Involutive Uninorm Logic with Fixed Point enjoys finite strong standard completeness

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Abstract
An algebraic proof is presented for the finite strong standard completeness of the Involutive Uninorm Logic with Fixed Point (IUL\(^{fp}\)). It may provide a first step towards settling the standard completeness problem for the Involutive Uninorm Logic (IUL, posed in G. Metcalfe, F. Montagna. (J Symb Log 72:834–864, 2007)) in an algebraic manner. The result is proved via an embedding theorem which is based on the structural description of the class of odd involutive FL\(_c\)-chains which have finitely many positive idempotent elements.

Keywords
Involutive residuated lattices · Substructural fuzzy logics · Standard completeness · Embedding

Mathematics Subject Classification 03B47 · 03B52 · 03G25

1 Introduction
Mathematical fuzzy logics have been introduced in [8], and the topic is a rapidly growing field ever since ([1, 4, 6]). Substructural fuzzy logics were introduced in [15] as substructural logics that are standard complete, that is, complete with respect to standard algebras which are the algebras where the real unit interval \([0, 1]\) is their lattice reduct, and standard completeness for several substructural logics, all are stronger than the there-introduced Uninorm Logic (UL), has been proven, too, with the notable exception of the Involutive Uninorm Logic (IUL). Its standard completeness has remained an open problem, which has stood against the attempts using the widely

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used embedding method of [13] or the density elimination technique of [15]. It is fair
to mention, however, that there is already a claimed proof for the standard complete-
ness of IUL in [18], but the presented proof-theoretic proof via density elimination
is very hard to read and even harder to check, and as a consequence, as of today, the
result is not generally recognized.\footnote{See the remark of the author himself in [18, second section in page 43].}

An algebraic semantics for IUL is the variety of bounded involutive FL\textsubscript{e}-chains. The structure of involutive FL\textsubscript{e}-chains is quite rich; even the integral algebras thereof
(the class of IMTL-chains) is a class containing, e.g., the connected rotations [12] of
all MTL-chains without zero divisors. This richness renders their algebraic description
a very hard task. Therefore, as a first step toward this goal, we focus on odd involutive
FL\textsubscript{e}-chains, which is a subclass of involutive FL\textsubscript{e}-chains. Bounded odd involutive
FL\textsubscript{e}-chains constitute the algebraic semantics for the Involutive Uninorm Logic with
Fixed Point (IUL\textsuperscript{fp}) [14]. Prominent examples of odd involutive FL\textsubscript{e}-algebras are
lattice-ordered abelian groups and odd Sugihara monoids. The former constitutes
an algebraic semantics for Abelian Logic [3, 16, 17] while the latter constitutes an
algebraic semantics for IUM\textsuperscript{L}\textsuperscript{e}, which is a logic at the intersection of relevance logic
and many-valued logic [7].

As defined in [15] substructural fuzzy logics are based on a countable propositional
language with formulas built inductively as usual from a set of propositional variables,
binary connectives \(\odot \to\), \(\land\), \(\lor\), and constants \(\bot\), \(\top\), \(f\), \(t\), with defined connectives:

\[
\neg A = A \to f \\
A \oplus B = \neg (\neg A \odot \neg B) \\
A \leftrightarrow B = (A \to B) \land (B \to A)
\]

MAILL consists of the following axioms and rules:

1. \(A \to A\)
2. \((A \to B) \to ((B \to C) \to (A \to C))\)
3. \((A \to (B \to C)) \to (B \to (A \to C))\)
4. \((A \odot (B \to C)) \leftrightarrow (A \to (B \to C))\)
5. \((A \land B) \to A\)
6. \((A \land B) \to B\)
7. \(((A \to B) \land (A \to C)) \to (A \to (B \land C))\)
8. \(A \to (A \lor B)\)
9. \(B \to (A \lor B)\)
10. \(((A \to C) \land (B \to C)) \to ((A \lor B) \to C)\)
11. \(A \leftrightarrow (t \to A)\)
12. \(\bot \to A\)
13. \(A \to \top\)

\[
\begin{array}{c}
A \to B \\
\hline
A \land B
\end{array} \quad \text{(mp)}
\begin{array}{c}
A \\
\hline
A \land B
\end{array} \quad \text{(adj)}
\]
Further substructural fuzzy logics are defined by extending $\text{MAIL}$. In particular, Uninorm Logic ($\text{UL}$) is the schematic extension of $\text{UL}$ by the axiom $((A \to B) \land t) \lor ((B \to A) \land t)$, Involutive Uninorm Logic ($\text{IUL}$) is the schematic extension of $\text{UL}$ by the axiom $\neg\neg A \to A$, and Involutive Uninorm Logic with Fixed Point ([14], $\text{IUL}_{fp}$) is the schematic extension of $\text{IUL}$ by the axiom $t \leftrightarrow f$.

In order to make a possible first step toward settling the standard completeness problem for $\text{IUL}$ in an algebraic manner, in this paper we prove the finite strong standard completeness of $\text{IUL}_{fp}$, a somewhat simpler logic than $\text{IUL}$. The key ingredient in our proof is constructing an embedding, based on a representation theorem of the class of odd involutive $\text{FL}_e$-chains which possess only finitely many positive idempotent elements, by means of totally ordered abelian groups and a modified version of the lexicographic product construction [9, 11].

As said, standard algebras for a mathematical fuzzy logic $L$ are the ones from the corresponding variety which universe is the real unit interval $[0, 1]$. A mathematical fuzzy logic $L$ enjoys finite strong standard completeness if the following conditions are equivalent for each formula $\varphi$ and finite theory $T$:

1. $T \models_L \varphi$,
2. for each standard $L$-algebra $A$ and each $A$-model $e$ of $T$, $e$ is an $A$-model of $\varphi$.

There are other alternatives for defining standard completeness of $L$. The same definition as above but without confining to finite theories yields the definition of strong standard completeness, whereas by setting $T = \emptyset$ we obtain the definition of (weak) standard completeness. A possible way of proving finite strong standard completeness is to embed finitely generated $L$-chains into standard $L$-chains, since this way one proves that any formula of $L$ which can be falsified in a totally ordered model of finitely many $L$-formulas (which is always a finitely generated $L$-chain) can also be falsified in a standard $L$-algebra. Knowing the facts that $\text{IUL}_{fp}$-chains constitute an algebraic semantics of $\text{IUL}_{fp}$, and that $\text{IUL}_{fp}$-chains are exactly non-trivial bounded odd involutive $\text{FL}_e$-chains, we shall prove that any non-trivial finitely generated bounded odd involutive $\text{FL}_e$-chain embeds into an odd involutive $\text{FL}_e$-chain over the real unit interval $[0, 1]$ such that its top element is mapped into 1 and its bottom element is mapped into 0.

2 Preliminaries

We start with the necessary definitions. Commutative residuated lattices are the $f$-free reducts of $\text{FL}_e$-algebras. An $\text{FL}_e$-algebra is a structure $(X, \land, \lor, \ast, \rightarrow_\ast, t, f)$ such that $(X, \land, \lor)$ is a lattice, $(X, \land, \lor, \ast, t)$ is a commutative, residuated monoid, and $f$ is an arbitrary constant. Being residuated means that there exists a binary operation $\rightarrow_\ast$, called the residual operation of $\ast$, such that $x \ast y \leq z$ if and only if $x \rightarrow_\ast z \geq y$. This equivalence is called adjointness condition, $(\ast, \rightarrow_\ast)$ is called an adjoint pair. Equivalently, for any $x, z$, the set $\{v \mid x \ast v \leq z\}$ has its greatest element, and $x \rightarrow_\ast z$, the residuum of $x$ and $z$, is defined as this element: $x \rightarrow_\ast z := \max\{v \mid x \ast v \leq z\}$; this is called the residuation condition. One defines $x' = x \rightarrow_\ast f$ and calls an $\text{FL}_e$-algebra involutive if $(x')' = x$ holds. In every involutive $\text{FL}_e$-algebra $x \rightarrow_\ast y = (x \ast y')'$
holds, hence for every term there exists an equivalent term which contains (besides \( \land \) and \( \lor \)) only the product and the residual complement operations but not the residual operation. An involutive FL_{e}\text{-algebra} is called \textit{odd} if \( t = f \) holds. As shown in [11], the \((X, \land, \lor, \ast, t)\)-reduct of an odd involutive FL_{e}\text{-algebra} is a lattice-ordered abelian group if and only if it has a single idempotent element, and it is if and only if \( \ast \) is cancellative. Therefore, one can loosely speak about subgroups of odd involutive FL_{e}\text{-algebras} by meaning a cancellative subalgebras. A totally ordered algebra is called discretely ordered if every element has its lower and upper cover. A totally ordered abelian group is called discrete if its unit element has an upper cover. Discrete totally ordered abelian groups are discretely ordered. Algebras will be denoted by bold capital letters, their underlying sets by the same regular letter. The lexicographic product of two totally ordered sets \( A = (A, \leq_1) \) and \( B = (B, \leq_2) \) is a totally ordered set \( A \times B = (A \times B, \leq) \), where \( A \times B \) is the Cartesian product of \( A \) and \( B \), and \( \leq \) is defined by \( \langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle \) if and only if \( a_1 <_1 a_2 \) or \( a_1 = a_2 \) and \( b_1 \leq_2 b_2 \). We shall view such a lexicographic product as an \textit{enlargement}: each element of \( A \) is replaced by a whole copy of \( B \). The lexicographic product \( \overset{\to}{A} \times \overset{\to}{B} \) of two FL_{e}\text{-chains} \( A \) and \( B \) is an FL_{e}\text{-chain} over the lexicographic product of their respective universes such that all operations are defined coordinatewise. For an FL_{e}\text{-algebra} \( X = (X, \land, \lor, \ast, \rightarrow_{\ast}, t, f) \), denote \( \tau(x) = x \rightarrow_{\ast} x \) for \( x \in X \) and let \( X_{gr} = \{x \in X : x \text{ is invertible} \} = \{x \in X : x \ast x' = t\} \). If \( X \) is odd then there is a cancellative subalgebra \( X_{gr} \) of \( X \) over \( X_{gr} \), call it the group part of \( X \).

**Lemma 1** [11] Let \( X = (X, \land, \lor, \ast, \rightarrow_{\ast}, t, f) \) be an involutive FL_{e}\text{-algebra}. The following statements hold true.

1. \( \text{Range}(\tau) = \{ \tau(x) : x \in X \} \) is equal to the set of positive idempotent elements of \( X \).
2. If \( X \) is totally ordered and odd and \( A \) is an \( X \)-term which contains only the operations \( \ast, \rightarrow_{\ast} \) and \( ' \) then for any evaluation \( e \) of the variables of \( A \) into \( X \), \( \tau(e(A)) \) equals the maximum of the \( \tau \)-values of the variables and constants of \( A \) under \( e \). Stated it in another way, for \( x, y \in X \), \( \tau(x') = \tau(x) \), \( \tau(x \ast y) = \max(\tau(x), \tau(y)) \), \( \tau(x \rightarrow_{\ast} y) = \max(\tau(x), \tau(y)) \).

The main theorem which we shall rely on in proving the main result of the present paper is Theorem 1 below. It says that every odd involutive FL_{e}\text{-chain} \( X \), which has only \( n \geq 1 \) positive idempotent elements can be built by starting with a totally ordered abelian group, and iteratively enlarging it in some way by other totally ordered abelian groups. Each such enlargement increases the number of positive idempotent elements of the enlarged algebra by the number of positive idempotent elements of the algebra we enlarge with. Since the totally ordered abelian group we start with has a single idempotent element (its unit element), and likewise each algebra we enlarge with does, after \( n - 1 \) enlargement we obtain \( X \). Odd involutive FL_{e}\text{-chains} (even with only finitely many positive idempotent elements) are rather complex mathematical objects, and this renders the description of their algebraic structure a hard task of [9, 11]. The price to pay for describing such complex structures by such simple ones as totally ordered abelian groups is rather complex mathematical objects, and this renders the description of their algebraic structure a hard task of [9, 11].
ordered abelian groups is that the way of enlargements is quite involved (including several parameterers), and this makes up for a correspondingly involved notation, too. These enlargements are described in Definition 1.

**Definition 1** [9, 11] (*Partial lex and sublex products*) Let \( Y = (Y, \leq_Y, *, \rightarrow_*, t_Y, f_Y) \) and \( G = (G, \leq_G, *, \rightarrow_*, t_G, f_G) \) be odd involutive FL-e-chains with residual complement operations \(*'\) and \( \star'\), respectively.

A. Add a new element \( \top \) to \( G \) as a top element and annihilator (for \( \star \)), then add a new element \( \bot \) to \( G \cup \{ \top \} \) as a bottom element and annihilator. Let

\[
Z \leq Y_{\text{gr}} \text{ and } H \leq Z \times G.
\]

Let

\[
Y_{Z \times G} = \left( Y_{Z \times G}, \leq, \rightarrow_*, (t_Y, t_G), (f_Y, f_G) \right)
\]

\[
Y_H(Z \times G) = \left( Y_H(Z \times G), \leq, \rightarrow_*, (t_Y, t_G), (f_Y, f_G) \right)
\]

where

\[
Y_{Z \times G} = (Z \times G) \cup (Z \times \{ \top \}) \cup (Y \times \{ \bot \}),
\]

\[
Y_H(Z \times G) = H \cup (Z \times \{ \top \}) \cup (Y \times \{ \bot \}),
\]

– \( \leq \) is the restriction of the lexicographical order of \( \leq_Y \) and \( \leq_{G \cup [\top, \bot]} \) to the respective universes,

– \( \rightarrow_\circ \) is defined coordinatewise, \( \rightarrow_\circ \) is (provisionally) the residual operation of \( \circ \), and the residual complement operation \( \circ' \) is given by

\[
(x, y)' = \begin{cases} (x'^*, y^\star) & \text{if } x \in Z \\ (x'^*, \bot) & \text{if } x \notin Z \end{cases}
\]

Call \( Y_{Z \times G} \) the *type I partial lexicographic extension (lex extension, for short)* of \( Y \) by \( Z \times G \), and \( Y_H(Z \times G) \) the *type I partial sublex extension* of \( Y \) by \( H \) (where \( H \) is in \( Z \times G \)). In the notations \( Y_{Z \times G} \) and \( Y_H(Z \times G) \) the arrow above the \( \times \) symbol refers to the lexicographic nature of the extension, and the two extra small lines attached to the arrow reminds that in this version two extra elements (\( \top \) and \( \bot \)) are used (compare it with the following construction below).

B. For a totally ordered set \( (Y, \leq_Y) \) for \( x \in Y \) let \( x_\downarrow = z \) if there exists \( z \in Y \) such that \( x \) covers \( z \), and let \( x_\uparrow = x \) otherwise; define \( x_\downarrow \) dually. We say for \( Z \subseteq Y \) that \( Z \) is
discretely embedded into $Y$ if for $x \in Z$ it holds true that $x \notin \{x_\uparrow, x_\downarrow\} \subseteq Z$. We shall also use the term discretely embedded for algebras if it holds true for their totally ordered set reducts. Assume that

$$Y_{\text{gr}} \text{ is discretely embedded into } Y \quad \text{and } H \leq Y_{\text{gr}} \times G,$$

(5)

Add a new element $\top$ to $G$ as a top element and annihilator. Let

$$Y_{Y_{\text{gr}} \times G} = \left( Y_{Y_{\text{gr}} \times G}, \leq, \oplus, \rightarrow_\oplus, (t_Y, t_G), (f_Y, f_G) \right)$$

$$Y_H(Y_{\text{gr}} \times G) = \left( Y_H(Y_{\text{gr}} \times G), \leq, \oplus, \rightarrow_\oplus, (t_Y, t_G), (f_Y, f_G) \right)$$

(6)

where

$$Y_{Y_{\text{gr}} \times G} = (Y_{gr} \times G) \cup (Y \times \{ \top \}),$$

$$Y_H(Y_{\text{gr}} \times G) = H \cup (Y \times \{ \top \}),$$

(7)

– $\leq$ is the restriction of the lexicographical order of $\leq_Y$ and $\leq_{G \cup \{\top\}}$ to the respective universes,

– $\oplus$ is defined coordinatewise, $\rightarrow_\oplus$ is (provisionally) the residual operation of $\oplus$, and the residual complement operation $'$ is given by

$$ (x, y)' = \begin{cases} 
(x^\oplus, y^\downarrow) & \text{if } x \in X_{\text{gr}} \text{ and } y \in Y \\
((x^\oplus)_\downarrow, \top) & \text{if } x \in X_{\text{gr}} \text{ and } y = \top \\
(x^\downarrow, \top) & \text{if } x \notin X_{\text{gr}} \text{ and } y = \top .
\end{cases}$$

(8)

Call $Y_{Y_{\text{gr}} \times G}$ the type II partial lex extension of $Y$ by $Y_{\text{gr}} \times G$, and call $Y_H(Y_{\text{gr}} \times G)$ the type II partial sublex extension of $Y$ by $H$ (where $H$ is in $Y_{\text{gr}} \times G$). In the notations $Y_{Y_{\text{gr}} \times G}$ and $Y_H(Y_{\text{gr}} \times G)$ the arrow above the $\times$ symbol refers to the lexicographic nature of the extension, and the single extra small line attached to the arrow reminds that in this version only one extra element ($\top$) is used.

**Remark 1** If the required condition of Definition 1 holds, namely, the conditions in (1) in case of a type I extension, or the conditions in (5) in case of a type II extension, then the respective algebra in (2) or in (6) is said to exist or said to be well-defined.

**Remark 2** The interested reader can find detailed, enlightening motivation for these constructions in [11, Sect. 3] along with related 3D plots of the graphs of the example operations.
Lemma 2 \[9, 11\] Type I and II partial lex and sublex products, as defined in Definition 1, are odd involutive FL\(_e\)-algebras such that

\[
Y_H(Z \times G) \leq Y_{Z \times G},
\]

\[
Y_H(Y_{gr \times G}) \leq Y_{Y_{gr \times G}}.
\]

\[\square\]

Remark 3 A type I partial lex product is a particular, simpler instance of a type I partial sublex product, and a type II partial lex product is a particular, simpler instance of a type II partial sublex product, where \(H = Z \times G\) and \(H = Y_{gr \times G}\), respectively.

Remark 4 About the two simpler notions: think of \(Y_{Z \times G}\) such that \(Z\) (which is in the group part of \(Y\)) is enlarged by \(G\) in a type I manner (as described in the first row of (3)), and think of \(Y_{Y_{gr \times G}}\) such that \(Y_{gr}\) (the group part of \(Y\)) is enlarged by \(G\) in a type II manner (as described in the first row of (7)).

Remark 5 Since \(Z\) is a subgroup and since the monoidal operation is defined coordinatewise, the group part of \(Y_{Z \times G}\) is \(Z \times G_{gr}\), the group part of \(Y_{Y_{gr \times G}}\) is \(Y_{gr} \times G_{gr}\), and the group part of both \(Y_H(Z \times G)\) and \(Y_H(Y_{gr \times G})\) is \(H\).

As said, the main theorem which we shall rely on in proving the main result of the present paper is Theorem 1. It asserts that up to isomorphism, every odd involutive FL\(_e\)-chain which has only finitely many positive idempotent elements can be built by starting with a totally ordered abelian group and iteratively enlarging it by other totally ordered abelian groups until we obtain the given algebra. In more technical terms, every odd involutive FL\(_e\)-chain which has only finitely many positive idempotent elements can be built by iterating finitely many times the type I and type II partial sublex product constructions using only totally ordered abelian groups, as building blocks. We shall also refer to this fact that every odd involutive FL\(_e\)-chain which has only finitely many positive idempotent elements has a partial sublex product group representation.

Theorem 1 \([9, 11]\) If \(Y\) is an odd involutive FL\(_e\)-chain, which has only \(n \in \mathbb{N}, n \geq 1\) positive idempotent elements then it has a partial sublex product group representation, that is, there exist a totally ordered abelian group \(H_1\) and for \(i = 2, \ldots, n\), totally ordered abelian groups \(H_i, G_i, Z_{i-1}\) along with \(\iota_i \in \{I, II\}\) such that \(Y \cong Y_n\), where for \(i \in \{2, \ldots, n\}\),

\[
Y_1 = H_1 \quad \text{and} \quad Y_i = \begin{cases} Y_{i-1 H_i(Z_{i-1} \times G_i)} & \text{if } \iota_i = I \\ Y_{i-1 H_i(Y_{i-1 gr \times G_i})} & \text{if } \iota_i = II \end{cases}.
\]

\[\square\]

\[\odot\] Springer
Notice that Theorem 1 claims isomorphism between $Y$ and $Y_n$ hence $Y_n$ and consequently for $i = n - 1, \ldots, 2$, the $Y_i$’s are claimed implicitly to exist (to be well defined). By Definition 1, since $(Y_i)_{gr} = H_i$ holds for $i \in \{1, \ldots, n\}$ (by assumption if $i = 1$, and by Remark 5 if $i \in \{2, \ldots, n\}$) therefore, it is necessarily that

- for $i = 2, \ldots, n$, $Z_{i-1} \leq H_{i-1}, Y_i \subseteq H_{i-1}$ and
- for $i = 2, \ldots, n$, if $i = 1$ then $H_{i-1}$ is discretely embedded into $Y_{i-1}$.

Let $Y$ be a non-trivial finitely generated bounded odd involutive FL$e$-chain. Roughly, our plan is first to show that $Y$ has a partial sublex product group representation, and then to take guidance from the way $Y$ is built iteratively from totally ordered abelian groups in constructing the embedding of $Y$ into $X^*$, which is an odd involutive FL$e$-chain over the unit interval, in a step-by-step iterative fashion, and guided by the iterative steps of the group representation of $Y$. Meanwhile we want the universes of the intermediated constructed $X_i^*$’s to stay as close to the topological structure of the unit interval as possible, to achieve in the end the universe of $X_n^* = X^*$ be order isomorphic to the unit interval. For this, we shall need and make use of the characterization in Lemma 3; note that the open unit interval and $\mathbb{R}$ are order isomorphic. A totally ordered set $(X, \leq)$ is called Dedekind complete if every non-empty subset of $X$ bounded from above has a supremum.

**Lemma 3** ([2, Theorem 2.29]) A totally ordered set $(K, \leq)$ is order isomorphic to $\mathbb{R}$ if and only if $(K, \leq)$ possesses the following four properties: $(K, \leq)$ has no least neither greatest element, $(K, \leq)$ is densely ordered, there exists a countable dense subset of $(K, \leq)$, and $(K, \leq)$ is Dedekind complete.

### 3 The proof of finite strong standard completeness

To ensure that Theorem 1 will be applicable to our problem, first we show that

**Proposition 1** Every finitely generated odd involutive FL$e$-chain has only finitely many positive idempotent elements.

**Proof** Since the order is total, $\tau(x \land y), \tau(x \lor y) \in \{\tau(x), \tau(y)\}$ holds. Therefore, using claim 2 in Lemma 1, an easy induction on the recursive structure of the term that generates a given element $x$ of the algebra shows that the $\tau$-value of $x$ coincides with the $\tau$-value of one of its generators or constants. Therefore, since the algebra is finitely generated, all elements of the algebra share only the (finitely many) $\tau$-values of the finitely many generators. Claim 1 in Lemma 1 concludes the proof. $\square$

Group representations are not unique, and hence even in a group representation of a finitely generated algebra the algebras which are employed in the iterative process in (9) (the $G_i$’s, the $Z_i$’s, the $H_i$’s, and the $Y_i$’s) are not necessarily finitely generated. More precisely they always are with the exception of the $G_i$’s. Our next aim is to achieve, by modifying the $G_i$’s in (11), that all the algebras are finitely generated in some group representation of $Y$. Later we shall make use of this property of the $G_i$’s.
(the modified groups which we shall iteratively extend with) and the enlarged parts (the $Z_i$’s). This will make it possible, e.g., to embed the $G_i$’s into appropriate odd involutive FL$_e$-algebras over the unit interval, a key intermediate step of our construction later in (21). Our aim will be achieved in Proposition 2 by defining a unique (or canonical) group representation for every odd involutive FL$_e$-chain.

**Proposition 2** If $Y$ is a finitely generated odd involutive FL$_e$-chain then there exists a group representation of $Y$ in which all the employed algebras are finitely generated.

**Proof** Let $Y$ be an odd involutive FL$_e$-chain which has finitely many positive idempotent elements, and let a representation of $Y$ by partial sublex products be given, according to Theorem 1, see it in (9). We modify the enlarging groups, and leave all the other algebras which are employed in (9) unchanged: for $i \in \{2, \ldots, n\}$ let

$$G_i' = pr_{G_i}(H_i) = \{g \in G_i : \exists a \in Z_{i-1} \text{ such that } (a, g) \in H_i\}. \quad (11)$$

Trivially, for $i \in \{2, \ldots, n\}$, $G_i' \leq G_i$ and

$$Y_1 = H_1 \text{ and } Y_i = \begin{cases} Y_{i-1}^{gr}_{H_i}(Z_{i-1} \times G_i') & \text{if } i \in \{2, \ldots, n\} \text{ and } \iota_i = I \\ Y_{i-1}^{gr}_{H_i}(Y_{i-1}^{gr} \times G_i') & \text{if } i \in \{2, \ldots, n\} \text{ and } \iota_i = II \end{cases}, \quad (12)$$

since the condition in (10) (the one with the $G_i$’s) holds and hence it trivially holds with the modified $G_i$’s, too, that is,

- for $i = 2, \ldots, n$, $Z_{i-1} \leq H_{i-1}, H_i \leq Z_{i-1} \times G_i'$ and
- for $i = 2, \ldots, n$, if $\iota_i = II$ then $H_{i-1}$ is discretely embedded into $Y_{i-1}$,

and for all $i \in \{2, \ldots, n\}$, the universe of $Y_{i-1}^{gr}_{H_i}(Z_{i-1} \times G_i')$ and the universe of $Y_{i-1}^{gr}_{H_i}(Y_{i-1}^{gr} \times G_i')$ (and hence all their operations, too) is equal to the universe of $Y_{i-1}^{gr}_{H_i}(Z_{i-1} \times G_i)$ and the universe of $Y_{i-1}^{gr}_{H_i}(Y_{i-1}^{gr} \times G_i)$, respectively.

We shall prove that all algebras used in the group representation of $Y$ in (12) are finitely generated. The statement is obvious if $n = 1$, since then the only algebra in the group representation of $Y$ is $Y$. Therefore, we may assume $n \geq 2$. Since $n \geq 2,$

$$Y_n = \begin{cases} Y_{n-1}^{gr}_{H_n}(Z_{n-1} \times G_n') & \text{if } \iota_n = I \\ Y_{n-1}^{gr}_{H_n}(Y_{n-1}^{gr} \times G_n') & \text{if } \iota_n = II \end{cases}$$

holds, and hence

$$Y_n = \begin{cases} H_n \cup (Z_{n-1} \times \{\top\}) \cup (Y_{n-1} \times \{\bot\}) & \text{if } \iota_n = I \\ H_n \cup (Y_{n-1} \times \{\top\}) & \text{if } \iota_n = II \end{cases}$$
holds, see (3) and (7). Therefore,

\[
Y_n \supseteq \begin{cases} 
Y_{n-1} \times \{\bot\} & \text{if } t_n = I \\
Y_{n-1} \times \{\top\} & \text{if } t_n = II.
\end{cases}
\]

If finitely many elements \((a_1, b_1), \ldots, (a_k, b_k)\) generate \(Y_n\), and in particular, generate the element \((a, \bot)\) if \(t_n = I\) (resp. \((a, \top)\) if \(t_n = II\)) in \(Y_{n-1} \times \{\bot\}\) (resp. in \(Y_{n-1} \times \{\top\}\)), then \((a_1, \bot), \ldots, (a_k, \bot)\) together with \((t_{y_{n-1}}, \bot)\) (resp. \((a_1, \top), \ldots, (a_k, \top)\) together with \((t_{y_{n-1}}, \top)\)) can generate the same element: use the equivalent term which contains only \(*\) and ‘, replace each generator and constant \((a_i, b_j)\) in it by \((a_i, \bot)\) (resp. \((a_i, \top)\)), and apply a multiplication by \((t_{y_{n-1}}, \bot)\) (resp. \((t_{y_{n-1}}, \top)\)) in the end. Since the multiplication is defined coordinatewise, and since at the residual complement operation the first coordinate of the result depends only on the first coordinate of the argument (see (4) and (8)), and since \(\bot\) (resp. \(\top\)) is an annihilator in the second coordinate, such a generation process trivially results in \((a, \bot)\) (resp. \((a, \top)\)). Therefore, \(Y_{n-1}\) is finitely generated. Summing up, if \(Y = Y_n\) is finitely generated then so is \(Y_{n-1}\). This way an easy downward induction shows that \(Y_i\) is finitely generated for \(i = n, n-1, \ldots, 1\). In particular, \(H_1 = Y_1\) is finitely generated, too. Trivially, if an element is invertible then its inverse is its residual complement, hence its residual complement is invertible, too. Also trivially, the product of two elements cannot be invertible if at least one of them is noninvertible. Therefore, if an element in \(Y_i\) is an invertible element is generated, then all the generator elements employed in the generating term (the equivalent one which does not contain the residual operation) must be invertible, too. Hence it follows that for \(i \in \{2, \ldots, n\}\), \(H_i = (Y_i)_{gr}\) is finitely generated. It is well-known that subgroups of finitely generated abelian groups are themselves finitely generated [5, page 80]. Hence \(Z_i\) for \(i \in \{1, \ldots, n-1\}\) is finitely generated, too, since by assumption, \(Z_{i-1} \leq H_{i-1}\) holds for \(i \in \{2, \ldots, n\}\). Trivially, for \(i \in \{2, \ldots, n\}\), \(G'_i\) is a homomorphic image of \(H_i\), see (11). Therefore, \(G'_i\) is finitely generated, too, since being finitely generated inherits to homomorphic images. \(\square\)

Our next aim is to make things simpler. The next proposition states that every finitely generated odd involutive FL_{e}-chain embeds into an odd involutive FL_{e}-chain, which has a group representation as in Proposition 2 with the additional specificity that instead of the partial sublex product construction, only the simpler partial lex product construction is used in the iterative steps.

**Proposition 3** Any finitely generated odd involutive FL_{e}-chain \(Y\) is embeddable into a finitely generated odd involutive FL_{e}-chain \(X\) which has a partial lex product representation in which all the algebras are finitely generated.

**Proof** Let \(Y\) be a finitely generated odd involutive FL_{e}-chain. Then \(Y\) has a partial sublex product representation by Proposition 2, see it in (12), such that all the algebras employed are finitely generated. Also (13) holds. Let

\[
X_1 = H_1, \quad X_i = \begin{cases} 
X_{i-1} \times Z_{i-1} \rightrightarrows G'_i & \text{if } i \in \{2, \ldots, n\} \text{ and } t_i = I \\
X_{i-1} \times X_{i-1, gr} \rightrightarrows G'_i & \text{if } i \in \{2, \ldots, n\} \text{ and } t_i = II,
\end{cases}
\]

and let \(X = X_n\). \(\square\)
Clearly, all the algebras employed in (14) are finitely generated because they are all inherited from (12).

To see that the $X_i$’s are well-defined, we need to verify that

- for $i = 2, \ldots, n$, $\mathbf{Z}_{i-1} \leq (\mathbf{X}_{i-1})_{gr}$ and
- for $i = 2, \ldots, n$, if $t_i = II$ then $(\mathbf{X}_{i-1})_{gr}$ is discretely embedded into $X_{i-1}$.

The first one is immediate since $\mathbf{Z}_1 \overset{(13)}{\leq} \mathbf{H}_1 = (\mathbf{H}_1)_{gr} \overset{(14)}{=} (\mathbf{X}_1)_{gr}$ and for $i \in \{3, \ldots, n\}$ $(\mathbf{X}_{i-1})_{gr} = \mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1}$ holds by (14) and Remark 5, yielding $\mathbf{Z}_{i-1} \overset{(13)}{\leq} \mathbf{H}_{i-1} = \mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1} = (\mathbf{X}_{i-1})_{gr}$.

As for the second row of (15), if $t_2 = II$ then $(\mathbf{X}_1)_{gr} = \mathbf{H}_1$ is discretely embedded into $\mathbf{Y}_1$ by (13), and $\mathbf{Y}_1 = \mathbf{X}_1$ by the construction. Let $t_i = II$ for some $i \in \{3, \ldots, n\}$. Then, since $i \geq 3$,

$$X_{i-1} = \begin{cases} \mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1} & \text{if } t_{i-1} = I \\ \mathbf{X}_{i-2} \times \mathbf{G}'_{i-1} & \text{if } t_{i-1} = II \end{cases}$$

holds by Remark 5. Now, $\mathbf{G}'_{i-1}$ is a finitely generated totally ordered abelian group by Proposition 2, and such groups are known to be either trivial, or discrete and hence discretely ordered. First we prove that for $i \in \{3, \ldots, n\}$, $\mathbf{G}'_{i-1}$ cannot be trivial. Since $t_i = II$, by (13) it holds that $H_{i-1}$ is discretely embedded into

$$Y_{i-1} = \begin{cases} Y_{i-2} H_{i-1}(\mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1}) = H_{i-1} \cup (\mathbf{Z}_{i-2} \times \{\top\}) \cup (\mathbf{Y}_{i-2} \times \{\bot\}) & \text{if } t_{i-1} = I \\ Y_{i-2} H_{i-1}(\mathbf{Y}_{i-2} \times \mathbf{G}'_{i-1}) = H_{i-1} \cup (\mathbf{Y}_{i-2} \times \{\top\}) & \text{if } t_{i-1} = II \end{cases},$$

see Definition 1. However, it cannot be that case if $\mathbf{G}'_{i-1}$ is trivial, since then the cover of $(a, 1) \in H_{i-1}$ in $Y_{i-1}$ would be $(a, \top)$, and $(a, \top)$ is not an element of $H_{i-1}$ since $(a, \top)$ is not invertible. This contradiction shows that for $i \in \{3, \ldots, n\}$, $\mathbf{G}'_{i-1}$ is discretely ordered. We claim that $X_{i-1}_{gr}$ is discretely embedded into $X_{i-1}$. Since $i \geq 3$,

$$X_{i-1} = \begin{cases} \mathbf{X}_{i-2} \mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1} & \text{if } t_{i-1} = I \\ \mathbf{X}_{i-2} \mathbf{X}_{i-2} \times \mathbf{G}'_{i-1} & \text{if } t_i = II \end{cases}$$

holds by (14), hence by (7),

$$X_{i-1} = \begin{cases} (\mathbf{Z}_{i-2} \times \mathbf{G}'_{i-1}) \cup (\mathbf{Z}_{i-2} \times \{\top\}) \cup (\mathbf{X}_{i-2} \times \{\bot\}) & \text{if } t_{i-1} = I \\ (\mathbf{X}_{i-2} \times \mathbf{G}'_{i-1}) \cup (\mathbf{X}_{i-2} \times \{\top\}) & \text{if } t_{i-1} = II \end{cases}.$$

Here $a \in \mathbf{Z}_{i-2}$ and 1 denotes the single element of $\mathbf{G}'_{i-1}$, c.f. the first line of (13).
For any element \((a, g)\) in \(X_{i-1, gr}\), its second coordinate \(g\) is an element of \(G'_{i-1}\) by (16), and since \(G'_{i-1}\) is discretely ordered, the upper neighbor \(s\) of \(g\) exists in \(G'_{i-1}\), hence \((a, s)\), which is the upper cover of \((a, g)\) in \(X_{i-1}\) by (18), is in \(X_{i-1, gr}\), too. An analogous argument works for the lower cover of \((a, g)\). The proof of the second row of (15) is concluded.

We claim that for \(i \in \{1, \ldots, n\}\), \(Y_i \leq X_i\) holds. Indeed, the claim being straightforward for \(i = 1\), assume \(Y_{i-1} \leq X_{i-1}\) for some \(i \in \{2, \ldots, n\}\). It follows that

\[
Y_i = Y_{i-1}Z_{i-1} \to G'_i \subseteq X_{i-1}Z_{i-1} \to G'_i = X_i
\]

and

\[
Y_i = Y_{i-1}Y_{i-1, gr} \to G'_i \subseteq X_{i-1}X_{i-1, gr} \to G'_i = X_i,
\]

compare the respective rows of (17) and (18). Therefore,

\[
Y_i = Y_{i-1}H_i(Z_{i-1} \times G'_i) \leq X_{i-1}Z_{i-1} \to G'_i = X_i
\]

if \(i = 1\)

\[
Y_i = Y_{i-1}H_i(Y_{i-1, gr} \times G'_i) \leq X_{i-1}X_{i-1, gr} \to G'_i = X_i
\]

if \(i = II\).

In particular, we have confirmed \(Y \equiv Y_n \leq X_n = X\).

Thanks to Proposition 3, in the rest of the paper it will be sufficient to work with partial lex products only. For this simpler construction a simpler notation will come handy in formulating the following three lemmas. Henceforth we shall denote \(A \times L\) by \(A_Z \times L\) and \(A_{gr} \times B\) by \(A \times B\). The first two assert associativity-like properties for partial lex product extensions. First we recall from [10] that two (and thus also finitely many) consecutive type II partial lex extensions can be replaced by a single type II partial lex extension, see Lemma 4, and that a type I partial lex extension followed by a type II partial lex extension can be replaced by a single type I partial lex extension, see Lemma 5.

**Lemma 4** ([10, Lemma 1]) *For any odd involutive FL-\(e\)-algebras \(A, B, C\), it holds true that*

\[
(A \to B) \to C \cong A \to (B \times C),
\]

*that is, if the algebra on one side is well-defined then the algebra on the other side is well-defined, too, and the two algebras are isomorphic.*

**Lemma 5** ([10, Lemma 2]) *For any odd involutive FL-\(e\)-algebras \(A, Z, L, B\) it holds true that*

\[
(A_Z \to L) \to B \cong A_Z \times (L \times B),
\]

*that is, if the algebra on one side is well-defined then the algebra on the other side is well-defined, too, and the two algebras are isomorphic.*
The last lemma here is about the preservation of the ‘order isomorphic to $\mathbb{R}$’ property.

**Lemma 6** ([10, Lemma 4]) Let $A$ and $D$ be odd involutive FL$_e$-chains which are order isomorphic to $\mathbb{R}$, $\mathbb{Z} \leq A_{\text{gr}}$. $\mathbb{Z}$ is countable. Then $A_{\mathbb{Z}} \times D$ is order isomorphic to $\mathbb{R}$. $\square$

It is well-known that among all totally ordered abelian groups up to isomorphism only the subgroups of $\mathbb{R}$ ($\mathbb{R}, \mathbb{Q}, \mathbb{Z}$, and $1$)$^3$ can be embedded qua totally ordered abelian groups into $\mathbb{R}$. However, since all totally ordered abelian groups are particular instances of odd involutive FL$_e$-chains, we have to be able to embed all finitely generated totally ordered abelian groups into $\mathbb{R}$. Surprisingly, it is possible qua odd involutive FL$_e$-algebras, and it is because we are not confined to use the usual addition of the real numbers. To this end in (21), we define three series of odd involutive FL$_e$-chains and first state some basic properties of them.

**Definition 2** For $j \in \mathbb{N}$, $j \geq 1$, let

$$
\begin{align*}
Z_0 &:= Z_1 := \mathbb{Z}, & Z_{j+1} := \mathbb{Z} \times Z_j, \\
Q_0 &:= Q_1 := \mathbb{Q}, & Q_{j+1} := \mathbb{Z} \times Q_j, \\
R_0 &:= R_1 := \mathbb{R}, & R_{j+1} := \mathbb{Z} \times R_{j-1}.
\end{align*}
$$

We shall denote the universes of $Z_j$, $Q_j$, and $R_j$ by $Z_j$, $Q_j$, and $R_j$, respectively, and for $j \geq 2$ we denote the new top element to be added here (according to Definition 1/B) to $Z_j$ (or $Q_j$ or $R_j$) by $\mathbb{J} \times \ldots \times \mathbb{J}$. $^4$

**Proposition 4** The following statements hold true.

(a) For $j \geq 2$, $Z_j = \{(x_1, \ldots, x_j) \in \mathbb{Z} \times (\mathbb{Z} \cup \{\mathbb{J}\}) \times \ldots \times (\mathbb{Z} \cup \{\mathbb{J}\}) \mid \text{if } x_i = \mathbb{J} \text{ for some } i \in \{2, \ldots, j\} \text{ then for all } l \leq j, x_l = \mathbb{J}\}$,

$$Q_j = \{(x_1, \ldots, x_j) \in \mathbb{Z} \times (\mathbb{Z} \cup \{\mathbb{J}\}) \times \ldots \times (\mathbb{Z} \cup \{\mathbb{J}\}) \times (\mathbb{Q} \cup \{\mathbb{J}\}) \mid \text{if } x_i = \mathbb{J} \text{ for some } i \in \{2, \ldots, j\} \text{ then for all } l \leq j, x_l = \mathbb{J}\},$$

$$R_j = \{(x_1, \ldots, x_j) \in \mathbb{Z} \times (\mathbb{Z} \cup \{\mathbb{J}\}) \times \ldots \times (\mathbb{Z} \cup \{\mathbb{J}\}) \times (\mathbb{R} \cup \{\mathbb{J}\}) \mid \text{if } x_i = \mathbb{J} \text{ for some } i \in \{2, \ldots, j\} \text{ then for all } l \leq j, x_l = \mathbb{J}\}.$$

$^3$ By abuse of notation, in the sequel we will use the symbols $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ for the totally ordered abelian groups, for the induced odd FL$_e$-algebras, for their universes, and also for the respective totally ordered sets.

$^4$ According to Definition 1/B the added new top element is denoted by $\mathbb{J}$. Strictly speaking here we consider an isomorphic structure when adding $\mathbb{J} \times \ldots \times \mathbb{J}$ to be that new top element. We do so in order to keep the “length” of all elements inside a given algebra $Z_j$, $Q_j$ or $R_j$ the same.
For \( j \in \{2, \ldots, j\} \) then for all \( i \leq l \leq j, x_l = T \).

(b) For \( j \geq 1, (Z_j)_{gr} = Z \times \ldots \times Z, (Q_j)_{gr} = Z \times \ldots \times Z \times Q, (R_j)_{gr} = Z \times \ldots \times Z \times \mathbb{R} \).

c) For \( j \in \mathbb{N}, (Z_j)_{gr} \) is discretely embedded into \( Z_j \), and for any \( x \in Z_j \\setminus (Z_j)_{gr} \) and \( x \neq y \in Z_j \), there exists \( z \in (Z_j)_{gr} \) such that \( z \) is strictly in between \( x \) and \( y \).

d) For \( j \in \mathbb{N}, Z_j \) and \( Q_j \) are countable.

e) For \( j \in \mathbb{N}, Z_j, Q_j, \) and \( R_j \) have no least neither greatest element.

(f) For \( j, k \geq 1, Z_j \times Z_k \cong Z_{j+k} \) and \( Z_j \times R_k \cong R_{j+k} \).

g) For \( j \in \mathbb{N}, Z_j \) and \( R_j \) are Dedekind complete.

(h) For \( j \in \mathbb{N}, R_j \) is order isomorphic to \( \mathbb{R} \).

**Proof** (a) We shall present the proof only for, e.g., \( R_j \) since the proof is analogous for \( Q_j \) and \( Z_j \). \( R_2 \) is equal to \( Z \times (\mathbb{R} \cup \{T\}) \) by the first row of (7), hence the statement holds for \( j = 2 \). Let \( j \geq 3 \) and assume that the statement holds for \( j - 1 \). By the first row of (7) we obtain \( R_j = Z \times (R_{j-1}) \cup \left(Z \times \{T \times \ldots \times T\}\right)_{j-1} \), which is clearly equal, using the induction hypothesis, to the set stated in (a).

(b) is obvious for \( j = 1 \), and follows from (a) for \( j \geq 2 \) since \( T \) is an annihilating element in the respective coordinate(s), and hence an element with a \( T \) coordinate it is never invertible.

(c) is obvious for \( j = 0, 1 \), and it follows in a straightforward manner for \( j \geq 2 \) from (a) and (b) using that \( Z \) is unbounded.

d) readily follows from (a) since \( Z \) and \( Q \) are countable.

(e) readily follows from (a) since \( Z \) is unbounded.

(f) readily follows from Lemma 4.

g) Dedekind completeness easily follows from (a), too: The statement clearly holds for \( j = 0, 1 \) since \( Z \) and \( R \) are Dedekind complete. Let \( j \geq 2 \) and assume that the statement holds for \( j - 1 \), that is, \( Z_{j-1} \) and \( R_{j-1} \) are Dedekind complete. Let a subset \( X \) of \( R_j \) (the proof for \( Z_j \) is analogous) with an upper bound \( m = (m_1, m_2) \) be given where \( m_1 \in Z \). Since the ordering on \( R_j \) is lexicographical, \( m_1 \) must be an upper bound of \( X_1 = \{x_1 \in Z : (x_1, x_2) \in X\} \), ensuring the existence of the supremum \( z_1 \) of \( X_1 \) since \( Z \) is Dedekind complete. In fact then \( z_1 \) is the greatest element of \( X_1 \) since if a subset of \( Z \) has a supremum then its supremum is necessarily its greatest element. If \( (z_1, T, \ldots, T) \) is an element of \( X \) then it is clearly the maximal element of \( X \) and we are done. If not then, referring to (a), for every \((z_1, x_2) \in X\), the first coordinate of \( x_2 \) must be in \( Z \), and hence \( x_2 \in R_{j-1} \). Therefore, the set \( X_2 = \{x_2 \in R_{j-1} : (z_1, x_2) \in X\} \) is nonempty. If the first coordinates of the elements of \( X_2 \) are unbounded from above then \((z_1, T, \ldots, T)\) is clearly the maximal element of \( X \) and we are done, whereas if the first coordinates of the elements of \( X_2 \) are bounded from above, say by
a, then \((a, \top, \ldots, \top) \in \mathbb{R}_{j-1}\) is an upper bound of \(X_2\). Then, by the induction hypothesis, there exists a supremum \(z_2\) of \(X_2\) in \(\mathbb{R}_{j-1}\), and hence \((z_1, z_2)\) is clearly the supremum of \(X\), so the proof of (g) is concluded.

(h) Finally, to prove (h), because of Lemma 3, in addition to (e) and (g) we only need to prove that \(\mathbb{R}_j\) is densely ordered, and that \(\mathbb{Q}_j\) is a dense subset of \(\mathbb{R}_j\); both are obvious from (a).

\[\Box\]

Remark 6 If \(G\) is a finitely generated totally ordered abelian group then \(G\) embeds (qua an ordered abelian group) into \(\mathbb{Z}^n := \mathbb{Z} \times \ldots \times \mathbb{Z}\) \((n\text{-times})\) for some positive integer \(n\). Indeed, since the divisible closure of \(G\) is a finite dimensional rational vector space, \(G\) has only a finite number \((\text{say} n)\) of convex subgroups. Therefore, by Hahn’s Theorem, \(G\) can be embedded (qua an ordered abelian group) into \(\mathbb{R}^n := \mathbb{R} \times \ldots \times \mathbb{R}\) \((n\text{-times})\). Since \(G\) is finitely generated, so is every (group-)homomorphic image of \(G\), is a finitely generated subgroup of \(\mathbb{R}\), and hence it is either trivial (in which case that index could have been omitted from the Hahn-product \(\mathbb{R}^n\)) or it is isomorphic to \(\mathbb{Z}\). Clearly, \(G\) is a subalgebra of the lexicographic product of its canonical projections, and so \(G\) embeds into \(\mathbb{Z}^n\), as stated.

We are ready to prove the main result of the paper.

Theorem 2 IUL\(^{fp}\) enjoys finite strong standard completeness.

Proof Let \(Y_1\) be a non-trivial finitely generated bounded odd involutive FL\(_e\)-chain. Hence it has at least three elements. Our plan is to embed \(Y_1\), guided by its group representation, into an odd involutive FL\(_e\)-chain over \([0, 1]\). As shown in [11, Proposition 5.3.], odd involutive FL\(_e\)-chains are diagonally strictly increasing, that is, \(x_1 < x\) and \(y_1 < y\) imply \(x_1 * y_1 < x * y\). Therefore the \(\top\) and the \(\bot\) of \(Y_1\) cannot arise as products of other elements (ones in between \(\bot\) and \(\top\)). The corresponding statement is obvious for the minimum, the maximum, and the residual complement operations, and hence also for the residual operation due to \(x \rightarrow * y = (x * y')'\). Therefore, there is a subalgebra \(Y\) over the universe of \(Y_1\) deprived its top and bottom elements. Since in every bounded odd involutive FL\(_e\)-algebra its bottom element is the zero element, and its top element acts like a zero element for all elements except the bottom one, it follows that also \(Y\) is a finitely generated (not necessarily bounded) odd involutive FL\(_e\)-chain. We shall embed \(Y\), guided by its group representation, into an odd involutive FL\(_e\)-chain \(X^*\) over a universe which is order isomorphic to \(\mathbb{R}\). Then, using the order isomorphism together with an order isomorphism between \(\mathbb{R}\) and \([0, 1]\), we can carry over the structure of \(X^*\) into \([0, 1]\), and finally we can add a top and a bottom element (as in item A in Definition 1) to get an odd involutive FL\(_e\)-chain over \([0, 1]\), in which \(Y_1\) embeds.\(^5\)

Since \(Y\) is finitely generated, by Proposition 3, \(Y\) embeds into an odd involutive FL\(_e\)-chain \(X\) which has a partial lex product representation in which all the employed algebras are finitely generated, see it in (14). In particular,

\(^5\) We shall work with \(Y\) rather than with \(Y_1\) because we do not want to bother with the embedding of the top and bottom elements throughout each step of the proof.
all the \( Z_i \)'s are finitely generated, \( \text{(20)} \)

and all groups \( H_1, G_i' \) \( (i = 2, \ldots, n) \) are finitely generated. By Remark 6, \( H_1 \hookrightarrow \mathbb{Z}_k^i \)
holds for some positive integer \( k_1 \), and for \( i \in \{2, \ldots, n\} \), \( G_i' \hookrightarrow \mathbb{Z}_k^i \) holds for some positive integer \( k_i \). So \( H_1 \) and the \( G_i' \)'s are countable, and unless equal to the one-

element group, they are discretely ordered. The embedding naturally extends to an
embedding between the induced odd involutive FL\(_e\)-chains. Referring to claims (b)
and (a) in Proposition 4, for \( i \in \{2, \ldots, n\} \), qua odd involutive FL\(_e\)-chains,

\[
H_1 \hookrightarrow \mathbb{Z}_k^1 = (\mathbb{Z}_k^1)_g \leq \mathbb{R}_{k_1} \quad \text{and} \quad G_i' \hookrightarrow \mathbb{Z}_k^i = (\mathbb{Z}_k^i)_g \leq \mathbb{R}_{k_i}.
\]  

(21)

Now, we are ready to define the embedding of

\( X \) into \( X^* \), where \( X^* \) is order isomorphic to \( \mathbb{R} \).

We shall inherit the types of the extensions (the \( \iota \) values) and the enlarged parts (the \( Z_i \)'s) from the representation of \( X \) in (14), meanwhile we enlarge the other algebras

and let \( X^* = X^*_n \). We shall prove by induction that for \( 1 \leq i \leq n \), \( X^*_i \) is well-defined, \( X^*_i \) embeds into \( X^*_i \), and \( X^*_n \) is order isomorphic to \( \mathbb{R} \), thus concluding the proof of the

theorem.

The statement holds if \( n = 1 \): \( X_1 \xrightarrow{(14)} H_1 \xrightarrow{(21)} R_{k_1} \xrightarrow{(22)} X^*_1 \), and \( R_{k_1} \) is order isomorphic to \( \mathbb{R} \) by claim (h) in Proposition 4. Hence in the sequel we assume \( n > 1 \).

**Claim 1** First we claim that for \( i \in \{1, \ldots, n\} \), \( X^*_i \) is well-defined and \( X^*_i \) embeds into \( X^*_i \).

(i) The statement holds for \( i = 1 \): \( X_1 \xrightarrow{(14)} H_1 \xrightarrow{(21)} R_{k_1} \xrightarrow{(22)} X^*_1 \), since \( X^*_1 \)
is equal to either \( \mathbb{Z}_k^1 \) or \( \mathbb{R}_{k_1} \) by (22). Induction hypothesis: assume that for some \( i \in \{2, \ldots, n\} \), \( X^*_i \) exists and \( X^*_i \) embeds into \( X^*_i \).

\(^6\) Intuitively, this is because if the coming extension is type II then we need to ensure that the group part of the algebra to be extended is discretely embedded in the algebra, whereas if the coming extension is not type II then we do not have such confinements, and hence in the preceeding extension we want to get as close to the topological structure of \( \mathbb{R} \) as possible.
– Assume $i_i = I$. To see that $X_i^*$ is well-defined, we need to verify that $Z_{i_i - 1}$ is a subalgebra of the group part of $X_{i_i - 1}^*$, see the third and fifth rows of (22). It readily follows since $Z_{i_i - 1}$ is a subalgebra of the group part of $X_{i_i - 1}$ by (14), and $X_{i_i - 1}$ embeds into $X_{i_i - 1}^*$ by the induction hypothesis (hence the group part of $X_{i_i - 1}$ clearly embeds into the group part of $X_{i_i - 1}^*$ since $X_{i_i - 1}$ and $X_{i_i - 1}^*$ share the same unit element and therefore the invertible elements in $X_{i_i - 1}$ are also invertible in $X_{i_i - 1}^*$).

Next we show that $X_i$ embeds into $X_i^*$. Since $i \neq 1$, by (14) and (22)

$$X_i = X_{i - 1} Z_{i - 1} \hookrightarrow G_i', \quad X_i^* = X_{i - 1}^* Z_{i - 1} \hookrightarrow Z_{k_i} \quad \text{if } i_{i + 1} = II,$$

$$X_i = X_{i - 1} Z_{i - 1} \hookrightarrow G_i', \quad X_i^* = X_{i - 1}^* Z_{i - 1} \hookrightarrow \mathbb{R}_{k_i} \quad \text{if } i_{i + 1} = I \text{ or } i = n, i_i = I.$$

Here, in each row the algebra on the left is embeddable into the algebra on the right. Indeed, by the first row of (3), the universes of

$$X_{i - 1} Z_{i - 1} \hookrightarrow G_i', X_{i - 1}^* Z_{i - 1} \hookrightarrow Z_{k_i}, X_{i - 1}^* Z_{i - 1} \hookrightarrow \mathbb{R}_{k_i}$$

are

$$(Z_{i - 1} \times G_i') \cup (Z_{i - 1} \times \{T\}) \cup (X_{i - 1} \times \{\perp\}),$$

$$(Z_{i - 1} \times Z_{k_i}) \cup (Z_{i - 1} \times \{T\}) \cup (X_{i - 1}^* \times \{\perp\}),$$

$$(Z_{i - 1} \times \mathbb{R}_{k_i}) \cup (Z_{i - 1} \times \{T\}) \cup (X_{i - 1}^* \times \{\perp\}),$$

respectively, and $X_{i - 1}$ embeds into $X_{i - 1}^*$ by the induction hypothesis and $G_i'$ embeds into $Z_{k_i}$ and into $\mathbb{R}_{k_i}$ by (21). Therefore, $X_i \hookrightarrow X_i^*$ holds, and the coordinatewise definition of the multiplication and the residual complement operation in Definition 1 shows that all the operations of $X_i^*$ are extensions of the corresponding operations of $X_i$. As noted before, the unit element of $X_i^*$ is the same as the unit element of $X_i$.

– Assume $i_i = II$. To see that $X_i^*$ is well-defined, we need to verify that $(X_{i_i - 1}^*)_{gr}$ is discretely embedded into $X_{i_i - 1}^*$, see the forth and sixth rows of (22).

If $i = 2 < n$ then $X_1^* = Z_{k_1}$ by the second row of (22), and $(Z_{k_1})_{gr}$ is discretely embedded into $Z_{k_1}$ by claim (c) of Proposition 4. If, for $i \in \{3, \ldots, n\}$, $i_i = II$ then by (22), $X_{i - 1}^*$ is equal to either $X_{i - 2} Z_{i - 2} \hookrightarrow Z_{k_i - 1}$ (if $i_{i - 1} = I$, see the third row) and then

$$X_{i - 1}^* = (Z_{i - 2} \times Z_{k_i - 1}) \cup (Z_{i - 2} \times \{T\}) \cup (X_{i - 2} \times \{\perp\}),$$

$$(X_{i - 1}^*)_{gr} = Z_{i - 2} \times (Z_{k_i - 1})_{gr}.$$

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or \( X^*_{i-2} \times \mathbb{Z}_{k_{i-1}} \) (if \( \iota_{i-1} = I \), see the forth row) and then
\[
X^*_{i-1} = ((X^*_{i-2})_{gr} \times \mathbb{Z}_{k_{i-1}}) \cup (X^*_{i-2} \times \{ \top \}).
\]
\[
(X^*_{i-1})_{gr} = (X^*_{i-2})_{gr} \times (\mathbb{Z}_{k_{i-1}})_{gr}.
\]

In both cases \((X^*_{i-1})_{gr}\) is clearly discretely embedded into \(X^*_{i-1}\) since \((\mathbb{Z}_{k_{i-1}})_{gr}\) is discretely embedded into \(\mathbb{Z}_{k_{i-1}}\) by item (c) in Proposition 4.

Next we show that \(X_i\) embeds into \(X_i^*\). Since \(i \neq 1\), by (14) and (22)
\[
X_i = X_{i-1} \times G_i', \quad X_i^* = X_{i-1}^* \times \mathbb{Z}_{k_i} \quad \text{if } \iota_{i+1} = I,
\]
\[
X_i = X_{i-1} \times G_i', \quad X_i^* = X_{i-1}^* \times \mathbb{R}_{k_i} \quad \text{if } \iota_{i+1} = I \text{ or } i = n, \iota_i = I.
\]

Here, in each row the algebra on the left is embeddable into the algebra on the right. Indeed, by the first row of (7), the universes of
\[
X_{i-1} \times G_i', X_{i-1}^* \times \mathbb{Z}_{k_i}, X_{i-1}^* \times \mathbb{R}_{k_i}
\]
are
\[
((X_{i-1})_{gr} \times G_i') \cup (X_{i-1} \times \{ \top \}),
\]
\[
((X_{i-1})_{gr} \times \mathbb{Z}_{k_i}) \cup (X_{i-1} \times \{ \top \}),
\]
\[
((X_{i-1})_{gr} \times \mathbb{R}_{k_i}) \cup (X_{i-1} \times \{ \top \}),
\]
respectively, and \(X_{i-1}\) embeds into \(X_{i-1}^*\) by the induction hypothesis and \(G_i'\) embeds into \(\mathbb{Z}_{k_i}\) and into \(\mathbb{R}_{k_i}\) by (21). Therefore, \(X_i \hookrightarrow X_i^*\) holds, and the coordinatewise definition of the multiplication and the residual complement operation in Definition 1 shows that all the operations of \(X_i^*\) are extensions of the corresponding operations of \(X_i\). As noted before, the unit element of \(X_i^*\) is the same as the unit element of \(X_i\).

**Claim 2** Finally, we prove that \(X_n^*\) is order isomorphic to \(\mathbb{R}\). Let
\[
M = \{ i \in \{1, \ldots, n-1\} \mid \iota_{i+1} = I \} \cup \{n\}.
\]
Since \(n \in M\), to conclude, it suffices to prove that for \(m \in M\), \(X_m^*\) is order-isomorphic to \(\mathbb{R}\). We proceed by induction.

Let \(m\) be the least element of \(M\). If \(m = 1\) then either \(n \geq 2\) and \(\iota_2 = I\) or \(m = n = 1\). In both cases, \(X_1^* = \mathbb{R}_{k_1}\) by the first row of (22), and \(\mathbb{R}_{k_1}\) is order isomorphic to \(\mathbb{R}\) by claim (h) in Proposition 4. If \(m > 1\) then for all \(1 < i \leq m\) it holds true that \(\iota_i = I\), and either \(m < n\) and \(\iota_{m+1} = I\) or \(m = n\). In both cases \(X_m^* = ((\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}) \times \ldots \times \mathbb{Z}_{k_{m-1}}) \times \mathbb{R}_{k_m}\) holds using the second row, then iteratively the forth row, and finally the last row of (22). Now, \(X_m^* \cong \mathbb{Z}_{k_1+k_2+\ldots+k_{m-1}} \times \mathbb{R}_{k_m} \cong \mathbb{R}_{k_1+k_2+\ldots+k_{m-1}+k_m}\) by Lemma 4 and claim (f) in Proposition 4, and hence \(X_m^*\) is order isomorphic to \(\mathbb{R}\) by claim (h) in Proposition 4, as stated.
Induction hypothesis: Assume that for \( m \in M \setminus \{n\}, X^*_m \) is order-isomorphic to \( R \), and let \( r = \min\{i \in M \mid i > m\} \) be the next element of \( M \) (it exists since \( m \neq n \)). Then \( r \geq 2 \).

- If \( r = m + 1 \) then either \( \iota_{r+1} = I \) or \( r = n \), and since \( m \in M \setminus \{n\}, \iota_{m+1} = \iota_r = I \) holds. Hence the fifth row of (22) applies yielding \( X^*_r = (X^*_m)_{Z_m} \times R_{k_r} \). Here, \( X^*_m \) and \( R_{k_r} \) are order isomorphic to \( R \) by the induction hypothesis and by claim (h) in Proposition 4, respectively, and \( Z_m \) is countable by (20). Therefore, Lemma 6 can be applied, which yields that \( X^*_r \) is order isomorphic to \( R \).

- If \( r > m + 1 \) then \( \iota_m = I \) and for all \( m < i \leq r \) it holds true that \( \iota_i = II \), and either \( r < n \) and \( \iota_{r+1} = I \) or \( r = n \). In both cases, using the third row, then (iteratively if \( r > m + 2 \)) the forth row, and finally the last row of (22),

\[
X^*_r = (((X^*_m)_{Z_m} \times Z_{k_{m+1}}) \times \ldots \times Z_{k_{r-1}}) \times R_{r_m}
\]

holds. By applying Lemmas 4 and 5 consecutively, it is isomorphic to

\[
(((X^*_m)_{Z_m} \times (Z_{k_{m+1}} \times Z_{k_{m+2}}) \times \ldots \times Z_{k_{r-1}}) \times R_{k_r}) \cong (X^*_m)_{Z_m} \times (Z_{k_{m+1}} \times Z_{k_{m+2}} \times \ldots \times Z_{k_{r-1}} \times R_{k_r})
\]

which is isomorphic to

\[
(X^*_m)_{Z_m} \times R_{k_{m+1} + \ldots + k_r}
\]

by claim (f) in Proposition 4. Here, \( X^*_m \) and \( R_{k_{m+1} + \ldots + k_r} \) are order isomorphic to \( R \) by the induction hypothesis and by claim (h) in Proposition 4, respectively, and \( Z_m \) is countable by (20). Therefore, Lemma 6 can be applied, which yields that \( X^*_r \) is order isomorphic to \( R \). \( \square \)

4 Conclusion

Substructural logics without the weakening rule are far less understood than substructural logics with it. In this paper a first example of a standard completeness proof of such a logic (\( IUL_{fp} \)) is presented. The proof is algebraic and is based on a construction of an embedding which, in turn, is based on the algebraic description of the related variety in [9, 11].
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