WHITNEY UMBRELLAS AND SWALLOWTAILS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his sixtieth birthday

Abstract. In this paper, we introduce the notions of map-germs of pedal unfolding type and normalized Legendrian map-germs; and then we show that the fundamental theorem of calculus provides a natural one to one correspondence between Whitney umbrellas of pedal unfolding type and normalized swallowtails.

1. Introduction

The map-germ

\[ f(x, y) = (xy, x^2, y) \]

is well-known as the normal form of Whitney umbrella after Whitney’s pioneer works \cite{10, 11}. For the map-germ \ref{1}, compose the following two coordinate transformations: \( h_s(x, y) = (x, x^2 + y) \) and \( h_t(X, Y, Z) = (X, -Z, -Y + Z) \) where \((X, Y, Z)\) is the standard coordinates of the target space \( \mathbb{R}^3 \). Then, we have the following:

\[ g(x, y) = h_t \circ f \circ h_s(x, y) = (x^3 + xy, -x^2 - y, y). \]

Put

\[ G(x, y) = \left( \int_0^x (x^3 + xy)dx, \int_0^x (-x^2 - y)dx, y \right) = \left( \frac{1}{4}x^4 + \frac{1}{2}x^2y, -\frac{1}{3}x^3 - xy, y \right). \]

For the map-germ \ref{3}, compose the following two scaling transformations: \( H_s(x, y) = (x, \frac{1}{6}y) \) and \( H_t(X, Y, Z) = (12X, 12Y, 6Z) \). Then, we have the following:

\[ H_t \circ G \circ H_s(x, y) = (3x^4 + x^2y, -4x^3 - 2xy, y). \]

The map-germ \ref{4} is well-known as the normal form of swallowtail (for instance, see \cite{2} p.129).

Two \( C^\infty \) map-germs \( \varphi, \psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) are said to be \( \mathcal{A} \)-equivalent if there exist germs of \( C^\infty \) diffeomorphisms \( h_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) and \( h_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0) \) such that \( \psi = h_t \circ \varphi \circ h_s \). A \( C^\infty \) map-germ \( \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) is called a Whitney umbrella (resp., swallowtail) if \( \varphi \) is \( \mathcal{A} \)-equivalent to \ref{1} (resp., \ref{4}). As above, the Whitney umbrella \ref{1} produces the swallowtail \ref{4} via \ref{2} and \ref{3}. By the converse procedure, the swallowtail \ref{4} produces the Whitney umbrella \ref{1}.

Note that it is impossible to produce a swallowtail by integrating \ref{1} directly. This is because the discriminant set of \ref{4} is not diffeomorphic to the discriminant.

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set of the following (4):

\[(x, y) \mapsto \left( \int_0^x y \, dx, \int_0^x x \, dx, y \right).\]

Note further that the form (2) may be written as follows:

\[g(x, y) = (x(x^2 + y), -(x^2 + y), y) = (b (-x, -(x^2 + y)), y),\]

where \(b(X, Y) = (XY, Y)\) (\(b\) stands for “the blow down”).

**Definition 1.**

(i) A \(C^\infty\) map-germ \(\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) having the form (6) is said to be of pedal unfolding type.

\[(6) \quad \varphi(x, y) = (n(x, y)p(x, y), p(x, y), y) = (b (n(x, y), p(x, y)), y)\]

where \(n : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) is a \(C^\infty\) function-germ such that \(\frac{\partial n}{\partial x}(0, 0) \neq 0\) and \(p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)\) is a \(C^\infty\) function-germ.

(ii) For a \(C^\infty\) map-germ of pedal unfolding type \(\varphi(x, y) = (n(x, y)p(x, y), p(x, y), y)\), put

\[I(\varphi) = \left( \int_0^x n(x, y)p(x, y) \, dx, \int_0^x p(x, y) \, dx, y \right).\]

The map-germ \(I(\varphi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) is called the integration of \(\varphi\).

(iii) A \(C^\infty\) map-germ \(\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+1}, 0)\) is called a Legendrian map-germ if there exists a germ of \(C^\infty\) vector field \(\nu_{\Phi} : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1}\) along \(\Phi\) such that the following conditions hold where the dot in the center stands for the scalar product of two vectors of \(T_{\Phi(x, y)}\mathbb{R}^{m+1}\) and \(T_1\mathbb{R}^{m+1}\) stands for the unit tangent bundle of \(\mathbb{R}^{m+1}\).

\[(a) \quad \frac{\partial \Phi}{\partial x_1}(x_1, \ldots, x_m) \cdot \nu_{\Phi}(x_1, \ldots, x_m) = \cdots = \frac{\partial \Phi}{\partial x_m}(x_1, \ldots, x_m) \cdot \nu_{\Phi}(x_1, \ldots, x_m) = 0.\]

\[(b) \quad \text{The map-germ } L_{\Phi} : (\mathbb{R}^m, 0) \to T_1\mathbb{R}^{m+1} \text{ defined by } L_{\Phi}(x_1, \ldots, x_m) = (\Phi(x_1, \ldots, x_m), \nu_{\Phi}(x_1, \ldots, x_m)) \text{ is non-singular. The map-germ } L_{\Phi} \text{ is called a Legendrian lift of } \Phi.\]

(iv) A Legendrian map-germ \(\Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) is said to be normalized if \(\Phi\) satisfies the following three conditions:

\[(a) \quad \Phi \text{ has the form } \Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y).\]

\[(b) \quad \frac{\partial \Phi_1}{\partial x}(0, 0) = 0.\]

\[(c) \quad \nu_{\Phi}(0, 0) \text{ is } \frac{\partial}{\partial x} \text{ or } -\frac{\partial}{\partial y}.\]

(v) For a normalized Legendrian map-germ \(\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)\), put

\[D(\Phi) = \left( \frac{\partial \Phi_1}{\partial x}(x, y), \frac{\partial \Phi_2}{\partial x}(x, y), y \right).\]

The map-germ \(D(\Phi) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) is called the differentiation of \(\Phi\).

Since any map-germ \(\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) which is a germ of one-parameter pedal unfolding of a spherical pedal curve has the form (5) (see [7]), it is reasonable that a map-germ \(\varphi\) having the form (6) is said to be of pedal unfolding type. As shown in [7], not only non-singular map-germs but also Whitney umbrellas may be realized as germs of one-parameter pedal unfoldings of spherical pedal curves. For details on Legendrian map-germs, see [11, 3, 12, 13]. Note that both (3) and (7) are normalized Legendrian map-germs.
Proposition 1.  (i) For a \( C^\infty \) map-germ of pedal unfolding type \( \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \), \( \mathcal{I}(\varphi) \) is a normalized Legendrian map-germ.

(ii) For a normalized Legendrian map-germ \( \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \), \( \mathcal{D}(\Phi) \) is a map-germ of pedal unfolding type.

Put
\[
\mathcal{W} = \{ \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{Whitney umbrella of pedal unfolding type} \},
\]
\[
\mathcal{S} = \{ \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{normalized swallowtail} \}.
\]

The main purpose of this paper is to show the following Theorem 1:

Theorem 1.  (i) The map \( \mathcal{I} : \mathcal{W} \to \mathcal{S} \) defined by \( \mathcal{W} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{S} \) is well-defined and bijective.

(ii) The map \( \mathcal{D} : \mathcal{S} \to \mathcal{W} \) defined by \( \mathcal{S} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{W} \) is well-defined and bijective.

Incidentally, we show the following Theorem 2. A \( C^\infty \) map-germ \( \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) is called a cuspidal edge if \( \Phi \) is \( \mathcal{A} \)-equivalent to the following (7):

\[
(x, y) \mapsto \left( \frac{1}{3}x^3, \frac{1}{2}x^2, y \right).
\]

Put
\[
\mathcal{N} = \{ \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{non-singular map-germ of pedal unfolding type} \},
\]
\[
\mathcal{C} = \{ \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \ \text{normalized cuspidal edge} \}.
\]

Theorem 2.  (i) The map \( \mathcal{I} : \mathcal{N} \to \mathcal{C} \) defined by \( \mathcal{N} \ni \varphi \mapsto \mathcal{I}(\varphi) \in \mathcal{C} \) is well-defined and bijective.

(ii) The map \( \mathcal{D} : \mathcal{C} \to \mathcal{N} \) defined by \( \mathcal{C} \ni \Phi \mapsto \mathcal{D}(\Phi) \in \mathcal{N} \) is well-defined and bijective.

Both the following two are well-known (for instance, see [1]).

(i) Any stable map-germ \( (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) is either a Whitney umbrella or non-singular.

(ii) Any Legendrian stable singularity \( (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) is either a swallowtail or a cuspidal edge.

Therefore, Theorems 1 and 2 may be regarded as the fundamental theorem of calculus for stable map-germs \( (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) and Legendrian stable singularities \( (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \).

Theorems 1 and 2 yield the following conjecture naturally.

Conjecture 1.  (i) Let \( \varphi_1, \varphi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be two \( C^\infty \) map-germs of pedal unfolding type. Suppose that \( \varphi_1 \) is \( \mathcal{A} \)-equivalent to \( \varphi_2 \). Then, \( \mathcal{I}(\varphi_1) \) is \( \mathcal{A} \)-equivalent to \( \mathcal{I}(\varphi_2) \).

(ii) Let \( \Phi_1, \Phi_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be two normalized Legendrian map-germs. Suppose that \( \Phi_1 \) is \( \mathcal{A} \)-equivalent to \( \Phi_2 \). Then, \( \mathcal{D}(\Phi_1) \) is \( \mathcal{A} \)-equivalent to \( \mathcal{D}(\Phi_2) \).

In [2] several preparations for the proofs of Theorems 1 and 2 and the proof of Proposition 1 are given. Theorems 1 and 2 are proved in [3] and [4] respectively.
2. Preliminaries

2.1. Function-germs with two variables and map-germs with two variables. Let $\mathcal{E}_2$ be the set of $C^\infty$ function-germs $(\mathbb{R}^2, 0) \to \mathbb{R}$ and let $m_2$ be the subset of $\mathcal{E}_2$ consisting of $C^\infty$ function-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$. The sets $\mathcal{E}_2$ have natural $\mathbb{R}$-algebra structures. For a $C^\infty$ map-germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, let $\varphi^* : \mathcal{E}_2 \to \mathcal{E}_2$ be the $\mathbb{R}$-algebra homomorphism defined by $\varphi^*(u) = u \circ \varphi$. Put $Q(\varphi) = \mathcal{E}_2/\varphi^*m_2\mathcal{E}_2$.

Then, $Q(\varphi)$ is an $\mathbb{R}$-algebra. The following Proposition 2 is a special case of theorem (2.1) of [5].

**Proposition 2.** Let $p : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a $C^\infty$ function-germ. Then, the following hold:

(i) The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x^2, y)$ if and only if $\frac{\partial p}{\partial x}(0, 0) = 0$ and $\frac{\partial^2 p}{\partial x \partial y}(0, 0) \neq 0$.

(ii) The $\mathbb{R}$-algebra $Q(p(x, y), y)$ is isomorphic to $Q(x, y)$ if and only if $(x, y) \mapsto (p(x, y), y)$ is a germ of $C^\infty$ diffeomorphism.

**Definition 2** (6). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two $C^\infty$ function-germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be $K^T$-equivalent if there exist a germ of $C^\infty$ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of the form $h \circ T = T \circ h$ and a $C^\infty$ function-germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R} - \{0\}$ of the form $M \circ T = M$ such that $p_1 \circ h(x, y) = M(x, y)p_2(x, y)$.

**Theorem 3** (6). Two $C^\infty$ map-germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) (i = 1, 2)$ of the following form

\[ \varphi_i(x, y) = (x p_i(x^2, y), x^2, y) \quad \text{where} \quad p_i(x^2, y) \notin m_2^\infty, \quad (i = 1, 2) \]

are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(x^2, y)$ are $K^T$-equivalent.

Here, $m_2^\infty = \{ q : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \mid \frac{\partial^{i+j}}{\partial x^i \partial y^j}(0, 0) = 0 \quad (\forall i, j \in \{0\} \cup \mathbb{N}) \}$. By Theorem 3 and the Malgrange preparation theorem (for instance, see [1]), the following holds:

**Corollary 1.** Two $C^\infty$ map-germs $\varphi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) (i = 1, 2)$ of the following form

\[ \varphi_i(x, y) = (n_i(x, y)p_i(x^2, y), x^2, y), \]

where $p_i(x^2, y) \notin m_2^\infty$ and $n_i(x, y)$ satisfies $\frac{\partial n_i}{\partial x}(0, 0) \neq 0$ for each $i \in \{1, 2\}$, are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(x^2, y)$ are $K^T$-equivalent.

2.2. Map-germs of pedal unfolding type. Let $\varphi : I \times J \to \mathbb{R}^3$ be a representative of a given $C^\infty$ map-germ of pedal unfolding type, where $I, J$ be a sufficiently small intervals containing the origin of $\mathbb{R}$. Then, we may put $\varphi(x, y) = (n(x, y)p(x, y), p(x, y))$. Put

\[ \Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) = \left( \int_0^x n(x, y)p(x, y)dx, \int_0^x p(x, y)dx, y \right) \]

and

\[ \tilde{\nu}_\Phi(x, y) = \frac{\partial}{\partial X} - n(x, y) \frac{\partial}{\partial Y}. \]

Since $\tilde{\nu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, for any fixed $y \in J$ we may define the map-germ $L_{\Phi,y} : (\mathbb{R}, 0) \to T_1 \mathbb{R}^2$ as

\[ L_{\Phi,y}(x) = \left( (\Phi_1(x, y), \Phi_2(x, y)), \frac{\tilde{\nu}_\Phi(x, y)}{||\tilde{\nu}_\Phi(x, y)||} \right), \]
where $T_1\mathbb{R}^2$ is the unit tangent bundle of $\mathbb{R}^2$. Then, since $\varphi$ is a representative of a map-germ of pedal unfolding type, we have the following:

**Lemma 2.1.** For any $y \in J$, $L_{\Phi, y} : (\mathbb{R}, 0) \to T_1\mathbb{R}^2$ is a Legendrian lift of the map-germ $x \mapsto (\Phi_1(x, y), \Phi_2(x, y))$.

By Lemma 2.1 we have the following:

**Lemma 2.2.** For any $y \in J$, the map-germ $\tilde{\Phi}_y : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ defined by $\tilde{\Phi}_y(x) = (\Phi_1(x, y), \Phi_2(x, y))$ is a Legendrian map-germ.

Next, put

$$\bar{\nu}_\Phi(x, y) = \tilde{\nu}_\Phi(x, y) - \left( \frac{\partial \Phi_1}{\partial y}(x, y) - n(x, y) \frac{\partial \Phi_2}{\partial y}(x, y) \right) \frac{\partial}{\partial Z}.$$  

Then, we have the following:

**Lemma 2.3.** For any $x \in I$ and $y \in J$,

$$\bar{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial x}(x, y) = 0, \quad \bar{\nu}_\Phi(x, y) \cdot \frac{\partial \Phi}{\partial y}(x, y) = 0.$$  

Since $\bar{\nu}_\Phi(x, y) \neq 0$ for any $x \in I$ and $y \in J$, we may define the map-germ $L_\Phi : (\mathbb{R}, 0) \to T_1\mathbb{R}^3$ as

$$L_\Phi(x, y) = \left( \Phi(x, y), \frac{\bar{\nu}_\Phi(x, y)}{||\bar{\nu}_\Phi(x, y)||} \right).$$

Then, by Lemma 2.3 we have the following:

**Lemma 2.4.** $L_\Phi : (\mathbb{R}, 0) \to T_1\mathbb{R}^3$ is a Legendrian lift of $\Phi$.

By Lemma 2.4 we have the following:

**Lemma 2.5.** $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a Legendrian map-germ.

### 2.3. Normalized Legendrian map-germs

Let $\Phi : U \to \mathbb{R}^3$ be a representative of a given normalized Legendrian map-germ $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$, where $U$ is a sufficiently small neighborhood of the origin of $\mathbb{R}^2$. We assume that the origin of $\mathbb{R}^2$ is a singular point of $\Phi$. By the condition (a) of the definition of normalized Legendrian map-germs, we may assume that $\Phi$ has the form $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y)$. Since $\Phi$ is a a representative of a Legendrian map-germ, we have the following:

**Lemma 2.6.** There exists a $C^\infty$ vector field $\nu_\Phi$ along $\Phi$,

$$\nu_\Phi(x, y) = n_1(x, y) \frac{\partial}{\partial X} + n_2(x, y) \frac{\partial}{\partial Y} + n_3(x, y) \frac{\partial}{\partial Z},$$

such that the following three hold:

(i) $n_1(x, y) \frac{\partial \Phi_1}{\partial x}(x, y) + n_2(x, y) \frac{\partial \Phi_2}{\partial x}(x, y) = 0$.

(ii) $n_1(x, y) \frac{\partial \Phi_1}{\partial y}(x, y) + n_2(x, y) \frac{\partial \Phi_2}{\partial y}(x, y) + n_3(x, y) = 0$.

(iii) The map $L_\Phi : U \to T_1\mathbb{R}^3$ defined by $L_\Phi(x, y) = (\Phi(x, y), \nu_\Phi(x, y))$ is an immersion.

By the condition (c) of the definition of normalized Legendrian map-germs, we have the following:

**Lemma 2.7.** For the vector field $\nu_\Phi$, $n_1(0, 0) \neq 0$ and $n_2(0, 0) = n_3(0, 0) = 0$. 
By the assertion (i) of Lemma 2.6 and Lemma 2.7, we have the following equality as function-germs.

\[
\frac{\partial \Phi_1}{\partial x}(x, y) = -\frac{n_2(x, y)}{n_1(x, y)} \frac{\partial \Phi_2}{\partial x}(x, y).
\]  

Then, by the condition (b) of the definition of normalized Legendrian maps and the equality (8), the following holds:

**Lemma 2.8.** The map-germ \( D(\Phi) \) maps the origin to the origin.

Put

\[ n(x, y) = -\frac{n_2(x, y)}{n_1(x, y)} \quad \text{and} \quad p(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y). \]

Then, we have clearly the following:

**Lemma 2.9.** Both function-germs \( n \) and \( p \) are of class \( C^\infty \) and the equality

\[ D(\Phi)(x, y) = (n(x, y)p(x, y), p(x, y), y) \]

holds.

Furthermore, we have the following:

**Lemma 2.10.** The function-germ \( n \) satisfies that \( n(0, 0) = 0 \) and \( \frac{\partial n}{\partial x}(0, 0) \neq 0 \).

**Proof.** By Lemma 2.7 we have that \( n(0, 0) = 0 \). Suppose that \( \frac{\partial n}{\partial x}(0, 0) = 0 \). Then, by differentiating both side of the equality in the assertion (ii) of Lemma 2.6 with respect to \( x \), we have the following equality:

\[ n_1(0, 0) \frac{\partial^2 \Phi_1}{\partial x \partial y}(0, 0) + \frac{\partial n_3}{\partial x}(0, 0) = 0. \]

Since we have assumed \( \frac{\partial n}{\partial x}(0, 0) = 0 \), we have that \( \frac{\partial n_3}{\partial x}(0, 0) = 0 \). Thus and since \( \Phi \) is a normalized Legendrian map-germ such that the origin of \( \mathbb{R}^2 \) is a singular point of \( \Phi \), we have that \( \frac{\partial n_3}{\partial x}(0, 0) \neq 0 \). Thus, we have that \( \frac{\partial^2 \Phi_1}{\partial x \partial y}(0, 0) \neq 0 \). Hence, by the condition (b) of the definition of normalized Legendrian maps, Lemma 2.7 and the equality (8), we have a contradiction. \( \square \)

**Definition 3.** Let \( \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a Legendrian map-germ and let \( \nu_\Phi \) be a unit normal vector field of \( \Phi \) given in the definition of Legendrian map-germs. The \( C^\infty \) function-germ \( LJ_\Phi : (\mathbb{R}^2, 0) \to \mathbb{R} \) defined by the following is called the Legendrian-Jacobian of \( \Phi \).

\[ LJ_\Phi(x, y) = \det \left( \frac{\partial \Phi}{\partial x}(x, y), \frac{\partial \Phi}{\partial y}(x, y), \nu_\Phi(x, y) \right). \]

Note that if \( \nu_\Phi \) satisfies the conditions of unit normal vector field of \( \Phi \), then \( -\nu_\Phi \) also satisfies them. Thus, the sign of \( LJ_\Phi(x, y) \) depends on the particular choice of unit normal vector field \( \nu_\Phi \). The Legendrian Jacobian of \( \Phi \) is called also the signed area density function (for instance, see [9]). Although it seems reasonable to call \( LJ_\Phi \) the area density function from the viewpoint of investigating the singular surface \( \Phi(U) \) \((U \) is a sufficiently small neighborhood of the origin of \( \mathbb{R}^2 \)), it seems reasonable to call it the Legendrian Jacobian from the viewpoint of investigating the singular map-germ \( \Phi \). Hence, we call \( LJ_\Phi \) the Legendrian Jacobian of \( \Phi \) in this paper.
Let \( \Phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a normalized Legendrian map-germ and \( \nu_\Phi \) is a unit normal vector field of \( \Phi \).

\[
\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y),
\]
\[
\nu_\Phi(x, y) = n_1(x, y) \frac{\partial}{\partial X} + n_2(x, y) \frac{\partial}{\partial Y} + n_3(x, y) \frac{\partial}{\partial Z}.
\]

By Lemma 2.7, we may put

\[
\bar{\nu}_\Phi(x, y) = \frac{\partial}{\partial X} + \frac{n_2(x, y)}{n_1(x, y)} \frac{\partial}{\partial Y} + \frac{n_3(x, y)}{n_1(x, y)} \frac{\partial}{\partial Z}.
\]

**Lemma 2.11.** The Legendrian Jacobian \( LJ_\Phi \) is expressed as follows:

\[
LJ_\Phi(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y) / n_1(x, y).
\]

**Proof.** Calculations show that

\[
\frac{\partial \Phi}{\partial x}(x, y) \times \frac{\partial \Phi}{\partial y}(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y) \bar{\nu}_\Phi(x, y)
\]

where the cross in the center stands for the vector product. It follows \( LJ_\Phi(x, y) = \frac{\partial \Phi_2}{\partial x}(x, y)/n_1(x, y) \).

\( \square \)

### 2.4. Proof of Proposition 1

**Proof of the assertion (i) of Proposition 1.**

Put \( I(\varphi) = \Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), y) \). Then, by Lemma 2.3, \( \Phi \) is a Legendrian map-germ. Thus, in order to complete the proof of the assertion (i) of Proposition 1, it is sufficient to show the conditions (b), (c) of the definition of normalized Legendrian map-germs are satisfied.

Put \( \varphi(x, y) = (n(x, y)p(x, y), p(x, y), y) \). Then, by the definition of map-germs of pedal unfolding type, we have that \( n(0, 0) = 0 \) and \( p(0, 0) = 0 \). It follows that \( \frac{\partial n_\Phi}{\partial x}(0, 0) = p(0, 0) = 0 \). Thus, the condition (b) is satisfied. By Lemma 2.4, the following \( L_\Phi \) is a germ of Legendrian lift of \( \Phi \):

\[
L_\Phi(x, y) = \left( \Phi(x, y), \bar{\nu}_\Phi(x, y) \right),
\]

where \( \bar{\nu}_\Phi(x, y) = \frac{\partial}{\partial y} - n(x, y) \frac{\partial}{\partial x} - \left( \frac{\partial n_\Phi}{\partial y}(x, y) - n(x, y) \frac{\partial n_\Phi}{\partial x}(x, y) \right) \frac{\partial}{\partial z} \). Since \( n(0, 0) = 0 \) and \( \frac{\partial n}{\partial y}(0, 0) = \int_0^x \frac{\partial n}{\partial y}(x, 0) dx = 0 \), we have

\[
\frac{\bar{\nu}_\Phi(0, 0)}{||\bar{\nu}_\Phi(0, 0)||} = \frac{\partial}{\partial X}.
\]

Thus the condition (c) is satisfied. \( \square \)

**Proof of the assertion (ii) of Proposition 1.** The assertion (ii) of Proposition 1 follows from Lemmas 2.8 and 2.10 \( \square \)

### 3. Proof of Theorem 1

Suppose that both \( I : W \to S \) and \( D : S \to W \) are well-defined. Then, by the fundamental theorem of calculus, the following hold:

\[
D \circ I(\varphi) = \varphi \quad \text{for any } \varphi \in W,
\]
\[
I \circ D(\Phi) = \Phi \quad \text{for any } \Phi \in S.
\]
Thus, both $\mathcal{I}$ and $\mathcal{D}$ are bijective. Therefore, in order to complete the proof, it is sufficient to show that both $\mathcal{I}$ and $\mathcal{D}$ are well-defined.

**Proof that $\mathcal{I} : \mathcal{W} \to \mathcal{S}$ is well-defined.**

Let $\varphi(x, y) = (m(x, y)p_\varphi(x, y), p_\varphi(x, y), y)$ be an element of $\mathcal{W}$. Put $\Phi = \mathcal{I}(\varphi)$. Then, $\Phi$ is a normalized Legendrian map-germ by Proposition 1 in §1. Let $g$ be the Whitney umbrella of pedal unfolding type (2) defined in §1:

$$g(x, y) = (xp_\varphi(x, y), p_\varphi(x, y), y) = (x(x^2 + y), -x^2 - y, y).$$

**Lemma 3.1.** There exists a germ of $C^\infty$ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $h$ has the form $h(x, y) = (h_1(x, y), h_2(y))$ and $p_\varphi \circ h(x, y)$ is $x^2 + y$ or $-(x^2 + y)$.

**Proof.** Since $\varphi$ is a Whitney umbrella of pedal unfolding type, we have the following:

$$Q(p_\varphi(x, y), y) \equiv Q(\varphi) \equiv Q(g) \equiv Q(x^2, y).$$

Thus, we may put $p_\varphi(x, 0) = a_2 x^2 + o(x^2)$ ($a_2 \neq 0$) by Proposition 2 in §2. By the Morse lemma with parameters (for instance, see [2]), there exists a germ of $C^\infty$ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $h$ has the form $h(x, y) = (h_1(x, y), h_2(y))$ and $p_\varphi \circ h(x, y) = \pm(x^2 + q(y))$ by a certain $C^\infty$ function-germ $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Since $\varphi$ is $\mathcal{A}$-equivalent to $g$, by Corollary 1 in [2] $\pm(x^2 + q(y))$ is $\mathcal{K}^T$-equivalent to $p_\varphi$ and thus $q : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a germ of $C^\infty$ diffeomorphism. Hence, Lemma 3.1 follows.

**Lemma 3.2.** The normalized Legendrian map-germ $\Phi$ is a swallowtail.

**Proof.** Put $G = \mathcal{I}(g)$. Then, $G$ has the form (3) in §1 which is a normalized swallowtail. Since $G$ is normalized, $\frac{\partial G}{\partial x}$ is the null vector field for $G$ defined in §1 [8], that is, $\frac{\partial G}{\partial x}(x, y) = 0$ holds for any $(x, y)$ which is a singular point of $G$. Thus and since $G$ is a swallowtail, the following three hold by corollary 2.5 of §3.

(i) $L_JG(0, 0) = \frac{\partial L_JG}{\partial x}(0, 0) = 0$,

(ii) $\frac{\partial^2 L_JG}{\partial x^2}(0, 0) \neq 0$,

(iii) $Q(L_JG, \frac{\partial L_JG}{\partial x}) \equiv Q(x, y)$.

On the other hand, by Lemmas 2.1. and §3.1 there exist a germ of $C^\infty$ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and a $C^\infty$ function-germ $\xi : (\mathbb{R}^2, 0) \to \mathbb{R}$ such that $h$ has the form $h(x, y) = (h_1(x, y), h_2(y))$, $\xi(0, 0) \neq 0$ and the following hold:

$$L_Jh \circ h(x, y) = \xi(x, y)L_JG(x, y).$$

Hence and since $\frac{\partial G}{\partial x}$ is the null vector field for $\Phi$ (this is because $\Phi$ is normalized), the following three hold for $L_J\Phi$.

(i) $L_J\Phi(0, 0) = \frac{\partial L_J\Phi}{\partial x}(0, 0) = 0$,

(ii) $\frac{\partial^2 L_J\Phi}{\partial x^2}(0, 0) \neq 0$,

(iii) $Q(L_J\Phi, \frac{\partial L_J\Phi}{\partial x}) \equiv Q(x, y)$.

Hence, $\Phi$ is a swallowtail by corollary 2.5 of §3.

**Proof that $\mathcal{D} : \mathcal{S} \to \mathcal{W}$ is well-defined.**

Let $\Phi$ be an element of $\mathcal{S}$. Then, by Proposition 2 in §2 $\mathcal{D}(\Phi)$ is of pedal unfolding type.

**Lemma 3.3.** For the Legendrian Jacobian $L_J\Phi$, the following three hold:
Since Lemma 4.2.

The normalized Legendrian map-germ \( h_{\mathcal{C}} \) and the Morse lemma with parameters, there exists a germ of Lemma 3.4.

Thus, \((\varphi(x,y))\) is non-singular and of pedal unfolding type, \( \Phi \) is a normalized Legendrian map-germ by Proposition 1.3 of [4]. Then, by Lemmas 2.11, 3.3 and the Morse lemma with parameters, there exists a germ of \( C^\infty \) diffeomorphism \( h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) such that \( h \) has the form \( h(x,y) = (h_1(x,y), h_2(y)) \) and \( p_\varphi \circ h(x,y) \) is \( x^2 + y \) or \( -(x^2 + y) \).

Therefore, by Corollary 1 in [2], \( D(\Phi) \) is \( \mathcal{A} \)-equivalent to \( g \).

**4. Proof of Theorem 2**

As same as Theorem 1 it is sufficient to show that both \( \mathcal{I} : \mathcal{N} \to \mathcal{C} \) and \( \mathcal{D} : \mathcal{C} \to \mathcal{N} \) are well-defined.

**Proof that \( \mathcal{I} : \mathcal{N} \to \mathcal{C} \) is well-defined.**

Let \( \varphi(x,y) = (n(x,y)p_\varphi(x,y), p_\varphi(x,y), y) \) be an element of \( \mathcal{N} \). Put \( \Phi = \mathcal{I}(\varphi) \).

Then, since \( \varphi \) is of pedal unfolding type, \( \Phi \) is a normalized Legendrian map-germ by Proposition 1 in [3]. Let \( g \) be the non-singular map-germ of pedal unfolding type defined by \( g(x,y) = (x^2, x, y) \).

**Lemma 4.1.** There exists a germ of \( C^\infty \) diffeomorphism \( h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) such that \( h \) has the form \( h(x,y) = (h_1(x,y), h_2(y)) \) and \( p_\varphi \circ h(x,y) = x \) holds.

**Proof.** Since \( \varphi \) is non-singular and of pedal unfolding type, we have the following:

\[
Q(p_\varphi(x,y), y) = Q(y) = Q(x,y).
\]

Thus, \( (p_\varphi(x,y), y) \) is a germ of \( C^\infty \) diffeomorphism by Proposition 1 in [2]. From the form of \( (p_\varphi(x,y), y) \), its inverse map-germ \( h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) has the form \( h(x,y) = (h_1(x,y), h_2(y)) \). Since \( h \) is the inverse map-germ of \( (p_\varphi(x,y), y) \), it follows that \( p_\varphi \circ h(x,y) = x \).

**Lemma 4.2.** The normalized Legendrian map-germ \( \Phi \) is a cuspidal edge.

**Proof.** Since \( \Phi \) is normalized, \( \frac{\partial}{\partial x} \) is the null vector field for \( \Phi \). By Lemmas 2.11, 4.1 we have that \( \frac{\partial L_{J_\Phi}}{\partial x}(0,0) \neq 0 \). Thus, the null vector field \( \frac{\partial}{\partial x} \) is transverse to \( \{(x,y) \mid L_{J_\Phi}(x,y) = 0\} \) at \( (0,0) \in \mathbb{R}^2 \). Hence, \( \Phi \) is a cuspidal edge by proposition 1.3 of [4].

**Proof that \( \mathcal{D} : \mathcal{C} \to \mathcal{N} \) is well-defined.**

Let \( \Phi \) be an element of \( \mathcal{C} \). Then, by Proposition 1 in [4], \( D(\Phi) \) is of pedal unfolding type.

**Lemma 4.3.** For the Legendrian Jacobian \( L_{J_\Phi} \), two properties \( L_{J_\Phi}(0,0) = 0 \) and \( \frac{\partial L_{J_\Phi}}{\partial x}(0,0) \neq 0 \) hold.
Proof. Since $\frac{\partial}{\partial x}$ is the null vector field for $\Phi$ and $\Phi$ is a cuspidal edge, Lemma 4.3 follows from corollary 2.5 of [8].

Lemma 4.4. For the $\Phi$, the map-germ of pedal unfolding type $D(\Phi)$ is non-singular and $D(\Phi)(0,0) = (0,0,0)$.

Proof. Since $D(\Phi)$ is of pedal unfolding type, there exists a $C^\infty$ function-germ $n : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ such that $\frac{\partial n}{\partial x}(0,0) \neq 0$ and $\frac{\partial n}{\partial x}(x,y) = n(x,y)\frac{\partial \Phi_2}{\partial x}(x,y)$ where $\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y), y)$. Put $p_\nu = \frac{\partial n}{\partial x}$. Then, by Lemmas 2.11 and the map-germ $(x,y) \mapsto (p_\nu(x,y), y)$ is a germ of $C^\infty$ diffeomorphism. Thus, $D(\Phi)$ is non-singular. \qed

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