NONLOCAL SCHRÖDINGER EQUATIONS FOR INTEGRO-DIFFERENTIAL OPERATORS WITH MEASURABLE KERNELS AND ASYMPTOTIC POTENTIALS.

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Abstract. In this paper, we investigate the existence of nonnegative solutions for the problem

$$-L_K u + V(x) u = f(u)$$

in $\mathbb{R}^n$, where $-L_K$ is an integro-differential operator with measurable kernel $K$ and $V$ is a continuous potential. Under appropriate hypothesis, we prove, using variational methods, that the above equation has solution.

1. Introduction

In this article we consider the class of integro-differential Schrödinger equations

$$(P) \quad -L_K u + V(x) u = f(u), \quad \text{in } \mathbb{R}^n,$$

where $-L_K$ is an integro-differential operator given by

$$-L_K u(x) = 2 \int_{\mathbb{R}^n} (u(x) - u(y)) K(x - y) dy$$

and $K$ satisfy general properties. This study leads both to nonlocal and to nonlinear difficulties. For example, we can not benefit from the $s$-harmonic extension of [10] or commutator properties (see [28]).

The study of nonlocal operators is important because they intervene in a quantity of applications and models. For example, we mention their use in phase transition models (see [1], [9]), image reconstruction problems (see [23]), obstacle problem, optimization, finance, phase transitions. Integro-differential equations arise naturally in the study of stochastic processes with jumps, and more precisely of Lévy processes.

This paper was motivated by [3], where the authors study the existence of positive solutions for the problem

$$\begin{cases}
-\Delta u + V(x) u = f(u), & \text{in } \mathbb{R}^n, \\
u \in D^{1,2} (\mathbb{R}^n)
\end{cases}$$

where $V$ and $f$ are continuous functions with $V$ being a nonnegative function and $f$ having a subcritical or critical growth. Our purpose is to study an analogously problem, considering the operator $-L_k$ instead of the Laplacian operator.

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Several papers have studied the problem \((P)\) when \(K(x) = \frac{C_{n,s}}{2} |x|^{n+2s}\), where
\[
C_{n,s} = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}} d\xi \right)^{-1},
\]
that is, when the operator \(-\mathcal{L}_k\) is the fractional Laplacian operator (see \cite{17}). Next, we will mention some of these papers. In \cite{4}, the author has proved the existence of positive solutions from \((P)\). In \cite{39}, the authors study the problem \((P)\) when \(V\) is a constant small enough. Also, in \cite{28}, the authors have shown the existence of solutions for \((P)\) when \(f\) is asymptotically linear and \(V\) is constant. In \cite{40}, the authors study the problem \((P)\) when \(V \in C^n(\mathbb{R}^n, \mathbb{R})\), \(V\) is positive and
\[
\lim_{n \to \infty} V(|x|) \in (0, \infty).
\]
In \cite{42}, the authors have studied \((P)\) when \(V\) and \(f\) are asymptotically periodic. When \(V = 1\), Felmer et al. has studied the existence, regularity and qualitative properties of ground states solutions for problem \((P)\) (see \cite{21}). In \cite{35}, Teng and He have shown the existence of solution for \((P)\) when \(f(x, u) = P(x)|u|^{p-2}u + Q(x)|u|^{2^*_s-2}u\), where \(2 < p < 2^*_s\) and the potential functions \(P(x)\) and \(Q(x)\) satisfy certain hypothesis. In \cite{39}, the authors have shown the existence of solution for \((P)\) when \(V \in C^n(\mathbb{R}^n, \mathbb{R})\) and there exists \(r_0 > 0\) such that, for any \(M > 0\),
\[
\text{meas}\{x \in B_{r_0}(y); V(x) \leq M\} \to 0 \text{ as } |y| \to \infty.
\]
In \cite{28} the problem \((P)\) was studied when \(V \in C^1(\mathbb{R}^n, \mathbb{R}), \liminf_{|x| \to \infty} V(x) \geq V_\infty\) where \(V_\infty\) is constant, and \(f \in C^1(\mathbb{R}^n, \mathbb{R})\). By method of the Nehari manifold, Sechi has shown that the problem \((P)\) has a solution if \(V \leq V_\infty\), but \(V\) is not identically equal to \(V_\infty\), where \(V_\infty\) is a constant. Also in \cite{28}, Sechi have obtained the existence of ground state solutions of \((P)\) for general \(s \in (0, 1)\) when \(V(x) \to \infty\) as \(|x| \to \infty\). In \cite{11}, the authors obtain the existence of a sequence of radial and non radial solutions for the problem \((P)\) when \(V\) and \(f\) are radial functions. Some other interesting studies by variational methods of the problem \((P)\) can be found in \cite{3}, \cite{7}, \cite{10}, \cite{14}, \cite{15}, \cite{20}, \cite{21}, \cite{22}, \cite{23}, \cite{24}, \cite{25}, \cite{27}, \cite{29}, \cite{30}, \cite{31}, \cite{33}, \cite{34}, \cite{37} and \cite{40}. Many of them use strong tools that we can not use here in our problem, as the \(s\)-harmonic extension and commutator properties.

Here, we will admit that the potential \(V\) is continuous and satisfies,
\begin{itemize}
  \item \((V_1)\) \(\inf_{x \in \mathbb{R}^n} V(x) > 0\);
  \item \((V_2)\) \(V(x) \leq V_\infty\) for some constant \(V_\infty > 0\) and for all \(x \in B_1(0)\).
\end{itemize}

Note that, \((V_1)\) implies that
\begin{itemize}
  \item \((V_4)\) There are \(R > 0\) and \(\Lambda > 0\) such that
  \[
  V(x) \geq \Lambda
  \]
  for all \(|x| \geq R\).
\end{itemize}

Also, we will assume that \(f \in C(\mathbb{R}, \mathbb{R})\) is a function satisfying:
\begin{itemize}
  \item \((f_1)\) \(|f(s)| \leq c_0 |s|^{p-1}\), for some constant \(C > 0\) and \(p \in (2, 2^*_s)\);
  \item \((f_2)\) There is \(\theta > 2\) such that
  \[
  \theta f(s) \leq sf(s)
  \]
  for all \(s > 0\);
  \item \((f_3)\) \(f(t) > 0\) for all \(t > 0\) and \(f(t) = 0\) for all \(t < 0\).
\end{itemize}
The kernel $K : \mathbb{R}^n \to (0, \infty)$ is a measurable function such that

- $(K_1)$ $K(x) = K(-x)$ for all $x \in \mathbb{R}^n$;
- $(K_2)$ There is $\lambda > 0$ and $s \in (0, 1)$ such that $\lambda \leq K(x)|x|^{n+2s}$ almost everywhere in $\mathbb{R}^n$;
- $(K_3)$ $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) = \min \{ |x|^2, 1 \}$.

Note that, when $K(x) = \frac{C_m}{|x|^2} |x|^{-(n+2s)}$ we have that $-\mathcal{L}_K$ is a fractional laplacian, $(-\Delta)^s$.

Our paper is organized as follows. In section 2, we will present some properties of the space in which we will study the problem $(P)$. In section 3, we will define an auxiliary problem and we will show that the functional energy associated with the auxiliary problem satisfies the condition of Palais-Smale. By difficulty nonlocal of the operator $-\mathcal{L}_K$, we will cannot use the same technique used in [3]. Therefore, we will present another technique to show this result. In section 4, we will prove that a general estimative for weak solution of

$$-\mathcal{L}_K u + b(x)u = g(x, u),$$

where $b \geq 0$, $|g(x, t)| \leq h(x)|t|$ and $h \in L^p(\mathbb{R}^n)$ with $q > \frac{np}{n-2s}$. We will show that there is $M = M(q, ||b||_{L^p})$ such that the solution $u$ satisfies

$$||u||_\infty \leq M||u||_{2^*_s}.$$

In [2], using the $s$-harmonic extension of [10], the authors has shown the same estimate when $-\mathcal{L}_K$ is the fractional Laplacian operator. In our case, we cannot use the $s$-harmonic extension, because we do not have an analogously extension for integro-differential operators. In section 5, we show our main result in this paper, the Theorem 5.2.

2. Preliminaries

Let $s \in (0, 1)$, we denote by $H^s(\mathbb{R}^n)$ the fractional Sobolev space. It is defined as

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n); \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dxdy < \infty \right\}.$$

The space $H^s(\mathbb{R}^n)$ is a Hilbert space with the norm

$$||u||_{H^s} = \left( \int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dxdy \right)^{\frac{1}{2}}.$$

We define $X$ as the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that any function $u$ in $X$ belongs to $L^2(\mathbb{R}^n)$ and the function

$$(x, y) \mapsto (u(x) - u(y)) \sqrt{K(x - y)}$$

is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. The function

$$||u||_X := \left( \int_{\mathbb{R}^n} u^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x - y) dxdy \right)^{\frac{1}{2}}$$

defines a norm in $X$ and $(X, || \cdot ||_X)$ is a Hilbert space. By $(K_2)$, the space $X$ is continuously embedded in $H^s(\mathbb{R}^n)$. Therefore, $X$ is continuously embedded in $L^p(\mathbb{R}^n)$ for $p \in [2, 2^*_s)$, where $2^*_s = \frac{2n}{n-2s}$. If $\Omega \subset \mathbb{R}^n$, we define

$$X_0(\Omega) = \{ u \in X; u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}.$$
The space $X_0(\Omega)$ is a Hilbert Space with the norm
\[
u \mapsto ||u||_{X_0(\Omega)} := \left( \int_{\Omega} u^2 \, dx + \int_{\Omega} (u(x) - u(y))^2 K(x - y) \, dx \, dy \right)^{\frac{1}{2}},
\]
where $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$ (see Lemma 7 in [30]). It is continuously embedded in $H_0^1(\mathbb{R}^n)$. For definition and properties of $H_0^1(\mathbb{R}^n)$ we indicate [17].

In the problem (P) we will consider the space $E$ defined as
\[(2.1)\]
\[E = \left\{ u \in X; \int_{\mathbb{R}^n} V(x) u^2 \, dx < \infty \right\}
\]
The space $E$ is a Hilbert space with the norm
\[
u \mapsto ||u|| := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x - y) \, dx \, dy + \int_{\mathbb{R}^n} V(x) u^2 \, dx \right)^{\frac{1}{2}}.
\]
If $u, v \in C_0^\infty(\mathbb{R}^n)$ then
\[-\mathcal{L}_k u, v \rangle_{L^2} = [u, v],
\]
where
\[[u, v] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy.
\]
Therefore, we say that $u \in E$ is a solution for the problem (P) if
\[[u, v] + \int_{\mathbb{R}^n} V(x) uv \, dx = \int_{\mathbb{R}^n} f(x)v \, dx
\]
for all $v \in E$, that is
\[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy + \int_{\mathbb{R}^n} V(x) uv \, dx = \int_{\mathbb{R}^n} f(x)v \, dx.
\]

Let $A, B \subset \mathbb{R}^n$ and $u, v \in X$. We will denote
\[[u, v]_{A \times B} = \int_A \int_B (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy
\]
and we will denote $[u, v]_{\mathbb{R}^n \times \mathbb{R}^n}$ by $[u, v]$.

The Euler-Lagrange functional associated with (P) is given by
\[I(u) = \frac{1}{2} ||u||^2 - \int_{\mathbb{R}^n} F(u) \, dx,
\]
where
\[F(t) = \int_0^t f(s) \, ds.
\]
From hypothesis about $f$, the functional is $C^1(E, \mathbb{R})$ and
\[F'(u)v = [u, v] + \int_{\mathbb{R}^n} V(x) uv \, dx - \int_{\mathbb{R}^n} f(u)v \, dx.
\]
We will denote by $B$ the unitary ball of $\mathbb{R}^n$. Define $I_0 : X_0(B) \rightarrow \mathbb{R}$ by
\[I_0(u) =: \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x - y) \, dx \, dy + \int_{\mathbb{R}^n} V_\infty u^2 \, dx - \int_{\mathbb{R}^n} F(u) \, dx,
\]
where $V_\infty$ is the constant of $V_2$. The functional $I_0$ has the mountain pass geometry. We will denote by $d$ the mountain pass level associated with $I_0$, that is
\[d = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_0(\gamma(t)),
\]
where
\begin{equation}
\Gamma = \{ \gamma \in C([0,1], X_0(\Omega)); \gamma(0) = 0 \text{ and } \gamma(1) = e \},
\end{equation}
with \( e \) fixed and verifying \( I_0(e) < 0 \). Note that \( d \) depends only on \( V_\infty, \theta \) and \( f \).

3. An Auxiliary Problem

According to \([3]\), we will modified the problem defining an auxiliary problem. But, as the operator \(-L_K\) is nonlocal, we can not use the same ideas of \([3]\) to prove that the functional associated the auxiliary problem satisfies the Palais-Smale condition. It is necessary that we use an another technics.

For \( k = \frac{2\theta}{\theta - 2} \) we consider
\begin{equation}
\tilde{f}(x, t) = \begin{cases}
  f(t) & \text{if } kf(t) \leq V(x)t \\
  V(x)k^{-1}t & \text{if } kf(t) > V(x)t
\end{cases}
\end{equation}
and
\begin{equation}
g(x, t) = \begin{cases}
  f(t) & \text{if } |x| \leq R \\
  f(x, t) & \text{if } |x| > R.
\end{cases}
\end{equation}

And we define the auxiliary problem
\begin{equation}
\begin{cases}
  -L_K u + V(x)u = g(x, u) & \text{in } \mathbb{R}^n \\
  u \in E
\end{cases}
\end{equation}

We have that, for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \)
\begin{enumerate}
  \item \( \tilde{f}(x, t) \leq f(t) \);
  \item \( g(x, t) \leq \frac{V(x)}{k}t \), se \( |x| \geq R \);
  \item \( G(x, t) = F(t) \), se \( |x| \leq R \)
  \item \( G(x, t) \leq \frac{V(x)}{2k}t^2 \), se \( |x| > R \);
\end{enumerate}
where
\[ G(x, t) = \int_0^t g(x, s)ds. \]

The Euler-Lagrange functional associated with the auxiliary problem is given by
\[ J(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^n} G(x, u)dx. \]

The functional \( J \in C^1(X, \mathbb{R}) \) and
\[ J'(u)v = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x) - u(y)) (v(x) - v(y)) K(x - y)dxdy + \int_{\mathbb{R}^n} V(x)uwdx - \int_{\mathbb{R}^n} g(x, u)vdx. \]

The functional \( J \) has the mountain pass geometry. Then, there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \) such that
\begin{equation}
J'(u_n) \to 0 \text{ and } J(u_n) \to c,
\end{equation}
where \( c > 0 \) is the mountain pass level associated with \( J \), that is
\begin{equation}
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))
\end{equation}
where
\[ \Gamma = \{ \gamma \in C([0,1], E); \gamma(0) = 0 \text{ and } \gamma(1) = e \}. \]
and \( e \) is the function fixed in \([2,2] \). By definition
\begin{equation}
c \leq d
\end{equation}
uniformly in $R > 0$.

**Lemma 3.1.** The sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded.

**Proof.** By (f2), (3) and (4)

\[
J(u) - \frac{1}{2} J'(u) u \\
= \left( \frac{\alpha}{2} \right) ||u||^2 + \frac{1}{4} ||u||^2 + \int_{\mathbb{R}^n} \frac{1}{2} g(x, u) u - G(x, u) dx \\
\geq \left( \frac{\alpha}{2} \right) ||u||^2 + \frac{1}{4} ||u||^2 + \int_{|x| > R} \frac{1}{2} g(x, u) u - \frac{1}{2} \int_{|x| > R} V(x) u^2 dx \\
\geq \left( \frac{1}{2\theta} \right) ||u||^2.
\]

Thereby

\[
(3.7) \quad |J(u)| + |J'(u) u| \geq \left( \frac{\theta - 2}{4\theta} \right) ||u||^2
\]

for all $u \in E$. This last inequality ensures that the sequence is bounded. \(\square\)

Let $r > R$ and $A = \{x \in \mathbb{R}^n: r < ||x|| < 2r\}$. Consider $\eta : \mathbb{R}^n \to \mathbb{R}$ a function such that $\eta = 1$ in $B_{3r}^c(0)$, $\eta = 0$ in $B_r(0)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| < \frac{1}{r}$. Note that

\[
(3.8) \quad (B_r \times B_r)^c = (B_r^c \times \mathbb{R}^n) \cup (B_r \times B_r^c),
\]

where $B_r = B_r(0)$ and $B_{2r} = B_{2r}(0)$. We will decompose

\[
(3.9) \quad B_r^c \times \mathbb{R}^n = (A \times \mathbb{R}^n) \cup (B_{2r}^c \times B_r) \cup (B_{2r} \times A) \cup (B_{2r} \times B_{2r}^c)
\]

and

\[
(3.10) \quad B_r \times B_r^c = (B_r \times A) \cup (B_r \times B_{2r}^c)
\]

**Lemma 3.2.** We have that

\[
\int_{B_r} \int_{B_{2r}^c} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x - y)dxdy \\
+ \int_{B_{2r}^c} \int_{B_r} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x - y)dxdy \\
\geq - \int_{B_r} \int_{B_{2r}^c} u_n(y)^2 K(x - y)dxdy
\]

**Proof.**

\[
\int_{B_r} \int_{B_{2r}^c} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x - y)dxdy \\
+ \int_{B_{2r}^c} \int_{B_r} (u_n(x) - u_n(y))(\eta(x)u_n(x) - \eta(y)u_n(y))K(x - y)dxdy \\
= \int_{B_r} \int_{B_{2r}^c} u_n(x)(u_n(x) - u_n(y))K(x - y)dxdy \\
- \int_{B_{2r}^c} \int_{B_r} u_n(y)(u_n(x) - u_n(y))K(x - y)dxdy \\
= \int_{B_r} \int_{B_{2r}^c} u_n(x)(u_n(x) - u_n(y))K(x - y)dxdy \\
- \int_{B_r} \int_{B_{2r}^c} u_n(y)(u_n(x) - u_n(y))K(x - y)dxdy \\
= \int_{B_r} \int_{B_{2r}^c} (u_n(x) - u_n(y))^2 K(x - y)dxdy \\
+ \int_{B_r} \int_{B_{2r}^c} u_n(x)^2 - u_n(y)^2 K(x - y)dxdy \\
\geq - \int_{B_r} \int_{B_{2r}^c} u_n(y)^2 K(x - y)dxdy
\]

\(\square\)

**Lemma 3.3.** Let $\epsilon > 0$. There is $r_0 > 0$ such that if $r > r_0$ then

\[
\int_{B_r} \int_{B_{2r}^c} u_n(y)^2 K(x - y)dxdy < \epsilon,
\]

for all $n \in \mathbb{N}$.
Proof. For each $y \in B_r(0)$

$$B_r(y) \subset B_{2r}(0).$$

Then

$$\int_{B_{2r}(0)} K(x-y)dx \leq \int_{B_r(y)} K(x-y)dx = \int_{B_r(0)} K(z)dz.$$

By Lemma 3.1 there is $L > 0$ such that $\|u_n\|^2_{L^2} < L$ for all $n \in \mathbb{N}$. By (K3), there is $r_0 > 0$ such that

$$\int_{B_r(0)} K(z)dz < \frac{\epsilon}{L},$$

for all $r > r_0$. Then by (3.11) for all $n \in \mathbb{N}$ and $r > r_0$

$$\int_{B_r(0)} \int_{B_{2r}(0)} u_n(y)^2K(x-y)dxdy = \int_{B_r(0)} \int_{B_{2r}(0)} u_n(y)^2K(x-y)dxdy \leq \int_{B_r(0)} \int_{B_{2r}(0)} u_n(y)^2K(z)dxdy \leq \epsilon.$$

□

Lemma 3.4. There are constants $K_1 > 0$ and $K_2 > 0$ such that

$$\int_A \int_{\mathbb{R}^n} |u_n(y)|^2|(u_n(x) - u_n(y))||(|\eta(x) - \eta(y)|)K(x-y)dxdy \leq \frac{2K_1}{\lambda \rho |\mathbb{R}^n|}\|u_n\|_{L^2(A)}[u_n, u_n]^{\frac{1}{2}} + K_2\|u_n\|_{L^2(A)}[u_n, u_n]^{\frac{1}{2}}.$$

Proof. Note that

$$\int_{\mathbb{R}^n} |\eta(x) - \eta(y)|^2K(x-y)dx = \int_{\mathbb{R}^n} |\eta(z + y) - \eta(y)|^2K(z)dz = \int_{B_1(0)} |\eta(z + y) - \eta(y)|^2K(z)dz + \int_{B_1(0)} |\eta(z + y) - \eta(y)|^2K(z)dz \leq \frac{4}{\lambda \rho |\mathbb{R}^n|} \int_{B_1(0)} |z|^2K(z)dz + 4 \int_{B_1(0)} K(z)dz \leq \frac{4}{\lambda \rho |\mathbb{R}^n|} P_1 + 4 P_2,$$

where

$$P_1 = \int_{B_1} |z|^2K(z)dz$$

and

$$P_2 = \int_{B_1} K(z)dz.$$

Let $K_1 = 2\sqrt{P_1}$ and $K_2 = 2\sqrt{P_2}$. Then, by Holder inequality

$$\int_A \int_{\mathbb{R}^n} |u_n(y)|^2|(u_n(x) - u_n(y))||(|\eta(x) - \eta(y)|)K(x-y)dxdy \leq (\frac{2K_1}{\lambda \rho |\mathbb{R}^n|} + 2\sqrt{P_2}) \int_A |u_n(y)| \left(\int_{\mathbb{R}^n} |(u_n(x) - u_n(y))|^2K(x-y)dx\right)^{\frac{1}{2}} dy \leq (K_1 + K_2)\|u_n\|_{L^2(A)}[u_n, u_n]^{\frac{1}{2}}.$$

□

Lemma 3.5. For the same constants $K_1 > 0$ and $K_2 > 0$ of the Lemma 3.4

$$\int_{B_r} \int_A |u_n(x) - u_n(y)||\eta(x)u_n(x) - \eta(y)u_n(y)|K(x-y)dxdy \leq \frac{K_1}{\lambda \rho |\mathbb{R}^n|}\|u_n\|_{L^2(A)}[u_n, u_n]^{\frac{1}{2}} + K_2\|u_n\|_{L^2(A)}[u_n, u_n]^{\frac{1}{2}}.$$
Proof. Indeed, by property $(K_1)$ of $K$
\[
\begin{align*}
\int_{B_r} \int_A |u_n(x) - u_n(y)| |\eta(x)u_n(x) - \eta(y)u_n(y)| K(x-y) \, dx \, dy \\
= \int_{B_r} \int_A |u_n(x)| |u_n(x) - u_n(y)| |\eta(x) - \eta(y)| K(x-y) \, dx \, dy \\
= \int_A \int_{B_r} |u_n(x)| |u_n(x) - u_n(y)| |n(x) - n(y)| K(x-y) \, dy \, dx \\
= \int_A \int_{B_r} |u_n(y)| |u_n(y) - u_n(x)| |n(y) - n(x)| K(y-x) \, dx \, dy
\end{align*}
\]
Using $(K_1)$ and Lemma 3.4, we prove this lemma.

Lemma 3.6. We have that
\[
- \int_{B^r_r} \int_A u_n(y)(u_n(x) - u_n(y))(\eta(x) - \eta(y)) K(x-y) \, dx \, dy
\leq \frac{1}{\lambda^2} \|u_n\|_{L^2(A)} [u_n, u_n]_{L^2} + K_2 \|u_n\|_{L^2(A)} [u_n, u_n]_{L^2}.
\]
Proof.
\[
\begin{align*}
- \int_{B^r_r} \int_A u_n(y)(u_n(x) - u_n(y))(\eta(x) - \eta(y)) K(x-y) \, dx \, dy \\
= \int_{B^r_r} \int_A u_n(x)(u_n(x) - u_n(y))^2 (\eta(x) - \eta(y)) K(x-y) \, dx \, dy \\
= \int_{B^r_r} \int_A u_n(x)(u_n(x) - u_n(y))^2 (n(x) - 1) K(x-y) \, dx \, dy \\
= \int_{B^r_r} \int_A u_n(x)(u_n(x) - u_n(y))(n(x) - n(y)) K(x-y) \, dx \, dy \\
\leq \int_{B^r_r} \int_A u_n(x)(u_n(x) - u_n(y))(n(x) - n(y)) K(x-y) \, dx \, dy \\
\leq \int_{B^r_r} \int_A [u_n(x)|u_n(y) - u_n(x)| |n(y) - n(x)| K(y-x) \, dx \, dy \\
= \int_A \int_{B^r_r} [u_n(x)|u_n(y) - u_n(x)| |n(y) - n(x)| K(y-x) \, dx \, dy \\
\leq K_1 \|u_n\|_{L^2(A)} [u_n, u_n]_{L^2} + K_2 \|u_n\|_{L^2(A)} [u_n, u_n]_{L^2}.
\end{align*}
\]
In the last inequality, we have used the Lemma 3.4 and $(K_1)$.

We will prove that the functional $J$ satisfies the Palais-Smale condition. The next proposition ensures the existence of solution in the level $c$ for the auxiliary problem (see 3.3). We emphasize that by a nonlocal difficulty, we can not repeat the same arguments used in 3 to show that the functional energy associated at the auxiliary problem satisfies the Palais-Smale condition, therefore we use another technique to show this.

Proposition 3.7. Suppose that $f$ and $V$ satisfy $(V_1)$, $(f_1)$ - $(f_3)$. Then the functional $J$ satisfies the Palais-Smale condition.

Proof. By Lemma 3.1 the Palais-Smale sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded. We can suppose that $\{u_n\}_{n \in \mathbb{N}}$ converges weakly to $u$. By $K$ properties we have that $\eta u_n \in X$ and $||\eta u_n|| \leq ||u_n||$ (see Lemma 5.1 in 17). Then, the sequence $\{\eta u_n\}$ is bounded in $X$. Therefore $I'(u_n)(\eta u_n) = o_n(1)$. That is

\[
[u_n, \eta u_n] + \int_{\mathbb{R}^n} V(x) u_n^2 \eta dx = \int_{\mathbb{R}^n} g(x, u_n) \eta u_n dx + o_n(1).
\]
But, note that $[u_n, \eta u_n] = [u_n, \eta u_n]_{C(B_r \times B_r)}$, because $\eta = 0$ in $B_r$. By 3.8 3.9 and 3.10 we have:
\[
[u_n, \eta u_n]_{A \times \mathbb{R}^n} + [u_n, \eta u_n]_{B^r, y \times A} + [u_n, \eta u_n]_{B^r, x \times B^r, y} + [u_n, \eta u_n]_{B^r, x \times A} + [u_n, \eta u_n]_{B^r, x \times B_r} + [u_n, \eta u_n]_{B^r, x \times B_r} + [u_n, \eta u_n]_{B^r, y \times A}
\]
\[
+ \int_{\mathbb{R}^n} V(x) u_n^2 \eta dx = \int_{\mathbb{R}^n} g(x, u_n) \eta u_n dx + o_n(1)
\]
From Lemmas 3.4, 3.5 and 3.6, we obtain constants $K$.

By Lemma 3.2, we have

\[ \int_{\mathbb{R}^n} V(x)u_n^2 \eta dx \leq \int_{\mathbb{R}^n} g(x, u_n)u_n \eta dx + \int_{B_2^r} u_n(y)^2 K(x-y)dx dy - [u_n, \eta u_n]_{B_2^r \times A} + o_n(1) \]

Above, we have used that $[u_n, \eta u_n]_{B_2^r \times B_2^r} = [u_n, u_n]_{B_2^r \times B_2^r} \geq 0$. Because $\eta = 1$ in $B_2^r$. If $C$ and $D$ are subsets of $\mathbb{R}^n$ and $u \in E$, then

\[ [u, \eta u]_{C \times D} = \int_C \int_D (u(x) - u(y))(\eta u(x) - \eta u(y))K(x-y)dxdy \]

\[ = \int_C \int_D \eta(x)(u(x) - u(y))^2 K(x-y)dxdy \]

\[ + \int_C \int_D u(y)(u(x) - u(y))(\eta(x) - \eta(y))K(x-y)dxdy. \]

Thereby,

\[ \int_{\mathbb{R}^n} g(x, u_n)u_n \eta dx + \int_{B_2^r} u_n(y)^2 K(x-y)dx dy - [u_n, \eta u_n]_{B_2^r \times A} \]

\[ = \int_{B_2^r} u_n(y)(\eta u_n(x) - \eta u_n(y))(\eta(x) - \eta(y))K(x-y)dx dy \]

\[ + \int_{B_2^r} \eta u_n(y)(u_n(x) - u_n(y))(\eta(x) - \eta(y))K(x-y)dx dy + o_n(1). \]

From Lemmas 3.4, 3.5 and 3.6, we obtain constants $K_1, K_2 > 0$ such that

\[ \int_{\mathbb{R}^n} V(x)u_n^2 \eta dx \]

\[ \leq \int_{B_2^r} u_n(y)^2 K(x-y)dx dy \]

\[ + \int_{B_2^r} \eta u_n(y)(u_n(x) - u_n(y))(\eta(x) - \eta(y))K(x-y)dx dy \]

\[ + \int_{B_2^r} \eta u_n(y)(u_n(x) - u_n(y))(\eta(x) - \eta(y))K(x-y)dx dy \]

\[ + \int_{B_2^r} u_n(y)(\eta u_n(x) - \eta u_n(y))(\eta(x) - \eta(y))K(x-y)dx dy + o_n(1). \]

By (2), (f3) and $r > R$ we have

\[ \int_{\mathbb{R}^n} g(x, u_n)\eta u_n dx \leq \frac{1}{k} \int_{\mathbb{R}^n} \eta V(x)u_n^2 dx. \]

Thereby,

\[ (1 - \frac{1}{k}) \int_{\mathbb{R}^n} V(x)u_n^2 \eta dx \]

\[ \leq \int_{B_2^r} u_n(y)^2 K(x-y)dx dy \]

\[ + \frac{K_1}{k} \|u_n\|_{L^2(A)}^2 + K_2 \|u_n\|_{L^2(A)}^2 + o_n(1) \]

By Lemma 3.1, there is $C_1 > 0$ such that $\|u_n\| \leq C_1$. Then, for some constant $C > 0$

\[ (1 - \frac{1}{k}) \int_{|x| > 2r} V(x)u_n^2 dx \]

\[ \leq \int_{B_2^r} u_n(y)^2 K(x-y)dx dy + C(\frac{1}{k} + 1)\|u_n\|_{L^2(A)} + o_n(1) \]

Let $\epsilon > 0$. By Lemma 3.3, we can take $r$, large enough, such that

\[ \int_{|x| > 2r} V(x)u_n^2 dx \leq \epsilon(\frac{k-1}{2r} + C(\frac{1}{k} + 1)\|u_n\|_{L^2(A)} + o_n(1) \]

for all $n \in \mathbb{N}$. Also, we can assume that

\[ \|u\|_{L^2(A)} \leq \frac{\epsilon(k-1)}{6C(\frac{1}{k} + 1)k} \]

By property (2) of $g$

\[ g(x, u_n) \leq \frac{V(x)}{k} u_n^2. \]

NONLOCAL SCHRODINGER EQUATIONS
for all $x$, with $|x| > 2r > R$. Therefore, by (5.12)
\[
\int_{|x| > 2r} g(x, u_n)u_n dx \leq \frac{\epsilon}{3} + C \left( \frac{1}{r} + 1 \right) \frac{k}{k-1} ||u_n||_{L^2(A)} + o_n(1).
\]
By (5.13) and compact embedding of the fractional Sobolev spaces, we can take $n_1 \in \mathbb{N}$ such that if $n > n_1$ then
\[
\int_{|x| > 2r} g(x, u_n)u_n dx \leq \frac{5\epsilon}{6}.
\]
Note that, we can suppose that $r > 0$ satisfies
\[
\int_{|x| > 2r} g(x, u) u dx \leq \frac{\epsilon}{12}.
\]
By compact embedding of fractional sobolev spaces we have that, there is, $n_0 \in \mathbb{N}$ with $n_0 > n_1$ and if $n > n_0$ then
\[
\left| \int_{|x| \leq 2r} g(x, u_n)u_n dx - \int_{|x| \leq 2r} g(x, u) u dx \right| < \frac{\epsilon}{12}.
\]
Thereby, for $n > n_0$ we have
\[
\left| \int_{\mathbb{R}^n} g(x, u_n)u_n dx - \int_{\mathbb{R}^n} g(x, u) u dx \right| < \epsilon
\]
that is
\[
\lim_{n \to \infty} \int_{\mathbb{R}^n} g(x, u_n)u_n dx = \int_{\mathbb{R}^n} g(x, u) u dx
\]
From $I'(u_n)u_n = o_n(1)$, we conclude that $||u_n|| \to ||u||$ and therefore $\{u_n\}$ converges to $u$ in $X$. 

\section*{Corollary 3.8.}
Suppose $(V_1), (f_1) - (f_3)$. Then, there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Moreover, $u \geq 0$ almost everywhere in $\mathbb{R}^n$.

\section*{Proof.}
By (3.4) and Proposition (3.7) there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Let $A = \{ x \in \mathbb{R}^n : |x| > R \} \cap \{ x \in \mathbb{R}^n : u(x) < 0 \}$. If $x \in A$, then $g(x, u(x)) = \frac{V(u)}{k} u(x)$ and in if $x \in A^c$, then $g(x, u) \geq 0$. We have
\[
0 \geq [u, u^-] + \int_{A^c} V(x) u^- dx = \left( \frac{1}{k} - 1 \right) \int_A V(x) u^- dx + \int_{A^c} g(x, u) u^- dx \geq 0
\]
where $u^-(x) = \max \{-u(x), 0\}$. Then
\[
[u, u^-] = 0
\]
This implies that $u^- = 0$ (see proof of Lemma 4.1 in [19]). 

As a consequence of inequalities (5.6) and (3.7) we have the following proposition.

\section*{Proposition 3.9.}
If $V$ and $f$ satisfies $(V_1), (V_2), (f_1) - (f_3)$, then the solution $u$ of the auxiliary problem satisfies
\[
||u||^2 \leq 2kd
\]
uniformly in $R > 0$. 

4. $L^\infty$ Estimative for Solution of Auxiliary Problem

In this section, we will prove a Brezis-Kato estimative. We will prove that, admitting some hypothesis, there is $M > 0$ such that the solution of the problem

\[
\begin{cases}
-\mathcal{L}_k v + b(x)v = g(x, v) & \text{in } \mathbb{R}^n
\end{cases}
\]

satisfies $\|u\|_{L^\infty(\mathbb{R}^n)} \leq M$ and $M$ does not depend on $\|u\|$ (see Proposition 4.5). We emphasize that, in the best of our knowledges, this result is being presented for the first time in the literature. In [2], the authors have shown this result when the operator $\mathcal{L}_K$ is the fractional Laplacian operator, that is, when $K(x) = C_n s |x|^{-n-2s}$. But, in our case, we can not use the same technique used in [2], because we do not have a version of the $s$-harmonic extension for more general integro-differential operators. Therefore, we present an another technique.

**Remark 4.1.** Let $\beta > 1$. Define the real functions

\[
m(x) := (\lambda - 1)(x^\beta + x^{-\beta}) - \lambda(x^{\beta - 1} + x^{1-\beta}) + 2
\]

and

\[
p(x) := (\lambda - 1)(x^\beta + x^{-\beta}) + \lambda(x^{\beta - 1} + x^{1-\beta}) - 2.
\]

where $\lambda := \frac{\beta^2}{2\beta^2 - 1}$. Then $m(x) \geq 0$ and $p(x) \geq 0$ for all $x > 0$.

We will omit the proof of the claim that appears in the Remark 4.1.

Let $\beta > 1$ and define

\[
f(x) = x|x|^{2(\beta-1)}
\]

and

\[
g(x) = x|x|^{\beta-1}.
\]

The functions $f$ and $g$ are continuous and differentiable for all $x \in \mathbb{R}$. Consider $x, y \in \mathbb{R}$ with $x \neq y$. By Mean Value Theorem, there are $\theta_1(x, y), \theta_2(x, y) \in \mathbb{R}$ such that

\[
f'(\theta_1(x, y)) = \frac{f(x) - f(y)}{x - y}
\]

and

\[
g'(\theta_2(x, y)) = \frac{g(x) - g(y)}{x - y},
\]

that is

\[
(2\beta - 1)|\theta_1(x, y)|^{2(\beta-1)} = \frac{x|x|^{2(\beta-1)} - y|y|^{2(\beta-1)}}{x - y}
\]

and

\[
\beta|\theta_2(x, y)|^{(\beta-1)} = \frac{x|x|^{(\beta-1)} - y|y|^{(\beta-1)}}{x - y}.
\]

Implying that

\[
|\theta_1(x, y)| = \left(\frac{1}{2\beta - 1} \frac{x|x|^{2(\beta-1)} - y|y|^{2(\beta-1)}}{x - y}\right)^{\frac{1}{2\beta - 1}}
\]

and

\[
|\theta_2(x, y)| = \left(\frac{1}{\beta} \frac{x|x|^{(\beta-1)} - y|y|^{(\beta-1)}}{x - y}\right)^{\frac{1}{\beta}}.
\]

We consider $\theta_1(x, x) = \theta_2(x, x) = 0$ for all $x \in \mathbb{R}$. 
Remark 4.2. Note that $|\theta_1(x, y)| = |\theta_1(y, x)|$ and $|\theta_2(x, y)| = |\theta_2(y, x)|$ for all $x, y \in \mathbb{R}$.

Lemma 4.3. With the same notation, if $x \neq 0$ then

$$|\theta_1(x, 0)| \geq |\theta_2(x, 0)|.$$  

Proof. By equalities 4.17 and 4.18 we have

$$|\theta_1(x, 0)| = \left( \frac{1}{2\beta - 1} \right) \frac{x|2x^{(\beta - 1)}|}{x^{2(\beta - 1)}} = \left( \frac{1}{2\beta - 1} \right) x^{2(\beta - 1)} |x|$$

and

$$|\theta_2(x, 0)| = \left( \frac{1}{\beta} \right) \frac{x|x|^{\beta - 1}}{x} = \left( \frac{1}{\beta} \right) x^{\beta - 1} |x|.$$  

Thereby,

$$|\theta_1(x, 0)| \geq |\theta_2(x, 0)|.$$

Lemma 4.4. If $x, y \in \mathbb{R}$, then

$$|\theta_1(x, y)| \geq |\theta_2(x, y)|.$$  

Proof. If $x = 0$ or $y = 0$ then the inequality was proved by Lemma 4.3 and Remark 4.2. The case $x = y$ is trivial. We can suppose that $x \neq y$, $x \neq 0$ and $y \neq 0$. By equalities 4.17 and 4.18 we have that

$$|\theta_1(x, y)| \geq |\theta_2(x, y)|$$

if, and only if

$$\left( \frac{1}{2\beta - 1} x|x|^{2(\beta - 1)} - y|y|^{2(\beta - 1)} \right)^{\frac{1}{\beta - 1}} \geq \left( \frac{1}{\beta} x|x|^{\beta - 1} - y|y|^{\beta - 1} \right)^{\frac{1}{\beta - 1}}.$$  

This last inequality is true if, and only if

$$\frac{1}{2\beta - 1} x|x|^{2(\beta - 1)} - y|y|^{2(\beta - 1)} \geq \frac{1}{\beta^2} \left( x|x|^{\beta - 1} - y|y|^{\beta - 1} \right)^2.$$  

This last inequality occurs if, and only if

$$x - y \geq \left( x|x|^{\beta - 1} - y|y|^{\beta - 1} \right)^2,$$

that is

$$\lambda(|x|^{2\beta} - xy|y|^{2(\beta - 1)} - y|x|^{2(\beta - 1)} + |y|^{2\beta}) \geq |x|^{2\beta} - 2xy|x|^{\beta - 1}|y|^{\beta - 1} + |y|^{2\beta}.$$  

But, we are supposing that $x \neq 0$ and $y \neq 0$, then the last inequality is equivalent to

$$(4.19)$$

$$\lambda \left[ \left( \frac{|x|}{|y|} \right)^\beta - \left( \frac{xy}{|y|} \right)^{\beta - 2} \right] \geq \left( \frac{|y|}{|x|} \right)^\beta - 2 \frac{x}{|x|} \frac{y}{|y|} + \left( \frac{|y|}{|x|} \right)^\beta.$$
We will prove that the inequality \[4.19\] is true. If \(x \cdot y > 0\), then
\[
\lambda \left[ \frac{|x|}{|y|} \right]^\beta - \left( \frac{|xy|}{|x|} \right)^{\beta - 2} - \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \left( \frac{|x|}{|y|} \right)^\beta + 2 \frac{x \cdot y}{|x|^2 |y|^2} - \left( \frac{|x|}{|y|} \right)^\beta
\]
\[
= \lambda \left[ \frac{|x|}{|y|} \right]^\beta - \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \left( \frac{|x|}{|y|} \right)^\beta - \left( \frac{|y|}{|x|} \right)^{\beta - 1} + \left( \frac{|x|}{|y|} \right)^\beta + 2 \frac{x \cdot y}{|x|^2 |y|^2} - \left( \frac{|x|}{|y|} \right)^\beta
\]
\[
= (\lambda - 1) \left[ \frac{|x|}{|y|} \right]^\beta + \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \lambda \left[ \frac{|x|}{|y|} \right]^\beta + \left( \frac{|x|}{|y|} \right)^{-\beta + 1} + 2
\]
\[
= m(\frac{|x|}{|y|}).
\]
If \(x \cdot y < 0\), then
\[
\lambda \left[ \frac{|x|}{|y|} \right]^\beta - \left( \frac{|xy|}{|x|} \right)^{\beta - 2} - \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \left( \frac{|x|}{|y|} \right)^\beta + 2 \frac{x \cdot y}{|x|^2 |y|^2} - \left( \frac{|x|}{|y|} \right)^\beta
\]
\[
= \lambda \left[ \frac{|x|}{|y|} \right]^\beta + \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \left( \frac{|x|}{|y|} \right)^\beta + \left( \frac{|y|}{|x|} \right)^{\beta - 1} - \left( \frac{|x|}{|y|} \right)^\beta - 2 \frac{x \cdot y}{|x|^2 |y|^2} - \left( \frac{|x|}{|y|} \right)^\beta
\]
\[
= (\lambda - 1) \left[ \frac{|x|}{|y|} \right]^\beta + \left( \frac{|y|}{|x|} \right)^{\beta - 1} + \lambda \left[ \frac{|x|}{|y|} \right]^\beta + \left( \frac{|x|}{|y|} \right)^{-\beta + 1} + 2
\]
\[
= p(\frac{|x|}{|y|}).
\]
By Remark 4.13, we have that \(m(\frac{|x|}{|y|}) \geq 0\) and \(p(\frac{|x|}{|y|}) \geq 0\). This proves the inequality \[4.19\] and the Lemma 4.13.

Our main result of this section will establishes an important estimate involving the \(L^\infty(\mathbb{R}^n)\) norm of the solution \(u\) of the auxiliary problem. It states that:

**Proposition 4.5.** Let \(h \in L^q(\mathbb{R}^n)\) with \(q > \frac{1}{2n}\), and \(v \in E \subset X\) be a weak solution of
\[
\{ -\mathcal{L} v + b(x)v = g(x,v) \text{ in } \mathbb{R}^n
\]
where \(g\) is a continuous functions satisfying
\[
|g(x,s)| \leq h(x)|s|
\]
for \(s \geq 0\), \(b\) is a positive function in \(\mathbb{R}^n\) and \(E\) is defined as in \[2.1\]. Then, there is a constant \(M = M(q, ||h||_{L^q})\) such that
\[
||v||_{L^\infty} \leq M||v||_{L^2}.
\]

**Proof.** Let \(\beta > 1\). For any \(n \in \mathbb{N}\) define
\[
A_n = \{ x \in \mathbb{R}^n; |v(x)|^{\beta - 1} \leq n \}
\]
and
\[
B_n := \mathbb{R}^n \setminus A_n.
\]
Consider
\[
f_n(t) := \begin{cases} \frac{t|t|^{2(\beta - 1)}}{n^\beta}, & t|^{\beta - 1} \leq n \\ \frac{nt}{n^\beta}, & t|^{\beta - 1} > n \end{cases}
\]
and
\[
g_n(t) := \begin{cases} \frac{t|t|^{(\beta - 1)}}{nt}, & t|^{\beta - 1} \leq n \\ \frac{nt}{n^\beta}, & t|^{\beta - 1} > n \end{cases}
\]
Note that \(f_n\) and \(g_n\) are continuous functions and they are differentiable with the exception of \(n \frac{|x|}{|x|} \) and \(-n \frac{|x|}{|x|}\) and its derivatives are limited. Then \(f_n\) and \(g_n\) are lipschitz. Therefore, setting
\[
v_n := f_n \circ v
\]
and

\[ w_n := g_n \circ v \]

we have that \( v_n, w_n \in E \). Note that

\[
\begin{align*}
[v, v_n] &= \int_{A_n} \int_{A_n} (v_n(x) - v_n(y))(v(x) - v(y))K(x - y)\,dxdy \\
&\quad + \int_{B_n} \int_{B_n} (v_n(x) - v_n(y))(v(x) - v(y))K(x - y)\,dxdy \\
&\quad + 2 \, [v, v_n]_{A_n \times B_n}.
\end{align*}
\]

By equality 4.15 if \( x, y \in A_n \) then

\[ v_n(x) - v_n(y) = f'_n(\theta_1(x, y))(v(x) - v(y)), \]

where we are denoting \( \theta_1(x, v(y)) \) by \( \theta_1(x, y) \). Thereby,

\[
\begin{align*}
[v, v_n] &= \int_{A_n} \int_{A_n} (2\beta - 1)|\theta_1(x, y)|^{2(\beta - 1)}(v(x) - v(y))^2K(x - y)\,dxdy \\
&\quad + n^2 \int_{B_n} \int_{B_n} (v(x) - v(y))^2K(x - y)\,dxdy + 2 \, [v, v_n]_{A_n \times B_n}.
\end{align*}
\]

Analogously, by 4.16

\[
\begin{align*}
[w_n, w_n] &= \int_{A_n} \int_{A_n} \beta^2|\theta_2(x, y)|^{2(\beta - 1)}(v(x) - v(y))^2K(x - y)\,dxdy \\
&\quad + n^2 \int_{B_n} \int_{B_n} (v(x) - v(y))^2K(x - y)\,dxdy + 2 \, [w_n, w_n]_{A_n \times B_n},
\end{align*}
\]

where we are denoting \( \theta_2(v(x), v(y)) \) by \( \theta_2(x, y) \). By Lemma 4.3

\[
[w_n, w_n] \leq \int_{A_n} \int_{A_n} \beta^2|\theta_1(x, y)|^{2(\beta - 1)}(v(x) - v(y))^2K(x - y)\,dxdy \\
&\quad + n^2 \int_{B_n} \int_{B_n} (v(x) - v(y))^2K(x - y)\,dxdy + 2 \, [w_n, w_n]_{A_n \times B_n}.
\]

This implies that

\[
\begin{align*}
[w_n, w_n] + \int_{\mathbb{R}^2} b(x)w_n^2\,dx - [v, v_n] - \int_{\mathbb{R}^2} b(x)v_n\,dx \\
&\leq (\beta - 1)^2 \int_{A_n} \int_{A_n} |\theta_1(x, y)|^{2(\beta - 1)}(v(x) - v(y))^2K(x - y)\,dxdy \\
&\quad + n^2 \int_{B_n} \int_{B_n} (v(x) - v(y))^2K(x - y)\,dxdy + 2 \, [w_n, w_n]_{A_n \times B_n}.
\end{align*}
\]

The equation 4.20 implies that

\[
\begin{align*}
[v, v_n] + \int_{\mathbb{R}^2} b(x)v_n\,dx - 2 \, [v, v_n]_{A_n \times B_n} \\
&\geq (2\beta - 1) \int_{A_n} \int_{A_n} |\theta_1(x, y)|^{2(\beta - 1)}(v(x) - v(y))^2K(x - y)\,dxdy,
\end{align*}
\]

because \( b(x)v_n = b(x)w_n^2 \geq 0 \). Replacing 4.22 in the inequality 4.21 we obtain

\[
\begin{align*}
[w_n, w_n] + \int_{\mathbb{R}^2} b(x)w_n^2\,dx - [v, v_n] - \int_{\mathbb{R}^2} b(x)v_n\,dx \\
&\leq \frac{(\beta - 1)^2}{2\gamma - 1} \left( [v, v_n] + \int_{\mathbb{R}^2} b(x)v_n\,dx \right) \\
&\quad + 2 \, [w_n, w_n]_{A_n \times B_n} + (-2 - \frac{2(\beta - 1)^2}{2\gamma - 1}) \, [v, v_n]_{A_n \times B_n},
\end{align*}
\]

that is

\[
\begin{align*}
[w_n, w_n] + \int_{\mathbb{R}^2} b(x)w_n^2\,dx \\
&\leq \left( \frac{(\beta - 1)^2}{2\gamma - 1} + 1 \right) \left( [v, v_n] + \int_{\mathbb{R}^2} b(x)v_n\,dx \right) \\
&\quad + 2 \, [w_n, w_n]_{A_n \times B_n} + (-2 - \frac{2(\beta - 1)^2}{2\gamma - 1}) \, [v, v_n]_{A_n \times B_n} \\
&= \frac{\beta^2}{2\gamma - 1} \left( [v, v_n] + \int_{\mathbb{R}^2} b(x)v_n\,dx \right) \\
&\quad + 2 \, [w_n, w_n]_{A_n \times B_n} + (-2 - \frac{2(\beta - 1)^2}{2\gamma - 1}) \, [v, v_n]_{A_n \times B_n}. \\
\end{align*}
\]

In short,

\[
\begin{align*}
[w_n, w_n] + \int_{\mathbb{R}^2} b(x)w_n^2\,dx &\leq \beta \int_{\mathbb{R}^2} g(x, v)v_n\,dx \\
&\quad + 2 \, [w_n, w_n]_{A_n \times B_n} + (-2 - \frac{2(\beta - 1)^2}{2\gamma - 1}) \, [v, v_n]_{A_n \times B_n}.
\end{align*}
\]
But, if $n \in \mathbb{N}$ and
\[ C = 2 + \frac{2(\beta - 1)^2}{2\beta - 1}, \]
then a simple calculation shows that the function
\[ r(s, t) = 2(ns - t|t|^{\beta-1})^2 - C(s - t)(n^2 s - t|t|^{2(\beta-1)}), \]
satisfies
\[ r(s, t) \leq 0 \]
for all $|s| > \frac{n}{\beta}$ and $|t| \leq \frac{n}{\beta}$. Then, taking $s = v(x)$ and $t = v(y)$ for $x \in B_n$ and $y \in A_n$ and replacing in (4.24) we obtain
\[ 2(w_n(x) - w_n(y))^2 - C(v(x) - v(y))(v_n(x) - v_n(y)) \leq 0. \]
Thereby
\[ +2[w_n, w_n]_{A_n \times B_n} + (-2 - \frac{2(\beta - 1)^2}{2\beta - 1})[v, v_n]_{A_n \times B_n} \leq 0. \]
By inequality 4.23
\[ ([w_n, w_n] + \int_{\mathbb{R}^n} b(x)w_n^2 dx) \leq \beta \int_{\mathbb{R}^n} g(x, v)v_n dx. \]
Let $S > 0$ be the best constant verifying
\[ \|u\|_2^2 \leq S\|u\|_X^2, \]
for all $u \in X$, that is
\[ S = \sup_{u \in X} \frac{\|u\|_2^2}{\|u\|_X^2}. \]
By inequality 4.25
\[ \left( \int_{A_n} |w_n|^{2^*} dx \right)^{\frac{2^*}{2^* - 1}} \leq \left( \int_{\mathbb{R}^n} |w_n|^{2^*} dx \right)^{\frac{2^*}{2^* - 1}} \]
\[ \leq S\|w_n\|^2 \]
\[ \leq S\beta \int_{\mathbb{R}^n} g(x, v(x))v_n dx \]
\[ \leq S\beta \int_{\mathbb{R}^n} h(x)v_n dx \]
\[ \leq S\beta\|h\|_q\|w_n\|_{2^*}. \]
But, we have that $|w_n(x)| \leq |v(x)|^{\beta}$ for all $x \in B_n$ and $|w_n(x)| = |v(x)|^{\beta}$ for all $x \in A_n$. Therefore,
\[ \left( \int_{A_n} |v|^{\beta 2^*} dx \right)^{\frac{2^*}{2^* - 1}} \leq S\beta\|h\|_q \left( \int_{\mathbb{R}^n} |v|^{2q_1} dx \right)^{\frac{2^* - 1}{2^* q_1}}. \]
By Monotone Convergence Theorem,
\[ ||v||_{2^* \beta} \leq (\beta S||h||_q)^{\frac{2^*}{2^* - 1}} ||v||_{2\beta q_1}, \]
where $q_1 = \frac{2^*}{q - 1}$. Define
\[ \eta := \frac{2^*}{2q_1}, \]
and note that $\eta > 1$. When $\beta = \eta$ we have that $2\beta q_1 = 2_*^*$. Then, by 4.26
\[ ||v||_{2\eta} \leq (\eta S||h||_q)^{\frac{2^*}{2^* - 1}} ||v||_{2^*}. \]
Taking $\beta = \eta^2$ in (4.26) we obtain

\[(4.28) \quad ||v||_{2^*\eta^2} \leq \eta^{\frac{1}{\eta^2}} (S||h||_q)^{\frac{1}{2\eta^2}} ||v||_{2^*}.\]

By inequalities (4.27) and (4.28) we have

\[||v||_{2^*\eta^2} \leq \eta^{\frac{1}{\eta^2} + \frac{1}{\eta} + \ldots + \frac{1}{m}} (S||h||_q)^{\frac{1}{2\eta^2} + \frac{1}{\eta^2} + \ldots + \frac{1}{m}} ||v||_{2^*}\]

Inductively, we can prove that

\[||v||_{2^*\eta^m} \leq \eta^{\frac{1}{\eta^2} + \frac{1}{\eta^2} + \ldots + \frac{1}{2\eta^2} + \ldots + \frac{1}{m}} (S||h||_q)^{\frac{1}{2\eta^2} + \frac{1}{\eta^2} + \ldots + \frac{1}{2\eta^2} + \ldots + \frac{1}{m}} ||v||_{2^*}\]

for all $m \in \mathbb{N}$. But,

\[\sum_{i=1}^{\infty} \frac{m}{2\eta^m} = \frac{1}{2(\eta - 1)^2}\]

and

\[\sum_{i=1}^{\infty} \frac{1}{2\eta^m} = \frac{1}{2(\eta - 1)}.\]

Thereby, for all $m > 0$

\[||v||_{2^*\eta^m} \leq \eta^{\frac{1}{2\eta^2(n)} (S||h||_q)^{\frac{1}{2\eta^2(n)}} ||v||_{2^*}\]

Recalling that

\[||v||_\infty = \lim_{n \to \infty} ||v||_p\]

and that $\eta > 1$ we have that

\[||v||_\infty \leq M||v||_{2^*}\]

for

\[M = \eta^{\frac{1}{2\eta^2(n-1)}} (S||h||_q)^{\frac{1}{2\eta^2(n-1)}}\]

and

\[\eta = \frac{n(q-1)}{q(n-2s)}\]

We conclude the proof of Proposition 4.5 noting that $M$ depends only on $q$, $||h||_q$. \hfill \Box

5. Solution for Problem (P)

In this section, we prove the main result, the Theorem 5.2. By Corollary 3.8 there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. We have the following estimate for $||u||_\infty$.

**Lemma 5.1.** The solution $u$ of the auxiliary problem satisfies

\[||u||_\infty \leq M(2Skd)^{\frac{1}{2}}\]

**Proof.** Consider the functions

\[H(x, t) = \begin{cases} f(t) & \text{if } |x| < R \text{ or } f(t) \leq \frac{V(x)}{k}t \\ 0 & \text{if } |x| \geq R \text{ and } f(t) > \frac{V(x)}{k}t \end{cases}\]

and

\[b(x) = \begin{cases} V(x) & \text{if } |x| < R \text{ or } f(u) \leq \frac{V(x)}{k}u \\ (1 - \frac{1}{k}) V(x) & \text{if } |x| \geq R \text{ and } f(u) > \frac{V(x)}{k}u. \end{cases}\]
Note that the function \( u \) is solution of
\[
\begin{cases}
-L_k u + b(x)u = H(x, u) \\
u \in E.
\end{cases}
\]
By \((f_1)\),
\[|H(x, t)| \leq c_0|t|^{p-1}\]
for \( p \in (2, 2^*_s) \). Thereby,
\[|H(x, u)| \leq h(x)|u|,\]
where \( h(x) = C_0|u|^{p-2} \). Note that \( h \in L^{2^{*}_s} \) with
\[\|h\|_{L^{2^*_s}({\mathbb{R}^n})} \leq C(2ksd)^{\frac{2^*_s}{2^*_s}}.\]
The number \( p \) satisfies
\[p < 2^*_s = 2 + \frac{2s}{n} 2^*_s.\]
Then
\[q = \frac{2^*_s}{p-2} > \frac{n}{2s}.\]
By Proposition 4.5 and sobolev embedding
\[\|u\|_{\infty} \leq M\|u\|_{2^*_s} \leq MS^\frac{p}{p}||u||,\]
where \( M = M(q, ||h||_q) \). By Proposition 3.9 we have
\[(5.29)\quad \|u\|_{\infty} \leq M(2ksd)^{\frac{p}{2^*_s}}.\]

**Theorem 5.2.** Suppose that \( V \) satisfies \((V_1)-(V_2)\) and that \( f \) satisfies \((f_1)-(f_3)\). There is \( \Lambda^* = \Lambda^*(V_\infty, \theta, p, c_0, s) > 0 \) such that if \( \Lambda > \Lambda^* \) in \((V_3)\), then the problem \((P)\) has a nonnegative nontrivial solution.

Indeed, let \( |x| \geq R \). If \( u(x) = 0 \) then by definition \( f(u(x)) = g(x, u(x)) \). If \( u(x) > 0 \) then
\[
\frac{f(u(x))}{u(x)} \leq c_0|u|^{p-2} \leq c_0||u||^2 \leq \frac{c_0\|u\|^2}{\Lambda} \leq \frac{c_0^{\frac{p}{2^*_s}}M^{p-2(2sd)^{\frac{p-2}{2^*_s}}}}{\Lambda} V(x)
\]
Define
\[\Lambda^* = k^{\frac{p}{2^*_s}}c_0M^{p-2(2sd)^{\frac{p-2}{2^*_s}}}.\]
If \( \Lambda > \Lambda^* \) then
\[f(u(x)) \leq \frac{V(x)}{k}.\]
By definition of \( g \) we have \( g(x, u(x)) = f(u(x)) \). Then \( g(x, u(x)) = f(u(x)) \) for all \( x \in {\mathbb{R}^n} \). Therefore, \( u \) is a solution nonnegative and nontrivial of problem \((P)\). \(\square\)
References

[1] G. Alberti and Bellettini G., A nonlocal anisotropic model for phase transitions. I. The optimal profile problem, Math. Ann. 310 (1998), 527-560.

[2] C. O. Alves and O. H. Miyagaki, A critical nonlinear fractional elliptic equation with saddle-like potential in $\mathbb{R}^N$, Journal of Mathematical Physics 57 (2016).

[3] C.O. Alves and M.A.S. Souto, Existence of solutions for a class of elliptic equations in $\mathbb{R}^n$ with vanishing potentials, Journal of Differential Equations 252 (2012), 5555-5568.

[4] V. Ambrosio, A Fractional Landesman-Lazer Type Problem set on $\mathbb{R}^N$, arXiv:1601.06281v1.

[5] V. Ambrosio, Ground state for superlinear fractional schrödinger equations in $\mathbb{R}^N$, Ann. Acad. Sci. Fenn. Math. 41 (2016), 745-756.

[6] A. L. Bertozzi, J. B. Garnett, and T. Laurent, Characterization of radially symmetric finite time blowup in multidimensional aggregation equations, SIAM Journal on Mathematical Analysis 44 (2012), 651-681.

[7] G.M. Bisci and V.D. Radulescu, Ground state solutions of scalar field fractional Schrödinger equations, Calculus of Variations 54 (2015), 2985-3008.

[8] C. Bucur, Some Observations on the Green Function for the Ball in the Fractional Laplace Framework, Comm. on Pure & App. Anal. 15 (2016), 657-699.

[9] X. Cabré and X. Sola-Morales, Layer solutions in a half-space for boundary reactions, Comm. Pure Appl. Math. 58 (2005), 1678-1732.

[10] L. Caffarelli and Silvestre L., An extension problem related to the fractional Laplacian, Communications in partial differential equations (2007).

[11] X. Chang, Ground States of some Fractional Schrödinger Equations on $\mathbb{R}^N$, Proceedings of the Edinburgh Mathematical Society 58 (2015), 305-321.

[12] X. Chang, Ground state solutions of asymptotically linear fractional Schrödinger equations, Journal of Mathematical Physics 54 (2013).

[13] C. Chen, Infinitely many solutions for fractional schrödinger equations in $\mathbb{R}^N$, Electronic Journal of Differential Equations 88 (2016), 1-15.

[14] M. Cheng, Bound state for the fractional Schrödinger equation with unbounded potential, Journal of Mathematical Physics 53 (2012).

[15] P. d’Avenia, M. Squassina, and M. Zenari, Fractional logarithmic Schrödinger equations, Mathematical Methods in the Applied Sciences 38 (2015), 5207-5216.

[16] A. Di Castro, T. Kuusi, and G. Palatucci, Local behavior of fractional p-minimizers, Annales de l’Institut Henri Poincare (C) Non Linear Analysis (2015).

[17] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 512-573.

[18] S. Dipierro, G. Palatucci, and E. Valdinoci, Existence and symmetry results for a Schrödinger type Problem involving the fractional laplacian, Le matematiche 68 (2013), 201-216.

[19] R. C. Duarte and M. A. S. Souto, Fractional Schrödinger-Poisson equations with general nonlinearities, Electron. J. Differential Equations 319 (2016), 1-19.

[20] M. M. Fall and E. Valdinoci, Uniqueness and nondegeneracy of positive solutions of $(-\Delta)^s u + u = u^p$ in $\mathbb{R}^N$ when $s$ is close to 1, Communications in Mathematical Physics 329 (2014), 383-404.

[21] P. Felmer, A. Quaas, and J. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian., Proc. R. Soc. Edinburgh Sect. A. 142 (2012), 1237-1262.

[22] G. Franzina and G Palatucci, Fractional p-eigenvalues, arXiv:1307.1789.

[23] G. Gillboa and S. Osher, Nonlocal operators with applications to image processing, Multiscale Modeling Simul. 7 (2008), 1005-1028.

[24] T. Gou and H. Sun, Solutions of nonlinear Schrödinger equation with fractional Laplacian without the Ambrosetti-Rabinowitzs condition, Applied Mathematics and Computation (2014).

[25] S. Khoutir and H. Chen, Existence of infinitely many high energy solutions for a fractional Schrödinger equation in $\mathbb{R}^N$, Applied Mathematics Letters 61 (2016), 156-162.

[26] R. Lehrer, L. A. Maia, and M. Squassina, Asymptotically linear fractional Schrödinger equations, arXiv:1401.2203.

[27] E. C. Oliveira, F. S. Costa, and J. Jr. Vaz, The fractional Schrödinger equation for delta potentials, Journal of Mathematical Physics 51 (2010).

[28] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$, Journal of Mathematical Physics 54 (2013).
[29] S. Secchi, On fractional Schroedinger equations in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition, Topological Methods in Nonlinear Analysis 47 (2016).
[30] R. Servadei and E. Valdinoci, Mountain Pass solutions for non-local elliptic operators, J. Math. Anal. Appl. 389 (2012), 887-898.
[31] R. Servadei and E. Valdinoci, Variational Methods for Non-local Operators of Elliptic Type, Discrete and Continuous Dynamical Systems 33 (2013), 2105-2137.
[32] D. Siegel and E. Talvila, Pointwise growth estimates of the Riesz potential, Dynamics of Continuous Discrete and Impulsive Systems 5 (1999), 185-194.
[33] M. Souza and Y. L. Araújo, On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth, Math. Nachr. 289 (2016), 610-625.
[34] K. Teng, Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$, Nonlinear Analysis: Real World Applications (2014).
[35] K. Teng and X. He, Ground state solutions for fractional Schrödinger equations with critical Sobolev exponent, Commun. Pure Appl. Anal. 15 (2016), 991-1008.
[36] Y. Wan and Z. Wang, Bound state for fractional Schrödinger equation with saturable nonlinearity, Applied Mathematics and Computation 273 (2016), 735-740.
[37] Q. Wang, D. Zhao, and K. Wang, Existence of solutions to nonlinear fractional schrödinger equations with singular potentials, Applied Mathematics Letters 218 (2016), 1-19.
[38] M. Willem, Minimax Theorems, Birkhauser (1986).
[39] J. Xu, Z. Wei, and W. Dong, Existence of weak solutions for a fractional Schrödinger equation, Communications in Nonlinear Science and Numerical Simulation 22 (2015), 1215-1222.
[40] L. Yang and Z. Liu, Multiplicity and concentration of solutions for fractional Schrödinger equation with sublinear perturbation and steep potential well, Computers and Mathematics with Applications (2016).
[41] W. Zhang, X. Tang, and J. Zhang, Infinitely many radial and non-radial solutions for a fractional Schrödinger equation, Computers and Mathematics with Applications (2015).
[42] H. Zhang, J. Xu, and F. Zhang, Existence and multiplicity of solutions for superlinear fractional Schrödinger equations in $\mathbb{R}^N$, Journal of Mathematical Physics 56 (2015).
[43] X. Zhang, B. Zhang, and D. Repovs, Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials, Nonlinear Analysis 142 (2016), 48-68.

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