ON THE CLASSIFICATION OF WEAKLY INTEGRAL MODULAR CATEGORIES

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Abstract. In this paper we classify all modular categories of dimension $4m$, where $m$ is an odd square-free integer, and all rank 6 and rank 7 weakly integral modular categories. This completes the classification of weakly integral modular categories through rank 7. In particular, our results imply that all integral modular categories of rank at most 7 are pointed (that is, every simple object has dimension 1). All the non-integral (but weakly integral) modular categories of ranks 6 and 7 have dimension $4m$, with $m$ an odd square free integer, so their classification is an application of our main result.

1 Introduction

In this paper we study weakly integral modular categories—that is, modular categories $C$ with $\dim C \in \mathbb{Z}$. We completely classify such categories when the number of simple objects (up to isomorphism) is 6 or 7. This is facilitated by a more general theorem in which we classify modular categories of dimension $4m$, where $m$ is an odd square-free integer.

A pivotal fusion category $C$ is called integral if the dimensions $d_X$ of simple objects $X$ are integers, whereas $C$ is weakly integral if and only if $(d_X)^2 \in \mathbb{Z}$, for all simple objects $X$. We will say that a pivotal fusion category is strictly weakly integral if it is weakly integral but not integral, i.e. if some simple object $X$ has $d_X \not\in \mathbb{Z}$. Interestingly, it is somewhat easier to classify strictly weakly integral modular categories of low rank than integral modular categories.

Our results are summarized in:

Theorem 1.1. Suppose that $C$ is a weakly integral modular category either of rank $\leq 7$ or of dimension $4m$ where $m$ is an odd square-free integer. Then $C$ is equivalent to a Deligne product of the following: pointed categories, Ising categories and $\mathbb{Z}_2$ de-equivariantizations of Tambara-Yamagami categories.

These classes of categories are described below.

2 Preliminaries

Several classes of modular categories will play a prominent role in this paper, so we establish notation for them:
(i) An Ising modular category $I$ is a non-pointed modular category with $\dim I = 4$.

(ii) A pointed modular category with fusion rules like those of $\text{Rep}(\mathbb{Z}_n)$ will be denoted $P_n$. Notice that $P_1 \cong \text{Vec}$.

(iii) A Tambara-Yamagami fusion category corresponding to an abelian group $A$, a symmetric bi-character $\chi$ and a sign choice $\nu$ will be denoted $TY(A, \chi, \nu)$.

(iv) A metaplectic modular category $C$ is a modular category with same fusion rules as the (weakly integral) modular category $SO(N)_2$, and will be denoted $M_N$.

Remark 2.1. There is an important distinction between the Ising category $SU(2)_2$ and an Ising category, which is any category with the same fusion rules as $SU(2)_2$. These are classified in [6, Appendix B]. It follows from [11] that, for $N$ odd, the $\mathbb{Z}_2$ de-equivariantization $TY(\mathbb{Z}_N, \chi, \nu)\mathbb{Z}_2$ is a metaplectic modular category.

Some well-known identities will be frequently used. Fix a labelling set $\text{Irr}(C) = \{X_0 = 1, X_1, \ldots, X_{r-1}\}$ for the isomorphism classes of simple objects in a rank $r$ modular category $C$. We denote by $S_{ij}$ the $(i,j)$-entry of the unnormalized $S$-matrix and by $d_i = \dim(X_i)$ the categorical dimension of the object $X_i$. The fusion coefficients are given by $N_{k}^{ij} := \dim \text{Hom}(X_i \otimes X_j, X_k)$ and the twists are denoted $\theta_i := \theta_{X_i}$.

(i) **Twist equation** (see [2, Equation 2.6]):

\[ p^+ S_{ij} = \theta_i \theta_j \sum_k S_{ki} S_{kj} \theta_k, \quad (2.1) \]

where $p^+ = \sum_i d_i^2 \theta_i$.

(ii) **Balancing equation** (see [2, Equation 2.3]):

\[ S_{ij} \theta_i \theta_j = \sum_k N_{i}^{kj} d_k \theta_k. \quad (2.2) \]

(iii) **Orthogonality:**

\[ SS^\dagger = \dim(C) I_r, \quad (2.3) \]

where $\dagger$ is the conjugate-transpose operation.

A fusion category $C$ is $G$-graded ($G$ a finite group) if $C = \bigoplus_{g \in G} C_g$ as an abelian category and $C_g \otimes C_h \subset C_{gh}$. If each $C_g$ is non-empty, the grading is called faithful. It was proved in [12, Theorem 3.5] that any fusion category $C$ is naturally graded by a group $U(C)$, called the universal grading group of $C$, and the adjoint subcategory $C_{ad}$ is the trivial component of this grading. Moreover, any other faithful grading of $C$ arises from a quotient of $U(C)$ [12, Corollary 3.7]. For a modular category, the universal grading group $U(C)$ is isomorphic to the group $G(C)$ of invertible simple objects of $C$ [12, Theorem 6.2]. The fusion subcategory generated by the group of invertible objects is the maximal pointed subcategory of $C$ and it is denoted $C_{pt}$. A strictly weakly integral fusion category is faithfully graded by an elementary abelian 2-group [12 Theorem 3.10]. Indeed, as each simple object has dimension $\sqrt{k}$, for some $k \in \mathbb{Z}$, one may partition the simple objects into finitely many non-empty sets of the form $A_n := \{X : d_X \in \sqrt{n} \mathbb{Z}\}$, where $n$ is square-free, and this partition induces a faithful grading by an elementary 2-group. The trivial component $C_e$ with respect to this grading is the subcategory $C_{int}$ generated by the simple objects of $C$ of integral dimension.
Two objects $X$ and $Y$ of a braided fusion category $\mathcal{C}$ (with braiding $c$) are said to centralize each other if $c_{Y,X}c_{X,Y} = \text{id}_{X\otimes Y}$. The centralizer $\mathcal{D}'$ of a subcategory $\mathcal{D} \subseteq \mathcal{C}$ is defined to be the full subcategory of objects of $\mathcal{C}$ that centralize every object of $\mathcal{D}$, that is

$$\mathcal{D}' = \{X \in \mathcal{C} | c_{Y,X}c_{X,Y} = \text{id}_{X\otimes Y}, \forall Y \in \mathcal{D}\}.$$  

The M"uger (or symmetric) center $Z_2(\mathcal{C})$ of $\mathcal{C}$ is the centralizer of $\mathcal{C}$, that is $Z_2(\mathcal{C}) = \mathcal{C}'$, which is a symmetric fusion subcategory of $\mathcal{C}$. A braided fusion category $\mathcal{C}$ is called symmetric when $Z_2(\mathcal{C}) = \mathcal{C}$. A premodular fusion category is called modular when $Z_2(\mathcal{C}) = \text{Vec}$.

Let $\mathcal{C}$ be a modular category with admissible modular data $(S, T)$. We define the Galois group $\text{Gal}(\mathcal{C})$ of $\mathcal{C}$ as $\text{Gal}(\mathcal{C}) = \text{Gal}(\mathbb{K}_C/\mathbb{Q}) = \text{Gal}(S)$, where $\mathbb{K}_C = \mathbb{Q}(\sqrt{S_{ij}}, | i, j \in I_C) = \mathbb{F}_S$ is the splitting field of the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$. Roughly speaking, the Galois group of the number field $\mathbb{K}_C$ is the abelian extension of $\mathbb{Q}$ obtained by adjoining all the entries of the matrix $S$. Notice that this group is isomorphic to an abelian subgroup of the symmetric group $S_n$, where $n$ is the rank of $\mathcal{C}$.

3 $\dim \mathcal{C} = 4m$, $m$ odd square-free

**Theorem 3.1.** Suppose that $\mathcal{C}$ is a modular category with $\dim \mathcal{C} = 4m$, with $m$ an odd square-free integer. Then either

(a) $\mathcal{C}$ contains an object of dimension $\sqrt{2}$ and $\mathcal{C} \cong \mathcal{I} \boxtimes \mathcal{P}_m$, with $\mathcal{I}$ an Ising modular category or

(b) $\mathcal{C}$ is non-integral and contains no objects of dimension $\sqrt{2}$ and $\mathcal{C} \cong TY(\mathbb{Z}_k, \chi, \nu)^{Z_2} \boxtimes \mathcal{P}_n$, where $1 \leq n = \frac{m}{k} \in \mathbb{Z}$ or

(c) $\mathcal{C}$ is pointed and equivalent to $\mathcal{P}_{4m}$ or $\mathcal{P}_{2} \boxtimes \mathcal{P}_{2m}$.

All equivalences are as balanced braided fusion categories.

**Proof.** Since spherical (balancing) structures are in 1-1 correspondence with invertible order 2 objects and may be chosen independently from the braiding on the category (see [2, Lemma 2.4]), we may assume we are given the unique spherical structure on $\mathcal{C}$ such that $\text{FPdim}(X) = \dim(X)$, for every object $X$.

First suppose that $\mathcal{C}$ is integral. Then every simple object must have dimension 1 or 2 (by [8, Lemma 1.2], [10, Proposition 8.27], [9, Proposition 2.11(i)]). If there are $s$ distinct isomorphism classes of simple objects of dimension 2, the equation $4m = \dim(\mathcal{C}) = \dim(C_{pt}) + 4s$ implies $\dim(C_{pt})$ is a multiple of 4. Therefore, with respect to the universal grading by $U(\mathcal{C})$, each component has odd dimension. In particular, the trivial component $\mathcal{C}_e = C_{ad}$ is pointed so that $\mathcal{C}$ is nilpotent. As such, by [8, Theorem 1.1], $\mathcal{C}$ decomposes as a Deligne product of braided fusion categories of prime-power dimension. This implies that $\mathcal{C}$ itself is pointed, since $m$ is square-free and any integral braided fusion category of dimension 4 is pointed. These categories are described in (c).

Therefore, we may assume that $\mathcal{C}$ contains a simple object $X_1$ with $\dim(X_1) \not\in \mathbb{Z}$. By considering the possible dimension of $\mathcal{C}_{pt}$, we see that $|U(\mathcal{C})| = 2^n a$ where $a \leq 2$ and $n | m$ is odd and square-free. If $a = 2$, by the above argument, we conclude that $\mathcal{C}$ is nilpotent and hence pointed, which can not happen because we assumed $\mathcal{C}$ not to be integral. By [12]
Theorem 3.10] $a \neq 0$, so we must have $a = 1$ and $U(C) \cong \mathbb{Z}_{2n}$. If $n = 1$, $\dim(C_{pt}) = 2$. Moreover, $C_{ad} = C_{int}$ and the only possible dimensions for the simple objects in it are 1 or 2. Then, $2m = \dim(C_{ad}) = 2 + 4b$, but this can not happen if $m > 1$. If $m = 1$, $C$ is an Ising modular category. Assume now that $n \neq 1$. Let $p \mid n$ be prime, and let $C_{pt}(p)$ be a pointed subcategory of dimension $p$. Observe that $C_{pt}(p)$ is either symmetric or modular since its Müger center is either all of $C_{pt}(p)$ or $Vec$. If $C_{pt}(p) \cong \text{Rep}(\mathbb{Z}_p)$ is symmetric then it is also Tannakian, since $p$ is odd [7, Corollary 2.50 (i)]. The corresponding $\mathbb{Z}_p$-de-equivariantization of $C$ is $\mathbb{Z}_p$-graded with trivial component $C^0_{\mathbb{Z}_p}$ must have dimension $\dim C/p^2$, contradicting $n$ square-free. Therefore $C_{pt}(p)$ is modular. Then, we have $C \cong C_{pt}(p) \boxtimes D$ for some modular subcategory $D \subset C$, by [13, Theorem 4.2], [7, Theorem 3.13]. By induction on the number of (distinct) primes factors of $n$, we have $C \cong D \boxtimes P_n$, where $D$ is modular with $\dim D = 4k$ with $\dim D_{pt} = 2$.

We now proceed to classify such $D$. First suppose that $m = n$. Then, $D$ is not pointed and $\dim D = 4$ so it is an Ising modular category [7, Appendix B]. In this case, $C$ corresponds to (a).

Now suppose that $m = nk$, where $1 < k \leq m$ (i.e. $1 \leq n < m$). Since $U(D) \cong \mathbb{Z}_2$, we have a $\mathbb{Z}_2$-grading with components $D_e = D_{ad}$ and $D_g$ where $g \in D$ is a (self-dual) invertible object of order 2. In this case, the universal grading coincides with the faithful grading described in [12, Theorem 3.10]. Then, any simple object $X_i$ in $D_g$ has $\dim(X_i) = \alpha_i \sqrt{\ell}$, with $\ell$ square-free and $\alpha_i \in \{1, 2\}$, since $D$ is modular and $k$ is square free. So, $D_e$ contains all of the objects of integral dimension, which consist of two 1-dimensional objects and $(k - 1)/2$ objects $Y_i$ of dimension 2. We claim that $\theta_g = 1$. To see this, first observe that if $Y$ is simple and $\dim(Y) = 2$ then, by dimension counting, $Y \otimes Y^* = 1 \oplus g \oplus Y'$, where $Y'$ is a 2-dimensional simple object. Thus, $N^g_{Y,Y'} = N^g_{Y,Y} = 1$ so that $Y \otimes g = Y$, for all 2-dimensional simple object $Y$. Next, note that the second row of the $S$-matrix must be the Galois conjugate of the first row since $S_{g,1} = 1$. That is, the second row is:

$$(1, 1, 2, \ldots, 2, -\alpha_1 \sqrt{\ell}, \ldots, -\alpha_\ell \sqrt{\ell}).$$

The balancing equation (2.2) gives:

$$2 = S_{g,Y} = \theta_g^{-1} \theta_{Y'}^{-1} N^g_{Y,Y} \dim(Y) \theta_Y = 2 \theta_g,$$

so that $\theta_g = 1$, as claimed.

Now, we use the balancing equation (2.2) again:

$$-\alpha_i \sqrt{\ell} = S_{g,X_i} = \theta_g \theta_{X_i}^{-1} \theta_{g \otimes X_i} \dim(g \otimes X_i) = \alpha_i \sqrt{\ell} \theta_{X_i}^{-1} \theta_{g \otimes X_i},$$

which implies that $\theta_{X_i} = -\theta_{g \otimes X_i}$. In particular, $g \otimes X_i \neq X_i$ and $g \otimes X_i \neq X_i^*$, since $\theta_{X_i} = 1$. Then $X_i$ is self-dual, for all $i$. We claim that $\text{rank}(D_g) = 2$. First, notice that $\text{rank}(D_g) \neq 1$ because $g$ does not fix any $X_i$. Let $X_1$ and $X_i$ be simple objects of dimension $\sqrt{\ell}$. Since $\dim(X_1 \otimes X_i) = \ell$ is odd, either $1$ or $g$ (and not both) is a subobject of $X_1 \otimes X_i$. Since all $X_i$ are self-dual, $N^1_{X_1,X_i} = 1$ implies $X_i = X_1$. If $N^g_{X_1,X_i} = 1$ then $X_i = g \otimes X_1$. Therefore, $X_1$ and $g \otimes X_1 \neq X_1$ represent the only two non-isomorphic simple objects in $D_g$. Thus, we have seen that $\ell = k$, and $D$ has two 1-dimensional objects $1, g; (k - 1)/2$ simple 2-dimensional objects $Y_1, \ldots Y_{k-1}$ and
two $\sqrt{k}$-dimensional objects $X_1$ and $X_2$. Moreover, since $\theta_g = 1$, the subcategory generated by $g$ is Tannakian. This induces an action of $\mathbb{Z}_2$ on $D$ by interchanging 1 and $g$, fixing all $Y_i$ and interchanging $X_1$ and $X_2$. Then, the $\mathbb{Z}_2$-de-equivariantization $D_{\mathbb{Z}_2}$ has $k$ objects of dimension 1 (one from $1 \oplus g$ and two from each $Y_i$) and one object of dimension $\sqrt{k}$ (from $X_1 \oplus X_2$). As a fusion category $D_{\mathbb{Z}_2}$ must be $TY(\mathbb{Z}_k, \chi, \nu)^{\mathbb{Z}_2}$. As these categories are described in (b), the proof is complete. □

**Remark 3.2.** An alternative description of the categories obtained in Theorem 3.1 (b) is as follows. The non-pointed factor $D := TY(\mathbb{Z}_k, \chi, \nu)^{\mathbb{Z}_2}$ can be recovered from the formula

$$D \boxtimes \mathcal{P}_k \cong \mathcal{Z}(TY(\mathbb{Z}_k, \chi, \nu)),$$

obtained from [4, Cor. 3.30]. Here $\mathcal{P}_n$ is the maximal pointed modular subcategory of the Drinfeld center of $TY(\mathbb{Z}_k, \chi, \nu)$. We see that there are a total of 8 possible categories up to balanced braided tensor equivalences: 4 for the choices of $\chi$ and $\nu$ (two each) and then an overall two choices of spherical structure.

The following result will be useful later:

**Lemma 3.3.** If $\mathcal{C}$ is a weakly integral modular category of rank $r \geq 3$ in which there is a unique simple isomorphism class of objects $X$ such that $\text{FPdim } X \notin \mathbb{Z}$ then $\mathcal{C}$ is equivalent to an Ising modular category.

**Proof.** Let $\mathcal{C}$ be such a category. Since $\text{FPdim } X = \sqrt{m}$, the Frobenius-Perron dimension of each component $\mathcal{C}_i$ in any faithful grading of $\mathcal{C}$ should be at least $m$. It follows from [12, Theorem 3.10] that the category $\mathcal{C}$ has a faithful $\mathbb{Z}_2$-grading, with $\mathcal{C}_0$ integral and the unique simple object in the non-trivial component $\mathcal{C}_1$ being $X$. Then, $FPdim(\mathcal{C}) = 2m$. Therefore, the universal grading group $U(\mathcal{C})$ must be of order 2. In particular, $G(\mathcal{C}) \simeq \mathbb{Z}_2$ [12, Theorem 6.2].

Clearly, both $X$ and the non-trivial invertible object $g$ are self-dual. Now, we look at the $S$-matrix associated to the modular data of $\mathcal{C}$. We consider the canonical positive spherical structure on $\mathcal{C}$, with respect to which categorical dimensions of simple objects coincide with their Frobenius-Perron dimensions [18, Proposition 8.23]. Then, the first column is given by the Frobenius-Perron dimensions of the simple objects of $\mathcal{C}$. We order the entries of the $S$-matrix in the following way: the index 0 corresponds to the unit object, the index 1 corresponds to $g$, the last index to the non-integral object $X$, and the middle ones to the non-invertible integral simple objects.

By [2, Lemma 4.9], the Galois automorphism $\sigma$ that sends $\sqrt{m}$ to $-\sqrt{m}$ interchanges the first two columns of the $S$-matrix. Since the matrix is symmetric we already know that the first two entries of the last column are $\pm \sqrt{m}$. Since the last column corresponds to the self-dual object $X$, all its entries are real numbers. But since the norm of each column is equal to $2m$, all the other entries of the last column must be equal to zero. Thus the $S$-matrix of $\mathcal{C}$ has
Applying the twist equation (2.1) to the entry $S_{X,X} = 0$, we get that $0 = \theta_X^2 \sum_i \theta_i S_{i,X}^2 = (m\theta_1 + m\theta_g)\theta_X^2$. We conclude that $\theta_g = -\theta_1 = -1$. Next, we apply the twist equation (2.1) to the entry $S_{g,X} = -\sqrt{m}$ obtaining:

$$p_+(-\sqrt{m}) = \theta_g \theta_X^2 \sum_i \theta_i S_{i,g} S_{i,X} = -\theta_X^2 \sqrt{m}.$$ 

It follows that $|p_+| = 2|\theta_X| = 2$ and $D^2 = 4$. Then, $\mathcal{C}$ is equivalent to an Ising modular category.

**Lemma 3.4.** Let $\mathcal{C}$ be a modular category of square-free (integral) Frobenius-Perron dimension. Then $\mathcal{C}$ is pointed.

**Proof.** The only possible integral FP-dimension of a simple object is 1, since FPdim($\mathcal{C}$) is square-free. Therefore, the integral subcategory $\mathcal{C}_{ad}$ is pointed, hence $\mathcal{C}$ is nilpotent. Thus, by [7, Theorem 1.1], $\mathcal{C}$ is a Deligne product of braided subcategories of prime dimension. Such categories are pointed, by [10, Corollary 8.30]. Then, $\mathcal{C}$ is pointed. \qed

### 4 Technical results

#### 4.1 G-grading of a modular category

Let $G = G(\mathcal{C})$ be the group of isomorphism classes of invertible objects of a modular category $\mathcal{C}$. Let $\hat{G}$ denote the character group of $G$. The modular category $\mathcal{C}$ admits a faithful $\hat{G}$-grading ([12, Theorem 6.2]) which is given by $\mathcal{C} = \bigoplus_{\chi \in \hat{G}} \mathcal{C}_\chi$, where the set of simple objects in $\mathcal{C}_\chi$ is given by:

$$\text{Irr}\mathcal{C}_\chi = \left\{ V_j \in \text{Irr}\mathcal{C} \mid \frac{S_{ij}}{d_i d_j} = \chi(i), \text{ for all } i \in G \right\}.$$ 

This natural $\hat{G}$-grading on $\mathcal{C}$ induces a canonical $\hat{H}$-grading on $\mathcal{C}$, for any subgroup $H$ of $G$. More precisely, for $\chi \in \hat{H}$, the set of simple objects in each component of the grading is:

$$\text{Irr}\mathcal{C}_\chi = \left\{ V_j \in \text{Irr}\mathcal{C} \mid \frac{S_{ij}}{d_i d_j} = \chi(i), \text{ for all } i \in H \right\}.$$ 

Since the restriction map $\hat{G} \to \hat{H}$ is surjective, this $H$-grading is also faithful and $\dim\mathcal{C}_\chi = \frac{\dim\mathcal{C}}{|H|}$, for all $\chi \in \hat{H}$. See [10, Proposition 8.20].

The group $G$ also acts on $\Pi_\mathcal{C}$ by tensor product. For $g \in G$ and $j \in \Pi_\mathcal{C}$, $g \cdot j \in \Pi_\mathcal{C}$ is defined by $V_{g \cdot j} \cong V_{g}^* \otimes V_j$. We are particular interested in the subgroups $H$ of $G$ which generates a symmetric full subcategory of $\mathcal{C}$. In this case, we will simply call $H$ a self-centralizing subgroup of $G$. 

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Remark 4.1. If $H$ is a self-centralizing subgroup of $G$ then $S_{g,h} = d_g d_h = 1$, for all $g, h \in H$. Therefore, $H \subseteq C_e$, where $C_e$ is the trivial component of the $\hat{H}$-grading of $C$ associated to the trivial character. If $H$ generates a Tannakian subcategory of $C$ then $H$ is a self-centralizing subgroup and $d_h = 1 = \theta_h$, for all $h \in H$.

Lemma 4.2. Let $H$ be a self-centralizing subgroup of $G$. Then $H \cdot j \subseteq \text{Irr}(C_\chi)$, for all $j \in \text{Irr}(C_\chi)$, for any $\chi \in \hat{H}$. If, in addition, $H$ generates a Tannakian subcategory of $C$ and $\chi$ is not trivial then $H \cdot j$ is not a singleton. In particular, $|H|$ divides $|\text{Irr}(C_\chi)|$ when $H$ has prime order and $\chi$ is not trivial.

Proof. Since $H \subseteq C_e$ and the action of $H$ is induced by the tensor product, it follows immediately from the definition of grading that $\hat{H} \cdot \text{Irr}(C_\chi) \subseteq \text{Irr}(C_\chi)$.

Now, we assume that $H$ generates a Tannakian subcategory of $C$. Thus $\theta_h = 1$, for all $h \in H$. Suppose that $j \in \text{Irr}(C_\chi)$ is fixed by $H$. Then $N_{h,j}^k = \delta_{k,j}$, $\forall h \in H$, $\forall k \in \Pi_C$. Using the balancing equation (2.2), we get that
\[
\chi(h) = \frac{S_{h,j}}{d_j} = \frac{\sum_k N_{h,j}^k d_k \theta_k}{\theta_h \theta_j d_j} = 1,
\]
since $d_h = 1$, $\forall h \in H$. Therefore, $\chi$ is trivial.

Notice that if $H$ is a group of prime order $p$ then $H$ generates a Tannakian subcategory of $C$, by [7 Corollary 2.50]. In addition, $|H| = p$ implies that each $H$-orbit in $\text{Irr}(C_\chi)$ has exactly $p$ simple objects. Therefore, the last statement follows.

Lemma 4.3. Suppose $H$ is a self-centralizing subgroup of $G$. If $\text{Irr}(C_\psi)$ consists of only one $H$-orbit, for some $\psi \in \hat{H}$, then $C_e$ is integral. If, in addition, $k \in \text{Irr}(C_\chi)$ is fixed by $H$ then $\frac{|H|}{|H_0|} \mid d_k$, where $H_0$ is the stabilizer of the $H$-orbit $\text{Irr}(C_\psi)$.

Proof. Since $H$ is self-centralizing then $H \subseteq C_e$, see Remark 4.1. Let $R = \sum_{i \in \text{Irr}_C} d_i V_i$ be the virtual regular object of $C$. Then $R = \sum_{\chi \in \hat{H}} R_\chi$, where $R_\chi$ is the regular object of the component $C_\chi$.

Suppose $\text{Irr}(C_\psi) = H \cdot j$, with $j \in \text{Irr}(C_\psi)$. Let $H_0$ be the stabilizer of the $H$-orbit $\text{Irr}(C_\psi)$. Note that the quotient group $H/H_0$ acts on $j$ as $h \cdot j = h \cdot j$, where $h \in H/H_0$. Thus $R_\psi = d_j \sum_{\pi \in H/H_0} V_{\pi,j}$, since $d_h = 1$ implies $d_{\pi,j} = d_j$. For $V_k \in \text{Irr}(C_e)$, $V_k \otimes R = d_k R$ and so $V_k \otimes R_\chi = d_k R_\chi$, for all $\chi \in \hat{H}$. On the other hand,
\[
V_k \otimes R_\psi = d_j \sum_{\pi \in H/H_0} V_{\pi,j} \otimes V_k = d_j \sum_{h,h' \in H/H_0} n_{h^{-1}h'} V_{h^{-1}h',j} \otimes V_{h^{-1}h',j} = d_j \sum_{h,h' \in H/H_0} n_{h^{-1}h'} V_{h^{-1}h',j} = n R_\psi, \quad (4.1)
\]
where $n = \sum_{\pi \in H/H_0} n_{\pi}$ is a non-negative integer. Therefore, $d_k = n \in \mathbb{Z}_+$ and $C_e$ is integral, as we stated.
If $k$ is fixed by $H$ then
\[
d_k R_\psi = V_k \otimes R_\psi = d_{j} \sum_{\overline{h} \in H/H_0} V^*_h \otimes V_j \otimes V_k = d_{j} \frac{|H|}{|H_0|} V_j \otimes V_k = d_{j} \frac{|H|}{|H_0|} \sum_{\overline{h} \in H/H_0} n_{\overline{h}} V^*_{\overline{h}} \otimes V_j \otimes V_k.
\]
Thus, $\frac{|H|}{|H_0|} n_{\overline{h}} = d_k$, for all $\overline{h} \in H/H_0$. In particular, $\frac{|H|}{|H_0|} | d_k$ and $n_{\overline{h}} = n_{\overline{h}'}$, $\forall h, h' \in H/H_0$. □

4.2 Support cycles Let $n$ be a positive integer. We define $v_p(n) = a$ if $p^a \parallel n$, and when $p$ is odd $k_p(n) = \frac{\varphi(p^n)}{2}$, where $\varphi$ is the Euler’s totient or phi function.

Suppose $p$ is a prime factor of $N = \text{FSexp}(C)$ of a modular category $C$ with $a = v_p(N)$. Let $N = p^a m$ and $\sigma \in \text{Gal}(Q_{ab})$, where $Q_{ab}$ is the abelian closure of $Q$ in $C$, such that $\sigma$ fixes $Q_{4q}$ and $\sigma|_{Q_{4q}}$ generates a maximal cyclic subgroup of $\text{Gal}(Q_{p^a}/Q)$. We call $\sigma$ a $p$-automorphism of $C$. Suppose $(s, t)$ is a normalized data of $C$ whose associated modular representation is of level $n$. By [3], $n$ can be chosen to be $N$ if $4 \nmid N$, and $2N$ otherwise. We simply call this type of normalized data of $C$ minimal.

A cycle $C$ in the disjoint cycle decomposition of $\hat{\sigma}$ is called a $p^i$-support cycle of $\sigma$ if $v_p(\text{ord}(t_j)) = \ell$ for some $j \in C$. If $a = v_p(n)$, a $p^a$-support cycle of $\sigma$ is called a maximal power support cycle of $\sigma$.

If $(s', t')$ is another normalized data of $C$, then $(s', t') = (sx^{-3}, tx)$ for some 12-th root of unity $x$. Therefore, all prime power support cycles, except 2, 4 are 3, independent of the choice of the normalized pair $(s, t)$.

**Lemma 4.4.** Suppose $C$ is a modular category, $p$ is an odd prime factor of $N = \text{FSexp}(C)$, and $\sigma$ is a $p$-automorphism of $C$. Let $(s, t)$ be a minimal normalized modular data of level $n = p^a q$. Then $\sigma$ admits a maximal power support cycle. For any $p^i$-support cycle $C$ of $\sigma$ with $1 \leq i \leq a$, we have
\[
\frac{\varphi(p^i)}{2} | \text{ord}(C) | 2k_p(N).
\]
Hence, $\frac{k_p(N)}{2} | \text{ord}(\hat{\sigma}) | 2k_p(N)$.

If 1 is in a $p^i$-support cycle $C$ of $\sigma$ and $t_1 = x \zeta$ for some primitive $p^i$-th root unity $\zeta$ and $x^q = 1$, $t_{\sigma^j(1)} = x \sigma^{2j}(\zeta)$ for all $j$. In particular, $v_p(\text{ord}(t_j)) = i$ for all $j \in C$.

**Proof.** Let $k = k_p(N)$. Since $\text{ord}(\sigma|_{Q_{p^a}}) = 2k$, $\text{ord}(\hat{\sigma}) | 2k$ and the length of each disjoint cycle of $\hat{\sigma}$ is a divisor of $2k$.

Since $\text{ord}(t) = p^a q$, there exists a simple object, say $V_{j_0}$ such that $\text{ord}(t_{j_0})$ is a multiple of $p^a$. Therefore, the disjoint cycle of $\hat{\sigma}$ containing $j_0$ is a $p^a$-support cycle of $\sigma$.

Suppose $C$ is a $p^i$-support cycle of $\sigma$, for some $i \leq a$, with $1 \in C$ and $t_1 = x \zeta$ for some $q$-th root of unity $x$ and a primitive $q^i$-th root unity $\zeta$. Let $k' = \varphi(p^i)/2$. Note that
\[
t_1, \sigma^2(t_1), \ldots, \sigma^{2(k'-1)}(t_1)
\]
are distinct, and $\sigma^{2k'}(t_1) = t_1$ By Galois symmetry, they are eigenvalues of $t$ and
\[
t_{\sigma^j(1)} = \sigma^{2j}(t_1) = x \sigma^{2j}(\zeta) \text{ for all } j.
\]
In particular, \(1, \sigma(1), \ldots, \sigma^{k'-1}(1)\) are distinct, and \(v_p(\text{ord}(t_{\sigma(1)})) = i\). Therefore, the length of the cycle \(C\) is a multiple \(k'\) and so

\[ k' \mid \text{ord}(C) \mid \text{ord}(\sigma) \mid 2k. \]

For \(i = a\), we find \(k \mid \text{ord}(C) \mid \text{ord}(\sigma) \mid 2k. \)

If the anomaly \(\alpha = \frac{p_+}{p_-}\) of \(C\) is such that \(3 \nmid \text{ord}(\alpha)\), one can choose a 6-th root \(\lambda\) of \(\alpha\) such that \(\frac{p_+}{\alpha} = D = \sqrt{\dim C}\) and \(3 \nmid \text{ord}(\lambda)\). The normalization \((s, t) = (\frac{S}{D}, \frac{T}{D})\) not be minimal but the 3-support cycle of a 3-automorphism of \(C\) will be independent of these normalization. Moreover, the level \(n\) of \((s, t)\) will satisfies \(\text{FSexp}(C) \mid n \mid 4 \text{FSexp}(C)\).

We are particularly interested in weakly integral modular categories as their anomaly \(\alpha = \frac{p_+}{p_-}\) can only be an 8-th root of unity. For any weakly integral modular category \(C\), we only consider the normalized modular data \((s, t) = (\frac{S}{D}, \frac{T}{D})\) with \(\lambda = \sqrt[4]{\alpha}\) where \(\sqrt[4]{D} = p_+\). Since \(\alpha \in \mathbb{Q}N\), the level of \((s, t)\) is a divisor of \(4N\). In particular, there is not any ambiguity for a 3-support cycle for a 3-automorphism of \(C\). We have a more refined statement for \(p^i\)-support cycles for a weakly integral modular category.

**Lemma 4.5.** Suppose \(\sigma\) is a \(p\)-automorphism of a weakly integral modular category \(C\) with \(p\) an odd prime, and \(\text{FSexp}(C) = p^d q\) for some integer \(q\) relative prime to \(p\). Then, integer \(i\) with \(1 \leq i \leq a\), the following statements are equivalent:

(i) \(C\) is a \(p^i\)-support cycle of \(\sigma\);

(ii) \(v_p(\text{ord}(\theta_j)) = p^i\) for some \(j \in C\);

(iii) \(v_p(\text{ord}(\theta_j)) = p^i\) for all \(j \in C\).

In particular, \(0 \notin C\). If \(C = (1, 2, \ldots, l)\) and \(\theta_1 = x\zeta\) for some primitive \(p^i\)-root of unity \(\zeta\) and \(x^4 = 1\), then \(\theta_j = x^{\sigma^{2j}}(\zeta)\).

**Proof.** Let \(t = \sqrt[p]{\sigma^5}T\) where \(\sqrt[p]{\sigma} = p_+/D\) is 16-th root of unity. Then \(v_p(\text{ord}(t_j)) = v_p(\text{ord}(\theta_j))\) for all \(j\), and so the equivalence of the three conditions follows. Note that \(t_0\) is a \(6\)-th root but \(p \mid \text{ord}(t_j)\) for any \(j\) in a \(p^i\)-support cycle \(C\) of \(\sigma\) \((i > 0)\). Therefore, \(0 \notin C\). If \(C = (1, 2, \ldots, l)\) and \(\theta_1 = x\zeta\) for some primitive \(p^i\)-root of unity \(\zeta\) and \(x^4 = 1\), then \(t_1 = \sqrt[p]{\sigma^5} x\zeta\) and so \(t_j = \sqrt[p]{\sigma^5} x^{\sigma^{2j}}(\zeta)\). Thus, \(\theta_j = x^{\sigma^{2j}}(\zeta)\).

**Theorem 4.6.** Let \(C\) be a weakly integral modular category, \(p\) an odd prime factor of \(N = \text{FSexp}(C)\) and \(\sigma\) a \(p\)-automorphism of \(C\). Suppose \(pq = N\) for some integer \(q\) relatively prime to \(p\), and \(\sigma\) has only one \(p\)-support cycle \(C_1 = (1, \ldots, l)\). Then:

(i) \(\dim C = \left(\frac{l}{p-1}\right)^2 d_1^4 p\), \(d_1 \in \mathcal{O}_q\), and \(\theta_1 = x\zeta\) for primitive \(p\)-th root of unity \(\zeta\) and \(x^{16} = 1\). In particular, if \(l = \frac{p-1}{2}\), then \(2 \mid d_1^2\).

(ii) If \(C\) is integral and \(\text{Gal}(C)\) is generated by \(\sigma\), then \(\sigma\) is cycle of length \(p - 1\) and \(C\) is pointed of rank \(p\).

(iii) If \(C\) is a strictly weakly integral and \(\text{Gal}(C)\) is generated by \(\sigma\), then \(\sigma = (0, 1)(2, \ldots, \frac{p+1}{2})\), up to relabeling of the simple objects, and \(C\) is a prime modular category of \(\dim C = 4p\). In particular, and of rank \(\frac{p+1}{2}\).
Proof. Let \( \theta_1 = x \zeta \) for some primitive \( p \)-root of unity \( \zeta \) and \( x^q = 1 \), then \( \theta_j = x \sigma^{2j}(\zeta) \) for \( j = 1, \ldots, l \). By Lemma 4.5, \( l = p - 1 \) or \( \frac{p-1}{2} \). Suppose \( \hat{\sigma} = C_0 C_1 C_2 \ldots C_m \) where \( C_0 \) is the cycle containing 0. Note that \( d_j = d_{C_i} \) for all \( j \in C_i \). In particular, and \( d_{C_1} = d_1 \). Since \( C_1 \) is the only \( p \)-support cycle of \( \sigma \), we also have \( \theta_j = \theta_{C_i} \in \mathcal{O}_q \) for all \( j \in C_i \) if \( i \neq 1 \). Let us denote \( l_i = \text{ord}(C_i) \) and \( C_i = (c_i, \hat{\sigma}(c_i), \sigma^2(c_i), \ldots) \). Now we consider the twist equation for any \( j \):

\[
p_{+} d_{j} \bar{\theta}_{j} = d_1 \sum_{r=1}^{l} S_{jr} \theta_{r} + \sum_{i \neq 1} d_{C_i} \theta_{C_i} \sum_{r \in C_i} S_{jr} = d_{1} \sum_{r=0}^{l-1} \frac{S_{j,1+r}}{d_1} \sigma^{2r}(\zeta) + \sum_{i \neq 1} d_{C_i} \theta_{C_i} \sum_{r \in C_i} \frac{S_{jr}}{d_r}.
\]

For any \( i \), we denote

\[
S_{j,C_i} = \sum_{r \in C_i} \frac{S_{jr}}{d_r} = \sum_{r=0}^{l-1} \sigma^{r-1} \left( \frac{S_{j,C_i}}{d_{C_i}} \right) \in \mathcal{O}_q,
\]

and so \( \sum_{i \neq 1} d_{C_i} \theta_{C_i} S_{j, C_i} \in \mathcal{O}_q \). Therefore, for \( j = 0 \), we have

\[
\sqrt{\alpha}D = p_{+} = d_{1} \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + \sum_{i \neq 1} d_{C_i} \theta_{C_i} l_i
\]

\[
= \frac{2 \sqrt{d_{1}} x}{p - 1} \left(-\frac{-1 \mp \sqrt{p}}{2}\right) + \sum_{i \neq 1} d_{C_i} \theta_{C_i} l_i
\]

\[
= \pm \frac{\sqrt{d_{1}} x}{p - 1} \sqrt{p} + \frac{- \sqrt{d_{1}} x}{p - 1} + \sum_{i \neq 1} d_{C_i} \theta_{C_i} l_i
\]

where \( \varepsilon = \sqrt{\left(\frac{-1}{p}\right)} \). Since \( \left\{1, \sqrt{p}\right\} \) is linearly independent over \( \mathbb{Q}_q \), it follows from (4.6) that

\[
\sqrt{\alpha}D = \pm \frac{\sqrt{d_{1}} x l}{p - 1} \sqrt{p} \quad \text{and} \quad \frac{\sqrt{d_{1}} x}{p - 1} = \sum_{i \neq 1} d_{C_i} \theta_{C_i} l_i.
\]

This implies \( \alpha = x^2 \left(\frac{-1}{p}\right), \quad \dim C = \left(\frac{1}{p-1}\right)^2 d_1^1 p, \quad \text{and} \quad \sigma(p_{+}) = -p_{+} \). Therefore,

\[
2p_{+} = d_{1} \sum_{r=0}^{l-1} \sigma^{2j}(\zeta) - \sigma^{2j+1}(\zeta) = 0.
\]

Let \( \mathcal{I} = \{i \mid d_{C_i} \in \mathbb{Z}\} \setminus \{1\} \). Suppose \( j \in C_i \) for some \( i \in \mathcal{I} \). By (4.3), we have

\[
p_{+} d_{j} \bar{\theta}_{j} = d_{1} \sum_{r=0}^{l-1} S_{j,r+1} \sigma^{2r}(\zeta) + \sum_{i \neq 1} d_{C_i} \theta_{C_i} S_{j, C_i}.
\]

Therefore,

\[
2p_{+} d_{j} \bar{\theta}_{j} = d_{1} \sum_{r=0}^{l-1} \sigma^{r} \left(\frac{S_{j,1}}{d_1}\right) \sigma^{2r}(\zeta) - \sigma^{r+1} \left(\frac{S_{j,1}}{d_1}\right) \sigma^{2r+1}(\zeta),
\]
and so

\begin{equation}
2l_i p_+ d_{C_i} \theta_{C_i} = d_{l_i} d_{C_i} x \sum_{r=0}^{l-1} \left( \sigma^r \left( \frac{S_{1,C_i}}{d_1} \right) \sigma^{2r}(\zeta) - \sigma^{r+1} \left( \frac{S_{1,C_i}}{d_1} \right) \sigma^{2r+1}(\zeta) \right) \tag{4.10}
\end{equation}

\begin{equation}
= d_{l_i} d_{C_i} x S_{1,C_i} \sum_{r=0}^{l-1} \left( \left( \frac{\sigma(d_1)}{d_1} \right)^r \sigma^{2r}(\zeta) - \left( \frac{\sigma(d_1)}{d_1} \right)^{r+1} \sigma^{2r+1}(\zeta) \right). \tag{4.11}
\end{equation}

Suppose \( d_1 \not\in \mathcal{O}_q \), then \( \sigma(d_1) = -d_1 \) as \( d_1 \) is square root of an integer. Since \( \sigma^2(d_i) = d_i \) for all \( i \), \( C_0 = (0,0) \) and so

\[ S_{1,C_0} = d_1 + S_{1,0} = d_1 - d_1 = 0. \]

By (4.11), this implies \( 2p_+ = 0 \), a contradiction. Therefore, \( d_1 \in \mathcal{O}_q \). This completes the proof of (i).

We now assume \( \text{Gal}(\mathcal{C}) = \langle \sigma \rangle \). Then, \( d_1, S_{C_i} \in \mathbb{Z} \) for all \( i \), and \( d^2 \theta_{C_i} S_{C_i} \in \mathcal{O}_q \) for \( i \neq 1 \).

Next, we show that \( \theta_{C_i} = 1 \) for \( i \in \mathcal{I} \). Now, (4.11) becomes

\begin{equation}
2l_i p_+ d_{C_i} \theta_{C_i} = d_{l_i} d_{C_i} x S_{1,C_i} \sum_{r=0}^{l-1} \left( \sigma^{2r}(\zeta) - \sigma^{2r+1}(\zeta) \right). \tag{4.12}
\end{equation}

By (4.8), we find \( S_{1,C_i} = l_i d_1 \theta_{C_i} \). Hence \( \theta_{C_i} = \pm 1 \) and \( S_{1,j} = d_1 d_j \theta_j \in \mathbb{Z} \) for \( j \in C_i \). Thus, \( S_{r,j} = S_{1,j} \) for all \( j \in C_i \) and \( 1 \leq r \leq l \).

Now, for \( a, b \in \mathcal{I}, j' \in C_a \) and \( j \in C_b \), we have the twist equation:

\[ p_+ S_{j,j'} = \theta_j \theta_{j'} \left( x \sum_{r=0}^{l-1} S_{j,1+r} S_{j',1+r} \sigma^{2r}(\zeta) + \sum_{i \neq 1} \sum_{r \in C_i} S_{j,r} S_{j',r} \theta_{C_i} \right). \]

Therefore,

\[
\sum_{j \in C_b} \sum_{j' \in C_a} p_+ S_{j,j'} = d_{C_a} l_b p_+ S_{c_b,C_a}
\]

\[
= \theta_{C_b} \theta_{C_a} \left( xd_{C_a} l_b \sum_{r=0}^{l-1} S_{1+r,C_b} S_{1+r,C_a} \sigma^{2r}(\zeta) + d_{C_a} d_{C_b} \sum_{i \neq 1} \sum_{r \in C_i} S_{r,C_b} S_{r,C_a} \theta_{C_i} \right)
\]

\[
= \theta_{C_b} \theta_{C_a} \left( xd_{C_a} d_{C_b} S_{1,C_b} S_{1,C_a} \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + d_{C_a} d_{C_b} \sum_{i \neq 1} l_i S_{c_i,C_b} S_{c_i,C_a} \theta_{C_i} \right)
\]

\[
= xd_{C_a} d_{C_b} l_b l_d l_i \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + \theta_{C_b} \theta_{C_a} d_{C_a} d_{C_b} \sum_{i \neq 1} l_i S_{c_i,C_b} S_{c_i,C_a} \theta_{C_i}
\]

Since \( S_{c_b,C_a} \in \mathbb{Z} \), we find

\[ 2d_{C_a} l_b p_+ S_{c_b,C_a} = xd_{C_a} d_{C_b} l_b l_d \sum_{j=0}^{l-1} \left( \sigma^{2j}(\zeta) - \sigma^{2j+1}(\zeta) \right) = 2p_+ d_{C_a} d_{C_b} l_a l_b \]
and hence $S_{cb,c_a} = l_a d_{cb}$. It follows from \cite{19} that

\[
\begin{align*}
l_{bp} d_{cb} \theta_{cb} &= d_1 d_{cb} x \sum_{r=0}^{l-1} S_{r,cb} \sigma^{2r}(\zeta) + \sum_{j \in cb \ i \neq 1} d_i^2 \theta_{ci} S_{j,ci} \\
&= d_1 d_{cb} x S_{1,cb} \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + l_b d_{cb} \sum_{i \neq 1} d_i^2 \theta_{ci} l_i \\
&= l_b d_i^2 d_{cb} \theta_{cb} x \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + l_b d_{cb} \sum_{i \neq 1} d_i^2 \theta_{ci} l_i.
\end{align*}
\]

Therefore,

\[
p_+ = d_i^2 x \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + \theta_{cb} \sum_{i \neq 1} d_i^2 \theta_{ci} l_i = d_i^2 x \sum_{r=0}^{l-1} \sigma^{2r}(\zeta) + \sum_{i \neq 1} d_i^2 \theta_{ci} l_i.
\]

This forces $\theta_{cb} = 1$ for all $b \in I$.

Let $I$ be the set of all simple objects of $C$ with integral dimension, $\tilde{I} = \{V_j \mid j \in C_i \text{ for some } i \in I\}$. In particular, $I = \{V_1, \ldots, V_l\} \cup \tilde{I}$. Our preceding paragraphs have shown $S_{V_j, V_r} = d_j d_r$ if $V_j \in \tilde{I}$ and $V_r \in I$. Let $\langle I \rangle$ be the tensor subcategory generated by $I$. Then the centralizer $V_j$ is the centralizer $C_C(\langle I \rangle)$ for all $V_j \in \tilde{I}$.

If $C$ is integral, then $\langle I \rangle = C$ and so $\tilde{I} = \{V_0\}$ by the modularity of $C$. Therefore, $I = \{V_0, \ldots, V_l\}$ and we have

\[
\left(\frac{l}{p-1}\right)^2 d_1^4 p = 1 + ld_1^2.
\]

The equation has no integral solution for $d_1$ if $l = \frac{p-1}{2}$. For $l = p - 1$, $d_1 = 1$ is the only integer solution. Thus, $C$ pointed of rank $p$, and the proof of (ii) is completed.

Now, we assume $C$ is strictly weakly integral. Since $d_j$ is a square root of a positive integer, say $C_0 = (0, l + 1)$. Moreover, $\dim \langle I \rangle = \dim C / 2$. By \cite{13}, $\dim C_C(\langle I \rangle) = 2$. Since $V_0, V_{i+1} \in C_C(\langle I \rangle)$ and $d_0^2 + d_{i+1}^2 = 2$, $\tilde{I} = \{V_0, V_{i+1}\}$, and so $I = \{V_0, \ldots, V_{i+1}\}$. Now, we have

\[
\left(\frac{l}{p-1}\right)^2 d_1^2 p = \frac{\dim C}{2} = 2 + ld_1^2.
\]

The equation has no integral solution for $l = p - 1$ $\langle I \rangle = C$ and so $\tilde{I} = \{V_0\}$ by the modularity of $C$. Therefore, $I = \{V_0, \ldots, V_l\}$ and we have

\[
\left(\frac{l}{p-1}\right)^2 d_1^4 p = 1 + ld_1^2.
\]

The equation has no integral solution for $d_1$ if $l = p - 1$. For $l = \frac{p-1}{2}$, $d_1 = 2$ is the only integer solution. Thus, $\dim C = 4p$. It follows from Theorem \cite{3.1} that $C$ is one of the four prime modular categories of dimension $4p$. This completes the proof of (iii). \hfill $$

**Lemma 4.7.** Suppose $C$ is a weakly modular category, $v_p(\text{FSexp}(C)) = 1$ for some odd prime $p$, and $\sigma$ is a $p$-automorphism. If $\sigma$ has exactly two $p$-support cycles and they both have length $k = \frac{p-1}{2}$, say $C_1 = (1, \ldots, k)$ and $C_2 = (k + 1, \ldots, 2k)$ then either
(i) $v_p(\dim C)$ is even and $d_1 = \cdots = d_{p-1}$, or

(ii) $v_p(\dim C)$ is odd and $\dim C = \frac{1}{4}(d_1^4 + d_2^4 + \epsilon d_1^2 d_2^2)p$ for some $\epsilon \in \{0, 1, -1\}$.

Proof. Suppose $\hat{\sigma} = (1, \ldots, k)(k+1, \ldots, 2k) \prod_i Z_i$ for some disjoint cycles $Z_i$. Since all for $Z_i$ are not support cycles of $\sigma$, $\sigma(\theta_j) = \theta_j$ for any $j$ in any of these cycles $Z_i$. Now we assume, $\theta_1 = x\zeta$ and $\theta_2 = y\zeta^k$ for some $q$-th roots of unity $x, y$, primitive $p$-th root of unity $\zeta$, and positive integer $\ell < p$. Let $u = \sum_{i=1}^{k-1} \sigma^2i(\zeta) = \frac{1-\epsilon\sqrt{y}}{2}, u' = \sum_{i=1}^{k-1} \sigma^2i(\zeta') = \frac{-1\pm(\frac{y}{x})\sqrt{y}}{2}$ where $\epsilon = \sqrt{\left(\frac{1}{p}\right)}$. Then

$$\sqrt{\alpha D} = p_+ = 1 + d_1^2 x u + d_2^2 y u' + \sum_{j>2k} d_j^2 \theta_j,$$

$$= \pm \frac{1}{2}(d_1^2 x + \left(\frac{\ell}{p}\right) d_2^2 y)\sqrt{p} - \frac{1}{2}(d_1^2 x + d_2^2 y) + 1 + \sum_{j>2k} d_j^2 \theta_j.$$

Note that $1 + \sum_{j>2k} d_j^2 \theta_j \in \mathbb{Q}_4$. If $v_p(\dim C)$ is even, then $p_+ \in \mathbb{Q}_4$, the $\mathbb{Q}_4$-linear independence of $\{1, \sqrt{p}\}$ implies $d_2^2 x + \left(\frac{\ell}{p}\right) d_2^2 y = 0$, and so $d_2^2 = d_2^2$. On the hand, If $v_p(\dim C)$ is odd, then $D \notin \mathbb{Q}_4$, and so

$$\sqrt{\alpha D} = \pm \frac{1}{2}(d_1^2 x + \left(\frac{\ell}{p}\right) d_2^2 y)\epsilon \sqrt{p}.$$

Hence, by considering the product of conjuagates, we have

$$\frac{4 \dim C}{p} = \left(d_1^2 + \left(\frac{\ell}{p}\right) d_2^2 \frac{y}{x}\right) \left(d_1^4 + \left(\frac{\ell}{p}\right) d_2^2 \frac{y}{x}\right) = d_1^4 + d_2^4 + \left(\frac{\ell}{p}\right) d_1^2 d_2^2 \left(\frac{y}{x} + \frac{x}{y}\right).$$

This implies $y/x$ is either a 4-th root or a 6-th root of unity and so we obtain

$$\frac{4 \dim C}{p} = d_1^4 + d_2^4 + \epsilon d_1^2 d_2^2,$$

where $\epsilon \in \{0, 1, -1\}$. \hfill \Box

## 5 Weakly integral modular categories of rank 6 and 7

We now apply the results of previous sections to the classification of weakly integral modular categories of rank 6 and 7.

### 5.1 Weakly integral modular categories of rank 6

**Theorem 5.1.** A weakly integral rank 6 modular category $C$ is equivalent (as balanced braided fusion category) to one of the following:

(a) $\mathcal{I} \boxtimes \mathcal{P}_2$, with $\mathcal{I}$ an Ising modular category,

(b) $\mathcal{I} \boxtimes \mathcal{P}_3$, with $\mathcal{I}$ an Ising modular category,

(c) $TY(\mathbb{Z}_5, \chi, \nu)^{\mathbb{Z}_2}$, or

(d) $\mathcal{P}_6$. 

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Proof. If $\mathcal{C}$ is integral, it follows from [3] Theorem 4.2 that $\mathcal{C}$ is in fact pointed (alternative (d)). Therefore, we may assume that there is an object with non-integral dimension. Moreover, by Lemma 3.3 there are at least two objects with non-integral dimensions. In addition, [12] Theorem 3.10 implies that $\mathcal{C}$ is faithfully graded by an elementary 2-group, so that there are at least two invertible objects. Thus, the potential universal grading groups $U(\mathcal{C})$ have order 2 or 4. The properties of a grading and the pigeonhole principle immediately imply that if $\dim C_{\text{pt}} = |U(\mathcal{C})| = 4$ then $\dim \mathcal{C} = 8$ and $\mathcal{C}$ has 4 invertible objects and 2 simple objects of dimension $\sqrt{2}$. In this case, $\mathcal{C}$ is a modular category with generalized Tambara-Yamagami fusion rules. More precisely, in this case $\mathcal{C}$ is equivalent to $\mathcal{I} \boxtimes \mathcal{P}_2$, by [14] Theorem 5.5. Then, by Theorem 5.1 it is enough to show that $\dim \mathcal{C} \in \{12, 20\}$ if $\mathcal{C}$ is strictly weakly integral and the group $U(\mathcal{C})$ has order 2.

If $\dim C_{\text{pt}} = |U(\mathcal{C})| = 2$ then $\mathcal{C}_{\text{ad}} = \mathcal{C}_e$ is an integral premodular category containing exactly two isomorphism classes of invertible objects. Then, the rank of $\mathcal{C}_{\text{ad}}$ is 3 or 4. In the first case, such categories have been classified by Ostrik and all have simple dimensions 1, 1, 2 $[16]$. In particular, $\dim \mathcal{C} = 12$. Now, assume that $\text{rank} \mathcal{C}_{\text{ad}} = 4$. The complete classification of these categories was given in [11] and all have simple dimensions 1, 1, 2, 2. It follows that $\dim \mathcal{C} = 20$. Now, the statement follows from Theorem 5.1. \qed

5.2 Weakly Integral Rank 7
The goal of this subsection is to give a classification, up to Grothendieck equivalences, of weakly integral rank 7 modular categories. Applying the results of subsection 4.1 we have:

Proposition 5.2. Suppose $\mathcal{C}$ is a modular category of rank 7. If $\mathcal{C}$ admits a Tannakian subcategory equivalent to $\text{Rep}(H)$, for some nontrivial subgroup $H$ of $G(\mathcal{C})$, then $\mathcal{C}$ is strictly weakly integral of dimension 28.

Proof. Since $\mathcal{C}$ has a faithful $\hat{H}$-grading, the pigeon principle implies that $|H| \leq 4$. If $|H| = 4$ then $|\text{Irr}(\mathcal{C}_\chi)| = 1$, for any non-trivial character $\chi$ of $H$, by Remark 4.2. In particular, $H$ fixes all the simple objects not in $\text{Irr}(\mathcal{C}_e)$, but this contradicts Lemma 4.2. Thus, $|H| = 2$ or 3. Lemma 4.2 implies that if $|H| = 3$ then $3 | |\text{Irr}(\mathcal{C}_\chi)|$, for each non-trivial character $\chi$ of $H$. However, this means $\text{rank} \mathcal{C} \geq 9$. Therefore, $|H| = 2$.

Suppose $H = \{0, h\}$ and $\hat{H} = \{e, \chi\}$. In particular, $\chi(h) = -1$. In view of Lemma 4.2 $\text{Irr}(\mathcal{C}_\chi)$ can only have 1 or 2 $H$-orbits.

We first show that $\text{Irr}(\mathcal{C}_\chi)$ must have only one $H$-orbit. Assume contrary. Let $\text{Irr}(\mathcal{C}_e) = \{1, h, V_1\}$ and $\text{Irr}(\mathcal{C}_\chi) = \{V_2, V_3, V_4, V_5\}$ with $h \cdot V_2 = V_3$ and $h \cdot V_4 = V_5$. Note that $V_1$ cannot be invertible and so $V_1$ is fixed by $H$. In particular, $d_1^2 - nd_1 - 2 = 0$, and $d_1 \neq \pm 1$. Moreover, $V_1$ is self-dual.

If $d_1 \notin \mathbb{Z}$, then $d_1$ is a square root of an integer and $x^2 - nx - 2$ must be the minimal polynomial of $d_1$. This implies $n = 0$ and $d_1 = \sqrt{2}$, and hence $h$ must be fermionic, a contradiction. Therefore, $d_1 \in \mathbb{Z}$. Since $n$ is a non-negative integer, the equation $d_1^2 - nd_1 - 2 = 0$ implies $d_2 = 2$ and $n = 1$. In particular, $\dim \mathcal{C}_\chi = 6 = \dim \mathcal{C}_\psi$. Hence, $\mathcal{C}$ is weakly integral and $3 = d_2^2 + d_1^2$. Up to renumbering, $(d_2^2, d_1^2) = (1, 2)$ and hence $G = G(\mathcal{C})$ has order 4. Thus, for $\psi \in \hat{G}$, the homogeneous component $\mathcal{C}_\psi$ has $\dim \mathcal{C}_\psi = 3$ but this is not possible as $d_2^2 = 4 > \dim \mathcal{C}_\psi$. Therefore, $\text{Irr}(\mathcal{C}_\chi)$ consists of a single $H$-orbit.
Let \( \text{Irr}(C_e) = \{1, h, V_1, V_2, V_3\} \) and \( \text{Irr}(C_\chi) = \{V_4, V_5\} \) with \( h \cdot V_4 = V_5 \). Then one of \( V_1, V_2, V_3 \) must be fixed by \( H \). We may assume \( V_2 \) is fixed by \( H \). Since the stabilizer of \( \text{Irr}(C_\chi) \) is trivial, it follows from Lemma 4.3 that \( d_2 = 2n_2 \) for some positive integer \( n_2 \). Moreover, \( d_1, d_3 \in \mathbb{Z}_+ \). Following the dimension equation, we find \( \text{dim}C = 4d_1^2, \quad n_2^2 \mid d_1^2 \) and
\[
2d_1^2 = 2 + d_1^2 + 4n_2^2 + d_3^2 \tag{5.1}
\]
The equation modulo 2 implies \( (d_1, d_3, d_3^2) \equiv (1, 1, 0) \mod 2 \).

We first show that \( (d_1, d_3, d_3^2) \neq (1, 1, 0) \mod 2 \) is not possible. Assume contrary. Then \( d_1, d_3 \) is odd and so \( d_1^2 \mid d_3^2 \). By Lemma 4.3 \( V_1 \) is not fixed by \( H \). Therefore, \( h \cdot V_1 = V_3 \) and hence \( d_1 = d_3 \). Now, (5.1) becomes
\[
d_1^4 = 1 + d_1^2 + 2n_2^2 \tag{5.2}
\]
and hence \( d_1^2 \) is even. Moreover, \( n_2^2 \) and \( d_1^2 \) are relatively prime, and \( d_1^2 n_2^2 = \text{lcm}(d_1^2, n_2^2) \). If \( d_1^2 \equiv 2 \mod 4 \), then \( n_2^2 \equiv 0 \mod 4 \) and hence \( 4 \mid d_1^2 \), a contradiction. Therefore, \( 4 \nmid d_1^2 \), and so \( n_2 \) must be odd. Now, we find
\[
4 \mid \frac{d_1^4}{d_1^2 n_2^2} = \frac{1}{d_1^2 n_2^2} + \frac{1}{n_2^2} + \frac{2}{d_1^2}.
\]
This forces \( n_2^2 = d_1^2 = 1 \) and hence \( d_1^2 = d_1^2 = 4 \) and \( \text{dim}C = 16 \). Thus, \( G = G(C) \) has order 4 and the homogeneous component \( C_\psi \) has \( \text{dim}C_\psi = 4 \) for all \( \psi \in \hat{G} \). In particular, \( |\text{Irr}(C_\psi)| = 1 \) for any non-trivial character \( \psi \) of \( G \). Therefore, \( V_4 \) must be fixed by \( G \), this contradicts that \( V_4 \) is not fixed by \( H \).

Now, we have \( (d_1, d_3, d_3^2) \equiv (0, 0, 1) \mod 2 \), we proceed to show that \( C \) is strictly weakly integral of \( \text{dim}C = 28 \). Let \( n_i = d_i/2 \) for \( i = 1, 2, 3 \). Then \( n_i^2 \mid d_3^2 \) and hence \( n_i \) is odd for \( i = 1, 2, 3 \). (5.1) becomes
\[
d_1^4 = 1 + 2n_1^2 + 2n_3^2 \tag{5.3}
\]
Now, this equation implies \( d_1^4 \equiv 7 \mod 8 \). Therefore, \( d_1 \not\in \mathbb{Z} \). Let \( l = \text{lcm}(n_1^2, n_2^2, n_3^2) \). Then \( l \) is the square of an odd integer and hence \( l \equiv 1 \mod 8 \). Since \( n_i^2 \mid d_3^2 \), \( m = \frac{d_3^2}{l} \equiv 7 \mod 8 \). Therefore,
\[
7 \leq m = \frac{d_3^2}{l} = \frac{1}{l} + \frac{2n_1^2}{l} + \frac{n_2^2}{l} + \frac{2n_3^2}{l} \leq 7.
\]
This forces \( n_1^2 = n_2^2 = n_3^2 = 1 \). Hence \( d_1 = d_2 = d_3 = 2 \), \( d_1^4 = 7 \) and \( \text{dim}C = 28 \).

From this we obtain:

**Theorem 5.3.** The only strictly weakly integral rank 7 categories are \( TY(\mathbb{Z}_7, \chi, \nu)^{\mathbb{Z}_2} \).

**Proof.** Assume that \( C \) is weakly integral of rank 7. By \([12, \text{Theorem 3.10}]\) we have that \( 2 \mid |G(C)| \) and by Lemma \([3, \text{Lemma 3.3}]\) \( |G(C)| \leq 5 \). So there are two cases to consider: \( G(C) = U(C) \cong \mathbb{Z}_2 \) or \( |G(mcC)| = 4 \).

First suppose \( D := C_{\text{pt}} \) has rank 2. Clearly \( D \) is not modular, as \([13]\) implies that \( C \) can have no modular subcategories. In particular \( D \) is premodular, and hence symmetric. If \( D \) is Tannakian, \( i.e. \ D \cong \text{Rep}(\mathbb{Z}_2) \) then Prop. \([5, \text{Proposition 5.2}]\) implies that \( \text{dim}C = 28 \) and we are done by Theorem \([3, \text{Theorem 3.1}]\). Otherwise \( D \cong s\text{Vec} \) and we have \( C_{\text{ad}} = D \) hence \( C_{\text{ad}} \) is slightly degenerate.
(9). In particular $C_{ad}$ must have even rank by [9] Cor. 2.7. It follows from [11] that $C_{ad}$ has dimension 10, so $\dim C = 20$. This is impossible in rank 7.

Now if $|G(C)| = 4$, consider the possible categories $C_{ad} = C_e$ corresponding to the universal grading. Clearly $C_{ad}$ has rank at least 2 and at most 4 (by the pigeonhole principle) and even dimension. Combining this with the classification of low-rank ribbon categories [1, 15, 17] we see that this is impossible.

It remains to consider integral modular categories of rank 7. For this we employ the methods of subsection 4.2.

**Lemma 5.4.** Let $C$ be a weakly integral modular category of rank $\leq 7$. If $p$ is an odd prime factor of $\text{FSexp}(C)$ then $p \leq 7$ and $v_p(\text{FSexp}(C)) = 1$. If rank $C = 6$, $2 < p \mid \text{FSexp}(C)$ implies $p \leq 5$.

**Proof.** Suppose $C$ is a weakly integral modular category of rank $\leq 7$, and $p$ is the largest odd prime factor of $\text{FSexp}(C)$, and $\sigma$ is a $p$-automorphism. Then, by Lemma 4.4, $p \leq 13$. For any prime $p = 5, 7, 11, 13$, $v_p(\text{FSexp}(C)) \leq 1$ otherwise $\hat{\sigma}$ admits a support cycle of length $(p^2 - p)/2 \leq 10$ by Lemma 4.4. This certainly won’t happen in ranks $\leq 7$.

If $p = 13$, then $\hat{\sigma} = (1, 2, 3, 4, 5, 6)$ by Lemma 4.5. Since the centralizer of $(1, 2, 3, 4, 5, 6)$ in $S_7$ is $\langle (1, 2, 3, 4, 5, 6) \rangle$, $\text{Gal}(C) = \langle (1, 2, 3, 4, 5, 6) \rangle$. However, this contradicts Theorem 4.6. Therefore, $p < 13$.

If $p = 11$, then $(1, 2, 3, 4, 5)$ is the unique $p$-support cycle of $\sigma$. Thus,

$$\hat{\sigma} = (1, 2, 3, 4, 5) \text{ or } (1, 2, 3, 4, 5)(0, 6).$$

By Theorem 4.6, $\text{Gal}(C) = \langle \hat{\sigma} \rangle$ implies rank $C = 11$ or 9. Therefore, $\langle \hat{\sigma} \rangle \subsetneq \text{Gal}(C)$ and $\dim C = \frac{11d^4}{4}$. Since the centralizer of $\langle \hat{\sigma} \rangle$ is $\langle (1, 2, 3, 4, 5), (0, 6) \rangle$, $\text{Gal}(C) = \langle (1, 2, 3, 4, 5), (0, 6) \rangle$, and we have

$$\frac{11d^4}{4} = 2 + 5d_1^2.
$$

However, the equation has no integral solution for $d_1^2$. Therefore, $p \leq 7$.

If rank $C = 6$ and $p = 7$, then $\sigma$ has a unique support cycle $(1, 2, 3)$ and $\langle \hat{\sigma} \rangle \subsetneq \text{Gal}(C)$ by Theorem 4.6. Thus, $\text{Gal}(C) = \langle (1, 2, 3), (0, 5) \rangle$ or $\langle (1, 2, 3), (4, 5) \rangle$. The second case implies $C$ is integral and

$$\frac{7d_1^4}{4} = 3d_1^2 + 2d_2^2 + 1,$$

by Theorem 4.6(i). However, the equation modulo 2 implies $0 \equiv 1 \mod 2$. Therefore, $\text{Gal}(C) = \langle (1, 2, 3), (0, 5) \rangle$ and

$$\frac{7d_1^4}{4} = 3d_1^2 + d_2^2 + 2.$$

By the lemma, $d_4 \in \mathbb{Z}$ otherwise $C$ is the Ising modular category. Thus, $d_1 \notin \mathbb{Z}$ and so $\frac{7d_1^4}{8} = 3d_1^2$ but this implies $d_1^2 = 0$ or $\frac{24}{7}$, a contradiction.

**Remark 5.5.** If $C$ is a weakly integral modular category but not pointed, then $d_{\text{max}}^2 = \max_i d_i^2$, then $\frac{\dim C}{d_{\text{max}}}$ is a positive integer strictly less than rank $C$. 

□
Lemma 5.6. Suppose $C$ is a weakly integral modular category of rank 7 such that $d_1 = d_2 = d_3 \leq d_4 = d_5 = d_6$. Then, one of the following statements holds:

(i) $(d_1, d_4) = (1, 1)$ and $\dim C = 7$.
(ii) $(d_1, d_4) = (1, \sqrt{2})$ and $\dim C = 10$.
(iii) $(d_1, d_4) = (1, 2)$ and $\dim C = 16$.

Proof. The assumption implies the equality

$$\dim C = 1 + 3d_1^2 + 3d_4^2.$$  \hfill (5.4)

Thus $d_1^2, d_4^2$ are relatively prime and so $d_1^2 d_4^2 \mid \dim C$. By the symmetry of the equation, we may assume $d_4 \geq d_1$. Then

$$d_2 \mid \frac{\dim C}{d_4^2} = \frac{1}{d_4^2} + \frac{3d_1^2}{d_4^2} \in \mathbb{Z}$$

implies $\frac{\dim C}{d_4^2} \leq 4$. The equality holds if, and only if, $d_1 = d_4^2$, and $d_i = 1$ for all $i$. In this case, $\dim C = 7$. If $\frac{\dim C}{d_4^2} \leq 3$, then $d_1^2 < d_4^2$ and so $(d_1^2, d_4^2) = (1, 4), (1, 2)$ by (5.4). The associated $\dim C$ are 16 and 10.

Proposition 5.7. If $C$ is a weakly integral modular category of rank 7 and $\dim C = 28$, then $C$ is either pointed or $\dim C = 28$.

Proof. Assume $C$ is not pointed. We first show that $\text{Gal}(C)$ does not contain any permutation of type $(1, 3, 3)$ or $(1, 6)$ with 0 fixed. Assume contrary. Then $d_1 = d_2 = d_3$, $d_4 = d_5 = d_6$ and we have

$$\dim C = 1 + 3d_1^2 + 3d_4^2.$$  \hfill (5.5)

By Lemma 5.6, $C$ must be pointed, a contradiction.

Let $p = 7$, $\sigma$ the $p$-automorphism of $C$ and $C_1$ a $p$-support cycle of $\sigma$. Then In view of Lemma 4.5, $\text{ord}(C_1) = 3$, say $C_1 = (1, 2, 3)$. Since all the 7-support cycle must have length $\geq 3$, $C_1$ is the unique 3-support cycle of $\sigma$ and all other disjoint cycles are of length less than 3. In view of Theorem 4.6, $\dim C = \frac{7d_1^4}{4}$ and $2 \mid d_1^2$. Moreover,

$$\hat{\sigma} = (1, 2, 3), (1, 2, 3)(4, 5), (0, 6)(1, 2, 3)(4, 5) \text{ or } (0, 6)(1, 2, 3)$$

up to renumbering the non-zero labels.

If $\hat{\sigma}$ were one of the first three cases, then $\langle \hat{\sigma} \rangle \neq \text{Gal}(C)$ otherwise it will contradicts Theorem 4.6. Since $\text{Gal}(C)$ is an abelian subgroup of $S_7$ containing $\hat{\sigma}$, there exists an element of the form $C_1^i(4, 5), (0, 6)C_1^i(4, 5)$ in $\text{Gal}(C)$ for some $i = 1, 2$. In particular, we have the equation

$$\frac{7d_1^4}{4} = 1 + 3d_1^2 + 2d_4^2 + d_6^2.$$  \hfill (5.5)

with $2 \mid d_1^2$.

We claim that if the dimensions the simple objects of $C$ satisfy (5.5), then $C$ is a prime modular category of dimension 28. We first observe that the equation (5.5) modulo 2 implies the parities of $\frac{d_1^2}{2}$ and $d_6^2$ are opposite. Since $d_6^2 \mid \frac{7d_1^4}{4}$, $d_6^2$ must be odd and $d_1^2/2$ is even. Thus,

$$28n_1^2 = 1 + 12n_1 + 2d_4^2 + d_6^2.$$  \hfill (5.6)
where \( d_1^2 = 4n_1 \) for some positive integer \( n_1 \). If \( d_6^2 \equiv 3 \pmod{4} \), then \( d_6 \not\in \mathbb{Z} \) and so there exists \( \tau \in \text{Gal}(C) \) which admits a transposition of the form \( (0, j) \), and in particular, \( d_j = 1 \). Thus, \( j \) can only be 4 or 5 but this does not balance the equation \((5.6)\) modulo 4. Therefore, \( d_6^2 \equiv 1 \mod 4 \) and hence \( d_4^2 \) is odd.

Since \( d_6^{max} \mid \text{dim} \ C = 28n_1^2 \) and \( \frac{\text{dim} \ C}{d_6^{max}} < 7 \), \( d_6^{max} \neq \frac{\text{dim} \ C}{2} \) and so \( d_6^{max} = d_4^2 \) or \( d_6^2 \). In particular, \( d_6^{max} \) is odd. Thus, \( \frac{\text{dim} \ C}{d_6^{max}} = 4 \) or \( \frac{\text{dim} \ C}{d_6^{max}} = 7n_1^2 \equiv 3 \mod 4 \). Hence, \( d_6^{max} = d_4^2 \) and \((5.6)\) becomes

\[
14n_1^2 = 1 + 12n_1 + d_6^2.
\]

In particular, \( d_6^2 \) and \( n_1 \) are relatively prime. Since \( d_6^2 | 28n_1^2 \) and \( d_6^2 \equiv 1 \mod 4, \) \( d_6^2 = 1 \) and so

\[
7n_1^2 - 6n_1 - 1 = 0.
\]

This equation forces \( n_1 = 1 \) and hence \((d_1^2, d_3^2, d_5^2) = (4, 7, 1) \). Therefore, \( C \) is a prime modular category of dimension 28, and this proves the claim.

As a consequence of the preceding claim, \( \hat{\sigma} = (0, 6)(1, 2, 3) \). To complete the proof, it suffices to show that \( \langle \hat{\sigma} \rangle = \text{Gal}(C) \). If not, then \( \text{Gal}(C) \) contains \((4, 5)\) and so the dimensions of \( C \) satisfy \((5.5)\) again. This implies \( C \) is a prime modular category of dimension 28 but then \( \text{Gal}(C) = \langle \hat{\sigma} \rangle \), a contradiction. \( \Box \)

**Theorem 5.8.** If \( C \) is an integral modular category of rank 7, then \( C \) is pointed.

**Proof.** Let \( C \) be an integral modular category of rank 7. By Lemma \( 5.4 \) the prime factors of \( \text{FSexp}(C) \) can only be 2, 3, 5, 7. By the Cauchy Theorem, the prime factors of \( \text{dim} \ C \) can only be 2, 3, 5, 7. In view of Proposition \( 5.7 \), it suffices to prove that \( 7 \mid \text{dim} \ C \). Equivalently, it enough to show none of 2, 3 or 5 is a prime factor of \( \text{dim} \ C \). In view of Proposition \( 5.2 \) it is not possible that only two the these primes are factors of \( \text{dim} \ C \). It suffices to show that \( \text{dim} \ C = 2^a 3^b 5^c \) with \( abc \geq 1 \) is not possible.

Suppose \( \text{dim} \ C = 2^a 3^b 5^c \) with \( abc \geq 1 \). Let \( \sigma \) be a 5-automorphism of \( C \), and \( C_1 \) a 5-support cycle of \( \sigma \). We first show that length of \( C_1 \) must be 2. If not, then \( \text{ord}(C_1) = 4 \) and so the dimensions of \( C \) satisfy the equation

\[
\text{dim} \ C = 1 + 4d_1^2 + d_3^2 + d_5^2
\]

where we simply assume \( C_1 = (1, 2, 3, 4) \). This equation modulo 2 implies

\[
0 \equiv 1 + d_3^2 + d_5^2 \mod 2.
\]

Without loss of generality, we may assume \( d_3 \) is odd and \( d_5 \) is even. In particular, \( 4 \mid \text{dim} \ C \) as \( d_6^2 \mid \text{dim} \ C \). However, we then find \( 0 \equiv 2 \mod 4 \), a contradiction. Therefore, the dimensions of \( C \) do not satisfy \((5.7)\), and so \( \text{Gal}(C) \) does not contain any permutation which admits a disjoint cycle of length \( \geq 4 \). In particular, \( \text{ord}(C_1) < 4 \)

Now, we may assume \( C_1 = (1, 2) \) and proceed to show that this is a unique 5-support cycle of \( \sigma \). Then \( \hat{\sigma} = (1, 2)(3, 4) \) or \( \hat{\sigma} = (1, 2)(3, 4)(5, 6) \). However, if \( \text{Gal}(C) \) contain any permutation of type \((1, 2, 2, 2)\) with 0 fixed, then the dimensions of \( C \) satisfy the equation:

\[
\text{dim} \ C = 1 + 2d_1^2 + 2d_3^2 + 2d_5^2 \equiv 1 \mod 2.
\]

This is not possible, and it also implies that \( \text{Gal}(C) \) does not contain any permutation of type \((1, 2, 2, 2)\) with 0 fixed. In particular, \( \hat{\sigma} = (1, 2)(3, 4) \).
If \((3, 4)\) is also a 5-support cycle of \(\hat{\sigma}\), then by Lemma 4.7 the dimensions of \(C\) satisfy (5.7) or
\[
1 + 2d_1^2 + 2d_2^2 + d_5^2 + d_6^2 = \frac{1}{4}(d_1^4 + d_4^4 + \epsilon d_1^2 d_4^2) \text{ with } 2 \mid d_1, d_2 \text{ and } \epsilon = 0, 1, -1. \tag{5.9}
\]
We have shown that the dimensions of \(C\) do not satisfy (5.7). By considering (5.9) modulo 2, we may assume that \(d_5\) is odd and \(d_6\) is even. Then the equation modulo 4 becomes
\[
0 \equiv 1 + d_5^2 \mod 4,
\]
but this is impossible. Therefore, \((3, 4)\) is not a 5-support cycle if \(\hat{\sigma} = (1, 2)(3, 4)\). In particular, \((1, 2)\) is the unique 5-support of \(\sigma\). By Theorem 4.6 we find \(\dim C = \frac{5d_1^2}{4}\) and \(d_1 = 2n_1\) for some positive integer \(n_1\).

Suppose there exists a permutation \(\tau \in \text{Gal}(C)\) which admits a cycle of length \(\geq 2\) and disjoint from \((1, 2)\), say \((3, 4, \ldots)\). Then the dimensions of \(C\) must satisfy the equation:
\[
20n_1^4 = 1 + 8n_2^2 + 2d_4^2 + d_5^2 + d_6^2. \tag{5.10}
\]
Then \(d_5\) or \(d_6\) must have opposite parities for otherwise the left hand side of (5.10) would be congruent to 1 modulo 2. We may assume \(d_5\) is odd and \(d_6 = 2n_6\) for some positive integer \(n_6\). Now, (5.10) modulo 4 yields
\[
0 \equiv 2 + 2d_4^2 \mod 4.
\]
This forces \(d_4\) to be odd, and we have
\[
\dim C = 4 + 4n_6^2 \mod 8.
\]
Thus, \(n_6\) must be odd, \(8 \mid \dim C\), and so \(n_1\) is also even. Let \(d_1 = 4m_1\) for some positive integer \(m_1\). Now, \(\dim C = \frac{20^5m_1^4}{d_j^4} > 7\) for \(j = 1, \ldots, 6\), but this contradicts that \(\frac{\dim C}{d_{\max}} \leq 7\).

Therefore, no permutation \(\tau \in \text{Gal}(C)\) admits a non-trivial cycle disjoint from \((1, 2)\). Hence, \(\text{Gal}(C) = \langle \hat{\sigma} \rangle\) and \(\hat{\sigma} = (1, 2)\), but this contradicts Theorem 4.6. \(\square\)

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