Energetic cooling below the BEC transition: a quantum kinetic description within the Bogoliubov approximation

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Abstract

The dynamics of Bose–Einstein condensation in a three-dimensional harmonic trap is studied explicitly including the Bogoliubov approximation for temperatures below the critical one. To model the evolution towards equilibrium at each cooling step, we derive quantum kinetic equations that describe the dynamics of the gas for temperatures above and below the transition temperature. These equations, valid in the Born and Markov approximations, consider the essential role of the chemical potential as the main parameter that signals the transition. The kinetic equation that describes the growth of the condensate below the transition temperature is derived within the Bogoliubov approximation. To illustrate our results we propose an energetic cooling protocol and simulate the whole sequence of the formation of a condensate.

Keywords: Bose–Einstein condensation, quantum kinetic equation, Bogoliubov approximation

(Some figures may appear in colour only in the online journal)

1. Introduction

The description of the dynamics of the formation of a Bose–Einstein condensate (BEC) remains as an unabated problem [1–14]. Among several issues, we point out mainly the consideration of the role of the interatomic collisions at different strengths and the proper description of the system when the condensate is present. In this work we readdress the kinetics of the formation of a BEC within a weak-coupling and Markov limits [1–5], but taking into account that for temperatures below the critical one \( T_c \), namely, when the chemical potential becomes zero in the dilute limit, the particles in the separate ground and excited states are not conserved. To consider this situation we derive a master equation, valid for those temperatures only, in which Bogoliubov approximation [14–16] is explicitly used. For temperatures above \( T_c \) we also derive a master equation that is essentially similar to previous ones [1–5].

In actual experiments, the observation of the BEC transition requires to perform several cooling stages in a thermally isolated Bose gas, in which a given equilibrium state is perturbed by evaporating atoms in highly excited states, thus reducing its energy and by irreversible collisional processes, the system lowers its temperature. This cooling is repeated until the system attains the critical temperature \( T_c \) at which the transition to condensation occurs. By contrast, in this article we propose an energetic cooling protocol, in which the temperature is decreased while maintaining the number of particles as a constant. The temperature at each cooling step is lowered by transferring particles from higher energy levels and distributing them in levels with lower energy. This protocol may represent an alternative route for the experimentalists to achieve the degeneration regime with a much larger number of atoms than usual. Although it is not the purpose of this article to describe how such a protocol should be implemented, we suggest that the transferring to lower energy levels may be achieved by means of impulsive anti-Stokes stimulated Raman scattering [17] with properly tailored ultrashort microwave pulses, similarly to techniques used with light in single-molecule wavepacket interferometry.
In this article we present numerical simulations of the obtained kinetic equations within the proposed energetic cooling protocol.

This work is organized as follows. In section 2 we present the derivation of the kinetic master equations. In section 3 we present numerical results that model the energetic cooling process as several stages are performed in a bosonic gas towards the formation of a condensate. Finally, in section 4 a summary of this work is given.

2. Quantum kinetic equations for the formation of a condensate

2.1. Kinetic theory for temperatures above Tc

The Hamiltonian of the system consists of an unperturbed Hamiltonian $H_0$ that takes into account an isotropic harmonic confinement of the Bose atoms, perturbed by a weakly interacting part $V$ that models their binary collisions. In a second quantized formalism these are

$$ H = \sum_i \epsilon_i a_i^\dagger a_i^\dagger + \frac{1}{2} \sum_{ij} \left\langle i j \right\rangle V_{i j} a_i^\dagger a_j^\dagger a_i a_j $$

$$ = H_0 + V, \tag{1} $$

where $\epsilon_i = \hbar \omega (i_x + i_y + i_z)$ are the single particle eigenenergies in a harmonic potential, with $\vec{r} = (i_x, i_y, i_z)$. $a_i^\dagger$ and $a_i$ are the operators that create and destroy particles in the state $\vec{r}$. We have omitted the ground state energy and as we recall below, this fixes the BEC transition at $\mu = 0$. The degeneracy of each energy is given by $g_i = \frac{1}{2}(i+1)(i+2)$. We shall assume that the ultracold gas is very diluted and consequently only s-wave binary collisions are relevant for the dynamics [20–22]. Thus, the interaction term can be written as

$$ \left\langle i j \right\rangle V_{i j} = g \int d^3 \delta_{\vec{r}_i} (\vec{r}) \delta_{\vec{r}_j} (\vec{r}) q_{ij} (\vec{r}) q_{ij} (\vec{r}) = g V_{ij}, $$

where $g = 4\pi \hbar^2 a_s / m$, with $a_s > 0$ the s-wave scattering length and $q_{ij}(\vec{r})$ are the eigenfunctions of the unperturbed Hamiltonian $H_0$ with eigenvalues $\epsilon_i$.

Our starting point is the derivation, in the weak coupling limit and the Markov approximation, of an irreversible master equation for the one body reduced density matrix operator [23–27]. By following a standard derivation procedure one arrives to the final form of the equations that satisfy the occupation numbers $n_i$

$$ \frac{\partial n_i (\tau)}{\partial \tau} = - \sum_{ijkl} \left\{ V_{i j k l}^2 \delta_{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l} \right\} $$

$$ \times \left[ n_i n_j (1 + n_i + n_j) - n_i n_j (1 + n_i + n_j) \right] $$

$$ + 2 V_{ij} n_i n_j \delta_{\epsilon_i + \epsilon_j} n_i n_j (n_i - n_j), \tag{2} $$

where $\tau = (8a_l^2 m\omega^2 / \hbar) \tau$ is a dimensionless time. It is of interest to note that the last term in the above equation typically does not appear in previous derivations [1–5]. We should point out that kinetic equations in the literature are the result of considering ergodic assumptions for the one body reduced density matrix, at the start of the derivations. Nevertheless such a starting point is actually valid in the case of an isotropic and homogeneous system [28]. It is important to emphasize that although this term should contribute in general, it does not when an ergodic approximation is made, as we shall see below.

By a simple substitution one can show that the steady state solution of equation (2), for all single particle states $\vec{n}$, including the ground state $\vec{n} = (0, 0, 0)$, is the Bose–Einstein distribution

$$ n_{\vec{n}} = \frac{1}{\exp[ (\epsilon_{\vec{n}} - \mu) / kT] - 1}, \tag{3} $$

where the temperature $\beta = 1 / kT$ and the chemical potential $\mu$ can be obtained by imposing the associated normalization conditions on the total energy $E = \sum_{\vec{n}} \epsilon_{\vec{n}} n_{\vec{n}}$ and on the total number of particles $N = \sum_{\vec{n}} n_{\vec{n}}$. However, if we consider $\mu = 0$ in the Bose–Einstein distribution (3) and substitute it in on the right-hand side of equation (2), we find that the left-hand side is zero if and only if $T = 0$. This result is inconsistent with the fact that a non-uniform ideal Bose gases can exhibit the Bose–Einstein transition in two and three dimensions. Therefore, taking into account the underlying physics, an alternative kinetic equation for $\mu = 0$ and $T < T_c$ must be derived to properly describe the time evolution and equilibration of the gas.

2.2. Kinetic theory for temperatures below $T_c$

The most important property of a dilute Bose gas that has experienced the transition to the condensate state is that as a result of having chemical potential equal to zero, the ground state becomes macroscopically populated. As a consequence of this fact the intensive thermodynamic variables, that adjust to the extensive ones, are exclusively determined by the particles occupying the excited states only. Perhaps a most relevant effect when a Bose gas has chemical potential equal to zero, is that the number of particles $N$ is no longer an independent thermodynamic variable. Although in the whole system the number of particles $N$ remains constant, the number of particles in the ground and excited states are not separately conserved since there is no energy cost in adding particles from the excited to the ground state and viceversa. Therefore, if one looks at the isolated fractions of particles in the ground and excited states, a continuous process of creation and annihilation of particles (having $c_i = 0$) takes place in each one of them.

The binary collisions are the key elements where the non conservation of particles can be taken into account. Since such collisions are the elementary mechanism to reach thermal equilibrium and they involve particles in the ground and excited states, we must have special care in each collision event. Up to now, each pair of particles coming from either the ground state, or the excited states, transform after the collision into a pair of particles that again can occupy the...
ground state or the excited states, in such a way that there is no distinction between the fractions of particles in the ground and excited states respectively. Considering the fact that the occupation number in the ground state becomes macroscopic \( N_0 \sim N \), we can proceed in a different way treating separately the ground state and the excited states. In particular, we can explicitly assume that below the critical temperature, the fraction of the particles in the excited states participating in the collision process is not conserved. This procedure lead us to derive a new kinetic equation for the excited states. Then, by considering the fact that the total number of particles is conserved we can obtain the number of particles in the ground state by subtracting the particles in the excited states from the total number of particles \( N \). To implement this effective non-conservation of particles, we shall consider that the creation and annihilation operators involving the ground state \( a_0^\dagger \) and \( a_0 \) can be substituted by \( a_0^\dagger = a_0 = \sqrt{N_0} \), with \( N_0 \) the number of particles in the condensate. This is the Bogoliubov approximation [29], which has already been considered previously in describing equilibrium phase transition of bosonic excitons [30].

To justify our suggestion let us consider one of the terms of the interaction potential, say for example

\[
V = 2 \sum_{jkl} V_{jkl} a_j^\dagger a_k^\dagger a_l^\dagger a_i^\dagger + \text{h.c.}
\]

this term represents the annihilation of particles in the states \( \vec{k} \) and \( \vec{l} \), and creation of particles in the states \( \vec{j} \) and \( \vec{0} \). If such a term is substituted by

\[
V = 2 \sqrt{N_0} \sum_{jkl} \sqrt{V_{jkl}} a_j^\dagger a_k^\dagger a_l^\dagger a_i^\dagger + \text{h.c.}
\]

the number of particles in the excited states is explicitly not conserved, although of course, the missing particle belongs to the ground state. After a straightforward algebra is done, we obtain

\[
\frac{\partial n_\eta (\tau)}{\partial \tau} = - \sum_{ijkl} \left\{ V_{ijkl}^2 \delta_{\eta i} \delta_{\eta j} \left[ n_\bar{k} \left( 1 + n_\bar{\eta} + n_\bar{\eta} \right) - n_\bar{k} n_\bar{\eta} (1 + n_\bar{\eta} + n_\bar{\eta}) \right] + 2 V_{ijkl}^2 \sqrt{V_{ijkl}} \delta_{\bar{\eta} \eta} n_\bar{k} n_\bar{\eta} (n_\bar{\eta} - n_\bar{\eta}) \right\} + N_0 \sum_{jkl} V_{ijkl}^2 \delta_{\eta j} \delta_{\eta k} n_\bar{\eta} (n_\bar{\eta} - n_\bar{\eta})
\]

\[
\times \left\{ 2 \delta_{\bar{\eta} \eta} \gamma + \delta_{\bar{\eta} \eta} \left[ n_\bar{\eta} \left( 1 + n_\bar{\eta} + n_\bar{\eta} \right) - n_\bar{\eta} n_\bar{\eta} \right] \right\} + \frac{n_\bar{\eta} n_\bar{\eta} - n_\bar{\eta} (1 + n_\bar{\eta} + n_\bar{\eta})}{n_\bar{\eta} n_\bar{\eta} - n_\bar{\eta} (1 + n_\bar{\eta} + n_\bar{\eta})}
\]

\[
+ 2 N_0 \sum_k \delta_{\eta i} \delta_{\eta k} V_{ijkl}^2 \frac{\gamma}{\Delta_{ij}} (n_k - n_\eta).
\]

(4)

with

\[
N_0(\tau) = N = \sum_{\eta \neq 0} n_\eta(\tau),
\]

being \( N_0 \) is the number of particles in the ground state. By substituting

\[
n_\eta = \frac{1}{\exp\left( e_\eta/kT \right) - 1},
\]

one can show that this is the correct stationary solution of equation (4) for the excited states, that is, the Bose–Einstein distribution for zero chemical potential.

Master equations using the Bogoliubov approach has already been considered [14]. In that approach the Bose gas is coupled to a thermal reservoir, and the thermalization is mediated by the interactions between the reservoir and the Bose gas and by interparticle interactions. Here we propose a scenario where the condensation takes place into an isolated system, and therefore, the inclusion of a reservoir is not needed.

3. Simulation of the cooling process

In this section we present numerical results that model the cooling process as several cooling stages are performed, both for \( T > T_c \) and \( T < T_c \). By solving equations (2) and (4) for \( \mu < 0 \) and \( \mu = 0 \) respectively. But before doing so, we shall also point out along the way the difficulties posed by a finite-size calculation. We use dimensionless units \( h = k_B = \omega = 1 \).

3.1. Numerical limitations

The sum of the states on the right hand side in the kinetic equations for \( \mu < 0 \) and \( \mu = 0 \) (equations (2) and (4)) extend to infinite, and therefore, an infinite number of coupled equations for the occupation numbers \( n_\eta \) must, in principle, be solved. In a real situation, however, prior to the first cooling stage, the gas has been precooled by some other methods and the atoms in the Bose gas do not occupy very high energy levels, namely, the ratio of their occupation numbers compared to those in the lowest levels is negligible. A simple calculation shows that for \( T \approx 10^{-6} \text{ K} \) and a typical trap frequency \( \omega \approx 2\pi \text{ (100) Hz, the maximum level occupied in the harmonic confining potential corresponds to} \mathcal{N} \approx 100 \text{ energy levels. In principle, considering such as the highest occupied level would be a good starting point for studying the dynamics of the condensation, and thus solving a finite number of equations. However,} \mathcal{N} \approx 100 \text{ is still an unattainable situation. As we discuss below, we shall use} \mathcal{N} = 40 \text{ as the maximum energy level within an ‘ergodic’ approximation.}

Two numerical tasks become impracticable as the highest energetic level in the sums of kinetic equations grows. One is the number of coupled kinetic equations that have to be solved and the other is the determination of the collision integrals \( V_{ijkl} \). Because the degeneracy grows with the energy level, the number of coupled equations to be solved increases exponentially. For instance, for \( \mathcal{N} = 100 \text{ energy levels, there are 171 700 states, while for} \mathcal{N} = 40 \text{ one finds 11 480 states, and thus, those are the number of linear coupled equations to solve in each case. On the other hand, since the right hand side of equations (2) and (4) involves an enormous number of sums, to be calculated at each time step, the calculation is even harder. For example, for} \mathcal{N} = 40 \text{, the number of operations per iteration is of the order of} 10^{15}. \text{ We have estimated the time needed for this in a standard workstation,}

\[ 3 \]
yielding the order of $10^7$ seconds. This poses a very stringent restriction in the number of states (and thus, levels) that we can practically include. Therefore, we shall adopt the ‘ergodic assumption’ proposed in [1] to deal with the dynamical description of the interacting Bose gas.

The basic idea of the ergodic assumption is that the number of particles in a state depends only on the energy of the state $\epsilon_i = \hbar \omega_i$ just as in thermal equilibrium. We then write $n_{m} \equiv n_{m}$, namely asserting that states with the same energy have the same population. To implement the ergodic hypothesis, we assume that the initial state is one of thermal equilibrium and then assume it remains ergodic throughout. Using this hypothesis we define an ergodic collision kernel [1] as

$$I_{\text{col}} = \delta_{\epsilon_{j} + \epsilon_{k} + \epsilon_{l} + \epsilon_{i}} \sum_{i, m, k, l} \frac{V_{j}^2}{m} \prod_{j} \delta_{\epsilon_{j}},$$

with $j$ standing for all values $(i, m, k, l)$. This collision kernel must be numerically obtained. Here we have calculated the kernel for $\epsilon_j \leq 40 \alpha$. The ergodic kinetic equation for $T > T_c$ is now

$$\frac{d n_{m}(c)}{dt} = -\sum_{i, m, k, l} \frac{V_{j}^2}{m} \prod_{j} \delta_{\epsilon_{j}} \left[ n_{m}(1 + n_{m} + n_{i}) - n_{i} n_{m}(1 + n_{k} + n_{l}) \right],$$

while the kinetic equation for $T < T_c$ becomes

$$\frac{d n_{m}(c)}{dt} = -\sum_{i, m, k, l} \frac{V_{j}^2}{m} \prod_{j} \delta_{\epsilon_{j}} \left[ n_{m}(1 + n_{m} + n_{i}) - n_{i} n_{m}(1 + n_{k} + n_{l}) \right] + \sum_{i} I_{\text{col}} \left\{ \delta_{\epsilon_{i}} n_{i} (1 + n_{m} + n_{i}) \right\} \times \left[ n_{k} (1 + n_{i} + n_{m}) - n_{i} n_{m} \right] + \delta_{\epsilon_{i} + \epsilon_{k} + \epsilon_{l} + \epsilon_{i}} \left[ n_{m} - n_{i} (1 + n_{i} + n_{m}) \right],$$

and we recall that the sums in the former case start with 0, while in the latter they begin with 1. Using the ergodic approximation lead us to reduce significantly the number of coupled equations. To the best of our knowledge, $N = 40$ levels for the calculation of the ergodic collision integrals $I_{\text{col}}$ is the highest performed so far.

### 3.2. Energetic cooling protocol

For our study we propose an energetic cooling protocol. It consists in maintaining fixed the number of particles during the whole cooling process while several reductions of energy lead the system to decrease its temperature. The basic idea of cooling consists in transferring particles from higher energy levels to lower ones. The system, being then out of equilibrium, will evolve to the equilibrium state via collisions as described by the kinetic equations. This process is successively repeated to lower the system temperature at each ‘evaporative energetic step’. Figure 1(a) illustrates a possible sequence of an energetic cooling process. The horizontal dashed (red) line shows a non-equilibrium process at constant number of particles $N$ that, starting from a temperature $T > T_c$ reaches BEC, by crossing the solid (black) line which represents the number of particles in the excited states $N_{\text{exc}}$, as a function of the critical temperature $T_c$, namely, at the onset of Bose–Einstein condensation. The latter is given by

$$N_{\text{exc}} = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_{i} - \mu)} - 1} \bigg|_{\mu=0}.$$

The numerical simulation is initiated by distributing $N$ particles following the Bose–Einstein statistics in the first 40 levels. In the first energetic cooling step we remove the particles lying, say, in level $m$, and distribute them uniformly in the level $m - 1$ for $m \leq 40$. The system is allowed to evolve through equation (7), for $\mu < 0$ which corresponds to $T < T_c$ until it reaches the equilibrium state associated to this first cooling stage. This process is successively repeated to follow the transition towards the condensation transition. Once the transition is crossed, the dynamics should follow that of equation (8), corresponding to $\mu = 0$ and $T < T_c$. As we shall discuss in detail, due to the finite number of particles, there is a clear crossover between the solutions of the two ergodic sets of equations, rather than a sharp, discontinuous transition, as expected in the thermodynamic limit.

To make use of the energetic cooling protocol we prepare the initial state considering the maximum energy level allowed by our computational capabilities. To ensure that all the particles participate in the dynamics during each energetic cooling step, it is necessary to select initial appropriate values of $\mu$ and $T$. In figure 1(b) we plot the occupation number as a function of energy level in a harmonic oscillator for three pairs of values of $T$ and $\mu$, indicated in the figure. As it can be seen from this plot, the number of particles below the level $N = 40$ depends on the values of $T$ and $\mu$. Given a fixed values of $T$ and $\mu$, one can estimate the relative number of particles above $N = 40$ through

$$\Delta N = \frac{N_{\text{exc}}(T, \mu) - N_{\text{exc}}(T, 0)}{N_{\text{exc}}(T, 0)}$$

with $N_{\text{exc}}(T, \mu) = \sum_{i=0}^{N} \frac{1}{e^{\beta(\epsilon_{i} - \mu)} - 1}$ and $N_{\text{exc}}(T, 0)$ the corresponding value of $N$ for an ‘infinite’ number of states $N \to \infty$.

We have selected two pairs of initial values of temperature $T$ and chemical potential $\mu$ to illustrate the predictions of the kinetic equations for $\mu < 0$ and $\mu = 0$ (equations (7) and (8)). These are, $(T = 1.5, \mu = -2.1)$ and $(T = 1.0, \mu = -0.25)$, ensuring that less than 6% and 8% of particles respectively will remain above the $N = 40$ level. The number of particles in each case is $N_{\text{exc}}(1.5, -2.1) = 100$ and $N_{\text{exc}}(1.0, -0.25) = 500$. The critical temperatures for

1. To solve the coupled system we use the Euler method, selecting an appropriate temporal step to ensure the conservation of the total energy and the total number of particles.
2. The criterion that we use to identify the situation of equilibrium at each cooling stage, is that in two consecutive time steps the value of each $n_{m}(r)$ does not change in more than $10^{-12}$. 

4
$N = 100$ and $N = 500$ are $T_c = 0.801$ and $T_c = 0.854$ respectively.

To analyze the predictions of equations (7) for $N = 100$, namely for the initial state $T_1.5$, $2$. In figure 2(a) we plot the growth of the normalized number of particles in the ground state $n_0 = N_0/N$, for $N = 100$, as a function of dimensionless time $\tau$, with a time step of $5 \times 10^{-5}$. As it can be seen from figure 2(a), the growth of $n_0$ as a function of time is not a linear function. At earlier cooling stages there is an efficient cooling. Then, once the system enters into the condensation transition the growth rate of particles in the ground state decreases. The cooling process modeled through the protocol allows us to follow the evolution of the temperature and chemical potential as consecutive energetic cooling stages are carried out. At each energetic cooling stage the gas thermalizes at well defined values of $\mu$ and $T$; these are obtained by fitting the stationary distributions $n_m$ at each stage to the Bose–Einstein distribution either for $\mu < 0$ or $\mu = 0$.

In figure 2(b) we plot the stationary distributions of the occupation numbers $n_m$. Different curves correspond to equilibrium values $T$ and $\mu$, at different cooling stages as indicated in the legend. The curves lying on the points correspond to the best Bose–Einstein distribution fit. As one can observe from this figure, as $T$ decreases and $\mu$ approaches zero from below, the number of particles in the condensate becomes the majority with respect to those associated to the excites states. As a matter of fact in figure 2(b) we have omitted the stationary distributions of $n_0$ associated to $T = 0.92$ and $0.0074 \mu = -2.1$, since they would appear out of scale. Different curves in figure 2(b) indicate the path towards the onset of degeneracy. Two important facts we stress here, first, that around $T \approx 0.9$, BEC seems to develop; but second, due to finite, actually rather small, number of particles, the chemical potential does not become zero, although it does becomes very small. That is, it is not possible to observe a sharp transition. This can be further observed in figure 3, where we plot the chemical potential $\mu$ as a function of $T$, for $N = 100$ and $N = 500$ number of particles. From this figure one can observe how for $N = 500$ the transition becomes more apparent and it should tend towards a true sharp transition in the thermodynamic limit $N \rightarrow \infty$ [31]. This behavior may seem to indicate that, since the chemical potential does not become zero unless in
the thermodynamic limit, the kinetic equations for $\mu < 0$ may suffice to describe the dynamics correctly, as done in [1, 5] for example. It is the purpose of this section to highlight the importance of the kinetic equations for $\mu = 0$.

To enquire the predictions of kinetic equations for $\mu = 0$ and then to contrast them with those associated to the equation for $\mu < 0$ we proceed in an analogous way to that described above. That is, we now solve equation (8) for $\mu = 0$. We illustrate our numerical results for $N = 500$ initiating at $T = 1.0$ ($T_c = 0.854$). Since the chemical potential in this case is zero we obtain the value of $T$ by fitting to the stationary occupation numbers the ideal Bose–Einstein distribution for $\mu = 0$.

In figure 4(a) we plot the time evolution of the condensate $n_0$ during the whole energetic cooling process. Once more, at the start, the cooling is very efficient, but as $T \to 0$, it slows down since there are very few particles in the excited states. In figure 4(b), we plot the number distribution as a function of the energy level, for different temperatures. BEC is clearly described for $T < T_c$ by using the corresponding equation (8) tailored for $\mu = 0$. This dynamics is very precise, less sensitive to finite size effects, and can be extended with no problem all the way to $T = 0$. We now compare the solutions both for $\mu < 0$ with $\mu = 0$, in the light of finite size effects and the limited number of levels we can use.

In figures 5(a) and (b) we illustrate in a diagram $n$ versus $T/T_c$ the stationary normalized number of particles in the ground $n_0$ and excited states $n_{exc}$ for $N = 500$ along the cooling process. Figure 5(a) shows a cooling process starting at a temperature $T > T_c$ and going to a temperature as low as possible, using the kinetic equations for $\mu < 0$, equation (7). Figure 5(b) shows a cooling initiating at $T > T_c$ but using equations (8) valid for $\mu = 0$ only and, again, going towards $T = 0$. The curves in black are the results in the thermodynamic limit, showing the above mentioned sharp transition. Several comments are in order. First of all, the solution for $\mu < 0$, while the BEC transition is rather smooth, simply due to finite size effects, it appears that it can be extended to very low temperatures, near $T = 0$, where it is supposed no to be quite valid. This robustness is because the chemical potential can become very small, though not quite zero, see figure 3. As we discuss below, the solution for $\mu < 0$ becomes numerically unstable as $T \to 0$. On the other hand, the kinetic equations for $\mu = 0$ become extremely good in the vicinity of $T = 0$, notwithstanding the small number of particles. This is so, because $\mathcal{N} = 40$ energy levels suffices to describe the part of the gas in the excited states. As expected, this solution becomes less confident as $T \to T_c$ from below, and similarly to the equations for $\mu < 0$ near absolute zero, the numerical description for $\mu = 0$ becomes unstable as BEC transition is approached. One expects that in the limit of very large number of particles, and large number of levels included in a numerical calculation, the limitations of both descriptions in the regions where are not supposed to be valid, should be more apparent.

Figures 6(a) and (b) show details of calculations near $T = 0$ and $\mu < 0$, equation (7), and for near $T_c$ and $\mu = 0$, equation (8), respectively. The purpose of these figures is to exemplify the numerical instabilities of the corresponding descriptions. Figure 6(a) shows the number of particles in the condensate, with $\mu < 0$, with a very large number of particles, $N = 10^5$, which can be performed due to the very small value of the temperature. One sees, first, that the solution deviates from that of the thermodynamic limit (solid line), then it becomes unstable. The simpler explanation is that, at these temperatures $\mu$ must be zero, but the fitted Bose–Einstein distribution given by equation (3) cannot take $\mu = 0$ for a divergence would occur. Although not shown here, the solution for $\mu = 0$ fits smoothly the solid curve all the way down to $T = 0$. On the other hand, figure 6(b) shows the stationary total occupation number in the excited states for $\mu = 0$ and $T < T_c$. Similarly to the previous case, as $T_c$ is approached, a numerical difficulty arises in the form of a double valued solution. Again, this exhibits that this solution requires not only more particles but more equations for higher energy levels. It is satisfactory, nevertheless, to find that, apart from near the vicinity of $T_c$, the solution for $\mu = 0$, which explicitly takes the Bogoliubov approximation into account, fits extremely well the curves of the thermodynamic limit.

4. Final remarks

We have studied an energetic cooling process that leads a low-interacting Bose gas confined in a harmonic potential to reach Bose–Einstein condensation, using different kinetic equations for temperatures above and below the critical one, $T_c$. While the equations for $T > T_c$ are similar to previous ones [4, 5, 11], the novelty of our work lies on the fact that, for $T > T_c$, we explicitly take into account the non-conservation of particles in the thermal, non-condensed part of the gas. This is done by implementing Bogoliubov approximation in the dynamics. With these equations we model an energetic cooling process that while removing energy from the system, keeps constant the number of particles, an essential difference from the typical
evaporative cooling procedure. In our numerical simulation we were able to handle with up to 40 energy quantum levels of the harmonic oscillator in 3D. To the best of our knowledge this is the highest energy level numerically computed up to now. We solved the system of coupled kinetic equations considering the ergodic assumption. The cooling sequence allowed us to follow the sequence of the growing of particles in the ground and excited states. It should be of interest to make some form of comparison between the proposed energetic cooling with the usual evaporative one. While it is difficult to attempt a direct testing between them, because the former is at constant number of

Figure 4. (a) Growth of the normalized number of particles in the ground state $n_0 = N_0/N$, for $N = 500$, as a function of dimensionless time $\tau$ along the whole sequence of energetic cooling, using the the kinetic equation (8) for $\mu = 0 (T < T_c)$, (b) Stationary distributions of the excited states for some energetic cooling steps. Different curves correspond to different values of $T$ as indicated in the labels.

Figure 5. Diagrams of the normalized number of particles $n$ as a function of $T$ along the whole sequence of energetic cooling. The (red) dotted curve that starts at $n = 0$ corresponds to the normalized number of particles in the ground state $n_0$, while the (red) dotted one that begins at $n = 1$ corresponds to the particles in the excited states $n_{\text{exc}}$. The (black) solid lines correspond to the thermodynamic limit. Figures (a) and (b) correspond to equations (7) and (8) respectively. In both cases, $N = 500$.

Figure 6. Details of the normalized number of particles $n$ as a function of $T$ along a reduced stage of the energetic cooling. (a) The (green) dotted line is the normalized number of particles in the ground state $n_0$, using equation (7) valid for $T > T_c$, with $N = 10^5$; (b) The (blue) dotted line is the normalized number of particles in the excited states $n_{\text{exc}}$, using equation (8), valid for $T < T_c$, with $N = 500$. The (black) solid curve corresponds to the thermodynamic limit. These calculations exemplify the limitations of the equations outside their range of validity, see text for further discussion.
particles and the latter reduces it at each cooling step, one can nevertheless compare typical times needed to reach condensation. In this regard, the characteristic times for the thermalization and evaporation processes have been experimentally determined [22, 32] and numerically verified [11]. For our energetic protocol those times can be estimated in 100–1000 ms for the thermalization process and 1–10 s for the whole energetic protocol process. These times were determined considering a proper time scale in the experiment, namely, the frequency of the harmonic trap. Such quantity is determined considering a proper time scale in the experiment, the characteristic times for the condensate fraction in the ground state with a thermal cloud never compare typical times needed to reach condensation. In this regard, the characteristic times for the condensate fraction in the ground state with a thermal cloud

\[ 0.1 < N \left( \frac{\hbar}{kT} \right)^3 < 10 \] [33]. One should recall that the influence of the harmonic environment leads to identify the harmonic density as \( N/\sqrt{V} \) and the thermodynamic limit as \( N \to \infty \) and \( V \to \infty \) with \( N/\sqrt{V} = \text{constant} \) [34]. Our numerical results are in the same order of magnitude with the current experiments. One must insist, however, that in the present case the Bose gas reaches a steady state given by a condensate fraction in the ground state with a thermal cloud of noninteracting particles, thus, further efforts should be pursued to obtain a weakly interacting gas in the steady state.

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