THE BLOCH GROUPS AND SPECIAL VALUES OF DEDEKIND ZETA FUNCTIONS

CHAOCHAO SUN AND LONG ZHANG

Abstract. In this paper, we compare the two definitions of Bloch group, and survey the elements in Bloch group. We confirm the Lichtenbaum conjecture on the field \( \mathbb{Q}(\zeta_p) \) under the assumption the truth of the base of the Bloch group of \( \mathbb{Q}(\zeta_p) \) and the relations of \( K_2 \) group. We also study the Lichtenbaum conjecture on non-Galois fields. By PARI, we get some equations of the zeta functions on special values and the structure of tame kernel of these fields.

1. Introduction

The special values of zeta function of number fields are an interesting field in number theory. When we consider the residue of zeta function, there is a class number formula as following

\[
R_1 = -\frac{w}{h} \lim_{s \to 0} \zeta_F(s) s^{-(r_1 + r_2 + 1)}
\]

where \( R_1 \) is the Dirichlet regulator of the field \( F \), \( \zeta_F(s) \) is the Dedekind’s zeta function, \( w \) is the root number of unity and \( h \) is the class number.

In order to generalize the formula (1.1) to higher K-theory, Borel [3] has introduced morphisms

\[
r : K_{2m-1}(\mathcal{O}_F) \to V_m
\]

where \( m \geq 2 \), \( \mathcal{O}_F \) is the algebraic integer of number field \( F \), \( V_m \) is a real vector space of dimension

\[
\dim_{\mathbb{R}} V_m = d_m = \begin{cases} 
    r_1 + r_2 & \text{if } m \text{ is odd}, m > 0, \\
    r_2 & \text{if } m \text{ is even}
\end{cases}
\]

where \( r_1 \) (respectively \( r_2 \)) is the number of real (respectively complex) places of \( F \). Borel has proved that \( r(K_{2m-1}(\mathcal{O}_F)) \) is a lattice of \( V_m \), so the rank of \( K_{2m-1}(\mathcal{O}_F) \) is \( d_m \). Let \( R_m(F) \) be a twisted version of the \( m \)th Borel regulator (see [4]), the twisted regulator map \( r_m \) being a map

\[
r_m : K_{2m-1}(\mathcal{O}_F) \to [(2\pi i)^{m-1}\mathbb{R}]^{d_m}
\]

Borel proved that, up to a rational factor, \( R_m(F) \) is equal to \( \zeta_F^*(1-m) \), the first non-vanishing Taylor coefficient of \( \zeta_F(s) \) at \( s = 1-m \). Lichtenbaum’s conjecture [13] (as

\[
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\]

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modified by Borel [4], tries to interpret this rational factor and asks whether for all number fields and for any integer $m \geq 2$ there is a relation of the form

$$\lim_{s \to 1^{-m}} \zeta_F(s)(s - 1 + m)^{-d} = \pm \frac{K_{2m-2}(\mathcal{O}_F)}{K_{2m-1}(\mathcal{O}_F)^\text{tor}} \cdot R_m(F).$$

For $m = 2$ and $F$ totally-real abelian it has been proved (up to a power of 2) by Mazur and Wiles [13] as a consequence of their proof of the main conjecture of Iwasawa theory. In [12], Kolster, Nguyen Quang Do and Fleckinger have proved a modified version of the conjecture (also up to a power of 2) for all abelian fields $F$ and $m \geq 2$.

When $m = 2$, Bloch [2] suggested and D.Burns, R.de Jeu, H.Gangl [8] finally proved that Borel’s regulator map can be given in terms of the Bloch-Wigner dilogarithm $D_2(z)$ as a map on the Bloch group $B(F)$. While for Bloch group, there exist two kinds of definitions, see [6] and [7]. In section 1, we compare the two definitions of Bloch of number field (Theorem 2.1) and discuss the relations of elements in Bloch group. Let $\tilde{R}_2(F)$ be the second dilogarithmic regulator (see [7]), $w_2(F)$ be the number of roots of unity in the compositum of all quadratic extensions of $F$. Then the Lichtenbaum conjecture can be read as follows

$$(1.2) \quad |\zeta_F^*(-1)| = \frac{\tilde{R}_2(F) \cdot K_2(\mathcal{O}_F)}{w_2(F)},$$

In section 2, assume two conjectures, we can prove the above version Lichtenbaum conjecture on the field $\mathbb{Q}(\zeta_p)$. In section 3, we get some fields which are not Galois. On these fields, we get some functional equations on zeta functions and the dilogarithm functions when $s = 2$ by comparing the numerical results. Using PARI, we get the structures of $K_2(\mathcal{O}_F)$.

2. The Bloch group

Let $F$ be a field of char $(F) = 0$, $\mathbb{Z}[F] := \bigoplus_{1 \neq a \in F^\times} \mathbb{Z}[a]$ be a free abelian group generated by base $[a]$. We have a natural map

$$\partial : \mathbb{Z}[F] \to F^\times \wedge F^\times := F^\times \otimes F^\times / (a \otimes (-a))$$

$$[a] \mapsto a \otimes (1 - a)$$

Let $H$ be the subgroup generated by the elements $[a] + [1 - a], [a] + [a^{-1}], [a] + [b] + [\frac{1-a}{1-ab}] + [1 - ab] + [\frac{1-b}{1-ab}], a, b \in F^\times \setminus \{1\}, ab \neq 1$. It is easy to check that $H \subseteq \ker \partial$ and the Bloch group of $F$ is defined to be

$$B(F) := \ker \partial / H$$

In Suslin’s paper [10], the Bloch group is defined by another form, we state it as follows. Let $\varphi$ be a map

$$\varphi : \mathbb{Z}[F] \to F^\times \otimes F^\times / (x \otimes y + y \otimes x)$$

$$[a] \mapsto a \otimes (1 - a)$$

First, we have (see [10])

$$\varphi([x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y}\right] + \left[\frac{1 - x}{1 - y}\right]) = x \otimes (\frac{1-x}{1-y}) + (\frac{1-x}{1-y}) \otimes x = 0.$$
Let $I$ be the subgroup of $\mathbb{Z}[F]$ generated by the elements $[x] - [y] + \frac{[x]}{[y]} - \frac{1-\frac{1}{x}}{1-\frac{1}{y}}$ and $[1-\frac{1}{y}]$. Then the Bloch group in Suslin’s paper is defined by
\[
B(F) := \ker \varphi/I
\]

Although the definitions of Bloch group are a little different, in fact, they differ at most by torsion. We have the following result

**Theorem 2.1.** The two kinds of Bloch groups $B(F)$ and $B(F)$ are different by torsion. Furthermore,
\[
B(F) \otimes \mathbb{Z}[\frac{1}{6}] \cong B(F) \otimes \mathbb{Z}[\frac{1}{6}].
\]

**Proof.** Since
\[
x \otimes y + y \otimes x = xy \otimes (-xy) - x \otimes (-x) - y \otimes (-y)
\]
we have
\[
2(a \otimes (-a)) = 2(a \otimes (-1)) + 2(a \otimes a) = a \otimes a + a \otimes a,
\]
where $a, x, y \in F^\times$. So, there exists the following inclusions
\[
(2.1) \quad \ker \varphi \subseteq \ker \partial, 2 \ker \partial \subseteq \ker \varphi.
\]

Another, we have $I \subseteq H$, we can show the inclusion from the generator:

\[
(2.2) \quad \frac{[x] - [y]}{x} + \frac{[y]}{x} - \frac{1-\frac{1}{x}}{1-\frac{1}{y}} + \frac{1-\frac{1}{x}}{1-\frac{1}{y}} = ([y^{-1}] + [yx^{-1}] + [\frac{1-\frac{1}{x}}{1-\frac{1}{y}}] + [1-\frac{1}{x^{-1}}] + [\frac{1-\frac{1}{x}}{1-\frac{1}{y}}] + ([x] + [x^{-1}]) + ([1-\frac{1}{x}] + [1-\frac{1}{y}]),
\]

By results in [16], we have
\[
(2.3) \quad 2([x] + [x^{-1}]), 6([x] + [1-x]) \in I.
\]

So, by (2.2), (2.3), we have $6H \subset I$.

Then there is an exact sequence
\[
(2.4) \quad 0 \to (\ker \varphi \cap H)/I \to \ker \varphi/I = B(F) \to B(F) = \ker \partial/H \to \ker \partial/\ker \varphi \to 0.
\]

Because $6(\ker \varphi \cap H)/I = 0$, we obtain that $B(k)$ and $B(k)$ are different by torsion. Tensor with $\mathbb{Z}[\frac{1}{6}]$ on (2.4), the flatness of $\mathbb{Z}[\frac{1}{6}]$ leads to get the isomorphism
\[
B(F) \otimes \mathbb{Z}[\frac{1}{6}] \cong B(F) \otimes \mathbb{Z}[\frac{1}{6}],
\]

The Bloch group are related with zeta function by Bloch-Wigner function. Now let us introduce the Bloch-Wigner function:
\[
D(z) = -\text{Im} \int_0^z \frac{\log(1-t)}{t} dt + \text{arg}(1-z) \cdot \log |z|, \quad z \in \mathbb{C}.
\]
$D(z)$ is real analytic on $\mathbb{C} \setminus \{0,1\}$ and continuous at $0,1$. It satisfies that
\[
D(z) = -D(z), \quad D(z) = -D(1-z) = -D(z^{-1})
\]
so, we have
\[
D(z_1) + D(z_2) + D\left(\frac{1-z_1}{1-z_1z_2}\right) + D(1-z_1z_2) + D\left(\frac{1-z_2}{1-z_1z_2}\right) = 0
\]
where $z,z_1, z_2 \in \mathbb{C}, z_1z_2 \neq 1$.

**Example 2.2.** Let $\zeta_p = e^{2\pi i/5}$. Then in the field $\mathbb{Q}(\zeta_3)$, $\zeta_3 = -\zeta_6$. So, we have
\[
D(\zeta_3) = \Im \left( \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^2} \right) = \frac{\sqrt{3}}{2} \sum_{n=1}^{\infty} \chi_3(n) \frac{1}{n^2} = \frac{\sqrt{3}}{2} L(\chi_3,2)
\]
where $\chi_3$ is the primitive Dirichlet character with conductor $3$.

Similarly, we have
\[
D(\zeta_6) = \frac{\sqrt{3}}{2} (L(\chi_6,2) + \frac{1}{4} L(\chi_3,2))
\]
where $\chi_6$ is the primitive Dirichlet character with conductor $6$.

On the other hand, we have
\[
L(\chi_3,2) = 1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \cdots
\]
\[
= (1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \cdots) - (\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{8^2} - \frac{1}{10^2} + \cdots)
\]
\[
= L(\chi_6,2) - \frac{1}{4} L(\chi_3,2)
\]
So, we have
\[
L(\chi_6,2)/L(\chi_3,2) = \frac{5}{4}
\]
By (2.5), (2.6) and (2.7), we have $D(\zeta_6)/D(\zeta_3) = \frac{5}{4}$. This relation can be reflected onto the elements of Bloch group. In fact, we have the following result

**Claim 1** \[ 2[\zeta_6] = 3[\zeta_3] \in B(\mathbb{Q}(\zeta_3)) \]

**Proof.** Suppose $\zeta_n \in F$, then we have $[x^n] = n([x] + [\zeta_n x] + \cdots + [\zeta_n^{n-1} x])$ in $B(F)$. That’s because
\[
x^n \otimes (1-x^n) = n(x \otimes ((1-x)\cdots(1-x^{n-1}x)))
\]
\[
= n(x \otimes (1-x) + \cdots + x \otimes (1-x^{n-1}x))
\]
\[
= n(x \otimes (1-x) + \cdots + \zeta_n^{n-1}x \otimes (1-x^{n-1}x)).
\]
Now, let $\zeta_n = -1, x = \zeta_6 \in \mathbb{Q}(\zeta_3)$. Then in $B(\mathbb{Q}(\zeta_3))$ we get
\[
2[\zeta_6] = [\zeta_3] - 2[-\zeta_6] = [\zeta_3] - 2[\zeta_3^{-1}] = 3[\zeta_3].
\]

**Example 2.3.** Let $\zeta_5 = \exp(2\pi i/5)$, $x_1 = 1 + \zeta_5 + \zeta_5^2, x_2 = -\zeta_5^3$. In [5], Browkin found that $b_1 := 2[x_1] + 4[x_2] \in B(\mathbb{Q}(\zeta_5))$. Let $\sigma$ be an automorphism of $B(\mathbb{Q}(\zeta_5))$ such that $\sigma(\zeta_5) = \zeta_5^2$. Then $b_2 := 2[\sigma(x_1)] + 4[\sigma(x_2)] \in B(\mathbb{Q}(\zeta_5))$. Let $a_1 = \ldots$
5[ζ_5], a_2 = 5[ζ_5^2] ∈ B(Q(ζ_5)). In fact, we have a_1 = b_1, a_2 = b_2 in B(Q(ζ_5)). Now, we show how to get it. Because (1 + ζ_5 + ζ_5^2)(1 + ζ_5^2) = 1, we get

\[
b_1 = 2\left(\frac{1}{1 + \zeta_5}\right) + 4[-\zeta_5^2] \\
= -2[1 + \zeta_5^3] + 2[\zeta_5^3] - 4[\zeta_5^4] \\
= 2[-\zeta_5^3] + 2[\zeta_5] - 4[\zeta_5^4] \\
= [\zeta_5^5] + 4[\zeta_5] = 5[\zeta_5] \\
= a_1
\]

Moreover, \(\zeta_p(-1) = 0.0248111839\). By the Lichtenbaum conjecture:

\[
|\zeta_p(-1)| = \frac{\widetilde{R}_2(F) \cdot \#K_2(O_F)}{w_2(F)}.
\]

It is easy to see that \(w_2(F) = 120\). Assuming dilogarithmic lattice \(\Lambda\) in \(R^2\) generated by the vectors

\[
(\widetilde{D}(b_1), \widetilde{D}(\sigma(b_1))) \quad \text{and} \quad (\widetilde{D}(b_2), \widetilde{D}(\sigma(b_2)))
\]

is full lattice, where \(\widetilde{D}(z) = \frac{1}{2}D(z)\). Substituting the above numerical data we get \(\#K_2(O_F) = 1\). It is proven in [20] that \(\#K_2(O_F) = 1\).

3. Special value of zeta function of \(Q(\zeta_p)\)

Now, we want to study the Lichtenbaum conjecture on the special value of zeta function at \(-1\) in the case \(F = Q(\zeta_p)\), where \(p\) be an odd prime number, \(\zeta_p\) be a primitive root of unity. It is easy to see that \(p[\zeta_p^i] \in B(Q(\zeta_p)), i = 1, \cdots, \frac{p-1}{2}\). The subgroup \(Z\) generated by \(p[\zeta_p^1], p[\zeta_p^2], \cdots, p[\zeta_p^{\frac{p-1}{2}}]\) is a finite index subgroup of the Bloch group \(B(Q(\zeta_p))\). The covolume of \(Z\) in \(R^{\frac{p-1}{2}}\) under the regulator map \(D\) is denoted by \(R\).

**Theorem 3.1.** The covolume of \(Z\) is

\[
R = (2\pi)^{\frac{3(p-1)}{2}} \prod_{\chi \text{odd char.}} |L(\chi, 2)|,
\]

where \(\chi\) is the character of the group \((\mathbb{Z}/p)^*\).

**Proof.** According to [12], P.713, we have

\[
R = |\det \alpha| = \prod_{\chi \text{odd char.}} |\sum_{a \in (\mathbb{Z}/p)^*} \frac{p\chi(a)D_2(e^{2\pi ia/p})}{2\pi}|,
\]

where \(D_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2}, \chi\) is the character of the group \((\mathbb{Z}/p)^*\).

On the other hand, by [11], P.12, we get

\[
|L(2, \chi)| = |g(\chi)| \cdot \sum_{a \in (\mathbb{Z}/p)^*} \chi^{-1}(a)D_2(e^{-2\pi ia/p}).
\]

By the definition of \(g(\chi)\) in [11] and the property of Gauss sum, we get \(|g(\chi^{-1})| = p^{-\frac{1}{2}}\), so from (3.2) we have

\[
|L(2, \chi^{-1})| = p^{-\frac{1}{2}} \sum_{a \in (\mathbb{Z}/p)^*} |\chi(a)D_2(e^{2\pi ia/p})|.
\]
Combining (3.1) and (3.3), we get
\[ R = (2\pi)^{\frac{1-\varepsilon}{2}} p^{\frac{3(p-1)}{2}} \prod_{\chi \text{odd char.}} |L(\chi, 2)|. \]

\[ \Box \]

**Conjecture 1** \[ p[\zeta_p^i] \in B(\mathbb{Q}(\zeta_p)), i = 1, \cdots, \frac{p-1}{2}, \] is the free part of Bloch group of \( B(\mathbb{Q}(\zeta_p)) \).

Another conjecture is related with the \( K_2 \) groups of the algebraic integers of \( \mathbb{Q}(\zeta_p) \) and \( \mathbb{Q}(\zeta_p)^+ \), which denotes the maximal real subfield of \( \mathbb{Q}(\zeta_p) \). Then we have the following conjecture.

**Conjecture 2** There is a natural exact sequence of \( F = \mathbb{Q}(\zeta_p) \)
\[ 0 \rightarrow \ker \psi \rightarrow K_2(\mathcal{O}_F^+) \xrightarrow{\psi} K_2(\mathcal{O}_F) \rightarrow 0, \]
and the order of \( \ker \psi \) is \( 2^\frac{p+1}{2} \).

**Remark** 3.2. Conjecture 2 is true for \( p = 3, 5 \). For \( p = 3 \), it is easy to check that this case is true. For \( p = 5 \), by [20] we know \( K_2(\mathcal{O}_F) = 0 \). Since \# \( K_2(\mathcal{O}_F^+) = 4 \), we get Conjecture 2 holds for \( p = 5 \).

**Theorem 3.3.** Assuming the above two conjecture, the zeta function of \( F = \mathbb{Q}(\zeta_p) \) has the following equation
\[ |\zeta_F^+(-1)| = \frac{\#K_2(\mathcal{O}_F)}{\mu_2(F)} \bar{R}_2(F) \]

**Proof.** In [2], Bloch has calculated that
\[ |\det(D(\sigma_1(\zeta_p))))| = 2^{\frac{1-\varepsilon}{2}} p^{\frac{3(p-1)}{2}} \prod_{\chi \text{odd char.}} |L(\chi, 2)|, \]
where \( \sigma_i := \sigma^i, \sigma(\zeta_p) = \zeta_p^2, \chi \) runs all the odd character of \( \mathbb{F}_p^\times \). Browkin defined \( \bar{R}_2(F) = \frac{1}{\mu_2} |\det(D(\sigma_i(\varepsilon_j))))| \) to be the second regulator, where \( \sigma_i \) is the complex places, \( \varepsilon_j \) is the base of Bloch group \( B(F), 1 \leq i, j \leq r_2 \). By Conjecture 1, \( \bar{R}_2(F) = R \) is the second regulator of \( F = \mathbb{Q}(\zeta_p) \). So, we have
\[ (3.5) \quad \bar{R}_2(F) = (2\pi)^{\frac{1-\varepsilon}{2}} p^{\frac{3(p-1)}{2}} \prod_{\chi \text{odd char.}} |L(\chi, 2)| \]

The absolute value of discriminant of \( F \) is \( |d_F| = p^{p-2} \). Decomposing the zeta function \( \zeta_F(s) \) into the Dirichlet \( L \)-function, and taking use of the fact \( \zeta_{F^+}(s) = \prod_{\chi \text{even char.}} L(\chi, s) \), we have
\[ (3.6) \quad \zeta_F(s) = \zeta_{F^+}(s) \cdot \prod_{\chi \text{odd char.}} L(\chi, s). \]

Using (3.5),(3.6), \( \Gamma^*(-1) = 1 \) and the function equation of \( \zeta_F(s) \), we get that
\[ (3.7) \quad |\zeta_F^+(-1)| = 2^{1-\varepsilon} p^{1-\varepsilon} \prod_{\chi \text{odd char.}} |L(\chi, 2)| \]

Using the function equation of \( \zeta_{F^+}(s) \) and \( |\Gamma(-\frac{1}{2})| = 2^\frac{1}{2} \), we have
\[ (3.8) \quad \zeta_{F^+}(2) = 2^{\frac{p-1}{2}} p^{p-1} |d_{F^+}| \frac{1}{2} |\zeta_{F^+}(-1)|, \]
where \( d_{F^+} \) is the discriminant of \( F^+ \).
So, combining (3.7) and (3.8), we have

\[(3.9) \quad \left| \zeta^*_{F'}(-1) \right| = 2^{\frac{1 + p}{2} - \frac{p + 3}{2} - \frac{p}{2}} |d_{F'}|^{-\frac{1}{2}} \left| \zeta_{F'}(-1) \right| \tilde{R}_2(F).\]

In fact, by Theorem 3.11 in [18], we obtain that \(d_{F'} = p^{\frac{p - 3}{2}}\), hence, \(\left| d_{F'} \right|^{-\frac{1}{2}} = p^\frac{3 - p}{2}\). So, from (3.9), we get

\[(3.10) \quad \left| \zeta^*_{F'}(-1) \right| = 2^\frac{1 - 2}{2} \left| \zeta_{F'}(-1) \right| \tilde{R}_2(F).\]

Wiles [19] has proved that the Birch-Tate conjecture is true for the abelian totally real field. So, for \(F^+\) we have

\[(3.11) \quad \left| \zeta_{F'}(-1) \right| = \frac{\#K_2(O_{F'})}{w_2(F^+)}\]

The method of calculating the number \(w_2(F)\) can be found in Weibel paper [17]. For \(\mathbb{Q}(\zeta_p)\), we get that

\[(3.12) \quad w_2(F) = w_2(F^+) = \begin{cases} 24, & p = 3; \\ 24p, & p \neq 3. \end{cases}\]

Hence, from (3.10), (3.11) and (3.12), we get

\[\left| \zeta^*_{F'}(-1) \right| = 2^{\frac{1 - p}{2} \frac{\#K_2(O_{F'})}{w_2(F)}} \tilde{R}_2(F).\]

By Conjecture 2, we know the Lichtenbaum conjecture is true for \(\mathbb{Q}(\zeta_p)\). \(\square\)

**Remark 3.4** Professor T. Nguyen Quang Do has recently told us that the Lichtenbaum conjecture has now been proved in full generality for abelian fields (see the literature [9] Chapter 9). We are grateful for his account of the status of the Lichtenbaum conjecture.

4. Lichtenbaum conjecture on non-Galois fields

Suppose \(F\) is a number field with \(r_2(F) = 1\), then the free part of \(B(F)\) is \(\mathbb{Z}\) module of rank 1. In this section, we list some fields \(F\) with \(r_2(F) = 1\). The elements of Bloch group \(B(F)\) are constructed in a flexible way. Assume the base of Bloch group and the Lichtenbaum conjecture, we get a conjectural order of the \(K_2\) group of \(O_F\).

In [10] P. 250, there is a theorem about the special value of Dedekind zeta function at 2 and the Borel regulator, which states as following

**Theorem 4.1.** Let \(\zeta_F(s)\) be the Dedekind zeta function of \(F\). Then there exist \(y_1, \ldots, y_{r_2} \in B(F)\)

such that

\[\zeta_F(2) = q \cdot \pi^{2(r_1 + r_2)} \cdot |d_F|^{1/2} \cdot \det |D(\sigma_{r_1 + j}(y_i))|\]

where \(1 \leq i, j \leq r_2\) and \(q\) is some rational number.

Using PARI, we find the equations between the special values of zeta function at 2 and the dilogarithm functions. These equations are expected to be given a proof. Using the computing program in [11], we compute that all the K-groups are confirmed with the conjectural order and give their structures.
Example 4.2. Consider the equations as follows
\[
\begin{aligned}
1 + y &= x \\
1 - y^{-1} &= -x^4 y^{-1}.
\end{aligned}
\]

We get that \(x^3 + x^2 + x + 2 = 0\). Let \(\alpha\) be a root of this equation. Then \(r_2(\mathbb{Q}(\alpha)) = 1\) and we can assume \(\mathbb{Q}(\alpha)\) is complex. Now we claim that \(-4[\alpha] - [\alpha - 1] = 4[-\beta] + [\beta^{-1}]\). So,
\[
\partial(4[-\beta] + [\beta^{-1}]) = 4((-\beta) \wedge (1 + \beta)) + \beta^{-1} \wedge (1 - \beta^{-1})
\]
\[
= (-\beta)^4 \wedge \alpha + \beta^{-1} \wedge (-\alpha^4 \beta^{-1})
\]
\[
= \beta^4 \wedge \alpha + \beta^{-1} \wedge \alpha^4 + \beta^{-1} \wedge (-\beta^{-1})
\]
\[
= 4\beta \wedge \alpha - 4\beta \wedge \alpha = 0.
\]

Hence, \(-4[\alpha] - [\alpha - 1] \in B(\mathbb{Q}(\alpha))\).

Assuming the Lichtenbaum conjecture and \(-4[\alpha] - [\alpha - 1] \in B(\mathbb{Q}(\alpha))\) being a base, we get the order of \(K_2(O_{\mathbb{Q}(\alpha)})\), i.e. \(#K_2(O_{\mathbb{Q}(\alpha)}) = 4\). In fact, let \(F = \mathbb{Q}(\alpha), \theta = 4[\alpha] + [\alpha - 1] \in B(F)\). Using PARI we have
\[
\alpha \approx 0.176604982099662 + 1.202820819285479 i,
\]
\[
R_2(F) := |D(\theta)| = D(\theta) \approx 4.415332477453866,
\]
\[
\zeta_F(2) \approx 1.516751720642021.
\]

Because \(r_1(F) = r_2(F) = 1, |\Gamma^*(-1)| = 1, |\Gamma(-\frac{1}{2})| = 2\sqrt{\pi}\), by the function equation of \(\zeta_F(s)\), we have
\[
\zeta_F(2) = 2^4 \pi^5 |d_F|^{-\frac{3}{2}} \zeta_F^*(1).
\]

Using the Lichtenbaum conjecture on \(F = \mathbb{Q}(\alpha)\), we get
\[
\zeta_F^*(-1) = \frac{#K_2(O_F) R_2(F)}{w_2(F) \pi}.
\]

By Proposition 20.22 in \([17]\), we know \(w_2(F) = 24\); By PARI, we find \(d_F = -83\). At last, we get
\[
#K_2(O_F) \approx 3 \cdot 83^\frac{2}{4} \zeta_F(2),
\]
\[
\frac{2\pi^4 R_2}{2}.
\]

Using PARI, we get
\[
#K_2(O_F) = 3.99999999999999 = 4.
\]

Using method in \([1]\), we have \(K_2(O_F)\) is isomorphic to \(\mathbb{Z}/2 \times \mathbb{Z}/2\).

Moreover, we get the equation as follows by the numerical method
\[
\zeta_F(2) = \frac{8\pi^4}{3 \cdot 83^\frac{2}{4}} D(\theta).
\]

Example 4.3. Consider the equations as follows
\[
\begin{aligned}
1 + y &= x \\
1 - y^{-1} &= x^2.
\end{aligned}
\]
We get that $x^3 - x^2 - x + 2 = 0$. Let $\alpha$ be its complex root and $F = \mathbb{Q}(\alpha)$. Then we know $r_1(F) = r_2(F) = 1$ and we claim that $-\alpha^2 - 2\alpha \in B(F)$. Let $\beta = \alpha - 1$. Then

$$\partial(-\alpha^2 - 2\alpha) = \partial(1 - \alpha^2 + 2(1 - \alpha))$$

$$= \partial((\beta^{-1}) + 2[-\beta])$$

$$= \beta^{-1} \wedge (1 - \beta^{-1}) + 2[(-\beta) \wedge (1 + \beta)]$$

$$= \beta^{-1} \wedge \alpha^2 + 2[(-\beta) \wedge \alpha]$$

$$= -2(\beta \wedge \alpha) + 2(\beta \wedge \alpha)$$

$$= 0.$$

Let $\theta = \alpha^2 + 2\alpha$. Computing by PARI, we get that $\alpha \approx 1.102784715200295 + 0.665456951152813 i$, $D(\theta) \approx 2.56897609936709$, $\zeta_F(2) \approx 1.472476780199297$, $d_F = -59$.

And we have $w_2(F) = 24$. Assuming the Lichtenbaum conjecture and $\theta$ the base of $B(F)$, then we have $\#K_2(O_F) = 3 \cdot 59^2 \cdot \zeta_F(2)$.

By PARI we have $\#K_2(O_F) = 4.000000000000000 = 4$. Using method in [1], we have $K_2(O_F)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. The equation of zeta function at 2 is

$$\zeta_F(2) = \frac{8\pi^4}{3 \cdot 59^2} D(\theta).$$

Example 4.4. Consider the equations as follows

$$\begin{cases} 1 + y = x, \\ 1 - y^{-1} = y^{-1}x^3. \end{cases}$$

We have $x^3 - x + 2 = 0$. Let $\alpha$ be complex root of it, we have $r_2(\mathbb{Q}(\alpha)) = 1$. Then, $\theta := -6[1 - \alpha] - 2[\frac{1}{\alpha - 1}] \in B(\mathbb{Q}(\alpha))$. In fact, let $\beta = \alpha - 1$. Then we have

$$\partial(-6[1 - \alpha] - 2[\frac{1}{\alpha - 1}]) = \partial(-6[1 - \alpha] - 2[\beta^{-1}])$$

$$= -6(-\beta \wedge \alpha) - 2(\beta^{-1} \wedge \beta^{-1} \alpha^3)$$

$$= -6(\beta \wedge \alpha) + 6(\beta \wedge \alpha)$$

$$= 0.$$

By PARI, we obtain $\alpha \approx 0.760689853402284 + 0.857873626595179 i$, $D(\theta) \approx 7.517689896474569$, $\zeta_F(2) \approx 1.841207016617394$, $d_F = -104$. 

Assuming Lichtenbaum conjecture and $\theta$ being the base of $B(F)$, since $w_2(F) = 24$, we have

$$\#K_2(O_F) = \frac{3 \cdot 104^2 \zeta_F(2)}{2\pi^4 D(\theta)} = 4.00000000000000. \quad (1)$$

Using method in [1], we have $K_2(O_F)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and the function equation is

$$\zeta_F(2) = \frac{8\pi^4}{3 \cdot 104^2} D(\theta). \quad (2)$$

**Example 4.5.** Consider the following equations

\begin{align*}
1 - y &= x^2 \\
1 - y^{-1} &= y^{-3}x^3.
\end{align*}

It is easy to get the equation $x^4 - 2x^2 + x + 1 = 0$. Let $\alpha$ be a complex root of this equation. Then we have $r_2(\mathbb{Q}(\alpha)) = 1$. Let $\beta = 1 - \alpha^2$. Then we have

$$\partial(3[\beta] + 2[\beta^{-1}]) = 3\beta \land \alpha^2 + 2\beta^{-1} \land \beta^{-3}\alpha^3 = 0.$$ 

Hence, $[\beta] = 3[\beta] + 2[\beta^{-1}] \in B(F)$, that is, $[\alpha^2] \in B(F)$. Using PARI, we have

\begin{align*}
\alpha &\approx 1.007552359378179 + 0.513115795597015i \\
D(\alpha^2) &\approx 0.98136828892232 \\
\zeta_F(2) &\approx 1.05694057499707 \\
d_F &\approx -283.
\end{align*}

Assuming Lichtenbaum conjecture and $[\alpha^2]$ being the base of $B(F)$, since $w_2(F) = 24$, we have

$$\#K_2(O_F) = \frac{3 \cdot 283^2 \zeta_F(2)}{2^2\pi^6 R_2}. \quad (3)$$

By PARI, we have $\#K_2(O_F) = 3.999999999999 = 4$. Using method in [1], we have $K_2(O_F)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and the function equation is

$$\zeta_F(2) = \frac{16\pi^6}{3 \cdot 283^2} D(\alpha^2). \quad (4)$$

**Example 4.6.** Considering the equations as follows

\begin{align*}
1 - y &= -x \\
1 - y^{-1} &= x^4,
\end{align*}

we have $x^4 + x^3 - 1 = 0$. Let $\alpha$ be complex root of it, we have $r_2(\mathbb{Q}(\alpha)) = 1$ and $3[-\alpha] \in B(\mathbb{Q}(\alpha))$. By PARI, we have

\begin{align*}
\alpha &\approx -0.219447472149275 - 0.914473662967726i, \\
R_2 &\approx 2^3D(-\alpha) = 2.944106486676696, \\
\zeta_F(2) &\approx 1.056940574599707, \\
d_F &\approx -283.
\end{align*}

Assuming Lichtenbaum conjecture and $3[-\alpha]$ being the base of $B(F)$, since $w_2(F) = 24$, we have

$$\#K_2(O_F) = \frac{3^2 \cdot 283^2 \zeta_F(2)}{2^2\pi^6 R_2} = 3.999999999999 = 4.$$
Using method in [11], we have $K_2(O_F)$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ and function equation is

$$\zeta_F(2) = \frac{16\pi^6}{3 \cdot 283^2} D(-\alpha).$$

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Corresponding author: Chaochao Sun

Chaochao Sun, School of Mathematics and Statistics, Linyi University, Linyi, China 276005

E-mail address: sunuso@163.com

Long Zhang, School of Mathematics and Statistics, Qingdao University, Qingdao, China, 266071

E-mail address: zhanglong_note@hotmail.com