On the analysis of multilevel rocking cores—A bioinspired analogy

Mark Grigorian1 | Mozhgan Kamizi2

1MGA Structural Engineering, Inc, Glendale, California
2Department of Civil Engineering, Faculty of Engineering, Golestan University, Gorgan, Iran

This article presents a simple analogy, with practical applications, between the human spine and multilevel rocking cores (MLRCs) under similar loading conditions. The use of energy dissipating rocking cores in general and MLRCs in particular is a relatively new concept for reducing earthquake damage in new and existing buildings. The literature on the subject is rather scant. There are neither official guidelines nor educational materials for practical design of MLRCs. The first step toward rational design of MLRCs is to understand their elastic state static/dynamic behavior as part of a gravity and/or earthquake resisting system. The purpose of the current paper is not to reiterate the merits of various rocking systems but to provide reliable formulae for the preliminary design of simple MLRCs. Suffice to note that the multitude of gap movements in MLRCs results in increased damping and elongated periods of vibrations. Several parametric examples have been provided to demonstrate the applications, the validity, and the simplicity of the proposed solutions. All solutions are exact within the bounds of the theoretical assumptions. All results have been verified by independent computer analysis.

KEYWORDS
constitutive equations, damage reduction, multilevel rocking cores, period analysis, static behavior

1INTRODUCTION

The human spine is a highly complex multifunctional bioelectromechanical system that has been evolved to perform fundamental biological tasks needed to maintain life as we know it. It is a naturally optimized three-dimensional load bearing structure that is capable of withstanding combined gravitational, torsional, and bending effects at any inclination with respect to any fixed line of reference.1 As a highly precise self-aligning configuration, the spine is also capable of damping dynamic effects due to normal locomotion and accidental impact.2 In the physical sense, the word structure implies arrangement of material parts in a purposeful manner and as such may apply to the human spine, the universe, as well as earthquake resisting frameworks. In the present context, structure is referred to manmade load bearing engineering frameworks. While there are countless numbers of natural systems and materials, there are only limited numbers of manmade earthquake resisting archetypes, consequently not all desirable features of natural systems could be incorporated into the design of all known forms of engineering structures. The purpose of the current article is to show that an understanding of the structural performance of the human spine can enhance the physical response of earthquake resisting frameworks. In general and that of multilevel rocking cores (MLRCs) in particular. It is also possible that the analogous performances of the natural and manmade structures may help explain the structural response of natural spines to similar loading scenarios. Design methodology transfer from natural objects to manmade frameworks is not new and...
has been successfully accomplished, among others by the senior authors.3,4 Here, the spinal column is looked upon an inspirational model for developing efficient earthquake resisting systems with a view to damage control and self-alignment. The most applicable components of the spine, from a structural engineering point of view, that may be utilized as the constitutive elements of MLRCs or similar systems can be listed as follows:

1. The vertebrae that compose the spinal column. The bony vertebrae are mimicked as the interstory solid cores of Figure 1 and constitute the building blocks of the proposed MLRC system of Figures 2C and 4C.

2. The intervertebral disks that prevent contact between vertebrae and act as natural shock absorbers due to normal locomotion. These jelly like cushions are replicated with energy absorbing devices such as resilient slip friction joints, elastoplastic yielding parts, or similar dampers as depicted in Figure 3C.

3. The facet joints that allow controlled articulation and load transfer between adjacent vertebrae. The mechanical functions of the facet joints are imitated by ordinary pin and socket joints as illustrated in Figures 1, 2, and 3.

4. The rib facets that hold the entire rib cage and its attachments together. The facets are replaced with short, relatively rigid link beams that connect the gravity structure to the MLRC.

5. The ligaments and tendons that stabilize, realign, and hold the spinal column together. These symmetrically positioned bands of toughened tissues are replaced with short lengths of slightly preloaded, axially extensible ties, rods, or similar members; see Figures 1A and 1F.

The spinal column is a three-dimensional system. However, its bioinspired counterpart was designed as a two-dimensional structure in order to avoid theoretical complications as earthquake resisting parts of commonly available building systems.
1.1 Rocking core technology

Rocking cores are relatively new technologies developed to reduce earthquake damage and to increase structural functionality after major seismic events.5-13 The early research on rocking systems can be credited, among others, to the works of Housner,14 Aslam et al,15 Priestley and Tao,16 Mander and Cheng,17 and Kurama et al.18 An excellent account of rocking frame innovations may be found in two comprehensive bibliographies.19,20 Short to medium rise MLRCs composed of stiffened plywood panels and precast concrete segments have been successfully tested by different groups of researchers. The most important innovations in this field are because of Wiebe and Christopoulos,21 Tao et al.,22 and Khanmohammadi and Heydari,23 who have shown through extensive dynamic analysis, that inclusion of multiple pivots along the height of rocking systems can reduce the contributions of higher modes to the overall response of the structure and lower the elastic shear and moment demands on the intermediate levels. Properly designed rocking systems can prevent soft story failure, minimize P-delta effects, and redistribute Drift Shift (DS) along the height of the system.24,25 Preventing soft story failure and compliance with drift limits are, perhaps, the two most important requirements of all contemporary codes of practice.26,27 Almost all performance based design methodologies emphasize the importance of drift control during linear as well nonlinear phases of loading. Soft story failure and DS in multistory buildings are influenced by many factors. Drift control and prevention of soft story failure can be addressed either theoretically or, by purpose, specific physical arrangements, leading to sustainable earthquake resisting systems.36-41 Relatively rigid rocking cores tend to impose straight-line drift profiles upon the parent structures and absorb the entire lateral force at failure. The scope of the current article is limited to the study of the behavior of some of the most common MLRCs under different loading profiles and boundary support conditions. DS is the change in drift ratios of two consecutive levels in multistory structures. Unacceptable levels of DS occur commonly in structures that have not been designed to withstand large inelastic displacements,42 DS and its effects can be more pronounced in frames designed in accordance with current codes of practice than those designed following principles of Collapse Prevention (CP) and Post-Earthquake Realignment and Repairs (PERR). It has been reported that a reduction in DS can help improve racking stability in all categories of moment frames in general and in rocking core enhanced systems in particular. Despite the encouraging surge of interest in computer-aided research on self-standing rocking systems, there are only a small number of short cut manual methods of displacement analysis for such frames as parts of new or existing building structures.44 However, recent studies suggest that it is possible to overcome the computational complexities associated with manual displacement analysis of certain classes of rocking systems during all phases of monotonic loading. Indeed, it is even possible to reduce the otherwise complicated task of nonlinear dynamic analysis to the study of the simplified version of the same structure under the same loading profile. While almost all rocking system cited above have passed tests of experiments and time-history analysis,12,45 they have rarely been examined as integral parts of actual buildings. Real buildings cannot be ideally centered unless specifically designed and detailed for CP and PERR. An appreciation of the pertinent design criteria and the conditions of desirable performance are a priori to establishing the forthcoming arguments.

2 ARCHETYPES AND COMPONENTS

The purpose of the current section is to define the scope and limitations of the proposed formulations, to clarify the popular nomenclature, and to describe the physical bases for the theoretical work presented in this paper. The main function of MLRCs and its modules is to add damping through gap opening, to prevent collapse, to reduce drift concentration, and to provide suitable supports for energy dissipating and self-centering devices. Depending upon their predesignated functions and base level boundary conditions, rocking cores may be categorized as either stepping or rocking. Stepping cores pivot alternatively about their corners, as in Figures 1C and 1E, whereas rocking systems rotate about their hinged supports, as in Figures 1A, 1B, and 1F. Figure 1D depicts a sliding or racking system. The pin ended links symbolize (shown in red) supplementary devices such as axially extensible ties or rods,46 resilient slip friction joints,47 or any other means devised to enhance and stabilize the system. These devices are commonly preloaded and are supposed to remain elastic during all phases of the cyclic loading. Continuous unbonded post-tensioned tendons are commonly used to assure CP and to assist proper recentering. The focus of the current article is on the mathematical treatment of MLRCs composed of primarily purely flexural or racking modules. In the present context, flexural and racking are synonymous with overturning and sliding, respectively. The study of stepping cores is not part of scope of the current study. If the combined unsupplemented core and parent structure are capable of maintaining a stable hysteretic response against cyclic loading, while at least one set of the supplementary devices, eg, the post-tensioned tendons, remains elastic, then the response of the entire
system would confirm to that of a flag shape hysteretic behavior, implying that the system can recenter itself with little to no residual effects. However, the meaningful design of any building frame-rocking core combination depends upon rational evaluation of the interacting forces acting between the two systems. Almost all code compliant buildings consist of combinations of gravity and lateral resisting structures, connected in parallel and signified by lateral stiffnesses $K_{\text{bldg}}$ and $K_{\text{frame}}$, respectively. Addition in parallel of a rocking core of lateral stiffness $K_{\text{core}}$ increases the total stiffness of the system to $K_{\text{total}} = K_{\text{bldg}} + K_{\text{frame}} + K_{\text{core}}$. This implies that, in the elastic range, each structural system would carry its own share of the external overturning moment $M_o$, ie,

$$
M_{o,\text{bldg}} = \frac{K_{\text{bldg}}}{K_{\text{total}}} M_o, \quad M_{o,\text{frame}} = \frac{K_{\text{frame}}}{K_{\text{total}}} M_o, \quad M_{o,\text{core}} = \frac{K_{\text{core}}}{K_{\text{total}}} M_o, \quad \text{and} \quad \phi = \frac{M_o}{K_{\text{total}}},
$$

where $\phi$ or $\phi_{\text{max}}$ may be looked upon as the maximum elastic drift allowed by the codes. However, if $\phi$ and recentering are to be achieved, then the MLRC should remain perfectly elastic beyond the plastic collapse load of the earthquake resisting frame. In other words, the MLRC should be sufficiently stiff and strong to withstand the entire seismic load after $M_o$ exceeds the plastic collapse capacity of the frame, in which case the global stiffness of the frame becomes zero, ie, $K_{\text{frame}} = 0$. The overwhelming majority of cases referred to in this and similar articles discuss the use of free-standing rocking/stepping cores in conjunction with perfectly articulated gravity systems where $K_{\text{bldg}} = K_{\text{frame}} = 0$ and $K_{\text{core}} \neq 0$ (Figures 2A to 2C). In general, two types of gravity buildings can be envisaged: continuous columns with hinged beams as in Figure 2B and continuous beams with hinged columns as in Figure 2G. In both cases, the assumption is that either the gravity system is totally articulated or the cumulative rigidity of the columns is negligible compared with that of the rocking core. Here, the trend is observed and extended to include the more common case, $K_{\text{bldg}} = 0$, $K_{\text{frame}} \neq 0$, and $K_{\text{core}} \neq 0$ (Figure 2F to 2I), where $F_l$ and $F_m$ define the magnitudes of the concentrated lateral forces at elevations $i$ and $m$, respectively. Similarly, $S_l$ and $Q_m$ refer to the magnitudes of the interactive forces acting between the core and the moment frame. In other words, instead of discussing the general case, $K_{\text{bldg}} \neq 0$, $K_{\text{frame}} \neq 0$, and $K_{\text{core}} \neq 0$, the two most practical cases, the free-standing MLRC of Figures 2A to 2C, and MLRC-moment fame combination of Figures 2F to 2I have been elaborated upon at some length. In both cases, lateral forces are transmitted to the cores via axially rigid, horizontal, pin ended link beams. An important utility of rocking modules, besides being parts of the MLRCs, is their use as earthquake resisting elements placed within the bays of existing or new buildings, as shown in Figure 3. Some of the more important attributes of such rocking modules as parts of existing or new building can be summarized as follows:

- seismic safety of incomplete buildings (before the seismic system is in place);
- reduction of additional base level footings (provided that the existing ones are adequate);
- elimination of special floor level drag members and link beams (as needed for continuous MLRCs);
- elimination of complicated rocking core-diaphragm interface details (to prevent floor slab failure);
- accessibility for installation, repairs, and/or replacements;
- possibility of relocation of modules if required;
- mass production for repetitive construction, etc.

### 2.1 Interactive forces

Consider the lateral resistance of the Gravity-MLRC combination of Figures 2A to 2C. Since in reality or for seismic design purposes $K_{\text{bldg}} = 0$ but $K_{\text{core}} \neq 0$, the gravity system is assumed to be a stable mechanism, then the core, standing upright as a vertical cantilever, will have to sustain the entire seismic force, see Figures 2D and 2E. While this is a conservative assumption for the practical design of the core, all other elements of the gravity system, especially the columns and drag members, should be checked for seismic safety. A complete formulation of the static/dynamic behavior of the freestanding MLRC of Figure 2C, under different loading profiles, is presented in the forthcoming sections. The lateral resisting behavior of the gravity-moment frame-MLRC combination of Figures 2F to 2I is similar to the preceding case with the exception that the primary earthquake resisting structure, ie, the moment frame, and the MLRC impose an indeterminate state of interactive forces upon each other, see Figure 2J. The physical behavior of rocking core moment frame combination can best be visualized by the frame restraining the MLRC in place, and the core imposing damping and controlled drift along the height of the frame. It has been shown that the stiffer the core, the more desirable the seismic response of the combined structure. However, MLRCs need not be overwhelmingly rigid; they should be stiff and strong enough to enforce minimum drift and prevent soft story failure. The ultimate strength of the MLRC should be greater than the total demand imposed upon the combined system. The stiffness of the MLRC should be selected in such a way as to reduce its
own maximum drift to less than a fraction of the prescribed design value. Under such circumstances, the MLRC behaves as an upright simply supported beam with end reactions $Q_m$ and $Q_0$ as shown in Figure 2J. A complete formulation of the static behavior of this particular case under three different types of lateral loading profiles is also presented in the forthcoming sections. For an exhaustive review of significant attributes of rocking cores, the interested reader is referred to the work of Grigorian et al.44

3 | THE CONSTITUTIVE EQUATIONS

This section aims to provide a number of simple, closed field solutions for the static response of practical MLRCs with different boundary support conditions under commonly considered distributions of lateral nodal forces. MLRCs are stable systems composed of discrete modules connected to each other at regular intervals.

Because of the discrete nature of the structure, resort has been made to finite difference calculus as the basis for the solutions presented in this paper. All symbols are defined as they first appear in the text. The static solution, dubbed as the Drift Projection Method, is based on the elastic compatibility of adjoining modules. Equilibrium is satisfied as part of the discrete field solutions influenced by the external loading conditions. As a prelude to formulating the characteristic equations of deformations of various modules of MLRCs, the reader is invited to consider the steps involved in the solution of the introductory Example 1 presented in the following, where the answer to the problem is worked out through a semigraphical solution that is ideally suited for both manual as well as spreadsheet analysis. Rigorous analysis of different types of rocking modules subjected to different loading profiles and combinations of boundary conditions is presented as the main body of the current paper. It may be seen from Figures 1 and 3 that most rocking modules consist of a relatively rigid core, a central or corner pivot, and stabilizing elements at each end. Theoretically, the hinges transfer shear and axial forces between interacting modules, and the supplementary devices provide damping and stability for the system. Since each module can rotate with respect to its neighboring modules, Figures 4C and 4D, the system sometimes is referred to as flexible or partially flexible in comparison with solid rocking core of Figure 4B. All rocking modules deform either in purely flexural, purely racking, purely axial, or combination of these modes, depending upon their configurations, rigidities, and material properties. The analysis of the combined case is rather cumbersome and sheds little to no light on the response of the system. For this reason, it was deemed instructive to develop the governing displacement equations separately. Obviously, the results of any two or three such solutions can be superimposed upon each other for final design purposes.

3.1 | The Drift Projection method

At first sight, the structural analysis of MLRCs appear complex and uncommon. MLRCs are relatively new and still unfamiliar earthquake resisting systems. However, two equally powerful methods of analysis for simple MLRCs are being proposed: (1) the closed-form finite difference solution for highly regular MLRCs and (2) the Drift Projection method, which is reminiscent of the classical Moment Area Theorem.48 Both methods result in identical solutions. The study of structure of Figure 4C, assuming no racking at this stage, can be presented as follows. Consider the deformations of the stiff and flexible rocking cores of Figures 4B and 4C, respectively, under the same distribution of lateral forces, Figure 4A. Assuming that the cores of the modules of MLRC of Figure 4C are infinitely rigid, then the local rotations and

![Figure 4](image-url)
displacements of any cantilevered module $x$ can be expressed as $\psi(x) = M(x)/K_x$ and $\delta(x) = \psi(x)h_x$, respectively, where $M(x)$ is the total external moment acting on the module, $K_x$ and $h_x$ stand for the rotational stiffness and height of the module, respectively. Next, if the global drift and sway of the same module are symbolized by $\phi(x)$ and $Y(x)$, respectively, then it may be deduced from Figure 4F or shown mathematically that

$$ Y(x) = \sum_{i=1}^{x} \psi(i) \sum_{j=1}^{x} h_j = \psi_1(h_1 + h_2 + h_3 + \cdots + h_x) + \psi_2(h_2 + h_3 + \cdots + h_x) + \cdots + \psi_x(h_1 + \cdots + h_x) + \cdots + \psi_x h_x \quad (2) $$

$$ Y(x-1) = \sum_{i=1}^{x-1} \psi(i) \sum_{j=1}^{x-1} h_j = \psi_1(h_1 + h_2 + \cdots + h_{x-1}) + \psi_2(h_2 + h_3 + \cdots + h_{x-1}) + \cdots + \psi_{x-1}(h_1 + \cdots + h_{x-1}) + \cdots + \psi_{x-1}h_{x-1} \quad (3) $$

$$ \phi(x) = \frac{Y(x) - Y(x - 1)}{h_x} = \sum_{i=1}^{x} \psi(i) \text{ and } Y(n)_{\text{flexible}} = \sum_{i=1}^{n} \psi(i) \sum_{j=1}^{n} h_j. \quad (4) $$

Expressions (2), (3), and (4) form the essence of the drift projection technique Figure 4F, which states the following.

The change of drift between two modules along the MLRC is equal to the sum of all local rotations $\psi$ between them, and that the deflection of any module at $x$, a distant $\hat{h}$ away from $x$, relative to the tangent at $(x)$ is equal to the product $\psi\hat{h}$.

It follows that, if $n = 1$, $h_1 = h$, and $K = K'$, then the moment-rotation relationship, $M = \phi$, and the maximum tip translation of the special case of rigid rocking core of Figure 4B can be expressed as

$$ \phi_{\text{rigid}} = \frac{n(n + 1)Fh}{2K'} \text{ and } Y(n)_{\text{rigid}} = \phi nh = \frac{n^2(n + 1)Fh^2}{2K'} \quad (5) $$

respectively. However, the variability of $K(x)$, $h_x$, and $n$ provides opportunities to customize the elements of the flexible structure to devise several response strategies for practical design of MLRCs. Some of the simpler design strategies regarding the selection of flexible cores can be summarized as follows:

(a) selecting $K(x) = K'$ or any other constant value for preliminary design and cost study purposes;
(b) selecting variations of $K(x)$ such that $\phi(x) \leq \phi_{\text{all}}$ along the height of the flexible rocking core;
(c) selecting variations of $K(x)$ such that maximum roof level displacement $Y(n) \leq \phi_{\text{all}}H$;
(d) using pre-programmed/smart devices to counter expected seismic demands;
(e) combining strategies (a), (b), and (c) with variations in $h_x$ and $n$ for similar purposes, etc; and
(f) selecting $K(x) = \infty$ for $x > 0$, such that $\phi(x) = \phi$ is constant along the height of the rocking system.

Example 1 addresses three different design strategies and has been worked out to demonstrate the inner workings of practical MLRCs under lateral loading.

### 3.1.1 Example 1

Given $F(x) = F$, $h_x = h$, and $n = 5$, select $K(x)$ in accordance with strategies (a), (c), and (d).

**Solution:** The moment-rotation relationship of the rigid core of height $nh$, Figure 4B can be expressed as

$$ \phi = \frac{M_{\text{max}}}{K'} = \sum_{x=1}^{n} Fx = n(n + 1)Fh/2K' = 15Fh/K'$

which leads to the maximum allowable tip displacement $Y_{\text{allow}} = \phi nh = 75Fh^2/K'$. The complete solution to all three strategies is summarized in Table 1.

In order to satisfy the requirements of strategy (c), let $Y_{\text{allow}} = 75Fh^2/K' = 140Fh^2/K$, which gives $K = 1.87K'$. Note that the effort involved in strategy (c) reduces the maximum roof level displacement from $Y_n = 140Fh^2/K$ to $Y_n = 75Fh^2/K$.

Similarly, the requirements of strategy (b) can be satisfied by equating maximum drift ratios described by Equations (4)

| Level ($x$) | Strategy (a) | Strategy (c) | Strategy (b) |
|------------|--------------|--------------|--------------|
| $M_x/(Fh)$ | 15, 10, 6, 3, 1 | 15, 10, 6, 3, 1 | 15, 10, 6, 3, 1 |
| $K / K'$ | 1, 1, 1, 1, 1 | 1.87, 1.87, 1.87, 1.87, 2.33 | 2.33, 2.33, 2.33, 2.33 |
| $\psi_x/(Fh/K)$ | 10, 15, 6, 3, 1 | 0.34, 0.54, 0.84, 1.14 | 0.14, 0.54, 0.84, 1.14 |
| $\delta_x/(Fh^2/K)$ | 15, 10, 6, 3, 1 | 0.54, 0.84, 1.14, 1.44, 1.74 | 0.14, 0.54, 0.84, 1.14 |
| $Y_{\text{allow}}/(Fh^2/K)$ | 15, 10, 6, 3, 1 | 0.54, 0.84, 1.14, 1.44, 1.74 | 0.14, 0.54, 0.84, 1.14 |
| $\phi_{\text{flexible}}/(Fh/K)$ | 15, 10, 6, 3, 1 | 0.54, 0.84, 1.14, 1.44, 1.74 | 0.14, 0.54, 0.84, 1.14 |
and (5), ie, \( \phi_{\text{allow}} = \sum_{i=1}^{n=5} \psi(i) = n(n + 1)Fh/2K \) or \( \phi_{\text{allow}} = 35Fh/K = 15Fh/K' \), which, in turn, gives \( K = 2.33K' \). As expected strategy (b), Table 1, results in the smallest DS along the height of the subject flexible core.

### 3.2 Basic force-deformation relationships

Depending upon their configuration, modules of MLRCs can experience different combinations of flexural, racking, and axial deformations as well as rigid body rotations. Considering the rotational changes of any two neighboring modules subjected to lateral and axial forces as in Figures 5A, 5B, and 5C, the total module rotation \( \phi(x) \) may be expressed in terms of total module deformations as

\[
\phi(x) = \psi_f(x) + \psi_r(x) + \psi_a(x) = \nabla_x [\theta(x) + \gamma(x) + \varphi(x)],
\]

where \( \theta(x), \gamma(x), \) and \( \varphi(x) \) are discrete functions describing the local rotations of the flexural, racking, and axial rotations, respectively, of module \( x \). \( \psi_f(x), \psi_r(x), \) and \( \psi_a(x) \) describe the relative rotations of two consecutive modules due to flexural, racking, and axial type deformations, respectively. \( Y(x) \) is the total lateral displacement of the module due to all deformation components. Subscripts \( f, r, \) and \( a \) refer to flexural (Figure 5A), racking or shear (Figure 5B), and axial (Figure 5C) modes of deformations, respectively. \( \Delta_x = (E_{x}^{-1} - 1) \) and \( \nabla_x = (1 - E_{x}^{-1}) \) are the finite difference forward and backward shift operators, respectively. The finite difference operator \( E_{x}^\pm \) performs the operation \( E_{x}^\pm f(x) = f(x \pm 1) \) on any function of the variable \( x \). Here, \( x \) is a nondimensional digit confined to the discrete range: \( x = 0, 1, 3, 4, \ldots, n \). Component rotations \( \theta(x), \gamma(x), \) and \( \varphi(x) \) can be related to their local displacement as follows:

\[
\begin{align*}
   h\theta_f(x) &= Y_f(x + 1) - Y_f(x) = (E_{x}^{+1} - 1) Y_f(x) = \Delta_x Y_f(x), \\
   h\gamma_r(x) &= Y_r(x + 1) - Y_r(x) = (E_{x}^{+1} - 1) Y_r(x) = \Delta_x Y_r(x), \\
   h\varphi_a(x) &= Y_a(x + 1) - Y_a(x) = (E_{x}^{+1} - 1) Y_a(x) = \Delta_x Y_a(x).
\end{align*}
\]

The change of drift angle \( \psi(x) \) between two consecutive modules can be estimated as

\[
\begin{align*}
   \psi_f(x) &= \theta(x) - \theta(x - 1) = (1 - E_{x}^{-1}) \theta(x) = \nabla_x \theta(x) = \frac{\nabla_x \Delta_x Y_f(x)}{h} = \frac{M(x)}{hK_f}, \\
   \psi_r(x) &= \gamma(x) - \gamma(x - 1) = (1 - E_{x}^{-1}) \gamma(x) = \nabla_x \gamma(x) = \frac{\nabla_x \Delta_x Y_r(x)}{h} = \frac{\Delta_x V(x)}{hK_r}, \\
   \psi_a(x) &= \varphi(x) - \varphi(x - 1) = (1 - E_{x}^{-1}) \varphi(x) = \nabla_x \varphi(x) = \frac{\nabla_x \Delta_x Y_a(x)}{h} = \frac{\Delta_x V(x)}{hK_a}.
\end{align*}
\]

where symbols \( F, V, \) and \( K \) stand for generalized lateral force, shear, and stiffness, respectively. Next, bearing in mind that the equivalent tributary \( P\)-delta moment absorbed by the module can be related to an imaginary force \( V(x) \), acting in the same sense and location as \( V(x) \), ie, \( \nabla V(x) = P \Delta_x \psi_f(x) = P \Delta_x \psi_r(x) \), then the total shear appearing in Equations (10),

**FIGURE 5** A, Flexural (rigid body) rotation; B, Racking/sliding deformations; C, Axial/(due to axial changes) deformations
(11), and (12) can be increased to \([M(x) + \nabla(x)h]\). \(P_x\) is the sum of all gravity forces acting on and above level \(x\). Although Equations (7) through (12) are applicable to all combinations of boundary support conditions, the current article is confined to the study of three sets of boundary conditions of practical interest, namely, cantilevered, propped cantilever, and simply supported MLRCs.

## 4 | CANTILEVERED FLEXURAL SYSTEMS \(Y_f\)

Figures 4B, 4C, and 4D illustrate three types of cantilevered or base supported rocking structures where the entire seismic force is absorbed by the elements of the rocking system. A generic scenario involving a cantilevered MLRC is shown in Figures 2B and 2C. In the particular case under study, Figure 5A, lateral displacements are caused by the local deformations of the energy absorbing devices that connect neighboring ends of consecutive rigid modules to each other. The deformed shape resembles that of an upright cantilever in bending, Figure 4F. Assuming no decompression of the energy absorbing devices, the drift angle \(\psi_f(x)\) and local displacement \(\delta_f(x)\) of the rigid modules can be expressed in terms of the stiffnesses of the energy absorbing devices \(K_f\), i.e.,

\[
\psi_f(x) = \frac{2\epsilon(x)}{d_f} = \frac{2T(x)L_f}{A_fE_fd_f} = \frac{2\left[M_f(x) + P_h\psi_f(x)\right]L_f}{A_fE_fd_f^2}, \text{ and } \delta_f(x) = \psi_f(x)h.
\]

where \(\epsilon(x)\) is the axial change of length of the generic energy absorbing devices shown in Figure 5A. \(d_f\) is the lateral distance between the two devices. \(T, A_f, E_f, \) and \(L_f\) stand for the axial tension, cross sectional area, modulus of elasticity, and the effective length of these devices, respectively. \(M_f(x)\) is the moment of all forces above joint \(x\). Now, if \(K_f = A_fE_fd_f^2/2L_f\), then Equation (13) can be rewritten as \(\psi_f(x) = [M_f(x) + P_h\psi_f]/K_f\), which simplifies to the familiar form

\[
\nabla_x \Delta_x Y_f(x) = \frac{M_f(x)h}{[1 - P_hh/K_f]K_f}.
\]

as the basic equation of deformations of the component \(Y_f\) of the modules of the rocking system, where \(\nabla_x \Delta_x = \Delta_x \nabla_x = (E_x^{-1} - 2 + E_x^{-1}) = h^2 \partial^2 / \partial x^2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \) is the symmetric central difference operator and can be identified as the second derivative of a continuous function with specific values at equal intervals \(h\). However, if initial device tension is inadequate and decompression can take place, then stiffness \(K_f\) will reduce to \(K_f = A_fE_fd_f^2/4L_fh^2\). The quantity \(1 - P_hh/K_f\) can be looked upon as the stability quotient that affects the response of the energy absorbing elements described above. The mathematical solution to Equation (14) is sensitive to the load profile and is contained in the double summation or integration

\[
Y_f(x) = \nabla_x^{-1} \Delta_x^{-1} \frac{M_f(x)h}{(1 - P_hh/K_f)K_f}. \tag{15}
\]

Three practical solutions of Equation (15), involving uniformly distributed, inverted triangular, and an arbitrarily located nodal lateral force, have been worked out as follows.

### 4.1 Cantilevered flexural system under uniform nodal lateral loading

Let \(K_f(x) = K_f = \text{(constant)}\). Measuring \(x\) from the base upwards, the general closed-form static solution of Equation (15) for combined gravity \(F\) and uniform distribution of lateral forces \(F\) can be expressed as

\[
F(x) = F \tag{16}
\]

\[
V(x) = \sum_{i=x}^{n} F(x) = (n + 1 - x)F \tag{17}
\]

\[
\delta_{f,x} = \frac{M(x)h}{f_{cr,f}K_f} = \frac{\sum_{i=x+1}^{n} V(i)h^2}{f_{cr,f}K_f} = \frac{(n-x)(n-x+1)Fh^2}{2f_{cr,f}K_f} \tag{18}
\]
The lateral displacement \( Y_f(x) \) of any module of the subject system at joint \( x \) is affected by the rotations and displacements of all other modules under level \( x \). In other words, since \( \psi_f(x) = \delta_f(x)/h \) and \( Y_f(x) = \psi_f(1)(x - 0)h + \psi_f(2)(x - 1)h + \cdots + \psi_f(i)(x + 1 - i)h + \cdots + \psi_f(x)(1)h \), then

\[
Y_f(x) = \sum_{i=1}^{x} \psi_f(i) \times (x + 1 - i)h = \frac{x(x + 1) \left[ 6n^2 + n(10 - 4x) + x^2 - 3x + 2 \right] Fh^2}{24f_{cr,f}K_f}
\]

(19)

\[
Y_{f,\text{max}} = \frac{n \left( 3n^3 + 10n^2 + 9n + 2 \right) Fh^2}{24f_{cr,f}K_f}.
\]

(20)

The seismic performance of the rocking systems depends largely on the relative rigidities of the system and the parent structure as well as the interactive forces due to supplementary devices. However, story drift is not sensitive to minor variations in the rigidity of the core. To establish a sufficiently high rigidity for the MLRC, its maximum drift ratio can be related to a fraction of the prescribed uniform drift of the corresponding rigid rocking core of the same height under similar loading conditions. Equating Equations (5) and (20), it yields

\[
\frac{K_f}{K} = \frac{(3n^3 + 10n^2 + 9n + 2)}{12(n^2 + n)}.
\]

(21)

Equation (21) can be used to obtain a numerical measure of the relative rigidities of the two systems.

### 4.1.1 Example 2

Compute the lateral displacements of a five-story, \( n = 5 \), regular constant stiffness \( K_f \), cantilevered flexural system under uniform distribution of lateral forces \( F \) and \( P = 0 \). Assume \( L_f = h/2 \).

**Solution:** The complete solution to Example 2 is presented in columns 1 and 2 of Table 2.

### 4.2 Cantilevered flexural system under inverted triangular loading

The general closed-form static solution of Equation (15) for combined gravity \( P \) and inverted triangular distribution of lateral forces \( caF(x) = Fx/n \) can be expressed as

\[
F(x) = \frac{Fx}{n},
\]

(22)

\[
V(x) = \sum_{i=x}^{n} \frac{Fx}{n} = \frac{(n + 1 - x)(n + x)F}{2n}
\]

(23)

\[
\delta_{f,x} = \frac{\sum_{i=x}^{n} V(i)h^2}{f_{cr,f}K_f} = \frac{\sum_{i=x}^{n}(n + 1 - i)(n + i)Fh^2}{2nf_{cr,f}K_f} = \frac{(n + 1 - x) \left[ 2n^2 - n(x - 4) - x(x - 2) \right] Fh^2}{6nf_{cr,f}K_f}.
\]

(24)

Next, following the same rationale leading to Equation (19), it gives

\[
Y_f(x) = \psi_f(1)(x - 0)h + \psi_f(2)(x - 1)h + \cdots + \psi_f(i)(x + 1 - i)h + \cdots + \psi_f(x)(1)h,
\]

then

\[
Y_f(x) = \sum_{i=1}^{x} \psi_f(i) \times (x + 1 - i)h = \frac{x(x + 1) \left[ 20n^3 - 10n^2(x - 2) - 10n(x - 2) + x^3 - x^2 - 4(x - 1) \right] Fh^2}{120f_{cr,f}K_f}
\]

(25)

\[
Y_{f,\text{max}} = \frac{(11n^4 + 40n^3 + 45n^2 + 20n + 4) Fh^2}{120f_{cr,f}K_f}.
\]

(26)

### 4.2.1 Example 3

Compute the lateral displacements of a five-story, \( n = 5 \), regular constant stiffness \( K_f \), cantilevered flexural system under inverted triangular distribution of lateral forces \( F(x) = Fx/n \) and \( P = 0 \).

**Solution:** The complete solution to Example 3 is presented in columns 3 and 4 of Table 2.
### Table 2: Solutions to Examples 2, 3, 4, 5, 6, and 7

| Level | Example 2 Uniform | Example 3 Triangular | Example 4 Concentrated | Example 5 Uniform | Example 6 Triangular | Example 7 Concentrated |
|-------|-------------------|----------------------|------------------------|-------------------|----------------------|-----------------------|
|       | 𝛿_\text{f} Y_\text{f} | 𝛿_\text{f} Y_\text{f} | 𝛿_\text{f} Y_\text{f} | 𝛿_\text{r} Y_\text{r} | 𝛿_\text{r} Y_\text{r} | 𝛿_\text{r} Y_\text{r} |
| 5     | 1 | 140 | 1 | 109.2 | 1 | 55 | 1 | 15 | 1.0 | 11.0 | 0 | 3 |
| 4     | 3 | 105 | 2.8 | 81.2 | 2 | 40 | 2 | 14 | 1.8 | 10.0 | 0 | 3 |
| 3     | 6 | 71 | 5.2 | 54.2 | 3 | 26 | 3 | 12 | 2.4 | 8.2 | 1 | 3 |
| 2     | 10 | 40 | 8 | 30 | 4 | 14 | 4 | 9 | 2.8 | 5.8 | 1 | 2 |
| 1     | 15 | 15 | 11 | 11 | 5 | 5 | 5 | 5 | 3.0 | 3.0 | 1 | 1 |
4.3 | Cantilevered flexural system under single joint force

For the particular combination of gravity $P$ and a single joint force $F$ acting on joint $s$, Equation (14) gives

$$F(x) = F \text{ for } x = s \text{ and } F(x) = 0 \text{ for } x \neq 0$$

(27)

$$V(x) = F \text{ for } x \leq s \text{ and } V(x) = 0 \text{ for } x > 0$$

(28)

$$\sum_{x=1}^{s} V(x) = (s + 1 - x)F \text{ for } s \geq x \geq 1 \text{ otherwise}$$

(29)

$$\delta_{f,x} = \frac{\sum_{x=1}^{s} V(x)}{f_{cr,f}K_f} \text{ for } s \geq x \geq 1 \text{ otherwise } \delta_{f,x} = \delta_{f,s}.$$  

(30)

Bearing in mind that $\psi_f(x) = \delta_{f,x}/h$ and following the rationale leading to Equation (19), it gives

$$Y_f(x) = \sum_{i=1}^{x} \psi_f(i) \times (s + 1 - i) = \frac{x(x + 1)(3s - x + 1)}{6f_{cr,f}K_f} Fh^2$$

(31)

$$Y_{f,\text{max}} = \frac{(n^2 + n)(3s + 1 - n)Fh^2}{6f_{cr,f}K_f}.$$  

(32)

4.3.1 | Example 4

Compute the lateral displacements of a five-story, $n = 5$, regular, constant stiffness $K_f$, cantilevered flexural rocking frame under a single lateral forces $F$ acting at level $s = n$. Assume $P = 0$.

**Solution:** The complete solution to Example 4 is presented in columns 5 and 6 of Table 2.

5 | CANTILEVERED RACKING DEFORMATIONS $Y_r$

The drift differential angle for the purely racking condition, Figure 5B, can also be expressed in terms of the racking stiffness $K_r$ and the net module shear force, ie,

$$h\psi(x) = \left[\Delta_x V(x) + \Delta_x \bar{V}(x)\right] h^2 = \frac{[F(x) + \Delta_x P_x \nabla_x Y_r(x)/h] h^2}{K_r},$$  

where $K_r = 24E_r/[I/h + (I/L)]$ and $\bar{V}(x)h = P_x h\psi_r(x) = P_x \nabla_x Y_r(x)$ represent the racking stiffness and the tributary equivalent P-delta moment acting on the module. Here, $I$ and $J$ stand for the moments of inertias of the beams and columns of the racking module, respectively. $E_r$ is the modulus of elasticity of the module. Next, substituting for $\gamma_r(x)$ from Equation (8) into Equation (11), ie, $\gamma_r(x) = \nabla_x \gamma_r(x) = \nabla_x \Delta_x Y_r(x)/h$, and rearranging it give

$$\nabla_x \Delta_x Y_r(x) = \frac{F(x)h^2}{[1 - P_x h/K_r]K_r}. $$

(34)

5.1 | Cantilevered racking system under uniform lateral forces

The general closed-form static solution of Equation (34) for combined gravity $P$ and uniform distribution of lateral forces $F$ can be expressed as

$$V_x = \sum_{x=1}^{n} F(x) = \sum_{x=1}^{n} (n + 1 - x)F$$

(35)

$$\delta_x = \frac{V(x)}{(K_r - P/h)} = \frac{(n + 1 - x)Fh^2}{(K_r - P/h)}$$

(36)

$$Y_{r,x} = \sum_{x=1}^{x} \delta_x = \frac{F}{(K_r - P/h)} \sum_{i=1}^{x} (n - i + 1) = [2(n + 1) - (x + 1)] \frac{Fh^2}{2(K_r - P/h)}$$

(37)

$$Y_{r,\text{max}} = \frac{n(n + 1)Fh^2}{2(K_r - P/h)}.$$  

(38)
5.1.1 Example 5
Compute the lateral displacements of a five-story, \( n = 5 \), regular constant stiffness \( K_r \), racking system under uniform distribution of lateral forces \( F \) and \( P = 0 \).

Solution: The complete solution to Example 4 is presented in columns 7 and 8 of Table 2.

5.2 Cantilevered racking system under inverted triangular forces
The general closed-form static solution of Equation (28) for combined gravity \( P \) and inverted triangular distribution of lateral forces \( F(x) = Fx/n \) can be expressed as

\[
V_x = \sum_{x=1}^{n} \frac{Fx}{n} - \sum_{i=1}^{x-1} \frac{Fx}{n} = (n + x)(n - x + 1) \frac{F}{2n} 
\]

\[
\delta_{r,x} = \frac{V_x}{(K_r - P/h)} = (n + x)(n - x + 1) \frac{Fh^2}{2n(K_r - P/h)} 
\]

\[
Y_{r,x} = \sum_{x=1}^{x} \delta_x = \frac{Fh^2}{2n(K_r - P/h)} \sum_{i=1}^{x} \left[ (n^2 + n) - i^2 + i \right] = \frac{Fh^2}{2n(K_r - P/h)} \left[ (n^2 + n)x - x(x + 1)(2x + 1) \frac{6}{2} + x(x + 1) \right].
\]

5.2.1 Example 6
Compute the lateral displacements of a five-story, \( n = 5 \), regular constant stiffness \( K_r \), cantilevered racking system under inverted triangular distribution of lateral forces \( F(x) = Fx/n \) and \( P = 0 \).

Solution: The complete solution to Example 6 is presented in columns 9 and 10 of Table 2.

5.3 Cantilevered racking system under single joint force
For the particular combination of gravity \( P \) and a single joint force \( F \) acting on joint \( s \), Equation (35) gives

\[
\delta_{r,x} = \frac{Fh^2}{(K_r - P/h)} \text{ and } Y_{r,x} = \frac{Fxh^2}{(K_r - P/h)} \text{ for } x \leq s, \text{ and}
\]

\[
\delta_{r,x} = \frac{Fsh^2}{(K_r - P/h)} \text{ and } Y_{r,x} = \frac{Fsh^2}{(K_r - P/h)} \text{ for } x > s.
\]

5.3.1 Example 7
Compute the lateral displacements of a five-story, \( n = 5 \), regular constant stiffness \( K_r \), cantilevered racking frame under a single lateral forces \( F \) acting at level \( s = 3 \). Assume \( P = 0 \).

Solution: The complete solution to Example 7 is presented in columns 11 and 12 of Table 2.

All six solutions presented in this section are correct and exact since they satisfy pertinent boundary conditions as well as the governing difference Equation (14).

6 CANTILEVERED AXIAL DEFORMATION \( Y_a \)

The case for axial deformations pertains to secondary axial stresses developed within the beams and columns of moment frames, eg, Figures 5B and 5C, or due to direct axial stresses developed within members of trussed frames as in Figure 3C. The axial component of the drift differential angle of the racking system of Figure 5B, as depicted in Figure 5C, can be expressed in terms of the member axial stiffness \( K_a \), ie,

\[
\psi_a(x) = \frac{Ma}{L^2 A_{col} E_{col}} \left( x - \frac{1}{2} \right) h^2 + P_s h \psi_a \left( x - \frac{1}{2} \right) \text{ and } \frac{1}{K_a} = \frac{1}{A_{col} E_{col} L^2}.
\]
\[ M_a(x - \frac{1}{2}) \] is the sum of moments of lateral forces acting above the inflexion point \((x - \frac{1}{2})\), about joint \(x\). Next, substituting for \(\phi(x)\) from Equation (4) into Equation (12) and the resulting \(\psi_a(x)\) into Equation (45), it gives, after simplifications,

\[
\nabla_x \Delta_y(x) = \frac{M_a \left( x - \frac{1}{2} \right)}{1 - P_x h / K_a} K_a.
\]  

(45)

Note that in order to relate the axial component of the deformation function to the mid-height of the modules, the integer \(x\) must be replaced with \((x - \frac{1}{2})\). In practical terms, \(Y_a(x)\) for uniformly distributed, inverted triangular, and concentrated nodal forces can be computed by substituting \((x - \frac{1}{2})\) for \(x\) in Equations (37), (41), and (35)-(36), respectively.

### 6.1 Total lateral displacements

In accordance with the principle of superposition, Equation (6) may be interpreted as the sum of the components of the total displacement \(Y(x) = Y_f(x) + Y_r(x) + Y_a(x)\); whence, the sum of Equations (14), (34), and (45) gives

\[
\nabla \Delta Y(x) = \begin{bmatrix} M_f(x) & F(x) & M_a \left( x - \frac{1}{2} \right) \end{bmatrix}.
\]  

(46)

as the basic difference equation of deformations of the discretized MLRC system under nodal lateral forces and accumulative \(P\)-delta effects from adjoining gravity structures.

### 7 PROPPED CANTILEVERED FLEXURAL SYSTEMS

In developing Equation (1), a rational assertion was made that the high rigidity of the MLRCs causes all restraining devices, including the equivalent rotational base spring to absorb proportional amounts of energy. Therefore, the moment absorbed in the base level rotational spring of the propped cantilevered MLRC of Figure 2I or Figure 4G would be equal to \(M(0) = K_f \phi\). The variables \(Q_m\) and \(Q_0\) may also be viewed as the end reactions of the core. Meanwhile, if CP and recentering are to be achieved, then the MLRC should remain perfectly elastic beyond the plastic collapse load of the earthquake resisting main frame. In other words, the MLRC should be sufficiently stiff and strong to withstand the entire seismic load after \(M_a\) exceeds the plastic collapse capacity of the frame. In theoretical terms, as \(M_a\) reaches the collapse capacity of the frame, the interactive forces \(S_i\) (Figure 4F) tend toward \(F_i\) (Figure 4J), thus ensuring a lower bound solution for the design of the subject MLRC. The solutions to the cantilevered MLRCs of Figure 4F were quickly derived because the slopes of the modules flared out with respect to the vertical tangent at base. Application of the Drift Projection method to the simply supported and propped cantilevered cases (Figure 4) involves a correction due to concentration of slopes at both ends of noncantilevered systems. The solution to the simply supported case, with no tendon attachments to the base is demonstrated as follows.

#### 7.1 Simply supported flexural system under uniform lateral loading

The simply supported case can be treated as a special case of the propped cantilever, \(K(0) = K(n) = 0\),

\[
F(x) = F
\]

(47)

\[
V(x) = \frac{(n - 1)F}{2} \sum_{i=1}^{x} F = \frac{(n - 1 - 2x)F}{2}
\]

(48)

\[
M(x) = \frac{(n - 1)F h x}{2} - F h \sum_{i=1}^{x-1} (x - 1) = \frac{(n-x)F h x}{2}.
\]

(49)
The projection of the upper end of the MLRC relative to the tangent at base is \( n h \psi_0 \), which is equal to the sum of all \( \psi(x)[n - x]h \) or \( \psi(x)h \) since both the MLRC and loading are symmetric. Thus,

\[
\psi_0 = \sum_{i=0}^{n} \frac{M(x)}{K} x = \frac{(n^3 - n)Fh}{24K}.
\]

The lateral displacement of the nodes of the MLRC at a point distance \( x \) from the base is made up of two parts: that due to \( \psi_0 x h \) and that due to curvature of the system over the length \( (xa) \), ie,

\[
Y(x) = \psi_0 x h - \sum_{i=1}^{x-1} \frac{M(x) x h}{K} = \frac{(n^3 - n) x h F^2}{24K} - \sum_{i=1}^{x-1} \frac{(n^2 - i^2) F h^2}{2K}, \text{ which simplifies to }
\]

\[
Y(x) = \frac{x(n^3 - n) h F^2}{24K} - (x^2 - x) [2n(2x - 1) - 3x(x - 1)] \frac{F h^2}{2K}.
\]

Note that Equation (51) readily satisfies the characteristic Equation (10).

### 7.1.1 Example 8

Compute the lateral displacements of a five-story, \( n = 5 \), regular, simply supported MLRC under uniformly distributed lateral nodal forces \( F \) and \( P = 0 \). Assume global rigid body rotation = 0.

**Solution:** For simply supported MLRCs, \( K_f(0) = K_f(n) = 0 \), \( K_f(x) = K_f \). Obviously, \( Y(0) = Y(n) = 0 \). Slope concentration at supported ends \( = \psi(0) = \psi(n) = (0+2+4+6+8)Fh/2K = 5Fh/K \). It follows therefore that \( Y(2) = \psi(0).2h - \psi(1) \).

\( h = (5Fh/K).2h - (2Fh/K)h = 8Fh^2/K \) and \( Y(1) = \psi(0).h = 5Fh^2/K \). The solution to Example 8 is presented in columns 1 and 2 of Table 3.

### 7.2 Propped cantilever flexural system under uniform lateral loading

The subject MLRC resembles an upright propped cantilever. It is externally indeterminate and can be treated as a regular cantilever with a support at the free end. Equation (20) describes the maximum tip deflection of a free-standing cantilever under uniformly distributed lateral nodal forces \( F(x) = F \). Substitution of \( s = n \) in Equation (32) results in the maximum tip deflection of the same cantilever under a single concentrated load \( Q \) at the free end. Depending on the relative magnitudes of \( F \) and \( Q \), the net tip displacement can be computed as

\[
Y_f(n) - Y_Q(n) = \frac{n(3n^3 + 10n^2 + 9n + 2)Fh^2}{24K_d} - \frac{n(n + 1)(2n + 1)Qh^2}{6} = nh\phi,
\]

where \( \phi \) is the rigid body rotation of the combined structure at incipient collapse. With \( \phi \) known, \( Q \) can be assessed directly from Equation (52).

### 7.2.1 Example 9

Compute the lateral displacements of a five-story, \( n = 5 \), free-standing cantilever MLRC under uniformly distributed lateral nodal forces \( F \) and \( P = 0 \).

**Solution:** The complete solution to Example 9 is numerically identical to Example 2; see Table 2.
7.2.2 | Example 10

Compute the lateral displacements of a five-story, \( n = 5 \), free-standing cantilever MLRC of Example 9 under a single tip force of magnitude \( Q \). Compute \( Q \) for \( \phi = 0 \). Assume \( P = 0 \).

Solution: Equating maximum tip deflections of Examples 9 and 10 gives \( Q = 28F/3 \). The complete solution to Example 10 is presented in Table 3.

7.3 | Simply supported flexural system under inverted triangular lateral loading

\[
F(x) = \frac{Fx}{n},
\]

\[
V(x) = \frac{(n^2 - 1)F}{6n} - \sum_{i=1}^{n} \frac{F_i}{n} = \frac{[(n^2 - 1) - 3(x^2 + x)]F}{6n}
\]

\[
M(x) = \frac{(n^2 - 1)Fh}{6n} - \sum_{i=1}^{n-1} \frac{Fhi(x-i)}{n} = \frac{(n^2x - x^3)Fh}{6n}
\]

\[
\psi_0 = \sum_{i=1}^{n} \frac{M(x)}{n^2hK} x = \frac{(n-1)(7n^3 + 7n^2 + 2n + 2)Fh}{360nK}
\]

\[
Y(x) = \psi_0 x h - \sum_{i=1}^{n} \frac{M(x)xh}{K} = \psi_0 x h - \sum_{i=1}^{n-1} \frac{(n^2i^2 - i^3)Fh^3}{6nK}
\]

which simplifies to

\[
Y(x) = \frac{\left( (n-1) \left[ 7(n^3 + n^2) + 2(n + 1) \right] - 2 \left[ (x - 1)(2x - 1)(3x^2 - 3x - 5n^2 - 1) \right] \right) Fh^2x}{360nK}.
\]

7.3.1 | Example 11

Compute the lateral displacements of a five-story, \( n = 5 \), regular, simply supported flexural MLRC under inverted triangular lateral nodal forces \( Fx/n \) and \( P = 0 \). Assume global rigid body rotation = 0.

Solution: For simply supported MLRCs, \( K_f(0) = K_f(n) = 0 \), \( K_f(x) = K_f \), ie, \( Y(0) = Y(n) = 0 \). Slope concentration at supported ends = \( \psi(0) = \psi(n) = (0 + 2 + 3 + 3 + 2 + 0)Fh/2K = 5Fh/K \). It follows that \( Y(t) = \psi(0)h - \psi(1)h = (5Fh/K).2h - (2Fh/K)h = 8Fh^2/K \) and \( Y(1) = \psi(0)h = 5Fh^2/K \). The complete solution to Example 8 is presented in columns 1 and 2 of Table 3.

8 | DYNAMICS

The short cut characteristic Equations (10), (11), and (12) together with the Drift Projection method constitute highly efficient means of static analysis for the type of MLRCs considered in this paper. However, they lack the linear \( WY(x)/g \) and rotary \( WR^2\dot{Y}(x)/gh \) inertia terms needed to formulate the corresponding equation of motion. The dotted symbols indicate differentiation with respect to time. \( R \) is the radius of gyration of the module and \( g \) stands for gravitational acceleration.

An efficient way of resolving this problem is by formulating the equation of motion based on the oscillatory equilibrium of two adjacent modules such as those shown in Figure 6. Here, displacement compatibility is satisfied as part of the discrete field solutions imposed by the pertaining boundary support conditions.

8.1 | Governing equation of motion of the purely flexural system

In flexural MLRCs, the lateral displacement of each level with respect to its own base is caused by the total moments acting at that level. Consider the momentary equilibrium of the lower module of the generic system of Figure 6B, subjected to gravity, inertia, and time-dependent lateral forces. The corresponding equations of motion may be expressed as

\[
\Delta_x M(x, t) + P \Delta_x Y_f(x, t) + V(x, t)h - \frac{WR^2\Delta_x Y_f(x, t)}{gh} = 0
\]

\[
\nabla_x V(x, t) + F(x, t) - \frac{W}{g} \dot{Y}_f(x, t) = 0.
\]
FIGURE 6  Free body diagrams. A, Purely racking system; B, Purely flexural system

TABLE 4  Eigenfrequencies of Example 12

| No. of levels | 2        | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |
|--------------|----------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( \omega^2/[gK_f/W_h^2] \) | 4.0000   | 1.0000 | 0.3432 | 0.1459 | 0.0718 | 0.0392 | 0.0232 | 0.0146 | 0.0096 |

Operating on both sides of Equation (58) with \( \Delta_x \) and eliminating the term \( \nabla_x V(x, t) \) from Equation (59), it gives

\[
\nabla_x \Delta_x \left[ M(x, t) + PY_f(x, t) - \frac{WR^2 \dot{Y}_f(x, t)}{gh} \right] - \left[ F(x, t) - \frac{W}{g} \ddot{Y}_f(x, t) \right] h = 0. \tag{60}
\]

Next, operating on both sides of (14) by the second central difference operator \( \nabla_x \Delta_x \), it gives

\[
\nabla_x \Delta_x V(x, t) = \frac{1}{hK_f} \nabla_x \Delta_x M(x, t). \tag{61}
\]

Substituting for \( \nabla_x \Delta_x M(x, t) \) from Equation (60) into Equation (61) results in the governing difference equation of motion of the flexural component of the structure

\[
\frac{K_f}{h^2} \nabla_x \Delta_x \nabla_x \Delta_x Y_f(x, t) + \nabla_x \Delta_x \left[ \frac{P}{h} Y_f(x, t) - \frac{WR^2}{gh^2} \ddot{Y}_f(x, t) \right] + \frac{W}{g} \ddot{Y}_f(x, t) - F(x, t) = 0. \tag{62}
\]

8.1.1 Example 12

Determine the dominant eigenfrequencies of the \( n \) story, regular, simply supported flexural MLRC of Figure 4G with equal masses \( (W/g) \) at each node. Compute \( \omega^2 \) for \( r = 1 \) and \( n = 2 \) to 10. Assume that axial and lateral forces, rotary inertia, as well as global rigid body rotation are zero.

**Solution:** Assume a solution of the form \( Y(x, t) = A_r e^{i\omega t} \sin \frac{r \pi x}{n} \) for the time-dependent deformations of the subject system. Substitution of \( Y(x, t) \) into Equation (62) gives, after some simplifications,

\[
\omega^2 = \frac{16gK_f}{Wh^2} \sin \frac{r \pi}{2n}. \tag{63}
\]

The complete solution to Example 12 is provided in Table 4.

8.2 Governing equation of motion of the purely racking system

The mathematical development of equation of motion of the racking module of Figure 1D can be immensely simplified by making use of the fact that the flexural distortions of its beams and columns, due to external moments, can be expressed in terms of a simple lateral nodal racking force. Consequently, the governing difference equation of motion of regular
TABLE 5  
Eigenfrequencies of Example 13

| No. of levels | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|---------------|------|------|------|------|------|------|------|------|------|------|
| \(\omega^2/ [gKf/Wh^2]\) | 1.00 | 0.3892 | 0.1981 | 0.1206 | 0.0810 | 0.0581 | 0.0437 | 0.0431 | 0.0273 | 0.0223 |

MLRCs, such as those shown in Figure 2E or Figure 4D can formulated in accordance with the rigid body sliding of the generic racking model of Figure 6A.

\[
V(x + 1, t)h - (K_f h - P)\Delta_x Y_f(x, t) = 0
\]  \((64)\)

\[
V(x, t)h - (K_f h - P) \nabla_x Y_f(x, t) = 0
\]  \((65)\)

\[
\Delta V(x, t) - F(x, t) + \frac{W}{g} \ddot{Y}(x, t) = 0
\]  \((66)\)

Operating on Equations (64) and (65) by forward and backward difference operators \(\Delta_x\) and \(\nabla_x\), respectively, and substituting in (66) for \(\Delta V(x)\), it gives

\[
\left( K_f - \frac{P}{h} \right) \Delta_x \nabla_x Y_f(x, t) - F(x, t) + \frac{W}{g} \ddot{Y}_f(x, t) = 0.
\]  \((67)\)

The complete solution of Equation (67) is sensitive to the drift profile and is contained in the double summation

\[
Y_r(x, t) = \nabla_x^{-1} \Delta_x^{-1} \left[ \frac{F(x) - W\ddot{Y}_r(x, t)/g}{(K_r - P/h)} \right].
\]  \((68)\)

Applications of Equation (67) are best demonstrated by the following example.

8.2.1  Example 13

Determine the dominant eigenfrequencies of the \(n\) story, regular, cantilevered racking MLRC of Figure 4D with equal masses \((W/g)\) at each node. Compute \(\omega^2\) for \(r = 1\) and \(n = 1\) to 10. Assume axial and lateral forces, rotary inertia, as well as global rigid body rotation are zero.

**Solution:** Assume a solution of the form \(Y(x, t) = A e^{i\omega t} \sin \frac{r \pi x}{2(n+1)}\) for the time-dependent deformations of the subject system. Substitution of \(Y(x, t)\) into Equation (62) gives, after simplification,

\[
\omega^2 = \frac{4gK_r}{Wh^2} \sin^2 \frac{r \pi}{4n + 2}.
\]  \((69)\)

The complete solution to Example 13 is provided in Table 5.

Obviously, all higher frequencies of the two systems discussed under Sections 8.1 and 8.2 can be computed by assigning the desired mode numbers to the variable \(r\) in Equations (63) and (69), respectively.

9  Verification

Solutions presented in this paper are all exact within the bounds of the theoretical assumptions. They satisfy all pertinent characteristic equations and the corresponding boundary support conditions. All numerical results have been verified by independent computer analysis. It may be instructive to note that combination of Equations (62) and (67) leads to the classical equation of motion of Timoshenko’s beam-column on elastic foundation.\(^{49}\) Operating on Equation (67) by the second central difference operator \(\Delta_x \nabla_x\), adding the result to Equation (62), and observing that \(Y(x, t) = Y_f(x, t) + Y_r(x, t)\), eventually gives

\[
\frac{K_f}{K_r} \left[ K_r - \frac{P}{h} \right] \Delta_x \nabla_x \Delta_x Y(x, t) + \frac{P}{h} \Delta_x \nabla_x Y(x, t) - \frac{K_f}{K_r} \Delta_x \nabla_x \left[ \frac{W}{g} \ddot{Y}_f(x, t) - F(x, t) \right] + \frac{W}{g} \left[ \Delta_x \nabla_x \frac{h^2}{R^2} - 1 \right] \ddot{Y}(x, t) - F(x, t) = 0.
\]  \((70)\)
Transforming the discrete Equation (70) into a continuous differential equation by using the identity \( \Delta_x \nabla_x \equiv h^2 \frac{\partial^2}{\partial x^2} \) as well as replacing \( EI \) with \( (Kf) \) and \( (kh) \) with \( K_r \) and ignoring the elastic foundation and rotary coupling, the two solutions coincide.

### 10 | CONCLUSIONS

This article has drawn a simple analogy, with practical applications, between the human spine and MLRCs under similar loading conditions.

Finite difference calculus has been used to study the subject matter as a regular discrete system. An alternative method of analysis based on continuum modeling\(^{50,51} \) for similar structures could have also been used to solve the same problem with sufficient accuracy for preliminary design purposes. The use of the proposed methodology can be extended to seismic upgrading\(^{52,53} \) of existing buildings.

The solutions presented in this paper provide solid mathematical basis for the theoretical development of regular multilevel rocking systems encountered in practice. It has been shown through parametric examples that the preliminary design of almost all practical MLRCs can be based on the formulations described in this work. Examples 12 and 13 introduce two generalized displacement profiles that can be used to conduct exact dynamic analysis involving pertinent higher modes of vibrations for cantilevered and top and bottom supported MLRCs. The list of different types of MLRCs presented is by no means complete. Neither all combinations of stiffnesses \( K_{\text{bldg}}, K_{\text{frame}}, \) and \( K_{\text{core}} \) nor combinations of loading profiles and boundary support conditions could be addressed in the limited space of the current report. However, sufficient basic information has been provided for the interested reader to be able to study the statics/dynamics of similar cases not elaborated upon in the current study. Over a dozen generic and numerical examples were provided to demonstrate the applications of the proposed solutions. In the interim, the use of the Drift Projection theorem was introduced as a basic tool for the analysis of MLRCs considered in this work.

The proposed structural schemes are neither perfect nor complete. They are still under development and need the test of time and scrutiny before they become widely accepted earthquake resisting systems. Hopefully, this article will motivate others to continue and improve the current research initiative and extend the use of the proposed methodologies to more efficient earthquake resistant archetypes.

**ORCID**

Mark Grigorian  [https://orcid.org/0000-0002-8508-1481](https://orcid.org/0000-0002-8508-1481)

Mozhgan Kamizi  [https://orcid.org/0000-0001-9878-6002](https://orcid.org/0000-0001-9878-6002)

**REFERENCES**

1. Benzel EC. *Biomechanics of Spine Stabilization*. New York, NY: McGraw-Hill; 1995.
2. Roberts SB, Chen PH. Elastostatic analysis of the human thoracic skeleton. *J Biomech*. 1970;3:527-545.
3. Grigorian M. Biomimicry and theory of structures-design methodology transfer from trees to moment frames. *J Bionic Eng*. 2014; 11(4):638-648.
4. Grigorian M. Performance control based on green tree behavior. *Asian J of Civil Eng*. 2014;15(6):897-922.
5. Ajrab JJ, Pekcan G, Mander JB. Rocking wall-frame structures with supplemental tendon systems. *J Struct Eng*. 2004;130(6):895-903. [http://doi.org/10.1061/(ASCE)0733-9445(2004)130:6(895)]
6. Filiatrault A, Restrepo J, Christopoulos C. Development of self-centering earthquake resisting systems. In: Proceedings of the 13th World Conference on Earthquake Engineering; 2004; Vancouver, Canada.
7. Roke D, Sause R, Ricles JM, Seo C-Y, Lee K-S. Self-centering seismic resistant steel concentrically-braced frames. In: Proceedings of the 8th US National Conference on Earthquake Engineering; 2006; San Francisco, CA.
8. Panian L, Steyer M, Tipping S. An innovative approach to earthquake safety and concrete construction. *J Post Tens Inst*. 2007;5(1):7-16.
9. Deierlein GG, Ma X, Eatherton M, Krawinkler H, Billington S, Hajjar JF. Collaborative research on development of innovative steel braced frame systems with controlled rocking and replaceable fuses. In: Proceedings of the 6th International Conference on Urban Earthquake Engineering; 2009; Tokyo, Japan.
10. Qu Z, Wada A, Motoyui S, Sakata H, Kishiki S. Pin supported walls for enhancing the seismic behavior of building structures. *Earthq Eng Struct Dyn*. 2012;41(14):2075-2091.
11. Janhunen B, Tipping S, Wolfe J, Mar D. Seismic retrofit of a 1960s steel moment-frame high-rise using a pivoting spine. In: Proceedings of the Annual Meeting of the Los Angeles Tall Buildings Structural Design Council; 2012; Los Angeles, CA.
12. Eatherton MR, Ma X, Krawinkler H, Deierlein GG, Hajjar JF. Quasi static cyclic behavior of controlled rocking steel frames. *J Struct Eng*. 2014;140(11):04014083. [https://doi.org/10.1061/(ASCE)ST.1943-541X.0001005]
AUTHOR BIOGRAPHIES

Mark Grigorian is an international consultant, investigator, and lecturer. He has more than 50 years of experience in structural design, education, code and standard development, forensic studies, and research and project management. He holds BSc and MSc degrees in Structural Engineering from the University of Manchester, Manchester, UK, and a DPhil degree in Engineering from the University of Oxford, Oxford, UK. He is a California licensed Civil and Structural Engineer and is an active member of the Structural Engineers Association of Southern California. He is a founding member and the first chairman of the Faculty of Structural Engineering (Sazeh) of Sharif University in Iran. Currently, he is the Chief Structural Engineer of MGA Structural Engineering Consultants, Inc, based in Glendale, California.

Mozhgan Kamizi holds BSc degree in Civil Engineering and MSc degree in Structural Engineering from the University of Golestan, Gorgan, Iran. Her research interests include bioinspiration, structures of uniform response, earthquake engineering in general, and low damage structural engineering in particular. She has published several scientific papers since graduation. She is currently working as coordinator and research assistant to the senior author of the current article.

How to cite this article: Grigorian M, Kamizi M. On the analysis of multilevel rocking cores—A bioinspired analogy. Engineering Reports. 2019;1:e12025. https://doi.org/10.1002/eng2.12025