Equivariant maps between sphere bundles over tori and $KO^*$-degree

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Abstract

We show a non-existence result for some class of equivariant maps between sphere bundles over tori. The notion of equivariant $KO^*$-degree is used in the proof. As an application to Seiberg-Witten theory we have a new inequality $b_2^+(X) \geq -\text{sign}(X)/8 + c(X) + \varepsilon(X)$ for a connected closed oriented spin 4-manifold $X$ with indefinite intersection form, where $c(X)$ is a non-negative integer determined by the quadruple cup product on $H^1(X; \mathbb{Z})$ and some maps in $KO$-theory induced from the Albanese map of $X$, and where $1 \leq \varepsilon(X) \leq 3$.

1 Introduction

Let $\tilde{R}$ be the 1-dimensional real line with the linear involution $-1$ and let $\tilde{R}^p = \mathbb{R}^\oplus p$. The involution on $\mathbb{R}$ then induces an involution on the $n$-dimensional torus $\tilde{T}^n = (\tilde{R}/\mathbb{Z})^n$, so it is a Real space $[2]$. Fix a non-negative integer $l$. We denote by $V_0$, $W_0$ the product vector bundles $\tilde{T}^n \times \tilde{R}^x$, $\tilde{T}^n \times \tilde{R}^{x+l}$ respectively. The diagonal actions on $V_0$, $W_0$ give lifts of the involution.

Let $V_1$ and $W_1$ be symplectic vector bundles over the Real space $\tilde{T}^n$ in the sense of Dupont [11]; each of them is defined to be a complex vector bundle together with an anti-linear lift of the involution such that its square is $-1$ on each fiber. Since the symplectic action defines a quaternionic structure on the fiber over a fixed point, the rank of symplectic vector bundles over $\tilde{T}^n$ should be even. We thus denote $\text{rank}_\mathbb{C}V_1 - \text{rank}_\mathbb{C}W_1 = 2k$.

Let $\text{Pin}^- (2)$ be the subgroup of $Sp(1)$ generated by $U(1)$ and $j$. Then the quotient $\text{Pin}^- (2)/U(1)$ is isomorphic to $\{\pm 1\}$, so we regard $V_0$, $W_0$ as $\text{Pin}^- (2)$-equivariant bundles. We define $\text{Pin}^- (2)$-actions on $V_1$, $W_1$ by using complex scalar multiplication and the symplectic action. Then our purpose of this paper is to give a necessary condition for the existence of a $\text{Pin}^- (2)$-equivariant
fiber-preserving map from the sphere bundle $S(V_0 \oplus V_1)$ to the sphere bundle $S(W_0 \oplus W_1)$, or equivalently the existence of a proper Pin$^-$-(2)-equivariant fiber-preserving map from $V_0 \oplus V_1$ to $W_0 \oplus W_1$, under some assumptions. To state our main theorem we prepare some notations.

Let $Ksp(\tilde{T}^n)$ be the Grothendieck group of symplectic vector bundles over $\tilde{T}^n$, so we consider the difference $[V_1] - [W_1]$ as an element in $Ksp(\tilde{T}^n)$. We now introduce a set of invariants to characterize it.

For each subset $S \subset \{1, \cdots, n\}$ we put $\tilde{R}^S = \text{Map}(S, \tilde{R})$ and induce the Real structure on $\tilde{R}^S$ from $\tilde{R}$. Then $\tilde{R}^S$ is isomorphic to $\tilde{R}^{|S|}$, where $|S|$ is the cardinal number of $S$. We also have the Real torus $\tilde{T}^S = \text{Map}(S, \tilde{T}^1)$, which is isomorphic to $\tilde{T}^{|S|}$. Let $\pi_S : \tilde{T}^n \to \tilde{T}^S$ be the projection and $i_S : \tilde{R}^S \to \tilde{T}^S$ an open $\{\pm 1\}$-equivariant embedding onto an open neighborhood of the origin. Then, as we shall show later, the following map is an isomorphism:

$$\sum_{S \subset \{1, \cdots, n\}} \pi_S^*(i_S)_! : \bigoplus_S Ksp(\tilde{R}^S) \to Ksp(\tilde{T}^n).$$

Thus $[V_1] - [W_1] \in Ksp(\tilde{T}^n)$ is determined by the corresponding components $a_S \in Ksp(\tilde{R}^S)$. By the Bott periodicity theorem in $Ksp$-theory [11] (see also Section 5), we see $Ksp(\tilde{R}^S) \cong \mathbb{Z}$ if $|S| \equiv 0, 4 \pmod{8}$, $Ksp(\tilde{R}^S) \cong \mathbb{Z}/2$ if $|S| \equiv 2$ or $3 \pmod{8}$, and in the other cases $Ksp(\tilde{R}^S) = \{0\}$.

In this paper we investigate the case $a_S = 0$ for any nonempty set $S$ with $|S| \equiv 0 \pmod{8}$. From now we assume it. Then the quantity $a_S$ for $S$ with $|S| \equiv 0, 4 \pmod{8}$, which we consider as an integer, can be detected by the Chern character:

$$\text{ch}(V_1) - \text{ch}(W_1) = 2k + \sum_{S \subset \{1, \cdots, n\}, |S| = 4} a_S \int d\xi^i \in H^*(\tilde{T}^n; \mathbb{Z}),$$

(1)

where $(\xi_1, \cdots, \xi_n)$ is the coordinate of $\tilde{R}^n$.

Let $\pi_S$ be the mod 2 reduction of $a_S$, i.e. $\overline{a}_S = a_S \pmod{2} \in Ksp(\tilde{R}^S) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$. Let $X_1, \cdots, X_n$ be indeterminates. For each $S \subset \{1, \cdots, n\}$ we define an integer $N_S$ by the equation

$$\prod_{|S| = 2, 3, 4, \pi_S \neq 0} (1 - 2 \prod_{i \in S} X_i) \equiv \sum_{S \subset \{1, \cdots, n\}} N_S \prod_{i \in S} X_i \pmod{\mathcal{I}},$$

where $\mathcal{I}$ is the ideal generated by $X_i^2 - X_i$ ($1 \leq i \leq n$). Combinatorially $N_S$ is the sum

$$N_S = \sum_{m \geq 0} N(S, m)(-2)^m,$$

2
where \( N(S,m) \) is the cardinal number of the set
\[
\{S_1, \cdots, S_m \mid |S_i| = 2, 3 \text{ or } 4, \bar{a}_{S_i} \neq 0 \quad (1 \leq i \leq m), \quad S = S_1 \cup \cdots \cup S_m \}.
\]
For \( S \) with \( N_S \neq 0 \) we denote by \( d_S \) the maximal power in 2 which divides \( N_S \).

**Theorem 1.** Assume \( l > 0 \) and \( a_S = 0 \) for any nonempty set \( S \) with \(|S| \equiv 0 \mod 8 \). Suppose there is a proper \( \text{Pin}^{-}(2) \)-equivariant fiber-preserving map from \( V_0 \oplus V_1 \) to \( W_0 \oplus W_1 \) which induces the identity on the base space \( \tilde{T}^n \) and whose restriction to \( V_0 \) is given by the standard linear inclusion \( \mathbb{R}^x \rightarrow \mathbb{R}^{x+l} \).

If \( S \subset \{1, \cdots, n\} \) satisfies \(|S| = \text{even}, N_S \neq 0 \) then we have
\[
l \geq 2k + |S| - 2d_S + \varepsilon(k + d_S, l),
\]
where
\[
\varepsilon(d, l) = \begin{cases} 3, & d' \equiv 0 \mod 4, \quad l \geq 4, \\ 1, & d' \equiv 0 \mod 4, \quad l < 4, \\ 2, & d' \equiv 1 \mod 4, \\ 3, & d' \equiv 2 \mod 4, \\ 4, & d' \equiv 3 \mod 4. \end{cases}
\]

We apply Theorem 1 to Seiberg-Witten theory \[26\] by using the argument in [12]. Then we have the following theorem, which is our motivation. The detail of the proof will be given in Section 10.

Let \( X \) be a connected closed oriented spin 4-manifold with indefinite intersection form. Let \( \text{Ind} \mathbb{D} \) be the index bundle of Dirac operators parameterized by the Jacobian torus \( J_X = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) \). We induce an involution on \( J_X \) from the linear involution on \( H^1(X; \mathbb{R}) \) and define a symplectic action on \( \text{Ind} \mathbb{D} \) by the simultaneous action of the involution on \( J_X \) and scalar multiplication by \( j \in H \) on the spinor bundle of \( X \). Thus we consider \( \text{Ind} \mathbb{D} \) as an element in \( Ksp(\tilde{T}^n) \). Take a basis \( x_1, \cdots, x_n \) of \( H^1(X; \mathbb{Z}) \) \((n = b_1(X))\) to identify \( J_X \) with the Real torus \( \tilde{T}^n \). From the above discussion we then obtain the invariant \( a_S \in Ksp(\mathbb{R}^S) \) for each \( S \subset \{1, \cdots, n\} \).

When \(|S| = 4\) the cohomological formula of the index theorem [9] shows
\[
a_S = \pm \prod_{i \in S} x_i, [X]. \tag{2}
\]

When \(|S| = 2 \text{ or } 3\) we can describe \( a_S \) by using the the Albanese map \( \rho : X \rightarrow T^n = \text{Hom}(H^1(X; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \): The composition of the maps
\[
\mathbb{Z} \equivKO^{(S)}(\mathbb{R}^S) \xrightarrow{(i_S)_!} KO^{(S)}(T^S) \xrightarrow{\pi^*_T} KO^{(S)}(T^n) \\
\xrightarrow{i^*_X} KO^{(S)}(X) \xrightarrow{i} KO^{(S)}(\text{pt}) \equiv \mathbb{Z}/2
\]
is zero if and only if \( \bar{a}_S = 0 \), where \( i : X \rightarrow \text{pt} \) is the constant map.
Theorem 2. Let $X$ be a connected closed oriented spin 4-manifold with indefinite intersection form. Let $n = b_1(X)$. Then for each $S \subset \{1, \cdots, n\}$ with $|S|$ even and $N_S \neq 0$, we have

$$b_2^+(X) \geq -\frac{\text{sign}(X)}{8} + |S| - 2d_S + \varepsilon \left( -\frac{\text{sign}(X)}{16} + d_S, b_2^+(X) \right).$$

We take $S = \emptyset$, then $N_\emptyset = 1$, so we get the corollary below:

Corollary 3. Let $X$ be a connected closed oriented spin 4-manifold with indefinite intersection form. Then we have

$$b_2^+(X) \geq -\frac{\text{sign}(X)}{8} + \varepsilon \left( -\frac{\text{sign}(X)}{16}, b_2^+(X) \right).$$

Remark 4. Corollary 3 is a small improvement of the inequality obtained in [12]. Such an improvement began in [13]. In the case when $-\text{sign}(X)/16 \equiv 2, 3 \mod 4$, Corollary 3 was first shown by N. Minami [21] and B. Schmidt [23] independently, in which they destabilize a Pin$^-$-(2)-equivariant map to appeal to a result by S. Stolz in [25], while our method is more direct. On the other hand, inspired by D. Ruberman and S. Stolz [22], we discussed its improvement when $b_1(X) > 0$ in [14]. Theorem 5 which will be given below, includes both improvements.

To get a stronger inequality than Corollary 3 we need to compute $d_S$ for a nonempty subset $S$, which includes calculation of some maps in $KO$-theory. As discussed in [7], this cannot be detected by ordinary cohomology theory. However we suggest a special case when we can avoid this calculation.

Let $mT^4$ be the connected sum of $m$-copies of the 4-dimensional torus $T^4 = (\mathbb{R}/\mathbb{Z})^4$. If we take the standard basis of $H^1(mT^4; \mathbb{Z})$, it is easy to see $\text{rank } H^4(mT^4; \mathbb{Z}) = 4m$ and $N(S_{\text{max}}, 4m) = 1$ for the maximal subset $S_{\text{max}} = \{1, \cdots, 4m\}$. Moreover since $S_{\text{max}}$ cannot be written as $S_{\text{max}} = S_1 \cup \cdots \cup S_{m'}$ ($2 \leq |S_i| \leq 4$) with $m' < m$, we see $d_{S_{\text{max}}} = m$ without calculating $a_S$ for $S$ which satisfies $|S| = 2$ or 3. Hence we get

Theorem 5. Let $X$ be a connected closed oriented spin 4-manifold with indefinite intersection form. If there is an injective homomorphism $\iota : H^1(mT^4; \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z})$ which satisfies

$$\langle \iota(x)\iota(y)\iota(z)\iota(w), [X] \rangle \equiv \langle xyzw, [mT^4] \rangle \mod 2$$

for any $x, y, z, w \in H^1(mT^4; \mathbb{Z})$, then we have

$$b_2^+(X) \geq -\frac{\text{sign}(X)}{8} + 2m + \varepsilon \left( -\frac{\text{sign}(X)}{16} + m, b_2^+(X) \right).$$
Corollary 6. Let $X$ be a connected closed oriented spin 4-manifold with indefinite intersection form. If $X$ decomposes as $X = X' \# (mT^4)$ by a closed 4-manifold $X'$ then the inequality $5m + \varepsilon(-\text{sign}(X)/16 + m, b_2^+(X) - 3m) - \text{sign}(X)/8 \leq b_2^+(X)$ holds.

This paper is organized as follows: In Section 2 we introduce equivariant $KO^*$-degree for $G$-equivariant fiber-preserving maps between spin $G$-vector bundles for a compact Lie group $G$. In Section 3 we give some formulae for the Euler class, which we need in our calculation. In Section 4 we discuss how to calculate the product of elements in $KO_G^*(pt)$. In Section 5 we recall Real $K$-theory and symplectic $K$-theory. Some relations between other $K$-theories are also discussed. In Section 6, 7 we carry out calculations in our case. The proof of Theorem 1 is given in Section 9, while we prove Theorem 2 in Section 10. We also show the equations (1), (2), (3) in Section 9 and 10.

2 Equivariant $KO^*$-degree

We first recall the equivariant $KO^*$-theoretic Bott class for spin bundles.

Let $G$ be a compact Lie group. For a compact $G$-space $B$ we let $KO_G(B)$, $K_G(B)$ and $KSp_G(B)$ be the Grothendieck groups of real, complex and quaternionic $G$-vector bundles over $B$ respectively. These definitions extend to a locally compact $G$-space $B$ in the usual way [24].

Suppose $V$ is a real spin $G$-vector bundle over a compact $G$-space $B$, i.e. a triple consisting of a real oriented $G$-vector bundle $V$, a spin structure $	ext{Spin}(V)$ on the principal bundle $SO(V)$ of oriented orthonormal frames (for some $G$-invariant Riemannian metric on $V$), and a lift of the $G$-action on $SO(V)$ to $\text{Spin}(V)$.

If rank $V = 4k$, the real spinor $G$-vector bundle $S(V) = S^+(V) \oplus S^-(V)$ is formed from $\text{Spin}(V)$ and the irreducible $\mathbb{Z}/2$-graded $\text{Cl}(\mathbb{R}^{4k})$-representation $\Delta_{4k} = \Delta_{4k}^+ \oplus \Delta_{4k}^-$, where $\text{Cl}^{(\mathbb{R}^n)}$ is the Clifford algebra (over $\mathbb{R}$) of the quadratic form $-x_1^2 + \cdots + x_n^2$ on $\mathbb{R}^n$. We follow [5 Section 6] to choose the $\mathbb{Z}/2$-grading of $\Delta_{4k}$. The Clifford multiplication then defines a $G$-equivariant bundle map

$$c : V \to \text{Hom}(S^+(V), S^-(V)).$$

For each vector $v \in V \setminus \{0\}$, $c(v)$ is an isomorphism.

Let $V$ be as above with arbitrary rank. Let $m$ be a non-negative integer satisfying $m + \text{rank} V \equiv 0 \text{ mod } 8$, and $\mathbb{R}^m$ the product bundle $B \times \mathbb{R}^m$. Then $V \oplus \mathbb{R}^m$ is naturally again a spin $G$-vector bundle via the inclusion $\text{Spin}(V) \subset \text{Spin}(V \oplus \mathbb{R}^m)$. The Bott class $\beta(V \oplus \mathbb{R}^m)$ is then an element of $KO_G(V \oplus \mathbb{R}^m) = KO_G^m(V)$ and defined by the triple

$$[(\pi^* S^+(V \oplus \mathbb{R}^m), \pi^* S^-(V \oplus \mathbb{R}^m), c)].$$
where $\pi : V \to B$ is the projection. By using the Bott periodicity theorem (or the Thom isomorphism theorem) in equivariant $KO$-theory \cite{4}, we have the definition of $KO^*_G$ for positive $*$ and $KO^{-m}_G(V)$ is identified with $KO^{\text{rank} V}_G(V)$ by this definition. With this identification we write 

$$\beta(V) \in KO^{\text{rank} V}_G(V)$$

for $\beta(V \oplus \mathbb{R}^m)$. Then the Bott periodicity theorem asserts $KO^*_G(V)$ is a free $KO^*_G(B)$-module generated by $\beta(V)$. By the restriction of $\beta(V)$ to $\{0\} \oplus \mathbb{R}^m$, we have the Euler class

$$e(V) \in KO^{\text{rank} V}_G(B).$$

Let $V$ be again as above with arbitrary rank. We next use a quaternionic multiplication on $\Delta^{8k+4}$ compatible with the action of $\text{Cl}^+(\mathbb{R}^{8k+4})$, so we take $m'$ to satisfy $m' + \text{rank} V \equiv 4 \pmod{8}$. Then the bundle $S^+(V \oplus \mathbb{R}^{m'}) \oplus S^-(V \oplus \mathbb{R}^{m'})$ has a quaternionic $G$-structure and the Clifford multiplication $c(v)$ for each vector $v \in V \setminus \{0\}$ is quaternionic linear. Thus the $G$-equivariant $KSp^*$-theoretic Bott class $\beta_H(V) \in KSp^{\text{rank} V+4}_G(V)$ is also defined by the triple $\Delta$ together with quaternionic scalar multiplication, if we replace $m$ by $m'$. Note that multiplication by $\beta_H(R^4)$ sends $\beta(V)$ to $\beta_H(V)$. However we use the explicit construction of $\beta_H(V)$ later.

Now we define $KO^*_G$-degree for a proper $G$-equivariant fiber-preserving map between spin $G$-vector bundles.

Suppose $V$ and $W$ are two spin $G$-vector bundles over a compact $G$-space $B$. Let $\varphi : V \to W$ be a proper $G$-equivariant fiber-preserving map which induces the identity on the base space $B$. The $KO^*_G$-degree $\alpha_\varphi$ of $\varphi$ is then the element of $KO^{\text{rank} W-\text{rank} V}_G(B)$ defined by the relation

$$\alpha_\varphi \beta(V) = \varphi^* \beta(W). \quad (5)$$

Let $G_0$ be a closed normal subgroup of $G$ such that $G_0$ acts trivially on $B$. Let $V_0$, $W_0$ be spin $G/G_0$-vector bundles, which we also regard spin $G$-vector bundles. Let $V_1$, $W_1$ be spin $G$-vector bundles whose actions of $G_0$ are free outside the zeros.

Suppose we have decompositions $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ as spin $G$-vector bundles. Then the fixed point sets of $V$ and $W$ for the $G_0$-actions are respectively $V_0$ and $W_0$, so we have a proper $G/G_0$-equivariant fiber-preserving map $\varphi_0 : V_0 \to W_0$ as a restriction of $\varphi$. Let $\alpha_{\varphi_0}$ be the $KO^*_G$-degree of $\varphi_0$:

$$\alpha_{\varphi_0} \beta(V_0) = \varphi_0^* \beta(W_0). \quad (6)$$

We regard $KO^*_G(B)$ as a $KO^*_G(G_0/B)$-module. We immediately obtain the following relations $\cite{8}$, $\cite{17}$.
Lemma 7.

\[ \alpha_{\varphi_0}e(V_0) = e(W_0) \in KO_G^{*}(G \alpha_0(B)), \]
\[ \alpha_{\varphi}e(V_1) = e(W_1)\alpha_{\varphi_0} \in KO_G^{*}(B). \]

Proof. Restrict \((\text{5})\) to \(B\) to obtain the first relation. Restrict \((\text{6})\) to \(V_0\) and use \((\text{7})\) to obtain the second relation. \(\Box\)

We want to apply this lemma to calculate the \(KO_G^{*}\)-degree. To do so we need to determine the Euler class and the product of elements in \(KO_G^{*}(B)\). In the next two sections we make preparations for it.

3 Some properties of the Euler class

This section consists of two observations on the Euler class. First we discuss spin structures on complex or quaternionic \(G\)-vector bundles and the Euler classes for them. Second we give a formula to calculate the Euler classes for complex \(G\)-vector bundles coming from the Bott class for a complex representation of \(G\). From now on we deal with operations between various equivariant \(K\)-theories. Thus we begin by introducing notation. The following operations are elementary:

1. Let \(V\) be a real \(G\)-vector bundle over \(B\). Denote by \(cV = C \otimes_\mathbb{R} V\) the complexification of \(V\). This operation induces a homomorphism \(c : KO_G(B) \to K_G(B)\).

2. Let \(V\) be a complex \(G\)-vector bundle over \(B\). Denote by \(qV = H \otimes_\mathbb{C} V\) the quaternionization of \(V\). This operation induces a homomorphism \(q : K_G(B) \to KSp_G(B)\).

3. Let \(V\) be a complex \(G\)-vector bundle over \(B\). Denote by \(rV\) the restriction of the scalars from \(\mathbb{C}\) to \(\mathbb{R}\). We denote by \(r : K_G(B) \to KO_G(B)\) the induced homomorphism.

4. Let \(V\) be a quaternionic \(G\)-vector bundle over \(B\). Denote by \(c'V\) the restriction of the scalars from \(\mathbb{H}\) to \(\mathbb{C}\). We denote by \(c' : KSp_G(B) \to K_G(B)\) the induced homomorphism.

5. Let \(V\) be a complex \(G\)-vector bundle over \(B\). Denote by \(tV = \overline{V}\) the complex conjugate of \(V\). This operation induces a homomorphism \(t : K_G(B) \to K_G(B)\).

6. Let \(V\) be a complex \(G\)-vector bundle over \(B\). Since the action of \(G\) commutes with the complex multiplication, \(V\) is naturally regarded as a complex \(G \times U(1)\)-vector bundle. We denote this complex \(G \times U(1)\)-vector bundle by \(bV\). Let \(b : K_G(B) \to K_{G \times U(1)}(B)\) be the induced homomorphism.
7. Let \( V \) be a quaternionic \( G \)-vector bundle \( V \) over \( B \). Since the action of \( G \) commutes with the quaternionic multiplication, \( rV \) is naturally regarded as a real \( G \times Sp(1) \)-vector bundle. We denote this real \( G \times Sp(1) \)-vector bundle by \( sV \). Let \( i : U(1) \to Sp(1) \) be the inclusion. Then it is obvious \( i^*sV = rV \). We let \( s : KSpG(B) \to KO_{G \times Sp(1)}(B) \) be the induced homomorphism.

Suppose \( V \) is a complex \( G \)-vector bundle over a compact \( G \)-space \( B \). Let \( \Lambda^*(V) = \Lambda^{even}(V) \oplus \Lambda^{odd}(V) \) be the exterior power of \( V \). Then in this paper the \( K_G \)-theoretic Bott class \( \beta_C(V) \in K_G(V) \) is the triple

\[
[(\pi^*\Lambda^{even}(V), \pi^*\Lambda^{odd}(V), \alpha)],
\]

where the bundle map

\[
\alpha : V \to \text{Hom}(\Lambda^{even}(V), \Lambda^{odd}(V))
\]
is given by \( \alpha(v) = v \wedge \cdot - v \cdot \cdot \cdot \). The Euler class \( e_C(V) \in K_G(B) \) is the restriction to the zero.

Let \( det V \) denote the determinant bundle of a complex \( G \)-vector bundle \( V \). If \( det V \) has a square root \( det(V)^{1/2} \) as a complex \( G \)-line bundle then the complex \( G \)-structure of \( V \) canonically induces a spin \( G \)-structure on the real \( G \)-vector bundle \( rV \) (see [5]). Then the Bott class \( \beta_C(V) \) and the complexification \( c(\beta(rV)) \) of the Bott class \( \beta(rV) \) are related by

\[
c(\beta(rV)) = det(V)^{-1/2}\beta_C(V) \in K_G(V),
\]
so we have

\[
c(e(rV)) = det(V)^{-1/2}e_C(V) \in K_G(B).
\]

The above argument is obviously applicable to quaternionic \( G \)-vector bundles. However it is insufficient in our application, so we will give its refinement.

Let \( V \) be a quaternionic \( G \)-vector bundle over a compact \( G \)-space \( B \) with rank \( V = n \). We first show that there is a natural spin \( G \times Sp(1) \)-structure on the real \( G \times Sp(1) \)-vector bundle \( sV \). Let \( Sp(V) \) be the principal \( Sp(n) \)-bundle of orthonormal frames for some \( G \)-invariant quaternionic metric on \( V \). The principal \( SO(4n) \)-bundle \( SO(sV) \) is then \( G \times Sp(1) \)-isomorphic to \( (Sp(V) \times Sp(1) \times SO(4n))/(Sp(n) \times Sp(1)) \), where the action of \( Sp(n) \times Sp(1) \) on \( SO(4n) \) is factoring through the homomorphism

\[
Sp(n) \times Sp(1) \to (Sp(n) \times Sp(1))/\{\pm 1\} \to SO(4n).
\]
The lift to \( Spin(4n) \) provides a spin \( G \times Sp(1) \)-structure on \( sV \).
On the other hand there is also a natural spin \( G \times U(1) \)-structure on the real \( G \times U(1) \)-vector bundle on \( rbcV \), since the principal \( SO(4n) \)-bundle \( SO(rbcV) \) is \( G \times U(1) \)-isomorphic to \((Sp(V) \times U(1) \times SO(4n))/((Sp(n) \times U(1)))\) and the action of \( Sp(n) \times U(1) \) on \( SO(4n) \) is factoring through the homomorphism

\[
Sp(n) \times U(1) \to SU(2n) \times U(1) \to (SU(2n) \times U(1))/\{\pm 1\} \to SO(4n).
\]

Let \( i : U(1) \to Sp(1) \) be the inclusion. Then from the above constructions the spin \( G \times U(1) \)-structure of \( i^*sV \) is obviously isomorphic to that of \( rbcV \). Moreover under the canonical trivialization \( \det c'V \cong B \times C_{2n}(t) \) the spin \( G \times U(1) \)-structure of \( c'V \) is isomorphic to the one obtained from the square root \( (\det c'V)^{1/2} = B \times C_{n}(t) \), where \( C_{n}(t) \) is the one-dimensional complex representation of \( U(1) \) with weight \( n \). Hence we use (5) to get

**Proposition 8.** Let \( V \) be a quaternionic \( G \)-vector bundle over \( B \) with rank \( H \) \( V = n \). Then we have

\[
c(e(i^*sV)) = C_{-n}(t)e_G(bc'V) \in K\times U(1)(B),
\]

where \( i : U(1) \to Sp(1) \) is the inclusion.

Now we let \( V \) be a complex representation of \( G \). Then the map \( \alpha \) gives an isomorphism between the product \( G \)-vector bundles \( V \setminus \{0\} \times \Lambda^{even}(V) \) and \( V \setminus \{0\} \times \Lambda^{odd}(V) \). If we regard \( \alpha \) as a trivialization of \( V \times \Lambda^{even}(V) \) over \( V \setminus \{0\} \) then we obtain a complex \( G \)-vector bundle \( V^+ \) as an extension of \( V \times \Lambda^{even}(V) \) to the one-point compactification \( V \cup \{\infty\} \) of \( V \), namely the \( G \)-sphere. If we exchange \( \Lambda^{even}(V) \) for \( \Lambda^{odd}(V) \) and replace \( \alpha \) by \( \alpha^{-1} \), we obtain another complex \( G \)-vector bundle \( V^- \) over \( V \cup \{\infty\} \). Under the canonical decomposition \( K_G(V \cup \{\infty\}) \cong K_G(\{\infty\}) \oplus K_G(V) \) these \( G \)-vector bundles satisfy

\[
[\hat{V}^+] = [\Lambda^{odd}(V)] + \beta_G(V), \quad [\hat{V}^-] = [\Lambda^{even}(V)] - \beta_G(V).
\]

**Definition 9.** Let \( x_0, x_1, \cdots, x_n \) be indeterminates. We define a polynomial \( \mu_n(x_0, x_1, \cdots, x_n) \) by the identity:

\[
\mu_n(x_0, x_1, \cdots, x_n) = \prod_{1 \leq i \leq n} (1 - x_i)
\]

\[
= \prod_{S \subseteq \{1, \cdots, n\}, |S|: even} (1 - x_0 \prod_{i \in S} x_i) - \prod_{S \subseteq \{1, \cdots, n\}, |S|: odd} (1 - x_0 \prod_{i \in S} x_i).
\]

For instance \( \mu_1 = \mu_2 = -x_0 \) and \( \mu_3 = -x_0 + x_0^2 x_1 x_2 x_3 \).
Proposition 10. Let $L_1, \ldots, L_n$ be complex 1-dimensional representations of $G$ and $V = L_1 \oplus \cdots \oplus L_n$. Let $L_0$ be another complex 1-dimensional representation of $G$. Then we have

$$e_C(L_0 \otimes \hat{V}^+) = e_C(L_0 \otimes \Lambda^{\text{odd}}(V)) + \mu_n(L_0, L_1, \cdots, L_n)\beta_C(V),$$

$$e_C(L_0 \otimes \hat{V}^-) = e_C(L_0 \otimes \Lambda^{\text{even}}(V)) - \mu_n(L_0, L_1, \cdots, L_n)\beta_C(V).$$

Proof. We introduce an action of $U(1)^{n+1}$ on $L_i$ via the $i$-th projection from $U(1)^{n+1}$ to $U(1)$ and multiplication ($0 \leq i \leq n$). Since the action of $G$ on $L_i$ factors through a homomorphism $G \rightarrow U(1) \rightarrow U(1)^{n+1}$, it is sufficient to show our formula in $K_{U(1)^{n+1}}(V \cup \{\infty\})$. By the Bott periodicity theorem $K_{U(1)^{n+1}}(V \cup \{\infty\})$ is freely generated by 1 and $\beta_C(V)$ as a $R(U(1)^{n+1})$-algebra, so we can write $e_C(L_0 \otimes \hat{V}^+)$ as

$$e_C(L_0 \otimes \hat{V}^+) = \alpha_n + \beta_n\beta_C(V)$$

for some $\alpha_n, \beta_n \in R(U(1)^{n+1})$. Restricting this equation to $\infty$ and 0, we get

$$\alpha_n = e_C(L_0 \otimes \Lambda^{\text{odd}}(V)),$$

$$\beta_n e_C(V) = e_C(L_0 \otimes \Lambda^{\text{even}}(V)) - e_C(L_0 \otimes \Lambda^{\text{odd}}(V)).$$

Since the representation ring $R(U(1)^{n+1})$ has no zero-divisor, we have $\beta_n = \mu_n(L_0, L_1, \cdots, L_n)$. We can compute the Euler class $e_C(L_0 \otimes \hat{V}^-)$ in the same way. \qed

Here we suggest a method to calculate the polynomial $\mu_n(x, 1, \cdots, 1)$, which will be appeared in our application. Although we shall not need its explicit form in it, it may be helpful in understanding this polynomial.

Definition 11. Let $\nu_n(x)$ be the polynomial defined by

$$\nu_n(x) = -(1 - x)^{2n-1}s_{n-1}(x), \quad s_0(x) = x/(1 - x), \quad s_n(x) = xs'_{n-1}(x).$$

Proposition 12. The identity $\mu_n(x, 1, \cdots, 1) = \nu_n(x)$ holds for any $n$.

We give a geometric proof although the identity is purely algebraic. Then the above proposition is clearly equivalent to

Proposition 13. Suppose $V$ is the trivial $n$-dimensional representation of $G$. Let $L$ be a complex 1-dimensional representation of $G$. Then we have

$$e_C(L \otimes \hat{V}^+) = e_C(L \otimes \Lambda^{\text{odd}}(V)) + \nu_n(L)\beta_C(V),$$

$$e_C(L \otimes \hat{V}^-) = e_C(L \otimes \Lambda^{\text{even}}(V)) - \nu_n(L)\beta_C(V).$$
Proof. We use some operations in $K$-theory (cf. [3]). Let $\lambda_t[L \otimes \hat{V}] = \sum_{i=0}^{n} t^i[L \otimes \Lambda^i(L \otimes \hat{V})] \in 1 + K_G(V \cup \{\infty\})[[t]].$ Then we have

$$e_C(L \otimes \hat{V}) = \lambda_{-1}[L \otimes \hat{V}] = (\lambda_t[L \otimes \Lambda^{odd}(V)]\lambda_t[L \otimes \beta_C(V)])_{t=-1}.$$ 

Let $\psi_k$ be the Adams operation. We write $W = L \otimes \beta_C(V).$ Then

$$\psi_k(W) = k^n L^k \beta_C(V)$$

and the formal power series $\psi_t(W) = \sum_{k=1}^{\infty} t^k \psi_k(W)$ is computed as $\psi_t(W) = s_n(tL)\beta_C(V)$ by using $s_n(x) = 1^n x + 2^n x^2 + \cdots + k^n x^k + \cdots.$ By the Newton formula (cf. Chapter 13 of [16]), the differential equation

$$\psi_t(W) = -t\lambda_t'(W)/\lambda_t(W), \quad \lambda_0(W) = 1$$

determines $\lambda_t(W)$ uniquely, so we obtain $\lambda_t(W) = 1 - s_{n-1}(-tL)\beta_C(V).$

By Proposition [10] the coefficient of $\beta_C(V)$ must be $-\nu_n(L)$ for $e_C(L_0 \otimes \hat{V}).$ 

4 Calculations of the products

Let $G$ be a compact Lie group. For a real irreducible representation space $V$ of $G$, the ring $\text{End}_G(V)$ of the $G$-invariant endomorphisms is a field. We have the following three cases.

- **Real case:** $\text{End}_G(V) \cong \mathbb{R}$. We denote by $Ir_{\mathbb{R}}$ the set consisting of these representations.
- **Complex case:** $\text{End}_G(V) \cong \mathbb{C}$. We denote by $Ir_{\mathbb{C}}$ the set consisting of these representations.
- **Quaternionic case:** $\text{End}_G(V) \cong \mathbb{H}$. We denote by $Ir_{\mathbb{H}}$ the set consisting of these representations.

Then any real representation $W$ of $\Gamma$ can be naturally decomposed as

$$\bigoplus_{V \in Ir_F, F = \mathbb{R}, \mathbb{C}, \mathbb{H}} V \otimes \text{End}_G(V) \text{Hom}^G(V,W) \cong \rightarrow U.$$

Thus this decomposition still holds for a $G$-vector bundle $W$ if the action of $G$ on the base space $B$ is trivial. Hence, as shown by Segal [24], if we fix an isomorphism $\text{End}_G(V) \cong F$ for each $V \in Ir_F$ and $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have a canonical decomposition:

$$KO^*_G(pt) \cong \bigoplus \mathbb{Z}Ir_{\mathbb{R}} \otimes KO^*_q(pt) \oplus \mathbb{Z}Ir_{\mathbb{C}} \otimes K^q(pt) \oplus \mathbb{Z}Ir_{\mathbb{H}} \otimes KSp^q(pt),$$

which extends to the case $q > 0$ by the Bott periodicity theorem.

We should remark that two different choices of isomorphisms $\text{End}_G(V) \cong F$ give different isomorphisms in (9) only when $F = \mathbb{C}$. From now on we fix an isomorphism $\text{End}_G(V) \cong \mathbb{C}$ for each $V \in Ir_{\mathbb{C}}.$
For our later use we fix generators of \( KO^q(pt) \), \( K^q(pt) \) and \( KSp^q(pt) \) in the following way: Put

\[
\beta_{R,8k} = \beta(R^{8k}) \in KO^{8k}(pt) \cong \mathbb{Z},
\]
\[
\beta_{C,2k} = \beta(C^{2k}) \in K^{2k}(pt) \cong \mathbb{Z},
\]
\[
\beta_{H,8k+4} = \beta(H(R^{8k+4})) \in KSp^{8k+4}(pt) \cong \mathbb{Z}.
\]

Then \( \beta_{R,8k+4} = r c \beta_{H,8k+4} \) and \( \beta_{H,8k} = q c \beta_{R,8k} \) are also generators of \( KO^{8k+4}(pt) \cong \mathbb{Z} \) and \( KSp^{8k}(pt) \cong \mathbb{Z} \) respectively.

Let \( \beta_{R,8k+i} \in KO^{8k+i}(pt) \cong \mathbb{Z}/2 \) \( (i = 6, 7) \), \( \beta_{H,8k+i} \in KSp^{8k+i}(pt) \cong \mathbb{Z}/2 \) \( (i = 2, 3) \) be the generators.

**Remark 14.** In our convention we have the following relations (cf. [5]).

\[
c_\beta_{R,8k} = \beta_{C,8k}, \quad c_\beta_{R,8k+4} = 2 \beta_{C,8k+4},
\]
\[
r \beta_{C,8k} = 2 \beta_{R,8k}, \quad r \beta_{C,8k+4} = \beta_{R,8k+4},
\]
\[
c'_\beta_{H,8k+4} = \beta_{C,8k+4}, \quad c'_\beta_{H,8k} = 2 \beta_{C,8k},
\]
\[
t \beta_{C,2k} = (-1)^k \beta_{C,2k},
\]
\[
\beta_{R,8k+6} = r \beta_{C,8k+6}, \quad \beta_{H,8k+2} = q \beta_{C,8k+2},
\]
\[
\beta_{H,8(k+1)+2} = \beta_{R,8k+6} \beta_{H,4}, \quad \beta_{H,8(k+1)+3} = \beta_{R,8k+7} \beta_{H,4},
\]
\[
\beta_{R,8(k+1)+6} = \beta_{R,8k+7} \beta_{R,7}, \quad \beta_{H,8(k+1)+2} = \beta_{R,8k+7} \beta_{H,3}.
\]

Let \( V \in Ir_F \). If \( KF^q(pt) \) is nontrivial, we define

\[
[V]_q = [V] \beta_{F,q} \in \mathbb{Z} Ir_F \otimes KF^q(pt) \subset KO^q_G(pt),
\]
(10)

where \( KR = KO \), \( KC = K \) and \( KH = KSp \). Then by the decomposition \([9]\) any element in \( KO^q_G(pt) \) can be uniquely written as a linear combination of \([V]_q \) for \( V \in Ir_F \) with \( KF^q(pt) \) nontrivial, where \( F = R, C \) or \( H \). We should note that

\[
[V]_2 = [V] q \beta_{C,2} = [c'V] \beta_{C,2} = [c'V]_2, \quad \text{if } F = H,
\]
\[
[V]_6 = [V] r \beta_{C,6} = [cV] \beta_{C,6} = [cV]_6, \quad \text{if } F = R.
\]

We want to calculate the product

\[
KO^q_G(pt) \times KO^q_G(pt) \rightarrow KO^{p+q}_G(pt).
\]

under the decomposition \([9]\). To do so we use the canonical isomorphisms \([1]\):

\[
i + r + rc'i : \mathbb{Z} Ir_R \oplus \mathbb{Z} Ir_C \oplus \mathbb{Z} Ir_H \rightarrow RO(G),
\]
\[
ci + i + c'i : \mathbb{Z} Ir_R \oplus \mathbb{Z} Ir_C \oplus \mathbb{Z} Ir_H \rightarrow R(G),
\]
\[
qci + qi + i : \mathbb{Z} Ir_R \oplus \mathbb{Z} Ir_C \oplus \mathbb{Z} Ir_H \rightarrow RSp(G),
\]
(11)

where \( i : \mathbb{Z} Ir_F \rightarrow RF(G) \) is the inclusion and \( RR = RO, RC = R, RH = RSp \). Note that we used the fixed isomorphism \( \text{End}^G(V) \cong C \) for \( V \in Ir_C \).
Lemma 15. The complexification map $c : KO_G^{2k}(pt) \to K_G^{2k}(pt) \cong R(G)$ is given by
\[
  c[V]_{2k} = [V] + (-1)^k[tV], \quad V \in Ir_C,
\]
\[
  c[V]_4 = \begin{cases} 
    2[cV], & V \in Ir_R, \\
    [c'V], & V \in Ir_H. 
  \end{cases}
\]
Moreover the kernel of $c$ coincides with the torsion subgroup.

Proof. Let $V \in Ir_F$. When $F = R$ or $C$, this easily follows from Remark 14. When $F = H$, we use the fact that $\mathbb{C} \otimes R \Delta_4 \cong \Lambda^4(\mathbb{C}^2) \otimes \mathbb{C} H$ as $\mathbb{Z}/2$-graded $\mathbb{C} \otimes R Cl(\mathbb{R}^4) \otimes \mathbb{R} H$-modules. (Compare the complex Spin(4) $\times$ Sp(1)-modules coming from the inclusions Spin(4) $\subset$ Cl(\mathbb{R}^4) and Sp(1) $\subset$ H.) Then we have $c([V] \beta_{4,4}) = [c'V] \beta_{4,4}$. The second isomorphism in (11) guarantees the kernel of $c$ is in the torsion subgroup.

This lemma asserts when $p + q \equiv 0 \pmod{4}$ the product can be computed from the complex representation ring of $G$. The other cases can be computed as follows:

We first note that the definition (10) obviously extends to any element $[V] \in R F(G)$ by using the tensor product over $F$, thus we get an element $[V]_q \in KO_G^{q}(pt)$, if $KF^q(pt)$ is nontrivial. To decompose $[V]_q$ according to (9) we again use the isomorphisms (11).

Lemma 16. Let $[V] \in R F(G)$ be decomposed as
\[
  [V] = \begin{cases} 
    V_R + rV_C + rd'V_H, & F = R, \\
    cV_R + tV_C + d'V_H, & F = C, \\
    qV_R + qV_C + V_H, & F = H, 
  \end{cases}
\]
where $V_R \in Z Ir_R$, $V_C, V_C' \in Z Ir_C$, $V_H \in Z Ir_H$. Then we have
\[
  [V]_2 = \begin{cases} 
    [V_C]_2 - [V_C']_2 + [V_H]_2, & F = C, \\
    [V_H]_2, & F = H, 
  \end{cases}
\]
\[
  [V]_3 = [V_H]_3, \quad F = H,
\]
\[
  [V]_6 = \begin{cases} 
    [V_R]_6, & F = R, \\
    [V_R]_6 + [V_C]_6 - [V_C']_6, & F = C, 
  \end{cases}
\]
\[
  [V]_7 = [V_R]_7, \quad F = R.
\]

Proof. From Remark 4 we see
\[
  [tV_C'] \beta_{C,2k} = t([tV_C'] \beta_{C,2k}) = (-1)^k[V_C']_{2k} \in KO_G^{2k}(pt),
\]
\[
  [c'V_H] \beta_{C,2} = [V_H] q \beta_{C,2} = [V_H]_2 \in KO_G^{2}(pt),
\]
\[
  [cV_R] \beta_{C,6} = [V_R] r \beta_{C,6} = [V_R]_6 \in KO_G^{6}(pt).
\]
Moreover since $c' : Ksp^2(\text{pt}) \to K^2(\text{pt})$, $c : KO^6(\text{pt}) \to K^6(\text{pt})$ are obviously the zero maps, we have

\[ [qV_C] \beta_{H,2} = [V_C] c' \beta_{H,2} = 0, \]
\[ [rV_C] \beta_{R,6} = [V_C] c \beta_{R,6} = 0. \]

Then the identities for $[V_2]$ and $[V_6]$ immediately follow. The other cases are trivial.

**Lemma 17.** Let $V_0 \in Ir_{F_0}$ and $V_1 \in Ir_{F_1}$. Then the products $[V_0][V_1]_2$, $[V_0][V_1]_6$, $[V_0][V_1]_2$ and $[V_0][V_1]_6$ are given by the tables below, where the symbol $\otimes$ in the tables means the tensor product over $C$.

**Table 1: $[V_0][V_1]_2$**

|          | $F_1 = C$ | $F_1 = H$ |
|----------|-----------|-----------|
| $F_0 = R$ | $cV_0 \otimes V_1$ | $cV_0 \otimes c'V_1$ |
| $F_0 = C$ | $[V_0 \otimes V_1]_2$ | $[V_0 \otimes cV_1]_2$ |
| $F_0 = H$ | $c'V_0 \otimes V_1$ | $c'V_0 \otimes c'V_1$ |

**Table 2: $[V_0][V_1]_6$**

|          | $F_1 = R$ | $F_1 = C$ |
|----------|-----------|-----------|
| $F_0 = R$ | $cV_0 \otimes cV_1$ | $cV_0 \otimes V_1$ |
| $F_0 = C$ | $[V_0 \otimes cV_1]_6$ | $[V_0 \otimes V_1]_6$ |
| $F_0 = H$ | $c'V_0 \otimes cV_1$ | $c'V_0 \otimes V_1$ |

**Table 3: $[V_0][V_1]_2$**

|          | $F_1 = C$ | $F_1 = H$ |
|----------|-----------|-----------|
| $F_0 = R$ | $2[cV_0 \otimes V_1]$ | $0$ |
| $F_0 = C$ | $[V_0 \otimes V_1]_6 + [tV_0 \otimes V_1]_6$ | $0$ |
| $F_0 = H$ | $[c'V_0 \otimes V_1]_6$ | $[c'V_0 \otimes c'V_1]_6$ |

**Proof.** Let $F_0 = F_1 = R$. Then

\[ [V_0][V_1]_6 = [V_0 \otimes R V_1] r \beta_{C,6} = [cV_0 \otimes C cV_1] \beta_{C,6} = [cV_0 \otimes C cV_1]_6. \]
The other cases in the tables of $[V_0]_0[V_1]_2$ and $[V_0]_0[V_1]_6$ also follow from the relations $\beta_{R,6} = r\beta_{C,6}$ and $\beta_{H,2} = q\beta_{C,2}$ in Remark 4.

We next consider the tables of $[V_0]_4[V_1]_2$ and $[V_0]_4[V_1]_6$. Now fix the canonical isomorphism $Cl(R^4) \otimes_R Cl(R^{4k+2}) \cong Cl(R^{4k+6})$ as $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras [5].

Let $F_0 = R$. Counting dimension shows

$$\Delta_4 \otimes_R \Lambda^*(C^{2k+1}) \cong \Lambda^*(C^{2k+3}) + \Lambda^*(C^{2k+3})$$

as $\mathbb{Z}/2$-graded $Cl(R^{4k+6}) \otimes_R C$-modules. Thus, for instance if $F_1 = R$, we obtain

$$[V_0]_4[V_1]_6 = [cV_0 \otimes C V_1] 2r\beta_{C,10} = 0.$$

Let $F_0 = H$. If we consider $\Delta_4$ as a $\mathbb{Z}/2$-graded quaternionic $Cl(R^4)$-module, we see

$$\Delta_4 \otimes_R \Lambda^*(C^{2k+1}) \cong H \otimes_C \Lambda^*(C^{2k+3})$$

as $\mathbb{Z}/2$-graded $H \otimes_R Cl(R^{4k+6}) \otimes_R C$-modules. (Compare the complex $Sp(1) \times \text{Spin}(4k + 6)$-modules coming from the inclusions $Sp(1) \subset H$ and $\text{Spin}(4k + 6) \subset Cl(R^{4k+6})$.) Thus, for instance if $F_1 = R$, we obtain

$$[V_0]_4[V_1]_6 = [V_0 \otimes R V_1] q\beta_{C,10} = [c'V_0 \otimes C cV_1] \beta_{C,10} = [c'V_0 \otimes C cV_1]_{10},$$

where we induce an $H$-action on $V_0 \otimes_R V_1$ from that on $V_0$.

Let $F_0 = C$. In this case we have

$$\Lambda^*((C')^2) \otimes_R \Lambda^*(C^{2k+1}) \cong C' \otimes_R \Lambda^*(R^2) \otimes_R \Lambda^*(C^{2k+1})$$

$$\cong C' \otimes_R \Lambda^*(C^{2k+3}) \cong \Lambda^*(C^{2k+3}) + t\Lambda^*(C^{2k+3})$$

as $\mathbb{Z}/2$-graded $C' \otimes_R Cl(R^{4k+6}) \otimes_R C$-modules, where $C'$ is a copy of $C$, and where $C$ acts on $t\Lambda^*(C^{2k+3})$ via scalar multiplication on $C$, while $C'$ acts on $t\Lambda^*(C^{2k+3})$ via complex conjugation. Thus if we identify $C'$ with $C$ by the complex conjugation then the map

$$\alpha : (C')^2 \oplus C^{2k+1} \to \text{Hom}(\Lambda^{\text{even}}(C^{2k+3}), \Lambda^{\text{odd}}(C^{2k+3}))$$
defined in Section 2 is complex linear. However the complex conjugation on
\((C')^2\) is homotopic to the identity in \(SO(\text{(C')})\). (This fact can be also used
to show \(t\beta_{C,-4} = \beta_{C,-4}\).) Hence, for instance if \(F_1 = C\), we have

\[
[V_0]_4[V_1]_{4k+2} = [V_0 \otimes C V_1]_4 \beta_{C,4k+6} + [tV_0 \otimes C V_1] \beta_{C,4k+6} = [V_0 \otimes C V_1]_{4k+6} + [tV_0 \otimes C V_1]_{4k+6}.
\]

The other cases also follow from the relations \(\beta_{R,6} = r\beta_{C,6}\) and \(\beta_{H,2} = q\beta_{C,2}\).

The only remaining nontrivial case to be calculated is

\[
KO_G^{4k+3}(pt) \times KO_G^{q}(pt) \to KO_G^{4k+3+q}(pt).
\]

**Lemma 18.** Let \(V_0 \in Ir_{F_0}\) and \(V_1 \in Ir_{F_1}\). Then the products \([V_0]_3[V_1]_q\) and \([V_0]_7[V_1]_q\) are given by the tables below, where if \(F_0 = F_1 = H\) then \(V_0 \otimes_H V_1\) is regarded as a real representation of \(G\), and if \((F_0,F_1) = (H,R)\) or \((F_0,F_1) = (R,H)\) then \(V_0 \otimes_R V_1\) is regarded as a quaternionic representation of \(G\).

**Table 5: \([V_0]_3[V_1]_q\)**

| \(F_0 = H\) | \(F_1 = R\) | \(F_1 = C\) | \(F_1 = H\) |
|---|---|---|---|
| \(q = 0\) | \([V_0]_3[V_1]_3\) | \([V_0]_3[V_1]_{r}V_1\] | \([V_0]_3[V_1]_{r}V_1\] |
| \(q = 3\) | 0 | 0 | \([V_0]_3[V_1]_{r}V_1\] |
| \(q = 4\) | 0 | 0 | \([V_0]_3[V_1]_{r}V_1\] |
| \(q = 7\) | \([V_0]_3[V_1]_3\) | 0 | 0 |

**Table 6: \([V_0]_7[V_1]_q\)**

| \(F_0 = R\) | \(F_1 = R\) | \(F_1 = C\) | \(F_1 = H\) |
|---|---|---|---|
| \(q = 0\) | \([V_0]_7[V_1]_7\) | \([V_0]_7[V_1]_{r}V_1\] | \([V_0]_7[V_1]_{r}V_1\] |
| \(q = 3\) | 0 | 0 | \([V_0]_7[V_1]_{r}V_1\] |
| \(q = 4\) | 0 | 0 | \([V_0]_7[V_1]_{r}V_1\] |
| \(q = 7\) | \([V_0]_7[V_1]_6\) | 0 | 0 |

**Proof.** We prove some in the tables and the others will be left to the reader.

Let \(F_0 = R\). Since \(\beta_{R,7} \beta_{R,7} = \beta_{R,8+6}, \beta_{R,7} \beta_{H,4} = \beta_{H,8+3}, \) and \(\beta_{R,7} \beta_{H,3} = \beta_{H,8+2}\) by Remark 14 if \(F_1 = R\) then we have \([V_0]_7[V_1]_7 = [V_0 \otimes R V_1]_6, \) and if \(F_1 = H\) then we have \([V_0]_7[V_1]_4 = [V_0 \otimes R V_1]_3, [V_0]_7[V_1]_3 = [V_0 \otimes R V_1]_2. \)
Let $F_0 = H$. We fix the canonical isomorphism $Cl(R^4) \otimes_R Cl(R^4) \cong Cl(R^8)$ as $\mathbb{Z}/2$-graded real algebras and consider $\Delta_4$ as a $\mathbb{Z}/2$-graded quaternionic $Cl(R^4)$-module. Then we have

$$\Delta_4 \otimes_R \Delta_4 \cong H \otimes_R \Delta_8$$

as $\mathbb{Z}/2$-graded $H \otimes_R Cl(R^8) \otimes_R H$-modules, where $H$ in the right hand side is regarded as the quaternionic bimodule by multiplication from the both sides. Thus if $F_1 = H$, we have $[V_0][V_1]_4 = [V_0 \otimes_H V_1]_7$ and $[V_0][V_1]_3 = [V_0 \otimes_H V_1]_6$ since $\beta_{R,7} \beta_{R,7} = \beta_{R,8+6}$.

## 5 Real and symplectic $K$-theory

In this section we briefly review Real and symplectic $K$-theory. We also discuss natural maps between other $K$-theories, which will be used to construct spin $\text{Pin}^{-}(2)$-vector bundles later.

Let $B$ be a compact Real space, i.e. a compact space with involution. By the definition in [2] a Real vector bundle $V$ is a complex vector bundle $V$ together with an anti-linear lift of the involution with its square equal to $1$. We let $KR(B)$ be the Grothendieck group of Real vector bundles over $B$. According to Dupont [11] a symplectic vector bundle $V$ is defined to be a complex vector bundle $V$ together with an anti-linear lift of the involution with its square equal to $-1$. We let $Ksp(B)$ be the Grothendieck group of symplectic vector bundles over $B$, so that $KM(B) = KR(B) \oplus Ksp(B)$ becomes a $\mathbb{Z}/2$-graded ring in the obvious way. These definitions then extend to a locally compact Real space $B$ in the usual way. In the following we denote by $-1$ the involution on $B$ and regard $B$ as a $\{\pm 1\}$-space.

1. Let $V$ be a quaternionic $\{\pm 1\}$-vector bundle over $B$. Then the simultaneous action of $-1$ and $j \in H$ on $V$ defines a symplectic structure on $V$. We denote by $\sigma V$ this symplectic vector bundle. Thus we obtain a natural homomorphism $\sigma : KSp_{\{\pm 1\}}(B) \rightarrow Ksp(B)$.

2. Let $V$ be a symplectic vector bundle $V$ over $B$. From the symplectic action and complex scalar multiplication on $V$ we obtain a real $\text{Pin}^{-}(2)$-action on $V$. We denote by $\vartheta V$ this real $\text{Pin}^{-}(2)$-vector bundle. Then we have a natural homomorphism $\vartheta : Ksp(B) \rightarrow KO_{\text{Pin}^{-}(2)}(B)$. Let $\tau : \text{Pin}^{-}(2) \rightarrow \{\pm 1\} \times Sp(1)$ be the inclusion defined by $\tau(t) = (1, t)$ and $\tau(j) = (-1, j)$. Then it is obvious $\tau^* sV = \vartheta \sigma V$ for a quaternionic $\{\pm 1\}$-vector bundle $V$.

3. Let $V$ be a symplectic vector bundle over $B$. If we ignore the symplectic structure on $V$, we get a complex vector bundle over $B$, which we write as $\kappa V$. Then we obtain a natural homomorphism $\kappa : Ksp(B) \rightarrow K(B)$. 

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**Remark 19.** The groups $KSp(B)$ and $Ksp(B)$ are similar in notation. However the latter is defined only for a Real space $B$, and when the involution on $B$ is trivial $KSp(B)$ is naturally isomorphic to $Ksp(B)$.

Let $R^{p,q} = R^p \oplus \tilde{R}^q$. We suppose $p \equiv q \pmod 8$ and denote by $\beta_C(R^{p,q}) \in KR(R^{p,q})$ the Bott class defined in [4]. Then the argument in [4] Theorem 6.3 shows that multiplication by $\beta_C(R^{p,q})$ induces an isomorphism $KM(B) \rightarrow KM(B \times R^{p,q})$.

Now we recall some results in [11]. Let $\beta_H(\tilde{R}^{8k+4}) \in KSp(\tilde{R}^{8k+4})$ be the Bott class, so $\sigma \beta_H(\tilde{R}^{8k+4}) \in Ksp(\tilde{R}^{8k+4})$. Then by the uniqueness (up to grading) of the irreducible $\mathbb{Z}/2$-graded module for the Real Clifford algebra associated to $\tilde{R}^{8k,0} = \tilde{R}^{8k}$ (see [2] Section 4), we see the square $(\sigma \beta_H(\tilde{R}^{8k+4}))^2$ coincides with $\beta_C(R^{16k+8,0})$. Thus multiplication by $\sigma \beta_H(\tilde{R}^{8k+4})$ induces isomorphisms

$$KR(B) \rightarrow Ksp(B \times \tilde{R}^{8k+4}), \quad Ksp(B) \rightarrow KR(B \times \tilde{R}^{8k+4}).$$

Combining them with the periodicity isomorphisms

$$KR(B \times \tilde{R}^q) \cong KR(B \times R^{8m,8m-q}) \cong KR(B \times R^{8m-q}) \cong KO^q(B),$$

we have the following lemma [11]:

**Lemma 20.** The group $Ksp(\tilde{R}^n)$ is non-trivial only in the cases

$$Ksp(\tilde{R}^{8k}) \cong Ksp(\tilde{R}^{8k+4}) \cong \mathbb{Z}, \quad Ksp(\tilde{R}^{8k+2}) \cong Ksp(\tilde{R}^{8k+3}) \cong \mathbb{Z}/2\mathbb{Z},$$

and they are respectively generated by

$$[H] \beta_C(R^{8k,0}), \sigma \beta_H(\tilde{R}^{8k+4}), i_{8k+2} \sigma \beta_H(\tilde{R}^{8k+4}), i_{8k+3} \sigma \beta_H(\tilde{R}^{8k+4}),$$

where $i_q : \tilde{R}^q \rightarrow 0 \rightarrow \tilde{R}^{8k+4}$ ($q = 8k + 2, 8k + 3$) is the inclusion. In addition, we have

$$\kappa \sigma \beta_H(\tilde{R}^{8k+4}) = \beta_{C,8k+4}.$$

### 6 Irreducible representations of $\Gamma$

In this section we list up some properties of real irreducible representations of the group $\Gamma$ obtained as a central extension of $\text{Pin}^-(2)$ by the group of order $2$.

**Definition 21.** Let $C_4 = \langle j \rangle$ be the cyclic group of order 4 generated by $j$. We define an action of $C_4$ on $U(1)$ by $j(t) = t^{-1}$ and denote by $G$ the semiproduct $C_4 \ltimes U(1)$. Then the quotient group $G/\{1, (j^2, -1)\}$ is isomorphic to $\text{Pin}^-(2)$. 

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We have the exact sequence

\[ 1 \to U(1) \to \Gamma \to C_4 \to 1 \]

and the center of \( \Gamma \) consists of the four elements \( \{\pm 1, (j^2, \pm 1)\} \). We write \( t = (1, t), j^p = (j^p, 1) \) below.

**Lemma 22.** The real irreducible representations of \( \Gamma \) are classified as follows:

1. \( \mathbb{R} \): the trivial 1-dimensional representation, on which we have
   \[ t = j = \text{id} \].

2. \( \tilde{\mathbb{R}} \): the non-trivial 1-dimensional representation, on which we have
   \[ t = \text{id}, \quad j = -\text{id} \].

3. \( C_0 \): the 2-dimensional irreducible representation of \( C_4 \), on which we have
   \[ t = \text{id}, \quad j = \sqrt{-1}\text{id} \],
   where we identify \( \mathbb{C} \) with the complex numbers.

4. \( D_m \) (\( m \geq 1 \)): the 2-dimensional representation that satisfies
   \[ \text{tr}(t|D_m) = t^m + t^{-m}, \quad \text{tr}(j|D_m) = 0, \quad j^2 = \text{id} \].

5. \( H_m \) (\( m \geq 1 \)): the 4-dimensional representation that satisfies
   \[ \text{tr}(t|H_m) = 2(t^m + t^{-m}), \quad \text{tr}(j|H_m) = 0, \quad j^2 = -\text{id} \].

For \( m = 0 \) we define \( D_0 \) and \( H_0 \) to be

\[ D_0 := \mathbb{R} \oplus \tilde{\mathbb{R}}, \quad H_0 := C_0 \oplus C_0 \]

so that the defining properties of \( C_m \) and \( H_m \) are still satisfied.

**Lemma 23.**

1. The real irreducible representations of \( \text{Pin}^{-}(2) \) are \( \mathbb{R}, \tilde{\mathbb{R}}, D_{2n} \) and \( H_{2n-1} \) for \( n \geq 1 \).

2. The real irreducible representations of \( C_4 \) are \( \mathbb{R}, \tilde{\mathbb{R}}, \) and \( C_0 \).

**Proof.** The latter part is obvious. The former part is a consequence of the following observation. The irreducible representation of \( \Gamma \) is that of \( \text{Pin}^{-}(2) \) if and only if the action of \( j^2 \) coincides with that of \( -1 \in U(1) \).

For an irreducible representation space \( V \), the ring \( \text{End}^\Gamma(V) \) of the \( \Gamma \)-invariant endomorphisms is a field. We have the following three cases.
• Real case: \( \text{End}^\Gamma(V) \cong \mathbb{R} \) for \( V = \mathbb{R}, \tilde{\mathbb{R}} \) and \( D_k \) \((k \geq 1)\). We denote by \( I_{R\mathbb{R}} \) the set consisting of these representations.

• Complex case: \( \text{End}^\Gamma(V) \cong \mathbb{C} \) for \( V = \mathbb{C}_0 \). We denote by \( I_{R\mathbb{C}} \) the set consisting of this representation.

• Quaternionic case: \( \text{End}^\Gamma(V) \cong \mathbb{H} \) for \( V = \mathbb{H}_k \) \((k \geq 1)\). We denote by \( I_{R\mathbb{H}} \) the set consisting of these representations.

To obtain the decomposition (9) for \( \Gamma \) we need to fix an isomorphism \( \text{End}^{\Gamma}(V) \cong \mathbb{C} \) for each \( V \in I_{R\mathbb{C}} \). In our case the only element in \( I_{R\mathbb{C}} \) is \( \mathbb{C}_0 \).

To fix an isomorphism \( \text{End}^{\Gamma}(\mathbb{C}_0) \cong \mathbb{C} \) we use the identification of \( \mathbb{C}_0 \) with the complex numbers in the definition of \( \mathbb{C}_0 \). Then we have

**Corollary 24.** There are canonical isomorphisms

\[
KO_{\Gamma}^0(\text{pt}) \cong \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}[\tilde{\mathbb{R}}] \oplus \bigoplus_{m=1}^{\infty} \mathbb{Z}[D_m] \oplus \mathbb{Z}[\mathbb{C}_0] \oplus \bigoplus_{n=1}^{\infty} \mathbb{Z}[\mathbb{H}_n],
\]

\[
KO_{\Gamma}^2(\text{pt}) \cong \mathbb{Z}[\mathbb{C}_0]^2 \oplus \bigoplus_{m=1}^{\infty} \mathbb{Z}/2[\mathbb{H}_m],
\]

\[
KO_{\Gamma}^4(\text{pt}) \cong \mathbb{Z}[\mathbb{R}]^4 \oplus \mathbb{Z}[\tilde{\mathbb{R}}]^4 \oplus \bigoplus_{m=1}^{\infty} \mathbb{Z}[D_m]^4 \oplus \mathbb{Z}[\mathbb{C}_0]^4 \oplus \bigoplus_{n=1}^{\infty} \mathbb{Z}[\mathbb{H}_n]^4,
\]

\[
KO_{\Gamma}^6(\text{pt}) \cong \mathbb{Z}/2[\mathbb{R}]^6 \oplus \mathbb{Z}/2[\tilde{\mathbb{R}}]^6 \oplus \bigoplus_{m=1}^{\infty} \mathbb{Z}/2[D_m]^6 \oplus \mathbb{Z}[\mathbb{C}_0]^6.
\]

The following first two tables describe the tensor product of the irreducible representations over \( \mathbb{R} \) or \( \mathbb{C} \), from which we use Lemma 16, 17 to get the table of the product of elements in \( KO_{\Gamma}^*(\text{pt}) \) below.

**Table 7: Tensor products over \( \mathbb{R} \)**

| \( V_0 \) | \( V_1 \) | \( V_0 \otimes_{\mathbb{R}} V_1 \) | \( rC_0 \) | \( D_m \) | \( rC_0 \) | \( rC_0 \) | \( r\tilde{C}_0 \) | \( D_m \) |
|---|---|---|---|---|---|---|---|---|
| \( \mathbb{R} \) | \( \tilde{\mathbb{R}} \) | \( \mathbb{R} \) | \( \tilde{\mathbb{R}} \) | \( \mathbb{R} \) | \( \mathbb{R} \) | | | |
| \( \mathbb{R} \) | \( rC_0 \) | \( rC_0 \) | \( \tilde{\mathbb{R}} \) | \( D_m \) | \( D_m \) | \( D_m \) | \( D_m \) | \( D_m \) |
| \( \mathbb{R} \) | \( \tilde{\mathbb{R}} \) | \( D_m \) | \( D_m \) | \( \tilde{\mathbb{R}} \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) |
| \( \mathbb{R} \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) | \( r\tilde{C}_0 \) |
| \( rC_0 \) | \( rC_0 \) | \( 2(\mathbb{R} \oplus \tilde{\mathbb{R}}) \) | \( 4D_m \) | \( 4D_m \) | \( 4D_m \) | \( 4D_m \) | \( 4D_m \) | \( 4D_m \) |

| \( r\tilde{C}_0 \) | \( D_m \) | \( D_m \) | | | | | | |
| \( r\tilde{C}_0 \) | \( \tilde{\mathbb{R}} \) | \( \mathbb{R} \) | | | | | | |
| \( r\tilde{C}_0 \) | \( \mathbb{R} \) | | | | | | | |
| \( r\tilde{C}_0 \) | | | | | | | | |

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Table 8: Tensor products over $\mathbb{C}$

| $V_0$ | $V_1$ | $V_0 \otimes_{\mathbb{C}} V_1$ |
|-------|-------|-----------------------------|
| $cR$  | $cR$  | $cR$                        |
| $c\tilde{R}$ | $C_0$ | $tC_0$                      |
| $c\tilde{R}$ | $tC_0$ | $C_0$                       |
| $cR$  | $cD_m$ | $cD_m$                      |
| $c\tilde{R}$ | $c' H_m$ | $c' H_m$                   |
| $C_0$  | $C_0$  | $cR$                        |
| $tC_0$ | $C_0$  | $cR$                        |

7 Calculations of the Euler classes

Let $C_{(i)}$ be the complex representation of $C_4$ that satisfies $tr(j|C_{(i)}) = tr(j|C_1)^i$. Then the irreducible complex representation of $C_4$ consists of $C_{(0)}$, $C_{(1)}$, $C_{(2)}$, and $C_{(3)}$.

**Definition 25.** We put a spin $C_4$-structure on $\tilde{R}^{2m} = \tilde{R}^{\otimes 2m}$ and a spin $\text{Pin}^{-}(2)$-structure on $H^m_1 = H^m_1$ in the following way:

- The representation $C_{(1)}$ gives a square root of the action $C_4$ on $C_{(2)}$, which induces a spin $C_4$-structure on $rC_{(2)} = \tilde{R}^2$.
- The $\text{Pin}^{-}(2)$-action on $H_1$ factors through the homomorphism $\text{Pin}^{-}(2) \subset Sp(1) = \text{Spin}(3) \to SO(4)$. Hence the lift to Spin(4) gives a spin $\text{Pin}^{-}(2)$-structure on $H_1$.
- We put a spin $C_4$-structure on $\tilde{R}^{2m}$ and a spin $\text{Pin}^{-}(2)$-structure on $H^m_1$ as the direct sums of $\tilde{R}^2$ and $H_1$ respectively.

Then by the formulae (7), (8) we have

$c(\beta(\tilde{R}^2)) = [C_{(3)}] \beta_C(C_{(2)}), \quad c(e(\tilde{R}^2)) = [C_{(3)}] - [C_{(1)}].$

Since $KO^2_{C_4}(pt) \cong \mathbb{Z}[C_0]_2$, $K^2_{C_4}(pt) \cong R(C_4)$ and $c[C_0]_2 = [C_{(1)}] - [C_{(3)}]$ by Lemma 15, we use Table 9 to obtain

**Lemma 26.** Consider the above spin $C_4$-structure on $\tilde{R}^{2m}$. Then we have
Table 9: Table of $[V_0]_p[V_1]_q$

| $[V_0]_p$ | $[V_1]_q$ | $[V_0]_p[V_1]_q$ | $[R]_4$ | $[R]_4$ | 4$[R]$ |
|----------|----------|-----------------|--------|--------|---------|
| $[R]$    | $[C_0]_2$ | $[C_0]_2$       | $[R]_4$ | $[R]_4$ | 4$[R]$  |
| $[R]$    | $[C_0]_2$ | $- [C_0]_2$     | $[R]_4$ | $[R]_4$ | 4$[R]$  |
| $[C_0]$  | $[C_0]_2$ | 0               | $[R]_4$ | $[C_0]_2$ | 2$[C_0]_6$ |
| $[D_n]$  | $[C_0]_2$ | $[H_m]_2$       | $[R]_4$ | $[C_0]_2$ | $-2[C_0]_6$ |
| $[H_n]$  | $[C_0]_2$ | 0               | $[C_0]_4$ | $[C_0]_2$ | 0       |
| $[R]$    | $[H_m]_2$ | $[H_m]_2$       | $[D_n]_4$ | $[C_0]_2$ | 0       |
| $[R]$    | $[H_m]_2$ | $[H_m]_2$       | $[H_n]_4$ | $[C_0]_2$ | 0       |
| $[C_0]$  | $[H_m]_2$ | 0               | $[R]_4$ | $[C_0]_6$ | 2$[C_0]_2$ |
| $[D_n]$  | $[H_m]_2$ | $[H_{n-m}]_2 + [H_{n+m}]_2$ | $[R]_4$ | $[C_0]_6$ | $-2[C_0]_2$ |
| $[C_0]_2$ | $[C_0]_2$ | $[R]_4 - [R]_4$ | $[C_0]_4$ | $[C_0]_6$ | 0       |
| $[R]_4$  | $[H_m]_4$ | $2[H_m]$       | $[D_n]_4$ | $[C_0]_6$ | 0       |
| $[R]_4$  | $[H_m]_4$ | $2[H_m]$       | $[H_n]_4$ | $[C_0]_6$ | 0       |

$c(\beta(\tilde{R}^{2m})) = [C_{(3)m}] \beta C(C^{(2m)}_2), c(e(\tilde{R}^{2m})) = ([C_{(3)}] - [C_{(1)}])^m$ and

$$e(\tilde{R}^{2m}) = \begin{cases} 
(-1)^{m/2}2^{m-1-1}(R)_{2m} - (\tilde{R})_{2m} & m \equiv 0 \mod 4, \\
(-1)^{(m+1)/2}2^{m-1}[-C_0]_{2m} & m \equiv 1 \mod 4, \\
(-1)^{m/2}2^{m-2}(R)_{2m} - (\tilde{R})_{2m} & m \equiv 2 \mod 4, \\
(-1)^{(m+1)/2}2^{m-1}[-C_0]_{2m} & m \equiv 3 \mod 4.
\end{cases}$$

On the other hand the complexification of $e(H_1)$ is

$$c(e(H_1)) = [\Lambda^{even}(c'H_1)] - [\Lambda^{odd}(c'H_1)] = 2[C_{(0)}] - [c'H_1],$$

by (3) and $c : KO^4_{Pin^{-}(2)}(pt) \rightarrow K^4_{Pin^{-}(2)}(pt)$ is injective by Lemma 15, so we obtain

**Lemma 27.** Consider the above spin $Pin^{-}(2)$-structure on $\tilde{H}^m$. Then we have $c(e(H_1)) = 2[C_{(0)}] - [c'H_1]$ and

$$e(H_1) = [R]_4 - [H_1]_4,$$

$$e(H_1)^{2m} = a_m[R] + b_m[\tilde{R}] + \sum_{1 \leq t < 2m, even} \binom{4m}{2m - t} [D_t]$$

$$- \frac{1}{2} \sum_{1 \leq t < 2m, odd} \binom{4m}{2m - t} [H_t] + [D_{2m}],$$

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where the integers \(a_m, b_m\) satisfy \(a_m - b_m = 2^{2m}, a_m + b_m = \left(\frac{4m}{2m}\right)\).

Let \(\tilde{H}^+(k)\) be the quaternionic \(\{\pm\}\)-vector bundle over the \(\{\pm\}\)-sphere \(\tilde{R}^{8k+4} \cup \{\infty\}\) which is the extension of the product quaternionic \(\{\pm\}\)-vector bundle \(\tilde{R}^{8k+4} \times \Delta^+_{8k+4}\) obtained by using the Clifford multiplication \(c\) as the trivialization of \(\tilde{R}^{8k+4} \times \Delta^+_{8k+4}\) over \(\tilde{R}^{8k+4} \setminus \{0\}\). If we exchange \(\Delta^+_{8k+4}\) for \(\Delta^-_{8k+4}\) and replace \(c\) by \(c^{-1}\) then we obtain another quaternionic \(\{\pm\}\)-vector bundle \(\tilde{H}^-(k)\). These bundles then satisfy

\[
[\tilde{H}^+(k)] = [\Delta^-_{8k+4}] + \beta_8(\tilde{R}^{8k+4}), \quad [\tilde{H}^-(k)] = [\Delta^+_{8k+4}] - \beta_8(\tilde{R}^{8k+4})
\]

as elements in \(KSp_{\{\pm\}}(\tilde{R}^{8k+4} \cup \{\infty\})\) (see Section 3). As shown in Section 3 we have a natural spin \(\{\pm\} \times Sp(1)\)-structure on \(s\tilde{H}^\pm(k)\), so we have a spin \(\text{Pin}^-(2)\)-structure on \(\partial s\tilde{H}^\pm(k) = \tau^* s\tilde{H}^\pm(k)\). In the following the Euler class \(e(\partial s\tilde{H}^\pm(k)) \in KO_{\text{Pin}^-(2)}(\tilde{R}^{8k+4} \cup \{\infty\})\) will be also denoted \(e(\tilde{H}^\pm(k))\) for simplicity.

**Remark 28.** In the above construction the group \(\text{Pin}^-(2)\) was defined as the subgroup of \(\{\pm\} \times Sp(1)\) generated by \(\{1\} \times U(1)\) and \((-1, j)\). However we also used the symbol \(j \in \text{Pin}^-(2)\) instead of \((-1, j)\). To avoid confusion in the rest of this section we denote by \(j_H\) the element \(j \in H\), while we denote by \(j_4\) the element \((-1, j_H) \in \text{Pin}^-(2)\). Thus \(C_4\) is the subgroup of \(\text{Pin}^-(2)\) generated by \(j_4\).

**Lemma 29.** Under the canonical decomposition \(KO_{\text{Pin}^-(2)}(\tilde{R}^{8k+4} \cup \{\infty\}) \cong KO_{\text{Pin}^-(2)}(\{\infty\}) \oplus KO_{\text{Pin}^-(2)}(\tilde{R}^{8k+4})\) we have

\[
e(\tilde{H}^+(k)) = (-[\tilde{R}]e(H_1))^2k + \gamma_k \beta(\tilde{R}^{8k+4}),
\]

\[
e(\tilde{H}^-(k)) = (-e(H_1))^2k - \gamma_k \beta(\tilde{R}^{8k+4}),
\]

where \(\gamma_0 = -[\tilde{R}]\), and \(\gamma_k \in KO_{\text{Pin}^-(2)}(\text{pt}) (k \geq 1)\) is in the subgroup generated by \([R]_4 + [\tilde{R}]_4, [C_0]_4, [D_{2m}]_4\) and \([H_{2m-1}]_4 (m \geq 1)\).

**Proof.** We first consider the action of the subgroup \(C_4\) in \(\text{Pin}^-(2)\). The action of \(j_H \in H\) then induces a complex \(\{\pm\}\)-structure on the quaternionic \(\{\pm\}\)-structure on \(\tilde{H}^+(k)\), so we denote by \(e''\tilde{H}^+(k)\) this complex \(\{\pm\}\)-vector bundle. Let \(i_{C_4} : C_4 \to \{\pm\} \times U(1)\) be the inclusion derived from the inclusion \(\tau : \text{Pin}^-(2) \to \{\pm\} \times Sp(1)\). Since the restriction \(\tau_\ell \Delta^+_{8k+4}\) of the scalars to \(C = R \oplus jR\) is isomorphic to \(\Lambda^* (C^{4k+2})\) as \(Z/2\)-graded complex \(Cl(\tilde{R}^{8k+4})\)-modules, we see \(i_{C_4} e''\tilde{H}^+(k)\) is isomorphic to \(C_{(1)} \otimes \hat{V}^+\) as complex \(C_4\)-vector bundles, where \(V = C^{(2)}_{(2)}\) and \(\hat{V}^+\) is the complex \(C_4\)-vector
bundle defined in Section 3. If we denote by \( j_{C_4} : C_4 \to \text{Pin}^{-}(2) \) the inclusion, then from Proposition 8 we obtain

\[
\tilde{j}_{C_4}^*(e(\vartheta \sigma \tilde{H}^{+}(k))) = [C(3)]^{2^{4k}} e(C(i_{C_4}^* bc' \tilde{H}^{+}(k))) = [C(3)]^{2^{4k}} \left( \epsilon C(C(3))^{2^{4k+1}} + \mu_{4k^2}(C(1), C(2), \cdots, C(2)) \beta C(C(2)^{2^{4k+2}}) \right) = \left( -[C(3)] e(C(1) \otimes C(3)) \right)^{2^{4k}} + \gamma_k' \beta C(C(2)^{2^{4k+2}}),
\]

where \( \gamma'_k = -1 \) and \( \gamma_k' \in R(C_4) \) is seen to satisfy \( tr(j_4|\gamma'_k) = 0 \) \( (k \geq 1) \).

We next consider the action of the subgroup \( U(1) \). Let \( i_{U(1)} : U(1) \to \{ \pm 1 \} \times U(1) \) be the inclusion. Then \( i_{U(1)}^* bc' \tilde{H}^{+}(k) \) is isomorphic to \( C(k(t) \otimes \tilde{V}^+) \) as complex \( U(1) \)-vector bundles, where \( V = C_0(t)^{2^{4k+2}} \). If we denote by \( j_{U(1)} : U(1) \to \text{Pin}^{-}(2) \) the inclusion, we use Proposition 8 to get

\[
\tilde{j}_{U(1)}^*(e(\vartheta \sigma \tilde{H}^{+}(k))) = [C -1(t)]^{2^{4k}} e(C(i_{U(1)}^* bc' \tilde{H}^{+}(k))) = \left( -e(C(1(t) \otimes C-1(t))) \right)^{2^{4k}} + \gamma_k'' \beta C(C_0(t)^{2^{4k+2}}),
\]

where \( \gamma'_k = -1 \) and \( \gamma''_k \in K0^4_{U(1)}(pt) \).

Thus we conclude \( \gamma_0 = -[\tilde{R}] \) by Lemma 26. When \( k \geq 1 \), Lemma 15 can be applied to show that the multiplicity of \( [\tilde{R}]_4 \) in \( \gamma_k \) should be equal to that of \( [\tilde{R}]_4 \), since \( tr(j_4|\gamma_k) = tr(j_4|\gamma'_k) = 0 \). The calculation of the Euler class \( e(\tilde{H}^{+}(k)) \) is similar. \( \square \)

If we apply Proposition 13 in the above proof, we can compute the quantity \( \gamma_k \in K0^4_{\text{Pin}^{-}(2)}(pt) \) \( (k \geq 1) \), thus the Euler classes \( e(\tilde{H}^{+}(k)) \) inductively. As mentioned in Section 3, we do not carry out it. However we need the following corollary obtained from Lemma 29 together with Table 9 and Lemma 26.

**Corollary 30.**

\[
e(\tilde{R}^2)e(\tilde{H}^\pm(0))^2 = e(\tilde{R}^2)e(H_1)^2 \in K0^2_{\Gamma}(\tilde{R}^{8k+4} \cup \{\infty\}),
\]

\[
e(\tilde{R}^2)e(\tilde{H}^\pm(k)) = e(\tilde{R}^2)e(H_1)^{2^{4k}} \in K0^2_{\Gamma}(\tilde{R}^{8k+4} \cup \{\infty\}) \quad (k \geq 1).
\]

### 8 \( K0^*_\Gamma(\tilde{T}^n) \)

Let \( A \) be a \( \Gamma \)-space. Suppose the \( \Gamma \)-action is not free. We take a fixed point \( a_0 \) in \( A \). Let \( S \) be a subset of \( \{1, \cdots, n\} \). Then the \( \Gamma \)-space \( A^S = \text{Map}(S, A) \) can be embedded \( \Gamma \)-equivariantly into \( A^n \) by the map

\[
h_S : A^S \xrightarrow{\cong} \{ (x_1, \cdots, x_n) \in A^n \mid x_i = x_0 (i \notin S) \} \subset A^n.
\]
Let \( \pi_S : A^n \to A^S \) be the projection. For a \( \Gamma \)-invariant open neighborhood \( U \) around \( \alpha_0 \) in \( A \) we let \( i_S : U^S \to A^S \) be the corresponding inclusion. For the subset \( S = \{1, \ldots, m\} \) (\( m \leq n \)) we write \( A^m = A^S, \pi = \pi_S : A^n \to A^m \) and \( h = h_S : A^m \to A^n \) for simplicity.

We recall that \( \text{Pin}^{-2} \) acts on \( \hat{R} \) via the map \( \text{Pin}^{-2} \to \{\pm 1\} \), hence \( \Gamma \) acts on \( \hat{R} \) via the projection \( \Gamma \to \text{Pin}^{-2} \), by which the torus \( \tilde{T}^n = (\hat{R}/\mathbb{Z})^n \) becomes a \( \Gamma \)-space. We now take a fixed point \( t_0 \) in \( \tilde{T}^1 \) and consider the following exact sequence

\[
\to KO_{\Gamma}^S((\tilde{T}^n, \tilde{T}^{n-1}) \times \hat{R}^p) \xrightarrow{j^*} KO_{\Gamma}^S(\tilde{T}^n \times \hat{R}^p) \xrightarrow{h_*} KO_{\Gamma}^S(\tilde{T}^{n-1} \times \hat{R}^p) \to,
\]

where the first term is \( KO_{\Gamma}^S((\tilde{T}^n, \tilde{T}^{n-1}) \times (\hat{R}^p \cup \{\infty\}, \{\infty\}) \), which is identified with \( KO_{\Gamma}^S(\tilde{T}^{n-1} \times \hat{R}^{1+p}) \) by excision. Then \( j^* \) is identified with the push-forward map \( i_* : KO_{\Gamma}^S(\tilde{T}^{n-1} \times \hat{R}^{1+p}) \to KO_{\Gamma}^S(\tilde{T}^n \times \hat{R}^p) \) induced from a \( \Gamma \)-open embedding \( i : \hat{R} \to \tilde{T}^1 \) onto a neighborhood of \( t_0 \). Since \( \pi^* \) gives a right-inverse of \( h_* \), the sequence is split and we obtain an isomorphism

\[ i! + \pi^* : KO_{\Gamma}^S(\tilde{T}^{n-1} \times \hat{R}^{1+p}) \oplus KO_{\Gamma}^S(\tilde{T}^{n-1} \times \hat{R}^p) \to KO_{\Gamma}^S(\tilde{T}^n \times \hat{R}^p). \]

By induction on the cardinal number \( |S| \) of \( S \) we see

**Lemma 31.** The following map is an isomorphism:

\[
\sum_{S \subset \{1, \ldots, n\}} \pi_S(i_S)! : \bigoplus_S KO_{\Gamma}^S(\hat{R}^S \times \hat{R}^p) \to KO_{\Gamma}^S(\tilde{T}^n \times \hat{R}^p). \tag{12}
\]

**Remark 32.** It is easy to see the decomposition of (12) still holds when we replace \( KO_{\Gamma} \) by other \( K \)-groups as \( KSp(\pm 1), KS p, K R \) and \( K \). We shall also use these isomorphisms later.

For each subset \( S \subset \{1, \ldots, n\} \) we fix an identification \( \hat{R}^S \) with \( \hat{R}^{|S|} \). Then it induces an identification \( \hat{R}^S \) with \( \hat{R}^{|S|} \). If \( |S| \) is even, we define the Bott class \( \beta(\hat{R}^S) \in KO_{\Gamma}^{|S|}(\hat{R}^S) \subset KO_{\Gamma}^{|S|}(\hat{R}^S) \) to be \( \beta(\hat{R}^S) \) under this identification. The Bott periodicity theorem then asserts that \( KO_{\Gamma}(\hat{R}^S) \) is freely generated by the Bott class \( \beta(\hat{R}^S) \) as \( KO_{\Gamma}^{-|S|}(\text{pt}) \)-algebra.

Let \( p \) be a non-negative integer and suppose \( |S| \) is even. Then by definition we have

\[
e(\hat{R}^p)\beta(\hat{R}^S) = i^* \beta(\hat{R}^p \oplus \hat{R}^S) \in KO_{\Gamma}^{p+|S|}(\hat{R}^S) \subset KO_{\Gamma}^{p+|S|}(\tilde{T}^n),
\]

where \( i : \hat{R}^S \to \hat{R}^p \oplus \hat{R}^S \) is the inclusion. More generally if \( p + |S| \) is even then the right hand side is still defined, so we may write it as \( e(\hat{R}^p)\beta(\hat{R}^S) \).

Then one can immediately see the following product formula:

\[
e(\hat{R}^p)\beta(\hat{R}^S)e(\hat{R}^{p'})\beta(\hat{R}^{S'}) = e(\hat{R}^{p+p'+|S \cap S'|})\beta(\hat{R}^{S \cup S'}) \tag{13}
\]
for any subsets $S, S' \subset \{1, \cdots, n\}$ and integers $p, p' \geq 0$ with $|S| + p, |S'| + p'$ even.

**Remark 33.** Even if $p$ or $|S|$ is not even it is possible to define $\beta(\tilde{R}^S)$ and $e(\tilde{R}^p)$ separately so that the above two (equivariant) product formulae hold, if one uses the notion of $KO$-group with local coefficients [10]. However in this paper we do not introduce this notion, since the above definition is sufficient in our calculation.

# 9 Proof of Theorem

Let $l$ be a positive even integer. Let $V_0 = \tilde{T}^n \times \tilde{R}^x, W_0 = \tilde{T}^n \times \tilde{R}^{x+l}$ be the product bundles over $\tilde{T}^n$, and let $V_1, W_1$ be symplectic bundles over $\tilde{T}^n$. We assume the difference $[V_1] - [W_1] \in K\sp(T^n)$ satisfies the condition of Theorem 1. Suppose we have a proper $\mbox{Pin}^{-}(2)$-equivariant fiber-preserving map $\varphi : V_0 \oplus W_0 \to V_1 \oplus W_1$ which induces the identity on the base space $\tilde{T}^n$ and whose restriction $\varphi_0 : V_0 \to W_0$ is given by the standard inclusion $\tilde{R}^x \to \tilde{R}^{x+l}$.

We use the projection $\Gamma \to \mbox{Pin}^{-}(2)$ to regard $V_1, W_1$ as $\Gamma$-equivariant bundles. Let $\Gamma_0 = U(1)$ be the subgroup in $\Gamma$, so the quotient $\Gamma/\Gamma_0$ is isomorphic to $C_4$. Using the projections $\Gamma \to C_4 \to \{\pm 1\}$ we regard $V_0, W_0$ as $C_4$-equivariant bundles, and also $\Gamma$-equivariant bundles. In this setting $\varphi$ is a $\Gamma$-equivariant map.

We first calculate $KO^{C_4}_{\star}\Gamma$-degree $\alpha_{\varphi_0}$ of $\varphi_0$. After stabilization by the identity map on $\tilde{T}^n \times \tilde{R}$ if necessary, we may assume that $x$ is even. Then $i^*\beta(\tilde{R}^{x+l}) = \beta(\tilde{R}^x)e(\tilde{R}^l)$, and so we have

$$\alpha_{\varphi_0} = e(\tilde{R}^l) \in KO^{C_4}_{-4k}(\tilde{T}^n).$$

We next calculate the $KO^\Gamma_{\star}$-degree $\alpha_{\varphi} \in KO^{\Gamma}_{-4k}(\tilde{T}^n)$. In the decomposition of $K\sp(\tilde{T}^n)$ we take $\sigma_2{_{\mathbb{H}}}(\tilde{R}^S)$ as a generator of $K\sp(\tilde{R}^S) \cong \mathbb{Z}$ when $|S| \equiv 4$ (mod 8), by which we identify $a_S$ with an integer. On the other hand when $|S| \equiv 2, 3$ (mod 8) we choose an integer $a_S$ as an lift of $\sigma_S \in \mathbb{Z}/2$. Then Lemma 20 implies that, after stabilization by the identity map on some symplectic bundle over $\tilde{T}^n$, we may assume that the symplectic bundles $V_1, W_1$
take of the form:

$$V_1 = \sigma \mathbb{H}^{p+k}, \quad A = \sum_{|S|=8p+2,8p+3,8p+4} |a_S|2^{4p},$$

$$W_1 = \sigma \mathbb{H}^p \oplus \bigoplus_{|S|=8p+4} \pi^*_S f^*_S(\sigma(\hat{H}^e(S)(p) \oplus \cdots \oplus \hat{H}^e(S)(p)))$$

$$\oplus \bigoplus_{|S|=8p+2,8p+3} \pi^*_S f^*_S \hat{\pi}^*_S(\sigma(\hat{H}^e(S)(p) \oplus \cdots \oplus \hat{H}^e(S)(p)))$$

where $\hat{\pi}_q (q = 8p + 2,8p + 3)$ is the extension of the inclusion $i_q : \hat{\mathbb{R}}^q \times \{0\} \to \hat{\mathbb{R}}^{8p+4}$ to the one-point compactifications, and where $f_S : T^S \to \hat{\mathbb{R}}^S \cup \{\infty\} \cong \hat{\mathbb{R}}^S \cup \{\infty\}$ is a \{±1\}-equivariant map obtained by shrinking the complement of a neighborhood of the fixed point $(t_0,\cdots ,t_0)$ in $T^S$ to the one point \{\infty\}, and where $\epsilon(S) = ”+”$ when $a_S > 0$, and $\epsilon(S) = ”-”$ when $a_S < 0$. By Corollary 30 the second equation of Lemma 7 becomes

$$\alpha \varphi e(\mathbb{H}_1)^{y+A+k} = e(\hat{\mathbb{R}}^l)e(\mathbb{H}_1)^{y+A-A_1} \prod_{|S|=4, \pi_S \neq 0} \pi^*_S f^*_S e(\hat{H}^e(S)(p))$$

$$\times \prod_{|S|=2,3, \pi_S \neq 0} \pi^*_S f^*_S \hat{\pi}^*_S e(\hat{H}^e(S)(p))$$

where $A_1$ is the cardinal number of subsets $S$ satisfying $|S| = 2,3,4$ and $\pi_S \neq 0$.

**Proposition 34.** Let $l$ be a positive even integer. Suppose $\alpha \varphi \in KO_{\Gamma}^{l-4k}(\hat{T}^n)$ satisfies the equation $[14]$. Let $(\alpha \varphi)_S \in KO_{\Gamma}^{l-4k}(\hat{\mathbb{R}}^S)$ be the component of $\alpha \varphi \in KO_{\Gamma}^{l-4k}(\hat{T}^n)$ in the decomposition of $KO_{\Gamma}^{l-4k}(\hat{T}^n)$. If $|S|$ is even then we have

$$(\alpha \varphi)_S = \begin{cases} \\
\varepsilon N_{S}2^{l/2-k-|S|/2-1}([\mathbb{R}]_d - [\mathbb{R}]_d)\beta(\hat{\mathbb{R}}^S), & d \equiv 0 \mod 8, \\
\varepsilon N_{S}2^{l/2-k-|S|/2-1}([C]_d + \text{(torsion)})\beta(\hat{\mathbb{R}}^S), & d \equiv 2 \mod 8, \\
\varepsilon N_{S}2^{l/2-k-|S|/2-2}([\mathbb{R}]_d - [\mathbb{R}]_d)\beta(\hat{\mathbb{R}}^S), & d \equiv 4 \mod 8, \\
\varepsilon N_{S}2^{l/2-k-|S|/2-1}([C]_d + \text{(torsion)})\beta(\hat{\mathbb{R}}^S), & d \equiv 6 \mod 8,
\end{cases}$$

where $d = l - 4k - |S|$ and $\varepsilon = \pm 1$. Moreover if $d \equiv 2 \mod 8$ and $l \geq 4$ then $(\alpha \varphi)_S$ has no component in the torsion subgroup and $N_{S}2^{l/2-k-|S|/2-2}$ is an integer.

To prove this we use the following lemma:
Lemma 35. Let $c \geq 0$ be even, $i \geq 0$ and $d$ even. Suppose $\alpha \in KO^{d+2i}_{\Gamma}(\mathbb{R}^{2i})$ satisfy the equation

$$ae(H_1)^c = \sum_{0 \leq n < c+d/4} a_n e(H_1)^n e(\mathbb{R}^{4c+d-4n}) \beta(\mathbb{R}^{2i}) \in KO^{4c+d+2i}_{\Gamma}(\mathbb{R}^{2i}),$$

where $a_n \in \mathbb{Z}$ ($0 \leq n < c+d/4$). Then we have

$$\alpha = \begin{cases} \sum a_n (-1)^{c+d/4-n/2} c+d/2-n-1 ([R]_d - [R]_d) \beta(\mathbb{R}^{2i}), & d \equiv 0 \pmod 8, \\ \sum a_n (-1)^{c+d/4-n+1/2} c+d/2-n-1 ([C_0]_d + \text{(torsion)}) \beta(\mathbb{R}^{2i}), & d \equiv 2 \pmod 8, \\ \sum a_n (-1)^{c+d/4-n/2} c+d/2-n-2 ([R]_d - [R]_d) \beta(\mathbb{R}^{2i}), & d \equiv 4 \pmod 8, \\ \sum a_n (-1)^{c+d/4-n+1/2} c+d/2-n-1 ([C_0]_d + \text{(torsion)}) \beta(\mathbb{R}^{2i}), & d \equiv 6 \pmod 8, \end{cases}$$

Moreover if $d \equiv 2 \pmod 8$ and $a_n = 0$ for all $n$ with $c+d/4-1 < n$ then $\alpha$ has no component in the torsion subgroup and $\sum a_n (-1)^{c+d/4-n+1/2} c+d/2-n-2$ is an integer.

Proof. From Lemma 26 we see the complexification $c(\alpha) \in K^{d+2i}_{\Gamma}(\text{pt}) \cong R(\Gamma)$ satisfies

$$tr(j^2|c(\alpha)) = tr(t|c(\alpha)) = 0, \quad tr(j|c(\alpha)) = \sum a_n (-2i)^{2c+d/2-2n} 2^{n-c},$$

since $c(e(H_1)) = 2[C_0] - [H_1]$ and $c(e(\mathbb{R}^2)) = [C_{(3)}] - [C_{(1)}]$. This implies $c(\alpha)$ is in the subgroup $R(C_4)$ and

$$c(\alpha) = \sum a_n ([C_{(3)}] - [C_{(1)}])^{2c+d/2-2n} ([C_0] - [C_{(2)}])^{n-c}.$$

Since the kernel of $c : KO^{d+2i}_{\Gamma}(\text{pt}) \to K^{d+2i}_{\Gamma}(\text{pt})$ is the torsion subgroup by Lemma 26 we get the equation for $\alpha$.

When $d \equiv 2 \pmod 8$ Corollary 24 shows the torsion subgroup is generated by $[H_m]_d$ ($m \geq 1$). Suppose $\alpha$ has a nontrivial $[H_m]_d$-component. Let $m_0$ be the maximal of such $m$. Note $[H_m]_d[D_c] = [H_{m_0-2c}]_d + [H_{m_0+2c}]_d$. It implies that $ae(H_1)^c$ has a nontrivial $[H_{m_0+2c}]_d$-component. On the other hand if we assume $a_n = 0$ for all $n$ with $c+d/4-1 < n$, then we see from Lemma 26 27 that

$$\sum_{0 \leq n < c+d/4} a_n e(H_1)^n e(\mathbb{R}^{4c+d-4n}) = \sum a_n (-1)^{c+d+4-n+1/2} 2^{c+d/2-n-1} [C_0]_{4c+d},$$

which is a contradiction.

Moreover if $\sum a_n (-1)^{c+d+4-n+1/2} 2^{c+d/2-n-1}$ were odd,

$$ae(H_1)^c = \sum a_n (-1)^{c+d+4-n+1/2} 2^{c+d/2-n-1} [C_0]_{4c+d} + [H_{4c}]_{4c+d},$$

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which is also a contradiction. \hfill \Box

Proof of Proposition 34 We consider the right hand side of (14) on $S \subset \{1, \ldots, n\}$. Since $-|R|e(R^2) = e(R^2)$, we use Proposition 34 and Corollary 30 to get the expansion:

$$\varepsilon \sum_S \sum_{m \geq 0} N(S, m)e(H_1)^{y+4m}e(R^{l+4m-|S|})\beta(R^S),$$

where $\varepsilon$ is 1 or $-1$. We may assume $y$ is large enough and $y + A + k$ is even. We then apply Lemma 35 for $c = y + A + k \geq 0$, $d = l - 4k - |S|$, $2i = |S|$, $n = y + A - m \geq 0$, $\alpha = N(S, m)$ and $\alpha = (\alpha')_S$. Since $c + d/2 - n = l/2 - k - |S|/2 + m$, $c + d/2 - n = l/4 - |S|/4 + m \geq l/4 > 0$, and $c + d/2 - n \geq 1$ for $l \geq 4$, we have the conclusion. \hfill \Box

Now we prove Theorem 1. Suppose the inequality were not satisfied. Then by the standard inclusion $R^{x+l} \to R^{x+l+z}$ for some $z \geq 0$ we may assume $l = 2k + |S| - 2d_S + \varepsilon(k + d_S) - 1$. Theorem 1 then immediately follows from Proposition 34 in the cases other than $k + d_S \equiv 2 \pmod 4$, so we consider this case. Let $\psi : H_1 \to \tilde{R}^3$ be the Pin$^-$-equivariant map defined by $\psi(q) = qi \bar{q}$, where Pin$^-$ acts on $H_1$ by right multiplication. Then by considering the direct sum $\varphi \oplus \psi$, we have the inequality $l + 3 \geq 2(k + 1) + |S| - 2d_S + 3$, since $(k + 1) + d_S \equiv 3 \pmod 4$. This completes the proof of Theorem 1.

The equation 11 in Introduction follows immediately since $\kappa \sigma \beta_H(R^4) = \beta_C(C^2)$ by Lemma 20 and

$$[V_1] - [W_1] = k\sigma [H] + \sum_{S \subset \{1, \ldots, n\}, |S| = 4} a_S \sigma \beta_H(R^S) + \text{(torsion)} \in Ksp(\tilde{T}^n).$$

10 Proof of Theorem 2

Let $X$ be a connected closed oriented spin 4-manifold with indefinite intersection form. Let $k = -\text{sign}(X)/16$ and $l = b^+(X) > 0$. We write the spinor bundle of $X$ as $S^+TX \oplus S^-TX = S^+ \oplus S^-$ for simplicity. Take a Riemannian metric $g$ on $X$. Then we consider the monopole equation 26 as the map:

$$\hat{\Phi} : \hat{\nu} = \text{Ker}(d^* : \Omega^1 \to \Omega^0) \oplus \Gamma(S^+) \to \hat{\nu} = \Omega^+ \oplus H^1(X; \mathbb{R}) \oplus \Gamma(S^-),$$

where $P$ is a linear map and $Q$ is a quadratic form which are constructed as follows: Let $\pi : \text{Ker}(d^* : \Omega^1 \to \Omega^0) \to H^1(X; \mathbb{R})$ be the projection to the harmonic part and $D : \Gamma(S^+) \to \Gamma(S^-)$ the Dirac operator of $X$. Then the maps $P$ and $Q$ are defined to be

$$P(a, s) = (d^+ a, \pi(a), Ds), \quad Q(a, s) = (q(s), 0, c(a)s),$$

where $\pi$ and $D$ are defined.
where \( q : S^+ \to \Lambda^+ \) is the quadratic map formed from a nontrivial quadratic Spin(4)-equivariant map \( \Delta^+_4 \to \Lambda^+(\mathbb{R}^4) \).

We should remark that if we regard \( a \) as a \( U(1) \)-connection on the trivial complex line bundle over \( X \) then the differential operator \( D_a : \Gamma(S^+) \to \Gamma(S^-) \) defined by \( D_0 s = Ds + c(a)s \) is the Dirac operator twisted by \( a \), and \( d^+ a \) is the self-dual part \( F^0_\mu \) of the curvature of \( a \).

The gauge symmetry of the monopole equation induces the following symmetry: Choose a base point \( x_0 \) in \( X \). Let \( \rho : X \to \text{Hom}(H^1(X; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \) be the Albanese map, which is given by \( x \in X \to \{ \omega \to \int_{x_0} \omega \text{ mod } \mathbb{Z} \} \). Let \( \hat{\rho} : H^1(X; \mathbb{Z}) \to C^\infty(X, \mathbb{R}/\mathbb{Z}) \) be the adjoint map of \( \rho \). Then the actions of \( H^1(X; \mathbb{Z}) \) on \( \Omega^1, \Gamma(S^+ \oplus S^-), \Omega^+ \) are respectively defined by

\[
h \cdot a = a + h, \quad h \cdot s = e^{2\pi \sqrt{-1} \hat{\rho}(h)} s, \quad h \cdot b = b \quad \text{for } h \in H^1(X; \mathbb{Z}). \tag{15}\]

We introduce the diagonal actions of \( H^1(X; \mathbb{Z}) \) on \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{W}} \). Since the projection \( \tilde{\mathcal{V}} \to H^1(X; \mathbb{R}) \) is \( H^1(X; \mathbb{Z}) \)-equivariant, it descends to the vector bundle \( \mathcal{V} = \mathcal{V}/H^1(X; \mathbb{Z}) \) over the Jacobian torus \( J_X = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) \) with the decomposition \( \mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \) into the product real bundle \( \mathcal{V}_0 = J_X \times K \) \( (K = \text{Ker } d^\ast \cap H^1(X; \mathbb{R})) \) and a complex vector bundle \( \mathcal{V}_1 \) over \( J_X \) with fiber \( \Gamma(S^+) \). Similarly we see the projection \( \tilde{\mathcal{W}} \to H^1(X; \mathbb{R}) \) descends to the vector bundle \( \mathcal{W} = \mathcal{W}/H^1(X; \mathbb{Z}) \) over \( J_X \) with the decomposition \( \mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_1 \) into the product real bundle \( \mathcal{W}_0 = J_X \times \Omega^+ \) and a complex vector bundle \( \mathcal{W}_1 \) over \( J_X \) with fiber \( \Gamma(S^-) \). Then \( \Phi \) induces a \( \Gamma \)-equivariant fiber-preserving map

\[
\Phi : \mathcal{V}_0 \oplus \mathcal{V}_1 \to \mathcal{W}_0 \oplus \mathcal{W}_1.
\]

We should note that the restriction \( \Phi_0 : \mathcal{V}_0 \to \mathcal{W}_0 \) of \( \Phi \) is given by the linear injective map \( d^+ : K \to \Omega^+ \).

The compactness of the moduli space implies that the inverse image \( \Phi^{-1}(0) \) of the zero section is compact.

Moreover the spin structure of \( X \) provides an extra symmetry [18, 26]: The linear involution on \( \Omega^1, \Omega^+ \) and quaternionic scalar multiplication on \( \Gamma(S^\pm) \) induce \( \text{Pin}^-(2) \)-actions on \( \tilde{\mathcal{V}}, \tilde{\mathcal{W}} \) via the projection \( \text{Pin}^-(2) \to \{ \pm 1 \} \) or the inclusion \( \text{Pin}^-(2) \to Sp(1) \), which respectively induce \( \text{Pin}^-(2) \)-actions on \( \mathcal{V}, \mathcal{W} \), so that \( \Phi \) is \( \text{Pin}^-(2) \)-equivariant.

By a finite-dimensional approximation [9, 12], we get a proper \( \text{Pin}^-(2) \)-equivariant fiber-preserving map:

\[
\varphi : \mathcal{V}_0 \oplus \mathcal{V}_1 \to \mathcal{W}_0 \oplus \mathcal{W}_1
\]

such that (i) \( \mathcal{V}_0 = J_X \times \tilde{\mathbb{R}}^x, \mathcal{W}_0 = J_X \times \tilde{\mathbb{R}}^{x+l} \) for some \( x \geq 0 \), (ii) \( \varphi \) induces the identity on the base space \( J_X \), (iii) the restriction \( \varphi_0 : \mathcal{V}_0 \to \mathcal{W}_0 \) of \( \varphi \) is given by the standard linear inclusion \( \tilde{\mathbb{R}}^x \to \tilde{\mathbb{R}}^{x+l} \), (iv) \( \varphi_1, \mathcal{W}_1 \) are symplectic.
bundles over $\tilde{T}^n$ whose difference $[V_1] - [W_1] \in Ksp(\tilde{T}^n)$ is the index bundle $\text{Ind} \mathbb{D}$ of the family of Dirac operators parameterized by $\mathcal{J}_X$:

$$\mathbb{D} : (H^1(X; \mathbb{R}) \oplus \Gamma(S^+) \oplus H^1(X; \mathbb{Z}) \rightarrow (H^1(X; \mathbb{R}) \oplus \Gamma(S^-) \oplus H^1(X; \mathbb{Z}),$$

$$[(a, s)] \mapsto [(a, D_n s)].$$

We next calculate $\text{Ind} \mathbb{D}$ by identifying it as the index bundle for a family introduced by G. Lusztig [20]: We take a basis $x_1, \ldots, x_n$ of $H^1(X; \mathbb{Z})$, so that $\mathcal{J}_X$ is identified with the Real torus $\tilde{T}^n = (\mathbb{R}/\mathbb{Z})^n$. Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be $n$-dimensional torus with the trivial involution. Let $\mathbb{Z}^n \times \mathbb{Z}^n$ act on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}$ by

$$(z_1, z_2) \cdot (r_1, r_2, c) = (r_1 + z_1, r_2 + z_2, e^{2\pi \sqrt{-1}(z_1, r_2)} c),$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. The orbit space $L = (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C})/(\mathbb{Z}^n \times \mathbb{Z}^n)$ is then a Real line bundle over $T^n \times \tilde{T}^n$ with the involution $-1$ defined by

$$(-1) \cdot [(r_1, r_2, c)] = [(r_1, -r_2, c)].$$

Put $L_X = (\rho \times \text{id}_{\tilde{T}^n})^* L$. If we denote by $\pi : X \times \tilde{T}^n \rightarrow \tilde{T}^n$ the projection, we have a $\mathbb{Z}/2$-graded symplectic bundle $(\pi^* S^+ \otimes L_X) \oplus (\pi^* S^- \otimes L_X)$ over $X \times \tilde{T}^n$ and a symplectic family of Dirac operators

$$\mathbb{D}' : \Gamma(\pi^* S^+ \otimes L_X) \rightarrow \Gamma(\pi^* S^- \otimes L_X)$$

parameterized by $\tilde{T}^n$. Then it may be obvious from the action (15) that $\text{Ind} \mathbb{D} = \text{Ind} \mathbb{D}' \in Ksp(\tilde{T}^n)$.

Hence, as shown in [20] and [22], the cohomological formula of the index theorem for families [6] indicates

$$\text{ch}(V_1) - \text{ch}(W_1) = 2k + \sum_{S \subseteq \{1, \ldots, n\}, |S| = 4} \langle \prod_{i \in S} x_i, [X] \rangle \bigwedge_{i \in S} d\xi_i.$$ 

The equation (2) then follows from (1) in Introduction.

We next describe how the map (3) in Introduction comes from the index $\text{Ind} \mathbb{D}$. Recall that, under the identification $\mathbb{R}^{1,1} = \mathbb{C}$, $\beta_C(\mathbb{R}^{1,1}) \in KR(\mathbb{R}^{1,1})$ is the Bott class $\beta_C(\mathbb{C})$ together with the involution given by complex conjugation [2]. Then it is easy to show

$$[L] = \prod_{1 \leq q \leq n} \pi_q^*((i_q)\{([C] + \beta_C(\mathbb{R}^{1,1})})) = \sum_{S \subseteq \{1, \ldots, n\}} \pi_S^*(i_S)\beta_C(\mathbb{R}^{S,S}) \in KR(T^n \times \tilde{T}^n),$$

(16)
where $R^{S,S} = \text{Map}(S,R^{1,1})$ and $\beta_C(R^{S,S}) \in KR(R^{S,S})$ is the Bott class corresponding to $\beta_C(R^{S|S|}) \in KR(R^{S|S|})$ under the identification $R^{S,S} \cong R^{S|S|}$ for each $S \subseteq \{1,\cdots, n\}$.

Since the symbol class of the Dirac operator $D$ is just the Bott class $\beta_H(TX) \in KSp(TX) = KSp(TX)$, we use the index theorem for families \cite{7} (see also \cite{19}, Section 16) to get

$$\text{Ind} D = (i \times \text{id}_{\tilde{T}^n})(\rho \times \text{id}_{\tilde{T}^n})^*[L] \in KSp(\tilde{T}^n) \cong KR^{-4}(\tilde{T}^n),$$

where $i : X \to \text{pt}$ is the constant map. Substituting \cite{10} for $[L]$ and applying the periodicity isomorphisms

$$KR(R^{S,S}) \cong KO^{S|S|}(R^S), \quad KR^{-4}(R^S) \cong KO^{S|S|+4}(\text{pt})$$

(see Section \cite{7}), we deduce that $a_S$ ($|S| = 2$ or $3$) is zero if and only if the map \cite{3} is zero.

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