Application of multiphase sequences with low sidelobe level in synthetic aperture radar

A N Leukhin, A A Voronin and A S Merzlyakov

Mari State University, 1, sq. Lenin, Yoshkar-Ola, Mari El Rep, 424000, Russia

E-mail: leukhinan@list.ru

Abstract. The construction of signals with a low level of side lobes of the impulse autocorrelation function is of great importance in the development of radar systems, and especially of synthetic aperture radars, where a bright point target can overlap a part of the radar image due to insufficiently low sidelobe levels. This paper presents an algorithm for constructing multiphase sequences of long lengths with a low level of side lobes of the impulse autocorrelation function. Using the proposed algorithm for lengths 680, 1024, 2048, 2730, 4096, polyphase pulse sequences were synthesized with the lowest known sidelobe levels.

1. Introduction

Let there be a unimodular sequence of the form:

$$U = \{u_0 = 1, u_1, \ldots, u_{N-1}\}$$

where $$u_n = e^{i\psi_n}$$, $$|u_n| = 1$$, $$n = 0, 1, \ldots, N - 1$$, $$i = \sqrt{-1}$$ - imaginary unit.

The phase vector defining the elements of the unimodular sequence $$U$$ is denoted as:

$$\Psi = \{\psi_0 = 0, \psi_1, \ldots, \psi_{N-1}\}.$$  

For a multiphase sequence phase value may be any value in the range $$\psi_n = [0, 2\pi]$$. For a binary sequence, the phase can take one of two values $$\psi_n = 0$$ or $$\psi_n = \pi$$.

The impulse autocorrelation function of the unimodular sequence has the form:

$$C_\tau = \sum_{n=0}^{N-1} u_n^* u_{n+\tau}, \quad \tau = 0, 1, \ldots, N - 1,$$  

where $$C_\tau$$ - an element of the autocorrelation function, $$\tau$$ - shift value, $$u_n^*$$ - complex conjugate operation.

The criterion for assessing the quality of unimodular sequences from the standpoint of the properties of impulse autocorrelation is the minimax criterion, according to which the maximum sidelobe level (PSL – peak sidelobe level):

$$PSL(C) = \max_{1 \leq \tau \leq N-1} |C_\tau|$$  

(2)
should be minimal

$$\min_{U} PSL(C).$$

The purpose of the work is to develop an algorithm for the synthesis of multiphase unimodular sequences with a low level of side lobes of the impulse autocorrelation function.

2. Barker codes

The best sequences from the standpoint of the criterion of the minimum-maximum level of the side lobe of the impulse autocorrelation function are Barker sequences, the sidelobe level of which does not exceed one \((PSL=1)\) \([1]\).

$$|C_\tau| \leq 1, \ \tau = \pm 1, \pm 2, \ldots, \pm(N-1).$$

In 1953, in work \([1]\), Barker synthesized optimal binary sequences with the minimum possible level of lateral \((PSL=1)\) for lengths \(N = 2, 3, 4, 5, 7, 11, 13\).

In 1963, Turin \([2]\) analytically proved that there are no binary Barker sequences of odd lengths \(N > 13\).

Author of work \([3]\), in 2009, established an upper bound for sequences of even lengths, below which he proved that there are no Barker sequences, except for the possible length:

$$N = 189 260 468 001 034 441 522 766 781 604.$$  

Unfortunately, the maximum known length of such sequences is small \((N = 13)\) and turns out to be insufficient for a large number of practical applications.

The next step in the development of approaches to constructing unimodular sequences that are optimal according to the minimax criterion was the idea to use an alphabet with a large volume compared to a binary one. In 1956, DeLong \([4]\) constructed three-phase Barker sequences, and then Welty \([5]\) in 1960 four-phase Barker sequences.

The idea of increasing the size of the alphabet for constructing Barker sequences was finally completed in 1965 by Golomb and Scholz in \([6]\). Golomb and Scholz introduced the concept of generalized Barker codes, defined as a multiphase sequence.

Along with the original (binary) Barker sequences \((u_n = \pm 1)\), Golomb and Scholz considered Barker \(M\) -phase sequences with rational phases of the form \([6]\):

$$u_n = \exp\left(\frac{2\pi i}{M} n \cdot r\right),$$

where \(M\) – the number of phase gradations, specifying the number of divisions of the circle into \(M\) equal parts, \(r \in [0,1,\ldots,M-1]\).

In particular, in \([6]\), references are given to \(M = 2, 3, 4\) -phase Barker sequences, and also for the first time six-phase \((M = 6)\) Barker sequences are considered. In this paper, it is shown that an increase in the volume of the alphabet makes it possible to increase the length of Barker sequences. For example, for four-phase Barker sequences, the maximum possible length equals \(N = 15\).

In 1989, Golomb and Zhang \([7]\) constructed Barker \(M = 60\) -phase sequences up to length \(N = 19\).

Further efforts in the search for Barker sequences led to the following results. In 1994, the list of Barker sequences was extended to length \(N = 31\) \([8]\). In 1996, in \([9]\), polyphase Barker codes were found up to the length \(N = 36\). In \([10]\) in 1998, multiphase Barker codes up to length \(N = 45\) were constructed. In 2005, in \([11]\), multiphase Barker codes up to length \(N = 63\) were constructed, and in \([12]\), multiphase Barker codes of length \(N = 64\) were constructed. In 2007, in \([13]\), multiphase sequences of length \(N = 65\). The multiphase Barker sequences with the largest lengths to date were obtained in 2009 in \([14]\) \(65 \leq N \leq 70\), \(N = 72\), \(N = 76\), \(N = 77\).
The authors of [13] searched for nonequivalent Barker sequences in the range of lengths \( N = [10,52] \). As a result, with the length \( N = 38 \), the maximum number of nonequivalent normalized polyphase Barker sequences was constructed, equal to 110. With an increase in length (\( N > 38 \)), the number of found nonequivalent sequences begins to decrease sharply, so the authors put forward the assumption that multiphase Barker sequences exist only for a certain finite length.

In [14,15,16,17,18], the authors developed an analytical approach to solving the problem of synthesizing impulse sequences based on solving a system of nonlinear equations and described its numerical solution. In [19], the number of found nonequivalent multiphase Barker codes in the range of length \( N = [3,52] \). The authors of works [14,15,16,17,18] experimentally showed that in the range of lengths, the number of constructed multiphase Barker sequences are several times higher than the number of sequences obtained in [13].

Thus, today the question of the existence of Barker codes of arbitrary length remains open; therefore, the problem of synthesizing multiphase sequences with low-level sidelobes of the impulse autocorrelation function is urgent.

3. Nonlinear optimization method for the synthesis of multiphase sequences with good impulse autocorrelation properties

In work [19] Bystrov, Zhukova and Chebotarev showed a method of nonlinear optimization, in which the minimization of the objective correlation function is ensured by the minimum of mean squares:

\[
\Phi = \sum_{\tau=1}^{N-2} |C_{\tau}|^2 \rightarrow \min
\]

where \( N \) – sequence length, \( C_{\tau} \) – values of the impulse autocorrelation function (except zero and maximum shifts).

At the same time, it was shown in [19] that the quadratic character gives great importance to minimizing the sidelobes of the impulse autocorrelation function.

In [19], the \( k \)-stage variant of minimizing the target function is proposed

\[
\Phi = \sum_{\tau=1}^{N-2} |C_{\tau}|^{2k} \rightarrow \min, \ k = 1,\ldots, K,
\]

in which after the first stage, at \( k = 1 \) the result of execution serves as the initial data for the next stage at \( k = 2 \), etc.

As a result of nonlinear optimization, the sidelobe levels of impulse autocorrelation function were assessed for the obtained sequences: when using the Frank sequence as the initial data, sequences were obtained in which the sidelobe level does not exceed \( PSL = 1.7(0.1\sqrt{N}) \) at length \( N = 256 \); when using random input data, the level is much higher, and, according to estimates, does not exceed \( PSL = 3.5 \) at length \( N = 256 \).

4. A method for the numerical solution of a system of nonlinear equations for the synthesis of multiphase sequences with a small level of side lobes of the impulse autocorrelation function

In [20], a method for the synthesis of multiphase sequences with good impulse autocorrelation function properties was proposed, for which the sidelobe level should be less than a given level \( \alpha \). This method is based on the numerical solution of a system of inequalities of the form:
where \( \alpha \) – predefined maximum level of sidelobe. In this case, the system of inequalities (3) can be transformed into a system of nonlinear equations of the form:

\[
\begin{align*}
C_1 \leq a, \\
C_2 \leq a, \\
\vdots \\
C_{N-1} \leq a,
\end{align*}
\]  

(3)

where \( f_\tau(\Psi) \) is the equation of the system obtained from expression (3) for the \( \tau \)-th side lobe. In general, the function \( f_\tau(\Psi) \) can be written as follows [20]:

\[
f_\tau(\Psi) = \left| \sum_{n=0}^\tau [\exp(-i \cdot \psi_n) \cdot \exp(i \cdot \psi_{n+\tau})] \right| - a, \quad \tau = 1, ..., N - 1
\]

For the numerical solution of a system of nonlinear equations of the form (4) in [20], the Levenberg-Marquardt method was used, which is used in mathematical packages such as MathCad, MatLab, Maple, and others. The method is based on the Newton-Gauss method and consists of successive approximation of the given initial values of the parameters to the initial local minimum.

At the beginning of the algorithm, an initial approximation is selected: the vector \( \Psi \). At each iteration step, the vector \( \Psi \) is replaced with a gradient increment:

\[
\Psi_{i+1} = \Psi + \Delta \Psi
\]

Initial increments are calculated according to the Levenberg-Marquardt method:

\[
\Delta \Psi = (J^T J + \lambda_k \text{diag}(J^T J))^{-1} J^T (a - f(\Psi_i))
\]

where \( J \) – Jacobian matrix, \( \lambda \) – regularisation parameter, \( a \) – the predefined maximum level of sidelobe.

The Jacobian matrix \( J \) of size \((N-1) \times (N-1)\) can be represented as:

\[
J = \begin{bmatrix}
\frac{\partial f_1(\Psi)}{\partial \psi_1} & \cdots & \frac{\partial f_1(\Psi)}{\partial \psi_{N-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_{N-1}(\Psi)}{\partial \psi_1} & \cdots & \frac{\partial f_{N-1}(\Psi)}{\partial \psi_{N-1}}
\end{bmatrix}
\]

In the general case, to optimize computations, one can make a finite-difference approximation of partial derivatives using the right-hand finite-difference formula:

\[
\frac{\partial f(\Psi)}{\partial \psi_j} \approx \frac{f(\Psi + \Phi_j) - f(\Psi)}{h}
\]

\( \Phi_j = \{\phi_0, \phi_1, \ldots, \phi_{N-1}\} \)
\[ \phi_n = \begin{cases} 0, & \text{if } n \neq j \text{ or } n = 0, \\ h, & \text{otherwise}, \end{cases} \]

where \( h \) – the difference interval of the function, \( n = 0,1,...,N - 1 \). Typically, the difference interval lies in the range \( 100 \cdot \varepsilon_M < h < 1000 \cdot \varepsilon_M \), where \( \varepsilon_M \) – machine accuracy, but in practice, the interval is chosen from taking into account the minimization of errors that arise when using finite difference approximations.

As an estimate of the iteration result, a check is made for the minimum of the function, which can be represented as a sum of squares:

\[ F(\Psi) = \sum_{\tau=1}^{N-1} k^2, \quad \text{if } f^2(\Psi)_{\tau} < \epsilon, \]
\[ = \text{otherwise}, \quad \text{if } f^2(\Psi)_{\tau} \geq \epsilon, \]

where \( F(\Psi) \leq \epsilon \) is the criterion for the residual \( \epsilon \), according to which the result is determined, both of the iteration and the entire iterative method, \( \epsilon \) is the accuracy of determining the results, \( k \) is a parameter that should be less than the calculation accuracy \( \epsilon \).

Based on this algorithm, in [20], polyphase sequences with a low sidelobe level of the impulse autocorrelation function sequence were synthesized for the following lengths: \( N = 128, PSL = 1.535; N = 256, PSL = 1.556; N = 512, PSL = 1.763 \). Also in this work, the predicted levels of sidelobe level of the impulse autocorrelation function for lengths are given: \( N = 1024, PSL = 1.997; N = 2048, PSL = 2.41; N = 4096, PSL = 2.832 \).

In [21], an example of the synthesis of a polyphase sequence with a length \( N = 1024 \) with a maximum sidelobe level is given \( PSL = 2.618 \).

5. Modification of the method for the synthesis of polyphase sequences with a low level of side lobes of the impulse autocorrelation function

In this paper, we propose a modification of the algorithm for the synthesis of multiphase sequences with a low sidelobe level of the impulse autocorrelation function, proposed in [21].

The method consists of a step-by-step decrease in the level of side lobes by solving a system of nonlinear equations of the form (4) and getting out of local minima by adding a noise component to the phase vector.

- **Step 1.** Calculate the impulse autocorrelation function of the resulting sequence \( \Psi^{(i)} \) and find the value \( PSL \).

- **Step 2.** We determine the value of \( a = PSL - \Delta \) and numerically solve the system of equations, in which we set the vector \( \Psi^{(i)} \) as the initial data.

- **Step 3.** If the new vector value \( \Psi^{(i+1)} \) is successfully found, repeat step 2 until the sequence meets the specified requirements.

- **Step 4.** In case of failure, decrease the step \( \Delta \) and repeat step 2.

- **Step 5.** If \( \Delta < \epsilon \) then we generate a noise sequence normally distributed with zero mean and variance \( \sigma^2 \) and add it to the sequence \( \Psi^{(i)} \):

\[ \Psi^{(i)} = \Psi^{(i)} + N, \]

(in this case, the sidelobe level will become higher than that of the original sequence for exiting the local minimum) and go to step 1.

**Step 6.** If step 5 did not result in lowering the sidelobe level, we generate a noise sequence with new parameters.
The well-known Frank, Zadoff-Chu, or P4-phase sequences can be taken as the initial sequences for the synthesis of polyphase sequences with a low level of side lobes of the impulse autocorrelation function.

6. Results
Based on the developed algorithm, polyphase sequences with a low level of sidelobes in impulse autocorrelation function for lengths 680, 1024, 2048, 2730, 4096 were synthesized.

Table 1 shows the values of the sidelobe level for the original (initial) sequences and newly synthesized sequences.

| Length, N | Type of sequence | PSL value |
|-----------|------------------|-----------|
|           |                  | Initial sequence | Synthesized sequence |
| 680       | Zadoff-Chu       | 12.534     | 1.900     |
| 1024      | Frank            | 10.202     | 2.193     |
| 2048      | P4               | 21.729     | 2.850     |
| 2730      | Zadoff-Chu       | 25.097     | 3.883     |
| 4096      | Frank            | 20.280     | 5.310     |

Table 1 shows that the synthesized sequences have a maximum sidelobe level of impulse autocorrelation function 10-16 dB lower than that of the original sequences.

Based on the developed algorithm, it was possible to synthesize a multiphase sequence of length with a lower sidelobe level than that of the sequence from [21].

A 4096 signal was applied in a synthetic aperture radar. The result of the survey of the area is shown in figure 1. A large number of anthropogenic objects with a high radar crosssection does not lead to the appearance of any artifacts associated with a high level of side lobes.

![Radar image](image1.png)

Figure 1. Radar image of the terrain obtained with a 4096 signal with a PSL level of 5.310.

7. Conclusion
This paper presents an algorithm for constructing long polyphase sequences with a low sidelobe level of the impulse autocorrelation function. Using the proposed algorithm for lengths 680, 1024, 2048, 2730, 4096, polyphase pulse sequences were synthesized with the current record low sidelobe levels. The 4096 signal has been shown to be effective in synthetic aperture radar.
Acknowledgments
The reported study was funded by RFBR, project number 19-31-90122.

References
[1] Barker R H 1953 Group Synchronizing of Binary Digital Systems Communication Theory 273-87
[2] Turyn R 1968 Sequences with small correlation Error correcting codes (New York: John Wiley and Sons) p 195-228
[3] Mossinghoff M J 2009 Wieferich pairs and Barker sequences Designs Codes and Cryptography 53(3) 149-63
[4] DeLong Jr D F 1959 Three-Phase Codes Massachusetts Inst of Tech Lexington Lincoln Lab
[5] Welti G 1960 Quaternary codes for pulsed radar IRE Transactions on Information Theory 6 400-8
[6] Golomb S W and Scholtz R A 1965 Generalized barker sequences IEEE Transactions on Information theory 11 533-7
[7] Zhang N and Golomb S W 1989 Sixty-phase generalized Barker sequences IEEE Transactions on Information theory 35 911-2
[8] Friese M and Zottmann H 1994 Polyphase Barker sequences up to length 31 Electronics letters 30 1930-1
[9] Friese M 1996 Polyphase Barker sequences up to length 36 IEEE Transactions on Information Theory 42 1248-50
[10] Brenner A R 1998 Polyphase Barker sequences up to length 45 with small alphabets Electronics 34 1576-7
[11] Borwein P and Ferguson R 2005 Polyphase sequences with low autocorrelation IEEE Transactions on Information Theory 51 1564-7
[12] Nunn C J 2005 Constrained optimization applied to pulse compression codes and filters IEEE International Radar Conference (Washington: IEEE) 190-4
[13] Nunn C J and Coxson G E 2009 Polyphase pulse compression codes with optimal peak and integrated sidelobes IEEE Transactions on Aerospace and Electronic Systems 45 775-81
[14] Leukhin A N 2009 Pulse phase-coded sequences with a single sidelobe level Proceedings of higher educational institutions of Russia. Radio electronics 6 13-8 (In Russian)
[15] Leukhin A N, Shuvalov A S and Petukhov A S 2010 Multiphase pulse sequences with a single-level autocorrelation Bulletin of the Mari State Technical University: Series of radiotechnical and infocommunication systems 2 77-90 (In Russian)
[16] Leukhin A N et al. 2012 Optimal pulse sequences Russian. In: Bulletin of the Mari State Technical University: Series of radiotechnical and infocommunication systems 1 37-46 (In Russian)
[17] Shuvalov A S, Leukhin A N and Gayvoronsky D V 2012 New search algorithm for polyphase Barker sequences Bulletin of the Mari State Technical University: Series of radiotechnical and infocommunication systems 2 19-23 (In Russian)
[18] Leukhin A N, Shuvalov A S and Vinogradov V O 2013 Multiphase Barker Sequences Scientific Notes of the Physics Department of Moscow University 1 24-5 (In Russian)
[19] Bystrov N E, Zhukova I N, and Chebotarev D V 2009 Synthesis of multiphase pulse signals by nonlinear optimization Proceedings of higher educational institutions of Russia. Radio electronics 6 13-8 (In Russian)
[20] Leukhin A N, Bezrodnyi V I, and Kokovikhina N A. 2017 Long polyphase sequences with low side lobes of the aperiodic autocorrelation function Scientific Notes of the Physics Department of Moscow University 6 1761001 (In Russian)
[21] Anatoly Nikolaevich Leukhin. 2017 A polyphase sequences with low autocorrelations Journal of Physics: Conference Series 859 012012