A step beyond Tsallis and Rényi entropies

Marco Masi*

Dipartimento di Fisica G. Galilei, Padova, Italy.

Abstract

Tsallis and Rényi entropy measures are two possible different generalizations of the Boltzmann-Gibbs entropy (or Shannon’s information) but are not generalizations of each others. It is however the Sharma-Mittal measure, which was already defined in 1975 (B.D. Sharma, D.P. Mittal, J.Math.Sci 10, 28) and which received attention only recently as an application in statistical mechanics (T.D. Frank & A. Daffertshofer, Physica A 285, 351 & T.D. Frank, A.R. Plastino, Eur. Phys. J., B 30, 543-549) that provides one possible unification. We will show how this generalization that unifies Rényi and Tsallis entropy in a coherent picture naturally comes into being if the q-formalism of generalized logarithm and exponential functions is used, how together with Sharma-Mittal’s measure another possible extension emerges which however does not obey a pseudo-additive law and lacks of other properties relevant for a generalized thermostatistics, and how the relation between all these information measures is best understood when described in terms of a particular logarithmic Kolmogorov-Nagumo average.

Key words: Generalized information entropy measures, Tsallis, Rényi, Sharma-Mittal

PACS: 05.70, 65.50, 89.70, 05.70.L

* Corresponding author.

Email address: marco.masi@spiro.fisica.unipd.it, marco.masi2@tin.it (Marco Masi).
1 Introduction

To gain a unified understanding of the different entropy measures and how they relate to each others in the frame of a generalized picture, it is first necessary to recall what characterizes "classical" entropies and emphasize some aspects which are important for the present paper.

1.1 The Boltzmann-Gibbs entropy and Shannon's information measure

As it is well known, given a probability distribution \( P = \{ p_i \}, \) \((i = 1, ..., N)\), with \( p_i \) representing the probability of the system to be in the \( i \)-th microstate, the Boltzmann-Gibbs (BG) entropy reads

\[
S_{BG}(P) = -k \sum_{i=1}^{N} p_i \log p_i ,
\]

where \( k \) is the Boltzmann constant and \( N \) the total number of possible configurations. If all states are equi-probable it leads to the famous Boltzmann principle \( S = k \log W \) \((N=W)\). BG entropy is equivalent to Shannon's expression if we set \( k = 1 \) (as we will do from now on) and use the immaterial base \( b \) for the logarithm function

\[
S_{S}(P) = - \sum_{i=1}^{N} p_i \log_b p_i .
\]

It is common to use the natural base for the BG entropy, while base 2 has the advantage to deliver information quantities in bits.

What characterizes BG and Shannon’s measure is additivity of information. Given two systems, described by two independent probability distributions \( A \) and \( B \) (i.e. \( P(A \cap B) = P(A)P(B) \)), using an additive information measure means that

\[
S_{S}(A \cap B) = S_{S}(A) + S_{S}(B|A) ,
\]

with

\[
S_{S}(B|A) = \sum_{i} p_i(A) S_{S}(B|A = A_i) ,
\]

being the conditional entropy. In this case we are talking about extensive systems, i.e. systems where the entropy is given by the sum of all the entropies of their parts, as it is customary to do in standard statistical mechanics. The unique function which assures additivity is the logarithm. Also in the axiomatic derivation of Shannon’s entropy performed by A.I. Khinchin (6), it is the additive property which leads to the appearance of the logarithm function.
This is the real reason that stands behind the ubiquitous presence of the logarithm function in information theory, and we can confidently say that every modification to it reflects a deviation from the additive law.

We will from now on use the natural base. Shannon’s entropy can be written in the form of a "linear" (the arithmetic) mean as

$$S_S(P) = \langle I_i \rangle_{\text{lin}} = \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{\text{lin}}, \quad (1.1)$$

where we will call the quantity

$$I_i = \log \left( \frac{1}{p_i} \right),$$

the elementary information gain associated to an event of probability $p_i$ (in information theory it is sometimes called the code length). The quantity $\frac{1}{p_i}$ is also called the surprise (less probable events are considered more "surprising" than more probable ones), and we will see that it is this quantity which is really measured in one way or another, not $-\log p_i$.

### 1.2 Tsallis’ entropy

Additivity is however not always preserved, especially in nonlinear complex systems, e.g. when we have to deal with long range forces, as it is in the case of the dynamic evolution of star clusters or in systems with long range microscopic memory, in fractal- or multifractal-like and self-organized critical systems, etc. We are dealing in this case with non-extensive systems; a case which received much attention in the last decade.\[15\]

A generalization of the BG entropy to non-extensive systems is known as Tsallis entropy.\[14\] C. Tsallis noted that if non-extensivity enters into the play things are described better by power law distributions, $p_i^q$, so called $q$-probabilities, i.e. by scaled probabilities where $q$ is a real parameter. This introduces the formal possibility not to set rare and common events on the same footing, as in BG or Shannon statistics, but it enhances or depresses them according to the parameter chosen (in complex systems rare events can have dramatic effects on the overall evolution).

With the introduction of the normalized $q$-probabilities it became customary to define so called escort- or zooming-distribution

$$\pi_i(P, q) = \frac{p_i^q}{\sum_{i=1}^N p_i^q}; \quad q > 0, \quad q \in \mathbb{R}.$$
In this frame Tsallis postulated his now famous generalization of Shannon’s entropy to non-extensivity (14):

\[
S_T(P, q) = \frac{\sum_{i=1}^{N} p_i^q - 1}{1 - q} = \frac{1}{q - 1} \sum_{i=1}^{N} p_i (1 - p_i^{q-1}).
\]  

(1.2)

For \( q \to 1 \), Shannon’s measure is recovered, i.e.: \( S_T(P, 1) = S_S(P) \).

Tsallis entropy extends to a pseudo-additive law

\[
S_T(A \cap B) = S_T(A) + S_T(B|A) + (1 - q)S_T(A)S_T(B|A),
\]

(1.3)

with

\[
S_T(B|A) = \sum_i \pi_i(A) S_T(B|A = A_i).
\]

Let us introduce the generalized \( q \)-logarithm function

\[
\log_q x = \frac{x^{1-q} - 1}{1 - q},
\]

(1.4)

which, for \( q = 1 \), becomes again the common natural logarithm. Its inverse is the generalized \( q \)-exponential function

\[
e_q^x = [1 + (1 - q)x]^{\frac{1}{1 - q}},
\]

(1.5)

which becomes the exponential function for \( q = 1 \). The importance of the \( q \)-logarithm is that it satisfies a pseudo-additive law

\[
\log_q xy = \log_q x + \log_q y + (1 - q)(\log_q x)(\log_q y).
\]

(1.6)

Then Tsallis entropy 1.2 can be written as the \( q \)-deformed Shannon entropy

\[
S_T(P, q) = -\sum_{i=1}^{N} p_i^q \log_q p_i = \sum_{i=1}^{N} p_i \log_q \left( \frac{1}{p_i} \right) = \langle \log_q \left( \frac{1}{p_i} \right) \rangle_{\text{lin}} = \langle I_i \rangle_{\text{lin}},
\]

(1.7)

with the last term resulting as the \( q \)-extension of 1.1. This reflects the non-extensive character of the system on the elementary information gains.

Note also that the classical power laws and the additivity rules for the logarithm and exponential do no longer hold in this generalized context. Except for \( q = 1 \), in general \( \log_q x^\alpha \neq \alpha \log_q x \), which explains why we keep writing throughout this paper Shannon’s elementary information gain as \( \langle \log \left( \frac{1}{p_i} \right) \rangle_{\text{lin}} \) instead of \( -\langle \log p_i \rangle_{\text{lin}} \). Useful for our purposes will be the equality

\[
e_q^{x+y+(1-q)xy} = e_q^x e_q^y.
\]

(1.8)
We will see how the q-deformed formalism fits naturally in the mathematical descriptions of generalized entropy measures.

1.3 Rényi’s entropy

Either in the case of BG as for Tsallis entropy, in 1.1 and 1.7, an entropy measure is the average $S$ obtained over many *elementary information gains* $I_i \equiv I_i(\frac{1}{p_i}) = \log_q \left( \frac{1}{p_i} \right)$ associated to the i-th event of probability $p_i$ (if the system is extensive, $q=1$).

Another possible generalization exists and has become commonplace throughout the literature, namely Rényi’s measure \cite{12}. A. Rényi maintained a still additive measure, as in BG entropy, but considered that another form of averaging is possible. His starting point was the generalized notion of average of A.N. Kolmogorov and M. Nagumo \cite{7,10}, who independently showed that, in the frame of the Kolmogorov axioms of probability theory, the definition of the average must be extended to the *quasi-arithmetic* or *quasi-linear mean* defined as

$$S = f^{-1} \left( \sum_{i=1}^{N} p_i f(I_i) \right), \quad (1.9)$$

where $f$ is a strictly monotone continuous and invertible function, the so called *Kolmogorov-Nagumo function* (KN function). On his side, Rényi showed that if we restrict to additive measures then only two possible KN functions exist. The first one is the common arithmetic mean and is associated with the KN function $f(x) = x$, and the second is the *exponential mean* with

$$f(x) = c_1 b^{(1-q)x} + c_2, \quad (1.10)$$

where $q$ is a real parameter, and $c_1$ and $c_2$ are two arbitrary constants.

The exponential mean leads to *Rényi’s information measure* or *Rényi’s entropy*

$$S_R(P, q) = \frac{1}{1-q} \log_b \sum_{i=1}^{N} p_i^q, \quad (1.11)$$

with $b$ the logarithm base (we will from now on assume the natural base, $b=e$, for Rényi’s entropy either). For $q \to 1$ Rényi’s measure becomes Shannon’s entropy.

It should be noted how P. Jizba and T. Arimitsu \cite{5} showed that Rényi’s measure can be obtained also extending the Shannon-Khinchin axioms to a quasi-linear conditional information

$$S_R(B|A) = f^{-1} \left( \sum_i \pi_i(A) f (S_R(B|A)) \right), \quad (1.12)$$
with $f$ as given in 1.10.

Therefore Shannon’s information measure is an averaged information in the ordinary sense, while Rényi’s measure represents an exponential mean over the same elementary information gains $\log \left( \frac{1}{p_i} \right)$.

### 2 The Sharma-Mittal and Supra-extensive entropy

#### 2.1 Generalizing with Kolmogorov-Nagumo means

It is important to understand that Tsallis and Rényi entropies are two different generalizations along two different paths. Tsallis generalized to non-extensive systems, while Rényi to quasi-linear means. But we can search for an entropy which generalizes to non-extensive sets and non-linear means containing Tsallis and Rényi measures as limiting cases.

Let us unify the picture of all the entropies considered here through KN averages (as J.Naudts and M.Czachor did [11], tough by a slightly different approach).

It is immediate to see from 1.7 and 1.9 how for Tsallis’s measure it is the KN function

$$f(x) = x$$  \hspace{1cm} (2.1)

which averages over the elementary information gain

$$I_i = \log_q \left( \frac{1}{p_i} \right).$$

This led us to write it as

$$S_T(P, q) = \left\langle \log_q \left( \frac{1}{p_i} \right) \right\rangle_{\text{lin}}.$$  

While, for Rényi’s measure, choose in 1.10, $c_1 = \frac{1}{1-q} = -c_2$ (remember 1.4), then the KN function takes the form

$$f(x) = \log_q e^x,$$  \hspace{1cm} (2.2)

which, applied on

$$I_i = \log \left( \frac{1}{p_i} \right),$$
in 1.9 \((f^{-1}(x) = \log_q e^x)\) leads us to rewrite 1.11 as

\[ S_R(P, q) = \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{\text{exp}}, \]

where, of course, \(\langle \cdot \rangle_{\text{exp}} \equiv \langle I_i \rangle_{\text{exp}}\) stands for the exponential mean defined by the KN function 2.2 over the elementary information \(I_i\).

But, what Tsallis and Rényi measures have in common is that in both cases

\[ f(I_i) = \log_q \left( \frac{1}{p_i} \right). \tag{2.3} \]

Then, for a further generalization, the simplest step beyond them would be that to generalize 2.1 and 2.2 with

\[ f(x) = \log_q e^x \tag{2.4} \]

and set

\[ I_i = \log_s \left( \frac{1}{p_i} \right), \]

where \(r, s\) are new parameters on the generalized exponential and logarithm functions. Maintaining constraint 2.3 implies \(s = r\). Then calculating 1.9 \((f^{-1}(x) = \log_q e^x)\), one obtains the Sharma-Mittal information measure \(\text{(13)}\)

\[ S_{SM}(P, \{q, r\}) = \log_r \sum_i p_i \log_q \left( \frac{1}{p_i} \right) = \]

\[ = \left\langle \log_r \left( \frac{1}{p_i} \right) \right\rangle_{q-\text{exp}} = \]

\[ = \frac{1}{1-r} \left( \left( \sum_i p_i^q \right)^{\frac{1-r}{q}} - 1 \right), \]

where \(\langle \cdot \rangle_{q-\text{exp}}\) stands for an average defined by the KN function 2.4 and that we will call the quasi-exponential mean.

We can see that for \(r \to 1\) Rényi’s measure, and for \(r \to q\) Tsallis measure, are recovered as limiting cases.

We will show in the next section that for two statistical independent systems \(A\) and \(B\) it is easy to check that

\[ S_{SM}(A \cap B) = S_{SM}(A) + S_{SM}(B|A) + (1-r)S_{SM}(A)S_{SM}(B|A), \]
i.e. a pseudo-additive law holds as in the case of Tsallis entropy.

Therefore Sharma-Mittal’s measure generalizes Rényi’s extensive entropy to non-extensivity, characterized by the r-logarithm. It is the parameter $r$ which determines the degree of non-extensivity, while $q$ is the deformation parameter of the probability distribution (however, when $r \to q$ the two parameters become intertwined and in Tsallis entropy it is $q$ which measures non-extensivity).

On information theoretic grounds, B.D. Sharma and D.P. Mittal (13), advanced already in 1975 this non-additive measure which shows to have a non-extensive character either. But it wasn’t until recently ((2), (3), and without mentioning it explicitly (11)) that Sharma-Mittal’s measure has been investigated in statistical mechanics.

### 2.2 Generalizing with $q$-logarithms and $q$-exponentials

At this point let us see how by using the q-deformed logarithm and exponential formalism, one could express in a much more compact form the same generalization path.

First of all recall a well known relationship which exists between Tsallis and Rényi entropies, namely

$$S_R(P, q) = \frac{1}{1 - q} \log \left[ 1 + (1 - q) S_T(P, q) \right]. \quad \text{(2.6)}$$

Here we can efficiently exploit the generalized logarithm and exponential functions 1.4 and 1.5, rewriting 2.6 in the more compact form

$$S_R(P, q) = \log q^{S_T(P, q)} , \quad \text{(2.7)}$$

from which follows immediately

$$S_T(P, q) = \log_q e^{S_R(P, q)} . \quad \text{(2.8)}$$

Looking at the structure of 2.7 and 2.8 we can ask if, given another parameter $r$, the following

$$S_{SM}(P, \{q, r\}) = \log_r e^{S_T(P, q)} = \frac{1}{1 - r} \left[ \left( \sum_i p_i^q \right)^{\frac{1}{r-1}} - 1 \right], \quad \text{(2.9)}$$
might then be other possible generalizations? 2.9 can be recognized immediately as Sharma-Mittal’s measure 2.5 and can be already accepted as an extension.

2.10 instead needs a closer look. For \( r \to q \) it obviously boils down to Rényi’s entropy. For \( r \to 1 \) we obtain Tsallis’ measure again. So, from a formal point of view it can be regarded as another generalization too. It is however not entirely clear what kind of statistics it expresses. Its particular status might be best evidenced expressing all the measures in terms of (logarithmic averaged) surprise quantities.

Indeed, notice that we can rewrite the quantity

\[
\left( \sum_i p_i^q \right)^{\frac{1}{1-q}} = \left( \sum_i p_i \left( \frac{1}{p_i} \right)^{1-q} \right)^{\frac{1}{1-q}} = \left\langle \left( \frac{1}{p_i} \right)^{1-q} \right\rangle_{\text{lin}}^{\frac{1}{1-q}} = e_q \left\langle \log_q \left( \frac{1}{p_i} \right) \right\rangle_{\text{lin}} = \left\langle \frac{1}{p_i} \right\rangle_{\log_q}^{\text{lin}}
\]

where we used the logarithmic mean \( \langle \cdot \rangle_{\log_q} \) defined by the KN function \( f(x) = \log_q x \). Then, from 1.7 and 1.11, and using 2.11, equations 2.7 to 2.10 can be rewritten as

\[
S_R(P, q) = \log \left\langle \frac{1}{p_i} \right\rangle_{\log_q}^{\text{lin}} ;
\]

\[
S_T(P, q) = \log_q \left\langle \frac{1}{p_i} \right\rangle_{\log_q} ;
\]

\[
S_SM(P, \{q, r\}) = \log_r \left\langle \frac{1}{p_i} \right\rangle_{\log_q} ;
\]

\[
S_SE(P, \{q, r\}) = \log_q e_r \left\langle \frac{1}{p_i} \right\rangle_{\log_q} .
\]

With the q-deformed logarithm and exponential formalism we could easily see the generalization path to follow and write all the measures into a more compact form (2.7 to 2.10). Moreover this makes it easier to recognize the behavior of the limits than in their explicit form (the r.h.s. of 2.9 and 2.10). With no or only few passages it is immediate to see how 2.9 reduces to Tsallis entropy for \( r \to q \), and for \( r \to 1 \) it reduces to Rényi’s entropy (without any
need to apply Hopital rule, first order approximations or whatever, insert 1.2 in 1.5).

The limit for $q \to 1$ for Sharma-Mittal measure is

$$
\lim_{q \to 1} S_{SM} = \lim_{q \to 1} \log_r e^s_{eq} = \log_r e^{sS} = \log_r \left( \frac{1}{p_i} \right) \log =
$$

$$
= e^{-(1-r) \sum p_i \log p_i} - \frac{1}{1 - r},
$$

which Frank and Daffertshofer used to call the *gaussian entropy*.

By the way

$$
\lim_{q \to 1} S_{SE} = \lim_{q \to 1} \log_q e^{sR} = \log e^{sS} = \log \left( \frac{1}{p_i} \log \right)
$$

$$
= \frac{1}{1 - r} \log \left( 1 - (1 - r) \sum p_i \log p_i \right).
$$

But the point is that when compared with 2.12, 2.13, 2.14, measure 2.15 seems to stand apart and does not correspond to some quasi-linear mean in the style of 1.9.

### 2.3 Comparing the supra-extensive entropy with Sharma-Mittal’s entropy

Let us then focus shortly on the separate nature of 2.15 (or 2.10) and some of its properties.

First of all note that it can be shown how for two statistical independent systems $A$ and $B$, similarly to Tsallis’ entropy, the Sharma-Mittal’s entropy obeys a pseudo-additive law and can be decomposed as in 1.3. It is almost immediate to see this by employing the generalized exponential formalism. Thanks to 1.3, 1.6, 1.8, starting from the middle term of 2.9 we can write

$$
S_{SM}(A \cap B) = \log_r e^{sT(A \cap B)} =
$$

$$
= \log_r e_q [S_T(A) + S_T(B|A) + (1 - q) S_T(A)S_T(B|A)] = \log_r \left( e^{sT(A) e^{sT(B|A)}} \right) =
$$

$$
= \log_r e_q^{sT(A)} + \log_r e_q^{sT(B|A)} + (1 - r) \log_r e_q^{sT(A)} \log_r e_q^{sT(B|A)} =
$$

$$
= S_{SM}(A) + S_{SM}(B|A) + (1 - r) S_{SM}(A)S_{SM}(B|A). 
$$

(2.16)
Proceeding in the same manner with 2.10 leads however not to the same decomposition. Because of Rényi’s measure additive character one can’t go further than

\[ S_{SE}(A \cap B) = \log_q e_r^{S_R(A \cap B)} = \log_q e_r [S_R(A) + S_R(B|A)] \]

with \( S_R(B|A) \) as given in 1.12.

Entropy 2.15 therefore obeys a new form of non-extensivity, we call supra-extensivity.

There are also other aspects which should be mentioned. Let us briefly recall the notions of concavity and stability applied to entropy measures.

Given two probability distributions \( P = \{p_1, ..., p_N\} \) and \( P' = \{p'_1, ..., p'_N\} \) and defining an intermediate distribution \( P'' = \{p''_1, ..., p''_N\} \) with

\[ p''_i \equiv \mu p_i + (1 - \mu) p'_i; \quad \forall \mu \in [0, 1], \]

\( S(P) \) is said to be a concave entropic functional if and only if

\[ S(P'') \geq \mu S(P) + (1 - \mu) S(P'). \]

Otherwise, \( S(P) \) is said to be convex. Concavity implies thermodynamic stability (e.g. thermal equilibrium between two initial temperatures in BG statistical mechanics).

Recall also the notion of stability (or experimental robustness, as Tsallis calls it, in order to avoid confusion with the previous form of thermodynamic stability) which implies that for arbitrary small variations of the probabilities \( p_i \) a statistical functional remains finite. That is, given a deformation

\[ ||p - p'|| = \sum_i |p_i - p'_i|, \]

such that \( ||p - p'|| < \delta_{\varepsilon} \), we obtain stability of \( S(P) \) if

\[ \Delta = \left| \frac{S(P) - S(P')}{S_{max}} \right| < \varepsilon; \quad \forall \delta_{\varepsilon} > 0, \forall \varepsilon > 0, \]

with \( S_{max} \) the maximum value \( S \) can attain and for all microstates \( i = 1, ..., N \).

Lesche claims \([8]\) that this is a necessary condition for an entropy measure to be a physical quantity and showed that, while BG entropy is always stable, Rényi’s measure is unstable for all \( q \neq 1 \). It is also known that BG entropy is always concave, while Rényi’s measure is concave only for \( q \leq 1 \), and can be either concave or convex for \( q > 1 \). More recently, Abe \([1]\) showed that Tsallis entropy is concave and stable for all positive values of \( q \). It might
also be worth mentioning that a physical entropy is not only expected to be
generically concave and Lesche-stable but should lead also to a finite entropy
production per unit time. BG and Tsallis entropies share all these properties.
Rényi entropy shares none. (4)

So, being an extension of it, it is clear that the properties of concavity and
stability and finite entropy production per unit time are generically violated
also in Sharma Mittal’s and the supra-extensive entropy.

Finally, it should also be underlined how Frank and Plastino showed (3) that
the Sharma-Mittal entropy is the only measure that allows for a pseudo-
additive decomposition and at the same time gives rise to a thermostatistics
based on escort mean energy values

\[ U = \frac{\sum p_i^q \epsilon_i}{\sum p_i^q} , \]

(\( \epsilon_i \) are the energy levels) admitting of a generalized partition function \( \tilde{Z} \) defined by \( \log \tilde{Z}_{SM} := \log \, Z_{SM} - \beta U \) with

\[ Z_{SM} = \left( \sum_i p_i^q \right) \frac{1}{\gamma} = \left\langle \frac{1}{p_i} \right\rangle_{\log q} \quad , \quad (2.17) \]

(\( Z_{SM} \) is the partition function which takes \( U \) while \( \tilde{Z}_{SM} \) takes zero as the
energy reference, and where \( \beta \) is an inverse temperature measure), that leads
to the usual expressions for the free energy

\[ F = U - T S_{SM} = -\frac{1}{\beta} \log \tilde{Z}_{SM} \]

and the mean energy

\[ U = -\frac{\partial}{\partial \beta} \log \tilde{Z}_{SM} . \]

We saw that the new measure we considered here does not allow for a pseudo-
additive decomposition like 2.16, and therefore it is to expect that the partition
function describing the free and mean energy cannot have the same structure
\( \tilde{Z}_{SM} \) common to Sharma-Mittal entropy unless, as can be shown (3) applying
the maximum entropy principle, one substitutes 2.17 with \( Z_{SE} = e_r \log \left\langle \frac{1}{p_i} \right\rangle_{\log q} \).

Summing up, the supra-extensive entropy, does no longer obey a pseudo-
additive statistics, if based on escort mean values the partition function must
take an intrinsically different form than Sharma-Mittals one, but what they
have in common with Rényi’s entropy is that, in general, they do not possess
the property of concavity, Lesche-stability and finite entropy production per
unit time.
3 Conclusion

We showed how the Sharma-Mittal and a new generalized entropy measure both unify Tsallis and Rényi entropies on two different paths in a way that appears natural and almost immediate when we make use of the generalized q-logarithm and q-exponentials as in 2.7, 2.8, 2.9 and 2.10. We underlined how the relationship among all measures becomes particularly clear using the logarithmic KN average 2.11 rewriting them as in 2.12, 2.13, 2.14 and 2.15. This path naturally leads to the supra-extensive entropy which does not emerge from the KN means approach alone and does not conform to a pseudo-additive law, lacks of concavity, Lesche-stability and finite entropy production. However, because the new measure here proposed emerges so naturally as another possible extension of Rényi and Tsallis entropy it is therefore worth of being mentioned. It is tempting to conclude that, while it might not have applications in a generalized thermostatistics, it nevertheless might be of some interest in the frame of information theory, cybernetics, control theory, etc.

Finally we obtained a way of understanding all these entropy measures in a unified picture that can be summarized in the following table and diagram.
| Entropy measure   | Explicit form                                                                 | KN-mean form                                                                 | KN_log-mean form                                  | log_q × exp_q-form |
|------------------|-------------------------------------------------------------------------------|------------------------------------------------------------------------------|--------------------------------------------------|--------------------|
| Supra-extensive  | $[1 + \frac{(1-r)}{1-q} \log \sum_i p_i^q]^{1-q} - 1$                         | $\log_q e_r \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{exp}$ | $\log_q e_r \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$ | $\log_q e_r^{S_R(P,q)}$ |
| Sharma-Mittal    | $\frac{1}{1-r} (\sum_i p_i^q)^{\frac{1}{1-q}} - 1$                          | $\left\langle \log_r \left( \frac{1}{p_i} \right) \right\rangle_{q-exp}$      | $\log_r \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$ | $\log_r e_r^{S_T(P,q)}$ |
| Tsallis          | $\frac{1}{1-q} \log \sum_i p_i^q$                                           | $\left\langle \log_q \left( \frac{1}{p_i} \right) \right\rangle_{lin}$        | $\log_q \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$ | $\log_q e_q^{S_R(P,q)}$ |
| Rényi            | $\frac{1}{1-q} \log \sum_i p_i^q$                                           | $\left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{exp}$          | $\log \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$ | $\log e_q^{S_T(P,q)}$ |
| BG-Shannon       | $-\sum_i p_i \log p_i$                                                       | $\left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{lin}$          | $\log \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$ | $\log e_q^{S_T(P)}$ |

**Diagram:**

- **Sharma-Mittal**
  - $\log_r \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$
  - $r \to q$
  - $r \to 1$

- **Supra-extensive**
  - $\log_q e_r \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$
  - $r \to q$
  - $r \to 1$

- **Rényi**
  - $\log \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{lin}$
  - $q \to 1$

- **Tsallis**
  - $\log_q \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log_q}$
  - $q \to 1$

- **Shannon (Boltzmann-Gibbs)**
  - $\log \left\langle \log \left( \frac{1}{p_i} \right) \right\rangle_{log}$
References

[1] S. Abe, *Stability of Tsallis entropy and instabilities of Rényi and normalized Tsallis entropies: A basis for q-exponential distributions*. Phys. Rev. E66, 046134 (2002).

[2] T.D. Frank, A. Daffertshofer, *Exact time-dependant solutions of the Rényi Fokker-Planck equation and the Fokker-Planck equations related to the entropies proposed by Sharma and Mittal*, Physica A 285, 351 (2000).

[3] T.D. Frank, A.R. Plastino, *Generalized thermostatistics based on the Sharma-Mittal entropy and escort mean values*, Eur. Phys. J. B 30, 543-549 (2002).

[4] See, for instance, the Preface of *Nonextensive Entropy - Interdisciplinary Applications*. M. Gell-Mann and C. Tsallis, Oxford University Press, New York (2004).

[5] P. Jizba, T. Arimitsu, *The world according to Rényi: thermodynamics of multifractal systems*, Annals of Physics, Volume 312, Issue 1, 17-59 (July 2004).

[6] A.I. Khinchin, *Mathematical Foundations of Information Theory*, Dover, New York (1957).

[7] A.N. Kolmogorov *Sur la notion de la moyenne*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. (6) 12, 388-391 (1930).

[8] B. Lesche, J. Stat. Phys., 27, 419 (1982).

[9] Author’s paper in preparation.

[10] M. Nagumo, *Über eine Klasse der Mittelwerte*, Japan. J. Math., 7, 71-79 (1930).

[11] J. Naudts, M. Czachor *Generalized thermostatistics and Kolmogorov-Nagumo averages*, arXiv:cond-mat/0110077 (3 Oct 2001).

[12] A. Rényi, *Probability theory*, North Holland, Amsterdam (1970); Selected Papers of Alfred Rényi, Vol.2 Akademia Kiado, Budapest (1976).

[13] B.D. Sharma and D.P. Mittal, J. Math. Sci. 10, 28 (1975). See also *New Nonadditive Measures of Relative Information*, J. Comb. Inform. and Syst. Sci., 2, 122-133 (1977).

[14] C. Tsallis, *Possible Generalization of Boltzmann-Gibbs Statistics*, J. Stat. Phys. 52, 479 (1988).

[15] For a general bibliography about nonextensive thermodynamics and updates see: [http://tsallis.cat.cbpf.br/biblio.htm](http://tsallis.cat.cbpf.br/biblio.htm)