FUSION RULES OF VIRASORO VERTEX OPERATOR ALGEBRAS

XIANZU LIN

College of Mathematics and Computer Science, Fujian Normal University, Fuzhou, 350108, China; Email: linxianzu@126.com

Abstract. In this paper we prove the fusion rules of Virasoro vertex operator algebras \(L(c_1, q, 0)\), for \(q \geq 1\). Roughly speaking, we consider \(L(c_1, q, 0)\) as the limit of \(L(c_{n, nq-1}, 0)\), for \(n \to \infty\), and the fusion rules of \(L(c_1, q, 0)\) follow as the limits of the fusion rules of \(L(c_{n, nq-1}, 0)\).

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1. INTRODUCTION

Among classical representation theory (of compact groups or semi-simple Lie algebras), the most important problems are,

(1) The classification problem: describe all the irreducible representations.

(2) The Clebsch-Gordon problem: given irreducible representations \(V\) and \(W\), describe the decomposition, with multiplicities, of the representation \(V \otimes W\).

For the representation theory of vertex operator algebras, the most important problem is also the classification of all the irreducible representations. The difference is that for two irreducible modules \(U\) and \(V\) over a vertex operator algebra \(A\), we can not define the tensor module of \(U\) and \(V\). Nevertheless, we still have the analogue of the Clebsch-Gordon problem via the notion of intertwining operators. In particular, for three irreducible modules \(U, V\) and \(W\) over a vertex operator algebra \(A\), we can define the fusion
rule $\mathcal{N}_{U,V}^W$, the analogue of the Clebsch-Gordon coefficient. As in the classical representation theory, the second most important problem in representation theory of vertex operator algebras is to determine the fusion rules $\mathcal{N}_{U,V}^W$.

The Virasoro vertex operator algebras constitute one of the most important classes of vertex operator algebras. In [15] it was proved that the vertex operator algebras $L(c_{p,q},0)$ are rational, where $(p,q) = 1$, $p, q > 1$ and $c_{p,q} = 13 - 6\left(\frac{2}{p} + \frac{2}{q}\right)$. Furthermore, the fusion rules of $L(c_{p,q},0)$ were proved in [15] using the Frenkel-Zhu’s formula (cf. [7]). In the case of $L(c_{1,q},0)$, we cannot prove the fusion rules as in [15], for Frenkel-Zhu’s formula cannot be applied to $L(c_{1,q},0)$ which is non-rational. The fusion rules of $L(c_{1,1},0)$ were first proved in [10] and further extended in [3].

In this paper we prove the fusion rules of $L(c_{1,q},0)$ for all $q \geq 1$. Our method is totally different from those of [3, 10]. Roughly speaking, we consider $L(c_{1,q},0)$ as the limit of $L(c_{n,nq-1},0)$, for $n \to \infty$, and the fusion rules of $L(c_{1,q},0)$ follow as the limits of the fusion rules of $L(c_{n,nq-1},0)$. Formally, the fusion ring of $L(c_{1,q},0)$ can be written as

$$L(c_{1,q}, h_{i_1,s_1}) \otimes L(c_{1,q}, h_{i_2,s_2}) = \bigoplus_{i \in A_{i_1,i_2,s_1,s_2}} L(c_{1,q}, h_{i,s}),$$

where $A_{m,n} = \{m+n-1, m+n-3, \cdots, |m-n|+1\}$ for $m,n > 0$. Many special cases of this result have already been applied in several papers (cf. [1, 2, 13]).

This paper is structured as follows: In Section 2 we give some preliminary results about the representation theory of Virasoro vertex operator algebras. In Section 3, using the easy part of Frenkel-Zhu’s formula, we get an upper bound for the fusion rules of $L(c_{1,q},0)$. In Section 4 we establish the fusion rules of $L(c_{1,q},0)$ by the limit method. In Section 5, we further extend the fusion rules of $L(c_{1,q},0)$ to include some other cases. Throughout this paper, we assume that the reader is familiar with the axiom theory vertex operator algebras and modules. For more information, see [6, 11].

2. VERTEX OPERATOR ALGEBRAS AND MODULES ASSOCIATED TO VIRASORO ALGEBRA

In this section, we give a short review of vertex operator algebras and modules associated to Virasoro algebra, details can be found in [4, 5, 8, 9, 11]. First, recall that Virasoro algebra is the Lie algebra $Vir$ with basis $\{L_n |
\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C \]

and

\[ [Vir, C] = \{0\}. \]

Define the following subalgebras of \( Vir \):

\[ Vir^\pm = \bigoplus_{n>0} C L_n; \quad \Vir^0 = L_0 \oplus C; \]

\[ \Vir^{\geq 0} = \Vir^+ \oplus \Vir^0; \quad \Vir^{\leq -1} = \Vir^{\geq 0} \oplus C L_{-1}. \]

Let \( c \) and \( h \) be two complex numbers and let \( \mathbb{C}v_{c,h} \) be the one dimensional \( Vir^{\geq 0} \)-module with \( C \) and \( L_0 \) acting as the scalars \( c \) and \( h \), and with \( Vir^+ \) acting trivially. Set

\[ M(c, h) = U(Vir) \otimes_{U(Vir^{\geq 0})} \mathbb{C}v_{c,h} \]

and call it the Verma module with central charge \( c \) and highest weight \( h \).

For any \( I = (1^{r_1} 2^{r_2} \cdots n^{r_n}) \in \mathcal{P}_n \), set

\[ e_I = L_{-n}^{r_n} \cdots L_2^{r_2} L_1^{r_1} \in U(Vir^-)_{-n}. \]

Then, \( \{e_I : v_{c,h} L_i \in \mathcal{P}_n \} \) forms a basis of the weight subspace \( M(c, h)_{h+n} \). Let \( M'(c, h) \) be the largest proper submodule of \( M(c, h) \). Then \( L(c, h) = M(c, h)/M'(c, h) \) is an irreducible \( Vir^- \)-module.

We recall the following proposition.

**Proposition 2.1.** (cf. [14]) Set \( n = \alpha \beta \), \( c = c(t) = 13 - 6t - 6t^{-1} \) and \( h = h_{\alpha, \beta}(t) = \frac{1}{3}(\alpha^2 - 1)t - \frac{1}{3}(\alpha \beta - 1) + \frac{1}{3}(\beta^2 - 1)t^{-1} \) for \( \alpha, \beta \in \mathbb{Z}_{>0} \). Then there exists

\[ S_n = \sum_{I \in \mathcal{P}_n} f_I(c, h)e_I \in U(Vir^-)_{-n} \]

such that \( S_n v_{c,h} \in M(c, h)_{h+n} \) is a singular vector, where \( f_I(x, y) \in \mathbb{C}[x, y] \) and \( f_{I_0}(x, y) = 1 \) for \( I_0 = (1^n) \).

Let \( p, q, r \) and \( s \) be positive integers, satisfying \( (p, q) = 1, r < p \) and \( s < q \). Let \( c = c_{p,q} = 13 - 6(\frac{2}{p} + \frac{2}{q}) \) and \( h = h_{p,q;r,s} = \frac{(sp-rq)^2-(p-q)^2}{4pq} \). Then by Proposition 2.1, \( M(c, h) \) has two singular vectors \( u^{r,s}_{p,q} \) and \( v^{r,s}_{p,q} \) of weights \( h+r,s \) and \( h+(p-r)(q-s) \) respectively. Moreover, the maximal submodule of \( M(c, h) \) is generated by \( u^{r,s}_{p,q} \) and \( v^{r,s}_{p,q} \).

Similarly, when \( p = 1 \) and \( h = h_{i,s} = \frac{(is^2-(q-1)^2}{4q} \) for some \( i > 0, 0 < s \leq q \), \( M(c_{1,q}, h) \) has a singular vector of weight \( h+i,s \) which generates the maximal
proper submodule of $M(c_{1,q}, h)$. Moreover, $M(c_{1,q}, h)$ is irreducible when $h \neq h_{i,s} = \frac{(iq-s)^2-(q-1)^2}{4q}$ for any $i > 0$, $0 < s \leq q$.

Now consider $\mathcal{C}$ as a $Vir$ module with $C$ acting as the scalar $c$, and with $Vir^+ \oplus L_0 \oplus L_{-1}$ acting trivially. Set

$$V_c = U(Vir) \otimes_{U(Vir_{\geq -1})} \mathbb{C}.$$ 

Then, it is well known that $V_c$ has a canonical structure of vertex operator algebra of central charge $c$ and with $\omega = L(-2)1$ as conformal vector. In this way, $M(c, h)$ and $L(c, h)$ are modules for $V_c$ viewed as a vertex algebra. Furthermore, $L(c, 0)$, as a quotient of $V_c$, is a simple vertex operator algebra. Note that $L(c, 0) = V_c$ when $c \neq c_{p,q}$, where $p, q > 1$ and $(p, q) = 1$. If $c = c_{p,q}$ for some $p, q$ as above, then $L(c, 0) \neq V_c$, and $L(c, h)$ is a $L(c, 0)$-module if and only if $h = h_{p,q;r,s} = \frac{(sp-rq)^2-(p-q)^2}{4pq}$ for some positive integer $r, s$ satisfying $r < p$ and $s < q$ (cf. [15]).

Now we introduce the definition of intertwining operator and fusion rule for a triple of modules of vertex operator algebra (cf. [6]).

**Definition 2.2.** Let $W_1$, $W_2$ and $W_3$ be three modules over a vertex operator algebra $V$. A linear map $W_1 \otimes W_2 \rightarrow W_3\{x\}$ or equivalently,

$$w \mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{Q}} w_n x^{-n-1} \quad (\text{where } w_n \in Hom(W_2, W_3))$$

is called an intertwining operator of type $(W_3 \quad W_1 W_2)$ if it satisfies:

1. (The truncation property) For any $w_1 \in W_1$, $w_2 \in W_2$, $(w_1)_n w_2 = 0$ for $n$ sufficiently large;
2. (The $L_{-1}$-derivative formula) For any $w \in W_1$,

$$\mathcal{Y}(L_{-1}w, x) = \frac{d}{dx} \mathcal{Y}(w, x);$$

3. (The Jacobi identity) For any $v \in V$ and $w_1 \in W_1$,

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \mathcal{Y}(v, x_1) \mathcal{Y}(w_1, x_2)$$

$$-x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_1, x_2) \mathcal{Y}(v, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(\mathcal{Y}(v, x_0) w_1, x_2).$$
Set \( I(\begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array}) \) to be the vector space of all intertwining operators of type \( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \), its dimension \( N_{W_i, W_j}^{W_k} \) is called the fusion rule of type \( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \).

The main result of this paper is the following:

**Theorem 2.3.** Let \( i_n > 0, 0 < s_n \leq q \) \((n=1, 2, 3)\), and \( A_{m,n} = \{m + n - 1, m + n - 3, \ldots, |m - n| + 1\} \). Then

\[
N_{L(c_1, q, h_{i_3, s_3}), L(c_1, q, h_{i_1, s_1}), L(c_1, q, h_{i_2, s_2})} \leq 1.
\]

The equality hold if and only if \( h_{i_3, s_3} = h_{i, s} \) for some \( i \in A_{i_1, i_2} \) and \( s \in A_{s_1, s_2} \).

3. **Frenkel-Zhu’s Formula**

Recall that to a vertex operator algebra \( V \), we can associate the Zhu’s algebra \( A(V) \), and for each lowest weight \( V \)-module \( M \), the lowest weight space \( M(0) \) has a natural structure of \( A(V) \)-module. More generally, for each \( V \)-module \( M \), define \( O(M) \subset M \) to be the linear span of elements of type

\[
\text{Res}_z(Y(a, z)^{(1+z)^{\deg a}} z^2 m)
\]

where \( a \in V \) and \( m \in M \), and let \( A(M) \) be the quotient space \( M/O(M) \), then \( A(M) \) has a natural structure of an \( A(V) \)-bimodule. We recall the following useful result (cf.\[7\]).

**Proposition 3.1.** For each submodule \( M_1 \) of \( M \), \( A(M_1) \) is a submodule of the \( A(V) \)-bimodule \( A(M) \), and the quotient \( A(M)/A(M_1) \) is isomorphic to the bimodule \( A(M/M_1) \).

In the case of Virasoro vertex operator algebras and Verma modules, we have the following results (cf.\[7, 8\]).

**Proposition 3.2.** Let \( \mathcal{L} \) to be the subalgebra of \( \text{Vir}^- \) spanned by

\[
L_{-n-2} + 2L_{-n-1} + L_{-n},
\]

for \( n \geq 1 \). Then \( O(V_c) = \mathcal{L} V_c \) and \( A(V_c) \cong H_0(\mathcal{L}, V_c) \). In the case of \( M(c, h) \) (resp. the irreducible quotient \( L(c, h) \)), we also have

\[
O(M(c, h)) = \mathcal{L} M(c, h)
\]

(resp. \( O(L(c, h)) = \mathcal{L} L(c, h) \))
and
\[ A(M(c, h)) \cong H_0(\mathcal{L}, M(c, h)). \]
(resp.\(A(L(c, h)) \cong H_0(\mathcal{L}, L(c, h)).\))

**Proposition 3.3.** We have an isomorphism of associative algebra:
\[ A(V_c) \cong \mathbb{C}[x]; \quad [\omega]^n \mapsto x^n, \quad n \in \mathbb{Z}_{\geq 0}. \]

For Verma module \(M(c, h)\), the \(A(V_c)\)-bimodule \(A(M(c, h))\) is isomorphic to \(\mathbb{C}[x, y]\), where the highest weight vector \(v_{c,h}\) represents \(1 \in \mathbb{C}[x, y]\), and the left and the right actions of \(A(V_c)\) are given by
\[ x \cdot f(x, y) = xf(x, y), \]
\[ f(x, y) \cdot x = yf(x, y), \]
for any \(f(x, y) \in \mathbb{C}[x, y]\).

**Proposition 3.4.** The left and right actions of \(A(V_c)\) on \(A(M(c, h))\) are given by
\[ [\omega][v] = [(L_{-2} + 2L_{-1} + L_0)v], \]
\[ [v][\omega] = [(L_{-2} + L_{-1})v], \]
for any \(v \in M(c, h)\), where \(\omega = L_{-2}1\).

From now on, \(W_i = \bigoplus_{n \in \mathbb{N}} W_i(n) \ (i = 1, 2, 3)\) will always be irreducible \(V\)-modules, where \(W_i(n)\) is the \(L_0\)-eigenspace of \(W_i\) with eigenvalue \(n + h_i\).

**Proposition 3.5.** (cf.[7]) Let \(\mathcal{Y}(\cdot, x)\) be an intertwining operator of type \(\begin{array}{c} W_3 \\ W_1 \ W_2 \end{array}\). Then \(\mathcal{Y}(\cdot, x)\) has the following form:
\[ \mathcal{Y}(w, x) = \sum_{n \in \mathbb{Z}} w(n)x^{-n-1}x^{-h_1-h_2+h_3}, \]
such that for any \(w \in W_1(k)\)
\[ w(n)W_2(m) \subset W_3(m + k - n - 1) \]

We need also the symmetry property of fusion rules, i.e., \(\mathcal{N}_{W_3}^{W_1, W_2} = \mathcal{N}_{W_2, W_1}^{W_3}\) (cf.[6]).

Let \(\mathcal{Y}(\cdot, x)\) be an intertwining operator of type \(\begin{array}{c} W_3 \\ W_1 \ W_2 \end{array}\). By Proposition 3.3, we can define a linear map \(o_Y\) from \(W_1 \otimes W_2(0)\) to \(W_3(0)\) by sending \(w_1 \otimes w_2 \ (w_1 \in W_1(n), \ w_2 \in W_2(0))\) to \(w_1(n-1)w_2\). It can be proved that \(w_1(n-1)w_2 = 0\) for \(w_1 \in O(W_1)\), and \(o_Y\) induces an \(A(V)\)-homomorphism
\[ \pi(\mathcal{Y}) : A(W_1) \otimes_{A(V)} W_2(0) \rightarrow W_3(0). \]
Thus we get a linear map:

$$\pi : I(\begin{array}{c} W_3 \\ W_1 W_2 \end{array}) \to Hom_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0))$$

The Frenkel-Zhu’s formula (cf. [7]) states that \( \pi \) is an isomorphism if \( W_i \) \((i = 1, 2, 3)\) are irreducible modules. It was pointed out in [12] that this formula only holds for rational vertex operator algebras, and for more general vertex operator algebras, we have the following proposition (cf. [12]).

**Proposition 3.6.** If \( W_3 \) is irreducible, then

$$\pi : I(\begin{array}{c} W_3 \\ W_1 W_2 \end{array}) \to Hom_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0))$$

is injective.

Now we follow the treatment in §9.3 of [8]. First, we consider the three \( L(c_{1,q}; 0) \)-modules \( L(c_{1,q}, h_{i_1,s_1}) \), where \( i_n > 0, 0 < s_n \leq q \) \((n=1,2,3)\). We want to compute the dimension of

$$Hom_{A(L(c_{1,q},0))}(A(L(c_{1,q}, h_{i_1,s_1}))) \otimes_{A(L(c_{1,q},0))} L(c_{1,q}, h_{i_2,s_2})(0), L(c_{1,q}, h_{i_3,s_3})(0)).$$

By Proposition 3 it is dual space is isomorphic to the simultaneous eigenspace of the left and right actions of \([\omega]\) on \(H_0(\mathcal{L}, L(c_{1,q}, h_{i_1,s_1}))^* = H^0(\mathcal{L}, L(c_{1,q}, h_{i_1,s_1}))^*\)

with the eigenvalues \(-h_{i_3,s_3}\) and \(-h_{i_2,s_2}\) respectively; denote this eigenspace by \(H^0(\mathcal{L}, L(c_{1,q}, h_{i_1,s_1}))^*(-h_{i_3,s_3}, -h_{i_2,s_2})\). Then the surjection

$$M(c_{1,q}, h_{i_1,s_1}) \to L(c_{1,q}, h_{i_1,s_1})$$

induces an injection

$$i : H^0(\mathcal{L}, L(c_{1,q}, h_{i_1,s_1}))^*(-h_{i_3,s_3}, -h_{i_2,s_2}) \hookrightarrow H^0(\mathcal{L}, M(c_{1,q}, h_{i_1,s_1}))^*(-h_{i_3,s_3}, -h_{i_2,s_2}).$$

The argument in §9.3 of [8] shows that \(H^0(\mathcal{L}, M(c_{1,q}, h_{i_1,s_1}))^*(-h_{i_3,s_3}, -h_{i_2,s_2})\) is one-dimensional, and \(i\) is an isomorphism if and only if

\[
P_{i_1,s_1}(-h_{i_2,s_2}, -h_{i_3,s_3} + h_{i_1,s_1}, q) = 0,
\]

where \(P_{\alpha,\beta}(a, b; \xi) \in \mathbb{C}[a, b, \xi, \xi^{-1}]\) satisfies

\[
P_{\alpha,\beta}(a, b; \xi)^2 = \prod_{k=0}^{\alpha-1} \prod_{l=0}^{\beta-1} Q_{k,l}^{\alpha,\beta}(a, b; \xi),
\]

\[
Q_{k,l}^{\alpha,\beta}(a, b; \xi) = \left\{(b-a) - (k\xi^{\frac{1}{2}} - l\xi^{-\frac{1}{2}})\right\}\left\{(\alpha - k)\xi^{\frac{1}{2}} - (\beta - l)\xi^{-\frac{1}{2}}\right\}\left\{(k+1)\xi^{\frac{1}{2}} - (l+1)\xi^{-\frac{1}{2}}\right\}\left\{(\alpha - k - 1)\xi^{\frac{1}{2}} - (\beta - l - 1)\xi^{-\frac{1}{2}}\right\}\left\{(\alpha - 2k - 1)\xi^{\frac{1}{2}} - (\beta - l - 1)\xi^{-\frac{1}{2}}\right\}^2 a.
\]
For each \( m, n > 0 \) the following two equations hold:

\[
\prod_{k=0}^{i_1-1} \prod_{l=0}^{s_1-1} (h_{i_3,s_3} - h_{i_1+i_2-2k-1,s_1+s_2-2l-1}) = 0
\]

Now combining the symmetry property of fusion rules and Proposition 3.6 yields \( \mathcal{N}_{L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q)} \leq 1 \), and \( \mathcal{N}_{L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q)} = 1 \) only if the following two equations hold:

\[
\prod_{k=0}^{i_1-1} \prod_{l=0}^{s_1-1} (h_{i_3,s_3} - h_{i_1+i_2-2k-1,s_1+s_2-2l-1}) = 0;
\]

\[
\prod_{k=0}^{i_2-1} \prod_{l=0}^{s_2-1} (h_{i_3,s_3} - h_{i_1+i_2-2k-1,s_1+s_2-2l-1}) = 0.
\]

For each \( m, n > 0 \), set \( A_{m,n} = \{ m + n - 1, m + n - 3, \ldots, |m - n| + 1 \} \). Then one checks that these two equations are equivalent to the existences of \( i \in A_{i_1,i_2}, s \in A_{s_1,s_2} \) such that \( h_{i_3,s_3} = h_{i,s} \).

To sum up, we have proved that \( \mathcal{N}_{L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q)} \leq 1 \) if there exists \( i \in A_{i_1,i_2}, s \in A_{s_1,s_2} \) such that \( h_{i_3,s_3} = h_{i,s} \), and \( \mathcal{N}_{L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q),L(c_1,q)} = 0 \) otherwise.

4. Construction of intertwining operators

In this section we always adopt the following convention:

**Convention 4.1.** We always fix a nonzero highest weight vector \( v_{c,h} \) in the Verma module \( M(c, h) \) for each \( c \) and \( h \), and identify \( U(Vir^-) \) with \( M(c, h) \) by sending \( e_3 \) to \( e_3 v_{c,h} \).

We fix three \( L(c_1,q),0 \)-modules \( L(c_1,q),h_{i_1,sn} \) \((n = 1, 2, 3)\), where \( i_n > 0 \), \( 0 < s_n \leq q \), and assume that there exists \( i \in A_{i_1,i_2}, s \in A_{s_1,s_2} \), satisfying \( h_{i_3,s_3} = h_{i,s} \). The purpose of this section is to construct a nonzero intertwining operator of type \( L(c_1,q),h_{i_3,s_3} \) \( L(c_1,q),h_{i_1,sn} \) \( L(c_1,q),h_{i_2,sn} \) \( L(c_1,q),h_{i_3,s_3} \). Set \( c_k = c_{k,kq-1} \), \( h^k_n = \frac{(i_n(kq-1)-s_n,k)^2-(kq-1-k)^2}{4k(kq-1)} \). By the fusion rules of \( L(c_k,0) \) when \( k \) is large enough, there exists a nontrivial intertwining operator \( \mathcal{Y}_k(\cdot, x) \) of type \( L(c_k,h^k_3) \) \( L(c_k,h^k_3) \) \( L(c_k,h^k_3) \) \( L(c_k,h^k_3) \). Our method is to get the desired intertwining operator from the limit of \( \mathcal{Y}_k(\cdot, x) \) as \( k \) approaches infinity. Hence from now on...
on, we always assume that $k$ is large enough when needed. We say that a sequence of monomials $\{a_k x^{n_k}\}$ converges to the limit $ax^n$

$$\lim_{k \to \infty} a_k x^{n_k} = ax^n$$

if $\{a_k\}$ and $\{n_k\}$ converge to the limits $a$ and $n$ respectively. The following proposition is crucial for our construction.

**Proposition 4.2.** As a left $A(L(c_k, 0))$-module, $A(L(c_k, h_n^k))$ is generated by $[v_{c_k, h_n^k}], [L_{-1} v_{c_k, h_n^k}], \ldots, [L_{-1}^{i_n s_n - 1} v_{c_k, h_n^k}]$.

**Proof.** Combining Proposition 3.2, 3.3 and 3.4 implies the formula $[L_{-n} v] = (-1)^n (ny - x + wt(v))[v]$ in $A(M(c_k, h_n^k))$ for each homogenous $v \in L(c_k, h_n^k)$ (recall the identification $A(M(c_k, h_n^k)) = \mathbb{C}[x, y]$ in Proposition 3.3). From this formula and Proposition 2.1 we have

$$[L_{-1}^m v_{c_k, h_n^k}] = (x - y)^m + \text{lower terms}$$

and

$$[S_{i_n s_n} v_{c_k, h_n^k}] = [v_{i_n s_n}] = (x - y)^{i_n s_n} + \text{lower terms}$$

in $A(M(c_k, h_n^k)) \cong \mathbb{C}[x, y]$. Now by Proposition 3.4 $[S_{i_n s_n} v_{c_k, h_n^k}]$ lies in the kernel of the surjective morphism $A(M(c_k, h_n^k)) \twoheadrightarrow A(L(c_k, h_n^k))$, hence $A(L(c_k, h_n^k))$ can be generated, as a left $A(L(c_k, 0))$-module by

$$[v_{c_k, h_n^k}], [L_{-1} v_{c_k, h_n^k}], \ldots, [L_{-1}^{i_n s_n - 1} v_{c_k, h_n^k}]$$

\[ \square \]

Now we are well prepared for the construction. It suffices to construct a bilinear pair $[\cdot, \cdot]$ (with value in $\mathbb{C}[x, y]$) between $(L(c_1, q; h_{i_3, s_3})^*)$ and $L(c_1, q; h_{i_2, s_2}) \otimes L(c_1, q; h_{i_1, s_1})$ that satisfies the corresponding properties. The construction is divided into several steps. In the following using Convention 4.1 we always identify Verma modules $M(c, h)$ for different pairs of $\{c, h\}$. For a $\Vir$-module $M$ of lowest weight $h$, we always use $M(n)$ to denote the weight subspace of weight $h + n$. The same notations are applied to submodules, quotient modules, and dual modules of $M$.

**Step 1.** Let $v' \in L(c_1, q; h_{i_3, s_3})^*(0) = M(c_1, q; h_{i_3, s_3})^*(0) = M(c_k, h_n^k)^*(0)$ be defined by $v'(v_{c_1, q; h_{i_3, s_3}}) = 1$. For each homogenous $v_1 \in M(c_1, q; h_{i_1, s_1})$ and $v_2 = v_{c_1, q; h_{i_2, s_2}} = v_{c_k, h_n^k}$, $[v', v_1 \otimes v_2]$ is defined as follows:

Set $a_1 = v_{c_1, q; h_{i_1, s_1}}, a_2 = L_{-1} v_{c_1, q; h_{i_1, s_1}}, \ldots, a_{i_1 s_1} = L_{-1}^{i_1 s_1 - 1} v_{c_1, q; h_{i_1, s_1}}$ in $M(c_1, q; h_{i_1, s_1}) = M(c_k, h_n^k)$. Then for each $k$, there is some $i$ such that $\langle v', \mathcal{Y}_k(a_i^k, x)v_2 \rangle \neq 0$, otherwise, by Proposition 3.6 and 4.2 $\mathcal{Y}_k$ will be zero. As $\langle v', \mathcal{Y}_k(a_{i+1}, x)v_2 \rangle$ is the derivation of $\langle v', \mathcal{Y}_k(a_i, x)v_2 \rangle$, hence $\langle v', \mathcal{Y}_k(a_i, x)v_2 \rangle \neq 0$ for each $k$. 

By replacing each $\mathcal{Y}_k$ by a nonzero multiple, there exists a subsequence of intertwining operators $\{\mathcal{Y}_{n_k}\}$ such that
\[
\langle v', \mathcal{Y}_{n_1}(a_1, x)v_2 \rangle, \langle v', \mathcal{Y}_{n_2}(a_1, x)v_2 \rangle, \cdots, \langle v', \mathcal{Y}_{n_k}(a_1, x)v_2 \rangle, \cdots
\]
converge to a nonzero monomial. Now assume that the sequence
\[
\langle v', \mathcal{Y}_{n_1}(v, x)v_2 \rangle, \langle v', \mathcal{Y}_{n_2}(v, x)v_2 \rangle, \cdots, \langle v', \mathcal{Y}_{n_k}(v, x)v_2 \rangle, \cdots
\]
converges for a homogeneous $v \in M(c_{1,q}, h_{i_1,s_1})$. Then the same result holds for $L_0v$ and $L_{-1}v$. As the left action of $A(V_c)$ on $A(M(c, h))$ is given by
\[
[w][v] = [(L_{-2} + 2L_{-1} + L_0)v],
\]
by the construction of the linear map $\pi$ in §3, we see that
\[
\langle v', \mathcal{Y}_{n_1}(L_{-2}v, x)v_2 \rangle, \langle v', \mathcal{Y}_{n_2}(L_{-2}v, x)v_2 \rangle, \cdots, \langle v', \mathcal{Y}_{n_k}(L_{-2}v, x)v_2 \rangle, \cdots
\]
also converges. By induction and the equality $[L_{-n}, L_{-1}] = (1-n)L_{-n-1}$, the same is true for $L_{-n}v$ ($n > 2$). As $M(c_{1,q}, h_{i_1,s_1})$ is generated by $a_1 = v_{c_{1,q}, h_{i_1}}$, we conclude by induction that for each homogeneous $v_1 \in M(c_{1,q}, h_{i_1,s_1})$, the sequence
\[
\langle v', \mathcal{Y}_{n_1}(v_1, x)v_2 \rangle, \langle v', \mathcal{Y}_{n_2}(v_1, x)v_2 \rangle, \cdots, \langle v', \mathcal{Y}_{n_k}(v_1, x)v_2 \rangle, \cdots
\]
converges. Let $[v', v_1 \otimes v_2]$ be the limit and Step 1 is complete.

**Step 2.** For any $v_1 \in M(c_{1,q}, h_{i_1,s_1})$ and $v_2 \in M(c_{1,q}, h_{i_2,s_2})$ we want to define $[v', v_1 \otimes v_2]$ as the limit of the sequence
\[
(2) \quad \langle v', \mathcal{Y}_{n_1}(v_1, x)v_2 \rangle, \langle v', \mathcal{Y}_{n_2}(v_1, x)v_2 \rangle, \cdots, \langle v', \mathcal{Y}_{n_k}(v_1, x)v_2 \rangle, \cdots
\]
Thus we need to show that the limit of the sequence (2) exists for each $v_1 \in M(c_{1,q}, h_{i_1,s_1})$ and $v_2 \in M(c_{1,q}, h_{i_2,s_2})$. Step 1 shows that when $v_2 = v_{c_{1,q}, h_{i_2}}$, the limit exists. Thus by induction, it suffices to prove that if the limit of the sequence (2) exists for any $v_1 \in M(c_{1,q}, h_{i_1,s_1})$ and a fixed $v_2 \in M(c_{1,q}, h_{i_2,s_2})$, then the same is true with $v_2$ replaced by $L_nv_2$ ($n > 0$). But this follows directly from the following identity:
\[
\langle v', \mathcal{Y}_k(v_1, x)L_nv_2 \rangle = \langle v', L_n\mathcal{Y}_k(v_1, x)v_2 \rangle
\]
\[
= \sum_{i=0}^{\infty} \binom{n+1}{i} x^{n+1-i} \langle v', \mathcal{Y}_k(L_{i-1}v_1, x)v_2 \rangle
\]
\[
- \sum_{i=0}^{\infty} \binom{n+1}{i} x^{n+1-i} \langle v', \mathcal{Y}_k(L_{i-1}v_1, x)v_2 \rangle.
\]
Hence we conclude that the limit of the sequence (2) exists for any homogeneous $v_2 \in M(c_{1,q}, h_{i_1,s_1})$. Set
\[
[v', v_1 \otimes v_2] = \lim_{k \to \infty} \langle v', \mathcal{Y}_{n_k}(v_1, x)v_2 \rangle,
\]
and Step 2 is complete.

**Step 3.** Now we want to define \([v'_3, v_1 \otimes v_2]\) for any homogeneous \(v_1 \in M(c_1, q, h_{1,s_1})\), \(v_2 \in M(c_1, q, h_{2,s_2})\) and \(v'_3 \in L(c_1, q, h_{i,s_3})^* \subset M(c_1, q, h_{i,s_3})^*\).

**Lemma 4.3.** For any \(I \in P_n\), \(\langle e_i v', J_{nk}(v_1, x)v_2 \rangle\) converges to a finite limit as \(k\) approaches infinity.

**Proof.** By induction on the length of \(I\), this lemma follows directly from the formula

\[
\langle L_n w', w \rangle = \langle w', L_{-n} w \rangle
\]

for any \(w' \in L(c_k, h_{k}^3)^*\) and \(w \in L(c_k, h_{k}^3)\).

By Convention 4.1 we can identify \(M(c_1, q, h_{i,s_3})^*\) with \(M(c_k, h_{k}^3)^*\). Under this identification, \(L(c_k, h_{k}^3)^*(n)\) converges to \(L(c_1, q, h_{i,s_3})^*(n)\) as \(k\) approaches infinity. As \(L(c_1, q, h_{i,s_3})^*\) is irreducible, \(L(c_1, q, h_{i,s_3})^*\) is generated by \(L(c_1, q, h_{i,s_3})^*(0) = \mathbb{C} v'\) as a module over \(Vir^-\), thus we can choose a subset \(\{I_1, \ldots, I_s\}\) of \(P_n\) such that \(e_{I_1} v', \ldots, e_{I_s} v'\) form a basis of \(L(c_1, q, h_{i,s_3})^*(n)\).

By our convention, both \(L(c_{l_k}, h_{k}^3)^*(n)\) and \(L(c_{e_k}, h_{k}^3)^*(n)\) are subspaces of \(M(c_1, q, h_{i,s_3})^*(n) = M(c_k, h_{k}^3)^*(n)\). Moreover, by Proposition 2.1,

\[
\lim_{k \to \infty} L(c_k, h_{k}^3)^*(n) = L(c_1, q, h_{i,s_3})^*(n).
\]

Hence it is easy to see that \(e_{I_1} v', \ldots, e_{I_s} v'\), as elements of \(M(c_k, h_{k}^3)^*\), form a basis of \(L(c_{l_k}, h_{k}^3)^*(n)\), and converge to \(e_{I_1} v', \ldots, e_{I_s} v'\) respectively in \(M(c_1, q, h_{i,s_3})^*\) as \(k\) approaches infinity.

Now for any homogeneous \(v'_3 \in L(c_1, q, h_{i,s_3})^*(n)\) we can choose a \(v'_{3,k} \in L(c_k, h_{k}^3)^*(n)\), such that the sequence

\[
\cdots v'_{3,k-2}, v'_{3,k-1}, v'_{3,k}, v'_{3,k+1}, v'_{3,k+2}, \ldots
\]

converges to \(v'_3\). If we write

\[
v'_3 = a_1 e_{I_1} v' + \cdots + a_s e_{I_s} v',
\]

and

\[
v'_{3,k} = a_{I_1,k} e_{I_1} v' + \cdots + a_{I_s,k} e_{I_s} v',
\]

then, for each \(i\) the sequence

\[
\cdots a_{i,k}, a_{i,k+1}, a_{i,k+2}, \ldots
\]

converges to \(a_i\). By Lemma 4.3 we can set

\[
[v'_3, v_1 \otimes v_2] = \lim_{k \to \infty} \langle v'_{3,n_k}, J_{nk}(v_1, x)v_2 \rangle,
\]
and it is easy to see that this setting is independent of the choice of the sequence
\[ \cdots v_{3,k}, v'_{3,k+1}, v'_{3,k+2} \cdots \]

**Step 4.** Now we check that the pairing \([\cdot, \cdot]\) induces an intertwining operator of type \(L(c_{1,q}, h_{i_3,s_3})\) \(M(c_{1,q}, h_{i_1,s_1}) \, M(c_{1,q}, h_{i_2,s_2})\).

It suffices to verify the \(L_{-1}\)-derivative formula
\[
\frac{d}{dx} [v'_{3}, v_1 \otimes v_2] = [v'_{3}, L_{-1}v_1 \otimes v_2]
\]
and the Jacobi identity
\[
x_0^{-1} \delta(\frac{x_1 - x}{x_0}) [Y(e^{x_1 L_1}(-x_1^{-2}L_0 v), x_1^{-1}) v'_{3}, v_1 \otimes v_2]
\]
\[
- x_0^{-1} \delta(\frac{x_1 - x}{x_0}) [v'_{3}, v_1 \otimes Y(v, x_1)v_2]
\]
\[
x^{-1} \delta(\frac{x_1 - x_0}{x}) [v'_{3}, Y(v, x_1)v_1 \otimes v_2],
\]
where \(v \in V_{c_{1,q}}\) (this form of Jacobi identity follows from the graded dual module structure of \(L(c_{1,q}, h_{i_3,s_3})^*\) defined in §5.2 of [6]), and the truncation property follows as a consequence.

The \(L_{-1}\)-derivative formula follows directly from our definition of \([\cdot, \cdot]\) and the fact that the derivation \(\frac{d}{dx}\) commutes with the limiting operation. In order to prove the Jacobi identity, we identify \(V_{c_{1,q}}\) with \(V_{c_k}\) by linear isomorphism \(V_{c_{1,q}} \rightarrow V_{c_k}\) which sending 1 to 1 and commutates with the action of \(Vir^\cdot\). Then, it is easy to see that the coefficients of \(Y(v, x_0)v_1\) (resp. \(Y(v, x_1)v_2\), as elements of \(M(c_{k}, h_{1}^{k})\) (resp. \(M(c_{k}, h_{2}^{k})\)), converge to the corresponding coefficients of \(Y(v, x_0)v_1\) (resp. \(Y(v, x_1)v_2\)), as elements of \(M(c_{1,q}, h_{i_1,s_1})\) (resp. \(M(c_{1,q}, h_{i_2,s_2})\)). If we choose, as in Step 3, a \(v'_{3,k} \in L(c_{k}, h_{3}^{k})^*\) for each \(k\), such that the sequence
\[
\cdots v'_{3,k}, v'_{3,k+1}, v'_{3,k+2} \cdots
\]
converges to \(v'_{3}\), then the coefficients of \(Y(e^{x_1 L_1}(-x_1^{-2}L_0 v), x_1^{-1}) v'_{3,k}\) converge to the corresponding coefficients of \(Y(e^{x_1 L_1}(-x_1^{-2}L_0 v), x_1^{-1}) v'_{3}\). Now the Jacobi identity of \([\cdot, \cdot]\) follows from the Jacobi identities of \(Y_k\) and Step 4 is complete.

**Step 5.** Show that \([v'_{3}, v_1 \otimes v_2] = 0\) if \(v_1\) lies in the maximal proper submodule \(M'(c_{1,q}, h_{i_1,s_1})\) of \(M(c_{1,q}, h_{i_1,s_1})\). Let \(M'(c_{k}, h_{1}^{k})\) be the maximal submodule of \(M(c_{k}, h_{1}^{k})\), then it is easy to see that \(M'(c_{k}, h_{1}^{k})(n)\) converges
to $M'(c_1, q, h_{i, s_1}) (n)$ for each $n$ as $k$ approaches infinity. Thus there exists $v_{1,k} \in M'(c_k, h_1^k)$ for each $k$, such that the sequence

$$\cdots v_{1,k}, v_{1,k+1}, v_{1,k+2} \cdots$$

converges to $v_1$. Using the argument in Step 3, we see that

$$[v'_3, v_1 \otimes v_2] = \lim_{k \to \infty} \langle v'_{3,n_k}, \mathcal{Y}_{n_k}(v_{1,n_k}, x)v_2 \rangle$$

where

$$\cdots v'_{3,k}, v'_{3,k+1}, v'_{3,k+2} \cdots$$

is the sequence converging to $v'_3$ in Step 3. Since $\mathcal{Y}_k(\cdot, x)$ is an intertwining operator of type $(L(c_k, h_3^k), L(c_k, h_1^k))$, this forces $[v'_3, v_1 \otimes v_2] = 0$.

**Step 6.** Show that $[v'_3, v_1 \otimes v_2] = 0$ if $v_2$ lies in the maximal proper submodule $M'(c_1, q, h_{i, s_2})$ of $M(c_1, q, h_{i, s_2})$. It suffices to repeat the argument in Step 5 and we omit the details.

From the above construction we see that the pairing $[\cdot, \cdot]$ induces a nonzero intertwining operator $\mathcal{Y}(\cdot, x)$ of type $(L(c_1, h_{i, s_3}), L(c_1, h_{i, s_1}))$ such that

$$[v'_3, v_1 \otimes v_2] = \langle v'_3, \mathcal{Y}(v_1, x)v_2 \rangle.$$ This finishes our construction. Hence the proof of Theorem 2.3 is complete.

**Remark 4.4.** The limit method is quite necessary, for we can not construct these intertwining operators by lattice vertex operator algebra as in [3, 10]. We hope to formalize this method and find more applications in future work.

## 5. Further extension

From Section 2, we see that the Verma module $M(c_1, q, h)$ is irreducible if and only if $h \neq h_{i, s} = \frac{(iq-s)^2-(q-1)^2}{4q}$ for any $i > 0$, $0 < s \leq q$. In this section, we consider the fusion rules of the type $(M(c_1, q, h'), L(c_1, q, h_{i, s_1}))$, $M(c_1, q, h)$ and $(L(c_1, q, h_{i, s_1}), L(c_1, q, h_{i, s_2}))$ where $M(c_1, q, h)$ and $M(c_1, q, h')$ are irreducible Verma modules and $i_1, s_1, i_2, s_2$ are as before. By Theorem 2.11 in [12], Frenkel-Zhu’s formula holds in the first case. Hence the argument of Section 3 directly implies

**Theorem 5.1.**

$$\mathcal{A}_{L(c_1, q, h_{i, s}), M(c_1, q, h)}^t M(c_1, q, h') \leq 1$$
where \( i > 0, 0 < s \leq q \), and \( M(c_{1,q}, h) \) and \( M(c_{1,q}, h') \) are irreducible Verma modules. Set 
\[
    h = \frac{s^2 - (q-1)^2}{4q}
\]
for some complex number \( s' \). Then the equality holds if and only if 
\[
    h' = \frac{(jq - s')^2 - (q-1)^2}{4q}
\]
for some \( j \in \{-i+1, -i+3, \cdots, i-1\} \) and \( t \in \{-s+1, -i+3, \cdots, s-1\} \).

Similarly, we have

**Theorem 5.2.**

\[
    \mathcal{N}^{M(c_{1,q}, h)}_{L(c_{1,q}, h_{i_1}, s_1), L(c_{1,q}, h_{i_2}, s_2)} = 0
\]

where \( i_1, i_2 > 0, 0 < s_1, s_2 \leq q \) and \( M(c_{1,q}, h) \) is an irreducible Verma module.

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