SOLUTIONS TO CHERN-SIMONS-SCHRÖDINGER SYSTEMS WITH EXTERNAL POTENTIAL

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ABSTRACT. In this paper, we consider the existence of static solutions to the nonlinear Chern-Simons-Schrödinger system

\[
\begin{align*}
-iD_0\Psi - (D_1D_1 + D_2D_2)\Psi + V\Psi &= |\Psi|^{p-2}\Psi, \\
\partial_0A_1 - \partial_1A_0 &= -\frac{i}{2}\lambda[\nabla D_2\Psi - \Psi\nabla\overline{\Psi}], \\
\partial_0A_2 - \partial_2A_0 &= \frac{i}{2}\lambda[\nabla D_1\Psi - \Psi\nabla\overline{\Psi}], \\
\partial_1A_2 - \partial_2A_1 &= -\frac{1}{2}\lambda|\Psi|^2.
\end{align*}
\]  

(1)

with an external potential \( V(x) \), where \( D_0 = \partial_t + i\lambda A_0 \) and \( D_k = \partial_{x_k} - i\lambda A_k \), \( k = 1, 2 \), for \( (x_1, x_2, t) \in \mathbb{R}^2, t \) are covariant derivatives, \( \lambda \) is the coupling number. Suppose that \( V(x) \) satisfies \( \lim_{|x| \to \infty} V(x) = +\infty \), we show for \( 2 < p < 4 \) that there exists \( \lambda^* > 0 \) such that if \( 0 < \lambda < \lambda^* \), problem (1) has two nontrivial static solutions \( (\Psi_\lambda, A_{0\lambda}, A_{1\lambda}, A_{2\lambda}) \). Moreover, there also exists \( \lambda^* > 0 \) such that if \( \lambda > \lambda^* \), problem (1) has no nontrivial solutions. While for \( p > 4 \) we assume in addition that \( x \cdot \nabla V(x) \geq 0 \), then problem (1) admits a mountain pass solution for all \( \lambda > 0 \).

1. Introduction. In this paper, we investigate the existence of solutions to Chern-Simons-Schrödinger systems. It is known that the Schrödinger equation

\[
i\frac{\partial \Psi(x, t)}{\partial t} = -\Delta \Psi(x, t) + V(x)\Psi(x, t) - |\Psi(x, t)|^{p-2}\Psi(x, t)
\]  

(2)

with \( p > 2 \) in \( \mathbb{R}^2 \times \mathbb{R}^+ \) can be introduced as the Euler-Lagrange equation of the Lagrange density

\[
\mathcal{L} = -\frac{1}{2}Re\{i\Psi(x, t)\frac{\partial \Psi(x, t)}{\partial t}\} + \frac{1}{2}|
abla \Psi(x, t)|^2 + \frac{1}{2}V(x)|\Psi(x, t)|^2 - \frac{1}{p}|\Psi(x, t)|^p,
\]

(3)

where \( V(x) \) is the external potential. A static solution, that is a solution \( \Psi(x, t) = u(x) \) of (2) which is independent of \( t \), satisfies the Schrödinger equation

\[
-\Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), \quad x \in \mathbb{R}^2.
\]

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Taking into account the interaction of the electromagnetic field and the matter field, one includes the Chern-Simons term into the Lagrangian density. The Lagrangian density then becomes
\[
\mathcal{L}_c = \frac{1}{4} \varepsilon^{\mu \nu \alpha \beta} A_\mu F_{\nu \alpha} - \frac{1}{4} Re \{ i\bar{\Psi}(x,t)D_\mu \Psi(x,t) \} + \frac{1}{2} |D \Psi(x,t)|^2 \\
+ \frac{1}{2} V(x)|\Psi(x,t)|^2 - \frac{1}{p} |\Psi(x,t)|^p,
\] (5)
where \( \Psi : \mathbb{R}^{2,1} \rightarrow \mathbb{C} \) is the complex scalar field, \( A_\mu : \mathbb{R}^{2,1} \rightarrow \mathbb{R} \), \( \mu = 0, 1, 2 \), are the gauge fields, which obey the Lorentz condition \( \sum_{\mu=0}^2 \partial_\mu A_\mu = 0 \). By \( D_\mu = \partial_\mu + i\lambda A_\mu \) and \( D_j = \partial_j - i\lambda A_j \), \( j = 1, 2 \), for \( (x_1, x_2, t) \in \mathbb{R}^{2,1} \) we denote the covariant derivatives, where \( \lambda \) is the coupling number and we set \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) for \( \mu, \nu = 0, 1, 2 \). Inside the Lagrangian density \( \mathcal{L}_c \) we denote by \( i \) the imaginary unit, and \(-\frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} A_\mu F_{\nu \alpha}\) the Chern-Simons term. The corresponding Euler-Lagrange system of \( \mathcal{L}_c \) is given as follows.
\[
\begin{align*}
-\partial_0 \Psi - & (D_1 D_1 + D_2 D_2) \Psi + V \Psi = |\Psi|^{p-2} \Psi, \\
\partial_\mu A_1 - & \partial_1 A_\mu = -\frac{1}{2} i\lambda [\bar{\Psi} D_\mu \Psi - \Psi \bar{D}_\mu \Psi], \\
\partial_\mu A_2 - & \partial_2 A_\mu = \frac{1}{2} i\lambda [\bar{\Psi} D_\mu \Psi - \Psi \bar{D}_\mu \Psi], \\
\partial_1 A_2 - & \partial_2 A_1 = -\frac{1}{2} \lambda |\Psi|^2.
\end{align*}
\] (6)
The system (6) is considered in \([4, 7, 8, 9]\), which describes the dynamics of large number of particles in an electromagnetic field. This model is important for the study of the high-temperature superconductor, fractional quantum Hall effect and Aharonov-Bohm scattering. System (6) is referred to be the Chern-Simons-Schrödinger system, it is invariant under the following gauge transformation
\[
\phi \rightarrow \phi e^{i\chi}, \quad A_\mu \rightarrow A_\mu - \partial_\mu \chi
\]
for arbitrary \( C^\infty \) function \( \chi : \mathbb{R}^{2,1} \rightarrow \mathbb{R} \).

Since system (6) is setting in the whole space, a problem of the loss of the compactness is then raised if the variational method applied. In order to avoid such a problem, in \([1]\) a particular form of solutions of (6)
\[
\Psi(t,x) = u(|x|) e^{i\omega t}, \quad A_0(t,x) = h_1(|x|), \\
A_1(t,x) = \frac{x_2}{|x|^2} h_2(|x|), \quad A_2(t,x) = \frac{x_1}{|x|^2} h_2(|x|),
\]
is considered without the potential \( V \), where \( \omega > 0 \) and \( u, h_1, h_2 \) are real value functions. Then, solutions are found in the radially symmetric space \( H^1_0(\mathbb{R}^2) \) as critical points of the associated functional
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla u|^2 + (\omega + \xi) u^2 + \frac{u^2}{|x|^2} \left( \int_0^{|x|} s^{-1/2} (s^{-1/2} \left. \frac{\partial u}{\partial s} \right|_{s=0} \right) ds \right\}^2 \right\} dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.
\]
However, it is quite involved in finding critical points of \( J \). Actually, such a problem was treated differently in accordance to the range of the exponent \( p \). Precisely, for \( p > 4 \) it is considered a minimization problem on the Nehari-Pohozaev manifold; while for \( 2 < p < 4 \), the minimization problem is constrained in \( L^2 \) sphere. Essentially, it is a nonlinear eigenvalue problem. For the case \( p = 4 \), a self-dual solution can be found by Liouville equations. Suppose that \( V \) is radially symmetric, it is studied in \([14]\) the existence, nonexistence and multiplicity of the same type of solutions for (6). While for the case \( V \) being non-radial, nontrivial solutions are found in \([15]\) under the assumption \( p > 4 \). Further results can be found in \([1, 2, 3, 6, 12, 13]\) and references therein.
In this paper, we consider the existence and nonexistence of static solutions of (6) for $p > 4$ and $2 < p < 4$ under the assumption that

(V) $V(x) \geq 0$ and $\lim_{|x| \to \infty} V(x) = \infty$;

(V₁) $x \cdot \nabla V(x) \geq 0$.

We remark that the harmonic potential $V(x) = |x|^2$ satisfies assumptions (V) and (V₁).

For static solutions of (6), the gauge field $(A₁, A₂)$ obeys the Coulomb condition

$\partial_1 A₁ + \partial_2 A₂ = 0$.

Moreover, a static solution $(u, A₀, A₁, A₂)$ satisfies

$\begin{align*}
-\Delta u + V(x)u + \lambda A₀ u + \lambda^2 (A₁² + A₂²) u &= |u|^{p-2} u, \\
\partial_1 A₀ &= \lambda^2 A₂ u², \\
\partial_2 A₀ &= -\lambda^2 A₁ u² \\
\partial_1 A₂ - \partial_2 A₁ &= -\frac{1}{2} \lambda |u|^², \\
\partial_1 A₁ + \partial_2 A₂ &= 0.
\end{align*}$

(7)

Let

$\mathcal{H} = \{ u \in H¹(\mathbb{R}²) : \int_{\mathbb{R}²} V(x) u² \, dx < \infty \}$

be the space with the norm

$||u||_H = \left( \int_{\mathbb{R}²} (|\nabla u|^2 + V(x)u²) \, dx \right)^{\frac{1}{2}}$.

We will find solutions of problem (7) by looking for critical points of the associated functional

$J_\lambda(u, A₀, A₁, A₂) = \frac{1}{2} \int_{\mathbb{R}²} (|\nabla u|^² + V(x)|u|^² + (\lambda A₀ + \lambda^2 A₁² + \lambda^2 A₂²)|u|^²) \, dx$

$+ \frac{1}{2} \int_{\mathbb{R}²} (A₀ F₁₂ + A₁ \partial₂ A₀ - A₂ \partial₁ A₀) \, dx - \frac{1}{p} \int_{\mathbb{R}²} |u|^p \, dx.$

(8)

Such a problem can be reduced, see section 2 for details, to find critical points of the functional

$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}²} (|\nabla u|^² + V(x)u²) \, dx + \frac{1}{2} \lambda^4 \int_{\mathbb{R}²} \left( \frac{1}{4\pi} \int_{\mathbb{R}²} \frac{x₂ - y₂}{|x - y|^²} u²(y) \, dy \right)^² u²(x) \, dx$

$+ \frac{1}{2} \lambda^4 \int_{\mathbb{R}²} \left( \frac{1}{4\pi} \int_{\mathbb{R}²} \frac{x₁ - y₁}{|x - y|^²} u²(y) \, dy \right)^² u²(x) \, dx - \frac{1}{p} \int_{\mathbb{R}²} |u|^p \, dx.$

(9)

In the case $p > 4$, we have the following existence result.

**Theorem 1.1.** Suppose that $p > 4$ and $V(x)$ satisfies (V) and (V₁). Then problem (7) has a nontrivial solution.

We may show that the functional $I_\lambda(u)$ has the mountain pass geometry, the mountain pass theorem without (PS) condition implies that there is a (PS)c sequence of $I_\lambda$. However, it is difficult to bound such a sequence. The information in the mountain pass theorem is not enough to do it. In order to obtain further information, we show that there is a (PS)c sequence of $I_\lambda$ near the Pohozaev manifold by a variant mountain pass theorem. With the help of the Pohozaev identity, the argument can be carried through.

Next, we consider the case $2 < p < 4$. In this case the functional $I_\lambda$ has different features from the case $p > 4$. The coupling number $\lambda$ now is taken into account as a parameter.

**Theorem 1.2.** Suppose that $2 < p < 4$ and $V(x)$ satisfies (V). Then there exists $\lambda^* > 0$ such that if $0 < \lambda < \lambda^*$, problem (7) has two nontrivial solutions.
In the case $p \in (2, 4)$, the essential difficulty is again to bound the $(PS)$ sequence. In the radial case with $V(x) = \omega$ being constant, it was proved in [12] that there is a threshold value $\omega_0$ in connection with $\omega$ such that, among other things, problem (6) has two solutions if $\omega \in (\omega_0, \bar{\omega})$, see [12] for details. The argument in [12] relies heavily on the radial symmetry of functions. The problem remains unsolved if the potential function $V(x)$ is not a constant. Similar problems arise in the Schrödinger-Poisson problem. Under certain conditions, it was proved in [10, 11] the existence and nonexistence of solutions for the Schrödinger-Poisson problem with $p \in (1, 2)$. Inspired of these works, we establish the existence of solutions for problem (7) in the case $2 < p < 4$. However, our situation differs from the Schrödinger-Poisson problem.

Finally, we have the following nonexistence result.

**Theorem 1.3.** Suppose that $2 < p < 4$ and $V(x)$ satisfies (V). Then there exists $\tilde{\lambda} > 0$ such that problem (7) has no nontrivial solutions if $\lambda > \tilde{\lambda}$.

In view of the coupling number $\lambda$ in the covariant derivatives, we remark that there exist static solutions of problem (7) if $0 < \lambda < \lambda^*$, while there is no solution provided that $\lambda > \lambda^*$. It is not clear whether $\lambda^* = \tilde{\lambda}$, or what is the threshold value of $\lambda$.

This paper is organized as follows. After some preparations in section 2, we prove existence and nonexistence results in section 3 for $p > 4$ and in section 4 for $2 < p < 4$.

2. Preliminaries. In this section, we present some fundamental facts for future reference.

Integrating by part in (8) we find

$$J_\varepsilon(u, A_0, A_1, A_2) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2 + (\lambda A_0 + \lambda^2 A_1^2 + \lambda^2 A_2^2)|u|^2) \, dx$$

$$+ \int_{\mathbb{R}^2} A_0 F_{12} \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx.$$  

Equation (7) implies

$$\int_{\mathbb{R}^2} A_0 F_{12} \, dx = -\frac{1}{2} \int_{\mathbb{R}^2} \lambda A_0 |u|^2 \, dx.$$  

Hence,

$$J_\lambda(u, A_0, A_1, A_2) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2 + \lambda^2 (A_1^2 + A_2^2)|u|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx.$$  

Next, the Coulomb condition $\partial_1 A_1 + \partial_2 A_2 = 0$ and the equation $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} \lambda |u|^2$ yield

$$\Delta A_1 = \frac{1}{2} \lambda \partial_2(|u|^2), \quad -\Delta A_2 = \frac{1}{2} \lambda \partial_1(|u|^2).$$  

Solving equation (12) for each $u \in \mathcal{H}$ we obtain

$$A_1 = A_1(u) = \frac{1}{2} \lambda K_2 * (|u|^2) = -\frac{1}{4\pi} \lambda \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} |u(y)|^2 \, dy$$  

and

$$A_2 = A_2(u) = -\frac{1}{2} \lambda K_1 * (|u|^2) = \frac{1}{4\pi} \lambda \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} |u(y)|^2 \, dy,$$  

where $K_1(x) = \frac{1}{2\pi} \ln |x|$, $K_2(x) = 2\pi |x|^{-1}$. The functions $K_1$ and $K_2$ are known as the fundamental solutions of the Laplace and Helmholtz equations, respectively.
where $K_i = \frac{-x_i}{\pi|x|^2}$, $i = 1, 2$ and $*$ denotes the convolution. Similarly, the equation
\[ \Delta A_0 = \lambda^2 (\partial_t (A_2 |u|^2) - \partial_2 (A_1 |u|^2)) \]
implies that
\[ A_0 = A_0(u) = \lambda^2 (K_1 \ast (A_2 |u|^2) - K_2 \ast (A_1 |u|^2)). \]  
Moreover, for $u \in H$ we have
\[ (A_k'(u), u) = 2A_k(u), \quad k = 1, 2. \]

Therefore, the functional $J_\lambda$ can be written as
\[ I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + V(x)u^2 \right) dx + \frac{1}{2} \lambda^4 \int_{\mathbb{R}^2} \left( -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|^2} u^2(y) dy \right) u^2(x) dx \]
\[ + \frac{1}{2} \lambda^4 \int_{\mathbb{R}^2} \left( \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} u^2(y) dy \right) u^2(x) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx. \]  
We know that $I_\lambda$ is well defined in $H$ and $I_\lambda \in C^1(H)$. If $u$ is a critical point of $I_\lambda$, defining $A_0, A_1, A_2$ through (13), (14) and (15), we may verify that $(u, A_0, A_1, A_2)$ is a solution of problem (7). In the following, we focus on finding critical points of the functional $I_\lambda$.

Define the integral operator $T$ by
\[ T u(x) = \int_{\mathbb{R}^2} \frac{u(y)}{|x - y|} \, dy. \]

The following result was proved in [15].

**Lemma 2.1.** Let $1 < s < 2$ and $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$.

(i) There is a positive constant $C$ depending only on $s$ and $q$ such that
\[ \left( \int_{\mathbb{R}^2} |Tu(x)|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^2} |u(x)|^s \, dx \right)^{\frac{1}{s}}. \]

(ii) If $u \in H^1(\mathbb{R}^2)$, then for $k = 1, 2$ we have
\[ \|A_k(u)\|_{L^s(\mathbb{R}^2)} \leq C \|u\|_{L^2(\mathbb{R}^2)}^2 \]
and
\[ \|A_0(u)\|_{L^s(\mathbb{R}^2)} \leq C \|u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^4(\mathbb{R}^2)}^2. \]

(iii) For $k = 1, 2$ we have
\[ \|A_k(u)u\|_{L^2(\mathbb{R}^2)} \leq C \|A_k(u)\|_{L^s(\mathbb{R}^2)} \|u\|_{L^2(\mathbb{R}^2)}^2. \]

The following result is a counterpart of Brézis-Lieb lemma for the nonlocal term, it can be found or proved as that in [15].

**Lemma 2.2.** Suppose that $u_n$ converges to $u$ a.e. in $\mathbb{R}^2$ and $u_n$ weakly converges to $u$ in $H^1(\mathbb{R}^2)$. Then $A_k(u_n(x)), k = 1, 2,$ converges to $A_k(u(x))$ a.e. in $\mathbb{R}^2$. Moreover, for $k = 1, 2,$
\[ \int_{\mathbb{R}^2} A_k^2(u_n - u)(u_n - u)^2 \, dx = \int_{\mathbb{R}^2} A_k^2(u_n^2 \, dx - \int_{\mathbb{R}^2} A_k^2(u) \, u^2 \, dx + o_n(1) \]
and for each $v \in L^2(\mathbb{R}^2)$,
\[ \int_{\mathbb{R}^2} (A_k^2(u_n - u)v) \, dx = \int_{\mathbb{R}^2} (A_k^2(u_n) - A_k^2(u))uv \, dx + o_n(1). \]
3. Existence for the case \( p > 4 \). In this section, we consider the existence of solutions for problem (7) in the case \( p > 4 \). In this case, the parameter \( \lambda \) is irrelevant, we set \( \lambda = 1 \).

First, we remark that the functional \( I = I_1 \) has a mountain pass geometry.

**Lemma 3.1.** There hold

(i) There is an \( e \in \mathcal{H} \) such that \( I(e) < 0 \).

(ii) There exist \( \delta > 0, r > 0 \) such that \( I(u) > \delta \) if \( \|u\|_{\mathcal{H}} = r \).

**Proof.** The proof of (ii) is standard. To show (i), fixed a \( u \in C_c^\infty(\mathbb{R}^2) \setminus \{0\} \), we set \( v_t(x) = tu(tx) \). Then,

\[
I(v_t(x)) = \frac{t^2}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(t^{-1}x)u^2 \, dx
+ \frac{t^2}{2} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2(x) \, dx - \frac{t^{p-2}}{p} \int_{\mathbb{R}^2} |u|^p \, dx.
\]

Hence, \( I_\lambda(v_t(x)) \to \infty \) is \( t \to +\infty \) provided that \( p > 4 \). The assertion follows. \( \square \)

Now we may define the mountain pass level for \( I \):

\[
e = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),
\]

where

\[
\Gamma = \{ \gamma \in C([0,1], \mathcal{H}) : \gamma(0) = 0, I(\gamma(1)) < 0 \}.
\]

In order to bound the \((PS)_c\) sequence of \( I \), we define for any \( u \in \mathcal{H} \) the function \( \Phi(s, u) = e^s u(e^s x), x \in \mathbb{R}^2 \). Note that \( A_k(\Phi(s, u)) = e^s A_k(u(e^s x)), k = 1, 2, \) we have

\[
I \circ \Phi(s, u) = \frac{1}{2} e^{2s} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(e^{-s}x)u^2 \, dx
+ \frac{1}{2} e^{2s} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2(x) \, dx - \frac{1}{p} e^{(p-2)s} \int_{\mathbb{R}^2} |u|^p \, dx
\]

defined on \( \mathbb{R} \times \mathcal{H} \). We may verify that

\[
\langle \frac{\partial I \circ \Phi(s, u)}{\partial u}, v \rangle = e^{2s} \int_{\mathbb{R}^2} \nabla u \nabla v \, dx + \int_{\mathbb{R}^2} V(e^{-s}x)uv \, dx
+ e^{2s} \int_{\mathbb{R}^2} (A_1(u)(A_1'(u), v) + A_2(u)(A_2'(u), v))u^2(x) \, dx
+ e^{2s} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))uv \, dx - e^{(p-2)s} \int_{\mathbb{R}^2} |u|^{p-2}uv \, dx
\]
for each $v \in \mathcal{H}$, and for $h \in \mathbb{R}$,
\[
\langle \frac{\partial I \circ \Phi(s,u)}{\partial s}, h \rangle = e^{2sh} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx - \frac{1}{2} e^{-sh} \int_{\mathbb{R}^2} x \cdot \nabla V(e^{-s}x)u^2 \, dx
\]
\[
+ e^{2sh} \int_{\mathbb{R}^2} (A^2_1(u) + A^2_2(u))u^2(x) \, dx - \frac{p-2}{p} he^{(p-2)s} \int_{\mathbb{R}^2} |u|^p \, dx.
\]  
(21)

In view of Lemma 3.1, we have $I \circ \Phi(s,u) > 0$ for $(s,u)$ with $|s|$ and $\|u\|_\mathcal{H}$ small, and $I \circ \Phi(0,c) < 0$. Hence, $I \circ \Phi(s,u)$ has the mountain pass geometry. We define the mountain pass level for $I \circ \Phi(s,u)$:
\[
b = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{\gamma \in \Gamma} I \circ \Phi(\tilde{\gamma}(t)),
\]  
(22)

where
\[
\tilde{\Gamma} = \{ \tilde{\gamma} \in C([0,1], \mathbb{R} \times \mathcal{H}) : \tilde{\gamma}(0) = (0,0), I \circ \Phi(\tilde{\gamma}(1)) < 0 \}.
\]

Since $\Gamma = \{ \Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma} \}$, we have $c = b$, where $c, b$ are defined in (18) and (22) respectively.

Now we will find a $(PS)_c$ sequence of $I$ nearby the Pohozaev manifold. The argument is based on the following general minimax principle which is Theorem 2.8 in [16].

**Proposition 1.** Let $X$ be a Banach space. Let $M_0$ be a closed subspace of the metric space $M$ and $\Gamma_0 \subset C(M_0, X)$. Define
\[
\Gamma = \{ \gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0 \}.
\]

If $\varphi \in C^1(X, \mathbb{R})$ satisfies
\[
\infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u))
\]
then, for every $\varepsilon \in (0, \frac{c-a}{2T})$, $\delta > 0$ and $\gamma \in \Gamma$ such that
\[
\sup_{\gamma \in \Gamma} \varphi \circ \gamma \leq c + \varepsilon,
\]
there exists $u \in X$ such that
\begin{enumerate}
\item $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,
\item $\text{dist}(u, \Gamma(M)) \leq 2\delta$,
\item $\|\varphi'(u)\| \leq 8\varepsilon/\delta$.
\end{enumerate}

Let
\[
Q(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 - \frac{1}{2} x \cdot \nabla V(x)u^2 + (A^2_1(u) + A^2_2(u))u^2) \, dx - \frac{p-2}{p} \int_{\mathbb{R}^2} |u|^p \, dx.
\]

An application of Proposition 1 to the functional $I \circ \Phi(s,u)$ yields the following result.

**Lemma 3.2.** There exists a sequence $\{u_n\} \subset \mathcal{H}$ such that
\[
I(u_n) \to c, \quad I'(u_n) \to 0, \quad Q(u_n) \to 0
\]  
(23)
as $n \to \infty$.

**Proof.** The result was given in [5] for the Schrödinger-Poisson equation. We sketch the proof here for reader’s convenience.

Since the functional $I \circ \Phi(s,u)$ has a mountain pass geometry, taking $M = [0,1]$ in Proposition 1, we find that there exists a sequence $\{(s_n, v_n)\}$ in $\mathbb{R} \times \mathcal{H}$ such that
\[
(I \circ \Phi)(s_n, v_n) \to c, \quad (I \circ \Phi)'(s_n, v_n) \to 0, \quad s_n \to 0
\]  
(24)
as $n \to \infty$. Therefore, for $(h, w) \in \mathbb{R} \times \mathcal{H}$,
\[
\langle (I \circ \Phi)'(s_n, v_n), (h, w) \rangle = (I'(\Phi(s_n, v_n)), \Phi(s_n, w)) + Q(\Phi(s_n, v_n))h. \tag{25}
\]
Taking $h = 1$, $w = 0$ in (25) we obtain
\[
Q(\Phi(s_n, v_n)) \to 0 \quad \text{as} \quad n \to \infty.
\]
Let $u_n = \Phi(s_n, v_n)$. Then
\[
Q(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
We may also verify that for each $v \in \mathcal{H}$,
\[
\langle I'(u_n), v \rangle \to 0 \quad \text{as} \quad n \to \infty.
\]
The assertion follows. \hfill \Box

Now we show that the $(PS)_c$ sequence of $I(u)$ is bounded.

**Lemma 3.3.** Any sequence $\{u_n\} \subset \mathcal{H}$ satisfying
\[
I(u_n) \to c, \quad I'(u_n) \to 0, \quad Q(u_n) \to 0 \tag{26}
\]
as $n \to \infty$, is bounded in $\mathcal{H}$.

**Proof.** By (26), we have
\[
I(u_n) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)u_n^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n))u_n^2(x) \, dx \\
- \frac{1}{p} \int_{\mathbb{R}^2} |u_n|^p \, dx = c + o(1) \tag{27}
\]
and
\[
Q(u_n) = \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^2} x \cdot \nabla V(x)u_n^2 \, dx + \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n))u_n^2(x) \, dx \\
- \frac{p-2}{p} \int_{\mathbb{R}^2} |u_n|^p \, dx = o(1) \tag{28}
\]
as $n \to \infty$. We deduce from (27) and (28) that
\[
\frac{p-4}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{p-2}{2} \int_{\mathbb{R}^2} V u_n^2 \, dx + \frac{p-4}{2} \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n))u_n^2(x) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^2} x \cdot \nabla V(x)u_n^2 \, dx = (p-2)c + o(1) \tag{29}
\]
Since $p > 4$, by assumption $(V_1)$, $\{\|u_n\|_{\mathcal{H}}\}$ is bounded. \hfill \Box

Finally, we show in this section that the Palais-Smale condition holds for each $c \in \mathbb{R}$.

**Proposition 2.** Suppose $(V)$ and $(V_1)$ hold for $p > 4$, and $(V)$ holds for $2 < p < 4$ respectively. Let $\{u_n\} \subset \mathcal{H}$ be a bounded $(PS)_c$ sequence of $I_\lambda$, that is,
\[
I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0
\]
as $n \to \infty$. Then $\{u_n\}$ contains a convergent subsequence.
Proof. Since \( \{u_n\} \) is bounded in \( \mathcal{H} \), so we may assume that
\[
  u_n \rightharpoonup u \quad \text{in} \quad \mathcal{H}, \quad u_n \to u \quad \text{in} \quad L^p(\mathbb{R}^2), \quad p \geq 2, \quad u_n \to u \quad \text{a.e.} \quad \mathbb{R}^2. \tag{30}
\]
Let
\[
  J_k(u) = \int_{\mathbb{R}^2} A_k^2(u)u^2\, dx, \quad k = 1, 2.
\]
We have for all \( v \in \mathcal{H} \) that
\[
  \langle J_k(u), v \rangle = 2 \int_{\mathbb{R}^2} A_k^2(u)uv + A_k \langle A'_k(u), v \rangle u^2\, dx.
\]
We claim that for all \( v \in \mathcal{H} \),
\[
  \langle J_k(u_n) - J_k(u), v \rangle = 2 \int_{\mathbb{R}^2} (A_k(u_n)(A'_k(u_n), v)u_n^2 + A_k^2(u_n)u_n v - A_k(u)(A'_k(u), v)u^2 - A_k^2(u)uv)\, dx
\]
\[
\to 0
\]
as \( n \to \infty \). Indeed,
\[
| \int_{\mathbb{R}^2} (A_k^2(u_n)u_n v - A_k^2(u)uv)\, dx | \leq | \int_{\mathbb{R}^2} (A_k^2(u_n)v - A_k^2(u_n)uv)\, dx | + | \int_{\mathbb{R}^2} (A_k^2(u_n) - A_k^2(u))uv\, dx |. \tag{32}
\]
By the Hölder and Hardy-Littlewood-Sobolev inequalities, we deduce that
\[
| \int_{\mathbb{R}^2} (A_k^2(u_n)u_n v - A_k^2(u)uv)\, dx |
\leq C\|A_k^2(u_n)\|_{L^{\frac{2}{q}}(\mathbb{R}^2)} \|v\|_{L^{2q'}} \|u_n - u\|_{L^{2q'}} \tag{33}
\]
as \( n \to \infty \), where \( q > 2, \frac{2}{q} + \frac{1}{q'} = 1 \) and \( 1 < s < 2, \frac{1}{s} - \frac{1}{q} = \frac{1}{2} \). Similarly, by Lemma 2.2 we find
\[
| \int_{\mathbb{R}^2} (A_k^2(u_n) - A_k^2(u))uv\, dx |
\leq C\|A_k^2(u_n) - u\|_{L^{\frac{2}{q}}(\mathbb{R}^2)} \|v\|_{L^{2q'}} \|u\|_{L^{2q'}} + o(1) \tag{34}
\]
as \( n \to \infty \).

Next, we show that
\[
\int_{\mathbb{R}^2} A_k(u_n)(A'_k(u_n), v)u_n^2\, dx \to \int_{\mathbb{R}^2} A_k(u)(A'_k(u), v)u^2\, dx \tag{35}
\]
as \( n \to \infty \) for \( v \in \mathcal{H} \). Since
\[
\int_{\mathbb{R}^2} A_k(u_n)(A'_k(u_n), v)u_n^2\, dx
\]
\[
= \int_{\mathbb{R}^2} A_k(u_n)(A'_k(u_n), v)(u_n^2 - u^2)\, dx + \int_{\mathbb{R}^2} A_k(u_n)(A'_k(u_n), v)u^2\, dx. \tag{36}
\]
First we show that
\[ \|A_k(u_n)\langle A'_k(u_n), v \rangle\|_{L^2} \leq C. \] (37)

Then, the weakly convergence of \( A_k(u_n)\langle A'_k(u_n), v \rangle \) implies for each \( v \in \mathcal{H} \) that
\[
\int_{\mathbb{R}^2} A_k(u_n)\langle A'_k(u_n), v \rangle u^2 \, dx \to \int_{\mathbb{R}^2} A_k(u)\langle A'_k(u), v \rangle u^2 \, dx
\] (38)
since \( u \in \mathcal{H} \). To prove (37), we note that
\[
|\langle A'_k(u_n), v \rangle| \leq C \int_{\mathbb{R}^2} \frac{|u_n|}{|x-y|} \, dx \leq (\int_{\mathbb{R}^2} \frac{|u_n|^2}{|x-y|^q} \, dx)^{\frac{p}{2}} (\int_{\mathbb{R}^2} \frac{|v|^2}{|x-y|^q} \, dx)^{\frac{1}{2}},
\]
that is,
\[
|\langle A'_k(u_n), v \rangle| \leq |Tu_n|^p |Tv|^\frac{q}{2}. \] (39)

It follows from the Hardy-Littlewood-Sobolev inequality that
\[
\left| \int_{\mathbb{R}^2} A_k(u_n)\langle A'_k(u_n), v \rangle u^2 \, dx \right| \\
\leq \|A_k(u_n)\|_{L^2} \|\langle A'_k(u_n), v \rangle\|_{L^2} \|u_n\|_{L^{2q'}} \|u\|_{L^q} \] (40)
\[
\leq C \|u_n\|_{L^{2q'}} \|Tu_n\|_{L^{q'}} \|Tv\| \to 0
\]
So (37) holds true.

Next, we prove that
\[
\int_{\mathbb{R}^2} A_k(u_n)\langle A'_k(u_n), v \rangle (u_n^2 - u^2) \, dx \to 0. \] (41)
Indeed, for \( s_1 = \frac{2q}{q+1} \),
\[
\left| \int_{\mathbb{R}^2} A_k(u_n)\langle A'_k(u_n), v \rangle (u_n^2 - u^2) \, dx \right| \\
\leq \|A_k(u_n)\|_{L^{s_1}} \|\langle A'_k(u_n), v \rangle\|_{L^{s_1}} \|u_n - u\|_{L^{2q'}} \|u_n + u\|_{L^{2q'}} \] (42)
\[
\leq C \|u_n\|_{L^{2q'}} \|Tu_n\|_{L^{2q'}} \|Tv\| \to 0
\]
as \( n \to \infty \). Consequently, the claim (31) holds true.

In the same way, we may infer from (38), (41) and the weak convergence of \( u_n \) in \( \mathcal{H} \) that
\[
\langle I'_k(u), v \rangle = 0, \quad \text{for each } v \in \mathcal{H}. \] (43)

We can also verify that
\[
\langle A'_k(u_n), u_n \rangle = 2A_k(u_n). \] (44)

Now, we are ready to show that \( u_n \) converges to \( u \) in \( \mathcal{H} \). By (44),
\[
(A_1(u_n), A'_1(u_n), u_n) + A_2(u_n, A'_2(u_n), u_n) u_n^2 = 2(A_1^2(u_n) + A_2^2(u_n)) u_n^2,
\]
and then
\[
\phi(1) = \langle I'_k(u_n), u_n \rangle \\
= \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(x) u_n^2) \, dx + 3\lambda^2 \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n)) u_n^2 \, dx - \int_{\mathbb{R}^2} |u_n|^p \, dx \] (45)
Similarly,
\[ 0 = \langle I'_*(u), u \rangle = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2) \, dx + 3\lambda^2 \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2 \, dx - \int_{\mathbb{R}^2} |u|^p \, dx. \]  
(46)

Let \( v_n = u_n - u \). By (45), (46) and Lemma 2.2,
\[ \int_{\mathbb{R}^2} (|\nabla v_n|^2 + V(x)v_n^2) \, dx + 3\lambda^2 \int_{\mathbb{R}^2} (A_1^2(v_n) + A_2^2(v_n))v_n^2 \, dx = o(1) \]
as \( n \to \infty \). Since
\[
\left| \int_{\mathbb{R}^2} (A_1^2(v_n) + A_2^2(v_n))v_n^2 \, dx \right| \\
\leq C \left( \|A_1^2(v_n)\|_{L^p} \|v_n^2\|_{L^q} + \|A_2^2(v_n)\|_{L^p} \|v_n^2\|_{L^q} \right) \\
\leq C \|v_n^2\|_{L^p}^2 \|v_n^2\|_{L^q}^2 \to 0
\]
as \( n \to \infty \), we conclude \( \|v_n\|_{H} \to 0 \) as \( n \to \infty \). The proof is complete. \( \square \)

**Proof of Theorem 1.1.** The result follows by Lemma 3.1, Lemma 3.2, Lemma 3.3 and Proposition 2. \( \square \)

4. **Existence and nonexistence for the case** \( 2 < p < 4 \). In this section, we show the existence and nonexistence of solutions for problem (7) when \( 2 < p < 4 \), that is, we will prove Theorem 1.2 and Theorem 1.3. We start with the following result.

**Proposition 3.** Suppose \( 2 < p < 4 \) and \((V)\) holds. If \( \{u_n\} \subset \mathcal{H} \) is such that \( I_\lambda(u_n) \leq C \), then \( \{u_n\} \) is uniformly bounded in \( \mathcal{H} \).

**Proof.** For each \( u \in \mathcal{H} \) solving equation (12), we obtain \( A_k = A_k(u), k = 1, 2 \), which satisfies
\[
\lambda \int_{\mathbb{R}^2} |u|^4 \, dx = \int_{\mathbb{R}^2} 2(\partial_2 A_1 - \partial_1 A_2)|u|^2 \, dx \\
= 4 \int_{\mathbb{R}^2} \left[ -A_1 \partial_2 \left( \frac{1}{2}|u|^2 \right) + A_2 \partial_1 \left( \frac{1}{2}|u|^2 \right) \right] \, dx \\
= 4\lambda^{-1} \int_{\mathbb{R}^2} ( - A_1 \Delta A_1 - A_2 \Delta A_2 ) \, dx \\
= 4\lambda^{-1} \int_{\mathbb{R}^2} (|\nabla A_1|^2 + |\nabla A_2|^2) \, dx.
\]

By (7),
\[
\int_{\mathbb{R}^2} (|\nabla u|^2 + \lambda^2 (A_1^2 + A_2^2)u^2) \, dx \\
= \int_{\mathbb{R}^2} (|\partial_1 u|^2 + |\partial_2 u|^2 + \lambda^2 (A_1^2 + A_2^2)u^2) \, dx \\
\geq 2\lambda \int_{\mathbb{R}^2} (A_2 \partial_1 \left( \frac{1}{2}|u|^2 \right) - A_1 \partial_2 \left( \frac{1}{2}|u|^2 \right) ) \, dx \\
= 2 \int_{\mathbb{R}^2} (|\nabla A_1|^2 + |\nabla A_2|^2) \, dx = \frac{1}{2} \lambda^2 \int_{\mathbb{R}^2} |u|^4 \, dx.
\]

(48)
For each $\varepsilon > 0$, there holds for $2 < p < 4$ that

$$\frac{1}{p} t^p \leq C t^2 + \varepsilon t^4.$$  

Hence, by the assumption and (48),

$$C \geq I_\lambda(n) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(x)u_n^2 + \lambda^2 A_1^n u_n^2 + \lambda^2 A_2^n u_n^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u_n|^p \, dx$$

$$\geq \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + \lambda A_1^n u_n^2 + \lambda A_2^n u_n^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)u_n^2 \, dx$$

$$+ \frac{1}{8} \lambda^2 \int_{\mathbb{R}^2} |u_n|^4 \, dx - C_\varepsilon \int_{\mathbb{R}^2} |u_n|^2 \, dx - \varepsilon \int_{\mathbb{R}^2} |u_n|^4 \, dx.$$  

Choosing $\varepsilon = \frac{1}{16} \lambda^2$, we obtain

$$C + C\|u_n\|_{L^2(\mathbb{R}^2)}^2 \geq \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + \lambda^2 A_1^n u_n^2 + \lambda^2 A_2^n u_n^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} V(x)u_n^2 \, dx$$

$$\geq \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + V(x)u_n^2 + \lambda A_1^n u_n^2 + \lambda A_2^n u_n^2) \, dx.$$  

(49)

Now, we claim that $\|u_n\|_{L^2(\mathbb{R}^2)} \leq C$ uniformly in $n$. Suppose on the contrary that $\|u_n\|_{L^2(\mathbb{R}^2)} \to \infty$ as $n \to \infty$. Let $v_n(x) = \frac{u_n(x)}{\|u_n\|_{L^2(\mathbb{R}^2)}}$. Then $v_n$ satisfies

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 \, dx + \lambda^2 \|u_n\|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} (A_1^n(v_n)v_n^2 + A_2^n(v_n)v_n^2) \, dx + \int_{\mathbb{R}^2} V(x)v_n^2 \, dx$$

$$\leq \frac{C}{\|u_n\|_{L^2(\mathbb{R}^2)}} + C.$$  

(50)

This implies that $\|v_n\|_{H^1} \leq C$. So we may assume that

$$v_n \rightharpoonup v \text{ in } H, \quad v_n \to v \text{ in } L^p(\mathbb{R}^2), \quad p \geq 2, \quad v_n \to v \text{ a.e. } \mathbb{R}^2.$$  

(52)

Therefore, $\int_{\mathbb{R}^2} v_n^2 \, dx = 1$. By Lemma 2.2, we have $A_k(v_n) \to A_k(v)$ a.e. $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} A_k^n(v_n)v_n^2 \, dx = \int_{\mathbb{R}^2} A_k^n(v_n - v)(v_n - v)^2 \, dx + \int_{\mathbb{R}^2} A_k^n(v)v^2 \, dx + o(1)$$  

(53)

as $n \to \infty$ for $k = 1, 2$. By the Hölder inequality,

$$\left| \int_{\mathbb{R}^2} A_k^n(v_n - v)(v_n - v)^2 \, dx \right| \leq \|A_k^n(v_n - v)\|_{L^\frac{q}{p}} \|v_n - v\|_{L^\frac{2q}{q'}} + o(1),$$  

(54)

where $\frac{q}{q'} + \frac{1}{q'} = 1$. The Hardy-Littlewood-Sobolev inequality yields

$$\|A_k^n(v_n - v)\|_{L^\frac{q}{p}} \leq C\|v_n - v\|_{L^q}^2,$$  

(55)

where $1 < s < 2$, $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$. We conclude from (52)-(55) that for $k = 1, 2$,

$$\int_{\mathbb{R}^2} A_k^n(v_n)v_n^2 \, dx \to \int_{\mathbb{R}^2} A_k^n(v)v^2 \, dx$$  

(56)

as $n \to \infty$. On the other hand, by (51),

$$\int_{\mathbb{R}^2} (A_1^n(v_n)v_n^2 + A_2^n(v_n)v_n^2) \, dx \to 0$$  

(57)
as \( n \to \infty \). Equations (56) and (57) imply that
\[
\int_{\mathbb{R}^2} (A_1^2(v) v^2 + A_2^2(v) v^2) \, dx = 0.
\] (58)

Thus,
\[ A_j(v) v = 0, \quad a.e. \quad \mathbb{R}^2, \quad \partial_2(A_1(v)^2) = 0, \quad \partial_1(A_2(v)^2) = 0 \quad a.e. \quad \mathbb{R}^2 \]
implying
\[
0 = \int_{\mathbb{R}^2} \partial_2(A_1(v)^2) \, dx = \int_{\mathbb{R}^2} \partial_2A_1(v)^2 + A_1 \partial_2 v^2 \, dx
\] (59)
and
\[
0 = \int_{\mathbb{R}^2} \partial_1(A_2(v)^2) \, dx = \int_{\mathbb{R}^2} \partial_1A_2(v)^2 + A_2 \partial_1 v^2 \, dx.
\] (60)

Namely,
\[
\int_{\mathbb{R}^2} (\partial_1 A_2(v) - \partial_1 A_2(v)) v^2 \, dx + 2 \int_{\mathbb{R}^2} (v A_2(v) \partial_1 v - v A_1 \partial_2 v) \, dx = 0.
\] (61)

As a result,
\[
\int_{\mathbb{R}^2} |v|^4 \, dx = 2 \int_{\mathbb{R}^2} (\partial_1 A_2(v) - \partial_1 A_2(v)) v^2 \, dx + 4 \int_{\mathbb{R}^2} (v A_2(v) \partial_1 v - v A_1 \partial_2 v) \, dx = 0.
\] (62)

Hence, \( v = 0 \) a.e. \( \mathbb{R}^2 \), which is a contradiction since
\[
1 = \int_{\mathbb{R}^2} |v_n|^2 \, dx \to \int_{\mathbb{R}^2} |v|^2 \, dx.
\]
The proof is completed. \( \square \)

Now, we prove the existence results.

Proof of Theorem 1.2. Consider the functional
\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x) u^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx.
\]
It is standard to verify that there are \( r > 0 \) and \( \alpha > 0 \) such that
\[
I_0(u) \geq \alpha \quad \text{if} \quad \|u\|_{\mathcal{H}} = r.
\]
We also can find an \( e \in \mathcal{H} \) such that \( \|e\|_{\mathcal{H}} > r \) and \( I_0(e) < 0 \). By the continuity of the function \( I_\lambda \) in \( \lambda \), there is \( \lambda^* > 0 \) such that for \( 0 < \lambda < \lambda^* \), \( I_\lambda(u) \geq \alpha \) if \( \|u\|_{\mathcal{H}} = r \) and \( I_\lambda(e) < 0 \). By Lemma 2.2, we know \( I_\lambda \) satisfies (PS) condition. The mountain pass theorem implies that \( I_\lambda(u) \) has a critical point \( u_c \) satisfies \( I_\lambda(u_c) > 0 \).

Now, we consider the minimization problem
\[
m_\lambda = \inf_{u \in \mathcal{H}} I_\lambda(u)
\]
for \( 0 < \lambda < \lambda^* \). Since \( I_\lambda(e) < 0 \), by Proposition 3, \( -\infty < m_\lambda < 0 \). By the Ekeland variational principle, there exists a sequence \( \{u_n\} \subset \mathcal{H} \) such that
\[
I_\lambda(u_n) \to m_\lambda, \quad I'_\lambda(u_n) \to 0
\]
as \( n \to \infty \). By Proposition 3, \( \{u_n\} \) is bounded in \( \mathcal{H} \). Proposition 2 implies that \( \{u_n\} \) has a convergent subsequence. Hence, we may assume \( u_n \to u_\lambda \) in \( \mathcal{H} \). As a result, we obtain that \( m_\lambda = I_\lambda(u_\lambda) \) and \( I'_\lambda(u_\lambda) = 0 \). The proof is complete. \( \square \)
Finally, we show the nonexistence result.

**Proof of Theorem 1.3.** Since

\[
\lambda \int_{\mathbb{R}^2} |u|^4 \, dx = 2 \int_{\mathbb{R}^2} (-u A_1 \partial_x u + u A_2 \partial_y u) \, dx
\]

we obtain

\[
\lambda^2 \int_{\mathbb{R}^2} (A_1^2 + A_2^2) u^2 \, dx \geq \frac{3}{2 \varepsilon^2} \int_{\mathbb{R}^2} |u|^4 \, dx - \frac{\lambda^2}{\varepsilon^4} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx.
\]

Let \( u = y_\lambda \neq 0 \) be a critical point of \( I_\lambda \) for every \( \lambda > 0 \). Then,

\[
0 = \langle I'_\lambda(u), u \rangle
\]

\[
= \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x) u^2) \, dx + 3\lambda^2 \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u)) u^2 \, dx - \int_{\mathbb{R}^2} |u|^p \, dx
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + \lambda_1 u^2) \, dx + \frac{3\lambda^2}{2 \varepsilon^2} \int_{\mathbb{R}^2} |u|^4 \, dx - \frac{3\lambda^2}{\varepsilon^4} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx - \int_{\mathbb{R}^2} |u|^p \, dx
\]

\[
= \left( \frac{1}{2} - \frac{3\lambda^2}{\varepsilon^4} \right) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \left( \frac{1}{2} \lambda_1 u^2 + \frac{3\lambda^3}{2 \varepsilon^2} |u|^4 - |u|^p \right) \, dx.
\]

Choosing \( \varepsilon > 0 \) such that \( \frac{1}{2} = \frac{3\lambda^2}{\varepsilon^2} \), we find

\[
0 = \langle I'_\lambda(u), u \rangle \geq \int_{\mathbb{R}^2} \left( \frac{1}{2} \lambda_1 u^2 + \frac{\sqrt{6} \lambda^2}{2} |u|^4 - |u|^p \right) \, dx > 0
\]

provided that \( \lambda > 0 \) large, a contradiction. The assertion follows.

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