Stochastic Variational Search for $^4\Lambda^4H$

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A four-body calculation of the $pn\Lambda\Lambda$ bound state, $^4\Lambda\Lambda^4H$, is performed using the stochastic variational method and phenomenological potentials. The $NN$, $\Lambda N$, and $\Lambda\Lambda$ potentials are taken from a recent Letter by Filikhin and Gal, PRL89, 172502 (2002). Although their Faddeev-Yakubovsky calculation found no bound-state solution over a wide range of $\Lambda\Lambda$ interaction strengths, the present variational calculation gives a bound-state energy, which is clearly lower than the $^3H + \Lambda$ threshold, even for a weak $\Lambda\Lambda$ interaction strength deduced from a recent experimental $B_{\Lambda\Lambda}(^3\Lambda\Lambda\text{He})$ value. The binding energies obtained are close to, and slightly larger than, the values obtained from the three-body $d\Lambda\Lambda$ model in the Letter.

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In a recent Physical Review Letter [1], Filikhin and Gal (FG) described systematic Faddeev-Yakubovsky (FY) calculations for the mass number $A = 4$, strangeness $S = -2$ problem, in which they searched for a particle-stable bound-state of $^4\Lambda\Lambda^4H$. They did not obtain a bound-state solution, even for a strongly attractive $\Lambda\Lambda$ interaction, the scattering length of which is about $a_{\Lambda\Lambda} \sim -3$ fm. On the other hand, they also studied the same system by using a three-body $d\Lambda\Lambda$ model, where the $Ad$ interaction was constructed to reproduce the low-energy parameters of a $\Lambda pn$ Faddeev calculation for both the spin-doublet and quartet states. In contrast with the four-body $pn\Lambda\Lambda$ calculation that produced no bound state, the three-body $d\Lambda\Lambda$ model produced a particle-stable bound-state. One may think that this incompatibility raises an interesting problem concerning “the formal relationship between these four-body and three-body models which do not share a common hamiltonian.” [1] However, we are doubtful that there is really no bound state in the four-body $pn\Lambda\Lambda$ calculation.

A recent experimental report [2] on the observation of $^6\Lambda^4\Lambda^4H$ in the KEK-E373 hybrid emulsion experiment has had a significant impact on strangeness nuclear physics. The Nagara event provides unambiguous identification of $^6\Lambda^4\Lambda^4H$ production, and suggests that the $\Lambda\Lambda$ interaction strength is rather weaker than that expected from an older experiment [3].

Before the publication of the Nagara event, we had already attempted to search for $^4\Lambda\Lambda^4H$ theoretically by performing a complete four-body calculation using a variational method [1, 4]. The $\Lambda\Lambda$ interaction used in those studies was strongly attractive with a scattering length of $a_{\Lambda\Lambda} \sim -3$ fm. We concluded that $^4\Lambda\Lambda^4H$ is particle stable provided that the $\Lambda\Lambda$ interaction is so strong.

The variational calculation gives an upper bound on the energy eigenvalue. In contrast with the FY formulation, although the variational calculation does not necessarily take the exact boundary condition into account, the variational principle guarantees that the energy obtained comes close to the exact value from above as the trial function is improved. Therefore, starting from an identical hamiltonian for the four-body system, if the bound-state solution is obtained in a variational calculation, the exact eigenenergy must be lower than that and the FY calculation should achieve this kind of solution.

In the calculation of this four-body system, determining the $\Lambda N$ interaction is very important. Particularly, the strength in the $^3S_1$ channel of the $\Lambda N$ interaction is crucial, as well as the strength of the $\Lambda\Lambda$ interaction, in determining whether $^4\Lambda\Lambda^4H$ is particle-stable.

The purpose of this article is twofold: One is to examine the recent result of the four-body calculation for $pn\Lambda\Lambda$ by FG. Our four-body calculation gives quite a different result from that of FG, and we discuss the structural aspects of $^4\Lambda\Lambda^4H$ as a four-body system. Another purpose is to clarify the importance of the choice of the $\Lambda N$ potential in searching for $^4\Lambda\Lambda^4H$.

In Ref. [1], FG used phenomenological $NN$, $\Lambda N$, and $\Lambda\Lambda$ potentials, which have functional forms of a three-range Gaussian. The $NN$ potential utilized in the $pn$ spin-triplet channel is consistent with the $^2H$ binding energy, and the $\Lambda N$ potential is parametrized by fitting the low-energy scattering parameters for the Nijmegen soft-core 97f (or 97e) potential. For the $\Lambda\Lambda$ interaction, since there is no direct information from experiments in free space, FG used various parameter sets. A promising one, deduced by reproducing the experimental binding energy of $^4\Lambda\Lambda^4\text{He}$ from an $\alpha + 2\Lambda$ three-body model, is weakly attractive with a scattering length of $a_{\Lambda\Lambda} = -0.77$ fm. For all of these interactions, the strength and range parameters were determined so as to be appropriate for $S$-wave interactions. We thus assume that these interactions are valid only for the even-partial wave component of the baryon-baryon interaction in the three- and four-body systems.

For systematic calculations of $^2H$, $^3\Lambda^3H$, $^3\Lambda^3H^+$, and $^4\Lambda\Lambda^4H$, we use the same sets of $NN$, $\Lambda N$, and $\Lambda\Lambda$ interactions as FG used. The Set A $\Lambda\Lambda$ potential from Ref. [2],
which has a different strength in the $^3S_1$ channel, is also used. The parameters of the Set A $ΛN$ potential were determined phenomenologically in order to reproduce the $A = 3, 4$ single-$Λ$ hypernuclei. Figure 1 shows the $ΛN$ $S$-wave phase shifts. In the low energy region, the $^1S_0$ phase shifts obtained from NSC97f(FG) and from Set A are almost identical. On the other hand, the $^3S_1$ interaction of the NSC97f(FG) is more attractive than that of Set A. As we show later, both $ΛN$ potentials reproduce the experimental $B_N(^3\Lambda\Lambda)$ value, because $B_N(^3\Lambda\Lambda)$ is sensitive to the strength of the $^1S_0$ $ΛN$ interaction, while it is insensitive to the $^3S_1$ strength of the $ΛN$ interaction. In other words, the experimental information for $A = 3$ cannot determine the $^3S_1$ strength of the $ΛN$ interaction. Therefore, the $ΛN$ interaction used in the calculation of $Λ^4\Lambda\Lambda$ has to be tested not only for $B_N(^3\Lambda\Lambda)$, but also for another $B_A$ which is sensitive to the strength of the $^3S_1$ $ΛN$ interaction; for example, one can use $B_A(^3\Lambda\Lambda)$ and $B_A(^3\Lambda\Lambda^*$). This is one of the most important points in this article, because the calculated $B_A$ value is very sensitive to the choice of the $ΛN$ interaction, particularly the strength in the $^3S_1$ channel.

In order to check the validity of the choice of the $ΛN$ potential, we calculate $A = 3, 4$ $Λ$ hypernuclei, using the NSC97f(FG) or the Set A $ΛN$ potential. Only for this task, the Minnesota potential [2] is used for the $NN$ interaction. The parameters of the Minnesota potential were determined so as to reproduce low-energy $NN$ scattering data. The Minnesota potential reproduces reasonably well both the binding energies and sizes of few-nucleon systems, such as $^2H$, $^3H$, $^3He$, and $^4He$ [7]. In this work, the few-body calculations of the various systems are performed using the stochastic variational method (SVM) with correlated Gaussian (CG) basis functions [8]. The trial function is given by a combination of basis functions:

\[
\Psi_{JM,T,M’} = \sum_{k=1}^{K} c_k A[G(x, A_k)\chi_{JM,T,M’}].
\]  

Here, $A$ is an antisymmetrizer acting on identical baryons, $x = (x_1, \ldots, x_{A-1})$ stands for a set of relative coordinates, and $\chi_{JM,T,M’}$ is the spin (isospin) function. The spatial part of the trial function, $G(x, A)$, is the CG, which is defined by

\[
G(x, A_k) = \exp \left\{ - \frac{1}{2} \sum_{i<j} A_{\alpha kij}(r_i - r_j)^2 \right\}
\]

\[
= \exp \left\{ - \frac{1}{2} \sum_{i<j=1}^{A-1} (A_k)_{ij} x_i \cdot x_j \right\}.
\]

The $(A - 1) \times (A - 1)$ symmetric matrix $A_k$ contains $A(A - 1)/2$ independent matrix elements, which characterizes the CG basis and is uniquely determined in terms of $\alpha_{\alpha kij}$. A set of linear variational parameters $(c_1, \ldots, c_K)$ is determined by using the Ritz variational principle. The variational parameters are optimized by a stochastic procedure. This is entirely the same as in a previous study [3]. The reader is referred to Refs. [3, 8] for details of the calculation.

Before showing the results of our four-body calculations for $Λ^4\Lambda\Lambda$, we report results for the binding energies of $^3H$, $^3\Lambda\Lambda$, and $^3H^*$ using the same potentials as were used by FG. Using the triplet $pn(FG)$ and NSC97f(FG) $ΛN$ potentials, the calculated binding energies were $B(^3H) = 2.250$ MeV, $B_A(^3\Lambda\Lambda) = 0.237$ MeV, and $B_A(^3\Lambda\Lambda^*) = 0.010$ MeV. These energies for the three-body systems are consistent with those quoted by FG, although each energy is actually slightly larger than that of FG. We think that these small discrepancies are due to the $s$-wave approximation of the Faddeev calculation. Note that both calculations for $^3\Lambda\Lambda^*$ produce a weakly bound state; this means that the SVM with CG basis functions and the Faddeev calculation with the $s$-wave approximation do work well even for the very weakly bound-state problem.

According to our previous studies [4, 5], $Λ^4\Lambda\Lambda$ should have a particle-stable bound state with an isospin of $I = 0$, and an angular momentum and parity such that $J^P = 1^+$, provided that a strongly attractive $ΛΛ$ interaction with a scattering length of $\alpha_{\Lambda\Lambda} \sim -3$ fm, is used. Using such a strong $ΛΛ$ interaction, we have obtained a bound-state solution for $Λ^4\Lambda\Lambda$ (see Fig. 2). The $B_{\Lambda\Lambda}(Λ^4\Lambda\Lambda)$ values ($\gtrsim 1.2$ MeV) obtained are more than two-times larger than the values obtained in our previous studies ($\sim 0.5$ MeV) with the Set A $ΛN$ potential. This is due to the difference in the strength of the $ΛN$ interaction in the $^3S_1$ channel (see Fig. 1).

The four-body calculation using a weaker $ΛΛ$ interaction ($\alpha_{\Lambda\Lambda} = -0.77$ fm) is a challenging problem, since the three-body $dΛ$ model by FG predicts a particle-stable bound-state with a very small binding energy ($B_{ΛΛ} \sim 0.3$ MeV). For such a weakly bound four-body calculation,
FIG. 2: Calculated $B_{\Lambda\Lambda}(^{4}\Lambda\Lambda)$ as a function of the scattering length, $a_{\Lambda\Lambda}$. The solid squares were obtained using the NSC97f(FG) $\Lambda N$ potential and the solid circles by the Set A potential. The open squares are the result of the $d\Lambda\Lambda$ three-body model, taken from Ref. [1]. The straight lines were drawn only for the sake of a guide to the reader.

through the convergence of the energy is rather slow, the energy obtained is clearly lower than the $^{3}\Lambda H + \Lambda$ threshold, and we found that the ground state is particle stable (see Fig. 3). This is a genuine four-body calculation, and the calculated $B_{\Lambda\Lambda}(^{4}\Lambda\Lambda) \sim 0.4$ MeV is slightly larger than that from the $d\Lambda\Lambda$ three-body calculation by FG, as shown in Fig. 2.

As can be seen in Fig. 2, the difference in the $B_{\Lambda\Lambda}$ values between the present four-body model and the FG three-body model becomes larger as the strength of the $\Lambda\Lambda$ interaction increases. Moreover, the two lines (labelled “$p\Lambda\Lambda\Lambda$” and “$d\Lambda\Lambda\Lambda$”) in Fig. 2 seem to meet each other at the point where $a_{\Lambda\Lambda} = 0$ fm. This means that the polarization of the $pn$ subsystem is small, and that the $d\Lambda\Lambda$ model is a good approximation if the $\Lambda\Lambda$ interaction is very weak. The polarization of the deuteron subsystem grows as the strength of the $\Lambda\Lambda$ interaction increases. Table II lists the energy expectation values for the proton and neutron subsystem in each ($p, n$) nucleus, and also the root-mean-square distances between a $p$ and an $n$, or between a nucleon and a $\Lambda$. Here, $T_c$ is the kinetic energy of the $\Lambda N$ subsystem, which is defined by $T_c = (p_1 - p_2)^2/4m_N$. The table shows that the influence of the $\Lambda$ particle upon the internal structure of the $pn$ subsystem becomes large as the $\Lambda$ particle comes close to the nucleon. Especially in the case of a strongly attractive $\Lambda\Lambda$ potential, the change in the internal energy ($E_c$) of the rms distance ($\langle r_{NN}^2 \rangle$) is significant.

As can be seen in Fig. 2, the $B_{\Lambda\Lambda}$ value is sensitive to the choice of the $\Lambda N$ potential. For the purpose of predicting whether $^{4}\Lambda\Lambda\Lambda$ exists as a particle-stable bound-state, the $\Lambda N$ potential has to be examined carefully. Table II compares the $B_{\Lambda}$ values of $A = 3, 4$ hypernuclei. The calculated $B_{\Lambda}$ value of the $A = 4$ system using NSC97f(FG) is larger than that using Set A, or larger than the experimental value. Particularly, the $^{3}\Lambda H$ strength of NSC97f(FG) is apparently too strong to reproduce the $B_{\Lambda}(^{4}\Lambda\Lambda)$ value, though the NSC97f(FG) reproduces reasonably well the $B_{\Lambda}(^{3}\Lambda H)$ value. It would, therefore, be rash to conclude that $^{3}\Lambda H$ has a particle-stable bound-state, though the present four-body calculation with the NSC97f(FG) gives a bound-state solution, even for a weaker $\Lambda\Lambda$ interaction, such as $a_{\Lambda\Lambda} = -0.77$ fm.

The present four-body calculation gives quite a different result from that of the FY study discussed in Ref. [1]. At present, we have no clear explanation for why the FY
search for $^{\Lambda \Lambda}_3H$ has not found a bound-state solution. We also checked the accuracy of the present variational calculation by examining the virial theorem. For an exact eigenstate of the hamiltonian, $H = T + V$, we have

$$
\langle T \rangle = \frac{1}{2} \langle W \rangle, \quad \text{with} \quad W = \sum_{i=1}^{A} r_i \cdot \frac{\partial V}{\partial r_i}.
$$

(4)

For the four-body calculation, we obtained the ratio $2\langle T \rangle / \langle W \rangle = 1.000016$ for a weak $\Lambda \Lambda$ potential ($a_{\Lambda \Lambda} = -0.77$ fm), and the ratio 0.999962 for a strong $\Lambda \Lambda$ potential ($a_{\Lambda \Lambda} = -2.8$ fm). Therefore, we think that the present four-body calculation gives a virtually exact eigenenergy, and that the $B_{\Lambda \Lambda}$ value obtained by a four-body calculation for $pn\Lambda\Lambda$ should be close to (and slightly larger than) the energy given by the $\Lambda\Lambda$ three-body model. In comparison with the $\Lambda\Lambda$ three-body model (in Fig. 2), the present result seems to be reasonable, in contrast to that of the FY four-body calculation in Ref. [1].

The contribution to the binding energy from the higher partial wave components are marginal. The present potentials are all central and have Gaussian form factors. This Gaussian radial form (e.g., $V e^{-\kappa(r_i - r_j)^2}$) is rewritten so as to be valid for each angular momentum in terms of nonlocal potentials:

$$
V e^{-\kappa(r_i - r_j)^2} = \int dr dr' \langle \delta(r_i - r_j - r') \rangle \delta(r_j - r_j - r) \times \sum_{l=0}^{\infty} V e^{-\kappa r^2} \delta(r - r) \sum_{lm} Y_{lm}(\hat{r}) Y_{lm}(\hat{r}).
$$

(5)

We also calculated the binding energies in which the $NN$, $\Lambda N$, and $\Lambda\Lambda$ potentials are restricted to be valid only for the $l = 0$ component. The binding energy calculated is $B(\Lambda\Lambda^4H) = 2.382$ MeV ($a_{\Lambda\Lambda} = -0.77$ fm), or $B(\Lambda\Lambda^4H) = 2.842$ MeV ($a_{\Lambda\Lambda} = -2.8$ fm). Each energy is still lower than the $\Lambda\Lambda^3H + \Lambda$ threshold. With $l = 0$ truncated interactions, we obtained $B(\Lambda\Lambda^3H) = 2.365$ MeV.

We should emphasize that in the study of $\Lambda\Lambda^3H$ the $^3S_1$ $\Lambda\Lambda N$ interaction has to be determined very carefully, since $B_{\Lambda\Lambda}$ is sensitive to the $^3S_1$ channel of the $\Lambda\Lambda N$ interaction. Therefore, a check of the $\Lambda\Lambda N$ potential concerning the observed binding energy of only the subsystem, $\Lambda\Lambda^3H$, is insufficient.

One might think that the spin-doublet structure of $A = 4 \Lambda$ hypernuclei is a means of determining the $^3S_1$ $\Lambda\Lambda N$ interaction. However, this strategy without any explicit $\Sigma$ admixture would lead us to a serious problem concerning the $A = 5$ anomaly. According to recent studies, taking account of the explicit $\Sigma$ degrees of freedom, the $\Lambda\Lambda N - \Sigma N$ coupling, plays a crucial role in the binding mechanism of $s$-shell $\Lambda$ hypernuclei. In other words, even the spin-doublet structure of $A = 4 \Lambda$ hypernuclei does not pin down the $^3S_1$ $\Lambda\Lambda N$ interaction, and the $\Lambda\Lambda N$ potential used in the study of $\Lambda\Lambda^3H$, has to be tested on a complete set of the observed $s$-shell $\Lambda$ hypernuclei. Moreover, the $\Lambda\Lambda - \Sigma N$ coupling effects should be explicitly taken into account, because the Pauli suppression effect in the $\Xi$ channel of $\Lambda\Lambda^3H$ is appreciably large. A theoretical search for $\Lambda\Lambda^4H$ is still an open subject.

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