Asymptotic Scaling in the Two-Dimensional $O(3)$ $\sigma$-Model at Correlation Length $10^5$

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Abstract

We carry out a high-precision Monte Carlo simulation of the two-dimensional $O(3)$-invariant $\sigma$-model at correlation lengths $\xi$ up to $\sim 10^5$. Our work employs a new and powerful method for extrapolating finite-volume Monte Carlo data to infinite volume, based on finite-size-scaling theory. We discuss carefully the systematic and statistical errors in this extrapolation. We then compare the extrapolated data to the renormalization-group predictions. The deviation from asymptotic scaling, which is $\approx 25\%$ at $\xi \sim 10^2$, decreases to $\approx 4\%$ at $\xi \sim 10^5$.

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Two-dimensional nonlinear $\sigma$-models are important “toy models” in elementary-particle physics because they share with four-dimensional nonabelian gauge theories the property of perturbative asymptotic freedom [1]. However, the nonperturbative validity of asymptotic freedom has been questioned [2]; and numerical tests of asymptotic scaling in the $O(3)$ $\sigma$-model at correlation lengths $\xi \sim 100$ have shown discrepancies of order 25% [3,4]. In this Letter we employ a new finite-size-scaling extrapolation method [5] (see also Lüscher et al. [6] and Kim [7] for related work [8]) to obtain high-precision estimates (errors $< 2\%$) in the $O(3)$ $\sigma$-model at correlation lengths $\xi$ up to $\sim 10^5$. We find that the discrepancy has decreased to $\approx 4\%$, in good agreement with the asymptotic-freedom predictions.

We study the lattice $\sigma$-model taking values in the unit sphere $S^{N-1} \subset \mathbb{R}^N$, with nearest-neighbor action $H(\sigma) = -\beta \sum \sigma_x \cdot \sigma_y$. Perturbative renormalization-group computations predict that the (infinite-volume) correlation lengths $\xi^{(exp)}$ and $\xi^{(2)}$ [9] behave as

$$\xi^#(\beta) = C_{\xi^#} e^{2\pi \beta/(N-2)} \left( \frac{2\pi \beta}{N-2} \right)^{-1/(N-2)} \left[ 1 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots \right]$$  \hfill (1)

as $\beta \to \infty$. Three-loop perturbation theory yields [11,12]

$$a_1 = -0.014127 + \left( \frac{1}{4} - \frac{5\pi}{48} \right)/(N - 2).$$  \hfill (2)

The nonperturbative constant $C_{\xi^{(exp)}}$ has been computed recently using the thermodynamic Bethe Ansatz [13]:

$$C_{\xi^{(exp)}} = 2^{-5/2} \left( \frac{e^{1-\pi/2}}{8} \right)^{1/(N-2)} \Gamma\left( 1 + \frac{1}{N-2} \right).$$  \hfill (3)

The remaining nonperturbative constant is known analytically only at large $N$ [14]:

$$C_{\xi^{(2)}}/C_{\xi^{(exp)}} = 1 - \frac{0.003225}{N} + O(1/N^2).$$  \hfill (4)

Previous Monte Carlo studies up to $\xi \sim 100$ agree with these predictions to within about 20–25% for $N = 3$, 6% for $N = 4$ and 2% for $N = 8$ [4,15].

Our extrapolation method [5] is based on the finite-size-scaling Ansatz

$$\frac{O(\beta, sL)}{O(\beta, L)} = F_{O}(\xi(\beta, L)/L ; s) + O(\xi^{-\omega}, L^{-\omega}),$$  \hfill (5)

where $O$ is any long-distance observable, $s$ is a fixed scale factor (usually $s = 2$), $L$ is the linear lattice size, $F_{O}$ is a universal function, and $\omega$ is a correction-to-scaling exponent. We make Monte Carlo runs at numerous pairs $(\beta, L)$ and $(\beta, sL)$; we then plot $O(\beta, sL)/O(\beta, L)$ versus $\xi(\beta, L)/L$, using those points satisfying both $\xi(\beta, L) \geq$ some value $\xi_{min}$ and $L \geq$ some value $L_{min}$. If all these points fall with good accuracy on a single curve, we choose a smooth fitting function $F_{O}$. Then, using the functions $F_\xi$ and $F_O$, we extrapolate the pair $(\xi, O)$ successively from
$L \rightarrow sL \rightarrow s^2L \rightarrow \ldots \rightarrow \infty$. See [5] for how to calculate statistical error bars on the extrapolated values.

We have chosen to use functions $F_\Omega$ of the form

$$F_\Omega(x) = 1 + a_1 e^{-1/x} + a_2 e^{-2/x} + \ldots + a_n e^{-n/x}.$$  \hfill (6)

This form is partially motivated by theory, which tells us that $F(x) \rightarrow 1$ exponentially fast as $x \rightarrow 0$ [16]. Typically a fit of order $3 \leq n \leq 12$ is sufficient; we increase $n$ until the $\chi^2$ of the fit becomes essentially constant. The resulting $\chi^2$ value provides a check on the systematic errors arising from corrections to scaling and/or from inadequacies of the form (6). The discrepancies between the extrapolated values from different $L$ at the same $\beta$ can also be subjected to a $\chi^2$ test. Further details on the method can be found in [5].

We simulated the two-dimensional $O(3)$ $\sigma$-model, using the Wolff embedding algorithm with standard Swendsen-Wang updates [17,18,10]; critical slowing-down appears to be completely eliminated. We ran on lattices $L = 32, 48, 64, 96, 128, 192, 256, 384, 512$ at 180 different pairs $(\beta, L)$ in the range $1.65 \leq \beta \leq 3.00$ (corresponding to $20 \lesssim \xi_\infty \lesssim 10^3$). Each run was between $10^5$ and $5 \times 10^5$ iterations, and the total CPU time was 7 years on an IBM RS-6000/370. The raw data will appear in [19].

Our data cover the range $0.15 \lesssim \xi(L)/L \lesssim 1.0$, and we found tentatively that a tenth-order fit (6) is indicated: see Table 1. Next we took $\xi_{\min} = 20$ and sought to choose $L_{\min}$ to avoid any detectable systematic error from corrections to scaling. There appear to be weak corrections to scaling ($\lesssim 1.5\%$) in the region $0.3 \lesssim \xi(L)/L \lesssim 0.7$ for lattices with $L \lesssim 64$–96: see the deviations plotted in Figure 1. We therefore investigated systematically the $\chi^2$ of the fits, allowing a different $L_{\min}$ for $\xi(L)/L \leq 0.7$ and $> 0.7$: see Table 1. A reasonable $\chi^2$ is obtained when $n \geq 9$ and $L_{\min} \geq (128, 64)$. Our preferred fit is $n = 10$ and $L_{\min} = (128, 64)$: see Figure 2, where we compare also with the perturbative prediction

$$F_\xi(x; s) = s \left[ 1 - \frac{aw_0 \log s}{2} x^{-2} - a^2 \left( w_1 \log s + \frac{w_0^2 \log^2 s}{8} \right) x^{-4} + O(x^{-6}) \right]$$  \hfill (7)

valid for $x \gg 1$, where $a = 1/(N - 1)$, $w_0 = (N - 2)/2\pi$ and $w_1 = (N - 2)/(2\pi)^2$.

The extrapolated values $\xi(2)$ from different lattice sizes at the same $\beta$ are consistent within statistical errors: only one of the 24 $\beta$ values has a $\chi^2$ too large at the 5% level; and summing all $\beta$ values we have $\chi^2 = 86.56$ (106 DF, level = 92%).

In Table 2 we show the extrapolated values $\xi(3)$ from our preferred fit and some alternative fits. The discrepancies between these values (if larger than the statistical errors) can serve as a rough estimate of the remaining systematic errors due to corrections to scaling. The statistical errors in our preferred fit are of order 0.2% (resp. 0.7%, 1.1%, 1.6%) at $\xi_\infty \approx 10^2$ (resp. $10^3$, $10^4$, $10^5$), and the systematic errors are of the same order or smaller. The statistical errors at different $\beta$ are strongly positively correlated.

In Figure 3 (points + and ×) we plot $\xi(2)_{\text{estimate}(128, 64)}$ divided by the two-loop and three-loop predictions (1)–(4). The discrepancy from three-loop asymptotic scaling, which is $\approx 16\%$ at $\beta = 2.0$ ($\xi \approx 200$), decreases to $\approx 4\%$ at $\beta = 3.0$ ($\xi \approx 10^5$).
This is roughly consistent with the expected $1/\beta^2$ corrections. The slight bump at $2.3 \lesssim \beta \lesssim 2.6$ is probably spurious, arising from correlated statistical or systematic errors.

We can also try an “improved expansion parameter” [20,4,12,19] based on the energy $E = \langle \sigma_0 \cdot \sigma_1 \rangle$. First we invert the perturbative expansion [21,12]

$$E(\beta) = 1 - \frac{N-1}{4\beta} - \frac{N-1}{32\beta^2} - \frac{0.005993(N-1)^2 + 0.007270(N-1)}{\beta^3} + O(1/\beta^4)$$

and substitute into (1); this gives a prediction for $\xi$ as a function of $1 - E$. For $E$ we use the value measured on the largest lattice; the statistical errors and finite-size corrections on $E$ are less than $5 \times 10^{-5}$, and therefore induce a negligible error (less than 0.5%) on the predicted $\xi$. The corresponding observed/predicted ratios are also shown in Figure 3 (points □ and ◦). The “improved” 3-loop prediction is in excellent agreement with the data.

Let us summarize the conceptual basis of our analysis. The main assumption is that if the Ansatz (5) with a given function $F_\xi$ is well satisfied by our data for $L_{\text{min}} \leq L \leq 256$ and $1.65 \leq \beta \leq 3$, then it will continue to be well satisfied for $L > 256$ and for $\beta > 3$. Obviously this assumption could fail, e.g. if [2] at some large correlation length ($\gtrsim 10^3$) the model crosses over to a new universality class associated with a finite-\(\beta\) critical point. In this respect our work is subject to the same caveats as any other Monte Carlo work on a finite lattice. However, it should be emphasized that our approach does not assume asymptotic scaling [eq. (1)], as $\beta$ plays no role in our extrapolation method. Thus, we can make an unbiased test of asymptotic scaling. The fact that we confirm (1) with the correct nonperturbative constant (3)/(4) is, we believe, good evidence in favor of the asymptotic-freedom picture. We are unable to imagine how, if there were in fact a finite-\(\beta\) critical point [2], the “presymptotic” region at $\beta \leq 3$ would mimic not only asymptotic freedom but also the nonperturbative constant predicted by the thermodynamic Bethe Ansatz.

Details of this work, including an analysis of the susceptibility $\chi$, will appear elsewhere [19].

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**References**

[1] A.M. Polyakov, Phys. Lett. **B59**, 79 (1975); E. Brézin and J. Zinn-Justin, Phys. Rev. **B14**, 3110 (1976); J.B. Kogut, Rev. Mod. Phys. **51**, 659 (1979), Section VIII.C.
Our method [5] is very closely related to that of Lüscher et al. [6]. The principal difference is that Lüscher et al. choose carefully their runs $(\beta, L)$ so as to produce only a few distinct values of $x \equiv \xi(L)/L$, while we attempt to cover an entire interval of $x$. The method of Kim [7] is also closely related, but he compares lattice size $L$ to $\infty$ rather than to $sL$; this is a disadvantage. Nevertheless, Kim has obtained extremely accurate estimates of $\xi_\infty(\beta)$ in the $O(3)$ $\sigma$-model: see Table 2 below. It should be emphasized that all three methods are completely general; they are not restricted to asymptotically free theories. Also, all three methods work only with observable quantities ($\xi$, $\mathcal{O}$ and $L$) and not with bare quantities ($\beta$). Therefore, they rely only on “scaling” and not on “asymptotic scaling”; and they differ from other FSS-based methods such as phenomenological renormalization.

[9] Here $\xi^{(exp)}$ is the exponential correlation length (= inverse mass gap), and $\xi^{(2)}$ is the second-moment correlation length defined by (4.11)–(4.13) of Ref. [10]. Note that $\xi^{(2)}$ is well-defined in finite volume as well as in infinite volume; where necessary we write $\xi^{(2)}(L)$ and $\xi^{(2)}_\infty$, respectively. In this paper, $\xi$ without a superscript denotes $\xi^{(2)}$.

[10] S. Caracciolo, R.G. Edwards, A. Pelissetto and A.D. Sokal, Nucl. Phys. B403, 475 (1993).

[11] M. Falciomi and A. Treves, Nucl. Phys. B265, 671 (1986); P. Weisz and M. Lüscher, unpublished, cited in [4].

[12] S. Caracciolo and A. Pelissetto, Nucl. Phys. B420, 141 (1994).

[13] P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. B245, 522 (1990); P. Hasenfratz and F. Niedermayer, Phys. Lett. B245, 529 (1990).
[14] H. Flyvbjerg, Nucl. Phys. B348, 714 (1991); P. Biscari, M. Campostrini and P. Rossi, Phys. Lett. B242, 225 (1990); S. Caracciolo and A. Pelissetto, in preparation.

[15] Steffen Meyer (private communication) has kindly supplied us with a high-precision Monte Carlo estimate of the universal ratio $C_{\langle\varphi^2\rangle}/C_{\langle\varphi^{\text{exp}}\rangle}$ in the $N = 3$ model: it is $0.9994 \pm 0.0008$ at $\beta = 1.7$ ($\xi_\infty \approx 35$), $L = 256$; and $0.9991 \pm 0.0009$ at $\beta = 1.8$ ($\xi_\infty \approx 65$), $L = 512$. This is in excellent agreement with the value 0.9989 obtained from the $1/N$ expansion, and is only marginally different from 1.

[16] H. Neuberger, Phys. Lett. B233, 183 (1989); S. Caracciolo and A. Pelissetto, in preparation.

[17] U. Wolff, Phys. Rev. Lett. 62, 361 (1989).

[18] R.H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 58, 86 (1987).

[19] S. Caracciolo, R.G. Edwards, A. Pelissetto and A.D. Sokal, in preparation.

[20] G. Martinelli, G. Parisi and R. Petronzio, Phys. Lett. B100, 485 (1981); S. Samuel, O. Martin and K. Moriarty, Phys. Lett. B153, 87 (1985); G.P. Lepage and P.B. Mackenzie, Phys. Rev. D48, 2250 (1993).

[21] M. Lüscher, unpublished, cited in [4].
Figure 1: Deviation of points from fit to $F_\xi$ with $s = 2$, $\xi_{\text{min}} = 20$, $L_{\text{min}} = 128$, $n = 10$. Symbols indicate $L = 32 (+), 48 (\circ)$, $64 (\times)$, $96 (\ast)$, $128 (\Box)$, $192 (\bigtriangleup)$, $256 (\bigtriangledown)$. Error bars are one standard deviation. Curves near zero indicate statistical error bars ($\pm$ one standard deviation) on the function $F_\xi(x)$. 

\[ \frac{\xi(2L)/L - \xi(L)/L}{\xi(L)/L} \]

\[ \frac{\xi(L)/L}{0.02} \]

\[ \frac{0.00}{0.01} \]

\[ \frac{0.01}{0.02} \]

\[ \frac{0.02}{0.03} \]

\[ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \]

\[ \frac{\xi(L)/L}{0.02} \]

\[ \frac{0.00}{0.01} \]

\[ \frac{0.01}{0.02} \]

\[ \frac{0.02}{0.03} \]

\[ 0.0 \ 0.2 \ 0.4 \ 0.6 \ 0.8 \ 1.0 \]
Figure 2: $\xi(\beta, 2L)/\xi(\beta, L)$ versus $\xi(\beta, L)/L$. Symbols indicate $L = 32$ (+), 48 ($\Phi$), 64 ($\times$), 96 ($\infty$), 128 ($\square$), 192 ($\blacksquare$), 256 ($\Diamond$). Error bars are one standard deviation. Solid curve is a tenth-order fit in (6), with $\xi_{\min} = 20$ and $L_{\min} = 128$ (resp. 64) for $\xi(L)/L \leq 0.7$ (resp. > 0.7). Dashed curve is the perturbative prediction (7).
Figure 3: $\xi^{(a)}_{\infty,\text{estimate}}(128,64)/\xi^{(a)}_{\infty,\text{theor}}$ versus $\beta$. Error bars are one standard deviation (statistical error only). There are four versions of $\xi^{(a)}_{\infty,\text{theor}}$: standard perturbation theory in $1/\beta$ gives points $+$ (2-loop) and $\times$ (3-loop); “improved” perturbation theory in $1 - E$ gives points $\Box$ (2-loop) and $\Diamond$ (3-loop).
| $L_{\text{min}}$ | DF  | $n = 7$  | $n = 8$  | $n = 9$  | $n = 10$ | $n = 11$ | $n = 12$ |
|-----------------|-----|---------|---------|---------|---------|---------|---------|
| (64,64)         | 108-n| 278.38  | 183.80  | 144.34  | 137.82  | 135.77  | 135.01  |
|                 |     | 0.0%    | 0.0%    | 0.2%    | 0.5%    | 0.6%    | 0.5%    |
| (96,32)         | 107-n| 228.85  | 164.46  | 129.38  | 124.87  | 122.15  | 120.48  |
|                 |     | 0.0%    | 0.0%    | 1.9%    | 3.0%    | 3.7%    | 4.0%    |
| (96,64)         | 97-n | 207.32  | 137.18  | 108.23  | 103.13  | 102.02  | 101.59  |
|                 |     | 0.0%    | 0.1%    | 7.1%    | 11.4%   | 11.5%   | 10.6%   |
| (96,96)         | 87-n | 190.61  | 115.05  | 100.99  | 93.90   | 93.89   | 93.73   |
|                 |     | 0.0%    | 0.5%    | 4.1%    | 9.2%    | 8.6%    | 7.1%    |
| (128,32)        | 93-n | 160.17  | 121.29  | 99.35   | 94.82   | 94.20   | 86.65   |
|                 |     | 0.0%    | 0.6%    | 12.1%   | 17.7%   | 16.8%   | 31.3%   |
| (128,64)        | 83-n | 139.60  | 95.94   | 78.23   | 72.91   | 72.89   | 68.43   |
|                 |     | 0.0%    | 5.2%    | 34.6%   | 48.1%   | 44.9%   | 56.4%   |
| (128,96)        | 73-n | 126.20  | 79.03   | 71.12   | 64.33   | 63.29   | 59.72   |
|                 |     | 0.0%    | 11.3%   | 25.3%   | 43.6%   | 43.1%   | 52.2%   |
| (128,128)       | 64-n | 101.05  | 63.45   | 61.96   | 59.70   | 59.28   | 52.89   |
|                 |     | 0.0%    | 23.1%   | 24.2%   | 27.6%   | 25.7%   | 43.9%   |
| (192,32)        | 75-n | 110.42  | 93.41   | 76.13   | 70.61   | 65.15   | 62.16   |
|                 |     | 0.1%    | 1.8%    | 18.5%   | 29.6%   | 43.6%   | 50.6%   |
| (192,64)        | 65-n | 90.60   | 69.57   | 55.03   | 47.60   | 45.12   | 43.74   |
|                 |     | 0.4%    | 12.3%   | 51.1%   | 75.0%   | 80.0%   | 81.4%   |
| (192,96)        | 57-n | 82.54   | 55.94   | 49.49   | 38.90   | 38.67   | 37.53   |
|                 |     | 0.3%    | 23.0%   | 41.4%   | 79.4%   | 77.0%   | 77.8%   |

Table 1: $\chi^2$ and confidence level for the fit (6) of $\xi(\beta, 2L)/\xi(\beta, L)$ versus $\xi(\beta, L)/L$. DF = number of degrees of freedom. The first (resp. second) $L_{\text{min}}$ value applies for $\xi(L)/L \leq 0.7$ (resp. $> 0.7$). In all cases $\xi_{\text{min}} = 20$. 

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| $L_{\text{min}}$ | 1.90 | 1.95 | 2.00 | 2.05 | 2.10 | 2.15 | 2.20 | 2.25 |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| (96,64)       | 122.43 (0.25) | 166.79 (0.36) | 228.37 (0.55) | 311.54 (0.93) | 420.52 (1.39) | 574.16 (2.51) | 774.24 (3.69) | 1039.1 (5.7) |
| (96,96)       | 122.55 (0.25) | 166.95 (0.37) | 228.93 (0.57) | 312.29 (0.93) | 421.61 (1.63) | 574.96 (2.51) | 776.03 (3.73) | 1038.2 (5.5) |
| (128,32)      | 122.34 (0.29) | 166.68 (0.42) | 228.50 (0.66) | 311.84 (1.09) | 422.67 (1.94) | 577.41 (3.13) | 779.33 (4.80) | 1048.9 (7.3) |
| (128,64)      | 122.34 (0.29) | 166.66 (0.43) | 228.54 (0.67) | 311.99 (1.10) | 422.73 (1.97) | 577.73 (3.12) | 780.64 (4.76) | 1048.7 (7.3) |
| (128,96)      | 122.25 (0.29) | 166.54 (0.43) | 228.11 (0.66) | 311.59 (1.10) | 421.71 (1.90) | 576.52 (3.06) | 778.40 (4.58) | 1045.9 (7.3) |
| (128,128)     | 122.36 (0.29) | 166.68 (0.43) | 228.59 (0.69) | 312.06 (1.13) | 422.89 (2.00) | 577.94 (3.09) | 781.23 (4.79) | 1046.7 (7.3) |
| (192,32)      | 122.40 (0.40) | 166.95 (0.60) | 229.05 (0.93) | 312.94 (1.49) | 424.90 (2.69) | 580.40 (4.39) | 784.04 (7.14) | 1057.7 (11.4) |
| (192,64)      | 122.41 (0.38) | 166.94 (0.57) | 229.15 (0.90) | 312.86 (1.44) | 425.42 (2.62) | 580.91 (4.41) | 785.39 (7.11) | 1057.3 (11.2) |
| (192,96)      | 122.43 (0.39) | 167.02 (0.58) | 229.30 (0.90) | 313.23 (1.45) | 426.08 (2.70) | 581.91 (4.44) | 787.63 (7.18) | 1057.9 (11.2) |
| **Kim**       | 122.0 (2.7)   | —     | 227.8 (3.2)  | 306.6 (3.9)  | 419 (5)       | 574 (8)      | 706 (7)      | —   |

| $L_{\text{min}}$ | 2.30 | 2.40 | 2.50 | 2.60 | 2.70 | 2.80 | 2.90 | 3.00 |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| (96,64)       | 1403.4 (8.3) | 2539.1 (17.9) | 4619.7 (38.6) | 8460.1 (81.7) | 15499 (172) | 28413 (362) | 51624 (746) | 93601 (1475) |
| (96,96)       | 1402.0 (8.4) | 2541.5 (19.2) | 4605.9 (44.5) | 8450.7 (101.2) | 15401 (218) | 28119 (455) | 51356 (934) | 93641 (1923) |
| (128,32)      | 1416.7 (10.6) | 2566.2 (20.8) | 4687.7 (41.3) | 8559.0 (81.1) | 15594 (161) | 28622 (322) | 51955 (651) | 94133 (1345) |
| (128,64)      | 1416.8 (10.5) | 2568.8 (21.2) | 4671.7 (43.9) | 8569.0 (91.6) | 15690 (189) | 28737 (389) | 52189 (779) | 94633 (1554) |
| (128,96)      | 1414.1 (10.8) | 2558.1 (22.8) | 4628.6 (48.4) | 8478.0 (104.3) | 15507 (226) | 28360 (470) | 51695 (961) | 94033 (1930) |
| (128,128)     | 1415.5 (11.2) | 2572.1 (26.2) | 4637.7 (62.0) | 8437.2 (143.2) | 15336 (311) | 27947 (666) | 51319 (1392) | 94627 (2922) |
| (192,32)      | 1425.3 (17.0) | 2584.4 (32.8) | 4716.9 (62.6) | 822.27 (118.4) | 15638 (225) | 28820 (432) | 52345 (843) | 94724 (1660) |
| (192,64)      | 1427.6 (17.0) | 2582.2 (32.9) | 4702.7 (62.7) | 825.0 (123.1) | 15862 (244) | 28952 (482) | 52562 (934) | 95266 (1819) |
| (192,96)      | 1427.0 (17.0) | 2584.2 (33.4) | 4688.2 (65.4) | 8599.8 (133.7) | 15660 (269) | 28663 (542) | 52314 (1082) | 95304 (2163) |
| **Kim**       | 1402 (22)    | 2499 (41)  | 4696 (128)  | 8022 (234)  | 15209 (449) | —     | —     | —     |

Table 2: Estimated correlation lengths $\xi_{\infty}^{(2)}$ as a function of $\beta$, from various extrapolations. Error bar is one standard deviation (statistical errors only). All extrapolations use $s = 2$, $\xi_{\text{min}} = 20$ and $n = 10$. The first (resp. second) $L_{\text{min}}$ value applies for $\xi(L)/L \leq 0.7$ (resp. $> 0.7$). Our preferred fit is $L_{\text{min}} = (128,64)$, shown in italics. Kim is the estimate from [7].