Twistor spinors on Lorentzian symmetric spaces

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Abstract

We solve the twistor equation on all indecomposable Lorentzian symmetric spaces explicitly.

1 Introduction

Let $(M^n, g)$ be an oriented semi-Riemannian spin manifold with the spinor bundle $S$. The twistor operator $D$ is defined as the composition of the spinor derivative $\nabla^S$ with the projection $p$ onto the kernel of the Clifford multiplication $\mu$

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \xrightarrow{\mu} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu).$$

The solutions of the conformally invariant equation $D\varphi = 0$ are called twistor spinors. Twistor spinors were introduced by R. Penrose in General Relativity (see [Pen67], [PRS86], [NWS84]). They are related to Killing vector fields in semi-Riemannian supergeometry (see [ACDS97]). In the last years essential results concerning the geometry of Riemannian spin manifolds admitting twistor spinors were obtained by A. Lichnerowicz, T. Friedrich, K. Habermann, H-B. Rademacher, W. Kuehnel and other authors. For a survey on the literature cf. [Fri97]. In the Lorentzian setting there was established a relation between a special class of solutions of the twistor equation and the Fefferman spaces occurring in CR-geometry (cf. [Lew91], [Bau97]).

Let $T(M^n, g)$ denote the space of all twistor spinors of $(M, g)$. It is known that

$$\dim T(M^n, g) \leq 2 \cdot 2^{\left\lfloor \frac{n}{2} \right\rfloor},$$

(see [BFGK91]). If $(M^n, g)$ is conformally flat and simply connected, then one has

$$\dim T(M^n, g) = 2 \cdot 2^{\left\lfloor \frac{n}{2} \right\rfloor}.$$ 

In the present paper we determine the twistor spinors on all indecomposable Lorentzian symmetric spaces explicitly. In particular, we prove:

1. If $(M^n, g)$ is an indecomposable non-conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$, then each twistor spinor is parallel and

$$\dim T(M^n, g) = q \cdot 2^{\left\lfloor \frac{n}{2} \right\rfloor},$$

where $q = \frac{1}{2},\frac{1}{4}$ or 0, depending on the fundamental group $\pi_1(M)$ and on the spin structure of $(M^n, g)$.

2. If $(M^n, g)$ is an indecomposable conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$ and non-constant sectional curvature, then

$$\dim T(M^n, g) = q \cdot 2^{\left\lfloor \frac{n}{2} \right\rfloor},$$

where $q = 2,\frac{3}{2},1,\frac{3}{4}$ or 0, depending on $\pi_1(M)$ and on the spin structure.
3. If \((M^n, g)\) is a Lorentzian symmetric spin manifold of dimension \(n \geq 3\) and constant sectional curvature, then \(\dim T(M^n, g) = q \cdot 2^\left\lfloor \frac{n}{2} \right\rfloor\), where \(q = 2, 1,\) or \(0\), depending on \(\pi_1(M)\) and on the spin structure.

The calculations are based on the fact, that the indecomposable Lorentzian symmetric spaces are completely classified (see [CW70]).

2 Lorentzian symmetric spaces

Let us first recall the description of Lorentzian symmetric spaces. A connected semi-Riemannian manifold \((M, g)\) is called indecomposable, if there is no proper, nondegenerate subspace of \(T_xM\) invariant under the action of the holonomy group \(\text{Hol}_x(g)\). Each simply connected semi-Riemannian symmetric space is isometric to a product \(M_0 \times \cdots \times M_r\), where \(M_i, i = 1, \ldots, r\), are indecomposable simply connected semi-Riemannian symmetric spaces of dimension \(\geq 2\) and \(M_0\) is semi-Euclidean.

Let \((M^n, g)\) be a Lorentzian symmetric space. By \(G(M)\) we denote the group of transvections of \((M^n, g)\) and by \(\mathfrak{g}\) its Lie algebra. One has the following structure result:

**Theorem 1** ([CW70])

Let \((M^n, g)\) be an indecomposable Lorentzian symmetric space of dimension \(n \geq 2\). Then the Lie algebra \(\mathfrak{g}\) of the transvection group of \((M^n, g)\) is either semi-simple or solvable.

Let \(\lambda = (\lambda_1, \ldots, \lambda_{n-2})\) be an \((n-2)\)-tupel of real numbers \(\lambda_j \in \mathbb{R}\setminus\{0\}\) and let us denote by \(M_\lambda^n\) the Lorentzian space \(M_\lambda^n := (\mathbb{R}^n, g_\lambda)\), where

\[
(g_\lambda)_{(s,t,x)} := 2ds \, dt + \sum_{j=1}^{n-2} \lambda_j x_j^2 \, ds^2 + \sum_{j=1}^{n-2} dx_j^2.
\]

If \(\lambda_\pi = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n-2)})\) is a permutation of \(\lambda\) and \(c > 0\), then \(M_\lambda^n\) is isometric to \(M_{c\lambda}^n\).

**Theorem 2** ([CW70], [CP80])

Let \((M^n, g)\) be an indecomposable solvable Lorentzian symmetric space of dimension \(n \geq 3\). Then \((M^n, g)\) is isometric to \(M^n_\lambda/A\), where \(\lambda \in (\mathbb{R}\setminus\{0\})^{n-2}\) and \(A\) is a discrete subgroup of the centralizer \(Z_{I(M_\lambda^n)}(G(M_\lambda^n))\) of the transvection group \(G(M_\lambda^n)\) in the isometry group \(I(M_\lambda^n)\) of \(M_\lambda^n\).

For the centralizer \(Z_\lambda := Z_{I(M_\lambda^n)}(G(M_\lambda^n))\) it is known:

**Theorem 3** ([CK78])

Let \(\lambda = (\lambda_1, \ldots, \lambda_{n-2})\) be a tupel of non-zero real numbers.

1. If there is a positive \(\lambda_i\) or if there are two numbers \(\lambda_i, \lambda_j\) such that \(\frac{\lambda_i}{\lambda_j} \notin \mathbb{Q}^2\), then \(Z_\lambda \simeq \mathbb{R}\) and \(\varphi \in Z_\lambda\) if and only if \(\varphi(s,t,x) = (s, t + \alpha, x), \alpha \in \mathbb{R}\).
2. Let $\lambda_i = -k_i^2 < 0$ and $\frac{k_i}{k_j} \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, n-2\}$. Then $\varphi \in Z_{\Delta}$ if and only if
$$\varphi(s, t, x) = (s + \beta, t + \alpha, (-1)^{m_1}x_1, \ldots, (-1)^{m_{n-2}}x_{n-2}),$$
where $\alpha \in \mathbb{R}$, $m_1, \ldots, m_{n-2} \in \mathbb{Z}$ and $\beta = \frac{m_i + \pi}{k_i}$ for all $i = 1, \ldots, n-2$.

Let us denote by $S^1_n(r)$ the pseudo-sphere
$$S^1_n(r) := \left\{ x \in \mathbb{R}^{n+1,1} \mid \langle x, x \rangle_{n+1,1} = -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = r^2 \right\} \subset \mathbb{R}^{n+1,1}$$
and by $H^1_n(r)$ the pseudo-hyperbolic space
$$H^1_n(r) := \left\{ x \in \mathbb{R}^{n+1,2} \mid \langle x, x \rangle_{n+1,2} = -x_1^2 - x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = -r^2 \right\} \subset \mathbb{R}^{n+1,2}$$
with the Lorentzian metrics induced by $\langle \cdot, \cdot \rangle_{n+1,1}$ and $\langle \cdot, \cdot \rangle_{n+1,2}$, respectively.

**Theorem 4** (CLP+91, Wol84)

Let $(M^n, g)$ be an indecomposable semi-simple Lorentzian symmetric space of dimension $n \geq 3$. Then $(M^n, g)$ has constant sectional curvature $k \neq 0$. Therefore, it is isometric to $S^1_n(r)/\{\pm 1\}$ or $S^1_n(r)$, $(k = \frac{1}{r^2} > 0)$, or to a Lorentzian covering of $H^1_n(r)/\{\pm 1\}$, $(k = -\frac{1}{r^2} < 0)$.

### 3 Spinor representation

For concrete calculations we will use the following realization of the spinor representation. Let $\text{Cliff}_{n,k}$ be the Clifford algebra of $(\mathbb{R}^n, -\langle \cdot, \cdot \rangle_k)$, where $\langle \cdot, \cdot \rangle_k$ is the scalar product $\langle x, y \rangle_k := -x_1y_1 - \ldots - x_ky_k + x_{k+1}y_{k+1} + \ldots + x_ny_n$. For the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ one has the following relations in $\text{Cliff}_{n,k}$: $e_i \cdot e_j + e_j \cdot e_i = -2\varepsilon_{ij}\delta_{ij}$, where $\varepsilon_j = \begin{cases} -1 & j \leq k \\ 1 & j > k \end{cases}$. Denote $\tau_j = \begin{cases} i & j \leq k \\ 1 & j > k \end{cases}$ and $U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

If $n = 2m$ is even, we have an isomorphism
$$\phi_{2m,k} : \text{Cliff}_{2m,k} \overset{\sim}{\longrightarrow} M(2^m; \mathbb{C})$$
given by the Kronecker product
\begin{align*}
\phi_{2m,k}(e_{2j-1}) &= \tau_{2j-1} E \otimes \ldots \otimes E \otimes U \otimes T \otimes \ldots \otimes T \\
\phi_{2m,k}(e_{2j}) &= \tau_{2j} E \otimes \ldots \otimes E \otimes V \otimes \underbrace{T \otimes \ldots \otimes T}_{j-1}.
\end{align*}

If $n = 2m + 1$ is odd and $k < n$, we have the isomorphism
$$\phi_{2m+1,k} : \text{Cliff}_{2m+1,k} \overset{\sim}{\longrightarrow} M(2^m; \mathbb{C}) \oplus M(2^m; \mathbb{C})$$
given by
\[
\Phi_{2m+1,k}(e_j) = (\Phi_{2m,k}(e_j), \Phi_{2m,k}(e_j)), \quad j = 1, \ldots, 2m
\]
\[
\Phi_{2m+1,k}(e_n) = \tau_n(i T \otimes \cdots \otimes T, -i T \otimes \cdots \otimes T).
\]
(2)

Let Spin\((n, k)\) \(\subset\) Cliff\(_{n,k}\) be the spin group. The spinor representation is given by
\[
\kappa_{n,k} = \hat{\phi}_{n,k}|_{\text{Spin}(n,k)} : \text{Spin}(n,k) \rightarrow \text{GL}(\mathbb{C}^{2^m}),
\]
where \(\hat{\phi}_{2m,k} = \phi_{2m,k}\) and \(\hat{\phi}_{2m+1,k} = \text{pr}_1 \circ \phi_{2m+1,k}\). We denote this representation by \(\Delta_{n,k}\). If \(n = 2m\), \(\Delta_{2m,k}\) splits into the sum \(\Delta_{2m,k} = \Delta^+_m \oplus \Delta^-_{2m,k}\), where \(\Delta^\pm_{2m,k}\) are the eigenspaces of the endomorphism \(\phi_{2m,k}(e_1 \cdots e_{2m})\) to the eigenvalue \(\pm i^{m+k}\).

Let us denote by \(u(\delta) \in \mathbb{C}^2\) the vector \(u(\delta) = \frac{1}{\sqrt{2}}(1 - \delta i)\), \(\delta = \pm 1\), and let
\[
u(\delta_1, \ldots, \delta_m) = u(\delta_1) \otimes \cdots \otimes u(\delta_m) \quad \delta_j = \pm 1.
\]
(3)

Then \((u(\delta_1, \ldots, \delta_m)| \prod_{j=1}^m \delta_j = \pm 1)\) is an orthonormal basis of \(\Delta^\pm_{2m,k}\) with respect to the standard scalar product of \(\mathbb{C}^{2^m}\).

4 General properties of twistor spinors

In this section we recall some properties of twistor spinors which we will need in the following calculations. For proofs see [BFGK91]. Let \((M^n, g)\) be an oriented semi-Riemannian spin manifold of dimension \(n \geq 3\). We denote by \(S\) the spinor bundle of \((M^n, g)\), by \(\nabla^S : \Gamma(S) \rightarrow \Gamma(T^*M \otimes S)\) the spinor derivative given by the Levi-Civita connection of \((M^n, g)\) and by \(D : \Gamma(S) \rightarrow \Gamma(S)\) the Dirac operator on \(S\). Let \(p : TM \otimes S \rightarrow TM \otimes S\) be the projection onto the kernel of the Clifford multiplication \(\mu\). The twistor operator
\[
D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu)
\]
is locally given by
\[
D\varphi = \sum_{j=1}^n \varepsilon_j s_j \otimes (\nabla_{s_j}^S \varphi + \frac{1}{n} s_j \cdot D\varphi),
\]
where \((s_1, \ldots, s_n)\) is a local orthonormal basis and \(\varepsilon_j = g(s_j, s_j) = \pm 1\). A spinor field \(\varphi \in \Gamma(S)\) is called twistor spinor if \(D\varphi = 0\).

**Proposition 1** For a spinor field \(\varphi \in \Gamma(S)\) the following conditions are equivalent:
1. \(\varphi\) is a twistor spinor.
2. \(\varphi\) satisfies the so-called twistor equation
\[
\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0
\]
(4)
3. There exists a spinor field $\psi \in \Gamma(S)$ such that
$$\psi = g(X, X)X \cdot \nabla^S_X \varphi$$
for all vector fields $X$ with $|g(X, X)| = 1$.

The dimension of the space $\mathcal{T}(M^n, g)$ of all twistor spinors is conformally invariant and bounded by
$$\dim \mathcal{T}(M^n, g) \leq 2 \cdot 2^{[\frac{n}{2}]}.$$
If $(M^n, g)$ is simply connected and conformally flat, $\dim \mathcal{T}(M^n, g) = 2 \cdot 2^{[\frac{n}{2}]}$.

In particular, the twistor spinors on the semi-Euclidean space $\mathbb{R}^{n,k} := (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{n,k})$ are
the functions
$$\mathcal{T}(\mathbb{R}^{n,k}) = \{ \varphi \in C^\infty(\mathbb{R}^{n,k}, \Delta_{n,k}) \mid \varphi(x) = u + x \cdot v; ~ u, v \in \Delta_{n,k} \}.$$ Let $R$ be the scalar curvature and $\text{Ric}$ the Ricci curvature of $(M^n, g)$.

$K : TM \to TM$ denotes the $(1,1)$-Schouten tensor of $(M^n, g)$
$$K(X) = \frac{1}{n-2} \left( \frac{R}{2(n-1)}X - \text{Ric}(X) \right).$$
Furthermore, let $W$ be the $(4,0)$-Weyl tensor of $(M, g)$ and let us denote by the same symbol the corresponding $(2,2)$-tensor field $W : \Lambda^2 M \to \Lambda^2 M$. Then we have

**Proposition 2** Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then
$$\nabla^S_X D\varphi = \frac{n}{2} K(X) \cdot \varphi,$$
$$W(\eta) \cdot \varphi = 0,$$
for all vector fields $X$ and 2-forms $\eta$.

Finally, we recall two possibilities to obtain new manifolds with twistor spinors from a given one.

Let $(\tilde{M}^n, \tilde{g})$ be a simply connected parallelizable semi-Riemannian manifold and let $A \subset I(\tilde{M}, \tilde{g})$ denote a discrete subgroup of orientation preserving isometries of $(\tilde{M}, \tilde{g})$.

We trivialize the spin structure of $(\tilde{M}, \tilde{g})$ with respect to a fixed global orthonormal basis field $a = (a_1, \ldots, a_n)$. For $\gamma \in A$ we denote by $\Gamma(x) \in SO(n, k)$ the matrix of $d\gamma_x$ with respect to $a(x)$ and $a(\gamma(x))$. Then there are two lifts $\tilde{\Gamma}^\pm$ of $\Gamma$ into $\text{Spin}(n, k)$
Let $\mathcal{E}(A)$ be the set of all left actions of $A$ on $\tilde{M} \times Spin(n,k)$ such that 

$$\varepsilon(\gamma)(x,a) = (\gamma(x), \varepsilon(\gamma,x) \cdot a) \quad \text{and} \quad \varepsilon(\gamma,x) = \tilde{\Gamma}(x)^{\pm}.$$ 

This set of left actions $\mathcal{E}(A)$ corresponds to the set of spin structures of the oriented semi-Riemannian manifold $M = \tilde{M}/A$. The spinor bundle on $M$ corresponding to the spin structure $\varepsilon \in \mathcal{E}(A)$ is given by 

$$S_{\varepsilon} = \left(\tilde{M} \times \Delta_{n,k}\right)/\varepsilon$$ 

where $\varepsilon(\gamma)(x,v) = (\gamma(x), \varepsilon(\gamma,x) \cdot v)$ for all $\gamma \in A$, $(x,v) \in \tilde{M} \times \Delta_{n,k}$. Hence, the spinor fields on $M$ corresponding to $\varepsilon \in \mathcal{E}(A)$ are given by the $\varepsilon$-invariant functions 

$$\Gamma(S_{\varepsilon}) = C^\infty(\tilde{M}, \Delta_{n,k})^{\varepsilon} := \left\{ \varphi \in C^\infty(\tilde{M}, \Delta_{n,k}) \mid \varphi(\gamma(x)) = \varepsilon(\gamma,x) \cdot \varphi(x) \right\},$$ 

and for the twistor spinors on $M$ with the spin structure $\varepsilon$ we have 

**Proposition 3** The twistor spinors on $M = \tilde{M}/A$ with respect to the spin structure $\varepsilon \in \mathcal{E}(A)$ are given by 

$$T((M,g),\varepsilon) = \{ \varphi \in T(\tilde{M},\tilde{g}) \mid \varphi \text{ ist } \varepsilon\text{-invariant} \}.$$ 

Let $(M^{n+1},g)$ be a semi-Riemannian spin manifold with spinor bundle $S_M$ and let $F^n \subset M^{n+1}$ be a non-degenerate oriented hypersurface in $M^{n+1}$. We denote by $\eta : F \to TM$ the Gauss map of $F$, $\kappa(\eta) := g(\eta,\eta) = \pm 1$. It is well known, that the spinor bundle $S_F$ of $(F,g|_F)$ with respect to the spin structure on $F$ induced by the embedding, is isomorphic to $S_M|_F$ in case of even dimension $n$ and to $S_M^\pm|_F$ in case of odd dimension $n$. Using this identification the Clifford multiplication and the spinor derivative are expressed by 

$$X \cdot (\varphi|_F) = (X \cdot \varphi)|_F$$

$$\nabla^S_F(X)(\varphi|_F) = \left(\nabla^{S(\pm)}_X \varphi + \frac{1}{2} \kappa(\eta) \nabla^M_X \eta \cdot \eta \cdot \varphi \right)|_F,$$

where $\varphi \in \Gamma(S^{\pm}_M), X \in TF$ and $\varphi|_F$ always means the spinor field in $\Gamma(S_F)$ corresponding to $\varphi$ with respect to the above mentioned isomorphism. 

**Proposition 4** If $F^n \subset M^{n+1}$ is an umbilic hypersurface and $\varphi \in \Gamma(S^{\pm}_M)$ is a twistor spinor on $M$, then $\varphi|_F \in \Gamma(S_F)$ is a twistor spinor on $F$. 

**Proof:** Let $\lambda \in C^\infty(F)$ be the function satisfying $\nabla^M_X \eta = \lambda X$. Then 

$$\nabla^S_F(X)(\varphi|_F) = \left(\nabla^{S(\pm)}_X \varphi + \frac{1}{2} \kappa(\eta) \lambda X \cdot \eta \cdot \varphi \right)|_F.$$ 

If $\varphi \in \Gamma(S^{\pm}_M)$ is a twistor spinor, from Proposition 1, (3), follows 

$$\kappa(X) X \cdot \nabla^S_X(\varphi|_F) = \frac{1}{n+1} D^{(\pm)}_M \varphi.$$
for each \( X \in TF \) with \( \kappa(X) = g(X, X) = \pm 1 \). Hence,

\[
\kappa(X) X \cdot \nabla^SF(\varphi|_F) = \left( \frac{1}{n+1} D^{(\pm)}_M \varphi - \frac{1}{2} \kappa(\eta) \lambda \eta \cdot \varphi \right)|_F.
\]

The right hand side is independent of \( X \in TF \). Therefore, according to Proposition 1, \( \varphi|_F \) is a twistor spinor on \( F \).

\( \square \)

5 Twistor spinors on indecomposable, non-conformally flat Lorentzian symmetric spaces

Let us first consider the simply connected solvable Lorentzian symmetric space \( M^n_\Lambda = (\mathbb{R}^n, g_\Lambda) \), where

\[
(g_\Lambda)_{(s,t,x^i)} = 2 \, ds \, dt + \sum_{j=1}^{n-2} \lambda_j x_j^2 \, ds^2 + \sum_{j=1}^{n-2} dx_j^2,
\]

and \( \Lambda = (\lambda_1,\ldots,\lambda_{n-2}) \), \( \lambda_i \in \mathbb{R} \setminus \{0\} \), \( n \geq 3 \). Let \( \Lambda_0 := -\sum_{j=1}^{n-2} \lambda_j \). We fix the following global orthonormal basis on \( M^n_\Lambda \):

\[
a_0(y) := \frac{\partial}{\partial s}(y) - \frac{1}{2} \left( \sum_{j=1}^{n-2} \lambda_j x_j^2 + 1 \right) \frac{\partial}{\partial t}(y),
\]

\[
a_0(y) := \frac{\partial}{\partial s}(y) - \frac{1}{2} \left( \sum_{j=1}^{n-2} \lambda_j x_j^2 - 1 \right) \frac{\partial}{\partial t}(y),
\]

\[
a_j(y) := \frac{\partial}{\partial x_j}(y) \quad j = 1,\ldots,n-2,
\]

where \( y = (s,t,x_1,\ldots,x_{n-2}) \in M^n_\Lambda \). The vector field \( V(y) := \frac{\partial}{\partial t}(y) \) is isotropic and parallel. The Ricci tensor of \( M^n_\Lambda \) is given by

\[
\text{Ric}(X) = \Lambda_0 \cdot g_\Lambda(X, V) V,
\]

the scalar curvature \( R \) vanishes. Therefore, the Schouten tensor satisfies

\[
K(X) = -\frac{1}{n-2} \Lambda_0 \cdot g_\Lambda(X, V) V. \tag{8}
\]

For the Weyl tensor \( W : \Lambda^2 M_\Lambda \rightarrow \Lambda^2 M_\Lambda \) one has

\[
W(a_\alpha \wedge a_j) = W(a_0 \wedge a_j) = (\lambda_j + \frac{1}{n-2} \Lambda_0) V \wedge a_j, \quad j = 1,\ldots,n-2 \tag{9}
\]

\[
W(a_\alpha \wedge a_\beta) = 0 \quad \text{for all other indices} \ \alpha, \beta,
\]

where \( TM_\Lambda \) is identified with \( T^* M_\Lambda \) using the metric \( g_\Lambda \). In particular, \( M_\Lambda \) is conformally flat if and only if \( \Lambda = (\lambda,\ldots,\lambda) \), \( \lambda \in \mathbb{R} \setminus \{0\} \).

Since \( M_\Lambda \) is simply connected, it has an uniquely determined spin structure. We trivialize this spin structure using the global orthonormal basis \( (a_0, a_0, a_1,\ldots,a_{n-2}) \) and identify the spinor fields with the smooth functions \( C^\infty(M_\Lambda, \Delta_{n,1}) \). The spinor derivative is defined by
\[ \nabla^S_X \varphi = X(\varphi) + \frac{1}{2} \sum_{1 \leq k < l \leq n} \varepsilon_k \varepsilon_l g(\nabla^C_X s_k, s_l) s_k \cdot s_l \cdot \varphi, \]

where \((s_1, \ldots, s_n)\) is a local orthonormal basis and \(\varepsilon_j = g(s_j, s_j) = \pm 1\). This gives for the spinor derivative on \(M_\Delta\)

\[ \nabla^S_X \frac{\partial}{\partial t} \varphi = \frac{\partial}{\partial t} \varphi \quad (10) \]

\[ \nabla^S_X \frac{\partial}{\partial s} \varphi = \frac{\partial}{\partial s} \varphi + \frac{1}{2} \sum_{j=1}^{n-2} \lambda^j x_j a_j \cdot V \cdot \varphi \quad (11) \]

\[ \nabla^S_X \frac{\partial}{\partial x_j} \varphi = \frac{\partial}{\partial x_j} \varphi \quad (12) \]

\[ \nabla^S_{a\bar{s}} \varphi = a_0(\varphi) + \frac{1}{2} \sum_{j=1}^{n-2} \lambda^j x_j a_j \cdot V \cdot \varphi \quad (13) \]

\[ \nabla^S_{a_s} \varphi = a_0(\varphi) + \frac{1}{2} \sum_{j=1}^{n-2} \lambda^j x_j a_j \cdot V \cdot \varphi. \quad (14) \]

The vector space \(\Delta_{n,1}\) is isomorphic to \(\Delta_{n-2,0} \otimes \mathbb{C}^2\). Let us denote by \(\Delta_V\) the subspace

\[ \Delta_V := \Delta_{n-2,0} \otimes \mathfrak{u}(-1) \subset \Delta_{n,1}. \]

Using the formulas (10), (11) and (12) one obtains that a spinor field \(\varphi \in C^\infty(M_\Delta, \Delta_{n,1})\) satisfies \(V \cdot \varphi = 0\) if and only if the image of \(\varphi\) lies in \(\Delta_V\).

**Proposition 5** The space \(\mathcal{P}(M_\Delta)\) of parallel spinors of \(M_\Delta\) is

\[ \mathcal{P}(M_\Delta) = \{ \varphi \in C^\infty(M_\Delta, \Delta_{n,1}) \mid \varphi = \text{constant} \in \Delta_V \}. \]

In particular, \(\dim \mathcal{P}(M_\Delta) = \frac{1}{2} \cdot 2^{\lfloor \frac{n}{2} \rfloor}\).

**Proof:** From (10) - (12) it follows that \(\varphi \in C^\infty(M_\Delta, \Delta_{n,1})\) is parallel if and only if \(\varphi\) depends only on \(s\) and

\[ \frac{\partial \varphi}{\partial s} = -\frac{1}{2} \sum_{j=1}^{n-2} \lambda^j x_j a_j \cdot V \cdot \varphi. \quad (15) \]

Therefore,

\[ 0 = \frac{\partial^2 \varphi}{\partial x_k \partial s} = -\frac{1}{2} \lambda_k a_k \cdot V \cdot \varphi. \]

Since \(\lambda_k \neq 0\) and \(a_k\) is space-like, this yields \(V \cdot \varphi = 0\). Hence, because of (15), \(\varphi\) has to be constant. \(\square\)

**Proposition 6** Let \(M_\Delta\) be non-conformally flat. Then each twistor spinor on \(M_\Delta\) is parallel. In particular,

\[ \dim \mathcal{T}(M_\Delta) = \frac{1}{2} \cdot 2^{\lfloor \frac{n}{2} \rfloor}. \]
Proof: Let $\varphi \in C^\infty(M_\Delta, \Delta_{n,1})$ be a twistor spinor. Then according to (7) of Proposition 2 $W(\eta) \cdot \varphi = 0$ for each 2-form $\eta$. Using (9) we obtain

$$\left(\lambda_j + \frac{1}{n-2} \Lambda_0\right) a_j \cdot V \cdot \varphi = 0 \quad j = 1, \ldots, n - 2.$$  

Since $M_\Delta$ is not conformally flat and $a_j$ is space-like, it follows

$$0 = V \cdot \varphi$$  

(16)

Furthermore, we have

$$\nabla^S_X \varphi = \frac{n}{2} K(X) \cdot \varphi.$$  

Using (8) and (16) we obtain

$$\nabla^S_X \varphi = -\frac{n}{2} \left(\sum_{j=1}^{n-2} \lambda_j x_j^2 - 1\right) \frac{\partial}{\partial s} (\varphi).$$

Therefore, $\varphi$ does not depend on $t$. Moreover, the twistor equation implies that

$$-a_0 \cdot \nabla^S_{\partial_s} \varphi = a_0 \cdot \nabla^S_{\partial_t} \varphi = \frac{1}{n} D\varphi.$$  

Then the formulas (13), (14) and (16) give

$$D\varphi = -n a_{0\gamma} \left(\frac{\partial}{\partial s} - \frac{1}{2} \left(\sum_{j=1}^{n-2} \lambda_j x_j^2 + 1\right) \frac{\partial}{\partial t}\right) (\varphi) = n a_0 \cdot \left(\frac{\partial}{\partial s} - \frac{1}{2} \left(\sum_{j=1}^{n-2} \lambda_j x_j^2 - 1\right) \frac{\partial}{\partial t}\right) (\varphi).$$

Since $\varphi$ does not depend on $t$ we obtain

$$2D\varphi = n(a_0 - a_{0\gamma}) \cdot \frac{\partial}{\partial t}(\varphi) = n V \cdot \frac{\partial}{\partial t}(\varphi) = n \frac{\partial}{\partial t}(V \cdot \varphi) \equiv 0.$$  

Therefore, $\varphi$ is harmonic and the twistor equation implies that $\varphi$ is parallel. \hfill \Box

Now, let $(M^n, g)$ be a non-conformally flat, non-simply connected, indecomposable Lorentzian symmetric space of dimension $n \geq 3$. Then, according to the Theorems 1, 2 and 4, $(M^n, g)$ is isometric to $M^n_{\Delta}/A$, where $A$ is a discrete subgroup of the centralizer $Z_\Delta := Z_I(M_\Delta)(G(M_\Delta))$.

1. **case:** There exist $i \in \{1, \ldots, n - 2\}$ such that $\lambda_i > 0$ or $(i, j)$ such that $\frac{\lambda_j}{\lambda_i} \notin \mathbb{Q}^2$. Then $Z_\Delta \simeq \mathbb{R} = \{\gamma_\alpha \mid \gamma_\alpha(s, t, x) = (s, t + \alpha, x), \alpha \in \mathbb{R}\}$.

Let $\gamma \in A$. With respect to the global basis $(a_{0\gamma}, a_0, a_1, \ldots, a_{n-2})$ the differential $d\gamma_y$ corresponds to the matrix $\Gamma(y) \equiv E \in SO(n, 1)$. Hence, $\Gamma^\pm(y) = \pm 1 \in Spin(n, 1)$. Therefore, we have 2 spin structures on $M = M_\Delta/A$ corresponding to the homomorphisms $\text{Hom}(A; \mathbb{Z}_2)$. If $\varepsilon \in \text{Hom}(A; \mathbb{Z}_2)$ is not trivial, there are no non-trivial
$\varepsilon$-invariant constant spinor fields on $M$. From the Propositions 3, 5 and 6 it follows that the twistor spinors on $M = M_\Delta / A$ are given by

$$\mathcal{T}(M_\Delta / A, \varepsilon) = \begin{cases} \{ \varphi \in C^\infty(M, \Delta_V) \mid \varphi \text{ const} \} & \varepsilon \text{ trivial} \\ \{0\} & \varepsilon \text{ non-trivial.} \end{cases}$$

2. case: Let $\lambda_j = -k_i^2 < 0$ and $k_i / k_j \in \mathbb{Q}$ for all $i, j = 1, \ldots, n - 2$. Then

$$Z_\Delta \simeq \left\{ \gamma_{m, \alpha} \mid \gamma_{m, \alpha}(s, t, x) := (s + \beta, t + \alpha, (-1)^{m_1}x_1, \ldots, (-1)^{m_{n-2}}x_{n-2}); \right. \left. \text{where } \alpha \in \mathbb{R}, m = (m_1, \ldots, m_{n-2}) \in \mathbb{Z}^{n-2}, \beta = \pi \cdot \frac{m_i}{k_i}, i = 1, \ldots, n - 2 \right\} \simeq \mathbb{Z} \oplus \mathbb{R}.$$

A discrete subgroup $A_{m, 0} \subset Z_\Delta$ is generated by $\gamma_{m, 0}$ and $\gamma_{0, \alpha}$. Let us suppose that $\sum_{i=1}^{n-2} m_i$ is even since otherwise $M_\Delta / A_{m, \alpha}$ is not orientable. $(d_{\gamma_{m, \alpha}})_{\gamma}$ corresponds to the matrix

$$\Gamma(y) = \begin{pmatrix} 1 & 0 \\ 1 & (-1)^{m_1} \\ & \ddots \\ & & (-1)^{m_{n-2}} \end{pmatrix}.$$ 

Hence $\tilde{\Gamma}^\pm(y) = \pm e_1^{m_1} \cdots e_{n-2}^{m_{n-2}}$. Let $m_1, \ldots, m_{n-2}$ be the odd elements in the tuple $m$ ($s \in 2\mathbb{Z}$), and let us denote by $\omega_m \in Spin(n-2, 0) \subset Spin(n, 1)$ the element

$$\omega_m = e_{i_1} \cdots e_{i_s}.$$ 

Then because of $\omega_m^2 = (-1)^{s/2}$, $\omega_m$ is an involution on $\Delta_{n-2, 0}$ if $s \equiv 0(4)$ and an almost complex structure if $s \equiv 2(4)$. The eigenspaces of $\omega_m$ to the eigenvalues $\pm 1$ and $\pm i$, respectively, have the same dimension (see formulas (3) and (2)).

The manifold $M_\Delta / A_{0, \alpha}, \alpha \neq 0$, has 2 spin structures and the twistor spinors are given as in case 1. $M_\Delta / A_{m, 0}, m \neq 0$, has 2 spin structures, described by the homomorphisms $\varepsilon_{\pm} \in \text{Hom}(A_{m, 0}, Spin(n-2, 0))$ given by $\varepsilon_{\pm}(\gamma_{m, 0}) = \pm \omega_m$. Then, according to the Propositions 3, 5 and 6 the twistor spinors on $M = M_\Delta / A_{m, 0}$ are

$$\mathcal{T}(M_\Delta / A_{m, 0}, \varepsilon_{\pm}) = \left\{ \varphi \in C^\infty(M_\Delta, \Delta_{n, 1}) \mid \varphi(y) \equiv v \otimes u(-1); \right. \left. \text{where } v \in \Delta_{n-2, 0} \text{ and } \omega_m \cdot v = \pm v \right\}.$$ 

Hence,

$$\dim \mathcal{T}(M_\Delta / A_{m, 0}, \varepsilon_{\pm}) = \begin{cases} 0 & \text{if } s \equiv 2(4) \\ \frac{1}{4} \cdot 2\left[\frac{s}{2}\right] & \text{if } s \equiv 0(4). \end{cases}$$

The manifold $M_\Delta / A_{m, \alpha}, m \neq 0, \alpha \neq 0$, has 4 spin structures corresponding to the homomorphisms $\varepsilon \in \text{Hom}(A_{m, \alpha}; Spin(n, 1))$ given by $\varepsilon(\gamma_{m, 0}) = \pm \omega_m, \varepsilon(\gamma_{0, \alpha}) = \pm 1$. 

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Hence,

\[ T(M_\Lambda/A_{m,0}, \varepsilon) = \begin{cases} 
\{0\} & \varepsilon(\gamma_{0,\alpha}) = -1 \\
T(M_\Lambda/A_{m,0}, \varepsilon) & \varepsilon(\gamma_{0,\alpha}) = 1 \\
\varepsilon(\gamma_{m,0}) = \pm \omega_m. 
\end{cases} \]

Summing up, we have

**Proposition 7** Let \((M^n, g)\) be an indecomposable, non-conformally flat Lorentzian symmetric spin manifold of dimension \(n \geq 3\). Then each twistor spinor is parallel and the dimension of the space of twistor spinors is

\[ \dim T(M^n, g) = q \cdot 2^{\lfloor \frac{n}{2} \rfloor}, \]

where \(q = 0, \frac{1}{4}, \) or \(\frac{1}{2}\), depending on the fundamental group \(\pi_1(M)\) and on the spin structure.

6 **Twistor spinors on indecomposable conformally flat Lorentzian symmetric spaces of non-constant sectional curvature**

According to the Theorems 1, 2 and 4 there are two isometry classes of indecomposable, conformally flat, simply connected Lorentzian symmetric spaces of dimension \(n \geq 3\) and non-constant sectional curvature, namely

\[ M_\pm^n := (\mathbb{R}^n, g_\pm), \]

where

\[ (g_\pm)(s,t,x) = 2ds\,dt \pm ||x||^2ds^2 + \sum_{j=1}^{n-2} dx_j^2. \]

Knowing that \(\dim T(M_\pm^n) = 2 \cdot 2^{\lfloor \frac{n}{2} \rfloor}\) we want to describe the twistor spinors explicitly. Let \(w_1, w_2, w_3, w_4 \in \Delta_{n-2,0}\) and let us denote by \(\varphi_{w_1, w_2, w_3, w_4} \in C^\infty(M_\pm^n, \Delta_{n,1})\) the following smooth functions

\[ \varphi_{w_1, w_2, w_3, w_4}(s, t, x) := \left( \mp f'(s)w_4 - f(s)w_4 + x \cdot w_1 \right) \otimes u(1) \]

\[ + \left[ -2w_1t + w_2 + x \cdot (f(s)w_3 + f'(s)w_4) \right] \otimes u(-1), \]

where

\[ f(s) = \begin{cases} 
\sinh(s) & \text{for } M_+^n \\
\sin(s) & \text{for } M_-^n. 
\end{cases} \]

**Proposition 8** The twistor spinors on \(M_\pm^n\) are

\[ T(M_\pm^n) = \{ \varphi_{w_1, w_2, w_3, w_4} \mid w_1, w_2, w_3, w_4 \in \Delta_{n-2,0} \}. \]
**Proof:** We use the identification

\[ \Delta_{n,1} \simeq \Delta_{n-2,0} \otimes \Delta_{2,1} \quad \overset{\sim}{\rightarrow} \quad \Delta_{n-2,0} \oplus \Delta_{n-2,0} \]

\[ \varphi = \varphi_1 \otimes u(1) + \varphi_2 \otimes u(-1) \quad \overset{\sim}{\rightarrow} \quad (\varphi_1, \varphi_2). \]

Then according to (\ref{eq:1}) and (\ref{eq:2}) the Clifford multiplication corresponds to

\[ X \cdot \varphi = (-X \cdot \varphi_1, X \cdot \varphi_2) \quad \text{if} \quad X \in \text{span} \{a_1, \ldots, a_{n-2}\}, \]

\[ a_\mp \cdot \varphi = (-\varphi_2, -\varphi_1) \quad \text{for} \quad a_0 \cdot \varphi = (-\varphi_2, \varphi_1) \quad \text{and} \quad V \cdot \varphi = (0, 2\varphi_1). \]

For the spinor derivative we obtain

\[ \nabla^S a_\mp \varphi = (a_\mp(\varphi_1), a_\mp(\varphi_2) \pm x \cdot \varphi_1), \quad \nabla^S_{a_0} \varphi = (a_0(\varphi_1), a_0(\varphi_2) \pm x \cdot \varphi_1), \]

\[ \nabla^S_{a_k} \varphi = (a_k(\varphi_1), a_k(\varphi_2)) \quad k = 1, \ldots, n-2. \]

Let \( \varphi = (\varphi_1, \varphi_2) \) be a twistor spinor on \( M^n_\pm \). Then, according to Proposition 1, there exists a spinor field \( \psi = (\psi_1, \psi_2) \) on \( M^n_\pm \) such that

\[ (\psi_1, \psi_2) = a_k \cdot \nabla^S_{a_k} \varphi = (-a_k \cdot a_k(\varphi_1), a_k \cdot a_k(\varphi_2)) \quad (17) \]

for each \( k = 1, 2, \ldots, n-2 \). Therefore, \( \varphi_1(s, t, \cdot) \) and \( \varphi_2(s, t, \cdot) \) are twistor spinors on the Euclidean space \( \mathbb{R}^{n-2} \). Hence,

\[ \varphi_i(s, t, x) = u_i(s, t) + x \cdot v_i(s, t) \quad , \quad i = 1, 2, \]

where \( u_i, v_i : \mathbb{R}^2 \rightarrow \Delta_{n-2,0} \). From (\ref{eq:17}) follows

\[ \psi_1(s, t, x) = v_1(s, t) \quad \text{and} \quad \psi_2(s, t, x) = -v_2(s, t). \]

Furthermore, \( \psi = (\psi_1, \psi_2) \) satisfies

\[ (\psi_1, \psi_2) = -a_\mp \cdot \nabla^S a_\mp \varphi = a_0 \cdot \nabla^S_{a_0} \varphi. \]

Therefore,

\[ v_1 = \left( \frac{\partial}{\partial x} - \frac{1}{2} (\pm ||x||^2 + 1) \frac{\partial}{\partial t} \right) (u_2 + x \cdot v_2) + x \cdot (u_1 + x \cdot v_1) \quad (18) \]

\[ v_1 = -\left( \frac{\partial}{\partial x} - \frac{1}{2} (\pm ||x||^2 - 1) \frac{\partial}{\partial t} \right) (u_2 + x \cdot v_2) + x \cdot (u_1 + x \cdot v_1) \quad (19) \]

\[ -v_2 = \left( \frac{\partial}{\partial x} - \frac{1}{2} (\pm ||x||^2 + 1) \frac{\partial}{\partial t} \right) (u_1 + x \cdot v_1) \quad (20) \]

\[ -v_2 = \left( \frac{\partial}{\partial x} - \frac{1}{2} (\pm ||x||^2 - 1) \frac{\partial}{\partial t} \right) (u_1 + x \cdot v_1). \quad (21) \]

(\ref{eq:18}) + (\ref{eq:19}) gives \( 2v_1 = -\frac{\partial}{\partial t}v_2 - x \cdot \frac{\partial}{\partial x}v_2 \) and after differentiation with respect to \( x_k \)

\( 0 = -a_k \cdot \frac{\partial}{\partial t}v_2 \). Hence,

\[ \frac{\partial}{\partial t}v_2 = 0 \quad \text{and} \quad v_1 = \frac{1}{2} \frac{\partial}{\partial x} v_2. \quad (22) \]

Using this, we obtain from (\ref{eq:19}) - (\ref{eq:18})

\[ 0 = \mp x \cdot u_1 - \frac{\partial}{\partial x} u_2 - x \cdot \frac{\partial}{\partial x} v_2. \quad (23) \]
Differentiation and (22) show that
\[ u_1 = \pm \frac{\partial}{\partial s} v_2 \quad \text{and} \quad \frac{\partial}{\partial s} u_1 = 0. \] (24)
Inserting this in (23) and using (22) we obtain
\[ \frac{\partial}{\partial s} u_2 = 0 \quad \text{and} \quad \frac{\partial}{\partial s} v_1 = 0. \] (25)
Hence, \( u_2 = u_2(t), v_1 = v_1(t), v_2 = v_2(s) \) and \( u_1 = u_1(s) \). (21) - (20) shows
\[ 0 = x \cdot \frac{\partial}{\partial t} v_1 + \frac{\partial}{\partial t} u_1 = x \cdot v_1'(t). \]
Therefore, we have \( v_1(t) \equiv w_1 \in \Delta_{n-2} \) and, because of (22), \( u_2(t) = -2tw_1 + w_2 \).
(21) + (20) yields
\[ -2v_2 = 2u_1'(s) \mp ||x||^2 x \cdot v_1'(t) = 2u_1'(s), \]
so that, regarding (24), \( v_2(s) = \pm v''_2(s) \). Therefore,
\[ v_2(s) = f(s)w_3 + f'(s)w_4 \quad \text{and} \quad u_1(s) = \mp f'(s)w_3 - f(s)w_4, \]
where
\[ f(s) = \begin{cases} \sinh(s) & \text{for } M^n_+ \\ \sin(s) & \text{for } M^n_- \end{cases} \]
Consequently, the twistor spinor \( \varphi \) is of the form \( \varphi = \varphi_{w_1,w_2,w_3,w_4} \).

Now, let \((M^n,g)\) be an indecomposable, conformally flat non-simply connected Lorentzian symmetric space of dimension \( n \geq 3 \) and non-constant sectional curvature. Then \((M^n,g)\) is isometric to \( M^n_+/A \) or to \( M^n_-/A \), where \( A \) is a discrete subgroup of
\[ Z_+ := Z_I(M_+)(G(M_+)) = \{ \varphi_\alpha \mid \varphi_\alpha(s,t,x) = (s,t + \alpha, x); \; \alpha \in \mathbb{R} \} \]
in the first and of
\[ Z_- := Z_I(M_-)(G(M_-)) = \left\{ \varphi_{m,\alpha} \mid \varphi_{m,\alpha}(s,t,x) = (s + m\pi, t + \alpha, (-1)^m x); \; m \in \mathbb{Z}, \; \alpha \in \mathbb{R} \right\} \]
in the second case.

1. case: \( M = M^n_+/A_\alpha, \; A_\alpha = \mathbb{Z}\varphi_\alpha \). Then there are 2 spin structures corresponding to \( \varepsilon \in \text{Hom}(A_\alpha,\mathbb{Z}_2) \). The Propositions 3 and 8 show
\[ \mathcal{T}(M,\varepsilon) = \begin{cases} \{ \varphi_{0,w_2,w_3,w_4} \mid w_2, w_3, w_4 \in \Delta_{n-2,0} \} & \varepsilon = 1 \\ \{0\} & \varepsilon \neq 1. \end{cases} \]
2. case: \( M = M^n_-/A_{m,\alpha}, \; A_{m,\alpha} = \langle \varphi_{m,0}, \varphi_{0,\alpha} \rangle \). If \( m \) is even and \( \alpha \neq 0 \), we have the same result as in case 1, since \( f(s) = \sin(s) \) is \( 2\pi \mathbb{Z} \)-invariant. If \( m \) is odd, \( M \)
is orientable only if $n$ is even. Then $M^{2k}$ has 2 spin structures if $\alpha = 0$ and 4 spin structures if $\alpha \neq 0$. The Propositions 3 and 8 show

\[
T(M^2_{\alpha}/A_{m,0}, \varepsilon) = \begin{cases} 
\{0\} & n = 2k \equiv 0(4) \\
\{\varphi_{w_1,w_2,w_3,w_4} \mid w_1, w_2 \in \Delta^\pm_{n-2,0}, w_3, w_4 \in \Delta^\mp_{n-2,0}\} & n = 2k \equiv 2(4) \\
\varepsilon(\varphi_{0,\alpha}) = -1 & \text{or } n = 2k \equiv 0(4)
\end{cases}
\]

\[
T(M^2_{\alpha}/A_{m,\alpha}, \varepsilon) = \begin{cases} 
\{0\} & \varepsilon(\varphi_{0,\alpha}) = -1 \\
\{\varphi_{0,w_2,w_3,w_4} \mid w_2 \in \Delta^\pm_{n-2,0}, w_3, w_4 \in \Delta^\mp_{n-2,0}\} & n = 2k \equiv 2(4) \\
\varepsilon(\varphi_{m,0}) = \pm e_1 \cdots e_{n-2} & \varepsilon(\varphi_{m,0}) = \pm e_1 \cdots e_{n-2}.
\end{cases}
\]

Summing up, we have in particular:

**Proposition 9** Let $(M^n, g)$ be an indecomposable, conformally flat Lorentzian spin manifold $(M^n, g)$ of non-constant sectional curvature and dimension $n \geq 3$. Then the dimension of the space of twistor spinors is

\[
\dim T(M^n, g) = q \cdot 2^{[\frac{n}{4}]},
\]

where $q = 0, \frac{3}{4}, 1, \frac{3}{2}$ or 2, depending on the fundamental group $\pi_1(M)$ and on the spin structure.

### 7 Twistor spinors on Lorentzian symmetric spaces of constant sectional curvature

Let $\psi_{u,v} \in C^\infty(\mathbb{R}^{n+1,k}, \Delta_{n+1,k})$ denote the twistor spinors on the pseudo-Euclidean space $\mathbb{R}^{n+1,k}$

\[
\psi_{u,v}(x) := u + x \cdot v, \quad u, v \in \Delta_{n+1,k}.
\]

The pseudo-sphere $S^n_1(r) \subset \mathbb{R}^{n+1,1}$ and the pseudo-hyperbolic space $H^n_1(r)$ are umbilic hypersurfaces. Using the identification of the spinor bundle of the hypersurface with that of the external space (see section 4) we obtain from Proposition 4:

**Proposition 10** The twistor spinors on $S^n_1(r)$ and $H^n_1(r)$ with the induced spin structure are

\[
T(S^n_1(r)) = \begin{cases} 
\psi_{u,v}|S^n_1(r) \mid u, v \in \Delta_{n+1,1}, \quad \begin{array}{ll}
\text{if } n \equiv 0(2) \\
\text{if } n \equiv 1(2)
\end{array}
\end{cases}
\]

\[
T(H^n_1(r)) = \begin{cases} 
\psi_{u,v}|H^n_1(r) \mid u, v \in \Delta_{n+1,2}, \quad \begin{array}{ll}
\text{if } n \equiv 0(2) \\
\text{if } n \equiv 1(2)
\end{array}
\end{cases}
\]

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The Lorentzian manifold $S^1_n(r)/\{\pm I\}$ is orientable if and only if $n$ is odd, hence let $n$ be odd. The volume form $\omega_{n+1,1} = e_1 \cdot \ldots \cdot e_{n+1} \in Spin(n+1,1)$ satisfies $\omega^2_{n+1,1} = (-1)^{n+1}$. Therefore, $S^1_n(r)/\{\pm I\}$ has no spin structure, if $n = 3(4)$ and 2 spin structures, if $n \equiv 1(4)$. The spinor fields to these different spin structures can be identified with the invariant functions $C^\infty(S^1_n(r), \Delta_{n+1,1}^+)^{\pm}$, where $\varepsilon_{\pm}$ is the $Z_2$-action given by

$$(\varepsilon_{\pm}(1)\varphi)(x) = \pm \omega_{n+1,1} \cdot \varphi(-x) = \pm \varphi(-x).$$

From the Propositions 3 and 10 follows

$$T(S^{4k+1}_1(r)/\{\pm I\}, \varepsilon) = \begin{cases} \{\psi_{u,+}|_{S^{4k+1}_1} \mid u^+ \in \Delta^+_n \} & \text{if } \varepsilon = \varepsilon_+ \\ \{\psi_{0,v-}|_{S^{4k+1}_1} \mid v^- \in \Delta^-_{n+2} \} & \text{if } \varepsilon = \varepsilon_-. \end{cases}$$

Now, let us consider a Lorentzian symmetric space $M^n$ of constant negative sectional curvature. Then $M^n$ is isometric to a Lorentzian covering of $H^1_n(r)/\{\pm I\}$. Let

$$\tilde{\pi} : \tilde{H}^1_n(r) = \mathbb{R} \times \mathbb{R}^{n-1} \longrightarrow H^1_n(r) \subset \mathbb{R}^{2,2} \times \mathbb{R}^{n-1}$$

be the universal Lorentzian covering of $H^1_n(r)$. Let $\tilde{Q}$ denote the reduction of the trivial spin structure $Q$ of $\mathbb{R}^{n+1,2}$ to the subgroup $Spin(n,1)$ given by the Gaus map. Then $\tilde{Q} := \tilde{\pi}^* \tilde{Q}$ is the uniquely determined spin structure of $\tilde{H}^1_n(r)$. The spinor fields of $\tilde{H}^1_n(r)$ can be identified with the smooth functions $C^\infty(\tilde{H}^1_n(r), \Delta_{n+1,2}^{(\pm)})$, the twistor spinors are given by

$$T(\tilde{H}^1_n(r)) = \{\tilde{\psi}_{u,v} := \psi_{u,v}|_{H^1_n(r)} \circ \tilde{\pi} \mid \psi_{u,v}|_{H^1_n(r)} \in T(\tilde{H}^1_n(r))\}.$$ 

Let

$$\pi_m : N^m_n \longrightarrow H^1_n(r)$$

$$(\sqrt{r} \cos t, \sqrt{r} \sin t, x) \longmapsto (\sqrt{r} \cos (mt), \sqrt{r} \sin (mt), x),$$

be the Lorentzian covering of $H^1_n(r)$ with respect to $m \mathbb{Z} \subset \pi_1(H^1_n(r)) = \mathbb{Z}, \ m = 1, 2, 3, \ldots$. The manifold $N^m_n$ has 2 spin structures. The corresponding spinor fields are given by the $\varepsilon_+^m$-invariant functions $C^\infty(\tilde{H}^1_n(r), \Delta_{n+1,2}^{(\pm)} e_+^m)$, where $e_+^m$ is the $m \mathbb{Z}$-action

$$(\varepsilon_{\pm}^m(mz, \varphi))(t, x) = (\pm 1)^2 \varphi(t + 2mz, x).$$

Therefore, the twistor spinors on $N^m_n$ are

$$T(N^m_n, \varepsilon) = \begin{cases} \{\psi_{u,v}|_{H^1_n(r)} \circ \pi_m \mid \psi_{u,v}|_{H^1_n(r)} \in T(\tilde{H}^1_n(r))\} & \text{if } \varepsilon = \varepsilon_+^m \\ \{0\} & \text{if } \varepsilon = \varepsilon_-^m. \end{cases}$$

Finally, let us consider the manifolds $N^m_n/\{\pm I\}$. Since $N^m_n/\{\pm I\}$ is orientable if and only if $n$ is odd, let $n$ be odd. For the volume form $\omega_{n+1,1} = e_1 \cdot \ldots \cdot e_{n+1} \in Spin(n+1,2)$ we have $\omega^2_{n+1,2} = (-1)^{n+1} + 2$. Therefore, there is no spin structure on $N^m_n/\{\pm I\}$ if
$n \equiv 1(4)$ and there are 4 spin structures in case $n \equiv 3(4)$. The spinor fields are given by the functions $C^\infty(\widetilde{H}^1_1(r), \Delta_{n+1,2}^{\mp}(\epsilon_m^\pm, \delta^\pm))$, invariant under the $m\mathbb{Z}$-action $\epsilon_m^\pm$ and the $\mathbb{Z}_2$-action $\delta^\pm$ given by
\[
(\delta^\pm(-1)\varphi)(t, x) = \pm\omega_{n+1,2} \cdot \varphi(t + m\pi, -x) \\
= \pm\varphi(t + m\pi, -x).
\]

Then, the twistor spinors are
\[
\mathcal{T}(N^{4k+3}_m/(\pm I), \epsilon) = \begin{cases}
\{0\} & \epsilon = (\epsilon_m^-, \delta^\pm) \text{ or } \epsilon = (\epsilon_m^+, \delta^-), \ m \equiv 0(2) \\
\{\tilde{\psi}_{u^+,0} | u^+ \in \Delta^{+}_{4k+4,2}\} & \epsilon = (\epsilon_m^+, \delta^+) \\
\{\tilde{\psi}_{0,v^-} | v^- \in \Delta^{+}_{4k+4,2}\} & \epsilon = (\epsilon_m^-, \delta^-), \ m \equiv 1(2).
\end{cases}
\]

Summing up, we have in particular

**Proposition 11** Let $(M^n, g)$ be a Lorentzian symmetric spin manifold of constant sectional curvature $k \neq 0$ and dimension $n \geq 3$, then the dimension of the space of twistor spinors is
\[
\dim \mathcal{T}(M^n, g) = q \cdot 2^{[\frac{n}{2}]},
\]
where $q = 0, 1, \text{ or } 2$ depending on $\pi_1(M)$ and on the spin structure.

**References**

[ACDS97] D.V. Alekseevsky, V. Cortes, C. Devchand, and U. Semmelmann. Killing spinors are Killing vector fields in Riemannian supergeometry. MPI-preprint 97-29, 1997.

[Bau97] H. Baum. Strictly pseudoconvex spin manifolds, Fefferman spaces and Lorentzian twistor spinors. SFB 288-Preprint No. 250, 1997.

[BFGK91] H. Baum, T. Friedrich, R. Grunewald, and I. Kath. *Twistors and Killing Spinors on Riemannian Manifolds*, volume 124 of *Teubner-Texte zur Mathematik*. Teubner-Verlag, Stuttgart/Leipzig, 1991.

[CK78] M. Cahen and Y. Kerbrat. Champs de vecteurs conformes et transformations conformes des spaces Lorentz symetriques. *J.Math. pures et appl.*, 57:99–132, 1978.

[CLP+90] M. Cahen, J. Leroy, M. Parker, F. Tricerri, and L. Vanhecke. Lorentzian manifolds modelled on a Lorentz symmetric space. *J. Geom. Phys.*, 7(4):571–581, 1990.

[CP80] M. Cahen and M. Parker. Pseudo-Riemannian symmetric spaces. *Mem. AMS*, 24(229):1–108, 1980.
[CW70] M. Cahen and N. Wallach. Lorentzian symmetric spaces. *Bull. AMS*, 76(3):585–591, 1970.

[Fri97] Th. Friedrich. *Dirac-Operatoren in der Riemannschen Geometrie*. Vieweg-Verlag, 1997.

[Lew91] J. Lewandowski. Twistor equation in a curved spacetime. *Class. Quant. Grav.*, 8:11–17, 1991.

[NW84] P. Nieuwenhuizen and N.P. Warner. Integrability conditions for Killing spinors. *Comm. Math. Phys.*, 93:277–284, 1984.

[Pen67] R. Penrose. Twistor algebra. *J. Math. Phys.*, 8:345–366, 1967.

[PR86] R. Penrose and W. Rindler. *Spinors and Space-time II*. Cambr. Univ. Press, 1986.

[Wol84] J.A. Wolf. *Spaces of constant curvature*. Publish or Perish, Inc., 1984.

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