One-step $G$-unimprovable numbers\textsuperscript{1}

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Abstract: The infinitude is established of the set $U_1$ of all integers $N > 5$ whose Gronwall numbers $G(N)$ do not increase when replacing $N$ by $N/q$ or $Np$, where $q,p$ are primes, $q | N$.

Keywords: Gronwall numbers, Ramanujan - Robin inequality, Caveney-Nicolas-Sondow Hypothesis

Bibliography: 6 items

1. Notations and problem setting

As usually let $\log x$ and $\sigma(n)$ stand (resp.) for the natural logarithm of a positive $x$ and the sum of all divisors of a positive integer $n$. In 1913 T. Gronwall established [1] the limit relationship which involves the Euler-Masceroni constant $\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \frac{1}{k} - \log n \right) = 0.577215..$:

$$\limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma = 1.78107.. \quad (1.1)$$

S. Ramanujan has noticed (in 1915, the first publication in 1997 [2]), that: if the Riemann Hypothesis on non-trivial zeros of $\zeta(s)$ holds true, then in addition to (1) for all sufficiently large $n$ the strict inequality $\sigma(n) < e^\gamma n \log \log n$; $n > n_0$, $\quad (1.2)$
is fulfilled. 70 years later G. Robin [3] proved a paramount assertion which in a sense is inverse to Ramanujan’s result, namely:

if (1.2) holds true for all integers $n > 5040$, then RH is valid.

Detailed discussion of historical aspects and adjacent questions may be found in a remarkable Caveney-Nicolas-Sondow paper [4], (which in fact was the initial point for this author’s research) where it is proved that RRI is in turn equivalent to the statement:

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Every integer \( N > 5 \) is \( G \)-improvable, i. e. either:

\[ I^l : \text{there is a prime } q \mid N, \text{ such that } G(N/q) > G(N), \quad \text{or:} \]

\[ I^x : \text{there is an integer } a > 1 \text{ such that } G(Na) > G(N). \]

Note that the numbers 3, 4 and 5 are not \( G \)-improvable.

We will consider a new class \( U_1 \) of all such integers \( N > 5 \) which cannot be \( G \)-improved neither by multiplication nor by division by any single prime.

**Definition.** An integer \( N > 5 \) is called 1-step \( G \)-umimprovable \((N \in U_1)\) if and only if the following two conditions hold:

\[ U^l_1 : \text{for any prime } q \mid N, \text{ one has } G(N/q) \leq G(N), \quad \text{and:} \]

\[ U^x_1 : \text{for any prime } p \text{ one has } G(Np) \leq G(N). \]

**Remark 1.** The condition \( U^l_1 \) (in [4], S. 5 it was studied under the name GA1) is exactly the negation of \( I^l \) whereas \( U^x_1 \) is essentially weaker than the negation of \( I^x \). Thus the CNSH is equivalent to \( U_1 \subset I^x \).

The purpose of this paper is to establish the infinitude of \( U_1 \) and to construct the explicit algorithm which successively calculates all elements of this class the least of them being equal

\[
N^*_1 = 2^5 \cdot 3^3 \cdot 5^2 
\cdot 7 
\cdot 11 
\cdot 13 
\cdot 17 
\cdot 19 
\cdot 23 = 160626866400; \ G(N^*_1) = 1.7374... \quad (1.3)
\]

The author believes this approach to be helpful for the proof of the CNSH.

**Preliminary results**

Let \( \mathbb{N} \) be the set of all positive integers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( (a, b) := \gcd(a, b) \) — the greatest common divisor of \( a, b \in \mathbb{N} \); \( \mathbb{P} := \{p_k\}_1^\infty \) — increasing sequence of all primes; the notation

\[
N \parallel p^\alpha, \ p \in \mathbb{P}, \ \alpha \in \mathbb{N}_0, \text{ means that } N = p^\alpha m, \ (m, p) = 1. \quad (2.1)
\]

First we will study how the Gronwall number changes if one replaces \( N \) by \( Np, \ p \in \mathbb{P} \). Let us denote

\[
\lambda \equiv \lambda(p, \alpha) := \frac{p^{\alpha+2} - 1}{p^{\alpha+2} - p} = 1 + \frac{1}{p^2 + \ldots + p^{\alpha+1}}; \quad (2.2)
\]

and let \( \xi := \xi(p, \alpha) \) be the unique positive root of the equation

\[
\xi^{\lambda(p, \alpha)} = \xi + \log p \iff \xi^{\lambda(p, \alpha) - 1} = 1 + \frac{\log p}{\xi} \iff \log \xi = (p + p^2 + \ldots + p^{\alpha+1}) \log \left( 1 + \frac{\log p}{\xi} \right). \quad (2.3)
\]
Lemma 1. Let \( N > 1, \ N \parallel p^\alpha; \) then three following conditions are equivalent (comp. the definition of \( U_1^x \)):

(i) \( G(Np) > G(N), \) (ii) \((\log N)\lambda > \log N + \log p, \) (iii) \( \log N > \xi(p, \alpha). \)

\( \triangle \) By virtue of the classical number theory formula (cf. [5], Ch 1, § 2d):

\[
(p, m) = 1 \Rightarrow \sigma(p^\alpha m) = \frac{p^{\alpha+1} - 1}{p - 1} \sigma(m); \tag{2.5}
\]

whence the identities follow:

\[
\frac{G(p^{\alpha+1}m)}{G(p^{\alpha}m)} = \frac{p^{\alpha+2} - 1}{p - 1} \cdot \frac{p - 1}{p^{\alpha+1} - 1} \cdot \frac{p^\alpha \log \log(p^\alpha m)}{p^{\alpha+1} \log \log(p^{\alpha+1}m)}
\]

\[
= \frac{\lambda \log \log(N)}{\log \log(N + \log p)} = \frac{(\log(N))\lambda}{\log \log(N + \log p)}.
\tag{2.6}
\]

Therefore the inequality \( G(Np) > G(N) \) is equivalent to

\[
\frac{G(p^{\alpha+1}m)}{G(p^{\alpha}m)} > 1 \iff (\log p^\alpha m)^\lambda > \log p^\alpha m + \log p; \tag{2.7}
\]

this proves the equivalence of (i) and (ii) in (2.4). It remains to note that since the function \( \lambda(p, \alpha) \) decreases as either \( p \) or \( \alpha \) increase (cf. (2.2)), the root \( \xi(p, \alpha) \) of (2.3) is an increasing function of both \( p \) and \( \alpha \)  \( \square \)

Lemma 2. Let the exponent \( \alpha \) in Lemma 1 be positive; then the following three conditions are equivalent (cp. \( U^x \)):

(i) \( G(N/p) \leq G(N), \) (ii) \((\log N - \log p)^\lambda(p, \alpha - 1) \geq \log N, \)

(iii) \( \log N \geq \log p + \xi(p, \alpha - 1). \)

\( \triangle \) It suffices to apply Lemma 1 to the numbers \( p \) and \( \tilde{N} := N/p \parallel p^{\alpha-1} \)  \( \square \)

Remark 2. The assertion of Lemma 2 in somewhat different form was given in [4, Proposition 15]. It was also observed that the integer

\[
\nu := 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 183 \, 783 \, 600 \tag{2.9}
\]

is the least \( N > 4 \) such that for all primes \( q \mid N, \) the inequality \( G(N/q) < G(N) \) is fulfilled. However \( G(19\nu) = 1.7238.. > G(\nu) = 1.7175.. \) and thus \( \nu \notin U_1. \)

Joining the two Lemmas yields the assertion helpful in the sequel.

Proposition 1. Let \( N \parallel p^\alpha, \alpha > 0; \) then following two properties are equivalent:
(i) $N$ is $G_p$-unimprovable, i.e. $G(N) \leq \min(G(N/p), G(Np))$,

(ii) $\log N \in \Delta_{p,\alpha} := [\xi(p, \alpha - 1) + \log p, \xi(p, \alpha)]$. \hfill (2.10)

For fixed prime $p$ the segments $\Delta_{p,\alpha}$, $\alpha \in \mathbb{N}$ are nonempty and disjoint; in the case when $\log N$ belongs to the junction interval $I_\alpha := (\xi(p, \alpha), \xi(p, \alpha) + \log p)$, the number $Np$ is $G_p$-unimprovable, i.e. $G(Np) \leq \min(G(N), G(Np^2))$.

We also need the bilateral estimates for the roots of equations (2.3) (more sharp as $\alpha = 0$) which are obtained by standard Analysis techniques.

Lemma 3. For all $\alpha \in \mathbb{N}_0$, $p \in \mathbb{P}$ the following inequalities are fulfilled:

(i) $p - \log p < \xi(p, 0) < p$;

(ii) $\frac{p^{\alpha+1}}{\alpha + 1} < \xi(p, \alpha) < 3 \frac{p^{\alpha+1}}{\alpha + 1}$, $\alpha \in \mathbb{N}$. \hfill (2.11)

\(\triangle \quad (i) \) The determining equation (2.3) $\xi^{1+\frac{1}{p}} = \xi + \log p$ for $\xi(p, 0)$ may be rewritten in the form:

$$f(\xi) := \frac{\log \xi}{p} - \log \left(1 + \frac{\log p}{\xi}\right) = 0,$$ \hfill (2.12)

where $f(\xi)$ is a strictly increasing function for $\xi > 0$. From the first inequality

(i) $\log(1 + z) < z$, ($z > -1$, $z \neq 0$),

(ii) $\log(1 + z) > z - 0.5z^2$ ($z > 0$) \hfill (2.13)

and (2.12) it follows that $f(p) > 0$. Further for $\xi_1 := p - \log p$ one obtains analogously

$$f(\xi_1) = \frac{\log(p - \log p)}{p} - \log \left(1 + \frac{\log p}{p - \log p}\right)$$

$$= - \log p + \left(1 + \frac{1}{p}\right) \log(p - \log p) = \frac{\log p}{p} + \left(1 + \frac{1}{p}\right) \log \left(1 - \frac{\log p}{p}\right)$$

$$< \frac{\log p}{p} - \left(1 + \frac{1}{p}\right) \frac{\log p}{p} = - \frac{\log p}{p^2} < 0 \quad \square$$ \hfill (2.14)

\(\triangle \quad (ii) \) Now let $\alpha > 0$; by introducing two new notations:

$$w = w(p, \alpha) := 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^\alpha}; \quad t := (\alpha + 1)\xi \quad \frac{p^{\alpha+1}}{p^{\alpha+1}}, \quad \text{i.e.} \quad \xi = \frac{tp^{\alpha+1}}{\alpha + 1}.$$ \hfill (2.15)
one can rewrite the determining equation (2.3) in the following equivalent form:

\[
g(t) := \log t - \log(\alpha + 1) + (\alpha + 1) \log p \]

\[-p^{\alpha+1}w \log \left(1 + \frac{(\alpha + 1) \log p}{t p^{\alpha+1}}\right) = 0. \tag{2.16}\]

For \(p, \alpha\) being fixed the function \(g(t)\) is strictly increasing; therefore the assertion (ii) of Lemma 3 is equivalent to two inequalities: \(g(3) > 0, g(1) < 0\).

To prove the first of them we again apply (2.12)(i), and taking into account that \(w < 2\), we obtain

\[
g(3) > \log 3 - \log(\alpha + 1) + (\alpha + 1) \log p - \frac{w}{3}(\alpha + 1) \log p \tag{2.17}\]

\[> \log 3 - \log(\alpha + 1) + \frac{\alpha + 1}{3} \log p \geq 1 + \log \log p \geq 1 + \log \log 2 = 0.633.\]

To demonstrate that \(g(1) < 0\), we take use of the inequality (2.12)(ii):

\[
g(1) < -\log(\alpha + 1) + (\alpha + 1) \log p - w(\alpha + 1) \log p \left(1 - \frac{(\alpha + 1) \log p}{2 p^{\alpha+1}}\right) \]

\[= -\log(\alpha + 1) + \left(1 - w\left(1 - \frac{(\alpha + 1) \log p}{2 p^{\alpha+1}}\right)\right)(\alpha + 1) \log p \tag{2.18}\]

\[< -\log(\alpha + 1) + \left(1 - \left(1 + \frac{1}{p}\right)\left(1 - \frac{(\alpha + 1) \log p}{2 p^{\alpha+1}}\right)\right)(\alpha + 1) \log p < 0.\]

Here we have taken into account the relationships valid for all \(p \geq 2, \alpha \geq 1\)

\[w \geq 1 + \frac{1}{p}; \quad \frac{(\alpha + 1) \log p}{2 p^{\alpha+1}} \leq \frac{\log p}{p^2} \implies \left(1 + \frac{1}{p}\right)\left(1 - \frac{\log p}{p^2}\right) > 1, \tag{2.19}\]

whence it follows that the large bracket in the last line (2.18) is also negative \(\square\)

The rather simple assertion concerning non-negative sequences will be also frequently helpful for us.

L e m m a 4. Let \(A > 0, B > 0\); denote by

\[\tau^* = \tau^*(A, B) := A + 0.5B(B + \sqrt{B^2 + 4A}) \tag{2.20}\]

the root of the equation \(\tau = A + B\sqrt{\tau}\); suppose that \(a_n \geq 0, a_{n+1} \leq A + B\sqrt{a_n}\) \(\forall n \in \mathbb{N}_0\); then if \(a_0 \leq \tau^*\), then \(a_n \leq \tau^*\) for all \(n\).

\(\triangle\) It suffices to consider the auxiliary sequence \(b_n \geq a_n\) defined recursively as \(b_0 := a_0, b_{n+1} = A + B\sqrt{b_n}\) which increases to its limit \(\tau^* \quad \square\)
3. Structure and properties of numbers from the class $U_1$

We will use the first and the second Chebyshev functions (cf. [6], 3.2, p. 104)

$$\theta(x) := \sum_{p \leq x} \log p; \quad \psi(x) := \sum_{p^k \leq x} \log p = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots,$$  \hspace{1cm} (3.1)

as well as Chebyshev products $T(p_n) := \exp(\theta(p_n)) = p_1 \cdot p_2 \cdot \ldots \cdot p_n$.

Let $P(N)$ stand for the greatest prime factor of an integer $N > 1$.

**Theorem 1.** Let the integer $N > 5$ have the canonical factorization

$$N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}, \quad \alpha_k > 0, \text{ i.e. } P(N) = p_k.$$  \hspace{1cm} (3.2)

Then (A): the belonging $N \in U_1$ is equivalent to the set of inequalities:

(i) $\xi(p_j, \alpha_j - 1) + \log p_j \leq \log N \quad \forall j \leq k$ such that $\alpha_j > 0$;

(ii) $\log N \leq \xi(p_i, \alpha_i) \quad \forall i \leq k$;

(iii) $\log N \leq \xi(p_{k+1}, 0).$  \hspace{1cm} (3.3)

(B1): the exponents $\alpha_j$ do not increase: $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$ and $\alpha_k = 1$.

$\Delta$ The equivalence of conditions $N \in U_1$ and (3.3)(i)-(iii) follows immediately from definitions of $U_1^/$ and $U_1^\times$ and assertions of lemmas 2 and 1 with taking into account that $\xi(p_n, 0) > \xi(p_{k+1}, 0)$ for all $n > k + 1$.

Further assuming that $\alpha_i < \alpha_j$ for some $i, j, 1 \leq i < j \leq k$, one obtains bearing in mind the monotone increase of $\xi(p, \alpha)$ with respect to both $p$ and $\alpha$

$$\xi(p_j, \alpha_j - 1) + \log p_j \geq \xi(p_j, \alpha_i) + \log p_j > \xi(p_i, \alpha_i),$$

and thus for all $N > 5$ the violation the exponents monotonicity would lead to the incompatibility of inequalities (3.3)(i) and (3.3)(ii) for $\log N$.

At last, if one supposes $\alpha_k \geq 2$, then from (3.3)(i), (2.11)(ii) it follows that

$$\log N \geq \xi(p_k, 1) + \log p_k > 0.5p_k^2 + \log p_k > p_{k+1} > \xi(p_{k+1}, 0),$$  \hspace{1cm} (3.4)

which is in contradiction with (3.3)(iii) for $k \geq 2$  \hspace{1cm} $\square$

4. Locally $G$-extremal numbers

We will start with the algorithm which generates the uniquely defined sequence of positive integers $V_k$, possessing some part of properties, prescribed by Theorem 1, and then we will prove that the class $U_1$ of all one-step $G$-unimprovable numbers, i. e. such that satisfy all inequalities (3.3), is its infinite subsequence $\{V_{k_m}\} =: \{N_m^*\}$, with indices $k_1 < k_2 < \ldots < k_m < \ldots$ also constructively calculated.
Theorem 2. (I) For any fixed $k \geq 4$ there exists the least (and thus unique) integer $V_k := p_1^{\alpha_{1,k}} \cdot \ldots \cdot p_k^{\alpha_{k,k}}$, where the exponents $\alpha_{j,k}, 1 \leq j \leq k$ satisfy the condition $(B_1)$ of Theorem 1, and the inequalities are fulfilled:

$$\xi(p_j, \alpha_{j,k} - 1) + \log p_j \leq \log V_k \leq \xi(p_j, \alpha_{j,k}); \quad 1 \leq j < k; \quad (4.1)$$

(II) If some other integer $\tilde{V}_k := p_1^{\tilde{\alpha}_{1,k}} \cdot \ldots \cdot p_k^{\tilde{\alpha}_{k,k}}$ possesses the same properties as $V_k$ (with replacing $V_k$ by $\tilde{V}_k$ and $\alpha_{j,k}$ by $\tilde{\alpha}_{j,k}$), then $V_k | \tilde{V}_k$.

(III) Moreover the relationship $\log V_k - \theta(p_k) = C_k \sqrt{p_k}$ holds where $0.5 < C_k < 3$.

$\triangle$ Step 1. According to Lemmas 1 and 2 the left and the right inequalities (4.1) are equivalent (respectively) to the fulfilment for the number $V_k$ of the conditions $U_1^\prime$ and $U_1^\times$ in which the primes $q, p$ are subject to the restrictions $q, p < p_k$; in other words in definition of $U_1$ the transitions $V_k \to V_k/p_k$ and $V_k \to V_k p_n, n \geq k$ are excluded.

Let us put $Y_k^{(0)} = T(p_k) := p_1 \cdot \ldots \cdot p_k$ (cf. (3.1)), and then define inductively for $s \in \mathbb{N}$:

$$\beta_{j,k,s} := \max \{ \beta \in \mathbb{N} : \xi(p_j, \beta - 1) + \log p_j \leq Y_k^{(s-1)} \}; \quad j < k;$$

$$\beta_{k,k,s} := 1; \quad Y_k^{(s)} := \prod_{j=1}^{k} p_j^{\beta_{j,k,s}}. \quad (4.2)$$

For $k \geq 4$ one has $\log Y_k^{(0)} = \theta(p_k) \geq \log 210 = 5.347.. > \xi(2, 1) + \log 2 = 2.56..$; therefore by virtue of Lemma 2 (cf. (2.8)(iii) the relationship $\beta_{1,k,1} \geq 2$ holds, and hence $Y_k^{(1)} > Y_k^{(0)}$. The increase of function $\xi(p, \alpha)$ with respect to both $p$ and $\alpha$ implies the condition $(B_1)$ of Theorem 1 for the exponents $\{\beta_{j,k,s}\}_{s=0}$, $j < k$; $k, s$ being fixed.

From the first line of defining formula (4.2) it follows that $\beta_{j,k,s} \geq \beta_{j,k,s-1}$, and thus $Y_k^{(s)} \geq Y_k^{(s-1)}$. Moreover, by virtue of the same formula one obtains for all $j \in \{1, 2, \ldots, k - 1\}, s \in \mathbb{N}$:

$$\xi(p_j, \beta_{j,k,s} - 1) + \log p_j \leq \log Y_k^{(s-1)} < \xi(p_j, \beta_{j,k,s}) + \log p_j. \quad (4.3)$$

Step 2. Further, let us denote $q_{m}^{(1)} := \max \{ p_j : \beta_{j,k,1} \geq m \}, m = 1, 2, \ldots, \beta_{1,k,1}$. It is clear that $q_1^{(1)} = p_k > q_2^{(1)} = \ldots \geq q_{\beta_{1,k,1}}^{(1)} = 2$. From the estimates (2.11)(ii) one may conclude that for all sufficiently large $k$ (for small $k$ the numbers $C_k$ in (III) are calculated directly) and for all $m > 1$ there holds $q_m^{(1)} = \tilde{C}_{k,m}^{(1)} p_k^{1/m}, 0.7 < \tilde{C}_{k,m}^{(1)} < 2; \quad \beta_{1,k,1}^{(1)} < 1.5 \log \log Y_k^{(0)}$, and hence:
\( \log Y_k^{(1)} = \log(Y_k^{(0)}) + C_k^{(1)} \sqrt{Y_k^{(0)}} + O(p_k^{1/3} \log \log Y_k^{(0)}) < \theta(p_k) + 2\sqrt{Y_k^{(0)}}. \) (4.4)

Going on to argue in the same way as in Step 1 (with replacing the upper index (1) by (s)), one obtains \( \log Y_k^{(s)} < \log Y_k^{(0)} + 2\sqrt{Y_k^{(s-1)}}. \)

According to Lemma 4 these relationships imply the upper estimate: \( \log Y_k^{(s)} < \theta(p_k) + 2 + 2\sqrt{1 + \theta(p_k)} \) for all \( s \in \mathbb{N} \), and because the sequence \( \{Y_k^{(s)}\}_{s=0}^{\infty} \) does not decrease and is integer-valued, it necessarily stabilizes beginning with certain number \( s_0 := s_0(k) \), and then, by virtue of definition (??), beginning with the same number the exponents sequences \( \beta_{j,k,s} \) stabilize as well, and the inequalities \( \xi(p_j, \beta_{j,k} - 1) + \log p_j \leq \log Y_k < \xi(p_j, \beta_{j,k}) + \log p_j \) hold true.

**Step 3.** Now introduce the set of indices \( E_0 := \{ j < k : \log Y_k > \xi(p_j, \beta_{j,k}) \} \). It will turn out to be empty then all conditions of Theorem 2 are fulfilled for \( V_k^{(0)} := Y_k \). Otherwise, i.e. if \( \log Y_k \in I_{p_j, \beta_{j,k}} \) for some \( j < k \) (cf. (2.10) ff), we put \( \alpha_{j,k}^{(0)} := \beta_{j,k} \), and in accordance with Proposition 1 define recurrently for \( s \in \mathbb{N} \):

\[
\alpha_{j,k}^{(s)} := \beta_{j,k} + 1, \ j \in E_{s-1}; \quad \alpha_{j,k}^{(s)} := \beta_{j,k}, \ j \notin E_{s-1};
\]

\[
V_k^{(s)} := \prod_{j=1}^{k} p_j^{\alpha_{j,k}^{(s)}}, \quad E_s := \{ j < k : \log V_k^{(s)} > \xi(p_j, \beta_{j,k}) \}. \quad (4.5)
\]

It’s clear that \( V_k^{(s)} \geq V_k^{(s-1)}, E_{s-1} \subset E_s, \alpha_{j,k}^{(s)} \geq \alpha_{j,k}^{(s-1)} \). It should be emphasized that no integer \( j \in E_{s-1} \) may repeatedly arise in \( E_s \setminus E_{s-1} \), because this would imply that \( \sum \{ \log p : p \in E_{s-1} \} > \xi(p_j, \beta_{j,k} + 1) - \xi(p_j, \beta_{j,k}) - \log p_j \) what is impossible by virtue of estimates adduced earlier.

Now using the arguments analogous to those of Step 2 one may deduce the estimate \( \log V_k^{(s)} \leq \log V_k^{(0)} + C_k \sqrt{\log V_k^{(s-1)}} \), and hence again follows the stabilization of \( V_k^{(s)} \), \( \alpha_{j,k}^{(s)} \), \( E_s \) for all \( s \geq s^* := s^*(k) \). By the very construction the established value \( V_k := V_k^{(s^*)} \) satisfies all inequalities (4.1).

**Step 4.** It is easy to check that all constructions of the Steps 1 – 3 have been accomplished by supplementing of minimally required prime factors to the initial number \( T(p_k) \). Therefore if for some \( \tilde{V}_k \) the assertions of the part (I) of the Theorem 2 hold true, then necessarily \( \tilde{V}_k \geq V_k \), and by virtue of these very inequalities (4.1) one has \( \tilde{\alpha}_{j,k} \geq \alpha_{j,k} \) for all \( j \leq k \), i.e. \( V_k \mid \tilde{V}_k \).

**Step 5.** The estimates of the part (III) were obtained during the proof.
The comparison of Theorems 1 and 2 (cf. (3.3), (4.1)) yields the assertion

**Proposition 2.** In order the number \( V_k, k \geq 4 \), constructed in Theorem 2, to be a one-step \( G \)-unimprovable \((V_k \in U_1)\) it is necessary and sufficient that in addition to (4.1) two more inequalities:

\[
\xi(p_k, 0) + \log p_k \leq \log V_k \leq \xi(p_{k+1}, 0)
\]  
(4.6)

hold true.

**Table 1:** The first 6 one-step \( G \)-unimprovable numbers computed by Maple - 13 according to algorithms given in Theorem 2 with taking into account the filtering relationships (4.6).

The last column contains the numbers in the estimates of part (III).

| \( m \) | \( k_m \) | \( N_m^* := V_{k_m} \) | \( G(N_m^*) \) | \( C_{k_m} \) |
|---|---|---|---|---|
| 1 | 9 | \( T(23) \cdot T(5) \cdot 3 \cdot 2^3 = 160 \, 626 \, 866 \, 400 \) | 1.7374.. | 1.37.. |
| 2 | 11 | \( T(31) \cdot T(7) \cdot 3^4 \cdot 2^4 = 2.02.. \cdot 10^{15} \) | 1.7368.. | 1.65.. |
| 3 | 16 | \( T(53) \cdot T(7) \cdot 3^2 \cdot 2^3 = 1.97.. \cdot 10^{24} \) | 1.7434.. | 1.51.. |
| 4 | 34 | \( T(139) \cdot T(13) \cdot T(5) \cdot 3^2 \cdot 2^6 = 5.19.. \cdot 10^{63} \) | 1.7582.. | 1.70.. |
| 5 | 99 | \( T(523) \cdot T(29) \cdot T(7) \cdot 5 \cdot 3^4 \cdot 2^8 = 4.08.. \cdot 10^{233} \) | 1.770728.. | 1.67.. |
| 6 | 101 | \( T(547) \cdot T(31) \cdot T(7) \cdot 5 \cdot 3^4 \cdot 2^8 = 3.75.. \cdot 10^{240} \) | 1.770765.. | 1.78.. |

5. The infinitude of the set \( U_1 \).

The key role in proving this assertion play simple sufficient conditions of \( G \)-improvability of the numbers \( V_k \) in terms of the values \( p_k, p_{k+1}, \theta(p_k) \) only.

**Proposition 3.** (i) For any \( n > 4 \)

\[
p_{n+1} < \theta(p_n) + 0.5 \sqrt{p_n} \Rightarrow G(V_n p_{n+1}) > G(V_n).
\]

(ii): For any \( m > 4 \)

\[
p_m > \theta(p_m) + 4 \sqrt{p_m} \Rightarrow G(V_m/p_m) > G(V_m).
\]

\( \Delta \) Indeed, in the case (i), taking into account of the part (III) of the Theorem 2 with the lower \( C_k > 0.5 \) one can conclude that:

\[
\log V_n > \theta(p_n) + 0.5 \sqrt{p_n} > p_{n+1} > \xi(p_{n+1}, 0),
\]

and then apply Lemma 1.

In the case (ii), analogously from the same relationship (III) with the upper estimate \( C_k < 4 \) we can derive the chain of inequalities

\[
\log V_m < \theta(p_m) + 4 \sqrt{p_m} < p_m < \xi(p_m, 0) + \log p_m,
\]

and then take use of Lemma 2 \( \square \)
Now we can formulate the main result of this paper.

Theorem 3. For any $M > 0$ there exists an integer $r$ such that $p_r > M$ and the number $V_r$ is one-step $G$-unimprovable, i.e. $V_r \in U_1$.

$\triangle$ According to the known Littlewood theorem (1914, cf. [6], Thm 6.20 and notations on p. (xi)) the difference $\psi(x) - x = \Omega_{\pm}(\sqrt{x \log \log \log x})$.

In particular, from here it follows (and that’s enough for us), that for arbitrary $M > 100$ there are the numbers $x, y, y > 2x > M$ such that

$$x < \theta(x) - 10\sqrt{x}, \quad y > \theta(y) + 10\sqrt{y}$$  \hspace{1cm} (5.5)

Let us denote $p_n := \max\{p \in \mathbb{P} : p \leq x\}, \ p_m := \min\{p \in \mathbb{P} : p \geq y\}$.

For any $A \in [-10, 10]$ the function $t - \theta(t) - A\sqrt{t}$ has the jumps $(- \log p_k)$ at the points $p_k$ and increases on every semi-segment $(p_k, p_{k+1}]$, $p_k > 100$; therefore for the numbers $p_n$ and $p_m$ the inequalities:

$$p_n < \theta(p_n) - 8\sqrt{p_n} < \theta(p_{n-1}); \quad p_m > \theta(p_m) + 6\sqrt{p_m}.$$  \hspace{1cm} (5.6)

hold true.

Now choose $r \in \{n - 1, \ldots, m\}$ such that

$$G(V_r) = \max\{G(V_k) : k \in \{n - 1, \ldots, m\}\}.$$  \hspace{1cm} (5.7)

Proposition 3 jointly with (5.6) yield the inequalities:

$$G(V_n) \geq G(V_{n-1}p_n) > G(V_{n-1}); \ G(V_{m-1}) \geq G(V_m/p_m) > G(V_m).$$  \hspace{1cm} (5.8)

Hence in turn it follows that $r$ is strictly in between of $n - 1$ and $m$, and thus $V_r$ is necessarily one-step $G$-unimprovable $\square$
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