The characteristic classes of Morita equivalent star products 
on symplectic manifolds

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Abstract

In this paper we give a complete characterization of Morita equivalent star products 
on symplectic manifolds in terms of their characteristic classes: two star products $\ast$ 
and $\ast'$ on $(M,\omega)$ are Morita equivalent if and only if there exists a symplectomorphism 
$\psi : M \to M$ such that the relative class $t(\ast, \psi^*(\ast'))$ is $2\pi i$-integral. For star products 
on cotangent bundles, we show that this integrality condition is related to Dirac’s 
quantization condition for magnetic charges.

1 Introduction

The concept of Morita equivalence has played an important role in different areas of math-
ematics (see [25] for an overview) since its introduction in the study of unital rings [26]. In 
applications of noncommutative geometry to $M$-theory [14], Morita equivalence was shown

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to be related to physical duality \[33\], motivating the study of Morita equivalence of quantum tori \[32\]. In this setting, the problem is to characterize constant Poisson structures \(\theta\) on the \(n\)-torus \(T^n\) that, after strict deformation quantization \[31\], give rise to Morita equivalent \(C^*\)-algebras \(T_\theta\).

In this paper we address the problem of characterizing Morita equivalent algebras obtained from formal deformation quantization of Poisson manifolds \[2\] (see \[19, 34, 36\] for surveys). In this approach to quantization, quantum algebras of observables are defined by formal associative deformations (in the sense of \[18\]) of classical Poisson algebras known as star products.

The problem of classifying Morita equivalent star products on a Poisson manifold \((M,\pi_0)\) can be phrased in terms of a canonical action \(\Phi\) of the Picard group \(\text{Pic}(M) \cong H^2(M,\mathbb{Z})\) on \(\text{Def}(M,\pi_0)\), the moduli space of equivalence classes of differential star products on \((M,\pi_0)\) \[10\]. The action \(\Phi\) is defined by deformation quantization of line bundles on \(M\) \[12\], and two star products \(\star, \star'\) are Morita equivalent (as unital \(C[\hbar]\)-algebras) if and only if there exists a Poisson diffeomorphism \(\psi : M \to M\) such that the classes \([\star]\) and \([\psi^*(\star')]\) lie in the same \(\Phi\)-orbit. The semiclassical limit of this action was described in \[10, \text{Thm. 5.11}\].

Let \((M,\omega)\) be a symplectic manifold. The main result of this paper is that, under the usual identification \[4, 27\]

\[
\text{Def}(M,\omega) \cong \frac{1}{i\lambda}[\omega] + H^2_{\text{dR}}(M)[[\lambda]],
\]

the action \(\Phi\) is given by

\[
\Phi_L([\omega_\lambda]) = [\omega_\lambda] + 2\pi i c_1(L), \quad (1.1)
\]

where \([\omega_\lambda] = (1/i\lambda)[\omega] + \sum_{r=0}^{\infty}[\omega_r] \lambda^r\), and \(c_1(L)\) is the Chern class of \(L\). It immediately follows from \(1.1\) that two star products \(\star, \star'\) on \(M\) are Morita equivalent if and only if there exists a symplectomorphism \(\psi : M \to M\) such that the relative class \(t(\star, \psi^*(\star'))\) is \(2\pi i\)-integral. The explicit computation of \(\Phi_L\) is based on a local description of deformed line bundles over \(M\), through deformed transition functions, and the Čech-cohomological approach to Deligne’s relative class developed in \[20\]. As it turns out, this result also gives a classification of Hermitian star products on \(M\) up to strong Morita equivalence, a purely algebraic generalization of the usual notion of strong Morita equivalence of \(C^*\)-algebras \[11, 13\].

By considering star products on cotangent bundles \(T^*Q\), we observe that the integrality condition coming from Morita equivalence can be interpreted as Dirac’s quantization condition for magnetic charges: We consider the star products \(\star_\kappa - \lambda B\), constructed in \[5\] out of a \(\kappa\)-ordered star product \(\star_\kappa\) on \(T^*Q\) and a magnetic field \(B \in \Omega^2(Q)[[\lambda]], dB = 0\), and show that \(\star_\kappa\) and \(\star_\kappa - \lambda B\) are Morita equivalent if and only if \((1/2\pi)B\) is an integral 2-form. In this case, well-known \(\star\)-representations of \(\star_\kappa - \lambda B\) on sections of line bundles \[5\] are obtained by means of Rieffel induction of the formal Schrödinger representation of \(\star_\kappa\).
After the conclusion of this work, [23] was brought to our attention; this paper addresses some related questions and introduces a similar local description of quantum line bundles.

We note that (1.1), when written in terms of formal Poisson structures, coincides with the expression of $\theta'$ in [23, pp. 3]. A detailed comparison between the approaches is in progress.

The paper is organized as follows. In Section 2 we recall the notions of star products, deformation quantization of vector bundles and Morita equivalence, and give a local description of deformed vector bundles in terms of quantum transition matrices, including Hermitian structures. In Section 3 we compute the relative class of Morita equivalent star products on symplectic manifolds and discuss the main results of the paper. In Section 4 we consider star products on cotangent bundles and discuss Morita equivalence in terms of Dirac’s condition for magnetic monopoles. We have included two appendices: Appendix A recalls some basic facts about $\star$-exponentials and logarithms used in the paper; Appendix B recalls the notions of algebraic Rieffel induction and strong Morita equivalence.

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2 Preliminaries

2.1 Star products, deformed vector bundles and Morita equivalence

Let $(M, \pi_0)$ be a Poisson manifold, where $\pi_0 \in \Gamma^\infty(\bigwedge^2 TM)$ denotes the Poisson tensor. The corresponding Poisson bracket is denoted by $\{f, g\} := \pi_0(df, dg)$. Let $C^\infty(M)$ be the algebra of complex-valued smooth functions on $M$. We recall the definition of star products [2].

**Definition 2.1** A star product on a Poisson manifold $(M, \pi_0)$ is a $\mathbb{C}[[\lambda]]$-bilinear associative product on $C^\infty(M)[[\lambda]]$ of the form

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g), \quad f, g \in C^\infty(M),$$

where each $C_r : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is a bidifferential operator, $C_0(f, g) = fg$ (pointwise product of functions) and $C_1(f, g) - C_1(g, f) = \{f, g\}$. It is often required that $f \star 1 = f = 1 \star f$.

For physical applications, $\lambda$ plays the role of Planck’s constant $\hbar$ as soon as the convergence of (2.1) can be established. The existence of star products on symplectic manifolds was proven in [16, 17, 28]; for arbitrary Poisson manifolds, it follows from Kontsevich’s formality theorem [24].
Two star products $\star$ and $\star'$ are called equivalent if there exist differential operators $T_r : C^\infty(M) \to C^\infty(M)$ so that $T = \text{id} + \sum_{r=1}^\infty \lambda^r T_r$ satisfies

$$T(f \star g) = T(f) \star T(g), \quad f, g \in C^\infty(M).$$  \hspace{1cm} (2.2)

The equivalence class of a star product $\star$ on $(M, \pi_0)$ will be denoted by $[\star]$. We let

$$\text{Def}(M, \pi_0) := \{[\star], \ast \text{ a star product on } (M, \pi_0)\}.$$  \hspace{1cm} (2.3)

For symplectic manifolds, the moduli space (2.3) admits a cohomological description (2.4) that will be recalled in Section 3.2. We note that the group of Poisson diffeomorphisms of $M$ acts naturally on $\text{Def}(M, \pi_0)$: $\ast' = \psi^*(\ast)$ is defined by $\psi^*(f \ast' g) = \psi^* f \ast \psi^* g$.

A classical result of Serre and Swan [1, Chap. XIV] asserts that finite dimensional complex vector bundles over $M$ naturally correspond to finitely generated projective modules over $C^\infty(M)$ (with equivalence functor $E \mapsto \Gamma^\infty(E)$). This motivates the following definition [2, Def. 3.1]: Let $E \to M$ be a $k$-dimensional complex vector bundle, and let $\ast$ be a star product on $M$.

**Definition 2.2** A deformation quantization of $E \to M$ with respect to $\ast$ is a $C[[\lambda]]$-bilinear map $\bullet : \Gamma^\infty(E)[[\lambda]] \times C^\infty(M)[[\lambda]] \to \Gamma^\infty(E)[[\lambda]]$ satisfying $s \bullet (f \ast g) = (s \bullet f) \ast g$ and so that

$$s \bullet f = \sum_{r=0}^\infty \lambda^r R_r(s, f),$$  \hspace{1cm} (2.4)

where each $R_r : \Gamma^\infty(E) \times C^\infty(M) \to \Gamma^\infty(E)$ is bidifferential and $R_0(s, f) = sf$ (pointwise multiplication of sections by functions).

Two deformations $\bullet$ and $\bullet'$ are called equivalent if there exist differential operators $T_r : \Gamma^\infty(E) \to \Gamma^\infty(E)$ so that $T = \text{id} + \sum_{r=1}^\infty \lambda^r T_r$ satisfies

$$T(s \bullet' f) = (Ts) \bullet f, \quad s \in \Gamma^\infty(E), \quad f \in C^\infty(M).$$  \hspace{1cm} (2.5)

The following result was proven in [2, Prop. 2.6].

**Proposition 2.3** Let $E \to M$ be a vector bundle, and let $\ast$ be a star product on $M$. Then there exists a deformation quantization $\bullet$ of $E$ with respect to $\ast$, which is unique up to equivalence. The right module $(\Gamma^\infty(E)[[\lambda]], \bullet)$ is finitely generated and projective over $(C^\infty(M)[[\lambda]], \ast)$, and any finitely generated projective module over this algebra arises in this way.

Let $E = \Gamma^\infty(E)$, considered as a right $C^\infty(M)$-module, and $E = (E[[\lambda]], \bullet)$, considered as a right $(C^\infty(M)[[\lambda]], \ast)$-module. We recall that $\text{End}(E) \cong \Gamma^\infty(\text{End } E)$, and $\text{End}(E)$ is isomorphic to $\Gamma^\infty(\text{End } E)[[\lambda]]$ as a $C[[\lambda]]$-module [2, Cor. 2.4].

If $E = L \to M$ is a complex line bundle, then $\Gamma^\infty(\text{End}(L)) \cong C^\infty(M)$, and any $C[[\lambda]]$-module isomorphism $\Gamma^\infty(\text{End}(L))[[\lambda]] \to \text{End}(E)$ determines a new star product $\ast'$ on
Proposition 2.4 was proven in [10, Thm. 4.1].

Two star products $\mathfrak{p}ic$ define an action of $L$ by $E_B M$ and $\star$ on $E = \Gamma(M) \cong \mathbb{H}^2(M, \mathbb{Z})$, where $\text{Pic}(M)$ is the Picard group of $M$. The following result was proven in [10, Thm. 4.1].

**Proposition 2.4** The map $\Phi : \text{Pic}(M) \times \text{Def}(M, \pi_0) \rightarrow \text{Def}(M, \pi_0)$, $([L], [\star]) \mapsto \Phi([\star])$ defines an action of $\text{Pic}(M)$ on the set of equivalence classes of star products on $M$, and two star products $\star$ and $\star'$ on $M$ are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \rightarrow M$ such that the classes $[\star]$ and $[\psi^*(\star)]$ lie in the same $\Phi$-orbit.

Recall that two unital algebras $\mathcal{A}, \mathcal{B}$ (over some ground ring $R$) are called *Morita equivalent* if they have equivalent categories of left modules [21]; alternatively, there must exist a full finitely generated projective right $\mathcal{A}$-module $\mathcal{E}_\mathcal{A}$ so that $\mathcal{B} \cong \text{End}(\mathcal{E}_\mathcal{A})$. The bimodule $\mathcal{E}_\mathcal{A}$ is called a $(\mathcal{B}, \mathcal{A})$-equivalence bimodule.

The Picard group of a unital $R$-algebra $\mathcal{A}$, $\text{Pic}(\mathcal{A})$, is defined as the set of isomorphism classes of $(\mathcal{A}, \mathcal{A})$-equivalence bimodules, with group operation given by tensor product. If $\mathcal{A} = (C^\infty(M))$, then the algebraic Picard group $\text{Pic}(C^\infty(M))$ can be identified with the geometric Picard group $\text{Pic}(M)$.

Let $\star$ be a star product on $M$, and $\mathcal{A} = (C^\infty(M)[[\lambda]], \star)$. We note that the isotropy group of $\Phi$ at $[\star]$ can be identified with a subgroup of $\text{Pic}(\mathcal{A})$. In Section 3.2, we will give an explicit description of $\text{Pic}(\mathcal{A})$ for certain star-product algebras on symplectic manifolds. We also describe the orbit space $\text{Def}(M, \pi_0)/\text{Pic}(M)$ for $\pi_0$ symplectic.

### 2.2 A local description of deformed vector bundles

Let $E \rightarrow M$ be a $k$-dimensional smooth complex vector bundle over a smooth manifold $M$, and let $\{\mathcal{O}_\alpha\}$ be a good cover of $M$. Let us fix $\{e_{\alpha,i}\}, i = 1 \ldots k$, basis of $\Gamma^\infty(E|_{\mathcal{O}_\alpha})$, and let $e_\alpha = (e_{\alpha,1}, \ldots, e_{\alpha,k})$ be the corresponding frame. Such a choice defines trivialization maps $\psi_\alpha : \Gamma^\infty(E|_{\mathcal{O}_\alpha}) \rightarrow C^\infty(\mathcal{O}_\alpha)^k$. On overlaps $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, we define transition matrices $\phi_{\alpha\beta} = \psi_\alpha \psi_\beta^{-1} \in M_k(C^\infty(\mathcal{O}_\alpha \cap \mathcal{O}_\beta))$. Clearly $\phi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$, and on triple intersections we have

$$\phi_{\alpha\beta} \phi_{\beta\gamma} \phi_{\gamma\alpha} = 1.$$ (2.7)

We will see that similar constructions can be carried out for deformed vector bundles (see also [22, 23]). Let $\star$ be a star product on $M$, and let us fix a deformation $\mathcal{E} = (\mathcal{E}[[\lambda]], \bullet)$, $\mathcal{E} = \Gamma^\infty(E)$, with respect to $\star$. A simple induction shows the following result.
Lemma 2.5 Let $e_{\alpha,i} = e_{\alpha,i} + \lambda e_{\alpha,i}^{(1)} + \cdots \in \Gamma^\infty(E|_{O_\alpha})[[\lambda]]$ be arbitrary deformations of the classical bases sections $e_{\alpha,i}$. Then for any global section $s \in \Gamma^\infty(E)[[\lambda]]$ there exist unique local functions $s^i_\alpha \in C^\infty(O_\alpha)[[\lambda]]$ such that

\[ s|_{O_\alpha} = \sum_{i=1}^k e_{\alpha,i} \cdot s^i_\alpha. \]  

(2.8)

We shall write $e_\alpha = (e_{\alpha,1}, \ldots, e_{\alpha,k})$ for the deformed frame, and $s_\alpha = (s^1_\alpha, \ldots, s^k_\alpha)$ for the deformed coefficient functions of a section $s$. As in the case of ordinary vector bundles, \[ \Psi \] induces $\mathbb{C}[\lambda]$-linear trivialization isomorphisms $\Psi_\alpha : \Gamma^\infty(E|_{O_\alpha})[[\lambda]] \to C^\infty(O_\alpha)[[\lambda]],$

\[ \Psi_\alpha = (\Psi^1_\alpha, \ldots, \Psi^k_\alpha), \quad \Psi^i_\alpha(s) = \Psi^i_\alpha(\sum_{j} e_{\alpha,j} \cdot s^j_\alpha) = s^i_\alpha, \]  

(2.9)

satisfying

\[ \Psi_\alpha(s \cdot f) = \Psi_\alpha(s) \ast f, \quad \text{for } f \in C^\infty(M). \]  

(2.10)

Clearly $\Psi_\alpha(s) = s_\alpha$. It is simple to check that $\Psi_\alpha$ deforms $\psi_\alpha$, i.e., $\Psi_\alpha = \psi_\alpha \mod \lambda$.

On overlaps $O_\alpha \cap O_\beta$, we define deformed transition matrices

\[ \Phi_{\alpha\beta} = \Psi_\alpha \circ \Psi^{-1}_\beta \in M_k(C^\infty(O_\alpha \cap O_\beta))[[\lambda]], \]  

(2.11)

satisfying $s_\alpha = \Phi_{\alpha\beta} \ast s_\beta$. We note that $\Phi_{\alpha\beta} = \Phi_{\beta\alpha}^{-1}$ (with respect to $\ast$), and the following deformed cocycle condition holds:

\[ \Phi_{\alpha\beta} \ast \Phi_{\beta\gamma} \ast \Phi_{\gamma\alpha} = 1. \]  

(2.12)

If $A \in \text{End}(E)$, then it is locally represented by a matrix $A_\alpha \in M_k(C^\infty(O_\alpha))[[\lambda]]$ satisfying $A(s)_\alpha = A_\alpha \ast s_\alpha$. On overlaps $O_\alpha \cap O_\beta$, we have

\[ A_\beta = \Phi_{\beta\alpha} \ast A_\alpha \ast \Phi_{\alpha\beta}. \]  

(2.13)

As in the classical case, a collection $\{A_\alpha\}$, $A_\alpha \in M_k(C^\infty(O_\alpha))[[\lambda]]$, satisfying (2.13) determines a global endomorphism of the deformed bundle. It is simple to see that the composition of endomorphisms corresponds locally to the deformed product of matrices:

\[ (A \circ B)_\alpha = A_\alpha \ast B_\alpha. \]  

(2.14)

Remark 2.6 One can define an explicit $\mathbb{C}[\lambda]$-module isomorphism $T : \Gamma^\infty(\text{End}(E))[[\lambda]] \to \text{End}(E)$ by patching local maps as follows. Let $\{\chi_\alpha\}$ be a quadratic partition of unity subordinated to $\{O_\alpha\}$ (i.e., $\text{supp} \chi_\alpha \subseteq O_\alpha$, and $\sum_\alpha \chi_\alpha = 1$). Then

\[ T_\alpha(A) = \sum_\gamma \Phi_{\alpha\gamma} \ast \chi_\gamma \ast A_\gamma \ast \chi_\gamma \ast \Phi_{\gamma\alpha} \]  

(2.15)

is well defined on $O_\alpha$. Here $A_\alpha$ are the local matrices of $A \in \Gamma^\infty(\text{End}(E))[[\lambda]]$ with respect to the undeformed trivialization maps $\psi_\alpha$. The collection $\{T_\alpha\}$ satisfies condition (2.13), and hence defines the desired global map $T$. In lowest order $T_\alpha(A)$ just reproduces $A_\alpha$. 

6
2.3 Hermitian structures

For completeness, we will briefly indicate how deformed Hermitian structures [12] can be treated locally. In this section, \( \star \) will be a Hermitian star product on \( M \), i.e. \( f \star g = \overline{g} \star \overline{f} \).

Let \( E \to M \) be equipped with a Hermitian fiber metric \( h_0 \) with respect to a deformation \( \bullet \) of \( E \) is a \( C^\infty(M)[[\lambda]] \)-valued Hermitian inner product \( h \) on \( \Gamma^\infty(E)[[\lambda]] \) (see Definition B.1) such that

$$h(s,s') = \sum_{r=0}^{\infty} \lambda^r h_r(s,s')$$

(2.16)

with bidifferential operators \( h_r : \Gamma^\infty(E) \times \Gamma^\infty(E) \to C^\infty(M) \).

Let \( \mathcal{E} \) denote the \( (C^\infty(M)[[\lambda]],\star) \)-module \( (\Gamma^\infty(E)[[\lambda]],\bullet) \). Two deformations \( h \) and \( h' \) are called isometric if there exists a module isomorphism

$$U = \text{id} + \sum_{r=1}^{\infty} \lambda^r U_r : \mathcal{E} \to \mathcal{E},$$

(2.17)

with differential operators \( U_r : \Gamma^\infty(E) \to \Gamma^\infty(E) \), so that

$$h(U s, U s') = h'(s,s')$$

(2.18)

for all \( s, s' \in \Gamma^\infty(E)[[\lambda]] \). From [12] we have the following result.

**Lemma 2.7** Let \( E \to M \) be a vector bundle with Hermitian fiber metric \( h_0 \), and let \( \bullet \) be a deformation quantization of \( E \). Then there exists a deformation quantization \( \mathcal{H} \) of \( h_0 \) and any two such deformations are isometric.

Let \( h \) be a deformation of \( h_0 \). We can construct local orthonormal frames \( e_\alpha \) with respect to \( h \):

**Lemma 2.8** Let \( \tilde{e}_\alpha \) be a local frame for \( \Gamma^\infty(E)[[\lambda]] \) such that the zeroth order is an orthonormal frame with respect to \( h_0 \). Then there exists a matrix \( V = \text{id} + \sum_{r=1}^{\infty} \lambda^r V_r \in M_k(C^\infty(O_\alpha)[[\lambda]]) \) such that \( e_\alpha := \tilde{e}_\alpha \bullet V \) is an orthonormal frame with respect to \( h \), i.e. one has

$$h(e_{\alpha,i},e_{\alpha,j}) = \delta_{ij}.$$  

(2.19)

**Proof:** Let \( H \) be the Hermitian matrix defined by \( H_{ij} = h(\tilde{e}_{\alpha,i},\tilde{e}_{\alpha,j}) \). Then \( H = \text{id} + \sum_{r=1}^{\infty} \lambda^r H_r \), since the zeroth order of \( \tilde{e}_\alpha \) is orthonormal with respect to \( h_0 \). From [12] Lem. 2.1] we know that there exists a matrix \( U = \text{id} + \sum_{r=1}^{\infty} \lambda^r U_r \) such that \( U^* \star U = H \). Then \( V = U^{-1} \) is the desired transformation.

Hence we can always assume that we have local orthonormal frames \( e_\alpha \) on each patch \( O_\alpha \). Obviously, the transition functions are unitary in this case:
Lemma 2.9 Let \( \{ e_\alpha \}_{\alpha \in I} \) be local orthonormal frames. Then we have
\[
\Phi^\alpha_\beta = \Phi^\beta_\alpha = \Phi^\alpha_\beta^{-1}
\] (2.20)
and \( h(s, s') = \langle s_\alpha, s'_\alpha \rangle \) is just the canonical Hermitian inner product on \( C^\infty(\mathcal{O}_\alpha)[[\lambda]] \) for the coefficient functions. If \( A \in \text{End}(E) \), then the local matrices of \( A \) and \( A^* \) are related by
\[
(A_\alpha)^* = (A^*_\alpha).
\] (2.21)
Note that, in this case, the isomorphism (2.13) is compatible with the *-structures.

3 Morita equivalent star products on symplectic manifolds

3.1 Deligne’s relative class (after Gutt and Rawnsley)

Let \( (M, \omega) \) be a symplectic manifold. In this case, it was shown in [4, 27] that there exists a bijection
\[
c : \text{Def}(M, \omega) \longrightarrow \frac{1}{1!\lambda}[\omega] + H^2_{\text{dir}}(M)[[\lambda]],
\] (3.1)
characterizing the moduli space of equivalence classes of star products on \( M \) in cohomological terms. For a star product \( \star \), \( c(\star) \) is called its characteristic class. Čech-cohomological description of these characteristic classes can be found in [15, 20].

For two star products \( \star, \star' \) on \( M \), their relative class is defined by
\[
t(\star', \star) = c(\star') - c(\star) \in H^2_{\text{dir}}(M)[[\lambda]].
\] (3.2)
A purely Čech-cohomological construction of \( t(\star', \star) \) was given in [20], and we will briefly recall it.

Let us fix a good cover \( \{ \mathcal{O}_\alpha \} \) of \( M \) and star products \( \star, \star' \). Then any two star products are equivalent on \( \mathcal{O}_\alpha \), see e.g. [20, Cor. 3.2]. Thus, for each \( \alpha \), we can find an equivalence transformation between \( \star \) and \( \star' \), \( T_\alpha = \text{id} + \sum_{r=1}^\infty \lambda^r T^{(r)}_\alpha \), where each \( T^{(r)}_\alpha \) is a differential operator on \( C^\infty(\mathcal{O}_\alpha) \). On the overlap \( \mathcal{O}_\alpha \cap \mathcal{O}_\beta \), the map \( T^{-1}_\alpha \circ T_\beta \) is a \( \star \)-automorphism starting with the identity. Since \( \mathcal{O}_\alpha \cap \mathcal{O}_\beta \) is contractible, the automorphism \( T^{-1}_\alpha \circ T_\beta \) is inner, and therefore there exists a function \( t_{\alpha\beta} \in C^\infty(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)[[\lambda]] \) such that (see Prop. A.1)
\[
T^{-1}_\alpha \circ T_\beta(f) = e^{[t_{\alpha\beta}]}(f) = \text{Exp}(t_{\alpha\beta}) \star f \star \text{Exp}(-t_{\alpha\beta}).
\] (3.3)
Since
\[
T^{-1}_\alpha \circ T_\beta \circ T^{-1}_\beta \circ T_\gamma \circ T^{-1}_\gamma \circ T_\alpha = \text{id},
\] (3.4)
the element \( \text{Exp}(t_{\alpha\beta}) \star \text{Exp}(t_{\beta\gamma}) \star \text{Exp}(t_{\gamma\alpha}) \) must be central. Thus
\[
t_{\alpha\beta\gamma} = t_{\alpha\beta} \circ t_{\beta\gamma} \circ t_{\gamma\alpha} \in \mathbb{C}[[\lambda]]
\] (3.5)
defines a Čech cochain on \( M \) with values in \( \mathbb{C}[[\lambda]] \). This cochain turns out to be a cocycle [20], and the Čech class \( [t_{\alpha\beta\gamma}] \) (viewed as a class in \( H^2_{\text{dir}}(M)[[\lambda]] \)) is the relative class \( t(\star', \star) \).
3.2 The relative class of Morita equivalent star products

We will now use the results in Sections 3.1 and 2.2 to compute the relative class of two Morita equivalent star products on a symplectic manifold \((M, \omega)\), providing an explicit description of the orbit space \(\text{Def}(M, \omega)/\text{Pic}(M)\).

**Theorem 3.1** Let \(L \to M\) be a complex line bundle over a symplectic manifold \((M, \omega)\). Suppose \(*, *'\) are star products on \(M\), with \(\Phi_L([*]) = [*'']\). Then \(t(*', *) = 2\pi i c_1(L)\), where \(c_1(L)\) is the Chern class of \(L\).

**Proof:** Let \(\{O_\alpha\}\) be a good cover of \(M\), and let us fix deformed trivialization maps \(\Psi_\alpha\) and transition functions \(\Phi_\alpha\beta\) as in Section 2.2. Let \(E = (\Gamma^\infty(L)[[\lambda]], *)\) be a deformation quantization of \(L\) with respect to \(*\). Let \(T : (\Gamma^\infty(M)[[\lambda]], *') \to \text{End}(E)\) be a \(\mathbb{C}[[\lambda]]\)-algebra isomorphism, that, by [12], can be chosen to preserve supports (see Remark 2.6). Such a \(T\) gives rise to a collection of local maps

\[ T_\alpha : C^\infty(O_\alpha)[[\lambda]] \to C^\infty(O_\alpha)[[\lambda]], \]

by \(T_\alpha(f) = T(f)_\alpha\), satisfying \(T_\alpha = \text{id} \mod \lambda\) and \(T_\alpha f \ast T_\alpha g = T_\alpha(f \ast' g)\) (by (2.14)). It follows from (2.13) that \(T_\beta^{-1}(f) = \Phi_\beta\alpha \ast T_\alpha(f) \ast \Phi_\alpha\beta\), and therefore

\[ T_\alpha^{-1}T_\beta(f) = \Phi_\alpha\beta \ast f \ast \Phi_\beta\alpha. \tag{3.6} \]

Since \(\phi_{\alpha\beta}\) is invertible and \(\Phi_{\alpha\beta} = \phi_{\alpha\beta} \mod \lambda\), we can write (see Appendix A)

\[ \Phi_{\alpha\beta} = \text{Exp}(t_{\alpha\beta}), \]

for some \(t_{\alpha\beta} = t^{(0)}_{\alpha\beta} + \sum_{r=1}^\infty \lambda^r \theta^{(r)}_{\alpha\beta} \in C^\infty(O_\alpha \cap O_\beta)[[\lambda]]\), and \(\phi_{\alpha\beta} = e^{t^{(0)}_{\alpha\beta}}\).

The deformed cocycle condition (2.12) and Prop. A.1 imply that, on triple intersections \(O_\alpha \cap O_\beta \cap O_\gamma\), the function \(t_{\alpha\beta\gamma} := t_{\alpha\beta} \circ_t t_{\beta\gamma} \circ_t t_{\gamma\alpha}\) must satisfy

\[ t_{\alpha\beta\gamma} = 2\pi i n_{\alpha\beta\gamma}, \quad \text{with } n_{\alpha\beta\gamma} \in \mathbb{Z}. \]

This shows that \(\frac{1}{2\pi i} t(*', *)\) is integral and does not depend on \(\lambda\). Since the classical limit of \(\circ_t\) is just the usual addition, we get

\[ t_{\alpha\beta\gamma} = t^{(0)}_{\alpha\beta} + t^{(0)}_{\beta\gamma} + t^{(0)}_{\gamma\alpha} = 2\pi i n_{\alpha\beta\gamma}. \]

But the complex Čech class defined by \(\frac{1}{2\pi i} (t^{(0)}_{\alpha\beta} + t^{(0)}_{\beta\gamma} + t^{(0)}_{\gamma\alpha})\), viewed as a de Rham class, is the Chern class of \(L\). Thus \(t(*', *) = 2\pi i c_1(L)\).

\[ \square \]

Let \(H^2_{\text{dir}}(M, \mathbb{Z})\) denote the image of the usual map \(i : \tilde{H}^2(M, \mathbb{Z}) \to H^2_{\text{dir}}(M, \mathbb{C})\).
**Corollary 3.2** Two star products \(\star, \star'\) on a symplectic manifold \(M\) are Morita equivalent if and only if there exists a symplectomorphism \(\psi : M \to M\) such that
\[
\frac{1}{2\pi i}(\psi', \psi^* \star) \in H^2_{dR}(M, \mathbb{Z}).
\] (3.7)

An immediate consequence of Theorem 3.1 is the following explicit expression for the action \(\Phi\) in terms of the characteristic classes of star products:
\[
\Phi_L([\omega_\lambda]) = [\omega_\lambda] + 2\pi ic_1(L),
\] (3.8)
where \([\omega_\lambda] = (1/i\lambda)[\omega] + \sum_{r=0}^{\infty} [\omega_r] \lambda^r\). The orbit space \(\text{Def}(M, \omega)/\text{Pic}(M)\) is just a trivial fibration over the torus \(H^2_{dR}(M, \mathbb{C})/H^2_{dR}(M, \mathbb{Z})\), with fiber \(H^2_{dR}(M, \mathbb{C})[[\lambda]]\).

It is clear that the isotropy group of \(\Phi\), for any \([\star] \in \text{Def}(M, \omega)\), is isomorphic to the subgroup \(\mathcal{T}(M) := \{[L] \in \text{Pic}(M), c_1(L) = 0\} \subseteq \text{Pic}(M)\) of flat line bundles.

Let \(\star\) be a star product on \((M, \omega)\) with \(c(\star) = [\omega]/i\lambda + O(\lambda)\) (i.e., \(c_0(\star) = 0\)). Since \(c(\psi^*(\star)) = \psi^*c(\star)\), it follows that in this case \(\text{Pic}(C^\infty(M)[[\lambda]], \star)\) is isomorphic to the isotropy group of \(\Phi\) at \([\star]\). Hence we have

**Corollary 3.3** Let \(\star\) be a star product on \((M, \omega)\) with \(c_0(\star) = 0\). Then the Picard group of the algebra \((C^\infty(M)[[\lambda]], \star)\) is isomorphic to \(\mathcal{T}(M)\).

Under the usual identification \(\text{Pic}(M) \cong \check{H}^2(M, \mathbb{Z})\), \(\mathcal{T}(M)\) correspond to torsion elements in \(\check{H}^2(M, \mathbb{Z})\). Hence if \(\check{H}^2(M, \mathbb{Z})\) is free, \(\Phi\) is faithful and the Picard groups of the deformed algebras with \(c_0(\star) = 0\) are trivial.

**Corollary 3.4** Let \(L \to M\) be a line bundle over \((M, \omega)\). Then \(\Gamma^\infty(L)[[\lambda]]\) has a \(\star\)-bimodule structure deforming the classical one if and only if \(L\) is flat.

### 3.3 Strong Morita equivalence of star products

We now observe that Theorem 3.1 also provides a complete classification of Hermitian star products up to strong Morita equivalence, see Appendix 3. The following lemma should be well-known.

**Lemma 3.5** Let \(A\) be a \(k\)-algebra, where \(k\) is a commutative ring with \(\mathbb{Q} \subseteq k\). Let \(D\) and \(T = \exp(\lambda D)\) be \(k[[\lambda]]\)-module endomorphisms of \(A[[\lambda]]\). If \(\star\) is a formal associative deformation for \(A\), then \(T\) is a \(\star\)-automorphism if and only if \(D\) is a \(\star\)-derivation.

**Proof:** If \(D\) is a \(\star\)-derivation, then \(T\) is clearly a \(\star\)-automorphism. For the converse, define \(E(a, b) = D(a \star b) - Da \star b - a \star Db\). It follows that
\[
D^k(a \star b) = \sum_{l=0}^{k} \binom{k}{l} D^l a \star D^{k-l} b + \sum_{r,s,t=0}^{k-1} c_{rst}^{(k)} D^r E(D^s a, D^t b)
\]

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with some rational coefficients \( c^{(k)}_{rst} \), obtained by recursion. From the fact that \( T \) is an automorphism, we obtain

\[
E(a, b) = -\lambda \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{k!} \sum_{r,s,t=0}^{k-1} c^{(k)}_{rst} D^r E(D^s a, D^t b).
\]

This equation can be seen as a fixed point condition for a \( k[\lambda] \)-linear operator acting on \( k[\lambda] \)-bilinear maps on \( A[\lambda] \), and this operator is clearly contracting in the \( \lambda \)-adic topology. Thus, by Banach’s fixed point theorem, there exists a unique fixed point, which must be 0 (see e.g. [7, App. A]). Therefore \( E = 0 \), and \( D \) is a derivation. \( \square \)

**Corollary 3.6** Let \( \ast, \ast' \) be Hermitian star products on a Poisson manifold \( M \). Then \( \ast \) is equivalent to \( \ast' \) if and only if \( \ast \) is \( \ast' \)-equivalent to \( \ast' \).

**Proof:** Let \( T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r \) be an equivalence, \( T(f \ast g) = Tf \ast' Tg \). Then \( f^\dagger := T^{-1}(T f) \) defines a new \( \ast' \)-involution for \( \ast \) of the form \( f^\dagger = SF \), where \( S = \text{id} + \sum_{r=1}^{\infty} \lambda^r S_r \) is a \( \ast \)-automorphism. We can write \( S = e^{iJD} \), where \( D \) is a real derivation of \( \ast \). Thus \( S^{1/2} \) is still a \( \ast \)-automorphism, and the map \( U = TS^{1/2} \) is a \( \ast' \)-equivalence between \( \ast \) and \( \ast' \). \( \square \)

**Theorem 3.7** Let \( \ast \) and \( \ast' \) be Hermitian star products on a Poisson manifold \( M \). Then \( \ast \) and \( \ast' \) are strongly Morita equivalent if and only if they are Morita equivalent.

**Proof:** Assume that \( \ast \) and \( \ast' \) are Morita equivalent via a line bundle \( L \). Equip \( L \) with a Hermitian fiber metric \( h_0 \), and let \( E = (\Gamma^\infty(L)[[\lambda]], \ast, h) \) be a quantization respect to \( \ast \). The endomorphisms \( \text{End}(E) \) form a \( \ast \)-algebra strongly Morita equivalent to \( (\Gamma^\infty(M)[[\lambda]], \ast) \), see [12]. This algebra is isomorphic to \( (\Gamma^\infty(M)[[\lambda]], \ast') \), and, by Lemma 3.4, we can chose the isomorphism to be a \( \ast' \)-isomorphism. Hence \( \ast \) and \( \ast' \) are strongly Morita equivalent. For the converse, see [13, Sec. 7]. \( \square \)

**Corollary 3.8** If \( M \) is symplectic, and \( \ast, \ast' \) are Hermitian star products, then they are strongly Morita equivalent if and only if there exists a symplectomorphism \( \psi : M \rightarrow M \) such that \( c(\psi^\ast(\ast')) - c(\ast) \) is \( 2\pi i \)-integral.

We note that a similar result holds for \( C^\ast \)-algebras [3]: two unital \( C^\ast \)-algebras are strongly Morita equivalent if and only if they are Morita equivalent as unital rings.

**4 Application**

In this section we shall consider star products on cotangent bundles \( \pi : T^*Q \rightarrow Q \), motivated by the importance of this class of symplectic manifolds in physical applications.
4.1 Star products on $T^*Q$

We will briefly recall the construction of star products on cotangent bundles in order to set up our notation. The reader is referred to [5–7] for details.

For $\gamma \in \Gamma^\infty(T^*Q)$, let $F(\gamma)$ be the differential operator

$$F(\gamma)f(\alpha_q) = \frac{d}{dt}f(\alpha_q + \gamma(q)) \bigg|_{t=0}$$

of fiber differentiation along $\gamma$, where $f \in C^\infty(T^*Q)$, $\alpha_q \in T^*_qQ$, and $q \in Q$. Since all the $F(\gamma)$ commute, $F$ can be extended uniquely to an injective algebra homomorphism from $\Gamma^\infty(\bigwedge^\cdot T^*Q)$ into the algebra of differential operators of $C^\infty(T^*Q)$, where zero forms $u \in C^\infty(Q)$ act by multiplication by $\pi^*u$.

Let $\nabla$ be a torsion-free connection on $Q$, and let $\mu \in \Gamma^\infty(\bigwedge^n|T^*Q)$ be a positive volume density. Using $\nabla$, we define the symmetrized covariant derivative

$$D : \Gamma^\infty(\bigwedge^\cdot T^*Q) \to \Gamma^\infty(\bigwedge^{\cdot+1}T^*Q),$$

which is a derivation of the $\vee$-product. Finally, let $\Delta$ be the Laplacian operator on $C^\infty(T^*Q)$ coming from the indefinite Riemannian metric on $T^*Q$ induced by the natural pairing of vertical and horizontal spaces with respect to $\nabla$. Locally, in a bundle chart, we have

$$\Delta = \sum_k \frac{\partial^2}{\partial p_k \partial q^k} + \sum_{k,l,j} p_l \pi^*\Gamma^l_{j,k} \frac{\partial^2}{\partial p_j \partial p_k} + \sum_{k,j} \pi^*\Gamma^j_{j,k} \frac{\partial}{\partial p_k},$$

where $\Gamma^l_{j,k}$ denote the Christoffel symbol of $\nabla$.

These operators provide a nice description of the usual (formal) differential operator calculus on $C^\infty(Q)$ in standard and in $\kappa$-ordering, see [3, Sect. 6] and [5, Sect. 2].

**Definition 4.1** The standard-ordered representation of a formal symbol $f \in C^\infty(T^*Q)[[\lambda]]$ acting as formal series of differential operators on a formal wave function $u \in C^\infty(Q)[[\lambda]]$ is defined by

$$\varrho_{\text{s}}(f)u = \iota^*F(\exp(-i\lambda D)u)f,$$

where $\iota : Q \hookrightarrow T^*Q$ is the zero-section embedding.

**Lemma 4.2** For a choice of $\nabla$ on $Q$, the expression

$$\varrho_{\text{s}}(f \star_{\text{s}} g) = \varrho_{\text{s}}(f)\varrho_{\text{s}}(g)$$

for $f, g \in C^\infty(T^*Q)[[\lambda]]$, defines a differential star product on $T^*Q$ of standard-order type, i.e. $(\pi^*u) \star_{\text{s}} f = (\pi^*u)f$. 

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The star product $\star_s$ in not Hermitian, but this can be fixed as follows. Let $\alpha \in \Gamma^\infty(T^*Q)$ be such that $\nabla_X \mu = \alpha(X)\mu$ for $X \in \Gamma^\infty(TQ)$, and consider the equivalence transformation

$$N_\kappa := e^{-i\kappa(\Delta + F(\alpha))}$$

(4.6)

for $\kappa \in \mathbb{R}$.

**Definition 4.3** The $\kappa$-ordered star product $\star_\kappa$ is defined by

$$f \star_\kappa g = N^{-1}_\kappa(N_\kappa f \star_s N_\kappa g),$$

(4.7)

and the corresponding $\kappa$-ordered representation on wave functions is defined by

$$q_\kappa(f)u = q_0(N_\kappa(f))u.$$ 

(4.8)

The Weyl-ordered star product is $\star_W = \star_{1/2}$, and the Schrödinger representation is $q_W = q_{1/2}$. We also set $N = N_{1/2}$.

One can check that $\star_W$ is Hermitian, and the Schrödinger representation $q_W$ yields a $^*$-representation of $(C^\infty(T^*Q)[[\lambda]], \star_W)$ on the pre-Hilbert space $C^\infty_0(Q)[[\lambda]]$ over $\mathbb{C}[[\lambda]]$ (see Definition [B.3]) with the usual $L^2$-inner product induced by $\mu$.

**Lemma 4.4** Let $u, v \in C^\infty(Q)[[\lambda]]$ and $f \in C^\infty(T^*Q)[[\lambda]]$. Then

$$\pi^* u \star_\kappa f = F(\exp(i\kappa \lambda D)u)f \quad \text{and} \quad f \star_\kappa \pi^* u = F(\exp(-i(1 - \kappa) \lambda D)u)f.$$ 

(4.9)

In particular, $\pi^* u \star_\kappa \pi^* v = \pi^*(uv)$ whence $\text{Exp}(\pi^* u) = \pi^* e^u$.

For $A \in \Gamma^\infty(T^*Q)[[\lambda]]$, let us define the operator

$$\delta_\kappa[A] = F\left(\frac{e^{i\kappa \lambda D} - e^{-i(1 - \kappa) \lambda D}}{D}A\right).$$

(4.10)

It is simple to check, using (4.9), that it provides a generalization of the $\star_\kappa$-commutator with a function $\pi^* u$, i.e.

$$\delta_\kappa[du] = \text{ad}_\kappa(\pi^* u).$$

(4.11)

Moreover, $A \mapsto \delta_\kappa[A]$ is linear and all $\delta_\kappa[A]$ commute.

**4.2 Deformed vector bundles over $T^*Q$ and magnetic monopoles**

We now consider deformation quantization of vector bundles over $T^*Q$ with respect to the star products $\star_\kappa$. As we will see, explicit formulas for the deformed structures are obtained in this case. We will restrict our attention to deformations of pulled-back vector bundles $\pi^* E \to T^*Q$, where $E \to Q$, since any vector bundle $F \to T^*Q$ is isomorphic to one of this
type. For the same reason, we assume that the Hermitian fiber metric on $\pi^*E$ is of the form $\pi^*h_0$, for a Hermitian fiber metric $h_0$ on $E$.

Let $\{O_\alpha\}$ be a good cover of $Q$, and $\{T^*O_\alpha\}$ be the corresponding good cover of $T^*Q$. We fix local frames $e_\alpha = \pi^*e_\alpha$ on $T^*O_\alpha$ induced by local frames $e_\alpha = (e_{\alpha,1}, \ldots, e_{\alpha,k})$ of $E$ on $O_\alpha$. Clearly, if $\varphi_{\alpha\beta} \in C^\infty(O_{\alpha\beta})$ are transition matrices for $E$, then $\phi_{\alpha\beta} = \pi^*\varphi_{\alpha\beta}$ are the transition matrices for $\pi^*E$ corresponding to the frames $e_\alpha$.

**Proposition 4.5** Let $E \to Q$ be a complex vector bundle and $\pi^*E \to T^*Q$ its pull-back to $T^*Q$. Then we have:

i.) The classical transition matrices $\phi_{\alpha\beta} = \pi^*\varphi_{\alpha\beta}$ satisfy the quantum cocycle condition

$$
\phi_{\alpha\beta} \star_\kappa \phi_{\beta\gamma} \star_\kappa \phi_{\gamma\alpha} = 1 \quad \text{and} \quad \phi_{\alpha\beta} \star_\kappa \phi_{\beta\alpha} = 1.
$$

(4.12)

ii.) For $s \in \Gamma^\infty(\pi^*E)[[\lambda]]$ and $f \in C^\infty(T^*Q)[[\lambda]]$,

$$
s \cdot_\kappa f|_{T^*O_\alpha} := e_\alpha(s_\alpha \star s N_\kappa(f)) = e_\alpha N_\kappa(N_\kappa^{-1}(s_\alpha) \star f) \quad (4.13)
$$

defines a global deformation quantization $\bullet_\kappa$ of $\pi^*E$ with respect to $\star_\kappa$ for all $\kappa$.

iii.) The quantum transition matrices $\Phi_{\alpha\beta}$ with respect to $\bullet_\kappa$ corresponding to the frame $e_\alpha = e_\alpha = \pi^*e_\alpha$ are $\Phi_{\alpha\beta} = \phi_{\alpha\beta} = \pi^*\varphi_{\alpha\beta}$, for all $\kappa$. The local quantum trivialization isomorphisms $\Psi^{(\kappa)}_\alpha$ are given by

$$
\Psi^{(\kappa)}_\alpha(s) = s^{(\kappa)}_\alpha = N_\kappa^{-1}(s_\alpha),
$$

(4.14)

where $s = e_\alpha s_{\alpha}$ locally.

**Proof:** The first part is clear. For the second part, let us first consider standard-order. In this case, $\phi_{\alpha\beta} \star s = \phi_{\alpha\beta}s_\beta$ by (1.3) whence (4.13) is well-defined for $\kappa = 0$. The general case follows from $s \cdot_\kappa f = s \cdot s N_\kappa(f)$. A local computation shows that (4.13) defines a deformation quantization of $\pi^*E$. The third part again follows from (4.13) and the fact that $N_\kappa \pi^* = \pi^*$.

In the Weyl-ordered case $\bullet_w = \bullet_{1/2}$, we can also deform the Hermitian metric $\pi^*h_0$ of $\pi^*E$. To this end we assume that the undeformed frames $e_\alpha = \pi^*e_\alpha$ are orthonormal with respect to $\pi^*h_0$.

**Lemma 4.6** Let $E \to Q$ be a Hermitian vector bundle with fiber metric $h_0$, and consider its pull back $(\pi^*E, \pi^*h_0)$. Assume that $e_\alpha = \pi^*e_\alpha$ are local orthonormal frames, and consider the Weyl-ordered deformation quantization $\bullet_w$ of $\pi^*E$. The following holds.
i.) For \( s, s' \in \Gamma^\infty(\pi^*E)[[\lambda]] \),
\[
\mathbf{h}(s, s')|_{T^*_\alpha} := \left( s_{\alpha}^{(w)} \right)^* \ast_w s'_{\alpha}^{(w)} = (N^{-1}s_{\alpha})^* \ast_w N^{-1}s'_\alpha
\]  
(4.15)
defines a global deformation quantization of \( \pi^*h_0 \) with respect to \( \bullet_w \). In particular, for pulled-back sections, one has \( \mathbf{h}(\pi^*\sigma, \pi^*\sigma') = \pi^*h_0(\sigma, \sigma') \).

ii.) The frames \( e_\alpha = \pi^*\epsilon_\alpha \) are orthonormal with respect to \( \mathbf{h} \), and hence the transition matrices are unitary:
\[
\phi_{\alpha\beta}^* \ast_w \phi_{\alpha\beta} = 1.
\]  
(4.16)

PROOF: Since (4.16) is obviously satisfied, (4.15) is globally defined. The remaining properties of a deformation quantization of \( \pi^*h_0 \) are easily verified from the local formula. Again \( N_\alpha\pi^* = \pi^* \) and (4.9) imply that \( \mathbf{h} \) coincides with \( \pi^*h_0 \) on pulled-back sections. Thus the \( e_\alpha \) are still orthonormal. 

Let us now consider a line bundle \( L \to Q \), with pull-back \( \pi^*L \to T^*Q \). In this case, we can describe the deformed endomorphisms (with respect to \( \bullet_\kappa \)) explicitly by using a connection \( \nabla^L \) on \( L \). The frame \( e_\alpha = \pi^*\epsilon_\alpha \) is a single non-vanishing local section of \( \pi^*L \), and \( \nabla^L \) determines local connection one-forms \( A_\alpha \in \Gamma^\infty(T^*O_\alpha) \) by
\[
\nabla^L_X \epsilon_\alpha = -iA_\alpha(X)\epsilon_\alpha,
\]  
(4.17)
where \( X \in \Gamma^\infty(TQ) \). Let \( B \) be the (global) curvature two-form,
\[
B = dA_\alpha.
\]  
(4.18)
We assume \( \nabla^L \) to be compatible with \( h_0 \), so that the forms \( A_\alpha \) and \( B \) are real. Using these local one-forms we can define local series of differential operators \( S^{(\kappa)}_\alpha \) by
\[
S^{(\kappa)}_\alpha(f) = e^{i\delta_\kappa[A_\alpha]}(f).
\]  
(4.19)
Note that the operator \( S^{(\kappa)}_\alpha \) is just the \( \kappa \)-ordered quantized fiber translation by the one-form \( \lambda A_\alpha \) in the sense of [3, Thm. 3.4].

**Lemma 4.7** For \( \Phi_{\alpha\beta} = \phi_{\alpha\beta} = \pi^*\varphi_{\alpha\beta} \) the relation
\[
\Phi_{\alpha\beta} \ast_\kappa f \ast_\kappa \Phi_{\beta\alpha} = e^{i\delta_\kappa[A_\alpha]} e^{-i\delta_\kappa[A_\beta]}(f)
\]  
(4.20)holds for all \( f \in C^\infty(T^*Q)[[\lambda]] \).
Proof: Choose local functions \( c_{\alpha \beta} \in C^\infty(\mathcal{O}_{\alpha \beta}) \) such that \( \varphi_{\alpha \beta} = e^{2\pi i c_{\alpha \beta}} \). Then we know that \( A_\alpha - A_\beta = 2\pi dc_{\alpha \beta} \) and (4.20) is a simple computation using (4.11), Lem. 4.4 and the commutativity of all \( \delta_\kappa \).

\[ \fbox{} \]

As a result, (2.13) is satisfied, and hence

\[ S_\kappa(f)_{|T^*\mathcal{O}_\alpha} := e_\alpha \cdot_\kappa \left( S_\alpha^{(\kappa)}(f) \cdot_\kappa S_\alpha^{(\kappa)} \right) \]  
(4.21)

defines a global endomorphism \( S_\kappa(f) \) of \( (\Gamma^\infty(\pi^*L)[[\lambda]], \cdot_\kappa) \) for any \( f \in C^\infty(T^*Q)[[\lambda]] \). Also observe that \( S_\alpha^{(\kappa)}(\pi^*u) = \pi^*u \).

Let \( \cdot^\prime_{\kappa} \) be the star product induced by the operator product of deformed endomorphisms,

\[ f \cdot^\prime_{\kappa} g = S_{\kappa}^{-1}(S_\kappa(f)S_\kappa(g)) = \left( S_\alpha^{(\kappa)} \right)^{-1} \left( S_\alpha^{(\kappa)}(f) \cdot_\kappa S_\alpha^{(\kappa)}(g) \right). \]  
(4.22)

It follows from the explicit form of the local equivalence transformations (4.19) and [5, Thm. 4.1 and Thm. 4.6] that the star product \( \cdot^\prime_{\kappa} \) coincides with the one constructed in [5]:

**Proposition 4.8** The star product \( \cdot^\prime_{\kappa} \) coincides with \( \cdot^{_{\kappa}}_{\Lambda^B} \) from [5, Thm. 4.1]. Its characteristic class is given by

\[ c \left( \cdot^{_{\kappa}}_{\Lambda^B} \right) = i[\pi^*B] = 2\pi ic_1(\pi^*L). \]  
(4.23)

Note that (4.23) is consistent with (3.8) since the characteristic class of \( \cdot_\kappa \) vanishes, see [6, Thm. 4.6].

**Remark 4.9** More generally [3], one can explicitly construct a star product \( \cdot_\kappa^B \), for any formal series of closed two-forms \( B \in \Gamma^\infty(\wedge^2 T^*Q)[[\lambda]], \) with \( c(\cdot^B) = \frac{1}{4\pi}[\pi^*B] \). In particular, any star product on \( T^*Q \) is equivalent to some \( \cdot_\kappa^B \).

The physical interpretation of the star products \( \cdot^{_{\kappa}}_{\Lambda^B} \) is discussed in [5]: they correspond to the quantization of a charged particle, with electric charge 1, moving in \( Q \) under the influence of a magnetic field \( B \). With this in mind, we can think of non-trivial characteristic classes of star products on \( T^*Q \) as corresponding to topologically non-trivial magnetic fields, i.e. to the presence of magnetic monopoles. The integral \( m = \frac{1}{4\pi} \int_{S^2} B \) gives the amount of ‘magnetic charge’ inside this 2-sphere \( S^2 \). Thus the integrality of \( B \) implies that \( 2m \in \mathbb{Z} \), which is Dirac’s integrality/quantization condition for magnetic charges \( m \). We summarize the discussion:

**Theorem 4.10** Let \( B \in \Gamma^\infty(\wedge^2 T^*Q)[[\lambda]] \) be a sequence of closed two-forms, and \( \cdot^{_{\kappa}}_{\Lambda^B} \) the star product in [5]. Then \( \cdot^{_{\kappa}}_{\Lambda^B} \) is Morita equivalent to \( \cdot_\kappa \) if and only if \( \frac{1}{4\pi}B \) is an integral two-form. In physical terms, the quantization with magnetic field \( B \) is Morita equivalent to the quantization without magnetic field if and only if Dirac’s integrality condition for the magnetic charge of \( B \) is fulfilled.
This theorem suggests the physical interpretation of characteristic classes of star products on arbitrary symplectic manifolds as ‘intrinsic magnetic monopole fields’, and of Morita equivalence as Dirac’s integrality condition for the ‘relative fields’.

4.3 Rieffel induction of the Schrödinger representation

Let $\star_w$ be the Weyl-ordered star product on $T^*Q$, and let $\varrho_w$ be the Schrödinger representation (4.8) of $\star_w$ on (formal) wave functions $\mathbf{f} = C^\infty_0(Q)[[\lambda]]$, with $L^2$-inner product coming from $\mu$, see [3, 4]. We now illustrate the consequences of Morita equivalence by constructing the $\star$-representation of $\star_w^{\lambda B}$ induced (in the sense of Rieffel induction) by $\varrho_w$.

Let $L \to Q$ be a Hermitian line bundle, and let $\pi^*L \to T^*Q$ be its pull-back, endowed with a quantization $\bullet_w$ and $\hbar$ as before. By fixing a compatible connection $\nabla_L$, we obtain a star product $\star_w^{\lambda B}$ by (4.24) such that $\Gamma^\infty(\pi^*L)[[\lambda]]$ has a bimodule structure with respect to $\star_w^{\lambda B}$ and $\star_w$. As shown in [3, Sect. 8 and 9], this data determines a $\star$-representation $\eta_w$ of $\star_w^{\lambda B}$ on $\Gamma^\infty_0(L)[[\lambda]]$, with $L^2$-inner product defined by $h_0$ and the volume density $\mu$. We have the following explicit local formula

$$\eta_w(f)(\epsilon_\alpha \sigma_\alpha) = \epsilon_\alpha \varrho_w\left(e^{i\delta_w[A_\alpha]} f\right) \sigma_\alpha, \quad (4.24)$$

where $\sigma = \epsilon_\alpha \sigma_\alpha \in \Gamma^\infty(L)[[\lambda]]$, see [3, Eq. (5.4) and Thm. 8.2] (The missing minus sign comes from a different convention for the Chern class of $L$.) We shall now show that $\eta_w$ is canonically unitarily equivalent to the Rieffel induction of the Schrödinger representation $\varrho_w$ of $\star_w$.

**Theorem 4.11** Let $(\mathcal{R}, \rho)$ be the $\star$-representation of $\star_w^{\lambda B}$ obtained by Rieffel induction of the Schrödinger representation $(\mathbf{f}, \varrho_w)$, using the equivalence bimodule $\mathcal{L} = \Gamma^\infty(\pi^*L)[[\lambda]]$. The following holds:

i.) Let $s \in \Gamma^\infty(\pi^*L)[[\lambda]]$ and $u \in C^\infty_0(Q)[[\lambda]]$. Then

$$\tilde{\mathcal{R}} = \mathcal{L} \otimes \star_w \mathbf{f} \ni s \otimes u \mapsto \epsilon_\alpha \varrho_w\left(s^{(w)}_\alpha\right) u \in \Gamma^\infty_0(L)[[\lambda]] \quad (4.25)$$

extends to a well-defined global $\mathbb{C}[[\lambda]]$-linear map $\tilde{U}$, which is isometric and surjective.

ii.) $\tilde{U}$ induces a unitary map $U : \mathcal{R} \to \Gamma^\infty_0(L)[[\lambda]]$.

iii.) $U$ is an intertwiner between $\rho$ and $\eta_w$.

**Proof:** Let $s = \epsilon_\alpha \bullet_w s^{(w)}_\alpha$. A straightforward computation shows that $\epsilon_\alpha \varrho_w(s^{(w)}_\alpha)u = \epsilon_\beta \varrho_w(s^{(w)}_\beta)u$, since $\phi_{\alpha\beta} = \pi^* \varphi_{\alpha\beta}$ and $\varrho_w$ is a representation satisfying $\varrho_w(\pi^*v) = v$. Thus the right hand side of (4.25) is a global section. A similar computation shows that $\tilde{U}(s \otimes \varrho_w(f)u) = \tilde{U}(s \otimes \varrho_w(f)u)$, whence $\tilde{U}$ is well-defined. From the fact that $\varrho_w$ is a $\star$-representation, one obtains for sections/functions with small enough support the relation

$$\int_Q h_0\left(\epsilon_\alpha \varrho_w\left(s^{(w)}_\alpha\right) u, \epsilon_\alpha \varrho_w\left(t^{(w)}_\alpha\right) v\right) \mu = \int_Q \pi \varrho_w(h(s,t)) v \mu. \quad (4.26)$$
Then a partition of unity argument implies that $\tilde{U}$ is isometric. Finally we choose $\sigma \in \Gamma_0^\infty(L)$ and $u$ such that $u = 1$ on supp $\sigma$. Then clearly $\tilde{U}(\pi^* \sigma \otimes u) = \sigma u = \sigma$ implies surjectivity. This shows the first part. The second part is trivial since $\mathcal{K}$ is the quotient of $\tilde{K}$ by the vectors of length zero. For the third part, we compute locally

$$U(\rho(f) s \otimes u)$$

$$= \epsilon_\alpha \varrho_W(S_{\alpha}(w)(f) \star_W s_{\alpha}(w)) u = \epsilon_\alpha \varrho_W \left( e^{i\delta_W[A_\alpha]}(f) \right) \varrho_W \left( s_{\alpha}(w) \right) u = \eta_W(f) U(s \otimes u),$$

which is sufficient since all representations are local.

The $\star$-representation $\eta_W$ is well-known, for instance, from geometric quantization [37, Sect. 8.4]: It is precisely the representation obtained if the symplectic form satisfies the integrality condition of pre-quantization. The difference is that we have treated $\hbar$ as a formal parameter $\lambda$, so the correction to the canonical symplectic form occurs in first order of $\lambda$. For a further discussion see also [5].

As we just saw, $\eta_W$ can be obtained as a result of Rieffel induction applied to the ordinary Schrödinger representation $\varrho_W$. We remark that, by Morita equivalence, $\star_W$ and $\star_{W-B}$ have equivalent categories of $\star$-representations, and the correspondence of $\varrho_W$ and $\eta_W$ is just one example of this more general fact. These considerations are based on the approach to quantization where primary objects are observable algebras, as opposed to specific $\star$-representations.

The results in this paper illustrate that several constructions and techniques present in more analytic approaches to quantization find counterparts in formal deformation quantization. It is interesting to investigate how far one can go without convergence.

A Star exponentials and star logarithms

In this appendix we recall a few properties of the star exponential [2] and the star logarithm (see [8, 35] for details).

Let $\star$ be a star product on a Poisson manifold $M$. Let $H = \sum_{r=0}^{\infty} \lambda^r H_r \in C^\infty(M)[[\lambda]]$ and consider the differential equation

$$\frac{d}{dt} f(t) = H \star f(t), \quad f(0) = 1,$$

(A.1)

for $t \in \mathbb{R}$ and $f(t) \in C^\infty(M)[[\lambda]]$. The next result follows from [8, Lem. 2.2, 2.3] and [35, Sect. 1.4.2]:

**Proposition A.1** For any $H \in C^\infty(M)[[\lambda]]$, the differential equation (A.1) has a unique solution, denoted by $t \mapsto \text{Exp}(tH)$, satisfying the following properties:

i.) $\text{Exp}(tH) = \sum_{r=0}^{\infty} \lambda^r \text{Exp}(tH)_r$, with $\text{Exp}(tH)_0 = e^{tH_0}$ and $\text{Exp}(H)_{r+1}$ equals $e^{H_0} H_{r+1}$ plus terms only depending on $H_0, \ldots, H_r$. 

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ii.) \( \text{Exp}(tH) \star H = H \star \text{Exp}(tH) \), and \( \text{Exp}((t + t')H) = \text{Exp}(tH) \star \text{Exp}(t'H) \).

iii.) If \( \star \) is a Hermitian star product, then \( \text{Exp}(tH) = \text{Exp}(tH) \).

iv.) \( \text{Exp}(H) = 1 \) if and only if \( H \) is constant on each connected component of \( M \) and equal to \( 2\pi i k \) for some \( k \in \mathbb{Z} \).

v.) For all \( f \in C^\infty(M)[[\lambda]] \) we have
\[
e^{\text{ad}(H)}(f) = \text{Exp}(H) \star f \star \text{Exp}(-H),
\]
where \( \text{ad}(H) = [H, \cdot] \), denotes the \( \star \)-commutator.

For \( f, g \in C^\infty(M)[[\lambda]] \), consider the Baker-Campbell-Hausdorff formula
\[
f \circ \star g = f + g + \frac{1}{2}[f, g] + \frac{1}{12}([f, [f, g]] + [g, [g, f]]) + \cdots
\]
Since in zeroth order the star commutator vanishes, the series \((A.3)\) is a well-defined formal power series in \( C^\infty(M)[[\lambda]] \), and one has
\[
\text{Exp}(f) \star \text{Exp}(g) = \text{Exp}(f \circ \star g).
\]
See e.g. [20, Lem. 4.1] for the properties of \( \circ \star \).

More generally, we define star logarithms in the following way. Let \( U \subseteq M \) be a contractible open subset, and let \( f = \sum_{r=0}^\infty \lambda^r f_r \in C^\infty(U)[[\lambda]] \). If \( f_0(x) \neq 0 \) for all \( x \in U \), then there exists a smooth logarithm \( H_0 = \ln(f_0) \in C^\infty(U) \) for the pointwise product, unique up to constants in \( 2\pi i \mathbb{Z} \). If we have fix the choice of the classical \( \ln \), then Prop. \((A.1)\) ensures that we can find \( H_1, H_2, \ldots \in C^\infty(U) \) by recursion such that \( \text{Exp}(H) = f \) for \( H = \sum_{r=0}^\infty \lambda^r H_r \). We write \( H = \text{Ln}(f) \), and call it (the/a) star logarithm of \( f \) corresponding to the choice of the classical \( \ln(f_0) \). Again \( H \) is unique up to constants in \( 2\pi i \mathbb{Z} \) and
\[
\text{Exp}(\text{Ln}(f)) = f \quad \text{and} \quad H = \text{Ln}(\text{Exp}(H)) \mod 2\pi i \mathbb{Z}.
\]

**B Rieffel induction and strong Morita equivalence**

This appendix recalls the notions of algebraic Rieffel induction and strong Morita equivalence for \( \star \)-algebras over an ordered ring. For simplicity, we assume \( \star \)-algebras to be unital. The reader is referred to [11, 13] for details.

Let \( \mathcal{R} \) be an ordered ring, and let \( \mathcal{C} = \mathcal{R}(i) \) with \( i^2 = -1 \). The main examples from deformation quantization are \( \mathcal{R} = \mathbb{R} \) and \( \mathcal{R} = \mathbb{R}[[\lambda]] \). We consider the following generalization of complex pre-Hilbert spaces.

**Definition B.1** Let \( \mathcal{H} \) be a \( \mathcal{C} \)-module. A Hermitian inner product on \( \mathcal{H} \) is a sesquilinear map \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{C} \) such that \( \langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle} \), and \( \langle \phi, \phi \rangle > 0 \) for all \( \phi \neq 0 \). The pair \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) is called a pre-Hilbert space over \( \mathcal{C} \).
Let $\mathcal{B}(\mathcal{H})$ be the $\ast$-algebra of adjointable $\mathbb{C}$-linear endomorphisms of $\mathcal{H}$. If $\mathcal{A}$ is a $\ast$-algebra over $\mathbb{C}$, a $\ast$-representation of $\mathcal{A}$ on $\mathcal{H}$ is a $\ast$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. We denote the category of nondegenerate (i.e. $\pi(1) = \text{id}$) $\ast$-representations of $\mathcal{A}$ by $\ast\text{-Rep}(\mathcal{A})$. Following the analogy with $\mathbb{C}^\ast$-algebras, we consider [9]:

**Definition B.2** A $\mathbb{C}$-linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is called positive if $\omega(A^\ast A) \geq 0$ for all $A \in \mathcal{A}$. An element $A \in \mathcal{A}$ is called positive if $\omega(A) \geq 0$ for all positive linear functionals $\omega$.

Elements of the form $A = b_1 B_1^\ast B_1 + \cdots + b_n B_n^\ast B_n$, with $b_i > 0$ and $B_i \in \mathcal{A}$, are necessarily positive, and called algebraically positive. These definitions recover the usual notions of positivity on $\mathbb{C}^\ast$-algebras. If $\mathcal{A} = C^\infty(M)$, then positive linear functionals are compactly supported positive measures, and positive elements are usual positive functions.

To describe Rieffel induction [29], we consider algebraic analogs of Hilbert modules.

**Definition B.3** Let $\mathcal{E}$ be a $\mathcal{A}$-right module. An $\mathcal{A}$-valued Hermitian inner product is a $\mathbb{C}$-sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that $\langle x, y \rangle = \langle y, x \rangle^\ast$, $\langle x, y \cdot A \rangle = \langle x, y \rangle A$, and $\langle x, x \rangle$ is positive in $\mathcal{A}$.

Suppose $\mathcal{E}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule, equipped with an $\mathcal{A}$-valued Hermitian inner product, so that

$$\langle B \cdot x, y \rangle = \langle x, B^\ast \cdot y \rangle. \quad (B.1)$$

Let $(\mathcal{B}, \pi)$ be a $\ast$-representation of $\mathcal{A}$. Consider the space $\widetilde{\mathcal{K}} = \mathcal{E} \otimes_\mathcal{A} \mathcal{B}$, endowed with its canonical $\mathcal{B}$-left module structure, and set

$$\langle x \otimes \phi, y \otimes \psi \rangle = \langle \phi, \pi(\langle x, y \rangle) \psi \rangle. \quad (B.2)$$

We assume that $\mathcal{E}$ is such that (B.2) defines a positive semi-definite inner product on $\widetilde{\mathcal{K}}$ for all $\ast$-representations (this is always the case for $\mathbb{C}^\ast$-algebras and for star product algebras if $\mathcal{E}$ is a deformation quantization of a Hermitian vector bundle [11]). Factoring $\widetilde{\mathcal{K}}$ by the vectors of length zero, we obtain a pre-Hilbert space $\mathcal{K}$ over $\mathbb{C}$ equipped with a $\ast$-representation of $\mathcal{B}$. This induced $\ast$-representation is denoted by $\mathcal{R}_\mathcal{E}(\mathcal{B}, \pi)$, and the induction process is functorial.

**Definition B.4** The functor $\mathcal{R}_\mathcal{E} : \ast\text{-Rep}(\mathcal{A}) \to \ast\text{-Rep}(\mathcal{B})$ is called Rieffel induction.

In order to get an equivalence of categories, we assume that $\mathcal{E}$ is, in addition, equipped with a $\mathcal{B}$-valued Hermitian inner product $\Theta_{-, -} : \mathcal{E} \times \mathcal{E} \to \mathcal{B}$ so that $\Theta_{x, y \cdot A} = \Theta_{x \cdot A^\ast, y}$. We require the compatibility

$$\Theta_{x, y} : z = x \cdot \langle y, z \rangle, \quad (B.3)$$

and assume that the following fullness conditions hold:

$$\mathcal{A} = \mathbb{C}\text{-span}\{\langle x, y \rangle \mid x, y \in \mathcal{E}\} \quad \mathcal{B} = \mathbb{C}\text{-span}\{\Theta_{x, y} \mid x, y \in \mathcal{E}\}. \quad (B.4)$$
Definition B.5 A \((\mathcal{B},\mathcal{A})\)-bimodule \(\mathcal{E}\) equipped with full \(\mathcal{A}\)- and \(\mathcal{B}\)-valued inner products satisfying the above properties is called an equivalence bimodule, and the \(*\)-algebras \(\mathcal{A}\) and \(\mathcal{B}\) are called strongly Morita equivalent as \(*\)-algebras over \(\mathcal{C}\).

Proposition B.6 Let \(\mathcal{A}, \mathcal{B}\) be strongly Morita equivalent unital \(*\)-algebras over \(\mathcal{C}\), with equivalence bimodule \(\mathcal{E}\). Then \(R_\mathcal{E}: \ast\text{-Rep}(\mathcal{A}) \to \ast\text{-Rep}(\mathcal{B})\) is an equivalence of categories.

Remark B.7 i.) The bimodule \(\mathcal{E}\) is also an equivalence bimodule in the purely ring theoretic sense of Morita equivalence \([13, \text{Sec. 7}]\). In particular, \(\mathcal{E}\) is finitely generated and projective over \(\mathcal{A}\) and \(\mathcal{B}\).

ii.) Analogous results hold for nonunital \(*\)-algebras. In particular, if \(\mathcal{A}\) and \(\mathcal{B}\) are \(\mathcal{C}^*\)-algebras, then they are strongly Morita equivalent (in the usual sense of operator algebras \([17]\)) if and only if their Pedersen ideals are strongly Morita equivalent in the sense of Definition [2, 3, 4, Sec. 3].

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