Harmonic cosmology: how much can we know about a universe before the big bang?

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Quantum gravity may remove classical space–time singularities and thus reveal what a universe at the big bang could be like. In loop quantum cosmology, an exactly solvable model is available, which allows one to address precise dynamical coherent states and their evolution in such a setting. It is shown here that quantum fluctuations before the big bang are generically unrelated to those after the big bang. While this is derived only in the solvable model, it presents the case of the strongest control on coherence properties; adding ingredients to a realistic model could only increase the complexity. A reliable determination of pre-big bang quantum fluctuations of geometry would thus require exceedingly precise observations.

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1. Introduction

Cosmology, as the physics of the Universe as a whole, places special limitations on scientific statements to be reasonably inferred from observations. On a general basis, this was discussed by McCrea (1970), who noted that some general properties seem to arise automatically during cosmological evolution. Their verification of our Universe then does not reveal anything about its possibly more complicated past. For instance, isotropization as shown by Misner (1968) demonstrates that the observation of a nearly isotropic present universe does not present much useful information on an initial state at much earlier times. Similarly, one may add decoherence to the list, which can be seen as doing the same for the observation of a nearly classical present universe. Most initial states would decohere and arrive at a semiclassical state; thus, its observation does not rule out stronger quantum behaviour at earlier stages. Given this, it is surprising to see recent discussions about a universe before the big bang in several approaches to quantum gravity. Not just statements about the possibility of the Universe having existed before the big bang are being made, which could, in principle, be inferred from a general analysis of equations of motion, but even assumptions on its classicality or claims about the form of its state such as its fluctuations at those times are put forward. Even if this may be possible theoretically, it raises the question of how much one can really learn about a pre-big bang universe from observations.

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A solvable toy model of quantum cosmology is used here to draw conclusions about the possible nature of the big bang and a universe preceding it. The viewpoint followed has been spelled out by Bojowald (2007b), where also some of the results have already been reported. This model does show a bounce at small volume instead of the classical singularity present in solutions of general relativity. We therefore use it to shed light on the general question posed above.

The promotion of this specific model is not intended as a statement that its bounce would be generic for quantum gravity even within the same framework. In fact, the conclusion of the bounce in this and related models available so far is based on several specific properties that prevent a generic statement about this form of singularity removal. We rather consider the following viewpoint: assume that there is a theoretical description of a bouncing universe; what implications can be derived for its pre-bounce state?

The specific model used is distinguished by a key fact that makes it suitable for addressing such a general question. As a solvable model, the system displays properties of its dynamical coherent states that, to some degree, are comparable with those of the harmonic oscillator. There is no back-reaction of quantum fluctuations or higher moments of a state on the time evolution of its expectation values. Their trajectory is thus independent of the spreading of a state if it occurs, a property that is well known for the harmonic oscillator (e.g. as a simple consequence of the Ehrenfest theorem).

This behaviour is certainly special compared with other quantum systems, but the precise derivation of properties of dynamical coherent states it allows is nevertheless important. For instance, a large part of theoretical quantum optics is devoted to a computation of fluctuations in squeezed coherent states of the harmonic oscillator. Similarly, the solvable model studied here allows detailed calculations of its dynamical coherent states that are of interest for quantum cosmology. In particular, the solvable model we will be using eliminates the classical big bang singularity by quantum effects. Dynamical coherent states highlight the behaviour of fluctuations of the state of the universe before and after the big bang.

Care is, however, needed for the physical interpretation of the results. While quantum optics allows one to prepare a desired state and perform measurements on it, quantum cosmology has to make use of the one universe state that is given to us. Unlike quantum optics, where states can be prepared to be close to harmonic oscillator states, we cannot realize states of the solvable quantum cosmological model. The real universe is certainly very different from anything solvable, and thus the availability of a certain feature or numerical result in a solvable model is unlikely to be related to a realistic property of the Universe. Thus, as spelled out by Bojowald (2007b), we focus on pessimism in our analysis of the solvable model: the inability of making certain predictions even in a fully controlled model is likely to be a reliable statement much more generally; adding complications to a real universe would only make those predictions even more difficult.

For solvable systems of this kind, it is easier by far to solve equations of motion for expectation values and fluctuations directly, rather than taking the detour of a specifically represented wave function. Properties of coherent states are then determined by selecting solutions of fluctuations that saturate uncertainty relations. We first illustrate this procedure for the harmonic oscillator with an emphasis on squeezed states. We present this brief review in §2 intended as an introduction to the methods then used in quantum cosmology.
The main part of this paper is an application of those methods to the solvable system of quantum cosmology, as well as a physical discussion. Instead of different cycles of a harmonic oscillator, in quantum cosmology, we will be dealing with the pre- and post-big bang phases of a universe. Just as fluctuations in a squeezed harmonic oscillator state can oscillate during the cycles, fluctuations in quantum cosmology may change from one phase to the next. Our main concern will be the reliability of predictions about the precise state of the Universe before the big bang, based on the knowledge we can achieve after the big bang.

2. Squeezed states

From the harmonic oscillator Hamiltonian $\hat{H} = (1/2m)\hat{p}^2 + (1/2)m\omega^2 \hat{q}^2$, we have equations of motion

$$\frac{d}{dt}\langle \hat{q} \rangle = \frac{1}{i\hbar}[\hat{q}, \hat{H}] = \frac{1}{m}\langle \hat{p} \rangle,$$

$$\frac{d}{dt}\langle \hat{p} \rangle = \frac{1}{i\hbar}[\hat{p}, \hat{H}] = -m\omega^2\langle \hat{q} \rangle,$$

for expectation values of operators in a given state. These are already in closed form and can thus be solved without knowing anything about fluctuations or other moments of the state. One obtains a trajectory in exact agreement with the classical one.

To find out more about the behaviour of the state while its expectation values follow the classical trajectory, we can derive and solve equations of motion

$$\frac{d}{dt}(\Delta q)^2 = \frac{\langle [\hat{q}^2, \hat{H}] \rangle}{\hbar} - 2\langle \hat{q} \rangle \frac{d\langle \hat{q} \rangle}{dt} = \frac{2}{m} G^{qp},$$

$$\frac{d}{dt} G^{qp} = -m\omega^2(\Delta q)^2 + \frac{1}{m}(\Delta p)^2,$$

$$\frac{d}{dt}(\Delta p)^2 = -2m\omega^2 G^{pp},$$

for fluctuations together with the covariance $G^{qp} = \langle (\hat{q}\hat{p} + \hat{p}\hat{q}) - \langle \hat{q} \rangle \langle \hat{p} \rangle \rangle$. In addition, these equations form a closed set and can thus be solved without requiring knowledge of higher moments, which we write in the general form

$$G^{O_1 \cdots O_n} := \langle (\hat{O}_1 - \langle \hat{O}_1 \rangle) \cdots (\hat{O}_n - \langle \hat{O}_n \rangle) \rangle_{\text{Weyl}}$$

$$= \frac{1}{n!} \sum_{\pi \in S_n} \langle (\hat{O}_{\pi(1)} - \langle \hat{O}_{\pi(1)} \rangle) \cdots (\hat{O}_{\pi(n)} - \langle \hat{O}_{\pi(n)} \rangle) \rangle,$$

as expectation values of Weyl-ordered (or totally symmetric) products of operators. (Totally symmetric ordering is achieved by summation over the permutation group $S_n$.) It is clear that to second-order fluctuations $G^{qq} = (\Delta q)^2$ and $G^{pp} = (\Delta p)^2$ result, as well as the covariance $G^{qp}$ used above. We thus determine exact state properties in this way and there is no need for an approximation. Approximations would, however, be necessary for anharmonic systems where equations of all the moments and of expectation values are
coupled with each other. Systematic treatments of Bojowald & Skirzewski (2006, 2007) give rise to effective equations for the motion of expectation values including quantum corrections from the back-reaction of moments on the expectation values.

To select coherent states, we need to saturate the uncertainty relation

\[ G^{qq} G^{pp} - (G^{qp})^2 \geq \frac{\hbar^2}{4}, \]  

i.e. determine quantum variables for which an equality results. Among all coherent states, those with vanishing correlations, \( G^{qp} = 0 \), have minimal uncertainties. For harmonic oscillator states, this requires \( \Delta p = m\omega \Delta q \) for \( G^{pp} \) to remain zero at all times according to (2.3). As the equations of motion (2.2) and (2.4) then show, the uncertainties must be constant in time. The state does not spread at all while its expectation values follow the classical trajectory. If we require the uncertainty relation to be saturated, fluctuations are determined uniquely for uncorrelated states. The corresponding state is the harmonic oscillator groundstate, where \( \Delta q = \sqrt{\hbar/2m\omega} \) and \( \Delta p = \sqrt{m\omega L/2} \). But for non-zero correlations, there are differently squeezed states, i.e. correlated states that still saturate the uncertainty relation, whose relation between \( \Delta q \) and \( \Delta p \) does not agree with that realized for the groundstate.

General squeezed states are thus obtained if we remain on the saturation surface but allow non-zero covariance. With \( G^{qp} \neq 0 \), fluctuations now depend on time in an oscillatory manner. These properties can all be derived without using a specific representation of states. But for comparison, one may easily check that, at any given time \( t_0 \), a state saturating the uncertainty relation is a Gaussian \( \psi(q, t_0) \propto \exp(-z_1 q^2 + z_2 q + z_3) \) in the position representation, with \( \text{Re} z_1 > 0 \) for normalizability. If \( z_1 \) is real, we have uncorrelated states at the absolute minimum of the saturation surface. Otherwise, the covariance is given by \( G^{qp} = -(\text{Im} z_1/2 \text{ Re} z_1) \hbar \).

3. Harmonic quantum cosmology

The simplest form of isotropic cosmology is based on one pair of canonical variables as functions of the scale factor \( a \), which determines the spatial size of a universe, and the derivative \( da/d\tau \) by the proper time \( \tau \). Dynamics is determined by the Friedmann equation

\[ \left( \frac{1}{a} \frac{da}{d\tau} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \]  

where the energy density \( \rho \) of matter provides the source and \( G \) is Newton’s constant. We will be especially interested in a spatially flat model, \( k = 0 \), with a matter source given by a minimally coupled, free massless scalar \( \phi \) with momentum \( p_\phi = a^3 \varphi/d\tau \). Its energy density is independent of \( \phi \) and takes the purely kinetic form

\[ \rho = \frac{p_\phi^2}{2a^6} = \frac{1}{2} \left( \frac{d\varphi}{d\tau} \right)^2. \]
As far as is known, massless scalar particles do not exist, but the choice of a massless scalar source together with the assumption that $k=0$ provides a solvable model playing the role of the harmonic oscillator for quantum cosmology. This specific choice is thus mathematically distinguished, as will become clear in what follows. Although many of its specific properties cannot be generalized to other models, some general conclusions can nevertheless be drawn along the lines sketched in §1. Matter potentials or a curvature term can be dealt with by perturbation theory around the solvable model used here (see Bojowald et al. 2007b; Bojowald 2008a). This will not affect the qualitative properties discussed here. We will be interested in upper bounds for the amount of spreading of states, which can only increase if deviations from a solvable model are introduced by interaction terms.

Since our aim is to study properties of dynamical coherent states, especially in their relation between the pre- and post-big bang phases, we use a quantization that eliminates the classical singularity occurring when solutions reach $a=0$. In isotropic models, this can generally be achieved in loop quantum cosmology (see Bojowald 2005, 2007a), where geometry is quantized and the spectra of geometrical operators become discrete. The form of the discreteness, i.e. whether it is the scale factor itself, which is equidistantly spaced in the basic representation of quantum operators, or rather some of its powers such as the volume $a^3$, is to be determined from the precise form of dynamics in quantum gravity. Fortunately, our model will not be sensitive to this detail, and we can thus leave the precise form of discreteness open in a parametrized way.

In gravity, momenta conjugate to metric components are given in terms of extrinsic curvature, which itself is related to the time derivative of metric components. For an isotropic metric, this results in $\{a^2, da/d\tau\} = 8\pi G/3$. Quantization sometimes depends on the specific choice of canonical variables that are turned into basic operators. To take into account some of the freedom that turns out to be important for loop quantum cosmology, we introduce new canonical variables

$$v = \frac{3}{8\pi Gf_0(1-x)} a^{2(1-x)}, \quad P = f_0 a^{2x} \frac{da}{d\tau}$$

such that $\{v, P\} = 1$, (3.3)

with two parameters $f_0$ and $x$. Classically, $f_0$ and $x$ have no influence whatsoever because changing them amounts to a canonical transformation. However, canonical transformations may not be implemented unitarily after quantization, which turns out to be the case here. We thus keep the parameters $f_0$ and $x$ in the general definition.

The first parameter, $f_0$, is arbitrary and included to make $P$ dimensionless while $v$ acquires the dimension of an action. This choice will simplify later equations, and our main results all refer to ratios of quantities that are independent of the value of $f_0$. Also, the value of $x$ is not crucial for results discussed here, but its effect on the variables is stronger than for $f_0$. In a quantization of these basic variables, it is $v$ that will be equidistantly spaced in the sense discussed below after equation (3.4). As $x$ is changed between different formulations of the model, the form of the spectrum for geometrical quantities takes on all possibilities of equidistant powers $a^{2(1-x)}$. From loop quantum gravity, one expects $x$ to lie within the range $0 > x > -1/2$ as argued by Bojowald.
(2006, 2008b), with phenomenological arguments favouring a value near $-1/2$
(see Ashtekar et al. 2006; Bojowald et al. 2007a; Bojowald & Hossain 2007, 2008;
Nelson & Sakellariadou 2007a, b).

A further implication of the discreteness of $\hat{v}$ in a quantization is that no
operator for $P$ itself can exist. This is similar to quantization on a circle, where
the angle as a local coordinate cannot be promoted to an operator but only its
periodic functions that are global functions on the configuration space. Similarly,
there is no operator for $P$ or $\dot{a}$ in loop quantum cosmology, but only quanti-
zations of $\sin \delta P$ and $\cos \delta P$ for real $\delta$ exist. Loop quantum cosmology is an
application of the methods of loop quantum gravity to symmetric models, which
preserves the discreteness of geometry seen in the full theory and demonstrates
some of its physical implications. In both cases, the discreteness is a direct
consequence of the quantum representation used, based on a compact space of
gravitational connections. In this way, the analogy to quantization on a circle
arises especially in isotropic loop quantum cosmology, which is a single canonical
pair $(v, P)$. But in contrast to quantization on a circle, there is no periodicity and
thus all $\delta \in \mathbb{R}$ are allowed for functions $\sin \delta P$ and $\cos \delta P$. The discreteness of the
$\hat{v}$-spectrum does not arise from a periodic identification but from a
compactification of the real line: the so-called Bohr compactification, which
contains the whole real line as a dense set but is itself compact. In fact, functions
$\sin \delta P$ and $\cos \delta P$ for all $\delta \in \mathbb{R}$ span the set of all continuous functions on the Bohr
compactification, which forms the quantum configuration space of loop quantum
cosmology, as analysed by Ashtekar et al. (2003). Accordingly, an orthonormal
basis can be written as $\{|\mu\rangle : \mu \in \mathbb{R}\}$ for this non-separable Hilbert space. The
basic operators act as

$$\hat{v}|\mu\rangle = \hbar \mu|\mu\rangle, \quad \exp(i\delta P)|\mu\rangle = |\mu + \delta\rangle,$$  \hfill (3.4)

with commutator

$$[\hat{v}, \exp(i\delta P)] = \delta \exp(i\delta P),$$

as appropriate for a canonical pair $(v, P)$. Although the spectrum of $\hat{v}$ is the
whole real line, $\hat{v}$ is equidistantly quantized in the sense that a given basic
operator $\exp(i\delta P)$ shifts the label $\mu$ by an amount independent of $\mu$. In terms of
$\alpha$, the quantity that is equidistant thus depends on $x$ as it appears in the
definition (3.3) of $v$.

Also the quantized Friedmann equation must make use of $\sin \delta P$ rather than
d$a/d\tau$ directly. At this point, $\delta$ would be fixed such that dynamics can be
restricted to a separable sector of the full Hilbert space. Qualitative properties
discussed here depend on neither the value of $\delta$, which we set equal to 1 for
simplicity, nor that of $x$ used in the definition of classical variables. The
construction of loop quantum cosmology shows that the term $a^4(d^2a/d\tau)^2$, which
as per equation (3.1) is proportional to $p_\phi^2$ for a free scalar, then takes the form
$v^2 \sin^2 P$ up to quantization ambiguities. Independently of $x$, it reduces to the
classical expression in the small curvature limit $P \ll 1$. (We do not write this as a
proper operator yet because we will later be led to a specific factor ordering that
is being kept free for now.) The dynamical equation combining (3.1) and (3.2)
can thus be written in the form

$$p_\phi = |v \sin P|.$$  \hfill (3.5)
(We drop a factor of $2\sqrt{4\pi G/3(1-x)}$ on the right-hand side in order to bring out the functional form of later expressions more clearly. The factor can simply be reinstated by redefining $\phi$. Note that all factors of $\hbar$, which are important for the following considerations, are written explicitly.)

If the matter value $\phi$ is interpreted as a time variable, this shows that the Hamiltonian in this internal time description is $H=|v\sin P|$. Using the internal time $\phi$, rather than the proper time $\tau$, is closer to the behaviour of the quantum wave function that depends on $\phi$, but not on space–time coordinates. Moreover, in our case, this has the advantage of showing more clearly that the system is explicitly solvable. For small $P$ and disregarding the absolute value, this would be a quadratic Hamiltonian $vP$ and we would have the same decoupling behaviour as in the harmonic oscillator. (This Berry–Keating–Connes Hamiltonian, as used in Connes (1996) and Berry & Keating (1999), plays a role in the context of the Riemann hypothesis.) However, the sine certainly makes the Hamiltonian non-quadratic in canonical variables.

Remarkably, the system is still solvable even with the sine. We have to change variables to non-canonical ones and specify a precise factor ordering of the equations of motion as in (2.1), which makes the system solvable and decouples fluctuations from expectation values: in equations of motion as in (2.1), $[\hat{O},\hat{H}]$ is linear in basic operators for any basic operator $\hat{O} \in \{\hat{v},\hat{J},\hat{J}^\dagger\}$. Here, our choice of model matters: with $k \neq 0$ or a matter potential, the Hamiltonian would not be linear in these variables.

With complex basic variables, we have to impose a reality condition making sure that $\hat{J}\hat{J}^\dagger = \hat{v}^2$. Taking an expectation value of this relation,

$$|\langle \hat{J} \rangle|^2 - (\langle \hat{v} \rangle + \frac{1}{2}\hbar)^2 = G^v - G^{JJ} + \frac{1}{4}\hbar^2,$$

(3.7)

shows that fluctuations and expectation values are related in a specific way. But this does not involve a coupling of all infinitely many moments. Moreover, one can impose (3.7) after having solved the equations of motion. Thus, no severe complication to solvable systems in real variables arises.

1 If desired, the proper time $\tau$ can be reintroduced at a later stage. The relation between $\phi$ and $\tau$ follows from the differential equation $p_\phi = a^2 d\phi / d\tau$ that determines $\phi(\tau)$ once $a(\phi)$ is known, using the fact that $p_\phi$ is constant for a free massless scalar.
We can thus efficiently make use of the same solution procedure introduced in §2 for the harmonic oscillator and determine properties of dynamical coherent states. Equations of motion for expectation values, fluctuations and covariances are

$$\begin{align*}
\frac{d}{d\phi} \langle \hat{\phi} \rangle &= \frac{1}{i\hbar} \langle [\hat{\phi}, \hat{\mathcal{H}}] \rangle = -\frac{1}{2} (\langle \hat{\mathcal{J}} \rangle + \langle \hat{\mathcal{J}}^\dagger \rangle), \\
\frac{d}{d\phi} \langle \hat{\mathcal{J}} \rangle &= \frac{1}{i\hbar} \langle [\hat{\mathcal{J}}, \hat{\mathcal{H}}] \rangle = -\langle \hat{\phi} \rangle - \frac{1}{2} \hbar = \frac{d}{d\phi} \langle \hat{\mathcal{J}}^\dagger \rangle,
\end{align*}$$

(3.8)

and

$$\begin{align*}
\dot{G}^{\mathcal{E} \mathcal{J}} &= -G^{\mathcal{E} \mathcal{J}} - G^{\mathcal{J} \mathcal{J}} , \\
\dot{G}^{\mathcal{J} \mathcal{J}} &= -2G^{\mathcal{J} \mathcal{J}} , \\
\dot{G}^{\mathcal{E} \mathcal{J}} &= -\frac{1}{2} G^{\mathcal{J} \mathcal{J}} - \frac{1}{2} G^{\mathcal{E} \mathcal{E}} - G^{\mathcal{J} \mathcal{J}} , \\
\dot{G}^{\mathcal{J} \mathcal{J}} &= -\frac{1}{2} G^{\mathcal{J} \mathcal{J}} - \frac{1}{2} G^{\mathcal{J} \mathcal{J}} - G^{\mathcal{E} \mathcal{E}} ,
\end{align*}$$

(3.9)

(3.10)

(3.11)

(3.12)

All equations can easily be solved. For expectation values, we obtain

$$\langle \hat{\phi} \rangle (\phi) = \frac{1}{2} (A e^{-\phi} + B e^{\phi}) - \frac{1}{2} \hbar,$$

(3.13)

$$\langle \hat{\mathcal{J}} \rangle (\phi) = \frac{1}{2} (A e^{-\phi} - B e^{\phi}) + iH,$$

(3.14)

where $\mathcal{H} := \langle \hat{\mathcal{H}} \rangle$. For these solutions, the reality condition (3.7) implies $AB = H^2 + c_1 - (1/4) \hbar^2$, with $c_1 := G^{\mathcal{J} \mathcal{J}} - G^{\mathcal{E} \mathcal{E}}$, which according to (3.9) and (3.12) is constant in time. For states that are semiclassical at large volume (or just at one time), we must have $c_1 \ll H^2$ and thus positive $AB$. This clearly shows the absence of the classical singularity since $\langle \hat{\phi} \rangle (\phi) = H \cosh(\phi - \epsilon) - (1/2) \hbar$, with $e^{2\epsilon} = A/B$, never reaches zero. Then, curvature invariants computed for an effective Friedmann–Robertson–Walker metric also remain finite. From now on, we set $A = B$ without loss of generality since it simply amounts to a shift in the origin of our internal time $\phi$.

For fluctuations and covariances, we have explicit solutions

$$G^{\mathcal{E} \mathcal{E}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} + c_4 e^{2\phi}) - \frac{1}{4} (c_1 + c_2),$$

(3.15)

$$G^{\mathcal{J} \mathcal{J}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4} (3c_2 - c_1) - i(c_5 e^{\phi} - c_6 e^{-\phi}),$$

(3.16)

$$G^{\mathcal{J} \mathcal{J}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4} (3c_2 - c_1) + i(c_5 e^{\phi} - c_6 e^{-\phi}),$$

(3.17)

$$G^{\mathcal{J} \mathcal{E}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} - c_4 e^{2\phi}) + \frac{i}{2} (c_5 e^{\phi} + c_6 e^{-\phi}),$$

(3.18)

$$G^{\mathcal{J} \mathcal{E}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} - c_4 e^{2\phi}) - \frac{i}{2} (c_5 e^{\phi} + c_6 e^{-\phi}),$$

(3.19)

$$G^{\mathcal{J} \mathcal{J}} (\phi) = \frac{1}{2} (c_3 e^{-2\phi} + c_4 e^{2\phi}) + \frac{1}{4} (3c_1 - c_2),$$

(3.20)

with constants of integration $c_i$ (see Bojowald 2007c for further details).
4. Dynamical coherent states

Depending on the integration constants $c_i$, there are obviously different possibilities in the relation of fluctuations before and after the big bang, defined as the minimum of $\langle \hat{v} \rangle (\phi)$. For special choices, one can arrange the fluctuations to be identical at positive and negative $\phi$, but not generically. The quantum state and its coherence will thus appear differently before and after the big bang, depending on what property of the state an observation might be sensitive to.

This behaviour of fluctuations is a direct consequence of the replacement of the classical singularity by a bounce, as it occurred due to the discreteness of geometry. Without taking into account the discreteness, we would have arrived at the Berry–Keating–Connes Hamiltonian quantizing $vP$ directly. This system differs from the harmonic oscillator (although it can be mapped by a canonical transformation to an upside-down harmonic oscillator) in the fact that it does not have constant or even bounded fluctuations in its coherent states. However, the ratio $Gvv/\langle \hat{v} \rangle^2$ is exactly preserved and ensures that an initial semiclassical state remains semiclassical. This behaviour is also true for the loop quantized model for small curvature $P$. But near the bounce, there are strong deviations that result in a non-constant ratio $Gvv/\langle \hat{v} \rangle^2$. One can see this directly from the equations of motion (3.8) and (3.9), from which we derive

$$\frac{d}{d\phi} Gvv(\langle \hat{v} \rangle^2) = -\frac{Gvv + G\hat{v}^{-1} Gvv \langle \hat{J} \rangle + \langle \hat{J} \rangle}{\langle \hat{v} \rangle^2} Gvv.$$ (4.1)

This is zero for $Gvv + G\hat{v}^{-1} Gvv \langle \hat{J} \rangle + \langle \hat{J} \rangle$, $Gvv \cos P = Gvv \cos P / \langle \hat{v} \rangle$. For small $P$, we can ignore the cosine and indeed derive that $Gvv/v^2$ is constant. But near the bounce, deviations of $\cos P$ from its low-curvature value are important and, in general, imply non-constant relative fluctuations through the bounce. Another possibility to satisfy the required relation is to have uncorrelated states, such that $Gvv \cos P \approx Gvv \cos P$ and $G\cos P \approx \langle \hat{v} \rangle \cos P$, which would also imply a constant $Gvv/\langle \hat{v} \rangle^2$ even through the bounce at high curvature. For squeezed states with correlations, however, relative fluctuations before and after the bounce can differ from each other, even though they are nearly constant in any pre- or post-bounce phase.

To make this precise, we determine explicit forms of the uncertainty relations in our variables. For any pair $(\hat{O}_1, \hat{O}_2)$ of self-adjoint operators, we have

$$G^{O_1, O_2} = \langle [\hat{O}_1, \hat{O}_2] \rangle^2 \geq -\frac{i}{4} \langle [\hat{O}_1, \hat{O}_2] \rangle^2,$$ (4.2)

so that in our case we can write three different inequalities as follows:

$$Gvv G^{J+\hat{J}, J+\hat{J}} - (G^{J+\hat{J}})^2 \geq \hbar^2 H^2,$$ (4.3)

for the pair $(\hat{\nu}, \hat{J} + \hat{\nu})$,

$$Gvv G^{(J-\hat{J}), i(J-\hat{J})} - (G^{i(J-\hat{J})})^2 \geq \frac{1}{4} \hbar^2 (\langle \hat{J} \rangle + \langle \hat{\nu} \rangle)^2,$$ (4.4)

for the pair $(\hat{\nu}, i(\hat{J} - \hat{\nu}))$ and

$$G^{J+\hat{J}, J+\hat{J}} G^{i(J-\hat{J}), i(J-\hat{J})} - (G^{J+\hat{J}, i(J-\hat{J})})^2 \geq \hbar^2 (2\langle \hat{\nu} \rangle + \hbar)^2,$$ (4.5)
for the pair \((\hat{J} + \hat{J}^\dagger, i(\hat{J} - \hat{J}^\dagger))\). Inserting solutions provides the relations

\[
4c_3c_4 - \frac{1}{4}(c_1 + c_2)^2 \geq \hbar^2 H^2, \tag{4.6}
\]

\[
(c_1 - c_2)(c_3e^{-2\phi} + c_4e^{2\phi}) + \frac{1}{2}(c_2^2 - c_1^2) - c_5^2e^{2\phi} - 2c_6^2 - c_6^2e^{-2\phi}
\geq \frac{1}{4} A^2 \hbar^2 (e^{-\phi} - e^{2\phi})^2, \tag{4.7}
\]

\[
4(c_1 - c_2)(c_3e^{-2\phi} + c_4e^{2\phi}) - 2(c_2^2 - c_1^2) - 4c_5^2e^{2\phi} + 8c_6^2 - 4c_6^2e^{-2\phi}
\geq A^2 \hbar^2 (e^{-\phi} + e^{\phi})^2, \tag{4.8}
\]

between the integration constants.

The last two relations (4.7) and (4.8) suppress the pre-bounce parameters of fluctuations by exponentials of \(\phi\) and thus by the total volume. This is huge at late times and fluctuation parameters from before the big bang can thus be ignored completely after the big bang. In other words, these two relations do not provide any constraints on the state before the big bang.

The first relation is different since any \(\phi\)-dependence is dropped out. It thus presents an uncertainty relation between the pre- and post-big bang fluctuation parameters \(c_3\) and \(c_4\), respectively. The covariance term \(-(1/4)(c_1 + c_2)^2\) in this relation can be seen to represent matter fluctuations. From the reality condition together with

\[
(\Delta H)^2 = -\frac{1}{4}(G^{J\dagger J} + G^{JJ}) + \frac{1}{2} G^{JJ} = \frac{1}{2}(c_1 - c_2), \tag{4.9}
\]

we have

\[
c_1 + c_2 = 2A^2 - 2H^2 - 2(\Delta H)^2 + \frac{1}{2} \hbar^2. \tag{4.10}
\]

In (4.6), however, \(\phi\)-dependent terms are dropped out due to cancellations between the fluctuation and covariance terms in \(G^{J\dagger J,J\dagger J} = 4G^{\dagger\dagger} + 2(c_1 + c_2)\) and \(G^{\dagger\dagger,J\dagger J} = G^{J\dagger J} + G^{\dagger\dagger J}\). If the model is changed slightly or one tries to use real observations that must be imprecise, those terms re-enter the game and lead to a dominant expression that involves only post-big bang fluctuations. But even with those cancellations, there is no strong relation between the pre- and post-big bang quantities, as shown here.

The fact that squeezing is responsible for the possible asymmetry of fluctuations around the bounce can be seen by directly computing the quantum variables in our basic set of operators for a Gaussian state. If this Gaussian is not completely squeezed, corresponding to a real \(z_1\) as defined in §2, then fluctuations before and after the bounce must be identical. A quick way to see this is to note that there are three independent parameters in an unsqueezed Gaussian, Re \(z_2\) and Im \(z_2\) that determine the peak position in phase space and Re \(z_1\), and two constants of motion for expectation values and fluctuations, \(\overline{H} = \langle \hat{H} \rangle\) and \(G^{HH}\). Comparing two Gaussian states, one before and one after the big bang along a dynamical trajectory, at equal volume fixes the free parameter. Thus, the spreads must also be equal at a given volume. Switching on general squeezing, however, introduces additional parameters and the conclusions are weaker.
Precise conditions can be found by solving the saturation equations explicitly. We use (4.6) directly, and obtain three more equations from the $e^{\pm 2\phi}$ and constant coefficients of (4.7) and (4.8), which are as follows:

$$4c_3c_4 = H^2h^2 + \frac{1}{4}(c_1 + c_2)^2,$$

(4.11)

$$(c_1 - c_2)c_3 - c_6^2 = \frac{1}{4}A^2h^2,$$

(4.12)

$$(c_1 - c_2)c_4 - c_5^2 = \frac{1}{4}A^2h^2,$$

(4.13)

$$4c_3c_6 = A^2h^2 + c_2^2 - c_1^2. $$

(4.14)

With (4.9) and (4.10), we can eliminate $c_1$ and $c_2$ in favour of $A$ and $\Delta H$ together with $H$. As a first result, subtraction of (4.12) and (4.13) shows that $v$-fluctuations can only be symmetric around the bounce, i.e. $c_3 = c_4$, if $|c_5| = |c_6|$ as noted by Bojowald (2007c). We have to look closer to see what range of deviations between $c_3$ and $c_4$ is allowed for coherent states.

We can directly solve (4.12) and (4.13) for $c_3$ and $c_4$ in terms of $c_5$ and $c_6$, and then insert these variables in (4.11). In combination with (4.14) to eliminate the resulting term $c_3^2c_6^2$, we then have

$$c_5^2 + c_6^2 = 4\frac{H^2}{A^2}(\Delta H)^4 - \frac{1}{2}A^2h^2 + (\Delta H)^2(c_1 + c_2)$$

$$= 4\left(\frac{H^2}{A^2} - 1\right)(\Delta H)^4 + 2\left(A^2 - H^2 + \frac{1}{4}h^2\right)(\Delta H)^2 - \frac{1}{2}A^2h^2. $$

(4.15)

Adding and subtracting $2c_5c_6$ from (4.14), we obtain

$$(c_5 + c_6)^2 = 4\left(\frac{H^2}{A^2} - 1\right)(\Delta H)^4 + 4\left(A^2 - H^2 + \frac{1}{4}h^2\right)(\Delta H)^2 - A^2h^2 = 4C^2$$

$$(c_5 - c_6)^2 = 4\frac{H^2}{A^2}(\Delta H)^4,$$

(4.16)

which finally provides

$$|c_3 - c_4| = \frac{1}{2(\Delta H)^2}\sqrt{(c_5 + c_6)^2(c_5 - c_6)^2} = 2\frac{H}{A}C$$

$$= 2\frac{H}{A}\sqrt{\left(\frac{H^2}{A^2} - 1\right)(\Delta H)^4 + \left(A^2 - H^2 + \frac{1}{4}h^2\right)(\Delta H)^2 - \frac{1}{4}A^2h^2},$$

(4.17)

as an explicit relation for the difference in pre- and post-big bang fluctuations of $v$.

This expression can easily be estimated for its order of magnitude, noting that the uncertainty relations, especially (4.7), indicate that $(\Delta H)^2 \sim Ah$ and, together with the reality condition, $A^2 \sim H^2 + O(Ah)$. Thus, the first term in the square root is smallest while the remaining terms are of the order $A^2h^2$. The
square root \( C \) and thus \(|c_3 - c_4|\) is of the same order \( \Delta h \) as the fluctuation \((\Delta H)^2\). The asymmetry vanishes for

\[
(\Delta H)^2 = \frac{A^2}{A^2 - H^2} \frac{\hbar^2}{4} \sim A h, \tag{4.18}
\]

which confirms the results of Bojowald (2007c) and is consistent with the orders of magnitude given above.

One can write the asymmetry more directly in fluctuation variables, providing a relation

\[
\lim_{\phi \to -\infty} \frac{G^{vv}}{\langle \hat{v} \rangle^2} - \lim_{\phi \to +\infty} \frac{G^{vv}}{\langle \hat{v} \rangle^2} = 2 \frac{|c_3 - c_4|}{A^2}
\]

\[
= 4 \frac{H}{A} \sqrt{1 - \frac{H^2}{A^2} + \frac{1}{4} \frac{\hbar^2}{A^2}} \frac{(\Delta H)^2}{A^2} \frac{1}{2} \frac{\hbar^2}{A^2} \frac{1}{4} \frac{\hbar^2}{A^2} + \left( \frac{H^2}{A^2} - 1 \right) \frac{(\Delta H)^4}{A^4}, \tag{4.19}
\]

which resembles estimates provided independently by A. Ashtekar et al. (2007, private communication). For any value of \( A, H \) and \( \Delta H \), this dimensionless number is of the small order \( \hbar/A \), and thus one could conclude that the asymmetry must be small and all coherent states are very nearly symmetric. However, this is not true. The order \( \hbar/A \) is what one already expects for a single relative fluctuation \( \lim_{\phi \to -\infty} G^{vv}/\langle \hat{v} \rangle^2 = c_3/A^2 \) since \( c_3 \sim \Delta h \), not just for the difference in the asymmetry. Fluctuations themselves are certainly small in a saturated state, and (4.19) gives only their magnitude but does not provide information on the asymmetry of those small quantities.

This can easily be corrected because we have full control on our dynamical coherent state parameters. Solving for \( c_4 \) from (4.13) and determining \( c_3 \) from (4.16), we derive

\[
1 - \frac{c_3}{c_4} = \frac{|c_4 - c_3|}{c_4} = \frac{2 H}{A} \frac{C}{2(\Delta H)^2} + \frac{H^2}{A^2} (\Delta H)^2 + \frac{1}{4} \frac{\Delta H}{(\Delta H)^2}, \tag{4.20}
\]

which is a measure for the asymmetry independent of the total size of fluctuations. The sign in the denominator depends on whether \( c_3 < c_4 \) (upper sign) or \( c_3 > c_4 \) (lower sign). As one can see, this vanishes for \( C=0 \), i.e. when (4.18) is satisfied. This is, however, only one special case and, in general, with \( C \sim A h \sim (\Delta H)^2 \), one can only infer that \( |1 - c_3/c_4| \) is of the order \( O(1) \), not necessarily close to zero. Generically, the asymmetry for coherent states is not restricted at all. Symmetry of fluctuations before and after the bounce is not generic even for states saturating the uncertainty relations.

One simple example that illustrates how large the asymmetry can be is obtained for \( C=(H/A)(\Delta H)^2 \), in which case the first three terms in the denominator of (4.20) cancel each other for the lower sign solution. For the constants, this implies that

\[
A^2 = H^2 \frac{(\Delta H)^2}{(\Delta H)^2 - \frac{1}{4} \hbar^2} + (\Delta H)^2 \sim H^2 + (\Delta H)^2, \tag{4.21}
\]

and thus \( c_1 \sim (\Delta H)^2 + (1/4) \hbar^2 \) and \( c_2 \sim -(\Delta H)^2 + (1/4) \hbar^2 \), as well as \( c_3 \sim (1/8) H \hbar + 2(\Delta H)^2 \), \( c_4 \sim (1/8) H \hbar \), \( c_5 = 0 \) and \( |c_6| \sim 2(\Delta H)^2 \). The only property that makes this case special is that \( c_5 = 0 \), but the other magnitudes for fluctuations are fully acceptable. With this \( C \), the relative asymmetry becomes \( |1 - c_3/c_4| \sim 16 \), which is certainly a large value.
In fact, although the denominator in (4.20) never becomes zero for real \( C \), the expression is unbounded from above for the lower sign (and reaches arbitrarily closely to \(|1 - c_3/c_4| = 1\) for the upper sign). Extremely large values require large fluctuations and thus, on the saturation surface, very high squeezing. But large values for \( c_3/c_4 \) can easily be reached under appropriate conditions on semiclassical states. At constant \( A \), the maximum of \(|1 - c_3/c_4|\) along varying \((\Delta H)^2\) is realized for

\[
(\Delta H)^2 = \frac{A^2}{H^2} \left(1 - \frac{H^2}{A^2}\right) + \frac{\hbar^2}{2(1 - H^2/A^2)},
\]

which results in a value

\[
\left|1 - \frac{c_3}{c_4}\right| = \frac{2}{\sqrt{1 + \frac{1}{4} \left(\frac{H^2\hbar^2}{A^2-H^2}\right)^2}} \pm 1.
\]

This is indeed unbounded for the lower sign, but to be large it requires values of \( A \) significantly larger than \( H \). Since previous relations showed that \( A^2 - H^2 \sim \hbar A \) for semiclassical behaviour, \( A \) should not differ too much from \( H \) and we can approximate the semiclassical maximum of the asymmetry by the value \( 4/(\sqrt{5} \pm 2) \). For the lower sign, this gives a value \(|1 - c_3/c_4| \approx 16.9 \) (or \(|1 - c_3/c_4| \approx 0.94\) for the upper sign) near the example found above.

One might think that a state with symmetric fluctuations makes the product \( c_3c_4 \) minimal, just as a non-squeezed state does for the usual uncertainty product in quantum mechanics according to (2.6). Interestingly, this is not necessarily the case: (4.11) shows that this product, for a given \( H \), is minimal for \( c_1 + c_2 = 0 \), and thus \((\Delta H)^2 = A^2 - H^2 + (1/4)\hbar^2 \). This is not the value (4.18) obtained above for symmetric fluctuations, unless we specialize the parameters further to \( A^2 - H^2 \sim (1/2)\hbar \). More generally, we rather have

\[
C^2 = \frac{H^2}{A^2} (\Delta H)^4 - \frac{1}{4} A^2 \hbar^2,
\]

and thus

\[
\left|1 - \frac{c_3}{c_4}\right| = \frac{2H}{A^2} C \left(\frac{\hbar}{A^2} (\Delta H)^2 \pm \frac{H}{A^2} C\right)^{-1/2} \sim \frac{2}{(1 - \frac{1}{4} A^2 \hbar^2 (A^2 - H^2)^{-2})^{-1/2}} \pm 1
\]

which can take values of a magnitude similar to those obtained above. It vanishes for \( A^2 - H^2 = (1/2)\sqrt{H^2 + (1/16)\hbar^2 + (1/8)\hbar^2} \sim (1/2)\hbar \), which corresponds to symmetric fluctuations, but increases very rapidly for slightly larger values and can become arbitrarily large (or very nearly one for the upper sign). For very large values, one needs a large difference \( A_2 - H_2 \), but even \( A^2 - H^2 \sim H\hbar \), which is not an extreme value compared with \((1/2)\hbar \hbar \) of the symmetric case, gives \(|1 - c_3/c_4| = 2\sqrt{3}/2 - \sqrt{3} \approx 13 \). As illustrated in figure 1, the asymmetry parameter \(|1 - c_3/c_4|\) increases very steeply from zero especially for large values of \( H \), i.e. universes with large matter content.\(^2\)

\(^2\)Inhomogeneous situations do not distribute the total energy content of matter in a single homogeneous patch and thus refer to smaller values of \( H \). This could lead to results less sensitive to the precise value of fluctuations, but it would also put the bounce into a much stronger quantum regime or possibly eliminate it altogether (see also Bojowald 2007d).
5. Interpretation

Uncertainty relations do provide a limit on the asymmetry provided that the size of fluctuations is known since the covariance and thus squeezing is limited for given fluctuations. One may thus hope to find a reliable bound on present fluctuations, by which one could restrict the squeezing of the state of the universe and thus also the pre-big bang fluctuations, which would be remarkable. But how can we find a strong upper bound on current fluctuations in the state of the universe?

Density fluctuations are tiny and were even smaller at the time of decoupling as we can see them through the cosmic microwave background. These are classical fluctuations in a very nearly classical phase and quantum fluctuations must be even smaller. This cannot give a good estimate for our purposes, however, because these are fluctuations of inhomogeneities. They were already assumed to vanish in the isotropic model used. By putting an observational bound on something already assumed to vanish, one can certainly not derive anything new. Fluctuations we are dealing with are quantum fluctuations even of the isotropic variables of a universe.

Moreover, if we wanted to use real observations to restrict the state of the universe before the big bang, we would have to consider decoherence. Decoherence is certainly acting in the real universe and is usually understood as making quantum states more classical by interactions with a large environment of weakly interacting degrees of freedom. The huge number of degrees of freedom it requires is also ignored in the model; to compare observations with results in the model, we would thus have to factor out decoherence processes.

These considerations bring us back to the introductory remarks: we can use a solvable model as the most optimistic option. Here, one can explicitly compute the general behaviour of states and how some properties (e.g. before the big bang) affect others (e.g. those after the big bang). Making this model more realistic can only complicate the derivation of any such properties, for which

\[ \frac{A}{H} \]

Figure 1. The asymmetry (4.23) for the minimal uncertainty product $c_3 c_4$ as a function $|1 - c_3/c_4| = 2/(1 - (1/4)(h^2/H^2)x^2(x^2 - 1)^{-1/2} - 1)$ of $x = A/H$. Curves are shown for $H=10$ (solid line), $H=15$ (long dashed line), $H=25$ (short dashed line), $H=50$ (dotted line), $H=100$ (long dot-dashed line) and $H=500$ (short dot-dashed line), with larger $H$ corresponding to steeper curves.
decoherence would be an example. Even if we take the model for real, there are limitations on knowledge of the precise form of the pre-big bang state, such as its fluctuations. The model clearly shows us that, using observations only after the big bang, we would have to determine the precise squeezing of the state to extrapolate all the way to the state before the big bang. Doing this would require so much control that, even disregarding decoherence, it should be considered hopeless, the existence of upper bounds on squeezing for given fluctuations notwithstanding.

What could one infer about the pre-big bang state in a hypothetical harmonic universe that follows the solutions provided here and does not show decoherence? In principle, this should give one access to properties of the wave function. Quantum variables such as $G^{\nu\nu}$ contain the pre-bounce quantities only with tiny suppression factors $e^{-2\phi}$ and thus do not provide insights. One would have to measure current fluctuations very precisely to infer the state parameters. Assuming that the state is coherent, (4.20) shows us what has to be measured to determine $C$. (If the state cannot be assumed coherent, changes between the pre- and post-bounce phases would be much more pronounced.)

We thus need to find the three parameters $A$, $H$ and $\Delta H$ from observations. This is subtle because the asymmetry depends on $A$ and $H$ only by the ratio $A/H$, which for purposes here can be assumed to be just unity, and $A^2 - H^2 \sim O(A\hbar)$. The latter is important for the precise magnitude of $C$, but arises as a small value from a near cancellation between two large quantities. Figure 1 clearly illustrates how sensitive the asymmetry is to small changes in the parameters, especially for the realistic case of large $H$. To understand the origin of this quantity, it is important to consider a solvable model with precise access to dynamical coherent states.

The Hamiltonian $H$ quantifies the matter content and can possibly be measured well. Similarly, matter fluctuations $\Delta H$ can be granted to be under good observational control. On the other hand, difficulties arise for the parameter $A$. It determines the scale $v$ of the universe at the bounce, but this can hardly be used for an observational determination. Alternatively, one could use the present $v$ and eliminate the factor $e^{\phi}$, the ratio of the present scale to the bounce scale, through the age of the universe. The internal time $\phi$ is not the proper time $\tau$, but they are related through $\tau(\phi) \propto \int_{0}^{\phi} \cosh^{3/2(1-\nu)}(z) \, dz$. Both potentially observable quantities, $v$ and $e^{\phi(\tau)}$, are huge and provide $A$ in their ratio. Relative uncertainties in each measurement will provide large absolute uncertainties for $A$, and yet we need to produce a subtle cancellation with $H$ to find the precise $C$.

This prevents sufficient observational control on the state in order to determine its pre-bounce fluctuations, even though the evolution is deterministic. We have presented details only for the moments of order 2, but all orders behave similarly. There are thus infinitely many variables lacking for a precise knowledge of the pre-bounce state, not just the fluctuation. Thus, the way the classical singularity is removed in loop quantum cosmology may present a well-defined universe scenario without divergences of energy density, but it does not allow us to know precisely what happened before the big bang: the past is shrouded by cosmic forgetfulness as introduced and discussed further by Bojowald (2007b).

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