Reduction with degenerate Gram matrix for one-loop integrals

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Abstract: An improved PV-reduction (Passarino-Veltman) method for one-loop integrals with auxiliary vector $R$ has been proposed in [1, 2]. It has also been shown that the new method is a self-completed method in [3]. Analytic reduction coefficients can be easily produced by recursion relations in this method, where the Gram determinant appears in denominators. The singularity caused by Gram determinant is a well-known fact and it is important to address these divergences in a given frame. In this paper, we propose a systematical algorithm to deal with this problem in our method. The key idea is that now the master integral of the highest topology will be decomposed into combinations of master integrals of lower topologies. By demanding the cancellation of divergence for obtained general reduction coefficients, we solve decomposition coefficients as a Taylor series of the Gram determinant. Moreover, the same idea can be applied to other kinds of divergences.

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1 Introduction

The calculation of scattering amplitudes at loop level is an important issue in High Energy Physics. A widely used method is to reduce a loop amplitude into a linear combination of some scalar integrals under dimensional regularization \[4–20\]

\[
F^{1\text{-loop}} = \sum_{i_{d_0+1}} C_{i_{d_0+1}} I_{d_0+1}^{i_{d_0+1}} + \sum_{i_{d_0}} C_{i_{d_0}} I_{d_0}^{i_{d_0}} + \cdots + \sum_{i_1} C_{i_1} I_1^{i_1},
\]

where \(i_s\) is the set of propagators appearing in the basis. The coefficient \(C_{i_s}\) \((s = 1, \cdots, d_0+1)\) is simply a rational function of some Lorentz invariant contractions of external momenta as well as the masses and spacetime dimension, while the terms \(I_s\) are the \(s\)-gon master integrals. With the general expansion (1.1), the computation of general one-loop amplitudes has been switched to determining those coefficients \(C_{i_s}\). Many tools have been invented to shovel the brambles, such as integration-by-parts (IBP) \([21, 22]\), PV (Passarino-Veltman) reduction \([6]\), OPP (Ossola-Papadopoulo-Pittau) reduction \([19, 23–25]\), unitarity cut method, etc \([17, 20, 26–32]\).

All these methods can be divided into two categories, i.e., the reduction at the integrand level or the integral level. For reduction at the integrand level, \([19]\) shows how to extract
the coefficients of the 4-, 3-, 2- and 1-point one-loop scalar integrals from the full one-loop integrand of arbitrary scattering processes in an algebraical way. For the reduction at the integral level, an efficient way is the unitarity cut method. The main idea is to compare the imaginary part of two sides of (1.1). However, since the loop-integral is well-defined only after using the dimensional regularization, the unitarity cut method in pure 4D needs to be generalized to $(4-2\epsilon)$-dimension, which has been done in [31, 33]. Based on this generalization, the analytic expressions for reduction coefficients (except the tadpole coefficients) have been derived in a series of papers [34–38].

Recently, we proposed a new framework for general one-loop tensor reduction by employing an auxiliary vector $R$ and two kinds of differential operators [1, 2]. Similar to other reduction methods, our method also suffers from divergences for vanishing Gram determinant, which appears as the inverse of Gram matrix in the recursion constructions of reduction coefficients. It is well known that the vanishing Gram determinant indicates external momenta are not completely independent of each other, thus the master integral of the highest topology will not be in the basis anymore, and it can be decomposed into combinations of master integrals of lower topologies. With this decomposition, the reduction coefficients will be reorganized so that the divergences between different terms will cancel with each other. To see this picture clearly, calculating analytic coefficients will be important, which is exactly the merit of our new method. In this paper, we will systematically study the degenerate case with vanishing Gram determinant using the results in [1, 2]. It turns out that the decomposition coefficients can easily be solved as a series in the Gram determinant.

This paper is organized as follows. In section 2, we will review the improved PV-reduction method with auxiliary vector $R$ in our previous work and briefly discuss the general structure of reduction coefficients. In section 3, based on the explicit computation for bubbles in the limit $K^2 \to 0$, we establish an algorithm to deal with the vanishing Gram determinant for general topologies. We demonstrate the algorithm with the triangle in the main text and the box and pentagon in appendix A. In section 4, an important consistency has been proved, i.e., our algorithm should be independent of the auxiliary vector $R$ in reduction coefficients and the choice of tensor ranks. Finally, we give a summary and some discussions in section 5.

2 Background

2.1 Review of improved PV-reduction

In this section, we briefly review how to reduce one-loop tensor integrals using our improved PV-reduction method. After that, we will explain how to deal with the degenerated Gram determinant in our framework.

1In this paper, the integrals of the highest topology refer to the integrals containing the maximum number of propagators while lower topology means the integrals containing fewer propagators, which is distinct from its definition in mathematics.
For a general one-loop $m$-rank tensor integral with $(n+1)$ propagators
\begin{equation}
I^{\mu_1 \mu_2 \ldots \mu_m}_{n+1} = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{\ell^{\mu_1} \ell^{\mu_2} \ldots \ell^{\mu_m}}{D_0 \prod_{j=1}^n D_j} = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{\ell^{\mu_1} \ell^{\mu_2} \ldots \ell^{\mu_m}}{(\ell^2 - M^2_0) \prod_{j=1}^n ((\ell - K_j)^2 - M^2_j)} \tag{2.1}
\end{equation}
the algorithm of traditional PV (Passarino-Veltman) reduction method \cite{6} is following. First, we use the external momenta to write down the matched general tensor structure. For example, with rank 3 tensor integral of the form (2.1), we write down (where permutation symmetry of $\mu_1, \mu_2, \mu_3$ is imposed)
\begin{equation}
I^{\mu_1 \mu_2 \mu_3}_{n+1} = a_i (g^{\mu_1 \mu_2} K_i^{\mu_3} + g^{\mu_1 \mu_3} K_i^{\mu_2} + g^{\mu_2 \mu_3} K_i^{\mu_1}) + b_{ijk} K^{\mu_1}_i K^{\mu_2}_j K^{\mu_3}_k \tag{2.2}
\end{equation}
with unknown expansion coefficients $a_i, b_{ijk}$. Secondly, we contract both sides of (2.2) with various $g^{\mu \nu}$ and $K_i$ to establish enough algebraic relations to solve these coefficients. It is easy to see that with higher and higher tensor rank, there will be more and more different tensor structures to be written down in (2.2) and more and more algebraic relations to be established to fix them.

A nice observation made in \cite{1, 2} is that the complicated tensor structure can be simply recovered from
\begin{equation}
I^{(m)}_{n+1} = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{(2R \cdot \ell)^m}{D_0 \prod_{j=1}^n D_j} \tag{2.3}
\end{equation}
yielding $R = \sum_{i=1}^m x_i R_i$ and extracting terms with coefficients $x_1 \ldots x_m$ (or taking the proper differentiation action). For example,
\begin{equation}
(\ell \cdot P)^2 \sim (\partial_R \cdot \partial_R)(P \cdot \partial_R)(2R \cdot \ell)^3 \tag{2.4}
\end{equation}
The vector $R$ in (2.3) is called auxiliary vector. Involving the auxiliary vector $R$ not only greatly simplified the tensor structure but also provided a very simple way to establish algebraic relations to solve the reduction problem, as we will review shortly.

The reduction of (2.3) can be denoted as
\begin{equation}
I^{(m)}_{n+1} = C^{(m)}_{n+1 \rightarrow n+1} I^{n+1} + C^{(m)}_{n+1 \rightarrow n+1; i} I_{n+1; i} + C^{(m)}_{n+1 \rightarrow n+1; ij} I_{n+1; ij} + \ldots \tag{2.5}
\end{equation}
where $I_{n+1; i \ldots n}$ represents the scalar integral got by removing propagators $D_i, D_{i+1}, \ldots, D_n$ from the integral (2.3). For example,
\begin{equation}
I_{8,0,2,5,3} = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{1}{D_1 \cdot D_3 \cdot D_4 \cdot D_7}; \quad I_{7,1,3,5} = \int \frac{d^D \ell}{i\pi^{D/2}} \frac{1}{D_0 \cdot D_2 \cdot D_4 \cdot D_6}. \tag{2.6}
\end{equation}
For later convenience, in this paper we define
\begin{equation}
s_{ij} = K_i \cdot K_j, \quad s_{0i} = R \cdot K_i, \quad f_i = M^2_0 - M^2_i + K^2_i, \tag{2.7}
\end{equation}
and the Gram matrix is given by $G = [G_{ij} = s_{ij}]$ and its determinant is denoted by $|G|$. Now we review how to solve reduction coefficients in (2.5). Without loss of generality, we only review how to calculate $C^{(m)}_{n+1 \rightarrow n+1; i \ldots i+2; \ldots; n}$. Other reduction coefficients can be
got by a label permutation and a proper momentum shift.\textsuperscript{2} Having introduced the $R$, we can construct the following two types of differential operators

$$D_i \equiv K_i \cdot \frac{\partial}{\partial R}, \quad i = 1, \ldots, n; \quad T \equiv \eta^\mu\nu \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu}. \quad (2.8)$$

Acting with these two operators on both sides of (2.5) and comparing the coefficients of each Master Integral,\textsuperscript{3} we get

$$TC^{(m)}_{n+1\rightarrow n+1;r+1,\ldots,n} = 4m(m-1)M_0^2 C^{(m-2)}_{n+1\rightarrow n+1;r+1,\ldots,n}, \quad (2.9)$$

and

$$D_i C^{(m)}_{n+1\rightarrow n+1;r+1,\ldots,n} = -mC^{(m-1)}_{n+1;i\rightarrow n+1;r+1,\ldots,n} + mf_i C^{(m-1)}_{n+1\rightarrow n+1;r+1,\ldots,n}. \quad (2.10)$$

where $C^{(m-1)}_{n+1;i\rightarrow n+1;r+1,\ldots,n-1}$ is the reduction coefficient of integral $I^{(m-1)}_{n+1;i\rightarrow n+1;r+1,\ldots,n}$. After expanding reduction coefficients according to its tensor structure

$$C^{(m)}_{n+1\rightarrow n+1;r+1,\ldots,n} = \sum_{2a_0 + \sum_{k=1}^n a_k = m} \left\{ c \left(0,1,\ldots,r \right) (m)(M_0^2)^{a_0+r-n} \prod_{k=0}^n \beta_{0k} \right\}, \quad (2.11)$$

equations (2.9) and (2.10) will lead to algebraic recursion relations for expansion coefficients $c \left(0,1,\ldots,r \right) (m)$ as a rational function of spacetime dimension $D$, rank $m$ and kinematics $(K_i \cdot K_j)$ and $M_0^2$. A very nice feature is that these algebraic recursion relations can be solved to give the recursive construction for expansion coefficients as following,

$$c \left(0,1,\ldots,r \right) (a_1, \ldots, a_n; m) = T^{-1} \hat{G}^{-1} O \left(0,1,\ldots,r \right) (a_1, \ldots, a_n; m), \quad (2.12)$$

$$\left[ (4 - \alpha^T \hat{G}^{-1} \alpha) c \left(0,1,\ldots,r \right) (0,2k) \bigg|_{n \text{ times}} + \alpha^T \hat{G}^{-1} c \left(0,1,\ldots,r \right) (0,2k) \bigg|_{n \text{ times}} \right],$$

where $\hat{G} = [\beta_{ij} = \frac{\delta_{ij}}{M_0^2}]$ is the $n \times n$ dimensionless Gram matrix and $T = \text{diag}(a_1 + 1, a_2 + 1, \ldots, a_n + 1)$ is a diagonal matrix. Other vectors are defined as

$$\alpha^T = \left( \frac{f_1}{M_0^2}, \frac{f_2}{M_0^2}, \ldots, \frac{f_n}{M_0^2} \right), \quad (2.14)$$

$$[c \left(0,1,\ldots,r \right) (a_1, \ldots, a_n; m)]_i = c \left(0,1,\ldots,r \right) (a_1, a_2, \ldots, a_{i+1}, \ldots, a_n; m), \quad (2.15)$$

\textsuperscript{2}More details can be found in [1, 2].

\textsuperscript{3}Note that Master Integral contains no $R$, the differential operators will directly act on reduction coefficients.
and

\[
[O^{(0, \cdots, r)}(a_1, \cdots, a_n; m)]_i = m\alpha_i c^{(0, \cdots, r)}_{a_1, \cdots, a_n}(m - 1) - m\delta_{\alpha_i} c^{(0, \cdots, r)}_{a_1, \cdots, a_i, \cdots, a_n}(m - 1; i) \\
- (m + 1 - \sum_{\ell=1}^{n} a_{\ell}) c^{(0, \cdots, r)}_{a_1, \cdots, a_{\ell-1}, \cdots, a_n}(m) \\
\]

\[
c_{(0, \cdots, r)}^{(0, \cdots, r), 0}(m) = \left(0, 0, \cdots, 0, c_{0, \cdots, 0}(m; r + 1), \cdots, c_{0, \cdots, 0}(m; n - 1)\right).
\]

(2.16)

With the known boundary conditions, one can get all reduction coefficients by applying the recursions (2.12), (2.13) iteratively. Above formulas are easy to be implemented into Mathematica and one can generate analytic expressions for reduction coefficients for any tensor rank (Exactly because of these benefits involving the auxiliary vector \(R\), we call the method as “improved PV-reduction method”). Knowing the analytic expressions for reduction coefficients, we can discuss many topics. For example, we can find the reduction for integrals with propagators having power higher than one as shown in [3]. Another important application is for the case where some reduction coefficients become divergent, as will be discussed in this paper.

Now we want to point out an important observation from (2.12), (2.13): with \(G^{-1}\) we will have \(|G|\) appearing in denominators of all reduction coefficients. When applying above PV-reduction method, we have made a hidden assumption that \(D \geq n\) and all \(K_i\)'s in propagators are generic (i.e., linear independent). With this hidden assumption, \(|G| \neq 0\) and above formulas are well-defined. However, in the practical applications, we do meet the degeneration of the Gram determinant, i.e., \(|G| = 0\), which means that these \(K_i\)'s are not linear independent anymore and the kinematics lives in a smaller space. For this case, the formulas (2.12) and (2.13) are ill-defined. Thus, for the completion of the improved PV-reduction method, we must provide an algorithm to solve this issue. Briefly, \(|G| = 0\) means that the scalar integrals of the highest topology are not in the basis anymore, and they can be reduced to a combination of lower topologies.

Later we need to use the divergence behavior when \(|G| \to 0\). Noticing that \(G^{-1} = G^* / |G|\), where \(G^*\) is the adjugate matrix of Gram matrix, there is a Gram determinant appearing in the denominator for each iteration, thus one can conclude that the reduction coefficients have the highest divergent part about the determinant of Gram matrix\(^4\)

\[
C_{(m)}^{(n+r)} \sim \begin{cases} 
\frac{1}{|G|}, & r = n, n-1 \\
\frac{1}{|G|^{m+n+r}}, & r < n-1.
\end{cases}
\]

(2.17)

\[^4\text{In fact, only those reduction coefficients with } m + r \geq n \text{ are nonzero. Because there are not enough } \ell \cdot R \text{ in the numerator to cancel } (n - r) \text{ propagators for } m + r < n.\]
3 Reduction for degenerate Gram matrix

In the previous section, we find that the reduction coefficients diverge when $|G| \to 0$, while $I_{n+1}^{(m)}$ is still well-defined. The naive conflict in (2.5) indicates that the expansion on the r.h.s. is not proper. In other words, the scalar integrals are not independent of each other anymore. As we will show in this section, these divergences tell us a lot of information about the degenerating behavior of the basis. We will use various examples, i.e., integrals of bubble, triangle, box, and pentagon, to demonstrate the strategy to deal with the divergences in our framework.

3.1 The reduction of scalar bubble with degenerate Gram matrix

Let us start with the simplest example, i.e., the reduction of the bubble with tensor rank 1. Our method gives

\[ I_2^{(1)} = C_{2 \to 2}^{(1)} I_2 + C_{2 \to 2,0}^{(1)} I_{2;0} + C_{2 \to 2,1}^{(1)} I_{2;1}, \]  

(3.1)

where

\[ C_{2 \to 2}^{(1)} = \frac{(M_0^2 - M_1^2 + s_{11}) s_{01}}{s_{11}}, \]

\[ C_{2 \to 2;0}^{(1)} = \frac{s_{01}}{s_{11}}, \quad C_{2 \to 2;1}^{(1)} = -\frac{s_{01}}{s_{11}}. \]  

(3.2)

It is easy to see that with $|G| = k_1 \cdot k_1 = s_{11} \to 0$, all coefficients in (3.2) become divergent. But $I_2^{(1)}$ is a definitely well-defined integral even with $s_{11} = 0$, so the divergence on the r.h.s. in (3.1) implies that the expansion is not proper. More explicitly, the three scalar integrals $I_2, I_{2;0}, I_{2;1}$ are not linearly independent anymore, i.e., we should have

\[ I_2 = \sum_{i=0}^{1} B_{2 \to 2;0}^{(a)} I_{2;0} = \sum_{i=0}^{\infty} \sum_{a=0}^{\infty} B_{2 \to 2;0}^{(a)} s_{a 11} I_{2;0}, \]  

(3.3)

where the decomposition coefficient $B_{2 \to 2;0}$ is expanded as a Taylor series of $|G| = s_{11}$.

Now we show how to use the (3.3) to make the r.h.s. of (3.1) well-defined under the limit $|G| \to 0$. Putting (3.2) and (3.3) to (3.1), the r.h.s. becomes

\[ \left( \frac{(M_0^2 - M_1^2 + s_{11}) s_{01}}{s_{11}} \sum_{a=0}^{\infty} B_{2 \to 2;0}^{(a)} s_{a 11} + \frac{s_{01}}{s_{11}} \right) I_{2;0} + \left( \frac{(M_0^2 - M_1^2 + s_{11}) s_{01}}{s_{11}} \sum_{a=0}^{\infty} B_{2 \to 2;1}^{(a)} s_{a 11} - \frac{s_{01}}{s_{11}} \right) I_{2;1}. \]  

(3.4)

To cancel the divergence, we just need

\[ B_{2 \to 2;0}^{(0)} = \frac{-1}{(M_0^2 - M_1^2)}, \quad B_{2 \to 2;1}^{(0)} = \frac{1}{(M_0^2 - M_1^2)}. \]  

(3.5)

Although we have only fixed the first coefficient in the above computation, the smoothness of r.h.s. in (3.1) does tell us the information of later coefficients. To get them, it is...
natural to consider the reduction of higher tensor rank. For rank 2, we have

\begin{align}
C^{(2)}_{2 \to 2} &= \frac{D (M_0^2 - M_1^2)}{(D - 1)s_1^2} + \frac{2 M_0^2 s_{01}^2 + 2 M_0^2 s_{00} + 2 M_1^2 s_{00}}{D - 1} + \frac{2 M_0^2 ((D - 2)s_{01}^2 + M_1^2 s_{00}) - 2 D M_1^2 s_{01}^2 - M_1^4 s_{00}}{(D - 1)s_{11}} - \frac{s_{11} s_{00}}{D - 1}, \\
C^{(2)}_{2 \to 2;\hat{M}} &= \frac{D (M_0^2 - M_1^2)}{(D - 1)s_{11}^2} + \frac{3 (D - 4)s_{01}^2 + M_0^2 (-s_{00}) + M_1^2 s_{00}}{(D - 1)s_{11}} + \frac{s_{00}}{D - 1}, \\
C^{(2)}_{2 \to 2;\hat{M}} &= \frac{D (M_0^2 - M_1^2)}{(D - 1)s_{11}^2} + \frac{-2 s_{01}^2 + M_0^2 s_{00} - M_1^2 s_{00}}{(D - 1)s_{11}} + \frac{s_{00}}{D - 1}. \tag{3.6}
\end{align}

Now the denominators have $s_{11}^2$ in (3.6). When combining (3.3) and (3.6), the smoothness under the limit requires the cancellation of poles $\frac{1}{s_{11}}$ and $\frac{1}{s_{11}}$. One can easily check that the result (3.5) will remove the pole $\frac{1}{s_{11}}$ automatically, while the cancellation of the pole $\frac{1}{s_{11}}$ gives

\begin{align}
B^{(1)}_{2 \to 2;\hat{M}} &= -\frac{D M_0^2 - (D - 4) M_1^2}{D (M_0 - M_1)^3 (M_0 + M_1)^3}, \\
B^{(1)}_{2 \to 2;\hat{M}} &= -\frac{(D - 4) M_0^2 - D M_1^2}{D (M_0 - M_1)^3 (M_0 + M_1)^3}. \tag{3.7}
\end{align}

Now the idea is clear. To get the higher-order decomposition coefficients $B^{(a)}_{2 \to 2;\hat{M}}$, we just need to impose the smoothness limit of tensor reduction with higher ranks. In fact, we can get expansion coefficients $B^{(a)}_{2 \to 2;\hat{M}}$ with $a \leq m$ for bubble through calculation with tensor rank $(m + 1)$.

Before ending this subsection, let us point out that the decomposition coefficients given in (3.5) and (3.7) have a nice symmetry between $M_0 \leftrightarrow M_1$ and they become singular under the limit $M_1 \to M_0$. In that limit, $I_{2;\hat{M}}$ and $I_{2;\hat{M}}$ are not linearly independent of each other, and we should expand

\begin{equation}
I_{2;\hat{M}} = \sum_{a=0}^{\infty} B^{(a)}_{(2;\hat{M}) \to (2;\hat{M})} (M_1^2 - M_0^2)^a I_{2;\hat{M}}. \tag{3.8}
\end{equation}

Putting (3.8) into (3.3) with known $B^{(a)}_{2 \to 2;\hat{M}}$ and requiring the smoothness under the limit $M_1 \to M_0$, we can calculate all the $B^{(a)}_{2 \to 2;\hat{M}}$. For example, with results (3.5) and (3.7), we get

\begin{equation}
B^{(0)}_{(2;\hat{M}) \to (2;\hat{M})} = 1, \quad B^{(1)}_{(2;\hat{M}) \to (2;\hat{M})} = \frac{D - 2}{2 M_0^2}, \quad B^{(2)}_{(2;\hat{M}) \to (2;\hat{M})} = \frac{D^2 - 6D + 8}{8 M_0^4}. \tag{3.9}
\end{equation}

### 3.2 Algorithm for general case

Now we are ready to discuss the degeneration of scalar $(n + 1)$-gon integral when the Gram determinant $|G| \to 0$. The Gram matrices corresponding to lower topologies don’t
degere generally with $|G| \to 0$, thus the scalar $(n+1)$-gon integral will become a linear combination of scalar $m$-gon integrals with $m \leq n$, i.e.,

$$I_{n+1} = \sum_{|b_j|=1}^{n} B_{n+1;b_j} I_{n+1;b_j} = \sum_{|b_j|=1}^{n} \sum_{a=0}^{\infty} B_{n+1;b_j}^{(a)} |G|^a I_{n+1;b_j},$$

(3.10)

where at the second equation, the Taylor expansion of $|G|$ is given explicitly. To determine these decomposition coefficients $B_{n+1;b_j}^{(a)}$, we put (3.10) to (2.5) and get

$$I_{n+1}^{(m)} = \sum_{|b_j|=0}^{n} C_{n+1\to n+1;b_j}^{(m)} I_{n+1;b_j} = C_{n+1\to n+1}^{(m)} I_{n+1;b_j} + \sum_{|b_j|=1}^{n} C_{n+1\to n+1;b_j}^{(m)} I_{n+1;b_j}$$

$$= \sum_{|b_j|=1}^{n} \left( C_{n+1\to n+1;b_j}^{(m)} + C_{n+1\to n+1}^{(m)} B_{n+1;b_j} \right) I_{n+1;b_j}.$$  

(3.11)

For later convenience, let us write

$$F_{n+1\to n+1;b_j}^{(m)} = \left( C_{n+1\to n+1;b_j}^{(m)} + C_{n+1\to n+1}^{(m)} B_{n+1;b_j} \right).$$

(3.12)

The smoothness of $I_{n+1}^{(m)}$ under the limit $|G| \to 0$ means that $F_{n+1\to n+1;b_j}^{(m)}$ is a Taylor series of $|G|$. Using this condition, we can get all the decomposition coefficients.

Now we show how to do it more directly instead of solving linear equations. Since $C_{n+1\to n+1;b_j}^{(m)}$ is a Laurent series about the Gram determinant $|G|$, for example,

$$C_{n+1\to n+1}^{(m;i)} = \sum_{i=-m}^{\infty} C_{n+1\to n+1}^{(m;i)} |G|^i,$$

(3.13)

using (3.12), we can write

$$B_{n+1;b_j} = \frac{F_{n+1\to n+1;b_j}^{(m)} - C_{n+1\to n+1;b_j}^{(m)}}{C_{n+1\to n+1}} = \frac{C_{n+1\to n+1;b_j}^{(m)}}{C_{n+1\to n+1}} + \frac{F_{n+1\to n+1;b_j}^{(m)}}{C_{n+1\to n+1}}.$$  

(3.14)

In (3.14), $F_{n+1\to n+1;b_j}^{(m)}$ is regular and $C_{n+1\to n+1}^{(m)} \sim |G|^{-m}$, so we just need to keep the first term in the numerator of (3.14) to get the coefficients $B_{n+1;b_j}^{(k<m)}$, i.e.,

$$B_{n+1;b_j} = -\frac{C_{n+1\to n+1;b_j}^{(m)}}{C_{n+1\to n+1}} + O(|G|^m).$$

(3.15)

Using (3.15), we can get

$$F_{n+1\to n+1;b_j}^{(m)} = \left( C_{n+1\to n+1;b_j}^{(m)} + \frac{C_{n+1\to n+1;b_j}^{(m)}}{C_{n+1\to n+1}} + O(|G|^m) \right) C_{n+1\to n+1}^{(m)}$$

$$= C_{n+1\to n+1;b_j}^{(m)} C_{n+1\to n+1}^{(m)} - C_{n+1\to n+1}^{(m)} C_{n+1\to n+1}^{(m)} + O(|G|^m - m).$$  

(3.16)
Here we use the fact $C^{(m)}_{n+1\to n+1} \sim |G|^{-m}$. To get the accurate expansion of $F^{(m)}_{n+1\to n+1,b_j}$ up to $k$-th order of $|G|$, we set $m’ - m = k + 1$, then we can write

$$F^{(m)}_{n+1\to n+1,b_j} = \frac{C^{(m)}_{n+1\to n+1,b_j} C^{(m+k+1)}_{n+1\to n+1} - C^{(m+k+1)}_{n+1\to n+1,b_j} C^{(m)}_{n+1\to n+1}}{C^{(m+k+1)}_{n+1\to n+1}} + O(|G|^{k+1}). \quad (3.17)$$

If we take $k = 0$ in (3.17), it gives

$$F^{(m)}_{n+1\to n+1,b_j} = \lim_{|G|\to 0} \left( \frac{C^{(m)}_{n+1\to n+1,b_j} C^{(m+1)}_{n+1\to n+1} - C^{(m+1)}_{n+1\to n+1,b_j} C^{(m)}_{n+1\to n+1}}{C^{(m+1)}_{n+1\to n+1}} \right), \quad (3.18)$$

which gives the reduction coefficients for $I^{(m)}_{n+1}$ when $|G| = 0$. The same result (3.18) has been obtained by methods in projective space in [39]. For the expansion (3.10), if we care about only the zeroth order, the result is also well known, which is given by [5, 19, 40] (see eq. (5.8) in [19])

$$|I_{n+1} - I_{n+1,\hat{0}} - I_{n+1,\hat{1}} \ldots - I_{n+1,\hat{n}}| = 0, \quad (3.19)$$

with

$$Y_{ij} = M_i^2 + N_j^2 - (K_i - K_j)^2, \quad i, j = 0, \ldots, n. \quad (3.20)$$

A good feature of the result (3.15) is that we can get the higher-order expansion, which may be helpful for a better numerical evaluation around the point $|G| = 0$.

Having established the general formula in (3.15) and (3.18), let us redo the example of the bubble in the previous subsection:

- Up to the order $s_{11}^1$: we have

$$B_{2,\hat{1}} = \frac{C_{2\to 2,\hat{1}}^{(1)}}{C_{2\to 2}^{(1)}} = \frac{1}{M_1^2 - M_1^2} + O(|G|),$$

$$B_{2,\hat{0}} = \frac{C_{2\to 2,\hat{0}}^{(1)}}{C_{2\to 2}^{(1)}} = \frac{1}{M_1^2 - M_0^2} + O(|G|). \quad (3.21)$$

after using the result (3.2).

- Up to the order $s_{11}^2$: we have

$$B_{2,\hat{1}} = \frac{C_{2\to 2,\hat{1}}^{(2)}}{C_{2\to 2}^{(2)}} = \frac{1}{M_0^2 - M_1^2} + \frac{(DM_1^2 - (D - 4)M_0^2)|G|}{D (M_0^2 - M_1^2)^3} + O(|G|^2),$$

$$B_{2,\hat{0}} = \frac{C_{2\to 2,\hat{0}}^{(2)}}{C_{2\to 2}^{(2)}} = \frac{1}{M_1^2 - M_0^2} + \frac{((D - 4)M_1^2 - DM_0^2)|G|}{D (M_0^2 - M_1^2)^3} + O(|G|^2). \quad (3.22)$$

after using the result (3.6).
• Up to the order $s_{11}^2$: we have

\[ B_{2;1} = -\frac{C_{3}^{(3)}(2)}{C_{3}^{(3)}(2)} = \frac{1}{M_0^2 - M_1^2} + \frac{\left(DM_1^2 - (D - 4)M_0^2\right)|G|}{D (M_0^2 - M_1^2)^3} + \frac{\left((D^2 - 10D + 24)M_0^4 - 2(D^2 - 4D - 12)M_1^4 + D(D + 2)M_0^4\right)|G|^2}{D(D + 2)(M_0^2 - M_1^2)^5} + \mathcal{O}(|G|^3), \]

\[ B_{2;0} = -\frac{C_{3}^{(3)}(2)}{C_{3}^{(3)}(2)} = \frac{1}{M_0^2 - M_1^2} + \frac{\left(DM_1^2 - DM_0^2\right)|G|}{D (M_0^2 - M_1^2)^3} - \frac{\left((D^2 - 10D + 24)M_0^4 - 2(D^2 - 4D - 12)M_1^4 + D(D + 2)M_0^4\right)|G|^2}{D(D + 2)(M_0^2 - M_1^2)^5} + \mathcal{O}(|G|^3), \]

(3.23)

after using the result in [2] (see eq. (4.11) to eq. (4.16)).

3.3 The degenerate triangle

In this subsection, we will present some results for the degenerate triangle. First, we parameterize the Gram determinant as

\[ G_{\text{tri}} = s_{11}s_{22} - s_{12}^2 = |G|, \]

(3.24)

where $s_{ij} = K_i \cdot K_j$ and the parameter $|G|$ controls the speed of degeneration. We expand the Master Integral $I_3$ as

\[ I_3 = \sum_{i_1=0}^{2} B_{3,i_{1}} \tilde{I}_{3,i_{1}} + \sum_{0 \leq i_1 < i_2} B_{3,i_{1}i_{2}} \tilde{I}_{3,i_{1}i_{2}}. \]

(3.25)

We will calculate several typical $B_{3;\hat{b}_j}$ by (3.15), while others can be obtained by proper permutation.

• $B_{3;2}$: using the tensor rank 1 we get

\[ B_{3;2} = \frac{s_{11}}{M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12}) + \mathcal{O}(|G|)}, \]

(3.26)

---

5One should be very careful about the kinematics. For example, in the calculation of tensor triangle, we cannot treat $s_{11}, s_{12}, s_{22}$ and $|G|$ as four independent Lorentz invariant combinations. Instead, we should replace, for example, every $s_{22}$ with $(|G| + s_{12}^2)/s_{11}$. 

---

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while using the tensor rank 2 we get

\[
\mathcal{B}_{3;\hat{2}} = \frac{s_{11}}{M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12})}
\]

\[
\times \frac{1}{s_{11}|G|} \left( - \left( D - 1 \right) (M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12}))^3 \times \left( s_{11} \left( D M_0^2 - D M_1^2 - 3 M_0^2 - 2 M_1^2 + M_2^2 \right) + (D - 1)s_{12}^2 M_1^4 + s_{11}^2 + s_{12}^2 (D M_0^2 + D M_1^2 + M_0^2 - M_1^2) + M_0^4 - 2 M_1^2 M_0^2 + (1 - D)s_{11}^2 \right) \right)
\]

\[
+ \mathcal{O} \left( |G|^2 \right). \tag{3.27}
\]

One can see that for zeroth order, results in (3.26) and (3.27) are same. Similar results hold for other coefficients.

- \( \mathcal{B}_{3;\hat{0}} \): using the tensor rank 1 we get

\[
\mathcal{B}_{3;\hat{0}} = \frac{s_{12} - s_{11}}{M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12})} + \mathcal{O} \left( |G| \right), \tag{3.28}
\]

while using the tensor rank 2 we get

\[
\mathcal{B}_{3;\hat{0}} = \frac{s_{12} - s_{11}}{M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12})}
\]

\[
\times \frac{1}{|G|} \left( - \left( D - 1 \right) (s_{11} - s_{12}) (M_0^2 (s_{11} - s_{12}) + M_2^2 s_{11} - s_{12} (M_1^2 - s_{11} + s_{12}))^3 \times \left( s_{11}^2 (D M_0^2 M_1^2 - D M_0^2 M_1^2 + D M_1^2 - D M_0^2 M_1^2 - M_1^2 - M_0^2 M_1^2 + 3 M_2^2 M_1^2 - 2 M_2^2 + M_0^2 M_1^2) + s_{12}^2 \left( - D M_1^2 + D M_0^2 - 3 M_0^2 - 5 M_2^2 \right) + s_{12}^2 s_{11} \left(D M_1^2 - D M_0^2 M_1^2 - D M_0^2 M_1^2 + D M_0^2 M_1^2 - M_1^4 + M_0^2 M_1^2 + M_2^2 M_1^2 - M_0^2 M_1^2 \right) + M_0^4 + 2 M_1^2 M_0^2 + (1 - D)s_{11}^2 \right) \right)
\]

\[
+ \mathcal{O} \left( |G|^2 \right). \tag{3.29}
\]

- \( \mathcal{B}_{3;\hat{2}} \): using the tensor rank 1 we get

\[
\mathcal{B}_{3;\hat{2}} = 0 + \mathcal{O} \left( |G| \right), \tag{3.30}
\]

while using the tensor rank 2 we get

\[
\mathcal{B}_{3;\hat{2}} = \frac{(D - 2)s_{11}|G|}{(D - 1)s_{12} (M_0^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_1^2 - s_{11} + s_{12}))^2} + \mathcal{O} \left( |G|^2 \right). \tag{3.31}
\]
Finally using the tensor rank 3 we get
\[
B_{3,\hat{i}2} = \frac{(D - 2)s_{11}|G|}{(D - 1)s_{12}(M_0^2(s_{11} - s_{12}) - M_2^2s_{11} + s_{12}(M_1^2 - s_{11} + s_{12}))^2} \\
\times \left(\frac{(D - 2)s_{11}|G|^2}{(D - 1)s_{12}(M_0^2(s_{11} - s_{12}) - M_2^2s_{11} + s_{12}(M_1^2 - s_{11} + s_{12}))^4}ight) \\
\times \left( -3DM_0^3 + 3DM_0^2 - 2M_0^2 + 2M_1^2 \right) + s_{12}(DM_0^3 - 2DM_0^2M_1^2 + DM_1^4 + 2M_0^4 - 4M_1^2M_0^2 + 2M_1^4) + s_{12}(4DM_0^3 - 2DM_0^2 - 4M_0^2 - 4M_1^2 - 4M_2^2) + s_{11}s_{12}(4DM_0^3 - 2DM_0^2 - 4M_0^2 - 4M_1^2 - 4M_2^2) + s_{11}s_{12}(-DM_0^4 + DM_1^4M_0^2 + DM_2^4M_1^2 - DM_1^2M_2^2) + (2D + 3)s_{12} + (-3D - 2)s_{11}s_{12} \\
+ (D + 2)s_{11}s_{12} + \left( M_0^4 - 2M_1^2M_0^2 + M_1^4 \right) s_{11}^2) + O\left(|G|^3\right). \tag{3.32}
\]

- \(B_{3,\hat{i}2}\): using the tensor rank 1 we get
\[
B_{3,\hat{i}2} = 0 + O\left(|G|\right), \tag{3.33}
\]
while using the tensor rank 2 we get
\[
B_{3,\hat{i}2} = \frac{(D - 2)s_{11}|G|}{(D - 1)(s_{11} - s_{12})(M_0^2(s_{12} - s_{11}) + M_0^2s_{11} - s_{12}(M_1^2 - s_{11} + s_{12}))^2} \\
+ O\left(|G|^2\right). \tag{3.34}
\]

Finally using the tensor rank 3 we get
\[
B_{3,\hat{i}2} = \frac{(D - 2)s_{11}|G|}{(D - 1)(s_{11} - s_{12})(M_0^2(s_{12} - s_{11}) + M_0^2s_{11} - s_{12}(M_1^2 - s_{11} + s_{12}))^2} \\
\times \left(\frac{(D - 2)s_{11}|G|^2}{(D - 1)(s_{11} - s_{12})(M_0^2(s_{12} - s_{11}) + M_0^2s_{11} - s_{12}(M_1^2 - s_{11} + s_{12}))^4}ight) \\
\times \left( s_{12}(-2DM_0^3 + 3DM_1^2M_0^2 + DM_2^2M_0^2 - DM_1^4 - DM_1^2M_2^2 - 4M_0^4 + 8M_1^2M_0^2 - 4M_1^4) + s_{11}\left( DM_0^3 - DM_0^2 - 2M_0^2 - 6M_1^2 + 4M_2^2 \right) + s_{11}(DM_0^3 - DM_1^2M_0^2 - DM_1^2M_2^2 + DM_2^4M_1^2 + 2M_0^4 - 4M_1^2M_0^2 + 3M_1^4 + M_2^4 - 2M_1^2M_2^2) \\
+ s_{12}(-5DM_0^2 + 2DM_2^2 + 3DM_1^2M_0^2 + 2M_0^4 + 14M_1^2 + 8M_2^2) + s_{12}(2M_1^2 - 3DM_0^2 + 3DM_2^2 - 2M_0^2) + s_{12}(7DM_0^2 - 5DM_0^2 - 2DM_1^2 + 2M_0^4 - 4M_2^2 - 10M_1^2) - (D + 10)s_{12}s_{11}^3 + (4D + 14)s_{12}s_{11}^3 + (5D + 10)s_{12}s_{11}^3 + 3s_{11}^4 \\
+ (2D + 3)s_{12}^2 + s_{12}\left( DM_0^3 - 2DM_1^2M_0^2 + DM_1^4 + 2M_0^4 - 4M_2^2 - M_0^4 \right) \right) \\
+ O\left(|G|^3\right). \tag{3.35}
\]

Note that the expansion of \(B_{n+1;\hat{i}j}\) is independent of \(R\) although the numerator and the denominator depend on \(R\) on the r.h.s. of (3.15). And it’s also independent of the tensor rank \(m\) up to the order \(|G|^k\) as long as \(m > k\). The above results confirm these nontrivial points, and we will give a proof in the later section.
The reduction coefficients $F^{(m)}_{3 \to 3; b_j}$ for $|G| = 0$ (see (3.18)) are given in following:

- $m = 0$:

$$F^{(0)}_{3 \to 3; 0} = B^{(0)}_{3 \to 3; 0} = \left( \frac{C^{(1)}_{3 \to 3; 0}}{C^{(1)}_{3 \to 3}} \right) |_{G = 0} = \frac{M_1^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_2^2 - s_{11} + s_{12})}{s_{12} - s_{11}}, \quad (3.36a)$$

$$F^{(0)}_{3 \to 3; 1} = \frac{M_1^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_2^2 - s_{11} + s_{12})}{s_{11}}, \quad (3.36b)$$

$$F^{(0)}_{3 \to 3; 2} = \frac{M_1^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_2^2 - s_{11} + s_{12})}{s_{12}}, \quad (3.36c)$$

$$F^{(0)}_{3 \to 3; 3} = F^{(0)}_{3 \to 3; 0} = F^{(0)}_{3 \to 3; 2} = 0. \quad (3.36d)$$

- $m = 1$:

$$F^{(1)}_{3 \to 3; 0} = \frac{1}{(D - 1) s_{11} (s_{11} - s_{12}) ((s_{12} - s_{11}) M_1^2 + M_1^2 s_{11} - s_{12} (M_1^2 - s_{11} + s_{12}))^2} \times$$

$$\left( s_{02} s_{11} + (-D M_2^2 + M_1^2 + D M_3^2 - M_2^3) s_{12} s_{11} + (D - 2) s_{12} s_{01} s_{11}^2 +
+ (-D M_2^2 + M_1^2 - 2 M_2^2 + D M_3^2 - 3 M_2^3) s_{02} s_{11} + (D - 5) s_{12} s_{02} s_{11}^2 +
+(7 - 3D) s_{12} s_{01} s_{11}^3 + (D M_3^2 - M_2^4 - D M_1^2 M_2^2 + M_1^2 M_2^2 - D M_2^2 M_3^2 + M_2^3 M_3^2 + D M_2^2 M_3^2 - M_1^3 M_2^2) s_{01} s_{11}^2 + (3 D M_2^2 - 3 M_2^4 - 2 D M_3^2 + 4 M_2^3 - D M_1^2 M_3^2 + 3 M_2^3 s_{12} s_{01} s_{11}^3 + (D M_1^2 - 2 M_2^3 - D M_1^2 M_2^2 + M_1^2 M_2^2 - D M_2^2 M_3^2 + 3 M_2^3 M_2^2 - 3 M_2^4 + D M_1^2 M_2^2 - M_1^3 M_3^2 - M_2^3 M_3^2) s_{12} s_{02} s_{11}^3 + (3 D M_2^2 - 3 M_2^4 + 4 M_2^3 + 7 M_2^3) s_{12} s_{02} s_{11}^3 + (-3 D M_1^2 + 3 M_2^3 + 3 D M_2^3 - 7 M_2^4 - 4 M_2^3) s_{12} s_{01} s_{11}^2 + (D M_1^2 - 2 M_2^3 - D M_1^2 M_2^2 + M_1^2 M_2^2 - D M_2^2 M_3^2 + 3 M_2^3 M_2^2 - M_2^4 + D M_1^2 M_2^2 - M_1^3 M_3^2 - M_2^3 M_3^2) s_{12} s_{01} s_{11}^2 + (3 D - 7) s_{12} s_{02} s_{11}^2 + (-3 D M_1^2 + 3 M_2^3 + D M_2^3 - 3 M_2^4 + 2 D M_3^2 - 4 M_2^3) s_{12} s_{02} s_{11}^2 + (-D M_1^2 + M_2^3 + D M_2^3 M_2^2 - M_1^3 M_2^2 + D M_3^2 M_2^2 - M_2^3 M_3^2 + M_1^2 M_2^2 + D M_2^2 M_3^2 - 3 M_2^3 M_2^2 - D M_3^2 + 2 M_2^3 + D M_3^2 M_3^2) s_{02} s_{11}^2 + (D M_1^2 - M_1^3 - D M_2^3 + 3 M_2^3 + 2 M_3^2) s_{12} s_{01} s_{11}^2 + (2 - D) s_{12} s_{02} s_{11}^2 + (D M_1^2 - M_1^3 - D M_2^3 + M_2^3) s_{12} s_{02} s_{11} - s_{12} s_{01} + (3 D - 9) s_{12} s_{01} s_{11}^2 \right), \quad (3.37a)$$

$$F^{(1)}_{3 \to 3; 1} = \frac{1}{(D - 1) s_{11} s_{12} (M_1^2 (s_{12} - s_{11}) + M_2^2 s_{11} - s_{12} (M_2^2 - s_{11} + s_{12}))^2} \times$$

$$\left( s_{11} s_{12} (D M_2^2 - D M_3^2 - M_2^2 + M_3^2) s_{02} + s_{11} s_{12} (2 D M_1^2 + 2 D M_3^2 - 4 M_3^2) s_{02} + s_{11} s_{12} s_{02} (D M_1^2 - D M_3^2 - M_2^2 + M_3^2) s_{02} + (2 - D) s_{11} s_{12} s_{02} \right). \quad (3.37b)$$
\[ F_{3 \rightarrow 3;2}^{(1)} = \frac{1}{(D - 1) \left( s_{11} - s_{12} \right) - M_3^2 s_{11} + s_{12} (M_2^2 - s_{11} + s_{12})} \times \left( s_{11}^2 (D M_1^2 - D M_2^2 - M_1^2 + M_2^2) s_{01} + s_{11} (D M_1^4 - D M_2^2 M_1^2 - D M_2^2 M_3^2 + D M_3^2 M_2^2 - M_1^4 + M_2^2 M_1^2 + M_3^2 M_1^2 - M_2^2 M_3^2) s_{01} + s_{11} (2 s_{12} (M_2^2 - D M_1^2 - D M_2^2 - M_1^2 + M_2^2) s_{01} + s_{12} (D M_1^2 - D M_2^2 - M_1^2 + M_2^2) s_{01} + s_{12} (-D M_1^2 + 2 D M_2^2 - 4 M_2^2) s_{01} + s_{12} (D M_1^2 - D M_2^2 - M_1^2 + M_2^2) s_{01} + s_{12} (-D M_1^2 + 2 D M_2^2 - 4 M_2^2) s_{01} + (2 - D) s_{12} s_{11} s_{01} + (D - 1) s_{12} s_{11} s_{01} + \left( -M_1^2 + 2 M_2^2 M_1^2 - M_3^2 \right) s_{11} s_{02} \right) \right) \right), \] (3.37c)

\[ F_{3 \rightarrow 3;12}^{(1)} = \frac{(D - 2) (s_{12} s_{01} - s_{11} s_{02})}{(D - 1) s_{12} (M_1^2 (s_{11} - s_{12}) - M_2^2 s_{11} + s_{12} (M_2^2 - s_{11} + s_{12}))}, \]

\[ F_{3 \rightarrow 3;01}^{(1)} = \frac{(D - 2) s_{11} (s_{11} s_{02} - s_{12} s_{01})}{(D - 1) (s_{11} - s_{12}) s_{12} (M_1^2 (s_{11} - s_{12}) + M_2^2 s_{11} - s_{12} (M_2^2 - s_{11} + s_{12}))}, \]

\[ F_{3 \rightarrow 3;02}^{(1)} = \frac{(D - 2) (s_{12} s_{01} - s_{11} s_{02})}{(D - 1) (s_{11} - s_{12}) (M_1^2 (s_{11} - s_{12}) + M_2^2 s_{11} - s_{12} (M_2^2 - s_{11} + s_{12}))}, \] (3.37d)

One interesting point is that expressions of the first three coefficients are complicated while the last three are relatively simple. The reason is that \( I_{n+1} \) is reduced to combinations of \( I_n \) only (see (3.19)) at zeroth order, thus coefficients of \( I_{m \leq n-1} \) are not affected.

In this section, we present examples up to the triangle. A complete result of \( B \) is listed in the Mathematica file in the supplementary material. For box and pentagon, we will give some numerical results in appendix A.

### 3.4 Two-loop example: sunset

Our idea to deal with degenerate Gram determinant is simple and it should apply to higher loops. To demonstrate this point, in this subsection we consider the simplest two-loop integrals, i.e., the integrals with sunset topology. Involving two auxiliary vectors \( R_1, R_2 \) we consider the tensor reduction of following integrals

\[ I_{r_1, r_2}^{(2)} = \int_0^{\ell_{1}^2 - M_1^2} d\ell_1 d\ell_2 \frac{(2 \ell_1 \cdot R_1)^r_1 (2 \ell_2 \cdot R_2)^r_2}{D_1^{a_1} D_2^{a_2} D_3^{a_3}}, \] (3.38)

where the propagators are

\[ D_1 = \ell_1^2 - M_1^2, \quad D_2 = \ell_2^2 - M_2^2, \quad D_3 = (\ell_1 + \ell_2 - K)^2 - M_3^2. \] (3.39)
It is found that the integrals (3.38) can be reduced to seven master integrals, \(^6\) i.e.,

\[
I_{1,1,1}^{(r_1,r_2)} = C_{(r_1,r_2)} \mathbf{J} = \sum_{i_1,i_2,j} s_{i_1} s_{i_2} s_j \frac{r_{1-i_1-j} s_{i_1} r_{2-i_2-j} s_{i_2}}{s_{i_1} s_{i_2}} \alpha_{i_1,i_2,j} \mathbf{J}.
\]

where the seven master integrals are (we have written \(C, \mathbf{J}\) to emphasize they are vector)

\[
\mathbf{J} = \left\{ \int \frac{1}{D_1D_2D_3}, \int \frac{(2\ell_1 \cdot K)}{D_1D_2D_3}, \int \frac{(2\ell_2 \cdot K)}{D_1D_2D_3}, \int \frac{(2\ell_1 \cdot K)(2\ell_2 \cdot K)}{D_1D_2D_3}, \int \frac{1}{D_1D_3}, \int \frac{1}{D_1D_2} \right\},
\]

and kinematic variables are

\[
s_{01} = R_1 \cdot K, \ s_{01}' = R_2 \cdot K, \ s_{11} = K^2, \ s_{00} = R_1^2, \ s_{00}' = R_2^2, \ s_{00}' = R_1 \cdot R_2.
\]

All the reduction coefficients could be found in [41], where we have generalized the improved PV-reduction method to the two-loop sunset topology. For example:

\[
C^{(1,0)} = \left\{ 0, \frac{s_{01}}{s_{11}}, 0, 0, 0, 0 \right\}, \\
C^{(0,1)} = \left\{ 0, 0, \frac{s_{01}}{s_{11}}, 0, 0, 0 \right\}.
\]

To study the reduction for degenerate Gram determinant (\(s_{11} \rightarrow 0\), we follow the same logic, i.e., the basis with the highest topology will be decomposed to the linear combinations of other integrals in the basis. However, there is something different between two-loop and one-loop integrals. For one-loop integrals, there is only one master integral with the highest topology in the basis. For two-loop integrals, in general, there are more than one highest-topology master integrals. For example, for the sunset topology, there are four master integrals \(J_1, J_2, J_3, J_4\) as given in (3.41). Now we face the problem: whether just one or several master integrals of them should be degenerate in the limit? In fact, from the tensor reduction results given in (3.43), we can see \(J_2\) and \(J_3\) should be degenerate. There is also a brute-force algorithm to determine the number of degenerate master integrals. Starting with just one degenerate master integral, we solve the decomposition coefficients using the idea given in the last subsection for various choices of tensor rank. If it is not the right degenerate pattern, we will find that it is impossible to cancel all divergences coming from \(s_{11} \rightarrow 0\). Then we take two master integrals to be degenerate, repeat the same computations for several tensor structures and check the consistency. Interacting the procedure until reaching the point that all divergences have been canceled consistently, we get the correct degenerate pattern.

For our sunset topology, because there are five independent Lorentz invariant contractions \(R_i \cdot R_j, R_i \cdot K\) comparing to two \(R^2, R \cdot K\) for one-loop bubble, there are three master

\(^6\)For our purpose, we consider only the case \(a_i = 1\).
integrals degenerated under the limit, which can be taken as $J_2, J_3, J_4$, i.e., we have the expansion

$$J_i = \sum_{k=0}^{\infty} B_{i \to j}^{(k)} s_{11}^k J_j, \quad i \in \text{deI}, \: j \in \text{reI},$$  \hfill (3.44)

where deI = \{2, 3, 4\} denotes the degenerate master integrals and reI = \{1, 5, 6, 7\} denotes the remaining master integrals. Expansion coefficients $B_{i \to j}^{(k)}$ are determined by requiring

$$F_j^{(r_1, r_2)} = \sum_{i \in \text{deI}} \sum_k C_i^{(r_1, r_2)} B_{i \to j}^{(k)} s_{11}^k + C_j^{(r_1, r_2)}, \quad \forall j \in \text{reI}.$$  \hfill (3.45)

to be finite. The $F_j^{(r_1, r_2)}$ is nothing but the reduction coefficients for the degenerate case. Above relation can be written more compactly as

$$C_i^{(r_1, r_2)} \bigg|_{s_{11}^{-k}} B_{i \to j}^{(k)} \bigg|_{s_{11}^{-k}} = -C_j^{(r_1, r_2)} \bigg|_{s_{11}^{-k}}$$ \hfill (3.46)

where we have treated $C_{j=1,5,6,7}$ and $C_{i=2,3,4}$ as row vector and $B$ as a matrix. The symbol $\bigg|_{s_{11}^{-k}}$ means to take the coefficients of the $k$-th pole $s_{11}^{-k}$.

Now we determine decomposition coefficients using the results from rank one to three given in [41].

- For the tensor rank level one, i.e., $r_1 + r_2 = 1$, there are two cases given in (3.43).

With the only divergent pole $s_{11}^{-1}$, by (3.46) we get following equations:

$$(1, 0) : \begin{pmatrix} s_{01} & 0 & 0 \end{pmatrix} \begin{pmatrix} B_{2 \to 1}^{(0)} & B_{2 \to 5}^{(0)} & B_{2 \to 7}^{(0)} \\ B_{3 \to 1}^{(0)} & B_{3 \to 5}^{(0)} & B_{3 \to 7}^{(0)} \\ B_{4 \to 1}^{(0)} & B_{4 \to 5}^{(0)} & B_{4 \to 7}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$  \hfill (3.47)

and

$$(0, 1) : \begin{pmatrix} 0 & s_{01} & 0 \end{pmatrix} \begin{pmatrix} B_{2 \to 1}^{(0)} & B_{2 \to 5}^{(0)} & B_{2 \to 7}^{(0)} \\ B_{3 \to 1}^{(0)} & B_{3 \to 5}^{(0)} & B_{3 \to 7}^{(0)} \\ B_{4 \to 1}^{(0)} & B_{4 \to 5}^{(0)} & B_{4 \to 7}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$  \hfill (3.48)

thus we can solve

$$B_{i \to j}^{(0)} = 0, \quad i = 2, 3; \quad j = 1, 5, 6, 7.$$  \hfill (3.49)

- For the tensor rank level two, i.e., $r_1 + r_2 = 2$, there are three cases $(r_1, r_2) = (2,0), (1,1), (0,2)$. There are also two divergent poles $\frac{1}{s_{11}}$ and $\frac{1}{s_{11}}$. Explicitly, we have

$$\begin{pmatrix} C_2^{(2,0)} & C_2^{(1,1)} & C_2^{(0,2)} \\ C_3^{(2,0)} & C_3^{(1,1)} & C_3^{(0,2)} \\ C_4^{(2,0)} & C_4^{(1,1)} & C_4^{(0,2)} \end{pmatrix} \bigg|_{s_{11}^{-2}} = \frac{D}{D-1} \begin{pmatrix} M_{123} s_{01}^2 & 0 & 2M_2 s_{01}^2 \\ 2M_2 s_{01}^2 & 0 & M_{123} s_{01}^2 \\ -2s_{01}^2 & -2s_{01}^2 & 2s_{01}^2 \end{pmatrix},$$  \hfill (3.50)

$$\begin{pmatrix} C_2^{(2,0)} & C_2^{(1,1)} & C_2^{(0,2)} \\ C_3^{(2,0)} & C_3^{(1,1)} & C_3^{(0,2)} \\ C_4^{(2,0)} & C_4^{(1,1)} & C_4^{(0,2)} \end{pmatrix} \bigg|_{s_{11}^{-1}} = \begin{pmatrix} 3D_{s_{01}}^2 M_{123} s_{01}^4 & -2s_{01}^2 s_{01} s_{01}^2 & 2s_{01}^2 s_{01}^2 \\ 2s_{01}^2 s_{01}^2 & 3D_{s_{01}}^2 M_{123} s_{01}^4 & -2s_{01}^2 s_{01}^2 \\ 2s_{01}^2 & 2s_{01}^2 & 3D_{s_{01}}^2 M_{123} s_{01}^4 \end{pmatrix},$$  \hfill (3.51)
where we have defined
\[
M_{12,3} \equiv M_1^2 + M_2^2 - M_3^2
\]
for the left hand side of (3.46). The corresponding data for the right hand side of (3.46) is following
\[
\begin{pmatrix}
C_1^{(2,0)}, C_1^{(1,1)}, C_1^{(0,2)} \\
C_5^{(2,0)}, C_5^{(1,1)}, C_5^{(0,2)} \\
C_6^{(2,0)}, C_6^{(1,1)}, C_6^{(0,2)} \\
C_7^{(2,0)}, C_7^{(1,1)}, C_7^{(0,2)}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Now we can use the (3.46) to solve wanted decomposition coefficients.

- For the pole $s_{11}^{-2}$ using the (3.46) we can solve
\[
\mathcal{B}_{i \rightarrow j}^{(0)} = 0, \quad i = 2, 3, 4.
\]

It is worth to notice that for $i = 2, 3$ it is consistent with the result (3.49), but for $i = 4$ it is the new solution.

- For the pole $s_{11}^{-1}$, using the data (3.51) and (3.53), by (3.46) we can solve
\[
\mathcal{B}_{2 \rightarrow 5}^{(1)} = \frac{2(DM_1^2 + DM_2^2 - 2M_1^2 - 2M_2^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{3 \rightarrow 5}^{(1)} = \frac{2(DM_1^2 + DM_2^2 - 2M_1^2 - 2M_2^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{2 \rightarrow 5}^{(2)} = \frac{2(DM_1^2 - 2M_1^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{3 \rightarrow 5}^{(2)} = \frac{2(DM_1^2 - 2M_1^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{2 \rightarrow 5}^{(3)} = \frac{2(DM_1^2 - 2M_1^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{3 \rightarrow 5}^{(3)} = \frac{2(DM_1^2 - 2M_1^2)}{D_2 - D_3 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2},
\]
\[
\mathcal{B}_{4 \rightarrow 5}^{(1)} = \frac{2M_1^2}{D}, \quad \mathcal{B}_{4 \rightarrow 5}^{(2)} = \frac{2M_1^2}{D}, \quad \mathcal{B}_{4 \rightarrow 5}^{(3)} = \frac{2M_1^2}{D}.
\]

There is one thing we need to emphasize. The expressions of $C$'s have different tensor structures of $R$ while the coefficients $\mathcal{B}$ are $R$ independent. Thus the
For the pole $s_{11}^{-3}$, using (3.46) we find the same results as given in (3.55).

For the pole $s_{11}^{-2}$, using (3.46) we find expansion coefficients $B_{i \rightarrow j}^{(2)}$. Since their expressions are very long, we have collected them into a Mathematica file in the supplementary material. To give a flavor, we present $B_{4 \rightarrow 1}^{(2)}$ as following

$$B_{4 \rightarrow 1}^{(2)} = \frac{1}{D^2(D + 2) \left( M_{12,3}^2 - 4M_2^2M_3^2 \right)^2} \left[ \left( 16D^3 - 208D^2 + 528D - 288 \right) M_2^2M_1^2M_{12,3}^2 
+ (4D^3 - 14D^2 + 28D - 48)M_{12,3}^4 
- (8D^3 - 40D^2 + 80D - 96)(M_2^2 + M_3^2)M_{12,3}^3 
+ (128D^2 - 416D + 96)(M_2^2M_1^4 + M_1^2M_3^4)M_{12,3} 
+ (384D - 96D^2)M_1^2M_3^4 \right].$$ (3.58)
Now we can calculate the final reduction coefficients by (3.45). For example, the reduction coefficients for rank (0, 1) is given by

\[
F^{(0,1)}_1 = \frac{2s_{01} \left( -2DM^2_{12,3} + DM^2_{12,3} + 6M^2_{12,3} - 2M^2_{12,3} - 4M^2_{1}M^2_{2} \right)}{D \left( M^2_{12,3} - 4M^2_{1}M^2_{2} \right)},
\[
F^{(1,0)}_1 = \frac{2s_{01} \left( 2DM^2_{12,3} - DM^2_{12,3} - 6M^2_{12,3} + 2M^2_{12,3} + 4M^2_{1}M^2_{2} \right)}{D \left( 4M^2_{1}M^2_{2} - M^2_{12,3} \right)},
\[
F^{(0,1)}_5 = \frac{2(D - 2)M_{12,3}s_{01}}{D \left( M^2_{12,3} - 4M^2_{1}M^2_{2} \right)}, \quad F^{(1,0)}_5 = \frac{4(D - 2)M^2_{1}M^2_{2}}{D \left( 4M^2_{1}M^2_{2} - M^2_{12,3} \right)},
\[
F^{(0,1)}_6 = \frac{4(D - 2)M^2_{1}M^2_{2} - M^2_{12,3}}{D \left( 4M^2_{1}M^2_{2} - M^2_{12,3} \right)}, \quad F^{(1,0)}_6 = \frac{2(D - 2)M_{12,3}s_{01}}{D \left( M^2_{12,3} - 4M^2_{1}M^2_{2} \right)},
\[
F^{(0,1)}_7 = \frac{2(D - 2)s_{01} \left( 2M^2_{12,3} - M^2_{12,3} \right)}{D \left( M^2_{12,3} - 4M^2_{1}M^2_{2} \right)}, \quad F^{(1,0)}_7 = \frac{2(D - 2)s_{01} \left( 2M^2_{12,3} - M^2_{12,3} \right)}{D \left( M^2_{12,3} - 4M^2_{1}M^2_{2} \right)}.
\] (3.59)

In the Mathematica file in the supplementary material we have present reduction coefficients for other tensor ranks up to rank level two.

4 Self-consistence

From the previous section we see that although the decomposition coefficients \( B_{n+1,b_j} \) are computed using the tensor reduction results in (3.15), by definition in (3.10), it should be independent of the auxiliary \( R \) and the tensor rank used in the formula. Thus to show the correctness of the formula (3.15), we should prove this point.

More explicitly, there are two facts we want to prove. The first one is that to get \( B_{n+1,b_j} \) up to \( |G|^k \) order, we require \( m \) in (3.15) to satisfy condition \( m > k \). One can choose different \( m_i \). As long as \( m_i > k \), result given by (3.15) should be the same up to \( |G|^k \). The second one is that although at the r.h.s. of (3.15), \( C^{(m)}_{n+1 \rightarrow n+1,b_j} \) are functions of \( R \), as long as \( m > k \), the series of \( |G| \) should be independent of \( R \) up to \( |G|^k \).

In this section, we will prove these crucial consistencies using two different methods.

4.1 Proof with recursion relation

The crucial input of our proof is the rather non-trivial relation for one-loop integrals\(^7\)

\[
I^{(m)}_n = \frac{1}{|G|} \left[ A_m I^{(m-1)}_n + B_m I^{(m-2)}_n + \text{Lower Terms} \right] = \frac{1}{|G|} \left[ A_m I^{(m-1)}_n + B_m I^{(m-2)}_n + \mathcal{O}(1) \right].
\] (4.1)

\(^7\)This relation is observed in [41]. Here we rephrase it in the language of projective space [39].
where “Lower Terms” means the contributions from lower topologies, we also denote it as $\mathcal{O}(1)$ for it contains no poles of $|G|$, and

$$A_m = a_m(VQ^*L) = \frac{2(D + 2m - n - 2)}{D + m - n - 1} (VQ^*L),$$

$$B_m = b_m \left[ (\det Q)R^2 - VQ^*V \right] = \frac{4(m - 1)((\det Q)R^2 - VQ^*V)}{D + m - n - 1}. \quad (4.2)$$

Here $Q^*$ is the adjoint matrix of $Q$ with component $Q_{ij} = \frac{M_j^2 + M_i^2 - (K_i - K_j)^2}{2}, i, j = 0, \ldots, n - 1, K_0 = 0$. The two vectors in the expression of $A_m, B_m$ are $L = \{1, 1, \ldots, 1\}, V = \{s_0, s_0, \ldots, s_0, n - 1\}$. One can prove the relation (4.1) by showing that it satisfies the $\mathcal{D}$-type and $\mathcal{T}$-type differential relations.\footnote{Since we have shown the reduction coefficients can be completely determined by $\mathcal{D}$-type and $\mathcal{T}$-type differential relations in [1, 2], checking relation (4.1) satisfies these two recursions will be sufficient for our proof.}

First, acting with $\mathcal{D}$-type operator, we have

$$\mathcal{D}_i I_n^{(m)} = m (f_i I_n^{(m-1)} + I_{n,0}^{(m)} - I_{n,2}^{(m)}) = m f_i I_n^{(m-1)} + \mathcal{O}(1) \quad (4.3)$$

at one side, while applying $\mathcal{D}_i$ on the r.h.s. of (4.1), we find

$$\frac{1}{|G|} \mathcal{D}_i \left[ A_m I_n^{(m-1)} + B_m I_n^{(m-2)} + \mathcal{O}(1) \right] = \frac{1}{|G|} \left[ (\mathcal{D}_i A_m) I_n^{(m-1)} + ((m - 1)f_i + \mathcal{D}_i B_m) I_n^{(m-2)} + B_m (m - 2)f_i I_n^{(m-3)} + \mathcal{O}(1) \right] \quad (4.4)$$

at another side. Identifying (4.4) and (4.3), we have

$$I_n^{(m-1)} = \frac{1}{mf_i |G| - \mathcal{D}_i A_m} \left[ ((m - 1)f_i + \mathcal{D}_i B_m) I_n^{(m-2)} + B_m (m - 2)f_i I_n^{(m-3)} + \mathcal{O}(1) \right]. \quad (4.5)$$

By our recursive assumption of the form (4.1), we just need to prove

$$(mf_i |G| - \mathcal{D}_i A_m) A_{m-1} = |G| ((m - 1)f_i + \mathcal{D}_i B_m),$$

$$(mf_i |G| - \mathcal{D}_i A_m) B_{m-1} = |G| B_m (m - 2)f_i, \quad (4.6)$$

which can be checked using the identities

$$\mathcal{D}_i A_m = \frac{2(D + 2m - n - 2)}{D + m - n - 1} \sum_{ab} s_{ia} Q_{ab} = \frac{2(D + 2m - n - 2)}{D + m - n - 1} f_i |G|,$$

$$\mathcal{D}_i B_m = \frac{4(m - 1)(2(\det Q)s_{0i} - 2 \sum_{jk} s_{ik} Q_{kj}^* s_{0j})}{D + m - n - 1} = \frac{4(m - 1)f_i (VQ^*L)}{D + m - n - 1}. \quad (4.7)$$

Next we consider the $\mathcal{T}$ operator. First, we have

$$\mathcal{T} I_n^{(m)} = 4m(m - 1) M_0^2 I_n^{(m-2)} + 4m(m - 2) I_n^{(m-2)} = 4m(m - 1) M_0^2 I_n^{(m-2)} + \mathcal{O}(1). \quad (4.8)$$
Acting with $\mathcal{T}$ on the r.h.s. of (4.1), we get

$$
\frac{\mathcal{T}}{|G|}\left[A_m I_n^{(m-1)} + B_m I_n^{(m-2)} + \mathcal{O}(1)\right]
$$

$$
= \frac{1}{|G|} \left[2a_m(m-1) \sum_{i,j} Q_{ij}^s f_i + 2(D + 2m - 4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^s \right] I_n^{(m-2)}
$$

$$
+ \frac{1}{|G|} \left[4(m-1)(m-2)M_0^2 A_m - 4b_m(m-2) \sum_{i,j=1}^{n-1} f_i Q_{ij}^s s_{0j} \right] I_n^{(m-3)}
$$

$$
+ \frac{1}{|G|} \left[4b_m(m-2)(m-3)M_0^2 (\det(Q) R^2 - (VQ^*V)) \right] I_n^{(m-4)}
$$

(4.9)

where we have used following algebraic results

$$
\mathcal{T} \left[(VQ^*L) I_n^{(m)}\right] = 2 \sum_{i,j} Q_{ij}^s f_i I_n^{(m-1)} + 4m(m-1)M_0^2 (VQ^*L) I_n^{(m-2)} + \mathcal{O}(1),
$$

$$
\mathcal{T} \left[R^2 I_n^{(m)}\right] = 2(D + 2m) I_n^{(m)} + 4m(m-1)M_0^2 R^2 I_n^{(m-2)} + \mathcal{O}(1),
$$

$$
\mathcal{T} \left[(VQ^*V) I_n^{(m)}\right] = 2 \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^s I_n^{(m)} + 4m \sum_{i,j=1}^{n-1} f_i Q_{ij}^s s_{0j} I_n^{(m-1)}
$$

$$
+ 4m(m-1)M_0^2 (VQ^*V) I_n^{(m-2)} + \mathcal{O}(1).
$$

(4.10)

Comparing both sides, we just need to prove

$$
4m(m-1)M_0^2 |G|
$$

$$
- \left[2a_m(m-1) \sum_{i,j} Q_{ij}^s f_i + 2(D + 2m - 4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^s \right] A_{m-2}
$$

$$
= - \frac{4b_m(m-2)}{|G|} \sum_{i,j=1}^{n-1} f_i Q_{ij}^s s_{0j}
$$

(4.11)

and

$$
4m(m-1)M_0^2 |G|
$$

$$
- \left[2a_m(m-1) \sum_{i,j} Q_{ij}^s f_i + 2(D + 2m - 4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^s \right] B_{m-2}
$$

$$
= \left[4a_m(m-1)(m-2)M_0^2 (VQ^*L) + 4b_m(m-2)(m-3)M_0^2 (\det(Q) R^2 - (VQ^*V)) \right].
$$

(4.12)

One can check the two equations after some algebra.

Now we employ (4.1) to prove self-consistence, i.e., to show that

$$
B_{n;b_j} = - \frac{C_{mm}^{(m)}}{C_{n-m}^{(m)}} + \mathcal{O}(|G|^m)
$$

(4.13)
is independent of the rank choice \( m \) and \( R \). First let us show the independence of rank, which is equivalent to the relation
\[
\frac{C^{(m+1)}_{n\to n_b}}{C^{(m+1)}_{n\to n}} - \frac{C^{(m)}_{n\to n_b}}{C^{(m)}_{n\to n}} = \mathcal{O}(|G|^m). \tag{4.14}
\]
To see it, first we rewrite (4.14) to
\[
\frac{C^{(m+1)}_{n\to n_b}}{C^{(m+1)}_{n\to n}} - \frac{C^{(m)}_{n\to n_b}}{C^{(m)}_{n\to n}} = \mathcal{O}(|G|^m). \tag{4.15}
\]
Using the divergence behavior of denominator
\[
C^{(m)}_{n\to n} = \mathcal{O}(|G|^{-2m-1}), \tag{4.16}
\]
we just need to show the leading divergent behavior of numerator is
\[
C^{(m)}_{n\to n} - C^{(m)}_{n\to n_b} = \mathcal{O}(|G|^{-m-1}). \tag{4.17}
\]
To prove it, we assume the equation holds for all \( m < m' \), while \( m' = 1, 2 \) can be checked explicitly, then we prove it holds for \( m = m' \). Using (4.1), the l.h.s. of (4.17) becomes
\[
\frac{C^{(m+1)}_{n\to n_b}}{C^{(m)}_{n\to n}} - \frac{C^{(m)}_{n\to n_b}}{C^{(m)}_{n\to n}} = \mathcal{O}(|G|^{-1}). \tag{4.18}
\]
Having shown the independence of the rank choice, now we prove the first \( m \) terms in (4.13) don’t rely on \( R \), i.e., the derivation over \( R \) is zero up to order \( \mathcal{O}(|G|^m) \):
\[
\frac{\partial}{\partial R^\mu} \left[ \frac{C^{(m)}_{n\to n_b}}{C^{(m)}_{n\to n}} \right] = \frac{\partial}{\partial R^\mu} \left[ \frac{C^{(m)}_{n\to n_b}}{C^{(m)}_{n\to n}} \right] \leq \mathcal{O}(|G|^m). \tag{4.19}
\]
Similarly, since the leading divergent behavior is \( |G|^{-2m} \), we just need to show that
\[
\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n_b} - C^{(m)}_{n\to n_b} = \mathcal{O}(|G|^{-m}). \tag{4.20}
\]
Due to reduction coefficients are functions of \( s_{0i}, s_{00} \), after acting \( \frac{\partial}{\partial R^\mu} \), the l.h.s. of (4.20) can be wrote as the form
\[
L^\mu \equiv \sum_{i=1}^{n-1} \tilde{\alpha}_i K^\mu_i + \tilde{\alpha}_n R^\mu. \tag{4.21}
\]
Noticing the limit $|G| \to 0$ means these $K_i$ are not linearly independent, thus the limit can be parameterized as

$$K_1 = \sum_{i=2}^{n-1} a_i K_i + t K_0$$

(4.22)

where $t$ is the same order of $|G|$ and $K_0$ is another linear independent momentum. Putting it back to (4.21) we get

$$L^\mu = \sum_{i=2}^{n-1} (\tilde{\alpha}_i + \tilde{\alpha}_1 a_i) R_0^\mu + \tilde{\alpha}_1 t K_0^\mu + \tilde{\alpha}_n R^\mu \equiv \sum_{i=2}^{n-1} \alpha_i K_i^\mu + \alpha_1 t K_0^\mu + \alpha_n R^\mu.$$  

(4.23)

Let us contract $L^\mu$ with $K_i, \ i = 2, \ldots, n-1$ and $R$, we will get $n$ equations in the matrix form as

$$G(K_0, K_2, \ldots, K_{n-1}, R) \begin{pmatrix} t \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$  

(4.24)

where $G(K_0, K_2, \ldots, K_{n-1}, R)$ is the non-degenerate Gram matrix and

$$b_1 = K_0^\mu \left[ \frac{\partial}{\partial R^\mu} C_{n\rightarrow n; b_j}^{(m)} - C_{n\rightarrow n; b_j}^{(m)} \frac{\partial}{\partial R^\mu} C_{n\rightarrow n}^{(m)} \right]$$

$$b_i = K_i^\mu \left[ \frac{\partial}{\partial R^\mu} C_{n\rightarrow n; b_j}^{(m)} - C_{n\rightarrow n; b_j}^{(m)} \frac{\partial}{\partial R^\mu} C_{n\rightarrow n}^{(m)} \right] = D_i C_{n\rightarrow n; b_j}^{(m)} - D_i C_{n\rightarrow n}^{(m)} = 0, i = 2, \ldots, n-1$$

$$b_n = R^\mu \left[ \frac{\partial}{\partial R^\mu} C_{n\rightarrow n; b_j}^{(m)} - C_{n\rightarrow n; b_j}^{(m)} \frac{\partial}{\partial R^\mu} C_{n\rightarrow n}^{(m)} \right] = 0$$  

(4.25)

where we have used the homogeneity of $R$ for $b_n$. Thus to show $t \alpha_1$ and all other $\alpha_i \leq O(|G|^{-m})$, we just need to show $b_i \leq O(|G|^{-m})$.

We will prove it recursively by assuming the equation holds for all $m < m'$, where the $m' = 1, 2$ can be checked explicitly. For the case $m = m'$, using

$$D_i C_{n\rightarrow n; b_j}^{(m)} = m f_i C_{n\rightarrow n; b_j}^{(m-1)} + \mathcal{O}(1),$$

(4.26)

we have

$$b_i = m f_i \left[ C_{n\rightarrow n; b_j}^{(m-1)} - C_{n\rightarrow n; b_j}^{(m)} \right] = \mathcal{O}(|G|^{-m')}$$

(4.27)

where in the second line we have used the result (4.17). Above argument has shown that $b_i \leq O(|G|^{-m})$ $i = 2, \ldots, n-1$ for $m = m'$. For $b_1$, we need to be more careful. Using (4.22), we have

$$b_1 = \frac{1}{t} \left( K_1 - \sum_{i=2}^{n-1} a_i K_i \right) \left[ \frac{\partial}{\partial R^\mu} C_{n\rightarrow n; b_j}^{(m)} - C_{n\rightarrow n; b_j}^{(m)} \frac{\partial}{\partial R^\mu} C_{n\rightarrow n}^{(m)} \right].$$

(4.28)
Naively we have $\frac{1}{t}$ in the front, which will make the divergence higher. However, when $t \to 0$, the combination $(K_1 - \sum_{i=2}^{n} a_i K_i) \sim t \to 0$, thus the divergent order estimation given in (4.27) is also true for $b_1$. Thus we finish the proof for $R$ independence.

### 4.2 Another proof

Now we will give another proof from a different point of view. For simplification, we will confine to the decomposition coefficient $B_{n+1, b_j}$ with $b_j = \{r + 1, \cdots, n\}$. For given tensor rank $m$, the reduction coefficients have the expansion

$$C^{(m)}_{n+1 \to n+1; r+1, \cdots, n} = \sum_{2a_0 + \sum_{k=1}^{n} a_k = m} \left\{ c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m) (M^2_0)^{a_0 + r - n} \prod_{k=0}^{n} s_k \right\}.$$

(4.29)

Thus we have the expansion of $F$ in (3.12)

$$F^{(m)}_{n+1 \to n+1; r+1, \cdots, n} = \sum_{2a_0 + \sum_{k=1}^{n} a_k = m} \left\{ c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m) + c^{(0, \cdots, n)}_{a_1, \cdots, a_n} (m) B_{n+1; r+1, \cdots, n} \right\} \times (R \cdot R)^{a_0} \prod_{i=1}^{n} (R \cdot K_i)^{a_i}.$$

(4.30)

Since $c^{(0, \cdots, n)}_{a_1, \cdots, a_n} (m) \sim |G| \left( \frac{m + \sum a_i}{2} \right)$, smoothness under the limit gives

$$B_{n+1; r+1, \cdots, n} = -\frac{c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m)}{c^{(0, \cdots, n)}_{a_1, \cdots, a_n} (m)} + O \left( |G|^{\frac{m + \sum a_i}{2}} \right).$$

(4.31)

The difference between (3.15) and (4.31) is that we have used expansion coefficients in (4.31) instead of the whole reduction coefficients in (3.15). Thus $B_{n+1; r+1, \cdots, n}$ from (4.31) will be independent of $R$ manifestly, but we need to show the same result will be obtained taking different $a'_i$ and $m$. Overall, the irrelevancy of the tensor rank $m$ and the expansion indices lead to following condition

$$\frac{c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m) - c^{(0, \cdots, n)}_{a_1, \cdots, a_n} (m)}{c^{(0, \cdots, r)}_{a'_1, \cdots, a'_n} (m')} \leq |G| \left( \min \left[ \frac{m + \sum a_i}{2}, \frac{m' + \sum a'_i}{2} \right] \right).$$

(4.32)

$$\Rightarrow c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m) c^{(0, \cdots, n)}_{a'_1, \cdots, a'_n} (m') - c^{(0, \cdots, n)}_{a_1, \cdots, a_n} (m) c^{(0, \cdots, r)}_{a'_1, \cdots, a'_n} (m') \leq |G| \left( \max \left[ \frac{m + \sum a_i}{2}, \frac{m' + \sum a'_i}{2} \right] \right).$$

holds for any nonzero $c^{(0, \cdots, r)}_{a_1, \cdots, a_n} (m), c^{(0, \cdots, r)}_{a'_1, \cdots, a'_n} (m')$. 

\[ -24 - \]
Before to prove (4.32), let us list several useful relations. Using the $\mathcal{T}$ operator [1, 2] we have relation

$$4m(m+1)c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1) = \left(m+1 - \sum_{k=1}^{n} a_k\right) \left(D + m + \sum_{k=1}^{n} a_k - 1\right) c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m+1)$$

$$+ \sum_{0 \leq i < j} 2(a_i + 1)(a_j + 1)\beta_{ij} c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_j+1,\ldots,a_n}(m+1)$$

$$+ \sum_{a=1}^{n} (a_i + 1)(a_i + 2)\beta_{ii} c^{(0,\ldots,r)}_{a_1,\ldots,a_i+2,\ldots,a_n}(m+2).$$  \[(4.33)\]

Using the $\mathcal{D}$-relation (2.12) twice, we have

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m+1) = \frac{(m+1) \left\{ 4m \cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1) - \alpha^T \hat{G}^{-1} O^{(0,\ldots,r)}(a_1, \ldots, a_n; m) \right\}}{(m+1 - \sum_{k=1}^{n} a_k)(D + m - n - 1)}.$$ \[(4.34)\]

where vector $O^{(0,\ldots,r)}(a_1, \ldots, a_n; m)$ is given in (2.16). Combining with the component form of (2.12) we have

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) = \frac{1}{a_i + 1} \sum_{k} \frac{\hat{G}_{ik}}{|G|} O^{(0,\ldots,r)}(a_1, \ldots, a_n; m).$$ \[(4.35)\]

From (4.34) and (4.35) we can read

$$c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) = \frac{N_i(a_1, \ldots, a_n; m)}{|G|} \left\{ A_i \cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1) + \sum_{k} B_{ik} \cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m) \right\},$$

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) = \frac{N_i(a_1, \ldots, a_n; m)}{|G|} \left\{ A_i \cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1) + \sum_{k} B_{ik} \cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m) \right\}$$

$$+ \frac{X_i^{\text{lower}}}{|G|},$$

$$c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1) = \frac{M(a_1, \ldots, a_n; m)}{|G|} \left\{ D \cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1) + \sum_{k} E_{ik} \cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m) \right\},$$

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m+1) = \frac{M(a_1, \ldots, a_n; m)}{|G|} \left\{ D \cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1) + \sum_{k} E_{ik} \cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m) \right\}$$

$$+ \frac{Y_i^{\text{lower}}}{|G|}.$$ \[(4.36)\]

where $A, B_k, D, E_k$ are finite under $|G| \sim |\hat{G}| \to 0$. $X_i^{\text{lower}}, Y_i^{\text{lower}}$ are those contributions from the reduction of lower topologies, which are also finite under $|G| \to 0$. $M_i, N$ are
constants decided by $a_1, \ldots, a_n, m$. We also need to mention that under $|G| \to 0$:

$$A_i : A_j = \left( \sum_\mu \bar{G}^*_{i\mu} \cdot \alpha_\mu \right) : \left( \sum_\mu \bar{G}^*_{j\mu} \cdot \alpha_\mu \right),$$

$$B_{ik} : B_{kj} = \bar{G}^*_{ik} : \bar{G}^*_{kj},$$

$$D : E_{k1} : E_{k2} = \left( \sum_\mu \alpha_\mu \cdot \bar{G}^*_{\mu k1} \cdot \alpha_\mu \right) : \left( \sum_\mu \bar{G}^*_{k1\mu} \cdot \alpha_\mu \right) : \left( \sum_\mu \bar{G}^*_{k2\mu} \cdot \alpha_\mu \right).$$

(4.37)

There are also some useful properties of $\bar{G}$. Since $|\bar{G}| \to 0$ means the corank of $\bar{G}$ is one, we have

$$\sum_\mu \bar{G}^*_{i\mu} \cdot v_\mu / \sum_\mu \bar{G}^*_{j\mu} \cdot v_\mu = \bar{G}^*_{ik} / \bar{G}^*_{jk},$$

$$\bar{G}^*_{ik} \bar{G}^*_{kj} - \bar{G}^*_{ik} \bar{G}^*_{kj} = 0,$$

$$\left( \sum_\mu \bar{G}^*_{i\mu} \cdot v_\mu \right) \cdot \left( \sum_\mu \bar{G}^*_{j\mu} \cdot v_\mu \right) = \bar{G}_{ij} \cdot \alpha^T \bar{G}^{-1} \alpha.$$  

(4.38)

Having above preparations, we will prove (4.32) by induction with index

$$\kappa = \text{Max} \left[ \frac{m+\sum a_i}{2}, \frac{m'+\sum a'_i}{2} \right].$$

Suppose it is true for $\kappa < \kappa'$, all we need to prove are following four relations:

- (A) With the same $m$ but different $a_i, a_j, a_0$: \(\frac{\Gamma_{c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)}{\bar{G}^{2m+\sum a_i+1}} \leq \frac{\Gamma_{c_{a_1, \ldots, a_i}^{(0, \ldots, r)}(m)c_{a_1, a_i}^{(0, \ldots, n)}(m)}{\bar{G}^{2m+\sum a_i+1}}. \)

Proof of (A).

Above relation is equal to

$$c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m) - c_{a_1, \ldots, a_i}^{(0, \ldots, r)}(m)c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)$$

$$\leq \frac{\Gamma_{c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)}{\bar{G}^{2m+\sum a_i+1}}. \quad \text{(4.39)}$$

Using (4.36), we have

$$c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m) - c_{a_1, \ldots, a_i}^{(0, \ldots, r)}(m)c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)$$

$$= \frac{N(c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)}{|G|}$$

$$\times \left\{ A_i \cdot c_{a_1, \ldots, a_i}^{(0, \ldots, r)}(m-1)c_{a_1, \ldots, a_i}^{(0, \ldots, n)}(m) - c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m-1)c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m) \right\}$$

$$+ \sum_k B_{ik} \cdot c_{a_1, \ldots, a_k}^{(0, \ldots, r)}(m)c_{a_1, \ldots, a_k}^{(0, \ldots, n)}(m) - c_{a_1, \ldots, a_k-1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, \ldots, a_k-1, \ldots, a_n}^{(0, \ldots, n)}(m)$$

$$+ \frac{\lambda^{\text{lower}} c_{a_1, \ldots, a_i}^{(0, \ldots, n)}(m)}{|G|} \quad \leq \frac{1}{|G|} \cdot \frac{\Gamma_{c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)}}{|G|^{2m+\sum a_i+1}} \leq \frac{1}{|G|} \cdot \frac{\Gamma_{c_{a_1, \ldots, a_i+1, \ldots, a_n}^{(0, \ldots, r)}(m)c_{a_1, a_i+1, \ldots, a_n}^{(0, \ldots, n)}(m)}}{|G|^{2m+\sum a_i+1}}. \quad \text{(4.40)}
Thus we have done the proof.

- (B) With same \(a_0\), but different \(m, a_i\):
  \[
  \frac{c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m)}{c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m)} - \frac{c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m-1)}{c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m-1)} \leq \frac{m + \sum_{a_i+1} a_i}{2G}
  \]

Proof of (B):
Change \(c_{a_1,\ldots,a_n}^{(0,\ldots,r)}/c_{a_1,\ldots,a_n}^{(0,\ldots,n)} (m)\) in (4.40) into \(c_{a_1,\ldots,a_n}^{(0,\ldots,r)}/c_{a_1,\ldots,a_n}^{(0,\ldots,n)} (m-1)\), we will reach the proof of (B).

- (C) With the same \(m, a_0\) but different \(a_i, a_j\):
  \[
  \frac{c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m)}{c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m)} - \frac{c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m)}{c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m)} \leq \frac{m + \sum_{a_i+1} a_i}{2G}
  \]

Proof of (C):
By (4.36), we have
\[
\begin{align*}
& c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m)c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m) - c_{a_1,\ldots,a_j + 1,a_n}^{(0,\ldots,r)}(m)c_{a_1,\ldots,a_i + 1,a_n}^{(0,\ldots,n)}(m) \\
& = N_l(a_1,\ldots,a_n; m)\frac{\sum_k [A_iB_{jk} - A_jB_{ik}]}{|G|^2} \times \left\{ \sum_k [A_iB_{jk} - A_jB_{ik}] ight\} \\
& \times \left[ c_{a_1,\ldots,a_n}^{(0,\ldots,n)}(m-1)c_{a_1,\ldots,a_k - 1,a_n}^{(0,\ldots,n)}(m) - c_{a_1,\ldots,a_n}^{(0,\ldots,r)}(m) \right] \\
& + \sum_{k_1 < k_2} [B_{ik_1}B_{jk_2} - B_{jk_1}B_{ik_2}] \\
& \times \left[ c_{a_1,\ldots,a_{k_1 - 1},a_n}^{(0,\ldots,r)}(m)c_{a_1,\ldots,a_{k_2} - 1,a_n}^{(0,\ldots,n)}(m) - c_{a_1,\ldots,a_{k_1},a_{k_2} - 1,a_n}^{(0,\ldots,r)}(m) \right] \\
& + \frac{1}{|G|} \times \left\{ X_i^{\text{lower}} c_{a_1,\ldots,a_i + 1,a_n}^{(0,\ldots,n)}(m) - X_j^{\text{lower}} c_{a_1,\ldots,a_j + 1,a_n}^{(0,\ldots,n)}(m) \right\}.
\end{align*}
\]

From (4.37) and (4.38), we know
\[
\lim_{|G|\to 0} (A_iB_{jk} - A_jB_{ik}) = \lim_{|G|\to 0} (B_{ik_1}B_{jk_2} - B_{jk_1}B_{ik_2}) = 0.
\]

For the last line of (4.41), using the following expression of \(X_i^{\text{lower}}\) and \(c_{a_1,\ldots,a_i + 1,a_n}^{(0,\ldots,n)}(m)\)
\[
\begin{align*}
& c_{a_1,\ldots,a_i + 1,a_n}^{(0,\ldots,n)}(m) = \frac{1}{(a_i + 1)|G|} \sum_{\mu} \hat{G}_{ij}^* \cdot \hat{O}_{\mu}^{(0,\ldots,n)}(a_1,\ldots,a_n; m), \\
& X_i^{\text{lower}} = -\frac{m}{a_i + 1} \sum_{\mu} \hat{G}_{ij}^* \cdot \delta_{0\mu} c_{a_1,\ldots,a_{i+1},a_n}^{(0,\ldots,r)}(m - 1; \mu),
\end{align*}
\]
then by the first line of (4.38), we know
\[
\begin{align*}
& X_i^{\text{lower}} c_{a_1,\ldots,a_j + 1,a_n}^{(0,\ldots,n)}(m) - X_j^{\text{lower}} c_{a_1,\ldots,a_i + 1,a_n}^{(0,\ldots,n)}(m) \\
& = 0 \cdot |G|^{-\frac{m + \sum_{a_i+1} a_i - 1}{2}} + O\left( |G|^{-\frac{m + \sum_{a_i+1} a_i - 1}{2}} \right).
\end{align*}
\]
Thus we have

\[
C_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m)c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m) - C_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m+1)c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m) \\
\leq \frac{1}{|G|^2} \cdot O(|G|) \cdot O\left(|G| - \frac{m+\sum a_k}{2}\right) + \frac{1}{|G|} \cdot O\left(|G| - \frac{m+\sum a_k}{2}\right) \sim |G|^{-\frac{m+\sum a_k}{2}}. 
\]

(4.45)

- (D) With the same \(a_0\), but different \(m, a_i\): \(m, a_i\) \(c_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m+1)\) \(c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m)\) \(c_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m+1)\) \(c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m)\) 

\[
|G|^{-\frac{m+\sum a_k}{2}}. 
\]

Proof of (D):

By \((4.36)\), we have

\[
e_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m) = \frac{N_1(a_1,\ldots,a_n; m)}{|G|^2} \times \left\{ \sum_{k_1<k_2} [E_{k_1}B_{k_2} - E_{k_2}B_{k_1}] \\
\times [c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, r)}(m)c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, n)}(m) - c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, r)}(m+1)c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, n)}(m)] \\
+ \sum_k [A_k E_k - D \cdot B_k] \\
\times [c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m-1)c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, n)}(m) - c_{a_1,\ldots,a_n}^{(0,\ldots, r)}(m-1)c_{a_1,\ldots,a_k-1,\ldots,a_n}^{(0,\ldots, n)}(m)] \right\} \\
+ \frac{1}{|G|} [Y_{lower}c_{a_1,\ldots,a_i+1,\ldots,a_n}^{(0,\ldots, n)}(m) - X_{lower}c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m+1)]. 
\]

(4.46)

Similar to the proof of (C), we have

\[
\lim_{|G|\to0} (E_{k_1}B_{k_2} - E_{k_2}B_{k_1}) = \lim_{|G|\to0} (A_k E_k - D \cdot B_k) = 0. 
\]

(4.47)

Then using the following expressions for \(Y_{lower}\) and \(c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m+1)\)

\[
Y_{lower} = \frac{m(m+1)}{(m+1 - \sum a_\alpha)(D + m - n - 1)} \cdot \sum_\mu \tilde{G}_\mu^\ast \cdot \delta_{a_\mu} c_{a_1,\ldots,a_\mu,\ldots,a_n}^{(0,\ldots, r)}(m-1; \mu) \\
c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m+1) = \frac{m+1}{(m+1 - \sum a_\alpha)(D + m - n - 1)|G|} \times \\
\left\{ 4m |\tilde{G}| \cdot c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(m-1) - \sum_\nu a_\nu \tilde{G}_\nu^\ast c_{a_1,\ldots,a_n}^{(0,\ldots, n)}(a_1, \ldots, a_n; m) \right\} 
\]

(4.48)
and combining with (4.43) and the property (4.38), the leading order of last line of (4.46) is
\[
Y_{\text{lower}}^{c_{a_1, \ldots, a_n+1, \ldots, a_n}(m)} - X_{i}^{\text{lower}}c_{a_1, \ldots, a_n+1, \ldots, a_n}(m+1) = 0 \cdot \left| G \right|^{m+\sum a_k+1} + O\left( \left| G \right|^{-\frac{m+\sum a_k-1}{2}} \right).
\]

(4.49)

thus we arrive
\[
c_{a_1, \ldots, a_n+1, \ldots, a_n}(m)c_{a_1, \ldots, a_n}(m+1) - c_{a_1, \ldots, a_n}(m+1)c_{a_1, \ldots, a_n+1, \ldots, a_n}(m)
\leq \frac{1}{|G|^2} \cdot O(\left| G \right|) \cdot O\left( \left| G \right|^{-\frac{m+\sum a_k-1}{2}} \right) + \frac{1}{|G|} \cdot O\left( \left| G \right|^{-\frac{m+\sum a_k-1}{2}} \right) \sim \left| G \right|^{-\frac{m+\sum a_k+1}{2}}.
\]

(4.50)

Having above four relations (A)(B)(C)(D), we can prove (4.32) iteratively. Eventually, we have proved that \( B_{n+1,b_j} \) is independent of the rank \( m \) and \( R \) by the recursion relation of \( c_{a_1, \ldots, a_n}(m) \).

5 Discussion

In this paper, we show how to do the reduction for one-loop tensor integrals when Gram determinant degenerates. In this case, the highest topological master basis will be reduced to an expansion of the basis with lower topologies. By demanding the cancellation of divergent parts, we can solve the decomposition coefficients. We have proved that the results are independent of the tensor rank \( m \) and the auxiliary vector \( R \), thus shown our method’s self-consistency.

One advantage of our method is that we have the series expansion around the degenerate point \( |G| \), which will be helpful for some situations, for example, the improvement of numerical accuracy. The same idea can also be used to deal with other kinds of singularities, such as soft limits or massless limits. Furthermore, as the improved PV-reduction method has been applied to two-loop sunset topology recently in [41], we show that the same method works well for the sunset topology in the limit \( K^2 \to 0 \) as given in the section 3.4.

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A Expansion for degenerate Gram determinant

In this appendix, we will give the decomposition for box and pentagon as the series of \( |G| \). Since the analytic expressions with higher and higher orders of \( |G| \) are complicated, we will present only numerical results to demonstrate the independence of decomposition coefficients for the rank choice and \( R \).
In following, we list some numerical results for box and pentagon. Here we set $K_0 = 0$, and choose the Gram Matrices as:

\[
\begin{align*}
\text{box : } G &= \frac{1}{71} \begin{pmatrix}
2 & 5 & 11 \\
5 & 17 & 23 \\
11 & 23 & 65 + \frac{357911}{9} t
\end{pmatrix}, \\
\text{pentagon : } G &= \frac{1}{71} \begin{pmatrix}
2 & 5 & 11 & 41 \\
5 & 17 & 23 & 47 \\
11 & 23 & 31 & 59 \\
41 & 47 & 59 & 1283 + \frac{25411681}{306} t
\end{pmatrix}, \quad (A.1)
\end{align*}
\]

where $G_{ij} = s_{ij} = K_i \cdot K_j$. With this parameterization, we have $\det(G) = t$. The chosen squares of masses are

\[
\begin{align*}
\text{box : } \left( M_0^2, M_1^2, M_2^2, M_3^2 \right) &= \frac{1}{71} \left( 3, 7, 13, 19 \right), \\
\text{pentagon : } \left( M_0^2, M_1^2, M_2^2, M_3^2, M_4^2 \right) &= \frac{1}{71} \left( 3, 7, 13, 19, 29 \right). \quad (A.2)
\end{align*}
\]

Firstly we present some reduction coefficients needed in (3.15) and (3.18). For $I^{(2)}_{4 \to 4}$ we have

\[
\begin{align*}
C^{(2)}_{4 \to 4} &= \frac{s_{03}s_{01}}{384300851763(D - 3) t^2} \times \left( -12041426688574 D t^2 - 32933538576 D t - 20155392 D + 29975466437514 t^2 + 73134102096 t + 40310784 \right) \\
&\quad + \frac{s_{03}^2}{128100283921(D - 3) t^2} \times \left( 128100283921 D t^2 + 463852656 D t + 419904 D - 256200567842 t^2 - 856838934 t - 839808 \right) \\
&\quad + \frac{s_{02}s_{03}}{384300851763(D - 3) t^2} \times \left( 2818206246262 D t^2 + 6493937184 D t + 2519424 D - 7686017035260 t^2 - 16273497348 t - 5038848 \right) \\
&\quad + \frac{s_{01}^2}{10376122997601(D - 3) t^2} \times \left( 2546761744633401 D t^2 + 4709032163712 D t + 2176782336 D + 7794245122136352 t^3 - 3722722351028181 t^2 - 8800212589632 t - 4353564672 \right) \\
&\quad + \frac{s_{02}^2 s_{01}}{10376122997601(D - 3) t^2} \times \left( -119210124216826 D t^2 - 1690742931120 D t - 544195584 D - 458485007184490310 t^4 + 1496467516765122 t^2 + 2876350319856 t + 1088391168 \right) \\
&\quad + \frac{s_{02}^2}{10376122997601(D - 3) t^2} \times \left( 139501209189969 D t^2 + 137764238832 D t + 34012224 D + 91697001436898062 t^3 - 25363856216358 t^2 - 69403953654 t - 68024448 \right) \\
&\quad + \frac{s_{00}}{228705129(D - 3) t} \times \left( -128100283921 t^2 - 534719034 t - 419904 \right), \quad (A.3a)
\end{align*}
\]
\[ \sum_{k=1}^{n} f(k) \]

\[ C^{(2)}_{4 \rightarrow 4;3} = \frac{s_{02}s_{01}}{16238064159(D-3)t^2} \times \left( 2145318534Dt + 839808D + 1281002839210t^2 - 3652839666t - 1679616 + \frac{(357911t + 648)s_{00}}{3457911(D-3)} \right) + \frac{s_{03}s_{01}}{1804229351(D-3)t^2} \times \left( 100930902Dt + 93312D - 251253522t - 186624 \right) + \frac{s_{02}^2}{16238064159(D-3)t^2} \times \left( - 183608343Dt - 52488D - 256200567842t^2 + 57981582t + 104976 \right) + \frac{s_{01}^2}{16238064159(D-3)t^2} \times \left( - 5411614320Dt - 3359232D - 2177704826657t^2 + 10436684760t + 6718464 \right) + \frac{s_{02}s_{03}}{1804229351(D-3)t^2} \times \left( -3221199Dt + 5832D + 6442398t + 11664 \right) + \frac{2s_{02}^2}{1804229351(D-3)t^2} \times \left( -3262126Dt - 11664D + 64423980t + 23328 \right) + \frac{s_{02}s_{03}}{1804229351(D-3)t^2} \times \left( -3262126Dt - 11664D + 64423980t + 23328 \right) + \frac{s_{02}s_{03}}{1804229351(D-3)t^2} \times \left( -3262126Dt - 11664D + 64423980t + 23328 \right)

\[ C^{(2)}_{4 \rightarrow 4;23} = \frac{s_{01}(48317985t + 10308)}{71t(715822t + 81)} - \frac{32s_{02}s_{01}}{71t} + \frac{s_{03}(715822t - 2592)}{71t} + \frac{2s_{02}^2}{71t} + \frac{162s_{03}}{71t(715822t + 81)} + \frac{4s_{02}s_{03}}{71t}. \]

For \( I_5^{(2)} \), we have

\[ C^{(2)}_{5 \rightarrow 5} = \frac{s_{04}s_{01}}{32933430093533811(D-4)t^2} \times \left( 14335728396558942Dt^2 + 23554798645968Dt + 801715968D - 514019810871625756t^2 - 65771935818336t - 2405147904 \right) + \frac{s_{04}}{645753531245761(D-4)t^2} \times \left( 645753531245761Dt^2 + 15003064624Dt + 8714304D - 1937260593737283t^2 - 29091292408t - 26142912 \right) + \frac{s_{00}}{552094181406(D-4)t^2} \times \left( 645753531245761t^2 + 309209334408t + 8714304 \right) + \frac{s_{04}^2}{32933430093533811(D-4)t^2} \times \left( 484315148434320750Dt^2 + 100670508862704Dt + 5158867968D - 1554974503239792488t^2 - 219630109481280t - 15476603904 \right) - \frac{s_{01}s_{04}}{10977810031177937(D-4)t^2} \times \left( 167895918123897860Dt^2 + 36157366074384Dt + 1934575488D - 52822638855032498t^2 - 75984992058960t - 5803726464 \right) + \frac{s_{01}s_{04}}{3023288825864038498(D-4)t^2} \times \left( 14321392665262383058Dt^2 + 13789009133462592Dt + 331910410752D - 1640968274064081113424t^3 - 55611777508599407112t - 43600316518303488t - 99573123256 \right) + \frac{s_{02}s_{01}}{1511644412932019249(D-4)t^2} \times \left( 483830833285886429250Dt^2 + 6765883293451296Dt + 2135771338752D + 8040744542913974577809t^3 - 1392037972235862072480t^2 - 183993225050551104t - 6407314016256 \right) \]
\[ C_{s_{03} s_{01}}^{(2)} = \frac{s_{03} s_{01}}{8!} \times \left( -\frac{55909340735257987380 D t^2 \times (D - 4) t^2}{-237128089551472 D t - 266971417344 D - 656387309625632454369643^2 + 172423941884993136132^2 + 22589794735742304 t + 809914252032) + \frac{s_{02}^2}{6!} \times \left( \frac{326912725193166562500 D t^2 + 595522136237504000 D t + 2748648533504 D - 96817181267807856920219^3 - 1150328292667937573096^2 - 1435853145074044608 t - 82459345600512) + \frac{s_{03}^2}{4!} \times \left( 39287644840992099924 D t^2 + 7793787771875520 D t + 3865281825024 D + 16409682740640811334247 t^3 - 1277220411366240783636 t^2 - 174378410197442208 t - 11595845747072) + \frac{s_{02} s_{03}}{2!} \times \left( -\frac{377765815778770185000 D t^2 - 7210919027533120 D t - 3435806066688 D + 16409682740640811334247 t^3 + 1264318255811950478856 t^2 + 166745623518606528 t + 1030741820064 t) , \right) \right) \] (A.4a)

\[ C_{s_{03} s_{01}}^{(2)} = \frac{s_{03} s_{01}}{6!} \times \left( \frac{3703193448768 D t + 112435776 D - 645753531245761 t^2 - 13919908971456 t - 337307328) + \frac{s_{02} s_{01}}{4!} \times \left( 33381800628840 D t + 144699952 D + 63283846062084578 t^2 - 113719508702928 t - 4340998656) + \frac{s_{03} s_{01}}{2!} \times \left( -\frac{4023685569540 D t - 180874944 D - 5166028249966088 t^2 + 13937595501432 t + 542624832) + \frac{s_{04} s_{01}}{0!} \times \left( 33848359092 D t + 1629504 D - 121366188456 t - 4888512) + \frac{s_{02}^2}{2!} \times \left( 122937240222144 D t + 9311127552 D - 3809945834399899 t^2 - 416021948215128 t - 27933382656) + \frac{s_{03}^2}{4!} \times \left( 15130216514124 D t + 1309377312 D + 645753531245761 t^2 - 49174347196548 t - 3928131936) + \frac{s_{04}^2}{1!} \times \left( 7775974386 D t + 903312 D - 23327923158 t - 2709936) + \frac{s_{02} s_{03}}{2!} \times \left( -\frac{14408118186828 D t - 1163890944 D + 645753531245761 t^2 + 47688373738392 t + 3491672832) + \frac{s_{02} s_{04}}{1!} \times \left( 114352564500 D t + 10485504 D - 367147967088 t - 31456512) + \frac{s_{03} s_{04}}{1!} \times \left( -1189266667080 D t - 11796192 D + 37416159104 t + 35388576) + \frac{s_{00} (25411681 t + 2952) \times \left( 25411681 (D - 4) t^2 , \right) \right) \right) \right) \] (A.4b)
\[ C^{(2)}_{5 \to 5;3} = \frac{s_{01}^2(1568612244768t - 8455536)}{85697t(25411681t + 443556)} + \frac{s_{02}s_{01}(-760215848796t - 108819072)}{85697t(25411681t + 443556)} + \frac{276s_{03}s_{01}}{85697t} + \frac{999s_{02}^2}{85697t} + \frac{1776s_{02}s_{03}}{85697t} + \frac{s_{04}s_{01}(-3659282064t - 1102896)}{5041t(25411681t + 443556)} + \frac{s_{02}s_{04}(457410258t - 7096896)}{5041t(25411681t + 443556)} + \frac{18s_{03}s_{04}}{5041t}. \] (A.4c)

When \( t = \det(G) \rightarrow 0 \), the basis with the highest typology is decomposed to the combinations of lower ones. Let us consider the coefficients \( B_{4;3;\hat{3}} \) and \( B_{5;4;\hat{4}} \) by (3.15) for different \( m \):

- \( B_{4;3;\hat{3}} \)
  - \( m = 2 \):
    \[ B_{4;3;\hat{3}} = -\frac{C^{(2)}_{4 \to 4;\hat{3}3}}{C^{(2)}_{4 \to 4} + O\left(t^2\right)} = \frac{71}{72} - \frac{25411681(36D - 61)t}{1679616(D - 2)} + O\left(t^2\right). \] (A.5)

- \( m = 3 \):
  \[ B_{4;3;\hat{3}} = -\frac{C^{(3)}_{4 \to 4;\hat{3}3}}{C^{(3)}_{4 \to 4} \hat{3}3} + O\left(t^3\right) = \frac{71}{72} - \frac{25411681(36D - 61)t}{1679616(D - 2)} + \frac{9095120158391(432D^2 - 468D + 121)t^2}{13060694016((D - 2)D)} + O\left(t^3\right). \] (A.6)

- \( m = 4 \):
  \[ B_{4;3;\hat{3}} = -\frac{C^{(4)}_{4 \to 4;\hat{3}3}}{C^{(4)}_{4 \to 4} \hat{3}3} + O\left(t^4\right) = \frac{71}{72} - \frac{25411681(36D - 61)t}{1679616(D - 2)} + \frac{9095120158391(432D^2 - 468D + 121)t^2}{13060694016((D - 2)D)} \]
  \[ - \frac{3255243551009881201(15552D^3 + 28512D^2 + 16596D + 50215)t^3}{30467987005248D(D^2 - 4)} + O\left(t^4\right). \] (A.7)

- \( B_{5;4;\hat{4}} \)
  - \( m = 2 \):
    \[ B_{5;4;\hat{4}} = -\frac{C^{(2)}_{5 \to 5;\hat{4}4}}{C^{(2)}_{5 \to 5} \hat{4}4} + O\left(t^2\right) = \frac{1207}{164} - \frac{30671898967(41D - 36)t}{19849248(D - 3)} + O\left(t^2\right). \] (A.8)
To make them look more intuitive, we choose $D = 4$, and here we list some results in tables 1 and 2.

From these numerical results, one can see that as the series of $t$, $B_{4; 3}$ and $B_{5; 4}$ are independent of the choice of rank $m$. And they are also independent of the auxiliary vector $R$, although $s_{0i} = R \cdot K_i$ and $s_{00} = R \cdot R$ appear in the expression of $C_{4 \rightarrow 4; 3}^{(3)}$, $C_{4 \rightarrow 4}^{(3)}$, etc.
We can also use (3.18) to derive $F_{n+1}^{(m)}$. Some results are:

\begin{align}
F_{4;3}^{(3)} &= \frac{(-1239447D^2 - 1774432D + 652916) s_{03}^2 s_{01}^2}{23328D(D^2 - 4)} + \frac{(-197583D - 206184) s_{03}^2 s_{01}^2}{46656D(D^2 - 4)} \\
&+ \frac{(862224D^2 + 1492491D - 274912) s_{02} s_{03} s_{01}}{23328D(D^2 - 4)} - \frac{94501 s_{03}^2}{559872D(D^2 - 4)} \\
&+ \frac{(274912D + 455323) s_{02} s_{03}^2}{186624D(D^2 - 4)} + \frac{(-1199616D^2 - 2674144D - 644325) s_{02}^2 s_{03}}{186624D(D^2 - 4)} \\
&+ \frac{(-7774713D^3 - 9053552D^2 + 13507608D + 2542936) s_{01}^3}{17496D(D^2 - 4)} \\
&+ \frac{(10816992D^3 + 12443673D^2 - 19600544D - 3195852) s_{02} s_{01}^2}{23328D(D^2 - 4)} \\
&+ \frac{(-7524864D^3 - 10046784D^2 + 11389039D + 2955304) s_{02}^2 s_{01}}{46656D(D^2 - 4)} \\
&+ \frac{(1046936D^3 + 17994240D^2 - 7538496D - 3393445) s_{02}^3}{559872D(D^2 - 4)} \\
&+ \frac{(759D - 242) s_{00} s_{01}}{108(D - 2)D} + \frac{(121 - 1056D) s_{00} s_{02}}{432(D - 2)D} + \frac{121 s_{00} s_{03}}{432(D - 2)D}, \quad (A.11a)
\end{align}

\begin{align}
F_{4;23}^{(3)} &= \frac{(-22265592D^2 + 225327659D + 51781152) s_{02} s_{03} s_{01}}{2916D(D^2 - 4)} \\
&+ \frac{(159295600D^2 - 46664537D - 85762533) s_{02} s_{03}^2}{23328D(D^2 - 4)} + \frac{(28045D - 22791) s_{00} s_{03}^2}{54(D - 2)D} \\
&+ \frac{(887216D^2 - 259919001D + 121362075) s_{02} s_{03}^2}{23328D(D^2 - 4)} + \frac{(781D - 22791) s_{00} s_{02}}{54(D - 2)D} \\
&+ \frac{(105390974D^3 - 444555708D^2 - 186234704D - 478975656) s_{01}^3}{2187D(D^2 - 4)} \\
&+ \frac{(-800067D^3 + 607223737D^2 - 1031570076D + 601955892) s_{02} s_{01}^2}{2916D(D^2 - 4)} \\
&+ \frac{(-752051667D^3 + 144661577D^2 + 657043940D - 122980236) s_{03} s_{01}^2}{2916D(D^2 - 4)} \\
&+ \frac{(5565264D^3 + 9169570D^2 + 362503351D - 556647384) s_{02} s_{01}}{5832D(D^2 - 4)} \\
&+ \frac{(269794320D^3 + 32842115D^2 - 349326177D + 38835864) s_{03} s_{01}}{5832D(D^2 - 4)} \\
&+ \frac{(-7742976D^3 - 15727920D^2 - 3271609D + 639173595) s_{02}^3}{69984D(D^2 - 4)} \\
&+ \frac{(-193574400D^3 - 97997040D^2 + 264828935D + 17799771) s_{03}^2}{69984D(D^2 - 4)} \\
&+ \frac{(91164 - 79378D) s_{00} s_{01}}{27(D - 2)D}, \quad (A.11b)
\end{align}
$$F^{(3)}_{4,123} = \frac{(117036897D^2 + 80887886D + 52970828) s_{01}^3}{486(D + 2)} - \frac{5041s_{00}s_{01}}{3D}$$

$$+ \frac{(-8231953D^2 + 40086032D - 66571446) s_{02}s_{01}^2}{648(D + 2)} + \frac{5041s_{00}s_{02}}{24D}$$

$$+ \frac{(-79814153D^2 - 62992336D + 13600618) s_{03}s_{01}^2}{648(D + 2)} + \frac{5041s_{00}s_{03}}{24D}$$

$$+ \frac{(22906304D^2 + 79098331D + 246242768) s_{02}s_{01}^2}{5184(D + 2)}$$

$$+ \frac{(114531520D^2 + 130637515D - 17179728) s_{03}s_{01}^2}{5184(D + 2)}$$

$$+ \frac{(-1206875892D^2 - 5693290277D - 9613489460) s_{02}^3}{2115072(D + 2)}$$

$$+ \frac{(-10269182412D^2 - 15210859589D - 1023625460) s_{03}^3}{8087040(D + 2)}$$

$$+ \frac{(-10021508D^2 + 107731211D + 75877132) s_{02}s_{03}^2}{41472(D + 2)}$$

$$+ \frac{(10021508D^2 + 56192027D - 107373300) s_{02}^2s_{03}}{41472(D + 2)}$$

$$+ \frac{(-101288813D - 22906304)s_{02}s_{03}s_{01}}{2592(D + 2)}$$

(A.11c)

$$F^{(3)}_{5,4} = \frac{s_{01}^3}{88197652332(D - 3)(D - 1)} \times (-1343847503375D^3 + 10417729721235D^2 -$$

$$17337620658673D - 1794280981875) + \frac{s_{02}s_{01}^3}{29399217444(D - 3)(D^2 - 1)} \times$$

$$(-1633291581025D^3 + 12077972337027D^2 - 68514754134455D +$$

$$93475183460901) + \frac{s_{03}s_{01}^2}{9799739148(D - 3)(D^2 - 1)} \times (392816962525D^3$$

$$- 4603678621302D^2 + 27041270862119D - 33850184121582)$$

$$+ \frac{(14623481275D^2 - 89044148172D + 94194083969) s_{03}s_{01}^2}{192151748(D - 3)(D^2 - 1)}$$

$$+ \frac{s_{02}s_{01}}{29399217444(D - 3)(D - 1)} \times (-1985077460015D^3 + 30582780834201D^2 -$$

$$243862525372933D + 854168652203259) + \frac{(580029 - 231855D)s_{00}s_{01}}{28577(D - 3)(D - 1)} \times$$

$$+ \frac{(501984059 - 1432168335D)s_{02}s_{01}^2}{565122(D - 3)(D^2 - 1)} + \frac{s_{03}s_{01}}{3266579716(D - 3)(D^2 - 1)} \times$$

$$(-114823491815D^3 + 2531886930939D^2 - 34017449607547D +$$

$$12323629454467) + \frac{s_{02}s_{03}s_{01}}{4899869574(D - 3)(D^2 - 1)} \times (477423692915D^3 -$$

$$8829762425970D^2 + 90569439300937D - 324098013014706)$$

$$+ \frac{(17773154165D^2 - 276974011536D + 97844996715)s_{02}s_{03}s_{01}}{96075874(D - 3)(D^2 - 1)}$$
\[ + \frac{(\mathcal{F}^{(3)})_{5,34}}{s_{03}^2 s_{04}^2 s_{01}} = \frac{36248334956758812852}{2272673007443171457382D^2 + 12750306686537356621621D} \times \left( \frac{829943177977556629537 D^3}{2272673007443171457382D^2 + 12750306686537356621621D} \right) \]

\[ + \frac{25457369478990513141690}{81640394046754083(D - 3)(D^2 - 1)} \times \left( \frac{284755975770333178999 D^3 - 90607604692549882710D^2 + 107185135379627366777D - 85969113572839070718}{2272673007443171457382D^2 + 12750306686537356621621D} \right) \]

\[ + \frac{138355148225398767416}{473834443879200168(D - 3)(D^2 - 1)} \times \left( \frac{2027743034533980625 D^3 - 47755513091825901372D^2 + 143945054906768879195D - 118509014188920876912}{2272673007443171457382D^2 + 12750306686537356621621D} \right) \]

\[ + \frac{4412994272797518(D - 3)(D^2 - 1)}{95273008629079876D^3 - 5459007779794224849D^2 + 34251474265548951092D - 68798575805871096567} \]

\[ + (\mathcal{F}^{(3)})_{5,34} \]
\(+ \frac{s_{01}^2 s_{01}}{217925643101112(D - 3)(D^2 - 1)} \times (14312658358171485D^3 - 308183885271005346D^2 + 2120141865244853789D - 4345132269866178904) + \frac{s_{01}^2 s_{01}}{9290871448611768(D - 3)(D^2 - 1)} \times (-840238182589725D^3 - 22834045823278810D^2 + 2175040781006931191D - 4392791821329079248) + \frac{(29997586789 - 30664084565)s_{00} s_{01}}{1906477614(D - 3)(D - 1)} + \frac{s_{00} s_{01}}{98066539395004(D - 3)(D^2 - 1)} \times (-178531580572981155D^3 + 276074479243558454D^2 - 1689939244552068877D + 34463496611388041614) + \frac{s_{00} s_{01}}{6403168160529732(D - 3)(D^2 - 1)} \times (4488869897440075D^3 - 228148167651259412D^2 + 17131774468846926949D - 34844020585587415026) + \frac{s_{00} s_{01}}{355731564473874(D - 3)(D^2 - 1)} \times (134329873732898840D^2 - 107585008373373373D + 2183705381853832549) + \frac{s_{01}^3}{178905173221521(D - 3)(D^2 - 1)} \times (10367549408254951D^3 - 244395249217448376D^2 + 2493217982690141465D - 669587407601568024) - \frac{s_{01}^3}{65443360664(D - 3)(D^2 - 1)} \times (12563664070467D^3 - 48025860687861D^2 + 11117705113357806D - 37576216670725964) + \frac{(286248073 - 76106817D)s_{00} s_{03}}{5725158(D - 3)(D - 1)} + \frac{s_{02}^3}{9290871448611768(D - 3)(D^2 - 1)} \times (591889764258953D^3 + 1861586631992809D^2 + 438017795639564794D - 2088530127407051460) + \frac{s_{02}^3}{5889882245076(D - 3)(D^2 - 1)} \times (470145534426423D^3 - 16661705329346940D^2 + 279904962791498257D - 883445043516938732) + \frac{s_{02}^3}{251104633746264(D - 3)(D^2 - 1)} \times (-186005811461743D^3 + 122956210936788D^2 + 23877566266263895D - 941769244106471412) - \frac{(-378033081851037D^2 - 28380809735275779D + 11955475493102352)s_{03} s_{04}}{27900514860696(D - 3)(D^2 - 1)} + \frac{s_{03} s_{04}}{53008940213784(D - 3)(D^2 - 1)} \times (-586444692942329D^3 + 183078476487862324D^2 - 235932772862219111D + 689558393538277052) + \frac{s_{03} s_{04}}{34611797886472(D - 3)(D^2 - 1)} \times (993713336121961D^3 - 104743917618759156D^2 + 219183989769601319D - 7146277328453200380) + \frac{(-467242505501283D^2 + 30910888602825155D - 116182101038199856)s_{03} s_{04}}{4273051825512(D - 3)(D^2 - 1)} \)
\[ F_{5,234}^{(3)} = \frac{s_{01}^2}{16806910847891893632(D^2 - 1)} \times \left( 45979283969009274300559 D^2 + 104703235731288452901 D - 46656230678599317469812 \right) \]
\[ + \frac{s_{02}s_{01}}{3376751499168(D^2 - 1)} \times \left( -7401414658284832 D^2 + 29506403105945531 D + 36745597196241525 \right) \]
\[ + \frac{s_{03}s_{01}}{177091429734144(D^2 - 1)} \times \left( -900085462820883747 D^2 - 17802547440638046161 D + 3064860211892055028 \right) \]
\[ + \frac{s_{04}s_{01}}{3723697983359232(D^2 - 1)} \times \left( 16379947016073091 D^2 + 59822244103749657 D - 68859179790347356 \right) \]
\[ + \frac{s_{02}s_{01}}{114079498308(D^2 - 1)} \times \left( -46832281406738 D^2 + 487441428989732 D - 919312210645983 \right) \]
\[ + \frac{s_{03}s_{01}}{10636152731136(D^2 - 1)} \times \left( 5962412169596571 D^2 + 57241427343714135 D - 108319299169698748 \right) \]
\[ + \frac{s_{04}s_{01}}{248246532239488(D^2 - 1)} \times \left( 20257440638046161 D + 3064860211892055028 \right) \]
\[ + \frac{s_{02}s_{01}}{248246532239488(D^2 - 1)} \times \left( 188034785817840 D^2 - 970393447937245 D + 1748833608329821 \right) \]
\[ + \frac{s_{03}s_{01}}{101403998496(D^2 - 1)} \times \left( -114853210406800 D^2 + 975404680624469 D - 1817277951053637 \right) \]
\[ + \frac{s_{04}s_{01}}{662108460768(D^2 - 1)} \times \left( 66634569689337 D^2 - 976812872646037 D + 1863724399749532 \right) \]
\[ + \frac{s_{02}s_{01}}{588540854016(D^2 - 1)} \times \left( 2271838998767 D^2 + 36591900080873 D - 87535035935325 \right) \]
\[ + \frac{s_{03}s_{01}}{4624844526(D^2 - 1)} \times \left( 11671435834413 D^2 - 1899510850309495 D + 5546345527961140 \right) \]
\[ + \frac{s_{04}s_{01}}{191642391552(D^2 - 1)} \times \left( 16098192433184611 D^2 + 144789759004827290 D - 56536411032193004 \right) \]
\[ + \frac{s_{03}s_{01}}{4964930644478976(D^2 - 1)} \times \left( -990342600288 D^2 + 33466366766187 D - 91942066376347 \right) \]
\[ + \frac{s_{02}s_{01}}{1218066048(D^2 - 1)} \times \left( 16181230755488 D^2 + 5007666963933835 D - 1664193721989627 \right) \]
\[ + \frac{s_{03}s_{01}}{88281280124(D^2 - 1)} \times \left( -31681674345033 D^2 - 481937156164675 D + 1698523705808332 \right) \]
\[ + \frac{s_{04}s_{01}}{784721138688(D^2 - 1)} \times \left( 772076303514 D^2 - 17465983786156 D + 44961327781741 \right) \]
\[ + \frac{s_{02}s_{01}}{685162152(D^2 - 1)} \times \left( -317871093958 D^2 + 16432322501716 D - 46449248275737 \right) \]
\[ + \frac{s_{04}s_{01}}{4473705816(D^2 - 1)} \times \left( 1961957761174202 D^2 - 64920756785680566 D + 228237749746197820 \right) \]
In this part, we give the related

\[ \frac{(82996618209D^2 + 3029142183827D - 96937318177204)}{7069559808(D^2 - 1)} + \frac{(94917907364339 - 31480868378483D)}{3976627392(D^2 - 1)}s_{02}s_{03}s_{04} = \frac{115943800s_{01}}{49284(D - 1)} \]

\[ - \frac{5041s_{02}s_{03}}{333(D - 1)} + \frac{5041s_{00}s_{03}}{296(D - 1)} - \frac{85697s_{00}s_{04}}{32856(D - 1)} \]  

(A.1f)

\[ F^{(3)}_{5,1234} = \frac{25411681(D - 2)(46s_{01} + 296s_{02} - 333s_{03} + 51s_{04})^3}{13384745856(D + 1)} \]  

(A.1g)

These results have been checked with FIRE6 [42].

B Divergence of reduction coefficients of two-loop sunset topology

In this part, we give the related \(C_i, C_j\) coefficients for rank level 3 with the chosen tensor structure, which have been used to solve \(B\) by (3.46).

- The pole of \(s_{11}^{-3}\)

\[
\begin{align*}
\begin{pmatrix} C_2^{(3,0)} \\ C_3^{(3,0)} \\ C_4^{(3,0)} \end{pmatrix} s_{11}^3 &= \begin{pmatrix} (D+2)(4(1-2D)M^2_{12,3}+(3D-2)M^2_{12,3}+8(D-1)M^2_{12,3})s_{01}^3 \\ (D-1)(3D-2) \\ (D-1)(3D-2) \end{pmatrix}, \\
\begin{pmatrix} C_2^{(2,1)} \\ C_3^{(2,1)} \\ C_4^{(2,1)} \end{pmatrix} s_{11}^3 &= \begin{pmatrix} 2(D+2)M^2_{12,3}(5D-4)M^2_{12,3}+4(1-2D)M^2_{12,3} \\ (D-1)(3D-2) \\ (D-1)(3D-2) \end{pmatrix}s_{01}s_{02}s_{03}, \\
\begin{pmatrix} C_2^{(1,2)} \\ C_3^{(1,2)} \\ C_4^{(1,2)} \end{pmatrix} s_{11}^3 &= \begin{pmatrix} 2(D+2)M^2_{12,3}(2D-1)M^2_{12,3} - 2(D-1)M^2_{12,3} \\ (D-1)(3D-2) \\ (D-1)(3D-2) \end{pmatrix}s_{01}s_{02}s_{03}, \\
\begin{pmatrix} C_2^{(0,3)} \\ C_3^{(0,3)} \\ C_4^{(0,3)} \end{pmatrix} s_{11}^3 &= \begin{pmatrix} 2(D+2)M^2_{12,3}(4D-1)M^2_{12,3} \\ (D-1)(3D-2) \\ (D-1)(3D-2) \end{pmatrix}s_{01}s_{02}s_{03}, \\
\begin{pmatrix} C_2^{(r_1,r_2)} \\ C_3^{(r_1,r_2)} \\ C_4^{(r_1,r_2)} \end{pmatrix} s_{11}^3 &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \forall r_1 + r_2 = 3. \end{align*}
\]  

(B.2)
The pole of $s_{11}^{-2}$

\[
\begin{pmatrix}
C_2^{(3,0)} \\
C_3^{(3,0)} \\
C_4^{(3,0)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{4((-3D^2-5D+2)M_1^2+D(5D-2)M_2^2-D(D+2)M_2^2)}{(D-1)(3D-2)} \\
\frac{2(D+2)(2(5D+4)M_2^2+(4-5D)M_2^2-3DM_2^2)}{(D-1)(3D-2)} \\
\frac{2(5D^2+6D-8)}{3D^2-5D+2}
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(2,1)} \\
C_3^{(2,1)} \\
C_4^{(2,1)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{2(D+2)((2D-1)M_1^2+(2-4D)M_2^2-M_3^2)}{(D-1)(3D-2)} \\
\frac{2(D^2-16D+12)M_2^2+(D+2)((4-3)M_2^2+M_1^2)}{(D-1)(3D-2)} \\
\frac{-5D^2-5D+2}{3D^2-5D+2}
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(1,2)} \\
C_3^{(1,2)} \\
C_4^{(1,2)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{2(4D^2+5D-6)M_1^2+(2D^2-16D+12)M_2^2+(D+2)M_2^2}{(D-1)(3D-2)} \\
\frac{2(D+2)((4-2D)M_2^2+(1-2D)M_2^2+M_1^2)}{(D-1)(3D-2)} \\
\frac{2(5D^2+6D-8)}{3D^2-5D+2}
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(0,3)} \\
C_3^{(0,3)} \\
C_4^{(0,3)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{-2(D+2)((5D-4)M_2^2+8-10D)M_2^2+3DM_2^2}{(D-1)(3D-2)} \\
\frac{-4(3D^2-5D+2)M_2^2+(D+2)M_2^2+(2-5D)M_2^2}{(D-1)(3D-2)} \\
\frac{-2(5D^2+6D-8)}{3D^2-5D+2}
\end{array} \right),
\]

\[\text{(B.3)}\]

\[
\begin{pmatrix}
C_2^{(3,0)} \\
C_3^{(3,0)} \\
C_4^{(3,0)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{-2(D+2)((7D-4)M_1^2+2M_2^2((D+4)M_2^2-(2-5D)M_3^2)-(M_2^2-M_3^2)((5D-4)M_2^2+3DM_2^2))}{(D-1)(3D-2)} \\
\frac{-4(D+2)((2D-1)M_2^2+2M_2^2(M_2^2+M_3^2))}{(D-1)(3D-2)} \\
\frac{-2(D+2)((3D^2+2D-4)M_2^2+D(4-5D)M_2^2-3DM_2^2)}{(D-1)(3D-2)} \\
\frac{2(D+2)((7D-10D+6)M_2^2+D(4-5D)M_2^2-3DM_2^2)}{(D-1)(3D-2)} \\
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(2,1)} \\
C_3^{(2,1)} \\
C_4^{(2,1)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{-2(D+2)((2D-1)M_1^2+2M_2^2((D+1)M_2^2+(2-4D)M_3^2)-(M_2^2-M_3^2)((4D-3)M_2^2+M_1^2))}{(D-1)(3D-2)} \\
\frac{2(D+2)((2D-1)M_2^2+2M_2^2(M_2^2+M_3^2))}{(D-1)(3D-2)} \\
\frac{-2(D+2)((4-5D)M_2^2+D(4-3)M_2^2+M_1^2)}{(D-1)(3D-2)} \\
\frac{2(D+2)(2D-1)M_2^2+(3-4D)M_2^2}{(D-1)(3D-2)} \\
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(1,2)} \\
C_3^{(1,2)} \\
C_4^{(1,2)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{-2(D+2)((4D-3)M_1^2+2M_2^2((D+1)M_2^2+(2-4D)M_3^2)+(1-2D)M_1^2+2DM_2^2M_2^2-M_3^2)}{(D-1)(3D-2)} \\
\frac{-2(D+2)((4D-3)M_1^2+4-5D)M_2^2+DM_2^2}{(D-1)(3D-2)} \\
\frac{-2(2D+1)((7D-4)M_1^2+D(1-2D)M_2^2+M_1^2)}{(D-1)(3D-2)} \\
\frac{2(D+2)((4D-3)M_1^2+(3-2D)M_2^2+M_1^2)}{(D-1)(3D-2)} \\
\end{array} \right),
\]

\[
\begin{pmatrix}
C_2^{(0,3)} \\
C_3^{(0,3)} \\
C_4^{(0,3)}
\end{pmatrix}
\bigg|_{s_{01}^3 s_{11}^{-2}} = \left( \begin{array}{c}
\frac{2(D+2)((5D-4)M_2^2+2M_2^2((D+4)M_2^2+(2-4D)M_3^2)-(M_2^2-M_3^2)((7D-4)M_2^2-3DM_2^2))}{(D-1)(3D-2)} \\
\frac{2(D+2)((3D^2-2D+4)M_2^2+3DM_2^2+D(5D-4)M_2^2)}{(D-1)(3D-2)} \\
\frac{4(D+2)(2D-1)(M_2^2-2M_2^2-M_3^2)}{(D-1)(3D-2)} \\
\frac{-2(D+2)((7D^2+10D-4)M_2^2+3DM_2^2+D(5D-4)M_2^2)}{(D-1)(3D-2)} \\
\end{array} \right),
\]

\[\text{(B.4)}\]
• The pole of $s_{11}^{-1}$

\[
\begin{bmatrix}
C_s^{(3,0)} \\
C_s^{(2,1)} \\
C_s^{(2,1)} \\
C_s^{(2,1)}
\end{bmatrix}
\begin{bmatrix}
s_{01}s_{11}^{-1}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{13D^2+16D-20}{(D-1)(3D-2)} \\
\frac{2(2D^2-9D+6)}{3D^2-5D+2} \\
\frac{2(2D^2-9D+6)}{3D^2-5D+2} \\
0
\end{bmatrix},
\]

(B.5)

\[
\begin{bmatrix}
C_s^{(3,0)} \\
C_s^{(3,0)} \\
C_s^{(3,0)} \\
C_s^{(3,0)}
\end{bmatrix}
\begin{bmatrix}
s_{01}s_{11}^{-1}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{2(5D^2+6D-8)}{(D-1)(3D-2)} \\
\frac{13D^2+16D-20}{(D-1)(3D-2)} \\
0
\end{bmatrix},
\]

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