Weighted dynamic Hardy-type inequalities involving many functions on arbitrary time scales

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Abstract
The objective of this paper is to prove some new dynamic inequalities of Hardy type on time scales which generalize and improve some recent results given in the literature. Further, we derive some new weighted Hardy dynamic inequalities involving many functions on time scales. As special cases, we get continuous and discrete inequalities.

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1 Introduction
In [1], Hardy showed that if \(\alpha > 1\) and \(\Psi(\zeta) \geq 0\) over the interval \((0, \infty)\) such that \(\int_0^{\infty} \Psi^\alpha(\zeta) \, d\zeta < \infty\), then
\[
\int_0^{\infty} \left( \frac{1}{x} \int_0^x \Psi(\zeta) \, d\zeta \right)^\alpha \, dx \leq \left( \frac{\alpha}{\alpha - 1} \right)^\alpha \int_0^{\infty} \Psi^\alpha(x) \, dx,
\]
where the constant \((\alpha/(\alpha - 1))^\alpha\) is sharp. In [2], Hardy obtained that if \(\alpha > 1\) and \(m > 1\), then
\[
\int_0^{\infty} \frac{1}{x^m} \left( \int_0^x \Psi(\zeta) \, d\zeta \right)^\alpha \, dx \leq \left( \frac{\alpha}{m - 1} \right)^\alpha \int_0^{\infty} \frac{1}{x^{m-\alpha}} \Psi^\alpha(x) \, dx.
\]
In [3], Levinson proved that if \(\alpha > 1\), \(\Psi(x) \geq 0\), \(f(x) > 0\) is an absolutely continuous function and
\[
\frac{\alpha}{\alpha - 1} + \frac{f'(x)}{f(x)} \geq \frac{1}{\beta} > 0, \quad \text{for all } x > 0,
\]
then
\[
\int_0^{\infty} \left( \frac{1}{xf(x)} \right)^\alpha \int_0^x f(\zeta) \Psi(\zeta) \, d\zeta \, dx \leq \beta^\alpha \int_0^{\infty} \Psi^\alpha(x) \, dx.
\]

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In [4], S. Hussan et al. proved that for any \( i = 1, 2, \ldots, n, n \in \mathbb{N}, f_i(\zeta) \geq 0 \) and integrable function on \((0, \infty)\) and \( \hat{w}, u_i, z_i \) are absolutely continuous functions with \( z_i' \) essentially bounded and positive, if \( u_i \) is increasing and

\[
1 + \frac{u_i(\zeta)\hat{w}'(\zeta)}{(1 - 2m)u_i'(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{2},
\]

\[
1 + \frac{u_i(\zeta)\hat{w}'(\zeta)}{(1 - 2m)u_i'(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\delta_i} > 0, \quad \text{for } m < \frac{1}{2},
\]

then

\[
\sum_{i=1}^{n} \int_{0}^{\infty} \hat{w}(\zeta)R_i(\zeta)R_{i+1}(\zeta) \, d\zeta \leq \sum_{i=1}^{n} \left( \frac{2\beta_i}{|2m-1|} \right)^2 \int_{0}^{\infty} \hat{w}(\zeta)g_i(\zeta) \, d\zeta,
\]

where

\[
R_i(\zeta) = \begin{cases} \frac{u_i'(\zeta)}{u_i(\zeta)} \int_{0}^{\infty} \frac{u_i(x)z_i'(x)}{z_i(x)} f_i(\zeta) \, dx, & m > \frac{1}{2}, \\ \frac{u_i'(\zeta)}{u_i(\zeta)} \int_{\infty}^{0} \frac{u_i(x)z_i'(x)}{z_i(x)} f_i(\zeta) \, dx, & m < \frac{1}{2}, \end{cases}
\]

\[
g_i(\zeta) = [u_i(\zeta)]^{\kappa \alpha_i(2m-2)} [z_i(\zeta)]^{\kappa \alpha_i} \left( \frac{1}{z_i'(\zeta)} \right)^{\kappa \alpha_i - 1} \text{ and } \beta_i = \max_{1 \leq i \leq n} (\lambda_i, \delta_i).
\]

Also in the same paper [4], the authors proved that for any \( i = 1, 2, \ldots, n, n \geq \kappa - 1, n, \kappa \in \mathbb{N}, \)
if \( \alpha_i > 1, \delta_i = \alpha_i/(\kappa \alpha_i - 1) \) and

\[
1 + \frac{u_i(\zeta)\hat{w}'(\zeta)}{(1 - \kappa \alpha_i m)u_i'(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{\kappa \alpha_i},
\]

\[
1 + \frac{u_i(\zeta)\hat{w}'(\zeta)}{(1 - \kappa \alpha_i m)u_i'(\zeta)\hat{w}(\zeta)} \geq \frac{1}{\delta_i} > 0, \quad \text{for } m < \frac{1}{\kappa \alpha_i},
\]

then

\[
\sum_{i=1}^{n-k+2} \int_{0}^{\infty} \hat{w}(\zeta) \left[ \prod_{j=1}^{i-1} R_j(\zeta) \right] \, d\zeta \leq \sum_{i=1}^{n} \left( \frac{\kappa \alpha_i \beta_i}{|\kappa \alpha_i m - 1|} \right)^{\kappa \alpha_i} \int_{0}^{\infty} \hat{w}(\zeta)g_i(\zeta) \, d\zeta,
\]

where

\[
R_i(\zeta) = \begin{cases} \frac{\kappa \alpha_i u_i'(\zeta)}{u_i(\zeta)} \int_{0}^{\infty} \frac{u_i(x)z_i'(x)}{z_i(x)} f_i(\zeta) \, dx, & m > \frac{1}{\kappa \alpha_i}, \\ \frac{\kappa \alpha_i u_i'(\zeta)}{u_i(\zeta)} \int_{\infty}^{0} \frac{u_i(x)z_i'(x)}{z_i(x)} f_i(\zeta) \, dx, & m < \frac{1}{\kappa \alpha_i}, \end{cases}
\]

\[
g_i(\zeta) = [u_i(\zeta)]^{\alpha_i(2m-2)} [z_i(\zeta)]^{\alpha_i} \left( \frac{1}{z_i'(\zeta)} \right)^{\alpha_i - 1} \text{ and } \beta_i = \max_{1 \leq i \leq n} (\lambda_i, \delta_i).
\]

The main aim is to establish some dynamic inequalities where the involved functions are defined on the \( \mathbb{T} \) domain. These results involve the classical discrete and continuous inequalities. For more details, we point the reader to the books [5, 6]. In [7], Řehak found
the time scale version of Hardy’s inequality. Especially, Řehák derived that if $\alpha > 1$ and 
$\Psi(\xi) \geq 0$ are such that $\int_{a}^{\infty} \Psi^\alpha(\xi) \Delta \xi < \infty$ then

$$\int_{a}^{\infty} \left( \frac{1}{\sigma(\xi) - a} \int_{a}^{\sigma(\xi)} \Psi(x) \Delta x \right)^{\alpha} \Delta \xi \leq \left( \frac{\alpha}{\alpha - 1} \right)^{\alpha} \int_{a}^{\infty} \Psi^\alpha(\xi) \Delta \xi.$$

In addition, if $\mu(\xi)/\xi \to 0$ as $\xi \to \infty$, then the constant $(\alpha/(\alpha - 1))^{\alpha}$ is sharp.

In [8], the authors showed that if $f(\xi) > 0$, $\Psi(\xi) \geq 0$ and $f^{\lambda}(\xi) \leq 0$ on $[0, \infty)_{T}$, $\alpha > 1$ and there exist constants $\kappa, \beta > 0$ such that $\xi/\sigma(\xi) \geq 1/\kappa$ and

$$\frac{\alpha}{\alpha - 1} + \frac{\kappa^{\alpha} \Phi(\xi)}{\Phi^{\alpha}(\xi)} \frac{f^{\lambda}(\xi)}{f^\alpha(\xi)} \geq \frac{1}{\beta}, \quad \text{for } \xi \in [0, \infty)_{T},$$

then

$$\int_{0}^{\infty} \frac{1}{\xi^{\alpha}} \left( \Phi^{\alpha}(\xi) \right)^{\alpha} \Delta \xi \leq \left( \beta \kappa^{\alpha} \right)^{\alpha} \int_{0}^{\infty} \left( \frac{f(\xi) \Psi(\xi)}{f^\alpha(\xi)} \right)^{\alpha} \Delta \xi,$$

where

$$\Phi(\xi) = \frac{1}{f(\xi)} \int_{0}^{\xi} f(x) \Psi(x) \Delta x, \quad \xi \in [0, \infty)_{T}.$$

The purpose of this manuscript is to establish some new Hardy-type inequalities on time scales $T$ involving many functions which generalize and improve some results in [4]. The following is the format of the paper: In Sect. 2, we begin with some background information about the delta derivative on $T$. Our main findings are obtained in Sect. 3.

### 2 Basic principles

A time scale $T$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We define the forward jump operator $\sigma : T \to T$ by $\sigma(\xi) = \inf \{ s \in T : s > \xi \}$ and define the backward jump operator $\rho : T \to T$ by $\rho(\xi) = \sup \{ s \in T : s < \xi \}$, respectively, where $\sup \emptyset = \inf T$.

A point $\xi \in T$ is called right-dense if $\sigma(\xi) = \xi$, left-dense if $\rho(\xi) = \xi$, right-scattered if $\sigma(\xi) > \xi$, and left-scattered if $\rho(\xi) < \xi$. If sup $T$ is finite and left-scattered, then $T^{k} = T \setminus \{ \sup T \}$, otherwise, $T^{k} = T$.

A function $f : T \to \mathbb{R}$ is a right-dense continuous (rd-continuous) if $f$ is continuous at right-dense points and its left-hand limits are finite at left-dense points in $T$.

Let $f : T \to \mathbb{R}$ be a real-valued function on $T$. Then for $\xi \in T^{k}$, we define $f^{\lambda}(\xi)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood $u$ of $\xi$ such that, for all $s \in u$, we have

$$|f(\sigma(\xi)) - f(s)| - f^{\lambda}(\xi)[\xi - s] | \leq \varepsilon |\sigma(\xi) - s|.$$

In this case, we say that $f$ is delta differentiable on $T^{k}$ provided $f(\xi)$ exists for all $\xi \in T^{k}$. If $f, g : T \to \mathbb{R}$ are delta differentiable at $\xi \in T$, then

$$(fg)^{\lambda} = f^{\lambda}g + f^{\sigma}g^{\lambda} = fg^{\lambda} + f^{\sigma}g^{\sigma}, \quad \text{where } f^{\sigma}(\xi) = (f \circ \sigma)(\xi) = f(\sigma(\xi)).$$

(6)
For $a, b \in \mathbb{T}$ and a delta differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$
\int_a^b f^\Delta(\xi) \Delta \xi = f(b) - f(a).
$$

The integration by parts formula on $\mathbb{T}$ is given by

$$
\int_a^b \Psi(\xi) \Phi^\Delta(\xi) \Delta \xi = (\Psi \Phi)(b) - (\Psi \Phi)(a) - \int_a^b \Phi^\Delta(\xi) \Phi^\sigma(\xi) \Delta \xi.
$$

\textbf{Theorem 6} (Hölder’s inequality \[10\]) Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$, we have

$$
\left( \int_a^b |f(\xi)g(\xi)| \Delta \xi \right)^{\frac{1}{q}} \leq \left( \int_a^b |f(\xi)|^p \Delta \xi \right)^{\frac{1}{p}} \left( \int_a^b |g(\xi)|^q \Delta \xi \right)^{\frac{1}{q}},
$$

where $p > 1$ and $q = \frac{p}{p-1}$.

3 Main results

Throughout this section, any time scale $\mathbb{T}$ is unbounded above with $a, b \in \mathbb{T}$. We will make the assumption that the functions $\hat{w}, u_i, z_i$ in the statements of the theorems are rd-continuous, nonnegative and increasing, and $f_i(\xi) > 0$ is an integrable function.

\textbf{Theorem 6} For any $1 \leq i \leq n$, $n \geq \kappa - 1$ and $n, k \in \mathbb{N}$, if there exist constants $\lambda_i > 0$, $\delta_i > 0$ such that

$$
1 - \frac{[u_i^\Delta(\xi)]^{am} \hat{w}^\Delta(\xi)}{(am - 1)[u_i(\xi)]^{am-1}u_i^\Delta(\xi) \hat{w}^\sigma(\xi)} \geq \frac{1}{\lambda_i}, \quad \text{for } m > \frac{1}{2}.
$$
\[
1 - \frac{u_i(\zeta) \hat{\omega}^\lambda(\zeta)}{(am - 1)u_i^\lambda(\zeta) \hat{\omega}^\lambda(\zeta)} \geq \frac{1}{\delta_i} > 0, \quad \text{for } m < \frac{1}{2},
\]  
\hspace{1cm} (14)

then
\[
\sum_{i=1}^{n} \int_{0}^{\infty} \hat{\omega}^\rho(\zeta)R_{\alpha}^\rho(\zeta)R_{i+1}^\rho(\zeta) \Delta \zeta \leq \sum_{i=1}^{n} \left( \frac{\alpha \beta_i}{|am - 1|} \right)^a \int_{0}^{\infty} \hat{\omega}^\rho(\zeta)g_i(\zeta) \Delta \zeta,
\]  
\hspace{1cm} (15)

where
\[
R_i(\zeta) = \begin{cases} 
\frac{u_i^\lambda(\zeta)}{u_i^\lambda(\zeta)^m} & \text{if } m > \frac{1}{2}, \alpha > 2, \\
\frac{u_i^\lambda(\zeta)}{u_i^\lambda(\zeta)^m} \int_{0}^{\infty} \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x, & \text{if } m < \frac{1}{2}, 1 \leq \alpha \leq 2,
\end{cases}
\]

\[
g_i(\zeta) = \begin{cases} 
\frac{u_i^{\rho-1}(\zeta)u_i^\lambda(\zeta)^{m-1}}{u_i^\lambda(\zeta)^m} & \text{if } m > \frac{1}{2}, \alpha > 2, \\
\frac{u_i^{\rho-1}(\zeta)u_i^\lambda(\zeta)^{m-1}}{u_i^\lambda(\zeta)^m} \int_{0}^{\infty} \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x, & \text{if } m < \frac{1}{2}, 1 \leq \alpha \leq 2,
\end{cases}
\]

and \(\beta_i = \max_{1 \leq i \leq n}(\lambda_i, \delta_i), u_i(\infty) = \infty.\)

**Proof** First, let us define for \(m > \frac{1}{2}, \alpha > 2,\) and \(0 < a < b < \infty,\)
\[
R_{il}(\zeta) = \left( \frac{u_i^\lambda(\zeta)}{u_i^\lambda(\zeta)^m} \right)^a \int_{0}^{\infty} \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x, \quad 1 \leq i \leq n,
\]
with \(R_{i0}(\zeta) = R_i(\zeta).\) Using (11) with \(\kappa = 2\) for \(C_i = R_{i+1}^\rho(\zeta),\) we get
\[
\sum_{i=1}^{n} R_{il}^\rho(\zeta)R_{i+1,0}^\rho(\zeta) \leq \sum_{i=1}^{n} R_{il}^\rho(\zeta).
\]  
\hspace{1cm} (16)

Multiplying (16) by \(\hat{\omega}^\rho(\zeta)\) and integrating from \(a\) to \(b,\) we have
\[
\sum_{i=1}^{n} \int_{a}^{b} \hat{\omega}^\rho(\zeta)R_{il}^\rho(\zeta)R_{i+1,0}^\rho(\zeta) \Delta \zeta \leq \sum_{i=1}^{n} \int_{a}^{b} \hat{\omega}^\rho(\zeta)R_{il}^\rho(\zeta) \Delta \zeta.
\]  
\hspace{1cm} (17)

Now,
\[
J = \int_{a}^{b} \hat{\omega}^\rho(\zeta)R_{il}^\rho(\zeta) \Delta \zeta
\]
\[
= \int_{a}^{b} \hat{\omega}^\rho(\zeta) \left( \frac{u_i^\lambda(\zeta)}{u_i^\lambda(\zeta)^m} \int_{0}^{\infty} \left( \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x \right)^a \right) \Delta \zeta
\]
\[
= \int_{a}^{b} \frac{u_i^\lambda(\zeta)}{(u_i^\lambda(\zeta))^m} \left( \int_{0}^{\infty} \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x \right)^a \Delta \zeta.
\]  
\hspace{1cm} (18)

Integrating (18) by parts using formula (7) with
\[
\Phi^\rho(\zeta) = \frac{u_i^\lambda(\zeta)}{(u_i^\lambda(\zeta))^m}, \quad \Psi^\rho(\zeta) = \left( \frac{\sqrt{\hat{\omega}^\rho(\zeta)}}{\int_{0}^{\infty} \frac{u_i(x)z_i(\zeta)}{z_i(x)} f_i(x) \Delta x} \right)^a,
\]
we obtain
\[
J = \left[ \Phi(\xi) \Psi(\xi) \right]_a^b + \int_a^b (-\Phi(\xi))(\Psi(\xi))^\Delta \Delta \xi \\
= \left[ -\Psi(\xi) \int_\xi^b \frac{u_1^\Delta(x)}{(u_1^\sigma)(x) \alpha m} \Delta x \right]_a^b + \int_a^b (-\Phi(\xi))(\Psi(\xi))^\Delta \Delta \xi \\
= \left[ -\Psi(b) \int_b^\infty \frac{u_1^\Delta(x)}{(u_1^\sigma)(x) \alpha m} \Delta x \right] + \int_a^b (-\Phi(\xi))(\Psi(\xi))^\Delta \Delta \xi \\
\leq \int_a^b (-\Phi(\xi))(\Psi(\xi))^\Delta \Delta \xi, \tag{19}
\]

where \( \Phi(\xi) = -\int_\xi^\infty \frac{u_1^\Delta(x)}{(u_1^\sigma)(x) \alpha m} \Delta x \) and \( (\Psi(\xi))^\Delta > 0 \). From (9), since \( u_1^\Delta(\xi) \geq 0 \) and \( c \in [\xi, \sigma(\xi)] \), we have
\[
\left[ u_1^{1-\alpha m}(\xi) \right]^\Delta = (1 - \alpha m)u_1^{1-\alpha m}(c)u_1^\Delta(\xi) \\
= (1 - \alpha m) \frac{u_1^\Delta(\xi)}{u_1^{\Delta m}(c)} \tag{20}
\]

Therefore, integrating (20) from \( \xi \) to \( \infty \) with respect to \( x \), we have
\[
-\Phi(\xi) \leq \frac{1}{\alpha m - 1} u_1^{1-\alpha m}(\xi). \tag{21}
\]

Combining (21) and (19), we get
\[
J \leq \frac{1}{\alpha m - 1} \int_a^b u_1^{1-\alpha m}(\xi)(\Psi(\xi))^\Delta \Delta \xi. \tag{22}
\]

Now, by applying (6) to \( \Psi(\xi) = \hat{\Psi}(\xi) \hat{\Psi}(\xi) \) and using (9), we obtain
\[
(\Psi(\xi))^\Delta = \hat{\Psi}(\xi) \hat{\Psi}(\xi)^\Delta = \hat{\Psi}(\xi) \hat{\Psi}(\xi)^\Delta + \alpha \hat{\Psi}(\xi) \hat{\Psi}(\xi)^{\alpha m-1}(\xi) \hat{\Psi}(\xi)^\Delta \\
\leq \hat{\Psi}(\xi) \hat{\Psi}(\xi)^\Delta + \alpha \hat{\Psi}(\xi) \frac{z_0(\xi)z_1^{\Delta}(\xi)}{z_1(\xi)} \hat{f}(\xi) \hat{y}(\xi)^{\alpha m-1}, \tag{23}
\]

where
\[
\hat{y}(\xi) = \int_a^\xi \frac{u_1(x)z_1^{\Delta}(x)}{z_1(\xi)} f(x) \Delta x.
\]

Substituting (23) into (22), we get
\[
J \leq \frac{1}{\alpha m - 1} \int_a^b u_1^{1-\alpha m}(\xi) \hat{\Psi}(\xi) \left( \int_a^\sigma(\xi) \frac{z_0(\xi)z_1^{\Delta}(\xi)}{z_1(\xi)} f(x) \Delta x \right)^\alpha \Delta \xi.
\]
From (13) and (24), we have

\[
\frac{\alpha}{am-1} \int_a^b \frac{u_i^{2-am}(\zeta) \hat{\omega}(\zeta) z_i^a(\zeta) f_i(\zeta)}{z_i(\zeta)} \Delta \zeta = \frac{1}{am-1} \int_a^b \frac{[u_i^a(\zeta)]^{am} \hat{\omega}(\zeta) \alpha}{u_i^{am-1}(\zeta) u_i^a(\zeta)} R_i^\alpha(\xi) \Delta \zeta
\]

Hence,

\[
\int_a^b \hat{\omega}(\zeta) R_i^\alpha(\xi) \Delta \zeta = \frac{\alpha}{am-1} \int_a^b \frac{u_i^{2-am}(\zeta) [u_i^a(\zeta)]^{m(\alpha-1)} z_i^a(\zeta) f_i(\zeta) \hat{\omega}(\zeta)}{z_i(\zeta) (u_i^a(\zeta))^{1+\frac{\alpha}{1-\alpha}}} \Delta \zeta \leq \frac{\alpha}{am-1} \int_a^b \frac{u_i^{2-am}(\zeta) [u_i^a(\zeta)]^{m(\alpha-1)} z_i^a(\zeta) f_i(\zeta) \hat{\omega}(\zeta)}{z_i(\zeta) (u_i^a(\zeta))^{1+\frac{\alpha}{1-\alpha}}} \Delta \zeta.
\]

Applying Hölder's inequality with \(\alpha\) and \(\alpha/(\alpha - 1)\), we have

\[
\int_a^b \hat{\omega}(\zeta) R_i^\alpha(\xi) \Delta \zeta = \frac{\alpha^\lambda_i}{am-1} \int_a^b \frac{u_i^{(2-am)}(\zeta) [u_i^a(\zeta)]^{m(\alpha-1)} z_i^a(\zeta) f_i^\alpha(\zeta) \hat{\omega}(\zeta)}{z_i^a(\zeta) (u_i^a(\zeta))^{1+\alpha} (\hat{\omega}(\zeta))^{1-\alpha}} \Delta \zeta \leq \frac{\alpha^\lambda_i}{am-1} \int_a^b \frac{u_i^{(2-am)}(\zeta) [u_i^a(\zeta)]^{m(\alpha-1)} z_i^a(\zeta) f_i^\alpha(\zeta) \hat{\omega}(\zeta)}{z_i^a(\zeta) (u_i^a(\zeta))^{1+\alpha} (\hat{\omega}(\zeta))^{1-\alpha}} \Delta \zeta.
\]

By letting \(a \to 0\), \(b \to \infty\) and from (25), (17), we have

\[
\sum_{i=1}^n \int_0^\infty \hat{\omega}(\zeta) R_i^\alpha(\xi) \Delta \zeta \leq \sum_{i=1}^n \left( \frac{\alpha^\beta_i}{am-1} \right)^a \int_0^\infty \hat{\omega}(\zeta) \xi_i(\zeta) \Delta \zeta.
\]

Second, let us define for \(m < \frac{1}{2}\), \(1 \leq \alpha \leq 2\), and \(0 < a < b < \infty\),

\[
R_{ib}(\xi) = \frac{\sqrt{u_i^a(\zeta)}}{u_i^a(\zeta)} \int_{\sigma(\xi)}^b \frac{u_i(x) z_i^a(\xi)}{z_i(x)} f_i(x) \Delta x, \quad 1 \leq i \leq n.
\]
Corollary 7 Let \( u(s) \in \mathbb{R}_+^\infty \), \( \hat{w}(s) \in \mathbb{R}_+^\infty \), and \( z(s) \in \mathbb{R}_+^\infty \) be increasing and nonnegative sequences.

For any \( 1 \leq i \leq n \), \( n \geq \kappa - 1 \), \( n, \kappa \in \mathbb{N} \), and

\[
1 - \frac{[u_i(s + 1)]^{m}[\hat{w}(s)]^{m-1}}{(am - 1)[u_i(s)]^{m-1}[\hat{w}(s + 1)]^{\Delta}u_i(s)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{2},
\]

we have

\[
\sum_{i=1}^{n} \left( \sum_{s=1}^{\infty} \hat{w}(s + 1) R_i^\frac{s}{i+1} R_i^\frac{s}{i+1} \right) \leq \sum_{i=1}^{n} \left( \frac{\alpha \beta_i}{|1 - am|} \right) \sum_{s=1}^{\infty} \hat{w}(s + 1) g_i(s),
\]

where

\[
R_i(s) = \left\{ \begin{array}{ll}
\sqrt[\frac{m}{u_i(s)}]{\frac{u_i(s)}{[u_i(s)]^m}} \sum_{r=1}^{\infty} \frac{u_i(r)\Delta_1(s)}{z_1(r)} f_i(r), & \text{for } m > \frac{1}{2}, \alpha \geq 2, \\
\sqrt[\frac{m}{u_i(s)}]{\frac{u_i(s)}{[u_i(s)]^m}} \sum_{r=1}^{\infty} \frac{u_i(r)\Delta_1(s)}{z_1(r)} f_i(r), & \text{for } m < \frac{1}{2}, 1 \leq \alpha \leq 2, \\
\frac{\sqrt[\frac{m}{u_i(s)}]{\frac{u_i(s)}{[u_i(s)]^m}} \sum_{r=1}^{\infty} \frac{u_i(r)\Delta_1(s)}{z_1(r)} f_i(r)}{\Delta y(s)} & \text{for } m > \frac{1}{2}, \alpha < 2, \\
\frac{\sqrt[\frac{m}{u_i(s)}]{\frac{u_i(s)}{[u_i(s)]^m}} \sum_{r=1}^{\infty} \frac{u_i(r)\Delta_1(s)}{z_1(r)} f_i(r)}{\Delta y(s)}, & \text{for } m < \frac{1}{2}, 1 \leq \alpha \leq 2,
\end{array} \right.
\]

with \( \Delta y(s) = y(s + 1) - y(s) \), \( \beta_i = \max_{1 \leq i \leq n} (\lambda_i, \delta_i) \), and \( u_i(\infty) = \infty \).

Remark 8 If we put \( T = \mathbb{R} \) and \( \alpha = 2 \), in Theorem 6, then (15) reduces to (4).

The next corollary follows from Theorem 6 by taking \( u_i(\xi) = z_i(\xi) = \xi, f_i(\xi) = f_{i+1}(\xi), m = 1, \) and \( \alpha = 2 \).

Corollary 9 For any \( 1 \leq i \leq n \), \( n \geq \kappa - 1 \) and \( \kappa \in \mathbb{N} \), if there exist \( \lambda_i > 0 \) such that

\[
1 - \frac{\sigma^2(\xi)\hat{w}(\xi)}{\xi \hat{w}(\xi)} \geq \frac{1}{\lambda_i} > 0,
\]

then

\[
\sum_{i=1}^{n} \int_0^\infty \hat{w}(\xi) \left[ \frac{1}{\sigma(\xi)} \int_0^{\sigma(\xi)} f_i(x) \, dx \right] \Delta \xi \leq \sum_{i=1}^{n} \left( 2\lambda_i \right)^2 \int_0^\infty \hat{w}(\xi) g_i(\xi) \Delta \xi,
\]
where
\[ g_i(\xi) = \frac{\sigma^2(\xi)\zeta^2(\xi)\hat{w}^2(\xi)}{\zeta^2(\hat{w}^2(\xi))^2} . \]

**Remark 10** Letting \( T = \mathbb{R} \) in Corollary 9, we have that \( \sigma(\xi) = \zeta \) and
\[ 1 - \frac{\xi \hat{w}'(\xi)}{\hat{w}(\xi)} \geq \frac{1}{\lambda_i} > 0. \]

Then
\[ \sum_{i=1}^{n} \int_0^\infty \hat{w}(\xi) \left[ \frac{1}{\xi} \int_0^\xi f_i(x) \, dx \right]^2 d\xi \leq \sum_{i=1}^{n} (2\lambda_i)^2 \int_0^\infty \hat{w}(\xi) f_i^2(\xi) \, d\xi , \]
which agrees with [4, Corollary 1].

**Theorem 11** For any \( 1 \leq i \leq n, n \geq \kappa - 1, \) and \( n, \kappa \in \mathbb{N}, \) if \( \alpha_i > 1, \delta_i = \alpha_i/(\kappa \alpha_i - 1) \) and there exist \( \lambda_i > 0, \delta_i > 0 \) such that
\[ 1 - \frac{[u_i^\alpha(\xi)]^{u_i} \hat{w}^\delta(\xi)}{(\kappa \alpha_i m - 1) u_i^{\alpha_i m - 1}(\xi) \hat{w}^\delta(\xi)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{\kappa \alpha_i} , \]
\[ 1 - \frac{u_i(\xi) \hat{w}^\delta(\xi)}{(\kappa \alpha_i m - 1) u_i^{\alpha_i}(\xi) \hat{w}^\delta(\xi)} \geq \frac{1}{\delta_i} > 0, \quad \text{for } m < \frac{1}{\kappa \alpha_i} , \]
then
\[ \sum_{i=1}^{n} \int_0^\infty \hat{w}^\delta(\xi) \left( \prod_{j=1}^{i-1} R_j^\alpha(\xi) \right) \Delta \xi \]
\[ \leq \sum_{i=1}^{n} \left( \frac{\kappa \alpha_i \beta_i}{|\kappa \alpha_i m - 1|} \right)^{\alpha_i} \int_0^\infty \hat{w}^\delta(\xi) g_i(\xi) \Delta \xi , \]
where
\[ \begin{align*}
R_i(\xi) &= \begin{cases} 
\frac{u_i^\alpha(\xi)}{|u_i^\alpha(\xi)|^m} \int_0^\sigma(\xi) u_i(x) \frac{x_i(x)}{z_i(x)} f_i(x) \Delta x, & \text{for } m > \frac{1}{\kappa \alpha_i} , \\
\frac{u_i^\alpha(\xi)}{|u_i^\alpha(\xi)|^m} \int_0^\sigma(\xi) u_i(x) \frac{x_i(x)}{z_i(x)} f_i(x) \Delta x, & \text{for } m < \frac{1}{\kappa \alpha_i} , 
\end{cases} \\
g_i(\xi) &= \begin{cases} 
\frac{u_i^{\alpha_i}(\xi)}{|u_i^{\alpha_i}(\xi)|^m} \int_{\xi_1}^{\xi_2} \frac{\xi^{\alpha_i}(\xi) u_i^{\alpha_i}(\xi) u_i^{\alpha_i - 1}(\xi) \hat{w}^\delta(\xi)}{z_i^{\alpha_i}(\xi) u_i^{\alpha_i}(\xi) \hat{w}^\delta(\xi)} f_i(x) \Delta x, & \text{for } m > \frac{1}{\kappa \alpha_i} , \\
\frac{u_i^{\alpha_i}(\xi)}{|u_i^{\alpha_i}(\xi)|^m} \int_{\xi_1}^{\xi_2} \frac{\xi^{\alpha_i}(\xi) u_i^{\alpha_i}(\xi) u_i^{\alpha_i - 1}(\xi) \hat{w}^\delta(\xi)}{z_i^{\alpha_i}(\xi) u_i^{\alpha_i}(\xi) \hat{w}^\delta(\xi)} f_i(x) \Delta x, & \text{for } m < \frac{1}{\kappa \alpha_i} , 
\end{cases} 
\end{align*} \]
and \( \beta_i = \max_{1 \leq i \leq n} (\lambda_i, \delta_i), \ u_i(\infty) = \infty. \)

**Proof** Let us define for \( m > \frac{1}{\kappa \alpha_i} \) and \( 0 < a < b < \infty, \)
\[ R_{ia}(\xi) = \frac{u_i^\alpha(\xi)}{|u_i^\alpha(\xi)|^m} \int_a^b u_i(x) \frac{z_i(x)}{z_i(x)} f_i(x) \Delta x, \quad 1 \leq i \leq n , \]
(32)
with $R_{i0}(\zeta) = R_i(\zeta)$. Using (11) with $C_i = R_{ia}^\alpha(\zeta)$, we get

$$\sum_{i=1}^{n-k+2} R_{ia}^\alpha(\zeta) R_{i+1,a}^\alpha(\zeta) \cdots R_{i+k-1,a}^\alpha(\zeta) \leq \sum_{i=1}^n R_{ia}^\alpha(\zeta).$$

(33)

Multiplying (33) by $\tilde{\nu}^\rho(\zeta)$ and integrating from $a$ to $b$, we have

$$\sum_{i=1}^{n-k+2} \int_a^b \tilde{\nu}^\rho(\zeta) \left( \prod_{j=1}^{i-1} R_{ja}^\alpha(\zeta) \right) \Delta \zeta \leq \sum_{i=1}^n \int_a^b \tilde{\nu}^\rho(\zeta) R_{ia}^\alpha(\zeta) \Delta \zeta. $$

(34)

Now,

$$J = \int_a^b \tilde{\nu}^\rho(\zeta) R_{ia}^\alpha(\zeta) \Delta \zeta$$

$$= \int_a^b \tilde{\nu}^\rho(\zeta) \left( \frac{x^{\alpha}(\zeta)}{|u_i^\rho(\zeta)|^m} \int_a^\sigma(x) \frac{\mu_i(x) z_1^\lambda(x)}{z_i(x)} f_i(x) \Delta x \right)^{\alpha_i} \Delta \zeta$$

$$= \int_a^b \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}} \left( \frac{x^{\alpha}(\zeta)}{|u_i^\rho(\zeta)|^m} \int_a^\sigma(x) \frac{\mu_i(x) z_1^\lambda(x)}{z_i(x)} f_i(x) \Delta x \right)^{\alpha_i} \Delta \zeta.$$

(35)

 Integrating (35) by parts using formula (7) with

$$\Phi^\lambda(\zeta) = \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}}, \quad \Psi^\sigma(\zeta) = \left( \frac{x^{\alpha}(\zeta)}{|u_i^\rho(\zeta)|^m} \int_a^\sigma(x) \frac{\mu_i(x) z_1^\lambda(x)}{z_i(x)} f_i(x) \Delta x \right)^{\alpha_i},$$

we obtain

$$J = \left[ \Phi(\zeta) \Psi(\zeta) \right]_a^b + \int_a^b \left( -\Phi(\zeta) \right) (\Psi(\zeta))^\Lambda \Delta \zeta$$

$$= \left[ -\Psi(\zeta) \int_a^\sigma(x) \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}} \Delta x \right]_a^b + \int_a^b \left( -\Phi(\zeta) \right) (\Psi(\zeta))^\Lambda \Delta \zeta$$

$$= \left[ -\Psi(b) \int_b^\sigma(x) \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}} \Delta x \right]_a^b + \int_a^b \left( -\Phi(\zeta) \right) (\Psi(\zeta))^\Lambda \Delta \zeta$$

$$\leq \int_a^b \left( -\Phi(\zeta) \right) (\Psi(\zeta))^\Lambda \Delta \zeta,$$

(36)

where $\Phi(\zeta) = -\int_a^\sigma(x) \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}} \Delta \zeta$ and $(\Psi(\zeta))^\Lambda > 0$. From (9), since $u_i^\lambda(\zeta) \geq 0$ and $c \in [\zeta, \sigma(\zeta)]$, we have

$$\left[ u_i^{-\alpha_i m}(\zeta) \right]^\Lambda = (1 - \kappa a, m) u_i^{-\alpha_i m}(\zeta) u_i^\lambda(\zeta)$$

$$= (1 - \kappa a, m) \frac{u_i^\lambda(\zeta)}{u_i^{-\alpha_i m}(\zeta)}$$

$$\leq (1 - \kappa a, m) \frac{u_i^\lambda(\zeta)}{|u_i^\rho(\zeta)|^{\alpha_i m}}.$$
Therefore, integrating (37) from \( \zeta \) to \( \infty \) with respect to \( x \), we have

\[
-\Phi(\zeta) \leq \frac{1}{\kappa \alpha_i m - 1} \mu_i 1^{1-\alpha_i m}(\zeta).
\] (38)

Combining (38) and (36), we have

\[
J \leq \frac{1}{\kappa \alpha_i m - 1} \int_a^b \mu_i 1^{1-\alpha_i m}(\zeta) (\Psi(\zeta))^{\Delta} \Delta \zeta.
\] (39)

Now, by applying (6) to \( \Psi(\zeta) = \hat{\psi}(\zeta) \hat{Y}^{\alpha_i}(\zeta) \) and using (9), we obtain

\[
(\Psi(\zeta))^{\Delta} = \hat{\psi}^{\Delta}(\zeta) [\hat{Y}^{\sigma}(\zeta)]^{x_{\alpha_i}} + \hat{\psi}(\zeta) [\hat{Y}^{x_{\alpha_i}}(\zeta)]^{\Delta}
\]

\[
= \hat{\psi}^{\Delta}(\zeta) [\hat{Y}^{\sigma}(\zeta)]^{x_{\alpha_i}} + \kappa \alpha_i \hat{\psi}(\zeta) \hat{Y}^{x_{\alpha_i-1}}(\zeta) \hat{Y}^{\Delta}(\zeta)
\]

\[
\leq \hat{\psi}^{\Delta}(\zeta) [\hat{Y}^{\sigma}(\zeta)]^{x_{\alpha_i}} + \kappa \alpha_i \hat{\psi}(\zeta) \frac{\mu_i(\zeta) z_{\alpha_i}(\zeta)}{z_i(\zeta)} f_{\zeta}(\zeta) [\hat{Y}^{\sigma}(\zeta)]^{x_{\alpha_i-1}},
\] (40)

where

\[
\hat{Y}(\zeta) = \int_a^\zeta \frac{\mu_i(x) z_{\alpha_i}(x)}{z_i(x)} f_{\zeta}(x) \Delta x.
\]

Substituting (40) into (39) and using (32), we get

\[
J \leq \frac{1}{\kappa \alpha_i m - 1} \int_a^b \mu_i 1^{1-\alpha_i m}(\zeta) \hat{\psi}^{\Delta}(\zeta) \left( \int_a^{x_{\alpha_i}} \frac{\mu_i(x) z_{\alpha_i}(x)}{z_i(x)} f_{\zeta}(x) \Delta x \right)^{x_{\alpha_i}} \Delta \zeta
\]

\[
+ \frac{\kappa \alpha_i}{\kappa \alpha_i m - 1} \int_a^b \mu_i 2^{1-\alpha_i m}(\zeta) \hat{\psi}(\zeta) z_{\alpha_i}(\zeta) f_{\zeta}(\zeta)
\]

\[
\times \left( \int_a^{x_{\alpha_i}} \frac{\mu_i(x) z_{\alpha_i}(x)}{z_i(x)} f_{\zeta}(x) \Delta x \right)^{x_{\alpha_i-1}} \Delta \zeta,
\]

\[
= \frac{1}{\kappa \alpha_i m - 1} \int_a^b \frac{\mu_i^2(\zeta) z_{\alpha_i}(\zeta)}{z_i(\zeta)} R^{x_{\alpha_i}}_{1a}(\zeta) \Delta \zeta
\]

\[
+ \frac{\kappa \alpha_i}{\kappa \alpha_i m - 1} \int_a^b \frac{\mu_i 2^{x_{\alpha_i m}}(\zeta) z_{\alpha_i}(\zeta)}{z_i(\zeta) (u_i^2(\zeta))^{1 - \frac{1}{x_{\alpha_i}}} R^{x_{\alpha_i-1}}_{1a}(\zeta) \Delta \zeta.
\]

Hence,

\[
\int_a^b \hat{\psi}^{\sigma}(\zeta) R^{x_{\alpha_i}}_{1a}(\zeta) \left( 1 - \frac{\mu_i^2(\zeta) z_{\alpha_i}(\zeta)}{(\kappa \alpha_i m - 1) u_i^{x_{\alpha_i m-1}}(\zeta) u_i^{\Delta}(\zeta) \hat{\psi}(\zeta) f_{\zeta}(\zeta) \Delta \zeta
\]

\[
\leq \frac{\kappa \alpha_i}{\kappa \alpha_i m - 1} \int_a^b \frac{\mu_i 2^{x_{\alpha_i m}}(\zeta) [u_i^2(\zeta)]^{x_{\alpha_i m-1}} z_{\alpha_i}(\zeta) f_{\zeta}(\zeta) \hat{\psi}(\zeta) R^{x_{\alpha_i-1}}_{1a}(\zeta) \Delta \zeta.
\] (41)
From (41) and (29), we have

$$
\int_a^b \hat{w}^\varphi(\zeta) R_{ia}^{\varphi(\zeta)}(\zeta) \Delta \zeta
\leq \frac{\kappa \alpha_1 \lambda_1}{\kappa \alpha_1 m - 1} \int_a^b \frac{u_i^{2 - \kappa \alpha_1 m}(\zeta) [u_i^\alpha(\zeta)]^{n(\kappa \alpha_1 - 1)} z_1^{\lambda}(\zeta) f_i(\zeta) \hat{w}(\zeta)}{z_i(\zeta)(u_i^\alpha(\zeta))^{1 - \frac{2}{\kappa \delta i}}} R_{ia}^{\varphi(\zeta)}(\zeta) \Delta \zeta
= \frac{\kappa \alpha_1 \lambda_1}{\kappa \alpha_1 m - 1} \int_a^b \left( \hat{w}^\varphi(\zeta) R_{ia}^{\varphi(\zeta)}(\zeta) \right)^{\frac{\kappa \delta i - 1}{\kappa \alpha_1}} \times \frac{u_i^{2 - \kappa \alpha_1 m}(\zeta) [u_i^\alpha(\zeta)]^{n(\kappa \alpha_1 - 1)} z_1^{\lambda}(\zeta) f_i(\zeta) \hat{w}(\zeta)}{z_i(\zeta)(u_i^\alpha(\zeta))^{1 - \frac{2}{\kappa \delta i}} (\hat{w}^\varphi(\zeta))^{1 - \frac{2}{\kappa \delta i}}} \Delta \zeta.
$$

Applying Hölder’s inequality with $\kappa \alpha_1$ and $\kappa \delta_i = \kappa \alpha_1 / (\kappa \alpha_1 - 1)$, we have

$$
\int_a^b \hat{w}^\varphi(\zeta) R_{ia}^{\varphi(\zeta)}(\zeta) \Delta \zeta
\leq \left( \frac{\kappa \alpha_1 \lambda_1}{\kappa \alpha_1 m - 1} \right)^{\frac{\kappa \alpha_1}{\kappa \delta_i}} \times \int_a^b \frac{u_i^{2 - \kappa \alpha_1 m}(\zeta) [u_i^\alpha(\zeta)]^{n(\kappa \alpha_1 - 1)} z_1^{\lambda}(\zeta) f_i(\zeta) \hat{w}(\zeta)}{z_i^{1 - \frac{2}{\kappa \delta i}}(\hat{w}^\varphi(\zeta))^{1 - \frac{2}{\kappa \delta i}}} \Delta \zeta
= \left( \frac{\kappa \alpha_1 \lambda_1}{\kappa \alpha_1 m - 1} \right)^{\frac{\kappa \alpha_1}{\kappa \delta_i}} \int_a^b \hat{w}^\varphi(\zeta) g_i(\zeta) \Delta \zeta.
$$

From (42) and (34), we have

$$
\sum_{i=1}^{n-k+2} \int_a^b \hat{w}^\varphi(\zeta) \left( \prod_{j=1}^{i+k-1} R_{ij}^{\varphi(\zeta)}(\zeta) \right) \Delta \zeta
\leq \sum_{i=1}^{n} \left( \frac{\kappa \alpha_1 \beta_i}{\kappa \alpha_1 m - 1} \right)^{\frac{\kappa \alpha_1}{\kappa \delta_i}} \int_a^b \hat{w}^\varphi(\zeta) g_i(\zeta) \Delta \zeta
\leq \sum_{i=1}^{n} \left( \frac{\kappa \alpha_1 \beta_i}{\kappa \alpha_1 m - 1} \right)^{\frac{\kappa \alpha_1}{\kappa \delta_i}} \int_a^\infty \hat{w}^\varphi(\zeta) g_i(\zeta) \Delta \zeta.
$$

Let us define for $m < 1 / \kappa \alpha_1$ and $0 < a < b < \infty$,

$$
R_{ib}(\zeta) = \frac{u_i^{\alpha}(\zeta)}{u_i^{\alpha}(\zeta)} \int_a^b \frac{u_i(x) z_i^{\lambda}(x)}{z_i(x)} f_i(x) \Delta x, \quad 1 \leq i \leq n,
$$

with $R_{i\infty}(\zeta) = R_i(\zeta)$. Following the same steps as in the proof of (43), we obtain

$$
\sum_{i=1}^{n-k+2} \int_a^b \hat{w}^\varphi(\zeta) \left( \prod_{j=1}^{i+k-1} R_{ij}^{\varphi(\zeta)}(\zeta) \right) \Delta \zeta
\leq \sum_{i=1}^{n} \left( \frac{\kappa \alpha_1 \beta_i}{1 - \kappa \alpha_1 m} \right)^{\frac{\kappa \alpha_1}{\kappa \delta_i}} \int_a^b \hat{w}^\varphi(\zeta) g_i(\zeta) \Delta \zeta.
$$
Corollary 12. We have then there exist for any and sequences  by letting \( = i \)

In Theorem 11, if we take \( T = \mathbb{N} \), then we have the following corollary.

**Corollary 12.** Let \( \{u(s)\}_{s=1}^{\infty} \), \( \{\hat{w}(s)\}_{s=1}^{\infty} \), and \( \{z(s)\}_{s=1}^{\infty} \) be increasing and nonnegative sequences. Then for any \( 1 \leq i \leq n \), \( n > \kappa - 1 \), \( \alpha_i > 1 \), \( \kappa \in \mathbb{N} \), \( \delta_i = \alpha_i/(\kappa \alpha_i - 1) \) and

\[
1 - \frac{[u_i(s + 1)]^{\alpha_i} \Delta \hat{w}(s)}{(\kappa \alpha_i - 1)[u_i(s)]^{\alpha_i} m \hat{w}(s + 1) \Delta u_i(s)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{2},
\]

we have

\[
\sum_{s=1}^{\infty} \frac{[u_i(s + 1)]^{\alpha_i} \Delta \hat{w}(s)}{(\kappa \alpha_i - 1)[u_i(s)]^{\alpha_i} m \hat{w}(s + 1) \Delta u_i(s)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{2},
\]

and \( \beta_i = \max_{1 \leq i \leq n}(\lambda_i, \beta_i) \), \( u_i(\infty) = \infty \).

**Remark 13.** If we put \( T = \mathbb{R} \) in Theorem 11, then (31) reduces to (5).

The next corollary follows from Theorem 11 by taking \( u_i(\zeta) = z_i(\zeta) = \zeta, f_i(\zeta) \rightarrow \zeta^{\alpha_i - 1}h_i, h_i = h_{i+1}, \alpha_i = \alpha_{i+1} \) and \( \kappa = 2 \).

**Corollary 14.** For any \( 1 \leq i \leq n \), \( n > \kappa - 1 \), \( \kappa \in \mathbb{N} \), if \( h_i \) are rd-continuous functions and there exist \( \lambda_i > 0 \) such that

\[
1 - \frac{\sigma_i^{2\alpha_i} h_i(z) \Delta \hat{w}(z)}{(2\alpha_i m - 1)\zeta^{2\alpha_i m - 1} \Delta \hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0,
\]

then

\[
\sum_{i=1}^{n} \int_{0}^{\infty} \hat{w}(\zeta) \left[ \frac{1}{\sigma_i^{m}(\zeta)} \int_{0}^{\sigma_i^{m}(\zeta)} x^{m-1} h_i(x) \Delta x \right]^{2\alpha_i} \Delta \zeta
\]

\[
\leq \sum_{i=1}^{n} \left( \frac{2\alpha_i^2}{2\alpha_i m - 1} \right)^{2\alpha_i} \int_{0}^{\infty} \hat{w}(\zeta) g_i(\zeta) \Delta \zeta,
\]
where
\[ g_i(\zeta) = \frac{\sigma^{2\alpha_i m(2\alpha_i-1)}(\zeta)\sigma^{2\alpha_i m(1-\alpha_i)} h_i^{2\alpha_i}(\zeta)\hat{\sigma}^{2\alpha_i}(\zeta)}{[\hat{\sigma}^{\alpha}(\zeta)]^{2\alpha_i}}, \quad \text{for } m > \frac{1}{2\alpha_i}. \]

**Remark 15** Letting \( T = \mathbb{R} \) in Corollary 14, we have that \( \sigma(\zeta) = \zeta \) and
\[ 1 - \frac{\zeta \hat{\sigma}(\zeta)}{(2\alpha_i m - 1)\hat{\sigma}(\zeta)} \geq \frac{1}{\lambda_i} > 0. \]

Then
\[
\sum_{i=1}^{n} \int_0^{\infty} \hat{\sigma}(\zeta) \left[ \frac{1}{m} \int_0^{\zeta} x^{m-1} h_i(x) \, dx \right]^{2\alpha_i} \, d\zeta \\
\leq \sum_{i=1}^{n} \left( \frac{2\alpha_i \lambda_i}{2\alpha_i m - 1} \right)^{2\alpha_i} \int_0^{\infty} \hat{\sigma}(\zeta) h_i^{2\alpha_i}(\zeta) \, d\zeta,
\]
which agrees with [4, Corollary 2].

**Theorem 16** For any \( 1 \leq i \leq n, n > \kappa - 1, \kappa \in \mathbb{N} \) and \( \frac{\zeta}{T} \), \( \frac{\sigma(\zeta)}{T} \in T \), if \( \alpha_i > 1, \delta_i = \alpha_i/(3\alpha_i - 1) \), and
\[ 1 + \frac{[u_i^\alpha(\zeta)]^{3\alpha_i m} h_i^{\alpha}(\zeta)}{(1 - 3\alpha_i m)u_i^{3\alpha_i m-1}(\zeta)u_i^{\alpha}(\zeta)\hat{\sigma}^{\alpha}(\zeta)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{3\alpha_i}, \]
then
\[ \sum_{i=1}^{n-1} \int_0^{\infty} \hat{\sigma}(\zeta) \Gamma_i(\zeta) \Delta \zeta \leq \sum_{i=1}^{n} \left( \frac{3\alpha_i \lambda_i}{3\alpha_i m - 1} \right)^{3\alpha_i} \int_0^{\infty} \hat{\sigma}(\zeta) g_i(\zeta) \Delta \zeta, \]
where
\[
\Gamma_i(\zeta) = \frac{3 \Delta^{\alpha_i}(\zeta)}{u_i^{\alpha}(\zeta)^{m}} \int_{\frac{\zeta}{T}}^{\sigma(\zeta)} \frac{u_i(\eta)z_i^\alpha(\eta)}{z_i(\eta)} f_i(\eta) \Delta \eta \\
\quad \text{and} \int_0^{\infty} u_i^{\alpha}(x) (u_i^{\alpha}(x))^{\alpha_i m} \Delta x < \infty, \\
\]
\[ g_i(\zeta) = \frac{[u_i^\alpha(\zeta)]^{3\alpha_i m - 1} h_i^{\alpha}(\zeta)}{u_i^{3\alpha_i m - 2}(\zeta) \sqrt{u_i^{\alpha}(\zeta) z_i(\zeta)}} \frac{1}{\sqrt{3\alpha_i}} f_i(\zeta) \
\quad - \frac{[u_i^\alpha(\zeta)]^{3\alpha_i m - 1} u_i(\zeta) z_i^\alpha(\xi)}{2u_i^{3\alpha_i m-1}(\zeta) z_i(\zeta) \sqrt{u_i^{\alpha}(\zeta)}} f_i(\xi) \frac{\xi}{2} \right]^{3\alpha_i}, \\
u_i(\infty) = \infty.
Proof Let us define for $m > \frac{1}{3\alpha_i}, 1 \leq i \leq n$,

$$\Gamma_i(\zeta) = \frac{3\sqrt{u_i^\alpha(\zeta)}}{[u_i^\alpha(\zeta)]^m} \int_{a_i(\eta)}^{\sigma_i(\eta)} \frac{u_i(\eta)z_i^\alpha(\eta)}{z_i(\eta)} f_i(\eta) \Delta \eta. \quad (47)$$

Using (11) with $\kappa = 3$, for $C_i = \Gamma_i^{3\alpha_i}(\zeta)$, we get

$$\sum_{i=1}^{n-1} \Gamma_i^{\alpha_i}(\zeta)\Gamma_{i+1}^{\alpha_{i+1}}(\zeta)\Gamma_{i+2}^{\alpha_{i+2}}(\zeta) \leq \sum_{i=1}^{n} \Gamma_i^{3\alpha_i}(\zeta). \quad (48)$$

Multiplying (48) by $\hat{w}^\sigma(\zeta)$ and integrating from 0 to $\infty$, we get

$$\sum_{i=1}^{n-1} \int_0^\infty \hat{w}^\sigma(\zeta)[\Gamma_i^{\alpha_i}(\zeta)\Gamma_{i+1}^{\alpha_{i+1}}(\zeta)\Gamma_{i+2}^{\alpha_{i+2}}(\zeta)] \Delta \zeta \leq \sum_{i=1}^{n} \int_0^\infty \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta) \Delta \zeta. \quad (49)$$

Now,

$$J = \int_0^\infty \hat{w}^\sigma(\zeta)\Gamma_i^{3\alpha_i}(\zeta) \Delta \zeta$$

$$= \int_0^\infty \hat{w}^\sigma(\zeta) \left[ \frac{3\sqrt{u_i^\alpha(\zeta)}}{[u_i^\alpha(\zeta)]^m} \int_{a_i(\eta)}^{\sigma_i(\eta)} \frac{u_i(\eta)z_i^\alpha(\eta)}{z_i(\eta)} f_i(\eta) \Delta \eta \right]^{3\alpha_i} \Delta \zeta$$

$$= \int_0^\infty \frac{u_i^\alpha(\zeta)}{[u_i^\alpha(\zeta)]^{3\alpha_i m}} \left[ \frac{3\sqrt{\hat{w}^\sigma(\zeta)}}{\int_{a_i(\eta)}^{\sigma_i(\eta)} f_i(\zeta) \Delta \eta} \right]^{3\alpha_i} \Delta \zeta. \quad (50)$$

Integrating (50) by parts using formula (7) with

$$\Phi^\Delta(\zeta) = \frac{u_i^\alpha(\zeta)}{[u_i^\alpha(\zeta)]^{3\alpha_i m}}, \quad \Psi^\sigma(\zeta) = \left[ \frac{3\sqrt{\hat{w}^\sigma(\zeta)}}{\int_{a_i(\eta)}^{\sigma_i(\eta)} f_i(\zeta) \Delta \eta} \right]^{3\alpha_i},$$

we obtain

$$J = \left[ \Phi(\zeta)\Psi(\zeta) \right]_0^\infty + \int_0^\infty \left( -\Phi(\zeta) \right) \left( \Psi(\zeta) \right)^\Delta \Delta \zeta$$

$$= \left[ -\Psi(\zeta) \int_{a_i(\eta)}^{\sigma_i(\eta)} \frac{u_i^\alpha(\eta)}{[u_i^\alpha(\eta)]^{3\alpha_i m}} \Delta \eta \right]_0^\infty + \int_0^\infty \left( -\Phi(\zeta) \right) \left( \Psi(\zeta) \right)^\Delta \Delta \zeta$$

$$= \left[ -\Psi(\zeta) \int_{a_i(\eta)}^{\sigma_i(\eta)} \frac{u_i^\alpha(\eta)}{[u_i^\alpha(\eta)]^{3\alpha_i m}} \Delta \eta \right]_0^\infty \int_0^\infty \left( -\Phi(\zeta) \right) \left( \Psi(\zeta) \right)^\Delta \Delta \zeta$$

$$\leq \int_0^\infty \left( -\Phi(\zeta) \right) \left( \Psi(\zeta) \right)^\Delta \Delta \Delta \zeta. \quad (51)$$
Therefore, integrating (52) from \( \zeta \) to \( \infty \) with respect to \( \eta \), we have

\[
-\Phi(\zeta) \leq \frac{1}{3\alpha_1m - 1} u_i^{1 - 3\alpha_1m}(\zeta). 
\]

Combining (53) and (51), we have

\[
J \leq \frac{1}{3\alpha_1m - 1} \int_0^\infty u_i^{1 - 3\alpha_1m}(\zeta) (\Psi(\zeta))^{\Delta} \Delta \zeta. 
\]
Hence,

\[
\int_0^x \hat{\omega}^{\sigma}(\zeta) \Gamma_{3\alpha_i}^{3\alpha i}(\zeta) \left(1 - \frac{[u_i^3(\zeta)\Gamma_{3\alpha_i}^{3\alpha i}(\zeta)]}{(3\alpha_i m - 1)u_i^{3\alpha_i m - 1}(\zeta)u_i^{\alpha}(\zeta)\hat{\omega}^{\lambda}(\zeta)}\right) \Delta \zeta
\]

\[
\leq \frac{3\alpha_i}{3\alpha_i m - 1} \int_0^x \frac{[u_i^3(\zeta)][3\alpha_i - 1]^{\alpha_i^{\lambda}}(\zeta)}{u_i^{3\alpha_i m - 2}(\zeta)\Gamma_{3\alpha_i}^{3\alpha_i^{\lambda}}(\zeta)z_i(\zeta)} f_i(\zeta) - \frac{[u_i^3(\zeta)][3\alpha_i - 1]^{\alpha_i^{\lambda}}u_i(\hat{\omega})z_i(\zeta)}{2u_i^{3\alpha_i m - 1}(\zeta)z_i(\zeta)\Gamma_{3\alpha_i}^{3\alpha_i^{\lambda}}(\zeta)f_i(\zeta)} \Delta \zeta
\]

From (56) and (45), we have

\[
\int_0^x \hat{\omega}^{\sigma}(\zeta) \Gamma_{3\alpha_i}^{3\alpha i}(\zeta) \Delta \zeta
\]

\[
\leq \frac{3\alpha_i \lambda_i}{3\alpha_i m - 1} \int_0^x \hat{\omega}^{\sigma}(\zeta) \Gamma_{3\alpha_i}^{3\alpha_i - 1}(\zeta)
\]

Applying Hölder’s inequality with 3\alpha_i and \alpha_i/(3\alpha_i - 1), we have

\[
\int_0^x \hat{\omega}^{\sigma}(\zeta) \Gamma_{3\alpha_i}^{3\alpha i}(\zeta) \Delta \zeta
\]

\[
\leq \left( \frac{3\alpha_i \lambda_i}{3\alpha_i m - 1} \right)^{3\alpha_i} \int_0^x \hat{\omega}^{\sigma}(\zeta)
\]

From (57) and (49), we get (46).

In Theorem 16, if we take T = N, then we obtain the following corollary.

**Corollary 17** For any \{u(s)\}_{i=1}^{\infty}, \{\hat{u}(s)\}_{i=1}^{\infty}, and \{z(s)\}_{i=1}^{\infty} increasing and nonnegative sequences, 1 \leq i \leq n, if \alpha_i > 1, \delta_i = \alpha_i/(3\alpha_i - 1), and

\[
1 + \frac{[u_i(s + 1)]^{3\alpha_i m} \Delta \hat{u}(s)}{(1 - 3\alpha_i m)u_i^{3\alpha_i m - 1}(s)\Delta u(s)} \geq \frac{1}{\lambda_i} > 0, \quad \text{for } m > \frac{1}{3\alpha_i},
\]
then

\[
\sum_{i=1}^{n-1} \left( \sum_{j=1}^{n-1} \tilde{w}(s+1) \left[ \Gamma_{ij}^{\alpha_1}(s) \Gamma_{ij+1}^{\alpha_2}(s) \Gamma_{ij+2}^{\alpha_3}(s) \right] \right) \\
\leq \sum_{i=1}^{n} \left( \frac{3\alpha_i \lambda_i}{3\alpha_i m - 1} \sum_{j=1}^{n-1} \tilde{w}(s+1) g_i(s), \right)
\]

where

\[
\Gamma_i(s) = \frac{3\sqrt{\Delta u_i(s)}}{u_i^{\alpha}(s+1)} \sum_{r=\frac{s}{4}}^{s} \frac{u_i(r) \Delta z_i(r)}{\Delta u_i(s)} f_i(r), \quad s + \frac{1}{2}, \frac{s}{2} \in \mathbb{N},
\]

\[
g_i(s) = \left[ \frac{[u_i(s+1)]^{3\alpha_i m-1} \Delta z_i(s) f_i(s)}{u_i^{3\alpha_i m-2}(s) 3\sqrt{\Delta u_i(s)} z_i(s)} - \frac{[u_i(s+1)]^{3\alpha_i m-1} u_i(z_i) \Delta z_i(z_i) \Delta u_i(s) f_i(s)}{2 u_i^{3\alpha_i m-1}(s) z_i(z_i) 3\sqrt{\Delta u_i(s)} f_i(s)} \right]^{3\alpha_i}
\]

and \( u_i(\infty) = \infty. \)

**Remark 18** Clearly, for \( T = \mathbb{R}, \) Theorem 16 reduces to [4, Theorem 3].

**Theorem 19** For any \( 1 \leq i \leq n, \) if \( \alpha > 1, \kappa \geq 1, \delta = \alpha / (\kappa \alpha - 1), \) and there exist \( \lambda_i > 0, m > 0 \) such that

\[
1 + \frac{u_i^{\alpha}(\zeta) \tilde{w}(\zeta)}{(1 + \kappa m u_i^{\alpha}(\zeta) \tilde{w}(\zeta))} \geq \frac{1}{\lambda_i} > 0,
\]

then

\[
\int_a^b \hat{w}(\zeta) \left( \sum_{i=1}^{n} \Gamma_{ia}(\zeta) \right)^{\frac{\kappa}{\alpha}} \Delta \zeta \leq \sum_{i=1}^{n} \left( \frac{-\kappa \alpha \lambda_i \sqrt{\kappa \Delta z_i}}{1 + \kappa m} \right)^{\frac{\kappa}{\alpha}} \int_a^b \hat{w}(\zeta) g_i(\zeta) \Delta \zeta,
\]

where

\[
\Gamma_i(\zeta) = \left[ u_i^{\alpha}(\zeta) \right]^{\frac{\kappa}{\alpha}} \frac{z_i^{\alpha}(\zeta)}{u_i^{\alpha}(\zeta) z_i(\zeta)} \int_0^x f_i(x) \Delta x, \quad \zeta \in [0, \infty)_T,
\]

and

\[
g_i(\zeta) = \left[ u_i^{\alpha}(\zeta) \right]^{\frac{\kappa}{\alpha}} \frac{\tilde{w}(\zeta) z_i^{\alpha}\alpha(\zeta)}{u_i^{\alpha}(\zeta) z_i^{\alpha}(\zeta)} f_i(x) \Delta x.
\]

**Proof** Let us define for \( 1 \leq i \leq n \) and \( 0 < a < b < \infty, \)

\[
\Gamma_{ia}(\zeta) = \left[ u_i^{\alpha}(\zeta) \right]^{\frac{\kappa}{\alpha}} \frac{z_i^{\alpha}(\zeta)}{u_i^{\alpha}(\zeta) z_i(\zeta)} \int_a^z f_i(x) \Delta x.
\]
Using (12), for $C_i = \Gamma_i(\xi)$ and $\kappa \to \kappa \alpha$, we get
\[
\left( \sum_{i=1}^{n} \Gamma_{i\alpha}(\xi) \right)^{\kappa \alpha} \leq n^{\alpha n - 1} \sum_{i=1}^{n} \Gamma_{i\alpha}(\xi). \tag{61}
\]

Multiplying (61) by $\hat{w}^\lambda(\xi)$ and integrating from $a$ to $b$, we get
\[
\int_{a}^{b} \hat{w}(\xi) \left( \sum_{i=1}^{n} \Gamma_{i\alpha}(\xi) \right)^{\kappa \alpha} \Delta \xi \leq n^{\alpha n - 1} \sum_{i=1}^{n} \int_{a}^{b} \hat{w}(\xi) \Gamma_{i\alpha}(\xi) \Delta \xi. \tag{62}
\]

Now,
\[
J = \int_{a}^{b} \hat{w}(\xi) \Gamma_{i\alpha}(\xi) \Delta \xi
\]
\[
= \int_{a}^{b} \hat{w}(\xi) \left[ \left[ u_1(\xi) \right]^{m \alpha} \left( \frac{\nu \hat{w}}{\mu} \right) \int_{a}^{b} \frac{z_1^{\alpha}(x)}{u_1^{\alpha}(x)} f(x) \Delta x \right]^{\kappa \alpha} \Delta \xi \tag{63}
\]
\[
= \int_{a}^{b} \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} \left( \frac{\nu \hat{w}}{\mu} \right) \int_{a}^{b} \frac{z_1^{\alpha}(x)}{u_1^{\alpha}(x)} f(x) \Delta x \right]^{\kappa \alpha} \Delta \xi.
\]

Integrating (63) by parts using formula (7) with
\[
\Phi^{\lambda}(\xi) = \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} \left( \frac{\nu \hat{w}}{\mu} \right) \int_{a}^{b} \frac{z_1^{\alpha}(x)}{u_1^{\alpha}(x)} f(x) \Delta x,
\]
\[
\Psi(\xi) = \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} \left( \frac{\nu \hat{w}}{\mu} \right) \int_{a}^{b} \frac{z_1^{\alpha}(x)}{u_1^{\alpha}(x)} f(x) \Delta x,
\]
we obtain
\[
J = \left[ \Phi(\xi) \Psi(\xi) \right]_{a}^{b} + \int_{a}^{b} \left( \Phi^{\lambda}(\xi) \right) \left( -\Psi(\xi) \right)^{\Delta \lambda} \Delta \xi
\]
\[
= \left[ -\Psi(\xi) \int_{a}^{b} \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} u_1^{\alpha}(x) \Delta x \right]_{a}^{b} + \int_{a}^{b} \left( \Phi^{\lambda}(\xi) \right) \left( -\Psi(\xi) \right)^{\Delta \lambda} \Delta \xi \tag{64}
\]
\[
= \left[ \Phi^{\lambda}(\xi) \right] \left( -\Psi(\xi) \right)^{\Delta \lambda} \Delta \xi,
\]

where $\Phi(\xi) = -\int_{a}^{b} \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} u_1^{\alpha}(x) \Delta x$ and $(\Psi(\xi))^\Delta > 0$. From (9), using $u_1^{\alpha}(\xi) \geq 0$ and $c \in [\xi, \sigma(\xi)]$, we have
\[
\left[ u_1^{\alpha}(\xi) \right]^{\Delta} = (1 + \kappa \alpha m) u_1^{\alpha}(c) u_1^{\alpha}(\xi)
\]
\[
\leq (1 + \kappa \alpha m) \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} u_1^{\alpha}(\xi).
\]

This implies that
\[
\frac{1}{1 + \kappa \alpha m} \int_{\xi}^{b} \left[ u_1^{\alpha}(\xi) \right]^{\Delta} \Delta x \leq \int_{\xi}^{b} \left[ u_1^{\alpha}(\xi) \right]^{m \alpha} u_1^{\alpha}(x) \Delta x = -\Phi(\xi).
\]
Therefore

\[
\Phi(\xi) \leq \frac{-1}{1 + \kappa \alpha m} \int_{\xi}^{b} \left[ u_{1}^{1 + \kappa \alpha m}(x) \right]^{\lambda} \Delta x \\
= \frac{-1}{1 + \kappa \alpha m} \left[ u_{1}^{1 + \kappa \alpha m}(b) - u_{1}^{1 + \kappa \alpha m}(\xi) \right] \\
= \frac{1}{1 + \kappa \alpha m} \left[ u_{1}^{1 + \kappa \alpha m}(\xi) - u_{1}^{1 + \kappa \alpha m}(b) \right] \\
\leq \frac{1}{1 + \kappa \alpha m} u_{1}^{1 + \kappa \alpha m}(\xi). \tag{65}
\]

Substituting (65) into (64), we have

\[
J \leq \frac{1}{1 + \kappa \alpha m} \int_{a}^{b} \left[ \mu_{\xi}(\xi) \right]^{1 + \kappa \alpha m} (-\Psi(\xi))^{\Delta} \Delta \zeta. \tag{66}
\]

Now, by applying (6) to \(\Psi(\xi) = \hat{\mu}(\xi) \hat{\gamma}^{\kappa \alpha}(\xi)\) and using (9), we obtain

\[
\left( \Psi(\xi) \right)^{\Delta} = \hat{\mu}^{\Delta}(\xi) \hat{\gamma}^{\kappa \alpha}(\xi) + \hat{\omega}^{\kappa}(\xi) \left[ \hat{\gamma}^{\kappa \alpha}(\xi) \right]^{\Delta} \\
= \hat{\mu}^{\Delta}(\xi) \hat{\gamma}^{\kappa \alpha}(\xi) + \kappa \alpha \hat{\mu}^{\kappa}(\xi) \hat{\gamma}^{\kappa \alpha - 1}(\xi) \hat{\gamma}^{\Delta}(\xi) \\
\geq \hat{\mu}^{\Delta}(\xi) \hat{\gamma}^{\kappa \alpha}(\xi) + \kappa \alpha \hat{\mu}^{\kappa}(\xi) \left[ \frac{z_{\xi}(\xi)}{u_{\xi}(\xi) \zeta_{i}(\xi)} f_{i}(\xi) \right] \hat{\gamma}^{\kappa \alpha - 1}(\xi). \tag{67}
\]

From (67) and (66), as well as using (60), we have

\[
J \leq \frac{-1}{1 + \kappa \alpha m} \int_{a}^{b} \left[ u_{\xi}(\xi) \right]^{1 + \kappa \alpha m} \hat{\mu}^{\Delta}(\xi) \left( \int_{a}^{\xi} \frac{z_{\xi}(x)}{u_{\xi}(x) \zeta_{i}(x)} f_{i}(x) \Delta x \right)^{\kappa \alpha} \Delta \zeta \\
+ \frac{-\kappa \alpha}{1 + \kappa \alpha m} \int_{a}^{b} \left[ u_{\xi}(\xi) \right]^{1 + \kappa \alpha m} \hat{\mu}^{\kappa}(\xi) \left( \int_{a}^{\xi} \frac{z_{\xi}(x)}{u_{\xi}(x) \zeta_{i}(x)} f_{i}(x) \Delta x \right)^{\kappa \alpha - 1} \Delta \zeta \\
\leq \frac{1}{1 + \kappa \alpha m} \int_{a}^{b} u_{\xi}(\xi) \hat{\mu}^{\Delta}(\xi) \frac{1}{1 + \kappa \alpha m} \hat{\mu}^{\kappa}(\xi) \Delta \zeta \\
+ \frac{-\kappa \alpha}{1 + \kappa \alpha m} \int_{a}^{b} \left[ u_{\xi}(\xi) \right]^{1 + m} \hat{\mu}^{\kappa}(\xi) \frac{1}{1 + \kappa \alpha m} \hat{\mu}^{\kappa - 1}(\xi) \left( \frac{z_{\xi}(\xi)}{u_{\xi}(\xi) \zeta_{i}(\xi)} f_{i}(\xi) \right) \Delta \zeta.
\]

Hence,

\[
\int_{a}^{b} \hat{\mu}(\xi) \hat{\gamma}^{\kappa \alpha}(\xi) \left( 1 + \frac{u_{\xi}(\xi) \hat{\mu}^{\Delta}(\xi)}{(1 + \kappa \alpha m) u_{\xi}(\xi) \hat{\mu}(\xi)} \right) \Delta \zeta \\
\leq \frac{-\kappa \alpha}{1 + \kappa \alpha m} \int_{a}^{b} \left[ u_{\xi}(\xi) \right]^{1 + m} \hat{\mu}^{\kappa}(\xi) \frac{1}{1 + \kappa \alpha m} \hat{\mu}^{\kappa - 1}(\xi) \left( \frac{z_{\xi}(\xi)}{u_{\xi}(\xi) \zeta_{i}(\xi)} f_{i}(\xi) \right) \Delta \zeta. \tag{68}
\]
From (68) and (58), we have

\[
\int_a^b \hat{w}(\xi) \Gamma_{\alpha \lambda_1}^{\kappa \alpha} (\xi) \Delta \xi \\
\leq \frac{-\kappa \alpha \lambda_1}{1 + \kappa \alpha m} \int_a^b \left( \frac{[u_i^\alpha(\xi)]^{\kappa \alpha}[\hat{w}^\alpha(\xi)]^{\kappa \alpha}[\hat{z}_i^\alpha(\xi)]^{\kappa \alpha}}{[u_i^\alpha(\xi)]^{\kappa \alpha}[\hat{w}^\alpha(\xi)]^{\kappa \alpha}[\hat{z}_i^\alpha(\xi)]^{\kappa \alpha}} \Gamma_1^{\kappa \alpha}(\xi) \Delta \xi. \right.
\]

Applying Hölder's inequality with \( \kappa \alpha \) and \( \alpha / (\kappa \alpha - 1) \), we have

\[
\int_a^b \hat{w}(\xi) \Gamma_{\alpha \lambda_1}^{\kappa \alpha} (\xi) \Delta \xi \\
\leq \left( \frac{-\kappa \alpha \lambda_1}{1 + \kappa \alpha m} \right)^{\kappa \alpha} \int_a^b \left( \frac{[u_i^\alpha(\xi)]^{\kappa \alpha}[\hat{w}^\alpha(\xi)]^{\kappa \alpha}[\hat{z}_i^\alpha(\xi)]^{\kappa \alpha}}{[u_i^\alpha(\xi)]^{\kappa \alpha}[\hat{w}^\alpha(\xi)]^{\kappa \alpha}[\hat{z}_i^\alpha(\xi)]^{\kappa \alpha}} \Gamma_1^{\kappa \alpha}(\xi) \Delta \xi. \right.
\]

From (69) and (62), we get (59).

In Theorem 19, if we take \( T = \mathbb{N} \), then we obtain the following corollary.

**Corollary 20** For any \( \{u(s)\}_{i=1}^\infty \), \( \{\hat{w}(s)\}_{i=1}^\infty \), and \( \{z(s)\}_{i=1}^\infty \) increasing and nonnegative sequences, if \( 1 \leq i \leq n, \alpha > 1, \kappa \geq 1, \delta = \alpha / (\kappa \alpha - 1) \), and

\[
1 + \frac{u_i(s + 1) \Delta \hat{w}(s)}{(1 + \kappa \alpha m) \hat{w}(s) \Delta u_i(s)} \geq \frac{1}{\lambda_i} > 0,
\]

then

\[
\sum_{i=1}^{r-1} \hat{w}(s + 1) \left( \sum_{i=1}^n \Gamma_{\alpha \lambda_1}(s) \right)^{\kappa \alpha} \leq \sum_{i=1}^n \left( \frac{-\kappa \alpha \lambda_1 \sqrt{\kappa}}{1 + \kappa \alpha m} \right)^{\kappa \alpha} \sum_{s=1}^{r-1} \hat{w}(s) g_i(s),
\]

where

\[
\Gamma_1(s) = \left[ u_i(s + 1) \right]^{\kappa \alpha} \sum_{q=1}^{s-1} \frac{\Delta z_i(q)}{u_i(q + 1) z_i(q)} f_i(q)
\]

and

\[
g_i(s) = \left[ u_i(s + 1) \right]^{\kappa \alpha} \left[ \frac{\hat{w}(s + 1) \Delta z_i(s)}{[\Delta u_i(s)]^{\delta} \hat{w}(s) z_i^{\kappa \alpha}(s)} \right]^{\kappa \alpha} f_i^{\kappa \alpha}(s).
\]

**Remark 21** Clearly, for \( T = \mathbb{R} \), Theorem 16 reduces to [4, Theorem 4].

The next corollary follows from Theorem 19 by taking \( u_i(\xi) = z_i(\xi) = \xi, f_i(\xi) \to \xi^{1-\gamma} h_1 \).

**Corollary 22** For any \( 1 \leq i \leq n \) and \( \alpha > 1 \), if \( h_1 \) are rd-continuous functions and

\[
1 + \frac{u_i(\xi) \Delta z_i(\xi)}{(\kappa \alpha m + 1) \hat{w}(\xi)} \geq \frac{1}{\lambda_i} > 0,
\]
then
\[
\int_a^b \hat{w}(\zeta) \left[ \sum_{i=1}^n \sigma^m(\zeta) \int_0^c \frac{x^{2-m}}{\sigma(x)} h_i(x) \Delta x \right]^{\kappa \alpha} \Delta \zeta
\]
\[
\leq n^{\alpha_{k-1}} \sum_{i=1}^n \left( \frac{-\kappa \alpha \lambda_i}{1 + \kappa \alpha m} \right)^{\kappa \alpha} \int_a^b \hat{w}(\zeta) g_i(\zeta) \Delta \zeta,
\]
where
\[
g_i(\zeta) = \left[ \sigma(\zeta)^{\kappa \alpha m} \hat{w}^\sigma(\zeta)^{\kappa \alpha} \xi^{\kappa \alpha - \kappa \alpha m} \hat{w}^{\kappa \alpha}(\zeta) \right]^{\kappa \alpha} h_i^{\kappa \alpha}(\zeta).
\]

Remark 23 Letting \( T = \mathbb{R} \) in Corollary 22, we have that \( \sigma(\zeta) = \zeta \) and
\[
1 + \frac{\zeta \hat{w}'(\zeta)}{(\kappa \alpha m + 1) \hat{w}(\zeta)} \geq \frac{1}{\lambda_i} > 0.
\]

Then
\[
\int_a^b \hat{w}(\zeta) \left[ \sum_{i=1}^n \xi^m \int_0^c \frac{1}{\xi^{1+m}} h_i(x) \, dx \right]^{\kappa \alpha} \, d\zeta
\]
\[
\leq n^{\alpha_{k-1}} \sum_{i=1}^n \left( \frac{-\kappa \alpha \lambda_i}{1 + \kappa \alpha m} \right)^{\kappa \alpha} \int_a^b \hat{w}(\zeta) h_i^{\kappa \alpha}(\zeta) \, d\zeta,
\]
which agrees with [4, Corollary 3].

4 Conclusion
In this work, we explored some new generalized inequalities involving many functions of Hardy type on time scales by using delta calculus. Further, we also applied our inequalities to discrete and continuous calculus to obtain some new Hardy inequalities as special cases. In a future work, we will continue to generalize more dynamic inequalities by conformable delta fractional calculus on time scales.

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