Abstract

We exactly calculated the parity-odd term of the effective action induced by the fermions in 2+1 dimensions at finite chemical potential and finite temperature. It shows that gauge invariance is still respected. A more general class of background configurations is considered. The knowledge of the reduced 1+1 determinant is required in order to draw exact conclusions about the gauge invariance of the parity-odd term in this latter case.

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Thanks to the exotic mathematical structure and the possible relevance to condensed matter physics in two space dimensions, Chern-Simons(CS) models have drawn much attention in the past decade.\[1\]\[2\](For a review, see \[3\]). The CS term can be either put in by hand, or more naturally, induced by fermion degrees, as a part of the original (effective) lagrangian. Two properties of the CS action are fundamental. One is that it is odd under parity transform due to the presence of three dimensional Levi-Civita symbol. The other is that it is invariant under small gauge transforms while non-invariant under large gauge transforms (those not to be continuously deformed to unity and thus carrying non-trivial winding numbers)\[4\]. In the free spacetime whose topology is trivial, the homotopy group $\pi_3$ is trivial in the Abelian case. But there may be nontrivial large gauge transformations if the gauge fields
are subject to non-trivial boundary conditions (for a more recent discussion see\cite{5}). In general, if there exists non-trivial $\pi_3$, the quantum theory is consistent only if the CS parameters are quantized. There then arises a problem: what happens to the quantized parameters by quantum corrections? In the zero temperature, the induced CS term is well-understood\cite{3}-\cite{14}. But at finite temperature, it was argued\cite{11} that the coefficient of the CS term in the effective action for the gauge field should remain unchanged at finite temperature. Yet, a naive perturbative calculation that mimics that at zero temperature leads to a CS term with a parameter continuously dependent on the temperature\cite{11}-\cite{12}. Therefore, the behavior under gauge transforms seems to be temperature-dependent. The problem of quantum corrections to the CS coefficient induced by fermions at finite temperature was re-examined\cite{13}, where it was concluded that, on gauge invariance grounds and in perturbation theory, the effective action for the gauge field can not contain a smoothly renormalized CS coefficient at non-zero temperature. Obviously, it is necessary to obtain some exact result in order to reconcile the contradiction. More recently, the effective action of a (0+1) analog of the 2+1 CS system was exactly calculated\cite{14}. It shows that in the analog, the exact finite $T$ effective action, which is non-extensive in temperature, has a well-defined behavior under a large gauge transformation, independent of the temperature, even though at any given finite order of a perturbation expansion, there is a temperature dependence. So it implies that the discussions of the gauge invariance of finite temperature effective actions and induced CS terms in higher dimensions requires consideration of the full perturbation series. Conversely, no sensible conclusions may be drawn by considering only the first finite number of terms in the expansion. The course of being exactly calculable is that the gauge field can be made constant by gauge-transformations. Employing this trick, Fosco et al. calculated exactly the parity breaking part of the fermion determinant in 2+1 dimensions with a particular background gauge field, for both Abelian and non-Abelian cases\cite{15}\cite{16}, and the result agrees with that from the $\zeta$-function method\cite{3}. More general background gauge fields were also considered\cite{17}. All these works show that (restricted to that particular ad hoc configuration) gauge invariance of the effective action is respected even when large gauge transformations are considered.

The effect of finite chemical potential should be taken into account whenever discussing the statistical physics of a grand canonical ensemble. It was shown that in 1+1 dimensions, the non-zero chemical potential may contribute a non-trivial phase factor to the partition function\cite{18}. As usual, gauge transform property of the effective action suffers some temperature-dependence. Therefore, it is worthwhile considering the problem by exact computation with some particular background. This is the topic of this paper.

As usual, the total effective action $\Gamma(A, m, \mu)$ is defined as

$$e^{-\Gamma(A, m, \mu)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\int_0^\beta d\tau \int d^2x \bar{\psi} (\partial^2 + ie A^\mu + m - \mu \gamma_3) \psi\right]$$

(1)

We are using Euclidean Dirac matrices in the representation $\gamma_\mu = \sigma_\mu$, and $\beta$ is the inverse temperature. It makes no difference whether the indices are lower or upper. The label 3 refers actually to the Euclidean time component. The fermion fields are subject to antiperiodic boundary conditions while the gauge field are periodic. Under parity transformation,

$$x^1 \to -x^1, x^2 \to x^2, x^3 \to x^3; \psi \to \gamma^1 \psi, \bar{\psi} \to -\bar{\psi} \gamma^1; A^1 \to -A^1, A^2 \to A^2, A^3 \to A^3$$

(2)

($\gamma$ matrices are kept intact). So only the mass term varies under the parity transformation. As in\cite{14}, the parity-odd part is defined as

$$2\Gamma(A, m, \mu)_{\text{odd}} = \Gamma(A, m, \mu) - \Gamma(A, -m, \mu)$$

(3)
It is not an easy task to calculate (3) for general configuration of the gauge field. A particular class of configurations of $A$ for which (3) can be exactly computed is that

$$A_3 = A_3(\tau), A_j = A_j(x), j = 1, 2$$

(4)

This class of gauge fields shares the same feature as in the 0+1 dimensions: the time dependence of the time component can be erased by gauge transformations. Therefore, the Euclidean action can be decoupled as a sum of an infinite 1+1 actions

$$e^{-\Gamma(A,m,\mu)} = \int \mathcal{D}\psi_n(x)\mathcal{D}\bar{\psi}_n(x) \exp\left\{-\frac{1}{\beta} \sum_{-\infty}^{+\infty} \int d^2x \bar{\psi}_n(x)[\mathcal{D} + m + i\gamma^3(\omega + e\bar{A}_3) - \mu\gamma^3]\psi_n(x)\right\}$$

(5)

where $\mathcal{D} = \gamma_j(\partial_j + ieA_j)$ is the 1+1 Dirac operator and $\bar{A}_3$ is the mean value of $A_3(\tau)$. It is seen that the chemical potential in 2+1 dimensions plays the role of a chiral potential in 1+1 dimensions. Let us introduce $\Omega_n$ for convenience, $\Omega_n = \omega_n + e\bar{A}_3$. Since

$$m + i\gamma^3\Omega_n - \mu\gamma^3 = \rho_n e^{i\gamma_3\phi_n}$$

(6)

where

$$e^{2i\phi_n} = \frac{m - \mu + i\Omega_n}{m + \mu - i\Omega_n}$$

(7)

and

$$\rho_n = \sqrt{(m + \mu - i\Omega_n)(m - \mu + i\Omega_n)}$$

(8)

we have therefore

$$\det(\mathcal{D} + \rho_n e^{i\gamma_3\phi_n}) = \prod_{n=-\infty}^{+\infty} \det[\mathcal{D} + \rho_n e^{i\gamma_3\phi_n}]$$

(9)

Explicitly, the 1+1 determinant for a given mode is a functional integral over 1+1 fermions

$$\det[\mathcal{D} + \rho_n e^{i\gamma_3\phi_n}] = \int \mathcal{D}\chi_n \mathcal{D}\bar{\chi}_n \exp\left\{-\int d^2x \bar{\chi}_n(x)(\mathcal{D} + \rho_n e^{i\gamma_3\phi_n})\chi_n(x)\right\}$$

(10)

After implementing a chiral rotation whose Jacobian is wellknown (the Fujikawa method applies also to complex chiral parameters), we obtain

$$\det[\mathcal{D} + m + i\gamma^3(\omega + e\bar{A}_3) - \mu\gamma^3] = J_n \det[\mathcal{D} + \rho_n]$$

(11)

where

$$J_n = \exp\left(-i\frac{e\phi_n}{2\pi} \int d^2x e^{ik\partial_j A_k}\right)$$

(12)

Note that the chiral anomalies, or the Jacobian $J$, depends on the boundary conditions as well. If the system is defined on a torus and the fields are subject to periodic boundary conditions, for instance $A_j(x,y) = A_j(x + L_x, y), A_j(x,y) = A_j(x, y + L_y)$, the trace of $\gamma_5$ in [20] is taken over discrete complete set instead of the continuous plane waves. Thus the momentum integral

$$\int \frac{d^2k}{(2\pi)^2} e^{-k^2} = \frac{1}{4\pi}$$

should be replaced by $\int \frac{L_x L_y}{L_x L_y} \sum_{n_1,n_2} \exp[-(\frac{2\pi}{L_x} n_1)^2 - (\frac{2\pi}{L_y} n_2)^2]$. Using the formula

$$\sum_{n=\infty}^{\infty} e^{-\pi z n^2} = \frac{1}{\sqrt{z}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi}{z} n^2}$$

which holds for any complex $z$ with $Re z > 0$. we have

$$\frac{1}{L_x L_y} \sum_{n_1,n_2} \exp[-(\frac{2\pi}{L_x} n_1)^2 - (\frac{2\pi}{L_y} n_2)^2] = \theta(L_x)\theta(L_y)$$

(13)
where \( \theta(L) = \frac{1}{\sqrt{4\pi}} \sum_{n=-\infty}^{+\infty} e^{-\frac{L^2}{4}n^2} \). In this case, (12) should be replaced by

\[
J_n = \exp(-2ie\phi_n \theta(L_x) \theta(L_y) \int_{L_x \times L_y} d^2x e^{ik \partial_j A_k})
\]

In the following, we only concentrate on the infinite space case since the conclusion on a torus can be obtained by a trivial substitution. Fortunately, we also have \( \rho_n(m) = \rho_n(-m) \) for finite chemical potential. Thus we have immediately

\[
\Gamma_{\text{odd}} = - \sum_{n=-\infty}^{+\infty} \ln J_n = i\frac{e}{2\pi} \sum_{n=-\infty}^{+\infty} \phi_n \int d^2x e^{ik \partial_j A_k}
\]

To calculate \( \sum_{n=-\infty}^{+\infty} \phi_n \), we need to compute \( \prod_{n=1}^{+\infty} \frac{m-\mu+i\Omega_n}{m+\mu-i\Omega_n} \). Using the formula \( \prod_{n=1}^{+\infty} \frac{1}{1-x} = \cos \pi x \) as in [1], we have (a = \( \tilde{A}_3 \))

\[
\prod_{n=-\infty}^{+\infty} e^{2i\phi_n} = \prod_{n=-\infty}^{+\infty} \frac{m-\mu+i\Omega_n}{m+\mu-i\Omega_n} = \frac{\text{ch}_{\beta}(m-\mu) + i\text{sh}_{\beta}(m-\mu)\text{tg}_{\beta}a}{\text{ch}_{\beta}(m+\mu) - i\text{sh}_{\beta}(m+\mu)\text{tg}_{\beta}a}
\]

Therefore

\[
\Gamma_{\text{odd}} = \frac{e}{4\pi} \ln \left[ \frac{\text{ch}_{\beta}(m-\mu) + i\text{sh}_{\beta}(m-\mu)\text{tg}_{\beta}a}{\text{ch}_{\beta}(m+\mu) - i\text{sh}_{\beta}(m+\mu)\text{tg}_{\beta}a} \right] \int d^2x e^{ik \partial_j A_k}
\]

which is quite different from the perturbative conclusion in [19]. (The formula eq(97) there is for an arbitrary background).

Now the low temperature limit can be obtained. It will depend on the relationship between \( m \) and \( \mu \).

(i). If \( m > \mu, m+\mu > 0 \)

\[
\lim_{\beta \to \infty} \Gamma_{\text{odd}} = \frac{e}{4\pi} \beta (ia - \mu) \int d^2x e^{ik \partial_j A_k}
\]

(ii). If \( m - \mu > 0, m+\mu < 0 \),

\[
\lim_{\beta \to \infty} \Gamma_{\text{odd}} = \frac{e}{4\pi} \beta m \int d^2x e^{ik \partial_j A_k}
\]

(iii). If \( m < \mu, m+\mu > 0 \)

\[
\lim_{\beta \to \infty} \Gamma_{\text{odd}} = \frac{e}{4\pi} (-\beta m) \int d^2x e^{ik \partial_j A_k}
\]

(iv). If \( m < \mu, m+\mu < 0 \)

\[
\lim_{\beta \to \infty} \Gamma_{\text{odd}} = \frac{e}{4\pi} \beta (\mu - ia) \int d^2x e^{ik \partial_j A_k}
\]

(v). If \( m = \mu \)

\[
\lim_{\beta \to \infty} \Gamma_{\text{odd}} = \frac{e}{4\pi} (-\beta m + i\beta a) \ln \cos \frac{\beta a}{2} \int d^2x e^{ik \partial_j A_k}
\]

It vanishes in the high temperature limit. It is obvious that the low temperature is very sensitive to the values of \( m \) and \( \mu \), as agrees with the results perturbatively obtained [19].

Since in the large-\( m \) limit (or in the low-density limit), the parity-odd part dominates over the
effective action, and the particle number in the ensemble is \(< N > = \frac{4}{\beta} \frac{\partial}{\partial \mu} \ln Z(\beta, \mu)\), we have from the limits (18) and (21) that the flux should be quantized,

\[
\Phi = < N > \frac{8\pi\hbar}{e}
\]

(23)

which implies that each particle carries flux \(\frac{8\pi\hbar}{e}\) and thus should be of fractional spin \(S_\phi = \frac{1}{8}\). This is different from the conclusion in [22].

The previous results can be extended to a more general class of configurations

\[
A_j = A_j(x), A_3 = A_{30}(x) + \sum_{n \neq 0} A_{3n} e^{\frac{2\pi in}{e}}
\]

(24)

with \(A_j\) are locally defined while \(A_{3n}\) is globally defined. \(A_{3n}\) are constant. The \(x\)-dependence of \(A_{30}\) results in the \(x\)-dependence of \(\rho_n\) and \(\phi_n\). In place of (11), we have in this case

\[
\text{det}[\vec{\rho} + m + i\gamma^3(\omega_n + e\vec{A}_3)(x) - \mu\gamma^3] = \tilde{J}_n \text{det}[\vec{\rho} + \rho_n(x)]
\]

(25)

where \(\vec{\rho}' = \vec{\rho} - \frac{i}{2} \hat{\phi}_n(x) \gamma^3\) which depends on the sign of \(m\) and thus contributes to \(\Gamma_{\text{odd}}\). The Jacobian of the corresponding chiral transformation is

\[
\tilde{J}_n' = \exp\left\{-i \frac{e}{2\pi} \int d^2x [\phi_n(x) e^{jk} \partial_j A_k + \frac{1}{4} \phi_n(x) \Delta \phi_n(x)]\right\}
\]

(26)

Now the contribution to \(\Gamma\) from \(\tilde{J}_n'\), denoted as \(\Gamma^J\) is

\[
\Gamma^J = -\sum_{n=-\infty}^{+\infty} \ln \tilde{J}_n' = i \frac{e}{2\pi} \int d^2x [\sum_{n=-\infty}^{+\infty} \phi_n(x) (e^{jk} \partial_j A_k + \frac{1}{4} \phi_n(x) \Delta \phi_n(x))]
\]

(27)

Now we need to calculate

\[
\int d^2x \sum_{n=-\infty}^{+\infty} \phi_n(x) \Delta \phi_n(x) = -\int d^2x \sum_{n=-\infty}^{+\infty} \partial^j \phi_n(x) \partial_j \phi_n(x)
\]

(28)

The equality holds because \(a(x)\) is globally defined and periodic. Denote \(z_1 = -(m - \mu + ia), z_2 = m + \mu - ia\), we have

\[
\sum_n \partial^j \phi_n \partial_j \phi_n = m^2 \partial^j a \partial_j a \sum \frac{1}{(i\omega_n - z_1)^2 (i\omega_n - z_2)^2}
\]

(29)

Suppose that \(m \pm \mu \neq 0\) so that \(z_1, z_2\) do not locate on the imaginary axis. Now the sum can be evaluated using the method analogous to that employed in [23]

\[
\sum \frac{1}{(i\omega_n - z_1)^2 (i\omega_n - z_2)^2} = -\frac{\beta}{2}\int_C \frac{dz}{e^{\beta z} + 1} \frac{1}{(z - z_1)^2 (z - z_2)^2}
\]

(30)

The contour \(C\) encircles the imaginary axis. Then by deforming \(C\) to one consisting of large arcs \(\Gamma_n : |z| = \frac{2\pi n}{\beta}\) which do not pass \(i\omega_n\) and the parts circumventing \(z_1\) and \(z_2\), we have
Therefore
\[
\sum \beta e^{\beta z_1} = \frac{2}{(e^{\beta z_1} + 1)^2(z_1 - z_2)^2} + \frac{2}{(e^{\beta z_1} + 1)(z_1 - z_2)^3} + \frac{2}{(e^{\beta z_2} + 1)^2(z_2 - z_1)^2} + \frac{2}{(e^{\beta z_2} + 1)(z_2 - z_1)^3} \quad (31)
\]

Hence we have
\[
\sum \beta e^{\beta z_1} = \frac{2}{(e^{\beta z_1} + 1)^2(z_1 - z_2)^2} + \frac{2}{(e^{\beta z_1} + 1)(z_1 - z_2)^3} + \frac{2}{(e^{\beta z_2} + 1)^2(z_2 - z_1)^2} + \frac{2}{(e^{\beta z_2} + 1)(z_2 - z_1)^3} \quad (32)
\]

From (29) we know that the quadratic part in (27) makes no contribution to the parity-odd part, therefore we have
\[
\int d^2 x \sum_{n=-\infty}^{+\infty} \phi_n(x) \Delta \phi_n(x) = \beta m^2 \int d^2 x \partial^i a \partial_j a
\]
\[
	imes \left[ \left( e^{\beta z_1} + 1 \right)^2(z_1 - z_2)^2 \right] + \frac{2}{(e^{\beta z_1} + 1)(z_1 - z_2)^3} + \frac{2}{(e^{\beta z_2} + 1)^2(z_2 - z_1)^2} + \frac{2}{(e^{\beta z_2} + 1)(z_2 - z_1)^3} \quad (33)
\]

From (29) we know that the quadratic part in (27) makes no contribution to the parity-odd part, therefore we have
\[
\Gamma_{\text{odd}}^{\text{J}} = i \frac{\beta e}{2\pi} \int d^2 x \sum_{n=-\infty}^{+\infty} \phi_n(x) e^{ik_j \partial_j A_k}
\]
\[
= \frac{e}{4\pi} \int d^2 x e^{ik_j \partial_j A_k} \ln \left[ \frac{\text{ch} \left( \frac{\beta}{\pi}(m - \mu) + i \text{sh} \left( \frac{\beta}{\pi}(m - \mu) \text{tg} \frac{\beta}{2} S_{\text{CS}} \right) \right]}{\text{ch} \left( \frac{\beta}{\pi}(m + \mu) - i \text{sh} \left( \frac{\beta}{\pi}(m + \mu) \text{tg} \frac{\beta}{2} S_{\text{CS}} \right) \right)} \right] \quad (34)
\]

It is seen from (17) that large gauge invariance is still respected at finite chemical potential as at the vanishing chemical potential for the background (4). For the more general class (24), (34) is also large gauge invariant, but in this case, the second factor in (25) contributes also to the parity-odd part. Therefore, it is necessary to consider this factor in order to investigate whether the gauge invariance of the total parity-odd part is still preserved.

In summary, we have in this letter discussed the parity-odd part of the induced effective action of fermions in 2+1 dimensions. For the class of background field discussed by Fosco et al., the parity-odd part can also be exactly calculated and is also gauge invariant. For a more general background field configuration, the total parity-odd part of the effective action can not be obtained exactly. The total contribution from the chiral rotation can be exactly calculated nevertheless. Accordingly, knowledge of the second factor, a massive fermion determinant, is required in order to draw rigorous conclusions about the total effective action. The generalizations to non-Abelian case as well as to higher odd dimensions are straightforward as in [15] in the case \( \mu = 0 \).

From (17) it is clear that we still have not obtained any exact knowledge for general background gauge field. Since (17) implies in the low-temperature limit, the average value \( a \) will multiply the two dimensional flux, it seems reasonable to make the following conjecture

**Conjecture** For a general background gauge field, the dominant part of the parity-odd term of the induced effective action in 2+1 dimensions is
\[
\Gamma_{\text{odd}} = \ln \left[ \frac{\text{ch} \left( \frac{\beta}{\pi}(m - \mu) + i \text{sh} \left( \frac{\beta}{\pi}(m - \mu) \text{tg} \frac{\beta}{2} S_{\text{CS}} \right) \right]}{\text{ch} \left( \frac{\beta}{\pi}(m + \mu) - i \text{sh} \left( \frac{\beta}{\pi}(m + \mu) \text{tg} \frac{\beta}{2} S_{\text{CS}} \right) \right)} \right] + \ldots \quad (35)
\]

where
\[
S_{\text{CS}} = \frac{\beta e^2}{4\pi} \int d^2 x e^{i\mu a} A_{\mu} \partial_{\nu} A_{\alpha} \quad (36)
\]
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