Tracking Problem with Consideration of Physical Constraints on Phase Variables and Control. Method of State Space Extension

S.I. Gulyukina, V.A. Utkin
V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences (ICS RAS), 65 Profsoyuznaya Street, Moscow, Russia
E-mail: gulyukina.s.i@mail.ru, vicutkin@ipu.ru

Abstract. The paper proposes a method for the synthesis of nonlinear systems invariant to the output variable, under the conditions of parametric and external perturbations, taking into account the constraints on phase variables and control. To solve the problem, the ideology of the block method using sat-functions as local feedback, and the disturbances observer on the sliding modes are applied.

1. Introduction
At present, the problems of synthesis of control systems considering the constraints on the state and control vector components are understudied. Basically, the account of signal constraints is carried out in some special cases using indirect methods [1-4].

In practice, the physical systems where constraints are imposed on phase variables and controls are widespread [5-15]. The common problem of taking into account the constraints on phase variables and controls is the fact that the mathematical models of control plants do not consider these constraints a priori. The standard procedures for control laws synthesis require applying some adjustments according to the specified constraints. Thus, it is necessary to make some corrections taking into account the specified constraints in the standard procedures of control laws synthesis.

A number of publications on the methods for nonlinear systems synthesis [16-18], as a rule, do not consider the whole complex of uncertainties of the control plant model.

The paper presents a common approach to the synthesis of relatively robust output variables of nonlinear systems under the conditions of parametric and external disturbances, with consideration of constraints on phase variables and control. To solve the problem, the ideology of the block method [19] is applied using sat-functions as local feedback [20], and the disturbance observer on sliding modes [21], sigma functions [22] or high-gains [23].

The paper is organized as follows. Section 2 provides a common model of the control and gives the problem description. Section 3 is dedicated to the synthesis of feedback with consideration of constraints on phase variables and controls. In Section 4, an observer of the state and disturbance vector is developed.

2. Problem statement
A non-linear SISO system in the form (1) is considered.
\[ \dot{x}_i = x_{i+1} + \eta_i(x', t), \quad i = 1, \ldots, n - 1, \quad \dot{x}_n = u + \eta_n(x^n, t), \]  

where the state vector \( x^T = (x_1, \ldots, x_n) \) is available for measurement, \( x_i \) is the output (controlled) variable, \( (x') = (x_1, \ldots, x_i) \), \( i = 1, \ldots, n \), \( \eta^T = (\eta_1, \ldots, \eta_n) \) is the disturbances and nonlinear uncertain vector.

The problem of output variable stabilization with specified accuracy is set.

In addition, the following assumptions are introduced.

The components of the state vector (starting from the second one) and control are considered limited:

\[ |x_i| < X_i, i = 2, n - 1, \quad |u| < U, \quad i = 1, n, \quad X_i, N_i, U = \text{const}. \]  

The disturbances \( \eta(.). \) include both unknown nonlinearities of the model and the external perturbations; they are not available for measurement and are limited together with their derivatives of the corresponding order

\[ |\eta^{(n-j)}| < N^{(n-j)}_i = \text{const}, \quad i = 1, n, \quad j = i, n + 1. \]  

The problem of synthesis of the feedback providing invariance of the output variable with respect to the external disturbances with the given accuracy is set:

\[ |e_i(t)| \leq \Delta_i = \text{const}, \quad \forall t > T_i > 0 \]  

with the constraints (2) satisfaction.

It should be noted that the problem of stabilizing the output variable with a given accuracy is reduced to the problem of tracking the output variable \( e_i = x_i - x_{id} \), where \( x_{id}(t) \) is the desired system output, limited together with its first derivative on the module \( |x_{id}(t)| \leq X_{id} = \text{const} \) and \( |\dot{x}_{id}(t)| \leq \dot{X}_{id} = \text{const} \). In this case, it is necessary to replace the first equation (1) with \( \dot{e}_i = x_i - \bar{\eta}_i \), where the external influence \( \bar{\eta}_i = \eta_i - \dot{x}_{id} \) is limited by the value \( |\bar{\eta}_i| = N_i + \dot{X}_{id} \leq \dot{N}_i = \text{const} \). Instead of solving the problem of stabilization (4), stabilization of a variable \( |e_i| \leq \Delta_i \) is provided with a given accuracy.

### 3. Feedback synthesis. Combined control

Let us present the solution of the problem within the block approach using combined feedback.

To take into account the constraints on the state vector and control, we introduce a nonsingular change of variables in the form

\[ s_i = x_i, s_i = x_i + M_{ji} \text{sat}(\bar{\eta}_j), \quad i = 2, n \]  

and formulate the control as a saturation function.

\[ u = -M_{jn} \text{sat}(\bar{\eta}_j), \]  

where the variables \( \bar{s}_i, i = 1, n \) are selected later.

By definition (Figure 1), \( M\text{sat}(s) = \min(M|s|)\text{sign}(s), \quad M = \text{const} > 0 \).
The idea is a follows. If we provide execution of the relations \( |\delta_i| \leq \Delta_i = \text{const} > 0, \ i = 2, n \) in the new variables (5), then the choice of amplitudes in the form \( M_{i+1} = X_i - \Delta_i \) with the requirement of \( u \leq M_n \leq U \) ensure fulfillment of the specified constraints on the state vector components and control \( |x_i| \leq \Delta_i + M_{i+1} = X_i, \ i = 2, n, \ |\eta| \leq M_u \leq U \).

A closed system (1), (6) in the variables (5) can be written as follows:

\[
\begin{align*}
\dot{s}_i &= -M_i \text{sat}(\overline{s}_i) + s_{z_i} + \eta_i, \ \ \dot{s}_u = -M_u \text{sat}(\overline{s}_u) + s_{z_u} + \eta_u + \frac{d}{dt} M_u \text{sat}(\overline{s}_u), \\
\dot{s}_u &= -M_u \text{sat}(\overline{s}_u) + \eta_u + \frac{d}{dt} M_u \text{sat}(\overline{s}_u),
\end{align*}
\]  

(7)

where \( \frac{d}{dt} M_u \text{sat}(\overline{s}_u) = \begin{cases} 
0, & |\overline{s}_i| \geq M_i, \\
\overline{s}_i, & |\overline{s}_i| < M_i,
\end{cases} \ i = 1, n.

Thus, the system stabilization problem (7) is formulated, the accuracy is set by the output variable, and other components will also tend to some areas of zero.

The problem under consideration is solved by a two-stage procedure of the control law synthesis (6). The developed procedure allows choosing the parameters of nonlinear change of variables (5) and control (6), namely the amplitudes and gain factors of linear functions with saturation.

The first stage consists in selecting the gains factors at the functioning of the closed system (8) in a linear zone irrespective of sat-functions’ amplitudes chosen at the second stage.

In the linear zones on all variables, (7) will take the form

\[
\dot{s}_i = -\overline{s}_i + s_{z_i} + \eta_i, \ \ \dot{s}_u = -\overline{s}_u + s_{z_u} + \eta_u + \hat{s}_{z_u}, \ i = 2, n - 1, \ \ \dot{s}_u = -\overline{s}_u + \eta_u + \hat{s}_{z_u},
\]  

(8)

where \( |\overline{s}_i| \leq M_i \).

We define the variables \( \overline{s}_i \) as follows:

\[
\overline{s}_i = k_i \hat{s}_i + \eta_i, \ i = 1, n,
\]  

(9)

where

\[
\eta_1 = \eta_i, \eta_2 = \eta_z + \hat{\eta}_1, \eta_3 = \eta_i + \hat{\eta}_2, \ldots, \eta_n = \eta_u + \hat{\eta}_n.
\]  

(10)

Note that the constraints on variables (10) can be recalculated using known constraints (3)

\[
|\overline{s}_i| < N_i, |\hat{s}_{z_u}| < N^{(i)}_{z_u}, \ i = 2, n, \ N_i, N^{(i)}_{z_u} = \text{const} > 0.
\]  

(11)
The system (7), when it functions in the linear zones on all variables, from top to bottom, is described by the equations of a kind:

\[
\begin{align*}
\dot{s}_1 &= -k_1s_1 + s_2 + \ldots + s_n, \\
\dot{s}_2 &= -k_2s_2 + s_3 + \ldots + s_n, \\
&\vdots \\
\dot{s}_n &= -k_ns_n + s_{n+1}.
\end{align*}
\]  
(12)

The system (12) is independent of the external disturbances and non-linearities, and by selecting the gain factors \(k_i > 0\), the problem of its stabilization is solved.

At the second stage, after selecting the gain factors, it is necessary to determine the amplitudes of the sat-functions \(M_i > 0\) to ensure that all the variables fall into the linear zones.

Let us write the system (7) relatively to the variables (9)

\[
\begin{align*}
\dot{s}_1 &= k_1[-M_{i} \text{sat}(\eta_i) + s_1 + \eta_i], \\
\dot{s}_2 &= k_2[-M_{i} \text{sat}(\eta_i) + s_2 + \eta_i], \\
&\vdots \\
\dot{s}_n &= k_n[-M_{i} \text{sat}(\eta_i) + s_n + \eta_i].
\end{align*}
\]  
(13)

Considering (5), we have

\[s_i = x_i + M_{i,i} \text{sat}(\tilde{\eta}_i) \Rightarrow |s_i| \leq 2X_i.
\]

The amplitudes are selected from top to bottom according to the following equations:

1. \(M_i > 2X_i + N_i + \frac{1}{k_i}N_i^{(1)}, |\tilde{\eta}_i| \leq k_i(M_i + 2X_i + N_i) + N_i^{(1)}; \ldots\)

i. \(M_i > 2X_i + N_i + k_{i-1}N_i^{(i-1)} + N_i^{(i-1)}, |\tilde{\eta}_i| \leq k_i(M_i + 2X_{i+1} + N_i + |\tilde{\eta}_{i+1}|) + N_i^{(i)}; i = \overline{2, n-2}; \ldots\)

n-1. \(M_{n-1} > 2U + N_{n-1} + k_{n-2}N_{n-2}^{(n-2)} + N_{n-2}^{(n-2)}, |\tilde{\eta}_{n-1}| \leq k_{n-1}(M_{n-1} + 2U + N_{n-1} + |\tilde{\eta}_{n-1}|) + N_{n-1}^{(n-1)}; \)

n. \(M_n > N_n + k_{n-1}|\tilde{\eta}_{n-1}| + N_{n-1}^{(n-1)}, |\tilde{\eta}_n| \leq k_n(M_n + N_n + |\tilde{\eta}_{n-1}|) + N_n^{(n)}; \)

The optimization problem of choosing the gain factors in sat-functions in order to minimize their amplitudes is not discussed here. Only sufficient conditions of solving the problem to prove its solvability are given.

To implement the proposed algorithm, information on the disturbance estimates of the variables (10) used in the control law is required.

In the following section, the problem of estimating the variables (10) used in the control law is solved.

4. Observer disturbances

Let us synthesize the disturbance observer, based on the system (1) for signals (10). It is easy to see that, to obtain the estimates of variables (10), it is enough to bring the system (1) to a canonical form

\[
\begin{align*}
\dot{\tilde{x}}_i &= \tilde{x}_{i+1}, \quad i = \overline{1, n-1}, \\
\dot{\tilde{x}}_n &= u + \tilde{\eta}_n,
\end{align*}
\]  
(14)

where \(\tilde{x}_1 = x_1, \tilde{x}_2 = x_2 + \tilde{\eta}_1, \tilde{x}_3 = x_3 + \tilde{\eta}_2, \ldots, \tilde{x}_{n-1} = x_{n-1} + \tilde{\eta}_{n-1}.\)

In the system (14), the variable \(\tilde{x}_i = x_i\) is available for measurement. Therefore, having solved the problem of obtaining the estimates of variables \(\tilde{x}_i\), it is possible to find the estimates of signals

\[
\tilde{\eta}_i = \tilde{x}_{i+1} - x_{i+1}, \quad i = \overline{1, n-1},
\]  
(15)

Let us construct an observer for the system (14) in the form

\[
\begin{align*}
\dot{\tilde{x}}_i &= \tilde{x}_{i+1}, \\
\dot{\tilde{\eta}}_i &= \tilde{x}_{i+1} - x_{i+1}, \quad i = \overline{1, n-1}.
\end{align*}
\]
\[ \dot{z}_i = z_2 + v_1, \ldots, \dot{z}_{n-1} = z_n + v_{n-1}, \dot{z}_n = u + v_n, \quad (16) \]

where corrective actions \( v_i, i = 1, n \) are selected later. Let us write the equations with regard to the observation errors \( e_i = \tilde{x}_i - z_i, i = 1, n \) according to (14) and (16):
\[ \dot{e}_i = e_2 - v_1, \ldots, \dot{e}_{n-1} = e_n - v_{n-1}, \dot{e}_n = \overline{\eta}_n - v_n. \quad (17) \]

We choose sequentially the corrective actions \( v_i, i = 1, n - 1 \) in the class of discontinuous functions based on the theory of sliding modes as follows: \( v_i = L_i \text{sign}(e_i), L_i = \text{const} > 0 \). After substitution in (17), the equation with the number \( i \) of the system takes the form \( \dot{e}_i = e_i - L_i \text{sign}(e_i) \). The choice of the amplitude \( L_i > 0 \) providing the sliding mode in a straight line \( e_i = 0 \) is determined on the basis of Lyapunov’s second method. We select a Lyapunov candidate function as \( V = 0.5 e_i^2 \).

Lyapunov’s derivative function has the form \( \dot{V} = e_i \dot{e}_i \leq \|e_i\| - L_i < 0 \), whence follows the inequality determining the choice of the amplitude \( L_i > \|e_i\| \). When the condition \( L_i > \|e_i\| \) is fulfilled, a sliding mode in a straight line \( e_i = 0 \) appears, thus \( \tilde{x}_i \rightarrow z_i \). The term \( \dot{e}_i = 0 \) is executed, from where we get the average value of the discontinuous signal \( v_{(i-1)\text{eq}} = e_i \).

The last step is to select a corrective action \( v_i = L_i \text{sign}(v_{(i-1)\text{eq}}) = L_i \text{sign}(e_i) \). We require the fulfillment of the relation for the choice of the amplitude, which follows from the second Lyapunov method \( L_i > \|\overline{\eta}_n\| \). When this ratio is fulfilled, the sliding mode in a straight line is provided \( e_i = 0 \Rightarrow v_{(i\text{eq})} = \overline{\eta}_i \).

Synthesis of corrective actions is carried out independently from each other from top to bottom. A sliding mode appears at each step and the average value of the discontinuous signal is used as an estimate of the next variable. At the last step, the equivalent value of the discontinuous signal gives an estimate of the unknown perturbation.

Note that the corresponding equivalent values of the corrective actions of the observer can be obtained at the output of the first order low-frequency filter \( \mu_i \dot{z}_i = -\tau_i + v_i, \mu_i > 0, i = 1, n \), \( v_{(i\text{eq})} = \tau_i \).

Thus, the problem of obtaining the estimates of the system state vector components (14) \( z_i \approx \overline{x}_i \), \( i = 1, n \) and disturbance estimate \( \overline{\eta}_n \) from (10) has been solved. Since the components of the initial system state vector (1) are available for measurement, we have the estimates \( \overline{x}_i, i = 1, n - 1 \) of unknown disturbances (15) that are used in the control algorithm (5), (6).

5. Conclusion
In the paper, the method of solving the output control problem for a non-linear system with a single input and a single output (SISO-system) in a robust setting and with the influence of external disturbances at the set constraints on the state vector and control was proposed.

It should be noted that the problem of accounting the constraints on the state and control vector is studied insufficiently in theory.

The main idea of this work is to implement the replacement of variables in the form of linear saturation functions in a control system model, which allows to automatically provide the specified constraints already at the stage of feedback synthesis.

The problem of ensuring the invariance of the control plant model to external disturbances and parametrical and functional uncertainties is solved by means of the “input-output” representation and the use of a special observer of states and disturbances.
In contrast to the asymptotic approach [24], where deep feedback is used, the authors of this paper use combined feedback by obtaining disturbance estimates (with the help of the state observer on the sliding modes). This approach allows us to solve the problem of stabilization at arbitrarily small gain factors in the synthesis of local feedback and control.

Acknowledgments
The research was partially supported by the Russian Foundation for Basic Research under the projects 18-01-00846A and 20-01-00363A.

References
[1] Zhu Z, Xia YQ and Fu MY 2011 Adaptive sliding mode control for attitude stabilization with actuator saturation IEEE Trans. Ind. Electron 58 pp 4898–4907.
[2] Yang J, Li SH and Yu X 2013 Sliding-mode control for systems with mismatched uncertainties via a disturbance observer IEEE Trans. Ind. Electron 60 pp 160–9.
[3] Ding S, Mei K and Li S 2019 A new second-order sliding mode and its application to nonlinear constrained systems IEEE Trans. Automat. Control 64 pp 2545-52,
[4] Yang J, Ding ZT and Li SH 2018 Continuous finite-time output regulation of nonlinear systems with unmatched time-varying disturbances IEEE Contr. Syst. Lett. 2 pp 97–102.
[5] Saravanathamizhan R, Paranthaman R and Balasubramanian N 2008 Research Tanks in series model for continuous stirred tank electrochemical reactor Industrial & Engineering Chemistry 47 pp 2976-84.
[6] Lagos B and Cipriano A 2015 Performance evaluation of a distributed MPC strategy applied to the continuous stirred tank reactor IEEE Latin America Transactions 13 pp 1921-26.
[7] P. Ignaciuk 2014 Nonlinear inventory control with discrete sliding modes in systems with uncertain delay IEEE Transactions on Industrial Informatics 10 pp 559-68.
[8] Ma H, Wu J and Xiong Z 2017 A Novel Exponential Reaching Law of Discrete-Time Sliding-Mode Control IEEE Transactions on Industrial Electronics 64 pp 3840-50.
[9] Ma H, Wu J and Xiong Z 2016 Discrete-time sliding-mode control with improved quasi-sliding-mode domain IEEE Transactions on Industrial Electronics 63 pp 6292-6304.
[10] Rios H, Efimov D, Moreno J, Perruquetti W and Rueda-Escobedo J 2017 Time-varying parameter identification algorithms: Finite and fixed-time convergence IEEE Transactions on Automatic Control 62 pp 3671-78.
[11] Flores-Tlacuahac A and Grossmann I 2006 Simultaneous cyclic scheduling and control of a multiproduct CSTR Industrial & engineering chemistry research 45 pp 6698-6712.
[12] Delbari M, Salahshoor K, Moshiri B 2010 Adaptive generalized predictive control and model reference adaptive control for CSTR reactor Int. Conf. Intelligent Control and Information Processing pp 165-9.
[13] F P de Mello 1991 Boiler models for system performance studies IEEE Transaction on Power Systems 6 pp 66-74.
[14] Flynn M E and O’Malley M J A 1999 Drum Boiler Model for Long Term Power System Dynamic Simulation IEEE Transactions on Power Systems 14 pp 209-17.
[15] Astrom K J and Bell R D 2000 Drum Boiler Dynamics Automatica 36 pp 363-78.
[16] Tan W, Marquez H J, Chen T and Liu J 2005 Analysis and control of a nonlinear boiler-turbine unit Journal of Process Control 15 pp 883-91.
[17] Liu H B, Li S Y and Chai T Y 2004 Intelligent coordinated control of power-plant main steam pressure and power output Journal of Systems Engineering and Electronics 15 pp 350-8.
[18] Li S 2014 DEB-oriented modelling and control of coal-fired power plant Proc. of 19th IFAC World Congress pp 413-418.
[19] Utkin V 2001 Invariance and Self-Sufficiency in Systems with Separable Automation and Remote Control 62 pp 1825–1843.
[20] S. A. Krasnova, V. A. Utkin, and A. V. Utkin 2017 Block Approach to Analysis and Design of
the Invariant Nonlinear Tracking Systems *Automation and Remote Control* 78 pp 2120–40.

[21] Utkin V, Guldner J and Shi J 2009 *Sliding Mode Control in Electro-Mechanical Systems* (Boca Raton: CRC Pres) p 503.

[22] S. A. Krasnova and A. V. Utkin 2016 Sigma function in observer design for states and perturbations *Automation and Remote Control* 77 pp 1676-88.

[23] Khalil H and Praly L 2014 High-gain observers in nonlinear feedback control *Int. J. Robust and Nonlinear Control* 24 pp 993–1015.

[24] Gulyukina S and Utkin V 2020 Tracking problem with consideration of physical restrictions on phase variables and controls, 15th Int. Conf. on Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference) (Russia: Moscow) pp 1–4.