Covariant Model for Relativistic Three-Body Systems

Philippe Droz-Vincent
LUTH, Observatoire de Paris-Meudon
Place Jules Janssen, F-92195 Meudon, France

Abstract

The system is described by three mass-shell constraints. When at least two masses are equal, this picture has a reasonable nonrelativistic limit. At first post-Galilean order and provided the interaction is not too much energy-dependent, the relativistic correction is tractable like a conventional perturbation problem. A covariant version of harmonic oscillator is given as a toy model.

A system of three particles can be covariantly described by three mass-shell constraints, involving an interaction term referred to as potential. These constraints must reduce to three independent Klein-Gordon (or Dirac) equations in the absence of potential. In any case, they determine the evolution of a wave function which depends on three four-dimensional arguments, say $p_a$ with $a, b = 1, 2, 3$, if we chose the momentum representation of quantum mechanics.

Naturally, the potential depends on both configuration and momentum variables, $q_a, p_b$, and must allow for mutual compatibility of the constraints. Moreover it happens that, just like in the Bethe-Salpeter approach, manifest covariance is paid by the presence of redundant degrees of freedom.

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which the elimination is by no means straightforward (in contrast to the two-body case). These two important issues have been considered earlier by H. Sazdjian [1] who aimed at solving the general \(n\)-body case and proposed an approximate solution.

Specially dealing with the three-boson case, we have recently exhibited in closed analytic form a new set of variables \(q'_a, p'_b\). In terms of these new variables, admissible expressions for the potential are explicitly available, and two superfluous degrees of freedom can be eliminated [2]. Setting \(P = \sum p\) we linearly introduce relative variables

\[
z_A = q_1 - q_A, \quad y_A = \frac{P}{3} - p_A, \quad A = 2, 3
\]

and similar formulas for \(z'_A, y'_B\) in terms of \(q'_a, p'_b\).

The mass-shell constraints can be equivalently replaced by their sum and differences; it is convenient to set

\[
\nu_A = \frac{1}{2}(m_1^2 - m_A^2).
\]

The difference equations, in their original form, yield no simplification. But we perform a quadratic change among the momenta, say \(p_a \rightarrow p'_a\), or equivalently \(P, y_A \rightarrow P', y'_A\). in order to ensure the elimination of two redundant degrees of freedom; this change is characterized by

\[
(p_1 - p_A)(p_1 + p_A) = (p'_1 - p'_A) \cdot P
\]

whereas \(P' = P\) and the transverse parts of the momenta remain unaffected, say \(\tilde{y}' = \tilde{y}\), where the tilde on any four-vector refers to its transverse part with respect to \(P\).

Of course, this procedure generates a change of canonical variables [2], in particular we obtain new configuration variables, \(z'_A\).

Three-dimensional Reduction
We impose a sharp value of the total linear momentum, it is a timelike vector \(k\), and we define \(k \cdot k = M^2\).

**Notations:** The hat on any vector refers to its transverse part with respect to \(k\).

**Underlining** any dynamical variable indicates that, in its expression, we substitute \(k\) for \(P\) and take into account equation the difference equations

\[
3y'_A \cdot k \Psi = (4\nu_A - 2\nu_B)c^2 \Psi
\]  

(1)
We factorize out the relative energies; as a result the sum equation becomes

\[
(3 \sum m^2 - M^2)c^2 \psi = 6(\hat{y}_2^2 + \hat{y}_3^2 + \hat{y}_2 \cdot \hat{y}_3)\psi + (6M^2c^2\Xi + 18V)\psi
\]  

(2)

for a reduced wave function \(\psi\) which depends only on the transverse relative momenta \(\hat{y}_A = \hat{y}_A\).

The meaning of \(\Xi\) is purely kinematic; this term depends only on the momenta and can be expressed in terms of their transverse part and \(P\). Here \(V\) denotes the relativistic potential; it may be phenomenological or motivated by considerations of field theory. In particular it may be formally constructed as a sum of two-body terms, like in equation (5) below; so doing one uses the shape of two-body potentials but (for the sake of compatibility) with the new three-body variables as arguments. Not only the total momentum \(P\) but also the new configuration variables \(z'_A\) mix the two-body clusters, which amounts to automatically incorporate three-body forces. Admissible potentials entail that \(V\) is a function of the new variables \(z'_2, z'_3\) and \(M^2c^2\).

The reduced equation (2) is actually a nonconventional eigenvalue problem, where the operator to be diagonalized explicitly depends on its eigenvalue. This situation is by no means a special drawback of our model, in fact it is common in relativistic quantum mechanics [3], but it would make a general treatment rather involved.

On the other hand, it is natural to expand the formulas in powers of \(1/c^2\) and to look for the nonrelativistic limit. For arbitrary masses, the term \(M^2c^2\Xi\) generally blows up, which leads to consider, instead of (2) an alternative combination of the mass-shell constraints.

**Two equal masses.**

Drastic simplifications arise when two masses are equal, say \(m_2 = m_3 = m\), equivalently \(\nu_2 = \nu_3 = \nu\). We find that the Galilean limit of our eigenvalue problem is a Schroedinger equation with effective (or Galilean) masses that are generally distinct from the constituent masses \(m_a\). However they still coincide with the constituent masses, at first order in the ”mass-dispersion index” \(\nu/m^2\).

**Three Equal Masses.**

When \(m_a = m\) for all particles, equation (2) can be written as follows, using the rest frame

\[
\lambda\psi = (y_2^2 + y_3^2 + y_2 \cdot y_3)\psi - 3V\psi - M^2c^2\Xi\psi
\]  

(3)
with $6\lambda = (M^2 - 9m^2)c^2$. Now the last term in (3) remains finite in the nonrelativistic limit. Indeed we can write $Mc^2\Xi = \frac{1}{M^2c^2}\Gamma(0) + O(1/c^4)$ where

$$\Gamma(0) = \frac{3}{4} \left\{ (\hat{y}_2')^2 + (\hat{y}_3')^2 + 4(\hat{y}_2 \cdot \hat{y}_3) + 2(\hat{y}_2^2 + \hat{y}_3^2)(\hat{y}_2 \cdot \hat{y}_3) - \hat{y}_2^2\hat{y}_3^2 \right\}$$ (4)

At first order in $1/c^2$ we can, in $\Xi$, replace $M^2$ by $9m^2$, which is independent from $\lambda$. Thus we replace $Mc^2\Xi$ by $\Gamma(0)/9m^2c^2$. If the relativistic ”potential” $V$ does not depend on $P^2$, or if this dependence is of higher order, equation (3) becomes a conventional eigenvalue problem, tractable by perturbation theory. The last term in (3) brings a negative correction to the value $\lambda_{NR}$ furnished by the nonrelativistic approximation, say

$$\lambda = \lambda_{NR} - <\Gamma(0)>/9m^2c^2$$

if $\lambda_{NR}$ corresponds to a nondegenerate level. One has to calculate $<\Gamma(0)>$ in the eigenstate solution of the nonrelativistic problem.

**Harmonic Oscillator**

A covariant version of the harmonic potential is given by

$$V = 2\kappa \left\{ (\hat{z}_2')^2 + (\hat{z}_3')^2 - \hat{z}_2' \cdot \hat{z}_3' \right\}$$ (5)

hence $\hat{V}$ in terms of $\hat{z}_A' \cdot \hat{z}_B' = -z_A'^2 \cdot z_B'^2$. In the nonrelativistic limit we recover the naive $SU_6$ invariant Schrödinger equation. At the first post-Galilean approximation, $M^2$ can be replaced by $9m^2$, neglecting the dependence on total energy in the reduced equation. At this stage, the eigenvalue problem amounts to diagonalize a nonrelativistic harmonic oscillator, with potential $V_{NR} = -3\hat{V}/m$, submitted to a momentum-dependent perturbation. Expressed in terms of Jacobi-like coordinates, namely $R_2 = -z_2' + z_3'$, $R_3 = (z_2' + z_3')/\sqrt{3}$ and their conjugate momenta, the unperturbed ground state is a Gaussian. If the unit of length is chosen such that $\kappa = \frac{2}{9}$, one finds $<\Gamma(0) >= 11 + 1/4$.

This approach is intended for applications to confining interactions; future work should implement spin and investigate a possible contact with recent developments [4] of the BS approach.
References

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