A NEW FAMILY OF EFFICIENT CONFORMING MIXED FINITE ELEMENTS ON BOTH RECTANGULAR AND CUBOID MESHES FOR LINEAR ELASTICITY IN THE SYMMETRIC FORMULATION

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Abstract. A new family of mixed finite elements is proposed for solving the classical Hellinger–Reissner mixed problem of the elasticity equations. For two dimensions, the normal stress of the matrix-valued stress field is approximated by an enriched Brezzi–Douglas–Fortin–Marini element of order \( k \), the shear stress by the serendipity element of order \( k \), and the displacement field by an enriched discontinuous vector-valued \( P_{k-1} \) element. The degrees of freedom on each element of the lowest order element, which is of first order, are 10 plus 4. For three dimensions, the normal stress is approximated by an enriched Raviart–Thomas element of order \( k \), each component of the shear stress by a product space of the serendipity element space of two variables and the space of polynomials of degree \( \leq k - 1 \) with respect to the rest variable, and the displacement field by an enriched discontinuous vector-valued \( Q_{k-1} \) element. The degrees of freedom on each element of the lowest order element, which is of first order, are 21 plus 6. A family of reduced elements is also proposed by dropping some interior bubble functions of the stress and employing the discontinuous vector-valued \( P_{k-1} \) (resp., \( Q_{k-1} \)) element for the displacement field on each element. As a result the lowest order elements have 8 plus 2 and 18 plus 3 degrees of freedom on each element for two and three dimensions, respectively. The well-posedness condition and the optimal a priori error estimate are proved for this family of finite elements. Numerical tests are presented to confirm the theoretical results.

Key words. mixed method, enriched Brezzi–Douglas–Fortin–Marini element, enriched Raviart–Thomas element, serendipity element

AMS subject classifications. 65N30, 65N15, 35J25

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1. Introduction. The first order system of equations, for the symmetric stress field \( \sigma \in \Sigma := H(\text{div},\Omega,\mathbb{S}) \) and the displacement field \( u \in V := L^2(\Omega,\mathbb{R}^n) \), reads as follows: Given \( f \in L^2(\Omega,\mathbb{R}^n) \) find \((\sigma,u) \in \Sigma \times V \) such that

\[
\begin{align*}
(A\sigma,\tau)_{L^2(\Omega)} + (\text{div} \tau, u)_{L^2(\Omega)} &= 0, \\
(\text{div} \sigma, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}
\end{align*}
\]

for any \((\tau,v) \in \Sigma \times V\). Here and throughout this paper, the compliance tensor \( A(x) : \mathbb{S} \rightarrow \mathbb{S} \) is bounded and symmetric positive definite uniformly for \( x \in \Omega \) with \( \mathbb{S} := \mathbb{R}^{n \times n}_{\text{sym}} \) the set of symmetric tensors. The space \( H(\text{div},\Omega,\mathbb{S}) \) is defined by

\[
H(\text{div},\Omega,\mathbb{S}) := \{ \tau \in L^2(\Omega,\mathbb{S}) \text{ and } \text{div} \tau \in L^2(\Omega,\mathbb{R}^n) \}
\]

equipped with the norm

\[
\| \tau \|_{H(\text{div},\Omega)}^2 := \| \tau \|_{L^2(\Omega)}^2 + \| \text{div} \tau \|_{L^2(\Omega)}^2.
\]

The stress-displacement formulation within the Hellinger–Reissner principle for the linear elasticity is one celebrated example of (1.1).
Compared with the mixed formulation of the Poisson equation (see, for instance, [16]), there is an additional symmetric requirement on the stress tensor. Such a constraint makes the stable discretization of the piecewise polynomials extremely difficult. Then one idea that may come up is to enforce the symmetry condition weakly, which in fact leads to Lagrange multiplier methods [2, 6, 13, 33, 34, 35, 36]. As an alternative method, composite elements were proposed by Johnson and Mercier [31], and Arnold, Douglas, Gupta, [7]. That idea might be motivated by the Hsieh–Clough–Tocher element for the biharmonic problem [21]. Indeed, there is an observation in [31] that the discrete divergence-free space therein is the range of the Airy stress function of the Hsieh–Clough–Tocher plate element space; see a similar observation in [7]. Given a scalar field \( q \), the Airy stress function reads

\[
Jq := \left( \frac{\partial^2 q}{\partial y^2}, \frac{-\partial^2 q}{\partial x\partial y}, \frac{-\partial^2 q}{\partial x^2} \right).
\]

Unfortunately, this observation was not further explored until more than 20 years later, when its importance was realized by Arnold and Winther [9]. In that landmark paper, it was found that to design a stable discrete scheme is to look for a discrete differential complex with the commuting diagram which reads, for two dimensions,

\[
\begin{align*}
0 & \longrightarrow P_1(\Omega) \overset{\subset}{\longrightarrow} C^\infty(\Omega) \overset{J}{\longrightarrow} C^\infty(\Omega, \mathbb{S}) \overset{\text{div}}{\longrightarrow} C^\infty(\Omega, \mathbb{R}^2) \overset{\Pi_h}{\longrightarrow} \Sigma_h \overset{\text{div}_h}{\longrightarrow} V_h \overset{P_h}{\longrightarrow} 0,
\end{align*}
\]

where \( Q_h \) is some conforming or nonconforming finite element space for the biharmonic equation; \( J_h \) and \( \text{div}_h \) are the discrete counterparts of the Airy operator \( J \) and the divergence operator \( \text{div} \), respectively, with respect to some regular triangulation \( T_h \) of \( \Omega \); \( \Sigma_h \) and \( V_h \) are some finite element approximations of \( \Sigma \) and \( V \), respectively; \( I_h \) and \( \Pi_h \) are canonical interpolation operators for the spaces \( Q_h \) and \( \Sigma_h \), respectively; and \( P_h \) is the \( L^2 \) projection operator from \( V \) onto \( V_h \). In particular, this commuting diagram implies the Fortin lemma [16]. See Arnold, Awanou, and Winther [5] for the corresponding theory in three dimensions. Based on those fundamental theories, conforming mixed finite elements of piecewise polynomials on both simplicial and product meshes can then be developed for both two and three dimensions [1, 3, 5, 9]; see [17, 18] for the implementation of the lowest order method of [9]. To avoid complexity of conforming mixed elements, several remedies are proposed; see [8, 22, 25, 26] for new weak-symmetry finite elements and [10, 24, 30, 32, 38] for nonconforming finite elements. See also [11, 19] for the enrichment of nonconforming elements of [30, 32] to conforming elements. In a recent paper [29], a family of first order nonconforming mixed finite elements on product meshes is proposed for the first order system of equations in any dimension, which was extended to a family of conforming mixed elements in [28].

This paper presents a family of conforming mixed elements for both two and three dimensions \( (n = 2, 3) \), which can be regarded as a generalization to any order of the first order methods from [28]. It is motivated by an observation that the conformity of the discrete methods on product meshes can be guaranteed by the \( H(\text{div}) \)-conformity of the normal stress and the \( H^1 \)-conformity of two corresponding variables for each component of the shear stress; see also [12] and [28, 29] for a similar observation in
two dimensions. For two dimensions, in these elements, an enriched Brezzi–Douglas–Fortin–Marini (BDFM) element of order \( k \) is proposed to approximate the normal stress, and the serendipity element of order \( k \) \([4, 14, 21]\) is used to approximate the shear stress. This discrete space for the stress and an enriched discontinuous \( P_{k-1} \) element for the displacement space are able to form a stable discretization of the two-dimensional problem under consideration. In the first order method which is the two-dimensional element of \([28]\) of this family, the total degrees of freedom are \(|E| + 6|K| + |P|\), with \(|E|\) the number of edges, \(|K|\) the number of elements, and \(|P|\) the number of vertices of the partition \( \mathcal{T}_h \). Note that the total degrees of freedom of the first order conforming mixed element method on rectangular meshes in \([19]\) are \(3|E| + 6|K| + |P|\). For three dimensions, an enriched Raviart–Thomas element of order \( k \) is constructed to approximate the normal stress, and each component of the shear stress is approximated by a product space of the serendipity element of order \( k \) with respect to two associated variables and the \( P_{k-1} \) element with respect to the rest of the variables. An enriched \( Q_{k-1} \) element space is taken as the space for the displacement. In the first order method which is the three-dimensional element of \([28]\) of this family, the total degrees of freedom are \(|E| + |F| + 9|K|\), with \(|E|\) the number of edges, \(|F|\) the number of faces, and \(|K|\) the number of elements of the partition \( \mathcal{T}_h \). Note that the total degrees of freedom of the first order conforming mixed element method on cuboid meshes in \([11]\) are \(2|E| + 8|F| + 18|K|\). A family of reduced elements is also proposed by dropping interior bubble functions on each element. As a result the lowest order elements have 8 plus 2 and 18 plus 3 degrees of freedom on each element for two and three dimensions, respectively, which were announced independently in \([20]\) after the first version of this paper was submitted.

These spaces of this paper are perfectly and tightly matched on each element. However, the analysis of the discrete inf-sup conditions for these elements has to overcome the difficulty of not using directly the Fortin lemma, the key ingredient for the stability analysis of the mixed finite element method for the elasticity problem; see, for instance, \([1, 3, 5, 9]\). For the pure displacement boundary problem, the remedy is an explicitly constructive proof of the discrete inf-sup condition, which can be regarded as a generalization to the more general case of the idea due to \([29]\); see also \([12]\) and \([28]\). For the more general case, in particular the pure traction boundary problem, we prove that the divergence space of the \( H(\text{div}) \) bubble function space is identical to the orthogonal complement space of the rigid motion space with respect to the discrete displacement space on each macroelement. As we shall see in section 4, the proof for such a result is very difficult and complicated. One important technique is to use two classes of orthogonal polynomials, namely, the Jacobi polynomials and the Legendre polynomials. As a second step, we construct a quasi–interpolation operator to control macroelementwise rigid motion for \( k > 1 \). Then the discrete inf-sup condition follows. For the first order methods with \( k = 1 \), we succeed in proposing a new macroelement technique to finally establish the discrete inf-sup condition, which can be regarded as an extension to the more general case of that from \([34, 37]\).

This paper is organized as follows. In the following two sections, we present the new mixed elements for two dimensions and analyze their properties, including the well-posedness. In section 4, we consider the pure traction boundary problem and prove the wellposedness of the discrete problem. In section 5 we define the new mixed elements for three dimensions. In section 6, we present a family of reduced elements by dropping some interior bubble functions on each element. In section 7 we briefly summarize the error estimates of the discrete solutions and present two
numerical examples, one for the pure displacement boundary problem and the other for the pure traction boundary problem.

2. Mixed finite element approximation in two dimensions. For approximating problem (1.1) by the finite element method, we introduce a rectangular triangulation $\mathcal{T}_h$ of the rectangular domain $\Omega \subset \mathbb{R}^2$ such that $\bigcup_{K \in \mathcal{T}_h} K = \Omega$, two distinct elements $K$ and $K'$ in $\mathcal{T}_h$ are either disjoint, or share the common edge $e$, or a common vertex. Let $\mathcal{E}$ denote the set of all edges in $\mathcal{T}_h$ with $E_{K,V}$ the two vertical edges of $K \in \mathcal{T}_h$ and $E_{K,H}$ the two horizontal edges. Let $\mathcal{E}_V^I$ and $\mathcal{E}_H^I$ denote the sets of all the interior vertical and horizontal edges of $\mathcal{T}_h$, respectively, and $\mathcal{V}^I$ be the set of all the internal vertices of $\mathcal{T}_h$. Given vertex $A \in \mathcal{V}^I$, let $E(A)$ be the set of edges that take $A$ as one of their endpoints. Given any edge $e \in \mathcal{E}$ we assign one fixed unit normal $\nu$ with $(\nu_1, \nu_2)^T$ its components and let $t = (-\nu_2, \nu_1)^T$ denote the tangential vector.

For each $K \in \mathcal{T}_h$, we introduce the affine invertible transformation

$$F_K : \hat{K} \to K, \quad x = \frac{h_{x,K}}{2} \xi + x_{0,K}, \quad y = \frac{h_{y,K}}{2} \eta + y_{0,K}$$

with the center $(x_{0,K}, y_{0,K})$, the horizontal and vertical edge lengths $h_{x,K}$ and $h_{y,K}$, respectively, and the reference element $\hat{K} = [-1,1]^2$. Given any integer $k$, let $P_k(\omega)$ denote the space of polynomials over $\omega$ of total degrees not greater than $k$, and let $Q_k(\omega)$ denote the space of polynomials of degree not greater than $k$ in each variable. Let $P_k(X)$ be the space of polynomials of degree not greater than $k$ with respect to the variable $X$, and let $P_k(X,Y)$ be the space of polynomials of degree not greater than $k$ with respect to the variables $X$ and $Y$.

For the symmetric fields $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \in \mathbb{S}$, we refer to $\sigma_n := (\sigma_{11}, \sigma_{22})^T$ as the normal stress and $\sigma_{12}$ as the shear stress.

Before defining the space for the stress, we introduce new mixed finite elements for the second order Poisson equation and the serendipity element of [4, 14, 21]. Given $K \in \mathcal{T}_h$ and an integer $k \geq 1$, the new mixed finite element space of order $k$ for the second order Poisson equation reads

$$H_k(K) := (P_k(K))^2 \setminus \text{span}\{(0,x^k)^T,(y^k,0)^T\} \oplus E_k(K),$$

where

$$E_k(K) := \text{span}\{(x^{k+1},0)^T,(0,y^{k+1})^T,(x^2y^{k-1},0)^T,(0,y^2x^{k-1})^T\}.$$  

To define the degrees of freedom of the space $H_k(K)$, we introduce the well-known Jacobi polynomials

$$J_\ell(\xi) := ((\ell + 1)!)^2 \sum_{s=0}^{\ell} \frac{1}{s!(\ell + 1 - s)!(s + 1)!(\ell - s)!} \left( \frac{\xi - 1}{2} \right)^{\ell-s} \left( \frac{\xi + 1}{2} \right)^{s}$$

for any $\xi \in [-1,1]$. The Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} (1 - \xi^2) J_\ell(\xi) J_m(\xi) d\xi = \frac{8}{2^{\ell + 3} (\ell + 3)!} \delta_{\ell m} \text{ with } \delta_{\ell m} = 1 \text{ if } \ell = m \text{ and } \delta_{\ell m} = 0 \text{ if } \ell \neq m.$$ 

We also need the Legendre polynomials

$$L_\ell(\xi) := \frac{1}{2^\ell \ell!} \frac{d^\ell (\xi^2 - 1)^\ell}{d\xi^\ell} \text{ for any } \xi \in [-1,1].$$
Therefore, by condition 2,
\[ \L_1 L_m(\xi)d\xi = \frac{2}{2l+1}\delta_{lm}. \]

**Lemma 2.1.** The vector-valued function \((\hat{q}_1, \hat{q}_2)^T =: \hat{q} \in H_k(\hat{K})\) can be uniquely determined by the following conditions:
1. \(\int_\hat{K} \hat{q}\cdot \hat{v}d\hat{s}\) for any \(\hat{p} \in P_{k-1}(\hat{e})\) and any \(\hat{e} \subset \partial \hat{K}\),
2. \(\int_\hat{K} \hat{q}_1 J_{k-1}(\xi)d\xi d\eta\), and \(\int_\hat{K} \hat{q}_2 J_{k-1}(\eta)d\xi d\eta\),
3. \(\int_\hat{K} \hat{q}_1 L_{k-1}(\eta)d\xi d\eta\), and \(\int_\hat{K} \hat{q}_2 L_{k-1}(\xi)d\xi d\eta\),
4. \(\int_\hat{K} \hat{q}\cdot \hat{p}d\hat{s}\) for any \(\hat{p} \in (P_{k-2}(\hat{K}))^2\).

**Proof.** Since the dimension of the space \(H_k(\hat{K})\) is equal to the number of these conditions, it suffices to prove that \(\hat{q} \equiv 0\) if these conditions vanish. Since \(\hat{q}\cdot \hat{v} \in P_{k-1}(\hat{e})\), condition 1 implies that
\[ \hat{q}_1 = (1-\xi^2)(\hat{g}_1+c_1 J_{k-1}(\xi)+b_1 L_{k-1}(\eta)), \]
and \(\hat{q}_2 = (1-\eta^2)(\hat{g}_2+c_2 J_{k-1}(\eta)+b_2 L_{k-1}(\xi))\), where \(\hat{g}_1, \hat{g}_2 \in P_{k-2}(\hat{K})\), and \(c_1, c_2, b_1, b_2\) are four interpolation parameters, and \(J_{k-1}\) and \(L_{k-1}\) are the Jacobi and Legendre polynomials of degree \(k-1\), respectively.

We first consider the case \(k \geq 2\). It follows from (2.2) that
\[ \int_\hat{K} (1-\xi^2)(\hat{g}_1+b_1 L_{k-1}(\eta))J_{k-1}(\xi)d\xi d\eta = 0 \]
and
\[ \int_\hat{K} (1-\eta^2)(\hat{g}_2+b_2 L_{k-1}(\xi))J_{k-1}(\eta)d\xi d\eta = 0. \]

Therefore, by condition 2,
\[ c_1 = c_2 = 0. \]

Condition (2.3) implies
\[ \int_\hat{K} (1-\xi^2)\hat{g}_1 L_{k-1}(\eta)d\xi d\eta = 0 \]
and
\[ \int_\hat{K} (1-\eta^2)\hat{g}_2 L_{k-1}(\xi)d\xi d\eta = 0. \]

This and condition 3 yield
\[ b_1 = b_2 = 0. \]

Hence the final result follows from condition 4. For the case \(k = 1\), condition 2 is identical to condition 3. A similar argument above completes the proof. \(\square\)

**Remark 2.2.** The space \(H_k(K)\) is an enrichment of the BDFM element space from [15]. Hence we call this new mixed element as the enriched BDFM element.

The global space of the enriched BDFM element reads
\[ H_k(T_h) := \{ q \in H(div, \Omega, \mathbb{R}^2), q|_K \in H_k(K) \text{ for any } K \in T_h \}. \]
Note that for any \( q \in H^k(T_h) \), the first component of \( q \) is continuous across the interior vertical edges of \( T_h \), while the second component of \( q \) is continuous across the interior horizontal edges of \( T_h \).

To get a stable pair of spaces, we propose to use the serendipity element of order \( k \) from [4, 14, 21] to approximate the shear stress, which reads

\[
S_k(x, y) := P_k(x, y) + \text{span}\{x^k y, xy^k\}.
\]

Given any \( \tau_{12} \in S_k(x, y) \), it can be uniquely determined by the following conditions [4]:

1. the values of \( \tau_{12} \) at four vertices of \( K \),
2. the values of \( \tau_{12} \) at \( k-1 \) distinct points in the interior of each edge of \( K \),
3. the moments \( \int_K \tau_{12} p \, dx \, dy \) for any \( p \in P_{k-4}(K) \).

The global space of the serendipity element of order \( k \) is defined as

\[
S_k(T_h) := \{ \tau_{12} \in H^1(\Omega), \tau_{12}\rvert_K \in S_k(x, y) \text{ for any } K \in T_h \}.
\]

Note that the space \( S_1(T_h) \) is the usual \( H^1 \)-conforming bilinear element space.

The discrete space of the element is combined from the enriched BDFM element space and the serendipity element space:

\[
\Sigma_k(K) := \{ \tau \in H^1(K, S), \tau_n \in H_k(K), \tau_{12} \in S_k(x, y) \}.
\]

The degrees of freedom are inherited from the enriched BDFM element and the serendipity element:

1. the moments of degree not greater than \( k-1 \) on the four edges of \( K \) for \( \sigma_n \cdot \nu \);
2. the moments of degree not greater than \( k-2 \) on \( K \) for \( \sigma_n \);
3. the moments \( \int_K (\sigma_n) \cdot J_{k-1}(2(x-x_0,K)/h_{x,K}) \, dx \, dy \), and \( \int_K (\sigma_n) \cdot J_{k-1}(2(y-y_0,K)/h_{y,K}) \, dx \, dy \), where \( (\sigma_n) \cdot \) is the first component of \( \sigma_n \), and \( (\sigma_n) \cdot \) is the second component of \( \sigma_n \);
4. the values \( \int_K (\sigma_n) \cdot L_{k-1}(2(y-y_0,K)/h_{y,K}) \, dx \, dy \), and \( \int_K (\sigma_n) \cdot L_{k-1}(2(x-x_0,K)/h_{x,K}) \, dx \, dy \);
5. the values of \( \sigma_{12} \) at four vertices of \( K \);
6. the values of \( \sigma_{12} \) at \( k-1 \) distinct points in the interior of each edge of \( K \);
7. the moments of degree not greater than \( k-4 \) on \( K \) for \( \sigma_{12} \).

The definitions of the enriched BDFM element and the serendipity element imply that these conditions are unisolvent for the space \( \Sigma_k(K) \). The degrees of freedom for the lowest order element are illustrated in Figure 1.

\[\text{Fig. 1. Element diagram for the lowest order stress and displacement.}\]
The global space of order $k$ is defined as
\begin{equation}
\Sigma_k(T_h) := \{ \tau \in \Sigma, \tau|_K \in \Sigma_k(K) \text{ for any } K \in T_h \}.
\end{equation}

On each element $K$, the space for the displacement is taken as
\begin{equation}
V_k(K) := (P_{k-1}(K))^2 \oplus \text{span}\{(x^k,0)^T, (0,y^k)^T, (xy^{k-1},0)^T, (0,x^{k-1}y)^T\}.
\end{equation}

Then the global space for the displacement reads
\begin{equation}
V_k(T_h) := \{ v \in V, v|_K \in V_k(K) \text{ for any } K \in T_h \}.
\end{equation}

**Remark 2.3.** The lowest order element ($k=1$) of this family has 10 stress and 4 displacement degrees of freedom per element, which is the two-dimensional element of [28]; see the degrees of freedom in Figure 1.

It follows from the definitions of the spaces $\Sigma_k(T_h)$ and $V_k(T_h)$ that $\text{div } \Sigma_k(T_h) \subset V_k(T_h)$; in the following section, we shall prove the converse $V_k(T_h) \subset \text{div } \Sigma_k(T_h)$. This indicates the well-posedness of this family of elements.

The mixed element methods can be stated as follows: Find $(\sigma_{k,h}, u_{k,h}) \in \Sigma_k(T_h) \times V_k(T_h)$ such that
\begin{equation}
\begin{aligned}
(A\sigma_{k,h}, \tau)_{L^2(\Omega)} + (\text{div } \tau, u_{k,h})_{L^2(\Omega)} &= 0, \\
(\text{div } \sigma_{k,h}, v)_{L^2(\Omega)} &= (f, v)_{L^2(\Omega)}
\end{aligned}
\end{equation}
for any $(\tau, v) \in \Sigma_k(T_h) \times V_k(T_h)$.

**3. Well-posedness of discrete problem for pure displacement boundary problem in two dimensions.** In this section, we analyze the well-posedness of the discrete problem (2.6). From the mixed theory of [16], we need the following two assumptions:

1. $K$-ellipticity. There exists a constant $C > 0$ independent of the meshsize such that
\begin{equation}
(A\tau, \tau)_{L^2(\Omega)} \geq C\|\tau\|^2_{H(\text{div}, \Omega)}
\end{equation}
for any
\begin{equation}
\tau \in Z_k(T_h) := \{ \tau \in \Sigma_k(T_h), (\text{div } \tau, v)_{L^2(\Omega)} = 0 \text{ for all } v \in V_k(T_h) \}.
\end{equation}

2. Discrete (inf-sup) condition. There exists a positive constant $C$ independent of the meshsize with
\begin{equation}
\sup_{0 \neq \tau \in \Sigma_k(T_h)} \frac{(\text{div } \tau, v)_{L^2(\Omega)}}{\|\tau\|_{H(\text{div}, \Omega)}} \geq C\|v\|_{L^2(\Omega)} \text{ for any } v \in V_k(T_h).
\end{equation}

Herein and throughout, $C$ denotes a generic positive constant, which may be different at the different occurrence but independent of the meshsize $h$. It follows from $\text{div } \Sigma_k(K) \subset V_k(K)$ for any $K \in T_h$ that $\text{div } \tau = 0$ for any $\tau \in Z_k(T_h)$. This implies the $K$-ellipticity condition.

To prove the discrete inf-sup condition, the usual idea in the literature is to use the Fortin lemma [16]. More precisely, a bounded interpolation operator $\Pi_K : H^1(K, S) \to \Sigma_k(K)$ is constructed such that the following commuting diagram property holds:
\begin{equation}
\text{div } \Pi_K \sigma = P_K \text{ div } \sigma \text{ for any } \sigma \in H^1(K, S),
\end{equation}
where $P_K$ is the projection operator from $L^2(K, \mathbb{R}^2)$ onto $V_k(K)$. So far, most stable mixed finite element methods for the linear elasticity problem within the Hellinger–Reissner principle are designed with such a property; see, for instance, [1, 3, 5, 9]. However, such a technique cannot be used directly herein since there are not enough local degrees of freedom for this family of elements under consideration. The idea is to make a construction proof. More precisely, given $v \in V_k(\mathcal{T}_h)$, we find explicitly $\tau \in \Sigma_k(\mathcal{T}_h)$ such that

$$
\text{div} \tau = v \quad \text{and} \quad \|\tau\|_{H(\text{div}, \Omega)} \leq C\|v\|_{L^2(\Omega)}.
$$

Such an idea is motivated by the stability analysis of the Raviart–Thomas element for the Poisson equation in one dimension, which is first explored to analyze the stability of a family of first order nonconforming mixed finite element methods on the product mesh for the linear elasticity problem with the stress-displacement formulation in any dimension in a recent paper [29]. Therein, the discrete displacement is a piecewise constant vector, which implies that the $\tau$ of (3.2) can be directly given so that $\text{div} \tau = v$ for any $v$. In this paper we use the form from [28] to construct $\tau$.

For convenience, suppose that the domain $\Omega$ is a unit square $[0, 1]^2$ which is triangulated evenly into $N^2$ elements, $\{K_{ij}\}$. This implies that $h_{x,K} = h_{y,K} = h := 1/N$ for any $K \in \mathcal{T}_h$. For any $v \in V_k(\mathcal{T}_h)$, it can be decomposed as a sum,

$$
v := (v_1, v_2)^T = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{ij} \varphi_{ij}(x),
$$

where $\varphi_{ij}(x)$ is the characteristic function on the element $K_{ij}$ and $v_{ij} = (v^{1}_{ij}, v^{2}_{ij})^T = v|_{K_{ij}}$. Before the construction of $\tau \in \Sigma_k(\mathcal{T}_h)$ with properties of (3.2), we need a decomposition of $v$. We define the space $V_{y,ij} := \text{span}\{1, y, \ldots, y^{k-1}\}$, which introduces the following decomposition:

$$
P_{k-1}(K_{ij}) \oplus \text{span}\{x, xy^{k-1}\} = V_{y,ij} \oplus (x - ih)\left( P_{k-2}(K_{ij}) \oplus \text{span}\{x^{k-1}, y^{k-1}\}\right).$$

This implies that there exist unique $v_{x,ij}^{1} \in P_{k-2}(K_{ij}) \oplus \text{span}\{x^{k-1}, y^{k-1}\}$ and $v_{y,ij}^{1} \in V_{y,ij}$ such that

$$
v_{ij}^{1} = (x - ih)v_{x,ij}^{1} + v_{y,ij}^{1}.
$$

**Theorem 3.1.** It holds that

$$
\sup_{0 \neq \tau \in \Sigma_k(\mathcal{T}_h)} \frac{(\text{div} \tau, v)_{L^2(\Omega)}}{\|\tau\|_{H(\text{div}, \Omega)}} \geq \sqrt{\frac{2}{3}}\|v\|_{L^2(\Omega)} \quad \text{for any} \quad v \in V_k(\mathcal{T}_h).
$$

**Proof.** Given $v = (v_1, v_2)^T \in V_k(\mathcal{T}_h)$, we define $\tau_{11}$ as the integration of $v_1$ along the rectangles along the $x$ direction:

$$
\tau_{11}(x,y) = \int_{0}^{x} v_1(t,y) dt.
$$

On the element $K_{ij} := [(i - 1)h, ih] \times [(j - 1)h, jh], 1 \leq i,j \leq N$, by (3.3) and (3.4), it is straightforward to see that

$$
\tau_{11}|_K \in P_k(K) \setminus \text{span}\{y^k\} \oplus \text{span}\{x^{k+1}, x^2 y^{k-1}\}.$$

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for \((x, y) \in K_{ij}\). Similarly we can define \(\tau_{22}\) as

\[
\tau_{22}(x, y) = \int_0^y v_2(x, t)dt.
\]

Then by (3.4), \(\tau_{11}\) is continuous in the \(x\) direction, and by (3.5), \(\tau_{22}\) is continuous in the \(y\) direction. Hence we get an \(H(\text{div})\) field

\[
\tau = \begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix} \in \Sigma_k(T_h).
\]

By the definition of \(\tau\), it follows that

\[
(3.6) \quad \text{div } \tau = v.
\]

It remains to bound the \(L^2\) norm of \(\tau\). We first consider the \(L^2\) norm of the first component \(\tau_{11}\):

\[
\|\tau_{11}\|_{L^2(\Omega)}^2 = \sum_{i=1}^N \sum_{j=1}^N \int_{K_{ij}} \left( \int_0^x \int_0 v_1 dt \right)^2 dxdy
\]

\[
\leq \sum_{i=1}^N \sum_{j=1}^N \int_0^{(j-1)h} \int_0^{(i-1)h} \left( x \int_0^x v_1^2 dt \right) dxdy
\]

\[
\leq \sum_{i=1}^N \sum_{j=1}^N \int_0^{(j-1)h} \int_0^{(i-1)h} \left( \int_0^x x \int_0^x v_1^2 dt \right) dy
\]

\[
= \sum_{i=1}^N \sum_{j=1}^N \sum_{\ell=i}^{j-1} \frac{h^2(2i - 1)}{2} \|v_1\|_{L^2(K_{ij})}^2
\]

\[
\leq h^2 \sum_{j=1}^N \sum_{i=1}^N \sum_{\ell=i}^{j-1} \|v_1\|_{L^2(K_{ij})}^2 \leq \frac{1}{2} \|v_1\|_{L^2(\Omega)}^2.
\]

A similar argument proves that

\[
\|\tau_{22}\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|v_2\|_{L^2(\Omega)}^2.
\]

Hence

\[
\|\tau\|_{H(\text{div}, \Omega)}^2 = \|\text{div } \tau\|_{L^2(\Omega)}^2 + \|\tau\|_{L^2(\Omega)}^2 \leq \frac{3}{2} \|v\|_{L^2(\Omega)}^2.
\]

This completes the proof.  \(\square\)

4. The pure traction boundary problem. This section considers the pure traction boundary problem, i.e., the stress space is subject to the zero Neumann boundary condition but no boundary condition on the displacement. In practice, part of the elasticity body should be located, i.e., the displacement has a Dirichlet boundary condition on some nonzero measure boundary. But the pure traction boundary
problem is the most difficult one in mathematical analysis. A similar proof for Theorems 4.5 and 4.7 can prove them for partial displacement boundary problems.

Let RM be the rigid motion space in two dimensions, which reads

\[ \text{RM} := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ y \\ -x \end{pmatrix} \right\}. \]

Consider a pure traction boundary problem:

\[ \begin{align*}
\text{div } \sigma &= f \quad \text{in } \Omega := (0,1)^2, \\
\sigma \nu &= 0 \quad \text{on } \partial \Omega, \\
(u,v) &= 0 \quad \text{for any } v \in \text{RM},
\end{align*} \]

where \( \sigma := A^{-1} \epsilon(u) \) for \( u \in H^1(\Omega, \mathbb{R}^2) \). By the same discretization of the uniform square grid \( T_h \) with \( h = 1/N \) as in the previous section, the finite element equations remain the same except the spaces are changed with boundary and rigid motion free conditions:

\[ \begin{align*}
(A \sigma_h, \tau)_{L^2(\Omega)} + (\text{div } \tau, u_h)_{L^2(\Omega)} &= 0 \quad \text{for all } \tau \in \Sigma_{k,0}(T_h), \\
(\text{div } \sigma_h, v)_{L^2(\Omega)} &= (f,v)_{L^2(\Omega)} \quad \text{for all } v \in V_{k,0}(T_h),
\end{align*} \]

where

\[ \begin{align*}
\Sigma_{k,0}(T_h) &= \left\{ \tau = \begin{pmatrix} \tau_{11} \\ \tau_{12} \\ \tau_{22} \end{pmatrix} \in \Sigma_k(T_h), \tau \nu = 0 \quad \text{on } \partial \Omega \right\}, \\
V_{k,0}(T_h) &= \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_k(T_h), (v,w)_{L^2(\Omega)} = 0 \quad \text{for all } w \in \text{RM} \right\}.
\end{align*} \]

The earlier analysis remains the same except the discrete inf-sup condition as the stress space \( \Sigma_{k,0}(T_h) \) is smaller than \( \Sigma_k(T_h) \). To prove the discrete inf-sup condition for the pair \( (\Sigma_{k,0}(T_h), V_{k,0}(T_h)) \), we introduce the concept of a macroelement, i.e., a union of four rectangles; see Figure 2.

Given a macroelement \( M \), we define finite element spaces

\[ \Sigma_{k,0}(M) := \{ \tau \in H(\text{div}, \Omega_M, \mathbb{S}), \tau|_K \in \Sigma_k(K) \text{ for any } K \subset M, \tau \nu = 0 \text{ on } \partial \Omega_M \} \]

and

\[ V_k(M) := \{ v \in L^2(\Omega_M, \mathbb{R}^2), v|_K \in V_k(K) \text{ for any } K \subset M \}, \]

where \( \Omega_M := \bigcup_{K \subset M} \text{int}(K) \). Define the orthogonal complement space of the rigid motion space \( \text{RM} \) with respect to \( V_k(M) \) by

\[ \text{RM}^{-1}(M) := \{ v \in V_k(M), (v,w)_{L^2(\Omega_M)} = 0 \text{ for any } w \in \text{RM} \}. \]
4.1. Discrete inf-sup conditions for higher order elements with $k \geq 2$.

In this subsection, we shall prove that, for $k \geq 2$,

$$\text{div} \Sigma_{k,0}(M) = \text{RM}^{0}(M),$$

which helps to establish the discrete inf-sup conditions. To this end, we define the discrete kernel space of the divergence operator on the macroelement $M$ by

$$N_{M} := \{v \in V_{k}(M), (\text{div} \tau, v)_{L^{2}(\Omega_{M})} = 0 \text{ for any } \tau \in \Sigma_{k,0}(M)\}.$$  

We shall show that $N_{M} = \text{RM}$ for $k \geq 2$ to accomplish our goal. The difficulty is how to explore the local degrees of freedom for the shear stress. One important technique is to invoke the Jacobi polynomials defined in (2.1), which need the following polynomials:

$$J_{i}(\xi) = \int_{1}^{\xi} J_{i}(s) ds, i \geq 0, \xi \in [-1, 1].$$

For the four elements $K_{i}$, $i = 1, \ldots, 4$, in the macroelement $M$ (see Figure 2), we recall the following affine mapping:

$$\xi_{i} = \frac{2x - 2x_{0,K_{i}}}{h_{x,K_{i}}}, \quad \eta_{i} = \frac{2y - 2y_{0,K_{i}}}{h_{y,K_{i}}}, (x, y) \in K_{i},$$

where $(x_{0,K_{i}}, y_{0,K_{i}})$ is the center of $K_{i}$, and $h_{x,K_{i}}$ and $h_{y,K_{i}}$ are the horizontal and vertical edge lengths of $K_{i}$, respectively. We also need the following spaces:

$$\Sigma_{12,0} := \left\{ \tau_{12} \in H^{1}_{0}(\Omega_{M}), \tau_{12} = q(1 - \eta_{i}^{2}) \times \left\{ \begin{array}{ll} (1 + \xi_{1}) \text{ on } K_{1}, & q \in p_{k-2}(\eta_{1}), \tau_{12} = 0 \text{ on } K_{3}, K_{4} \end{array} \right. \right\},$$

$$\Sigma_{12,0} := \left\{ \tau_{12} \in H^{1}_{0}(\Omega_{M}), \tau_{12} = q(1 - \eta_{i}^{2}) \times \left\{ \begin{array}{ll} (1 + \eta_{2}) \text{ on } K_{2}, & q \in p_{k-2}(\eta_{2}), \tau_{12} = 0 \text{ on } K_{1}, K_{3}, K_{4} \end{array} \right. \right\},$$

$$\Sigma_{12,0} := \left\{ \tau_{12} \in H^{1}_{0}(\Omega_{M}), \tau_{12} = q(1 - \eta_{i}^{2}) \times \left\{ \begin{array}{ll} (1 - \xi_{3}) \text{ on } K_{3}, & q \in p_{k-2}(\eta_{3}), \tau_{12} = 0 \text{ on } K_{1}, K_{2}, K_{4} \end{array} \right. \right\},$$

$$\Sigma_{12,0} := \left\{ \tau_{12} \in H^{1}_{0}(\Omega_{M}), \tau_{12} = q(1 - \eta_{i}^{2}) \times \left\{ \begin{array}{ll} (1 - \eta_{4}) \text{ on } K_{4}, & q \in p_{k-2}(\eta_{4}), \tau_{12} = 0 \text{ on } K_{1}, K_{2}, K_{3} \end{array} \right. \right\}.$$  

The restriction space on $M$ of the space $S_{k}(T_{h})$ of the serendipity element reads

$$S_{k,0}(M) := \{\tau_{12} \in H^{1}_{0}(M), \tau_{12}|_{K_{i}} \in S_{k}(x, y), i = 1, \ldots, 4\}.$$  

Note that $\Sigma_{12,0} \subset S_{k,0}(M)$, $i = 1, 2, 3, 4$.

**Lemma 4.1.** For $k \geq 4$, suppose that $(v_{1}, v_{2})^{T} \in V_{k}(M)$ is of the form

$$v_{1}|_{K_{i}} = a_{-1,i} + \sum_{\ell=0}^{k-2} a_{\ell}(J_{\ell}(\eta_{i}) - (\delta_{11} + \delta_{12})J_{\ell}(1)) \quad \text{and}$$

$$v_{2}|_{K_{i}} = b_{-1,i} + \sum_{\ell=0}^{k-2} b_{\ell}(J_{\ell}(\xi_{i}) - (\delta_{11} + \delta_{14})J_{\ell}(1))$$

with $a_{-1,1} = a_{-1,2}, a_{-1,3} = a_{-1,4}, b_{-1,1} = b_{-1,4}, b_{-1,2} = b_{-1,3}$, and that

$$\int_{\Omega_{M}} \frac{\partial \tau_{12}}{\partial y} v_{1} + \frac{\partial \tau_{12}}{\partial x} v_{2} dx dy = 0 \text{ for any } \tau_{12} \in S_{k,0}(M);$$

then

$$a_{-1,1} = a_{-1,2} = a_{-1,3} = a_{-1,4}, b_{-1,1} = b_{-1,2} = b_{-1,3} = b_{-1,4},$$

$$\frac{2a_{0}}{h_{y,K_{i}}} = \frac{2b_{0}}{h_{x,K_{i}}}, a_{\ell} = b_{\ell} = 0, \ell = 1, \ldots, k-2.$$
Proof. An integration by parts yields
\[ 0 = \int_{\Omega_M} \frac{\partial \tau_{12}}{\partial y} \psi_1 + \frac{\partial \tau_{12}}{\partial x} \psi_2 \, dxdy = - \sum_{i=1}^{4} \int_{K_i} \tau_{12} \left( \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x} \right) \, dxdy \]
\[ + \int_{\tau_2 \cup \varepsilon_4} \tau_{12}(a_{-1,1} - a_{-1,4}) \, dx \]
\[ + \int_{\tau_1 \cup \varepsilon_3} \tau_{12}(b_{-1,1} - b_{-1,2}) \, dy. \]
(4.7)

We take \( \tau_{12} \) in (4.7) such that
\[ \tau_{12}|_{K_i} \in (1 - \xi_i^2)(1 - \eta_i^2) \text{ span}\{J_0(\eta_i), \ldots, J_{k-4}(\eta_i), J_1(\xi_i), \ldots, J_{k-4}(\xi_i)\}. \]

This leads to
\[ \frac{2a_0}{h_{y,K_i}} = - \frac{2b_0}{h_{x,K_i}}, a_\ell = b_\ell = 0, \ell = 1, \ldots, k - 4. \]

To show these four parameters \( a_{k-3}, a_{k-2}, b_{k-3}, \) and \( b_{k-2} \) to be zero, we turn to the case where \( k = 4 \). Since \( J_0(\xi_i) = J_0(\eta_i) = 1, J_1(\xi_i) = 2\xi_i, \) and \( J_1(\eta_i) = 2\eta_i, \) we take \( \tau_{12} \in \Sigma_{12,\varepsilon_4} \) with \( q = J_1(\xi_i) \) in (4.7). Since \( \int_{\epsilon_1}(1 - \eta_i^2)(1 + \xi_i)J_1(\eta_i) \, dy = 0, \) this yields \( a_1 = 0. \)

Similarly, the choice of \( \tau_{12} \in \Sigma_{12,\varepsilon_4} \) with \( q = J_1(\xi_i) \) shows \( b_1 = 0. \)

Then the choice of \( \tau_{12} \in \Sigma_{12,\varepsilon_1} \) with \( q = J_0(\eta_i) \) in (4.7) yields \( b_{-1,1} = b_{-1,2}, \) while the choice of \( \tau_{12} \in \Sigma_{12,\varepsilon_4} \) with \( q = J_0(\xi_i) \) in (4.7) leads to \( a_{-1,1} = a_{-1,4}. \)

Hence we choose
\[ \tau_{12} \in \Sigma_{12,\varepsilon_1} \text{ with } q = J_2(\eta_i), \]
\[ \tau_{12} \in \Sigma_{12,\varepsilon_4} \text{ with } q = J_2(\xi_i) \]
in (4.7), respectively, to show \( a_2 = b_2 = 0. \)

Next we consider the case where \( k = 4 \) which allows us to take \( \tau_{12} \in \Sigma_{12,\varepsilon_1} \) with \( q = 1 \) in (4.7).

A similar argument with \( \tau_{12} \in \Sigma_{12,\varepsilon_4} \) and \( q = 1 \) gets \( a_{-1,1} = a_{-1,4}. \)

Therefore, the choices of
\[ \tau_{12} \in \Sigma_{12,\varepsilon_1} \text{ with } q = J_{k-2}(\eta_i) \text{ and } q = J_{k-3}(\eta_i), \]
\[ \tau_{12} \in \Sigma_{12,\varepsilon_4} \text{ with } q = J_{k-2}(\xi_i) \text{ and } q = J_{k-3}(\xi_i) \]
in (4.7), respectively, prove
\[ a_{k-3} = b_{k-3} = a_{k-2} = b_{k-2} = 0. \]

This completes the proof. \( \square \)

**Lemma 4.2.** For \( k = 2, 3, \) suppose that \( (\nu_1, \nu_2)^T \in V_k(M) \) is of the form
\[ \begin{align*}
\nu_1|_{K_i} &= a_{-1,i} + \sum_{\ell=0}^{k-2} a_\ell (J_\ell(\eta_i) - (\delta_{\ell 1} + \delta_{\ell 2})J_\ell(1)) \quad \text{and} \\
\nu_2|_{K_i} &= b_{-1,i} + \sum_{\ell=0}^{k-2} b_\ell (J_\ell(\xi_i) - (\delta_{\ell 1} + \delta_{\ell 4})J_\ell(1)),
\end{align*} \]
(4.8)
with \( a_{-1,1} = a_{-1,2}, a_{-1,3} = a_{-1,4}, b_{-1,1} = b_{-1,4}, b_{-1,2} = b_{-1,3} \), and that
\[
\int_{\Omega_M} \frac{\partial \tau_{12}}{\partial y} v_1 + \frac{\partial \tau_{12}}{\partial x} v_2 \, dx \, dy = 0 \text{ for any } \tau_{12} \in S_{k,0}(M);
\]
then
\[
a_{-1,1} = a_{-1,2} = a_{-1,3} = a_{-1,4}, b_{-1,1} = b_{-1,2} = b_{-1,3} = b_{-1,4},
\]
\[
\frac{2a_0}{h_{y,K_i}} = -\frac{2b_0}{h_{x,K_i}}, a_\ell = b_\ell = 0, \ell = 1, \ldots, k - 2.
\]

Proof. We only present the details for the case where \( k = 3 \) since the proof for the case \( k = 2 \) is similar and simple. An integration by parts yields
\[
0 = \int_{\Omega_M} \frac{\partial \tau_{12}}{\partial y} v_1 + \frac{\partial \tau_{12}}{\partial x} v_2 \, dx \, dy - \sum_{i=1}^4 \int_{K_i} \tau_{12} \left( \frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) \, dx \, dy
\]
\[
+ \int_{\Xi_2 \cup \Xi_4} \tau_{12} (a_{-1,1} - a_{-1,4}) \, dx
\]
\[
+ \int_{\Xi_1 \cup \Xi_3} \tau_{12} (b_{-1,1} - b_{-1,2}) \, dy.
\]
For such a case, we have
\[
\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} |_{K_2} = \frac{2a_0}{h_{y,K_i}} + \frac{2b_0}{h_{x,K_i}} + \frac{2a_1 J_1(\eta_1)}{h_{y,K_i}} + \frac{2b_1 J_1(\xi_4)}{h_{x,K_i}}.
\]

Let \( \tau_{12} \in \Sigma_{12,\epsilon_1} \) with \( q = J_1(\eta_1) \) and \( \tau_{12} \in \Sigma_{12,\epsilon_4} \) with \( q = J_1(\xi_4) \) in (4.9), respectively.
This yields \( a_1 = 0 \) and \( b_1 = 0 \), respectively. The choices of \( \tau_{12} \in \Sigma_{12,\epsilon_1} \) with \( q = J_0(\eta_1) \) and \( \tau_{12} \in \Sigma_{12,\epsilon_4} \) with \( q = J_0(\xi_4) \) yield, respectively,
\[
\frac{2a_0}{h_{y,K_i}} + \frac{2b_0}{h_{x,K_i}} + \frac{b_{-1,2} - b_{-1,1}}{h_{x,K_i}} = 0 \quad \text{and} \quad \frac{2a_0}{h_{y,K_i}} + \frac{2b_0}{h_{x,K_i}} + \frac{a_{-1,4} - a_{-1,1}}{h_{y,K_i}} = 0.
\]
Now we let \( \tau_{12} |_{K_i} = (1 + \xi_4)(1 + \eta_1) \) (with appropriate definitions in \( K_2, K_3 \), and \( K_4 \)) in (4.9) to obtain
\[
\frac{2a_0}{h_{y,K_i}} + \frac{2b_0}{h_{x,K_i}} + \frac{b_{-1,2} - b_{-1,1}}{h_{x,K_i}} + \frac{a_{-1,4} - a_{-1,1}}{h_{y,K_i}} = 0.
\]
Finally we solve these three equations to show the desired result. \( \Box \)

LEMMA 4.3. It holds, for \( k \geq 2 \), that
\[
\text{div} \Sigma_{k,0}(M) = RM^1(M).
\]

Proof. Since it is straightforward to see that \( \text{div} \Sigma_{k,0}(M) \subseteq RM^1(M) \), we only need to prove that
\[
N_M = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}.
\]
Any \( v = (v_1, v_2)^T \in V_k(M) \) can be expressed as, for \( \ell = 1, \ldots, 4 \),
\[
v_1 |_{K_\ell} = \sum_{i+j \leq k-1} a_{i,j}^{(\ell)} x^i y^j + a_{k,0}^{(\ell)} x^k + a_{1,k-1}^{(\ell)} x y^{k-1}
\]
and
\[ v_2|_{K_\ell} = \sum_{i+j\leq k-1} b_{i,j}^{(\ell)} x^i y^j + b_{0,k}^{(\ell)} y^k + b_{k-1,i}^{(\ell)} y x^{k-1}. \]

We choose \( \tau \) such that \( \tau_{12} = \tau_{22} = 0 \) and
\[ \tau_{11}|_{K_\ell} \circ F_{K_\ell}^{-1} \in (1 - \xi^2) \times \hat{\Sigma}_{11, K_\ell} \]  
where \( \hat{\Sigma}_{11, K_\ell} := \text{span}\{J_{k-1}(\xi), L_{k-1}(\eta)\} \oplus P_{k-2}(K), \ell = 1, \ldots, 4. \)
The condition for \( N_M \) implies that
\[ 0 = \left( \frac{\partial \tau_{11}}{\partial x}, v_1 \right)_{L^2(\Omega_M)} = -\sum_{\ell=1}^4 \left( \tau_{11}|_{K_\ell} \frac{\partial (v_1|_{K_\ell})}{\partial x} \right)_{L^2(K_\ell)}. \]
Since \( \frac{\partial (v_1|_{K_\ell})}{\partial x} \circ F_{K_\ell}^{-1} \in \hat{\Sigma}_{11, K_\ell} \), this yields
\[ a_{i,j}^{(\ell)} = a_{i,1}^{(\ell)} = a_{1,j}^{(\ell)} = 0 \]  
for all \( i \geq 1 \) with \( i + j \leq k - 1 \).

Hence \( v_1 \) is of the form
\[ (4.11) \quad v_1|_{K_\ell} = \sum_{j\leq k-1} a_{0,j}^{(\ell)} x^j. \]

The continuity and degrees of \( \tau_{11} \) across the edges \( e_1 \) and \( e_3 \) show (using the moments of degree not greater than \( k - 1 \) on \( e_1 \) and \( e_3 \) for \( \tau_{11} \))
\[ (4.12) \quad a_{0,j}^{(1)} = a_{0,j}^{(2)} \quad \text{and} \quad a_{0,j}^{(3)} = a_{0,j}^{(4)}, \quad j = 0, \ldots, k - 1. \]

A similar argument for \( v_2 \) shows that \( v_2 \) is of the form
\[ (4.13) \quad v_2|_{K_\ell} = \sum_{i\leq k-1} b_{i,0}^{(\ell)} x^i \]
and
\[ (4.14) \quad b_{1,0}^{(1)} = b_{1,0}^{(4)} \quad \text{and} \quad b_{1,0}^{(2)} = b_{1,0}^{(3)}, \quad i = 0, \ldots, k - 1. \]

To decide these parameters \( a_{0,j}^{(\ell)} \) and \( b_{i,0}^{(\ell)} \), we propose to use the degrees of freedom for the shear stress component \( \tau_{12} \). We choose \( \tau \) such that \( \tau_{11} = \tau_{22} = 0 \) and \( \tau_{12} = \tau_{12}^{(\ell)} \in \Sigma_{12, e_\ell}, \ell = 1, \ldots, 4. \) The condition for \( N_M \), and (4.11)–(4.14), produce
\[ 0 = \int_{e_\ell} \tau_{12}^{(\ell)} [v((\ell+1 \mod 2))]|ds - \int_{K_\ell} \tau_{12}^{(\ell)} \left( \frac{\partial v_1|_{K_\ell}}{\partial y} + \frac{\partial v_2|_{K_\ell}}{\partial x} \right) dxdy \]
\[ -\int_{K_\ell} \tau_{12}^{(\ell)} \left( \frac{\partial v_1|_{K_{(\ell+1 \mod 4)}}}{\partial y} + \frac{\partial v_2|_{K_{(\ell+1 \mod 4)}}}{\partial x} \right) dxdy, \]
where \([\cdot]\) denotes the jump of piecewise functions across edge \( e_\ell \) with \( v_0 = v_2 \) and \( K_0 = K_4 \). Since \( v_1 \) (resp., \( v_2 \)) is a piecewise polynomial with respect to variable \( y \) (resp., \( x \)), the symmetries of \( \tau_{12}^{(\ell)} \) with the edges \( e_\ell, \ell = 1, \ldots, 4 \), lead to
\[ (4.16) \quad \int_{K_1} \tau_{12}^{(1)} \frac{\partial v_1|_{K_1}}{\partial y} dxdy = \int_{K_2} \tau_{12}^{(2)} \frac{\partial v_1|_{K_2}}{\partial y} dxdy, \quad \int_{K_3} \tau_{12}^{(3)} \frac{\partial v_1|_{K_3}}{\partial y} dxdy = \int_{K_4} \tau_{12}^{(4)} \frac{\partial v_1|_{K_4}}{\partial y} dxdy \]
and
\[
\int_{K_{2}} \frac{\partial v_2}{\partial x} K_2 \, dx \, dy = \int_{K_{3}} \frac{\partial v_2}{\partial x} K_3 \, dx \, dy, \quad \int_{K_{4}} \frac{\partial v_2}{\partial x} K_4 \, dx \, dy = \int_{K_{1}} \frac{\partial v_2}{\partial x} K_1 \, dx \, dy.
\]

Next let all \( \tau_{12}^{(\ell)} \in \Sigma_{12, e_{\ell}}, \ \ell = 1, \ldots, 4, \) be defined by, up to the variable \( x \) or \( y, \) and some transformation(s), the same polynomials \( q \) of one variable of degree \( \leq k - 2. \)

Since \([v_1]|_{e_2} = [v_1]|_{e_4} \) and \([v_2]|_{e_1} = [v_2]|_{e_3}, \) the symmetries of \( \tau_{12}^{(\ell)} \) imply additionally that
\[
\int_{e_1} \tau_{12}^{(1)} [v_2] \, ds = \int_{e_3} \tau_{12}^{(3)} [v_2] \, ds \quad \text{and} \quad \int_{e_2} \tau_{12}^{(2)} [v_1] \, ds = \int_{e_4} \tau_{12}^{(4)} [v_1] \, ds.
\]

A substitution of (4.16) through (4.18) into (4.15) shows that
\[
\int_{K_{1}} \frac{\partial v_1}{\partial y} K_1 \, dx \, dy = \int_{K_{2}} \frac{\partial v_2}{\partial x} K_1 \, dx \, dy = \int_{K_{3}} \frac{\partial v_2}{\partial x} K_3 \, dx \, dy = \int_{K_{4}} \frac{\partial v_1}{\partial y} K_4 \, dx \, dy
\]
and
\[
\int_{K_{2}} \frac{\partial v_2}{\partial x} K_2 \, dx \, dy = \int_{K_{3}} \frac{\partial v_2}{\partial x} K_3 \, dx \, dy = \int_{K_{4}} \frac{\partial v_2}{\partial x} K_4 \, dx \, dy = \int_{K_{1}} \frac{\partial v_1}{\partial y} K_1 \, dx \, dy.
\]

Since both \( \frac{\partial v_1}{\partial y} K_1 \) and \( \frac{\partial v_2}{\partial x} K_4 \) are polynomials of degree \( \leq k - 2 \) with respect to \( y, \) the conditions of (4.19) for all \( \tau_{12}^{(1)} \in \Sigma_{12, e_{1}} \) and \( \tau_{12}^{(3)} \in \Sigma_{12, e_{3}}, \) the conditions of (4.20) for all \( \tau_{12}^{(2)} \in \Sigma_{12, e_{2}} \) and \( \tau_{12}^{(4)} \in \Sigma_{12, e_{4}}, \) show that \( (v_1, v_2)^T \in V_k(M) \) is of the form
\[
v_1|_{K_i} = a_{-1, i} + \sum_{\ell=0}^{k-2} a_{\ell} (\tilde{J}_\ell(\eta) - (\delta_{i1} + \delta_{i2})\tilde{J}_1(1)) \quad \text{and} \quad v_2|_{K_i} = b_{-1, i} + \sum_{\ell=0}^{k-2} b_{\ell} (\tilde{J}_\ell(\xi) - (\delta_{i1} + \delta_{i4})\tilde{J}_1(1)), \quad i = 1, \ldots, 4,
\]
with \( a_{-1, 1} = a_{-1, 2}, a_{-1, 3} = a_{-1, 4}, b_{-1, 1} = b_{-1, 4}, b_{-1, 2} = b_{-1, 3}. \) Hence it follows from Lemmas 4.1 and 4.2 that
\[
\frac{2a_0}{h_{y, k_i}} = -\frac{2b_0}{h_{x, k_i}}, \quad a_{\ell} = b_{\ell} = 0, \ \ell = 1, \ldots, k - 2.
\]

This completes the proof. \( \square \)

**Lemma 4.4.** For any \( v_h \in V_{k, 0}(T_h), \) there exists a \( \tau_h \in \Sigma_{k, 0}(T_h) \) such that
\[
\int_{M} (\nabla \tau_h - v_h) \cdot wdx = 0 \quad \text{for any} \ w \in RM \quad \text{and any macroelement} \ M
\]
and
\[
\|\tau_h\|_{H^{1}(\Omega)} \leq C\|v_h\|_{L^2(\Omega)}.
\]

**Proof.** It is standard that there exists a \( \tau := \left(\begin{array}{c} \tau_{11} \\ \tau_{12} \\ \tau_{22} \end{array}\right) \in H^1_0(\Omega, \mathbb{S}) \) such that
\[
\div \tau = v_h \quad \text{and} \quad \|\tau\|_{H^1(\Omega)} \leq C\|v_h\|_{L^2(\Omega)}.
\]
It follows from the degrees of freedom for the enriched BDFM element in Lemma 2.1 and for the serendipity element that there exist $(\tau_{11,h}, \tau_{22,h})^T \in H_k(T_h)$ and $\tau_{12,h} \in S_k(T_h)$ such that, for edges of macroelement $M$ (see Figure 2 for notation),

\[
\begin{align*}
\int_{e_\tau \cup e_\varsigma} (\tau_{11} - \tau_{11,h})pdy &= \int_{e_{11} \cup e_{12}} (\tau_{11} - \tau_{11,h})qdy = 0 \text{ for any } p \in P_1(e_7 \cup e_8), q \in P_1(e_{11} \cup e_{12}), \\
\int_{e_5 \cup e_6} (\tau_{22} - \tau_{22,h})pdx &= \int_{e_9 \cup e_{10}} (\tau_{22} - \tau_{22,h})qdx = 0 \text{ for any } p \in P_1(e_5 \cup e_6), q \in P_1(e_9 \cup e_{10}), \\
\int_{e_{11} \cup e_{12}} (\tau_{12} - \tau_{12,h})dy &= \int_{e_5 \cup e_6} (\tau_{12} - \tau_{12,h})dx = \int_{e_9 \cup e_{10}} (\tau_{12} - \tau_{12,h})dx = 0.
\end{align*}
\]

Let $\tau_h = \begin{pmatrix} \tau_{11,h} \\ \tau_{12,h} \\ \tau_{22,h} \end{pmatrix}$. We additionally have

$$\|\tau_h\|_{H(\text{div}, \Omega)} \leq C\|\tau\|_{H^1(\Omega)}.$$ 

This completes the proof. \hfill \Box

We are now ready to establish the following inf-sup condition.

**Theorem 4.5.** For $k \geq 2$, there exists a positive constant $C$ independent of the meshsize with

$$\sup_{0 \neq \tau \in \Sigma_{k,0}(T_h)} \frac{(\text{div } \tau, v)_{L^2(\Omega)}}{\|\tau\|_{H(\text{div}, \Omega)}} \geq C\|v\|_{L^2(\Omega)} \quad \text{for any } v \in V_{k,0}(T_h).$$

**Proof.** Given $v \in V_{k,0}(T_h)$, it follows from Lemma 4.4 that there exists a $\tau_1 \in \Sigma_{k,0}(T_h)$ such that

\[(4.25) \quad \int_M (\text{div } \tau_1 - v) \cdot wdx = 0 \text{ for any } w \in \text{RM and any macroelement } M \]

and

\[(4.26) \quad \|\tau_1\|_{H(\text{div}, \Omega)} \leq C\|v\|_{L^2(\Omega)}.\]

By Lemma 4.3, there exists a $\tau_2 \in \Sigma_{k,0}(T_h)$ such that

\[(4.27) \quad \text{div } \tau_2 = \text{div } \tau_1 - v \text{ and } \|\tau_2\|_{H(\text{div}, \Omega)} \leq C\|\text{div } \tau_1 - v\|_{L^2(\Omega)}.\]

Then we have $\text{div } (\tau_1 + \tau_2) = v$ and $\|\tau\|_{H(\text{div}, \Omega)} \leq C\|v\|_{L^2(\Omega)}. \hfill \Box$

**4.2. Discrete inf-sup condition for the first order element with $k = 1$.**

Since the analysis in the previous subsection cannot be applied to the current case, it needs a separate analysis. The ingredient is a modified macroelement technique. We also note that the macroelement technique from [34] cannot be used directly here since the seminorm $| \cdot |_{1,h,M}$ there is not equivalent to the seminorm $| \cdot |_{M}$ there for
the present case; see [34, Theorem 4.1]. To overcome this difficulty, for \( v \in V_1(M) \), we propose the following mesh dependent seminorm (see Figure 2 for notation):

\[
|v|^2_{1,h,M} = \sum_{i=1}^{4} \|\epsilon(v)\|^2_{0,K_i} + h_{e_1}^{-1}\||v_1||^2_{L^2(e_1)} + h_{e_2}^{-1}\||v_1||^2_{L^2(e_2)} + h_{e_3}^{-1}\||v_2||^2_{L^2(e_3)} + h_{e_4}^{-1}\||v_2||^2_{L^2(e_4)}
\]

(4.28)

\[
+ ((v_1|_{K_1} - v_1|_{K_2})(\mathcal{M}(e_4)) + (v_1|_{K_2} - v_1|_{K_3})(\mathcal{M}(e_2)) + (v_2|_{K_1} - v_2|_{K_2})(\mathcal{M}(e_1)) + (v_2|_{K_4} - v_2|_{K_3})(\mathcal{M}(e_3)))^2,
\]

where \( \mathcal{M}(e_i), i = 1, \ldots, 4 \), denote the midpoints of edges \( e_i \), and \( |\cdot| \) denote the jump of piecewise functions over edge.

Define a global seminorm

\[
|v|^2_{1,h} = \sum_M |v|^2_{1,h,M}
\]

for all macroelements consisting of four elements like that in Figure 2.

It is straightforward to see that \(|\cdot|_{1,h}\) defines a norm over \( V_{1,0}(T_h) \). For \( \tau \in \Sigma_{1,0}(T_h) \), we define the following mesh dependent norm:

\[
\|\tau\|^2_{0,h} = \|\tau\|^2_{L^2(\Omega)} + \sum_{\epsilon \in \mathcal{E}_h^I} h_{\epsilon}\|\gamma_{\epsilon}\|^2_{L^2(e)} + \sum_{\epsilon \in \mathcal{E}_h} h_{\epsilon}\|\gamma_{\epsilon}\|^2_{L^2(e)} + \sum_{A \in \mathcal{V}^I} \sum_{\epsilon \in \mathcal{E}(A)} h_{\epsilon}\|\gamma_{\epsilon}\|^2_{L^2(e)}.
\]

**Lemma 4.6.** For any macroelement \( M \) illustrated in Figure 2, it holds that

\[
N_M = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \epsilon_{y,-x} \right\},
\]

where

\[
\epsilon_{y,-x} := \begin{cases} 
(-1) & \text{on } K_1, \\
(1) & \text{on } K_2, \\
(-1) & \text{on } K_3, \\
(1) & \text{on } K_4.
\end{cases}
\]

**Proof.** Any \( v = (v_1, v_2)^T \in V_1(M) \) can be expressed as, for \( \ell = 1, \ldots, 4 \),

\[
v_1|_{K_\ell} = a_0^{(\ell)} + a_1^{(\ell)}x
\]

and

\[
v_2|_{K_\ell} = b_0^{(\ell)} + b_1^{(\ell)}y.
\]

We choose \( \tau \) such that \( \tau_{12} = \tau_{22} = 0 \) and

\[
\tau_{11}|_{K_\ell} \circ F_{K_\ell}^{-1} = 1 - \xi^2, \ell = 1, \ldots, 4.
\]
The condition for $N_M$ implies that

$$a^{(ell)}_1 = 0, \ell = 1, \ldots, 4.$$  

Similarly,

$$b^{(ell)}_1 = 0, \ell = 1, \ldots, 4.$$  

Hence we can use the degrees on the edges $e_1$ and $e_3$ for $\tau_{11}$ (using the moments of degree zero on $e_1$ and $e_3$ for $\tau_{11}$) to show that

$$a^{(1)}_0 = a^{(2)}_0 \text{ and } a^{(3)}_0 = a^{(4)}_0.$$  

A similar argument proves

$$b^{(1)}_0 = b^{(4)}_0 \text{ and } b^{(2)}_0 = b^{(3)}_0.$$  

At the end we use the degree of $\tau_{12}$ at the interior vertex of $M$ to complete the proof.

We need another seminorm for the space $V_1(M)$:

$$|v|_M = \sup_{0 \neq \tau \in \Sigma_{1,0}(M)} \frac{(\text{div } \tau, v)_{L_2(\Omega_M)}}{||\tau||_{0,h,M}}.$$  

It follows from Lemma 4.6 that the seminorm $| \cdot |_{1,h,M}$ is equivalent to the seminorm $| \cdot |_M$. This allows for following a similar argument of [34] and the references therein to prove the discrete inf-sup condition.

**Theorem 4.7.** There exists a positive constant $C$ independent of the meshsize with

$$\sup_{0 \neq \tau \in \Sigma_{k,x}(\Omega)} \frac{(\text{div } \tau, v)_{L_2(\Omega)}}{||\tau||_{0,h}} \geq C|v|_{1,h} \text{ for any } v \in V_{k,0}(\mathcal{T}_h).$$

5. Mixed finite element for three dimensions. We define a family of conforming mixed finite element methods in three dimensions in this section. To this end, let $\mathcal{T}_h$ be a cuboid triangulation of the cuboid domain $\Omega \subset \mathbb{R}^3$ such that $\bigcup_{K \in \mathcal{T}_h} K = \Omega$. On element $K \in \mathcal{T}_h$, for $k \geq 1$, we define an enriched Raviart–Thomas element space by

$$H_k(K) = (P_{k,k-1,k-1}(K) \oplus E_{k,x}) \times (P_{k-1,k,k-1}(K) \oplus E_{k,y}) \times (P_{k-1,k-1,k}(K) \oplus E_{k,z}),$$

where

$$P_{k,k-1,k-1}(K) = P_k(x) \times P_{k-1}(y) \times P_{k-1}(z),$$

$$P_{k-1,k,k-1}(K) = P_{k-1}(x) \times P_k(y) \times P_{k-1}(z),$$

$$P_{k-1,k-1,k}(K) = P_{k-1}(x) \times P_{k-1}(y) \times P_k(z)$$

and

$$E_{k,x} = x^{k+1}(P_{k-1}(y) + P_{k-1}(z)),$$

$$E_{k,y} = y^{k+1}(P_{k-1}(z) + P_{k-1}(x)),$$

$$E_{k,z} = z^{k+1}(P_{k-1}(x) + P_{k-1}(y)).$$
To construct the degrees of freedom of the space $H_k(K)$, we define

$$
\Psi_{k-1}(K) := P_{k-2,k-1,k-1}(K) \times P_{k-1,k-2,k-1}(K) \times P_{k-1,k-1,k-2}(K),
$$

$$
\mathcal{J}_{k-1}(\xi) := J_{k-1}(\xi)(P_{k-1}(\eta) + P_{k-1}(\zeta)),
$$

$$
\mathcal{J}_{k-1}(\eta) := J_{k-1}(\eta)(P_{k-1}(\xi) + P_{k-1}(\zeta)),
$$

$$
\mathcal{J}_{k-1}(\zeta) := J_{k-1}(\zeta)(P_{k-1}(\xi) + P_{k-1}(\eta))
$$

for any $(\xi, \eta, \zeta) \in K := [-1, 1]^3$. We recall that $J_{k-1}(\xi)$ is the Jacobi polynomial of degree $k-1$ with respect to $\xi$, and $L_{k-1}(\xi)$ is the Legendre polynomial of degree $k-1$ with respect to $\xi$.

**Lemma 5.1.** The vector-valued function $(\hat{q}_1, \hat{q}_2, \hat{q}_3)^T =: \hat{q} \in H_k(K)$ can be uniquely determined by the following conditions:

1. $\int_K \hat{q} \cdot \nu d\hat{s}$ for any $\hat{p} \in Q_{k-1}(\hat{e})$ and any $\hat{e} \subset \partial K$,
2. $\int_K \hat{q}_1 \hat{p} d\hat{\xi} d\hat{\eta} d\hat{\zeta}$ for any $\hat{p} \in \mathcal{J}_{k-1}(\xi)$,
3. $\int_K \hat{q}_2 \hat{p} d\hat{\xi} d\hat{\eta} d\hat{\zeta}$ for any $\hat{p} \in \mathcal{J}_{k-1}(\eta)$,
4. $\int_K \hat{q}_3 \hat{p} d\hat{\xi} d\hat{\eta} d\hat{\zeta}$ for any $\hat{p} \in \mathcal{J}_{k-1}(\zeta)$,
5. $\int_K \hat{q} \cdot \hat{p} d\hat{\xi} d\hat{\eta} d\hat{\zeta}$ for any $\hat{p} \in \Psi_{k-1}(K)$.

**Proof.** Since the dimensions of the space $H_k(K)$ are equal to the number of these conditions, it suffices to prove that $\hat{q} \equiv 0$ if these conditions vanish. Since $\hat{q} \cdot \nu \in Q_{k-1}(\hat{e})$, condition 1 implies that

$$
\hat{q}_1 = (1 - \xi^2)(\hat{g}_1 + \hat{f}_1),
$$

$$
\hat{q}_2 = (1 - \eta^2)(\hat{g}_2 + \hat{f}_2),
$$

$$
\hat{q}_2 = (1 - \zeta^2)(\hat{g}_3 + \hat{f}_3),
$$

where $(\hat{g}_1, \hat{g}_2, \hat{g}_3)^T \in \Psi_{k-1}(K)$, $\hat{f}_1 \in \mathcal{J}_{k-1}(\xi)$, $\hat{f}_2 \in \mathcal{J}_{k-1}(\eta)$, and $\hat{f}_3 \in \mathcal{J}_{k-1}(\zeta)$. Note that

$$
\int_K (1 - \xi^2)(\hat{g}_1 \hat{p} \hat{d}\xi \hat{d}\eta \hat{d}\zeta) = 0
$$

for any $\hat{p} \in \mathcal{J}_{k-1}(\xi)$.

By condition 2, this shows $\hat{f}_1 = 0$. A similar argument (using conditions 3–4) yields

$$
\hat{f}_2 = \hat{f}_3 = 0.
$$

Finally, condition 5 proves $\hat{g}_1 = \hat{g}_2 = \hat{g}_3 = 0$, which completes the proof.

To design finite element spaces for the components $\sigma_{12}$, $\sigma_{13}$, and $\sigma_{23}$ of the shear stress, we need the following space:

$$
S_k(X,Y) \times P_{k-1}(Z) \text{ for any } (X,Y,Z) \in K := [-1, 1]^3.
$$

**Lemma 5.2.** Given any $\tau_{12} \in S_k(X,Y) \times P_{k-1}(Z)$, it can be uniquely determined by the following conditions:

1. the values of $\tau_{12}$ at $k$ distinct points on each edge of $K$ that is perpendicular to the $(X,Y)$-plane,
2. the values of $\tau_{12}$ at $k(k-1)$ distinct points in the interior of each face of $K$ that parallel the $Z$-axis,
3. the moments $\int_K \tau_{12} p_{k-4} dX dY dZ$ for any $p_{k-4} \in P_{k-4}(X,Y) \times P_{k-1}(Z)$. 

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Here the points in the second term are chosen in this way so that they lie in the same plane as the points in the first term.

Proof. Since $S_k(X,Y)$ is the space of the serendipity element of order $k$ with respect to the variables $X$ and $Y$, on each rectangle that parallels the $(X,Y)$-plane, $\tau_{12}$ can be uniquely determined by

the values of $\tau_{12}$ at four vertices of the rectangle,

the values of $\tau_{12}$ at $k - 1$ distinct points in the interior of each edge of the rectangle,

the moments of order $k - 4$ of $\tau_{12}$ on the rectangle.

Then the desired result follows from the fact that $S_k(X,Y) \times P_{k-1}(Z)$ is a product space.

Then, on element $K$, the space for the stress can be defined as

$$
\Sigma_k(K) := \{\sigma \in H(\text{div}, K, \mathbb{S}) | \sigma_n \in H_k(K), \sigma_{12} \in S_k(x,y) \times P_{k-1}(z), \sigma_{13} \in S_k(x,z) \times P_{k-1}(y), \sigma_{23} \in S_k(y,z) \times P_{k-1}(x)\}.
$$

The global space is defined as

$$
\Sigma_k(T_h) := \{\tau \in \Sigma, \tau|_K \in \Sigma_k(K) \text{ for any } K \in T_h\}.
$$

On each element $K$, the space for the displacement is taken as

$$
V_k(K) := (Q_{k-1}(K))^3 \oplus \{(Q_{k,x}, 0, 0)^T\} \oplus \{(0, Q_{k,y}, 0)^T\} \oplus \{(0, 0, Q_{k,z})^T\},
$$

where

$$
Q_{k,x} = x^k(P_{k-1}(y) + P_{k-1}(z)),
$$

$$
Q_{k,y} = y^k(P_{k-1}(z) + P_{k-1}(x)),
$$

$$
Q_{k,z} = z^k(P_{k-1}(x) + P_{k-1}(y)).
$$

Then the global space for the displacement reads

$$
V_k(T_h) := \{v \in V, v|_K \in V_k(K) \text{ for any } K \in T_h\}.
$$

Remark 5.3. The lowest order element ($k = 1$) of this family has 21 stress and 6 displacement degrees of freedom per element, which is the three-dimensional element of [28]; see some degrees of freedom in Figure 3.

To discretize the pure traction boundary problem, we introduce the rigid motion space

$$
\text{RM} := \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} \right\},
$$

$$
\sigma_{11} : 1, x, x^2 \quad \sigma_{12} : 1, x, y, xy \quad \sigma_{23} : 1, y, z, yz \quad u_1 : 1, x
$$

\text{FIG. 3. Some nodal degrees of freedom.}
which defines
\[
\begin{align*}
\Sigma_{k,0}(T_h) &= \{ \tau \in \Sigma_k(T_h) \mid \tau \nu = 0 \quad \text{on } \partial \Omega \}, \\
V_{k,0}(T_h) &= \{ v \in V_k(T_h) \mid (v, w)_{L^2(\Omega)} = 0 \quad \text{for all } w \in \mathbb{R}M \}.
\end{align*}
\] (5.2)

It follows from the definitions of the spaces \(\Sigma_k(T_h)\) (resp., \(\Sigma_{k,0}(T_h)\)) and \(V_k(T_h)\) (resp., \(V_{k,0}(T_h)\)) that \(\text{div} \Sigma_k(T_h) \subset V_k(T_h)\) (resp., \(\text{div} \Sigma_{k,0}(T_h) \subset V_{k,0}(T_h)\)). Similar arguments of Theorems 3.1, 4.5, and 4.7 can prove the converses \(V_k(T_h) \subset \text{div} \Sigma_k(T_h)\) and \(V_{k,0}(T_h) \subset \text{div} \Sigma_{k,0}(T_h)\), respectively. In fact, to extend the result of Theorem 3.1 to the present case, we only need essentially three one-dimensional arguments used in Theorem 3.1, while to get a generalization of Theorems 4.5 and 4.7, we only need essentially three two-dimensional arguments used in Theorems 4.5 and 4.7. In particular, in this way, we can get three two-dimensional rigid motion spaces, which proves, on a macroelement consisting of eight elements, the kernel space \(N_M\) (see (4.4) for the definition in two dimensions) is the rigid motion space in three dimension. This in turn implies a similar result of Lemma 4.3. Finally, this indicates the well-posedness of this family of elements.

6. Reduced elements in both two and three dimensions. In this section we present a family of reduced elements for these in sections 2 and 5. To this end, we introduce Airy’s stress function for a scalar field \(q(X,Y)\) as follows:
\[
J_{X,Y}(q(X,Y)) := \left( \begin{array}{c}
\frac{\partial^2 q}{\partial Y^2} & -\frac{\partial^2 q}{\partial X \partial Y} \\
-\frac{\partial^2 q}{\partial X \partial Y} & \frac{\partial^2 q}{\partial X^2}
\end{array} \right)
\] Throughout this section we let \((X,Y,Z)\) denote permutations of \((x,y,z)\).

6.1. The reduced elements in two dimensions. We define the shape function space for the BDFM element \([15]\) as
\[
BDFM_k(K) := (P_k(K))^2 \backslash \text{span}\{(0,x^k)^T,(y^k,0)^T\}.
\]

The stress space of the reduced element of order \(k\) is defined as
\[
\Sigma^R_k(K) := \{ \tau \in \mathbb{S}, \tau_n \in BDFM_k(K), \tau_{12} \in P_k(x,y) \} \oplus E_k(K),
\]
where
\[
E_k(K) := \text{span}\{J_{x,y}(x^{k+1}y^2), J_{x,y}(x^2y^{k+1})\}.
\]
The degrees of freedom for the stress are inherited from the BDFM element and the serendipity element:
1. the moments of degree not greater than \(k-1\) on the four edges of \(K\) for \(\sigma_n \cdot \nu\),
2. the moments of degree not greater than \(k-2\) on \(K\) for \(\sigma_n\),
3. the values of \(\sigma_{12}\) at four vertices of \(K\),
4. the values of \(\sigma_{12}\) at \(k-1\) distinct points in the interior of each edge of \(K\),
5. the moments of degree not greater than \(k-4\) on \(K\) for \(\sigma_{12}\).
The global space for the stress of order $k$ is defined as

\begin{equation}
\Sigma^R_k(\mathcal{T}_h) := \{ \tau \in \Sigma, \tau|_K \in \Sigma^R_k(K) \text{ for any } K \in \mathcal{T}_h \}.
\end{equation}

On each element $K$, the space for the displacement is taken as

\[ V^R_k(K) := (P_{k-1}(K))^2. \]

Then the global space for the displacement reads

\begin{equation}
V^R_k(\mathcal{T}_h) := \{ v \in V, v|_K \in V^R_k(K) \text{ for any } K \in \mathcal{T}_h \}.
\end{equation}

**Remark 6.1.** Let $RT_k(K) = RT_k(\mathcal{T}_h) = P_{k-1}(K) \times P_{k-1}(K)$. One can also define the space for the stress as

\[ \hat{\Sigma}_k(K) := \{ \tau \in \mathcal{S}, \tau_n \in RT_k(K), \tau_{12} \in P_k(x, y) \} \oplus E_k(K). \]

The space for the displacement in this case is

\[ \hat{V}_k(K) := (Q_{k-1}(K))^2. \]

**6.2. The reduced elements in three dimensions.** On element $K \in \mathcal{T}_h$, for $k \geq 1$, we define the Raviart–Thomas element space by

\[ RT_k(K) = P_{k,k-1,k-1}(K) \times P_{k-1,k,k-1}(K) \times P_{k-1,k-1,k}(K), \]

where

\begin{align*}
P_{k,k-1,k-1}(K) &= P_k(x) \times P_{k-1}(y) \times P_{k-1}(z), \\
P_{k-1,k,k-1}(K) &= P_{k-1}(x) \times P_k(y) \times P_{k-1}(z), \\
P_{k-1,k-1,k}(K) &= P_{k-1}(x) \times P_{k-1}(y) \times P_k(z).
\end{align*}

Given a scalar field $q(X,Y)$ and the corresponding Airy’s function $J_{X,Y}(q(X,Y))$, we define $\tau(J_{X,Y}(q(X,Y))) \in H(\text{div}, K, \mathcal{S})$ such that

\[ \tau_{x,x} = (J_{X,Y}(q(X,Y)))_{x,x}, \tau_{y,y} = (J_{X,Y}(q(X,Y)))_{y,y}, \tau_{x,y} = (J_{X,Y}(q(X,Y)))_{x,y}, \tau_{y,x} = (J_{X,Y}(q(X,Y)))_{y,x} \]

and the rest of the entries are zero. This notation allows us to define

\[ E_k(K) := \text{span} \left\{ \tau(J_{x,y}(x^{k+1}y^2)), \tau(J_{x,y}(x^{k+1}y^{k+1})) P_{k-1}(z) \right\} \oplus \text{span} \left\{ \tau(J_{x,z}(x^{k+1}z^2)), \tau(J_{x,z}(x^{k+1}z^{k+1})) P_{k-1}(y) \right\} \oplus \text{span} \left\{ \tau(J_{y,z}(y^{k+1}z^2)), \tau(J_{y,z}(y^{k+1}z^{k+1})) P_{k-1}(x) \right\}. \]

Then, on element $K$, the space for the stress can be defined as

\begin{equation}
\Sigma^R_k(K) := \{ \sigma \in H(\text{div}, K, \mathcal{S}) | \sigma_n \in RT_k(K), \sigma_{12} \in P_k(x, y) \times P_{k-1}(z), \sigma_{13} \in P_k(x, z) \times P_{k-1}(y), \sigma_{23} \in P_k(y, z) \times P_{k-1}(x) \} \oplus E_k(K).
\end{equation}

The stress $\tau \in \Sigma^R_k(K)$ can be uniquely determined by the following conditions:

1. $\int_e \tau_n \cdot \nu ds$ for any $p \in Q_{k-1}(e)$ and any $e \subset \partial K$,
2. \( \int_K \tau_h \cdot p \, dx \, dy \, dz \) for any \( p \in \Psi_{k-1}(K) \),
3. the values of \( \tau_{XY} \) at \( k \) distinct points on each edge of \( K \) that is perpendicular to the \((X,Y)\)-plane,
4. the values of \( \tau_{XY} \) at \( k(k-1) \) distinct points in the interior of each face of \( K \) that parallels the \( Z \)-axis,
5. the moments \( \int_K \tau_{XY} p_{k-4} \, dx \, dy \, dz \) for any \( p_{k-4} \in P_{k-4}(X,Y) \times P_{k-1}(Z) \).

The proof for unisolvence of these degrees of freedom follows directly from those of the Raviart–Thomas element and the serendipity element and is omitted herein; see similar proofs in Lemmas 5.1 and 5.2.

The global space is defined as

\[
\Sigma_k^R(T_h) := \{ \tau \in \Sigma, \tau|_K \in \Sigma_k^R(K) \text{ for any } K \in T_h \}.
\]

On each element \( K \), the space for the displacement is taken as

\[
V_k^R(K) := (Q_{k-1}(K))^3.
\]

Then the global space for the displacement reads

\[
V_k^R(T_h) := \{ v \in V, v|_K \in V_k^R(K) \text{ for any } K \in T_h \}.
\]

To discretize the pure traction boundary problem, we define

\[
\begin{align*}
\Sigma_{k,0}^R(T_h) &= \{ \tau \in \Sigma_{k,0}^R(T_h) \mid \tau \nu = 0 \text{ on } \partial \Omega \}, \\
V_{k,0}^R(T_h) &= \{ v \in V_k^R(T_h) \mid (v, w)_{L^2(\Omega)} = 0 \text{ for all } w \in R^m \}.
\end{align*}
\]

It follows from the definitions of the spaces \( \Sigma_k^R(T_h) \) (resp., \( \Sigma_{k,0}^R(T_h) \)) and \( V_k^R(T_h) \) (resp., \( V_{k,0}^R(T_h) \)) that \( \text{div} \Sigma_k^R(T_h) \subset V_k^R(T_h) \) (resp., \( \text{div} \Sigma_{k,0}^R(T_h) \subset V_{k,0}^R(T_h) \)). Similar arguments of Theorems 3.1, 4.5, and 4.7 can prove the converses \( V_k^R(T_h) \subset \text{div} \Sigma_k^R(T_h) \) and \( V_{k,0}^R(T_h) \subset \text{div} \Sigma_{k,0}^R(T_h) \), respectively. This indicates the well-posedness of this family of elements.

**Remark 6.2.** The lowest order element \( (k=1) \) of this family has 8 stress and 2 displacement and 18 stress and 3 displacement degrees of freedom per element for two and three dimensions, respectively, which were announced independently by Chen [20] after the first version of the paper was submitted.

7. **The error estimate and numerical results.**

7.1. **The error estimate.** The section is devoted to the error analysis of the approximation defined by (2.6). It follows from (1.1) and (2.6) that

\[
\begin{align*}
(A(\sigma - \sigma_{k,h}), \tau_h)_{L^2(\Omega)} + (\text{div} \tau_h, (u - u_{k,h}))_{L^2(\Omega)} &= 0 \text{ for any } \tau_h \in \Sigma_k(T_h), \\
(\text{div}(\sigma - \sigma_{k,h}), v_h)_{L^2(\Omega)} &= 0 \text{ for any } v_h \in V_k(T_h).
\end{align*}
\]

Let \( P_h \) be the \( L^2 \) projection operator from \( L^2(\Omega; \mathbb{R}^n) \) onto \( V_k(T_h) \). Since \( \text{div} \sigma_{k,h} \in V_k(T_h) \), the second equation of (7.1) yields

\[
\| \text{div} (\sigma - \sigma_{k,h}) \|_{L^2(\Omega)} = \| \text{div} \sigma - P_h \text{div} \sigma \|_{L^2(\Omega)} \leq Ch^m \| \text{div} \sigma \|_{H^m(\Omega)} \text{ for any } 0 \leq m \leq k.
\]
It follows from the $K$-ellipticity, Theorem 3.1, and the approximation properties of $\Sigma_{k,h}$ and $V_{k,h}$ that
\begin{equation}
\|\sigma - \sigma_{k,h}\|_{L^2(\Omega)} + \|u - u_{k,h}\|_{L^2(\Omega)} \\
\leq C \left( \inf_{\tau_h \in \Sigma_{k,h}} \|\sigma - \tau_h\|_{H(\text{div}, \Omega)} + \inf_{v_h \in V_{k,h}} \|u - v_h\|_{L^2(\Omega)} \right)
\end{equation}
(7.3)

A similar error estimate holds for the pure traction boundary problem studied in section 4 and the reduced elements in section 6.

7.2. The numerical result. The first example is presented to demonstrate the second order method (with $k = 2$) for the pure displacement boundary problem with a homogeneous boundary condition that $u \equiv 0$ on $\partial \Omega$; see [28] for numerical examples for $k = 1$. Assume the material is isotropic in the sense that
\begin{equation}
A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + 2\lambda} \text{tr}(\sigma)\delta \right),
\end{equation}
where $\delta$ is the identity matrix, and $\mu$ and $\lambda$ are the Lamé constants such that $0 < \mu_1 \leq \mu \leq \mu_2$ and $0 < \lambda < \infty$. In the numerical example, these parameters are chosen as
\begin{equation}
\lambda = 1, \quad \mu = \frac{1}{2}.
\end{equation}

Let the exact solution on the unit square $[0, 1]^2$ be
\begin{equation}
(7.4) \quad u = (\sin \pi x \sin \pi y, \sin \pi x \sin \pi y)^T.
\end{equation}

In the computation, the level one grid is the given domain, a unit square or a unit cube. Each grid is refined into a half-size grid uniformly; to get a higher level grid, see the first column in Table 1.

As the second example, we compute the pure traction boundary problem with the exact solution
\begin{equation}
(7.5) \quad u = \begin{bmatrix} 100x^2(1-x)^2y^2(1-y)^2 - \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix},
\end{equation}
The matrix $A$ is same as that in the first example. Our new finite element has no problem in solving the pure traction boundary problems. The convergence results are listed in Table 2.

| $\|u - u_{2,h}\|_0$ | Rate | $\|\sigma - \sigma_{2,h}\|_0$ | Rate | $\|\text{div}(\sigma - \sigma_{2,h})\|_0$ | Rate |
|----------------------|------|----------------------|------|---------------------------------|------|
| 1                    | 0.3156 | 0.0                  | 2.0116 | 0.0                            | 7.8083 | 0.0 |
| 2                    | 0.0693 | 2.2                  | 0.4465 | 2.2                            | 1.9752 | 2.0 |
| 3                    | 0.0166 | 2.1                  | 0.1134 | 2.0                            | 0.4760 | 2.1 |
| 4                    | 0.0041 | 2.0                  | 0.0285 | 2.0                            | 0.1175 | 2.0 |
| 5                    | 0.0010 | 2.0                  | 0.0071 | 2.0                            | 0.0293 | 2.0 |
| 6                    | 2.5408e-004 | 2.0         | 0.0018 | 1.9                            | 0.0073 | 2.0 |
| 7                    | 6.3500e-005 | 2.0        | 4.4605e-004 | 2.0 | 0.0018 | 2.0 |
Table 2

The errors and the order of convergence for the pure traction boundary problem

|     | \|u - u^h_2\|_0 | Rate | \|\sigma - \sigma^h_2\|_0 | Rate | \|\text{div}(\sigma - \sigma^h_2)\|_0 | Rate |
|-----|----------------|------|----------------|------|----------------|------|
| 2   | 0.0264         | 0.0  | 0.2516         | 0.0  | 2.4645         | 0.0  |
| 3   | 0.0107         | 1.3  | 0.0804         | 1.6  | 0.7090         | 1.8  |
| 4   | 0.0029         | 1.9  | 0.0211         | 2.0  | 0.1807         | 2.0  |
| 5   | 7.294e-004     | 2.0  | 0.0054         | 2.0  | 0.0453         | 2.0  |
| 6   | 1.8315e-004    | 2.0  | 3.3684e-004    | 2.0  | 0.0113         | 2.0  |
| 7   | 4.5836e-005    | 2.0  | 7.2940e-004    | 2.0  | 0.0028         | 2.0  |

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