Effects on the eigenvalues of the quantum bouncer due to dissipation

G. López and G. González

Departamento de Física de la Universidad de Guadalajara
Apartado Postal 4-137
44410 Guadalajara, Jalisco, México

Nov 2003

PACS 03.20.+i 03.65.Ca
keywords: dissipation, constant of motion, quantum bouncer

ABSTRACT

Effects on the spectra of the quantum bouncer due to dissipation are given when a linear or quadratic dissipation is taken into account. Classical constant of motions and Hamiltonians are deduced for these systems and their quantized eigenvalues are estimated through perturbation theory. We found some differences when we compare the eigenvalues of these two quantities.
I. Introduction.

Dissipative systems has been one of the must subtle and difficult topics to deal with in classical [1] and quantum physics [2]. In general, to construct a consistent Lagrangian and Hamiltonian formulation for a given dissipative system can be a really challenge [3]. There are basically two approaches to study dissipative systems. The first one tries to bring about the dissipation as a results of averaging over all the coordinates of the batch system, where one considers the whole system as composed of two parts, our original conservative system and the batch system which interacts with the conservative system and causes the dissipation (of energy) on it [4]. This approach has its own value and will not be followed or discussed here. The second approach considers that the bath system produces on our initially conservative system and average effect which is expressed as an additional external velocity depending force acting on the conservative system and transforming it into a dissipative system with this velocity depending force, the resulting classical dissipative system contains then this phenomenological (or theoretical) velocity depending force. Then, the question arises over its consistent Lagrangian and Hamiltonian formalism and the consequences of its quantization. This approach, in addition, allows us to study and test the Hamiltonian approach for quantum mechanics and its consistence [5] and is the approach we will follow in this paper. A system which has bring our attention for dissipation study through the above approach is the quantum bouncer [6]. The quantum bouncer [6] is the quantization of the motion of a particle which is attracted by the constant gravity force, that is, close to surface of the earth. This particle hits a perfectly reflexing surface, producing the bouncing effect. This system with an additional dissipation force has particular importance because of its potential experimental realization. This dissipative system has been studied very little and using so far the first approach mentioned above [7].

In this paper we considers the second approach to dissipation systems and will assume that the external velocity depending force has linear and quadratic dependency with respect the velocity. This approach gives us the opportunity to check the nature of quantization via Hamiltonian or constant of motion associated to the system, that is, using the usual quantization of the linear generalized momentum or using the quantization of the velocity. This consideration is particularly interesting in dissipative systems since one can not always have a Hamiltonian in an explicit form as a function of the variables position and linear momentum [8], that is, the velocity "v" can not always be known explicitly in terms of the linear momentum "p" and position "x" of the particle through the relation \( p = \partial L/\partial v \),
where $L$ is the Lagrangian of the system. This paper is organized as follows: we present the classical study for the dissipative system considering the linear and quadratic velocity depending force. The constant of motion, the Lagrangian, and the Hamiltonian of the system are derived, and we give their expressions up to second order in the dissipation parameter. We present the modification of the eigenvalues for the quantum bouncer, when this dissipation is taken into account, for the above approximated (weak dissipation) constant of motion and Hamiltonian using quantum perturbation theory. Finally, conclusions and some discussions of our results are made.

II. Classical linear dissipation.

The motion of a particle of mass $m$ under a constant gravitational force and a linear dissipative force is described by the equation

$$m \frac{d^2 x}{dt^2} = -mg - \alpha v,$$  

(1)

where $x$ is the position of the particle, $g$ is the constant acceleration due to earth gravity, $\alpha$ is the parameter which characterizes the dissipation, and $v = dx/dt$ is the velocity of the particle. A constant of motion of the autonomous system (1) is a function $K_\alpha = K_\alpha(x, v)$ satisfying the equation [9]

$$v \frac{\partial K_\alpha}{\partial x} - (g + \frac{\alpha}{m} v) \frac{\partial K_\alpha}{v} = 0.$$  

(2)

The solution of this equation such that $\lim_{\alpha \to 0} K_\alpha = mv^2/2 + mgx$ (the usual total energy of the non dissipative system) is given by

$$K_\alpha = \frac{m^2 g v}{\alpha} - m \left( \frac{mg}{\alpha} \right)^2 \ln(1 + \frac{\alpha v}{mg}) + mgx.$$  

(3)

The Lagrangian associated to (1) can be obtained using the known expression [5]

$$L_\alpha = v \int \frac{K_\alpha(x, v)}{v^2} dv,$$  

(4)

bringing about the following Lagrangian

$$L_\alpha = \frac{m^2 g v}{\alpha} \ln \left(1 + \frac{\alpha v}{mg}\right) + m \left( \frac{mg}{\alpha} \right)^2 \ln \left(1 + \frac{\alpha v}{mg}\right) - mgx - \frac{m^2 g v}{\alpha}.$$  

(5)

Therefore, the generalized linear momentum and Hamiltonian are given by

$$p_\alpha = \frac{m^2 g}{\alpha} \ln \left(1 + \frac{\alpha v}{mg}\right)$$  

(6)
and
\[ H_\alpha = m \left( \frac{mg}{\alpha} \right)^2 \left( \exp \left( \frac{\alpha p_\alpha}{m^2 g} \right) - 1 \right) - \frac{mg}{\alpha} p_\alpha + mgx. \] (7)

At two orders in the dissipation parameter \( \alpha \), one has the constant of motion, the Lagrangian, the generalized linear momentum, and Hamiltonian given as

\[ K = \frac{1}{2} mv^2 + mgx - \frac{\alpha}{3g} v^3 + \frac{\alpha^2}{4mg^2} v^4, \] (8a)

\[ L = \frac{1}{2} mv^2 - mgx - \frac{\alpha}{6g} v^3 + \frac{\alpha^2}{12mg^2} v^4, \] (8b)

\[ p = mv - \frac{\alpha}{2g} v^2 + \frac{\alpha^2}{3mg} v^3, \] (8c)

and
\[ H = \frac{p^2}{2m} + mgx + \frac{\alpha}{6mg} p^3 + \frac{\alpha^2}{24m^5 g^2} p^4. \] (8d)

The constant of motion (3) or (8a) and the Hamiltonian (7) or (8d) bring about the damping bouncing effect on the spaces \((x, v)\) and \((x, p)\). The dissipative parameter \( \alpha \) can be determined by measuring the velocity \( v_o \) at the reflexing surface \((x = 0)\) and then measuring its maximum displacement \( x_{max} \) \((v = 0)\). Equaling the value of the constant of motion on both situations, one gets the expression

\[ \frac{m^2 g v_o}{\alpha} - m \left( \frac{mg}{\alpha} \right)^2 \ln(1 + \frac{\alpha v_o}{mg}) = mg x_{max}, \] (9)

from the parameter \( \alpha \) can be gotten.

**III. Classical quadratic dissipation.**

In this case, the motion of the particle is described by the equation

\[ m \frac{d^2 x}{dt^2} = -mg - \gamma v|v|, \] (10)

where \( \gamma \) represents a dissipation constant which, of course, is different from the previous case. Proceeding in the same way as we did for the linear case, the constant of motion, Lagrangian, generalized linear momentum, and Hamiltonian are given by

\[ K_\pm = \frac{1}{2} mv^2 \exp \left( \frac{\pm 2\gamma x}{m} \right) \pm \frac{m^2 g}{2\gamma} \left( \exp \left( \frac{\pm 2\gamma x}{m} \right) - 1 \right), \] (11a)

\[ L_\pm = \frac{1}{2} mv^2 \exp \left( \frac{\pm 2\gamma x}{m} \right) \mp \frac{m^2 g}{2\gamma} \left( \exp \left( \frac{\pm 2\gamma x}{m} \right) - 1 \right), \] (11b)
\[ p_\pm = mv \exp\left( \pm \frac{2\gamma x}{m} \right) , \quad \tag{11c} \]

and

\[ H_\pm = \frac{p_\pm^2}{2m} \exp\left( \pm \frac{2\gamma x}{m} \right) \pm \frac{m^2 g}{2\gamma} \left( \exp\left( \pm \frac{2\gamma x}{m} \right) - 1 \right) , \quad \tag{11d} \]

where the upper sign corresponds to the case \( v \geq 0 \), and the lower sign corresponds to the case \( v < 0 \). These equations were already given in reference [10]. The damping effect of the bouncing particle in the space \((x,v)\) can be traced in the following way: starting with the initial condition \( x_o = 0 \) and \( v_o > 0 \), for example, the constant of motion \( K_+ \) is determined, \( K_+ = \frac{mv_o^2}{2} \). Then, the maximum distance \( x_{max} (v = 0) \) is calculated from the expression \( K_+ = (m^2 g/2\gamma)(\exp(2\gamma x_{max}/m) - 1) \) which helps to calculate the constant \( K_- \), \( K_- = -(m^2 g/2\gamma)(\exp(-2\gamma x_{max}/m) - 1) \). This \( K_- \) is used now to calculate the velocity at the return point \((x = 0)\), \( v_1^* = -\sqrt{2 K_-/m} \). Due to perfectly reflecting surface, the velocity of the bouncing particle for the next cycle is \( v_1 = -v_1^* (v_1 < v_o) \), and the above cycle is reproduced again, and so on. Starting with the same initial conditions, the trajectories in this space are one below the other at any time, as the damping factor is greater. The damping effect in the space \((x,p)\) through the Hamiltonian approach can be analyzed similarly. However, the trajectories starting with the same initial conditions on this space are not below the other all the time, as the damping factor is greater. This strange effect is due to change in sign in (11d) with respect to (11a), produced by the position and velocity dependence of the expression (11c).

To determine the constant \( \gamma \) through the constant of motion, one can start with the initial conditions \((x_o = 0, v_o > 0)\) and can determine the constant of motion \( K_+ = \frac{mv_o^2}{2} \). Then, one can measure the maximum displacement \( x_{max} (v = 0) \) and to solve \( \gamma \) from the equation

\[ \frac{1}{2} mv_o^2 = \frac{m^2 g}{2\gamma} \left( \exp\left( \frac{2\gamma x_{max}}{m} \right) - 1 \right) . \quad \tag{12} \]

Up to second order in the dissipation parameter, one has from (11a) to (11d) the constant of motion, the Lagrangian, the generalized linear momentum, and the Hamiltonian given by

\[ K_\pm = \frac{1}{2} mv^2 + mgx \pm \gamma[v^2 x + gx^2] + \gamma^2[v^2 x^2/m + 2gx^3/3m] , \quad \tag{13a} \]

\[ L_\pm = \frac{1}{2} mv^2 - mgx \pm \gamma[v^2 x - gx^2] + \gamma^2[v^2 x^2/m - 2gx^3/3m] , \quad \tag{13b} \]

\[ p_\pm = mv \pm \gamma[2vx] + \gamma^2[2vx^2/m] , \quad \tag{13c} \]

\[ H_\pm = \frac{p_\pm^2}{2m} + mgx \pm \gamma[p^2 x/m^2 - gx^2] + \gamma^2[p^2 x^2/m^3 + 2gx^3/3m] . \quad \tag{13d} \]
IV. Quantization of the constant of motion.

Eqs. (8a) and (13a) can be written as

\[ K(x,v) = K_o(x,v) + V(x,v) , \]

where \( K_o \) is the constant of motion without dissipation

\[ K_o(x,v) = \frac{1}{2}mv^2 + mgx , \]

and \( V \) takes into account the dissipation factors

\[
V(x,v) = \begin{cases} 
-\alpha \left( \frac{v^3}{3g} \right) + \alpha^2 \left( \frac{v^4}{4mg^2} \right) & \text{(linear case)} \\
\mp \gamma \left[ v^2 x + gx^2 \right] + \gamma^2 \left[ \frac{v^2 x^2}{m} + \frac{2gx^3}{3m} \right] & \text{(quadratic case)} 
\end{cases}
\]

The quantization of (14) can be carried out through the associated Schrödinger’s equation of this constant of motion

\[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{K}(\hat{x}, \hat{v}) \Psi , \]

where \( \Psi = \Psi(x,t) \) is the wave function, \( \hbar \) is the Plank constant divided by \( 2\pi \), \( \hat{K} = \hat{K}_o + \hat{V} \) is a Hermitian operator associated to (17), and \( \hat{v} \) is the velocity operator defined as

\[ \hat{v} = -\frac{i\hbar}{m} \frac{\partial}{\partial x} . \]

Since Eq. (16) represents an stationary problem, the usual proposition \( \Psi(x,t) = \exp(-iE^K t/\hbar) \psi(x) \) transforms (16) to an eigenvalue problem

\[ (\hat{K}_o + \hat{V})\psi = E^K \psi . \]

Considering now the operator \( \hat{V} \) as a perturbation of the constant of motion \( K_o \), one can calculate an approximated solution to the problem (18) through perturbation theory . The solution of the eigenvalue problem

\[ \hat{K}_o \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)} \]

is well known [6], with \( \psi_n^{(0)} \) being the eigenfunction given by

\[ \psi_n^{(0)} = \frac{Ai(z - z_n)}{|Ai'(z_n)|} , \]
where $Ai$ and $Ai'$ are the Airy function and its differentiation, and $z_n$ is its $n$th-zero ($Ai(-z_n) = 0$) which occurs for negative argument only. $z$ is the normalized variable $z = x/l_g$ with $l_g = (\hbar^2 / 2m^2 g)^{1/3}$, and $z_n$ is related with the eigenvalue $E_n^{(0)}$ through the expression

$$z_n = \frac{E_n^{(0)}}{mgl_g}.$$  

Up to second order in perturbation theory, the eigenvalues of (18) are given (in Dirac notation [11]) as

$$E_n^K = E_n^{(0)} + \langle n|\hat{V}|n\rangle + \sum_{k\neq n} \left|\langle n|\hat{V}|k\rangle\right|^2 \left(\frac{E_k^{(0)} - E_n^{(0)}}{E_k^{(0)} - E_n^{(0)}}\right)^2,$$  

where $\langle z|n\rangle = \psi_n^{(0)}$. Using the Hermitian operators $\hat{v}^2x = (\hat{v}^2x + \hat{v}x\hat{v} + \hat{v}^2x)/3$ and $\hat{v}^2x^2 = (\hat{v}^2x^2 + \hat{v}x^2\hat{v} + x^2\hat{v}^2 + x\hat{v}x\hat{v} + \hat{v}x^2x)/6$ for the associated expressions on (25b), and using the relations $\langle n|x^s|k\rangle = l_g^s\langle n|x^s|k\rangle$ and $\langle n|d^s/dx^s|k\rangle = l_g^{-s}\langle n|d^s/dx^s|k\rangle$ for any integer $s$, one has (see appendix for a list of matrix elements)

$$E_n^K = E_n^{(0)} + \left\{ \alpha^2 \left[ \frac{l_g^2 z_n^2}{3m} + \frac{8}{9}g l_g^3 \sum_{k\neq n} \frac{|1/2 + mgl_g/(E_k^{(0)} - E_n^{(0)})|^2}{E_k^{(0)} - E_n^{(0)}} \right] \right\} \left(\text{linear}\right)$$

$$\pm \gamma \frac{12g l_g^2 z_n^2}{15} + \gamma^2 \left[ \left( -\frac{1}{2} + \frac{56z_n^3}{105} \right) \frac{2gl_g^3}{m} + 4g^2 l_g^4 \sum_{k\neq n} a_{nk} \right], \quad (\text{quadratic})$$

where $a_{nk}$ are real number given by

$$a_{nk} = \frac{|12 - 2z_k(z_n - z_k)^2 + (z_n - z_k)^3|^2}{(z_k - z_n)^9}.$$  

Note that for the linear dissipation case, there is not real contribution a first approximation, and for the quadratic dissipative case, the first order contribution depends on whether the particle is moving (-) or down (-). Within a full cycle, this first order is cancelled out and the second order contribution remains. Of course, for the approximation (3a) to be valid, one must have that the second term in this expression must be much less than $E_n^{(0)}$ which restricts the value of the dissipative parameter.
V. Quantization of the Hamiltonian.

Eqs. (8d) and (13d) can be written as

\[ H(x, p) = H_o(x, p) + W(x, p), \quad (24) \]

where \( H_o \) is the Hamiltonian without dissipation,

\[ H_o(x, p) = \frac{p^2}{2m} + mgx, \quad (25a) \]

and \( W \) has the dissipation terms,

\[ W(x, p) = \begin{cases} \alpha \left( \frac{p^3}{6mg} \right) + \alpha^2 \left( \frac{p^4}{24m^5g^2} \right) & \text{(linear)} \\ \pm \gamma \left[ \frac{p^2x}{m^2} - gx^2 \right] + \gamma^2 \left[ \frac{p^2x^2}{m^3} + \frac{2gx^3}{3m} \right] & \text{(quadratic)} \end{cases} \quad (25b) \]

It is necessary to mention that the quantization of some systems for quadratic dissipation has been solved by different authors \[10,12\] but perfectly reflecting wall potential,

\[ \tilde{V}(x) = \begin{cases} \infty & \text{for } x < 0 \\ mgx & \text{for } x \geq 0 \end{cases}. \quad (26) \]

Moreover, the solution given in reference \[10\] is singular for the dissipation parameter equal to zero. Therefore, we think it worths to make the analysis of the quantization for small orders in the parameter \( \gamma \). For the usual Shrödinger quantization approach, one has the stationary equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}(x, \hat{p}) \Psi, \quad (27) \]

where \( \hat{H} \) is the Hamiltonian operator associated to (24), and \( \hat{p} \) is the usual linear momentum operator \( \hat{p} = -i\hbar \partial/\partial x \). Eq. (27) is transformed to an eigenvalue problem, \( \hat{H} \psi(x) = E^H \psi(x) \), through the proposition \( \Psi(x, t) = \exp \left( -iE^H t/\hbar \right) \psi(x) \). Since the Hamiltonian \( \hat{H} \) is given by \( \hat{H} = \hat{H}_o + \tilde{W} \), where the solution of the equation

\[ \hat{H}_o \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)} \quad (28) \]

is well known (as before), being \( \psi_n^{(0)} \) and \( E_n^{(0)} \) given by (20) and (21), perturbation theory can be used to determine the approximated values of the eigenvalues \( E_n^H \) (similarly as done with expression (22)). Using the Hermitian operators \( \hat{p}^2 = (\hat{p}^2 + \hat{p}x\hat{p} + x\hat{p}^2)/3 \)
and $\vec{p}^2x^2 = (\vec{p}^2x^2 + \vec{p}x^2\vec{v} + x^2\vec{p}^2 + x\vec{p}x\vec{p} + \vec{p}x\vec{p})/6$ for the associated expressions on (25b), one gets

$$E_n^H = E_n^{(0)} + \begin{cases} \alpha^2 \left[ \frac{l_g^2z_n^2}{3m} + \frac{4}{9} g l_g^3 \sum_{k \neq n} |1/2 + m l_g/(E_k^{(0)} - E_n^{(0)})|^2 \right] & \text{(linear)} \\ \pm \gamma \frac{4 g l_g^2 z_n^2}{15} + \gamma^2 \left[ \left( -\frac{1}{2} + \frac{56 z_n^3}{105} \right) \frac{2 g l_g^3}{m} + 4 g^2 l_g^4 \sum_{k \neq n} a_{nk} \right] & \text{(quadratic)} \end{cases}$$

(29)

where $a_{nk}$ is given by (23b). As one can see from (23) and (29), there is a difference with the eigenvalues associated to the constant of motion and Hamiltonian quantizations. Their relative differences, $\delta E_n = (E_n^H - E_n^K)/E_n^{(0)}$, is given by

$$\frac{\delta E_n}{E_n^{(0)}} = \begin{cases} \alpha^2 \left[ -\frac{1}{6} g z_n^2 + \frac{4}{9} g l_g^2 \sum_{k \neq n} |1/2 + m l_g/(E_k^{(0)} - E_n^{(0)})|^2 \right] & \text{(linear)} \\ \pm \gamma \frac{16 g l_g^2 z_n^2}{15} & \text{(quadratic)} \end{cases}$$

(30)

VI. Conclusion.

We have presented the classical and quantum analysis of a particle attracted by constant gravity and velocity depending dissipative forces which bounce on a perfectly reflecting surface. We have considered linear and quadratic velocity depending dissipative cases and have deduced their constant of motion and Hamiltonians. Expression (3) and (11a) gives us the expected damping behavior of the particle on the space $(x,v)$, but the expression (7) and (11d) show us the damping with unexpected behavior in the space $(x,p)$ (two trajectories on this space, for dissipative parameter one bigger than other, do not follow one under the other all the time). For the quantum case, we have analyzed the eigenvalues of the assigned operators pertubedly and up to second order on the dissipative parameters. Relation (30) tells us that that there is a difference whether the constant of motion or the Hamiltonian is quantized, and it suggests that one could see this difference experimentally. In this way, one could see whether nature prefers to follow constant of motions rather than Hamiltonians for dissipative systems. Finally, one must observe that for the full linear case (7), it is possible to solve exactly the Schrödinger equation in the momentum representation, and this will be analyzed on a future paper.
APPENDIX

We show here a list of some matrix elements from reference [6] and some other calculated from the same reference (an correction of a sign has been made to some matrix elements). Given the functions (20) and \( n \neq k \), one has

\[
\langle n|k \rangle = \delta_{nk} \tag{A_1}
\]

\[
\langle n|z|n \rangle = \frac{2}{3} z_n \quad \langle n|z|k \rangle = \frac{2(-1)^{n+k+1}}{(z_n - z_k)^2} \tag{A_2}
\]

\[
\langle n|z^2|n \rangle = \frac{8}{15} z_n^2 \quad \langle n|z^2|k \rangle = \frac{24(-1)^{n+k+1}}{(z_n - z_k)^4} \tag{A_3}
\]

\[
\langle n|z^3|n \rangle = \frac{3}{7} + \frac{48}{105} z_n^3 \quad \langle n|z^3|k \rangle = \frac{24(z_n + z_k)(-1)^{n+k+1}}{(z_n - z_k)^4} \tag{A_4}
\]

\[
\langle n|\frac{d}{dz}|n \rangle = 0 \quad \langle n|\frac{d}{dz}|k \rangle = \frac{(-1)^{n+k}}{z_n - z_k} \tag{A_5}
\]

\[
\langle n|\frac{d^2}{dz^2}|n \rangle = -\frac{1}{3} z_n \quad \langle n|\frac{d^2}{dz^2}|k \rangle = \frac{2(-1)^{n+k}}{(z_n - z_k)^2} \tag{A_6}
\]

\[
\langle n|\frac{d^3}{dz^3}|n \rangle = \frac{1}{2} \quad \langle n|\frac{d^3}{dz^3}|k \rangle = \left(\frac{1}{2} + \frac{1}{z_k - z_n}\right) (-1)^{n+k} \tag{A_7}
\]

\[
\langle n|\frac{d^4}{dz^4}|n \rangle = \frac{1}{5} z_n^2 \quad \langle n|\frac{d^4}{dz^4}|k \rangle = -\frac{2(z_k - z_n) + 24 - 2z_k(z_k - z_n)^2}{(z_k - z_n)^4} (-1)^{n+k} \tag{A_8}
\]
References

[1] R. Glauber and V.I. Man’ko, Sov. Phys. JEPT 60 (1984) 450.
V.V. Dodonov, Hadron J., 4 (1981) 173.
G. López, M. Murgía and M. Sosa, Mod. Phys. Lett. B, 11, 14 (1997) 625.
[2] M. Razavy, Can. J. Phys. 50 (1972) 2037.
S. Okubo, Phys. Rev. A, 23 (1981) 2776.
G. López, M. Murgía and M. Sosa, Mod. Phys. Lett. B, 15, 22 (2001) 965.
[3] M. Mijatovic, B. Veljanoski and D. Hajdukovic, Hadronic J. 7, 5 (1984) 1207.
G. López, Ann. Phys., 251, 2 (1996) 372.
G. López and G. González, IL Nuovo Cim. B, 118, 2 (2003) 107.
[4] A.O. Caldeira and A.T. Leggett, Physica A, 121 (1983) 587.
W.G. Unruh and W.H. Zurek, phys. Rev. D, 40 (1989) 1071.
B.L. Hu, J.P. Paz and Y. Zhang, Phys. Rev. D, 45 (1992) 2843.
G.P. Berman, F. Borgonovi, G.V. López and V.I. Tsifrinovich, Phys. Rev. A, 68 (2003) 012102.
[5] G. López, Rev. Mex. Fis., 48, 1 (2002) 10.
[6] J. Gean-Banacloche, Am. J. Phys. 67, 9 (1999) 776.
D.M. Goodmanson, Am. J. Phys. 68, 9 (2000) 866.
[7] R. Onofrio and L. Viola, quant-ph/9606024 v1 (1996).
[8] G. López, Rev. Mex. Fis., 45 (1999) 1817.
[9] G. López, ”Partial Differential Equations of First Order and Their Applications to Physics”, World Scientific (1999) page 37.
[10] F. Negro and A. Tartaglia, Phys. Lett. A, 77 (1980) 1.
F. Negro and A. Tartaglia, Phys. Rev. A, 1981 (23) 1591.
[11] P.A.M. Dirac, ”The Principles of Quantum Mechanics”, Oxford Science Publications, 1992.
[12] J.S. Borges, L.N. Epele and H. Fanchiotti, Phys. Rev. A, 38, 6 (1988) 3101.
C. Stuckeno and D.H. Kobe, Phys. Rev. A, 34, 5 (1986) 3565.
M. Mijatovic, B. Veljanoski and D. Hajdukovic, Hadronic J., 7, 5(1984) 1207.
B.W. Huang, Z.V. Gu and S.W. Qiam, Phys. Lett A, 142, 4-5 (1989) 203.
M. Razavy, Hadronic J., 6, 2 (1983) 406.