QUANTUM MECHANICS OF THE ELECTRIC CHARGE

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ABSTRACT

A simple argument against the existence of magnetic monopoles is given. The argument is an important part of the quantum theory of the electric charge developed by the author.

“The same modification of the (Maxwell –Lorentz) theory which contains $e$ as a consequence, will also have the quantum structure of radiation as a consequence.”

Albert Einstein
(Phys. Zeit. 10 (1909) 192)

1. Introduction

This paper is dedicated to Professor Yakir Aharonov on the occasion of his 60th birthday. The subject of the paper, quantum mechanics of the electric charge, is based on the notion of phase, this elusive concept which has always fascinated Professor Aharonov.

The electric charge $Q$ and the phase $S(x)$ of a (second quantized) charged system are canonically conjugated variables:

$$[Q, S(x)] = i\epsilon, \quad (\hbar = 1 = c) \quad (1)$$

$\epsilon$ being the elementary charge. Proof of this theorem is given in $^1$. Here I will make only two rather obvious comments.

Eq.(1) does explain quantization of the electric charge $Q$ in units equal to the constant $\epsilon$:

$$Q = n\epsilon, \quad n = 0, \pm 1, \pm 2, \ldots$$

It does not, however, explain the universality of the electric charge i.e. the fact that e.g. the electric charge of the electron seems to be mathematically equal to the electric charge of the proton. Indeed, since the constant $\epsilon$ in Eq.(1) is arbitrary, we cannot exclude theoretically a situation in which $\epsilon = \epsilon_1$ for one charged system and $\epsilon = \epsilon_2 \neq \epsilon_1$ for another system.

2. The phase $S(x)$ can be uniquely determined at the spatial infinity

$x$ in Eq.(1) is an arbitrary spatio-temporal point. Let us imagine that $x$
tends to the spatial infinity:
\[ xx \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \to -\infty. \]

Mathematically-minded readers will object that we are not allowed to fix, even in the form of a limit, the argument of an operator-valued distribution. True. The argument which follows is physical rather than mathematical, it constitutes a piece of theoretical rather than mathematical physics.

At the spatial infinity there is only one function which can possibly play the role of phase. This function must be equal to
\[ S(x) = -e x^\mu A_\mu(x), \tag{2} \]
where \( e \) is a constant proportionality factor and \( A_\mu(x) \) is the electromagnetic potential. To see this one has to note that at the spatial infinity the electromagnetic field is free,
\[ \partial^\mu F_{\mu\nu} \equiv 4\pi j_\nu = 0 \]
and homogeneous of degree \(-2\), \( F_{\mu\nu}(\lambda x) = \lambda^{-2} F_{\mu\nu}(x) \) for each \( \lambda > 0 \). The field is free because the electric current \( j_\nu \), being carried by massive particles, must be confined to the future and past light cone. It must be homogeneous of degree \(-2\) because, as seen e.g. in the static case, the charge generated monopole term dominates dipole and higher terms.

Consider a classical electromagnetic field which is free and homogeneous of degree \(-2\); assume that its potential is homogeneous of degree \(-1\), which is natural. Let us form two vectors,
\[ F_{\mu\nu}(x) x^\nu \quad \text{and} \quad \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} x_\nu F_{\rho\sigma}(x), \]
where \( x \) is the radius vector in the Lorentzian reference frame in which the homogeneity condition holds.

The two vectors given above determine the tensor \( F_{\mu\nu} \) in a purely algebraic way. Both these vectors are gradients of homogeneous of degree zero functions:
\[ F_{\mu\nu}(x) x^\nu = \partial_\nu e(x), \quad \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} x_\nu F_{\rho\sigma}(x) = \partial^\mu m(x). \]

\( e(x) \) and \( m(x) \) denote “electric” and “magnetic” parts respectively. \( e(x) \) can be easily calculated:
\[ F_{\mu\nu}(x) x^\nu = [\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)] x^\nu = \partial_\mu [A_\nu(x) x^\nu] - \delta^\nu_\nu A_\nu(x) - x^\nu \partial_\nu A_\mu(x) = \partial_\mu [x^\nu A_\nu(x)] \]
because
\[ x^\nu \partial_\nu A_\mu(x) = -A_\mu(x) \]
from the Euler theorem on homogeneous functions.
I maintain that \( m(x) \) must be a constant. This is an argument against the existence of magnetic monopoles which, to the best of my knowledge, has never been put forward before. (The argument given by Dr. Herdegen\(^3\) is different.)

To see this let us calculate the Lagrangian density

\[
dx^0 dx^1 dx^2 dx^3 F_{\mu\nu} F^{\mu\nu}
\]

for a homogeneous of degree \(-2\) field \( F_{\mu\nu} \), using the spherical coordinates

\[
\begin{align*}
x^0 &= \xi^0 \sinh \xi^1, \\
x^1 &= \xi^0 \cosh \xi^1 \sin \xi^2 \cos \xi^3, \\
x^2 &= \xi^0 \cosh \xi^1 \sin \xi^2 \sin \xi^3, \\
x^3 &= \xi^0 \cosh \xi^1 \cos \xi^2,
\end{align*}
\]

\(0 < \xi^0 < \infty, -\infty < \xi^1 < +\infty, 0 \leq \xi^2 \leq \pi, 0 \leq \xi^3 < 2\pi.\)

These coordinates cover in an obvious way the spatial infinity we are interested in. Note that \( \xi^0 \) is a space-like coordinate while \( \xi^1 \) is a time-like coordinate. A simple calculation gives

\[
dx^0 dx^1 dx^2 dx^3 F_{\mu\nu} F^{\mu\nu} = 2 \frac{d\xi^0}{\xi^0} \sqrt{g} d\xi^1 d\xi^2 d\xi^3 \left( -\frac{\partial}{\partial \xi^0} e \frac{\partial}{\partial \xi^0} e + \frac{\partial}{\partial \xi^i} m \frac{\partial}{\partial \xi^k} m \right).
\]

Here

\[
g_{ik} = (\xi^0)^{-2} g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^k}, \quad i, k = 1, 2, 3,
\]

is the metric on the spatial infinity.

The Lagrangian density (3) is seen to be a difference of two identical Lagrangian densities. Thus only one of them can have the correct sign i.e. the sign which, upon quantization, would give a positive definite inner product. The part with the right sign is called electric, the part with the wrong sign is called magnetic and must be put equal to zero.

Now, the Gauss theorem says that the total charge \( Q \) is determined by the electromagnetic field at the spatial infinity. In the quantum theory the charge operator \( Q \) must have its canonically conjugated variable \( S(x) \). Thus \( S(x) \) must have a “tail” which does not vanish even at the spatial infinity. We have seen, however, that there is exactly one function, namely \( x^\mu A_\mu(x) \), which can play the role of the “tail”. Hence, there must exist a constant \( e \) such that at the spatial infinity

\[
S(x) = -e x^\mu A_\mu(x).
\]

The constant \( e \) in this equation is identical with the constant \( e \) in Eq.(1). This is a hypothesis substantiated in the next section.

3. The proportionality factor in the phase

The two equations
constitute together a closed theory, the quantum mechanics of the electric charge. It is important to understand correctly the epistemological status of both equations. The first equation is simply a theorem in the Q.E.D. which, by continuity, is assumed to hold also at the spatial infinity. The second equation is a hypothesis; one can give several arguments supporting Eq.(2) but all those arguments do not amount to a proof. Here are two simple arguments, to be added to those which I have given elsewhere.

Take the Coulomb field of the charge \( Q \) at rest:

\[
A_0 = \frac{Q}{r}, \quad A_1 = A_2 = A_3 = 0.
\]

Its phase, according to Eq.(2), is

\[
S(x) = -e \frac{Q}{r} t = -eQ \frac{t}{r}.
\]

During the eternity of time available at the spatial infinity,

\[-r < t < r,
\]

the phase \( S(x) \) changes from \( eQ \) to \( -eQ \). Take now the hydrogen atom with the nuclear charge \( Q \) and the electron charge \( e \) and assume that the radius of its circular orbit tends to infinity. During the eternity of time available,

\[-r < t < r,
\]

the electromagnetic phase of the electron wave function,

\[-e \int A_\mu(x) dx^\mu,
\]

will change by the same amount:

\[-e \int_{-r}^{r} \frac{Q}{r} dt = -2eQ.
\]

Thus the phase given by Eq.(2) changes as the true phase of the electron wave function in an infinitely large hydrogen atom.

The phase of the Coulomb field,

\[
S(x) = -\frac{eQ}{r} t
\]
may be compared with the phase of the wave function of a stationary state, $-Et$, $E$ being the energy of the stationary state. Thus $S(x)$ looks like the phase of a stationary state driven by the Coulomb energy $eQ/r$. Again, this is not a proof but a heuristic argument supporting Eq.(2).

Equations (1) and (2) together do allow to explain the universality of the electric charge. To be more precise, they allow to prove the following theorem: the total charge of the universe is always a multiple of a single constant. To apply this to the electron or to the proton one must be able to estimate the accuracy with which, under specific observational circumstances, they can be considered as isolated universes. The experimental equality of electron’s and proton’s charge shows that this accuracy is indeed extremely high.

4. Acknowledgement

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5. References

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