Vibration of rotating circular cylindrical shells with distributed springs

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ABSTRACT

Free vibrations of rotating cylindrical shells with distributed springs were studied. Based on the Flügge shell theory, the governing equations of rotating cylindrical shells with distributed springs were derived under typical boundary conditions. Multicomponent modal functions were used to satisfy the distributed springs around the circumference. The natural responses were analyzed using the Galerkin method. The effects of parameters, rotation speed, stiffness, and ratios of thickness/radius and length/radius, on natural response were also examined.

KEYWORDS: vibration, rotating, distributed spring, stiffness

1. INTRODUCTION

Rotating circular cylindrical shells have been used in wide applications, such as missile technology, tires, turbines and planetary gear trains [1–3]. Vibration analysis is of great importance for the safety and life of these applications.

Cylindrical shells and rotating cylindrical shells have been studied in a body of works. Qin et al. [4] investigated the free vibration characteristics of cylindrical shells with arbitrary boundary conditions. A unified solution for the three different types of expansion functions is developed using the Rayleigh–Ritz method. Bryan [5] studied the rotating shells and discovered the traveling mode phenomenon of a rotating thin ring. The influence of Coriolis effect on free vibrations of rotating cylindrical shells have been studied in [6–9], and the results show that Coriolis forces have a significant impact on the natural response of cylindrical shells. The influences of boundary conditions on the natural frequencies of rotating cylindrical shells were also investigated in earlier works [10, 11]. Furthermore, Sun et al. [12] employed the Rayleigh–Ritz method to solve the free vibration of a rotating cylindrical shell having arbitrary boundary conditions. Qin et al. [13] studied the vibration of a rotating cylindrical shell coupled with an annular plate, and the Rayleigh–Ritz method was employed to derive the motion equations for the rotating shell–plate combination. Ren et al. [14] used the Galerkin method to study the vibration and instability of rotating thin-walled composite shafts with internal damping. Senjanovic et al. [15] studied rotating cylindrical shells with pre-stress using a finite strip method. Qin et al. [16] studied traveling wave motions of rotating multilayered functionally graded graphene platelet-reinforced composite cylindrical shells under general boundary conditions. Donnell shell theory and artificial spring technique were used to obtain theoretical equations.

Intermediate supports on cylindrical shells are a common way to increase their stiffness and improve the system’s robustness. There are few published references available concerning the vibration of cylindrical shells with supports. Irie et al. [17] studied free vibrations of a circular shell that is elastically restrained by axially spaced springs using the transfer matrix method. Dynamic analyses of elastic ring-stiffened cylindrical shells have been carried out by Forsberg [18], Al-Najafi and Warburton [19], and Beskos and Oates [20]. Yang and Du [21] studied the free vibration of cylindrical shells with discrete stiffnesses. Swadiaudhipong et al. [22] used the Ritz method to investigate the vibration of cylindrical shells with rigid intermediate supports. Loy and Lam [23] and Xiang et al. [24] studied the vibration of cylindrical shells with ring supports. Qin et al. [25] provided a general approach for the free vibration analysis of functionally graded carbon nanotube-reinforced composite cylindrical shells with arbitrary boundary conditions. Wang et al. [26] proposed the Ritz method for vibration analysis of cylindrical shells with ring stiffeners. Chen et al. [27] studied free vibrations of ring-stiffened cylindrical shells with intermediate large frame ribs by using the wave method. For free vibration analysis of shells, numerical simulations, such as the finite element method [28], Ritz method [29] and Galerkin method [30], are usually used. Fletcher [31] systematically presented computational Galerkin methods. Araki and Samejima [32] used the Galerkin method to study the vibration of cylindrical shells. Mahmoudkhani [33] studied the nonlinear vibration of cylindrical shells using the Galerkin method. Cooley and Parker [34]...
used the Galerkin method to study the vibration of rotating rings.

As mentioned earlier, the Coriolis and centrifugal accelerations due to rotation have a significant influence on the natural response of cylindrical shells. The intermediate supports, which change the stiffness of cylindrical shells, also have a great influence on the vibration properties. In the literature, elastic supports of rotating cylindrical shells are mainly arranged in the axial direction. However, to the authors’ knowledge, the vibration of rotating cylindrical shells with distributed circumferential support springs has not been studied so far. The main subject of this work is to study the free vibration of rotating cylindrical shells with distributed springs in the circumferential direction. Based on the Flügge shell theory, the governing equations of rotating cylindrical shells with distributed springs were derived under typical boundary conditions. The distributed stiffnesses are considered as external forces by Dirac delta functions. To express the displacement constraint caused by distributed springs, multi-component modal functions were used to represent the circumferential mode. The natural responses, i.e. natural frequencies and natural modes, were analyzed by the Galerkin method. The effects of parameters, such as rotation speed, stiffness, and ratios of thickness/radius and length/radius, on natural response were also examined.

2. FORMULATIONS

Consider a rotating thin circular cylindrical shell with Young’s modulus $E$, Poisson’s ratio $\mu$, mass density $\rho$, mid-surface radius $R$, length $L$ and thickness $h$, as shown in Fig. 1. The shell is supported by distributed springs and rotating at a constant speed $\Omega$ about its longitudinal axis. The mid-surface displacements in the axial, circumferential and radial directions are denoted by $u(x, \theta, t)$, $v(x, \theta, t)$ and $w(x, \theta, t)$, respectively; $x$ is the coordinate along its longitudinal axis, $\theta$ is the coordinate along the circumferential direction and $t$ is time. The springs are distributed around the circumference with stiffness $k_i$, $i = 1, 2, \ldots, N_s$, where $N_s$ is the number of springs. $\alpha$ is the angle between the radial direction and a spring.

An infinitesimal element $P$ in the mid-surface of the thin cylindrical shell, with arc length $d\alpha$ and $R \, d\alpha$, is analyzed to establish the governing equations. According to the kinetic principle, the velocity of $P$ is

$$ v_P = v_O + \Omega \bar{j} \times r_P + v_P $$

$$ = R\Omega \bar{k} + \Omega \bar{i} \times (\bar{u} \bar{j} + \bar{w} \bar{k}) + (\bar{u} \bar{j} + \bar{w} \bar{k}) $$

$$ = \bar{u} \bar{j} + (\bar{w} - v\Omega) \bar{k} + (\bar{v} + R\Omega + w\Omega) \bar{k}, \quad (1) $$

where $v_O$ is the velocity of the origin $O$ of the coordinates $i, j, k$, $\Omega$ is the rotation speed of the shell, and $r_P$ and $v_P$ are the position vector and velocity of $P$ in the $i, j, k$ coordinates, respectively. Therefore, the inertia force $F_i$ per unit area of the cylindrical shell is

$$ F_i = -\rho h v_P = -\rho h \left[ \bar{u} \bar{j} + (\bar{w} - 2\Omega \bar{v} - \Omega^2 \bar{w} + R\Omega^2) \bar{k} \right] $$

$$ + (\bar{v} + 2\Omega \bar{w} - \Omega^2 \bar{v}) \bar{k}, \quad (2) $$

where $\Omega^2 \bar{v}$ and $\Omega^2 \bar{w}$ are the centrifugal terms, and $2\Omega \bar{v}$ and $2\Omega \bar{w}$ are the Coriolis terms. Notice that $\rho h R \Omega^2$ is a term of constant centrifugal force, which causes a static deflection in the $w$ direction. Thus, this static deflection is taken as the equilibrium
Table 2 Comparison of frequencies (Hz) of a cylindrical shell with experiment and analytical solutions \((m = 1, \rho = 7900 \ \text{kg/m}^3, \ E = 193 \ \text{GPa}, \ \mu = 0.315, \ h = 0.178 \ \text{mm})\).

| \(n\) | Experiment \([39]\) | Analytical solutions \([40]\) | Current |
|---|---|---|---|
| Case I \((L = 1.067 \ \text{m}, R = 0.356 \ \text{m})\) | | | |
| 2 | 1.0 | 0.9 | |
| 3 | 3.9 | 2.6 | 2.5 |
| 4 | 6.3 | 4.9 | 4.9 |
| 5 | 9.1 | 8.2 | 7.9 |
| 6 | 12.1 | 11.8 | 11.6 |
| 7 | 16.4 | 16.3 | 16.1 |
| 8 | 20.9 | 20.9 | 20.9 |
| 9 | 26.8 | 26.8 | 26.7 |
| 10 | 32.8 | 32.9 | 32.9 |

| Case II \((L = 0.457 \ \text{m}, R = 0.254 \ \text{m})\) | | | |
|---|---|---|---|
| 2 | 1.9 | 1.8 | |
| 3 | 6.6 | 6.4 | 5.9 |
| 4 | 10.9 | 10.8 | 9.8 |
| 5 | 17.4 | 17.5 | 15.7 |
| 6 | 23.1 | 22.8 | 22.7 |
| 7 | 33.0 | 32.9 | 31.3 |
| 8 | 43.1 | 43.2 | 41.2 |
| 9 | 52.7 | 52.3 | 52.4 |
| 10 | 65.5 | 65.2 | 65.0 |

The forces of distributed springs acting on the shell can be considered as external forces and are decomposed as radial forces \(F_v\) and circumferential forces \(F_w\). Since \(F_v\) and \(F_w\) are discrete, they are represented by using the Dirac delta function as

\[
F_v = \sum_{i=1}^{N_v} k_i \left[ v(x, \theta_i, t) \sin^2 \alpha + w(x, \theta_i, t) \sin \alpha \cos \alpha \delta(\theta - \theta_i) \right],
\]

\[
F_w = \sum_{i=1}^{N_w} k_i \left[ v(x, \theta_i, t) \sin \alpha \cos \alpha + w(x, \theta_i, t) \cos^2 \alpha \delta(\theta - \theta_i) \right].
\]

where \(\theta_i\) is the spring's position angle and \(\delta(\cdot)\) is the Dirac delta function. Notice that \(F_v\) and \(F_w\) are discrete, and have value only at the position \(\theta_i\).

Considering the equilibrium of the shell element \(P\) under the inertia forces in Eq. (2), the internal forces and moments in Eq. (3), and external forces in Eq. (5), the motion equations of the rotating cylindrical shell with distributed springs can be obtained according to the derivation method after \([35]\):

\[
L_x(u, v, w) - \rho h \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{(6a)}
\]

\[
L_\theta(u, v, w) - F_v - \rho h \left( \frac{\partial^2 v}{\partial t^2} + 2\Omega \frac{\partial w}{\partial t} - \Omega^2 v \right) = 0, \quad \text{(6b)}
\]

\[
L_w(u, v, w) - F_w - \rho h \left( \frac{\partial^2 w}{\partial t^2} - 2\Omega \frac{\partial v}{\partial t} - \Omega^2 w \right) = 0. \quad \text{(6c)}
\]

The differential operators \(L_x, L_\theta\) and \(L_w\) in Eq. (6) are given as follows:

\[
L_x = \frac{\partial^2 N_x}{\partial x^2} + \frac{\partial N_{\theta x}}{R \partial \theta} + \frac{N_{\theta}}{R^2} \left( \frac{\partial^2 w}{\partial \theta^2} - R \frac{\partial w}{\partial x} \right), \quad \text{(7a)}
\]
Figure 2 Normalized natural frequencies with respect to rotation speed \( L/R = 5, h/R = 0.05, \mu = 0.3, m = 1 \): (a) \( k_i = 0 \) and (b) \( k_i = 0.03 \). \( O_i \) in the legend denotes frequency order \( i \).

\[ L_{\theta} = \frac{\partial N_{\theta}}{\partial x} + \frac{\partial N_\vartheta}{\partial \vartheta} + \frac{Q_\vartheta}{R} + \tilde{N}_\theta \left( \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial w}{\partial \vartheta} \right), \quad \text{(7b)} \]

\[ L_w = -\frac{N_\vartheta}{R} + \frac{\partial Q_\vartheta}{\partial x} + \frac{\partial Q_{\vartheta}}{\partial \vartheta} + \tilde{N}_\vartheta \left( -R \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \vartheta^2} \right). \quad \text{(7c)} \]

By substituting Eqs (2)–(5) into Eq. (6), the governing equations yield

\[ \frac{\partial^2 u}{\partial x^2} + \frac{1 - \mu}{2R^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1 + \mu}{2R} \frac{\partial^2 v}{\partial x \partial \theta} + \frac{\mu}{K} \frac{\partial w}{\partial x} + \kappa \left( \frac{1 - \mu}{2R^2} \frac{\partial^2 w}{\partial \theta^2} - R \frac{\partial^3 w}{\partial x^3} + \frac{1 - \mu}{2R} \frac{\partial^3 w}{\partial x \partial \theta^2} \right) + \frac{\rho h \Omega^2}{K} \left( \frac{\partial^2 u}{\partial \theta^2} - R \frac{\partial w}{\partial x} \right) = \frac{\rho h \partial^2 u}{K \partial t^2}, \quad \text{(8a)} \]
Vibration modes of thin-walled cylindrical panels with distributed springs are expressed as the superposition of axial, circumferential, and radial modes. Therefore, functions of the complex Fourier series are adopted for cylindrical shells without circumferential support [5-10].

Vibration modes are represented by the separation of axial and circumferential vibrations. The circumferential vibration is expressed as multicomponent forms to satisfy the asymmetry caused by distributed springs. Therefore, the displacements of the cylindrical shell are expressed as

\[ u = \psi_\varepsilon(x) \phi_\varepsilon(\theta) \cos \omega t = \psi_\varepsilon(x) \sum_{n=-N_w}^{N_w} A_n e^{i\kappa t} \cos \omega t, \]  \hspace{1cm} (9a)

\[ v = \psi_\varepsilon(x) \phi_\varepsilon(\theta) \cos \omega t = \psi_\varepsilon(x) \sum_{n=-N_w}^{N_w} B_n e^{i\kappa t} \cos \omega t, \]  \hspace{1cm} (9b)

\[ w = \psi_\varepsilon(x) \phi_\varepsilon(\theta) \cos \omega t = \psi_\varepsilon(x) \sum_{n=-N_w}^{N_w} C_n e^{i\kappa t} \cos \omega t, \]  \hspace{1cm} (9c)

where \( \omega \) is the natural frequency, the subscript \( u, v \) and \( w \) represent the axial, circumferential and radial directions, and \( \psi_\varepsilon(x), \psi_\varepsilon(x) \) and \( \psi_\varepsilon(x) \) are the axial modal functions. \( \phi_\varepsilon(\theta), \phi_\varepsilon(\theta) \) and \( \phi_\varepsilon(\theta) \) are the modal functions in the circumferential direction. Due to the discreteness of distributed springs, the circumferential mode cannot use a single sine or cosine function as done for cylindrical shells without circumferential support [5-10].

Therefore, functions of the complex Fourier series are adopted to represent the circumferential modal functions, that is \( \phi_\varepsilon(\theta), \phi_\varepsilon(\theta) \) and \( \phi_\varepsilon(\theta) \) as shown in Eq. (9), to satisfy the asymmetric property of the cylindrical shells with distributed stiffnesses. \( l, k \) and \( n \) are the order numbers of complex Fourier series, respectively; \( A_n, B_n \) and \( C_n \) are the corresponding Fourier coefficients; and \( N_u, N_v, N_w, N_u \) and \( N_w \) are the orders of the truncated Fourier series of \( \phi_\varepsilon(\theta), \phi_\varepsilon(\theta) \) and \( \phi_\varepsilon(\theta) \) in the positive and negative directions. \( N_u, N_v, N_w, N_u \) and \( N_w \) should be infinite, but usually finite orders are enough for an approximation analysis as shown in Section 3.

The beam functions are adopted to express the axial modal functions [37]. \( \psi_\varepsilon(x) \) is the partial derivative of \( \psi_\varepsilon(x) \) and \( \psi_\varepsilon(x) \) with respect to \( x \):

\[ \psi_\varepsilon(x) = \frac{\partial \psi_\varepsilon(x)}{\partial x} = \frac{\partial \psi_\varepsilon(x)}{\partial x}. \]  \hspace{1cm} (10)
Springs. They are expressed by beam functions [37] as

\[ \psi(x) = \sin \left( \frac{m \pi x}{L} \right), \quad m = 1, 2, 3, \ldots \]  

(11a)

C-C, C-S, C-F:

\[ \psi(x) = \cosh \left( \frac{\lambda_m x}{L} \right) - \cos \left( \frac{\lambda_m x}{L} \right) \]
\[ - \varsigma_m \left[ \sinh \left( \frac{\lambda_m x}{L} \right) - \sin \left( \frac{\lambda_m x}{L} \right) \right], \]  

(11b)

F-F:

\[ \psi(x) = \cosh \left( \frac{\lambda_m x}{L} \right) + \cos \left( \frac{\lambda_m x}{L} \right) \]
\[ - \varsigma_m \left[ \sinh \left( \frac{\lambda_m x}{L} \right) + \sin \left( \frac{\lambda_m x}{L} \right) \right], \]  

(11c)

where

\[ \varsigma_m = \frac{\cosh (\lambda_m) - \cos (\lambda_m)}{\sinh (\lambda_m) - \sin (\lambda_m)}, \quad \cos \lambda_m \cos \lambda_m = 1 \]  

for C-C, F-F.  

(11d)
\[
\frac{\cosh(\lambda_m) + \cos(\lambda_m)}{\sinh(\lambda_m) + \sin(\lambda_m)}, \quad \tan \lambda_m = \tanh \lambda_m \text{ for } C-S
\]
and
\[
\frac{\sinh(\lambda_m) - \sin(\lambda_m)}{\cosh(\lambda_m) + \cos(\lambda_m)}, \quad \cos \lambda_m \cosh \lambda_m
\]
\[
= -1 \text{ for } C-F.
\]
Here, S denotes a simply supported end, C denotes a clamped end and F denotes a free end. \(\lambda_m\) can be determined by solving the transcendental equations numerically; \(m\) is the axial half wavenumber. The governing equations are normalized by parameters
\[
\begin{align*}
(u, v, w) &= h(\bar{u}, \bar{v}, \bar{w}), \\
x &= L\bar{x}, \\
\gamma &= \frac{R}{L}, \\
\zeta &= \frac{h}{R}, \\
k_i &= \frac{Eh}{R^2 (1 - \mu^2)} \bar{k}_i, \\
\omega_0 &= \sqrt{\frac{E}{\rho R^2 (1 - \mu^2)}}, \\
i &= \omega_0, \\
\dot{\omega}, \dot{\Omega} &= \frac{\Omega}{\omega_0}
\end{align*}
\]
to get
\[
\begin{align*}
\gamma^2 \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial w}{\partial x} + \frac{1 - \mu}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1 + \mu}{2} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial \theta} \\
+ \kappa \left( \frac{1 - \mu}{2} \frac{\partial^2 u}{\partial \theta^2} - \frac{\gamma^2 \frac{\partial^4 w}{\partial x^2}}{2} + \frac{1 - \mu}{2} \frac{\gamma^2 \frac{\partial^4 w}{\partial x^2}}{2} \right) \\
+ \Omega^2 \left( \frac{\partial^4 u}{\partial \theta^4} + \frac{\partial w}{\partial \theta} \right) &= -\omega^2 \bar{u}, \\
+ \frac{1}{2} \frac{\mu}{\partial \theta} \frac{\partial^4 v}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \\
+ \kappa \left( \frac{3 (1 - \mu)}{2} \frac{\gamma^2 \frac{\partial^4 v}{\partial \theta^2}}{2} - \frac{3 - \mu}{2} \frac{\gamma^2 \frac{\partial^4 v}{\partial \theta^2}}{2} \right) \\
+ \dot{\Omega}^2 \left( \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \right) &= -\omega^2 \bar{v} - 2\Omega \omega \bar{\omega} - \Omega^2 \bar{v}, \\
- \mu \frac{\partial u}{\partial x} \frac{\partial v}{\partial \theta} - \kappa \left( \frac{\gamma^2 \frac{\partial^4 w}{\partial x^2}}{2} + \frac{\partial v}{\partial \theta} + \frac{2 \gamma^2 \frac{\partial^4 w}{\partial x^2 \partial \theta^2}}{2} \right) \\
- \kappa \left[ -\frac{\gamma^2 \frac{\partial^3 u}{\partial x^2}}{2} + \frac{1 - \mu}{2} \frac{\gamma \frac{\partial^3 u}{\partial \theta^2}}{2} - \frac{3 - \mu}{2} \frac{\gamma^2 \frac{\partial^3 v}{\partial \theta^2}}{2} \right] \\
+ \omega \frac{2 \gamma^2 \frac{\partial^2 w}{\partial \theta^2}}{2} - \Omega^2 \left( -\frac{\gamma \frac{\partial u}{\partial x} + \gamma \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}}{2} \right) \\
= -\omega^2 \bar{w} - 2\Omega \omega \bar{\omega} - \Omega^2 \bar{w}.
\end{align*}
\]
Substitute Eq. (9) into Eq. (13) for different boundary conditions in Eq. (11) for a determined axial modal order \(m\). The order numbers of Fourier series \(N_m, N_e, \text{ and } N_w\) in Eq. (9) are set to the same number \(N_{saw}\) for simplification. Thus, the total order number of \(\phi_n(\theta), \phi_e(\theta) \text{ or } \phi_w(\theta)\) is \(N = 2N_{saw} + 1\) since they have both negative and positive components. After that, each equation in Eq. (13) is multiplied by the conjugate function of corresponding Fourier basis functions \(e^{i\beta \theta}, e^{i\xi \theta} \text{ and } e^{i\theta_0}\) and then integrated by the Galerkin method. Accordingly, \(3N \times 3N\) equations can be obtained with \(A_i, B_i \text{ and } C_n\) as unknowns, which is written in the matrix form as
\[
(\dot{\omega}^2 M + \omega G + K) X = 0,
\]
where \(M, G \text{ and } K\) are \(3N \times 3N\) matrices and \(X\) is a \(3N \times 1\) vector. The details of the matrices in Eq. (14) and how to calculate the elements of the matrices are listed in the Appendix. The natural frequencies can be calculated by Eq. (14), and the vibration modes can be obtained by Eqs (9) and (14).

3. RESULTS

To validate the present method, natural frequencies of a cylindrical shell without distributed springs were compared to experimental results in [38] for simply supported and free–free end conditions. The results were also compared to both experimental results [39] and analytical solutions [40] for free–free end conditions. The same parameters as the references were adopted to calculate the natural frequencies, and the results are listed in Tables 1 and 2. It shows that the results obtained by the current method agree well with those in the literature. In the following analysis, distributed springs are assumed equally distributed in the circumference and having the same stiffness with typical \(N_s = 3\). Simply supported end condition is assumed and \(l, k \text{ and } n\) are set to the same number.

Figure 2 shows normalized natural frequencies with respect to rotation speed without and with distributed springs. It indicates that frequencies separate for nonzero speed, which corresponds to forward and backward waves in both cases. The forward one represents the vibration wave propagating in the positive \(\theta\) direction and the backward one represents the vibration wave propagating in the negative \(\theta\) direction. As shown in Fig. 2b, distributed springs will increase natural frequencies, especially for lower order natural frequencies. On the other hand, because the distributed spring only affects the circumferential mode intermittently, for the high-frequency waveform, a limited number of circumferentially distributed springs only affect few waveforms, so it can be seen that the influence of distributed springs on high frequency is very small. There are multiple regions of frequency veering for cylindrical shells without or with distributed springs, such as regions V1 and V2 in Fig. 2b. However, the frequency veering in Fig. 2b is more complex in the lower frequency areas than Fig. 2a. At lower order frequencies, e.g. V1, veering regions are large and wide. At higher order frequencies, e.g. V2, veering regions are sharp and narrow. This means the sensitivity of natural frequencies in these areas is much higher for shells with distributed springs than those without distributed springs.

Figures 3 and 4 show the vibration modes for the natural frequencies \(\omega = 0.1879\) and \(\omega = 0.4476\) with the rotation speed \(\Omega = 0.05\). In Fig. 3, the component of \(n = 1\) is almost equal to the component of \(n = 2\) for \(A_i\), and the components of \(n = 2\) for \(B_i \text{ and } C_n\) are higher than the components of \(n = 1\). In Fig. 4.
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Figure 6 Vibration mode of $\dot{\omega} = 0.1401$ ($m = 1, L/R = 5, h/R = 0.05, \mu = 0.3, k_i = 0.03, \bar{\Omega} = 0.06$): (a) end face view; (b) 3D view; and (c) coefficients of $\phi_u, \phi_v$, and $\phi_w$.

Figure 7 Vibration mode of $\dot{\omega} = 0.2027$ ($m = 1, L/R = 5, h/R = 0.05, \mu = 0.3, k_i = 0.03, \bar{\Omega} = 0.12$): (a) end face view; (b) 3D view; and (c) coefficients of $\phi_u, \phi_v$, and $\phi_w$.

the component of $n = 1$ is dominant for $A_0$; the component of $n = 5$ is dominant for $C_n$; and the components of $n = 1$ and $n = 5$ are equivalent for $B_k$. Other components, such as $n = 2, n = 4$ and $n = 7$, also exist and cannot be ignored. It indicates that the vibration mode is complicated and $\phi_u, \phi_v$, and $\phi_w$ contain multiple components instead of a single component compared to the case without distributed stiffnesses. Thus, it is difficult to nominate a vibration mode by circumferential wavenumber as done for free cylindrical shells. Bifurcation appears for some frequencies when the rotation speed is low, such as S1 and S2 in Fig. 2b. For instance, before point S1 with the rotation speed $\bar{\Omega} = 0.05$, there are two groups of forward and backward waves that have similar vibration modes as shown in Fig. 5a and b for the natural frequencies $\dot{\omega} = 0.1879$ and $\dot{\omega} = 0.2187$. For the two natural frequencies, their vibration modes contain both positive wave ($n = 3$) and negative wave ($n = -3$), and the positive and negative waves are equivalent. Figure 5c shows the vibration mode of forward wave after the point S1 at $\bar{\Omega} = 0.07$ for the natural frequency $\dot{\omega} = 0.2418$. It can be found that after the bifurcation the negative wave ($n = -3$) is higher than the positive wave ($n = 3$), which means the vibration mode of the shell changes.
Vibration modes change greatly at some regions, such as region V1 in Fig. 2b. Figures 6–9 illustrate vibration mode changing at the region V1. It can be seen that at point A (\(\bar{\omega} = 0.1401\)) main component for \(C_n\) is \(n = 2\); the components of \(n = 1\) and \(n = 2\) for \(A_i\) and \(B_k\) are equivalent. Because first-order natural frequency and second-order natural frequency are close as shown in Fig. 2b, their main components of vibration modes are hard to distinguish. At point B (\(\bar{\omega} = 0.2027\)), which is close to the veering point, the component of \(n = 1\) becomes larger than that of \(n = 2\) for \(A_i\) and \(B_k\). At point C (\(\bar{\omega} = 0.2172\)), which is away from the veering point, the component of \(n = 1\) becomes larger than that of \(n = 2\) for the vibration modes of three directions. At point D (\(\bar{\omega} = 0.25\)), which is far away from the veering point, the main component of the vibration mode is \(n = 1\). Therefore, at veering regions, vibration modes are complicated, while at the regions away from the veering region, vibration modes are dominated by a certain component. By the analysis of the other typical boundary
conditions, a similar phenomenon occurs as mentioned above.

Figure 10 shows the effect of stiffness of distributed springs on the natural frequencies at \( \bar{\Omega} = 0.1 \). It indicates that with the increase of stiffness, most natural frequencies increase while a few natural frequencies decrease. In some regions, frequency veering occurs, which means that the stiffness variation of distributed springs can change the natural frequency and vibration modes greatly.

Figure 11 shows the effect of \( L/R \) on the natural frequencies at \( \bar{\Omega} = 0.1 \). It can be seen that the natural frequencies will decrease with the increase of \( L/R \), especially for lower values of \( L/R \). Besides, frequency veering occurs in some regions as explained in Fig. 2. Figure 12 shows the effect of \( h/R \) on the natural frequencies at \( \bar{\Omega} = 0.1 \). The results show that the increase of \( h/R \) has less effect on the lower order frequencies but significantly affects the higher order frequencies.

4. CONCLUSIONS

Free vibration of rotating thin cylindrical shells with distributed springs was studied. The governing equations were derived based on the Flügge shell theory for typical end conditions. The beam and exponential functions are adopted to express the vibration modes. Natural responses were analyzed by discretizing the governing equations using the Galerkin method. The effects of parameters on natural frequencies were also examined. The results show that distributed springs
Figure 12 Normalized natural frequencies with respect to \( h/R \) \((L/R = 5, k_i = 0.03, \mu = 0.3, m = 1, \bar{\Omega}/\omega_1 = 0.1)\).

increase the natural frequencies and make the vibration mode complicated. With the increase of rotation speed, frequency veering occurs at some rotation speed. With the increase of stiffness, most natural frequencies increase while a few natural frequencies decrease. The natural frequencies will decrease with the increase of \( L/R \), especially for the lower value of \( L/R \). The increase of \( h/R \) has less effect on the lower order frequencies but significantly affects the higher order frequencies.

### NOMENCLATURE

- \( E \) = Young's modulus (Pa)
- \( \mu \) = Poisson's ratio
- \( \rho \) = mass density (kg/m\(^3\))
- \( R \) = mid-surface radius (m)
- \( L \) = length (m)
- \( H \) = thickness (m)
- \( \Omega \) = rotation speed (rad/s)
- \( u, v, w \) = mid-surface displacements in the axial, circumferential and radial directions (m)
- \( k_i \) = stiffness (N/m)
- \( \omega \) = natural frequency (rad/s)
- \( \phi_u, \phi_v, \phi_w \) = the modal functions of circumferential direction
- \( \psi(x) \) = the axial modal function

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### APPENDIX

\[ M = \begin{bmatrix} M_{AA} & 0 & 0 \\ 0 & M_{BB} & 0 \\ 0 & 0 & M_{CC} \end{bmatrix}, \]

\[ G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G_{BC} \\ 0 & G_{CB} & 0 \end{bmatrix}, \]

\[ K = \begin{bmatrix} K_{AA} & K_{AB} & K_{AC} \\ K_{BA} & K_{BB} & K_{BC} \\ K_{CA} & K_{CB} & K_{CC} \end{bmatrix}, \]

\[ X = \begin{bmatrix} [A] \\ [B] \\ [C] \end{bmatrix}, \]

\[ M^m_{AA} = \alpha_1, \quad M^m_{BB} = 1, \quad M^m_{CC} = 1, \]

\[ G^m_{BC} = 2\bar{\Omega}, \quad G^{m}_{CB} = 2\bar{\Omega}, \]

\[ K^m_{AA} = \gamma^2\alpha_3 - \left[ 1 - \frac{\mu}{2} \left( 1 + \frac{\zeta^2}{12} + \bar{\Omega}^2 \right) \right]^2 n^2\alpha_1, \]

\[ K^m_{AB} = \frac{(1 + \mu)}{2} n^2\gamma\alpha_1, \]
K_{BC}^{\text{en}} = \mu \gamma \alpha_1 - \frac{\gamma^2}{12} \left( \gamma^3 \alpha_3 + \frac{1 - \mu}{2} \gamma n^2 \alpha_1 \right) - \gamma \Omega_0^2 \alpha_1,

K_{BA}^{\text{en}} = j \left( 1 + \frac{\mu}{2} \right) \gamma n \alpha_2,

K_{BB}^{\text{en}} = \frac{1 - \mu}{2} \gamma^2 \alpha_2 - n^2 + \frac{3 (1 - \mu) \gamma^2}{24} \gamma \alpha_2

\sum_{i=1}^{N_i} \left( \hat{k}_i \sum_{n=-N}^{N} e^{i(n-n')0} \sin^2 \alpha \right).

K_{BC}^{\text{en}} = j n - j \frac{(3 - \mu) \gamma^2}{24} n \gamma^2 \alpha_2 + j n \Omega_0^2

\sum_{i=1}^{N_i} \left( \hat{k}_i \sum_{n=-N}^{N} e^{i(n-n')0} \sin \alpha \cos \alpha \right).

K_{CA}^{\text{en}} = -\mu \gamma \alpha_2 + \frac{\gamma^2}{12} \left( \gamma^3 \alpha_4 + \frac{1 - \mu}{2} \gamma n^2 \alpha_2 \right) + \gamma \Omega_0^2 \alpha_2,

K_{CB}^{\text{en}} = -j n + j \frac{(3 - \mu) \gamma^2}{24} n \gamma^2 \alpha_2 - j n \Omega_0^2

\sum_{i=1}^{N_i} \left( \hat{k}_i \sum_{n=-N}^{N} e^{i(n-n')0} \sin \alpha \cos \alpha \right).

K_{CC}^{\text{en}} = -1 - \frac{\gamma^2}{12} \left[ \gamma^4 \alpha_4 - 2n^2 \left( \gamma^2 \alpha_4 + 1 \right) + n^4 + 1 \right]

\sum_{i=1}^{N_i} \left( \hat{k}_i \sum_{n=-N}^{N} e^{i(n-n')0} \cos^2 \alpha \right).

\alpha_1 = \int_L \psi' \psi' \, dz, \quad \alpha_2 = \int_L \psi'' \psi \, dz, \quad \alpha_3 = \int_L \psi''' \psi \, dz, \quad \alpha_4 = \int_L \psi'' \psi' \, dz.

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