The Bohr Inequality for the Generalized Cesáro Averaging Operators

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Abstract. The main aim of this paper is to prove a generalization of the classical Bohr theorem and as an application, we obtain a counterpart of Bohr theorem for the generalized Cesáro operator.

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1. Introduction and Preliminaries

This work is connected with one of classical results known as Bohr’s theorem for the class \( B \) of analytic self mappings of the unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \). Harold Bohr’s initial result of 1914 has sharpened by several prominent mathematicians. Since then it has been a source of investigations in numerous other function spaces. The Bohr theorem in its final form says the following.

**Theorem A** [11, H. Bohr, 1914]. If \( f \in B \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then \( \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \) for \( r \leq 1/3 \) and the constant \( 1/3 \) cannot be improved.

The constant \( 1/3 \) in this theorem is called the Bohr radius. Few other proofs of this result were also given (see [29]). It is also true [24] that \( \sum_{n=0}^{\infty} |a_n| (1/3)^n = 1 \) if and only if \( f \) is a constant function. However, there are a lot of generalizations and extensions of this theorem (cf. [12–14,29]). The interest on this topic was revived due to the discovery of extensions to domains in \( \mathbb{C}^n \) and to more general abstract setting in various contexts, due mainly to works of Aizenberg, Boas, Khavinson, and others (cf. [2–4,8,10,25]). In [2,4], multidimensional analogues of Bohr’s inequality in which the unit disk \( \mathbb{D} \) is replaced by a domain in \( \mathbb{C}^n \) were considered. One can also find some information about it, for example, in the survey by Abu-Muhanna et al. [6], [15, Chapter 8] and the monograph [21].
Another widely discussed problem is the investigation of the asymptotical behaviour of the Bohr sum. In this connection, a natural question is to ask for the best constant $C(r) \geq 1$ such that for $f \in \mathcal{B}$ we have

$$\sum_{n=0}^{\infty} |a_n| r^n \leq C(r).$$

Indeed, Bombieri [12] proved that

$$\sum_{n=0}^{\infty} |a_n| r^n \leq \frac{3 - \sqrt{8(1 - r^2)}}{r} \quad \text{for } 1/3 \leq r \leq 1/\sqrt{2}.$$

Later in [13], Bombieri and Bourgain proved that

$$\sum_{n=0}^{\infty} |a_n| r^n < \frac{1}{\sqrt{1 - r^2}} \quad \text{for } r > 1/\sqrt{2}$$

so that $C(r) \asymp (1 - r^2)^{-1/2}$ as $r \to 1$. In the same paper they also obtained a lower bound. Namely, they proved that for $\varepsilon > 0$ there exists a constant $c = c(\varepsilon)$ such that

$$\sum_{n=0}^{\infty} |a_n| r^n \geq (1 - r^2)^{-1/2} - \left(c \log \frac{1}{1 - r}\right)^{3/2+\varepsilon} \quad \text{as } r \to 1.$$

Some recent results on the topic can be found in [7,17–19,22,23,26,27].

The article is organized as follows. First, we consider a natural generalization of Theorem A and make it applicable to many situations (see Theorem 1). Secondly, in Sect. 3 as an application, we investigate a convolution counterpart of Bohr radius and also the operator counterpart of the so-called one parameter family of averaging Cesáro operator $C_1^\alpha f$, discussed for example in [1,28]. Finally, in Theorem 6, we discuss asymptotic Bohr radius for $C_1^1 f$.

2. Bohr Radius in General Form

Let $\{\varphi_k(r)\}_{k=0}^{\infty}$ be a sequence of nonnegative continuous functions in $[0,1)$ such that the series $\sum_{k=0}^{\infty} \varphi_k(r)$ converges locally uniformly with respect to $r \in [0,1)$.

**Theorem 1.** Let $f \in \mathcal{B}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $p \in (0,2]$. If

$$\varphi_0(r) > \frac{2}{p} \sum_{k=1}^{\infty} \varphi_k(r) \quad \text{for } r \in [0,R),$$

where $R$ is the minimal positive root of the equation

$$\varphi_0(x) = \frac{2}{p} \sum_{k=1}^{\infty} \varphi_k(x),$$

then the following sharp inequality holds:

$$B_f(\varphi,p,r) := |a_0|^p \varphi_0(r) + \sum_{k=1}^{\infty} |a_k| \varphi_k(r) \leq \varphi_0(r) \quad \text{for all } r \leq R.$$
In the case when \( \varphi_0(x) < \frac{2}{p} \sum_{k=1}^{\infty} \varphi_k(x) \) in some interval \((R, R + \varepsilon)\), the number \( R \) cannot be improved. If the functions \( \varphi_k(x) \) \((k \geq 0)\) are smooth functions then the last condition is equivalent to the inequality

\[
\varphi'_0(R) < \frac{2}{p} \sum_{k=1}^{\infty} \varphi'_k(R)
\]

**Proof.** For \( f \in \mathcal{B} \), an application of Schwarz–Pick lemma gives the inequality

\[
|a_k| \leq 1 - |a|^2
\]

for all \( k \geq 1 \) and thus, we get that

\[
\mathcal{B}_f(\varphi, p, r) \leq |a_0|^p \varphi_0(r) + (1 - |a_0|^2) \sum_{k=1}^{\infty} \varphi_k(r)
\]

\[
= \varphi_0(r) + (1 - |a_0|^2) \left[ \sum_{k=1}^{\infty} \varphi_k(r) - \left( \frac{1 - |a_0|^p}{1 - |a_0|^2} \right) \varphi_0(r) \right]
\]

\[
\leq \varphi_0(r) + (1 - |a_0|^2) \left[ \sum_{k=1}^{\infty} \varphi_k(r) - \frac{p}{2} \varphi_0(r) \right]
\]

\[
\leq \varphi_0(r), \quad \text{by Eq. (1),}
\]

for all \( r \leq R \), by the definition of \( R \). This proves the desired inequality (2). In the third inequality above, we have used the following fact:

\[
A(x) = \frac{1 - x^p}{1 - x^2} \geq \frac{p}{2} \quad \text{for all } x \in [0, 1)
\]

and there is nothing to prove for \( p = 2 \). This inequality is easy to verify. Indeed,

\[
A'(x) = -\frac{xM(x)}{(1 - x^2)^2}, \quad M(x) = (2 - p)x^p + px^{p-2} - 2,
\]

and, since \( M'(x) = -p(2 - p)x^{p-3}(1 - x^2) < 0 \) for \( x \in (0, 1) \) and for each \( p \in (0, 2) \), it follows that \( M(x) > M(1) = 0 \) and thus, \( A(x) \) is decreasing on \([0, 1)\). Hence \( A(x) \geq \lim_{x \to 1^-} A(x) = p/2 \), as desired.

Now let us prove that \( R \) is an optimal number. We consider the function

\[
f(z) = \frac{z - a}{1 - az}
\]

with \( a \in [0, 1) \). For this function we have
\[ |a_0|^p \varphi_0(r) + \sum_{k=1}^{\infty} |a_k| \varphi_k(r) \]
\[ = a^p \varphi_0(r) + (1 - a^2) \sum_{k=1}^{\infty} a^{k-1} \varphi_k(r) \]
\[ = \varphi_0(r) + (1 - a) \left[ 2 \sum_{k=1}^{\infty} a^{k-1} \varphi_k(r) - p \varphi_0(r) \right] \]
\[ - (1 - a) \left[ (1 - a) \sum_{k=1}^{\infty} a^{k-1} \varphi_k(r) + \left( \frac{1 - a^p}{1 - a} - p \right) \varphi_0(r) \right] \]
\[ = \varphi_0(r) + (1 - a) \left[ 2 \sum_{k=1}^{\infty} a^{k-1} \varphi_k(r) - p \varphi_0(r) \right] + O((1 - a)^2) \]
as \( a \to 1^- \). Now it is easy to see that the last number is \( > \varphi_0(r) \) when \( a \) is close to 1. The proof of the theorem is complete. □

**Remark 1.** Clearly for \( p > 2 \), we see that \( 1 \leq A(x) < p/2 \) for \( x \in [0, 1) \), and thus, in this case, Theorem 1 holds by replacing the factor \( 2/p \) in Eq. (1) by 1 and also at the other three places in the statement. The most important cases are at \( p = 1, 2 \).

**Example 1.** Suppose that \( f \in \mathcal{B} \), \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( p \in (0, 2] \). Then Theorem 1 gives the following:

1. For \( \varphi_k(r) = r^k (k \geq 0) \), we easily have (see [9, Proposition 1.4] and [26, Remark 1])
\[ |a_0|^p + \sum_{k=1}^{\infty} |a_k| r^k \leq 1 \quad \text{for} \quad r \leq R_1(p) = \frac{p}{2 + p} \]
and the constant \( R_1(p) \) cannot be improved. The case \( p = 1 \) is the classical Bohr inequality. The case \( p = 2 \) is due to [24] and the inequality in this case does play a special role. We remark that for \( p > 2 \), \( R_1(p) \) should be taken as 1/2.

2. For \( \varphi_k(r) = (k+1)r^k (k \geq 0) \), we easily have the sharp inequality
\[ |a_0|^p + \sum_{k=1}^{\infty} (k+1)|a_k| r^k \leq 1 \quad \text{for} \quad r \leq R_2(p) = 1 - \sqrt{\frac{2}{2 + p}}. \]

3. For \( \varphi_0(r) = 1 \) and \( \varphi_k(r) = k^\alpha r^k (k \geq 1) \), the condition (1) reduces to \( p \geq 2 \sum_{k=1}^{\infty} k^{\alpha} r^k \). In particular, as
\[ \sum_{k=1}^{\infty} k r^k = \frac{r}{(1 - r)^2} \quad \text{and} \quad \sum_{k=1}^{\infty} k^2 r^k = \frac{r(1 + r)}{(1 - r)^3}, \]
it can be easily seen that the following sharp inequalities (with \( \alpha = 1, 2 \)) hold:
\[ |a_0|^p + \sum_{k=1}^{\infty} k|a_k| r^k \leq 1 \quad \text{for} \quad r \leq R_3(p) = \frac{p + 1 - \sqrt{2p + 1}}{p} \]
\[ |a_0|^p + \sum_{k=1}^{\infty} k^2 |a_k|^r^k \leq 1 \quad \text{for } r \leq R_4(p), \]

where \( R_4(p) \) is the minimal positive root of the equation \( p(1-r)^3 - 2r(1+r) = 0 \).

### 3. Convolution Counterpart of Bohr Radius

For two analytic functions \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) in \( \mathbb{D} \), we define the Hadamard product (or convolution) \( f \ast g \) of \( f \) and \( g \) by the power series

\[
(f \ast g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in \mathbb{D}.
\]

Clearly, \( f \ast g = g \ast f \).

As an application of Theorem 1 we consider first the convolution operator of the form

\[
(F \ast f)(z) = \sum_{k=0}^{\infty} \gamma_k a_k z^k,
\]

where \( F(z) := 2F_1(a, b; c; z) = F(a, b; c; z) \) denotes the Gaussian hypergeometric function defined by the power series expansion

\[
F(z) = \sum_{k=0}^{\infty} \gamma_k z^k, \quad \gamma_k = \frac{(a)_k (b)_k}{(c)_k (1)_k}.
\]

Clearly, \( F(a, b; c; z) \) is analytic in \( \mathbb{D} \) and in particular, \( F(a, 1; 1; z) = (1-z)^{-a} \).

Here \( a, b, c \) are complex numbers such that \( c \neq -m, m = 0, 1, 2, \ldots \), and \( (a)_k \) is the shifted factorial defined by Appel’s symbol

\[
(a)_k := a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad k \in \mathbb{N},
\]

and \( (a)_0 = 1 \) for \( a \neq 0 \). In the exceptional case \( c = -m, m = 0, 1, 2, \ldots \), \( F(a, b; c; z) \) is defined if \( a = -j \) or \( b = -j \), where \( j = 0, 1, 2, \ldots \) and \( j \leq m \). It is clear that if \( a = -m \), a negative integer, then \( F(a, b; c; z) \) becomes a polynomial of degree \( m \) in \( z \).

**Theorem 2.** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) belong to \( \mathcal{B} \) and \( p \in (0, 2] \). Assume that \( a, b, c > -1 \) such that all \( \gamma_k \) have the same sign for \( k \geq 1 \). Then

\[ |a_0|^p + \sum_{k=1}^{\infty} |\gamma_k| |a_k|r^k \leq 1 \quad \text{for all } r \leq R, \]

where \( R \) is the minimal positive root of the equation \( |F(a, b; c; x) - 1| = p/2 \), and the number \( R \) cannot be improved.
Proof. We apply Theorem 1. Set \( \varphi_k(r) = |\gamma_k|r^k \) and remark that \( \gamma_0 = 1 \). Let us also note that all \( \gamma_k \) have the same sign. Therefore, we have

\[
|F(a, b; c; r) - 1| = \sum_{k=1}^{\infty} \varphi_k(r).
\]

Now the statement of Theorem 1 concludes the proof. \( \square \)

Sometimes the Bohr radius can be found explicitly. For instance, let us set \( b = c = 1 \). In this case, we have

\[
F(z) = (1 - z)^{-a} \quad \text{and hence,}
\]

\[
|F(a, b; c; r) - 1| = (1 - r)^{-a} - 1 = \frac{p}{2}, \quad \text{i.e., } R = 1 - \left( \frac{2}{2 + p} \right)^{1/a}
\]

which in the cases \( a = 1 \) and \( p = 1 \) coincide with the classical Bohr radius. Note that the case \( a = 2 \) is dealt also in Example 1(2).

4. \( \alpha \)-Cesáro Operators

For any \( \alpha \in \mathbb{C} \) with \( \Re \alpha > -1 \), we consider

\[
\frac{1}{(1 - z)^{\alpha + 1}} = \sum_{k=0}^{\infty} A_k^\alpha z^k, \quad A_k^\alpha = \frac{(\alpha + 1)_n}{(1)_n}.
\]

Next, by comparing the coefficient of \( z^n \) on both sides of the identity

\[
\frac{1}{(1 - z)^{\alpha + 1}} \cdot \frac{1}{1 - z} = \frac{1}{(1 - z)^{\alpha + 2}},
\]

it follows that

\[
A_{n+1}^\alpha = \sum_{k=0}^{n} A_k^\alpha, \quad \text{i.e., } \frac{1}{A_{n+1}^\alpha} \sum_{k=0}^{n} A_k^\alpha = 1. \quad (3)
\]

With this principle, the Cesáro operator of order \( \alpha \) or \( \alpha \)-Cesáro operator (see Stempak [28]) on the space of analytic functions \( f \) in the unit disk \( \mathbb{D} \) is therefore defined by

\[
\mathcal{C}^\alpha f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{A_{n+1}^\alpha} \sum_{k=0}^{n} A_k^\alpha a_k \right) z^n, \quad (4)
\]

where \( f(z) = \sum_{k=0}^{\infty} a_k z^k \). In terms of convolution, we can write this as

\[
\mathcal{C}^\alpha f(z) = \frac{f(z)}{(1 - z)^{\alpha + 1}} \ast F(1, 1; \alpha + 2; z)
\]

and thus, we have (cf. [1,28]) the following integral form

\[
\mathcal{C}^\alpha f(z) = (\alpha + 1) \int_{0}^{1} f(tz) \frac{(1 - t)^{\alpha}}{(1 - tz)^{\alpha + 1}} \, dt,
\]

where \( \Re \alpha > -1 \). This for \( \alpha = 0 \) gives

\[
\mathcal{C}^0 f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} \sum_{k=0}^{n} a_k \right) z^n,
\]
which is simply the classical Cesáro operator considered by Hardy–Littlewood in 1932 [16]. Several authors have studied the boundedness property of these operators on different function spaces (see, for example, [5]). In [20], the present authors established a theorem giving an analog of Bohr theorem for the classical Cesáro operator $C^0 f$. It also considered the asymptotical behaviour of the Bohr sum in this case.

For $f \in B$, we define a counterpart of Bohr sum for $\alpha > -1$ as

$$C^\alpha_f(z) := \sum_{n=0}^{\infty} \left( \frac{1}{A_n^{\alpha+1}} \sum_{k=0}^{n} A_{n-k}^\alpha |a_k| \right) r^n.$$ 

Before writing a counterpart of Bohr theorem we need the following estimate.

**Theorem 3.** For $f \in B$ and $\alpha > -1$, we have

$$|C^\alpha_f(z)| \leq (\alpha + 1) \Phi(r, 1, \alpha + 1) = \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \frac{t^\alpha}{1-t} \, dt,$$

where $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n (n+a)^{-s}$ is the Lerch transcedent function.

**Proof.** We may represent $C^\alpha_f(z)$ as

$$C^\alpha_f(z) = (\alpha + 1) \int_0^1 \frac{1}{(t-1)z+1} f \left( \frac{tz}{(t-1)z+1} \right) (1-t)^\alpha \, dt$$

and thus,

$$|C^\alpha_f(z)| \leq (\alpha + 1) \int_0^1 \frac{(1-t)^\alpha}{(t-1)z+1} \left| f \left( \frac{tz}{(t-1)z+1} \right) \right| \, dt$$

$$\leq (\alpha + 1) \int_0^1 \frac{(1-t)^\alpha}{(t-1)r+1} \, dt = C^\alpha_{f_0}(r), \quad f_0(z) = 1.$$

This integral is not easy to calculate and therefore, we may return to the standard series representation and obtain

$$C^\alpha_{f_0}(r) = \sum_{n=0}^{\infty} \frac{A_n^\alpha}{A_n^{\alpha+1}} r^n = (\alpha + 1) \sum_{n=0}^{\infty} \frac{r^n}{\alpha + n + 1},$$

and the proof is complete. \qed

Now we are ready to prove the counterpart of Bohr theorem for the $\alpha$-Cesáro operator.

**Theorem 4.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to $B$ and $\alpha > -1$. Then

$$C^\alpha_f(r) \leq (\alpha + 1) \sum_{n=0}^{\infty} \frac{r^n}{n + \alpha + 1} = \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \frac{t^\alpha}{1-t} \, dt \text{ for all } r \leq R, \quad (5)$$
where $R = R(\alpha)$ is the minimal positive root of the equation
\[
3(1 + \alpha) \sum_0^\infty \frac{x^n}{n + \alpha + 1} = \frac{2}{1 - x}, \quad \text{i.e.,} \quad \sum_0^\infty \frac{\alpha + 1 - 2n}{n + \alpha + 1} x^n = 0.
\]

The number $R$ cannot be replaced by a larger constant. Note that $R(0) = 0.5335$.

Proof. We apply Theorem 1 with $p = 1$. If we write
\[
C^\alpha(f)(z) = \sum_0^\infty a_n \phi_n(z),
\]
then collecting the terms involving only $a_n$ in the right hand side of (4) we find that
\[
\phi_n(z) = \sum_k^\infty \frac{A_k^n}{A_k^\alpha} z^k,
\]
so that for the $\alpha$-Cesáro operators $C^\alpha(f)$ we have
\[
\varphi_0(x) = \sum_k^\infty \frac{A_k^n}{A_k^\alpha} x^k = (\alpha + 1) \sum_k^\infty \frac{x^k}{k + \alpha + 1}, \quad x \in [0, 1),
\]
by the definition of $A_k^n$. Moreover, by setting $f(z) = 1/(1 - z)$ in (6), it is not difficult to find from (3) and (4) that
\[
\sum_0^\infty \phi_n(r) = C^\alpha \left( \frac{1}{1 - r} \right) = \frac{1}{1 - r}
\]
and thus, Eq. (1) for $p = 1$ takes the form
\[
3\varphi_0(r) > 2 \sum_0^\infty \phi_n(r), \quad \text{i.e.,} \quad 3(1 + \alpha) \sum_0^\infty \frac{r^n}{n + \alpha + 1} > \frac{2}{1 - r}.
\]
The desired inequality (5) follows from Theorem 1 and the sharpness part also follows. \[\square\]

In what follows $pF_q$ represent the generalized hypergeometric function defined by
\[
pF_q(a_1, \ldots, a_p; c_1, \ldots, c_q; z) = \sum_0^\infty \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n} \frac{z^n}{n!}.
\]
We remark that in the interesting case where $p = q + 1$, the series converges for $|z| < 1$. If $\text{Re} \left( \sum_{j=1}^q c_j - \sum_{j=1}^{q+1} a_j \right) > 0$, then $q+1F_q$ converges also at the point $z = 1$. 
Theorem 5. For $f \in B$ and $\alpha > -1$, the inequality $C_f^\alpha(r) \leq S_\alpha(r)$ holds for all $r \in [0,1)$, where $S_\alpha(r)$ is equal to

\[
\begin{align*}
\frac{\alpha + 1}{1 - r^2} \sqrt{\frac{\Gamma''(1 + \alpha)}{\Gamma(1 + \alpha)} - \left(\frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)}\right)^2} - r^2 \Phi(r^2, 2, 1 + \alpha) \\
\frac{1}{1 - r^2} \sqrt{3 F_2(1, 1, 1; 2 + \alpha, 2 + \alpha; 1) - r^2 3 F_2(1, 1, 1; 2 + \alpha, 2 + \alpha; r^2)} \\
\frac{1}{1 - r^2} \sqrt{3 F_2(1, 1, 1; 1.5, 1.5; r^2)}
\end{align*}
\]

for $\alpha \geq 0$, for $-\frac{1}{2} < \alpha < 0$, and for $\alpha = -\frac{1}{2}$, respectively. Here $\Phi(z, s, a)$ is the Lerch transcendent function.

Proof. First of all, we represent $C_f^\alpha(r)$ as

\[
C_f^\alpha(r) = \sum_{n=0}^{\infty} \frac{1}{A_{n+1}^{\alpha}} \left(\sum_{k=0}^{n} A_{n-k}^{\alpha} |a_k|\right) r^n = \sum_{n=0}^{\infty} |a_n| \phi_n(r),
\]

where $\phi_n(r)$ is defined by (7). Using the triangle inequality and the fact that $\sum_{n=0}^{\infty} |a_n|^2 \leq 1$ (for $f \in B$), we can estimate

\[
C_f^\alpha(r) \leq \sqrt{\sum_{n=0}^{\infty} |a_n|^2} \cdot \sqrt{\sum_{n=0}^{\infty} \phi_n^2(r)} \leq \sqrt{\sum_{n=0}^{\infty} \phi_n^2(r)}.
\]

For $\alpha > -1/2$, we use the triangle inequality one more time and obtain

\[
\phi_n^2(r) = \left(\sum_{k=n}^{\infty} \frac{A_{k-n}^{\alpha}}{A_{k+1}^{\alpha}} r^k\right)^2 \leq \sum_{k=n}^{\infty} \left(\frac{A_{k-n}^{\alpha}}{A_{k+1}^{\alpha}}\right)^2 \sum_{k=n}^{\infty} r^{2k} = \frac{r^{2n}}{1 - r^2} \sum_{k=n}^{\infty} \left(\frac{A_{k-n}^{\alpha}}{A_{k+1}^{\alpha}}\right)^2.
\]

To estimate the second term on the right, we first observe that

\[
A_k^{\alpha} = \left(\frac{k + 1}{\alpha + k + 1}\right) A_{k+1}^{\alpha} \quad \text{for all } \alpha > -1 \text{ and } k \geq 0.
\]

(8)

First we see that $A_k^{\alpha} \leq A_{k+1}^{\alpha}$ for all $\alpha \geq 0$ and $k \geq 0$. As a consequence, one can see that for $\alpha > 0$ and for $k \geq n$,

\[
\frac{A_{k-n}^{\alpha}}{A_k^{\alpha+1}} \leq \frac{A_{k-n}^{\alpha}}{A_{k}^{\alpha+1}} = \frac{1 + \alpha}{1 + k + \alpha}
\]

which gives

\[
\phi_n^2(r) \leq \frac{r^{2n}}{1 - r^2} \sum_{k=n}^{\infty} \frac{(\alpha + 1)^2}{(1 + k + \alpha)^2}
\]
so that
\[ \sum_{n=0}^{\infty} \phi_n^2(r) \leq \frac{(\alpha + 1)^2}{1 - r^2} \sum_{n=0}^{\infty} r^{2n} \sum_{k=n}^{\infty} \frac{1}{(1 + k + \alpha)^2} \]
\[ = \frac{(\alpha + 1)^2}{1 - r^2} \sum_{k=0}^{\infty} \frac{1}{(1 + k + \alpha)^2} \sum_{n=0}^{k} r^{2n} \]
\[ = \frac{(\alpha + 1)^2}{(1 - r^2)^2} \sum_{k=0}^{\infty} \frac{1}{(1 + k + \alpha)^2} \sum_{n=0}^{k} r^{2n} \]
\[ = \frac{(\alpha + 1)^2}{(1 - r^2)^2} \left( \frac{\Gamma''(1 + \alpha)}{\Gamma(1 + \alpha)} \left( \frac{\Gamma'(1 + \alpha)}{\Gamma(1 + \alpha)} \right)^2 - r^2 \Phi(r^2, 2, 1 + \alpha) \right). \quad (9) \]

Secondly, by (8), we find that
\[ A_{\alpha}^k \leq A_{\alpha}^{k+1} \] for all \(-1 < \alpha < 0\) and \(k \geq 0\), and thus,
\[ \frac{A_{\alpha}^{k-n}}{A_{\alpha}^{k+1}} \leq \frac{A_{\alpha}^0}{A_{\alpha}^{k+1}} = \frac{1}{A_{\alpha}^{k+1}} = \frac{(1)_k}{(\alpha + 2)_k} \quad (10) \]
which gives
\[ \phi_n^2(r) \leq \frac{r^{2n}}{1 - r^2} \sum_{k=n}^{\infty} \left( \frac{(1)_k}{(\alpha + 2)_k} \right)^2. \]

Hence, as before, we can easily deduce that
\[ \sum_{n=0}^{\infty} \phi_n^2(r) \leq \frac{1}{(1 - r^2)^2} \sum_{k=0}^{\infty} \left( \frac{(1)_k}{(\alpha + 2)_k} \right)^2 (1 - r^{2(k+1)}) \]
\[ = \frac{1}{(1 - r^2)^2} \left( 3 F_2 \left( 1, 1, 2 + \alpha, 2 + \alpha; 1 \right) - r^2 3 F_2 \left( 1, 1, 2 + \alpha, 2 + \alpha; r^2 \right) \right) \]
and the last expression converges for \(-0.5 < \alpha < 0\).

Thirdly, for \(\alpha = -0.5\) we use the inequality (10) and obtain
\[ \frac{A_{-1/2}^{k-n}}{A_{1/2}^{k-1/2}} \leq \frac{A_{-1/2}^0}{A_{1/2}^{k-1/2}} = \frac{1}{A_{1/2}^{k-1/2}} = \frac{(1)_n}{(3/2)_n} - \frac{\sqrt{\pi} \Gamma(n + 1)}{2 \Gamma(n + 3/2)}. \]
In this case
\[ \sum_{n=0}^{\infty} \phi_n^2(r) \leq \sum_{n=0}^{\infty} \frac{\pi \Gamma^2(n + 1)}{4 \Gamma^2(n + 1.5)} \left( \sum_{k=n}^{\infty} r^k \right)^2 = \frac{\pi}{4(1 - r^2)} \sum_{n=0}^{\infty} \frac{\Gamma^2(n + 1)}{\Gamma^2(n + 1.5)} r^{2n} \]
\[ = \frac{1}{(1 - r^2)^2} 3 F_2 \left( 1, 1, 1.5, 1.5; r^2 \right). \]

Finally, for \(\alpha < -0.5\), we write the quotient as
\[ \frac{A_{\alpha}^{k-n}}{A_{\alpha}^{k+1}} = \frac{(\alpha + 1)_{k-n}}{(1)_{k-n}} \cdot \frac{(1)_k}{(\alpha + 2)_k} \]
and estimate $\phi_n^2(r)$ using the triangle inequality in the following way:

$$\phi_n^2(r) = \left( \sum_{k=n}^{\infty} \frac{(\alpha + 1)_{k-n}}{(1)_{k-n}} \cdot \frac{(1)_k}{(\alpha + 2)_k} r^k \right)^2 \leq \sum_{k=n}^{\infty} \left( \frac{(\alpha + 1)_{k-n}}{(1)_{k-n}} \right)^2 \sum_{k=n}^{\infty} \left( \frac{(1)_k}{(\alpha + 2)_k} \right)^2 r^{2k}$$

$$= 2F_1(\alpha + 1, \alpha + 1; 1; 1) \sum_{k=n}^{\infty} \left( \frac{(1)_k}{(\alpha + 2)_k} \right)^2 r^{2k},$$

where the first sum in the second step converges to

$$2F_1(\alpha + 1, \alpha + 1; 1; 1) = \frac{\Gamma(-1 - 2\alpha)}{\Gamma^2(-\alpha)}$$

for $-1 < \alpha < -0.5$.

Here we have used the well-known formula

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} < \infty \quad \text{for } c > a + b.$$ 

To estimate the other series, we use the well-known inequality

$$\frac{(1)_n}{(s)_n} \leq \frac{\Gamma(s)}{(n + 1)^{s-1}},$$

which holds for any natural $n$ and $0 \leq s \leq 1$. Therefore,

$$\frac{(1)_k}{(\alpha + 2)_k} = \left( \frac{\alpha + 1}{\alpha + k + 1} \right) \frac{(1)_k}{(\alpha + 1)_k} \leq \frac{\Gamma(\alpha + 2)}{(\alpha + k + 1)} \frac{1}{(k + 1)^\alpha}.$$ 

It follows that

$$\sum_{n=0}^{\infty} \phi_n^2(r) \leq 2F_1(\alpha + 1, \alpha + 1; 1; 1) \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(1)_{k-n}}{(k + \alpha + 1)^2} r^{2k}$$

$$\leq \frac{\Gamma(1 - 2\alpha)\Gamma^2(\alpha + 2)}{\Gamma^2(-\alpha)} \sum_{n=0}^{\infty} \frac{(n + 1)^{1-2\alpha}}{(n + \alpha + 1)^2} r^{2n}.$$

This finishes the proof. \qed

5. Asymptotic Bohr Radius for $C^1_f$

Let us study the order of the estimate for $\alpha = 1$. We recall the following equality:

$$Li_2(x) + Li_2(1 - x) = \frac{\pi^2}{6} - \log x \log(1 - x).$$
Then the estimate looks like

$$\frac{1}{1-r^2} \cdot 2\sqrt{\frac{\pi^2}{6} - \frac{\text{Li}_2(r^2)}{r^2}}$$

$$= 2\sqrt{\frac{\pi^2}{6} - \frac{1}{r^2} \left( \frac{\pi^2}{6} - 2 \log r \log(1 - r^2) - \text{Li}_2(1 - r^2) \right)} \frac{1}{1-r^2},$$

where $\text{Li}_2(z) = \sum_{k=1}^{\infty} (1/k^2)z^k$ is a polylogarithm function. Moreover,

$$\text{Li}_2(1 - r^2) \to 0 \quad \text{and} \quad \log(1 - r) = 1 \quad \text{as} \quad r \to 1,$$

and so we obtain

$$C_j^1(r) \leq 2\sqrt{2 \log r \log(1 - r^2) + o(1)} \sim \sqrt{2} \cdot \sqrt{\log \frac{1}{1-r^2}}.$$

We shall prove

**Theorem 6.** There exists an $f \in B$ such that

$$C_j^1(r) \sim \frac{4\sqrt{2q}}{(3 + q)\sqrt{1-r}} \approx \frac{1.47217}{\sqrt{1-r}},$$

where $q \approx 7.57736 \ldots$ is the root of the equation $3q = (3 + q) \log(1 + q)$.

**Proof.** Consider the function $\phi_n(r)$ defined by (7) with $\alpha = 1$ and calculate it using definition of $A_k^\alpha$. This gives that

$$\phi_n(r) = \sum_{k=n}^{\infty} A_k^\alpha r^k = 2 \sum_{k=n}^{\infty} \frac{k - n + 1}{(k+2)(k+1)} r^k.$$

As

$$\frac{r^{k+2}}{(k+2)(k+1)} = \int_0^r \int_0^\rho s^k \, ds \, d\rho,$$

it follows that

$$\phi_n(r) = \frac{2}{r^2} \int_0^r \int_0^\rho s^n \sum_{k=n}^{\infty} (k - n + 1)s^{k-n} \, ds \, d\rho = \frac{2}{r^2} \int_0^r \int_0^\rho \frac{s^n}{(1-s)^2} \, ds \, d\rho. \quad (11)$$

In [13], Bombieri and Bourgain showed how one can build functions $h(z) = \sum_{k=0}^{\infty} h_k z^k$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in the family $B$ such that
(i) \(|h_k| = t^k \sqrt{1 - t^2}\)

(ii) \(\|f - h\|_2 := \sqrt{\sum_{k=0}^{\infty} |h_k - a_k|^2} < \sqrt{1 - t^2} \sqrt{\log \frac{1}{1 - t}},\)

where \(0 \leq t \leq 1\) is some number.

Accordingly, using their idea, we obtain the following estimate

\[C_1^1(f) = \sum_{k=0}^{\infty} |h_k + a_k - h_k| \phi_k(r) \geq \sum_{k=0}^{\infty} |h_k| \phi_k(r) - \sum_{k=0}^{\infty} |h_k - a_k| \phi_k(r)\]

\[\geq \sqrt{1 - t^2} \sum_{k=0}^{\infty} t^k \phi_k(r) - \sqrt{1 - t^2} \sqrt{\log \frac{1}{1 - t}} \sqrt{\sum_{k=0}^{\infty} \phi_k^2(r)}. \quad (12)\]

Using (11) and the above consideration, the first term in (12) can be calculated directly. Note that

\[\sum_{k=0}^{\infty} t^k \phi_k(r) = \frac{2}{r^2} \sum_{k=0}^{\infty} \int_{0}^{\rho} \frac{t^k s^k}{(1 - s)^2} ds d\rho = \frac{2}{r^2} \int_{0}^{\rho} ds d\rho \int_{0}^{\frac{1}{1 - t}} \frac{ds}{(1 - s)^2 (1 - ts)}\]

and so to compute the integral on the right, we write

\[\frac{1}{(1 - s)^2 (1 - ts)} = \frac{1}{1 - t} \cdot \frac{1}{(1 - s)^2} - \frac{t}{(1 - t)^2} \cdot \frac{1}{1 - s} + \frac{t^2}{(1 - t)^2} \cdot \frac{1}{1 - ts},\]

which gives by integration

\[\int_{0}^{\rho} \frac{ds}{(1 - s)^2 (1 - ts)} = \frac{1}{1 - t} \cdot \frac{\rho}{1 - \rho} + \frac{t}{(1 - t)^2} \left[\log(1 - \rho) - \log(1 - t \rho)\right]\]

and hence we can easily obtain by integrating it again

\[\int_{0}^{\rho} \int_{0}^{\rho} \frac{ds d\rho}{(1 - s)^2 (1 - ts)} = \frac{2[\rho (1 - t) + (1 - rt) (\log(1 - rt) - \log(1 - r))]}{r^2 (1 - t)^2}.\]

The second term in (12) can be estimated using the result from the first part of Theorem 5, i.e., Eq. (9). For \(\alpha = 1\), we have

\[\sqrt{\sum_{k=0}^{\infty} \phi_k^2(r)} \leq \frac{2}{1 - r^2} \sqrt{\frac{\Gamma''(2)}{\Gamma(2)} - \left(\frac{\Gamma'(2)}{\Gamma(2)}\right)^2} - r^2 \Phi(r^2, 2, 2)\]

\[= \frac{2}{1 - r^2} \sqrt{\frac{\pi^2}{6} - \frac{\text{Li}_2(r^2)}{r^2}}.\]

Therefore, the above discussion shows that \(C_1^1(f)\) is not less than

\[\frac{1}{\sqrt{1 - r}} \left(\frac{2 \sqrt{1 - t^2} \sqrt{1 - r} [-r (1 - t) + (1 - rt) (\log(1 - rt) - \log(1 - r))]}{r^2 (1 - t)^2}\right)\]

\[- 2 \sqrt{1 - t^2} \sqrt{\log \frac{1}{1 - t}} \cdot \frac{1}{\sqrt{1 - r} (1 + r)} \sqrt{\frac{\pi^2}{6} - \frac{\text{Li}_2(r^2)}{r^2}}.\]
Let \( t = r^q \) for some \( q \). Since
\[
\lim_{r \to 1} \sqrt{1 - r^{2q}} \sqrt{\frac{\log \frac{1}{1 - r^q}}{1 - r^q} \cdot \frac{1}{\sqrt{1 - r^q}} \sqrt{\frac{\pi^2}{6} - \frac{\text{Li}_2(r^2)}{r^2}}} = 0,
\]
we have to consider only the first term and calculate the limit
\[
\lim_{r \to 1} \frac{2\sqrt{1 - r^{2q}}}{\sqrt{1 - r}} \left[ -r(1 - r^q) + (1 - r^{q+1})(\log(1 - r^{q+1}) - \log(1 - r)) \right] \frac{r^2(1 - r^q)^2}{(q/2)^3} = -q + (q + 1) \log(q + 1).
\]
The point of maximum is the root of the equation \( 3q = (3 + q) \log(1 + q) \). Calculation shows that \( q \approx 7.57736 \ldots \). Therefore,
\[
C_j^1(r) \sim \frac{4\sqrt{2q}}{(3 + q)\sqrt{1 - r}} \approx 1.47217 \frac{\sqrt{1 - r}}{\sqrt{1 - r}}.
\]
\( \square \)

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