Inferring Program Transformations from Type Transformations for Partitioning of Ordered Sets

Wim Vanderbauwhede

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Abstract

In this paper I introduce a mechanism to derive program transformations from order-preserving transformations of vector types. The purpose of this work is to allow automatic generation of correct-by-construction instances of programs in a streaming data processing paradigm suitable for FPGA processing. We show that for it is possible to automatically derive instances for programs based on combinations of opaque element-processing functions combined using foldl and map, purely from the type transformations.

1 Introduction

In this discussion paper I want to introduce a set of type transformations on vector types. In this work we will use a simple form of dependent types [Bove and Dybjer, 2009], but the concept can be generalised to transformations on other types, including session types [Honda et al., 2008]. The overall idea is to introduce the transformations and then explore the effect of transforming the types in a program on the program itself, i.e. what are the required corresponding functions that will transform the types of the computations while preserving the results of the computations.

The purpose of this work is to allow automatic generation of correct-by-construction instances of programs in a streaming data processing paradigm suitable for data processing using FPGAs (Field Programmable Gate Arrays, [Vanderbauwhede and Benkrid, 2013]). Using an optimisation technique such as simulated annealing [Aarts and Korst, 1988] and a cost model for the FPGA implementation, the best instance can be automatically selected.

2 Preliminaries

2.1 Type Variables

\(a\) is a type variable representing a nullary type constructor. We will call this kind of type variable atomic.
are general type variables. We call the set of these type variables $B$

$k, l, m, n$ are non-zero natural numbers, i.e. $k, m, n \in \mathbb{N}_{>0}$. We will call these sizes.

$p, q$ are (unary) type constructor variables, i.e. types that apply to other types and can depend on non-zero natural numbers, e.g. $p\, n\, a$ or $p\, k\, q\, q\, b$. Note that we use unary (i.e. one type and one size), right-associative type constructors purely to simplify the discussion, not as a fundamental limitation. The crucial point is however that these are dependent types.

$F, G, H$ are general functions operating on types (we could call them type transformers). We assume the functions take a single type as argument and are right associative. Consequently, we can write $G\,(F\, a)$ as $G\, F\, a$

$S, M, R$ and $I$ are specific functions operating on types, to be defined later.

2.2 Notations and Definitions

total size The total size of a type is the product of all sizes:

$\mathcal{N}(p_1\, n_1\, p_2\, n_2\, \ldots\, p_i\, n_i\, \ldots\, p_k\, n_k\, a) = \prod_{i=1}^{k} n_i$  

type transformation A type transformation is the application of a function from one type to another to a type, i.e. $F\, b = c$

$i, j$ are used as subscripts to distinguish between type variables of the same class, so that we can write $p_1\, p_2\, \ldots\, p_i\, \ldots\, p_k\, a$

3 Restrictions on Type Transformations

Given a type expression of the form $p_1\, n_1\, p_2\, n_2\, \ldots\, p_i\, n_i\, \ldots\, p_k\, n_k\, a$, the type transformations we want to consider must obey following restrictions:

1. The transformations do not remove any atomic type variables or introduce fresh atomic type variables. It follows that atomic type variables cannot be modified either.

2. The transformations can only remove or add one or more outer type constructors.

3. The type transformations can transform the sizes, but the total size of the type is an invariant.

Although the types and transformations are more general, our focus is on transformations of types describing ordered sets. The above restrictions intend to reflect that the type transformations should not alter the number or nature of the elements of the set, but only the way the set is partitioned.
4 Vector Types

For general types $p_i$, it may be hard to prove that the above rules do not alter the number or nature of the elements of the set, but only the way the set is partitioned. However, if we assume a single type representing a vector, then what these restrictions say is that a vector can only be reshaped but not modified in terms of its type or size. This is of course not a sufficient condition to guarantee that the type transformations will not change the computations, but it is a necessary one.

We introduce the vector type $v_k b$ where $k$ is a non-zero positive integer and $b$ is an arbitrary type. This the type representing a vector of length $k$ containing values of type $b$. Specifically we define

$$b \triangleq v 0 b$$

Given an atomic type $a$, we can generate the set of all vector types $V(a)$ for $a$ as follows:

$$\begin{cases}
a \in V(a) \\
\forall b \in V(a), \forall k \in \mathbb{N}_{>0}|v_k b \in V(a)
\end{cases}$$

For convenience, we introduce the following notation:

$$v_k b \triangleq [b][k]$$

And the shorthand:

$$[...[b][k_1][k_2]...(k_n)] \triangleq [b][k_1][k_2]...(k_n)$$

5 Transformations on Vector Types

For the rest of the paper we consider a specific case of types and type transformations: transformations on vector types. I posit three fundamental transformations, each with a corresponding inverse:

- converting a type to a singleton vector type
- applying a type transformation to the type variable of a vector type (mapping)
- reshaping a vector type, i.e. modifying the sizes in a vector type such that the total size is remains invariant

We can formalise each of these transformations:

5.1 Singleton vector type

The purpose of this operation is to change the dimensionality of a vector.

$$S b \triangleq [b][1]$$

The inverse operation (reducing dimensionality) is defined trivially as
\[ S^{-1} [b|1] \triangleq b \]

So that
\[ S S^{-1} b = b \]
\[ S S^{-1} [b|1] = [b|1] \]

Repeated application of S leads to higher-dimensional singleton vectors:
\[ S S b = [[b|1]|1] \]

We introduce a convenient notation
\[ S^k b = [b|1]^k \]

5.2 Mapping

Mapping applies a transformation to the type argument of a vector type.

This operation is independent of the size, so I have omitted it:
\[ M F [b] \triangleq [F b] \]

Note that the inverse operation is the application of the inverse of F, not of M:
\[ M F^{-1} [F b] = [b] \]

Although of course we can define purely notationally
\[ M^{-1} F \triangleq M F^{-1} \]

Repeated application of M has two cases. The first case is applying different transformations to a 1-D vector
\[ (M F) (M G) [b] = [F G b] \]

We can rewrite the lhs as
\[ (M F) (M G) [b] = M (F G) [b] \]

The second case is applying a single transformation to a multi-dimensional vector
\[ M (M F) [[b]] = [M F [b]] = [[F b]] \]

We can rewrite the lhs as
\[ M (M F) [[b]] = M^2 F [[b]] \]

5.3 Reshaping

The purpose of this operation is to re-partition a vector. The operation works on a 2-D vector.
\[ R m [[b\langle n_1 \rangle | n_2]] \triangleq [[b\langle n_1, m \rangle | n_2/m]] \]

The condition on m is of course that \( n/m \) is a natural number, i.e. \( n_2 \) is a multiple of \( m \).

The inverse operation can again be defined notationally:
\[ R_{m^{-1}} [[b\langle n_1 \rangle | n_2]] = [[b\langle n_1/m \rangle | n_2.m]] \]

and
\[ R_{m^{-1}}^{-1} m \triangleq R_{m^{-1}} \]
5.4 Identity Operation

We define \( I b \triangleq b \) for completeness. I contend (but have not formally proven) that the set of operations \( S,M,R,I \) form a group over \( V(a) \). Each of the operations is associative and can be inverted, and any combination of operations on a vector type results in a vector type, i.e. it is closed as well. By adding \( I \), the conditions for a group are satisfied.

In fact, as we shall show below, \( S,M,R,I \) form a group over a particular finite subset of \( V(a) \):

- define \( V(a,n) \) as the subset of \( V(a) \) where, for any given vector \( \langle a \rangle \langle k_1 \rangle \langle k_2 \rangle \ldots \langle k_m \rangle \),
  \[ \prod_{i=1}^{m} k_i = n. \]
- then \( S,M,R,I \) form a group over \( V(a,n), \forall a, n \)

In the next section, we will give a proof of the closure constraint.

5.5 Operations on Atomic Types

We define mapping on or reshaping of an atomic type as identity operations:

\[
MF a \triangleq a
\]
\[
R k a \triangleq a
\]

5.6 Vector Creation

For what follows, we will need an invertible operation \( V \) to create vector types:

\[
V k b \triangleq \left[ b \langle k \rangle \right]
\]
with its inverse

\[
V^{-1} k \left[ b \langle k \rangle \right] \triangleq b
\]

In other words, \( V \) is equivalent to the vector type constructor but has an inverse. We use this operation to formally extract the argument of a vector type from the constructor. The dependent variable \( k \) is not strictly speaking necessary:

\[
V^{-1} k (v k b) = b
\]
However, for any vector type \( c \), \( V^{-1} k c \) will result in a type error unless \( c = v k b \).

5.7 Theorem: \( V(a,n) \) is closed and complete under \( S,M,R \)

**Theorem 1.** Any type transformation on any vector type in \( V(a,N) \) that observes the rules from Section 3:

1. can be expressed as a combination of the operations \( S, M \) and \( R \), and
2. results in a vector type in \( V(a,N) \).
Proof.

• The most general expression for a type in our system is and multidimensional vector of type \(a\), where the size in every dimension is different. We consider two instances of this type:

\[
T_1 = [a] \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_k \rangle \\
T_2 = [a] \langle m_1 \rangle \langle m_2 \rangle \ldots \langle m_j \rangle \ldots \langle m_l \rangle
\]

where

\[
\prod_{i=1}^{k} n_i = \prod_{j=1}^{l} m_j = N
\]

and in general \(k \neq l\) and the various \(n_i\) and \(m_j\) values can be different.

• We aim to show that \(T_1\) can be transformed into \(T_2\) through application of a combination of the operations \(S, M\) and \(R\). Our approach is to first reduce \(T_1\) to a one-dimensional vector of size \(N\), and then transform this vector into \(T_2\).

• First we reduce the expression for \(T_1\) to \([a] \langle N \rangle\)as follows

1. First reshape the outer two vectors through application of \(R\):

\[
R^{-1} m_{k-1} T_1 \\
= R^{-1} m_{k-1} [a] \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_{k-1} \rangle \langle n_k \rangle \\
= [a] \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle 1 \rangle \langle n_{k-1} \rangle \langle n_k \rangle
\]

which can also be written as

\[
\left[ \left[ \langle a \rangle \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_{k-2} \rangle \langle 1 \rangle \langle n_{k-1} \rangle \langle n_k \rangle \right]\right]
\]

2. Then apply \(S^{-1}\) to the type of the outer vector:

\[
MS^{-1} \left[ \left[ \langle a \rangle \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_{k-2} \rangle \langle 1 \rangle \langle n_{k-1} \rangle \langle n_k \rangle \right]\right]
\]

\[
= \left[ S^{-1} \left[ \langle a \rangle \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_{k-2} \rangle \langle 1 \rangle \langle n_{k-1} \rangle \langle n_k \rangle \right]\right]
\]

\[
= \left[ \langle a \rangle \langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_i \rangle \ldots \langle n_{k-2} \rangle \langle n_{k-1} \rangle \langle n_k \rangle \right]
\]

3. Repeating these steps results in

\[
[a] \langle n_1, n_2, \ldots, n_i, \ldots, n_k \rangle
\]

which can be written as

\[
[a] \langle N \rangle
\]
Then we perform the reverse process to obtain $T_2$:

1. First increase the dimensionality by calling $S$ on the type of the outer vector

$$M S [a] \langle N \rangle = [[a](1)] \langle N \rangle$$

2. Then reshape the outer two vectors using $R$

$$R m_1 [[a](1)] \langle N \rangle = [[a] \langle m_1 \rangle] \langle N/m_1 \rangle$$

3. Repeating these steps results in

$$[a] \langle m_1 \rangle \langle m_2 \rangle \ldots \langle m_j \rangle \ldots \langle m_l \rangle = T_2$$

Corollary 1. An type transformation consisting of a combination of the transformations $S$, $M$ and $R$ is reversible.

6 Program Transformations

In this section we want to explore how transforming a top-level type impacts on the program. The context is the FPGA architecture developed for the TyTra project\(^1\), which is similar to the MORA architecture [Chalamalasetti et al., 2009], and we consider a simple pipeline of computations. Like MORA, and indeed most FPGA architectures, TyTra assumes that data is streamed, and we model this as a map over a vector.

I will use the notation $t_s$ or $t(s)$ to mean “the type of s”, and denote type-transformed functions and variables using a prime, e.g. $s'$.

I will use the notation $\{b\}$ to indicate “b may have to be transformed” during the inference process.

6.1 General Assumption on the Program

The general assumption is that the program is built entirely out of

- a set of functions on atomic types $f_j : a_i \rightarrow a_k$
- the fold operation \(^2\)[Hutton, 1999]
- the cons ($\cdot$) operation
- tuple construction ($,$)

\(^1\)http://tytra.org.uk/
However, we immediately note that \textit{foldl} can be defined in terms of \textit{fold} and the identity function \textit{id}:

\[
\text{foldl} f v x s = \text{foldr} \left( \lambda x g \rightarrow \left( \lambda a \rightarrow g \left( f a x \right) \right) \right) \text{id} x s v
\]

and \textit{map} in terms of \textit{fold} and \textit{cons}:

\[
\text{map} f = \text{foldr} \left( \lambda x \rightarrow \left( x : v \right) \right)
\]

and the same goes for all basic list operations, so we take those as given.

Essentially, our purpose is to split the program in computational functions and functions which describe the communication. Based on the type transformations, we aim to derive the transformation of those higher-order functions. We start with a few exploratory examples using \textit{map} and \textit{foldl}.

### 6.2 Increasing dimensionality – map

We assume a very simple program, we use Haskell syntax [Hudak et al., 1992] augmented with the \langle N \rangle notation to indicate sizes.

\[
\begin{align*}
s &:: [a] \langle N \rangle 
g &:: [a] \langle N \rangle \rightarrow [a] \langle N \rangle 
r &:: [b] \langle N \rangle 
f &:: a \rightarrow a 
r &\equiv g \ s 
g &\equiv \text{map} \ f
\end{align*}
\]

For completeness:

\[
\begin{align*}
\text{map} &:: (t_1 \rightarrow t_2) \rightarrow [t_1] \langle n \rangle \rightarrow [t_2] \langle n \rangle
\end{align*}
\]

We transform the top-level type:

\[
t'_s = R \ k \ M \ S \ t_s = R \ k \ M \ S \ [a] \langle N \rangle = [[a] \langle k \rangle] \langle N/k \rangle
\]

As \( g \) is applied to \( s \), this leads to a transformation of \( g \):

\[
t'_g = t'_s \rightarrow t'_r
\]

We assume that we only explicitly transform each of the arguments of \( g \).

Then we get:

\[
\begin{align*}
g' &:: t'_s \rightarrow [b] \langle N \rangle 
g' &\equiv \text{map} \ f'
\end{align*}
\]

We can substitute \( t_1 \) by the actual type of \( s' \) in the \textit{map} inside \( g' \):

\[
r' = g' \ s' = \text{map}_{g'} \ f' \ s'
\]

\[
\text{map}_{g'} :: \left( V^{-1} \ (N/k) \ t_{s'} \right) \rightarrow \{ b \} \rightarrow t_{s'} \rightarrow \{ [b] \langle N \rangle \}
\]

Clearly, this type can’t work for \textit{map} because the return type \([b] \langle N \rangle\) has a different size from \( t_{s'} \). So we need to transform that type:

\[
t'_{s'} = R \ k \ M \ S \ [b] \langle N \rangle = [b \langle k \rangle] \langle N/k \rangle
\]

This means that the signature for the map in \( g' \) becomes
\[
\text{map} g' :: (V^{-1}(N/k) t_s' \rightarrow V^{-1}(N/k) t_2') \rightarrow t_{s'} \rightarrow t_{2'}
\]

So that
\[
g' :: t_{s'} \rightarrow t_{2'}
\]

Consequently
\[
t_{s'} = t(g' s') = t_{2'}
\]

Rewriting the above in a more systematic way:
\[
g' s' = \text{map } f' s'
\]

\[
\text{map} g' :: (t_1 \rightarrow t_2) \rightarrow [t_1] \rightarrow [t_2]
\]

\[
\text{map} g' :: ((a) \rightarrow ([b]<k>) \rightarrow [[a]<k]<N/k> \rightarrow {{b<N}})
\]

\[
\text{map} g' :: (fV^{-1}N/k \ [[a]<k><N/k> \rightarrow ([b]<k>) \rightarrow [[a]<k><N/k> \rightarrow {b<N}})
\]

\[
\text{map} g' :: ([a]<k> \rightarrow ([b]<k>) \rightarrow [a]<k*m> \rightarrow R k M S {b<N})
\]

\[
\text{map} g' :: ([a]<k> \rightarrow ([b]<k>) \rightarrow [a]<k*m> \rightarrow [b]<k*m>
\]

\[
\Rightarrow f' :: [a]<k> \rightarrow b<k>
\]

\[
\Rightarrow r' :: [b]<k>N/k
\]

In other words, we can infer the return type from the single type transformation.

What we have so far is
\[
s' :: [a'] <N'>
\]
\[
g' :: [a'] <N'> \rightarrow [b'] <N'>
\]
\[
r' :: [b'] <N'>
\]
\[
f' :: a' \rightarrow b'
\]
\[
r' = g' s'
\]
\[
g' = \text{map } f'
\]

where

\[
\text{type } a' = [a] <k>
\]
\[
\text{type } b' = [b] <k>
\]
\[
N' = N/k
\]

What we need now is the transformations between \( s \) and \( s' \) and \( f \) and \( f' \)

To transform \( s \):
\[
s' = \text{reshapeTo } k s
\]

where

\[
\text{reshapeTo} :: \text{Int } k \Rightarrow k \rightarrow [a] <n> \rightarrow [[a] <k> <n/k>
\]
We define the inverse for further use:

\[ \text{reshapeFrom} :: \text{Int} \, k \Rightarrow k \rightarrow [a \langle k \rangle] \langle n \rangle \rightarrow [a] \langle n.k \rangle \]

So the \( R \, k \, M \, S \, t(s) \) type transformation maps directly to \( \text{reshapeTo} \, k \, s \).

The transformation from \( f \) to \( f' \) is even more straightforward, because the transformation of the original type raises the dimensionality

\[ f' = \text{map} \, f \]

In general, the original map is replaced by maps over both dimensions.

\[ g' = \text{map} \, \text{map} \, f \]

and in full, the transformed program becomes

\[ r = (\text{reshapeFrom} \, k) \cdot (\text{map map} \, f) \cdot (\text{reshapeTo} \, k) \, s \]

### 6.3 Reducing the dimensionality – map

Assume we have

\[ s :: \[[a \langle k \rangle]\langle m \rangle\] \]
\[ g :: \[[a \langle k \rangle]\langle m \rangle \rightarrow \[[b \langle k \rangle]\langle m \rangle\] \]
\[ g = \text{map} \, f \]
\[ f :: [a \langle k \rangle] \rightarrow [b \langle k \rangle] \]
\[ r = g \, s \]

And we apply the transformation \( M \, S^{-1} \, R^{-1} \, k \) to \( s \):

\[ R^{-1} \, k \, t(s) = [[a \langle 1 \rangle] \langle k.m \rangle] \]
\[ M \, S^{-1} \, [[a \langle 1 \rangle] \langle k.m \rangle] = [a] \langle k.m \rangle \]
\[ t(s') = [a \langle k.m \rangle] \]

So we obtain

\[ s' :: = [a \langle k.m \rangle] \]

As \( g' \) is applied to \( s' \), we obtain

\[ g' :: [a \langle k.m \rangle] \rightarrow \{[[b \langle k \rangle]\langle m \rangle]\} \]

Now we use inference on \( \text{map} \):

\[ g' \, s' = \text{map} \, f' \, s' \]
\[ \text{map}_{g'} :: (t_1 \rightarrow t_2) \rightarrow [t_1] \rightarrow [t_2] \]
\[ \text{map}_{g'} :: (a \rightarrow \{[b] \langle k \rangle\}) \rightarrow [a \langle k.m \rangle] \rightarrow \{[[b] \langle k \rangle]\langle m \rangle\} \]
\[ \text{map}_{g'} :: (a \rightarrow \{[b] \langle k \rangle\}) \rightarrow [a \langle k.m \rangle] \rightarrow M \, S^{-1} \, R^{-1} \, k \, \{[[b] \langle k \rangle]\langle m \rangle\} \]
\[ \text{map}_{g'} :: (a \rightarrow \{[b] \langle k \rangle\}) \rightarrow [a \langle k.m \rangle] \rightarrow \{[b \langle k.m \rangle]\} \]
\[ \text{map}_{g'} :: (a \rightarrow \{V^{-1} \, k \, m \, (b \langle k.m \rangle)\}) \rightarrow [a \langle k.m \rangle] \rightarrow [b \langle k.m \rangle] \]
\[ \text{map}_{g'} :: (a \rightarrow b) \rightarrow [a \langle k.m \rangle] \rightarrow [b \langle k.m \rangle] \]
\[ \Rightarrow f' :: a \rightarrow b \]
\[ \Rightarrow r' :: [b \langle k.m \rangle] \]
to express $f'$ as a function of $f$, we need a `toVector k x` function

```haskell
toVector :: Int k ⇒ k → a → [a]⟨k⟩
```

The most intuitive implementation seems to be

```haskell
toVector :: k x = replicate k x
```

Similarly, we need `fromVector k x` (although we don’t really need $k$)

```haskell
fromVector :: Int k ⇒ k → [a]⟨k⟩ → a
```

The most intuitive implementation seems to be

```haskell
fromVector k (x:_ ) = x
```

With these, we simply say

$$f' x = fromVector k (f (toVector k x))$$

### 6.3.1 Correctness condition

In general, the above transformation does not necessarily preserve the computation. However, we can see that a sufficient condition to preserves the computation is that `map f' = f`:

**Lemma 1.** Mapping $f'$ over $s'$ preserves the computation of mapping $f$ over $s$ iff

$$f = map h$$

**Proof.**

1. Observe that $s' = reshapeFrom k s$ and we must show that

$$r' = g' s' = reshapeFrom k r = reshapeFrom k g s$$

2. We show that $map f' s' = map h s'$

   (a) Mapping $f'$ to $s'$:
   
   $$r' = g' s' = map f' s'$$
   
   $$= [f' x1,f' x2,...,f' zk,f' y1,f' y2,...,f' yk,...,f' z1,f' z2,...,f' zk]$$

   (b) $f'$ is identical to $h$:
   
   $$f' x$$
   
   $$= fromVector k (f (toVector k x))$$
   
   $$= head (f [x])$$
   
   $$= head (map h [x])$$
   
   $$= head [h x]$$
   
   $$= h x$$
   
   $$⇒ f' = h$$
3. Mapping \( f \) to \( s \):

\[
r = g \ s = \text{map} \ f \ \ s
\]

\[
= \text{map} \ f \ [[x_1,x_2,...,x_k],[y_1,y_2,...,y_k],...,[z_1,z_2,...,z_k]]
\]

\[
= [[f \ x_1,x_2,...,x_k],[f \ y_1,y_2,...,y_k],...,[f \ z_1,z_2,...,z_k]]
\]

\[
= [\text{map} \ h \ [x_1,x_2,...,x_k],[\text{map} \ h \ [y_1,y_2,...,y_k],...,[\text{map} \ h \ [z_1,z_2,...,z_k]]
\]

\[
= [[h \ x_1,h \ x_2,...,h \ x_k],[h \ y_1,h \ y_2,...,h \ y_k],...,[h \ z_1,h \ z_2,...,h \ z_k]]
\]

4. Finally, transforming \( r \) to \( r' \):

\[
\text{reshapeFrom} \ k \ r
\]

\[
= \text{reshapeFrom} \ k [[h \ x_1,h \ x_2,...,h \ x_k],[h \ y_1,h \ y_2,...,h \ y_k],...,[h \ z_1,h \ z_2,...,h \ z_k]]
\]

\[
= [h \ x_1,h \ x_2,...,h \ x_k,h \ y_1,h \ y_2,...,h \ y_k,...,h \ z_1,h \ z_2,...,h \ z_k]
\]

\[
= r'
\]

6.4 Preserving the dimensionality – \( \text{map} \)

With the same example as above, we apply the transformation \( R n R^{-1} k \) to \( s \):

\[
R^{-1} k \ t(s) = [[a \langle 1 \rangle \langle k.m \rangle]
\]

\[
R n [[a \langle 1 \rangle \langle k.m \rangle] = [a \langle n \rangle \langle k.m/n \rangle]
\]

\[
t(s') = [a \langle n \rangle \langle k.m/n \rangle]
\]

So we obtain

\[
s' : = [a \langle n \rangle \langle k.m/n \rangle]
\]

As \( g' \) is applied to \( s' \), we obtain

\[
g' : = [a \langle n \rangle \langle k.m/n \rangle \rightarrow \{[b \langle k \rangle \langle m \rangle}\}
\]

Again we use inference on \( \text{map} \):

\[
g' \ s' = \text{map} \ f' \ s'
\]

\[
\text{map}'_g : (t_1 \rightarrow t_2) \rightarrow [t_1] \rightarrow [t_2]
\]

\[
\text{map}'_g : (a<n> \rightarrow \{[b]<k>\}) \rightarrow [a<n>]<k.m/n> \rightarrow \{[b]<k>\}<m>
\]

\[
\text{map}'_g : (a<n> \rightarrow \{[b]<k>\}) \rightarrow [a]<k.m> \rightarrow R n R^{-1} k \ \{[b]<k>\}<m>
\]

\[
\text{map}'_g : (a<n> \rightarrow \{[b]<k>\}) \rightarrow [a]<k.m> \rightarrow [b]<k.m/n>\rightarrow [a]<k.m> \rightarrow [b]<k.m>
\]

\[
\Rightarrow f' : \text{a<n> \rightarrow b<n> \rightarrow [a]<k.m> \rightarrow [b]<k.m>}
\]

\[
\Rightarrow r' : [b<n>]<k.m/n>
\]

As \( \text{map} \) is independent of the size of the vector, we have

\[
f' = f
\]

Consequently, the computation will always be preserved.
6.5 Increasing dimensionality – fold

We can easily show that if the operation on \( f \) is a fold, then increasing the dimensionality results in applying the fold to every dimension.

**Lemma 2.** Repeated application of fold to a nested list is equivalent to applying fold to the flattened list

\[
\text{fold} (\text{fold} f) \text{ acc } [x_1,x_2,...,x_k],[y_1,y_2,...,yk],... = \text{fold} f \text{ acc } [x_1,x_2,...,x_k,y_1,y_2,...,yk,...,z_1,z_2,...,zk]
\]

**Proof.**

\[
\text{fold} (\text{fold} f) \text{ acc } [x_1,x_2,...,x_k],[y_1,y_2,...,y],[z_1,z_2,...,zk]
\]

\[
= (\text{fold} f \ldots (\text{fold} f (\text{fold} f \text{ acc } [x_1,x_2,...,x_k]) [y_1,y_2,...,yk]) \ldots [z_1,z_2,...,zk])
\]

\[
= (\text{fold} f \ldots (\text{fold} f (f \ldots (f (f \text{ acc } x_1) x_2) \ldots x_k) [y_1,y_2,...,yk]) \ldots [z_1,z_2,...,zk])
\]

\[
= (f \ldots (f \ldots (f \ldots (f (f \text{ acc } x_1) x_2) \ldots x_k) y_1) y_2) \ldots y_k)
\]

\[
= \text{fold} f \text{ acc } [x_1,x_2,...,x_k,y_1,y_2,...,yk,...,z_1,z_2,...,zk]
\]

Furthermore, as we consider a streaming operations, we only consider the left fold (\( \text{foldl} \)).

We assume the same program as for \( \text{map} \) above:

\[
s :: [a] \langle n \rangle
\]

\[
g :: [a] (\langle n \rangle \rightarrow b)
\]

\[
r :: b
\]

\[
f :: b \rightarrow a \rightarrow b
\]

\[
\text{acc} :: b
\]

\[
r = gs
\]

\[
g = \text{fold} f \text{ acc}
\]

For completeness:

\[
\text{fold} :: (t_2 \rightarrow t_1 \rightarrow t_2) \rightarrow t_2 \rightarrow [t_1] \langle m \rangle \rightarrow t_2
\]

We transform the top-level type:

\[
t' = R k M S t_s = R k M S [a]<n> = [\langle a\rangle<\langle k\rangle><n/k>]
\]

So we obtain

\[
s' :: [a \langle k\rangle] \langle n/k \rangle
\]

As \( g' \) is applied to \( s' \), we obtain

\[
g' :: [a \langle n\rangle] \langle k.m/n \rangle \rightarrow \{b\}
\]

Using inference on \( \text{fold} \):

\[
g' s' = \text{fold} f' \text{ acc } s'
\]

\[
\text{fold}_{g'} :: (t_2 \rightarrow t_1 \rightarrow t_2) \rightarrow t_2 \rightarrow [t_1]<m> \rightarrow t_2
\]

\[
\text{fold}_{g'} :: ([b] \rightarrow \{a\} \rightarrow \{b\}) \rightarrow \{b\} \rightarrow [a]<\langle k\rangle<\langle n/k\rangle> \rightarrow \{b\}
\]

\[
\text{fold}_{g'} :: (\langle b\rangle \rightarrow \langle V^{-1}.n/k \langle a\rangle<\langle k\rangle<\langle n/k\rangle\rangle \rightarrow \{b\} \rangle \rightarrow \{a\}<\langle k\rangle<\langle n/k\rangle> \rightarrow \{b\})
\]

\[
\text{fold}_{g'} :: (\{b\} \rightarrow [a]<\langle k\rangle \rightarrow \{b\} \rangle \rightarrow [a]<\langle k\rangle<\langle n/k\rangle> \rightarrow \{b\})
\]
At this point, the types are valid, so no further transformation is required

\[
\text{fold}_g :: (b \to [a]_{<k>} \to b) \to [a]_{<k>\langle n/k \rangle} \to b \\
\Rightarrow f' :: b \to [a]_{<k>} \to b \\
\Rightarrow r' :: b \\
\Rightarrow \text{acc'} :: b
\]

To transform \( f \) into \( f' \):

\[ f' = \text{fold} \]

### 6.6 Decreasing dimensionality – fold

We assume the same program as for map above:

\[
s :: [a \langle k \rangle \langle m \rangle] \\
g :: [a \langle k \rangle \langle m \rangle] \to b \\
r :: b \\
f :: b \to a \langle k \rangle \to b \\
\text{acc} :: b \\
r = gs \\
g = \text{fold} f \text{acc}
\]

For completeness:

\[
\text{fold} :: (t_2 \to t_1 \to t_2) \to t_2 \to [t_1 \langle m \rangle] \to t_2
\]

We transform the top-level type:

\[
t'_s = MS^{-1}R^{-1}k \quad ts = MS^{-1}R^{-1}k [a]_{<k>\langle m \rangle} = [a]_{<k,m>}
\]

So we obtain

\[
s' :: [a]_{\langle k,m \rangle} \\
s' = \text{flatten} s
\]

As \( g' \) is applied to \( s' \), we obtain

\[
g' :: [a]_{\langle k,m \rangle} \to \{b\}
\]

Using inference on \( \text{fold} \):

\[
g' \ s' = \text{fold} f' \ \text{acc} \ s' \\
\text{fold}_{g'} :: (t_2 \to t_1 \to t_2) \to t_2 \to [t_1\langle m \rangle] \to t_2 \\
\text{fold}_{g'} :: (\{b\} \to \{a\}_{<k>\langle m \rangle} \to \{b\}) \to \{b\} \to [a]_{<k,m>} \to \{b\} \\
\text{fold}_{g'} :: (\{b\} \to (V^{-1} k.m \ [a]_{<k,m>} \to \{b\}) \to \{b\} \to [a]_{<k,m>} \to \{b\} \\
\text{fold}_{g'} :: (\{b\} \to a \to \{b\}) \to [a]_{<k,m>} \to \{b\}
\]

At this point, the types are valid, so no further transformation is required
fold_{g'} :: (b -> a -> b) -> [a]\<k*m> -> b
⇒ f' :: b -> a -> b
⇒ r' :: b
⇒ acc' :: b

To transform \( f \) into \( f' \):

\[
f' \text{acc} x = f \text{acc} (\text{toVector} k x)
\]

### 6.6.1 Correctness condition

In the case of fold, “preserves the computation” means “produces an identical result”, as from the perspective of the type transformation, the type \( b \) is opaque. In general, folding \( f' \) over \( s' \) is not equal to folding \( f \) over \( s \). However, a sufficient condition for equality is this:

**Lemma 3.** Folding \( f' \) over \( s' \) is equal to folding \( f \) over \( s \) iff

\[
f = \text{fold} h
\]

**Proof.**

1. \( \text{foldl} f' \text{acc} s' = \text{foldl} h \text{acc} s' \)

   \[
   \text{foldl} f' \text{acc} [x_1,x_2,...,x_k,y_1,y_2,...,y_k,...,z_1,z_2,...,z_k]
   \]
   \[\text{def. of } f'\]
   \[
   = \text{foldl} (\lambda x \to f \text{acc} (\text{toVector} k x)) \text{acc} [x_1,x_2,...,x_k,y_1,y_2,...,y_k,...,z_1,z_2,...,z_k]
   \]
   \[\text{def of toVector}\]
   \[
   = \text{foldl} (\lambda x \to f \text{acc} [x]) \text{acc} [x_1,x_2,...,x_k,y_1,y_2,...,y_k,...,z_1,z_2,...,z_k]
   \]
   \[\text{def of foldl}\]
   \[
   = \text{foldl} h \text{acc} [x] = h \text{acc} x
   \]
   \[\eta\text{ conversion}\]
   \[
   = \text{foldl} h \text{acc} [x_1,x_2,...,x_k,y_1,y_2,...,y_k,...,z_1,z_2,...,z_k]
   \]
   \[\text{def. of foldl}\]
   \[
   = (h (...) (h (h (...) (h (h (h \text{acc} x_1) x_2) ... x_k) y_1) y_2) ...
   \]
   \[\text{... } y_k) ...) z_1) z_2) ... ) z_k)
   \]

2. \( \text{foldl} h \text{acc} s' = \text{foldl} f \text{acc} s \)

   \[
   \text{foldl} f \text{acc} s
   \]
   \[\text{def. of } f\]
   \[
   = \text{foldl} (\text{foldl} h) \text{acc} s
   \]
   \[\text{Lemma } 2 + \text{ def. of } s'\]
   \[
   = \text{foldl} h \text{acc} s'
   \]

\[\square\]
6.7 About zip and unzip

We use zip and unzip to change nested lists of tuples into tuples of nested lists.

\[
\begin{align*}
\text{zip} & : \ [a]^{<n>} \rightarrow [b]^{<n>} \rightarrow [(a,b)]^{<n>} \\
\text{unzip} & : \ [(a,b)]^{<n>} \rightarrow ([a]^{<n>}, [b]^{<n>})
\end{align*}
\]

The same type transformation must be applied to both arguments, e.g. for \( R \ k \ [a]^{<n>} \). In order to preserve the computation, it is quite clear that

\[
\text{zip'} : \ [(a,k)]^{<n/k>} \rightarrow [(b,k)]^{<n/k>} \rightarrow [[(a,b)]^{<k>}]^{<n/k>}
\]

can be implemented in terms of zip as

\[
\text{zip'} \ x's' \ y's' = \text{map} \ (\lambda (x,y) \rightarrow \text{zip} \ x \ y) \ (\text{zip} \ x's' \ y's')
\]

and similar for unzip.

To simplify the discussion, we introduce a variant of zip, zipt, which takes a single tuple as argument, and a corresponding unzipt.

\[
\begin{align*}
\text{zipt} & : \ ([a]^{<n>}, [b]^{<n>}) \rightarrow [(a,b)]^{<n>} \\
\text{zipt} \ (xs,ys) & = \text{zip} \ xs \ ys
\end{align*}
\]

and

\[
\begin{align*}
\text{unzipt} & : \ [(a,b)]^{<n>} \rightarrow ([a]^{<n>}, [b]^{<n>}) \\
\text{unzipt} \ ltups & = \text{map} \ \text{fst} \ ltups, \ \text{map} \ \text{snd} \ ltups
\end{align*}
\]

then zipt’ becomes

\[
\begin{align*}
\text{zipt'} : \ ([a]^{<k>^{<n/k>}}, [b]^{<k>^{<n/k>}}) \rightarrow [(a,b)]^{<k>^{<n/k>}} \\
\text{zipt'} \ \text{tup} & = \text{map} \ \text{zipt} \ (\text{zipt} \ \text{tup})
\end{align*}
\]

and similar for unzipt.

7 Conclusion

The approach described allows to transform programs consisting of combinations of map, foldl and zip based on transformation of the types of the vectors on which the map or fold acts.

We have shown that the the set \( V(a,n) \) of vectors of type \( a \) and size \( n \) is closed under the proposed operations for transforming the vector types, \( S, M \) and \( R \), with the corollary that every combination of the transformations is reversible.

We have shown that, for programs consisting of opaque functions and the operations map, foldl and zip, the program transformations can be automatically derived from the type transformations.

This mechanism allows to generate correct-by-construction variants of the programs. The purpose of this works is to allow automatic selection of the variant most suitable for a given platform through optimisation against a platform cost model.

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