ON TRIVIAL ZEROS OF PERRIN-RIOU’S $L$-FUNCTIONS

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Abstract. In the previous paper [Ben2] we generalized Greenberg’s construction of the $L$-invariant to semistable $p$-adic representations. Here we prove that this construction is compatible with Perrin-Riou’s theory of $p$-adic $L$-functions. Namely, using Nekovář’s machinery of Selmer complexes we prove that our $L$-invariant appears as an additional factor in the Bloch-Kato type formula for special values of Perrin-Riou’s Iwasawa $L$-function.

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Introduction

0.1. In [Ben2], using ideas of Colmez [C4] we defined a natural generalization of Greenberg’s $L$-invariant $[G]$ to pseudo-geometric representations $V$ of $\text{Gal}(\overline{Q}/Q)$ which are semistable at $p$.

More precisely, assume that $V$ satisfies the following conditions:

1) $H^0(V) = H^0(V^*(1)) = 0$ and $H^1_f(V) = H^1_f(V^*(1)) = 0$;
2) $V$ is semistable at $p$ and the map $1 - p^{-1} \varphi^{-1}$ acts semisimply on $D_{st}(V)$.
3) $D_{st}(V)^{\varphi=1} = 0$.

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The $(\phi, \Gamma)$-module $D_{rig}^+(V)$ has no crystalline subquotient of the form
\[ 0 \rightarrow R(|x|^k) \rightarrow U \rightarrow R \rightarrow 0, \quad k \geq 1. \]

See sections 1.1, 2.1 and 3.1 for unexplained notations and further details. Remark that $R$ denotes the Robba ring over $\mathbb{Q}_p$ and 4) is a direct generalization of Hypothesis U of [G]. Let $t_V(\mathbb{Q}_p) = \text{Fil}^0 D_{st}(V)$ denote the tangent space of $V$ at $p$. We say that a $\mathbb{Q}_p$-subspace $D \subset D_{st}(V)$ is admissible if it is stable under the action of $\varphi$ and the natural projection $D \rightarrow t_V(\mathbb{Q}_p)$ is an isomorphism. The main construction of [Ben2] associates to $(\ell, D)$ a $p$-adic number $\mathcal{L}(\ell, D) \in \mathbb{Q}_p$ which coincides with the Greenberg’s $\mathcal{L}$-invariant if $\ell$ is ordinary at $p$ and $D = D_{st}(F^1V)$ where $F^1V$ denotes the canonical filtration of $V$ provided by ordinarity.

0.2. The goal of the present paper is to show that this definition is compatible with Perrin-Riou’s theory of $p$-adic L-functions. For a profinite group $G$ and a continuous $G$-module $X$ we denote by $C^\bullet(G, X)$ the standard complex of continuous cochains. Let $S$ be a finite set of primes containing $p$. Denote by $G_S$ the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. Set $R\Gamma_S(X) = C^\bullet(G_S, X)$ and $R\Gamma(\mathbb{Q}_p, X) = C^\bullet(G_v, X)$, where $G_v$ is the absolute Galois group of $\mathbb{Q}_v$. Let $R\Gamma_c(V)$ denote the complex sitting in the distinguished triangle
\[ R\Gamma_c(V) \rightarrow R\Gamma_S(V) \rightarrow \bigoplus_{v \in S \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, V). \]

The Euler-Poincaré line of $V$ is defined by $\Delta_{EP}(V) = \det_{\mathbb{Q}_p}^{-1} R\Gamma_c(V)$.

Now assume that $V$ is the $p$-adic realization of a pure motive $M/\mathbb{Q}$. Let $M_B$ and $M_{dR}$ denote the Betti and the de Rham realizations of $M$ and let $t_M(\mathbb{Q}) = M_{dR}/\text{Fil}^0 M_{dR}$ denote the tangent space of $M$. Fixing non zero elements $\omega_B \in \det_{\mathbb{Q}} M_B^+$ and $\omega_t \in \det_{\mathbb{Q}} t_M(\mathbb{Q})$ one can define a canonical trivialization
\[ i_{\omega_t, \omega_B, p} : \Delta_{EP}(V) \rightarrow \mathbb{Q}_p. \]

Let $T$ be a $G_S$-stable lattice of $V$. According to the conjecture of Bloch and Kato [BK] in the form of Fontaine and Perrin-Riou [F3]
\[ i_{\omega_t, \omega_B, p}(\Delta_{EP}(T)) = \frac{L(M, 0)}{\Omega_\infty(\omega_t, \omega_B)} \mathbb{Z}_p, \]

where $\Omega_\infty(\omega_t, \omega_B)$ is the Deligne period. Assume in addition that $V$ is crystalline at $p$. Fix an admissible subspace $D$ of $D_{cris}(V)$ and a $\mathbb{Z}_p$-lattice $N$ of $D$. From the semisimplicity of $\varphi$ we deduce the decomposition $D \simeq D_{-1} \oplus D^{\varphi = p^{-1}}$ where $D_{-1} = (\varphi - p^{-1})D$. Set $\Gamma = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$, $\Gamma_1 = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}(\zeta_p))$ and $\Lambda = \mathbb{Z}_p[[\Gamma_1]]$. Fix a topological generator $\gamma_1 \in \Gamma_1$ and denote by $\mathcal{H}$ the ring of operators $f(\gamma_1 - 1)$ where $f(X) = \sum_{n=0}^{\infty} a_n X^n \in \mathbb{Q}_p[[X]]$ converges on the $p$-adic open unit disk. Let $K$ be the field of fractions of $\mathcal{H}$. Fix $h \geq 1$ such that $\text{Fil}^{-h} D_{cris}(V) = D_{cris}(V)$. Perrin-Riou’s theory [PR2] associates to $(T, N)$ a free $\Lambda$-module $L_{tw, h}^{(\omega_0)}(N, T) \subset K$ Fix a generator $f(\gamma_1 - 1)$ of $L_{tw, h}^{(\omega_0)}(N, T)$ and define a meromorphic $p$-adic function
\[ L_{tw, h}(T, N, s) = f(\chi(\gamma_1)^s - 1), \]

where $\chi : \Gamma \rightarrow \mathbb{Z}_p\$ is the cyclotomic character. Let $\omega_N$ be a generator of $\det_{\mathbb{Z}_p}(N)$. The isomorphism $D \simeq t_V(\mathbb{Q}_p)$ allows us to consider $\omega_N$ as a basis of $\det_{\mathbb{Q}_p} t_V(\mathbb{Q}_p)$. We also fix a generator $\omega_T \in \det_{\mathbb{Z}_p} T^+$ and define the $p$-adic period $\Omega_p(\omega_T, \omega_B) \in \mathbb{Q}_p$ by $\omega_B = \Omega_p(\omega_T, \omega_B) \omega_T$. Our main result can be stated as follows.
Theorem 0.3. Let $V$ be a pseudo-geometric $p$-adic representation which is crystalline at $p$. Assume that it satisfies conditions 1-4). Let $D$ be an admissible subspace of $D_{\text{cris}}(V)$. If $\mathcal{L}(D, V) \neq 0$ then

i) $L_{1w,h}(T, N, s)$ is a meromorphic $p$-adic function which has a zero at $s = 0$ of order $e = \dim_{\mathbb{Q}_p}(D^{=p^{-1}})$.

ii) Let $L_{1w,h}^*(T, N, 0) = \lim_{s \to 0} s^{-e} L_{1w,h}(T, N, s)$ be the special value of $L_{1w,h}(T, N, s)$ at $s = 0$. Then

$$L_{1w,h}^*(T, N, 0) \overset{\mathcal{L}}{=} \Gamma(h)^{d^+(V)} \mathcal{L}(D, V) E_p^*(V, 1) \det_{\mathbb{Q}_p} \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) |D_{-1} \Omega_p(\omega_T, \omega_B) i_{\omega_{N, \omega_{B, p}}}(\Delta_{\text{EP}}(T)),$$

where $\Gamma(h) = (h-1)!$, $d^+(V) = \dim_{\mathbb{Q}_p}(V^+)$, $E_p(V, t) = \det(1 - \varphi t | D_{\text{cris}}(V))$ is the Euler factor at $p$ and $E_p^*(V, t) = E_p(V, t) \left(1 - \frac{t}{p}\right)^{-e}$.

Remarks 0.4. 1) Assume that $V$ is an arbitrary pseudo-geometric representation which is crystalline at $p$ and such that $D_{\text{cris}}(V)^{\varphi=1} = D_{\text{cris}}(V)^{\varphi=p^{-1}} = 0$. In this case the $p$-adic $L$-function has no trivial zeros (if exists) and a very general Iwasawa-theoretic descent result is proved in [PR2], Chapitre III. If $V$ satisfies 1-4) and $D_{\text{cris}}(V)^{\varphi=p^{-1}} = 0$, it is easy to see that $\mathcal{L}(D, V) = 1$ and Theorem 0.3 is a particular case of this result, but our goal here is to study the case of trivial zeros.

2) Let $E/\mathbb{Q}$ be an elliptic curve having good reduction at $p$. Consider the $p$-adic representation $V = \text{Sym}^2(T_p(E)) \otimes \mathbb{Q}_p$, where $T_p(E)$ is the $p$-adic Tate module of $E$. It is easy to see that $D = D_{\text{cris}}(V)^{\varphi=p^{-1}}$ is one dimensional. In this case some versions of Theorem 0.3 were proved in [PR3] and [D] with an ad hoc definition of the $\mathcal{L}$-invariant. Remark that $p$-adic $L$-functions attached to the symmetric square of a newform were constructed by Dabrowski and Delbourgo [DD].

3) This theorem suggests that one should exist an analytic $p$-adic $L$-function $L_{\text{an}}(T, N, s)$ such that

- $L_{\text{an}}(T, N, s)$ has a zero of order $e - d^+(V)$ at $s = 0$;

- $L_{\text{an}}^*(T, N, 0) \overset{\mathcal{L}}{=} \mathcal{L}(D, V) E_p^*(V, 1) \det_{\mathbb{Q}_p} \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) |D_{-1} \Omega_p(\omega_T, \omega_B) i_{\omega_{N, \omega_{B, p}}}(\Delta_{\text{EP}}(T))$. $L(M, 0)$.

0.5. The organization of the paper is as follows. In §1 we review the theory of $(\varphi, \Gamma)$-modules, in particular, the computation of cohomology of $(\varphi, \Gamma)$-modules of rank 1 following [C4]. In §2 we recall preliminaries on the Bloch-Kato exponential map and review the construction of the large exponential map of Perrin-Riou given by Berger [Ber3]. In §3 we review the definition of the $\mathcal{L}$-invariant given in [Ben2] and interpret it in terms of the Bockstein homomorphism associated to the large exponential map. In §4 we prove Theorem 0.3 using the main result of §3 and Nekovář’s Iwasawa-theoretic descent techniques. In Appendix we prove derived versions of the well known computation of the local Galois cohomology in terms of $(\varphi, \Gamma)$-modules [H1], [CC2].

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§1. Preliminaries

1.1. (ϕ, Γ)-modules.

1.1.1. The Robba ring (see [Ber1],[C3]). In this section $K$ is a finite unramified extension of $\mathbb{Q}_p$ with residue field $k_K$, $O_K$ its ring of integers, and $\sigma$ the absolute Frobenius of $K$. Let $\overline{K}$ an algebraic closure of $K$, $G_K = \text{Gal}(\overline{K}/K)$ and $C$ the completion of $\overline{K}$. Let $v_p : C \to \mathbb{R} \cup \{\infty\}$ denote the $p$-adic valuation normalized so that $v_p(p) = 1$ and set $|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$. Write $B(r, 1)$ for the $p$-adic annulus $B(r, 1) = \{x \in C \mid r \leq |x| < 1\}$. As usually, $\mu_{p^n}$ denotes the group of $p^n$-th roots of unity. Fix a system of primitive roots of unity $\pi_k$ for the $p$-adic annulus $B(r, 1) = \{x \in C \mid 0 < |x| < 1\}$. Set $\hat{\mathbb{A}} = \mathbb{A}/(\mathbb{A}^+)^\mu$ equipped with the valuation $v_E(x) = v_p(x^{(0)})$ is a complete local ring of characteristic $p$ with residue field $\hat{k}_K$. Moreover it is integrally closed in its field of fractions $\hat{\mathbb{E}} = \text{Fr}(\mathbb{E}^+)$. Let $\tilde{\mathbb{A}} = W(\tilde{\mathbb{E}})$ be the ring of Witt vectors with coefficients in $\tilde{\mathbb{E}}$. Denote by $[\cdot] : \tilde{\mathbb{E}} \to W(\tilde{\mathbb{E}})$ the Teichmuller lift. Any $u = (u_0, u_1, \ldots) \in \tilde{\mathbb{A}}$ can be written in the form

$$u = \sum_{n=0}^{\infty} [u^{p^{-n}}] p^n.$$

Set $\pi = [\varepsilon] - 1$, $\mathbb{A}_K = O_K[[\pi]]$ and denote by $\mathbb{A}_K$ the $p$-adic completion of $\mathbb{A}_K^{+}[1/\pi]$. Let $\hat{\mathbb{B}} = \tilde{\mathbb{A}}[(1/p)]$, $B_K = \mathbb{A}_K[1/p]$ and let $B$ denote the completion of the maximal unramified extension of $B_K$ in $\hat{\mathbb{B}}$. Set $\mathbb{A} = B \cap \hat{\mathbb{A}}$, $\mathbb{A}^{+} = W(\mathbb{E}^{+})$, $\mathbb{A}^{+} = \hat{\mathbb{A}}^{+} \cap \mathbb{A}$ and $B^{+} = \mathbb{A}^{+}[1/p]$. All these rings are endowed with natural actions of the Galois group $G_K$ and Frobenius $\varphi$.

Set $A_K = A^{HK}$ and $B_K = A_K[1/p]$. Remark that $\Gamma$ and $\varphi$ act on $B_K$ by

$$\tau(\pi) = (1 + \pi)^{\chi(\tau)} - 1, \quad \tau \in \Gamma$$

$$\varphi(\pi) = (1 + \pi)p - 1.$$

For any $r > 0$ define

$$\hat{\mathbb{B}}^{1,r} = \left\{ x \in \hat{\mathbb{B}} \mid \lim_{k \to +\infty} \left( v_E(x_k) + \frac{pr}{p-1} k \right) = +\infty \right\}.$$
Set \( \mathcal{R}(K) = \bigcup_{r \geq p} \mathcal{B}_{\text{rig},r}^{+} \) and \( \mathcal{R}^{+}(K) = \mathcal{R}(K) \cap K[[\pi]] \). It is not difficult to check that these rings are stable under \( \Gamma \) and \( \varphi \). To simplify notations we will write \( \mathcal{R} = \mathcal{R}(\mathbb{Q}_p) \) and \( \mathcal{R}^{+} = \mathcal{R}^{+}(\mathbb{Q}_p) \).

1.1.2. \((\varphi, \Gamma)\)-modules (see [F2], [CC1]). Let \( A \) be either \( \mathcal{B}_{\text{rig},r}^{+} \) or \( \mathcal{R}(K) \). \((\varphi, \Gamma)\)-module over \( A \) is a finitely generated free \( A \)-module \( D \) equipped with semilinear actions of \( \varphi \) and \( \Gamma \) commuting to each other and such that the induced linear map \( \varphi : A \otimes_{K} D \to D \) is an isomorphism. Such a module is said to be etale if it admits a \( \mathcal{A}_{K}^{-} \)-lattice \( N \) stable under \( \varphi \) and \( \Gamma \) and such that \( \varphi : \mathcal{A}_{K}^{-} \otimes_{K} N \to N \) is an isomorphism. The functor \( D \mapsto \mathcal{R}(K) \otimes_{\mathcal{B}_{\text{rig},r}^{+}} D \) induces an equivalence between the category of etale \((\varphi, \Gamma)\)-modules over \( \mathcal{B}_{\text{rig},r}^{+} \) and the category of \((\varphi, \Gamma)\)-modules over \( \mathcal{R}(K) \) which are of slope 0 in the sense of Kedlaya’s theory ([Ke] and [C5], Corollary 1.5). Then Fontaine’s classification of \( p \)-adic representations [F2] together with the main result of [CC1] lead to the following statement.

**Proposition 1.1.3.** i) The functor

\[ D^{\dagger} : V \mapsto D^{\dagger}(V) = (B^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{K}}} \]

establishes an equivalence between the category of \( p \)-adic representations of \( G_{\mathbb{K}} \) and the category of etale \((\varphi, \Gamma)\)-modules over \( \mathcal{B}_{\text{rig},r}^{+} \).

ii) The functor \( D^{\dagger}_{\text{rig}}(V) = \mathcal{R}(K) \otimes_{\mathcal{B}_{\text{rig},r}^{+}} D^{\dagger}(V) \) gives an equivalence between the category of \( p \)-adic representations of \( G_{\mathbb{K}} \) and the category of \((\varphi, \Gamma)\)-modules over \( \mathcal{R}(K) \) of slope 0.

**Proof.** see [C4], Proposition 1.7.

1.1.4. Cohomology of \((\varphi, \Gamma)\)-modules (see [H1], [H2], [Li]). Fix a generator \( \gamma \) of \( \Gamma \). If \( D \) is a \((\varphi, \Gamma)\)-module over \( A \), we denote by \( C_{\varphi, \gamma}(D) \) the complex

\[ C_{\varphi, \gamma}(D) : 0 \xrightarrow{f} D \xrightarrow{g} D \oplus D \xrightarrow{g} D \to 0 \]

where \( f(x) = ((\varphi - 1) x, (\gamma - 1) x) \) and \( g(y, z) = (\gamma - 1) y - (\varphi - 1) z \). Set \( H^{i}(D) = H^{i}(C_{\varphi, \gamma}(D)) \).

A short exact sequence of \((\varphi, \Gamma)\)-modules

\[ 0 \to D' \to D \to D'' \to 0 \]

gives rise to an exact cohomology sequence:

\[ 0 \to H^{0}(D') \to H^{0}(D) \to H^{0}(D'') \to H^{1}(D') \to \cdots \to H^{2}(D'') \to 0. \]

**Proposition 1.1.5.** Let \( V \) be a \( p \)-adic representation of \( G_{\mathbb{K}} \). Then

i) The complexes \( R\Gamma(K, V), C_{\varphi, \gamma}(D^{\dagger}(V)) \) and \( C_{\varphi, \gamma}(D^{\dagger}_{\text{rig}}(V)) \) are isomorphic in the derived category of \( \mathbb{Q}_p \)-vector spaces \( \mathcal{D}(\mathbb{Q}_p) \).

**Proof.** This is a derived version of Herr’s computation of Galois cohomology [H1]. The proof is given in the Appendix, Propositions A.3 and Corollary A.4.

1.1.6. Recall that \( \Lambda \) denotes the Iwasawa algebra of \( \Gamma_{1} \), \( \Delta = \text{Gal}(K_{1}/K) \) and \( \Lambda(\Gamma) = \mathbb{Z}_{p}[\Delta] \otimes_{\mathbb{Z}_{p}} \Lambda \). Let \( \iota : \Lambda(\Gamma) \to \Lambda(\Gamma) \) denote the involution defined by \( \iota(g) = g^{-1} \), \( g \in \Gamma \). If \( T \) is a \( \mathbb{Z}_{p} \)-adic representation of \( G_{\mathbb{K}} \), then the induced module \( \text{Ind}_{K_{\infty}/K}(T) \) is isomorphic to \( (\Lambda(\Gamma) \otimes_{\mathbb{Z}_{p}} T)^{\iota} \) and we set

\[ R\Gamma_{\text{Iw}}(K, T) = R\Gamma(K, \text{Ind}_{K_{\infty}/K}(T)). \]
Write \( H_{Iw}^i(K,T) \) for the Iwasawa cohomology

\[
H_{Iw}^i(K,T) = \lim_{\text{cor}_K} H^i(K_n,T).
\]

Recall that there are canonical and functorial isomorphisms

\[
\begin{align*}
\mathbf{R}^i \Gamma_{Iw}(K,T) &\simeq H_{Iw}^i(K,T), & i &\geq 0, \\
\mathbf{R} \Gamma_{Iw}(K,T) \otimes_{\Lambda(\Gamma)}^L \mathbb{Z}_p[G_n] &\simeq \mathbf{R} \Gamma(K_n,T)
\end{align*}
\]

(see [N2], Proposition 8.4.22). The interpretation of the Iwasawa cohomology in terms of \((\varphi, \Gamma)\)-modules was found by Fontaine (unpublished but see [CC2]). We give here the derived version of this result. Let \( \psi : \mathbb{B} \to \mathbb{B} \) be the operator defined by the formula \( \psi(x) = \frac{1}{p} \varphi^{-1}(\text{Tr}_{\mathbb{B}/\mathbb{B}(\varphi)}(x)) \).

We see immediately that \( \psi \circ \varphi = \text{id} \). Moreover \( \psi \) commutes with the action of \( G_K \) and \( \psi(A^\dagger) = A^\dagger \). Consider the complexes

\[
\begin{align*}
C_{Iw,\varphi}(T) & : D(T) \xrightarrow{\psi-1} D(T), \\
C_{Iw,\psi}(T) & : D^\dagger(T) \xrightarrow{\psi-1} D^\dagger(T).
\end{align*}
\]

**Proposition 1.1.7.** i) The complexes \( \mathbf{R} \Gamma_{Iw}(K,T) \), \( C_{Iw,\varphi}(T) \) and \( C_{Iw,\psi}(T) \) are naturally isomorphic in the derived category \( \mathcal{D}(\Lambda(\Gamma)) \) of \( \Lambda(\Gamma) \)-modules.

**Proof.** See Proposition A.7 and Corollary A.8.

1.1.8. Finally, recall the computation of the cohomology of \((\varphi, \Gamma)\)-modules of rank 1 following Colmez [C4]. As in [C4], we consider the case \( K = \mathbb{Q}_p \) and put \( \mathcal{R} = \mathbb{B}_{rig,\mathbb{Q}_p}^\dagger \) and \( \mathcal{R}^+ = \mathbb{B}_{rig,\mathbb{Q}_p}^+ \). The differential operator \( \partial = (1 + \pi) \frac{d}{d\pi} \) acts on \( \mathcal{R} \) and \( \mathcal{R}^+ \). If \( \delta : \mathbb{Q}_p^* \to \mathbb{Q}_p^* \) is a continuous character, we write \( \mathcal{R}(\delta) \) for the \((\varphi, \Gamma)\)-module \( \mathcal{R}e_\delta \) defined by \( \varphi(e_\delta) = \delta(p)e_\delta \) and \( \gamma(e_\delta) = \delta(\chi(\pi))e_\delta \). Let \( x \) denote the character induced by the natural inclusion of \( \mathbb{Q}_p \) in \( L \) and \( |x| \) the character defined by \( |x| = p^{-v_p(x)} \).

**Proposition 1.1.9.** Let \( \delta : \mathbb{Q}_p^* \to \mathbb{Q}_p^* \) be a continuous character. Then:

i) \[
H^0(\mathcal{R}(\delta)) = \begin{cases} 
\mathbb{Q}_p t^m & \text{if } \delta = x^{-m}, \ k \in \mathbb{N} \\
0 & \text{otherwise.}
\end{cases}
\]

ii) \[
\dim_{\mathbb{Q}_p}(H^1(\mathcal{R}(\delta))) = \begin{cases} 
2 & \text{if either } \delta(x) = x^{-m}, \ m \geq 0 \text{ or } \delta(x) = |x|x^m, \ k \geq 1, \\
1 & \text{otherwise.}
\end{cases}
\]

iii) Assume that \( \delta(x) = x^{-m}, \ m \geq 0 \). The classes \( \text{cl}(t^m,0)e_\delta \) and \( \text{cl}(0,t^m)e_\delta \) form a basis of \( H^1(\mathcal{R}(x^{-m})) \).

iv) Assume that \( \delta(x) = |x|x^m, \ m \geq 1 \). Then \( H^1(\mathcal{R}(|x|x^m)), m \geq 1 \) is generated by \( \text{cl}(\alpha_m) \) and \( \text{cl}(\beta_m) \) where

\[
\begin{align*}
\alpha_m &= \frac{(-1)^{m-1}}{(m-1)!} \partial^{m-1} \left( \frac{1}{\pi} + \frac{1}{2} \right) a e_\delta, & (1 - \varphi) a &= (1 - \chi(\gamma) \gamma) \left( \frac{1}{\pi} + \frac{1}{2} \right), \\
\beta_m &= \frac{(-1)^{m-1}}{(m-1)!} \partial^{m-1} \left( b, \frac{1}{\pi} \right) e_\delta, & (1 - \varphi) \left( \frac{1}{\pi} \right) &= (1 - \chi(\gamma) \gamma) b.
\end{align*}
\]
Proof. See [C4], sections 2.3-2.5.

1.2. Crystalline representations.

1.2.1. The rings $B_{cris}$ and $B_{dR}$ (see [F1], [F4]). Let $\theta_0 : A^+ \to O_C$ be the map given by the formula

$$\theta_0 \left( \sum_{n=0}^{\infty} u_n p^n \right) = \sum_{n=0}^{\infty} u_n(0) p^n.$$ 

It can be shown that $\theta_0$ is a surjective ring homomorphism and that $\ker(\theta_0)$ is the principal ideal generated by $\omega = \sum_{i=0}^{p-1} [\epsilon]_{i/p}$. By linearity, $\theta_0$ can be extended to a map $\theta : \hat{B}^+ \to C$. The ring $B_{dR}^+$ is defined to be the completion of $\hat{B}^+$ for the $\ker(\theta)$-adic topology:

$$B_{dR}^+ = \lim_{\to} \hat{B}^+/\ker(\theta)^n.$$ 

This is a complete discrete valuation ring with residue field $C$ equipped with a natural action of $G_K$. Moreover, there exists a canonical embedding $\hat{K} \subset B_{dR}^+$. The series $t = \sum_{n=0}^{\infty} (-1)^{n-1} \pi^n / n$ converges in the topology of $B_{dR}^+$ and it is easy to see that $t$ generates the maximal ideal of $B_{dR}^+$. The Galois group acts on $t$ by the formula $g(t) = \chi(g) t$. Let $B_{dR} = B_{dR}^+[t^{-1}]$ be the field of fractions of $B_{dR}^+$. This is a complete discrete valuation field equipped with a $G_K$-action and an exhaustive separated decreasing filtration $\Fil^i B_{dR} = t^i B_{dR}^+$. As $G_K$-module, $\Fil^i B_{dR}/\Fil^{i+1} B_{dR} \simeq C(i)$ and $B_{dR}^{G_K} = K$.

Consider the $PD$-envelope of $A^+$ with respect to the map $\theta_0$

$$A^{PD} = A^+ \left[ \frac{\omega^2}{2!}, \frac{\omega^3}{3!}, \ldots, \frac{\omega^n}{n!}, \ldots \right]$$ 

and denote by $A_{cris}^+$ its $p$-adic completion. Let $B_{cris}^+ = A_{cris}^+ \otimes_{Z_p} \mathbb{Q}_p$ and $B_{cris} = B_{cris}^+[t^{-1}]$. Then $B_{cris}$ is a subring of $B_{dR}$ endowed with the induced filtration and Galois action. Moreover, it is equipped with a continuous Frobenius $\varphi$, extending the map $\varphi : A^+ \to A^+$. One has $\varphi(t) = pt$.

1.2.2. Crystalline representations (see [F5], [Ber1], [Ber2]).

Let $L$ be a finite extension of $\mathbb{Q}_p$. Denote by $K$ its maximal unramified subextension. A filtered Dieudonné module over $L$ is a finite dimensional $K$-vector space $M$ equipped with the following structures:

- a $\sigma$-semilinear bijective map $\varphi : M \to M$;
- an exhaustive decreasing filtration $(\Fil^i M_L)$ on the $L$-vector space $M_L = L \otimes_K M$.

A $K$-linear map $f : M \to M'$ is said to be a morphism of filtered modules if

- $f(\varphi(d)) = \varphi(f(d))$, for all $d \in M$;
- $f(\Fil^i M_L) \subset \Fil^i M_{L}'$, for all $i \in \mathbb{Z}$.

The category $\MF_L^\varphi$ of filtered Dieudonné modules is additive, has kernels and cokernels but is not abelian. Denote by $1$ the vector space $K_0$ with the natural action of $\sigma$ and the filtration given by

$$\Fil^i 1 = \begin{cases} K, & \text{if } i \leq 0, \\ 0, & \text{if } i > 0. \end{cases}$$

Then $1$ is a unit object of $\MF_L^\varphi$ i.e. $M \otimes 1 \simeq 1 \otimes M \simeq M$ for any $M$.

If $M$ is a one dimensional Dieudonné module and $d$ is a basis vector of $M$, then $\varphi(d) = \alpha d$ for some $\alpha \in K$. Set $t_N(M) = v_p(\alpha)$ and denote by $t_H(M)$ the unique filtration jump of $M$. If $M$ is
of an arbitrary finite dimension \(d\), set \(s_N(M) = t_N^{d}(\wedge M)\) and \(s_H(M) = t_H^{d}(\wedge M)\). A Dieudonné module \(M\) is said to be weakly admissible if \(t_H(M) = t_N(M)\) and if \(t_H(M') \leq t_N(M')\) for any \(\varphi\)-submodule \(M' \subset M\) equipped with the induced filtration. Weakly admissible modules form a subcategory of \(\text{MF}_L\) which we denote by \(\text{MF}_L^{\varphi,f}\).

If \(V\) is a \(p\)-adic representation of \(G_L\), define \(\text{D}_\text{dr}(V) = (\mathcal{B}_\text{dr} \otimes V)^{G_L}\). Then \(\text{D}_\text{dr}(V)\) is a \(L\)-vector space equipped with the decreasing filtration \(\text{Fil}^i \text{D}_\text{dr}(V) = (\text{Fil}^i \mathcal{B}_\text{dr} \otimes V)^{G_L}\). One has \(\dim_L \text{D}_\text{dr}(V) \leq \dim_{\mathbb{Q}_p}(V)\) and \(V\) is said to be de Rham if \(\dim_L \text{D}_\text{dr}(V) = \dim_{\mathbb{Q}_p}(V)\). Analogously one defines \(\text{D}_\text{cris}(V) = (\mathcal{B}_\text{cris} \otimes V)^{G_L}\). Then \(\text{D}_\text{cris}(V)\) is a filtered Dieudonné module over \(L\) of dimension \(\dim_K \text{D}_\text{cris}(V) \leq \dim_{\mathbb{Q}_p}(V)\) and \(V\) is said to be crystalline if the equality holds here. In particular, for crystalline representations one has \(\text{D}_\text{dr}(V) = \text{D}_\text{cris}(V) \otimes_K L\). By the theorem of Colmez-Fontaine \([CF]\), the functor \(\text{D}_\text{cris}\) establishes an equivalence between the category of crystalline representations of \(G_L\) and \(\text{MF}_L^{\varphi,f}\). Its quasi-inverse \(\text{V}_\text{cris}\) is given by \(\text{V}_\text{cris}(D) = \text{Fil}^0(D \otimes_{\mathbb{Q}_K} \mathcal{B}_\text{cris}, \varphi = 1)\).

An important result of Berger ([Ber 1], Theorem 0.2) says that \(\text{D}_\text{cris}(V)\) can be recovered from the \((\varphi, \Gamma)\)-module \(\text{D}^\times_{\text{rig}}(V)\). The situation is particularly simple if \(L/\mathbb{Q}_p\) is unramified. In this case set \(\text{D}^+(V) = (V \otimes_{\mathbb{Q}_p} \mathcal{B}^+)^{H_K}\) and \(\text{D}^+_{\text{rig}}(V) = \mathcal{R}^+(K) \otimes_{\mathcal{B}_K^+} \text{D}^+(V)\). Then

\[
\text{D}_\text{cris}(V) = \left(\text{D}^+_{\text{rig}}(V) \left[\frac{1}{\ell}\right]\right)^{\Gamma}
\]

(see [Ber2], Proposition 3.4).

§2. The exponential map

2.1. The Bloch-Kato exponential map ([BK], [N1], [FP]).

2.1.1. Let \(L\) be a finite extension of \(\mathbb{Q}_p\). Recall that we denote by \(\text{MF}_L^{\varphi}\) the category of filtered Dieudonné modules over \(L\). If \(M\) is an object of \(\text{MF}_L^{\varphi}\), define

\[
H^i(L, M) = \text{Ext}^i_{\text{MF}_L^{\varphi}}(1, M), \quad i = 0, 1.
\]

Remark that \(H^*(L, M)\) can be computed explicitly as the cohomology of the complex

\[
C^\bullet(M) : M \xrightarrow{f} (M_L/\text{Fil}^0 M_L) \oplus M
\]

where the modules are placed in degrees 0 and 1 and \(f(d) = (d \mod \text{Fil}^0 M_L, (1 - \varphi)(d))\) ([N1],[FP]). Remark that if \(M\) is weakly admissible then each extension \(0 \rightarrow M \rightarrow M' \rightarrow 1 \rightarrow 0\) is weakly admissible too and we can write \(H^i(L, M) = \text{Ext}^i_{\text{MF}_L^{\varphi,f}}(1, M)\).

2.1.2. Let \(\text{Rep}_\text{cris}(G_K)\) denote the category of crystalline representations of \(G_K\). For any object \(V\) of \(\text{Rep}_\text{cris}(G_K)\) define

\[
H^i_j(K, V) = \text{Ext}^i_{\text{Re}p_\text{cris}(G_K)}(\mathbb{Q}_p(0), V).
\]

An easy computation shows that

\[
H^i_j(K, V) = \begin{cases} 
H^0(K, V), & \text{if } i = 0, \\
\ker (H^1(K, V) \rightarrow H^1(K, V \otimes B_{\text{cris}})), & \text{if } i = 1, \\
0, & \text{if } i \geq 2.
\end{cases}
\]
Let \( t_V(K) = D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V) \) denote the tangent space of \( V \). The rings \( B_{\text{dR}} \) and \( B_{\text{cris}} \) are related to each other via the fundamental exact sequence

\[
0 \to \mathbb{Q}_p \to B_{\text{cris}} \xrightarrow{f} B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}} \oplus B_{\text{cris}} \to 0
\]

where \( f(x) = (x \mod \text{Fil}^0 B_{\text{dR}}), (1 - \varphi) x \) (see [BK], §4). Tensoring this sequence with \( V \) and taking cohomology one obtains an exact sequence

\[
0 \to H^0(K, V) \to D_{\text{cris}}(V) \to t_V(K) \oplus D_{\text{cris}}(V) \to H^1_f(K, V) \to 0.
\]

The last map of this sequence gives rise to the Bloch-Kato exponential map

\[
\exp_{V, K} : t_V(K) \oplus D_{\text{cris}}(V) \to H^1(K, V).
\]

Following [F3] set

\[
R\Gamma_f(K, V) = C^\bullet(D_{\text{cris}}(V)) = \left[ D_{\text{cris}}(V) \xrightarrow{f} t_V(K) \oplus D_{\text{cris}}(V) \right].
\]

From the classification of crystalline representations in terms of Dieudonné modules it follows that the functor \( V_{\text{cris}} \) induces natural isomorphisms

\[
r_{V,p}^i : R^i\Gamma_f(K, V) \to H^i_f(K, V), \quad i = 0, 1.
\]

The composite homomorphism

\[
t_K(V) \oplus D_{\text{cris}}(V) \to R^1\Gamma_f(K, V) \xrightarrow{r_{V,p}^1} H^1(K, V)
\]

coincides with the Bloch-Kato exponential map \( \exp_{V, K} \) ([N1], Proposition 1.21).

2.1.3. Let \( g : B^\bullet \to C^\bullet \) be a morphism of complexes. We denote by \( \text{Tot}^n(g) \) the complex \( \text{Tot}^n(g) = C^{n-1} \oplus B^n \) with differentials \( d^n : \text{Tot}^n(g) \to \text{Tot}^{n+1}(g) \) defined by the formula \( d^n(c, b) = ((-1)^n g^n(b) + d^{n-1}(c), d^n(b)) \). It is well known that if \( 0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\phi} C^\bullet \to 0 \) is an exact sequence of complexes, then \( f \) induces a quasi isomorphism \( A^\bullet \sim \text{Tot}^\bullet(f) \). In particular, tensoring the fundamental exact sequence with \( V \), we obtain an exact sequence of complexes

\[
0 \to R\Gamma(K, V) \to C^\bullet_c(G_K, V \otimes B_{\text{cris}}) \xrightarrow{f} C^\bullet_c(G_K, (V \otimes (B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}})) \oplus (V \otimes B_{\text{cris}})) \to 0
\]

which gives a quasi isomorphism \( R\Gamma(K, V) \sim \text{Tot}^\bullet(f) \). Since \( R\Gamma_f(K, V) \) coincides tautologically with the complex

\[
C^0_c(G_K, V \otimes B_{\text{cris}}) \xrightarrow{f} C^0_c(G_K, (V \otimes (B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}})) \oplus (V \otimes B_{\text{cris}}))
\]

we obtain a diagram

\[
\begin{array}{ccc}
R\Gamma(K, V) & \sim & \text{Tot}^\bullet(f) \\
\downarrow & & \\
R\Gamma_f(K, V)
\end{array}
\]
which defines a morphism $\mathbf{R} \Gamma_f(K,V) \to \mathbf{R} \Gamma(K,V)$ in $\mathcal{D}(\mathbb{Q}_p)$ (see [BF], Proposition 1.17). Remark that the induced homomorphisms $\mathbf{R} \Gamma_f(K,V) \to H^i(K,V)$ ($i = 0, 1$) coincide with the composition of $r^i_{f,p}$ with natural embeddings $H^i_f(K,V) \to H^i(K,V)$.

2.1.4. In this subsection we define an analogue of the exponential map for crystalline $(\varphi, \Gamma)$-modules. Let $K/\mathbb{Q}_p$ be an unramified extension. If $D$ is a $(\varphi, \Gamma)$-module over $\mathcal{R}(K)$ define

$$\mathcal{D}_{\text{cris}}(D) = (D[1/t])^\Gamma.$$ 

It can be shown that $\mathcal{D}_{\text{cris}}(D)$ is a finite dimensional $K$-vector space equipped with a natural decreasing filtration $\text{Fil}^i\mathcal{D}_{\text{cris}}(D)$ and a semilinear action of $\varphi$. One says that $D$ is crystalline if

$$\dim_K(\mathcal{D}_{\text{cris}}(D)) = \text{rg}(D).$$

(see [BC]). From [Ber4], Théorème A it follows that the functor $D \mapsto \mathcal{D}_{\text{cris}}(D)$ is an equivalence between the category of crystalline $(\varphi, \Gamma)$-modules and $\text{MF}_K^\varphi$. Remark that if $V$ is a $p$-adic representation of $G_K$ then $\mathcal{D}_{\text{cris}}(V) = \mathcal{D}_{\text{cris}}(\mathcal{D}_{\text{rig}}(V))$ and $V$ is crystalline if and only if $\mathcal{D}_{\text{rig}}(V)$ is.

Let $D$ be a $(\varphi, \Gamma)$-module. To any cocycle $\alpha = (a,b) \in Z^1(C_{\varphi, \gamma}(D))$ one can associate the extension

$$0 \to D \to D_\alpha \to \mathcal{R}(K) \to 0$$

defined by

$$D_\alpha = D \oplus \mathcal{R}(K)e, \quad (\varphi - 1)e = a, \quad (\gamma - 1)e = b.$$ 

As usually, this gives rise to an isomorphism $H^1(D) \simeq \text{Ext}^1_{\mathcal{R}}(\mathcal{R}(K), D)$. We say that $\text{cl}(\alpha)$ is crystalline if $\dim_K(\mathcal{D}_{\text{cris}}(D_{\alpha})) = \dim_K(\mathcal{D}_{\text{cris}}(D)) + 1$ and define

$$H^1_{\text{cris}}(D) = \{ \text{cl}(\alpha) \in H^1(D) \mid \text{cl}(\alpha) \text{ is crystalline} \}$$

(see [Ben2], section 1.4.1). If $D$ is crystalline (or more generally potentially semistable) one has a natural isomorphism

$$H^1(K, \mathcal{D}_{\text{cris}}(D)) \to H^1_{\text{cris}}(D).$$

Set $t_D = \mathcal{D}_{\text{cris}}(D)/\text{Fil}^0\mathcal{D}_{\text{cris}}(D)$ and denote by $\exp_D : t_D \oplus \mathcal{D}_{\text{cris}}(D) \to H^1(D)$ the composition of this isomorphism with the projection $t_D \oplus \mathcal{D}_{\text{cris}}(D) \to H^1(K, \mathcal{D}_{\text{cris}}(D))$ and the embedding $H^1_{\text{cris}}(D) \hookrightarrow H^1(D)$.

2.1.5. Assume that $K = \mathbb{Q}_p$. To simplify notation we will write $D_m$ for $\mathcal{R}(|x|^{1/m})$ and $e_m$ for its canonical basis. Then $\mathcal{D}_{\text{cris}}(D_m)$ is the one dimensional $\mathbb{Q}_p$-vector space generated by $t^{-m}e_m$. As in [Ben2], we normalize the basis $(\text{cl}(\alpha_m), \text{cl}(\beta_m))$ of $H^1(D_m)$ putting $\alpha_m = (1 - 1/p) \text{cl}(\alpha_m)$ and $\beta_m = (1 - 1/p) \log(\chi(\gamma)) \text{cl}(\beta_m)$.

**Proposition 2.1.6.** i) $H^1_{\text{cris}}(D_m)$ is the one-dimensional $\mathbb{Q}_p$-vector space generated by $\alpha_m^*$. ii) The exponential map

$$\exp_{D_m} : t_{D_m} \to H^1(D_m)$$

sends $t^{-m}w_m$ to $-\alpha_m^*$.

*Proof.* This is a reformulation of [Ben2], Proposition 1.5.8 ii).
2.2. The large exponential map.

2.2.1. In this section $p$ is an odd prime number, $K$ is a finite unramified extension of $\mathbb{Q}_p$ and $\sigma$ the absolute Frobenius acting on $K$. Recall that $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup_{n=1}^{\infty} K_n$. We set $\Gamma = \text{Gal}(K_\infty/K)$, $\Gamma_n = \text{Gal}(K_\infty/K_n)$ and $\Delta = \text{Gal}(K_1/K)$. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and $\Lambda(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda$. We will consider the following operators acting on the ring $K[[X]]$ of formal power series with coefficients in $K$:

- The ring homomorphism $\sigma : K[[X]] \rightarrow K[[X]]$ defined by $\sigma \left( \sum_{i=0}^{\infty} a_i X^i \right) = \sum_{i=0}^{\infty} \sigma(a_i) X^i$;
- The ring homomorphism $\varphi : K[[X]] \rightarrow K[[X]]$ defined by $\varphi \left( \sum_{i=0}^{\infty} a_i X^i \right) = \sum_{i=0}^{\infty} \sigma(a_i) \varphi(X)^i$, $\varphi(X) = (1 + X)^p - 1$.

- The differential operator $\partial = (1 + X) \frac{d}{dX}$. One has $\partial \circ \varphi = p \varphi \circ \partial$.

- The operator $\psi : K[[X]] \rightarrow K[[X]]$ defined by $\psi(f(X)) = \frac{1}{p} \varphi^{-1} \left( \sum_{i=0}^{\infty} f((1 + X)\zeta - 1) \right)$.

It is easy to see that $\psi$ is a left inverse to $\varphi$, i.e. that $\psi \circ \varphi = \text{id}$.

- An action of $\Gamma$ given by $\gamma \left( \sum_{i=0}^{\infty} a_i X^i \right) = \sum_{i=0}^{\infty} a_i \gamma(X)^i$, $\gamma(X) = (1 + X)^{\chi(\gamma)} - 1$.

Remark that these formulas are compatible with the definitions from sections 1.1.1 and 1.1.6.

Fix a generator $\gamma_1 \in \Gamma_1$ and define

$$\mathcal{H} = \{ f(\gamma_1 - 1) \mid f \in \mathbb{Q}_p[[X]] \text{ is holomorphic on } B(0,1) \}, \quad \mathcal{H}(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathcal{H}.$$

2.2.2. It is well known that $\mathbb{Z}_p[[X]]^{\psi=0}$ is a free $\Lambda$-module generated by $(1 + X)$ and the operator $\partial$ is bijective on $\mathbb{Z}_p[[X]]^{\psi=0}$. If $V$ is a crystalline representation of $G_K$ put $\mathcal{D}(V) = \mathcal{D}_{\text{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]^{\psi=0}$. Let $\Xi_{V,n} : \mathcal{D}(V)_{\Gamma_n[-1]} \rightarrow \mathbf{R} \mathcal{F}_f(K_n, V)$ be the map defined by

$$\Xi_{V,n}^\varepsilon(\alpha) = \begin{cases} p^{-n} (\sum_{k=1}^{n} (\sigma \otimes \varphi)^{-k} \alpha(\zeta_{p^k} - 1), -\alpha(0)) & \text{if } n \geq 1, \\ \text{Tr}_{K_1/K} (\Xi_{V,1}^\varepsilon(\alpha)) & \text{if } n = 0. \end{cases}$$

An easy computation shows that $\Xi_{V,0} : \mathcal{D}_{\text{cris}}(V)[-1] \rightarrow \mathbf{R} \mathcal{F}_f(K, V)$ is given by the formula

$$\Xi_{V,0}^\varepsilon(a) = \frac{1}{p} (-\varphi^{-1}(a), -(p - 1) a).$$

In particular, it is homotopic to the map $a \mapsto -(0, (1 - p^{-1} \varphi^{-1}) a)$. Write

$$\Xi_{V,n}^\varepsilon : \mathcal{D}(V) \rightarrow \mathbf{R} \mathcal{F}_f(K_n, V) = \frac{t_V(K_n) \oplus \mathcal{D}_{\text{cris}}(V)}{\mathcal{D}_{\text{cris}}(V) / V^{G_K}}$$

denote the homomorphism induced by $\Xi_{V,n}^\varepsilon$. Then

$$\Xi_{V,0}^\varepsilon(a) = -(0, (1 - p^{-1} \varphi^{-1}) a) \pmod{\mathcal{D}_{\text{cris}}(V) / V^{G_K}}.$$
If $D_{\text{cris}}(V)^{\varphi=1} = 0$ the operator $1 - \varphi$ is invertible on $D_{\text{cris}}(V)$ and we can write

$$
\Xi_{V,0}^\varepsilon(a) = \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi}, a, 0 \right) \pmod{D_{\text{cris}}(V)/V^{G_K}}.
$$

(2.1)

For any $i \in \mathbb{Z}$ let $\Delta_i : D(V) \rightarrow \frac{D_{\text{cris}}(V)}{(1 - p^i\varphi)}D_{\text{cris}}(V) \otimes \mathbb{Q}_p(i)$ be the map given by

$$
\Delta_i(\alpha(X)) = \partial^i\alpha(0) \otimes \varepsilon^{\otimes i} \pmod{(1 - p^i\varphi)D_{\text{cris}}(V)}.
$$

Set $\Delta = \bigoplus_{i \in \mathbb{Z}} \Delta_i$. If $\alpha \in D(V)^{\Delta=0}$, then by [PR1], Proposition 2.2.1 there exists $F \in D_{\text{cris}}(V) \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p[[X]]$ which converges on the open unit disk and such that $(1 - \varphi)F = \alpha$. A short computation shows that

$$
\Xi_{V,n}^\varepsilon(\alpha) = p^{-n}((\sigma \otimes \varphi)^{-n}(F)(\zeta_p^n - 1), 0) \pmod{D_{\text{cris}}(V)/V^{G_K}}, \quad \text{if } n \geq 1
$$

(see [BB], Lemma 4.9).

2.2.3. As $\mathbb{Z}_p[[X]][1/p]$ is a principal ideal domain and $\mathcal{H}$ is $\mathbb{Z}_p[[X]][1/p]$-torsion free, $\mathcal{H}$ is flat. Thus

$$
C^+_{1w,\psi}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma) = C^+_{1w,\psi}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma) = \left[ \mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D^+_{\text{rig}}(V) \xrightarrow{\psi^{-1}} \mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D^+(V) \right].
$$

By proposition 1.1.7 on has an isomorphism in $D(\mathcal{H}(\Gamma))$

$$
R\Gamma_{1w}(K, V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma) \simeq C^+_{1w,\psi}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma).
$$

The action of $\mathcal{H}(\Gamma)$ on $D^+(V)^{\psi=1}$ induces an injection $\mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D^+_1(V)^{\psi=1} \hookrightarrow D^+_{\text{rig}}(V)^{\psi=1}$. Composing this map with the canonical isomorphism $H^1_{1w}(K, V) \simeq D^+_1(V)^{\psi=1}$ we obtain a map $\mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} H^1_{1w}(K, V) \hookrightarrow D^+_1(V)^{\psi=1}$. For any $k \in \mathbb{Z}$ set $\nabla_k = t\partial - k = t \frac{d}{dt} - k$. An easy induction shows that $\nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0 = t^k \partial^k$.

Fix $h \geq 1$ such that $\text{Fil}^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V)$ and $V(-h)^{G_K} = 0$. For any $\alpha \in D(V)^{\Delta=0}$ define

$$
\Omega_{V,h}^\varepsilon(\alpha) = (-1)^{h-1} \frac{\log \chi(\gamma_1)}{p} \nabla_{h-1} \circ \nabla_{h-2} \circ \cdots \circ \nabla_0(F(\pi)),
$$

where $F \in \mathcal{H}(V)$ is such that $(1 - \varphi)F = \alpha$. It is easy to see that $\Omega_{V,h}^\varepsilon(\alpha) \in D^+_1(V)^{\psi=1}$. In [Ber3] Berger shows that $\Omega_{V,h}^\varepsilon(\alpha) \in \mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} D^+(V)^{\psi=1}$ and therefore gives rise to a map

$$
\text{Exp}_{V,h}^\varepsilon : D(V)^{\Delta=0} [-1] \rightarrow R\Gamma_{1w}(K, V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma)
$$

Let

$$
\text{Exp}_{V,h}^\varepsilon : D(V)^{\Delta=0} \rightarrow \mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} H^1_{1w}(K, V)
$$

denote the map induced by $\text{Exp}_{V,h}^\varepsilon$ in degree 1. The following theorem is a reformulation of the construction of the large exponential map given by Berger in [Ber3].
Theorem 2.2.4. Let 

$$\text{Exp}^c_{V,n} : D(V)^{\Delta=0}_{\Gamma_n} \to R\Gamma_{I_w}(K,V) \otimes_{A_{\mathbb{Q}_p}} \mathbb{Q}_p[G_n]$$

denote the map induced by $\text{Exp}^c_{V,h}$. Then for any $n \geq 0$ the following diagram in $D(\mathbb{Q}_p[G_n])$ is commutative:

$$
\begin{array}{ccc}
D(V)^{\Delta=0}_{\Gamma_n} & \xrightarrow{\text{Exp}^c_{V,n}} & R\Gamma_{I_w}(K,V) \otimes_{A_{\mathbb{Q}_p}} \mathbb{Q}_p[G_n] \\
\downarrow{\Xi}_{V,n} & & \downarrow{\cong} \\
R\Gamma_f(K_n,V) & \xrightarrow{(h-1)! \text{exp}_{V,K_n}} & R\Gamma(K_n,V).
\end{array}
$$

In particular, $\text{Exp}^c_{V,h}$ coincides with the large exponential map of Perrin-Riou.

Proof. Passing to cohomology in the previous diagram one obtains the diagram

$$
\begin{array}{ccc}
D(V)^{\Delta=0}_{\Gamma_n} & \xrightarrow{\text{Exp}^c_{V,h}} & H(\Gamma) \otimes_{A_{\mathbb{Q}_p}} H^1_{I_w}(K,V) \\
\downarrow{\Xi}_{V,n} & & \downarrow{\text{pr}_{V,n}} \\
\mathbb{D}_{\text{dR}/K_n}(V) \oplus \mathbb{D}_{\text{cris}}(V) & \xrightarrow{(h-1)! \text{exp}_{V,K_n}} & H^1(K_n,V)
\end{array}
$$

which is exactly the definition of the large exponential map. Its commutativity is proved in [Ber3], Theorem II.13. Now, the theorem is an immediate consequence of the following remark. Let $D$ be a free $A$-module and let $f_1, f_2 : D[-1] \to K^\bullet$ be two maps from $D[-1]$ to a complex of $A$-modules such that the induced maps $h_1(f_1)$ and $h(f_2) : D \to H^1(K^\bullet)$ coincide. Then $f_1$ and $f_2$ are homotopic.

§3. The $L$-invariant

3.1. Definition of the $L$-invariant ([Ben2]).

3.1.1. In this section we recall the definition of the $L$-invariant for the case of crystalline representations. For further details and proofs see [Ben2], §2. Let $S$ be a finite set of primes of $\mathbb{Q}$ containing $p$ and $G_S$ the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside $S \cup \{\infty\}$. For each place $v$ we denote by $G_v$ the decomposition at $v$ group and by $I_v$ and $f_v$ the inertia subgroup and Frobenius automorphism respectively. Let $V$ be a $p$-adic pseudo-geometric representation of $G_S$. Thus $V$ is a de Rham at $p$. For any $v \notin \{p, \infty\}$ set

$$R\Gamma_f(Q_v,V) = \left[V^{I_v} \frac{1-f_v}{f_v}, V^{I_v}\right],$$

where the terms are placed in degrees 0 and 1 (see [F3], [BF]). Observe that there is a natural quasi-isomorphism $R\Gamma_f(Q_v,V) \simeq C^*_c(G_v/I_v,V^{I_v})$. In particular, $R^0\Gamma(Q_v,V) = H^0(Q_v,V)$ and $R^1\Gamma_f(Q_v,V) = H^1_f(Q_v,V)$ where

$$H^1_f(Q_v,V) = \ker(H^1(Q_v,V) \to H^1(Q_v^{ur},V)).$$

For $v = p$ the complex $R\Gamma_f(Q_v,V)$ was defined in section 2.1.2. To simplify notation write $H^1_S(V) = H^1(G_S,V)$. The Selmer group of $V$ is defined by

$$H^1_f(V) = \ker\left(H^1_S(V) \to \bigoplus_{v \in S} H^1_f(Q_v,V)\right).$$
3.1.2. Assume that $V$ satisfies the following conditions:

C1) $H^1(D(V) = H^{1}(V^{*}(1)) = 0$.
C2) $H^2(D(V) = H^{2}(V^{*}(1)) = 0$.
C3) $V$ is crystalline at $p$, $D_{cris}(V)\phi = 0$ and the linear map $1 - p^{-1}\phi^{-1} : D_{cris}(V) \rightarrow D_{cris}(V)$ is semisimple.
C4) The $(\phi, \Gamma)$-module $D_{rig}^{\dagger}(V)$ has no crystalline subquotient of the form

$$0 \rightarrow \mathcal{R}(x, x^k) \rightarrow U \rightarrow \mathcal{R} \rightarrow 0, \quad k \geq 1.$$

Write $c$ for the complex conjugation and set $d_c(V) = \dim(V^{c=\pm 1})$. From the Poitou-Tate exact sequence it follows that $\dim_{\mathbb{Q}_p} t_V(\mathbb{Q}_p) = d_c(V)$. We say that a $\mathbb{Q}_p$-subspace $D \subset D_{cris}(V)$ is admissible if it is stable under $\phi$ and the natural projection $D \rightarrow t_V(\mathbb{Q}_p)$ is an isomorphism.

3.1.3. Let $D$ be an admissible subspace of $D_{cris}(V)$. As $1 - p^{-1}\phi^{-1}$ acts semisimply, one has a decomposition $D \simeq D_{-1} \oplus D_{\phi=p^{-1}}$ where $D_{-1} = (\phi - p^{-1}) D$ is stable under $\phi$ and $(D_{-1})^\phi = p^{-1} = 0$. Consider the filtration $(D_i)$ on $D_{cris}(V)$ defined by

$$D_i = \begin{cases} 0 & \text{if } i = -2, \\ D_{-1} & \text{if } i = -1, \\ D & \text{if } i = 0, \\ D_{cris}(V) & \text{if } i = 1. \end{cases}$$

By Berger’s theory [Ber4] $(D_i)$ induces a filtration on $D_{rig}^{\dagger}(V)$:

$$0 \subset F_{-1}D_{rig}^{\dagger}(V) \subset F_0D_{rig}^{\dagger}(V) \subset F_1D_{rig}^{\dagger}(V) = D_{rig}^{\dagger}(V).$$

Explicitly $F_0D_{rig}^{\dagger}(V) = D_{rig}^{\dagger}(V) \cap (D_i \otimes_{\mathbb{Q}_p} \mathcal{R}[1/l]) ([BC]$, section 2.4.2). Set $gr_1D_{rig}^{\dagger}(V) = F_0D_{rig}^{\dagger}(V)/F_{-1}D_{rig}^{\dagger}(V)$. By [Ben2], Corollary 1.4.6 the exact sequence

$$0 \rightarrow F_0D_{rig}^{\dagger}(V) \rightarrow D_{rig}^{\dagger}(V) \rightarrow gr_1D_{rig}^{\dagger}(V) \rightarrow 0$$
gives rise to exact sequences

$$\cdots \rightarrow H^0(gr_1D_{rig}^{\dagger}(V)) \rightarrow H^1(F_0D_{rig}^{\dagger}(V)) \rightarrow H^1(D_{rig}^{\dagger}(V)) \rightarrow H^1(gr_1D_{rig}^{\dagger}(V)) \rightarrow \cdots$$

and

$$\cdots \rightarrow H^0(gr_1D_{rig}^{\dagger}(V)) \rightarrow H^1(F_0D_{rig}^{\dagger}(V)) \rightarrow H^1(D_{rig}^{\dagger}(V)) \rightarrow H^1(gr_1D_{rig}^{\dagger}(V)) \rightarrow 0$$

The condition C3) implies that $D_{cris}(gr_1D_{rig}^{\dagger}(V))^{\phi = 1} = 0$. Since $D$ is admissible, the Hodge-Tate weights of $gr_1D_{rig}^{\dagger}(V)$ are $\leq 0$ and by Proposition 1.4.4 of [Ben2] $H^0(gr_1D_{rig}^{\dagger}(V)) = 0$ and $H^1(gr_1D_{rig}^{\dagger}(V)) = 0$. This shows that $H^1(F_0D_{rig}^{\dagger}(V))$ injects into $H^1(D_{rig}^{\dagger}(V)) \simeq H^1(\mathbb{Q}_p, V)$ and that

$$H^1(F_0D_{rig}^{\dagger}(V)) \simeq H^1(D_{rig}^{\dagger}(V)) \simeq H^1(\mathbb{Q}_p, V).$$
Now, consider the short exact sequence

\[ 0 \to F_1^\dagger D_{\text{rig}}(V) \to F_0^\dagger D_{\text{rig}}(V) \to \text{gr}_0 D_{\text{rig}}^\dagger(V) \to 0. \]

Since \( \mathcal{D}_{\text{cris}}(F_1^\dagger D_{\text{rig}}(V))_{\varphi = p^{-1}} = 0 \) and Hodge-Tate weights of \( F_1^\dagger D_{\text{rig}}(V) \) are positive, we have \( H^1_f(F_1^\dagger D_{\text{rig}}(V)) = H^1(F_1^\dagger D_{\text{rig}}(V)) \) by [Ben2], Proposition 1.4.4. As \( \mathcal{D}_{\text{cris}}((F_1^\dagger D_{\text{rig}}(V))^\dagger(\chi)) \) is dual to \( \mathcal{D}_{\text{cris}}(F_1^\dagger D_{\text{rig}}(V)) \), the map \( 1 - \varphi \) is bijective on \( \mathcal{D}_{\text{cris}}((F_1^\dagger D_{\text{rig}}(V))^\dagger(\chi)) \) and \( H^0((F_1^\dagger D_{\text{rig}}(V))^\dagger(\chi)) = 0 \). Using the local duality [Li] we obtain that \( H^2(F_1^\dagger D_{\text{rig}}(V)) \) = 0.

Finally \( \mathcal{D}_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^\dagger(V))_{\varphi = 1} = 0 \) implies that \( H^0(\text{gr}_0 D_{\text{rig}}^\dagger(V)) = 0 \). Thus we have exact sequences

\[ 0 \to H^1(F_1^\dagger D_{\text{rig}}(V)) \to H^1(F_0^\dagger D_{\text{rig}}(V)) \to H^1(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \to 0, \]
\[ 0 \to H^1(F_1^\dagger D_{\text{rig}}(V)) \to H^1(F_0^\dagger D_{\text{rig}}(V)) \to H^1(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \to 0. \]

Therefore

\[ H^1(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \simeq \frac{H^1(F_0^\dagger D_{\text{rig}}(V))}{H^1(F_1^\dagger D_{\text{rig}}(V))} \rightarrow \frac{H^1(\mathbb{Q}_p, V)}{H^1(F_1^\dagger D_{\text{rig}}(V))}, \]

and

\[ H^1_j(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \simeq \frac{H^1(F_0^\dagger D_{\text{rig}}(V))}{H^1(F_1^\dagger D_{\text{rig}}(V))}. \]

As \( \mathcal{D}_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^\dagger(V))_{\varphi = p^{-1}} = \mathcal{D}_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \), Proposition 1.5.9 of [Ben2] implies that

\[ \text{gr}_0 D_{\text{rig}}^\dagger(V) \simeq \bigoplus_{i=1}^e D_{m_i}, \quad e = \dim_{\mathbb{Q}_p}(D^\varphi = p^{-1}) \]

where \( D_{m_i} = R(|x| x^{m_i}), m_i \geq 1 \). By Proposition 2.1.6 \( H^1_f(D_m) \) is generated by \( \alpha_m^* \) and we denote by \( H^1_c(D_m) \) the subspace generated by \( \beta_m^* \). This gives a decomposition

\[ H^1(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \simeq H^1_f(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \oplus H^1_c(\text{gr}_0 D_{\text{rig}}^\dagger(V)). \]

In particular, \( H^1_f(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \) and \( H^1_c(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \) are \( \mathbb{Q}_p \)-vector spaces of dimension \( e \). Further, fixing the basis \( \alpha_m^*, \beta_m^* \) of \( H^1(D_m) \) we fixe isomorphisms

\[ i_{D,f} : \mathcal{D}_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \simeq H^1_f(\text{gr}_0 D_{\text{rig}}^\dagger(V)), \quad i_{D,c} : \mathcal{D}_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^\dagger(V)) \simeq H^1_c(\text{gr}_0 D_{\text{rig}}^\dagger(V)). \]

The condition C1 together with the Poitou-Tate exact sequence implies that

\[ H^1_S(V) \simeq \bigoplus_{v \in S} \frac{H^1(\mathbb{Q}_v, V)}{H^1_f(\mathbb{Q}_v, V)}. \]

Let \( H^1_S(D, V) \) be the subspace of \( H^1_S(V) \) whose image under this isomorphism is \( H^1(F_0^\dagger D_{\text{rig}}(V))/H^1_f(\mathbb{Q}_p, V) \). The localization map \( H^1_S(D, V) \to \frac{H^1(\mathbb{Q}_p, V)}{H^1_f(\mathbb{Q}_p, V)} \) is injective.
and its image is contained in $H^1(\text{gr}_0 D_{\text{rig}}^+(V))$. Hence, we have a diagram

$$\begin{array}{ccc}
D_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^+(V)) & \xrightarrow{\rho_{D,f}} & H^1(\text{gr}_0 D_{\text{rig}}^+(V)) \\
\downarrow \rho_{D,c} & & \downarrow \rho_{D,c} \\
D_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^+(V)) & \xrightarrow{\rho_{D,c}^{-1}} & H^1(\text{gr}_0 D_{\text{rig}}^+(V))
\end{array}$$

where $\rho_{D,f}$ and $\rho_{D,c}$ are defined as the unique maps making this diagram commute. From the definition of $H^1_S(D,V)$ it follows that $\rho_{D,c}$ is an isomorphism.

**Definition 3.1.4.** The determinant

$$L(V,D) = \det \left( \rho_{D,f} \circ \rho_{D,c}^{-1} \mid D_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^+(V)) \right)$$

will be called the $L$-invariant associated to $V$ and $D$.

**3.2. The Bockstein homomorphism.**

**3.2.1.** In this section we interpret $L(D,V)$ in terms of the Bockstein homomorphism associated to the large exponential map. This interpretation is crucial for the proof of the main theorem of this paper. Recall that $H^1(Q_p, H(\Gamma) \otimes Q_p V) = H(\Gamma) \otimes_{\Lambda(\Gamma)} H^1_{\text{Iw}}(Q_p, V)$ injects into $D^\dagger_{\text{rig}}(V)$. Set

$$F_i H^1(Q_p, H(\Gamma) \otimes Q_p V) = F_i D^\dagger_{\text{rig}}(V) \cap H^1(Q_p, H(\Gamma) \otimes Q_p V).$$

As in section 2.2 we fix a generator $\gamma \in \Gamma$.

**Proposition 3.2.2.** Let $D$ be an admissible subspace of $D_{\text{cris}}(V)$. For any $a \in D^{\varphi=p^{-1}}$ let $\alpha \in D(V)$ be such that $\alpha(0) = a$. Then

i) There exists a unique $\beta \in F_0 H^1(Q_p, H(\Gamma) \otimes V)$ such that

$$(\gamma - 1) \beta = \text{Exp}_{V,h}^\varphi(\alpha).$$

ii) The composition map

$$\delta_{D,h} : D^{\varphi=p^{-1}} \to F_0 H^1(Q_p, H(\Gamma) \otimes V) \to H^1(\text{gr}_0 D_{\text{rig}}^+(V))$$

is given explicitly by the following formula:

$$\delta_{D,h}(a) = \beta \pmod{H^1(F_{-1} D_{\text{rig}}^+(V))} = -(h - 1)! \left(1 - \frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} i_{D,c}(\alpha).$$

**Proof.** Since $D_{\text{cris}}(V)^{\varphi=1} = 0$, the operator $1 - \varphi$ is invertible on $D_{\text{cris}}(V)$ and we have a diagram

$$\begin{array}{ccc}
D(V)^{\Delta=0} & \xrightarrow{\text{Exp}_{V,h}} & H^1(Q_p, H(\Gamma) \otimes V) \\
\downarrow \text{Exp} & & \downarrow \text{pr}_V \\
D_{\text{cris}}(V) & \xrightarrow{(h-1)! \text{Exp}} & H^1(Q_p, V).
\end{array}$$
Thus from the congruence \( \tilde{\beta} \equiv \beta \pmod{\mathbb{Z}} \) (see (2.1)). If \( \alpha \in D^{p-1} \otimes \mathbb{Z}_p[[X]]^{\psi=0} \), then \( \Xi_{V,0}(\alpha) = 0 \) and \( \text{pr}_V \left( \text{Exp}^\varepsilon_{V,h}(\alpha) \right) = 0 \). On the other hand, as \( D_{\text{cris}}(V)^{\psi=1} = 0 \), we have \( V^{G_K} = 0 \) and the map \( (\mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\text{Iw}}(\mathbb{Q}_p, V))_1 \to H^1(\mathbb{Q}_p, V) \) is injective. Thus there exists a unique \( \beta \in \mathcal{H}(\Gamma) \otimes_{\Lambda} H^1_{\text{Iw}}(\mathbb{Q}_p, T) \) such that \( \text{Exp}^\varepsilon_{V,h}(\alpha) = (\gamma - 1) \beta \). Now take \( a \in D^{p-1} \) and set

\[
 f = a \otimes \ell \left( \frac{(1 + X)^{\chi(\gamma)} - 1}{X} \right),
\]

where \( \ell(g) = \frac{1}{p} \log \left( \frac{g^p}{\varphi(g)} \right) \). An easy computation shows that

\[
\sum_{\zeta_p = 1} \ell \left( \frac{\zeta(\gamma)(1 + X)^{\chi(\gamma)} - 1}{\zeta(1 + X) - 1} \right) = 0.
\]

Thus \( f \in D^{p-1} \otimes \mathbb{Z}_p[[X]]^{\psi=0} \). Write \( \alpha \) in the form \( \alpha = (1 - \varphi)(1 - \gamma)(a \otimes \log(X)) \). Then

\[
\Omega_{V,h}(\alpha) = (-1)^{h-1} \frac{\log \chi(\gamma)}{p} t^h \partial^h((\gamma - 1)(a \log(\pi))) = \frac{\log \chi(\gamma)}{p}(\gamma - 1) \beta
\]

where

\[
\beta = (-1)^{h-1} t^h \partial^h(a \log(\pi)) = (-1)^{h-1}at^h \partial^h \left( \frac{1 + \pi}{\pi} \right).
\]

It implies immediately that \( \beta \in E_0 D_{\text{rig}}^h(V) \). On the other hand \( D^{p-1} = D_{\text{cris}}(\text{gr}_0 D_{\text{rig}}^h(V)) \). Write \( \tilde{a} \) for the image of \( a \) in \( \text{gr}_0 D_{\text{rig}}^h(V) [1/t] \) and \( e_m \) for the canonical base of \( D_m \). Since \( \text{gr}_0 D_{\text{rig}}^h(V) \simeq \bigoplus D_m, \) without lost of generality we may assume that \( \tilde{a} = t^{-m}e_m \) for some \( i \).

Let \( \tilde{\beta} \) be the image of \( \beta \) in \( \text{gr}_0 D_{\text{rig}}^h(V)^{\psi=1} \) and let \( h^1_0 : \text{gr}_0 D_{\text{rig}}^h(V)^{\psi=1} \to H^1(\text{gr}_0 D_{\text{rig}}^h(V)) \) be the canonical map furnished by Proposition 1.1.7. Recall that \( h^1_0(\tilde{\beta}) = \text{cl}(c, \tilde{\beta}) \) where \( (1 - \gamma)c = (1 - \varphi)\tilde{\beta} \). Then \( \tilde{\beta} = (-1)^{h-1}t^h m_i \partial^h \log(\pi) \). By Lemma 1.5.1 of [CC1] there exists a unique \( b_0 \in B_{Q_p}^{\psi=0} \) such that \( (\gamma - 1)b_0 = \ell(X) \). This implies that

\[
(1 - \gamma)(t^h m_i \partial^h b_0 e_m) = (1 - \varphi)(t^h m_i \partial^h \log(\pi)e_m) = (-1)^{h-1}(1 - \varphi)\tilde{\beta}.
\]

Thus \( c = (-1)^{h-1}t^h m_i \partial^h b_0 e_m \) and \( \text{res}(ct^m - dt) = (-1)^{h-1}\text{res}(t^h m_i \partial^h b_0 dt)e_m = 0 \). Next from the congruence \( \tilde{\beta} \equiv (h - 1)!e_m \pmod{Q_p[[\pi]]e_m} \), it follows that \( \text{res}(\tilde{\beta}t^m - dt) = (h - 1)!e_m \). Therefore by [Ben2], Corollary 1.5.6 we have

\[
\text{cl}(c, \tilde{\beta}) = (h - 1)!\text{cl}(\beta m) = (h - 1)! \frac{p}{\log \chi(\gamma)} \text{cl}(\text{gr}_0 D_{\text{rig}}^h(V), c)(a).
\]

On the other hand

\[
\alpha(0) = a \otimes \ell \left( \frac{(1 + X)^{\chi(\gamma)} - 1}{X} \right) \bigg|_{X=0} = a \left( 1 - \frac{1}{p} \right) \log(\chi(\gamma)).
\]
Theses formulas imply that
\[ \delta_{D,h}(\alpha) = (h - 1)! \left(1 - \frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} \text{gr}_0(D^\dagger_{\text{rig}}(V)). \]
and the proposition is proved.

3.2.3. Define
\[ H^1_{f,(p)}(V) = \ker \left( H^1_S(V) \to \bigoplus_{v \in \mathcal{S} - \{\infty\}} \frac{H^1(Q_p, V)}{H^1_f(Q_p, V)} \right). \]

From the definition of \( H^1_S(D, V) \) we immediately obtain isomorphisms
\[ \frac{H^1(Q_p, V)}{H^1_{f,(p)}(V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))} \cong \frac{H^1(F_0D^\dagger_{\text{rig}}(V))}{H^1(S(D, V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))}. \]

Thus, the map \( \delta_{D,h} \) constructed in Proposition 3.3.2 induces a map
\[ D^\varphi = p^{-1} \to \frac{H^1(Q_p, V)}{H^1_{f,(p)}(V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))} \]
which we will denote again by \( \delta_{D,h} \). On the other hand, we have isomorphisms
\[ D^\varphi = p^{-1} \xrightarrow{\exp_V} \frac{H^1(Q_p, V)}{H^1_f(V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))} \cong \frac{H^1(Q_p, V)}{H^1_{f,(p)}(V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))}. \]

**Proposition 3.2.4.** Let \( \lambda_D : D^\varphi = p^{-1} \to D^\varepsilon = p^{-1} \) denote the homomorphism making the diagram
\[ \begin{array}{ccc}
D^\varphi = p^{-1} & \xrightarrow{\delta_{D,h}} & D^\varepsilon = p^{-1} \\
\downarrow \quad \lambda_D & & \quad \downarrow \quad \exp_V \\
H^1(Q_p, V) & \cong & H^1_{f,(p)}(V) + H^1(F_{-1}D^\dagger_{\text{rig}}(V))
\end{array} \]
commute. Then
\[ \det \left( \lambda_D \mid D^\varphi = p^{-1} \right) = (\log \chi(\gamma))^{-e} \left(1 - \frac{1}{p}\right)^{-e} \mathcal{L}(D, V). \]

**Proof.** The proposition follows from Proposition 2.1.6, Proposition 3.2.2 and the following elementary fact. Let \( U = U_1 \oplus U_2 \) be the decomposition of a vector space \( U \) of dimension \( 2e \) into the direct sum of two subspaces of dimension \( e \). Let \( W \subset U \) be a subspace of dimension \( e \) such that \( W \cap U_1 = \{0\} \). Consider the diagrams
\[
\begin{array}{ccc}
W & \xrightarrow{p_1} & U_1 \\
p_2 & \xrightarrow{f} & U_2
\end{array} \quad \quad \begin{array}{ccc}
U/W & \xrightarrow{i_1} & U_1 \\
i_2 & \xrightarrow{g} & U_2
\end{array}
\]
where \( p_k \) and \( i_k \) are induced by natural projections and inclusions. Then \( f = -g \). Applying this remark to \( U = H^1(\text{gr}_0D^\dagger_{\text{rig}}(V)), W = H^1_S(D, V), U_1 = H^1(\text{gr}_0D^\dagger_{\text{rig}}(V)), U_2 = H^1_c(\text{gr}_0D^\dagger_{\text{rig}}(V)) \) and taking determinants we obtain the proposition.
§4. Special values of $p$-adic $L$-functions

4.1. The Bloch-Kato conjecture (see [F3], [FP],[BF]).

4.1.1. Let $V$ be a $p$-adic pseudo-geometric representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus $V$ is a finite-dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of the Galois group $G_S$ for a suitable finite set of places $S$ containing $p$. Write $\mathbf{R} \Gamma_S(V) = C^*_c(G_S, V)$ and define

$$\mathbf{R} \Gamma_{S,c}(V) = \text{cone} \left( \mathbf{R} \Gamma_S(V) \rightarrow \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma(Q_v, V) \right) [-1].$$

Fix a $\mathbb{Z}_p$-lattice $T$ of $V$ stable under the action of $G_S$ and set $\Delta_S(V) = \det_{\mathbb{Q}_p}^{-1} \mathbf{R} \Gamma_{S,c}(V)$ and $\Delta_S(T) = \det_{\mathbb{Z}_p}^{-1} \mathbf{R} \Gamma_{S,c}(T)$. Then $\Delta_S(T)$ is a $\mathbb{Z}_p$-lattice of the one-dimensional $\mathbb{Q}_p$-vector space $\Delta_S(V)$ which does not depend on the choice of $T$. Therefore it defines a $p$-adic norm on $\Delta_S(V)$ which we denote by $\| \cdot \|_S$. Moreover, $(\Delta_S(V), \| \cdot \|_S)$ does not depend on the choice of $S$. More precisely, if $\Sigma$ is a finite set of places which contains $S$, then there exists a natural isomorphism $\Delta_S(V) \rightarrow \Delta_{\Sigma}(V)$ such that $\| \cdot \|_{\Sigma} = \| \cdot \|_S$. It allows to define the Euler-Poincaré line $\Delta_{EP}(V)$ as $(\Delta_S(V), \| \cdot \|_S)$ where $S$ is sufficiently large. Recall that for any finite place $v \in S$ we defined

$$\mathbf{R} \Gamma_f(Q_v, V) = \begin{cases} [V^I, \frac{1-f_v}{V^I}] & \text{if } v \neq p \\ D_{\text{cris}}(V) \left( \frac{p}{1-\nu} \right) t_V(Q_p) \oplus D_{\text{cris}}(V) & \text{if } v = p. \end{cases}$$

At $v = \infty$ we set $\mathbf{R} \Gamma_f(\mathbb{R}, V) = [V^+ \rightarrow 0]$, where the first term is placed in degree 0. Thus $\mathbf{R} \Gamma_f(\mathbb{R}, V) \simeq \mathbf{R} \Gamma(\mathbb{R}, V)$. For any $v$ we have a canonical morphism $\text{loc}_v : \mathbf{R} \Gamma_f(Q_v, V) \rightarrow \mathbf{R} \Gamma(Q_v, V)$ which can be viewed as a local condition in the sense of [N2]. Consider the diagram

$$\begin{array}{ccc} \mathbf{R} \Gamma_S(V) & \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma(Q_v, V) & \mathbf{R} \Gamma_f(Q_v, V) \\ \downarrow & \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma_f(Q_v, V) & \\ \mathbf{R} \Gamma_f(V) \end{array}$$

and define

$$\mathbf{R} \Gamma_f(V) = \text{cone} \left( \mathbf{R} \Gamma_S(V) \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma_f(Q_v, V) \right) \rightarrow \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma(Q_v, V) [-1].$$

Thus, we have a distinguished triangle

$$\mathbf{R} \Gamma_f(V) \rightarrow \mathbf{R} \Gamma_S(V) \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma_f(Q_v, V) \rightarrow \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma(Q_v, V). \quad (4.1)$$

Set

$$\Delta_f(V) = \det_{\mathbb{Q}_p}^{-1} \mathbf{R} \Gamma_f(V) \otimes \det_{\mathbb{Q}_p}^{-1} t_V(Q_p) \otimes \det_{\mathbb{Q}_p} V^+. $$

It is easy to see that $\mathbf{R} \Gamma_f(V)$ and $\Delta_f(V)$ do not depend on the choice of $S$. Consider the distinguished triangle

$$\mathbf{R} \Gamma_{S,c}(V) \rightarrow \mathbf{R} \Gamma_f(V) \rightarrow \bigoplus_{v \in S \cup \{ \infty \}} \mathbf{R} \Gamma_f(Q_v, V).$$
Since $\det_{\mathbb{Q}_p} R \Gamma_f(\mathbb{Q}_p, V) \simeq \det_{\mathbb{Q}_p}^{-1} t_V(\mathbb{Q}_p)$ and $\det_{\mathbb{Q}_p} R \Gamma_f(\mathbb{R}, V) = \det_{\mathbb{Q}_p} V^+$ tautologically, we obtain canonical isomorphisms

$$\Delta_f(V) \simeq \det_{\mathbb{Q}_p}^{-1} R \Gamma_{S,c}(V) \simeq \Delta_{EP}(V).$$

The cohomology of $R \Gamma_f(V)$ is as follow:

$$\begin{align*}
R^0 \Gamma_f(V) &= H^0_S(V), & R^1 \Gamma_f(V) &= H^1_f(V), & R^2 \Gamma_f(V) &\simeq H^1_f(\mathbb{Q}^*(1))^*, \\
R^3 \Gamma_f(V) &= \ker \left( H^2_S(V) \to \bigoplus_{v \in S} H^2(\mathbb{Q}_v, V) \right) \simeq H^0_S(V^*(1))^*. \\
\end{align*}$$

These groups seat in the following exact sequence:

$$0 \to R^1 \Gamma(V) \to H^2_S(V) \to \bigoplus_{v \in S} H^1(\mathbb{Q}_v, V) \to R^2 \Gamma_f(V) \to H^2_S(V) \to \bigoplus_{v \in S} H^2(\mathbb{Q}_v, V) \to R^3 \Gamma_f(V) \to 0.$$ 

The $L$-function of $V$ is defined as the Euler product

$$L(V, s) = \prod_v E_v(V, (Nv)^{-s})^{-1}$$

where

$$E_v(V, t) = \begin{cases}
\det (1 - f_v t | V^{t, v}), & \text{if } v \neq p \\
\det (1 - \varphi t | D_{\text{cris}}(V)), & \text{if } v = p.
\end{cases}$$

4.1.2. In this paper we treat motives in the formal sense and assume all conjectures about the category of mixed motives $\mathcal{MM}$ over $\mathbb{Q}$ which are necessary to state the Bloch-Kato conjecture (see [F3], [FP]). If $M$ is a pure motive over $\mathbb{Q}$ we denote by $M_v$ its $v$-adic realizations. Assume that the groups $H^i(M) = \text{Ext}^i_{\mathcal{MM}}(\mathbb{Q}(0), M)$ are well defined and vanish for $i \neq 0, 1$. It should be possible to define a $\mathbb{Q}$-subspace $H^i_f(M)$ of $H^i(M)$ consisting of "integral" classes of extensions which is expected to be finite dimensional. It is convenient to set $H^0_f(M) = H^0(M)$. Then we assume that for any finite place $v$ the regulator map induces isomorphisms

$$H^i_f(M) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq H^i_f(M_v), \quad i = 0, 1.$$ 

Let $M$ be a motive satisfying the following condition

**M)** $H^i_f(M) = H^i_f(M^*(1))$ for $i = 0, 1$.

Let $M_{\text{dR}}$ and $M_{\text{B}}$ denote the de Rham and the Betti realizations of $M$ respectively and let $t_M(\mathbb{Q}) = M_{\text{dR}}/\text{Fil}^0 M_{\text{dR}}$ be the tangent space of $M$. The complex conjugation $c$ acts on $M_{\text{B}}$ and $M_{\text{B}} = M_{\text{B}}^+ \oplus M_{\text{B}}^-$. The comparision isomorphism $M_{\text{B}} \otimes_{\mathbb{Q}_p} \mathbb{R} \simeq M_{\text{dR}} \otimes_{\mathbb{Q}_p} \mathbb{R}$ induces a map

$$M_{\text{B}}^+ \otimes_{\mathbb{Q}} \mathbb{R} \to t_M(\mathbb{R})$$

which is expected to be an isomorphism. Assuming this, we can define a natural injective map

$$\Omega_{\infty} : \det_{\mathbb{Q}_p}^{-1} t_M(\mathbb{Q}) \otimes \det_{\mathbb{Q}_p} M_{\text{B}}^+ \to \mathbb{R}.$$ 

Fix $\omega_\ell \in \det_{\mathbb{Q}} t_M(\mathbb{Q})$ and $\omega_B \in \det_{\mathbb{Q}} M_{\text{B}}^+$ and set $\Omega_{\infty}(\omega_\ell, \omega_B) = \Omega_{\infty}(\omega_\ell^{-1} \otimes \omega_B)$. It is conjectured that the $L$-function $L(M_v, s)$ does not depend of $v$. It will be denoted by $L(M, s)$. 


**Conjecture (Deligne).** Let $M$ be a motive satisfying $\mathbf{M}$. Then

$$
\frac{L(M,0)}{\Omega_\infty(\omega_t,\omega_B)} \in \mathbb{Q}^*.
$$

4.1.3. Let $p$ be a prime number and let $M_p$ denote the $p$-adic realization of $M$. From $\mathbf{M}$ and (4.3) it follows that $H^0(M_p) = H^0(M_p^e(1)) = 0$ and $H^1_f(M_p) = H^1_f(M_p(1)) = 0$. Hence $\mathbf{R}\Gamma_f(M_p)$ is acyclic. Fix $\omega_t$ and $\omega_B$ and define a map

$$
i_{\omega_t,\omega_B,p} : \Delta_{EP}(M_p) \xrightarrow{\sim} \det_{\mathbb{Q}_p}^{-1} \iota_M(\mathbb{Q}_p) \otimes \mathbb{Q}_B^+ \to \mathbb{Q}_p
$$

by $x = i_{\omega_t,\omega_B,p}(x)(\omega_t^{-1} \otimes \omega_B)$. The Bloch-Kato conjecture states as follows:

**Conjecture (Bloch-Kato).** Let $T_p$ be a $\mathbb{Z}_p$-lattice of $M_p$ stable under the action of $G_S$. Then

$$
i_{\omega_t,\omega_B,p}(\Delta_{EP}(T_p)) = \frac{L(M,0)}{\Omega_\infty(\omega_t,\omega_B)} \mathbb{Z}_p.
$$

4.2. The complex $\mathbf{R}\Gamma_{1w,h}(D, V)$.

4.2.1. Let $\Gamma$ denote the Galois group of $\mathbb{Q}(\zeta_p^\infty)/\mathbb{Q}$ and $\Gamma_n = \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_{p^n}))$. Set $\Lambda = \mathbb{Z}_p[\Gamma_1]$ and $\Lambda(\Delta) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda$. For any character $\eta \in X(\Delta)$ put

$$
e_\eta = \frac{1}{|\Delta|} \sum_{g \in \Delta} \eta^{-1}(g)g.
$$

Then $\Lambda(\Delta) = \bigoplus_{\eta \in X(\Delta)} \Lambda(\Delta)^{(\eta)}$ where $\Lambda(\Delta)^{(\eta)} = \Lambda e_\eta$ and for any $\Lambda(\Delta)$-module $M$ one has a canonical decomposition

$$M \cong \bigoplus_{\eta \in X(\Delta)} M^{(\eta)}, \quad M^{(\eta)} = e_\eta(M).
$$

We write $\eta_0$ for the trivial character of $\Delta$ and identify $\Lambda$ with $\Lambda(\Delta)e_{\eta_0}$.

Let $V$ be a $p$-adic pseudo-geometric representation unramified outside $S$. Set $d(V) = \dim(V)$ and $d'_\pm(V) = \dim(V_{c=\pm1})$. Fix a $\mathbb{Z}_p$-lattice $T$ of $V$ stable under the action of $G_S$. Let $\iota : \Lambda(\Delta) \to \Lambda(\Delta)$ denote the canonical involution $g \mapsto g^{-1}$. Recall that the induced module $\text{Ind}_{\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}}(T)$ is isomorphic to $(\Lambda(\Delta) \otimes_{\mathbb{Z}_p} T)^\iota$ ([N2], section 8.1). Define

$$
H^1_{1w,S}(T) = H^1_S((\Lambda(\Delta) \otimes_{\mathbb{Z}_p} T)^\iota),
$$

$$
H^1_{1w}(\mathbb{Q}_v,T) = H^1(\mathbb{Q}_v, (\Lambda(\Delta) \otimes_{\mathbb{Z}_p} T)^\iota) \quad \text{for any finite place } v.
$$

From Shapiro’s lemma it follows immediately that

$$
H^i_{1w,S}(T) = \lim_{\text{cores}} H^i_S(\mathbb{Q}(\zeta_{p^n}), T), \quad H^i_{1w}(\mathbb{Q}_p,T) = \lim_{\text{cores}} H^i(\mathbb{Q}_p(\zeta_{p^n}), T).
$$

Set $H^i_{1w,S}(V) = H^i_{1w,S}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $H^i_{1w}(\mathbb{Q}_v,V) = H^i_{1w}(\mathbb{Q}_v,T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In [PR2] Perrin-Riou proved the following results about the structure of these modules.
i) \( H^1_{Iw,S}(V) = 0 \) and \( H^1_{Iw}(\mathbb{Q}_v, T) = 0 \) if \( i \neq 1, 2 \);

ii) If \( v \neq p \), then for each \( \eta \in X(\Delta) \) the \( \eta \)-component \( H^1_{Iw}(\mathbb{Q}_v, T)^{(\eta)} \) is a finitely generated torsion \( \Lambda \)-module. In particular, \( H^1_{Iw}(\mathbb{Q}_v, T) \simeq H^1(\mathbb{Q}_v^\text{ur}/\mathbb{Q}_v, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \text{Iw}^\text{Iw})^\vee) \).

iii) If \( v = p \) then \( H^2_{Iw}(\mathbb{Q}_p, T)^{(\eta)} \) are finitely generated torsion \( \Lambda \)-modules. Moreover, for each \( \eta \in X(\Delta) \)

\[ \text{rg}_\Lambda \left( H^1_{Iw}(\mathbb{Q}_p, T)^{(\eta)} \right) = d, \quad H^1_{Iw}(\mathbb{Q}_p, T)^{(\eta)} \simeq H^0(\mathbb{Q}_p(\zeta_{p^\infty}), T)^{(\eta)}. \]

Remark that by local duality \( H^2_{Iw}(\mathbb{Q}_p, T) \simeq H^0(\mathbb{Q}_p(\zeta_{p^\infty}), V^*(1)/T^*(1)) \).

iv) If the weak Leopoldt conjecture holds for the pair \( (V, \eta) \) i.e. if \( H^2_S(\mathbb{Q}(\zeta_{p^\infty}), V/T)^{(\eta)} = 0 \) then \( H^2_{Iw,S}(T)^{(\eta)} \) is \( \Lambda \)-torsion and

\[ \text{rank}_\Lambda \left( H^1_{Iw,S}(T)^{(\eta)} \right) = \begin{cases} \text{d}_-(V), & \text{if } \eta(c) = 1 \\ \text{d}_+(V), & \text{if } \eta(c) = -1. \end{cases} \]

Passing to the projective limit in the Poitou-Tate exact sequence one obtains an exact sequence

\[ 0 \rightarrow H^2_S(\mathbb{Q}(\zeta_{p^\infty}), V^*(1)/T^*(1))^\wedge \rightarrow H^1_{Iw,S}(T) \rightarrow \bigoplus_{v \in S} H^1_{Iw}(\mathbb{Q}_v, T) \rightarrow H^1_S(\mathbb{Q}(\zeta_{p^\infty}), V^*(1)/T^*(1))^\wedge \]

\[ \rightarrow H^2_{Iw,S}(T) \rightarrow \bigoplus_{v \in S} H^2_{Iw}(\mathbb{Q}_v, T) \rightarrow H^0_S(\mathbb{Q}(\zeta_{p^\infty}), V^*(1)/T^*(1))^\wedge \rightarrow 0. \quad (4.4) \]

Define

\[ \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_p, T) = C^\bullet_c(\mathbb{G}_S, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \text{Iw}^\text{Iw})^\vee), \]

\[ \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_v, T) = C^\bullet_c(\mathbb{G}_v, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \text{Iw}^\text{Iw})^\vee), \]

\[ \text{RG}_S(\mathbb{Q}(\zeta_{p^\infty}), V^*(1)/T^*(1)) = C^\bullet_c(\mathbb{G}_S, \text{Hom}_{\mathbb{Z}_p}(\Lambda(\Gamma), V^*(1)/T^*(1))). \]

Then the sequence (4.3) is induced by the distinguished triangle

\[ \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_p, T) \rightarrow \bigoplus_{v \in S} \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_v, T) \rightarrow (\text{RG}_S(\mathbb{Q}(\zeta_{p^\infty}), V^*(1)/T^*(1))^\vee)^\wedge \left[-2\right] \]

([N2], Theorem 8.5.6). Finally, we have usual descent formulas

\[ \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_p, T) \otimes_{\Lambda} \mathbb{Z}_p \simeq \text{RG}_S(T), \quad \text{RG}_1 \subseteq \mathbb{E} \mathcal{S}(\mathbb{Q}_v, T) \otimes_{\Lambda} \mathbb{Z}_p \simeq \text{RG}(\mathbb{Q}_v, T) \]

([N2], Proposition 8.4.21).

4.2.2. For the remainder of this chapter we assume that \( V \) satisfies the conditions \( \text{C1-5) of section 3.1.2 where C2) is replaced by the following stronger condition} \)

\[ \text{C2*) } H^0(\mathbb{Q}_p, V) = H^0(\mathbb{Q}_p, V^*(1)) = 0. \]

Remark that \( \text{C1) and C2*) guarantee that the weak Leopoldt conjecture holds for } (V, \eta_0) \text{ and } (V^*(1), \eta_0) \) (Proposition B.5 of [PR2]). To simplify notations we write \( \mathcal{H} \) for \( \mathcal{H}(\Gamma_1) \). In this subsection we interpret Perrin-Riou’s construction of the module of \( p \)-adic \( L \)-functions in terms
Theorem 4.2.3. is the following theorem. It is easy to see that it does not depend on the choice of $S$. Consider the associated Selmer complex $D$ of $[N2]$. Fix an admissible subspace $D$ of $\mathbb{D}_{\text{cris}}(V)$ and a $\mathbb{Z}_p$-lattice $N$ of $D$. Set $\mathcal{D}_p(N,T)^{\eta_0} = N \otimes \mathbb{Z}_p \Lambda, \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_p, N, T) = D_p(N, T)^{\eta_0}[-1]$ and $\mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_p, D, V) = \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_p, N, T) \otimes \mathbb{Z}_p \mathbb{Q}_p$. Consider the map

$\text{Exp}^\text{\textit{f}}_{v, \eta} : \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_p, T) \otimes \Lambda \mathcal{H} \to \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_p, T) \otimes \Lambda \mathcal{H}$

which will be viewed as a local condition at $p$. If $v \neq p$ the inertia group $I_v$ acts trivially on $\Lambda$

$\mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_v, N, T) = \left[ T_{I_v} \otimes \Lambda \overset{1-f_v}{\longrightarrow} T_{I_v} \otimes \Lambda \right]$

where the first term is placed in degree 0. We have a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}^1 \Gamma_{1w,S}(T) \otimes \Lambda \mathcal{H} & \longrightarrow & \bigoplus_{v \in S} \mathbb{R}^1 \Gamma_{1w}(\mathbb{Q}_v, T) \otimes \Lambda \mathcal{H} \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S} \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_v, N, T) \otimes \Lambda \mathcal{H}
\end{array}$$ (4.5)

Consider the associated Selmer complex

$$\mathbb{R}^1 \Gamma_{1w,h}(D, V) = \text{cone} \left[ \left( \mathbb{R}^1 \Gamma_{1w,S}(T) \oplus \left( \bigoplus_{v \in S} \mathbb{R}^1 \Gamma_{1w,f}(\mathbb{Q}_v, N, T) \right) \right) \otimes \Lambda \mathcal{H} \to \bigoplus_{v \in S} \mathbb{R}^1 \Gamma_{1w}(\mathbb{Q}_v, T) \otimes \Lambda \mathcal{H} \right] [-1]$$

It is easy to see that it does not depend on the choice of $S$. Our main result about this complex is the following theorem.

**Theorem 4.2.3.** Assume that $V$ satisfies the conditions C1-5). Let $D$ be an admissible subspace of $\mathbb{D}_{\text{cris}}(V)$. Assume that $\mathcal{L}(V, D) \neq 0$. Then

i) $\mathbb{R}^1 \Gamma_{1w,h}(D, V)$ are $\mathcal{H}$-torsion modules for all $i$.

ii) $\mathbb{R}^1 \Gamma_{1w,h}(D, V) = 0$ for $i \neq 2, 3$ and

$$\mathbb{R}^3 \Gamma_{1w,h}(D, V) \simeq \left( H^0(\mathbb{Q}(\zeta_{p^\infty}), V^*(1))^{\eta_0} \right) \otimes \Lambda \mathcal{H}.$$

iii) The complex $\mathbb{R}^1 \Gamma_{1w,h}(D, V)$ is semisimple i.e. for each $i$ the natural map

$$\mathbb{R}^i \Gamma_{1w,h}(D, V) \Gamma \to \mathbb{R}^i \Gamma_{1w,h}(D, V)\Gamma$$

is an isomorphism.

**4.2.4. Proof of Proposition 4.2.3.** We leave the proof of the following lemma as an easy exercise.

**Lemma 4.2.4.1.** Let $A$ and $B$ be two submodules of a finitely generated free $\mathcal{H}$-module $M$. Assume that the natural maps $A_{\Gamma_1} \to M_{\Gamma_1}$ and $B_{\Gamma_1} \to M_{\Gamma_1}$ are both injective. Then $A_{\Gamma_1} \cap B_{\Gamma_1} = \{0\}$ implies that $A \cap B = \{0\}$. 

4.2.4.2. Since $H^{0}_{iw,S}(V)$ and $H^{0}_{iw}(Q_{v},V)$ are zero, we have $R^{1}\Gamma_{iw,h}^{(\nu_{0})}(D,V) = 0$. Next, by definition $R^{1}\Gamma_{iw,h}(D,V)^{(\nu_{0})} = \ker(f)$ where

$$f : \left( H^{1}_{iw,S}(T)^{(\nu_{0})} \oplus D_{p}(N,T)^{(\nu_{0})} \oplus \bigoplus_{v \in S - \{p\}} H^{1}_{iw,f}(Q_{v},T)^{(\nu_{0})} \right) \otimes H \to \bigoplus_{v \in S} H^{1}_{iw}(Q_{v},T)^{(\nu_{0})} \otimes H$$

is the map induced by (4.5). If $v \in S - \{p\}$ one has

$$H^{1}_{iw,f}(Q_{v},T)^{(\nu_{0})} = H^{1}_{iw}(Q_{v},T)^{(\nu_{0})} = H^{1}(Q_{v} / Q_{v}, (\Lambda \otimes T^{f})^{c}).$$

Thus

$$R^{1}\Gamma_{iw,h}(D,V) = \left( H^{1}_{iw,S}(T)^{(\nu_{0})} \otimes \Lambda \ H \right) \cap \left( \text{Exp}_{V,h}^{\varepsilon} \left( D_{p}(D,T)^{(\nu_{0})} \right) \otimes \Lambda \ H \right)$$

in $H^{1}_{iw}(Q_{p},T)^{(\nu_{0})} \otimes \Lambda \ H$. Put

$$A = \text{Exp}_{V,h}^{\varepsilon}(D_{-1} \otimes \Lambda \ H) \oplus X^{-1}\text{Exp}_{V,h}^{\varepsilon}(D^{\varphi = p^{-1}} \otimes \Lambda \ H) \subset H^{1}_{iw}(Q_{p},T)^{(\nu_{0})} \otimes \Lambda \ H.$$ 

By Theorem 2.2.4 and Proposition 3.2.2 $A_{\Gamma_{1}}$ injects into $H^{1}(Q_{p},V)$. The $\Lambda$-module $M = \left( \frac{H^{1}_{iw}(Q_{p},T)^{(\nu_{0})}}{T^{H_{Q_{p}}}} \right) \otimes \Lambda \ H$ is free and $A \hookrightarrow M$. Since $T^{G_{Q_{p}}} = 0$ by $C2^{*}$, one has $M_{\Gamma_{1}} = H^{1}_{iw}(Q_{p},V)_{\Gamma} \subset H^{1}(Q_{p},V)$ and we obtain that $A_{\Gamma_{1}}$ injects into $M_{\Gamma_{1}}$.

Set $B = \left( \frac{H^{1}_{iw,S}(T)^{(\nu_{0})}}{T^{H_{S}}} \right) \otimes \Lambda \ H$. The weak Leopoldt conjecture for $(V^{*}(1), \nu_{0})$ together with the fact that $H^{1}_{iw}(Q_{v},T)$ are $\Lambda$-torsion for $v \in S - \{p\}$ imply that $B \hookrightarrow M$. Since the image of $H^{1}_{iw}(Q_{v},V)_{\Gamma} \subset H^{1}(Q_{v},V)$ is contained in $H^{1}_{f}(Q_{v},V)$, the image of $H^{1}_{iw,S}(V)_{\Gamma}$ in $H^{1}_{S}(V)$ is in fact contained in

$$H_{\Gamma_{1}}^{1}(V) = \ker \left( H^{1}_{S}(V) \to \bigoplus_{v \in S - \{p\}} H^{1}(Q_{v},V) \right).$$

Because $H^{1}_{f}(V) = 0$, the group $H_{\Gamma_{1}}^{1}(V)$ injects into $H^{1}(Q_{p},V)$ and we have

$$H^{1}_{iw,S}(V)_{\Gamma_{1}} = H^{1}_{iw,S}(V)_{\Gamma} \hookrightarrow H_{\Gamma_{1}}^{1}(V) \hookrightarrow H^{1}(Q_{p},V).$$

Thus $B_{\Gamma_{1}} \subset M_{\Gamma_{1}}$. We shall prove that $R^{1}\Gamma_{iw,h}^{(\nu_{0})}(D,V) = 0$. By Lemma 4.2.4.1 it suffices to show that $A_{\Gamma_{1}} \cap B_{\Gamma_{1}} = \{0\}$. Now we claim that $A_{\Gamma_{1}} \cap H^{1}_{\Gamma_{1}}(V) = \{0\}$. First remark that

$$H^{1}_{\Gamma_{1}}(V) \hookrightarrow \frac{H^{1}(Q_{p},V)}{H^{1}(F_{-1}D_{\text{rig}}^{1}(V))}.$$ 

On the other hand, from Theorem 2.2.4 it follows that

$$\text{Exp}_{V,h}^{\varepsilon}(D_{-1} \otimes \Lambda \ H)_{\Gamma_{1}} = \text{exp}_{V,Q_{p}}(D_{-1}) \subset H^{1}(F_{-1}D_{\text{rig}}^{1}(V)).$$

Therefore, Proposition 3.2.2 implies that the image of $A_{\Gamma_{1}}$ in $\frac{H^{1}(Q_{p},V)}{H^{1}(F_{-1}D_{\text{rig}}^{1}(V))}$ coincides with $H_{c}^{1}((\gr_{0}D^{1}_{\text{rig}}(V)))$. But $L(D,V) \not= 0$ if and only if $H^{1}_{S}(D,V) \cap H^{1}_{c}((\gr_{0}D^{1}_{\text{rig}}(V))) = 0$ where $H^{1}_{S}(D,V)$
denotes the inverse image of $H^1(\text{gr}_0 D_{\text{rig}}^+(V))$ in $H^1_{f,\{p\}}(V)$. This proves the claim and implies that $R^1\Gamma_{1w,h}(D, V) = 0$.

4.2.4.3. We shall show that $R^2\Gamma_{1w,h}(D, V)$ is $\mathcal{H}$-torsion. By definition, we have an exact sequence

$$0 \to \text{coker}(f) \to R^2\Gamma_{1w,h}(D, V) \to \mathcal{W}^2_{1w,h}(V)^{(\eta_0)} \otimes \Lambda_{\mathbb{Q}_p} \mathcal{H} \to 0,$$

where

$$\mathcal{W}^2_{1w,h}(V) = \ker \left( H^2_{1w,h}(V) \to \bigoplus_{v \in S} H^2_{1w}(\mathbb{Q}_v, V) \right).$$

It follows from the weak Leopoldt conjecture that $\mathcal{W}^2_{1w,h}(V)$ is $\Lambda_{\mathbb{Q}_p}$-torsion. On the other hand, as $\mathcal{H}$ is a Beilinson ring $[\mathcal{L}_a]$, the formulas

$$\text{rank}_\Lambda H^1_{1w,h}(T)^{(\eta_0)} = d_-(V), \quad \text{rank}_\Lambda H^1_{1w}(\mathbb{Q}_p, T)^{(\eta_0)} = d(V), \quad \text{rank}_\Lambda \mathcal{D}_p(N, T) = d_+(V)$$

together with the fact that $R^1\Gamma_{1w,h}(D, V) = 0$ imply that $\text{coker}(f)$ is $\mathcal{H}$-torsion. We have therefore proved that $R^2\Gamma_{1w,h}(D, V)$ is $\mathcal{H}$-torsion. Finally, the Poitou-Tate exact sequence gives that

$$R^3\Gamma_{1w,h}(D, V) = (H^0(\mathbb{Q}(\zeta_{p^\infty}), V^*(1))^*)^{(\eta_0)} \otimes \Lambda_{\mathbb{Q}_p} \mathcal{H}$$

is also $\mathcal{H}$-torsion. The proposition is proved.

4.2.4.4. Now we prove the semisimplicity of $R^1\Gamma_{1w,h}(D, V)$. First, remark that $C2^*$ implies that $H^1_{1w}(\mathbb{Q}_p, V)^\Gamma = 0$ and $H^1_{1w}(\mathbb{Q}_p, V) = H^1(\mathbb{Q}_p, V)$. Next, $H^1_{1w,h}(V)^{(\eta_0)} \simeq \Lambda_{\mathbb{Q}_p} \otimes H^1_{1w,h}(V)$, since $H^1_{1w,h}(V)^{(\eta_0)} \subset V^{\mathcal{H}_{\mathbb{Q}_p}}$, we have $(H^1_{1w,h}(V)^{(\eta_0)})^\Gamma = 0$ by the snake lemma. Thus $\dim_{\mathbb{Q}_p} H^1_{1w,h}(V)^{(\eta_0)} = d_-(V)$. On the other hand $\dim_{\mathbb{Q}_p} H^1_{f,\{p\}}(V) = \dim_{\mathbb{Q}_p} H^1(\mathbb{Q}_p, V) - \dim_{\mathbb{Q}_p} t_V = d_-(V)$. Since $H^1_{1w,h}(V)^{(\eta_0)}$ injects into $H^1_{f,\{p\}}(V)$ this proves that $H^1_{1w,h}(V)^{(\eta_0)} = H^1_{f,\{p\}}(V)$. Consider the exact sequence

$$0 \to \left( H^1_{1w,h}(T)^{(\eta_0)} \oplus \mathcal{D}_p(N, T)^{(\eta_0)} \right) \otimes \mathcal{H} \to H^1_{1w}(\mathbb{Q}_p, T)^{(\eta_0)} \otimes \mathcal{H} \to \text{coker}(f) \to 0.$$

Recall that $\text{Exp}_{t, h, 0} : D \to H^1_{1w}(\mathbb{Q}_p, V)^\Gamma$ denotes the homomorphism induced by the large exponential map. Applying the snake lemma, and taking into account that $\text{Im}(\text{Exp}_{t, h, 0}) = \text{Exp}_{t, h, 0}(D_{-1}) = H^1(F_{\epsilon} D_{\text{rig}}^+(V))$ and $\ker(\text{Exp}_{t, h, 0}) = D^{\varphi = p^{-1}}$ (see for example [BB], Propositions 4.17 and 4.18 or the proof of Proposition 3.3.2) we obtain

$$\text{coker}(f)^\Gamma = \ker \left( H^1_{f,\{p\}}(V) \oplus D \xrightarrow{\text{Exp}_{t, h, 0}} H^1(\mathbb{Q}_p, V) \right) = D^{\varphi = p^{-1}},$$

$$\text{coker}(f)^\Gamma = \frac{H^1(\mathbb{Q}_p, V)}{H^1_{f,\{p\}}(V) + H^1(F_{\epsilon} D_{\text{rig}}^+(V))}.$$
Corollary 4.2.5. The exponential map induces an isomorphism of $D^{\varphi=p^{-1}}$ onto $\text{coker}(f)_{\Gamma} \simeq \mathbf{R}^{2}\Gamma^{(q_0)}_{1w,h}(D, V)_{\Gamma}$ and the diagram

$$
\begin{array}{ccc}
D^{\varphi=p^{-1}} & \xrightarrow{\sim} & \mathbf{R}^{2}\Gamma^{(q_0)}_{1w,h}(D, V)_{\Gamma} \\
\downarrow \lambda_D & & \downarrow (h-1)! \exp_V \\
D^{\varphi=p^{-1}} & \xrightarrow{(h-1)! \exp_V} & \mathbf{R}^{2}\Gamma^{(q_0)}_{1w,h}(D, V)_{\Gamma}
\end{array}
$$
in which the map $\lambda_D$ is defined in Proposition 3.2.4, commutes.

4.3. The module of $p$-adic $L$-functions.

4.3.1. We conserve the notation and conventions of section 4.2. Let $D$ be an admissible subspace of $D_{\text{cris}}(V)$ and assume that $\mathcal{L}(V,D) \neq 0$. We review the definition of the module of $p$-adic $L$-functions using the formalism of Selmer complexes. Set

$$
\Delta_{1w,h}(D,V) = \det_{\Lambda_{\mathbb{Q}_p}}^{-1} \left( R\Gamma_{1w,S}(D,V) \oplus \left( \bigoplus_{v \in S} R\Gamma_{1w,f}(\mathbb{Q}_v,D,V) \right) \right) \otimes \det_{\Lambda_{\mathbb{Q}_p}} \left( \bigoplus_{v \in S} R\Gamma_{1w}(\mathbb{Q}_v,V) \right).
$$

The exact triangle

$$
R\Gamma_{1w,S}(D,V) \rightarrow \left( R\Gamma_{1w,S}(V) \oplus \left( \bigoplus_{v \in S} R\Gamma_{1w,f}(\mathbb{Q}_v,D,V) \right) \right) \otimes \mathcal{H} \rightarrow \left( \bigoplus_{v \in S} R\Gamma_{1w}(\mathbb{Q}_v,V) \right) \otimes \mathcal{H}
$$

gives an isomorphism $\Delta_{1w,h}(D,V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H} \simeq \det_{\mathcal{H}}^{-1} R\Gamma_{1w,S}(D,V)$. Let $\mathcal{K}$ denote the field of fractions of $\mathcal{H}$. By Theorem 4.2.3, all $R\Gamma_{1w,S}(D,V)$ are $\mathcal{H}$-torsion and we have a canonical map.

$$
\det_{\mathcal{H}}^{-1} R\Gamma_{1w,S}(D,V) \simeq \bigotimes_{i \in \{2,3\}} \det_{\mathcal{H}} \left( -1 \right)^{i+1} R\Gamma_{1w,S}(D,V) \hookrightarrow \mathcal{K}.
$$

The composition of these maps gives a trivialization $i_{V,1w,h} : \Delta_{1w,h}(D,V) \hookrightarrow \mathcal{K}$. Fix a $\mathbb{Z}_p$-lattice $N$ of $D$ and set

$$
\Delta_{1w,h}(N,T) = \det_{\Lambda}^{-1} \left( R\Gamma_{1w,S}(T) \oplus \left( \bigoplus_{v \in S} R\Gamma_{1w,f}(\mathbb{Q}_v,N,T) \right) \right) \otimes \det_{\Lambda} \left( \bigoplus_{v \in S} R\Gamma_{1w}(\mathbb{Q}_v,T) \right).
$$

Perrin-Riou [PR2] defined the module of $p$-adic $L$-functions associated to $(N,T)$ as

$$
L_{1w,h}^{(n)}(N,T) = i_{V,1w,h} \left( \Delta_{1w,h}(N,T) \right) \subset \mathcal{K}.
$$

Fix a generator $f(\gamma - 1)$ of $L_{1w,h}^{(n)}(N,T)$ and define a meromorphic $p$-adic function

$$
L_{1w,h}(T,N,s) = f(\chi(\gamma)^s - 1).
$$

Let $\omega_N$ be a generator of $\text{det}_{\mathbb{Z}_p}(N)$. The isomorphism $D \simeq t_V(\mathbb{Q}_p)$ allows us to consider $\omega_N$ as a basis of $\text{det}_{\mathbb{Q}_p} t_V(\mathbb{Q}_p)$. We also fix a generator $\omega_T$ of $\text{det}_{\mathbb{Z}_p} T^+$ and define the $p$-adic period $\Omega_p(\omega_N,\omega_T) \in \mathbb{Q}_p$ by $\omega_B = \Omega_p(\omega_T,\omega_B)\omega_T$. Now we can state the main result of this paper.

**Theorem 4.3.2.** Assume that a pseudo-geometric representation $V$ satisfies $\textbf{C1-5})$. Let $D$ be an admissible subspace of $D_{\text{cris}}(V)$. Fix a $G_{\mathbb{Q}}$-stable lattice $T$ of $V$ and a lattice $N$ of $D$. Assume that $\mathcal{L}(D,V) \neq 0$. Then

i) $L_{1w,h}(T,N,s)$ is a meromorphic $p$-adic function which has a zero at $s = 0$ of order $e = \dim_{\mathbb{Q}_p}(D^{\varphi = p^{-1}}).

ii) Let $L_{1w,h}^*(T,N,0) = \lim_{s \rightarrow 0} s^{-e} L_{1w,h}(T,N,s)$ be the special value of $L_{1w,h}(T,N,s)$ at $s = 0$. Then

$$
L_{1w,h}^*(T,N,0) \sim \Gamma(h)^{d_+(V)} L(D,V) E_p^*(V,1) \det_{\mathbb{Q}_p} \left( \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} \right) D_{-1} \frac{i_{\omega_N,\omega_B,p}(\Delta_{\text{EF}}(T))}{\Omega_p(\omega_T,\omega_B)},
$$

where $D_{-1}$ is a constant depending on $\mathcal{L}(D,V) \neq 0$. The factor on the right-hand side involves a regularized period $i_{\omega_N,\omega_B,p}(\Delta_{\text{EF}}(T))$. This theorem generalizes the work of Perrin-Riou and provides a formula for the special value of the $L$-function at the critical point.
where \( E_p(V, t) = E_p^*(V, t) \left( 1 - \frac{t}{p} \right)^e \) and \( \Gamma(h) = (h - 1)! \).

4.3.3. Proof of Theorem 4.3.2.
4.3.3.1. First recall the formalism of Iwasawa descent which will be used in the proof. The result we need is proved in [BG]. This is a particular case of Nekovář’s descent theory [N2]. Let \( C^\bullet \) be a perfect complex of \( \mathcal{H} \)-modules and let \( C^\bullet_0 = C^\bullet \otimes_{\mathcal{H}} \mathbb{Q}_p \). We have a natural distinguished triangle
\[
C^\bullet \xrightarrow{X} C^\bullet \to C^\bullet_0,
\]
where \( X = \gamma_1 - 1 \). In each degree this triangle gives a short exact sequence
\[
0 \to H^n(C^\bullet)_{\gamma_1} \to H^n(C^\bullet_0) \to H^{n+1}(C^\bullet)_{\gamma_1} \to 0.
\]
One says that \( C^\bullet \) is semisimple if the natural map
\[
H^n(C^\bullet)_{\gamma_1} \to H^n(C^\bullet) \to H^n(C^\bullet)_{\gamma_1}
\]
is an isomorphism in all degrees. If \( C^\bullet \) is semisimple, there exists a natural trivialisation of \( \det_{\mathbb{Q}_p} C^\bullet_0 \), namely
\[
\vartheta : \det_{\mathbb{Q}_p} C^\bullet_0 \simeq \otimes_{n \in \mathbb{Z}} (\det_{\mathbb{Q}_p}(-1)^n H^n(C^\bullet_0) \simeq \otimes_{n \in \mathbb{Z}} \left( \det_{\mathbb{Q}_p}(-1)^n H^n(C^\bullet)_{\gamma_1} \otimes \det_{\mathbb{Q}_p}(-1)^{n+1} H^{n+1}(C^\bullet)_{\gamma_1} \right)
\]
\[
\simeq \otimes_{n \in \mathbb{Z}} \left( \det_{\mathbb{Q}_p}(-1)^n H^n(C^\bullet)_{\gamma_1} \otimes \det_{\mathbb{Q}_p}(-1)^{n-1} H^{n-1}(C^\bullet)_{\gamma_1} \right) \simeq \mathbb{Q}_p
\]
where the last map is induced by (4.7). We now suppose that \( C \otimes_{\mathcal{H}} \mathcal{K} \) is acyclic and write \( i_{\infty} : \det_{\mathcal{H}} C^\bullet \to \mathcal{K} \) for the associated morphism in \( \mathcal{P}(\mathcal{K}) \). Then \( i_{\infty}(\det_{\mathcal{H}} C^\bullet) = f\mathcal{H} \), where \( f \in \mathcal{K} \). Let \( r \) be the unique integer such that \( X^{-r}f \) is a unit of the localization \( \mathcal{H}_0 \) of \( \mathcal{H} \) with respect to the principal ideal \( X\mathcal{H} \).

**Lemma 4.3.3.2.** Assume that \( C^\bullet \) is semisimple. Then \( r = \sum_{n \in \mathbb{Z}} (-1)^{n+1} \dim_{\mathbb{Q}_p} H^n(C^\bullet)_{\gamma_1} \) and there exists a commutative diagram
\[
\begin{array}{ccc}
\det_{\mathcal{H}} C^\bullet & \xrightarrow{X^{-r}i_{\infty}} & \mathcal{H}_0 \\
\otimes_{\mathbb{Q}_p} \downarrow & & \downarrow \\
\det_{\mathbb{Q}_p} C^\bullet_0 & \xrightarrow{\vartheta} & \mathbb{Q}_p
\end{array}
\]
in which the right vertical arrow is the augmentation map.

**Proof.** See [BG], Lemma 8.1. Remark that Burns and Greither consider complexes over \( \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) but since \( \mathcal{H} \) is a Bézout ring, all their arguments work in our case and are omitted here.

4.3.3.3. By Theorem 4.2.3 the complex \( R\Gamma^{(\emptyset)}_{Iw, h}(D, V) \) is semisimple and the first assertion follows from Lemma 4.3.3.2 together with Corollary 4.2.5.

4.3.3.4. In this subsection we compare the Bloch-Kato local condition at \( p \) with the local condition coming from Perrin-Riou’s theory. Set \( R\Gamma_f(\mathbb{Q}_p, D, V) = D[-1] \) and define
\[
S = \text{cone} \left( \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} : R\Gamma_f(\mathbb{Q}_p, D, V) \to R\Gamma_f(\mathbb{Q}_p, V) \right) [-1].
\]
Thus, explicitly

\[ S = [D \oplus D_{\text{cris}}(V) \to D_{\text{cris}}(V) \oplus t_{V}(Q_p)] [-1] \simeq [D \oplus D_{\text{cris}}(V) \to D_{\text{cris}}(V) \oplus D] [-1], \]

where the unique non-trivial map is given by

\[(x, y) \mapsto \left( (1 - \varphi) y, \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} x + y \right) \mod \text{Fil}^0 D_{\text{cris}}(V). \]

Thus \( H^1(S) = D^{\varphi = p^{-1}} \) and \( H^2(S) = \frac{t_{V}(Q_p)}{(1 - p^{-1}\varphi^{-1})D} \simeq \frac{D}{(1 - p^{-1}\varphi^{-1})D}. \) From the semi-simplicity of \( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \) it follows that the natural projection \( H^1(S) \to H^2(S) \) is an isomorphism and we have a canonical trivialization \( \theta_S : \det_{Q_p} S \simeq \det_{Q_p}^{-1} H^1(S) \otimes \det_{Q_p} H^2(S) \simeq Q_p. \) Hence the distinguished triangle

\[ S \to R\Gamma_f(Q_p, D, V) \to R\Gamma_f(Q_p, V) \to S[1] \]

induces isomorphisms

\[ \det_{Q_p} R\Gamma_f(Q_p, V) \simeq R\Gamma_f(Q_p, D, V) \otimes \det_{Q_p}^{-1} S \simeq \det_{Q_p} \det_{Q_p} R\Gamma_f(Q_p, D, V). \]

**Lemma 4.3.3.5.** i) Let \( f : W \to W \) be a semi-simple endomorphism of a finitely dimensional \( k \)-vector space \( W. \) The canonical projection \( \ker(f) \to \text{coker}(f) \) is an isomorphism and the tautological exact sequence

\[ 0 \to \ker(f) \to W \xrightarrow{f} W \to \text{coker}(f) \to 0 \]

induces an isomorphism

\[ \det^* f : \det_k(W) \to \det_k(W) \otimes \det_k(\ker(f)) \otimes \det_k^{-1}(\text{coker}(f)) \to \det_k(W). \]

Then \( \det^* f(x) = \det(f \mid \text{coker}(f)). \)

ii) The diagram

\[
\begin{array}{ccc}
\det_{Q_p} R\Gamma_f(Q_p, V) & \xrightarrow{\det_{Q_p}^{-1} t_{V}(Q_p)} & \det_{Q_p} R\Gamma_f(Q_p, D, V) \\
\downarrow & & \downarrow \\
\det_{Q_p}^{-1} t_{V}(Q_p) & \xrightarrow{\det^* \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) D_{E_p(V, 1)}} & \det_{Q_p}^{-1} D \\
\downarrow & & \downarrow \\
\det_{Q_p}^{-1} D & \xrightarrow{\det^* \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) D_{E_p(V, 1)}} & \det_{Q_p}^{-1} D
\end{array}
\]

in which the bottom map is the multiplication by \( \det^* \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} \right) D_{E_p(V, 1)}, \) commutes.

**Proof.** The proof of i) is straightforward and is omitted here. Next, ii) follows from i) applied to \( W = D \) and the fact what \( E_p(V, 1) = \det(1 - \varphi \mid D_{\text{cris}}(V)). \)
4.3.3.6. Now we can prove Theorem 4.3.2. Define

$$R\Gamma_f(Q_v, N, T) = R\Gamma^{(\eta_0)}_{Iw, f}(Q_v, N, T) \otimes_\Lambda \mathbb{Z}_p,$$

$$R\Gamma_f(Q_v, D, V) = R\Gamma_f(Q_v, N, T) \otimes Z_p Q_p.$$ Remark that for $$v = p$$ this definition coincides with the definition given in 4.3.3.4. Applying $$\otimes_\mathcal{H} Q_p$$ to the map $$R\Gamma^{(\eta_0)}_{Iw, f}(Q_v, D, V) \to R\Gamma^{(\eta_0)}_{Iw}(Q_v, T) \otimes_\Lambda \mathcal{H}$$ we obtain a morphism

$$R\Gamma_f(Q_v, D, V) \to R\Gamma(Q_v, V).$$

If $$v \neq p$$, then $$R\Gamma_f(Q_v, D, V) = R\Gamma_f(Q_v, V)$$ and this morphism coincides with the natural map $$R\Gamma_f(Q_v, V) \to R\Gamma(Q_v, V)$$. If $$v = p$$, then $$R\Gamma_f(Q_v, D, V) = D[-1]$$ and by Theorem 2.2.4 it coincides with the composition

$$D \xrightarrow{1-p-1-q-1} D_{cris}(V) \xrightarrow{(h-1)! \exp_{V, Q_p}} H^1(Q_v, V).$$

Let $$R\Gamma_{f,h}(D, V)$$ denote the Selmer complex associated to the diagram

$$R\Gamma_S(V) \oplus R\Gamma(Q_v, V) \oplus R\Gamma_f(Q_v, D, V)$$

Then we have a distinguished triangle

$$R\Gamma_{f,h}(D, V) \to R\Gamma_S(V) \oplus \left( \bigoplus_{v \in S} R\Gamma_f(Q_v, D, V) \right) \to \bigoplus_{v \in S} R\Gamma(Q_v, V)$$ (4.9)

which induces isomorphisms

$$\det_{Q_p}^{-1} R\Gamma_S(V) \otimes_{Q_p} \left( \bigoplus_{v \in S} R\Gamma(Q_v, V) \right) \otimes_{Q_p} D \xrightarrow{\sim} \det_{Q_p}^{-1} R\Gamma_{f,h}(D, V),$$

$$\xi_{D, h} : \Delta_{EP}(V) \otimes_{Q_p} \left( \det_{Q_p} D \otimes \det_{Q_p}^{-1} V^+ \right) \xrightarrow{\sim} \det_{Q_p}^{-1} R\Gamma_{f,h}(D, V).$$

Next, $$R\Gamma_{f,h}(D, V) = R\Gamma_{(\eta_0)}^{(\eta_0)}_{Iw, h}(D, V) \otimes \mathcal{H} Q_p$$ and for any $$i$$ one has an exact sequence

$$0 \to R^i\Gamma_{Iw, h}^{(\eta_0)}(D, V) \to R^i\Gamma_{f,h}(D, V) \to R^{i+1}\Gamma_{Iw, h}^{(\eta_0)}(D, V) \to 0.$$ From Theorem 4.2.3 it follows that

$$R^i\Gamma_{f,h}(D, V) = \begin{cases} R^2\Gamma_{Iw, h}^{(\eta_0)}(D, V) & \text{if } i = 1 \\ R^2\Gamma_{Iw, h}^{(\eta_0)}(D, V) & \text{if } i = 2 \\ 0 & \text{if } i \neq 1, 2. \end{cases}$$

Therefore, the isomorphism $$R^2\Gamma_{Iw, h}(D, V)^\Gamma \to R^2\Gamma_{Iw, h}(D, V)^\Gamma$$ induces a canonical trivialization

$$d_{D,h} : \det_{Q_p} R\Gamma_{f,h}(D, V) \xrightarrow{\sim} Q_p.$$
By Lemma 4.3.3.2 we have a commutative diagram

\[
\begin{array}{ccc}
\det^{-1} H \Gamma_{f,h}^{(p)}(D, V) & \xrightarrow{X^{-1} \tau_{V,\text{Iw,h}}} & H_0 \\
\downarrow & & \\
\det^{-1} Q_p H \Gamma_{f,h} (D, V) & \xrightarrow{\vartheta_{D,h}^{-1}} & Q_p.
\end{array}
\]

Since

\[
\Delta_{\text{Iw,h}}(N, T) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \Delta_{\text{EP}}(T) \otimes_{\mathbb{Z}_p} \omega_N \otimes_{\mathbb{Z}_p} \omega_T^{-1}
\]

it implies that

\[
\vartheta_{D,h}^{-1} \circ \xi_{D,h}(\Delta_{\text{EP}}(T) \otimes_{\mathbb{Z}_p} \omega_N \otimes_{\mathbb{Z}_p} \omega_T^{-1}) = \log(\chi(\gamma))^{-e} L_{\text{Iw,h}}^*(T, N, 0) \mathbb{Z}_p.
\]

(4.10)

Consider the diagram

\[
\begin{array}{ccc}
\mathbb{R} \Gamma_f(V) & \rightarrow & \mathbb{R} \Gamma_s(V) \oplus \bigoplus_{v \in S \cup \{\infty\}} \mathbb{R} \Gamma_f(Q_v, V) \oplus \bigoplus_{v \in S \cup \{\infty\}} \mathbb{R} \Gamma(Q_v, V) \\
\downarrow & & \downarrow \\
\mathbb{R} \Gamma_{f,h}(D, V) & \rightarrow & \mathbb{R} \Gamma_s(V) \oplus \bigoplus_{v \in S} \mathbb{R} \Gamma_f(Q_v, D, V) \oplus \bigoplus_{v \in S} \mathbb{R} \Gamma(Q_v, V)
\end{array}
\]

(4.11)

in which \( L = \text{cone}(\mathbb{R} \Gamma_{f,h}(D, V) \rightarrow \mathbb{R} \Gamma_f(V)) \) and the upper and middle rows coincide with (4.1) and (4.9) up to the following modification: the map \( \text{loc}_p : \mathbb{R} \Gamma_f(Q_p, V) \rightarrow \mathbb{R} \Gamma(Q_p, V) \) is replaced by \( \Gamma(h) \text{loc}_p \). It follows from C1-5) that \( \mathbb{R} \Gamma_f(V) \) is acyclic. Hence in the derived category \( D^p(Q_p) \) the composition \( \alpha : S \rightarrow L \rightarrow \mathbb{R} \Gamma_{f,h}(D, V) \) is an isomorphism. An easy diagram search shows that \( H^1(S) \simeq \mathbb{R}^1 \Gamma_{f,h}(D, V) \) coincides with \( \text{id} : D^p_{\varphi=p^{-1}} \rightarrow D^p_{\varphi=p^{-1}} \) and that \( H^2(S) \simeq \mathbb{R}^2 \Gamma_{f,h}(D, V) \) coincides with \( \Gamma(h) \exp_{V, Q_p} \). Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
\det Q_p S & \xrightarrow{\alpha} & \det Q_p \mathbb{R} \Gamma_{f,h}(D, V) \\
\downarrow & \vartheta_{D,h} & \downarrow \vartheta_{D,h} \\
Q_p & \xrightarrow{\kappa} & Q_p
\end{array}
\]

there \( \kappa \) can be written as the composition

\[
\mathbb{Q}_p \rightarrow \det^{-1} H^1(S) \otimes \det Q_p H^2(S) \rightarrow \det^{-1} \mathbb{R}^1 \Gamma_{f,h}(D, V) \otimes \det Q_p \mathbb{R}^2 \Gamma_{f,h}(D, V) \rightarrow \mathbb{Q}_p
\]

From Proposition 3.2.4 and Corollary 4.2.5 we obtain immediately that

\[
\kappa = (\log \chi(\gamma))^e \left(1 - \frac{1}{p}\right)^e L(D, V)^{-1} \text{id}_{Q_p}.
\]

(4.12)
Passing to determinants in the diagram (4.11) we obtain a commutative diagram

\[
\begin{array}{cccccc}
\Delta_{EP}(V) \otimes (\det(t_{V}(\mathbb{Q}_{p})) \otimes \det^{-1}V^+) & \longrightarrow & \det^{-1}R\Gamma_{f}(V) & \longrightarrow & \mathbb{Q}_{p} \\
\downarrow f & & \downarrow & & \\
\Delta_{EP}(V) \otimes (\det D \otimes \det^{-1}V^+) \otimes \det S & \longrightarrow & \det^{-1}R\Gamma_{f,h}(D, V) \otimes \det R\Gamma_{f,h}(D, V) & \longrightarrow & \mathbb{Q}_{p}
\end{array}
\]

in which the map \(f\) is induced by (4.8). The upper row of this diagram sends \(\Delta_{EP}(T) \otimes (\omega_{N} \otimes \omega_{B}^{-1})\) onto

\[
\Gamma(h)^{d_{+}(V)}i_{\omega_{N},\omega_{B},p}(\Delta_{EP}(T)).
\]  

(4.13)

From Lemma 4.3.3.5 it follows that

\[
(id \otimes \vartheta_{S}) \circ f = \det^{*}(1 - p^{-1} \varphi^{-1} | D)^{-1} E_{p}(V, 1)^{-1} \text{id}
\]  

(4.14)

Next, (4.10) and (4.12) give

\[
\vartheta_{D,h}^{-1} \circ (\xi_{D,h} \otimes \kappa)(\Delta_{EP}(T) \otimes_{\mathbb{Z}_{p}} \omega_{N} \otimes_{\mathbb{Z}_{p}} \omega_{T}^{-1}) = \left(1 - \frac{1}{p}\right)^{e} \mathcal{L}(D, V)^{-1} L_{i_{w,h}}^{*}(T, N, 0) \mathbb{Z}_{p}.
\]  

(4.15)

Putting together (4.13), (4.14) and (4.15) we obtain that

\[
\text{L}_{i_{w,h}}^{*}(T, N, 0) \sim \Gamma(h)^{d_{+}(V)} \mathcal{L}(D, V) E_{p}^{*}(V, 1) \det_{\mathbb{Q}_{p}}^{*} \left(1 - p^{-1} \varphi^{-1} | D\right) \frac{i_{\omega_{N},\omega_{B},p}(\Delta_{EP}(T))}{\Omega_{p}(\omega_{T}, \omega_{B})}.
\]

The theorem is proved.

**Appendix. Galois cohomology of \(p\)-adic representations**

**A.1.** Let \(K\) be a finite extension of \(\mathbb{Q}_{p}\) and \(T\) a \(p\)-adic representation of \(G_{K}\). Fix a topological generator \(\gamma\) of \(\Gamma\). Let \(D(T) = (T \otimes_{\mathbb{Z}_{p}} \mathbb{A})^{H_{K}}\) be the \((\varphi, \Gamma)\)-module associated to \(T\) by Fontaine’s theory [F2]. Consider the complex

\[
C_{\varphi,\gamma}(D(T)) = \left[ D(T) \xrightarrow{f} D(T) \otimes D(T) \xrightarrow{g} D(T) \right]
\]

where the modules are placed in degrees 0, 1 and 2 and the maps \(f\) and \(g\) are given by

\[
f(x) = ((\varphi - 1)x, (\gamma - 1)x), \quad g(y, z) = (\gamma - 1)y - (\varphi - 1)z.
\]
Proposition A.2. There are canonical and functorial isomorphisms

\[ h^i : H^i(C_{\varphi,\gamma}(D(T))) \cong H^i(K, T) \]

which can be described explicitly by the following formulas:

i) If \( i = 0 \), then \( h^0 \) coincides with the natural isomorphism

\[ D(T)^{\varphi=1,\gamma=1} = H^0(K, T \otimes_{\mathbb{Z}_p} A^{\varphi=1}) = H^0(K, T). \]

ii) Let \( \alpha, \beta \in D(T) \) be such that \((\gamma - 1)\alpha = (1 - \varphi)\beta\). Then \( h^1 \) sends \( \text{cl}(\alpha, \beta) \) to the class of the cocycle

\[ \mu_1(g) = (g - 1)x + \frac{g - 1}{\gamma - 1}\beta, \]

where \( x \in D(T) \otimes_{\mathbb{A}_K} A \) is a solution of the equation \((1 - \varphi)x = \alpha\).

iii) Let \( \hat{\gamma} \in G_K \) be a lifting of \( g \in \Gamma \) and let \( x \) be a solution of \((\varphi - 1)x = \alpha\). Then \( h^2 \) sends \( \alpha \) to the class of the 2-cocycle

\[ \mu_2(g_1, g_2) = \hat{\gamma}^{k_1}(h_1 - 1) (\frac{\hat{\gamma}^{k_2} - 1}{\gamma - 1}x) \]

where \( g_i = \hat{\gamma}^{k_i}h_i, h_i \in H_K \).

Proof. The isomorphisms \( h^i \) were constructed in [H1], Theorem 2.1. Remark that i) follows directly from this construction (see [H1], p.573) and that ii) is proved in [Ben1], Proposition 1.3.2 and [CC2], Proposition I.4.1. The proof of iii) follows along exactly the same lines. Namely, it is enough to prove this formula modulo \( p^n \) for each \( n \). Let \( \alpha \in D(T)/p^nD(T) \). By Proposition 2.4 of [H1] there exists \( r \geq 0 \) and \( y \in D(T)/p^nD(T) \) such that \((\varphi - 1)\alpha = (\gamma - 1)^r\beta \). Let

\[ N_x = (D(T)/p^nD(T)) \oplus (\oplus_{i=1}^{r-1}(A_K/p^nA_K) t_i), \]

where \( \varphi(t_i) = t_i + (\gamma - 1)^{r-i}(\alpha) \) and \( \gamma(t_i) = t_i + t_{i-1} \). Then \( N_x \) is a \((\varphi, \Gamma)\)-module and we have a short exact sequence

\[ 0 \rightarrow D \rightarrow N_x \rightarrow X \rightarrow 0 \]

where \( X = N_x/M \cong \oplus_{i=1}^{r-1}A_K/p^nA_K t_i \). An easy diagram search shows that the connecting homomorphism \( \delta^1_D : H^1(C_{\varphi,\gamma}(D(X))) \rightarrow H^2(C_{\varphi,\gamma}(D(T))) \) sends \( \text{cl}(0, \tilde{t}_r) \) to \(-\text{cl}(\alpha)\). The functor \( V(D) = (D \otimes_{A_K} A)^{\varphi=1} \) is a quasi-inverse to \( D \). Thus one has an exact sequence of Galois modules

\[ 0 \rightarrow T/p^nT \rightarrow T_x \rightarrow V(X) \rightarrow 0 \]

where \( T_x = V(N_x) \). From the definition of \( X \) it follows immediately that \( t_r - x \in T_x \). By ii), \( h^1(\text{cl}(0, \tilde{t}_r)) \) can be represented by the cocycle \( c(g) = \frac{g - 1}{\gamma - 1} \tilde{t}_r \) and we fix its lifting \( \hat{c} : G_K \rightarrow N_x \) putting \( \hat{c}(g) = \frac{g - 1}{\gamma - 1} (t_r - x) \). As \( g_1\hat{c}(g_2) - \hat{c}(g_1g_2) + \hat{c}(g_1) = -\mu_2(g_1, g_2) \), the connecting map \( \delta^1_L : H^1(K, V(X)) \rightarrow H^2(K, T/p^nT) \) sends \( \text{cl}(c) \) to \(-\text{cl}(\mu_2) \) and iii) follows from the commutativity of the diagram

\[
\begin{array}{ccc}
H^1(C_{\varphi,\gamma}(X)) & \xrightarrow{\delta^1_D} & H^2(C_{\varphi,\gamma}(T/p^nT)) \\
\downarrow h^1 & & \downarrow h^2 \\
H^1(K, V(X)) & \xrightarrow{\delta^1_L} & H^2(K, T/p^nT).
\end{array}
\]
Proposition A.3. The complexes $R\Gamma(K, T)$ and $C_{\varphi, \gamma}(T)$ are isomorphic in $D(\mathbb{Z}_p)$.

Proof. The proof is standard (see for example [BF], proof of Proposition 1.17). The exact sequence

$$0 \to T \to D(T) \otimes_{A_K} A \xrightarrow{\varphi^{-1}} D(T) \otimes_{A_K} A \to 0$$

gives rise to an exact sequence of complexes

$$0 \to C^*_c(G_K, T) \to C^*_c(G_K, D(T) \otimes_{A_K} A) \xrightarrow{\varphi^{-1}} C^*_c(G_K, D(T) \otimes_{A_K} A) \to 0$$

Thus $R\Gamma(K, T)$ is quasi-isomorphic to the total complex

$$K^*(T) = \text{Tot}^*(C^*_c(G_K, D(T) \otimes_{A_K} A) \xrightarrow{\varphi^{-1}} C^*_c(G_K, D(T) \otimes_{A_K} A)).$$

On the other hand $C_{\varphi, \gamma}(T) = \text{Tot}^*(A^*(T) \xrightarrow{\varphi^{-1}} A^*(T))$, where $A^*(T) = [D(T) \xrightarrow{\gamma^{-1}} D(T)]$. Consider the following commutative diagram of complexes

$$\begin{array}{cccccccc}
D(T) & \xrightarrow{\gamma^{-1}} & D(T) & \xrightarrow{0} & \cdots \\
\downarrow{\beta_0} & & \downarrow{\beta_1} & & \\
C^0(G_K, D(T) \otimes_{A_K} A) & \xrightarrow{\varphi^{-1}} & C^1(G_K, D(T) \otimes_{A_K} A) & \xrightarrow{\varphi^{-1}} & C^2(G_K, D(T) \otimes_{A_K} A) & \xrightarrow{\varphi^{-1}} & \cdots \\
\end{array}$$

in which $\beta_0(x) = x$ viewed as a constant function on $G_K$ and $\beta_1(x)$ denotes the map $G_K \to D(T) \otimes_{A_K} A$ defined by $(\beta_1(x))(g) = \frac{g - 1}{\gamma - 1} x$. This diagram induces a map $\text{Tot}^*(A^*(T) \xrightarrow{\varphi^{-1}} A^*(T)) \to K^*(T)$ and we obtain a diagram

$$C_{\varphi, \gamma}(T) \to K^*(T) \leftarrow R\Gamma(K, T)$$

where the right map is a quasi-isomorphism. Then for each $i$ one has a map

$$H^i(C_{\varphi, \gamma}(T)) \to H^i(K^*(T)) \simeq H^i(K, T)$$

and an easy diagram search shows that it coincides with $h^i$. The proposition is proved.

Corollary A.4. Let $V$ be a $p$-adic representation of $G_K$. Then the complexes $R\Gamma(K, V)$, $C_{\varphi, \gamma}(D^!(V))$ and $C_{\varphi, \gamma}(D^!_{rig}(V))$ are isomorphic in $D(\mathbb{Q}_p)$.

Proof. This follows from Theorem 1.1 of [Li] together with Proposition A.2.

A.5. Recall that $K^\infty/K$ denotes the cyclotomic extension obtained by adjoining all $p^n$-th roots of unity. Let $\Gamma = \text{Gal}(K^\infty/K)$ and let $\Lambda(\Gamma) = \mathbb{Z}_p[[\Gamma]]$ denote the Iwasawa algebra of $\Gamma$. For any $\mathbb{Z}_p$-adic representation $T$ of $G_K$ the induced representation $\text{Ind}_{K^\infty/K}^T$ is isomorphic to $(T \otimes_{\mathbb{Z}_p} \Lambda(\Gamma))^i$ and we set $R\Gamma_{\text{Iw}}(K, T) = C^*_c(G_K, \text{Ind}_{K^\infty/K}^T)$. Consider the complex

$$C_{\text{Iw}, \psi}(T) = [D(T) \xrightarrow{\psi^{-1}} D(T)]$$

in which the first term is placed in degree 1.
Proposition A.6. There are canonical and functorial isomorphisms
\[ h^1_{Iw} : H^i(C_{Iw, \psi}(T)) \to H^i_{Iw}(K, T) \]
which can be described explicitly by the following formulas:

i) Let \( \alpha \in \mathbf{D}(T)^{\psi = 0} \). Then \( (\varphi - 1) \alpha \in \mathbf{D}(T)^{\psi = 0} \) and for any \( n \) there exists a unique \( \beta_n \in \mathbf{D}(T) \) such that \( (\gamma_n - 1) \beta_n = (\varphi - 1) \alpha \). The map \( h^1_{Iw} \) sends \( \text{cl}(\alpha) \) to \( (h^1_{Iw}(\text{cl}(\beta_n, \alpha)))_{n \in \mathbb{N}} \in H^1_{Iw}(K_n, T) \).

ii) If \( \alpha \in \mathbf{D}(T) \), then \( h^2_{Iw}(\text{cl}(\alpha)) = -(h^2_{Iw}(\varphi(\alpha)))_{n \in \mathbb{N}} \).

Proof. The proposition follows from Theorem II.1.3 and Remark II.3.2 of [CC2] together with Proposition A.2.

Proposition A.7. The complexes \( \mathbf{R} \Gamma_{Iw}(K, T) \) and \( C_{Iw, \psi}(T) \) are isomorphic in the derived category \( \mathcal{D}(\Lambda(\Gamma)) \).

Proof. We repeat the arguments used in the proof of Proposition A.1.2 with some modifications. For any \( n \geq 1 \) one has an exact sequence
\[ 0 \to \text{Ind}_{K_n/K} T \to (\mathbf{D}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_n]^t) \otimes_{A_K} A \xrightarrow{\varphi - 1} (\mathbf{D}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_n]^t) \otimes_{A_K} A \to 0. \]
Set \( \mathbf{D}({\text{Ind}}_{K_{\infty}/K} T) = (\mathbf{D}(T) \otimes_{\mathbb{Z}_p} A(\Gamma))' \) and
\[ \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A = \varprojlim_n (\mathbf{D}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_n]^t) \otimes_{A_K} A. \]
As \( \text{Ind}_{K_n/K} T \) are compact, taking projective limit one obtains an exact sequence
\[ 0 \to \text{Ind}_{K_{\infty}/K} T \to \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A \xrightarrow{\varphi - 1} \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A \to 0. \]
Thus \( \mathbf{R} \Gamma_{Iw}(K, T) \) is quasi-isomorphic to
\[ K_{Iw}^*(T) = \text{Tot}^* \left(C^*_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \xrightarrow{\varphi - 1} C^*_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \right). \]

We construct a quasi-isomorphism \( f_* : C_{Iw, \psi}(T) \to K_{Iw}^*(T) \). Any \( x \in \mathbf{D}(T) \) can be written in the form \( x = (1 - \varphi) x + \varphi x \) where \( \psi(1 - \varphi) x = 0 \). Then for each \( n \geq 0 \) the equation \( (\gamma_n - 1) y_n = (\varphi - 1) x \) has a unique solution \( y_n \in \mathbf{D}(T)^{\psi 0} \) ([CC2], Proposition I.5.1). In particular, \( y_n = \frac{(\gamma_n + 1 - 1)}{\gamma_n - 1} y_{n+1} \) and we have a compatible system of elements
\[ Y_n = \sum_{k=0}^{\lfloor G_n - 1 \rfloor} \gamma^k \otimes \gamma^k(y_n) \in \mathbf{D}(T) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_n]^t. \]
Put \( Y = (Y_n)_{n \geq 0} \in \mathbf{D}(\text{Ind}_{K_{\infty}/K} T) \). Then
\[ (\gamma_n - 1) Y_n = (\gamma - 1) Y \mod \mathbf{D}(\text{Ind}_{K_{\infty}/K} T). \]
Let \( \eta_x \in C^1_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \) be the map defined by \( \eta_x(g) = \frac{g - 1}{\gamma - 1} (1 \otimes x) \). Define \( f_1 : \mathbf{D}(T) \to K^1_{Iw}(T) = C^0_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \oplus C^1_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \) by \( f_1(x) = (Y, \eta_x) \) and \( f_2 : \mathbf{D}(T) \to C^1_c(G_K, \mathbf{D}(\text{Ind}_{K_{\infty}/K}(T)) \otimes_{A_K} A) \subset K^2_{Iw}(T) \) by \( f_2(z) = -\eta_{\psi(z)} \). It is easy to check that \( f_* \) is a morphism of complexes. This gives a diagram
\[ C_{Iw, \psi}(T) \to K^1_{Iw}(T) \to \mathbf{R} \Gamma_{Iw}(K, T) \]
in which the right map is a quasi-isomorphism. Using Proposition A.1.4 it is not difficult to check that for each \( i \) the induced map
\[ H^i(C_{Iw, \psi}(T)) \to H^i(K^1_{Iw}(T)) \simeq H^i_{Iw}(K, T) \]
coincides with \( h^i_{Iw} \). The proposition is proved.
Corollary A.8. The complexes $R\Gamma_{1w}(K,T)$ and $C^{1}_{1w,\psi}(T)$ are isomorphic in $D(\Lambda(\Gamma))$.

Proof. One has $D^{1}(T)^{\psi=1} = D(T)^{\psi=1}$ ([CC1], Proposition 3.3.2) and $D^{1}(T)/(\psi-1) = D(T)/(\psi-1)$ ([Li], Lemma 3.6). This shows that the inclusion $C^{1}_{1w,\psi}(T) \to D(T)^{\psi=1}$ is a quasi-isomorphism.

Remark A.9. These results can be slightly improved. Namely, set $r_{n} = (p-1)p^{n-1}$. The method used in the proof of Proposition III.2.1 [CC2] allows to show that $\psi(D^{1+r_{n}}(T)) \subset D^{1+r_{n}-1}(T)$ for $n \gg 0$. Moreover, for any $a \in D^{1+r_{n}}(T)$ the solutions of the equation $(\psi-1)x = a$ are in $D^{1+r_{n}}(T)$. Thus $C^{1}_{1w,\psi}(T) = [D^{1+r_{n}}(T) \overset{\psi=1}{\to} D^{1+r_{n}}(T)]$, $n \gg 0$ is a well-defined complex which is quasi-isomorphic to $C^{1}_{1w,\psi}(T)$. Further, as $\varphi(A^{1+r/p}) = A^{1+r}$ we can consider the complex

$$C^{1,r_{n}}_{\varphi,\gamma}(T) = [D^{1+r_{n}-1}(T) \overset{f}{\to} D^{1+r_{n}}(T) \oplus D^{1+r_{n}-2}(T) \overset{g}{\to} D^{1+r_{n}}(T)], \quad n \gg 0$$

in which $f$ and $g$ are defined by the same formulas as before. Then the inclusion $C^{1,r_{n}}_{\varphi,\gamma}(T) \to C_{\varphi,\gamma}(T)$ is a quasi-isomorphism.

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