A Product Version of the Hilton-Milner-Frankl Theorem

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Abstract

Two families $F, G$ of $k$-subsets of $\{1, 2, \ldots, n\}$ are called non-trivial cross $t$-intersecting if $|F \cap G| \geq t$ for all $F \in F, G \in G$ and $|\{F: F \in F\}| < t$, $|\{G: G \in G\}| < t$. In the present paper, we determine the maximum product of the sizes of two non-trivial cross $t$-intersecting families of $k$-subsets of $\{1, 2, \ldots, n\}$ for $n \geq 4(t + 2)^2k^2$, $k \geq 5$, which is a product version of the Hilton-Milner-Frankl Theorem.

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1 Introduction

Let $n > k > t$ be positive integers and let $[n] = \{1, 2, \ldots, n\}$ be the standard $n$-element set. Let $\binom{[n]}{k}$ denote the collection of all $k$-subsets of $[n]$. Subsets of $\binom{[n]}{k}$ are called $k$-uniform hypergraphs or $k$-graphs for short. A $k$-graph $F$ is called $t$-intersecting if $|F \cap F'| \geq t$ for all $F, F' \in F$.

One of the most important results in extremal set theory is the following:

\textbf{Erdős-Ko-Rado Theorem} ([3]). Suppose that $n \geq n_0(k, t)$ and $F \subset \binom{[n]}{k}$ is $t$-intersecting. Then

$$|F| \leq \binom{n-t}{k-t}.$$ (1.1)

\textbf{Remark.} For $t = 1$ the exact value $n_0(k, t) = (k - t + 1)(t + 1)$ was proved in [3]. For $t \geq 15$ it is due to [4]. Finally Wilson [17] closed the gap $2 \leq t \leq 14$ with a proof valid for all $t$.

Pyber [16] proved a product version of the Erdős-Ko-Rado Theorem for $t = 1$.

\textbf{Theorem 1.1 (Pyber [16]).} Suppose that $F, G \subset \binom{[n]}{k}$ are cross-intersecting, $n \geq 2k$ then

$$|F||G| \leq \binom{n-1}{k-1}^2.$$ (1.2)
A \( t \)-intersecting family \( \mathcal{F} \subset \binom{[n]}{k} \) is called \emph{non-trivial} if \(| \cap \{ F : F \in \mathcal{F} \}| < t \).

**Example 1.2.** Define two families
\[
\mathcal{H}(n, k, t) = \left\{ H \in \binom{[n]}{k} : |t| < H, H \cap |t+1, k+1| = \emptyset \right\} \cup \{(k+1) \setminus \{j\} : 1 \leq j \leq t\},
\]
\[
\mathcal{A}(n, k, t) = \left\{ A \in \binom{[n]}{k} : |A \cap |t+2| \geq t+1 \right\}.
\]

It is easy to see that both \( \mathcal{H}(n, k, t) \) and \( \mathcal{A}(n, k, t) \) are non-trivial \( t \)-intersecting families.

**Hilton-Milner-Frankl Theorem \((\text{[12]} \text{[3]}\).** Suppose that \( \mathcal{F} \subset \binom{[n]}{k} \) is non-trivial \( t \)-intersecting, \( n \geq (k-t+1)(t+1) \). Then
\[
\text{(1.3)} \quad |\mathcal{F}| \leq \max \left\{ |\mathcal{A}(n, k, t)|, |\mathcal{H}(n, k, t)| \right\}.
\]

Two families \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) are called \emph{cross} \( t \)-intersecting if \( |F \cap G| \geq t \) for any \( F \in \mathcal{F}, G \in \mathcal{G} \). If \( \mathcal{A} \subset \binom{[n]}{k} \) is \( t \)-intersecting, then \( \mathcal{F} = \mathcal{A}, \mathcal{G} = \mathcal{A} \) are cross \( t \)-intersecting.

In the present paper, we prove a product version of the Hilton-Milner-Frankl Theorem.

**Theorem 1.3.** Suppose that \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) are non-trivial cross \( t \)-intersecting. If \( n \geq 4(t+2)^2k^2 \) and \( k \geq 5 \), then
\[
\text{(1.4)} \quad |\mathcal{F}| |\mathcal{G}| \leq \max \left\{ |\mathcal{A}(n, k, t)|^2, |\mathcal{H}(n, k, t)|^2 \right\}.
\]

It should be mentioned that the case \( t = 1 \) of Theorem 1.3 was proved for \( n \geq 9k \) and \( k \geq 6 \) in \([9]\). Thus we always assume \( t \geq 2 \) in this paper. Since we are concerned with the maximum of \( |\mathcal{F}| |\mathcal{G}| \), without further mention we are assuming throughout the paper that \( \mathcal{F} \) and \( \mathcal{G} \) form a saturated pair, that is, adding an extra \( k \)-set to either of the families would destroy the cross \( t \)-intersecting property.

It should also be mentioned that a similar result to Theorem 1.3 was obtained recently by Cao, Lu, Lv and Wang \([1]\), in which a different notion of non-triviality was considered. In their paper, the non-triviality of two cross \( t \)-intersecting families \( \mathcal{F}, \mathcal{G} \) means that \( \mathcal{F} \cup \mathcal{G} \) is not a \( t \)-star. However, to the best of our knowledge \( 1.4 \) is the first one that implies the Hilton-Milner-Frankl Theorem (just set \( \mathcal{F} = \mathcal{G} \)).

For \( \mathcal{A}, \mathcal{B} \subset \binom{[n]}{k} \), we say that \( \mathcal{A}, \mathcal{B} \) are \emph{exact} \( t \)-intersecting if \( |A \cap B| = t \) for every \( A \in \mathcal{A}, B \in \mathcal{B} \). For \( \mathcal{F} \subset \binom{[n]}{k} \), the \( t \)-covering number \( \tau_t(\mathcal{F}) \) of \( \mathcal{F} \) is defined as
\[
\tau_t(\mathcal{F}) = \min \{|T| : |T \cap F| \geq t \text{ for all } F \in \mathcal{F}|\}
\]

For the proofs, we need a result concerning exact cross \( t \)-intersecting \((t+1)\)-uniform families.

**Proposition 1.4.** Let \( n > k > t \geq 2 \) and let \( \mathcal{A}, \mathcal{B} \subset \binom{[n]}{t+1} \) be non-empty exact cross \( t \)-intersecting. If both \( \mathcal{A} \) and \( \mathcal{B} \) do not contain a sunflower of \( k-t+2 \) petals with center of size \( t \), then one of the following holds:

(i) either \( |A| \leq 2, |B| \leq k+1, \tau_t(\mathcal{B}) \geq t+1 \) or \( |A| \leq 2, |B| \leq 2, \tau_t(\mathcal{A}) \geq t+1 \).

(ii) \( \mathcal{A} \cup \mathcal{B} \) is a sunflower with center of size \( t \).

(iii) \( |A||B| \leq \frac{(t+2)^2}{2} \).
Let us present some inequalities and notations needed in the proofs.

**Proposition 1.5.** Let \( n, k, i \) be positive integers. Then

\[
\left( \frac{n-i}{k} \right) \geq \frac{n-ik}{n} \left( \frac{n}{k} \right), \text{ for } n > ik. 
\]

(1.5)

**Proof.** It is easy to check for all \( b > a > 0 \) that

\[
ba > (b+1)(a-1) \text{ holds.} 
\]

(1.6)

Note that

\[
\left( \frac{n-i}{k} \right) = \frac{(n-k)(n-k-1)\ldots(n-k-(i-1))}{n(n-1)\ldots(n-(i-1))}. 
\]

Applying (1.6) repeatedly we see that the numerator is greater than \( (n-1)(n-2)\ldots(n-(i-1))(n-ki) \) implying

\[
\left( \frac{n-i}{k} \right) \geq \frac{1-ik}{n}. 
\]

Thus (1.5) holds. \( \square \)

By (1.5), we obtain that for \( c > 1 \) and \( n \geq c(k-t)^2 + (t+1) \)

\[
\left( \frac{n-t-1}{k-t-1} \right) \leq \frac{n-t-1}{n-t-1-(k-t)(k-t-1)} \left( \frac{n-k-1}{k-t-1} \right) 
\]

\[
\leq \frac{n-t-1}{n-t-1-(k-t)^2} \left( \frac{n-k-1}{k-t-1} \right) 
\]

\[
\leq \frac{c}{c-1} \left( \frac{n-k-1}{k-t-1} \right). 
\]

(1.7)

Moreover, for \( c > 2 \) we have

\[
\left( \frac{n-t-1}{k-t-1} \right)^2 \leq \left( \frac{c}{c-1} \right)^2 \left( \frac{n-k-1}{k-t-1} \right)^2 
\]

\[
= \frac{c^2}{c^2-2c+1} \left( \frac{n-k-1}{k-t-1} \right)^2 
\]

\[
\leq \frac{c}{c-2} \left( \frac{n-k-1}{k-t-1} \right)^2. 
\]

(1.8)

Similarly, we can show that for \( n \geq ck \) and \( c > 2 \),

\[
\left( \frac{n-t-1}{k-t-1} \right)^2 \leq \frac{c}{c-2} \left( \frac{n-t-2}{k-t-1} \right)^2. 
\]

(1.9)

Let us recall the following common notations:

\( \mathcal{F}(i) = \{ F \setminus \{i\} : i \in F \in \mathcal{F} \} \), \( \mathcal{F}(\bar{i}) = \{ F \in \mathcal{F} : i \notin F \} \).

Note that \( |\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})| \). For \( P \subset Q \subset [n] \), let

\( \mathcal{F}(Q) = \{ F \setminus Q : Q \subset F \in \mathcal{F} \} \), \( \mathcal{F}(P, Q) = \{ F \setminus Q : F \cap Q = P, F \in \mathcal{F} \} \).
We also use $\mathcal{F}(Q)$ to denote $\mathcal{F}(\emptyset, Q)$. For $\mathcal{F}([i], Q)$ we simply write $\mathcal{F}(i, Q)$.

Define the family of $\ell$-th $t$-transversals of $\mathcal{F} \subset \binom{[n]}{k}$:

$$\mathcal{T}_\ell^{(t)}(\mathcal{F}) = \{T \subset [n]: |T| = \ell, |T \cap F| \geq t \text{ for all } F \in \mathcal{F}\}$$

and define the family of $t$-transversals of $\mathcal{F}$:

$$\mathcal{T}_t(\mathcal{F}) = \bigcup_{t \leq \ell \leq k} \mathcal{T}_\ell^{(t)}(\mathcal{F}).$$

Clearly, if $\mathcal{F}, \mathcal{G}$ are cross $t$-intersecting then $\mathcal{F} \subset \mathcal{T}_t^{(k)}(\mathcal{G})$ and $\mathcal{G} \subset \mathcal{T}_t^{(k)}(\mathcal{F})$.

The rest of the paper is organized as follows. In Section 2, we recall some inequalities concerning cross-intersecting families that are needed in the proofs. In Section 3, we determine the maximum product size of non-trivial cross $t$-intersecting families with a common $t$-transversal of size $t + 1$. In Section 4, we define a notion of basis for cross $t$-intersecting families and establish an upper bound on the size of the basis. In Section 5, we prove Theorem 1.3.

## 2 Some inequalities concerning cross-intersecting families

In this section, we recall several useful inequalities concerning cross-intersecting families. We also give a proof of a result of Hilton via the Hilton-Milner-Frankl Theorem.

An important tool for proving the results concerning cross-intersecting families is the Kruskal-Katona Theorem (15, 13, cf. [6] or [14] for short proofs of it).

Daykin [2] was the first to show that the Kruskal-Katona Theorem implies the Kruskal-Katona Theorem [11]. Let $\{a\}$ be pairwise cross-intersecting and $|A| = 2, 3$. For two distinct sets $F, G \in \binom{[n]}{k}$ we say that $F$ precedes $G$ if

$$\min\{i: i \in F \setminus G\} < \min\{i: i \in G \setminus F\}.$$ 

E.g., $\{1, 7\}$ precedes $\{2, 3\}$. For a positive integer $b$, let $\mathcal{L}(n, b, m)$ denote the first $m$ members of $\binom{[n]}{b}$ in lexicographic order.

**Hilton’s Lemma** ([10]). Let $n, a, b$ be positive integers, $n \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[a]}{a}$ and $\mathcal{B} \subset \binom{[b]}{b}$ are cross-intersecting. Then $\mathcal{L}(n, a, |\mathcal{A}|)$ and $\mathcal{L}(n, b, |\mathcal{B}|)$ are cross-intersecting as well.

One can deduce a result of Hilton [11] from the Hilton-Milner-Frankl Theorem.

**Theorem 2.1** ([11]). Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_{t+1} \subset \binom{X}{k}$ be pairwise cross-intersecting and $|X| = m \geq (t + 1)a$. If at least two of them are non-empty, then

$$\sum_{i=1}^{t+1} |\mathcal{K}_i| \leq \max \left\{(t + 1)\binom{m-1}{a-1}, \binom{m}{a} - \binom{m-a}{a} + t\right\}.$$  

**Proof.** Set $n = m + t + 1, k = a + t, X = [t + 2, n]$. Define the family $\mathcal{K} = \mathcal{K}_0 \cup \ldots \cup \mathcal{K}_{t+1} \subset \binom{[n]}{k}$ via

$$\mathcal{K}_0 = \left\{[t + 1] \cup K: K \in \binom{X}{k-t-1}\right\},$$

$$\mathcal{K}_i = \left\{([t + 1] \setminus \{i\}) \cup K: K \in \mathcal{K}_i\right\}, \ 1 \leq i \leq t + 1.$$
It is easy to check that $\mathcal{K}$ is $t$-intersecting. Let $I = \cap \{K : K \in \mathcal{K}\}$. Let us show $|I| < t$. Without loss of generality assume that $\mathcal{K}_1, \mathcal{K}_2$ are non-empty. It follows that $1, 2 \notin I$. For any $t + 2 \leq x \leq n$, $x \notin \cap \{K : K \in \mathcal{K}_0\} \supset I$. Thus $\mathcal{K}$ is non-trivial. Applying the Hilton-Milner-Frankl Theorem, for $(m + t + 1) \geq (a + 1)(t + 1)$ we have

$$|\mathcal{K}| = \binom{n - t - 1}{k - t - 1} + \sum_{1 \leq i \leq t+1} |\tilde{\mathcal{K}}_i|$$

$$\leq \max\{|A(n, k, t)|, |\mathcal{H}(n, k, t)|\}$$

$$= \max\left\{ (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t}{k-t} + \binom{n}{k} - \binom{n-k-1}{k} \right\}$$

$$= \binom{n-t-1}{k-t-1} + \max\left\{ (t+1)\left( \binom{m-1}{a-1}, \binom{m}{a} - \binom{m-a}{a} \right) + t \right\},$$

implying (2.1).

**Remark.** It should be mentioned that if $\mathcal{F}$ is non-trivial $t$-intersecting and has a $t$-transversal of size $t+1$, then one can also deduce the Hilton-Milner-Frankl Theorem from Theorem 2.1

**Corollary 2.2.** Let $\mathcal{K}_1, \mathcal{K}_2 \subset \binom{X}{a}$ be non-empty cross-intersecting and $|X| = m \geq (t+1)a$. If $|\mathcal{K}_1| \geq |\mathcal{K}_2|$, then

$$|\mathcal{K}_1| + t|\mathcal{K}_2| \leq \max\left\{ (t+1)\left( \binom{m-1}{a-1}, \binom{m}{a} - \binom{m-a}{a} \right) + t \right\}. \quad (2.2)$$

**Proof.** By Hilton’s Lemma, we may assume that $\mathcal{K}_i$ consists of the first $|\mathcal{K}_i|$ members of $\binom{X}{a}$ in lexicographic order. Clearly $|\mathcal{K}_2| \leq \binom{m-1}{a-1}$, implying that $\mathcal{K}_2$ is intersecting. Then $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_2$ are all non-empty and pairwise cross-intersecting. By (2.1) the corollary follows.

We should mention that Hilton’s result holds for the full range $m \geq 2a$, but we only use it in the range that we proved it.

**Theorem 2.3 ([7]).** Let $\mathcal{K}_1, \mathcal{K}_2 \subset \binom{X}{a}$ be non-empty cross-intersecting and $|X| = m \geq 2a$. If $|\mathcal{K}_1| \geq |\mathcal{K}_2| \geq \binom{m-2}{a-2}$, then

$$|\mathcal{K}_1| + |\mathcal{K}_2| \leq 2\binom{m-1}{a-1}. \quad (2.3)$$

### 3 Non-trivial cross $t$-intersecting families with common trasversals

In this section, we determine the maximum product sizes of two non-trivial cross $t$-intersecting families with a common $t$-transversal of size $t + 1$.

**Proposition 3.1.** Suppose that $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are non-trivial cross $t$-intersecting and

$$\mathcal{T}_t^{(t+1)}(\mathcal{F}) \cap \mathcal{T}_t^{(t+1)}(\mathcal{G}) \neq \emptyset.$$

If $n \geq 6(t+1)k^2$, then

$$|\mathcal{F}||\mathcal{G}| \leq \max\left\{ |A(n, k, t)|^2, |\mathcal{H}(n, k, t)|^2 \right\}. \quad (3.1)$$
Proof. Without loss of generality, assume that \([t + 1] \in \mathcal{T}_t^{(t+1)}(\mathcal{F}) \cap \mathcal{T}_t^{(t+1)}(\mathcal{G})\). Since \([t + 1]\) is a common \(t\)-transversal, for \(\mathcal{H} = \mathcal{F}\) or \(\mathcal{G}\)

\[
|\mathcal{H}| = \sum_{1 \leq i \leq t+1} |\mathcal{H}([t + 1] \setminus \{i\}, [t + 1])| + |\mathcal{H}([t + 1])|.
\]

Assume that \(\mathcal{F}\) and \(\mathcal{G}\) form a saturated pair. Then

\[
\mathcal{H}([t + 1]) = \left\{ H \setminus [t + 1] : [t + 1] \subset H \in \binom{[n]}{k} \right\}.
\]

I.e., the last term in (3.2) is \(\binom{n-t-1}{k-t-1}\).

For \(1 \leq i \leq t + 1\) set \(\mathcal{H}_i = \mathcal{H}([t + 1] \setminus \{i\}, [t + 1])\). For convenience let \(X = [t + 2, n]\).

Then \(\mathcal{H}_i \subset \binom{X}{k-1}\). Note that (3.2) can be rewritten as

\[
|\mathcal{H}| = \binom{n-t-1}{k-t-1} + \sum_{1 \leq i \leq t+1} |\mathcal{H}_i|.
\]

Note also that \(\mathcal{F}_i, \mathcal{G}_j\) are cross-intersecting for \(i \neq j\).

Fact 3.2. If \(\mathcal{F}_i, \mathcal{G}_j\) are cross-intersecting and \(\mathcal{G}_j\) is non-empty, then

\[
|\mathcal{F}_i| \leq (k-t) \binom{n-t-1}{k-t-1}.
\]

Proof. Let \(G \in \mathcal{G}_j\). Since each member in \(\mathcal{F}_i\) intersects \(G\) and \(|G| = k-t\), we infer

\[
|\mathcal{F}_i| \leq \binom{|X|}{k-t} - \binom{|X| - |G|}{k-t} \leq (k-t) \binom{|X| - 1}{k-t-1} \leq (k-t) \binom{n-t-1}{k-t-1}.
\]

By Hilton’s Lemma, we may assume that \(\mathcal{H}_i\) consists of the first \(|\mathcal{H}_i|\) members of \(\binom{X}{k-1}\) in lexicographic order. By symmetry assume that

\[
|\mathcal{F}_1| \geq |\mathcal{F}_2| \geq \ldots \geq |\mathcal{F}_{t+1}|\]

and thereby \(\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_{t+1}\).

Let \(\mathcal{G}^*\) be the one of \(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{t+1}\) with the maximum size.

Fact 3.3. We may assume that \(\min\{|\mathcal{F}_1|, |\mathcal{G}^*|\} \geq \binom{n-t-3}{k-t-2}\).

Proof. By non-triviality of \(\mathcal{F}, \mathcal{G}\), we see that both \(\mathcal{F}_1\) and \(\mathcal{G}^*\) are non-empty. Then by Fact 3.2

\[
|\mathcal{F}_1| \leq (k-t) \binom{n-t-1}{k-t-1} \text{ and } |\mathcal{G}^*| \leq (k-t) \binom{n-t-1}{k-t-1}.
\]

If \(\min\{|\mathcal{F}_1|, |\mathcal{G}^*|\} < \binom{n-t-3}{k-t-2}\), without loss of generality assume that \(|\mathcal{G}^*| < \binom{n-t-3}{k-t-2}\).

Then for \(n \geq 4(t+1)^2(k-t+1) + t + 1\) we obtain that

\[
|\mathcal{F}_1| \leq \left( \binom{n-t-1}{k-t-1} + (t+1)|\mathcal{F}_1| \right) \left( \binom{n-t-1}{k-t-1} + (t+1)|\mathcal{G}^*| \right)
\leq \left( \binom{n-t-1}{k-t-1} + (t+1)(k-t) \binom{n-t-1}{k-t-1} \right) \left( \binom{n-t-1}{k-t-1} + (t+1) \binom{n-t-3}{k-t-2} \right)
\leq (t+1)(k-t+1) \binom{n-t-1}{k-t-1}^2 + ((t+1)(k-t)+1)(t+1) \binom{n-t-2}{k-t-2} \binom{n-t-1}{k-t-1}
\leq (t+1)(k-t+1) \binom{n-t-1}{k-t-1}^2 + \frac{(t+1)^2(k-t+1)^2}{n-t-1} \binom{n-t-1}{k-t-1}^2
\leq \left( t + \frac{5}{4} \right) (k-t+1) \binom{n-t-1}{k-t-1}^2.
\]
Then apply \( (3.3) \) with \( c = 4(t+1) \) and note that \((t + \frac{5}{4})(t+1) < (t + \frac{1}{2})(t+2) \) for \( t \geq 2 \), we obtain that
\[
|\mathcal{F}| |\mathcal{G}| \leq \left( t + \frac{5}{4} \right) \frac{4t+4}{4t+2} (k-t+1) \left( \frac{n-k-1}{k-t} \right)^2
\leq (t+2)(k-t+1) \left( \frac{n-k-1}{k-t} \right)^2
\leq \max \{ |A(n,k,t)|^2, |H(n,k,t)|^2 \}.
\]

Thus we may assume that \( \min \{ |\mathcal{F}|, |\mathcal{G}|^* \} \geq \binom{n-t-3}{k-t-2} \).

Now we distinguish two cases.

**Case 1.** \( |\mathcal{G}_1| \geq \max_{2 \leq i \leq t+1} |\mathcal{G}_i| \).

Without loss of generality, assume that \( |\mathcal{G}_2| = \max_{2 \leq i \leq t+1} |\mathcal{G}_i| \). Then \( \mathcal{G}_j \subset \mathcal{G}_2 \) for \( 3 \leq j \leq t+1 \). Since \( \mathcal{F} \) is non-trivial, we know that \( |\mathcal{F}| \neq 0 \). For otherwise \( \mathcal{F}_2 = \mathcal{F}_3 = \ldots = \mathcal{F}_{t+1} = \emptyset \), it follows that \( \mathcal{F} \subset \{ F \in \binom{[n]}{k} : [2, t+1] \subset F \} \), contradicting the non-triviality of \( \mathcal{F} \). Similarly \( \mathcal{G}_2 \neq \emptyset \).

**Subcase 1.1.** \( |\mathcal{F}_1| \geq |\mathcal{G}_2| \) and \( |\mathcal{G}_1| \geq |\mathcal{F}_2| \).

Since \( \mathcal{F}_1, \mathcal{G}_2 \) is cross-intersecting and \( |\mathcal{F}_1| \geq |\mathcal{G}_2| \), by \( (2.2) \) we obtain that
\[
|\mathcal{F}_1| + t|\mathcal{G}_2| \leq \max \left\{ (t+1) \binom{|X| - 1}{k-t-1}, \binom{|X|}{k-t} - \binom{|X| - (k-t)}{k-t} + t \right\}.
\]

Similarly,
\[
|\mathcal{G}_1| + t|\mathcal{F}_2| \leq \max \left\{ (t+1) \binom{|X| - 1}{k-t-1}, \binom{|X|}{k-t} - \binom{|X| - (k-t)}{k-t} + t \right\}.
\]

By \( (3.3), (3.4) \) and \( (3.5) \), we arrive at
\[
|\mathcal{F}| + |\mathcal{G}| = 2 \binom{n-t-1}{k-t-1} + \sum_{1 \leq i \leq t+1} |\mathcal{F}_i| + \sum_{1 \leq i \leq t+1} |\mathcal{G}_i|
\leq 2 \binom{n-t-1}{k-t-1} + 2 \max \left\{ (t+1) \binom{|X| - 1}{k-t-1}, \binom{|X|}{k-t} - \binom{|X| - (k-t)}{k-t} + t \right\}
= 2 \max \left\{ (t+2) \binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2} + t + \binom{n-t}{k-t} - \binom{n-k-1}{k-t} \right\}
\]
and \( (3.1) \) follows from \( |\mathcal{F}| |\mathcal{G}| \leq \left( \frac{|\mathcal{F}| + |\mathcal{G}|}{2} \right)^2 \).

**Subcase 1.2.** \( |\mathcal{G}_1| < |\mathcal{F}_2| \) or \( |\mathcal{F}_1| < |\mathcal{G}_2| \).

By symmetry, we may assume that \( |\mathcal{G}_1| < |\mathcal{F}_2| \). By Fact \( (3.3) \) we see \( |\mathcal{G}_1| \geq \binom{n-t-3}{k-t-2} \). Since \( \mathcal{G}_1, \mathcal{F}_2 \) are cross-intersecting, by \( (2.3) \) we have
\[
|\mathcal{G}_1| + |\mathcal{F}_2| \leq 2 \binom{|X| - 1}{k-t-1},
\]
and thereby \( |\mathcal{G}_1| \leq \binom{n-t-2}{k-t-1} \).
Subcase 1.2.1. \(|G_2| \geq \binom{n-t-3}{k-t-2}\).

Since \(G_2, F_1\) are cross-intersecting and \(|F_1| \geq |G_2| \geq \binom{n-t-3}{k-t-2}\), by (2.3)

\[(3.7) \quad |F_1| + |G_2| \leq 2\left|X\right| - 1.
\]

By (3.6) and (3.7), we obtain that

\[
|F| + |G| \leq 2\left|X\right| + t(|G_1| + |F_2|) + (|F_1| + |G_2|)
\]

\[
\leq 2\left|X\right| + (2t + 2)\left|X\right| - 1
\]

\[
= 2(t + 2)\left|X\right| - 2
\]

\[
= 2|A(n, k, t)|
\]

and (3.1) follows.

Subcase 1.2.2. \(|G_2| < \binom{n-t-3}{k-t-2} < \binom{n-t-2}{k-t-2}\).

By Fact 3.2 we have \(|F_1| \leq (k-t)\binom{n-t-1}{k-t-1}\). Since \(F_2, G_1\) are cross-intersecting, by (1.2)

we obtain that

\[(3.8) \quad |F_2||G_1| \leq \left(\frac{n-t-2}{k-t-1}\right)^2 \leq \left(\frac{n-t-1}{k-t-1}\right)^2.
\]

Therefore,

\[
|F||G| \leq \left(\frac{n-t-1}{k-t-1} + t|F_2| + |F_1|\right) \left(\frac{n-t-1}{k-t-1} + t|G_2| + |G_1|\right)
\]

\[
\leq \left(k-t+1\right)\left(\frac{n-t-1}{k-t-1} + t|F_2|\right) \left(\frac{n-t-1}{k-t-1} + t\left(\frac{n-t-2}{k-t-2}\right) + |G_1|\right)
\]

\[
= \left(k-t+1\right)\left(1 + t\frac{k-t-1}{n-t-1}\right)\left(\frac{n-t-1}{k-t-1}\right)^2 + t|F_2||G_1|
\]

\[
+ \left(\frac{n-t-1}{k-t-1}\right) \left(1 + t\frac{k-t-1}{n-t-1}\right) t|F_2| + \left(k-t+1\right)|G_1|.
\]

By (3.8), it follows that

\[
|F||G| \leq (k+1)\left(\frac{n-t-1}{k-t-1}\right)^2 + \frac{t(k-t)^2}{n-t-1}\left(\frac{n-t-1}{k-t-1}\right)^2
\]

\[
+ \left(\frac{n-t-1}{k-t-1}\right) \left(1 + t\frac{k-t-1}{n-t-1}\right) t|F_2| + \left(k-t+1\right)|G_1|.
\]
By (3.6) and \(|G_1| \leq \binom{n-t-1}{k-t-1}\), we have
\[
\left(1 + \frac{t(k-t-1)}{n-t-1}\right) t |F_2| + (k-t+1)|G_1|
\]
\[
< \left(1 + \frac{t(k-t)}{n-t-1}\right) t \left(2 \cdot \binom{n-t-1}{k-t-1} - |G_1|\right) + (k-t+1)|G_1|
\]
(3.10)
\[
\leq \max \left\{2t + \frac{2t^2(k-t)}{n-t-1}, k+1 + \frac{t^2(k-t)}{n-t-1}\right\} \binom{n-t-1}{k-t-1}.
\]
Combining (3.10) and (3.9), we arrive at
\[
|F||G| \leq \left(\frac{t(k-t)}{n-t-1}\right) + \max \left\{k+1 + \frac{2t^2(k-t)}{n-t-1}, 2(k+1) + \frac{t^2(k-t)}{n-t-1}\right\} \binom{n-t-1}{k-t-1}.
\]
Note that
\[
t(k-t)^2 + 2t^2(k-t) = t(k-t)(k+t) = t(k^2 - t^2) \leq tk^2.
\]
For \(n \geq tk^2 + (t+1)\),
\[
|F||G| < \left(\frac{n-t-1}{k-t-1}\right)^2 \max \left\{k+2t+1 + \frac{tk^2}{n-t-1}, 2(k+1) + \frac{tk^2}{n-t-1}\right\}
\]
(3.11)
\[
\leq \left(\frac{n-t-1}{k-t-1}\right)^2 \max \{k+2t+2, 2k+3\}.
\]
If \(k \geq 2t+3\), then \(k+2t+2 < 2k+3\) and \(2k+3 = 4(k - \frac{k}{2} + \frac{3}{4}) < 4(k-t+1)\). By applying (3.8) with \(c = 6\), we obtain for \(n \geq 6(k-t)^2 + t + 1\),
\[
|F||G| \leq (2k+3) \binom{n-t-1}{k-t-1}^2 \leq 4(k-t+1) \binom{n-t-1}{k-t-1}^2 \leq 6(k-t+1) \binom{n-k-1}{k-t-1}^2.
\]
Since \(k \geq 2t+3\) implies \(k-t+1 \geq t+4 \geq 6\), we have
\[
|F||G| \leq 6(k-t+1) \binom{n-k-1}{k-t-1}^2 \leq (k-t+1)^2 \binom{n-k-1}{k-t-1}^2 < |H(n, k, t)|^2.
\]
If \(2t \leq k \leq 2t+2\), then \(k+2t+2 \leq 2k+3\). By (3.11) we have
\[
|F||G| \leq (2k+3) \binom{n-t-1}{k-t-1}^2 \leq (4t+7) \binom{n-t-1}{k-t-1}^2.
\]
Applying (1.9) with \(c = 8(t+2)\), we arrive at
\[
|F||G| \leq (4t+7) \frac{8(t+2)}{8(t+2) - 2} \binom{n-t-2}{k-t-1}^2 = 4(t+2) \binom{n-t-2}{k-t-1}^2 < |A(n, k, t)|^2.
\]
If \(k \leq 2t-1\), then \(k+2t+2 \geq 2k+3\). By (3.11) we have
\[
|F||G| \leq (k+2t+2) \binom{n-t-1}{k-t-1}^2 < (4t+4) \binom{n-t-1}{k-t-1}^2.
\]
Applying (1.9) with \( c = 2(t+2) \), we arrive at
\[
|\mathcal{F}||\mathcal{G}| \leq 4(t+1)\frac{2(t+2)}{2(t+2)-2} \left( \frac{n-t-2}{k-t-1} \right)^2 = 4(t+2) \left( \frac{n-t-2}{k-t-1} \right)^2 < |\mathcal{A}(n,k,t)|^2.
\]

**Case 2.** \( |\mathcal{G}_1| \leq \max_{2 \leq i \leq t+1} |\mathcal{G}_i| \).

Without loss of generality, assume that \( |\mathcal{G}_2| = \max_{2 \leq i \leq t+1} |\mathcal{G}_i| \). Then \( |\mathcal{G}_2| = \max_{1 \leq i \leq t+1} |\mathcal{G}_i| \) and \( \mathcal{G}_i \subset \mathcal{G}_2 \) for all \( 1 \leq i \leq t+1 \). Since \( \mathcal{F}_i \subset \mathcal{F}_1 \), \( \mathcal{G}_i \subset \mathcal{G}_2 \), \( \mathcal{F}_i \) and \( \mathcal{G}_i \) are cross-intersecting for \( 1 \leq i \leq t+1 \), by saturatedness we may assume that \( |\mathcal{F}_i| = |\mathcal{F}_1| \) and \( |\mathcal{G}_i| = |\mathcal{G}_2| \) for all \( i \in [t+1] \). Then (3.11) is equivalent to

\[
(3.12) \\
\left( \frac{n-t-1}{k-t-1} + (t+1)|\mathcal{F}_1| \right) \left( \frac{n-t-1}{k-t-1} + (t+1)|\mathcal{G}_2| \right) \leq \max\{|\mathcal{A}(n,k,t)|^2, |\mathcal{H}(n,k,t)|^2\}.
\]

Without loss of generality, assume that \( |\mathcal{F}_1| \leq |\mathcal{G}_2| \). By Fact 5.3 we have \( |\mathcal{F}_1| \geq \binom{n-t-3}{k-t-2} \).

Since \( \mathcal{F}_1, \mathcal{G}_2 \) are cross-intersecting, by (2.3) we have \( |\mathcal{F}_1| + |\mathcal{G}_2| \leq 2\binom{|X|-1}{k-1} \). By (1.2) \( |\mathcal{F}_1||\mathcal{G}_2| \leq \binom{|X|-1}{k-1}^2 \). Then expanding (3.12):

\[
LHS \leq \frac{(n-t-1)^2}{k-t-1} + 2\frac{(n-t-1)}{k-t-1}(t+1)\frac{(n-t-2)}{k-t-1} + \left( (t+1)\frac{(n-t-2)}{k-t-1} \right)^2
\]

\[= |\mathcal{A}(n,k,t)|^2, \]

and the proposition is proven. \( \square \)

**4 The basis of cross \( t \)-intersecting families**

In this section, we prove an inequality concerning the size of basis of cross \( t \)-intersecting families by a branching process.

We need the following notion of basis. Let \( \mathcal{B}(\mathcal{F}) \) be the family of minimal (for containment) sets in \( \mathcal{T}_i(\mathcal{G}) \) and let \( \mathcal{B}(\mathcal{G}) \) be the family of minimal sets in \( \mathcal{T}_i(\mathcal{F}) \).

Let us prove some properties of the basis.

**Lemma 4.1.** Suppose that \( \mathcal{F}, \mathcal{G} \subset \binom{|X|}{k} \) form a saturated pair of cross \( t \)-intersecting families. Then (i) and (ii) hold.

(i) Both \( \mathcal{B}(\mathcal{F}) \) and \( \mathcal{B}(\mathcal{G}) \) are antichains, and \( \mathcal{B}(\mathcal{F}), \mathcal{B}(\mathcal{G}) \) are cross \( t \)-intersecting,

(ii) \( \mathcal{F} = \left\{ F \in \binom{|X|}{k} : \exists B \in \mathcal{B}(\mathcal{F}), B \subset F \right\} \) and \( \mathcal{G} = \left\{ G \in \binom{|X|}{k} : \exists B \in \mathcal{B}(\mathcal{G}), B \subset G \right\} \).

**Proof.** (i) Clearly, \( \mathcal{B}(\mathcal{F}) \) and \( \mathcal{B}(\mathcal{G}) \) are both antichains. Suppose for contradiction that \( B \in \mathcal{B}(\mathcal{F}), B' \in \mathcal{B}(\mathcal{G}) \) but \( |B \cap B'| < t \). If \( |B| = |B'| = k \), then \( B \in \mathcal{F}, B' \in \mathcal{G} \) follow from saturatedness, a contradiction. If \( |B| < k \), then there exists \( F \supseteq B \) such that \( |F| = k \) and \( |F \cap B'| = |B \cap B'| < t \). By definition \( F \in \mathcal{T}_i(\mathcal{G}) \). Since \( \mathcal{F}, \mathcal{G} \) are saturated, we see that \( F \in \mathcal{F} \). But this contradicts the assumption that \( B' \) is a \( t \)-transversal of \( \mathcal{F} \). Since \( \mathcal{F}, \mathcal{G} \) are saturated, (ii) is immediate from the definition of \( \mathcal{B}(\mathcal{F}) \) and \( \mathcal{B}(\mathcal{G}) \). \( \square \)
Let \( s(\mathcal{B}) = \min \{|B| : B \in \mathcal{B} \} \). For any \( \ell \) with \( s(\mathcal{B}) \leq \ell \leq k \), define

\[
\mathcal{B}^{(\ell)} = \{ B \in \mathcal{B} : |B| = \ell \} \quad \text{and} \quad \mathcal{B}^{(\leq \ell)} = \bigcup_{i=s(\mathcal{B})}^{\ell} \mathcal{B}^{(i)}.
\]

It is easy to see that \( s(\mathcal{B}(\mathcal{G})) = \tau_1(\mathcal{F}) \).

By a branching process, we establish an inequality concerning the size of the basis.

**Lemma 4.2.** Suppose that \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) are a saturated pair of cross t-intersecting families. Let \( \mathcal{B}_1 = \mathcal{B}(\mathcal{F}) \) and \( \mathcal{B}_2 = \mathcal{B}(\mathcal{G}) \). For each \( i = 1, 2 \), if \( s(\mathcal{B}_i) \geq t + 1 \) and there exists \( r_i \geq s(\mathcal{B}_i) \) such that \( \tau_1(\mathcal{B}_i^{(\leq r_i)}) \geq t + 1 \) then

\[
(4.1) \quad \sum_{r_i \leq \ell \leq k} \binom{s(\mathcal{B}_i)}{t}\ell^\ell \leq |\mathcal{B}_3^{(\leq 1)}| \leq 1
\]

and

\[
(4.2) \quad \sum_{s_i \leq \ell \leq k} \binom{s(\mathcal{B}_i)}{t} \ell^\ell \leq |\mathcal{B}_3^{(\leq 1)}| \leq 1.
\]

**Proof.** By symmetry, it is sufficient to prove the lemma only for \( i = 1 \). For the proof we use a branching process. During the proof a sequence \( S = (x_1, x_2, \ldots, x_t) \) is an ordered sequence of distinct elements of \([n]\) and we use \( \hat{S} \) to denote the underlying unordered set \( \{x_1, x_2, \ldots, x_t\} \). At the beginning, we assign weight 1 to the empty sequence \( S_0 \).

At the first stage, we choose \( B_{1,1} \in \mathcal{B}_1 \) with \( |B_{1,1}| = s(\mathcal{B}_1) \geq t + 1 \). For any \( t \)-subset \( \{x_1, x_2, \ldots, x_t\} \subset B_{1,1} \), define one sequence \((x_1, x_2, \ldots, x_t)\) and assign weight \( \left(\frac{s(\mathcal{B}_1)}{t}\right)^{-1} \) to it.

At the second stage, since \( \tau_1(\mathcal{B}_1^{(\leq r_1)}) \geq t + 1 \), for each sequence \( S = (x_1, \ldots, x_t) \) we may choose \( B_{1,t+1} \in \mathcal{B}_1^{(\leq r_1)} \) such that \( |\hat{S} \cap B_{1,t+1}| < t \). Then we replace \( S = (x_1, \ldots, x_t) \) by \( |B_{1,t+1} \setminus \hat{S}| \) sequences of the form \((x_1, \ldots, x_t, y)\) with \( y \in B_{1,t+1} \setminus \hat{S} \) and weight \( \frac{w(S)}{|B_{1,t+1} \setminus \hat{S}|} \).

In each subsequent stage, we pick a sequence \( S = (x_1, \ldots, x_p) \) and denote its weight by \( w(S) \). If \( |\hat{S} \cap B_1| \geq t \) holds for all \( B_1 \in \mathcal{B}_1 \) then we do nothing. Otherwise we pick \( B_1 \in \mathcal{B}_1 \) satisfying \( |\hat{S} \cap B_1| < t \) and replace \( S \) by the \( |B_1 \setminus \hat{S}| \) sequences \((x_1, \ldots, x_p, y)\) with \( y \in B_1 \setminus \hat{S} \) and assign weight \( \frac{w(S)}{|B_1 \setminus \hat{S}|} \) to each of them. Clearly, the total weight is always 1.

We continue until \( |\hat{S} \cap B_1| \geq t \) for all sequences and all \( B_1 \in \mathcal{B}_1 \). Since \([n]\) is finite, each sequence has length at most \( n \) and eventually the process stops. Let \( \mathcal{S} \) be the collection of sequences that survived in the end of the branching process and let \( \mathcal{S}^{(\ell)} \) be the collection of sequences in \( \mathcal{S} \) with length \( \ell \).

**Claim 1.** To each \( B_2 \in \mathcal{B}_2^{(\ell)} \) with \( \ell \geq r_1 \) there is some sequence \( S \in \mathcal{S}^{(\ell)} \) with \( \hat{S} = B_2 \).

**Proof.** Let us suppose the contrary and let \( S = (x_1, \ldots, x_p) \) be a sequence of maximal length that occurred at some stage of the branching process satisfying \( \hat{S} \subsetneq B_2 \). Since \( B_1, B_2 \) are cross \( t \)-intersecting, \( |B_1 \cap B_2| \geq t \), implying that such an \( S \) exists. By the choice of \( S \) we see that \( p \geq t \). Since \( \hat{S} \) is a proper subset of \( B_2 \), it follows that \( \hat{S} \notin \mathcal{B}(\mathcal{G}) \subset \mathcal{T}(\mathcal{F}) \). Thereby there exists \( F \in \mathcal{F} \) with \( |\hat{S} \cap F| < t \). In view of Lemma 1.1 (ii), we can find \( B'_1 \in \mathcal{B}_1 \) such that \( |\hat{S} \cap B'_1| < t \). Thus at some point we picked \( S \) and some \( B_1 \in \mathcal{B}_1 \) with \( |\hat{S} \cap B_1| < t \). Since \( B_1, B_2 \) are cross \( t \)-intersecting, \( |B_2 \cap B_1| \geq t \). Consequently, for each \( y \in B_2 \cap B_1 \) the sequence \((x_1, \ldots, x_p, y)\) occurred in
the branching process. This contradicts the maximality of \( p \). Hence there is an \( S \) at some stage satisfying \( \hat{S} = B_2 \). Since \( B_1, B_2 \) are cross \( t \)-intersecting, \( |\hat{S} \cap B'_1| = |B_2 \cap B'_1| \geq t \) for all \( B'_1 \in B_1 \). Thus \( \hat{S} \in S \) and the claim holds.

By Claim 1, we see that \( |B_2^{(\ell)}| \leq |S^{(\ell)}| \) for all \( \ell \geq r_1 \). Let \( S = (x_1, \ldots, x_t) \in S^{(\ell)} \) and let \( S_i = (x_1, \ldots, x_i) \) for \( i = t, \ldots, \ell \). At the first stage, \( w(S_t) = 1/(s(B_1)) \). Assume that \( B_{1,i} \) is the selected set when replacing \( S_i = S_{i-1} \) and the branching process for \( i = t+1, \ldots, \ell \). Clearly, \( x_i \in B_{1,i} \) for \( i \geq t + 1 \), \( B_{1,i+1} \in B_1^{(r_1)} \) and

\[
    w(S) = \left( s(B_1) \right)^{-1} \prod_{i=t+1}^{\ell} \frac{1}{|B_{1,i} \setminus S_{i-1}|}.
\]

Note that \( |B_{1,i+1} \setminus \hat{S}_i| \leq r_1 \) and \( |B_{1,i} \setminus \hat{S}_{i-1}| \leq k \) for \( i \geq t + 2 \). It follows that

\[
    w(S) \geq \left( \frac{s(B_1)}{t} r_1 k^{\ell-t-1} \right)^{-1}.
\]

Thus we obtain that

\[
    \sum_{r_1 \leq t \leq k} \left( \frac{s(B_1)}{t} r_1 k^{\ell-t-1} \right)^{-1} \leq \sum_{r_1 \leq t \leq k} \sum_{S \in S^{(\ell)}} w(S) \leq \sum_{S \in S} w(S) = 1
\]

and (4.1) holds.

If we omit the second stage in the branching process, then by the similar argument we can obtain (4.2).

### 5 The proof of the main theorem

In this section, we determine the maximum product of the sizes of two non-trivial cross \( t \)-intersecting families.

**Lemma 5.1.** Let \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) be a saturated pair of non-trivial cross \( t \)-intersecting families. Set \( B_1 = B(\mathcal{F}), B_2 = B(\mathcal{G}) \). Then neither \( B_1 \) nor \( B_2 \) contains a sunflower of \( k-t+2 \) petals with center of size \( t \).

**Proof.** Suppose for contradiction that there exists a sunflower of \( k-t+2 \) petals with center of size \( t \) in \( B_1 \). Without loss of generality, assume that \( [t] \cup A_1, \ldots, [t] \cup A_{k-t+2} \) is such a sunflower. Then for each \( G \in \mathcal{G} \), by the definition of \( B(\mathcal{F}) \) we have \( |G \cap ([t] \cup A_j)| \geq t \) for \( j = 1, \ldots, k-t+2 \). It follows that \( [t] \subset G \), contradicting the non-triviality of \( \mathcal{G} \).

**Lemma 5.2.** Let \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) be a saturated pair of non-trivial cross \( t \)-intersecting families. Set \( \mathcal{A}_1 = T_t^{(t+1)}(\mathcal{G}), \mathcal{A}_2 = T_t^{(t+1)}(\mathcal{F}) \). If \( s(B(\mathcal{F})) = s \), then

\[
    |\mathcal{A}_2| \leq \left( \frac{s}{t} \right) (k-t+1).
\]

If \( \mathcal{A}_1 \neq \emptyset \) and \( \mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset \), then

\[
    |\mathcal{A}_2| \leq 2(k-t+1).
\]
Proof. Let us prove (5.1) first. Since \( s(\mathcal{B}(\mathcal{F})) = s \), there exists an \( S \in \mathcal{B}(\mathcal{F}) \) with \( |S| = s \). Note that by Lemma 4.1 (i) we have \(|A \cap S| \geq t\) for each \( A \in \mathcal{A}_2\). Then

\[
|A_2| \leq \sum_{T \in \binom{S}{t}} |A_2(T)|.
\]

By Lemma 5.1 we see that \(|A_2(T)| \leq k - t + 1\) and (5.1) follows.

Now we prove (5.2). Without loss of generality, let \([t + 1] \in A_1\). Then \(|D \cap [t + 1]| \geq t\) for all \( D \in \mathcal{A}_2\). Should \(|F \cap [t + 1]| \geq t\) hold for all \( F \in \mathcal{F} \), we get \([t + 1] \in A_1 \cap \mathcal{A}_2\), contradicting \( A_1 \cap \mathcal{A}_2 = \emptyset \). I.e., we may fix \( F_0 \) with \(|F_0 \cap [t + 1]| \leq t - 1\). Let \( \mathcal{A}_2 = \{D_1, \ldots, D_\ell\} \) and define \( x_i \) by \( D_1 \setminus [t + 1] = \{x_i\} \). Note that

\[
t \leq |F_0 \cap D_1| \leq |F_0 \cap [t + 1]| + |F_0 \cap \{x_i\}| \leq t + 1.
\]

Thus \(|F_0 \cap [t + 1]| = t - 1, x_i \in F_0\) and \( F_0 \cap [t + 1] \subset D_i\). Since \(|D_i \cap [t + 1]| = t\), it follows that there are only two choices for \( D_i \cap [t + 1]\). Then by Lemma 5.1 we conclude that \(|A_2| \leq 2(k - t + 1)\). \(\square\)

Proof of Proposition 1.4. Let us prove the following two facts.

Fact 5.3. For any \( A, A' \in \mathcal{A}, |A \cap A'| = t - 1 \) or \( t \).

Proof. Indeed, if \( A \cap A' = D \) with \(|D| \leq t - 2\), then for any \( B \in \mathcal{B} \) the fact \(|B \cap (A' \setminus D)| \geq 2\) implies

\[
|B \cap (A \cup (A' \setminus D))| = |B \cap A| + |B \cap (A' \setminus D)| \geq t + 2,
\]

contradicting \(|B| = t + 1\). \(\square\)

Fact 5.4. Suppose that \(|A \cap A'| = t - 1\) then \( A \cap A' \subset B \) for all \( B \in \mathcal{B}\).

Proof. Suppose \(|B \cap (A \cap A')| \leq t - 2\). Then \( B \supset A \setminus A', B \supset A' \setminus A \) and \(|B \cap A \cap A'| = t - 2\) follow. Consequently \(|B| = t + 2\), contradiction. \(\square\)

If \(|A \cap A'| = t\) for all \( A, A' \in \mathcal{A}\) and also \( B \cap B' = t\) for all \( B, B' \in \mathcal{B}\), then \( \mathcal{A} \cup \mathcal{B} \) is either a subset of \(\binom{[t + 2]}{t + 1}\) up to isomorphism or a sunflower with center of size \( t \). If \( \mathcal{A} \cup \mathcal{B} \) is a sunflower with center of size \( t \), then (ii) holds. If \( \mathcal{A} \cup \mathcal{B} \) is a subset of \(\binom{[t + 2]}{t + 1}\), note that the exact cross \( t \)-intersection implies \( \mathcal{A} \cap B = \emptyset \), then \(|A| + |B| \leq t + 2\) and thereby \(|\mathcal{A}| |\mathcal{B}| \leq \frac{(t+2)^2}{2}\). Thus (iii) holds.

In the rest of the proof, we assume that there exist \( A, A' \in \mathcal{A} \) such that \(|A \cap A'| = t - 1\). By symmetry we assume that \([t + 1], [t - 1] \cup \{t + 2, t + 3\} \in \mathcal{A}\). By Fact 5.3 \([t - 1] \subset B\) for all \( B \in \mathcal{B}\). We claim that \(|B| \leq 4\). Indeed, for every \( B \in \mathcal{B}, |B \cap [t + 1]| = t\) and \(|B \cap ([t - 1] \cup \{t + 2, t + 3\})| = t\) imply that \( B = [t - 1] \cup \{x, y\}\) with \( x \in \{t, t + 1\}, y \in \{t + 2, t + 3\}\). Now we distinguish two cases.

Case 1. \( \exists B, B' \in \mathcal{B} \) with \(|B \cap B'| = t - 1\).

Necessarily, \( B \cap B' = [t - 1]\). Consequently \([t - 1] \subset A\) for all \( A \in \mathcal{A}\) and even \( A \setminus [t - 1] \subset [t, t + 3]\). Now \( \mathcal{A}(t - 1) \) and \( \mathcal{B}([t - 1]) \) are cross-intersecting 2-graphs on the four vertices \( t, t + 1, t + 2, t + 3 \). We infer \(\min\{|A|, |B|\} = 2\). It follows that \(|\mathcal{A}| |\mathcal{B}| \leq 8 \leq \frac{(t+2)^2}{2}\) and (iii) holds.

Case 2. \(|B \cap B'| = t\) for every \( B, B' \in \mathcal{B}\).
Case 1. Similarly, we have for \( |A| \leq t \). Hence \( |A| \leq t \). Since \( A \) does not contain a sunflower of size \( k - t + 1 \) petals with center of size \( t \), we infer that \( |A_0| \leq k - t + 1 \). If \( |A_0| \leq 2 \), then \( |A| \leq 2(k - t + 2) \leq \frac{(k + 2)^2}{2} \), (iii) holds. If \( |A_0| \geq 3 \), since \( |t - 1| \leq 2 \), \( |A| \leq k + 1 \), (i) holds. \( \square \)

Proof of Theorem \( 4 \). Let \( B_1 = B(F), B_2 = B(G), A_1 = B_1^{(t+1)}, A_2 = B_2^{(t+1)} \) and let \( s_1 = s(B_1), s_2 = s(B_2) \). Let us partition \( F \) into \( F^{(s_1)} \cup \ldots \cup F^{(s_2)} \) where \( F \in F^{(s)} \) if \( \max\{|B|: B \in B_1, B \subset F\} = s \). Similarly, partition \( G \) into \( G^{(s_2)} \cup \ldots \cup G^{(k)} \). By non-triviality of \( F \) and \( G \), we know \( s_1 \geq t + 1 \) and \( s_2 \geq t + 1 \). For every \( m \in [s_1, k] \), define

\[
F^{(\geq m)} = \bigcup_{m \leq \ell \leq k} F^{(\ell)}.
\]

For \( m \in [s_2, k] \), define

\[
G^{(\geq m)} = \bigcup_{m \leq \ell \leq k} G^{(\ell)}.
\]

Let \( \alpha_\ell = \binom{s_2}{\ell} k^{\ell - t} |B_1^{(\ell)}|^{-1} \). By (4.2) we have

\[
\sum_{m \leq \ell \leq k} \alpha_\ell \leq \sum_{s_1 \leq \ell \leq k} \alpha_\ell \leq 1.
\]  

Let \( f(n, k, \ell) = \binom{s_2}{\ell} k^{\ell - t} (\frac{n}{k_\ell} - \ell) \). For \( n \geq k^2 \), we have

\[
\frac{f(n, k, \ell) + 1}{f(n, k, \ell)} = \frac{k^{\ell+1-t}(\frac{n-k-1}{k_{\ell-1}})}{k^{\ell-t}(\frac{n-k\ell}{k_{\ell-1}})} = \frac{k(k-\ell)}{n-\ell} \leq 1.
\]  

Then by (5.3) and (5.4) we have for \( m \geq s_1 \)

\[
|F^{(\geq m)}| \leq \sum_{m \leq \ell \leq k} |B_1^{(\ell)}|(\frac{n-\ell}{k-\ell}) = \sum_{m \leq \ell \leq k} \alpha_\ell f(n, k, \ell)
\]

\[
\leq f(n, k, m)
\]  

\[
(\binom{s_2}{t}) k^{m-t} (\frac{n-m}{k-m})
\]

Similarly, we have for \( m \geq s_2 \)

\[
|G^{(\geq m)}| \leq \binom{s_1}{t} k^{m-t} (\frac{n-m}{k-m})
\]

Without loss of generality we assume that \( s_1 \geq s_2 \). Now we distinguish four cases.

Case 1. \( s_1 \geq s_2 \geq t + 2 \).

Applying (5.5) with \( m = s_1 \), we obtain that

\[
|F| \leq \binom{s_2}{t} k^{s_1-t} (\frac{n-s_1}{k-s_1})
\]
Applying (5.6) with \( m = s_2 \), we obtain that

\begin{equation}
(5.8) \quad |G| \leq \binom{s_1}{t} k^{s_2-t} \binom{n-s_2}{k-s_2}.
\end{equation}

Then

\begin{equation}
|F||G| \leq \binom{s_1}{t} k^{s_1-t} \binom{n-s_1}{k-s_1} \binom{s_2}{t} k^{s_2-t} \binom{n-s_2}{k-s_2}.
\end{equation}

Let \( g_i(n, k, s) = \binom{s}{t} k^{s_1-t} \binom{n-s}{k-s_1}, \) \( i = 1, 2 \). Since \( s_i \geq t + 2 \) and for \( n \geq tk^2 \),

\begin{equation}
(5.9) \quad \frac{g_i(n, k, s_1 + 1)}{g_i(n, k, s_i)} = \frac{(s_i+1)_t}{(s_i)_t} \frac{k^{s_1-t} \binom{n-s_1-1}{k-s_1}}{k^{s_i-t} \binom{n-s_i}{k-s_i}} = \frac{s_i + 1}{s_i + 1 - t} \frac{k(k-s_i)}{n-s_i} \leq \frac{t+3}{3} \frac{k(k-s_i)}{n-s_i} < 1,
\end{equation}

it follows that for \( n \geq (t+2)^2k^2 + t + 1 \),

\[ |F||G| \leq g_1(n, k, s_1) g_2(n, k, s_2) \leq g_1(n, k, t+2) g_2(n, k, t+2) \]

\[ = \binom{t+2}{t} k^4 \binom{n-t-2}{k-t-2} \]

\[ \leq \frac{(t+2)^4 k^4 (k-t-1)^2}{4(n-t-1)^2} \binom{n-t-1}{k-t-1} \]

\[ \leq \frac{(k-t+1)^2}{4} \binom{n-t-1}{k-t-1}. \]

Apply (1.8) with \( c = 4 \), we conclude that

\[ |F||G| \leq \frac{(k-t+1)^2}{2} \binom{n-k-1}{k-t-1} < |H(n, k, t)|^2. \]

Case 2. \( s_1 \geq t + 3 \) and \( s_2 = t + 1 \).

By (5.1) we have \( |A_2| \leq \binom{s_1}{t} (k-t+1) \). By (5.6),

\[ |F| \leq (t+1) k^{s_1-t} \binom{n-s_1}{k-s_1}. \]

By (5.6),

\[ |G(\geq t+2)| \leq \binom{s_1}{t} k^2 \binom{n-t-2}{k-t-2}. \]

It follows that

\[ |F||G| \leq (t+1) k^{s_1-t} \binom{n-s_1}{k-s_1} \left( |A_2| \binom{n-t-1}{k-t-1} + \binom{s_1}{t} k^2 \binom{n-t-2}{k-t-2} \right) \]

\[ \leq (t+1) k^{s_1-t} \binom{n-s_1}{k-s_1} \left( \binom{s_1}{t} (k-t+1) \binom{n-t-1}{k-t-1} + \binom{s_1}{t} k^2 \binom{n-t-2}{k-t-2} \right) \]

\[ \leq (t+1) g_1(n, k, s_1) \left( (k-t+1) \binom{n-t-1}{k-t-1} + k^2 \binom{n-t-2}{k-t-2} \right). \]
By (5.9) we see that \( g_1(n,k,s_1) \) is a decreasing function of \( s_1 \) and \( s_1 \geq t + 3 \). Thus, for \( n \geq (t + 2)^2 k^2 + t + 1 \) we have

\[
|\mathcal{F}||\mathcal{G}| \leq (t + 1) g_1(n,k,t + 3) \left( (k - t + 1) \binom{n - t - 1}{k - t - 1} + k^2 \binom{n - t - 2}{k - t - 2} \right)
\]

\[
\leq (t + 1) \binom{t + 3}{3} k^3 \binom{n - t - 3}{k - t - 3} \left( (k - t + 1) \binom{n - t - 1}{k - t - 1} + k^2 \binom{n - t - 2}{k - t - 2} \right)
\]

\[
\leq \frac{(t + 1)^2(t + 2)(t + 3)k^3(k - t - 1)^2}{6(n - t - 1)^2} \left( (k - t + 1) + \frac{k^2(k - t - 1)}{n - t - 1} \right) \left( n - t - 1 \right)^2
\]

\[
\leq \frac{k - t + 1}{6} (k - t + 1 + k - t + 1) \left( n - t - 1 \right)^2 \left( k - t - 1 \right)
\]

\[
= \frac{(k - t + 1)^2}{3} \left( n - t - 1 \right)^2 \left( k - t - 1 \right).
\]

Apply (1.8) with \( c = 4 \), we conclude that

\[
|\mathcal{F}| |\mathcal{G}| < \frac{2(k - t + 1)^2}{3} \left( n - k - 1 \right)^2 \leq |\mathcal{H}(n,k,t)|^2.
\]

**Case 3.** \( s_1 = t + 2 \) and \( s_2 = t + 1 \).

By (5.3) and (5.6), we obtain that

\[
|\mathcal{F}(\geq t+3)| \leq (t + 1) k^3 \left( \binom{n - t - 3}{k - t - 3} \right)
\]

and

\[
|\mathcal{G}(\geq t+2)| \leq \binom{t + 2}{t} k^2 \left( \binom{n - t - 2}{k - t - 2} \right) \leq \frac{(t + 2)^2}{2} k^2 \left( \binom{n - t - 2}{k - t - 2} \right).
\]

Fix some \( S \in B_1^{(t+2)} \). By Lemma 4.1(i), we see that \( |A_2 \cap S| \geq t \) for all \( A_2 \in A_2 \). For any \( T \in \binom{S}{t} \), define

\[
N(T) = \{x: T \cup \{x\} \in A_2\} \quad \text{and} \quad \Gamma = \left\{ T \in \binom{S}{t}: |N(T)| \geq 4 \right\}.
\]

By Lemma 5.1 we see that \( |N(T)| \leq k - t + 1 \). If \( |N(T)| \geq 4 \), then we claim that \( T \subseteq S' \) for all \( S' \in B_1^{(t+2)} \). Indeed, if \( |S' \cap T| \leq t \), then for every \( x \in N(T) \), \( |S' \cap (T \cup \{x\})| \geq t \) implies \( |S' \cap T| = t - 1 \) and \( x \in S' \). It follows that \( |S'| \geq t - 1 + 4 = t + 3 \), a contradiction.

**Subcase 3.1.** There exist \( T, T' \in \Gamma \) such that \( T \cup T' = S \).

Then we have \( T' \cup T \subset S' \) for all \( S' \in B_1^{(t+2)} \). It follows that \( |B_1^{(t+2)}| = 1 \). By (5.1) we
have $|A_2| \leq \binom{t+2}{t}(k-t+1)$. By (5.10) and (5.11) we obtain that

$$|\mathcal{F}| \leq \left( \binom{n-t-2}{k-t-2} + (t+1)k^3\binom{n-t-3}{k-t-3} \right) \cdot \left( \frac{k-t-1}{n-t-1} + \binom{t+2}{k-t-2} \right) \cdot \left( \frac{n-t-1}{k-t-1} \right).$$

Therefore, using $n \geq (t+2)^2k^2 + t + 1$ and $k \geq 5$ we arrive at

$$|\mathcal{F}| \leq \left( \frac{(t+2)^2(k-t-1)}{n-t-1} + \frac{(t+1)(t+2)k^3(k-t-1)^2}{(n-t-1)^2} \right) \cdot \left( \frac{k-t+1}{2(n-t-1)} \right) \cdot \left( \frac{n-t-1}{k-t-1} \right)^2 \leq \frac{3}{8}(k-t+1)^2 \left( \frac{n-t-1}{k-t-1} \right)^2.$$

Apply (1.8) with $c = 4$, we conclude that

$$|\mathcal{F}| \leq \frac{3}{4}(k-t+1)^2 \left( \frac{n-t-1}{k-t-1} \right)^2 < |\mathcal{H}(n,k,t)|^2.$$

**Subcase 3.2.** $T \cup T' \subseteq S$ for every $T, T' \in \Gamma$.

Note that $T \cup T' \subseteq S$ implies $|T \cap T'| = t - 1$ for every $T, T' \in \Gamma$. Fix some $T, T' \in \Gamma$ and let $C = T \cap T', T' \setminus C = \{x\}, T \setminus C = \{y\}$. Recall that $\Gamma \subset \binom{S}{t}$ and $|S| = t+2$. If there exists $T'' \in \Gamma \setminus \{T, T'\}$ such that $C \subset T''$, then we infer that $|\Gamma| = 3$. If $|T'' \cap C| \leq t - 1$ for all $T'' \in \Gamma \setminus \{T, T'\}$, then $|T'' \cap T| = t - 1$ and $|T'' \cap T'| = t - 1$ imply that $x, y \in T''$ and $|T'' \cap C| = t - 1$. Hence there are at most $t - 1$ possibilities for $T''$ and $|\Gamma| \leq t + 1$. Thus $|\Gamma| \leq t + 1$ and it follows that

$$|A_2| \leq |\Gamma|(k-t+1) + 3 \left( \frac{t+2}{t} - |\Gamma| \right) \leq (t+1)(k-t-2) + 3 \left( \frac{t+2}{2} \right).$$

By (5.5) we have

$$|\mathcal{F}| \leq (t+1)k^2 \binom{n-t-2}{k-t-2}.$$
Therefore, by (5.11), (5.12) and (5.13) we obtain that
\[
|\mathcal{F}| |\mathcal{G}| \leq (t+1)k^2 \binom{n-t-2}{k-t-2} \left( |\mathcal{A}_2| \binom{n-t-1}{k-t-1} + \frac{(t+2)^2}{2} k^2 \binom{n-t-2}{k-t-2} \right)
\]
\[
\leq (t+1)k^2 \binom{n-t-2}{k-t-2} \left( (t+1)(k-t-2) + 3 \binom{t+2}{2} \binom{n-t-1}{k-t-1} \right)
\]
\[
+ (t+1)k^2 \binom{n-t-2}{k-t-2} \frac{(t+2)^2}{2} k^2 \binom{n-t-2}{k-t-2}
\]
\[
\leq \left( \frac{(t+1)^2 k^2 (k-t-1)^2}{n-t-1} + \frac{3(t+1)(k-t+1)^2}{k-t-1} \right) \binom{n-t-1}{k-t-1}
\]
\[
+ \frac{(t+1)(t+2)^2 k^4 (k-t-1)^2}{2(n-t-1)^2} \binom{n-t-1}{k-t-1}.
\]
For \(n \geq 4(t+2)^2 k^2\), we arrive at
\[
|\mathcal{F}| |\mathcal{G}| < \left( \frac{(k-t-1)^2}{4} + \frac{3(t+1)(k-t+1)}{8} + \frac{(k-t+1)^2}{8} \right) \binom{n-t-1}{k-t-1}^2
\]
\[
\leq \frac{3}{4} \max \{ (t+2)^2, (k-t+1)^2 \} \binom{n-t-1}{k-t-1}^2.
\]
Now apply (1.8) with \(c = 8\), we conclude that
\[
|\mathcal{F}| |\mathcal{G}| < \max \{ (t+2)^2, (k-t-1)^2 \} \binom{n-t-1}{k-t-1}^2 < \max \{|\mathcal{A}(n,k,t)|^2, |\mathcal{H}(n,k,t)|^2\}.
\]

**Case 4.** \(s_1 = t+1\) and \(s_2 = t+1\).

Recall that \(\mathcal{A}_1 = T_{t+1}^t(\mathcal{G})\) and \(\mathcal{A}_2 = T_{t+1}^t(\mathcal{F})\). By the assumption, we have \(\mathcal{A}_1 \neq \emptyset \neq \mathcal{A}_2\). By (5.9) and (5.6), we have
\[
|\mathcal{F}| \leq |\mathcal{A}_1| \binom{n-t-1}{k-t-1} + (t+1)k^2 \binom{n-t-2}{k-t-2}
\]
and
\[
|\mathcal{G}| \leq |\mathcal{A}_2| \binom{n-t-1}{k-t-1} + (t+1)k^2 \binom{n-t-2}{k-t-2}.
\]
By Proposition 3.1, we may assume that \(\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset\). Then (5.2) implies \(|\mathcal{A}_1| + |\mathcal{A}_2| \leq 4(k-t-1)\).

If \(|\mathcal{A}_1|, |\mathcal{A}_2| \leq \max\{ (t+2)^2, (k-t+1)^2 \}/2\), then by multiplying (5.14) and (5.15) we get
\[
|\mathcal{F}| |\mathcal{G}| \leq |\mathcal{A}_1| |\mathcal{A}_2| \binom{n-t-1}{k-t-1}^2 + (t+1)k^2 \binom{n-t-2}{k-t-2} \binom{n-t-1}{k-t-1} (|\mathcal{A}_1| + |\mathcal{A}_2|)
\]
\[
+ (t+1)^2 k^4 \binom{n-t-2}{k-t-2}^2
\]
\[
\leq \frac{1}{2} \max \{ (t+2)^2, (k-t+1)^2 \} \binom{n-t-1}{k-t-1}^2 + \frac{4(t+1)k^2 (k-t+1)^2}{n-t-1} \binom{n-t-1}{k-t-1}^2
\]
\[
+ \frac{(t+1)^2 k^4 (k-t-1)^2}{(n-t-1)^2} \binom{n-t-1}{k-t-1}^2.
\]

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For $n \geq 4(t + 2)^2k^2$, we arrive at

\[
|\mathcal{F}| |\mathcal{G}| \leq \left( \frac{1}{2} \max \left\{ (t + 2)^2, (k - t + 1)^2 \right\} + \frac{(k - t + 1)^2}{4} + \frac{(k - t + 1)^2}{8} \right) \left( \frac{n - t - 1}{k - t - 1} \right)^2
\]

\[
\leq \frac{7}{8} \max \left\{ (t + 2)^2, (k - t + 1)^2 \right\} \left( \frac{n - t - 1}{k - t - 1} \right)^2.
\]

Applying (1.8) with $c = 16$, we obtain that

\[
|\mathcal{F}| |\mathcal{G}| \leq \frac{7}{8} \times \frac{16}{16 - 2} \max \left\{ (t + 2)^2, (k - t + 1)^2 \right\} \left( \frac{n - k - 1}{k - t - 1} \right)^2
\]

\[
= \max \left\{ (t + 2)^2, (k - t + 1)^2 \right\} \left( \frac{n - k - 1}{k - t - 1} \right)^2
\]

\[
\leq \max \left\{ |A(n, k, t)|^2, |H(n, k, t)|^2 \right\}.
\]

Thus in the rest of the proof we may assume that

\[
(5.16) \quad |A_1| |A_2| > \frac{1}{2} \max \left\{ (t + 2)^2, (k - t + 1)^2 \right\}.
\]

Since $A_1 \cap A_2 = \emptyset$, we see that $A_1, A_2$ are non-empty exact cross $t$-intersecting. Applying Proposition 1.4 with $A = A_1, B = A_2$, we see that one of (i), (ii), (iii) in Proposition 1.4 holds. If (iii) holds, then $|A_1| |A_2| \leq \frac{(t + 2)^2}{2}$, contradicting (5.16). Thus, either (i) or (ii) of Proposition 1.4 holds.

**Subcase 4.1.** Either $|A_1| \leq 2$, $|A_2| \leq k + 1$, $τ_t(A_2) \geq t + 1$ or $|A_2| \leq 2$, $|A_1| \leq k + 1$, $τ_t(A_1) \geq t + 1$.

By symmetry, assume that $|A_1| \leq 2$, $|A_2| \leq k + 1$ and $τ_t(A_2) \geq t + 1$. Let $r_2$ be the minimum integer such that $τ_t(B_2^{(\leq r_2)}) \geq t + 1$. Then clearly $s_2 = r_2 = t + 1$. Let $α'_t = ((t + 1)^2k^{t-1})^{-1} |B_1^{(t)}|$. By (5.1) we have

\[
(5.17) \quad \sum_{t+2 \leq \ell \leq k} α'_t \leq \sum_{t+1 \leq \ell \leq k} α'_t \leq 1.
\]

Let $h(n, k, \ell) = (t + 1)^2k^{t-1} \binom{n-\ell}{k-\ell}$. Since

\[
\frac{h(n, k, \ell + 1)}{h(n, k, \ell)} = \frac{k^{\ell-t} \binom{n-\ell-1}{k-\ell-1}}{k^{\ell-t-1} \binom{n-\ell}{k-\ell-1}} = \frac{k^{t-1}}{n-\ell} < 1,
\]

by (5.17) we infer

\[
\sum_{t+2 \leq \ell \leq k} |\mathcal{F}(\ell)| \leq \sum_{t+2 \leq \ell \leq k} α'_t h(n, k, \ell) \leq h(n, k, t + 2) = (t + 1)^2k \binom{n-t}{k-t-2}.
\]

It follows that

\[
|\mathcal{F}| = |\mathcal{F}^{(t+1)}| + \sum_{t+2 \leq \ell \leq k} |\mathcal{F}(\ell)| \leq |A_1| \binom{n-t-1}{k-t-1} + (t + 1)^2k \binom{n-t-2}{k-t-2}
\]

\[
\leq 2 \binom{n-t-1}{k-t-1} + (t + 1)^2k \binom{n-t-2}{k-t-2}.
\]

(5.18)
If $k = t + 1$ then
\[ |\mathcal{F}| |\mathcal{G}| = |A_1||A_2| \leq 2(k + 1) \leq 2(t + 2) < |\mathcal{A}(n, t+1, t)|^2 \]

and we are done. Hence we may assume that $k \geq t + 2$. Note that $k \geq t + 2$ and $k \geq 5$ imply $2(k + 1) \leq (t + 1)(k - t + 1)$. Therefore, by (5.15) and (5.16) we obtain that
\[
|\mathcal{F}| |\mathcal{G}| \leq \left(2 \binom{n-t-1}{k-t-1} + (t+1)^2k \binom{n-t-2}{k-t-2}\right) \\
\quad \cdot \left((k+1) \binom{n-t-1}{k-t-1} + (t+1)k^2 \binom{n-t-2}{k-t-2}\right) \\
\leq 2(k+1) \binom{n-t-1}{k-t-1}^2 + (2(t+1)^2(k+1)) \binom{n-t-2}{k-t-2} \binom{n-t-1}{k-t-1} \\
+ (t+1)^3k^3 \binom{n-t-2}{k-t-2}^2.
\]

Note that $(t+1)(k+1) = tk + k + t + 1 \leq tk + 2k = (t+2)k$ and thereby

\[ 2(t+1)^2 + 2(t+1)^2k + 2(t+1)(t+2)^2 = 2(t+1)(t+3)k^2. \]

Then for $n \geq 4(t+2)^2k^2$,
\[
|\mathcal{F}| |\mathcal{G}| < (t+1)(k-t+1) \binom{n-t-1}{k-t-1}^2 + \frac{2(t+1)(t+3)k^2 \binom{n-t-1}{k-t-1}^2}{n-t-1} \\
\quad + \frac{(t+1)^3k^3 \binom{k-t-1}^2}{(n-t-1)^2} \binom{n-t-1}{k-t-1}^2 \\
\quad < (t+1)(k-t+1) \binom{n-t-1}{k-t-1}^2 + \frac{k-t-1}{2} \binom{n-t-1}{k-t-1}^2 + \frac{k-t-1}{4} \binom{n-t-1}{k-t-1}^2 \\
\quad = \left(\frac{t+7}{4}\right)(k-t+1) \binom{n-t-1}{k-t-1}^2.
\]

Now applying (1.18) with $c = 8(t+2)$, we conclude that
\[
|\mathcal{F}| |\mathcal{G}| < \left(\frac{t+7}{4}\right)(k-t+1) \frac{8(t+2)}{8(t+2)-2} \binom{n-t-1}{k-t-1}^2 = (t+2)(k-t+1) \binom{n-k-1}{k-t-1}^2 \\
\quad \leq \max \{ |\mathcal{A}(n,k,t)|^2, |\mathcal{H}(n,k,t)|^2 \}.
\]

**Subcase 4.2.** $A_1 \cup A_2$ is a sunflower with center of size $t$.

Without loss of generality, assume that $A_1 = \{[t] \cup \{a_1\}, \ldots, [t] \cup \{a_p\}\}$ and $A_2 = \{[t] \cup \{b_1\}, \ldots, [t] \cup \{b_q\}\}$. By Lemma 5.1 we see $p \leq k - t + 1$ and $q \leq k - t + 1$. If $p \leq 2$ or $q \leq 2$ holds, then
\[
|A_1||A_2| \leq 2(k-t+1) \leq \frac{1}{2}(t+2)(k-t+1) \leq \frac{1}{2} \max \{ (t+2)^2, (k-t+1)^2 \},
\]
contradicting (5.16). Thus we further assume that $p, q \geq 3$. Let $\mathcal{F}_0 = \mathcal{F}([t]), \mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0,$
$\mathcal{G}_0 = \mathcal{G}([t])$ and $\mathcal{G}_1 = \mathcal{G} \setminus \mathcal{G}_0$. 

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Claim 2. For each \( F \in \mathcal{F}_1 \), \(|F \cap [t]| = t - 1\) and \( \{b_1, \ldots, b_q\} \subset F \). Similarly, for each \( G \in \mathcal{G}_1 \), \(|G \cap [t]| = t - 1\) and \( \{a_1, \ldots, a_p\} \subset G \).

Proof. Indeed, simply note that \( |F \cap ([t] \cup \{b_j\})| \geq t \) for \( j = 1, \ldots, q \) and \([t] \not\subset F\), we see that \( |F \cap [t]| = t - 1\) and \( \{b_1, \ldots, b_q\} \subset F \).

By Claim 2, we see that

\[
|\mathcal{F}_1| \leq t \left( \frac{n - t - q + 1}{k - t - q + 1} \right) \leq t \left( \frac{n - t - 2}{k - t - 2} \right) \leq t \left( \frac{n - t - p + 1}{k - t - p + 1} \right) \leq t \left( \frac{n - t - 2}{k - t - 2} \right).
\]

By non-triviality, we know that \( \mathcal{F}_1 \neq \emptyset \neq \mathcal{G}_1 \). Fix some \( F \in \mathcal{F}_1 \). Then for every \( G_0 \in \mathcal{G}_0 \), since \(|G_0 \cap F| \geq t\) and \(|F \cap [t]| = t - 1\), \( G_0 \cap (F \setminus [t]) \neq \emptyset \). Therefore,

\[
|\mathcal{G}_0| \leq \left( \frac{n - t}{k - t} \right) - \left( \frac{n - t - |F \setminus [t]|}{k - t} \right) = \left( \frac{n - k - 1}{k - t} \right) = |H(n, k, t)| - t.
\]

Similarly,

\[
|\mathcal{F}_0| \leq \left( \frac{n - t}{k - t} \right) - \left( \frac{n - k - 1}{k - t} \right) = |H(n, k, t)| - t.
\]

Let \( \mathcal{K}_1 = \{F \setminus [t]: F \in \mathcal{F}_1\} \) and \( \mathcal{K}_2 = \{G \setminus [t]: G \in \mathcal{G}_1\} \). If \(|\mathcal{K}_1| = |\mathcal{K}_2| = 1\), then Claim 2 implies that \(|\mathcal{F}_1|, |\mathcal{G}_1| \leq t\). It follows that \(|\mathcal{F}| \mid \mathcal{G} \mid \leq |H(n, k, t)|^2\) and we are done. If at least one of \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) has size greater than one, without loss of generality, assume that \(|\mathcal{K}_1| \geq 2\) and let \( K_1, K_2 \in \mathcal{K}_1 \). Then \( F \cap K_i \neq \emptyset \) for \( i = 1, 2\) and for each \( F \in \mathcal{F}_0\). Therefore \( F \cap (K_1 \cap K_2) \neq \emptyset \) or \( F \cap (K_1 \setminus K_2) \neq \emptyset \), \( F \cap (K_2 \setminus K_1) \neq \emptyset \). It follows that

\[
|\mathcal{F}_0| \leq \left( \frac{n - t}{k - t} \right) - \left( \frac{n - k + 1 \mid K_1 \cap K_2 \mid}{k - t} \right) + |K_2 \setminus K_1||K_1 \setminus K_2| \left( \frac{n - t - 2}{k - t - 2} \right).
\]

Let \(|K_2 \setminus K_1| = x\). Note that \(|K_1| = |K_2| = k - t + 1\) implies that \( 1 \leq x \leq k - t \). Define

\[
\varphi(x) = \left( \frac{n - t}{k - t} \right) - \left( \frac{n - k - 1 + x}{k - t} \right) + x^{2} \left( \frac{n - t - 2}{k - t - 2} \right).
\]

Note that for \( n \geq 4k^2 \),

\[
\varphi'(x) = - \left( \frac{n - k - 1 + x}{k - t} \right) \sum_{i=0}^{k-t-1} \frac{1}{n - k - 1 + x - i} + 2x \left( \frac{n - t - 2}{k - t - 2} \right)
\]

\[
\leq - \left( \frac{n - k}{k - t} \right) \frac{k - t}{n - t} + 2(k - t) \left( \frac{n - t - 2}{k - t - 2} \right)
\]

\[
\leq - \frac{n - t - (k - t)^2}{n - t} \left( \frac{n - t}{k - t} \right) \frac{k - t}{n - t} + 2(k - t) \left( \frac{n - t - 2}{k - t - 2} \right)
\]

\[
\leq - \frac{1}{2} \left( \frac{n - t - 1}{k - t - 1} \right) + 2(k - t) \left( \frac{n - t - 2}{k - t - 2} \right).
\]

\[
< 0.
\]
By $x \geq 1$, we infer
\[
|\mathcal{F}_0| \leq \varphi(1) = \binom{n-t}{k-t} - \binom{n-k}{k-t} + \binom{n-t-2}{k-t-2}
\]
(5.22)
\[= |\mathcal{H}(n,k,t)| - \binom{n-k-1}{k-t-1} + \binom{n-t-2}{k-t-2}.\]

Thus by (5.22), (5.19) and (5.20) we have
\[
|\mathcal{F}| G = (|\mathcal{F}_0| + |\mathcal{F}_1|) (|\mathcal{G}_0| + |\mathcal{G}_1|)
\]
\[\leq \left( |\mathcal{H}(n,k,t)| - \binom{n-k-1}{k-t-1} + (t+1) \binom{n-t-2}{k-t-2} \right)
\[\cdot \left( |\mathcal{H}(n,k,t)| + t \binom{n-t-2}{k-t-2} \right)
\[= |\mathcal{H}(n,k,t)|^2 + |\mathcal{H}(n,k,t)| \left( (2t+1) \binom{n-t-2}{k-t-2} - \binom{n-k-1}{k-t-1} \right)
\[+ t(t+1) \binom{n-t-2}{k-t-2}^2 - t \binom{n-k-1}{k-t-1} \binom{n-t-2}{k-t-2}.\]
(5.23)

Applying (1.7) with $c = 2$, we have
\[
\binom{n-k-1}{k-t-1} \geq \frac{c-1}{c} \binom{n-t-1}{k-t-1} = \frac{1}{2} \binom{n-t-1}{k-t-1}.
\]

Then for $n \geq 4(t+1)k$,
\[
(2t+1) \binom{n-t-2}{k-t-2} - \binom{n-k-1}{k-t-1} \leq (2t+1) \binom{n-t-2}{k-t-2} - \frac{1}{2} \binom{n-t-1}{k-t-1}
\leq \left( \frac{n-t-2}{k-t-2} \right) \left( 2t + 1 - \frac{n-t-1}{2(k-t-1)} \right)
\[< 0,
\]
(5.24)

and
\[
t(t+1) \binom{n-t-2}{k-t-2}^2 - t \binom{n-k-1}{k-t-1} \binom{n-t-2}{k-t-2}
\leq t(t+1) \binom{n-t-2}{k-t-2}^2 - \frac{1}{2} \binom{n-t-1}{k-t-1} \binom{n-t-2}{k-t-2}
\geq t \left( \frac{n-t-2}{k-t-2} \right)^2 \left( t+1 - \frac{n-t-1}{2(k-t-1)} \right)
\[< 0.
\]
(5.25)

By (5.23), (5.24) and (5.25), we conclude that $|\mathcal{F}| G < |\mathcal{H}(n,k,t)|^2$ and the theorem is proven.

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