Central Limit Theorem for Peaks of a Random Permutation in a Fixed Conjugacy Class of $S_n$

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Abstract. The number of peaks of a random permutation is known to be asymptotically normal. We give a new proof of this and prove a central limit theorem for the distribution of peaks in a fixed conjugacy class of the symmetric group. Our technique is to apply analytic combinatorics to study a complicated but exact generating function for peaks in a given conjugacy class.

1. Introduction

We say that a permutation on $n$ symbols has a descent at position $i$ if $\pi(i) > \pi(i+1)$, and we let $d(\pi)$ denote the number of descents of $\pi$. For example, the permutation $143265$ has descents at positions 2, 3 and 5, and has $d(\pi) = 3$. Descents appear in numerous parts of mathematics. For example, see Knuth [16] for connections of descents with the theory of sorting and the theory of runs in permutations and see Bayer and Diaconis [1] for applications of descents to card shuffling. The number $A(n, k)$ of permutations on $n$ symbols with $k$ descents is called an Eulerian number, and there is an entire book devoted to their study [18].

It is well known that the distribution of descents $d(\pi)$ in $S_n$ is asymptotically normal with mean $(n - 1)/2$ and variance $(n + 1)/12$, in the sense that for $n \geq 2$, $d(\pi)$ has the prescribed mean and variance and its normalization $(d(\pi) - \mathbb{E}[d(\pi)])/\sqrt{\text{Var}(d(\pi))}$ converges in distribution to the standard normal distribution. There are many proofs of this:

(a) Pitman [20] uses real-rootedness of the Eulerian polynomials

$$ A_n(t) := \sum_{\pi \in S_n} t^{d(\pi)+1} $$

(b) The generating function $A_n(t)$ can be expressed as an Eulerian polynomial in $t$, which is real-rooted.

(c) The central limit theorem for descents follows from the fact that the descents of a random permutation are negatively correlated.

The number of peaks of a permutation is also asymptotically normal, as proved in [12]. However, the number of peaks in a fixed conjugacy class is more complicated to study, and our goal is to prove a central limit theorem for this distribution using analytic combinatorics.

Our technique is to apply analytic combinatorics to study a complicated but exact generating function for peaks in a given conjugacy class. This involves proving a number of new results about the combinatorics of permutations in fixed conjugacy classes, which we believe are of independent interest.
(Some authors have exponent $d(\pi)$ instead of $d(\pi) + 1$).

(b) David and Barton [3] use the method of moments.

(c) Tanny [25] uses the fact that if $U_1, \ldots, U_n$ are independent uniform [0, 1] random variables, then for all integers $k$,

$$\mathbb{P} \left( k \leq \sum_{i=1}^{n} U_i < k + 1 \right) = A(n, k)/n!$$

(d) Fulman [8] uses Stein’s method.

There is also interesting literature on the joint distribution of descents and cycles. Gessel and Reutenauer [11] use symmetric function theory to enumerate permutations with a given cycle structure and descent set, and Diaconis, McGrath, and Pitman [7] interpret this in the context of card shuffling. We regard these exact results as a miracle, and they enable one to write down an exact (but quite complicated) generating function for descents of permutations in a given conjugacy class. These exact generating functions make it possible to prove central limit theorems for the number of descents in fixed conjugacy classes of the symmetric group. Fulman [9] proved a central limit theorem when the conjugacy classes consist of large cycles. Almost twenty years later, Kim [14] proved a central limit for descents in random fixed point-free involutions. Quite recently, Kim and Lee [15] proved a central limit theorem for arbitrary conjugacy classes. These results would be very difficult to obtain without exact generating functions.

Given the above discussion, it is natural to ask if there are other permutation statistics for which there is exact information about the joint distribution with cycle structure. In their work on casino shuffling machines, Diaconis, Fulman, and Holmes [5] discovered that there is an exact generating function for the number of peaks of a permutation enumerated according to cycle structure. Let us describe their result. We say that a permutation $\pi \in S_n$ has a peak at position $1 < i < n - 1$ if $\pi(i - 1) < \pi(i) > \pi(i + 1)$, and we let $p(\pi)$ be the number of peaks of $\pi$. Thus $\pi = 1426753$ has peaks at positions 2 and 5, so that $p(\pi) = 2$. Letting $\lambda$ be a partition of $n$ with $n_i$ parts of size $i$, Corollary 3.8 of [5] gives that

$$\sum_{\pi \in C_\lambda} \left( \frac{4t}{(1+t)^2} \right)^{p(\pi)+1} = 2 \left( \frac{1-t}{1+t} \right) \sum_{a=1}^{\infty} t^a \prod_{i} [x_i^{n_i}] \left( \frac{1+x_i}{1-x_i} \right)^{f_{a,i}}. \quad (1.1)$$

Here, $C_\lambda$ denotes the elements of $S_n$ of cycle type $\lambda$, and $[x_i^{n_i}] g(x_i)$ denotes the coefficient of $x_i^{n_i}$ in the function $g(x_i)$, and

$$f_{a,i} := \frac{1}{2i} \sum_{d \mid i \text{ odd}} \mu(d)(2a)^{i/d},$$

where $\mu$ is the Möbius function of the lattice of integers ordered by divisibility. (The result of [5] actually deals with valleys rather than peaks, but the joint generating function with cycle structure is the same as can be seen by conjugating by the longest permutation $n \ldots 21$). The reader will agree that the generating function (1.1) looks hard to deal with (it need not be real-rooted),
and our main insight is that we can adapt the methods of Kim and Lee \[15\] to analyze it.

To close the introduction, we mention that the number of peaks of a permutation is a feature of interest. The paper \[5\] uses peaks to analyze casino shelf-shuffling machines. The number of peaks is classically used as a test of randomness for time series; see Warren and Seneta \[27\] and their references, which also include a central limit theorem for the number of peaks for a uniform random permutation. Permutations with no peaks are called unimodal (usually unimodal refers to no valleys but these are equivalent for our purposes), and are of interest in social choice theory through Coombs’s “unfolding hypothesis” (see Chapter 6 of \[4\]). They also appear in dynamical systems and magic tricks (see Chapter 5 of \[6\]).

Finally, we note that peaks have been widely studied by combinatorialists; see Petersen \[19\], Stembridge \[24\], Nyman \[17\], Schocker \[23\] and a paper of Billey, Burdzy, and Sagan \[2\], for a small sample of combinatorial work on peaks.

1.1. Main results

To motivate the readers, we first demonstrate a numerical simulation result. Figure 1 is a histogram of peaks of $10^5$ permutations drawn from the conjugacy class $C_{2^{250}4^{125}} \subset S_{1000}$.

The histogram suggests that the number of peaks of permutations in $C_{2^{250}4^{125}}$ are normally distributed, and indeed, the p.d.f. of $\mathcal{N}\left(\frac{n-2}{3}, \frac{2(n+1)}{45}\right)$ with $n = 1000$ fits very well. This suggests that the behavior of peaks for a particular conjugacy class is mostly the same as that of peaks for $S_n$. This does turn out to be true for conjugacy classes with no fixed points, as the following main theorem states that the asymptotic distribution of peaks in conjugacy classes is normal, where the asymptotic mean and variance depend only on the density of fixed points.
Theorem 1.1. Let \( C_n \) be a conjugacy class of \( S_n \) for each \( n \geq 1 \). Denote by \( \alpha_1(C_n) \) the fraction of fixed points of each element of \( C_n \). Suppose that \( \pi_n \) is chosen uniformly at random from \( C_n \) and that \( \alpha_1(C_n) \) converges to some \( \alpha \in [0, 1] \) as \( n \to \infty \). Then, as \( n \to \infty \),

\[
\frac{p(\pi_n) - \left( \frac{1 - \alpha_1(C_n)}{3} \right) n}{\sqrt{n}} \text{ converges in distribution to } N\left(0, \frac{2}{45} + \frac{\alpha^3}{9} - \frac{3\alpha^5}{5} + \frac{4\alpha^6}{9}\right).
\]

Our main strategy is to adopt the modified Curtiss’ theorem from [15], which relates convergence in distribution of random variables to the pointwise convergence of their moment generating functions on an open set. In this regard, the main theorem is a direct consequence of the following technical theorem:

Theorem 1.2. For each \( s > 0 \), there exists a universal constant \( C = C(s) > 0 \), depending only on \( s \), such that the following is true: Let \( C_\lambda \subset S_n \) be the conjugacy class of cycle type \( \lambda = 1^{n_1}2^{n_2} \ldots \) and \( \pi \) be chosen uniformly at random from \( C_\lambda \). Denote by \( \alpha_1 = n_1/n \) the density of fixed points. Then,

\[
E\left[ e^{-sp(\pi)/\sqrt{n}} \right] = \exp \left\{ - \left( \frac{1 - \alpha_1}{3} \right) s \sqrt{n} + \left( \frac{1}{45} + \frac{\alpha_1^3}{18} - \frac{3\alpha_1^5}{10} + \frac{2\alpha_1^6}{9} \right) s^2 + E_{\lambda,s} \right\},
\]

where \( |E_{\lambda,s}| \leq Cn^{-1/4}\sqrt{\log n} \).

This theorem is interesting in its own right because the uniform estimate allows us to generalize the scope of the main theorem to a broader class of sequences \( (C_n) \). More precisely, the statement of Theorem 1.1 readily extends to the case where each \( C_n \) is a conjugacy-invariant subset of \( S_n \) such that every element of \( C_n \) has the same number of fixed points. For example, if we consider the set of all elements of \( S_n \) with zero fixed points, we would obtain a central limit theorem for peaks of derangements.

1.2. Notation

Consider families of functions \((f_a)_{a \in A}\) and \((g_a)_{a \in A}\) from a set \( D \) to \( \mathbb{R} \), parametrized by the index set \( A \). Then we write

\[
f_a(x) = O_a(g_a(x)), \quad \text{or equivalently} \quad f_a(x) \lesssim_a g_a(x)
\]

in the range \( x \in D \) if there exists a constant \( C(a) \in (0, \infty) \), depending only on the parameter \( a \), such that \( |f_a(x)| \leq C(a)g_a(x) \) for any \( x \in D \). When \( A \) is a singleton so that each family contains exactly one function, we suppress the parameter from notation.
2. Central limit theorem for peaks of a random permutation in $S_n$

Recall that

$$A_n(t) = \sum_{\pi \in S_n} t^{d(\pi)+1}.$$ 

Define the peak generating function by

$$W_n(t) := \sum_{\pi \in S_n} t^{p(\pi)+1}.$$ 

Then it is well known [24, p. 779] that $A_n(t)$ and $W_n(t)$ are related by the identity

$$W_n \left( \frac{4t}{(1+t)^2} \right) = \left( \frac{2}{1+t} \right)^{n+1} A_n(t). \quad (2.1)$$

Our aim in this section is to identify the asymptotic distribution of peaks of a random permutation in $S_n$ using (2.1).

2.1. Computing mean and variance of peaks in $S_n$

We will compute the mean and variance of the number of peaks of a random permutation in $S_n$. The next result will be useful for this purpose.

**Lemma 2.1.** For each $n \geq 4$, the peak generating function $W_n$ satisfies

$$\frac{W_n'(1)}{n!} = \frac{n+1}{3} \quad \text{and} \quad \frac{W_n''(1)}{n!} = \frac{5n^2 - 3n - 8}{45}.$$

**Proof.** It is well known that the Eulerian polynomials satisfy the identity

$$A_n(t) = (1 - t)^{n+1} \sum_{a=0}^{\infty} a^n t^a.$$ 

In addition, recall that the Stirling numbers of the second kind $\{n\_k\}$ count the number of partitions of an $n$-element set into $k$ blocks. Then by using the expansion of $a^n$ in terms of falling factorials,

$$a^n = \sum_{k=0}^{n} \binom{n}{k} \frac{a!}{(a-k)!},$$

we obtain

$$\sum_{a=0}^{\infty} a^n t^a = \sum_{k=0}^{n} \binom{n}{k} \left( \sum_{a=0}^{\infty} \frac{a!}{(a-k)!} t^a \right) = \sum_{k=0}^{n} \binom{n}{k} \frac{k! t^k}{(1-t)^{k+1}}.$$ 

Plugging this into the above identity for the Eulerian polynomial $A_n$,

$$A_n(t) = \sum_{k=0}^{n} k! \binom{n}{k} t^k (1-t)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (n-k)! \binom{n}{n-k} t^{n-k} (1-t)^k.$$
Now we substitute \( t = \frac{1 - \varepsilon}{1 + \varepsilon} \) into (2.1) and into the above identity. Using the relations \( \frac{2}{1 + t} = 1 + \varepsilon, \) \( 1 - t = \frac{2\varepsilon}{1 + \varepsilon}, \) and \( \frac{4t}{(1 + t)^2} = 1 - \varepsilon^2 \), the identity (2.1) reduces to

\[
W_n(1 - \varepsilon^2) = \sum_{k=0}^{n} (n-k)! \binom{n}{n-k} (1 + \varepsilon)(1 - \varepsilon)^{n-k}(2\varepsilon)^k.
\]

From this, for each positive integer \( p \), we get

\[
W_n^{(p)}(1) = p!(-1)^p \left( [\varepsilon^{2p}]W_n(1 - \varepsilon^2) \right)
= p!(-1)^p \sum_{k=0}^{n} (n-k)! \binom{n}{n-k} (2^k)[\varepsilon^{2p-k}](1 + \varepsilon)(1 - \varepsilon)^{n-k}
= p!(-1)^p \sum_{k=0}^{2p} (n-k)! \binom{n}{n-k}(-2^k)\left( \frac{n-k}{2p-k} - \frac{n-k}{2p-k-1} \right).
\]

So it remains to find the values of \( \binom{n}{n-k} \)'s for \( k = 0, \ldots, 4 \). From their combinatorial interpretation, we easily obtain \( \binom{n}{n} = 1 \) and \( \binom{n}{n-1} = \binom{n}{2} \). For other values of \( k \), they can be systematically computed by employing the relationship between the Stirling numbers of the second kind and Eulerian numbers of the second kind (see equation (6.43) of [12]). The \( \binom{n}{n-k} \)'s relevant to us are

\[
\begin{align*}
\binom{n}{n-2} &= 2\binom{n}{4} + \binom{n+1}{4}, \\
\binom{n}{n-3} &= 6\binom{n}{6} + 8\binom{n+1}{6} + \binom{n+2}{6}, \text{ and} \\
\binom{n}{n-4} &= 24\binom{n}{8} + 58\binom{n+1}{8} + 22\binom{n+2}{8} + \binom{n+3}{8}.
\end{align*}
\]

Plugging these back into the expansion of \( W_n(1 - \varepsilon^2) \) proves the desired lemma. \( \square \)

Now we return to the computation of the mean and variance of \( p(\pi) \) in \( S_n \). Since the distribution for \( p(\pi) \) is related to \( W_n \) by the identity \( W_n(z) = n!E[z^{p(\pi)+1}] \), we obtain \( W_n'(1) = n!E[p(\pi) + 1] \) and \( W_n''(1) = n!E[(p(\pi) + 1)p(\pi)] \). Then for \( n \geq 4 \), it follows that

\[
E[p(\pi)] = \frac{W_n'(1)}{n!} - 1 = \frac{n-2}{3}
\]

and

\[
\text{Var}(p(\pi)) = \frac{W_n''(1)}{n!} + \frac{W_n'(1)}{n!} - \left( \frac{W_n'(1)}{n!} \right)^2 = \frac{2(n+1)}{45}.
\]

At this point, it is worth noting (2.1) implies that, like \( A_n(t) \), \( W_n(t) \) has only real roots, and so, by Harper’s method [13], we can obtain a central limit theorem for peaks of a random permutation in \( S_n \). In the upcoming section, we give a new proof of this central limit theorem by using analytic combinatorics and will go further to prove a central limit theorem for peaks in arbitrary
conjugacy classes of $S_n$, where the asymptotic mean and variance depend only on the density of fixed points in the conjugacy classes.

### 2.2. Establishing the asymptotic normality of peaks in $S_n$

Kim and Lee [15] proved the following modification of Curtiss’ theorem:

**Theorem 2.2.** Let $X_n$ be random vectors in $\mathbb{R}^d$ for each $n \in \mathbb{N} \cup \{\infty\}$ and $M_{X_n}(s) = \mathbb{E}[e^{sX_n}]$ be the moment generating function (m.g.f.) of $X_n$. Suppose there is a non-empty open subset $U \subseteq \mathbb{R}^d$ such that $\lim_{n \to \infty} M_{X_n}(s) = M_{X_\infty}(s)$ for all $s \in U$. Then, $X_n$ converges in distribution to $X_\infty$.

This theorem will be used in this subsection to prove a central limit theorem about peaks of permutations chosen, uniformly at random, from $S_n$, and in Section 3 to prove an analogous theorem about peaks of permutations chosen, uniformly at random, from arbitrary conjugacy classes, where the asymptotic mean and variance are functions of only $\alpha$, the density of fixed points in the conjugacy classes.

**Theorem 2.3.** Let $\pi_n$ be chosen uniformly at random from $S_n$. Then $p(\pi_n)$ is asymptotically normal with mean $\frac{n-2}{3}$ and variance $\frac{2(n+1)}{45}$. More precisely, for $n \geq 4$, $p(\pi_n)$ has the aforementioned mean and variance, and as $n \to \infty$, $\frac{p(\pi_n) - \frac{n-2}{3}}{\sqrt{n}}$ converges in distribution to $\mathcal{N}(0, \frac{2}{45})$.

**Proof.** Let $X_n = (p(\pi_n) - \frac{n-2}{3}) / \sqrt{n}$. In light of Theorem 2.2, it suffices to show that $M_{X_n}$ converges pointwise to the m.g.f. of $\mathcal{N}(0, \frac{2}{45})$ on some open interval. Assume $0 < t < 1$. Then by a simple comparison, we get

$$t \left( \frac{n!}{\log^{n+1}(1/t)} \right) = \int_0^\infty a^n t^{a+1} \, da \leq \sum_{a=0}^\infty a^n t^a$$

$$\leq \int_0^\infty a^n t^{a-1} \, da = \frac{1}{t} \left( \frac{n!}{\log^{n+1}(1/t)} \right).$$

Plugging this into the peak generating function (2.1) and using the identity for the Eulerian polynomial $A_n$, we obtain

$$\frac{1}{n!} W_n \left( \frac{4t}{(1+t)^2} \right) = \frac{1}{n!} \left( \frac{2(1-t)}{1+t} \right)^{n+1} \sum_{a=0}^\infty a^n t^a$$

$$= e^{O(\log(1/t))} \left( \frac{2(1-t)}{(1+t) \log(1/t)} \right)^{n+1}.$$

Now, let $s > 0$ and choose $t = t_n$ as the unique solution of $\frac{4t}{(1+t)^2} = e^{-s/\sqrt{n}}$ in the range $(0, 1)$, which is explicitly given by

$$t_n = \frac{1 - \sqrt{1 - e^{-s/\sqrt{n}}}}{1 + \sqrt{1 - e^{-s/\sqrt{n}}}} = 1 - \frac{2s^{1/2}}{n^{1/4}} + \frac{2s}{n^{1/2}} - \frac{3s^{3/2}}{2n^{3/4}} + \frac{s^2}{n} + O_s(n^{-5/4}).$$

(2.2)
From this asymptotic expansion, we have
\[
\log(1/t_n) = \frac{2s^{1/2}}{n^{1/4}} + \frac{s^{3/2}}{6n^{3/4}} + \mathcal{O}(n^{-5/4})
\] (2.3)
and
\[
\log\left(\frac{2(1 - t_n)}{(1 + t_n)\log(1/t_n)}\right) = -\frac{s}{3\sqrt{n}} + \frac{s^2}{45n} + \mathcal{O}(n^{-5/4}).
\] (2.4)
Plugging these estimates into \(M_{X_n}(-s)\), we see that
\[
M_{X_n}(-s) = \mathbb{E}[e^{-sX_n}] = \frac{1}{n!} W_n(e^{-s/\sqrt{n}}) \exp\left\{\frac{s^2}{45} + \mathcal{O}_s(n^{-1/4})\right\}.
\]
Then the desired conclusion follows since \(e^{s^2/45}\) is the m.g.f. of the \(\mathcal{N}\left(0, \frac{2}{45}\right)\).

3. Central limit theorem for peaks of a random permutation in a fixed conjugacy class of \(S_n\)

Let \(C_\lambda\) denote the set of all permutations of \(S_n\) of cycle type \(\lambda = 1^{n_1}2^{n_2} \ldots\) of \(n\). It is well known that \(|C_\lambda| = n!/(\prod_i n_i!^{n_i})\). Recall that the peak generating function over \(C_\lambda\) has an explicit formula (1.1), which involves the quantity \(f_{a,i}\) defined in the introduction. In the course of the proof of the main theorem, it is important to know a precise estimation of \(f_{a,i}\). Define \(g_{a,i}\) by
\[
g_{a,i} := \frac{2i}{(2a)^i} f_{a,i}.
\]
For large \(a\), it is reasonable to expect that \(f_{a,i}\) behaves much like its leading term \((2a)^i/(2i)\). Then the quantity \(g_{a,i}\) captures the relative difference between these two. The next lemma reveals that \(g_{a,i}\) is close to 1 in a uniform manner.

**Lemma 3.1.** There exists a universal constant \(c_1 > 0\) such that
\[
e^{-c_1(2a)^{-2i/3}} \leq g_{a,i} \leq e^{c_1(2a)^{-2i/3}}
\] (3.1)
for all \(a \geq 1\) and \(i \geq 1\). Moreover, we have
\[
e^{-c_1/4a^2} \leq g_{a,i} \leq e^{c_1/4a^2}.
\] (3.2)
Although the intermediate step of the proof will show that the explicit choice \(c_1 = 4\) works, we prefer to leave it as a named constant. This is because its value is not important for the argument and its presence will clarify the way we utilize this lemma.
Proof. Recall that $f_{a,i} = \frac{1}{2i} \sum \mu(d)(2a)^i/d$, where the sum is over $d$, the positive odd divisors of $i$. From this, we see that $g_{a,i} = 1$ when $i$ is either 1 or 2, and so, it suffices to assume that $i \geq 3$. For such $i \geq 3$,

$$(2a)^i |g_{a,i} - 1| \leq \sum_{d_i | i \text{ odd}, d \neq 1} (2a)^{i/d} \leq \sum_{k=1}^{|i/3|} (2a)^k = \frac{2a}{2a-1} ((2a)^{|i/3|} - 1) \leq 2(2a)^{i/3}.$$ 

Rearranging, it follows that

$$1 - 2(2a)^{-2i/3} \leq g_{a,i} \leq 1 + 2(2a)^{-2i/3}.$$ 

Since $a \geq 1$ and $i \geq 3$, we have $2(2a)^{-2i/3} \leq \frac{1}{2}$. Then, applying the inequalities $e^{-2x} \leq 1 - x$ and $1 + x \leq e^{2x}$, which are valid for $0 \leq x \leq \frac{1}{2}$, proves the first equality (3.1) with the choice $c_1 = 4$. Then (3.2) is a simple consequence of the fact that $(2a)^{-2i/3} \leq (2a)^{-2}$ for $i \geq 3$. □

Remark 3.2. The quantity $f_{a,i}$ is a positive integer. In the special case when $a$ is a power of 2, this follows from Lemma 1.3.16 of [10], which enumerates monic, irreducible, self-conjugate polynomials of degree $2i$ over a finite field of size $2a$.

For general $a$, the quantity $f_{a,i}$ enumerates what Victor Reiner calls “nowhere-zero primitive twisted necklaces” with values in

$$A = \{+1, -1, +2, -2, \ldots, +a, -a\}$$

having $i$ entries. To define this notion, let the cyclic group $C_{2i}$ act on $i$-tuples of words $(b_1, \ldots, b_i)$ where the $b_k$’s take values in $A$, and the generator of $C_{2i}$ acts by

$$g(b_1, \ldots, b_i) = (b_2, \ldots, b_i, -b_1).$$

An orbit $P$ of this action is called a twisted necklace, and $P$ primitive means that the $C_{2i}$ action is free (i.e. no non-trivial group element fixes any vector in the orbit $P$). Arguing as in the proof of Theorem 4.2 of [21] shows that $f_{a,i}$ does indeed enumerate nowhere-zero primitive twisted necklaces. We thank Victor Reiner for this observation.

3.1. Heuristics and main idea

We begin by focusing on the product of the coefficients appearing in the formula of the peak generating function (1.1) for a given cycle type. More specifically, we seek a formula for each of the coefficients that is more manageable for estimation. Applying the binomial series,

$$[x_i^{n_i}] (1 + x_i)_{f_{a,i}} = [x_i^{n_i}] ((1 + x_i)^{f_{a,i}} (1 - x_i)^{-f_{a,i}})$$

$$= \sum_{k=0}^{\infty} \binom{f_{a,i}}{k} \binom{f_{a,i} - 1 + n_i - k}{f_{a,i} - 1} = \frac{(2f_{a,i})^{n_i}}{n_i!} K_{a,i}, \quad (3.3)$$
where $K_{a,i}$ is the quantity defined by

$$K_{a,i} := \sum_{k=0}^{n_i} \frac{1}{2^{n_i}} \binom{n_i}{k} \frac{(f_{a,i} - 1 + n_i - k)!}{(f_{a,i} - k)! f_{n_i}^{n_i-1}}.$$

In applying (3.3), we will later observe that the main contribution for the expansion of the peak generating function (1.1) comes from the range in which $a$ is at least of the order of $n_i^{5/4}$. In addition, $K_{a,i}$'s are approximately 1 when $f_{a,i}$ is considerably larger than $n_i$. So, arguing heuristically,

$$\frac{1}{|\mathcal{C}_\lambda|} \sum_{\pi \in \mathcal{C}_\lambda} \left( \frac{4t}{(1 + t)^2} \right)^{p(\pi) + 1} \approx \left( \prod_i n_i ! i^{n_i} \right) \cdot 2 \left( \frac{1 - t}{1 + t} \right)^{n+1} \int_0^\infty \frac{t^a \prod_i \left( \frac{(2a)^i}{i} \right)^{n_i}}{n_i !} da$$

$$= \frac{1}{n!} \left( \frac{2(1 - t)}{1 + t} \right)^{n+1} \int_0^\infty t^x x^{n} dx$$

$$= \left( \frac{2(1 - t)}{(1 + t) \log(1/t)} \right)^{n+1}.$$

The final result is the same as what appears in the proof of the asymptotic normality of peaks over $S_n$. This leads to a naive guess that the peaks over $\mathcal{C}_\lambda$ have asymptotically the same normal distribution as the peaks over $S_n$. Of course, we must test the validity of this claim. One major concern is that the alleged asymptotic behavior of (3.3) may fail for small $i$'s. Such phenomenon is already observed in the case of descents [15], where the asymptotic distribution of descents for a fixed cycle type is parametrized by the density of fixed points. Indeed, it will be verified later that corrections are also needed for the peak distribution due to the presence of fixed points. In summary, we need to

- precisely control error terms appearing in various approximations, and
- investigate how the presence of fixed points affects the asymptotic formula for the peak generating function.

From this point forward, let $s > 0$ be a fixed positive real number. Then, $t_n$ is chosen as in the Eq. (2.2) and represents the unique solution of the equation $4t_n/(1 + t_n)^2 = e^{-s}/\sqrt{n}$ in the interval $(0, 1)$. As the first step of rigorization, we mimic the heuristic computation, but now without hand-waving
approximations. Applying \((3.3)\) to the peak generating function \((1.1)\),
\[
\frac{1}{|C_\lambda|} \sum_{\pi \in C_\lambda} e^{-\frac{\phi_{\lambda}(p(\pi)+1)}{\sqrt{n}}} = \frac{2}{n!} \left( \frac{1-t_n}{1+t_n} \right)^{n+1} \sum_{a=1}^{n+1} t_n^a \prod_{i=1}^{n} n_i! \left[ x_i^{n_i} \right] \frac{1+x_i}{1-x_i} f_{a,i}
\]
\[
= \frac{2}{n!} \left( \frac{1-t_n}{1+t_n} \right)^{n+1} \sum_{a=1}^{n+1} t_n^a \prod_{i=1}^{n} (2a)^{n_i} g_{a,i} \cdot K_{a,i}
\]
\[
= \left( \frac{2(1-t_n)}{(1+t_n) \log(1/t_n)} \right)^{n+1} \times \left[ \frac{\log^{n+1}(1/t_n)}{n!} \sum_{a=1}^{n} \frac{a^n t_n^a \prod_{i=1}^{n} g_{a,i} \cdot K_{a,i}}{n^a} \right].
\]
For the sake of conciseness, for each \(A \subseteq \mathbb{R}\), we define the quantity \(L_A\) by
\[
L_A := \frac{\log^{n+1}(1/t_n)}{n!} \sum_{a=1}^{n} a^n t_n^a \prod_{i=1}^{n} g_{a,i} \cdot K_{a,i}.
\]
Then the above computation simplifies to
\[
\frac{1}{|C_\lambda|} \sum_{\pi \in C_\lambda} e^{-\frac{\phi_{\lambda}(p(\pi)+1)}{\sqrt{n}}} = \left( \frac{2(1-t_n)}{(1+t_n) \log(1/t_n)} \right)^{n+1} L_{[1,\infty)}.
\]
As in the heuristic computation, the quantity \(L_{[1,\infty)}\) will be approximated by its integral analogue. In doing so, it is convenient to split the sum into two parts at a certain threshold. The primary reason is that the aforementioned approximation works poorly in the regime of “small” \(a\)’s, and a different kind of estimation is required there. To describe this threshold, define \(\delta_0\) by
\[
\delta_0 := \left[ \sup_{n \geq 1} \left( n^{1/4} \log(1/t_n) e^{(c_1/4)+1} \right) \right]^{-1}.
\]
We note that \(\delta_0\) is strictly positive. This follows from the asymptotic formula \((2.3)\) for \(\log(1/t_n)\). Then for each \(\delta \in (0, \delta_0)\), the sum \(L_{[1,\infty)}\) will be split into two parts,
\[
L_{[1,\delta n^{5/4}]} + L_{(\delta n^{5/4},\infty)},
\]
with the former corresponding the small range for \(a\) and the latter to the large range for \(a\).

3.2. Estimation of the small range
We will first focus on the range \(a \leq \delta n^{5/4}\) for \(\delta \in (0, \delta_0)\). The main goal of this subsection is to show that the contribution arising from this range is negligible.

Lemma 3.3. Let \(\delta > 0\). Then we have
\[
\prod_{i=1}^{n} K_{a,i} \leq \left( \frac{\delta n^{5/4}}{a} \right)^n e^{O_{\delta}(n^{3/4})}
\]
whenever \(a \leq \delta n^{5/4}\) holds.
Proof. If \(0 \leq k \leq n_i\), then
\[
\frac{(f_{a,i} - 1 + n_i - k)!}{(f_{a,i} - k)!f_{a,i}^{n_i-1}} = \prod_{j=1}^{n_i-1} \left| 1 + \frac{j - k}{f_{a,i}} \right| \leq \left( 1 + \frac{n_i}{f_{a,i}} \right)^{n_i}.
\]
Since this upper bound does not depend on \(k\), plugging this into the definition of \(K_{a,i}\) shows that the same upper bound also works for \(K_{a,i}\), that is, \(K_{a,i} \leq (1 + n_i / f_{a,i})^{n_i}\).

Now we assume \(a \leq \delta n^{5/4}\), and let \(r = \delta n^{5/4} / a \geq 1\). Then by using the inequalities \(1 + x \leq e^x\) and (3.2), we can further bound \(K_{a,i}\) by
\[
K_{a,i} \leq r^{in_i} \left( 1 + \frac{n_i}{f_{a,i} r^i} \right)^{n_i} \leq r^{in_i} \exp \left\{ \frac{n_i^2}{f_{a,i} r^i} \right\} \\
= r^{in_i} \exp \left\{ \frac{2in_i^2}{g_{a,i}(2ar)^i} \right\} \leq r^{in_i} \exp \left\{ \frac{2in_i}{e^{c_1/4}(2ar)} \right\} = r^{in_i} \exp \left\{ \frac{cin_i}{n^{1/4}} \right\},
\]
where \(c = e^{c_1/4}/\delta\). Now using \(\sum_{i=1}^{n} in_i = n\),
\[
\prod_{i=1}^{n} K_{a,i} \leq \prod_{i=1}^{n} r^{in_i} \exp \left\{ \frac{cin_i}{n^{1/4}} \right\} = r^n e^{cn^{3/4}},
\]
and, therefore, the desired inequality follows with the implicit bound chosen as \(c\).

Now we are ready to formulate the precise statement for the “small-ness” of the contribution from the range \(a \leq \delta n^{5/4}\).

**Proposition 3.4.** Let \(\delta \in (0, \delta_0)\). Then
\[
L_{[1,\delta n^{5/4}]} \leq (\delta / \delta_0)^n e^{O_\delta(n^{3/4})}.
\]

**Proof.** Lemma 3.1 and \(\sum_{i=1}^{n} n_i \leq n\) yield \(\prod_{i=1}^{n} g_{a,i}^{n_i} \leq e^{c_1 n/4 a^2}\). Then by Lemma 3.3,
\[
\sum_{a \in [1,\delta n^{5/4}] \cap \mathbb{N}} a^n t_n^{a} \prod_{i=1}^{n} g_{a,i}^{n_i} K_{a,i} \leq \left( \sum_{a \in [1,\delta n^{5/4}] \cap \mathbb{N}} t_n a e^{c_1 n/4 a^2} \right) (\delta n^{5/4})^n e^{O_\delta(n^{3/4})} \\
\leq (\delta n^{5/4})^{n+1} e^{c_1 n/4} e^{O_\delta(n^{3/4})},
\]
where the second step follows by bounding \(t_n a e^{c_1 n/4 a^2}\) from above by \(e^{c_1 n/4}\). In addition, by the definition of \(\delta_0\), we have
\[
\log(1 / t_n) \leq \frac{1}{\delta_0 e^{(c_1/4)+1} n^{1/4}}.
\]
Furthermore, a quantitative form of the Stirling’s formula [22] tells us that \(n! \geq \sqrt{2\pi} n^{n+1/2} e^{-n}\). Combining altogether,
\[
L_{[1,\delta n^{5/4}]} \leq (\delta / \delta_0)^n+1 e^{O_\delta(n^{3/4})} \left( \frac{n^{1/2}}{(2\pi)^{1/2} e^{(c_1/4)+1}} \right),
\]
and the conclusion follows by absorbing all the subexponential contributions into \(e^{O_\delta(n^{3/4})}\). \(\square\)
3.3. Estimation of the large range

We now turn our attention to the range \( a > \delta n^{5/4} \) for \( \delta \in (0, \delta_0) \). We start with the lemma that resolves the contribution of the \( K_{a,i} \)'s for \( i \geq 2 \).

**Lemma 3.5.** We have

\[
\prod_{i \geq 2} K_{a,i} = e^{O(n^2/a^2)}
\]

in the range \( a \geq e^{c_1/8} n^{1/2} \).

**Proof.** Assume \( a \geq e^{c_1/8} n^{1/2} \). For \( i \geq 2 \), Lemma 3.1 gives

\[
\frac{n_i}{f_{a,i}} = \frac{2i n_i}{g_{a,i}(2a)^i} \leq \frac{2i n_i}{e^{-c_1/4}(2a)^2} \leq \frac{i n_i}{2n} \leq \frac{1}{2}.
\]

Moreover, for any \( 0 \leq k \leq n_i \) and \( 1 \leq j \leq n_i - 1 \), we have \( |j - k| \leq n_i \). From this and the inequality \(|\log(1 + x)| \leq 2|x|\), valid for \(|x| \leq \frac{1}{2}\), we get

\[
\left| \log \left( \frac{(f_{a,i} - 1 + n_i - k)!}{(f_{a,i} - k)! f_{a,i}^{n_i-1}} \right) \right| \leq \sum_{j=1}^{n_i-1} \left| \log \left( 1 + \frac{j - k}{f_{a,i}} \right) \right| \leq \frac{2n_i^2}{f_{a,i}} \leq e^{c_1/4} \frac{i n_i^2}{a^2}.
\]

Since this bound does not depend on \( k \), it follows that \( \log K_{a,i} \) satisfies the same bound. So by summing this inequality for \( i = 2, \ldots, n \) and utilizing the bound \( \sum_i i n_i^2 \leq n^2 \),

\[
\sum_{i=2}^{n} |\log K_{a,i}| \leq \sum_{i=2}^{n} e^{c_1/4} \frac{i n_i^2}{a^2} \leq e^{c_1/4} \frac{n^2}{a^2}.
\]

Therefore, the desired conclusion follows with the implicit bound chosen as \( e^{c_1/4} \). \( \square \)

Given that the uninteresting part has been treated, we move on to establishing a detailed asymptotic expansion of \( K_{a,1} \). The relevant statement is as follows.

**Lemma 3.6.** Let \( \delta \in (0, \delta_0) \). Then,

\[
K_{a,1} = \exp \left\{ \frac{n_1^3}{12a^2} - \frac{3n_1^5}{160a^4} + O_\delta(n^{-1/4} \sqrt{\log n}) \right\}
\]

(3.6)

holds in the range \( a \geq \max\{\delta n^{5/4}, 2n\} \).

This is arguably the most complicated part of this paper. As the first step, we introduce several technical estimations that will facilitate the proof of Lemma 3.6. Since the proofs of these lemmas are isolated from the main argument, they are postponed to the end of the paper.

We begin by stating the lemma that allows us to approximate the sum of a piecewise-monotone function by its integral analog.
Lemma 3.7. Let $I$ be a bounded interval, and let $f : I \rightarrow \mathbb{R}$ be continuous and bounded. Suppose $I$ is the union of $p$ disjoint subintervals $I_1, I_2, \ldots, I_p$ such that $f$ is monotone on each subinterval $I_i$. Then
\[
\left| \sum_{k \in I \cap \mathbb{Z}} f(k) - \int_I f(x) \, dx \right| \leq (p + 1) \sup_{x \in I} |f(x)|.
\]

As a corollary of this lemma, we obtain the following estimates useful for adapting Laplace’s method to sums.

Lemma 3.8. The following estimates hold in the range $k_0 \in \mathbb{R}$, $\sigma > 0$, and $\lambda > 0$:
\[
\sum_{|k - k_0| \leq \lambda \sigma} e^{-\frac{1}{2\sigma^2} (k - k_0)^2} = \sqrt{2\pi \sigma} \left( 1 + O(\sigma^{-1}) + O(\lambda^{-1} e^{-\frac{\lambda}{2} \lambda^2}) \right) \quad (3.7)
\]
\[
\sum_{|k - k_0| > \lambda \sigma} e^{-\frac{1}{2\sigma^2} (k - k_0)^2} \leq (4 + 2\sigma/\lambda) e^{-\frac{\lambda}{4} \lambda^2} \quad (3.8)
\]

The next lemma provides local-limit-theorem-type estimates for the binomial coefficients. Even though the inequality (3.9) is an overkill for the purpose of proving Lemma 3.6, it is included because it can be easily obtained as a by-product of the proof of (3.10).

Lemma 3.9. Let $n \geq 1$ and $0 \leq k \leq n$ be integers. Then with $\varphi = \frac{k - (n/2)}{\sqrt{n/2}}$, we have
\[
\frac{1}{2^n} \binom{n}{k} \leq \frac{e}{\sqrt{\pi n/2}} e^{-\varphi^2/2}. \quad (3.9)
\]
Moreover, for each given $c > 0$,
\[
\frac{1}{2^n} \binom{n}{k} = \frac{1}{\sqrt{\pi n/2}} \exp \left\{ - \frac{\varphi^2}{2} - \frac{\varphi^4}{12n} + O_c \left( \frac{\varphi^2}{n} \right) \right\} \quad (3.10)
\]
holds for $|\varphi| \leq \min\{cn^{1/4}, \sqrt{n/2}\}$.

Lemma 3.10. Let $a \geq 2n$ and $0 \leq k \leq n$. Then with $\varphi = \frac{k - (n/2)}{\sqrt{n/2}}$, we have
\[
\frac{(a - 1 + n - k)!}{(a - k)!a^{k-1}} \leq e^2 \exp \left\{ - \frac{n^{3/2} \varphi}{2a} - \left( \frac{n^3}{24a^2} + \frac{n^{7/2} \varphi}{24a^3} \right) \right\}. \quad (3.11)
\]
Moreover, for each given $\delta > 0$ and $c > 0$,
\[
\frac{(a - 1 + n - k)!}{(a - k)!a^{k-1}} = \exp \left\{ - \left( \frac{n^{3/2} \varphi}{2a} + \frac{n^2 \varphi^2}{8a^2} \right) - \left( \frac{n^3}{24a^2} + \frac{n^{7/2} \varphi}{24a^3} \right) - \frac{n^5}{320a^4} + O_{\delta,c} (a^{-1/5}) \right\} \quad (3.12)
\]
holds for $a \geq \max\{\delta n^{5/4}, 2n\}$ and $|\varphi| \leq ca^{1/5}$.

Now we return to the proof of Lemma 3.6.
Proof of Lemma 3.6. Let \( a \geq \max\{\delta n^{5/4}, 2n\} \). From \( f_{a,1} = a \), the sum defining \( K_{a,1} \) reduces to

\[
K_{a,1} = \sum_{k=0}^{n_1} \frac{1}{2^{n_1}} \binom{n_1}{k} q(k), \quad \text{where} \quad q(k) := \frac{(a-k+n_1-1)!}{(a-k)! a^{n_1-1}}. \tag{3.13}
\]

**Case 1.** We first separate the case of small \( n_1 \)'s from the general argument. Suppose \( n_1 < n^{3/4} \). Then arguing as in the proof of Lemma 3.5, we have

\[
\log q(k) = \sum_{j=1}^{n_1-1} \frac{j-k}{a} + O\left(\frac{n_1^3}{a^2}\right) = -\frac{(2k-n)(n-1)}{2a} + O\left(\frac{n_1^3}{a^2}\right)
\]

and hence

\[
K_{a,1} = \left[ \cosh\left(\frac{n_1-1}{2a}\right) \right]^{n_1} e^{O(n_1^3/a^2)} = e^{O(n_1^3/a^2)}.
\]

So it follows that \( K_{a,1} = e^{O(n_1^3/a^2)} \). On the other hand, using \( a \geq 2n_1 \) and \( n_1 < n^{3/4} \),

\[
\left| \frac{n_1^3}{12a^2} - \frac{3n_1^5}{160a^4} \right| \lesssim \frac{n_1^3}{a^2} \leq \frac{(n^{3/4})^3}{\delta^2 n_1^{5/2}} \lesssim n^{-1/4}.
\]

Hence the estimate (3.6) holds trivially in this case.

**Case 2.** Now we assume that \( n_1 \geq n^{3/4} \). Similarly as in Lemmas 3.9 and 3.10, we introduce the parameter \( \varphi \) defined by

\[
\varphi = \frac{k-n_1/2}{\sqrt{n_1}/2}, \quad \text{or equivalently,} \quad k = \frac{n_1}{2} + \frac{\sqrt{n_1}}{2} \varphi.
\]

Since \( 0 \leq k \leq n_1 \), we have \( |\varphi| \leq \sqrt{n_1} \). Then it will turn out that \( q(k) \) behaves like an exponential tilting to the gaussian density function approximating the binomial coefficient in \( K_{a,1} \). Overall, the summand will be approximately the gaussian density function of the form

\[
\frac{1}{2^{n_1}} \binom{n_1}{k} q(k) \approx e^{-\frac{1}{2}(\varphi - \varphi_0)^2 + \text{[function of } a \text{ and } n_1]} \quad \text{where} \quad \varphi_0 := -\frac{n_1^{3/2}}{2a}.
\]

In order to make this argument precise, we will split the sum in (3.13) into three parts based on the behavior of the summand. Let \( \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \) be the sets of indices defined by

\[
\mathcal{I}_1 = \left\{ k \in [0, n_1] \cap \mathbb{Z} : |\varphi - \varphi_0| \leq \sqrt{\log n} \right\},
\]

\[
\mathcal{I}_2 = \left\{ k \in [0, n_1] \cap \mathbb{Z} : \sqrt{\log n} < |\varphi - \varphi_0| \leq n^{1/4} \right\},
\]

\[
\mathcal{I}_3 = \left\{ k \in [0, n_1] \cap \mathbb{Z} : |\varphi - \varphi_0| > n^{1/4} \right\}.
\]

We remark that the split points \( \sqrt{\log n} \) and \( n^{1/4} \) are functions of \( n \) rather than of \( n_1 \). We will investigate the behavior of the sum on each of \( \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \).
We first estimate the sum on the range $\mathcal{I}_3$. Applying (3.9) and (3.11) to $\frac{1}{2^{n_1}} \binom{n_1}{k}$ and $q(k)$ respectively and then utilizing the bound $|\varphi| \leq \sqrt{n_1}$, we obtain

$$
\sum_{k \in \mathcal{I}_3} \frac{1}{2^{n_1}} \binom{n_1}{k} q(k) \lesssim \frac{1}{\sqrt{n_1}} \sum_{k \in \mathcal{I}_3} \exp \left\{ -\frac{\varphi^2}{2} - \frac{n_1^{3/2} \varphi}{2a} - \frac{n_1^3}{24a^2} - \frac{n_1^{7/2} \varphi}{24a^3} \right\} 
\lesssim \frac{1}{\sqrt{n_1}} \left( \sum_{k \in \mathcal{I}_3} e^{-\frac{1}{2}(\varphi - \varphi_0)^2} \right) \exp \left\{ \frac{n_1^3}{12a^2} + \frac{n_1^4}{24a^3} \right\}.
$$

Now applying (3.8) to the sum in the last step with $\sigma = \sqrt{n_1}/2$ and $\lambda = n^{1/4}$ and then noting that $n \geq n_1$ and $a \geq \delta n^{5/4} \geq \delta n_1^{5/4}$,

$$
\sum_{k \in \mathcal{I}_3} \frac{1}{2^{n_1}} \binom{n_1}{k} q(k) \lesssim \frac{4 + \sqrt{n_1}/n^{1/4}}{\sqrt{n_1}} e^{-\sqrt{n_1}/2} \exp \left\{ \frac{n_1^3}{12a^2} + O_\delta(n^{1/4}) \right\}
\lesssim e^{-\sqrt{n_1}/2 + O_\delta(n^{1/4})} \exp \left\{ \frac{n_1^3}{12a^2} - \frac{3n_1^5}{160a^4} \right\}, \quad (3.14)
$$

So the sum on the range $\mathcal{I}_3$ is negligible compared to the right-hand side of (3.6) and will be absorbed into the error term of (3.6).

Next, the sum on the range $\mathcal{I}_2$ can be estimated in a similar manner. Starting as in the case of $\mathcal{I}_3$ but now using the estimation

$$
|\varphi| \leq |\varphi - \varphi_0| + |\varphi_0| \leq n^{1/4} + \frac{n_1^{3/2}}{2\delta n^{5/4}} \lesssim n^{1/4},
$$

which is valid for $k \in \mathcal{I}_2$, we have

$$
\sum_{k \in \mathcal{I}_2} \frac{1}{2^{n_1}} \binom{n_1}{k} q(k) \lesssim \frac{1}{\sqrt{n_1}} \left( \sum_{k \in \mathcal{I}_2} e^{-\frac{1}{2}(\varphi - \varphi_0)^2} \right) \exp \left\{ \frac{n_1^3}{12a^2} + O_\delta \left( \frac{n_1^{7/2} n^{1/4}}{a^3} \right) \right\}.
$$

Then by applying (3.8) with $\sigma = \sqrt{n_1}/2$ and $\lambda = \sqrt{\log n}$ and arguing similarly as before,

$$
\sum_{k \in \mathcal{I}_2} \frac{1}{2^{n_1}} \binom{n_1}{k} q(k) \lesssim \frac{1 + \sqrt{n_1/\log n}}{\sqrt{n_1}} e^{-(\log n)/2} \exp \left\{ \frac{n_1^3}{12a^2} + O_\delta(1) \right\}
\lesssim \frac{1}{\sqrt{n}} \exp \left\{ \frac{n_1^3}{12a^2} - \frac{3n_1^5}{160a^4} \right\}, \quad (3.15)
$$

Again, this will be absorbed into the error term of (3.6).

Finally, we consider the sum on the range $\mathcal{I}_1$. Note first that $a \geq \max\{\delta n_1^{5/4}, 2n_1\}$ is obvious from the assumption on $a$ and $n \geq n_1$. In addition, recall that we are assuming $n_1 \geq n^{3/4}$. Then, for $k \in \mathcal{I}_1$, we have

$$
|\varphi| \leq |\varphi - \varphi_0| + |\varphi_0| \leq \sqrt{\log n} + \frac{n_1^{3/2}}{2a} \leq \sqrt{\frac{4}{3} \log n} + \frac{n_1^{3/2}}{2\delta n_1^{5/4}} \lesssim n_1^{1/4} \lesssim a^{1/5}, \quad (3.16)
$$

Hence we know that (3.10) and (3.12) are applicable to $\frac{1}{2^{n_1}} \binom{n_1}{k}$ and $q(k)$, respectively, by choosing the parameters $c$ as the implicit constants for the
estimates $|\varphi| \lesssim \delta n^{1/4}$ and $|\varphi| \lesssim \delta a^{1/5}$ in (3.16), respectively. We remark that, since the implicit bounds in (3.16) depend on $\delta$, the error terms in both (3.10) and (3.12) depend on $\delta$ as well. Then, from $\frac{\varphi^2}{n_1} \lesssim \delta \frac{(n^{1/4})^2}{n^{3/4}} = n^{-1/4}$, we get

$$\frac{1}{2n_1} \left( \frac{n_1}{k} \right) = \frac{1}{\sqrt{\pi} n_1/2} \exp\left\{ -\frac{\varphi^4}{2} - \frac{\varphi^4}{12n_1} + O_{\delta}(n^{-1/4}) \right\}$$

and

$$q(k) = \exp\left\{ -\left( \frac{n_1^3/2 \varphi}{2a} + \frac{n_1^2 \varphi^2}{8a^2} \right) - \left( \frac{n_1^3}{24a^2} + \frac{n_1^7/2 \varphi}{24a^3} \right) - \frac{n_1^5}{320a^4} + O_{\delta}(n^{-1/4}) \right\}.$$  

To simplify this, write $\varepsilon = \varphi - \varphi_0$ and note that $|\varepsilon| \leq \sqrt{\log n}$. Then $\varphi = -\frac{n_1^3}{2a} + \varepsilon$, and so,

$$-\frac{\varphi^4}{12n_1} = -\frac{n_1^5}{192a^4} + \left( \frac{n_1^{7/2}}{24a^3} \frac{\varepsilon}{8a^2} - \frac{n_1^2}{8a^2} \varepsilon^2 + \frac{n_1^{1/2}}{6a} \varepsilon^3 - \frac{1}{12n_1} \varepsilon^4 \right)$$

$$= -\frac{n_1^5}{192a^4} + O_{\delta}(n^{-1/4} \sqrt{\log n}),$$

$$-\frac{n_1^2 \varphi^2}{8a^2} = -\frac{n_1^5}{32a^4} + \left( \frac{n_1^{7/2}}{8a^3} \frac{\varepsilon}{8a^2} - \frac{n_1^2}{8a^2} \varepsilon^2 \right)$$

$$= -\frac{n_1^5}{32a^4} + O_{\delta}(n^{-1/4} \sqrt{\log n}),$$

$$-\frac{n_1^{7/2} \varphi}{24a^3} = \frac{n_1^5}{48a^3} - \frac{n_1^{5/2}}{24a^2} \varepsilon$$

$$= \frac{n_1^5}{48a^3} + O_{\delta}(n^{-1/4} \sqrt{\log n}).$$

Combining all the estimates, we get

$$\sum_{k \in I_1} \frac{1}{2n_1} \left( \frac{n_1}{k} \right) q(k) = \left( \frac{1}{\sqrt{\pi} n_1/2} \sum_{k \in I_1} e^{-\frac{1}{2}(\varphi - \varphi_0)^2} \right)$$

$$\times \exp\left\{ \frac{n_1^3}{12a^2} - \frac{3n_1^5}{160a^4} + O_{\delta}(n^{-1/4} \sqrt{\log n}) \right\}.$$ 

Then by applying (3.7) with $\sigma = \sqrt{n_1}/2$ and $\lambda = \sqrt{\log n}$, we get

$$\frac{1}{\sqrt{\pi} n_1/2} \sum_{k \in I_1} e^{-\frac{1}{2}(\varphi - \varphi_0)^2} = 1 + O(n_1^{-1/2}) + O((n \log n)^{-1/2}) = e^{O(n^{-1/4} \sqrt{\log n})}.$$ 

Therefore, the desired estimate (3.6) follows. □

3.4. Estimation of the peak generating function

**Proposition 3.11.** Let $\delta \in (0, \delta_0)$ and write $\alpha_1 = n_1/n$ for the density of fixed points. Then

$$L_{(\delta n^{3/4}, \infty)} = \exp\left\{ \frac{\alpha_1^3}{3} s \sqrt{n} + \left( \frac{\alpha_1^3}{18} - \frac{3\alpha_1^5}{10} + \frac{2\alpha_1^7}{9} \right) s^2 + O_{\delta, s}(n^{-1/4} \sqrt{\log n}) \right\}.$$
Following Kim and Lee’s method [15], we will utilize Laplace’s method to approximate the sum by the integral of a certain gaussian density function and show that the relative error due to this approximation can be controlled in an explicit and uniform manner. The following simple lemma is useful for this purpose.

**Lemma 3.12.** Write \( x_+ = \max\{0, x\} \) for \( x \in \mathbb{R} \). Then for \( x > -1 \),

\[
\log(1 + x) \leq x - \frac{x^2}{2(1 + x_+)}.
\]

This lemma can be easily proved via differentiation, and the proof is included at the end of the paper. Now we return to the proof of the main claim of this section.

**Proof of Proposition 3.11.** Choose \( N_1 = N_1(\delta) \) such that \( \delta n^{5/4} \geq \max\{e^{c_1/8}n^{1/2}, 2n\} \) whenever \( n \geq N_1 \). Then by Lemmas 3.1 and 3.5, for \( n \geq N_1 \) and \( a \geq \delta n^{5/4} \) we have

\[
\left| \log \prod_{i=1}^{n} g_{a,i}^{n_i} \right| \lesssim \frac{1}{a^2} \sum_{i=1}^{n} n_i \lesssim \delta n^{-3/2} \quad \text{and} \quad \left| \log \prod_{i=2}^{n} K_{a,i} \right| \lesssim \frac{n^2}{a^2} \lesssim \delta n^{-1/2}.
\]

In addition, recall that \( t_n \) is defined by (2.2) and is the unique solution of \( \frac{4t_n}{1 + t_n} = e^{-s/\sqrt{n}} \) in the range \( t_n \in (0, 1) \). For simplicity, write \( r_n = \log(1/t_n) \). Then, by the estimations above and Lemma 3.6,

\[
L(\delta n^{5/4}, \infty) = e^{O_\delta(n^{-1/4}\sqrt{\log n})} \frac{r_{n+1}^n}{n!} \sum_{a > \delta n^{5/4}} a^n e^{-ar_n} \exp\left\{ \frac{n^3}{12a^2} - \frac{3n^5}{160a^4} \right\}.
\]

(3.17)

Now we adapt Laplace’s method to analyze the behavior of the sum in (3.17). The key observation is that, similarly as before, the factor \( K_{a,1} \) will behave like an exponential tilting to the integral analog of the sum in (3.17). To properly capture this behavior, we substitute

\[
w = \tilde{w}(a) = \frac{a - n/r_n}{\sqrt{n/r_n}}, \quad \text{or equivalently} \quad a = \tilde{w}^{-1}(w) = \frac{n}{r_n} + \sqrt{n} w,
\]

where \( \tilde{w}(a) \) is the function of \( a \) defined by the above formula. In addition, we define \( \alpha_1 \) and \( w_0 \) by

\[
\alpha_1 = \frac{n_1}{n} \quad \text{and} \quad w_0 = -\alpha_1^3 \sqrt{n/r_n}.
\]

Then by noting that

\[
\log\left( \frac{n^3}{12a^2} - \frac{3n^5}{160a^4} \right)
= (n + 1) \log r_n - \log(n!) + n \log a - ar_n
= \log r_n - \log\left( \frac{n!}{(n/e)^n} \right) + n \log\left( 1 + \frac{w}{\sqrt{n}} \right) - w \sqrt{n}
= -\log(\sqrt{2\pi n/r_n}) + n \log\left( 1 + \frac{w}{\sqrt{n}} \right) - w \sqrt{n} + O(n^{-1}),
\]
the formula (3.17) for \( L_{(\delta n^{5/4}, \infty)} \) can be recast as

\[
L_{(\delta n^{5/4}, \infty)} = \left( \frac{1}{\sqrt{2\pi n/r_n}} \sum_{a > \delta n^{5/4}} e^{f_n(w)} \right) \exp \left\{ \frac{n_1^3}{12(n/r_n)^2} + O_\delta(n^{-1/4}\sqrt{\log n}) \right\},
\]

(3.18)

where \( f_n \) is defined by

\[
f_n(w) = n \log \left( 1 + \frac{w}{\sqrt{n}} \right) - w\sqrt{n} + \left( \frac{n_1^3}{12a^2} - \frac{n_1^3}{12(n/r_n)^2} \right) - \frac{3n_1^5}{160a^4}.
\]

We will estimate the sum appearing in (3.18) by splitting it into various sums capturing different asymptotic behaviors of \( f_n \). To this end, define the sets of indices \( \mathcal{I}_1, \ldots, \mathcal{I}_4 \) by

\[
\mathcal{I}_1 = \{ a \in (\delta n^{5/4}, \infty) \cap \mathbb{N} : w - w_0 > \log n \},
\]

\[
\mathcal{I}_2 = \{ a \in (\delta n^{5/4}, \infty) \cap \mathbb{N} : w - w_0 < -n^{1/4}\sqrt{\log n} \},
\]

\[
\mathcal{I}_3 = \{ a \in (\delta n^{5/4}, \infty) \cap \mathbb{N} : -n^{1/4}\sqrt{\log n} \leq w - w_0 < -\log n \},
\]

\[
\mathcal{I}_4 = \{ a \in (\delta n^{5/4}, \infty) \cap \mathbb{N} : |w - w_0| \leq \log n \},
\]

The main contribution will come from the sum over \( \mathcal{I}_4 \), and the other sums will be absorbed into the error term.

**Case 1.** We start with the sum over \( \mathcal{I}_1 \). Choose \( N_2 = N_2(s) \) such that \( \log n \geq 1 + \sqrt{n}w_0^2 \) for all \( n \geq N_2 \). This is possible because \( \sqrt{n}r_n^2 = O_\delta(1) \) by (2.3). Then, for \( n \geq N_2 \) and \( a \in \mathcal{I}_1 \), Lemma 3.12 gives

\[
f_n(w) \leq -\frac{w^2}{2(1 + w/\sqrt{n})} \leq -w/4,
\]

where we utilized the fact that \( \frac{x}{1+x/\sqrt{n}} \geq \frac{1}{2} \) whenever \( x \geq 1 \) and \( n \geq 1 \), and \( w > w_0 + \log n \geq 1 \). Then, bounding the sum from above by its integral analogue using Lemma 3.7,

\[
\frac{1}{\sqrt{2\pi n/r_n}} \sum_{a \in \mathcal{I}_1} e^{f_n(w)} \leq \frac{1}{\sqrt{2\pi n/r_n}} \left( \int_{\tilde{w}(a) > w_0 + \log n} e^{-\tilde{w}(a)/4} da + 2e^{-(w_0 + \log n)/4} \right).
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{w_0 + \log n}^{\infty} e^{-w/4} dw + \frac{2}{\sqrt{2\pi n/r_n}} e^{-(w_0 + \log n)/4} \lesssim_s n^{-1/4}.
\]

(3.19)

**Case 2.** Next we study the sum over \( \mathcal{I}_2 \). For \( a \in \mathcal{I}_2 \), Lemma 3.12 gives

\[
f_n(w) \leq -\frac{w^2}{2} + \frac{n_1^3}{12(\delta n^{5/4})^2} \leq -\frac{w^2}{2} + O_\delta(n^{1/2}).
\]

So by invoking Lemma 3.8 with \( \sigma = \sqrt{n/r_n} \) and \( \lambda = n^{1/4}\sqrt{\log n} - w_0 \),

\[
\frac{1}{\sqrt{2\pi n/r_n}} \sum_{a \in \mathcal{I}_2} e^{f_n(w)} \leq e^{O_\delta(n^{1/2})} \frac{1}{\sqrt{2\pi n/r_n}} \left( 2 + \frac{\sqrt{n/r_n}}{n^{1/4}\sqrt{\log n} - w_0} \right) e^{-\frac{1}{2}(n^{1/4}\sqrt{\log n} - w_0)^2} \leq e^{-\frac{1}{2}n^{1/2}\log n + O_{\delta,s}(n^{1/2})}.
\]

(3.20)
Case 3. Now we estimate the sum over $I_3$. Noting that $|w| \leq n^{1/4} \sqrt{\log n} + O_s(1)$ for $a \in I_3$, we can expand the factor $\frac{n_1^3}{12a^2}$ as

$$\frac{n_1^3}{12a^2} = \frac{\alpha_1^3 n r_n^2}{12(1 + w/\sqrt{n})^2} - \frac{\alpha_1^3 n r_n^2}{12}$$

$$= -\frac{\alpha_1^3 \sqrt{n} r_n^2}{6} w + O_s\left(\frac{\alpha_1^3 n r_n^2}{12} \frac{w^2}{n}\right)$$

$$= w_0 w + O_s(\log n).$$

From this and Lemma 3.12 altogether, we obtain the following bound for $f_n$ on $I_3$:

$$f_n(w) \leq -\frac{1}{2}(w - w_0)^2 + O_s(\log n)$$

Then by applying Lemma 3.8 with $\sigma = \sqrt{n}/r_n$ and $\lambda = \log n$,

$$\frac{1}{\sqrt{2\pi n}/r_n} \sum_{a \in I_3} e^{f_n(w)} \leq \frac{1}{\sqrt{2\pi n}/r_n} \left(2 + \frac{\sqrt{n}/r_n}{\log n}\right) e^{-\frac{1}{2}(\log n)^2 + O_s(\log n)}$$

$$\leq e^{-\frac{1}{2}(\log n)^2 + O_s(\log n)}.$$ (3.21)

Case 4. Finally, we investigate the sum over $I_4$. We first check that there exists $N_3 = N_3(\delta, s)$ such that $I_4$ contains all the integer $a$ satisfying $|\tilde{w}(a) - w_0| \leq \log n$ whenever $n \geq N_3$. This amounts to showing that $\delta n^{5/4} \leq \tilde{w}(-\log n)$ for all large $n$’s, or equivalently, $\delta r_n n^{1/4} \leq 1 - \frac{\log n}{\sqrt{n}}$. Then the claim follows by noting that $\sup_{n \geq 1} \delta r_n n^{1/4} = (\delta/\delta_0)e^{-(c_1/4) - 1} < 1$. Now let $n \geq N_3$ and $a \in I_4$. Then $|w| \leq \log n + O_s(1)$, and so,

$$f_n(w) = n \left[\frac{w}{\sqrt{n}} - \frac{w^2}{2n} + O_s\left(\log^3 n^{3/2}\right)\right] - w \sqrt{n}$$

$$+ \frac{\alpha_1^3 n r_n^2}{12} \left[1 - \frac{2w}{\sqrt{n}} + O_s\left(\log^2 n\right)\right] - \frac{\alpha_1^3 n r_n^2}{12}$$

$$- \frac{3\alpha_1^5 n r_n^4}{160} \left[1 + O_s\left(\log n / \sqrt{n}\right)\right]$$

$$= -\frac{w^2}{2} + w_0 w - \frac{3\alpha_1^5 n r_n^4}{160} + O_s\left(\log^3 n / n^{1/2}\right),$$

where the asymptotic relation $r_n \sim 2s^{1/2}/n^{1/4}$ is utilized in the last line. Then by Lemma 3.8 with $\sigma = \sqrt{n}/r_n$ and $\lambda = \log n$,

$$\frac{1}{\sqrt{2\pi n}/r_n} \sum_{a \in I_4} e^{f_n(w)} = \left(1 + O_s(n^{-1/4})\right) \exp\left\{-\frac{3\alpha_1^5 n r_n^4}{160} + \frac{w_0^2}{2}\right\}. \quad (3.22)$$

Now we are ready to prove the claim of Lemma 3.11. Plugging all the estimates (3.19)–(3.22) into the formula (3.18), we obtain

$$L_{(\delta n^{5/4}, \infty)} = \exp\left\{\frac{\alpha_1^3 n r_n^2}{12} - \frac{3\alpha_1^5 n r_n^4}{160} + \frac{\alpha_1^6 n r_n^4}{72} + O_{\delta, s}(n^{-1/4} \sqrt{\log n})\right\} \quad (3.23)$$
for all \( n \geq \max\{N_1, N_2, N_3\} \). Furthermore, the asymptotic expansion (2.3) of \( r_n \) shows that

\[
r_n^2 = \frac{4s}{\sqrt{n}} + \frac{2s^2}{3n} + O_s(n^{-3/2}) \quad \text{and} \quad r_n^4 = \frac{16s^2}{n} + O_s(n^{-3/2}).
\]

Therefore, the desired conclusion follows by plugging these into (3.23). \( \square \)

3.5. Conclusion

With all the ingredients ready, we immediately obtain the proof of Theorem 1.2.

Proof of Theorem 1.2. In the course of the proof of Theorem 2.3, we established the asymptotic formula (2.4) that is given by

\[
\left( \frac{2(1 - t_n)}{(1 + t_n) \log(1/t_n)} \right)^{n+1} = \exp \left\{ -\frac{s}{3} \sqrt{n} + \frac{1}{45} s^2 + O_s(n^{-1/4}) \right\}.
\]

Now we fix \( \delta \in (0, \delta_0) \). Then, for a random permutation \( \pi \) from the conjugacy class \( C_\lambda \) of the cycle type \( \lambda = 1^{n_1}2^{n_2}\ldots \) and with the density \( \alpha_1 = n_1/n \) of the fixed points, together Proposition 3.4 and 3.11 show that

\[
E\left[ e^{-sp(\pi)/\sqrt{n}} \right] = e^{s/\sqrt{n}} \left( \frac{2(1 - t_n)}{(1 + t_n) \log(1/t_n)} \right)^{n+1} \left( L_{[1,\delta n^{5/4}]} + L_{[\delta n^{5/4}, \infty)} \right)
\]

\[
\exp \left\{ -\frac{s}{3} \sqrt{n} + \frac{1}{45} s^2 + O_s(n^{-1/4}) \right\}
\]

\[
\times \left[ O\left( (\delta/\delta_0)^{n+O_s(n^{3/4})} \right) + \exp \left\{ \frac{\alpha_1^3}{3} s\sqrt{n} + \left( \frac{\alpha_1^3}{18} - \frac{3\alpha_1^5}{10} - \frac{2\alpha_1^6}{9} \right) s^2 + O_{\delta,s}(n^{-1/4}\sqrt{\log n}) \right\} \right].
\]

Finally, the exponentially decaying term from the small range \( L_{[1,\delta n^{5/4}]} \) may be absorbed into the error term \( O_{\delta,s}(n^{-1/4}\sqrt{\log n}) \). Then we obtain the desired bound for the term \( E_{\lambda,s} \) appearing in the statement of Theorem 1.2, completing the proof. \( \square \)

4. Proof of Lemmas

Proof of Lemma 3.7

Let \( B(x) = x - [x] - \frac{1}{2} \) and \( M = \sup_{x \in I} |f(x)| \). Then invoking the Lebesgue–Stieltjes integral and performing integration by parts, for each closed subinterval \( [a, b] \subseteq I \),

\[
\sum_{k \in [a, b] \cap \mathbb{Z}} f(k) - \int_{[a, b]} f(x) \, dx = \int_{[a, b]} f(x) \, dB(x)
\]

\[
= [f(b)B(b) - f(a)B(a^-)] - \sum_{i=1}^{p} \int_{[a_i, b_i] \cap I_i} B(x) \, df(x).
\]
By noting that $|B(x)| \leq \frac{1}{2}$, each of the $p + 1$ terms in the last line is bounded by $M$. Then the desired claim follows by passing to the limit as $[a, b] \uparrow I$. □

**Proof of Lemma 3.8**

Let $k_0 \in \mathbb{R}$ and $\sigma, \lambda > 0$. In addition, note that we have the following tail estimate for the gaussian density function:

$$
\int_{|k-k_0|>\lambda \sigma} e^{-(k-k_0)^2/2\sigma^2} \, dk = 2\sigma \int_{\lambda}^{\infty} e^{-x^2/2} \, dx \leq 2\sigma \int_{\lambda}^{\infty} \frac{x}{\lambda} e^{-x^2/2} \, dx
$$

Then by Lemma 3.7,

$$
\sum_{k \in \mathbb{Z}} e^{-(k-k_0)^2/2\sigma^2} = \int_{|k-k_0|\leq \lambda \sigma} e^{-(k-k_0)^2/2\sigma^2} \, dk + \mathcal{O} \left( \sup_{|k-k_0|\leq \lambda \sigma} e^{-(k-k_0)^2/2\sigma^2} \right)
$$

$$
= \sqrt{2\pi} \sigma + \mathcal{O} \left( \frac{\sigma}{\lambda} e^{-\lambda^2/2} \right) + \mathcal{O}(1).
$$

This proves (3.7). Similarly, applying Lemma 3.7 to the sum on the range $\lambda \sigma < |k-k_0| < K$ and passing to the limit as $K \to \infty$,

$$
\sum_{k \in \mathbb{Z}} e^{-(k-k_0)^2/2\sigma^2} \leq \int_{|k-k_0|>\lambda \sigma} e^{-(k-k_0)^2/2\sigma^2} \, dk + \mathcal{O} \left( \sup_{|k-k_0|>\lambda \sigma} e^{-(k-k_0)^2/2\sigma^2} \right)
$$

$$
\leq 2\sigma e^{-\lambda^2/2} + 4 e^{-\lambda^2/2},
$$

and, therefore, (3.8) is established. □

**Proof of Lemma 3.9**

Throughout the proof, the factorials will be identified with the values of the gamma function via the relation $z! = \Gamma(z + 1)$. Using the inequality $\log \cos x \leq -x^2/2$ for $|x| < \frac{\pi}{2}$, we obtain the following well known bound for the central binomial coefficient:

$$
\frac{1}{2^n} \binom{n}{n/2} = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos^n x \, dx \leq \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-nx^2/2} \, dx = \frac{1}{\sqrt{\pi n/2}}
$$
Next, write \( a = \frac{n}{2} \) and \( b = \sqrt{n} \frac{\varphi}{2} \) for simplicity. Then \( k = a + b \) and \( n - k = a - b \), and so,

\[
\frac{\binom{n}{k}}{\binom{n}{n/2}} = \frac{a!^2}{(a + b)!(a - b)!} = \frac{(a + m)!^2}{(a + b + m)!(a - b + m)!} \prod_{i=1}^{m} \frac{(a + b + i)(a - b + i)}{(a + i)^2}.
\]

For the limit of the ratio of factorials, by the Stirling’s approximation, we easily check that

\[
\lim_{m \to \infty} \frac{(a + m)!^2}{(a + b + m)!(a - b + m)!} = 1.
\]

This leads to the following identity

\[
\frac{\binom{n}{k}}{\binom{n}{n/2}} = \prod_{i=1}^{\infty} \frac{(a + b + i)(a - b + i)}{(a + i)^2} = \prod_{i=1}^{\infty} \left( 1 - \frac{b^2}{(a + i)^2} \right).
\]

Then by using the inequalities \( \log(1 + x) \leq x \) and \( |b| \leq a \), we get

\[
\log \left( \frac{\binom{n}{k}}{\binom{n}{n/2}} \right) \leq -\sum_{i=1}^{\infty} \frac{b^2}{(a + i)^2} \leq -\sum_{i=1}^{\infty} \frac{b^2}{(a + i)(a + i + 1)} = -\frac{b^2}{a + 1} \leq 1 - \frac{b^2}{a} = 1 - \frac{\varphi^2}{2}.
\]

Combining this and the upper bound for \( \binom{n}{n/2} \) altogether, we obtain (3.9).

Now we move on to deriving the second part, (3.10). Using Stirling’s approximation of the form \( n! = \sqrt{2\pi n} (n/e)^n (1 + O(n^{-1})) \), it is easy to verify that the central binomial coefficient admits the asymptotic formula of the form

\[
\frac{1}{2^n} \binom{n}{n/2} = \frac{1 + O(n^{-1})}{\sqrt{\pi n/2}}.
\]

Next, we assume that \( |\varphi| \leq \sqrt{n/2} \) holds. Then \( n \varphi^2 \binom{n}{n+2i} \) is bounded by \( \frac{1}{2} \) for any \( i \geq 1 \). So by (4.1),

\[
\frac{\binom{n}{k}}{\binom{n}{n/2}} = \exp \left\{ \sum_{i=1}^{\infty} \log \left( 1 - \frac{n \varphi^2}{(n + 2i)^2} \right) \right\} = \exp \left\{ -\sum_{i=1}^{\infty} \left( \frac{n \varphi^2}{(n + 2i)^2} + \frac{n^2 \varphi^4}{2(n + 2i)^4} + O\left( \frac{n^3 \varphi^6}{(n + 2i)^6} \right) \right) \right\}.
\]

Using Lemma 3.7, we have \( \sum_{i=1}^{\infty} \frac{1}{(n + 2i)^p} = \frac{1}{2(p-1)n^{p-1}} + O(n^{-p}) \) for each \( p > 1 \), and so, the above expansion further simplifies to

\[
\exp \left\{ -\left[ \frac{\varphi^2}{2} + O\left( \frac{\varphi^2}{n} \right) \right] - \left[ \frac{\varphi^4}{12n} + O\left( \frac{\varphi^4}{n^2} \right) \right] + O\left( \frac{\varphi^6}{n^2} \right) \right\}.
\]

All these error terms reduce to \( O_c(\frac{\varphi^2}{n}) \) when \( |\varphi| \leq cn^{1/4} \), hence the formula (3.10) follows.
Proof of Lemma 3.10
Assume \( a \geq 2n \) holds. In addition, write
\[
q(k) = \frac{(a - 1 + n - k)!}{(a - k)!a^{k-1}} = \prod_{j=1}^{k} \left(1 + \frac{j - k}{a}\right).
\]
Then by Lemma 3.7 applied to the monotone function \( j \mapsto \log \left(1 + \frac{j - k}{a}\right) \) for \( j \in [0, n) \),
\[
\left| \log q(k) - \int_0^n \log \left(1 + \frac{s - k}{a}\right) \, ds \right| \leq 2 \sup_{s \in [0, n)} \left| \log \left(1 + \frac{s - k}{a}\right) \right| \leq \frac{4n}{a} \leq 2.
\]
Here, we utilized the inequality \( \left| \log(1 + x) \right| \leq 2|x| \), valid for \( |x| \leq \frac{1}{2} \). Furthermore, we have
\[
\int_0^n \log \left(1 + \frac{s - k}{a}\right) \, ds = \int_{-\frac{n}{2}}^{\frac{n}{2}} \log \left(1 - \sqrt{n\varphi} + \frac{s}{a}\right) \, ds
\]
\[
= n \log \left(1 - \sqrt{n\varphi} + \frac{s^2}{(a - \sqrt{n\varphi/2})^2}\right) \, ds.
\]
Then by bounding each of the logarithmic terms in (4.2) using \( \log(1 + x) \leq x \) and then applying \( (1 - x)^{-2} \geq 1 + 2x \), valid for \( |x| < 1 \), we get
\[
\log q(k) \leq -\frac{n^{3/2}\varphi}{2a} - \int_0^{\frac{n}{2}} \frac{s^2}{a^2} \left(1 + \sqrt{n\varphi} + \frac{s}{a}\right) \, ds + 2.
\]
Evaluating the last integral, we conclude that (3.11) holds.

Now we turn to proving (3.12). This time, we expand each of the logarithmic terms in (4.2) using Taylor expansions for \( \log(1 + x) \) and \( (1 - x)^{-p} \) in the range \( |x| \leq \frac{1}{2} \). Then by noting that both \( \sqrt{n\varphi} \) and \( \frac{s^2}{(a - \sqrt{n\varphi/2})^2} \) are bounded by \( \frac{1}{2} \) whenever \( a \geq 2n \), we get
\[
\log q(k) = -n \left[ \sum_{j=1}^{2} \frac{1}{j} \left(\frac{\sqrt{n\varphi}}{2a}\right)^j + \mathcal{O} \left(\frac{n^{3/2}\varphi^3}{a^3}\right)\right]
\]
\[
- \left[ \sum_{j=1}^{2} \frac{(n/2)^{2j+1}}{j(2j+1)a^{2j}} \left(1 - \frac{\sqrt{n\varphi}}{2a}\right)^{-2j} + \mathcal{O} \left(\frac{n^7}{a^6}\right)\right] + \mathcal{O} \left(\frac{n}{a}\right)
\]
\[
= -\left[ \frac{n^{3/2}\varphi}{a} + \frac{n^2\varphi}{8a^3} + \mathcal{O} \left(\frac{n^{5/2}\varphi^3}{a^3}\right)\right]
\]
\[
- \left[ \frac{n^3}{24a^2} + \frac{n^7/2\varphi}{24a^4} + \mathcal{O} \left(\frac{n^4\varphi^2}{a^4}\right)\right] - \left[ \frac{n^5}{320a^4}\right]
\]
\[+ \mathcal{O} \left(\frac{n^{11/2}\varphi}{a^5}\right) + \mathcal{O} \left(\frac{n^7}{a^6}\right) + \mathcal{O} \left(\frac{n}{a}\right).
\]
If we fix $\delta, c > 0$ and assume that both $a \geq \delta n^{5/4}$ and $|\varphi| \leq ca^{1/5}$ hold, then each of the error terms above reduces to $O_{\delta, c}(a^{-1/5})$ and hence (3.12) follows. □

Proof of Lemma 3.12

The inequality follows by noting that $f(x) = x - \frac{x^2}{2(1 + x)} - \log(1 + x)$ achieves its global minimum at $x = 0$ with the value $f(0) = 0$, which itself is the consequence of the computation

$$f'(x) = \begin{cases} -\frac{x^2}{1+x^2} < 0, & x < 0, \\ \frac{x^2}{2(1+x)^2} > 0, & x > 0. \end{cases}$$

□

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References

[1] Bayer, D. and Diaconis, P., Trailing the dovetail shuffle to its lair, Ann. Appl. Probab. 2 (1992), 294–313.

[2] Billey, S., Burdzy, K. and Sagan, B., Permutations with given peak set, J. Integer Seq. 16 (2013), Article 13.6.1.

[3] David, F. and Barton, D., Combinatorial chance, Hafner Publishing Co., 1962.

[4] Diaconis, P., Group representations in probability and statistics, Institute of Mathematical Statistics, Hayward, CA, 1988.

[5] Diaconis, P., Fulman, J. and Holmes, S., Analysis of casino shelf shuffling machines, Annals Appl. Probab. 23 (2013), 1692–1720.

[6] Diaconis, P. and Graham, R., Magical mathematics. The mathematical ideas that animate great magic tricks, Princeton University Press, 2012.

[7] Diaconis, P., McGrath, M. and Pitman, J., Riffle shuffles, cycles, and descents, Combinatorica 15 (1995), 11–29.

[8] Fulman, J., Stein’s method and non-reversible Markov chains, in: Stein’s method: expository lectures and applications, 69–77, IMS Lecture Notes Monogr. Ser., 46, Inst. Math. Statist., 2004.

[9] Fulman, J., The distribution of descents in fixed conjugacy classes of the symmetric groups, J. Combin. Theory Ser. A 84 (1998), 171–180.
[10] Fulman, J., Neumann, P. and Praeger, C., A generating function approach to the enumeration of matrices in classical groups over finite fields, *Mem. Amer. Math. Soc.* **176** (2005), no. 830.

[11] Gessel, I. and Reutenauer, C., Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A* **64** (1993), 189–215.

[12] Graham, R.L., Knuth, D.E., Patashnik, O., Concrete mathematics: a foundation for computer science, 2nd ed. Addison-Wesley, Reading, Mass. (1994).

[13] Harper, L., Stirling behavior is asymptotically normal, *Ann. Math. Stat.* **38** (1966), 410–414.

[14] Kim, G., Distribution of descents in matchings, *Annals Combin.* **23** (2019), 73–87.

[15] Kim, G. and Lee, S., Central limit theorems for descents in conjugacy classes of $S_n$, *J. Combin. Theory Ser. A* **169** (2020), 105123.

[16] Knuth, D., The art of computer programming, Volume 3. Sorting and searching, Addison-Wesley, 1973.

[17] Nyman, K., The peak algebra of the symmetric group, *J. Algebraic Combin.* **17** (2003), 309–322.

[18] Petersen, K., Eulerian numbers, Birkhauser, 2015.

[19] Petersen, K., Enriched $P$-partitions and peak algebras, *Adv. Math.* **209** (2007), 561–610.

[20] Pitman, J., Probabilistic bounds on the coefficients of polynomials with only real zeros, *J. Combin. Theory Ser. A* **77** (1997), 279–303.

[21] Reiner, V., Signed permutation statistics and cycle type, *Europ. J. Combin.* **14** (1993), 569–579.

[22] Robbins, H. “A Remark on Stirling’s Formula.” *The American Mathematical Monthly* **62**, no. 1 (1955), 26–29.

[23] Schocker, M., The peak algebra of the symmetric group revisited, *Adv. Math.* **192** (2005), 259–309.

[24] Stembridge, J., Enriched $P$-partitions, *Trans. Amer. Math. Soc.* **349** (1997), 763–788.

[25] Tanny, S., A probabilistic interpretation of Eulerian numbers, *Duke Math. J.* **40** (1973), 717–722.

[26] Vershynin, R., High-Dimensional Probability. Cambridge University Press, 2018.

[27] Warren, D. and Seneta, E., Peaks and Eulerian numbers in a random sequence, *J. Appl. Probab.* **33** (1996), 101–114.
