Fitting the Power-law Distribution to the Mexican Stock Market index data.

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Abstract. In the spirit of the emergent field of econophysics, a goodness-of-fit test for the Power-Law distribution, based on the Empirical Distribution Function (EDF) is presented, and related problems are discussed. An analysis of the tail behaviour of the daily logarithmic variation of the Mexican Stock Market Index (IPC), showed distributional properties which are consistent with previous studies.

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1 Introduction

The Power-law distribution, also called Pareto distribution describes phenomena presented in fields as Social Sciences, i.e. Economics and Finance (individuals income distribution, stock market price variation distribution, etc) or Physics (phase transitions, nonlinear dynamics, and disordered systems). Although with a tradition of more than a hundred years [1,2,3], interest of physics community in the complex behavior of Financial Markets has strongly increased in the last years boosted by the availability of huge worldwide economical data electronically recorded, giving rise to a complementary or non orthodox under the point of view of traditional Economic Theory way to attack problems based in the empirical data analysis rather than in the traditional economical analysis. The collection of methods and techniques originally developed to study problems arising from physics and that currently are being used to understand financial complex systems, is called Econophysics and appears as an emergent branch of Physics by itself [3,4,5,6,7,8].

About 100 years ago, the Power-Law distribution was proposed by Pareto to describe the distribution of income of individuals. More recently, based in Mandelbrot’s pioneer work [9] and later of Mantegna and Stanley [10], analyses of price distribution variations of leading stock markets [11,12,13,14] and individual companies [15,16] have been reported; all of them showing Pareto tails, with $\alpha \approx 3$ for the stock market case.

In section 4 of this work, we analyze the distribution of daily logarithmic differences of the Mexican IPC stock index, defined as $S(t) := \log Z(t + 1) - \log Z(t)$; for IPC values of $Z(t)$ recorded during an almost 10 years period, from April 14, 1990 until December 31, 2002.

A goodness-of-fit test based on the Empirical Distribution Function (EDF) is introduced in sections 2 to 4. Section 5 explains the test procedures and section 6 shows results of a Monte Carlo study used to investigate the small sample distribution of the test statistics and the speed of convergence to their asymptotic distribution.

2 Empirical Distribution Function (EDF).

Let $Y_1, \ldots, Y_n$ be a random sample from an absolutely continuous distribution $F$ and suppose that we are interested in testing the null hypothesis that the sample was drawn from the distribution

$$F(y) = 1 - \left( \frac{y}{\gamma} \right)^\alpha$$

(1)

with support on $0 < \gamma \leq y$ for $\alpha > 0$.

The distribution (1) is known as the Power-law distribution. A test of fit can be based on measures of discrepancy between the empirical distribution function $F_n$ and the hypothesized distribution $F$. Such test statistics are referred to as empirical distribution function based statistics or simply EDF-statistics. Here we will consider statistics within the class

$$Q_n = n \int_{-\infty}^{\infty} [F_n - F]^2 \psi(dF)$$

(2)

When $\psi(.) = 1$, $Q_n$ is known as the Cramér von-Mises $W^2$ statistic and, for $\psi(\nu) = \{[F(\nu)][1 - F(\nu)]\}^{-1}$, it is known as the Anderson-Darling $A^2$ statistic.

The test are based on the quantities $Z_t = F(Y_t; \gamma, \alpha)$, the Probability Integral Transformation which, under the null hypothesis produces observations uniformly distributed on
(0, 1). To obtain computational formulas for the test statistics, the expression in (2) can be written in terms of the observed discrepancy between the empirical distribution function and the uniform cumulative distribution function; i.e.,

\[ Q_n = n \int_0^1 \left[ F_n^2(z) - z \right] \psi'(z) dz \]  

(3)

where \( \psi'(z) = 1 \) for \( W^2 \) and \( \psi'(z) = [z(1-z)]^{-1} \) for the Anderson-Darling statistic \( A^2 \).

Computation formulas for these statistics involve the ordered sample values \( Z(1) < \ldots < Z(n) \):

\[ W^2 = \frac{n}{n} \sum_{i=1}^{n} \left[ (z_i - (2i - 1)/(2n))^2 + 1/(12n) \right] \]  

(4)

\[ A^2 = -n - (1/n) \sum_{i=1}^{n} (2i - 1) \left[ \log Z(i) + \log \{ 1 - Z(i) \} \right] \]  

(5)

3 Estimation

Given observed values \( y_1, \ldots, y_n \) of a random sample from the distribution (1), the log-likelihood is

\[ \lambda(\alpha, \gamma) = n \log \alpha + na \log \gamma - (\alpha + 1) \sum_{i=1}^{n} \log y_i \]  

(6)

When \( \gamma \) is known, the maximum-likelihood estimator of the parameter \( \alpha \) is

\[ \hat{\alpha} = \left[ \frac{1}{n} \sum_{i=1}^{n} \log (y_i/\gamma) \right]^{-1} \]  

(7)

If \( \gamma \) is unknown, we use an approach proposed by [17] substituting (7) into (6) to obtain the profile log-likelihood for \( \gamma \); namely

\[ \lambda^*(\gamma) = n \ln(n) - n \ln \left[ -n \ln(\gamma) + \sum_{i=1}^{n} \ln y_i \right] - n - \sum_{i=1}^{n} \ln y_i \]  

(8)

It can be seen that the above function is increasing in the range \( 0 < \gamma < \exp(\sum_{i=1}^{n} \ln y_i/n) \). In fact, since \( d\lambda^*(\gamma)/d\gamma = na/\gamma \), the derivative is positive in it’s range of admissible values. Thus, the maximum-profile-likelihood estimator for \( \gamma \) is \( \hat{\gamma} = \gamma(1) \), the minimum sample value.

When \( \gamma(i) = \gamma \), \( i = 1, \ldots, n \), is unknown, the estimate \( \hat{\gamma} = Y(1) \) is super-efficient in the sense that it’s variance tends to zero faster than \( 1/n \). Using this estimate, we "loosely" one sample observation and there are computational problems for calculating \( A^2 \). Since, in this case,

| Parameters          | 0.250  | 0.15  | 0.10  | 0.05  | 0.025 | 0.010 |
|---------------------|--------|-------|-------|-------|-------|-------|
| \( W^2 \)           | 0.209  | 0.284 | 0.347 | 0.461 | 0.581 | 0.743 |
| \( A^2 \)           | 1.248  | 1.610 | 1.933 | 2.492 | 3.070 | 3.880 |

\( \alpha \) unknown

| Parameters          | 0.25  | 0.15  | 0.10  | 0.05  | 0.025 | 0.010 |
|---------------------|-------|-------|-------|-------|-------|-------|
| \( W^2 \)           | 0.116 | 0.148 | 0.175 | 0.222 | 0.271 | 0.338 |
| \( A^2 \)           | 0.736 | 0.916 | 1.062 | 1.321 | 1.591 | 1.959 |

\( z(1) = 0 \), we suggest to estimate the parameter \( \gamma \) by finding the values \( \gamma_i \) of \( \gamma \), which satisfies

\[ \gamma - y(i) \left[ 1 - \{ na(\gamma) \}^{-1} \right] = 0 \]  

(9)

where \( a(\gamma) \) is defined by (7). Thus, starting with \( \gamma = y(1) \), we search for the solution over the interval \((0, y(1))\). This method of estimation does not seem to have a significant effect over the sampling distributions of the test statistics as it will be indicated from the results of the section 4.

4 Asymptotic distributions

EDF asymptotic distribution statistics was obtained applying the theory in [19]. The process \( \sqrt{n} \{ F_n(x) - F(x) \} \) evaluated at \( t = F(x) \), converges weakly to \( \{ Y(t), t \in (0, 1) \} \), a Gaussian process with zero mean and covariance function \( \rho(s, t) \) which depends on the parameters estimated and \( F \). The statistics \( W^2 \) and \( A^2 \) are asymptotically functions of \( Y(t) \); namely, \( W^2 \rightarrow_{D} \frac{1}{n} Y^2(t) dt, A^2 \rightarrow_{D} \int_0^1 a^2(t) dt \), where \( a(t) = \frac{Y(t)}{\sqrt{1-t}} \). Let \( \rho^*(s, t) \) denote the covariance function for a given statistic. The limiting distribution of the test statistic is that of \( \lambda_1, \lambda_2, \ldots \) are independent \( \chi^2(1) \) random variables and \( \lambda_1, \lambda_2, \ldots \) are the eigenvalues of the integral equation

\[ \int_0^1 \rho^*(s, t) f_j(s) ds = \lambda_j f_j(t) \]  

(10)

When the parameters are known, the covariance function is given by \( \rho(s, t) = \min(s, t) - st \). When \( \gamma \) is known, and \( \alpha \) is estimated using (6), the covariance function for the limiting process becomes \( \rho^*(s, t) = \rho(s, t) - (1-s) (1-t) \log(1-s) \log(1-t) \) which corresponds to the same covariance function as the resulting one when testing fit to the exponential distribution for unknown \( \alpha \), so the asymptotic distribution of the test statistics is the same for testing exponentially or fit to the Power-law distribution with \( \alpha \) unknown. Tables 4.2 and 4.11 in [18], give selected percentage points for the case of known parameters, and for testing exponentially with unknown \( \alpha \), respectively. They are summarized in table 1 for quick reference.
5 Test procedures

1. Given the ordered sample values \( y_1 \leq \ldots \leq y_n \), the test statistics are computed from the values \( z_{(i)} = F(y_{(i)}; \alpha, \gamma) \) where the values of the parameters \( \alpha \) and/or \( \gamma \) can be replaced by their estimates if they are not known.

2. Compute the value of the test statistic \( A^2 \) using (3) or \( W^2 \) from (1).

3. Refer to table 1 for the appropriate case and significance level.

4. If the value of the test statistic exceeds that from the table, reject the Power-law model at the corresponding significance level.

When \( \gamma \) is known, the distribution of \( X = \log(y/\gamma) \) is exponential with parameter \( \alpha \); i.e., \( F_X(x) = 1 - \exp(-\alpha x) \), and the problem of testing fit to the Power-law distribution is equivalent to that of testing the null hypothesis that the transformed observations \( x_1, \ldots, x_n \) were drawn from an exponential distribution. If \( Y_1, \ldots, Y_n \) is a sample of \( n \) independent values from Power-law distributions with parameters \( \gamma_1, \ldots, \gamma_n \) and the same value of the parameter \( \alpha \), a test of fit can be carried out by transforming \( X_i = \log(Y_i/\gamma_i) \) and, again, test that the transformed values come from an exponential population. It is important to remark that this procedure is computationally equivalent to the procedure described in the previous paragraph, due to the fact that \( F_X(x_i) = F_Y(y_i) \).

6 Monte Carlo study

A simulation study was conducted to investigate the speed of convergence of the empirical percentage points of the test statistics to their asymptotic values. Five thousand samples of size \( n = 20(200)20 \) were simulated. In the simulation, values of \( \gamma \) and \( \alpha \) ranging from 1/4 to 10 were used to verify that there was no effect of the parameter values on the speed of convergence; the results showed little or no effect at all. The empirical percentage points presented in tables 2 and 3 show typical results obtained for \( \gamma = 1 \) and \( \alpha = 5 \).

Table 2 shows the empirical percentage points for the test-statistics using the known value of \( \gamma \) and estimating \( \alpha \) using (4). These results clearly suggest that the asymptotic percentage points can be used even for small values of \( n \). When the parameter \( \gamma \) is estimated using (6), the speed of convergence of the empirical percentage points, to those corresponding to the asymptotic distribution, is slower. As it is shown in table 3, it is recommended the use the Monte Carlo percentage points for \( n \leq 100 \). It is worth to note that the results in table 3 are very close to those reported in table 4.15 from \( [\ ] \) for testing exponentially with origin and scale parameters unknown, which is referred as an external check of these results.

7 Application to the analysis of stock price variations

The data analyzed consist of two data sets; the first corresponding to the 126 largest differences \( d_1 \), and the second one corresponding to the absolute values of the 144 smallest differences \( d_2 \), from the series \( S(t) = \log Z(t + 1) - \log Z(t) \); where \( Z(t) \) denotes daily values of the Mexican IPC stock index, between the April 14, 1990 and December 31, 2002.

Following Gopikrishnan et al. [11][12], Liu et al. [13], and Mantegna and Stanley [8], we wish to test statistically, the null hypothesis that these extreme observations are consistent with the Power-law distribution.

For the data set \( d_1, \ldots, d_{126} \), the estimates of the parameters are \( \hat{\gamma} = 0.0313 \) and \( \hat{\alpha} = 3.31 \); for which the values of the test statistics were found to be \( A^2 = 0.5778 \) and \( W^2 = 0.0995 \).

The second data set, consisting of the absolute values \( d_2^* \), of the smallest differences, we find \( \hat{\gamma} = 0.0275 \),
they are based, relating it to the case of the exponential distribution. When the threshold parameter \( \gamma \) is known, simulations results suggest that the asymptotic percentage points of the Anderson-Darling and Cramér von-Mises statistics can be used with good accuracy even for small \( n \). For the case of \( \gamma \) unknown, an estimator was proposed. From Monte Carlo results, we conclude that such an estimator is useful for goodness-of-fit purposes in the sense that it allows the calculation of the test statistics, preserving the asymptotic distribution, although the speed of convergence appears to be slower in this case. The proposed test was shown to be useful in analyzing stock price variations, where it is required a significance test of fit for the power-law distribution. The results obtained here for describing the changes in the Mexican stock exchange index IPC, are consistent with previous studies where the power-law distribution with shape parameter \( \alpha \simeq 3 \) has been proposed.

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References

1. V. Pareto, Course d’Economie Politique, [Lausanne and Paris, 1897].
2. L. Bachelier, Ph.D. Thesis, Théorie de la Spéculation, Annales Scientifiques de l’Ecole Normale Supérieure III-17, (1900) 21-86.
3. Bertrand M. Roehner, Patterns of Speculation. A Study in Observational Econophysics (Cambridge University Press, United Kingdom 2002) 25-35.
4. Proceedings of the Workshop “Empirical Science of Financial Fluctuations. The Advent of Econophysics”, edited by Hideki Takayasu (Workshop Organized by Nihon Keizai Shimbun, Tokyo 2002).
5. Jean-Philippe Bouchaud, Physica A 313,(2002) 238-251.
6. H. E. Stanley et al, Physica A 269,(1999) 156-159.
7. Dietrich Stauffer, Int. J. Mod. Phys. C 11 (2000) 1081-1087.
8. R. N. Mantegna, and H. E. Stanley, An Introduction to Econophysics (Cambridge University Press, United Kingdom 2000).
9. B. B. Mandelbrot, J. Business 36, (1963) 394-419.
10. R. N. Mantegna, H. E. Stanley, Nature 376 (1995) 46-49.
11. P. Gopikrishnan et al, Eur. Phys. J.B 3, (1998) 139-140.
12. P. Gopikrishnan et al, Phys.Rev. E 60, (1999) 5305-5316.
13. Y. Liu, P. Gopikrishnan et al, Phys.Rev. E 60,2, (1999) 6517-6529.
14. T. Lux, Applied Financial Economics 6, (1996) 463-475
15. Liu, Y. et al., Physical Review E, 60, 2, (1999) 1390-1400.
16. V. Plerou, P. Gopikrishnan, L. A. Amaral, M. Meyer, H. E. Stanley, Phys.Rev. E 60,6, (1999) 6519-6529.
17. Lockhart, R.A. and Stephens, M.A., Estimation and Tests of fit for the three-parameter Weibull distribution. Research Report 92-10, (Department of Mathematics and Statistics, Simon Fraser University 1992).
18. D’Agostino, R.B. and Stephens, M.A., Goodness-of-fit Techniques, (Marcel Dekker, New York 1986).
19. Durbin, J., Distribution theory for tests based on the sample distribution function, Regional Conference Series in Applied Mathematics, 9 (Philadelphia, SIAM, 1973).