HYPERCENTRAL UNIT GROUPS AND THE HYPERBOLICITY OF A MODULAR GROUP ALGEBRA

E. IWAKI AND S.O. JURIAANS

Abstract. We classify groups $G$ such that the unit group $U_1(ZG)$ is hypercentral. In the second part, we classify groups $G$ whose modular group algebra has hyperbolic unit groups $V(KG)$.

1. Introduction

We denote by $\Gamma = U_1(ZG)$ the group of units of augmentation one of the integral group ring $ZG$ of $G$. $Z_n(\Gamma)$ will denote the $n$-th centre of $\Gamma$ e we define $Z_{\infty}(\Gamma) = \bigcup_{n \in \mathbb{N}} Z_n(\Gamma)$. An element in $Z_{\infty}(\Gamma)$ is called an hypercentral unit.

In the finite group case, Arora, Hales and Passi in [1] showed that the central height of $\Gamma$ is at most 2, that is, $Z_{\infty}(\Gamma) = Z_2(\Gamma)$. Arora and Passi in [2] then proved that $Z_{\infty}(\Gamma) = Z(\Gamma)T$, where $T$ denotes the torsion subgroup of $Z_{\infty}(\Gamma)$. These results were extended to torsion groups by Li [10] and Li, Parmenter [11]. In [12], [13] they presented some contributions to the problem for non-periodic groups. In [6, Chapter VI] Hertweck extended these results to group rings $RG$ of periodic groups $G$ over $G$–adapted rings $R$. From its exposition is clear that the containment of $Z_{\infty}(\Gamma)$ in the normalizer $N_{\Gamma}(G)$ is an important property. We present our contribution to the study of the hypercentral units in [8]. Among many other results in [8] it is proved that the containment of $Z_{\infty}(\Gamma)$ in $N_{\Gamma}(G)$ holds for an arbitrary group $G$. The support of an hypercentral unit is investigated and it is proved that the normal closure of the group generated by an hypercentral unit is a polycyclic-by-finite group (in case, $G$ is finitely generated).

In [14], Polcino Milies classified finite groups such that the unit group of an integral group ring is nilpotent. This result was extended to arbitrary groups by Sehgal-Zassenhaus in [17]. Since nilpotent groups are hypercentral it is natural to consider the question of classify groups $G$ such that the group of units of an integral group ring $U_1(ZG)$ is hypercentral. This problem was posed by several leading experts in the field. In the second section we completely solve it as a natural consequence of our research about the hypercentral units of an integral group ring done in [8].

In sections 3, 4, 5 we deal with the topic of hyperbolic unit groups. In the context of hyperbolic unit groups, Juriaans, Passi, Prasad in [9] studied the groups $G$ whose unit group $U(ZG)$ is hyperbolic, classified the torsion subgroups of $U(ZG)$ and the polycyclic-by-finite subgroups of $G$. We consider the natural question of classifying groups $G$ for which the group of units with augmentation one of a modular group algebra, $V(KG)$, is hyperbolic.

Research partially supported by FAPESP-Brazil.

2000 Mathematics Subject Classification. Primary 16S34, 16U60, 20C07

Keywords and Phrases. group ring, unit, normalizer, hypercenter.
Notation is mostly standard and the reader is referred to [15, 16] for general results on group rings. For the theory of hyperbolic groups, we refer the reader to the reference [5].

2. Groups with Hypercentral Unit Group

Unless otherwise stated explicitly $G$ will always denote an arbitrary group $G$.

Firstly we recall a result proved in [8] which we will need in our investigations. It will also appear in [7].

**Lemma 2.1.** Let $u \in Z_n(\Gamma)$ and $v$ an element of finite order in $\Gamma$. If $c = [u, v] \neq 1$ then $u^{-1}vu = v^{-1}$, $v^2 \in G \cap Z_{n+1}(\Gamma) \subseteq Z_{n+1}(G)$, $o(v) = 2^m$, $m \leq n$, $v^{2^{n-1}}$ is central and if $n = 2$ then $m = 2$. In particular, elements of $\Gamma$ that are of finite order and whose order is not a power of 2 commute with $Z_\infty(\Gamma)$, and $C^2_\infty(\Gamma) \subseteq C_1(T(G))$, where $C_1(T(G))$ denotes the centralizer of $T(G)$ in $\Gamma$ and $T(G)$ denotes the set of torsion elements of $G$.

We need the following result proved in the context of nilpotent unit groups by Sehgal-Zassenhaus in [17].

**Lemma 2.2.** Suppose that $\Gamma$ is hypercentral and let $t, t_1, t_2 \in T = T(G), g \in G$.

1. Every finite subgroup of $G$ is normal in $G$.
2. If $g^{-1}tg \neq t$ then $g^{-1}tg = t^{-1}$.
3. If $t$ has odd order then $gt = tg$.
4. If $1 \neq t_1$ has odd order, $t_2$ has even order then $T$ is a central subgroup of $G$.

**Proof.**

Observe initially that since $\Gamma$ is hypercentral, we have that $G$ is hypercentral. Since $G \subseteq \Gamma = Z_\infty(\Gamma)$ it follows, by Lemma 2.1, that $g^{-1}tg \in \langle t \rangle$, for all $g \in G, t \in T$. Since every subgroup of $T$ is normal, $T$ is an abelian subgroup or an Hamiltonian subgroup.

2 and 3 follow immediately from Lemma 2.1.

4. Suppose $g \in G$ such that $g^{-1}t_1g = t_1$ and $g^{-1}t_2g = t_2^{-1}$. It follows that $g^{-1}t_1t_2g = t_1t_2^{-1}$ which should be equal to $t_1t_2$ or $(t_1t_2)^{-1}$. This implies that $t_1t_2^{-1} = t_1t_2$ and $t_2^2 = 1$. Hence $g^{-1}t_2g = t_2$. Finishing the proof.

We now state the main result of this section.

**Theorem 2.3.** $\Gamma = U_1(ZG)$ is hypercentral if and only if $G$ is hypercentral and the torsion subgroup $T$ of $G$ satisfies one of the following conditions:

(a) $T$ is central in $G$.
(b) $T$ is an abelian 2-group and for $g \in G, t \in T$

$$g^{-1}tg = t^{\delta(g)}, \delta(g) = \pm 1.$$  

(c) $T = K_8 \times E_2$, where $K_8$ denotes the quaternion group of order 8, $E_2$ is an elementary abelian 2-group. Moreover, $E_2$ is central and conjugation by $g \in G$ induces on $K_8$ one of the four inner automorphisms.

**Proof.**

$\Rightarrow$) Suppose that $\Gamma = Z_\infty(\Gamma)$. By Lemma 2.2 every subgroup of $T$ is normal in $G$ and $T$ is an abelian subgroup or a Hamiltonian subgroup.
Suppose that $\Gamma$ is hypercentral and $T$ is not central. In any case $T$ is abelian or a Hamiltonian group.

Suppose firstly that $T$ is a non-central Hamiltonian group with an element $x$ of odd order. Consider the subgroup $H = K \times \langle x \rangle$. In this case it follows by item (4) of Lemma 2.2 that $H$ is a central subgroup of $G$. Contradiction.

Suppose that $T$ is an abelian, non-central subgroup of $G$ and that $g^{-1}t_1g = t_1$, $g^{-1}t_2g = t_2^{-1}$ for some $t_1, t_2 \in T$. Then $g^{-1}t_1t_2g = t_1t_2^{-1}$. But by Lemma 2.1, $g^{-1}t_1t_2g$ must be equal to either $t_1t_2$ or $t_1^{-1}t_2^{-1}$. Hence either $t_1^2 = 1$ and $g^{-1}t_1g = t_2^{-1}$, $i = 1, 2$ or $t_2^2 = 1$ and $g^{-1}t_1g = t_1$. In any case we obtain that $g^{-1}t_1g = t_1^{(g)}$ and $g^{-1}t_2g = t_2^{(g)}$, $\delta(g) = \pm 1$. Also $T(G)$ is an abelian $2$–group by Lemma 2.2.

Denote by $K_8 = \langle i, j : i^2 = j^2 = u, u^2 = 1, ji = ij \rangle$. Every $g \in G$ maps every subgroup of $K_8$ onto itself and induces the identity in $K_8/(i^2)$. In fact only one of the four inner automorphisms

\begin{enumerate}
\item $i \rightarrow i, j \rightarrow j, ij \rightarrow ij$.
\item $i \rightarrow i, j \rightarrow ju, ij \rightarrow iju$.
\item $i \rightarrow iu, j \rightarrow ju, ij \rightarrow ij$.
\item $i \rightarrow iu, j \rightarrow j, ij \rightarrow iju$.
\end{enumerate}

arises.

$\quad (\Leftarrow)$ Under the hypothesis of the Theorem we must prove that $\Gamma$ is hypercentral. Since $G/T$ is an ordered group and $Q/T$ have no nilpotent elements, it follows by Theorem 45.7 of [16] that $\Gamma = U_1(ZT)G$.

We must consider three cases separately.

(1) Suppose (a) holds. In this case, since $U_1(ZT)$ is central and $G$ is hypercentral, the result follows.

(2) Suppose (b) holds. We claim that $U_1(ZT) \subseteq Z_\infty(\Gamma)$.

Let $u \in U_1(ZT), \; v = \tau x \in U, \; \tau \in U_1(ZT), \; x \in G$. Then

\[ [u, v] = [u, \tau x] = [u, x] = u^{-1}u^x =: \gamma, \quad \text{where} \quad \gamma = 1 \quad \text{or} \quad u^{-1}u^* . \]

To see this let $u = \sum_{g \in G} \alpha_g g \in U_1(ZT), x \in G$. So $u^x = \sum_{g \in G} \alpha_g x^{-1}gx$. Since $x$ centralizes the elements of $T$ or $x$ acts by inversion on the elements of $T$ we obtain in the first case that $u^x = u, \gamma = 1$ and in the second case we obtain that $u^x = u^*, \gamma = u^{-1}u^*$. Since $T$ is abelian, it follows that $\gamma^* = \gamma^{-1}$ and by Proposition 1.3 of [16] $\gamma = \pm t$, for some $t \in T$. Since $\gamma$ has augmentation 1, $\gamma \in T$. We conclude that

\[ [U_1(ZT), \Gamma] \subseteq T. \]

So

\[ [U_1(ZT), \Gamma, \Gamma] \subseteq [T, \Gamma] = [T, G]. \]

and

\[ [U_1(ZT), \Gamma, \Gamma] \subseteq [T, G, G]. \]

Continuing this process and using the fact that $G$ is a hypercentral group, we conclude that $U_1(ZT) \subseteq Z_\infty(\Gamma)$ and $\Gamma$ is hypercentral.
(3) Suppose \( (c) \). In this case, since \( T \) is an Hamiltonian 2–group. It is well known that in this case \( \mathcal{U}_1(ZT) \) has only trivial units. It follows that \( \mathcal{U}_1(ZT) = T \). Consequently, \( \Gamma = G \) is hypercentral.

\[ \square \]

3. Modular Group Algebras with Hyperbolic \( V(KG) \)

Let \( \mathbb{Z}^2 \) denote the free Abelian group of rank two, \( p \) be a rational prime, \( GF(p^n) \) will denote the Galois Field with \( p^n \) elements, \( tr.deg(K) \) denotes the transcendence degree of the field \( K \) over \( GF(p) \), \( \mathcal{U}(KG) \) denotes the group of units of \( KG \), \( V(KG) \) denotes the group of units of \( KG \) with augmentation one.

**Lemma 3.1.** Let \( G \) be an arbitrary group, \( K \) a field with \( char(K) = p > 0 \) and \( tr.deg(K) \geq 1 \). Suppose that \( g_0 \) is a torsion element of \( G \) and \( p \nmid o(g) \). Then \( \mathbb{Z}^2 \) embeds in \( V(KG) \) and consequently, \( V(KG) \) is not hyperbolic.

In what follow we investigate under which conditions the group of units of a modular group algebra of an arbitrary (non-trivial) group \( G \) is hyperbolic.

We denote by \( J(KG) \) the Jacobson Radical of \( KG \) and \( \omega(G) \) represents the augmentation ideal of \( KG \).

**Lemma 3.2.** Suppose that \( G \) is a finite (non-trivial) group and \( K \) is a field, \( char(K) = p > 0 \), \( tr.deg(K) \geq 1 \). Then \( V(KG) \) is not hyperbolic.

**Theorem 3.3.** Let \( G \) be a finite (non-trivial) group, \( K \) be a field, \( char(K) = p > 0 \). Under these conditions, \( V(KG) \) is hyperbolic if and only if \( K \) is a finite field.

**Theorem 3.4.** Let \( G \) be an arbitrary group with torsion, \( K \) be a field, \( char(K) = p > 0 \). If \( V(KG) \) is hyperbolic then \( K \) is algebraic over \( GF(p) \).

Our next Theorem considers the case in which \( G \) is an arbitrary (non-trivial) group, \( K \) a field of \( char(K) = p > 0 \) under the hypothesis that \( \mathcal{U}(KG) \) is hyperbolic.

**Theorem 3.5.** Let \( G \) be an arbitrary (non-trivial) group, \( K \) a field of \( char(K) = p > 0 \). If \( \mathcal{U}(KG) \) is hyperbolic then \( K \) is finite.

Acknowledgements: This work is part of the first authors Ph.D thesis. He would like to thank his thesis supervisor, Prof. Dr. Stanley Orlando Juriaans, for his guidance during this work.

**References**

[1] S. R. Arora, A. W. Hales, I. B. S. Passi, *Jordan decomposition and hypercentral units in integral group rings*, Comm. Algebra 21(1993), no.1,25-35.

[2] S. R. Arora, I. B. S. Passi, *Central height of the unit group of an integral group ring*, Comm. Algebra 21(1993), no.10, 3673-3683.

[3] A. A. Bovdi, *The periodic normal divisors of the multiplicative group of a group ring I*, Sibirsk Mat. Z. 9,(1968), no.3, 495-498.

[4] A. A. Bovdi, *The periodic normal divisors of the multiplicative group of a group ring II*, Sibirsk Mat. Z. 11,(1970), no.3, 492-511.

[5] Gromov, M.: *Hyperbolic groups*, In: Essays in group theory (S. M. Gersten, Ed.), Springer Verlag, MSRI Publ. 8, 1997, 75-263. MR 89e:20070.

[6] M. Hertweck, *Contributions to the integral representation theory of groups*, [http://elib.uni-stuttgart.de/opus](http://elib.uni-stuttgart.de/opus) 2003, Habilitationsschrift (autographed copy donated by the author).

[7] Martin Hertweck, E. Iwaki, E. Jespers and S. O. Juriaans, *On hypercentral units of integral group rings*, 2006, submitted.
[8] E. Iwaki, *Unidades hipercentrais em anéis de grupo inteiro e a hiperbolicidade do grupo de unidades de uma álgebra de grupo modular*, Ph.D. Thesis, IME-USP, 2006.

[9] S. O. Juriaans, I. B. S. Passi, D. Prasad., *Hyperbolic unit groups*, Proc. A.M.S 133, no. 2, (2005), 415-423.

[10] Y. Li, *The hypercentre and the n-centre of the unit group of an integral group ring*, Canad. J. Math 50(1998), no. 2, 401-411.

[11] Y. Li, M. M. Parmenter, *Hypercentral units in integral group rings*, Proc. Amer. Math. Soc. 129(2001), no. 8, 2235-2238 (electronic).

[12] Y. Li, M. M. Parmenter, *Some results on hypercentral units in integral group rings*, Comm. Algebra, 31(2003), no. 7, 3207-3217.

[13] Y. Li, M. M. Parmenter, *The upper central series of the unit group of an integral group ring*, Comm. Algebra, 33(2005), 1409-1415.

[14] C. P. Milies, *Integral group rings with nilpotent unit groups*, Canad. J. Math., 28:954-960, 1976.

[15] D. S. Passman, *The algebraic structure of group rings*, Robert E. Krieger Publishing Company, Malabar, Florida. Orig. Ed 1977, Reprint Ed. 1985 with corrections and appendix.

[16] S. K. Sehgal, *Units in integral group rings*, Longman Scientific & Technial, Harlow, 1993, With an appendix by A. Weiss.

[17] S. K. Sehgal, H. Zassenhaus., *Integral group rings with nilpotent unit groups*, Comm. Alg. 5:101-111, 1977.

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970 - Brazil

E-mail address: iwaki@ime.usp.br

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970 - Brazil

E-mail address: ostanley@ime.usp.br