The purpose of this paper is to construct $p$-adic Dedekind sums and Hardy-Berndt type sums. We also construct generating function of the twisted Bernoulli polynomials and functions. Furthermore, we give some discussions on elliptic analogue of the Apostol-Dedekind sums.

1. Introduction, Definitions and Notations

Let $(h, k) = 1$ with $k > 0$. The classical Dedekind sums are defined by

$$s(h, k) = \sum_{a=1}^{k-1} \left( \frac{a}{k} \right) \left( \frac{ha}{k} \right),$$

where $((x)) = x - [x]_{\mathbb{Z}} - \frac{1}{k}$, if $x \notin \mathbb{Z}$, $((x)) = 0$, $x \in \mathbb{Z}$, where $[x]_{\mathbb{Z}}$ is the largest integer $\leq x$ cf. \([3, 5, 9, 11, 12, 13]\).

In this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$, $\mathbb{C}$ and $\mathbb{Z}$, respectively, denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$ normalized by $|p|_p = p^{-1}$, and the complex field and integer numbers. Let $q$ be an indeterminate such that if $q \in \mathbb{C}$, then $|q| < 1$ and if $q \in \mathbb{C}_p$, then $|1 - q|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log_q(q))$ for $|x|_p \leq 1$. Let $[x] = [x : q] = \frac{1 - q^{-x}}{1 - q^{-1}}$. We note that $\lim_{q \to 1} [x] = x$. The $p$-adic $q$-Volkenborn integral is originally constructed by Kim \([7, 8]\), which is defined as follows: for $g \in UD(Z_p, \mathbb{C}_p) = \{g \mid g : Z_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the $p$-adic $q$-Volkenborn integral is defined by

$$I_q(g) = \int_{Z_p} g(x) \, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{q^{N+1} - 1} \sum_{x=0}^{q^{N-1} - 1} g(x) q^x,$$

where $\mu_q(x + p^N Z_p) = \frac{q^x}{[p^N : q]}$. Note that $I_1(g) = \lim_{q \to 1} I_q(g)$. If $g_1(x) = g(x + 1)$, then

$$I_1(g_1) = I_1(g) + g'(0) \text{ cf. } (7, 6), \quad (1.1)$$

$$qI_q(g_1) - I_q(g) = (q - 1)g(0) + \frac{q - 1}{\log q} g'(0) \text{ cf. } [8], \quad (1.2)$$

where $g'(0) = \left. \frac{d}{dx} g(x) \right|_{x=0}$. The $q$-deformed $p$-adic invariant integral on $\mathbb{Z}_p$, in the fermionic sense, is defined by

$$I_{-1}(f) = \lim_{q \to 1} I_q(f) = \int_{Z_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} (-1)^x f(x) \text{ cf. } (7, 8). \quad (1.3)$$

In \([7, 8]\), by using $p$-adic $q$-integral on $\mathbb{Z}_p$, Kim defined

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \quad (1.4)$$

For applications of the $p$-adic $q$-integral on $\mathbb{Z}_p$ see also cf. \([4, 14]\).
2. \( p \)-adIC HARDY-BERNDT TYPE SUMS

In this section, by using the \( q \)-deformed \( p \)-adic invariant integral on \( \mathbb{Z}_p \), in the fermionic sense, we construct \( p \)-adic Hardy-Berndt type sums.

\[
\int_{\mathbb{Z}_p} \sin(bx) \, d\mu_{-1}(x) = -\tan\left(\frac{b}{2}\right) \text{ cf. } [7].
\] (2.1)

Multiplying both sides of (2.1) by \( \frac{1}{\pi n} \), and replacing \( b \) by \( 2\pi ny \), and then summing over \( n = 1, 2, \ldots, \infty \), we have

\[
\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{Z}_p} \sin(2\pi n y x) \, d\mu_{-1}(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \tan(\pi n y) \tag{2.2}
\]

After some elementary calculations, we get

\[
\int_{\mathbb{Z}_p} \left( \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n y x) \right) \, d\mu_{-1}(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \tan(\pi n y),
\]

where \( (y) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n y)}{n} \) cf. [3], [4], [12], [15]. Thus we arrive at the following result:

Lemma 1.

\[
\int_{\mathbb{Z}_p} (y x) \, d\mu_{-1}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \tan(\pi n y). \]

By using Lemma 1 we construct \( p \)-adic Hardy-Berndt type sums as follows:

Theorem 1. Let \( h,k \in \mathbb{Z} \), \( (h,k) = 1 \). If \( h \) is odd and \( k \) is even, then we have

\[
S_2(h,k) = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\tan(\pi n h / k)}{n} \tag{2.3}
\]

2n \( \not\equiv 0 \) (mod \( k \))

By using Lemma 1 with \( y = \frac{h}{k} \) with \( (h,k) = 1 \), we have

\[
\int_{\mathbb{Z}_p} \left( \frac{h x}{k} \right) \, d\mu_{-1}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \tan(\pi n h / k). \tag{2.4}
\]

By using (2.2) and (2.3), with \( 2n \not\equiv 0 \) (mod \( k \)), after some elementary calculations, we arrive at the desired result.

In [3], Bernd and Goldberg defined Hardy sums, \( S_3(h,k) \) as follows: if \( h \) is odd and \( k \) is even, then

\[
S_3(h,k) = \frac{1}{\pi} \sum_{n=1}^{\infty} \tan(\frac{\pi h n}{k}). \tag{2.5}
\]

By using (2.4) and Theorem 1 we obtain the following corollary:

Corollary 1. Let \( h,k \in \mathbb{Z} \), \( (h,k) = 1 \). If \( k \) is odd, then we have

\[
S_3(h,k) = \int_{\mathbb{Z}_p} \left( \frac{h x}{k} \right) \, d\mu_{-1}(x). \tag{2.6}
\]

Multiplying both sides of (2.3) by \( \frac{4}{\pi n \sin^{n+1} \left( \frac{\pi h(2n-1)}{2k} \right)} \), and replacing \( b \) by \( \frac{\pi h(2n-1)}{2k} \), with \( (h,k) = 1 \), and then summing over \( n = 1, 2, \ldots, \infty \), we have

\[
\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \int_{\mathbb{Z}_p} \sin \left( \frac{\pi h(2n-1)x}{2k} \right) \, d\mu_{-1}(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \tan \left( \frac{\pi h(2n-1)}{2k} \right)
\]

After some elementary calculations, we get

\[
\int_{\mathbb{Z}_p} \left( \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \left( \frac{\pi h(2n-1)x}{2k} \right) \right) \, d\mu_{-1}(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \tan \left( \frac{\pi h(2n-1)}{2k} \right). \tag{2.7}
\]
where
\[ (-1)^{k^{\text{twisted}}} = \frac{4}{\pi} \sum_{n=1}^{\infty} \sin((2n-1)\pi x) \quad \text{cf. } [7, 3, 12, 15]. \]

If \( h \) and \( k \) are odd, then we have
\[ S_0(h, k) = 2 \quad \sum_{n=1}^{\infty} \frac{\tan\left(\frac{\pi h(2n-1)}{2k}\right)}{2n-1} \quad \text{cf. } [3]. \]  

(2.6)

By using (2.5) and (2.6), we arrive at the following theorem:

**Theorem 2.** Let \( h, k \in \mathbb{Z}, (h, k) = 1 \). If \( k \) and \( k \) are odd, then we have
\[ S_0(h, k) = -2 \int_{\mathbb{Z}_p} (-1)^{k^{\text{twisted}}} d\mu_{-1}(x). \]

Let \( h \) and \( k \) denote relatively prime integers with \( k > 0 \). If \( h + k \) is odd, then
\[ S(h, k) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\tan\left(\frac{\pi h(2n-1)}{2k}\right)}{2n-1} \quad \text{cf. } [3]. \]

(2.7)

By using (2.5) and (2.7), we easily arrive at the following corollary:

**Corollary 2.** Let \( h, k \in \mathbb{Z}, (h, k) = 1 \) with \( k > 0 \). If \( h + k \) is odd, then
\[ S(h, k) = -\int_{\mathbb{Z}_p} (-1)^{k^{\text{twisted}}} d\mu_{-1}(x). \]

Note that for detail on Hardy-Berndt sums see also cf. ([3], [12], [15], [4]).

3. Twisted Dedekind Sums

In this section, by using \( q \)-Volkenborn integral, we construct a new generating function of twisted \( q \)-Bernoulli polynomials. We define twisted new approach \( q \)-Bernoulli functions. We also construct \( p \)-adic twisted \( q \)-Dedekind type sums. Let \( T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} \), where \( C_{p^n} = \{ w : w^n = 1 \} \) is the cyclic group of order \( p^n \). For \( w \in T_p \), the function \( x \mapsto w^x \) is a locally constant function from \( \mathbb{Z}_p \) to \( C_p \) cf. ([7], [13]). If we take \( f(x) = w^x q^x e^{tx} \) in (1.2), then we define twisted \( q \)-Bernoulli numbers by means of the following generating function:
\[ F_{q,w}(t) = I_q(w^x q^x e^{tx}) = \left( \frac{q-1}{\log q} \right) t^2 + t = \sum_{n=0}^{\infty} b_n^{w,q}(q) \frac{t^n}{n!} \]

where the numbers \( b_n^{w,q}(q) \) are called twisted \( q \)-Bernoulli numbers. By using Taylor series of \( e^{tx} \) in the above, we get
\[ b_n^{w,q}(q) = \int_{\mathbb{Z}_p} w^x q^x x^n d\mu_q(x). \]  

(3.1)

We define twisted \( q \)-Bernoulli polynomials by means of the following generating function:
\[ F_{q,w}(t, z) = F_{q,w}(t) e^{tz} = \sum_{n=0}^{\infty} b_n^{w,z,q}(z, q) \frac{t^n}{n!}, \]

(3.2)

where the numbers \( b_n^{w,z,q}(z, q) \) are called twisted \( q \)-Bernoulli polynomials. By using Cauchy product in the above, we have
\[ b_n^{w,z,q}(z, q) = \int_{\mathbb{Z}_p} w^x q^x (x + z)^n d\mu_q(x) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} b_k^{w,z,q}(q). \]

Observe that if \( q \to 1 \), then (1.2) reduces to (1.1). See also cf. ([3], [12], [11]).

We need the following definitions. \( \overline{B}_n(x) \) is denoted the \( n \)th Bernoulli function, which is defined as follows:
\[ \overline{B}_n(x) = \begin{cases} B_n(x), & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer and } n = 1, \end{cases} \]  

(3.3)
where \( \{ x \} \) denotes the fractional part of a real number \( x \), \( B_n(x) \) is the Bernoulli polynomial or \( \overline{B}_n(x) = B_n(x - \lfloor x \rfloor) \) cf. \( [11, 13, 12, 15] \). Apostol \( [1] \) generalized Dedekind sums \( s(h, k, n) \) as follows

\[
s(h, k, n) = \sum_{a=1}^{k-1} \frac{a}{k} \overline{B}_n \left( \frac{ha}{k} \right),
\]

(3.4)

where \( n, h, k \) are positive integers. By using \( (3.1), (3.2) \) and \( (3.3) \), we have

\[
\overline{b}^{n, w}_{n, w} \left( \frac{jb}{k}, q \right) = \int_{\mathbb{Z}_p} w^s q^x \left( x + \left\{ \frac{jb}{k} \right\} \right)^n \, d\mu_q(x).
\]

(3.5)

By using \( (3.3) \) and \( (3.4) \), we construct twisted \( p \)-adic \( q \)-higher order Dedekind type sums by the following theorem:

**Theorem 3.** Let \( h, k \in \mathbb{Z} \), \( (h, k) = 1 \), and let \( p \) be an odd prime such that \( p | k \). For \( w \in \mathbb{Z}_p \), we have

\[
s_w(h, k, m, q) = \sum_{j=0}^{k-1} \sum_{k=1}^{\infty} \frac{1}{k} \int_{\mathbb{Z}_p} w^s q^x \left( x + \left\{ \frac{jb}{k} \right\} \right)^n \, d\mu_q(x),
\]

or

\[
s_w(h, k, m, q) = \sum_{j=0}^{k-1} \sum_{k=1}^{\infty} \frac{1}{k} \int_{\mathbb{Z}_p} w^s q^x \left( x + \left\{ \frac{jb}{k} \right\} \right)^n \, d\mu_q(x).
\]

Observe that when \( q \to 1 \) and \( w \to 1 \), the sum \( s_w(a, b, m, q) \) reduces to \( k^m s(h, k, m + 1) \) in \( (3.4) \). Different type \( p \)-adic Dedekind and Hardy type sums were defined see for detail \( [9, 11, 13, 15, 4] \).

**Remark 1.** Recently, elliptic Apostol-Dedekind sums have studied by many authors in the different areas. Bayad \( [4] \), constructed multiple elliptic Dedekind sums as an elliptic analogue of Zagier’s sums multiple Dedekind sums. Machide \( [12] \) defined elliptic analogue of the generalized Dedekind-Rademacher sums, which involve an elliptic analogue of the classical Bernoulli functions.

Find elliptic analogue and reciprocity law of the sum \( s_w(h, k, m, q) \).

**Acknowledgement.** This paper was supported by the Scientific Research Project Administration Akdeniz University.

**References**

[1] T. M. Apostol, Generalized Dedekind sums and transformation formulæ of certain Lambert series, Duke Math. J. 17 (1950), 147-157.
[2] A. Bayad, Sommes elliptiques multiples d’Apostol-Dedekind-Zagier, Comptes Rendus Mathematique, 339(7) (2004), 457-462.
[3] B. C. Berndt and L. A. Goldberg, Analytic properties of arithmetic sums arising in the theory of the classical theta-functions, SIAM J. Math. Anal. 15 (1984), 143-150.
[4] M. Cenkci, Y. Simsek, M. Can and V. Kurt, Twisted Dedekind type sums associated with Barnes’ type multiple Frobenius-Euler l-functions, submitted.
[5] T. Kim, A new approach to \( q \)-zeta function, Adv. Stud. Contemp. Math. 11(2) (2005), 157-162.
[6] T. Machide, An elliptic analogue of the generalized Dedekind-Rademacher sums, J. Number Theory, in press.
[7] T.-M. Apostol, Generalized Dedekind sums and transformation formulæ of certain Lambert series, Duke Math. J. 17 (1950), 147-157.
[8] A. Bayad, Sommes elliptiques multiples d’Apostol-Dedekind-Zagier, Comptes Rendus Mathematique, 339(7) (2004), 457-462.
[9] B. C. Berndt and L. A. Goldberg, Analytic properties of arithmetic sums arising in the theory of the classical theta-functions, SIAM J. Math. Anal. 15 (1984), 143-150.
[10] M. Cenkci, Y. Simsek, M. Can and V. Kurt, Twisted Dedekind type sums associated with Barnes’ type multiple Frobenius-Euler l-functions, submitted.
[11] T. Kim, A new approach to \( q \)-zeta function, Adv. Stud. Contemp. Math. 11(2) (2005), 157-162.
[12] T. Machide, An elliptic analogue of the generalized Dedekind-Rademacher sums, J. Number Theory, in press.
[13] T.-M. Apostol, Generalized Dedekind sums and transformation formulæ of certain Lambert series, Duke Math. J. 17 (1950), 147-157.