Motivation
Categories of $L$-Relations
Fuzzy Sharpness Theorem

Sharpness in the Fuzzy World

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RAMiCS 2020, October 28, 2020
Content

- Motivation.
- Categories of $L$-Fuzzy Relations.
- Fuzzy Sharpness Theorem.
Sharpness problem of relational products

\[ Q \Join R := Q; \pi \sqcap R; \rho \sqcap S \Join T := \pi; S \sqcap \rho; T \]

\[ (Q \Join R); (S \Join T) = Q; S \sqcap R; T \]
Sharpness problem of relational products

\[ Q \otimes R := Q; \pi^- \sqcap R; \rho^- \quad S \otimes T := \pi; S \sqcap \rho; T \]

\[(Q \otimes R); (S \otimes T) = Q; S \sqcap R; T\]

- Easy to verify for concrete relations.
Sharpness problem of relational products

$$Q \otimes R := Q; \pi \cap R; \rho \quad S \otimes T := \pi; S \cap \rho; T$$

$$(Q \otimes R); (S \otimes T) = Q; S \cap R; T$$

- Easy to verify for concrete relations.
- Not true in all relational categories (R. Maddux).
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**Sharpness problem of relational products**

$Q \ominus R := Q; \pi \sqcap R; \rho \sqcup S \ominus T := \pi; S \sqcap \rho; T$

$(Q \ominus R); (S \ominus T) = Q; S \sqcap R; T$

- Easy to verify for concrete relations.
- Not true in all relational catgeories (R. Maddux).
- Related to the representation problem.
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Motivation I

Sharpness problem of relational products

\[ Q \otimes R := Q; \pi \sqcap R; \rho \sqcap S \otimes T := \pi; S \sqcap \rho; T \]

\[ (Q \otimes R); (S \otimes T) = Q; S \sqcap R; T \]

- Easy to verify for concrete relations.
- Not true in all relational categories (R. Maddux).
- Related to the representation problem.
- Can be shown with additional assumptions, e.g.,
  - certain relations are univalent,
Sharpness problem of relational products

\[ Q \otimes R := Q; \pi \; \cap; \; R; \; \rho \; S \otimes T := \pi; \; S \; \cap; \; \rho; \; T \]

\[ (Q \otimes R); (S \otimes T) = Q; S \cap R; T \]

- Easy to verify for concrete relations.
- Not true in all relational categories (R. Maddux).
- Related to the representation problem.
- Can be shown with additional assumptions, e.g.,
  - certain relations are univalent,
  - additional products exist (H. Zierer and J. Desharnais),
Sharpness problem of relational products

\[ Q \otimes R := Q; \pi \cap R; \rho \quad S \otimes T := \pi; S \cap \rho; T \]

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- Easy to verify for concrete relations.
- Not true in all relational categories (R. Maddux).
- Related to the representation problem.
- Can be shown with additional assumptions, e.g.,
  - certain relations are univalent,
  - additional products exist (H. Zierer and J. Desharnais),
  - one relation is of type \( B \rightarrow \mathcal{P}(B) \) (G. Schmidt and M. Winter).
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Motivation II

Sharpness problem in the Fuzzy World

$Q \otimes_* R := Q; \pi \ast R; \rho \ast S \otimes_* T := \pi; S \ast \rho; T$

$(Q \otimes_* R) \ast (S \otimes T) = Q \ast S \ast R \ast T$

For concrete relations and a $t$-norm like operation $\ast$:

$(U \ast V)(x, y) := U(x, y) \ast V(x, y)$

$(Q \ast S)(x, z) := \bigvee_y Q(x, y) \ast S(y, z)$
Motivation II

Sharpness problem in the Fuzzy World

\[ Q \odot_* R := Q; \pi^{-1} \ast R; \rho^{-1} \quad S \odot_* T := \pi; S \ast \rho; T \]

\[ (Q \odot_* R) \ast (S \odot_* T) = Q \ast S \ast R \ast T \]

For concrete relations and a \( t \)-norm like operation \(*\):

\[ (U \ast V)(x, y) := U(x, y) \ast V(x, y) \]

\[ (Q \ast S)(x, z) := \bigvee_y Q(x, y) \ast S(y, z) \]

- Again easy to verify for concrete relations.
Sharpness problem in the Fuzzy World

\[ Q \ominus_* R := Q; \pi \star R; \rho \star S \ominus_* T := \pi; S \star \rho ; T \]

\[ (Q \ominus_* R) \star (S \ominus_* T) = Q \star S \star R \star T \]

For concrete relations and a \( t \)-norm like operation \( \star \):

\[ (U \star V)(x, y) := U(x, y) \star V(x, y) \]

\[ (Q \star S)(x, z) := \bigvee_y Q(x, y) \star S(y, z) \]

- Again easy to verify for concrete relations.
- Trivial: Not true in all relational categories (Regular sharpness is special case).

\[ A \rightarrow C \rightarrow \triangleright_A \times \triangleright_B \rightarrow D \]

\[ Q \rightarrow A \rightarrow C \]

\[ S \rightarrow D \rightarrow B \]

\[ \pi \rightarrow A \rightarrow B \]

\[ \rho \rightarrow C \rightarrow D \]

\[ \sigma \rightarrow C \rightarrow B \]

\[ \tau \rightarrow A \rightarrow D \]

\[ \tau \rightarrow B \rightarrow D \]
Sharpness problem in the Fuzzy World

\[ Q \Delta_* R := Q ; \pi \sim * R ; \rho \sim \quad S \Delta_* T := \pi ; S * \rho ; T \]

\[(Q \Delta_* R) \Delta_*(S \Delta T) = Q * S * R * T\]

For concrete relations and a \( t \)-norm like operation \( * \):

\[(U * V)(x, y) := U(x, y) * V(x, y)\]

\[(Q * S)(x, z) := \bigvee_y Q(x, y) * S(y, z)\]

- Again easy to verify for concrete relations.
- Trivial: Not true in all relational categories (Regular sharpness is special case).
- Additional challenge: Less properties available since \( * \) is not idempotent.
Sharpness problem in the Fuzzy World

Possible approaches to solve the problem:

- Generalize H. Zierers’s theorem to the fuzzy case.
Sharpness problem in the Fuzzy World

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- Generalize H. Zierers’s theorem to the fuzzy case.
  - The proof relies on the fact that a composition with a map distributes over meet. The corresponding property for $*$ and $\ast$ is only valid for crisp maps.
Motivation III

Sharpness problem in the Fuzzy World

Possible approaches to solve the problem:

- Generalize H. Zierers’s theorem to the fuzzy case.
  - The proof relies on the fact that a composition with a map distributes over meet. The corresponding property for $\ast$, and $\ast$ is only valid for crisp maps.

- Generalize J. Desharnais’s theorem to the fuzzy case.
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Motivation III

Sharpness problem in the Fuzzy World

Possible approaches to solve the problem:

- Generalize H. Zierers’s theorem to the fuzzy case.
  - The proof relies on the fact that a composition with a map distributes over meet. The corresponding property for \( * \) and \( * \) is only valid for crisp maps.

- Generalize J. Desharnais’s theorem to the fuzzy case.
  - The proof uses the modular inclusion in its full generality multiples times. Only weaker versions of the modular inclusion for \( * \) and \( * \) are valid.

- Generalize G. Schmidt’s and M. Winter’s theorem to the fuzzy case.
Motivation III

Sharpness problem in the Fuzzy World

Possible approaches to solve the problem:

- Generalize H. Zierers’s theorem to the fuzzy case.
  - The proof relies on the fact that a composition with a map distributes over meet. The corresponding property for $\ast$, and $\ast$ is only valid for crisp maps.

- Generalize J. Desharnais’s theorem to the fuzzy case.
  - The proof uses the modular inclusion in its full generality multiples times. Only weaker versions of the modular inclusion for $\ast$, and $\ast$ are valid.

- Generalize G. Schmidt’s and M. Winter’s theorem to the fuzzy case.
  - We will show that this is possible.
  - This version of the theorem will then be used to reduce the fuzzy sharpness problem to a regular one.
Definition

A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects $A$ and $B$ the collection $\mathcal{R}[A,B]$ is a complete distributive lattice with operations $\cap, \sqcup, \sqsubseteq, \perp_{AB}, \sqcap_{AB}$.

2. There is a monotone operation $\sim$ (called conversion) so that for all relations $Q : A \rightarrow B$ and $R : B \rightarrow C$:

$$
(Q; R) \sim = R \sim; Q \sim, \quad (Q \sim) \sim = Q.
$$

3. For all relations $Q : A \rightarrow B, R : B \rightarrow C$ and $S : A \rightarrow C$ the modular law holds:

$$
Q; R \cap S \sqsubseteq Q; (R \cap Q \sim; S).
$$

4. For all relations $R : B \rightarrow C$ and $S : A \rightarrow C$ there is a relation $S/R : A \rightarrow B$ (called the left residual of $S$ and $R$) so that for all $Q : A \rightarrow B$ the following holds:

$$
Q; R \sqsubseteq S \iff Q \sqsubseteq S/R.
$$
Special $L$-fuzzy Relations

Relations corresponding to elements in $L$: 

\[
\begin{pmatrix}
a & a & a \\
a & 0 & 0 \\
a & 0 & 0
\end{pmatrix}
\]
Special $L$-fuzzy Relations

Relations corresponding to elements in $L$:

\[
\begin{pmatrix}
    a & a & a \\
    a & a & a \\
\end{pmatrix}
\]

ideal relations

\[
\begin{pmatrix}
    a & 0 & 0 \\
    0 & a & 0 \\
    0 & 0 & a \\
\end{pmatrix}
\]

scalar relations
Special $L$-fuzzy Relations

Relations corresponding to elements in $L$:

\[
\begin{pmatrix}
a & a & a \\ a & a & a
\end{pmatrix}
\quad \quad
\begin{pmatrix}
a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a
\end{pmatrix}
\]

ideal relations  
scalar relations

**Definition**

1. A relation $J : A \to B$ is called an ideal iff $\pi_{AA}J; \pi_{BB} = J$.
2. A relation $\alpha : A \to A$ is called a scalar on $A$ iff $\alpha \subseteq I_A$ and $\pi_{AA}; \alpha = \alpha; \pi_{AA}$.
Arrow Categories

Definition

An arrow category \( \mathcal{A} \) is a Dedekind category with \( \pi_{AB} \neq \square_{AB} \) for all objects \( A \) and \( B \) together with two operations \( \uparrow \) and \( \downarrow \) satisfying the following:

1. \( R\uparrow, R\downarrow : A \to B \) for all \( R : A \to B \).
2. \( (\uparrow, \downarrow) \) is a Galois correspondence.
3. \( (R^{-}; S\downarrow)^\uparrow = R^{-}; S\downarrow \) for all \( R : B \to A \) and \( S : B \to C \).
4. \( (Q \cap R\downarrow)^\uparrow = Q^\uparrow \cap R\downarrow \) for all \( Q, R : A \to B \).
5. If \( \alpha_A \neq \square_{AA} \) is a non-zero scalar, then \( \alpha_A^\uparrow = \mathbb{I}_A \).
Definition

A fuzzy category $\mathcal{F}$ is an arrow category together with two operations $*$ and $\ast$, so that the following holds:

1. $*$ maps two relations $Q : A \to B$ and $R : A \to B$ to a relation $Q * R : A \to B$.
2. $*$ is associative, commutative and continuous.
3. $Q * R^\perp = Q \cap R^\perp$ for all $Q, R : A \to B$.
4. $(Q * R)^\perp = Q^\perp * R^\perp$ for all $Q, R : A \to B$.
5. $\ast$ maps two relations $Q : A \to B$ and $R : B \to C$ to a relation $Q \ast R : A \to C$.
6. $\ast$ is associative and continuous.
7. $Q \ast R^\perp = Q^\perp ; R^\perp$ for all $Q : A \to B$ and $R : B \to C$.
8. $(Q \ast R)^\perp = R^\perp \ast Q^\perp$ for all $Q : A \to B$ and $R : B \to C$. 

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Fuzzy Categories

Definition

The exchange inclusion \((Q \ast R) \ast (S \ast T) \subseteq Q \ast S \ast R \ast T\) is valid for all \(Q, R : A \rightarrow B\) and \(S, T : B \rightarrow C\).

The following versions of the modular inclusion are valid:

1. \(Q; R \ast S \subseteq Q; (R \cap Q^\sim \ast S)\),
2. \(Q \ast R \ast S \subseteq Q; (R \ast Q^\sim \ast S)\),
3. \((P \ast Q) \ast R \ast S \subseteq P \ast (R \ast Q^\sim \ast S)\),

for all \(P, Q : A \rightarrow B, R : B \rightarrow C\), and \(S : A \rightarrow C\).
Relational Products

Definition

An object $A \times B$ together with relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a relational product of $A$ and $B$ iff

$$
\pi, \rho \text{ are crisp, } \pi \subseteq \mathbb{I}_A, \quad \rho \subseteq \mathbb{I}_B, \quad \pi \cap \rho = \mathbb{I}_{AB}, \quad \pi \cap \rho = \mathbb{I}_{A \times B}.
$$
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Relational Products

Definition

An object $A \times B$ together with relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a relational product of $A$ and $B$ iff

$$
\pi, \rho \text{ are crisp, } \pi \sqsubseteq I_A, \rho \sqsubseteq I_B, \pi; \rho = \Pi_{AB}, \pi; \rho \sqcap \rho; \rho = I_{A \times B}.
$$

$$
\pi = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 1
\end{pmatrix}, \quad
\rho = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$
Relational Powers

Definition
An object $\mathcal{P}(A)$ together with a relation $\varepsilon : A \rightarrow \mathcal{P}(A)$ is called a relational power iff

$$\text{syQ}(\varepsilon, \varepsilon)^\downarrow = \mathbb{I}_{\mathcal{P}(A)} \quad \text{and} \quad \text{syQ}(R, \varepsilon)^\downarrow \quad \text{is total for every} \quad R : A \rightarrow B.$$
Relational Powers

Definition

An object $\mathcal{P}(A)$ together with a relation $\varepsilon : A \rightarrow \mathcal{P}(A)$ is called a relational power iff

$$\text{syQ}(\varepsilon, \varepsilon)^\uparrow = \mathbb{I}_{\mathcal{P}(A)}$$

and $\text{syQ}(R, \varepsilon)^\uparrow$ is total for every $R : A \rightarrow B$.

$$\varepsilon = \begin{pmatrix}
0 & 0 & 0 & a & a & a & 1 & 1 & 1 \\
0 & a & 1 & 0 & a & 1 & 0 & a & 1
\end{pmatrix}$$
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Fuzzy Sharpness for Concrete Relations

Theorem

If $Q : C \to A$, $R : C \to B$, $S : A \to D$ and $T : B \to D$ are $L$-relations and
$
\pi : A \times B \to A$, $\rho : A \times B \to B$ the standard (concrete) projection relations, then we have:

$$(Q \otimes_* R) \ast_S (S \otimes_* T) = Q \ast S \ast R \ast T.$$
**Theorem**

Let $\mathcal{F}$ be a fuzzy category, $A \times B$ the relational product of $A$ and $B$, $Q : C \to A$, $R : D \to B$, $S : A \to E$, $T : B \to E$ be relations, and $f : C \to D$ be crisp and univalent. If there is an injective relation $U : D \to B$ such that $f; U$ is total, then we have:

$$(Q \otimes_f^*; R) \ast (S \otimes^*_T) = Q \ast S \ast f; (R \ast T).$$
Fuzzy Sharpness Theorem II

\[ \begin{array}{ccc}
B & \overset{\mathcal{I}_B}{\leftarrow} & B \\
\downarrow^{Q \otimes_* R} & & \downarrow^{S \otimes_* T} \\
A \times \mathcal{P}(B) & \overset{\pi}{\leftarrow} & D \\
\downarrow^{R} & & \downarrow^{T} \\
\mathcal{P}(B) & \overset{s y Q(\mathcal{I}_B, \varepsilon)}{\leftarrow} & \\
\end{array} \]

Corollary

Let \( \mathcal{F} \) be a fuzzy category, \( \mathcal{P}(B) \) the relational power of \( B \), \( A \times \mathcal{P}(B) \) the relational product of \( A \) and \( \mathcal{P}(B) \), and \( Q : B \to A, R : B \to \mathcal{P}(B), S : A \to D, T : \mathcal{P}(B) \to D \) be relations. Then we have

\[
(Q \otimes_* R) \ast (S \otimes T) = Q \ast S \ast R \ast T.
\]
Please note that the previous corollary immediately implies

\[(\varepsilon \oplus \varepsilon) \ast (S \ominus T) = \varepsilon \ast S \ast \varepsilon \ast T\]

for appropriate relations \(S\) and \(T\).
Fuzzy Sharpness Theorem IV

\[ \begin{array}{c}
C \xrightarrow{Q \otimes R} A \times B \\
\downarrow R \downarrow \pi \\
\downarrow B \\
B
\end{array} \quad \begin{array}{c}
A \\
\uparrow \pi \\
S \\
A \times B \xrightarrow{S \otimes T} D
\end{array} \]
Fuzzy Sharpness Theorem IV

\[ Q \xrightarrow{\pi} S \]

\[ C \xrightarrow{Q \otimes R} A \times B \xrightarrow{S \otimes T} D \]

\[ \mathcal{P}(C) \xrightarrow{\text{syQ}(\varepsilon,Q)\downarrow} A \xrightarrow{\text{syQ}(S^-,\varepsilon)\downarrow} \mathcal{P}(D) \]

\[ C \xrightarrow{\varepsilon \otimes^* \varepsilon} \mathcal{P}(C) \times \mathcal{P}(C) \xrightarrow{X} A \times B \xrightarrow{Y} \mathcal{P}(D) \times \mathcal{P}(D) \xrightarrow{\varepsilon^- \otimes^* \varepsilon^-} D \]

\[ \mathcal{P}(C) \xrightarrow{\text{syQ}(\varepsilon,R)\downarrow} B \xrightarrow{\text{syQ}(T^-,\varepsilon)\downarrow} \mathcal{P}(D) \]
Fuzzy Sharpness Theorem V

Theorem

Let $\mathcal{F}$ be a fuzzy category, and be $A \times B$ the relational product of $A$ and $B$. If the objects $\mathcal{P}(C), \mathcal{P}(D), \mathcal{P}(C) \times \mathcal{P}(C), \mathcal{P}(D) \times \mathcal{P}(D)$ exist, and if the product $A \times B$ is sharp for all crisp relations $Q' : \mathcal{P}(C) \times \mathcal{P}(C) \to A$, $R' : \mathcal{P}(C) \times \mathcal{P}(C) \to B$, $S' : A \to \mathcal{P}(D) \times \mathcal{P}(D)$, and $T' : B \to \mathcal{P}(D) \times \mathcal{P}(D)$, i.e., we have

$$(Q' \ominus R') ; (S' \ominus T') = Q'; S' \sqcap R'; T',$$

then also the fuzzy version of sharpness holds, i.e., we have

$$(Q \ominus R) \hat{*} (S \ominus T) = Q \hat{*} S \ast R \ast T.$$

for all $Q : C \to A, R : C \to B, S : A \to D$, and $T : B \to D$. 
Corollary

Let $\mathcal{F}$ be a fuzzy category with relational products and powers. Then fuzzy sharpness holds.
Thank you for your attention!

Questions?