq 0 \), and estimate (5.7) for this choice of \( J \) follows exactly in the same way as in (5.15).
Finally, we can take $C^* = \max\{Q_2, Q_6, Q_9, B + 4C_3\}$ and $Q^* = \max\{Q_5, Q_8\}$, to cover all of the above cases. \hfill \square

6 Proof of Theorem 1 when $A_\infty = 0$

Since (K3) is a more restrictive condition than (K3) with small $V_\infty$ and (K3'), it is enough to prove the inequalities under the conditions $A_\infty = 0$, the first estimate in (K3) and (K4) for $V_x(d(x, y)) \leq 1$, and (K3').

The proof in this case is similar to the proof under conditions (K1)-(K4) and $A_\infty \neq 0$ but with some appropriate modifications. Most parts of the proof follow immediately given the above assumptions. We will mention the main changes as well as the related proof.

First we prove that (K1') still holds, since it is needed to get (3.10), the Adams inequality on measure spaces with finite measure. Let $0 \leq V_x(r_1) \leq V_x(r_2) \leq 1$, then by (K1), (K3) for small balls, (K3') and (K2) with $A_\infty = 0$,

\[
\int_{V_x(r_1)}^{V_x(r_2)} (k^*(x, u))^{\beta} du \leq \int_{r_1 \leq d(x,y) \leq r_2} |k(x, y)|^{\beta} d\mu(y) + \int_{d(x,y) \leq r_1, \ |k(x,y)| \leq k^*(x,V_x(r_1))} |k(x, y)|^{\beta} d\mu(y)
\]

\[
+ V_x(r_2) \sup_{r_2 \leq d(x,y) \leq r_x(1)} |k(x, y)|^{\beta} + \int_{d(x,y) \geq r_x(1)} |k(x, y)|^{\beta} d\mu(y) \leq \int_{r_1 \leq d(x,y) \leq r_2} |k(x, y)|^{\beta} d\mu(y) + C. \tag{6.1}
\]

Next, the definition of $r_j$ in (4.16) should be modified since $A_\infty = 0$. If $V_x(R) \leq 1$, let $N$ be the same as in definition (4.12). Otherwise $N$ is defined by replacing all the $V_x(R)$ by 1 in (4.12), that is

\[
N = \begin{cases} 
0 & \text{if } 1 \leq \tau \\
1 & \text{if } \tau < 1 \leq \tau e^{2\beta'} \\
\left[\frac{1}{\beta'} \log \frac{1}{\tau} \right] - 1 & \text{if } \tau > \tau e^{2\beta'}
\end{cases} \tag{6.2}
\]

Let

\[
r_j = \begin{cases} 
R_0, & \text{if } j = 0 \\
\sup\{r \leq R_j : \int_{r_{j-1} \leq d(x,y) \leq r} |k(x, y)|^{\beta} d\mu(y) \leq \beta' A_0\}, & \text{if } 1 \leq j \leq N - 1 \text{ if } N \geq 2 \\
R, & \text{if } j = N,
\end{cases} \tag{6.3}
\]

where $R_j$ is defined in (4.13), that is, $R_j = r_x(\tau e^{\beta'j})$. Note that

\[
V_x(r_j) \leq 1, \text{ if } 0 \leq j \leq N - 1, \text{ } N \geq 1, \tag{6.4}
\]

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and $V_x(r_N) > 1$ if and only if $\tau > 1$ or $V_x(R) > 1$.

Since we have (K1) and (K3) for small balls, by the same argument in Appendix B, we get (4.19) for $0 \leq j \leq N - 1$, and (4.20) for $0 \leq j \leq N - 2$ with slightly different constants depending only on $\beta'$, $B$, $A_0$. On the other hand, by the definition of $R_j, r_j$ for $j = N - 1, N$, we have

$$V_x(r_N) \geq C V_x(R_N), \quad V_x(r_N) - V_x(r_{N-1}) \geq C V_x(R_{N-1}).$$

(6.5)

Recall that $m$ is defined in (4.22), and we will assume $m = 1$ as before. The inequalities that were derived by using (4.19), (4.20), (6.5) are (5.15), (D.2) and (D.8). It is easy to verify that they still hold.

The inequalities (3.17), (5.2), which were proved using (K3), still hold by (K1) and (K2) with $A_\infty = 0$.

For the rest of the proof in sections 4, 5, we always take $j_1 = N$ by definition and $A_\infty = 0$.

Lastly, we mention that (D.6) still holds. We can apply (K4) for $V_x(r) \leq 1$ since by (6.4) we have $V_x(r_{N-1}) \leq 1$ and clearly $\xi \in B(x, r_{N-1})$.

\[ \Box \]

7 Proof of Theorem 2

For simplicity, in this proof we will let $V(r) = V_{x_0}(r)$. Let $B(x_0, \epsilon_0)$ be the largest ball with zero volume, and given any $\epsilon > \epsilon_0$ we will let $\epsilon' = r_{x_0}(\frac{1}{2} V(\epsilon))$, i.e. $B(x_0, \epsilon')$ is the smallest ball inside $B(x_0, \epsilon)$ with

$$V(\epsilon) = 2V(\epsilon').$$

In the notation of (2.15), (2.16) let

$$P(x, y) = \sum_{j=0}^{m-1} k_j(x, x_0)p_j(y, x_0), \quad Q(y, z) = \sum_{j=0}^{m-1} v_j(y, x_0)v_j(z, x_0)$$

(7.1)

i.e. $Q(y, z)$ is the kernel of the orthogonal projection of $L^2(B(x_0, r))$ to the space spanned by the $p_j(\cdot, x_0)$ restricted to the ball $B(x_0, r)$. Using (2.16) we get

$$|Q(y, z)| \leq \frac{C}{V(r)}, \quad y, z \in B(x_0, r).$$

(7.2)

It’s easy to check that (2.3) implies

$$\int_{d(x_0, y) > \epsilon} |(k(x, y) - k(x_0, y))|k(x_0, y)|^{\beta - 1}d\mu(y) \leq C, \quad x \in B(x_0, \epsilon')$$

(7.3)

(note that the above estimate implies (K4), if valid for all $x_0$).

Given $\epsilon_0 < \epsilon < r$, let

$$\phi_{\epsilon, r}(y) = k(x_0, y)k(x_0, y)|^{\beta - 2} \chi_{B(x_0, r) \setminus B(x_0, \epsilon)}(y),$$

(7.4)

and let

$$\tilde{\phi}_{\epsilon, r}(y) = \phi_{\epsilon, r}(y) - Q\phi_{\epsilon, r}(y)$$

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where
\[ Q\phi_{\epsilon,r}(y) = \int_{B(x_0, r)} Q(y,z)\phi_{\epsilon,r}(z)\,d\mu(z). \] (7.5)

Clearly \( \tilde{\phi}_{\epsilon,r} \) is orthogonal to \( P(x, \cdot) \), for any \( x \in B(x_0, r) \). By (K3), (7.2), and (2.1) we have
\[ |Q\phi_{\epsilon,r}(y)| \leq \frac{C}{V(r)} \int_{B(x_0, r)} |k(x_0, z)|^{\beta'}\,d\mu(z) \leq CV(r)^{-1/\beta'}. \] (7.6)

We now prove the following estimates:
\[ \log \frac{V(r)}{V(\epsilon)} - C \leq \|\phi_{\epsilon,r}\|_{\beta'}^\beta \leq A \log \frac{V(r)}{V(\epsilon)} + C. \] (7.7)
\[ |T\phi_{\epsilon,r}(x)| \geq A \log \frac{V(r)}{V(\epsilon)} - C, \quad x \in B(x_0, \epsilon') \] (7.8)
\[ \|T\phi_{\epsilon,r}\|_{p'}^p \leq CV(r). \] (7.9)

where in (7.7), (7.8) we have where \( A = A_0 \) if \( k(x,y) \) is proper on the diagonal and \( 0 < V(\epsilon) < V(r) \leq 1 \), and \( A = A_\infty \) if \( k(x,y) \) is proper at infinity, with \( V(r) > V(\epsilon) \geq 1 \).

To prove (7.7), using (7.6)
\[ |\phi_{\epsilon,r}| - CV(r)^{-1/\beta'} \leq |\tilde{\phi}_{\epsilon,r}| \leq |\phi_{\epsilon,r}| + CV(r)^{-1/\beta'}, \]

hence
\[ |\phi_{\epsilon,r}|^\beta - C|\phi_{\epsilon,r}|^{\beta-1}V(r)^{-1/\beta'} \leq |\tilde{\phi}_{\epsilon,r}|^\beta \leq |\phi_{\epsilon,r}|^\beta + CV(r)^{-1/\beta'} + CV(r)^{-1}. \]

(for the first inequality use that if \( c > a - b \) then \( c^p > a^p - pba^{p-1} \), for \( a, b, c \geq 0, p > 1 \), for the second, use \( (a+b)^p \leq a^p + p^2b^{p-1}(a^{p-1}b + b^p) \)). So by (K1), (K2), (2.13), (2.14) and (K3) we obtain (7.7).

Note that if \( x \in B(x_0, R) \), then by (7.6), (K3) (or (2.1))
\[ |TQ\phi_{\epsilon,r}(x)| \leq CV(r)^{-1/\beta'} \int_{B(x_0, r)} |k(x,y)|\,d\mu(y) \leq C \] (7.10)

(recall that \( Q(y,z) \) is supported in \( B(x_0, r) \times B(x_0, r) \)).

Estimate (7.8) follows from
\[ |T\tilde{\phi}_{\epsilon,r}(x)| \geq \int_{\epsilon \leq d(x_0,y) \leq r} k(x,y)k(x_0, y)|k(x_0, y)|^{\beta-2}\,d\mu(y) - C \]
\[ \geq \int_{\epsilon \leq d(x_0,y) \leq r} |k(x_0, y)|^{\beta}\,d\mu(y) - \int_{\epsilon \leq d(x_0,y) \leq r} (k(x,y) - k(x_0, y))k(x_0, y)|k(x_0, y)|^{\beta-2}\,d\mu(y) - C \]
\[ \geq A \log \frac{V(r)}{V(\epsilon)} - C \] (7.11)
where the first inequality is by (7.10) and in the last inequality we used (7.3).

Finally, to prove (7.9), take any $R > 0$ such that $V(R) = 2V(r)$. For any $x \in B(x_0, R)$ using the orthogonality property of $\tilde{\phi}_{\epsilon,r}$ we have

$$T\tilde{\phi}_{\epsilon,r}(x) = \int_{B(x_0,r)} (k(x,y) - P(x,y))\tilde{\phi}_{\epsilon,r}(y) d\mu(y).$$

Assuming $p > (1 + p/\beta')(1 + \eta)^{-1}$ and using (2.15), (2.1) we get, for $y \in B(x_0, r)$,

$$\int_{d(x_0,x) > R} |k(x,y) - P(x,y)|^p d\mu(x) \leq C(V(r))^{p} \int_{B(x_0,r)^c} (V(d(x_0,x))^{-p(\eta + 1/\beta)}) d\mu(x) \leq C(V(r))^{1-p/\beta}.$$

Therefore by Minkowski’s inequality, (7.12), (7.6)

$$\int_{d(x_0,x) > R} |T\tilde{\phi}_{\epsilon,r}(x)|^p d\mu(x) \leq \int_{d(x_0,x) > R} \left( \int_{B(x_0,r)} |k(x,y) - P(x,y)||\tilde{\phi}_{\epsilon,r}(y)| d\mu(y) \right)^p d\mu(x) \leq C(V(r))^{1-p/\beta} \int_{B(x_0,r)} (|k(x,y)|^{\beta-1} + |Q\phi_{\epsilon,r}(y)|) d\mu(y) \leq CV(r).$$

On the other hand, if $x \in B(x_0, R)$, by (K.1), \( \|\phi_{\epsilon,r}\|_q \leq CV(r)^{-1/\beta + 1/q} \) if $q < \beta'$. So by taking $q' \geq p$, $q < \beta'$ such that $(q')^{-1} = q^{-1} - (\beta')^{-1}$ and applying the Hardy-Littlewood-Sobolev inequality for $T\phi_{\epsilon,r}$, (see [A2]) we have

$$\left( \int_{d(x_0,x) \leq R} |T\phi_{\epsilon,r}(x)|^p d\mu(x) \right)^{1/p} \leq CV(R)^{1-p-1/q'} \left( \int_{d(x_0,x) \leq R} |T\phi_{\epsilon,r}(x)|^{q'} d\mu(x) \right)^{1/q'} \leq CV(r)^{1/p-1/q'} \|\phi_{\epsilon,r}\|_q \leq CV(r)^{1/p}.$$  

Therefore

$$\int_{d(x_0,x) < R} |T\tilde{\phi}_{\epsilon,r}(x)|^p d\mu(x) \leq C \int_{d(x_0,x) < R} (|T\phi_{\epsilon,r}(x)|^p + |TQ\phi_{\epsilon,r}(x)|^p) d\mu(x) \leq V(r).$$

Proof of (a). Suppose that $k(x,y)$ is proper on the diagonal. Let $r_0 = r_{x_0}(1)$, i.e. $B(x_0, r_0)$ is the smallest ball with volume $V(r_0) = 1$, and recall that $\epsilon_0 \geq 0$ is the largest $\epsilon$ with $V(\epsilon) = 0$. For $\epsilon_0 < \epsilon < r_0$ let

$$\psi_{\epsilon} = \frac{\tilde{\phi}_{\epsilon,r_0}}{\|\tilde{\phi}_{\epsilon,r_0}\|_q^{q'}}.$$

Clearly $\|\psi_{\epsilon}\|_{q'} = 1$ and from (7.7), (7.8) and (7.9)

$$|T\psi_{\epsilon}(x)|^\beta \geq A_0 \log \frac{1}{V(\epsilon)} - C, \quad x \in B(x_0, \epsilon'),$$

where the first inequality is by (7.10) and in the last inequality we used (7.3).
(recall: \( V(\epsilon) = 2V(\epsilon') \)) and
\[
\| T\psi_\epsilon \|_p^p \leq C \left( \log \frac{1}{V(\epsilon)} \right)^{-p/\beta'} \to 0, \quad \epsilon \to \epsilon_0. \tag{7.17}
\]

For the sharpness in (2.9), note that for \( \theta > 1 \)
\[
\int_{B(x_0,\epsilon')} \frac{\exp \left[ \frac{\theta}{A_0} |T\psi_\epsilon|^\beta \right]}{1 + |T\psi_\epsilon|^\beta_{\theta p/\beta'}} d\mu(x) \geq CV(\epsilon')V(\epsilon)^{-\theta} \left( \log \frac{1}{V(\epsilon)} \right)^{-p/\beta'}
\]
\[
= CV(\epsilon)^{1-\theta} \left( \log \frac{1}{V(\epsilon)} \right)^{-p/\beta'} \to +\infty, \quad \epsilon \to \epsilon_0.
\tag{7.18}
\]

whereas for \( 0 < \theta < 1 \)
\[
\frac{1}{\| T\psi_\epsilon \|_p^p} \int_{B(x_0,\epsilon')} \frac{\exp \left[ \frac{1}{A_0} |T\psi_\epsilon|^\beta \right]}{1 + |T\psi_\epsilon|^\beta_{\theta p/\beta'}} d\mu(x) \geq C \left( \log \frac{1}{V(\epsilon)} \right)^{(1-\theta)p/\beta'} \to +\infty, \quad \epsilon \to \epsilon_0. \tag{7.19}
\]

To prove the sharpness for (2.11), let
\[
\psi_\epsilon = \tilde{\phi}_\epsilon \left( \| \tilde{\phi}_\epsilon \|_{\beta'} + \kappa \| T\tilde{\phi}_\epsilon \|_{\beta'}^{1/\beta'} \right)^{-1/\beta'}, \tag{7.20}
\]
then it’s easy to check that (7.16)-(7.19) still hold.

For the sharpness of (2.12), let \( \psi_\epsilon \) be defined as in (7.20). If \( \theta > 1 \), by (7.16), (7.17) we get
\[
\int_{B(x_0,\epsilon')} \exp \left[ \frac{\theta}{A_0} |T\psi_\epsilon|^\beta \right] d\mu(x) \geq CV(\epsilon')V(\epsilon)^{-\theta} = CV(\epsilon)^{1-\theta} \to \infty, \quad \epsilon \to \epsilon_0. \tag{7.21}
\]

Proof of (b). Suppose that \( k(x, y) \) is proper at infinity, and let \( r_0 \) be as above. Let \( 0 < r_1 < \frac{r_0}{r_0} < r \) with \( 1 = V(r_0) = 2V(r_1) \). Let
\[
\psi_r = \tilde{\phi}_{r_0,r} \left( \| \tilde{\phi}_{r_0,r} \|_{\beta'} \right)^{-1/\beta'},
\]
then from (7.7), (7.8), (7.9)
\[
|T\psi_r(x)|^\beta \geq A_\infty \log V(r) - C \quad x \in B(x_0, r_1), \tag{7.22}
\]
and
\[
\| T\psi_r \|_p^p \leq CV(r) \left( \log V(r) \right)^{-p/\beta'}. \tag{7.23}
\]

Therefore
\[
\frac{1}{\| T\psi_r \|_p^p} \int_{B(x_0,r_1)} \frac{\exp \left[ \frac{\theta}{A_\infty} |T\psi_r|^\beta \right]}{1 + |T\psi_r|^\beta_{\theta p/\beta'}} d\mu(x) \geq CV(r)^{\theta_1 - 1} \left( \log V(r) \right)^{(1-\theta_2)p/\beta'}. \tag{7.24}
\]
Hence if either $\theta_1 > 1, \theta_2 = 1$ or $\theta_1 = 1, \theta_2 < 1$, then we have

$$
\frac{1}{\|T\psi_r\|_p^p} \int_{B(x_0, r_1)} \exp \left[ \frac{\theta_1}{A_\infty^{\beta}} |T\psi_r|^{\beta} \right] \frac{\theta_2^{2\beta/\beta'}}{1 + |T\psi_r|^{\theta_2 p\beta/\beta'}} d\mu(x) \to \infty, \quad r \to \infty.
$$

(7.25)

\[\square\]

**Remark 9.** An example where $A_\infty > A_0$ and inequality (2.9) is sharp.

Consider a Riesz-like kernel on $\mathbb{R}^n$ such that $k \in C^\alpha(\mathbb{R}^n \setminus 0)$ and

$$
k(x, y) = k(\|x - y\|) = \begin{cases} |x - y|^{\alpha-n} & \text{if } |x - y| \leq 1 \\ 2|x - y|^{\alpha-n} & \text{if } |x - y| \geq 2. \end{cases}
$$

We will verify that the above kernel satisfies the conditions of Theorem 2 with $\beta = n/(n - \alpha), \beta' = n/\alpha, A_0 = |B_1|, A_\infty = 2^\beta |B_1|.

It is clear that (K1), (K2) and (K3) are satisfied with the given $A_0, A_\infty$. To verify (K4), note that $V_x(R) \geq (1 + \delta)V_x(r)$ implies $R \geq (1 + \delta')r$, and hence $c_1 R \leq |x' - y| \leq c_2 R$ for all $x' \in B(x, r), y \in B(x, R)$. So we have $||x' - y|^{\alpha-n} - |x - y|^{\alpha-n}| \leq C|x' - x||y|^{\alpha-n-1} \leq C r |y|^{\alpha-n-1}$ and (K4) follows.

Let $x_0 = 0$. It is clear that (2.3) and (2.14) hold. Since $k(x, y)$ is a $C^n$ function, we can take its Taylor expansion of order $n - 1$ so that (2.15) holds with $\eta = 1, m = n$.

## 8 Existence of ground state solutions for quasilinear equations

In [MS2] Masmoudi and Sani showed that their inequality (1.1) in the case $\alpha = 1$ can be applied to show the existence of a radial ground state solution for the following quasilinear equation in $\mathbb{R}^n, n \geq 2$:

$$
-\Delta_n u + V_0 |u|^{p-2} u = f(u)
$$

(8.1)

where

$$
\Delta_n u = \text{div}(|\nabla u|^{n-2} \nabla u)
$$

is the $n$-Laplacian operator, $V_0$ is a positive constant, and the allowed maximal growth on the nonlinear term $f$ is exponential.

In this section, by making use of an improved version of (1.1) we prove an existence result for the following more general quasilinear elliptic equation:

$$
\begin{cases}
-\Delta_n u + V_0 |u|^{p-2} u = f(u) & p > 1, \ n \geq 2 \\
||\nabla u||_{n} < \infty, \ ||u||_{p} < \infty.
\end{cases}
$$

(8.2)

First, let’s observe that as an immediate consequence of Theorem 1, we obtain the improved MSI given in (1.3), simply by writing $u \in C^\infty_c(\mathbb{R}^n)$ in terms of the Riesz potential.
Then for each $V$ the following hold:

\[
\nabla \gamma_{n,\alpha} = \begin{cases} 
\frac{c_{\alpha} n^\alpha}{|B_1|} & \text{if } \alpha \text{ even} \\
((n - \alpha - 1)c_{\alpha + 1})^{-\frac{\alpha}{n - \alpha}} |B_1| & \text{if } \alpha \text{ odd,}
\end{cases}
\]

(8.3)

and where $|B_1|$ is the measure of the unit ball.

In order to extend the validity of (1.3) to a wider space, we define the mixed Sobolev space

\[
W^{\alpha,q,p}(\mathbb{R}^n) := \{ u \in L^p(\mathbb{R}^n) : |\nabla^\alpha u| \in L^q(\mathbb{R}^n) \}
\]

(8.4)

which can be easily seen to be a reflexive Banach space, under the norm $\|u\|_p + \|\nabla^\alpha u\|_q$, and obviously $W^{\alpha,q,q}(\mathbb{R}^n)$ coincides with the usual Sobolev space $W^{\alpha,q}(\mathbb{R}^n)$. It is also easy to check using standard arguments (for example as in [AF, Thm. 3.22]) that $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $W^{\alpha,q,p}(\mathbb{R}^n)$.

We record the full statement of the improved MSI, equivalent to (1.3), here:

**Theorem 3.** Let $0 < \alpha < n$ be an integer, and let $p \geq 1$. There exists $C$ such that for every $u \in W^{\alpha,q,p}(\mathbb{R}^n)$ with $\|\nabla^\alpha u\|_{n/\alpha} \leq 1$ we have

\[
\int_{\mathbb{R}^n} \exp\left[\frac{n^\alpha - 1}{n} \frac{\gamma_{n,\alpha} |u|^n}{n - \alpha} \right] \frac{dx}{1 + |u|^n} \leq C \|u\|_p^p
\]

(8.5)

where $\gamma_{n,\alpha}$ is given in (8.3) and it is sharp.

In particular, we have, for all $u \in W^{1,n,p}(\mathbb{R}^n)$ such that $|\nabla u| \leq 1$ and $p \geq 1$

\[
\int_{|u| \geq 1} \frac{\exp[\gamma_{n,1} |u|^{n/(n-1)}]}{|u|^{p/(n-1)}} dx \leq C \|u\|_p^p.
\]

(8.6)

and using this we obtain that $W^{1,n,p}$ is continuously embedded in $W^{1,n}$ for $1 \leq p < n$, and $W^{1,n}$ continuously embedded in $W^{1,n,p}$ if $p > n$ (consider the cases $|u| < 1$ and $|u| \geq 1$). Similar such embeddings can be obtained for general $\alpha$, using (8.5).

The main theorem of this section is the following:

**Theorem 4.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying $f(0) = 0$ and such that the following hold:

There exists $\mu > p$ such that $0 < F(t) := \int_0^t f(s) ds \leq \frac{t}{\mu} f(t)$ for all $t \in \mathbb{R} \setminus \{0\}$, (f1)

There exist $t_0, M_0 > 0$ such that $F(t) \leq M_0 f(t)$ for all $t \geq t_0$, (f2)

There exist $\gamma_0 \geq 0$ such that

\[
\lim_{t \to \infty} \exp[-\gamma t^{n/(n-1)}] f(t) = \begin{cases} 
0 & \text{if } \gamma > \gamma_0, \\
\infty & \text{if } \gamma < \gamma_0.
\end{cases}
\]

(f3)

\[
\lim_{t \to t_0^{-}} t^{p/(n-1)} \exp[-\gamma_0 t^{n/(n-1)}] F(t) = \infty.
\]

(f4)

Then for each $V_0 > 0$, (8.2) has a positive radial solution $u_0 \in W^{1,n,p}(\mathbb{R}^n)$ which is also a ground state solution.
In [MS2] the above theorem was proved for \( p = n \). A sufficient condition for \((f_4)\) (from l’Hospital rule) is
\[
\lim_{t \to \infty} t^{(p-1)/(n-1)} \exp[-\gamma_0 t^{n/(n-1)}] f(t) = \infty. \tag{f_5}
\]
Given any \( \gamma_0 > 0 \), a typical function \( f \) for which all of the above conditions are satisfied is given by
\[
f(t) = \begin{cases} 
  t^{-\epsilon} e^{\gamma_0 t^{n/(n-1)}} & \text{if } t \geq t_0 \\
  t^{\mu-1} & \text{if } 0 < t < t_0 
\end{cases} \tag{8.7}
\]
where \( \mu > p \), \( 0 \leq \epsilon < \frac{p-1}{n-1} \), and where \( t_0 \) is any positive number.

In general conditions \((f_1)-(f_3)\) imply
\[
F(t) \leq Ct^\mu, \quad 0 < t < 1 \tag{8.8}
\]
\[
F(t) \leq C_\gamma t^{-\frac{p}{n-1}} e^{\gamma_0 t^{n/(n-1)}}, \quad \gamma > \gamma_0, \quad t \geq 1 \tag{8.9}
\]
while \((f_4)\) can be rewritten as
\[
F(t) = C_0(t) t^{-\frac{p}{n-1}} e^{\gamma_0 t^{n/(n-1)}}, \quad \lim_{t \to +\infty} C_0(t) = +\infty. \tag{8.10}
\]

**Remark 10.** The estimate in \((8.9)\) can be refined a little: \( F(t) \leq C_\gamma(t) t^{-\frac{p}{n-1}} e^{\gamma_0 t^{n/(n-1)}} \), where \( C_\gamma(t) \to 0 \) as \( t \to +\infty \).

The proof of Theorem 4 is essentially the same as the proof of [MS2], with some small changes. We will therefore outline the main steps of the argument, highlighting the changes and the role of the crucial inequality in Theorem 8.6.

**Step 1.** Consider the functional associated to the variational formulation of \((8.2)\) i.e.
\[
I(u) = \frac{1}{n} \|\nabla u\|_n^n + \frac{V_0}{p} \|u\|_p^p - \int_{\mathbb{R}^n} F(u) dx, \quad u \in W^{1,n,p}.
\]
Any critical point of \( I(u) \) is a solution to the quasilinear equation \((8.2)\).

A solution \( u_0 \) is called a ground state solution if it satisfies
\[
I(u_0) = m := \inf \{ I(u) : u \in W^{1,n,p} \setminus \{0\}, u \text{ a solution of (8.2)} \},
\]
i.e. \( u_0 \) is a a solution with the least energy among all solutions of \((8.2)\).

**Step 2.** Observe that if \( u \) is a solution of \((8.2)\), then it must satisfy the so-called Pohozaev identity, which in this case is given as
\[
V_0 \|u\|_p^p - p \int_{\mathbb{R}^n} F(u) dx = 0. \tag{8.11}
\]
The identity can be proved by arguing along the same lines as in [BL, Proposition 1]. In the case \( u \) radial (which is the case of interest here) the proof is easier: multiply \((8.2)\) by \( r^\alpha u'(r) \) and integrate by parts. This step is not necessary, strictly speaking, but it provides a motivation for the next steps.
Step 3. Observe that there exists at least one \( u \in W^{1,n,p}(\mathbb{R}^n) \setminus \{0\} \) such that the Pohozaev identity is verified and that if there exists one such \( u_0 \) solving the constrained minimization problem

\[
I(u_0) = m_0 := \inf \left\{ I(u) : u \in W^{1,n,p} \setminus \{0\}, \text{ and (}8.11\text{) holds} \right\} \\
= \inf \left\{ \frac{1}{n} ||\nabla u||_n^n : u \in W^{1,n,p} \setminus \{0\}, \frac{p}{||u||_p} \int_{\mathbb{R}^n} F(u) dx = V_0 \right\} \leq m
\]

(8.12)

then \( \tilde{u}_0(x) = u_0(x(1-p\theta)^{-1/n}) \) is a ground state solution, where \( \theta \) is a Lagrange multiplier for the constrained problem. Hence the problem is reduced to showing that \( m_0 \) is attained at a radial function. Note that using the Pólya-Szego inequality, in the above constrained minimization problem it is enough to consider those \( u \) which are radially decreasing.

Step 4.

Define for \( D > 0 \)

\[
Q(D) := \sup \left\{ \frac{p}{||u||_p} \int_{\mathbb{R}^n} F(u) dx : ||\nabla u||_n \leq D, ||u||_p < \infty, u \neq 0 \right\}
\]

and show that for \( \gamma_0 > 0 \)

\[
\sup \{D > 0 : Q(D) < \infty\} = \left(\frac{\gamma_{n,1}}{\gamma_0}\right)^{\frac{n}{n-1}} := D^*.
\]

(8.13)

To see this, note that from (8.8), (8.9) and the improved Moser-Trudinger inequality inequality with exact growth condition (8.6), we have for each fixed \( D < D^* = (\gamma_{n,1}/\gamma_0)^{(n-1)/n} \)

\[
\int_{\mathbb{R}^n} F(u) dx \leq C \int_{|u| \leq 1} |u|^p + C_D \int_{|u| \geq 1} \frac{\exp[\gamma_{n,1}D_{n-1}^{-n/(n-1)}|u|^{n/(n-1)}]}{|u|^{p/(n-1)}} dx \leq C||u||_p^p
\]

whenever \( ||\nabla u||_n \leq D \) and \( ||u||_p < \infty \). Hence we can deduce that \( Q(D) < \infty \) when \( D < D^* \). On the other hand, if \( D = D^* \), using (8.10) and the usual Moser extremal sequence \( u_{\epsilon} \) (which is constant on \( B(0, \epsilon) \) and goes to \( +\infty \) as \( \epsilon \to 0 \)), we have that

\[
\frac{p}{||u_{\epsilon}||_p^p} \int_{\mathbb{R}^n} F(u_{\epsilon}) dx \geq \frac{p}{||u_{\epsilon}||_p^p} \int_{|x| < \epsilon} C_0(|u_{\epsilon}|) \exp[\gamma_0|u_{\epsilon}|^{n/(n-1)}] dx \to \infty \quad \text{as} \quad \epsilon \to 0^+
\]

with \( ||u_{\epsilon}||_p \leq 1 \) and \( ||\nabla u_{\epsilon}||_n = D^* \). Therefore \( Q(D^*) = +\infty \) and (8.13) holds.

Step 5. Show that for any \( V_0 > 0 \)

\[
m_0 < \left(\frac{D^*}{n}\right)^n
\]

(8.14)

To prove (8.14), first note that since \( Q(D^*) = \infty \), then there exists \( u_0 \neq 0 \) with \( ||u_0||_p < \infty \) and \( ||\nabla u||_n \leq D^* \) such that

\[
V_0 < \frac{p}{||u_0||_p^p} \int_{\mathbb{R}^n} F(u_0) dx,
\]

(8.15)

and using this estimate the proof of (8.14) is exactly the same as in [MS2, Proposition 7.2].
Step 6. Take a sequence of radially decreasing functions \( \{u_k\} \) which is minimizing for (8.12), i.e.
\[
\lim_{k \to +\infty} \frac{1}{n} \| \nabla u_k \|^n_n = m_0, \quad \|u_k\|_p = 1, \quad p \int_{\mathbb{R}^n} F(u_k) dx = V_0
\]
and such that, moreover, \( u_k \) converges weakly to \( u_0 \) in \( W^{1,n,p}(\mathbb{R}^n) \). Then prove that \( m_0 > 0 \) and
\[
\frac{1}{n} \| \nabla u_0 \|^n_n = m_0, \quad p \int_{\mathbb{R}^n} F(u_0) = V_0 \|u_0\|_p^p. \tag{8.16}
\]
The key estimate here is again based on the improved MSI (8.6). In particular, since
\[
\| \nabla u_k \|^n_n \to nm_0 < (D^*)^n = \left( \frac{\gamma_{n,1}}{\gamma_0} \right)^{n-1}
\]
then we can assume that for some \( D > nm_0 \) and for all \( k \)
\[
nm_0 \leq \| \nabla u_k \|^n_n < D^n < \left( \frac{\gamma_{n,1}}{\gamma_0} \right)^{n-1}.
\]
Hence, Remark 10 applied to \( \gamma = \gamma_{n,1}/D^{n/(n-1)} > \gamma_0 \), gives that for any \( L > 0 \)
\[
\int_{|u_k|>L} F(u_k) dx \leq \int_{|u_k|>L} C_D(|u_k|) \ exp \left[ \gamma_{n,1} D^{-n/(n-1)} |u_k|^{n/(n-1)} \right] dx \tag{8.17}
\]
where \( C_D(t) \to 0 \) as \( t \to +\infty \). This means, using (8.6), that for each \( \epsilon > 0 \)
\[
\int_{|u_k|>L} F(u_k) dx \leq C \epsilon
\]
is \( L \) is chosen large enough, independently of \( k \), and the same is true for \( u \). Arguing as in [MS2, Sect. 5], one can then show that \( \int_{\mathbb{R}^n} F(u_k) \to \int_{\mathbb{R}^n} F(u) = V_0/n \) and complete the proof arguing as in [MS2, Prop. 7.1].

Note that in the above proof we assumed \( \gamma_0 > 0 \). If \( \gamma_0 = 0 \) then \( Q(D) < \infty \) for all \( D > 0 \) hence the supremum in (8.13) is \( D^* = +\infty \) and the proof still works. Also note that the condition (9.4) is only needed to guarantee \( Q(D^*) = +\infty \), in case \( \gamma_0 > 0 \). Without assuming (9.4) one can still conclude that a radial ground state solution exists when \( 0 < V_0 < Q(D^*) \), when \( Q(D^*) < \infty \), where \( D^* \) is given still as in (8.13) (see [MS2, Thm. 7.4]).

9 Moser-Trudinger inequalities with exact growth conditions on Heisenberg group

The Heisenberg group will be denoted as \( H^n \), with elements \( x = (z,t) \in \mathbb{C}^n \times \mathbb{R} \) and endowed with the usual group law, dilation, and Haar measure \( dx = dz dt \). The norm in \( H^n \) is denoted as \(|(z,t)| = (|z|^4 + t^2)^{1/4}\) and the homogeneous dimension is \( Q = 2n + 2 \). The unit sphere is denoted as \( \Sigma = \{ x \in H^n : |x| = 1 \} \). Given any \( x \in H^n \) we will let
\[
x^* = (z^*, t^*) = \left( \frac{x}{|x|}, \frac{t}{|x|^2} \right) \in \Sigma.
\]
The sublaplacian on $\mathbb{H}^n$ is defined as

$$L = -\frac{1}{4} \sum_{j=1}^{n} (X_j^2 + Y_j^2),$$

where $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ is the usual basis of the left-invariant vector fields on $\mathbb{H}^n$, and the horizontal gradient is

$$\nabla_{\mathbb{H}} = (X_1, Y_1, \ldots, X_n, Y_n).$$

For $1 \leq p, q < \infty$, and $\alpha \geq 0$ an even integer we define mixed Sobolev space on $\mathbb{H}^n$ as

$$W^{\alpha,q,p}(\mathbb{H}^n) := \{ u \in L^p(\mathbb{H}^n) : L^{\alpha/2}u \in L^q(\mathbb{H}^n) \},$$

with the graph norm $\|u\|_{p} + \|L^{\alpha/2}u\|_{q}$. This defines a Banach space, which coincides with the usual Sobolev space $W^{\alpha,q}(\mathbb{H}^n)$ if $p = q$. Also, it’s possible to check that $C^\infty_c(\mathbb{H}^n)$ is dense in $W^{\alpha,q,p}(\mathbb{H}^n)$, by the same arguments one can use for $W^{\alpha,q,p}(\mathbb{R}^n)$ (see Section 8).

If $\alpha < Q$ the fundamental solution of the powers of the $\mathbb{H}^n$ sublaplacian, is homogeneous of degree $\alpha - Q$, smooth away from 0, and can be written as

$$L^{-\alpha/2}(x,0) = g_\alpha(\theta)|x|^\alpha - Q$$

where

$$\theta = \theta(t^*) = \arg \frac{|z|^2 + it}{|x|^2} \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].$$

The function $g_\alpha(\theta)$, which is bounded, was computed explicitly (in integral form except a few cases) in [BDR], [CT], and when $\alpha = 2$ reduces to $g_2(\theta) = 2^{n-2}\Gamma(n/2)^2\pi^{-n-1}$, derived by Folland in [Fol3].

**Theorem 5.** Let $0 < \alpha < Q$ be an even integer, and let $p \geq 1$. There exists $C$ such that for every $u \in W^{\alpha,\frac{Q}{\alpha},p}(\mathbb{H}^n)$ with $\|L^{\alpha/2}u\|_{Q/\alpha} \leq 1$ we have

$$\int_{\mathbb{H}^n} \exp \left[ p^{-\alpha/2} - 1 \right] \frac{1}{A_\alpha |u|^\frac{Q}{\alpha}} dx \leq C \|u\|_p^p$$

where

$$A_\alpha = \frac{1}{Q} \int_{\Sigma} |g_\alpha(\theta)|^{\frac{Q}{\alpha}} d\theta^*.$$ (9.3)

Moreover for every $u \in W^{\alpha,\frac{Q}{\alpha}}(\mathbb{H}^n)$ with

$$\|u\|_{Q/\alpha} + \|L^{\alpha/2}u\|_{Q/\alpha} \leq 1$$

we have

$$\int_{\mathbb{H}^n} \exp \left[ \frac{\alpha}{Q} - 2 \right] \frac{1}{A_\alpha |u|^\frac{Q}{\alpha}} dx \leq C$$

The exponential constant $A_\alpha^{-1}$ in (9.2) and (9.5) is sharp, i.e. it cannot be replaced by any larger number.

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Note that the relation between the surface measure $dx^*$ in the above (9.3) and $\theta$ is as follows [BFM, (2.4)]

$$\int_\Sigma \phi dx^* = \omega_{2n-1} \int_{-\pi/2}^{\pi/2} \phi(\cos \theta)^{n-1} d\theta,$$

where $\phi : [-\pi/2, \pi/2] \to \mathbb{R}$ is a measurable function.

In the case of subgradient, let

$$W^{1,q,p}(H^n) := \left\{ u \in L^p(H^n) : |\nabla_{H^n} u| \in L^q(H^n) \right\},$$

with the graph norm $||f||_p + ||\nabla_{H^n} u||_q$. Once again this is a Banach space, which coincides with the usual Sobolev space $W^{1,q}(H^n)$ if $p = q$, and which contains $C_c^\infty(H^n)$ as a dense subspace.

**Theorem 6.** For every $p \geq 1$, there exists $C$ such that for every $u \in W^{1,q,p}(H^n)$ with $||\nabla_{H^n} u||_Q \leq 1$ we have

$$\int_{H^n} \exp[p^{Q-1} \int \frac{1}{A} |u|^{\frac{Q}{Q-1}}] dx \leq C||u||^p_p$$

(9.6)

where

$$A = \frac{1}{Q} \left( \int_\Sigma |z^*|Q dx^* \right)^{-\frac{1}{Q-1}} = \frac{1}{Q} \left( \frac{\Gamma(\frac{Q}{2})\Gamma(n)}{2\pi^{n+1/2}\Gamma(\frac{Q-1}{2})} \right)^{\frac{1}{Q-1}},$$

(9.7)

and the exponential constant $A^{-1}$ is sharp, i.e. it cannot be replaced by any larger number.

For the computation of the constant $A$ in (9.7), see [CL].

**Proof of Theorem 5.** The inequalities are immediate consequences of Theorem 1 as applied to the operator $T$ with kernel $k(x, y) = K(y^{-1}x)$, where $K(x) = g_\alpha(\theta)|x|^{\alpha-Q}$, and with $\beta = \frac{Q}{Q-\alpha}$. Indeed, it is enough to prove the inequalities for $u$ smooth and compactly supported, in which case we can write $u = Tf$ with $f = L^\frac{\alpha}{Q} u$ in $L^{Q/\alpha}$ and compactly supported. All one needs to check is that the conditions (K1)-(K4) are verified. Estimate (K3) is obvious since $g_\alpha(\theta)$ is bounded and $|B(0, r)| = C r^Q$. Using polar coordinates one readily gets

$$\int_{r_1 < |x| < r_2} |K(x)|^\frac{Q}{Q-1} dx = A_\alpha \log \frac{|B(0, r_2)|}{|B(0, r_1)|}, \quad r_1 < r_2$$

(9.8)

which gives (K1) and (K2). Regarding (K4) we in fact have both (2.4) and (2.15) (and also (2.16)) simply from Taylor’s formula and the fact that

$$|D_z h^\ell K(x)| \leq C|x|^{\alpha-Q-|\ell|-2\ell}$$

for any multi-index $h = (h_1, ..., h_{2n})$ with $|h| = h_1 + ... + h_n$ (and the obvious notation for $D_z^{\ell}$).
The sharpness statements follow from Theorem 2 and the pointwise regularity estimates on $K$. Indeed one just has to consider the family of functions $u_\epsilon = T\psi_\epsilon$, where $\psi_\epsilon$ is defined as in (7.15).

**Proof of Theorem 6.** Let $x = (z, t)$, $y = (z', t')$. In the case of subgradient, the following representation formula was derived by [CL]:

$$u(x) = -\frac{1}{4c_Q} \int_{\mathbb{H}^n} \frac{|z|^Q - 2}{|y|^{2Q}} \nabla_{\mathbb{H}^n}(|y|^4) \cdot \nabla_{\mathbb{H}^n} F(y^{-1} x) dy$$

(9.9)

where "·" denotes the standard inner product of two vectors in $\mathbb{R}^{2n}$, $c_Q = \int_{\Sigma} |z^*|^Q dx^*$ and $u \in C_0^\infty(\mathbb{H}^n)$. Hence, we let

$$K(x) = -\frac{1}{4c_Q} \frac{|z|^Q - 2}{|x|^{2Q}} \nabla_{\mathbb{H}^n}(|x|^4), \quad x \neq 0,$$

and since $|\nabla_{\mathbb{H}^n}(|x|^4)| = 4|z||x|^2$ (see [CL, Lemma 2.1]) then

$$|K(x)| = c_Q^{-1} |z|^Q |x|^{-2} = c_Q^{-1} |z^*|^Q |x|^{1-Q} : = g(x^*)|x|^{1-Q}.$$

Therefore, arguing as in (9.8) via polar coordinates

$$\int_{r_1 < |x| < r_2} |K(x)| \frac{Q-1}{Q} dx \leq A \log \frac{|B(0, r_2)|}{|B(0, r_1)|}, \quad r_1 < r_2$$

(9.10)

with

$$A = \frac{1}{Q} \int_{\Sigma} |g(x^*)| \frac{Q-1}{Q} dx^* = \frac{1}{Q} (c_Q)^{-\frac{Q-1}{Q}} \int_{\Sigma} |z^*|^{(Q-1)} \frac{Q-1}{Q} dx^* = \frac{1}{Q} (c_Q)^{-\frac{1}{Q-1}}.$$ 

The inequality in (9.6) follows now from Theorem 1, applied to the kernel $|K(x)|$, which also satisfies (K4) since it is of the same type discussed in the previous proof.

To prove the sharpness, we use the same extremal family of functions as in [CL]. We consider the function

$$v_\epsilon(y) = \begin{cases} \log(1/|y|) & \text{for } \epsilon \leq |y| \leq 1 \\ \log(1/\epsilon) & \text{for } |y| \leq \epsilon. \end{cases}$$

Then [CL] computed that

$$||\nabla_{\mathbb{H}^n} v_\epsilon||_Q = c_Q \log \frac{1}{\epsilon},$$

(9.11)

and it is clear that $||v_\epsilon||_{\sigma Q} \leq C$. Let

$$u_\epsilon = \frac{v_\epsilon}{||\nabla_{\mathbb{H}^n} v_\epsilon||_Q},$$

so that $||\nabla_{\mathbb{H}^n} u_\epsilon||_Q = 1$

$$||u_\epsilon||_p^p \leq C \left( \log \frac{1}{\epsilon} \right)^{p/Q} \rightarrow 0, \quad \epsilon \rightarrow 0^+,$$

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and
\[ |u_\epsilon|^{\frac{Q}{Q-1}} \geq c_Q^{-\frac{1}{Q-1}} \log \frac{1}{\epsilon} \quad \text{for } |y| \leq \epsilon. \]

If \( \theta > 1 \) then
\[
\int_\mathbb{R} \exp[|u_\epsilon|^{\frac{Q}{Q-1}}] \frac{1}{1 + |u_\epsilon|^{\frac{Q}{Q-1}}} \, dy \geq \int_{|y| \leq \epsilon} \exp[\theta A^{-1} |u_\epsilon|^{\frac{Q}{Q-1}}] \frac{1}{1 + |u_\epsilon|^{\frac{Q}{Q-1}}} \, dy \geq C\epsilon^Q \frac{\exp[\theta A^{-1} |u_\epsilon|^{\frac{Q}{Q-1}}]}{1 + C\left(\log \frac{1}{\epsilon}\right)^{p/Q}} \]
\[
\geq C\epsilon^{(1-\theta)Q} \left(\log \frac{1}{\epsilon}\right)^{-p/Q} \rightarrow \infty
\]
as \( \epsilon \to 0^+ \).

\[
\square
\]

10 Masmoudi-Sani inequalities on Riemannian manifolds with negative curvature

Let \( M \) be a complete, connected, smooth Riemannian manifold with metric tensor \( g \). The geodesic distance between two points \( x, y \in M \) is denoted by \( d(x, y) \), which gives a metric in \( M \). The Riemannian measure \( \mu \) associated to \( g \) is defined in local coordinates as \( d\mu = \sqrt{|g|} \, dm \), where \( |g| = \det(g^{ij}) \), \( g = (g^{ij}) \), and \( dm \) is the Lebesgue measure.

For vector fields \( Z, W \) we will let
\[
\langle Z, W \rangle = g(Z, W), \quad |Z| = \langle Z, Z \rangle^{1/2}.
\]

Covariant derivatives of order \( k \) of a smooth function \( u \) are denoted as \( \nabla^k u \), and their components in a local chart are denoted as \( (\nabla^k u)_{j_1...j_k} \). In particular \( \nabla^0 u = u \), \( (\nabla^1 u)_j = \partial_j u \).

Define
\[
|\nabla^k u|^2 = g^{i_1...i_k} (\nabla^k u)_{i_1...i_k} (\nabla^k u)_{j_1...j_k}
\]
The gradient and the Laplacian of a smooth function \( u \) on \( M \) are given as
\[
\nabla u = g^{ij} \partial_j u \partial_i, \quad \Delta u = g^{ij} (\nabla^2 u)_{ij} = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j u \right).
\]

and
\[
|\nabla u|^2 = g(\nabla u, \nabla u) = g_{ij} (\nabla u)^i (\nabla u)^j = g^{ij} \partial_i u \partial_j u = |\nabla^1 u|^2.
\]

Let us define
\[
D^\alpha = \begin{cases} (-\Delta)^{\frac{\alpha}{2}} & \text{for } \alpha \text{ even} \\ \nabla(-\Delta)^{\frac{\alpha-1}{2}} & \text{for } \alpha \text{ odd} \end{cases}
\]

The sectional curvature in the metric \( g \) will be denoted as \( K \) and the Ricci curvature as \( \text{Ric} \).

For \( u \in C^\infty \) define \( \|u\|_{W^{k,q}} = \sum_0^k \left( \int_M |\nabla^k u|^q d\mu \right)^{1/q} \), and let \( C^{k,q}(M) = \{ u \in C^\infty(M) : \|u\|_{W^{k,q}} < \infty \} \). The Sobolev space \( W^{k,q}(M) \) is defined to be the completion of \( C^{k,q}(M) \),
relative to the norm $\| \cdot \|_{W^{k,q}}$, viewed as a subspace of $L^q(M)$. Endowed with the obvious norm, still denoted as $\| \cdot \|_{W^{k,q}}$, the space $W^{k,q}$ is a Banach space. The Sobolev space $W^{k,q}_0(M)$ is the closure of $C^\infty(M)$ in $W^{k,q}(M)$. As it turns out $W^{k,q}_0(M) = W^{k,q}(M)$ for $k = 1$, and for $k \geq 2$ if $|\nabla^j\text{Ric}| \leq C$ for all $j \leq k - 2$ (see e.g. [H]).

**Theorem 7.** Let $(M,g)$ be an $n$-dimensional, complete, simply connected, Riemannian manifold such that $-b^2 \leq K \leq -a^2$ for some $a,b > 0$. For any integer $0 < \alpha < n$, there exists a constant $C$ such that for all $u \in W^{\alpha,n/\alpha}_0(M)$ with

$$\|D^\alpha u\|_{n/\alpha}^{n/\alpha} \leq 1$$

we have

$$\int_M \exp\left[\frac{\alpha}{n-\alpha} - 1\right]\frac{[\gamma_{n,\alpha}|u|^n]}{1 + |u|^{n/\alpha}} d\mu \leq C\|u\|_{n/\alpha}^{n/\alpha},$$

where $\gamma_{n,\alpha}$ is defined in (8.3) and it is sharp.

The above theorem in the special case $\alpha = 1$ and $K = -1$ (hyperbolic space) was derived in [LTZ]. The classical version of the above inequality, without the norm on the right-hand side, and without denominator, was derived in [BS], extending earlier results in [FM3] on the hyperbolic space. Observe that in [BS] the result is stated under $\text{Ric} \geq -(n-1)b^2$, $K \leq -a^2$, for some $a,b > 0$, which is however equivalent to $-b^2 \leq K \leq -a^2$ for some $a,b > 0$.

**Remark 11.** A version of Theorem 7 can be given with an inequality like (8.5), for any $p \geq 1$. For simplicity we stated it only in the case $p = n/\alpha$.

**Proof:** It suffices to prove (10.3) for $u \in C^\infty_c(M)$. If $G_\alpha(x,y)$ denotes the Green function of $D^\alpha = (-\Delta)^{\alpha/2}$ for $\alpha$ even, then for each fixed $x \in M$

$$u(x) = \int_M G_\alpha(x,y)D^\alpha u(y)d\mu(y),$$

for $\alpha$ even, and

$$u(x) = \int_M \langle \nabla_y G_{\alpha+1}(x,y), \nabla_y D^{\alpha-1}u(y) \rangle d\mu(y),$$

for $\alpha$ odd, where $\nabla_y$ denotes the gradient in the $y$ variable.

We let

$$K_\alpha(x,y) = \begin{cases} G_\alpha(x,y) & \text{for } \alpha \text{ even} \\ \nabla_y G_{\alpha+1}(x,y) & \text{for } \alpha \text{ odd.} \end{cases}$$

Note that $|K_\alpha(x,y)|$ is symmetric in $x,y$ for $\alpha$ even, but not necessarily so for $\alpha$ odd.

We would like to apply Theorem 1, where $T$ is the integral operator defined as in (10.5), (10.6). One observation to make here is that when $\alpha$ is odd the inner product in (10.6) and the norms $|K_\alpha(x,y)|, |D^\alpha u(y)|$ are with respect to the metric $g$, and not the standard Euclidean metric, in any given chart, so Remark 2 following Theorem 1, does not apply directly. However, we can use a single coordinate chart $(M,\phi)$, a Hadamard manifold, and
in these coordinates write the metric tensor as \((g_{ij}(y)) = R^T_y R_y\), for some invertible matrix \(R_y\), for each \(y \in M\). Hence the operator in (10.6) for \(\alpha\) odd can be written as

\[
u(x) = T_0 f(x) = \int_{\mathbb{R}^n} R_y K_\alpha(x, y) \cdot R_y f(y) \sqrt{|g(y)|} dm(\xi), \tag{10.8}\]

where \(y = \phi^{-1} \xi\) and \(f = \nabla_y D^{\alpha - 1} u(y) = D^\alpha f(y)\), and Remark 2 can be applied to the kernel \(R_y K_\alpha\) and the function \(R_y f\), which clearly satisfy \(|K_\alpha| = ((R_y K_\alpha) \cdot (R_y K_\alpha))^{1/2}\), and \(|f| = |D^\alpha u| = ((R_y f) \cdot (R_y f))^{1/2}\). With this observation in mind we will then just check all the Riesz-like kernel conditions using the norm in the metric \(g\), when \(\alpha\) is odd, and the usual absolute value when \(\alpha\) is even. We will show that (K1), (K2) hold, with \(A_0 = \gamma_{n, \alpha}^{-1}\) and \(A_\infty = 0\). We will also show that the first estimate of (K3) holds for \(V_x(d(x, y) \leq 1\) and that (K3) holds. Finally, we will show that (K4) holds whenever \(V(r) \leq 1\).

Sandeep and Bertrand [BS, Lemma 4.1, Theorem 4.2] proved that, given the assumptions on the curvature

\[
|K_\alpha(x, y)| \leq \begin{cases} 
    \left|B_1\right| \gamma_{n, \alpha}^{-\frac{n-\alpha}{2n}} d(x, y)^{\alpha-n} + C d(x, y)^{\alpha-n+\frac{1}{2}} & \text{if } d(x, y) < 1 \\
    C e^{-\delta d(x, y)} & \text{if } d(x, y) \geq 1,
\end{cases} \tag{10.9}
\]

for some \(C > 0\) and \(\delta > 0\), and that there are \(C, \delta' > 0\) such that for any \(x \in M\), and any \(\sigma \in (0, 1)\), and for any \(x \in M\)

\[
|K_\alpha|^*(x, t) \leq \begin{cases} 
    \gamma_{n, \alpha}^{-\frac{n-\alpha}{2n}} t^{-\frac{n-\alpha}{2n}} + C t^{-\frac{n-\alpha}{2n} + \delta'} & \text{for } 0 < t \leq 1 \\
    C(\sigma) t^{-\sigma} & \text{for } t > 1,
\end{cases} \tag{10.11}
\]

where \(C(\sigma)\) is a positive constant depending on \(\sigma\) (see Remark 12 below regarding these estimates).

By the classical comparison theorems, denoting \(V^\lambda(r)\) to be the volume of (any) ball of radius \(r\) in the \(n\)-dimensional space form of constant sectional curvature \(\lambda\), we have (since \(\text{Ric} \geq -(n-1)\lambda^2\))

\[
V^0_x(r) = |B_1| r^n \leq V_x(r) \leq V^{-b^2}(r) = n |B_1| b^{-n} \int_0^{b r} (\sinh u)^{n-1} du \tag{10.13}
\]

so that if \(r_0 > 0\) there is \(C(r_0)\) such that for all \(r \in [0, r_0]\) we have

\[
|B_1| r^n \leq V_x(r) \leq C(r_0) r^n. \tag{10.14}
\]

Hence for \(V_x(d(x, y)) \leq 1\) we have \(d(x, y) \leq r_0 := |B_1|^{1/n} \) and \(d(x, y) \geq d_0 V_x(r)^{1/n}\), some \(\delta_0 > 0\) (independent of \(x\), and

\[
|K_\alpha(x, y)| \leq C d(x, y)^{\alpha-n} \leq C V_x(d(x, y))^{-\alpha} \tag{10.15}
\]

which is the first condition in (K3), for balls with small volumes. Also, with the definition of \(r_x(\tau)\) given in (3.14) we have \(0 < \delta_0 \leq r_x(1) \leq r_0\), and (10.9), (10.10) imply that \(|K_\alpha(x, y)| \leq C\) if \(V(d(x, y)) \geq 1\) (since in that case \(d(x, y) \geq r_x(1) \geq \delta_0\)).
Clearly (10.11) implies \( [K1] \) and hence, together with (10.15), it also implies \( [K1] \) with \( A_0 = \gamma_{n,\alpha}^{-1} \) (see Remark 14 in Appendix A). Additionally, \( [K2] \) holds with \( A_\infty = 0 \) (critical integrability), then \( [K2] \) also holds with \( A_\infty = 0 \), since \( |K_\alpha(x,y)| \) bounded when \( V_x(d(x,y)) \geq 1 \) (again by Remark 14).

The first estimate in \( (K3') \), for \( \sup_x (|K_\alpha(x, \cdot)|^*) (t) \), follows directly from (10.11), (10.12). The second estimate in \( (K3') \), for \( \sup_y (|K_\alpha(\cdot, y)|^*) (t) \), follows from the symmetry in \( x, y \) of \( |K_\alpha(x,y)| \), when \( \alpha \) is even, however it needs to be justified when \( \alpha \) is odd, since symmetry is lost, in general. It’s enough to argue that the estimates (10.11), (10.12) are also valid for \( (|K_\alpha(\cdot, y)|^*) (t) \), following the proof of (10.11), (10.12) given in [BS]. Estimate (10.11) follows directly from (10.9), however the same method does not work for (10.12) (see Remark 12).

The way this was done in [BS] was to first prove

\[
|G_2|^*(x,t) \leq C \begin{cases} t^{-\frac{n-2}{n}} & \text{for } 0 < t \leq 1 \\ t^{-1} & \text{for } t > 1, \end{cases} \tag{10.16}
\]

and then prove (10.12) for all \( \alpha \) by induction, via O’Nei’s Lemma.

Estimate (10.16) for \( t \leq 1 \) follows clearly from (10.11), whereas the one for \( t > 1 \) is a consequence of the following distribution function estimate ([BS, eq. (3.25))]:

\[
\mu \{ y : G_2(x,y) > s \} \leq \frac{C_{n,\alpha}}{s}, \quad s > 0 \tag{10.18}
\]

(where \( C_{n,\alpha} \) is some explicit constant). Estimate (10.17) follows from the classical gradient estimate for positive harmonic functions (see e.g. [Li, Thm. 6.1]) applied to \( G_2(x,y) \), in the \( y \) variable inside \( B(y,d(x,y)) \)

\[
|\nabla_y G_2(x,y)| \leq C \left( 1 + \frac{1}{d(x,y)} \right) G_2(x,y) \leq C \begin{cases} d(x,y)^{1-n} & \text{if } d(x,y) \leq 1 \\ G_2(x,y) & \text{if } d(x,y) > 1 \end{cases} \tag{10.19}
\]

combined with (10.14) and (10.18). It is clear then that since the bound in (10.19) is symmetric in \( x \) and \( y \) then one can get the same estimate in (10.17), and hence \( (K3') \), for \( (|\nabla_y G_2(\cdot, y)|^*) (t) \), for any \( y \in M \).

Now we will prove that \( (K4) \) holds for \( V_x(r) \leq 1 \). Let us then assume \( V_x(r) \leq 1 \) and \( V_x(R) \geq (1 + \delta) V_x(r), x' \in B(x,r) \) and \( y \in B(x,R') \). We then have \( r \leq r_x(1) \) since on a Riemannian manifold \( V_x(r) \) is strictly increasing, and we know that \( r_x(1) \leq r_0 = |B_1|^{1/n} \). Assume first \( R > 2r_0 \), in which case \( d(x',y) \geq d(x,y) - d(x,x') \geq R - r \geq r_0 \geq r_x(1) \), and since \( (K2) \) holds (with \( A_\infty = 0 \)) we have

\[
\int_{d(x,y) \geq R} |K_\alpha(x',y) - K_\alpha(x,y)|^{\frac{n}{n-\alpha}} d\mu(y) \leq C \int_{d(x,y) \geq R} |K_\alpha(x',y)|^{\frac{n}{n-\alpha}} d\mu(y) + C \int_{d(x,y) \geq R} |K_\alpha(x,y)|^{\frac{n}{n-\alpha}} d\mu(y)
\]

\[
\leq C \int_{d(x,y) \geq r_x(1)} |K_\alpha(x',y)|^{\frac{n}{n-\alpha}} d\mu(y) + C \int_{d(x,y) \geq r_x(1)} |K_\alpha(x,y)|^{\frac{n}{n-\alpha}} d\mu(y) \leq C. \tag{10.20}
\]
On the other hand, if \( R \leq 2r_0 \) the following pointwise condition holds for all \( x \in M \) all if \( r, R \) such that \( V_x(r) \leq 1 \) and \( V_x(R) \geq (1 + \delta)V_x(r) \):

\[
|K_\alpha(x', y) - K_\alpha(x, y)| \leq C_\delta d(x, x')d(x, y)\alpha^{n-1}, \quad \forall x' \in B(x, r), \; \forall y \notin B(x, R), \tag{10.21}
\]

from which \((2.3)\) and hence \((K4)\) follow. We will prove the above only for \( \alpha = 1, 2 \); the proof for general \( \alpha \) is obtained easily using the integral formula that gives of \( K_\alpha \) in terms of \( K_{\alpha-1} \), and the details are left to the reader.

First, note that the condition \( V_x(R) \geq (1 + \delta)V_x(r) \) implies the existence of \( c_\delta > 0 \), independent of \( x, r, R \) such that \( R \geq (1 + c_\delta)r \). Indeed from the standard comparison theorems \( V_x(r)/V^{n-2}(r) \) is decreasing in \( r \) (and tends to 1 as \( r \to 0 \)). Hence, since \( R > r \), \( V_x(R)/V^{n-2}(R) \leq V_x(r)/V^{n-2}(r) \) and from this inequality we get

\[
V_x(R) - V_x(r) \leq V^{n-2}(R) - V^{n-2}(r) = n|B_1|b^{-n}\iint_{br} (\sinh u)^{n-1} du \leq C_0(R^n - r^n) \tag{10.22}
\]

with \( C_0 \) independent of \( x \), where in the last inequality we used \( R \leq 2r_0 \). The result follows since \( \delta|B_1|r^n \leq \delta V_x(r) \leq V_x(R) - V_x(r) \).

Integrating along the geodesic from \( x \) to \( x' \), inside \( B(x, r) \), we get

\[
|G_2(x', y) - G_2(x, y)| \leq d(x, x') \sup_{z \in B(x, r)} |\nabla_z G_2(z, y)| \leq Cd(x, x') \sup_{z \in B(x, r)} d(z, y)^{1-n} \leq Cd(x, x')\left(\frac{c_\delta}{1 + c_\delta} d(x, y)\right)^{1-n} = C_\delta d(x, x')d(x, y)^{1-n}, \tag{10.23}
\]

which gives \((10.21)\) for \( \alpha = 2 \).

To prove \((10.21)\) for \( \alpha = 1 \), for each \( y \) in a given coordinate chart let \((g_{ij}(y)) = R_y^T R_y\) for some invertible matrix \( R_y \), and let \( U_j(z, y) = (R_y \nabla G(z, y))^j \), so that

\[
|\nabla_y G(x', y) - \nabla_y G(x, y)|^2 = \sum_{j=1}^n (U_j(x', y) - U_j(x, y))^2 \leq d(x, x')^2 \sum_{j=1}^n \sup_{z \in B(x, r)} |\nabla_z U_j(z, y)|^2. \tag{10.24}
\]

Now note that for each \( y \) in the chart and any \( j \), the function \( U_j(z, y) \) is harmonic in \( z \) inside \( B(x, d(x, y)) \), in particular if \( \sigma \in (0, 1) \) is so that \( r < \sigma d(x, y) < d(x, y) \) then

\[
\min_{z \in B(x, \sigma d(x, y))} U_j(z, y) = U_j(z_y, y), \quad \text{some } z_y \in \partial B(x, \sigma d(x, y)). \tag{10.25}
\]

It’s easy to see, using the definition of Green function, that \( U_j(\cdot, y) \) cannot be constant in \( B(x, \sigma d(x, y)) \), hence one can apply the gradient estimate to the positive harmonic function \( U_j(z, y) - U_j(z_y, y) \) on the ball \( B(x, \sigma d(x, y)) \) and obtain

\[
|\nabla_z U_j(z, y)| = |\nabla_z (R_y \nabla G(z, y))| \leq C \left(1 + \frac{1}{d(x, y)}\right) |U_j(z, y) - U_j(z_y, y)| \leq Cd(x, y)^{-1}d(z, y)^{1-n} + d(z_y, y)^{1-n} \leq Cd(x, y)^{-n}, \tag{10.26}
\]

where in the second inequality we used \( |U_j(z, y)| \leq |\nabla G(z, y)| \leq Cd(z, y)^{1-n} \) (for example from \((10.9); (10.10)\)). Combining \((10.24)\) and \((10.26)\) gives \((10.21)\) for \( \alpha = 1 \).
To prove the sharpness, consider a smoothing of the following function
\[ v_\epsilon(y) = \begin{cases} \log \frac{1}{d(x_0,y)} & \text{for } \epsilon \leq d(x_0,y) \leq 1 \\ \log \frac{1}{\epsilon} & \text{for } d(x_0,y) \leq \epsilon, \end{cases} \]
where \( x_0 \) is a fixed point in \( M \). Then by [F, Proposition 3.6], we have
\[ \|\nabla^\alpha v_\epsilon\|_{n/\alpha} = \frac{\gamma_{n,\alpha}}{n} \left( \log \frac{1}{\epsilon} \right)^{n/\alpha} + O(1). \] (10.27)
Also, note that \( \|v_\epsilon\|_{n/\alpha} \leq C \). Take \( u_\epsilon = v_\epsilon(\|\nabla^\alpha v_\epsilon\|_{n/\alpha})^{-1} \). We have
\[ \|\nabla^\alpha u_\epsilon\|_{n/\alpha} \leq 1, \quad \|u_\epsilon\|_{n/\alpha} \leq C(\log \frac{1}{\epsilon})^{-1}, \]
and
\[ |u_\epsilon|^{n/\alpha} \geq \gamma_{n,\alpha}^{-1} \log \frac{1}{\epsilon} \quad \text{for } d(x_0,y) \leq \epsilon. \]

For the sharpness of the exponential constant, we take \( \theta > 1 \) and estimate
\[
\int_\mathcal{M} \exp \left[ \frac{n/\alpha - 1}{1 + |u_\epsilon|^{n/\alpha}} \right] d\mu(y) \geq \int_{d(x_0,y) \leq \epsilon} \exp \left[ \frac{\theta \gamma_{n,\alpha} |u_\epsilon|^{n/\alpha}}{1 + |u_\epsilon|^{n/\alpha}} \right] dy \\
\geq C e^n \frac{\theta \log \frac{1}{\epsilon} + C}{1 + C \log \frac{1}{\epsilon}} \to \infty \quad \text{as } \epsilon \to 0^+. \]

**Remark 12.** Estimate (10.10) can be refined as
\[
C_{b,n} \rho_{\alpha/2} e^{-b(n-1)d(x,y)} \leq |K_\alpha(x,y)| \leq C_{a,n} \rho_{\alpha/2} e^{-a(n-1)d(x,y)}, \quad d(x,y) \geq 1 \quad (10.28)
\]
where \( C_{a,n}, C_{b,n} \) are two positive constants depending on \( b, n \). This bound can be obtained directly from the heat kernel comparison theorem due to Debiard-Gaveau-Mazet [DGM, Théorème 1]
\[
p^{(-b^2)}(t, x, y) \leq p(t, x, y) \leq p^{(-a^2)}(t, x, y), \quad t > 0, x, y \in \mathcal{M} \quad (10.29)
\]
where \( p(t, x, y) \) is the heat kernel for the Laplace-Beltrami operator on \( \mathcal{M} \) and \( p^{(\lambda)}(t, x, y) \) the one on the space form of constant sectional curvature \( \lambda \).

We note that when \( \alpha = 2 \) the above refined bound is not enough to prove (10.17), unless \( a \) and \( b \) are close enough. In fact (10.17) suggests a bound of type
\[
|G_2(x, y)| \leq C V_x \left( d(x, y) \right)^{-1}, \quad (10.30)
\]
which is certainly true when \( K \) is constant, but not known in general.

Also, it is possible to refine (10.12) as follows
\[
|K_\alpha|^n(x, t) \leq C \left( \log t \right)^{\frac{n}{2} - 1} t, \quad t > 1 \quad (10.31)
\]
with the same strategy as in [BS, Thm. 4.2], using induction and O’Neil’s Lemma.
Appendices

A Proof of the equivalency of (K1), (K2) and (K3), (K2')

It is enough to prove that assuming (K3), there exists \( C \) such that

\[
\left| \int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) - \int_{V_x(r_2)} (k^*(x,t))^\beta dt \right| \leq C \quad \forall x \in M, \tag{A.1}
\]

where \( 0 < V_x(r_1) < V_x(r_2) \).

Fix \( x \in M \). By the same argument used to find \( E_\tau, F_\tau \) in (3.11), (3.13), we can find two sets \( H_i, i = 1, 2 \) such that

\[
\bigg\{ \{ y : |k(x,y)| > k^*(x,V_x(r_i)) \} \subseteq H_i \subseteq \{ y : |k(x,y)| \geq k^*(x,V_x(r_i)) \} \bigg\}
\]

\[
\mu(H_i) = V_x(r_i),
\]

Note that \( H_1 \subseteq H_2 \) by construction and the fact that \( k^*(x,V_x(r_1)) \geq k^*(x,V_x(r_2)) \).

We can write

\[
\int_{V_x(r_2)} (k^*(x,t))^\beta dt = \int_{H_2 \setminus H_1} |k(x,y)|^\beta d\mu(y) \leq \int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y)
\]

\[
+ \int_{(H_2 \setminus H_1) \cap B(x,r_1)} |k(x,y)|^\beta d\mu(y) + \int_{(H_2 \setminus H_1) \cap B(x,r_2)^c} |k(x,y)|^\beta d\mu(y)
\]

\[
\leq \int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) + V(r_1)(k^*(x,V_x(r_1)))^\beta + CV_x(r_2) \sup_{d(x,y) \geq r_2} \frac{1}{V_x(d(x,y))}
\]

\[
\leq \int_{r_1 \leq d(x,y) \leq r_2} |k(x,y)|^\beta d\mu(y) + C,
\]

where for the last inequality we used (K3) and (K3') (which follows from (K3)).

Remark 13. If \( A_\infty = 0 \) then (K1) implies (K1') provided that the first inequality in (K3) holds for \( V_x(d(x,y)) \leq 1 \), and that (K3') holds. Indeed, we argue as in (A.2) but we estimate

\[
\int_{(H_2 \setminus H_1) \cap B(x,r_2)^c} |k(x,y)|^\beta d\mu(y) \leq \mu(H_2 \setminus H_1) \sup_{r_2 \leq d(x,y) \leq r_x(1)} |k(x,y)|^\beta
\]

\[
+ \int_{d(x,y) \geq r_x(1)} |k(x,y)|^\beta d\mu(y) \leq C \tag{A.3}
\]
Note that if $k$ is symmetric, then we do not even need to assume (K3) since it follows from the above hypothesis. Indeed critical integrability implies, for all $s > 0$,

$$
\mu\{ y : |k(x, y)| > s, d(x, y) \geq r_x(1) \} \leq Cs^{-\beta}
$$

(A.4)

and the validity of (K3) for volumes less then or equal 1 implies

$$
\mu\{ y : |k(x, y)| > s, d(x, y) < r_x(1) \} \leq Cs^{-\beta}
$$

(A.5)

from which it follows $k^*_1(t) = k^*_2(t) \leq Ct^{-1/\beta}$.

Next let $\tilde{k}(x, y) = k(x, y)\chi_{(r_1 \leq d(x, y) \leq r_2)}$. Then clearly $\tilde{k}^*(x, t) \leq k^*(x, t)$, and by (K3) we have $|\tilde{k}(x, y)|^\beta \leq CV_x(r_1)^{-1}$, hence

$$
\tilde{k}^*(x, t)^\beta \leq CV_x(r_1)^{-1}.
$$

So we have that

$$
\int_{r_1 \leq d(x, y) \leq r_2} |k(x, y)|^\beta d\mu(y) \leq \int_0^{V_x(r_2)} |\tilde{k}^*(x, t)|^\beta dt = \int_{V_x(r_1)}^{V_x(r_2)} |\tilde{k}^*(x, t)|^\beta dt + \int_0^{V_x(r_1)} |\tilde{k}^*(x, t)|^\beta dt \leq \int_{V_x(r_1)}^{V_x(r_2)} |k^*(x, t)|^\beta dt + C \int_0^{V_x(r_1)} (V_x(r_1))^{-1} dt \leq \int_{V_x(r_1)}^{V_x(r_2)} |k^*(x, t)|^\beta dt + C. \tag{A.6}
$$

which shows that (K1) and (K2) follows from (K1') and (K2'), assuming (K3).

**Remark 14.** Under the assumption that the first inequality in (K3) holds when $V_x(d(x, y)) \leq 1$ the argument in (A.6) is still valid, to show how (K1) follows from (K1'). If, additionally, (K2') holds with $A_\infty = 0$ and $k$ is bounded for large volumes, then (K2') holds, with $A_\infty = 0$. Indeed, if $|k(x, y)| \leq C_0$ for $V_x(d(x, y)) \geq 1$ then taking $r_1 = r_x(1)$ we get $\tilde{k}^*(x, t)^\beta \leq C_0$ and as in (A.6) we deduce

$$
\int_{d(x, y) \geq r_x(1)} |k(x, y)|^\beta d\mu(y) \leq \int_1^{\infty} (k^*(x, t))^\beta dt + \int_0^1 C_0^\beta du \leq C. \tag{A.7}
$$

**B** Proof of (4.19), (4.20)

First note that $V_x(R_{j+1}) = e^{\beta(j+1)} = e^{\beta}V_x(R_j)$. If $j \leq N - 1$ then $r_j \leq R_j$, hence $V_x(r_j) \leq V_x(R_j)$. From (4.14) we have

$$
V_x(R_N) \leq V_x(r_N) \leq e^{\beta}V_x(r_N)
$$
and
\[(e^{\beta'} - 1)V_x(R_{N-1}) \leq V_x(R_N) - V_x(R_{N-1}) \leq V_x(r_N) - V_x(r_{N-1}) \leq V_x(r_N) \leq e^{\beta'}V_x(R_N) = e^{2\beta'}V_x(R_{N-1}).\]

Now let us prove that for any \( j = 0, 1, ..., N-1 \) there is \( C > 0 \) such that \( V_x(r_j) \geq CV_x(R_j) \). Clearly this is true if \( r_j = R_j \). If \( r_j < R_j \) then, since \( r_0 = R_0 \), there is \( i \) with \( 0 \leq i < j \), \( r_i = R_i \) and \( r_k < R_k \) for \( i < k \leq j \).

If \( i < j_1 < j \) then from Remark \# following the definition of \( r_j \) in (4.16), and using conditions (K1), (K2) we get
\[
(j_1 - i)\beta'A_0 = \int_{r_i \leq d(x,y) \leq r_{j_1}} |k(x,y)|^{\beta}d\mu(y) \leq A_0 \log \frac{V_x(r_{j_1})}{V_x(r_i)} + B, \tag{B.1}
\]
and
\[
(j - j_1)\beta'A_\infty = \int_{r_{j_1} \leq d(x,y) \leq r_j} |k(x,y)|^{\beta}d\mu(y) \leq A_\infty \log \frac{V_x(r_j)}{V_x(r_{j_1})} + B, \tag{B.2}
\]
therefore
\[
V_x(r_j) \geq e^{-\frac{\beta'}{A_\infty} - \frac{\beta}{A_0}} e^{\beta'(j-i)}V_x(r_{j_1}) = e^{-\frac{\beta'}{A_0} - \frac{\beta}{A_\infty}} e^{\beta'(j-i)}V_x(R_j) = e^{-\frac{\beta'}{A_0} - \frac{\beta}{A_\infty}} V_x(R_j). \tag{B.3}
\]
If \( j_1 \leq i \) then the proof is identical except we only use (B.2) with \( i \) in place of \( j_1 \), and if \( j_1 \geq j \) we only use (B.1) with \( j \) in place of \( j_1 \).

We need to show now that there is \( C > 0 \) such that \( V_x(r_{j+1}) - V_x(r_j) \geq CV_x(R_j) \) for \( j = 0, 1, 2, ..., N-2 \). If \( r_{j+1} = R_{j+1} \) then
\[
V_x(r_{j+1}) - V_x(r_j) = V_x(R_{j+1}) - V_x(r_j) \geq V_x(R_{j+1}) - V_x(R_j) = (e^{\beta'} - 1)V_x(R_j).
\]
If \( r_{j+1} < R_{j+1} \), then by definition of \( r_j \) and Remark \# we have that
\[
\int_{r_j \leq d(x,y) \leq r_{j+1}} |k(x,y)|^{\beta}d\mu(y) \geq \beta' \min\{A_0, A_\infty\}.
\]
On the other hand, we can use condition (K3) to get
\[
\int_{r_j \leq d(x,y) \leq r_{j+1}} |k(x,y)|^{\beta}d\mu(y) \leq \frac{B}{V_x(r_j)}(V_x(r_{j+1}) - V_x(r_j))
\]
and therefore,
\[
V_x(r_{j+1}) - V_x(r_j) \geq \frac{\beta' \min\{A_0, A_\infty\}}{B} V_x(r_j) \geq \frac{\beta' \min\{A_0, A_\infty\}}{B} CV_x(R_j). \tag{B.4}
\]
\[\square\]
C Proof of Lemma 2

Suppose first that \( \lambda_k = 1 \) for all \( k \). We can assume WLOG that the sequences \( a = \{a_k\} \) in the definition of \( \mu(h) \) in (4.4) are nonnegative.

It is enough to show the result for \( h \geq 1 \). Indeed, if \( 0 < h_0 \leq h \leq 1 \) and \( \|a\|_1 = h \), then we have \( \|a\|_{\beta'} \leq \|a\|_1 = h \), and \( \mu(h) = h^p \mu(1) \) so (4.5) follows in this case.

By dilation it’s easy to see that \( \mu(h) \) increases as \( h > 0 \) increases, so it is enough to show that for any integer \( N \geq 1 \), the following holds

\[
C e^{\beta' N} \frac{N^p}{N^{p/\beta'}} \leq \mu(N^{1/\beta}) \leq C' e^{\beta' N} \frac{N^p}{N^{p/\beta'}}.
\]  

(C.1)

Consider the sequence \( a_k = N^{-1/\beta'} \) for \( k = 0, \ldots, N - 1 \), and \( a_k = 0 \) otherwise. Then \( \|a\|_1 = N^{1/\beta}, \|a\|_{\beta'} = 1 \) and

\[
\mu(N^{1/\beta}) \leq \sum_{k=0}^{\infty} a_k^{p/\beta'} e^{\beta' k} = N^{-p/\beta'} e^{N\beta'} - e^{\beta'} e^{\beta' N} - e^{\beta'} e^{\beta' N} \leq C e^{N\beta'} \frac{N^p}{N^{p/\beta'}}.
\]

So we are left to prove that for any nonnegative sequence such that

\[
\|a\|_1 = N^{1/\beta}, \|a\|_{\beta'} \leq 1 \tag{C.2}
\]

we have

\[
\|a_k^{p/\beta'} e^k\|_{\beta'} = \sum_{k=0}^{\infty} a_k^{p/\beta'} e^{\beta' k} \geq C e^{\beta' N} \frac{N^p}{N^{p/\beta'}} \tag{C.3}
\]

If

\[
a_{N-1} \geq \frac{1}{2\beta} N^{-1/\beta'} \tag{C.4}
\]

then (C.3) follows using

\[
\|a_k^{p/\beta'} e^k\|_{\beta'} \geq a_{N-1}^{p/\beta'} e^{\beta' (N-1)} \geq (2\beta)^{-p} e^{-\beta'} e^{\beta' N} \frac{N^p}{N^{p/\beta'}} \tag{C.5}
\]

Assume then that

\[
a_{N-1} < \frac{1}{2\beta} N^{-1/\beta'} \tag{C.6}
\]

and let \( \Lambda \) be defined so that

\[
\Lambda^p = \|a_k^{p/\beta'} e^k\|_{\beta'} \frac{N^p}{e^{\beta' N}} \tag{C.7}
\]

Let us also define \( a_{-1} = 0 \) so that the proof below works out also for \( N = 1 \).

On one hand we have, from Hölder’s inequality and (C.2)

\[
\left( \sum_{j=0}^{N-2} a_j \right)^\beta \leq (N - 1) \left( \sum_{j=0}^{N-2} a_j^{\beta'/\beta} \right)^{\beta/\beta'} \leq N - 1 \tag{C.8}
\]
On the other hand,

\[ \sum_{j=0}^{N-2} a_j = N^{1/\beta} - a_{N-1} - \sum_{j=N}^{\infty} a_j \]  

(C.9)

since \( a_j \leq \|a_k^{p/\beta''}e^{\beta''/p}e^{-j\beta'/p} \) we have

\[ \sum_{j=N}^{\infty} a_j \leq \sum_{j=N}^{\infty} \|a_k^{p/\beta''}e^{\beta''/p}e^{-j\beta'/p} \|_{\beta''} = \|a_k^{p/\beta''}e^{\beta''/p} \|_{\beta''} \frac{e^{-N\beta'/p}}{1 - e^{-\beta'/p}} = C_0 N^{-1/\beta'} \]  

(C.10)

so that from (C.9) and (C.6) we get

\[ \sum_{j=0}^{N-2} a_j \geq N^{1/\beta} - \frac{1}{2\beta} N^{1/\beta'} - C_0 N^{-1/\beta'} = N^{1/\beta} \left( 1 - \left( \frac{1}{2\beta} + C_0 \Lambda \right) N^{-1} \right) \]  

(C.11)

If \( \frac{1}{2\beta} + C_0 \Lambda \geq 1 \) then \( \Lambda \geq C_0^{-1} \left( 1 - \frac{1}{2\beta} \right) > 0 \) and we are done. If instead \( \frac{1}{2\beta} + C_0 \Lambda < 1 \) then using the inequality \((1 - x)\beta > 1 - \beta x\) for \(0 < x < 1\), we get

\[ \left( \sum_{j=0}^{N-2} a_j \right)^{\beta} > N - \left( \frac{1}{2} + \beta C_0 \Lambda \right) \]  

(C.12)

which together with (C.3) yields \( N -1 > N - \frac{1}{2} - \beta C_0 \Lambda\), i.e.

\[ \Lambda > \frac{1}{2\beta C_0}. \]  

(C.13)

To settle the more general case, let \( \lambda = \{\lambda_k\} \) be so that \( \lambda_k \in (0,1] \) with \( \lambda_k < 1 \), for finitely many \( k \), and let

\[ \mu'(h) = \inf \left\{ \sum_{k=0}^{\infty} a_k^p e^{\beta' k} : \|a\|_1 = h, \|\lambda a\|_{\beta'} \leq 1 \right\}. \]  

(C.14)

We will prove (4.5) by showing that for some \( C_3 > 0 \)

\[ C_3 e^{-L\beta'} \mu(h) \leq \mu'(h) \leq \mu(h). \]  

(C.15)

It is clear that \( \mu'(h) \leq \mu(h) \), since any sequence satisfying \( \|a\|_{\beta'} \leq 1 \) also satisfies \( \|\lambda a\|_{\beta'} \leq 1 \).

It is then enough to show that for any sequence \( a \) such that \( \|a\|_1 = h, \|\lambda a\|_{\beta'} \leq 1 \) there is a nonnegative sequence \( b \) with \( \|b\|_1 = h, \|b\|_{\beta'} \leq 1 \) and

\[ C_3 e^{-L\beta'} \sum_{k=0}^{\infty} b_k^p e^{\beta' k} \leq \sum_{k=0}^{\infty} a_k^p e^{\beta' k}. \]  

(C.16)

Define

\[ J_k = \begin{cases} \left\lceil \lambda_k^{-\beta} \right\rceil & \text{if } k \geq 0 \\ 0 & \text{if } k = -1, \end{cases} \]
and

\[
  b_j = \begin{cases} 
    a_0/J_0, & 0 \leq j \leq J_0 - 1 \\
    a_1/J_1, & J_0 \leq j \leq J_0 + J_1 - 1 \\
    \ldots \\
    a_k/J_k, & \sum_{\ell=0}^{k-1} J_\ell \leq j \leq \sum_{\ell=0}^{k} J_\ell - 1 \\
    \ldots 
  \end{cases}
\]

Then we have

\[
  \|b\|_1 = \frac{a_0}{J_0} J_0 + \ldots + \frac{a_k}{J_k} J_k + \ldots = \|a\|_1 = h
\]

\[
  \|b\|_{\beta'}^\beta = \left( \frac{a_0}{J_0} \right)^{\beta'} J_0 + \ldots + \left( \frac{a_k}{J_k} \right)^{\beta'} J_k + \ldots = \sum_{k=0}^{\infty} J_k^{1-\beta'} a_k^{\beta'} \leq \sum_{k=0}^{\infty} \lambda_k^{\beta'} a_k^{\beta'} \leq 1,
\]

and

\[
  \sum_{j=0}^{\infty} \beta_j^P = \left( \frac{a_0}{J_0} \right)^P (1 + \ldots + e^{\beta'(J_0-1)}) + \ldots
\]

\[
  + \left( \frac{a_k}{J_k} \right)^P (e^{\beta'(J_0 + \ldots + J_k-1)} + \ldots + e^{\beta'(J_0 + \ldots + J_k-1)}) + \ldots
\]

\[
  \leq \sum_{k=0}^{\infty} a_k^P e^{\beta'} \sum_{\ell=0}^{k-1} J_\ell \sum_{j=0}^{J_k-1} e^{\beta' k} \leq \sum_{k=0}^{\infty} a_k^P e^{\beta'} \sum_{\ell=0}^{k-1} J_\ell \sum_{j=0}^{J_k-1} e^{\beta' k} \leq \frac{1}{e^{\beta'} - 1} \sum_{k=0}^{\infty} a_k^P e^{\beta' k} e^{\beta' (J_k-1)} \leq \frac{e^\beta}{e^{\beta'} - 1} \sum_{k=0}^{\infty} a_k^P e^{\beta' k},
\]

which gives (C.16).

\[\square\]

**D   Proof of Lemma 3**

**Proof of (5.16):** Recall (4.19), (4.20), (4.22) and that \(\mu(E_\tau) \leq \tau\). Using (K3) we get, for fixed \(y \in M\),

\[
  \int_{(D_{j+1} \setminus D_j) \setminus E_\tau} |k(\xi, y)| d\mu(\xi) \leq \int_0^{\mu(D_{j+1} \setminus D_j)} k_2(t, y) dt \leq C \int_0^{\mu(D_{j+1} \setminus D_j)} t^{-1/\beta} dt = C \mu(D_{j+1} \setminus D_j)^{1-1/\beta}.
\]

(D.1)

Hence we have

\[
  \int_{(D_{j+1} \setminus D_j) \setminus E_\tau} |Tf| d\mu \leq \frac{C}{\mu(D_{j+1} \setminus D_j)} \int_{M} |f(y)| \int_{(D_{j+1} \setminus D_j) \setminus E_\tau} |k(\xi, y)| d\mu(\xi) d\mu(y)
\]

\[
  \leq C \mu(D_{j+1} \setminus D_j)^{-1/\beta} \int_{M} |f(y)| d\mu(y) \leq CV_x(R_j)^{-1/\beta} \int_{M} |f| \chi_{F_j} d\mu
\]

\[
  \leq CV_x(R_j)^{-1/\beta} \mu(F_j)^{1/\beta} \|f\|_{\beta'} = Ce^{-j^\beta/\beta'} \|f\|_{\beta'} \leq C_3 e^{-j(\beta-1)/\beta}.
\]

(D.2)
Proof of (5.17): We will show that in fact for every $\xi \in D_{j+1} \setminus D_j$ we have
\[ |TS_{j+2}f'_\tau(\xi) - TS_{j+1}f'_\tau(x)| \leq C_{\beta_j+1}, \] (D.3)
which obviously implies (5.17). First write
\[ |TS_{j+2}f'_\tau(\xi) - TS_{j+1}f'_\tau(x)| \leq |TS_{j+2}f'_\tau(\xi) - TS_{j+2}f'_\tau(x)| + |T(S_{j+1} - S_{j+2})f'_\tau(x)|. \] (D.4)
From (5.4)
\[ |T(S_{j+1} - S_{j+2})f'_\tau(x)| \leq C_{\alpha_j+1} \leq C\beta_j+1. \] (D.5)
Now let us estimate the first term in (D.4):
\[ |TS_{j+2}f'_\tau(\xi) - TS_{j+2}f'_\tau(x)| = \left| \int_{D_{j+2}^c \setminus D_0'} (K(\xi, y) - K(x, y)) f'_\tau(y) d\mu(y) \right| \]
\[ \leq \left( \int_{D_{j+2}^c \setminus D_0'} |k(\xi, y) - k(x, y)|^{\beta} d\mu(y) \right)^{1/\beta} \left( \int_{D_{j+2}^c \setminus D_0'} |f'_\tau(y)|^{\beta} d\mu(y) \right)^{1/\beta} \]
\[ \leq \left( \int_{d(x,y) \geq r_{m+j+1}} |k(\xi, y) - k(x, y)|^{\beta} d\mu(y) \right)^{1/\beta} \left( \int_{D_{j+1}^c \setminus D_0} |f'_\tau(y)|^{\beta} d\mu(y) \right)^{1/\beta} \leq C\beta_j+1. \] (D.6)
where the last inequality follows from (K.4) since $x, \xi \in B(x, r_{j+1})$ and there is $\delta > 0$
(depending only from $k, \beta$) such that
\[ V_x(r_{j+2}) \geq (1 + \delta)V_x(r_{j+1}), \] (D.7)
which can be deduced from (4.19), (4.20).

Proof of (5.18): We use (D.1) to get, for any $k$
\[ \left| \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} T(S_k - S_{k+1})f'_\tau(\xi) d\mu(\xi) \right| = \left| \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} \int_{(D_{k+1} \setminus D_k) \setminus D_0} k(\xi, y)f'_\tau(y) d\mu(y) d\mu(\xi) \right| \]
\[ \leq C\mu(D_{j+1} \setminus D_j)^{-1/\beta} \int_{(D_{k+1} \setminus D_k) \setminus D_0} |f'_\tau(y)| d\mu(y) \]
\[ \leq CV_x(R_j)^{-1/\beta}V_x(R_k)^{1/\beta} \alpha_k = Ce^{-\beta(j-k)}\alpha_k. \] (D.8)
Therefore,
\[ \left| \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} T(S_0 - S_{j+2})f'_\tau(\xi) d\mu(\xi) \right| = \left| \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} \sum_{k=0}^{j+1} T(S_k - S_{k+1})f'_\tau(\xi) d\mu(\xi) \right| \]
\[ \leq \sum_{k=0}^{j+1} \left| \int_{(D_{j+1} \setminus D_j) \setminus E_{\tau}} T(S_k - S_{k+1})f'_\tau(\xi) d\mu(\xi) \right| \leq C\sum_{k=0}^{j+1} e^{-\beta(j-k)}\alpha_k \leq C\alpha_{j+1}. \] (D.9)

Proof of (5.19) and (5.20): Both (5.19) and (5.20) can be derived exactly the same way as we did in proof of (5.16), using $f'_\tau \chi_{D_0 \setminus D_0'}$ and $f'_\tau \chi_{D_0'}$ in place of $f'_\tau$. \[ \square \]
E Some results on $(Tf)^\circ$ for the Riesz potential on $\mathbb{R}^n$

First let us remark that for an operator of type

$$Tf(x) = \int_{\mathbb{R}^n} K(|x-y|)f(y)dy$$

if both $K$ and $f$ are nonnegative and radially decreasing, then $Tf$ is also nonnegative and radially decreasing, hence $E_{\tau} = F_{\tau} = B(0,r)$, up to a set of 0 measure, where $|B(0,r)| = \tau$. Also, we have that $(Tf)^\circ(\tau) = T(f\chi_{B(0,r)^c})(0)$, i.e. the ess sup in the definition of $(Tf)^\circ$ is a max attained at $x = 0$. This is because for $|x| \leq r$

$$T(f\chi_{B(0,r)^c\cap B(x,r)^c})(x) = \int_{\mathbb{R}^n} (K\chi_{B(0,r)^c})(x-y)(f\chi_{B(0,r)^c})(y)dy$$

$$\leq \int_0^\infty (K\chi_{B(0,r)^c})^*(t)(f\chi_{B(0,r)^c})^*(t)$$

$$= \int_\tau^\infty K^*(t)f^*(t)dt = \int_{B(0,r)^c} K(y)f(y)dy = T(f\chi_{B(0,r)^c})(0).$$

Let $I_\alpha f$ be the Riesz potential on $\mathbb{R}^n$ i.e.

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n}f(y)dy$$

and let, for a vector-valued function $f$,

$$\tilde{I}_1 f(x) = \int_{\mathbb{R}^n} \nabla_y |x-y|^{2-n}f(y) = (n-2) \int_{\mathbb{R}^n} |x-y|^{-n}(x-y) \cdot f(y)dy.$$

**Proposition 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be nonnegative, radially decreasing, and with compact support. For any $\tau > 0$ the following hold:

a) If $0 < \alpha \leq 2$, then $(I_\alpha f)^\circ(\tau) \leq (I_\alpha f)^*(\tau)$ with strict inequality if $f > 0$ near 0.

b) If $\alpha > 2$, $f$ bounded and positive near 0, then $(I_\alpha f)^\circ(\tau) > (I_\alpha f)^*(\tau)$ for all $\tau$ small enough.

If $f$ is vector-valued and $|f|$ is radially decreasing with compact support, then for every $\tau > 0$ we have $(\tilde{I}_1 f)^\circ(\tau) = (\tilde{I}_1 f)^*(\tau)$.

**Proof.** We have

$$I_\alpha f(x) = \int_0^\infty f(\rho)\rho^{\alpha-1}d\rho \int_{\mathbb{S}^{n-1}} |\rho^{-1}x - y|^\alpha dy^*$$

(E.2)

and due to invariance under rotation we can assume $x = re_1$. We will write $f(r) = f(re_1)$ and $I_\alpha f(r) = I_\alpha f(re_1)$.

Note that we have

$$\int_{\mathbb{S}^{n-1}} |z - y|^2\rho^{\alpha-1}dy^* = \begin{cases} |z|^{2-n} & \text{if } |z| > 1 \\ 1 & \text{if } |z| < 1 \end{cases}$$

(E.3)

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and for $0 < \alpha < 2$
\[
\int_{S^{n-1}} |z - y|^\alpha y^\ast d\gamma \geq \begin{cases} 
|z|^{2-n} & \text{if } |z| > 1 \\
1 & \text{if } |z| < 1
\end{cases}
\] (E.4)

while the inequality in (E.4) is reversed if $\alpha > 2$. The above are due to the fact that for fixed $z$ with $|z| > 1$ the function $|z - y|^\alpha y^\ast$ in the unit ball is harmonic for $\alpha = 2$, subharmonic for $0 < \alpha < 2$, and superharmonic for $\alpha > 2$. If $|z| < 1$ then $|z - y|^\alpha y^\ast = |z^\ast - |y||$ and the function $|z^\ast - |y||$ defined for $y$ in the unit ball is harmonic for $\alpha = 2$, subharmonic for $0 < \alpha < 2$ and superharmonic for $\alpha > 2$, and continuous up to the boundary.

When $0 < \alpha \leq 2$ we then have
\[
(I_\alpha f)^0(\tau) = I_\alpha(f \chi_{B(0,r)})(0) = \omega_{n-1} \int_r^\infty f(\rho) \rho^{\alpha-1} d\rho \leq \int_0^\infty f(\rho) \rho^{\alpha-1} d\rho \int_{S^{n-1}} |r \rho^{-1} e_1 - y|^\alpha y^\ast d\gamma
\]
with strict inequality in the last inequality if $f > 0$ around 0.

Let $\alpha > 2$. Assume further that $f$ is bounded. First we estimate
\[
|\rho^{-1} e_1 - y^\ast|^{\alpha-n} = (1 - 2 \rho^{-1} r y_1^* + \rho^{-2} r^2)^{(\alpha-n)/2}
= 1 + \frac{n - \alpha}{2}(2 \rho^{-1} r y_1^* - \rho^{-2} r^2) + \frac{n - \alpha}{2}(n - \alpha + 2) \rho^{-2} r^2 (y_1^*)^2 + O(\rho^{-3} r^3),
\] (E.6)

so that
\[
\int_{S^{n-1}} |\rho^{-1} e_1 - y|^\alpha y^\ast d\gamma = \omega_{n-1} - \frac{n - \alpha}{2} \rho^{-2} r^2 \int_{S^{n-1}} (1 - (n - \alpha + 2) (y_1^*)^2) d\gamma + O(\rho^{-3} r^3)
= \omega_{n-1} - \omega_{n-1}(\rho^{-2} r^2) \frac{(n - \alpha)(\alpha - 2)}{2n} + O(\rho^{-3} r^3),
\]

where we used the identities $\int_{S^{n-1}} y_1^* d\gamma = 0$ and $\int_{S^{n-1}} (y_1^*)^2 d\gamma = 1/n$. Therefore,
\[
(I_\alpha f)^*(\tau) = I_\alpha f(r) = \int_{S^{n-1}} |r e_1 - y|^\alpha f(y) d\gamma = \int_0^\infty \rho^{\alpha-1} f(\rho) \int_{S^{n-1}} |\rho^{-1} e_1 - y|^\alpha y^\ast d\rho d\rho
= \int_0^\infty \rho^{\alpha-1} f(\rho) (\omega_{n-1} - \omega_{n-1}(\rho^{-2} r^2) \frac{(n - \alpha)(\alpha - 2)}{2n} + O(\rho^{-3} r^3)) d\rho
= \omega_{n-1} \int_0^r \rho^{\alpha-1} f(\rho) d\rho + \omega_{n-1} \int_r^\infty \rho^{\alpha-1} f(\rho) d\rho - C r^2 + O(r^3)
= \omega_{n-1} \int_0^r \rho^{\alpha-1} f(\rho) d\rho + C' r^\alpha - C r^2 + O(r^3),
\] (E.7)

where $C, C'$ and the second last equality is by the fact that $f$ is bounded and $\alpha > 2$. Hence for $r$ small enough, we have
\[
(I_\alpha f)^0(\tau) - (I_\alpha f)^*(\tau) = Cr^2 - C' r^\alpha > 0.
\]
Finally, let $|f|$ be radially decreasing with compact support, and write

$$
\tilde{I}_1 f(x) = (n - 2) \int_0^\infty |f(\rho)| d\rho \int_{S^{n-1}} |\rho^{-1} x - y^*|^{-n} (\rho^{-1} x \cdot y^* - 1) dy^*. \quad (E.8)
$$

If $H(z)$ denotes the LHS of (E.3) it’s easy to see that

$$
H(z) - \frac{|z|}{2 - n} \frac{\partial H}{\partial |z|} = \int_{S^{n-1}} |z - y^*|^{-n} (1 - z \cdot y^*) dy^* = \begin{cases} 0 & \text{if } |z| > 1 \\ 1 & \text{if } |z| < 1 \end{cases} \quad (E.9)
$$

which gives

$$
\tilde{I}_1 f(x) = -(n - 2) \omega_{n-1} \int_{|x|}^\infty |f(\rho)| d\rho. \quad (E.10)
$$

Hence both $|f|$ and $\tilde{I}_1 f|$ are radially decreasing so that $E_x = F_\tau = B(0, r)$.

Next, note that

$$
(x - \rho y^*) \cdot |f(y)| = (x \cdot y^* - \rho)|f|(\rho) \leq 0, \quad \text{for } |x| \leq r, \ |y| = \rho \geq r,
$$

so that

$$
0 \leq -\tilde{I}_1 (f \chi_{B(0,r)^c \cap B(x,r)^c})(x) \leq -\tilde{I}_1 (f \chi_{B(0,r)^c})(x), \quad \forall x \in B(0, r). \quad (E.11)
$$

Using (E.9) and (E.10) we get that for every $x \in B(0, r)$

$$
|\tilde{I}_1 (f \chi_{B(0,r)^c})(x)| = (n - 2) \int_{|y| \geq r} |x - y|^{-n} (y - x) \cdot f(y) dy
$$

$$
= (n - 2) \int_r^\infty |f(\rho)| d\rho \int_{S^{n-1}} |\rho^{-1} x - y^*|^{-n} (1 - \rho^{-1} x \cdot y^*) dy^* = |\tilde{I}_1 f(r)| = (\tilde{I}_1 f)^*(\tau) \quad (E.12)
$$

Therefore combining (E.11) and (E.12) we have

$$
(\tilde{I}_1 f)^*(\tau) = \tilde{I}_1 (f \chi_{B(0,r)^c})(0) = (\tilde{I}_1 f)^*(\tau), \quad \forall \tau > 0. \quad (E.13)
$$

\[ \square \]

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