RINGS ADDITIVELY GENERATED BY IDEMPOTENTS AND NILPOTENTS

HUANYIN CHEN AND MARJAN SHEIBANI

Abstract. A ring $R$ is a strongly 2-nil-clean if every element in $R$ is the sum of two idempotents and a nilpotent that commute. A ring $R$ is feebly clean if every element in $R$ is the sum of two orthogonal idempotents and a unit. In this paper, strongly 2-nil-clean rings are studied with an emphasis on their relations with feebly clean rings. This work shows new interesting connections between strongly 2-nil-clean rings and weakly exchange rings.

1. Introduction

Throughout, all rings are associative with an identity. An element $a$ in a ring $R$ is strongly nil-clean provided that every element in $R$ is the sum of an idempotent and a nilpotent that commute (see [9]). A ring $R$ is a strongly 2-nil-clean if every element in $R$ is the sum of two idempotents and a nilpotent that commute. As is well known, A ring $R$ is strongly 2-nil-clean if and only if every element in $R$ is the sum of a tripotent and a nilpotent that commute (see [3, Theorem 2.8]).

A ring $R$ is an exchange ring provided that for any $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. A ring $R$ is a weakly exchange ring provided that for any $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R \cup (1 + a)R$. Such rings have been studied extensively by many authors (see [5, 12]). In [9, Corollary 2.15], it was proved

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that a ring $R$ is strongly nil-clean if and only if $R$ is an exchange ring and a UU ring. Here, a ring $R$ is a UU ring if every unit in $R$ is a unipotent.

A ring $R$ is feebly clean if every element in $R$ is the sum of two orthogonal idempotents and a unit. Commutative feebly clean rings were extensively investigated by [1], motivated by the work on continuous function rings (see [1]). In this paper, strongly 2-nil-clean rings are studied with an emphasis on their relations with feebly clean rings. This work shows new interesting connections between strongly 2-nil-clean rings and weakly exchange rings. The Danchev’s problem [?] was thereby answered.

We use $N(R)$ to denote the set of all nilpotent elements in $R$ and $J(R)$ the Jacobson radical of $R$. An element $u \in R$ is a unipotent if $1 - u \in N(R)$. Two idempotents $e, f \in R$ are orthogonal if $ef = fe = 0$. N stands for the set of all natural numbers.

2. Feebly Clean Rings

The aim of this section is to characterize strongly 2-nil-clean rings by means of feeble cleanness. Recall that a ring $R$ is 2-UU if $u^2$ is a unipotent for all $u \in U(R)$. We begin with

**Lemma 2.1.** Let $R$ be a feebly clean 2-UU ring. Then $6 \in R$ is nilpotent.

**Proof.** Write $3 = e - f + u$ where $e, f$ are orthogonal idempotents and $u$ is a unit. Set $g = e - f$. Then $g = g^3$. Since $R$ is a 2-UU ring, $u^2 = 1 + w$ for some $w \in N(R)$. Then

$$8(g + u) = 24 = 3^3 - 3 = (g + u)^3 - (g + u) = 3g^2u + 3gu^2 + v,$$

where $v = u^3 - u = u(u^2 - 1) = uw$. We note that $uw = u(u^2 - 1) = (u^2 - 1)u = wu$. Hence, $v = uw \in N(R)$. Multiplying both sides by $gu$, we get $8(g^2u + gu^2) = 3(g^2u + gu^2) + t$ for some $t \in N(R)$, and so $5(g^2u + gu^2) = t$. Thus, $2^3 \cdot 3 \cdot 5 = 5 \cdot 24 = 5 \cdot (3g^2u + 3gu^2 + v) = 3 \cdot 5(g^2u + gu^2) + 5v = 3t + 5v \in N(R)$. Therefore $2 \times 3 \times 5 \in N(R)$. Write $2^m \cdot 3^m \cdot 5^m = 0$. Then $R \cong R/2^mR \times R/3^mR \times R/5^mR$. Set $R_3 = R/5^mR$. Then $5 \in N(R_3)$, and so $4 = 5 - 1 \in U(R_3)$. This implies that $2 \in U(R_3)$. By hypothesis, we easily see that $R_3$ is a 2-UU ring. Thus, $2^2 \in 1 + N(R_3)$; hence, $3 \in N(R_3)$. As
(3^m, 5^n) = 1, we see that 5 ∈ U(R), a contradiction. Therefore
R ∼= R/2^n R, or R/3^n R, or the product of such rings. This implies
that 6 ∈ N(R), as asserted. □

**Lemma 2.2.** Let R be a feebly clean 2-UU ring. Then J(R) is nil.

*Proof.* By Lemma 2.1, 6 is in N(R). Say 6^n = 0. Then R ∼= R_1 × R_2,
where R_1 ∼= R/2^n R, R_2 ∼= R/3^n R. As 2 ∈ N(R_1), We have
a = e + u
for some u ∈ U(R_1), as R is 2-UU ring then so is R_1 so u^2 = 1 + w
for some w ∈ N(R_1), also 2 ∈ N(R_1) then, (u − 1)(w+1)+2(1−u) ∈
N(R_1), this implies that (u − 1) ∈ N(R_1). We get a = e + v + 1
for some v ∈ N(R_1). We deduce that R_1 is strongly 2-nil-clean.
According to [3, Theorem 3.3], J(R_1) is nil. Let x ∈ J(R_2). As
R_2 is a 2-UU ring, (1 + x)^2 = 1 + w for some w ∈ N(R_2), i.e.,
x(x + 2) = w. As 3 ∈ N(R_2), we see that 2 = 3 − 1 ∈ U(R_2) and so
x+2 = 2(1+2^{-1}x) ∈ U(R_2). By applying (x+2)^{-1}w = w(x+2)−1,
we deduce that x = w(x + 2)−1 ∈ N(R_2), and so J(R_2) is nil.
Accordingly, J(R) is nil, hence the result. □

**Lemma 2.3.** Let I be an ideal of a feebly clean 2-UU ring R. If
R/I is a domain and 3 ∈ R is nilpotent, then every unit lifts modulo I.

*Proof.* Take a ∈ U(R/I). Since R is feebly clean, we can find
orthogonal idempotents e, f ∈ R and u ∈ U(R) such that a =
e − f + u. Since R/I is a domain, \{e, f\} ⊆ \{e, f\} in R/I.
As R is a 2-UU ring, u^2 = 1 + w for some w ∈ N(R). Hence,
π^2 = π in R/I, and so π ± 1.
Case I. e − f ≡ 0 (mod I). Then a ≡ u (mod I).
Case II. e − f ≡ 1 (mod I). If π = 1, then a − 2 ∈ I with
2 ∈ U(R). If π = 1, then a ∈ I, a contradiction.
Case III. e − f ≡ −1 (mod I). If π = 1, then a ∈ I, a contradic-
tion. If π = 1, then a + 2 ∈ I with 2 ∈ U(R).
Therefore we complete the proof. □

Recall that a ring R is clean if every element in R is the sum of
an idempotent and a unit (see [2]). We have

**Lemma 2.4.** A ring R is strongly 2-nil-clean if and only if
(1) $R$ is feebly clean;
(2) $R$ is a 2-UU ring.
(3) $N(R)$ forms an ideal of $R$.

Proof. $\implies$ (1) is obvious as every strongly 2-nil-clean ring is clean and so feebly clean.
(2) Let $u \in U(R)$, as $R$ is strongly 2-nil-clean, in view of [3, Theorem 2.8], there exist $p^3 = p \in R$ and $w \in N(R)$ such that $u = p + w, pw = wp$, then $u^2 = p^2 + v$ for some $v \in U(R)$ so $u^2 - v = p^2$ and $p^4 = p^2$, this implies that $p^2 = 1$ and so $u^2 = 1 + w$ is a unipotent.
(3) follows from [3, Theorem 3.6].

$\impliedby$ By Lemma 2.1, $6 \in N(R)$. Say $6^n = 0$. Then $R \cong R_1 \times R_2$, where $R_1 \cong R/2^n R$, $R_2 \cong R/3^n R$. Clearly, $R_i$ is feebly clean, $R_i$ is a 2-UU ring and $N(R_i)$ forms an ideal of $R_i$ for $i = 1, 2$.

Step 1. Let $a \in R_1$. Then there exists orthogonal idempotents $e, f \in R$ and a unit $u \in R$ such that $a = e - f + u$. Hence, $a = (e + f) + 2(f + u)$. Clearly, $(e + f)^2 = e + f$. As $2 \in N(R_1)$, we see that $2f + u = (2fu^{-1} + 1)u$ is invertible. Thus, $R_1$ is clean. We have $a = e + u$ for some $u \in U(R_1)$, as $R$ is 2-UU ring then so is $R_1$ so $u^2 = 1 + w$ for some $w \in N(R_1)$, also $2 \in N(R_1)$ then, $(u - 1)(u + 1) + 2(1 - u) \in N(R_1)$, this implies that $(u - 1) \in N(R_1)$. We get $a = e + v + 1$ for some $v \in N(R_1)$. We deduce that $R_1$ is strongly 2-nil-clean.

Step 2. Suppose that $\overline{x^2} = \overline{0}$ in $R_2/J(R_2)$. Then $x^2 \in J(R_2)$. In view of Lemma 2.2, $J(R_2)$ is nil; hence, $x \in N(R_2)$. By hypothesis, $x \in J(R_2)$. This shows that $R_2/J(R_2)$ is reduced. In light of [10, Theorem 12.7], it is the subdirect product of domains $S_i$. This, there exists epimorphisms $\varphi_i : R_2/J(R_2) \to S_i$ such that $\bigcap Ker(\varphi_i) = 0$.

Since $R_2$ is feebly clean, then so is $R_2/J(R_2)$. As every unit lifts modulo $J(R_2)$, we see that $R_2/J(R_2)$ is a 2-UU ring. Thus, $R_2/J(R_2)$ is a feebly clean 2-UU ring with $3 \in R_2/J(R_2)$ is nilpotent.

As $S_i$ is domain and $S_i \cong R_2/J(R_2)/Ker(\varphi_i)$. It follows by Lemma 2.3 that every unit modulo $Ker(\varphi)$. It follows that $S_i$ is a 2-UU ring. But $S_i$ is a domain, we see that $U(S_i) = \{-1, 1\}$. 

Since $S_i$ is a homomorphic image of $R_2/J(R_2)$, we see that $S_i$ is feebly clean. But all idempotents in $S_i$ are $0, 1$, and so $S_i = \{-2, -1, 0, 1, 2\}$. This implies that $S_i$ is commutative.

Since $R_2/J(R_2)$ is the subdirect product of $S_i$, it is isomorphic to the subring of $\Pi S_i$, and so $R_2/J(R_2)$ is commutative. Thus, $R_2/J(R_2)$ is strongly feebly clean. According to [3, Lemma 2.2], $R_2$ is strongly 2-nil-clean, and so the result is proved. □

We have accumulated all the information necessary to prove the following.

**Theorem 2.5.** A ring $R$ is strongly 2-nil-clean if and only if

1. $R$ is feebly clean;
2. $J(R)$ is nil;
3. $U(R/J(R))$ has exponent $\leq 2$.

**Proof.** $\Longrightarrow$ (1) and (2) are obvious by [3, Lemma 2.2] and [3, Theorem 3.3]. Let $\overline{u} \in U(R/J(R))$. Then $u \in U(R)$. $u^2 = 1 + w$, where $w \in N(R) \subseteq J(R)$. Thus, $\overline{u}^2 = \overline{1}$, as desired.

$\Longleftarrow$ As in the proof of Lemma 2.4, $R \cong R_1 \times R_2$ with $2 \in N(R_1)$ and $3 \in N(R_2)$. Clearly, each $S_i$ is feebly clean, $J(S_i)$ is nil and $U(S_i/J(S_i))$ has exponent $\leq 2$.

Step 1. Let $u \in U(R_1)$. Then $u^2 = 1 + r$ for some $r \in J(R_1)$, and so $r \in N(R_1)$. Thus, $R_1$ is a 2-UU ring. As $2 \in N(R_1)$, we see that $R_1$ is clean. Thus, $R_1$ is strongly 2-nil-clean as we see in the proof of Lemma 2.4.

Step 2. Let $a \in N(R_2)$. Then $1 + a \in U(R_2)$. Hence, $(1 + a)^2 = 1 + w$ for some $w \in J(R_2)$. Hence, $a(a + 2) = w$. As $2 \in U(R_2)$, we see that $a + 2 \in U(R_2)$. Therefore $a = w(a + 2)^{-1} \in J(R_2)$. Therefore $N(R_2) = J(R_2)$ is an ideal of $R_2$. Let $u \in U(R_2)$. Then $\overline{u} \in U(R_2/J(R_2))$; hence, $\overline{u}^2 = \overline{1}$; whence, $u^2 \in 1 + J(R_2) \subseteq 1 + N(R_2)$. Thus, $R_2$ is a 2-UU ring. In light of Lemma 2.4, $R_2$ is strongly 2-nil-clean.

Therefore $R$ is strongly 2-nil-clean, as asserted. □

**Corollary 2.6.** A ring $R$ is strongly nil-clean if and only if

1. $R$ is feebly clean;
(2) \( J(R) \) is nil;
(3) \( U(R) = 1 + J(R) \).

**Proof.** \( \Rightarrow \) This is obvious, by [?????????].

\( \Leftarrow \) Clearly, \( U(R/J(R)) \) has exponent \( \leq 2 \). In view of Theorem 2.5, \( R \) is strongly 2-nil-clean. As \(-1 \in 1 + J(R)\), we see that \( 2 \in J(R) \) is nil. According to [3, Theorem 2.11], \( R \) is strongly nil-clean. \( \square \)

**Example 2.7.** Let \( R = \mathbb{Z}(2) \cap \mathbb{Z}(3) = \{\frac{m}{n} \mid (m, n) = 1, m, n \in \mathbb{Z}, 2, 3 \nmid n\} \). Then \( R \) is feebly clean and \( U(R/J(R)) \) has exponent \( \leq 2 \), but \( R \) is not strongly 2-nil-clean.

**Proof.** In view of [1, Example 3.3], \( R \) is feebly clean. Since \( J(R) = 2R \cap 3R \),
\[
\frac{R}{J(R)} \cong \frac{R}{2R} \times \frac{R}{3R} \cong \mathbb{Z}_2 \times \mathbb{Z}_3;
\]
hence, \( U(R/J(R)) = \{(1, 1), (1, -1)\} \), which has exponent \( \leq 2 \). But \( R \) is not strongly 2-nil-clean, as \( J(R) \) is not nil. \( \square \)

### 3. Weakly Exchange Properties

The goal of this section is to characterize strongly 2-nil-clean rings by means of weakly exchange rings. In fact we extend the results in [9] from exchange rings to weakly exchange rings. An element \( a \in R \) is exchange if there exists an idempotent \( e \in aR \) such that \( v1 - e \in (1 - a)R \). We have

**Lemma 3.1.** Let \( R \) be weakly exchange. If \( R \) is a 2-UU ring, then \( 6 \in R \) is nilpotent.

**Proof.** Since \( R \) is weakly exchange, then 3 or \(-3\) is exchange.

Case 1. \( 3 \in R \) is exchange. Then there exists an idempotent \( e \in R \) such that \( e \in 3R \) and \( (1 - e) \in (1 - 3)R \). There exist \( a, b \in R \) such that \( e = 3a \) and \( (1 - e) = -2b \), where \( ae = a \) and \( b(1 - e) = b \). Now \( 3 = (1 - e) + (3 - (1 - e)) \). It is easy to prove that \( 3 - (1 - e) \) is a unit with inverse \( a - b \).

Case 2. \(-3\) is exchange. By the similar argument above, we can find an idempotent \( e \in R \) and a unit \( v \in R \) such that \(-3 = e + v \). Hence, \( 3 = -e - v \).
Accordingly, $3 \in R$ is feebly clean. As in the proof of Lemma 2.1, we conclude that $6 \in N(R)$. □

Recall that a ring $R$ is weakly clean if every element in $R$ is the sum or difference of a nilpotent and an idempotent (see [7]).

**Lemma 3.2.** A ring $R$ is strongly 2-nil-clean if and only if

1. $R$ is weakly clean;
2. $J(R)$ is nil;
3. $U(R/J(R))$ has exponent $\leq 2$.

**Proof.** $\Longrightarrow$ As $R$ is strongly 2-nil-clean, by [3, Proposition 3.5] it is strongly clean and then it is weakly clean. (2), (3) follow from Theorem 2.5.

$\Longleftarrow$ Clearly, $R$ is feebly clean and so the result follows from Theorem 2.5. □

**Theorem 3.3.** A ring $R$ is strongly 2-nil-clean if and only if

1. $R$ is weakly exchange;
2. $J(R)$ is nil;
3. $U(R/J(R))$ has exponent $\leq 2$.

**Proof.** $\Longrightarrow$ In view of [3, Proposition 3.5], every strongly 2-nil-clean ring ring is clean, and so it is weakly exchange. Thus, this implication is obtained by Theorem 2.5.

$\Longleftarrow$ Let $u \in U(R)$. Then $u^2 = 1$ in $R/J(R)$. Hence, $u^2 - 1 \in J(R) \subseteq N(R)$, and so $R$ is a 2-UU ring. In view of Lemma 4.1, $6 \in N(R)$. Write $2^n3^n = 0$. Then $R \cong R_1 \times R_2$ where $R_1 = R/2^nR$ and $R_2 = R/3^nR$. Obviously, each $S_i$ is weakly exchange, $J(S_i)$ is nil and $U(S_i/J(S_i))$ has exponent $\leq 2$.

Step 1. Let $u \in U(R_1)$. Then $u^2 = 1 + r$ for some $r \in J(R_1)$, and so $r \in N(R_1)$. Thus, $R_1$ is a 2-UU ring.

Since $2 \in N(R_1)$, we see that $2 \in J(R_1)$. In light of [5, Theorem 2.2], $R_1$ is exchange. Therefore $R_1$ is strongly 2-nil-clean, as we see in the proof of Lemma 2.4.

Step 2. Let $a \in N(R_2/J(R_2))$. Since $J(R_2)$ is nil, we see that $a \in N(R_2)$, and so $1 + a \in U(R_2)$. Hence, we can find some $w \in J(R_2)$ such that $(1 + a)^2 = 1 + w$. This shows that $a(2 + a) = w$. As $3 \in N(R_2)$, we see that $2 + a \in U(R_2)$, and so $a = (2 + a)^{-1}w \in$
Thus, \( R_2/J(R_2) \) is reduced, and so it is abelian. Clearly, \( R_2/J(R_2) \) is weakly exchangeable. In light of [7, 8], \( R_2/J(R_2) \) is feebly clean. Obviously, \( J(R/J(R_2)) = 0 \) and \( U(R_2/J(R_2)) \) has exponent \( \leq 2 \). Applying Theorem 2.5 to \( R_2/J(R_2) \), we see that \( R_2/J(R_2) \) is strongly 2-nil-clean. Since \( J(R_2) \) is nil, we show that \( R_2 \) is strongly 2-nil-clean, by [3, Lemma 3.1].

Therefore \( R \) is strongly 2-nil-clean, as asserted. \( \square \)

**Corollary 3.4.** A ring \( R \) is strongly nil-clean if and only if

1. \( R \) is weakly exchange;
2. \( J(R) \) is nil;
3. \( R \) is a UU ring.

**Proof.** \( \implies \) This is obvious.

\( \Leftarrow \) in light of Theorem 3.3, \( R \) is strongly 2-nil-clean. By the UU property of \( R \), according to [9, Theorem 2.8] \( 2 \in N(R) \). Then by applying [3, Theorem 2.11], \( R \) is strongly nil-clean. \( \square \)

A ring \( R \) is strongly weakly nil-clean if every element in \( R \) is the sum or difference of a nilpotent and an idempotent that commute (see [4]). We now turn to describe strongly weakly clean rings and thereby answer the Danchev’s problem.

**Lemma 3.5.** A ring \( R \) is strongly weakly nil-clean if and only if

1. \( R \) has no homomorphic image \( \mathbb{Z}_3 \times \mathbb{Z}_3 \);
2. \( R \) is strongly 2-nil-clean.

**Proof.** \( \implies \) If \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) is a homomorphic image of \( R \), then it is strongly weakly nil-clean. But \( (1, -1) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \) is not strongly weakly nil-clean. This gives a contradiction. Thus proving (1).

Let \( a \in R \). In view of [4, Theorem 2.1], \( a + a^2 \in N(R) \). Hence, \( a^2 - a^4 = (a - a^2)(a + a^2) \in N(R) \). Then \( a - a^3 \in N(R) \), and thus proving (2) by [3, Theorem 2.3].

\( \Leftarrow \) In view of [3, Lemma 4.1], \( R/J(R) \) is isomorphic to a Boolean ring, a Yaqub ring, or the product of such rings. By (1), \( R/J(R) \) is isomorphic to a Boolean ring, \( \mathbb{Z}_3 \) or the product of such rings. In light of [4, Corollary 3.2], \( R \) is strongly weakly nil-clean. \( \square \)
Recall that a ring $R$ is WUU if for any unit $u \in R$, $1 \pm u \in R$ is a unipotent. We now describe weakly exchange WUU ring and extend [9, Corollary 2.15] as follows.

**Theorem 3.6.** A ring $R$ is strongly weakly nil-clean if and only if

1. $R$ is weakly exchange;
2. $R$ is WUU.

*Proof.* $\implies$ Clearly, every exchange ring is weakly exchange. Thus, this implication is obtained by [9, Corollary 2.15].

$\Longleftarrow$ Since $R$ is WUU, it is 2-UU. By virtue of Theorem 3.3, $R$ is strongly 2-nil-clean. If $R$ has homomorphic image $\mathbb{Z}_3 \times \mathbb{Z}_3$, then $\mathbb{Z}_3 \times \mathbb{Z}_3$ is WUU. But $(-1,1) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ is invertible, but $(-1,1)-(1,1)$ and $(-1,1)+(1,1)$ are not nilpotent, a contradiction. Therefore $R$ is strongly weakly nil-clean, by Lemma 3.5. □

**Corollary 3.7.** A ring $R$ is strongly weakly nil-clean if and only if

1. $R$ is weakly exchange;
2. every unit in $R$ is strongly weakly nil-clean.

*Proof.* $\implies$ This is clear.

$\Longleftarrow$ Let $u \in U(R)$. Then there exists an idempotent $e \in R$ such that $w := u \pm e \in N(R)$ and $ue = eu$. Hence, $e = w - u$ or $u - w$. Thus, $e \in U(R)$, and so $e = 1$. This shows that $u \in \pm 1 + N(R)$, i.e., $R$ is WUU. This completes the proof, by Theorem 3.6. □

The next observation is a generalization of [9, Theorem ??? and Corollary 2.15].

**Corollary 3.8.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is strongly nil-clean.
2. $R$ is a weakly exchange UU ring.
3. $R$ is a weakly exchange in which every unit is strongly nil-clean.

*Proof.* (1) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (2) Let $u \in U(R)$. Then there exists an idempotent $e \in R$ and $w \in N(R)$ such that $u = e - w$. Hence, $e = u + w \in U(R)$. This implies that $e = 1$, and so $u = 1 + w$. Thus, $R$ is UU, as desired.
In view of Theorem 3.6, \( R \) is strongly weakly nil-clean rings. As \( R \) is a UU ring, \(-1 \in 1 + N(R)\), and so \( 2 \in N(R) \). This completes the proof by \([4]\).

\[\square\]

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