Geometric singular perturbation analysis to a perturbed (1 + 1)-dimensional dispersive long wave equation

Hang Zheng\textsuperscript{1,2} Y. H. Xia\textsuperscript{1}\textsuperscript{*}

1. Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China
   yhxia@zjnu.cn; xiadoc@163.outlook.com

2. Department of Mathematics and Computer, Wuyi University, Wuyishan, 354300, China
   zhenghang513@zjnu.edu.cn; zhenghwyxy@163.com

Abstract

The existence of the solitary wave and the nonexistence of kink (anti-kink) wave solutions are studied for a perturbed (1 + 1)-dimensional dispersive long wave equation. The methods are based on the geometric singular perturbation (GSP, for short) approach, Melnikov method and bifurcation analysis. The results show that the solitary wave solution with a suitable wave speed $c$ and parameter $\kappa$ exists under the small singular perturbation. Interestingly, unlike solitary wave solutions, the kink (anti-kink) wave solution doesn’t persist because the corresponding Melnikov function has no zeros. Further, numerical simulations are utilized to verify the correctness of our analytical results.

Key Words: Wave equation, travelling wave solution, geometric singular perturbation.

\textsuperscript{*}Corresponding author: Y. H. Xia, Email: xiadoc@outlook.com; yhxia@zjnu.cn.
1 Introduction

1.1 History

The study of water wave equations and their travelling wave solutions have attracted a lot of interest among scholars (see, e.g., [1–13]). For instance, Korteweg-de Vries (KdV) equation, Benjamin-Bona-Mahony (BBM) equation, Degasperis-Procesi (DP) equation and Camassa-Holm (CH) equation, etc. These equations have been extensively used to describe dynamical behaviour of long waves in shallow water. Since the interaction of nonlinear and dispersion factors, long wave in shallow water would admit many characteristics. One of the important property of long waves, is that, they retain their shapes and forms after mutual interactions and collisions. In 1996, Wu and Zhang [14] derived an equation describing nonlinear dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth format formulated as

\[
\begin{align*}
\theta_t + \theta \theta_x + u \theta_y + v_x &= 0, \\
u_t + \theta u_x + u u_y + v_y &= 0, \\
v_t + (\theta v)_x + (uv)_y + \frac{1}{3} (\theta_{xxx} + \theta_{xyy} + u_{xxy} + u_{yyy}) &= 0.
\end{align*}
\]

(1)

where \(\theta\) (resp., \(u\)) is the surface velocity of water along the \(x\) (resp., \(y\)) direction and \(v\) is the elevation of the water wave. Eq. (1) can be reduced to the following \((1+1)\)-dimensional dispersive long wave equation (Wu-Zhang equation) by symmetry reduction and scale transition:

\[
\begin{align*}
u_t &= -u u_x - v_x, \\
v_t &= -v u_x - u v_x - \frac{1}{3} u_{xxx}.
\end{align*}
\]

(2)

Due to it models the nonlinear water wave available, Wu-Zhang equation (2) is often applied to coastal design and harbor construction. So far, a lot of articles have been concerned with the exact solutions of Wu-Zhang equation (2) by using various methods (see, e.g., [15–21]). However, to the best of our knowledge, there are few works on the perturbed Wu-Zhang equation. In this paper, we study a perturbed Wu-Zhang equation.
described by
\[
\begin{aligned}
  u_t &= -uu_x - v_x, \\
  v_t &= -vu_x - uv_x - \frac{1}{3} u_{xxx} + \varepsilon (u_{xx} + u_{xxxx}),
\end{aligned}
\]
(3)

where \(0 < \varepsilon \ll 1\) (a sufficiently small parameter), \(u_{xx}\) is backward diffusion and \(u_{xx}\) represents a dissipation term. In fact, it is better to consider the small perturbations in the evolution equation to describe the real situation. To study the existence of traveling wave solutions for these perturbed equations is of great importance in hydrodynamics. There exists an extensive literature on the use of geometric singular perturbation theory \[22, 23\] to deal with these issues including KdV equation \[24, 25\], generalized KdV equation \[26–28\], delayed CH equation \[29\], generalized CH equation \[30\], CH Kuramoto-Sivashinsky equation \[31\], perturbed BBM equation \[32\], generalized BBM equation \[33\], random systems \[34–36\], biological model \[37, 38\], and so on.

Based on the geometric singular perturbation approach to track invariant manifolds of corresponding ordinary differential equations (ODEs), Du and Qiao \[39\] explained existence of traveling wave solutions in a Belousov-Zhabotinskii system with delay by combing Fredholm orthogonality and asymptotic theory. Du et al. \[40\] also confirmed the persistence of solitary wave solution of a generalized Keller-Segel system with small cell diffusion by Poincaré-Bendixson theorem. Clearly, the idea of detecting monotonicity of the ratio of Abelian integrals (MRAI) to prove existence of traveling wave solutions have been valid. If the velocity of traveling wave can be represented by the RAI, the theory of analytic functions and algebraic geometry which reported by \[41–43\] would be employed to study these systems. For instance, Derks and Gils \[24\], Ogawa \[25\] computed the MRAI of perturbed KdV equation. Chen et al. \[26, 32, 44\] detected the MRAI of perturbed BBM equation, perturbed defocusing mKdV equation and perturbed generalized KdV equation by the similar method. Du et al. \[45\] considered MRAI of a generalized Nizhnik-Novikov-Veselov equation with diffusion term. Then they all proved the existence of traveling wave solutions. Different from them, Sun et al. \[46\] employed the Chebyshev criterion to detect the MRAI of a shallow water fluid to analyze the coexistence of the solitary and periodic wave solutions. Sun and Yu \[33\] also illustrated
the existence and uniqueness of periodic waves of a generalized BBM equation based on
the same technique. Recently, the CH equation with a special local and nonlocal delay
convolution kernel were concerned by Du et al. [29]. They obtained the results that
solitary wave solution will persist for \( c > 2k \), since Melnikov integrals of model with local
and non-local delay are monotone about \( c \) for \( c > 2k \).

1.2 Motivation and novelty

Except for detecting the MRAI of perturbed equations, another effective method is to
directly compute the explicit expression of Melnikov function which, in turn, the existence
of traveling wave solution is proved. Qiu et al. [30], Zhu et al. [47] used the explicit
expression of Melnikov function to discuss the existence of solitary waves in a perturbed
generalized BBM and generalized CH equation, respectively. Cheng and Li [48] computed
the expression of Melnikov function for the delayed DP equation. Based on Melnikov
function, Wen [49] and Xu et al. [50] proved the existence of kink (anti-kink) wave for the
perturbed Gardner equation and delayed Schrödinger equation, respectively. Combining
bifurcation theory of limit cycles [51–53], Zhang et al. [28, 54] demonstrated the existence
of the solitary wave solutions of the perturbed mKdV and mK(3, 1) equations.

Inspired by aforementioned works, the main purpose of this paper is to investigate
the existence of traveling wave solutions for Wu-Zhang equation under a small singular
perturbation. Firstly, we introduce a new parameter \( \kappa \) to obtain the exact solutions
including solitary wave and kink (anti-kink) wave solutions of the unperturbed Wu-
Zhang equation. Then, the geometric singular perturbation approach is used to construct
homoclinic or heteroclinic orbits by tracking invariant manifolds of corresponding ODEs.
Finally, by Melnikov method (see, e.g., [55, 56]) and bifurcation theorem, we analyze
conditions such as the wave speed and parameter for the existence of traveling wave
solutions. The highlights of this work can be enumerated as:

(i) Not only the analytical expression of Melnikov integral is given for perturbed
Wu-Zhang equation, but also the analytical expression of speed \( c \) is obtained.

(ii) The non-persistence of kink (anti-kink) wave solutions are proved by analytical
Melnikov method and bifurcation theorem. Most of the literature emphasized the existence of the solitary wave by GSP approach, but there are few works applying the GSP approach to discuss the nonexistence of kink (anti-kink) wave solutions. We prove the nonexistence of kink (anti-kink) wave solutions by GSP approach.

(iii) Numerical simulations are utilized to verify the theoretical results.

1.3 The outline of the paper

The outline of the paper is as follows. Some preliminaries including geometric singular perturbation theory are introduced in section 2. We obtain the bifurcations of phase portraits for the unperturbed Wu-Zhang equation based on dynamical system method in section 3. By introducing a new parameter $\kappa$, the solitary wave and kink wave solution are determined in section 4. In section 5, combining the GSP approach, Melnikov method and bifurcation theorem, the existence of traveling wave solutions for a perturbed Wu-Zhang equation is investigated. In section 6, numerical simulations are carried out to show the effectiveness of previous theoretical results. Finally a conclusion ends our work.

2 Preliminaries

In this section, we introduce some known results on the theory of geometric singular perturbation (see. e.g. [23, 29, 48]). Consider the system

$$\begin{cases}
  \dot{x}_1' = f(x_1, x_2, \epsilon), \\
  \dot{x}_2' = \epsilon g(x_1, x_2, \epsilon),
\end{cases}$$  \hspace{1cm} (4)

where $' = \frac{d}{dt}$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^l$ and $\epsilon$ is a positive real parameter, $U \subseteq \mathbb{R}^{n+l}$ is open subset, and $I$ is an open subset of $\mathbb{R}$, containing 0; $f$ and $g$ are $C^\infty$ on a set $U \times I$. Moreover, the $x_1$ (resp., $x_2$) variables are called fast (resp., slow) variables. Letting $\tau = \epsilon t$ which gives the following equivalent system

$$\begin{cases}
  \epsilon \dot{x}_1 = f(x_1, x_2, \epsilon), \\
  \dot{x}_2 = g(x_1, x_2, \epsilon),
\end{cases}$$  \hspace{1cm} (5)
where \( \cdot = \frac{d}{d\tau} \). We refer to \( t \) (resp., \( \tau \)) as the fast time scale or fast time (resp., slow time scale or slow time). Each of the scalings is naturally associated with a limit as \( \epsilon \) tend to zero. These limits are respectively given by

\[
\begin{align*}
\begin{cases}
x'_1 = f(x_1, x_2, 0), \\
x'_2 = 0,
\end{cases}
\end{align*}
\]

(6)

and

\[
\begin{align*}
\begin{cases}
0 = f(x_1, x_2, 0), \\
x'_2 = g(x_1, x_2, 0).
\end{cases}
\end{align*}
\]

(7)

System (6) is called the layer problem and system (7) is reduced system.

\textbf{Definition 1} (see [23, 29, 48]) A manifold \( M_0 \) on which \( f(x_1, x_2, 0) = 0 \) is called a critical manifold or slow manifold. A critical manifold \( M_0 \) is said to be normally hyperbolic if the linearization of system (4) at each point in \( M_0 \) has exactly \( l \) eigenvalues on the imaginary axis \( \text{Re}(\lambda) = 0 \).

\textbf{Definition 2} (see [23, 29, 48]) A set \( M \) is locally invariant under the flow of system (4) if it has neighborhood \( V \) so that no trajectory can leave \( M \) without also leaving \( V \). In other words, it is locally invariant if for all \( x_1 \in M \), \( x_1 \cdot [0, t] \subseteq V \) implies that \( x_1 \in M \), \( x_1 \cdot [0, t] \subseteq M \), similarly with \( [0, t] \) replaced by \( [t, 0] \) when \( t < 0 \), where \( x_1 \cdot [0, t] \) denotes the application of a flow after time \( t \) to the initial condition \( x_1 \).

\textbf{Lemma 1} (see [23, 29, 48]) Let \( M_0 \) be a compact, normally hyperbolic critical manifold given as a graph \( \{(x_1, x_2) : y = h^0(x_2)\} \). Then for sufficiently small positive \( \epsilon \) and any \( 0 < r < +\infty \),

- there exists a manifold \( M_\epsilon \), which is locally invariant under the flow of system (4) and \( C^r \) in \( x_1, x_2, \epsilon \). Moreover, \( M_\epsilon \) is given as graph:

\[
M_\epsilon = \{(x_1, x_2) : x = h^\epsilon(x_2)\}
\]

for some \( C^r \) function \( h^\epsilon(x_2) \);
• $M_{\epsilon}$ possesses locally invariant stable and unstable manifold $W^s(M_{\epsilon})$ and $W^u(M_{\epsilon})$ lying within $O(\epsilon)$ and being $C^r$ diffeomorphic to the stable and unstable manifold $W^s(M_0)$ and $W^u(M_0)$ of the critical manifold $M_0$;

• $W^s(M_{\epsilon})$ is partitioned by moving invariant submanifolds $F^s(p_{\epsilon})$, which are $O(\epsilon)$ close and diffeomorphic to $F^s(p_0)$, with base point $p_{\epsilon}$ belonging to $M_{\epsilon}$. Moreover, they are $C^r$ with respect to $p$ and $\epsilon$. Moving invariance means the submanifold $F^s(p_{\epsilon})$ is mapped under the time $t$ flow to another submanifold $F^s(p_{\epsilon} \cdot t)$ whose base point is the time $t$ evolution image of the taken base point $p_{\epsilon}$;

• the dynamics on $M_{\epsilon}$ is a regular perturbation of that generated by system (7).

3 Bifurcation of phase portraits for the unperturbed Wu-Zhang equation (3).

In this section, firstly, we consider the exact solutions of the unperturbed Wu-Zhang equation (3) by dynamical system method.

By introducing the following transformations:

$$u(x,t) = \phi(\xi), \quad v(x,t) = \psi(\xi), \quad \xi = x - ct, \quad (8)$$

we obtain

$$\frac{\partial u(x,t)}{\partial t} = -c\phi_\xi, \quad \frac{\partial v(x,t)}{\partial t} = -c\psi_\xi,$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \phi_{\xi\xi}, \quad \frac{\partial^3 u(x,t)}{\partial x^3} = \phi_{\xi\xi\xi}, \quad \frac{\partial^4 u(x,t)}{\partial x^4} = \phi_{\xi\xi\xi\xi}, \quad (9)$$

where $\phi_\xi$ and $\psi_\xi$ are the first order derivative with respect to $\xi$. And $\phi_{\xi\xi}$, $\phi_{\xi\xi\xi}$ and $\phi_{\xi\xi\xi\xi}$ means the second, third and fourth order derivative with respect to $\xi$, respectively.

Substituting (8) and (9) into (3), then system (3) is given by

$$\begin{cases}
  c\phi_\xi = \phi\phi_\xi + \psi_\xi, \\
  c\psi_\xi = \psi\phi_\xi + \phi\psi_\xi + \frac{1}{3}\phi_{\xi\xi\xi} - \epsilon(\phi_{\xi\xi} + \phi_{\xi\xi\xi}).
\end{cases} \quad (10)$$

Integrating both sides of the first equation of system (10) once with respect to $\xi$ and letting the integration constant be zero yields

$$\psi(\xi) = c\phi(\xi) - \frac{1}{2}\phi^2(\xi). \quad (11)$$
We substitute (11) into the second equation of (10), then the coupled system (10) becomes the following ODE:

\[ \frac{1}{3} \phi_{\xi\xi\xi} - \frac{3}{2} \phi^2 \phi_{\xi} + 3c\phi \phi_{\xi} - c^2 \phi_{\xi} - \varepsilon(\phi_{\xi\xi} + \phi_{\xi\xi\xi\xi}) = 0. \]  

(12)

Integrating both sides on (12) once with respect to \( \xi \) and rescaling \( \varepsilon = 3\varepsilon \), it follows that

\[ \phi_{\xi\xi} - \frac{3}{2} \phi^3 + \frac{9}{2} c\phi^2 - 3c^2 \phi - \varepsilon(\phi_{\xi\xi} + \phi_{\xi\xi\xi\xi}) = g, \]  

(13)

where \( g \) is an integration constant (\( g \in \mathbb{R} \)).

Introducing new variables \( \zeta = c\xi, \ z = \frac{\phi}{\varepsilon} \) and \( G = \frac{g}{\varepsilon^2} \), eq. (13) is equivalent to

\[ -3z + \frac{9}{2} z^2 - \frac{3}{2} z^3 + \frac{d^2 z}{d\zeta^2} - \varepsilon(\frac{1}{c} \frac{dz}{d\zeta} + \frac{d^3 z}{d\zeta^3}) = G. \]  

(14)

Obviously, eq. (14) reduces to a planar system:

\[
\begin{cases}
\frac{dz}{d\zeta} = y, \\
\frac{dy}{d\zeta} = w, \\
\varepsilon c \frac{dw}{d\eta} = -3z + \frac{9}{2} z^2 - \frac{3}{2} z^3 + w - \frac{\varepsilon}{c} y - G.
\end{cases}
\]  

(15)

where \( \varepsilon \) is a sufficiently small parameter such that \( 0 < \varepsilon \ll 1 \). Therefore, the traveling wave solutions of eq. (3) can be obtained by studying the corresponding orbits of system (15).

Obviously, system (15) is a singularly perturbed system described as “slow system”. Rescaling \( \eta = c\zeta \), we have the following equivalent “fast system”:

\[
\begin{cases}
\frac{dz}{d\eta} = \varepsilon y, \\
\frac{dy}{d\eta} = \varepsilon w, \\
c \frac{d^2 w}{d\eta^2} = -3z + \frac{9}{2} z^2 - \frac{3}{2} z^3 + w - \frac{\varepsilon}{c} y - G.
\end{cases}
\]  

(16)

Let \( \varepsilon = 0 \) in system (15) and (16). Then, the corresponding reduced system is given by

\[
\begin{cases}
\frac{dz}{d\zeta} = y, \\
\frac{dy}{d\zeta} = w, \\
0 = -3z + \frac{9}{2} z^2 - \frac{3}{2} z^3 + w - G,
\end{cases}
\]  

(17)
and the layer system is
\[
\begin{align*}
\frac{dz}{d\eta} &= 0, \\
\frac{dy}{d\eta} &= 0, \\
-c\frac{dw}{d\eta} &= -3z + \frac{9}{2}z^2 - \frac{3}{2}z^3 + w - G.
\end{align*}
\]
which admits a two-dimensional critical manifold as follows
\[
M_0 = \{(z, y, w) \in \mathbb{R}^3 | w = 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 + G\}.
\]
Suppose $A$ to be the linearized matrix of the fast system (16), then it is given by
\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-3z + \frac{9}{2}z^2 & c & 0 \\
\end{pmatrix},
\]
that the eigenvalues of $A$ are 0, 0 and $\frac{1}{c}$. Thus, $M_0$ is a normally hyperbolic invariant manifold (see Definition[1] and Definition[2]). There exists a two-dimensional submanifold $M_\epsilon$ which is a differentiable homeomorphism and locates near $M_0$ at a distance $O(\epsilon)$ (see Lemma[1]). The invariant submanifold $M_\epsilon$ is represented by
\[
M_\epsilon = \{(z, y, w) \in \mathbb{R}^3 | w = 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 + G + \epsilon[cy(\frac{9}{2}z^2 - 9z + 3 + \frac{1}{c^2})] + O(\epsilon^2)\}.
\]
The dynamical behaviour of slow system (15) or fast system (16) is governed by
\[
\begin{align*}
\frac{dz}{dt} &= y, \\
\frac{dw}{dt} &= 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 + G + \epsilon[cy(\frac{9}{2}z^2 - 9z + 3 + \frac{1}{c^2})].
\end{align*}
\]
For $\epsilon = 0$, the unperturbed system admits homoclinic, heteroclinic and periodic orbits with the parameter $G$ taking different values. Here, we consider more general case for solitary and kink (anti-kink) wave solutions of eq. (22).

**Proposition 1** By the bifurcation theory of planar dynamical systems (see [3, 52]), we have
(i) If $|G| > \frac{3\sqrt{3}}{4}$, there exists a single saddle point (see Fig.1 (a) or (g)).
(ii) If $|G| = \sqrt{3}$, there exist a saddle point and a cusp point (see Fig.1 (b) or (f)).

(iii) If $G = 0$, there exist two saddle points $(0,0)$ and $(2,0)$, and a center point $(1,0)$. It has two heteroclinic orbits surrounding the center point $(1,0)$ to two saddle points $(0,0)$ and $(2,0)$ (see Fig.1 (d)).

(iv) If $0 < |G| < \sqrt{3}$, there exist two saddle points and a center point. It has a homoclinic orbit surrounding the center point to one saddle point (see Fig.1 (c) or (e)).

Assume that $z(\zeta)$ is a solution of eq. (14) satisfying $\lim_{\zeta \to \infty} z(\zeta) = \kappa$, thus $u(x,t) = cz(\zeta) = cz[c(x - ct)]$ is a solitary wave solution or a kink wave solution of eq. (14).

For $-\sqrt{3} < G \leq 0$ and $0 \leq G < \sqrt{3}$, the unperturbed system (22) with $\epsilon = 0$ admits a saddle point defined by $S_0(\kappa,0)$. We know that $G = -(3\kappa - \frac{9}{2}\kappa^2 + \frac{3}{2}\kappa^3)$, then we obtain $1 + \frac{\sqrt{3}}{3} < \kappa \leq 2$ for $G \in (-\sqrt{3},0]$ and $0 < \kappa < 1 - \frac{\sqrt{3}}{3}$ for $G \in [0,\sqrt{3})$, respectively. We can regard $\kappa$ as a new parameter to find the homoclinic and heteroclinic orbits of system (22). Therefore, we rewrite system (22), that is:

$$
\begin{aligned}
\frac{dz}{d\zeta} &= y, \\
\frac{dy}{d\zeta} &= 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 - (3\kappa - \frac{9}{2}\kappa^2 + \frac{3}{2}\kappa^3) + \epsilon[cy(\frac{9}{2}z^2 - 9z + 3 + \frac{1}{\epsilon})].
\end{aligned}
$$

Fig. 1 The bifurcation and phase portraits of system (22) $|\epsilon|=0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The bifurcation and phase portraits of system (22) $|\epsilon|=0$.}
\end{figure}
4  Traveling wave solution of eq. \( (3) \mid _{\varepsilon=0} \)

In this section, we study the solitary and kink (anti-kink) wave solutions of eq. \( (3) \mid _{\varepsilon=0} \). By dynamical method, we consider the traveling wave solution of system \( (23) \mid _{\varepsilon=0} \) with \( \kappa \in [0, 1 - \frac{\sqrt{3}}{3}] \cup (1 + \frac{\sqrt{3}}{3}, 2] \). The unperturbed system \( (23) \) is of the form:

\[
\begin{align*}
\frac{dz}{d\zeta} &= y, \\
\frac{dy}{d\zeta} &= 3z - \frac{9}{2}z^2 + \frac{3}{2}z^3 - (3\kappa - \frac{9}{2}\kappa^2 + \frac{3}{2}\kappa^3).
\end{align*}
\]

It is easy to obtain the first integral of system \( (24) \), it is given by

\[
H(z, y) = \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 + \left(\frac{3}{2}\kappa^3 - \frac{9}{2}\kappa^2 + 3\kappa\right)z.
\]

The homoclinic orbit \( \Gamma(\kappa) \) to the saddle \((\kappa, 0)\) is defined by

\[
H(z, y) = \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 + \left(\frac{3}{2}\kappa^3 - \frac{9}{2}\kappa^2 + 3\kappa\right)z
\]

And the heteroclinic orbits \( \Upsilon(\kappa)_{\pm} \) (where \( \pm \) represents upper and lower branch curves) to the two saddle points \((0, 0)\) and \((2, 0)\), namely, \( \kappa = 0 \) or \( \kappa = 2 \), the heteroclinic curve is determined by

\[
H(z, y) = \frac{1}{2}y^2 - \frac{3}{8}z^4 + \frac{3}{2}z^3 - \frac{3}{2}z^2 = 0.
\]

4.1 Solitary wave solution of eq. \( (3) \mid _{\varepsilon=0} \)

By \( (26) \) and Proposition \( (iv) \), we can obtain the expression of \( y \) as follows:

\[
y = \pm \sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2 - (3\kappa^3 - 9\kappa^2 + 6\kappa)z + \frac{9}{8}\kappa^4 - 6\kappa^3 + 3\kappa^2}
\]

where \( z_{\pm} = -\kappa + 2 \pm \sqrt{-2\kappa^2 + 4\kappa} \). It can be seen that \( z_{-} < z_{+} < \kappa \) for \( 1 + \frac{\sqrt{3}}{3} < \kappa < 2 \) (see Fig. 1(c)) and \( \kappa < z_{-} < z_{+} \) for \( 0 < \kappa < 1 - \frac{\sqrt{3}}{3} \) (see Fig. 1(e)).

Since \( \frac{dz}{d\kappa} = y \), it implies

\[
\frac{dz}{d\zeta} = \pm \sqrt{\frac{3}{4}(z - \kappa)^2(z - z_{-})(z - z_{+})},
\]
which yields
\[ \zeta = \pm \int_{z}^{z} \frac{1}{\sqrt{\frac{3}{2} (s - \kappa)^2 (s - z_+) (s - z_-)}} ds, \quad (1 + \frac{\sqrt{3}}{3} < \kappa < 2), \] 
(30)
and
\[ \zeta = \pm \int_{z}^{z} \frac{1}{\sqrt{\frac{3}{2} (s - \kappa)^2 (z_+ - s) (z_- - s)}} ds, \quad (0 < \kappa < 1 - \frac{\sqrt{3}}{3}). \] 
(31)

Thus, for \( 1 + \frac{\sqrt{3}}{3} < \kappa < 2 \), the parametric representation of homoclinic orbit \( \Gamma(\kappa) \) can be obtained by
\[ z(\zeta) = -\frac{2(6\kappa^2 - 12\kappa + 4)}{2\cosh(\frac{1}{2}\sqrt{18\kappa^2 - 36\kappa + 12}\zeta)\sqrt{-2\kappa^2 + 4\kappa + 4\kappa - 4}}. \] 
(32)

Then, it corresponds to a dark solitary wave solution of eq. (3) \( |\varepsilon = 0 \) (see Fig. 2(a)) is given by:
\[ u(x,t) = -\frac{2(6\kappa^2 - 12\kappa + 4)c}{2\cosh\left(\frac{1}{2}\sqrt{18\kappa^2 - 36\kappa + 12c(x - ct)}\right)\sqrt{-2\kappa^2 + 4\kappa + 4\kappa - 4}}. \] 
(33)

For \( 0 < \kappa < 1 - \frac{\sqrt{3}}{3} \), the expression of homoclinic orbit \( \Gamma(\kappa) \) can be obtained by
\[ z(\zeta) = -\frac{2(6\kappa^2 - 12\kappa + 4)}{4\kappa - 4 - 2\cosh(\frac{1}{2}\sqrt{18\kappa^2 - 36\kappa + 12}\zeta)\sqrt{-2\kappa^2 + 4\kappa}}. \] 
(34)

The slight solitary wave solution of eq. (3) \( |\varepsilon = 0 \) (see Fig. 2(b)) is of the form
\[ u(x,t) = -\frac{2(6\kappa^2 - 12\kappa + 4)c}{4\kappa - 4 - 2\cosh\left(\frac{1}{2}\sqrt{18\kappa^2 - 36\kappa + 12c(x - ct)}\right)\sqrt{-2\kappa^2 + 4\kappa}}. \] 
(35)
Fig. 2 Dark and Bright solitary wave solutions of eq. (3) |\varepsilon=0| for c = 0.5, where (a) \(\kappa = 1.7\), (b) \(\kappa = 0.3\).

4.2 Kink and anti-kink wave solution of eq. (3) |\varepsilon=0|

According to (27) and (iii) of Proposition 1, the expression of y is:

\[
y = \pm \sqrt{\frac{3}{4}z^4 - 3z^3 + 3z^2} = \pm \sqrt{\frac{3}{4}(z - 0)^2(2 - z)^2} = \pm \frac{\sqrt{3}}{2}z(z - 2),
\]

Similarly, due to \(\frac{dz}{d\zeta} = y\), we have

\[
\zeta = \pm \int_{z_1}^{z} \frac{1}{\sqrt{3}z(s - 2)} ds,
\]

Therefore, we obtain the following kink and anti-kink wave solutions (see Fig. 3(a) and (b)):

\[
u(x, t) = -\frac{2c}{1 + e^{\pm\sqrt{3}c(x-ct)}}.
\]

(a) Kink wave (b) Anti-kink wave

Fig. 3 Kink and anti-kink of eq. (3) |\varepsilon=0| for c = 0.5.

In summary, we give the theorems as follows.

**Theorem 1** For any wave speed c and \(\kappa \in (0, 1 - \frac{\sqrt{3}}{3}) \cup (1 + \frac{\sqrt{3}}{3}, 2)\), eq. (3) |\varepsilon=0| has solitary wave solutions given by (33) or (35).

**Theorem 2** For any wave speed c and \(\kappa = 0\) or \(\kappa = 2\), eq. (3) |\varepsilon=0| has kink or anti-kink solutions given by (38).
5 Existence of traveling wave solutions for eq. (3)

Firstly, let us consider the case of solitary wave solutions. Melnikov method is employed to detect the existence of solitary wave solutions under small perturbation. The homoclinic Melnikov function of system \(^{(23)}\) is defined by

\[
M_{\text{hom}}(c, \kappa) = \oint_{\Gamma(\kappa)} \left( \frac{9}{2} z^2 - 9z + 3 + \frac{1}{c^2} \right) y^2 d\zeta = \frac{1}{c^2} I(\kappa) + J(\kappa),
\]

where \(I(\kappa) = \oint_{\Gamma(\kappa)} y^2 d\zeta\) and \(J(\kappa) = \oint_{\Gamma(\kappa)} \left( \frac{9}{2} z^2 - 9z + 3 \right) y^2 d\zeta\).

**Lemma 2** For any \(\kappa \in (0, 1 - \sqrt{3} \sqrt{3}) \cup (1 + \sqrt{3} \sqrt{3}, 2)\), there exists a positive root \(c = c(\kappa)\) of \(M_{\text{hom}}(c, \kappa) = 0\). Moreover, \(\frac{\partial M_{\text{hom}}}{\partial c} \mid_{c=c(\kappa)} \neq 0\).

**Proof.** Obviously, we know that \(I(\kappa) = \oint_{\Gamma(\kappa)} y^2 d\zeta > 0\). Because \(J(\kappa)\) integrates along the homoclinic orbit \(\Gamma(\kappa)\), then

\[
y'' = \left( \frac{9}{2} z^2 - 9z + 3 \right) z' = \left( \frac{9}{2} z^2 - 9z + 3 \right) y,
\]

which yields

\[
dy' = \left( \frac{9}{2} z^2 - 9z + 3 \right) z' d\zeta = \left( \frac{9}{2} z^2 - 9z + 3 \right) y d\zeta.
\]

By integration of parts, it is not difficult to have

\[
J(\kappa) = \oint_{\Gamma(\kappa)} \left( \frac{9}{2} z^2 - 9z + 3 \right) y^2 d\zeta = \oint_{\Gamma(\kappa)} y dy'
\]  
\[
- \int_{R} (y')^2 d\zeta < 0.
\]

Hence, \(\left. \frac{I(\kappa)}{J(\kappa)} \right|_{c=c(\kappa)} > 0\), there exists a single positive root \(c = c(\kappa)\) of \(M_{\text{hom}}(c, \kappa) = 0\) that

\[
c(\kappa) = \sqrt{- \frac{I(\kappa)}{J(\kappa)}}.
\]

On the other hand, it follows from \(^{(39)}\) that

\[
\left. \frac{\partial M_{\text{hom}}(c, \kappa)}{\partial c} \right|_{c=c(\kappa)} = -2 \frac{1}{c^2} I(\kappa) = -2 I(\kappa) \sqrt{- \frac{J(\kappa)}{I(\kappa)}} > 0.
\]

The proof is completed.

In fact, the analytical expressions of \(I(\kappa)\) and \(J(\kappa)\) can be by dividing them in two cases to obtain.
Case (I): $1 + \sqrt{3} \frac{3}{4} < \kappa < 2$.

Firstly, since $I(\kappa)$ and $J(\kappa)$ are integrated over a closed curve, and the time variable $\zeta$ can be represented by the state variable $z$ on homoclinic or heteroclinic orbit. Thus, by (28), we have

$$I(\kappa) = \oint_{\Gamma(\kappa)} y^2 d\zeta = \oint_{\Gamma(\kappa)} y dz$$

$$= 2 \int_{z_+}^{\kappa} \sqrt{\frac{3}{4} z^4 - 3z^3 + 3z^2 - (3\kappa^3 - 9\kappa^2 + 6\kappa)} z + \frac{9}{4} \kappa^4 - 6\kappa^3 + 3\kappa^2 dz$$

$$= 2 \int_{z_+}^{\kappa} \sqrt{\frac{3}{4} (z - \kappa)^2 (z - z_+)(z - z_-)} dz$$

$$= \sqrt{3} \left[ \int_{z_+}^{\kappa} \kappa \sqrt{(z - z_+)(z - z_-)} dz - \int_{z_+}^{\kappa} z \sqrt{(z - z_+)(z - z_-)} dz \right]$$

$$= \sqrt{3} \left[ \kappa I_1(\kappa) - I_2(\kappa) \right],$$

where $I_1(\kappa) = \int_{z_+}^{\kappa} \sqrt{(z - z_+)(z - z_-)} dz$ and $I_2(\kappa) = \int_{z_+}^{\kappa} z \sqrt{(z - z_+)(z - z_-)} dz$.

Checking the integral book [58], by the formulas 2.261, 2.262 (1) and 2.262 (2), it gives us the following formulas:

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c_1}} \ln(2\sqrt{c_1 R} + 2c_1 x + b), \quad (c_1 > 0, \ 2c_1 x + b > \sqrt{-\Delta}, \ \Delta < 0),$$

$$\int \sqrt{R} dx = \frac{(2c_1 x + b)\sqrt{R}}{4c_1} + \frac{\Delta}{8c_1} \int \frac{dx}{\sqrt{R}},$$

$$\int x\sqrt{R} dx = \frac{\sqrt{R}^3}{3c_1} - \frac{(2c_1 x + b)\sqrt{R}}{8c_1} - \frac{b\Delta}{16c_1} \int \frac{dx}{\sqrt{R}} \quad (46)$$

where $R = a + bx + c_1 x^2$ and $\Delta = 4ac_1 - b^2$.

Combining (45) and (46), it follows that

$$I(\kappa) = \frac{\sqrt{3}}{3} \left[ 6\ln 2\kappa^3 + 6\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 18\ln 2\kappa^2 - 18\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 12\ln 2\kappa + 12\ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 2\sqrt{6\kappa^2 - 12\kappa + 4} \right]$$

$$- \sqrt{3}\kappa \left[ 2\ln(2\kappa - \kappa^2) + 3\ln 2\kappa^2 - 3\kappa \ln(2\kappa - \kappa^2) - 9\ln 2\kappa + 2\ln(2\kappa - \kappa^2) + 6\ln 2 \right]. \quad (47)$$
Secondly,

\[ J(\kappa) = \int_{\Gamma(\kappa)} \left( \frac{9}{2} z^2 - 9z + 3 \right) y^2 d\zeta = \int_{\Gamma(\kappa)} \left( \frac{9}{2} z^2 - 9z + 3 \right) y dz \]

\[ = 2 \int_{z_+}^{\kappa} \left( \frac{9}{2} z^2 - 9z + 3 \right) \sqrt{\frac{2}{\kappa} - 3z^3 + 3z^2 - (3\kappa^2 + 6\kappa)z + \frac{9}{2}\kappa^2 - 6\kappa^3 + 3\kappa^2} dz \]

\[ = \sqrt{3} \int_{z_+}^{\kappa} \left( \frac{9}{2} z^2 - 9z + 3 \right) \sqrt{(\kappa - z)^2 (z - z_+) (z - z_-)} dz \]

\[ = \sqrt{3} \int_{z_+}^{\kappa} \left[ - \frac{9}{2} z^3 + \left( \frac{9}{2} \kappa + 9 \right) z^2 - (9\kappa + 3)z + 3\kappa \right] \sqrt{(z - z_+) (z - z_-)} dz \]

\[ = \sqrt{3} \left[ - \frac{9}{2} J_1(\kappa) + \left( \frac{9}{2} \kappa + 9 \right) J_2(\kappa) - (9\kappa + 3) I_2(\kappa) + 3\kappa I_1(\kappa) \right]. \]

where \( J_1(\kappa) = \int_{z_+}^{\kappa} z^3 \sqrt{(z - z_+) (z - z_-)} dz \), \( J_2(\kappa) = \int_{z_+}^{\kappa} z^2 \sqrt{(z - z_+) (z - z_-)} dz \).

Similarly, by the formulas 2.262 (3) and 2.262 (4) in \[58\], they are expressed by:

\[ \int x^2 \sqrt{R} dx = \left( \frac{x}{4c_1} - \frac{\Delta b}{24c_1^2} \right) \sqrt{R^3} + \left( \frac{5b^2}{16c_1^2} - \frac{9}{4c_1} \right) \frac{(2c_1 x + b) \sqrt{R}}{4c_1} + \left( \frac{5b^2}{16c_1^2} - \frac{9}{4c_1} \right) \frac{\Delta}{8c_1} \int \frac{dx}{\sqrt{R}} \]

\[ \int x^3 \sqrt{R} dx = \left( \frac{x^2}{5c_1} - \frac{7bx}{40c_1^2} + \frac{7b^2}{80c_1^3} - \frac{2a}{15c_1} \right) \sqrt{R^3} - \left( \frac{7b^3}{32c_1^3} - \frac{3ab}{8c_1^4} \right) \frac{(2c_1 x + b) \sqrt{R}}{4c_1} \]

\[ - \left( \frac{7b^3}{32c_1^3} - \frac{3ab}{8c_1^4} \right) \frac{\Delta}{8c_1} \int \frac{dx}{\sqrt{R}} \]

Using \[46\], \[48\] and \[49\], we have

\[ J(\kappa) = -\frac{\sqrt{3}}{9} \left[ 27\kappa^4 \sqrt{6\kappa^2 - 12\kappa + 4} + 15\kappa^3 \ln(2\kappa - \kappa^2) - 198\kappa^3 \sqrt{6\kappa^2 - 12\kappa + 4} \right. \]

\[ - 30\kappa^3 \ln(2\kappa + 4 + 2\kappa - 2) + 15\ln2\kappa^3 - 45\kappa^2 \ln(2\kappa - \kappa^2) \]

\[ + 90\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 30\sqrt{2\kappa} \sqrt{(3\kappa^2 - 6\kappa + 2)^3} \]

\[ + 294\kappa^2 \sqrt{6\kappa^2 - 12\kappa + 4} - 45\ln2\kappa^2 + 30\ln(2\kappa - \kappa^2) \]

\[ - 60\ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 10\sqrt{2} \sqrt{(3\kappa^2 - 6\kappa + 2)^3} \]

\[ - 72\kappa \sqrt{6\kappa^2 - 12\kappa + 4} + 30\ln2\kappa - 18\sqrt{6\kappa^2 - 12\kappa + 4} \].

Case (II): \( 0 < \kappa < \frac{\sqrt{3}}{3} \).
Same calculation method as case (I), we obtain

\[
I(\kappa) = \oint_{\Gamma(\kappa)} y^2 d\zeta = 2 \int_{z_-}^{z_+} \cdots dz
\]

\[
= \frac{1}{3} [6\ln 2\kappa^3 + 6\kappa^3 \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 18\ln 2\kappa^2
- 18\kappa^2 \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 12\ln 2\kappa
+ 12\kappa \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 2\sqrt{6\kappa^2 - 12\kappa + 4}]
- \sqrt{3} [2\kappa^2 \ln (-\sqrt{2\kappa - \kappa^2}) + 3\ln 2\kappa^2 - 6\kappa \ln (-\sqrt{2\kappa - \kappa^2}) - 9\ln 2\kappa
+ 4\ln (-\sqrt{2\kappa - \kappa^2}) + 6\ln 2],
\]

(51)

and

\[
J(\kappa) = \oint_{\Gamma(\kappa)} \left(\frac{9}{2} z^2 - 9z + 3\right) y^2 d\zeta = 2 \int_{z_-}^{z_+} \cdots dz
\]

\[
= -\frac{\sqrt{3}}{5} [27\kappa^4 \sqrt{6\kappa^2 - 12\kappa + 4} + 30\kappa^3 \ln (-\sqrt{2\kappa - \kappa^2})
+ 60\kappa \ln (-\sqrt{2\kappa - \kappa^2})
- 30\kappa^3 \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 15\ln 2\kappa^3 - 90\kappa^2 \ln (-\sqrt{2\kappa - \kappa^2})
+ 90\kappa^2 \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 30\sqrt{2\kappa} \sqrt{(3\kappa^2 - 6\kappa + 2)^3}
+ 294\kappa^2 \sqrt{6\kappa^2 - 12\kappa + 4} - 45\ln 2\kappa^2 - 198\kappa^3 \sqrt{6\kappa^2 - 12\kappa + 4}
- 60\kappa \ln (\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 10\sqrt{2} \sqrt{(3\kappa^2 - 6\kappa + 2)^3}
- 72\sqrt{6\kappa^2 - 12\kappa + 4} + 30\ln 2\kappa - 18\sqrt{6\kappa^2 - 12\kappa + 4}].
\]

(52)

We plot the algebraic curve of \(I(\kappa)\) with respect to \(\kappa\) and \(J(\kappa)\) with respect to \(\kappa\), respectively. (see Fig. 4 and Fig. 5)

![Graphs](image-url)

Fig. 4 The algebraic curves of \(I(\kappa)\) and \(J(\kappa)\) with respect to \(\kappa\) for \(1 + \sqrt{3} < \kappa < 2\).
Fig. 5  The algebraic curves of $I(\kappa)$ and $J(\kappa)$ with respect to $\kappa$ for $0 < \kappa < 1 - \frac{\sqrt{3}}{3}$.

We can see that $I(\kappa) > 0$ and $J(\kappa) < 0$ for $1 + \frac{\sqrt{3}}{3} < \kappa < 2$ in Fig. 4, $I(\kappa) > 0$ and $J(\kappa) < 0$ for $0 < \kappa < 1 - \frac{\sqrt{3}}{3}$ in Fig. 5. It is shown that a good agreement with the proof of Lemma 2.

Hence, we substitute (47) and (50) into (39), and the expression of the homoclinic Melnikov function of system (23) for $1 + \frac{\sqrt{3}}{3} < \kappa < 2$ can be rewritten as

$$M_{\text{hom}}(c, \kappa) = \frac{1}{c^2} \left[ 6\ln 2 \kappa^3 + 6\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 18\ln 2\kappa^2 
- 18\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 12\ln 2\kappa 
+ 12\kappa \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 2\sqrt{6\kappa^2 - 12\kappa + 4} \right] 
- \sqrt{3}\kappa \left[ \kappa^2 \ln(2\kappa - \kappa^2) + 3\ln 2\kappa^2 - 3\kappa \ln(2\kappa - \kappa^2) - 9\ln 2\kappa 
+ 2\ln(2\kappa - \kappa^2) + 6\ln 2 \right] - \frac{\sqrt{3}}{27} \left[ 27\kappa^4 \sqrt{6\kappa^2 - 12\kappa + 4} + 15\kappa^3 \ln(2\kappa - \kappa^2) - 198\kappa^3 \sqrt{6\kappa^2 - 12\kappa + 4} 
- 30\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 15\ln 2\kappa^3 - 45\kappa^2 \ln(2\kappa - \kappa^2) 
+ 90\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 30\sqrt{2}\kappa \sqrt{(3\kappa^2 - 6\kappa + 2)^3} 
+ 294\kappa^2 \sqrt{6\kappa^2 - 12\kappa + 4} - 45\ln 2\kappa^2 + 30\kappa \ln(2\kappa - \kappa^2) 
- 60\ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 10\sqrt{2} \sqrt{(3\kappa^2 - 6\kappa + 2)^3} 
- 72\kappa \sqrt{6\kappa^2 - 12\kappa + 4} + 30\ln 2\kappa - 18\sqrt{6\kappa^2 - 12\kappa + 4} \right].$$

(53)
Consequently,

\[
c = \frac{\sqrt{15}}{3} \left[ (6\ln 2\kappa^3 + 6\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 18\ln 2\kappa^2 \\
- 18\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 12\ln 2\kappa \\
+ 12\ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 2\sqrt{6\kappa^2 - 12\kappa + 4} \right] \\
- \sqrt{3} \kappa \left[ 2\kappa^2 \ln(\sqrt{2\kappa - \kappa^2}) + 3\ln 2\kappa^2 - 6\kappa \ln(\sqrt{2\kappa - \kappa^2}) - 9\ln 2\kappa \\
+ 4\ln(\sqrt{2\kappa - \kappa^2}) + 6\ln 2 \right] - \frac{\sqrt{3}}{3} \left[ 27\kappa^4 \sqrt{6\kappa^2 - 12\kappa + 4} \\
+ 30\kappa^3 \ln(-\sqrt{2\kappa - \kappa^2}) + 60\kappa \ln(-\sqrt{2\kappa - \kappa^2}) + 15\ln 2\kappa^3 \\
- 30\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 90\kappa^2 \ln(-\sqrt{2\kappa - \kappa^2}) \\
+ 90\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 30\sqrt{2\kappa} \sqrt{(3\kappa^2 - 6\kappa + 2)^3} \\
+ 294\kappa^2 \sqrt{6\kappa^2 - 12\kappa + 4} + 45\ln 2\kappa^2 - 198\kappa^3 \sqrt{6\kappa^2 - 12\kappa + 4} \right]^{\frac{1}{2}}. \tag{54}
\]

Similarly, the expression of the homoclinic Melnikov function of system (23) for \(0 < \kappa < \frac{\sqrt{3}}{3}\) is

\[
M_{\text{hom}}(c, \kappa) = \frac{1}{c} \frac{\sqrt{3}}{3} \left[ (6\ln 2\kappa^3 + 6\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 18\ln 2\kappa^2 \\
- 18\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 12\ln 2\kappa \\
+ 12\ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 2\sqrt{6\kappa^2 - 12\kappa + 4} \right] \\
- \sqrt{3} \kappa \left[ 2\kappa^2 \ln(\sqrt{2\kappa - \kappa^2}) + 3\ln 2\kappa^2 - 6\kappa \ln(\sqrt{2\kappa - \kappa^2}) - 9\ln 2\kappa \\
+ 4\ln(\sqrt{2\kappa - \kappa^2}) + 6\ln 2 \right] - \frac{\sqrt{3}}{3} \left[ 27\kappa^4 \sqrt{6\kappa^2 - 12\kappa + 4} \\
+ 30\kappa^3 \ln(-\sqrt{2\kappa - \kappa^2}) + 60\kappa \ln(-\sqrt{2\kappa - \kappa^2}) + 15\ln 2\kappa^3 \\
- 30\kappa^3 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) - 90\kappa^2 \ln(-\sqrt{2\kappa - \kappa^2}) \\
+ 90\kappa^2 \ln(\sqrt{6\kappa^2 - 12\kappa + 4} + 2\kappa - 2) + 30\sqrt{2\kappa} \sqrt{(3\kappa^2 - 6\kappa + 2)^3} \\
+ 294\kappa^2 \sqrt{6\kappa^2 - 12\kappa + 4} + 45\ln 2\kappa^2 - 198\kappa^3 \sqrt{6\kappa^2 - 12\kappa + 4} \right]^{\frac{1}{2}}. \tag{55}
\]
and

$$c = \frac{\sqrt{3}}{\kappa} \left[ (6\ln 2\kappa^3 + 6\kappa^3 \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2)) - 18\ln 2\kappa^2 \
-18\kappa^2 \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2) + 12\ln 2\kappa \
+12\kappa \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2) + 2\sqrt{6\kappa^2} - 12\kappa + 4 \right] 
- \sqrt{3} \kappa [ 2\kappa^2 \ln(-\sqrt{2\kappa - \kappa^2}) + 3\ln 2\kappa^2 - 6\kappa \ln(-\sqrt{2\kappa - \kappa^2}) - 9\ln 2\kappa 
+4\ln(-\sqrt{2\kappa - \kappa^2}) + 6\ln 2)/(27\kappa^4 \sqrt{6\kappa^2} - 12\kappa + 4 
+30\kappa^3 \ln(-\sqrt{2\kappa - \kappa^2}) + 60\kappa \ln(-\sqrt{2\kappa - \kappa^2}) + 15\ln 2\kappa^3 
-30\kappa^3 \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2) - 90\kappa^2 \ln(-\sqrt{2\kappa - \kappa^2}) 
+90\kappa^2 \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2) + 30\sqrt{2}\kappa \sqrt{3\kappa^2 - 6\kappa + 2}^3 
+294\kappa^2 \sqrt{6\kappa^2} - 12\kappa + 4 - 45\ln 2\kappa^2 - 198\kappa^3 \sqrt{6\kappa^2} - 12\kappa + 4 
-60\kappa \ln(\sqrt{6\kappa^2} - 12\kappa + 4 + 2\kappa - 2) + 10\sqrt{2}\kappa \sqrt{3\kappa^2 - 6\kappa + 2}^3 
-72\kappa \sqrt{6\kappa^2} - 12\kappa + 4 + 30\ln 2\kappa - 18\sqrt{6\kappa^2} - 12\kappa + 4 ] \right]^{\frac{1}{2}}.$$

According to above analysis, we have the following conclusions.

**Proposition 2** For sufficiently small $\epsilon$ $(0 < \epsilon \ll 1)$ and $\kappa \in (0, 1 - \frac{\sqrt{3}}{3}) \cup (1 + \frac{\sqrt{3}}{3}, 2)$, there is $c(\kappa, \epsilon) = c(\kappa) + O(\epsilon)$ such that system (23) with $c = c(\kappa, \epsilon)$ possesses a homoclinic orbit near $\Gamma(\kappa)$.

**Proof.** According to [51, Chapter 4, Theorem 4.2.1] and Lemma [2] that system (23) with $c(\kappa, \epsilon) = c(\kappa) + O(\epsilon)$ has a homoclinic orbit near $\Gamma(\kappa)$. This completes the proof.

**Theorem 3** For any wave speed $c$ and $\kappa \in (0, 1 - \frac{\sqrt{3}}{3}) \cup (1 + \frac{\sqrt{3}}{3}, 2)$, there is $c(\kappa, \epsilon) = c(\kappa) + O(\epsilon)$ such that eq. (3) has a solitary wave solution $u = u(x, t, \kappa, \epsilon)$ with a wave speed $c = c(\kappa, \epsilon)$. Moreover, if $\epsilon$ trends to 0, the solitary wave solution $u = u(x, t, \kappa, \epsilon)$ converges to the solitary wave solution (33) or (35) of the unperturbed eq. (3) $|\epsilon = 0$ with $c = c(\kappa)$.

Then, the Melnikov method is also employed to detect the existence of kink wave solutions under small perturbation. The heteroclinic Melnikov function of system (23) is
defined as follows:

\[
M_{\text{het}}(c,\kappa) = \pm \int_{\Upsilon(\kappa)})^{\pm} \left( \frac{9}{2} z^2 - 9z + 3 + \frac{1}{c^2} \right) y^2 d\zeta = \pm \int_{-\infty}^{+\infty} \left( \frac{9}{2} z^2 - 9z + 3 + \frac{1}{c^2} \right) y^2 d\zeta
\]

\[
= \pm \int_{0}^{2} \left( \frac{9}{2} z^2 - 9z + 3 + \frac{1}{c^2} \right) y dz = \pm \frac{\sqrt{3}}{2} \int_{0}^{2} \left( \frac{9}{2} z^2 - 9z + 3 + \frac{1}{c^2} \right) z(2-z)dz
\]

\[
= \pm \left( \frac{13\sqrt{3}}{2} + \frac{2\sqrt{3}}{3} \frac{1}{c^2} \right).
\]

(57)

**Lemma 3** For any wave speed \(c \neq 0\) and \(\kappa = 0\) or \(\kappa = 2\), \(M_{\text{het}}(c,\kappa) = 0\) has no zeros.

**Proof.** Obviously, it directly follows form (57) that \(M_{\text{het}}(c,\kappa) \neq 0\) for any \(c \neq 0\).

**Proposition 3** For any \(\kappa = 0\) or \(\kappa = 2\), there does not exist a function \(c(\kappa,\epsilon) = c(\kappa) + O(\epsilon)\) such that system (23) with \(c = c(\kappa,\epsilon)\) has a heteroclinic loop near \(\Upsilon(\kappa)\).

**Proof.** It follows directly from [53, Theorem 2.5] and Lemma 3 that system (23) with \(c(\kappa,\epsilon) = c(\kappa) + O(\epsilon)\) doesn't have a heteroclinic loop near \(\Upsilon(\kappa)\). The proof is completed.

**Theorem 4** For any wave speed \(c \neq 0\) and \(\kappa = 0\) or \(\kappa = 2\), there does not exist a function \(c(\kappa,\epsilon) = c(\kappa) + O(\epsilon)\) such that eq. (3) admits a kink (anti-kink) wave solution \(u = u(x,t,\kappa,\epsilon)\) with \(c = c(\kappa,\epsilon)\).

### 6 Numerical analysis

Numerical simulations are employed to confirm the theoretical results derived in previous sections. Here, maple software 18.0 is used.

Firstly, we simulate the existence of solitary wave solution of perturbed Wu-zhang equation (3). Let \(\kappa = 1.7\) and \(\epsilon = 0.01\). By (47) and (50), we obtain \(I(1.7) \approx 0.0648882124\) and \(J(1.7) \approx -0.0279080120\) such that \(c(1.7) = \sqrt{-\frac{J(1.7)}{I(1.7)}} \approx 1.524819859\).

Then, according to Theorem 3 taking \(c = c(1.7) + 0.01 \approx 1.524819859 + 0.01\) and \(c = c(1.7) - 0.01 \approx 1.524819859 - 0.01\), respectively. Set the initial value to be \(z(0), y(0) = (z_+, 0.001, 0)\) which the homoclinic orbit would pass through. The phase portraits \((z, y)\), time history curves \((\zeta, z)\) and \((\zeta, y)\) of system (23) are plotted in Fig. 6 and Fig 7.
Fig. 6 (a) Phase portraits with $c = c(1.7) + 0.01$, (b) Time history curves of $(\zeta, z)$ and (c) Time history curves of $(\zeta, y)$ of system (23) for $\epsilon = 0.01$, $\kappa = 1.7$ and initial value $z(0), y(0) = (z_+ + 0.001, 0)$.

Fig. 7 (a) Phase portraits with $c = c(1.7) - 0.01$, (b) Time history curves of $(\zeta, z)$ and (c) Time history curves of $(\zeta, y)$ of system (23) for $\epsilon = 0.01$, $\kappa = 1.7$ and initial value $z(0), y(0) = (z_+ + 0.001, 0)$.

One can see that the homoclinic orbit and dark solitary wave solution of system (23) still persist after a small singular perturbation.

Secondly, set $\kappa = 0.3$, $\epsilon = 0.01$. By (51) and (52), we have $I(0.3) \approx 0.0648882229$ and $J(0.3) \approx -0.02790784012$ such that $c(0.3) = \sqrt{-\frac{I(0.3)}{J(0.3)}} \approx 1.524824377$. Taking $c = c(0.3) + 0.01 \approx 1.524824377 + 0.01$ and $c = c(0.3) - 0.01 \approx 1.524824377 - 0.01$ the initial value to be $z(0), y(0) = (z_- - 0.001, 0)$ which the homoclinic orbit would pass through. The phase portraits $(z, y)$, time history curves $(\zeta, z)$ and $(\zeta, y)$ of system (23) are plotted in Fig. 8 and Fig 9.
Fig. 8 (a) Phase portraits with $c = c(0.3) + 0.01$, (b) Time history curves of $(\zeta, z)$ and (c) Time history curves of $(\zeta, y)$ of system (23) for $\epsilon = 0.01$, $\kappa = 0.3$ and initial value $z(0), y(0) = (z_-, 0.001, 0)$.

(a) $z = c(0.3) + 0.01$  
(b) $(\zeta, z)$  
(c) $(\zeta, y)$

Fig. 9 (a) Phase portraits with $c = c(0.3) - 0.01$, (b) Time history curves of $(\zeta, z)$ and (c) Time history curves of $(\zeta, y)$ of system (23) for $\epsilon = 0.01$, $\kappa = 0.3$ and initial value $z(0), y(0) = (z_-, 0.001, 0)$.

(a) $z = c(0.3) - 0.01$  
(b) $(\zeta, z)$  
(c) $(\zeta, y)$

We see that the homoclinic orbit and bright solitary wave solution of system (23) still exist after a small singular perturbation which verifies the Theorem 3.

Next, we take a look at the existence of kink (anti-kink) wave solution of system (23). Giving $\kappa = 2$, $\epsilon = 0.01$ and initial value $z(0), y(0) = (1, \pm \frac{\sqrt{2}}{2} + 0.005)$. We draw the phase portraits of system (23) after taking $c = 0.5$ and $c = 2.4$ in Fig. 10. From the Fig. 10, it is shown that the heteroclinic orbits are broken which implies the kink (anti-kink) wave solution would not exist. Similarly, it is consistent with Theorem 4.
Fig. 10 Phase portraits with $c = 0.5$ and $c = 2.4$ for $\epsilon = 0.01$, $\kappa = 2$ and initial value $z(0), y(0) = (1, \pm \sqrt{3} + 0.005)$.

7 Conclusion

This paper mainly studies the existence of traveling wave solution for a perturbed Wu-Zhang equation by using GSP approach, Melnikov method and bifurcation theorem. We obtain the exact traveling wave solutions of unperturbed Wu-Zhang equation by introducing a new parameter $\kappa$, then we prove that the solitary wave solution will persist under a suitable $c = c(\kappa, \epsilon)$ for any $\kappa \in (0, 1 - \frac{\sqrt{3}}{3}) \cup (1 + \frac{\sqrt{3}}{3}, 2)$. But, for $\kappa = 0$ or $\kappa = 2$, the kink (anti-kink) wave solution can not exist since the corresponding Melnikov function has no simple zeros. Whereafter, with the help of Maple, numerical simulations illustrate the previous theoretical results.

8 Fund Acknowledgement

This work was jointly supported by the National Natural Science Foundation of China under Grant (No. 11931016, 11671176), Natural Science Foundation of Zhejiang Province under Grant (No. LY20A010016), Natural Science Foundation of Fujian Province under Grant (No. 2021J011148), Fujian Province Young Middle-Aged teachers education scientific research project (No. JAT210454) and Teacher and Student Scientific Team Fund
9 Conflict of Interest

The authors declare that they have no conflict of interest.

10 Data Availability Statement

My manuscript has no associated data.

References

[1] A Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998) 229–243.

[2] A Constantin, On the inverse spectral problem for the Camassa-Holm equation, J. Funct. Anal. 155 (1998) 352–363.

[3] Y. Fu, Y. Liu, C. Z. Qu, On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equations, J. Funct. Anal. 262 (2012) 3125–3158.

[4] L. L. Fan, H. J. Gao, Y. Liu, On the rotation-two-component Camassa-Holm system modelling the equatorial water waves, Adv. Math. 291 (2016) 59–89.

[5] J. B. Li, Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact Solutions, Beijing: Science, 2013.

[6] W. S. Liu, V. E. Van, Turning points and traveling waves in Fitzhugh-Nagumo type equations, J. Differential Equations 225 (2006) 381–410.

[7] G. Gui, Y. Liu, J. Sun, A nonlocal shallow-water model arising the Full Water Waves with Coriolis effect, J. Math. Fluid Mech. 21 (2018) 27.
[8] G. Gui, Y. Liu, T. Luo, Model Equations and Traveling Wave Solutions for Shallow-Water Waves with the Coriolis Effect, J. Nonlinear Sci. 29 (2019) 993–1039.

[9] R. Chen, G. Gui, Y. Liu, On a shallow-water approximation to the Green-Naghdi equations with the Coriolis effect, Adv. Math. 340 (2018) 106–137.

[10] T. Luo, Y. Liu, Y. Mi, B. Moon, On a shallow-water model with the Coriolis effect, J. Differential Equations 267 (2019) 3232–3270.

[11] J. Chu, Y. Yang, Constant vorticity water flows in the equatorial β-plane approximation with centripetal forces, J. Differential Equations 269 (2020) 9336–9347.

[12] J. Chu, L. Wang, Analyticity of rotational travelling gravity two layer waves, Stud. Appl. Math. 146 (3) (2021) 605–634.

[13] R. Ma, Y. Zhang, B. F. Feng, Short wave limit of the Novikov equation and its integrable semi-discretizations, J. Phys. A 54 (2021) 495701.

[14] T. Wu, J. Zhang, On modeling nonlinear long wave, in: L.P. Cook, V. Roytbhurd, M. Tulin (Eds.), Mathematics is for Solving Problems, SIAM, 1996, pp. 233.

[15] C. Chen, X. Tang, S. Lou, Solutions of a (2 + 1)-dimensional dispersive long wave equation, Phys. Rev. E 66 (2002) 036605.

[16] A. Pickering, A new truncation in Painleve analysis, J Phys A 26 (1993) 4395–4405.

[17] Q. Wang, Y. Chen, H. Q. Zhang, A new Jacobi elliptic function rational expansion method and its application to (1 + 1)-dimensional dispersive long wave equation, Chaos Soliton Fract. 23 (2005) 477–483.

[18] J. Weiss, M. Tabor, Carnevale G. The Painlevé property for partial differential equations, J. Math. Phys. 24 (1983) 522–526.

[19] X. D. Zheng, Y. Chen, H. Q. Zhang, Generalized extended tanh-function method and its application to (1 + 1)-dimensional dispersive long wave equation, Phys. Lett. A 311 (2003) 145–157.
[20] X. Zeng, D. S. Wang. A generalized extended rational expansion method and its application to (1+1)-dimensional dispersive long wave equation, Appl. Math. Comput. 212 (2) (2009), 296-304.

[21] E. G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212–218.

[22] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations 31 (1979) 53–98.

[23] C.K.R. Jones, Geometric Singular Perturbation Theory Dynamical Systems, in: Lecture Notes Math., vol. 1609, Springer, Berlin, 1995, pp. 44–120.

[24] G. Derks, S. Gils, On the uniqueness of traveling waves in perturbed Korteweg-de Vries equations, Jpn. J. Ind. Appl. Math. 10 (1993) 413–430.

[25] T. Ogama, Travelling wave solutions to a perturbed Korteweg-de Vries equation, Hiroshima Math. J. 24 (1994) 401–422.

[26] A. Y. Chen, C. Zhang, W. T. Huang, Monotonicity of limit wave speed of traveling wave solutions for a perturbed generalized KdV equation, Appl. Math. Lett. 121 (2021) 107381.

[27] W. F. Yan, Z. R. Liu, Y. Liang, Existence of solitary waves and periodic waves to a perturbed generalized KdV equation, Math. Model. Anal. 19 (2014) 537–555.

[28] L. J. Zhang, J. D. Wang, E. Shchepakina, V. Sobolev, New type of solitary wave solution with coexisting crest and trough for a perturbed wave equation, Nonlinear Dyn. 106 (202) 3479–3493.

[29] Z. J. Du, J. Li, X. W. Li, The existence of solitary wave solutions of delayed Camassa-Holm equation via a geometric approach, J. Funct. Anal. 275 (2018) 988–1007.

[30] H. M. Qiu, L. N. Zhong, J. H. Shen, Traveling waves in a generalized Camassa-Holm equation involving dual-power law nonlinearities. Commun. Nonlinear Sci. Numer. Simul. 106 (2022) 106106.
[31] Z. J. Du, J. Li, Geometric singular perturbation analysis to Camassa-Holm Kuramoto-Sivashinsky equation, J. Differential Equations 306 (2022) 418–438.

[32] A. Y. Chen, L. N. Guo, X. J. Deng, Existence of solitary waves and periodic waves for a perturbed generalized BBM equation, J. Differential Equations 261 (2016) 5324–5349.

[33] X. B. Sun, P. Yu, Periodic traveling waves in a generalized BBM equation with weak backward diffusion and dissipation terms, Discrete Contin. Dyn. Syst. Ser. B 24 (2019) 965-987.

[34] J. Li, K. N. Lu, P.W. Bates, Normally hyperbolic invariant manifolds for random dynamical systems, Trans. Amer. Math. Soc. 365 (2013) 5933–5966.

[35] J. Li, K. N. Lu, P. W. Bates, Invariant foliations for random dynamical systems, Discrete Contin. Dyn. Syst. 34 (2014) 3639-3666.

[36] J. Li, K. Lu, P.W. Bates, Geometric singular perturbation theory with real noise, J. Differential Equations 259 (2015) 5137-5167.

[37] X. F. Chen, X. Zhang. Dynamics of the predator-prey model with the Sigmoid functional response, Stud. Appl. Math. 147(1) (2021) 300–318.

[38] C. Wang, X. Zhang. Canards, heteroclinic and homoclinic orbits for a slow-fast predator-prey model of generalized Holling type III, J Differential Equations 267 (2019) 3397–3441.

[39] Z. J. Du, Q. Qiao, The dynamics of traveling waves for a nonlinear Belousov-Zhabotinskii system, J. Differential Equations 269 (2020) 7214–7230.

[40] Z. J. Du, J. Liu, Y. L. Ren, Traveling pulse solutions of a generalized Keller-Segel system with small cell diffusion via a geometric approach, J. Differential Equations 270 (2020) 1019–1042.

[41] J. Carr, J. K. Hale, Abelian integrals and bifurcation theory, J. Differential Equations 59 (1985) 413–436.
[42] S. N. Chow, J. A. Sanders, On the number of critical points of the period, J. Differential Equations 64 (1986) 51–66.

[43] R. Cushman, J. A. Sanders, A codimension two bifurcations with a third order Picard-Fuchs equation, J. Differential Equations 59 (1985) 243–256.

[44] A. Y. Chen, L. N. Guo, W. T. Huang, Existence of Kink Waves and Periodic Waves for a Perturbed Defocusing mKdV Equation, Qual. Theory Dyn. Syst. 17 (2018) 495–517.

[45] Z. J. Du, X. J. Lin, S. S. Yu. Solitary wave and periodic wave for a generalized Nizhnik-Novikov-Veselov equation with diffusion term (in Chinese). Sci. Sin. Math. 50 (2020) 1–22.

[46] X. B. Sun, W. T. Huang, J. N. Cai, Coexistence of the solitary and periodic waves in convecting shallow water fluid, Nonlinear Anal. Real World Appl. 53 (2020) 103067.

[47] K. Zhu, Y. H. Wu, Z. P. Yu, J. H. Shen, New solitary wave solutions in a perturbed generalized BBM equation. Nonlinear Dyn. 97 (2019) 2413–2423.

[48] F. F. Cheng, J. Li, Geometric singular perturbation analysis of Degasperis-Procesi equation with distributed delay, Discrete Contin. Dyn. Syst. 41 (2021) 967–985.

[49] Z. S. Wen, On existence of kink and antikink wave solutions of singularly perturbed Gardner equation, Math. Methods Appl. Sci. 43 (7) (2020) 4422–4427.

[50] C. H. Xu, Y. H. Wu, L. X. Tian, B. L. Guo, On kink and anti-kink wave solutions of Schrodinger equation with distributed delay, J. Appl. Anal. Comput. 8 (2018) 1385–1395.

[51] M. A. Han, Bifurcation theory of limit cycles, Beijing: Science press, 2013, pp. 252–289.

[52] H. B. Chen, Y. H. Xia, M. A. Han, Limit cycles of a Liénard system with symmetry allowing for discontinuity, J. Math. Anal. Appl. 468 (2018) 799–816.
[53] M. A. Han, D. J. Luo, D. M. Zhu, Uniqueness of limit cycles bifurcating from a singular closed orbit (III) (Chinese), Acta Math. Sinica. 35(5) (1992) 541–548.

[54] L. J. Zhang, M. A. Han, M. J. Zhang, C. M. Khalique, A new type of solitary wave solution of the mKdV equation under singular perturbations, Int. J. Bifurcat. Chaos 30 (2020) 1–14.

[55] V. Melnikov, On the stability of the center for time-periodic perturbations, Trans. Mosc. Math. Soc. 12 (1963) 3–52.

[56] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag, New York, 1997, pp. 184–193.

[57] P. Szmolyan, Transversal heteroclinic and homoclinic orbits in singular perturbation problems, J. Differential Equations 92 (1991) 252–281.

[58] I. S. Gradshteyn, I. M. Ryzhik, Table of integrals, series, and products. Amsterdam: Elsevier/Academic Press, Amsterdam, Translation edited and with a preface by Daniel Zwillinger and Victor Moll. Eighth edition, 2015, pp. 94–95.