Boundary Triplets, Tensor Products
and Point Contacts to Reservoirs

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Abstract. We consider symmetric operators of the form $S := A \otimes I_T + I_H \otimes T$, where $A$ is symmetric and $T = T^\ast$ is (in general) unbounded. Such operators naturally arise in problems of simulating point contacts to reservoirs. We construct a boundary triplet $\Pi_S$ for $S^\ast$ preserving the tensor structure. The corresponding $\gamma$-field and Weyl function are expressed by means of the $\gamma$-field and Weyl function corresponding to the boundary triplet $\Pi_A$ for $A^\ast$ and the spectral measure of $T$. An application to 1-D Schrödinger operators is given. A model of electron transport through a quantum dot assisted by cavity photons is proposed. In this model the boundary operator is chosen to be the well-known Jaynes–Cummings operator which is regarded as the Hamiltonian of the quantum dot.

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1. Introduction

In the following we are interested in the description of point contacts of quantum systems to quantum reservoirs. Let us recall the general philosophy of modeling of point contacts in quantum mechanics. Let \( \{ H, A_0 \} \) be a quantum system where \( A_0 \) is a self-adjoint operator acting on the separable Hilbert space \( H \). To describe point interactions one restricts the self-adjoint operator \( A_0 \) to a densely defined closed symmetric operator \( A \) and extends it subsequently to another self-adjoint operator \( S' \). The new self-adjoint operator \( S' \) is regarded as the Hamiltonian taking into account point interactions. Which extension one has to choose depends on the physical problem. Typical examples for instance are \( \delta \)- and \( \delta' \)-point interactions, cf. [4]. From the mathematical point of view it is interesting to note that the problem of describing point interactions fits into the framework of extension theory for symmetric operators.

To describe point contacts of a quantum system with a reservoir one has to specify the approach. At first, one considers the compound system consisting of the quantum system \( \{ H, A_0 \} \) and the reservoir \( \{ \Sigma, T \} \) where \( A_0 \) and \( T \) are self-adjoint operators on the separable Hilbert spaces \( H \) and \( T \), respectively. Its Hamiltonian is given by the self-adjoint operator

\[
S_0 := A_0 \otimes I_T + I_H \otimes T, \tag{1.1}
\]

where \( S_0 \) acts in the Hilbert space \( H \otimes \Sigma \). To model a contact to the quantum reservoir, the Hamiltonian \( S_0 \) is usually additively perturbed in a suitable manner, cf. [21–23]. On the other hand, to model point contacts to reservoirs we use the restricting-extension procedure: We restrict the operator \( A_0 = A_0^* \) to a densely defined closed symmetric operator \( A \) and consider the closed symmetric operator

\[
S := A \otimes I_T + I_H \otimes T \subset S_0. \tag{1.2}
\]

From the physical point of view, the restriction of \( A_0 \) to \( A \) and the subsequent extension to a self-adjoint operator \( S' \) can be regarded as the opening of the quantum system \( \{ H, A_0 \} \) and the subsequent coupling of it to a reservoir. The self-adjoint extension \( S' \) should be different from \( S_0 \). However, self-adjoint extensions preserving the tensor product form \( \tilde{S} = \tilde{A} \otimes I_T + I_H \otimes T \) with \( \tilde{A} = \tilde{A}^* \) being an extension of \( A \) do not describe any interaction with the reservoir. From the physical point of view it is very important to describe all those extensions, which really describe point interactions with the reservoir.
In this paper we investigate operator (1.2) in the framework of boundary triplets. This is a relatively new approach to the extension theory of symmetric operators that has been developed during the last three decades (see, e.g., [24–26,29,36,53]).

A boundary triplet \( \Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\} \) for the adjoint operator \( A^* \) of a densely defined closed symmetric operator \( A \) consists of an auxiliary Hilbert space \( \mathcal{H}^A \) and linear mappings \( \Gamma_0^A, \Gamma_1^A : \text{dom}(A^*) \rightarrow \mathcal{H}^A \) such that the abstract Green’s identity
\[
(A^* f, g) - (f, A^* g) = (\Gamma_1^A f, \Gamma_0^A g) - (\Gamma_0^A f, \Gamma_1^A g), \quad f, g \in \text{dom}(A^*),
\]
holds and the mapping
\[
\Gamma^A := \begin{pmatrix} \Gamma_0^A \\ \Gamma_1^A \end{pmatrix} : \text{dom}(A^*) \rightarrow \mathcal{H}^A 
\]
is surjective.

A boundary triplet for \( A^* \) exists whenever \( A \) has equal deficiency indices. It plays the role of a “coordinate system” for the quotient space \( \text{dom}(A^*)/\text{dom}(A) \) and leads to a natural parametrization of the self-adjoint extensions of \( A \) by means of self-adjoint linear relations (multi-valued operators) in \( \mathcal{H} \), see [29,53] for details. More precisely, any self-adjoint extension \( \widetilde{A} \) of \( A \) defines a self-adjoint relation \( \Theta := \Gamma^A \text{dom}(\widetilde{A}) \) in \( \mathcal{H}^A \) and vice versa. We write \( A_\Theta = \widetilde{A}, \) i.e.,
\[
\text{dom}(\widetilde{A}) = \text{dom}(A_\Theta) := \{f \in \text{dom}(A^*) : \Gamma^A f \in \Theta\}.
\]
If \( \Theta \) is an operator \( \Theta = B \), this relation takes the form
\[
\text{dom}(\widetilde{A}) = \text{dom}(A_B) := \{f \in \text{dom}(A^*) : \Gamma_1^A f = B \Gamma_0^A f\} \tag{1.3}
\]
and looks like an abstract boundary condition. Among all self-adjoint extensions there are two particular (classical) ones: \( A_0 := A^* \upharpoonright \ker(\Gamma_0^A) \) and \( A_1 := A^* \upharpoonright \ker(\Gamma_1^A) \) which correspond to the self-adjoint relations \( \Theta_0 := (0, f)^t \) and \( \Theta_1 := (f, 0)^t, \ f \in \mathcal{H}, \) respectively. Clearly, \( \Theta_1^{-1} = \Theta_0. \)

The main analytical tool in this approach is the abstract Weyl function \( M^A(\cdot) \) which was introduced and studied in [26]. This abstract Weyl function \( M^A(\cdot) \) plays a similar role in the theory of boundary triplets as the classical Weyl–Titchmarsh function does it in the theory of Sturm–Liouville operators. In particular, it allows one to investigate spectral properties of extensions (see [14,26,42,44]). The Weyl function is defined by
\[
M^A(z) := \Gamma_1^A \gamma^A(z), \quad z \in \rho(A_0),
\]
where
\[
\gamma^A(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1}, \quad \mathfrak{N}_z := \ker(A^* - z), \quad z \in \rho(A_0).
\]
Here \( \gamma^A(\cdot) \) is the so-called \( \gamma \)-field, the second important quantity related to a boundary triplet \( \Pi_A. \)
We emphasize that a boundary triplet for $A^*$ is not unique. Its role in extension theory is similar to that of a coordinate system in analytic geometry. The problem is to construct an adequate ("good") boundary triplet such that the corresponding Weyl function and the boundary operator corresponding to the extension of interest have "good" properties. To demonstrate the latter point we mention an application to the 1-D Schrödinger operator $H_{X,\alpha}$ with $\delta$-interactions on a discrete set $X = \{x_n\}$. Namely, a special boundary triplet for the direct sum of maximal 1-D Schrödinger operators $-d^2/\Delta^2 + q$ in $L^2[x_{n-1},x_n]$ was constructed in [38], where the boundary operator corresponding to $H_{X,\alpha}$ in this triplet is a Jacobi matrix. A similar result was obtained in [19] for 1-D Dirac operators with point interactions on the line.

This approach has successfully been applied to the characterization of the absolutely continuous spectrum of self-adjoint realizations [14,44], as well as to the investigation of the spectral properties of 1-D Schrödinger and 1-D Dirac operators with point interactions [19,37,38], 3-D Schrödinger operators with point interactions [46] and elliptic boundary value problems in domains with compact boundaries [7,8,15,16,30–32,43], to the scattering theory [9,10], etc. In particular, we mention the works [28,29,45] (see also the literature quoted therein) where the Sturm–Liouville operator with unbounded operator potential was treated as an operator admitting the tensor structure (1.2).

Our goal in this paper is to apply the boundary triplet approach to the problem of coupling of a quantum system to a reservoir by point interactions. More precisely, the mathematically rigorous problem is: Given a boundary triplet $\Pi_A = \{H_A, \Gamma^A_0, \Gamma^A_1\}$ for $A^*$ construct an adequate ("good") boundary triplet $\Pi_S = \{H_S, \Gamma^S_0, \Gamma^S_1\}$ for $S^*$ with $S$ given by (1.2) and such that

$$S_0 := S^* \mid \ker(\Gamma^S_0) = A_0 \otimes I_T + I_H \otimes T$$

and compute the corresponding Weyl function and $\gamma$-field.

So, starting with a given boundary triplet $\Pi_A$ for $A^*$ a "good" candidate for a boundary triplet $\Pi_S := \{H_S, \Gamma^S_0, \Gamma^S_1\}$ for $S^*$ would be

$$H_S = H_A \otimes \Sigma, \quad \Gamma^S_0 := \Gamma^A_0 \otimes I_\Sigma, \quad \Gamma^S_1 := \Gamma^A_1 \otimes I_\Sigma.$$  

(1.5)

This triplet feels the tensor structure of the problem. For instance, according to (1.3) and (1.5) an extension $S' = A' \otimes I_\Sigma + I_\Sigma \otimes T \in \text{Ext}_S$ admits a representation $S' = S_{B'_S}$ with the boundary operator $B'_S = (B'_S)^*$ having the tensor form, $B'_S = B'_A \otimes I_\Sigma$, where $B'_A = (B'_A)^*$ is the boundary operator of $A' = A'^* \in \text{Ext}_A$, i.e., $A' = A_{B'_A}$. In particular, formula (1.4) holds. Hence, any extension $S_{B'} = S_{B'} \in \text{Ext}_S$ with the boundary operator $B'$ not admitting tensor structure can be regarded as a Hamiltonian describing a point interaction with the reservoir. In applications below $A$ is a Schrödinger operator on the line, $T$ is the boson Hamiltonian, and $B$ is a special boundary operator having a physical meaning.

It is shown in [13,45] that triplet (1.5) is a boundary triplet for $S^*$ whenever $T$ is bounded. However, this fails for unbounded $T$, a case naturally arising in physical problems. This case requires new ideas and is much more technically involved.
Let $\Pi_A = \{\cal H^A, \Gamma^A_0, \Gamma^A_1\}$ be a boundary triplet for $A^*$, and let $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Using the regularization procedure introduced and developed in [19,38,45] we construct a special boundary triplet $\Pi_S = \{\cal H^S, \Gamma^S_0, \Gamma^S_1\}$ for $S^*$ such that $S_0 := S^* \upharpoonright \ker (\Gamma^S_0) = A_0 \otimes I_{\bar \tau} + I_{\bar \delta} \otimes T$. Moreover, we show in Theorem 4.8 that the corresponding $\gamma$-field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ are given by
\[ \gamma^S(z)f := \int_{\mathbb{R}} \left( \gamma^A(z - \lambda) \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_{\bar \tau} \right) \widehat{E}_T(d\lambda)f, \quad (1.6) \]
and
\[ M^S(z)f := \int_{\mathbb{R}} \left( L^A(z - \lambda, i - \lambda) \otimes I_{\bar \tau} \right) \widehat{E}_T(d\lambda)f \quad z \in \mathbb{C}_+, \quad (1.7) \]
where
\[ L^A(z, \zeta) := \frac{1}{\sqrt{\text{Im}(M^A(z))}}(M^A(z) - \text{Re}(M^A(\zeta))) \frac{1}{\sqrt{\text{Im}(M^A(\zeta))}}, \]
$z \in \mathbb{C}_+$, $\zeta \in \mathbb{C}_+$, and $\widehat{E}_T(\cdot) := I_{\cal H^A} \otimes E_T(\cdot)$, where $E_T(\cdot)$ is the spectral measure of $T = T^*$. Here both improper integrals exist for every $f \in \cal H^A \otimes \bar \Sigma$.

We apply formula (1.7) for $M^S(\cdot)$ to show that for nonnegative $A \geq 0$ and $T \geq 0$ the Friedrichs’ and Krein’s extensions of $S \geq 0$ are given by
\[ \widehat{S}_F = \widehat{A}_F \otimes I_{\bar \tau} + I_{\bar \delta} \otimes T, \quad \widehat{S}_K = \widehat{A}_K \otimes I_{\bar \tau} + I_{\bar \delta} \otimes T, \quad (1.8) \]
where $\widehat{A}_F$ and $\widehat{A}_K$ are the Friedrichs’ and Krein’s extensions of $A$, respectively. In turn, we apply these formulas to show that if $T \in \cal B(\bar \Sigma)$, then the operator $S$ has LSB-property (each semi-bounded boundary operator $B$ defines a semi-bounded extension $S_B$ of $S$) if and only if the operator $A \geq 0$ has the LSB-property.

This approach can be used to propose a model describing rigorously the electron transport through a quantum dot assisted by photons, a topic which is of great interest for physicists, cf. [1,33,39,52]. In this case we start from operator (1.1) with $A_0$ being a Sturm–Liouville operator on the line with piecewise constant potential and unbounded $T$ given by $T = b^*b$, where $b^*$ and $b$ are the creation and annihilation operators, respectively. We define $A$ as a restriction of $A_0$ to the domain $\text{dom}(A) = W^{2,2}_0(\mathbb{R}_-) \oplus W^{2,2}_0(\mathbb{R}_+)$ and then define the operator $S$ by (1.2). We construct a boundary triplet for $S^*$, which feels the tensor structure (1.2), and compute the corresponding Weyl function [a special case of (1.7)]

An interesting feature of our approach is the following: It allows us to define models whose point contacts to the reservoir are physically grounded. To this end, one regards the boundary operator as the Hamiltonian of a quantum dot to which the compound system $\{\cal R, S_0\}$, cf. (1.1), is coupled by virtue of the $\Gamma$-maps of the boundary triplet $\Pi_S$. In the present paper the Jaynes–Cummings operator known from quantum optics, cf. [34],
\[ C_{JC} = B \otimes I_{\bar \tau} + I_{\cal H^A} \otimes T + \tau V_{JC}, \quad \tau \in \mathbb{R}, \]
is chosen to be the boundary operator.
In this connection let us mention the papers by B. Pavlov [49–51] treating solvable physical models in the framework of extension theory.

In a forthcoming paper we plan to express the scattering matrix for a naturally related scattering system by means of the Weyl function, using results from [9,10]. Explicit knowledge of the scattering matrix allows one to calculate the current going through the quantum dot using the so-called Landauer–Büttiker formula invented in [18,41], see also [5,20] for a mathematically rigorous proof of this formula. Using this approach our final goal is to compute explicitly the electron current going through the quantum dot as well as the photon current.

The paper is organized as follows: In Sect. 2 we give a short introduction into the boundary triplet approach. In particular, we consider the case of a direct sum of symmetric operators. In Sect. 3 we develop the functional calculus for operator-spectral integrals which are systematically used in the sequel. In Sect. 4 we consider boundary triplets for tensor products. First, we compute explicitly the Weyl function and \( \gamma \)-field for the triplet (1.5) with a bounded \( T \in B(\mathcal{H}) \) by using the functional calculus developed in Sect. 3. In Sect. 4.2 we construct an adequate boundary triplet for \( S^* \) assuming \( T \) to be unbounded and prove formulas (1.7) and (1.6). Section 5 is devoted to the case of nonnegative operators \( A \) and \( T \), a situation typical in physics. In particular, formulas (1.8) are proved here. In Sect. 6 we illustrate the abstract results of Sects. 4 and 5 by a typical physical example. We consider the Schrödinger operator on bounded and semi-bounded intervals as well as a boson reservoir. Finally, in Sect. 7 we use the previous example to propose a simple model describing a photon-assisted electronic transport through a quantum dot.

**Notation.** Let \( \mathcal{H}_1, \mathcal{H}_2 \) be separable Hilbert spaces. By \( \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) \) we denote the set of closed (bounded) linear operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \); \( \mathcal{C}(\mathcal{H}) := \mathcal{C}(\mathcal{H}, \mathcal{H}) \), \( \mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H}) \). By \( \mathcal{S}_p(\mathcal{H}), p \in (0, \infty] \), we denote the Schatten–Neumann ideals of order \( p \) on \( \mathcal{H} \); in particular, \( \mathcal{S}_\infty(\mathcal{H}) \) is the ideal of compact operators on \( \mathcal{H} \). By \( \text{dom}(T), \text{ran}(T), \rho(T) \) and \( \sigma(T) \) we denote the domain, range, resolvent set and spectrum of the operator \( T \), respectively. Denote by \( \hat{\rho}(S) \) the set of points of regular type of the symmetric operator \( S \) (see [2]).

**2. Preliminaries**

**2.1. Linear Relations**

A linear relation \( \Theta \) in \( \mathcal{H} \) is a closed linear subspace of \( \mathcal{H} \oplus \mathcal{H} \). The set of all linear relations in \( \mathcal{H} \) is denoted by \( \mathcal{C}(\mathcal{H}) \). Denote also by \( \mathcal{C}(\mathcal{H}) \) the set of all closed linear (not necessarily densely defined) operators in \( \mathcal{H} \). Identifying an operator \( T \in \mathcal{C}(\mathcal{H}) \) with its graph \( \text{gr}(T) \) we regard \( \mathcal{C}(\mathcal{H}) \) as a subset of \( \mathcal{C}(\mathcal{H}) \).

The role of the set \( \mathcal{C}(\mathcal{H}) \) in extension theory becomes clear from Proposition 2.3 below. However, its role in the operator theory is substantially motivated by the following circumstances: In contrast to \( \mathcal{C}(\mathcal{H}) \), the set \( \mathcal{C}(\mathcal{H}) \) is closed with respect to taking inverse and adjoint relations \( \Theta^{-1} \) and \( \Theta^* \). Here
\( \Theta^{-1} = \{ \{g, f\} : \{f, g\} \in \Theta \} \) and

\[
\Theta^* = \left\{ \begin{pmatrix} k \\ k' \end{pmatrix} : (h', k) = (h, k') \text{ for all } (h, k') \in \Theta \right\}.
\]

A linear relation \( \Theta \) is called symmetric if \( \Theta \subset \Theta^* \) and self-adjoint if \( \Theta = \Theta^* \).

### 2.2. Boundary Triplets and Proper Extensions

Following [26, 27, 29], we briefly recall some basic facts on boundary triplets. Let \( S \) be a closed densely defined symmetric operator in a separable Hilbert space \( \mathcal{H} \), and let \( n_\pm(S) := \dim(\mathcal{N}_\pm) \) be its deficiency indices, where \( \mathcal{N}_z := \ker(S^* - z) \), \( z \in \mathbb{C}_\pm \), is the defect subspace of \( S \).

**Definition 2.1.** A closed extension \( \hat{S} \) of \( S \) is called a proper extension if \( \text{dom}(S) \subsetneq \text{dom}(\hat{S}) \subsetneq \text{dom}(S^*) \).

We denote by \( \text{Ext}_{S} \) the set of all proper extensions of \( S \) completed by the non-proper extensions \( S \) and \( S^* \). Any dissipative (accumulative), in particular symmetric, extension of \( S \) is either proper or equal to \( S \) (see [27]).

**Definition 2.2** (cf. [29]). A triplet \( \Pi_S = \{H^S, \Gamma_0^S, \Gamma_1^S\} \), where \( H^S \) is an auxiliary Hilbert space and \( \Gamma_0^S, \Gamma_1^S : \text{dom}(S^*) \to H^S \) are linear mappings, is called a boundary triplet for \( S^* \) if the “abstract Green’s identity,”

\[(S^* f, g) - (f, S^* g) = (\Gamma_1^S f, \Gamma_1^S g) - (\Gamma_0^S f, \Gamma_0^S g), \quad f, g \in \text{dom}(S^*), \quad (2.1)\]

holds and the mapping \( \Gamma^S := (\Gamma_0^S, \Gamma_1^S)^t : \text{dom}(S^*) \to (H^S \oplus H^S)^t \) is surjective, i.e., \( \text{ran}(\Gamma^S) = (H^S \oplus H^S)^t \).

In the following it will appear different kinds of generalized “boundary triplets.” To distinguish between these kinds of triplets and that one of Definition 2.2, we will sometimes write “ordinary boundary triplet” instead of “boundary triplet.”

A boundary triplet \( \Pi_S \) for \( S^* \) always exists whenever \( n_+(S) = n_-(S) \). Note also that \( n_\pm(S) = \dim(H^S) \) and \( \ker(\Gamma_0^S) \cap \ker(\Gamma_1^S) = \text{dom}(S) \).

The linear maps \( \Gamma_j^S : \text{dom}(S^*) \to H^S \), \( j \in \{0, 1\} \), are neither bounded nor closable. However, equipping the domain \( \text{dom}(S^*) \) with the graph norm

\[
\|f\|_{S^*}^2 := \|S^* f\|^2 + \|f\|^2, \quad f \in \text{dom}(S^*),
\]

one obtains a Hilbert space \( \mathcal{H}_+ \).

It turns out that the mappings \( \Gamma_j^S : \text{dom}(S^*) \to H^S \), \( j \in \{0, 1\} \), regarded as mappings from \( \mathcal{H}_+ \) into \( H^S \) are bounded.

In what follows, we denote by \( \widehat{\Gamma}_j^S \) the operator \( \Gamma_j^S \) treated as the mapping \( \widehat{\Gamma}_j^S : \mathcal{H}_+ \to H^S \), \( j \in \{0, 1\} \). If \( J_{S^*} : \mathcal{H}_+ \to \text{dom}(S^*) \) denotes the embedding operator, then \( \widehat{\Gamma}_j^S = \Gamma_j^S J_{S^*} \), \( j \in \{0, 1\} \). It follows from Definition 2.2 that \( \text{ran}(\widehat{\Gamma}_j^S) = H^S \oplus H^S \), where \( \widehat{\Gamma}_j^S := (\widehat{\Gamma}_0^S, \widehat{\Gamma}_1^S)^t \). Notice that the abstract Green’s identity (2.1) can be rewritten as

\[
(S^* J_{S^*} f, J_{S^*} g) - (J_{S^*} f, S^* J_{S^*} g) = (\widehat{\Gamma}_1^S f, \widehat{\Gamma}_0^S g) - (\widehat{\Gamma}_0^S f, \widehat{\Gamma}_1^S g),
\]
Proposition 2.3. (cf. [26,42]) Let \( \Pi^S = \{ \mathcal{H}^S, \Gamma_0^S, \Gamma_1^S \} \) be a boundary triplet for \( S^* \). Then the mapping

\[
\text{Ext}_S \ni \tilde{S} \rightarrow \Gamma^S \text{dom}(\tilde{S}) = \{(\Gamma_0^S f, \Gamma_1^S f)^\dagger : f \in \text{dom}(\tilde{S})\} =: \Theta \in \tilde{C}(\mathcal{H}^S) \quad (2.2)
\]

establishes a bijective correspondence between the sets \( \text{Ext}_S \) and \( \tilde{C}(\mathcal{H}^S) \). We write \( \tilde{S} = S_\Theta \) if \( \tilde{S} \) corresponds to \( \Theta \) by (2.2). Moreover, the following holds:

(i) \( S_\Theta^* = S_\Theta \), in particular, \( S_\Theta^* = S_\Theta \) if and only if \( \Theta^* = \Theta \);

(ii) \( S_\Theta \) is symmetric if and only if so is \( \Theta \). Moreover, \( n_+(S_\Theta) = n_+(\Theta) \).

(iii) \( S_\Theta \) is disjoint with \( S_0 \), i.e., \( \text{dom}(S_\Theta) \cap \text{dom}(S_0) = \text{dom}(S) \), if and only if \( \Theta \) is a graph of an operator, \( \Theta = \text{gr}B \). In this case extension \( S_\Theta \) is denoted by \( S_B \) and relation (2.2) determining \( S_\Theta \) is simplified to

\[
S_\Theta := S_B := S^* \upharpoonright \text{dom}(S_B),
\]

\[
\text{dom}(S_B) := \{ f \in \text{dom}(S^*) : \Gamma_1^S f = B \Gamma_0^S f \}. \quad (2.3)
\]

The operator \( B \) in (2.3) is called the boundary operator of the extension \( S_\Theta \) in the triplet \( \Pi^S \).

In particular, \( S_j := S^* \upharpoonright \ker(\Gamma_j^S) = S_{\Theta_j}, \ j \in \{0,1\} \), where \( \Theta_0 := \begin{pmatrix} 0 \\ \mathcal{H}^S \end{pmatrix} \) and \( \Theta_1 := \begin{pmatrix} \mathcal{H}^S \\ \{0\} \end{pmatrix} = \text{gr}(\mathcal{O}) \) where \( \mathcal{O} \) denotes the zero operator in \( \mathcal{H}^S \). Note also that the trivial linear relations \( \{0\} \times \{0\} \) and \( \mathcal{H}^S \times \mathcal{H}^S \in \tilde{C}(\mathcal{H}^S) \) parameterize the extensions \( S \) and \( S^* \), respectively, in any triplet \( \Pi^S \).

2.3. \( \gamma \)-Field and Weyl Function

It is well known that the Weyl function is an important tool in the direct and inverse spectral theory of Sturm–Liouville operators. In [26,27] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator \( S \) with \( n_+(S) = n_-(S) \leq \infty \). Following [26], [27, Chapter 7] we briefly recall basic facts on Weyl functions and \( \gamma \)-fields, associated with a boundary triplet \( \Pi \). For further properties and applications see [17,25–27,53] (and references therein).

Definition 2.4 (cf. [26]). Let \( \Pi^S = \{ \mathcal{H}^S, \Gamma_0^S, \Gamma_1^S \} \) be a boundary triplet for \( S^* \) and \( S_0 = S^* \upharpoonright \ker(\Gamma_0^S) \). The operator-valued functions \( \gamma^S(\cdot) : \rho(S_0) \rightarrow \mathcal{B}(\mathcal{H}^S, \mathcal{F}) \) and \( M^S(\cdot) : \rho(S_0) \rightarrow \mathcal{B}(\mathcal{H}^S) \) defined by

\[
\gamma^S(z) := (\Gamma_0^S \upharpoonright \mathcal{F}_z)^{-1} \quad \text{and} \quad M^S(z) := \Gamma_1^S \gamma^S(z), \quad z \in \rho(S_0), \quad (2.4)
\]

are called the \( \gamma \)-field and the Weyl function, respectively, corresponding to the boundary triplet \( \Pi^S \).
Note that with certain positive constants $C_1, C_2 > 0$ the following estimate holds

$$
\frac{C_1}{1 + |\lambda|^2} \leq \frac{1 + t^2}{(t - \lambda)^2 + 1} \leq C_2(1 + |\lambda|^2), \quad \lambda \in \mathbb{R}.
$$

Combining these estimates with the identity $\text{Im}M(i) = \int_\mathbb{R}(1 + t^2)^{-1}d\Sigma_S(t)$ one derives from (2.8) that

$$
C_1(1 + |\lambda|^2)^{-1}\text{Im}M(i) \leq \text{Im}M^S(i - \lambda) \leq C_2(1 + |\lambda|^2)\text{Im}M(i), \quad \lambda \in \mathbb{R}. \quad (2.9)
$$
Emphasize that since the proof of estimates (2.9) is based only on the integral representation (2.7), these estimates are valid for any $R[\mathcal{H}]$-function not necessarily being a Weyl function.

### 2.4. Krein-Type Formula for Resolvents

Let $\Pi_S = \{\mathcal{H}_S, \Gamma_0^S, \Gamma_1^S\}$ be a boundary triplet for $S^*$, and $M^S(\cdot)$ and $\gamma^S(\cdot)$ the corresponding Weyl function and $\gamma$-field, respectively. For any proper (not necessarily self-adjoint) extension $S_\Theta \in \text{Ext}_S$ with non-empty resolvent set $\rho(S_\Theta)$ the following Krein-type formula holds (cf. [25,26])

$$
(S_\Theta - z)^{-1} - (S_0 - z)^{-1} = \gamma^S(z)(\Theta - M^S(z))^{-1}(\gamma^S(z))^*,
$$

(2.10)

$z \in \rho(S_0) \cap \rho(S_\Theta)$. Formula (2.10) extends the known Krein formula for canonical resolvents to the case of any $S_\Theta \in \text{Ext}_S$ with $\rho(S_\Theta) \neq \emptyset$. Moreover, due to relations (2.2) and (2.4) all objects in formula (2.10) are expressed by means of the boundary triplet $\Pi_S$. We emphasize that this connection makes it possible to apply the Krein-type formula (2.10) to boundary value problems.

### 2.5. Normalized Boundary Triplets

Let $S_n$ be a densely defined closed symmetric operator in $\mathcal{H}_n$, $n \in \mathbb{Z}$, and let $S := \bigoplus_{n \in \mathbb{Z}} S_n$. Clearly,

$$
S^* = \bigoplus_{n \in \mathbb{Z}} S_n^*,
$$

$$
dom(S^*) = \left\{ f = \bigoplus_{n \in \mathbb{Z}} f_n \in \mathcal{H} : f_n \in dom(S_n^*), \sum_{n=-\infty}^{\infty} \|S_n^* f_n\|^2 < \infty \right\}.
$$

Let $\Pi_{S_n} = \{\mathcal{H}_{S_n}, \Gamma_0^{S_n}, \Gamma_1^{S_n}\}$ be a boundary triplet for $S_n^*$, $n \in \mathbb{Z}$. Define mappings $\Gamma_j^S$, $j \in \{0, 1\}$, by setting

$$
\Gamma_j^S := \bigoplus_{n \in \mathbb{Z}} \Gamma_j^{S_n},
$$

$$
dom(\Gamma_j^S) := \left\{ \bigoplus_{n \in \mathbb{Z}} f_n \in \text{dom}(S^*) : \sum_{n \in \mathbb{Z}} \|\Gamma_j^{S_n} f_n\|^2 < \infty \right\}.
$$

(2.11)

**Definition 2.5.** Let $\Gamma_j^S$ be given by (2.11) and $\mathcal{H}^S := \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^{S_n}$. A collection $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ is called a direct sum of boundary triplets and is assigned as $\Pi_S = \bigoplus_{n \in \mathbb{Z}} \Pi_{S_n}$.

It was first discovered by Kochubei [36] that the direct sum $\bigoplus \Pi_n$ of boundary triplets $\Pi_n$ is not a boundary triplet in general. Later on, simple counterexamples were constructed in [19,38,45]. Moreover, it was shown in [38, Theorem 3.2] that $\Pi_S$ is only a generalized “boundary triplet” (a unitary boundary relation in the sense of [24]). Moreover, according to [24] $\Pi_S$ is also called an ES-generalized boundary triplet for $S^*$, since the operator $S_0 := S^* \upharpoonright \ker(\Gamma_0^S)$ is essentially self-adjoint.

The reason is that the domain $\text{dom}(\Gamma_j^S)$, $j \in \{0, 1\}$, might be narrower than $\text{dom}(S^*)$ and the range of the mapping $\Gamma^S := (\Gamma_0^S, \Gamma_1^S)^t : \text{dom}(S^*) \to \mathcal{H}^S$. 

might be a proper subset of $(\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Nevertheless, dom $(\Gamma^S_j)$, 
$j \in \{0, 1\}$, is always dense in $\mathfrak{S}_+(S^*)$ and its range ran $(\Gamma^S)$ is dense in $(\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Moreover, by \cite{24}, $\Pi_S$ is a boundary triplet whenever ran $(\Gamma^S) = (\mathcal{H}^S \oplus \mathcal{H}^S)^t$. Besides, in accordance with \cite{38, Proposition 3.8} the conditions

$$\sum_{n \in \mathbb{Z}} \|f_n^S\|^2 < \infty, \quad f = \bigoplus_{n \in \mathbb{Z}} f_n \in \text{dom } (S^*), \quad j \in \{0, 1\}, \quad (2.12)$$

imply that $\Pi_S = \bigoplus_{n \in \mathbb{Z}} \Pi_{S_n}$ is an ordinary boundary triplet, i.e., a boundary triplet in the sense of Definition 2.2, while the sole first condition in (2.12) (with $j = 0$) ensures only that $\Pi_S$ is a $B$-generalized boundary triplet in the sense of \cite{24, 25}.

The regularization procedure described below was first proposed in \cite{45} and was applied to construct a boundary triplet for the Sturm–Liouville operator

$$-d^2/dx^2 \otimes I_\Sigma + I_\mathfrak{S} \otimes T, \quad \mathfrak{S} = L^2(\mathbb{R}_+; \Sigma) = L^2(\mathbb{R}_+) \otimes \Sigma,$$

with unbounded potential $T = T^* \in \mathcal{C}(\Sigma)$. Further generalizations of regularization procedures as well as applications to Schrödinger and Dirac operators with δ-interactions were obtained in \cite{19, 38}, respectively.

Let $\Pi_S = \{\mathcal{H}^S, \Gamma^S_0, \Gamma^S_1\}$ be a boundary triplet for $S^*$ with Weyl function $M^S(\cdot)$. We call $\Pi_S$ a normalized boundary triplet for $S^*$ if the condition $M^S(i) = i\mathcal{H}^S$ is satisfied.

**Lemma 2.6** (\cite{45}). Let $\Pi_S = \{\mathcal{H}^S, \Gamma^S_0, \Gamma^S_1\}$ be a boundary triplet for $S^*$, and let $\gamma^S(\cdot)$ and $M^S(\cdot)$ be the $\gamma(\cdot)$-field and Weyl function, respectively. Let $R_S := \sqrt{\text{Im}(M^S(i))}$ and $Q_S := \text{Re}(M^S(i))$. Then $\tilde{\Pi}_S = \{\tilde{\mathcal{H}}^S, \tilde{\Gamma}^S_0, \tilde{\Gamma}^S_1\}$, where

$$\tilde{\mathcal{H}}^S := \mathcal{H}^S, \quad \tilde{\Gamma}^S_0 := R_S \Gamma^S_0 \quad \text{and} \quad \tilde{\Gamma}^S_1 := R^{-1}_S (\Gamma^S_1 - Q_S \Gamma^S_0), \quad (2.13)$$

is a normalized boundary triplet for $S^*$ such that

$$S_0 := S^* \upharpoonright \text{ker } (\Gamma^S_0) = S^* \upharpoonright \text{ker } (\tilde{\Gamma}^S_0).$$

The $\gamma$-field $\tilde{\gamma}^S(\cdot)$ and Weyl function $\tilde{M}^S(\cdot)$ corresponding to the triplet $\tilde{\Pi}_S$ are given by

$$\tilde{\gamma}^S(z) = \gamma^S(z) R^{-1}_S \quad \text{and} \quad \tilde{M}^S(z) = R^{-1}_S (M^S(z) - Q_S) R^{-1}_S, \quad z \in \mathbb{C}_\pm.$$

Lemma 2.6 shows that with any boundary triplet one can associate a normalized boundary triplet such that $S_0$ remains unchanged. The following theorem presents a regularization procedure for direct sums $\Pi_S = \bigoplus_{n \in \mathbb{Z}} \Pi_{S_n}$ to define an ordinary boundary triplet.

**Theorem 2.7** (Theorem 3.3, \cite{45}). Let $S_n$ be a densely defined closed symmetric operator in $\mathfrak{S}_n$, $n \in \mathbb{Z}$, and $S := \bigoplus_{n \in \mathbb{Z}} S_n$. Let $\Pi_{S_n} = \{\mathcal{H}^{S_n}, \Gamma^{S_n}_0, \Gamma^{S_n}_1\}$ be a boundary triplet for $S^*_n$, $S_{0n} := S^*_n \upharpoonright \text{ker } (\Gamma^{S_n}_0)$, $n \in \mathbb{Z}$, and let $\gamma^{S_n}(\cdot)$ and $M^{S_n}(\cdot)$ be the corresponding $\gamma$-field and Weyl function, respectively. Finally,
let $R_{S_n} := \sqrt{\text{Im}(M^{S_n}(i))}$ and $Q_{S_n} := \text{Re}(M^{S_n}(i))$, $n \in \mathbb{Z}$. Then the triplet
\[ \tilde{\Pi}_S = \{ \tilde{\mathcal{H}}^S, \tilde{\Gamma}_0^S, \tilde{\Gamma}_1^S \} \]
with
\[ \tilde{\mathcal{H}}^S := \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^{S_n}, \quad \tilde{\Gamma}_0^S := \bigoplus_{n \in \mathbb{Z}} R_{S_n} \Gamma_0^{S_n}, \]
\[ \tilde{\Gamma}_1^S := \bigoplus_{n \in \mathbb{Z}} R_{S_n}^{-1} \left( \Gamma_1^{S_n} - Q_{S_n} \Gamma_0^{S_n} \right) \]  
(2.14)
is a (normalized) boundary triplet for $S^*$ satisfying
\[ \tilde{S}_0 = S^* \upharpoonright \ker(\tilde{\Gamma}_0^S) = \bigoplus_{n \in \mathbb{Z}} \tilde{S}_{0n} = \bigoplus_{n \in \mathbb{Z}} S_{0n}, \quad \tilde{S}_{0n} = S_{0n}^* \upharpoonright \ker(\tilde{\Gamma}_0^{S_n}). \]
Moreover, the $\gamma$-field $\tilde{\gamma}^S(\cdot)$ and Weyl function $\tilde{M}^S(\cdot)$ corresponding to $\tilde{\Pi}_S$ are given by
\[ \tilde{\gamma}^S(z) = \bigoplus_{n \in \mathbb{Z}} \gamma^{S_n}(z) R_{S_n}^{-1} \quad \text{and} \]
\[ \tilde{M}^S(z) = \bigoplus_{n \in \mathbb{Z}} R_{S_n}^{-1} (M^{S_n}(z) - Q_{S_n}) R_{S_n}^{-1}, \quad z \in \mathbb{C}_\pm. \]  
(2.15)

Next we assume that the operator $S = \bigoplus_{n = -\infty}^{\infty} S_n$ has a real regular-type point $a = \bar{a} \in \hat{\rho}(S)$. The latter means that there exists $\varepsilon > 0$ such that
\[ (a - \varepsilon, a + \varepsilon) \subset \bigcap_{n = -\infty}^{\infty} \hat{\rho}(S_n). \]  
(2.16)
We emphasize that the condition $a \in \bigcap_{n = -\infty}^{\infty} \hat{\rho}(S_n)$ is not sufficient for the inclusion $a \in \hat{\rho}(S)$.

It is known (see, e.g., [2,40]) that under condition (2.16) for every $k \in \mathbb{Z}$ there exists a self-adjoint extension $\tilde{S}_k = \tilde{S}_k^*$ of $S_k$ preserving the gap $(a - \varepsilon, a + \varepsilon)$. The latter amounts to saying that the Weyl function of the pair $\{ S_k, \tilde{S}_k \}$ is regular within the gap $(a - \varepsilon, a + \varepsilon)$.

For operators $S = \bigoplus_{n = -\infty}^{\infty} S_n$ satisfying (2.16) we complement Theorem 2.7 by presenting a regularization procedure for $\Pi = \bigoplus_{n = -\infty}^{\infty} \Pi_n$ leading to a boundary triplet (cf. [38, Theorem 3.13], [19, Theorem 2.12 and Corollary 2.13]). In applications to symmetric operators with a gap this regularization is more appropriate and simpler than the one described in Theorem 2.7.

**Proposition 2.8 ([19,38]).** Let $\{S_n\}_{n = -\infty}^{\infty}$ be a sequence of symmetric operators satisfying (2.16). Let also $\Pi_n = \{ \mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)} \}$ be a boundary triplet for $S_n^*$ such that $(a - \varepsilon, a + \varepsilon) \subset \rho(S_{0n})$, $S_{0n} = S_{0n}^* \upharpoonright \ker(\Gamma_0^{(n)})$. Let also $\gamma^{S_n}(\cdot)$ and $M_n(\cdot) := M^{S_n}(\cdot)$ be the corresponding $\gamma$-field and Weyl function, respectively. Assume also that for some operators $R_n$ such that $R_n, R_n^{-1} \in \mathcal{B}(\mathcal{H}_n)$, the following conditions are satisfied
\[ \sup_n \| R_n^{-1}(M_n'(a))(R_n^{-1})^* \|_{\mathcal{H}_n} < \infty \quad \text{and} \]
\[ \sup_n \| R_n^*(M_n'(a))^{-1} R_n \|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{Z}. \]
Then the direct sum \( \tilde{\Pi}_s = \bigoplus_{n=-\infty}^{\infty} \tilde{\Pi}_n \) of boundary triplets where
\[
\tilde{\Pi}_n = \{ \mathcal{H}_n, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)} \},
\]
\[
\tilde{\Gamma}_0^{(n)} := R_n \Gamma_0^{(n)} \quad \text{and} \quad \tilde{\Gamma}_1^{(n)} := (R_n^{-1})^\ast (\Gamma_1^{(n)} - M_n(a) \Gamma_0^{(n)}),
\]
forms a boundary triplet for \( S^* = \bigoplus_{n=-\infty}^{\infty} S_n^* \).

Moreover, the corresponding \( \gamma \)-field \( \tilde{\gamma}^S(\cdot) \) and Weyl function \( \tilde{M}^S(\cdot) \) are given by
\[
\tilde{\gamma}^S(z) = \bigoplus_{n \in \mathbb{Z}} \gamma^S_n(z) R_n^{-1} \quad \text{and} \quad \tilde{M}^S(z) = \bigoplus_{n \in \mathbb{Z}} (R_n^{-1})^\ast (M_n(z) - M_n(a)) R_n^{-1}, \quad z \in \mathbb{C}_\pm.
\]

In particular, one can set \( R_n = \sqrt{M_n'(a)} \), \( n \in \mathbb{Z} \).

We emphasize that \( M_n'(a) \) is a positive definite operator whenever \( a \in \rho(S_{0n}) \).

### 3. Operator-Spectral Integrals

Let \( F(\cdot) \) be an orthogonal operator measure with compact support \( \text{supp}(F) \subseteq \Delta := [a,b] \), \( -\infty < a < b < \infty \), and with values in \( \mathcal{B}(\mathcal{H}) \). Further, let \( \Omega(\cdot) : [a,b] \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}_1) \) be an operator-valued function. We consider partitions \( \mathcal{Z} \) of \( [a,b] \) of the form \( [a,b] = [\lambda_0, \lambda_1) \cup [\lambda_1, \lambda_2) \cup \ldots \cup [\lambda_{n-1}, \lambda_n) \), \( \lambda_0 = a \), \( \lambda_n = b \) and set \( \Delta_m := [\lambda_{m-1}, \lambda_m), \ m = 1, 2, \ldots, n \). Thus, \( [a,b] = \bigcup_{m=1}^{n} \Delta_m \) and the \( \Delta_m \) are pairwise disjoint. Let \( |\mathcal{Z}| := \max_{m=1,2,\ldots,n} |\Delta_m| \), where \( |\Delta_m| := \lambda_m - \lambda_{m-1} \). Let \( \{x_m\}_{m=1}^{n} \) be an arbitrary but fixed family satisfying \( x_m \in \Delta_m \) for \( m = 1, 2, \ldots, n \). We define the operator \( \Sigma_{\mathcal{Z}} \Omega \) by
\[
\Sigma_{\mathcal{Z}} \Omega = \sum_{m=1}^{n} \Omega(x_m) F(\Delta_m).
\]

The sum \( \Sigma_{\mathcal{Z}} \Omega \) is called the Riemann–Stieltjes sum of \( \Omega(\cdot) \) with respect to the operator measure \( F(\cdot) \). If there is an operator \( \Sigma_0 \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1) \) such that \( \lim_{|\mathcal{Z}| \to 0} \| \Sigma_{\mathcal{Z}} \Omega - \Sigma_0 \| = 0 \) independent of the special choice of \( \mathcal{Z} \) and \( \{x_m\}_{m=1}^{n} \), then \( \Sigma_0 \) is called the operator-spectral integral of \( \Omega(\cdot) \) with respect to \( F(\cdot) \) and is denoted by
\[
\Sigma_0 := \int_{\Delta} \Omega(\lambda) F(d\lambda).
\]

Obviously, in a similar way one can define for operator-valued functions \( \Omega : \Delta \rightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{H}) \) the operator-spectral integral \( \int_{\Delta} F(d\lambda) \Omega(\lambda) \) as the limit of the Riemann–Stieltjes sums \( \sum_{m} F(\Delta_m) \Omega(x_m) \). It is clear that the operator-spectral integral is linear with respect to \( \Omega(\cdot) \). If \( B \) is a bounded operator, then
\[
B \int_{\Delta} \Omega(\lambda) F(d\lambda) = \int_{\Delta} B \Omega(\lambda) F(d\lambda).
\]
Definition 3.1. The operator-valued mapping \( \Omega : (a, b) \rightarrow \mathcal{B}(H) \) will be called \( F \)-admissible, if the integral \( \int_{\Delta} \Omega(\lambda)F(d\lambda) \) exists and
\[
\Omega(\lambda)F(\delta) = F(\delta)\Omega(\lambda)F(\delta), \quad \delta \in \mathcal{B}([a, b]), \quad \lambda \in \Delta,
\]
where \( \mathcal{B}([a, b]) \) denotes the Borel sets of \([a, b]\).

Proposition 3.2. Let \( \Omega : (a, b) \rightarrow \mathcal{B}(H) \) be \( F \)-admissible, \( \Omega_1 : (a, b) \rightarrow \mathcal{B}(\mathcal{H}, \mathcal{H}_1) \), and assume that \( \int_{\Delta} \Omega_1(\lambda)F(d\lambda) \) exists. Then \( \int_{\Delta} \Omega_1(\lambda)\Omega(\lambda)F(d\lambda) \) exists and
\[
\int_{\Delta} \Omega_1(\lambda)\Omega(\lambda)F(d\lambda) = \int_{\Delta} \Omega_1(\lambda)F(d\lambda) \int_{\Delta} \Omega(\mu)F(d\mu).
\]

Proof. It is easily seen that
\[
\Sigma_{Z} \Omega_1 \Sigma_{Z} \Omega \longrightarrow \int_{\Delta} \Omega_1(\lambda)F(d\lambda) \int_{\Delta} \Omega(\mu)F(d\mu) \quad \text{as} \quad |Z| \longrightarrow 0.
\]
On the other hand, since the measure \( F(\cdot) \) is orthogonal, \( F(\Delta_j)F(\Delta_k) = F(\Delta_j)\delta_{jk}, \quad j, k \in \{1, \ldots, m\} \). Combining these relations with the \( F \)-admissibility of \( \Omega \) yields
\[
\Sigma_{Z} \Omega_1 \Sigma_{Z} \Omega = \sum_{m, m' = 1}^{n} \Omega_1(x_m)F(\Delta_m)\Omega(x_{m'})F(\Delta_{m'})
\]
\[
= \sum_{m, m' = 1}^{n} \Omega_1(x_m)F(\Delta_m)F(\Delta_{m'})\Omega(x_{m'})F(\Delta_{m'})
\]
\[
= \sum_{m = 1}^{n} \Omega_1(x_m)F(\Delta_m)\Omega(x_m)F(\Delta_m)
\]
\[
= \sum_{m = 1}^{n} \Omega_1(x_m)\Omega(x_m)F(\Delta_m) \longrightarrow \int_{\Delta} \Omega_1(\lambda)\Omega(\lambda)F(d\lambda)
\]
as \( |Z| \longrightarrow 0. \) Combining both relations one completes the proof. \( \Box \)

In what follows, we assume that \( \mathcal{H} = \mathcal{H}_1 \).

Proposition 3.3. Let \( X : (a, b) \rightarrow \mathcal{B}(H) \) be an \( F \)-admissible function, and assume, in addition, that there exist real numbers \( c_1, c_2 \), such that \( X(\lambda) \) is self-adjoint and \( c_1 \leq X(\lambda) \leq c_2, \quad \lambda \in \Delta \). Let \( \varphi \in C[c_1, c_2] \). Then the following holds:
\[
\begin{align*}
\text{(i) } & \text{The operator } \hat{X} := \int_{\Delta} X(\lambda)F(d\lambda) \text{ is self-adjoint and satisfies } c_1 \leq \hat{X} \leq c_2. \\
\text{(ii) } & \text{The estimate } \| \varphi(\hat{X}) \| \leq \| \varphi \|_{\infty} \text{ holds.} \\
\text{(iii) } & \text{The operator-valued function } \varphi(X(\cdot)) \text{ is } F \text{-admissible and} \\
& \int_{\Delta} \varphi(X(\lambda))F(d\lambda) = \varphi(\hat{X}).
\end{align*}
\]
Proof. (i) Let $Z$ be any partition as above. Then for any $h \in \mathcal{H}$ one gets

$$\langle \Sigma_Z X h, h \rangle = \sum_{m=1}^{n} \langle F(\Delta_m) X(x_m) F(\Delta_m) h, h \rangle \geq \sum_{m=1}^{n} c_1 \| F(\Delta_m) h \|^2.$$  

Thus, $\langle \Sigma_Z h, h \rangle \in \mathbb{R}$ and $\langle \Sigma_Z h, h \rangle \geq c_1 \| h \|^2$. In the same way, one shows that $\langle \Sigma_Z h, h \rangle \leq c_2 \| h \|^2$. By passing to the limit, as $|Z| \rightarrow 0$, we get that $\langle \tilde{X} h, h \rangle \in [c_1 \| h \|^2, c_2 \| h \|^2]$ for every $h \in \mathcal{H}$, and (i) is proved.

(ii) By the functional calculus, both inequalities $\| \varphi(\hat{X}) \| \leq \| \varphi \|_\infty$ and $\| \varphi(X(\lambda)) \| \leq \| \varphi \|_\infty$ hold for every $\lambda \in \Delta$ and each continuous function $\varphi \in C[c_1,c_2]$.

(iii) First we prove, by induction, the assertion (iii) in the special case, when $\varphi(\lambda) = \lambda^n$. By the assumption, the assertion is true for $n = 1$. Suppose that it is true for $n = k$. Let us prove it for $n = k + 1$. One has

$$X^{k+1}(\lambda) F(\delta) = X(\lambda) F(\delta) X^k(\lambda) F(\delta)$$

$$= F(\delta) X(\lambda) F(\delta) \cdot X^k(\lambda) F(\delta)$$

$$= F(\delta) X^{k+1}(\lambda) F(\delta),$$  

(3.4)  

$\lambda \in \Delta, \delta \in \mathcal{B}(\Delta)$. Therefore, Proposition 3.2 ensures that the integral $\int_\Delta X^{k+1}(\lambda) F(d\lambda)$ exists and

$$\int_\Delta X^{k+1}(\lambda) F(d\lambda) = \hat{X}^{k+1}.$$  

By linearity, these equalities are easily extended for polynomials in $\lambda$.

Let $\varphi$ be a continuous function, $\varphi \in C[c_1,c_2]$. By the Weierstrass theorem, there exists a sequence $\{p_k\}_{k=1}^\infty$ of polynomials approaching $\varphi$ in $C[c_1,c_2]$. In accordance with the functional calculus for self-adjoint operators, $\| \varphi(\hat{X}) - p_k(\hat{X}) \| \leq \| \varphi - p_k \|_\infty \rightarrow 0$ as $k \rightarrow \infty$

and

$$\| \varphi(X(\lambda)) - p_k(X(\lambda)) \| \leq \| \varphi - p_k \|_\infty \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty, \quad \lambda \in \Delta.$$  

Combining this relation with equalities (3.4) for polynomials we obtain that for every $\lambda \in \Delta$ and any Borel subset $\delta \subset \Delta$ the following holds

$$\varphi(X(\lambda)) F(\delta) = \lim_{k \rightarrow \infty} p_k(X(\lambda)) F(\delta)$$

$$= \lim_{k \rightarrow \infty} F(\delta) p_k(X(\lambda)) F(\delta) = F(\delta) \varphi(X(\lambda)) F(\delta), \quad \lambda \in \Delta.$$  

This relation means that the function $\varphi(X(\cdot))$ satisfies commutation relation (3.2). To prove its $F$-admissibility it remains to prove the existence of the integral $\int_\Delta \varphi(X(\lambda)) F(d\lambda)$. We prove it together with relation (3.3). To this end, for each partition $Z$ of $[a,b]$ we prove the following estimate

$$\| \Sigma_Z (\varphi(X) - p_k(X)) \| \leq \| \varphi - p_k \|_\infty.$$
Since the measure $F$ is orthogonal, one gets
\[
\| \Sigma \varphi(X)f - \Sigma p_k(X)f \|^2 = \left\| \sum_{m=1}^{n} (\varphi - p_k)(X)(x_m)F(\Delta_m)f \right\|^2
\]
\[
= \sum_{m=1}^{n} \| (\varphi - p_k)(X)(x_m)F(\Delta_m)f \|^2
\]
\[
\leq \sum_{m=1}^{n} \| \varphi - p_k \|^2_{\infty} \| F(\Delta_m)f \|^2 = \| \varphi - p_k \|^2_{\infty} \| f \|^2 .
\]

Let \{Z_j\} be a sequence of partitions satisfying \(|Z_j| \to 0\), and let \(\varepsilon > 0\). Choose \(k\) such that \(\| \varphi - p_k \|_{\infty} < \varepsilon\) and \(j_0\) such that
\[
\| \Sigma Z_j p_k(X) - \Sigma Z_j p_k(X) \| < \varepsilon, \quad j, j' \geq j_0,
\]
and hence \(\| \Sigma Z_j \varphi(X) - \Sigma Z_{j'} \varphi(X) \| < 3\varepsilon\) for all \(j, j' \geq j_0\). Thus, the limit \(\lim_{j \to \infty} \Sigma Z_j \varphi(X)\) exists, and
\[
\| \lim_{j \to \infty} \Sigma Z_j \varphi(X) - \varphi(\hat{X}) \|
\]
\[
\leq \| \lim_{j \to \infty} \Sigma Z_j (\varphi(X) - p_k(X)) \| + \| p_k(\hat{X}) - \varphi(\hat{X}) \| < 2\varepsilon . \tag{3.5}
\]
Since \(\varepsilon > 0\) is arbitrary, this inequality ensures the existence of the integral \(\int_{\Delta} \varphi(X(\lambda))F(\mathrm{d}\lambda)\) and thus proves \(F\)-admissibility of \(\varphi(X(\cdot))\). Moreover, estimate (3.5) proves equality (3.3). \(\square\)

Denote by \(dm\) the Lebesgue measure on \(\mathbb{R}\).

**Corollary 3.4.** Assume that \(\Omega(\cdot) = \Omega(\cdot)^{\ast}\) is a self-adjoint \(\mathcal{B}(\mathcal{H})\)-valued Lipschitz function in \(\Delta = [a, b]\) and \(c_1 \leq \Omega(\cdot) \leq c_2\). Assume also that \(F(\cdot)\) is a spectral measure in \(\mathcal{H}\) with compact support, \(\text{supp}(F) \subseteq \Delta := [a, b]\), and \(\varphi \in C[c_1, c_2]\). If, in addition, commutation relation (3.2) holds, then the operator-valued function \(\varphi(\Omega(\cdot))\) is \(F\)-admissible and
\[
\int_{\Delta} \varphi(\Omega(\lambda))F(\mathrm{d}\lambda) = \varphi(\hat{X}). \tag{3.6}
\]

**Proof.** It is shown in [3, Lemma 7.2] that the integral (3.1) exists whenever \(\Omega(\cdot)\) is Lipschitz function. By Proposition 3.3(iii) \(\varphi(\Omega(\cdot))\) is \(F\)-admissible and equality (3.6) holds. \(\square\)

**Corollary 3.5.** Let \(\Omega(\cdot) = \Omega(\cdot)^{\ast}\) be differentiable with respect to the operator norm \(m\)-almost everywhere in \(\Delta = [a, b]\), \(c_1 \leq \Omega(\cdot) \leq c_2\), and let \(\Omega(\cdot)\) be expressed by means of its derivative \(\Omega'(\cdot)\) via the Bochner integral on \([a, b]\), i.e.,
\[
\Omega(\lambda) = \Omega(a) + \int_{a}^{\lambda} \Omega'(x)dx, \quad \lambda \in [a, b]. \tag{3.7}
\]
Assume also that \(F(\cdot)\) is a spectral measure in \(\mathcal{H}\) with compact support, \(\text{supp}(F) \subseteq \Delta := [a, b]\), and \(\varphi \in C[c_1, c_2]\). Assume also that commutation
relation (3.2) holds. Then the operator-valued function \( \varphi(\Omega(\cdot)) \) is \( F \)-admissible and
\[
\int_{\Delta} \varphi(\Omega(\lambda)) F(d\lambda) = \varphi(\hat{X}).
\] (3.8)

**Proof.** It is known (see [6, Proposition 5.1.4]) that the integral (3.1) exists whenever \( \Omega(\cdot) \) admits representation (3.7). By Proposition 3.3(iii) \( \varphi(X(\cdot)) \) is \( F \)-admissible and equality (3.8) holds. \( \square \)

**Remark 3.6.** Absolute continuity of \( \Omega(\cdot) \) (and even its Lipschitz property) does not ensure representation (3.7) (see [54, Chapter 5]). Thus, the conditions in both corollaries are different.

If \( F(\cdot) \) is a spectral measure on \( \mathbb{R} \) with non-compact support, then we define improper operator-spectral integrals by
\[
\int_{\mathbb{R}} \Omega(\lambda) F(d\lambda) := \lim_{b \to +\infty, a \to -\infty} \int_{[a,b)} \Omega(\lambda) F(d\lambda).
\]

Obviously, the improper operator-spectral integral \( \int_{\mathbb{R}} \Omega(\lambda) F(d\lambda) \) exists if and only if the following conditions
\[
s-\lim_{b \to \infty} \int_{b}^{b+\varepsilon} \Omega(\lambda) F(d\lambda) = 0 \quad \text{and} \quad s-\lim_{a \to -\infty} \int_{a-\varepsilon}^{a} \Omega(\lambda) F(d\lambda) = 0
\]
are satisfied for any \( \varepsilon > 0 \). Similar results hold true for \( \int_{\mathbb{R}} F(d\lambda) \Omega(\lambda) \).

**Proposition 3.7.** Let \( \Omega : \mathbb{R} \to \mathcal{B}(\mathcal{H}) \). Assume that \( \Omega \mid \Delta \) is \( F \)-admissible for every compact interval \( \Delta \) and
\[
\| \Omega(\lambda) \| \leq C_0 (1 + |\lambda|)^\alpha, \quad \lambda \in \mathbb{R},
\]
for some constants \( \alpha \geq 0, C_0 > 0 \). Then the improper-spectral integral \( \int_{\mathbb{R}} \Omega(\lambda) F(d\lambda) f \) exists for any \( f \in \mathcal{H} \) satisfying
\[
\int_{\mathbb{R}} |\lambda|^{2\alpha} d \| F(\lambda) f \|^2 < \infty.
\] (3.9)

**Proof.** Let \( b, c > 0 \). Let \( n \in \mathbb{N} \). Put \( x_m := b + \frac{m-1}{n} c, \Delta_m := [x_m, x_m + \frac{c}{n}] \), \( Z := \bigcup_{m=1}^{n} \Delta_m \). Then
\[
\| \Sigma_Z \Omega f \|^2 = \sum_{m=1}^{n} \| \Omega(x_m) F(\Delta_m) f \|^2
\leq C_0^2 \sum_{m=1}^{n} (1 + x_m)^{2\alpha} \| F(\Delta_m) f \|^2
\leq C_0^2 \int_{[b,b+c)} (1 + \lambda)^{2\alpha} d \| F(\lambda) f \|^2.
\]

Passing to the limit, as \( n \) tends to infinity, we get that
\[
\left\| \int_{[b,b+c)} \Omega(\lambda) F(d\lambda) f \right\|^2 \leq C_0^2 \int_{[b,b+c)} (1 + \lambda)^{2\alpha} d \| F(\lambda) f \|^2.
\]
The integral on the right-hand side tends to zero, as $b$ tends to infinity, provided (3.9) holds. The case $a \rightarrow -\infty$ is treated similarly. \hfill \Box

# 4. Boundary Triplets for Tensor Products

## 4.1. Bounded Case

Let $A$ be a densely defined symmetric operator with equal deficiency indices acting in the separable Hilbert space $\mathcal{H}$, and let $T$ be a bounded self-adjoint operator acting on the separable Hilbert space $\mathcal{T}$. Let us consider the closed symmetric operator $S := A \otimes I_\mathcal{T} + I_\mathcal{H} \otimes T$ in $\mathcal{H} \otimes \mathcal{T}$. We recall that the operator $S$ is defined as the closure of $S_0 := A \otimes I_\mathcal{T} + I_\mathcal{H} \otimes T$,

$$\text{dom} (A \otimes I_\mathcal{T} + I_\mathcal{H} \otimes T) := \left\{ f = \sum_{k=1}^{n} g_k \otimes h_k : g_k \in \text{dom} (A), h_k \in \mathcal{T} \right\}$$

and

$$S_0 f := \sum_{k=1}^{n} (Ag_k \otimes h_k + g_k \otimes Th_k), \quad f \in \text{dom} (A \otimes I_\mathcal{T} + I_\mathcal{H} \otimes T).$$

Obviously, the operator $S_0$ is densely defined and symmetric.

Let $\Pi_A = \{ \mathcal{H}^A, \Gamma_0^A, \Gamma_1^A \}$ be a boundary triplet for $A^*$ with $\gamma$-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Let $J_{A^*}$ be the embedding operator $J_{A^*}: \mathcal{H}_+(A^*) \rightarrow \text{dom}(A^*)$. Obviously, $\text{ran}(J_{A^*}) = \text{dom}(A^*)$ and $\ker(J_{A^*}) = \{0\}$ as well as $\Gamma_0^A = \hat{\Gamma}_j J_{A^*}^{-1}, \ j = 0, 1$. Notice that $\mathcal{H}_+(\mathcal{A} \otimes I_\mathcal{T})^* = \mathcal{H}_+(A^* \otimes I_\mathcal{T}) = \mathcal{H}_+(A^*) \otimes \mathcal{T}$ and $J_{(A \otimes I_\mathcal{T})^*} = J_{A^* \otimes I_\mathcal{T}}$. Moreover, one has

$$\text{ran}(J_{(A \otimes I_\mathcal{T})^*}) = \text{dom}((A \otimes I_\mathcal{T})^*) = \text{dom}(A^* \otimes I_\mathcal{T}).$$

We set

$$(\Gamma_j \hat{I}_\mathcal{T}) f := (\hat{\Gamma}_j \otimes I_\mathcal{T}) J_{(A \otimes I_\mathcal{T})^*}, \quad j \in \{0, 1\}, \quad f \in \text{dom}(A^* \otimes I_\mathcal{T}). \quad (4.1)$$

It turns out that $\Pi_A \hat{I}_\mathcal{T} := \{ \mathcal{H}^A \otimes \mathcal{T}, \Gamma_0^A \otimes I_\mathcal{T}, \Gamma_1^A \otimes I_\mathcal{T} \}$ is a boundary triplet for $(A \otimes I_\mathcal{T})^* = A^* \otimes I_\mathcal{T}$.

**Theorem 4.1.** Let $\Pi_A = \{ \mathcal{H}^A, \Gamma_0^A, \Gamma_1^A \}$ be a boundary triplet for $A^*$ with $\gamma$-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Let also $T = T^* \in \mathcal{B}(\mathcal{T})$, and let $\Delta$ be the smallest closed interval containing the spectrum $\sigma(T)$. Finally, let $\hat{E}_T(\delta) := I_{\mathcal{H}^T} \otimes E_T(\delta)$, $\delta \in \mathcal{B}(\mathbb{R})$, where $E_T(\cdot)$ is the spectral measure of $T$. Then:

(i) $\Pi_S = \{ \mathcal{H}^S, \Gamma_0^S, \Gamma_1^S \} := \Pi_A \hat{I}_\mathcal{T}$ is a boundary triplet for $S^*$ such that $S_0 := S^* \upharpoonright \ker(\Gamma_0^S) = A_0 \otimes I_\mathcal{T} + I_\mathcal{H} \otimes T$.

(ii) The $\gamma$-field $\gamma^S(\cdot)$ and the Weyl function $M^S(\cdot)$ of $\Pi_S$ admit the following representations

$$\gamma^S(z) = \int_{\Delta} \left( \gamma^A(z - \lambda) \otimes I_\mathcal{T} \right) \hat{E}_T(d\lambda), \quad z \in \mathbb{C}_\pm. \quad (4.2)$$
\[
M^S(z) = \int_\Delta \hat{E}_T(d\lambda) \left( M^A(z - \lambda) \otimes I_{\bar{\mathcal{T}}} \right) \\
= \int_\Delta \left( M^A(z - \lambda) \otimes I_{\bar{\mathcal{T}}} \right) \hat{E}_T(d\lambda), \quad z \in \mathbb{C}_\pm. \quad (4.3)
\]

In particular,
\[
\text{ran} \left( \int_\Delta \left( \gamma^A(z - \lambda) \otimes I_{\bar{\mathcal{T}}} \right) \hat{E}_T(d\lambda) \right) = \mathfrak{M}_z(S^*) = \ker(S^* - z). \quad (4.4)
\]

(iii) If the Weyl function \( M^A(\cdot) \) is of scalar type, \( M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}_\Lambda} \), then
\[
M^S(z) = I_{\mathcal{H}_A} \otimes m^A(z - T), \quad z \in \mathbb{C}_\pm,
\]
where \( m^A(z - T) \) is defined by the functional calculus. In particular, the latter holds whenever \( n_\pm(A) = 1 \).

Note that integrals (4.3) and (4.2) exist due to Corollary 3.4 since both the Weyl function \( M^S(z - \cdot) \) and \( \gamma^S(z - \cdot) \) are holomorphic in \( \lambda \), hence Lipschitz functions.

**Proof.** (i) The proof is straightforward.

(ii) In accordance with [3, Lemma 7.2] both integrals (4.2) and (4.3) exist since \( \gamma^A(\cdot) \) and \( M^A(\cdot) \) are Lipschitz. Let \( \pi = \{ a = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n = b \} \) be a partition of \( \Delta = [a, b] \), \( \Delta_k := [\lambda_{k-1}, \lambda_k) \), and let \( x_k \in \Delta_k \),
\[
T_k := x_k E(\Delta_k), \quad T_\pi := \bigoplus_{k=1}^n T_k = \sum_{k=1}^n x_k E_T(\Delta_k),
\]
\[
S_\pi := A \otimes I_{\bar{\mathcal{T}}} + I_{\bar{\mathcal{S}}} \otimes T_\pi,
\]
and \( \bar{\mathcal{T}}_k := \text{ran } E(\Delta_k) \). \( T_k \) is regarded as an operator in \( \bar{\mathcal{T}}_k \). It is easily seen that \( \bar{\mathcal{T}} = \bigoplus_1^n \bar{\mathcal{T}}_k \) and
\[
S_\pi = \bigoplus_{k=1}^n S_k, \quad S_k := A \otimes I_{\bar{\mathcal{T}}_k} + I_{\bar{\mathcal{S}}} \otimes T_k \in \mathcal{C}(\bar{\mathcal{S}} \otimes \bar{\mathcal{T}}_k).
\]

Clearly, \( S_k^* := A^* \otimes I_{\bar{\mathcal{T}}_k} + I_{\bar{\mathcal{S}}} \otimes T_k \). Moreover, for every \( k \) such that \( \bar{\mathcal{T}}_k \neq \{0\} \) we have \( \sigma(T_k) = \{ x_k \} \) and hence \( \sigma(S_k^*) = \sigma(A^* \otimes I_{\bar{\mathcal{T}}_k}) + x_k \) and \( \mathfrak{M}_z(S_k) = \mathfrak{N}_{z-x_k}(A) \otimes \bar{\mathcal{T}}_k \). Clearly,
\[
S_\pi^* = A^* \otimes \left( \bigoplus_{k=1}^n E_T(\Delta_k) \right) + I_{\bar{\mathcal{S}}} \otimes \left( \bigoplus_{k=1}^n x_k E_T(\Delta_k) \right)
\]
\[
= \bigoplus_{k=1}^n (A^* + x_k I_{\bar{\mathcal{S}}}) \otimes E_T(\Delta_k).
\]

Hence, \( \mathfrak{M}_z(S_\pi) = \ker(S_\pi^* - z I_{\bar{\mathcal{S}}}) = \text{ran} \left( \sum_{k=1}^n \gamma^A(z - x_k) \otimes E_T(\Delta_k) \right), \quad z \in \mathbb{C}_\pm. \)
Noting that $\Gamma_{0}^{S_{\pi}} = \Gamma_{0}^{S} = \Gamma_{0}^{A} \otimes I_{\mathcal{F}}$ and using definition (2.4) one gets
\[
\Gamma_{0}^{S_{\pi}} \left( \sum_{k=1}^{n} \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k}) \right) = \sum_{k=1}^{n} \Gamma_{0}^{A} \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k})
\]
\[
= \sum_{k=1}^{n} I_{\mathcal{H}^{A}} \otimes E_{T}(\Delta_{k}) = I_{\mathcal{H}^{A}} \otimes I_{\mathcal{F}} = I_{\mathcal{H}^{A} \otimes \mathcal{F}} \tag{4.5}
\]
for $z \in \mathbb{C}_{\pm}$. Combining this relation with definition (2.4) of the $\gamma$-field one derives
\[
\gamma^{S_{\pi}}(z) = \left( \Gamma_{0}^{S_{\pi}} \restriction_{\mathcal{M}_{z}(S_{\pi})} \right)^{-1} = \sum_{k=1}^{n} \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k}).
\]
Applying operator $\Gamma_{1}$ to this equality and using Definition 2.4 we arrive at the Weyl function $M_{\pi}^{S_{\pi}}(\cdot)$ corresponding to the triplet $\Pi_{\pi}^{S_{\pi}}$ of $S_{\pi}^{*}$,
\[
M_{\pi}^{S_{\pi}}(z) = \Gamma_{1}^{S_{\pi}} \gamma^{S_{\pi}}(z) = \sum_{k=1}^{n} \Gamma_{1}^{A} \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k})
\]
\[
= \sum_{k=1}^{n} M^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k}).
\]
Since integrals (5.1) and (5.2) exist, the convergence
\[
\gamma^{S_{\pi}}(z) \to \int_{\Delta} \left( \gamma^{A}(z - \lambda) \otimes I_{\mathcal{F}} \right) \hat{E}_{T}(d\lambda) =: \tilde{\gamma}^{S}(z) \quad \text{as} \quad |\pi| \to 0, \tag{4.6}
\]
and
\[
M^{S_{\pi}}(z) \to \int_{\Delta} \left( M^{A}(z - \lambda) \otimes I_{\mathcal{F}} \right) \hat{E}_{T}(d\lambda) =: \tilde{M}^{S}(z) \quad \text{as} \quad |\pi| \to 0, \tag{4.7}
\]
holds in the operator norm for each $z \in \mathbb{C}_{\pm}$, where as usual $|\pi| = \max_{k=1,2,...,n} |\Delta_{k}|$.

Next we show that $\tilde{\gamma}^{S}(z) = \gamma^{S}(z)$ and $\tilde{M}^{S}(z) = M^{S}(z)$ for $z \in \mathbb{C}_{\pm}$. One gets
\[
((A^{*} - z) \otimes I_{\mathcal{F}}) \gamma^{S_{\pi}}(z)g = \sum_{k=1}^{n} (A^{*} - z) \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k})g
\]
\[
= - \sum_{k=1}^{n} x_{k} \gamma^{A}(z - x_{k}) \otimes E_{T}(\Delta_{k})g \to - \int_{\Delta} (\lambda \gamma^{A}(z - \lambda) \otimes I_{\mathcal{F}}) \hat{E}_{T}(d\lambda)g
\]
as $|\pi| \to 0$. Since $A^{*}$ is closed, one gets by combining this relation with (4.6) that $\int_{\Delta} (\lambda \gamma^{A}(z - \lambda) \otimes I_{\mathcal{F}}) \hat{E}_{T}(d\lambda)g \in \text{dom} \ (A^{*} \otimes I_{\mathcal{F}})$ for each $g \in \mathcal{H}^{A} \otimes \mathcal{F}$ and
\[
((A^{*} - z) \otimes I_{\mathcal{F}}) \int_{\Delta} (\gamma^{A}(z - \lambda) \otimes I_{\mathcal{F}}) \hat{E}_{T}(d\lambda) = - \int_{\Delta} (\lambda \gamma^{A}(z - \lambda) \otimes I_{\mathcal{F}}) \hat{E}_{T}(d\lambda).
\]
In turn, using this relation and applying Proposition 3.2 we derive
\[(S^* - z) \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda)\]
\[= ((A^* - z) \otimes I) \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda)\]
\[+ \int_\Delta \lambda \hat{E}_T(d\lambda) \cdot \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda)\]
\[= - \int_\Delta (\lambda \gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda) + \int_\Delta (\lambda \gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda) = 0.\]

It follows that \(\text{ran} \left( \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda) \right) \subset \mathfrak{N}_z(S^*) = \ker(S^* - z).\)

Let us show that the convergence in (4.6) holds in \(\mathfrak{H}_+(S),\) i.e., in the graph norm.

Choose a sequence \(\{\pi_n\}_{n=1}^\infty\) of partitions of \([a, b]\) such that \(\lim_{n \to \infty} |\pi_n| = 0.\) Since the convergence in (4.6) is uniform, there exists a constant \(C(z) > 0\) depending on \(z\) and not depending on \(n\) such that \(\|\gamma S_{\pi_n}(z)\| \leq C(z)\) for all \(n.\) Besides, for any \(\varepsilon > 0\) there exists \(N = N(\varepsilon) \in \mathbb{N}\) such that \(\|T_{\pi_n} - T\| \leq \varepsilon\) for \(n \geq N.\) Taking these relations into account one gets
\[\|S^* - z\gamma S_{\pi_n}(z)g\| = \|S^* - z\gamma S_{\pi_n}(z)g - (S^*_\pi - z)\gamma S_{\pi_n}(z)g\|\]
\[= \|(I \otimes (T - T_\pi))\gamma S_{\pi_n}(z)g\| \leq \varepsilon \|\gamma S_{\pi_n}(z)\| \cdot \|g\| \leq \varepsilon C(z)\|g\|\]
for any \(\pi \in \{\pi_n\}_{n=N}^\infty,\) hence \(\lim_{n \to \infty} (S^* - z)\gamma S_{\pi_n}(z)g = 0\) for any \(g \in \mathcal{H}^A \otimes \mathcal{T}.\) In turn, combining this relation with (4.6) yields
\[\lim_{n \to \infty} \gamma S_{\pi_n}(z) - \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathcal{F}) \hat{E}_T(d\lambda) = 0. \tag{4.8}\]

It follows from (4.5) that \(\Gamma_0^S \gamma S_{\pi_n}(z) = \Gamma_0^S \gamma S_{\pi_n}(z) = I_{\mathcal{H}^A} \otimes I_\mathcal{F} \to I_{\mathcal{H}^A} \otimes I_\mathcal{F}\) as \(|\pi| \to 0.\) Therefore, relation (4.8) implies
\[\Gamma_0^S \gamma S(z) = I_{\mathcal{H}^A} \otimes I_\mathcal{F},\]
i.e., \(\gamma S(z) = \gamma S(z).\) This proves (4.2). In turn, (4.2) implies (4.4).

Further, combining the just established relation \(\tilde{\gamma} S(\cdot) = \gamma S(\cdot)\) with relation (4.8) and using the boundedness of the operator \(\Gamma_1^S \in [\mathfrak{H}_+(S), \mathcal{H}^S]\) we obtain
\[\lim_{n \to \infty} M^{S_{\pi_n}}(z) = \lim_{n \to \infty} \Gamma_1^S \gamma S_{\pi_n}(z)\]
\[= \lim_{n \to \infty} \Gamma_1^S \gamma S_{\pi_n}(z) = \Gamma_1^S \gamma S(z) = M^S(z), \quad z \in \mathbb{C}_\pm,\]
where the convergence takes place w.r.t. the operator norm. In turn, combining this relation with (4.7) yields (4.3).

\[\square\]

Remark 4.2. Another proof of Theorem 4.1 can be found in [13, cf. Proposition 3.1 and 3.2]. Our proof based on functional calculus for operator integrals developed in Propositions 3.2 and 3.3 is more transparent. That reasoning is also used in the following, in particular, in the proofs of Proposition 4.4 and Lemma 4.6.
Example 4.3. Let us illustrate the theorem above. To this end, we consider the case that \( A \) is a closed symmetric operator with deficiency indices \( n_{\pm} = 2 \). In particular, let \( \Pi_A = \{ \mathcal{H}^A, \Gamma_0^A, \Gamma_1^A \} \) where \( \mathcal{H}^A = (\mathcal{H}_1^A \oplus \mathcal{H}_2^A)^I, \mathcal{H}_j^A = \mathbb{C} \), \( j = 1, 2 \). We use the representation

\[
\Gamma_j^A = \begin{pmatrix} \Gamma_{j1}^A \cr \Gamma_{j2}^A \end{pmatrix} : \text{dom} (A^*) \longrightarrow \mathcal{H}_1^A \oplus \mathcal{H}_2^A, \quad j = 0, 1.
\]

For the gamma field \( \gamma^A(\cdot) \) we use the representation \( \gamma^A(z) = (\gamma_{1j}^A(z), \gamma_{2j}^A(z)), \gamma_j^A(z) : \mathcal{H}_j^A \longrightarrow \mathfrak{H}, j = 1, 2, z \in \mathbb{C}_\pm \). The Weyl function \( M^A(\cdot) \) admits the representation

\[
M^A(z) = \begin{pmatrix} m_{11}^A(z) & m_{12}^A(z) \\
m_{21}^A(z) & m_{22}^A(z) \end{pmatrix}, \quad z \in \mathbb{C}_\pm,
\]

where \( m_{ij}^A(\cdot) \) are holomorphic functions in \( \mathbb{C}_\pm \).

We consider the closed symmetric operator \( S = A \otimes I_\mathbb{F} + I_\mathfrak{H} \otimes T \), where \( T \) is bounded and self-adjoint. Let \( \Pi_S = \Pi_A \otimes I_\mathbb{F} \), cf. Theorem 4.1 (i). Obviously, the boundary value space \( \mathcal{H}^S = \mathcal{H}^A \otimes \mathbb{F} \) can be decomposed by \( \mathcal{H}^S = (\mathcal{H}_1^S \oplus \mathcal{H}_2^S)^I, \mathcal{H}_j^S := \mathbb{F}, j = 1, 2 \). The boundary value maps \( \Gamma_0^S = \Gamma_0^A \otimes I_\mathbb{F} \) and \( \Gamma_1^S = \Gamma_1^A \otimes I_\mathbb{F} \) will be represented by

\[
\Gamma_0^S = \begin{pmatrix} \Gamma_{01}^S \\
\Gamma_{02}^S \end{pmatrix} : \text{dom} (S^*) \longrightarrow \mathcal{H}_1^S \oplus \mathcal{H}_2^S, \quad \text{and} \quad \Gamma_1^S = \begin{pmatrix} \Gamma_{11}^S \\
\Gamma_{12}^S \end{pmatrix} : \text{dom} (S^*) \longrightarrow \mathcal{H}_1^S \oplus \mathcal{H}_2^S,
\]

where \( \Gamma_{0j}^S := \Gamma_{0j}^A \otimes I_\mathbb{F} \) and \( \Gamma_{1j}^S := \Gamma_{1j}^A \otimes I_\mathbb{F} \), \( j = 0, 1 \). From (4.2) we get the representation \( \gamma_j^S(z) = (\gamma_{1j}^S(z), \gamma_{2j}^S(z)), z \in \mathbb{C}_\pm \), where \( \gamma_{1j}^S(z) : \mathcal{H}_j^S \longrightarrow \mathfrak{H}, \gamma_{2j}^S(z) \in \mathcal{H}_j^S \) is the smallest closed interval containing the spectrum \( \sigma(T) \), and let \( \Pi_S = \Pi_A \otimes I_\mathbb{F} \). Finally, let \( \hat{E}_T(\delta) := I_{\mathcal{H}^A} \otimes E_T(\delta), \delta \in \mathfrak{B}(\mathbb{R}) \), where \( E_T(\cdot) \) is the spectral measure of \( T \). Then:

Proposition 4.4. Let \( \Pi_A = \{ \mathcal{H}^A, \Gamma_0^A, \Gamma_1^A \} \) be a boundary triplet for \( A^* \) with the \( \gamma \)-field \( \gamma^A(\cdot) \) and Weyl function \( M^A(\cdot) \). Let also \( A_0 := A^* \uparrow \ker (\Gamma_0^A), T = T^* \in \mathcal{B}(\mathbb{F}) \), and let \( \Delta \) be the smallest closed interval containing the spectrum \( \sigma(T) \), and let \( \Pi_S = \Pi_A \otimes I_\mathbb{F} \). Finally, let \( \hat{E}_T(\delta) := I_{\mathcal{H}^A} \otimes E_T(\delta), \delta \in \mathfrak{B}(\mathbb{R}) \), where \( E_T(\cdot) \) is the spectral measure of \( T \). Then:
The Weyl function

By Theorem 4.1, 

First we note that both integrals in (4.14) exist since the operator-valued functions \( \text{Im}(M^A(i - \lambda)) \) and \( \text{Re}(M^A(i - \lambda)) \) are Lipschitz (see [3]). Moreover, since the spectral measure \( \hat{E}_T = I_\Sigma \otimes E_T \) commutes with \( M^A(z - \lambda) \otimes I_\Sigma \), both functions \( \text{Im}(M^A(i - \lambda)) \otimes I_\Sigma \) and \( \text{Re}(M^A(i - \lambda)) \otimes I_\Sigma \) are \( \hat{E}_T \)-admissible. Noting that \( M^A(\cdot) \) is holomorphic on \( \mathbb{C}_+ \) and \( 0 \in \rho(\text{Im}M(z)) \) for \( z \in \mathbb{C}_+ \), one easily concludes that the operator-valued functions \( \text{Im}(M^A(i - \lambda)) \otimes I_\Sigma \),
Re(M^A(i - \cdot)) \otimes I_\Gamma, and (Im(M^A(i - \cdot)))^{-1} \otimes I_\Gamma are bounded on the compact set \Delta and with some constants \(c_1, c_2 > 0\) the following estimates hold

\[ 0 < c_1 \leq \text{Im}(M^A(i - \lambda)) \otimes I_\Gamma \leq c_2 \quad \text{and} \quad c_2^{-1} \leq (\text{Im}(M^A(i - \lambda)))^{-1} \otimes I_\Gamma \leq c_1^{-1}, \quad \lambda \in \Delta. \]

Since the function \(\varphi(\cdot) = \sqrt{\cdot}\) is continuous on \(\mathbb{R}_+\), then in accordance with Proposition 3.3(iii) the compositions \((\text{Im}(M^A(i - \lambda)))^{1/2} \otimes I_\Gamma\) and \((\text{Im}(M^A(i - \lambda)))^{-1/2} \otimes I_\Gamma\) are \(\hat{E}_T\)-admissible and

\[
R := \sqrt{\text{Im}(M^S(i))} = \int_\Delta \left( \sqrt{\text{Im}(M^A(i - \lambda))} \otimes I_\Gamma \right) \hat{E}_T(d\lambda),
\]

\[
R^{-1} = \frac{1}{\sqrt{\text{Im}(M^S(i))}} = \int_\Delta \left( \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_\Gamma \right) \hat{E}_T(d\lambda). \tag{4.15}
\]

Combining the second formula in (4.15) with formula (4.3) and applying Proposition 3.2 one arrives at

\[
R^{-1}Q := R^{-1}\text{Re}(M^S(i))
= \int_\Delta \left( \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \text{Re}(M^A(i - \lambda)) \otimes I_\Gamma \right) \hat{E}_T(d\lambda). \tag{4.16}
\]

Now it follows from Lemma 2.6 (see formula (2.13)) that a triplet \(\tilde{\Pi}_S = \{\mathcal{H}_S, \tilde{\Gamma}_0^S, \tilde{\Gamma}_1^S\}\), where

\[ \tilde{\Gamma}_0^S = \sqrt{\text{Im}(M^S(i))}\Gamma_0^S \quad \text{and} \quad \tilde{\Gamma}_1^S = \frac{1}{\sqrt{\text{Im}(M^S(i))}}(\Gamma_1^S - \text{Re}(M^S(i))\Gamma_0^S), \]

is a (normalized) boundary triplet for \(S^\ast\). Combining these formulas with formulas (4.15) yields (4.9).

(ii) Combining (4.2) with the second identity in (4.15) and applying Proposition 3.2 we arrive at

\[
\tilde{\gamma}^S(z) = \gamma^S(z)R^{-1}
= \int_\Delta (\gamma^A(z - \lambda) \otimes I_\Gamma) \hat{E}_T(d\lambda) \cdot \int_\Delta \left( \frac{1}{\sqrt{\text{Im}(M^A(i - \mu))}} \otimes I_\Gamma \right) \hat{E}_T(d\mu)
= \int_\Delta \left( \gamma^A(z - \lambda) \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_\Gamma \right) \hat{E}_T(d\lambda), \quad z \in \mathbb{C}_\pm,
\]

which proves (4.10).
Similarly, combining formula (4.3) with the third formula in (4.15) and applying Proposition 3.2 implies
\[
\frac{1}{\sqrt{\text{Im}(M^S(i))}} \left( M^S(z) - \text{Re}(M^S(i)) \right) \frac{1}{\sqrt{\text{Im}(M^S(i))}} = \int_{\Delta} \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \times \left( M^A(z - \lambda) - \text{Re}(M^A(i - \lambda)) \right) \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \hat{E}_T(d\lambda),
\]
z \in \mathbb{C}_\pm. This proves (4.11). Moreover, inserting in (4.11) \(z = i\) one easily gets the equality \(\tilde{M}^S(i) = i(I_{H^A} \otimes I_T) = iI_{H^S}\) meaning that the triplet \(\tilde{\Pi}_S\) is normalized.

(iii) Representation (4.13) is immediate from (4.11). \(\square\)

4.2. Unbounded Case
Let \(A\) be a closed densely defined symmetric operator with equal deficiency indices in \(\mathcal{H}\), and let \(T\) be an unbounded self-adjoint operator in \(\mathcal{F}\). First, we introduce an operator \(S' := A \otimes I_T + I_H \otimes T\) by setting (cf. \([53, \text{Chapter 7.5.2}]\))
\[
S' f := A \otimes I_T f + I_H \otimes T f := \sum_{k=1}^l (Ag_k \otimes h_k) + \sum_{k=1}^l (g_k \otimes Th_k),
\]
\[
f = \sum_{k=1}^l g_k \otimes h_k \in \text{dom (}S'\text{),}
\]
\[
\text{dom (}S'\text{)} := \left\{ f = \sum_{k=1}^l g_k \otimes h_k : g_k \in \text{dom (}A\text{), } h_k \in \text{dom (}T\text{), } l \in \mathbb{N} \right\}.
\]
Clearly, \(S'\) is a densely defined symmetric operator. Further, we define the operator \(S := A \otimes I_T + I_H \otimes T\) on \(\mathcal{R} := \mathcal{H} \otimes \mathcal{F}\) as the closure of \(S'\), i.e.,
\[
S := \overline{S'} := \overline{A \otimes I_T + I_H \otimes T}.
\]
Denote by \(\mathcal{H}_+(A)\) the Hilbert space obtained by equipping the domain \(\text{dom (}A\text{)}\) with the graph norm. Let \(J_A : \mathcal{H}_+(A) \rightarrow \mathcal{H}\) be the embedding operator. Then \(\text{dom (}A \otimes I_T\text{)} = (J_A \otimes I_T)(\mathcal{H}_+(A) \otimes \mathcal{F})\). By \([53, \text{Proposition 7.26}]\), \((A \otimes I_T)^* = A^* \otimes I_T\) and \(\text{dom (}A^* \otimes I_T\text{)} = (J_A^* \otimes I_T)(\mathcal{H}_+(A^*) \otimes \mathcal{F})\).
The operator \(I_H \otimes T = I_H \otimes T\) is unbounded and self-adjoint. Moreover, one has \(S = A \otimes I_T + I_H \otimes T\) and \(\text{dom (}S\text{)} \supseteq \mathcal{D} := \text{dom (}A \otimes I_T\text{)} \cap \text{dom (}I_H \otimes T\text{)}\). Clearly, \(\mathcal{D}\) is a core for \(S\), i.e., \(S = \overline{S \upharpoonright \mathcal{D}}\).
Further, setting \(T_n := E_T([n, n+1])T\) and \(\mathcal{F}_n := E_T([n, n+1])\mathcal{F}\), \(n \in \mathbb{Z}\), one arrives at the orthogonal decomposition
\[
T = \bigoplus_{n \in \mathbb{Z}} T_n, \quad \mathcal{F} := \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n,
\]
(4.17)
where $T_n = T_n^* \in \mathcal{B}(\mathfrak{T}_n)$. Let $\mathfrak{R}_n := \mathfrak{H} \otimes \mathfrak{T}_n$, $n \in \mathbb{Z}$. Clearly, $\mathfrak{R} := \mathfrak{H} \otimes \mathfrak{T} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{R}_n$. We set $S_n := A \otimes I_{\mathfrak{T}_n} + I_{\mathfrak{H}} \otimes T_n$, $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ the operator $S_n$ is a well-defined closed symmetric operator in $\mathfrak{R}_n$.

**Lemma 4.5.** Let $A$ and $T$ be as above. Let $T = \bigoplus_{n \in \mathbb{Z}} T_n$ be an orthogonal decomposition of $T$, where $T_n = T_n^* \in \mathcal{B}(\mathfrak{T}_n)$. Then

$$S = \bigoplus_{n \in \mathbb{Z}} S_n, \quad S_n := A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T_n.$$

**Proof.** The proof is obvious. $\square$

We emphasize that the orthogonal decomposition (4.17) is not the only possibility in Lemma 4.5. If $T$ has a pure point spectrum, $\{\lambda_n\}_{n \in \mathbb{Z}}$ is a sequence of its eigenvalues, and $\mathfrak{T}_n = E_T(\{\lambda_n\})\mathfrak{T}$, then we can chose $T_n = \lambda_n I_{\mathfrak{T}_n}$. In this case we get $S = \bigoplus_{n \in \mathbb{Z}} S_n$ where $S_n = A \otimes I_{\mathfrak{T}_n} + \lambda_n I_{\mathfrak{R}_n}$, $\mathfrak{R}_n := \mathfrak{H} \otimes \mathfrak{T}_n$.

In general, for any self-adjoint extension $S_0$ of $S$ there is a boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ such that $S_0 = S^* \upharpoonright \ker (\Gamma_0^S)$. Moreover, in accordance with Lemma 2.6 it is always possible starting with a $\Pi_S$ to define a normalized boundary triplet $\tilde{\Pi}_S$. In particular, we can find a boundary triplet $\Pi_S$ for $S^*$, $S = A \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$, such that $S_0 := A_0 \otimes I_{\mathfrak{T}} + I_{\mathfrak{H}} \otimes T$. However, in applications we need a special boundary triplet feeling the tensor structure of the operators $S$ and $S^*$ and leading to simple forms of the corresponding Weyl function and $\gamma$-field.

Therefore, in what follows we choose a different strategy. Let $\Pi_A$ be a boundary triplet for $A^*$ with the corresponding $\gamma$-field $\gamma^A(\cdot)$ and Weyl function $M^A(\cdot)$. Starting with this boundary triplet for $A^*$ we construct a normalized boundary triplet $\Pi_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\}$ for $S^*$ such that $S_0 = S^* \upharpoonright \ker (\Gamma_0^S)$ and the corresponding $\gamma$-field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ can be explicitly computed by means of $\gamma^A(\cdot)$ and $M^A(\cdot)$ (cf. the proof of Theorem 4.8).

**Lemma 4.6.** Let $A$ be a densely defined closed symmetric operator in $\mathfrak{H}$. Let also $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for $A^*$, and let $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Further, let $T$ be a self-adjoint operator on $\mathfrak{T}$ with spectral measure $E_T(\cdot)$ and let $\hat{E}_T(\cdot) := I_{\mathcal{H}^A} \otimes E_T(\cdot)$. Then the following improper-spectral integrals

$$G_0 f := \int_{\mathbb{R}} \hat{E}_T(d\lambda) \left( \sqrt{\text{Im}(M^A(i-\lambda))} \otimes I_{\mathfrak{T}} \right) f$$

$$= \int_{\mathbb{R}} \left( \sqrt{\text{Im}(M^A(i-\lambda))} \otimes I_{\mathfrak{T}} \right) \hat{E}_T(d\lambda) f$$

$$G_1 f := \int_{\mathbb{R}} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{\text{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}} \right) f$$

$$= \int_{\mathbb{R}} \left( \frac{1}{\sqrt{\text{Im}(M^A(i-\lambda))}} \otimes I_{\mathfrak{T}} \right) \hat{E}_T(d\lambda) f,$$
\[ G_2 f := \int_{\mathbb{R}} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \operatorname{Re}(M^A(i - \lambda)) \otimes I_{\mathcal{T}} \right) f \]
\[ = \int_{\mathbb{R}} \left( \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \operatorname{Re}(M^A(i - \lambda)) \otimes I_{\mathcal{T}} \right) \hat{E}_T(d\lambda) f \]
(4.20)

exist for each \( f \in \text{dom} \left( I_{\mathcal{H}^A} \otimes T \right) \). Moreover, the following improper-spectral integrals
\[ G(z) f := \int_{\mathbb{R}} \left( \gamma^A(z - \lambda) \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_{\mathcal{T}} \right) \hat{E}_T(d\lambda) f, \]
(4.21)
and
\[ M(z) f := \int_{\mathbb{R}} (L^A(z - \lambda, i - \lambda) \otimes I_{\mathcal{T}}) \hat{E}_T(d\lambda) f \]
\[ = \int_{\mathbb{R}} \hat{E}_T(d\lambda) (L^A(z - \lambda, i - \lambda) \otimes I_{\mathcal{T}}) f, \quad z \in \mathbb{C}_{\pm}, \]
(4.22)
exist for every \( f \in \mathcal{H}^A \otimes \mathcal{T} \), where \( L^A(z, \zeta) \), \( z \in \mathbb{C}_{\pm}, \zeta \in \mathbb{C}_{+} \), is given by (4.12).

**Proof.** We divide the proof in several steps. (i) Let \( f \in \text{dom} \left( I_{\mathcal{H}^A} \otimes T \right) \). Then
\[ \int_{\mathbb{R}} \lambda^2 d \| \hat{E}_T(\lambda)f \|^2 < \infty. \]

Note that in accordance with (2.9),
\[ \| (\text{Im}(M^A(i - \lambda)))^{1/2} \otimes I_{\mathcal{T}} \| = O(|\lambda|) \quad \text{and} \]
\[ \| (\text{Im}(M^A(i - \lambda)))^{-1/2} \otimes I_{\mathcal{T}} \| = O(|\lambda|) \quad \text{as} \quad \lambda \to \infty. \]

Therefore, the convergence of the integrals in (4.18) and (4.19) is immediate from Proposition 3.7 with \( \alpha = 1 \).

(ii) To prove (4.20) it suffices to show that
\[ \| (\text{Im}(M^A(i - \lambda)))^{-1/2} \operatorname{Re}(M^A(i - \lambda)) \| = O(|\lambda|) \quad \text{as} \quad \lambda \to \infty. \]
(4.23)

Noting that
\[ (\text{Im}(M^A(i - \lambda)))^{-1/2} M^A(i - \lambda) \]
\[ = (\text{Im}(M^A(i - \lambda)))^{-1/2} \operatorname{Re}(M^A(i - \lambda)) + i(\text{Im}(M^A(i - \lambda)))^{1/2} \]
and taking estimate (2.9) into account one concludes that the required estimate (4.23) is equivalent to the following one
\[ \| (\text{Im}(M^A(i - \lambda)))^{-1/2} M^A(i - \lambda) \| = O(|\lambda|) \quad \text{as} \quad \lambda \to \infty. \]
(4.24)

Further, in accordance with (2.6)
\[ \text{Im}(M^A(i - \lambda)) = -\text{Im}(M^A(-i - \lambda)) = \gamma^A(-i - \lambda)^* \gamma^A(-i - \lambda) \]
\[ = \gamma^A(i - \lambda)^* \gamma^A(i - \lambda), \quad \lambda \in \mathbb{R}. \]
Hence, there exists a family of isometries $V(\lambda \pm i)$ mapping $\mathcal{H}^A$ onto $\mathcal{N}_A(\pm i - \lambda) = \ker (A^* + \lambda \mp i)$ and such that

$$V(\lambda \pm i)(\text{Im}(M^A(i - \lambda)))^{1/2} = \gamma^A(\pm i - \lambda), \quad \lambda \in \mathbb{R}. \quad (4.25)$$

Using (2.6), we get

$$M^A(i - \lambda) - M^A(i)^* = (2i - \lambda)\gamma^A(-i)\gamma^A(-i)$$

$$= (2i - \lambda)(\text{Im}(M^A(i - \lambda)))^{1/2}V(\lambda)^*\gamma^A(-i).$$

Thus,

$$\text{Im}(M^A(i - \lambda))^{-1/2}M^A(i - \lambda)$$

$$= (2i - \lambda)V(\lambda)^*\gamma^A(-i) + \text{Im}(M^A(i - \lambda))^{-1/2}M^A(-i).$$

Combining this relation with estimate (2.9) yields (4.24) as well as

$$\|\text{Im}(M^A(i - \lambda))^{-1/2}\text{Re}(M^A(i - \lambda))\| = O(|\lambda|).$$

To prove (4.20) it remains to apply Proposition 3.7 with $\alpha = 1$.

(iii) To prove the convergence of integral (4.21) it suffices to show that

$$\|\gamma^A(z - \lambda)(\text{Im}(M^A(i - \lambda)))^{-1/2}\| \leq \varkappa(z), \quad \lambda \in \mathbb{R}, \quad (4.26)$$

with some positive constant $\varkappa(z) > 0$. In accordance with (2.5)

$$\gamma^A(z - \lambda) = (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1}\gamma^A(i - \lambda), \quad z \in \mathbb{C}_+, \quad \lambda \in \mathbb{R}.$$ 

Moreover, it follows from (4.25) that

$$(\text{Im}(M^A(i - \lambda)))^{-1/2} = (\gamma^A(i - \lambda))^{-1}V(i + \lambda), \quad \lambda \in \mathbb{R}.$$ 

Combining these relations yields

$$\gamma^A(z - \lambda)(\text{Im}(M^A(i - \lambda)))^{-1/2} = (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1}V(i + \lambda), \quad (4.27)$$

$z \in \mathbb{C}_+, \lambda \in \mathbb{R}$. On the other hand,

$$\| (A_0 + \lambda - i)(A_0 + \lambda - z)^{-1} \| = \| I + (z - i)(A_0 + \lambda - z)^{-1} \|$$

$$\leq 1 + \frac{|z - i|}{|\text{Im}z|} := \varkappa(z).$$

Combining this estimate with identity (4.27) we arrive at estimate (4.26).

Proposition 3.7 with $\alpha = 1$ completes the proof.

(iv) To prove the existence of integral (4.22) it suffices to show that for each fixed $z \in \mathbb{C}_+$

$$\| L^A(z - \lambda, i - \lambda) \| = O(1) \quad \text{as} \quad \lambda \to \infty \quad (4.28)$$

and apply Proposition 3.7. It follows from (4.12) and identity (2.6) that

$L^A(z - \lambda, i - \lambda)$

$$= \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}}(M^A(z - \lambda) - M^A(i - \lambda))\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} + iI_H$

$$= (z - i)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}}\gamma^A(-i - \lambda)\gamma^A(z - \lambda)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} + iI_H,$$
$z \in \mathbb{C}_\pm$, $\lambda \in \mathbb{R}$. Inserting in this identity instead of $\gamma^A(-i - \lambda)^*$ its expression from (4.25) one gets
\[ L^A(z - \lambda, i - \lambda) = (z - i)V(\lambda - i)^*\gamma^A(z - \lambda)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} + i\mathcal{I}_\mathcal{H}. \]

Finally, combining this identity with (4.26) implies (4.28).

**Remark 4.7.** Combining estimates (4.23) and (2.9) we obtain
\[
\|\text{Re}(M^A(i - \lambda))\| \leq \|\text{Im}(M^A(i - \lambda))\|^{1/2}\]
\[
\times \|\text{Re}(M^A(i - \lambda))^{-1/2}\text{Re}(M^A(i - \lambda))\| = O(|\lambda|^2)
\]
as $\lambda \to \infty$. Simple examples show that even for a scalar Nevanlinna function $f$ the function $\|\text{Im}(f(i - \lambda))^{-1/2}\text{Re}(f(i - \lambda))\|$ is not necessarily bounded.

**Theorem 4.8.** Let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for $A^*$, and let $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Let also $T = T^* \in \mathcal{C}(\mathcal{E}) \setminus \mathcal{B}(\mathcal{E})$ and $S := A \otimes I_\mathcal{H} + I_\mathcal{H} \otimes T$. Then:

(i) There exists a normalized boundary triplet $\tilde{\Pi}_S = \{\tilde{\mathcal{H}}^S, \tilde{\Gamma}_0^S, \tilde{\Gamma}_1^S\}$ for $S^*$ such that $\tilde{\mathcal{H}}^S := \mathcal{H}^A \otimes \mathcal{E}$ and $S_0 := S^* \uparrow \ker(\tilde{\Gamma}_0^S) = A_0 \otimes I_\mathcal{H} + I_\mathcal{H} \otimes T$, and for any $f \in \mathcal{D} := \text{dom}(S^*) \cap \text{dom}(I_\mathcal{H} \otimes T)(\subseteq \text{dom}(S^*))$
\[
\tilde{\Gamma}_0^S f := \left(\int_{\mathbb{R}} \tilde{E}_T(d\lambda)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_\mathcal{H}\right) \cdot (\mathcal{T}_0^A \otimes I_\mathcal{H})f,
\]
\[
\tilde{\Gamma}_1^S f := \left(\int_{\mathbb{R}} \tilde{E}_T(d\lambda)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_\mathcal{H}\right) \cdot (\mathcal{T}_1^A \otimes I_\mathcal{H})f
\]
\[
- \left(\int_{\mathbb{R}} \tilde{E}_T(d\lambda)\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \text{Re}(M^A(i - \lambda)) \otimes I_\mathcal{H}\right) \cdot (\mathcal{T}_0^A \otimes I_\mathcal{H})f.
\]

(ii) The $\gamma$-field $\tilde{\gamma}^S(\cdot)$ and Weyl function $\tilde{M}^S(\cdot)$ corresponding to the triplet $\tilde{\Pi}_S$ are given by
\[
\tilde{\gamma}^S(z) = G(z) \quad \text{and} \quad \tilde{M}^S(z) = M(z), \quad z \in \mathbb{C}_\pm,
\]
where $G(\cdot)$ and $M(\cdot)$ are defined by (4.21) and (4.22), respectively.

(iii) If $M^A(\cdot)$ is of scalar type, i.e., $M^A(\cdot) = m^A(\cdot) I_{\mathcal{H}^A}$, then representation (4.13) remains true.

**Proof.** (i) Clearly, $f \in \text{dom}(A^* \otimes I_\mathcal{H})$. Let $\Delta_n := [n, n + 1)$, $n \in \mathbb{Z}$. We set $\mathcal{E}_n := E_T(\Delta_n)\mathcal{E}$ and $T_n := T E_T(\Delta_n)$, $n \in \mathbb{Z}$. Notice that $\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{E}_n$ and $T = \bigoplus_{n \in \mathbb{Z}} T_n$. Let also $R_{S_n} := \sqrt{\text{Im}(M^S_n(i))}$ and $Q_{S_n} := \text{Re}(M^S_n(i))$, $n \in \mathbb{Z}$.

Then, by Proposition 4.4, a triplet $\tilde{\Pi}_{S_n} = \{\mathcal{H}^S_{S_n}, \tilde{\Gamma}_0^S, \tilde{\Gamma}_1^S\}$ with $\mathcal{H}^S_{S_n} := \mathcal{H}^A \otimes \mathcal{E}_n$, $\tilde{\Gamma}_0^S = R_{S_n}(T^A \otimes I_\mathcal{H}_n)$, and $\tilde{\Gamma}_1^S = R_{S_n}^{-1}\left(\Gamma_1^S - Q_{S_n} \Gamma_0^S\right) = R_{S_n}^{-1} \Gamma_1^S - R_{S_n}^{-1} Q_{S_n} \Gamma_0^S$,
is a normalized boundary triplet for $S^*_n$ for each $n \in \mathbb{Z}$. In turn, Theorem 2.7 ensures that the direct sum $\tilde{\Pi}_S := \bigoplus_{n \in \mathbb{Z}} \tilde{\Pi}_{S_n} = \{\tilde{\mathcal{H}}^S, \tilde{\Gamma}^S_0, \tilde{\Gamma}^S_1\}$ of boundary triplets is an ordinary (normalized) boundary triplet for $S^* = \bigoplus_{n \in \mathbb{Z}} S^*_n$.

Setting $R := \bigoplus_n R_{S_n}$, applying formula (4.15) and noting that, by Lemma 4.6, the improper-spectral integral (4.18) exists one gets that for any $h = \bigoplus_n h_n \in \text{dom}(I_{\mathcal{H}_A} \otimes T) = \bigoplus_n \text{dom}(I_{\mathcal{H}_A} \otimes T_n)$

$$Rh = \bigoplus_{n \in \mathbb{Z}} R_{S_n} h_n = \bigoplus_{n \in \mathbb{Z}} \sqrt{\text{Im}(M^{S_n}(i))} h_n$$

$$= \bigoplus_{n \in \mathbb{Z}} \int_{[n,n+1]} \tilde{E}_{T_n}(\lambda) \left(\sqrt{\text{Im}(M^A(i - \lambda)) \otimes I_{\Xi_n}}\right) h_n$$

$$= \text{s- lim}_{p \to -\infty} \int_{[q,p)} \tilde{E}_T(\lambda) \left(\sqrt{\text{Im}(M^A(i - \lambda)) \otimes I_{\Xi}}\right) h$$

$$= \int_{\mathbb{R}} \tilde{E}_T(\lambda) \left(\sqrt{\text{Im}(M^A(i - \lambda)) \otimes I_{\Xi}}\right) = G_0 h.$$  \hspace{1cm} (4.31)

Note that applying formula (4.15) we have replaced the integral $\int_{[n,n+1]}$ by $\int_{[n,n+1]}$. The latter is possible since $n + 1 \not\in \sigma_p(T_n)$ for each $n \in \mathbb{Z}$.

Next, similarly to (4.31) and using the convergence of the improper-spectral integral (4.19) one gets from (4.15)

$$R^{-1}h = \bigoplus_{n \in \mathbb{Z}} R_{S_n}^{-1} h_n = \bigoplus_{n \in \mathbb{Z}} \left(\sqrt{\text{Im}(M^{S_n}(i))}\right)^{-1} h_n$$

$$= \int_{\mathbb{R}} \tilde{E}_T(d\lambda) \left(\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_{\Xi}\right) h = G_1 h.$$  \hspace{1cm} (4.32)

Further, setting $Q := \bigoplus_n Q_{S_n} := \bigoplus_n \text{Re}(M^{S_n}(i))$, applying formula (4.16) with $\Delta_n$ in place of $\Delta$, and noting that by Lemma 4.6 the improper-spectral integral (4.20) exists, we derive

$$R^{-1}Qh = \bigoplus_{n \in \mathbb{Z}} R_{S_n}^{-1} \text{Re}(M^{S_n}(i)) h_n$$

$$= \bigoplus_{n \in \mathbb{Z}} \int_{[n,n+1]} \tilde{E}_{T_n}(\lambda) \left(\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \text{Re}(M^A(i - \lambda)) \otimes I_{\Xi_n}\right) h_n$$

$$= \int_{\mathbb{R}} \tilde{E}_T(\lambda) \left(\frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \text{Re}(M^A(i - \lambda)) \otimes I_{\Xi}\right) h = G_2 h.$$  \hspace{1cm} (4.33)

Further, let $f = \{f_n\}_{n \in \mathbb{Z}} \in \mathcal{D} \subseteq \text{dom}(\mathcal{A}^* \otimes I_{\Xi})$, $f_n \in \mathcal{H}_A \otimes \Xi_n$, $n \in \mathbb{Z}$. Note that $f \in \text{dom}(\tilde{\Gamma}^A_0 \otimes I_{\Xi}) \cap \text{dom}(\Gamma^A_1 \otimes I_{\Xi})$ because $f \in \text{dom}(\mathcal{A}^* \otimes I_{\Xi})$. Hence,

$$\left(\Gamma^A_0 \otimes I_{\Xi}\right)f = \bigoplus_{n \in \mathbb{Z}} \left(\Gamma^A_0 \otimes I_{\Xi_n}\right)f_n \quad \text{and} \quad \left(\Gamma^A_1 \otimes I_{\Xi}\right)f = \bigoplus_{n \in \mathbb{Z}} \left(\Gamma^A_1 \otimes I_{\Xi_n}\right)f_n.$$
On the other hand, by Theorem 2.7 (see formula (2.14))
\[ \widetilde{\Gamma}^S_0 f = R(\Gamma^A_1 \otimes I_\Sigma)f \quad \text{and} \quad \widetilde{\Gamma}^S_1 f = R^{-1}(\Gamma^A_1 \otimes I_\Sigma)f + R^{-1}Q(\Gamma^A_0 \otimes I_\Sigma)f, \]
f \in \mathcal{D}. Inserting in these relations instead of \( R, R^{-1}, \) and \( R^{-1}Q \) their expressions from (4.31)–(4.33), one arrives at formulas (4.29).

(ii) In accordance with Proposition 4.4(ii) the \( \gamma \)-field and Weyl function corresponding to the triplet \( \tilde{\Pi}^{S_n} = \{ \mathcal{H}^{S_n}, \widetilde{\Gamma}^S_0, \widetilde{\Gamma}^S_1 \} \) are given by
\[ \widetilde{\gamma}^{S_n}(z) = \int_{[n,n+1]} \left( \gamma^A(z - \lambda) \frac{1}{\sqrt{\text{Im}(M^A(i - \lambda))}} \otimes I_{\Sigma_n} \right) \hat{E}_T(d\lambda), \quad z \in \mathbb{C}_{\pm}, \]
and
\[ \widetilde{M}^{S_n}(z) = \int_{[n,n+1]} \left( L^A(z - \lambda, i - \lambda) \otimes I_{\Sigma_n} \right) \hat{E}_T(d\lambda) \]
\[ = \int_{[n,n+1]} \hat{E}_T(d\lambda) \left( L^A(z - \lambda, i - \lambda) \otimes I_{\Sigma_n} \right), \quad z \in \mathbb{C}_{\pm}, \]
respectively. Here \( L^A(z, \zeta) \) is given by (4.12). Further, applying Theorem 2.7 (see formula (2.15)) and taking into account formulas (4.21) and (4.22), we arrive at (4.30).

(iii) This statement is now immediate from formula (4.13) and the representation \( T = \bigoplus_{n \in \mathbb{Z}} T_n \) with \( T_n \in \mathcal{B}(\Sigma_n) \). \( \square \)

**Remark 4.9.** (i) If \( T \) is pure point, \( \sigma(T) = \sigma_{pp}(T) = \{ \lambda_k \}_{k \in \mathbb{Z}} \), then the boundary space \( \mathcal{H}^S \) admits the representation \( \mathcal{H}^S = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k \), where \( \mathcal{H}_k = \mathcal{H}^A \otimes \Sigma_k \) and \( \Sigma_k \) is the eigenspace which corresponds to \( \lambda_k \). One easily checks that the Weyl function admits the representation
\[ M^S(z) = \bigoplus_{k \in \mathbb{Z}} (L(z - \lambda_k, i - \lambda_k) \otimes I_{\Sigma_k}), \quad z \in \mathbb{C}_{\pm}. \]

(ii) The set \( \mathcal{D} := \text{dom}(S^*) \cap \text{dom}(I_{\mathcal{H}_A} \otimes T) \subseteq \text{dom}(S^*) \) is a core for \( S^* \). Equivalently, this means that \( \mathcal{D} \) regarded as a subset \( \tilde{\mathcal{D}} \) of \( \mathcal{H}_+(S^*) \) is dense in the Hilbert space \( \mathcal{H}_+(S^*) \). Let \( J_{S^*} : \mathcal{H}_+(S^*) \rightarrow \mathcal{H}_+ = \mathcal{H}^A \otimes \Sigma \) be the embedding operator. We set \( \tilde{\Gamma}^S_j := \Gamma^S_j J_{S^*} : \mathcal{H}_+(S^*) \rightarrow \mathcal{H}^S, j \in \{0, 1\} \). The operator \( \tilde{\Gamma}^S_j, j \in \{0, 1\} \), is bounded. Hence,
\[ \tilde{\Gamma}^S_j = \Gamma^S_j J_{S^*} \mid \tilde{\mathcal{D}}, \quad j \in \{0, 1\}. \]
In other words, the closure of the operator \( \Gamma^S_j \mid \mathcal{D}, j \in \{0, 1\} \), with respect to the topology of \( \mathcal{H}_+(S^*) \) gives \( \Gamma^S_j, j \in \{0, 1\} \).

**Remark 4.10.** The case of a scalar-type Weyl function can be slightly extended. Let us assume that there is a boundary triplet \( \Pi_A = \{ \mathcal{H}^A, \Gamma^A_0, \Gamma^A_1 \} \) of \( A^* \) such that \( \mathcal{H}^A = \bigoplus_{k=1}^{n(A)} \mathcal{H}^A_k, \mathcal{H}^A_k := \mathbb{C}, n(A) := n_{\pm}(A) \). With respect to this
decomposition we suppose that the Weyl function $M^A(\cdot)$ is diagonal, that is, it admits the representation

$$M^A(z) = \text{diag}(m_1(z), m_2(z), \ldots, m_{n(A)}(z))$$

where $m_k(\cdot)$, $k = 1, 2, \ldots, n(A)$, are scalar Nevanlinna functions. If the Weyl function of a boundary triplet has this structure, then it is called to be of quasi-scalar type. We are going to compute the boundary triplet $\Pi_S$ as well as the $\gamma$-field $\gamma^S(\cdot)$ and the Weyl function $M^S(\cdot)$ for the quasi-scalar-type case. We set

$$\Gamma^A_{jk} := P^A_{\mathcal{T}^A} \Gamma_j : \text{dom}(A^*) \to H^A_k, \quad j = 0, 1, \quad k = 1, 2, \ldots, n(A).$$

Obviously, we have

$$\Gamma^A_{1k}f_z = m_k(z) \Gamma^A_{0k}f_z, \quad f_z \in \ker(A^* - z), \quad k = 1, 2, \ldots, n(A).$$

Let us introduce the operator $\Gamma^A_j \otimes I_{\mathcal{T}} : \text{dom}(A^* \otimes I_{\mathcal{T}}) \to H^S_k := H^A_k \otimes \mathcal{T} = \mathcal{T}$, $j = 0, 1$, $k = 1, 2, \ldots, n(A)$. Notice that

$$\Gamma^A_j \hat{\otimes} I_{\mathcal{T}} = \begin{pmatrix} \Gamma^A_{j1} \otimes I_{\mathcal{T}} \\ \Gamma^A_{j2} \otimes I_{\mathcal{T}} \\ \vdots \\ \Gamma^A_{jn(A)} \hat{\otimes} I_{\mathcal{T}} \end{pmatrix} : \text{dom}(A^* \otimes I_{\mathcal{T}}) \to H^S_k,$$

Notice that $H^S = H^A \otimes \mathcal{T} = \bigoplus_{k=1}^{n(A)} H^A_k$. Setting $\Gamma^S_{jk} := P^S_{\mathcal{T}^A} \Gamma^S_{jk}$, $j \in \{0, 1\}$, $k \in \{1, 2, \ldots, n(A)\}$, we get $\Gamma^S_{2j} = (\Gamma^S_{j1}, \Gamma^S_{j2}, \ldots, \Gamma^S_{jn(A)})^t$, $j \in \{0, 1\}$. Using (4.29) we get

$$\Gamma^S_{0k}f = \frac{\sqrt{\text{Im}(m_k(i - T))}(\Gamma^A_{0k} \otimes I_{\mathcal{T}})f}{\Gamma^S_{1k}f = \frac{1}{\sqrt{\text{Im}(m_k(i - T))}} \left( \frac{\sqrt{\text{Im}(m_k(i - T))}(\Gamma^A_{1k} \otimes I_{\mathcal{T}} - \text{Re}(m_k(i - T))(\Gamma^A_{0k} \otimes I_{\mathcal{T}}))f}{\right)},$$

$$f \in \text{dom}(A^* \otimes I_{\mathcal{T}}) \cap \text{dom}(I_{\mathcal{D}^A} \otimes T), \quad k \in \{1, 2, \ldots, n(A)\}.$$

To compute the $\gamma$-field we set

$$\gamma^A_k(\cdot) := \gamma^A(\cdot) \upharpoonright H^A_k, \quad \gamma^A(\cdot) = (\gamma^A_1(\cdot), \gamma^A_2(\cdot), \ldots, \gamma^A_{n(A)}(\cdot)),$$

$z \in \mathbb{C}_\pm$, and

$$\gamma^S_k(\cdot) = \gamma^S(\cdot) \upharpoonright H^S_k, \quad \gamma^S(\cdot) = (\gamma^S_1(\cdot), \gamma^S_2(\cdot), \ldots, \gamma^S_{n(A)}(\cdot)),$$
$z \in \mathbb{C}_\pm$, where $\mathcal{H}_k^S := \mathcal{H}_k^A \otimes \mathfrak{F} = \mathfrak{F}$, $k \in \{1, 2, \ldots, n(A)\}$. From (4.21) we find
\[
\gamma_k^S(z) = \frac{1}{\sqrt{\text{Im}(m_k(i-T))}}, \quad z \in \mathbb{C}_\pm, \quad k \in \{1, 2, \ldots, n(A)\}
\]
Finally, the Weyl function takes the form
\[
M^S(z) = \begin{pmatrix}
\frac{m_1^A(z-T) - \text{Re}(m_1(i-T))}{\text{Im}(m_1(i-T))} & \cdots & \frac{m_{n(A)}^A(z-T) - \text{Re}(m_{n(A)}(i-T))}{\text{Im}(m_{n(A)}(i-T))}
\end{pmatrix}
\]
$z \in \mathbb{C}_\pm$.

5. Sums of Tensor Products with Nonnegative Summands

5.1. Boundary Triplets in the Case of Nonnegative Operators $A$ and $T$

Here we complete previous results assuming the operators $A$ and $T$ to be nonnegative. We denote by $\hat{A}_F$ and $\hat{A}_K$ the Friedrichs’ and Krein’s extension of $A$, respectively. We recall that $\Pi_A \otimes I_\mathfrak{F} = \{\mathcal{H}_0^A \otimes \mathfrak{F}, \Gamma_0^A \otimes I_\mathfrak{F}, \Gamma_1^A \otimes I_\mathfrak{F}\}$ where the maps $\Gamma_0^A \otimes I_\mathfrak{F}$ and $\Gamma_1^A \otimes I_\mathfrak{F}$ are defined by (4.1).

**Theorem 5.1.** Let $A$ be a nonnegative symmetric operator in $\mathfrak{F}$, and let $\Pi_A = \{\mathcal{H}_0^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for $A^*$ such that $A_0 := A^* \upharpoonright \ker(I_0^A) = \hat{A}_F$. Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Let also $T = T^* \in \mathcal{B}(\mathfrak{F})$, $T \geq 0$, and let $S = A \otimes I_\mathfrak{F} + I_\mathfrak{F} \otimes T$. Finally, let $\hat{E}_T(\cdot) := I_{\mathcal{H}_0^A} \otimes E_T(\cdot)$, where $E_T(\cdot)$ is the spectral measure of $T$. Then:

(i) $\Pi_S = \{\mathcal{H}_0^S, \Gamma_0^S, \Gamma_1^S\} := \Pi_A \otimes I_\mathfrak{F} := \{\mathcal{H}_0^A \otimes \mathfrak{F}, \Gamma_0^A \otimes I_\mathfrak{F}, \Gamma_1^A \otimes I_\mathfrak{F}\}$ is a boundary triplet for $S^*$ such that
\[
S_0 := S^* \upharpoonright \ker(I_0^S) = \hat{S}_F = \hat{A}_F \otimes I_\mathfrak{F} + I_\mathfrak{F} \otimes T.
\]

(ii) The $\gamma$-field $\gamma^S(\cdot)$ and Weyl function $M^S(\cdot)$ of $\Pi_S$ admit the following representations
\[
\gamma^S(z) = \int_\Delta (\gamma^A(z - \lambda) \otimes I_\mathfrak{F}) \hat{E}_T(d\lambda), \quad z \in \mathbb{C} \setminus \Delta, \tag{5.1}
\]
and
\[
M^S(z) = \int_\Delta \hat{E}_T(d\lambda) (M^A(z - \lambda) \otimes I_\mathfrak{F})
= \int_\Delta (M^A(z - \lambda) \otimes I_\mathfrak{F}) \hat{E}_T(d\lambda), \quad z \in \mathbb{C} \setminus \Delta, \tag{5.2}
\]
where $\Delta$ is the smallest closed interval containing the spectrum $\sigma(T)$.

(iii) If the Weyl function $M^A(\cdot)$ is of scalar type, $M^A(\cdot) = m^A(\cdot)I_{\mathcal{H}_0^A}$, then
\[
M^S(z) = I_{\mathcal{H}_0^A} \otimes m^A(z-T), \quad z \in \mathbb{C}_\pm.
\]
In particular, the latter holds whenever $n_\pm(A) = 1$. 
Lemma 5.2. Let $A$ be a densely defined closed nonnegative symmetric operator in $\mathfrak{S}$, and let $\Pi_A = \{H^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for $A^*$ and let $A_0 \geq 0$. Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the corresponding Weyl function and $\gamma$-field, respectively. Further, let $T$ be a nonnegative self-adjoint operator on $\mathfrak{S}$, let $E_T(\cdot)$ be its spectral measure, and let $\hat{E}_T(\cdot) := I_{H^A} \otimes E_T(\cdot)$. Then the following improper-spectral integrals

\[
G_0^+ f := \int_{\mathbb{R}_+} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{(M^A)'(a-\lambda)}} \otimes I_{\mathfrak{I}} \right) f, \quad a < 0, \tag{5.3}
\]

\[
G_1^+ f := \int_{\mathbb{R}_+} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{(M^A)'(a-\lambda)}} M^A(a-\lambda) \otimes I_{\mathfrak{I}} \right) f, \quad a < 0, \tag{5.4}
\]

\[
G_2^+ f := \int_{\mathbb{R}_+} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{(M^A)'(a-\lambda)}} M^A(a-\lambda) \otimes I_{\mathfrak{I}} \right) f, \quad a < 0, \tag{5.5}
\]

exist for each $f \in \text{dom} (I_{H^A} \otimes T)$. Moreover, the following improper-spectral integrals

\[
G(z)f := \int_{\mathbb{R}_+} \left( \gamma^A(z-\lambda) \frac{1}{\sqrt{(M^A)'(a-\lambda)}} \otimes I_{\mathfrak{I}} \right) \hat{E}_T(d\lambda)f, \quad \lambda < 0, \tag{5.6}
\]

\[
M(z)f := \int_{\mathbb{R}_+} (L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{I}}) \hat{E}_T(d\lambda)f
\]

\[
= \int_{\mathbb{R}_+} \hat{E}_T(d\lambda) \left( L^A(z-\lambda, a-\lambda) \otimes I_{\mathfrak{I}} \right) f, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{5.7}
\]

converge for every $f \in H^A \otimes \mathfrak{S}$, where

\[
L^A(z, a) := \frac{1}{\sqrt{(M^A)'(a)}} (M^A(z) - M^A(a)) \frac{1}{\sqrt{(M^A)'(a)}}, \tag{5.8}
\]

\[
z \in \rho(A_0), \quad a \in \mathbb{R}_-.
\]

Proof. (i) First we prove the convergence of the integral in (5.6). It follows from (2.6) that $(M^A)'(z) = \gamma^A(\overline{z})^* \gamma^A(z)$. Hence,

\[
(M^A)'(a-\lambda) = \gamma^A(a-\lambda)^* \gamma^A(a-\lambda), \quad \lambda \in \mathbb{R}_+, \quad a < 0. \tag{5.9}
\]

This identity implies the existence of an isometry $V(a-\lambda)$ mapping $H$ onto $\mathfrak{H}_{a-\lambda}(A)$ and such that

\[
V(a-\lambda) \sqrt{(M^A)'(a-\lambda)} = \gamma^A(a-\lambda), \quad \lambda \in \mathbb{R}_+. \tag{5.10}
\]

Further, in accordance with (2.5)

\[
\gamma^A(z-\lambda) = (A_0 - a + \lambda)(A_0 - z + \lambda)^{-1} \gamma^A(a-\lambda)
= U(a-\lambda, z-\lambda) \gamma^A(a-\lambda), \tag{5.11}
\]

Proof. (i) It is immediate from the definition that $S_0 = S^* \upharpoonright \ker (\Gamma_0^S) = A_0 \otimes I_{\mathfrak{I}} + I_\mathfrak{S} \otimes T$. It remains to apply Proposition 4.4.

Statements (ii) and (iii) are immediate from Theorem 4.1. \hfill \Box
where $U(a - \lambda, z - \lambda) := (A_0 - a + \lambda)(A_0 - z + \lambda)^{-1} \mid \mathfrak{N}_{a - \lambda}(A)$. It is easily checked that $U(a - \lambda, z - \lambda)$ isomorphically maps $\mathfrak{N}_{a - \lambda}(A)$ onto $\mathfrak{N}_{z - \lambda}(A)$. Combining relation (5.11) with (5.10) yields
\[
\|\gamma^A(z - \lambda)((M^A)'(a - \lambda))^{-1/2}\| = \|U(a - \lambda, z - \lambda)V(a - \lambda)\|
\]
\[
= \|I + (z - a)(A_0 - z + \lambda)^{-1}\| \leq \begin{cases} 1 + |z - a| \cdot |\text{Im} z|^{-1}, & z \in \mathbb{C} \setminus \mathbb{R}, \\ 1 + |x - a| \cdot |x|^{-1}, & x \in \mathbb{R}_-. \end{cases}
\]
(5.12)

Here we have taken into account that $|x| \leq |x - \lambda| = |x| + \lambda$. The latter estimate implies boundedness of the integrand in (5.6) for each $z \in \mathbb{C} \setminus \mathbb{R}_+$. It remains to apply Proposition 3.7 with $\alpha = 0$.

(ii) Let us prove that
\[
C(z, a) := \sup_{\lambda \in \mathbb{R}_+} \|L^A(z - \lambda, a - \lambda)\| < \infty \quad \text{for each} \quad z \in \mathbb{C} \setminus \mathbb{R}_+ \quad \text{and} \quad a < 0.
\]
(5.13)

Combining identity (2.6) with (5.11) yields
\[
M^A(z - \lambda) - M^A(a - \lambda) = (z - a)\gamma^A(a - \lambda)^*\gamma^A(z - \lambda)
\]
\[
= (z - a)\gamma^A(a - \lambda)^*U(a - \lambda, z - \lambda)\gamma^A(a - \lambda).
\]
In turn, inserting this identity in (5.8) and using (5.10) one derives
\[
L^A(z - \lambda, a - \lambda) = (z - a) \frac{1}{\sqrt{(M^A)'(a - \lambda)}} \gamma^A(a - \lambda)^* \times 
\]
\[
\times U(a - \lambda, z - \lambda)\gamma^A(a - \lambda) \frac{1}{\sqrt{(M^A)'(a - \lambda)}}
\]
\[
= (z - a)V(a - \lambda)^*U(a - \lambda, z - \lambda)V(a - \lambda), \quad \lambda \in \mathbb{R}_+.
\]
(5.14)

Noting that $V(a - \lambda)$ is an isometry for each $\lambda \in \mathbb{R}_+$ and using estimate (5.12) one arrives at estimate (5.13). To prove convergence of integral (5.7) for each $f \in \mathcal{H}^A \otimes \mathfrak{F}$, it remains to apply Proposition 3.7 with $\alpha = 0$.

(iii) Let us prove convergence of integrals (5.3) and (5.4). Since $A_0 \geq 0$, integral representation (2.7) implies
\[
(M^A)'(a - \lambda) = \int_{\mathbb{R}_+} \frac{d\Sigma_A(t)}{(t - a + \lambda)^2}, \quad \lambda \in \mathbb{R}_+, \quad a < 0.
\]
(5.15)

Using this representation instead of (2.8) one proves the following analog of estimate (2.9)
\[
C_1(1 + |\lambda|^2)^{-1}\text{Im} M(i) \leq (M^A)'(a - \lambda) \leq C_2(1 + |\lambda|^2)\text{Im} M(i),
\]
(5.16)
\[
\lambda \in \mathbb{R}_+. \text{ Combining this estimate with inequality} \int_{\mathbb{R}_+} \lambda^2 d \| \hat{E}_T(\lambda)f \|^2 < \infty \text{ characterizing} \ f \in \text{dom} (I_{\mathcal{H}^A} \otimes T), \text{ and applying Proposition 3.7 with} \ \alpha = 1, \text{ yields convergence of both integrals (5.3) and (5.4).}
that

In accordance with (2.6)

\[ M^A(a - \lambda) = M^A(a) - \lambda \gamma^A(a - \lambda)^* \gamma^A(a). \]  

(5.18)

Combining this identity with (5.9) we derive

\[
\left\| ((M^A)'(a - \lambda))^{-1/2} M^A(a - \lambda) \right\| 
\leq \| M^A(a) \| \cdot \| ((M^A)'(a - \lambda))^{-1/2} \| + |\lambda| \cdot \| V^*(a - \lambda) \cdot \gamma^A(a) \|.
\]  

(5.19)

Noting that \( V(a - \lambda) \) is an isometry and taking (5.16) into account we arrive at estimate (5.17).

\[ \square \]

**Theorem 5.3.** Let \( A \) be a nonnegative densely defined closed symmetric operator, and let \( \Pi_A = \{H^A, \Gamma^A_0, \Gamma^A_1\} \) be a boundary triplet for \( A^* \), \( A_0 := \tilde{A}^* \upharpoonright \ker(\Gamma^A_0) \), and let \( M^A(\cdot) \) and \( \gamma^A(\cdot) \) be the corresponding Weyl function and \( \gamma \)-field, respectively. Let also \( T = T^* \in C(\tilde{\Sigma}) \) be an unbounded nonnegative self-adjoint operator in \( \tilde{\Sigma} \) and \( S := A \otimes I_{\tilde{\Sigma}} + I_\Sigma \otimes T \). Then:

(i) There exists a boundary triplet \( \tilde{\Pi}_S = \{\hat{H}^S, \Gamma^S_0, \Gamma^S_1\} \) for \( S^* \) such that \( \hat{H}^S := H^A \otimes \tilde{\Sigma} \) and \( S_0 := S^* \upharpoonright \ker(\Gamma^S_0) = A_0 \otimes I_{\tilde{\Sigma}} + I_\Sigma \otimes T \). If \( f \in \mathcal{D} := \text{dom}(S^*) \cap \text{dom}(I_\Sigma \otimes T) \subseteq \text{dom}(S^*) \), then \( f \in \text{dom}(S^*) \cap \text{dom}(A^* \otimes I_{\tilde{\Sigma}}) \) and

\[
\begin{align*}
\Gamma^S_0 f &:= \left( \int_{\mathbb{R}^+} \hat{E}_T(d\lambda) \frac{1}{\sqrt{(M^A)'(a - \lambda)}} \otimes I_{\tilde{\Sigma}} \right) (\Gamma^A_0 \hat{\otimes} I_{\tilde{\Sigma}}) f, \\
\Gamma^S_1 f &:= \left( \int_{\mathbb{R}^+} \hat{E}_T(d\lambda) \left( \frac{1}{\sqrt{(M^A)'(a - \lambda)}} M^A(a - \lambda) \otimes I_{\tilde{\Sigma}} \right) \right) \cdot (\Gamma^A_0 \hat{\otimes} I_{\tilde{\Sigma}}) f,
\end{align*}
\]  

(5.20)

where \( a < 0 \).

(ii) The \( \gamma \)-field \( \hat{\gamma}^S(\cdot) \) and Weyl function \( \hat{M}^S(\cdot) \) corresponding to \( \tilde{\Pi}_S \) are given by

\[
\hat{\gamma}^S(z) = G(z) \quad \text{and} \quad \hat{M}^S(z) = M(z), \quad z \in \rho(S_0),
\]  

(5.21)

where \( G(\cdot) \) and \( M(\cdot) \) are defined by (5.6) and (5.7), respectively.

(iii) If \( M^A(\cdot) \) is a scalar-type function, i.e., \( M^A(\cdot) = m^A(\cdot) I_{H^A} \), then representation (4.13) remains true.

**Proof.** (i) First we let \( \Delta_n := [n - 1, n) \), \( \Sigma_n := E_T(\Delta_n) \Sigma \), and \( T_n = T E_T(\Delta_n) \), \( n \in \mathbb{N} := \{1, 2, \ldots\} \). Clearly, \( \tilde{\Sigma} = \bigoplus_{n \in \mathbb{N}} \tilde{\Sigma}_n \) and \( T = \bigoplus_{n \in \mathbb{N}} T_n \). We also put \( S_n := A \otimes I_{\Sigma_n} + I_\Sigma \otimes T_n \in C(\Sigma \otimes \tilde{\Sigma}_n) \). Clearly, each \( T_n \) is bounded and \( \sigma(T_n) \subseteq [n - 1, n] \).
By Theorem 5.1, \( \Pi_{S_n} = \{ \mathcal{H}^{S_n}, \Gamma^{S_n}_0, \Gamma^{S_n}_1 \} := \Pi_A \otimes I_{\mathcal{F}_n} := \{ \mathcal{H}^A \otimes \mathcal{F}_n, \Gamma^A_0 \otimes I_{\mathcal{F}_n}, \Gamma^A_1 \otimes I_{\mathcal{F}_n} \} \) is a boundary triplet for \( S_n^* \) such that

\[
S_{0n} := S^* \mid \ker (\Gamma^{S_n}_0) = A_0 \otimes I_{\mathcal{F}_n} + I_{\mathcal{F}} \otimes T_n, \quad n \in \mathbb{N}.
\]

Let also \( M^{S_n}(\cdot) \) be the corresponding Weyl function. It follows from (5.2) that

\[
(M^{S_n})'(z) = \int_{\Delta_n} ((M^A)'(z - \lambda) \otimes I_{\mathcal{F}_n}) \hat{E}_T(d\lambda), \quad z \in \mathbb{C} \setminus \Delta_n. \tag{5.22}
\]

Since the function \( \varphi(\cdot) = \sqrt{\cdot} \) is continuous on \( \mathbb{R}_+ \), then in accordance with Proposition 3.3(iii) the compositions \( ((M^A)'(a - \lambda))^{1/2} \otimes I_{\mathcal{F}_n} \) and \( ((M^A)'(a - \lambda))^{-1/2} \otimes I_{\mathcal{F}_n} \) are \( \hat{E}_T \)-admissible. Therefore, combining representation (5.22) with Proposition 3.3(iii) yields

\[
R_n := \sqrt{(M^{S_n})'(a)} = \int_{\Delta_n} \left( \sqrt{(M^A)'(a - \lambda)} \otimes I_{\mathcal{F}_n} \right) \hat{E}_T(d\lambda),
\]

\[
R^{-1}_n = \frac{1}{\sqrt{(M^{S_n})'(a)}} = \int_{\Delta_n} \left( \frac{1}{\sqrt{(M^A)'(a - \lambda)}} \otimes I_{\mathcal{F}_n} \right) \hat{E}_T(d\lambda),
\]

\( a < 0 \). Similarly, using representations (5.2) and (5.23) and applying Proposition 3.2 yields

\[
R^{-1}_n M^A_n(a) = \frac{1}{\sqrt{(M^{S_n})'(a)}} M^A_n(a)
\]

\[
= \int_{\Delta_n} \left( \frac{1}{\sqrt{(M^A)'(a - \lambda)}} M^A(a - \lambda) \otimes I_{\mathcal{F}_n} \right) \hat{E}_T(d\lambda), \quad a < 0. \tag{5.24}
\]

Setting \( \mathcal{H}_{S_n} := \mathcal{H}^A \otimes \mathcal{F}_n \),

\[
\tilde{\Pi}_{S_n}^0 = \sqrt{(M^{S_n})'(a)} \Gamma^{S_n}_0 \quad \text{and} \quad \tilde{\Pi}_{S_n}^1 = \frac{1}{\sqrt{(M^{S_n})'(a)}} (\Gamma^{S_n}_1 - M^{S_n}(a)) \Gamma^{S_n}_0 \tag{5.25},
\]

we obtain an ordinary boundary triplet \( \tilde{\Pi}_{S_n} = \{ \mathcal{H}_{S_n}, \tilde{\Pi}_{S_n}^0, \tilde{\Pi}_{S_n}^1 \} \) for \( S_n^* \). Inserting formulas (5.23) and (5.22) in (5.20) yields (5.20) with \( \Delta_n \) in place of \( \mathbb{R}_+ \). Now applying Proposition 2.8 (see formula (2.17)) one gets that the direct sum \( \tilde{\Pi} := \bigoplus_{n \in \mathbb{N}} \tilde{\Pi}_{S_n} \) is an ordinary boundary triplet for \( S^* \). In particular, for any \( f \in \mathcal{D} = \text{dom} (S^*) \cap \text{dom} (A^* \otimes I_{\mathcal{F}}) \)

\[
\tilde{\Pi}^S_0 f := \bigoplus_{n=0}^{\infty} \tilde{\Pi}_{S_n}^0 f
\]

\[
= \bigoplus_{n=0}^{\infty} \left( \int_{[n,n+1]} \left( \sqrt{(M^A)'(a - \lambda)} \otimes I_{\mathcal{F}_n} \right) \hat{E}_{T_n}(d\lambda) \right) \cdot (\Gamma^A_0 \otimes I_{\mathcal{F}_n})f
\]

\[
= \left( \int_{\mathbb{R}_+} \left( \sqrt{(M^A)'(a - \lambda)} \otimes I_{\mathcal{F}} \right) \hat{E}_{T_n}(d\lambda) \right) \cdot (\Gamma^A_0 \otimes I_{\mathcal{F}})f, \tag{5.26}
\]
which proves the first formula in (5.20). Note that convergence of the last integral for every $f \in \mathcal{D}$ (cf. (5.3)) is guaranteed by Lemma 5.2. Formula (5.20) for $\tilde{\Gamma}^S_1$ is proved similarly.

(ii) It easily follows from (5.25) that the Weyl function $\tilde{M}^{S_n}(\cdot)$ corresponding to the triplet $\tilde{\Pi}_{S_n}$ is

$$
\tilde{M}^{S_n}(z) = R_n^{-1} \left( M^{S_n}(z) - M^{S_n}(a) \right) R_n^{-1} = \frac{1}{\sqrt{(M^{S_n})'(a)}} \left( M^{S_n}(z) - M^{S_n}(a) \right) \frac{1}{\sqrt{(M^{S_n})'(a)}}.
$$

Inserting formulas (5.23) and (5.2) into (5.27) and applying Proposition 3.2 we arrive at the following representation

$$
\tilde{M}^{S_n}(z) = \int_{\Delta_n} \left( \frac{1}{\sqrt{(M^{A})'(a - \lambda)}} \right) \times (M^{A}(z - \lambda) - M^{A}(a - \lambda)) \frac{1}{\sqrt{(M^{A})'(a - \lambda)}} \hat{E}_T(d\lambda),
$$

$z \in \mathbb{C}_+$. Finally, applying Proposition 2.8 and taking notation (5.8) into account we arrive at the following formula for the Weyl function $\tilde{M}^S(\cdot)$ corresponding to $\tilde{\Pi}_S$:

$$
\tilde{M}^S(z) f = \bigoplus_{n \in \mathbb{N}} \tilde{M}^{S_n}(z) f = \bigoplus_{n \in \mathbb{N}} \int_{\Delta_n} (L^{A}(z - \lambda, a - \lambda) \otimes I_{\mathcal{T}}) \hat{E}_T(d\lambda) f
$$

$$
= \int_{\mathbb{R}_+} (L^{A}(z - \lambda, a - \lambda) \otimes I_{\mathcal{T}}) \hat{E}_T(d\lambda) f, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
$$

exists for every $f \in \mathcal{H}^A \otimes \mathfrak{T}$ and any $z \in \mathbb{C} \setminus \mathbb{R}_+$. Note that Lemma 5.2 ensures convergence of the last integral for every $f \in \mathcal{H}^A \otimes \mathfrak{T}$. Comparison with (5.7) proves the second equality in (5.21). The first one is extracted by combining the first formula in (2.15) with (5.23) and applying Proposition 3.2. \qed

5.2. Friedrichs’ and Krein’s Extensions of $S := A \otimes I_{\mathcal{T}} + I_{\mathfrak{S}} \otimes T$

In this section we assume that both the symmetric operator $A \in C(\mathfrak{S})$ and the operator $T = T^*$ are nonnegative. Then the set $\text{Ext}_A[0, \infty)$ of nonnegative self-adjoint extensions of $A$ is non-empty (see [2,11,35]). Moreover, according to the Krein result [40] the set $\text{Ext}_A[0, \infty)$ contains two extremal extensions: a maximal nonnegative extension $\hat{A}_F$ (also called Friedrichs’ or hard extension) and a minimal nonnegative extension $\hat{A}_K$ (Krein’s or soft extension). The latter are uniquely determined by the following inequalities

$$(\hat{A}_F + x)^{-1} \leq (\hat{A} + x)^{-1} \leq (\hat{A}_K + x)^{-1}, \quad x \in (0, \infty), \quad \hat{A} \in \text{Ext}_A(0, \infty).$$

(For details we refer the reader to [2,35].)

Recall the following statements.

Proposition 5.4 ([26]). Let $A \geq 0$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$ such that $A_0 = A^* \upharpoonright \ker \Gamma_0 \geq 0$. Let $M(\cdot)$ be the corresponding Weyl function. Then $A_0 = \hat{A}_F$ $(A_0 = \hat{A}_K)$ if and only if
\[
\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty, \quad \lim_{x \uparrow 0} (M(x)f, f) = +\infty, \quad f \in \mathcal{H}\setminus\{0\}. \tag{5.28}
\]

Next we describe the Friedichs’ extension \( \hat{S}_F \) of \( S \) by means of the extension \( \hat{A}_F \) of \( A \). We start with the following simple algebraic lemma.

**Lemma 5.5.** Let \( \{X_k\}_{k=1}^n \) be a sequence of positive definite operators in \( \mathcal{H} \), \( X_k \geq dI_\mathcal{H} > 0 \), \( d > 0 \), and let \( E_T(\cdot) \) be a spectral measure of the self-adjoint operator \( T \in \mathcal{B}(\Sigma) \) which is not necessarily nonnegative. Then for any partition \( \{\Delta_k\}_{k=1}^n \) of \([a, b]\) one has

\[
X := \sum_k X_k \otimes E_T(\Delta_k) \geq dI_\mathcal{H} \otimes \Sigma. \tag{5.29}
\]

**Proof.** Since \( X_k \geq dI_\mathcal{H} > 0 \), the operator \((X_k - dI_\mathcal{H}) \otimes E_T(\Delta_k)\) is nonnegative. Hence,

\[
X = \sum_k X_k \otimes E_T(\Delta_k) \geq d \sum_k I_\mathcal{H} \otimes E_T(\Delta_k)
= dI_\mathcal{H} \otimes \left( \sum_k E_T(\Delta_k) \right) = dI_\mathcal{H} \otimes I_\Sigma = dI_\mathcal{H} \otimes \Sigma.
\]

This inequality proves the result. \( \square \)

**Proposition 5.6.** Let \( A \) be a nonnegative symmetric operator in \( \mathcal{S} \), let \( T = T^* \geq 0 \), and let \( S := A \otimes I_\Sigma + I_\mathcal{S} \otimes T \). Then:

\[
\hat{S}_F = \hat{A}_F \otimes I_\Sigma + I_\mathcal{S} \otimes T \quad \text{and} \quad \hat{S}_K = \hat{A}_K \otimes I_\Sigma + I_\mathcal{S} \otimes T. \tag{5.30}
\]

**Proof.** (i) Assume for the beginning that \( T \) is bounded, \( T \in \mathcal{B}(\Sigma) \). Let \( \Pi_A = \{A, A^*, \Gamma_A, \Gamma_A^*\} \) be a boundary triplet for \( A^* \) such that \( A_0 = A_F \). Then, by Theorem 5.1, \( \Pi_S = \{\mathcal{H}, \Gamma_S, \Gamma_S^*\} := \Pi_A \otimes I_\Sigma \) is a boundary triplet for \( S \), satisfying \( S_0 := S^* \downarrow \ker(\Gamma_S^*) = A_0 \otimes I_\Sigma + I_\mathcal{S} \otimes T \), and the corresponding Weyl function \( M_S(\cdot) \) is given by (5.2).

To prove the first relation in (5.30) it suffices to check condition (5.28) for \( M_S(\cdot) \). Let \( h := \sum_{j=1}^n h_j' \otimes h_j'' \), where \( h_j' \in \mathcal{H}, h_j'' \in \Sigma \), let \( \mathcal{H}_n^A := \text{span}\{h_j' : 1 \leq j \leq n\} \), and let \( P_n \) be the orthogonal projection on \( \mathcal{H}_n^A \in \mathcal{H}^A \).

Since \( A_0 = A_F \), the Weyl function \( M^A(\cdot) \) satisfies condition (5.28). Setting \( M_n^A(\cdot) = P_n M^A(\cdot) \upharpoonright \mathcal{H}_n^A \) we note that due to the compactness of the finite-dimensional ball condition (5.28) is uniform on each \( \mathcal{H}_n^A \). In other words, for each \( N > 0 \) there exists \( x_N < 0 \) such that

\[
-M_n^A(x) \geq N \quad \text{for} \quad x \leq x_N. \tag{5.31}
\]

Since \( A_0 \geq 0 \), Theorem 5.1 ensures that the Weyl function \( M^A(\cdot) \) being holomorphic in \( \mathbb{C} \setminus \mathbb{R}_+ \) admits the integral representation (5.2) for any \( z = x < 0 \) and \( \lambda > 0 \). Let \( \pi = \{\Delta_k\}_{k=1}^p \) be a partition of \( \Delta = [a, b] \), let \( \lambda_k \in \Delta_k \), and let

\[
S_p(\pi) := \sum_{k=1}^p M^A(x_N - \lambda_k) \otimes E_T(\Delta_k) \tag{5.32}
\]
be an integral sum for integral (5.2) with \( x = x_N \). Setting \( Y_k = M_n^A(x_N - \lambda_k) \), \( k \in \{1, \ldots, p\} \), one gets

\[
(P_n \otimes I_{\mathcal{T}})S_p(\pi)h = \sum_{k=1}^{p} \sum_{j=1}^{n} P_n M_n^A(x_N - \lambda_k)h'_j \otimes E_T(\Delta_k)h''_j = \sum_{k=1}^{p} \sum_{j=1}^{n} Y_k h'_j \otimes E_T(\Delta_k)h''_j = \sum_{k=1}^{p} (Y_k \otimes E_T(\Delta_k))h. \tag{5.33}
\]

Combining this relation with (5.31) and noting that \( h \in \mathcal{H}_n^A \otimes \mathcal{I} \) and \( x_N - \lambda_k < x_N \) one gets from Lemma 5.5 that

\[
(S_p(\pi)h, h) = ((P_n \otimes I_{\mathcal{T}})S_p(\pi)h, h) \leq -N.
\]

Passing here to the limit as the diameter \( |\pi| \) of partition \( \pi \) tends to zero and taking formula (5.2) for the Weyl function into account and setting \( M_n^S(\cdot) = (P_n \otimes I_{\mathcal{T}})M(\cdot) |\mathcal{H}_n^A \otimes I_{\mathcal{T}} \), one derives

\[
(M^S(x), h, h) = (M_n^S(x)h, h) \leq -N \quad \text{for} \quad x \leq x_N.
\]

Since finite tensors \( h = \sum_{j=1}^{n} h'_j \otimes h''_j \) are dense in \( \mathcal{H}_n^A \otimes \mathcal{I} \), this inequality yields condition (5.28) for \( M(\cdot) = M^S(\cdot) \) and arbitrary \( h \in \mathcal{H}_n^A \otimes \mathcal{I} \).

(ii) Let \( T \in \mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H}) \). Then \( T \) admits a decomposition

\[
T = \bigoplus_{n \in \mathbb{N}} T_n,
\]

where \( T_n := TE_T[n - 1, n] \in \mathcal{B}(\mathcal{I}_n) \) and \( \mathcal{I}_n := E_T([n - 1, n]) \). Hence,

\[
S = \bigoplus_{n \in \mathbb{N}} S_n \quad \text{where} \quad S_n := A \otimes I_{\mathcal{I}_n} + I_{\mathcal{F}} \otimes T_n. \tag{5.34}
\]

Clearly, \( S_n \) is a nonnegative symmetric operator in \( \mathcal{F} \otimes \mathcal{I}_n \). According to [45, Corollary 3.10]

\[
\hat{S}_F = \bigoplus_{n \in \mathbb{N}} \hat{S}_{n,F} \quad \text{and} \quad \hat{S}_K = \bigoplus_{n \in \mathbb{N}} \hat{S}_{n,K}, \tag{5.35}
\]

where \( \hat{S}_{n,F} \) and \( \hat{S}_{n,K} \) denote the Friedrichs’ and Krein’s extensions of the symmetric nonnegative operator \( S_n \), respectively. Combining representations (5.35) with representations (5.30) with bounded \( T_n \in \mathcal{B}(\mathcal{I}_n) \) in place of \( T \in \mathcal{B}(\mathcal{I}) \) proved at the previous step implies

\[
\hat{S}_F = \bigoplus_{n \in \mathbb{N}} \hat{S}_{n,F} = \bigoplus_{n \in \mathbb{N}} (\hat{A}_F \otimes I_{\mathcal{H}_n} + I_{\mathcal{F}} \otimes T_n) = \hat{A}_F \otimes I_{\mathcal{H}} + \bigoplus_{n \in \mathbb{N}} (I_{\mathcal{F}} \otimes T_n) = \hat{A}_F \otimes I_{\mathcal{H}} + I_{\mathcal{F}} \otimes T.
\]

The representation for \( S_K \) is proved similarly. \( \square \)

Next we are going to discuss semi-bounded extensions of the operator \( S = A \otimes I_{\mathcal{T}} + I_{\mathcal{F}} \otimes T \). It is known that under the conditions of Proposition 5.4 the following implication holds: If \( \hat{A} = \hat{A}^* = A_{\Theta} \) is semi-bounded below then \( \Theta \) is semi-bounded below. The equivalence does not hold in general.
Definition 5.7. Let $A \geq 0$ be a nonnegative symmetric operator in $\mathcal{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$ such that $A_0 = \hat{A}_F$. We say that $A$ satisfies LSB-property (abbreviation of lower semi-boundedness) if the following equivalence holds:

$$A_\Theta = A_{\Theta}^*$$

is lower semi-bounded $\iff\Theta = \Theta^*$ is lower semi-bounded.

To describe the operators with LSB-property we introduce the following definition.

Definition 5.8 ([26]). It is said that $M(\cdot)$ uniformly tends to $-\infty$ (in symbols $M(\cdot) \rightrightarrows -\infty$) if for any $N > 0$ there exists $x_N$ such that

$$M(x)h, h \leq -N \cdot \|h\|^2 \quad \text{for} \quad x \leq x_N, \quad h \in \mathcal{H}.$$  \hspace{1cm} (5.36)

Clearly, (5.36) implies (5.28), but not vice versa.

Proposition 5.9 ([26]). Let $A \geq 0$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$ such that $A_0 = \hat{A}_F$. Then the following statements are equivalent:

(i) $A$ satisfies LSB-property;

(ii) $M(x) \rightrightarrows -\infty$ as $x \to -\infty$.

Proposition 5.10. Let $A$ be a nonnegative symmetric operator in $\mathcal{H}$, and let $\Pi_A = \{\mathcal{H}^A, \Gamma_0^A, \Gamma_1^A\}$ be a boundary triplet for $A^*$ such that $A_0 := A^* \upharpoonright \ker (\Gamma_0^A) = \hat{A}_F$. Let also $\pi = T^* \in B(\mathcal{E})$, $T \geq 0$, and let $S = A \otimes I_{\pi} + I_{\delta} \otimes T$. If $A$ satisfies the LSB-property, then the operator $S$ also satisfies the LSB-property.

Proof. Consider a boundary triplet $\Pi_S = \{\hat{\mathcal{H}}^S, \hat{\Gamma}_0^S, \hat{\Gamma}_1^S\}$ for $S^*$ given by (5.20). By Theorem 5.1(i),

$$S_0 = S^* \upharpoonright \ker (\Gamma_0^S) = \hat{S}_F = \hat{A}_F \otimes I_{\pi} + I_{\delta} \otimes T.$$  

Let also $M^A(\cdot)$ and $\gamma^A(\cdot)$ be the Weyl function and $\gamma$-field, respectively, corresponding to the triplet $\Pi_A$. Since $A$ satisfies the LSB-property and $A_0 = \hat{A}_F$, Theorem 5.9 ensures that the Weyl function $M^A(\cdot)$ tends to $-\infty$ uniformly, i.e., $M^A(x) \rightrightarrows -\infty$ as $x \to -\infty$. In other words, for each $N > 0$ there exists $x_N < 0$ such that $-M^A(x) \geq N$ for $x \leq x_N$.

By Theorem 5.1(i) the Weyl function $M^S(\cdot)$ corresponding to $\Pi_S$ is given by (5.2). Let $\pi = \{\Delta_k\}_{k=1}^p$ be a partition of $\Delta = [a, b]$, and let $\lambda_k \in \Delta_k$. Then applying Lemma 5.5 to the integral sum (5.32) we get

$$-S_p(\pi) = -\sum_{k=1}^p M^A(x_N - \lambda_k) \otimes E_T(\Delta_k) \geq N.$$  \hspace{1cm} (5.37)

Passing here to the limit as $|\pi| \to 0$ one obtains

$$-M^S(x) = \int_{\Delta} \hat{E}_T(d\lambda) (M^A(z - \lambda) \otimes I_{\pi}) \geq N \quad \text{for} \quad x \leq x_N.$$  

The latter amounts to saying that $M^S(x) \rightrightarrows -\infty$ as $x \to -\infty$. By Theorem 5.9 this property implies (in fact is equivalent to) the LSB-property of $S$. \hfill $\Box$
6. Schrödinger Operators and Bosons in 1-D

In what follows, \( T \) is an arbitrary (not necessarily bounded) self-adjoint operator acting on a separable Hilbert space \( \mathcal{F} \). Here we illustrate the considerations above by considering a simple example which is useful in applications in Sect. 7.

### 6.1. Schrödinger Operators on Half-Lines

Let \( v_r \in \mathbb{R}, b \in \mathbb{R} \), and let \( H_r = -\frac{d^2}{dx^2} + v_r \) denote the minimal operator in \( \mathcal{F}_r := L^2(\Delta_r), \Delta_r = (b, \infty) \). Clearly, \( \text{dom}(H_r) = W^2,2(\Delta_r) := \{ f \in W^2_2((b, \infty)) : f(b) = f'(b) = 0 \} \) and \( H_r \) is a closed densely defined symmetric operator with \( n_{+}(H_r) = 1 \). The adjoint operator is given by the same expression \( H_r^* = -\frac{d^2}{dx^2} + v_r \) on the domain \( \text{dom}(H_r^*) = W^2,2(\Delta_r) \). One easily checks that a triplet \( \Pi_{H_r} = \{ \mathcal{H}^{H_r}, \Gamma^{H_r}_0, \Gamma^{H_r}_1 \} \) with

\[
\mathcal{H}^{H_r} := \mathbb{C}, \quad \Gamma^{H_r}_0 f = f(b), \quad \text{and} \quad \Gamma^{H_r}_1 f = f'(b), \quad f \in \text{dom}(H_r^*),
\]

is a boundary triplet for \( H_r^* \). The corresponding \( \gamma \)-field \( \gamma^{H_r}(\cdot) \) and Weyl function \( M^{H_r}(\cdot) \) are given by

\[
(\gamma^{H_r}(z)\xi)(x) = e^{i\sqrt{z-v_r}(x-b)}\xi, \quad \xi \in \mathbb{C}, \quad x \in \Delta_r, \quad z \in \mathbb{C}_{\pm},
\]

and

\[
M^{H_r}(z) = m^{H_r}(z) = i\sqrt{z-v_r}, \quad z \in \mathbb{C}_{\pm}, \tag{6.1}
\]

respectively. The function \( \sqrt{\cdot} \) is defined on \( \mathbb{C} \) with the cut along the positive semi-axis \( \mathbb{R}_+ \). Its branch is fixed by the condition \( \sqrt{-1} = i \). Clearly, the Weyl function \( M^{H_r}(\cdot) \) is a scalar function.

Let us consider the closed densely defined symmetric operator

\[
S_r = H_r \otimes I_{\mathbb{F}} + I_{\mathcal{F}_r} \otimes T \tag{6.2}
\]

on the Hilbert space \( \mathfrak{F}_r := \mathcal{F}_r \otimes \mathbb{F} = L_2(\Delta_r, \mathbb{F}) \). In the following we use the notation \( \tilde{f}(x), x \in \Delta_r \) for elements of \( \mathfrak{F}_r = L_2(\Delta_r, \mathbb{F}) \). In accordance with Theorem 4.8(iii) there is a boundary triplet \( \Pi_{S_r} = \{ \mathcal{H}^{S_r}, \Gamma^{S_r}_0, \Gamma^{S_r}_1 \} \) for \( S_r^* \) such that \( \mathcal{H}^{S_r} = \mathcal{H}^{H_r} \otimes \mathbb{F} = \mathbb{F}, \Gamma^{S_r}_0 \) is a function \( \Gamma^{S_r}_0(\cdot) \) and Weyl function \( M^{S_r}(\cdot) \) are given by

\[
(\gamma^{S_r}\xi)(x) = e^{i\sqrt{z-v_r-T}(x-b)}\frac{1}{\sqrt{\text{Im}(m^{H_r}(i-T))}}\xi, \quad \xi \in \mathbb{F}, \quad x \in \Delta_r,
\]

and

\[
M^{S_r}(z) = \frac{m^{H_r}(z-T) - \text{Re}(m^{H_r}(i-T))}{\text{Im}(m^{H_r}(i-T))}, \quad z \in \mathbb{C}_{\pm}. \tag{6.4}
\]
Of course, the considerations are similar for the interval $\Delta_l = (-\infty, a)$, $a \in \mathbb{R}$. Let $H_l = -\frac{d^2}{dx^2} + v_l$, $v_l \in \mathbb{R}$, with domain $\text{dom}(H_l) := W^{2,2}(_0) \Delta_l$ defined on $\mathcal{H}_l := L^2(\Delta_l, \Sigma)$. One checks that $\Pi_{H_l} = \{H_l^H, \Gamma_{H_l}^H, \Gamma_{H_l}^T\}$,

$$\mathcal{H}_{H_l} := \mathbb{C}, \quad \Gamma_{H_l}^H f = f(a), \quad \text{and} \quad \Gamma_{H_l}^T f = -f'(a), \quad f \in \text{dom}(H_l^*)$$

is a boundary triplet for $H_l^T$. The $\gamma$-field and Weyl function are given by

$$\varphi^{H_l}(z) = e^{i\sqrt{z-v_l(a-x)}} \xi, \quad \xi \in \mathbb{C}, \quad x \in \Delta_l, \quad z \in \mathbb{C}_\pm,$$

and

$$M^{H_l}(z) = m^{H_l}(z) = i\sqrt{z-v_l}, \quad z \in \mathbb{C}_\pm. \quad (6.5)$$

Let us consider the closed densely defined symmetric operator $S_l := H_l \otimes I_\Sigma + I_{\mathcal{H}_l} \otimes T$ acting in $\mathcal{H}_l := \mathcal{H}_l \otimes \Sigma = L^2(\Delta_l, \Sigma)$. As above one finds

$$\begin{align*}
\varphi^{S_l} f &= \sqrt{\text{Im}(m^{H_l}(i-T))} \varphi^T(a), \\
\varphi^{S_l} f &= \frac{1}{\sqrt{\text{Im}(m^{H_l}(i-T))}} \left( -\bar{\varphi^T}(a) - \text{Re}(m^{H_l}(i-T)) \varphi^T(a) \right)
\end{align*} \quad (6.6)$$

$\varphi^T \in \text{dom}(H_l^* \otimes I_\Sigma) \cap \text{dom}(I_{\mathcal{H}_l} \otimes T) = W^{2,2}(\Delta_l, \Sigma) \cap \text{dom}(I_{\mathcal{H}_l} \otimes T) \subseteq \text{dom}(S_l^*)$ as well as

$$\varphi^{S_l}(x) = e^{i\sqrt{x-v_l-T(a-x)}} \frac{1}{\sqrt{\text{Im}(m^{H_l}(i-T))}} \xi, \quad \xi \in \Sigma, \quad x \in \Delta_r,$$

and

$$M^{S_l}(z) = \frac{m^{H_l}(z-T) - \text{Re}(m^{H_l}(i-T))}{\text{Im}(m^{H_l}(i-T))}, \quad z \in \mathbb{C}_\pm. \quad (6.7)$$

### 6.2. Schrödinger Operators on Bounded Intervals

Let $\Delta_c = (a, b)$ and $v_c \in \mathbb{R}$. Consider a minimal Sturm–Liouville operator $H_c$ in $\mathcal{H}_c := L^2(\Delta_c)$ given by

$$(H_c f)(x) = -\frac{d^2}{dx^2} f(x) + v_c f(x), \quad x \in \Delta_c,$$

$f \in \text{dom}(H_c) = \left\{ f \in W^{2,2}(\Delta_c) : f(a) = f(b) = 0 \right\}.$

Clearly, $H_c$ is a closed symmetric operator with the deficiency indices $n_{\pm}(A) = 2$. Its adjoint $H_c^*$ is given by

$$(H_c^* f)(x) = -\frac{d^2}{dx^2} f(x) + v_c f(x), \quad f \in \text{dom}(H_c^*) = W^{2,2}(\Delta_c).$$

Consider the extension (Dirichlet realization) $H_c^D$ of the minimal operator $H_c$ defined by

$$H_c^D = -\frac{d^2}{dx^2} + v_c, \quad \text{dom}(H_c^D) = \left\{ f \in W^{2,2}(\Delta_c) : f(a) = f(b) = 0 \right\}.$$

The Neumann extension (realization) $H_c^N$ is fixed by

$$H_c^N = -\frac{d^2}{dx^2} + v_c, \quad \text{dom}(H_c^N) = \left\{ f \in W^{2,2}(\Delta_c) : f'(a) = f'(b) = 0 \right\}.$$
One easily checks that the triplet $\Pi_{H_c} := \{\mathcal{H}_{H_c}, \Gamma_0^{H_c}, \Gamma_1^{H_c}\}$ with

$$\mathcal{H}_{H_c} := \mathbb{C}^2, \quad \Gamma_0^{H_c} f = \frac{1}{\sqrt{2}} \left( f(a) + f(b) \right), \quad \Gamma_1^{H_c} f = \frac{1}{\sqrt{2}} \left( f'(a) + f'(b) \right),$$

$f \in \text{dom}(H^c)$, is a boundary triplet for $H^c$. Clearly, $H^c_0 = H^c_0 \cap \ker(\Gamma_0^{H_c})$ and $H^c_N = H^c_0 \cap \ker(\Gamma_1^{H_c})$. The corresponding $\gamma$-field $\gamma^{H_c}(\cdot)$ and Weyl function $M^{H_c}(\cdot)$ are given by

$$(\gamma^{H_c}(z)\xi)(x) = \frac{1}{\sqrt{2}} \left( \cos(\sqrt{z-v_c}(x-v)) - \sin(\sqrt{z-v_c}(x-v)) \right), \quad (\xi_1, \xi_2),$$

$z \in \mathbb{C}_\pm, x \in (a,b), \nu := \frac{a+b}{2}, d := \frac{b-a}{2}$, and

$$M^{H_c}(z) = \begin{pmatrix} m_1^{H_c}(z) & 0 \\ 0 & m_2^{H_c}(z) \end{pmatrix}, \quad z \in \mathbb{C}_\pm,$$

where

$$m_1^{H_c}(z) := \sqrt{z-v_c} \tan(\sqrt{z-v_c} d), \quad m_2^{H_c}(z) := -\sqrt{z-v_c} \cot(\sqrt{z-v_c} d), \quad z \in \mathbb{C}_\pm.$$

Notice that the Weyl function $M^{H_c}(\cdot)$ is of quasi-scalar type.

We consider the closed densely defined symmetric operator

$$S_c := H^c_c \otimes I_\Xi + I_{\mathcal{S}_H} \otimes T \quad (6.8)$$

defined on $\mathcal{K}_c := \mathcal{S}_c \otimes \Xi = L^2(\Delta_c, \Xi)$. Elements of $L^2(\Delta_c, \Xi)$ are denoted by $\tilde{f}(x), x \in \Delta_c$. Obviously, the self-adjoint operators $S_c^D := H^c_c \otimes I_\Xi + I_{\mathcal{S}_H} \otimes T$ and $S_c^N := H^c_c \otimes I_\Xi + I_{\mathcal{S}_H} \otimes T$ are self-adjoint extensions of $S_c$.

Let us introduce the subspaces $\mathcal{H}^{H_c}_1 := \mathbb{C}$ and $\mathcal{H}^{H_c}_2 := \mathbb{C}$. Notice that $\mathcal{H}^{H_c} = (\mathcal{H}^{H_c}_1 \oplus \mathcal{H}^{H_c}_2)$. It follows from (4.34) that there is a boundary triplet $\Pi_{S_c} = \{\mathcal{H}^{S_c}, \Gamma_0^{S_c}, \Gamma_1^{S_c}\}$ for $S_c^*$ such that

$$\mathcal{H}^{S_c} = \mathcal{H}^{H_c} \otimes \Xi \oplus \mathcal{H}^{H_c}_1 \otimes \Xi \oplus \mathcal{H}^{H_c}_2$$

and

$$\Gamma_0^{S_c} \tilde{f} = \frac{1}{\sqrt{2}} \left( \sqrt{\text{Im}(m_1^{H_c}(i-T))(\tilde{f}(a) + \tilde{f}(b))} \right),$$

$$\Gamma_1^{S_c}(z) \tilde{f} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{\text{Im}(m_1^{H_c}(i-T))}} \left( \tilde{f}(a) - \tilde{f}(b) - \text{Re}(m_1^{H_c}(i-T))(\tilde{f}(a) + \tilde{f}(b)) \right) \right),$$

$$\frac{1}{\sqrt{\text{Im}(m_2^{H_c}(i-T))}} \left( \tilde{f}(a) + \tilde{f}(b) - \text{Re}(m_2^{H_c}(i-T))(\tilde{f}(a) - \tilde{f}(b)) \right),$$

where

$$m_1^{H_c}(z) := \sqrt{z-v_c} \tan(\sqrt{z-v_c} d), \quad m_2^{H_c}(z) := -\sqrt{z-v_c} \cot(\sqrt{z-v_c} d), \quad z \in \mathbb{C}_\pm.$$
\( \tilde{f} \in \text{dom}(H^*_c \otimes I_\Xi) \cap \text{dom}(I_{\delta_c} \otimes T) = W^{2,2}(\Delta_c, \Xi) \cap \text{dom}(I_{\delta_c} \otimes T) \subseteq \text{dom}(S^*_c) \). From (4.35) we get the \( \gamma \)-field \( \gamma^{S_c}(\cdot) : (\mathcal{H}^{S_c} \oplus \mathcal{H}^{S_c}_2)^t \rightarrow \mathbb{R} \),

\[
(\gamma^{S_c}(z)\tilde{\xi})(x) = \frac{\cos(\sqrt{z-T-v_c(x-\nu)})}{\sqrt{2} \cos(\sqrt{z-T-v_c d}) \sqrt{\text{Im}(m^{H_c}(i-T))}} \xi_1
- \frac{\sin(\sqrt{z-T-v_c(x-\nu)})}{\sqrt{2} \sin(\sqrt{z-T-v_c d}) \sqrt{\text{Im}(m^{H_c}(i-T))}} \xi_2,
\]

\( z \in \mathbb{C}_\pm \). Finally, from (4.36) the Weyl function \( M^{S_c}(\cdot) : (\mathcal{H}^{S_c} \oplus \mathcal{H}^{S_c}_2)^t \rightarrow (\mathcal{H}^{S_c} \oplus \mathcal{H}^{S_c}_2)^t \) is computed by

\[
M^{S_c}(z) = \begin{pmatrix}
\frac{m^{H_c}(z-T) - \text{Re}(m^{H_c}(i-T))}{\text{Im}(m^{H_c}(i-T))} & 0 \\
0 & \frac{m^{H_c}(z-T) - \text{Re}(m^{H_c}(i-T))}{\text{Im}(m^{H_c}(i-T))}
\end{pmatrix}, \quad z \in \mathbb{C}_\pm.
\]

**Remark 6.1.** Sturm–Liouville operators \( S_c \) with operator-valued potential \( T = T^* \in C(\Xi) \) have first been treated on a finite interval in the pioneering paper by M.L. Gorbachuk [28]. Clearly, the corresponding minimal operator \( S_c \) admits representation (6.8). In particular, a boundary triplet for \( S^*_c \) was first constructed in [28] (see also [29]). A construction of a boundary triplet for \( S^*_r \) in the case of semi-axis has first been proposed in [26, Section 9]. However, our construction (6.3) of the boundary triplet for \( S^*_c \) is borrowed from [45] where a representation of \( S \) as a direct sum \( S = \bigoplus_j S_j \) with \( S_j := H_c \otimes I_\Xi + I_{\delta_H} \otimes T_j \) and bounded \( T_j = T^*_j \) in place of unbounded \( T = T^* \) was first proposed and the regularization procedure for direct sums was invented and applied to the operator \( S_r \).

After appearance of the work [28] the spectral theory of self-adjoint and dissipative extensions of \( S_r \) in \( L^2(\Delta_c, \Xi) \) has intensively been investigated. The results are summarized in [29, Chapter 4] where one finds, in particular, criteria for discreteness of the spectra, asymptotic formulas for the eigenvalues, resolvent comparability results, etc. Spectral properties of self-adjoint extensions of \( S_r \) have been investigated in [45], see also [44]. In particular, a criterion for all self-adjoint extensions of \( S_r \) to have absolutely continuous nonnegative spectra has also been obtained there.

**Remark 6.2.** A similar treatment of the Dirac operator on a half-line as well as on bounded intervals can also be done (see [12,19]).

### 7. A Model for Electronic Transport Through a Boson Cavity

Let us propose a simple model describing the electronic transport through an optical cavity, cf. [47,48]. We consider the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+) \), where \( \mathbb{R}_- = (-\infty, 0) \) and \( \mathbb{R}_+ = (0, \infty) \). On the subspaces \( \mathcal{H}_l := L^2(\mathbb{R}_-) \) and \( \mathcal{H}_r := L^2(\mathbb{R}_+) \) we consider the closed symmetric operators \( H_l = -\frac{d^2}{dx^2} + v_l \) and \( H_r = -\frac{d^2}{dx^2} + v_r \) of Sect. 6.1. We set \( \mathcal{H} := \mathcal{H}_l \oplus \mathcal{H}_r = L^2(\mathbb{R}) \) and \( A := H_l \oplus H_r \).
Notice that $A$ can be regarded as the symmetric operator
\[ A = -\frac{d^2}{dx^2} + v(x), \quad v(x) := \begin{cases} v_l & x \in \mathbb{R}_- \\ v_r & x \in \mathbb{R}_+ \end{cases} \]
with domain $\text{dom} (A) = W^{2,2}_0(\mathbb{R}) := \{ f \in W^{2,2}(\mathbb{R}) : f(0) = f'(0) = 0 \}$. The operator $A$ is symmetric and has deficiency indices $n_\pm(A) = 2$. For simplicity, we assume that
\[ 0 \leq v_r \leq v_l. \]
We consider the extension $A^D = H^D_i \oplus H^D_r$, where $H^D_i$ and $H^D_r$ are the extensions of $H_i$ and $H_r$, respectively, with Dirichlet boundary conditions at zero. One easily checks that the triple $\Pi_A = \{ \mathcal{H}^A, \Gamma_0^A, \Gamma_1^A \}$ with
\[
\mathcal{H}^A := \mathcal{H}^H_i \oplus \mathcal{H}^H_r \supseteq \mathbb{C}, \quad \Gamma_0^A := \begin{pmatrix} f(-0) \\ f'(0) \end{pmatrix}, \quad \Gamma_1^A := \begin{pmatrix} -f'(-0) \\ f'(0) \end{pmatrix}, \quad (7.1)
\]
defines a boundary triplet for $A^*$, cf. Sect. 6.1. The Weyl function $M^A(z)$ corresponding to the boundary triplet $\Pi_A$ is given by
\[
M^A(z) = \begin{pmatrix} m^{H_i}(z) & 0 \\ 0 & m^{H_r}(z) \end{pmatrix} = \begin{pmatrix} i\sqrt{z - v_l} & 0 \\ 0 & i\sqrt{z - v_r} \end{pmatrix}, \quad z \in \rho(H^D),
\]
where $A^D = A_0 := A^* \upharpoonright \ker (\Gamma_0^A)$. Any self-adjoint extension $\tilde{A} \in \text{Ext}_A$ disjoint with $A_0$ is given by formula (2.3) with a boundary operator $B = B^* \in \mathbb{C}^{2 \times 2}$, i.e.,
\[
A_B := A^* \upharpoonright \ker (\Gamma_1^A - B\Gamma_0^A), \quad \text{where} \quad B = \begin{pmatrix} \alpha & \gamma \\ \overline{\gamma} & \beta \end{pmatrix} = B^*, \quad (7.2)
\]
cf. Proposition 2.3(iii), $\alpha, \beta \in \mathbb{R}$. Using formulas (7.1) one rewrites the extension $A_B$ by means of the boundary conditions
\[
\begin{aligned}
f'(-0) &= -\alpha f(-0) - \gamma f'(0) \\
f'(0) &= \gamma f(-0) + \beta f'(0)
\end{aligned}, \quad f \in \text{dom} (A^*) = W^{2,2}(\mathbb{R}_-) \oplus W^{2,2}(\mathbb{R}_+).
\]
If the matrix $B$ is diagonal, then there is no coupling between the left and right quantum systems, i.e., the operator $A_B$ decomposes into a direct sum of two self-adjoint operators acting on $\mathcal{H}_i$ and $\mathcal{H}_r$, respectively. If $\gamma \neq 0$, then in some sense the left and right systems interact.

Let us view the point zero as a quantum dot or quantum cavity. In particular, the Hilbert space $\mathcal{H}^A = \mathbb{C}^2$ is viewed as the state space of the quantum dot and the self-adjoint operator $B$ as the Hamiltonian of the dot. The Hamiltonian $B$ describes a two-level system to which we are going to couple bosons. The state space of the bosons is the Hilbert space $\mathfrak{F} = l_2(\mathbb{N}_0)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The boson operator $T$ is given by
\[
T\xi = T\{\xi_k\}_{k \in \mathbb{N}_0} = \{k\xi_k\}_{k \in \mathbb{N}_0},
\]
\[
\xi = \{\xi_k\}_{k \in \mathbb{N}_0} \in \text{dom} (T) := \{\{\xi_k\}_{k \in \mathbb{N}_0} \in l_2(\mathbb{N}_0) : \{k\xi_k\}_{k \in \mathbb{N}_0} \in l_2(\mathbb{N}_0)\}.\]
The Hamiltonian \( T \) describes a system of bosons which do not interact mutually. The number of bosons is not fixed and varies from zero to infinity. The Hilbert space \( \mathcal{F} \) has a natural basis given by \( e_k = \{ \delta_{kj} \}_{j \in \mathbb{N}_0} \). Let us introduce the creation and annihilation operators \( b^* \) and \( b \), respectively, defined by

\[
b^* e_k = \sqrt{k + 1} e_{k+1}, \quad k \in \mathbb{N}_0, \quad \text{and} \quad b e_k = \sqrt{k} e_{k-1}, \quad k \in \mathbb{N}_0,
\]

where \( e_{-1} = 0 \). One easily checks that \( T = b^* b \).

Let us consider the compound system \( \{ \mathcal{H}^B, C \} \) consisting of the two-level quantum system \( \{ \mathcal{H}^A, B \} \), with \( B \) given by (7.2), and of the boson system \( \{ l_2(\mathbb{N}_0), T \} \). Its state space is given by \( \mathcal{H}^C := \mathcal{H}^A \otimes l_2(\mathbb{N}_0) = l_2(\mathbb{N}_0, \mathcal{H}^A) = l_2(\mathbb{N}_0, \mathbb{C}^2) \), and the compound Hamiltonian is

\[
C := B \otimes I_\mathcal{F} + I_{\mathcal{H}^A} \otimes T, \quad \text{dom} \ (C) = \text{dom} \ (I_{\mathcal{H}^A} \otimes T). \quad (7.3)
\]

The Hamiltonian \( C \) does not describe any interaction between the system \( \{ \mathcal{H}^A, B \} \) and the bosons. To introduce such an interaction we consider the so-called Jaynes–Cummings Hamiltonian, cf. [34]. To this end, we introduced the eigenvalues \( \lambda^B_0 \) and \( \lambda^B_1 \) of \( B \) and the corresponding normalized eigenvectors \( e^B_0 \) and \( e^B_1 \). Therefore, \( B \) admits the representation

\[
B = \lambda^B_0 (\cdot, e^B_0) e^B_0 + \lambda^B_1 (\cdot, e^B_1) e^B_1,
\]

where it is assumed that \( \lambda^B_0 < \lambda^B_1 \). Let us define the matrices

\[
\sigma^B_+ e^B_0 = e^B_1, \quad \sigma^B_+ e^B_1 = 0 \\
\sigma^B_- e^B_0 = 0, \quad \sigma^B_- e^B_1 = e^B_0.
\]

Clearly,

\[
B = \lambda^B_1 \sigma^B_+ \sigma^B_- + \lambda^B_0 \sigma^B_- \sigma^B_+ \quad (7.4)
\]

We set

\[
V_{JC} := \sigma^B_+ \otimes b + \sigma^B_- \otimes b^*, \quad \text{dom} \ (V_{JC}) = \text{dom} \ (I_{\mathcal{H}^A} \otimes \sqrt{T}) \quad (7.5)
\]

and define the Jaynes–Cummings Hamiltonian \( C_{JC} \) by setting

\[
C_{JC} = B \otimes I_\mathcal{F} + I_{\mathcal{H}^A} \otimes T + \tau V_{JC}, \quad \tau \in \mathbb{R}. \quad (7.6)
\]

One easily checks that the perturbation \( V_{JC} \) is infinitesimally small with respect to \( I_{\mathcal{H}^A} \otimes T \), i.e., \( V_{JC} \) is \( I_\mathcal{F} \otimes T \)-bounded with zero \( I_\mathcal{F} \otimes T \)-bound. In particular, this yields that \( C_{JC} \) with \( \text{dom} \ (C_{JC}) = \text{dom} \ (I_{\mathcal{H}^A} \otimes T) \) is self-adjoint.

The Jaynes–Cummings model (JC-model) is a theoretical model in quantum optics. It describes the system of a two-level atom characterized by the Hamiltonian \( B \) interacting with a quantized mode of an optical cavity described by the Hamiltonian \( T \), with or without the presence of light (in the form of a bath of electromagnetic radiation that can cause spontaneous emission and absorption). One easily checks that

\[
V_{JC}(e_0^B \otimes e_n) = \sqrt{n}(e_1^B \otimes e_{n-1}) \quad \text{and} \quad V_{JC}(e_1^B \otimes e_n) = \sqrt{n+1}(e_0^B \otimes e_{n+1}).
\]

From the physical point of view this means that in the first case the electron jumps into the state \( e_1^B \) with the higher energy \( \lambda^B_1 > \lambda^B_0 \) by absorbing a boson.
In the second case the electron jumps into the state $e_0^B$ with lower energy $\lambda_0^B$ by emitting a boson. This model was originally proposed in 1963 by Edwin Jaynes and Fred Cummings in [34].

Let us consider a closed symmetric operator

$$S := A \otimes I_{\mathcal{F}} + I_{\mathcal{S}} \otimes T$$

in the Hilbert space $\mathcal{K} := \mathcal{H} \otimes \mathcal{X}$. Setting

$$\mathcal{H}_l := \mathcal{H}_l \otimes \mathcal{X}, \quad S_l := H_l \otimes I_{\mathcal{F}} + I_{\mathcal{S}_l} \otimes T,$$

$$\mathcal{H}_r := \mathcal{H}_r \otimes \mathcal{X}, \quad S_r := H_r \otimes I_{\mathcal{F}} + I_{\mathcal{S}_r} \otimes T,$$

we obtain

$$\mathcal{K} = \mathcal{H}_l \oplus \mathcal{H}_r \quad \text{and} \quad S = S_l \oplus S_r.$$  

We propose to define the operator $\tilde{S} = \tilde{S}^*$ describing the point contact of the quantum system $\{\mathcal{H}, A_B\}$ with $A_B$ given by (7.2), to the boson reservoir $\{\mathcal{X}, T\}$ by treating $C_{JC}$ defined by (7.6) as the boundary operator with respect to a triplet $\Pi_S = \Pi_A \widehat{\otimes} I_{\mathcal{F}}$, i.e., by setting $\tilde{S} := S_{C_{JC}}$,

$$S_{C_{JC}} := S^* \upharpoonright \text{dom} (S_{C_{JC}}),$$

$$\text{dom} (S_{C_{JC}}) := \{ f \in \mathcal{D} : (\Gamma_1^A \widehat{\otimes} I_{\mathcal{F}}) f = C_{JC}(\Gamma_0^A \widehat{\otimes} I_{\mathcal{F}}) f \}, \quad (7.7)$$

where $\mathcal{D} = \text{dom} (S^*) \cap \text{dom} (I_{\mathcal{S}} \otimes T)$. The arising model is called the JCL-model (Jaynes–Cummings-leads model), see also [47, 48] for a similar model. In contrast to the traditional JC-model in quantum optics now additionally leads are coupled to the JC-model which makes it a model either for solar cells or for light-emitting diodes.

From the physical point of view the JCL-model can be interpreted as follows. The left- and right-hand systems $\{\mathcal{H}_l, S_l\}$ and $\{\mathcal{H}_r, S_r\}$, where $S_l := H_l \otimes I_{\mathcal{F}} + I_{\mathcal{S}_l} \otimes T$ and $S_r := H_r \otimes I_{\mathcal{F}} + I_{\mathcal{S}_r} \otimes T$, are viewed as leads for electrons, see Landauer–Büttiker [18, 41], where each electron can be accompanied by a number of bosons. The leads are coupled to the JC-model $\{\mathcal{H}^A \widehat{\otimes} \mathcal{X}, C_{JC}\}$ by virtue of the $\Gamma$-maps of the boundary triplet $\Pi_S = \{\mathcal{H}^A \widehat{\otimes} \mathcal{X}, \Gamma_0^A \widehat{\otimes} I_{\mathcal{F}}, \Gamma_1^A \widehat{\otimes} I_{\mathcal{F}}\}$ which shows that the coupling is really point-like.

However, Definition (7.7) is only heuristic because the triplet $\Pi_S = \Pi_A \widehat{\otimes} I_{\mathcal{F}} = \Pi_{H_l} \widehat{\otimes} I_{\mathcal{F}} \oplus \Pi_{H_r} \widehat{\otimes} I_{\mathcal{F}}$ is, in general, not a boundary triplet for $S^*$ in the sense of Definition 2.2 whenever $T$ is unbounded. Therefore, it is unclear whether the extension $S_{C_{JC}}$ of $S$ given by (7.7) is self-adjoint. To get a boundary triplet for $S^*$ one should regularize the triplet $\Pi_S$ in accordance with Theorem 4.8. This leads to the boundary triplet $\tilde{\Pi}_S = \{\mathcal{H}^S, \Gamma_0^S, \Gamma_1^S\} = S_{\mathcal{S}_l} \oplus S_{\mathcal{S}_r}$, where $\Pi_{S_l}$ and $\Pi_{S_r}$ are the normalized boundary triplets defined by (6.3) and (6.6), respectively. The corresponding Weyl function $M^S(\cdot)$ is

$$M^S(z) = \begin{pmatrix} M^{S_l}(z) & 0 \\ 0 & M^{S_r}(z) \end{pmatrix}, \quad z \in \rho (S_0), \quad S_0 = S_l^D \oplus S_r^D,$$

where $M^{S_l}(\cdot)$ and $M^{S_r}(\cdot)$ are given by (6.4) and (6.7), respectively. Moreover, $S_l^D = H_l^D \otimes I_{\mathcal{F}} + I_{\mathcal{S}_l} \otimes T$ and $S_r^D = H_r^D \otimes I_{\mathcal{F}} + I_{\mathcal{S}_r} \otimes T$. 
With respect to regularized boundary triplet \( \tilde{\Pi}_S \) extension (7.7) (in fact, its closure) is given by \( \tilde{S} = S_{\tilde{\mathcal{C}}_{JC}} \) where, in accordance with formulas (2.14), the boundary operator \( \tilde{\mathcal{C}}_{JC} \) of the extension \( \tilde{S} \) is determined by

\[
\tilde{\mathcal{C}}_{JC} := R^{-1}(C_{JC} - Q)R^{-1}, \quad \text{dom}(\tilde{\mathcal{C}}_{JC}) = \text{dom}(T^{3/2}),
\]

(7.8)

with

\[
Q := \begin{pmatrix} Q_l & 0 \\ 0 & Q_r \end{pmatrix} = \begin{pmatrix} \text{Re}(m^{H_i}(i - T)) & 0 \\ 0 & \text{Re}(m^{H_r}(i - T)) \end{pmatrix}
\]

(7.9)

and

\[
R := \begin{pmatrix} R_l & 0 \\ 0 & R_r \end{pmatrix} = \begin{pmatrix} \sqrt{\text{Im}(m^{H_i}(i - T))} & 0 \\ 0 & \sqrt{\text{Im}(m^{H_r}(i - T))} \end{pmatrix},
\]

(7.10)

where \( m^{H_r}(\cdot) \) and \( m^{H_i}(\cdot) \) are given by (6.1) and (6.5). Let us check that the operator \( \tilde{\mathcal{C}}_{JC} \) is well defined on \( \text{dom}(\tilde{\mathcal{C}}_{JC}) = \text{dom}(T^{3/2}) \oplus \text{dom}(T^{3/2}) \) and self-adjoint. Setting

\[
\tilde{C} := R^{-1}(C - Q)R^{-1} \quad \text{and} \quad \tilde{V}_{JC} := R^{-1}V_{JC}R^{-1},
\]

(7.11)

where \( C \) is given by (7.3), we arrive at the following representation of the boundary operator (7.8):

\[
\tilde{\mathcal{C}}_{JC} = \tilde{B} + \tilde{T} + \tau \tilde{V}_{JC} = \tilde{C} + \tau \tilde{V}_{JC}.
\]

Here

\( \tilde{C} := \tilde{B} + \tilde{T} \) and \( \tilde{B} := R^{-1}(B \otimes I_\mathcal{F})R^{-1}, \ \tilde{T} := R^{-1}(I_{\mathcal{H}^A} \otimes T - Q)R^{-1} \).

It follows with account of (7.9) and (7.10) that

\[
\tilde{T} := R^{-1}(I_{\mathcal{H}^A} \otimes T - Q)R^{-1}
\]

\[
= \begin{pmatrix} R^{-1}_l(T - Q_l)R^{-1}_l & 0 \\ 0 & R^{-1}_r(T - Q_r)R^{-1}_r \end{pmatrix}
\]

(7.12)

Let us introduce the operators

\[
Z_l := \sqrt{I_\mathcal{F} + (T + v_l)^2 + T + v_l} \geq I_\mathcal{F}, \quad Z_r := \sqrt{I_\mathcal{F} + (T + v_r)^2 + T + v_r} \geq I_\mathcal{F}.
\]

(7.13)

Clearly, these operators are self-adjoint and \( \text{dom}(Z_l) = \text{dom}(Z_r) = \text{dom}(T^{1/2}) \).

It follows from (7.10) with account of (6.1) and (6.5) that

\[
R = \frac{1}{\sqrt{2}} \begin{pmatrix} Z_l^{-1/2} & 0 \\ 0 & Z_r^{-1/2} \end{pmatrix} \quad \text{and} \quad Q = -\frac{1}{\sqrt{2}} \begin{pmatrix} Z_l & 0 \\ 0 & Z_r \end{pmatrix}.
\]

(7.14)

Combining these relations with (7.12) yields

\[
\tilde{T} = \begin{pmatrix} \sqrt{2}TZ_l + Z_l^2 & 0 \\ 0 & \sqrt{2}TZ_r + Z_r^2 \end{pmatrix}.
\]

It is easily seen that the operator \( \tilde{T} \) is well defined on the domain \( \text{dom}(\tilde{T}) = \text{dom}(T^{3/2}) \oplus \text{dom}(T^{3/2}) \) and self-adjoint, i.e., \( \tilde{T} = \tilde{T}^* \). Moreover, inequalities
(7.13) imply \( \tilde{T} \geq I_{H^A \otimes \tau} \). Further, combining formula (7.2) for the matrix \( B \) with formula (7.14) for \( R \) implies

\[
\tilde{B} = R^{-1}(B \otimes I_{\tau})R^{-1} = \sqrt{2} \begin{pmatrix} \alpha Z_{l} & \gamma Z_{l}^{1/2}Z_{r}^{1/2} \\ \gamma Z_{l}^{1/2}Z_{l}^{1/2} & \beta Z_{r} \end{pmatrix}.
\]

This operator is symmetric on the natural domain \( \text{dom} \left( T^{1/2} \right) \oplus \text{dom} \left( T^{1/2} \right) \). Moreover, one checks that \( \tilde{B} \) is infinitesimally small with respect to \( \tilde{T} \), because \( T^{1/2} \) is infinitesimally small with respect to \( T^{3/2} \). Hence, by the Kato–Rellich theorem, the operator \( \tilde{\mathcal{C}} := \tilde{T} + \tilde{B} \) is self-adjoint on the domain \( \text{dom} \left( T^{3/2} \right) \oplus \text{dom} \left( T^{3/2} \right) \). Furthermore, a straightforward computation shows that the operator \( \tilde{V}_{JC} \) is symmetric on \( \text{dom} \left( T \right) \). Moreover, it follows from (7.5) and (7.11) that the operator \( \tilde{V}_{JC} \) is infinitesimally small with respect to \( \tilde{B} + \tilde{T} \). Therefore, by the Kato–Rellich theorem, the operator \( \tilde{C}_{JC} = \tilde{B} + \tilde{T} + \tau \tilde{V}_{JC} \) is well defined on \( \text{dom} \left( T^{3/2} \right) \oplus \text{dom} \left( T^{3/2} \right) \) and self-adjoint for any \( \tau \in \mathbb{R} \).

According to formulas (2.14) and (4.29) the mappings \( \tilde{\Gamma}^{S}_{0} \) and \( \tilde{\Gamma}^{S}_{1} \) for the boundary triplet \( \tilde{\Pi}^{S} := \{ H^{S}, \tilde{\Gamma}^{S}_{0}, \tilde{\Gamma}^{S}_{1} \} \) read as follows

\[
\tilde{\Gamma}^{S}_{0} f = R(\Gamma^{A}_{0} \otimes I_{\tau})f \quad \text{and} \quad \tilde{\Gamma}^{S}_{1} f = R^{-1}(\Gamma^{A}_{0} \otimes I_{\tau} - Q \Gamma^{H}_{0} \otimes I_{\tau})f, \quad (7.15)
\]

for \( f \in \mathcal{D} = \text{dom} \left( S^{*} \right) \cap \text{dom} \left( I_{\tilde{S}} \otimes T \right) \). Hence, the Hamiltonian \( \tilde{S} \) describing the contact to the reservoir is given by

\[
\tilde{S} = S_{\tilde{C}_{JC}} := S^{*} \upharpoonright \text{dom} \left( S_{\tilde{C}_{JC}} \right), \\
\text{dom} \left( S_{\tilde{C}_{JC}} \right) := \{ f \in \text{dom} \left( S^{*} \right) : \tilde{\Gamma}^{S}_{1} f = \tilde{C}_{JC} \tilde{\Gamma}^{S}_{0} f \}. \quad (7.16)
\]

Inserting (7.15) into (7.16) one rewrites the proceeding relation as

\[
\text{dom} \left( S_{\tilde{C}_{JC}} \right) \cap \mathcal{D} = \{ f \in \mathcal{D} : \Gamma^{H}_{1} \otimes I_{\tau} f = C_{JC} \Gamma^{H}_{0} \otimes I_{\tau} f \} \quad (7.17)
\]

which coincides with (7.7). Thus, Hamiltonian \( S_{\tilde{C}_{JC}} \) is just a mathematically rigorous representation of heuristically defined Hamiltonian (7.7).

The system \( \{ \mathfrak{A}, S_{\tilde{C}_{JC}} \} \) can be regarded as the Jaynes–Cummings model coupled to leads. It describes the electronic transport through a dot or cavity where the electrons interact with bosons. Notice that the interaction of electrons to bosons is restricted only to one point.

\textbf{Remark 7.1.} Let us make some comments.

(i) The system \( \{ \mathfrak{A}, S_{0} \} \), \( S_{0} := S^{*} \upharpoonright \ker (\Gamma^{S}_{0}) = A^{D} \otimes I_{\tau} + I_{\mathcal{B}_{H}} \otimes T \) describes a situation where the left system \( \{ \mathfrak{A}_{l}, S^{D}_{l} \} \), \( S^{D}_{l} := S^{*}_{l} \upharpoonright \ker (\Gamma^{S}_{0}) \), and right system \( \{ \mathfrak{A}_{r}, S^{D}_{r} \} \), \( S^{D}_{r} := S^{*}_{r} \upharpoonright \ker (\Gamma^{S}_{0}) \) are completely decoupled.

(ii) If \( \tau = 0 \), then \( \tilde{C}_{JC} = \tilde{C} \). The system \( \{ \mathfrak{A}, S_{\tilde{C}} \} \) describes a situation where the left and right systems \( \{ \mathfrak{A}_{l}, S^{D}_{l} \} \) and \( \{ \mathfrak{A}_{r}, S^{D}_{r} \} \) are coupled by a point interaction at zero, but not by a boson–electron interaction. In particular, if \( B \) is diagonal, then again the left and right systems \( \{ \mathfrak{A}_{l}, S^{D}_{l} \} \) and \( \{ \mathfrak{A}_{r}, S^{D}_{r} \} \) are decoupled.

(iii) If \( \tau \neq 0 \), then the system \( \{ \mathfrak{A}, S_{\tilde{C}_{JC}} \} \) can be viewed as fully coupled: The left and right systems are coupled by point interaction at zero as well as by a boson–electron interaction at zero. Since the systems \( \{ \mathfrak{A}_{l}, S^{D}_{l} \} \)
and \{\mathfrak{r}, S_r^D\} are considered as leads in the Landauer–Büttiker theory [18,41], see also [5,20], and \{\mathcal{C}^2 \otimes \mathfrak{T}, C_{\text{JC}}\} is the Jaynes–Cummings model of quantum optics the whole system \{\mathfrak{r}, S_{\text{JC}}\} can be seen as the coupling of leads to the Jaynes–Cummings model (JC-model). So it makes sense to call the whole system \{\mathfrak{r}, S_{\text{JC}}\} the Jaynes–Cummings–leads model (JCL-model), see also [1,33].

(iv) If \(\tau \neq 0\) and \(B\) is diagonal, then the left and right systems are only coupled by the boson–electron interaction at zero.

(v) The model above can be viewed as a simple model of a solar cell or a light-emitting diode (LED).

(vi) Using the result of [12] one can introduce a similar model where the Schrödinger operators \(H_l\) and \(H_r\) are replaced by Dirac operators \(D_l\) and \(D_r\), respectively.

(vii) Finally, we mention that the self-adjoint operators \(C_{\text{JC}}\) and \(\tilde{C}_{\text{JC}}\) have Jacobi structure in a basis formed by the orthogonal systems \(\{e^B_0 \otimes e_k, e^B_1 \otimes e_k\}_{k \in \mathbb{N}_0}\) in \(\mathcal{H}^A \otimes \mathfrak{T}\). To see this we note first that the subspaces spanned by \(\{e^B_0 \otimes e_{k+1}, e^B_1 \otimes e_k\}_{k \in \mathbb{N}_0}\) leave the self-adjoint operators \(T+\tau V_{\text{JC}}\) and \(\tilde{T}+\tau \tilde{V}_{\text{JC}}\) invariant, respectively. Second, the operators \(B \otimes I_\mathfrak{T}\) and \(\tilde{B}\) leave the subspace spanned by the systems \(\{e^B_0 \otimes e_k\}_{k \in \mathbb{N}_0}\) and \(\{e^B_1 \otimes e_k\}_{k \in \mathbb{N}_0}\) invariant, respectively. Both statements ensure the Jacobi structure of both operators \(C_{\text{JC}}\) and \(\tilde{C}_{\text{JC}}\).

(viii) There are several models describing the boson-assisted electron transport through a cavity or a quantum dot, see [1,33,39,52]. However, in the present paper for the first time a cavity is considered to which leads are coupled point-like. A similar model was considered in [47,48]; however, in contrast to here lead Hamiltonians were “discrete” Schrödinger operators. The present choice of “continuous” Schrödinger operators is more realistic and closer to the ideas of Landauer and Büttiker, see [18,41]. From experience one knows that models with point interactions can be analyzed to a large extent explicitly. In a subsequent paper we will see that this rule confirms for the JCL-model as well.

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