Homogenization of Stable-like Feller processes

Qiao Huang\textsuperscript{1}, Jinqiao Duan\textsuperscript{2}, Renming Song\textsuperscript{3}

Abstract

We study the homogenization for a class of non-symmetric stable-like Feller processes, and demonstrate its usage in the numerical approximation of first exit time. The jump intensity involves periodic and aperiodic constituents, as well as oscillating and non-oscillating constituents. This means that the noise can come both from the underlying periodic medium and from external environments, and is allowed to have different scales. It turns out that the stable-like process converges in distribution, as the scaling parameter goes to zero, to a Lévy process. As special cases of our result, some homogenization problems studied in previous works can be recovered. Moreover, we present some numerical experiments to illustrate our result by demonstrating that the first exit time of the original process can be approximated by that of the limit process.

AMS 2020 Mathematics Subject Classification: 35B27, 60G53, 35R09, 60F17.

Keywords and Phrases: Homogenization, Feller processes, weak convergence, stable-like processes, nonlocal operators, first exit time.

1 Introduction

As a subclass of Markov processes, Feller processes possess lots of nice properties in both probabilistic and analytic aspects \cite{1, 7, 14, 23}. The generator of a Feller process is in general a nonlocal operator. It looks locally like the generator of a Lévy process, in the sense that it is given by a Lévy-Khintchine type representation with an $x$-dependent Lévy triplet $(b(x), a(x), \eta(x, \cdot))$. For this reason, Feller processes are sometimes called Lévy-type processes or jump-diffusions, their generators are called Lévy-type operators. Feller processes with no diffusion parts at all, i.e., $a \equiv 0$, are called pure jump processes. If, in addition, for every $x$, the jump measure $\eta(x, \cdot)$ is a stable Lévy measure \cite{29}, the Feller process is called a stable-like process.

Homogenization problems arise from the study of porous media, composite materials and other physical and engineering systems \cite{3, 10, 11}. Generally speaking, in a periodic structure, such as medium or material, the heterogeneities are relatively small compared to its global dimension. Thus, two scales characterize the motion of particles in the structure: the microscopic one describing the heterogeneities, and the macroscopic one describing the global behavior of particles. The aim of homogenization is precisely to give the macroscopic properties of the particles by taking into account the properties of the microscopic structure.

In this paper, we focus on the homogenization of stable-like processes, periodic in space and locally periodic in noise. Consider a pure jump process in a periodic medium. The drift $b(x)$ and the jump measure $\eta(x, \cdot)$ are periodic and have a small scale in the spatial variable $x$, due to heterogeneities. In mathematical formulation, the small scale is represented by a small parameter $\epsilon > 0$. From the realistic point of view, the noise may occur not only from the underlying periodic medium, but also from external environments. So we assume the jump measure $\eta(x, dz)$ have mixed scales in the noise variable $z$. These suggest that the generators of stable-like processes in periodic medium will take the following form:

\begin{equation}
\mathcal{A}_\epsilon f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+z) - f(x) - z \cdot \nabla f(x) \mathbb{1}_{[1,2]}(\alpha) \mathbb{1}_B(z) \right] \kappa(x/\epsilon, z, z/\epsilon) J(z) dz + \left( \frac{1}{\epsilon} b(x/\epsilon) + c(x/\epsilon) \right) \cdot \nabla f(x) \mathbb{1}_{[1,2]}(\alpha).
\end{equation}

\textsuperscript{1}School of Mathematics and Statistics & Center for Mathematical Sciences, Huazhong University of Science and Technology, Wuhan, Hubei 430074, P.R. China. Email: hq932309@alumni.hust.edu.cn
\textsuperscript{2}Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA. Email: duan@iit.edu
\textsuperscript{3}Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. Email: rsong@illinois.edu
Here and after, we denote by $B$ the unit open ball in $\mathbb{R}^d$, and by $S := \partial B$ the unit sphere. We use Einstein’s convention that the repeated indices in a product will be summed automatically. Besides, the drift functions $b, c : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are Borel measurable and periodic, $J : \mathbb{R}^d \setminus \{0\} \rightarrow [0, \infty)$ is an $\alpha$-stable Lévy density with $\alpha \in (0, 2)$.

The jump intensity $\kappa : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\}) \rightarrow [0, \infty)$ is a Borel measurable function, periodic in its first variable. The normal scale of $\kappa$ in $z$ corresponds to the noise constituent coming from external environments. Furthermore, we assume that there exists a function $\kappa^* : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\}) \rightarrow [0, \infty)$ periodic in its first and third variables, such that $\kappa(x, z, u) = \kappa^*(x, z, u, u)$. In this case, the jump intensity $\kappa(x/\epsilon, z, z/\epsilon)$ in (1.1) is locally periodic in noise variable $z$. It means that the small noise scale can be decomposed into two constituents, corresponding to the periodic medium and the external environments respectively.

Under some regularity assumptions (see the next section), each Lévy-type operator $\mathcal{A}$ can generate a Feller process on $\mathbb{R}^d$, say $X^\epsilon$. Our aim is to identify the limit of $X^\epsilon$, when the scaling parameter $\epsilon$ goes to zero. It turns out that (see Theorem 3.2), the limit process $X$, in the sense of convergence in distribution, is a Lévy process with the following generator:

$$
\mathcal{A}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + z) - f(x) - z \cdot \nabla f(x) \mathbf{1}_{[1,2]}(\alpha) \mathbf{1}_{\{|z| < 1\}} \right] \tilde{\kappa}(z) J(z) dz + \tilde{c} \cdot \nabla f(x) \mathbf{1}_{(1,2)}(\alpha),
$$

where $\tilde{\kappa}$ is a homogenized jump intensity related to the function $\kappa$, $\tilde{c}$ is a homogenized constant.

Homogenization of Feller processes with jumps has been investigated by a number of authors. To name a few, the paper [20] considered the one-dimensional pure-jump case, and [15] studied the homogenization of stable-like processes with variable order. See also [32] for a multi-dimensional generalization with diffusion terms involved. The paper [30] investigated the homogenization problem of a class of pure-jump Lévy processes using a pure analytical approach — Mosco convergence. Recently, the authors studied in [21] the periodic homogenization of SDEs with jump noise and corresponding nonlocal PDEs. Under the setting of the present paper, many homogenization problems in the literature mentioned above can be recovered. See Section 4 for comparisons.

The sequel of this paper is organized as follows. In the next section, we list our main assumptions, and prove some preliminary results such as the well-posedness of martingale problems, invariance and ergodicity, etc. Some technical results will be left into appendices, with no affect to the smoothness of reading. In Section 3, we prove our main result to identify the homogenization limit. Some examples of resolving the homogenization problems in previous works are presented in Section 4. Section 5 is devoted to some numerical experiments and an application to the first exit time.

## 2 General assumptions and preliminary results

In the section, we collect general assumptions and some results we need. The most crucial results are Corollary 2.6 and 2.8. The former allow us to pass the functional convergence in the main theorem in next section, while the latter is the well-posedness for a Poisson equation. Most proofs in this section are quite short. We put other auxiliary but technical results into Appendix.

Firstly, we list our assumptions on the functions $b, c$, $\kappa$ (or $\kappa^*$) and $J$.

### Assumption H1

The functions $b, c$ are in the Hölder class $C^\beta$ for some $\beta \in (0, 1)$, and they are periodic of period 1.

### Assumption H2

The function $(x, z, u, v) \rightarrow \kappa^*(x, z, u, v)$ is periodic of period 1 in $x$ and $u$; there exist constants $\kappa_1, \kappa_2, \kappa_3 > 0$, $\beta \in (0, 1)$, such that for all $x, x_1, x_2 \in \mathbb{R}^d$ and $z, u \in \mathbb{R}^d \setminus \{0\}$,

$$
\kappa_1 \leq \kappa(x, z, u) \leq \kappa_2,
$$

$$
|\kappa(x_1, z, u) - \kappa(x_2, z, u)| \leq \kappa_3 |x_1 - x_2|^\beta.
$$

There exists a function $\kappa_0 : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\}) \rightarrow [0, \infty)$ such that for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \setminus \{0\}$,

$$
|\kappa^*(x, z, z/\epsilon, z/\epsilon) - \kappa_0(x, z, z/\epsilon)| \rightarrow 0, \quad \epsilon \rightarrow 0^+.
$$
We assume also that there exists a function $κ_1 : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$ such that for all $z \in \mathbb{R}^d \setminus \{0\}$,
\[
κ(x, εz, z) \to κ_1(x, z) \text{ uniformly in } x, \quad ε \to 0^+.
\] (2.4)

In the case $α ∈ (1, 2)$ and $b \neq 0$, we assume further that for all $z \in \mathbb{R}^d \setminus \{0\}$,
\[
\frac{1}{c_α−1}|κ(x, εz, z) − κ_1(x, z)| \to 0 \text{ uniformly in } x, \quad ε \to 0^+.
\] (2.5)

Assumption H3. Assume that the Lévy density $J$ is positive homogeneous of degree $−(d + α)$ for some $α ∈ (0, 2)$, that is,
\[
J(rz) = r^{−(d+α)}J(z), \quad r > 0, z \in \mathbb{R}^d \setminus \{0\},
\] (2.6)

and is bounded between two positive constants on the unit sphere $S$, i.e., there exist constants $j_1, j_2 > 0$, such that for all $ξ ∈ S$
\[
j_1 ≤ J(ξ) ≤ j_2.
\] (2.7)

In the case $α = 1$, we assume additionally that for each $x ∈ \mathbb{R}^d$ and $r_1, r_2 ∈ (0, ∞),$
\[
∫_S J_κ(x, r_1ξ, r_2ξ)J(ξ)dξ = 0.
\] (2.8)

We denote by $C^k(\mathbb{T}^d)$ with integer $k ≥ 0$ the space of continuous functions on $\mathbb{T}^d$ possessing derivatives of orders not greater than $k$, endowed with the norm $∥f∥_k := \sup_{x \in \mathbb{T}^d} |f(x)| + \sum_{|α| ≤ k} \sup_{x \in \mathbb{T}^d} |∇^α f(x)|$. When $k = 0$, we simply denote by $C(\mathbb{T}^d) := C^0(\mathbb{T}^d)$. For a non-integer $γ > 0$, the Hölder space $C^γ(\mathbb{T}^d)$ are defined as the subspaces of $C^{[γ]}$ consisting of functions whose $[γ]$-th order partial derivatives are locally Hölder continuous with exponent $γ − [γ]$. The spaces $C^γ(\mathbb{T}^d)$ is a Banach space endowed with the norm $∥f∥_γ = ∥f∥_{[γ]} + ∥∇^α f∥_{γ−[γ]}$, where the seminorm $|f|^γ$ with $0 < γ < 1$ is defined as $|f|^γ := \sup_{x,y \in \mathbb{T}^d, x ≠ y} \frac{|f(x)−f(y)|}{|x−y|^γ}$.

Remark 2.1. (i). We shall always identify a periodic function on $\mathbb{R}^d$ of period 1 with its restriction on the flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Then Assumption H1 amounts to saying that $b, c ∈ C^2(\mathbb{T}^d)$.

(ii). The relationship (2.3) ensures that the function $κ_0(x, z, u)$ is also periodic in $x$ and $u$ and satisfies (2.1). The relationship (2.4) or (2.5) ensures that the function $κ_1(x, z)$ is periodic in $x$.

(iii). A typical example for assumptions (2.4) and (2.5) to hold is that $κ(x, z, u)$ can be written as the quotient of two positive homogeneous functions in $z$. In the case that $α ∈ (1, 2)$ and $b ≠ 0$, the convergence (2.5) implies (2.4). In this case, there involves singularity in the drift coefficient $\frac{1}{c_α−1}b$, so that we need more regularities for $κ$ to cancel the singularity in the drift, as we will see in the proof of Theorem 3.2.

(iv). The positive homogeneity assumption on $J$ is equivalent to saying that $J$ is an $α$-stable Lévy density (cf. [29, Theorem 14.3]). By (2.6), $J(z) = J(|z|, \frac{|z|}{|z|^2}) = |z|^{−(d+α)}J(\frac{|z|}{|z|^2})$. Then assumption (2.7) implies
\[
j_1|z|^{−(d+α)} ≤ J(z) ≤ j_2|z|^{−(d+α)}, \quad z ∈ \mathbb{R}^d \setminus \{0\}.
\] (2.9)

(v). It is easy to verify that the assumptions for $κ$ and $J$, (2.1), (2.2) and (2.6)-(2.8) ensure all assumptions in [18] for $α ∈ (0, 1) ∪ (1, 2)$ and in [31] for $α = 1$. We will use the results therein in the sequel.

We need some auxiliary operators and processes. In fact, we will rescale the operator $A^ε$ and its canonical process in a effective fashion. For this purpose, we define the following nonlocal operators for $f ∈ C^∞(\mathbb{T}^d)$, the space of all smooth functions on the flat torus $\mathbb{T}^d$ (i.e., smooth periodic functions of period 1),
\[
\tilde{A}^ε f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+z) − f(x) − z \cdot ∇ f(x) \left( 1_{(1]}(α)1_B(z) + 1_{(1,2]}(α)1_B(εz) \right) \right ] κ(x, εz, z) J(z) dz
\]
\[+ \left( b(x) + c_α−1c(x) \right ) \cdot ∇ f(x) 1_{(1,2]}(α), \quad ε > 0,
\]
\[
\tilde{A}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+z) − f(x) − z \cdot ∇ f(x) \left( 1_{(1]}(α)1_B(z) + 1_{(1,2]}(α) \right) \right ] κ_1(x, z) J(z) dz
\]
\[+ b(x) \cdot ∇ f(x) 1_{(1,2]}(α).
\]

For notational simplicity, we shall allow the parameter $ε$ to be zero so that $\tilde{A}^0 := \tilde{A}$. The periodicity and continuity of the function $x → κ(x, z, u)$ and (2.1), (2.8) and (2.9) imply that $\tilde{A}^ε, \ ε ≥ 0$, are all well-defined
unbounded operators on \((C(T^d), \| \cdot \|_0)\), whose domains contain \(C^\infty(T^d)\). Moreover, it is easy to verify by (2.6) that for each \(\epsilon > 0\), the operator \(\tilde{A}^\epsilon\) is a rescaling of \(A^\epsilon\) in the sense
\[
\tilde{A}^\epsilon f(x) = \epsilon^\alpha (A^\epsilon f_\epsilon)(\epsilon x), \quad f \in C^\infty(T^d).
\] (2.10)

Here and after, we denote \(f_\epsilon(x) := f(x/\epsilon)\).

Denote by \(\mathcal{D} = \mathcal{D}(\mathbb{R}_+; \mathbb{R}^d)\) \((\mathcal{D}_{\text{per}} = \mathcal{D}(\mathbb{R}_+; T^d))\) the space of all \(\mathbb{R}^d\)-valued càdlàg functions on \(\mathbb{R}_+\), equipped with the Skorokhod topology. The following theorem tells us that the operators \(\alpha\) \(\alpha\) Remark 1.5], the assertions for the case \(\alpha = 1\) can be found in [31, Theorem 1.1, Theorem 1.3, Theorem 1.4]. All the assertions for the case \(\alpha = 1\) can be found in [31, Theorem 1.1, Theorem 1.3, Theorem 1.4]. Therefore, we only need to prove the results for \(\alpha \in (1, 2)\). The lower bound estimates of \(\bar{\nu}\) can be found in [9, Theorem 1.5], we also put them into Appendix for the reader’s convenience, see Proposition A.2. The proofs of the rest parts are tedious, especially that \(C^\infty(T^d)\) is the core of the generators, and we leave them into Appendix, see Proposition A.4 and Corollary B.1.

Of course, each of the processes \(X^\epsilon\), \(X\) and \(\tilde{X}^\epsilon\) is defined on its own stochastic basis. However, by taking product of the probability spaces, it is always possible to assume that:

\[X^\epsilon, \tilde{X} \text{ and } \tilde{X}^\epsilon, \epsilon > 0, \text{ are all defined on the same probability space } (\Omega, \mathcal{F}, \mathbb{P}).\]

We also assume for simplicity that

\[X^\epsilon_0 = \tilde{X}^\epsilon_0 = 0.\]

If we identify a periodic function on \(\mathbb{R}^d\) of period \(\epsilon\) with its restriction on the \(\epsilon\)-torus \(T^d_\epsilon := \epsilon T^d\), then each \(A^\epsilon\) maps \(C^\infty(T^d_\epsilon)\) into \(C^\infty(T^d_\epsilon)\) by virtue of the periodicity of \(\kappa\) in \(x\). In view of this, the canonical process \(X^\epsilon\) can also be treated as a processes taking values on \(T^d_\epsilon\), via the quotient map \(\mathbb{R}^d \to T^d_\epsilon\). We will use this treatment only in the rest of this section, the benefit is the following relation, which follows from the well-known fact that Feller semigroups and Feller processes are in one-to-one correspondence if we identify the processes that have same finite-dimensional distributions (see, e.g., [7]).

**Lemma 2.3.** We have the following identity in law

\[
\{\tilde{X}^\epsilon_t\}_{t \geq 0} \overset{d}{=} \left\{ \frac{1}{\epsilon} X^\epsilon_{\epsilon t} \right\}_{t \geq 0}, \quad \text{for each } \epsilon > 0.
\]

**Proof.** We derive the generator for the Feller process \(\left\{ \frac{1}{\epsilon} X^\epsilon_{\epsilon t} \right\}_{t \geq 0}\). For \(f \in C^\infty(T^d)\), by (2.10),

\[
\lim_{t \downarrow 0} \frac{\mathbb{E}^\epsilon_{\epsilon x} \left[ f \left( \frac{1}{\epsilon} X^\epsilon_{\epsilon t} \right) \right] - f(x)}{t} = \epsilon^\alpha \lim_{t \downarrow 0} \frac{\mathbb{E}^\epsilon_{\epsilon x} \left[ f \left( X^\epsilon_{\epsilon t} \right) \right] - f(x)}{\epsilon^\alpha t} = \epsilon^\alpha (A^\epsilon f_{\epsilon})(\epsilon x) = \tilde{A}^\epsilon f(x).
\]

Therefore, the Feller semigroup associated to \(\left\{ \frac{1}{\epsilon} X^\epsilon_{\epsilon t} \right\}_{t \geq 0}\) is also generated by the closure of \((\tilde{A}^\epsilon, C^\infty(T^d))\).
Denote by \( \{\hat{P}_t^\epsilon\}_{t \geq 0} \) (or \( \{\hat{P}_t\}_{t \geq 0} \)) the Feller semigroup of \( \hat{X}^\epsilon \) (or \( \hat{X} \)). Let \( \hat{X}^0 = \hat{X} \) and \( \hat{P}_0^\epsilon = \hat{P}_t \). Now utilizing the lower bound of the transition probability density \( \tilde{p}'(t; x, y) \), we obtain the following exponential ergodicity.

**Proposition 2.4.** For each \( \epsilon \geq 0 \), the Feller process \( \hat{X}^\epsilon \) possesses a unique invariant probability distribution \( \mu_\epsilon \) on \( \mathbb{T}^d \). Moreover, there exist positive constants \( C \) and \( \rho \) such that for each periodic bounded Borel function \( f \) on \( \mathbb{R}^d \) (i.e., \( f \) is bounded Borel on \( \mathbb{T}^d \)),

\[
\sup_{x \in \mathbb{T}^d} \left| \hat{P}_t^\epsilon f(x) - \int_{\mathbb{T}^d} f(y)\mu_\epsilon(dy) \right| \leq C\|f\|_{\infty}e^{-\rho t}
\]

for every \( t \geq 0 \).

**Proof.** The proof follows the lines of [3, Theorem 3.3.2]. Due to the Doeblin-type results in [3, Theorem 3.3.1], it is enough to ensure that for every \( t > 0 \), the map \( (x, y) \rightarrow \tilde{p}'(t; x, y) \) is bounded from below by a positive constant, which follows immediately from the transition density estimates in Proposition 2.2 and the compactness of the state space \( \mathbb{T}^d \).

Denote by \( \mu = \mu_0 \) the invariant probability measure for \( \hat{X} \).

**Lemma 2.5.** As \( \epsilon \to 0^+ \), we have \( \mu_\epsilon \to \mu \).

**Proof.** By the argument in [19, Lemma 2.4], we only need to show that \( \hat{P}_t^\epsilon f \to \hat{P}_t f \) in \( \mathcal{C}(\mathbb{T}^d) \) as \( \epsilon \to 0^+ \) for each \( f \in \mathcal{C}(\mathbb{T}^d) \) and \( t \geq 0 \).

By Proposition 2.2, we know that \( \mathcal{C}^\infty(\mathbb{T}^d) \) is a core for each \( \hat{A}^\epsilon \), \( \epsilon \geq 0 \). Now fix an arbitrary \( f \in \mathcal{C}^\infty(\mathbb{T}^d) \). If \( \alpha \in (0, 1] \), then \( \|\hat{A}^\epsilon f - \hat{A} f\|_0 \) as \( \epsilon \to 0^+ \) by dominated convergence and (2.4). For the case \( \alpha \in (1, 2) \), we have

\[
|\hat{A}^\epsilon f(x) - \hat{A} f(x)| \leq |\epsilon^{-1}c(x) \cdot \nabla f(x)| + \left| \int_{\mathbb{R}^d \setminus \{0\}} z \cdot \nabla f(x) (1 - 1_{B(\epsilon z)}) \kappa(x, \epsilon z, z) J(z) dz \right|
\]

\[
+ \left| \int_{\mathbb{R}^d \setminus \{0\}} [f(x + z) - f(x) - z \cdot \nabla f(x)] (\kappa(x, \epsilon z, z) - \kappa_1(x, z)) J(z) dz \right|
\]

\[
\leq \epsilon^{-1}\|c\|_0\|f\|_1 + \kappa_2 j_2 \|f\|_1 \int_{|z| \geq 1/\epsilon} \frac{dz}{|z|^{d+a-1}}
\]

\[
+ 2\kappa_2 j_2 \|f\|_1 \int_{\mathbb{R}^d \setminus \{0\}} \{1 \wedge |z|\} |\kappa(x, \epsilon z, z) - \kappa_1(x, z)| \frac{dz}{|z|^{d+a}},
\]

which converges to zero, uniformly in \( x \), as \( \epsilon \to 0^+ \), by (2.4) and dominated convergence. Now by the Trotter-Kato approximation theorem (see [13, Theorem III.4.8]), \( \hat{P}_t^\epsilon f \to \hat{P}_t f \) in \( \mathcal{C}(\mathbb{T}^d) \) as \( \epsilon \to 0^+ \) for all \( f \in \mathcal{C}(\mathbb{T}^d) \), uniformly for \( t \) in compact intervals.

Now using Proposition 2.4 and Lemma 2.5, we can obtain a useful functional convergence theorem.

**Corollary 2.6.** Let \( f \) be a bounded Borel function on \( \mathbb{T}^d \). Then for every \( t > 0 \),

\[
\mathbb{E} \left[ \left( \int_0^t \left| f \left( \frac{X_s^\epsilon}{\epsilon} \right) - \int_{\mathbb{T}^d} f(y)\mu(dy) \right| ds \right)^2 \right] \to 0, \quad \text{as } \epsilon \to 0^+.
\]

In particular, for every \( T > 0 \),

\[
\sup_{0 \leq t \leq T} \left| \int_0^t f \left( \frac{X_s^\epsilon}{\epsilon} \right) ds - t \int_{\mathbb{T}^d} f(y)\mu(dy) \right| \to 0, \quad \text{in probability } \mathbb{P}, \quad \text{as } \epsilon \to 0^+.
\]
Proof. We follow the lines of [26, Proposition 2.4]. Fix \( \epsilon > 0 \). Let \( \tilde{f}_\epsilon := f - \int_{\mathbb{R}^d} f(x) \mu_\epsilon(dx) \). By virtue of Lemma 2.5, it suffices to prove that \( \int_0^s |\tilde{f}_\epsilon(x)/\epsilon|ds \to 0 \) in \( L^2(\Omega, \mathcal{P}) \). Using Lemma 2.3, it is easy to get \( \int_0^s |\tilde{f}_\epsilon(x)/\epsilon|ds = \epsilon^{\alpha} \int_0^{\epsilon^{-\alpha} t} |\tilde{f}_\epsilon(x)|ds \). By Proposition 2.4, for \( 0 \leq s < t \) we have
\[
\mathbf{E} \left( |\tilde{f}_\epsilon(X_t^\epsilon)| \right) = \int_{\mathbb{R}^d} \tilde{f}_\epsilon(y) \left[ \tilde{p}(t-s, X_s^\epsilon, y) \right] d\mu_\epsilon(dy) \leq 2C\|f\|_0 e^{-\rho(t-s)},
\]
and then by the Markov property,
\[
\mathbf{E}|\tilde{f}_\epsilon(X_s^\epsilon)| = \mathbf{E} \left[ \left| \tilde{f}_\epsilon(X_s^\epsilon) \right| \mathbf{E} \left( \left| \tilde{f}_\epsilon(X_t^\epsilon) \right| \right) \right] \leq 2C\|\tilde{f}\|_0^2 e^{-\rho(t-s)}.
\]
Hence,
\[
\mathbf{E} \left[ \left( \int_0^t |\tilde{f}_\epsilon(X_s^\epsilon)/\epsilon|ds \right)^2 \right] = 2e^{2\alpha} \int_0^t e^{-\alpha s} \int_0^s \mathbf{E}|\tilde{f}_\epsilon(X_s^\epsilon)|^2 ds dr \leq 4Ce^{2\alpha} \|f\|_0^2 \int_0^t e^{-\rho(r-s)} ds dr = 4Ce^{2\alpha} \|f\|_0^2 \rho^{-2} (-1 + \rho e^{-\alpha} t + e^{-\rho e^{-\alpha} t}) \to 0,
\]
as \( \epsilon \to 0^+ \). The results follow. \( \square \)

Using the exponential ergodicity, we can also obtain the well-posedness of the nonlocal Poisson equation. Denote by \( \mathcal{C}_\mu(\mathbb{T}^d) = \mathcal{D}(\mathcal{A}_\mu) \), \( \gamma > 0 \), the class of all \( f \in \mathcal{C}_\mu(\mathbb{T}^d) \) which are centered with respect to the invariant measure \( \mu \) in the sense that \( \int_{\mathbb{T}^d} f(x) \mu(dx) = 0 \). It is easy to check that \( \mathcal{C}_\mu(\mathbb{T}^d) \) is a closed subset, and hence a sub-Banach space of \( \mathcal{C}(\mathbb{T}^d) \) under the norm \( \| \cdot \|_0 \).

**Lemma 2.7.** The restrictions \( \{ \tilde{P}_t^\mu := \tilde{P}_t|\mathcal{C}_\mu(\mathbb{T}^d) \}_{t \geq 0} \) form a \( C_0 \) semigroup on the Banach space \( (\mathcal{C}_\mu(\mathbb{T}^d), \| \cdot \|_0) \), with generator \( (\mathcal{A}_\mu, \mathcal{D}(\mathcal{A}_\mu)) := (\mathcal{A}, \mathcal{C}_\mu^\infty(\mathbb{T}^d)) \). Moreover, the set \( \{ z \in \mathbb{C} \mid \text{Re} z > -\rho \} \) is contained in the resolvent set of \( \mathcal{A}_\mu \).

**Proof.** Since \( \mu \) is invariant with respect to \( \{ \tilde{P}_t \}_{t \geq 0} \), it is easy to see that \( \mathcal{C}_\mu(\mathbb{T}^d) \) is \( \{ \tilde{P}_t \}_{t \geq 0} \)-invariant, in the sense that \( \tilde{P}_t(\mathcal{C}_\mu(\mathbb{T}^d)) \subset \mathcal{C}_\mu(\mathbb{T}^d) \) for all \( t \geq 0 \). The first part of the lemma then follows from the corollary in [13, Subsection II.2.3]. By the exponential ergodicity result in Proposition 2.4, we have
\[
\| \tilde{P}_t^\mu f \|_0 \leq C\|f\|_0 e^{-\rho t}
\]
for all \( f \in \mathcal{C}_\mu(\mathbb{T}^d) \) and \( t \geq 0 \). This yields the second part of the lemma, using [13, Theorem II.1.10.(ii)]. \( \square \)

**Corollary 2.8.** Let \( \alpha \in (1, 2) \). For every \( f \in \mathcal{C}_\mu^0(\mathbb{T}^d) \), there exists a unique solution in \( \mathcal{C}_\mu^{\alpha+\beta}(\mathbb{T}^d) \) to the Poisson equation
\[
\mathcal{A}u + f = 0.
\]

**Proof.** If \( u \in \mathcal{C}_\mu^{\alpha+\beta}(\mathbb{T}^d) \) is a solution, then by (2.12),
\[
\int_0^\infty \tilde{P}_t^\mu f dt = -\int_0^\infty \tilde{P}_t^\mu \mathcal{A}u dt = -\int_0^\infty \frac{d}{dt} \tilde{P}_t^\mu u dt = u - \lim_{t \to \infty} \tilde{P}_t^\mu u = u.
\]
This yields the uniqueness. Thanks to Corollary B.3, the existence follows from a standard argument of Fredholm alternative ([17, Section 5.3]). \( \square \)

According to the terminology of periodic homogenization, we will refer to equation (2.13) as the cell problem.
3 Homogenization result

In this section we will prove our homogenization result. Before that, some preparations are needed.

Firstly, we need a convergence lemma for locally periodic function.

**Lemma 3.1.** Let \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, (x, y) \mapsto \phi(x, y) \) be a function periodic in \( y \) with period 1.

(i). Let Assumption H4. The function \( \phi \) is uniquely determined by \( \frac{1}{\bar{b}} \).

(ii). Suppose for each \( x \in \mathbb{R}^d, \phi(x, \cdot) \in L^p([0,1]^d) \), and for each \( y \in \mathbb{R}^d, \phi(\cdot, y) \in L^p_{\text{loc}}(\mathbb{R}^d) \), where \( p' \) is the conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then for every compact set \( K \subset \mathbb{R}^d \), we have

\[
\lim_{\epsilon \to 0^+} \int_K \phi \left( x, \frac{X}{\epsilon} \right) dx = \int_K \int_{\mathbb{T}^d} \phi(x, y) dy dx.
\]

In the case that the function \( \phi \) is separable, that is, \( \phi \) is of the form \( \phi(x, y) = f(x)g(y) \) with \( g \) periodic, the conclusions of the lemma can be found in [10, Theorem 2.6]. The general case can be achieved via a standard Fourier decomposition of \( \phi \) in \( y \), i.e., \( \phi(x, y) = \sum_{k=0}^\infty \phi_k(x) \sin(2\pi k y) + \phi_k(x) \cos(2\pi k y) \).

Now we are in the position to prove the homogenization result. To get rid of the singularity in the coefficient \( \frac{1}{\bar{b}} \) in the case \( \alpha \in (1, 2) \), we need one more assumption on \( \bar{b} \).

**Assumption H4.** The function \( \bar{b} \) satisfies the centering condition,

\[
\int_{\mathbb{T}^d} \bar{b}(x) \mu(dx) = 0.
\]

By virtue of Assumptions H1, H4 and Corollary 2.8, when \( \alpha \in (1, 2) \) there exists a function \( \hat{b} \in \mathcal{C}_{\mu+}^{\alpha+}(\mathbb{T}^d) \) that uniquely solves the Poisson equation

\[
\hat{\Delta} \hat{b} + \hat{b} = 0.
\]

**Theorem 3.2.** Suppose that Assumptions H1-H4 hold. In the sense of weak convergence on the space \( \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d) \), we have

\[
X^\epsilon \Rightarrow \hat{X}, \quad \text{as } \epsilon \to 0^+.
\]

The limit process \( \hat{X} \) is a Lévy process starting from 0 with Lévy triplet \((\hat{b}, 0, \hat{\nu})\) given by

\[
\begin{cases}
\hat{b} = 1_{(0,1)}(\alpha) \int_{B^c \setminus \{0\}} \kappa(z) J(z) dz + 1_{(1,2)}(\alpha) \hat{c}, \\
\hat{\nu}(dz) = \kappa(z) J(z) dz,
\end{cases}
\]

with homogenized coefficients

\[
\kappa(z) := \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \kappa_0(x, z, u) d\mu(dx),
\]

\[
\hat{c} := \int_{\mathbb{T}^d} \left( I + \nabla \hat{b}(x) \right) \cdot c(x) \mu(dx) + \int_{\mathbb{T}^d} z \cdot \left( \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla \hat{b}(x) \kappa_0(x, z, u) d\mu(dx) \right) J(z) dz,
\]

where \( \hat{b} \) is uniquely determined by (3.1).

**Proof.** (i). We first prove the theorem for the case that \( b \equiv 0 \) or \( \alpha \in (0, 1] \). By [7, Theorem 2.44], we know that the semimartingale characteristics of \( X^\epsilon \) relative to the truncation function \( 1_B \) are \((B^\epsilon, 0, \nu^\epsilon)\), where

\[
\begin{cases}
B^\epsilon_t = 1_{(0,1)}(\alpha) \int_0^t \int_{B^c \setminus \{0\}} z \kappa^\epsilon \left( \frac{X^\epsilon_s}{\epsilon}, \frac{z}{\epsilon}, \frac{z}{\epsilon} \right) J(z) dz ds + 1_{(1,2)}(\alpha) \int_0^t \frac{X^\epsilon_s}{\epsilon} dz ds, \\
\nu^\epsilon(dz, dt) = \kappa^\epsilon \left( \frac{X^\epsilon_s}{\epsilon}, \frac{z}{\epsilon}, \frac{z}{\epsilon} \right) J(z) dz dt.
\end{cases}
\]
By applying the functional central limit theorem in [22, Theorem VIII.2.17], we only need to show that for all \( t \in \mathbb{R}^+ \) and every bounded continuous function \( f : \mathbb{R}^d \to \mathbb{R} \) vanishing in a neighbourhood of the origin, the following convergences hold in probability when \( \epsilon \to 0^+ \):

\[
\sup_{0 \leq s \leq t} |B^\epsilon_s - \tilde{b}_s| \to 0,
\]

(3.3)

\[
\int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(z) \mu^\epsilon(dz, ds) \to t \int_{\mathbb{R}^d \setminus \{0\}} f(z) \nu(dz).
\]

(3.4)

Clearly, by Corollary 2.6 we have

\[
\int_0^t c \left( \frac{X^\epsilon_t}{\epsilon} \right) ds \to t \int_{\mathbb{T}^d} c(x) \mu(dx), \text{ in probability, as } \epsilon \to 0^+.
\]

(3.5)

We also have the following convergence in probability when \( \alpha \in (0,1) \),

\[
\int_0^t \int_{B \setminus \{0\}} z\kappa^\alpha \left( \frac{X^\epsilon_s}{\epsilon}, z, \frac{z}{\epsilon^2}, \frac{z}{\epsilon} \right) J(z) dz ds
\]

\[
\frac{\epsilon_1 \to 0^+}{\epsilon_2 \to 0^+} \to \frac{t}{\int_{B \setminus \{0\}}} \left[ \int_{\mathbb{T}^d} \kappa^\alpha \left( x, z, \frac{z}{\epsilon^2}, \frac{z}{\epsilon} \right) \mu(dx) \right] z J(z) dz \quad \text{(by Corollary 2.6)}
\]

\[
\frac{\epsilon_2 \to 1}{\epsilon_1 \to 0^+} \to \frac{t}{\epsilon_1} \int_{B \setminus \{0\}} \left[ \int_{\mathbb{T}^d} \kappa_0 \left( x, z, \frac{z}{\epsilon^2}, \frac{z}{\epsilon} \right) \mu(dx) \right] z J(z) dz \quad \text{(by (2.3) and dominated convergence)}
\]

\[
\frac{\epsilon_1 \to 0^+}{\epsilon_2 \to 0^+} \to \frac{t}{\epsilon_1} \int_{B \setminus \{0\}} \left[ \int_{\mathbb{T}^d} \kappa_0 \left( x, z, u \right) du \mu(dx) \right] z J(z) dz \quad \text{(by Lemma 3.1.(i))}
\]

In the second arrow, to apply Lemma 3.1.(i) we take \( K = B \), and \( \phi(z,u) = \kappa_0(x,z,u)z )J(z) \) for fixed \( x \). Choose \( p' \in (1, \frac{d}{d-1}) \), then it is easy to verify from (2.1) and (2.9) that for each \( u \), \( \phi(\cdot,u) \in L^{p'}(K) \), and for each \( z \), \( \phi(z,\cdot) \in L^p([0,1]^d) \). Moreover, all convergences above are uniform with respect to \( t \) in closed intervals. This proves the assertion (3.3). The assertion (3.4) follows in a similar fashion but with Lemma 3.1.(ii) in place of Lemma 3.1.(i) and letting \( \phi(z,u) = \kappa_0(x,z,u) z )J(z) \).

(ii). We prove the general case that \( \hat{b} \neq 0 \) and \( \alpha \in (1,2) \). Define \( \hat{X}^\epsilon_t := X^\epsilon_t + \hat{b}_t \left( X^\epsilon_t \right) \), the boundedness of \( \hat{b} \) yields that \( \hat{X}^\epsilon_t \) and \( X^\epsilon_t \) have the same limit. Applying Corollary B.1, Lemma B.2 and (2.10), we have

\[
\hat{X}^\epsilon_t = \int_0^t c \left( \frac{X^\epsilon_s}{\epsilon} \right) ds + \int_0^t \frac{1}{\epsilon^{\alpha-1}} \left( \hat{A}^\epsilon \hat{b} - \hat{A} \hat{b} \right) \left( \frac{X^\epsilon_s}{\epsilon} \right) ds
\]

\[
\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} c \left[ \hat{b}_t \left( X^\epsilon_s + 1_{0,\kappa(X^\epsilon_s,\epsilon^2,\epsilon)}(r) \right) \right] \tilde{N}(dz,dr,ds)
\]

\[
\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_{0,\kappa(X^\epsilon_s,\epsilon^2,\epsilon,\epsilon)}(r) z \tilde{N}(dz,dr,ds)
\]

\[
\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} 1_{0,\kappa(X^\epsilon_s,\epsilon^2,\epsilon,\epsilon)}(r) z N(dz,dr,ds)
\]

\[
=: I_1^\epsilon(t) + I_2^\epsilon(t) + I_3^\epsilon(t) + I_4^\epsilon(t) + I_5^\epsilon(t),
\]

where \( N \) is a Poisson random measure on \( \mathbb{R}^d \times [0,\infty) \times [0,\infty) \) with intensity measure \( J(z)dz \times m \times m \) and \( \tilde{N} \) is the associated compensated Poisson random measure. The convergence of \( I_1^\epsilon \) is shown in (3.5). For \( I_2^\epsilon \), we derive in a similar way as (2.11),

\[
\frac{1}{\epsilon^{\alpha-2}} \left( \hat{A}^\epsilon \hat{b} - \hat{A} \hat{b} \right) \left( x/\epsilon \right)
\]

\[
= \frac{1}{\epsilon^{\alpha-1}} \int_{\mathbb{R}^d \setminus \{0\}} \left[ \hat{b}(x/\epsilon + z) - \hat{b}(x/\epsilon) - z \cdot \nabla \hat{b}(x/\epsilon) \right] \left( \kappa(x/\epsilon,\epsilon z, z) - \kappa_1(x/\epsilon, z) \right) J(z) dz
\]

\[
+ \left[ c(x/\epsilon) + \int_{\mathbb{R}^d} z \kappa(x/\epsilon, z, \epsilon z) J(z) dz \right] \cdot \nabla \hat{b}(x/\epsilon),
\]
where the first term at right hand side converges to 0 uniformly in $x$ as $\epsilon \to 0^+$, by assumption (2.5) and the following fact

$$|\tilde{b}(x/\epsilon + z) - \tilde{b}(x/\epsilon) - z \cdot \nabla \tilde{b}(x/\epsilon)| \leq \frac{||\tilde{b}||_{\alpha + \beta}}{\alpha + \beta} |x|^\alpha |z|^\beta.$$

Using the same argument as in the proof of (3.3), we also have the following locally uniform convergence in $t$ in probability, as $\epsilon \to 0^+$,

$$I_2(t) \sim \int_0^t \left\{ \frac{c}{\epsilon} \left( X_s^\epsilon \right) + \int_{B^c} z \kappa \left( \frac{X_s^\epsilon}{\epsilon}, z, \frac{z}{\epsilon} \right) J(z)dz \right\} ds \to t \left[ \int_{T^d} c(x) \cdot \nabla \tilde{b}(x) \mu(dx) + \int_{T^d} \left( \int_{T^d} \nabla \tilde{b}(x) \kappa_0(x, z, u) du \mu(dx) \right) J(z)dz \right].$$

For $I_3$, we use Itô’s isometry to get

$$\mathbb{E}(|I_3(t)|^2) = \mathbb{E} \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left| \epsilon \left[ \tilde{b}_\epsilon (X_{s-}^\epsilon + 1_{[0, \kappa(X_{s-}^\epsilon, z, z/\epsilon)]}(r)z) - \tilde{b}_\epsilon (X_{s-}^\epsilon) \right] \right|^2 J(dz)dr ds$$

$$= \mathbb{E} \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \epsilon^2 \left| \tilde{b}_\epsilon (X_{s-}^\epsilon + z) - \tilde{b}_\epsilon (X_{s-}^\epsilon) \right|^2 \kappa \left( \frac{X_{s-}^\epsilon}{\epsilon}, z, \frac{z}{\epsilon} \right) J(dz)ds$$

$$\leq \kappa_2 j_2 t \left( 4 ||\tilde{b}_\epsilon||^2_\alpha + \frac{||\tilde{b}_\epsilon||^2_\alpha}{\epsilon^2} \right) \epsilon^2.$$
Consider the following SDE 

\[ X_t^{x,\epsilon} = x + \int_0^t \left( \frac{1}{\epsilon^a} b \left( \frac{X_s^{x,\epsilon}}{\epsilon} \right) + c \left( \frac{X_s^{x,\epsilon}}{\epsilon} \right) \right) ds 
+ \int_0^t \int_{B(\epsilon)} \sigma \left( \frac{X_s^{x,\epsilon}}{\epsilon}, y \right) \tilde{N}^\alpha(dy, ds) + \int_0^t \int_{B^c} \sigma \left( \frac{X_s^{x,\epsilon}}{\epsilon}, y \right) N^\alpha(dy, ds), \]

(4.1)

where the functions \( b, c \) are all periodic of period 1, the function \( \sigma(x, y) \) is periodic in \( x \) of period 1, and odd in \( y \) in the sense that \( \sigma(x, -y) = -\sigma(x, y) \) for all \( x, y \in \mathbb{R}^d \). We assume that \( \sigma \in C^{1,2}(\mathbb{R}^d \times \mathbb{R}^d) \) and there exists constants \( C_1 > 0, C_2 > 1 \), such that for all \( x_1, x_2, x, y \in \mathbb{R}^d \),

\[ |\sigma(x_1, y) - \sigma(x_2, y)| \leq C_1|x_1 - x_2||y|, \quad C_2^{-1}|y| \leq |\sigma(x, y)| \leq C_2|y|. \]

Assume in addition that for every \( x \), \( \sigma(x, \cdot) \) is uniformly continuous and is \( C^2 \)-diffeomorphism with inverse \( \tau(x, \cdot) := \sigma(x, \cdot)^{-1} \). Then we know that (4.1) possesses a unique strong solution which is a Feller process, for each \( \epsilon > 0 \), see [21, Theorem 4.2, Corollary 4.3].

Now the generator of the solution processes \( X^{x,\epsilon} \) restricted to \( C^\infty(\mathbb{T}^d) \) is

\[ \mathcal{A}_\epsilon f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f \left( x + \sigma \left( \frac{x}{\epsilon}, y \right) \right) - f(x) - \sigma \left( \frac{x}{\epsilon}, y \right) \cdot \nabla f(x)1_B(y) \right] \nu^\alpha(dy) 
+ \left[ \frac{1}{\epsilon^{a-1}} b \left( \frac{x}{\epsilon} \right) + c \left( \frac{x}{\epsilon} \right) \right] \cdot \nabla f(x). \]

We can rewrite it, by a change of variables and the oddness of \( \sigma(x, y) \), to the form in (1.1) with

\[ \kappa(x, z, u) \equiv \kappa(x, z) := |\det \nabla_z \tau(x, z)| \frac{|z|^{d+a}}{|\tau(x, z)|^{d+a}}, \]

(4.2)

that is,

\[ \int_A \kappa(x, z) \frac{dz}{|z|^{d+a}} = \int_{\mathbb{R}^{d\setminus\{0\}}} 1_A(\sigma(x, y)) \nu^\alpha(dy), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \]

(4.3)

Then the function \( \kappa \) satisfies assumptions (2.1) and (2.2) (see [21, Assumption H3, Lemma 2.3, Proposition 2.5]), as well as assumption (2.3) with \( \kappa_0(x, z, u) \equiv \kappa(x, z) \). Note that for each \( x \), the oddness of \( \sigma(x, \cdot) \) implies the oddness of \( \tau(x, \cdot) \), and further the symmetry of \( \kappa(x, \cdot) \) in the sense that

\[ \kappa(x, z) = \kappa(x, -z) \quad \text{for all} \quad x, z. \]

We assume further that (cf. [21, Assumption H5]),

\[ \frac{1}{\epsilon} \sigma(x, \epsilon y) \to \nabla_y \sigma(x, 0) \cdot y, \quad \text{uniformly in} \ x \text{ and} \ y, \quad \text{as} \ \epsilon \to 0^+, \]

Then we can prove easily (e.g., by [28, Theorem 7.17]) that for each \( z \),

\[ \frac{1}{\epsilon} \tau(x, \epsilon z) \to \nabla_z \tau(x, 0) \cdot z \quad \text{and} \quad \nabla_z \tau(x, \epsilon z) \to \nabla_z \tau(x, 0), \quad \text{uniformly in} \ x, y, \quad \text{as} \ \epsilon \to 0^+. \]

Hence, we conclude that the function \( \kappa \) defined in (4.2) satisfies assumption (2.5) with

\[ \kappa_1(x, z) \equiv \kappa_1(x) := |\det \nabla_z \tau(x, 0)| \frac{1}{|\nabla_z \tau(x, 0)|^{d+a}}. \]

Applying Theorem 3.2, we know that the sequence of solutions \( X^{x,\epsilon} \) converges in distribution to a Lévy process \( \tilde{X}^x \) starting from \( x \) with Lévy triplet \((\tilde{b}, 0, \nu)\) given in (3.2). By the symmetry of \( \kappa \) and \( \nu^\alpha \), the homogenized constant \( \tilde{b} = \int_{\mathbb{T}^d} (1 + \nabla b(x)) \cdot c(x) \mu(dx) \), where \( \mu \) is the invariant measure of the Feller process generated by

\[ \tilde{\mathcal{A}} \alpha f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f \left( x + \nabla_y \sigma(x, 0) \cdot y \right) - f(x) - y \cdot \nabla_y \sigma(x, 0) \cdot \nabla f(x)1_B(y) \right] \nu^\alpha(dy) + b(x) \cdot \nabla f(x), \]

10
and \( \hat{b} \) is the unique solution to the Poisson equation \( \hat{A}_\alpha \hat{b} = b \). Moreover, the homogenized function is \( \overline{\kappa}(z) = \int_{\mathbb{R}^d} \kappa_0(x, z) \mu(dx) \). This coincides with the result in [21, Theorem 5.2]. To see this, we derive \( \overline{\nu}(A) \) for \( A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \) by (4.3),

\[
\overline{\nu}(A) = \int_A \overline{\kappa}(z) \frac{dz}{|z|^{d+\alpha}} = \int_A \int_{\mathbb{T}^d} \kappa_0(x, z) \mu(dx) \frac{dz}{|z|^{d+\alpha}} = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{T}^d} 1_A(\sigma(x, y)) \mu(dx) \nu^\alpha(dy).
\]

In particular, this also generalizes the result in [16], where the author consider the special case \( \sigma(x, y) = \sigma_0(x)y \).

\[\square\]

**Example 4.3** (Stable-like processes with variable order). Let \( \alpha \in C(\mathbb{R}^d) \) be a periodic function of period 1, taking values in \((0, 2)\). Define a family of measures

\[
\eta(x, A) = \int_S \rho(x, \xi) d\xi \int_0^\infty 1_A(r\xi) \frac{dr}{r^{1+\alpha(x)}} \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),
\]

(4.4)

where \( \rho : \mathbb{R}^d \times S \to [0, \infty) \) is periodic of period 1 in the first variable. We refer to the function \( \rho(x, \xi) \) as the spectral measure. In general, we do not assume the function \( \rho \) to be symmetric in the second variable in the sense that \( \rho(x, \xi) = \rho(x, -\xi) \) for all \( x \) and \( \xi \), while this is assumed in [15]. Note that for each \( x \in \mathbb{R}^d \), \( \eta(x, \cdot) \) is a stable Lévy measure with stability index \( \alpha(x) \). Consider the following pseudo-differential operator defined for \( f \in C^\infty(\mathbb{T}^d) \),

\[
A^\eta f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + z) - f(x) - z \cdot \nabla f(x) 1_{[1,2]}(\alpha) 1_B(z) \right] \eta(x, dz).
\]

(4.5)

To write the operator \( A^\eta \) to the form (1.1), let \( \alpha_0 := \inf_{x \in \mathbb{R}^d} \alpha(x) \) (the infimum is attainable since \( \alpha \) is continuous and periodic), and define

\[
\kappa^*(x, z, u, v) \equiv \kappa^*(x, v) := \frac{\rho(x, v/|v|)}{J_0(|v|)|v|^{\alpha_0 - \alpha(x)}},
\]

(4.6)

where \( J_0 \) is a Lévy density related to \( \alpha_0 \) in the manner of Assumption H3. Then for \( A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \),

\[
\int_A \kappa^*(x, z) J_0(z) dz = \int_0^\infty \int_S 1_A(r\xi) \left( \frac{\rho(x, \xi) J_0(|\xi|)}{J_0(|\xi|)|\xi|^{\alpha_0 - \alpha(x)}} \right) J_0(r\xi) r^{d-1} d\xi dr = \eta(x, A).
\]

We assume that the function \( \kappa^* \) in (4.6) satisfies assumptions (2.1) and (2.2), so that the operator \( A^\eta \) generates a Feller process \( X \) on \( \mathbb{T}^d \). This process is called stable-like, since locally it looks like a stable processes (see [15] and the literatures therein). It is easy to check that for each \( x \) and \( z \),

\[
\kappa^*(x, z/\epsilon) \to \frac{\rho(x, z/|z|)}{J_0(z/|z|)} 1_{\{\alpha = \alpha_0\}}(x), \quad \epsilon \to 0^+.
\]

Thus, assumption (2.3) holds by taking

\[
\kappa_0(x, z, u) \equiv \kappa_0(x, z) := \frac{\rho(x, z/|z|)}{J_0(z/|z|)} 1_{\{\alpha = \alpha_0\}}(x),
\]

while assumptions (2.4) and (2.5) hold trivially with \( \kappa_1 = \kappa \). Let \( \eta^e(x, dz) := \kappa^*(x/\epsilon, z/\epsilon) J_0(\epsilon) dz \). Then \( \eta^e \) is of the form (4.4) with \( \alpha \) and \( \rho \) replace by

\[
\alpha_\epsilon(x) := \alpha(x/\epsilon), \quad \rho^e(x, \xi) := \epsilon^{\alpha(x/\epsilon) - \alpha_0} \rho(x/\epsilon, \xi).
\]

Let \( A^{\eta^e} \) be the operator associated to \( \eta^e \) as in (4.5), denote by \( X^e \) the Feller process generated by \( A^{\eta^e} \). Then Theorem 3.2 implies that the processes \( X^e \) converge in distribution to a Lévy process \( \hat{X} \) with Lévy triplet \((\hat{b}, 0, \hat{\nu})\), where for \( A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \),

\[
\hat{\nu}(A) = \int_A \left( \int_{\mathbb{T}^d} \kappa_0(x, z) \mu(dx) \right) J_0(z) dz
\]

\[
= \int_{\mathbb{T}^d} \int_0^\infty \int_S 1_A(r\xi) \frac{\rho(x, \xi) J_0(|\xi|) r^{-(1+\alpha_0)} d\xi dr}{r^{1+\alpha_0}} \mu(dx)
\]

\[
= \int_{\mathbb{T}^d} 1_{\{\alpha = \alpha_0\}}(x) \eta(x, A) \mu(dx),
\]
the measure $\mu$ is the invariant measure of the Feller process generated by

$$\tilde{\mathcal{A}}^\eta f(x) = \int_{\mathbb{R}\setminus\{0\}} [f(x + z) - f(x) - z \cdot \nabla f(x) \left(\mathbf{1}_{(1)}(\alpha)\mathbf{1}_B(z) + \mathbf{1}_{(1,2)}(\alpha)\right)] \eta(x, dz).$$

In the special case that $\rho(x, \cdot)$ is symmetric for each $x$, we have $\tilde{\mathcal{A}}^\eta = \mathcal{A}^\eta$, $\bar{b} = 0$, and the result coincides with that in [15, Theorem 1]. Last but not least, the limit Lévy process $\bar{X}$ is $\alpha_0$-stable, its spectral density is

$$\tilde{\rho}(\xi) = \int_{\mathbb{T}^d} \mathbf{1}_{(\alpha = \alpha_0)}(x)\rho(x, \xi)\mu(dx).$$

\[\square\]

**Example 4.4** (One-dimensional stable-like Feller processes). Consider the one-dimensional case with $\alpha \in (1, 2)$, $c \equiv 0$ and $\kappa^*(x, z, u, v) \equiv \kappa^*(x, v)$, that is,

$$\mathcal{A}^\eta_{1d} f(x) = \int_{-\infty}^{+\infty} [f(x + z) - f(x) - zf'(x)\mathbf{1}_{(|z| < 1)}(z)] \kappa^*(\xi, \xi, \eta) J(z) dz + \frac{1}{c(x)} b(\xi) f'(x).$$

Here $J$ is the $\alpha$-stable Lévy density on $\mathbb{R} \setminus\{0\}$ (see [29, Remark 14.4]), that is,

$$J(z) = j^+ z^{-(1+\alpha)}\mathbf{1}_{(0, +\infty)}(z) + j^- |z|^{-(1+\alpha)}\mathbf{1}_{(-\infty, 0)},$$

with constants $j^+, j^- > 0$, so that assumption (2.7) is fulfilled.

Besides assumptions (2.1), (2.2), H1 and H4, we assume further that there exists two functions $\kappa^+, \kappa^-$ : $\mathbb{R}^d \setminus\{0\} \to [0, \infty)$ such that for each $x$,

$$\lim_{y \to \pm \infty} y^{-1}\int_0^y \kappa^*(x, v) dv = \kappa_0^+(x).$$

Note that this is the type of assumption in [20]. Then by L'Hôpital's rule, we have

$$\lim_{v \to \pm \infty} \kappa^*(x, v) = \kappa_0^+(x).$$

Thus, our assumption (2.3) is fulfilled by letting

$$\kappa_0^0(x, z, u) \equiv \kappa_0^0(x, z) := \kappa_0^+(x)\mathbf{1}_{(0, +\infty)}(z) + \kappa_0^-(x)\mathbf{1}_{(-\infty, 0)}(z).$$

And assumption (2.4) hold trivially. Now using Theorem 3.2, the Feller process generated by $\mathcal{A}^\eta_{1d}$ converges in distribution, as $\epsilon \to 0^+$, to a one-dimensional $\alpha$-stable Lévy process $\bar{X}$ with Lévy triplet $(\bar{b}, 0, \bar{\nu})$ in (3.2).

Let $\mu$ be the invariant measure of the Feller process generated by

$$\tilde{\mathcal{A}}_{1d} f(x) = \int_{-\infty}^{+\infty} [f(x + z) - f(x) - zf'(x)] \kappa^*(x, z) J(z) dz + b(x)f'(x).$$

Then the homogenized drift $\bar{b}$ is

$$\bar{b} = \frac{1}{\alpha - 1} \int_0^1 (j^+ \kappa^+(x) + j^- \kappa^-(x)) \hat{b}'(x) \mu(dx),$$

where $\hat{b}$ is the unique solution to the Poisson equation $\tilde{\mathcal{A}}_{1d} \hat{b} = \bar{b}$. Define two constants $\bar{\kappa}^\pm := \int_{\mathbb{T}^d} \kappa_0^\pm (x) \mu(dx)$, then

$$\bar{\kappa}(z) = \bar{\kappa}^+(\mathbf{1}_{(0, +\infty)}(z) + \bar{\kappa}^-(\mathbf{1}_{(-\infty, 0)}(z).$$

Note that the authors in [20] consider the operators of the form $\tilde{\mathcal{A}}_{1d}$ with $\kappa^*(\xi, \xi)$ and $\frac{1}{c(x)} b(\xi)$ in place of $\kappa^*(x, z)$ and $b(x)$, which is slightly different from $\mathcal{A}^\eta_{1d}$, but the homogenized jump measure therein coincides with $\bar{\nu}$.
5 Numerical simulations

In this section, we will present a numerical experiment to give some visualization for the homogenization result. Besides, for a jump particle in a periodic structure, a typical question in practical applications is the distribution property of the first exit time for the particle to escape a given domain. We will also give some visualization for the empirical mean of the first exit time. For simplicity, we will use the stable-like processes $X^\epsilon$ with variable order in Example 4.3 for instance.

5.1 Numerical scheme

Set the dimension $d = 2$, and

$$\alpha(x) = 1 + \frac{1}{4} \left(1_{\{|x_1-\frac{1}{2}| > \frac{1}{4}\}} \cos(x_1) - 1_{\{|x_1-\frac{1}{2}| \leq \frac{1}{4}\}} + 1_{\{|x_2-\frac{1}{2}| > \frac{1}{4}\}} \cos(x_2) - 1_{\{|x_2-\frac{1}{2}| \leq \frac{1}{4}\}} \right), \quad x \in [0,1]^2,$$

$$\rho(x,d\xi) = \sum_{i=1}^{4} \delta_{e_i}(d\xi), \quad \xi \in S^1.$$

where $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (-1,0)$, $e_4 = (0,-1)$, they form an orthonormal basis for $\mathbb{R}^2$. Then the minimum of $\alpha$ is $\alpha_0 = \frac{1}{2}$, and $\rho$ is symmetric in $\xi$. The spectral measure of the process $X^\epsilon$ is $\rho'(x,d\xi) = e^{\alpha(x/\epsilon)} \rho(d\xi)$, while the spectral measure of the limit process $\bar{X}$ is $\bar{\rho}(d\xi) = \mu(\alpha^{-1}(\frac{1}{2})) \rho(d\xi)$.

We use the method in [25] to simulate the ‘one-step’ stable random vectors, and then use the method in [6] to simulate each Feller process $X^\epsilon$ by gluing all ‘one-step’ stable random vectors together. Note that the distribution of each ‘one-step’ stable random vector depends on the position of the previous step. As for the limit Lévy process $\bar{X}$, the ‘one-step’ stable random vectors are independent of the previous positions. The following figure (Fig. 1) shows the sample paths on the plane for the processes $X^\epsilon$ with $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and the limit process $\bar{X}$. As we can see, when the scaling parameter $\epsilon$ gets smaller and smaller, the path of $X^\epsilon$ is getting more and more concentrated into some small clusters.

![Sample paths for $X^\epsilon$ and $\bar{X}$](image.png)

Fig. 1. Sample paths for $X^\epsilon$ and $\bar{X}$ on the time interval $[0,10]$ with time-step size 0.01. The coordinates represent the particle positions in $\mathbb{R}^2$. 

13
5.2 Simulations of first exit time

For $x \in \mathcal{D}$ and $r > 0$, define

$$S_r(x) := \inf\{t \geq 0 : |x(t)| \geq r \text{ or } |x(t^-)| \geq r\},$$
$$S_{r+}(x) := \inf\{t \geq 0 : |x(t)| > r \text{ or } |x(t^-)| > r\},$$
$$V(x) := \{r > 0 : S_r(x) < S_{r+}(x)\}.$$

It is easy to see that

$$S_r(x) = \inf\left\{t \geq 0 : \sup_{0 \leq s \leq t} |x(s)| \geq r \right\},$$

which is exact the first exit time for the path $x$ to escape the ball of radius $r$. In order to simplify the notation as before, we denote $X^0 := \bar{X}$. Using [22, Lemma VI.2.10], we know that for all $\epsilon \geq 0$ and $\omega \in \Omega$, $V(X^\epsilon(\omega))$ is an at most countable subset of $\mathbb{R}_+$. It follows that each set

$$U^\epsilon = \{r > 0 : \mathbb{P}(r \in V(X^\epsilon)) = 0\},$$

has full measure in $\mathbb{R}_+$, and thus we have

**Lemma 5.1.** The set $\cap_{\epsilon > 0} U^\epsilon$ also has full measure in $\mathbb{R}_+$.

Now for each $r \in \cap_{\epsilon > 0} U^\epsilon$, the mapping $X^\epsilon \mapsto S_r(X^\epsilon)$ is continuous for all $\epsilon \geq 0$, by virtue of [22, Proposition VI.2.11]. Hence, by the continuous mapping theorem (see, e.g., [14, Corollary 3.1.9]), we have the following corollary, of which the second statement follows from [4, Theorem 25.12].

**Corollary 5.2.** For each $r \in \cap_{\epsilon > 0} U^\epsilon$, $S_r(X^\epsilon) \Rightarrow S_r(\bar{X})$ as $\epsilon \to 0^+$. If in addition, the family $\{S_r(X^\epsilon)\}_{\epsilon > 0}$ is uniformly integrable, then $E(S_r(X^\epsilon)) \to E(S_r(\bar{X}))$.

We choose $r = \pi$. The following figure (Fig. 2) shows the empirical mean of the first exit time to escape the ball of radius $\pi$ for the processes $X^\epsilon$, $\epsilon = 10^{-6}, 10^{-12}, 10^{-18}, 10^{-24}, 10^{-30}$, and the limit process $\bar{X}$. Through this figure, we can see that as $\epsilon$ gets smaller, the convergence rate of the empirical mean with respect to the number of samples is getting slower.

The next figure (Fig. 3) shows the trend of the empirical mean of $S_{\pi}(X^\epsilon)$, $\epsilon = 10^{-n}$, with respect to $n$. It follows that the difference of empirical mean of $S_{\pi}(X^\epsilon)$ with that of $S_{\pi}(\bar{X})$ is almost inversely proportional to $n$, so as to be proportional to $\frac{1}{\log(\epsilon^{-1})}$. Even though this rate is not strict, we can still conclude from the figure that the convergence rate of the mean first exit time with respect to $\epsilon$ is very slow.

Therefore, the way of simulating the first exit time of a particle in a periodic structure by choosing a very small $\epsilon$ is quite expensive (in computational time) and not precise in general. The advantage of our homogenization result is that we can use directly the limit process we have just identified to study its distribution properties, instead of using approximations.

**Appendix A** Properties of the semigroups and generators

As we have seen in Section 2, we need the Feller nature of the semigroups and the properties of the generators, in order to study the ergodicity of the canonical Feller processes, see Proposition 2.2 and Proposition 2.4. We devote this section to investigate the semigroups and generators. As corollaries, we can also obtain the solvability of the Possion equations with zeroth-order terms and the generalized Itô’s formula, which are used in Corollary 2.8 and the proof of our main result Theorem 3.2.

We consider the following operator

$$\mathcal{L}f(x) = \mathcal{L}^{b,\eta}f(x) := b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} [f(x + z) - f(x) - z \cdot \nabla f(x) \mathbf{1}_B(z)] \eta^d(x, z) J(z) dz,$$  \hspace{1cm} (A.1)

with $\eta(x, dz) := \kappa^d(x, z) J(z) dz$. We suppose that the vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the Hölder class $C^\beta$ with some $\beta \in (0, 1)$ and periodic of period 1, $J$ satisfies (2.9), that is,

$$j_1|z|^{-(d+\alpha)} \leq J(z) \leq j_2|z|^{-(d+\alpha)}, \quad z \in \mathbb{R}^d \setminus \{0\},$$

for some $j_1, j_2 > 0$.
Fig. 2. Empirical mean of the first exit time for $X^\epsilon$ and $\bar{X}$. The horizontal and vertical coordinates indicate the number of test samples and the average to the present, respectively. The labeled points is the values of empirical mean of 200 samples with time-step size 0.01.

Fig. 3. Empirical mean of the first exit time for $X^\epsilon$ with $\epsilon = 10^{-n}$, $n = 1, 2, \ldots, 30$. The horizontal coordinate indicates the parameter $n$ and the vertical coordinate indicate the empirical mean of $S_\pi(X^\epsilon)$. The dashed line is the reference of empirical mean of $S_\pi(\bar{X})$. All empirical mean are simulated with 200 samples with time-step size 0.01.

with $\alpha \in (1, 2)$, $\kappa^\alpha(x, z)$ is periodic in $x$ of period 1 and satisfies similar conditions as (2.1) and (2.2), that
is, for all \(x, x_1, x_2, z \in \mathbb{R}^d\),
\[
\kappa_1 \leq \kappa^d(x, z) \leq \kappa_2,
\]
|\(\kappa^d(x_1, z) - \kappa^d(x_2, z)\)| \(\leq \kappa_3|x_1 - x_2|^\beta\).

Note that the operators \(\tilde{A}, A^t\) and \(\hat{A}^t\) are all of the form (A.1) by choosing appropriate \(\kappa^d\). It is easy to verify that \(L^{b, \eta}f \in C(\mathbb{T}^d)\) for each \(f \in C^{1+\gamma} (\mathbb{T}^d)\) with \(1 + \gamma > \alpha\). Now we treat \(L^{b, \eta}\) as a perturbation of \(L^n := L^{b, \eta}\) by the gradient operator \(L^b := L^{b, 0} = b \cdot \nabla\), and follow [5, 8, 18] to investigate the heat kernel for \(L^{b, \eta}\).

We introduce the following functions on \((0, \infty) \times \mathbb{R}^d\) for later use:
\[
\theta_{\gamma}(t; x) := t^{\gamma/\alpha} \left( t^{-(d+\alpha)/\alpha} \wedge |x|^{-(d+\alpha)} \right), \quad \gamma \in \mathbb{R}.
\]
For abbreviation we write \(c_0\) for the set of constants \((d, \alpha, \beta, \kappa_1, \kappa_2, \kappa_3, j_1, j_2)\). Before investigating the semigroups generated by \(L^{b, \eta}\), we need some facts for the heat kernels of \(L^n\) and \(L^{b, \eta}\).

By virtue of the periodicity assumptions on the coefficients, we can choose the underlying space to be \(\mathbb{T}^d\) instead of \(\mathbb{R}^d\) (cf. [3, Section 3.3.2]). Indeed, if \(q^n(t; x, y) : [0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}\) is the fundamental solution of \(L^n\), then it must be periodic in \(x\) due to the periodicity of all coefficients in \(L^n\). Now we define \(q^n(t; x, y) := \sum_{l \in \mathbb{Z}^d} q^n_l(t; x, y + l)\), then \(q^n\) is periodic in both \(x\) and \(y\). Therefore, we can treat \(q^n\) as a function from \([0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d\) to \(\mathbb{R}\), which is exactly the fundamental solution of \(L^n\) on the state space \(\mathbb{T}^d\). The same arguments hold for the operator \(L^{b, \eta}\). Keeping these in mind, the following facts for the operator \(L^n\) are adapted from [18, Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4, Remark 1.5, Lemma 3.17].

**Proposition A.1.** (i). The fundamental solution \(q^n(t; x, y) : [0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}\) of \(L^n\) has the following properties: for all \((t, y) \in (0, \infty) \times \mathbb{T}^d\), the function \(x \to q^n(t; x, y)\) is differentiable and the derivative \(\nabla_x q^n(t; x, y)\) is jointly continuous on \((0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d\); the integral in \(\mathbb{T}^d q^n(t; x, y)\) is absolutely integrable and the function \(\mathbb{T}^d q^n(t; x, y)\) is jointly continuous on \((0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d\). For every \(T > 0\), there exist a constant \(C = C(c_0, T) > 0\), such that for all \(t \in (0, T]\) and \(x, y \in \mathbb{T}^d\),
\[
|\nabla_x q^n(t; x, y)| \leq C \sum_{l \in \mathbb{Z}^d} \theta_{\alpha-1}(t; x - y + l),
\]
\[
|\mathbb{T}^d q^n(t; x, y)| \leq C \sum_{l \in \mathbb{Z}^d} \theta_0(t; x - y + l).
\]

(ii). Define a family of operators by
\[
T^n_t f(x) = \int_{\mathbb{T}^d} f(y) q^n(t; x, y) dy, \quad f \in C(\mathbb{T}^d),
\]
then \(\{T^n_t\}_{t \geq 0}\) forms a Feller semigroup on the Banach space \((C(\mathbb{T}^d), \| \cdot \|_0)\) with generator the closure of \((L^n, C^\infty(\mathbb{T}^d))\). The domain of the generator contains \(C^{1+\gamma}(\mathbb{T}^d)\) with \(1 + \gamma > \alpha\), on which the restriction of the generator is \(L^n\).

Note that the joint continuity of \(\nabla_x q^n(t; x, y)\) is not mentioned explicitly in the previous references, but it is a consequence of [18, Lemma 3.1, Lemma 3.5, Theorem 3.7, Lemma 3.10, Eq. (59)]. In addition, it is only shown in the above reference that \(C^2(\mathbb{T}^d)\) is contained in the domain of the generator, but we can easily generalize to our case, using the same argument as the proofs of [18, Theorem 1.3.(3a), Proposition 4.9] and the fact that \(L^n f \in C(\mathbb{T}^d)\) for each \(f \in C^{1+\gamma}(\mathbb{T}^d)\) with \(1 + \gamma > \alpha\).

For notational simplicity, the summation over the lattice \(\mathbb{Z}^d\) will be omitted in all coming results. Keep in mind that there will be a summation over \(\mathbb{Z}^d\) when the letter \(l\) involved in the expression without ambiguity. The following facts for the heat kernel of \(L^{b, \eta}\) are adapted from [9, Theorem 1.5].

**Proposition A.2.** There is a unique function \(q^{b, \eta}(t; x, y)\) which is jointly continuous on \((0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d\) and solves the following variation of parameters formula (or Duhamel’s formula)
\[
q^{b, \eta}(t; x, y) = q^n(t; x, y) + \int_0^t \int_{\mathbb{T}^d} q^{b, \eta}(t - s; x, z) b(z) \cdot \nabla_x q^n(s; z, y) dz ds,
\]
\[
(A.5)
\]
and satisfying that for every $T > 0$, there is a constant $C = C(c_0, T, \|b\|_0) > 0$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$, 

$$|q^{b,n}(t; x, y)| \leq C\eta(t; x - y + 1).$$

Moreover, $q^{b,n}$ enjoys the following properties.

(i) (Conservativeness). For all $t > 0$, $x \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} q^{b,n}(t; x, y)dy = 1$.

(ii) (Chapman-Kolmogorov equation). For all $s, t > 0$, $x, y \in \mathbb{T}^d$,

$$\int_{\mathbb{T}^d} q^{b,n}(t; x, z)q^{b,n}(s; z, y)dz = q^{b,n}(t + s; x, y).$$

(iii) (Two-sided estimate). For every $T > 0$, there are constants $c_1, c_2 > 0$ depending only on $c_0, T, \|b\|_0$, such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,

$$c_1\eta(t; x - y + 1) \leq q^{b,n}(t; x, y) \leq c_2\eta(t; x - y + 1),$$

(iv) (Gradient estimate). The function $\nabla x q^{b,n}(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$. For every $T > 0$, there is a constant $C = C(c_0, T, \|b\|_0) > 0$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,

$$|\nabla x q^{b,n}(t; x, y)| \leq C\eta_{-1}(t; x - y + 1).$$

**Corollary A.3.** The following version of variation of parameters formula holds,

$$q^{b,n}(t; x, y) = q^n(t; x, y) + \int_0^t \int_{\mathbb{T}^d} q^n(t - s; x, z)b(z) \cdot \nabla x q^{b,n}(s; z, y)dzds.$$  \hspace{1cm} (A.7)

The function $L^b_xq^{b,n}(t; x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{T}^d \times \mathbb{T}^d$. For every $T > 0$, there is a constant $C = C(c_0, T, \|b\|_0) > 0$ such that on $(0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,

$$|L^b_xq^{b,n}(t; x, y)| \leq C\eta(t; x - y + 1).$$  \hspace{1cm} (A.8)

**Proof.** The equation (A.7) follows from a similar argument as the proof of (A.5), cf. [8, Theorem 4.2]. We prove (A.8). Recall that $L^b_x = L^n + b \cdot \nabla$ and $\alpha > 1$. By (A.2), (A.3) and [18, Eq. (92)], we have for all $(t, x, y) \in (0, T] \times \mathbb{T}^d \times \mathbb{T}^d$,

$$|L^b_xq^{b,n}(t; x, y)| \leq C(c_0, T, \|b\|_0)\eta(t; x - y + 1).$$

It follows from (A.6) and [18, Eq. (92), Lemma 5.17(c)] that

$$\int_0^t \int_{\mathbb{T}^d} \left| L^b_x q^{b,n}(t - s; x, z)b(z) \cdot \nabla x q^{b,n}(s; z, y) \right| dzds \\
\leq C(c_0, T, \|b\|_0) \int_0^t \int_{\mathbb{R}^d} \frac{(t - s)\eta_{-1}(t - s; x - z + 1)s \eta_{-1}(s; z - y)}{(t - s)^{1/2}} dzds \\
\leq C(c_0, T, \|b\|_0) B\left(1 - \frac{a_1}{2}, \frac{1}{2}\right)\eta_{-1}(t; x - y + 1) \\
\leq C(c_0, T, \|b\|_0)\eta(t; x - y + 1).$$

Combining these estimates with (A.7), we get (A.8). The joint continuity of $L^b_xq^{b,n}(t; x, y)$ follows from the jointly continuity of $L^b_xq^{n}(t; x, y)$, $\nabla x q^n(t; x, y)$ and $\nabla x q^{b,n}(t; x, y)$ and (A.7). \Box

Define a family of operators

$$T^b_{f}(t; \cdot, y)f(y)dy, \quad f \in C(\mathbb{T}^d).$$  \hspace{1cm} (A.9)

By Proposition A.2, $\{T^b_{f}(t; \cdot, y)f(y)dy\}_{t \geq 0}$ forms a (one-parameter operator) semigroup which is Markovian (positivity preserving, conservative and sub-Markovian) and Feller (each $T^b_{f}(t; \cdot, y)f(y)dy$ map $C(\mathbb{T}^d)$ to $C(\mathbb{T}^d)$). We can also prove the strong continuity. Hence, we have
Proposition A.4. The family of operators \( \{ T^b,\eta \} \) forms a Feller semigroup on \( C(\mathbb{T}^d) \). Let \( (\hat{\mathcal{L}}^b,\eta, D(\hat{\mathcal{L}}^b,\eta)) \) be the generator, then for all \( \gamma > \alpha - 1 \), \( C^{1+\gamma}(\mathbb{T}^d) \subset D(\hat{\mathcal{L}}^b,\eta) \) and \( \hat{\mathcal{L}}^b,\eta = \mathcal{L}^b,\eta \) on \( C^{1+\gamma}(\mathbb{T}^d) \). Moreover, \( C^\infty(\mathbb{T}^d) \) is a core of \( \mathcal{L}^b,\eta \).

Proof. (i). Fix \( f \in C(\mathbb{T}^d) \). For every \( \epsilon > 0 \), there is a constant \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) with \( |x - y| < \delta \), \( x, y \in \mathbb{T}^d \). Then by Proposition A.2.(i) and (iii),

\[
\sup_x \left| T^b,\eta f(x) - f(x) \right| \leq \sup_x \int_{\mathbb{T}^d} q^{b,\eta}(t; x, y) |f(y) - f(x)| dy \\
\leq \epsilon \sup_x \int_{|x-y|<\delta} q^{b,\eta}(t; x, y) dy + 2\|f\|_0 \sup_x \int_{|x-y|\geq\delta} \varrho_\alpha(t; x - y + l) dy \\
\leq \epsilon + 2\|f\|_0 \int_{|z|\geq\delta} \left( t^{-(d+\alpha)/\alpha} \wedge |z|^{-(d+\alpha)} \right) dz.
\]

When \( t \to 0^+ \),

\[
\int_{|z|\geq\delta} \left( t^{-(d+\alpha)/\alpha} \wedge |z|^{-(d+\alpha)} \right) dz \leq \int_{|z|\geq\delta} |z|^{-(d+\alpha)} dz < \infty,
\]

and then \( \| T^b,\eta f - f \|_0 \to 0 \). This proves that \( \{ T^b,\eta \} \) is strongly continuous on \( C(\mathbb{T}^d) \). Thus, \( \{ T^{b,\eta} \} \) is a Feller semigroup.

(ii). To identify the generator of \( \{ T^{b,\eta} \} \), we fix \( f \in C^{1+\gamma}(\mathbb{T}^d) \) with \( 1 + \gamma > \alpha \). We claim that for every \( g \in C^\infty(\mathbb{T}^d) \),

\[
\lim_{t \to 0} \frac{1}{t} \left( T^{b,\eta} f(x) - f(x) \right) g(x) dx = \int_{\mathbb{T}^d} \mathcal{L}^{b,\eta} f(x) g(x) dx. \tag{A.10}
\]

Then using [12, Theorem 1.24] and the fact that \( C^\infty(\mathbb{T}^d) \) is vaguely (i.e., weak-* dense in the space \( \mathcal{M}_b(\mathbb{T}^d) \) of all bounded signed Radon measures on \( \mathbb{T}^d \), which is the topological dual of \( C(\mathbb{T}^d) \)), we get that \( C^{1+\gamma}(\mathbb{T}^d) \) is contained in the domain of \( \mathcal{L}^{b,\eta} \), and the restriction of \( \mathcal{L}^{b,\eta} \) on \( C^{1+\gamma}(\mathbb{T}^d) \) equals to \( \mathcal{L}^{b,\eta} \).

Now we prove the claim (A.10). By (A.4), (A.9) and (A.5) we have

\[
\int_{\mathbb{T}^d} 1_t \left( T^{b,\eta} f(x) - f(x) \right) g(x) dx - \int_{\mathbb{T}^d} \mathcal{L}^{b,\eta} f(x) g(x) dx \\
= \int_{\mathbb{T}^d} \left[ \frac{1}{t} (T^{b,\eta} f(x) - f(x)) - \mathcal{L}^b f(x) \right] g(x) dx \\
+ \frac{1}{t} \int_{\mathbb{T}^d} \int_0^t \int_{\mathbb{T}^d} q^{b,\eta}(t-s; x, z) b(z) \cdot \nabla z q^{\eta}(z; y) f(y) dy dz ds dy - b(x) \cdot \nabla f(x) g(x) dx \\
=: I + II.
\]

The term \( I \) goes to zero, by Proposition A.1.(ii), as \( t \to 0 \). For the term \( II \), we use Fubini’s theorem and integration by parts which we can do by the periodicity of \( b, f, g, x \to q^{\eta}(t; x, y) \) and \( x \to q^{b,\eta}(t; x, y) \), then we get

\[
II = \frac{1}{t} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} q^{b,\eta}(t-s; x, z) g(x) \left[ b(z) \cdot \left( \int_{\mathbb{T}^d} \nabla z q^{\eta}(s; z, y) f(y) dy \right) - b(x) \cdot \nabla f(x) \right] dx dz ds \\
= \frac{1}{t} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} q^{b,\eta}(t-s; x, z) g(x) \int_{\mathbb{T}^d} [b(z) \cdot \nabla z q^{\eta}(s; z, y) - b(x) \cdot \nabla z q^{\eta}(s; x, y)] f(y) dy dz ds dx \\
+ \frac{1}{t} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} q^{b,\eta}(t-s; x, z) g(x) b(x) \cdot \nabla x \left[ \int_{\mathbb{T}^d} q^{\eta}(s; x, y) f(y) dy - f(x) \right] dx dz ds \\
=: II_1 + II_2.
\]

Since the function \( (s, x, y) \to b(x) \cdot \nabla x q^{\eta}(s; x, y) \) is uniformly continuous on \([0, t] \times \mathbb{T}^d \times \mathbb{T}^d\), there exists a constant \( C > 0 \), such that \( |b(x) \cdot \nabla x q^{\eta}(s; x, y)| < C \) for all \( (s, x, y) \in [0, t] \times \mathbb{T}^d \times \mathbb{T}^d \); and for every \( \epsilon > 0 \),
there is \( \delta > 0 \) such that \( |b(z) \cdot \nabla_z q^0(s, z, y) - b(x) \cdot \nabla_x q^0(s, x, y)| < \epsilon \) for \( |x - z| < \delta \). Then by Proposition A.2.(iii), for \( t \to 0 \),

\[
|II_1| \leq \|f\|_0 \|g\|_0 \left( \frac{1}{t} \int_0^t \int_{|x - z| < \delta} g^{b,\eta}(t - s; s, z) dx dz ds + 2C \frac{1}{t} \int_0^t \int_{|x - z| \geq \delta} q^{b,\eta}(t - s; s, z) dx dz ds \right)
\]

\[
\leq \|f\|_0 \|g\|_0 \left( \epsilon + 2C \frac{1}{t} \int_0^t \int_{|y| \geq \delta} \rho_\eta(t; y) dy dz ds \right)
\]

\[
\leq \|f\|_0 \|g\|_0 \left( \epsilon + 2C t \int_{|y| \geq \delta} |y|^{-(d + \alpha)} dy \right)
\]

\[
\to \epsilon \|f\|_0 \|g\|_0.
\]

Since \( \epsilon > 0 \) is arbitrary, \( II_1 \to 0 \) as \( t \to 0 \). Moreover, the strong continuity of the semigroup \( \{T_t\}_{t \geq 0} \) and dominated convergence imply that \( II_2 \to 0 \) as \( t \to 0 \). Thus, we get (A.10).

(iii). Finally, we prove that \( C^\infty(T^d) \) is a core of the generator. We divide this proof into three steps.

**Step 1:** We prove that for every \( f \in C(T^d) \) and all \( t > 0 \), \( T_t^{b,\eta} f \) is differentiable and the integral in \( L^{b,\eta} T_t^{b,\eta} f \in C(T^d) \) is absolutely integrable, and for all \( x \in T^d \),

\[
\nabla T_t^{b,\eta} f(x) = \int_{T^d} \nabla x^{b,\eta}(t; x, y) f(y) dy,
\]

(A.11)

\[
L^{b,\eta} T_t^{b,\eta} f(x) = \int_{T^d} L^{b,\eta} x^{b,\eta}(t; x, y) f(y) dy.
\]

(A.12)

Using the estimate (A.6) and writting the derivative as the limit of difference quotients, then (A.11) follows from the dominated convergence. Further, (A.12) follows from (A.11) and Fubini’s theorem. The continuity of the function \( L^{b,\eta} T_t^{b,\eta} f \) follows from the joint continuity of \( L^{b,\eta} q^{b,\eta}(t; x, y) \) and (A.12).

**Step 2:** The Feller nature of the semigroup \( \{T_t^{b,\eta}\} \) yields that the operator \( (L^{b,\eta}, C^\infty(T^d)) \) is closable in \( C(T^d) \) (cf. [13, Proposition II.3.14]). Denote \( (\tilde{L}^{b,\eta}, D(\tilde{L}^{b,\eta})) := (L^{b,\eta}, C^\infty(T^d)) \). We show that for every \( f \in C(T^d) \) and all \( t > 0 \), \( T_t^{b,\eta} f \in D(\tilde{L}^{b,\eta}) \) and \( \tilde{L}^{b,\eta} T_t^{b,\eta} f = L^{b,\eta} T_t^{b,\eta} f \). Let \( \{\phi_n\}_{n \in \N} \) be a standard mollifier such that \( \text{supp}(\phi_n) \subset B(0, 1/n) \). Then \( T_t^{b,\eta} f * \phi_n \in C^\infty(T^d) \) and \( \|T_t^{b,\eta} f * \phi_n - T_t^{b,\eta} f\|_0 \to 0 \) as \( n \to \infty \).

Using (A.11), (A.12), (A.6), (A.8), [18, Lemma 5.17.(a)] and Fubini’s theorem, we have

\[
\left| L^{b,\eta} (T_t^{b,\eta} f * \phi_n)(x) - L^{b,\eta} T_t^{b,\eta} f \right| \phi_n(x) \right|
\]

\[
\leq \left| \int_{\R^d} (b(x) - b(y)) \cdot \nabla T_t^{b,\eta} f(x - y) \phi_n(y) dy \right|
\]

\[
+ \left| \int_{\R^d} \int_{\R^d \setminus \{0\}} \left[ T_t^{b,\eta} f(x - y + z) - T_t^{b,\eta} f(x - y) - z \cdot \nabla T_t^{b,\eta} f(x - y) \right] \phi_n(y) dy dz \right|
\]

\[
\leq \frac{1}{\eta^\beta} \|b\|_\beta \|\nabla T_t^{b,\eta} f\|_0 + \frac{1}{\eta^\beta} \int_{\R^d} \int_{\R^d \setminus \{0\}} \left| \nabla T_t^{b,\eta} f(x - y) \right| \phi_n(y) dy dz
\]

\[
\leq \frac{1}{\eta^\beta} C(\epsilon_0, T, \|b\|_0, \|f\|_0) \left( \frac{1}{\eta^\beta} \int_{\R^d} \int_{\R^d \setminus \{0\}} \left| \nabla T_t^{b,\eta} f\right| \phi_n(y) dy dz \right)
\]

Let \( n \to \infty \), we get \( \|L^{b,\eta} (T_t^{b,\eta} f * \phi_n) - L^{b,\eta} T_t^{b,\eta} f\|_0 \to 0 \). Since \( L^{b,\eta} T_t^{b,\eta} f \in C(T^d) \) by Step 1, \( L^{b,\eta} T_t^{b,\eta} f * \phi_n \to L^{b,\eta} T_t^{b,\eta} f \) in \( C(T^d) \) as \( n \to \infty \). Thus, we have \( \|L^{b,\eta} (T_t^{b,\eta} f * \phi_n) - L^{b,\eta} T_t^{b,\eta} f\|_0 \to 0 \) as \( n \to \infty \) which ends the proof.

**Step 3:** It is obvious that \( \tilde{L}^{b,\eta} \subset L^{b,\eta} \), i.e., \( D(\tilde{L}^{b,\eta}) \subset D(L^{b,\eta}) \) and \( \tilde{L}^{b,\eta} |_{D(\tilde{L}^{b,\eta})} = L^{b,\eta} \). We will show the converse. For an arbitrary \( f \in D(\tilde{L}^{b,\eta}) \), let \( f_n = T_{1/n}^{b,\eta} f \). Then by Step 2, \( f_n \in D(L^{b,\eta}) \). We also have \( \|f_n - f\|_0 \to 0 \) and

\[
\|\tilde{L}^{b,\eta} f_n - \tilde{L}^{b,\eta} f\|_0 = \|L^{b,\eta} f_n - L^{b,\eta} f\|_0 = \|T_{1/n}^{b,\eta} \tilde{L}^{b,\eta} f - \tilde{L}^{b,\eta} f\|_0 \to 0.
\]
This gives $\mathcal{L}^{b,\eta} = \mathcal{L}^{b,\eta}$ and we complete the whole proof. □

Appendix B  SDEs and nonlocal PDEs

The following result is a consequence of the nature of Feller semigroups (see [24, Theorem 2.3, Corollary 2.5] and [14, Theorem 4.4.1, Proposition 4.1.7]).

Corollary B.1. The canonical Feller process $(X^{b,\eta};(\Omega,\mathcal{F},P))$ corresponding to $(T^b_t)_{t \geq 0}$ with càdlàg trajectories is the unique solution to the martingale problem for $(L^{b,\eta},P \circ (X^{b,\eta})^{-1})$, and also the unique weak solution to the following SDE

$$dX_t = b(X_t)dt + \int_0^\infty \int_{\mathbb{R}^d \setminus \{0\}} 1_{[0,\kappa^t(X_{t-},s)]}(r)z\tilde{N}(dz,dr,dt) + \int_0^\infty \int_{\mathbb{R}^d} 1_{[0,\kappa^t(X_{t-},s)]}(r)zN(dz,dr,dt), \quad (B.1)$$

where $N$ is a Poisson random measure on $\mathbb{R}^d \times [0,\infty) \times [0,\infty)$ with intensity measure $J(z)dz \times m \times m$ and $\tilde{N}$ is the associated compensated Poisson random measure.

We have a generalized version of Itô’s formula as following. The proof is similar with that of [33, Lemma 3.4] and shall be omitted.

Lemma B.2. Let $f \in C^{1+\gamma}(\mathbb{T}^d)$ with $1 + \gamma > \alpha$. If $X$ satisfies the SDE (B.1), then

$$f(X_t) - f(X_0) = \int_0^t \mathcal{L}^{b,\eta}f(X_s)ds + \int_0^t \int_0^\infty \int_{\mathbb{R}^d \setminus \{0\}} [f(X_{s-} + 1_{[0,\kappa^t(X_{s-},z)]}(r)z) - f(X_{s-})] \tilde{N}(dz,dr,ds). \quad (B.3)$$

We can solve the nonlocal Poisson with zeroth-order term, using the semigroup representation.

Corollary B.3. For every $f \in C^{\beta}(\mathbb{T}^d)$ and $\lambda > 0$, there exists a unique classical solution $u \in C^{\alpha+\beta}(\mathbb{T}^d)$ to the Poisson equation

$$\lambda u - \mathcal{L}^{b,\eta}u = f. \quad (B.2)$$

Proof. We first prove that if $u_\lambda \in C^{\alpha+\beta}(\mathbb{T}^d)$ is a solution of (B.2), then $u_\lambda$ must have the following representation

$$u_\lambda(x) = \int_0^\infty e^{-\lambda t}T_t^{b,\eta}f(x)dt, \quad (B.3)$$

and there exists a constant $C = C(c_0,\|b\|_0,\lambda) > 0$ not depending on $f$ such that

$$\|u_\lambda\|_{\alpha+\beta} \leq C\|f\|_{\beta}. \quad (B.4)$$

Since the restriction of the generator $\mathcal{L}^{b,\eta}$ on $C^{\alpha+\beta}(\mathbb{T}^d)$ is $\mathcal{L}^{b,\eta}$, we have

$$\int_0^\infty e^{-\lambda t}T_t^{b,\eta}f dt = \int_0^\infty e^{-\lambda t}T_t^{b,\eta}(\lambda u_\lambda - \mathcal{L}^{b,\eta}u_\lambda)dt = -\int_0^\infty \frac{d}{dt} \left(e^{-\lambda t}T_t^{b,\eta}u_\lambda\right)dt = u_\lambda - \lim_{t \to \infty} e^{-\lambda t}T_t^{b,\eta}u_\lambda = u_\lambda,$$

where we have used the fact that $\|e^{-\lambda t}T_t^{b,\eta}u_\lambda\|_0 \leq e^{-\lambda t}\|u_\lambda\|_0 \to 0$ as $t \to \infty$. This gives (B.3) and the uniqueness follows. Further, using the Schauder-type estimates in [2, Theorem 7.1, Theorem 7.2], there exist a constant $C = C(c_0,\|b\|_0,\lambda) > 0$ such that

$$\|u_\lambda\|_{\alpha+\beta} \leq C(\|u_\lambda\|_0 + \|f\|_{\beta}).$$

The representation (B.3) yields that

$$\|u_\lambda\|_0 \leq \|f\|_0 \int_0^\infty e^{-\lambda t}dt = \frac{1}{\lambda}\|f\|_0.$$

The estimate (B.4) follows.

Moreover, it is shown in [27, Theorem 3.4] that when the function $\kappa^t$ is a constant, the existence and uniqueness hold in $C^{\alpha+\beta}(\mathbb{T}^d)$. We can now obtain the existence of (B.2) by the energy estimate (B.4) and the method of continuity, see [17, Section 5.2]), also cf. [21, Theorem 3.2]. □
Acknowledgements. The research of J. Duan was partly supported by the NSF grant 1620449. The research of Q. Huang was partly supported by China Scholarship Council (CSC), and NSFC grants 11531006 and 11771449. The research of R. Song is supported in part by a grant from the Simons Foundation (#429343, Renming Song). We would like to thank Dr. Yanjie Zhang for useful discussions.

References

[1] D. Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[2] R.F. Bass. Regularity results for stable-like operators. *J. Funct. Anal.*, 8(257):2693–2722, 2009.

[3] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, vol. 5. North-Holland Publishing Company Amsterdam, 1978.

[4] Patrick Billingsley. *Probability and measure*. John Wiley & Sons, 2nd ed., 1986.

[5] K. Bogdan and T. Jakubowski. Estimates of heat kernel of fractional laplacian perturbed by gradient operators. *Commun. Math. Phys.*, 271(1):179–198, 2007.

[6] B. Böttcher. Feller processes: The next generation in modeling. Brownian motion, Lévy processes and beyond. *PLoS One*, 5(12):e15102, 2010.

[7] B. Böttcher, R.L. Schilling, and J. Wang. *Lévy matters III: Lévy-type processes: construction, approximation and sample path properties*, vol. 1. Springer, 2013.

[8] Z.-Q. Chen and E. Hu. Heat kernel estimates for $\delta + \delta a/2$ under gradient perturbation. *Stoch. Process. Their Appl.*, 125(7):2603–2642, 2015.

[9] Z.-Q. Chen and X. Zhang. Heat kernels for time-dependent non-symmetric stable-like operators. *J. Math. Anal. Appl.*, 465(1):1–21, 2018.

[10] D. Cioranescu and P. Donato. *An introduction to homogenization*, vol. 17. Oxford university press, 1999.

[11] D. Cioranescu and J.S.J. Paulin. *Homogenization of reticulated structures*, vol. 136. Springer-Verlag New York, 1999.

[12] E.B. Davies. *One-parameter semigroups*, vol. 15. Academic Pr, 1980.

[13] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, vol. 194. Springer-Verlag New York, 2000.

[14] S.N. Ethier and T.G. Kurtz. *Markov processes: characterization and convergence*, vol. 282. John Wiley & Sons, 2009.

[15] B. Franke. The scaling limit behaviour of periodic stable-like processes. *Bernoulli*, 12(3):551–570, 2006.

[16] B. Franke. A functional non-central limit theorem for jump-diffusions with periodic coefficients driven by stable Lévy-noise. *J. Theor. Probab.*, 20(4):1087–1100, 2007.

[17] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*, vol. 224. Springer Science & Business Media, 2001.

[18] T. Grzywny and K. Szczypkowski. Heat kernels of non-symmetric Lévy-type operators. *arXiv preprint arXiv:1804.01313*, 2018.

[19] M. Hairer and É. Pardoux. Homogenization of periodic linear degenerate PDEs. *J. Funct. Anal.*, 255(9):2462–2487, 2008.

[20] M. Horie, T. Inuzuka, and H. Tanaka. Homogenization of certain one-dimensional discontinuous markov processes. *Hiroshima Math. J.*, 7(2):629–641, 1977.

[21] Q. Huang, J. Duan, and R. Song. Homogenization of nonlocal partial differential equations related to stochastic differential equations with Lévy noise. *Bernoulli*, 28(3):1648–1674, 2022.

[22] J. Jacod and A. Shiryaev. *Limit theorems for stochastic processes*, vol. 288. Springer-Verlag Berlin Heidelberg, 2nd ed., 1987.

[23] O. Kallenberg. *Foundations of modern probability*. Springer Science & Business Media, 2006.

[24] T.G. Kurtz. Equivalence of stochastic equations and martingale problems. In *Stochastic analysis 2010*, pp. 113–130. Springer, 2011.

[25] R. Modarres and J.P. Nolan. A method for simulating stable random vectors. *Comput. Stat.*, 9(1):11–19, 1994.
É. Pardoux. Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: a probabilistic approach. *J. Funct. Anal.*, 167(2):498–520, 1999.

E. Priola. Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka J. Math.*, 49(2):421–447, 2012.

W. Rudin. *Principles of mathematical analysis*, vol. 3. McGraw-hill New York, 1976.

K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge university press, 1999.

R.L. Schilling and T. Uemura. Homogenization of symmetric Lévy processes on $\mathbb{R}^d$. *arXiv preprint arXiv:1808.01667*, 2018.

K. Szczypkowski. Fundamental solution for super-critical non-symmetric Lévy-type operators. *arXiv preprint arXiv:1807.04257*, 2018.

M. Tomisaki. Homogenization of càdlàg processes. *J. Math. Soc. Jpn.*, 44(2):281–305, 1992.

L. Xie. Singular SDEs with critical non-local and non-symmetric Lévy type generator. *Stoch. Process. Their Appl.*, 127 (11):3792–3824, 2017.