HIGHER DIMENSIONAL FORMAL ORBIFOLDS AND ORBIFOLD
BUNDLES IN POSITIVE CHARACTERISTIC

INDRANIL BISWAS, MANISH KUMAR, AND A. J. PARAMESWARAN

ABSTRACT. In [5], the last two authors introduced formal orbifold curves defined over an
algebraically closed field of positive characteristics. They studied both étale and Nori fun-
damental group schemes associated to such objects. Our aim here is to study the higher
dimensional analog of these objects objects and their fundamental groups.

1. Introduction

Given a quasiprojective variety $X$ defined over an algebraically closed field $k$ of positive
characteristic, and a base point $x_0 \in X$, the Nori fundamental group $\pi^N(X, x_0)$ is defined
using the torsors on $X$ for finite $k$–group schemes. This construction gives the étale funda-
mental group of $X$ if we restrict to the reduced group schemes. When $X$ is complete,
$\pi^N(X, x_0)$ has a Tannakian description using the essentially finite bundles on $X$ introduced
in [7]. The homomorphisms between the fundamental groups induced by étale morphisms of
varieties are well understood. The paper [5], which is a predecessor of the present work, origin-
ated from attempts to understand the homomorphisms between the fundamental groups
induced by ramified maps between curves.

We quickly recall the aspects of [5] that connect it to the present work. Given a finite
morphism $f : X \rightarrow Y$ between curves, consider all finite morphisms $g : Z \rightarrow Y$ that
are locally dominated by $f$. This will form an inverse system, and by taking corresponding
Galois extensions, it is possible define a group obtained by the inverse limits. This is made
precise by introducing a branch data on $Y$ and a condition in terms of this branch data
is imposed on these coverings of $Y$. Those branch data coming from a global finite map
are referred to as geometrical branch data. In [5], a class of bundles on those coverings are
defined, and it is shown there that they form a tensor abelian category; the Tannakian dual
of this tensor abelian category is called orbifold fundamental group with respect the orbifold
structure defined by the branch data.

Here we consider the questions addressed in [5] in the set-up of higher dimensional varieties.
We recall the definition of formal orbifolds $(X, P)$, where $X$ is a normal proper variety defined
over $k$, and $P$ is a branch data on $X$ (see Section 2). Associated to $(X, P)$ is a tensor abelian
category $\text{Vect}^f(X, P)$ (see [21]). It is defined by taking equivariant essentially finite bundles
on suitable ramified Galois coverings $Y$ of $X$ whose ramifications are are controlled by $P$.

After fixing a base point $x \in X$ outside $P$, this tensor abelian category produces a proalgebraic group scheme which is denoted by $\pi^N((X, P), x)$. Our first theorem is the
following (see Theorem 3.1):

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category.
Theorem 1.1. Let \( f : (Y, O) \rightarrow (X, P) \) be an étale \( \Gamma \)-Galois cover of projective formal orbifolds. Take a point \( y \in Y \). There is a natural exact sequence
\[
1 \rightarrow \pi^N(Y, y) \xrightarrow{i} \pi^N((X, P), f(y)) \xrightarrow{q} \Gamma \rightarrow 1.
\]

Let \( X^o \) be an open dense subset of a normal projective variety, we define (3.3) a fundamental group scheme \( \pi_n((X, P), x) \) as inverse limit of \( \pi^N(X, P) \) where the limit is taken over branch data \( P \) whose branch locus is disjoint from \( X^o \). We observe that \( \pi_n((X, P), x) \) is a quotient of the Nori fundamental group \( \pi^N(X, P) \) (Proposition 3.4). We also show that \( \pi_n((X, P), x) \) classifies finite group scheme torsors over \( X^o \) which étale locally extends to \( X \) (Proposition 3.5).

Our second theorem (Theorem 4.1) identified the kernel of the natural homomorphism \( \pi^N((X, P), x) \rightarrow \pi^\text{et}_1((X, P), x) \).

2. Formal orbifold and Orbifold bundles

Let \( X \) be a normal variety \( X \) defined over a perfect field \( k \). We recall from [6, Section 3] the definition of a branch data on \( X \). Let \( x \in X \) be a point of codimension at least one, and let \( U \) be an affine open connected neighborhood of \( x \); we note that \( U \) is integral because \( X \) is normal. Again normality of \( X \) implies that the completion \( \widehat{\mathcal{O}_X(U)} \) of the coordinate ring \( \mathcal{O}_X(U) \) along \( x \) is an integral domain. Let \( \mathcal{K}_X(U) \) denote the field of fractions for \( \widehat{\mathcal{O}_X(U)} \). The fraction field of \( \widehat{\mathcal{O}_X,x} \) will be denoted by \( \mathcal{K}_X,x \).

A quasi branch data \( P \) on \( X \) assigns to every such pair \((x, U)\) a finite Galois extension of \( \mathcal{K}_X(U) \) in a fixed algebraic closure of \( \mathcal{K}_X(U) \), which is denoted by \( P(x, U) \), such that the following compatibility conditions hold:

1. \( P(x_1, U) = P(x_2, U)\mathcal{K}_X(x_2)(U) \), where \( x_1 \in \{ x_2 \} \), and \( U \) is an affine open connected neighborhood of \( x_1 \) and \( x_2 \).
2. For \( x \in V \subset U \subset X \), with \( U \) and \( V \) affine open connected subsets, we have \( P(x, V) = P(x, U)\mathcal{K}_X(V) \).

Define \( P(x) := P(x, U)\mathcal{K}_X,x \); note that \( P(x) \) is independent of the choice of \( U \). Also, define
\[
\text{BL}(P) := \{ x \in X \mid \widehat{\mathcal{O}_X,x} \text{ is branched in } \mathcal{K}_X,x \}.
\]
A quasi branch data \( P \) is called a branch data if \( \text{BL}(P) \) is a closed subset of \( X \) of codimension at least one. This \( \text{BL}(P) \) is called the branch locus of \( P \).

Note that if \( \dim X = 1 \), then \( P(x, U) = P(x) \) (i.e., it is independent of \( U \)), and hence it agrees with the notion in [5].

The branch data in which all the Galois extensions are trivial is called the trivial branch data, and it is denoted by \( O \). For a finite morphism \( f : Y \rightarrow X \) of normal varieties, the natural branch data associated to \( f \) will be denoted by \( B_f \).

We recall the definition of formal orbifolds from [6]. As before, \( k \) is a perfect field. A formal orbifold over \( k \) is a pair \((X, P)\), where \( X \) is a normal finite type scheme over \( k \) and \( P \) is a branch data on \( X \).
A morphism of formal orbifolds \( f: (Y, Q) \to (X, P) \) is a quasi-finite dominant separable morphism \( f: Y \to X \) such that for all points \( y \in Y \) of codimension at least one and some affine open neighborhood \( U \) of \( f(y) \), we have
\[
Q(y, f^{-1}(U)) \supset P(f(y), U).
\]
It is said to be étale if the extension \( Q(y)/P(f(y)) \) is unramified, for all \( y \in Y \) of codimension at least one. Moreover, \( f \) is called a covering morphism (or simply a covering) if it is also proper.

A formal orbifold \((X, P)\) is called geometric if there exist an étale cover \((Y, O) \to (X, P)\) and in this case \( P \) is called a geometric branch data [5].

Let \((Y, O) \to (X, P)\) be an étale \( \Gamma \)-Galois covering of formal orbifolds. Like in [5], we define vector bundles on \((X, P)\) as the \( \Gamma \)-equivariant bundles on \( Y \), while morphisms between two vector bundles on \((X, P)\) are defined to be the \( \Gamma \)-equivariant homomorphisms between the corresponding \( \Gamma \)-bundles on \( Y \). For the case of curves, it was shown in [5] that this definition does not depend on the choice of the étale cover. The key point is that if \((Y_i, O) \to (X, P)\) are étale \( \Gamma_i \)-covers for \( i = 1, 2 \), then take an étale \( \Gamma \)-cover \((Y, O) \to (X, P)\) that dominates these two covers (for instance \( Y \) can be the normalized fiber product of \( Y_1 \) and \( Y_2 \)). It follows that \( Y \to Y_i \) are Galois étale covers, and then using Galois descent it is shown that the pullback functor defines an equivalence of category of \( \Gamma \)-bundles of \( Y \) and the category of \( \Gamma_i \)-bundles on \( Y_i \). (See [5, Lemma 3.3, Lemma 3.4, Proposition 3.6] for the proof.) It should be clarified that the proofs of these results in [5] do not use the hypothesis in [5] that \( Y \) is a curve.

Now assume the base field \( k \) to be algebraically closed. Let \( X \) be a smooth proper variety over \( k \). A vector bundle on \((X, P)\) is called stable (respectively, semi-stable) if the corresponding \( \Gamma \)-equivariant bundle on \( Y \) is equivariantly stable (respectively, equivariantly semistable). However, an equivariant vector bundle is equivariantly semistable if and only if the underlying vector bundle is semistable. Similarly, a vector bundle on \((X, P)\) is called essentially finite if the corresponding \( \Gamma \)-equivariant bundle on \( Y \) is essentially finite. However, an equivariant vector bundle is equivariantly

The tensor product and duals of vector bundles on \((X, P)\) are defined in the usual way. This makes the category
\[
\text{Vect}^\Gamma(X, P)
\]
of essentially finite bundles a Tannakian category, and any closed point \( x \in X \) outside support of \( P \) defines a fiber functor from \( \text{Vect}^\Gamma(X, P) \) to the category of \( k \)-vector spaces. Hence we define \( \pi^N((X, P), x) \) to be the automorphism of this fiber functor. Note that if \( P \) is the trivial branch data, then \( \pi^N((X, P), x) \) is the fundamental group \( \pi^N(X, x) \) corresponding to the essentially finite bundles [7], [8] (its definition is recalled in Section 3).

### 3. Basic properties of \( \pi^N(X, P) \)

Let \( f: (Y, O) \to (X, P) \) be an étale \( \Gamma \)-Galois cover of projective formal orbifolds.

**Theorem 3.1.** Take a point \( y \in Y \). There is a natural exact sequence
\[
1 \to \pi^N(Y, y) \to i^* \pi^N((X, P), f(y)) \to \Gamma \to 1.
\]
Proof. Let $E$ be a $\pi^N((X, P), y)$–module, meaning it is an essentially finite vector bundle on $(X, P)$. So $E$ is also an essentially finite vector bundle on $Y$. Hence we have a homomorphism

$$i : \pi^N(Y) \longrightarrow \pi^N(X, P)$$

(3.1)

(the base point is suppressed). We note that any essentially finite vector bundle $F$ on $Y$ is a sub-bundle of the $\Gamma$–equivariant bundle $\bigoplus_{\gamma \in \Gamma} \gamma^*F$ on $Y$; this direct sum $\bigoplus_{\gamma \in \Gamma} \gamma^*F$ is essentially finite because $F$ is so. Consequently, $\bigoplus_{\gamma \in \Gamma} \gamma^*F$ is an essentially finite vector bundle on $(X, P)$. Hence the homomorphism $i$ in (3.1) is a closed immersion [2, p. 139, Proposition 2.21(b)].

Given a $\Gamma$–module $V$, we have the $\Gamma$–equivariant vector bundle

$$Y(V) := Y \times V \longrightarrow Y;$$

here $\Gamma$ acts diagonally on $Y \times V$ using its actions on $Y$ and $V$. Since $Y(V)$ is essentially finite, it defines an essentially finite bundle on $(X, P)$. This construction produces a homomorphism

$$q : \pi^N((X, P)) \longrightarrow \Gamma$$

(the base point is suppressed). This $q$ is surjective because the above functor from $\Gamma$–modules to $\text{Vect}^f(X, P)$ (defined in (2.1)) is fully faithful [2, p. 139, Proposition 2.21(a)].

The composition $q \circ i$ is evidently trivial, because the vector bundle underlying the $\Gamma$–equivariant bundle $Y(V)$ is trivial.

The inclusion homomorphism $\ker(q) \hookrightarrow \pi^N((X, P))$ corresponds to the forgetful functor that simply forgets the $\Gamma$–action on a $\Gamma$–equivariant vector bundle on $Y$. From this it follows that $\ker(q) = \text{image}(i)$. This completes the proof. $\square$

Let $k$ be an algebraically closed field of positive characteristic. Take a reduced and connected $k$–scheme $X$, and fix a rational point $x \in X$. We recall from [8] the construction of a profinite group-scheme over $k$ associated to the pair $(X, x)$. Consider all quadruples of the form $(G, f, y)$, where

- $G$ is a finite group-scheme defined over $k$,
- $f : Y \longrightarrow X$ is a $G$–torsor, and
- $y \in Y$ is a rational point such that $f(y) = x$.

A morphism $(G, f, y) \longrightarrow (G', f', y')$ between two such quadruples is a pair of the form $(\rho, \varphi)$, where $\rho : G \longrightarrow G'$ is a homomorphism of $k$–group-schemes and $\varphi : Y \longrightarrow Y'$ is a morphism, such that

- $f' \circ \varphi = f$,
- $\varphi(y) = y'$,
- the morphism $\varphi$ is $G$–equivariant, for the action of $G$ on the $G'$–torsor $Y'$ given by $\rho$.

Let $N(X, x)$ denote the category constructed using these quadruples and morphisms between them.

The category $N(X, x)$ forms an inverse system. Nori proved that the inverse limit

$$\lim \limits_{\longrightarrow} \frac{\text{lim}}{N(X, x)} G$$
exists as a profinite group-scheme over $k$ [8, Chapter 2, Proposition 2]. This inverse limit will be denoted by $\pi^N(X, x)$. When $X$ is a projective variety, this profinite group-scheme $\pi^N(X, x)$ coincides with the Tannaka dual of the category of essentially finite vector bundles on $X$ [8, Chapter 1, Proposition 3.11].

**Definition 3.2.** Let $X^o$ be an open dense subset of a normal projective variety $X$. Let $G$ be a finite group scheme, and let $Z^o \rightarrow X^o$ be a $G$–torsor. We say that this $G$–torsor $Z^o$ étale locally extends to $X$ if there exist a connected étale cover

$$\phi : U \rightarrow X^o$$

such that the $G$–torsor $\phi^*Z^o$ extends to the normalization of $X$ in the function field $k(U)$.

**Definition 3.3.** Let $X^o \subset X$ be a dense open subset. Define

$$\pi^n(X^o, x) := \lim_{\text{BL}(P) \cap X^o = \emptyset} \pi^N((X, P), x)$$

to be the inverse limit.

**Proposition 3.4.** As before, $X^o \subset X$ is a dense open subset. There is a natural homomorphism $\pi^N(X^o, x) \rightarrow \pi^n(X^o, x)$, which is surjective.

**Proof.** Let $G$ be a finite group scheme and $f : \pi^n(X^o) \rightarrow G$ a surjection. Then $f$ defines a functor $\text{Rep}(G) \rightarrow \text{Rep}(\pi^N(X, P))$ for some geometric branch data $P$ on $X$ such that $\text{BL}(P) \cap X^o = \emptyset$. But $\text{Rep}(\pi^N((X, P)))$ is same as $\text{Vect}^f(Y, P)$ which is same as $\text{Vect}^f(Y)$, where

$$f : (Y, O) \rightarrow (X, P)$$

is an étale $\Gamma$–cover. By Nori’s result a functor $\text{Rep}(G) \rightarrow \text{Vect}^f(Y)$ defines a $G$–torsor $W$ on $Y$ which is $\Gamma$–equivariant. Set $Y^o := f^{-1}(X^o)$, and let $W^o$ be the preimage of $Y^o$ in $W$. This $W^o$ is a $G$–torsor on $Y^o$ which is $\Gamma$–equivariant. But $Y^o \rightarrow X^o$ is an étale $\Gamma$–cover, and hence $W^o$ descends to a $G$–torsor on $X^o$. Therefore, it defines a surjection $\pi^N(X^o) \rightarrow G$. This construction is compatible with epimorphism of finite group schemes, and $\pi^n(X^o)$ is the inverse limit of its finite group scheme quotients. Consequently, this construction gives a surjection from $\pi^N(X^o) \rightarrow \pi^n(X^o)$. \qed

**Proposition 3.5.** Let $G$ be a finite group scheme, and let $Z^o \rightarrow X^o$ be a $G$–torsor which étale locally extends to $X$. Then $G$ is a quotient of $\pi^n(X^o)$. Conversely, given a surjection $\pi^n((X, P)) \rightarrow G$, where $P$ is such that $\text{BL}(P) \cap X^o = \emptyset$ and $G$ is a finite group scheme, the associated $G$–torsor $Z^o \rightarrow X^o$ étale locally extends to $X$.

**Proof.** Let $Y^o \rightarrow X^o$ be a connected étale cover such that the pullback of the $G$–torsor $Z^o$ to $Y^o$ extends to the normalization of $X$ in $k(Y^o)$ (the unique normal proper model of $Y^o$ finite over $X$). By passing to the Galois closure, we may assume that $f : Y \rightarrow X$ is a Galois cover; the Galois group for $f$ will be denoted by $\Gamma$. Let $P$ be the branch data on $X$ associated to $f$, i.e., $P = B_f$ in the notation of [5]. Then $f : (Y, O) \rightarrow (X, P)$ is an étale $\Gamma$–cover. Also, the pull back of the $G$–torsor $Z^o \rightarrow X^o$ to $Y^o$ and its extension to $Y$ is a $\Gamma$–equivariant $G$–torsor. Now a representation $V$ of $G$ induces an essentially finite $\Gamma$–equivariant
bundle $V$ on $Y$. The Tannaka subcategory generated by $V$ in the Tannaka category of $\Gamma$-equivariant essentially finite bundles on $Y$ induces a surjection $\pi^N((X, P)) \rightarrow G$. Hence we get a surjection $\pi^n(X^o) \rightarrow G$.

For the converse, first note that since $G$ is a finite group scheme, a surjection $\pi^n(X^o)$ factors through $\pi_N((X, P))$ for some branch data $P$ such that $BL(P) \cap X^o = \emptyset$.

Let $f : (Y, O) \rightarrow (X, P)$ be an étale $\Gamma$-Galois cover of formal orbifolds. The surjection $\pi^N((X, P)) \rightarrow G$ by Tannaka formalism yields a finite collection $S$ of essentially finite $\Gamma$-equivariant bundle on $Y$ such that the Tannaka dual of the Tannaka subcategory generated by $S$ is $G$. This by an equivariant version of Nori’s reconstruction, [1, Section 2], yields a $\Gamma$-equivariant $G$-torsor on $Y$. This torsor restricts to a $\Gamma$-equivariant $G$-torsor on $Y^o = f^{-1}(X^o)$. But $Y^o \rightarrow X^o$ is an étale $\Gamma$ cover. Hence by Galois descent we get a $G$-torsor on $X^o$ and by construction it étale locally extends to $X$. □

Let $P$ and $Q$ be two branch data on a normal variety $X$. We say that $P \geq Q$ if for all points $x \in X$ of codimension at least one and for every affine connected open neighborhood $U$ of $x$,

$$P(x, U) \supset Q(x, U).$$

**Proposition 3.6.** Let $X$ be a smooth projective variety over $k$, and let $P \geq Q$ be two geometric branch data on $X$. There is a fully faithful functor

$$\text{Vect}^f(X, Q) \rightarrow \text{Vect}^f(X, P)$$

that makes $\text{Vect}^f(X, Q)$ into a Tannakian subcategory of $\text{Vect}^f(X, P)$. In particular, this functor induces an epimorphism $\pi^N((X, P)) \rightarrow \pi^N((X, Q))$.

**Proof.** This is proved in [5, Theorem 3.7]. We note that although [5, Theorem 3.7] is stated for curves, its proof works, without any change, for all dimensions. □

4. THE KERNEL OF PROJECTION FROM $\pi^N((X, P))$ TO $\pi^N_1((X, P))$

Let $\mathcal{X} = (X, P)$ be a proper formal orbifold, and let $\text{Vect}^f(\mathcal{X})$ be the Tannakian category of essentially finite vector bundles on $\mathcal{X}$. We now define a new category $\text{Vect}^f_1(\mathcal{X})$.

An object of this category is a pair $\{f : (Y, Q) \rightarrow (X, P), V\}$, where $f$ is étale and $V$ is an object of $\text{Vect}^f(Y, Q)$ (i.e., $V$ is an essentially finite vector bundle on $(Y, Q)$). Let $\{f_i : (Y_i, Q_i) \rightarrow (X, P), V_i\}$, $i = 1, 2$, be two objects, and let $f : (Y, Q) \rightarrow (X, P)$ be any étale morphism dominating $f_1$ and $f_2$; let $g_i : Y \rightarrow Y_i$ be the morphisms through which $f$ factors. Define

$$\text{Hom}((f_1, V_1), (f_2, V_2)) := \lim_{f : (Y, Q) \rightarrow (X, P)} \text{Hom}_{\text{Vect}^f_1(Y, Q)}(g_1^* V_1, g_2^* V_2);$$
here the limit is over all étale morphisms $f$ dominating $f_1$ and $f_2$. Since $\text{Vect}^f(Y, Q)$ is an abelian category for any proper formal orbifold $(Y, Q)$, the category $\text{Vect}_{\text{et}}^f(\mathcal{X})$ is also abelian.

The tensor product $(f_1, V_1) \otimes (f_2, V_2)$ is defined as follows: let $f : (Y, Q) \to (X, P)$ be a dominating connected component of the fiber product of $f_1$ and $f_2$, and let $p_1$ and $p_2$ be the natural projection morphisms from this fiber product. Then

$$(f_1, V_1) \otimes (f_2, V_2) = (f, p_1^*V_1 \otimes p_2^*V_2).$$

The dual of $(f_1, V_1)$ is $(f_1, V_1^\vee)$. So $\text{Vect}_{\text{et}}^f(\mathcal{X})$ is a rigid tensor abelian category.

Let $x$ be a closed point of the complement $X \setminus \text{BL}(P)$. Let $\tilde{x}$ be a point of the universal cover $\tilde{\mathcal{X}}$ of $\mathcal{X}$; this means that for every finite étale connected cover $(Y, Q) \to (X, P)$ we choose a point in $x$ over $\tilde{x}$ in a compatible way. The point $\tilde{x}$ defines a fiber functor $\mathcal{F}_{\tilde{x},\mathcal{X}}$ from $\text{Vect}^f(\mathcal{X})$ to the category of vector spaces $\text{Vect}_k$ by sending $\{f : (Y, Q) \to (X, P), V\}$ to the stalk of $V$ at the image of $\tilde{x}$ in $Y$. This makes $\text{Vect}_{\text{et}}^f(\mathcal{X})$ into a Tannakian category. Let corresponding proalgebraic group scheme will be denoted by $S(X, P)$.

**Theorem 4.1.** Let $\mathcal{X} = (X, P)$ be a projective smooth formal orbifold. The dual group of the Tannakian category $(\text{Vect}_{\text{et}}^f(X, P), \mathcal{F}_{\tilde{x},\mathcal{X}})$ is the kernel

$$K(X, P) := \ker(\pi^N(\mathcal{X}, x) \to \pi^N(x, x)).$$

**Proof.** Let $S(X, P)$ denote the Tannaka dual of the category $(\text{Vect}_{\text{et}}^f(X, P), \mathcal{F}_{\tilde{x},\mathcal{X}})$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a finite connected étale cover with $y \in \mathcal{Y}$ being the image of $\tilde{x}$. Note that there is a natural functor of Tannakian categories

$$I_{\mathcal{Y}} : \text{Vect}^f(\mathcal{Y}) \to \text{Vect}_{\text{et}}^f(\mathcal{X})$$

that sends an essentially finite vector bundle $V$ on $\mathcal{Y}$ to $(f, V)$. This functor is a full embedding. Note that $\mathcal{F}_{\tilde{x},\mathcal{Y}} := \mathcal{F}_{\tilde{x},\mathcal{X}} \circ I_{\mathcal{Y}}$ is a fiber functor from $\text{Vect}^f(\mathcal{Y})$ to the category $\text{Vect}_k$ of $k$-vector spaces. The functor $I_{\mathcal{Y}}$ in (4.1) induces a homomorphism of the duals

$$S(X, P) \to \pi^N(\mathcal{Y}, y).$$

Also the pullback $f^*$ defines a functor $\text{Vect}^f(\mathcal{X}) \to \text{Vect}^f(\mathcal{Y})$, and we have an isomorphism of the functors $I_{\mathcal{X}}$ and $I_{\mathcal{Y}} \circ f^*$. Hence the homomorphism $S(X, P) \to \pi^N(\mathcal{X}, x)$, constructed using the homomorphisms in (4.2), factors through $S(X, P) \to \pi^N(\mathcal{Y}, y)$ for every finite étale cover $\mathcal{Y} \to \mathcal{X}$. Consequently, the image of $S(X, P)$ in $\pi^N(\mathcal{X}, x)$ lies in $K(X, P)$.

Let $\{f : \mathcal{Y} \to \mathcal{X}, V\}$ be an object of $\text{Vect}^f_{\text{et}}(\mathcal{X})$. Then $V$ embeds into $f^*f_*V$. Also for a vector bundle $W$ on $\mathcal{X}$ the objects $\{f : \mathcal{Y} \to \mathcal{X}, f^*W\}$ and $\{\text{id} : \mathcal{X} \to \mathcal{X}, W\}$ are isomorphic. Hence $\{f : \mathcal{Y} \to \mathcal{X}, V\}$ is a subobject of $I_{\mathcal{X}}(V)$. So an automorphism of $\mathcal{F}_{\tilde{x},\mathcal{X}}$ which restricts to identity automorphism on the category $\text{Vect}^f(\mathcal{X})$ must be identity. Hence the induced homomorphism $S(X, P) \to K(X, P)$ is injective.

Let $\Phi$ be an automorphism of the fiber functor $F_x : \text{Vect}^f(\mathcal{X}) \to \text{Vect}_k$ such that its image in $\pi^N_{\text{et}}(\mathcal{X}, x)$ is trivial. So $\Phi \in \pi^N(\mathcal{Y}, y)$ for every étale connected covering $\mathcal{Y} \to \mathcal{X}$ and any point $y \in Y$ lying above $x$. Therefore, $\Phi$ is an automorphism of the fiber functor $F_y : \text{Vect}^f(\mathcal{Y}) \to \text{Vect}_k$. 

Let \( O := \{ f : Y \to X, V \} \) be an object of \( \text{Vect}_l^f(X) \); define a map \( \tilde{\Phi} \) from \( \mathcal{F}_x(X, Q) \) to itself to be the map \( \Phi \) from \( F_y(V) \) to itself. Note that \( \tilde{\Phi} \) defines an automorphism of the fiber functor \( \mathcal{F}_x(X) \) whose restriction to \( F_x \) is \( \Phi \). Hence the natural map \( S(X, P) \to K(X, P) \) is also surjective and so it, being injective also, is an isomorphism.

**Corollary 4.2.** Let \( X \) be a projective normal variety, and let \( P \geq Q \) be two geometric branch data on \( X \). Then we have the following morphism of exact sequences in which all the vertical arrows are surjective:

\[
\begin{array}{c}
1 \longrightarrow K(X, P) \longrightarrow \pi^N((X, P)) \longrightarrow \pi^e_1((X, P)) \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow K(X, Q) \longrightarrow \pi^N((X, Q)) \longrightarrow \pi^e_1((X, Q)) \longrightarrow 1
\end{array}
\]

**Proof.** The surjectivity of the third arrow follows from \([6]\). The surjectivity of the first (respectively, second) arrow follows from the observation that \( \text{Vect}_l^f(X, Q) \) (respectively, \( \text{Vect}_l^e(X, Q) \)) is a fully faithful subcategory of \( \text{Vect}_l^f(X, P) \) (respectively, \( \text{Vect}_l^e(X, P) \)). \(\square\)

**Corollary 4.3.** Let \( X \) be a projective normal variety and \( X^o \) be an open subset of \( X \) such that \( X \setminus X^o \) is a normal crossing divisor. Then we have the following morphism of exact sequences in which all the vertical arrows are surjective:

\[
\begin{array}{c}
1 \longrightarrow K^o \longrightarrow \pi^a(X^o) \longrightarrow \pi^e_1(X^o) \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow K \longrightarrow \pi^N(X) \longrightarrow \pi^e_1(X) \longrightarrow 1
\end{array}
\]

**Proof.** This follows from Corollary 4.2 by taking \( Q \) to be the trivial branch data and taking the inverse limit over the branch data \( P \) whose branch locus lie in \( X \setminus X^o \). \(\square\)

**Remark 4.4.** Let \( K^t \) be the kernel of the epimorphism \( \pi^N(X^o) \to \pi^a_1(X^o) \). It is not clear to us whether \( K^t \) is trivial. But the image of \( K^t \) in \( K \) under the natural homomorphism induced from \( \pi^N(X^o) \to \pi^N(X) \) is trivial, since \( \pi^N(X^o) \to \pi^N(X) \) factors through \( \pi^a_1(X^o) \).

**Example 4.5.** Let \( X = \mathbb{P}^1, Q = O, P \) to be tame ramification at four points of \( \mathbb{P}^1 \) of order 2 (i.e., characteristic \( p \neq 2 \)). Let \( E \to X \) be a \( \mathbb{Z}/2\mathbb{Z} \)-cover by an elliptic curve of \( X = \mathbb{P}^1 \). Let \( V \) be a non-trivial Frobenius-trivial \( \mathbb{Z}/2\mathbb{Z} \)-equivariant bundle on the elliptic curve. This can be constructed by starting with a non-trivial Frobenius-trivial bundle \( L \) on \( E \) (for instance take the bundle associated to \( \mu_p \) torsor which arises from the kernel of the Frobenius morphism). Let \( V = L \oplus g^*L \) where \( g \in \mathbb{Z}/2\mathbb{Z} \) is the nontrivial element. This shows that \( K(\mathbb{P}^1, P) \) is non-trivial but \( K(\mathbb{P}^1, Q) \) is trivial (as \( \pi^N(\mathbb{P}^1, Q) = \pi^N(\mathbb{P}^1) \) is trivial). Hence \( K(X, P) \to K(X, Q) \) is not an isomorphism. In particular, the map \( K^o \to K \) in the above corollary need not be an isomorphism. This also demonstrates that \( \pi^N((X, P)) \not\cong \pi^N(X) \times \pi^e_1(X) \). \(\pi^e_1((X, P)) \) in general.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, 
Bombay 400005, India

Email address: indranil@math.tifr.res.in

Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India

Email address: manish@isibang.ac.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, 
Bombay 400005, India

Email address: param@math.tifr.res.in