EXPLICIT QUATERNIONIC CONTACT STRUCTURES AND METRICS WITH SPECIAL HOLONOMY

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ABSTRACT. We construct explicit left invariant quaternionic contact structures on Lie groups with zero and non-zero torsion, and with non-vanishing quaternionic contact conformal curvature tensor, thus showing the existence of quaternionic contact manifolds not locally quaternionic contact conformal to the quaternionic sphere. We present a left invariant quaternionic contact structure on a seven dimensional non-nilpotent Lie group, and show that this structure is locally quaternionic contact conformal to the flat quaternionic contact structure on the quaternionic Heisenberg group. On the product of a seven dimensional Lie group, equipped with a quaternionic contact structure, with the real line we determine explicit complete quaternionic Kähler metrics and Spin(7)-holonomy metrics which seem to be new. We give explicit complete non-compact eight dimensional almost quaternion hermitian manifolds with closed fundamental four form which are not quaternionic Kähler.

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1. Introduction

It is well known that the sphere at infinity of a non-compact symmetric space $M$ of rank one carries a natural Carnot-Carathéodory structure (see [51, 54]). Quaternionic contact structures were introduced by Biquard in [7, 8], and they appear naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. Such structures are also relevant for the quaternionic contact Yamabe problem which is naturally connected with the extremals and the best constant in an associated Sobolev-type (Folland-Stein [25]) embedding on the quaternionic Heisenberg group [58, 38, 39].

A quaternionic contact structure (qc structure) on a real $(4n + 3)$-dimensional manifold $M$, following Biquard, is a codimension three distribution $H$ locally given as the kernel of a $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$, such that, the three 2-forms $d\eta_i|_H$ are the fundamental forms of a quaternionic structure on $H$. The 1-form $\eta$ is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$, and therefore $H$ is equipped with a conformal class $[g]$ of Riemannian metrics. The transformations preserving a given quaternionic contact structure $\eta$, i.e. $\tilde{\eta} = \mu \Psi \eta$ for a positive smooth function $\mu$ and a non-constant $SO(3)$ matrix $\Psi$ are called quaternionic contact conformal (qc conformal for short) transformations. If the function $\mu$ is constant we have quaternionic contact homothetic (qc-homothetic) transformations. To every metric in the fixed conformal class $[g]$ on $H$ one can associate a linear connection preserving the qc structure, see [7], which we shall call the Biquard connection. The Biquard connection is invariant under qc homothetic transformations but changes in a non-trivial way under qc conformal transformations.

The quaternionic Heisenberg group $G(\mathbb{H})$ with its standard left-invariant qc structure is the unique (up to a $SO(3)$-action) example of a qc structure with flat Biquard connection [40]. The quaternionic Cayley transform is a quaternionic contact conformal equivalence between the standard 3-Sasakian structure on the $(4n + 3)$-dimensional sphere $S^{4n+3}$ minus a point and the flat qc structure on $G(\mathbb{H})$ [38]. All qc structures locally qc conformal to $G(\mathbb{H})$ and $S^{4n+3}$ are characterized in [40] by the vanishing of a tensor invariant, the qc-conformal curvature $W^{qc}$ defined in terms of the curvature and torsion of the Biquard connection.

Examples of qc manifolds arising from quaternionic Kähler deformations are given in [7, 8, 21]. A totally umbilic hypersurface of a quaternionic Kähler or hyperKähler manifold carries such a structure. A basic example is provided by any 3-Sasakian manifold which can be defined as a $(4n+3)$-dimensional Riemannian manifold whose Riemannian cone is a hyperKähler manifold. It was shown in [38] that the torsion endomorphism of the Biquard connection is the obstruction for a given qc-structure to be locally qc homothetic to a 3-Sasakian structure provided the scalar curvature of the Biquard connection is positive. Duchemin shows [21] that for any qc manifold there exists a quaternionic Kähler manifold such that the qc manifold is realized as a hypersurface. However, the embedding in his construction is not isometric and it is difficult to write an explicit expression of the quaternionic Kähler metric except the 3-Sasakian case where the cone metric is hyperKähler.

One purpose of this paper is to find new explicit examples of qc structures. We construct explicit left invariant qc structures on seven dimensional Lie groups with zero and non-zero torsion of the Biquard connection for which the qc-conformal curvature tensor does not vanish, $W^{qc} \neq 0$ thus showing the existence of qc manifolds not locally qc conformal to the quaternionic Heisenberg group $G(\mathbb{H})$. We present a left invariant qc structure with zero torsion of the Biquard connection on a seven dimensional non-nilpotent Lie group $G_1$. Surprisingly, we obtain that this qc structure is locally qc conformal to the flat qc structure on the two-step nilpotent quaternionic Heisenberg group $G(\mathbb{H})$ showing that the qc conformal curvature is zero and applying the main result in [40]. Consequently, this fact yields the existence of a local function $\mu$ such that the qc conformal transformation $\tilde{\eta} = \mu \eta$ preserves the vanishing of the torsion of the Biquard connection.

The second goal of the paper is to construct explicit quaternionic Kähler and Spin(7)-holonomy metrics, i.e metrics with holonomy $Sp(n)Sp(1)$ and $Spin(7)$, respectively, on a product of a qc manifold with a real line. We generalize the notion of a qc structure, namely, we define $Sp(n)Sp(1)$-hypo structures on a $(4n+3)$-dimensional manifold as structures induced on a hypersurface of a quaternionic Kähler manifold. We present explicit complete non-compact quaternionic Kähler metrics and Spin(7)-holonomy metrics on the product of the locally qc conformally flat quaternionic contact structure on the seven dimensional Lie group $G_1$ with the real line some of which seem to be new, see Section 5 and 6.
It is well known that in dimension eight an almost quaternion hermitian structure with closed fundamental four form is not necessarily quaternionic Kähler [36]. This fact was confirmed by Salamon constructing in [55] a compact example of an almost quaternion hermitian manifold with closed fundamental four form which is not Einstein, and therefore it is not a quaternionic Kähler. We give a three parameter family of explicit complete non-compact eight dimensional almost quaternion hermitian manifolds with closed fundamental four form which are not quaternionic Kähler. We also check that these examples are not Einstein as well.

To the best of our knowledge there is not known example of an almost quaternion hermitian eight dimensional manifold with closed fundamental four form which is Einstein but not quaternionic Kähler.

In dimension four, we recover some of the known hyper Kähler metrics known as gravitational instantons (Bianchi-type metrics). Furthermore, we give explicit hyper-symplectic (hyper para Kähler) metrics of signature (2,2). A hyper symplectic structure in dimension four underlines an anti-self-dual Ricci-flat neutral (Bianchi-type metrics). Furthermore, we give explicit hyper-symplectic (hyper para Kähler) metrics of dimensional manifold with closed fundamental four form which is not Einstein, and therefore it is not a quaternionic Kähler. We give a three parameter family of explicit examples of almost quaternion hermitian manifolds with closed fundamental four form which are not quaternionic Kähler. We also check that these examples are not Einstein as well.

Conventions

1.1. Conventions

a) We shall use $X, Y, Z, U$ to denote horizontal vector fields, i.e. $X, Y, Z, U \in H$;

b) \{\epsilon_1, \ldots, \epsilon_{4n}\} denotes a local orthonormal basis of the horizontal space $H$;

c) The summation convention over repeated vectors from the basis \{\epsilon_1, \ldots, \epsilon_{4n}\} is used. For example, the formula $k = P(\epsilon_b, \epsilon_a, \epsilon_a, \epsilon_b)$ means $k = \sum_{a,b=1}^{4n} P(\epsilon_b, \epsilon_a, \epsilon_a, \epsilon_b)$.

d) The triple $(i, j, k)$ denotes any cyclic permutation of $(1, 2, 3)$.

e) $s$ will be any number from the set $\{1, 2, 3\}$.

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2. Quaternionic contact manifolds

In this section we will briefly review the basic notions of quaternionic contact geometry and recall some results from [7], [38] and [40] which we will use in this paper.

2.1. qc structures and the Biquard connection. A quaternionic contact (qc) manifold $(M, g, \mathbb{Q})$ is a $4n + 3$-dimensional manifold $M$ with a codimension three distribution $H$ satisfying

i) $H$ has an $Sp(n)Sp(1)$ structure, that is it is equipped with a Riemannian metric $g$ and a rank-three bundle $\mathbb{Q}$ consisting of $(1,1)$-tensors on $H$ locally generated by three almost complex structures $I_1, I_2, I_3$ on $H$ satisfying the identities of the imaginary unit quaternions, $I_1I_2 = -I_2I_1 = I_3, \ I_1I_2I_3 = -id|_{\mu}$ which are hermitian compatible with the metric $g(I_s, I_s) = g(., .), s = 1, 2, 3$, i.e., $H$ has an almost quaternion hermitian structure.

ii) $H$ is locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds $2g(I_sX, Y) = d\eta_s(X, Y), \ s = 1, 2, 3, \ X, Y \in H$.

A special phenomena here, noted in [7], is that the contact form $\eta$ determines the quaternionic structure and the metric on the horizontal distribution in a unique way.

Correspondingly, given a quaternionic contact manifold we shall denote with $\eta$ any associated contact form. The associated contact form is determined up to an $SO(3)$-action, namely if $\Psi \in SO(3)$ then $\Psi \eta$ is again a contact form satisfying the above compatibility condition (rotating also the almost complex structures). On the other hand, if we consider the conformal class $[g]$ on $H$, the associated contact forms are determined up to a multiplication with a positive conformal factor $\mu$ and an $SO(3)$-action, namely if $\Psi \in SO(3)$ then $\mu \Psi \eta$ is a contact form associated with a metric in the conformal class $[g]$ on $H$. A qc manifold $(M, \bar{g}, \mathbb{Q})$ is called
qc conformal to \((M, g, Q)\) if \(\bar{g} \in [g]\). In that case, if \(\bar{g}\) is a corresponding associated 1-form with complex structures \(I_s\), \(s = 1, 2, 3\), we have \(\bar{g} = \mu \Psi \eta\) for some \(\Psi \in SO(3)\) with smooth functions as entries and a positive function \(\mu\). In particular, starting with a qc manifold \((M, \eta)\) and defining \(\bar{g} = \mu \eta\) we obtain a qc manifold \((M, \bar{g})\) qc conformal to the original one.

If the first Pontryagin class of \(M\) vanishes then the 2-sphere bundle of \(\mathbb{R}^3\)-valued 1-forms is trivial \([1]\), i.e. there is a globally defined form \(\eta\) that annihilates \(H\), we denote the corresponding qc manifold \((M, \eta)\). In this case the 2-sphere of associated almost complex structures is also globally defined on \(H\).

Any endomorphism \(\Psi\) of \(H\) decomposes with respect to the quaternionic structure \((Q, g)\) uniquely into \(Sp(n)\)-invariant parts as follows \(\Psi = \Psi^{+++} + \Psi^{++-} + \Psi^{+-+} + \Psi^{-++}\), where \(\Psi^{+++}\) commutes with all three \(I_i\), \(\Psi^{+-+}\) commutes with \(I_1\) and anti-commutes with the other two and etc. The two \(Sp(n)Sp(1)\)-invariant components are given by \(\Psi_{[3]} = \Psi^{+++}\), \(\Psi_{[-1]} = \Psi^{++-} + \Psi^{-++} + \Psi^{+--}\). Denoting the corresponding \((0,2)\) tensor via \(g\) by the same letter one sees that the \(Sp(n)Sp(1)\)-invariant components are the projections on the eigenspaces of the Casimir operator \(\bar{\tau} = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3\) corresponding, respectively, to the eigenvalues \(3\) and \(-1\), see \([16]\). If \(n = 1\) then the space of symmetric endomorphisms commuting with all \(I_i\) is 1-dimensional, i.e. the \([3]\)-component of any symmetric endomorphism \(\Psi\) on \(H\) is proportional to the identity, \(\Psi_{[3]} = \frac{tr(\Psi)}{|H|} Id|_H\).

On a quaternionic contact manifold there exists a canonical connection defined in \([7]\) when the dimension \((4n + 3) > 7\), and in \([20]\) in the 7-dimensional case.

**Theorem 2.1.** \([7]\) Let \((M, g, Q)\) be a quaternionic contact manifold of dimension \(4n + 3 > 7\) and a fixed metric \(g\) on \(H\) in the conformal class \([g]\). Then there exists a unique connection \(\nabla\) with torsion \(T\) on \(M^{4n+3}\) and a unique supplementary subspace \(V\) to \(H\) in \(TM\), such that:

\begin{enumerate}
  \item \(\nabla\) preserves the decomposition \(H \oplus V\) and the metric \(g\);
  \item for \(X, Y \in H\), one has \(T(X, Y) = -[X, Y]|_V\);
  \item \(\nabla\) preserves the \(Sp(n)Sp(1)\) structure on \(H\), i.e. \(\nabla g = 0, \nabla \sigma \in \Gamma(Q)\) for a section \(\sigma \in \Gamma(Q)\);
  \item for \(\xi \in V\), the endomorphism \(T(\xi, \cdot)|_H\) of \(H\) lies in \((sp(n) \oplus sp(1))\) into \(gl(4n)\);
  \item the connection on \(V\) is induced by the natural identification \(\varphi\) of \(V\) with the subspace \(sp(1)\) of the endomorphisms of \(H\), i.e. \(\nabla \varphi = 0\).
\end{enumerate}

In iv), the inner product \(<,>\) of \(End(H)\) is given by \(<A, B> = \sum_{k=1}^{4n} g(A(e_i), B(e_i))\), for \(A, B \in End(H)\).

We shall call the above connection the Biquard connection. Biquard \([7]\) also described the supplementary subspace \(V\), namely, locally \(V\) is generated by vector fields \(\{\xi_1, \xi_2, \xi_3\}\), such that

\begin{equation}
\eta_k(\xi_k) = \delta_{sk}, \quad (\xi_s \cdot d\eta_k)|_H = 0,
\end{equation}

\begin{equation}
(\xi_s \cdot d\eta_k)|_H = - (\xi_k \cdot d\eta_s)|_H,
\end{equation}

where \(\cdot\) denotes the interior multiplication. The vector fields \(\xi_1, \xi_2, \xi_3\) are called Reeb vector fields.

If the dimension of \(M\) is seven, there might be no vector fields satisfying (2.1). Duchemin shows in \([20]\) that if we assume, in addition, the existence of Reeb vector fields as in (2.1), then Theorem 2.1 holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (2.1).

Notice that equations (2.1) are invariant under the natural \(SO(3)\) action. Using the triple of Reeb vector fields we extend \(g\) to a metric on \(M\) by requiring \(span\{\xi_1, \xi_2, \xi_3\} = V \perp H\) and \(g(\xi_s, \xi_k) = \delta_{sk}\). The extended metric does not depend on the action of \(SO(3)\) on \(V\), but it changes in an obvious manner if \(\eta\) is multiplied by a conformal factor. Clearly, the Biquard connection preserves the extended metric on \(TM, \nabla g = 0\).

The covariant derivative of the qc structure with respect to the Biquard connection and the covariant derivative of the distribution \(V\) are given by

\begin{equation}
\nabla I_i = - \alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \xi_i = - \alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j,
\end{equation}

where the \(sp(1)\)-connection 1-forms \(\alpha_s\) on \(H\) are given by \([7]\)

\begin{equation}
\alpha_i(X) = d\eta_k(\xi_j, X) = - d\eta_j(\xi_k, X), \quad X \in H, \quad \xi_i \in V,
\end{equation}
while the $sp(1)$-connection 1-forms $\alpha_s$ on the vertical space $V$ are calculated in [38]
\begin{equation}
\alpha_i(\xi_s) = d\eta_i(\xi_j, \xi_k) - \delta_{is} \left( \frac{S}{2} + \frac{1}{2} (d\eta_1(\xi_2, \xi_3) + d\eta_2(\xi_3, \xi_1) + d\eta_1(\xi_1, \xi_2)) \right),
\end{equation}
where $s \in \{1, 2, 3\}$ and $S$ is the normalized qc scalar curvature defined below in (2.5). The vanishing of the normalized qc scalar curvature $\frac{S}{2}$ implies the vanishing of the torsion endomorphism of the Biquard connection (see [38]).

The fundamental 2-forms $\omega_i, i = 1, 2, 3$ [7] are defined by $2\omega_i|_H = d\eta_i|_H$, $\xi.\omega_i = 0$, $\xi \in V$. The properties of the Biquard connection are encoded in the properties of the torsion endomorphism $T_\xi = T(\xi, \cdot): H \to H$, $\xi \in V$. Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^1$ into its symmetric part $T_\xi^0$ and skew-symmetric part $b_\xi, T_\xi^0 = T_\xi^0 + b_\xi$, O. Biquard in [7] shows that the torsion $T_\xi$ is completely trace-free, $tr T_\xi = tr T_\xi \circ I = 0$, $I \in Q$, its symmetric part has the properties $T_\xi^0 I_1 = -I_1 T_\xi^0$, $I_2(T_\xi^0)_{1-3-3}^0 = I_1(T_\xi^0)_{2-3-3}^0$, $I_3(T_\xi^0)_{2-3-3}^0 = I_0(T_\xi^0)^{1-3-3}$ and the skew-symmetric part can be represented as $b_\xi = I_u$, where $u$ is a traceless symmetric (1,1)-tensor on $H$ which commutes with $I_1, I_2, I_3$. If $n = 1$ then the tensor $u$ vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T_\xi^0$.

Any 3-Sasakian manifold has zero torsion endomorphism, and the converse is true if in addition the normalized qc scalar curvature $S$ (see (2.5)) is a positive constant [38]. We remark that a $(4n+3)$-dimensional Riemannian manifold $(M, g)$ is called 3-Sasakian if the cone metric $g_c = t^2 g + dt^2$ on $C = M \times \mathbb{R}^+$ is a hyper Kähler metric, namely, it has holonomy contained in $Sp(n + 1)$ [10]. A 3-Sasakian manifold of dimension $(4n + 3)$ is Einstein with positive Riemannian scalar curvature $(4n + 2)(4n + 3)$ [45] and if complete it is compact with a finite fundamental group, due to Mayer’s theorem (see [9] for a nice overview of 3-Sasakian spaces).

2.2. Torsion and curvature. Let $R = [\nabla, \nabla] - \nabla[\nabla]$ be the curvature tensor of $\nabla$ and the dimension is $4n + 3$. We denote the curvature tensor of type (0,4) by the same letter, $R(A, B, C, D) = g(R(A, B)C, D)$, $A, B, C, D \in \Gamma(TM)$. The Ricci 2-forms and the normalized scalar curvature of the Biquard connection, called qc-Ricci forms and normalized qc-scalar curvature, respectively, are defined by
\begin{equation}
4n \rho_s(X, Y) = R(X, Y, e_a, I_s e_a), \quad 8n(n + 2) S = R(e_b, e_a, e_a, e_b).
\end{equation}
The $sp(1)$-part of $R$ is determined by the Ricci 2-forms and the connection 1-forms by
\begin{equation}
R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B), \quad A, B \in \Gamma(TM).
\end{equation}
The structure equations of a qc structure, discovered in [41], read
\begin{equation}
2\omega_i = d\eta_i + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_j + S \eta_j \wedge \eta_k
\end{equation}
and the qc structure is 3-Sasakian exactly when
\begin{equation}
2\omega_i = d\eta_i - 2\eta_j \wedge \eta_k,
\end{equation}
for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. The two $Sp(n)Sp(1)$-invariant trace-free symmetric 2-tensors $T^0, U$ on $H$ are introduced in [38] as follows $T^0(X, Y) \overset{def}{=} g((T^0_{I_1} I_1 + T^0_{I_2} I_2 + T^0_{I_3} I_3) X, Y)$, $U(X, Y) \overset{def}{=} g(uX, Y)$. The tensor $T^0$ belongs to the [-1]-eigenspace while $U$ is in the [3]-eigenspace of the operator $\hat{t}$, i.e., they have the properties:
\begin{equation}
T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0,
\end{equation}
\begin{equation}
U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y).
\end{equation}
In dimension seven ($n = 1$), the tensor $U$ vanishes identically, $U = 0$.

We shall need the following identity taken from [40, Proposition 2.3]
\begin{equation}
4g(T^0(\xi_s, I_s X, Y), Y) = T^0(X, Y) - T^0(I_s X, I_s Y)
\end{equation}
The horizontal Ricci 2-forms can be expressed in terms of the torsion of the Biquard connection [38] (see also [39, 40]). We collect the necessary facts from [38, Theorem 1.3, Theorem 3.12, Corollary 3.14, Proposition 4.3 and Proposition 4.4] with slight modification presented in [40].
Theorem 2.2. [38] On a (4n + 3)-dimensional qc manifold \((M, \eta, Q)\) the next formulas hold
\begin{equation}
2\rho_s(X, I_sY) = -T^0(X, Y) - T^0(I_sX, I_sY) - 4U(X, Y) - 2Sg(X, Y),
\end{equation}
\begin{equation}
T(\xi_1, \xi_2) = -S\xi_k - [\xi_1, \xi_2]_H.
\end{equation}

The vanishing of the trace-free part of the Ricci 2-forms is equivalent to the vanishing of the torsion endomorphism of the Biquard connection. In this case the vertical distribution is integrable, the (normalized) qc scalar curvature \(S\) is constant and if \(S > 0\) then there locally exists an SO(3)-matrix \(\Psi\) with smooth entries depending on an auxiliary parameter such the (local) qc structure \((\hat{\omega}, \Psi, \mathbb{Q})\) is 3-Sasakian.

If dimension is bigger than seven it turns out that the vanishing of the torsion endomorphism of the Biquard connection is equivalent the 4-form \(\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3\) to be closed [41].

2.3. The qc conformal curvature. The qc conformal curvature tensor \(W^{qc}\) introduced in [40] is the obstruction for a qc structure to be locally qc conformal to the flat structure on the quaternionic Heisenberg group \(G(\mathbb{H})\). In terms of the torsion and curvature of the Biquard connection \(W^{qc}\) is defined in [40] by
\begin{equation}
W^{qc}(X, Y, Z, V) = \frac{1}{4} \left[ R(X, Y, Z, V) + \sum_{s=1}^{3} R(I_sX, I_sY, Z, V) \right]
+ (g \otimes U)(X, Y, Z, V) + \sum_{s=1}^{3} (\omega_s \otimes I_sU)(X, Y, Z, V) - \frac{1}{2} \sum_{s=1}^{3} \omega_s(Z, V) \left[ T^0(X, I_sY) - T^0(I_sX, Y) \right]
+ \frac{S}{4} \left[ (g \otimes g)(X, Y, Z, V) + \sum_{s=1}^{3} (\omega_s \otimes \omega_s)(X, Y, Z, V) \right],
\end{equation}
where \(I_sU(X, Y) = -U(X, I_sY)\) and \(\otimes\) is the Kulkarni-Nomizu product of \((0,2)\) tensors, for example,
\[(\omega_s \otimes U)(X, Y, Z, V) := \omega_s(X, Z)U(Y, V) + \omega_s(Y, V)U(X, Z) - \omega_s(Y, Z)U(X, V) - \omega_s(X, V)U(Y, Z).\]

The main result from [40] can be stated as follows

Theorem 2.3. [40] A qc structure on a (4n + 3)-dimensional smooth manifold is locally quaternionic contact conformal to the standard flat qc structure on the quaternionic Heisenberg group \(G(\mathbb{H})\) if and only if the qc conformal curvature vanishes, \(W^{qc} = 0\). In this case, we call the qc structure a qc conformally flat structure.

Denote \(L_0 = \frac{1}{2}T^0 + U\). For computational purposes we use the fact established in [40] that \(W^{qc} = 0\) exactly when the tensor \(WR = 0\), where
\begin{equation}
WR(X, Y, Z, V) = R(X, Y, Z, V) + (g \otimes L_0)(X, Y, Z, V) + \sum_{s=1}^{3} (\omega_s \otimes I_sL_0)(X, Y, Z, V)
- \frac{1}{2} \sum_{s=1}^{3} \omega_s(X, Y) \left\{ T^0(Z, I_sV) - T^0(I_sZ, V) \right\} + \omega_s(Z, V) \left\{ T^0(X, I_sY) - T^0(I_sX, Y) - 4U(X, I_sY) \right\}
+ \frac{S}{4} \left[ (g \otimes g)(X, Y, Z, V) + \sum_{s=1}^{3} (\omega_s \otimes \omega_s)(X, Y, Z, V) + 4\omega_s(X, Y)\omega_s(Z, V) \right].
\end{equation}
We also recall that as a manifold \(G(\mathbb{H}) = \mathbb{H}^n \times \text{Im} \mathbb{H}\), while the group multiplication is given by \((q', \omega') = (q_0, \omega_0) \circ (q, \omega) = (q_0 + q, \omega + \omega_0 + 2 \text{ Im } q_0 \bar{q},\) where \(q, q_0 \in \mathbb{H}^n\) and \(\omega, \omega_0 \in \text{Im } \mathbb{H}\). The standard flat quaternionic contact structure is defined by the left-invariant quaternionic contact form \(\hat{\Theta} = (\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Theta}_3) = \frac{1}{2}(d\omega - q^1 \cdot dq^2 + dq^2 \cdot q^3)\), where \(\cdot\) denotes the quaternion multiplication. As a Lie group it can be characterized by the following structure equations. Denote by \(e^a, 1 \leq a \leq (4n + 3)\) the basis of the left invariant 1-forms, and by \(e^jo\) the wedge product \(e^i \wedge e^j\). The \((4n + 3)\)-dimensional quaternionic
Heisenberg Lie algebra is the 2-step nilpotent Lie algebra defined by:
\[ de^a = 0, \quad 1 \leq a \leq 4n, \]
\[ d\eta_1 = de^{4n+1} = 2(e^{12} + e^{34} + \cdots + e^{(4n-3)(4n-2)} + e^{(4n-1)4n}), \]
\[ d\eta_2 = de^{4n+2} = 2(e^{13} + e^{42} + \cdots + e^{(4n-3)(4n-1)} + e^{4n(4n-2)}), \]
\[ d\eta_3 = de^{4n+3} = 2(e^{14} + e^{23} + \cdots + e^{(4n-3)4n} + e^{(4n-2)(4n-1)}). \]

3. Examples

In this section we give explicit examples of qc structures in dimension seven satisfying the compatibility conditions (2.1). The first example has zero torsion and is locally qc conformal to the quaternionic Heisenberg group. The second example has zero torsion while the third is with non-vanishing torsion, and both are not locally qc conformal to the quaternionic Heisenberg group.

Clearly, a qc conformally flat structure is locally qc conformal to a 3-Sasakian structure due to the local qc conformal equivalence of the standard 3-Sasakian structure on the 4n + 3-dimensional sphere and the quaternionic Heisenberg group.

Remark 3.1. We note explicitly that the vanishing of the torsion endomorphism implies that, locally, the structure is homothetic to a 3-Sasakian structure if the qc scalar curvature is positive. In the seven dimensional examples below the qc scalar curvature is a negative constant. In that respect, as pointed by Charles Boyer, there are no compact invariant with respect to translations 3-Sasakian Lie groups of dimension seven.

3.1. Zero torsion qc-flat-Example 1. Denote \( \{ \tilde{\epsilon}^l, 1 \leq l \leq 7 \} \) the basis of the left invariant 1-forms and consider the simply connected Lie group with Lie algebra \( L_1 \) defined by the following equations:
\[ de^1 = 0, \quad de^2 = \tilde{\epsilon}^{34}, \quad de^3 = -e^{24}, \quad de^4 = e^{23}, \quad de^5 = -2\tilde{\epsilon}^{14} + \tilde{\epsilon}^{15} + \tilde{\epsilon}^{26} - \tilde{\epsilon}^{37}, \]
\[ de^6 = -2\epsilon^{13} - 2\tilde{\epsilon}^{42} + \tilde{\epsilon}^{16} - \epsilon^{25} + \epsilon^{47}, \quad de^7 = -2\epsilon^{12} - 2\epsilon^{34} + \epsilon^{17} + \epsilon^{35} - \epsilon^{46}. \]
Let \( L_1 \) be the Lie algebra isomorphic to (3.1) described by
\[ de^1 = 0, \quad de^2 = -e^{12} - 2e^{34} - \frac{1}{2}e^{37} + \frac{1}{2}e^{46}, \]
\[ de^3 = -e^{13} + 2e^{24} + \frac{1}{2}e^{27} - \frac{1}{2}e^{45}, \quad de^4 = -e^{14} - 2e^{23} - \frac{1}{2}e^{26} + \frac{1}{2}e^{35} \]
\[ de^5 = 2e^{12} + 2e^{34} - \frac{1}{2}e^{67}, \quad de^6 = 2e^{13} + 2e^{42} + \frac{1}{2}e^{67}, \quad de^7 = 2e^{14} + 2e^{23} - \frac{1}{2}e^{56}. \]
and \( e_i, 1 \leq l \leq 7 \) be the left invariant vector field dual to the 1-forms \( e^i, 1 \leq i \leq 7 \), respectively. We define a global qc structure on \( L_1 \) by setting
\[ \eta_1 = e^5, \quad \eta_2 = e^6, \quad \eta_3 = e^7, \quad H = \text{span}\{ e^1, \ldots, e^4 \}, \]
\[ \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = e^{14} + e^{23}. \]
It is straightforward to check from (3.2) that the vector fields \( \xi_1 = e_5, \xi_2 = e_6, \xi_3 = e_7 \) satisfy the Duchemin compatibility conditions (2.1) and therefore the Biquard connection exists and \( \xi_i \) are the Reeb vector fields.

Theorem 3.2. Let \((G_1, \eta, Q)\) be the simply connected Lie group with Lie algebra \( L_1 \) equipped with the left invariant qc structure \((\eta, Q)\) defined above. Then
a) The torsion endomorphism of the Biquard connection is zero and the normalized qc scalar curvature is a negative constant, \( S = -\frac{1}{2} \).

b) The qc conformal curvature is zero, \( W^{qc} = 0 \), and therefore \((G_1, \eta, Q)\) is locally qc conformally flat.

Proof. We compute the connection 1-forms and the horizontal Ricci forms of the Biquard connection. The Lie algebra structure equations (3.2) together with (2.3), (2.4) and (2.6) imply
\[ \alpha_i = \left( \frac{1}{4} - \frac{S}{2} \right) \eta_i, \quad \rho_i(X, Y) = \frac{1}{2} \alpha_i(X, Y) = \left( \frac{1}{4} - \frac{S}{2} \right) \omega_i(X, Y). \]
Compare (3.4) with (2.11) to conclude that the torsion is zero and the normalized qc scalar $S = -\frac{1}{2}$ and Theorem 2.2 completes the proof of a).

In view of Theorem 2.3, to prove b) we have to show $W^qc = 0$. We claim $WR = 0$. Indeed, since the torsion of the Biquard connection vanishes and $S = -\frac{1}{2}$, (2.13) takes the form

$$\begin{align*}
WR(X,Y,Z,V) &= R(X,Y,Z,V) \\
&= -\frac{1}{8} \left[ (g \otimes g)(X,Y,Z,V) + \sum_{s=1}^{3} \left( (\omega_s \otimes \omega_s)(X,Y,Z,V) + 4\omega_s(X,Y)\omega_s(Z,V) \right) \right].
\end{align*}$$

Let $A,B,C \in \Gamma(TG_1)$. Since the Biquard connection preserves the whole metric, it is connected with the Levi-Civita connection $\nabla^g$ of the metric $g$ by the general formula

$$g(\nabla_A B, C) = g(\nabla^g_A B, C) + \frac{1}{2} \left[ g(T(A,B),C) - g(T(B,C),A) + g(T(C,A),B) \right].$$

The Koszul formula for a left-invariant vector fields reads

$$g(\nabla^g_{e_a} e_b, e_c) = \frac{1}{2} \left[ g([e_a, e_b], e_c) - g([e_b, e_c], e_a) + g([e_c, e_a], e_b) \right].$$

Theorem 2.1 supplies the formula

$$T(X,Y) = 2 \sum_{s=1}^{3} \omega_s(X,Y)\xi_s.$$

Using (3.8), (3.7), (3.6) and the structure equations (3.2) we found that the non zero Christoffel symbols for the Biquard connection (defined by $\nabla_{e_a} e_b = \sum_c \Gamma^c_{ab} e_c$) are:

$$1 = \Gamma^1_{23} = \Gamma^4_{23} = \Gamma^1_{33} = \Gamma^2_{34} = \Gamma^3_{42} = \Gamma^1_{44} = -\Gamma^2_{21} = -\Gamma^3_{42} = -\Gamma^4_{31} = -\Gamma^4_{41} = -\Gamma^2_{23},$$

$$\frac{1}{2} = \Gamma^4_{53} = \Gamma^7_{56} = \Gamma^2_{64} = \Gamma^5_{67} = \Gamma^3_{72} = \Gamma^6_{75} = -\Gamma^3_{54} = -\Gamma^6_{57} = -\Gamma^6_{62} = -\Gamma^7_{63} = -\Gamma^7_{73} = -\Gamma^5_{76}.$$

And the non zero coefficients of the curvature tensor are $R(e_a, e_b, e_a, e_b) = -R(e_a, e_b, e_b, e_a) = 1$, $a, b = 1, \ldots, 4$, $a \neq b$. Now (3.5) yields $WR(e_a, e_b, e_c, e_d) = R(e_a, e_b, e_c, e_d) = 0$, when there are three different indices in $a, b, c, d$. For the indices repeated in pairs we have

$$WR(e_a, e_b, e_a, e_b) = R(e_a, e_b, e_a, e_b) = -\frac{1}{8} (g \otimes g)(e_a, e_b, e_a, e_b)$$

$$= -\frac{1}{8} \left[ \sum_{s=1}^{3} (\omega_s \otimes \omega_s)(e_a, e_b, e_a, e_b) + 4\omega_s(e_a, e_b)\omega_s(e_a, e_b) \right] = 1 - \frac{1}{8} \cdot 2 - \frac{1}{8} \cdot 6 = 0$$

Then Theorem 2.3 completes the proof.

**3.2. Zero torsion qc-non-flat-Example 2.** Consider the simply connected Lie group $L_2$ with Lie algebra defined by the equations:

$$\begin{align*}
de^{1} &= 0, \quad de^{2} = -e^{12} + e^{34}, \quad de^{3} = -\frac{1}{2}e^{13}, \quad de^{4} = -\frac{1}{2}e^{14}, \\
de^{5} &= 2e^{12} + 2e^{34} + e^{37} = e^{46} + \frac{1}{4}e^{67}, \quad de^{6} = 2e^{13} - 2e^{24} - \frac{1}{2}e^{27} + e^{45} - \frac{1}{4}e^{57}, \\
de^{7} &= 2e^{14} + 2e^{23} + \frac{1}{2}e^{26} - e^{35} + \frac{1}{4}e^{56}.
\end{align*}$$

A global qc structure on $L_2$ is defined by setting

$$\eta_1 = e^{5}, \quad \eta_2 = e^{6}, \quad \eta_3 = e^{7}, \quad \xi_1 = e^{5}, \quad \xi_2 = e^{6}, \quad \xi_3 = e^{7},$$

$$\mathbb{H} = \text{span}\{ e^{1}, \ldots, e^{4} \}, \quad \omega_1 = e^{12} + e^{34}, \quad \omega_2 = e^{13} + e^{42} \quad \omega_3 = e^{14} + e^{23},$$

It is straightforward to check from (3.9) that the triple $\{\xi_1, \xi_2, \xi_3\}$ forms the Reeb vector fields satisfying (2.1) and therefore the Biquard connection do exists.
Theorem 3.3. Let \((G_2, \eta, \mathbb{Q})\) be the simply connected Lie group with Lie algebra \(L_2\) equipped with the left invariant qc structure \((\eta, \mathbb{Q})\) defined above. Then:

a) The torsion endomorphism of the Biquard connection is zero and the normalized qc scalar curvature is a negative constant, \(S = -\frac{1}{4}\).

b) The qc conformal curvature is not zero, \(W^Q \neq 0\) and therefore \((G_2, \eta, \mathbb{Q})\) is not locally qc conformally flat.

Proof. We compute the connection 1-forms and the horizontal Ricci forms of the Biquard connection. The Lie algebra structure equations (3.9) together with (2.3), (2.4) and (2.6) imply

\[
\begin{align*}
\alpha_1 &= -\frac{1}{2} e^2 - \frac{1}{8} (1 + \frac{S}{2}) \eta_1, \quad \alpha_2 = -e^3 - (\frac{1}{8} + \frac{S}{2}) \eta_2, \quad \alpha_3 = -e^4 - (\frac{1}{8} + \frac{S}{2}) \eta_3, \\
\rho_l(X,Y) &= (\frac{1}{8} - \frac{S}{2}) \omega_l(X,Y).
\end{align*}
\]

Compare (3.11) with (2.11) to conclude that the torsion is zero and the normalized qc scalar \(S = -\frac{1}{4}\). Theorem 2.2 completes the proof of a).

In view of the proof of Theorem 3.2, to get b) we have to show \(WR(e_1, e_2, e_3, e_4) = R(e_1, e_2, e_3, e_4) \neq 0\).

Indeed, using (3.8), (3.7), (3.6) and the structure equations (3.9) we found that the non zero Christoffel symbols for the Biquard connection are

\[
\begin{align*}
1 = -\Gamma_{21} = \Gamma_{12} = \Gamma_{43} = -\Gamma_{34} = -\Gamma_{42} = \Gamma_{24} = -\Gamma_{41} = \Gamma_{32} = -\Gamma_{43} = \Gamma_{14}, \\
1 = \Gamma_{53} = -\Gamma_{35} = \Gamma_{75} = -\Gamma_{26} = -\Gamma_{62} = -\Gamma_{56} = -\Gamma_{67} = \Gamma_{37} = -\Gamma_{73} = -\Gamma_{57} = \Gamma_{65},
\end{align*}
\]

and then \(R(e_1, e_2, e_3, e_4) = -\frac{1}{2} \neq 0\). Theorem 2.3 completes the proof. \(\square\)

3.3. Non-zero torsion qc-non-flat-Example 3. Consider the Lie algebra defined by the equations:

\[
\begin{align*}
d\tilde{e}^1 &= \tilde{e}^{13} - \tilde{e}^{24}, & d\tilde{e}^2 &= \tilde{e}^{14} + \tilde{e}^{23}, & d\tilde{e}^3 &= d\tilde{e}^4 = 0; \\
d\tilde{e}^5 &= -2\tilde{e}^{12} - \tilde{e}^{34} + \frac{1}{2} \tilde{e}^{17} + \frac{1}{2} \tilde{e}^{26} - \tilde{e}^{35} - \frac{1}{8} \tilde{e}^{67}; \\
d\tilde{e}^6 &= -2\tilde{e}^{13} + \tilde{e}^{24} - \frac{1}{2} \tilde{e}^{36} + \frac{1}{2} \tilde{e}^{47}; \\
d\tilde{e}^7 &= -2\tilde{e}^{14} - \tilde{e}^{23} - \frac{1}{2} \tilde{e}^{37} - \frac{1}{2} \tilde{e}^{46}.
\end{align*}
\]

Let \(L_3\) be the Lie algebra isomorphic to (3.12) described by

\[
\begin{align*}
de^1 &= \frac{3}{2} \tilde{e}^{13} + \frac{3}{2} \tilde{e}^{24} - \frac{3}{4} \tilde{e}^{25} - \frac{1}{4} \tilde{e}^{36} + \frac{1}{4} \tilde{e}^{47} + \frac{1}{8} \tilde{e}^{57}, \\
de^2 &= \frac{3}{2} \tilde{e}^{14} + \frac{3}{2} \tilde{e}^{23} - \frac{3}{4} \tilde{e}^{15} + \frac{1}{4} \tilde{e}^{46} - \frac{1}{4} \tilde{e}^{56}, \\
de^3 &= 0, & de^4 &= \tilde{e}^{12} + \tilde{e}^{34} + \frac{1}{2} \tilde{e}^{17} - \frac{1}{2} \tilde{e}^{26} + \frac{1}{4} \tilde{e}^{67}, \\
de^5 &= 2\tilde{e}^{12} + 2\tilde{e}^{34} + \tilde{e}^{17} - \tilde{e}^{26} + \frac{1}{2} \tilde{e}^{67}, \\
de^6 &= 2\tilde{e}^{13} + 2\tilde{e}^{24} + \tilde{e}^{15}, & de^7 &= 2\tilde{e}^{14} + 2\tilde{e}^{23} - \tilde{e}^{15},
\end{align*}
\]

and \(e_1, 1 \leq l \leq 7\) be the left invariant vector field dual to the 1-forms \(e^i, 1 \leq i \leq 7\), respectively. We define a global qc structure on \(L_3\) by setting

\[
\begin{align*}
\eta_1 &= e^5, & \eta_2 &= e^6, & \eta_3 &= e^7, & \xi_1 &= e_5, & \xi_2 &= e_6, & \xi_3 &= e_7, \\
H &= \text{span}\{e^1, \ldots, e^4\}, & \omega_1 &= e^{12} + e^{34}, & \omega_2 &= e^{13} + e^{42} & \omega_3 &= e^{14} + e^{23}.
\end{align*}
\]

It is straightforward to check from (3.12) that the triple \(\{\xi_1, \xi_2, \xi_3\}\) forms the Reeb vector fields satisfying (2.1) and therefore the Biquard connection do exists.
Theorem 3.4. Let \((G_3, \eta, Q)\) be the simply connected Lie group with Lie algebra \(L_3\) equipped with the left invariant \(qc\) structure \((\eta, Q)\) defined by (3.14). Then

1. The torsion endomorphism of the Biquard connection is not zero and therefore \((G_3, \eta, Q)\) is not locally \(qc\) homothetic to a 3-Sasaki manifold. The normalized \(qc\) scalar curvature is negative, \(S = -1\).

2. The \(qc\) conformal curvature is not zero, \(W^{qc} \neq 0\), and therefore \((G_3, \eta, Q)\) is not locally \(qc\) conformally flat.

Proof. It is clear from (3.13) that the vertical distribution spanned by \(\{\xi_1, \xi_2, \xi_3\}\) is not integrable. Consequently, the torsion of the Biquard connection is not zero due to [38, Theorem 3.1] which proves the first part of a).

To prove \(S = -1\) we compute the torsion. The Lie algebra structure equations (3.13) together with (2.3), (2.4) imply

\[(3.15) \quad \alpha_1 = \left(\frac{1}{4} - \frac{S}{2}\right)\eta_1, \quad \alpha_2 = -e^1 - \left(\frac{1}{4} + \frac{S}{2}\right)\eta_2, \quad \alpha_3 = -e^2 - \left(\frac{1}{4} + \frac{S}{2}\right)\eta_3.\]

Now, (3.15), (3.13) and (2.6) yield

\[(3.16) \quad \rho_1(X, Y) = \frac{1}{2} \left(\frac{1}{2} - S\right)(e^{12} + e^{34}) + e^{12}, \quad \rho_2(X, Y) = \frac{1}{2} \left(\frac{3}{2} e^{13} + e^{24}\right)(X, Y) - \frac{1}{2} + S\left(e^{13} - e^{24}\right)(X, Y), \quad \rho_3(X, Y) = \frac{1}{2} \left(\frac{3}{2} e^{14} + e^{23}\right)(X, Y) - \frac{1}{2} + S\left(e^{14} + e^{23}\right)(X, Y).\]

Compare (3.16) with (2.11) to conclude

\[(3.17) \quad T^0(X, I_1 Y) - T^0(I_1 X, Y) = \frac{1}{2}(e^{12} - e^{34})(X, Y), \quad S = -1, \quad T^0(X, I_2 Y) - T^0(I_2 X, Y) = 0, \quad T^0(X, I_3 Y) - T^0(I_3 X, Y) = 0.\]

To prove b) we compute the tensor \(WR\). Denote \(\psi = -\frac{1}{4}(e^{12} - e^{34})\) and compare (3.17) with (2.9) and (2.10) to obtain

\[(3.18) \quad T^0(X, Y) = \psi(X, I_1 Y), \quad g(T(\xi_5, X), Y) = -\frac{1}{4}(\psi(I_5 X, I_1 Y) + \psi(X, I_1 I_5 Y)).\]

Using \(U = 0\) and (2.9) we conclude from (2.13) that \(WR(e_1, e_2, e_3, e_4) = R(e_1, e_2, e_3, e_4)\) since other terms on the right hand side of (2.13) vanish on the quadruple \(\{e_1, e_2 = -I_1 e_1, e_3 = -I_2 e_1, e_4 = -I_3 e_1\}\).

We calculate \(R(e_1, e_2, e_3, e_4)\) using (3.8), (3.7), (3.8), (3.13) and (3.18). We have

\[
\begin{align*}
\frac{3}{2} & = \Gamma^1_{13} = -\Gamma^3_{11} = -\Gamma^3_{22} = -\Gamma^2_{23}, & \frac{1}{2} & = -\Gamma^4_{12} = \Gamma^2_{14} = \Gamma^4_{21} = -\Gamma^1_{24}, \\
\frac{3}{4} & = \Gamma^2_{51} = -\Gamma^1_{52} = \Gamma^5_{56} = -\Gamma^6_{57}, & \frac{1}{8} & = -\Gamma^3_{61} = \Gamma^1_{63} = -\Gamma^3_{72} = \Gamma^2_{73}, \\
\frac{1}{4} & = -\Gamma^5_{65} = \Gamma^5_{67} = \Gamma^5_{75} = -\Gamma^5_{76}, & \frac{3}{8} & = -\Gamma^4_{62} = \Gamma^2_{64} = \Gamma^4_{71} = -\Gamma^4_{74}, \\
1 & = \Gamma^1_{15} = -\Gamma^1_{17} = -\Gamma^6_{26} = -\Gamma^5_{25} = -\Gamma^6_{36} = -\Gamma^5_{35} = -\Gamma^6_{46} = -\Gamma^5_{45} = -\Gamma^6_{56} = -\Gamma^5_{56} = -\Gamma^6_{65} = -\Gamma^5_{75} = -\Gamma^6_{76} = -\Gamma^5_{76}.
\end{align*}
\]

This gives \(WR(e_1, e_2, e_3, e_4) = R(e_1, e_2, e_3, e_4) = -\frac{1}{7} \neq 0\) and Theorem 2.3 completes the proof.

4. \(Sp(n)Sp(1)\)-Hypo structures and hypersurfaces in quaternionic Kähler manifolds

Guided by the Examples 1-3, we relax the definition of a qc structure dropping the “contact condition” \(d\eta_{s,H} = 2\omega_s\) and come to an \(Sp(n)Sp(1)\) structure (almost 3-contact structure see [46]). The purpose is to get a structure which possibly may induce an explicit quaternionic Kähler metric on a product with a real line.

Definition 4.1. An \(Sp(n)Sp(1)\) structure on a \((4n + 3)\)-dimensional Riemannian manifold \((M, g)\) is a codimension three distribution \(H\) satisfying
i) H has an Sp\(n)Sp(1)\) structure, that is it is equipped with a Riemannian metric \(g\) and a rank-three bundle \(Q\) consisting of \((1,1)\)-tensors on \(H\) locally generated by three almost complex structures \(I_1, I_2, I_3\) on \(H\) satisfying the identities of the imaginary unit quaternions, \(I_1I_2 = -I_3 I_1 = I_2\), \(I_1I_2I_3 = -id_H\) which are hermitian compatible with the metric \(g(I_1, I_2) = g(., .), s = 1, 2, 3\), i.e. \(H\) has an almost quaternion hermitian structure.

ii) \(H\) is locally given as the kernel of a 1-form \(\eta = (\eta_1, \eta_2, \eta_3)\) with values in \(\mathbb{R}^3\).

The local fundamental 2-forms are defined on \(H\) as usual by \(\omega_s(X, Y) = g(I_s X, Y)\).

**Definition 4.2.** We define a global \(Sp(n)Sp(1)\)-invariant 4-form of an \(Sp(n)Sp(1)\) structure \((M, g, Q)\) on a \((4n + 3)\)-dimensional manifold \(M\) by the expression

\[
\Omega = \omega_1^2 + \omega_2^2 + \omega_3^2 + 2\omega_1 \land \eta_2 \land \eta_3 + 2\omega_2 \land \eta_3 \land \eta_1 + 2\omega_3 \land \eta_1 \land \eta_2.
\]

Let \(M^{4n+4}\) be a \((4n + 4)\)-dimensional manifold equipped with an \(Sp(n + 1)Sp(1)\) structure, i.e. \((M^{4n+4}, g, J_1, J_2, J_3)\) is an almost quaternion hermitian manifold with local Kähler forms \(F_i = g(J_i\ldots)\). The fundamental 4-form

\[
\Phi = F_1 \land F_1 + F_2 \land F_2 + F_3 \land F_3
\]

is globally defined and encodes fundamental properties of the structure. If the holonomy of the Levi-Civita connection is contained in \(Sp(n + 1)Sp(1)\) then the manifold is a quaternionic Kähler manifold which is consequently an Einstein manifold. Equivalent conditions are either that

\[
dF_i \in \text{span}\{F_i, F_j, F_k\}
\]

or the fundamental 4-form \(\Phi\) is parallel with respect to the Levi-Civita connection. The latter is equivalent to the fact that the fundamental 4-form is closed (\(d\Phi = 0\)) provided the dimension is strictly bigger than eight \([56, 55]\) with a counter-example in dimension eight constructed by Salamon in \([55]\).

Let \(f : N^{4n+3} \longrightarrow M^{4n+4}\) be an oriented hypersurface of \(M^{4n+4}\) and denote by \(N\) the unit normal vector field. Then \(Sp(n + 1)Sp(1)\) structure on \(M\) induces an \(Sp(n)Sp(1)\) structure on \(N^{4n+3}\) locally given by \((\eta_s, \omega_s)\) defined by the equalities

\[
\eta_s = N \cdot F_s, \quad \omega_i = f^* F_i - \eta_j \land \eta_k,
\]

for any cyclic permutation \((i, j, k)\) of \((1, 2, 3)\). The fundamental four form \(\Phi\) on \(M\) restricts to the fundamental four form \(\Omega\) on \(N\).

\[
\Omega = f^* \Phi = (f^* F_1)^2 + (f^* F_2)^2 + (f^* F_3)^2.
\]

Suppose that \((M^{4n+4}, g)\) has holonomy contained in \(Sp(n + 1)Sp(1)\). Then \(d\Phi = 0, (4.5)\) and \((4.4)\) imply that the \(Sp(n)Sp(1)\) structure induced on \(N^{4n+3}\) satisfies the equation

\[
d\Omega = 0,
\]

since \(d\) commutes with \(f^*, df^* = f^* d\).

**Definition 4.3.** An \(Sp(n)Sp(1)\) structure \((M, g, Q)\) on a \((4n + 3)\)-dimensional manifold is called \(Sp(n)Sp(1)\) - hypo if its fundamental 4-form is closed, \(d\Omega = 0\).

Hence, any oriented hypersurface \(N^{4n+3}\) of a quaternionic Kähler \(M^{4n+4}\) is naturally endowed with an \(Sp(n)Sp(1)\)-hypo structure.

Vice versa, a \((4n + 3)\)-manifold \(N^{4n+3}\) with an \(Sp(n)Sp(1)\) structure \((\eta_s, \omega_s)\) induces an \(Sp(n + 1)Sp(1)\) structure \((F_s)\) on \(N^{4n+3} \times \mathbb{R}\) defined by

\[
F_i = \omega_i + \eta_j \land \eta_k - \eta_i \land dt,
\]

where \(t\) is a coordinate on \(\mathbb{R}\).

Consider \(Sp(n)Sp(1)\) structures \((\eta_s(t), \omega_s(t))\) on \(N^{4n+3}\) depending on a real parameter \(t \in \mathbb{R}\), and the corresponding \(Sp(n + 1)Sp(1)\) structures \(F_s(t)\) on \(N^{4n+3} \times \mathbb{R}\). We have
Proposition 4.4. An $Sp(n)Sp(1)$ structure $(\eta_s, \omega_s; 1 \leq s \leq 3)$ on $N^{4n+3}$ can be lifted to a quaternionic Kähler structure $(F_s(t))$ on $N^{4n+3} \times \mathbb{R}$ defined by (4.7) if and only if it is an $Sp(n)Sp(1)$-hypo structure which generates a 1-parameter family of $Sp(n)Sp(1)$-hypo structures $(\eta_s(t), \omega_s(t))$ satisfying the following evolution $Sp(n)Sp(1)$-hypo equations

$$\partial_t \Omega(t) = d\left[6\eta_1(t) \wedge \eta_2(t) \wedge \eta_3(t) + 2\omega_1(t) \wedge \eta_1(t) + 2\omega_2(t) \wedge \eta_2(t) + 2\omega_3(t) \wedge \eta_3(t)\right],$$

where $d$ is the exterior derivative on $N$.

Proof. If we apply (4.7) to (4.2) and then take the exterior derivative in the obtained equation we see that the equality $d\Phi = 0$ holds precisely when (4.6) and (4.8) are fulfilled.

It remains to show that the equations (4.8) imply that (4.6) hold for each $t$. Indeed, using (4.8), we calculate

$$\partial_t d\Omega = d^2\left[6\eta_1(t) \wedge \eta_2(t) \wedge \eta_3(t) + 2\omega_1(t) \wedge \eta_1(t) + 2\omega_2(t) \wedge \eta_2(t) + 2\omega_3(t) \wedge \eta_3(t)\right] = 0.$$

Hence, the equalities (4.6) are independent of $t$ and therefore valid for all $t$ since it holds in the beginning for $t = 0$.

Solutions to the (4.6) are given in the case of 3-Sasakian manifolds in [59]. In the next section we construct explicit examples relying on the properties of the qc structures.

In general, a question remains.

Question 1. Does the converse of Proposition 4.4 hold?, i.e. is it true that any $Sp(n)Sp(1)$-hypo structure on $N^{4n+3}$ can be lifted to a quaternionic Kähler structure on $N^{4n+3} \times \mathbb{R}$?

4.1. $Sp(n)$-hypo structures and hypersurfaces in hyper Kähler manifolds. Suppose that $M^{4n+4}$ has holonomy contained in $Sp(n+1)$, that is the $Sp(n+1)Sp(1)$ structure $(F_s)$ is globally defined and integrable (i.e. hyper-Kähler) or, equivalently due to Hitchin [33],

$$dF_s = 0.$$

Then, (4.9) and (4.4) imply that the $Sp(n)$ structure $(\eta_s, \omega_s)$ induced on $N^{4n+3}$ satisfies the equations

$$d(\omega_i + \eta_j \wedge \eta_k) = 0,$$

since $d$ commutes with $f^*$, $df^* = f^*d$.

Definition 4.5. An $Sp(n)$ structure determined by $(\eta_s, \omega_s)$ on a $(4n+3)$-dimensional manifold is called $Sp(n)$-hypo if it satisfies the equations (4.10).

Hence, any oriented hypersurface $N^{4n+3}$ of a hyper Kähler $M^{4n+4}$ is naturally endowed with an $Sp(n)$-hypo structure.

Vice versa, a $(4n+3)$-manifold $N^{4n+3}$ with an $Sp(n)$ structure $(\eta_s, \omega_s)$ induces an $Sp(n+1)$ structure $(F_s)$ on $N^{4n+3} \times \mathbb{R}$ defined by (4.7).

Consider $Sp(n)$ structures $(\eta_s(t), \omega_s(t))$ on $N^{4n+3}$ depending on a real parameter $t \in \mathbb{R}$, and the corresponding $Sp(n+1)$ structures $F_s(t)$ on $N^{4n+3} \times \mathbb{R}$. We have

Proposition 4.6. An $Sp(n)$ structure $(\eta_s, \omega_s; 1 \leq s \leq 3)$ on $N^{4n+3}$ can be lifted to a hyper Kähler structure $(F_s(t))$ on $N^{4n+3} \times \mathbb{R}$ defined by (4.7) if and only if it is an $Sp(n)$-hypo structure which generates an 1-parameter family of $Sp(n)$ structures $(\eta_s(t), \omega_s(t))$ satisfying the following evolution $Sp(n)$-hypo equations

$$\partial_t (\omega_i + \eta_j \wedge \eta_k) = d\eta_i.$$

Proof. Taking the exterior derivatives in (4.7) shows that the equalities $dF_s = 0$ hold precisely when (4.10) and (4.11) are fulfilled.

It remains to show that the equations (4.11) imply that (4.10) hold for each $t$. Indeed, using (4.11), we calculate

$$\partial_t \left[d(\omega_i + \eta_j \wedge \eta_k)\right] = d^2\eta_i = 0.$$

Hence, the equalities (4.10) are independent of $t$ and therefore valid for all $t$ since it holds in the beginning for $t = 0$. 

□
It is known, [10], that the cone over a 3-Sasakian manifold is hyper-Kähler, i.e., there is a solution to (4.11). Indeed, for a 3-Sasakian manifold we have [38] $S = 2, d\eta_t(\xi_j, \xi_k) = 2, \alpha_s = -2\eta_s$ and the structure equations (2.7) of a 3-Sasakian manifold become (2.8). A solution to (4.11) is given by $F_i(t) = t^2\omega_i + t^2\eta_j \wedge \eta_k - t\eta_i \wedge dt$.

In general, a question remains.

**Question 2.** Does the converse of Proposition 4.6 hold?, i.e. is it true that any $Sp(n)$-hypo structure on $N^{4n+3}$ can be lifted to a hyper Kähler structure on $N^{4n+3} \times \mathbb{R}$?

**Remark 4.7.** Question 2 is an embedding problem analogous to the (hypo) $SU(n)$ embedding problem solved in [15, 14]. Here, we consider hyper Kähler manifolds instead of Calabi-Yau manifolds. Since $Sp(n)$ is contained in $SU(2n)$, it follows that an $Sp(n)$ structure $(\omega, \eta)$ on a $(4n+3)$-dimensional manifold induces an $SU(2n)$ structure $(\eta_t, F, \Omega)$ where the 2-form $F$ and the complex $(2n+1)$-form $\Omega$ are defined by $F = \omega + \eta_t \wedge \eta_3$, $\Omega = (\omega_2 + \sqrt{-1} \omega_3)^n \wedge (\eta_2 + \sqrt{-1} \eta_3)$. Direct computations show that the $Sp(n)$-hypo conditions (4.10) yield the $SU(2n)$-hypo conditions $dF = 0, d(\eta_t \wedge \Omega) = 0$ which, in the real analytic case, imply an embedding into a Calabi-Yau manifold [15, 14]. Hence, it follows that any real analytic $(4n+3)$-manifold with an $Sp(n)$-hypo structure can be embedded in a Calabi-Yau manifold. However, it is not clear whether this Calabi-Yau structure is a hyper Kähler.

A proof of the embedding property could be achieved following the considerations in the recent paper by Diego Conti [14]. Consider $Sp(n)$ as a subgroup of $SO(4n)$, one has to show the existence of an $\mathbb{H}^{Sp(n)}$-ordinary flag in the sense of [14].

5. EXAMPLES OF QUATERNIONIC KähLER STRUCTURES

In this section we suppose that $M$ is a Riemannian manifold of dimension $4n+3$ equipped with an $Sp(n)/Sp(1)$ structure as in Definition 4.1. We shall denote with $g_H$ the metric on the horizontal distribution $H$. In addition, we assume that for some constant $\tau$ the following structure equations hold

$$d\eta_t = 2\omega_i + 2\tau\eta_j \wedge \eta_k,$$

for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. Examples of such manifolds are provided by the following quaternionic contact manifolds: i) the quaternionic Heisenberg group, where $\tau = 0$; ii) any 3-Sasakian manifold, where $\tau = 1$ (see [41] where it is proved that these structure equations characterize the 3-Sasakian quaternionic contact manifolds); and iii) the zero torsion qc-flat group $G_1$ defined in Theorem 3.2 with the structure equations described in (3.2), where $\tau = -1/4$. Actually, this is the only Lie group satisfying the structure equation (5.1) for some (necessarily) negative constant $\tau$. We prefer to include the parameter $\tau$ since it describes qc structures homothetic to each other. In particular, for $\tau < 0$ (resp. $\tau > 0$), the qc homothety $\eta_t \mapsto -2\tau\eta_t$ (resp. $\eta_t \mapsto \tau\eta_t$) brings the qc structure (3.3) on the lie algebra (3.2) (resp. a 3-Sasakain structure) to one satisfying (5.1). On the other hand, this one parameter family of homothetic to each other qc structures lead to different special holonomy metrics, which we construct next, when we take the product with a real line.

**Theorem 5.1.** Let $M$ be a smooth manifold of dimension $4n+3$ equipped with an $Sp(n)/Sp(1)$ structure such that, for some constant $\tau$, the structure equations (5.1) hold for any cyclic permutation $(i, j, k)$ of $(1, 2, 3)$. For any constant $\alpha$, the manifold $M \times \mathbb{R}$ has a quaternionic Kähler structure given by the following metric and fundamental 4-form

$$g = u g_H + (\tau u + au^2)(\eta_j^2 + \eta_k^2 + \eta_l^2) + \frac{1}{4(\tau u + au^2)}(du)^2, \quad \tau u + au^2 > 0,$$

$$\Phi = F_i \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3,$$

where locally

$$F_i(u) = u \omega_i + (au^2 + \tau u) \eta_j \wedge \eta_k - \frac{1}{2} \eta_t \wedge du.$$

**Proof.** Let $h$ and $f$ be some functions of the unknown $t$ and $F_i(t) = f(t)\omega_i + h^2(t)\eta_j \wedge \eta_k - h(t)\eta_t \wedge dt$ and $\Phi$ be as in (5.2). A direct calculation shows that $(\Sigma_{(ijk)}$ means the cyclic sum)$

$$d\Phi = \Sigma_{(ijk)} \left[ ((f^2)^' - 4fh) \omega_i \wedge \omega_j \wedge dt + \left( 2(fh^2)^' + 4fh - 12h^3 \right) \omega_i \wedge \eta_j \wedge \eta_k \wedge dt \right].$$
Thus, if we take $h = \frac{1}{2} f'$ we come to
\[ d\Phi = f' \Sigma_{(ijk)} (-f'^2 + f f'' + 2\tau f) \omega_i \wedge \eta_j \wedge \eta_k \wedge dt, \]
which shows that $\Phi$ is closed when
\[ (5.4) \]
\[ f f'' - f'^2 + 2\tau f = 0, \quad h = \frac{1}{2} f'. \]
With the help of the substitution $v = -\ln f$ we see that $\left( \frac{dv}{df} \right)^2 = 4\tau e^v + 4a$ for any constant $a$. This shows that
\[ (5.5) \]
\[ \left( \frac{dv}{df} \right)^2 = \left( \frac{du}{df} \right)^2 \frac{1}{4(\tau f + af^2)} > 0 \]
and $h^2 = \tau f + af^2$. Renaming $f$ to $u$ gives the quaternionic structure in the local form (5.3) and the metric in (5.2). In order to see that $\langle F_1, F_2, F_3 \rangle$ is a differential ideal we need to compute the differentials $dF_i$. A small calculation shows
\[ (5.6) \]
\[ dF_i = f f'' (f f'' - f'^2 + 2\tau f) \eta_j \wedge \eta_k \wedge dt \quad \text{mod} \quad \langle F_1, F_2, F_3 \rangle, \]
i.e. (4.3) hold. This proves that the defined structure is quaternionic Kähler taking into account the differential equation (5.4) satisfied by $f$, which completes the proof. \hfill \Box

With the help of the above theorem we obtain the following one parameter families of quaternionic Kähler structures.

i) **Quaternionic Kähler metrics from the quaternionic Heisenberg group, $\tau = 0$.** Consider the $(4n + 3)$-dimensional quaternionic Heisenberg group $G^n$, viewed as a quaternionic contact structure. The metric
\[ (5.7) \]
\[ g = e^{at} \left( (e^1)^2 + \cdots + (e^{4n})^2 \right) + \frac{a^2}{4} e^{2at} \left( (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 \right) + dt^2 \]
is a complete quaternionic Kähler metric in dimensions $4n + 4$ with $n \geq 1$. The Einstein constant is negative equal to $-16na^2$. This complete Einstein metric has been found in dimension eight as an Einstein metric on a $T^3$ bundle over $T^4$ in [29, equation (148)].

ii) **Quaternionic Kähler metrics from a 3-Sasakian structure, $\tau = 1$.** The metric
\[ (5.8) \]
\[ g = u g_H + \frac{u + au^2}{4} \left( \eta_1^2 + \eta_2^2 + \eta_3^2 \right) + \frac{1}{4(u + au^2)} du^2 \]
is a quaternionic Kähler, and in the case of $a = 0$ is the hyper-Kähler cone over the 3-Sasakian manifold. These metrics have been found earlier in [59, Theorem 5.2].

5.1. **Explicit non quaternionic Kähler structures with closed four form in dimension 8.** As it is well known [56] in dimension $4n$, $n > 2$, the condition that the fundamental 4-form is closed is equivalent to the fundamental 4-form being parallel which is not true in dimension eight. Salamon constructed in [55] a compact example of an almost quaternion hermitian manifold with closed fundamental four form which is not Einstein, and therefore it is not quaternionic Kähler. We give below explicit complete non-compact examples of that kind inspired by the following

**Remark 5.2.** In dimension seven, due to the relations $\omega_i \wedge \omega_j = 0$, $i \neq j$, a more general evolution than the one considered in the proof of Theorem 5.1 can be handled. We consider the evolution
\[ (5.9) \]
\[ \omega_s(t) = f(t) \omega_s, \quad \eta_s(t) = f_s(t) \eta_s, \quad s = 1, 2, 3, \]
where $f, f_1, f_2, f_3$ are smooth function of $t$. Using the structure equations (5.1) one easily obtain that the equation $d\Omega = 0$ is satisfied and (4.8) is equivalent to the system
\[ (5.10) \]
\[ 3f' - 2(f_1 + f_2 + f_3) = 0, \]
\[ (f f_2 f_3)' - 2\tau f (f_1 - f_2 - f_3) - 6f_1 f_2 f_3 = 0, \]
\[ (f f_1 f_3)' - 2\tau f (-f_1 + f_2 - f_3) - 6f_1 f_2 f_3 = 0, \]
\[ (f f_1 f_2)' - 2\tau f (-f_1 - f_2 + f_3) - 6f_1 f_2 f_3 = 0. \]
On the other hand, $\langle F_1, F_2, F_3 \rangle$ is a differential ideal iff the following system holds
\[ (5.11) \]
\[ f (f f_j)' - f' f_1 f_j + 2f_1 f_2 f_3 - 2f_1 f_j (f_1 + f_j) + 2\tau f f_1 f_j - 2\tau f f_k = 0, \]
for any cyclic permutation \((i,j,k)\) of \((1,2,3)\). This claim follows from the fact that working \(\text{mod } <F_1,F_2,F_3>\) we have

\[
\frac{dF_i}{f} = \frac{1}{f} \left( f(f_i f_j)' - f' f_i f_j + 2 f_1 f_2 f_3 - 2 f_i^2 f_j - 2 f_i^2 f_j - 2 \tau f f_i f_j - 2 \tau f f_i f_j \right) \eta_j \wedge \eta_k \wedge dt.
\]

Taking \(f_1 = f_2 = f_3 = h\) in (5.8) we come, correspondingly, to the case considered in Theorem 5.1.

We integrate the system (5.8) completely when \(\tau = 0\). This is achieved by introducing the new variable \(du = f_1 f_2 f_3 dt\), which allow to determine \(f_1 f_2 f_3\) as a constant and \((i,j,k)\) is a permutation of \((1,2,3)\). Thus \(f_i = \frac{f}{(u+a_1)(u+a_2)(u+a_3)}\), \(j = 1,2,3\). With the help of these three equations and the first equation of (5.8) we come to

\[
\frac{dF_i}{f} = \frac{du}{f} \left( \frac{1}{u+a_1} + \frac{1}{u+a_2} + \frac{1}{u+a_3} \right),
\]

and hence \(f^9 = C^9 (u+a_1)(u+a_2)(u+a_3)\) for some constant \(C\). Now, the equations \(f_1 f_2 f_3 = 6(u+a_k)\) yield

\[
F_i = \sqrt{\frac{6}{C}} \left( \frac{(u+a_j)^4(u+a_k)^4}{(u+a_i)^2} \right)^{1/9}
\]

and then the definition of \(u\) shows

\[
dt = (C/6)^{3/2} \frac{du}{(u+a_1)(u+a_2)(u+a_3))^{1/3}}.
\]

If we impose also the system (5.9), in which we substitute \(2(f_i + f_j) = 3f' - 2f_k\) and \(f(f_i f_j)' = 6 f_1 f_2 f_3 - 2 \frac{3}{4} (f_1 + f_2 + f_3) f_i f_j\) (using the equations of (5.8) and \(\tau = 0\)), we see that

\[
\frac{dF_i}{f} = \frac{10 f_j f_k}{3f} (2 f_j - f_i - f_j) \eta_j \wedge \eta_k \wedge dt \mod <F_1,F_2,F_3>.
\]

Thus, \(d\Phi = 0\) and \(<F_1,F_2,F_3>\) is a differential ideal if and only if \(f_1 = f_2 = f_3\) which yield.

**Proposition 5.3.** The metric on the product of the seven dimensional quaternionic Heisenberg group with the real line defined (on \(\mathbb{R}^8\)) by

\[
g = C \left( (u+a_1)(u+a_2)(u+a_3) \right)^{1/9} (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + \frac{6}{C} \left( \frac{(u+a_2)^8(u+a_3)^3}{(u+a_1)^{10}} \right)^{1/9} (dx_5 + 2 x_1 dx_2 + x_3 dx_4)^2 + \frac{6}{C} \left( \frac{(u+a_3)^8(u+a_1)^3}{(u+a_2)^{10}} \right)^{1/9} (dx_6 + 2 x_1 dx_3 + x_4 dx_2)^2 + \frac{6}{C} \left( \frac{(u+a_1)^8(u+a_2)^3}{(u+a_3)^{10}} \right)^{1/9} (dx_7 + 2 x_1 dx_4 + x_2 dx_3)^2 + \left( \frac{C}{6} \right)^3 \frac{du^2}{(u+a_1)(u+a_2)(u+a_3)^2/3},
\]

where \(a_1, a_2\) and \(a_3\) are three constants not all of them equal to each other, supports an almost quaternion hermitian structure which has closed fundamental form, but is not quaternionic Kähler.

**Remark 5.4.** Using a suitable computer program one can check the the metrics (5.10) are Einstein exactly when \(f_1 = f_2 = f_3\), i.e. when are quaternionic Kähler.

We note that one of the arbitrary constants in (5.10) is unnecessary since a translation of the unknown \(u\) does not change the metric.

Let us also remark that the quaternionic Kähler metric (5.6) is obtained from the general family (5.10) by taking \(\frac{C}{6} = \frac{a^2}{4}\) and \(v = e^{at}\) when the constants are the same \(a_1 = a_2 = a_3\) and we use \(u + a_1\) as a variable, which is denoted also by \(u\).

If one takes a solution of the system (5.9) which does not satisfy the system (5.8), one could obtain a non quaternionic Kahler manifold with an almost quaternion hermitian structure such that \(<F_1,F_2,F_3>\) is a differential ideal and the fundamental four form is non-parallel (see also the paragraph after [50, Corollary 2.4]).
5.2. New quaternionic Kähler metrics from the zero-torsion qc-flat qc structure on \( G_1 \). Here we consider the Lie group defined by the structure equations (3.2), which can be described in local coordinates \( \{t, x, y, z, x_5, x_6, x_7\} \) as follows

\[
\begin{align*}
    e^1 &= -dt, \\
    e^2 &= \frac{1}{2} x_6 \, dx + \frac{1}{2} x_5 \cos x \, dy + \left( \frac{1}{2} x_6 \cos y + \frac{1}{2} x_5 \sin y \sin x \right) dz - \frac{1}{2} x_7 \, dt + \frac{1}{2} dx_7, \\
    e^3 &= -\frac{1}{2} x_7 \, dx + \frac{1}{2} x_5 \sin x \, dy + \left( -\frac{1}{2} x_7 \cos y - \frac{1}{2} x_5 \sin y \cos x \right) dz - \frac{1}{2} x_6 \, dt + \frac{1}{2} dx_6, \\
    e^4 &= \left( -\frac{1}{2} x_7 \cos x - \frac{1}{2} x_5 \sin x \right) dy - \frac{1}{2} \sin y \left( -x_6 \cos x + x_7 \sin x \right) dz - \frac{1}{2} x_5 \, dt + \frac{1}{2} dx_5, \\
    \eta_1 &= e^5 = -x_6 \, dx + (-x_5 \cos x - 2 \sin x) \, dy \\
    &\quad + (-x_6 \cos y - \sin y \sin x \, x_5 + 2 \sin y \cos x) \, dz + x_7 \, dt - dx_7, \\
    \eta_2 &= e^6 = x_7 \, dx + (2 \cos x - x_5 \sin x) \, dy \\
    &\quad + (x_7 \cos y + 2 \sin y \sin x + x_5 \sin y \cos x) \, dz + x_6 \, dt - dx_6, \\
    \eta_3 &= e^7 = -2 \, dx + (\cos x \, x_7 + x_5 \sin x) \, dy \\
    &\quad + (-2 \cos y + x_7 \sin y \sin x - x_6 \sin y \cos x) \, dz + x_5 \, dt - dx_5.
\end{align*}
\]

In this case \( \tau = -\frac{1}{4} \) in (5.2), and the corresponding quaternionic Kähler metric on \( G_1 \) is (using \( a/4 \) as a constant)

\[
g = u \left( (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \right) + \frac{au^2 - u}{4} \left( (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 \right) + \frac{1}{au^2 - u} \, du^2,
\]

for \( au^2 - u > 0 \). The Ricci tensor is given by \( Ric = -4ag \).

The metric (5.12) seems to be a new explicit quaternionic Kähler metric. In local coordinates \( \{v^1 = t, v^2 = x, v^3 = y, v^4 = z, v^5 = x_5, v^6 = x_6, v^7 = x_7, v^8 = u\} \) the metric has the expression written in Appendix 1.

6. \( Sp(1)Sp(1) \) structures and \( Spin(7) \)-holonomy metrics

An \( Sp(1)Sp(1) \) structure on a seven dimensional manifold \( M^7 \), whose 2-forms \( \omega_i \) and 1-forms \( \eta_j \) are globally defined, induces a \( G_2 \)-form \( \phi \) given by

\[
\phi = 2 \omega_1 \wedge \eta_1 + 2 \omega_2 \wedge \eta_2 - 2 \omega_3 \wedge \eta_3 + 2 \eta_1 \wedge \eta_2 \wedge \eta_3.
\]

The Hodge dual \( *\phi \) is

\[
*\phi = -\left( \omega_1 \wedge \omega_1 + 2 \omega_1 \wedge \eta_2 \wedge \eta_3 + 2 \omega_2 \wedge \eta_3 \wedge \eta_1 - 2 \omega_3 \wedge \eta_1 \wedge \eta_2 \right).
\]

Consider the \( Spin(7) \)-form \( \Psi \) on \( M^7 \times \mathbb{R} \) defined by [11]

\[
\Psi = F_1 \wedge F_1 + F_2 \wedge F_2 - F_3 \wedge F_3 = -*\phi - \phi \wedge dt,
\]

where the 2-forms \( F_1, F_2, F_3 \) are given by (4.7).

Following Hitchin, [35], the \( Spin(7) \)-form \( \Psi \) is closed if and only if the \( G_2 \) structure is cocalibrated, \( d*\phi = 0 \), and the Hitchin flow equations \( \partial_t(*\phi) = -d\phi \) are satisfied, i.e.

\[
d(*\phi) = 0, \quad \partial_t(*\phi) = -d\phi.
\]

**Theorem 6.1.** Let \( M \) be a smooth seven dimensional manifold equipped with an \( Sp(1)Sp(1) \) structure such that the 3-form \( \phi \) determined by (6.1) is globally defined and, for some constant \( \tau \neq 0 \), the structure equations (5.1) hold for any cyclic permutation \( (i, j, k) \) of \( (1, 2, 3) \). For any constant \( a \), the manifold \( M \times I \), where \( I \subset \mathbb{R} \), has a parallel \( Spin(7) \) structure given by the following metric and fundamental 4-form

\[
\begin{align*}
g &= ug_{\mu} + \frac{\tau u^{5/3} - a}{5u^{2/3}} \left( (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 \right) + \frac{5u^{2/3}}{36(\tau u^{5/3} - a)} \, du^2, \\
\psi &= F_1 \wedge F_1 + F_2 \wedge F_2 - F_3 \wedge F_3,
\end{align*}
\]

where \( u=g_{\mu} \).
where locally

\( F_i(u) = u\omega_i - \epsilon_i \frac{\tau u^{5/3} - a}{5u^{2/3}} \eta_j \wedge \eta_k - \epsilon_i \frac{1}{6} \eta_i \wedge du. \)

where \( \epsilon_1 = \epsilon_2 = 1 \) and \( \epsilon_3 = -1 \).

**Proof.** We evolve the structure as in (5.7). Using the structure equations (5.1) one easily obtains that the equation \( d(\ast \phi) = 0 \) is satisfied, and the second equation of the system (6.4) is equivalent to the system

\[
\begin{align*}
(f f_2 f_3)' - 2\tau f(f_1 - f_2 + f_3) - 2f_1 f_2 f_3 &= 0, \\
(f f_1 f_3)' - 2\tau f(-f_1 + f_2 + f_3) - 2f_1 f_2 f_3 &= 0, \\
(f f_1 f_2)' - 2\tau f(f_1 + f_2 + f_3) + 2f_1 f_2 f_3 &= 0.
\end{align*}
\]

Taking \( f_1 = f_2 = -f_3 \) in (6.7) we come to the ODE system

\[
3f f'' + (f')^2 - 18\tau f = 0, \quad f_1 = f_2 = -f_3 = \frac{1}{6} f'.
\]

To solve this differential equation, we use \( v = f^{4/3} \) as a variable. Equation (6.8) shows that \( \left( \frac{dv}{df} \right)^2 = \frac{64(\tau v^{5/3} - a)}{5} \), where \( a \) is a constant. Hence, \( \left( \frac{dv}{df} \right)^2 = \left( \frac{dv}{du} \right)^2 \left( \frac{du}{df} \right)^2 = \frac{5f^{2/3}}{36(\tau f^{5/3} - a)} \), which implies that \( f_1^2 = f_2^2 = f_3^2 = \frac{4}{5}(f')^2 = \frac{3f^{5/3}}{5f^{2/3} - a} \). Recall that we also have \( F_i(t) = f(t)\omega_i + f_j(t) f_k(t) \eta_j \wedge \eta_k - f_i(t) \eta_i \wedge dt \). Renaming \( f \) to \( u \) gives the metric and the \( \text{Spin}(7) \) form \( \psi \) in (6.5) with (6.6).

i) **\( \text{Spin}(7) \)-holonomy metrics from the quaternionic Heisenberg group.** Using the seven dimensional quaternionic Heisenberg group with structure equations (2.14), taken for \( n = 1 \), the corresponding eight dimensional \( \text{Spin}(7) \)-holonomy metric written with respect to the parameter \( u = (at + b)^{1/4} \) is

\[
g = u^3 ((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{a^2}{16} u^{-2} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{4}{a^2} u^6 du^2.
\]

These \( \text{Spin}(7) \)-holonomy metrics are found in [29, Section 4.3.1].

ii) **\( \text{Spin}(7) \)-holonomy metrics from a 3-Sasakian manifold.** This case was investigated in general in [5] and explicit solutions in particular cases are known (see [5] and references therein). We use again only the particular solution to (6.7) found above. Thus, starting with a 3-Sasakian manifold with structure equations (2.8) the resulting metric is

\[
g = u ((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2) + \frac{u^{5/3} - a}{5u^{2/3}} ((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2) + \frac{4a^{2/3}}{36(u^{5/3} - a)} du^2.
\]

This is the (first) complete metric with holonomy \( \text{Spin}(7) \) constructed by Bryant and Salamon [12, 30].

6.1. **New \( \text{Spin}(7) \)-holonomy metrics from the quaternionic Heisenberg group.** New metrics can be obtained similarly to the derivation of (5.10). Namely, we integrate the system (6.7) when \( \tau = 0 \) to obtain the next family of \( \text{Spin}(7) \)-holonomy metrics which seems to be new

\[
g = C ((u + a_1)(u + a_2)(a_3 - u)) ((dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + \frac{2}{C (u + a_1)} (dx_5 + 2x_1 dx_2 + 2x_3 dx_4)^2 + \frac{2}{C (u + a_2)} (dx_6 + 2x_1 dx_3 + 2x_4 dx_2)^2 + \frac{2}{C (a_3 - u)} (dx_7 + 2x_1 dx_4 + 2x_2 dx_3)^2 + \frac{C^3}{8} (u + a_1)^2 (u + a_2)^2 (a_3 - u)^2 du^2.
\]

Taking \( a_2 = -a_3 = a_1 \) into (6.10) one gets the \( \text{Spin}(7) \)-holonomy metrics (6.9). Since the coefficients of the metrics (6.10) are continuous with respect to the parameters, and since the holonomy is equal to \( \text{Spin}(7) \) for \((a_2, a_3) = (a_1, -a_1)\) then the same holds for any \((a_2, a_3)\) in a neighbourhood of \((a_1, -a_1)\). Thus, we get a three parameter family of metrics with holonomy equal to \( \text{Spin}(7) \) which seem to be new.
6.2. New $Spin(7)$-holonomy metrics from a zero-torsion qc-flat qc structure on $G_1$. Consider the 7-dimensional Lie group defined in (3.2). From Theorem 6.1 we obtain the metrics

\begin{equation}
\begin{aligned}
g &= u \left((e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2\right) + \frac{(a - u^{5/3})}{20u^{2/3}} \left((\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2\right) + \frac{5u^{2/3}}{9(a - u^{5/3})} du^2.
\end{aligned}
\end{equation}

These metrics have holonomy equal to $Spin(7)$. In local coordinates $\{v^1 = t, v^2 = x, v^3 = y, v^4 = z, v^5 = x_5, v^6 = x_6, v^7 = x_7, v^8 = u\}$ the $Spin(7)$-holonomy metric is written in Appendix 2.

7. Hyper Kähler metrics in dimension four

In this section we recover some of the known Ricci-flat gravitational instantons in dimension four applying our method from the preceding section lifting the $sp(0)$-hypo structures on the non-Euclidean Bianchi type groups of class $\Lambda$.

Let $G_3$ is a three dimensional Lie group with Lie algebra $g_3$ and $e^1, e^2, e^3$ be a basis of left invariant 1-forms. We consider the $Sp(1)$ structure on $g_3 \times \mathbb{R}^+$ defined by the following 2-forms

\begin{equation}
\begin{aligned}
F_1(t) &= e^1(t) \wedge e^2(t) + e^3(t) \wedge f(t) dt, \\
F_2(t) &= e^1(t) \wedge e^3(t) - e^2(t) \wedge f(t) dt, \\
F_3(t) &= e^2(t) \wedge e^3(t) + e^1(t) \wedge f(t) dt,
\end{aligned}
\end{equation}

where $f(t)$ is a function of $t$ and $e^i(t)$ depend on $t$. With the help of Hitchin’s theorem, it is straightforward to prove the next

**Proposition 7.1.** The $Sp(1)$ structure $(F_1, F_2, F_3)$ is hyper Kähler if and only if

\begin{equation}
d\eta^{12} = d\eta^{13} = d\eta^{23} = 0
\end{equation}

and the following evolution equations hold

\begin{equation}
\frac{\partial}{\partial t} e^{ij}(t) = -f(t) d\eta^{kl}(t).
\end{equation}

The hyper Kähler metric is given by

\begin{equation}
g = (e^1(t))^2 + (e^2(t))^2 + (e^3(t))^2 + f^2(t) dt^2.
\end{equation}

**The group $SU(2)$, Bianchi type IX.** Let $G_3 = SU(2) = S^3$ be described by the structure equations

\begin{equation}
d\eta = -\eta^{jk}.
\end{equation}

In terms of Euler angles the left invariant forms $e^i$ are given by

\begin{equation}
e^1 = \sin \psi d\theta - \cos \psi \sin \theta d\phi, \quad e^2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad e^3 = d\psi + \cos \theta d\phi.
\end{equation}

Clearly (7.2) are satisfied. We evolve the $SU(2)$ structure as

\begin{equation}
e^s(t) = f_s(t) e^s, \quad s = 1, 2, 3, \quad (\text{no summation on } s)
\end{equation}

where $f_s$ are functions of $t$.

Using the structure equations (7.5) we reduce the evolution equations (7.3) to the following system of ODEs

\begin{equation}
\frac{d}{dt}(f_1 f_2) = f f_3, \quad \frac{d}{dt}(f_1 f_3) = f f_2, \quad \frac{d}{dt}(f_2 f_3) = f f_1.
\end{equation}

The system (7.8) is equivalent to the following ‘BGPP’ [6] system

\begin{equation}
\frac{d}{dt} f_1 = \frac{f_2^2 + f_3^2 - f_1^2}{2f_2 f_3}, \quad \frac{d}{dt} f_2 = \frac{f_3^2 + f_1^2 - f_2^2}{2f_1 f_3}, \quad \frac{d}{dt} f_3 = \frac{f_2^2 + f_3^2 - f_1^2}{2f_1 f_2}.
\end{equation}

The equations (7.9) admit the triaxial Bianchi IX BGPP [6] hyper Kähler metrics by taking $f = f_1 f_2 f_3$ and all $f_i$ different (see also [31]) and Eguchi-Hanson [24] hyper Kähler metric when two of the functions are equal.
7.0.1. The general solution. With the substitution \( x_i = (f_j f_k)^2 \), the system (7.8) becomes

\[
\frac{dx_i}{dr} = 2(x_1 x_2 x_3)^{1/4},
\]

in terms of the parameter \( dr = f dt \). Hence the functions \( x_i \) differ by a constant, i.e., there is a function \( x(r) \) such that \( x(r) = x_1 + a_1 = x_2 + a_2 = x_3 + a_3 \). The equation for \( x(r) \) is

\[
\frac{dx}{dr} = 2 ((x - a_1)(x - a_2)(x - a_3))^{1/4}, \quad \text{i.e.,} \quad dr = \frac{1}{2} ((x - a_1)(x - a_2)(x - a_3))^{1/4} dx.
\]

If we let \( g(x) = \frac{1}{2} ((x - a_1)(x - a_2)(x - a_3))^{-1/4} \) and take into account \( x_i = (f_j f_k)^2 \), we see from (7.8) that the functions \( f_i(x) \) satisfy

\[
\frac{d}{dx} (x_i^{1/2}) = g(x) f_i.
\]

Solving for \( f_i \) we showed that the general solution of (7.8) is

\[
f_1(x) = \frac{(x - a_1)^{1/4}(x - a_2)^{1/4}}{(x - a_3)^{1/4}}, \quad f(t) = g(x(t)) x'(t), \quad g(x) = \frac{1}{2} ((x - a_1)(x - a_2)(x - a_3))^{-1/4},
\]

where \( a_1, a_2 \) and \( a_3 \) are constants, and \( x \) is an auxiliary independent variable (substituting any function \( x = x(t) \) gives a solution of (7.8) in terms of \( t \)).

7.0.2. Eguchi-Hanson instantons. A particular solution to (7.8) is obtained by taking \( x = (t/2)^4 \) and \( a_1 = a_2 = \frac{1}{2} \), \( a_3 = 0 \), which gives

\[
f_1 = f_2 = \frac{t}{2}, \quad f_3 = \frac{t}{2} \sqrt{\frac{t^4 - a}{t^4}}, \quad f = \sqrt{\frac{t^4}{t^4 - a}}.
\]

This is the Eguchi-Hanson instanton [24] with the metric given by

\[
g = \frac{t^2}{4} \left[ (e^1)^2 + (e^2)^2 + \left( 1 - \frac{a}{t^4} \right)(e^3)^2 + \left( 1 - \frac{a}{t^4} \right)^{-1} (dt)^2 \right].
\]

7.0.3. Triaxial Bianchi type IX BGPP metrics [6]. The substitution \( x = t^4, a_1 = a^4, a_2 = b^4 \) and \( a_3 = c^4 \) gives

\[
f_1(t) = \frac{(t^4 - b^4)^{1/2} (t^4 - c^4)^{1/2}}{(t^4 - a^4)^{1/2}}, \quad f_2(t) = \frac{(t^4 - a^4)^{1/2} (t^4 - c^4)^{1/2}}{(t^4 - b^4)^{1/2}}, \quad f_3(t) = \frac{(t^4 - a^4)^{1/2} (t^4 - b^4)^{1/2}}{(t^4 - c^4)^{1/2}}, \quad f(t) = \frac{2t^3}{(t^4 - a^4)^{1/2} (t^4 - b^4)^{1/2} (t^4 - c^4)^{1/2}}.
\]

These are the triaxial Bianchi IX metrics discovered in [6] (see also [31, 26, 27]), which do not have any tri-holomorphic U(1) isometries [27]. In the derivation above we avoided the use of elliptic functions.

The group SU(1,1)-Bianchi type VIII. Bianchi type VIII are investigated in [48, 49, 47].

Let \( G_3 = SU(1,1) \) be described by the structure equations

\[
de^1 = -e^{23}, \quad de^2 = e^{31}, \quad de^3 = -e^{12}.
\]

In terms of local coordinates the left invariant forms \( e^i \) are given by

\[
e^1 = \sin \theta \psi + \cosh \psi \sin \theta \phi, \quad e^2 = \cosh \psi \phi + \sinh \psi \sin \theta \phi, \quad e^3 = d\psi - \cos \theta \phi.
\]

Clearly (7.2) are satisfied. We evolve the SU(1,1) structure as in (7.7). Using the structure equations (7.15) we reduce the evolution equations (7.3) to the following system of ODEs

\[
\frac{\partial}{\partial t} (f_1 f_2) = ff_3, \quad \frac{\partial}{\partial t} (f_1 f_3) = -ff_2, \quad \frac{\partial}{\partial t} (f_2 f_3) = ff_1.
\]

Solutions to the above system yield corresponding hyper Kähler metrics (7.4) indicated in [6].
**Triaxial Bianchi type VIII metrics.** Working as in 7.0.1 we obtain the following system for the functions $x_i$

$$\frac{dx_i}{dr} = 2(x_1x_2x_3)^{1/4}, \quad \frac{dx_2}{dr} = -2(x_1x_2x_3)^{1/4}. $$

Solving for $f_i$, as in the derivation (7.11), we find the general solution of (7.16) is

$$f_1(x) = \frac{(x - a_3)^{1/4}(a_2 - x)^{1/4}}{(x - a_1)^{1/4}}, \quad f_2(x) = \frac{(x - a_1)^{1/4}(a_3 - x)^{1/4}}{(a_2 - x)^{1/4}}, \quad f_3(x) = \frac{(x - a_2)^{1/4}(a_1 - x)^{1/4}}{(a_3 - x)^{1/4}}.$$

(7.17)

where $a_1$, $a_2$, and $a_3$ are constants, and $x$ is an auxiliary independent variable (substituting any function $x = x(t)$ gives a solution of (7.8) in terms of $t$).

Taking $f = f_1f_2f_3$ and all $f_i$ different, we obtain explicit expression of the triaxial Bianchi VIII solutions indicated in [6].

A particular solution is obtained by letting $a_1 = a_3 = 0, a_2 = \frac{\sqrt{2}}{16}$ which gives

$$f_1 = f_3 = \frac{1}{2}(a - t^4)^{1/4}, \quad f_2 = \frac{t^2}{2}(a - t^4)^{-1/4}, \quad f = t(a - t^4)^{-1/4}, \quad -a < t^4 < a.$$ 

The resulting hyper Kähler metric is given by

$$g = \frac{1}{2}(a - t^4)^{1/4}\left((e_1)^2 + \frac{t^2}{(a - t^4)^1/4}(e_2)^2 + (e_3)^2 + \frac{2t}{(a - t^4)^{1/4}}dt^2\right),$$

where the forms $e^i$ are given by (7.15).

**The Heisenberg group $H^3$, Bianchi type II, Gibbons-Hawking class.** Consider the two-step nilpotent Heisenberg group $H^3$ defined by the structure equations

$$(7.18) \quad de^1 = de^2 = 0, \quad de^3 = -e^2; \quad e^1 = dx, \quad e^2 = dy, \quad e^3 = dz - \frac{1}{2}xdy + \frac{1}{2}ydx.$$

The necessary conditions (7.2) are satisfied. We evolve the structure according to (7.7). The structure equations (7.18) reduce the evolution equations (7.3) to the following system of ODEs

$$\frac{\partial}{\partial t}(f_1f_2) = f_3f_5, \quad \frac{\partial}{\partial t}(f_1f_3) = 0, \quad \frac{\partial}{\partial t}(f_2f_3) = 0.$$

(7.19)

Working as in the previous example, i.e., using the same substitutions we see that the function $x_i$ satisfy the system

$$\frac{dx_1}{dr} = 2(x_1x_2x_3)^{1/4}, \quad \frac{dx_2}{dr} = \frac{dx_2}{dr} = 0.$$ 

The general solution of thus system is

$$\frac{dx_1}{dr} = 2(x_1x_2x_3)^{1/4}, \quad \frac{dx_2}{dr} = \frac{dx_2}{dr} = 0.$$

(7.20)

where $a$, $b$, and $c$ are constants. Therefore, using again $f_i = \left(\frac{e_i}{e_i}ight)^{1/4}$, the general solution of (7.19) is

$$f_1 = \left(\frac{b}{a}\right)^{1/4}\left(\frac{3}{2}(ab)^{1/4}r + c\right)^{1/3}, \quad f_2 = \left(\frac{a}{b}\right)^{1/4}\left(\frac{3}{2}(ab)^{1/4}r + c\right)^{1/3}, \quad f_3 = \frac{(ab)^{1/4}}{\left(\frac{3}{2}(ab)^{1/4}r + c\right)^{1/3}}.$$ 

(7.21)

A particular solution is obtained by taking $c = 0$ and $a = b = 1$, which gives

$$f_1 = f_2 = \lambda t^{1/3}, \quad f_3 = f_3^{-1},$$

with $\lambda = \left(\frac{4}{9}\right)^{1/3}$. The substitution $t = \lambda^2t^{1/3}$ gives $f_1 = f_2 = f = t^{1/4}, \quad f_3 = t^{1/2}$. This is the hyper Kähler metric, first written in [47, 48],

$$g = t\left[dt^2 + dx^2 + dy^2\right] + \frac{1}{t}\left[dz - \frac{1}{2}xdy + \frac{1}{2}ydx\right]^2$$

where
belonging to the Gibbons-Hawking class \cite{gibbonshawking} with an $S^1$-action and known also as Heisenberg metric \cite{heisenberg} (see also \cite{heisenberg, heisenberg, heisenberg, heisenberg, heisenberg}).

**Rigid motions of euclidean 2-plane-Bianchi VII$_0$ metrics.** We consider the group $E_2$ of rigid motions of Euclidean 2-plane defined by the structure equations

\begin{equation}
7.22 \quad de^1 = 0, \quad de^2 = e^{13}, \quad de^3 = -e^{12}; \quad e^1 = d\phi, \quad e^2 = \sin \phi dx - \cos \phi dy, \quad e^3 = \cos \phi dx + \sin \phi dy.
\end{equation}

Clearly (7.2) are satisfied. We evolve the structure as in (7.7). Using the structure equations (7.22) we reduce the evolution equations (7.3) to the following system of ODE

\begin{equation}
7.23 \quad \frac{\partial}{\partial t} (f_1 f_2) = ff_3, \quad \frac{\partial}{\partial t} (f_1 f_3) = ff_2, \quad \frac{\partial}{\partial t} (f_2 f_3) = 0.
\end{equation}

With the substitution $x_i = (f_j f_k)^2$, the above system becomes

\begin{equation}
7.24 \quad \frac{dx_1}{dr} = 0, \quad \frac{dx_2}{dr} = \frac{dx_3}{dr} = 2(x_1 x_2 x_3)^{1/4},
\end{equation}

in terms of the parameter $dr = f dt$. Hence, there is a function $x(r)$ and three constants $a_1$, $a_2$, $a_3$, such that, $x(r) = x_2 + a_2 = x_3 + a_3, x_1 = a_1$. The equation for $x(r)$ is

\begin{equation}
7.25 \quad \frac{dx}{dr} = 2(a_1(x-a_2)(x-a_3))^{1/4}, \quad i.e., \quad dr = \frac{1}{2(a_1(x-a_2)(x-a_3))^{1/4}} dx.
\end{equation}

If we let $g(x) = \frac{1}{2}((a_1(x-a_2)(x-a_3))^{-1/4}$, and take into account $x_i = (f_j f_k)^2$, we see from (7.23) that the functions $f_i(x)$ satisfy

\begin{equation}
\frac{d}{dx} \left( (x-a_i)^{1/2} \right) = g(x)f_i, \quad i = 2, 3.
\end{equation}

Solving for $f_i$ we show that the general solution of (7.8) is

\begin{equation}
7.26 \quad f_1(x) = \frac{(x-a_2)^{1/4}(x-a_3)^{1/4}}{a_1^{1/4}}, \quad f_2(x) = \frac{a_1^{1/4}(x-a_3)^{1/4}}{(x-a_2)^{1/4}}, \quad f_3(x) = \frac{a_1^{1/4}(x-a_2)^{1/4}}{(x-a_3)^{1/4}},
\end{equation}

where $a_1$, $a_2$ and $a_3$ are constants, and $x$ is an auxiliary independent variable (substituting any function $x = x(t)$ gives a solution of (7.23) in terms of $t$).

**Vacuum solutions of Bianchi type VII$_0$.** When $f_2 = f_3^{-1}$, $f_1 = f$ we have

\begin{equation}
\frac{\partial}{\partial t} (f f_3^{-1}) = ff_3, \quad \frac{\partial}{\partial t} (f f_3) = ff_3^{-1},
\end{equation}

with solution of the form $f f_3 + f f_3^{-1} = A e^t, \quad f f_3^{-1} - f f_3 = B e^{-t}$. Hence,

\begin{equation}
7.27 \quad f = f_1 = \frac{1}{2}(A e^t + B e^{-t})^{1/2}(A e^t - B e^{-t})^{1/2}, \quad f_3 = f_2^{-1} = (A e^t + B e^{-t})^{-1/2}(A e^t - B e^{-t})^{-1/2},
\end{equation}

and the hyper Kähler metric is

\begin{equation}
\frac{1}{4} \left( A^2 e^{2t} - B^2 e^{-2t} \right) \left( dt + d\phi^2 + \frac{4}{(A e^t - B e^{-t})^2} (e^2)^2 + \frac{4}{(A e^t + B e^{-t})^2} (e^3)^2 \right),
\end{equation}

where $e^2, e^3$ are given by (7.22).

In particular, setting $A = B$ in (7.26) we obtain

\begin{equation}
g = \frac{A^2}{2} \sinh 2t \left( dt^2 + d\phi^2 \right) + \coth t(e^2)^2 + \tanh t(e^3)^2,
\end{equation}

which is the vacuum solutions of Bianchi type VII$_0$ \cite{heisenberg, heisenberg} with group of isometries $E_2$ \cite{heisenberg}, (see also \cite{heisenberg}).

**Rigid motions of Lorentzian 2-plane-Bianchi VII$_0$ metrics.** Now we consider the group of rigid motions $E(1,1)$ of Lorentzian 2-plane defined by the structure equations and coordinates as follows

\begin{equation}
7.27 \quad de^1 = 0, de^2 = e^{13}, de^3 = e^{12}; \quad e^1 = d\phi, \quad e^2 = \sinh \phi dx + \cosh \phi dy, \quad e^3 = \cosh \phi dx + \sin \phi dy.
\end{equation}
We evolve the structure as in (7.7). Using the structure equations (7.27), the evolution equations (7.3) turn into the next system of ODEs
\[
\frac{\partial}{\partial t}(f_1 f_2) = -f f_3, \quad \frac{\partial}{\partial t}(f_1 f_3) = f f_2, \quad \frac{\partial}{\partial t}(f_2 f_3) = 0.
\]
(7.28)

The general solution of (7.28) is
\[
f_1(x) = \frac{(x - a_2)^{1/4}(a_3 - x)^{1/4}}{a_1^{1/4}}, \quad f_2(x) = \frac{a_1^{1/4}(a_3 - x)^{1/4}}{(x - a_2)^{1/4}}, \quad f_3(x) = \frac{a_2^{1/4}(x - a_2)^{1/4}}{(a_3 - x)^{1/4}},
\]
(7.29)
where \(a_1, a_2, a_3\) are constants, and \(x\) is an auxiliary independent variable (substituting any function \(x = x(t)\) gives a solution of (7.23) in terms of \(t\)). When \(f_2 = f_3^{-1}, f_1 = f\) we have \(\frac{\partial}{\partial t}(f f_3^{-1}) = -f f_3, \quad \frac{\partial}{\partial t}(f f_3) = f f_3^{-1}\) with solution of the form
\[
f = f_1 = \frac{1}{2}(a \cos t + b \sin t)^{\frac{3}{2}}(a \cos t - b \sin t)^{\frac{3}{2}}, \quad f_2 = f_3^{-1} = (a \cos t + b \sin t)^{\frac{1}{2}}(a \sin t - b \cos t)^{\frac{1}{2}},
\]
and the hyper Kähler metric is given by
\[
g = \frac{1}{4}(a^2 \sin^2 t - b^2 \cos^2 t)\left(dt^2 + d\phi^2 + \frac{4}{(a \sin t + b \cos t)^2}(e^2)^2 + \frac{4}{(a \sin t - b \cos t)^2}(e^3)^2\right),
\]
(7.30)
where \(e^2, e^3\) are given by (7.27). Introducing \(t_0\) and \(r_0\) by letting \(r_0 = \sqrt{a^2 + b^2}, \cos t_0 = a/\sqrt{a^2 + b^2}\) and \(\sin t_0 = b/\sqrt{a^2 + b^2}\) the above metric can be put in the form
\[
g = \frac{1}{4}\left(r_0^2 \sin(t + t_0) \sin(t - t_0)\right)\left(dt^2 + d\phi^2 + \frac{4}{r_0^2 \sin^2(t + t_0)}(e^2)^2 + \frac{4}{r_0^2 \sin^2(t - t_0)}(e^3)^2\right).
\]
(7.31)

**Bianchi type** \(VI_0\) In particular, setting \(a = b\) in (7.31) we obtain \(r_0^2 = 2a^2\), \(\sin t_0 = \cos t_0 = \frac{\sqrt{2}}{2}\). Taking \(\tau = t + \frac{\phi}{4}\), the metric (7.31) takes the form
\[
g = \frac{a^2}{4} \sin 2\tau \left(\frac{g}{4} \right) + \cot \tau (e^2)^2 + \tan \tau (e^3)^2,
\]
which is the vacuum solutions of Bianchi type \(VI_0\) [47, 48] with group of isometries \(E(1, 1)\) [32], (see also [57]).

**8. Hyper symplectic (hyper para Kähler) metrics in dimension 4**

In this section, following the method of the preceding section, we present explicit hyper symplectic (hyper para Kähler) metrics in dimension four of signature (2,2). The construction gives a kind of duality between hyper Kähler instantons and hyper para Kähler structures.

We recall that an almost hyper paracomplex structure on a 4\(n\) dimensional space is a triple \((J, P_1, P_2)\) satisfying the paraquaternionionic identities
\[
J^2 = -P_1^2 = -P_2^2 = -1, \quad JP_1 = -P_1 J = P_2.
\]
A compatible metric \(g\) satisfies
\[
g(J, J) = -g(P_1, P_1) = -g(P_2, P_2) = g(\ldots)
\]
and is necessarily of neutral signature \((2n, 2n)\). The fundamental 2-forms are defined by
\[
\Omega_1 = g(\ldots, J), \quad \omega_2 = g(\ldots, P_1), \quad \Omega_3 = g(\ldots, P_2).
\]
When these forms are closed the structure is said to be hypersymplectic [34]. This implies (adapting the computations of Atiyah-Hitchin [2] for hyper Kähler manifolds) that the structures are integrable and parallel with respect to the Levi-Civita connection [34, 18]. Sometimes a hyper symplectic structure is called also neutral hyper Kähler [44], hyper para Kähler [42]. In dimension 4 an almost hyper paracomplex structure is locally equivalent to an oriented neutral conformal structure, or an \(Sp(1, \mathbb{R})\) structure, and the integrability
implies the anti-self-duality of the corresponding neutral conformal structure [44, 42]. In particular, a hyper symplectic structure in dimension four underlines an anti-self-dual Ricci-flat neutral metric. For this reason such structures have been used in string theory [53, 36, 43, 3, 37, 13] and integrable systems [22, 4, 28].

Let \( G_3 \) be a three dimensional Lie group with Lie algebra \( g_3 \) and \( e^1, e^2, e^3 \) be a basis of left invariant 1-forms. We consider the \( Sp(1, \mathbb{R}) \) structure on \( g_3 \times \mathbb{R}^+ \) defined by the following 2-forms

\[
\begin{align*}
\Omega_1(t) &= -e^1(t) \wedge e^2(t) + e^3(t) \wedge f(t)dt, \\
\Omega_2(t) &= e^1(t) \wedge e^3(t) - e^2(t) \wedge f(t)dt, \\
\Omega_3(t) &= e^2(t) \wedge e^3(t) + e^1(t) \wedge f(t)dt,
\end{align*}
\]

(8.1)

where \( f(t) \) is a function of \( t \) and \( e^i(t) \) depend on \( t \).

With the help of Hitchin’s theorem [34], it is straightforward to prove the next

**Proposition 8.1.** The \( Sp(1, \mathbb{R}) \) structure \((\Omega_1, \Omega_2, \Omega_3)\) is hyper para K"ahler if and only if

\[
d e^{12} = d e^{13} = d e^{23} = 0,
\]

(8.2)

and the following evolution equations hold

\[
\frac{\partial}{\partial t} e^{12}(t) = f(t)de^3(t), \quad \frac{\partial}{\partial t} e^{13}(t) = f(t)de^2(t), \quad \frac{\partial}{\partial t} e^{23}(t) = -f(t)de^1(t).
\]

(8.3)

The hyper para K"ahler metric is given by

\[
g = (e^1)^2 + (e^2)^2 - (e^3)^2 - f^2(t)dt^2.
\]

(8.4)

**The group \( SU(2) \).** Let \( G_3 = SU(2) = S^3 \) be described by the structure equations (7.5). Clearly (8.2) are satisfied. We evolve the \( SU(2) \) structure according to (7.7).

Using the structure equations (7.5), we reduce the evolution equations (8.3) to the following system of ODEs

\[
\frac{d}{dt}(f_1 f_2) = -f f_3, \quad \frac{d}{dt}(f_1 f_3) = f f_2, \quad \frac{d}{dt}(f_2 f_3) = f f_1,
\]

(8.5)

which is equivalent to the system (7.16) after interchanging \( f_2 \) with \( f_3 \). The general solution is given by (7.17).

Taking \( f = f_1 f_2 f_3 \) in (7.17) and all \( f_i \) different we obtain explicit expression of a triaxial neutral hyper para K"ahler metric

\[
g = f_2^2(e_1)^2 + f_3^2(e_2)^2 - f_1^2(e_3)^2 - f^2 dt^2,
\]

where the forms \( e^i \) are given by (7.6).

A particular solution is obtained by letting \( a_1 = a_3 = 0, c_2 = \frac{\pi}{16} \) in (7.17) which gives

\[
f_1 = f_3 = \frac{1}{2}(a - r^4)^{\frac{1}{4}}, \quad f_2 = \frac{r}{2}(a - r^4)^{-\frac{1}{4}}, \quad f = r(a - r^4)^{-\frac{1}{4}}, \quad -a < t^4 < a.
\]

The resulting neutral hyper para K"ahler metric is

\[
g = \frac{1}{2}(a - r^4)^{\frac{1}{4}}\left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - \frac{r^2}{2(a - r^4)^{\frac{3}{4}}} \left( d\psi + \cos \theta d\phi \right)^2 - \frac{r}{(a - r^4)^{\frac{5}{4}}} dt^2.
\]

**The group \( SU(1, 1) \).** Let \( G_3 = SU(1, 1) \) be defined by the structure equations

\[
d e^1 = -e^{23}, \quad d e^2 = -e^{31}, \quad d e^3 = e^{12}.
\]

(8.6)

In terms of local coordinates the left invariant forms \( e^i \) are given by

\[
e^1 = d\psi - \cos \theta d\phi, \quad e^2 = \sinh \psi d\theta + \cosh \psi \sin \theta d\phi, \quad e^3 = \cosh \psi d\theta + \sinh \psi \sin \theta d\phi.
\]

Clearly (8.2) are satisfied. We consider the \( SU(1, 1) \) structure as in (7.7). Using the structure equations (8.7), the evolution equations (8.3) reduce to the already solved system (7.8) with a general solution of the form (7.11).
A particular solution to (7.8) is given by (7.12), which results in a neutral hyper para Kähler metric in Eguchi-Hanson form given by
\[
g = \frac{t^2}{4} \left[ (d\psi - \cos \theta d\phi)^2 + \left( \sinh \psi d\theta + \cosh \psi \sin \theta d\phi \right)^2 \right]
- \frac{t^2}{4} \left( 1 - \frac{a}{t^4} \right) \left( \cosh \psi d\theta + \sinh \psi \sin \theta d\phi \right)^2 - \left( 1 - \frac{a}{t^4} \right)^{-1} (dt)^2.
\]

Setting \( f = -\frac{t^2}{4} \) one obtains another neutral hyper para Kähler. Triaxial neutral hyper para Kähler metric can be obtained with the help of (7.13).

The Heisenberg group \( H^3 \). Consider the two-step nilpotent Heisenberg group \( H^3 \) defined by the structure equations (7.18). The structure equations (7.18) reduce the evolution equations (8.3) to the already solved system (7.19) with a general solution (7.21).

A particular solution is \( f_1 = f_2 = f = t^2 \), \( f_3 = -t^{-\frac{3}{2}} \). This is the neutral hyper para Kähler metric
\[
g = t \left[ -dt^2 + dx^2 + dy^2 \right] - \frac{1}{t} \left[ dz - \frac{1}{2} x dy + \frac{1}{2} y dx \right]^2.
\]

Rigid motions of the Euclidean 2-plane. We consider the group \( \mathbb{E}_2 \) of rigid motions of Euclidean 2-plane defined by the structure equations (7.22). Clearly (7.2) are satisfied. We evolve the structure as in (7.7). Using the structure equations (7.22), the evolution equations (8.3) take the form of the already solved system of ODEs (7.28) with a general solution (7.29).

When \( f_2 = f_3^{-1} \), \( f_1 = f \) we have
\[
f = f_1 = \frac{1}{2} (a \cos t + b \sin t)^{\frac{1}{2}} (a \cos t - b \sin t)^{\frac{1}{2}}, \quad f_3 = f_2^{-1} = (a \cos t + b \sin t)^{\frac{1}{2}} (a \sin t - b \cos t)^{-\frac{1}{2}}.
\]

Introducing \( t_0 \) and \( r_0 \) by letting \( r_0 = \sqrt{a^2 + b^2} \), \( \cos t_0 = a/\sqrt{a^2 + b^2} \) and \( \sin t_0 = b/\sqrt{a^2 + b^2} \), the resulting neutral hyper para Kähler metric can be put in the form
\[
g = \frac{1}{4} \left( r_0^2 \sin(t + t_0) \sin(t - t_0) \right) \left( -dt^2 + d\phi^2 + \frac{4}{r_0^2 \sin^2(t + t_0)} (e^2)^2 - \frac{4}{r_0^2 \sin^2(t - t_0)} (e^3)^2 \right),
\]
where \( e^2, e^3 \) are given by (7.22).

In particular, setting \( a = b \) in (7.31) we obtain \( r_0^2 = 2a^2 \), \( \sin t_0 = \cos t_0 = \frac{\sqrt{2}}{2} \). Taking \( \tau = t + \frac{\pi}{4} \), the metric (8.8) can be written as
\[
g = \frac{a^2}{4} \sin 2\tau \left( -d\tau^2 + d\phi^2 \right) + \cot \tau \left( \sin \phi dx - \cos \phi dy \right)^2 - \tan \tau \left( \cos \phi dx + \sin \phi dy \right)^2.
\]

Rigid motions of Lorentzian 2-plane-Bianchi \( VI_0 \) metrics. Now we consider the group \( \mathbb{E}(1,1) \) of rigid motions \( E(1,1) \) of Lorentzian 2-plane defined by the structure equations (7.27). We evolve the structure as in (7.7). Using the structure equations (7.27), the evolution equations (8.3) turn into the solved system of ODEs (7.23) with the general solution given by (7.25).

When \( f_2 = f_3^{-1} \), \( f_1 = f \) we have
\[
f = f_1 = \frac{1}{2} (Ae^t + Be^{-t})^{\frac{1}{2}} (Ae^t - Be^{-t})^{\frac{1}{2}}, \quad f_3 = f_2^{-1} = (Ae^t + Be^{-t})^{-\frac{1}{2}} (Ae^t - Be^{-t})^{\frac{1}{2}},
\]
and the neutral hyper para Kähler metric is
\[
g = \frac{1}{4} \left( A^2 e^{2t} - B^2 e^{-2t} \right) \left( -dt^2 + d\phi^2 + \frac{4}{(Ae^t - Be^{-t})^2} (e^2)^2 - \frac{4}{(Ae^t + Be^{-t})^2} (e^3)^2 \right),
\]
where \( e^2, e^3 \) are given by (7.27).

In particular, setting \( A = B \) in (8.9) we obtain
\[
g = \frac{A^2}{2} \sinh 2t \left( -dt^2 + d\phi^2 \right) + \coth t \left( \sinh \phi dx + \cosh \phi dy \right)^2 - \tanh t \left( \cosh \phi dx + \sinh \phi dy \right)^2.
\]
9. Hyper Kähler structures in dimension eight

In this section we apply our method from Section 4.1.

Let $G_7$ be the seven dimensional solvable non-nilpotent Lie group defined by the following structure equations

\[
\begin{align*}
    de^1 &= e^{17} + e^{27}, \\
    de^2 &= -e^{17} - e^{27}, \\
    de^3 &= -e^{15} + e^{16} - e^{25} + e^{26}, \\
    de^4 &= -e^{16} - e^{15} - e^{25} - e^{26}, \\
    de^5 &= e^{13} + e^{14} + e^{23} + e^{24}, \\
    de^6 &= -e^{13} + e^{14} - e^{23} + e^{24}, \\
    de^7 &= 2e^{12}.
\end{align*}
\]

(9.1)

This is a solvable non nilpotent Lie algebra because $[g,g] = g_1$ is generated by $e_1 - e_2, e_3, e_4, e_5, e_6, e_7, [g,g] = g_1$ and $[g_1, g_1] = 0$. The $Sp(2)$-hypo structure is determined by the equalities

\[
\begin{align*}
    d(e^{12} + e^{34} + e^{56}) &= 0, \\
    d(e^{13} - e^{24} + e^{57}) &= 0, \\
    d(e^{14} + e^{23} + e^{67}) &= 0.
\end{align*}
\]

We consider the $Sp(2)$ structure on $g_3 \times \mathbb{R}^+$ defined by the following 2-forms

\[
\begin{align*}
    F_1(t) &= e^1(t) \wedge e^2(t) + e^3(t) \wedge e^4(t) + e^5(t) \wedge e^6(t) + f(t) dt, \\
    F_2(t) &= e^1(t) \wedge e^3(t) - e^2(t) \wedge e^4(t) + e^5(t) \wedge e^7(t) - e^6(t) \wedge f(t) dt, \\
    F_3(t) &= e^1(t) \wedge e^4(t) + e^2(t) \wedge e^3(t) + e^6(t) \wedge e^7(t) + e^5(t) \wedge f(t) dt.
\end{align*}
\]

(9.2)

where $f(t)$ is a function of $t$ and $e^i(t)$ depend on $t$. A direct calculation shows that for the evolution

\[
e^1(t) = -e(t+1) e^2, \quad e^2(t) = -(t+1) e^1 - e^2, \quad e^3(t) = e^3, \quad a = 3, \ldots, 7,
\]

the corresponding forms $F_1(t), F_2(t), F_3(t)$ are closed.

We consider the basis

\[
\begin{align*}
    e^1 &= \sqrt{2}(e^1 + e^2), \quad e^2 = e^2, \quad e^3 = e^3 + e^4, \quad e^4 = e^3 - e^4, \quad e^5 = \sqrt{2} e^5, \quad e^6 = \sqrt{2} e^6, \quad e^7 = \frac{1}{\sqrt{2}} e^7.
\end{align*}
\]

(9.4)

In this basis the structure equations (9.1) take the form

\[
\begin{align*}
    de^1 &= 0, \\
    de^2 &= -e^{17}, \\
    de^3 &= -e^{15}, \\
    de^4 &= e^{16}, \\
    de^5 &= e^{13}, \\
    de^6 &= -e^{14}, \\
    de^7 &= e^{12}.
\end{align*}
\]

(9.5)

Considering the triples $(e^1, e^2, e^7), (e^1, e^3, e^5), (e^1, e^4, e^6)$, we obtain

\[
\begin{align*}
    e^1 &= dx^1, \\
    e^2 &= \cos x^1 dx^2 - \sin x^1 dx^7, \\
    e^3 &= -(\sin x^1 dx^5 + \cos x^1 dx^3), \\
    e^4 &= \cos x^1 dx^6 + \cos x^1 dx^4, \\
    e^5 &= \cos x^1 dx^5 - \sin x^1 dx^3, \\
    e^6 &= \cos x^1 dx^6 - \sin x^1 dx^4.
\end{align*}
\]

(9.6)

For the hyper Kähler metric on $G_7 \times \mathbb{R}$ given by $g = \sum_{r=1}^7 e^r(t)^2 + dt^2$ the equations (9.4) and (9.6) yield

\[
g = (t^2 + t + 1/2)(dx^1)^2 + 2(dx^2)^2 + 2(dx^7)^2 - \sqrt{2} \cos x^1 dx^1 dx^2 + \sqrt{2} \sin x^1 dx^1 dx^7 + \sum_{s=3}^6 (dx^s)^2 + dt^2.
\]

When $t = -1/2$ the metric degenerates ($e_1 - e_2$ is of zero length). The above metric is of constant zero curvature, but it is not complete. The 8-dimensional manifold becomes a product of the Euclidean $\mathbb{R}^4$ with a four dimensional manifold $M$ of vanishing curvature.

One can consider also the following $Sp(2)$ structure on $G_7 \times \mathbb{R}^+$

\[
\begin{align*}
    F_1^1 &= e^1(t) \wedge e^2(t) + e^3(t) \wedge e^4(t) - e^5(t) \wedge e^6(t) + h(t) dt, \\
    F_2^2 &= e^1(t) \wedge e^3(t) - e^2(t) \wedge e^4(t) - e^6(t) \wedge e^7(t) + h(t) dt, \\
    F_3^3 &= e^1(t) \wedge e^4(t) + e^2(t) \wedge e^3(t) - e^5(t) \wedge e^7(t) - e^6(t) \wedge h(t) dt,
\end{align*}
\]

(9.7)

where $h(t)$ is a function of $t$ and $e^i(t)$ depend on $t$. A direct calculation shows that for the evolution

\[
e^1(t) = h_1(t), e^a(t) = e^a, \quad a = 2, \ldots, 7, \quad h_1 = -h
\]

(9.8)

the corresponding forms $F_1^1(t), F_2^2(t), F_3^3(t)$ are closed. The corresponding hyper Kähler metric $g = \sum_{r=1}^7 e^r(t)^2 + dt^2$ is flat having the expression ($u = h_1(t)$)

\[
g = u^2(dx^1)^2 + (du)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 + (dx^5)^2 + (dx^6)^2 + (dx^7)^2.
\]
10. Appendix 1. Explicit quaternionic Kähler metric

Substituting in (5.12) the equations (5.11) we obtain the following expression for the metric coefficients of the quaternionic Kähler metric (5.12) in coordinates
\{v^1 = t, v^2 = x, v^3 = y, v^4 = z, v^5 = x_3, v^6 = x_6, v^7 = x_7, v^8 = u\}:

\[
\begin{align*}
g_{11} &= \frac{1}{4}u \left( au \left( x_5^2 + x_6^2 + x_7^2 \right) + 4 \right), \quad g_{12} = -\frac{1}{4}u(au-1)x_5, \\
g_{13} &= \frac{1}{2}u(au-1) \left( x_6 \cos x - x_7 \sin x \right), \\
g_{14} &= -\frac{1}{4}u(au-1) \left( x_5 \cos y - \sin y \left( x_6 \sin x + x_7 \cos x \right) \right), \quad g_{15} = -\frac{1}{4}au^2x_5, \\
g_{16} &= -\frac{1}{4}au^2x_6, \quad g_{17} = -\frac{1}{4}au^2x_7, \\
g_{22} &= \frac{1}{4}u \left( au \left( x_5^2 + x_6^2 + x_7^2 \right) + 4 \right), \quad g_{23} = \frac{1}{4}au^2x_5 \left( x_6 \cos x - x_7 \sin x \right), \\
g_{24} &= \frac{1}{4}u \left( aux_5 \sin y \left( x_6 \sin x + x_7 \cos x \right) + \cos y \left( au \left( x_5^2 + x_6^2 + x_7^2 \right) + 4 \right) \right), \\
g_{25} &= \frac{1}{2}u(au-1), \quad g_{26} = -\frac{1}{4}au^2x_7, \quad g_{27} = \frac{1}{4}au^2x_6, \\
g_{33} &= \frac{1}{4}u \left( 2aux_5^2 + aux_6^2 + aux_7^2 + 8au + 2aux_6x_7 \sin 2x \right. \\
&\quad \left. - au \cos 2x \left( x_6^2 - x_7^2 \right) - 8 \right), \\
g_{34} &= \frac{1}{8}au^2 \left( x_6 \cos x - x_7 \sin x \right) \left( 2x_5 \cos y - 2 \left( x_6 \sin x + x_7 \cos x \right) \sin y \right), \\
g_{35} &= -\frac{1}{4}au^2 \left( x_6 \sin x + x_7 \cos x \right), \quad g_{36} = \frac{1}{4}u \left( 2 - 2au \right) \cos x + aux_5 \sin x, \\
g_{37} &= \frac{1}{4}u \left( 2(au-1) \sin x + aux_5 \cos x \right), \\
g_{44} &= \frac{1}{4}u \left( \left( au \left( x_5^2 + x_6^2 + x_7^2 \right) + 4 \right) \cos^2 y + 4au \sin^2 x \sin^2 y - 4\sin^2 x \sin^2 y ight. \\
&\quad +aux_5^2 \sin^2 x \sin^2 x + aux_6^2 \sin^2 x \sin^2 x + aux_7^2 \sin^2 x \sin^2 x + aux_5x_6 \sin x \sin 2y \\
&\quad + \cos^2 x \sin^2 y \left( au \left( x_5^2 + x_6^2 + x_7^2 \right) + 4 \right) + aux_5x_7 \cos x \sin 2y \\
&\quad -aux_6x_7 \sin 2x \sin^2 y \right), \\
g_{45} &= \frac{1}{4}u \left( 2(au-1) \cos y + aux y \left( x_6 \cos x - x_7 \sin x \right) \right), \\
g_{46} &= -\frac{1}{4}u \left( 2(au-1) \sin x \sin y + aux \left( x_5 \cos x \sin y + x_7 \cos(y) \right) \right), \\
g_{47} &= \frac{1}{4}u \left( au \left( x_5 \sin x \sin y + x_6 \cos y \right) - 2(au-1) \cos x \sin y \right), \\
g_{55} = g_{66} = g_{77} = \frac{au^2}{4}, \quad g_{88} = \frac{1}{4(au-1)}. \end{align*}
\]
11. Appendix 2. Explicit $\text{Spin}(7)$-holonomy metric

Substituting in (6.11) the equations (5.11) we obtain the following expression for the metric coefficients of the $\text{Spin}(7)$ metric (6.11) in coordinates 
\[ \{v^1 = t, v^2 = x, v^3 = y, v^4 = z, v^5 = x_5, v^6 = x_6, v^7 = x_7, v^8 = u\} : \]
\[
g_{11} = \frac{20u^{5/3} + (9u^{5/3} + 4a)(x_5^2 + x_6^2 + x_7^2)}{20u^{2/3}}, \quad g_{12} = -\frac{2(u^{5/3} + a)x_5}{5u^{2/3}}, \\
g_{13} = -\frac{2(u^{5/3} + a)(x_5 \cos x_5 - x_7 \sin x_7)}{5u^{2/3}}, \quad g_{14} = -\frac{2(u^{5/3} + a)(x_5 \cos y - \sin y(x_6 \sin x_5 + x_7 \cos x_7))}{5u^{2/3}}, \\
g_{15} = \frac{(9u^{5/3} + 4a)x_6}{20u^{2/3}}, \quad g_{16} = -\frac{(9u^{5/3} + 4a)x_6}{20u^{2/3}}, \quad g_{17} = -\frac{(9u^{5/3} + 4a)x_7}{20u^{2/3}}, \\
g_{22} = \frac{16(u^{5/3} + a)(x_5^2 + x_6^2 + x_7^2)}{20u^{2/3}}, \quad g_{23} = \frac{(9u^{5/3} + 4a)x_5(x_6 \cos x_5 - x_7 \sin x_7)}{20u^{2/3}}, \\
g_{24} = \frac{2(u^{5/3} + a)}{5u^{2/3}}, \quad g_{26} = -\frac{(9u^{5/3} + 4a)x_7}{20u^{2/3}}, \quad g_{27} = \frac{(9u^{5/3} + 4a)x_7}{20u^{2/3}}, \\
g_{33} = \frac{16(u^{5/3} + a)(x_5^2 + x_6^2 + x_7^2) - (x_6 \cos x_5 - x_7 \sin x_7)^2}{20u^{2/3}}, \\
g_{34} = -\frac{(9u^{5/3} + 4a)(x_6 \cos x_5 - x_7 \sin x_7)(-x_5 \cos y + x_6 \sin x_5 \sin y + x_7 \cos x_5 \sin(y))}{20u^{2/3}}, \\
g_{35} = -\frac{(9u^{5/3} + 4a)(x_6 \sin x_5 \sin x_7 \cos x_7)}{20u^{2/3}}, \quad g_{36} = \frac{(9u^{5/3} + 4a)x_5 \sin x_5 - 8(u^{5/3} + a) \cos x_7}{20u^{2/3}}, \\
g_{44} = \frac{8(u^{5/3} + a) \sin x_5 + (9u^{5/3} + 4a)x_5 \cos x_5}{20u^{2/3}}, \\
g_{45} = \frac{4(u^{5/3} + a)(x_5 \sin x_5 + 2y(u_6 \sin x_5 - x_7 \cos x_7))}{20u^{2/3}} - \frac{(9u^{5/3} + 4a)((x_5^2 + x_7^2) \cos x_5 + (x_5 \cos x_5 - x_7 \sin x_7)^2) \sin^2 y}{20u^{2/3}}, \\
g_{46} = \frac{2(u^{5/3} + a) \cos y + (9u^{5/3} + 4a)(x_5 \cos x_5 - x_7 \sin x_7)}{20u^{2/3}}, \quad g_{47} = \frac{(9u^{5/3} + 4a)(x_6 \sin x_5 \sin x_7 \cos x_7)}{20u^{2/3}}, \quad g_{48} = \frac{2(u^{5/3} + a) \cos x_5 \sin y}{5u^{2/3}} - \frac{(9u^{5/3} + 4a)(x_6 \cos x_5 \sin x_7 \cos x_7)}{20u^{2/3}}, \\
g_{55} = g_{66} = g_{77} = \frac{9u^{5/3} + 4a}{20u^{2/3}}, \quad g_{88} = \frac{5u^{2/3}}{36(u^{5/3} + a)}.
\]

11.1. Holonomy of the $\text{Spin}(7)$ metrics. Let us consider the Lie group (3.2) and the metric 
\[ g = u \left( (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \right) + \frac{\left( (u - u^{5/3}) \right)^2}{20u^{2/3}} \left( (\eta_1)^2 + (\eta_2)^2 + (\eta_3)^2 \right) + \frac{5u^{2/3}}{9(u - u^{5/3})} \mathrm{d}u^2. \]

Since $\eta_1 = e^5$, $\eta_2 = e^6$ and $\eta_3 = e^7$, the metric can be written as 
\[ g = \sqrt{u} e^1 + \sqrt{u} e^2 + \sqrt{u} e^3 + \sqrt{u} e^4 + \sqrt{u} e^5 + \sqrt{u} e^6 + \sqrt{u} e^7 + \sqrt{u} e^8 = \sqrt{u} e^1 + \sqrt{u} e^2 + \sqrt{u} e^3 + \sqrt{u} e^4 + \sqrt{u} e^5 + \sqrt{u} e^6 + \sqrt{u} e^7 + \sqrt{u} e^8 = g(u) \left( \frac{1}{6 g(u)} \right) \mathrm{d}u^2, \]

where the function $g(u)$ is given by $g(u) = \sqrt{\frac{(u - u^{5/3})}{20u^{2/3}}}$. From now on, we shall work with the orthonormal basis 
\[ \{\gamma^1 = \sqrt{u} e^1, \gamma^2 = \sqrt{u} e^2, \gamma^3 = \sqrt{u} e^3, \gamma^4 = \sqrt{u} e^4, \gamma^5 = g(u) e^5, \gamma^6 = g(u) e^6, \gamma^7 = g(u) e^7, \gamma^8 = \frac{du}{6 g(u)} \}. \]

The curvature 2-forms $\Omega^i_j$ of the Levi-Civita connection with respect to the basis $\{\gamma^1, \ldots, \gamma^8\}$ are:
\[
\Omega_2 = - \frac{u+12g(u)^2}{u^2} - \frac{6g(u)(2ug(u) - g(u))}{u^2} + \frac{1}{u^2} - \frac{u+4g(u)^2}{u^2} \gamma^{57}
\]
\[
\Omega_3 = - \frac{u+12g(u)^2}{u^2} + \frac{u+4g(u)^2}{u^2} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{57} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{68}
\]
\[
\Omega_4 = - \frac{u+12g(u)^2}{u^2} + \frac{u+4g(u)^2}{u^2} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{57} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{78}
\]
\[
\Omega_5 = \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{15} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{37} + \frac{u+4g(u)^2}{u^2} \gamma^{46}
\]
\[
\Omega_6 = \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{16} + \frac{u+4g(u)^2}{u^2} \gamma^{27} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{45}
\]
\[
\Omega_7 = \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{17} + \frac{u+4g(u)^2}{u^2} \gamma^{26} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{35} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{48}
\]
\[
\Omega_8 = \frac{9g(u)(2ug(u) - g(u))}{u^2} \gamma^{18} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{25} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{36} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{47}
\]
\[
\Omega_9 = - \frac{u+12g(u)^2}{u^2} \gamma^{23} + \frac{u+4g(u)^2}{u^2} \gamma^{56} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{78}
\]
\[
\Omega_{10} = - \frac{u+12g(u)^2}{u^2} \gamma^{24} + \frac{u+4g(u)^2}{u^2} \gamma^{57} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{68}
\]
\[
\Omega_{11} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{18} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{25} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{36} - \frac{u+4g(u)^2}{u^2} \gamma^{47}
\]
\[
\Omega_{12} = \frac{u+4g(u)^2}{u^2} \gamma^{17} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{26} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{35} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{48}
\]
\[
\Omega_{13} = - \frac{u+4g(u)^2}{u^2} \gamma^{16} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{27} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{38} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{45}
\]
\[
\Omega_{14} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{15} - \frac{9g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{37} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{46}
\]
\[
\Omega_{15} = \frac{u+12g(u)^2}{u^2} \gamma^{34} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{58} - \frac{u+4g(u)^2}{u^2} \gamma^{67}
\]
\[
\Omega_{16} = \frac{u+4g(u)^2}{u^2} \gamma^{17} + \frac{u+4g(u)^2}{u^2} \gamma^{26} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{35} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{48}
\]
\[
\Omega_{17} = \frac{u+4g(u)^2}{u^2} \gamma^{16} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{25} - \frac{1}{u^2} + \frac{1}{u^2} \gamma^{36} - \frac{u+4g(u)^2}{u^2} \gamma^{47}
\]
\[
\Omega_{18} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{15} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{37} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{46}
\]
\[
\Omega_{19} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{16} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{27} + \frac{9g(u)(2ug(u) - g(u))}{u^2} \gamma^{38} - \frac{5g(u)(2ug(u) - g(u))}{u^2} \gamma^{45}
\]
\[
\Omega_{20} = \frac{u+4g(u)^2}{u^2} \gamma^{16} + \frac{u+4g(u)^2}{u^2} \gamma^{27} - \frac{3g(u)(18ug(u) - g(u))}{u^2} \gamma^{38} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{45}
\]
\[
\Omega_{21} = \frac{1}{u^2} + \frac{1}{u^2} \gamma^{37} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{38} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{46}
\]
\[
\Omega_{22} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{18} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{35} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{48}
\]
\[
\Omega_{23} = \frac{u+4g(u)^2}{u^2} \gamma^{15} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{28} + \frac{1}{u^2} + \frac{1}{u^2} \gamma^{36} - \frac{g(u)(18ug(u) - g(u))}{u^2} \gamma^{47}
\]
\[
\Omega_{24} = \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{17} - \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{26} + \frac{3g(u)(2ug(u) - g(u))}{u^2} \gamma^{35} - \frac{9g(u)(2ug(u) - g(u))}{u^2} \gamma^{48}
\]
\[
\Omega_{25} = \frac{u+4g(u)^2}{u^2} \gamma^{14} - \frac{1}{u^2} + \frac{1}{u^2} \gamma^{23} - (24g(u)g(u)^2(u) + \lambda^2 + \mu^2)(24g(u)g(u)^2(u) - \lambda^2 - \mu^2) \gamma^{56}
\]
\[
\Omega_{26} = \frac{u+4g(u)^2}{u^2} \gamma^{13} - \frac{1}{u^2} + \frac{1}{u^2} \gamma^{24} - (24g(u)g(u)^2(u) + \lambda^2 + \mu^2)(24g(u)g(u)^2(u) - \lambda^2 - \mu^2) \gamma^{57}
\]
\[
\Omega_{27} = \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{12} + \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{34} - 36g(u)^2 + g(u)g''(u) \gamma^{58}
\]
\[
\Omega_{28} = \frac{u+4g(u)^2}{u^2} \gamma^{12} - \frac{1}{u^2} + \frac{1}{u^2} \gamma^{34} - (24g(u)g(u)^2(u) + \lambda^2 + \mu^2)(24g(u)g(u)^2(u) - \lambda^2 - \mu^2) \gamma^{67}
\]
\[
\Omega^6 = \frac{6g'(u)(2ug(u) - g(u))}{u^2} \gamma^{11} - \frac{6g(u)(2ug(u) - g(u))}{u^2} \gamma^{24} - 36(g'(u)^2 + g(u)g''(u))\gamma^{68}
\]
\[
\Omega^7 = \frac{6g(u)(2ug'(u) - g(u))}{u^2} \gamma^{14} + \frac{6g(u)(2ug'(u) - g(u))}{u^2} \gamma^{23} - 36(g'(u)^2 + g(u)g''(u))\gamma^{78}.
\]

First of all, using that \(g(u) = \sqrt{\frac{1(a-u^5/3)}{20u^2/3}}\), from these expressions one can check directly that the metric is Ricci flat because

\[
\text{Ric}(X_i, X_j) = \Omega^1_j(X_1, X_i) + \cdots + \Omega^8_j(X_8, X_i) = 0,
\]

for any \(i, j = 1, \ldots, 8\) and for any \(a\), where \(\{X_1, \ldots, X_8\}\) denotes the dual basis of \(\{\gamma^1, \ldots, \gamma^8\}\). Now, one can evaluate the coefficients above using that \(g(u)\). It turns out that all the coefficients above are nonzero when \(a \neq 0\) and \(\lambda^2 + \mu^2 \neq 0\). It is clear that the first 9 curvature forms, i.e from \(\Omega^1_6\) to \(\Omega^7_8\), are independent. The form \(\Omega^2_7\) is independent from the previous ones, except possibly for \(\Omega^1_6\). But if \(\Omega^1_6\) and \(\Omega^2_6\) were proportional then, from the coefficient in \(\gamma^{18}\), the factor of proportionality should be equal to 3 and this is not the case for the coefficients in \(\gamma^{25}\). So we conclude that \(\Omega^2_8\) is independent from the previous ones. Similar argument allows to prove that \(\Omega^1_6, \Omega^2_7\) and \(\Omega^2_8\) are also independent from the previous ones. The form \(\Omega^1_6\) is clearly independent from the previous ones. So, at this moment we have 14 independent curvature forms. Let us consider now the curvature form \(\Omega^0_6\). This form could be dependent only of \(\Omega^1_6, \Omega^2_7\) and \(\Omega^2_8\). Suppose that \(a \Omega^1_6 + \beta \Omega^2_7 = \frac{1}{4} \Omega^2_8\) for some \(a, \beta\). Then, from the coefficients of \(\gamma^{58}\) in these curvature forms we get that \(\beta = -a\), but then from the coefficients of \(\gamma^{67}\) we conclude that \(\Omega^0_6\) is independent from the previous ones. A similar argument can be applied to \(\Omega^6_6\) to get another independent form.

Therefore there are at least 16 independent curvature forms and this implies that the holonomy is equal to \(\text{Spin}(7)\).

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