Hamiltonian analysis of $n$-dimensional Palatini gravity with matter

Muxin Han*, Yongge Ma†, You Ding‡, and Li Qin§
Department of Physics, Beijing Normal University, Beijing 100875, CHINA

July 7, 2018

Abstract

We consider the Palatini formalism of gravity with cosmological constant $\Lambda$ coupled to a scalar field $\phi$ in $n$-dimensions. The $n$-dimensional Einstein equations with $\Lambda$ can be derived by the variation of the coupled Palatini action provided $n > 2$. The Hamiltonian analysis of the coupled action is carried out by a $1 + (n - 1)$ decomposition of the spacetime. It turns out that both Palatini action and Hilbert action lead to the same geometric dynamics in the presence of $\Lambda$ and $\phi$. While, the $n$-dimensional Palatini action could not give a connection dynamics formalism directly.

Keywords: Palatini action, high dimensional gravity, Hamiltonian analysis

PACS number(s): 04.50.+h, 04.20.Fy

1 Introduction

In Palatini formalism Lorentz connection becomes one of the basic dynamical variables. This feature causes great interest in the study of non-perturbative quantum gravity[1][2][3], modified gravity theories[4][5] and their cosmological applications[6][7]. On the other hand, high dimensional gravitational theories, such as Kaluza-Klein theory, are widely investigated[8][9][10][11] since they provide the possibility to unify gravity with gauge fields by certain higher dimensional geometry. Also the matter and black holes in high dimensional gravity[12] and the concept of energy in high dimensional spacetime[13][14] are fully investigated. The $n$-dimensional Palatini action, in the case $n > 2$, can reproduce $n$-dimensional vacuum Einstein equations[15]. Although the Hamiltonian formalism of 4(or 3)-dimensional Palatini gravity has been fully studied[16][2], the

*Email address: hamsyncolor@hotmail.com
†Email address: mayg@bnu.edu.cn
‡Email address: dingyou@hotmail.com
§Email address: qinli0510163.com
full Hamiltonian analysis of higher dimensional Palatini formalism is still lacking. One may ask the question whether one could derive a connection dynamics formalism from n-dimensional Palatini action, as connection variables are the foundation to apply the technique of loop quantum gravity.

In this paper, we consider the Palatini formalism of n-dimensional gravity with cosmological constant $\Lambda$ coupled to a Klein-Gordon field $\phi$. A straightforward calculation show that the coupled Palatini action can still reproduce n-dimensional Einstein’s equations even in the presence of $\Lambda$ and $\phi$ provided $n > 2$. We then derive the corresponding Hamiltonian formulation by carrying out a $1 + (n - 1)$ decomposition of the underlying n-dimensional spacetime. It is well known that in the Legendre transform of 4-dimensional Palatini formalism, besides the expected first-class constraints representing the internal gauge symmetry and spacetime symmetry, there appear also second-class constraints to account for the degrees of freedom of the theory. A complicated step in the Hamiltonian analysis is then to solve those second-class constraints. While, in higher dimensional Palatini formalism, another problem arises since the construction of the anticipated constraints responsible for the degrees of freedom is not so obvious as that in 4-dimensional case. Let alone to solve the constraints. We deal with the two troublesome problems by one trick, namely gauge fixing a unit time-like internal vector $n^\mu$ and solving the boost part of the Gauss constraint relative to $n^\mu$. It turns out that both first order Palatini action and second order Einstein-Hilbert action lead to the same geometric dynamics in $n$-dimensions ($n > 2$) in the presence of $\Lambda$ and $\phi$. Thus, as in 4-dimensional case, one could not obtain connection dynamics directly from the n-dimensional Palatini action.

2 Lagrangian Formalism

Consider an $n$-manifold ($n > 2$), $M$, on which the basic dynamical variables in the Palatini framework are $e^a_\mu$ and $so(1, n - 1)$-valued connection $\bar{\omega}^{\mu\nu}_a$ (not necessarily torsion-free), where the Greek indices $\mu, \nu, ...$ denote the internal $SO(1, n - 1)$ group and the Latin indices $a, b, ...$ denote the ”spacetime indices”. The internal space is equipped with a Minkowskian metric $\eta_{\mu\nu}$ (of signature $- + \ldots +$), fixed once for all, such that the spacetime metric reads:

$$g_{ab} = \eta_{\mu\nu} e^a_\mu e^b_\nu.$$ 

The coupled Palatini action in which we are interested is given by:

$$S_p[e^b_x, \bar{\omega}^{\mu\nu}_a, \phi] = \frac{1}{2} \int_M d^n x(e)[e^a_\mu e^b_\nu \Omega_{ab}^{\mu\nu} + 2\Lambda]$$
$$- \frac{\alpha_M}{2} \int_M d^n x(e)[\eta^\mu\nu e^a_\mu e^b_\nu (\bar{\partial}_a \phi) \bar{\partial}_b \phi + m^2 \phi^2],$$

where $e$ is the square root of the determinant of the $n$-metric $g_{ab}$. $\Lambda$ is the n-dimensional cosmological constant, $\alpha_M$ is the coupling constant, $\bar{\partial}_a$ is a flat
derivative operator on \( M \), and the \( so(1, n - 1) \)-valued curvature 2-form \( \Omega_{ab}^{\mu \nu} \) of the connection \( \bar{\omega}_{a}^{\mu \nu} \) reads: 
\[
\bar{\Omega}_{ab}^{\mu \nu} = 2 \bar{D}_{[a} \bar{\omega}_{b]}^{\mu \nu} \equiv (d\bar{\omega}^{\mu \nu})_{ab} + \bar{\omega}_{a}^{\mu \sigma} \wedge \bar{\omega}_{b\sigma}^{\nu} .
\]

The gravitational field equations are obtained by varying this action with respect to \( e_{a}^{\mu} \) and \( \bar{\omega}_{a}^{\mu \nu} \). To carry out the variation with respect to the connection, it is convenient to introduce the unique (torsion-free) generalized covariant derivative \( \nabla_{a} \) on both space-time and internal indices determined by the bases \( e_{a} \) via:
\[
\nabla_{a} e_{\mu}^{b} = \bar{\nabla}_{a} e_{\mu}^{b} + \bar{\Gamma}_{a e_{\mu}}^{b} = 0 ,
\]
where \( \bar{\Gamma}_{a e}^{b} \) and \( \bar{\Gamma}_{a \mu}^{\nu} \) are respectively the Levi-Civita connection and spin connection on \( M \). Hence the difference between the curvatures of \( \bar{\omega}_{a}^{\mu \nu} \) and \( \bar{\Gamma}_{a}^{\mu \nu} \) is a covariant generalized tensor field with respect to both internal and spacetime indices defined by:
\[
\bar{C}_{a}^{\mu \nu} \equiv \bar{\omega}_{a}^{\mu \nu} - \bar{\Gamma}_{a}^{\mu \nu} .
\]
Hence the difference between the curvatures of \( \bar{\omega} \) and \( \bar{\Gamma} \) is given by:
\[
\bar{\Omega}_{ab}^{\mu \nu} - \bar{\Omega}_{ab}^{\mu \nu} = 2 \bar{D}_{[a} \bar{C}_{b]}^{\mu \nu} + 2 \bar{C}_{[a}^{\mu \rho} \bar{C}_{b] \rho}^{\nu} ,
\]
where \( \bar{\Omega}_{ab}^{\mu \nu} \) is the curvature 2-form of \( \nabla_{a} \). Note that the variation of the action \( \mathcal{S}_{\mu \nu}^{(a)} \) with respect to \( \bar{\omega}_{a}^{\mu \nu} \) (keeping the basis fixed) is the same as its variation with respect to \( \bar{C}_{a}^{\mu \nu} \). Using Eq. \( (5) \), the action \( \mathcal{S}_{\mu \nu}^{(a)} \) becomes:
\[
\mathcal{S}_{\mu \nu}^{(a)} = \frac{1}{2} \int_{M} d^{n}x(e) \left[ e_{n}^{a} \Gamma_{a}^{\mu \nu} \left( \bar{\Omega}_{ab}^{\mu \nu} + 2 \bar{D}_{[a} \bar{C}_{b]}^{\mu \nu} + 2 \bar{C}_{[a}^{\mu \rho} \bar{C}_{b] \rho}^{\nu} \right) + 2 \mathcal{A} \right] - \frac{\alpha M}{2} \int_{M} d^{n}x(e) \left( \eta^{\mu \nu} e_{\mu}^{a} (\bar{\nabla}_{a} \phi) \bar{\nabla}_{a} \phi + m^{2} \phi^{2} \right) .
\]
By varying this action with respect to \( \bar{C}_{a}^{\mu \nu} \), one obtains:
\[
(e_{n}^{a} \delta_{(b}^{[\mu \nu] \sigma} \bar{D}_{(c}^{\sigma}) \bar{C}_{b}^{\tau}) = 0 ,
\]
which implies:
\[
(n - 2) \bar{C}_{a}^{\mu \nu} e_{\mu}^{a} = 0 .
\]
This yields \( \bar{C}_{a}^{\mu \nu} = 0 \), when \( n \neq 2 \). Using this result, Eq. \( (5) \) leads to
\[
\bar{C}_{a}^{\mu \nu} = \bar{C}_{(a \mu \nu)} .
\]
Thus, \( \bar{C}_{a}^{\mu \nu} \) is symmetric in its first two indices. Since \( \bar{C}_{a}^{\mu \nu} = \bar{C}_{(a \mu \nu)} \), we can successively interchange the indices to show \( \bar{C}_{a}^{\mu \nu} = 0 \). This is the desired result. Thus, the equation of motion for the connection \( \bar{\omega}_{a}^{\mu \nu} \) is simply that it equals \( \bar{\Gamma}_{a}^{\mu \nu} \). Thus the connection \( \bar{\omega}_{a}^{\mu \nu} \) is completely determined by the bases. By carrying out the variation of action \( \mathcal{S}_{\mu \nu}^{(a)} \) with respect to the bases, one obtains:
\[
e_{n}^{\mu} \bar{\Omega}_{cb}^{\mu \nu} - \frac{1}{2} \bar{\Omega}_{cd}^{\sigma \rho} \bar{e}_{c} e_{\rho}^{a} e_{\sigma}^{b} - \Lambda e_{b}^{\mu} = \alpha M \left( \eta^{\mu \nu} e_{\mu}^{a} (\bar{\nabla}_{a} \phi) \bar{\nabla}_{a} \phi + m^{2} \phi^{2} \right) .
\]
Using the fact that $\Omega_{ab}^{\mu\nu} = \bar{\Omega}_{ab}^{\mu\nu}$ and the curvature 2-form of $\nabla_a$ is related to its space-time curvature by $\bar{\Omega}_{ab\nu} = \bar{\Omega}_{abc} \epsilon_c^{\nu}$, and multiplying Eq. (9) by $e_{\nu a}$, it follows that the Einstein equations holds. It is obvious that the variation of action (1) with respect to $\phi$ will still give the Klein-Gordon equation.

3 Hamiltonian Analysis

To carry out the Hamiltonian analysis of action (1), suppose the spacetime $M$ is topologically $\Sigma \times R$ for some $(n-1)$-manifold $\Sigma$. We introduce a foliation and a time-evolution vector field $t^a$ in $M$, where $t^a$ can be decomposed with respect to the unit normal vector $n^a$ of $\Sigma$ as:

$$t^a = N n^a + N^a,$$

where $N$ is called the lapse function and $N^a$ called the shift vector. Denote $S_p = S_G + S_{KG}$. Then the action of gravity and matter can be respectively decomposed as:

$$S_G = \frac{1}{2} \int_{\Sigma \times R} d^n x(E) |NE^a_{[\mu}E^b_{\nu]}\epsilon^{\mu\nu} + 2n_{[\mu}E^a_{\nu]}D_\nu \omega_i^{\mu\nu} - 2n_{[\mu}E^a_{\nu]}\omega_i^{\mu\nu} + 2N n_{[\mu}E^b_{\nu]} \Omega_{ab}^{\mu\nu} + 2NA|,$$

$$S_{KG} = -\frac{\alpha M}{2} \int_{\Sigma \times R} d^n x(E)N |\epsilon^{\mu\nu\rho\sigma}E^a_{[\mu}E^b_{\nu]}(\partial_\rho \phi) \partial_\sigma \phi - \frac{1}{N^2}(\dot{\phi} - N^a \partial_a \phi)^2 + m^2 \phi^2 |,$$

where $E_{[\mu}^a \equiv \epsilon_{\mu}^b \epsilon_{\nu}^c \epsilon_{\sigma}^d \epsilon_{[\rho}^e \epsilon_{\sigma}^f \epsilon_{\nu}^g \epsilon_{\mu}^h | \epsilon_{\sigma}^i \epsilon_{[\rho}^j |$, $E^a_{\mu} = \epsilon^b_{[\mu} \epsilon^a_{\nu]}$, $E = \sqrt{q}$, $N = \mathcal{V}_x\equiv \mathcal{V}_{\phi \omega_i}$, $\Omega_{ab}^{\mu\nu}$ is the corresponding spatial $so(1, n-1)$-valued curvature 2-form, $\omega_i^{\mu\nu}$ is the Lie derivative of $\omega_i^{\mu\nu}$ with respect to $t^a$ (treating the internal indices as scalars), and $\partial_a$ is the derivative operator on $\Sigma$ reduced from $\partial_t$. The internal normal vector is defined as $n_{[\mu} \equiv n_{[\mu}, \phi(0, 0, 0, 0, 0, 0)$. By a gauge fixing $n_{[\mu} = (1, 0, 0, 0, 0, 0)$, one can split the internal indices $\mu, \nu, \sigma, ...$ into $0, i, j, ...$. Note that the gauge fixing puts no restriction on our real dynamics.

Let $\mathcal{N} \equiv N/E$ be the densitized lapse scalar of weight $-1$ and $\tilde{E}^a_i \equiv (E_i)E^a_i$ the densitized spatial field of weight 1. The action of gravitational field can then be decomposed as:

$$S_G = \int_{\Sigma \times R} d^nx|\mathcal{N}\tilde{E}^a_i \tilde{E}^b_j (D_{[\mu} \omega_i^{\mu\nu} + K_{[\mu}^{ij} D_{\nu]} \omega_i^{\mu\nu}) - \phi^{ij} D_a \tilde{E}^a_i + \tilde{E}^a_i K_{[\mu}^{ij} D_{\nu]} \omega_i^{\mu\nu} - 2N^a \tilde{E}^b_j D_a K_{[\mu}^{ij} D_{\nu]} \omega_i^{\mu\nu} + (E^2)^{ij}\mathcal{N}A|,$$

where $K_{[\mu}^{ij} \equiv \omega_{[\mu}^{ij}$, and $D_a$ is the $SO(n-1)$ generalized covariant derivative operator with respect to $\omega_i^{\mu\nu}$. The unique torsion-free $SO(n-1)$ generalized
covariant derivative operator annihilating $E^b_i$ is defined as:

$$\nabla_a E^b_i = \partial_a E^b_i + \Gamma^b_{ac} E^c_i + \Gamma^b_{ai} E^b_j = 0,$$

(14)

where $\Gamma^b_{ac}$ and $\Gamma^b_{ai}$ are respectively the Levi-Civita connection and the spin connection on $\Sigma$. Let $C^{ij}_a$ be the difference between $\omega^{ij}_a$ and $\Gamma^{ij}_a$, i.e.,

$$\omega^{ij}_a = \Gamma^{ij}_a + C^{ij}_a.$$

(15)

Then the constraint equation with respect to the Lagrangian multiplier $\tilde{\omega}^{ij}_a$ reads:

$$\frac{\delta S_p}{\delta \omega^{ij}_a} = D_a \tilde{E}^a_i = \nabla_a \tilde{E}^a_i + C^{ij}_a \tilde{E}^a_j = (E) C^{ij}_a = 0,$$

(16)

which means that $C^{ij}_a = -C^{ji}_a = 0$ in the reduced phase space. So in the reduced phase space determined by Eq. (16), the action of gravitational field reads

$$S_G = \int_{\Sigma \times R} d^n x [\frac{1}{2} N \tilde{E}^a_i \tilde{E}^b_i R^{ij}_{ab} - \frac{1}{2} (E)^2 N C^{ij}_a \Gamma^k_j + N \tilde{E}^a_i \tilde{E}^b_i K^i_a K^j_b + \tilde{\omega}^{ij}_a \tilde{E}^a_i K^{[ij]}_b + E^a_i \nabla_{[a} K^{ij]}_b + N a C^{ij}_a \tilde{E}^a_i K^{ij}_b + (E)^2 N \Lambda],$$

(17)

where $R^{ij}_{ab} = 2 \nabla_{[a} \Gamma^{ij}_b]^{ij}$ is the Riemann curvature 2-form compatible with the spatial basis $E^a_i$. Then the variation of $S_p$ respect to $C^{ijk}_j$ gives:

$$\frac{\delta S_p}{\delta C^{ijk}_j} = \frac{1}{2} (E)^2 N (C^{[ik]}_j + C^{[kj]}_i) + N \tilde{E}^a_i K^{[ij]}_b = 0.$$

(18)

While, the constraint equation determined by the Lagrangian multiplier $\tilde{\omega}^{ij}_a$ reads:

$$\frac{\delta S_p}{\delta \tilde{\omega}^{ij}_a} = \tilde{E}^a_i K^{[ij]}_b = 0.$$

(19)

Since $C^{ijk}_j = C^{[jk]}_i$, by substituting Eq. (19) into Eq. (18), one gets:

$$C^{[jk]}_i = 0,$$

(20)

which leads to $C^{ijk}_j = 0$ by using the same trick in the previous section. Hence, the action of gravitational field can be reduced to:

$$S_G = \int_{\Sigma \times R} d^n x [\tilde{E}^a_i K^{ij}_a + \frac{1}{2} (E)^2 (R + 2 \Lambda) + \tilde{E}^a_i \tilde{E}^b_i K^i_a K^j_b + \tilde{\omega}^{ij}_a \tilde{E}^a_i K^{[ij]}_b - 2 N \tilde{E}^a_i \nabla_{[a} K^{ij]}_b].$$

(21)

On the other hand, the action of the scalar field can be written via the internal gauge fixing as:

$$S_{KG} = \int_{\Sigma \times R} d^n x [\delta^{ij} \frac{N}{2} \tilde{E}^a_i \tilde{E}^b_i (\partial_a \phi) \partial_b \phi - \frac{1}{N} (\partial_a \phi - N a \partial_a \phi)^2 + (E)^2 N m^2 \phi^2].$$

(22)
The canonical momentum conjugating to $\phi$ reads:

$$\tilde{\pi} = \frac{\partial L}{\partial \dot{\phi}} = \frac{\alpha_M}{N} (\phi - N^a \partial_a \phi). \quad (23)$$

Hence the scalar field action can be expressed as:

$$S_{KG} = \int_{\Sigma \times R} d^n x [\tilde{\pi} \dot{\phi} - N\left(\frac{\alpha_M}{2} \delta^{ij} \tilde{E}_i^a \tilde{E}_j^b (\partial_a \phi) (\partial_b \phi) + \frac{1}{2\alpha_M} \tilde{\pi}^2 + \frac{\alpha_M}{2} (E^2 m^2 \phi^2)\right) - N^a \tilde{\pi} \partial_a \phi]. \quad (24)$$

Thus, combining Eqs. (21) and (24), the total Hamiltonian and all constraint equations of $n$-dimensional Palatini gravity with $\Lambda$ coupled to $\phi$ can be summarized as:

$$H_{tot} = \int_{\Sigma \times R} d^n x [\tilde{\pi} \dot{\phi} - N\left(\frac{\alpha_M}{2} \delta^{ij} \tilde{E}_i^a \tilde{E}_j^b (\partial_a \phi) (\partial_b \phi) + \frac{1}{2\alpha_M} \tilde{\pi}^2 + \frac{\alpha_M}{2} (E^2 m^2 \phi^2)\right) - N^a \tilde{\pi} \partial_a \phi], \quad (25)$$

$$G_{ij} = \tilde{E}_i^a K_{(i} |a| j), \quad (26)$$

$$\mathcal{H} = \frac{1}{2} (E^2 (R + 2\Lambda) + \tilde{E}_i^a \tilde{E}_j^b K_{i}^{a} K_{j}^{b}] - \left[\frac{\alpha_M}{2} \delta^{ij} \tilde{E}_i^a \tilde{E}_j^b (\partial_a \phi) (\partial_b \phi) + \frac{1}{2\alpha_M} \tilde{\pi}^2 + \frac{\alpha_M}{2} (E^2 m^2 \phi^2)\right], \quad (27)$$

$$\mathcal{V}_a = -2\tilde{E}_j^b \nabla^{[a} K_{j}^{b]} - \tilde{\pi} \partial_a \phi. \quad (28)$$

They have the same form as those in the ADM Hamiltonian formalism. Hence the constraints (26)-(28) also comprise a first-class system.

In conclusion, for arbitrary $n > 2$ dimensional spacetime, the Palatini action and the Einstein-Hilbert action lead to the same classical dynamics in the presence of cosmological constant and Klein-Gordon field. Thus, as an alternative approach, one may study the dynamics of higher dimensional gravitation with matter fields, such as the brane world theory, in Palatini formalism as well. While, the $n$-dimensional Palatini action could not give a connection dynamics formalism directly.

**Acknowledgments**

This work is supported in part by NSFC (10205002) and YSRF for ROC S, SEM. Muxin Han, You Ding and Li Qin would also like to acknowledge support from Undergraduate Research Foundation of BNU.

**References**

[1] A. Ashtekar, Lectures on Non-perturbative Canonical Gravity, (World Scientific, Singapore, 1991).

[2] D. Romano, Gen. Rel. Grav. **759**, 25 (1993).
[3] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53 (2004).
[4] E. E. Flanagan, Phys. Rev. Lett. 92, 071101 (2004).
[5] D. N. Vollick, Phys. Rev. D 69, 064030 (2004).
[6] X. Meng and P. Wang, Class. Quant. Grav. 21, 2029 (2004).
[7] G. M. Kremer and D. S. N. Alves, Phys. Rev. D 70, 023503 (2004).
[8] P.S. Wesson, Space-Time-Matter: Modern Kaluza-Klein Theory, (World Scientific Publishing, 1999).
[9] Y. Ma and J. Wu, Int. J. Mod. Phys. A 19, 5043 (2004).
[10] X. Yang, Y. Ma, J. Shao, and W. Zhou, Phys. Rev. D 68, 024006 (2003).
[11] L. Qiang, Y. Ma, M. Han, and D. Yu, "5-dimensional Brans-Dicke theory and cosmic acceleration", accepted for publication in Phys. Rev. D (Rapid Communication), gr-qc/0411066.
[12] L. Sokolowski, B. Carr, Phys. Lett. B176, 334 (1986).
[13] W.N. Sajko, P.S. Wesson, Mod. Phys. Lett. A 16, 627 (2001).
[14] L. Bombelli, R. Koul, G. Kunstatter, J. Lee, and R.D. Sorkin, Nucl. Phys. B289, 735 (1987).
[15] Y. Ding, Y. Ma, M. Han, and J. Shao, Mod. Phys. Lett. A 20, 345 (2005).
[16] R. M. Wald, General Relativity, (The University of Chicago Press, 1984).
[17] C. Liang, Introductory Differential Geometry and General Relativity I, II, (Beijing Normal University Press, 2000, in Chinese).