Research Article

On Ostrowski Type Inequalities for Generalized Integral Operators

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It is well known that mathematical inequalities have played a very important role in solving both theoretical and practical problems. In this paper, we show some new results related to Ostrowski type inequalities for generalized integral operators.

1. Introduction

Mathematical inequalities have been present in the development and consolidation of Science. Nowadays, inequalities are essential tools in multiple applications to different problems, since they are involved in the basis of the processes of approximation, estimation, interpolation, and extrapolation, and in general, they appear in the models used in the study of applied problems.

The formalization of mathematical inequalities begins in the 18th century, essentially, with the works of the so-called “Prince of Mathematics” Johann Carl Friedrich Gauss (1777–1855); passing through the investigations and applications of inequalities to Mathematical Analysis developed by Augustin-Louis Cauchy (1789–1857) and Pafnuti Lvovich Chebyshev (1821–1894). It would be unfair not to mention among the formalizations of mathematical inequalities to Viktor Yakovlevich Bunyakovsky (1804–1889). This remarkable Russian mathematician received all possible mathematical influence from his thesis advisor Augustin-Louis Cauchy. This remarkable scientist is credited with having proved in 1859, many years before Hermann Schwarz, the well-known Cauchy–Schwarz Inequality for the infinite-dimensional case. It is worth noting that in many texts the famous inequality is known as: Cauchy–Bunyakovsky–Schwarz.

The proof of Hardy’s famous inequality involved an important group of prominent mathematicians of his time such as Edmund Hermann Landau (1887–1938), George Pólya (1887–1985), and Issai Schur (1875–1941), and Marcel Riesz (1886–1969), among others. It is worth noting the coordinating role played by Godfrey Harold Hardy (1887–1947) in the study of inequalities; his work has been very significant, fundamentally, for the systematization and application of the Theory of Mathematical Inequalities. Hardy was the founder of the Journal of the London Mathematical Society, a suitable publication for many articles on inequalities. In addition, along with Littlewood and Pólya, Hardy was the editor of the volume Inequalities see [1], which was the first monograph, on inequalities and immediately used as the basis for the later development of mathematical inequalities. For more information on the epistemological evolution of the Theory of Mathematical Inequalities see [2].
In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. For the most recent works, we recommend the reader [3–5]. Some of the inequalities studied are Gronwall, Chebyshev, Hermite–Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [6–13]). Other types of inequalities such as Hilbert-type inequalities and Hadamard-type inequalities have also been recently studied in the context of integral inequalities and generalized mapping on fractal sets, [14, 15].

In this paper, we show some new results related to Ostrowski-type inequalities via conformable and non-conformable operators.

Ostrowski-type inequalities have significant contributions to the area of numerical analysis since they provide estimates of the error of many quadrature rules, for example, the midpoint rule, Simpson’s rule, the trapezoidal rule, and other generalized fractional integrals. They also have many powerful results and a large number of applications in Probability Theory and Statistics, Information Theory, and Integral Operator Theory. For further discussions, we refer the reader to the book by Dragomir and Rassias (see [16]).

2. On Generalized Ostrowski’s Inequality

Alexander Ostrowski (1893–1986) was an important mathematician born in Kyiv, the former Russian empire, today the capital of Ukraine. From a mathematical point of view, he was directly influenced by great mathematicians such as Hensel, Hilbert, Klein, and Landau. Another tribute, in addition to the inequality that bears his name, is the well-known Ostrowski Prize that is jointly sponsored by several renowned universities and the Academies of Science of the Netherlands and Denmark.

Ostrowski proved in [17] an integral inequality associated with a real differentiable function that establishes an upper bound for the difference between the function evaluated at any interior point of some interval and the average of the function over the same interval.

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function which is differentiable on \((a, b)\). If \( f' \in L^\infty[a, b] \), then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f'(t) \, dt \right| \leq \frac{1}{b-a} \left[ \frac{(b-a)^2}{2} + \left( x - \frac{a+b}{2} \right)^2 \right] \| f' \|_{\infty}.
\]

Since then, there are a lot of generalizations and applications of this inequality (see, e.g., [16]). In particular, Dragomir and Wang generalized this inequality to \( L^p[a, b] \) \((p > 1)\) in [18] as follows:

**Theorem 2.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function which is differentiable on \((a, b)\). If \( p > 1 \), \((1/p) + (1/q) = 1\) and \( f' \in L^p[a, b] \), then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f'(t) \, dt \right| \leq \frac{1}{b-a} \left[ \frac{(b-a)^q}{q} + \left( x - \frac{a+b}{2} \right) \right] \| f' \|_{\infty}.
\]

In this paper, we prove the following two weighted versions of this inequality. The main improvement is to consider general weights, but also, we prove the inequality for a larger class of functions, and we include the case \( p = 1 \). Furthermore, we prove that our inequality is sharp for every weight.

**Theorem 3.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function, and \( w : [a, b] \rightarrow [0, \infty) \) an integrable function with \( \int_a^b w(t) \, dt > 0 \).

1. If \( 1 < p \leq \infty \) and \((1/p) + (1/q) = 1\), then

\[
\left| f(x) - \frac{1}{\int_a^b w(t) \, dt} \int_a^b f(t) w(t) \, dt \right| \leq \left( \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^{1/q} \| f' \|_p.
\]

2. If \( p = 1 \), then

\[
\left| f(x) - \frac{1}{\int_a^b w(t) \, dt} \int_a^b f(t) w(t) \, dt \right| \leq \| f' \|_1.
\]

Theorem 3 provides simple bounds, but they do not depend on the weight \( w \). This theorem can be improved by the following bounds involving the weight.

**Theorem 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function, and \( w : [a, b] \rightarrow [0, \infty) \) an integrable function with \( \int_a^b w(t) \, dt > 0 \).

1. If \( 1 < p \leq \infty \) and \((1/p) + (1/q) = 1\), then

\[
\left| f(x) - \frac{1}{\int_a^b w(t) \, dt} \int_a^b f(t) w(t) \, dt \right| \leq \left( \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right)^{1/q} \| f' \|_p.
\]

2. If \( p = 1 \), then

\[
\left| f(x) - \frac{1}{\int_a^b w(t) \, dt} \int_a^b f(t) w(t) \, dt \right| \leq \| f' \|_1.
\]

3. For each weight \( w \), \( 1 < p < \infty \) and \( x \in [a, b] \), there exists an absolutely continuous function \( f \) with \( f' \in L^p[a, b] \) such that the equality in the inequality holds.
3. Proofs of the Inequalities

Recall that a function \( f : [a, b] \rightarrow \mathbb{R} \) is absolutely continuous if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if a finite sequence of pairwise disjoint sub-intervals \( \{[x_k, y_k]\}_k \subset [a, b] \) satisfies
\[
\sum_k (y_k - x_k) < \delta,
\]
then
\[
\sum_k |f(y_k) - f(x_k)| < \varepsilon.
\]

It is well-known that \( f \) is absolutely continuous on \([a, b]\) if and only if it has a derivative \( f' \) almost everywhere, the derivative is integrable, and \( f(x) = f(a) + \int_a^x f'(t)dt \) for every \( x \in [a, b] \).

**Proof of Theorem 3.** First of all, note that \( f \omega \in L^1[a, b] \), since \( f \in L^\infty[a, b] \) and \( \omega \in L^1[a, b] \). We can assume that \( f' \in L^p[a, b] \), since otherwise the inequality trivially holds.

Let us define \( m \) and \( M \) as the minimum and maximum values of \( f' \) on \([a, b]\), respectively. Thus, we have
\[
m \leq \frac{1}{f'(t)dt} \int_a^b f(t)\omega(t)dt \leq M.
\]

The intermediate values theorem gives that there exists \( x_0 \in [a, b] \) with
\[
f(x_0) = \frac{1}{f'(t)dt} \int_a^{b} f(t)\omega(t)dt.
\]

Assume first \( 1 < p < \infty \). Hölder inequality gives
\[
\left| f(x) - \frac{1}{\int_a^b f(t)\omega(t)dt} \int_a^b f(t)\omega(t)dt \right| = \left| f(x) - f(x_0) \right| = \left| \int_{x_0}^x f'(t)dt \right| \leq \int_{x_0}^x |f'(t)|^p dt \left( \int_{x_0}^x 1^q dt \right)^{1/q} = |x - x_0|^{1/q} \int_{x_0}^x |f'(t)|^p dt \leq \max|x - a, b - x|^{1/q} \|f'\|_p.
\]

The desired inequality holds since
\[
\max|x - a, b - x| = \frac{b - a}{2} + \frac{x - a + b}{2}.
\]

If \( p = 1 \) or \( p = \infty \), then a similar and simpler argument gives the inequalities.

**Proof of Theorem 4.** We can assume that \( f' \in L^p[a, b] \), since otherwise the inequality trivially holds. Since \( \omega \in L^1[a, b] \), the function \( \int_a^x \omega(t)dt \) is absolutely continuous on \([a, b]\) for every \( A \in [a, b] \).

Since the integration by parts rule holds for absolutely continuous functions, we have
\[
\int_{a}^{b} \left( \int_{a}^{t} \omega(s)ds \right) f'(t)dt = \left[ f(t) \int_{a}^{t} \omega(s)ds \right]_{a}^{b} - \int_{a}^{b} f(t)\omega(t)dt,
\]
and so,
\[
f(x) \int_{a}^{b} \omega(t)dt - \int_{a}^{b} f(t)\omega(t)dt = \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right) f'(t)dt.
\]

Assume first \( 1 < p < \infty \). Hölder inequality gives
\[
\left| \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right) f'(t)dt \right| \leq \left( \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right)^q dt \right)^{1/q} \left( \int_{a}^{x} |f'(t)|^p dt \right)^{1/q}.
\]

Thus, discrete Hölder inequality \( a_1b_1 + a_2b_2 \leq (a_1^2 + a_2^2)^{1/2} (b_1^2 + b_2^2)^{1/2} \) gives
\[
\left( \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right)f'(t)dt \right)^{1/p} + \left( \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right)f'(t)dt \right)^{1/q} \leq \left( \int_{a}^{x} \left( \int_{a}^{t} \omega(s)ds \right)^q dt \right)^{1/q} \left( \int_{a}^{x} |f'(t)|^p dt \right)^{1/p}.
\]

and the inequality holds.

Assume now that \( p = 1 \). Note that, since \( \omega \geq 0 \),
Finally, let us prove (3). Fix $p$, $0 < p < \infty$ and $x \in [a, b]$, and define

$$f(t) = \begin{cases} \int_x^b (\int_a^r w(s) \mathrm{d}s) \frac{1}{r^{(p-1)}} \mathrm{d}s, & \text{if } t \in [a, x], \\ -\int_x^b (\int_s^b w(t) \mathrm{d}t) \frac{1}{r^{(p-1)}} \mathrm{d}s, & \text{if } t \in [x, b]. \end{cases}$$

(18)

Since $w \geq 0$ and $w \in L^1 [a, b]$, we have

$$\frac{\%}{\%}\left(\int_a^b w(r) \mathrm{d}r\right)^{1/(p-1)} \in C[a, x], \left(\int_a^b w(r) \mathrm{d}r\right)^{1/(p-1)} \in C[x, b],$$

(19)

and the inequality also holds in this case. If $p = \infty$, then a similar argument gives the inequality.

Finally, let us prove (3). Fix $w$, $1 < p < \infty$ and $x \in [a, b]$, and define

$$\left|\int_a^x \left(\int_a^t w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t + \int_x^b \left(\int_t^b w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t\right|$$

$$\leq \left(\int_a^x \left(\int_a^t w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t + \int_x^b \left(\int_t^b w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t\right)$$

$$\leq \max\left\{\int_a^x w(t) \mathrm{d}t, \int_x^b w(t) \mathrm{d}t\right\} \left|\int_a^x f'(t) \mathrm{d}t + \int_x^b f'(t) \mathrm{d}t\right|$$

$$= \max\left\{\int_a^x w(t) \mathrm{d}t, \int_x^b w(t) \mathrm{d}t\right\} \left\|f'\right\|_p,$$

(17)

and so, $f$ is an absolutely continuous function on $[a, b]$ and $f' \in L^\infty [a, b]$.

The argument in the proof of item (1) shows that it suffices to check that

$$\int_a^x \left(\int_a^t w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t + \int_x^b \left(\int_t^b w(s) \mathrm{d}s\right) f'(t) \mathrm{d}t$$

$$= \left(\int_a^x \left(\int_a^t w(s) \mathrm{d}s\right) \frac{1}{r^{(p-1)}} \mathrm{d}r + \int_x^b \left(\int_t^b w(s) \mathrm{d}s\right) \frac{1}{r^{(p-1)}} \mathrm{d}r\right) \left\|f'\right\|_p,$$

(20)

Note that,
and so, the equality in the inequality in the first item is attained for this choice of $f$.

Note that, if we substitute the weight $w$ by the constant function 1 in Theorem 4, then we get the classical inequality described in Theorem 2.

\[ \square \]

4. On the Ostrowski Inequality in Conformable and Nonconformable Context

The evolution of many physical processes can be described in a more precise way by using fractional derivatives (see, e.g., [19–24]). Usually, it suffices to replace the time derivative in a given evolution equation by a fractional derivative. There is a solid mathematical basis for proceeding this way (see, e.g., [23–26]). Recent developments on evolution fractional calculus and its applications can be found in [27–30].

In several papers (see, e.g., [23, 31–33]) are defined local fractional derivatives in the following way. Given a function $f(t)$, $a \in (0,1]$ and a kernel $T(t,a)$, the derivative of $f$ of order $a$ at the point $t$ with respect to the kernel $T$ is defined by the following equation:

\[ G^a_T f(t) = \lim_{h \to 0} \frac{f(t)-f(t-hT(t,a))}{h}. \]  

(22)

Let $I$ be an (open or not) interval $I \subseteq \mathbb{R}$. The generalized derivative $G^a_T$ is said conformable if $G^a_T f(t) = f'(t)$ for every $t \in I$ or, equivalently, if $T(t,1) = 1$ for every $t \in I$.

Let $I$ be an interval $I \subseteq \mathbb{R}$, $a \in (0,1]$ and $T$ a positive continuous function on $I \times (0,1]$. In [33] the integral operator $J^a_{T,a}$ is defined for every locally integrable function $f$ on $I$ as follows:

\[ J^a_{T,a}(f)(t) = \int_0^1 f(s)h(s,a)\frac{d\lambda}{\lambda} \]  

(23)

The following basic properties related to the operator $G^a_T$ appear in [34].

Theorem 5 (see [34], Theorem 2.4). Let $I$ be an interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ and $a \in \mathbb{R}^+$.

1. If there exists $D^a_0 f$ at the point $t \in I$, then $f$ is $G^a_T$-differentiable at $t$ and $G^a_T f(t) = T(t,a)D^a_0 f(t)$.

2. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable at $t \in I$ if and only if $f$ is differentiable at $t$; in this case, we have $G^a_T f(t) = T(t,a)f'(t)$.

Theorem 6 (see [34] Theorem 2.5). Let $I$ be an interval $I \subseteq \mathbb{R}$, $f, g : I \to \mathbb{R}$ and $a \in \mathbb{R}^+$. Assume that $f$ and $g$ are $G^a_T$-differentiable functions at $t \in I$. Then the following statements hold:

1. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

2. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

3. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

4. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

5. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

6. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

7. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

8. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

9. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

10. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

11. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

12. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

13. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

14. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

15. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

16. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

17. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

18. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

19. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

20. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

21. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

22. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

23. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

24. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

25. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

26. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

27. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

28. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

29. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

30. If $a \in (0,1]$, then $f$ is $G^a_T$-differentiable and $G^a_T f(t) = T(t,a)f'(t)$.

For further information about this integral operator and its applications, we refer the readers to [25, 33–35].

If we take $u(t) = (1/T)(t,a)$ in Theorem 4 with $I = [a,b]$, we can obtain inequalities involving integral operators, conformable with $T(t,a) = (t-a)^{-a}$ (see Proposition 3) and nonconformable with $T(t,a) = e^{-a(t-a)}$ (see Propositions 4 and 5).

Recall that the incomplete beta functions $B_1$ and $B_2$ are defined, respectively, as follows:
Proof. Note that \( w(x) = (x - a)^{\alpha - 1} \in L^1[a, b] \), since \( \alpha > 0 \).

We have

\[
\int_b^a w(t)dt = \frac{(b-a)^{\alpha}}{\alpha},
\]

\[
\int_a^b \left( \int_a^t w(s)ds \right)^q dt = \int_a^b \left( \int_s^a w(h)dh \right)^q dt = \frac{(x-a)^{\alpha q+1}}{\alpha^q (aq+1)}.
\]

By making the change of variables \( u^{1/\alpha} = (t-a)/(b-a) \), we obtain

\[
\left| f(x) - \frac{\alpha}{e^{\alpha(b-a)} - 1} \int_a^b f(t)e^{\alpha(t-a)} dt \right| \leq \frac{\alpha^q}{e^{\alpha(b-a)} - 1} \left( B_2(e^{-\alpha(x-a)}; q, q+1) + e^{\alpha q(b-a)} B_2(e^{\alpha(x-b)}; 0, q+1) \right) \|f\|_p^q.
\]  

(2) If \( p = 1 \), then

\[
\left| f(x) - \frac{\alpha}{e^{\alpha(b-a)} - 1} \int_a^b f(t)e^{\alpha(t-a)} dt \right| \leq \frac{1}{e^{\alpha(b-a)} - 1} \max\left\{ e^{\alpha(x-a)} - 1, e^{\alpha(b-a)} - e^{\alpha(x-a)} \right\} \|f\|_1.
\]

Thus, Theorem 4 gives the inequalities. \( \Box \)

Proposition 4. Let \( f: [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function, and \( \alpha > 0 \).

(1) If \( 1 < p \leq \infty \) and \( (1/p) + (1/q) = 1 \), then

\[
\int_a^b \left( \int_a^t w(s)ds \right)^q dt = \int_a^b \left( (b-a)^{\alpha} - (t-a)^{\alpha} \right)^q dt.
\]

(2) If \( p = 1 \), then

\[
\int_a^b w(t)dt = \frac{e^{\alpha(b-a)} - 1}{\alpha}.
\]

By making the change of variables \( u = e^{-\alpha(t-a)} \), we obtain

\[
\left| f(x) - \frac{\alpha}{e^{\alpha(b-a)} - 1} \int_a^b f(t)e^{\alpha(t-a)} dt \right| \leq \frac{1}{e^{\alpha(b-a)} - 1} \max\left\{ e^{\alpha(x-a)} - 1, e^{\alpha(b-a)} - e^{\alpha(x-a)} \right\} \|f\|_1.
\]

Proof. Note that \( w(x) = e^{\alpha(x-a)} \in L^1[a, b] \). We have
\[
\int_a^b \left( \int_a^t w(s) \, ds \right)^q \, dt = \int_a^b \left( \frac{e^{\alpha(t-a)} - 1}{\alpha} \right)^q \, dt \\
= \int_a^b \frac{e^{\alpha(t-a)}}{\alpha} \left( 1 - e^{-\alpha(t-a)} \right)^q \, dt \\
= \frac{1}{\alpha^{q+1}} \int_{e^{\alpha(t-a)}}^1 u^{q-1} (1 - u)^q \, du \\
= \frac{1}{\alpha^{q+1}} B_2 \left( e^{-\alpha(t-a)} ; q, q + 1 \right). 
\]

(36)

By making the change of variables \( u = e^{\alpha(t-b)} \), we have

\[
\int_x^b \left( \int_t^b w(s) \, ds \right)^q \, dt = \int_x^b \frac{e^{\alpha(t-a)} - e^{\alpha(t-b)}}{\alpha} \, dt \\
= \frac{e^{\alpha(t-b)}}{\alpha^{q+1}} \int_x^b \left( 1 - e^{\alpha(t-b)} \right)^q e^{-\alpha(t-b)} \, dt \\
= \frac{e^{\alpha(t-b)}}{\alpha^{q+1}} \int_{e^{\alpha(t-b)}}^1 (1 - u)^q u^{-1} \, du \\
= \frac{e^{\alpha(t-b)}}{\alpha^{q+1}} B_2 \left( e^{\alpha(x-b)} ; 0, q + 1 \right). 
\]

(37)

Therefore, Theorem 4 gives the inequalities. \( \Box \)

**Proposition 5.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function, and \( \alpha < 0 \).

1. If \( 1 < p \leq \infty \) and \( (1/p) + (1/q) = 1 \), then

\[
\left| f(x) - \frac{|a|}{1 - e^{\alpha(b-a)}} \int_a^b f(t) e^{\alpha(t-a)} \, dt \right| \\
\leq \left| \frac{|a|}{1 - e^{\alpha(b-a)}} B_2 \left( e^{\alpha(x-a)} ; 0, q + 1 \right) \right| + e^{\alpha(b-a)} B_2 \left( e^{-\alpha(x-b)} ; q, q + 1 \right) \| f' \|_p. 
\]

(38)

2. If \( p = 1 \), then

\[
\left| f(x) - \frac{|a|}{1 - e^{\alpha(b-a)}} \int_a^b f(t) e^{\alpha(t-a)} \, dt \right| \\
\leq \frac{1}{1 - e^{\alpha(b-a)}} \max \left\{ 1 - e^{\alpha(x-a)}, e^{\alpha(x-a)} - e^{\alpha(b-a)} \right\} \| f' \|_1. 
\]

(39)

**Proof.** Note that, \( w(x) = e^{\alpha(x-a)} \in L^1 [a, b] \). We have

\[
\int_a^b w(t) \, dt = \frac{e^{\alpha(b-a)} - 1}{\alpha} = \frac{1 - e^{\alpha(b-a)}}{|a|}. 
\]

(40)

By making the change of variables \( u = e^{\alpha(t-a)} \), we obtain

\[
\int_a^b \left( \int_a^t w(s) \, ds \right)^q \, dt = \int_a^b \left( 1 - e^{\alpha(t-a)} \right)^q \, dt \\
= \int_a^b \frac{e^{\alpha(t-a)}}{|a|^{q+1}} (1 - \alpha) e^{\alpha(t-a)} \, dt \\
= \frac{1}{|a|^{q+1}} \int_{e^{\alpha(t-a)}}^1 u^{-1} (1 - u)^q \, du \\
= \frac{1}{|a|^{q+1}} B_2 \left( e^{\alpha(x-a)} ; 0, q + 1 \right). 
\]

(41)

By making the change of variables \( u = e^{\alpha(t-b)} \), we have

\[
\int_x^b \left( \int_t^b w(s) \, ds \right)^q \, dt = \int_x^b \frac{e^{\alpha(t-a)} - e^{\alpha(t-b)}}{|a|} \, dt \\
= \frac{e^{\alpha(t-b)}}{|a|^{q+1}} \int_x^b (1 - e^{\alpha(t-b)})^q e^{\alpha(t-b)} \, dt \\
= \frac{e^{\alpha(t-b)}}{|a|^{q+1}} \int_{e^{\alpha(t-b)}}^1 (1 - u)^q u^{-1} \, du \\
= \frac{e^{\alpha(b-a)}}{|a|^{q+1}} B_2 \left( e^{\alpha(x-b)} ; q, q + 1 \right). 
\]

(42)

thus, Theorem 4 gives the inequalities. \( \Box \)

**5. Conclusions**

In this article, we continue with the study and development of an important topic in mathematics which are inequalities, particularly inequalities in fractional context. We prove two weighted versions of the generalized Ostrowsky inequality with a weight function \( w(t) \), as a consequence of these results we prove conformable and nonconformable fractional versions of this inequality, with the choice of the function \( w(t) = (1/(t-a)^\alpha) \) for the conformable case and \( w(t) = (1/e^{\alpha(t-a)}) \) for the nonconformable case, these functions have the form \((1/T(t,a))\), where \( T(t,a) \) represents the kernel of the fractional integral operator \( f^\alpha_T \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.
Authors’ Contributions

The authors contributed equally to this work.

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