On Short Cuts
or
Fencing in Rectangular Strips

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Abstract. In this paper we consider an isoperimetric inequality for the free perimeter of a planar shape inside a rectangular domain, the free perimeter being the length of the shape boundary that does not touch the border of the domain.

1 Introduction

The isoperimetric inequality for shapes in \( \mathbb{R}^2 \) states that the area enclosed by a simple closed curve is at most that of a circle of the same length, and that equality occurs only for circles. This immediately implies that among all simple closed curves enclosing a given area, a circle is the shortest.

Several variations on the isoperimetric inequality were considered in the literature (see e.g. [7]). In this paper we shall discuss inequalities involving the notion of "free perimeter" for a shape \( S \), located inside a simple, bounded, planar domain \( D \). We may assume that there is a border or wall surrounding this domain, or alternatively that this domain is an island surrounded by water. A simple shape \( S \) inside this domain will be defined by a boundary curve, some portion of which may touch and even follow the border (wall / shoreline) of the domain / island. The free perimeter of the shape will be defined as the length of the boundary curve of \( S \) that does not overlap with, or trace, the border of the enclosing domain \( D \).

The problem that we can pose with these definitions is the following: given the domain \( D \), determine the shape with the shortest free-perimeter that has a given area \( A \). This problem is, of course, that of determining the way to cut out a shape of a total area \( A \) from \( D \) with the least effort of cutting, i.e. with the shortest cut. Equivalently, this is the problem of determining the shortest length "fence" that can separate a contiguous region of area \( A \) inside the domain \( D \).

This interpretation clearly explains the totally misleading title of our paper, in which we do not take any short-cuts and of course we do not discuss fencing as a sport that happens to be played on a rectangular, strip-shaped, “ring”.

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A related problem is that of finding the connected shape of largest area that can be “lifted out” of \( D \) with a total length of “cuts” or “fences” less than or equal to \( L \).

In this paper we solve the problem raised above when the region \( D \) is a rectangle. We prove that the shortest cut, i.e. the minimum free perimeter, that separates a shape with half of the area of \( D \) has, as expected, the length of the shorter side of the rectangle. We then provide the shortest free perimeter for all \( \frac{\text{Area}(S)}{\text{Area}(D)} \) ratios from 0 to 1.

We note that the problem we discuss is closely related to the problem A26, “Dividing up a piece of land by a short fence”, discussed in the book “Unsolved Problems in Geometry” [4]. The challenge posed there is that of dividing a convex shape into two equal-area parts. We refer the interested reader to [4] and to some recent follow-up papers [5, 6].

**Fig. 1.** An illustration of the notion of “free perimeter”. The area of the shape \( S \) equals \( A \), while its free perimeter equals \( l_1 + l_2 \).

### 2 Free Perimeter of Half Area Shapes in Rectangles

Let \( D[ X, Y ] \) be a bounding rectangle of dimensions \( X \) and \( Y \), with \( X \leq Y \). Let \( A \) be the area we want to enclose with a region of shape \( S \), and denote by \( l_{FP}(S) \) the length of the free perimeter of the shape \( S \). Let us denote by \( l^*(A) \) the length of the free perimeter of a shape \( S \) with area \( A \), such that \( S \) has the smallest value of \( l_{FP}(S) \) out of all the shapes of area \( A \). Namely:

\[
l^*(A) \triangleq \min_{\text{Area}(S)=A} \{ l_{FP}(S) \}
\]
We shall be interested in determining the value of \( l^*(A) \) for \( A \in [0, XY] \). For this, we shall first prove the following result:

**Theorem 1.**

\[
l^* \left( A = \frac{1}{2} XY \right) = X
\]

**Proof.** To prove the above stated, and rather natural and hardly surprising result, we shall need to combine several simple facts.

**Fact 1 The Classical Planar Isoperimetric Inequality**

Given any shape of area \( A \) in the plane, and perimeter of length \( l \) we have:

\[
l \geq 2\sqrt{\pi \sqrt{A}} = \sqrt{4\pi A}
\]

with equality achieved for a circle.

**Fact 2 The Half-Plane Isoperimetric Inequality**

Given any shape \( S \) of area \( A \) in a half plain domain, with free perimeter of \( l_{FP}(S) \) we have:

\[
l_{FP} \geq \sqrt{2\pi A}
\]

**Proof.** If \( S \) touches the boundary of the half-plane, let us reflect it along the boundary line, thereby generating a (symmetric) shape of area \( 2A \) in the plane. For this “double shape” \( S' \) we have:

\[
l_{FP}(S') = 2l_{FP}(S)
\]

and with the classical isoperimetric inequality of Fact 1 we obtain:

\[
l_{FP}(S') \geq 2\sqrt{\pi \sqrt{2A}}
\]

hence:

\[
l_{FP}(S) = \frac{1}{2} l_{FP}(S') \geq \sqrt{\pi \sqrt{2A}}
\]

**Fact 3 The Quarter-Plane Isoperimetric Inequality**

Given any shape \( S \) of area \( A \) in a quarter plain domain, with free perimeter of \( l_{FP}(S) \) we have:

\[
l_{FP} \geq \sqrt{\pi A}
\]

**Proof.** If \( S \) touches the two orthogonal boundaries of the quarter-plane, let us reflect it symmetrically into the three quarters plane domain boundary, generating a shape \( S' \) in the plane, of area \( 4A \). For \( S' \) we have:

\[
l_{FP}(S') = 4l_{FP}(S)
\]

and with the classical isoperimetric inequality of Fact 1 we obtain:

\[
l_{FP}(S') \geq 2\sqrt{\pi \sqrt{4A}}
\]

yielding:

\[
l_{FP}(S) = \frac{1}{4} l_{FP}(S') \geq \sqrt{\pi \sqrt{A}}
\]
A shape \( S \subset D[X,Y] \) may touch the sides of the boundary of the rectangle \( D(X,Y) \) in several ways. We may have \( S \) that touches 0, 1, 2, 3 or 4 sides. Let us consider these cases separately:

**Case 0**: \( S \) touches 0 sides of \( D[X,Y] \). In this case, the classical isoperimetric inequality of Fact 1 yields:

\[
l_{FP}(S) \geq 2 \sqrt{\pi} \sqrt{\frac{1}{2}XY} \geq \sqrt{\frac{1}{2} \pi \cdot X} > X
\]

**Case 1**: \( S \) touches 1 of the sides of \( D[X,Y] \). In this case, Fact 2 yields:

\[
l_{FP}(S) \geq \sqrt{\frac{1}{2} \pi \cdot X} > X
\]

**Case 2**: \( S \) touches 2 of the sides of \( D[X,Y] \). In this case we have either \( S \) touches two opposite sides, yielding \( l_{FP}(S) \geq 2 \min\{X,Y\} \geq 2X \), or \( S \) touches two adjacent sides, in which case Fact 3 provides:

\[
l_{FP}(S) \geq \sqrt{\frac{1}{2} \pi \cdot X} \geq \sqrt{\frac{1}{2} \pi \cdot X} > X \quad \text{(since } Y \geq X)\]

**Case 3**: \( S \) touches 3 of the sides of \( D[X,Y] \). In this case we have \( l_{FP}(S) \geq \min\{X,Y\} \geq X \), since any of the portions of the boundary of \( S \) will have to join parts on opposite sides of \( D[X,Y] \).

**Case 4**: \( S \) touches all four sides of \( D[X,Y] \). In this case we have a connected shape \( S \) which is continuous (i.e. connected), whose complement \( S^C \triangleq D[X,Y] \setminus S \) might be a set of disconnected regions \( S^C_1, S^C_2, S^C_3, \ldots, S^C_k \), of areas \( A_1, A_2, A_3, \ldots, A_k \), which all belong to \( D[X,Y] \), and for which we have:

\[
\sum A_i = \frac{1}{2}XY
\]

We also have that:

\[
\sum l_{FP}(S^C_i) = l_{FP}(S^C) = l_{FP}(S)
\]

Notice that for all \( i, S^C_i \) cannot touch more than 2 sides of the rectangle \( D[X,Y] \), since this would imply that \( S \) is disconnected.

By Facts 1, 2 and 3 we therefore have:

\[
f_{FP}(S^C_i) \geq \min\{\sqrt{\pi}, \sqrt{2\pi}, \sqrt{4\pi}\} \cdot \sqrt{A_i} = \sqrt{\pi} \sqrt{A_i}
\]

and subsequently:

\[
f_{FP}(S) = f_{FP}(S^C) = \sum_{i=1}^{k} l_{FP}(S^C_i) \geq \sqrt{\pi} \sum_{i=1}^{k} \sqrt{A_i}
\]
Notice that:

\[
\left( \sum_{i=1}^{k} \sqrt{A_i} \right)^2 = \sum_{i=1}^{k} A_i + \sum_{i\neq j} \sqrt{A_i} \sqrt{A_j}
\]

Hence:

\[
\sum_{i=1}^{k} \sqrt{A_i} \geq \sqrt{A}
\]

and therefore:

\[
l_{FP}(S) \geq \sqrt{\pi} \sum_{i=1}^{k} \sqrt{A_i} \geq \sqrt{\pi} \sqrt{\frac{1}{2} XY} \geq X
\]

It is important to note that although:

\[
l_{FP}(S) \geq \sqrt{\pi} \sum_{i=1}^{k} \sqrt{A_i} \geq \sqrt{\pi} \sqrt{\text{Area}(D) - A}
\]

(which is the same in this case, as here \(\text{Area}(D) = 2A\)).

We have shown that in all cases, \(l_{FP}(S) \geq X\). It is easy to see that when \(S\) is defined as the half-rectangle \(X \times \frac{1}{2} Y\), the free perimeter obtained is exactly \(X\). Therefore, we have shown that \(l^*(\frac{1}{2} XY) = X\).

In fact, we have shown something stronger that just \(l^*(\frac{1}{2} XY) = X\). In all cases where \(S\) touches 0, 1, 2 or 4 sides of the rectangle, its free perimeter \(l_{FP}(S)\) was strictly higher than \(X\), by factors of \(\sqrt{2\pi} > 2 > \sqrt{\pi} > \sqrt{\frac{\pi}{2}} > 1\).

Interestingly, note that \(\sum_{i=1}^{k} \sqrt{A_i}\) is maximized where \(\forall i, A_i = \frac{A}{k}\):

**Proof.** Let us define:

\[
\Psi = \sum_{i=1}^{k} \sqrt{A_i} + \lambda \left( \sum_{i=1}^{k} A_i - A \right)
\]

In order for \(\frac{\partial \Psi}{\partial A_i} = 0\) we must have \(\frac{1}{2} \frac{1}{\sqrt{A_i}} + \lambda = 0\). Namely:

\[
\forall i \quad A_i = \frac{1}{4\lambda^2}
\]

In other words:

\[
A = \sum_{i=1}^{k} A_i = k \frac{1}{4\lambda^2}
\]
and subsequently:

$$\lambda = \frac{1}{2} \sqrt{\frac{k}{A}}$$

Assigning \( \lambda \) back to \( A_i \) yields:

$$\forall i \ A_i = A$$

3 The Free Perimeter \( l^*(A) \) for \( A < \frac{1}{2}XY \)

From the proof of Theorem 1 we saw that cutting the rectangle \( D[X,Y] \) into two equal pieces by a cut parallel to the short side of length \( X \) of \( D[X,Y] \) is optimal w.r.t the length of the free perimeter. The results we have, in fact, state that if a shape \( S \) of an area \( A \) is to be separated by a short fence in \( D[X,Y] \) we shall have:

$$f_{FP}(S) \geq 2\sqrt{\pi \sqrt{A}} \quad \text{if } S \text{ touches 0 sides}$$
$$f_{FP}(S) \geq \sqrt{2\pi \sqrt{A}} \quad \text{if } S \text{ touches 1 sides}$$
$$f_{FP}(S) \geq \sqrt{\pi \sqrt{A}} \quad \text{if } S \text{ touches 2 adjacent sides}$$
$$f_{FP}(S) \geq 2X \quad \text{if } S \text{ touches 2 opposite sides}$$
$$f_{FP}(S) \geq X \quad \text{if } S \text{ touches 3 sides}$$
$$f_{FP}(S) \geq \sqrt{\pi \sqrt{XY - A}} \quad \text{if } S \text{ touches 4 sides}$$

We shall now ask what happens when \( A < \frac{1}{2}XY \), and as \( A \to 0 \). It is clear that for any \( A \) we can separate a shape of area \( A \) with a cut of size \( X \), hence for every value of \( A < \frac{1}{2}XY \) it holds that \( l^*(A) \leq X \).
Contemplating the above inequalities we realize that while $A$ is such that $\sqrt{\pi} \sqrt{A}$ is not less than $X$ we cannot hope to find a better cut! Hence, if:

$$\sqrt{\pi} \sqrt{A} \geq X$$

namely, if:

$$A \geq \frac{X^2}{\pi} \text{ then}$$

we shall have:

$$l^*(A) \geq X$$

This can also be obtained using a quarter of a circle of radius $r = \frac{2X}{\pi}$.

What happens when $A < \frac{X^2}{\pi}$? It can be seen that from this point it pays to use quarter-circular of smaller and smaller radii, that will achieve the bound of $l^*(A) = \sqrt{\pi} \sqrt{A}$. We therefore get the following result:

**Theorem 2.**

$$l^*(A) = \begin{cases} 
\frac{X}{\sqrt{\pi A}} & \text{for } \frac{X^2}{\pi} \leq A \leq \frac{1}{2} XY \\
\sqrt{\pi} \sqrt{(XY - A)} & \text{for } XY - \frac{X^2}{\pi} \leq A \leq XY
\end{cases}$$

4 The Free Perimeter $l^*(A)$ for $A > \frac{1}{2} XY$

Due to symmetry considerations, we can see that for any shape $S$ of area larger than $\frac{1}{2} XY$ we can simply analyze the combined free perimeters of the shapes that comprise the complement $S^C \triangleq D[X,Y] \setminus S = S_1^C, S_2^C, S_3^C, \ldots, S_k^C$, as it clearly equals the free perimeter of $S$. From the results shown in the previous section, we already know that the free perimeter of $S$ is minimized when $S^C$ is in fact a single connected shape, that touches either two adjacent sides of the rectangle, or three of its sides (depending on the area of $S$). In other words, $S^C$ is either a portion of the rectangle that is generated using a cut which is parallel to its shorter side, or a quarter of a circle of radius $r \leq \frac{2X}{\pi}$.

We can now complete our bound concerning the free perimeter for shapes of area larger than $\frac{1}{2} XY$, as follows:

**Theorem 3.**

$$l^*(A) = \begin{cases} 
\frac{X}{\sqrt{\pi (XY - A)}} & \text{for } \frac{1}{2} XY \leq A \leq XY - \frac{X^2}{\pi} \\
\sqrt{\pi} \sqrt{(XY - A)} & \text{for } XY - \frac{X^2}{\pi} \leq A \leq XY
\end{cases}$$

5 Concluding Remarks

In this paper we have completely analyzed the free perimeter isoperimetric inequality for a rectangular ambient domain. It would be very interesting to do so for various other domains as well, such as a circular domain or an annular region, and in fact any regular polygon. Our motivation for this study was a problem
that arose in designing good strategies for cooperative search of smart targets using swarm of robots [2]. As is obvious from the list of references, such problems are of great interest both from a purely geometric point of view, and in conjunction with some interesting robotics / multi agents search applications [1–3].

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