Alpha-robust investment-reinsurance strategy for a mean-variance insurer with delay

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Abstract

In this paper, a robust optimal reinsurance-investment problem with delay is studied under the $\alpha$-maxmin mean-variance criterion. The surplus process of an insurance company approximates Brownian motion with drift. The financial market consists of a risk-free asset and a risky asset that obeys geometric Brownian motion. Using the principle of dynamic programming and Hamilton-Jacobi-Bellman (HJB) equation, the specific expression of optimal strategy and the explicit solution of the corresponding HJB equation are obtained. In addition, a verification theorem is provided to ensure that the value function is indeed the solution of the HJB equation. Finally, some numerical examples and graphs are given to illustrate the results, and the influence of some important parameters in the model on the optimal strategy is discussed.

Keywords: Mean–variance criterion, Delay, $\alpha$-maxmin utility, Robust investment and reinsurance, Equilibrium strategy.

1. Introduction

Since reinsurance is an effective way to diversify risks, and investment is becoming more and more important in insurance business, the problems of optimal investment and reinsurance of insurers have received much attention in recent years. There are many research objectives for the optimal reinsurance-investment problem of insurers, such as minimizing the probability of ruin (see Schmidli, 2001; Liang (2007); Chen et al., 2010; Azcue and Muler, 2013), and maximizing the expected utility of terminal wealth (see Liu and Ma, 2009; Bai and Guo, 2010; Liang and Bayraktar, 2014; Liang and Yuen, 2016), and the mean-variance criterion (see Zeng and Li, 2011; Zeng et al., 2013; Bi et al., 2014; Lin and Qian, 2016).

In this paper, we study the optimal investment and reinsurance strategy for the insurer under the $\alpha$-maxmin mean-variance criterion, such as Li et al. (2016). However, we need to solve the problem of time inconsistency in the mean-variance criterion.
There are two ways to solve the time inconsistency problem. One way is to study the
 corresponding pre-commitment problem. For example, Bäuerle (2005) considered a clas-
 sic Cramér–Lundberg model with dynamic proportional reinsurance and found the best
 reinsurance strategy. Bai and Zhang (2008) studied the optimal strategy for the optimal
 reinsurance-investment problem in the classical risk model and the diffusion model. Bi
 et al. (2014) considered the insurer’s optimal investment-reinsurance problem under the
 mean-variance criterion, that is, the insurer’s wealth process is not allowed to fall below
 zero at any time. Another way is to formulate the problem in the framework of game
 theory. Björk and Murgoci (2010) and Björk et al. (2014) looked for the perfect equi-
 librium point of the Nash subgame by observing the problem within the framework of
 game theory. Li and Li (2013) studied the optimal time consistency strategy of insurer
 under the mean-variance criterion with state-related risk aversion. For insurers whose
 surplus is controlled by the compound Poisson risk model, Lin and Qian (2016) consider
 the choice of optimal time consistency reinsurance investment strategy.

Most of the existing literatures considering optimal reinsurance only determine the
 corresponding optimal strategy based on the information at that time, without taking
 into account the information of the past information. In fact, past information will affect
 the decisions of investors or insurers. For example, when investing in risky assets such
 as stocks, investors are not only concerned about the current price of stocks, but also
 about the price trend of stocks and other relevant information in the past period of time.
 If stock prices continue to rise over a period of time, investors tend to buy more stocks.
 Conversely, if share prices have fallen, investors will be more likely to sell their shares
 and invest in other assets. Therefore, incorporating some of the past information into
 the model can help us make more reasonable decisions. For example, Elsanousi et al.
 (2000) use the principle of dynamic programming to provide a relevant theoretical basis
 for solving stochastic control problems with delays. Chang et al. (2011) considered the
 Merton’s type portfolio problem, in which the return on risky assets is related to the
 return history. The problem is modeled by a stochastic system with delay. Shen et al.
 (2014) study a class of mean-field stochastic optimal control problems with delay in which
 the controlled state process is controlled by mean-field jump-diffusion stochastic delay
 differential equation. A and Li (2015) considered the problem of optimal investment and
 excess-of-loss reinsurance for the insurer under Heston’s stochastic volatility model. Li

3
and Min (2019) analyzed the optimal portfolio and consumption problem with stochastic factor and delay over a finite time range.

In addition, it is difficult to accurately estimate the return level of risky assets in the real market, so scholars have considered some alternative models. A common way to consider model uncertainty in an optimal investment model is to treat it as a robust control problem. For instance, Yi et al. (2013) considered the robust optimal reinsurance and investment problem for ambiguity-averse insurers under Heston’s stochastic volatility model. Huang et al. (2017) studied the robust optimal investment-reinsurance problem of an insurer, and used the techniques of stochastic control theory to give the closed-form expressions of the robust optimal investment-reinsurance strategies and the corresponding value function. In a financial market consisting of a risk-free asset and a credit default swap (CDS), Zhao et al. (2019) studied the robust equilibrium reinsurance and investment strategy of the ambiguity-averse insurer under the dynamic mean-variance criterion. Yang et al. (2020) studied the robust portfolio optimization problem under the multi-factor volatility model.

Inspired by the above research, this paper studies the robust optimal investment-reinsurance problem with delay. We use the arithmetic Brownian motion as a diffusion approximation for the insurer surplus process and assume that the financial market consists of a risk-free asset and a risky asset. We assume that there is capital inflow or outflow from the current wealth of the insurer, and that the amount of capital inflow/outflow is directly proportional to the wealth statement of the insurer in the past. In order to solve the problem of time inconsistency, we discuss this problem under the framework of game theory and find the corresponding sub-game Nash equilibrium strategy. In this paper, the optimal portfolio problem of the insurer is studied by combining delay, robustness and $\alpha$-maxmin mean-variance criterion. The corresponding HJB equation is obtained by using dynamic programming method, and the explicit expression of value function and optimal strategy are further obtained.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 derives the optimal strategy of investment and reinsurance and the explicit expression of the value function. Section 4 considers several special cases. Section 5 provides some numerical examples to illustrate our results. Section 6 concludes this paper. The proof of the verification theorem is postponed to the Appendix.
2. Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$, where $T$ is a positive finite constant representing the time horizon. Let $\{\mathcal{F}_t\}_{t \in [0,T]}$ be the right-continuous, $\mathbb{P}$-complete filtration generated by two standard Brownian motions $\{W_1(t)\}$ and $\{W_2(t)\}$. Throughout this paper, let $\mathbb{P}$ be the real world probability motions $\{W_1(t)\}$ and $\{W_2(t)\}$. Throughout this paper, let $\mathbb{P}$ be the real world probability measure, and $\mathbb{E}[]$ and $\text{Var}[]$ be the expectation and the variance under $\mathbb{P}$, respectively. We assume that there exists a martingale probability measure $\mathbb{Q}$ (or the risk neutral measure) equivalent to measure $\mathbb{P}$.

We first introduce the insurance risk model, we model the claim process $\{C(t)\}_{t \in [0,T]}$ as the following arithmetic Brownian motion

$$dC(t) = adt - \sigma_1 dW_1(t), \quad (2.1)$$

where $a$ and $\sigma_1$ are positive constants. Furthermore, we assume that the premium is paid continuously at the rate of $c = (1 + \theta)a$, where $\theta \geq 0$ is the relative security loading of the insurer. Therefore, the surplus process of an insurer without reinsurance and investment is modeled by a diffusion approximation model

$$dR_0(t) = cdt - dC(t) = \theta adt + \sigma_1 dW_1(t), \quad R_0(0) > 0, \quad (2.2)$$

The insurer can purchase proportional reinsurance or acquire new business to adjust the exposure to insurance risk. The proportional reinsurance or new business level level is associated with the risk exposure $\pi_q(t)$ at time $t$. When $\pi_q(t) \in [0,1]$, it corresponds to a proportional reinsurance cover. In this case, the cedent should divert part of the premium to the reinsurer at the rate of $(1 + \eta)(1 - \pi_q(t))a$, where $\eta$ is the safety loading of the reinsurer satisfying $\eta > \theta > 0$. Then, the cedent pays $100\pi_q(t)\%$ of the claim while the reinsurer pays $100(1 - \pi_q(t))\%$ of the claim. Note that $\pi_q(t) \in (1, +\infty)$ refers to acquiring new business. After taking into account such a reinsurance strategy $\pi_q(t)$, $t \in (0,T)$, the insurer’s surplus process can be described by

$$dR(t) = (1 + \theta)adt - (1 + \eta)(1 - \pi_q(t)) adt - \pi_q(t)dC(t)$$

$$= [\theta - \eta + \eta \pi_q(t)] adt + \pi_q(t)\sigma_1 dW_1(t) \quad (2.3)$$
We assume that the financial market consists of a risk-free bond and a risky stock. The price process of the risk-free bond $B(t)$ under measure $\mathbb{P}$ follows

$$dB(t) = r_0 B(t) dt,$$

where $r_0 > 0$ denotes the risk-free interest rate. The price process of the stock under measure $\mathbb{P}$ follows

$$dS(t) = S(t) \mu dt + \sigma_2 S(t) dW_2(t),$$

where $\mu \in \mathbb{R}$, $\sigma_2 > 0$, and $\{W_2(t)\}_{t \geq 0}$ is another standard Brownian motion which is independent of $W_1$.

We denote $\pi_s(t)$ the amount of money the insurer invested in the stock at time $t$, and $X^\pi(t)$ the insurance surplus process under the reinsurance-investment strategy $\pi := \{\pi_q(t), \pi_s(t)\}_{t \in [0, T]}$. Apart from the investment on the stock, the insurer invest the rest of the surplus $X^\pi(t) - \pi_s(t)$ to the risk-free bond. The dynamics of $X^\pi(t)$ follows

$$dX^\pi(t) = \pi_s(t) \frac{dS(t)}{S(t)} + (X^\pi(t) - \pi_s(t)) r_0 dt + dR(t)$$

$$= [r_0 X^\pi(t) + (\mu - r_0) \pi_s(t) + (\theta - \eta + \eta \pi_q(t)) a] dt$$

$$+ \pi_q(t) \sigma_1 dW_1(t) + \pi_s(t) \sigma_2 dW_2(t). \quad (2.4)$$

Next, we consider a delayed wealth process caused by instantaneous capital inflows or outflows. A little abuse of the symbol, we use $X^\pi(t)$ to represent the delayed wealth process of the insurer. Assuming that the function $f(t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - Z^\pi(t))$ represents the total capital inflow/outflow, the wealth process of the insurer is governed by

$$dX^\pi(t) = \pi_s(t) \frac{dS(t)}{S(t)} + (X^\pi(t) - \pi_s(t)) r_0 dt + dR(t)$$

$$= [r_0 X^\pi(t) + (\mu - r_0) \pi_s(t) + (\theta - \eta + \eta \pi_q(t)) a] dt$$

$$- f(t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - Z^\pi(t)) dt + \pi_q(t) \sigma_1 dW_1(t) + \pi_s(t) \sigma_2 dW_2(t). \quad (2.5)$$

To make the problem tractable, we consider the following linear instantaneous capital
inflow/outflow function

\[ f \left( t, X^\pi(t) - \bar{Y}^\pi(t), X^\pi(t) - Z^\pi(t) \right) = \alpha_1 \left( X^\pi(t) - \bar{Y}^\pi(t) \right) + \alpha_2 \left( X^\pi(t) - Z^\pi(t) \right) \]

\[ = \alpha_1 \left( X^\pi(t) - \frac{Y^\pi(t)}{\int_{-h}^0 e^{\delta \theta} d\theta} \right) + \alpha_2 \left( X^\pi(t) - Z^\pi(t) \right) \]

\[ = (\alpha_1 + \alpha_2) X^\pi(t) - \tilde{\alpha}_1 Y^\pi(t) - \alpha_2 Z^\pi(t), \]

where

\[ \tilde{\alpha}_1 = \frac{\alpha_1}{\int_{-h}^0 e^{\delta \theta} d\theta}. \]

While \( \alpha_1 (\geq 0) \) and \( \alpha_2 (\geq 0) \) are constants, and

\[ Y^\pi(t) = \int_{-h}^0 e^{\delta \theta} X(t + \theta) d\theta, \quad \bar{Y}^\pi(t) = \frac{Y^\pi(t)}{\int_{-h}^0 e^{\delta \theta} d\theta}, \quad Z^\pi(t) = X^\pi(t - h), \]

where \( \delta > 0 \) is a constant average parameter and \( h > 0 \) is the delay parameter. The \( e^{\delta \theta} \) in the integral term represents the exponential decay factor. \( Y^\pi(t), \bar{Y}^\pi(t) \) and \( Z^\pi(t) \) respectively reflect the integrated, average and pointwise delayed information of the wealth process in the past period \([t-h,t]\). \( X^\pi(t) - \bar{Y}^\pi(t) \) reflects the average performance of wealth from \( t-h \) to \( t \), and \( X^\pi(t) - Z^\pi(t) \) reflects the absolute performance of wealth in the interval \([t-h,t]\).

Substituting (2.6) into (2.5), the delayed wealth equation becomes

\[
\begin{align*}
\frac{dX^\pi(t)}{dt} &= \left[ r_1 X^\pi(t) + \tilde{\alpha}_1 Y^\pi(t) + \alpha_2 Z^\pi(t) + (\mu - r_0)\pi_s(t) + (\theta - \eta + \eta\pi_q(t))a \right] dt \\
&\quad + \pi_q(t)\sigma_1 dW_1(t) + \pi_s(t)\sigma_2 dW_2(t), \quad \forall t \in [0,T],
\end{align*}
\]

where \( r_1 = r_0 - \alpha_1 - \alpha_2 \). We assume that when \( t \in [-h,0] \), \( X(t) = x_0 > 0 \), in other words, the insurance company already has wealth \( x_0 \) at time \( t \), and has not conducted any business operations during the time period \([-h,0]\).

In order to introduce the ambiguity on the insurance and financial risks, we define a set of prior probability measures as below. We call the probability distortion function
\[ \phi := (\phi_1(t), \phi_2(t))_{t \in [0, T]} \in \Theta, \] if \( \phi_1(t), \phi_2(t) \) are deterministic functions of \( t \) and satisfy

\[
\exp \left\{ \int_t^T \frac{\phi_1(s)^2 + \phi_2(s)^2}{2} \, ds \right\} < \infty
\]

for any \( t \in [0, T] \). Each probability distortion function \( \phi \in \Theta \) is associated with a probability measure \( \mathbb{Q}^\phi \sim \mathbb{P} \). And the Radon-Nikodym derivative process \( \frac{d\mathbb{Q}^\phi}{d\mathbb{P}} \bigg|_{\mathcal{F}(t)} := \Lambda^\phi(t) \) is given by

\[
\Lambda^\phi(t) = \exp \left\{ -\int_0^t \phi_1(s) \, dW_1(s) - \frac{1}{2} \int_0^t \phi_1(s)^2 \, ds \\
- \int_0^t \phi_2(s) \, dW_2(s) - \frac{1}{2} \int_0^t \phi_2(s)^2 \, ds \right\}.
\]

Then we define a set of prior probability measures by

\[ \mathcal{Q} = \{ \mathbb{Q}^\phi : \phi \in \Theta \} \] is established.

Applying Girsanov’s Theorem (see, for example, Øksendal and Sulem (2007), it is clear that

\[
dW_1^\phi(t) = dW_1(t) + \phi_1(t)dt \quad \text{and} \quad dW_2^\phi(t) = dW_2(t) + \phi_2(t)dt
\]  

(2.9)

are \( \mathbb{Q}^\phi \)-Brownian motions.

Using (2.9), it is easy to see that the dynamics of the surplus process \( X^\pi(t) \) under \( \mathbb{Q}^\phi \) is governed by

\[
dX^\pi(t) = \left[ r_1 X^\pi(t) + \bar{\alpha}_1 Y^\pi(t) + \alpha_2 Z^\pi(t) + (\mu - r_0)\pi_s(t) + (\theta - \eta + \eta \pi_q(t))a \\
- \pi_q(t)\sigma_1 \phi_1(t) - \pi_s(t)\sigma_2 \phi_2(t) \right] dt + \pi_q(t)\sigma_1 dW_1^\phi(t) + \pi_s(t)\sigma_2 dW_2^\phi(t).
\]  

(2.10)

In addition, with simple calculations, we can obtain

\[
dY^\pi(t) = \left[ X^\pi(t) - \delta Y^\pi(t) - e^{-\delta \bar{h}} Z^\pi(t) \right] dt.
\]  

(2.11)

**Definition 2.1.** A trading strategy \( \pi = \{(\pi_s(t), \pi_q(t))\}_{t \in [0, T]} \) is said to be admissible if it satisfies the following conditions:

1. \((\pi_s(t), \pi_q(t))\) is \( \mathbb{Q} \)-predictable;
∀t ∈ [0, T], πₜ(𝑡) ≥ 0;

(3) $\mathbb{E}_{t,x,y}^\phi \left[ \int_0^T (\pi_s(t)^2 + \pi_q(t)^2) dt \right] < \infty$ and $\mathbb{E}_{t,x,y}^{\phi^\pi} \left[ \int_0^T (\pi_s(t)^2 + \pi_q(t)^2) dt \right] < \infty$ for any $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$; 

(4) $(\pi, X^\pi)$ is the unique strong solution to the stochastic differential equation (2.10).

For any initial condition $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, the corresponding set of all admissible strategies is denoted by Π$(t, x, y)$.

Now, we consider a mean-variance optimal problem with delay for the insurer purchasing reinsurance or acquiring new business and investing in the risk-free asset and the stock. Similar to Bin et al. (2016), we define the $\alpha$-robust mean-variance criterion for a controlled surplus process $X^\pi$ by

$$J_\alpha^\pi(t, x, y) = \alpha \inf_{\phi \in \Theta} J_{\phi}^\pi(t, x, y) + \tilde{\alpha} \sup_{\phi \in \Theta} \tilde{J}_{\phi}^\pi(t, x, y)$$

(2.12)

where $\alpha \in [0, 1]$, 

$$J_{\phi}^\pi(t, x, y) = E_{t,x,y}^\phi [X^\pi(T) + \lambda Y^\pi(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^\phi [X^\pi(T) + \lambda Y^\pi(T)] + \int_t^T h_\beta(\phi(s))ds,$$  

(2.13)

and

$$\tilde{J}_{\phi}^\pi(t, x, y) = E_{t,x,y}^\phi [X^\pi(T) + \lambda Y^\pi(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^\phi [X^\pi(T) + \lambda Y^\pi(T)] - \int_t^T h_\beta(\phi(s))ds.$$  

(2.14)

We write

$$\mathbb{E}_{t,x,y}^\phi [\cdot] = \mathbb{E}^{\phi} [\cdot | X^\pi(t) = x, Y^\pi(t) = y], \text{Var}_{t,x,y}^\phi [\cdot] = \text{Var}^{\phi} [\cdot | X^\pi(t) = x, Y^\pi(t) = y],$$

and $\gamma > 0$ is the insurer’s risk aversion coefficient. Here the constant $\lambda(\geq 0)$ is the weight of $X^\pi(T)$ and $Y^\pi(T)$. By introducing the weight $\lambda(\geq 0)$, taking into account the terminal wealth $X^\pi(T)$ and the accumulated wealth $Y^\pi(T)$ in the time period $[T - h, T]$, this can help us make relevant decisions more reasonably. And $\phi^\pi$ and $\phi^\pi$ respectively represents the probability distortion functions to achieve the infimum and supremum in
(2.12), respectively. And the penalty function is selected to

\[
h_\beta(\phi(s)) = \frac{\phi_1(s)^2}{2\beta_1} + \frac{\phi_2(s)^2}{2\beta_2}.
\]  

(2.15)

The main purpose of this paper is to study the \(\alpha\)-robust reinsurance investment problem of a class of time-consistent mean-variance insurers, that is,

\[
\sup_{\pi \in \Pi} J_\alpha^\pi(t, x, y),
\]

(2.16)
in which the equilibrium strategy are defined below.

**Definition 2.2.** For any fixed initial state \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}\), consider an admissible strategy \(\pi^*(t, x, y)\). Choose three fixed numbers \(\tilde{\pi}_s, \tilde{\pi}_q \in \mathbb{R}, \tilde{\pi}_q \in \mathbb{R}_+\) and \(\epsilon \in \mathbb{R}_+\) and define the following strategy:

\[
\pi^\epsilon(s) := \begin{cases} 
(\tilde{\pi}_s, \tilde{\pi}_q), & \text{for } s \in [t, t + \epsilon), \\
\pi^*(s), & \text{for } s \in [t + \epsilon, T].
\end{cases}
\]

If

\[
\liminf_{\epsilon \to 0} \frac{J_\alpha^\pi^*(t, x, y) - J_\alpha^\pi^\epsilon(t, x, y)}{\epsilon} \geq 0
\]

for all \((\tilde{\pi}_s, \tilde{\pi}_q) \in \mathbb{R} \times \mathbb{R}_+\) and \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}\), \(\pi^*(t, x, y)\) is called an equilibrium strategy and \(J_\alpha^\pi^*(t, x, y)\) is the associated equilibrium value function.

The equilibrium strategy is time-consistent and hereafter we call the equilibrium strategy \(\pi^*\) and the corresponding equilibrium value function \(J_\alpha^\pi^*(t, x, y)\) the optimal time-consistent strategy and the value function for (2.16), respectively. Before giving the verification theorem, we denote by

\[
C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R}) := \{ \varphi(t, x, y) \mid \varphi(\cdot, x, y) \text{ is continuously differentiable on } [0, T], \\
\varphi(t, \cdot, y) \text{ is twice continuously differentiable on } \mathbb{R}, \\
\text{and } \varphi(t, x, \cdot) \text{ is continuously differentiable on } \mathbb{R} \},
\]

and for any \(\psi(t, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R})\) and \(\pi \in \Pi(t, x, y)\), define the infinitesimal
generator

\[ A^{\pi, \phi} \psi(t, x, y) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}_{t, x, y}^\phi [\psi(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon))] - \psi(t, x, y)}{\epsilon} \]

\[ = \psi_t + \left[ r_1 x + \alpha_1 y + \alpha_2 z + \pi_s (\mu - r_0) + (\theta - \eta + \eta \pi) a - \phi \sigma_2 \phi_1(t) - \pi_s \sigma_2 \phi_2(t) \right] \psi_x \]

\[ + (x - \delta y - e^{-\delta h} z) \psi_y + \left( \frac{1}{2} \pi^2 \sigma_1^2 + \frac{1}{2} \pi^2 \sigma_2^2 \right) \psi_{xx}. \]

(2.17)

**Theorem 2.1.** (Verification theorem) For problem (2.16), if there exist \( V(t, x, y), g(t, x, y), \bar{g}(t, x, y) \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R}) \) satisfy the following conditions:

1. For any \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},\)

\[ 0 = \sup_{\pi \in \Pi} \left\{ \alpha \inf_{\phi \in \Theta} \left[ A^{\pi, \phi} V(t, x, y) - \frac{\gamma}{2} A^{\pi, \phi} g^2(t, x, y) + \gamma g(t, x, y) A^{\pi, \phi} g(t, x, y) + h_\beta(\phi(t)) \right] + \alpha \sup_{\phi \in \Theta} \left[ A^{\pi, \phi} V(t, x, y) - \frac{\gamma}{2} A^{\pi, \phi} \bar{g}^2(t, x, y) + \gamma \bar{g}(t, x, y) A^{\pi, \phi} \bar{g}(t, x, y) - h_\beta(\phi(t)) \right] \right\}, \]

and \((\pi^*, \phi^*, \bar{\phi}^*)\) denote the optimal values to achieve the supremum in \( \pi, \) infimum and supremum in \( \phi, \) respectively.

2. For any \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},\)

\[ \begin{cases} V(T, x, y) = x + \lambda y, \\
A^{\pi^*, \phi^*} g(t, x, y) = A^{\pi^*, \bar{\phi}^*} \bar{g}(t, x, y) = 0, \\
\bar{g}(T, x, y) = \bar{g}(T, x, y) = x + \lambda y. \end{cases} \]

(2.19)

3. For any \((t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \pi^*(t), \phi^*(t), \bar{\phi}^*(t), A^{\pi^*, \phi^*} V(t, x, y), A^{\pi^*, \bar{\phi}^*} V(t, x, y), A^{\pi^*, \phi^*} g^2(t, x, y) \) and \( A^{\pi^*, \bar{\phi}^*} \bar{g}^2(t, x, y) \) are all deterministic functions of \( t \) and independent of \( x \) and \( y. \)

4. \( \hat{\phi}^* = \hat{\phi}^{\pi^*} \) and \( \bar{\phi}^* = \bar{\phi}^{\pi^*}. \)

Then \( \pi^* \) is the equilibrium strategy and \( V(t, x, y) = J^\pi_{\alpha}(t, x, y) \) is the equilibrium value function to the \( \alpha \)-robust reinsurance-investment problem (2.16). Besides, \( g(t, x, y) = \mathbb{E}_{t, x, y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] \) and \( \bar{g}(t, x, y) = \mathbb{E}_{t, x, y}^{\bar{\phi}^*} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)]. \)

The proof of the verification theorem is postponed to the Appendix.
3. Solution to the optimization problem

In this section, we derive the optimal time-consistent reinsurance and investment strategies and the corresponding equilibrium value functions for problem (2.16). Suppose that $V(t, x, y), g(t, x, y), \tilde{g}(t, x, y), \phi^*, \tilde{\phi}^*$ satisfying conditions (1) and (2) of Theorem 2.1.

Then, with some calculations, we can rewrite (2.18) as

$$0 = \sup_{\pi \in \Pi} \left\{ V_t + \left[ r_1 x + \tilde{\alpha}_1 y + \alpha_2 z + (\mu - r_0) \pi_s + (\theta - \eta + \eta \pi_q) a \right] V_x ight. \\
+ (x - \delta y - e^{-\delta h z}) V_y + \frac{1}{2} \left( \pi_q^2 \sigma_1^2 + \pi_s^2 \sigma_2^2 \right) \left( V_{xx} - \alpha \gamma g_x^2 - \tilde{\alpha} \gamma \tilde{g}_x^2 \right) \\
\left. + \alpha \inf_{\phi \in \Theta} \left\{ -(\pi_q \sigma_1 \phi_1 + \pi_s \sigma_2 \phi_2) V_x + h_\beta(\phi) \right\} \\
+ \tilde{\alpha} \sup_{\phi \in \Theta} \left\{ -(\pi_q \sigma_1 \phi_1 + \pi_s \sigma_2 \phi_2) V_x - h_\beta(\phi) \right\} \right\}, \quad (3.1)$$

Applying the first-order condition on (3.1) with respect to $\phi$, and obtain the following infimum and supremum of $\phi$ respectively.

$$\begin{align*}
\phi^*_1 &= \beta_1 \pi_q \sigma_1 V_x, \\
\phi^*_2 &= \beta_2 \pi_s \sigma_2 V_x,
\end{align*} \quad (3.2)$$

and

$$\begin{align*}
\bar{\phi}^*_1 &= -\beta_1 \pi_q \sigma_1 V_x, \\
\bar{\phi}^*_2 &= -\beta_2 \pi_s \sigma_2 V_x.
\end{align*} \quad (3.3)$$

Substituting (3.2) and (3.3) back into (3.1) yields

$$0 = \sup_{\pi \in \Pi} \left\{ V_t + \left[ r_1 x + \tilde{\alpha}_1 y + \alpha_2 z + (\mu - r_0) \pi_s + (\theta - \eta + \eta \pi_q) a \right] V_x ight. \\
+ (x - \delta y - e^{-\delta h z}) V_y + \frac{1}{2} \left( \pi_q^2 \sigma_1^2 + \pi_s^2 \sigma_2^2 \right) \left( V_{xx} - \alpha \gamma g_x^2 - \tilde{\alpha} \gamma \tilde{g}_x^2 \right) \\
\left. + \frac{1 - 2\alpha}{2} \left( \beta_1 \pi_q^2 \sigma_1^2 + \beta_2 \pi_s^2 \sigma_2^2 \right) V_x^2 \right\}.
\quad (3.4)$$

By the first-order condition and differentiating with respect to $\pi_q(t)$ and $\pi_s(t)$ in (3.4), respectively, we obtain $\pi^*_q(t)$ and $\pi^*_s(t)$ as follows,

$$\pi^*_q = \frac{\eta a V_x}{\sigma_1^2 \left[ \alpha \gamma g_x^2 + \tilde{\alpha} \gamma \tilde{g}_x^2 - V_{xx} + (2\alpha - 1) \beta_1 V_x^2 \right]}, \quad (3.5)$$
and
\[
\pi^* = \frac{(\mu - r_0)V_x}{\sigma^2 \left[ \alpha \gamma \rho_x^2 + \dot{\alpha} \gamma \bar{g}_x^2 - V_{xx} + (2\alpha - 1)\beta_2 V_x^2 \right]}.
\] (3.6)

Inspired by the boundary conditions (2.19), we assume the solutions of (3.4) of the forms
\[
\begin{align*}
V(t, x, y) &= A(t)(x + \lambda y) + B(t), \\
q(t, x, y) &= a(t)(x + \lambda y) + b(t), \\
\bar{g}(t, x, y) &= \bar{a}(t)(x + \lambda y) + \bar{b}(t),
\end{align*}
\] (3.7)

where \(A(t), B(t), a(t), b(t), \bar{a}(t), \bar{b}(t)\) are functions of \(t\). By the first and the third relation of (2.19), the boundary conditions are given by
\[
A(T) = a(T) = 1 \text{ and } B(T) = b(T) = 0.
\]

Substituting (3.7) into (3.4) yields
\[
0 = A'(x + \lambda y) + B' + \left[ (r_1 + \lambda)x + (\ddot{\alpha} - \lambda \delta)y + (\alpha_2 - \lambda e^{-\delta h})z \right] A
+ \left[ (\mu - r_0)\pi^* + (\theta - \eta + \eta \pi^*)a \right] A
- \frac{1}{2} \left( \pi^* \sigma_1^2 + \pi^* \sigma_2^2 \right) \left( \alpha \gamma \rho_x^2 + \dot{\alpha} \gamma \bar{g}_x^2 \right)
+ \frac{1 - 2\alpha}{2} \left[ \beta_1 \pi^* \sigma_1^2 + \beta_2 \pi^* \sigma_2^2 \right] A^2.
\] (3.8)

By virtue of Pang and Hussain (2015), we assume the value function is independent on \(z\). Suppose
\[
\alpha_2 = \lambda e^{-\delta h},
\] (3.9)

and
\[
\ddot{\alpha} - \delta e^{\delta h} \alpha_2 = (r_1 + e^{\delta h} \alpha_2)e^{\delta h} \alpha_2,
\] (3.10)

let \(x + \lambda y = u\), we can rewrite (3.8) as
\[
0 = A'u + B' + \left[ (r_1 + \lambda)u + (\mu - r_0)\pi^* + (\theta - \eta + \eta \pi^*)a \right] A
- \frac{1}{2} \left( \pi^* \sigma_1^2 + \pi^* \sigma_2^2 \right) \left( \alpha \gamma \rho_x^2 + \dot{\alpha} \gamma \bar{g}_x^2 \right)
+ \frac{1 - 2\alpha}{2} \left[ \beta_1 \pi^* \sigma_1^2 + \beta_2 \pi^* \sigma_2^2 \right] A^2.
\] (3.11)
Substituting (3.5) and (3.6) into (3.11) yields

\[ 0 = A'u + B' + [(r_1 + \lambda)u + (\theta - \eta)a] \]
\[ + \frac{\eta^2 a^2 A^2}{2\sigma^2_1 [\alpha \gamma a^2 + \dot{\alpha} \gamma a^2 + (2\alpha - 1)\beta_1 A^2]} + \frac{(\mu - r_0)^2 A^2}{2\sigma^2_2 [\alpha \gamma a^2 + \dot{\alpha} \gamma a^2 + (2\alpha - 1)\beta_2 A^2]}. \]  

(3.12)

Similarly, substituting (3.5), (3.6) and (3.7) into the second relation of (2.19) yields

\[ 0 = a'u + b' + [(r_1 + \lambda)u + (\mu - r_0)\pi_1^* + (\theta - \eta + \eta \pi_1^*)a - \pi_1^* \sigma_1^2 \beta_1 A - \pi_1^* \sigma_2^2 \beta_2 A] a, \]

(3.13)

and

\[ 0 = \overline{a}'u + \overline{b}' + [(r_1 + \lambda)u + (\mu - r_0)\pi_1^* + (\theta - \eta + \eta \pi_1^*)a + \pi_1^* \sigma_1^2 \beta_1 A + \pi_1^* \sigma_2^2 \beta_2 A] \overline{a}. \]

(3.14)

By matching the coefficients of the terms of \( u \), we get

\[ A(t)' + (r_1 + \lambda)A(t) = a(t)' + (r_1 + \lambda)a(t) = \overline{a}(t)' + (r_1 + \lambda)\overline{a}(t) = 0. \]

(3.15)

We use boundary conditions \( A(T) = a(T) = \overline{a}(T) = 1 \) to get

\[ A(t) = a(t) = \overline{a}(t) = e^{(r_1 + \lambda)(T - t)}, \quad t \in [0, T]. \]

(3.16)

Substituting (3.16) back into (3.12)-(3.14), and using the boundary condition \( B(T) = b(T) = \overline{b}(T) = 0 \), we can easily obtain the explicit solution of \( B(t), b(t), \) and \( \overline{b}(t) \).

\[ B(t) = \frac{a(\theta - \eta)}{(r_1 + \lambda)}(e^{(r_1 + \lambda)(T - t)} - 1) + \frac{\eta^2 a^2 (T - t)}{2\sigma^2_1 [\gamma + (2\alpha - 1)\beta_1]} + \frac{(\mu - r_0)^2 (T - t)}{2\sigma^2_2 [\gamma + (2\alpha - 1)\beta_2]}, \]

(3.17)

and the expressions of \( b(t) \) and \( \overline{b}(t) \) are given by

\[ b(t) = \frac{a(\theta - \eta)}{(r_1 + \lambda)}(e^{(r_1 + \lambda)(T - t)} - 1) + \frac{\eta^2 a^2 [\gamma + (2\alpha - 2)\beta_1] (T - t)}{\sigma^2_1 [\gamma + (2\alpha - 1)\beta_1]^2} + \frac{(\mu - r_0)^2 [\gamma + (2\alpha - 2)\beta_2] (T - t)}{\sigma^2_2 [\gamma + (2\alpha - 1)\beta_2]^2}, \]

(3.18)
and
\[
\bar{b}(t) = \frac{a(\theta - \eta)}{(r_1 + \lambda)}(e^{(r_1 + \lambda)(T-t)} - 1) + \frac{\eta^2 a^2 (\gamma + 2\alpha \beta_1)(T-t)}{\sigma_1^2 [\gamma + (2\alpha - 1)\beta_1]^2} \\
+ \frac{(\mu - r_0)^2 (\gamma + 2\alpha \beta_2)(T-t)}{\sigma_2^2 [\gamma + (2\alpha - 1)\beta_2]^2}.
\] (3.19)

Thus, the explicit expression of the value function is obtained. We state the equilibrium strategy for optimal control problem (2.16) in the following theorem.

**Theorem 3.1.** For the $\alpha$-robust reinsurance-investment problem (2.16), the optimal reinsurance and investment strategies for the insurer are as follows.

1. The equilibrium reinsurance strategy $\pi^*_q(t)$ is given by
\[
\pi^*_q(t) = \frac{\eta a}{\sigma_1^2 [\gamma + (2\alpha - 1)\beta_1]} e^{(r_1 + \lambda)(T-t)},
\] (3.20)

and the equilibrium investment strategy $\pi^*_s(t)$ is given by
\[
\pi^*_s(t) = \frac{\mu - r_0}{\sigma_2^2 [\gamma + (2\alpha - 1)\beta_2]} e^{(r_1 + \lambda)(T-t)},
\] (3.21)

2. The corresponding equilibrium value function is given by
\[
J^\pi_\alpha(t, x, y) = e^{(r_1 + \lambda)(T-t)}(x + \lambda y) + B(t),
\]
where $B(t)$ is given by (3.17).

3. The associated probability distortion functions of the extremely ambiguity-averse measure and the extremely ambiguity-seeking measure are given respectively by
\[
\begin{align*}
\phi^*_1(t) &= \frac{\eta a \beta_1}{\sigma_1 [\gamma + (2\alpha - 1)\beta_1]}, \\
\phi^*_2(t) &= \frac{(\mu - r_0) \beta_2}{\sigma_2 [\gamma + (2\alpha - 1)\beta_2]},
\end{align*}
\] (3.22)

and
\[
\begin{align*}
\tilde{\phi}^*_1(t) &= -\frac{\eta a \beta_1}{\sigma_1 [\gamma + (2\alpha - 1)\beta_1]}, \\
\tilde{\phi}^*_2(t) &= -\frac{(\mu - r_0) \beta_2}{\sigma_2 [\gamma + (2\alpha - 1)\beta_2]}.
\end{align*}
\] (3.23)

4. Special cases

In the section, we consider some special cases of our model.
Special case 1: model without robustness. Then the model degenerates into the optimal reinsurance-investment problem with delay.

Then the equilibrium strategy $\pi_q^*(t)$ and $\pi_s^*(t)$ are given by

$$
\pi_q^*(t) = \eta a \frac{\sigma^2}{\sigma_1^2 \gamma e^{(\gamma + \lambda)(T-t)}},
$$

and

$$
\pi_s^*(t) = \frac{\mu - r_0}{\sigma_2^2 \gamma e^{(\gamma + \lambda)(T-t)}}.
$$

The value function is given by

$$
J_\pi^*(t, x, y) = e^{(\gamma + \lambda)(T-t)}(x + \lambda y)
+ \frac{a(\theta - \eta)}{(r_1 + \lambda)}(e^{(\gamma + \lambda)(T-t)} - 1) + \frac{\eta^2 a^2}{2\sigma_1^2 \gamma} + \frac{(\mu - r_0)^2}{2\sigma_2^2 \gamma}.
$$

Special case 2: model without delay. Then the model degenerates into the $\alpha$-robust reinsurance-investment problem.

Then the equilibrium strategy $\pi_q^*(t)$ and $\pi_s^*(t)$ are given by

$$
\pi_q^*(t) = \frac{\eta a}{\sigma_1^2 [\gamma + (2\alpha - 1)\beta_1] e^{r_0(T-t)}},
$$

and

$$
\pi_s^*(t) = \frac{\mu - r_0}{\sigma_2^2 [\gamma + (2\alpha - 1)\beta_2] e^{r_0(T-t)}}.
$$

The value function is given by

$$
J_\alpha^*(t, x) = e^{r_0(T-t)} x + \frac{a(\theta - \eta)}{r_0}(e^{r_0(T-t)} - 1)
+ \frac{\eta^2 a^2}{2\sigma_1^2 [\gamma + (2\alpha - 1)\beta_1]} + \frac{(\mu - r_0)^2}{2\sigma_2^2 [\gamma + (2\alpha - 1)\beta_2]}.
$$

Special case 3: model without reinsurance. Then the model degenerates into the $\alpha$-robust investment problem.

Then the equilibrium investment strategy $\pi_s^*(t)$ is given by

$$
\pi_s^*(t) = \frac{\mu - r_0}{\sigma_2^2 [\gamma + (2\alpha - 1)\beta_2] e^{(r_1 + \lambda)(T-t)}}.
$$
where \( \lambda \) given in (3.9). The value function is given by

\[
J^*_\alpha(t, x, y) = e^{(r_1 + \lambda)(T-t)}(x + \lambda y) + \frac{(\mu - r_0)^2(T-t)}{2\sigma^2_2 [\gamma + (2\alpha - 1)\beta_2]}.
\] (4.8)

5. Numerical analysis

In this section, we present some numerical examples to illustrate the effects of model parameters on the results derived in the previous section. For numerical illustrations, unless otherwise stated, the basic parameters are follows: \( \alpha = 0.8, \gamma = 0.5, \eta = 0.2, \beta_1 = 1, \beta_2 = 3, \mu = 0.1, r_0 = 0.08, \lambda = 0.05, \sigma_1 = 0.5, \sigma_2 = 0.2, a = 1, \delta = 0.5, h = 1, t = 0 \) and \( T = 10 \). Note that \( \alpha_2 = e^{-\delta h} \lambda, \alpha_1 = (r_0 - \alpha_2 + \lambda + \delta) \frac{\lambda}{1 + \lambda} \int_0^h e^{\delta \theta} d\theta \) and \( r_1 = r_0 - \alpha_1 - \alpha_2 \).

The optimal reinsurance strategy \( \pi^*_q(t) \) and the optimal investment strategy \( \pi^*_s(t) \) can be rewritten as follows:

\[
\pi^*_q(t) = \frac{\eta a}{\sigma^2_1 [\gamma + (2\alpha - 1)\beta_1]} e^{(r_1 + \lambda)(T-t)},
\] (5.1)

and

\[
\pi^*_s(t) = \frac{\mu - r_0}{\sigma^2_2 [\gamma + (2\alpha - 1)\beta_2]} e^{(r_1 + \lambda)(T-t)}.
\] (5.2)

5.1. Analysis of the optimal reinsurance strategy

This subsection introduces some numerical examples and sensitivity analysis of optimal reinsurance strategies \( \pi^*_q(t) \). According (5.1), we can derive that \( \frac{\partial \pi^*_q(t)}{\partial \gamma} < 0, \frac{\partial \pi^*_q(t)}{\partial \alpha} < 0, \frac{\partial \pi^*_q(t)}{\partial r_0} < 0, \frac{\partial \pi^*_q(t)}{\partial \eta} < 0, \frac{\partial \pi^*_q(t)}{\partial h} > 0, \frac{\partial \pi^*_q(t)}{\partial \eta} > 0, \frac{\partial \pi^*_q(t)}{\partial a} > 0 \), which imply that the optimal reinsurance strategy \( \pi^*_q(t) \) decreases w.r.t the insurance liability ambiguity \( \beta_1 \), the coefficient of risk aversion of the insurer \( \gamma \), the ambiguity attitude \( \alpha \), the risk-free interest rate \( r_0 \) and the volatility of the surplus process of the insurer \( \sigma_1 \), but increases w.r.t the premium return rate of the reinsurer \( \eta \), time \( t \) and insurance claim amount \( a \).

Figure 1 shows that when the the level of ambiguity towards insurance liability \( \beta_1 \) decreases or the relative safety loading of the reinsurer \( \eta \) increases, the insurer will purchase less reinsurance or acquire more new business. As \( \beta_3 \) increase, the insurance liability is more ambiguous, then to manage the risk exposure, the insurer will purchase more reinsurance or acquire less new business. With the increase of \( \eta \), in order to reduce
the costly compensation of reinsurance, insurers are more willing to undertake more insurance business and increase the retention rate of reinsurance. In addition, as time $t$ goes by, insurers will retain more insurance business by buying less reinsurance or acquiring more new business.

Figure 2 shows that the optimal reinsurance strategy $\pi^*_q(t)$ decreases with ambiguity attitude $\alpha$ and risk coefficient $\gamma$, while increases with the transformed weight $\lambda$ and insurance claim amount $a$. As $\lambda$ increases, the average delayed wealth $\bar{Y}(T)$ takes up a larger weight in the final performance measure. Therefore, the average effect will definitely reduce the overall risk of the insurance company’s terminal wealth. That is, to achieve the same expected terminal wealth level, when $\lambda$ is large, the risk can be controlled at a low level. As $\alpha$ and $\gamma$ increase, the insurer is more ambiguity averse and
risk averse, the insurer will be more conservative to undertake risk and thus purchase more reinsurance or acquire less new business.

Figure 3a: Effects of $\delta$ and $h$ on $\pi^*_{q}$

Figure 3b: Effects of $r_0$ and $\sigma_1$ on $\pi^*_{q}$

Figure 3: Effects of parameters $\delta$, $h$, $r_0$ and $\sigma_1$ on $\pi^*_{q}$

Figure 3 shows that the optimal reinsurance strategy $\pi^*_{q}(t)$ decreases with average parameter $\delta$, the risk-free interest rate $r_0$ and the volatility of the surplus process $\sigma_1$, while increases with the delay parameter $h$. From the definition of the average delayed wealth $\bar{Y}(t)$, we can see that the larger $\delta$ is, the smaller the weight of early wealth. In other words, wealth closer to the current time is more important in constructing the average delayed wealth $\bar{Y}(t)$. As $h$ increases, the average time range becomes longer, so the average delayed wealth is more stable.

5.2. Analysis of the optimal investment strategy

Figure 4a: Effects of $\mu$ and $\alpha$ on $\pi^*_{s}$

Figure 4b: Effects of $\sigma_2$ and $r_0$ on $\pi^*_{s}$

Figure 4: Effects of parameters $\mu$, $\alpha$, $\sigma_2$ and $r_0$ on $\pi^*_{s}$
In this subsection, we provide some sensitivity analyses on the effect of the parameters on the optimal investment strategies \( \pi_s(t) \). According to (5.2), we can derive that \( \frac{\partial \pi_s(t)}{\partial \alpha} < 0, \frac{\partial \pi_s(t)}{\partial \beta_2} < 0, \frac{\partial \pi_s(t)}{\partial \sigma_2^2} < 0, \frac{\partial \pi_s(t)}{\partial \gamma} < 0, \frac{\partial \pi_s(t)}{\partial \mu} > 0, \frac{\partial \pi_s(t)}{\partial t} > 0 \), which imply that the optimal investment strategy \( \pi_s(t) \) decreases w.r.t the ambiguity attitude \( \alpha \), the stock return ambiguity \( \beta_2 \), the volatility of the stock volatility \( \sigma_2^2 \) and the coefficient of risk aversion of the insurer \( \gamma \) but increases w.r.t the stock returns \( \mu \) and time \( t \).

Figure 4 imply that the optimal investment strategy \( \pi_s(t) \) increases with \( \mu \), while decreases with \( \alpha, r_0 \) and \( \sigma_2 \). This shows that the higher the expected return on stocks, the smaller the volatility, the greater the amount invested in stocks. When \( r_0 \) and \( \alpha \) increases, the amount invested in stocks decreases.

When the average parameter \( \delta \) decreases or the delay parameter \( h \) increases, the
insures will invest more money in the risky asset, as showed in Figure 5a. The insurer will invest more money in the risky asset as the transformed weight $\lambda$ increases. This is showed in Figure 5b.

Figure 6 imply that the optimal investment strategy $\pi_s^*(t)$ increases with $t$ and $\lambda$, decreases with $\beta_2$ and $\gamma$. The insurer will invest less money in the stock if she/he becomes more risk aversion or the stock return is more ambiguous.

6. Conclusion

This paper studies the robust optimal reinsurance-investment problem for insurers with delay. Assume that the financial market consists of a risk-free asset and a risky asset. The explicit solution of the optimal reinsurance and investment strategy and value function is derived, and the proof of the verification theorem is given. The influence of parameters on the optimal strategy is analyzed through numerical examples. There are some related issues that deserve further study. On the one hand, more complex stock models can be considered, although explicit solutions may not be possible. On the other hand, we can extend the risk aversion coefficient in the objective function of this paper from constant to more general cases, such as dependence on market state or current surplus. In addition, we only consider the optimal time-consistent strategy for the mean-variance problem, which can be extended to consider the optimal reinsurance and investment problems with other objectives in future studies.

Appendix

The proof of Theorem 2.1

In this section, we deduce the proof of Theorem 2.1. The results of Theorem 2.1 are divided into two parts, which are proved in Proposition 6.1 and Proposition 6.2, respectively.

Lemma 6.1. Consider a deterministic strategy $\pi = \{\pi_q(t), \pi_s(t)\}_{t \in [0,T]}$.

1. The discounted process $\{e^{-\gamma_1 + \lambda}t \left( X^\pi(t) + \lambda Y^\pi(t) \right) \}_{t \in [0,T]}$ has independent increments under any $Q^\phi \in \mathcal{Q}$.

2. The function $J^\pi, \tilde{\omega}(t, x, y) - \tilde{J}^\pi, \omega(t, x, y)$ is independent of $x$ and $y$.

3. For any $0 \leq t \leq w \leq T$, the probability distortion functions corresponding to reaching the supremum and infimum in (2.12) are the same as the probability distortion
functions attaining the infimum in
\[
\inf_{\phi \in \Theta} \left\{ \mathbb{E}^\phi \left[ e^{(r_1 + \lambda)(T - w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T - t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] \right. \\
- \frac{\gamma}{2} \text{Var}^\phi \left[ e^{(r_1 + \lambda)(T - w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T - t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] \\
+ \int_t^w h_\beta(\phi(s))ds \right\},
\]
and supremum in
\[
\sup_{\phi \in \Theta} \left\{ \mathbb{E}^\phi \left[ e^{(r_1 + \lambda)(T - w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T - t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] \right. \\
- \frac{\gamma}{2} \text{Var}^\phi \left[ e^{(r_1 + \lambda)(T - w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T - t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] \\
- \int_t^w h_\beta(\phi(s))ds \right\},
\]
which are given respectively by
\[
\begin{cases}
\phi^\pi_1(t) = \beta_1 \pi_q(t)\sigma_1 e^{(r_1 + \lambda)(T - t)}, \\
\phi^\pi_2(t) = \beta_2 \pi_s(t)\sigma_2 e^{(r_1 + \lambda)(T - t)},
\end{cases}
\tag{6.1}
\]
and
\[
\begin{cases}
\bar{\phi}^\pi_1(t) = -\beta_1 \pi_q(t)\sigma_1 e^{(r_1 + \lambda)(T - t)}, \\
\bar{\phi}^\pi_2(t) = -\beta_2 \pi_s(t)\sigma_2 e^{(r_1 + \lambda)(T - t)}. 
\end{cases}
\tag{6.2}
\]

Proof. (1) By applying Ito’s formula to (2.10), we have
\[
e^{-r_1 T}(X^\pi(t) + \lambda Y^\pi(t)) - e^{-r_1 T}(X^\pi(t) + \lambda Y^\pi(t)) = \\
\int_t^T e^{-(r_1 + \lambda)s} \left[ (\mu - r_0)\pi_s(s) + (\theta - \eta + \eta \pi_q(s))a \right] ds \\
+ \int_t^T e^{-(r_1 + \lambda)s} \left[ -\pi_q(s)\sigma_1 \phi_1(s) - \pi_s(s)\sigma_2 \phi_2(s) \right] ds \\
+ \int_t^T e^{-(r_1 + \lambda)s} \pi_q(s)\sigma_1 dW_1^\phi(s) + \int_t^T e^{-(r_1 + \lambda)s} \pi_s(s)\sigma_2 dW_2^\phi(s) \\
\tag{6.3}
\]
Because \(\pi\) and \(\phi\) are determined, it is easy to get the independent incremental property.
(2) By subtracting (2.14) from (2.13), and using $\bar{\phi}^\pi, \bar{\phi}^\pi \in \Theta$, we have

\[
\bar{J}^\pi, \bar{\phi}^\pi(t, x, y) - \bar{J}^{\pi, \phi^\pi}(t, x, y) = \mathbb{E}^\phi_t \left[ (X^\pi(t) + \lambda Y^\pi(t)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] - \mathbb{E}^{\bar{\phi}^\pi}_t \left[ (X^\pi(t) + \lambda Y^\pi(t)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] - \frac{\gamma}{2} \text{Var}^\phi \left[ (X^\pi(t) + \lambda Y^\pi(t)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] + \frac{\gamma}{2} \text{Var}^{\bar{\phi}^\pi} \left[ (X^\pi(t) + \lambda Y^\pi(t)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] - \int_t^T h_\beta \left( \bar{\phi}(s) \right) ds - \int_t^T h_\beta \left( \phi^\pi(s) \right) ds,
\]

which is independent of $x$ and $y$. Since $\bar{J}^\pi, \bar{\phi}^\pi(t, x, y) - \bar{J}^{\pi, \phi^\pi}(t, x, y)$ is independent of $x$ and $y$, we write it as $\bar{J}^\pi, \bar{\phi}^\pi(t, \cdot, \cdot) - \bar{J}^{\pi, \phi^\pi}(t, \cdot, \cdot)$. In the following, we use the same notation for functions that are independent of $x$ and $y$.

(3) It follows from (6.3) and (2.15) that

\[
\mathbb{E}^\phi \left[ e^{(r_1 + \lambda)(T-w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] - \frac{\gamma}{2} \text{Var}^\phi \left[ e^{(r_1 + \lambda)(T-w)}(X^\pi(w) + \lambda Y^\pi(w)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t)) \right] \\
\pm \int_t^w h_\beta(\phi(s))ds \\
= \int_t^w e^{(r_1 + \lambda)(T-s)} \left[ (\mu - r_0)\pi_s(s) + (\theta - \eta + \eta\pi_q(s))a - \pi_q(s)\sigma_1\phi_1(s) - \pi_s(s)\sigma_2\phi_2(s) \right] ds \\
- \frac{\gamma}{2} \int_t^w e^{2(r_1 + \lambda)(T-s)} \left[ \pi_q(s)^2\sigma_1^2 + \pi_s(s)^2\sigma_2^2 \right] ds \pm \int_t^w \phi_1(s)^2 + \frac{\phi_2(s)^2}{2}\beta_2 ds.
\]

The variance formula of $X$ can be found in Remark 1.18 of Øksendal and Sulem (2007). Applying the first-order condition to (6.6) yields the expressions for $\phi^\pi$ and $\bar{\phi}^\pi$ given in

\[
J^\pi, \phi^\pi(t, x, y) - J^{\pi, \phi^\pi}(t, x, y) = \mathbb{E}_t \left[ J^\pi, \phi^\pi(t, x, y) - J^{\pi, \phi^\pi}(t, x, y) \right],
\]

which are independent of $x$ and $y$. Since $J^\pi, \phi^\pi(t, x, y) - J^{\pi, \phi^\pi}(t, x, y)$ is independent of $x$ and $y$, we write it as $J^\pi, \phi^\pi(t, \cdot, \cdot) - J^{\pi, \phi^\pi}(t, \cdot, \cdot)$.
and (6.2), respectively. And both expressions of $\phi^\pi$ and $\overline{\phi}^\pi$ are independent of $w$. By letting $w = T$ in (6.6), we have

$$
\mathbb{E}^{\phi} [(X^\pi(T) + \lambda Y^\pi(T)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t))]
$$

$$
- \frac{\gamma}{2} \text{Var}^{\phi} [(X^\pi(T) + \lambda Y^\pi(T)) - e^{(r_1 + \lambda)(T-t)}(X^\pi(t) + \lambda Y^\pi(t))]
$$

$$
\pm \int_t^T h_\beta(\phi(s))ds
$$

$$
= E_{t,x,y}^{\phi^\pi} [X^\pi(T) + \lambda Y^\pi(T)] - e^{(r_1 + \lambda)(T-t)}(X + \lambda y) - \frac{\gamma}{2} \text{Var}_{t,x,y}^{\phi^\pi} [X^\pi(T) + \lambda Y^\pi(T)]
$$

$$
\pm \int_t^T h_\beta(\phi(s))ds.
$$

(6.7)

We get the conclusion that $\phi^\pi$ and $\overline{\phi}^\pi$ given in (6.1) and (6.2) also reach the infimum and supremum in (2.12), respectively.

The following proposition proves the first part of Theorem 2.1, which is the representation of the value functions $g, \overline{g}$, and $V$.

**Proposition 6.1.** Under all the conditions of Theorem 2.1, we have

\[
\begin{aligned}
V(t, x, y) &= J^\pi_\alpha(t, x, y), \\
\underline{g}(t, x, y) &= \mathbb{E}_{t,x,y}^{\phi^\pi} [X^\pi(T) + \lambda Y^\pi(T)], \\
\overline{g}(t, x, y) &= \mathbb{E}_{t,x,y}^{\overline{\phi}^\pi} [X^\pi(T) + \lambda Y^\pi(T)].
\end{aligned}
\]  

(6.8)

**Proof.** According to (2.17), Dynkin’s formula and the second relationship of (2.19), we have

$$
\mathbb{E}_{t,x,y}^{\phi^\pi} [g(T, X^\pi(T), Y^\pi(T))] = g(t, x, y) + \mathbb{E}_{t,x,y}^{\phi^\pi} \left[ \int_t^T A^{\pi^*, \phi^*} g(s, X^\pi(s), Y^\pi(s)) ds \right]
$$

$$
= g(t, x, y).
$$

Then through the third relation of (2.19) it is easy to get

$$
\underline{g}(t, x, y) = \mathbb{E}_{t,x,y}^{\phi^\pi} [g(T, X^\pi(T), Y^\pi(T))] = \mathbb{E}_{t,x,y}^{\phi^\pi} [X^\pi(T) + \lambda Y^\pi(T)].
$$

Similarly, we can obtain $\overline{g}(t, x, y) = \mathbb{E}_{t,x,y}^{\overline{\phi}^\pi} [X^\pi(T) + \lambda Y^\pi(T)]$.

Next, we need to prove $V(t, x, y) = J^\pi_\alpha(t, x, y)$. Since the optimal values in (2.18) is in $(\pi^*, \underline{\phi}^*, \overline{\phi}^*)$. Use the condition (3) of Theorem 2.1 and the second relation of (2.19),
we can rewrite (2.18) as
\[
0 = \alpha \left\{ A_{\pi^*}^{-1} \phi^* V(t, \cdot, \cdot) - \frac{\gamma}{2} A_{\pi^*}^{-1} g^2(t, \cdot, \cdot) + h_\beta (\phi^*(t)) \right\} \\
+ \hat{\alpha} \left\{ A_{\pi^*}^{-1} \phi^* V(t, \cdot, \cdot) - \frac{\gamma}{2} A_{\pi^*}^{-1} g^2(t, \cdot, \cdot) - h_\beta (\bar{\phi}(t)) \right\}.
\]
(6.9)

From the first relation of (2.19), Dynkin’s formula and condition (3), we get
\[
\mathbb{E}_{t,x,y}^{\phi^*} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] = \mathbb{E}_{t,x,y}^{\bar{\phi}} [V(T, X^{\pi^*}(T), Y^{\pi^*}(T))]
\[
= V(t, x, y) + \int_t^T A_{\pi^*}^{-1} \phi^* V(s, \cdot, \cdot) ds,
\]
\[
\mathbb{E}_{t,x,y}^{\bar{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] = \mathbb{E}_{t,x,y}^{\bar{\phi}} [V(T, X^{\pi^*}(T), Y^{\pi^*}(T))]
\[
= V(t, x, y) + \int_t^T A_{\pi^*}^{-1} \bar{\phi} V(s, \cdot, \cdot) ds.
\]

The linear combination of the above two equations yields
\[
\alpha \mathbb{E}_{t,x,y}^{\phi^*} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] + \hat{\alpha} \mathbb{E}_{t,x,y}^{\bar{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)]
\]
\[
= V(t, x, y) + \alpha \int_t^T A_{\pi^*}^{-1} \phi^* V(s, \cdot, \cdot) ds + \hat{\alpha} \int_t^T A_{\pi^*}^{-1} \bar{\phi} V(s, \cdot, \cdot) ds.
\]

Substituting (6.9) into the last equation, we have
\[
V(t, x, y)
\]
\[
= \alpha \left\{ \mathbb{E}_{t,x,y}^{\phi^*} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \int_t^T A_{\pi^*}^{-1} \phi^* g^2(s, \cdot, \cdot) ds + \int_t^T h_\beta (\phi^*(s)) ds \right\} \\
+ \hat{\alpha} \left\{ \mathbb{E}_{t,x,y}^{\bar{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \int_t^T A_{\pi^*}^{-1} \bar{\phi} g^2(s, \cdot, \cdot) ds - \int_t^T h_\beta (\bar{\phi}(s)) ds \right\}.
\]
(6.10)

By the third relation of (2.19), Dynkin’s formula and condition (3) of Theorem 2.1, we
have

\[
\alpha \mathbb{E}_{t,x,y}^\phi [(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T))^2] + \hat{\alpha} \mathbb{E}_{t,x,y}^{\tilde{\phi}} [(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T))^2]
\]
\[
= \alpha \mathbb{E}_{t,x,y}^\phi \left[ g^2(T, X^{\pi^*}(T), Y^{\pi^*}(T)) \right] + \hat{\alpha} \mathbb{E}_{t,x,y}^{\tilde{\phi}} \left[ \tilde{g}^2(T, X^{\pi^*}(T), Y^{\pi^*}(T)) \right]
\]
\[
= \alpha g^2(t, x, y) + \alpha \int_t^T \mathcal{A}^{\pi^*} \tilde{g}^2(s, \cdot, \cdot)ds + \hat{\alpha} \tilde{g}^2(t, x, y) + \hat{\alpha} \int_t^T \mathcal{A}^{\pi^*} \tilde{\phi^2}(s, \cdot, \cdot)ds
\]
\[
= \alpha \left( \mathbb{E}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] \right)^2 + \alpha \int_t^T \mathcal{A}^{\pi^*} \tilde{g}^2(s, \cdot, \cdot)ds + \hat{\alpha} \left( \mathbb{E}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] \right)^2 + \hat{\alpha} \int_t^T \mathcal{A}^{\pi^*} \tilde{\phi^2}(s, \cdot, \cdot)ds,
\]
\[\text{(6.11)}\]

And (6.11) is equivalent to

\[
\alpha \text{Var}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] + \hat{\alpha} \text{Var}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)]
\]
\[
= \alpha \int_t^T \mathcal{A}^{\pi^*} \tilde{g}^2(s, \cdot, \cdot)ds + \hat{\alpha} \int_t^T \mathcal{A}^{\pi^*} \tilde{\phi^2}(s, \cdot, \cdot)ds.
\]
\[\text{(6.12)}\]

Finally, using condition (4) of Theorem 2.1 and substituting (6.12) into (6.10) yields

\[
V(t, x, y)
\]
\[
= \alpha \left\{ \mathbb{E}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] + \int_t^T h_\beta(\phi^*(s))ds \right\}
\]
\[
+ \hat{\alpha} \left\{ \mathbb{E}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \int_t^T h_\beta(\tilde{\phi}^*(s))ds \right\}
\]
\[
= \alpha \left\{ \mathbb{E}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^\phi [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] + \int_t^T h_\beta(\phi^*(s))ds \right\}
\]
\[
+ \hat{\alpha} \left\{ \mathbb{E}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \frac{\gamma}{2} \text{Var}_{t,x,y}^{\tilde{\phi}} [X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)] - \int_t^T h_\beta(\tilde{\phi}^*(s))ds \right\}
\]
\[
= J_{\alpha}^\pi (t, x, y).
\]
\[\text{(6.13)}\]

Next we prove that $\pi^*$ is an equilibrium strategy, we first show some properties of the perturbed strategy $\pi^\epsilon$ in Definition 2.1.

**Lemma 6.2.** Under conditions (3) and (4) of Theorem 2.1, consider the deterministic
Further, recall from part (1) of Lemma 6.1 that \((\bar{\pi}, \bar{X})\) and \((\tilde{\pi}, \tilde{X})\) are given respectively by

\[
\begin{cases}
\tilde{\pi}(t), & s \in [t, t + \epsilon), \\
\bar{\pi}(t), & s \in [t + \epsilon, T].
\end{cases}
\]

where \(\tilde{\pi} = (\tilde{\pi}_t, \tilde{\pi}_s) \in [0, +\infty) \times \mathbb{R}, t \in [0, T], \) and \(\epsilon > 0.\)

(1) The corresponding probability distortion functions attaining the infimum and supremum in (2.12) are given respectively by

\[
\phi^{\pi^*}(s) = \begin{cases}
\phi^{\hat{\pi}}(s), & s \in [t, t + \epsilon), \\
\phi^{\tilde{\pi}}(s), & s \in [t + \epsilon, T],
\end{cases}
\quad \text{and} \quad
\phi^{-\pi^*}(s) = \begin{cases}
\phi^{-\hat{\pi}}(s), & s \in [t, t + \epsilon), \\
\phi^{-\tilde{\pi}}(s), & s \in [t + \epsilon, T].
\end{cases}
\] (6.14)

(2) For any measurable function \(f : \mathbb{R} \to \mathbb{R}\) such that the following conditional expectations are finite, we have

\[
\mathbb{E}^{\phi^{\pi^*}}_{t,x,y} \left[ f(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) \right] = \mathbb{E}^{\phi^{\hat{\pi}}}_{t,x,y} \left[ \mathbb{E}^{\phi^{\tilde{\pi}}}_{t+\epsilon,x^* (t+\epsilon),y^* (t+\epsilon)} \left[ f(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) \right] \right],
\] (6.15)

and

\[
\mathbb{E}^{\phi^{-\pi^*}}_{t,x,y} \left[ f(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) \right] = \mathbb{E}^{\phi^{-\hat{\pi}}}_{t,x,y} \left[ \mathbb{E}^{\phi^{-\tilde{\pi}}}_{t+\epsilon,x^* (t+\epsilon),y^* (t+\epsilon)} \left[ f(X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) \right] \right].
\] (6.16)

Proof. (1) From the definition of the strategy \(\pi^*\) and (6.3), under any \(\mathbb{Q}^\phi \in \mathcal{Q}\), we have

\[
X^{\pi^*}(t + \epsilon) + \lambda Y^{\pi^*}(t + \epsilon) \overset{d}{=} X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon), \quad \text{given} \ X^{\pi^*}(t) = X^\pi(t), Y^{\pi^*}(t) = Y^\pi(t).
\] (6.17)

and

\[
e^{-(r_1 + \lambda)T} (X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) - e^{-(r_1 + \lambda)(t + \epsilon)} (X^{\pi^*}(t + \epsilon) + \lambda Y^{\pi^*}(t + \epsilon)) \\
\overset{d}{=} e^{-(r_1 + \lambda)T} (X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) - e^{-(r_1 + \lambda)(t + \epsilon)} (X^{\pi^*}(t + \epsilon) + \lambda Y^{\pi^*}(t + \epsilon)).
\] (6.18)

Further, recall from part (1) of Lemma 6.1 that \((X^{\pi^*}(T) + \lambda Y^{\pi^*}(T)) - e^{(r_1 + \lambda)(T-t-\epsilon)} (X^{\pi^*}(t+\epsilon) + \lambda Y^{\pi^*}(t+\epsilon))\) and \((X^{\pi^*}(t+\epsilon) + \lambda Y^{\pi^*}(t+\epsilon))\) are independent under any \(\mathbb{Q}^\phi \in \mathcal{Q}\).
This together with (6.17) and (6.18) implies

\[
\mathcal{J}^{\pi^\star, \phi}(t, x, y) = \mathbb{E}^{\phi}_{t,x,y} \left[ X^{\pi^\star}(T) + \lambda Y^{\pi^\star}(T) \right] - \frac{\gamma}{2} \operatorname{Var}^{\phi}_{t,x,y} \left[ X^{\pi^\star}(T) + \lambda Y^{\pi^\star}(T) \right] \\
+ \int_{t}^{T} h_\beta(\phi(s))ds 
\]

(6.19)

where

\[
F^\phi(t + \varepsilon, T) := \mathbb{E}^{\phi} \left[ X^{\pi^\star}(T) + \lambda Y^{\pi^\star}(T) - e^{(r_1 + \lambda)(T-t-\varepsilon)}(X^{\pi^\star}(t + \varepsilon) + \lambda Y^{\pi^\star}(t + \varepsilon)) \right] \\
- \frac{\gamma}{2} \operatorname{Var}^{\phi} \left[ X^{\pi^\star}(T) + \lambda Y^{\pi^\star}(T) - e^{(r_1 + \lambda)(T-t-\varepsilon)}(X^{\pi^\star}(t + \varepsilon) + \lambda Y^{\pi^\star}(t + \varepsilon)) \right] \\
+ \int_{t+\varepsilon}^{T} h_\beta(\phi(s))ds 
\]

and

\[
F^\phi(t, t + \varepsilon) := \mathbb{E}^{\phi}_{t,x,y} \left[ e^{(r_1 + \lambda)(T-t-\varepsilon)}(X^{\pi^\star}(t + \varepsilon) + \lambda Y^{\pi^\star}(t + \varepsilon)) \right] \\
- \frac{\gamma}{2} \operatorname{Var}^{\phi}_{t,x,y} \left[ e^{(r_1 + \lambda)(T-t-\varepsilon)}(X^{\pi^\star}(t + \varepsilon) + \lambda Y^{\pi^\star}(t + \varepsilon)) \right] + \int_{t}^{t+\varepsilon} h_\beta(\phi(s))ds 
\]

By choosing \(w = T\) and \(t = t + \varepsilon\) in part (3) of Lemma 6.1, we have

\[
\left\{ \phi^{\pi^\star}(s) \right\}_{s \in [t+\varepsilon, T]} = \operatorname{argmin}_{\phi \in \Theta} F^\phi(t + \varepsilon, T),
\]

Similarly, by choosing \(w = t + \varepsilon\) and \(t = t\) in part (3) of Lemma 6.1, we have

\[
\left\{ \phi^{*\pi}(s) \right\}_{s \in [t, t+\varepsilon]} = \operatorname{argmin}_{\phi \in \Theta} F^\phi(t, t + \varepsilon).
\]

The first relation of (6.14) follows immediately from (6.19), and the second relation can
be proved in the same way.

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function such that all the involved conditional expectations are finite. It follows from (6.14) that

\[
\mathbb{E}^{\phi^\pi}_{t, x, y} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \right] = \mathbb{E}^{\phi^\pi} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \mid X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) \right] = \mathbb{E}^{\phi^\pi} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \mid X^\pi(t + \epsilon) = x, Y^\pi(t) = y \right].
\]

Recall from (6.17) that given \( X^\pi(t) + \lambda Y^\pi(t) = X^\pi(t) + \lambda Y^\pi(t) = x + \lambda y \), we have \( X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) \overset{d}{=} X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) \). Thus, by the strong Markov property,

\[
\mathbb{E}^{\phi^\pi}_{t, x, y} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \right] = \mathbb{E}^{\phi^\pi} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \mid X^\pi(t) = x, Y^\pi(t) = y \right].
\]

Given \( X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) = X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) = X^\pi(t + \epsilon) + \lambda Y^\pi(t + \epsilon) \), since the dynamics of \( X^\pi(s) + \lambda Y^\pi(s) \) and \( X^\pi(s) + \lambda Y^\pi(s) \) are the same for \( s \in (t + \epsilon, T] \) we have

\[
\mathbb{E}^{\phi^\pi}_{t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \right] = \mathbb{E}^{\phi^\pi}_{t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)} \left[ f \left( X^\pi(t) + \lambda Y^\pi(t) \right) \right].
\]

Substituting the last equation into (6.20) yields (6.15). Equation (6.16) can be obtained symmetrically. \( \square \)

Now we prove that \( \pi^* \) is an equilibrium strategy.

**Proposition 6.2.** Under all the conditions of Theorem 2.1, \( \pi^* \) is an equilibrium strategy.

**Proof.** Consider the perturbed strategy \( \pi^\epsilon \) in Lemma 6.2, in order to prove that \( \pi^* \) is an equilibrium strategy, that is, to prove

\[
J_{\pi^*}^{\pi^\epsilon}(t, x, y) - J_{\pi^*}^{\pi^\epsilon}(t, x, y) = O(\epsilon).
\]
Therefore, in the following proof, we first derive the expression of $J_\alpha^\pi(t, x, y) - J_\alpha^\pi(t, x, y)$, and then show that it is bounded by $o(\epsilon)$. By (2.13), (6.8) and (6.15),

$$
J_\alpha^\pi (t, x, y) = \mathbb{E} \phi(t, x, y) \bigg[ (X^\pi(T) + \lambda Y^\pi(T)) \bigg] - \frac{\gamma}{2} (X^\pi(T) + \lambda Y^\pi(T))^2 
$$

$$
+ \frac{\gamma}{2} \bigg( \mathbb{E} \phi(s) \bigg[ X^\pi(T) + \lambda Y^\pi(T) \bigg] \bigg)^2 + \int_t^T h_\beta \left( \phi(s) \right) ds 
$$

$$
= \mathbb{E} \phi(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \bigg[ (X^\pi(T) + \lambda Y^\pi(T)) \bigg] - \frac{\gamma}{2} \mathbb{E} \phi(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon))^2 
$$

$$
+ \frac{\gamma}{2} \bigg( \mathbb{E} \phi(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \bigg)^2 + \int_t^{t+\epsilon} h_\beta \left( \phi(s) \right) ds. 
$$

Similarly, we can prove that

$$
J_\alpha^\pi (t, x, y) = \mathbb{E} \bar{\phi}(t, x, y) \bigg[ (X^\pi(T) + \lambda Y^\pi(T)) \bigg] - \frac{\gamma}{2} \mathbb{E} \bar{\phi}(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon))^2 
$$

$$
+ \frac{\gamma}{2} \bigg( \mathbb{E} \bar{\phi}(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \bigg)^2 - \int_t^{t+\epsilon} h_\beta \left( \bar{\phi}(s) \right) ds. 
$$
A linear combination of (6.21) and (6.22) yields

\[
J^{\pi^*}_{\alpha}(t, x, y) = \alpha J^{\pi^*, \varphi^*}_{\alpha}(t, x, y) + \hat{\alpha} \mathcal{J}^{\pi^*, \bar{\varphi}^*}_{\alpha}(t, x, y)
\]

where

\[
\begin{align*}
\frac{\alpha}{2} \mathbb{E}_{t,x,y}^{\pi^*} \left[ J^{\pi^*, \varphi^*}_{\alpha}(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] &+ \frac{\hat{\alpha}}{2} \mathbb{E}_{t,x,y}^{\pi^*} \left[ \mathcal{J}^{\pi^*, \bar{\varphi}^*}_{\alpha}(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] \\
- \frac{\alpha \gamma}{2} \mathbb{E}_{t,x,y}^{\pi^*} \left[ g^2(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] &+ \frac{\hat{\alpha} \gamma}{2} \mathbb{E}_{t,x,y}^{\pi^*} \left[ \mathcal{g}^2(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] \\
+ \alpha \int_t^{t+\epsilon} h_\beta \left( \varphi_{\bar{\pi}}(s) \right) ds - \hat{\alpha} \int_t^{t+\epsilon} h_\beta \left( \bar{\varphi}(s) \right) ds
\end{align*}
\]

(6.23)

By subtracting \(J^{\pi^*}_{\alpha}(t, x, y)\) to both sides of (6.23), using condition (4) of Theorem 2.1, and inserting the extra terms of \(g^2(t, x, y)\) and \(\mathcal{g}^2(t, x, y)\), we obtain

\[
J^{\pi^*}_{\alpha}(t, x, y) - J^{\pi^*}_{\alpha}(t, x, y) = J^{\pi^*}_{\alpha}(t, x, y) - \alpha J^{\pi^*, \varphi^*}_{\alpha}(t, x, y) - \hat{\alpha} \mathcal{J}^{\pi^*, \bar{\varphi}^*}_{\alpha}(t, x, y) := H_\epsilon
\]

where

\[
H_\epsilon := \alpha \left\{ \mathbb{E}_{t,x,y}^{\pi^*} \left[ J^{\pi^*, \varphi^*}_{\alpha}(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] - J^{\pi^*, \varphi^*}_{\alpha}(t, x, y) \right\} \\
+ \hat{\alpha} \left\{ \mathbb{E}_{t,x,y}^{\pi^*} \left[ \mathcal{J}^{\pi^*, \bar{\varphi}^*}_{\alpha}(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] - \mathcal{J}^{\pi^*, \bar{\varphi}^*}_{\alpha}(t, x, y) \right\} \\
- \frac{\alpha \gamma}{2} \left\{ \mathbb{E}_{t,x,y}^{\pi^*} \left[ g^2(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] - g^2(t, x, y) \right\} \\
+ \frac{\hat{\alpha} \gamma}{2} \left\{ \mathbb{E}_{t,x,y}^{\pi^*} \left[ \mathcal{g}^2(t + \epsilon, X^{\bar{\pi}}(t + \epsilon), Y^{\bar{\pi}}(t + \epsilon)) \right] - \mathcal{g}^2(t, x, y) \right\} \\
+ \alpha \int_t^{t+\epsilon} h_\beta \left( \varphi_{\bar{\pi}}(s) \right) ds - \hat{\alpha} \int_t^{t+\epsilon} h_\beta \left( \bar{\varphi}(s) \right) ds
\]

(6.24)

Next we show that \(H_\epsilon \leq o(\epsilon)\). For ease of notation, we define the operator

\[
\mathcal{A}^{\pi^*, \varphi}_{\epsilon}(t, x, y) := \mathbb{E}_{t,x,y}^{\pi^*} \left[ \psi \left( t + \epsilon, Y^{\pi}(t + \epsilon) \right) \right] - \psi(t, x, y)
\]
where $u \in \Pi, \phi \in \Theta, \epsilon > 0$ is a small constant, and $\psi \in C^{1,2}([0, T] \times \mathbb{R})$. By Lemma 6.2, we have

$$
\mathcal{A}_{\epsilon}^{\pi, \phi^*} \psi(t, x, y) = \mathcal{A}_{\epsilon}^{\pi, \phi^*} \psi(t, x, y) \quad \text{and} \quad \mathcal{A}_{\epsilon}^{\psi, \phi^*} \psi(t, x, y) = \mathcal{A}_{\epsilon}^{\psi, \phi^*} \psi(t, x, y)
$$

(6.25)

which further implies from (2.17) that

$$
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathcal{A}_{\epsilon}^{\pi, \phi^*} \psi(t, x, y) = \mathcal{A}_{\epsilon}^{\pi, \phi^*} \psi(t, x, y) \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathcal{A}_{\epsilon}^{\psi, \phi^*} \psi(t, x, y) = \mathcal{A}_{\epsilon}^{\psi, \phi^*} \psi(t, x, y)
$$

(6.26)

So we can rewrite $H_\epsilon$ in (6.24) as

$$
H_\epsilon = \alpha \mathcal{A}_{\epsilon}^{\pi, \phi^*} \mathcal{J}^{\pi, \phi^*}(t, x, y) - \frac{\alpha \gamma}{2} \mathcal{A}_{\epsilon}^{\pi, \phi^*} g_2(t, x, y)
$$

$$
+ \frac{\alpha \gamma}{2} \left\{ \left( \mathbb{E}_{t, x, y}^{\phi^*} \left[ g(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \right] \right)^2 - g_2^2(t, x, y) \right\}
$$

$$
+ \alpha \int_t^{t+\epsilon} h_\beta \left( \psi^\pi(s) \right) ds + \hat{\alpha} \mathcal{A}_{\epsilon}^{\pi, \phi^*} \mathcal{J}^{\pi, \phi^*}(t, x, y) - \frac{\hat{\alpha} \gamma}{2} \mathcal{A}_{\epsilon}^{\pi, \phi^*} \overline{g}^2(t, x, y)
$$

$$
+ \frac{\hat{\alpha} \gamma}{2} \left\{ \left( \mathbb{E}_{t, x, y}^{\phi^*} \left[ \overline{g}(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \right] \right)^2 - \overline{g}^2(t, x, y) \right\} - \hat{\alpha} \int_t^{t+\epsilon} h_\beta \left( \psi^\pi(s) \right) ds.
$$

(6.27)

By Dynkin's formula, we have

$$
\mathbb{E}_{t, x, y}^{\phi^*} \left[ g(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \right] = g(t, x, y) + \mathbb{E}_{t, x, y}^{\phi^*} \left[ \int_t^{t+\epsilon} \mathcal{A}_{\epsilon}^{\pi, \phi^*} g(s, X^\pi(s), Y^\pi(s)) ds \right],
$$

which implies

$$
\left[ \mathbb{E}_{t, x, y}^{\phi^*} \left( g(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \right) \right]^2 - g_2^2(t, x, y) = 2g(t, x, y) \mathbb{E}_{t, x, y}^{\phi^*} \left[ \int_t^{t+\epsilon} \mathcal{A}_{\epsilon}^{\pi, \phi^*} g(s, X^\pi(s), Y^\pi(s)) ds \right] + o(\epsilon).
$$

(6.28)

Similarly,

$$
\left[ \mathbb{E}_{t, x, y}^{\phi^*} \left( \overline{g}(t + \epsilon, X^\pi(t + \epsilon), Y^\pi(t + \epsilon)) \right) \right]^2 - \overline{g}^2(t, x, y) = 2\overline{g}(t, x, y) \mathbb{E}_{t, x, y}^{\phi^*} \left[ \int_t^{t+\epsilon} \mathcal{A}_{\epsilon}^{\pi, \phi^*} \overline{g}(s, X^\pi(s), Y^\pi(s)) ds \right] + o(\epsilon).
$$

(6.29)
Substituting (6.28) and (6.29) into (6.27) yields

\[
H_\epsilon - o(\epsilon) = \alpha A^{\bar{\tau}, \bar{\phi}}_t J^{\pi^{\ast}, \bar{\phi}^{\ast}}_{t,x,y}(t, x, y) - \frac{\alpha \gamma}{2} A^{\bar{\tau}, \bar{\phi}}_t g^2(t, x, y) \\
+ \alpha g(t, x, y) E^\phi_{t,x,y} \left[ \int_t^{t+\epsilon} A^{\bar{\tau}, \bar{\phi}}_s g(s, X^\pi(s), Y^\pi(s)) \, ds \right] \\
+ \alpha \int_t^{t+\epsilon} h_\beta(\bar{\phi}^\pi(s)) \, ds + \hat{\alpha} A^{\bar{\tau}, \bar{\phi}}_t J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) - \frac{\hat{\alpha} \gamma}{2} A^{\bar{\tau}, \bar{\phi}}_t g^2(t, x, y) \\
+ \hat{\alpha} g(t, x, y) E^{\bar{\phi}^\pi}_{t,x,y} \left[ \int_t^{t+\epsilon} A^{\bar{\tau}, \bar{\phi}}_s g(s, X^\pi(s), Y^\pi(s)) \, ds \right] + o(\epsilon) \\
- \hat{\alpha} \int_t^{t+\epsilon} h_\beta(\bar{\phi}^{\ast}(s)) \, ds.
\]

It is only left to show that the right-hand side of (6.30) is bounded above by \(o(\epsilon)\). By the extended HJB (2.18), we have

\[
0 \geq \alpha \left[ A^{\bar{\tau}, \bar{\phi}} V(t, x, y) - \frac{\gamma}{2} A^{\bar{\tau}, \bar{\phi}} g^2(t, x, y) + g(t, x, y) A^{\bar{\tau}, \bar{\phi}} g(t, x, y) + h_\beta(\bar{\phi}^\pi(t)) \right] \\
+ \hat{\alpha} \left[ A^{\bar{\tau}, \bar{\phi}} V(t, x, y) - \frac{\gamma}{2} A^{\bar{\tau}, \bar{\phi}} g^2(t, x, y) + g(t, x, y) A^{\bar{\tau}, \bar{\phi}} g(t, x, y) - h_\beta(\bar{\phi}^\pi(t)) \right].
\]

(6.31)

It follows from (6.8) that

\[
\alpha A^{\bar{\tau}, \bar{\phi}} V(t, x, y) + \hat{\alpha} A^{\bar{\tau}, \bar{\phi}} V(t, x, y) \\
= \alpha A^{\bar{\tau}, \bar{\phi}} \left[ \alpha J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) + \hat{\alpha} J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) \right] + \hat{\alpha} A^{\bar{\tau}, \bar{\phi}} \left[ \alpha J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) + \hat{\alpha} J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) \right] \\
= \alpha \hat{\alpha} \left( A^{\bar{\tau}, \bar{\phi}} - A^{\bar{\tau}, \bar{\phi}} \right) \left[ J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \right] \\
+ \alpha A^{\bar{\tau}, \bar{\phi}} J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y) + \hat{\alpha} A^{\bar{\tau}, \bar{\phi}} J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, x, y).
\]

(6.32)

Recall from part (1) of Lemma 6.1 that \( J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \) is independent of \( x \) and \( y \). Thus, it is easy to see from (2.17) that

\[
A^{\bar{\tau}, \bar{\phi}} \left( J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \right) - A^{\bar{\tau}, \bar{\phi}} \left( J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \right) \\
= \frac{d}{dt} \left( J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \right) - \frac{d}{dt} \left( J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) - J^{\pi^{\ast}, \bar{\phi}^{\ast}}(t, \cdot, \cdot) \right)
\]

\[
= 0.
\]

(6.33)
Substituting (6.33) into (6.32) yields

$$\alpha A^\pi,\phi V(t, x, y) + \hat{\alpha} A^\pi,\phi \bar{V}(t, x, y) = \alpha A^\pi,\phi J^\pi,\phi^* (t, x, y) + \hat{\alpha} A^\pi,\phi \bar{J}^\pi,\phi^* (t, x, y). \quad (6.34)$$

We further substitute the last equation into (6.31) and then obtain

$$0 \geq \alpha \left[ A^\pi,\phi J^\pi,\phi^* (t, x, y) - \frac{\gamma}{2} A^\pi,\phi \bar{g}^2 (t, x, y) + \gamma g(t, x, y) A^\pi,\phi \bar{g}(t, x, y) + h_\beta (\phi^*(t)) \right]$$

$$+ \hat{\alpha} \left[ A^\pi,\phi \bar{J}^\pi,\phi^* (t, x, y) - \frac{\gamma}{2} A^\pi,\phi \bar{g}^2 (t, x, y) + \gamma \bar{g}(t, x, y) A^\pi,\phi \bar{g}(t, x, y) - h_\beta (\phi^*(t)) \right].$$

This together with (6.26) implies that

$$o(\epsilon) \geq \alpha A^\pi,\phi J^\pi,\phi^* (t, x, y) - \frac{\alpha \gamma}{2} A^\pi,\phi \bar{g}^2 (t, x, y)$$

$$+ \alpha \gamma g(t, x, y) \mathbb{E}_{t,x,y} \left[ \int_t^{t+\epsilon} A^\pi,\phi \bar{g} (s, X^\pi(s), Y^\pi(s)) \, ds \right]$$

$$+ \alpha \int_t^{t+\epsilon} h_\beta (\phi^*(s)) \, ds + \hat{\alpha} A^\pi,\phi \bar{J}^\pi,\phi^* (t, x, y) - \frac{\hat{\alpha} \gamma}{2} A^\pi,\phi \bar{g}^2 (t, x, y)$$

$$+ \hat{\alpha} \gamma \bar{g}(t, x, y) \mathbb{E}_{t,x,y} \left[ \int_t^{t+\epsilon} A^\pi,\phi \bar{g} (s, X^\pi(s), Y^\pi(s)) \, ds \right] - \hat{\alpha} \int_t^{t+\epsilon} h_\beta (\phi^*(s)) \, ds.$$

Finally, substituting (6.36) into (6.30) leads to

$$J^\pi_\alpha (t, x, y) - J^\pi_{\alpha^*} (t, x, y) = H_\epsilon \leq o(\epsilon),$$

which implies that $\pi^*$ is an equilibrium strategy. \qed
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