Modules with Cosupport and Injective Functors

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Abstract Several authors have studied the filtered colimit closure $\varinjlim B$ of a class $B$ of finitely presented modules. Lenzing called $\varinjlim B$ the category of modules with support in $B$, and proved that it is equivalent to the category of flat objects in the functor category $(B^{\text{op}}, \text{Ab})$. In this paper, we study the category $(\text{Mod-}R)^B$ of modules with cosupport in $B$. We show that $(\text{Mod-}R)^B$ is equivalent to the category of injective objects in $(B, \text{Ab})$, and thus recover a classical result by Jensen-Lenzing on pure injective modules. Works of Angeleri-Hügel, Enochs, Krause, Rada, and Saorín make it easy to discuss covering and enveloping properties of $(\text{Mod-}R)^B$, and furthermore we compare the naturally associated notions of $B$-coherence and $B$-noetherianness. Finally, we prove a number of stability results for $\varinjlim B$ and $(\text{Mod-}R)^B$. Our applications include a generalization of a result by Gruson-Jensen and Enochs on pure injective envelopes of flat modules.

Keywords Algebraically compact · Coherent · Contravariantly finite · Cosupport · Cotorsion pairs · Covariantly finite · Covers · Direct limits · Envelopes · Equivalence · Filtered colimits · Flat functors · Functor category · Injective functors · Noetherian · Pure injective · Stability · Support

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1 Introduction

Let $B$ be a finitely presented left module over a ring $R$, and let $\Lambda$ be its endomorphism ring. Since $B$ is a left-$\Lambda$-left-$R$-bimodule, one can consider the functors

\[
\begin{array}{ccl}
\text{Mod-} \Lambda & \xleftarrow{\text{Hom}_\Lambda(B,-)} & \text{R-Mod} \\
\xrightarrow{-\otimes_\Lambda B} & & \\
\end{array}
\]

An important observation in Auslander’s work on representation theory for Artin algebras is that these functors give an equivalence between $\text{add} B$ and $\text{proj-}\Lambda$; see Notation 2.2. Actually, it follows by Lazard [35] that the functors above also induce an equivalence between $\lim^{-}\,(\text{add} B)$ and $\text{Flat-}\Lambda$. In [36] Lenzing generalizes this result even further by proving that for any additive category $B$ of finitely presented left $R$-modules, the Yoneda functor,

\[
\begin{array}{ccl}
\text{Mod-} R & \xrightarrow{\text{R-Mod}} & (B^{\text{op}}, \text{Ab}) \\
M \mapsto \text{Hom}_R(\_, M)|_B \\
\end{array}
\]

restricts to an equivalence between $\lim^{-}B$ and the category $\text{Flat}(B^{\text{op}}, \text{Ab})$ of flat functors in the sense of Oberst-Röhrl [39] and Stenström [42]. The category $\lim^{-}B$ has several nice properties, and it has been studied in great detail by e.g. the authors of [3–5, 13, 15, 34], and [36].

In this paper, we study the category of modules with cosupport in $B$,

\[
(\text{Mod-} R)^B = \text{Prod}(\text{Hom}_B(B, \mathbb{Q}/\mathbb{Z}) \mid B \in B).
\]

The main theorem of Section 3 is a result dual to that of Lenzing [36, prop. 2.4].

**Theorem 1.1** The tensor evaluation functor,

\[
\begin{array}{ccl}
\text{Mod-} R & \xrightarrow{\text{R-Mod}} & (B^{\text{op}}, \text{Ab}) \\
N \mapsto (N \otimes_R \_)|_B \\
\end{array}
\]

restricts to an equivalence between $(\text{Mod-} R)^B$ and $\text{Inj}(B, \text{Ab})$.

Two special cases of Theorem 1.1 are worth mentioning: If $B = \text{add} B$ for some finitely presented module $B$ with endomorphism ring $\Lambda$, it follows that the functors

\[
\begin{array}{ccl}
\text{Mod-} R & \xleftarrow{\text{Hom}_\Lambda(B,-)} & \text{\Lambda-Mod} \\
\xrightarrow{-\otimes_\Lambda B} & & \\
\end{array}
\]

induce an equivalence between $\text{Prod}(\text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z}))$ and $\Lambda\text{-Inj}$. For $B = R\text{-mod}$ we get an equivalence between the category of pure injective right $R$-modules and $\text{Inj}(R\text{-mod}, \text{Ab})$. We refer to Jensen-Lenzing [30, thm. B.16] for this classical result.

\footnote{Unfortunately, the proof of Jensen-Lenzing [30, thm. B.16] does not apply to give a proof of Theorem 1.1, as one key ingredient in their argument is the fact that the tensor evaluation functor

\[
\begin{array}{ccl}
\text{Mod-} R & \xrightarrow{\text{R-Mod}} & (R\text{-mod}, \text{Ab}) \\
N \mapsto (N \otimes_R \_)|_{R\text{-mod}} \\
\end{array}
\]

is fully faithful. If $R \not\in B$ the tensor evaluation functor in Theorem 1.1 is, in general, neither full nor faithful as Example 3.3 shows. Our proof of Theorem 1.1 uses techniques—such as tensor products of functors—different from those found in the proof of [30, thm. B.16].}