BIHARMONIC HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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Abstract
We study biharmonic hypersurfaces in a generic Riemannian manifold. We first derive an invariant equation for such hypersurfaces generalizing the biharmonic hypersurface equation in space forms studied in [15], [7], [5], [6]. We then apply the equation to show that the generalized Chen’s conjecture is true for totally umbilical biharmonic hypersurfaces in an Einstein space, and construct a (2-parameter) family of conformally flat metrics and a (4-parameter) family of multiply warped product metrics each of which turns the foliation of an upper-half space of $\mathbb{R}^m$ by parallel hyperplanes into a foliation with each leaf a proper biharmonic hypersurface. We also study the biharmonicity of Hopf cylinders of a Riemannian submersion.

1. Biharmonic maps and submanifolds

All manifolds, maps, and tensor fields that appear in this paper are supposed to be smooth unless there is an otherwise statement.

A biharmonic map is a map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional
$$E^2(\varphi, \Omega) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2 \, dx$$
for every compact subset $\Omega$ of $M$, where $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ is the tension field of $\varphi$. The Euler-Lagrange equation of this functional gives the biharmonic map equation ([14])
$$\tau^2(\varphi) := \text{Trace}_g (\nabla^\varphi \nabla^\varphi - \nabla^\varphi_{\nabla^\varphi})\tau(\varphi) - \text{Trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

* Supported by Texas A & M University-Commerce “Faculty Development Program” (2008).
which states the fact that the map \( \varphi \) is biharmonic if and only if its bitension field \( \tau^2(\varphi) \) vanishes identically. In the above equation we have used \( R^N \) to denote the curvature operator of \((N,h)\) defined by

\[
R^N(X,Y)Z = [\nabla^N_X,\nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.
\]

Clearly, it follows from (1) that any harmonic map is biharmonic and we call those non-harmonic biharmonic maps **proper biharmonic maps**.

For a submanifold \( M^m \) of Euclidean space \( \mathbb{R}^n \) with the mean curvature vector \( H \) viewed as a map \( H : M \rightarrow \mathbb{R}^n \), B. Y. Chen [7] called it a biharmonic submanifold if \( \Delta H = (\Delta H^1, \ldots, \Delta H^n) = 0 \), where \( \Delta \) is the Beltrami-Laplace operator of the induced metric on \( M^m \). Note that if we use \( i : M \rightarrow \mathbb{R}^n \) to denote the inclusion map of the submanifold, then the tension field of the inclusion map \( i \) is given by \( \tau(i) = \Delta i = mH \) and hence the submanifold \( M^m \subset \mathbb{R}^n \) is biharmonic if and only if \( \Delta H = \Delta(\frac{1}{m}\Delta i) = \frac{1}{m}\Delta^2 i = \frac{1}{m}\tau^2(i) = 0 \), i.e., the inclusion map is a biharmonic map. In general, a submanifold \( M \) of \((N,h)\) is called a **biharmonic submanifold** if the inclusion map \( i : (M,i^*h) \rightarrow (N,h) \) is biharmonic isometric immersion. It is well-known that an isometric immersion is minimal if and only if it is harmonic. So a minimal submanifold is trivially biharmonic and we call a non-minimal biharmonic submanifold a **proper biharmonic submanifold**.

Here are some known facts about biharmonic submanifolds:

1. **Biharmonic submanifolds in Euclidean spaces**: Jiang [15], Chen-Ishikawa [8] proved that any biharmonic submanifold in \( \mathbb{R}^3 \) is minimal; Dimitrić [9] showed that any biharmonic curves in \( \mathbb{R}^n \) is a part of a straight line, any biharmonic submanifold of finite type in \( \mathbb{R}^n \) is minimal, any pseudo-umbilical submanifolds \( M^m \subset \mathbb{R}^n \) with \( m \neq 4 \) is minimal, and any biharmonic hypersurface in \( \mathbb{R}^n \) with at most two distinct principal curvatures is minimal; it is proved in [12] that any biharmonic hypersurface in \( \mathbb{R}^4 \) is minimal. Based on these, B. Y. Chen [7] proposed the conjecture: any biharmonic submanifold of Euclidean space is minimal, which is still open.

2. **Biharmonic submanifolds in hyperbolic space forms**: Caddeo, Montaldo and Oniciuc [6] showed that any biharmonic submanifold in hyperbolic 3-space \( H^3(-1) \) is minimal, and pseudo-umbilical biharmonic submanifold \( M^m \subset H^n \) with \( m \neq 4 \) is minimal. It is shown in [4] that any biharmonic hypersurface of \( H^n \) with at most two distinct principal curvatures is minimal. Based on these, Caddeo, Montaldo and Oniciuc [5] extended Chen’s conjecture to be the **generalized Chen’s conjecture**: any biharmonic submanifold in \((N,h)\) with \( \text{Riem}^N \leq 0 \) is minimal.
3. Biharmonic submanifolds in spheres: The first example of proper biharmonic submanifold in $S^{n+1}$ was found ([16]) to be the generalized Clifford torus $S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}})$ with $p \neq q, p + q = n$. The second type of the proper biharmonic submanifolds in $S^{n+1}$ was found in [5] to be hypersphere $S^n(\frac{1}{\sqrt{2}})$. The authors in [5] also gave a complete classification of biharmonic submanifolds in $S^3$. It was proved in [4] that any pseudo-umbilical biharmonic submanifold $M^m \subset S^{n+1}$ with $m \neq 4$ has constant mean curvature whilst in [4] the same authors showed that a hypersurface $M^n \subset S^{n+1}$ with at most two distinct principal curvatures (which, for $n > 3$, is equivalent to saying that $M$ is a quasi-umbilical or conformally flat hypersurface in $S^{n+1}$ [19]) is biharmonic, then $M$ is an open part of the hypersphere $S^n(\frac{1}{\sqrt{2}})$, or the generalized Clifford torus $S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}})$ with $p \neq q, p + q = n$. Some example of proper biharmonic real hypersurfaces in $CP^n$ were found and all proper biharmonic tori $T^{n+1} = S^1(r_1) \times S^1(r_2) \times \ldots \times S^1(r_{n+1})$ in $S^{2n+1}$ were determined in [27]. All the known examples of biharmonic submanifolds in spheres lead to the conjecture [4]: any biharmonic submanifold in sphere has constant mean curvature; and any proper biharmonic hypersurface in $S^{n+1}$ is an open part of the hypersphere $S^n(\frac{1}{\sqrt{2}})$, or the generalized Clifford torus $S^p(\frac{1}{\sqrt{2}}) \times S^q(\frac{1}{\sqrt{2}})$ with $p \neq q, p + q = n$.

4. Biharmonic submanifolds in other model spaces: For the study of biharmonic curves in various model spaces we refer the readers to the survey article [18], and for special biharmonic submanifolds in contact manifolds or Sasakian space forms see recent works [1], [13], [10], [11], [24], and [25]. Some constructions and classifications of biharmonic surfaces in three-dimensional geometries will appear in [23].

5. Biharmonic submanifolds in other senses: We would like to point out that some authors (as in [26]) use $\Delta H = 0$ to define a “biharmonic submanifold” in a Riemannian manifold, which agree with our notion of biharmonic submanifold only if the ambient space is flat. For conformal biharmonic submanifolds (i.e., conformal biharmonic immersions) see [22].

In this paper, we study biharmonic hypersurfaces in a generic Riemannian manifold. In Section 2, we derive an invariant equation for biharmonic hypersurfaces in a Riemannian manifold that involves the mean curvature function, the norm of the second fundamental form, the shape operator of the hypersurface, and the Ricci curvature of the ambient space, and prove that the generalized Chen’s conjecture is true for totally umbilical hypersurfaces in an Einstein space.
Section 3 is devoted to construct a family of conformally flat metrics and a family of multiply warped product metrics each of which turns the foliation of an upper-half space of $\mathbb{R}^m$ by parallel hyperplanes into a foliation with each leaf a proper biharmonic hypersurface. These are accomplished by starting with hyperplanes in Euclidean space then looking for a special type of conformally flat or multiply warped product metrics on the ambient space that reduce the biharmonic hypersurface equation into ordinary differential equations whose solutions give the metrics that render the inclusion maps proper biharmonic isometric immersions. Finally, we study biharmonicity of Hopf cylinders given by a Riemannian submersion from a complete 3-manifold in Section 4. Our method shows that there is no proper biharmonic Hopf cylinder in $S^3$ which recovers Proposition 3.1 in [13].

2. THE EQUATIONS OF BIHARMONIC HYPERSURFACES

Recall that if $\varphi : M \rightarrow (N, h)$ is the inclusion map of a submanifold, or more generally, an isometric immersion, then we have an orthogonal decomposition of the vector bundle $\varphi^{-1}TN = \tau M \oplus \nu M$ into the tangent and normal bundles. We use $d\varphi$ to identify $TM$ with its image $\tau M$ in $\varphi^{-1}TN$. Then, for any $X, Y \in \Gamma(TM)$ we have $\nabla^\varphi_X(d\varphi(Y)) = \nabla^\varphi_X Y$, whereas $d\varphi(\nabla^\varphi_N Y)$ equals the tangential component of $\nabla^\varphi_N Y$. It follows that

$$\nabla d\varphi(X, Y) = \nabla^\varphi_X(d\varphi(Y)) - d\varphi(\nabla^\varphi_N Y) = B(X, Y),$$

i.e., the second fundamental form $\nabla d\varphi(X, Y)$ of the isometric immersion $\varphi$ agrees with the second fundamental form $B(X, Y)$ of the immersed submanifold $\varphi(M)$ in $N$ (see [17], Chapter 7, also [2], Example 3.2.3 for details). From (2) we see that the tension field $\tau(\varphi)$ of an isometric immersion and the mean curvature vector field $\eta$ of the submanifold are related by

$$\tau(\varphi) = m\eta.$$

For a hypersurface, i.e., a codimensional one isometric immersion $\varphi : M^m \rightarrow N^{m+1}$, we can choose a local unit normal vector field $\xi$ to $\varphi(M) \subset N$. Then, $\eta = H\xi$ with $H$ being the mean curvature function, and we can write $B(X, Y) = b(X, Y)\xi$, where $b : TM \times TM \rightarrow C^\infty(M)$ is the function-valued second fundamental form. The relationship between the shape operator $A$ of the hypersurface with respect to the unit normal vector field $\xi$ and the second fundamental form is given by

$$B(X, Y) = \langle \nabla^\varphi_N Y, \xi \rangle \xi = -\langle Y, \nabla^\varphi_N \xi \rangle \xi = \langle AX, Y \rangle \xi,$$

$$\langle AX, Y \rangle = \langle B(X, Y), \xi \rangle = \langle b(X, Y)\xi, \xi \rangle = b(X, Y).$$
Theorem 2.1. Let $\varphi : M^m \rightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then $\varphi$ is biharmonic if and only if:

$$
\begin{aligned}
\Delta H - H|A|^2 + H\text{Ric}^N(\xi, \xi) &= 0, \\
2A(\text{grad} H) + \frac{m}{2}\text{grad} H^2 - 2 H (\text{Ric}^N(\xi))^\top &= 0,
\end{aligned}
$$

where $\text{Ric}^N : T_qN \rightarrow T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W)$ and $A$ is the shape operator of the hypersurface with respect to the unit normal vector $\xi$.

Proof. Choose a local orthonormal frame $\{e_i\}_{i=1,...,m}$ on $M$ so that $\{d\varphi(e_1), \ldots, d\varphi(e_m), \xi\}$ is an adapted orthonormal frame of the ambient space defined on the hypersurface. Identifying $d\varphi(X) = X$, $\nabla^\varphi WX = \nabla^NWX$ and noting that the tension field of $\varphi$ is $\tau(\varphi) = mH\xi$ we can compute the bitension field of $\varphi$ as:

$$
\tau^2(\varphi) = m\sum_{i=1}^m \{\nabla^\varphi e_i (mH\xi) - \nabla^\varphi e_i, e_i(mH\xi) - \text{Ric}^N(d\varphi(e_i), mH\xi)d\varphi(e_i)\} = m\sum_{i=1}^m \{e_ie_i(H)\xi + 2e_i(H)\nabla^N e_i\xi + H\nabla^N e_i\nabla^N e_i\xi - (\nabla^\varphi e_i)(H)\xi - H\nabla^N e_i e_i\xi\}
$$

$$
= \frac{m}{2}(-\text{Ric}^N(\xi, \xi), \xi) - mH\sum_{i=1}^m \text{Ric}^N(d\varphi(e_i), \xi)d\varphi(e_i) - [\text{Ric}^N(\xi, e_k)]e_k = -(\text{Ric}^N(\xi, \xi))^\top,
$$

and

$$
\sum_{i,k=1}^m \langle \text{Ric}^N(d\varphi(e_i), \xi)d\varphi(e_i), e_k \rangle e_k = \sum_{i,k=1}^m \langle \text{Ric}^N(\xi, e_k) \rangle e_k = \sum_{i,k=1}^m \langle \text{Ric}^N(\xi, e_k) \rangle e_k = \sum_{i,k=1}^m \langle \nabla^N e_i e_i, \nabla^N e_i e_i \rangle e_k.
$$

To find the normal part of $\Delta^\varphi \xi$ we compute:

$$
\langle \Delta^\varphi \xi, \xi \rangle = \sum_{i=1}^m \langle -\nabla^N e_i e_i e_i e_i \xi + \nabla^N e_i e_i e_i e_i, \xi \rangle = \sum_{i=1}^m \langle \nabla^N e_i e_i e_i e_i, \nabla^N e_i e_i \rangle e_k.
$$

On the other hand, using (4) and (5) we have

$$
|A|^2 = \sum_{i,j=1}^m \langle Ae_i, e_j \rangle^2 = \sum_{i,j=1}^m \langle \nabla^N e_i e_i e_i, \nabla^N e_i e_i, e_j e_j \rangle = \sum_{i=1}^m \langle \nabla^N e_i e_i e_i, \nabla^N e_i e_i \rangle e_k,
$$
which, together with (10), implies that

\[(\Delta^2 \varphi) \perp = \langle \Delta^2 \varphi, \xi \rangle \xi = \sum_{i=1}^{m} \langle \nabla^N_{e_i} \xi, \nabla^N_{e_i} \xi \rangle \xi = |A|^2 \xi. \]

A straightforward computation gives the tangential part of \(\Delta^2 \varphi\) as

\[(\Delta^2 \varphi) \perp = \sum_{i,k=1}^{m} \langle -\nabla^{N}_{e_i} \nabla^{N}_{e_i} \xi + \nabla^{N}_{e_i e_i} \xi, e_k \rangle e_k = \sum_{i,k=1}^{m} \left[ (\nabla_{e_i} b)(e_k, e_i) \right] e_k. \]

Substituting Codazzi-Mainardi equation for a hypersurface:

\[\langle R^{N}_{e_i e_k} \xi, \xi \rangle = \sum_{i,k=1}^{m} \left[ (\nabla_{e_i} b)(e_k, e_i) \right] e_k = \sum_{i,k=1}^{m} \left[ m \sum_{i=1}^{m} (\nabla_{e_i} b)(e_i, e_i) - \operatorname{Ric}(\xi, e_k) \right] e_k = m \operatorname{grad}(H) - [\operatorname{Ric}(\xi, e_k)] e_k. \]

Therefore, by collecting all the tangent and normal parts of the bitension field separately, we have

\[(\Delta^2 \varphi) \perp = \sum_{i,k=1}^{m} \left[ (\nabla_{e_i} b)(e_k, e_i) \right] e_k = \sum_{i,k=1}^{m} \left[ m \sum_{i=1}^{m} (\nabla_{e_i} b)(e_i, e_i) - \operatorname{Ric}(\xi, e_k) \right] e_k = m \operatorname{grad}(H) - \sum_{i=1}^{m} (\nabla_{e_i} b)(e_i, e_k) \right] e_k = m \operatorname{grad}(H) - [\operatorname{Ric}(\xi, e_k)] e_k. \]

from which the theorem follows. \(\Box\)

As an immediate consequence of Theorem 2.1 we have

\[\text{Corollary 2.2. A constant mean curvature hypersurface in a Riemannian manifold is biharmonic if and only if it is minimal or, } \operatorname{Ric}^N(\xi, \xi) = |A|^2 \text{ and } (\operatorname{Ric}^N(\xi, \xi))^\perp = 0. \text{ In particular, we recovered Proposition 2.4 in [21] which states that a constant mean curvature hypersurface in a Riemannian manifold } (N^{m+1}, h) \text{ with nonpositive Ricci curvature is biharmonic if and only if it is minimal.}\]

\[\]
Corollary 2.3. A hypersurface in an Einstein space \((N^{m+1}, h)\) is biharmonic if and only if its mean curvature function \(H\) is a solution of the following PDEs

\[
\begin{align*}
\Delta H - H |A|^2 + \frac{rH}{m+1} &= 0, \\
2A (\text{grad } H) + \frac{m}{2} \text{ grad } H^2 &= 0,
\end{align*}
\]

where \(r\) is the scalar curvature of the ambient space. In particular, a hypersurface \(\varphi : (M^m, g) \to (N^{m+1}(C), h)\) in a space of constant sectional curvature \(C\) is biharmonic if and only if its mean curvature function \(H\) is a solution of the following PDEs which was obtained by different authors in several steps (see [15], [7] and [6])

\[
\begin{align*}
\Delta H - H |A|^2 + mCH &= 0, \\
2A (\text{grad } H) + \frac{m}{2} \text{ grad } H^2 &= 0.
\end{align*}
\]

Proof. It is well known that if \((N^{m+1}, h)\) is an Einstein manifold then \(\text{Ric}^N(Z, W) = \frac{r}{m+1} h(Z, W)\) for any \(Z, W \in TN\) and hence \((\text{Ric}^N(\xi))^\top = 0\) and \(\text{Ric}^N(\xi, \xi) = \frac{r}{m+1}\). From these and Equation (6) we obtain Equation (17). When \((N^{m+1}(C), h)\) is a space of constant sectional curvature \(C\), then it is an Einstein space with the scalar curvature \(r = m(m+1)C\). Substituting this into (17) we obtain (18). \(\square\)

Theorem 2.4. A totally umbilical hypersurface in an Einstein space with non-positive scalar curvature is biharmonic if and only if it is minimal.

Proof. Take an orthonormal frame \(\{e_1, \ldots, e_m, \xi\}\) of \((N^{m+1}, h)\) adapted to the hypersurface \(M\) such that \(Ae_i = \lambda_i e_i\), where \(A\) is the Weingarten map of the hypersurface and \(\lambda_i\) is the principal curvature in the direction \(e_i\). Since \(M\) is supposed to be totally umbilical, i.e., all principal normal curvatures at any point \(p \in M\) are equal to the same number \(\lambda(p)\). It follows that

\[
H = \frac{1}{m} \sum_{i=1}^m \langle Ae_i, e_i \rangle = \lambda,
\]

\[
A(\text{grad } H) = A(\sum_{i=1}^m (e_i \lambda) e_i) = \frac{1}{2} \text{grad } \lambda^2,
\]

\[
|A|^2 = m\lambda^2.
\]

The biharmonic hypersurface equation (17) becomes

\[
\begin{align*}
\Delta \lambda - m\lambda^3 + \frac{r\lambda}{m+1} &= 0, \\
(2 + m) \text{grad } \lambda^2 &= 0.
\end{align*}
\]
Solving the equation we have either \( \lambda = 0 \) and hence \( H = 0 \), or \( \lambda = \pm \sqrt{\frac{r}{m(m+1)}} \) is a constant and this happens only if the scalar curvature is nonnegative, from which we obtain the theorem. \( \square \)

**Remark 1.** Our Theorem 2.4 generalizes the results of [3], [6] and [9] about the totally umbilical biharmonic hypersurfaces in a space form. It also implies that the generalized B. Y. Chen’s conjecture is true for totally umbilical hypersurfaces in an Einstein space with non-positive scalar curvature. Note that non-positive scalar curvature is a much weaker condition than non-positive sectional curvature.

**Corollary 2.5.** Any totally umbilical biharmonic hypersurface in a Ricci flat manifold is minimal.

**Proof.** These follows from Theorem 2.4 and the fact that a Ricci flat manifold is an Einstein space with zero scalar curvature. \( \square \)

### 3. Proper Biharmonic foliations of codimension one

In general, proper biharmonic maps as local solutions of a system of 4-th order PDEs are extremely difficult to unearth. Even in the case of biharmonic submanifolds (viewed as biharmonic maps with geometric constraints) few examples have been found. In this section, we construct families of metrics that turns some foliations of hypersurfaces into proper biharmonic foliations thus providing infinitely many proper biharmonic hypersurfaces.

**Theorem 3.1.** For any constant \( C \), let \( N = \{(x_1, \ldots, x_m, z) \in \mathbb{R}^{m+1} | z > -C \} \) denote the upper half space. Then, the conformally flat space \( (N, h = f^{-2}(z)(\sum_{i=1}^{m} dx_i^2 + dz^2)) \) is foliated by proper biharmonic hyperplanes \( z = k \) (\( k \in \mathbb{R}, k > -C \)) if and only if \( f(z) = \frac{D}{2+z^2}, \) where \( E \geq C \) and \( D \in \mathbb{R} \setminus \{0\} \).

**Proof.** Consider the isometric immersion \( \varphi : (\mathbb{R}^{m}, g) \longrightarrow (\mathbb{R}^{m+1}, h = f^{-2}(z)(\sum_{i=1}^{m} dx_i^2 + dz^2)) \) with \( \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, k) \) and \( k \) being a constant, where the induced metric \( g \) with respect to the natural frame \( \partial_i = \frac{\partial}{\partial x_i}, i = 1, 2, \ldots, m, \partial_{m+1} = \frac{\partial}{\partial z} \) has components

\[
g_{ij} = g(\partial_i, \partial_j) = h(d\varphi(\partial_i), d\varphi(\partial_j)) = \begin{cases} f^{-2}(k), & i = j, \\ 0 & i \neq j. \end{cases}
\]

One can check that \( e_A = f(z)\partial_A \ (A = 1, 2, \ldots, m, m + 1) \) constitute a local orthonormal frame on \( \mathbb{R}^{m+1} \) adapted to the hypersurface \( z = k \) with \( \xi = e_{m+1} \) being the unit normal vector field. A straightforward computation using Koszul’s formula gives the coefficients of the Levi-Civita connection of the ambient space as

\[
\begin{align*}

\end{align*}
\]
\[ \nabla_{e_A} e_B = \begin{pmatrix} f'e_{m+1} & 0 & \ldots & 0 & -f'e_1 \\ 0 & f'e_{m+1} & \ldots & 0 & -f'e_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & f'e_{m+1} & -f'e_m \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{(m+1) \times (m+1)}. \]

Noting that \( \xi = e_{m+1} \) is the unit normal vector field we can easily compute the components of the second fundamental form as

\[ h(e_i, e_j) = \langle \nabla_{e_i} e_j, e_{m+1} \rangle = \begin{cases} f', & i = j = 1, 2, \ldots, m; \\ 0, & \text{for all other cases}. \end{cases} \]

from which we conclude that each of the hyperplane \( z = k \) is a totally umbilical hypersurface in the conformally flat space.

We compute the mean curvature of the hypersurface to have

\[ H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i) = f', \]

and the norm of the second fundamental form is given by

\[ |A|^2 = \sum_{i=1}^{m} \left| h(e_i, e_i) \right|^2 = mf'^2. \]

Since \( H \) depends only on \( z \) we have

\[ \operatorname{grad}_g H = \sum_{i=1}^{m} e_i(H) e_i = 0 \]

and hence \( \Delta_g H = \text{div}(\operatorname{grad}_g H) = 0 \). Therefore, by Theorem 2.1, the biharmonic equation of the isometric immersion reduces to the following system

\[ \begin{cases} - |A|^2 + \operatorname{Ric}^N(\xi, \xi) = 0, \\ \sum_{i=1}^{m} (\operatorname{Ric}^N(\xi, e_i)) e_i = 0. \end{cases} \]

We can compute the Ricci curvature of the ambient space to have

\[ \operatorname{Ric}(e_i, \xi) = \operatorname{Ric}(e_i, e_{m+1}) = \sum_{j=1}^{m} \langle R(e_{m+1}, e_j) e_j, e_i \rangle = 0, \quad \forall \ i = 1, 2, \ldots, m. \]

\[ \operatorname{Ric}(\xi, \xi) = \operatorname{Ric}(e_{m+1}, e_{m+1}) = \sum_{j=1}^{m} \langle R(e_{m+1}, e_j) e_j, e_{m+1} \rangle = mf'' - mf'^2. \]

Substitute these into Equation (20) we conclude that all isometric immersions \( \varphi : \mathbb{R}^m \rightarrow (\mathbb{R}^{m+1}, h = f^{-2}(z)(\sum_{i=1}^{m} dx_i^2 + dz^2)) \) with \( \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, k) \)
are biharmonic if and only if
\[ ff'' - 2f'^2 = 0. \]
This equation can be written as
\[
\left( \frac{f'}{f} \right)' - \left( \frac{f'}{f} \right)^2 = 0.
\]
Solving this ordinary differential equation we obtain the solutions \( f(z) = \frac{D}{z + C} \) where \( C, D \) are constants. Since the mean curvature of the hypersurface \( H = f'(k) \) is never zero we conclude that each of the hyperplanes \( z = k \) \((k \neq -C)\) is a proper biharmonic hypersurface in the conformally flat space \((N, h = (z^2 + C)^2(\sum_{i=1}^m dx_i^2 + dz^2))\). This completes the proof of the theorem.

**Theorem 3.2.** The isometric immersion \( \varphi : \mathbb{R}^2 \rightarrow (\mathbb{R}^3, h = e^{2p(z)}dx^2 + e^{2q(z)}dy^2 + dz^2) \) with \( \varphi(x, y) = (x, y, c) \) is biharmonic if and only if
\[
p'' + 2p'^2 + q'' + 2q'^2 = 0.
\]
In particular, for any positive constants \( A, B, C, D \), the upper half space \((\mathbb{R}^3_+ = \{(x, y, z)|z > 0\})\) with the metric \( h = (Az + B)d\xi^2 + (Cz + D)d\zeta^2 + dz^2 \) is foliated by proper biharmonic planes \( z = c \) constant.

**Proof.** Consider the isometric immersion \( \varphi : \mathbb{R}^2 \rightarrow (\mathbb{R}^3_+, h = e^{2p(z)}dx^2 + e^{2q(z)}dy^2 + dz^2) \) with \( \varphi(x, y) = (x, y, c) \) and \( c > 0 \) being a constant. Using the notations \( \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z} \) we can easily check that the induced metric is given by
\[
\left\{ \begin{array}{l}
g_{11} = g(\partial_1, \partial_1) = h(d\varphi(\partial_1), d\varphi(\partial_1)) \circ \varphi = e^{2p(c)},
g_{12} = g(\partial_1, \partial_2) = h(d\varphi(\partial_1), d\varphi(\partial_2)) \circ \varphi = 0,
g_{22} = g(\partial_2, \partial_2) = h(d\varphi(\partial_2), d\varphi(\partial_2)) \circ \varphi = e^{2q(c)}. \end{array} \right.
\]
One can also check that \( e_1 = e^{-p(z)}\partial_1, \ e_2 = e^{-q(z)}\partial_2, \ e_3 = \partial_3 \) constitute an orthonormal frame on \( \mathbb{R}^3_+ \) adapted to the surface \( z = c \) with \( \xi = e_3 \) being the unit normal vector field. A further computation gives the following Lie brackets
\[
[e_1, e_2] = 0, \ [e_1, e_3] = p'e_1, \ [e_2, e_3] = q'e_2,
\]
and the coefficients of the Levi-Civita connection
\[
\begin{align*}
\nabla_{e_1}e_1 &= -p'e_3, & \nabla_{e_1}e_2 &= 0, & \nabla_{e_1}e_3 &= p'e_1, \\
\nabla_{e_2}e_1 &= 0, & \nabla_{e_2}e_2 &= -q'e_3, & \nabla_{e_2}e_3 &= q'e_2, \\
\nabla_{e_3}e_1 &= 0, & \nabla_{e_3}e_2 &= 0, & \nabla_{e_3}e_3 &= 0.
\end{align*}
\]
Noting that $\xi = e_3$ is the unit normal vector field we can compute the components of the second fundamental form as

$$h(e_1, e_1) = \langle \nabla_{e_1} e_1, e_3 \rangle = -p',$$

$$h(e_1, e_2) = \langle \nabla_{e_1} e_2, e_3 \rangle = 0,$$

$$h(e_2, e_2) = \langle \nabla_{e_2} e_2, e_3 \rangle = -q'.$$

From these we obtain the mean curvature of the isometric immersion

$$H = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2)) = -(p' + q')/2,$$

and the norm of the second fundamental form

$$|A|^2 = \sum_{i=1}^2 |h(e_i, e_i)|^2 = p'^2 + q'^2.$$

Since $H$ depends only on $z$ we have $\text{grad}_g H = e_1(H)e_1 + e_2(H)e_2 = 0$ and hence $\Delta_g H = \text{div}(\text{grad}_g H) = 0$. Therefore, by Theorem 2.1 the biharmonic equation of the isometric immersion reduces to Equation (20) with $m = 2$. To compute the Ricci curvature of the ambient space we can use (22) and (23) to have

$$\text{Ric}(e_1, \xi) = \text{Ric}(e_1, e_3) = \langle R(e_3, e_2)e_2, e_1 \rangle = 0,$$

$$\text{Ric}(e_2, \xi) = \text{Ric}(e_2, e_3) = \langle R(e_3, e_1)e_1, e_3 \rangle = 0,$$

$$\text{Ric}(\xi, \xi) = \text{Ric}(e_3, e_3) = \langle R(e_3, e_1)e_1, e_3 \rangle + \langle R(e_3, e_2)e_2, e_3 \rangle$$

$$= -p'' - p'^2 - q'' - q'^2.$$

Substitute these into Equation (20) with $m = 2$ we conclude that the isometric immersion $\varphi : \mathbb{R}^2 \longrightarrow (\mathbb{R}^3, h = e^{2p(z)}dx^2 + e^{2q(z)}dy^2 + dz^2)$ with $\varphi(x, y) = (x, y, c)$ is biharmonic if and only if Equation (21) holds, which gives the first statement of the Theorem. The second statement of the theorem is obtained by looking for the solutions of (21) satisfying $p'' + 2p'^2 = 0$ and $q'' + 2q'^2 = 0$. In fact, we have special solutions $p(z) = \frac{1}{2} \ln(Az + B)$ and $q(z) = \frac{1}{2} \ln(Cz + D)$ with positive constants $A, B, C, D$. By (21) and the choice of positive constants $A, B, C, D$ we see that the mean curvature of the surface $z = c$ is

$$H = -\frac{2ACz + AD + BC}{(Az + B)(Cz + D)} \neq 0$$

and hence each such surface is a non-minimal biharmonic surface. This completes the proof of the theorem.

Remark 2. One can check that Theorem 3.2 has a generalization to a higher dimensional space $\mathbb{R}^{m+1}$ for $m > 3$.

Example 1. Let $\lambda(t) = \sqrt{At + B}$ with positive constants $A, B$. Then, the warped product space $N = (S^2 \times \mathbb{R}^+, h = \lambda^2(t)g^{S^2} + dt^2)$ is foliated by the spheres $(S^2 \times \{t\}, \lambda^2(t)g^{S^2})$ each of which is a totally umbilical proper biharmonic surface.
Consider the isometric immersion $\varphi$ reduces to Equation (20) with $m = 2$, therefore, by Theorem 2.1, the proper biharmonic equation of the isometric immersion

$\nabla^2 \varphi = 0$

Then, the standard metric can be written as $g^{S^2} = d\rho^2 + \sin^2 \rho \, d\theta^2$ and hence the warped product metric on $N$ takes the form $h = \lambda^2(t) \, d\rho^2 + \lambda^2(t) \sin^2 \rho \, d\theta^2 + dt^2$. Consider the isometric immersion $\varphi : S^2 \rightarrow (\mathbb{R}^+ \times S^2, dt^2 + \lambda^2(t) g^{S^2})$ with $\varphi(\rho, \theta) = (\rho, \theta, c)$ and $c > 0$ being a constant. Using the notations $\partial_1 = \frac{\partial}{\partial \rho}$, $\partial_2 = \frac{\partial}{\partial \theta}$, and $\partial_3 = \frac{\partial}{\partial t}$ we can easily check that the induced metric is given by

$$
\begin{cases}
g_{11} = g(\partial_1, \partial_1) = h(d\varphi(\partial_1), d\varphi(\partial_1)) \circ \varphi = \lambda^2(c), \\
g_{12} = g(\partial_1, \partial_2) = h(d\varphi(\partial_1), d\varphi(\partial_2)) \circ \varphi = 0, \\
g_{22} = g(\partial_2, \partial_2) = h(d\varphi(\partial_2), d\varphi(\partial_2)) \circ \varphi = \lambda^2(c) \sin^2 \rho.
\end{cases}
$$

Using the orthonormal frame $e_1 = \lambda^{-1}(t) \partial_1$, $e_2 = (\lambda(t) \sin \rho)^{-1} \partial_2$, $e_3 = \partial_3$ we have the Lie brackets

$$
[e_1, e_2] = -\frac{\cot \rho}{\lambda} e_2, \quad [e_1, e_3] = f e_1, \quad [e_2, e_3] = f e_2,
$$

where and in the sequel we use the notation $f = (\ln \lambda)' = \frac{\lambda'}{\lambda}$. Clearly, $e_1, e_2, \xi = e_3 = \partial_3$ constitute a local orthonormal frame of $N$ adapted to the surface with $\xi = e_3 = \partial_3$ being the unit normal vector field of the surface. We can use the Koszul formula to compute the components of the second fundamental form as

$$
h(e_1, e_1) = \langle \nabla_{e_1} e_1, \xi \rangle = \langle \nabla_{e_1} e_1, e_3 \rangle = \frac{1}{2} (-\langle e_1, [e_1, e_3] \rangle - \langle e_1, [e_1, e_3] \rangle + \langle e_3, [e_1, e_1] \rangle) = -f,
$$

$$
h(e_1, e_2) = \langle \nabla_{e_1} e_2, \xi \rangle = \langle \nabla_{e_1} e_2, e_3 \rangle = 0,
$$

$$
h(e_2, e_2) = \langle \nabla_{e_2} e_2, \xi \rangle = \langle \nabla_{e_2} e_2, e_3 \rangle = -f,
$$

from which we conclude that each of such spheres is totally umbilical surface in $N$.

Notice that the mean curvature of the isometric immersion is $H = \frac{1}{2} (h(e_1, e_1) + h(e_2, e_2)) = -f$, and the norm of the second fundamental form $|A|^2 = \sum_{i=1}^2 |h(e_i, e_i)|^2 = 2f^2$, which depend only on $t$. It follows that $\text{grad}_g H = 0$ and $\Delta_g H = 0$. Therefore, by Theorem 2.4 the proper biharmonic equation of the isometric immersion reduces to Equation (20) with $m = 2$. 
On the other hand, using the Ricci curvature formula (see e.g., [?]) of the warped product $M = B \times_\lambda F$ we have
\[
\text{Ric}(e_1, \xi) = \text{Ric}(e_1, e_3) = 0, \quad \text{Ric}(e_2, \xi) = \text{Ric}(e_2, e_3) = 0,
\]
\[
\text{Ric}(\xi, \xi) = \text{Ric}(e_3, e_3) = \text{Ric}(e_3, e_3) - \frac{2}{\lambda} \text{Hess}_\lambda(e_3, e_3)
\]
\[
= -2 \lambda (e_3 \lambda) - d\lambda(\nabla e_3 e_3) = \frac{-2\lambda''}{\lambda}.
\]
Substitute these into Equation (20) with $m = 2$ we conclude that the isometric immersion $\varphi : S^2 \longrightarrow (S^2 \times \mathbb{R}^+, \lambda^2(t)g_{S^2} + dt^2)$ with $\varphi(\rho, \theta) = (\rho, \theta, c)$ is biharmonic if and only if
\[
-2 \left( \frac{\lambda'}{\lambda} \right)^2 - \frac{2\lambda''}{\lambda} = 0.
\]
Solving this final equation we have $\lambda(t) = \sqrt{At + B}$ and from which we obtain the proposition.

Remark 3. The author would like to thank the referee for informing him that the biharmonicity of the inclusion maps in Example 1 can be obtained as a particular case of Corollary 3.4 in [3] which was proved by a different method.

4. Biharmonic cylinders of a Riemannian submersion

Let $\pi : (M^3, g) \longrightarrow (N^2, h)$ be a Riemannian submersion with totally geodesic fibers from a complete manifold. Let $\alpha : I \longrightarrow (N^2, h)$ be an immersed regular curve parametrized by arclength. Then $\Sigma = \bigcup_{t \in I} \pi^{-1}(\alpha(t))$ is a surface in $M$ which can be viewed as a disjoint union of all horizontal lifts of the curve $\alpha$. Let $\{\bar{X}, \bar{\xi}\}$ be a Frenet frame along $\alpha$ and $\bar{\kappa}$ be the geodesic curvature of the curve. Then, the Frenet formula for $\alpha$ is give by
\[
\begin{cases}
\nabla_{\bar{X}} \bar{X} = \bar{\kappa} \bar{\xi}, \\
\nabla_{\bar{X}} \bar{\xi} = -\bar{\kappa} \bar{X},
\end{cases}
\]
where $\nabla$ denote the Levi-Civita connection of $(N, h)$. Let $\beta : I \longrightarrow (M^3, g)$ be a horizontal lift of $\alpha$. Let $X$ and $\xi$ be the horizontal lifts of $\bar{X}$ and $\bar{\xi}$ respectively. Let $V$ be the unit vector field tangent to the fibers of the submersion $\pi$. Then $\{X, \xi, V\}$ form an orthonormal frame of $M$ adapted to the surface with $\xi$ being the unit normal vector of the surface. Notice that the restriction of this frame to the curve $\beta$ is the Frenet frame along $\beta$. Therefore, we have the Frenet formula
along $\beta$ given by

\[
\begin{align*}
\nabla_X X &= \kappa \xi, \\
\nabla_X \xi &= -\kappa X + \tau V, \\
\nabla_X V &= -\tau \xi,
\end{align*}
\]

(25)

where $\nabla$ denotes the Levi-Civita connection of $(M, g)$. Since a Riemannian submersion preserves the inner product of horizontal vector fields we can check that $\kappa = \bar{\kappa} \circ \pi$ and $\tau = \langle \nabla_X \xi, V \rangle = \langle A_X \xi, V \rangle$ (where, $A$ is the $A$-tensor of the Riemannian submersion, c.f. [20]) is the torsion of the horizontal lift which vanishes if the Riemannian submersion has integrable horizontal distribution. In what follows we are going to use the orthonormal frame $\{X, \xi, V\}$ to compute the mean curvature, second fundamental form, and other terms that appear in the biharmonic equation of the surface $\Sigma$. Using (25) we have

\[
\begin{align*}
A(X) &= -\langle \nabla_X \xi, X \rangle X - \langle \nabla_X \xi, V \rangle V = \kappa X - \tau V; \\
A(V) &= -\langle \nabla_V \xi, X \rangle X - \langle \nabla_V \xi, V \rangle V = -\tau X; \\
b(X, X) &= \langle A(X), X \rangle = \kappa, \quad b(X, V) = \langle A(X), V \rangle = -\tau; \\
b(V, X) &= \langle A(V), X \rangle = -\tau, \quad b(V, V) = \langle A(V), V \rangle = 0; \\
H &= \frac{1}{2}(b(X, X) + b(V, V)) = \frac{\kappa}{2}, \\
A(\text{grad } H) &= A(X(\frac{\kappa}{2}) X + V(\frac{\kappa}{2}) V) = X(\frac{\kappa}{2}) A(X) = \frac{\kappa'}{2}(\kappa X - \tau V); \\
\Delta H &= XX(H) - \langle \nabla_X X \rangle H + VV(H) - \langle \nabla_V V \rangle H = \frac{\kappa''}{2}; \\
|A|^2 &= (b(X, X))^2 + (b(X, V))^2 + (b(V, X))^2 + (b(V, V))^2 = \kappa^2 + 2\tau^2.
\end{align*}
\]

Substituting these into the biharmonic hypersurface Equation (6) we conclude that the surface $\Sigma$ is biharmonic in $(M^3, g)$ if and only if

\[
\begin{align*}
\frac{\kappa''}{2} - \frac{\kappa}{2}(\kappa^2 + 2\tau^2) + \frac{\kappa}{2}\text{Ric}^M(\xi, \xi) &= 0, \\
\kappa'(\kappa X - \tau V) + \frac{\kappa}{2}X - \kappa\text{Ric}^M(\xi, X)X - \kappa\text{Ric}^M(\xi, V)V &= 0,
\end{align*}
\]

which are equivalent to

\[
\begin{align*}
\kappa'' - \kappa(\kappa^2 + 2\tau^2) + \kappa\text{Ric}^M(\xi, \xi) &= 0, \\
3\kappa'\kappa - 2\kappa\text{Ric}^M(\xi, X) &= 0, \\
\kappa'\tau + \kappa\text{Ric}^M(\xi, V) &= 0.
\end{align*}
\]

(26)
Applying Equation (26) to Hopf fiberation \( \pi: S^3 \rightarrow S^2 \) we have the following corollary which recovers Proposition 3.1 in [13].

Corollary 4.1. There is no proper biharmonic Hopf cylinder in \( S^3 \).

Lastly, applying Equation (26) to submersions \( \pi: S^2 \times \mathbb{R} \rightarrow S^2 \) and \( \pi: H^2 \times \mathbb{R} \rightarrow H^2 \) we can have

\[
\text{Corollary 4.2. (1) The Hopf cylinder } \Sigma = \bigcup_{t \in I} \pi^{-1}(\alpha(t)) \text{ is a proper biharmonic surface in } S^2 \times \mathbb{R} \text{ if and only if the directrix } \alpha: I \rightarrow (S^2, h) \text{ is (a part of) a circle in } S^2 \text{ with radius } \sqrt{2}/2; \\
(2) \text{ The Hopf cylinder } \Sigma = \bigcup_{t \in I} \pi^{-1}(\alpha(t)) \text{ is biharmonic in } H^2 \times \mathbb{R} \text{ if and only if it is minimal.}
\]

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