GLORIOUS PAIRS OF ROOTS AND ABELIAN IDEALS OF A BOREL SUBALGEBRA

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ABSTRACT. Let $\mathfrak{g}$ be a simple Lie algebra with a Borel subalgebra $\mathfrak{b}$. Let $\Delta^+$ be the corresponding (po)set of positive roots and $\theta$ the highest root. A pair $\{\eta, \eta'\} \subset \Delta^+$ is said to be glorious, if $\eta, \eta'$ are incomparable and $\eta + \eta' = \theta$. Using the theory of abelian ideals of $\mathfrak{b}$, we (1) establish a relationship of $\eta, \eta'$ to certain abelian ideals associated with long simple roots, (2) provide a natural bijection between the glorious pairs and the pairs of adjacent long simple roots (i.e., some edges of the Dynkin diagram), and (3) point out a simple transform connecting two glorious pairs corresponding to the incident edges in the Dynkin digram. In types $\text{DE}$, we prove that if $\{\eta, \eta'\}$ corresponds to the edge through the branching node of the Dynkin diagram, then the meet $\eta \land \eta'$ is the unique maximal non-commutative root. There is also an analogue of this property for all other types except type $\text{A}$. As an application, we describe the minimal non-abelian ideals of $\mathfrak{b}$.

INTRODUCTION

Let $\mathfrak{g}$ be a complex simple Lie algebra, with a Cartan subalgebra $\mathfrak{t}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Here $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is a fixed Borel subalgebra. In this note, we present new combinatorial properties of root systems and abelian ideals of $\mathfrak{b}$. Our setting is always combinatorial, i.e., the abelian ideals of $\mathfrak{b}$, which are sums of root spaces of $\mathfrak{u}$, are identified with the corresponding sets of positive roots.

Let $\Delta$ be the root system of $(\mathfrak{g}, \mathfrak{t})$ in the $\mathbb{R}$-vector space $V = \mathfrak{t}_\mathbb{R}^*, \Delta^+$ the set of positive roots in $\Delta$ corresponding to $\mathfrak{u}$, $\Pi$ the set of simple roots in $\Delta^+$, and $\theta$ the highest root in $\Delta^+$. We regard $\Delta^+$ as poset with the usual partial ordering $\preceq$. An ideal of $(\Delta^+, \preceq)$ is a subset $I \subset \Delta^+$ such that if $\gamma \in I, \nu \in \Delta^+$, and $\nu + \gamma \in \Delta^+$, then $\nu + \gamma \in I$. The set of minimal elements of $I$ is denoted by $\min(I)$. Write also $\max(\Delta^+ \setminus I)$ for the set of maximal elements of $\Delta^+ \setminus I$. An ideal $I$ is abelian, if $\gamma' + \gamma'' \not\in \Delta^+$ for all $\gamma', \gamma'' \in I$.

Let $\mathfrak{Ad}$ (resp. $\mathfrak{Ab}$) be the set of all (resp. all abelian) ideals of $\Delta^+$. Then $\mathfrak{Ab} \subset \mathfrak{Ad}$ and we think of both sets as posets with respect to inclusion. The ideal generated by $\gamma \in \Delta^+$ is $I(\preceq \gamma) = \{\nu \in \Delta^+ | \nu \succeq \gamma\} \subset \mathfrak{Ad}$, and $\gamma$ is said to be commutative, if $I(\preceq \gamma) \in \mathfrak{Ab}$. Write $\Delta^+_{\text{com}}$ for the set of all commutative roots. For any $\Delta$, this subset is explicitly described in [9, Theorem 4.4]. Note that $\Delta^+_{\text{com}} \subset \mathfrak{Ad}$.

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A pair \( \{ \eta, \eta' \} \subset \Delta^+ \) is said to be glorious, if \( \eta, \eta' \) are incomparable in the poset \((\Delta^+, \supseteq)\), i.e., \( \{ \eta, \eta' \} \) is an antichain, and \( \eta + \eta' = \theta \). Our goal is to explore properties of glorious pairs in connection with the theory of abelian ideals. We prove that

1. these \( \eta, \eta' \) are necessarily long and commutative;
2. there is a natural bijection between the glorious pairs and the pairs of adjacent long simple roots, i.e., certain edges of the Dynkin diagram, see Theorem 3.5;
3. there is a simple transformation connecting two glorious pairs corresponding to a triple of consecutive long simple roots, i.e., two incident edges, see Theorem 3.11.
4. if \( I \in \mathfrak{A} \) and \( \#I < h^* \), where \( h^* \) is the dual Coxeter number of \( \Delta \), then \( I \in \mathfrak{A} \), see Lemma 2.2.

Let \( \Pi_t \) be the set of long simple roots. (In the ADE-case, all roots are assumed to be long and hence \( \Pi_t = \Pi \).) Since \( \Pi_t \) is connected in the Dynkin diagram, we have \( \#\{ \text{glorious pairs} \} = \#\Pi_t - 1 \), and the above bijection (2) can be understood as a bijection with the edges of the sub-diagram \( \Pi_t \). In the non-simply laced case, there is a unique pair of adjacent simple roots of different length. This gives rise to a similar pair of roots (the semi-glorious pair), which is discussed in Section 4.2. To state more properties of glorious pairs, we need more notation.

Let \( \mathfrak{A}^{\text{o}} \) be the set of nonempty abelian ideals and \( \Delta_t^+ \) the set of long positive roots. In [8, Sect. 2], we constructed a map \( \tau : \mathfrak{A}^{\text{o}} \to \Delta_t^+ \), which is onto (see Section 1). If \( \tau(I) = \mu \), then \( \mu \in \Delta_t^+ \) is called the rootlet of \( I \), also denoted \( rt(I) \). Letting \( \mathfrak{A}_\mu = \tau^{-1}(\mu) \), we get a partition of \( \mathfrak{A}^{\text{o}} \) parameterised by \( \Delta_t^+ \). Each \( \mathfrak{A}_\mu \) is a subposet of \( \mathfrak{A} \) and, moreover, \( \mathfrak{A}_\mu \) has a unique minimal and unique maximal element (ideal) [8, Sect. 3]. These extreme elements of \( \mathfrak{A}_\mu \) are denoted by \( I(\mu)_{\text{min}} \) and \( I(\mu)_{\text{max}} \).

- We show that the glorious pairs are related to the abelian ideals \( I(\alpha)_{\text{min}} \) with \( \alpha \in \Pi_t \). If a glorious pair \( \{ \eta, \eta' \} \) gives rise to adjacent \( \alpha, \alpha' \in \Pi_t \), then \( \eta \in \min(I(\alpha)_{\text{min}}) \), \( \eta' \in \min(I(\alpha')_{\text{min}}) \) and also \( \text{cl}(\eta) = \alpha' \) and \( \text{cl}(\eta') = \alpha \), see Prop. 3.2. Here \( \text{cl}(\cdot) \) is a natural map from \( \Delta^+_{\text{com}} \) to \( \bar{\Pi} \), the set of affine simple roots, see Section 1 for details.
- Conversely, given adjacent \( \alpha, \alpha' \in \Pi_t \), the corresponding glorious pair is the unique pair in \( I(\alpha)_{\text{min}} \cup I(\alpha')_{\text{min}} \) that sums to \( \theta \). The constructions of \( \{ \eta, \eta' \} \) from \( \{ \alpha, \alpha' \} \) and vice versa heavily relies on properties of \( I(\alpha)_{\text{min}}, I(\alpha')_{\text{max}} \) and the minuscule elements of the affine Weyl group \( \hat{W} \), as it was developed in [8, 10].
- It is noticed in [9, Sect. 4] and conceptually proved in [13, Sect. 3] that if \( \Delta \) is not of type \( A_n \), then \( \Delta^+_{\text{nc}} := \Delta^+ \setminus \Delta^+_{\text{com}} \) has a unique maximal element, denoted \( \lceil \theta/2 \rceil \). Then \( \lceil \theta/2 \rceil := \theta - \lceil \theta/2 \rceil \in \Delta^+ \) and \( \lceil \theta/2 \rceil \leq \lceil \theta/2 \rceil \). In Section 4, we prove that if \( \Delta \in \{ D-E \} \), then the interval \([\lceil \theta/2 \rceil, \lceil \theta/2 \rceil]\) in \( \Delta^+ \), which is a cube \( B^3 \) [13, Sect. 5], is related to the glorious pairs associated with the edges through the branching node, \( \alpha \), of the Dynkin diagram. This is deduced from some properties of the shortest element of \( W \) that takes \( \theta \) to \( \alpha \), see Theorem 4.3. If \( \Delta \in \{ B-C-F-G \} \), then \( \lceil \theta/2 \rceil = \lceil \theta/2 \rceil \in \Pi \) and \( \{ \lceil \theta/2 \rceil, \lceil \theta/2 \rceil \} \) is just the
unique semi-glorious pair associated with the adjacent simple roots of different length.

- In Section 5, we prove that all minimal non-abelian ideals of $\mathfrak{b}$ are related to either the glorious or semi-glorious pairs. All these ideals have the same cardinality, $h^*$.

We refer to [1], [5, §3.1] for standard results on root systems and Weyl groups. The numbering of simple roots for the exceptional Lie algebras follows [5, Tables].

1. Preliminaries

Our main objects are $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, the vector space $V = \bigoplus_{i=1}^n \mathbb{R} \alpha_i$, the Weyl group $W$ generated by simple reflections $s_\alpha$ ($\alpha \in \Pi$), and a $W$-invariant inner product $(\cdot, \cdot)$ on $V$. Set $\rho = \frac{1}{2} \sum_{\nu \in \Delta^+} \nu$. The partial ordering $'\leq' in \Delta^+$ is defined by the rule that $\mu \leq \nu$ if $\nu - \mu$ is a non-negative integral linear combination of simple roots. If $\mu = \sum_{i=1}^n c_i \alpha_i \in \Delta^+$, then $\text{ht}(\mu) := \sum_{i=1}^n c_i$ and $\{\mu : \alpha_i\} := c_i$.

1.1. The theory of abelian ideals of $\mathfrak{b}$ is built on their relationship, due to D. Peterson, with the minuscule elements of the affine Weyl group $\widehat{W}$, see Kostant’s account in [6]; another approach is presented in [2]. Recall the necessary setup. Letting $\widehat{V} = V \oplus \mathbb{R} \delta \oplus \mathbb{R} \lambda$, we extend the inner product $(\cdot, \cdot)$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_0 = \delta - \theta$, where $\theta$ is the highest root in $\Delta^+$. Then

\[ \widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\} \] is the set of affine (real) roots;
\[ \widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\} \] is the set of positive affine roots;
\[ \widehat{\Pi} = \Pi \cup \{\alpha_0\} \] is the corresponding set of affine simple roots;
\[ \mu^\vee = 2\mu/(\mu, \mu) \] is the coroot corresponding to $\mu \in \widehat{\Delta}$;

For each $\alpha \in \widehat{\Pi}$, let $s_\alpha$ denote the corresponding reflection in $GL(\widehat{V})$. That is, $s_\alpha(x) = x - (x, \alpha)\alpha^\vee$ for any $x \in \widehat{V}$. The affine Weyl group, $\widehat{W}$, is the subgroup of $GL(\widehat{V})$ generated by the reflections $s_\alpha$ ($\alpha \in \widehat{\Pi}$). The extended inner product $(\cdot, \cdot)$ on $\widehat{V}$ is $\widehat{W}$-invariant. The inversion set of $w \in \widehat{W}$ is $N(w) = \{\nu \in \widehat{\Delta}^+ \mid w(\nu) \in -\widehat{\Delta}^+\}$.

1.2. Following D. Peterson, we say that $\dot{w} \in \widehat{W}$ is minuscule, if $N(\dot{w}) = \{\delta - \gamma \mid \gamma \in I_{\dot{w}}\}$ for some $I_{\dot{w}} \subset \Delta$, see [6, 2.2]. One then proves that (i) $I_{\dot{w}} \subset \Delta^+$ and it is an abelian ideal, (ii) the assignment $\dot{w} \mapsto I_{\dot{w}}$ yields a bijection between the minuscule elements of $\widehat{W}$ and the abelian ideals, see [6], [2, Prop. 2.8]. Accordingly, if $I \in \mathfrak{Ab}$, then $\dot{w}_I$ denotes the corresponding minuscule element of $\widehat{W}$. Obviously, $#I = #N(\dot{w}_I) = \ell(\dot{w}_I)$, where $\ell$ is the usual length function on $\widehat{W}$.

**Proposition 1.1** (cf. [8, Thm. 2.2 & Thm. 2.4]). Let $J \in \mathfrak{Ab}$.

1) For $\gamma \in J$, we have $\gamma \in \text{min}(J)$ if and only if $\dot{w}_J(\delta - \gamma) \in -\widehat{\Pi}$;
2) if $\eta \notin J$, then $J \cup \{\eta\} \in \mathfrak{Ab}$ if and only if $\dot{w}_J(\delta - \gamma) =: \beta \in \widehat{\Pi}$. If this is the case, then $\dot{w}_{J \cup \{\gamma\}} = s_\beta \dot{w}_J$. 


1.3. Using the above assertion, one defines a natural map \( \text{cl} : \Delta^+_{\text{com}} \to \hat{\Pi} \) as follows. Given \( \gamma \in \Delta^+_{\text{com}} \), take any \( I \in \mathfrak{Ab} \) such that \( \gamma \in \min(I) \). (For instance, \( I = I(\gamma \gamma) \) will do.) Then \( \hat{w}_I(-\delta + \gamma) = \beta \in \hat{\Pi} \) and we set \( \text{cl}(\gamma) = \beta \). The point is that this \( \beta \) does not depend on the choice of \( I \), see \([9, \text{Theorem 4.5}]\). We say that \( \text{cl}(\gamma) \) is the class of \( \gamma \). It can be shown that \( \text{cl}(\gamma) \in \Pi \), whenever \( \gamma \in \mathcal{H} \setminus \{\theta\} \).

1.4. For any nontrivial minuscule \( \hat{w}_I \), we have \( \hat{w}_I(\alpha_0) + \delta = \hat{w}_I(2\delta - \theta) \in \Delta^+_i \) \([8, \text{Prop. 2.5}]\). This yields the mapping \( \tau : \mathfrak{Ab}^o \to \Delta^+_i \), which is onto \([8, \text{Theorem 2.6(1)}]\). That is, \( \text{rt}(I) = \hat{w}_I(2\delta - \theta) \). The Heisenberg ideal

\[
\mathcal{H} := \{ \gamma \in \Delta^+ \mid (\gamma, \theta) \neq 0 \} = \{ \gamma \in \Delta^+ \mid (\gamma, \theta) > 0 \} \in \mathfrak{Ab}
\]

plays a prominent role in the theory of abelian ideals and posets \( \mathfrak{Ab}_\mu = \tau^{-1}(\mu) \):

- \( I = \text{I}(\mu)_{\text{min}} \) for some \( \mu \in \Delta^+_i \) if and only if \( I \subset \mathcal{H} \) \([8, \text{Theorem 4.3}]\);
- \( \#I(\mu)_{\text{min}} = (\rho, \theta^\vee - \mu^\vee) + 1 \) \([8, \text{Theorem 4.2(4)}]\);
- For \( I \in \mathfrak{Ab}^o \), we have \( I \in \mathfrak{Ab}_\mu \) if and only if \( I \cap \mathcal{H} = I(\mu)_{\text{min}} \) \([10, \text{Prop. 3.2}]\);

If \( I = I(\mu)_{\text{min}} \), then \( \hat{w}_I \) has the following simple description. First, there is a unique element of minimal length in \( W \) that takes \( \theta \) to \( \mu \) \([8, \text{Theorem 4.1}]\). Writing \( w_\mu \) for this element, one has \( \ell(w_\mu) = (\rho, \theta^\vee - \mu^\vee) \) and \( \hat{w}_I = w_\mu s_{\alpha_0} \) \([8, \text{Theorem 4.2}]\).

1.5. Generalising Peterson’s idea, Cellini–Papi associate a canonical element of \( \hat{W} \) to any \( I \in \mathfrak{Ab} \), see \([2, \text{Sect. 2}]\). Set \( I^k := I^{k-1} + I \) for \( k \geq 2 \), where \( J + I = \{ \nu + \eta \in \Delta^+ \mid \nu \in J, \eta \in I \} \). If \( m \in \mathbb{N} \) is the smallest integer such that \( I^m = \emptyset \), then \( \hat{w}_I \) is the unique element of \( \hat{W} \) such that \( \mathcal{N}(\hat{w}_I) = \bigcup_{k=1}^{m-1} (k\delta - I^k) \). For the abelian ideals, \( I^2 = \emptyset \) and one recovers the definition of minuscule elements.

Some basic results on abelian ideals of \( \mathfrak{b} \) that explore other directions of investigations can be found in \([3, 4, 7, 12, 14]\).

2. SOME AUXILIARY RESULTS

We begin with some properties of arbitrary ideals in \((\Delta^+, \preceq)\). Recall that \( \#\mathcal{H} = 2h^* - 3 \), where \( h^* := (\rho, \theta^\vee) + 1 \) is the dual Coxeter number of \( \Delta \).

**Lemma 2.1.** If \( I \subset \Delta^+ \) is a non-abelian ideal, then there are \( \gamma_1, \gamma_2 \in I \) such that \( \gamma_1 + \gamma_2 = \theta \).

**Proof.** If \( \gamma_1 + \gamma_2 = \mu \) for some \( \gamma_1, \gamma_2 \in I \) and \( \mu \neq \theta \), then there is \( \alpha \in \Pi \) such that \( \mu + \alpha \) is a root. In this case, \( \gamma_1 + \alpha \) or \( \gamma_2 + \alpha \) is a root, necessarily in \( I \), which provides an induction step. \( \square \)

**Lemma 2.2.** If \( \hat{I} \in \mathfrak{Ab} \) is non-abelian, then \( \#\hat{I} \geq h^* \). Conversely, if \( I \subset \mathcal{H} \) is an ideal and \( \#I \geq h^* \), then \( I \) is not abelian.
Proof. 1) If \( \hat{\mathcal{I}} \) is non-abelian, then so is \( \hat{\mathcal{I}} \cap \mathcal{H} \) (Lemma 2.1). Therefore, we may assume that \( \hat{\mathcal{I}} \subset \mathcal{H} \) and \( \gamma_1 + \gamma_2 = \theta \) for some \( \gamma_1, \gamma_2 \in \hat{\mathcal{I}} \). Let \( \hat{w}_\mathcal{I} \in \hat{\mathcal{W}} \) be the canonical element associated with \( \hat{\mathcal{I}} \), see Section 1.5. Since \( \hat{\mathcal{I}}^2 = \{ \theta \} \) and \( \hat{\mathcal{I}}^3 = \emptyset \), we have

\[
\mathcal{N}(\hat{w}_\mathcal{I}) = \{ \delta - \gamma \mid \gamma \in \hat{\mathcal{I}} \} \cup \{ 2\delta - \theta \}.
\]

Since \( \hat{\Delta}^+ \setminus \mathcal{N}(\hat{w}_\mathcal{I}) \) is a closed under addition subset of \( \hat{\Delta}^+ \), \( \mathcal{H} \setminus \hat{\mathcal{I}} \) does not contain pairs of summable (to \( \theta \)) elements. Hence \( \mathcal{H} \setminus \hat{\mathcal{I}} \) is contained in a Lagrangian half of \( \mathcal{H} \setminus \{ \theta \} \) and is strictly less (because \( \hat{\mathcal{I}} \) contains a summable pair \( \gamma_1, \gamma_2 \)). Thus,

\[
\#(\mathcal{H} \setminus \hat{\mathcal{I}}) = \#\mathcal{H} - \#\hat{\mathcal{I}} < (\#\mathcal{H} - 1)/2.
\]

Hence \( \#\hat{\mathcal{I}} > (\#\mathcal{H} + 1)/2 = h^* - 1 \).

2) The set \( \mathcal{H} \setminus \{ \theta \} \) consists of \( h^* - 2 \) different pairs of roots that sums to \( \theta \). Therefore, if \( \#\mathcal{I} \geq h^* \), then \( \mathcal{I} \) contains at least one such summable pair. \( \square \)

**Remark.** There are abelian ideals \( \mathcal{I} \) with \( \#\mathcal{I} \geq h^* \). Such ideals are not contained in \( \mathcal{H} \).

**Lemma 2.3.** If \( \mathcal{I} \subset \mathcal{H} \) is an abelian ideal and \( \mathcal{I} \cup \{ \gamma \} \) is a non-abelian ideal for some \( \gamma \in \Delta^+ \), then (i) \( \gamma \in \mathcal{H} \), (ii) \( \mathcal{I} = \mathcal{I}(\alpha)_{\min} \) for some \( \alpha \in \Pi \), and (iii) \( \#\mathcal{I} = h^* - 1 \).

**Proof.** Since \( \mathcal{I} \) is abelian, whereas \( \mathcal{I} \cup \{ \gamma \} \) is not, there is \( \mu \in \mathcal{I} \) such that \( \gamma + \mu = \theta \) (use Lemma 2.1). Hence \( \gamma \in \mathcal{H} \). By [8, Prop. 4.2(4)], if \( \mathcal{I} \) is abelian, then \( \#\mathcal{I} \leq (\rho, \theta^\gamma) = h^* - 1 \). Furthermore, \( \#\mathcal{I} = h^* - 1 \) if and only if \( \mathcal{I} = \mathcal{I}(\alpha)_{\min} \) for some \( \alpha \in \Pi \). Now, since \( \mathcal{I} \cup \{ \gamma \} \) is non-abelian, we have \( \#(\mathcal{I} \cup \{ \gamma \}) \geq h^* \). Thus, \( \#\mathcal{I} = h^* - 1 \) and we are done.

If \( \eta + \eta' = \theta \), then we set \( \hat{\mathcal{I}} = I(\geq \eta, \eta') = \{ \nu \in \Delta^+ \mid \nu \succ \eta \) or \( \nu \succeq \eta' \} \). Then \( \hat{\mathcal{I}} \) is not abelian and \( \min(\hat{\mathcal{I}}) \subset \{ \eta, \eta' \} \). Clearly, both roots \( \eta, \eta' \) are either long or short.

**Lemma 2.4.** If \( \eta, \eta' \in \Delta_{\max}^+ \) and \( \eta + \eta' = \theta \), then

(i) these two roots are incomparable and hence \( \min(\hat{\mathcal{I}}) = \{ \eta, \eta' \} \);

(ii) the ideals \( \hat{\mathcal{I}} \setminus \{ \eta \} \) and \( \hat{\mathcal{I}} \setminus \{ \eta' \} \) are abelian;

(iii) \( \eta, \eta' \in \Delta_\mathcal{I} \).

**Proof.** (i) Obvious.

(ii) Assume that \( \hat{\mathcal{I}} \setminus \{ \eta \} \) is non-abelian. Then there are \( \gamma_1, \gamma_2 \in \hat{\mathcal{I}} \setminus \{ \eta \} \) such that \( \gamma_1 + \gamma_2 = \theta \). Since both \( \mathcal{I}(\geq \eta) \) and \( \mathcal{I}(\geq \eta') \) are abelian, up to permutation of indices, one has

\[
\gamma_1 \in \mathcal{I}(\geq \eta') \text{ and } \gamma_2 \in \mathcal{I}(\geq \eta) \setminus \{ \eta \},
\]

while \( \gamma_1 \notin \mathcal{I}(\geq \eta) \) and \( \gamma_2 \notin \mathcal{I}(\geq \eta') \).

This means that \( \text{ht}(\gamma_1) \geq \text{ht}(\eta') \) and \( \text{ht}(\gamma_2) > \text{ht}(\eta) \). Hence \( \text{ht}(\gamma_1 + \gamma_2) > \text{ht}(\theta) \). A contradiction! Therefore, \( \hat{\mathcal{I}} \setminus \{ \eta \} \in \mathfrak{A} \mathfrak{b} \) and likewise for \( \hat{\mathcal{I}} \setminus \{ \eta' \} \).

(iii) Now, \( \mathcal{J} = \hat{\mathcal{I}} \setminus \{ \eta, \eta' \} \) is an abelian ideal that has two abelian extensions: \( \mathcal{J} \mapsto \mathcal{J} \cup \{ \eta \} \), \( \mathcal{J} \mapsto \mathcal{J} \cup \{ \eta' \} \). Let \( \hat{w}_J \in \hat{\mathcal{W}} \) be the minuscule element corresponding to \( \mathcal{J} \). The first
extension means that \( \hat{w}_J(\delta - \eta) = \beta \) for some \( \beta \in \hat{\Pi} \). Likewise, \( \hat{w}_J(\delta - \eta') = \beta' \in \hat{\Pi} \), see Prop. 1.1. Since \( \eta + \eta' = \theta \), we have \( \hat{w}_J(2\delta - \theta) = \hat{w}_J(2\delta - \eta - \eta') = \beta + \beta' \). On the other hand, \( \hat{w}_J(2\delta - \theta) = \tau(J) \), and it is an element of \( \Delta^+_I \), see Section 1.4. Hence \( \beta, \beta' \) are adjacent simple roots of \( \Delta \). Finally, because \( \theta \) is long, then so is \( \beta + \beta' \). This is only possible if \( \beta, \beta' \) are long and hence \( \eta, \eta' \) are also long. \( \square \)

**Remark.** All the ideals \( I(\alpha)_{\min} (\alpha \in \Pi_I) \) have the same cardinality, \( h^* - 1 = (\rho, \theta') \); while this is not the case for \( I(\alpha)_{\max} \). A general formula for \( \#(I(\alpha)_{\max}) \) is given in [3].

### 3. Glorious pairs, adjacent long simple roots, and incident edges of the Dynkin diagram

**Definition 1.** A pair \( \{\eta, \eta'\} \subset \Delta^+ \) is said to be **glorious**, if these two roots are incomparable and \( \eta + \eta' = \theta \).

Set \( [\theta/2] = \sum_{\alpha \in \Pi} [\theta : \alpha]/2 \alpha \). If \( \Delta \) is of type \( A_n \), then \( [\theta/2] = 0 \). For all other types, one can prove uniformly that \( [\theta/2] \in \Delta^+ \), \( [\theta/2] \) is not commutative, and \( [\theta/2] \) is the unique maximal element of \( \Delta^+_\text{com} := \Delta^+ \setminus \Delta^+_{\text{com}} \), see [13, Section 3].

**Lemma 3.1.** Suppose that \( \eta + \eta' = \theta \). Then \( \eta, \eta' \) are incomparable \( \iff \) they are commutative.

**Proof.** The implication `\( \Leftarrow \)` is obvious (cf. Lemma 2.4(i)). To prove the other implication, assume that \( \eta \not\in \Delta^+_\text{com} \). Then there are \( \gamma_1, \gamma_2 \supset \eta \) such that \( \gamma_1 + \gamma_2 = \theta \). Therefore, \( \theta \not\geq \eta \). In other words, \( \eta \not\leq [\theta/2] \). This clearly implies that \( \eta' \not\geq [\theta/2] \) and hence \( \eta' \not\geq [\theta/2] \supset \eta \). \( \square \)

Note that both elements of a glorious pair belong to \( \mathcal{H} \) and they are long in view of Lemma 2.4(iii).

#### 3.1. From glorious pairs to adjacent simple roots.

**Proposition 3.2.** For a glorious pair \( \{\eta, \eta'\} \) and \( \hat{I} = I(\gg \eta, \eta') \) as in Section 2, we have

(i) \( \#\hat{I} = h^* \) and \( \hat{I} = I(\alpha)_{\min} \cup \{\eta\} = I(\alpha')_{\min} \cup \{\eta\} \) for some \( \alpha, \alpha' \in \Pi_I \);

(ii) the long simple roots \( \alpha, \alpha' \) are adjacent in the Dynkin diagram;

(iii) \( \text{cl}(\eta) = \alpha' \) and \( \text{cl}(\eta') = \alpha \);

(iv) \( \eta \in \text{min}(I(\alpha)_{\min}) \) and \( \eta' \in \text{min}(I(\alpha')_{\min}) \).

**Proof.** (i) This follows from Lemmata 2.3 and 2.4(ii).

(ii) Using \( J := \hat{I} \setminus \{\eta, \eta'\} \), the minuscule element \( \hat{w}_J \in \hat{\mathcal{W}} \), and two abelian extensions of \( J \), as in the proof of Lemma 2.4(iii), we obtain

\[
\hat{w}_J(\delta - \eta) = \beta \in \Pi_I , \quad \hat{w}_J(\delta - \eta') = \beta' \in \Pi_I , \quad \text{and} \quad \text{rt}(J) = \beta + \beta'.
\]
This also means that \( \hat{w}_{J \cup \{ \eta \}} = s_\beta \hat{w}_J \) and \( \hat{w}_{J \cup \{ \eta' \}} = s_\beta' \hat{w}_J \), see Prop. 1.1. Then

\[
\begin{align*}
\alpha &= \text{rt}(I(\alpha)_{\min}) = \text{rt}(J \cup \{ \eta \}) = s_\beta \hat{w}_J(2\delta - \theta) = s_\beta (\beta + \beta') = \beta', \\
\alpha' &= \text{rt}(I(\alpha')_{\min}) = \text{rt}(J \cup \{ \eta' \}) = s_\beta' \hat{w}_J(2\delta - \theta) = s_\beta (\beta + \beta') = \beta.
\end{align*}
\]

(iii) By the previous part of the proof, we have \( \hat{w}_{J \cup \{ \eta \}} = s_\alpha \hat{w}_J \) and \( \hat{w}_{J \cup \{ \eta' \}} = s_\alpha \hat{w}_J \), and by definition of the class we have

\[
\begin{align*}
\text{cl}(\eta) &= -s_\alpha \hat{w}_J(\delta - \eta) = -s_\alpha (\alpha) = \alpha', \\
\text{cl}(\eta') &= -s_\alpha \hat{w}_J(\delta - \eta') = -s_\alpha (\alpha) = \alpha.
\end{align*}
\]

(iv) This follows from (i). \( \square \)

**Remark.** Part (ii) of the proposition already appears in [8, Prop. 3.4(1)]. However, we essentially reproduced a proof in order to have the necessary notation for Part (iii).

It follows from Proposition 3.2 that one associates an edge of the Dynkin (sub-)diagram of \( \Pi_l \) to any glorious pair. Below, we prove that this map is actually a bijection and obtain some more properties for the quadruple \( (\eta, \eta', \alpha, \alpha') \) of Proposition 3.2. To achieve this goal, one should be able to construct a glorious pair starting from a pair of adjacent long simple roots.

### 3.2. From adjacent simple roots to glorious pairs

Let \( \alpha \in \Pi_l \). By [8, Prop. 4.6], there is a natural bijection between \( \min(I(\alpha)_{\min}) \) and the simple roots adjacent to \( \alpha \), and this correspondence respects the length. More precisely, let \( w_\alpha \in W \) be the unique shortest element taking \( \theta \) to \( \alpha \). If \( \alpha' \in \Pi \) is adjacent to \( \alpha \), then \( \eta := w_\alpha^{-1}(\alpha + \alpha') \in \min(I(\alpha)_{\min}) \) corresponds to \( \alpha' \). In particular, \( \eta \in \Delta^\text{com}. \) (Here \( \alpha' \) is not necessarily long and \( \|\alpha'\| = \|\eta\| \).

Suppose now that \( \alpha' \) is long. Then so is \( \eta \). Furthermore, \( (\theta, \eta) = (\alpha, \alpha + \alpha') > 0 \), hence \( \eta' := \theta - \eta \) is a long root, too.

**Proposition 3.3.** If \( \alpha, \alpha' \in \Pi_l \) are adjacent and \( \eta = w_\alpha^{-1}(\alpha + \alpha') \), then \( \eta' = w_\alpha^{-1}(\alpha + \alpha') \) has the property that \( \eta + \eta' = \theta \). Furthermore, \( \text{cl}(\eta) = \alpha' \) and \( \text{cl}(\eta') = \alpha \).

**Proof.** 1) Recall that \( \ell(w_{\alpha + \alpha'}) = (\rho, \theta' - (\alpha + \alpha')') = h^* - 3 \), see Section 1.4. Then \( \ell(s_\alpha w_{\alpha + \alpha'}) \leq h^* - 2 = (\rho, \theta' - (\alpha')') \) and \( s_\alpha w_{\alpha + \alpha'} \) takes \( \theta \) to \( \alpha' \). Hence one must have \( w_{\alpha'} = s_\alpha w_{\alpha + \alpha'} \) and also \( w_{\alpha} = s_\alpha w_{\alpha + \alpha'} \). (The first equality means that a “shortest reflection route” inside \( \Delta^+_l \) from \( \theta \) to \( \alpha' \) can be realised as a shortest route from \( \theta \) to \( \alpha + \alpha' \) and then the last step \( \alpha + \alpha' \mapsto \alpha' \).) Now \( w_{\alpha}^{-1}(\alpha + \alpha') = w_{\alpha + \alpha'}^{-1}(\alpha') \). Hence \( \begin{align*}
w_{\alpha}^{-1}(\alpha + \alpha') + w_{\alpha}^{-1}(\alpha + \alpha') &= w_{\alpha + \alpha'}^{-1}(\alpha + \alpha') = \theta.
\end{align*} \)

2) To compute the class, we need the fact that the minuscule element of \( I(\alpha)_{\min} \) in \( \tilde{W} \) is \( w_{\alpha} s_{\alpha_{\alpha}} \). (This element already appeared in Section 3.1 as \( s_{\alpha'} \hat{w}_J \), which is the same, because \( \hat{w}_J = w_{\alpha + \alpha'} s_{\alpha_{\alpha}} \) and \( s_{\alpha'} w_{\alpha + \alpha'} = w_{\alpha} \).) Therefore, by definition,

\[
\text{cl}(\eta) = w_{\alpha} s_{\alpha_{\alpha}}(\delta - \eta) = -w_{\alpha}(\theta - \eta) = -\alpha + (\alpha + \alpha') = \alpha'.
\]
And likewise for \( \eta' \).

Thus, both \( \eta \) and \( \eta' \) are long, commutative, and sum to \( \theta \), i.e., we have obtained a glorious pair associated with \((\alpha, \alpha')\). The above result shows that the construction of \( \eta \) and \( \eta' \) is symmetric w.r.t. \( \alpha \) and \( \alpha' \), and the classes of \( \eta \) and \( \eta' \) are computable.

**Remark 3.4.** Proposition 3.3 provides an explicit formula for \( \eta, \eta' \) via adjacent \( \alpha, \alpha' \in \Pi_i \).

There is also another approach. By [8, Theorem 2.1], for two adjacent long simple roots, \( I(\alpha)_{\min} \cap I(\alpha')_{\min} = I(\alpha + \alpha')_{\min} \). Furthermore, \( \#I(\alpha + \alpha')_{\min} = (\rho, \theta^\vee - (\alpha + \alpha')^\vee) + 1 = h^* - 2 \), see Section 1.4. Therefore, \( \#(I(\alpha)_{\min} \setminus I(\alpha + \alpha')_{\min}) = 1 \) and likewise for \( \alpha' \). Hence \( I(\alpha)_{\min} \setminus I(\alpha + \alpha')_{\min} = \{\eta\} \) and \( I(\alpha')_{\min} \setminus I(\alpha + \alpha')_{\min} = \{\eta'\} \). Since \( \#(I(\alpha)_{\min} \cup I(\alpha')_{\min}) = h^* \) and \( I(\alpha)_{\min} \cup I(\alpha')_{\min} \subset \mathcal{H} \), this ideal cannot be abelian, i.e., \( \{\eta, \eta'\} \) is a glorious pair.

### 3.3. Bijections and examples

It is easily seen that the constructions of Sections 3.1 and 3.2 are mutually inverse. This yields one of the main results of the article.

**Theorem 3.5.** There is a natural bijection

\[
\{\text{the glorious pairs in } \Delta^+\} \leftrightarrow \{\text{the pairs of adjacent roots in } \Pi_i\},
\]

\[
(\eta, \eta') \leftrightarrow (\alpha, \alpha')
\]

such that \( \eta \in \min(I(\alpha)_{\min}), \eta' \in \min(I(\alpha')_{\min}), \text{cl}(\eta) = \alpha', \text{ and cl}(\eta') = \alpha. \)

**Corollary 3.6.** The number of glorious pairs equals \( \#\Pi_i - 1 \).

This provides an *a priori* proof for Theorem 4.8(ii) in [11], where glorious pairs appear disguised as “summable 2-antichains” in \( \Delta(1) := \mathcal{H} \setminus \{\theta\} \).

**Corollary 3.7.** Every ordered glorious pair \( (\eta, \eta') \) arises as an element of the set \( \min(I(\alpha)_{\min}) \times \max(\Delta^+ \setminus I(\alpha)_{\max}) \) for a unique \( \alpha \in \Pi_i \).

**Proof.** Given a glorious pair \( (\eta, \eta') \in \min(I(\alpha)_{\min}) \times \min(I(\alpha')_{\min}) \), one can apply [10, Theorem 4.7] in two different ways. For, one can say that either \( \eta' = \theta - \eta \in \max(\Delta^+ \setminus I(\alpha)_{\max}) \) or \( \eta = \theta - \eta' \in \max(\Delta^+ \setminus I(\alpha')_{\max}) \). Therefore,

\[
(\eta, \eta') \in \min(I(\alpha)_{\min}) \times \max(\Delta^+ \setminus I(\alpha)_{\max}),
\]

\[
(\eta, \eta') \in \min(I(\alpha')_{\min}) \times \max(\Delta^+ \setminus I(\alpha')_{\max}). \quad \square
\]

**Example 3.8.** Let \( \Delta \) be of type \( \text{D}_{n} \), with \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) (\( 1 \leq i \leq n - 1 \)) and \( \alpha_n = \varepsilon_{n-1} + \varepsilon_n \).

Then \( \theta = \varepsilon_1 + \varepsilon_2 \) and the quadruples \( (\alpha, \alpha', \eta, \eta') \) are given in the table.

| \( \alpha \) | \( \alpha' \) | \( \eta \) | \( \eta' \) |
|----------------|----------------|----------------|----------------|
| \( \alpha_i \) | \( \alpha_{i+1} \) | \( \varepsilon_1 - \varepsilon_i + \varepsilon_{i+2} \) | \( \varepsilon_2 + \varepsilon_{i+2} \) (\( 1 \leq i \leq n - 3 \)) |
| \( \alpha_{n-2} \) | \( \alpha_{n-1} \) | \( \varepsilon_1 + \varepsilon_n \) | \( \varepsilon_2 - \varepsilon_n \) |
| \( \alpha_{n-2} \) | \( \alpha_n \) | \( \varepsilon_1 - \varepsilon_n \) | \( \varepsilon_2 + \varepsilon_n \) |
Using these data, one easily determines $\min(I(\alpha)_{\min})$ for all $\alpha \in \Pi$. Given $\alpha_i \in \Pi$, we look at all its occurrences in the first two columns an pick the related roots $\eta$, if $\alpha_i = \alpha$, and $\eta'$, if $\alpha_i = \alpha'$. This yields $\min(I(\alpha_i)_{\min})$. For instance, $\min(I(\alpha_{n-2})_{\min}) = \{\varepsilon_2 + \varepsilon_{n-1}, \varepsilon_1 + \varepsilon_n, \varepsilon_1 - \varepsilon_n\}$.

Example 3.9. Let $\Delta$ be of type $B_n$, with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq n)$, where $\varepsilon_{n+1} = 0$. Then $\Pi_l = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $\theta = \varepsilon_1 + \varepsilon_2$, and the quadruples $(\alpha, \alpha', \eta, \eta')$ are

$\begin{array}{cccc}
\alpha & \alpha' & \eta & \eta' \\
\alpha_i & \alpha_{i+1} & \varepsilon_1 - \varepsilon_{i+2} & \varepsilon_2 + \varepsilon_{i+2} & (1 \leq i \leq n-2) \\
\end{array}$

Example 3.10. In the following table, we illustrate the bijection of Theorem 3.5 and Corollary 3.7 with data for $\Delta$ of type $E_6$. The second column contains 10 different roots, which form 5 glorious pairs. For each root there, one has to pick the unique root such that their sum equals $\theta = \left(\begin{smallmatrix} 12321 \\ 2 \end{smallmatrix}\right)$. The same 10 roots occur in the third column, and reading the table along the rows, we meet all 10 ordered glorious pairs.

| $\alpha$ | $\alpha'$ | $\eta$ | $\eta'$ |
|---------|---------|-------|-------|
| $\alpha_1$ | 11100 | 01221 |
| $\alpha_2$ | 11110, 01221 | 01211, 11100 |
| $\alpha_3$ | 01211, 11111, 11210 | 11110, 01210, 01111 |
| $\alpha_4$ | 01111, 12210 | 11210, 00111 |
| $\alpha_5$ | 00111 | 12210 |
| $\alpha_6$ | 01210 | 11111 |

Remark. There are no glorious pairs for $C_n$ and $G_2$, since $\#\Pi_l = 1$ in these cases.

3.4. Adjacent glorious pairs. Two glorious pairs are said to be adjacent if the corresponding edges are incident in the Dynkin diagram, i.e., have a common node (in $\Pi_l$). Here we prove that there is a simple transform connecting two adjacent glorious pairs. First, we need some notation.

Let $\alpha_i, \alpha_j, \alpha_k$ be consecutive long simple roots in the Dynkin diagram, see figure

$\begin{array}{c}
\ldots \quad \alpha_i \quad \alpha_j \quad \alpha_k \quad \ldots \\
\end{array}$

It is not forbidden here that one of the nodes is the branching node (if there is any).

To handle this situation, we slightly modify our notation. The glorious pair associated with $(\alpha_i, \alpha_j)$ is denoted by $(\eta_{ij}, \eta_{ji})$ and likewise for $(\alpha_j, \alpha_k)$. Here $\eta_{ij} \in \min(I(\alpha_i)_{\min})$, $\text{cl}(\eta_{ij}) = \alpha_j$, etc. Set $w_i = w_{\alpha_i}$, the shortest element of $W$ taking $\theta$ to $\alpha_i$, and likewise for
the other admissible sets of indices; e.g. \( w_{jk} \) (resp. \( w_{ijk} \)) is the shortest element taking \( \theta \) to \( \alpha_j + \alpha_k \) (resp. \( \alpha_i + \alpha_j + \alpha_k \)). Then \( \eta_{ij} = w_i^{-1}(\alpha_i + \alpha_j) \) and \( \eta_{ji} = w_j^{-1}(\alpha_i + \alpha_j) \), cf. Section 3.2.

**Theorem 3.11.** If \( \alpha_i, \alpha_j, \alpha_k \) are consecutive long simple roots, then

\[
(\eta_{jk}, \eta_{kj}) = (\eta_{ij} - \gamma, \eta_{ji} + \gamma),
\]

where \( \gamma = w_{ijk}^{-1}(\alpha_j) \in \Pi_l \) and \( (\gamma, \theta) = 0 \).

**Proof.** It is easily seen that \( w_i = s_j s_k w_{ijk} \), \( w_j = (s_is_k) w_{ijk} \), \( w_k = s_j s_i w_{ijk} \), cf. the proof of Proposition 3.3(1). Then

\[
\eta_{ij} = w_{ijk}^{-1}(\alpha_i) \quad \text{and} \quad \eta_{ji} = w_{ijk}^{-1}(\alpha_j + \alpha_k).
\]

Thus

\[
\eta_{jk} - \eta_{ij} = w_{ijk}^{-1}(\alpha_j) = \eta_{ji} - \eta_{kj}.
\]

Since \( (\alpha_i + \alpha_j + \alpha_k, \alpha_j) = 0 \), it follows from [12, Lemma 2.1] that \( w_{ijk}^{-1}(\alpha_j) \in \Pi \) and \( (w_{ijk}^{-1}(\alpha_j), \theta) = 0 \). \( \square \)

We will say that this \( \gamma \in \Pi \) is the transition root for two incident edges \( (\alpha_i, \alpha_j) \) and \( (\alpha_j, \alpha_k) \). If there is a longer chain of long simple roots, say \( \cdots - \alpha_i - \alpha_j - \alpha_k - \alpha_l - \cdots \), then one obtains two associated transition roots \( \gamma_j = w_{ijk}^{-1}(\alpha_j) \) and \( \gamma_k = w_{jkl}^{-1}(\alpha_k) \). A useful complement to the previous theorem is

**Proposition 3.12.** The simple roots \( \gamma_j, \gamma_k \) are adjacent in the Dynkin diagram, i.e., \( (\gamma_j, \gamma_k) < 0 \), and \( \eta_{kl} - \eta_{ij} = \gamma_j + \gamma_k = \eta_{ji} - \eta_{lk} \in \Delta^+ \).

**Proof.** Using the obvious notation, one has \( w_{ij} = s_i w_{ijkl} \) and \( w_{jkl} = s_i w_{ijkl} \). Therefore

\[
\gamma_j = w_{ijk}^{-1}(\alpha_j) = w_{ijkl}^{-1}(\alpha_j) \quad \text{and} \quad \gamma_k = w_{jkl}^{-1}(\alpha_k) = w_{ijkl}^{-1}(\alpha_k).
\]

Hence \( (\gamma_j, \gamma_k) = (\alpha_j, \alpha_k) < 0 \). To prove the second relation, use Eq. (3.2) and its analogue for \( (jkl) \). \( \square \)

**Remark 3.13.** For a chain of \( m \) consecutive long simple roots, say \( \alpha_1, \ldots, \alpha_m \), one similarly obtains \( m - 2 \) transition roots \( \gamma_j, j = 2, \ldots, m - 1 \). Here \( \gamma_2 = w_{123}^{-1}(\alpha_2) \), etc. It follows from Proposition 3.12 that \( \gamma_i \) and \( \gamma_{i+1} \) are adjacent in the Dynkin diagram. Moreover, arguing by induction on \( m \geq 4 \), one easily proves that all \( \gamma_i \)'s are different, i.e., they form a genuine chain in \( \Pi \). This provides a curious bijection in the ADE-case. Namely, Theorem 3.11 asserts that each pair of incident edges in the Dynkin diagram defines a \( \gamma \in \Pi_l \) such that \( (\gamma, \theta) = 0 \). For \( D_n \) and \( E_n \), there is \( n - 1 \) pair of incident edges in the Dynkin diagram and also \( n - 1 \) simple roots orthogonal to \( \theta \). For \( A_n \), the same phenomenon occurs with \( n - 2 \) in place of \( n - 1 \). Thus, one obtains natural bijections related to the Dynkin diagram:

\[
\left\{ \text{triples of consecutive simple roots} \right\} \leftrightarrow \{ \text{pairs of incident edges} \} \leftrightarrow \{ \gamma \in \Pi \mid (\gamma, \theta) = 0 \}.
\]
That is, every simple root orthogonal to $\theta$ occurs as the transition root for a unique pair of incident edges. In A.3, we provide this correspondence for $E_6$, $E_7$, $E_8$, and $D_8$.

This fails, however, in the non-simply laced case.

4. GLORIOUS PAIRS RELATED TO THE INTERVAL $[[\theta/2], [\theta/2]]$, TAILS, AND ELEMENTS OF $W$

Recall that $[\theta/2] = \sum_{\alpha \in \Pi} [\{\theta : \alpha\}/2] \alpha$ and if $\Delta \neq A_n$, then $[\theta/2] \in \Delta^+$. Outside type $A$, $\theta$ is a multiple of a fundamental weight and there is a unique $\alpha_\theta \in \Pi$ such that $(\theta, \alpha_\theta) \neq 0$. Here $\{\theta : \alpha_\theta\} = 2$, hence $[\theta/2] : \alpha_\theta = 1$ and $[\theta/2] \in \mathcal{H}$. Then $[\theta/2] := \theta - [\theta/2]$ also belongs to $\mathcal{H}$ and $[\theta/2] \leq \theta / 2$. Set $\mathcal{J} = \{\gamma \in \Delta^+ : |\theta/2| < \gamma < [\theta/2]\}$. By [13, Sect. 5], if $\Delta \in \{D-E\}$, then $\mathcal{J} \simeq B^3$ (boolean cube, see Figure 2); and if $\Delta \in \{B-C-F-G\}$, then $\mathcal{J} = \{[\theta/2], [\theta/2]\}$.

Write $|M|$ for the sum of elements of a subset $M \subset \Pi$. Recall that $|M| \in \Delta^+$ if and only if $M$ is connected. Our results below exploit the following fact for $\Delta$ that is not of type $A$ (see [13, Prop. 4.6]):

\[(4.1) \quad \text{there is a unique } \hat{\alpha} \in \Pi \text{ such that } |\Pi_i| + \hat{\alpha} \in \Delta^+ \text{ and then } w_{|\Pi_i|}^{-1}(\hat{\alpha}) = -[\theta/2].\]

Set $\Omega = \{\alpha \in \Pi \mid \{\theta : \alpha\} \text{ is odd}\}$. The roots in $\Omega$ are also said to be odd. It follows from the definition of $[\theta/2]$ that

\[(4.2) \quad \theta - 2[\theta/2] = [\theta/2] - [\theta/2] = |\Omega|.\]

4.1. The glorious pairs related to $\mathcal{J}$ in the $\{D-E\}$-case. In types $D$ and $E$, there are exactly three odd roots. They are denoted by $\beta_1, \beta_2, \beta_3$. In this case, $\hat{\alpha} \in \Pi$ of Eq. (4.1) is the branching node in the Dynkin diagram and $\Pi \setminus \{\hat{\alpha}\}$ has three connected components, which are called tails. Each tail $\tilde{T}_i$ is a chain, and it is also regarded as a subset of $\Pi$. Then $|\mathcal{T}_i| \in \Delta^+$, $\Pi \setminus \{\hat{\alpha}\} = \sqcup_{i=1}^3 \mathcal{T}_i$, and

\[(4.3) \quad |\Pi| = \sum_{i=1}^3 |\mathcal{T}_i| + \hat{\alpha}.\]

Set $\tilde{T}_i = \mathcal{T}_i \cup \{\hat{\alpha}\}$. Then $|\tilde{T}_i| \in \Delta^+$ and $\Pi \setminus \tilde{T}_i$ is a disconnected subset of the Dynkin diagram. Therefore, $|\Pi| - |\tilde{T}_i| \notin \Delta^+$ and hence $(|\Pi|, |\tilde{T}_i|) \geq 0$. Since $N(w_{W}^{-1}) = \{\gamma \in \Delta^+ : (\gamma, \mu^{-1}) = -1\}$ for any $\mu \in \Delta^+$ [8, Theorem 4.1], this means that $|\tilde{T}_i| \not\in N(w_{|\Pi|}^{-1})$, i.e., $w_{|\Pi|}^{-1}(|\tilde{T}_i|) \in \Delta^+$. Moreover, the following is true.

**Theorem 4.1.** After a suitable renumbering of the $\beta_i$’s, one has

$w_{|\Pi|}^{-1}(|\tilde{T}_i|) = \beta_i$ for $i = 1, 2, 3$. 
Proof. It follows from Eq. (4.3) that \(|\Pi| = \sum_{i=1}^{3} |\hat{T}_i| - 2\hat{\alpha}|. By (4.1), we also have \(w_{|\Pi|}^{-1}(\hat{\alpha}) = -[\theta/2]\). Therefore,

\[
\theta = w_{|\Pi|}^{-1}(|\Pi|) = \sum_{i=1}^{3} w_{|\Pi|}^{-1}(|\hat{T}_i|) + 2[\theta/2].
\]

Using Eq. (4.2), we obtain \(\beta_1 + \beta_2 + \beta_3 = \sum_{i=1}^{3} w_{|\Pi|}^{-1}(|\hat{T}_i|)\), and each summand in the right-hand side is a positive root. Hence the assertion! \(\square\)

**Corollary 4.2.** \(w_{|\Pi|}^{-1}(|\tau_i|) = [\theta/2] + \beta_i\).

It is assumed below that numbering of the tails and odd roots is matched as prescribed by Theorem 4.1. The explicit correspondence \(T_i \leftrightarrow \beta_i\) is pointed out in A.2.

**Theorem 4.3.** Let \(\nu_i \in T_i\) be the unique root adjacent to \(\hat{\alpha}\). Then \(w_{\hat{\alpha}}^{-1}(\nu_i) = -[\theta/2] - \beta_i\).

Proof. For each tail \(T_i\), we define the element \(w(T_i) \in W\) as the ordered product of simple reflections \(s_{\alpha_i}\), where the \(\alpha_i\)'s are taken along the chain \(T_i\) and \(s_{\nu_i}\) is the initial factor on the left, see figure:

```
\[\begin{array}{c}
\ldots \quad \nu_1 \quad \hat{\alpha} \quad \nu_2 \quad \ldots \\
T_1: \nu_3
\end{array}\]
```

**Figure 1. Tails**

Note that \(w(T_i)\) and \(w(T_j)\) commute, \(w(T_i)^{-1}(\nu_i) = -|\tau_i|\), and \(w(T_j)^{-1}(\nu_i) = \nu_i\) if \(i \neq j\). We can write \(w_{\hat{\alpha}} = w(T_1) w(T_2) w(T_3) w_{|\Pi|}\). Indeed, \(w(T_i)|\Pi| = |\Pi| - |T_i|\) and hence the element in the RHS takes \(\theta\) to \(|\Pi| - \sum_{i=1}^{3} |T_i| = \hat{\alpha}\). Moreover, its length is at most \((\rho, \theta\nu) - 1 = (\rho, \theta\nu - \hat{\alpha}\nu)\). Hence it must be the unique element of minimal length taking \(\theta\) to \(\hat{\alpha}\). Then

\[
w_{\hat{\alpha}}^{-1}(\nu_i) = w_{|\Pi|}^{-1} \prod_{j=1}^{3} w(T_j)^{-1}(\nu_i) = w_{|\Pi|}w(T_i)^{-1}(\nu_i) = w_{|\Pi|}^{-1}(-|\tau_i|) = -[\theta/2] - \beta_i. \quad \square
\]

We assume below that \(\{i, j, k\} = \{1, 2, 3\}\).

**Corollary 4.4.** For any \(i \in \{1, 2, 3\}\),

(i) \(w_{\hat{\alpha}}^{-1}(\hat{\alpha} + \nu_i) = [\theta/2] - \beta_i = [\theta/2] + \beta_j + \beta_k\);
(ii) \(w_{\nu_i}^{-1}(\hat{\alpha} + \nu_i) = [\theta/2] + \beta_i\).

Proof. (i) Use the equality \(w_{\hat{\alpha}}^{-1}(\hat{\alpha}) = \theta\).

(ii) This follows from (i) and Proposition 3.3 with \(\alpha = \hat{\alpha}\) and \(\alpha' = \nu_i\). \(\square\)
It follows that the six roots strictly between \( |\theta/2| \) and \( [\theta/2] \) form the three glorious pairs corresponding to the edges of the Dynkin diagram through \( \hat{\alpha} \). Namely, the pair \( \{[\theta/2] - \beta_i, [\theta/2] + \beta_i\} \) corresponds to the edge \( \{\hat{\alpha}, \nu_i\} \).

![Diagram](https://via.placeholder.com/150)

**Figure 2.** The interval between \( [\theta/2] \) and \( [\theta/2] \) for \( D_n \) and \( E_n \)

By [10, Appendix], the join ‘\( \lor \)’ of two elements of the poset \( (\Delta^+, \succeq) \) always exists. In [13, Sect. 2], we proved that the meet ‘\( \land \)’ of two elements of \( (\Delta^+, \succeq) \) exists if and only if their supports are not disjoint. More precisely, if \( \text{supp}(\eta) := \{\alpha \in \Pi \mid \{\eta : \alpha\} \neq 0\} \) and \( \text{supp}(\eta) \cap \text{supp}(\eta') \neq \emptyset \), then

\[
\eta \land \eta' = \sum_{\alpha \in \Pi} \min\{\{\eta : \alpha\}, \{\eta' : \alpha\}\}\alpha.
\]

In this case, \( \eta \lor \eta' = \sum_{\alpha \in \Pi} \max\{\{\eta : \alpha\}, \{\eta' : \alpha\}\}\alpha \). Hence \( \eta + \eta' = \eta \land \eta' + \eta \lor \eta' \). Therefore, for any glorious pair, we have \( \eta \land \eta' + \eta \lor \eta' = \theta \).

**Corollary 4.5.** If \( \{\eta, \eta'\} \) is the glorious pair corresponding to \( \{\hat{\alpha}, \nu_i\} \), then \( \eta \land \eta' = [\theta/2] \) and \( \eta \lor \eta' = [\theta/2] \).

**Proof.** \( ([\theta/2] + \beta_i) \land ([\theta/2] + \beta_j + \beta_k) = [\theta/2] \) and \( ([\theta/2] - \beta_j - \beta_k) \lor ([\theta/2] - \beta_i) = [\theta/2] \). □

**Remark 4.6.** For any glorious pair \( \{\eta, \eta'\} \), the meet \( \eta \land \eta' \) exists and belongs to \( \mathcal{H} \) whenever \( \Delta \) is not of type \( A_n \), see [13, Remark 2.6]. Since \( \eta, \eta' \in I(\succeq(\eta \land \eta')) \), we have \( \eta \land \eta' \in \Delta^+_{nc} \) and hence \( \eta \land \eta' \prec [\theta/2] \). It is just proved that this upper bound is attained for the edges through the branching node \( \hat{\alpha} \in \Pi \). Using Theorem 3.11, Proposition 3.12, and Remark 3.13, one can prove that, for all other edges, the strict inequality \( \eta \land \eta' \prec [\theta/2] \) holds. More precisely, let us say that the edges through \( \hat{\alpha} \) are central. For any other edge \( \mathcal{E} \) of the subdiagram \( \Pi_l \), we consider the natural distance, \( d(\mathcal{E}) \), from \( \mathcal{E} \) to the closest central
are commutative, hence their classes are well-defined.

3.3 associates a $\alpha$ and Corollary 2.

\[ \gamma \]

Dynkin diagram, see \[ \text{Remark 4.7} \]

\[ \text{ht} \]

We have Proposition 4.8.

adjacent simple roots of different length, and one can construct a certain pair of roots as in the simply laced case. But, in the non-simply laced cases, there is exactly one pair of glorious pair to any pair of adjacent long simple roots. This leaves nothing more to say 4.2.

The interval $\gamma$ in Figure 2 are commutative, hence their classes are well-defined.

\[ \text{Proposition 4.8.} \] We have $\text{cl}([\theta/2] + \beta_i) = \hat{\alpha}$, $\text{cl}([\theta/2] + \beta_j + \beta_k) = \nu_i$, and $\text{cl}([\theta/2]) = \hat{\alpha}$.

Proof. Since the roots between $\gamma/2$ and $\gamma/2$ comprise three glorious pairs, the first two equalities follow from Proposition 3.3 and Corollary 4.4.

The root $\gamma/2$ is adjacent to the roots $\gamma/2 - \beta_i (i = 1, 2, 3)$ in the Hasse diagram of $\Delta^+$, see Figure 2. Therefore, $\text{cl}([\theta/2])$ is adjacent to the $\text{cl}([\theta/2] - \beta_i) = \nu_i (i = 1, 2, 3)$ in the Dynkin diagram, see [9, Lemma 4.6]. Hence the only possibility for $\text{cl}([\theta/2])$ is $\hat{\alpha}$. \[ \square \]

Remark. The roots of class $\hat{\alpha}$ are coloured in Figure 2.

4.2. The interval $\gamma$ in the $\{\text{B-C-F-G}\}$-case. The construction in Section 3.2 associates a glorious pair to any pair of adjacent long simple roots. This leaves nothing more to say in the simply laced case. But, in the non-simply laced cases, there is exactly one pair of adjacent simple roots of different length, and one can construct a certain pair of roots as follows.

Let $\alpha \in \Pi_I$, $\alpha' \in \Pi_s := \Pi \setminus \Pi_I$, and $(\alpha, \alpha') < 0$. Then $\alpha' = \hat{\alpha}$ and $\eta = w_{\alpha}^{-1}(\alpha + \alpha') \in \text{min}(I(\alpha)_{\text{min}}$ is a short root in $\mathcal{H}$. Hence $\eta \in \Delta^+_{\text{com}}$ and $(\eta, \theta) = (\alpha, \alpha + \alpha') > 0$. But $\eta' = \theta - \eta \notin \Delta^+_{\text{com}}$ in view of Lemma 2.4(iii). For this reason, we say that the pair $(\eta, \eta')$ is the (unique) semi-glorious pair.

\[ \text{Theorem 4.9.} \] Suppose that $\Delta$ is non-simply laced and $(\eta, \eta')$ is the semi-glorious pair associated with the adjacent $\alpha \in \Pi_I$ and $\alpha' \in \Pi_s$, as above. Then

(i) $I(\eta') \setminus \{\eta'\} = I(\alpha)_{\text{min}}$ and $\eta - \eta' \in \Pi$
(ii) \( \text{cl}(\eta) = \alpha' \);
(iii) \( (\eta, \eta') = ([\theta/2], [\theta/2]) \).

Proof. (i) Since \( I(\geq \eta') \) is not abelian, there are \( \mu, \mu' \geq \eta \) such that \( \mu + \mu' = \theta \) (Lemma 2.1). Because \( \eta \in \min(I(\alpha)_{\text{min}}) \), it follows from [10, Theorem 4.7] that \( \eta' = \theta - \eta \in \max(\Delta^+ \setminus I(\alpha)_{\text{max}}) \). Hence \( I(\geq \eta') \setminus \{\eta'\} \) lies in \( I(\alpha)_{\text{max}} \) and thereby is abelian. Therefore, the only possibility is \( \mu = \eta, \mu' = \eta' \). In this case, we have \( \eta' < \eta \).

Furthermore, since \( I(\geq \eta') \setminus \{\eta'\} \) is abelian, while \( I(\geq \eta') \) is not, Lemma 2.3 implies that the former ideal is \( I(\beta)_{\text{min}} \) for some \( \beta \in \Pi \). By the construction, \( I(\geq \eta') \setminus \{\eta'\} \subseteq I(\alpha)_{\text{max}} \). Now, using the theory developed in [10, Sect. 3], we obtain

\[
I(\alpha)_{\text{min}} = (I(\alpha)_{\text{max}} \cap \mathcal{I}) \supset \left(I(\geq \eta') \setminus \{\eta'\}\right) = I(\beta)_{\text{min}},
\]

which implies that \( \beta = \alpha \). Thus, \( I(\geq \eta') \setminus \{\eta'\} = I(\alpha)_{\text{min}} \) contains \( \eta \) as a minimal element. Since all minimal elements of \( I(\geq \eta') \setminus \{\eta'\} \) cover \( \eta' \), we have \( \eta - \eta' \in \Pi \).

(ii) The computation of Eq. (3.1) applies here as well.

(iii) Set \( \gamma = |\Pi| \), the sum of all roots in \( \Pi \). Then \( \gamma \in \Delta^+ \) and also \( \gamma + \alpha' \in \Delta^+ \). That is, \( \alpha' = \alpha \) occurs in Eq. (4.1). Therefore, \( w_\alpha^{-1}(\alpha') = -[\theta/2] \). Let \( \mathcal{T} \) be the connected component of \( \Pi \setminus \{\alpha\} \) that contains long roots. Then \( \mathcal{T} \) is a chain, and if \( \mathcal{T} \) consists of consecutive (long) simple roots \( \alpha_{i_1}, \ldots, \alpha_{i_n} \), where \( \alpha_{i_s} \) is adjacent to \( \alpha \), then we consider the element \( w = s_{i_s} \cdots s_{i_1} w_\gamma \in W \). Since \( w_\gamma(\theta) = \gamma = \alpha_{i_1} + \cdots + \alpha_{i_s} + \alpha \), we have \( w(\theta) = \alpha \).

Because \( \ell(w) \leq \ell(w_\gamma) + |\Pi| - 1 = (\rho, \theta') - 1 \), we see that \( \ell(w) = (\rho, \theta') - 1 \) and \( w = w_\alpha \). Since \( s_{i_j}(\alpha') = \alpha' \) for each \( i_j \), we have \( w_\alpha^{-1}(\alpha') = w_\gamma^{-1}(\alpha') = -[\theta/2] \). Therefore, \( \eta = w_\alpha^{-1}(\alpha + \alpha') = \theta - [\theta/2] = [\theta/2] \).

The complete information on the semi-glorious pairs is presented in the following table:

| \( B_n \) | \( C_n \) | \( F_4 \) | \( G_2 \) |
|---|---|---|---|
| \( \alpha_{n-1} \) | \( \alpha_n \) | \( \alpha_3 \) | \( \alpha_2 \) |
| \( \varepsilon_1 \) | \( \varepsilon_2 \) | (1211) | (21) |
| \( \alpha_{n-1} \) | \( \alpha_n \) | \( \alpha_3 \) | \( \alpha_2 \) |
| \( \varepsilon_1 \) | \( \varepsilon_2 \) | (1211) | (21) |
| \( \alpha_{n-1} \) | \( \alpha_n \) | \( \alpha_3 \) | \( \alpha_2 \) |
| \( \varepsilon_1 + \varepsilon_n \) | \( \varepsilon_2 - \varepsilon_n \) | (211) | (11) |

The numbering of simple roots follows [5, Tables], and the notation, say, (1221) stands for \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \), and so on.

Remark 4.10. Our result that \( [\theta/2] - [\theta/2] \in \Pi \) yields a proof for the fact that \( \#\emptyset = 1 \) in the BCFG-case, cf. Eq. (4.2).

There is an analogue of Theorem 3.11 that involves semi-glorious pairs. Let \( \alpha_i, \alpha_j, \alpha_k \) be consecutive simple roots such that \( \alpha_i, \alpha_j \in \Pi_l \) and \( \alpha_k \in \Pi_s \), cf. Section 3.4. (This only happens for \( B_n \) and \( F_4 \).) Then we still have the shortest elements \( w_i \) and \( w_j \), but
not $w_k$. Recall that $\eta_{ij} = w_i^{-1}(\alpha_i + \alpha_j)$, $\eta_{ji} = w_j^{-1}(\alpha_i + \alpha_j)$, and also $\eta_{jk} = w_j^{-1}(\alpha_j + \alpha_k)$ is the commutative root in the semi-glorious pair. It follows from Theorem 4.9(iii) that $\eta_{jk} = \lceil \theta/2 \rceil$, but this is not needed now.

**Theorem 4.11.** $\eta_{jk} - \eta_{ij} =: \gamma$ is a short simple root and $(\gamma, \theta) = 0$.

**Proof.** In this situation, $\|\alpha_j\|^2/\|\alpha_k\|^2 = 2$ and therefore $\alpha_i + \alpha_j + 2\alpha_k \in \Delta^+_T$. Write $w_{ijk}$ for the shortest element of $W$ taking $\theta$ to $\alpha_i + \alpha_j + 2\alpha_k$. Then $w_i = s_j s_k w_{ijk}$ and $w_j = (s_i s_k) w_{ijk}$. Therefore

$$
\eta_{jk} = w_{ijk}^{-1}(\alpha_j + \alpha_k) = w_{ijk}^{-1}(\alpha_i + \alpha_j + \alpha_k), \\
\eta_{ij} = w_{ijk}^{-1}s_k s_j(\alpha_i + \alpha_j) = w_{ijk}^{-1}(\alpha_i).
$$

Hence $\eta_{jk} - \eta_{ij} = w_{ijk}^{-1}(\alpha_j + \alpha_k) = \gamma$ is a short root. Here $(\gamma, \theta) = (\alpha_i + \alpha_j + 2\alpha_k, \alpha_j + \alpha_k) = 0$, but we cannot blindly apply [12, Lemma 2.1], as in Theorem 3.11, because $\alpha_j + \alpha_k$ is not simple. Nevertheless, that proof can be adapted to this situation, see below. \hfill \Box

**Lemma 4.12.** Under the above notation, $\gamma := w_{ijk}^{-1}(\alpha_j + \alpha_k)$ is a simple root.

**Proof.** Set $\mu = \alpha_i + \alpha_j + 2\alpha_k$. Since $\mathcal{N}(w_{\mu}^{-1}) = \{ \nu \in \Delta^+ \mid \langle \nu, \mu^\vee \rangle = -1 \}$ [8, Theorem 4.1(2)] and $(\mu, \alpha_j + \alpha_k) = 0$, we have $\gamma \in \Delta^+$. Assume that $w_{\mu}^{-1}(\alpha_j + \alpha_k) = \gamma_1 + \gamma_2$ is a sum of positive roots. Then $\alpha_j + \alpha_k = w_{\mu}(\gamma_1) + w_{\mu}(\gamma_2)$.

If both summands in the RHS are positive, then $\alpha_j = w_{\mu}(\gamma_1)$, which is impossible, because $(\alpha_j, \mu^\vee) = -1$ and hence $w_{\mu}^{-1}(\alpha_j)$ is negative. Therefore, without loss of generality, one may assume that $-\nu_1 := w_{\mu}(\gamma_1)$ is negative. Then $\nu_1 \in \mathcal{N}(w_{\mu}^{-1})$, hence $(-\nu_1, \mu^\vee) = 1$. Consequently, $(\gamma_1, \theta^\vee) = 1$. On the other hand, $0 = (\mu, \alpha_j + \alpha_k) = (\theta, \gamma_1 + \gamma_2)$ and therefore $(\theta, \gamma_2) < 0$, which is impossible. Thus, $w_{\mu}^{-1}(\alpha_j + \alpha_k)$ must be simple. \hfill \Box

**Remark 4.13.** If $(\eta, \eta')$ is the unique semi-glorious pair, then $\eta \vee \eta' = \eta = \lceil \theta/2 \rceil$; whereas for all genuine glorious pairs, one has $\eta \vee \eta' \succ \lceil \theta/2 \rceil$. Again, if a glorious pair $(\eta, \eta')$ corresponds to the edge $E$ and $d(E)$ is the distance from $E$ to the edge of the semi-glorious pair, then $\eta \vee \eta' = \lceil \theta/2 \rceil + \sum_{i=1}^d \gamma_i$ and $\text{ht}(\eta \vee \eta') = \text{ht}(\lceil \theta/2 \rceil) + d(E)$, cf. Remark 4.6.

5. THE MINIMAL NON-ABELIAN IDEALS OF $\mathfrak{h}$

As an application of previous theory, we characterise the minimal non-abelian ideals in $\mathfrak{A} \mathfrak{d}$ and point out the associated canonical elements in $\widetilde{W}$ (cf. Section 1.5). By Lemma 2.2, if $\mathfrak{J} \in \mathfrak{A} \mathfrak{d} \setminus \mathfrak{A} \mathfrak{b}$ and $\# \mathfrak{J} = h^*$, then $\mathfrak{J}$ is minimal non-abelian. The precise assertion is:

**Theorem 5.1.** An ideal $\mathfrak{J} \in \mathfrak{A} \mathfrak{d}$ is minimal non-abelian if and only if either $\text{min}(\mathfrak{J})$ is a glorious pair or $\Delta$ is non-simply laced and $\mathfrak{J} = I(\lceil \theta/2 \rceil)$. For all these cases, $\mathfrak{J} \subset \mathcal{H}$, $\# \mathfrak{J} = h^*$, and the unique summable (to $\theta$) pair of roots in $\mathfrak{J}$ is either a glorious or semi-glorious pair.
GLORIOUS PAIRS AND ABELIAN IDEALS

Proof. If $\mathcal{J}$ is non-abelian, then there are $\eta, \eta' \in \mathcal{J}$ such that $\eta + \eta' = \theta$. Hence $I(\geqslant \eta, \eta') \subset \mathcal{J}$, and there must be equality if $\mathcal{J}$ is minimal. There are two possibilities now.

(a) If $\eta, \eta'$ are incomparable, then $\{\eta, \eta'\}$ is a glorious pair and also $\#(\mathcal{J}) = h^*$, see Proposition 3.2(i). Here $\mathcal{J} = I(\alpha)_{\min} \cup \{\eta'\} = I(\alpha')_{\min} \cup \{\eta\}$.

(b) Suppose that $\eta' \preccurlyeq \eta$. Then $\mathcal{J} = I(\geqslant \eta')$ and also $\eta' \in \Delta_{nc}^+$. (This already excludes the $A_n$-case, where all 1-generated ideals are abelian.) Then $\eta' \preccurlyeq [\theta/2]$ and the minimality of $\mathcal{J}$ implies that $\eta' = [\theta/2]$. However, in the DE-case the non-abelian ideal $I(\geqslant [\theta/2])$ is not minimal, since it properly contains three smaller non-abelian ideals generated by the glorious pairs related to the central edges of the Dynkin diagram, see Corollary 4.4 and Figure 2. In the BCFG-case, we do obtain a minimal non-abelian ideal of cardinality $h^*$, as follows from Theorem 4.9(i). For, then $I(\geqslant \eta') \setminus \{\eta'\} = I(\alpha)_{\min}$ and $\#I(\alpha)_{\min} = h^* - 1$. □

A nice uniform formulation is that the minimal non-abelian ideals are in a one-to-one correspondence with the edges of the Dynkin diagram that are incident to the long simple roots.

Remark 5.2. For a minimal non-abelian ideal $\mathcal{J}$, the canonical element $\hat{w}_\mathcal{J} \in \widehat{W}$ (see 1.5) can explicitly be described. Note that in both cases below, $\ell(\hat{w}_\mathcal{J}) = \#(\mathcal{J}) + \#(\mathcal{J}^2) = h^* + 1$.

(a) Suppose that $\mathcal{J} = I(\geqslant \eta, \eta')$, where $\eta, \eta'$ is the glorious pair corresponding to adjacent $\alpha, \alpha' \in \Pi_t$. Then

$$\hat{w}_\mathcal{J} = s_{\alpha'} s_{\alpha} w_{\alpha} s_0 = s_{\alpha} s_{\alpha'} w_{\alpha} s_0.$$  

For, $w_{\alpha} s_0$ is the minuscule element for $I(\alpha)_{\min}$ (Section 1.4). Hence $N(w_{\alpha} s_0) = \{\delta - \gamma \mid \gamma \in I(\alpha)_{\min}\}$. Then $N(s_{\alpha} w_{\alpha} s_0) = N(w_{\alpha} s_0) \cup \{(w_{\alpha} s_0)^{-1}(\alpha)\} = N(w_{\alpha} s_0) \cup \{2\delta - \theta\}$, and

$$N(s_{\alpha'} s_{\alpha} w_{\alpha} s_0) = N(s_{\alpha} w_{\alpha} s_0) \cup \{\delta - \eta'\} = \{\delta - \gamma \mid \gamma \in \mathcal{J}\} \cup \{2\delta - \theta\}.$$  

For the second equality, use the relations $s_{\alpha} w_{\alpha+\alpha'} = w_{\alpha'}, s_{\alpha'} w_{\alpha+\alpha'} = w_{\alpha}$, and $(s_{\alpha} s_{\alpha'})^3 = 1$.

(b) If $\mathcal{J} = I(\geqslant [\theta/2])$, then $\mathcal{J}$ can be thought of as the ideal related to the semi-glorious pair $([\theta/2], [\theta/2])$. If $\alpha \in \Pi_t$ and $\alpha' \in \Pi_s$ are the related adjacent roots, then similarly

$$\hat{w}_\mathcal{J} = s_{\alpha'} s_{\alpha} w_{\alpha} s_0.$$  

Note that here $w_{\alpha'}$ does not exist and $(s_{\alpha} s_{\alpha'})^3 \neq 1$.

APPENDIX A. DATA FOR THE EXCEPTIONAL LIE ALGEBRAS

A.1. For $E_6$, $E_7$, and $E_8$, the numbering of the simple roots is

$$1-2-3-4-5, \quad 1-2-3-4-5-6, \quad \text{and} \quad 1-2-3-4-5-6-7,$$  

respectively.

The glorious pairs $(\eta, \eta')$ associated with adjacent simple roots $(\alpha, \alpha')$ are given below. Recall that here $\eta \in \min(I(\alpha)_{\min})$ and $\cl(\eta) = \alpha'$.
A.2. For $D_n$ and $E_n$, the explicit correspondence between tails and odd roots is presented below. The numbering of tails follows Figure 1. That is, according to the chosen numbering of $\Pi$, see Example 3.8 and figures in A.1, $T_1$ contains $\alpha_1$, $T_2$ contains $\alpha_{n-1}$, and $T_3$ contains $\alpha_n$. In particular, we always have $\# T_3 = 1$. Then the odd root $\beta_i$ corresponding to $T_i$ is:

| $\alpha$ | $\alpha'$ | $\eta$ | $\eta'$ |
|----------|----------|--------|--------|
| $\alpha_1$ | $\alpha_2$ | 111111 | 012321 |
| $\alpha_2$ | $\alpha_3$ | 111111 | 012321 |
| $\alpha_3$ | $\alpha_4$ | 112211 | 012221 |
| $\alpha_4$ | $\alpha_5$ | 112211 | 012221 |
| $\alpha_5$ | $\alpha_6$ | 112211 | 001221 |

| $E_7$: | $E_8$: | $E_7$: |
|--------|--------|--------|
| $\alpha_1$ | $\alpha_2$ | 122210 | 1123432 |
| $\alpha_2$ | $\alpha_3$ | 122221 | 1123431 |
| $\alpha_3$ | $\alpha_4$ | 122221 | 1123421 |
| $\alpha_4$ | $\alpha_5$ | 122231 | 1123321 |
| $\alpha_5$ | $\alpha_6$ | 122332 | 1123321 |
| $\alpha_6$ | $\alpha_7$ | 123332 | 1112321 |
| $\beta_1$ | $\beta_2$ | $\beta_3$ | $\hat{\alpha}$ |
| $D_n$ | $\alpha_1$ | $\alpha_{n-1}$ | $\alpha_n$ | $\alpha_{n-2}$ |
| $E_6$ | $\alpha_2$ | $\alpha_5$ | $\alpha_3$ | $\alpha_3$ |
| $E_7$ | $\alpha_1$ | $\alpha_5$ | $\alpha_3$ | $\alpha_4$ |
| $E_7$ | $\alpha_2$ | $\alpha_8$ | $\alpha_4$ | $\alpha_5$ |

A.3. Here we provide the transition roots for the pairs of incident edges in types $E_6$, $E_7$, and $E_8$, and thereby explicitly present the bijection of Remark 3.13.


REFERENCES

[1] N. Bourbaki. “Groupes et algèbres de Lie”, Chapitres 4,5 et 6, Paris: Hermann 1975.
[2] P. Cellini and P. Papi. ad-nilpotent ideals of a Borel subalgebra, *J. Algebra*, 225 (2000), 130–141.
[3] P. Cellini and P. Papi. Abelian ideals of Borel subalgebras and affine Weyl groups, *Adv. Math.*, 187 (2004), 320–361.
[4] P. Cellini, P. Möseneder Frajria and P. Papi. Symmetries of the poset of abelian ideals in a Borel subalgebra, *J. Lie Theory* 24, no. 1 (2014), 199–224.
[5] V.V. Gorbatsevich, A.L. Onishchik and E.B. Vinberg. “Lie Groups and Lie Algebras” III (Encyclopaedia Math. Sci., vol. 41) Berlin Heidelberg New York: Springer 1994.
[6] B. Kostant. The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations, *Intern. Math. Res. Notices*, (1998), no. 5, 225–252.
[7] B. Kostant. Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra, *Invent. Math.*, 158, no. 1 (2004), 181–226.
[8] D. Panyushev. Abelian ideals of a Borel subalgebra and long positive roots, *Intern. Math. Res. Notices*, (2003), no. 35, 1889–1913.

[9] D. Panyushev. The poset of positive roots and its relatives, *J. Algebraic Combin.*, 23 (2006), 79–101.

[10] D. Panyushev. Abelian ideals of a Borel subalgebra and root systems, *J. Eur. Math. Soc.*, 16, no. 12 (2014), 2693–2708.

[11] D. Panyushev. Antichains in weight posets associated with gradings of simple Lie algebras, *Math. Zeitschrift*, 281, no. 3-4 (2015), 1191–1214.

[12] D. Panyushev. Normalisers of abelian ideals of Borel subalgebras and $\mathbb{Z}$-gradings of a simple Lie algebra, *J. Lie Theory*, 26 (2016), 659–672.

[13] D. Panyushev. Abelian ideals of a Borel subalgebra and root systems, II (arxiv:1805.02057v1 [math.RT], 12 pp.)

[14] R. Suter. Abelian ideals in a Borel subalgebra of a complex simple Lie algebra, *Invent. Math.*, 156 (2004), 175–221.

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