Conservation of Quantum Correlations in Multimode Systems with \( U(1) \) Symmetry

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We present a theoretical investigation of the properties of quantum correlation functions in a dissipative multimode system. We define a total \( m^{th} \) order equal-time correlation function, summed over all modes, which is shown to be conserved if the Hamiltonian possesses \( U(1) \) symmetry, provided any dissipation processes are linear in the system operators. As examples, we demonstrate this conservation using numerical simulations of a coupled cavity system and the Jaynes-Cummings model.

In this letter, we investigate theoretically the properties of number operator correlation functions in a multimode system with dissipation, where the Hamiltonian has \( U(1) \) symmetry [1], or equivalently commutes with the total number operator for all the modes in the system. We define a total \( m^{th} \) order equal-time correlation function, summed over all modes, which we show to be conserved in such a system, with certain conditions on the linearity of the dissipation processes. As examples, we demonstrate this conservation using numerical simulations of a coupled cavity system and the Jaynes-Cummings model. We show that the second order correlation functions for individual modes are not conserved, but the total correlation function is constant.

Equal time correlation functions, particularly the second order, equal time, correlation function (SOC), \( g^{(2)}(t,t) \) (often referred to as \( g^{(2)}(0) \) in a steady state), are important tools, both theoretically and experimentally for investigating the statistical properties of fluctuations in a quantum system [2]. The SOC characterizes the variance in the occupation of a mode. It is defined in such a way that for a classical field \( g^{(2)}(0) \geq 1 \), so that a value less than one is regarded as a clear signature of ‘quantum’ behavior, while the smallness of \( g^{(2)}(0) \) is an important figure of merit for a single photon source. Second, and higher order correlation functions are also relevant in a range of photon counting experiments, such as the Hanbury Brown - Twiss interferometer [3]. Understanding the dynamics of correlation functions is important in the investigation of phase transitions, entanglement and other quantum properties of optical and condensed matter systems.

In the present work, we investigate the properties of quantum correlation functions in a multimode system with dissipation. In this case there are numerous correlation functions, describing the fluctuations in each mode individually, but also the cross-correlations between different modes. We define a total, \( m^{th} \)-order, correlation function for such a system, summing over all the different modes. We show that this total correlation function is conserved if the Hamiltonian governing the dynamics exhibits \( U(1) \) symmetry and any Lindblad term, describing non-Hamiltonian dissipative effects [4], is linear in the system operators. A Hamiltonian which possesses \( U(1) \) symmetry will preserve the total number of excitations in a non-dissipative system; dissipation will, of course, mean that the number is not conserved.

The Total Correlation Function. For a multimode system, there are multiple second order quantum correlation function corresponding to the different modes which are measured. At equal times, these have the form

\[
g^{(2)}_{ij}(t,t) = \frac{\langle c_i^\dagger(t)c_j^\dagger(t)c_i(t)c_j(t) \rangle}{\langle c_i^\dagger(t)c_i(t) \rangle \langle c_j^\dagger(t)c_j(t) \rangle},
\]

where \( c_i^\dagger \) and \( c_j^\dagger \) are the creation and annihilation operators for modes \( i \) and \( j \), which may be bosonic or fermionic. If \( i = j \), this quantifies the fluctuations within a mode, otherwise it describes correlations between the fluctuations in different modes. This definition can be extended to higher order correlations: the \( m^{th} \) order function contains \( m \) creation and annihilation operators, potentially for \( m \) different modes.

We define the total \( m^{th} \)-order correlation function for a multimode system as

\[
g^{(m)}_{\text{tot}}(t,t) = \frac{\langle J \rangle}{\langle N \rangle^m},
\]

where

\[
\langle J \rangle = \langle \sum_i c_i^\dagger(t)c_i(t)^m \rangle
\]

and

\[
\langle N \rangle = \sum_i \langle c_i^\dagger(t)c_i(t) \rangle
\]

is the total number operator for the system. \( \langle \ldots \rangle \) indicates the normal ordering of the enclosed operators, and the sum extended over every mode in the system. Note that this is not the same as summing the \( g^{(m)}_{ij} \) over all the modes because every term is normalized by the \( m^{th} \) power of the total occupation of the system. However, we regard our definition as a more natural sum, because it weights the contribution of each mode according to its occupation. Furthermore, in a system where all the modes are photonic with equal external coupling, it represents the \( m^{th} \)-order correlation function which would
be obtained if the total emission were to be measured, without resolving the individual modes. In the case of a single mode and \( m = 2 \) it is identical to the standard SOC, given by Eq. (1) with \( i = j \).

Conservation of \( g(m)_{t,t} \). In order to determine the condition that the total correlation function is conserved, we equate its time derivative to zero, which gives

\[
\frac{d}{dt} \langle J \rangle \langle N \rangle^m - \left( m \langle N \rangle^{m-1} \frac{d}{dt} \langle N \rangle \right) \langle J \rangle = 0. \tag{5}
\]

Apart from the trivial solutions, \( \langle J \rangle = 0 \) or \( \langle N \rangle = 0 \), this is true if:

\[
\frac{d}{dt} \langle N \rangle = \gamma \langle N \rangle \tag{6}
\]

\[
\frac{d}{dt} \langle J \rangle = m \gamma \langle J \rangle \tag{7}
\]

where \( \gamma \) is an arbitrary constant.

The time derivative of the mean value of an operator, using the Schrödinger picture, can be expressed as:

\[
\frac{\partial}{\partial t} \langle A \rangle = \text{Tr} \left\{ \frac{\partial}{\partial t} (\rho A) \right\} = \text{Tr} \left\{ \left[ A, \frac{\partial \rho}{\partial t} \right] \right\}. \tag{8}
\]

In the presence of dissipation, the derivative of \( \rho \) is determined by Lindblad operator [5] using the master equation formalism:

\[
\frac{\partial \rho}{\partial t} = -i[H, \rho] + D(\rho) \tag{9}
\]

where

\[
D(\rho) = \sum_i \gamma_i (2F_i^\dagger \rho F_i - F_i^\dagger F_i \rho - \rho F_i^\dagger F_i) \tag{10}
\]

with \( F_i \) Lindblad operators [6], which characterize the processes generated by the interaction of the system with the environment.

From this, the conditions for the conservation of the total correlation function in Eq. (6) become

\[
-i \langle [H, N] \rangle + \sum_i \langle (2F_i^\dagger NF_i - F_i^\dagger F_iN - NF_i^\dagger F_i) \rangle = \gamma \langle N \rangle \tag{11}
\]

\[
-i \langle [H, J] \rangle + \sum_i \langle (2F_i^\dagger JF_i - F_i^\dagger F_iJ - JF_i^\dagger F_i) \rangle = \gamma m \langle J \rangle \tag{12}
\]

We consider first the case of a Lindblad that is linear in the basis of the field operators \( F_i^\dagger = c_i^\dagger \). In this case, the non-unitary terms in Eqs. (11), (12) are equal to \( \gamma N \) and \( \gamma mJ \), with \( \gamma = \sum_i \gamma_i \). Hence, for the correlation function to be conserved, the unitary part needs to be zero, i.e. \( [H, N] = 0 \) and \( [H, J] = 0 \). The normally ordered operator \( J \) can be written as a power expansion in the total number operator, \( J = \sum_k d_k N^k \), where \( d_k \) are numerical coefficients related to the order of the expansion

\[
\frac{d}{dt} \langle a^\dagger a \rangle = -i f \left( 2 \langle a^\dagger a \rangle - 2 \langle a^\dagger a^\dagger a \rangle + \langle a^\dagger \rangle - \langle a \rangle - 2 \gamma \langle a^\dagger a \rangle \right) \tag{14}
\]

which corresponds to a global \( U(1) \) symmetry of the Hamiltonian \( H \).

This result can be summarized in a general theorem:

**Theorem:** For any dissipative system with an arbitrary number of modes, described by an Hamiltonian \( H \), and with a linear dissipator \( D(\rho) \), iff \( H \) globally possess a \( U(1) \) symmetry, i.e. iff \( [H, N] = 0 \), where \( N \) is the total number operator, then the total \( m \)th order equal time correlation function \( g(m)_{t,t} \) is a conserved quantity.

We now consider briefly the case where the dissipation terms are non linear, that is \( F_i \propto c_i^\gamma \), with \( \gamma > 1 \). The non-linearity of the Lindbald implies that the conditions in Eqs. (11), (12) are not automatically satisfied when \( [H, N] = 0 \). For example, if we consider the SOC function \( (m = 2) \) for a single bosonic mode with a non-linear absorber, \( \langle a^\dagger a \rangle \) (2 [8]):

\[
D(\rho) = -\gamma (2a^\dagger^2 a^2 - a^\dagger^2 a^2 \rho - \rho a^\dagger^2 a^2), \tag{13}
\]

we find that \( dJ/dt = -4\gamma J \), and \( dN/dt = -4\gamma J \) so the derivative of \( g(2)(t, t) \) is not zero. Intuitively, this happens because a higher order Lindblad distinguishes between single and multiphoton channels, generating significant changes in the statistics of the emitted particles. It follows that in such systems it is not possible to formulate a general conservation theorem for the \( m \)th order quantum correlation functions.

In the following, we provide detailed examples, both numerical and analytical, to better explain the implications of the theorem stated above.

**Single mode quantum field.** We consider first the dynamics of the SOC function for a single mode photonic field \( (a, a^\dagger) \). This may be linear, but we note that there may also be self-interaction terms, such as a Kerr non-linearity, provided they depend only on the number operator \( a^\dagger a \). Such a Hamiltonian will commute with the number operator, so, provided the dissipator, representing loss or gain, is linear, the conditions for our theorem are satisfied and the SOC is conserved during the dynamics of the system.

This result changes when we add a classical driving term like a continuous wave (CW) resonant pump of the form \( \Omega = f(a + a^\dagger) \). In this case the Hamiltonian does not possess the \( U(1) \) symmetry, and this implies that extra terms appear in the derivative of the mean values. It can be shown that

\[
\frac{d}{dt} \langle a^\dagger a \rangle = -i f \left( 2 \langle a^\dagger a \rangle - 2 \langle a^\dagger a^\dagger a \rangle + \langle a^\dagger \rangle - \langle a \rangle - 2 \gamma \langle a^\dagger a \rangle \right) \tag{14}
\]
while the derivative of the number of particles, $\langle n \rangle$, is
\[
\frac{d}{dt} \langle n \rangle = -i \tau \langle \alpha^\dagger \alpha \rangle - \gamma \langle n \rangle.
\] (15)

The time derivative of the SOC function is thus
\[
\frac{d}{dt} g^{(2)}(t, t) = -i \frac{f}{\langle n \rangle} \left[ 2 \langle a^\dagger a \rangle - \langle a^\dagger a^\dagger a a \rangle \right] + \left( \langle a \rangle - \langle a^\dagger \rangle \right) \left( 1 - 2 \langle n \rangle g^{(2)} \right),
\] (16)
so the emitted statistics undergoes a dynamical evolution. The additional terms, arising in the derivative from the commutator, express the correlation between the classical field of the CW pump and the field describing the physical system. This quantity can be experimentally measured and, as demonstrated by Carmichael et al. [9], can be directly linked with quantum phenomena like squeezing and antibunching. However, it should be noted that in a linear system, the steady state of this equation has $g^{(2)}(t, t) = 1$, corresponding to a coherent state of the mode, so we cannot generate non-classical behaviour by classical pumping a linear oscillator.

Two modes quantum field. To underline the link between the global $U(1)$ symmetry of the Hamiltonian and the conservation of the SOC function, we next consider the case of two undriven linear bosonic modes, coherently coupled. This system can be described by a Hamiltonian of the form
\[
H_{tm} = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \tau (a_1^\dagger a_2 + a_2^\dagger a_1),
\]
where $\omega_i$ are the energies of each mode and $\tau$ is the coupling strength between modes. Again we can add nonlinear terms with no effect provided they commute with the number operators for each mode, so this discussion could equally apply to a two-mode Bose-Hubbard model.

In this case we need to evaluate the time derivative for three correlators: the two auto-correlation functions $(g^{(2)}_1, g^{(2)}_2)$ and the cross correlation function $(g^{(2)}_{1, 2} = g^{(2)}_2)$. Using the master equation approach it is easy to show the dependence of the auto-correlation, $(a_1^\dagger a_1 a_1^\dagger a_1)$, from the hopping terms:
\[
\frac{d}{dt} \langle a_1^\dagger a_1 a_1^\dagger a_1 \rangle = -i \tau (2 \langle a_1^\dagger a_1 a_1^\dagger a_2 \rangle - 2 \langle a_1^\dagger a_1 a_2 a_2 \rangle + \langle a_2^\dagger a_2 \rangle - 2 \gamma \langle a_1^\dagger a_1 \rangle) - \gamma \langle a_1^\dagger a_1 \rangle,
\] (17)
while the derivative of the number of particles for mode 1, $\langle n_1 \rangle$, is
\[
\frac{d}{dt} \langle n_1 \rangle = -i \tau \langle a_1^\dagger a_2 - a_2^\dagger a_1 \rangle - \gamma \langle n_1 \rangle,
\] (18)
with an equivalent form for mode 2. Relation (17) is analogous to the one obtained in the classically driven single mode (14), where instead of the c-number field $f$, we now have a quantum field $a_2$ as a source of the dynamical change of the correlation function for the mode.

Therefore, the autocorrelation function for mode 1 (and analogously for mode 2) is not conserved anymore. This consideration shows an important conceptual point: the origin of the change in the statistics of a system is directly related to the coupling to another system, whether it is an external pump or a coupled quantum mode.

Considering now our Theorem, we note that though the Hamiltonian does not commute with the number operator for each mode ($[H_{tm}, n_{1, 2}] \neq 0$), i.e. the $U(1)$ symmetry is locally broken, it does commute with the total number operator $N = n_1 + n_2$. This is a consequence of the global $U(1)$ symmetry of the Hamiltonian. Therefore, it can be expected that total SOC function, $g^{(2)}_{tot}$, is conserved. To show this explicitly, we evaluate the derivative of the cross correlation between the two modes, and see that it also depends on the hopping term, and is structurally identical to the derivative of the autocorrelations:
\[
\frac{d}{dt} \langle a_1^\dagger a_2 a_2^\dagger a_1 \rangle = i \tau (\langle a_1^\dagger a_1 a_2 a_2^\dagger \rangle + \langle a_1^\dagger a_2 a_2^\dagger a_1 \rangle - \langle a_1^\dagger a_2 a_2^\dagger a_1 \rangle - \langle a_1^\dagger a_1 a_2 a_2^\dagger \rangle - \gamma (\langle a_1^\dagger a_1 a_2 a_2^\dagger \rangle - \langle a_1^\dagger a_1 a_2 a_2^\dagger \rangle).
\] (19)
Substituting the above expression, together with the derivative of the total number of excitation in (6), it can be verified that $g^{(2)}_{tot}$ is conserved.

We now demonstrate our analytical results numerically, using the positive-P representation developed by Drummond et al. [10]. This allows us to transform the master equation, Eq. (9), into a set of stochastic differential equations which are solved using Monte-Carlo methods. We consider first a system of two interacting bosonic
modes described by the Hamiltonian $H_{nm}$, with the addition of on-site non-linearities $g(a_i^2 a_j^2 + a_i^+ a_j^2)$, in a dissipative regime and prepared in an initial coherent state, $|\psi(0)\rangle = |\alpha_1, \alpha_2\rangle$, with $|\alpha_1|^2$ particles in one mode and $|\alpha_2|^2$ particles in the other mode. We evaluate the normalized second order correlation function for each mode and for the cross-correlations. While the particles move between cavities and disappear due to dissipation (FIG.1-Top), the quantities $G^{(2)}_{i,j} = \langle a_i^+ a_i a_j a_j \rangle / (n_i + n_j)^2$, describing the correlation functions normalized to the total number of particles, undergo a dynamical evolution. However, it can be clearly seen that the total correlation function is constant in the whole time interval, as our theorem requires.

By increasing the number of modes, the above argument can easily encompass the 1D Bose-Hubbard Hamiltonian. Moreover, it can be also extended to bosonic quantum networks [11], which can describe a huge variety of physical systems, like photonic or polaritonic lattices (Lieb [12], Kagome [13] and Graphene [14]), quantum networks and collective phenomena involving parametric processes (like parametric up/down conversion). The correlation function for each mode of the network changes dynamically, as a consequence of the interaction with other modes and this change is dependent on the structure of the network itself, and on the strength of the coupling terms ($\tau$ in the two mode case). This leads to the possibility of controlling the statistics of the emitted photons by engineering the geometry of the quantum networks, as it has been shown for photonic molecules in [15].

Jaynes-Cummings Hamiltonian. It is also interesting to consider the case of mixed bosonic and fermionic systems, for example, the Jaynes-Cummings-Hubbard model [16], the spin-boson network model [17], and light-matter coupled systems [18]. The simplest case is the Jaynes Cummings model, for a single-mode cavity, containing one two-level atom:

$$
H_{JC} = \omega_0 a^+ a + \omega_a \sigma^+ \sigma^- + \eta(a^+ \sigma^- + a \sigma^+),
$$

(20)

where $\omega_0$ and $\omega_a$ are the energies of the mode and atom, $\eta$ is the vacuum Rabi frequency that characterizes the photon-atom interaction strength and $\sigma^\pm$ are the atomic raising and lowering operators. We have written this Hamiltonian using the rotating wave approximation (RWA), since it then commutes with the total number of excitations, $[H_{JC}, N] = 0$, where $N = n_{fm} + n_{bs}$, $n_{fm} = \sigma^+ \sigma^-$ is the number operator for fermions, and $n_{bs} = a^+ a$ is the number operator for bosons. In Fig.2 we show numerical calculations of the dynamics of the Jaynes-Cummings model described by $H_{JC}$, preparing the system in an initial state $|\psi(0)\rangle = |\alpha_0, e\rangle$, with $|\alpha_0|^2$ particles in the field mode, and the two level system in the excited state $|e\rangle$. As expected, we find that the atomic and the field correlations change in time, while $g^{(2)}_{tot}(t, t)$ remains constant. However, each component changes dynamically since the $U(1)$ symmetry is locally broken, $[H_{JC}, n_{fm}] \neq 0$ and $[H_{JC}, n_{bs}] \neq 0$. In this configuration, as in the coupled mode case, it is sufficient to break the symmetry with respect to the total number operator to lose the stationarity, for example having a coherent driving term like $\Omega_\tau(t) = h(t)(\sigma^+ + \sigma^-)$ or $\Omega_\tau(t) = h(t)(a^+ + a)$ [19]. We note here that the Jaynes Cummings model beyond the rotating wave approximation contains terms that break the $U(1)$ symmetry, as a spontaneous creation and destruction of particles occurs in the system. This generates a dynamics of the total SOC functions. Similar behavior should be seen in every system which breaks a $U(1)$ symmetry while undergoing a phase transition, for example, in the case of Hubbard Hamiltonians inside a cavity [20].

Continuum limit. Until now we have always considered a discrete collection of quantum systems. However, our previous Theorem is implicitly valid also in the case of a system with a continuum of modes, for example, the optical field in a waveguide. If the Hamiltonian commutes with the total number operator [21],

$$
N = \int dx a^+ (x) a(x),
$$

(21)

then the total SOC,

$$
g_{tot}^{(2)} = \frac{1}{N^2} \int_{-\infty}^{+\infty} dx dx' a^+ (x) a^+ (x') a(x) a(x')
$$

(22)

is stationary. This is true even when individual point-like components, $g^{(2)}(x, x')$, experience temporal dynamics. In order to induce a change in the statistics of the total emission, it is sufficient to break the commutation with the number operator with a driving term, either point-like or with a spatial profile.
Conclusion. In conclusion we have shown that the $m^{th}$ order quantum correlation function is a constant of motion, for systems where the Hamiltonian possess global $U(1)$ symmetry and any dissipation is linear in the system operators. For a composite system, the $m^{th}$ order quantum correlation function dynamically changes in each of its subsystem, due to their mutual interactions, since the $U(1)$ symmetry is locally broken. However, the statistics of the total collected light is still a constant of motion. Our theorem suggests that the total correlation function may be an interesting parameter to measure in systems which undergo phase transitions characterized by the breaking of a global $U(1)$ symmetry. It also demonstrates that care is necessary when using the SOC as a probe for non-classical physics: in systems with $U(1)$ symmetry, it may be necessary to isolate light from individual modes, rather than looking at the statistics of the total emitted light.

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