An optical approach to the dynamical Casimir effect

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Abstract

We have recently proposed a new approach to analyse the parametric resonance in a vibrating cavity based on the analysis of classical optical paths. This approach is used to examine various models of one-dimensional cavities with moving walls. We prove that our method is useful to extract easily basic physical outcomes.

1. Introduction

The derivation of the Casimir effect in its original context [1] was the calculation of the electromagnetic force between parallel perfectly conducting plates placed in a vacuum. The geometry of a cavity formed by two parallel plates is also the best context to manifest the non-stationary (dynamical) Casimir effect. The theoretical calculations were first carried out by Moore [2]. The force between plates is dynamically modified, but one encounters a more amazing phenomenon, that of the vacuum emission of photons with nonclassical properties. The latter topic became the subject of intensive studies since it was recognized that the production of photons could be significantly enhanced under the parametric resonance conditions [3, 4]. From the experimental point of view, hopes for observing vacuum radiation are based on the belief that vibrating cavities can be adjusted to parametric resonance conditions.

Quantum parametric resonance is a complex process and traditional and well-tried methods for calculations in quantum electrodynamics are obsolete here. We must examine open quantum systems under the action of external periodic perturbations. Moreover, the focus of our interest is unstable systems. If we outline the physics of resonant systems, then the corresponding mathematical functions for physical quantities inevitably yield narrow peaks or steep stairs. A naive approach to make perturbations in small amplitudes of cavity vibrations breaks down. Therefore, we must go beyond the standard mathematical treatment known from QFT and pursue some new machinery. Most ways of tackling the problem follow the quantization scheme described by Moore [2] in the case of one-dimensional cavities. Then, special mathematical techniques are used to trace the evolution of the quantum system with
time-dependent boundary conditions. Dodonov and his collaborators [5] reduced the analysis of field equations in a cavity system to some infinite set of coupled and time-dependent oscillators. Another idea based on a concept of a renormalization group was proposed in [6, 7]. Our suggestion [8] is to trace the evolution of the quantum fields in a vibrating cavity system following classical optical paths. This is related to the iteration procedure described in [9] and techniques developed for classical field theory in [10, 11]. In fact, the resonant evolution of the physical system is similar in classical and quantum models. From the mathematical point of view, such approaches are in fact modifications of the d’Alembert method of characteristics for solving the wave equation [12–15]. Recently, a similar idea to ground calculations on the analysis of classical optical paths has been advocated for a stationary Casimir effect in [16–19].

In the present paper, we apply our technique [8] to examine cavities with various kinds of their motions and different types of boundary conditions. The calculations are involved in determining the essential physical outcome. In particular, the conditions necessary to keep the parametric resonance and the critical exponents describing resonant evolution are derived. All results are in agreement with those obtained for cavities with moving walls using other techniques. We give more information about resonance windows and other new details. More subtle structure of energy densities is recounted in case either amplitudes increase or the adjustment of resonance parameters gets worse. As new results, we will show how the well-known picture of travelling peaks of field energy densities changes with increasing amplitudes of oscillations. There will be given exact formulae for cumulative Doppler factors, which enable us to read the evolution of the energy–momentum tensor. The mathematical correspondence between cavity models with one oscillating wall and two oscillating walls will be established.

2. A cavity with one oscillating wall

Most research on vibrating cavities has concentrated on a model of a cavity formed by a static wall and a moving one. The static wall is fixed at $x = 0$, while the moving one follows its trajectory $x = L(t)$. The obvious requirement $L(t) > 0$ is assumed. It is convenient to make use of the billiard function $f(t)$ [20] defined by the following condition:

$$f(t + L(t)) = t - L(t).$$  
(1)

This function contains full information about scattering of massless particles from the moving wall. It can be also understood as the trajectory of the wall in light-cone coordinates. Its derivatives are recognized as retarded Doppler factors (the retardation relation is $t = t^* + L(t^*)$):

$$\dot{f}(t) = \frac{1 - L(t^*)}{1 + L(t^*)}.$$  
(2)

Consider now a classical scalar field $A(t, x)$ in our one-dimensional cavity. It obeys the wave equation

$$\left(\partial_t^2 - \partial_x^2\right)A(t, x) = 0.$$  
(3)

The field is subject to some boundary conditions. Usually, we assume either Dirichlet’s boundary conditions:

$$A(t, 0) = A(t, L(t)) = 0,$$  
(4)

or Neumann’s boundary conditions:

$$\partial_x A(t, 0) = (\dot{L}(t)\partial_t + \partial_x)A(t, L(t)) = 0.$$  
(5)
We have adopted here the relativistically covariant generalization of Neumann’s boundary conditions. The field derivatives are assumed to vanish at the instantaneous rest frame of the moving cavity wall. In both cases of boundary conditions, the classical solutions are given in the following common form:

$$A(t, x) = \varphi(t + x) \mp \varphi(t - x),$$  \hspace{1cm} (6)

where the upper sign corresponds to Dirichlet’s case, while the lower one to Neumann’s case. The profile function is subject to the following relation:

$$\varphi(\tau) = \varphi(f(\tau)).$$  \hspace{1cm} (7)

The energy density of the classical wave packet and its total energy are given by:

$$T_{00}(t, x) = \frac{1}{2} (\partial_t A)^2 + \frac{1}{2} (\partial_x A)^2 = \varphi(t + x) + \varphi(t - x),$$  \hspace{1cm} (8)

$$E(t) = \int_{L(t)}^{L(t)} dx T_{00}(t, x) = \int_{-L(t)}^{L(t)} dt \varphi(\tau),$$  \hspace{1cm} (9)

where $\varphi(\tau) = \dot{\varphi}^2(\tau)$. Classical optical paths [8] are described by functions $T_n(\tau) = (f^{-1})^n(\tau)$ and $T^*_n(\tau)$ to be determined from the retardation relations $T^*_n(\tau) + L(T_n^*(\tau)) = T_n(\tau)$. The notation $(f^{-1})^n(\tau)$ means here and throughout this paper $n$-fold composition $f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}$. The parametric resonance is related to the existence of periodic particle trajectories [11, 8]. Each periodic trajectory obeys the following condition (for any non-negative integer $n$):

$$T_n(\tau_0) = \tau_0 + nT,$$  \hspace{1cm} (10)

where $\tau_0$ is a starting point in time and $T$ is a period. The retarded times are respectively $T^*_n(\tau_0) = T_n(\tau_0) - T/2$. Any periodic particle trajectory appears on condition that the mirror trajectory has return points:

$$L(T^*_n(\tau_0)) = T/2.$$  \hspace{1cm} (11)

It is usually assumed that $T/2$ refers to the length of the static cavity, so that $T/2 \equiv L \equiv L(0)$. The crucial characteristic of a particle trajectory is the cumulative Doppler factor [8, 13]:

$$D_n(\tau) = \frac{1}{T_n(\tau)} = \prod_{k=1}^{n} f(T_n(\tau)) = \prod_{k=1}^{n} \frac{1 - L(T^*_k(\tau))}{1 + L(T^*_k(\tau))}.$$  \hspace{1cm} (12)

The above function has a direct physical interpretation. If a massless particle (‘photon’) starts at time $\tau$ moving right inside the cavity, then after $n$ rebounds from both cavity walls its kinetic energy changes by the factor $D_n(\tau)$. A trajectory is called positive (or negative respectively) if its cumulative Doppler factor tends to infinity (or zero) with increasing $n$. We denote by $\tau^+$ and $\tau^-$ starting points for positive and negative periodic trajectories. If positive trajectories exist, then any trajectory approaches one of the positive trajectories for long times. The same is true for negative trajectories if we consider the evolution backward in time. In other words, $T_n(\tau) \to T_n(\tau_+) \text{ and } T^{-1}_n(\tau) \to T^{-1}_n(\tau_-)$ for large $n$.

The evolution of the energy density and the total energy can be deduced from the following formulae [8],

$$\varphi(T_n(\tau)) = \varphi(\tau)D_n^2(\tau),$$  \hspace{1cm} (13)

$$E(T^*_n(\tau_0)) = \int_{T_{n-1}(\tau_0)}^{T_n(\tau_0)} d\tau \varphi(\tau) = \int_{f(\tau_0)}^{\tau_0} d\tau \varphi(\tau)D_n(\tau).$$  \hspace{1cm} (14)
together with the asymptotic approximation for long times (i.e. $n \gg 1$):

$$
\varrho(nT + \tau_\pm + \varepsilon D_n^{-1}(\tau_\pm)) \approx \varrho(\tau_\pm + \varepsilon) D_n^2(\tau_\pm + \varepsilon),
$$

$$(15)$$

$$
\varrho(nT + \tau_\pm - \varepsilon)
\approx \varrho(\tau_\pm - \varepsilon D_n(\tau_\pm)) D_n^2(\tau_\pm - \varepsilon D_n(\tau_\pm)).
$$

$$(16)$$

The above formulae explain the formation of travelling narrow packets in the energy density $T_{00}(t, x)$ [3, 5, 21]. The profile function $\varrho(\tau)$ has peaks located at the spots of positive periodic trajectories $\tau_\pm$. The height of a peak grows like $D_n^2(\tau_\pm)$, while its width diminishes like $D_n^{-1}(\tau_\pm)$. The value of the energy density near the peak decreases like $D_n^2(\tau_\pm)$.

The basic model of vibrating cavities specified by the sinusoidal cavity wall motion was analysed with the help of classical optical paths in [8]:

$$
L(t) = L + \Delta L \sin(\omega t),
$$

$$(17)$$

where $\Delta L < L$ and $\omega \Delta L < 1$. The parametric resonance frequencies are $\omega_N = N\pi/L$, where $N$ is the order of the resonance. The cumulative Doppler factors at spots of periodic trajectories are for $\Delta L \ll L$ [8]:

$$
D_n(\tau \pm m) = \left( \frac{1 \pm \omega_N \Delta L}{1 \mp \omega_N \Delta L} \right)^n \approx \exp(\pm 2n\omega_N \Delta L).
$$

$$(18)$$

We conclude from the above formula that $N$ identical travelling peaks develop in the energy density. The average distances between peaks are equal and the total energy grows exponentially with time. These classical results agree with the results of the corresponding quantized cavity model [4]. The above well-known picture is only true for small amplitudes of cavity oscillations. Strictly speaking, we need $\Delta L/L < N$. For larger amplitudes, the behaviour of the cavity system is slightly modified. If $\Delta L/L > M/N$ ($M = 1, 2, 3, \ldots$), then there appear additional positive trajectories corresponding to the following return points [11]:

$$
\omega_N \Delta L \sin(\omega_N \tau_M \pm) = M\pi,
$$

$$(19)$$

and the corresponding cumulative Doppler factors are:

$$
D_n(\tau_M \pm) = \left( \frac{1 \pm \sqrt{(\omega \Delta L)^2 - (M\pi)^2} }{1 \mp \sqrt{(\omega \Delta L)^2 - (M\pi)^2}} \right)^n.
$$

$$(20)$$

It follows that besides the first series of $N$ peaks in the energy density, there appear next $M$ series of $N$ peaks each. The peaks of additional series are smaller and located between higher peaks.

For practical purposes, it is important to consider off-resonant behaviour of the cavity system. The instability in the field theory appears not only for finely tuned frequencies. We are usually dealing with some band structure. To establish the resonance window, we move the resonant frequency: $\omega = \omega_N + \Delta \omega$. The resonance condition is $\Delta \omega / \omega < \Delta L / L$ and the cumulative Doppler factors at starting points of periodic trajectories are [8]:

$$
D_n(\tau \pm m) = \left( \frac{1 \pm \sqrt{(\omega \Delta L)^2 - (L\Delta \omega)^2} }{1 \mp \sqrt{(\omega \Delta L)^2 - (L\Delta \omega)^2}} \right)^n.
$$

$$(21)$$

Again, the above picture is exact provided that $\Delta L/L + (1 + 1/N) \Delta \omega / \omega < 1/N$. Otherwise, the next series of smaller peaks in the energy density will appear.

Now, we recall basic quantum formulae. The vacuum expectation value of the energy density is given by [2]:

$$
\langle T_{00}(t, x) \rangle = \varrho(t + x) + \varrho(t - x).
$$

$$(22)$$
where \( \varrho(\tau) = -\pi/48 R^2(\tau) - 1/24\pi S[R](\tau) \). \( R(\tau) \) is Moore’s function and \( S[R] \) is its Schwartzian derivative. The quantum formulae for the evolution of the energy density and the total energy are given by [8]:

\[
\varrho(T_n(\tau)) = \varrho(\tau) D_n^2(\tau) + A_n(\tau) D_n^3(\tau),
\]

where the cumulative Doppler factor is defined again by the same formula (12) and the definition of the quantum term called the cumulative conformal anomaly contribution [8] is given below:

\[
A_n(\tau) = \frac{1}{24\pi} S[T_n](\tau) = -\frac{1}{24\pi} \sum_{k=1}^{n} D_{-2}^{k-2}(\tau) S[f(T_k(\tau))).
\]

The quantum anomaly implies the difference between classical and quantum formulae for the evolution of the energy. However, under the resonance conditions the leading role is played by the cumulative Doppler factor. This fact explains why most of classical results match quantum ones. It refers to the band structure of resonance frequencies, the formation and the shape of travelling packets in the energy density, the exponential growth of the total energy. To make predictions for these features, the classical theory could be good enough. But it is not true for the lowest resonance channel. In this case, the anomalous mechanism of energy growth clashes with the resonant enhancement of the initial vacuum fluctuations [8].

3. A cavity with two oscillating walls

A natural generalization of the considerations of the previous section is to allow both walls to oscillate [5, 7, 25]. Consider now that the right cavity wall moves along the trajectory \( x = L_1(t) \), while the left one follows \( x = -L_2(t) \). We assume also that the cavity was static in the past, so that \( L_1(t) = L_2(t) = L/2 \) for \( t < 0 \), and \( L \) corresponds to the static cavity length. The general formulae given in this section will be rigorously justified if we assure that \( L_1(t) > 0 \) and \( L_2(t) > 0 \) (the walls never collide) and the wall velocities are never close to the velocity of light. The billiard functions are implemented for both trajectories \( i = 1, 2 \):

\[
f_i(t + L_i(t)) = t - L_i(t).
\]

Moreover, it is convenient to define an additional pair of billiard functions:

\[
f_L = f_2 \circ f_1, \quad f_R = f_1 \circ f_2.
\]

They describe double reflections of massless particles from the cavity walls. The set of classical optical paths is now more complex. Consider again a massless particle that starts at time \( \tau \) moving from the position \( x = 0 \). Denote by \( T_{L,n}(\tau) \) (and \( T_{R,n}(\tau) \) respectively) times when the particle returns to its initial position \( x = 0 \) after \( n \) rebounds from both walls assuming that the first collision is with the left (or with the right) wall. It is easy to find that \( T_{L,R,n}(\tau) = (f_{L,R}^{-1})^{(n)}(\tau) \). Further, \( T^*_{L,R,n}(\tau) \) and \( T^{**}_{L,R,n}(\tau) \) are times of successive collisions
with the left (right) and the right (left) walls for both types of trajectories respectively. They can be derived from the following relations:

\[ T_{\star LR,0}^\tau + L_{1(2)}(T_{\star LR,0}^\tau(\tau)) = T_{\star LR,0}(\tau), \]  
\[ T_{\star**, LR,0}^\tau + L_{2(1)}(T_{\star**, LR,0}^\tau(\tau)) = f_{1(2)}(T_{\star LR,0}(\tau)). \]  

(28)  

(29)

For both types of trajectories cumulative Doppler factors are defined:

\[ D_{\star LR,0}^\tau(\tau) = \prod_{k=1}^n \frac{1 - L_{1(2)}(T_{\star LR,k}^\tau(\tau))}{1 + L_{1(2)}(T_{\star LR,k}^\tau(\tau))}. \]  

(30)

Classical solutions of a wave equation inside a cavity with two moving walls are given by:

\[ A(t, x) = \varphi_L(t + x) \mp \varphi_R(t - x), \]  

(31)

where again the upper sign corresponds to Dirichlet’s boundary conditions and the lower one to Neumann’s boundary conditions on the walls. The profile functions in both cases obey the same relations:

\[ \varphi_L(\tau) = \varphi_R(f_1(\tau)), \quad \varphi_R(\tau) = \varphi_L(f_2(\tau)). \]  

(32)

The energy density of the classical wave packet and its total energy can be represented as:

\[ T_{00}(t, x) = \varrho_L(t + x) \pm \varrho_R(t - x), \quad \varrho_{\star LR}(\tau) = \varrho_{\star LR}^2(\tau), \]  

(33)

\[ E(t) = \int_{-L(t)}^{L(t)} dx \ T_{00}(t, x) = \int_{-L_{1(2)}(t)}^{L_{1(2)}(t)} d\tau \ \varrho_L(\tau) + \int_{-L_{1(2)}(t)}^{L_{1(2)}(t)} d\tau \ \varrho_R(\tau). \]  

(34)

The classical evolution of the system can be traced with the help of the following formulae:

\[ \varrho_{\star LR}(T_{\star LR,0}(\tau)) = \varrho_{\star LR}(\tau) D_{\star LR,0}^2(\tau). \]  

(35)

\[ E(t) = \int_{f_1^R(t_2)}^{f_1^R(t_1)} d\tau \ \varrho_L(\tau) D_{\star LR}(\tau) + \int_{f_2^L(t_1)}^{f_2^L(t_2)} d\tau \ \varrho_R(\tau) D_{\star LR}(\tau), \]  

(36)

where \( r_{1(2)} \equiv f_{1(2)}^\tau(t + L_{1(2)}(t)). \) If we take \( n \) big enough to support that \( r_{1(2)} \leq L/2 \), then we obtain the formula that allows us to calculate the evolution of the total energy from the knowledge of Doppler factors and the energy distribution in the initial state.

Before we turn to examples of cavity systems, we explain the reasons why many specific systems of cavities with two oscillating walls just reproduce results similar to those obtained in cavity models with one oscillating wall. The major reason is that billiard functions \( f_L \) correspond to some physical trajectories in two dimensions. This means that we can find a trajectory \( x = L(t) \) which describes some realistic motion of the wall such that

\[ f_{LR}(t + L(t)) = t - L(t). \]  

(37)

We can verify that \( L(t) > 0, L(t) = L \) for \( t < 0 \) and \( L(t) \) is never close to the speed of light. Obviously, in general such ‘effective’ trajectories are different for \( f_L \) and \( f_R \), but in both cases
the following composition rules are satisfied:

\[ L(t) = L_1(t_1) + L_2(t_2), \]
\[ 1 - \frac{\dot{L}(t)}{1 + \dot{L}(t)} = \frac{1 - \dot{L}_1(t_1) - \dot{L}_2(t_2)}{1 + \dot{L}_1(t_1) + \dot{L}_2(t_2)}, \]

where \( t \pm L(t) = t_1 \pm L_1(t_1) \) and \( t_1 \mp L_1(t_1) = t_2 \pm L_2(t_2) \) for \( f_L \) (the upper sign) and \( f_R \) respectively.

The latter relation (39) states just that \( \dot{L}(t) \) is the relativistic sum of velocities \( \dot{L}_1(t_1) \) and \( \dot{L}_2(t_2) \). However, this composition rule has nothing to do with any relative motion since there exists no inertial frame where we could make respective time moments synchronous. Nevertheless, both left- and right-movers in cavities with two oscillating walls correspond to some models of cavities with a single oscillating wall. In other words, each mode of a cavity system with two moving walls can be described by a simpler model discussed in the previous section.

Let us discuss specific types of symmetric and antisymmetric oscillations of cavities. In the paper of Dalvit and Mazzitelli [7], these types of cavity oscillations are called breathing modes (‘electromagnetic antishaker’) and translational modes (‘electromagnetic shaker’). For symmetric oscillations we have \( L_1(t) = L_2(t) \) and the cavity oscillates symmetrically with respect to the centre of the cavity. The billiard functions and Doppler factors are identical, namely \( f_1 \equiv f_2, f_1 \equiv f_R \) and \( D_L \equiv D_R \). The model is equivalent to the cavity model with one oscillating wall, where the prescribed trajectory of the wall is defined by the billiard function \( f_1 \circ f_1 \). For antisymmetric oscillations we have \( L_1(t) + L_2(t) = L \), and the cavity is oscillating as a whole with its length kept constant in the laboratory frame. Then, \( f_{1L}(\tau) \equiv f_{2L}(\tau - L) - L \) and \( f_{1R}(\tau) \equiv f_{1L}(\tau - L) - L \). It is interesting to note that for periodic translational cavity oscillations with \( L \) being a period, the billiard functions \( f_1 \) and \( f_R \) correspond to static ones. It follows that there is no effect of the cavity movement on the energy distribution of fields and the whole cavity system is equivalent to the static one.

Finally, we analyse harmonic oscillations of cavity walls:

\[ L_1(t) = \frac{L}{2} + \Delta L_1 \sin(\omega_R t), \]
\[ L_2(t) = \frac{L}{2} + \Delta L_2 \sin(\omega_R t - \delta) + \Delta L_2 \sin \delta. \]

Again, we want to establish resonance windows, find the evolution of the total energy and describe the shape of the local energy density. We must first detect positive and negative periodic trajectories of massless particles moving inside the cavity. They can be derived from the generalization of equation (11):

\[ L_1(t_1) + L_2 \left( t_1 \pm \frac{T}{2} \right) = \frac{T}{2}, \]

where the upper (lower) sign corresponds to the ‘left’ \( T_{L,R} \) (‘right’ \( T_{R,R} \)) trajectories. The cumulative Doppler factors at the starting points of periodic trajectories can be calculated then:

\[ D_{L,R,n} = \left( \frac{1 - \omega_R \Delta L_1 \cos(\omega_R t_1)}{1 + \omega_R \Delta L_1 \cos(\omega_R t_1)} \right)^n \left( \frac{1 - \omega_R \Delta L_2 \cos(\omega_R t_1 \mp \omega_R T/2 - \delta)}{1 + \omega_R \Delta L_2 \cos(\omega_R t_1 \mp \omega_R T/2 - \delta)} \right)^n \]

\[ \equiv \exp \left( -2n[\omega_R \Delta L_1 \cos(\omega_R t_1) + \omega_R \Delta L_2 \cos(\omega_R t_1 \mp \omega_R T/2 - \delta)] \right) \text{ for } \Delta L_i \ll L. \]

First, consider resonant oscillations with \( \omega_L = \omega_R = \omega_N \equiv N\pi/L \). If there is no dephasing \( \delta = 0 \), the system is equivalent to the cavity with one oscillating wall. The oscillation
frequency is $\omega_N$, the effective cavity length is $L$ and the effective amplitude of oscillations is $\Delta L_1 + (-1)^N \Delta L_2$. For $\Delta L_1 = \Delta L_2$ and $N$ odd the effective cavity system is static. The same is true for $\Delta L_1 = \Delta L_2$, $\delta = \pi$ and $N$ even. These results agree with [7]. If the amplitudes of oscillations are equal $\Delta L_1 = \Delta L_2 \equiv \Delta L$ and $\delta \neq 0, \pi$, then the solutions of equation (41) are $t_1 = (\delta L/\pi + 2k)L/N$ and $t_1 = (N - 2k - 1)L/N$, where $k = 0, 1, \ldots, N - 1$. The respective cumulative Doppler factors are:

$$D_n(\tau_{k, \pm}) \cong \exp(\pm 2n\omega N \Delta L(1 + (-1)^N \cos \delta)).$$

(43)

There are $N$ positive and $N$ negative particle trajectories. It follows that there are $N$ peaks in the energy density. The calculations for $\Delta L_1 \neq \Delta L_2$ are a bit more complicated. However, if $\Delta L_1 + (-1)^N \Delta L_2 \cos \delta \neq 0$ then there exist $N$ positive and $N$ negative particle trajectories. The corresponding cumulative Doppler factors are given by

$$D_n \equiv \exp(\pm 2n\omega_N(\Delta L_1 + (-1)^N \Delta L_2 \cos \delta)).$$

(44)

For $\Delta L_1 + (-1)^N \Delta L_2 \cos \delta = 0$ there is no exponential instability in the cavity system.

To establish the resonance window we assume $\omega_L = \omega_R \equiv \omega_N + \Delta \omega$. Relation (41) yields

$$\Delta L_1 \sin(\omega t_1) + (-1)^N \Delta L_2 \sin(\omega t_1 - \delta) + \Delta L_2 \sin \delta = -L \Delta \omega / \omega.$$  \hspace{1cm} (45)

To keep the parametric resonance conditions for the vibrating cavity equation (40) it is enough to support that

$$\frac{\Delta \omega}{\omega} < \frac{\Delta L_2 \sin \delta + \sqrt{(\Delta L_1)^2 + (\Delta L_2)^2 + 2(-1)^N \Delta L_1 \Delta L_2 \cos \delta}}{L}. \hspace{1cm} (46)$$

The quantum formulae for cavities with two oscillating walls include anomaly contributions in the respective way. Since the modifications are straightforward, we skip their discussion here.

4. Conclusions

An optical approach to the dynamical Casimir effect based on the construction of classical optical paths is useful. In particular, it is a powerful technique to analyse resonance systems. We can trace the behaviour of a cavity system with moving walls when we increase amplitudes or loose fine tuning for resonance eigenfrequencies. We can calculate modified critical exponents, find more subtle structure of travelling peaks in energy densities and control the conditions for the instability of the system. We have established some ‘composition laws’ that come into effect if we want a more complicated model of a cavity with two moving walls to reduce to the analysis of a cavity with one moving wall. Moreover, our understanding of the mechanism of parametric resonance in cavities with moving walls and the implications of quantum field theory is much better. For example, we can judge when some semi-classical approximation is accurate or we can explain why some resonances are not observed at the quantum level. It is further remarkable that we can distinguish between exponential and power-like instability and observe the role of symmetries at the quantum level [24].

We have seen in several examples that many qualitative or even quantitative conclusions can be drawn from classical versions of respective cavity models. Presumably, the parametric resonance mechanism and evolution laws are similar in both cases. However, we should be careful about quantum anomalies that can significantly modify our predictions. A subtle role of quantum modifications can be traced within our approach.

The results derived for Dirichlet’s and Neumann’s boundary conditions for one-dimensional cavities are the same. This point was already made in [26, 27]. The cavities with
two oscillating walls possess a double set of characteristic functions, namely two families of both cumulative Doppler factors and cumulative anomaly contributions. This is related to the fact that from a given spacetime point there are two independent directions to follow trajectories of massless particles. In three dimensions, such functions depend also on localizations and orientations.

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