The Algebra of the Pseudo-Observables I: Why Quantum Mechanics is the ultimate description of Reality

Edoardo Piparo
Liceo Scientifico Statale “Archimede”, Viale Regina Margherita 3, I-98121 Messina, Italy
E-mail: edoardo.piparo@istruzione.it

Abstract. This paper is the first of several parts introducing a new powerful algebra: the algebra of the pseudo-observables. This is a C*-algebra whose set is formed by formal expressions involving observables. The algebra is constructed by applying the Occam’s razor principle, in order to obtain the minimal description of physical reality.

Proceeding in such a manner, every aspect of quantum mechanics acquires a clear physical interpretation or a logical explanation, providing, for instance, in a natural way the reason for the structure of complex algebra and the matrix structure of Werner Heisenberg’s formulation of quantum mechanics.

Last but not least, the very general hypotheses assumed, allow one to state that quantum mechanics is the unique minimal description of physical reality.

Keywords: Foundations of quantum mechanics, interpretation of quantum mechanics, quantum measurement problem

PACS numbers: 03.65.Ta

‡ A.I.F. Associazione per l’Insegnamento della Fisica - Gruppo Storia della Fisica: http://www.lfns.it/STORIA/index.php/it/chi-siamo
1. Introduction

1.1. A brief historical background

After a long and hard working, started from the formula proposed by Planck in 1900 for the black-body radiation and Einstein’s ideas on light quanta and those of Bohr on the constitution of atoms, quantum mechanics suddenly arises between 1925 and 1927, thanks to some brilliant scientists such as Heisenberg, Born, Jordan, Schrödinger, Dirac, Pauli. The new formalism has quickly found a resounding success in justifying and predicting the phenomena of the atomic world, but it also immediately proves itself much more abstract than that of classical physics, and therefore more difficult to be acquired.

A few months after their publication in 1926, the four Schrödinger’s works on quantization as an eigenvalue problem, collected in one volume together with others of the same author, formed the first manual of wave mechanics. It was, however, only a manual for a few specialists, soon followed by new ones, among which outstanding is the fundamental text of Dirac, written so definitive that the first edition of 1930 has remained virtually unchanged for ten of its twelve chapters until the fourth and final version of 1958 and reprinted for the seventh time in 1976. Numerous presentations of quantum mechanics, the most comprehensive and educationally oriented, have followed in the now long period of time that separates us from those glorious years, in which the intellectual liveliness and mathematics knowledge of some young researchers, together with the genius and the experience of more mature scientists have produced one of the greatest revolutions in the history of thought.

The reorientation of perspective introduced by quantum mechanics in the way of thinking the natural phenomena affects not only the attitude of people of science, but in the intimate touches the mentality of the whole mankind. Accustomed to nineteenth-century physics - the so-called classical physics based on the mechanics of Galileo and Newton - one could conceive of the entire universe in mechanistic evolution, according to a deterministic cause and effect chain of relations. Emblematic of this is the reductionist position of Laplace, that is unexpectedly denied by the conclusions reached by the physicists in building a theory able to account for the new experimental data accumulated at the end of the Nineteenth and in the first quarter of the Twentieth. Heisenberg discovery of limitations on observation, which prevent one to determine exactly the initial conditions from which depend the time evolution of classical physics, puts on evidence the participation of the observer itself to the construction of the phenomenon and it is a stimulus for philosophical insights which resized the human-nature relationship (the first part of this short historical survey is partly based on the preface of the Boffi’s book).

Despite the successes of quantum mechanics in explaining physical phenomena, the interpretation of quantum mechanics, however, was a much more controversial issue, that may be regarded also now as an opened one. An interpretation of quantum mechanics is a set of statements which attempts to explain how quantum mechanics informs our understanding of nature. Although quantum mechanics has held up to rigorous and thorough experimental testing, many of these experiments are open to different interpretations.

A particularly problematic aspect of the matter is that it seems to be more in the fields of interest of philosophers of physics than of those of physicists themselves. The latter however felt and feel the need for an interpretation of the mathematical formalism of quantum mechanics, specifying the physical meaning of the mathematical entities of the theory.

However the issue is so slippery that even Dirac, in the new edition of 1958 of his fundamental treatment, states: «the main object of physical science is not the provision of pictures, but is the
 formulation of laws to the discovery of new phenomena», that sounds like a capitulation declaration.

Nevertheless, several researchers fronted the issue, giving rise to a number of contending schools of thought, differing over whether quantum mechanics can be understood to be deterministic, which elements of quantum mechanics can be considered “real” and other matters.

The definition of quantum theorists’ terms, such as wavefunctions and matrix mechanics, progressed through many stages. For instance, Erwin Schrödinger originally viewed the electron’s wavefunction as its charge density smeared across the field, whereas Max Born reinterpreted it as the electron’s probability density distributed across the field.

Among the plethora of interpretations, I limit myself here to recall only a few of them.

The Copenhagen interpretation is the “standard” interpretation of quantum mechanics formulated by Niels Bohr and Werner Heisenberg while collaborating in Copenhagen around 1927. According to John G. Cramer, “Despite an extensive literature which refers to, discusses, and criticizes the Copenhagen interpretation of quantum mechanics, nowhere does there seem to be any concise statement which defines the full Copenhagen interpretation.”

The many-worlds interpretation is an interpretation of quantum mechanics in which the phenomena associated with measurement are claimed to be explained by decoherence, which occurs when states interact with the environment producing entanglement, repeatedly splitting the universe into mutually unobservable alternative histories-distinct universes within a greater multiverse. The original formulation is due to Hugh Everett in 1957. Later, this interpretation was popularized and renamed many-worlds by Bryce Seligman DeWitt in the 1960s and 1970s.

The consistent histories approach is intended to give a modern interpretation of quantum mechanics, generalizing the conventional Copenhagen interpretation. This interpretation of quantum mechanics is based on a consistency criterion that then allows probabilities to be assigned to various alternative histories of a system such that the probabilities for each history obey the rules of classical probability while being consistent with the Schrödinger equation. In contrast to some interpretations of quantum mechanics, particularly the Copenhagen interpretation, the framework does not include “wavefunction collapse” as a relevant description of any physical process, and emphasizes that measurement theory is not a fundamental ingredient of quantum mechanics.

The relational quantum mechanics is an interpretation of quantum mechanics which treats the state of a quantum system as being observer-dependent, that is, the state is the relation between the observer and the system. This interpretation was first delineated by Carlo Rovelli, and has since been expanded upon by a number of theorists. The essential idea behind it, inspired by special relativity, is that different observers may give different accounts of the same series of events: for example, to one observer at a given point in time, a system may be in a single, “collapsed” eigenstate, while to another observer at the same time, it may be in a superposition of two or more states. Consequently, if quantum mechanics is to be a complete theory, relational quantum mechanics argues that the notion of “state” describes not the observed system itself, but the relationship, or correlation, between the system and its observer(s). The state vector of conventional quantum mechanics becomes a description of the correlation of some degrees of freedom in the observer - intended here as a generic physical object, whether or not conscious or macroscopic - with respect to the observed system. Any “measurement event” is seen simply as an ordinary physical interaction, an establishment of the above sort of correlation. Thus the physical content of the theory has to do not with objects themselves, but the relations between them.

The objective collapse theories differ from the Copenhagen interpretation in regarding both the wavefunction and the process of collapse as ontologically objective. The most well-known examples of such theories are: the Ghirardi-Rimini-Weber theory - first reported in 1985 - in which,
as an attempt to avoid the measurement problem in quantum mechanics, is proposed that wave function collapse happens spontaneously; and the Penrose interpretation, in which it is proposed that a quantum state remains in superposition until the difference of space-time curvature attains a significant level[15].

1.2. The quest for a new formulation

The proliferation of interpretations proposed is a clear symptom of the fact that, in contrast to what stated by Dirac, the laws which rule the phenomena are not sufficient to provide a complete understanding of Nature. For this to happen, such laws must not contain any “gray area”, i.e. shall not to depend on entities or processes whose nature is not clear. But the traditional formulation of quantum mechanics actually depends on these. Wavefunctions, or, more generally speaking, vector-states, haven’t a clear and complete physical interpretation, and it’s not even clear if they are or aren’t ontologically real. The measurement process is account only for his final effect on vector states, without any reference to its time-development, and is not describable as a physical process. These issues are related to the choose of the fundamental entities and of the fundamental laws of quantum mechanics, i.e., ultimately, to its mathematical formulation.

The only possibility to resolve this is therefore to look for a new formulation of quantum mechanics.

The earliest versions of quantum mechanics were formulated in the first decade of the 20th century, as theories of matter and electromagnetic radiation, in order to front the problems arisen when the atomic theory and the corpuscular theory of light first came to be widely accepted as scientific facts.

Early quantum theory was significantly reformulated in 1925-1926 following a dual path. Werner Heisenberg[16], Max Born and Pascual Jordan[17, 18] gave rise to matrix mechanics; Louis de Broglie[19] and Erwin Schrödinger[20] created wave mechanics.

Although Schrödinger himself after a year proved the equivalence of his wave-mechanics and Heisenberg’s matrix mechanics, the reconciliation of the two approaches and their modern abstraction as motions in Hilbert space is generally attributed to Paul Dirac, who wrote a lucid account in his 1930 classic *The Principles of Quantum Mechanics*[6]. Dirac introduced the bra-ket notation, together with an abstract formulation in terms of the Hilbert space used in functional analysis; he showed that Schrödinger’s and Heisenberg’s approaches were two different representations of the same theory, and found a third, more general one, which represented the dynamics of the system.

The first complete mathematical formulation of this approach, known as the Dirac–von Neumann axioms, is generally credited to John von Neumann’s 1932 book *Mathematical Foundations of Quantum Mechanics*[21], although Hermann Weyl had already referred to Hilbert spaces (which he called unitary spaces) in his 1927 classic paper[22] and book[23].

A new formulation of quantum mechanics had to wait until 1948, when Richard Feynman, recovering an idea presented by Dirac in his 1933 paper[24], introduced his path integral formulation[25], in which a quantum-mechanical amplitude is considered as a sum over all possible classical and non-classical paths between the initial and final states. This is the quantum-mechanical counterpart of the action principle in classical mechanics and allows a quantum mechanical formulation based on Lagrangian rather than Hamiltonian, which is more desirable in certain theoretical contexts.

All of these formulation, however, rely on the same axiomatic context and share the “gray
areas” proper of the Copenhagen interpretation, that is their common framework. The issues arisen by the von Neumann’s postulate on the collapse of the wavefunction, however, stimulated the development, in the intervening 70 years, of new formulations of quantum mechanics in which the overcoming of the problem of measurement was central. Many-worlds interpretation, consistent histories approach, objective collapse theories and quantum logic arose in such a context. Quantum logic is a set of rules for reasoning about propositions which takes the principles of quantum theory into account. This research area and its name originated in the 1936 paper by Garrett Birkhoff and John von Neumann[26], who attempted to reconcile the apparent inconsistency of classical logic with the facts concerning the measurement of complementary variables in quantum mechanics, such as position and momentum.

On another side, some of the originators of quantum theory (notably Einstein and Schrödinger) were unhappy with what they thought were the philosophical implications of quantum mechanics. In particular, Einstein took the position that quantum mechanics must be incomplete, which motivated research into so-called hidden-variable theories. In his 1926 paper[3], Max Born, was the first to clearly enunciate the probabilistic interpretation of the quantum wavefunction, which had been introduced by Erwin Schrödinger earlier in the same year. Born’s interpretation of the wavefunction was criticized by Schrödinger, who had previously attempted to interpret it in real physical terms, but Albert Einstein’s response[27], contained in a letter sent to Max Born in 1926, became one of the earliest and most famous assertions that quantum mechanics is incomplete:

Quantum mechanics is certainly imposing. But an inner voice tells me that this is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the ‘old one’. I, in any rate, am convinced that

Shortly after making his famous comment, Einstein attempted to formulate a deterministic counterproposal to quantum mechanics. On 5 May 1927 he read a paper to the Prussian Academy of Sciences in Berlin entitled: “Bestimmt Schrödinger’s Wellenmechanik die Bewegung eines Systems vollständig oder nur im Sinne der Statistik?” (“Does Schrödinger’s wave mechanics determine the motion of a system completely or only in the statistical sense?”); but, just a few weeks later, as the paper was being prepared for publication in the academy’s journal, he decided to withdraw it, possibly because he discovered that implied non-separability of entangled systems could not be eliminated, as he had hoped[28].

At the Fifth Solvay Congress, held in Belgium in October 1927, Louis de Broglie presented his own version of a deterministic hidden-variable theory[29], apparently unaware of Einstein’s aborted attempt earlier in the year. In his theory, every particle had an associated, hidden “pilot wave” which served to guide its trajectory through space. de Broglie generalized the pilot-wave dynamics to many-body system, presenting also some elementary applications (interference, scattering). The theory was discussed extensively, with reactions, comments and critics of Pauli, Born, Brillouin, Einstein, Kramer, Lorentz, Schrödinger and others; but de Broglie defended it fairly well[30]. However he abandoned the theory shortly thereafter.

The general interest on this approach was lost when von Neumann, in his 1932 book[21], claimed to prove the impossibility of theories which, by using the so called hidden variables, attempt to give a deterministic explanation of quantum mechanical behaviors.

It’s worth recalling that in 1935 Grete Hermann criticized the von Neumann proof on a fundamental point[31]. Her work, however, remained substantially unnoticed by the physics community for several decades.

In the meantime, in 1952, David Bohm, dissatisfied with the prevailing orthodoxy, developed a theory[32, 33] that is essentially the same as de Broglie’s pilot wave theory, the only difference being
that de Broglie’s dynamics is formulated in terms of velocity rather than acceleration. However the
general interest in the theory was little, till when, in 1966. John Bell[34] rediscovered the results
of Grete Hermann’s work, i.e. that “von Neumann’s no-hidden-variables proof was based on an
assumption that can only be described as silly”[35]. The von Neumann argumentation, in facts,
demonstrates only that hidden-variables theories must be nonlocal.

Unlike Hermann’s one, Bell’s critique had a great foundational impact and Bell himself, during
the sixties, the seventies and the eighties, became the principal proponent of the now called de
Broglie-Bohm theory, to which his 1987 book[36] contains an yet unsurpassed introduction.

The theory, anyway, remains still controversial. In August 2011, Roger Colbeck and Renato
Renner proved[37] that, under the assumption that measurements can be chosen freely, no extension
of quantum theory, whether using hidden variables or otherwise, can give more information about
the outcomes of future measurements than quantum theory itself. In January 2013, however,
Giancarlo Ghirardi and Raffaele Romano described a model which, “under a different free choice
assumption [. . . ] violates [the statement by Colbeck and Renner] for almost all states of a bipartite
two-level system, in a possibly experimentally testable way”[38].

Moreover, if on one side, it seems to resolve the measurement problem[39], on the other it still
relies on the framework of the Dirac-von Neumann axioms and on the concept of wavefunction,
which is also regarded as ontologically real.

Another interesting line of research in the field of quantum mechanics formulations is that
of algebraic theories[40]. Algebraic quantum mechanics is an abstraction and generalization of the
Hilbert space formulation. In the development of the algebraic approach a major role was played by
von Neumann. His joint paper with Jordan and Wigner[41] was, in facts, one of the first attempts to
go beyond Hilbert space of state vectors. Subsequently, von Neumann developed the mathematical
theory of operator algebras in a series of outstanding papers[42–46]. Notwithstanding his efforts,
von Neumann’s attempts to apply this theory to quantum mechanics were unsuccessfully[47]. The
operator algebras that he introduced, now called von Neumann algebras, however, still play a central
role in the algebraic approach to quantum theory.

In 1943, Gelfand and Naimark[48] introduced the notion of $C^*$-algebras (the term “$C^*$-algebra”
was, however, introduced by Irving Segal in 1947[49]), a generalization of von Neumann ones, freeing
themselves from the need to do reference to operators on a Hilbert space. The framework of a $C^*$-
algebra appears as an ideal one to find a new formulation to quantum mechanics and is already
nowadays largely utilized to represent the actual one based on the Dirac-von Neumann axioms.

An original solution to the problem of the nature of wavefunctions is to try to reformulate
quantum mechanics in a way such that these entities are no more fundamentals ones, but only
mathematical instruments. A significant upgrade in this direction was made with what that its
creators called “QBism” - a particular form of Quantum Bayesianism, i.e. a formulation that uses
a Bayesian approach to the probabilities that appear in quantum theory[50]. According to QBism,
wavefunctions are solely a mathematical tool that an observer uses to assign his personal belief that
a quantum system will have a specific property. In this conception, a wavefunction isn’t ontologically
real but merely reflects an individual’s subjective mental state. The proponents of QBism embrace
the notion that until an experiment is performed, its outcome simply does not exist[51].

One of the criticisms on QBism is that it is unable to explain complex macroscopic phenomena
in terms of more primitive microscopic ones, in the way that conventional quantum mechanics does.
In order to obtain this, QBism needs again a reformulation of quantum mechanics, based on a new
set of axioms.
1.3. What it has been done in this paper

It would have to been clear, at this point, how it is important to find a new formulation of quantum mechanics, based on a new set of axioms and on a new interpretation. At this regards, I agree both with Rovelli[13] and Fuchs thesis[52]. The point is that quantum mechanics was formulated making reference to the notions of “observer” and “measurement” but without a real analysis about them, that were, as a matter of facts, taken as primitive. In order to overcome this weakness, it arose the idea to remove the “observer” from the theory as quickly as possible and, to do this, the general strategy was to reify or objectify all the mathematical symbols of the theory, disregarding that this would have implied an unclear physical interpretation of them.

In this paper I propose a really new formulation of quantum mechanics, that, in my opinion, is free of “gray areas” and fully physically transparent in its axioms.

The new formulation is carried out as an algebraic theory and turns to be expressed in terms of a C*-algebra.

The starting point was a deep analysis of fundamental concepts, as “physical reality”, “observers”, “physical quantities” and “measurements”. On the basis of this analysis and the assumptions therein implied, I constructed an algebraic structure for “observables”, that turned to be a C*-algebra, making use essentially only of the Occam’s razor principle, that in this context has to be meant as follows: in building the formulation of a physical theory it is necessary to avoid, as much as possible, the introduction of entities not reducible to measurable physical properties. The aim is to obtain the minimal description of physical reality; whereas the introduction of unnecessary unobservable entities leads to descriptions depending on a potentially enormous number of arbitrary parameters.

Proceeding in such a manner, every aspect of quantum mechanics acquires a clear physical interpretation or a logical explanation, providing, for instance, in a natural way the reason for the structure of complex algebra and the matrix structure of Werner Heisenberg’s formulation of quantum mechanics.

Last but not least, the very general hypotheses assumed, allow one to state that quantum mechanics is the unique minimal description of physical reality.

2. Spaces of Observables

2.1. Reality and observers

For a big part, the human doing is based on the assumption of the existence of an “objective reality”. Such an hypothesis may considered the most fundamental archetype of our minds, upon which depend also the fundamental concepts of space, meant as the “container” of reality elements, and of time, meant as the ordered of evolution of reality elements.

The objectivity of reality implies the total independence of the external world from the observer. This assumption of independence is, however, largely unfounded.

The knowledge of reality, indeed, is based on subjective experiences. The «objectivity» dimension rises only by knowledge mediated by language. A language, however, is based on a shared set of symbols species specific, subject, moreover, to variations as time goes on. Such a kind of «objectivity» is, therefore, conventional and strongly tied to the observers.

In the last analysis, we can convincingly state that «reality» is nothing more than a coherent and shared reconstruction of a complex of subjective experiences. The objectivity of reality, therefore,
doesn’t transcend the subjective level, but merely implies to take into account only a convenient part of it.

Ultimately, then, the so-called “physical reality” is not distinguishable from the mental representation that a group of coherent observers make of it. In such a context, the “laws of Physics” reflect rules and logical relations rather more proper of the observer than of the observed. Then, therefore, I think that one can only agree with the Bohr’s quotation: It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature and with what states Fuchs in:

With every quantum measurement set by an experimenter’s free will, the world is shaped just a little as it participates in a kind of moment of birth.

So, I am convinced that by looking at the Universe the Human Being can see the reflex of his most intimate nature. In fact, I believe in the truthiness of Weyl’s words:

The world exists only as met with by an ego, as one appearing to a consciousness; the consciousness in this function does not belong to the world, but stands out against the being as the sphere of vision, of meaning, of image, or however else one may call it.

2.2. Physical quantities and observables

The aim of Physics is the study of the evolution of a portion of the world (the physical system) that surrounds us, characterized by a certain number of measurable properties, whatever is the system state (on the concept of state I will return in section 2.3, called, as usual in quantum mechanics, observables. Observables the measurements of which results in measures belonging to the same set, and therefore comparable, will be of the same type.

It is easily verified that this is an equivalence relation. It is, then, reasonable to identify a physical quantity with an equivalence class formed by observables of the same type. A physical quantity is, therefore, a measurable property relative to a set of physical systems, like, for instance, the distance between two points, whereas the term observable will be reserved to the same property relative to a specific physical system.

By these definitions, it is possible to state that a physical quantity is characterized by the set of the possible outcomes of an its measurement. Since this set is the result of a series of measurement processes, it will contain a finite number of possible outcomes. This number can, eventually, tend to infinite, in the ideal limit of measurement instruments whose resolution capability tends to zero or whose range tends to infinite. These are, however, only mathematical limits, unattainable for every observer. As already pointed out, the problem is that in the Dirac formulation of quantum mechanics there is an implied platonic vision on the nature of the world, with the implicit assumption that it must exist a sort of “world of ideas”, in the form of an abstract mathematical model, disregarding the possibility of an effective measurability or interpretability of the entities therein introduced.

It is worth, therefore, to follow a different way, starting from an accurate physical analysis of physical quantities and observables, in order to try to build a fully physically interpretable conceptual picture.
2.3. Observables as stochastic variables

Imagine one having to measure a certain observable relative to a given system. Due to issues related to the operating limits of the measuring instrument, to fluctuations in the environmental conditions and in the system itself, a single measurement won’t be sufficiently reliable.

In ideal conditions, what one could do is to consider a set (ensemble) of faithful copies of the system under examination, all in the same conditions, i.e., in the same state, and to perform the measurement simultaneously in all the copies. Obviously the outcomes will be, generally, different and therefore, in the so outlined measurement process, the observable assumes the connotation of a stochastic variable.

In what follows, observables will be usually labeled with uppercase letters, for instance $X$, whereas their possible outcomes will be denoted with the corresponding lowercase letter, in our example $x$. The set of all possible outcomes of the measurement of a given observable will be called the observable spectrum.

Particularly important are the constant observables, that are observables for which there is only a single possible measurement outcome $c$. These observables will be labeled with the name or the value of their constant outcome, that will be briefly called constant's value. Among them, remarkable examples are the null observable, whose measurement outcome is always 0, and the unit observable, whose measurement outcome is always 1.

Two observables will be called compatible if it is possible to measure both simultaneously. Evidently, each observable is compatible with itself and a constant observable is compatible with every observable.

It is, finally, important to point out that an observable relative to a certain time instant $t$ isn’t generally compatible with the same observable evolved to a different time instant $t'$, since they cannot be measured simultaneously and independently.

2.4. Spaces of compatible observables

Given two compatible observables, $A$ and $B$, it is easy to define their sum or product: it suffices to consider an observable which has as possible outcomes of measuring, respectively, the sum and the product of the possible outcomes of measurement of the two observables. One can similarly define the difference of two observables, and their quotient, provided, however, for the latter, that the observable in the denominator doesn’t admit 0 as a possible outcome of measurement. The observables obtained with these operations are clearly compatible with starting ones.

According to the definition, the sum of two compatible observables satisfies the commutative property, the associative one, the existence of a neutral element (the null observable) and opposite element (the observable whose possible outcomes of measurement are the opposites of the one considered). For the product are valid the commutative property, the associative one and the existence of neutral element (the unit observable). The multiplication also distributes over addition.

Not all observables different from the null one, however, admit the inverse element (those do not admit it are non-constant observables that contain the zero in their spectrum).

The set of observables compatible with a given observable $A$, equipped with the operations of sum and product thus defined, has, therefore, the algebraic structure of a commutative unitary ring. This ring is not, however, an integral domain, since, in general, the zero-product property isn’t valid.

If we, then, identify constant observables with their values, it is easy to see that every set consisting of mutually compatible observables, provided with the operations of addition and
multiplication by a constant coefficient and closed with respect to these operations, has the structure of a vector space over the field $\mathbb{R}$ of the real numbers. This will be called a space of compatible observables.

It is also possible to introduce an equivalence criterion between observables: two observables are equivalent if their difference is the null observable. It is easily verified that the reflexive, symmetric and transitive properties hold for the so-defined criterion, as required for an equivalence relation. Equivalent observables are physically indistinguishable and will be, therefore, identified as the same observable.

It is, finally, possible to introduce a partial order relation between two compatible observables. We will say that between two compatible observables of the same type $A$ and $B$ it holds the relation $A \leq B$, if for every simultaneous measurement of the two observable, of outcomes respectively $a$ and $b$, one has always $a \leq b$.

2.5. Projectors

In the theory of observables that I am outlining, of a particular importance are the projectors (the name is given in analogy to that of the projection operators of the Dirac formulation, to which they correspond). A projector is an observable whose possible measurement outcomes are only 0 and 1. The null and the unit observables may be considered as extreme forms of projectors and are, obviously, the unique constant projectors. For a generic projector $I$, it always results:

$$0 \leq I \leq 1 .$$

It will now prove the following important theorem: necessary and sufficient condition in order to an observable $I$ be a projector is that it is idempotent ($I^2 = I$). In fact, if $I$ is a projector, its possible measurement outcomes are 0 and 1. If the outcome of a measurement of $I$ is 0, also the outcome for $I^2$ will be 0; whereas if the outcome for $I$ is 1, the same will be for $I^2$. So, for the equivalence criterion, it results:

$$I^2 = I .$$

Conversely, if the idempotency relation holds, one has:

$$I (1 - I) = 0 .$$

In a measurement of the observable in the left-hand side of the, due to the zero-product property, that holds for measures, the only possible outcomes for $I$ are 0 or 1. The observable $I$ is therefore a projector.

We can now demonstrate that if $I$ is a projector, $1 - I$ is also a projector. In fact, it results:

$$(1 - I)^2 = 1 - 2I + I^2 = 1 - 2I + I = 1 - I .$$

The projector $1 - I$ will be called complementary of $I$. It is worth noting that if $I$ is not constant, its complementary too isn’t, besides, by virtue of [3], the product of $I$ and its complementary is null even if no one of the factors is. This demonstrates that, in the ring of observables, in general the zero-product property is not valid.

Let’s now demonstrate that the product of two compatible projectors, $I_1$ and $I_2$, is also a projector. In fact it results:

$$(I_1I_2)^2 = (I_1I_2)(I_2I_1) = I_1^2I_2^2 = I_1I_2 .$$
Note that the result is true if and only if the commutative and the associative properties of the product both hold. It is immediate to verify that the product of two compatible projectors is always less or equal to both of them. Besides, the following property holds:

\[ I_1 \leq I_2 \Rightarrow I_1 I_2 = I_1 \]  
(5)

Finally, we will prove that if \( I_1 \leq I_2 \), the difference: \( I_2 - I_1 \) is also a projector. In fact, making use of (5), one has:

\[ (I_2 - I_1)^2 = I_2^2 - 2I_1 I_2 + I_1^2 = I_2 - 2I_1 + I_1 = I_2 - I_1 \]

Two projectors, \( I_1 \) and \( I_2 \), will be called mutually exclusive if their product is null:

\[ I_1 I_2 = 0 \]  
(6)

Therefore a projector and its complementary are always mutually exclusive. We can demonstrate that the sum of two mutually exclusive projectors is still a projector. In fact, it results:

\[ (I_1 + I_2)^2 = I_1^2 + I_2^2 + 2I_1 I_2 = I_1 + I_2 \]

It is easily verified that both \( I_1 \) and \( I_2 \) are less or equal to their sum.

I define an elementary projector a non-null projector that cannot be expressed as sum of non-null mutually exclusive projectors. Elementary projectors correspond in the Dirac formulation to pure state projection operator, i.e. operators of the form \( |\psi\rangle \langle \psi| \), where \( |\psi\rangle \) is vector state.

It will be proved that an elementary projector can be major or equal only of the null projector. If, in fact, an elementary projector \( I_2 \) were major or equal of a non-null projector, \( I_1 \), for what above demonstrated, the observable \( I_3 = I_2 - I_1 \) would be a non-null projector and, besides, by virtue of (6), it would be:

\[ I_1 I_3 = I_1 (I_2 - I_1) = I_1 I_2 - I_1^2 = I_1 - I_1 = 0 \]

Therefore \( I_2 \) would be equal to the sum of two non-null mutually exclusive projectors, that is a contradiction.

An important consequence of this property of the elementary projectors is that the product of an elementary projector for another compatible projector is or 0 or equal to the elementary projector itself (otherwise one would fall in the previous contradiction).

It is, now, possible to define a projector basis \( \{I_j\} \), as a set of non-null mutually exclusive elementary projectors that satisfy the closure relation:

\[ \sum_j I_j = 1 \]  
(7)

The projectors that form such a basis are linearly independent. A linear combination of them, in fact, using coefficients \( \alpha_j \), gives as result an observable whose possible outcomes of measurement are the coefficients themselves. This, therefore, will be equal to the null observable if and only if all the coefficients \( \alpha_j \) are equal to zero.
2.6. Projectors and events

In order to start to clarify the physical meaning of the projectors, it is helpful to introduce the concept of event associated projector. We will say that \( I_{E} \) is the projector associated to the event \( E \), if the outcome of measurement of \( I_{E} \) is equal to 1 if and only if the event \( E \) occurs. With this definition, one can state that the null observable is the projector associated to the impossible event, whereas the unit observable is the projector associated to the certain event.

The complementary projector is, moreover, associated with the event complementary of that associated to the starting projector and mutually exclusive projectors are associated to mutually exclusive events, i.e. such that the occurrence of one precludes the occurrence of the others. An important consequence of this last association is that, therefore, mutually exclusive projectors are compatibles among each other, since the verify on the events implies the simultaneous measuring of all of the projectors.

The sum of mutually exclusive projectors is then associated to the union of the events associated to each of them, the compound event which occurs if one of these events occurs.

The product of two compatible projectors gives as result the projector associated to the intersection of the events associated to the factors, namely the event that occurs when both events occur.

The elementary projectors, finally, are associated to elementary events, not reducible to the union of simpler events.

2.7. Observable decomposition

Now let’s consider an observable \( O \), having the spectrum \( \{o_j\} \). If we denote with \( O = o_j \) the event that occurs when by measuring \( O \) one obtains \( o_j \) as outcome, it can be proved that it results:

\[
O = \sum_j o_j I_{O= o_j} .
\] (8)

The outcomes of measurement of the observables in both sides of (8), indeed, coincide in all measurements. In fact, if, by measuring \( O \), one obtains a certain outcome \( o_k \), the measurement of the projectors in the right-hand side of (8) will give 0 as outcome except for the projector \( I_{O= o_k} \), for which one will get 1, and therefore the outcome of measurement of the sum will be equal to: \( o_k 1 + 0 = o_k \).

Note that the projectors \( I_{O= o} \) are mutually exclusive and satisfy the closure relation (7), since the union of the events associated gives as result the certain event (the measurement of \( O \) must have as outcome one of \( o_j \)). These, therefore, form a projector basis uniquely determined, apart from the order, by the observable \( O \). We will call it projector basis associated to the observable \( O \). Notice, however, that, generally speaking, the projectors of this basis aren’t elementary.

Consider, now, two compatible observables \( O_1 \) and \( O_2 \). According to what above proved, they may be written in the form:

\[
O_1 = \sum_j o_{1,j} I_{O_1= o_{1,j}} ,
\]

\[
O_2 = \sum_k o_{2,k} I_{O_2= o_{2,k}} .
\]
One should observe that the compatibility of $O_1$ and $O_2$ implies that also the projectors $I_{O_1=0_1,j}$ and $I_{O_2=0_2,k}$, that appear in the two above decompositions, must be compatible among themselves, since their measurability depends on that of $O_1$ and $O_2$, respectively.

By applying the closure relation (7) to the projector bases associated to the two observables, one obtains:

$$O_1 = O_11 = \sum_j o_{1,j} I_{O_1=0_1,j} \sum_k I_{O_2=0_2,k} = \sum_j o_{1,j} I_{O_1=0_1,j} I_{O_2=0_2,k} ;$$

$$O_2 = O_21 = \sum_k o_{2,k} I_{O_2=0_2,k} \sum_j I_{O_1=0_1,j} = \sum_j o_{2,k} I_{O_1=0_1,j} I_{O_2=0_2,k} .$$

It is, therefore, possible to express both the observables as linear combinations of the same set of projectors:

$$I_{O_1=0_1,j} I_{O_2=0_2,k} = I_{O_1=0_1,j \land O_2=0_2,k} .$$

The projectors of this set are clearly mutually exclusive and satisfy the closure relation:

$$\sum_{j,k} I_{O_1=0_1,j \land O_2=0_2,k} = \sum_{j,k} I_{O_1=0_1,j} I_{O_2=0_2,k} = \sum_j I_{O_1=0_1,j} \sum_k I_{O_2=0_2,k} = 1 .$$

They thus form a new projector basis.

In general, if $\{I_j\}$ is a projector basis and $O$ is an observable that can be put in the form of a linear combination of them of the type:

$$O = \sum_j o_j I_j$$

where the coefficients $o_j$ aren’t necessarily all distinct, we will call the (9) decomposition of the observable $O$ according to the projector basis $\{I_j\}$. The coefficients $o_j$, will be called, besides, spectral coefficients of the observable $O$ with respect to the projector basis $\{I_j\}$. Clearly, the spectral coefficients of an observable are the same numbers of its possible measurement outcomes.

By multiplying both sides of the (9) for a generic projector $I_{j_0}$ of the basis, one obtains the following important relation:

$$OI_{j_0} = o_{j_0} I_{j_0}$$

which characterizes the value of the spectral coefficient $o_{j_0}$ of a given observable $O$, relative to the projector $I_{j_0}$.

By adding further compatible observables, therefore, one obtains sets of mutually exclusive projectors, all satisfying the closure relation and always minor or equal to the starting ones. The procedure comes to an end when all the projectors obtained are elementary: at this point the addiction of a further observable can no more alter the set of projectors.

The set, that will be assumed as finite, of observables $\{O_r\}$, independent of each other, that are needed to individuate, by means of the described procedure, a basis of elementary projectors, will be called a complete set of compatible observables.\footnote{The analogies with the complete sets of commuting observables, introduced by Dirac in his book [6], page 57, are clear. One should, however, note the difference in the point of view: here there is no reference to eigenstates.}

The above outlined procedure proves that every observable is an element of at least one complete set of compatible observables. If, in fact,
the projectors of the basis associated to an observable $O$ were not elementary, they, by definition, would be expressible as a sum of mutually exclusive elementary projectors, each of which would be compatible with $O$ (in fact $O$ results to be a linear combination of them and other projectors that are mutually exclusive with them). The set of compatible observables formed by $O$ and by such projectors would be, therefore, complete.

Note, besides, that if $I_j$ and $I_k$ ($j \neq k$) are two distinct elementary projectors of the basis, according to the basis of elementary projectors construction procedure, the set of coefficients $\{o_{r,j}\}$ associated to the first projector and that of those, $\{o_{r,k}\}$, associated to the second must differ for at least one element.

We can now define a function of an observable $O$ as an observable whose measurement have as outcomes the images, under a certain function $f$, of the outcomes of measurement of the observable $O$. It can, therefore, be expressed, in terms of the projector basis associated to $O$, in the form:

$$f(O) := \sum_j f(o_j) I_{O = o_j}. \quad (11)$$

More generally, if an observable $O$ is decomposed according to a projector basis $\{I_j\}$, in the form of (9), it immediately follows by definition, making reference to (11), that it results:

$$f(O) = \sum_j f(o_j) I_j \quad (12)$$

for every function of the observable $O$. Clearly all functions of an observable are compatible among each other.

It is worth noting that, by defining the Kronecker $\delta$ function as usual:

$$\delta(x) := \begin{cases} 
0 & \text{for } x \neq 0 \\
1 & \text{for } x = 0 
\end{cases} \quad (13)$$

chosen an index $j$, one has:

$$I_{O = o_j} = \delta(O - o_j). \quad (14)$$

So the projectors of a basis associated to a certain observable may be considered as functions of the observable itself. Projectors of bases associated to compatible observables, therefore, are compatible among each other. This proves what stated before.

The definition of a function of an observable, given above, can easily be extended also to the case of functions of several observables compatible among each other. If one denotes with $O$ the $n$-tuple of the compatible observables:

$$O := (O_1, \ldots, O_r, \ldots)$$

and with $o_j$ the $n$-tuple of the corresponding coefficients associated to the elementary projector $I_j$ of the basis:

$$o_j := (o_{1,j}, \ldots, o_{r,j}, \ldots)$$

we can define a function $f$ of the observables $O$ as:

$$f(O) := \sum_j f(o_j) I_j. \quad (15)$$
If an observable $A$ is compatible with a complete set of compatible observables, that individuates a basis $\{I_j\}$ of elementary projectors, for what outlined before, it will be decomposable, according to the projector basis $\{I_j\}$, in the form (9), with spectral coefficients $a_j$. We will now prove that the observable $A$ is a function of the observables $O$ of the complete set. In fact, after observing that, by changing the index $j$, the $n$-tuple $o_j$ are distinct among each other, it suffices to put: $f(o_j) := a_j$.

We will call a complete space of compatible observables, a space of compatible observables containing a complete set of compatible observables. For them it holds the following fundamental theorem of uniqueness of the basis of elementary projectors: every complete space of compatible observables admits an unique basis of elementary projectors. Let’s suppose, in fact, for absurd, that a complete space of compatible observables admits two distinct bases of elementary projectors: $\{I_j\}$ and $\{J_k\}$. For a generic projector $I_{j_0}$ of the first basis, one would have:

$$I_{j_0} = I_{j_0} 1 = I_{j_0} \sum_k J_k = \sum_k I_{j_0} J_k .$$

Since $I_{j_0}$ is elementary, the products in the last summation should cancel out all but one, corresponding to a certain index $k_0$, and so it should result $I_{j_0} = I_{j_0} J_{k_0}$. But being $J_{k_0}$ also elementary, the product $I_{j_0} J_{k_0}$, not being equal to 0, should be equal to $J_{k_0}$, and so, by transitivity, it should result: $I_{j_0} = J_{k_0}$. Every projector of the first basis, therefore, will be equal to one of the second and, by symmetry, every projector of the second basis too will be equal to one of the first. The two bases will so be identical.

For what above proved, every observable $O$, belonging to a complete space of compatible observables with the basis $\{I_j\}$ of elementary projectors, will be decomposable, according to this basis, in the form (9). This decomposition, besides, is unique, in force of the theorem of uniqueness of the basis of elementary projectors.

We will define multiplicity of a given outcome of measurement $o$ of an observable $O$, the number of times that this coefficient appears in the decomposition according to the basis of elementary projectors.

It is worth, now, however, to point out an important clarification. The circumstance that a given set of independent compatible observables, relative to a certain physical system, is complete, is, actually, a characterization of the representation that one assume to adopt for the physical system itself. It is therefore more correct to state that the assumption that a given set of compatible observable is complete characterize a model of the physical system in study, model that, if necessary, may be also subsequently amended by adding new independent observable to the starting “complete” set.

3. Incompatible Observables

3.1. The issue of the incompatible observables

The existence of incompatible observables complicates the conceptual framework introduced so far, but, at the same time, enriches it also greatly.

First of all, it is to be noted that the compatibility between observables is not an equivalence relation, since, in general, it is not valid the transitive property. However, different complete spaces of compatible observables have always the constants as common elements. The set of observables belonging to both spaces forms a vector subspace, that it is called the intersection space. The intersection space contains the constants and it is therefore definitely not empty. However, it does
not admit a basis of elementary projectors, except in the trivial case in which the two complete spaces coincide.

The introduction of operations between incompatible observables is far from being trivial. The underlying issue is that you cannot even give an adequate operational definition of sum or product, in the sense that it is not possible to express the results of these operations solely in terms of the spectra of the observables themselves. The reason for this difficulty lies in the concept of non-compatibility of observables: not being possible a simultaneously measure the two observables on the same “copy” (ensemble member) of the system in the given state, it is not permissible to combine the results of two successive measurements for obtain the measurement of a new quantity, as the second measurement is influenced by the information arising from the first (conditioned events).

The operations involving incompatible observables shall, therefore, be treated by following a different approach. The aim is to define them, in a such a way that, in the case of compatible observables, the sum and the product are reduced to what previously defined. In general, however, the result will be only a formal expression corresponding to a non-observable entity, that we will call pseudo-observable.

In defining the operations, we will try to preserve, as much as possible, their fundamental properties. In particular, the operations will be defined in such a way that the set of the pseudo-observables retains, with respect to these operations, the algebraic structure of vector space and unitary ring. It will be assumed, therefore, that for the addition the commutative property, the associative property, the existence of the additive identity (obviously the constant 0) and existence of the additive inverse all hold. For the multiplication, we will assume that they hold the associative property, but not the commutative one, and both the left and right distributivity over addition. The constant 1, besides, will be the multiplicative identity.

The renunciation of the commutative property is necessary in order to preserve the possibility that two observables may be incompatible. Let, in fact, be $A$ and $B$ two incompatible observables. Let, besides, be $\mathfrak{A}$ a complete space, having the basis $\{I_{\mathfrak{A},j}\}$ of elementary projectors, containing the observable $A$ and be $\mathfrak{B}$ a complete space, with the basis $\{I_{\mathfrak{B},k}\}$ of elementary projectors, containing the observable $B$. If the product of two observables were always commutative and gave as result another observable, the product of elementary projectors $I_{\mathfrak{A},j}I_{\mathfrak{B},k}$, according to (4), would be a projector too. Following the same lines of reasoning used to prove the theorem of uniqueness of the basis of elementary projectors, one would conclude that the bases $\{I_{\mathfrak{A},j}\}$ and $\{I_{\mathfrak{B},k}\}$ should coincide and therefore all their linear combination, included the observables $A$ and $B$, should be compatible, which is a contradiction.

3.2. Hermitian transposition of pseudo-observables

But what physical meaning has to be given to an operation involving incompatible observables? Since two incompatible observables cannot be measured simultaneously, the observation order must be relevant. With the term “observation order” I mean the order in which the observer delivers to eventually measure the various observables in study, which is obviously relevant in the case of incompatible observables. It is worth to point out that the observation order is independent of the eventual temporal ordering of the observables, since it is experimentally possible to measure before an observable relative to a time posterior with respect to the others, by recording, for instance exploiting the entanglement mechanism, the information in different instants of time for a subsequent measurement in a whatever order.

We will then adopt the convention that in an operation the rightmost observable has to be
measured before the leftmost.

It is besides clear that an observable, defined through operations on other observables, does not change if one inverts the observation order.

With these ideas in mind, we can now introduce the Hermitian transposition operator ($\dagger$), which switches from an expression to that obtained by reversing the observation order of the observables involved in it. The Hermitian transposition operator, acting on the space of the pseudo-observables $\mathcal{P}$, is defined by the following properties:

(i) For each pseudo-observable $P$ one has: $(P^\dagger)^\dagger = P$.

(ii) A pseudo-observable $P$ is an observable iff $P^\dagger = P$ (maximum physical significance assumption).

(iii) For each pair of pseudo-observables $P$ and $Q$ one has: $(P + Q)^\dagger = P^\dagger + Q^\dagger$.

(iv) For each pair of pseudo-observables $P$ and $Q$ one has: $(PQ)^\dagger = Q^\dagger P^\dagger$.

(v) For each pseudo-observable $P$ one has: $PP^\dagger \geq 0$.

(vi) For a pseudo-observable $P$ one has $PP^\dagger = 0$ iff $P = 0$.

The first property follows from the fact that reversing twice the observation order one comes back to the starting situation.

The second property gives the maximum possible physical meaning to the transposition, by connecting it directly to the observability of a given expression.

The third and fourth properties derive directly from the definitions and from the assumption that the sum of observables is commutative.

The fifth property extends to pseudo-observables the condition of reality of an observable, according to which the square of an observable is always positive or zero. The expression to the first member is constructed in such a manner to ensure that it is an observable.

The sixth property is the extension to the case of pseudo-observables of a similar property on the square of the observables, easily proved by considering the possible outcomes of measurement.

The Hermitian transposition operator allows us to generalize to the case of pseudo-observables the idempotency relation [2], characteristic of the projectors. In fact, it is easy to prove that necessary and sufficient condition in order a pseudo-observable $I$ is a projector is that it results:

$$II^\dagger = I$$  \hfill (16)

Indeed, if $I$ is a projector, due to the second property of the Hermitian transposition, the (16) is equivalent to (2). If, instead, $I$ satisfies the (16), $I$ is an observable, being the left hand side invariant for Hermitian transposition, and so it satisfies the idempotency relation (2), to which (16) reduces itself in the case of observables. $I$ is therefore a projector.

3.3. Complex form of the pseudo-observables

Let’s consider now a generic pseudo-observable $P$, generated by addition and multiplication of compatible and incompatible observables. Note that it results:

$$P = P_S + P_A \quad \text{with: } P_S = \frac{P + P^\dagger}{2} \quad \text{and } P = \frac{P - P^\dagger}{2}$$  \hfill (17)

∥ The reference to the Hermitian conjugate of the operators associated to the observables in the Dirac-von Neumann formulation of quantum mechanics is obvious.
where, since $P_S^\dagger = P_S$, $P_S$ is an observable, while $P_A^\dagger = -P_A$. We will call $P_S$ the symmetric part of the pseudo-observable $P$ and $P_A$ its antisymmetric part.

In order to give to the pseudo-observables a much more meaningful form than (17), we now introduce a non-null pseudo-observable $i$, whose product with any other pseudo-observable is commutative and such that it results:

$$i^\dagger = -i$$  \hfill (18)

or, in other words, whose symmetric part is equal to the null observable.

To fully define the pseudo-observable $i$, we first observe that, for the fifth and the sixth properties of the Hermitian transposition operator, it results:

$$ii^\dagger = -i^2 > 0 \Rightarrow i^2 < 0$$

so that $i^2$ turns out to be a negative observable (for “negative observable” we mean an observable whose possible outcomes of measurement are all negative). One is, therefore, allowed to put:

$$i^2 := -1$$  \hfill (19)

The pseudo-observable $i$, then, behaves like the imaginary unit of the complex numbers, with which, in what it follows, it will be identified.

It is now worth observing that, for the antisymmetric $P_A$ of a pseudo-observable $P$, it results:

$$(-iP_A)^\dagger = -iP_A^\dagger = i(-P_A) = -iP_A$$  \hfill (20)

which proves that the product of the antisymmetric part of a pseudo-observable $P$ and the opposite of $i$ is an observable, that will be called imaginary part, $P_I$, of $P$:

$$P_I := -iP_A$$  \hfill (21)

Making use of (21), (17) and (19), we can finally give the following complex representation of a pseudo-observable:

$$P = P_S + P_A = P_R + iP_I$$  \hfill (22)

where we have put $P_R := P_S$, the real part of $P$.

The introduction of the pseudo-observable $i$, according to (22), allows to associate every pseudo-observable to a pair of observables, its real and imaginary part. In this manner the unique non-physical entity needed to express a pseudo-observable is just the imaginary unit $i$. It is also necessary, since if the imaginary part of each pseudo-observable were the null observable, then every of them, reducing itself to its symmetric part, should be an observable. In particular, in such hypothesis, the product of each pair of observables would be an observable. But this would imply, by virtue of the fourth and the second properties of the Hermitian transposition operator, that every pair of observables commutes. But in this case, as it will be demonstrated in section “Transformations” of the third paper, no transformation would be allowed and therefore no evolution: an Universe of only compatible observables would be static, definitely dead! Therefore an evolving, living Universe needs to be described by complex representable pseudo-observables.

According to (22) and (18), one has:

$$P^\dagger = (P_R + iP_I)^\dagger = P_R - iP_I$$  \hfill (23)

that proves the analogous of a well known property of the Hermitian conjugate of an operator in the Dirac formulation of quantum mechanics.
In what follows, we will call complex constants the pseudo-observables of the form:

\[ \gamma = \alpha + i\beta \]  

(24)

where \(\alpha\) and \(\beta\) are two constant observables. They will be identified with their complex value, i.e. with the complex number having the value of their real part as real part and the value of their imaginary part as imaginary part. In such a context, we will refer to constant observables with the term of real constants.

By this definition, for each complex constant \(\gamma\), it results:

\[ \gamma^\dagger = \gamma^* \]  

(25)

where the star (*) indicates the complex conjugate.

The introduction of the pseudo-observable \(i\) allows us to define the set \(\mathcal{P}\) of the pseudo-observables as that formed by every expression of the form:

\[ \mathcal{P} := A + iB \]

where \(A\) and \(B\) are two generic observables (compatible or incompatible). For what above discussed, this set, with the operations of addition and multiplication, has the algebraic structure of a ring.

3.4. Commutator and compatibility

As already seen, the non-commutativity of the product of two observables is tied in a deep way to their incompatibility.

In order to make this point more clear, it is useful to introduce the commutator of two observables \(A\) and \(B\), defined, as well known, as follows:

\[ [A, B] := AB - BA . \]  

(26)

This definition can be extended, in an obvious manner, to the more general case of two pseudo-observables.

If \(A\) and \(B\) are two generic pseudo-observables, it is immediate to prove the following properties of the commutator:

(i) The commutator is anti-commutative: \([A, B] = -[B, A]\).

(ii) The commutator is nilpotent: \([A, A] = 0\).

(iii) The commutator is a bilinear form:

\[ [c_1 A + c_2 B, C] = c_1 [A, C] + c_2 [B, C] \]

\[ [A, c_1 B + c_2 C] = c_1 [A, B] + c_2 [A, C] \]

where \(c_1\) and \(c_2\) are two generic constants (real or complex).

(iv) The commutator satisfies the following forms of the Leibniz rule:

\[ [A, BC] = [A, B] C + B[A, C] \]

\[ AB, C] = A[B, C] + [A, C] B . \]

(v) The commutator satisfies the following Jacobi identity:

\[ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 . \]
In analogy with the Dirac formulation, we will say that two observables, or pseudo-observables, **commute** if their commutator is equal to 0.

Consider two generic observables, \( A \) and \( B \), which commute with each other: 
\[
[A,B] = 0
\]
We will prove that it results \( [A^n, B] = 0 \) for every natural exponent \( n \). This is trivially true for \( n = 1 \). Assumed true for the exponent \( n - 1 \), moreover one has:
\[
[A^n, B] = [A, [A^{n-1}, B]] = [A, A^{n-1}B] + [A, B]A^{n-1} = 0
\]
The property is so demonstrated by induction. Applying the same reasoning to the powers of \( B \), more generally, one obtains:
\[
[A^n, B^m] = 0 \quad \forall n, m \in \mathbb{N}
\]
Besides, due to the bilinearity of the commutator, each polynomial expression in \( A \) commutes with any polynomial expression in \( B \), i.e. if \( p \) and \( q \) are two whatever polynomials, one has:
\[
[p(A), q(B)] = 0
\]
(27)
Having premised that, we now can demonstrate that necessary and sufficient condition in order that two observables are compatible is that they commute.

We begin to prove that two compatible observables commute. Let \( A \) and \( B \) two compatible observables and so belonging to the same complete space of compatible observables, having the basis \( \{I_j\} \) of elementary projectors. By decomposing the observables \( A \) and \( B \) according to this basis, one has:
\[
A = \sum_j a_j I_j \quad \text{and} \quad B = \sum_k b_k I_k
\]
and therefore for (2) and (6):
\[
AB = \sum_{j,k} a_j b_k I_j I_k = \sum_j a_j b_j I_j = \sum_{j,k} b_k a_j I_k I_j = BA
\]
that implies: \( [A, B] = 0 \).

The proof of the sufficiency of the condition is more complex. Suppose, therefore, that \( A \) and \( B \) are two observables that commute, and therefore such that it results:
\[
[A, B] = 0
\]
(28)
By decomposing these observables according to the bases associated with them, for the (8), one has:
\[
A = \sum_j a_j I_{A=a_j} \quad \text{and} \quad B = \sum_k b_k I_{B=b_k}
\]
(29)
where both the coefficients \( a_j \) and \( b_k \) are distinct among themselves.

Since the observables \( A \) and \( B \) commute, by virtue of the property (27), the (29) and (15), considered two whatever polynomial expressions \( p(A) \) and \( q(B) \), respectively in \( A \) and \( B \), it results:
\[
[p(A), q(B)] = \sum_{j,k} p(a_j) q(b_k) [I_{A=a_j}, I_{B=b_k}] = 0
\]
(30)
Chosen arbitrarily two indexes, \( j' \) and \( k' \), we now put:
\[
p(A) = \prod_{j \neq j'} (A - a_j) \quad \text{and} \quad q(B) = \prod_{k \neq k'} (B - b_k)
\]
(31)
Note that, by construction, one has \( p(a_j) = 0 \) for each index \( j \neq j' \) and \( q(b_k) = 0 \) for each index \( k \neq k' \). It also results:

\[
p(a_j') = \prod_{j \neq j'} (a_j' - a_j) \neq 0
\]

since the coefficients \( a_j \) are all each other distinct. For the same reason, it also holds: \( q(b_k') \neq 0 \).

By applying these relations to (30) and exploiting the arbitrariness in the choice of the indexes \( j' \) and \( k' \), one can therefore state that, for each pair of indexes \( j \neq k \), it has to result:

\[
[I_{A=a_j}, I_{B=b_k}] = 0.
\] (32)

But if two projectors commute, their product is an observable and is still a projector. It is then easy to verify that the set of the products of projectors thus obtained satisfies the relation of closure (7) and consists of mutually exclusive projectors, for which it holds (6). According to the link between projectors and events, therefore, these projectors are each other compatible, since they are associated with events mutually exclusive and therefore necessarily observable simultaneously, and form a basis.

According to (29) and to the closure relation (7), it then follows:

\[
A = A_1 = \sum_j a_j I_{A=a_j} \sum_k I_{B=b_k} = \sum_{j,k} a_j I_{A=a_j} I_{B=b_k} \, ,
\]
\[
B = B_1 = \sum_j b_k I_{A=a_j} \sum_{k} I_{B=b_k} = \sum_{j,k} b_k I_{A=a_j} I_{B=b_k} \, .
\]

The observables \( A \) and \( B \), therefore, being expressible as linear combinations of the projectors of the same base, are each other compatible.

An important consequence of the theorem just proved is that if two observables commute, also every function of one commutes with whatever function of the other. If, in fact, the two observables commute each other, they are both expressible as a linear combination of the projectors obtained by multiplying the projectors of the bases associated with the two observables and discarding the null products. These projectors form a basis. Each function of the two observables, therefore, for (12), will be expressible as a linear combination of the projectors of the same base, and then each pair of functions of the two observables will be constituted by compatible observables which commute among themselves.

Another important consequence is given by the following theorem: if an observable commutes with the elements of a basis of elementary projectors, then the observable is expressible as a linear combination of the projectors of the basis. If, in fact, an observable \( A \) commutes with the elements of a basis \( \{I_k\} \) of elementary projectors, after having written the observable \( A \) in the form of the first of the (29), it should first be noted that also every polynomial expression in \( A \), in particular those of the form of the first of the (31), commutes with each projector of the basis. This implies, as seen before, that also every projector \( I_{A=a_j} \), which appears in the decomposition of \( A \), commutes with each projector \( I_k \) of the basis. The product \( I_{A=a_j} I_k \) is therefore a projector and, for the properties of the elementary projectors, it is or null or equal to \( I_k \) and mutually exclusive with all other products. The thesis then follows by observing that it results:

\[
A = A_1 = \sum_j a_j I_{A=a_j} \sum_k I_k = \sum_k a_{jk} I_k
\]
where $j_k$ is the only index for which one has:

$$I_{A=a_{j_k}} = I_k .$$

It is, finally, worth noting that the imaginary part of the product of two observables $A$ and $B$:

$$(AB)_1 = \frac{1}{i} [A, B]$$

is an observable which, for what above proved, gives a measure of their grade of incompatibility.

4. Dyads

4.1. Dyadic forms

The set $\mathcal{P}$ of the pseudo-observables constitutes, with respect to the operations of addition and multiplication by a constant, a vector space over the field of complex numbers $\mathbb{C}$. In this space, by virtue of (25) and of the properties of the Hermitian transposition operator, for each pair of constants $\gamma_1$ and $\gamma_2$ and for each pair of pseudo-observables $Z_1$ and $Z_2$, the following relation holds:

$$\left(\gamma_1 Z_1 + \gamma_2 Z_2\right)^\dagger = \gamma_1^* Z_1^\dagger + \gamma_2^* Z_2^\dagger$$

which is expressed by saying that the Hermitian transposition operator is antilinear.

Consider a space of compatible observables having the basis $\{I_j\}$ of elementary projectors. We will show how it is possible to construct a set of generators for the space $\mathcal{P}$ starting from the projectors $\{I_j\}$. In order to do this, one has to observe that, if $C$ is a whatever observable, it results:

$$C = 1 C 1 = \sum_{j,k} I_j CI_k .$$

According to (35), any observable $C$ is then given by a linear combination of the pseudo-observables obtained by multiplying the observable to the right and to the left for an element of the basis of elementary projectors. This result can be immediately extended to pseudo-observables also, by virtue of (22). It can thus be said that the space of the pseudo-observables is spanned by the set $\{I_j CI_k\}$, where one should consider the products for every observable $C$. However, in fact, most of the elements of this set are linearly dependent on the others.

Initially we consider a product of the form: $I_j CI_j$, that will be called, for a reason that will be explained shortly, projection of $C$ according the projector $I_j$. It is immediate, first of all, to prove that this expression is an observable that commutes with each of the elementary projectors of the basis $\{I_k\}$. By the theorem proved at the end of section 3.4. the projection is expressible as a linear combination of the projectors of the basis $\{I_k\}$. Applying the associative property of the product of observables and the fact that the projectors of the basis are mutually exclusive, it is concluded finally that the projection of $C$ according the projector $I_j$ is proportional to the only projector $I_j$:

$$I_j CI_j = c_j I_j$$

being $c_j$ a suitable real constant, which we will call the component of the observable $C$ according the elementary projector $I_j$. It is possible to give a physical interpretation to this formula, by noting that the measurement of the projection of $C$ according the projector $I_j$ gives as outcome $c_j$ when it occurs the event associated to the projector $I_j$. In such a case the measurement of this projector
The Algebra of the Pseudo-Observables I

gives as outcome 1. According to (36), one is allowed, besides, to state that each product \( I_j C I_j \) is linearly dependent on the projector \( I_j \).

It is interesting to observe that the relation (36) is characteristic of elementary projectors. Be, in fact, \( I \) a projector such that for each observable \( C \) it results:

\[
IC = cI
\]

(37)

where \( c \) is a suitable real constant. Assuming that \( I \) is not elementary, they must exist two mutually exclusive projectors, \( J_1 \) and \( J_2 \), such that it results:

\[
I = J_1 + J_2 .
\]

(38)

By applying (37) to both \( J_1 \) and \( J_2 \), moreover, one has:

\[
IJ_1 = y_1 I \quad \text{and} \quad IJ_2 = y_2 I
\]

where \( y_1 \) and \( y_2 \) are two suitable real constants. By substituting these relations in (38), finally we get:

\[
J_1 = y_1 I \quad \text{and} \quad J_2 = y_2 I
\]

that necessarily implies that one between \( J_1 \) and \( J_2 \) must be the null observable and, therefore, that the projector \( I \) is elementary.

Let us now consider a product of the form: \( I_j C I_k \), where \( C \) is a generic observable. It will be called dyadic form relative to the pair of elementary projectors \((I_j, I_k)\) having core \( C \). We state the following postulate on the dyadic forms\(\dagger\) considered a pair of elementary projectors \((I_j, I_k)\) there is always an observable \( C_{jk} \) such that the dyadic form relative to the pair of elementary projectors and having core \( C_{jk} \) is not the null observable; furthermore, given a whatever observable \( C \), there exists a complex constant \( \alpha \) such that it is:

\[
I_j C I_k = \alpha I_j C_{jk} I_k .
\]

(39)

It is worth observing that, in the case where the indexes \( j \) and \( k \) are equal, the postulate on the dyadic forms reduces itself to what before has been proved about the projections according elementary projectors. Note, also, that the relation (39) is trivially satisfied if the observable \( C \) is compatible, and therefore commutes, with \( I_j \) or with \( I_k \), because, if \( j \neq k \), it results \( I_j C I_k = 0 \). The postulate, therefore, actually concerns only those cases in which we consider products between incompatible observables that do not have as result an observable, for which, up to this point, there was provided no actual definition.

The postulate on the dyadic forms, therefore, completes the definition of the product of pseudo-observables, compared to what was done in section 2.4 for compatible observables. This definition allows to minimize the number of not actually measurable pseudo-observables in the space \( \mathcal{P} \) and therefore represents a further application of the Occam’s razor principle.

By virtue of the postulate on the dyadic forms and according to (35) we can now state that the pseudo-observable space \( \mathcal{P} \) is spanned by the set of dyadic forms \( \{ \Phi_{jk} := I_j C_{jk} I_k \} \), where the observables \( C_{jk} \) are chosen in such a way that they make not null the dyadic forms of which they

\(\dagger\) In the Dirac formulation of quantum mechanics, the dyadic forms correspond to operators of the type: \( \gamma |j⟩⟨k| \), where the indexes \( j \) and \( k \) denote distinct eigenvectors, normalized to one, of a certain linear Hermitian operator and \( \gamma \) is a suitable complex constant, dependent on the core. If \( \hat{C} \) is the operator associated to the observable \( C \), in fact, it results: \( \gamma = ⟨j|\hat{C}|k⟩ \).
are core. If, in fact, \( P \) is a generic pseudo-observable, whose real part is \( P_R \) and the imaginary part is \( P_I \), one has:

\[
P = \sum_{j,k} \left( I_j P_R I_k + iI_j P_I I_k \right).
\]

(40)

For each pair of indexes \( j \) and \( k \), according to (39), exist two complex constants, \( \alpha_{jk} \) and \( \beta_{jk} \), such that it results:

\[
I_j P_R I_k = \alpha_{jk} \Phi_{jk} \quad \text{and} \quad I_j P_I I_k = \beta_{jk} \Phi_{jk}
\]

which, once replaced in (40), allow to obtain:

\[
P = \sum_{j,k} \left( \alpha_{jk} + i\beta_{jk} \right) \Phi_{jk} = \sum_{j,k} \varpi_{jk} \Phi_{jk}
\]

(41)

where we have put: \( \varpi_{jk} := \alpha_{jk} + i\beta_{jk} \).

The dyadic forms of the set \( \{ \Phi_{jk} \} \) are, also, linearly independent. If, in fact, we consider a linear combination of them, by means of a set of complex constants \( \{ \alpha_{jk} \} \), and set it to be equal to the null observable:

\[
\sum_{j,k} \alpha_{jk} \Phi_{jk} = 0
\]

(42)

after chosen any two indexes, \( j' \) and \( k' \), and multiplying the relation (42) on the left by \( I_{j'} \) and on the right by \( I_{k'} \), according to (6), one has:

\[
\alpha_{j' k'} \Phi_{j' k'} = 0 \Rightarrow \alpha_{j' k'} = 0
\]

from which follows the linear independence of the dyadic forms.

The set of dyadic forms \( \{ \Phi_{jk} \} \) therefore constitutes a basis, that, in our assumptions, is finite or countable to the limit, for the pseudo-observable space. This basis is equipotent to the Cartesian product of the basis of elementary projectors by itself. Since the bases of a vector space are mutually equipotent, an important consequence of this result is that also the bases of elementary projectors associated with the various spaces of compatible observables must be each other equipotent.

Since the set of dyadic forms \( \{ \Phi_{jk} \} \) constitutes a basis for the space \( \mathcal{P} \), each coefficient \( \varpi_{jk} \) of the decomposition (41) of a generic pseudo-observable \( P \) is, therefore, uniquely determined and will be called component of the pseudo-observable according the dyadic form \( \Phi_{jk} \).

It is now worth observing that if \( K \) is a generic pseudo-observable such that the product \( I_j K I_k \) is not null, according to (41), it must exist a non-zero complex constant \( \kappa_{jk} \) such that it results \( I_j K I_k = \kappa_{jk} \Phi_{jk} \) and therefore:

\[
\Phi_{jk} = I_j \left( \frac{1}{\kappa_{jk}} K \right) I_k
\]

that implies that one can safely assume that the cores of the dyadic form are pseudo-observables.

Note, finally, that, if \( k \neq l \), by virtue of (6), it is always:

\[
\Phi_{jk} \Phi_{lm} = 0
\]

(43)
4.2. Dyad bases

We will now prove that it is possible to choose the dyadic form pseudo-observable cores of the basis in such a manner to obtain a set of normalized dyadic forms or **dyads** that will be denoted by

\[ \Gamma_{jk} \]

satisfying the following conditions:

(i) \( \Gamma_{jj} = I_j \)

(ii) \( \Gamma_{kj} = \Gamma_{jk}^\dagger \)

(iii) \( \Gamma_{jl} \Gamma_{l \ell} = \delta_{l,l} \Gamma_{jk} \)

where the Kronecker symbol \( \delta_{l,l} \) is, as usual, equal to 1 if the subscript indexes are equal and 0 otherwise. In order to do this, called \( C_{jk} \) the core of the dyad \( \Gamma_{jk} \), one can observe that the first condition is immediately satisfied by putting:

\[ C_{jj} = I_j \]

whereas for the second, observed that, by transposition, it results:

\[ I_k C_{jk} I_j \neq 0 \iff I_k C_{jk}^\dagger I_k \neq 0 \]

one can put:

\[ C_{kj} = C_{jk}^\dagger \]

For what regards the third condition, fixed a value \( k_0 \) of the second index, for each index \( j \neq k_0 \) one can choose a pseudo-observable \( A_j \) such that it results:

\[ \Phi_{jk_0}^\dagger I_j I_k = 0 \]

Observe now that, by the fifth property of the transposition, it results:

\[ \Phi_{jk_0}^\dagger I_j I_k \geq 0 \]

and therefore, by virtue of \( \Phi_{jk} \), one has:

\[ \Phi_{jk_0}^\dagger I_j I_k = a_j^2 I_j \]

where \( a_j \) is a suitable real positive constant. We now put:

\[ C_{jk_0} = a_j^{-1} A_j \]

So one has:

\[ \Gamma_{jk_0} \Gamma_{k_0 j} = \Gamma_{jk_0} \Gamma_{k_0 j} \]

\[ = a_j^{-2} \Phi_{jk_0}^\dagger I_j I_k = I_j = \Gamma_{jj} \quad \text{(44)} \]

Observe, now, that, by virtue of \( \Gamma_{jk_0} \), one has:

\[ \Gamma_{k_0 j} \Gamma_{jk_0} = \Gamma_{k_0 j} \Gamma_{jk_0} \]

\[ = I_k C_{jk_0}^\dagger I_j C_{jk_0} I_k = g_j I_k = \Gamma_{k_0 k_0} \quad \text{(45)} \]

where \( g_j \) is a suitable real positive constant. By the associative property of the product, besides, it results:

\[ (\Gamma_{k_0 j} \Gamma_{jk_0}) \Gamma_{k_0 j} = \Gamma_{k_0 j} (\Gamma_{jk_0} \Gamma_{k_0 j}) \]

that, by substituting relations \( \text{(44)} \) and \( \text{(45)} \), allows one to obtain:

\[ g_j I_k \Gamma_{k_0 j} = \Gamma_{k_0 j} I_j \Rightarrow g_j = 1 \]

by which it follows:

\[ \Gamma_{k_0 j} \Gamma_{jk_0} = I_k = \Gamma_{k_0 k_0} \quad \text{(46)} \]

Dyads correspond in the Dirac formulation of quantum mechanics to operators of the form: \( |j\rangle \langle k| \), where the indexes \( j \) and \( k \) denote distinct eigenvectors normalized to one, of a given linear Hermitian operator. The name of dyad will find justification in subsection “State vectors” of the second paper.
The Algebra of the Pseudo-Observables I

Let's put, now, for \( j \neq k, j \neq k_0 \) and \( k \neq k_0 \):

\[
C_{jk} = C_{jko} I_{k_0} C_{kk_0}^\dagger
\]

that immediately implies:

\[
\Gamma_{jk} = \Gamma_{jko} \Gamma_{k_0 k}.
\] (47)

The relation (48) applies without any restriction over the indexes \( j \) and \( k \). For \( j = k \), in fact, the relation reduces itself to (44); whereas for \( j = k_0 \) and \( k = k_0 \), it immediately follows from the fact that it is \( \Gamma_{k_0 k_0} = I_{k_0} \). One should, besides, note that the definition (47) is well-posed, since, according to (44) and (46), it results:

\[
(\Gamma_{jko} \Gamma_{k_0 k}) (\Gamma_{jko} \Gamma_{k_0 k})^\dagger = \Gamma_{jko} \Gamma_{k_0 k} \Gamma_{kk_0} \Gamma_{k_0 j} = \Gamma_{jko} I_{k_0} \Gamma_{k_0 j} = \Gamma_{jko} \Gamma_{k_0 k} = \Gamma_{jk}
\]

and therefore, by the sixth property of the transposition, \( \Gamma_{jko} \Gamma_{k_0 k} \neq 0 \). The dyadic forms with core given by (47), finally, verify also the third condition of the dyad bases. In fact, one has:

\[
\Gamma_{jl} \Gamma_{lk} = \Gamma_{jk} \Gamma_{l_0 k} \Gamma_{k_0 k} \Gamma_{k_0 l} = \Gamma_{jk} I_{k_0} \Gamma_{k_0 l} = \Gamma_{jk} \Gamma_{k_0 k} = \Gamma_{jk}
\]

and the property immediately follows on the basis of the orthogonality relation (43).

The dyad bases are the natural extensions to the whole space of pseudo-observables of the bases of elementary projectors and allow one to make explicit, in terms of components, the operations of sum and of product of pseudo-observables, completing, as already stated, their definition. Let, in fact, be \( P \) a pseudo-observable and \( \{ \Gamma_{jk} \} \) a dyad basis. By decomposing the pseudo-observable \( P \) according the dyad basis, as in (41), one has:

\[
P = \sum_{j,k} \omega_{jk} \Gamma_{jk}
\] (49)

where the coefficients \( \omega_{jk} \), the dyadic components, are suitable complex constants. The components \( \omega_{jk} \) may be thought, in the finite-dimensional cases, as elements of a matrix that will be called matrix associated to the pseudo-observable with respect to the dyad basis. On the basis of this interpretation, we will call diagonal components the dyadic components of the form \( \omega_{jj} \).

The concept of matrix associated will be used also in infinite-dimensional spaces, that would be treated as limiting cases of finite-dimensional ones.

Observe, now, that for a pseudo-observable \( P \) it results \( P^\dagger = P \) if and only if:

\[
\omega_{kj} = \omega_{jk}^*.
\] (50)

The matrices associated to such pseudo-observables are, therefore, Hermitian. We will, accordingly, call Hermitian pseudo-observables those satisfying the condition \( P^\dagger = P \). The second property of the transposition, section 3.2, can so be rephrased by stating that the observables are all and only the Hermitian pseudo-observables.

By multiplying both sides of (49) for the elementary projectors of the basis from which the dyads are deduced, one obtains the following important relation:

\[
I_{j} P I_{k} = \omega_{jk} \Gamma_{jk}
\] (51)

that generalizes relation (10) and supplies a characterization of the components of a pseudo-observables with respect to a dyad basis.
If $Q$ is another pseudo-observable, decomposed according to the dyad basis as follows:

$$Q = \sum_{j,k} \vartheta_{jk} \Gamma_{jk}$$

where $\vartheta_{jk}$ are suitable complex constants, one has:

$$P + Q = \sum_{j,k} (\varpi_{jk} + \vartheta_{jk}) \Gamma_{jk} \quad (52)$$

and

$$PQ = \sum_{j,k} \left( \sum_l \varpi_{jl} \vartheta_{lk} \right) \Gamma_{jk} \quad (53)$$

that allow one to calculate the sum and the product of two pseudo-observables, being known the components of the pseudo-observables involved according a given dyad basis.

It is to be observed that, according to (52) and (53), the matrix associated to the sum of pseudo-observables is equal to the sum of the matrices associated to the addends, while the matrix associated to the product of pseudo-observables is equal to the matrix product of the matrices associated to the factors. Fixed a dyad basis, there is, therefore, an isomorphism between the ring of pseudo-observables and appropriate matrix rings. This isomorphism makes, in particular, observables correspond to Hermitian matrices. In this manner it is recovered and clarified the link, characteristic of the Dirac formalism of quantum mechanics, between these two entities.

It will be, now, clarified better the tie between dyad basis and basis of elementary projectors. We will say, first of all, that a dyad basis $\{\Gamma_{jk}\}$ is associated to the basis $\{I_j\}$ of elementary projectors, if all the dyadic forms of the first are relative to pairs of projectors of the basis $\{I_j\}$.

This stated, let $\{\Gamma_{jk}\}$ and $\{\tilde{\Gamma}_{jk}\}$ be two dyad bases associated to the same basis $\{I_j\}$ of elementary projectors. By virtue of the postulate on the dyadic forms, introduced in section 4.1, there must exist complex constants $\gamma_{jk}$ such that it results:

$$\tilde{\Gamma}_{jk} = \gamma_{jk} \Gamma_{jk} . \quad (54)$$

The second property of dyads implies:

$$\gamma_{jk}^* = \gamma_{kj} . \quad (55)$$

By virtue of this relation and of the second and third property of the dyads, one, besides, has:

$$I_j = \tilde{\Gamma}_{jk} \left( \tilde{\Gamma}_{jk} \right)^\dagger = |\gamma_{jk}|^2 I_j \Rightarrow |\gamma_{jk}|^2 = 1$$

where $|\gamma_{jk}|^2 := \gamma_{jk}^* \gamma_{jk}$ is the square of the modulus of the complex constant $\gamma_{jk}$. This last relation implies that the coefficients $\gamma_{jk}$ are phase factors, i.e. complex constants of the form:

$$\gamma_{jk} = e^{i\vartheta_{jk}} \quad (56)$$

in which the real constants $\vartheta_{jk}$, by virtue of (55), satisfy the relation:

$$\vartheta_{kj} = -\vartheta_{jk} . \quad (57)$$

Having arbitrarily chosen an index $k_0$ and put:

$$\vartheta_j := \vartheta_{j_{k0}} \quad (58)$$
by the second and third property of the dyads and the relations \([55]\), \([56]\) and \([58]\), one has:

\[
\tilde{\Gamma}_{jk} = \tilde{\Gamma}_{jk0} \left( \tilde{\Gamma}_{kk0} \right)^\dagger = e^{i(\vartheta_j - \vartheta_k)} \Gamma_{jk}
\]  

(59)

that compared to \([54]\), by means of \([56]\), allows one to set:

\[
\vartheta_{jk} = \vartheta_j - \vartheta_k
\]  

(60)

where the \textit{phases} \(\vartheta_j\) are arbitrary real constants. In general we will call \textit{equivalent} two dyad bases associated to the same basis of elementary projectors.

4.3. Change of dyad basis

We will now show as to do a change of dyad basis, i.e. as to express the dyads of a given basis \(\{\Gamma_{jk}\}\), associated to the basis \(\{I_j\}\) of elementary projectors, in terms of the dyads of another basis \(\{\tilde{\Gamma}_{jk'}\}\), associated to the basis \(\{\tilde{I}_j\}\) of elementary projectors. At this aim, one has to start to choose an index \(k_0\). Since it results:

\[
I_{k_0} = \sum_{k'} I_{k_0} \tilde{I}_{k_0}' \neq 0
\]

one may choose an index \(k_0'\) such that it is \(I_{k_0} \tilde{I}_{k_0'} \neq 0\). The pseudo-observable \(I_{k_0} \tilde{I}_{k_0'}\), besides, may be expressed, according the \([49]\), as a linear combination of the dyads of the basis \(\{\Gamma_{jk}\}\). By virtue of the \([51]\) one finds that it results:

\[
I_{k_0} \tilde{I}_{k_0'} = \sum_l \alpha_l \Gamma_{kl} \tilde{I}_{k_0'}
\]

(61)

where the \(\alpha_l\) are suitable non-null complex constants. By this relation and the third property of the dyads, one obtains:

\[
\Gamma_{jk_0} \tilde{I}_{k_0'} = \Gamma_{jk_0} \left( I_{k_0} \tilde{I}_{k_0'} \right) = \sum_l \alpha_l \Gamma_{jl} \tilde{I}_{k_0'}.
\]

Note, also, that it results:

\[
\left( \Gamma_{jk_0} \tilde{I}_{k_0'} \right) \left( \Gamma_{kk_0} \tilde{I}_{k_0} \right)^\dagger = \sum_{l,l'} \alpha_l \alpha_{l'}^* \Gamma_{jl} \Gamma_{l'k} = \sum_l |\alpha_l|^2 \Gamma_{jk}
\]

by which, after putting \(\sum_l |\alpha_l|^2 := 1/g^2\), one obtains:

\[
\Gamma_{jk} = g^2 \left( \Gamma_{jk_0} \tilde{I}_{k_0'} \right) \left( \Gamma_{kk_0} \tilde{I}_{k_0} \right)^\dagger.
\]

(61)

Decompose, now, the products \(\Gamma_{jk_0} \tilde{I}_{k_0'}\) according the dyads of the basis \(\{\tilde{\Gamma}_{jk'}\}\):

\[
\Gamma_{jk_0} \tilde{I}_{k_0'} = \sum_l \beta_{lj} \tilde{\Gamma}_{lk_0'}
\]

(62)

where \(\beta_{lj}\) are suitable complex constants and having used \([51]\) and the relationships among the projectors of a same basis. By substituting \([62]\) in \([61]\), therefore, one has:

\[
\Gamma_{jk} = \sum_{l,l'} (g \beta_{lj}) (g \beta_{l'k})^* \tilde{\Gamma}_{lk_0} \tilde{\Gamma}_{k_0'l'} = \sum_{l,l'} \omega_{lj} \omega_{l'k}^* \tilde{\Gamma}_{l'l'}
\]

(63)
having set: $\omega_{lj} := g_{lj}$. Equation (63), expressing the relation among the dyads of a basis and those of another basis, may be put in a form of particular interest by introducing the pseudo-observable $\Omega$ of the change from the basis $\{\Gamma_{jk}\}$ to the basis $\{\tilde{\Gamma}_{jk}\}$:

$$\Omega := \sum_{l,m} \omega_{lm} \tilde{\Gamma}_{lm}.$$  

(64)

It is easily verified that:

$$\Gamma_{jk} = \Omega \tilde{\Gamma}_{jk} \Omega^\dagger.$$  

(65)

Due to the symmetry between the two basis, the relation (65) must be invertible, and this implies that the pseudo-observable $\Omega$ admits left inverse. By putting $k = j$ in (65) and remembering the first property of the dyads, it results:

$$I_j = \Gamma_{jj} = \Omega \tilde{\Gamma}_{jj} \Omega^\dagger = \Omega \tilde{I}_j \Omega^\dagger$$  

(66)

by summation over the index $j$ and recalling the closure relation (7), one obtains:

$$\Omega \Omega^\dagger = 1.$$  

The change of basis pseudo-observable so admits right inverse too and it is therefore invertible. Its inverse, besides, due to the previous relation, coincides with its transposition:

$$\Omega^{-1} = \Omega^\dagger \Rightarrow \Omega^\dagger \Omega = \Omega \Omega^\dagger = 1.$$  

(67)

The relations (67), expressed in terms of components, are equivalent, by virtue of (64) and (53), to the following:

$$\sum_l \omega^*_{lj} \omega_{lk} = \sum_l \omega_{lj} \omega^*_{kl} = \delta_{jk}.$$  

(68)

The matrix associated to $\Omega$ is, therefore, unitary. We will then call unitary every pseudo-observable satisfying (67).

Some useful properties, of immediate verification, of unitary pseudo-observables are the following:

(i) If $\Omega$ is an unitary pseudo-observable, its transposition, $\Omega^\dagger$, and its inverse, $\Omega^{-1}$, which, moreover, coincide with each other, are unitary too.

(ii) If $\Omega_1$ and $\Omega_2$ are two unitary pseudo-observables, their product, $\Omega_1 \Omega_2$, is an unitary pseudo-observable also.

As first important example of change of basis, we will regard that induced by the pseudo-observable:

$$S_{j_0j_1} := 1 - I_{j_0} - I_{j_1} + \Gamma_{j_0j_1} + \Gamma_{j_1j_0}.$$  

(69)

It is immediately verified that $S_{j_0j_1}$ is Hermitian and unitary. The change of basis associated to it exchanges in the dyads the projector $I_{j_0}$ with $I_{j_1}$.

Another interesting change of basis is that associated to the pseudo-observable:

$$\tilde{\Omega} := \sum_j e^{i\theta_j} I_j$$  

(70)

where the $\theta_j$ are real constants. It is immediate to verify that $\tilde{\Omega}$ is unitary and that the relative change of basis corresponds to one that makes it to pass from a dyad basis to an equivalent one, according to what seen at the end of section 4.2.

In such a manner the matrix structure of Heisenberg’s formulation of quantum mechanics is fully recovered and justified.
5. Conclusions

5.1. Summary of Postulates

At this point it is necessary to make a summary of the postulates introduced, commenting them appropriately to better understand their meaning:

(i) Each physical system is characterized by a set of measurable properties, called observables. An observable is characterized by all the possible outcomes of its measurement (the observable spectrum).

(ii) Two observables are said to be compatible if you can measure them simultaneously and independently, in the sense that measurement of one does not affect the measurement outcomes of the other. There do exist incompatible observables.

(iii) For two given compatible observables it is possible to define their sum (if they are homogeneous among themselves) and their product in terms of the their spectra. For every observable it is possible to determine at least a set of observables, of which it is a member, that is complete, i.e. such that each observable compatible with all the observables of the set may be expressed as a function of them. The complete set of compatible observables admits an unique basis of elementary projectors, such that each observable compatible with the observables of this set may be expressed as linear combinations of the projectors of the basis.

(iv) It is not possible to give a definition of the sum or the product of two incompatible observables based on physical based arguments. One can, however, consider the space $\mathcal{P}$, formed by the formal expressions obtained by sum and product of observables (compatible or not). Such expressions, which have physical meaning only when they involve solely observables compatible among themselves, are called pseudo-observables. The space of the pseudo-observables is an algebraic structure whose construction is bound only by the condition of being consistent with the properties of the algebra of compatible observables. From a physical point of view it is appropriate to follow a criterion of maximum significance, in the spirit of the Occam razor principle, according to which hypotheses will be taken to utmost reduce the entities without immediate physical interpretation. To this end, it is assumed that the space has a unitary ring structure with respect to sum and product operations. The existence of incompatible observables implies that the product is generally not commutative.

(v) In accordance with the criterion of maximum significance, a transposition operator ($\dagger$) is introduced, which reverses the observation order of a set of observables, characterized by the six properties (assumed by hypothesis) presented and commented at the beginning of the subsection 3.2.

(vi) It is introduced the pseudo-observable $i$, formally defined as a pseudo-observable that commutes with every other one and such that it results: $i^\dagger = -i$ and $i^2 = -1$. By means of the pseudo-observable $i$ it is possible write down the pseudo-observables in complex form and so put them in biunivocal correspondence with pairs of observable (real and imaginary part).

(vii) It is, last, assumed the validity of postulate on the dyadic forms, according to which, for each pair of elementary projectors of a basis, there is a non null associated dyadic form; every other dyadic form associated with the same pair of elementary projectors is obtained by multiplying that dyadic form by a suitable complex constant (i.e., dyadic forms relative to the same pair of elementary projectors of the same basis are linearly dependent on each other). The postulate on the dyadic forms completes, in a coherent manner, the definition of product among non
compatible observables and allows to extend the bases of elementary projectors, that span complete subspaces of compatible observables, to dyad bases, that span the whole space $\mathcal{Q}$ of the pseudo-observables. The decomposition of the pseudo-observables according the dyads of a basis allows, finally, one to write explicit expressions for sums and products of observables, compatible or not, or pseudo-observables.

5.2. Observers and meta-observers

In this paper it was shown how the algebraic structure of quantum mechanics is the unique one by means of which it is possible to give a coherent description of observing a set of properties, not all compatible with each other, and that introduces the minimum possible number of non-observable entities. The incomprehensible oddness of quantum mechanics thus becomes a necessary characteristic of the logic structure of a “reality” understood as a coherent construction derived from a set of observations, or as a result of a dialectical process of exchange of information between all the observers.

The measurement problem, transformations and time evolution in the framework of the algebra of the pseudo-observables will be treated in the second paper, where it will be also revealed the full structure of C*-algebra of this new algebra, and in the third.

In conclusion of this paper, we want to discuss about the immutability, in space and time, of the laws of physics.

Even if the problem of the eternal validity of physical laws ultimately refers to the historicizing of the evolution of the Universe and makes obvious reference to the historical becoming of things; it is also true that there is no History without Observers and without Observers able to communicate with each other through the space and the time. But communication aggregates the Observers, generating others of higher level (meta-observers). Thus the whole Humanity, in its historical becoming, acquires the status of a meta-observer: the result of the observational process of such a meta-observer is what we call (our) “physical reality”.

An important point to clarify, is that the aggregation of the observers into higher-level ones does not imply that the lasts sum up the perspective of their “parts”. As Anderson well stated in 1972 [56], “More Is Different”, and so it happens for the perspective of higher-level observer. To make a simple example, it is well known that our mind arises “in some manner” by the neuron activity. Each neuron receive inputs and send output, and there is no doubt that may be consider a simple observer. Our mind is the result of the exchange of information of our neurons, and so is an observer of higher level with respect to them, but cannot access the perspective of any single neuron: its perspective is different!

Even inanimate matter can act as a “passive” observer, bringing in its structure the traces of its own becoming. In the act in which a researcher reads and interprets those traces, an one-way communication is established, which still give rise to a meta-observer, of higher level respect to its components.

If we therefore consider all possible interactions between observers, active or passive, throughout the course of historical becoming, we finally find a single entity that we will call the universal observer (or super-observer), of level higher than any other observer, necessarily atemporal and delocalized.

In conclusion:

(i) What we call “reality” is an emergent property that arises by the internal communications of a closed network of observers.
The logical structure of a such “reality” reflects the inner logical structure of the observers, and this explains “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” quoted by Wigner in 1960.

The hierarchical structure of the observers, aggregated in higher and higher meta-levels till to that of the universal (meta-)observer, justifies the validity of the laws of physics in every place of the Universe and at all times.

Acknowledgments

I want to kindly thank David Edwards for a useful discussion that gave me the opportunity to better clarify the role of the observers in the theory and their hierarchical relationships.

References

[1] E. Schrödinger. Quantisierung als Eigenwertproblem (Erste Mitteilung). Ann. Phys., 79:361–376, 1926.
[2] E. Schrödinger. Quantisierung als Eigenwertproblem (Zweite Mitteilung). Ann. Phys., 79:489–527, 1926.
[3] E. Schrödinger. Quantisierung als Eigenwertproblem (Dritte Mitteilung: Störungstheorie, mit Anwendung auf den Starkeffekt der Balmerlinien). Ann. Phys., 80:437–490, 1926.
[4] E. Schrödinger. Quantisierung als Eigenwertproblem (Vierte Mitteilung). Ann. Phys., 81:109–139, 1926.
[5] E. Schrödinger. Abhandlungen zur wellenmechanik. J.A. Barth, Leipzig, 1927.
[6] P. A. M. Dirac. The Principles of Quantum Mechanics. Clarendon Press, Oxford, 1930.
[7] S. Boffi. Da Laplace a Heisenberg: un’introduzione alla meccanica quantistica e alle sue applicazioni. Biblioteca Delle Scienze, Pavia, 2010.
[8] M. Born. Zur Quantenmechanik der Stoßvorgänge. Z. Phys., 37:863, 1926.
[9] J. G. Cramer. The Transactional Interpretation of Quantum Mechanics. Rev. Mod. Phys., 58(3):649, 1986.
[10] H. Everett. “Relative State” Formulation of Quantum Mechanics. Rev. Mod. Phys., 29:454–462, 1957.
[11] B. S. DeWitt. Quantum Mechanics and Reality: Could the solution to the dilemma of indeterminism be a universe in which all possible outcomes of an experiment actually occur? Phys. Today, 23(9):30–40, 1970.
[12] R. B. Griffiths. Consistent histories and the interpretation of quantum mechanics. J. Stat. Phys., 36(1-2):219–272, 1984.
[13] C. Rovelli. Relational quantum mechanics. Int. J. Theor. Phys., 35:1637–1678, 1996.
[14] G.C. Ghirardi, A. Rimini, and T. Weber. A Model for a Unified Quantum Description of Macroscopic and Microscopic Systems. In L. Accardi et al., editor, Quantum Probability and Applications. Springer, Berlin, 1985.
[15] R. Penrose. The Emperor’s New Mind: Concerning Computers, Minds, and the Laws of Physics (Popular Science), pages 475–481. Oxford University Press, 1989.
REFERENCES

[16] W. Heisenberg. Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen. Z. Phys., 33:879–893, 1925.
[17] M. Born and P. Jordan. Zur Quantenmechanik. Z. Phys., 34:858–888, 1925.
[18] M. Born, W. Heisenberg, and P. Jordan. Zur Quantenmechanik II. Z. Phys., 35:557–615, 1925.
[19] L. de Broglie. Recherches sur la théorie des quanta. Ann. Phys., 10(3):22–128, 1925.
[20] E. Schrödinger. An Undulatory Theory of the Mechanics of Atoms and Molecules. Phys. Rev., 28(6):1049–1070, 1926.
[21] J. von Neumann. Mathematische Grundlagen der Quantenmechanik. Springer, Berlin, 1932.
[22] H. Weyl. Quantenmechanik und Gruppentheorie. Z. Phys., 46:1–46, 1927.
[23] H. Weyl. Gruppentheorie und Quantenmechanik. S. Hirzel, Leipzig, 1928. (Translation from Robertson, H.P., The Theory of Groups and Quantum Mechanics, Dover Publications, 1950).
[24] P. A. M. Dirac. The Lagrangian in Quantum Mechanics. Phys. Z. Sowjetunion, 3:64–72, 1933.
[25] R. P. Feynman. Space-Time Approach to Non-Relativistic Quantum Mechanics. Rev. Mod. Phys., 20(2):367–387, 1948.
[26] G. Birkhoff and J. von Neumann. The Logic of Quantum Mechanics. Ann. Math., 37:823–843, 1936.
[27] A. Einstein. In The Born-Einstein letters: correspondence between Albert Einstein and Max and Hedwig Born from 1916-1955, with commentaries by Max Born, page 91. Macmillan, 1971.
[28] J. Baggott. The Quantum Story: A History in 40 Moments, pages 116–117. Oxford University Press, New York, 2011.
[29] L. de Broglie. La nouvelle dynamique des quanta. In Paris Gauthier-Villars, editor, Électrons et photons: Rapports et discussions du cinquième Conseil de physique tenu à Bruxelles du 24 au 29 octobre 1927 sous les auspices de l’Institut international de physique Solvay, pages 103–132, 1928.
[30] G. Bacciagaluppi and A. Valentini. Quantum theory at the crossroads: Reconsidering the 1927 Solvay Conference. Cambridge University Press, 2013.
[31] G. Hermann. Die naturphilosophischen Grundlagen der Quantenmechanik. Abhandlungen der Fries’schen Schule, 6(2):69–152, 1935.
[32] D. Bohm. A suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, I. Phys. Rev., 85(2):166–179, 1952.
[33] D. Bohm. A suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, II. Phys. Rev., 85(2):180–193, 1952.
[34] J. S. Bell. On the Problem of Hidden Variables in Quantum Mechanics. Rev. Mod. Phys., 38(3):447–452, July 1966.
[35] N. D. Mermin. Hidden variables and the two theorems of John Bell. Rev. Mod. Phys., 65:803–815, 1993.
[36] J. S. Bell. Speakable and Unspeakable in Quantum Mechanics. Cambridge University Press, 1987.
[37] R. Colbeck and R. Renner. No extension of quantum theory can have improved predictive power. Nat. Commun., 2(1):1–5, 2011.
REFERENCES

[38] G. Ghirardi and R. Romano. Ontological Models Predictively Inequivalent to Quantum Theory. Phys. Rev. Lett., 110:170404, 2013.

[39] T. Maudlin. Why Bohm’s Theory Solves the Measurement Problem. Philos. Sci., 62(3):479–483, 1995.

[40] N. P. Landsman. Algebraic quantum mechanics. In K. Hentschel D. Greenberger and F. Weinert, editors, Compendium of Quantum Physics: Concepts, Experiments, History and Philosophy, pages 6–9. Springer, Berlin, 2009.

[41] P. Jordan and E. von Neumann. On an algebraic generalization of the quantum mechanical formalism. Ann. Math., 35:29–64, 1934.

[42] F. J. Murray and J. von Neumann. On Rings of Operators I. Ann. Math., 37:116–229, 1936.

[43] F. J. Murray and J. von Neumann. On Rings of Operators II. Trans. Amer. Math. Soc., 41:208–248, 1937.

[44] J. von Neumann. On Rings of Operators III. Ann. Math., 41:94–161, 1940.

[45] F. J. Murray and J. von Neumann. On Rings of Operators IV. Ann. Math., 44:716–808, 1943.

[46] J. von Neumann. On Rings of Operators V. Ann. Math., 50:401–485, 1949.

[47] M. Redei. Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead). Stud. Hist. Phil. Mod. Phys., 27:493–510, 1996.

[48] I. M. Gelfand and M. A. Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. Rec. Math. [Mat. Sbornik] N.S., 12(54):197–217, 1943.

[49] I. E. Segal. Irreducible representations of operator algebras. Bull. Amer. Math. Soc., 53(2):73–88, 1947.

[50] C. A. Fuchs and R. Schack. A Quantum-Bayesian Route to Quantum-State Space. Found. Phys., 41(3):345–356, 2011.

[51] C. A. Fuchs, N. D. Mermin, and R. Schack. An Introduction to QBism with an Application to the Locality of Quantum Mechanics. Am. J. Phys., 82(8):749–754, 2014.

[52] C. A. Fuchs. QBism, the Perimeter of Quantum Bayesianism. (arXiv:1003.5209)

[53] A. Petersen. The Philosophy of Niels Bohr. Bulletin of the Atomic Scientists, XIX(7):12, 1963.

[54] C. A. Fuchs. Interview with a Quantum Bayesian. (arXiv:1207.2141)

[55] H. Weyl. Mind and Nature: Selected Writings on Philosophy, Mathematics, and Physics., volume 17. 2009.

[56] P. W. Anderson. More is different. Science, 177(4047):393–396, 1972.

[57] E.P. Wigner. The Unreasonable Effectiveness of Mathematics in the Natural Sciences. Richard Courant Lecture in Mathematical Sciences delivered at New York University, may 11, 1959. Communications on Pure and Applied Mathematics, 13(1):1–14, 1960.