Bowtie-free graphs have a Ramsey lift

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Abstract

A bowtie is a graph consisting of two triangles with one vertex identified. We show that the class of all (finite) graphs not containing a bowtie as a subgraph have a Ramsey lift (expansion). This solves one of the old problems in the area and it is the first Ramsey class with a non-trivial algebraic closure.

1 Introduction

A bowtie graph is formed by two triangles intersecting in a single vertex (see Figure 1). We denote by $\mathcal{B}$ the class of all finite graphs not containing a bowtie as a (not necessarily induced) subgraph. The class $\mathcal{B}$ seems to be a rather special class. However, it appears that it plays a key role in the context of both Ramsey theory and model theory in the area related to universality and homogeneity. It is the interplay of these two fields which makes this example interesting and important. We briefly explain both sides and their interplay in this introduction.

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1This poetic name seems to be first used in [22], see also [14], butterfly graph or hourglass graph are other names used; it is “\bowtie” in TeX and sign for “natural join” in databases.
1.1 Ramsey Theory

Ramsey theory (see [8, 16] for background information) is established in the context of several mathematical areas. Structural Ramsey theory is interested in generalisations of the Ramsey Theorem to as wide class of structures as possible. The key notion in this area is the Ramsey class. To make this paper self-contained, we introduce it in the following notation (which is by now standard, see e.g. [16]).

Let $C$ be a class of structures endowed with embeddings. The class is usually understood from the context. Let $A, B \in C$. Then by $(B \uparrow A)$ we denote the set of all sub-objects $\tilde{A}$ of $B$ isomorphic to $A$. (By a sub-object we mean that the inclusion is an embedding.) Using this notation the definition of Ramsey class gets the following form: A class $C$ is a Ramsey class if for every two objects $A, B \in C$ and for every positive integer $k$ there exists object $C \in C$ such that for every partition of $(B \uparrow A)$ in $k$ classes there exists $\tilde{B} \in (C \uparrow B)$ such that $(\tilde{B} \uparrow A)$ belongs to one class of the partition. It is usual to shorten the last part of the definition as $C \rightarrow (B \uparrow A)^k$.

The Ramsey classes originated in 70’s (see [16]) as the top of the line of Ramsey properties and examples found present the backbone of the structural Ramsey theory, see [18, 15, 16].

In most instances, a class is not Ramsey for some easily formulated reason and all one needs is to add some more information such as ordering or colouring of distinguished parts. For example, all finite graphs form a Ramsey class if we add an ordering of vertices, bipartite graphs need an ordering respecting the bipartition and colouring distinguishing the parts, disjoint unions of complete graphs (or equivalences) needs an ordering respecting components. This additional information is usually called an expansion, or in a combinatorial setting a lift, of the original structure (see the next section). Such lifts are usually easy to define, however the example of bowtie-free graphs we present here is an example where the lift is quite complex and uses an intricate system of relations. In fact, for this class it is the first explicitly defined ultrahomogeneous lift.

1.2 Model theory

It is important to realise that for more complicated Ramsey questions (even when related to graphs) one needs to deal with more general structures.

A language $L$ is a set of relational symbols $R \in L$, each associated with natural number $a(R)$ called arity. A (relational) $L$-structure $A$ is a pair $(A, (R^A_A; R \in L))$ where $R^A_A \subseteq A^{a(R)}$ (i.e. $R^A_A$ is a $a(R)$-ary relation on $A$). The set $A$ is called the vertex set or the domain of $A$ and elements of $A$ are vertices. The language is usually fixed and understood from the context (and it is in most cases denoted by $L$). However it is the essence of this paper that the languages considered are complex and we consider an interplay of several of them. This will be carefully described. If set $A$ is finite we call $A$ finite structure. We consider only structures with finitely or countably many vertices. The class of all (finite or countable) relational $L$-structures will be denoted by Rel($L$).

We consider graphs as a special case of relational structure with one binary relation. We use bold letters $A, B, \ldots$ to denote structures and normal letters $G, H, \ldots$ for graphs. The following are standard graph theoretic notions re-stated in the language of model theory. A homomorphism $f : A \rightarrow B =$
$$(B, (R_B; R \in L))$$ is a mapping $$f : A \to B$$ satisfying for every $$R \in L$$ the implication $$(x_1, x_2, \ldots, x_n) \in R_A \implies (f(x_1), f(x_2), \ldots, f(x_n)) \in R_B$$. (For a subset $$A' \subseteq A$$ we denote by $$f(A')$$ the set $$\{f(x); x \in A'\}$$ and by $$f(A)$$ the homomorphic image of a structure.) If $$f$$ is injective, then $$f$$ is called a monomorphism. A monomorphism is called embedding if the above implication is equivalence, i.e. if for every $$R \in L$$ we have $$(x_1, x_2, \ldots, x_n) \in R_A \iff (f(x_1), f(x_2), \ldots, f(x_n)) \in R_B$$. If $$f$$ is an embedding which is an inclusion then $$A$$ is a substructure (or subobject) of $$B$$. Note that substructures correspond to induced subgraphs. For an embedding $$f : A \to B$$ we say that $$A$$ is isomorphic to $$f(A)$$ and $$f(A)$$ is also called a copy of $$A$$ in $$B$$. Thus $$(B)$$ is defined as the set of all copies of $$A$$ in $$B$$.

Using the language of model theory we can conveniently define the concept of lift discussed informally in the previous section. Let $$L^+$$ be a language containing language $$L$$. By this we mean $$L \subseteq L^+$$ and the arities of the relations both in $$L$$ and $$L^+$$ are the same. Then every structure $$X = (X, (R_X; R \in L^+)) \in \text{Rel}(L^+)$$ may be viewed as a structure $$A = (X, (R_X; R \in L)) \in \text{Rel}(L)$$ together with some additional relations $$R_X$$ for $$R \in L^+ \setminus L$$. We call $$X$$ a lift (and in the model theory context usually expansion) of $$A$$. In this situation the structure $$A$$ is called the shadow (or alternatively the reduct) of $$X$$. In this sense the class $$\text{Rel}(L^+)$$ is the class of all lifts of $$\text{Rel}(L)$$. Conversely, $$\text{Rel}(L)$$ is the class of all shadows of $$\text{Rel}(L^+)$$.

In this paper the languages $$L$$ and $$L^+$$ will always be finite, we speak about finite lifts.

Two notions are related to our main result: A (countable) structure $$A$$ is said to be universal for a class $$C$$ of (finite or countably infinite) structures if every structure $$B \in C$$ embeds to $$A$$. A relational structure $$A$$ is called ultrahomogeneous if every isomorphism between two induced finite substructures of $$A$$ can be extended to an automorphism of $$A$$.

It is a classical result of model theory that ultrahomogeneous structures may be alternatively described as Fraïssé limits of amalgamation classes of finite structures (see e.g. [10]). Here amalgamation class $$C$$ is a hereditary class of structures containing only countably many mutually non-isomorphic structures which satisfy:

1. (Joint embedding property) For every $$A, B \in C$$ there exists $$C \in C$$ such that $$C$$ contains both $$A$$ and $$B$$ as substructures.

2. (Amalgamation property) For $$A, B_1, B_2 \in C$$ and $$\alpha_1$$ embedding of $$A$$ into $$B_1$$, $$\alpha_2$$ embedding of $$A$$ into $$B_2$$, there is $$C \in C$$ with embeddings $$\beta_1 : B_1 \to C$$ and $$\beta_2 : B_2 \to C$$ such that $$\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$$. Every such structure $$C$$ is called an amalgamation of $$B_1$$ and $$B_2$$ over $$A$$ with respect to $$\alpha_1$$ and $$\alpha_2$$.

We say that an amalgamation is strong when $$\beta_1(x_1) = \beta_2(x_2)$$ if and only if $$x_1 \in \alpha_1(A)$$ and $$x_2 \in \alpha_2(A)$$. Less formally, a strong amalgamation glues together $$B_1$$ and $$B_2$$ with an overlap no greater than the copy of $$A$$ itself. A strong amalgamation is free if there are no tuples in any relations of $$C$$ spanning both vertices of $$\beta_1(B_1 \setminus \alpha_1(A))$$ and $$\beta_2(B_2 \setminus \alpha_2(A))$$.

For a structure $$A$$ the age of $$A$$, denoted by $$\text{Age}(A)$$, is the class of all finite structures which have embedding to $$A$$. Thus every homogeneous structure $$A$$ is determined by $$\text{Age}(A)$$ which forms an amalgamation class (see [10]).
Let $A$ be an $L$-relational structure and $S$ a finite subset of $A$. The algebraic closure of $S$ in $A$, denoted by $Acl_A(S)$, is the set all vertices $v \in A$ for which there is a formula $\phi$ in the language $L$ with $|S| + 1$ variables such that $\phi(\vec{S}, v)$ is true and there are only finitely many vertices $v' \in A$ such that $\phi(\vec{S}, v')$ is also true. (Here $\vec{S}$ is an arbitrary ordering of vertices of $S$.)

Algebraic closure is, of course, related to amalgamation: For example, it is easy to see that if an ultrahomogeneous structure $H$ has trivial closure (i.e. $Acl_H(S) = S$ for every $S \subseteq H$) then its age is closed for strong amalgamation [5].

### 1.3 Ramsey classes, ultrahomogeneity and universality

The link between Ramsey classes and ultrahomogeneous structures was established in [15]: Under a mild assumption any Ramsey class is an amalgamation class and thus it is an age of an ultrahomogeneous structure. This was used in [19] to completely characterise hereditary Ramsey classes of undirected graphs. (Essentially, all Ramsey classes of graphs were known earlier, [16].) This connection of Ramsey classes proved to be fruitful and led to the characterisation programme for Ramsey classes [17] and to important connection of Ramsey classes with topological dynamics and ergodic theory [13].

As we indicated above a given class $C$ is often not Ramsey but $C$ may have an easy lift $C^+$ which is Ramsey and thus it leads to the age of an ultrahomogeneous structure $U^+$. This in turn means that the shadow $U$ of $U^+$ is universal for $C$. In this sense the universality is the first test for the existence of a Ramsey lift.

The existence of universal objects is a difficult question in its own and this is also where the class of all bowtie-free graphs played a vital role. Here is a brief history: It starts with a (somewhat surprising) result of Komjáth [14] that the class of bowtie-free graphs contains a universal graph, i.e. there exists a (countably infinite) bowtie-free graph $U$ such that every finite or countably infinite bowtie free graph $G$ has an embedding into $U$ (in other words, $G$ is isomorphic to an induced subgraph of $U$). In a sense this obscurely looking example was (and is) a key case for further development (see e.g. [1, 5, 7, 2, 4, 6, 3]).

Note that the problem of characterising universal graphs seem to be far from being solved even in the following case: Given a finite set of finite graphs $F$, denote by Forb$_M(F)$ the class of all finite or countable infinite graphs which do not contain any $F \in F$ as a (not necessarily induced) subgraph. For which $F$ does the class Forb$_M(F)$ have a universal graph? (non-induced subgraphs correspond to monomorphism and $M$ in Forb$_M(F)$ stands for monomorphism.) The answer is positive for the bowtie graph while, for example, the answer is negative for the rectangle $C_4$. It is not even known whether this question, for a general finite $F$, is decidable. [2].

The problem was recast in the model theory setting by Cherlin et al. [5]. They narrowed the search for universal graphs to more structured ultrahomogeneous and $\omega$-categorical graphs (and structures) and in [5] they provided a structural characterisation of such universal structures: There is an $\omega$-categorical universal graph in Forb$_M(F)$ if and only if the class of existentially complete graphs in Forb$_M(F)$ has a locally finite algebraic closure. Bowtie-free graphs fall in this category.
The rest of the paper is organised as follows: Section 2 contains a detailed description of the structure and of the lifts of bowtie-free graphs. This leads to an explicit homogenisation of these graphs which will be used (Section 3). In Section 4 we give a simpler (reduced) variant of our lift. In Section 5 we review the basic properties of Ramsey classes used in our proof. The proof of Ramsey property splits into two parts: In Section 6 we prove the Ramsey property for incomplete lifts, and finally in Section 7 we combine this to obtain the final result:

**Theorem 1.1.** The class $B$ has a finite Ramsey lift.

In a more detailed way this is formulated as Theorem 7.1 below. This Ramsey lift includes special “admissible” orderings. As an explanation of this and as an application of Theorem 1.1 we then prove the lift property for the class of orderings in Section 8 (lift property is introduced there). The final section contains some remarks and open problems.

## 2 Structure of bowtie-free graphs

In order to prove the Ramsey property one has to understand the lift very well and the lift has to be explicit. We start to develop the structure of bowtie-free graphs by means of the following concepts which will describe the structure of triangles in bowtie-free graphs. From now on, in this section, $G = (V,E)$ is a finite bowtie-free graph.

**Definition 2.1 (Chimneys).** For $n \geq 2$, an $n$-chimney graph, $Ch_n$, is a free amalgamation of $n$ triangles over one common edge. A chimney graph is any graph $Ch_n$ for some $n \geq 2$.

Chimneys together with $K_4$ (a clique on 4 vertices) will form the only components of bowtie-free graphs formed by triangles. The assumption $n \geq 2$ for chimney is a technical assumption to avoid isolated triangles. Note also that $Ch_2$ is not an induced subgraph of $K_4$.

**Definition 2.2 (Good bowtie-free graphs).** A bowtie-free graph $G = (V,E)$ is good if every vertex is contained either in a copy of chimney or a copy of the complete graph $K_4$.

The structure of bowtie-free graphs is captured by means of the following three lemmas:

**Lemma 2.3.** Every bowtie-free graph $G$ is an induced subgraph of some good bowtie-free graph $G'$.

**Proof.** Graph $G$ can be extended in the following way:

1. For every vertex $v$ not contained in a triangle add a new copy of $Ch_2$ and identify vertex $v$ with one of vertices of $Ch_2$.

2. For every triangle $v_1, v_2, v_3$ that is not part of a 2-chimney nor $K_4$ add a new vertex $v_4$ and triangle $v_1, v_2, v_4$ turning the original triangle into $Ch_2$. 

It is easy to see that step 1 cannot introduce new bowtie.

Assume, to the contrary, that step 2 introduced a new bowtie. Further assume that $v_1$ is the unique vertex of degree 4 of this new bowtie and consequently there is another triangle on vertex $v_1$ in $G$. Because $G$ is bowtie-free, this triangle must share a common edge with triangle $v_1, v_2, v_3$ and therefore triangle $v_1, v_2, v_3$ is already part of $K_4$ or a 2-chimney in the original graph $G$. A contradiction.

For a graph $G = (V,E)$ we split its edge set into two types: $E_0 = E_0(G)$ consisting of all edges in triangles and $E_1 = E_1(G)$ consisting of all remaining edges. We also speak about edges of type 0 and edges of type 1. Put also $G_0 = (V,E_0)$

**Lemma 2.4.** For every good bowtie-free graph $G = (V,E)$ the graph $G_0$ is a disjoint union of copies of chimneys and $K_4$.

**Proof.** This follows directly from the fact that $Ch_n$, $n \geq 1$, and $K_4$ is a complete listing of connected bowtie-free graphs with every vertex and edge in a triangle.

**Lemma 2.5.** Let $G$ be a graph with every vertex contained either in a copy of chimney or a copy of the complete graph $K_4$. If graph $G_0 = (V,E_0)$ is as described in Lemma 2.4 and the remaining edges of $G$ (i.e. edges in $E_1$) do not form a triangle, then $G$ is a good bowtie-free graph.

**Proof.** A bowtie in $G$ must be a bowtie in $G_0$ and $G_0$ is a bowtie-free by the assumption.

It follows that bowtie-free graphs are made of chimneys and $K_4$’s (forming the edge set $E_0$) and a triangle free graph (with the edge set $E_1$).

An example of a good bowtie-free graph is depicted in Figure 2. Type 0 edges are depicted as solid lines, type 1 edges are dashed. We will use this graph as our reference graph thorough the paper.
A homogenisation is a technique which provides an ultrahomogeneous lift for a non-ultrahomogeneous structure. The special structure of good bowtie-free graphs indicates that we have vertices of various types and that the Ramsey lift will have to be defined carefully (to distinguish all possible combination of types). In this section we shall define three lifts (with languages $L_0$, $L_1$ and $L_2$) and use them to define an amalgamation class (see Corollary 3.7). We start with the definition of central vertices.

Let $G$ be a good bowtie-free graph. Then the centre of $G$, denoted by $c(G)$, is a subgraph induced by all vertices contained in two or more triangles. The centre of a vertex $v$, denoted by $c(v)$, is a subgraph of $G$ induced by all vertices in two or more triangles which are in the same connectivity component of $(V,E_0)$ as the vertex $v$. (We define centre for good bowtie-free graphs only, so every vertex has a centre.)

Remark. Note that in the language of model theory centre of a vertex is a definable set and thus bowtie-free graphs have a nontrivial algebraic closure (and as explained above this was one of the motivations for a study of this particular example, see e.g. [2]).

For a good bowtie-free graph $G$, the centre of its subgraph $H$ is a subgraph of $G$ induced by the union of all centres of vertices of $H$. Note that the centre of $H$ is not necessarily a subgraph of $H$.

We also call a vertex central if it appears in its centre. Other vertices are non-central.

Example. Our reference graph depicted in Figure 2 has central vertices labelled $u$ and non-central $v$. There are 4 centres of vertices: $\{u_1^1, u_1^r\}$, $\{u_2^2, u_2^r\}$, $\{u_3^3, u_3^r\}$, and $\{u_4^1, u_4^2, u_4^3, u_4^4\}$. The centre of vertex $v_1^1$ is $\{u_2^2, u_2^r\}$. The centre of $u_4^1$ is $\{u_4^1, u_4^2, u_4^3, u_4^4\}$.

We start with the following (easy and optimistic) statement:

Lemma 3.1 (Central amalgamation). Let $G$ and $G'$ be good bowtie-free graphs and $f$ an isomorphism from $c(G)$ to $c(G')$. Then the free amalgamation of $G$ and $G'$ over their centres (with respect to $f$) is a good bowtie-free graph.

Proof. Without loss of generality we can assume that $f$ is an identity and vertex sets of $G = (V,E)$ and $G' = (V',E')$ intersect only on vertices of $c(G)$. The free amalgamation is a graph $G'' = (V \cup V', E \cup E')$.

All triangles of $G''$ are clearly either triangles in $G$ or $G'$ (or both).

$G''$ is good because all copies of $K_4$ are also part of centres and thus identified. For vertices contained in chimneys, some vertices of chimney $Ch_n$ of $G$ gets identified with some vertices of chimney $Ch_m$ of $G'$ if an only if centres of the chimneys are the same. This produce a chimney $Ch_{n+m}$ in $G''$.

Finally $G''$ is bowtie-free by Lemma 2.5.

An example of the central amalgamation is depicted in Figure 3.
Figure 3: Structures A and B and their amalgamation over the common centre.

Definition 3.2 (Lift $L_0$). Given a good bowtie-free graph $G = (V, E)$, an ordered good bowtie-free graph is a structure $\mathbf{G} = (V, R^0_{\mathbf{G}}, R^1_{\mathbf{G}}, \leq_{\mathbf{G}})$ where $R^0_{\mathbf{G}} = E_0(G)$, $R^1_{\mathbf{G}} = E_1(G)$ and $\leq_{\mathbf{G}}$ is a linear order of $V$ such that

1. vertices of every centre of every vertex $v \in G$ form an interval of $\leq_{\mathbf{G}}$,
2. all centres of chimneys are before vertices in copies of $K_4$,
3. all central vertices are before non-central vertices, and
4. non-central vertices belonging to a given chimney form an interval. The relative order of these intervals corresponding to given centres follows the order of the relative order of the centres.

Such ordering is called an admissible ordering. We denote by $L_0$ the language of ordered good bowtie-free graphs and by $B_0$ the class of all ordered good bowtie-free graphs. By an abuse of notation, for a good bowtie-free graph $G$ we also denote $\mathbf{G} = L_0(G)$ the corresponding ordered good bowtie-free graph (i.e. $L_0$ lift of $G$).

Example. One of admissible orderings of our reference graph in Figure 2 is: $u^1_r, u^1_l, u^2_r, u^2_l, u^3_r, u^3_l, u^4_r, u^4_l, v^1_1, v^1_2, v^1_3, v^2_1, v^2_2, v^3_1, v^3_2, v^3_3$. There are four centres of a vertex in the graph: $\{u^1_r, u^1_l\}$, $\{u^2_r, u^2_l\}$, $\{u^3_r, u^3_l\}$, and $\{u^4_r, u^4_l, u^4_r, u^4_l\}$ each of them forms an interval (to satisfy 1). $\{u^1_r, u^2_r, u^3_r, u^4_r\}$ is after all centres of vertices in a chimney (to satisfy 2). The non-central vertices associated with each chimney forms an interval: $\{v^1_1, v^1_2, v^1_3\}$, $\{v^2_1, v^2_2\}$, $\{v^3_1, v^3_2, v^3_3\}$ and their relative order corresponds to the order of their centres, as required by 4.

We introduce two more lifts of good bowtie-free graphs. The lift $L_1$ is introducing unary relations and the lift $L_2$ in addition binary relations. It is
$L_0 \subset L_1 \subset L_2$. The hereditary class defined by the lift $L_2$ will form our Ramsey class.

**Definition 3.3 (Lift $L_1$).** Let $G$ be an ordered good bowtie-free graph. $A = L_1(G)$ is an lift of $G$ adding new unary relations $R^1_A, R^2_A, R^3_A, R^4_A$ and $R^5_A$ such that:

1. for every pair $u, v$ forming the centre of a chimney of $G$, $u <_G v$, we put $(u) \in R^1_A$ and $(v) \in R^3_A$;
2. for every $a <_G b <_G c <_G d$ that are vertices of a copy of $K_4$ in $G$ we put $(a) \in R^2_A, (b) \in R^4_A, (c) \in R^5_A, (d) \in R^6_A$.

We denote by $L_1$ the language of this lift. For a given $G \in B_0$ we denote $L_1(G)$ the corresponding lift of $G$. By $B_1$ we then denote the the class of all structures $L_1(G)$, $G \in B_0$.

**Example.** The unary relations of the $L_1$-lifts of our reference graph in Figure 2 are indicated by labels of the $u$ vertices.

Advancing the definition of the lift $L_2$ we first note that we shall sometimes consider rooted structures (with either one or two roots). Isomorphisms (and embeddings) are, of course, defined as root preserving isomorphisms (and embeddings). If, for example, the structures $G$ and $G'$ are considered with roots $u$ and $v$ and $u'$ and $v'$ then these structures are called isomorphic if there is an isomorphism $f$ from $G$ to $G'$ such that $f(u) = u'$ and $f(v) = v'$.

Given ordered good bowtie-free graph $G$ and two vertices $u, v, u \neq v$, we denote by $t(u, v)$ the isomorphism type of the structure induced by $L_1(G)$ on the set $\{u, v\} \cup c(u) \cup c(v)$ rooted in $(u, v)$. We fix an enumeration $\ell_1, \ell_2, \ldots, \ell_N$ of all the possible such types. Clearly there are only finitely many possibilities for types (as $t(u, v)$ is an isomorphism type of a graph with at most 8 vertices). In this situation we define binary relations $R^{\ell_1}, R^{\ell_2}, \ldots, R^{\ell_N}$ as follows:

**Definition 3.4 (Homogenising lift $L_2$).** Let $G \in B_0$ be an ordered good bowtie-free graph. $A = L_2(G)$ is the lift consisting from $L_1$-structure $L_1(G)$ and in addition from new binary relations $R^{\ell_1}, R^{\ell_2}, \ldots, R^{\ell_N}$. For $u <_G v$ we put $(u, v) \in R^{R^{\ell_1}}$. We denote by $L_2$ the language of this lift and by $B_2$ the the class of all structures $L_2(G)$, $G \in B_0$.

If $(u, v) \in R^{R^{\ell_1}}$ then $t(u, v)$ is called the type of pair $(u, v)$.

Denote by $B_2$ the class of all $L_2$-lifts $L_2(G)$, $G \in B_0$. Observe that structures in $B_2$ have the property that every two distinct vertices are in a tuple of some relation (called irreducible bellow).

**Example.** Some types of pairs in our reference graph depicted in Figure 2 are depicted in Figure 3.

**Remark.** The relations $R^{\ell_1}$ of the lift $L_2(G)$ form a natural homogenisation of ordered good bowtie-free graphs as the new binary relations introduced describe necessary orbits of the automorphism group of a universal graph for class $B_1$. On the other hand the relations in lift $L_1(G)$ (i.e. unary relations) are not necessary from the point of view of $\omega$-categoricity. The universal graph in Figure 3 has an automorphisms exchanging vertices within centres of vertices. We however
consider ordered graphs and the order on every centre prevents any non-trivial automorphism within it. It is also interesting to observe that \([5]\) leads to homogenisation of the existentially complete universal graph which needs relations of unbounded arity. Our ordered lift has only unary and binary relations. This is in agreement with \([9]\) where we show that the relational complexity of \(\omega\)-categorical and existentially complete bowtie-free graph is infinite, while the relational complexity of the ordered \(\omega\)-categorical and existentially complete bowtie-free graph is 2.

**Definition 3.5.** Denote by \(\mathcal{B}\) the class of all substructures of \(\mathcal{B}_2\). (Thus \(\mathcal{B}\) is the hereditary closure of \(\mathcal{B}_2\).) For structure \(A \in \mathcal{B}\) an ordered good bowtie-free graph \(G \in \mathcal{B}_0\) is called a witness of \(A\) if \(A\) is induced on \(A\) by \(L^2(G)\).

It follows directly that \(A \in \mathcal{B}\) if and only if there exists a witness \(G\) of \(A\).

The lift \(L^2(G)\) encodes enough information so that for every substructure \(A\) of \(L^2(G)\) it is possible to uniquely reconstruct the type of its centre (a precise procedure for this appears in proof of Theorem 3.8). Lemma 3.1 extends to the amalgamation property of \(\mathcal{B}\):

**Lemma 3.6 (Amalgamation of lifts).** \(\mathcal{B}\) is an amalgamation class.

*Proof.* Fix \(A, B_1\) and \(B_2\) from \(\mathcal{B}\) such that identity is an embedding from \(A\) to \(B_1\) and \(B_2\). We will construct an amalgamation of \(B_1\) and \(B_2\) over \(A\).

Let \(G_{B_1}\) and \(G_{B_2}\) be witnesses of \(B_1\) and \(B_2\) respectively. Denote by \(G_A\) the \(L_0\) shadow of \(A\) (that is an ordered bowtie-free graph). By the construction of the lift, the centre of \(G_A\) in \(G_{B_1}\) is isomorphic to the centre of \(G_A\) in \(G_{B_2}\) and moreover there is an isomorphism that is an identity on the central vertices of \(A\). It is now possible to extend \(G_{B_1}\) to \(G'_{B_1}\) and \(G_{B_2}\) to \(G'_{B_2}\) (by possibly adding centres) in a way that centres \(c(G'_{B_1})\) and \(c(G'_{B_2})\) are isomorphic with fixing vertices of \(A\).

By Lemma 3.1 we get ordered good bowtie-free graph \(G_D\) that is an amalgamation of \(G'_{B_1}\) and \(G'_{B_2}\) over \(G_A\). It is easy to verify that \(L^2(G_D)\) is as well an amalgamation of \(B_1\) and \(B_2\) over \(A\), since the type of every pair of vertices in \(B_1\) or \(B_2\) is preserved and thus \(B_1\) and \(B_2\) are induced substructures of \(L^2(G_D)\). \(\square\)
Observe that in this amalgamation the non-central vertices of $B_1$ and $B_2$ are identified if and only if they belong to $A$ (so amalgamation is “strong on non-central vertices”) Consequently by the standard Fraïssé argument we get:

**Corollary 3.7.** The class of all finite structures in $\overline{B}$ is the age of an ultrahomogeneous structure and its shadow is a universal graph for class $B$.

**Remark.** This, of course, follows also from Proposition 1 in [5] where the existence of $\omega$-categorical universal object is established. However, here we provided an explicit construction by means of lifts (which form a finite lift of bowtie-free graphs). In fact this is the first such explicit lift (compare [5]) and this is of independent interest [11, 12].

Thus we succeeded to satisfy the first universality criteria for Ramsey class (see Section 1.3). Nevertheless for Ramsey results we need more properties, gaining more information about $B$. Particularly, we need the following alternative description of $B$ by means of forbidden substructures. This will allow us to use strong Ramsey properties proved in [20].

Recall that a structure $A$ is called irreducible if every pair of distinct vertices belong to a relation of $A$. In the context of Ramsey theory it is often convenient to consider the lift adding linear order alone. Given a structure $A \in \overline{B}$ we call its shadow $\text{Sh}(A)$ in the language $L_2 \setminus \{\leq\}$ pure $L_2$-lift. $L_2$-structure $A$ is pure irreducible if its pure $L_2$-lift is irreducible. While the linear order is present in $\text{Sh}(A)$ implicitly this will allow us to describe class $\overline{B}$ by means of forbidden pure irreducible substructures.

Given family of finite $L$-structures $F$ we denote by $\text{Forb}(F)$ the class of finite $L$-structures not containing any structure $F \in F$ as a substructure. We sometimes write $\text{Forb}_L(F)$ to denote explicitly the language of structures we are considering.

**Theorem 3.8.** There exists finite set $T$ of $L_2$-structures such that every pure irreducible structure in $\text{Forb}_L(T)$ is a lift of bowtie-free graph and the class $\overline{B}$ is precisely the class of all finite admissibly ordered pure irreducible structures in $\text{Forb}_L(T)$. Moreover $T$ can be chosen to contain only pure irreducible structures with at most 3 vertices.

**Proof.** Consider structure $A \in \overline{B}$. It easily follows from Definition 3.5 that every pair of vertices $(u, v), u \leq_A v$, is in some binary relation $R_A^{u,v}$ and thus $\overline{B}$ is a class of pure irreducible structures.

Because $\overline{B}$ is closed on substructures it remains to show that every pure irreducible structure $A \notin \overline{B}$ contains substructure $A' \notin \overline{B}$ that consist of at most 3 vertices.

We give an effective procedure that attempts to construct, for a given ordered structure $A$, an ordered good bowtie-free graph (a witness) $G$ such that $A$ is an induced substructure of $L_2(G)$. The existence of witness $G$ proves that $A \in \overline{B}$.

Then we analyse cases where such procedure fails and show that these failures all correspond structures $F$ on at most 3 vertices. All those structures will have property that $F \notin \overline{B}$.

Denote by $A^0$ the $L_1$-shadow of $A$. Enumerate all pairs of vertices $u, v, u <_A v$ in $A$ as $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$. For every pair $(u_i, v_i), 1 \leq i \leq n$, we construct $A^i$ inductively from $A^{i-1}$ based on the type of $(u_i, v_i)$ in $A$. This involves the following elementary steps:
1. addition of new vertices to represent centres of \( u \) and \( v \) if they are not already present in \( A^{i-1} \),

2. addition of the new vertices into the corresponding unary relations \( R^0_{A^i}, R^1_{A^i}, R^2_{A^i}, R^3_{A^i} \), and \( R^4_{A^i} \), as required by the type,

3. addition of new edges of type 0 or 1 from \( u \) and \( v \) to the newly added vertices,

4. addition of edges of type 0 or 1 between the newly added vertices,

5. extension of the linear order \( \leq_{A^{i-1}} \) in a way consistent with the definition of ordered good bowtie-free graphs (Definition 3.2) and the type of the pair \( (u,v) \).

Because vertices of centres of a given vertex \( v \) are uniquely determined by the unary relations in \( L_1 \), there is (up to isomorphism) unique way of doing so (if it exist at all).

This procedure may fail if the extension is impossible. Assume that \( (u_i,v_i) \) is the first pair such that \( A^i \) can not be constructed. We consider individual cases that may happen and show that such failure scenarios all imply existence of forbidden substructures in \( A \) with at most 3 vertices:

1. Some or all vertices of the centre of \( u_i \) already exists in \( A^{i-1} \) and they are in conflict with the centre required by the type of pair \( (u_i,v_i) \).

   For example there is a vertex \( u' \) connected by edge of type 0 to \( u_i \) that is in \( R^0_{A^i} \) while the centre of \( u_i \) required is a copy of \( K_4 \) and thus the vertex should be in \( R^1_{A^i}, R^2_{A^i}, \text{ or } R^3_{A^i} \) instead.

   In this case let \( u' \) be such vertex. If \( u' \) is in \( A \) then the structure induced on \( u_i,v_i,u' \) must be forbidden: pair \( (u_i,v_i) \) require \( u_i \) to have its centre of one type, while pair \( (u_i,u') \) require its centre of a different type (or if \( u_i = u' \) then unary the relation on \( u_i \) must be already in conflict). This is not possible in structure in \( B \).

   If \( u' \) is not in \( A \) then it was introduced when defining the centre of vertex \( u'' \) and then \( u_i,v_i,u'' \) induce the forbidden substructure for the same reason.

2. The centre of \( v_i \) is already present in the structure and differs from one required by the type.

   This case follows in complete analogy to 1.

3. Vertices \( u_i \) and \( v_i \) are connected or ordered differently than required by the type. In this case the structure induced on \( u_i,v_i \) is forbidden.

4. Edges or orders in between already defined parts of centres \( u_i \) and \( v_i \) are different then required by the type.

   Denote by \( u' \) and \( v' \) the conflicting vertices of the the centre of \( u_i \) and \( v_i \) respectively. Now put \( u'' = u' \) if \( u' \in A \) or put \( u'' \) to be a vertex of \( A \) whose centre contains \( u' \). Similarly put \( v'' = v_i \) if \( v' \in A \) or \( v'' \) to a vertex of \( A \) whose centre contains \( v' \). Now structure induced on \( u_i,v_i,u'' \) or \( u',v_i,v'' \) is forbidden and moreover at least one of \( u_i,v_i,u'' \) or \( u',v_i,v'' \) must be forbidden.
We have shown that the procedure of adding centres can always be completed for all pairs \((u_i, v_i)\), \(i = 1, 2, \ldots, n\) if all substructures of \(A\) on at most 3 vertices are in \(\mathcal{B}\). Denote by \(G = G^n\) the resulting ordered graph (i.e. \(L_0\) shadow of \(A^n\)). We shall verify that \(G\) is bowtie-free. We use Lemma 2.5. Assume, to the contrary, that \(G\) contains a triangle \(u, v, w\) such that \(u, v\) is an edge in \(R_{G}^{E_1}\) and other edges are in \(R_{G}^{E_0}\) or \(R_{G}^{E_1}\). Consider a triple of vertices \(u', v', w' \in A\) such that \(u, v, w\) are either equivalent to \(u', v', w'\) or present in their centres. \(u', v', w'\) again induce a forbidden substructure of \(A\).

It follows that we can characterise lifts \(A\) such that there exists \(G\) (described above) that is an ordered good bowtie-free graph. Because \(A\) is substructure of \(L_2(G)\) (and thus \(A \in \mathcal{B}\)) the statement follows.

4 Reduced structures

To simplify our future analysis, we now invoke another modification of \(L_2\)-structures. It is easy to see that the \(L_2\)-lift was created in such a way that all edges in between two centres of a vertex, \(C_1\) and \(C_2\), are in fact encoded by type of any pair of vertices \(v_1 \in C_1\) and \(v_2 \in C_2\) which is explicitely represented in \(L_2\)-lifts. We can thus safely omit all but one vertex from every centre of a vertex without losing any information about a good bowtie-free \(L_2\)-structure:

**Definition 4.1 (Reduced structures).** \(A^*\) is a reduction of \(L_2\)-structure \(A\) if it is created from \(A\) by removing all vertices \(v \in R_{A}^r, R_{A}^2, R_{A}^3, R_{A}^4\).

We modify the language \(L_2\) correspondingly into \(L^*\) and denote by \(B^*\) the class of all reduced structures \(A^*\) where \(A \in B_2\) and by \(\overline{B}^*\) for all reduced structures \(A^*\) where \(A \in \overline{B}\). Accordingly we modify the other definitions (such as the definition of pure-irreducible structures).

By comparing the corresponding definitions we have that reduced structures are still described by a set of forbidden substructures with at most 3 vertices (in a sense of Theorem 3.8):

**Theorem 4.2.** \(\overline{B}^*\) is the class of all finite admissibly ordered pure irreducible structures in \(\text{Forb}(L^*)\) where \(T^*\) is a finite set of pure irreducible structures with at most 3 vertices.

**Proof.** \(T^*\) is the subset of \(T\) defined in Theorem 3.8 containing \(L^*\) shadows of all structures \(A \in T\) such that all relations \(R_{A}^r, R_{A}^2, R_{A}^3, R_{A}^4\) are empty.

5 Ramsey structures

The following strong Ramsey theorem is a variant of the main result of [20], see also [16]. It will be used repeatedly (for example in Sections 6 and 7). This result deals with relational structures in language \(K\) with a given ordering: ordered \(K\)-structure is a lift of a \(K\)-structure adding binary relation \(\leq\) representing the order.

**Theorem 5.1 ([20]).** Let \(K\) be a finite relational language involving unary relations \(U_1, U_2, \ldots, U_N\). Let \(F\) be a set of irreducible \(K\)-structures. Let \(C\) be
the class of all ordered $K$-structures $A$ where every vertex is in exactly one of
unary relations $R_{A}^{i}$, and the ordering $\leq_{A}$ satisfies
\[ x < y \text{ whenever } (x) \in R_{A}^{i} \text{ and } (y) \in R_{A}^{j} \text{ and } 1 \leq i < j \leq N. \]
We call such an ordering an admissible ordering. Then the class $C \cap \text{Forb}(F)$
together with admissible orderings is a Ramsey class.

In fact, this is an $N$-partite version of the main result of [20]. It follows easily either from the (Partite Construction) proof, or directly by a product argument. The later can be outlined as follows: Given admissibly ordered $A,B \in C \cap \text{Forb}(F)$, by [20] there exists $C \in \text{Forb}(F)$ with $C \rightarrow (B)^{A}_{2}$. C is ordered but this ordering may not have to be admissible. Then it is possible to re-order vertices of C lexicographically first by unary relations they belong to and second by the original order of C. It is easy to see that this new order is admissible and preserves all copies of (admissibly ordered) structure B.

We apply Theorem 4.2 to the set $T^{*}$ of irreducible $L^{*}$-structures defined in the Theorem 4.2. We consider the structures in $\text{Forb}_{L^{*}}(T^{*})$ with admissible orderings defined above. Theorem 5.1 then specialises to the following:

**Theorem 5.2.** The class $\text{Forb}_{L^{*}}(T^{*})$ is a Ramsey class.

Explicitly: For every pair of $L^{*}$-structures $A,B$ in $\text{Forb}_{L^{*}}(T^{*})$ there exists an $L^{*}$-structure $C \in \text{Forb}_{L^{*}}(T^{*})$ such that
\[ C \rightarrow (B)^{A}_{2}. \]

**Proof.** Indeed this is just a specialisation of Theorem 5.1. 

However note that even when $A$ and $B$ are pure irreducible structures in $\text{Forb}_{L^{*}}(T^{*})$ the structure $C$ in $\text{Forb}_{L^{*}}(T^{*})$ is not necessarily pure irreducible and thus it may not correspond to the reduction of $L_{2}$-lift of a good bowtie-free graph. (As there are forbidden configurations we cannot complete $C$ to an pure irreducible structure "freely".)

Theorem 5.2 is just the first step in proving the main result.

### 6 Star Equivalences are Ramsey

The key feature of bowtie-free graphs is the partition to chimneys with each class of the partition “rooted” in the centre (the root being its algebraic closure). In this section we prove the Theorem 6.6 which extends the Theorem 5.2 to structures with such “rooted equivalences”. This brings us closer to the main result (which is proved in the next section).

**Definition 6.1 (Chimney equivalence).** For a $L^{*}$-structure $A \in B^{*}$ (i.e. which is the reduction of the $L_{2}$-lift of a good bowtie-free graph) denote by $\sim_{A}$ the equivalence expressing that two vertices belong to the same chimney (contracted central vertices are included in this). $\sim_{A}$ is called the chimney equivalence of $A$.

Note that each equivalence class of $\sim_{A}$ contains a distinguished vertex $x$ which is the (reduced) centre of the corresponding chimney or a copy of $K_{4}$. Moreover all other vertices of this equivalence class are related to $x$ by edges
belonging to $R^{E_0}_A$ that corresponds to a (spanning) star and there are no other vertices joined to $x$ by $R^{E_0}_A$ edges. Thus the equivalence $\sim_A$ is described by a star forest formed by $R^{E_0}_A$ edges. (Star is a complete bipartite graph $K_{1,k}$, $k \geq 0$. Star forest is any graph created as a disjoint union of stars.) This leads us to the following definition which make sense for reducible structures in $\text{Forb}_L^\bullet(T^*)$:

**Definition 6.2** (Star equivalence). For an $L^\bullet$-structure $A \in \text{Forb}_L^\bullet(T^*)$ assume that the edges $R^{E_0}_A$ form a star forest. Denote by $\approx_A$ (called star equivalence) the equivalence expressing the component structure of this star forest.

The equivalence $\approx_A$ for pure reducible structures will play the role of the chimney equivalence for pure irreducible structure.

**Definition 6.3.** Denote by $\text{Forb}^\square_A(T^*)$ the class of all (not necessarily pure irreducible) structures $A \in \text{Forb}^\bullet_L(T^*)$ where $\approx_A$ is a star equivalence (that is edges $R^{E_0}_A$ forms a star forest) and such that all vertices that appear in centres of stars (possibly degenerated to 1 vertex) are either in $R^{E_0}_1$ or $R^{E_0}_A$.

In this section we aim to prove Theorem 6.6 which gives Ramsey property for structures with star equivalences. Advancing this we modify the key part of proof of Theorem 5.1.

We shall stress the fact that $\text{Forb}^\square_A(T^*)$ can not be expressed as a class $\text{Forb}_1^\bullet(T')$ where $T'$ is a set of pure irreducible structures. There is no way to express the fact that no vertex can be connected to centres of two different stars. Consequently we can not apply Theorem 5.1 or 5.2 directly. The proof bellow uses a variant of the Partite Construction [19]. We modify its core part—Partite Lemma—in order to satisfy the additional equivalence condition.

The following is the main definition of this section.

**Definition 6.4** (A-partite structure). Let $A$ be a $L^\bullet$-structure. Assume $A = \{1, 2, \ldots, a\}$ with the natural ordering. An A-partite $L^\bullet$-structure is a tuple $(A, \lambda_{A}, B)$ where $B$ is an $L^\bullet$-structure and $\lambda_{A} = \{X^1_{B}, X^2_{B}, \ldots, X^a_{B}\}$ partitions vertex set of $B$ into $a$ classes ($X^i_{B}$ are called parts of $B$) such that

1. the ordering of $B$ is lexicographic induced by the ordering of $A$ and of parts $X^i_{B}$ (particularly it satisfies $X^1_{B} < X^2_{B} < \ldots < X^a_{B}$);
2. mapping $\pi$ which maps every $x \in X^i_{B}$ to $i$ ($i = 1, 2, \ldots, a$) is a homomorphism $B \to A$ in $L^\bullet$ ($\pi$ is called the projection);
3. every tuple in every relation of $B$ meets every class $X^i_{B}$ in at most one element (i.e. these tuples are transversal with respect to the partition).

The isomorphisms and embeddings of $A$-partite structures, say of $B$ into $B'$ are defined as the isomorphisms and embeddings of $L^\bullet$-structures together with the condition that all parts are preserved (i.e. the part $X^i_{B}$ is mapped to $X^i_{B'}$ for every $i = 1, 2, \ldots, a$).

In the following we will consider $L^\bullet$-structure $A$ to be also an $A$-partite structure, where each class of the partitions $X^1_{A}, X^2_{A}, \ldots, X^a_{A}$ consist of single vertex. For brevity, given a class of $L^\bullet$ structures $\mathcal{K}$ and an $A$-partite structure $B = (A, \lambda_{A}, B')$, we will also write $B \in \mathcal{K}$ with the meaning $B' \in \mathcal{K}$. We start by proving the following modification of the Partite Lemma [19]:

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**Lemma 6.5 (Partite Lemma).** Let $A \in B^*$ be an $L^*$-structure with star equivalence $\approx_A$ induced by $R^0_A$ edges. Assume without loss of generality $A = \{1,2,\ldots,a\}$ with the natural ordering. Let $B \in B \in \text{Forb}_{L^*}(T^*)$ be an $A$-partite $L^*$-structure with parts $X_B = \{X_B^1, X_B^2, \ldots, X_B^n\}$ and star equivalence $\approx_B$ induced by $R^0_B$ edges. Further assume that every vertex of $B$ is contained in a copy of $A$. Then there exists an $A$-partite $L^*$-structure $C$ with parts $X_C = \{X_C^1, X_C^2, \ldots, X_C^n\}$ where $R^0_C$ form a star forest defining star equivalence $\approx_C$ such that

$$C \rightarrow (B)^2_A.$$ 

Explicitly: For every 2-colouring of all $A$-partite substructures of $C$ which are isomorphic to $A$ there exists a substructure $\tilde{B}$ of $C$, $\tilde{B}$ isomorphic to $B$, such that all the substructures of $\tilde{B}$ which are isomorphic to $A$ are all monochromatic. Particularly, the isomorphism of $\tilde{B}$ and $B$ (which is an embedding of $B$ into $C$) and thus maps $R^0_B$ to $R^0_C$ and therefore also maps the equivalence $\approx_B$ to $\approx_C$.

**Proof.** Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_t$ be the enumeration of all substructures of $B$ which are isomorphic to $A$.

We take $N$ sufficiently large (that will be defined later) and construct an $A$-partite $L^*$-structure $C$ with parts $X_C = \{X_C^1, X_C^2, \ldots, X_C^n\}$ as follows:

1. For every $1 \leq i \leq a$ set $X_i^C$ is the set of all functions $f : \{1,2,\ldots,N\} \rightarrow X_B^i$.

2. The ordering $\leq_C$ of $C$ is defined lexicographically as an extension of all orderings of $A$ and $\{1,2,\ldots,N\}$.

3. For every relational symbol $R \in L^*$, $(f_1, f_2, \ldots, f_r) \in R_C$ if and only if one of the following occur:

   (a) There exists function $u : \{1,2,\ldots,N\} \rightarrow \{1,2,\ldots,t\}$ such that for every $1 \leq l \leq N$ the tuple $(f_1(l), f_2(l), \ldots, f_r(l))$ is in $R^0_{A_u(l)}$.

   (b) There exists $\omega \subseteq \{1,2,\ldots,N\}$ and function $u : \{1,2,\ldots,N\} \setminus \omega \rightarrow \{1,2,\ldots,t\}$ such that functions $f_1, f_2, \ldots, f_r$ are all constant on $\omega$ and:

   i. $f_1(l), f_2(l), \ldots, f_r(l)$ are all vertices of $\tilde{A}_u(l)$ in $B$, for every $l \in \{1,2,\ldots,N\} \setminus \omega$.

   ii. $(f_1(l), f_2(l), \ldots, f_r(l)) \in R^0_B$ and there is no copy of $A$ in $B$ containing all vertices $f_1(l), f_2(l), \ldots, f_r(l)$, for $l \in \omega$.

If vertex $v$ of $B$ is contained in the star $S$, denote by $s_B(v)$ the centre of $S$ and put $s_B(v) = v$ otherwise. Define the equivalence $\equiv$ on $C$ as follows: $f \equiv g$ if and only if $s_B(f(l)) = s_B(g(l))$ for every $1 \leq l \leq N$.

Observe that $C \in \text{Forb}_{L^*}(T^*)$ (as $A \in \text{Forb}_{L^*}(T^*)$ and the projection $\pi : C \rightarrow A$ is a homomorphism). We shall check that indeed $C$ is an $A$-partite $L^*$-structure (and thus again $C \in \text{Forb}_{L^*}(T^*)$) with parts $X_C = \{X_C^1, X_C^2, \ldots, X_C^n\}$ and finally we prove that the edges $R^0_C$ form a star forest and that the star equivalence $\approx_C$ coincides with $\equiv$. Most of this follows immediately from the definition. We pay extra attention to checking that $\equiv$ give the star equivalence. This is the main difference from [19].
It is easy to see that $\equiv$ is indeed an equivalence. We show that the $\equiv$ is the star equivalence of $\approx_C$ induced by edges $R^E_C$. First observe that $u \approx_B v$ if and only if $s_B(u) = s_B(v)$. Moreover, because $A$ corresponds to a reduced good bowtie-free structure and because every vertex of $B$ is contained in a copy of $A$ we know that every edge of type 0 in $B$ is an edge of a copy of $A$. It follows that vertices $f,g$ of $C$ are connected by edge of type 0 if and only if for each $1 \leq l \leq N$ we have an edge of type 0 in between $f(l)$ and $g(l)$, and consequently we have $s_B(f(l)) = s_B(g(l))$. Thus $\approx_C$ and $\equiv$ coincide. This proves that $C \in \text{Forb}_{T^*}^L (T^*)$.

For completeness we check that $C \rightarrow (B)^2_2$. Let $N$ be the Hales-Jewett number guaranteeing a monochromatic line in any 2-colouring of $N$-dimensional cube over alphabet $\{1,2,\ldots,t\}$.

Now assume that we have a 2-colouring of all copies of $A$ in $C$. Using the definition of $C$ we see that among these copies of $A$ are copies induced by an $N$-tuple $(\tilde{A}_{u(1)}, \tilde{A}_{u(2)}, \ldots, \tilde{A}_{u(N)})$ copies of $A$ for every function $u : \{1,2,\ldots,N\} \rightarrow \{1,2,\ldots,t\}$. However such copies are coded by the elements of the cube $\{1,2,\ldots,l\}^N$ and thus there is a monochromatic combinatorial line. This line in turn will lead to a copy $\tilde{B}$ of $B$ in $C$ with all edges of the form (a), (b) described above.

We can now invoke the Partite Construction [19] [16] in its standard form. We prove:

**Theorem 6.6.** Let $A, B \in B^*$ be $L^*$-structures with star equivalences $\approx_A$ and $\approx_B$ (induced by $R^E_A$ and $R^E_B$). (Thus also $A, B \in \text{Forb}_{L^*}^C (T^*)$ for $T^*$ defined in Theorem 3.8 and Theorem 4.2). Then there exists $L^*$-structure $C \in \text{Forb}_{L^*}^C (T^*)$ with the star equivalence $\approx_C$ induced by star forest $R^E_C$ such that

$$C \rightarrow (B)^2_2$$

with respect to embeddings preserving the equivalences.

**Proof.** Fix structures $A, B$. Using Theorem 5.2 obtain $C_0 \in \text{Forb}_{L^*}^C (T^*)$ (i.e. without the star forest condition) that satisfies $C_0 \rightarrow (B)^2_2$. Assume without loss of generality that $C_0 = \{1,2,\ldots,c\}$. Enumerate all copies of $A$ in $C_0$ as $\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_b\}$. We shall define $C_0$-partite structures $P_0, P_1, \ldots, P_b$ which, as we shall show, will all belong to $\text{Forb}_{L^*}^C (T^*)$ with the star equivalence which induces $\approx_P$. Putting $C = P_0$ we shall have the desired Ramsey property $C \rightarrow (B)^2_2$.

We denote the parts of $C_0$-partite structures $P_i$ as $X_{P_i} = \{X_{i1}, X_{i2}, \ldots, X_{ic}\}$. As usual the structures $P_i$ are called *pictures*. Pictures will be constructed by induction on $i$.

The picture $P_0$ is constructed as a disjoint union of copies of $B$: for every copy $B$ of $B$ in $C_0$ we consider a new isomorphic and disjoint copy $B'$ in $P_0$ which intersects part $X_{i0}$ if and only if the image of the projection $B$ contains vertex $l$ (so the projection restricted to $B'$ is $B$). Clearly $P_0 \in \text{Forb}_{L^*}^C (T^*)$. The star forest of $P_0$ is induced by the star forest of all copies $B'$.

Let the picture $P_i \in \text{Forb}_{L^*}^C (T^*)$ be already constructed. Let $\tilde{A}_i = \{x_1, x_2, \ldots, x_a\}$ be the vertices of $\tilde{A}_i$ (in the order of $C_0$). Let $B_i$ be the substructure of $P_i$ induced by $P_i$ on the union of vertices of those copies of $A$ which projects to $\tilde{A}_i$. (Note that $B_i$ need not contain all vertices of $P_i$ in parts...
In this situation we use Partite Lemma 6.5 to obtain an A-partite structures C_{i+1} with parts X_{1}, X_{2}, \ldots, X_{n}. Now consider all substructures of C_{i+1} which are isomorphic to B, and extend each of these structures to a copy of P_{i} (thus some new vertices may be added even in parts X_{i}). These copies are disjoint outside C_{i+1}, however in this extension we preserve the parts of all the copies. The result of this multiple amalgamation of copies of P_{i} is P_{i+1}. The star forest of P_{i+1} is defined as an amalgamation of star forest of copies of P_{i}. Of course we have to check bellow that this indeed results in a star forest.

Put C = P_{b}. It follows easily (by now by a standard argument cf. [13, 14]) that C \rightarrow (B)^{2}_{A}: by a backward induction one proves that in any 2-colouring of (C) there exists a copy P of P_{0} such that the colour of a copy of A in P depends only on its projection. As this in turn induces colouring of copies of A in C_{0}, we obtain a monochromatic copy of B.

We have to check that the edges of R^{E_{0}}_{B} form a star forest and that C belongs to Forb_{L}^{\infty}(T^{*}). To do so we proceed by an induction on i = 0, 1, 2, \ldots, b. The statement is clear for picture P_{0}. In the induction step (i \Rightarrow i + 1) we have to inspect the amalgamation of copies of P_{i} along the copies of structures B_{i} in C_{i+1}. It is clear that P_{i+1} belongs again to Forb_{L}^{\infty}(T^{*}) (as the forbidden substructures in T^{*} are all pure irreducible). It remains to show that \approx_{P_{i+1}} is a star equivalence of P_{i+1}. Because A is assumed to be a good bowtie-free structure and because B, has every vertex in a copy of A, we know that every star with leaf in B, also contains its centre in B, . By Lemma 6.5 we a get a star equivalence on C_{i+1}. The star equivalence is preserved by the free amalgamations of P_{i} over C_{i+1} because every time we unify leaves of a star we also unify the centre. Consequently the edges of R^{E_{0}}_{P_{i+1}} form a star forest inducing in P_{i+1} a star equivalence \approx_{P_{i+1}}.

7 Putting it together: Bowtie-free graphs have a Ramsey lift

In this section we prove the main Theorem 1.1 in the following form (\mathcal{B} is defined in Definition 3.5):

**Theorem 7.1.** The class \mathcal{B} is a Ramsey class.

**Proof.** Let A, B ∈ \mathcal{B} be fixed. We prove the existence of Ramsey object C ∈ \mathcal{B} in several steps.

Without loss of generality we can assume that B ∈ B_{2}. For A ∈ \mathcal{B} there exists, up to isomorphism, a unique minimal L_{2}-structure \tilde{A} ∈ B_{2} which correspond to a good bowtie-free graph such that A is a substructure of \tilde{A}. This just means that we complete each centre c(A) to a “full” centre by possibly adding to every vertex left-, or right-vertex or completing some vertices to K_{4}. This correspondence A → \tilde{A} is functorial in the sense that (for any L_{2}-structure B ∈ B_{2}) the correspondence of (B_{A}) and (B_{\tilde{A}}) is one to one. (This is another consequence of the algebraic closure.) Thus we may assume that both A and B are L_{2}-structure (in the sense of Definition 3.5).

Let A^{*}, B^{*} ∈ \mathcal{B} be reduced structures of A and B with star forests inducing equivalences \approx_{A} and \approx_{B}. By Theorem 6.6 there exists D^{*} ∈ Forb_{L}^{\infty}(T^{*})
satisfying

\[ D^* \rightarrow (B^*)^A_2. \]

with \( R^E_D \) forming a star forest and defining star equivalence \( \approx_D \). Without loss of generality we assume that all vertices and tuples in relations of \( D^* \) are contained in a copy of \( B^* \).

We use \( D^* \) to reconstruct \( C \in B_2 \subseteq \overline{E} \) which is a “completion” of \( D^* \): \( C \) will contain \( D^* \) as a non-induced substructure in a way that every copy of \( B^* \) in \( D^* \) can be extended to induced copy of \( B \) in \( C \).

\( D^* \) is a reduced structure. First we reverse the operation. Denote by \( D \) a structure created from \( D^* \) by adding, for every vertex \((u) \in R_0^E \) a new vertex \( v \), adding \((v) \) to \( R^E_D \), and connecting every vertex \( u', u' \approx_D u \), to \( v \) by an edge in \( R^E_D \). The result of this operation is a chimney. Similarly, for every \((u) \in R_1^E \), introduce the additional 3 vertices to form the clique. Finally add the edges \( R^E_D \) connecting the newly introduced vertices to vertices of \( D^* \) as described by the existing tuples in \( R^t_D \). Because we assume that every vertex and every tuple of every relation of \( D^* \) is in a copy of \( B^* \) and every vertex in \( D^* \) is a part of a star equivalence where all copies share the centre, there is unique way of doing so: consider \( a_1 \approx_D a_2 \in D^* \) and \( b_1 \approx_D b_2 \) such that \((a_1, b_1) \in R^t_D \), and \((a_2, b_2) \in R^t_D \). We have a copy \( \tilde{B}_1^* \) containing \( \{a_1, b_1\} \) and a copy \( \tilde{B}_2^* \) containing \( \{a_2, b_2\} \). Because \( B^* \) is a good structure, we know the amalgamation is possible (Lemma 3.6). It follows that types \( t_i \) and \( t_j \) must agree on the newly introduced edges.

Next we check that the \( L_1 \)-shadow \( \text{Sh}(D) \) is a good ordered bowtie-free graph. \( \text{Sh}(D^*) \) is triangle-free and thus every triangle in \( D \) contains at least one new central vertex. Every new triangle is a part of a copy of \( \text{Sh}(B) \) (which was created by expansion of a copy of \( B^* \)) and thus it consist of edges of type 0 only. Consequently we only need to verify that edges in \( R^E_D \) do not form a bowtie. It is easy to verify that those edges however forms cliques \( C_4 \) (which were introduced by expanding vertex in \( R^t_D \) ) and chimneys spanning vertices of each star equivalence class of \( B^* \) which centre is in \( R^t_D \).

The ordering of \( D^* \) is not necessarily admissible. The use of contracted structures along with the notion of admissible ordering in the sense of Theorem 5.1 makes the order of \( D^* \) satisfy conditions 1, 2, and 3 of Definition 3.2.

The order of non-central vertices is however free. It is not difficult to see that the non-central vertices can be reordered according their centres preserving relative order of vertices with the same centre. This order is admissible and does preserve all embeddings of admissibly ordered structures we need. The order of \( D \) can then be defined from the admissible order of \( D^* \) in a natural way.

We have constructed good ordered \( L_1 \)-structure \( \text{Sh}(D) \) where every copy of shadow \( \text{Sh}(B^*) \) was extended to a copy of \( \text{Sh}(B) \). Finally we put \( C = L_2(\text{Sh}(D)) \in E \). Because the lift \( L_2 \) is constructed in an unique way which preserves substructures, we have

\[ C \rightarrow (B)^A_2. \]
8 The lift property

We say that a lifted class \( K^+ \) has the \textit{lift property relative to} \( K \) (where \( K \) is shadow of \( K^+ \)) if and only if for every \( A \in K \) there is \( B \in K \) such that for every \( A^+, B^+ \in K^+ \) such that the shadow of \( A^+ \) is \( A \) and the shadow of \( B^+ \) is \( B \) there is an embedding from \( A^+ \) to \( B^+ \).

The lift property is a generalisation of the ordering property [18]. Structures that have both Ramsey and ordering property play important role in [13] where they are used to obtain universal minimal flows. [21] defines expansion property (called here the lift property) and gives analogous results in the setting of Ramsey classes with the lift property. To apply these results to the class \( B \) we now show that lifts constructed in this paper have the lift property.

**Theorem 8.1.** \( \overline{B} \) has the lift property relative to \( B \).

**Proof.** This result follows in an analogy to [18] (where it is shown that the ordered edge Ramsey property implies ordering property). Essentially we only need to deal with additional unary and binary relations present in our lifts.

Fix a bowtie-free graph \( A \). We will give explicit construction of \( B \) needed for the lift property. Consider all possible extensions of \( A \) into a good bowtie-free graphs that are minimal in the sense that removing any non-empty set of vertices from the extension makes the graph either not good or not containing \( A \). (Clearly this is finite set.) Now consider all admissible orderings (Definition 3.2) of these extensions. Denote these ordered good bowtie-free graphs by \( A_1, A_2, \ldots, A_N \).

Now we extend graphs \( A_1, A_2, \ldots, A_N \) to graphs \( A'_1, A'_2, \ldots, A'_N \) by simple gadgets that will allow us to use the Ramsey property to ensure the order. For that we consider all vertices \( v \) of \( A_i \) \((1 \leq i \leq N)\) with the following properties:

1. \( M_I = \{v; (v) \in R_{L_2(A_i)}^I\} \),

2. \( M_{II} = \{v; (v) \in R_{L_2(A_i)}^I\} \),

3. \( M_{III} = \{v; (v) \text{ is in no unary relations of } L_2(A_i)\} \).

Sets \( M_I, M_{II} \) and \( M_{III} \) are chosen in a way that the orders of sets \( M_I, M_{II} \) and \( M_{III} \) together with admissibility of order (Definition 3.2) determine total ordering of all vertices.

Consider pair of vertices \( u \neq v \) of \( A_i \) where both \( u \) and \( v \) belong to one of the sets \( M_I, M_{II} \) or where both \( u \) and \( v \) belong to set \( M_{III} \) and have the same centre. For each such pair extend ordered good bowtie-free graph \( A_i \) in a way so there is a vertex \( w(u, v) \) that belongs to same class in the lift as \( u \) and \( v \), it is not connected by an edge to \( u \) nor \( v \), and it is in between \( u \) and \( v \) in the order of \( \leq A^+_i \). Such a vertex can always be added in a way that the result is an ordered good bowtie-free graph (by possibly introducing new chimney or a copy of \( K_4 \)).

We denote by \( A'_i \) an ordered good bowtie-free graph having vertex \( w(u, v) \) for every possible choice of \( u \) and \( v \) in \( A_i \).

Denote by \( \overline{A'} \) the disjoint union of graphs \( A'_1, A'_2, \ldots, A'_N \). Now for every pair of \( u <_A w \) for which we introduced \( w(u, v) \) consider substructures induced on \( \{u, w(u, v)\} \) and \( \{w(u, v), v\} \) by \( L_2(A') \).
These structures are always isomorphic. (Recall that $R^\ell, R^r, R^1, R^2, R^3, R^4$ are the only unary relations in $L_1$.) Denote isomorphism types of those structures by $E_I, E_{II}$ and $E_{III}$. Now we use Theorem 6.6 to get:

\[
\begin{align*}
C_I &\rightarrow (L_2(A'))_2^{E_I}, \\
C_{II} &\rightarrow (C_{II})_2^{E_{II}}, \\
C_{III} &\rightarrow (C_{III})_2^{E_{III}},
\end{align*}
\]

where all $C_I, C_{II}, C_{III}$ belong to $\mathfrak{B}$. Let $B \in \mathfrak{B}_0$ be the shadow of $C_{III}$. We claim that $B$ is a good bowtie-free graph with the lift property for $A$.

Let $B^+ \in \mathfrak{B}$ be a $L_2$-lift of any admissibly ordered structure $B$. We now assign colours to copies of $E_I, E_{II}$, and $E_{III}$ in $C_{III}$ by comparing order of vertices in $B^+$ and $C_{III}$. (If the order agrees the colour is red and blue otherwise.) By Ramsey property we obtain a copy of $A'$ such all copies of $E_I, E_{II}$, $E_{III}$ are monochromatic. This means that within the copies of $A$, the relative order of vertices within sets $M_I, M_{II}$ and set $M_{III}$ vertices assigned to a given centre is either the same as in $B^+$ or opposite (independently in each class). It is easy to see that any admissible orderings of $A$ can be turned to another admissible ordering of $A$ by reversing orders within each of the classes (and possibly adjusting order in between intervals assigned to each centre). It is thus possible to find $A_i$ within $B^+$ that is ordered the same way.

Remark. Theorem 8.1 is the only place in the paper that actually needs conditions given on admissible ordering by Definition 3.2. There are many possible choices of orderings of good bowtie-free graphs; completely free ordering, ordering by unary relations, ordering by corresponding centre, etc. Good graphs ordered freely (as well as other cases) can also be shown to have Ramsey lift constructed the same way as our lift. It is easy to see that such class however fails to have the lift property: Consider a good bowtie-free graph $A$ consisting of two chimneys where order of non-central vertices does not follow the order of centres. For any choice of $B$ it is possible to give an admissible ordering in the sense of Definition 3.2 giving a lift of $B$ that does not include the given lift of $A$ (because the order of $A$ is not admissible).

Theorem 8.1 can thus be understood as an argument why among possible Ramsey lifts of $\mathfrak{B}$ the one given here is the optimal one.

9 Concluding remarks

It is conjectured in [2] that for classes defined by forbidden monomorphisms from one forbidden graph the algebraic closure operator is either unary or there are no universal $\omega$-categorical graph at all. We believe that all such classes with unary closure operator can be proved to be Ramsey by a generalisation of a proof presented here. On the other hand a simple example is given in [2] showing that the closure does not need to be unary for classes defined by forbidden homomorphism from more than one connected graph. Our techniques do not seem to directly generalise for this case.

Other important case is the situation where the amalgamation is not free over closed sets. Several such classes with strong amalgamation have been proved to be Ramsey by means of Partite Construction (among those the classes mentioned in the introduction: partial orders, metric spaces and classes $\text{Forb}_{III}(\mathcal{F})$).
We hope that it is possible to combine both techniques to obtain Ramsey results on even more restricted classes of graphs.

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