SEMI-CONTINUOUS CONVOLUTIONS ON WEAKLY PERIODIC LEBESGUE SPACES

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ABSTRACT. We deduce mixed quasi-norm estimates of Lebesgue types on semi-continuous convolutions between sequences and functions which may be periodic or possess a weaker form of periodicity in certain directions. In these directions, the Lebesgue quasi-norms are applied on the period instead of the whole axes.

0. Introduction

Continuous, discrete and semi-continuous convolutions appear naturally when searching for estimates between short-time Fourier transforms with different window functions. By straightforward application of Fourier’s inversion formula, the short-time Fourier transform $V_{\phi}f$ of the function or (ultra-)distribution $f$ with window function $\phi$ is linked to $V_{\phi_0}f$ by

$$|V_{\phi}f| \lesssim |V_{\phi_0}| * |V_{\phi_0}f|$$

(cf. e.g. [8, Chapter 11]). Here * denotes the usual (continuous) convolution and it is assumed that the window functions $\phi$ and $\phi_0$ are fixed and belongs to suitable classes (see [8, 10] and Section 1 for notations).

Modulation spaces appear by imposing norm or quasi-norm estimates on the short-time Fourier transforms of (ultra-)distributions in Fourier-invariant spaces. In most situations these (quasi-)norms are mixed norms of (weighted) Lebesgue types. More precisely, let $B$ be a mixed quasi-Banach space of Lebesgue type with functions defined on the phase space, and let $\omega$ be a moderate weight. Then the modulation space $M(\omega, B)$ consists of all ultra-distributions $f$ such that

$$\|f\|_{M(\omega, B)} \equiv \|V_{\phi}f \cdot \omega\|_B$$

is finite.

If $B$ is a Banach space of mixed Lebesgue type, then the inequality (0.1) can be used to deduce:

1. that $M(\omega, B)$ is invariant of the choice of window function $\phi$ in (0.2), and that different $\phi$ give rise to equivalent norms.

2. that $M(\omega, B)$ increases with the Lebesgue exponents.

3. that $M(\omega, B)$ is complete.

Essential parts of these basic properties for modulation spaces were established in the pioneering paper [3], but some tracks goes back to
The theory has thereafter been developed in different ways, see e.g. [4–6, 8].

A more complicated situation appears when some of the Lebesgue parameters for $B$ above are strictly smaller than one, since $B$ is then merely a quasi-Banach space, but not a Banach space, since only a weaker form of the triangle inequality holds true. In such situations, $B$ even fails to be a local convex topological vector space, and the analysis based on (0.1) to reach (1)–(3) in their full strength above seems not work. (Some partial properties can be achieved if for example it is required that the Fourier transform of $\phi$ and $\phi_0$ should be compactly supported, see e.g. [13].)

In [7], the more discrete approach is used to handle this situation, where a Gabor expansion of $\phi$ with $\phi_0$ as Gabor window leads to that $|V_{\phi} f|$ can be estimated by

$$|V_{\phi} f| \lesssim a *_{[E]} |V_{\phi_0} f|,$$

for some non-negative sequence $a$ with enough rapid decay towards zero at infinity. Here $*_{[E]}$ denotes the semi-continuous convolution

$$a *_{[E]} F \equiv \sum_{j \in \Lambda_E} F(\cdot - j)a(j)$$

with respect to the basis $E$, between functions $F$ and sequences $a$, and $\Lambda_E$ is the lattice spanned by $E$. It follows that $*_{[E]}$ is similar to discrete convolutions.

For the discrete convolution $*$ both the classical Young’s inequality

$$\|a * b\|_{\ell^p_E} \leq \|a\|_{\ell^p_1} \|b\|_{\ell^p_2}, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_0}, \quad p_j \in [1, \infty],$$

as well as

$$\|a * b\|_{\ell^r} \leq \|a\|_{\ell^r} \|b\|_{\ell^r}, \quad r \leq \min(1, p), \quad p, r \in (0, \infty],$$

hold true, and it is proved in [7] and extended in [11] that similar facts hold true for semi-continuous convolutions. In the end the following restatement of [11, Proposition 2.1] is deduced. The result also extends [7, Lemma 2.6].

**Theorem 0.1.** Let $E$ be an ordered basis of $\mathbb{R}^d$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate, and let $p, r \in (0, \infty]^d$ be such that

$$r_k \leq \min_{m \leq k}(1, p_m).$$

Also let $f$ be measurable. Then the map $(a, f) \mapsto a *_{[E]} f$ from $\ell_0(\Lambda_E) \times \Sigma_1(\mathbb{R}^d)$ to $L^p_{E,(\omega)}(\mathbb{R}^d)$ extends uniquely to a linear and continuous map from $\ell^r_{E,(v)}(\Lambda_E) \times L^p_{E,(\omega)}(\mathbb{R}^d)$ to $L^p_{E,(\omega)}(\mathbb{R}^d)$, and

$$\|a *_{[E]} f\|_{L^p_{E,(\omega)}} \lesssim \|a\|_{\ell^r_{E,(v)}} \|f\|_{L^p_{E,(\omega)}},$$

(0.6)
In [7], (0.3) in combination with [7, ??] is used to show that (1)–(3) still hold when $B = \mathcal{L}^{p,q}$ and $\omega$ is a moderate weight of polynomial type. In [11], (0.3) in combination with Theorem 0.1 are used to show (1)–(3) for an even broader class of mixed Lebesgue spaces $B$ and weight functions $\omega$.

The aim of the paper is to extend Theorem 0.1, so that $f$ in some directions (variables) is allowed to be periodic, or a weaker form of periodicity, called echo-periodic functions. Such functions appear for example when applying the short-time Fourier transform on periodic or quasi-periodic functions. In fact, if $f$ is $E$-periodic, then $x \mapsto |\mathcal{V}_\phi f(x, \xi)|$ is $E$-periodic for every $\xi$. A function or distribution $F(x, \xi)$ is called quasi-periodic of order $\rho > 0$, if

$$F(x + \rho k, \xi) = e^{2\pi i \rho \langle k, \xi \rangle} F(x, \xi), \quad k \in \mathbb{Z}^d,$$

and by straightforward computations it follows that

$$|(\mathcal{V}_\phi F)(x + \rho k, \xi, \eta, y)| = |(\mathcal{V}_\phi F)(x, \xi, \eta, y - 2\pi k)|, \quad k \in \mathbb{Z}^d, \quad (0.7)$$

$$|(\mathcal{V}_\phi F)(x + \kappa/\rho, \eta, y)| = |(\mathcal{V}_\phi F)(x, \xi, \eta, y)|, \quad \kappa \in \mathbb{Z}^d,$$

for such $F$.

It is expected that the achieved extensions will be useful when performing local investigations of short-time Fourier transforms of periodic and quasi-periodic functions, e.g. in [12].

1. Preliminaries

In this section we recall some basic facts and introduce some notations. In the first part we recall the notion of weight functions. Thereafter we discuss mixed quasi-norm spaces of Lebesgue types. Finally we consider periodic functions and distributions, and introduce the notion of echo-periodic functions, which is a weaker form of periodicity which at the same time also include the notion of quasi-periodicity.

1.1. Weight functions. A weight on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ such that $1/\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. A usual condition on $\omega$ is that it should be moderate, or $v$-moderate for some positive function $v \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. This means that

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d. \quad (1.1)$$

We note that (1.1) implies that $\omega$ fulfills the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbb{R}^d. \quad (1.2)$$

We let $\mathcal{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$.

It can be proved that if $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, then $\omega$ is $v$-moderate for some $v(x) = e^{r|x|}$, provided the positive constant $r$ is large enough (cf. [9]).
In particular, (1.2) shows that for any $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, there is a constant $r > 0$ such that
\[
e^{-r|x|} \preceq \omega(x) \preceq e^{r|x|}, \quad x \in \mathbb{R}^d.
\]

We say that $v$ is submultiplicative if $v$ is even and (1.1) holds with $\omega = v$. In the sequel, $v$ and $v_j$ for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

1.2. Spaces of mixed quasi-norm spaces of Lebesgue types. Our discussions on periodicity are done in terms of suitable bases.

**Definition 1.1.** Let $E$ be an (ordered) basis $e_1, \ldots, e_d$ to $\mathbb{R}^d$. Then
\[\Lambda_E = \{ n_1 e_1 + \cdots + n_d e_d ; (n_1, \ldots, n_d) \in \mathbb{Z}^d \}\]
is the corresponding lattices.

Evidently, if $E$ is the same as in Definition 1.1, then there is a matrix $T_E$ with $E$ as the image of the standard basis in $\mathbb{R}^d$. Then $E'$ is the image of the standard basis under the map $T_{E'} = 2\pi(T_E^{-1})'$.

**Definition 1.2.** Let $E$ be a basis of $\mathbb{R}^d$, $\kappa(E)$ be the parallelepiped spanned by $E$, $\omega \in \mathcal{P}_E(\mathbb{R}^d)$ $p = (p_1, \ldots, p_d) \in (0, \infty]^d$ and $r = \min(1, p)$. If $f \in L^r_{\text{loc}}(\mathbb{R}^d)$, then
\[
\|f\|_{L^r_{E,(\omega)}(\mathbb{R}^d)} \equiv \|g_{d-1}\|_{L^p(\mathbb{R})}
\]
where $g_k(z_k)$, $z_k \in \mathbb{R}^{d-k}$, $k = 0, \ldots, d - 1$, are inductively defined as
\[
g_0(x_1, \ldots, x_d) \equiv |f(x_1 e_1 + \cdots + x_d e_d)\omega(x_1 e_1 + \cdots + x_d e_d)|,
\]
and
\[
g_k(z_k) \equiv \|g_{k-1}(\cdot, z_k)\|_{L^p(\mathbb{R})}, \quad k = 1, \ldots, d - 1.
\]

1. If $\Omega \subseteq \mathbb{R}^d$ is measurable, then $L^r_{E,(\omega)}(\Omega)$ consists of all $f \in L^r_{\text{loc}}(\Omega)$ with finite quasi-norm
\[
\|f\|_{L^r_{E,(\omega)}(\Omega)} \equiv \|f_\Omega\|_{L^r_{E,(\omega)}(\mathbb{R}^d)}, \quad f_\Omega(x) \equiv \begin{cases} f(x), & \text{when } x \in \Omega \\ 0, & \text{when } x \not\in \Omega. \end{cases}
\]
The space $L^r_{E,(\omega)}(\Omega)$ is called $E$-split Lebesgue space (with respect to $\omega$, $p$, $\Omega$ and $E$);

2. If $\Lambda \subseteq \mathbb{R}^d$ is a lattice such that $\Lambda_E \subseteq \Lambda$, then the quasi-Banach space $\ell^r_{E,(\omega)}(\Lambda)$ consists of all $a \in \ell^r_0(\Lambda)$ such that
\[
\|a\|_{\ell^r_{E,(\omega)}(\Lambda)} \equiv \| \sum_{j \in \Lambda} a(j) \chi_{j+\kappa(E)} \|_{L^r_{E,(\omega)}(\mathbb{R}^d)}
\]
is finite. The space $\ell^r_{E,(\omega)}(\Lambda_E)$ is called the discrete version of $L^r_{E,(\omega)}(\mathbb{R}^d)$. 4
Evidently, $L^p_E, (\omega)$ and $\ell^p_E, (\Lambda)$ in Definition 1.2 are quasi-Banach spaces of order $\min(p, 1)$. We set

$$L^p_E = L^p_{E, (\omega)} \quad \text{and} \quad \ell^p_E = \ell^p_{E, (\omega)}$$

when $\omega = 1$, and if $p = (p, \ldots, p)$ for some $p \in (0, \infty]$, then

$$L^p_{E, (\omega)} = L^p_{E, (\omega)}; \quad \ell^p_E = \ell^p_E; \quad \ell^p_{E, (\omega)} = \ell^p_{E, (\omega)} \quad \text{and} \quad \ell^p_E = \ell^p_E$$

agree with

$$L^p_{(\omega)}, \quad L^p, \quad \ell^p_{(\omega)} \quad \text{and} \quad \ell^p,$$

respectively, with equivalent quasi-norms.

1.3. Periodic and echo-periodic functions. We recall that if $E = \{e_1, \ldots, e_d\}$ is an ordered basis of $\mathbb{R}^d$, then the function or distribution $f$ on $\mathbb{R}^d$ is called $E$-periodic, if $f(\cdot + v) = f$ for every $v \in E$. More generally, if $E_0 \subseteq E$, then $f$ above is called $E_0$-periodic, if $f(\cdot + v) = f$ for every $v \in E_0$. We shall consider functions that possess weaker periodic like conditions, which appear when dealing with e.g. quasi-periodic functions and their short-time Fourier transforms.

**Definition 1.3.** Let $E = \{e_1, \ldots, e_d\}$ be an ordered basis of $\mathbb{R}^d$, $E_0 \subseteq E$ and let $f$ be a (complex-valued) function on $\mathbb{R}^d$. For every $k \in \{1, \ldots, d\}$, let $M_k$ be the set of all $l \in \{1, \ldots, k\}$ such that $e_l \in E \setminus E_0$. Then $f$ is called an *echo-periodic function with respect to* $E_0$, if for every $e_k \in E_0$, there is a vector

$$v_k = \sum_{l \in M_k} v_{k,l} e_l$$

such that

$$|f(\cdot + e_k)| = |f(\cdot + v_k)|. \quad (1.3)$$

We notice that in (1.7), relations of the form (1.3) appears.

**Remark 1.4.** Let $E$, $E_0$ and $M_k$ be the same as in Definition 1.3 and let $f$ be a (complex-valued) function on $\mathbb{R}^d$ such that (1.3) holds true. Also let

$$J_k = \begin{cases} \mathbb{R}, & k \in M_d; \\ [0, 1], & k \notin M_d; \end{cases}$$

$$I_k = \{xe_k; x \in J_k\}, \quad k \in \{1, \ldots, d\}$$

and

$$I = \{x_1e_1 + \cdots + x_de_d; x_k \in J_k, \ k = 1, \ldots, d\} \simeq I_1 \times \cdots \times I_d.$$
Then evidently, \(|f(\cdot + ne_k)| = |f(\cdot + nv_k)|\) for every integer \(n\). Hence, if \(f\) is measurable and echo-periodic with respect to \(E_0\), and \(p \in (0, \infty]^d\), then it follows by straight-forward computations that

\[
\|f(\cdot + ne_k)\|_{L^p_E(I)} = \|f\|_{L^p_E(I)}
\]

for every integer \(n\) and \(e_k \in E_0\).

**Definition 1.5.** Let \(E, E_0\) and \(I \subseteq \mathbb{R}^d\) be the same as in Remark 1.4, \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\) and let \(p \in (0, \infty]^d\). Then \(L^p_{E, E_0}(\mathbb{R}^d)\) denotes the set of all complex-valued measurable echo-periodic functions \(f\) with respect to \(E_0\) such that

\[
\|f\|_{L^p_{E, E_0}(\mathbb{R}^d)} = \|f\|_{L^p_{E, (\omega)}(I)}
\]

is finite.

In the next section we shall deduce weighted \(L^p_{E, E_0}(\mathbb{R}^d)\) estimates of the semi-discrete convolution

\[
(a *_{[E]} f)(x) = \sum_{j \in \Lambda_E} a(j)f(x - j), \quad (1.4)
\]

of the measurable function \(f\) on \(\mathbb{R}^d\) and \(a \in \ell_0(\Lambda_E)\), with respect to the ordered basis \(E\).

2. **Weighted Lebesgue estimates on semi-discrete convolutions**

In this section we extend Theorem 0.1 from the introduction such that \(L^p_{E, E_0}(\mathbb{R}^d)\)-estimates of echo-periodic functions are included.

Let \(E, E_0, M_k\) and \(J_k, k = 1, \ldots, d\), be the same as in Remark 1.4. In what follows we let \(\Sigma^E_{1}(\mathbb{R}^d)\) be the set of all \(E_0\)-periodic \(f \in C^\infty(\mathbb{R}^d)\) such that if

\[
g(x_1, \ldots, x_d) = f(x_1e_1 + \cdots + x_de_d),
\]

then

\[
\sup_{\alpha, \beta \in \mathbb{N}^d} \frac{\|x^\alpha D^\beta g\|_{L^\infty(I)}}{h^{\alpha + \beta} \alpha! \beta!}
\]

is finite for every \(h > 0\). By the assumptions and basic properties due to \([\Pi]\) it follows that \(\Sigma^E_{1}(\mathbb{R}^d) \subseteq L^p_{E, E_0}(\mathbb{R}^d)\) for every choice of \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\) and \(p \in (0, \infty]^d\) such that

\[
\omega(x) = \omega(x_0) \quad \text{when} \quad x = \sum_{k=1}^d x_ke_k, \quad x_0 = \sum_{k \in M_d} x_ke_k. \quad (2.1)
\]

Our extension of Theorem 0.1 to include echo-periodic functions is the following, which is also our main result.
Theorem 2.1. Let $E$ be an ordered basis of $\mathbb{R}^d$, $E_0 \subseteq E$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate and satisfy (2.1), and let $p, r \in (0, \infty]^d$ be such that

$$r_k \leq \min_{m \leq k} (1, p_m).$$

Also let $f$ be measurable echo-periodic function with respect to $E_0$, and let $I \subseteq \mathbb{R}^d$ be as in Remark 1.4. Then the map $(a, f) \mapsto a *_{[E]} f$ from $\ell_0(\Lambda_E) \times \Sigma^E_1(\mathbb{R}^d)$ to $L^p_{E,(\omega)}(I)$ extends uniquely to a linear and continuous map from $r^p_{E,(\omega)}(\Lambda_E) \times r^p_{E,(\omega)}(\mathbb{R}^d)$ to $L^p_{E,(\omega)}(I)$, and

$$\|a *_{[E]} f\|_{L^p_{E,(\omega)}(I)} \leq \|a\|_{r^p_{E,(\omega)}(\Lambda_E)} \|f\|_{L^p_{E,(\omega)}(I)}.$$  \hspace{1cm} (2.2)

For the proof we recall that

$$\left(\sum_{j \in I} |b(j)|^r\right)^{1/r} \leq \sum_{j \in I} |b(j)|, \quad 0 < r \leq 1,$$  \hspace{1cm} (2.3)

for any sequence $b$ and countable set $I$.

Proof. By letting

$$f_0(x_1, \ldots, x_d) = |f(x_1 e_1 + \cdots + x_d e_d)\omega(x_1 e_1 + \cdots + x_d e_d)|,$$

$$a_0(l_1, \ldots, l_d) = |a(l_1 e_1 + \cdots + l_d e_d)\omega(l_1 e_1 + \cdots + l_d e_d)|$$

and using the inequality

$$|a *_{[E]} f \cdot \omega| \leq a *_{[E]} f \omega,$$

we reduce ourselves to the case when $E$ is the standard basis, $\omega = v = 1$, and $f, a \geq 0$. This implies that we may identify $I_k$ in Remark 1.4 with $J_k$ for every $k$.

Let

$$z_k = (x_{k+1}, \ldots, x_d) \in \mathbb{R}^{d-k}, \quad m_k = (l_{k+1}, \ldots, l_d) \in \mathbb{Z}^{d-k}$$

for $k = 0, \ldots, d - 1$, and let

$$f_0 = f; \quad a_0 = a; \quad g_0 = a *_{[E]} f.$$

Then $z_k = (x_k, z_k)$ and $m_k = (l_k, m_k)$. It follows that $x_k \in I_k$ when applying the mixed quasi-norms of Lebesgue types, and that

$$0 \leq (a *_{[E]} f)(x_1, \ldots, x_d) \leq \sum_{m_0 \in \mathbb{Z}^d} f(x_1 - \varphi_1(m_0), \ldots, x_d - \varphi_d(m_{d-1}))a(m_0),$$  \hspace{1cm} (2.4)

for some linear functions $\varphi_k$ from $\mathbb{R}^{d+1-k}$ to $\mathbb{R}$, which satisfy

$$\varphi_k(z_{k-1}) = \begin{cases} x_k + \psi_k(z_k), & J_k = \mathbb{R}, \\ 0, & J_k = [0, 1], \end{cases}$$  \hspace{1cm} (2.5)

for some linear forms $\psi_k$ on $\mathbb{R}^{d-k}, k = 1, \ldots, d$. 

Define inductively

\[ f_k(z_k) = \|f_{k-1}(\cdot, z_k)\|_{L^p(J_k)}, \quad a_k(m_k) = \|a_{k-1}(\cdot, m_k)\|_{\ell^p(Z)}, \]

and

\[ g_k(z_k) = \|g_{k-1}(\cdot, z_k)\|_{L^p(J_k)}, \quad k = 1, \ldots d. \]

Also let

\[ \varphi_k(z_k) = (\varphi_{k+1}(z_k), \ldots, \varphi_d(z_{d-1})), \quad k = 0, \ldots, d - 1. \]

Then (2.4) is the same as

\[ 0 \leq (a *_{[E]} f)(x_1, \ldots, x_d) \leq \sum_{m_0 \in \mathbb{Z}^d} f(z_0 - \varphi_0(m_0))a(m_0), \quad (2.6) \]

We claim

\[ g_k(z_k) \lesssim \left( \sum_{m_k} f_k(x_{k+1} - \varphi_{k+1}(m_k), \ldots, x_d - \varphi_d(m_{d-1}))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right)^{1/p_{0,k}}, \]

which in view of the links between (2.4) and (2.6) is the same as

\[ g_k(z_k) \lesssim \left( \sum_{m_k} f_k(z_k - \varphi_k(m_k))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right)^{1/p_{0,k}} \quad (2.7) \]

when \( k = 0, \ldots, d \). Here we set \( p_{0,0} = 1 \), and interprete \( f_d, a_d, g_d \) and the right-hand side of (2.7) as \( \|f\|_{L^p_p(I)}, \|a\|_{\ell^p_p(Z)}, \|g_0\|_{L^p_p(I)} \) and \( \|f\|_{L^p_p(I)} \|a\|_{\ell^p_p(Z)} \), respectively. The result then follows by letting \( k = d \) in (2.7).

We shall prove (2.7) by induction. The result is evidently true when \( k = 0 \). Suppose it is true for \( k - 1, \ k \in \{1, \ldots, d - 1\} \). We shall consider the cases when \( p_k \geq p_{0,k-1} \) or \( p_k \leq p_{0,k-1} \), and \( J_k = \mathbb{R} \) or \( J_k = [0, 1] \) separately, and for conveniency we set \( p_{0,k-1} = p \) and \( f_{k-1} = h \).

First assume that \( p_k \geq p_{0,k-1} \). Then \( p_{0,k} = p_{0,k-1} \). Also suppose \( J_k = \mathbb{R} \). Then it follows from the induction hypothesis that

\[ g_k(z_k) \lesssim \left( \int_{-\infty}^{\infty} \left( \sum_{l_k} h(x_k - \varphi_k(m_{k-1}), z_k - \varphi_k(m_k))^{p_{k-1}} a_{k-1}(l_k, m_k)^{p_{k-1}} \right)^{p_k} \right)^{1/p_k} \]

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where the sum is taken over all \((l_k, m_k) \in \mathbb{Z} \times \mathbb{Z}^{d-k}\). By Minkowski’s inequality, the right-hand side can be estimated by

\[
\left( \sum \left( \int_{-\infty}^{\infty} h(x_k - \varphi_k(m_{k-1}), z_k - \varphi_k(m_k))^p dx_k \right)^{\frac{1}{p}} a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum \left( \int_{-\infty}^{\infty} h(x_k, z_k - \varphi_k(m_k))^p dx_k \right)^{\frac{1}{p}} a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum f_k(z_k - \varphi_k(m_k))^p a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{m_k \in \mathbb{Z}^{d-k}} f_k(z_k - \varphi_k(m_k))^p \left( \sum \frac{a_k^{-1}(l_k, m_k)^p}{p_k} \right) \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{m_k \in \mathbb{Z}^{d-k}} f_k(z_k - \varphi_k(m_k))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right)^{\frac{1}{p_{0,k}}},
\]

and \((2.7)\) follows in the case \(p_k \geq p_{0,k-1}\) and \(J_k = \mathbb{R}\) by combining these estimates.

Next we consider the case when \(p_k \geq p_{0,k-1}\) and \(J_k = [0,1]\). Then \(\varphi_k(m_{k-1}) = 0\), and by the induction hypothesis and Minkowski’s inequality we get

\[
g_k(z_k)
\]

\[
\leq \left( \int_0^1 \left( \sum_{m_{k-1}} h(x_k, z_k - \varphi_k(m_k))^p a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}} dx_k \right)^{\frac{1}{p}}
\]

\[
\leq \left( \sum_{m_{k-1}} \left( \int_0^1 h(x_k, z_k - \varphi_k(m_k))^p dx_k \right)^{\frac{1}{p}} a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum f_k(z_k - \varphi_k(m_k))^p a_k^{-1}(l_k, m_k)^p \right)^{\frac{1}{p}}
\]

\[
= \left( \sum_{m_k \in \mathbb{Z}^{d-k}} f_k(z_k - \varphi_k(m_k))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right)^{\frac{1}{p_{0,k}}},
\]

and \((2.7)\) follows in the case \(p_k \geq p_{0,k-1}\) and \(J_k = [0,1]\) as well.
Next assume that $p_k \leq p_{0,k-1}$ and $J_k = \mathbb{R}$. Then

$$p_k/p_{0,k-1} = p_k/p \leq 1 \quad \text{and} \quad p_{0,k} = p_k,$$

and (2.3) gives

$$g_k(z_k) \lesssim \left( \int_{-\infty}^{\infty} \left( \sum_{m_{k-1}} h(x_k - \varphi_k(m_{k-1}), z_k - \varphi_k(m_k)) a_{k-1}(l_k, m_k)^p \right)^{\frac{p_k}{p}} \, dx_k \right)^{\frac{1}{p_k}}$$

$$\leq \left( \int_{-\infty}^{\infty} \sum_{m_{k-1}} (h(x_k - \varphi_k(m_{k-1}), z_k - \varphi_k(m_k)) a_{k-1}(l_k, m_k)^p)^{\frac{p_k}{p}} \, dx_k \right)^{\frac{1}{p_k}}$$

$$= \left( \sum_{m_{k-1}} \left( \int_{-\infty}^{\infty} h(x_k, z_k - \varphi_k(m_k)) a_{k-1}(l_k, m_k)^p \, dx_k \right)^{\frac{1}{p_k}} \, a_{k-1}(l_k, m_k)^p \right)^{\frac{1}{p_k}}$$

$$= \left( \sum_{m_k} f_k(z_k - \varphi_k(m_k))^{p_k} \left( \sum_{l_k} a_{k-1}(l_k, m_k)^p \right)^{\frac{1}{p_k}} \right)\left( \sum_{m_k} f_k(z_k - \varphi_k(m_k))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right)^{\frac{1}{p_{0,k}}},$$

and (2.7) follows in this case as well.

It remain to consider the case $p_k \leq p_{0,k-1}$ and $J_k = [0,1]$. Then $\varphi_k(m_{k-1}) = 0$, and by similar arguments as above we get

$$g_k(z_k) \lesssim \left( \int_0^1 \left( \sum_{m_{k-1}} h(x_k, z_k - \varphi_k(m_k)) a_{k-1}(l_k, m_k)^p \right)^{\frac{p_k}{p}} \, dx_k \right)^{\frac{1}{p_k}}$$

$$\leq \left( \int_0^1 \sum_{m_{k-1}} (h(x_k, z_k - \varphi_k(m_k)) a_{k-1}(l_k, m_k)^p)^{\frac{p_k}{p}} \, dx_k \right)^{\frac{1}{p_k}}.$$
\[
\sum_{m_{k-1}} \left( \int_0^1 h(x_k, z_k - \varphi_k(m_k))^{p_k} \, dx_k \right)^{1/p_k} \left( \int_0^1 h(x_k, z_k - \varphi_k(m_k))^{p_k} \, dx_k \right)^{1/p_k} = \sum_{m_k} f_k(z_k - \varphi_k(m_k))^{p_{0,k}} a_k(m_k)^{p_{0,k}} \right) \right)^{1/p_{0,k}},
\]

and (2.7), and thereby the result follow. \(\square\)

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