Riemann-Roch and Riemann-Hurwitz theorems for global fields

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In this paper, we use counting theorems from the geometry of numbers to extend the Riemann-Roch theorem and the Riemann-Hurwitz formula to global fields of arbitrary characteristic.

In the first part of the paper (cf. §1), we review some basic definitions and results from the theory of valuations on global fields.

In the following section (cf. §2), we define a divisor on a global field $K$ to be a formal linear combination of equivalence classes of valuations on $K$ (cf. Definition 2.1). If the characteristic of $K$ is zero, this definition coincides with the notion of an Arakelov divisor. If the characteristic of $K$ is positive, the definition coincides with the ordinary notion of a divisor on the complete nonsingular curve with function field $K$. We then deviate from standard fare, by defining the degree of a divisor $D$ multiplicatively (cf. Definition 2.1). (The usual notion of the degree of $D$ is simply recovered by taking the logarithm to a suitable base.)

For a divisor $D$, we define its set of multiples $H^0(D)$ in a standard manner (cf. Definition 2.2). However, we cannot define $h^0(D)$ in the usual way, as the dimension of $H^0(D)$, since this would not make sense when the characteristic of $K$ is zero. Instead, we declare that $h^0(D)$ is the cardinality of $H^0(D)$.

With these definitions, we are in a position to formulate a Riemann-Roch type result for global fields (cf. Theorem 2.1), using a counting theorem for metrized modules, proved by H. Gillet and C. Soulé (cf. [1]) in the characteristic zero case. The statement is that there exists a divisor $\omega_K$ on the global field $K$ such that

$$\frac{1}{C(K)} \leq \frac{h^0(D)}{h^0(\omega_K - D)} \cdot \frac{\sqrt{\deg \omega_K}}{\deg D} \leq C(K),$$

for any divisor $D$ on $K$. 

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In the inequalities above, \( C(K) \) denotes a quantity depending only on the archimedean valuations on \( K \). In particular, \( C(K) = 1 \) iff \( K \) has no archimedean valuations, in which case we recover the classical Riemann-Roch theorem by taking logarithms. We also use a theorem from [3] to conclude that there exists a function \( i \) on the set of divisors on \( K \) such that \( i(\cdot) \to 1 \) as \( \deg \cdot \to \infty \), and such that

\[
\frac{h^0(D)}{i(D)} \cdot \frac{\sqrt{\deg \omega_K}}{\deg D} = B(K),
\]

for any divisor \( D \) on \( K \). Here \( B(K) \) denotes a quantity depending only on the archimedean valuations on \( K \), such that \( B(K) = 1 \) iff \( K \) has no archimedean valuations. Once again, we recover the classical Riemann-Roch theorem in this case.

In the final section (cf. §3), we associate to a finite separable extension of global fields \( L/K \), a divisor \( R_{L/K} \) on \( L \). For a special choice of \( K \) (cf. Definition 1.2), this gives us a "canonical divisor" \( \omega_L \) on \( L \) that can be used in Theorem 2.1. Finally, we note that the divisors so defined satisfy a formula

\[
\deg \omega_L = \deg \omega_K[L:K] \cdot \deg R_{L/K},
\]

which coincides with the classical Riemann-Hurwitz formula when the global fields have positive characteristic.

1 Global fields and their valuations

In this section, we introduce some notation, and give a brief review of the theory of valuations on global fields.

**Definition 1.1** Let \( K \) be a field. A valuation on \( K \) is a mapping \( \varphi : K \to \mathbb{R}^+ \), such that

1. \( \varphi(\alpha) = 0 \) if and only if \( \alpha = 0 \),
2. \( \varphi(\alpha \beta) = \varphi(\alpha) \varphi(\beta) \),
3. \( \varphi(\alpha + \beta) \leq \varphi(\alpha) + \varphi(\beta) \),

for all \( \alpha, \beta \in K \). A valuation \( \varphi \) on \( K \) is said to be archimedean if

\[
\varphi(\alpha + \beta) > \max(\varphi(\alpha), \varphi(\beta))
\]

for some \( \alpha, \beta \in K \). \( \square \)
We define an equivalence relation $\sim$ on the set of valuations on $K$, by declaring that $\varphi \sim \phi$ if and only if
\[ \varphi(\alpha) < 1 \Leftrightarrow \phi(\alpha) < 1, \quad \text{for all } \alpha \in K. \]
We denote by $\sum_K$ the set of equivalence classes under this relation.

It is easily verified that whenever a valuation $\varphi$ is archimedean, $\varphi \sim \phi$ implies that the valuation $\phi$ is also archimedean. We say that an element $P \in \sum_K$ is archimedean if $P$ contains an archimedean valuation, and we denote by $\sum_\infty K$ the set of archimedean elements in $\sum_K$.

**Definition 1.2** By a global field we mean either
- a finite extension of the field $\mathbb{Q}$, or
- a finite extension of a field $\mathbb{F}_q(t)$ of rational functions in an indeterminate $t$ over a finite field $\mathbb{F}_q$.

For a global field $K$, we set
\[ K_0 = \begin{cases} \mathbb{Q}, & \text{if } \text{char}(K) = 0, \\ \mathbb{F}_q(t), & \text{if } \text{char}(K) > 0. \end{cases} \]

**Remark 1.1** If $K$ is a global field of characteristic zero, the extension $K/K_0$ is separable (cf. [2], Corollary 6.12 in §V.6). For a global field $K$ of positive characteristic, this is not always the case for an arbitrary choice of $t$. However, there is at least one choice of $t$ which makes the extension $K/K_0$ separable (cf. [2], Proposition 4.9 in §VIII.4), and in the sequel, we shall assume that such a choice is made in Definition 1.2.

Let $K$ be a global field, and choose a representative $\phi_P \in P$, for each $P \in \sum_K$. Set
\[ A_P = \{ \alpha \in K; \phi_P(\alpha) \leq 1 \}, \quad M_P = \{ \alpha \in K; \phi_P(\alpha) < 1 \}. \]
Denote by $\hat{K}_P$ the completion of $K$ with respect to $\phi_P$. We define a function $N: \sum_K \to [1, \infty) \subset \mathbb{R}$, by letting
\[ N(P) = \begin{cases} e^{\text{dim}_K \hat{K}_P}, & \text{if } P \in \sum_\infty K, \\ \# (A_P/M_P), & \text{if } P \in \sum_K \setminus \sum_\infty K. \end{cases} \]
Indeed, it is easily seen that both $\hat{K}_P$ and $A_P/M_P$ are independent of the choice of $\phi_P \in P$. Since $K$ is global, the residue field $A_P/M_P$ is finite for all $P \in \sum_K \setminus \sum_K^\infty$ (cf. [4]). For $P \in \sum_K^\infty$, the completion $\hat{K}_P$ is either $\mathbb{R}$ or $\mathbb{C}$.

We define the integers $S_i(K) = i \cdot \# \{(P \in \sum_K^\infty; \log N(P) = i)\}, i \in \{1, 2\}$, determined by the archimedean valuations on $K$.

**Remark 1.2** With this notation, $S_1(K) = S_2(K) = 0$ if $\text{char}(K) > 0$. If $\text{char}(K) = 0$, then $S_1(K)$ is the number of real embeddings of $K$, and $S_2(K)$ is the number of complex embeddings of $K$. 

For $P \in \sum_K \setminus \sum_K^\infty$, we define the normalized valuation $\varphi_P \in P$ by requiring that

$$-\log_{N(P)} \varphi_P(K) = \mathbb{Z} \cup \{\infty\}.$$ 

If $P \in \sum_K^\infty$, there is a unique embedding $\theta_P : K \to \hat{K}_P$ corresponding to $P$. We let $|\cdot|$ be the usual absolute value on $\hat{K}_P (= \mathbb{R}$ or $\mathbb{C})$, and define

$$\varphi_P = |\theta_P|^{\dim \hat{K}_P}.$$ 

We have the following product formula (cf. [4]).

**Theorem 1.1** If $K$ is a global field, then

$$\prod_{P \in \sum_K} \varphi_P(\alpha) = 1,$$

for all $\alpha \in K^\ast$. 

Consider a global field $L$, and assume that $K$ is another global field such that the extension $L/K$ is finite and separable. To each element $Q \in \sum_L$, we shall now associate an integer $e_Q$ and a real number $r_Q$, depending on this extension.

For $Q \in \sum_L$, we denote by $P_Q$ the element in $\sum_K$ that contains the restriction $\varphi_Q|_K$. We let $B_{P_Q}$ be the integral closure of $A_{P_Q}$ in $L$, and denote by $\hat{B}_{P_Q}$ and $\hat{A}_{P_Q}$ the corresponding completed rings.
Definition 1.3 The ramification index of \( Q \) relative to the extension \( L/K \), is the integer

\[
e_Q = \begin{cases} 
\lceil \hat{L}_Q : \hat{R}_P \rceil, & \text{if } Q \in \sum_L^\infty, \\
(\phi_Q(L^*) : \phi_P(Q)\hat{K}^*), & \text{if } Q \in \sum_L \setminus \sum_L^\infty.
\end{cases}
\]

For \( Q \in \sum_L^\infty \), we define

\[
r_Q = \begin{cases} 
- \log \log N(Q), & \text{if } e_Q \neq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

For \( Q \in \sum_L \setminus \sum_L^\infty \), we define \( r_Q \) to be the exponent of \( \hat{M}_Q \) in the different of \( \hat{B}_{P_Q} \) over \( \hat{A}_{P_Q} \) (cf. [6], §3 in Chapter III). \( \square \)

Remark 1.3 When \( \text{char}(\mathbb{K}) = 0 \), we obtain with this definition

\[
\prod_{P \in \sum_K \setminus \sum_K^\infty} N(P)^{r_Q} = |\text{Disc}_K|,
\]

where \( \text{Disc}_K \) denotes the discriminant of the number field \( \mathbb{K} \). This follows from Proposition 6 and Proposition 10 in Chapter III of [6]. \( \square \)

Throughout the paper, we employ the conventions of letting empty sums equal 0, and letting empty products equal 1.

2 Divisors on global fields

In this section, we define and study divisors on global fields. In particular, we consider the set of multiples of a divisor \( D \), and relate its cardinality to the degree of \( D \) (cf. Theorem 2.1).
Definition 2.1 Let $\mathbb{K}$ be a global field. A divisor $D$ on $\mathbb{K}$ is a formal finite sum

\[ D = \sum_{P \in \mathbb{K}} a_P \cdot P, \]

where $a_P \in \mathbb{R}$ if $P \in \mathbb{K}^\infty$, and $a_P \in \mathbb{Z}$ otherwise. The degree of $D$ is the real number

\[ \deg D = \prod_{P \in \mathbb{K}} N(P)^{a_P}. \]

The divisor $D$ is said to be principal if there exists an $\alpha \in \mathbb{K}^*$, such that

\[ N(P)^{a_P} = \varphi_P(\alpha), \]

for all $P \in \mathbb{K}^\infty$. \hfill \Box

We may now state Theorem 1.1 by saying that a principal divisor has degree 1. It follows that the identity element in the group defined by the degree homomorphism

\[ \deg : \{\text{divisors on } \mathbb{K}\} \to \mathbb{R}, \]

is the class containing the principal divisors. The inverse of the class containing a divisor $D = \sum a_P \cdot P$, is the class containing the divisor

\[ -D = \sum (-a_P) \cdot P. \]

Definition 2.2 Let $D = \sum a_P \cdot P$ be a divisor on a global field $\mathbb{K}$. The space of multiples of $D$ is the set

\[ H^0(D) = \{ \alpha \in \mathbb{K}; \varphi_P(\alpha) \leq N(P)^{a_P}, \text{ for all } P \in \mathbb{K}' \}. \]

We denote by $h^0(D)$ the cardinality of $H^0(D)$. \hfill \Box

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Theorem 2.1 Let $D$ be a divisor on a global field $K$.

(i) There exists a divisor $\omega_K$, depending only on $K$, such that

$$\frac{1}{C(S_1(K), S_2(K))} \leq \frac{h^0(D)}{h^0(\omega_K - D)} \cdot \frac{\sqrt{\deg \omega_K \deg D}}{\deg D} \leq C(S_1(K), S_2(K)),$$

with

$$C(S_1(K), S_2(K)) = \frac{6^{S_1(K)} + S_2(K) \cdot (S_1(K) + S_2(K))!}{2^{S_1(K)} \cdot (\pi/2)^{S_2(K)}}.$$

(ii) There exists a function $i : \{\text{divisors on } K\} \to \mathbb{R}$, such that $i(\cdot) \to 1$ when $\deg \cdot \to \infty$, and

$$\frac{h^0(D)}{i(D)} \cdot \frac{\sqrt{\deg \omega_K \deg D}}{\deg D} = 2^{S_1(K)} \cdot (2\pi)^{S_2(K)/2}.$$

Proof Assume first that $\text{char}(K) = 0$, and denote by $O_K$ the integral closure of $\mathbb{Z}$ in $K$. Consider the $O_K$-module $\text{Hom}_\mathbb{Z}(O_K, \mathbb{Z})$, metrized by defining $|\text{Tr} P| = \log N(P)$, for $P \in \sum_K \sum_K \mathbb{Z}$ (cf. [1], §2.4).

If a divisor $\omega_K$ on $K$ is chosen such that $\deg \omega_K = |\text{Disc}_K|^2$, the corresponding metrized $O_K$-module will be isometrically isomorphic to $\text{Hom}_\mathbb{Z}(O_K, \mathbb{Z})$, metrized as above (cf. [3], Theorem 4.5 in Chapter III).

Hence one obtains (i) from Theorem 2 in [1]. However, note Remark 2.1 on the value of $C(S_1(K), S_2(K))$.

For a divisor $D = \sum a_P \cdot P$, denote by $\chi(D)$ the Euler-Minkowski characteristic (cf. [3], Definition 3.1 in §3 of Chapter III) of the fractional ideal

$$\prod_{P \in \sum_K \setminus \sum_K} (O_K \cap M_P)^{-a_P}.$$

Setting

$$i(D) = \frac{h^0(D) \cdot e^{-\chi(D)}}{2^{S_1(K)} \cdot (\pi/2)^{S_2(K)/2}},$$

one obtains (ii) as a slight reformulation of Theorem 3.9 (Chapter III, §3) in [3].

Now assume that $\text{char}(K) > 0$. In this case $S_1(K) = S_2(K) = 0$, and

$$\log_q h^0(D) = \text{dim}_{\mathbb{F}_q} H^0(D).$$
Let $g$ be the genus of the complete non-singular curve determined by $\mathbb{K}$, and choose $\omega_\mathbb{K}$ from the class of divisors of degree $q^2 g - 2$. Then (i) is a multiplicative formulation of the Riemann-Roch theorem (cf. [3], Theorem 5.4 in Chapter 5). Setting $i(D) = h^0(\omega_\mathbb{K} - D)$, (ii) follows from (i), since $h^0(\omega_\mathbb{K} - D) = 1$ whenever $\deg D > \deg \omega_\mathbb{K}$ (cf. [3], Corollary 4 in Chapter 5).

Remark 2.1 We make a minor correction to the proof of Theorem 2 in [1]. Numbers in bold-face refer to lines or pages in [1]. The quantity $C(r_1, r_2, N)$ is defined on line (26) (pg. 355) as

$$- \log \mu(K^*) + N(r_1 + 2r_2) \log(6),$$

where $\mu(K^*)$ is the euclidean volume of the set of $(y_i, z_j) \in (\mathbb{R}^N)^{r_1} \times (\mathbb{C}^N)^{r_2}$ such that

$$\sum_{i=1}^{r_1} |y_i| + 2 \sum_{j=1}^{r_2} |z_j| \leq 1.$$

However, the value of $C(r_1, r_2, N)$ stated in Theorem 2 in [1] is incorrect, due to a missing minus sign in the computation of $\mu(K^*)$ on line (24) (pg. 355). The correct value is

$$C(r_1, r_2, N) = \log \left( \frac{6^{N(r_1 + 2r_2)}}{\mu(K^*)} \right) = \log \left( \frac{(N(r_1 + 2r_2))! \cdot 2^{2N} r_2 \cdot 6^{N(r_1 + 2r_2)}}{(V(B_N))^{r_1} \cdot (V(B_{2N}))(2N)!^{r_2}} \right).$$

The value of $C(S_1(\mathbb{K}), S_2(\mathbb{K}))$ in Theorem 2.7 is simply $e^{C(r_1, r_2, 1)}$.

3 A canonical divisor

In this section, we describe a divisor $\omega'_\mathbb{K}$ that is determined by the global field $\mathbb{K}$, and show that Theorem 2.1 holds with $\omega_\mathbb{K} = \omega'_\mathbb{K}$. We also show that for a finite separable extension $L/\mathbb{K}$ of global fields, the corresponding divisors $\omega'_L$ and $\omega'_\mathbb{K}$ satisfy a Riemann-Hurwitz type formula (cf. Theorem 3.1).

We begin by considering a divisor that is determined by a finite separable extension of global fields. Recall the real numbers $r_Q$ from Definition 1.3.
Definition 3.1 Let \( L/K \) be a finite separable extension of global fields. The ramification divisor relative to the extension \( L/K \) is the divisor
\[
R_{L/K} = \sum_{Q \in L} r_Q \cdot Q.
\]

If \( \text{char}(K) > 0 \), we denote by \( P_0 \) a fixed element of \( \sum_{K_0} \setminus \sum_{K_0}^\infty \) such that \( N(P_0) = q \). If \( \text{char}(K) = 0 \), we let \( P_0 \) be an arbitrary fixed element of \( \sum_{K_0} \setminus \sum_{K_0}^\infty \). In both cases, we denote by \( S_0 \) the set of \( P \in \sum_K \) such that the restriction \( \varphi_P|_{K_0} \) is contained in \( P_0 \).

If \( \text{char}(K) = 0 \), we choose in addition an element \( P_\infty \in \sum_{K_0}^\infty \), and set
\[
a_\infty = \sum_{P \in S_0} 2e_P \log N(P_\infty) N(P),
\]
where \( e_P \) denotes the ramification index of \( P \) relative to the extension \( K/K_0 \) (cf. Definition 1.3).

Recall that the extension \( K/K_0 \) is separable by definition (cf. Remark 1.2), and consider the divisor
\[
\omega'_K = R_{K/K_0} - \sum_{P \in S_0} 2e_P \cdot P + a_\infty \cdot P_\infty.
\]

Proposition 3.1 Theorem 2.1 holds with \( \omega_K = \omega'_K \).

Proof It suffices to verify that
\[
\deg \omega'_K = \begin{cases} |\text{Disc}_K| \cdot 2^{-s_2(K)}, & \text{if } \text{char}(K) = 0, \\ q^{2g_K - 2}, & \text{if } \text{char}(K) > 0. \end{cases}
\]

If \( \text{char}(K) = 0 \), one has
\[
\deg \omega'_K = e^{-s_2(K)} \log 2 \prod_{Q \in \sum_K \setminus \sum_{K_0}^\infty} N(Q)^{r_Q} = \frac{|\text{Disc}_K|}{2^{s_2(K)}},
\]
where Remark 1.3 is used to obtain the last equality.

If \( \text{char}(K) > 0 \), one has \( g_{K_0} = 0 \). Hence
\[
\deg R_{K/K_0} = q^{2g_K - 2 + 2[ K : K_0 ]},
\]
by the Riemann-Hurwitz formula for function fields (cf. [3], Theorem 7.16 in Chapter 7).
Since the extension $\mathbb{K}/\mathbb{K}_0$ is separable, one has

$$\prod_{P|P_0} N(P)^{e_P} = q^{[\mathbb{K}:\mathbb{K}_0]},$$

for any choice of $P_0 \in \mathbb{K}_0$ such that $N(P_0) = q$ (cf. [6], Proposition 10 in §4 of Chapter I). This completes the proof.

**Theorem 3.1** If $L/\mathbb{K}$ is a finite separable extension of global fields, and if $L_0 = \mathbb{K}_0$, then

$$\deg \omega'_L = \deg \omega'_K [L:K] \cdot \deg R_{L/K}.$$

**Proof** Assume first that $\text{char}(\mathbb{K}) = 0$. Denote the restrictions of $R_{L/K}$, $\omega'_L$ and $\omega'_K$ to the archimedean classes by $R_{L/\mathbb{K}}^\infty$, $\omega'_L^\infty$ and $\omega'_K^\infty$, respectively. Note that by the construction of $R_{L/\mathbb{K}}$, $\omega'_L$ and $\omega'_K$:

1. $\log_{1/4} \deg R_{L/\mathbb{K}}^\infty$ is the number of elements $Q \in \sum_{L}^\infty$ with $\log N(Q) = 2$ extending elements $P \in \sum_{K}^\infty$ with $\log N(P) = 1$,

2. $[L:K] \cdot \log_{1/4} \deg \omega_K^\infty$ is the number of elements $Q \in \sum_{K}^\infty$ with $\log N(Q) = 2$ extending elements $P \in \sum_{K}^\infty$ with $\log N(P) = 2$,

3. $\log_{1/4} \deg \omega_L^\infty$ is the total number of elements $Q \in \sum_{L}^\infty$ with $\log N(Q) = 2$.

From these remarks, we obtain the equality

$$\deg \omega_L^\infty = \deg \omega_K^\infty [L:K] \cdot \deg R_{L/\mathbb{K}}^\infty.$$

The corresponding equality for $R_{L/\mathbb{K}} - R_{L/\mathbb{K}}^\infty$, $\omega'_L - \omega'_L^\infty$ and $\omega'_K - \omega'_K^\infty$ follows from the transitivity of the different in a tower of finite separable extensions of fields (cf. [6], Proposition 8 in §4 of Chapter III).

When $\text{char}(\mathbb{K}) > 0$, the statement in the theorem is a multiplicative formulation of the Riemann-Hurwitz formula for function fields (cf. [6], Theorem 7.16 in Chapter 7).
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