Poisson reduction

Juan-Pablo Ortega$^1$ and Tudor S. Ratiu$^2$

Abstract

This encyclopedia article briefly reviews without proofs some of the main results in Poisson reduction. The article recalls most the necessary prerequisites to understand the main results.

The Poisson reduction techniques allow the construction of new Poisson structures out of a given one by combination of two operations: restriction to submanifolds that satisfy certain compatibility assumptions and passage to a quotient space where certain degeneracies have been eliminated. For certain kinds of reduction it is necessary to pass first to a submanifold and then take a quotient. Before making this more explicit we introduce the notations that will be used in this article. All manifolds in this article are finite dimensional.

Poisson manifolds. A Poisson manifold is a pair $(M, \{\cdot, \cdot\})$, where $M$ is a manifold and $\{\cdot, \cdot\}$ is a bilinear operation on $C^\infty(M)$ such that $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra and $\{\cdot, \cdot\}$ is a derivation (that is, the Leibniz identity holds) in each argument. The pair $(C^\infty(M), \{\cdot, \cdot\})$ is also called a Poisson algebra. The functions in the center $\mathcal{C}(M)$ of the Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$ are called Casimir functions. From the natural isomorphism between derivations on $\mathcal{C}(M)$ and vector fields on $M$, it follows that each $h \in C^\infty(M)$ induces a vector field on $M$ via the expression $X_h = \{\cdot, h\}$, called the Hamiltonian vector field associated to the Hamiltonian function $h$. The triplet $(M, \{\cdot, \cdot\}, h)$ is called a Poisson dynamical system. Any Hamiltonian system on a symplectic manifold is a Poisson dynamical system relative to the Poisson bracket induced by the symplectic structure. Given a Poisson dynamical system $(M, \{\cdot, \cdot\}, h)$, its integrals of motion or conserved quantities are defined as the centralizer of $h$ in $(C^\infty(M), \{\cdot, \cdot\})$ that is, the subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$ consisting of the functions $f \in C^\infty(M)$ such that $\{f, h\} = 0$. Note that the terminology is justified since, by Hamilton’s equations in Poisson bracket form, we have $f = X_h[f] = \{f, h\} = 0$, that is, $f$ is constant on the flow of $X_h$. A smooth mapping $\varphi : M_1 \to M_2$, between the two Poisson manifolds $(M_1, \{\cdot, \cdot\}_1)$ and $(M_2, \{\cdot, \cdot\}_2)$ is called canonical or Poisson if for all $g, h \in C^\infty(M_2)$ we have $\varphi^*\{g, h\}_2 = \{\varphi^*g, \varphi^*h\}_1$. If $\varphi : M_1 \to M_2$ is a smooth map between two Poisson manifolds $(M_1, \{\cdot, \cdot\}_1)$ and $(M_2, \{\cdot, \cdot\}_2)$ then $\varphi$ is a Poisson map if and only if $T\varphi \circ X_{h\circ\varphi} = X_{h\circ\varphi}$ for any $h \in C^\infty(M_2)$, where $T\varphi : TM_1 \to TM_2$ denotes the tangent map (or derivative) of $\varphi$.

Let $(S, \{\cdot, \cdot\}_S)$ and $(M, \{\cdot, \cdot\}_M)$ be two Poisson manifolds such that $S \subset M$ and the inclusion $i_S : S \to M$ is an immersion. The Poisson manifold $(S, \{\cdot, \cdot\}_S)$ is called a Poisson submanifold of $(M, \{\cdot, \cdot\}_M)$ if $i_S$ is a canonical map. An immersed submanifold $Q$ of $M$ is called a quasi Poisson submanifold of $(M, \{\cdot, \cdot\}_M)$ if for any $q \in Q$, any open neighborhood $U$ of $q$ in $M$, and any $f \in C^\infty(U)$ we have
the relation for any \( \{ \cdot, \cdot \} \)

Canonical Lie group actions. Let \((M, \{ \cdot, \cdot \})\) be a Poisson manifold and let \(G\) be a Lie group acting canonically on \(M\) via the map \(\Phi : G \times M \to M\). An action is called canonical if for any \(h \in G\) and \(f, g \in C^\infty(M)\) one has

\[
\{ f \circ \Phi_h, g \circ \Phi_h \} = \{ f, g \} \circ \Phi_h.
\]

If the \(G\)-action is free and proper then the orbit space \(M/G\) is a smooth regular quotient manifold. Moreover, it is also a Poisson manifold with the Poisson bracket \(\{ \cdot, \cdot \}^{M/G}\), uniquely characterized by the relation

\[
\{ f, g \}^{M/G}(\pi(m)) = \{ f \circ \pi, g \circ \pi \}(m), \tag{1.1}
\]

for any \(m \in M\) and where \(f, g : M/G \to \mathbb{R}\) are two arbitrary smooth functions. This bracket is appropriate for the reduction of Hamiltonian dynamics in the sense that if \(h \in C^\infty(M)^G\) is a \(G\)-invariant smooth function on \(M\), then the Hamiltonian flow \(F_t\) of \(X_h\) commutes with the \(G\)-action, so it induces a flow \(F_t^{M/G}\) on \(M/G\) that is Hamiltonian on \((M/G, \{ \cdot, \cdot \}^{M/G})\) for the reduced Hamiltonian function \([h] \in C^\infty(M/G)\) defined by \([h] \circ \pi = h\).

If the Poisson manifold \((M, \{ \cdot, \cdot \})\) is actually symplectic with form \(\omega\) and the \(G\)-action has an associated momentum map \(J : M \to g^*\), then the symplectic leaves of \((M/G, \{ \cdot, \cdot \}^{M/G})\) are given by the spaces \(M_{\omega^G} := G \cdot J^{-1}(\mu)^c/G\), where \(J^{-1}(\mu)^c\) is a connected component of the fiber \(J^{-1}(\mu)\) and \(\omega^G\) is the restriction to \(M_{\omega^G}\) of the symplectic form \(\omega^G\) of the symplectic orbit reduced space \(M_{\omega^G}\) (see \(\Box\)). If, additionally, \(G\) is compact, \(M\) is connected, and the momentum map \(J\) is proper then \(M_{\omega^G} = M_{\omega^G}\).

In the remainder of this section we characterize the situations in which new Poisson manifolds can be obtained out of a given one by a combination of restriction to a submanifold and passage to the
Definition 1.1 Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold and \(D \subset TM\) a smooth distribution on \(M\). The distribution \(D\) is called \textbf{Poisson} or \textbf{canonical}, if the condition \(df|_D = dg|_D = 0\), for any \(f, g \in C^\infty(U)\) and any open subset \(U \subset P\), implies that \(d\{f, g\}|_D = 0\).

Unless strong regularity assumptions are invoked, the passage to the leaf space of a canonical distribution destroys the smoothness of the quotient topological space. In such situations the Poisson algebra of functions is too small and the notion of \textbf{presheaf of Poisson algebras} is needed.

Definition 1.2 Let \(M\) be a topological space with a presheaf \(\mathcal{F}\) of smooth functions. A \textbf{presheaf of Poisson algebras} on \((M, \mathcal{F})\) is a map \(\{\cdot, \cdot\}\) that assigns to each open set \(U \subset M\) a bilinear operation \(\{\cdot, \cdot\}_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)\) such that the pair \((\mathcal{F}(U), \{\cdot, \cdot\}_U)\) is a Poisson algebra. A presheaf of Poisson algebras is denoted as a triple \((M, \mathcal{F}, \{\cdot, \cdot\})\). The presheaf of Poisson algebras \((M, \mathcal{F}, \{\cdot, \cdot\})\) is said to be \textbf{non-degenerate} if the following condition holds: if \(f \in \mathcal{F}(U)\) is such that \(\{f, g\}_U|_V = 0\), for any \(g \in \mathcal{F}(V)\) and any open set of \(V\), then \(f\) is constant on the connected components of \(U\).

Any Poisson manifold \((M, \{\cdot, \cdot\})\) has a natural presheaf of Poisson algebras on its presheaf of smooth functions that associates to any open subset \(U \subset M\) the restriction \(\{\cdot, \cdot\}|_U\) of \(\{\cdot, \cdot\}\) to \(C^\infty(U) \times C^\infty(U)\).

Definition 1.3 Let \(P\) be a topological space and \(Z = \{S_i\}_{i \in I}\) a locally finite partition of \(P\) into smooth manifolds \(S_i \subset P\), \(i \in I\), that are locally closed topological subspaces of \(P\) (hence their manifold topology is the relative one induced by \(P\)). The pair \((P, Z)\) is called a \textbf{decomposition} of \(P\) with \textbf{pieces} in \(Z\), or a \textbf{decomposed space}, if the following \textbf{frontier condition} holds:

\[(DS)\] If \(R, S \in Z\) are such that \(R \cap S \neq \emptyset\), then \(R \subset S\). In this case we write \(R \preceq S\). If, in addition, \(R \neq S\) we say that \(R\) is \textbf{incident} to \(S\) or that it is a \textbf{boundary piece} of \(S\) and write \(R \prec S\).

Definition 1.4 Let \(M\) be a differentiable manifold and \(S \subset M\) a decomposed subset of \(M\). Let \(\{S_i\}_{i \in I}\) be the pieces of this decomposition. The topology of \(S\) is not necessarily the relative topology as a subset of \(M\). Then \(D \subset TM|_S\) is called a \textbf{smooth distribution on} \(S\) \textbf{adapted to the decomposition} \(\{S_i\}_{i \in I}\), if \(D \cap TS_i\) is a smooth distribution on \(S_i\) for all \(i \in I\). The distribution \(D\) is said to be \textbf{integrable} if \(D \cap TS_i\) is integrable for each \(i \in I\).

In the situation described by the previous definition and if \(D\) is integrable, the integrability of the distributions \(D_{S_i} := D \cap TS_i\) on \(S_i\) allows us to partition each \(S_i\) into the corresponding maximal integral manifolds. Thus, there is an equivalence relation on \(S_i\) whose equivalence classes are precisely these maximal integral manifolds. Doing this on each \(S_i\), we obtain an equivalence relation \(D_S\) on the whole set \(S\) by taking the union of the different equivalence classes corresponding to all the \(D_{S_i}\). Define the quotient space \(S/D_S\) by

\[S/D_S := \bigcup_{i \in I} S_i/D_{S_i}\]

and let \(\pi_{D_S} : S \to S/D_S\) be the natural projection.

The \textbf{presheaf of smooth functions on} \(S/D_S\). Define the presheaf of smooth functions \(C^\infty_{S/D_S}\) on \(S/D_S\) as the map that associates to any open subset \(V\) of \(S/D_S\) the set of functions \(C^\infty_{S/D_S}(V)\)
characterized by the following property: \( f \in C^\infty_S(V) \) if and only if for any \( z \in V \) there exists \( m \in \pi^{-1}_D(V) \), \( U_m \) open neighborhood of \( m \) in \( M \), and \( F \in C^\infty(U_m) \) such that

\[
f \circ \pi_D \big|_{\pi^{-1}_D(V) \cap U_m} = F \big|_{\pi^{-1}_D(V) \cap U_m};
\]

(1.2)

\( F \) is called a \textit{local extension} of \( f \circ \pi_D \) at the point \( m \in \pi^{-1}_D(V) \). When the distribution \( D \) is trivial, the presheaf \( C^\infty_S \) coincides with the presheaf of \textit{Whitney smooth functions} \( C^\infty_S \) on \( S \) induced by the smooth functions on \( M \).

The presheaf \( C^\infty_S \) is said to have the \((D,D)\)-\textit{local extension property} when the topology of \( S \) is stronger than the relative topology and, at the same time, the local extensions of \( f \circ \pi_D \) defined in (1.2) can always be chosen to satisfy

\[
dF(n)|_{D(n)} = 0, \quad \text{for any } n \in \pi^{-1}_D(V) \cap U_m;
\]

\( F \) is called a \textit{local }\( D \)-invariant extension of \( f \circ \pi_D \) at the point \( m \in \pi^{-1}_D(V) \). If \( S \) is a smooth embedded submanifold of \( M \) and \( D \) is a smooth, integrable, and regular distribution on \( S \), then the presheaf \( C^\infty_S \) coincides with the presheaf of smooth functions on \( S/D \) as considered as a regular quotient manifold.

The following definition spells out what we mean by obtaining a bracket via reduction.

\textbf{Definition 1.5} \( (M,\{\cdot,\cdot\}) \) be a Poisson manifold, \( S \) a decomposed subset of \( M \), and \( D \subset TM|_S \) a Poisson integrable generalization distribution adapted to the decomposition of \( S \). Assume that \( C^\infty_S \) has the \((D,D)\)-local extension property. Then \( (M,\{\cdot,\cdot\},D,S) \) is said to be \textit{Poisson reducible} if \((S/D_S,\{\cdot,\cdot\}_S)\) is a well-defined presheaf of Poisson algebras where, for any open set \( V \subset S/D \), the bracket \( \{\cdot,\cdot\}_V : C^\infty_{S/D}(V) \times C^\infty_{S/D}(V) \rightarrow C^\infty_{S/D}(V) \) is given by

\[
\{f,g\}_{S/D} = \pi_{D_S}^{-1}(m) := \{F,G\}(m),
\]

for any \( m \in \pi^{-1}_D(V) \) for local \( D \)-invariant extensions \( F,G \) at \( m \) of \( f \circ \pi_D \) and \( g \circ \pi_D \), respectively.

\textbf{Theorem 1.6} \( (M,\{\cdot,\cdot\}) \) be a Poisson manifold with associated Poisson tensor \( B \in \Lambda^2(T^*M) \), \( S \) a decomposed space, and \( D \subset TM|_S \) a Poisson integrable generalization distribution adapted to the decomposition of \( S \) (see Definitions 1.4 and 1.5). Assume that \( C^\infty_S \) has the \((D,D)\)-local extension property. Then \( (M,\{\cdot,\cdot\},D,S) \) is Poisson reducible if for any \( m \in S \)

\[
B^2(\Delta_m) \subset [\Delta_m]^0
\]

(1.3)

where \( \Delta_m := \{dF(m) \mid F \in C^\infty(U_m), dF(z)|_{D(z)} = 0, \text{ for all } z \in U_m \cap S, \text{ and for any open neighborhood } U_m \text{ of } m \text{ in } M \} \) and \( \Delta_m^0 := \{dF(m) \in \Delta_m \mid F|_{U_m \cap V_m} \text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } M \text{ and an open neighborhood } V_m \text{ of } m \text{ in } S \} \).

If \( S \) is endowed with the relative topology then \( \Delta_m^0 := \{dF(m) \in \Delta_m \mid F|_{U_m \cap V_m} \text{ is constant for an open neighborhood } U_m \text{ of } m \text{ in } M \} \).

\textbf{Reduction by regular canonical distributions}. Let \( (M,\{\cdot,\cdot\}) \) be a Poisson manifold and \( S \) an embedded submanifold of \( M \). Let \( D \subset TM|_S \) be a subbundle of the tangent bundle of \( M \) restricted to \( S \) such that \( D_S := D \cap TS \) is a smooth, integrable, regular distribution on \( S \) and \( D \) is canonical.

\textbf{Theorem 1.7} \textit{With the above hypotheses, }\( (M,\{\cdot,\cdot\},D,S) \) \textit{is Poisson reducible if and only if}

\[
B^2(D^0) \subset TS + D.
\]

(1.4)
2 Applications of the Poisson Reduction Theorem

Reduction of coisotropic submanifolds. Let \((M,\{\cdot,\cdot\})\) be a Poisson manifold with associated Poisson tensor \(B \in \Lambda^2(T^*M)\) and \(S\) an immersed smooth submanifold of \(M\). Denote by \((TS)^o := \{\alpha_s \in T^*_s M \mid \langle \alpha_s, v_s \rangle = 0, \ \forall s \in S, v_s \in T_s S\} \subset T^*M\) the conormal bundle of the manifold \(S\); it is a vector subbundle of \(T^*M|_S\). The manifold \(S\) is called coisotropic if \((TS)^o \subset TS\).

In the physics literature, coisotropic submanifolds appear sometimes under the name of first class constraints. The following are equivalent:

(i) \(S\) is coisotropic;
(ii) if \(f \in C^\infty(M)\) satisfies \(f|_S = 0\) then \(X_f|_S \in \mathfrak{x}(S)\);
(iii) for any \(s \in S\), any open neighborhood \(U_s\) of \(s\) in \(M\), and any function \(g \in C^\infty(U_s)\) such that \(X_g(s) \in T_s S\), if \(f \in C^\infty(U_s)\) satisfies \(\{f, g\}(s) = 0\), it follows that \(X_f(s) \in T_s S\);
(iv) the subalgebra \(\{ f \in C^\infty(M) \mid f|_S = 0 \}\) is a Poisson subalgebra of \((C^\infty(M),\{\cdot,\cdot\})\).

The following proposition shows how to endow the coisotropic submanifolds of a Poisson manifold with a Poisson structure by using the Reduction Theorem 1.6.

**Proposition 2.1** Let \((M,\{\cdot,\cdot\})\) be a Poisson manifold with associated Poisson tensor \(B \in \Lambda^2(T^*M)\). Let \(S\) be an embedded coisotropic submanifold of \(M\) and \(D := B^2((TS)^o)\). Then

(i) \(D = D \cap TS = D_S\) is a smooth generalized distribution on \(S\).
(ii) \(D\) is integrable.
(iii) If \(C^\infty_{S,D_S}\) has the \((D,D_S)\)-local extension property then \((M,\{\cdot,\cdot\},D,S)\) is Poisson reducible.

Coisotropic submanifolds usually appear as the level sets of integrals in involution. Let \((M,\{\cdot,\cdot\})\) be a Poisson manifold with Poisson tensor \(B\) and let \(f_1, \ldots, f_k \in C^\infty(M)\) be \(k\) smooth functions in involution, that is, \(\{f_i, f_j\} = 0\), for any \(i, j \in \{1, \ldots, k\}\). Assume that \(0 \in \mathbb{R}^k\) is a regular value of the function \(F := (f_1, \ldots, f_k) : M \to \mathbb{R}^k\) and let \(S := F^{-1}(0)\). Since for any \(s \in S\), \(\text{span}\{df_1(s), \ldots, df_k(s)\} \subset (T_s S)^o\) and the dimensions of both sides of this inclusion are equal it follows that \(\text{span}\{df_1(s), \ldots, df_k(s)\} = (T_s S)^o\). Hence \(B^i(s)((T_s S)^o) = \text{span}\{X_{f_1}(s), \ldots, X_{f_k}(s)\}\) and \(B^i(s)((T_s S)^o) \subset T_s S\) by the involutivity of the components of \(F\). Consequently, \(S\) is a coisotropic submanifold of \((M,\{\cdot,\cdot\})\).

Cosymplectic submanifolds and Dirac’s constraints formula. The Poisson Reduction Theorem 1.7 allows us to define Poisson structures on certain embedded submanifolds that are not Poisson submanifolds.

**Definition 2.2** Let \((M,\{\cdot,\cdot\})\) be a Poisson manifold and let \(B \in \Lambda^2(T^*M)\) be the corresponding Poisson tensor. An embedded submanifold \(S \subset M\) is called cosymplectic if

(i) \(B^2((TS)^o) \cap TS = \{0\}\).
(ii) \(T_s S + T_s L_s = T_s M\),

for any \(s \in S\) and \(L_s\) the symplectic leaf of \((M,\{\cdot,\cdot\})\) containing \(s \in S\).
The cosymplectic submanifolds of a symplectic manifold \((M, \omega)\) are its symplectic submanifolds. Cosymplectic submanifolds appear in the physics literature under the name of second class constraints.

**Proposition 2.3** Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold, \(B \in \Lambda^2(T^*M)\) the corresponding Poisson tensor, and \(S\) a cosymplectic submanifold of \(M\). Then for any \(s \in S\)

(i) \(T_sL_s = (T_sS \cap T_sL_s) \oplus B^2(s)((T_sS)^\circ)\), where \(L_s\) is the symplectic leaf of \((M, \{\cdot, \cdot\})\) that contains \(s \in S\).

(ii) \((T_sS)^\circ \cap \ker B^2(s) = \{0\}\).

(iii) \(T_sM = B^2(s)((T_sS)^\circ) \oplus T_sS\).

(iv) \(B^2((TS)^\circ)\) is a subbundle of \(TM|_S\) and hence \(TM|_S = B^2((TS)^\circ) \oplus TS\).

(v) The symplectic leaves of \((M, \{\cdot, \cdot\})\) intersect \(S\) transversely and hence \(S \cap L\) is an initial submanifold of \(S\), for any symplectic leaf \(L\) of \((M, \{\cdot, \cdot\})\).

**Theorem 2.4 (The Poisson structure of a cosymplectic submanifold)** Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold, \(B \in \Lambda^2(T^*M)\) the corresponding Poisson tensor, and \(S\) a cosymplectic submanifold of \(M\). Let \(D := B^2((TS)^\circ) \subset TM|_S\). Then

(i) \((M, \{\cdot, \cdot\}, D, S)\) is Poisson reducible.

(ii) The corresponding quotient manifold equals \(S\) and the reduced bracket \(\{\cdot, \cdot\}^S\) is given by

\[
\{f, g\}^S(s) = \{F, G\}(s),
\]

where \(f, g \in C^\infty_{S,M}(V)\) are arbitrary and \(F, G \in C^\infty(U)\) are local \(D\)-invariant extensions of \(f\) and \(g\) around \(s \in S\), respectively.

(iii) The Hamiltonian vector field \(X_f\) of an arbitrary function \(f \in C^\infty_{S,M}(V)\) is given either by

\[
Ti \circ X_f = X_F \circ i,
\]

where \(F \in C^\infty(U)\) is a local \(D\)-invariant extension of \(f\) and \(i : S \hookrightarrow M\) is the inclusion, or by

\[
Ti \circ X_f = \pi_S \circ X_{\overline{f}} \circ i,
\]

where \(\overline{f} \in C^\infty(U)\) is an arbitrary local extension of \(f\) and \(\pi_S : TM|_S \rightarrow TS\) is the projection induced by the Whitney sum decomposition \(TM|_S = B^2((TS)^\circ) \oplus TS\) of \(TM|_S\).

(iv) The symplectic leaves of \((S, \{\cdot, \cdot\}^S)\) are the connected components of the intersections \(S \cap \mathcal{L}\), where \(\mathcal{L}\) is a symplectic leaf of \((M, \{\cdot, \cdot\})\). Any symplectic leaf of \((S, \{\cdot, \cdot\}^S)\) is a symplectic submanifold of the symplectic leaf of \((M, \{\cdot, \cdot\})\) that contains it.

(v) Let \(\mathcal{L}_s\) and \(\mathcal{L}_s^S\) be the symplectic leaves of \((M, \{\cdot, \cdot\})\) and \((S, \{\cdot, \cdot\}^S)\), respectively, that contain the point \(s \in S\). Let \(\omega_{\mathcal{L}_s}\) and \(\omega_{\mathcal{L}_s^S}\) be the corresponding symplectic forms. Then \(B^2(s)((T_sS)^\circ)\) is a symplectic subspace of \(T_s\mathcal{L}_s\) and

\[
B^2(s)((T_sS)^\circ) = (T_s\mathcal{L}_s^S)^{\omega_{\mathcal{L}_s}(s)},
\]

where \((T_s\mathcal{L}_s^S)^{\omega_{\mathcal{L}_s}(s)}\) denotes the \(\omega_{\mathcal{L}_s}(s)\)-orthogonal complement of \(T_s\mathcal{L}_s^S\) in \(T_s\mathcal{L}_s\).
(vi) Let $B_S \in \Lambda^2(T^*S)$ be the Poisson tensor associated to $(S, \{\cdot, \cdot\}^S)$. Then

$$B_S^k = \pi_S \circ B^2 \circ \pi_S^*,$$

(2.5)

where $\pi_S^* : T^*S \to T^*M|_S$ is the dual of $\pi_S : TM|_S \to TS$.

The **Dirac constraints formula** is the expression in coordinates for the bracket of a cosymplectic submanifold. Let $(M, \{\cdot, \cdot\})$ be a $n$-dimensional Poisson manifold and let $S$ be a $k$-dimensional cosymplectic submanifold of $M$. Let $z_0$ be an arbitrary point in $S$ and $(U, \pi)$ a submanifold chart around $z_0$ such that $\pi = (\pi, \psi) : U \to V_1 \times V_2$, where $V_1$ and $V_2$ are two open neighborhoods of the origin in two Euclidean spaces such that $\pi(z_0) = (\pi(z_0), \psi(z_0)) = (0, 0)$ and

$$\pi(U \cap S) = V_1 \times \{0\}.$$  

(2.6)

Let $\pi = (\pi^1, \ldots, \pi^k)$ be the components of $\pi$ and define $\hat{\varphi}^1 := \pi^1|_{U \cap S}, \ldots, \hat{\varphi}^k := \pi^k|_{U \cap S}$. Extend $\hat{\varphi}^1, \ldots, \hat{\varphi}^k$ to $D$-invariant functions $\varphi^1, \ldots, \varphi^k$ on $U$. Since the differentials $d\varphi^1(s), \ldots, d\varphi^k(s)$ are linearly independent for any $s \in U \cap S$, we can assume (by shrinking $U$ if necessary) that $d\varphi^1(z), \ldots, d\varphi^k(z)$ are also linearly independent for any $z \in U$. Consequently, $(U, \kappa)$ with $\kappa := (\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$, is a submanifold chart for $M$ around $z_0$ with respect to $S$ such that, by construction,

$$d\varphi^1(s)|_{BH(s)(T_sS)^o} = \cdots = d\varphi^k(s)|_{BH(s)(T_sS)^o} = 0,$$

for any $s \in U \cap S$. This implies that for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n-k\}$, and $s \in S$

$$\{\varphi^i, \psi^j\}(s) = d\varphi^i(s)\left(X_{\varphi^i}(s)\right) = 0$$

since $d\psi^j(s) \in (T_sS)^o$ by (2.5) and hence

$$X_{\varphi^i}(s) \in B^2(s)((T_sS)^o).$$  

(2.7)

Additionally, since the functions $\varphi^1, \ldots, \varphi^k$ are $D$-invariant, by (2.2), it follows that

$$X_{\varphi^1}(s) = X_{\hat{\varphi}^1(s)}(s) \in T_sS, \ldots, X_{\varphi^k}(s) = X_{\hat{\varphi}^k(s)}(s) \in T_sS,$$

for any $s \in S$. Consequently, $\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s), X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\}$ spans $T_sL_s$ with

$$\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s)\} \subset T_sS \cap T_sL_s$$

and

$$\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\} \subset B^2(s)((T_sS)^o).$$

By Proposition (2.3)(i),

$$\text{span}\{X_{\varphi^1}(s), \ldots, X_{\varphi^k}(s)\} = T_sS \cap T_sL_s$$

and

$$\text{span}\{X_{\psi^1}(s), \ldots, X_{\psi^{n-k}}(s)\} = B^2(s)((T_sS)^o).$$

Since $\dim \left(B^2(s)((T_sS)^o)\right) = n - k$ by Proposition (2.3)(iii), it follows that $\{X_{\varphi^1}(s), \ldots, X_{\psi^{n-k}}(s)\}$ is a basis of $B^2(s)((T_sS)^o)$.
Since $B^\#(T_sS)^\circ$ is a symplectic subspace of $T_sL_s$ by Theorem 2.4(v), there exists some $r \in \mathbb{N}$ such that $n - k = 2r$ and, additionally, the matrix $C(s)$ with entries

$$C^{ij}(s) := \{\psi^i, \psi^j\}(s), \quad i, j \in \{1, \ldots, n - k\}$$

is invertible. Therefore, in the coordinates $(\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$, the matrix associated to the Poisson tensor $B(s)$ is

$$B(s) = \begin{pmatrix} B_S(s) & 0 \\ 0 & C(s) \end{pmatrix},$$

where $B_S \in \Lambda^2(T^*S)$ is the Poisson tensor associated to $(S, \{\cdot, \cdot\}^S)$. Let $C_{ij}(s)$ be the entries of the matrix $C(s)^{-1}$.

**Proposition 2.5 (Dirac formulas)** In the coordinate neighborhood $(\varphi^1, \ldots, \varphi^k, \psi^1, \ldots, \psi^{n-k})$ constructed above and for $s \in S$ we have, for any $f, g \in C^\infty_S(M(V))$:

$$X_f(s) = X_F(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s)C_{ij}(s)X_{\psi^j}(s) \quad (2.8)$$

and

$$\{f, g\}^S(s) = \{F, G\}(s) - \sum_{i,j=1}^{n-k} \{F, \psi^i\}(s)C_{ij}(s)\{\psi^j, G\}(s), \quad (2.9)$$

where $F, G \in C^\infty(U)$ are arbitrary local extensions of $f$ and $g$, respectively, around $s \in S$.

**References**

[1] Abraham, R., and Marsden, J.E. [1978] *Foundations of Mechanics*. Second edition, Addison–Wesley.

[2] Casati, P. and Pedroni, M. [1992] Drinfeld-Sokolov reduction on a simple Lie algebra from the bi-Hamiltonian point of view. *Lett. Math. Phys.*, 25(2), 89–101.

[3] Castrillón-López, M. and J. E. Marsden [2003] Some remarks on Lagrangian and Poisson reduction for field theories. *J. Geom. and Physics* 48, 52–83.

[4] Cendra, H., J. E. Marsden, and T. S. Ratiu [2003] Cocycles, compatibility, and Poisson brackets for complex fluids. In Capriz, G. and P. Mariano, editors, *Advances in Multifield Theories of Continua with Substructures*, Memoirs, pages 51–73. Aarhus Univ., Aarhus.

[5] Faybusovich, L. [1995] A Hamiltonian structure for generalized affine-scaling vector fields. *J. of Nonlinear Science*, 5, no 1, pp. 11-28

[6] Faybusovich, L. [1991] Hamiltonian structure of dynamical systems which solve linear programming problems. *Physica D*, 53, pp. 217-232.

[7] Lewis, D., J. E. Marsden, R. Montgomery, and T. S. Ratiu [1986] The Hamiltonian structure for dynamic free boundary problems. *Physica D*, 18, 391–404.

[8] Gotay, M.J., Nester, M.J., and Hinds, G. [1978] Presymplectic Manifolds and the Dirac-Bergmann Theory of Constraints. *J. Math. Phys.* 19, 2388–2399.

[9] Krishnaprasad, P. S. and J. E. Marsden [1987] Hamiltonian structure and stability for rigid bodies with flexible attachments. *Arch. Rational Mech. Anal.* 98, 137–158.
[10] Lu, J.-H. and Weinstein, A. [1990], Poisson Lie groups, dressing transformations and Bruhat decompositions. *J. Diff. Geom.* 31, 510–526.

[11] Marsden, J.E. and Ratiu, T.S. [1986] Reduction of Poisson manifolds. *Letters in Mathematical Physics*, 11, 161–169.

[12] Marsden, J.E. and Ratiu, T.S. [2003] *Introduction to Mechanics and Symmetry*, second ed., second printing. First ed. [1994]. *Texts in Applied Mathematics*, volume 17. Springer-Verlag, New York.

[13] Ortega, J.–P. and Ratiu, T.S. [1998] Singular reduction of Poisson manifolds. *Letters in Mathematical Physics*, 46, 359–372.

[14] Ortega, J.–P. and Ratiu, T. S. [2003] *Momentum Maps and Hamiltonian Reduction. Progress in Math.*, 222. Birkhäuser, Boston.

[15] Ortega, J.–P. and Ratiu, T. S. [2005] Cotangent bundle reduction. Article in the *Encyclopedia of Mathematical Physics*. Jean-Pierre Françoise, Greg Naber, Tsou Sheung Tsun (editors). Elsevier.

[16] Ortega, J.–P. and Ratiu, T. S. [2005] Symmetry and symplectic reduction. Article in the *Encyclopedia of Mathematical Physics*. Jean-Pierre Françoise, Greg Naber, Tsou Sheung Tsun (editors). Elsevier.

[17] Pedroni, M. [1995] Equivalence of the Drinfeld-Sokolov reduction to a bi-Hamiltonian reduction. *Lett. Math. Phys.*, 35(4), 291–302.

[18] Sundermeyer, K. [1982] *Constrained Dynamics. Lecture Notes in Physics*, 169, Springer-Verlag, New York.

[19] Weinstein, A. [1983] The local structure of Poisson manifolds. *J. Differential Geom.*, 18, 523–557. Errata and addenda: *J. Differential Geom.*, 22(2), 255 (1985).

[20] Zaalani, N. [1999] Phase space reduction and Poisson structure. *J. Math. Phys.* 40, 3431–3438.