CONTINUED FRACTIONS
WITH ODD PARTIAL QUOTIENTS

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ABSTRACT. Consider the representation of a rational number as a continued fraction, associated with ”odd” Euclidean algorithm. In this paper we prove certain properties for the limit distribution function for sequences of rationals with bounded sum of partial quotients.

1 Introduction and main results

The classical Euclidean algorithm leads to ordinary continued fraction expansion of a real number

\[ x = [b_0; b_1, b_2, \ldots, b_l, \ldots] = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ldots + \frac{1}{b_l + \ldots}}}, \tag{1} \]

where \( b_0 \in \mathbb{Z}, b_j \in \mathbb{N} \) for \( j \geq 1 \). For rational \( x \) this representation is finite.

There are different kinds of Euclidean algorithms (for example, ”by-excess”, ”centered”, ”odd” Euclidean algorithms). Each of them is associated with a kind of continued fraction expansion of a real number (such fractions can be found in the book [6] by O.Perron).

Cases of ”by-excess” and ”centered” Euclidean algorithms were considered in papers [3], [4] correspondingly. In this paper we consider ”odd” Euclidean algorithm. This algorithm uses ”odd” division, i.e.

\[ a = bq + r, \quad q = 2 \left\lfloor \frac{a}{2b} \right\rfloor + 1, \quad -b < r \leq b, \]

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and leads to the following representation of a number $x \in \mathbb{Q} \cap [0, 1]$

$$x = \left[1; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}\right] = 1 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \ldots + \frac{\varepsilon_l}{a_l}}}, \quad (2)$$

where all $a_i$ are odd, $\varepsilon_i = \pm 1$ ($\varepsilon_1 = -1$) and $a_j + \varepsilon_{j+1} \geq 2$ for $j \geq 1$. If the last partial quotient is $a_l = 1$, then $\varepsilon_l = 1$ for uniqueness of the representation. For irrational $x$ representation $(2)$ is infinite. We’ll call this representation odd continued fraction. One can find these fractions in paper [7].

Let us call $F(x)$ by limit distributional function of sequence $\mathcal{M}_n$, where $\mathcal{M}_n$ is a final subset of segment $[0, 1]$, if

$$\lim_{n \to \infty} F_n(x) = F(x),$$

where

$$F_n(x) = \frac{\# \{\xi \in \mathcal{M}_n : \xi \leq x\}}{\# \mathcal{M}_n}, \quad x \in [0, 1].$$

We denote the sum of all partial quotients of representation $(2)$ of a rational number $x \in [0, 1]$ by

$$S(x) = \sum_{j=1}^{l} a_j,$$

and put

$$\mathcal{M}_n = \{x \in \mathbb{Q} \cap [0, 1] : S(x) \leq n + 1\}.$$

In such a way limit distributional function can be defined for any kind of continued fraction representation. For ordinary continued fractions function $F(x)$ coincides with famous Minkowski’s question mark function $\exists(x)$ (properties of $\exists(x)$ were investigated in [1], [8]). For regular reduced continued fractions (“by-excess” Euclidean algorithm) and for continued fractions with minimal reminders (“centered” Euclidean algorithm) functions $F(x)$ were described in papers [3, 4] correspondingly. In present paper we consider the function $F(x)$ for odd continued fractions.

The main result is the following theorem.
Theorem 1. Suppose that \( x \in [0, 1] \) is represented in the form (2), then

\[
F(x) = 1 - \sum_{i=1}^{\infty} \frac{E_i}{\lambda A_i},
\]

where

\[
E_i = \prod_{j=1}^{i} (-\epsilon_j), \quad A_i = \sum_{j=1}^{i} a_j - 1,
\]

and \( \lambda \) is the unique real root of the equation

\[
\lambda^3 - \lambda^2 - \lambda - 1 = 0.
\]

For rational \( x \) the sum in formula (3) is finite.

As a consequence of Theorem 1 we prove a formula for \( F(x) \) in terms of partial quotients of ordinary continued fraction.

Corollary 1. Suppose that \( x \in [0, 1] \) is represented in the form (1), then we have

\[
F(x) = 1 - \sum_{i=1}^{\infty} (-1)^{i+1} \frac{c_i}{\lambda S_i(x) - 1},
\]

where

\[
c_i = \begin{cases} 
1, & b_i \text{ is odd} \\
1 + \frac{1}{\lambda}, & b_i \text{ is even}
\end{cases}
\]

\[
S_i(x) = b_1 + \ldots + b_i + \#\{j \leq i : b_j \text{ is even}\}.
\]

For rational \( x \) the sum in formula (4) is finite.

In this paper we also prove the following result.

Proposition 1. For \( x \in [0, 1] \), \( n \in \mathbb{N} \) function \( F(x) \) satisfies following functional equations

\[
\frac{1 - F(1 - x)}{\lambda^{2n-1}} = \frac{1}{\lambda^{2n-2}} - 1 + F \left( 1 - \frac{1}{2n - 1 + x} \right),
\]

\[
\frac{1 - F(1 - x)}{\lambda^{2n}} = 1 - F \left( 1 - \frac{1}{2n + \frac{1}{x}} \right).
\]
It is more convenient for us to consider representation of $x \in \mathbb{Q} \cap [0, 1]$ with the first partial quotient $a_0 = 0$ instead of 1:

\[
\left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] := 1 - \left[ 1; \frac{-\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] = \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots + \frac{\varepsilon_l}{a_l}}},
\]

(5)

where $\left[ 1; \frac{-\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right]$ is representation of number $1 - x$ in the form (2).

The limit distributional function corresponding to this representation we denote by $F^0(x)$. For function $F^0(x)$ we prove the following results.

**Theorem 2.** Suppose that $x \in [0, 1]$ is represented in the form (5), then

\[
F^0(x) = -\sum_{i=1}^{\infty} \frac{E_i}{\lambda^{A_i}},
\]

(6)

where

\[
E_i = \prod_{j=1}^{i} (-\varepsilon_j), \quad A_i = \sum_{j=1}^{i} a_j - 1,
\]

and $\lambda$ is the unique real root of the equation

\[
\lambda^3 - \lambda^2 - \lambda - 1 = 0.
\]

For rational $x$ the sum in formula (6) is finite.

**Proposition 2.** For $x \in [0, 1]$, $n \in \mathbb{N}$ function $F^0(x)$ satisfies following functional equations

\[
\frac{F^0(x)}{\lambda^{2n-1}} = \frac{1}{\lambda^{2n-2}} - F^0\left(\frac{1}{2n-1+x}\right),
\]

\[
\frac{F^0(x)}{\lambda^{2n}} = F^0\left(\frac{1}{2n+x}\right).
\]

**Proposition 3.** For all $x \in [0, 1]$ we have

\[
F(x) = 1 - F^0(1 - x).
\]
Thus, Theorem 1 and Proposition 1 follow immediately from Theorem 2, Proposition 2 and Proposition 3. So our main aim is to prove results for function \( F^0(x) \).

In the end of the paper we prove the following theorem.

**Theorem 3.** Let for \( x \in [0, 1] \) the derivative \( F'(x) \) (finite or infinite) exists. Then either \( F'(x) = 0 \) or \( F'(x) = \infty \).

As function \( F(x) \) is monotonic, then by Lebesgue’s theorem, the derivative \( F'(x) \) exists and is finite almost everywhere (in the sense of Lebesgue measure). That is why \( F'(x) = 0 \) almost everywhere. In other words, \( F(x) \) is a singular function.

## 2 Auxiliary results

Let us denote by \( S^0(x) \) sum of partial quotients of representation (5) of a number \( x \in \mathbb{Q} \cap [0, 1] \). We define sequences of sets \( Y_n \) and \( X_n \) in the following way:

\[
Y_n := \{ x \in \mathbb{Q} \cap [0, 1] : S^0(x) \leq n + 1 \}, \\
X_k = \{ x \in \mathbb{Q} \cap [0, 1] : S^0(x) = k + 1 \},
\]

where \( n, k \geq 1 \).

It is clear that

\[
Y_n = \bigcup_{1 \leq k \leq n} X_k.
\]

Suppose that the elements of \( Y_k \) are arranged in increasing order. The number of elements of \( Y_n, X_n \) we denote by \( Y_n, X_n \) correspondingly.

Particularly, \( X_1 = \{ \frac{1}{2} \}, X_2 = \{ \frac{1}{3}, \frac{2}{3} \}, X_3 = \{ \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \}, X_4 = \{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5} \} \).

So \( X_1 = 1, X_2 = 2, X_3 = 3, X_4 = 6 \).

**Lemma 1.** For \( n \geq 1 \) we have

\[
X_{n+3} = X_{n+2} + X_{n+1} + X_n.
\]

**Proof.** We construct one-to-one correspondence \( \Phi \) between elements of sets \( X_{n+2} \cup X_{n+1} \cup X_n \) and \( X_{n+3} \).

Let \( x \in X_{n+2} \cup X_{n+1} \cup X_n, x = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] \), we define \( \Phi(x) : X_{n+2} \cup X_{n+1} \cup X_n \rightarrow X_{n+3} \) in the following way:
• In case \( x \in X_{n+2} \) if \( a_l = 1 \), then
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-2}}{a_{l-2}}, \frac{\varepsilon_{l-1}}{a_{l-1} + 2} \right] \in X_{n+3},
\]
else
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] \in X_{n+3}.
\]
• In case \( x \in X_{n+1} \) if \( a_l = 1 \), then \( \varepsilon_l = 1 \) and
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{1}{1}, \frac{1}{1} \right] \in X_{n+3},
\]
else
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}, \frac{-1}{1}, \frac{1}{1} \right] \in X_{n+3}.
\]
• In case \( x \in X_n \) if \( a_l = 1 \), then
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-1}}{a_{l-1}}, \frac{1}{1}, \frac{1}{1} \right] \in X_{n+3},
\]
else
\[
\Phi(x) = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}, \frac{-1}{1}, \frac{1}{1} \right] \in X_{n+3}.
\]

The correspondence \( \Phi(x) \) is injective by the construction. Let us show that it is surjective. For any \( y \in X_{n+3} \), \( y = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] \) we find the preimage \( x \) of \( y \).

• If \( a_l > 1 \) then
\[
x = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l - 2}, \frac{1}{1} \right] \in X_{n+2}.
\]
• If $a_l = 1$ and $a_{l-1} > 1$, then

$$x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-1}}{a_{l-1}} \right] \in \mathcal{X}_{n+2}.$$ 

• If $a_l = a_{l-1} = 1$, $\varepsilon_{l-1} = -1$, then $a_{l-2} > 1$, therefore

$$x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-2}}{a_{l-2}} \right] \in \mathcal{X}_{n+1}.$$ 

• If $a_l = a_{l-1} = 1$, $\varepsilon_{l-1} = 1$, then either $a_{l-2} > 1$ and

$$x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-2}}{a_{l-2}} - \frac{1}{2} \right] \in \mathcal{X}_n$$

or $a_{l-2} = 1$. In this case either $\varepsilon_{l-2} = 1$ and

$$x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-3}}{a_{l-3}} \right] \in \mathcal{X}_n$$

or $\varepsilon_{l-2} = -1$, then $a_{l-3} > 1$ and

$$x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-3}}{a_{l-3}} \right] \in \mathcal{X}_n$$

Lemma is proved. ■

Lemma 2. For $n \geq 1$ we have

$$Y_{n+3} = Y_{n+2} + Y_{n+1} + Y_n + 2.$$ \hspace{2cm} (7)

Proof. By the definition of $\mathcal{Y}_n$ and Lemma 1 we get

$$Y_{n+2} + Y_{n+1} + Y_n =$$

$$= (X_1 + \ldots + X_{n+2}) + (X_1 + \ldots + X_{n+1}) + (X_1 + \ldots + X_n) =$$

$$= X_1 + X_2 + X_3 + X_4 + \ldots + X_{n+3} + (X_1 - X_3) = Y_{n+3} - 2.$$ ■
We remind the definition of the Stern-Brocot sequences $F_n$, $n = 0, 1, 2, \ldots$.

Consider two-point set $F_0 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$. Let $n \geq 0$ and

$$F_n = \left\{ 0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1 \right\},$$

where $x_{j,n} = \frac{p_{j,n}}{q_{j,n}}$, $(p_{j,n}, q_{j,n}) = 1$, $j = 0, \ldots, N(n)$ and $N(n) = 2^n + 1$.

Then

$$F_{n+1} = F_n \cup Q_{n+1}$$

with

$$Q_{n+1} = \{ x_{j-1,n} \oplus x_{j,n}, \quad j = 1, \ldots, N(n) \}.$$

Here

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a + b}{c + d}$$

is the mediant of fractions $\frac{a}{b}$ and $\frac{c}{d}$.

It is convenient to represent sequences $F_n$ by means of the binary tree $D^{[0]}$ (Figure 1). This tree is called Stern-Brocot’s tree. Nodes of the tree are labeled by rationals from interval $(0, 1)$ and partitioned into levels by the following rule: $n$-th level consists of nodes, labeled by numbers from $Q_n$.

It is possible to distribute nodes of the tree into levels by another way. For example, we can use such a rule: $n$-th level consists of nodes labeled by numbers $x$, such that $S^0(x) = n + 1$. We denote this tree by $D$ (Figure 2).
Example.

\[
\begin{align*}
\frac{1}{2} &= \left[0; \frac{1}{1}, \frac{1}{1}\right], \\
\frac{1}{3} &= \left[0; \frac{1}{3}, \frac{1}{1}\right], \\
\frac{3}{5} &= \left[0; \frac{1}{1}, \frac{1}{1}, \frac{1}{1}\right], \\
\frac{2}{5} &= \left[0; \frac{1}{3}, \frac{-1}{1}, \frac{1}{1}\right].
\end{align*}
\]

Any node \(\xi\) of the tree \(\mathcal{D}\) is a root of a subtree, which we denote by \(\mathcal{D}(\xi)\) \((\mathcal{D} = \mathcal{D}^{(1/2)}\)). We denote by \(D_n^{(\xi)}\) the number of nodes of \(\mathcal{D}(\xi)\) from the level 1 to the level \(n\) (particularly, \(D_n^{(1/2)} = \sharp \mathcal{D}_n\)).

Let us consider more detailed structure of the tree \(\mathcal{D}\). From every node \(\xi\) of \(\mathcal{D}\) we issue two arrows: the left one and the right one. The left one goes to the node labeled by \(\xi^l\) and the right one goes to node labeled by \(\xi^r\). Note that if \(\xi = x \oplus y\), where \(x, y\) are consecutive elements of \(F_n\), then \(\xi^l = x \oplus \xi\), \(\xi^r = \xi \oplus y\) (let us call \(\xi^l\) and \(\xi^r\) successor of \(\xi\)). There are arrows of two kinds: short and long. Short arrow from \(\xi\) to \(\eta\), where \(\eta \in \{\xi^l, \xi^r\}\), means that

\[
S^0(\eta) - S^0(\xi) = 1,
\]

and long arrow means that

\[
S^0(\eta) - S^0(\xi) = 2. \quad (8)
\]
Let us call a node with two short arrows by node of first type, and a node with one short and one long arrow by node of second type.

**Proposition 4.** Suppose that \( \xi = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] \). If \( a_l = 1 \), then \( \xi \) is a node of first type and if \( a_l > 1 \), then \( \xi \) is a node of second type.

**Proof.** If \( a_l = 1 \), then

\[
\xi^l, \xi^r \in \left\{ \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{l-1}}{a_{l-1}} + 2 \right], \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} + 1 \right] \right\}.
\]

If \( a_l > 1 \), then

\[
\xi^l, \xi^r \in \left\{ \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} + 1 \right], \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}, -1, 1 \right] \right\}.
\]

So the equality (8) occurs only in the last case. □

From Proposition 4 and construction of the tree \( D \) we deduce the following statement.

**Lemma 3.** Suppose that \( \xi = \left[ 0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] \), then

\[
D^{(\ell)}_n = \begin{cases} 
D_{n-S_0(\xi)+2}^{(1/2)}, & \text{if } a_l = 1 \\
D_{n-S_0(\xi)+3}^{(1/3)}, & \text{if } a_l > 1.
\end{cases}
\]

Note that \( D_n^{(1/2)} = Y_n \). For brevity we put \( D_{n+1}^{(1/3)} = Z_n \). From construction of the tree \( D \) it is clear that

\[
Y_n = Y_{n-1} + Z_{n-1} + 1. \tag{9}
\]

For \( Y_n \) we have recurrence formula (7). Using equality (9) it is easy to prove a similar formula for \( Z_n \):

\[
Z_{n+3} = Z_{n+2} + Z_{n+1} + Z_n + 2.
\]

In paper [4] the author obtained the following result.

**Proposition 5.** Let \( \lambda \) be the unique real root of the equation

\[
\lambda^3 - \lambda^2 - \lambda - 1 = 0,
\]

Then

\[
\lim_{n \to \infty} \frac{Y_n}{Y_{n+1}} = \lim_{n \to \infty} \frac{Z_n}{Z_{n+1}} = \frac{1}{\lambda}, \quad \frac{Y_n}{Z_n} = \frac{1}{\lambda - 1}.
\]

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3 Proof of results for function $F^0(x)$

In order to prove Theorem 2 we need the following lemma.

**Lemma 4.** Suppose that $x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}, \ldots\right]$, then

$$F_n^0(x) = -\left(\sum_{i=1}^{\infty} E_i D_n^\left[\frac{0,\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, 1\right]\right)/D_n^{1/2},$$

where

$$E_i = \prod_{j=1}^{i}(-\varepsilon_j).$$

**Proof.** We will prove the lemma by induction on the length $l$ of odd continued fraction representation of $x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l}\right]$.

For $l = 1$ we have

$$F_n^0(\left[0; 1/a_1\right]) = \frac{\#\{\xi \in Y_n : \xi \leq 1/a_1\}}{Y_n} = \frac{D_n^{0/1+1/a_1}}{D_n^{1/2}} = \frac{D_n^{\left[0; 1/a_1, 1\right]}}{D_n^{1/2}}.$$

Now suppose that $l = i + 1$ and the lemma is true for $l \leq i$.

Put $a = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, \frac{\varepsilon_{i+1}}{a_i+1}\right]$, $b = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}\right]$, $c = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_{i-1}}{a_{i-1}}\right]$ (if the last partial quotient of $a$, $b$ or $c$ is $-1/1$, we replace it by $1/1$ and decrease previous partial quotient by 2). Suppose that $c < b$ (in case $c < b$ the proof is analogously). By assumption of induction we have

$$F_n^0(b) - F_n^0(c) = \frac{D_n^{\left[0,\frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, 1\right]}}{D_n^{1/2}}.$$

Taking into account the fact that $a$ and $b$ are consecutive elements of an element of Stern-Brocot sequence (as consecutive convergents of number $x$) we have

$$a \oplus b = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, \frac{\varepsilon_{i+1}}{a_i+1}, 1\right].$$
By definition of $F_n$

$$F_n^0(\max (a, b)) - F_n^0(\min (a, b)) = \frac{\sharp \{ \xi \in Y_n : \min (a, b) < \xi \leq \max (a, b) \}}{Y_n} = \frac{D_n^{a \oplus b}}{D_n^{1/2}} = \frac{D_n^{\left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, 1/a_{i+1}, 1\right]}}{D_n^{1/2}}.$$  

For $\varepsilon_{i+1} = 1$ we have $a \in (c, b)$, so

$$\sharp \{ a < \xi \leq b \} = \sharp \{ c < \xi \leq b \} - \sharp \{ a < \xi \leq b \}$$

and

$$F_n^0(a) - F_n^0(c) = \frac{D_n^{\left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, 1/a_{i+1}, 1\right]}}{D_n^{1/2}}.$$

For $\varepsilon_{i+1} = -1$ we have $a > b$, so

$$\sharp \{ b < \xi \leq a \} = \sharp \{ c < \xi \leq b \} + \sharp \{ b < \xi \leq a \}$$

and

$$F_n^0(a) - F_n^0(c) = \frac{D_n^{\left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, 1/a_{i+1}, 1\right]} + D_n^{\left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_i}{a_i}, -1/a_{i+1}, 1\right]}}{D_n^{1/2}}.$$  

**Proof of the Theorem 2.** By Lemma 3 and Proposition 5 we have

$$\frac{D_n^{[0: \varepsilon_1/a_1, \ldots, \varepsilon_i/a_i, 1/1]} - D_n^{[0: \varepsilon_1/a_1, \ldots, \varepsilon_i/a_i, 1/1]}}{D_n^{1/2}} = Y_n - (\sum_{j=1}^{i} a_j + 2) = \lambda^{-1}.$$  

So formula (6) follows immediately from Lemma 4. 

**Proof of Proposition 2.** Suppose that $x = \left[0; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_n}{a_n}, \ldots \right]$ is representation of $x$ in the form (5). Then we have

$$\frac{1}{2n - 1 + x} = \left[0; \frac{1}{2n - 1} \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_n}{a_n}, \ldots \right]$$

and

$$\frac{1}{2n + \frac{1}{x}} = \left[0; \frac{\varepsilon_1}{2n + a_1}, \frac{\varepsilon_2}{a_2}, \ldots, \frac{\varepsilon_n}{a_n}, \ldots \right].$$
Now it is only left to apply Theorem 1 to these numbers.

**Proof of Proposition 3.** Suppose that \( x \in \mathbb{Q} \) and

\[
x = \left[ 1; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_n}{a_n} \right]
\]

is representation of \( x \) in the form (2).

Representation (5) is connected with (2) in the following way

\[
1 - \left[ 1; \frac{\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right] = \left[ 0; \frac{-\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_l}{a_l} \right].
\]

So we have

\[
1 - x = \left[ 0; \frac{-\varepsilon_1}{a_1}, \ldots, \frac{\varepsilon_n}{a_n} \right].
\]

Consequently

\[
S(x) = 1 + a_1 + \ldots + a_n = 1 + S^0(1 - x).
\]

That is why we have

\[
F_n(x) = \frac{\# \{ \xi \in \mathcal{M}_n : \xi \leq x \}}{\# \{ \xi \in \mathcal{M}_n \}} = \frac{\# \{ 1 - \xi \in \mathcal{Y}_{n-1} : 1 - \xi \geq 1 - x \}}{Y_{n-1}} = 1 - F^0_n(1 - x).
\]

For irrational \( x \in [0, 1] \) we should take into account continuity of considered functions.

**4 Proof of Corollary 1**

Function \( F(x) \) can be also expressed in terms of partial quotients of ordinary continued fraction.

Suppose that \( x \in \mathbb{Q} \cap [0, 1] \) is represented in the form of ordinary continued fraction:

\[
x = [0; b_1, \ldots, b_n].
\]

We describe the algorithm for converting this fraction into a fraction of the form (2). For the fist \( i \) such that \( b_i \) is even we use one of the following
identities:
\[
b_i + \frac{1}{b_{i+1} + \alpha} = b_i + 1 - \frac{1}{1 + \frac{1}{b_{i+1} - 1 + \alpha}},
\]
(10)
\[
b_i + \frac{1}{1 + \frac{1}{b_{i+2} + \alpha}} = b_i + 1 - \frac{1}{b_{i+2} + 1 + \alpha},
\]
(11)
where \(\alpha\) is the "tail" of the fraction.

In case \(b_{i+1} > 1\) we use identity (10) whereas in case \(b_{i+1} = 1\) we use identity (11). Then we apply the same procedure to the obtained fraction.

To prove Corollary 1 we should apply Theorem 1 to the result of the procedure described above. Taking into account the fact that in case \(b_i\) is even, if \(b_{i+1} > 1\) we have
\[
\frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) - 1} + \frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) + 1 - 1} =
\]
\[
= \frac{1}{\lambda b_1 + \ldots + b_{i-1} + b_{i+1} - 1 - 1} \left( 1 + \frac{1}{\lambda} \right) - \frac{1}{\lambda b_1 + \ldots + b_{i-1} + b_{i+1} + 1 - 1}
\]
and if \(b_{i+1} = 1\) then
\[
\frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) - 1} + \frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+2}) - 1} =
\]
\[
= \frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) - 1} \left( 1 + \frac{1}{\lambda} \right) - \frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) + 1 - 1} +
\]
\[+ \frac{1}{\lambda b_1 + \ldots + b_{i-1} + (b_{i+1}) + 1 + b_{i+2} - 1}.
\]
So we get formula (4).

5 Singularity of the function \(F(x)\)

In this section we prove Theorem 3. At first we consider the case \(x \in \mathbb{Q}\).

Lemma 5. For rational \(x \in [0, 1]\) we have \(F'(x) = 0\).
Proof. As \( x \in \mathbb{Q} \), so there exists such \( n \), that \( a/b \in \mathcal{X}_n \). By \( p/q, p'/q' \) we denote the left and the right neighbouring to \( a/b \) elements in \( \mathcal{Y}_n \) correspondingly. Sequences of mediants
\[
\xi_k = \frac{ka + p}{kb + q}, \quad \xi'_k = \frac{ka + p'}{kb + q'}
\]
converge to \( a/b \) from the left and from the right correspondingly as \( k \to \infty \).

For consecutive elements \( x, y \) of \( \mathcal{Z}_n \) the ration
\[
(F(x \oplus y) - F(x)) : (F(y) - F(x \oplus y))
\]
can take values \( \{\frac{\lambda}{\lambda+1}, \frac{1}{\lambda+1}, \frac{\lambda-1}{\lambda}, \frac{1}{\lambda}\} \), so
\[
0 \leq F(x) - F(\xi_k) \leq \left( \max \left\{ \frac{\lambda}{\lambda+1}, \frac{1}{\lambda+1}, \frac{\lambda-1}{\lambda}, \frac{1}{\lambda} \right\} \right)^k (F(x) - F(p/q)) = \left( \frac{\lambda}{\lambda+1} \right)^k (F(x) - F(p/q)).
\]

Analogously
\[
0 \leq F(\xi'_k) - F(x) \leq \left( \frac{\lambda}{\lambda+1} \right)^k (F(p'/q') - F(x)).
\]

Taking into account the fact that
\[
\xi'_k - \xi_k = \frac{2k + 1}{(kb + q')(kb + q)} \geq \frac{2k + 1}{k^2(b + p')(b + p)}
\]
we have
\[
0 \leq \lim_{k \to \infty} F(\xi'_k) - F(\xi_k) \leq \lim_{k \to \infty} \left( \frac{\lambda}{\lambda+1} \right)^k \frac{(F(p'/q') - F(p/q))}{2k + 1} = 0
\]

Now we should prove the Theorem for irrational \( x \in [0, 1] \).

Given \( n \) we can find two consecutive elements \( p_n/q_n < p'_n/q'_n \) from the set \( \mathcal{F}_n \) such that \( p_n/q_n < x < p'_n/q'_n \). In such a way we obtain an infinite sequence of pairs of elements \( \{p_n/q_n, p'_n/q'_n\} \), converging to \( x \) from the left and from the right correspondingly. Note that \( p_n/q_n, p'_n/q'_n \) are always among the intermediate and convergent fractions to \( x \) in the sense of ordinary continued fraction.
Lemma 6. There are two possibilities

1. There are infinitely many \(i\) such that for three consecutive pairs \(\{p_i, q_i\}, \{p_{i+1}, q_{i+1}\}\), \(\{p_{i+2}, q_{i+2}\}\), the following equalities hold:

\[
F\left(\frac{p_{i+1}}{q_{i+1}}\right) - F\left(\frac{p_i}{q_i}\right) \in \left\{\lambda - 1, \frac{1}{\lambda}\right\}
\]

(12)

\[
F\left(\frac{p_{i+2}}{q_{i+2}}\right) - F\left(\frac{p_{i+1}}{q_{i+1}}\right) \in \left\{\lambda - 1, \frac{1}{\lambda}\right\}
\]

(13)

2. There are infinitely many \(i\) such that for three consecutive pairs \(\{p_i, q_i\}, \{p_{i+1}, q_{i+1}\}\), \(\{p_{i+2}, q_{i+2}\}\), the following equalities hold:

\[
F\left(\frac{p_{i+1}}{q_{i+1}}\right) - F\left(\frac{p_{i+1}}{q_{i+1}}\right) \in \left\{\frac{\lambda}{\lambda + 1}, \frac{1}{\lambda + 1}\right\}
\]

(14)

\[
F\left(\frac{p_{i+2}}{q_{i+2}}\right) - F\left(\frac{p_{i+1}}{q_{i+1}}\right) \in \left\{\lambda - 1, \frac{1}{\lambda}\right\}
\]

(15)

Proof of the Lemma. Let \(x, y\) be consecutive elements of \(F_n\) for some \(n\). Then by Proposition

\[
(F(x \oplus y) - F(x)) : (F(y) - F(x \oplus y)) = \begin{cases} 
\lambda - 1 \quad \text{or} \quad \frac{1}{\lambda - 1}, & \text{if } x \oplus y \text{ is a vertex of the first type}, \\
\lambda \quad \text{or} \quad \frac{1}{\lambda}, & \text{if } x \oplus y \text{ is a vertex of the second type}.
\end{cases}
\]

Suppose that first case of the Lemma is not hold true. Then there are infinitely many \(i\) such that \(14\) holds. But if \(\frac{p_i}{q_i} \oplus \frac{p_i'}{q_i'}\) is a vertex of second type, then both \(\left(\frac{p_i}{q_i} \oplus \frac{p_i'}{q_i'}\right)^l\) and \(\left(\frac{p_i}{q_i} \oplus \frac{p_i'}{q_i'}\right)^r\) are vertexes of first type. That is why \(15\) holds.
Proof of the Theorem. Suppose that \( F'(x) = a \), where \( a \) is finite and \( a \neq 0 \). By definition of derivative we have

\[
\lim_{i \to \infty} \frac{F\left(\frac{p'_i}{q'_i}\right) - F\left(\frac{p_i}{q_i}\right)}{\frac{p'_i}{q'_i} - \frac{p_i}{q_i}} = a \neq 0,
\]

consequently

\[
\frac{F\left(\frac{p'_{i+1}}{q'_{i+1}}\right) - F\left(\frac{p_{i+1}}{q_{i+1}}\right)}{F\left(\frac{p'_i}{q'_i}\right) - F\left(\frac{p_i}{q_i}\right)} \sim \frac{\frac{p'_{i+1}}{q'_{i+1}} - \frac{p_{i+1}}{q_{i+1}}}{\frac{p'_i}{q'_i} - \frac{p_i}{q_i}}. \tag{16}
\]

And since \( \left\{\frac{p_{i+1}}{q_{i+1}}, \frac{p'_{i+1}}{q'_{i+1}}\right\} \) is either \( \left\{\frac{p_i}{q_i}, \frac{p_i + p'_i}{q_i + q'_i}\right\} \) or \( \left\{\frac{p_i + p'_i}{q_i + q'_i}, \frac{p'_i}{q'_i}\right\} \) then

\[
\frac{\frac{p'_{i+1}}{q'_{i+1}} - \frac{p_{i+1}}{q_{i+1}}}{\frac{p'_i}{q'_i} - \frac{p_i}{q_i}} = \frac{1}{q_{i+1}q_i} \in \left\{\frac{q'_i}{q_i + q'_i}, \frac{q_i}{q_i + q'_i}\right\}.
\]

As

\[
\frac{q'_{i_k}}{q_{i_k} + q'_{i_k}} + \frac{q_{i_k}}{q_{i_k} + q'_{i_k}} = 1
\]

so if the first case of lemma 6 holds for a sequence \( \{i_k\}_{k=1}^{\infty} \), then either

\[
\lim_{i_k \to \infty} \frac{q'_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{\lambda - 1}{\lambda} \quad \text{and} \quad \lim_{i_k \to \infty} \frac{q_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{1}{\lambda} \tag{17}
\]

or

\[
\lim_{i_k \to \infty} \frac{q'_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{1}{\lambda} \quad \text{and} \quad \lim_{i_k \to \infty} \frac{q_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{\lambda - 1}{\lambda}. \tag{18}
\]

If the second case holds, then either

\[
\lim_{i_k \to \infty} \frac{q'_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{\lambda}{\lambda + 1} \quad \text{and} \quad \lim_{i_k \to \infty} \frac{q_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{1}{\lambda + 1} \tag{19}
\]

or

\[
\lim_{i_k \to \infty} \frac{q'_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{1}{\lambda + 1} \quad \text{and} \quad \lim_{i_k \to \infty} \frac{q_{i_k}}{q_{i_k} + q'_{i_k}} = \frac{\lambda}{\lambda + 1}. \tag{20}
\]
Analogously the pair \( \left\{ \frac{p_{i+2}+p'}{q_{i+2}}, \frac{p_{i+2}}{q_{i+2}} \right\} \) can take one of values \( \left\{ \frac{p_{i}+2p'}{q_{i}+2q'}, \frac{p_{i}+2p'}{q_{i}+2q'} \right\} \), \( \frac{p_{i}+p'}{q_{i}+q'} \) or \( \frac{p_{i}+2p'}{q_{i}+2q'}, \frac{p_{i}+p'}{q_{i}+q'} \). So

\[
\frac{p_{i+2}+p'}{q_{i+2}+2q'} - \frac{p_{i+2}}{q_{i+2}+2q'} = \frac{1}{q_{i+2}+2q'} = \begin{cases} \frac{q_{i}}{2q_{i}+q'} & \text{or} \frac{q_{i}+q'}{2q_{i}+q'}, \text{if} \; x < \frac{p_{i}+p'}{q_{i}+q'} \\ \frac{q_{i}}{q_{i}+2q'} & \text{or} \frac{q_{i}+q'}{q_{i}+2q'}, \text{if} \; x > \frac{p_{i}+p'}{q_{i}+q'} \end{cases}
\]

So in the first case of lemma \( \text{[6]} \) for \( x < \frac{p_{i}+p'}{q_{i}+q'} \) we have either

\[
\lim_{i_{k} \to \infty} \frac{q_{i_{k}}}{2q_{i_{k}}+q_{i_{k}}'} = \frac{\lambda-1}{\lambda} \quad \text{and} \quad \lim_{i_{k} \to \infty} \frac{q_{i_{k}}+q_{i_{k}}'}{2q_{i_{k}}+q_{i_{k}}'} = \frac{1}{\lambda} \quad (21)
\]

or

\[
\lim_{i_{k} \to \infty} \frac{q_{i_{k}}}{2q_{i_{k}}+q_{i_{k}}'} = \frac{1}{\lambda} \quad \text{and} \quad \lim_{i_{k} \to \infty} \frac{q_{i_{k}}+q_{i_{k}}'}{2q_{i_{k}}+q_{i_{k}}'} = \frac{\lambda-1}{\lambda} \quad (22)
\]

and for \( x > \frac{p_{i}+p'}{q_{i}+q'} \) we have either

\[
\lim_{i_{k} \to \infty} \frac{q_{i_{k}}+q_{i_{k}}'}{q_{i_{k}}+2q_{i_{k}}'} = \frac{\lambda-1}{\lambda} \quad \text{and} \quad \lim_{i_{k} \to \infty} \frac{q_{i_{k}}'+q_{i_{k}}'}{q_{i_{k}}+2q_{i_{k}}'} = \frac{1}{\lambda} \quad (23)
\]

or

\[
\lim_{i_{k} \to \infty} \frac{q_{i_{k}}+q_{i_{k}}'}{q_{i_{k}}+2q_{i_{k}}'} = \frac{1}{\lambda} \quad \text{and} \quad \lim_{i_{k} \to \infty} \frac{q_{i_{k}}'+q_{i_{k}}'}{q_{i_{k}}+2q_{i_{k}}'} = \frac{\lambda-1}{\lambda} \quad (24)
\]

Analogously in the second case.

By lemma \( \text{[6]} \) we can find sequence \( \{i_{k}\}_{k=1}^{\infty} \) such that for all \( i_{k} \) one of the following cases is realized:

1. Case 1 of Lemma holds, \( x < \frac{p_{i_{k}}+p'}{q_{i_{k}}+q_{i_{k}}'} \).
2. Case 1 of Lemma holds, \( x > \frac{p_{i_{k}}+p'}{q_{i_{k}}+q_{i_{k}}'} \).
3. Case 2 of Lemma holds, \( x < \frac{p_{i_{k}}+p'}{q_{i_{k}}+q_{i_{k}}'} \).
4. Case 2 of Lemma holds, \( x > \frac{p_{i_{k}}+p'}{q_{i_{k}}+q_{i_{k}}'} \).
Let us consider for example first of these cases. As case 1 of lemma holds, then one of equalities (17), (18) is satisfied. As $x < \frac{p_{ik} + p'_{ik}}{q_{ik} + q'_{ik}}$, then one of equalities (21), (22) is satisfied. In such way we get four variants:

- $\lim_{i_k \to \infty} \frac{q_{ik}'}{q_{ik} + q'_{ik}} = \frac{\lambda - 1}{\lambda}, \quad \lim_{i_k \to \infty} \frac{q_{ik}}{2q_{ik} + q'_{ik}} = \frac{\lambda - 1}{\lambda}. \quad (25)$

  Since $\frac{2q_{ik} + q'_{ik}}{q_{ik}} = 1 + \frac{q_{ik} + q'_{ik}}{q_{ik}'}$ we get from (25) incorrect equality

  $$\frac{\lambda}{\lambda - 1} = 1 + \frac{\lambda}{\lambda - 1}.$$  

- $\lim_{i_k \to \infty} \frac{q_{ik}'}{q_{ik} + q'_{ik}} = \frac{1}{\lambda}, \quad \lim_{i_k \to \infty} \frac{q_{ik}}{2q_{ik} + q'_{ik}} = \frac{\lambda - 1}{\lambda}. \quad (26)$

  Analogously to the previous case we get from (26) following equation:

  $$\lambda^2 - \lambda - 1 = 0.$$  

  But $\lambda$ is a root of irreducible equation of degree 3.

- $\lim_{i_k \to \infty} \frac{q_{ik}'}{q_{ik} + q'_{ik}} = \frac{\lambda - 1}{\lambda}, \quad \lim_{i_k \to \infty} \frac{q_{ik}}{2q_{ik} + q'_{ik}} = \frac{1}{\lambda}. \quad (27)$

  From (27) we get following equation:

  $$\lambda^2 - 3\lambda + 1 = 0.$$  

  But $\lambda$ is a root of irreducible equation of degree 3.

- $\lim_{i_k \to \infty} \frac{q_{ik}'}{q_{ik} + q'_{ik}} = \frac{1}{\lambda}, \quad \lim_{i_k \to \infty} \frac{q_{ik}}{2q_{ik} + q'_{ik}} = \frac{1}{\lambda}. \quad (28)$

  From (28) we get incorrect equation:

  $$\lambda = 1 + \lambda.$$  

So we get contradiction in all of examined cases. The rest of the cases can be examined analogously. ■
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