ON THE LERAY PROBLEM FOR STEADY FLOWS IN
TWO-DIMENSIONAL INFINITELY LONG CHANNELS WITH SLIP
BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate the Leray problem for steady Navier-Stokes system with full slip boundary conditions in a two-dimensional channel with straight outlets. The existence of solutions with arbitrary flux in a general channel supplemented with slip boundary conditions, which tend to the associated shear flows at far fields, is established. Furthermore, if the flux is suitably small, the solution is proved to be unique. One of the crucial ingredients is to construct an appropriate flux carrier and to show a Hardy type inequality for flows with full slip boundary conditions.

1. Introduction

An interesting and important problem in mathematical fluid mechanics is to study the solutions of the steady Navier-Stokes system

\[
\begin{aligned}
- \Delta u + u \cdot \nabla u + \nabla p &= 0 \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{aligned}
\]

in a channel domain \(\Omega\), where the unknown function \(u = (u_1, \ldots, u_N)\) is the velocity and \(p\) is the pressure. If the boundary condition \(u \cdot n = 0\) is prescribed, then the flux

\[
\Phi = \int_{\Sigma} u \cdot \nu \, ds
\]

is a conserved quantity along each cross section \(\Sigma\) of the channel, where \(\nu\) is the unit normal of \(\Sigma\) pointing to the same direction. If \(\Omega\) is a channel type domain with straight outlets at far fields, in 1950s, Leray proposed the problem to look for solutions for Navier-Stokes system (1) with no-slip boundary conditions under the constraint that

\[
u \to U \quad \text{at far fields},
\]

where \(U\) is the shear flow solution of Navier-Stokes system with flux \(\Phi\) in the corresponding straight channel with no-slip boundary conditions. The problem is called Leray problem.

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nowadays. Without loss of generality, the flux $\Phi$ is always assumed to be nonnegative in this paper.

The major breakthrough on the Leray problem in infinitely long channels was made by Amick [2, 3], Ladyzhenskaya and Solonnikov [27]. It was proved in [2, 27] that Leray problem in a channel is solvable as long as the flux is small. Actually, the existence of solutions with arbitrary flux was also proved in [27]. However, the far field behavior and uniqueness of such solutions are not clear when the flux is large. The far field behavior of solutions was studied in [4]. One can refer to [16, 24, 34, 35] for the further studies on far field behavior of flows and the detailed progress on Leray problem. To the best of our knowledge, there is no result on the far field behavior of solutions of steady Navier-Stokes system with large flux except for the axisymmetric solutions in a pipe studied in [39].

For viscous flows near solid boundary, besides the no-slip boundary condition, the Navier boundary conditions

$$
\begin{align*}
    u \cdot n &= 0, \\
    (n \cdot D(u) + \alpha u) \cdot \tau &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

are also usually used, which were suggested by Navier [33] for the first time. Here $D(u)$ is the strain tensor defined by

$$(D(u))_{ij} = (\partial_{x_j}u_i + \partial_{x_i}u_j)/2,$$

and $\alpha \geq 0$ is the friction coefficient which measures the tendency of a fluid to slip over the boundary. $\tau$ and $n$ are the unit tangent and outer normal vector on the boundary $\partial \Omega$, respectively. If $\alpha = 0$, (3) is also called the full slip boundary conditions. If $\alpha \to \infty$, the boundary conditions (3) formally reduces to the classical no-slip boundary conditions.

The Navier-Stokes system with Navier slip boundary condition has been widely studied in various aspects. One may refer to [5, 8, 12, 14, 17, 20, 21, 23, 28, 38, 39] for some important results on nonstationary problem. For the stationary problem, the existence and regularity of the solutions were first studied in [37], where the Dirichlet condition and the full slip condition are imposed on different parts of the boundary of a three-dimensional interior or exterior domain. It is noteworthy that the existence and the regularity for solutions of a generalized Stokes system with Navier boundary conditions were investigated in [7] in some regular domain. The existence and uniqueness of very weak, weak, and strong solutions have been proved in appropriate Banach spaces in [10]. In [6], the existence, uniqueness, and regularity of solutions to the stationary Stokes system and also to the Navier-Stokes system with the full slip condition in both Hilbert space and $L^p$ space has been investigated. Recently, the stationary Stokes and Navier-Stokes system with nonhomogeneous Navier boundary conditions in a bounded three-dimensional domain were studied in [1], where the existence
and uniqueness for weak and strong solutions in $W^{1,p}$ and $W^{2,p}$ spaces have been established, respectively, even when the friction coefficient $\alpha$ is generalized to a function. Furthermore, the behavior of these solutions was also investigated when $\alpha$ tends to infinity ([1]). For more issues on the Navier slip boundary conditions, one may refer to [9, 13, 22].

For flows in a nozzle with Navier-slip boundary condition, the flux across each section is also a constant, and the associated Leray problem has been studied by [22, 26, 30, 32] and references therein. In the case of three-dimensional pipes with straight outlets, a weak solution of the Navier-Stokes system with arbitrary flux has been obtained in [26], which satisfies mixed boundary condition and the far field behavior (2). Very recently, Leray problem for flows in a pipe with Navier boundary condition was solved in [22], as long as the flux $\Phi$ is small and the nozzle becomes straight at large distance.

For flows in general two-dimensional channels with straight outlets, it was also proved in [31] that the Navier-Stokes system has a smooth solution with arbitrary flux if

$$\|\alpha - 2\chi\|_{L^\infty(\partial\Omega)} \leq C(\Omega),$$

where $\chi$ is the curvature of the boundary and $C(\Omega)$ is a constant depending only on $\Omega$. However, the far behavior is not known even when the flux is small. In [30], Leray problem (1)-(3) with friction coefficient $\alpha = 0$ was solved for any flux provided that the two-dimensional channel has straight upper boundary and coincides with the straight channel at far field. Then the exponential convergence rate of the velocity was studied in [32]. It’s worth noting that the Dirichlet norm of the solution is finite since the corresponding shear flow $U$ is a constant flow in the case $\alpha = 0$. The existence of solutions in a general two-dimensional channel, which may even have unbounded width, with Navier-slip boundary conditions was established in [36] when the friction coefficient $\alpha$ is positive. When the channel tends to be flat at far fields, the uniqueness and asymptotic behavior of solutions was also established when the flux is sufficiently small ([36]).

In this paper, we study Leray problem with full slip boundary conditions, i.e.

$$u \cdot n = 0, \quad n \cdot D(u) \cdot \tau = 0 \text{ on } \partial \Omega,$$

in a more general two-dimensional channel $\Omega$ (See Figure 1) of the form

$$\Omega = \{(x_1, x_2) : x_1 \in \mathbb{R}, \ f_1(x_1) < x_2 < f_2(x_1)\}.$$

Without loss of generality, assume that $f_1$ and $f_2$ are smooth functions satisfying

$$f_2(t) = 1 \quad \text{and} \quad f_1(t) = -1 \quad \text{for any } |t| \geq L,$$
where \( L \) is a constant. The straightforward computations show that the shear flows \( U \) in \( \Omega = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\} \) with full slip boundary conditions and flux \( \Phi \), i.e.,
\[
\int_{\Sigma} U \cdot \nu \, ds = \Phi,
\]
are of the form \( U = \frac{\Phi}{2} e_1 \).

![Diagram of the channel \( \Omega \)](image)

**Figure 1.** The channel \( \Omega \)

We consider the solution of the form \( u = v + g \), where \( v \in H^1(\Omega) \) and \( g \) is a smooth vector field satisfying
\[
\begin{align*}
\text{div} \ g &= 0 \quad \text{in} \ \Omega, \\
g \cdot n &= 0, \ n \cdot D(g) \cdot \tau = 0 \quad \text{on} \ \partial \Omega, \\
g \to U &= \frac{\Phi}{2} e_1 \quad \text{as} \ |x_1| \to \infty.
\end{align*}
\]

Using (1)-(2), (4), and (6), one has that \( v = u - g \) satisfies
\[
\begin{align*}
- \Delta v + v \cdot \nabla g + g \cdot \nabla v + v \cdot \nabla v + \nabla p &= \Delta g - g \cdot \nabla g \quad \text{in} \ \Omega, \\
\text{div} \ v &= 0 \quad \text{in} \ \Omega, \\
v \cdot n &= 0, \ n \cdot D(v) \cdot \tau = 0 \quad \text{on} \ \partial \Omega, \\
v &\to 0 \quad \text{as} \ |x_1| \to \infty.
\end{align*}
\]

Before giving the main results of this paper, the definitions of some function spaces and the weak solution are introduced.

**Definition 1.1.** Given a domain \( D \subseteq \mathbb{R}^2 \), denote
\[
L^2_0(D) = \left\{ w(x) : w \in L^2(D), \int_D w(x) \, dx = 0 \right\}.
\]
Given \( \Omega \) defined in \((5)\), define
\[
\mathcal{C}(\Omega) = \{ u \in C^\infty_c(\Omega) : u \cdot n = 0 \text{ on } \partial \Omega \}
\]
and
\[
\mathcal{C}_\sigma(\Omega) = \{ u \in \mathcal{C}(\Omega) : \text{div } u = 0 \}.
\]
Let \( \mathcal{H}(\Omega) \) and \( \mathcal{H}_\sigma(\Omega) \) be the completions of \( \mathcal{C}(\Omega) \) and \( \mathcal{C}_\sigma(\Omega) \) under \( H^1 \) norm, respectively.

Furthermore, for any constants \( a < b \) and \( 0 < T < \infty \), denote
\[
\Omega_{a,b} = \{(x_1, x_2) \in \Omega : a < x_1 < b \} \quad \text{and} \quad \Omega_T = \Omega_{-T,T}.
\]
Define
\[
\mathcal{C}(\Omega_{a,b}) = \left\{ u|_{\Omega_{a,b}} : u \in C^\infty_c(\Omega), \ u = 0 \text{ in } \Omega \setminus \Omega_{a,b}, \ u \cdot n = 0 \text{ on } \partial \Omega_{a,b} \cap \partial \Omega \right\}
\]
and
\[
\mathcal{C}_\sigma(\Omega_{a,b}) = \{ u \in \mathcal{C}(\Omega_{a,b}) : \text{div } u = 0 \text{ in } \Omega_{a,b} \}.
\]
Let \( \mathcal{H}(\Omega_{a,b}) \) and \( \mathcal{H}_\sigma(\Omega_{a,b}) \) be the completions of \( \mathcal{C}(\Omega_{a,b}) \) and \( \mathcal{C}_\sigma(\Omega_{a,b}) \) under \( H^1 \) norm, respectively.

Finally, denote \( H^1_+(\Omega_{a,b}) \) to be the set of functions in \( H^1(\Omega_{a,b}) \) with zero flux, i.e., for any \( v \in H^1_+(\Omega_{a,b}) \), one has
\[
\left(8\right) \quad \int_{f_1(x_1)}^{f_2(x_1)} v_1(x_1, x_2) \, dx_2 = 0 \quad \text{for any } x_1 \in (a, b).
\]

**Definition 1.2.** Assume that \( g \) is a smooth vector field satisfying \((6)\). Then a vector field \( u = g + v \) with \( v \in \mathcal{H}_\sigma(\Omega) \) is said to be a weak solution of the problem \((1)\), \((2)\), and \((4)\) if for any \( \phi \in \mathcal{H}_\sigma(\Omega) \), \( v \) satisfies
\[
\left(9\right) \quad \int_\Omega 2D(v) : D(\phi) + (v \cdot \nabla g + (g + v) \cdot \nabla v) \cdot \phi \, dx = \int_\Omega \Delta g \cdot \phi - g \cdot \nabla g \cdot \phi \, dx.
\]

Then the main results of this paper can be stated as follows.

**Theorem 1.1.** Let \( \Omega \) be the domain given in \((5)\). Given any flux \( \Phi \geq 0 \), the Navier-Stokes system \((1)\), \((2)\), and \((4)\) has a solution \( u = g + v \), where \( g \) is a smooth vector field satisfying \((6)\) and \( v \in \mathcal{H}_\sigma(\Omega) \) satisfies
\[
\| v \|_{H^1(\Omega)} \leq C_1.
\]

Furthermore, there exist positive constants \( C_2 \) and \( C_3 \) independent of \( T \) such that for sufficiently large \( T \), one has
\[
\| u - U \|_{H^1(\Omega \cap \{|x_1| > T\})} \leq C_3 e^{-C_2^{-1}T}.
\]
Finally, there exists a $\Phi_0 > 0$ such that if the flux $\Phi \in [0, \Phi_0)$, the solution $u$ is unique in the class

$$S = \{ w \in H^1_{\text{loc}}(\Omega) : \liminf_{t \to \infty} t^{-3} \| \nabla w \|^2_{L^2(\Omega_t)} = 0 \}.$$ 

There are a few remarks in order.

**Remark 1.1.** The constants $C_1$, $C_2$, and $C_3$ depend only on the flux $\Phi$ and the domain $\Omega$.

**Remark 1.2.** Theorem 1.1 provides a positive answer to Leray problem with full slip boundary condition and arbitrary flux.

**Remark 1.3.** Theorem 1.1 also holds if the channel is not flat at far field. Suppose that there exist $\gamma_1 < \gamma_2$, $\beta$, and $L$ such that

\begin{align}
(10) & \quad f_1(t) = -1, \quad f_2(t) = 1 \quad \text{for any } t \geq L \\
(11) & \quad f_1(t) = \beta t + \gamma_1, \quad f_2(t) = \beta t + \gamma_2 \quad \text{for any } t \leq -L.
\end{align}

We can also construct the flux carrier $g$, see Remark 3.1. Then the existence, far field behavior, and uniqueness of the solutions to the problem (1), (2), and (4) in these channels can be proved in a similar way.

**Remark 1.4.** When the paper has been finished, we got to know that a similar result has been obtained in [22] independently. Although there are some overlaps between the results in [22] and that in [36] and this paper, the analysis is different in many aspects.

The rest of the paper is organized as follows. In Section 2, we give some important lemmas which are used here and there in the paper. Section 3 devotes to the construction of the flux carrier. In Section 4, the existence of solutions to the problem (1), (2), and (4) is proved by Leray-Schauder fixed point theorem. The exponential convergence rate of the $H^1$ norm of the solutions is also given in Section 4. In Section 5, we show that the solutions obtained in Section 4 is unique in $S$ provided that the flux is suitably small.

2. Preliminaries

In this section, we collect some elementary but important lemmas. We first give the Poincaré type inequality and embedding inequality in channels, whose proof could be found in [36].
Lemma 2.1. For any $v \in H^1_a(\Omega_{a,b})$ satisfying $v \cdot n = 0$ on $\partial \Omega_{a,b} \cap \partial \Omega$, one has

$$
\|v\|_{L^2(\Omega_{a,b})} \leq M_1(\Omega_{a,b}) \|\nabla v\|_{L^2(\Omega_{a,b})},
$$

where

$$
M_1(\Omega_{a,b}) = C\|f\|_{L^\infty(a,b)} \cdot (1 + \|f'_2\|_{L^\infty(a,b)}).
$$

Lemma 2.2. Assume that $f(x_1) = f_2(x_1) - f_1(x_1) \geq d_{a,b} > 0$ for any $x_1 \in (a,b)$. Then for any $v \in H^1_a(\Omega_{a,b})$ satisfying $v \cdot n = 0$ on $\partial \Omega_{a,b} \cap \partial \Omega$, one has

$$
\|v\|_{L^4(\Omega_{a,b})} \leq M_4(\Omega_{a,b}) \|\nabla v\|_{L^2(\Omega_{a,b})},
$$

where

$$
M_4(\Omega_{a,b}) = C(1 + \|(f'_1, f'_2)\|_{L^\infty(a,b)}^2) \left( \frac{M_1}{b-a} + 1 \right)^{\frac{1}{2}} (|\Omega_{a,b}| + (b-a)d_{a,b})^{\frac{1}{4}} \left( 1 + \frac{M_1}{d_{a,b}} \right)
$$

with a universal constant $C$ and $M_1 = M_1(\Omega_{a,b})$ defined in (13).

Then we give the Korn inequality in the channel $\Omega$.

Lemma 2.3. Assume that $T > L + 1$. There exists a constant $c > 0$ such that for any $v \in H^1_\sigma(\Omega_T)$, it holds that

$$
c \|\nabla v\|_{L^2(\Omega_T)}^2 \leq 2\|D(v)\|_{L^2(\Omega_T)}^2,
$$

where $c$ is a constant independent of $T$.

Proof. Without loss of generality, we assume that $v \in C_\sigma(\Omega_T)$ satisfying

$$
\int_{f_1(x_1)}^{f_2(x_1)} v_1(x_1, x_2) \, dx_2 = 0 \quad \text{for any } |x_1| < T.
$$

According to the formula

$$
\Delta v = 2 \text{div} D(v),
$$

integration by parts yields

$$
\int_{\Omega_T} |\nabla v|^2 \, dx - \int_{\partial \Omega_T \cap \partial \Omega} n \cdot \nabla v \cdot v \, ds
$$

$$
= \int_{\Omega_T} -\Delta v \cdot v \, dx = \int_{\Omega_T} -2 \text{div} D(v) \cdot v \, dx
$$

$$
= \int_{\Omega_T} 2|D(v)|^2 \, dx - \int_{\partial \Omega_T \cap \partial \Omega} 2n \cdot D(v) \cdot v \, ds.
$$
Therefore, one has
\[
\int_{\Omega} |\nabla v|^2 \, dx = \int_{\Omega} 2|D(v)|^2 \, dx - \int_{\partial\Omega} 2n \cdot D(v) \cdot v - n \cdot \nabla v \cdot v \, ds.
\]
Note that
\[
n \cdot \nabla v \cdot v = 2n \cdot D(v) \cdot v - n \cdot \nabla v \cdot v.
\]
The boundary condition \(v \cdot n = 0\) also implies that \(\partial_r (v \cdot n) = 0\) on the boundary \(\partial \Omega\). Hence one has
\[
\begin{align*}
n \cdot \nabla v \cdot v &= (v \cdot \tau) \partial_r v \cdot n + (v \cdot n) \partial_n v \cdot n \\
&= (v \cdot \tau) [\partial_r (v \cdot n) - v \cdot \partial_r n] \\
&= - (v \cdot \tau) (v \cdot \partial_r n) \quad \text{on } \partial \Omega.
\end{align*}
\]
Since \(\partial_r n = 0\) on \(\partial \Omega \setminus \partial \Omega_{L+1}\), it holds that
\[
\begin{align*}
\int_{\Omega_T} |\nabla v|^2 \, dx &= \int_{\Omega_T} 2|D(v)|^2 \, dx - \int_{\partial \Omega_T \cap \partial \Omega} (v \cdot \tau) (v \cdot \partial_r n) \, ds \\
&\leq 2\|D(v)\|_{L^2(\Omega_T)}^2 + \int_{\partial \Omega_T \cap \partial \Omega} |v|^2 |\partial_r n| \, ds \\
&\leq 2\|D(v)\|_{L^2(\Omega_T)}^2 + C_4\|v\|_{L^2(\partial \Omega_{L+1} \cap \partial \Omega)}^2,
\end{align*}
\]
where
\[
C_4 = \|\partial_r n\|_{L^\infty(\partial \Omega)}.
\]
Next, we claim that there exists a constant \(C_5\) such that
\[
C_4\|v\|_{L^2(\partial \Omega_{L+1} \cap \partial \Omega)}^2 \leq \frac{1}{2}\|\nabla v\|_{L^2(\Omega_{L+1})}^2 + C_5\|D(v)\|_{L^2(\Omega_{L+1})}^2.
\]
Otherwise, there exists a sequence \(\{v^m\} \subset H_T(\Omega_T)\) satisfying
\[
C_4\|v^m\|_{L^2(\partial \Omega_{L+1} \cap \partial \Omega)}^2 > \frac{1}{2}\|\nabla v^m\|_{L^2(\Omega_{L+1})}^2 + m\|D(v^m)\|_{L^2(\Omega_{L+1})}^2.
\]
Define
\[
u^m := \frac{v^m}{\|v^m\|_{L^2(\partial \Omega_{L+1} \cap \partial \Omega)}}.
\]
One has
\[
\|\nu^m\|_{L^2(\partial \Omega_{L+1} \cap \partial \Omega)} = 1, \quad \|\nabla \nu^m\|_{L^2(\Omega_{L+1})} < 2C_4 \quad \text{and} \quad \|D(\nu^m)\|_{L^2(\Omega_{L+1})} \leq \frac{C_4}{m}.
\]
It follows from Lemma 2.1 that \(\{\nu^m\}\) is also bounded in \(H^1(\Omega_{L+1})\). Hence one can choose a subsequence stilled labelled by \(\{\nu^m\}\), which converges weakly in \(H^1(\Omega_{L+1})\) and strongly in
$L^2(\partial\Omega_{L+1} \cap \partial\Omega)$ to a vector field $u^* \in H^1(\Omega_{L+1})$. Clearly, one has

$$\|u^*\|_{L^2(\partial\Omega_{L+1} \cap \partial\Omega)} = 1, \quad \|D(u^*)\|_{L^2(\Omega_{L+1})} = 0, \quad \int_{f_1(x_1)}^{f_2(x_1)} u^*_1 \, dx_2 = 0.$$  

In particular, one has

$$\partial_1 u^*_1 = \partial_2 u^*_2 = 0, \quad \partial_1 u^*_2 + \partial_2 u^*_1 = 0.$$  

Therefore, $u^*$ takes the form $u^*_1 = ax_2 + b_1$, $u^*_2 = -ax_1 + b_2$ for some $a \in \mathbb{R}$.

On the other hand, on the boundary $\partial\Omega_{L, L+1} \cap \partial\Omega = \{(x_1, x_2) : x_1 \in (L, L + 1), \, x_2 = \pm 1\}$, one has $u^* \cdot n = u^*_2 = 0$ so that $a = b_1 = 0$. This contradicts with the first property in (22).

Finally, one combines (19)-(21) to conclude (15) with

$$c = \frac{1}{2 + C_5}.$$  

This finishes the proof of the lemma. \qed

Remark 2.1. It is noteworthy that the constant $c$ depends only on the subdomain $\Omega_{L+1}$.

The following lemma on the solvability of the divergence equation is used to give the estimates involving pressure. For the proof, one may refer to [16, Theorem III.3.1] and [11].

Lemma 2.4. Let $D \subset \mathbb{R}^n$ be a locally Lipschitz domain. Then there exists a constant $M_5$ such that for any $w \in L^2_0(D)$, the problem

$$\begin{cases} \text{div} \, a = w \quad \text{in} \, D, \\ a = 0 \quad \text{on} \, \partial D \end{cases}$$

has a solution $a \in H^1_0(D)$ satisfying

$$\|\nabla a\|_{L^2(D)} \leq M_5(D)\|w\|_{L^2(D)}.$$  

In particular, if the domain $D$ is star-like with respect to some open ball $B$ with $\overline{B} \subset D$, then the constant $M_5(D)$ admits the following estimate

$$M_5(D) \leq C \left( \frac{R_0}{R} \right)^n \left( 1 + \frac{R_0}{R} \right),$$

where $R_0$ is the diameter of the domain $D$ and $R$ is the radius of the ball $B$.

Remark 2.2. In particular, for $D = \Omega_{t-1, t}$ or $\Omega_{-t, -t+1}$, $t > L + 1$, the constant $M_5(D)$ is independent of $t$ since $D$ is a star-like domain with respect to a ball with radius $\frac{1}{4}$. 


We next recall a differential inequality (cf. [27]), which plays the key role in establishing the uniqueness of the solutions.

**Lemma 2.5.** Let \( z(t) \) be a nondecreasing and nonnegative function, which is not identically equal to zero. Assume that \( \Psi(\tau) \) is a monotonically increasing function, which equals to zero at \( \tau = 0 \) and tends to \( \infty \) as \( \tau \to \infty \). Suppose that there exist \( m > 1, t_0 \geq 0, \tau_1 \geq 0, c_0 > 0 \) such that
\[
z(t) \leq \Psi(z'(t)) \quad \text{for any } t \geq t_0 \quad \text{and} \quad \Psi(\tau) \leq c_0 \tau^m \quad \text{for any } \tau \geq \tau_1,
\]
then it holds that
\[
\lim \inf_{t \to \infty} t^{-\frac{m-1}{m}} z(t) > 0.
\]

With the aid of the differential inequality for the Dirichlet norm on approximate domain \( \Omega_t \), one has that either it is trivial or it grows faster that \( t^\frac{m}{m-1} \).

### 3. Flux carrier

In this section, we construct the so called flux carrier \( g = (g_1, g_2) \), which is a smooth vector field satisfying
\[
\begin{aligned}
\operatorname{div} g &= 0 \\
g \cdot \nu &= 0, \quad \nu \cdot \mathbf{D}(g) \cdot \tau = 0 \quad \text{on } \partial \Omega, \\
g \to U = \frac{\Phi}{2} e_1 &\quad \text{as } |x_1| \to \infty.
\end{aligned}
\]

Inspired by [2], [30], we introduce two smooth functions \( \mu(t; \varepsilon) : [0, \infty) \to [0, 1] \) and \( \pi(s; \mathcal{D}) : \mathbb{R} \to [0, 1] \) satisfying
\[
\mu(t; \varepsilon) = \begin{cases} 
1, & \text{if } t \text{ near } 0, \\
0, & \text{if } t \geq \varepsilon,
\end{cases}
\quad \pi(s; \mathcal{D}) = \begin{cases} 
0, & \text{if } |s| \leq \frac{5\mathcal{D}}{4}, \\
1, & \text{if } |s| \geq \frac{7\mathcal{D}}{4},
\end{cases}
\]
and
\[
0 \leq -\mu'(t; \varepsilon) \leq \frac{\varepsilon}{t}, \quad 0 \leq \pi'(s; \mathcal{D}) \leq \frac{4}{\mathcal{D}}, \quad 0 \leq \pi''(s; \mathcal{D}) \leq \frac{16}{\mathcal{D}^2},
\]
where \( \varepsilon \) and \( \mathcal{D} > L \) are two parameters to be determined. One can refer to [2, Lemma 2.6] for the detailed construction of \( \mu(t; \varepsilon) \). Define \( g = (g_1, g_2) \) as
\[
\begin{aligned}
g_1(x_1, x_2) &= \partial_{x_2} G(x_1, x_2; \varepsilon) + \left( \frac{\Phi}{2} - \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \pi(x_1; \mathcal{D})
\end{aligned}
\]

[...remaining content of the document...]
and

\begin{equation}
    g_2(x_1, x_2) = \begin{cases} 
        -\partial_{x_1} G(x_1, x_2; \varepsilon) & \text{if } |x_1| < \mathcal{D}, \\
        \pi'(x_1; \mathcal{D}) \left( G(x_1, x_2; \varepsilon) - \frac{\Phi}{2}(x_2 + 1) \right) & \text{if } |x_1| \geq \mathcal{D},
    \end{cases}
\end{equation}

where

\begin{equation}
    G(x_1, x_2; \varepsilon) = \Phi \mu(f_2(x_1) - x_2; \varepsilon).
\end{equation}

Denote

\begin{equation}
    \Sigma(x_1) = \{(x_1, x_2) : f_1(x_1) < x_2 < f_2(x_1)\}.
\end{equation}

In order to show that \( g \in C^\infty(\overline{\Omega}) \), it’s sufficient to verify the smoothness of \( g_2 \) near \( \Sigma(\pm \mathcal{D}) \) since both \( \pi \) and \( \mu \) are smooth. Actually, it holds that \( f_2(x_1) = 1 \) for any \( |x_1| > L \) and then the function

\[ G(x_1, x_2; \varepsilon) = \mu(f_2(x_1) - x_2; \varepsilon, \mathcal{D}) = \Phi \mu(1 - x_2; \varepsilon) \]

depends only on \( x_2 \) in the subdomain \( \Omega \setminus \Omega_L \). Therefore, for any \( x \in \Omega \) with \( L \leq |x_1| < \mathcal{D} \), one has

\[ g_2(x_1, x_2) = -\partial_{x_1} G(x_1, x_2; \varepsilon) = -\partial_{x_1} \mu(1 - x_2; \varepsilon, \mathcal{D}) = 0. \]

On the other hand, (26), together with (29), implies that \( g_2(x_1, x_2) \equiv 0 \) for any \( x \in \Omega \) with \( \mathcal{D} \leq |x_1| < \frac{5\mathcal{D}}{4} \). Hence \( g \in C^\infty(\overline{\Omega}) \).

Next, note that

\[ G(x_1, x_2; \varepsilon) = \begin{cases} 
        \Phi, & \text{if } x_2 \text{ near } f_2(x_1), \\
        0, & \text{if } x_2 \leq f_2(x_1) - \varepsilon,
    \end{cases} \]

and

\[ g = (\partial_{x_2} G, -\partial_{x_1} G) \text{ in } \Omega_{\mathcal{D}}. \]

Hence \( g \) is a solenoidal vector field with flux \( \Phi \) in \( \Omega_{\mathcal{D}} \). In particular, \( g \) vanishes near the boundary \( \partial \Omega \cap \partial \Omega_{\mathcal{D}} \).

In the subdomain \( \Omega \setminus \Omega_{\mathcal{D}} \), since \( f_2(x_1) = 1 \) and \( f_1(x_1) = -1 \) for any \( |x_1| \geq L \), one has \( \partial_{x_1} G = 0 \). It follows from straightforward computations that one has

\[ \text{div } g = \partial_{x_1} g_1 + \partial_{x_2} g_2 = 0 \]
and
\[
\int_{\Sigma(x_1)} g_1(x_1, x_2) \, dx_2 = \int_{-1}^{1} \partial_{x_2} G(x_1, x_2; \varepsilon) + \left( \frac{\Phi}{2} - \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \pi(x_1; \mathcal{D}) \, dx_2 \\
= \Phi + \pi(x_1; \mathcal{D}) \int_{-1}^{1} \frac{\Phi}{2} - \partial_{x_2} G(x_1, x_2; \varepsilon) \, dx_2 \\
= \Phi.
\]

Moreover, at the upper boundary
\[
S_{2; \mathcal{D}} = \{ x \in \partial \Omega : x_2 = 1, \ |x_1| > \mathcal{D} \},
\]
one has \( \tau = (1, 0), \ n = (0, 1) \). Note also that \( G(x_1, x_2; \varepsilon)|_{x_2 = f_2(x_1)} = \Phi \) and \( \partial_{x_2} G(x_1, x_2; \varepsilon) \) vanishes near the boundary \( \partial \Omega \). Hence it holds that
\[
g \cdot n = g_2(x_1, x_2)|_{x_2 = f_2(x_1)} = \pi'(x_1; \mathcal{D})(G(x_1, 1; \varepsilon) - \Phi) = 0
\]
and
\[
n \cdot D(g) \cdot \tau = \frac{1}{2} (\partial_{x_2} g_1 + \partial_{x_1} g_2)(x_1, x_2)|_{x_2 = f_2(x_1)} \\
= \frac{1}{2} \left( \partial_{x_2}^2 G(x_1, 1; \varepsilon) - \partial_{x_2}^2 G(x_1, 1; \varepsilon) \pi(x_1; \mathcal{D}) + \pi''(x_1; \mathcal{D})(G(x_1, 1; \varepsilon) - \Phi) \right) \\
= 0.
\]

Similarly, at the lower boundary
\[
S_{1; \mathcal{D}} = \{ x \in \partial \Omega : x_2 = -1, \ |x_1| > \mathcal{D} \},
\]
one has also \( \tau = (1, 0), \ n = (0, -1) \) and \( G(x_1, x_2; \varepsilon)|_{x_2 = f_1(x_1)} = 0 \). Therefore, one has
\[
g \cdot n = -g_2(x_1, x_2; \varepsilon, \mathcal{D})|_{x_2 = f_1(x_1)} = \pi'(x_1; \mathcal{D})G(x_1, -1; \varepsilon) = 0
\]
and
\[
n \cdot D(g) \cdot \tau = -\frac{1}{2} (\partial_{x_2} g_1 + \partial_{x_1} g_2)(x_1, x_2)|_{x_2 = f_1(x_1)} \\
= -\frac{1}{2} \left( \partial_{x_2}^2 G(x_1, -1; \varepsilon) - \partial_{x_2}^2 G(x_1, -1; \varepsilon) \pi(x_1; \mathcal{D}) + \pi''(x_1; \mathcal{D})G(x_1, -1; \varepsilon) \right) \\
= 0.
\]

Finally, noting \( \pi(x_1; \mathcal{D}) = 1 \) for any \( |x_1| \geq \frac{7D}{4} \), one has
\[
g \equiv \left( \frac{\Phi}{2}, 0 \right) \quad \text{in} \quad \Omega \setminus \Omega_{\frac{7D}{4}}.
\]

Hence \( g \) satisfies \((25)\) in \( \Omega \).
Remark 3.1. For a more general channel domain $\Omega$ with (10)–(11), one could also construct the corresponding flux carrier $g$ via some modifications. Assume $\beta > 0$. For any $x \in \Omega_{-L,\infty}$, we define $g = (g_1, g_2)$ as the same form of (28) and (29), i.e.,

$$g_1(x_1, x_2) = \partial_{x_2}G(x_1, x_2; \varepsilon) + \left(\frac{\Phi}{2} - \partial_{x_2}G(x_1, x_2; \varepsilon)\right)\pi(x_1; \mathcal{D}) \text{ if } x_1 \geq -L$$

and

$$g_2(x_1, x_2) = \begin{cases} -\partial_{x_1}G(x_1, x_2; \varepsilon), & \text{if } -L \leq x_1 < \mathcal{D}, \\ \pi'(x_1; \mathcal{D})\left(G(x_1, x_2; \varepsilon) - \frac{\Phi}{2}(x_2 + 1)\right), & \text{if } x_1 \geq \mathcal{D}. \end{cases}$$

On the other hand, let

$$\tilde{x}_1 = x_1 \cos \theta + x_2 \sin \theta \quad \text{and} \quad \tilde{x}_2 = -x_1 \sin \theta + x_2 \cos \theta,$$

with $\theta = \arctan \beta$, which transforms the outlet $\Omega_{-\infty,-L}$ into a flat outlet $\tilde{\Omega}_{-\infty,-L}$ in the new coordinate $(\tilde{x}_1, \tilde{x}_2)$. More precisely,

$$\tilde{\Omega}_{-\infty,-L} = \left\{(\tilde{x}_1, \tilde{x}_2) : \tilde{x}_2 \in (\gamma_1 \cos \theta, \gamma_2 \cos \theta), \tilde{x}_1 < \tilde{x}_2 \tan \theta - \frac{L}{\cos \theta}\right\}.$$

In the flat outlet $\tilde{\Omega}_{-\infty,-L}$, one could construct the vector field $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$ in a way similar to (28)–(29),

$$\tilde{g}_1(\tilde{x}_1, \tilde{x}_2) = \partial_{\tilde{x}_2}G(\tilde{x}_1, \tilde{x}_2) + \left(\frac{\Phi}{(\gamma_2 - \gamma_1) \cos \theta} - \partial_{\tilde{x}_2}G(\tilde{x}_1, \tilde{x}_2)\right)\pi(\tilde{x}_1 - \tilde{L}_1 - \mathcal{D}; \mathcal{D}) \quad \text{if } \tilde{x}_1 < \tilde{L}_2$$

and

$$\tilde{g}_2(\tilde{x}_1, \tilde{x}_2) = \begin{cases} -\partial_{\tilde{x}_1}G(\tilde{x}_1, \tilde{x}_2), & \text{if } \tilde{L}_1 \leq \tilde{x}_1 < \tilde{L}_2, \\ \pi'(\tilde{x}_1 - \tilde{L}_1 - \mathcal{D}; \mathcal{D})\left(\tilde{G}(\tilde{x}_1, \tilde{x}_2) - \frac{\Phi(\tilde{x}_2 - \gamma_1 \cos \theta)}{(\gamma_2 - \gamma_1) \cos \theta}\right), & \text{if } \tilde{x}_1 < \tilde{L}_1, \end{cases}$$

\[\text{Figure 2. Rotation transformation}\]
where
\[ \tilde{L}_2 = \gamma_2 \sin \theta - \frac{L}{\cos \theta}, \quad \tilde{L}_1 = \gamma_1 \sin \theta - \frac{L}{\cos \theta}, \]
and
\[ \tilde{G}(\tilde{x}_1, \tilde{x}_2) = \Phi \mu \left( \frac{\gamma_2 \cos \theta - \tilde{x}_2; \varepsilon}{\cos \theta} \right). \]
For \( x_1 < -L \), define
\[ g_1(x_1, x_2) = \tilde{g}_1(\tilde{x}_1, \tilde{x}_2) \cos \theta - \tilde{g}_2(\tilde{x}_1, \tilde{x}_2) \sin \theta, \quad g_2(x_1, x_2) = \tilde{g}_1(\tilde{x}_1, \tilde{x}_2) \sin \theta + \tilde{g}_2(\tilde{x}_1, \tilde{x}_2) \cos \theta, \]
where the relation between \((x_1, x_2)\) and \((\tilde{x}_1, \tilde{x}_2)\) is given in (32). Then the straightforward computations show that \( g = (g_1, g_2) \) is smooth near \( \Sigma(-L) \) and thus is smooth in \( \Omega \). Furthermore, \( g \) is divergence free, satisfies the slip boundary condition on the channel boundary, and tends to the associated shear flows with flux \( \Phi \) far fields. Hence \( g \) is a flux carrier.

The following two lemmas give the crucial properties of the flux carrier \( g \), which plays an important role in the energy estimates.

**Lemma 3.1.** The function \( G(x_1, x_2; \varepsilon) \) defined in (30) satisfies
\[ |\nabla G(x_1, x_2; \varepsilon)| + |\nabla^2 G(x_1, x_2; \varepsilon)| \leq C(\varepsilon) \Phi. \]
Furthermore, for any function \( w \in H^1(\Omega_{a,b}) \) satisfying \( w = 0 \) on the upper boundary \( S_{2,a,b} := \{x \in \partial \Omega : x_2 = f_2(x_1), a < x_1 < b\} \), it holds that
\[ \int_{\Omega_{a,b}} w^2 |\partial_{x_2} G|^2 \, dx \leq C \Phi^2 \varepsilon^2 \int_{\Omega_{a,b}} |\partial_{x_2} w|^2 \, dx, \]
where \( C(\varepsilon) \) is a constant depending only on \( \varepsilon \) and \( C \) is a uniform constant independent of \( \varepsilon \).

**Proof.** Recall the definition for \( G(x_1, x_2; \varepsilon) \) in (30). It follows from direct computations that one has
\[ (33) \quad \partial_{x_1} G(x_1, x_2; \varepsilon) = \Phi \mu'(f_2(x_1) - x_2; \varepsilon)f'_2(x_1), \quad \partial_{x_2} G(x_1, x_2; \varepsilon) = -\Phi \mu'(f_2(x_1) - x_2; \varepsilon). \]
Furthermore,
\[ (34) \quad \partial^2_{x_1x_2} G(x_1, x_2; \varepsilon) = -\Phi \mu''(f_2(x_1) - x_2; \varepsilon)f'_2(x_1), \]
\[ (35) \quad \partial^2_{x_2} G(x_1, x_2; \varepsilon) = \Phi \mu''(f_2(x_1) - x_2; \varepsilon) \]
and
\[ (36) \quad \partial^2_{x_1} G(x_1, x_2; \varepsilon) = \Phi \mu'(f_2(x_1) - x_2; \varepsilon)f''_2(x_1) + \Phi \mu''(f_2(x_1) - x_2; \varepsilon)|f'_2(x_1)|^2. \]
Noting \( \mu(t; \varepsilon) \) is smooth and \( \text{supp} \mu' \subset [0, \varepsilon] \), one has
\[ |\mu'(t; \varepsilon)|, |\mu''(t; \varepsilon)| \leq C(\varepsilon). \]
Moreover, since $f_2(x_1) = 1$ for any $|x_1| \geq L$, one has also

$$|f'_2(x_1)|, |f''_2(x_1)| \leq C.$$  

Then it follows that

$$|\nabla G(x_1, x_2; \varepsilon)| + |\nabla^2 G(x_1, x_2; \varepsilon)| \leq C(\varepsilon)\Phi.$$  

Next, one has

$$\int_{\Omega_{a,b}} w^2 |\partial_{x_2} G|^2 \, dx = \int_{\Omega_{a,b}} \Phi^2 (\mu'(f_2(x_1) - x_2; \varepsilon))^2 w^2 \, dx$$

$$\leq C\Phi^2 \varepsilon^2 \int_a^b dx_1 \int_{f_1(x_1)}^{f_2(x_1)} \frac{w^2}{(f_2(x_1) - x_2)^2} \, dx_2$$

$$\leq C\Phi^2 \varepsilon^2 \int_{\Omega_{a,b}} |\partial_{x_2} w|^2 \, dx,$$

where (27) and the Hardy inequality (19) have been used. This finishes the proof of the lemma.

**Lemma 3.2.** The flux carrier $g$ satisfies

$$\int_\Omega |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx \leq C(\varepsilon, \mathcal{D})(\Phi^2 + \Phi^4),$$

where $C(\varepsilon, \mathcal{D})$ is a constant depending only on $\varepsilon$ and $\mathcal{D}$. Moreover, for any $\delta > 0$, there exist $\varepsilon$ and $\mathcal{D}$ such that for any $\mathbf{v} \in H_\sigma(\Omega)$, it holds that

$$\left| \int_\Omega \mathbf{v} \cdot \nabla g \cdot \mathbf{v} \, dx \right| \leq \delta \| \nabla \mathbf{v} \|_{L^2}^2.$$  

**Proof.** Noting $g = \frac{\Phi}{2} \mathbf{e}_1$ for any $\mathbf{x} \in \Omega$ with $|x_1| \geq 2\mathcal{D}$, one has

$$\int_\Omega |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx = \int_{\Omega_{2\mathcal{D}}} |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx \leq \| \nabla \mathbf{v} \|_{L^2}^2.$$  

Using (28)-(29) and Lemma 3.1, one has

$$\sup_{x \in \Omega_{2\mathcal{D}}} (|\nabla g|^2 + |g \cdot \nabla g|^2) \leq C(\varepsilon, \mathcal{D})(\Phi^2 + \Phi^4).$$

This, together with (38), gives (37). Next, from (33), one has the following equality

$$\partial_{x_1} G(x_1, x_2; \varepsilon) = -f'_2(x_1) \partial_{x_2} G(x_1, x_2; \varepsilon).$$
Using (28) gives
\[
\int_{\Omega} \mathbf{v} \cdot \nabla g_1 v_1 \, dx = \int_{\Omega} (v_1 \partial_{x_1} + v_2 \partial_{x_2} ) \left( \partial_{x_2} G(x_1, x_2; \varepsilon) + \frac{\Phi}{2} - \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \pi(x_1; \mathcal{D}) \, v_1 \, dx \\
= \int_{\Omega} (v_1 \partial_{x_1} + v_2 \partial_{x_2} ) \left( (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} G(x_1, x_2; \varepsilon) + \frac{\Phi}{2} \pi(x_1; \mathcal{D}) \right) \, v_1 \, dx \\
= \int_{\Omega} (v_1^2 \partial_{x_1} + v_1 v_2 \partial_{x_2} ) \left( (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \, dx + \int_{\Omega} \frac{\Phi}{2} v_1^2 \pi'(x_1; \mathcal{D}) \, dx.
\]

It follows from (26) and Lemma 2.1 that
\[
\int_{\Omega} \frac{\Phi}{2} v_1^2 \pi'(x_1; \mathcal{D}) \, dx \leq \frac{C \Phi}{\mathcal{D}} \| \mathbf{v} \|_{L^2(\Omega)}^2 \leq \frac{C \Phi}{\mathcal{D}} \| \nabla \mathbf{v} \|_{L^2(\Omega)}^2.
\]

Noting \( \partial_{x_2} G(x_1, x_2; \varepsilon) \) vanishes near the boundary \( \partial \Omega \) and \( \mathbf{v} \) is divergence free in \( \Omega \), one uses integration by parts to obtain
\[
\int_{\Omega} (v_1^2 \partial_{x_1} + v_1 v_2 \partial_{x_2} ) \left( (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \, dx \\
= - \int_{\Omega} (v_1 \partial_{x_1} v_1 + v_2 \partial_{x_2} v_1) (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} G(x_1, x_2; \varepsilon) \, dx \\
= - \int_{\Omega} (1 - \pi(x_1; \mathcal{D})) \left( v_1 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) + v_2 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \, dx \\
= - \int_{\Omega} \left( (1 - \pi(x_1; \mathcal{D})) \left( v_1 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) - v_1 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \right) \, dx \\
= - \int_{\Omega} \left( (1 - \pi(x_1; \mathcal{D})) \left( v_2 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) + v_1 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \right) \, dx \\
= - \int_{\Omega} \frac{1}{2} (1 - \pi(x_1; \mathcal{D})) \left( \partial_{x_2} (v_1^2 \partial_{x_2} G(x_1, x_2; \varepsilon) - \partial_{x_2} (v_1^2 \partial_{x_2} G(x_1, x_2; \varepsilon)) \right) \, dx \\
= - \int_{\Omega} (1 - \pi(x_1; \mathcal{D})) \left( v_2 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) + v_1 \partial_{x_2} v_1 \partial_{x_2} G(x_1, x_2; \varepsilon) \right) \, dx \\
= - \int_{\Omega} (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} v_1 (v_2 \partial_{x_2} G(x_1, x_2; \varepsilon) + v_1 \partial_{x_1} G(x_1, x_2; \varepsilon)) \, dx \\
= - \int_{\Omega} \frac{1}{2} \pi'(x_1; \mathcal{D}) v_1^2 \partial_{x_2} G(x_1, x_2; \varepsilon) \, dx \\
= - \int_{\Omega} (1 - \pi(x_1; \mathcal{D})) \partial_{x_2} v_1 (v_2 - v_1 f_2(x_1)) \partial_{x_2} G(x_1, x_2; \varepsilon) \, dx \\
= - \int_{\Omega} \frac{1}{2} \pi'(x_1; \mathcal{D}) v_1^2 \partial_{x_2} G(x_1, x_2; \varepsilon) \, dx,
where the equality (39) has been used to get the last equality. Note that on the upper boundary $S_2 = \{x \in \partial \Omega : x_1 \in \mathbb{R}, x_2 = f_2(x_1)\}$, the impermeability condition $\nu \cdot n = 0$ can be written as

$$v_2(x_1, f_2(x_1)) - f_2'(x_1)v_1(x_1, f_2(x_1)) = 0.$$  

Then applying Cauchy-Schwarz inequality and Lemma 3.1 gives

$$\left| \int_{\Omega} (1 - \pi(x_1; \mathcal{D})) \partial_{x_2}v_1(v_2 - v_1f_2'(x_1))\partial_{x_2}G(x_1, x_2; \varepsilon) \, dx \right| \leq \|\partial_{x_2}v_1\|_{L^2(\Omega)} \left( \int_{\Omega} |(v_2 - v_1f_2'(x_1))\partial_{x_2}G(x_1, x_2; \varepsilon)|^2 \, dx \right)^{\frac{1}{2}} \leq C\varepsilon \Phi \|\partial_{x_2}v_1\|_{L^2(\Omega)} \|\partial_{x_2}(v_2 - v_1f_2'(x_1))\|_{L^2(\Omega)} \leq C\varepsilon \Phi \|\nabla \nu\|_{L^2(\Omega)}^2.$$  

Lemmas 2.1 and 3.1 together with (27), yield

$$\left| \int_{\Omega} \frac{1}{2} \pi'(x_1; \mathcal{D})v_1^2\partial_{x_2}G(x_1, x_2; \varepsilon) \, dx \right| \leq \frac{C(\varepsilon)\Phi}{\mathcal{D}} \|\nu\|_{L^2(\Omega)}^2 \leq \frac{C(\varepsilon)\Phi}{\mathcal{D}} \|\nabla \nu\|_{L^2(\Omega)}^2.$$  

On the other hand, with the aid of the explicit form in (29), one has

$$\int_{\Omega} \nu \cdot \nabla g_2v_2 \, dx = \int_{\Omega^D} -(v_2v_1\partial_{x_1} + v_2^2\partial_{x_2})\partial_{x_1}G(x_1, x_2; \varepsilon) \, dx$$

$$+ \int_{\Omega \cap \Omega^D} (v_1v_2\partial_{x_1} + v_2^2\partial_{x_2}) \left[ \pi'(x_1; \mathcal{D}) \left( G(x_1, x_2; \varepsilon) - \frac{\Phi}{2} (x_2 + 1) \right) \right] \, dx.$$  

Since $\partial_{x_1}G(x_1, x_2; \varepsilon) = \Phi\partial_{x_1}\mu(1 - x_2; \varepsilon) = 0$ near $\Sigma(\pm \mathcal{D})$ and $\partial_{x_1}G(x_1, x_2; \varepsilon)$ vanishes near the boundary $\partial \Omega \cap \partial \Omega^D$, the integration by parts together with (39) gives

$$\int_{\Omega_D} -(v_2v_1\partial_{x_1} + v_2^2\partial_{x_2})\partial_{x_1}G(x_1, x_2; \varepsilon) \, dx$$

$$= \int_{\Omega_D} (v_1\partial_{x_1}v_2 + v_2\partial_{x_2}v_2)\partial_{x_1}G(x_1, x_2; \varepsilon) \, dx$$

$$= \int_{\Omega_D} v_1\partial_{x_1}v_2\partial_{x_1}G(x_1, x_2; \varepsilon) + v_2\partial_{x_1}v_2\partial_{x_2}G(x_1, x_2; \varepsilon) \, dx$$

$$+ \int_{\Omega_D} \partial_{x_2} \left( \frac{v_2^2}{2} \right) \partial_{x_1}G(x_1, x_2; \varepsilon) - \partial_{x_1} \left( \frac{v_2^2}{2} \right) \partial_{x_2}G(x_1, x_2; \varepsilon) \, dx$$

$$= \int_{\Omega_D} \partial_{x_1}v_2 \left[ v_1\partial_{x_1}G(x_1, x_2; \varepsilon) + v_2\partial_{x_2}G(x_1, x_2; \varepsilon) \right] \, dx$$

$$= \int_{\Omega_D} \partial_{x_1}v_2(v_2 - f_2'(x_1)v_1)\partial_{x_2}G(x_1, x_2; \varepsilon) \, dx.$$
Therefore, similar to (43), one uses Cauchy-Schwarz inequality and Lemma 3.1 to conclude

\[
\left| \int_{\Omega_{\partial}} -\left( v_{2}v_{1}\partial_{x_{1}} + v_{2}^{2}\partial_{x_{2}} \right) \partial_{x_{1}}G(x_{1}, x_{2}; \varepsilon) \, dx \right|
\]

\[
\leq \| \partial_{x_{1}} v_{2} \|_{L^{2}(\Omega_{\partial})} \left( \int_{\Omega_{\partial}} |(v_{2} - v_{1}f_{2}(x_{1}))\partial_{x_{2}}G(x_{1}, x_{2}; \varepsilon)|^{2} \, dx \right)^{\frac{1}{2}}
\]

\[
\leq C\varepsilon \Phi \| \partial_{x_{1}} v_{2} \|_{L^{2}(\Omega_{\partial})} \| \partial_{x_{2}}(v_{2} - v_{1}f_{2}(x_{1})) \|_{L^{2}(\Omega_{\partial})}
\]

\[
\leq C\varepsilon \Phi \| \nabla v \|_{L^{2}(\Omega_{\partial})}^{2}.
\]

Note that the function \( G(x_{1}, x_{2}; \varepsilon) = \Phi\mu(1 - x_{2}; \varepsilon) \) depends only on \( x_{2} \) in the straight outlets \( \Omega \setminus \Omega_{\partial} \). Hence one has

\[
\int_{\Omega \setminus \Omega_{\partial}} (v_{1}v_{2}\partial_{x_{1}} + v_{2}^{2}\partial_{x_{2}}) \left[ \pi'(x_{1}; \mathcal{D}) \left( G(x_{1}, x_{2}; \varepsilon) - \frac{\Phi}{2}(x_{2} + 1) \right) \right] \, dx
\]

\[
= \int_{\Omega \setminus \Omega_{\partial}} v_{1}v_{2}\pi''(x_{1}; \mathcal{D}) \left( G(x_{1}, x_{2}; \varepsilon) - \frac{\Phi}{2}(x_{2} + 1) \right) + v_{2}^{2}\pi'(x_{1}; \mathcal{D}) \left( \partial_{x_{2}}G(x_{1}, x_{2}; \varepsilon) - \frac{\Phi}{2} \right) \, dx.
\]

It follows from (26) and Lemmas 2.1 that one has

\[
\left| \int_{\Omega \setminus \Omega_{\partial}} (v_{1}v_{2}\partial_{x_{1}} + v_{2}^{2}\partial_{x_{2}}) \left[ \pi'(x_{1}; \mathcal{D}) \left( G(x_{1}, x_{2}; \varepsilon) - \frac{\Phi}{2}(x_{2} + 1) \right) \right] \, dx \right|
\]

\[
\leq \frac{C\Phi}{\mathcal{D}^{2}} \int_{\Omega \setminus \Omega_{\partial}} |v_{1}v_{2}| \, dx + \frac{C(\varepsilon)\Phi}{\mathcal{D}} \int_{\Omega \setminus \Omega_{\partial}} v_{2}^{2} \, dx
\]

\[
\leq \frac{C(\varepsilon)\Phi}{\mathcal{D}} \| v \|_{L^{2}(\Omega \setminus \Omega_{\partial})}^{2}
\]

\[
\leq \frac{C(\varepsilon)\Phi}{\mathcal{D}} \| \nabla v \|_{L^{2}(\Omega \setminus \Omega_{\partial})}^{2}.
\]

Combining (40) and (47) gives

\[
\left| \int_{\Omega} v \cdot \nabla g \cdot v \, dx \right| = \left| \int_{\Omega} v \cdot \nabla g_{1}v_{1} \, dx + \int_{\Omega} v \cdot \nabla g_{2}v_{2} \, dx \right|
\]

\[
\leq \frac{C(\varepsilon)\Phi}{\mathcal{D}} \| \nabla v \|_{L^{2}(\Omega)}^{2} + C\varepsilon \Phi \| \nabla v \|_{L^{2}(\Omega)}^{2}.
\]

Then for any \( \delta > 0 \) and \( \Phi \), one can choose sufficiently small \( \varepsilon \) and sufficiently large \( \mathcal{D} \) such that

\[
\left| \int_{\Omega} v \cdot \nabla g \cdot v \, dx \right| \leq \delta \| \nabla v \|_{L^{2}(\Omega)}^{2}.
\]

This finishes the proof of the lemma. \( \square \)
4. Existence and Far Field Behavior of the Solutions

As long as the flux carrier \( g \) has been constructed in Section 3, we prove the existence of solutions to the problem (7) in this section. More precisely, we seek for the solutions to problem (7) as the limit of the solutions of the following approximate problem on the bounded domain \( \Omega_T \),

\[
\begin{aligned}
- \Delta v + v \cdot \nabla g + g \cdot \nabla v + v \cdot \nabla p &= \Delta g - g \cdot \nabla g & \text{in } \Omega_T, \\
\text{div } v &= 0 & \text{in } \Omega_T, \\
v \cdot n &= 0, \quad n \cdot D(v) \cdot \tau &= 0 & \text{on } \partial \Omega_T \cap \partial \Omega, \\
v &= 0 & \text{on } \Sigma(\pm T). \\
\end{aligned}
\]  

(48)

The corresponding linearized problem of (48) is

\[
\begin{aligned}
- \Delta v + v \cdot \nabla g + g \cdot \nabla v + \nabla p &= h & \text{in } \Omega_T, \\
\text{div } v &= 0 & \text{in } \Omega_T, \\
v \cdot n &= 0, \quad n \cdot D(v) \cdot \tau &= 0 & \text{on } \partial \Omega_T \cap \partial \Omega, \\
v &= 0 & \text{on } \Sigma(\pm T). \\
\end{aligned}
\]  

(49)

The weak solutions of problems (48) and (49) can be defined as follows.

**Definition 4.1.** A vector field \( v \in H_\sigma(\Omega_T) \) is a weak solution of the problem (48) and (49) if for any \( \phi \in H_\sigma(\Omega_T) \), \( v \) satisfies

\[
\begin{aligned}
\int_{\Omega_T} 2D(v) : D(\phi) + (v \cdot \nabla g + (g + v) \cdot \nabla v) \cdot \phi \, dx &= \int_{\Omega_T} \Delta g \cdot \phi - g \cdot \nabla g \cdot \phi \, dx, \\
\int_{\Omega_T} 2D(v) : D(\phi) + (v \cdot \nabla g + g \cdot \nabla v) \cdot \phi \, dx &= \int_{\Omega_T} h \cdot \phi \, dx,
\end{aligned}
\]  

(50) and (51) respectively.

Next, we use Leray-Schauder fixed point theorem (cf. [18, Theorem 11.3]) to prove the existence of solutions to the approximate problem (48). To this end, the existence of solutions to the linearized problem (49) is first established by the following lemma.

**Lemma 4.1.** For any \( T > L + 1 \) and any \( h \in L^4(\Omega_T) \), there exists a unique \( v \in H_\sigma(\Omega_T) \) such that for any \( \phi \in H_\sigma(\Omega_T) \), it holds that

\[
\begin{aligned}
\int_{\Omega_T} 2D(v) : D(\phi) + (v \cdot \nabla g + g \cdot \nabla v) \cdot \phi \, dx &= \int_{\Omega_T} h \cdot \phi \, dx.
\end{aligned}
\]  

(52)
Proof. The proof is based on Lax-Milgram theorem. For any \( v, u \in H_\sigma(\Omega_T) \), define the bilinear functional on \( H_\sigma(\Omega_T) \)

\[
B[v, u] = \int_{\Omega_T} 2D(v) : D(u) + (v \cdot \nabla g + g \cdot \nabla v) \cdot u \, dx.
\]

Since \( g \) is bounded on \( \Omega \), using Hölder inequality yields

\[
|B[v, u]| \leq C \|v\|_{H^1(\Omega_T)} \|u\|_{H^1(\Omega_T)}.
\]

According to Lemma 2.3, it holds that

\[
c \|\nabla v\|_{L^2(\Omega_T)}^2 \leq 2 \|D(v)\|_{L^2(\Omega_T)}^2,
\]

where \( c \) is independent of \( T \), and is given in Lemma 2.3. For any \( v \in H_\sigma(\Omega_T) \), one has also \( v \in H_\sigma(\Omega) \) by extending \( v \) to the whole channel \( \Omega \) by zero. Using Lemma 3.2 and setting \( \delta = \frac{c}{2} \), for arbitrary flux \( \Phi \), one choose sufficiently small \( \varepsilon \) and sufficiently large \( D \) such that

\[
\left| \int_{\Omega_T} v \cdot \nabla g \cdot v \, dx \right| \leq \frac{c}{2} \|\nabla v\|_{L^2(\Omega_T)}^2.
\]

Moreover, using integration by parts gives

\[
\int_{\Omega_T} g \cdot \nabla v \cdot v \, dx = 0.
\]

Therefore, combining (53) and (55)-(57), and using Lemma 2.1 one has

\[
B[v, v] \geq \frac{c}{2(1 + M_1^2)} \|v\|_{H^1(\Omega_T)}^2.
\]

By Lemma 2.1, the constant \( M_1 \) is uniformly bounded for any \( T \).

For any \( \Phi \in H_\sigma(\Omega_T) \), one uses Hölder inequality and Lemma 2.2 to obtain

\[
\left| \int_{\Omega_T} h \cdot \Phi \, dx \right| \leq \|h\|_{L^4(\Omega_T)} \|\Phi\|_{L^4(\Omega_T)} \leq C \|h\|_{L^4(\Omega_T)} \|\nabla \Phi\|_{L^2(\Omega_T)}.
\]

It follows from (54), (58)-(59), and Lax-Milgram theorem that there exists a unique \( v \in H_\sigma(\Omega_T) \) such that (52) holds for any \( \Phi \in H_\sigma(\Omega_T) \). This finishes the proof of the lemma. \( \square \)

Now we are ready to prove the existence of solutions for the approximate problem (48).

**Proposition 4.2.** For any \( T > L + 1 \), the problem (48) has a weak solution \( v \in H_\sigma(\Omega_T) \) satisfying

\[
\|v\|^2_{H^1(\Omega_T)} \leq C_0 \int_{\Omega_T} |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx,
\]

where the constant \( C_0 \) is independent of \( T \).
Proof. Lemma 4.1 defines a map $T$ which maps $h \in L^4(\Omega)$ to $v \in H_\sigma(\Omega_T)$. For any $w \in H_\sigma(\Omega_T)$, using H"older inequality and Lemma 2.2 gives

$$\|w \cdot \nabla w\|_{L^4} \leq \|w\|_{L^4(\Omega)} \|\nabla w\|_{L^2(\Omega)} \leq C\|\nabla w\|_{L^2(\Omega_T)}^2.$$ 

Note that $\Delta g - g \cdot \nabla g \in L^4(\Omega)$. Hence $h = \Delta g - g \cdot \nabla g - w \cdot \nabla w \in L^4(\Omega_T)$ and one could define the map

$$K(w) := T(\Delta g - g \cdot \nabla g - w \cdot \nabla w).$$

It follows from Lemma 4.1 that $K$ is a map from $H_\sigma(\Omega_T)$ to $H_\sigma(\Omega_T)$. Solving the problem (48) is equivalent to finding a fixed point for

$$K(v) = v.$$ 

In order to apply Leray-Schauder fixed point theorem, we show that $K : H_\sigma(\Omega_T) \rightarrow H_\sigma(\Omega_T)$ is continuous and compact. First, for any $v^1, v^2 \in H_\sigma(\Omega_T)$, integration by parts yields

$$\left| \int_{\Omega_T} (v^1 \cdot \nabla v^1 - v^2 \cdot \nabla v^2) \cdot \phi \, dx \right|$$

$$= \left| \int_{\Omega_T} v^1 \cdot \nabla \phi \cdot v^1 - v^2 \cdot \nabla \phi \cdot v^2 \, dx \right|$$

$$= \left| \int_{\Omega_T} v^1 \cdot \nabla \phi \cdot (v^1 - v^2) + (v^2 - v^1) \cdot \nabla \phi \cdot v^2 \, dx \right|$$

$$\leq C(\|v^1\|_{L^4(\Omega_T)} + \|v^2\|_{L^4(\Omega_T)}) \|v^1 - v^2\|_{L^4(\Omega_T)} \|\phi\|_{H^1(\Omega_T)}.$$ 

Hence it holds that

$$\|K(v^1) - K(v^2)\|_{H^1(\Omega_T)} \leq C\|T(v^1 \cdot \nabla v^1 - v^2 \cdot \nabla v^2)\|_{H^1(\Omega_T)}$$

$$\leq C(\|v^1\|_{L^4(\Omega_T)} + \|v^2\|_{L^4(\Omega_T)}) \|v^1 - v^2\|_{L^4(\Omega_T)}.$$ 

This implies that $K$ is a continuous map from $H_\sigma(\Omega_T)$ into itself. Moreover, the compactness of $K$ follows from the compactness of the Sobolev embedding $H^1(\Omega_T) \hookrightarrow L^4(\Omega_T)$.

Finally, if $v \in H_\sigma(\Omega_T)$ satisfies $v = \sigma K(v)$ with $\sigma \in [0, 1]$, then for any $\phi \in H_\sigma(\Omega_T)$,

$$\int_{\Omega_T} 2D(v) : D(\phi) + (v \cdot \nabla g + g \cdot \nabla v) \cdot \phi \, dx = \sigma \int_{\Omega_T} (\Delta g - g \cdot \nabla g - v \cdot \nabla v) \cdot \phi \, dx.$$ 

In particular, taking $\phi = v$ in (61) yields

$$\int_{\Omega_T} 2|D(v)|^2 + (v \cdot \nabla g + g \cdot \nabla v) \cdot v \, dx = \sigma \int_{\Omega_T} (\Delta g - g \cdot \nabla g - v \cdot \nabla v) \cdot v \, dx.$$ 

Noting that $g \cdot n = v \cdot n = 0$ on $\partial \Omega \cap \partial \Omega_T$, and $v = 0$ on $\Sigma(\pm T)$, one uses integration by parts to obtain

$$
\left| \int_{\Omega_T} (\Delta g - g \cdot \nabla g - v \cdot \nabla v) \cdot v \, dx \right| = \left| \int_{\Omega_T} -2D(g) : D(v) - g \cdot \nabla g \cdot v \, dx \right|
$$

$$
\leq C \left( \int_{\Omega_T} |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx \right)^{\frac{1}{2}} \|\nabla v\|_{L^2(\Omega_T)}.
$$

This, together with (58) and (62), gives

$$
\|v\|_{H^1(\Omega_T)}^2 \leq C_0 \int_{\Omega_T} |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx.
$$

Then Leray-Schauder fixed point theorem shows that there exists a solution $v \in H_\sigma(\Omega_T)$ of the problem $v = K(v)$. Hence the proof of the proposition is completed. \square

For $\Omega_T$ with $T \in \mathbb{Z}^+$ and $T > L + 1$, let $v^T$ be the solution of the approximate problem (48), which is obtained in Proposition 4.2. In particular, $v^T \in H_\sigma(\Omega)$ if we extend $v^T$ by zero to the whole channel $\Omega$. By Proposition 4.2, $\{v^T\}$ is a bounded sequence in $H_\sigma(\Omega)$. Hence there exists a subsequence, which converges weakly in $H_\sigma(\Omega)$ to the solution $v$ of the problem (7). Moreover, $v$ satisfies the estimate

$$
\|v\|_{H^1(\Omega)} \leq C \left( \int_{\Omega} |\nabla g|^2 + |g \cdot \nabla g|^2 \, dx \right)^{\frac{1}{2}} =: C_1,
$$

where the constant $C_1$ is a constant depending only on the flux $\Phi$ and $\Omega$. Then we conclude the existence of the solutions to the problem (1), (2), and (4).

**Proposition 4.3.** The problem (1), (2), and (4) has a solution $u = g + v$ satisfying $v \in H_\sigma(\Omega)$ and

$$
\|v\|_{H^1(\Omega)} \leq C_1.
$$

In particular, the constant $C_1$ goes to zero of the same order of $\Phi$ when $\Phi \to 0$.

When the existence of weak solutions is established, one can further obtain the corresponding pressure by using the following lemma, whose proof can be found in [16, Theorem III.5.3].
Proposition 4.4. The vector field \( v \in H_\sigma(\Omega) \) is a weak solution of the problem (7) if and only if there exists a function \( p \in L^2_{\text{loc}}(\Omega) \) such that for any \( \phi \in H(\Omega) \), it holds that

\[
\int_\Omega 2D(v) : D(\phi) + (v \cdot \nabla g + (g + v) \cdot \nabla v) \cdot \phi \, dx - \int_\Omega p \, \text{div} \phi \, dx = \int_\Omega (\Delta g - g \cdot \nabla g) \cdot \phi \, dx.
\]

If the boundary \( \partial \Omega \) is smooth, we can improve the global regularity of the weak solutions \((u, p)\) obtained in Propositions 4.3-4.4 and obtain the following regularity theorem. One may refer to [30, Theorem C] for the details of the proof.

Proposition 4.5. For \( C^\infty\)-smooth functions \( f_1, f_2 \), the solution \((u, p)\) to the problem (1), (2), and (4), which is obtained in Propositions 4.3 and 4.4, belongs to \( C^\infty(\Omega) \).

The boundedness of the \( H^1\)-norm of \( v = u - g \) implies the convergence of \( u \) to \( U \) at far field. In particular, we can show the exponential convergence rate of the solution \( u \) as follows.

Proposition 4.6. Let \( u = v + g \) be a solution to the problem (1), (2), and (4), which is obtained in Proposition 4.3. Then there exist constants \( C_2 \) and \( C_3 \) such that for any \( T \geq 2D + 1 \), it holds that

\[
\|u - U\|_{H^1(\Omega \cap \{|x_1| > T\})} \leq C_3 e^{-C_2^{-1}T}.
\]

Proof. For any \( t \geq 1 + 2D \), if \( k \) is much larger than \( t \), we introduce the truncating function

\[
\zeta_k^+(x_1, t) = \begin{cases} 
0 & \text{if } x_1 \in (-\infty, t-1), \\
x_1 - t + 1 & \text{if } x_1 \in [t-1, t], \\
1 & \text{if } x_1 \in (t, k), \\
k + 1 - x_1 & \text{if } x_1 \in [k, k+1], \\
0 & \text{if } x_1 \in (k+1, \infty).
\end{cases}
\]

Denote

\[
E^+ = \{x \in \Omega : x_1 \in (t-1, t)\}.
\]

Clearly, \( |\partial_{x_1} \zeta_k^+| = 1 \) in \( E^+ \) and \( \Omega_{k,k+1} \).
According to the formula (16), one uses integration by parts to obtain

$$
\int_{\Omega} \zeta_k^+ |\nabla v|^2 + \partial_{x_1} \zeta_k^+ \partial_{x_1} v \cdot v \, dx - \int_{\partial \Omega} \zeta_k^+ n \cdot \nabla v \cdot v \, ds
= \int_{\Omega} -\Delta v \cdot (\zeta_k^+ v) \, dx = \int_{\Omega} -2\text{div}(v) \cdot (\zeta_k^+ v) \, dx
$$

(66)

Therefore, one has

$$
\int_{\Omega} \zeta_k^+ |\nabla v|^2 \, dx = \int_{\Omega} 2D(v) : D(\zeta_k^+ v) \, dx - \int_{E^+} \partial_{x_1} v \cdot v \, dx + \int_{\Omega_{k,k+1}} \partial_{x_1} v \cdot v \, dx
+ \int_{\partial \Omega} \zeta_k^+ n \cdot \nabla v \cdot v \, ds.
$$

(67)

The boundary condition $v \cdot n = 0$ also implies that $\partial_{\tau}(v \cdot n) = 0$ on the boundary $\partial \Omega$. Then one has

$$
\zeta_k^+ n \cdot \nabla v \cdot v = 2\zeta_k^+ n \cdot D(v) \cdot v - \zeta_k^+ v \cdot \nabla v \cdot n
= -\zeta_k^+(v \cdot \tau)[\partial_{\tau}(v \cdot n) - v \cdot \partial_{\tau} n]
= \zeta_k^+(v \cdot \tau)(v \cdot \partial_{\tau} n)
$$

(68)

Noting that $\partial_{\tau} n = 0$ on $\text{supp} \ z_k^+ = \Omega_{t-1,k+1}$, one combines (67) and (68) to obtain

$$
\int_{\Omega} \zeta_k^+ |\nabla v|^2 \, dx = \int_{\Omega} 2D(v) : D(\zeta_k^+ v) \, dx - \int_{E^+} \partial_{x_1} v \cdot v \, dx + \int_{\Omega_{k,k+1}} \partial_{x_1} v \cdot v \, dx.
$$

(69)

This, together with Lemma 2.1, gives

$$
\int_{\Omega} \zeta_k^+ |\nabla v|^2 \, dx \leq \int_{\Omega} 2D(v) : D(\zeta_k^+ v) \, dx + \|v\|_{L^2(E^+ \cup \Omega_{k,k+1})} \|\nabla v\|_{L^2(E^+ \cup \Omega_{k,k+1})}
\leq \int_{\Omega} 2D(v) : D(\zeta_k^+ v) \, dx + C\|\nabla v\|_{L^2(E^+)}^2 + C\|\nabla v\|_{L^2(\Omega_{k,k+1})}^2.
$$

(70)

Taking the test function $\phi = \zeta_k^+ v$ in (64) and noting $\nabla g = 0$ in $\text{supp} \ z_k^+ = \Omega_{t-1,k+1}$, one has

$$
\int_{\Omega} 2D(v) : D(\zeta_k^+ v) + (g + v) \cdot \nabla v \cdot (\zeta_k^+ v) \, dx - \int_{\Omega} p \text{div}(\zeta_k^+ v) \, dx = 0.
$$

(71)
Moreover, using integration by parts and Lemmas 2.1, 2.2 gives

\[
\frac{1}{2} \int_\Omega |(g + \nu) \cdot \nabla \nu \cdot (\zeta^k_1 \nu)|^2 \, dx = \int_\Omega \frac{1}{2} \partial_{x_1} \zeta_1^k (g_1 + \nu_1) \nu_1^2 \, dx
\]

(72)

\[
\leq \frac{\Phi}{4} \| \nu \|_{L^2(E^+ \cup \Omega_{h,k+1})}^2 + \frac{1}{2} \| \nu_1 \|_{L^2(E^+)}^2 + \frac{1}{2} \| \nu_1 \|_{L^2(\Omega_{h,k+1})}^2 \| \nu \|_{L^2(\Omega_{h,k+1})}^2
\]

\[
\leq \frac{\Phi}{4} \| \nabla \nu \|_{L^2(E^+ \cup \Omega_{h,k+1})}^2 + C \| \nabla \nu \|_{L^2(E^+)}^3 + C \| \nabla \nu \|_{L^2(\Omega_{h,k+1})}^3
\]

\[
\leq C \| \nabla \nu \|_{L^2(E^+ \cup \Omega_{h,k+1})}^2,
\]

where the boundedness

\[
\| \nu \|_{L^2(E^+ \cup \Omega_{h,k+1})} \leq \| \nu \|_{H^1(\Omega)} \leq C_1
\]

has been used in the last inequality.

The most troublesome term involves the pressure \( p \). Here we adapt a method introduced in [27], by making use of the Bogovskii map. Note

\[
\int_\Omega p \text{div}(\zeta^k_1 \nu) \, dx = \int_\Omega p \nu_1 \partial_{x_1} \zeta_1^k \, dx = \int_{E^+} p \nu_1 \, dx - \int_{\Omega_{h,k+1}} p \nu_1 \, dx.
\]

Since \( \nu_1 \in L^2(\Omega_{h,k+1}) \), it follows from Lemma 2.4 that there exists a vector field \( \alpha \in H^1_0(\Omega_{h,k+1}) \) satisfying

\[
\text{div} \, \alpha = \nu_1 \quad \text{in} \quad E^+
\]

and

\[
\| \nabla \alpha \|_{L^2(E^+)} \leq M_5 \| \nu_1 \|_{L^2(E^+)},
\]

Here \( M_5 = M_5(E^+) \) is a uniform constant since each \( E^+ \) is a star-like domain with respect to a ball with radius \( \frac{1}{2} \). One uses integration by parts and the equality (64) with \( \phi = \alpha \) to obtain

\[
\int_{E^+} p \nu_1 \, dx = \int_{E^+} p \text{div} \, \alpha \, dx
\]

\[
= \int_{E^+} 2D(\nu) : D(\alpha) + (g + \nu) \cdot \nabla \nu \cdot \alpha \, dx
\]

\[
= \int_{E^+} 2D(\nu) : D(\alpha) - (g + \nu) \cdot \nabla \alpha \cdot \nu \, dx
\]

\[
\leq C \left( \| \nabla \nu \|_{L^2(E^+)} + \| \nu \|_{L^2(E^+)} + \| \nu \|_{L^2(E^+)}^2 \right) \| \nabla \alpha \|_{L^2(E^+)}
\]

\[
\leq C \left( \| \nabla \nu \|_{L^2(E^+)} + \| \nu \|_{L^2(E^+)} + \| \nu \|_{L^2(E^+)}^2 \right) \| \nu \|_{L^2(E^+)} \leq C \| \nabla \nu \|_{L^2(E^+)}^2,
\]
where Lemmas 2.1 and 2.2, and Proposition 4.3 have been used to get the last inequality. Similarly, one can prove that
\[ \left| \int_{\Omega_{k,k+1}} pv_1 \, dx \right| \leq C \| \nabla v \|_{L^2(\Omega_{k,k+1})}^2. \]

Hence
\[ (73) \quad \left| \int_{\Omega} p \text{div}(\zeta_k^+ v) \, dx \right| \leq C \| \nabla v \|_{L^2(E^+ \cup \Omega_{k,k+1})}^2. \]

Combining (71) and (70)-(73) gives
\[ (74) \quad \int_{\Omega} \zeta_k^+ |\nabla v|^2 \, dx \leq C \| \nabla v \|_{L^2(E^+ \cup \Omega_{k,k+1})}^2. \]

Let \( k \) go to +\( \infty \), one has
\[ (75) \quad \int_{\Omega} \zeta^+ |\nabla v|^2 \, dx \leq C_2 \| \nabla v \|_{L^2(E^+)}^2, \]

where
\[ \zeta^+(x_1,t) = \begin{cases} 0 & \text{if } x_1 \in (-\infty, t-1), \\ x_1 - t + 1 & \text{if } x_1 \in [t-1, t], \\ 1 & \text{if } x_1 \in (t, \infty). \end{cases} \]

Define
\[ y^+(t) = \int_{\Omega} \zeta^+ |\nabla v|^2 \, dx. \]

The straightforward computations give
\[ (y^+)'(t) = \int_{\Omega} \partial_t \zeta^+ |\nabla v|^2 \, dx = - \int_{E^+} |\nabla v|^2 \, dx. \]

Hence the energy inequality (75) can be rewritten as
\[ y^+(t) \leq -C_2 (y^+)'(t). \]

Integrating the inequality with respect to \( t \) over \([2D + 1, T]\) for any \( T > 2D + 1 \) and using Proposition 4.3 yield
\[ y^+(T) \leq e^{C_2(2D+1)T} y^+(2D + 1) e^{-C_2^{-1}T} \leq C_3 e^{-C_2^{-1}T}. \]

This, together with Lemma 2.1, implies that
\[ \| u - U \|_{H^1(\Omega \cap \{x_1 > T\})} = \| v \|_{H^1(\Omega \cap \{x_1 > T\})} \leq y^+(T) \leq C_3 e^{-C_2^{-1}T}. \]

Similarly, one can also prove
\[ \| u - U \|_{H^1(\Omega \cap \{x_1 < -T\})} \leq C_3 e^{-C_2^{-1}T}. \]
Hence the proof of the proposition is completed. □

5. Uniqueness of solutions

In this section, the uniqueness of the solution obtained in Proposition 4.3 is proved. We first show that the Dirichlet norm of the solution \( u \) is uniformly bounded in any sub-domain \( \Omega_{t-1,t} \).

**Lemma 5.1.** Let \( u \) be the solution obtained in Proposition 4.3. Then there exists a constant \( C_6 \) such that for any \( t \in \mathbb{R} \), it holds that

\[
\|u\|_{H^1(\Omega_{t-1,t})} + \|u\|_{L^4(\Omega_{t-1,t})} \leq C_6
\]

and

\[
\|\nabla u\|_{L^2(\Omega_{t})} \leq C_6.
\]

In particular, there exists a constant \( \Phi_1 \) such that if \( \Phi \in [0, \Phi_1) \), then

\[
C_6 \leq C\Phi.
\]

**Proof.** Write \( u = g + v \) with \( v \in \mathcal{H}_\sigma(\Omega) \). By Proposition 4.3 one has

\[
\|\nabla v\|_{L^2(\Omega_{t-1,t})} \leq \|v\|_{H^1(\Omega)} \leq C_1.
\]

Using Lemma 2.2 one has

\[
\|v\|_{L^3(\Omega_{t-1,t})} \leq C\|\nabla v\|_{L^2(\Omega_{t-1,t})} \leq C.
\]

On the other hand, it follows from the definition (28) and (29) of \( g \) that one has

\[
|g| + |\nabla g| \leq C(\varepsilon, \mathcal{D})\Phi.
\]

In particular, the constant \( C(\varepsilon, \mathcal{D})\Phi \) goes to zero of the same order of \( \Phi \) as \( \Phi \to 0 \). Thus,

\[
\|g\|_{H^1(\Omega_{t-1,t})} + \|g\|_{L^4(\Omega_{t-1,t})} \leq C(\varepsilon, \mathcal{D})\Phi
\]

and

\[
\|\nabla g\|_{L^2(\Omega_{t})} \leq C(\varepsilon, \mathcal{D})\Phi.
\]

Combining (76)-(79), we finish the proof of this lemma. □

With the help of the uniform estimate obtained in Lemma 5.1, we can prove the uniqueness of the solution when the flux is sufficiently small.
Proposition 5.2. Let \( \mathbf{u} \) be the solution obtained in Theorem 1.4. Assume that \( \tilde{\mathbf{u}} \) is also a smooth solution of problem (1), (2), and (4) satisfying

\[
\liminf_{t \to \infty} t^{-3}\| \nabla \tilde{\mathbf{u}} \|_{L^2(\Omega_t)}^2 = 0.
\]

There exists a constant \( \Phi_0 > 0 \) such that if \( \Phi \in [0, \Phi_0) \), then \( \mathbf{u} = \tilde{\mathbf{u}} \).

Proof. We divide the proof into five steps.

Step 1. Set up. The straightforward computations show that \( \mathbf{w} := \tilde{\mathbf{u}} - \mathbf{u} \) is a solution to the equations

\[
\begin{cases}
- \Delta \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla p = 0 & \text{in } \Omega, \\
\text{div } \mathbf{w} = 0 & \text{in } \Omega, \\
\mathbf{w} \cdot \mathbf{n} = 0, \quad \mathbf{n} \cdot \mathbf{D}(\mathbf{w}) \cdot \tau = 0 & \text{on } \partial \Omega, \\
\int_{\Sigma(x_1)} \mathbf{w} \cdot \mathbf{n} \, ds = 0 & \text{for any } x_1 \in \mathbb{R}.
\end{cases}
\]

(80)

Then we introduce the truncating function \( \zeta(x,t) \) with \( t \geq L + 2 \) on \( \Omega \) as follows.

\[
\zeta(x,t) = \begin{cases}
1, & \text{if } x_1 \in (-t + 1, t - 1), \\
0, & \text{if } x_1 \in (-\infty, -t) \cup (t, \infty), \\
t - x_1, & \text{if } x_1 \in [t - 1, t], \\
t + x_1, & \text{if } x_1 \in [-t, -t + 1].
\end{cases}
\]

Clearly, \( \zeta \) depends only on \( t \) and \( x_1 \). Furthermore, \( \partial_t \zeta = |\partial_{x_1} \zeta| = 1 \) in \( E = E^+ \cup E^- \), where

\[
E^- = \{ \mathbf{x} \in \Omega : x_1 \in (-t, -t + 1) \} \quad \text{and} \quad E^+ = \{ \mathbf{x} \in \Omega : x_1 \in (t - 1, t) \}.
\]

Step 3. Energy estimates. Multiply the first equation in (80) by \( \zeta \mathbf{w} \) and integrating the result equation over \( \Omega \). Using integration by parts, one has

\[
\int_{\Omega} 2\mathbf{D}(\mathbf{w}) : \mathbf{D}(\zeta \mathbf{w}) + (\mathbf{w} \cdot \nabla \mathbf{u} + (\mathbf{u} + \mathbf{w}) \cdot \nabla \mathbf{w}) \cdot \zeta \mathbf{w} - pw_1 \partial_{x_1} \zeta \, dx = 0.
\]

(81)

Similar to the proof of the equality (69) in Proposition 4.6, one can also obtain

\[
\int_{\Omega} \zeta |\nabla \mathbf{w}|^2 \, dx = \int_{\Omega} 2\mathbf{D}(\mathbf{w}) : \mathbf{D}(\zeta \mathbf{w}) \, dx - \int_E \partial_{x_1} \mathbf{w} \cdot \mathbf{w} \, dx + \int_{\partial \Omega} \zeta (\mathbf{w} \cdot \tau)(\mathbf{w} \cdot \partial_n) \, ds.
\]

(82)

Noting that \( \partial_n \mathbf{n} = 0 \) on \( \partial \Omega \setminus \partial \Omega_{L+1} \) and \( \zeta = 1 \) in \( \Omega_{L+1} \), it follows from (82) that one has

\[
\int_{\Omega} \zeta |\nabla \mathbf{w}|^2 \, dx \leq \int_{\Omega} 2\mathbf{D}(\mathbf{w}) : \mathbf{D}(\zeta \mathbf{w}) \, dx - \int_E \partial_{x_1} \mathbf{w} \cdot \mathbf{w} \, dx + C_4 \int_{\partial \Omega \setminus \partial \Omega_{L+1}} |\mathbf{w}|^2 \, ds,
\]

(83)
where \( C_4 \) is defined in (20). Following the proof of (21) in Lemma 2.3 one has

\[
C_4 \int_{\partial \Omega \cap \partial \Omega_{L+1}} |w|^2 \, ds \leq \frac{1}{2} \| \nabla w \|_{L^2(\Omega_{L+1})}^2 + C_5 \| \mathbf{D}(w) \|_{L^2(\Omega_{L+1})}^2;
\]

where \( C_5 \) is a constant independent of \( t \). This, together with (83) and Lemma 2.1 gives

\[
\frac{1}{2} \int_{\Omega} \zeta |\nabla w|^2 \, dx \leq (2 + C_5) \int_{\Omega} \mathbf{D}(w) : \mathbf{D}(\zeta w) \, dx + C \| \nabla w \|_{L^2(E)} \| w \|_{L^2(E)}
\]

Hence one has

\[
(84) \quad \mathfrak{c} \int_{\Omega} \zeta |\nabla w|^2 \, dx \leq \int_{\Omega} 2 \mathbf{D}(w) : \mathbf{D}(\zeta w) \, dx + C \| \nabla w \|_{L^2(E)}^2,
\]

where \( \mathfrak{c} \) is defined in (23). Moreover, one uses integration by parts, Lemmas 2.1-2.2 and Proposition 5.1 to obtain

\[
(85) \quad - \int_{\Omega} (u \cdot \nabla w + w \cdot \nabla w) \cdot (\zeta w) \, dx = \int_{E} \frac{1}{2} |w|^2 (u_1 + w_1) \partial x_1 \zeta \, dx
\]

\[
\leq \| w \|_{L^4(E)}^2 (\| w \|_{L^2(E)} + \| u \|_{L^2(E)})
\]

\[
\leq C \| \nabla w \|_{L^2(E)}^3 + C \| \nabla w \|_{L^2(E)}^2
\]

and

\[
(86) \quad - \int_{\Omega} w \cdot \nabla u \cdot (\zeta w) \, dx
\]

\[
= \int_{\Omega} \zeta w \cdot \nabla w \cdot u \, dx + \int_{E} (w \cdot u) w_1 \partial x_1 \zeta \, dx
\]

\[
= \int_{\Omega_{t-1}} w \cdot \nabla w \cdot u \, dx + \int_{E} \zeta w \cdot \nabla w \cdot u + (w \cdot u) w_1 \partial x_1 \zeta \, dx
\]

\[
\leq \int_{\Omega_{t-1}} w \cdot \nabla w \cdot u \, dx + (\| \nabla w \|_{L^2(E)} + \| w \|_{L^2(E)}) \| w \|_{L^2(E)} \| u \|_{L^2(E)}
\]

\[
\leq \int_{\Omega_{t-1}} w \cdot \nabla w \cdot u \, dx + C \| \nabla w \|_{L^2(E)}^2.
\]

Decompose \( \Omega_{t-1} \) into several parts \( D_i^t = \{ x \in \Omega : x_1 \in (A_{i-1}, A_i) \} \), where \(-t + 1 = A_0 \leq A_1 \leq \cdots \leq A_{N(t)} = t - 1 \) and \( \frac{1}{2} \leq A_i - A_{i-1} \leq 1 \) for every \( i \). By Lemma 2.2 and Lemma 5.1
one has
\[
\int_{\Omega_{t-1}} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u} \, dx \leq \sum_{i=1}^{N(t)} \int_{D_i^t} |\mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u}| \, dx
\]
\[
\leq \sum_{i=1}^{N(t)} \|\nabla \mathbf{w}\|_{L^2(D_i^t)} \|\mathbf{w}\|_{L^4(D_i^t)} \|\mathbf{u}\|_{L^4(D_i^t)}
\]
\[
\leq C_7 \sum_{i=1}^{N(t)} \|\nabla \mathbf{w}\|_{L^2(D_i^t)}^2
\]
\[
= C_7 \int_{\Omega_{t-1}} |\nabla \mathbf{w}|^2 \, dx.
\]

By virtue of Lemma 5.1, the constant $|C_7| \leq C\Phi$ if $\Phi$ is sufficiently small. Then there exists a $\Phi_0 > 0$ such that for any $\Phi \in [0, \Phi_0)$, one has
\[
(87) \quad \int_{\Omega_{t-1}} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u} \, dx \leq \frac{c}{2} \int_{\Omega} |\nabla \mathbf{w}|^2 \, dx.
\]

**Step 4. Estimate for pressure term.** For the term involving pressure, similar to the proof of Proposition 4.6, there exists a vector field $\mathbf{a} \in H_0^1(E^\pm)$ satisfying
\[
\text{div} \, \mathbf{a} = w_1 \quad \text{in} \ E^\pm
\]
and
\[
\|\nabla \mathbf{a}\|_{L^2(E^\pm)} \leq M_5 \|w_1\|_{L^2(E^\pm)}.
\]

Then one uses integration by parts and the equation (80) to obtain
\[
\left| \int_{E^\pm} pw_1 \partial_{x_1} \zeta \, dx \right| = \left| \int_{E^\pm} pw_1 \, dx \right| = \left| \int_{E^\pm} p \text{div} \, \mathbf{a} \, dx \right|
\]
\[
= \left| \int_{E^\pm} (-\Delta \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w}) \cdot \mathbf{a} \, dx \right|
\]
\[
= \left| \int_{E^\pm} \nabla \mathbf{w} : \nabla \mathbf{a} - \mathbf{w} \cdot \nabla \mathbf{a} \cdot \mathbf{u} - (\mathbf{u} + \mathbf{w}) \cdot \nabla \mathbf{a} \cdot \mathbf{w} \, dx \right|
\]
\[
\leq C \left( \|\nabla \mathbf{w}\|_{L^2(E^\pm)} + \|\mathbf{w}\|_{L^4(E^\pm)} \|\mathbf{u}\|_{L^4(E^\pm)} + \|\mathbf{w}\|_{L^4(E^\pm)}^2 \right) \|\nabla \mathbf{a}\|_{L^2(E^\pm)}
\]
\[
\leq C \left( \|\nabla \mathbf{w}\|_{L^2(E^\pm)} + \|\mathbf{w}\|_{L^4(E^\pm)} \|\mathbf{u}\|_{L^4(E^\pm)} + \|\mathbf{w}\|_{L^4(E^\pm)}^2 \right) \|w_1\|_{L^2(E^\pm)}.
\]

Using Lemmas 2.1, 2.2, and 5.1, one has
\[
(88) \quad \left| \int_{E^\pm} pw_1 \partial_{x_1} \zeta \, dx \right| \leq C \|\nabla \mathbf{w}\|_{L^2(E^\pm)}^2 + C \|\nabla \mathbf{w}\|_{L^2(E^\pm)}^3.
\]
Combining (81) and (84)-(88) gives

\[ \frac{c}{2} \int_{\Omega} \zeta |\nabla w|^2 \, dx \leq C \| \nabla w \|_{L^2(E)}^2 + C \| \nabla w \|_{L^2(E)}^3. \]

**Step 5. Growth estimate.** Define

\[ y(t) = \int_{\Omega} \zeta |\nabla w|^2 \, dx. \]

The straightforward computations give

\[ y'(t) = \int_{\Omega} \partial_t \zeta |\nabla w|^2 \, dx = \int_{E} |\nabla w|^2 \, dx. \]

Then the energy inequality (89) can also be written as

\[ y(t) \leq C_8 \left\{ y'(t) + \left[ y'(t) \right]^\frac{3}{2} \right\}. \]

Set

\[ \Psi(\tau) = C_8 (t + t^\frac{3}{2}) \quad \text{and} \quad m = \frac{3}{2}. \]

It follows from Lemma 2.5 that either \( w = 0 \) or

\[ \liminf_{t \to +\infty} \frac{y(t)}{t^3} > 0. \]

This finishes the proof of the proposition. \( \square \)

Combining Propositions 4.3, 4.6, and 5.2, we finish the proof of Theorem 1.1.

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**References**

[1] P. Acevedo Tapia, C. Amrouche, C. Conca, and A. Ghosh, Stokes and Navier-Stokes equations with Navier boundary conditions, *J. Differ. Equ.*, 285(2014), 1515–1547.

[2] C. J. Amick, Steady solutions of the Navier-Stokes equations in unbounded channels and pipes, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 4(1977), 473–513.

[3] C. J. Amick, Properties of steady Navier-Stokes solutions for certain unbounded channels and pipes, *Nonlinear Anal.*, 2(1978), 689–720.

[4] C. J. Amick and L. E. Fraenkel, Steady solutions of the Navier-Stokes equations representing plane flow in channels of various types, *Acta Math.*, 144(1980), 83–151.

[5] C. Amrouche, P. Penel and N. Seloula, Some remarks on the boundary conditions in the theory of Navier-Stokes equations, *Ann. Math. Blaise Pascal*, 20(2013), no. 1, 37–73.
[6] C. Amrouche and A. Rejaiba, $L^p$-theory for Stokes and Navier-Stokes equations with Navier boundary condition, *J. Differ. Equ.*. 256 (2014), no. 4, 1515–1547.

[7] H. Beirão da Veiga, Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions, *Adv. Differ. Equ.*. 9 (2004), no. 9-10, 1079–1114.

[8] H. Beirão da Veiga, Remarks on the Navier-Stokes evolution equations under slip type boundary conditions with linear friction, *Port. Math.*, 64 (2007), 377–387.

[9] H. Beirão da Veiga, On the regularity of flows with Ladyzhenskaya shear-dependent viscosity and slip or nonslip boundary conditions, *Comm. Pure Appl. Math.*, 58 (2005), no. 4, 552–577.

[10] L.C. Berselli, An elementary approach to the 3D Navier-Stokes equations with Navier boundary conditions: existence and uniqueness of various classes of solutions in the flat boundary case, *Discrete Contin. Dyn. Syst., Ser. S*, 3 (2010), no. 2, 199–219.

[11] M. E. Bogovskiǐ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, *Dokl. Akad. Nauk SSSR*, 248 (1979), no. 5, 1037–1040.

[12] T. Clopeau, A. Mikelić, R. Robert, On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions, *Nonlinearity*, 11 (1998), 1625–1636.

[13] C. Conca, On the application of the homogenization theory to a class of problems arising in fluid mechanics, *J. Math. Pures Appl.*, 64 (1985), no. 1, 31–75.

[14] S. Ding and Z. Lin, Stability for two-dimensional plane Couette flow to the incompressible Navier-Stokes equations with Navier boundary conditions, *Commun. Math. Sci.*, 18 (2020), no. 5, 1233–1258.

[15] S. Ding, Q. Li and Z. Xin, Stability analysis for the incompressible Navier-Stokes equations with Navier boundary conditions, *J. Math. Fluid Mech.*, 20 (2018), no. 2, 603–629.

[16] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*. Springer, New-York, 2011.

[17] G. P. Galdi and W. J. Layton, Approximation of the larger eddies in fluid motions. II. A model for space-filtered flow, *Math. Models Methods Appl. Sci.*, 10 (2000), no. 3, 343–350.

[18] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*. 2nd Ed. Springer-Verlag: Berlin.

[19] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd edition, Cambridge University Press, 1952.

[20] D. Iftimie, F. Sueur, Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions, *Arch. Ration. Mech. Anal.*, 199 (2011), no. 1, 145–175.

[21] W. Jäger, A. Mikelić, On the roughness-induced effective boundary conditions for an incompressible viscous flow, *J. Differ. Equ.*, 170 (2001), no. 1, 96–122.

[22] Z. Li, X. Pan, and J. Yang, On Leray’s problem in an infinite-long pipe with the Navier-slip boundary condition, arXiv:2204.10578.

[23] H. Li and X. Zhang, Stability of plane Couette flow for the compressible Navier-Stokes equations with Navier-slip boundary, *J. Differ. Equ.*, 263 (2017), no. 2, 1160–1187.

[24] L. V. Kapitanski and K. I. Piletskas, Spaces of solenoidal vector fields and boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries. (Russian) Boundary value problems of mathematical physics, 12. Trudy Mat. Inst. Steklov. 159 (1983), 5–36.

[25] J. P. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane, *SIAM J. Math. Anal.*, 38 (2006), no. 1, 210–232.
[26] P. Konieczny, On a steady flow in a three-dimensional infinite pipe, \textit{Colloq. Math.}, \textbf{104}(2006), 33-56.
[27] O. A. Ladyzhenskaja and V. A. Solonnikov, Determination of solutions of boundary value problems for stationary Stokes and Navier-Stokes equations having an unbounded Dirichlet integral, \textit{Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)}, \textbf{96}(1980), 117–160.
[28] N. Masmoudi, F. Rousset, Uniform regularity for the Navier-Stokes equation with Navier boundary condition, \textit{Arch. Ration. Mech. Anal.}, \textbf{203}(2012), no. 2, 529–575.
[29] D. Medková, One problem of the Navier type for the Stokes system in planar domains, \textit{J. Differ. Equ.}, \textbf{261}(2016), no. 10, 5670–5689.
[30] P. B. Mucha, On Navier-Stokes equations with slip boundary conditions in an infinite pipe, \textit{Acta Appl. Math.}, \textbf{76}(2003), 1–15.
[31] P. B. Mucha, The Navier-Stokes equations and the maximum principle, \textit{Int. Math. Res. Not.}, \textbf{67}(2004), 3585–3605.
[32] P. B. Mucha, Asymptotic behavior of a steady flow in a two-dimensional pipe, \textit{Studia Math.}, \textbf{158}(2003), no. 1, 39–58.
[33] C. L. M. H. Navier, Mémoire sur les Lois du Mouvement des Fluides, \textit{Mem. Acad. Sci. Inst. de France}, \textbf{6}(1823), 389–440.
[34] S. A. Nazarov and K. I. Piletskas, Behavior of solutions of Stokes and Navier-Stokes systems in domains with periodically changing cross-section. (Russian) Boundary value problems of mathematical physics, 12. \textit{Trudy Mat. Inst. Steklov.}, \textbf{159} (1983), 95–102.
[35] S. A. Nazarov and K. I. Piletskas, The Reynolds flow of a fluid in a thin three-dimensional channel, \textit{Litovsk. Mat. Sb.}, \textbf{30} (1990), no. 4, 772–783.
[36] K. Sha, Y. Wang, and C. Xie, On the Steady Navier-Stokes system with Navier slip boundary conditions in two-dimensional channels, preprint, 2022, \texttt{arXiv:2210.15204}.
[37] V. A. Solonnikov and V. E. Ščadilov, On a boundary value problem for a stationary system of Navier-Stokes equations, \textit{Trudy Mat. Inst. Steklov.} \textbf{125}(1973), 196–210.
[38] X. Wang, Y. Wang and Z. Xin, Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit, \textit{Commun. Math. Sci.}, \textbf{8}(2010), no. 4, 965–998.
[39] Y. Wang and C. Xie, Existence and asymptotic behavior of large axisymmetric solutions for steady Navier-Stokes system in a pipe, \textit{Arch. Ration. Mech. Anal.}, \textbf{243}(2022), no. 3, 1325–1360.
[40] Y. Xiao and Z. Xin, On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, \textit{Comm. Pure Appl. Math.}, \textbf{60}(2007), no. 7, 1027–1055.
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