Equivalence of effects for a limiting stress state

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Abstract. The article considers the possibility of an equivalent substitution of the volume forces for a limiting stress state by a surface loading. For any reason causing the stresses in a slope, in the final analysis these effects are considered in the system of equations for an under-limiting stress state with the help of forced strains, surface and volume forces and in the system of equations for a limiting stress state - with the help of surface and volume forces. The determination of forced strain values, values of surface and volume forces require a thorough analysis of engineering geological situation at the slope and the performance of some simple calculations. Depending on the problem, there can be no influence from the part of forced strains and surface forces. As to the volume forces, they always act as the own weight forces even if the slope is not subject to any process.

Methods. The methodology of physical modeling of stresses in a non-uniform slope is stated.

Result. The problem of determination of limiting stresses caused by volume forces in the slope is reduced to the problem of determination of stresses caused by fictitious surface forces. Considered a non-uniform slope containing n layers, the volume masses (weights) of which equal \( \gamma_n \). The boundaries between the layers are not plane. The volume forces are piecewise-constant. The solution of the problem is represented as the sum of solutions of n problems.

Conclusion. The solution obtained through the developed methodology identically coincides with the known solution. This fact testifies to the truth of the developed methodology.

1. Introduction

When solving problems of mechanics of a deformable body to the side, especially when using experimental methods, effectively apply conditions of equivalence of impacts. Stanley equivalence effects for problems of mechanics of deformable bodies makes possible the solution of many important problems. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

For an under-limiting stress state of the slope, we can perform an equivalent substitution of one effect by the other one through the following formulas [11, 12, 13, 14, 15, 16, 17]

\[
\frac{\partial P(t)}{\partial t} = F(t), \quad \xi(t) = \frac{1 - 2\nu}{E} P(t). \tag{1}
\]

Then the stresses in the slope, caused by these effects, are connected by the formulas

\[
\sigma_y^{(F)}(t) = \sigma_y^{(P)(}\(t) - \sigma_y^{(s)}(t) - \delta_y P(t). \tag{2}
\]
2. Methods

Now determine the conditions of equivalence of effects for a limiting stress state of the slope. Let all the slope points have a limiting stress state under the influence of volume forces \( F_i(t) \). Then the stresses in the slope caused by volume forces satisfy the following equations

\[
\sum_j \frac{\partial \sigma_y^{(F)}(t)}{\partial j} + F_i(t) = 0, \quad (3)
\]

\[
\frac{\sigma_1^{(F)}(t) - \sigma_3^{(F)}(t)}{\sigma_1^{(F)}(t) + \sigma_3^{(F)}(t) + 2c(t)\cot \varphi(t)} = \sin \varphi(t), \quad (4)
\]

\[
\sum_j \sigma_y^{(F)}(t)n_j = 0. \quad (5)
\]

Denote through \( \sigma_y^{(F)}(t) \) the stresses caused in the slope by fictitious surface forces \( P(t) \), distributed on the boundary surface and directed normally to this surface. In this case, it is obvious that these stresses satisfy the equations

\[
\sum_j \frac{\partial \sigma_y^{(F)}(t)}{\partial j} = 0, \quad (6)
\]

\[
\sum_j \sigma_y^{(F)}(t)n_j = P(t). \quad (7)
\]

It is easy to make sure that the stresses in the slope caused by volume forces and presented in the form

\[
\sigma_y^{(F)}(t) = \sigma_y^{(p)}(t) - P\delta_y\quad (8)
\]

satisfy the boundary conditions (5). Substituting the formula for stresses from (8) into (3), we obtain

\[
\frac{\partial P(t)}{\partial t} = F_i(t). \quad (9)
\]

Thus, the problem of determination of limiting stresses caused by volume forces in the slope is reduced to the problem of determination of stresses caused by fictitious surface forces determined by the formula (9).

3. Main part

The condition of limiting equilibrium (4) in case of a plane problem takes the form

\[
\left[ \sigma_x^{(F)}(t) - \sigma_z^{(F)}(t) \right]^2 + 4\left[ \tau_{xz}^{(F)}(t) \right]^2 = \left[ \sigma_x^{(F)}(t) + \sigma_z^{(F)}(t) + 2c(t)\cot \varphi(t) \right]^2 \sin^2 \varphi(t) \quad (10)
\]

With consideration of (8), the condition of limiting equilibrium (10) in stresses depending on a fictitious surface force takes the form

\[
\left[ \sigma_x^{(p)}(t) - \sigma_z^{(p)}(t) \right]^2 + 4\left[ \tau_{xz}^{(p)}(t) \right]^2 = \left[ \sigma_x^{(p)}(t) + \sigma_z^{(p)}(t) - 2P(t) + 2c(t)\cot \varphi(t) \right]^2 \sin^2 \varphi(t) \quad (11)
\]

Determining the stresses caused by a fictitious surface loading, we check the slope stability through the formulas (11) or (10), where the stress values caused by volume forces are determined with the help of the formula (8), which connects the stress values caused by equivalent effects.

Therefore, if the slope is in one stress state (limiting or under-limiting) under the influence of the loading, it may be in other stress state under the influence of an equivalent loading. This conclusion is very important for the use of experimental methods. It allows us to analyze the limiting stress state of the slope through the results of the study of its under-limiting stress state.

Comparing the equivalence conditions (1) and the formula (2) for the stresses for an under-limiting stress state of the slope with the equivalence condition (9) and the formula (8) for the stresses for a limiting stress state of the slope, we note that the last formulas coincide with the first ones if we take \( \nu = 0.5 \). The explanation is like that: when there are limiting stress states in all the slope points, the
plastic deformations develop limitlessly in all the slope points, i.e. develops the flow of the whole slope. It is obvious that the lateral deformation coefficient for the slope ground in the flow state equals 0.5. Hence the condition of equivalence of effects (9) and the formula connecting the stresses caused by different influence types (8) could be obtained from the simplest considerations, taking into account the lateral deformation coefficient \( \nu = 0.5 \) in corresponding formulas.

As the problem can be solved for a ground slope, we take (following the studies by S.R. Meschyan [17]) that the ground skeleton satisfies the law of linear hereditary creep theory.

The assumption of a constant Poisson’s ratio turns to be true for many grounds for the compacting pressures up to 300 kPa, which is convincingly shown in the experiments by S.R. Meschyan.

In this case, in accordance with the elastic analogy [18], the stresses \( \sigma_{ij}^*(t) \) obtained with consideration of the creep agree with momentary elastic ones: \( \sigma_{ij}^*(t) = \sigma_{ij} \).

Let us consider a non-uniform slope containing \( n \) layers, the volume masses (weights) of which equal \( \gamma_n \). The boundaries between the layers are not plane.

In this case, the volume forces are piecewise-constant
\[
F_i = \gamma_n \quad \text{and} \quad F_i = F_j = 0,
\]
and within the limits of each layer they are potential, i.e. the slope is under the influence of piecewise-potential volume forces.

The solution of the problem we present as the sum of solutions of \( n \) problems, in which the action of volume forces equals zero in the slope area over the roof of the \( n \)-th layer, and the slope area lower than this level is under the influence of volume forces.

\[
F_i = \gamma_n - \gamma_{n-1} \quad \text{and} \quad F_j = F_z = 0
\]

Thus, the solution of the problem is reduced to the solution of \( n \) problems of the same type. For the solution of these problems, the models of the slope areas under the influence of volume forces are made from an opto-polarization material. Further, the stresses corresponding to the acting volume forces are “frozen” in these models.

For the creation of volume forces (13) in the model areas we can use the centrifuge method. But not all laboratories are equipped with centrifugal machines. In such a case, we can use the equivalence of effects, in accordance with which the stresses caused by volume forces (13) in the model areas (due to their potentiality) may be presented in the form
\[
\sigma_{ij}^{(F)} = \sigma_{ij}^{(P)} - \sigma_{ij}^{(\xi)} - \sigma_{ij},
\]

where \( P = (\gamma_n - \gamma_{n-1}) \zeta \) - loading normal to the model surface;
\[
\xi = \frac{1 - 2\nu}{E} (\gamma_n - \gamma_{n-1}) \zeta \quad \text{forced strains;}
\]
\[
\sigma_{ij} = \sigma_{ij} \left( \gamma_n - \gamma_{n-1} \right) \zeta.
\]

Further, to the “frozen” models the corresponding models of the slope areas are stuck, in which the action of volume forces equals zero, i.e. they are in their natural non-deformed state. Then the “annealing” of the compiled models is performed.

In accordance with the modeling criteria, the full-scale stress values in the \( n \)-th problem are determined by the following formulas:

In case of the use of the centrifuge method
\[
\sigma_{ij}^{(n)} = \frac{K_y \left( \gamma_n - \gamma_{n-1} \right)}{K_y} \left( \frac{\sigma_{mod} - \sigma_{mod}}{\sigma_{mod} - \sigma_{mod}} \right),
\]

or in case of the use of the equivalence of effects
\[
\sigma_{ij}^{(n)} = K_y \left( \frac{\sigma_{mod} - \sigma_{mod}}{\sigma_{mod} - \sigma_{mod}} \right) K_y \left( \frac{\sigma_{mod} - \sigma_{mod}}{\sigma_{mod} - \sigma_{mod}} \right) K_y \left( \frac{\sigma_{mod} - \sigma_{mod}}{\sigma_{mod} - \sigma_{mod}} \right).
\]
where

\[ K_i = \frac{l_{\text{fc}}}{l_{\text{mod}}}; \quad K_c = \frac{R \Omega^2}{g}; \quad K_p = \frac{P_{\text{fc}}}{P_{\text{mod}}}; \quad K_\sigma = \frac{(\gamma_n - \gamma_{n-1}) z_{\text{fc}}}{(1 - 2 \nu) E_{\text{mod}} z_{\text{mod}}}; \]

\[ K_\sigma = \frac{\sigma_{\text{fc}}}{\sigma_{\text{mod}}} = \frac{(\gamma_n - \gamma_{n-1}) z_{\text{fc}}}{(1 - 2 \nu) E_{\text{mod}} z_{\text{mod}}}; \]

- \( \Omega \) - centrifuge angular velocity;
- \( R \) – centrifuge radius;
- \( g \) – free fall acceleration;
- \( \gamma_{\text{mod}} \) - volume mass of the model materials;
- \( \sigma_{\text{mod}} \) - stresses in the model areas where the volume forces act;
- \( \sigma_{\text{mod}} \) - stresses in the model compiled from the model areas after the “annealing”.

Then the stress values in the initial problem are determined by the formula

\[ \sigma_{ij} = \sum \sigma_{ij}^{(n)}. \]  \hspace{1cm} (17)

Let us illustrate the suggested methodology and consider a ground medium consisting of \( n \) layers, each with the thickness of \( h_n \) and the volume mass of \( \gamma_n \).

In this case, the volume forces are piecewise-constant

\[ F_x = \gamma_n \quad \text{and} \quad F_y = F_z = 0, \]  \hspace{1cm} (18)

and they are potential within each ground layer, i.e. the ground medium is under the influence of piecewise-potential volume forces.

The stress values in this problem with the conditions \( \varepsilon_x = \varepsilon_z = 0 \) and \( \sigma_y = \sigma_z \) at the depth of \( x = \sum h_n \) are determined, as it is known, through the formulas [19, 20]

\[ \sigma_x = -\sum \gamma_i h_i, \quad \sigma_y = \sigma_z = -\frac{\nu}{1-\nu} \sum \gamma_i h_i, \]  \hspace{1cm} (19)

where \( h_n \) denotes the distance from the roof of the layer to the point where the stress values are determined.

Now consider the solution of this problem through the suggested methodology. The solution of the problem we present as the sum of solutions of \( n \) problem. In these problems, the action of volume forces in the ground medium area over the roof of the \( n \)-th layer equals zero, and the area lower than that level is under the influence of volume forces

\[ F_x = \gamma_n - \gamma_{n-1} \quad \text{and} \quad F_y = F_z = 0. \]  \hspace{1cm} (20)

Thus, the solution of the problem is reduced to the solution of \( n \) problems of the same type.

In accordance with the developed methodology, the solutions of these problems in the models of the areas under the influence of volume forces (20) require the “freezing” of corresponding stresses.

The expressions for these stresses in the \( n \)-th problem have the following form:

\[ \sigma_x^{(n)} = -\left( \gamma_n - \gamma_{n-1} \right) \left( x - \sum_{k=0}^{n-1} h_k \right), \]  \hspace{1cm} (21)
\[ \sigma_y^{(n)} = \sigma_z^{(n)} = -\frac{\nu}{1-\nu} \left( \gamma_n - \gamma_{n-1} \right) \left( x - \sum_{k=0}^{n-1} h_k \right). \]  

(22)

It is obvious that the stresses \( \sigma_x^{(n)} \), \( \sigma_y^{(n)} \), and \( \sigma_z^{(n)} \) in the points of the surfaces \( x = \sum_{k=0}^{n-1} h_k \) equal zero as these surfaces are planes.

So the stresses \( \bar{\sigma}_x^{(n)} \), \( \bar{\sigma}_y^{(n)} \), and \( \bar{\sigma}_z^{(n)} \) appearing in the process of “de-freezing” of compiled models with stresses \( \bar{\sigma}_x^{(n)} \), \( \bar{\sigma}_y^{(n)} \), and \( \bar{\sigma}_z^{(n)} \), which are stuck together with the models of areas with zero volume forces in their natural states, will be equal to zero as well.

Therefore, it is expedient to carry out a “mental” experiment for this problem.

In this case, in accordance with the developed methodology, the solution of the n-th problem has the form

\[ \sigma_i^{(n)} = \bar{\sigma}_i^{(n)} - \bar{\sigma}_i^{(n)} = \begin{cases} 0 & \text{when } x \leq \sum_{k=0}^{n-1} h_k \\ \bar{\sigma}_i^{(n)} & \text{when } x > \sum_{k=0}^{n-1} h_k \end{cases}, \]  

(23)

where \( i = x, y, z \).

Then for the final solution we obtain

\[ \sigma_x = \sum \sigma_x^{(n)} = -\sum \gamma_n h_n, \]

\[ \sigma_y = \sigma_z = \sum \sigma_y^{(n)} = \sum \sigma_z^{(n)} = -\frac{\nu}{1-\nu} \sum \gamma_n h_n. \]  

(24)

4. Conclusions

Thus, the solution obtained through the developed methodology (24) identically coincides with the known solution (19). This fact testifies to the truth of the developed methodology.

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