DE RHAM PRISMATIC CRYSTALS OVER $\mathcal{O}_K$

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Abstract. We study de Rham prismatic crystals on $(\mathcal{O}_K)_{\Delta}$. We show that a de Rham crystal is controlled by a sequence of matrices $(A_m)_{m \geq 0}$ with $A_{0,1}$ "nilpotent". Using this, we prove that the natural functor from the category of de Rham crystals over $(\mathcal{O}_K)_{\Delta}$ to the category of nearly de Rham representations is fully faithful. The key ingredient is a Sen style decompletion theorem for $B_{dR}^{+}$-representations of $G_K$.

Contents

1. Introduction 1
2. de Rham prismatic crystals 8
  2.1. Stratification of de Rham prismatic crystals 8
  2.2. Calculation in the commutative case 14
  2.3. Evidence of the conjecture 21
3. Absolute prismatic cohomology of de Rham crystals 23
4. Locally analytic vectors and Decompletion theorem 27
  4.1. First descent step 27
  4.2. Sen theory over $B_{dR}^{+}$: decompletion along Kummer tower 29
  4.3. Kummer-Sen operator 33
5. Representations associated to de Rham prismatic crystals 38
  5.1. De Rham crystals and $B_{dR}^{+}$-representations 38
  5.2. Full faithfulness of the restriction functor 39
  5.3. Essential image of the restriction functor 42
References 44

1. Introduction

Once constructed by Bhatt and Scholze in [BS19], prismatic cohomology theory plays a central role in the study of $p$-adic cohomologies as it can naturally serve as a bridge connecting various classical $p$-adic cohomology theories via specialization along different directions. Motivated by such "universal" property, it is natural to expect that various coefficients in the prismatic site of a $p$-adic formal scheme could recover some coefficients in other $p$-adic cohomology theories. Recently, there are fruitful results in this direction. Let us give a quick review here.

Let $X$ be a $p$-adic formal scheme. Consider the absolute prismatic site $X_{\Delta}$. Recall that prismatic crystals with various coefficients can be defined in a universal way:

For $\ast \in \{ \mathcal{O}_{\Delta}, \overline{\mathcal{O}}_{\Delta}, \overline{\mathcal{O}}_{\Delta}[\frac{1}{p}], (\mathcal{O}_{\Delta}[\frac{1}{p}])^{\ast}_{2} \}$,

**Definition 1.1.** An abelian sheaf $\mathcal{F}$ on $X_{\Delta}$ is called a (\ast)-crystal if for any $(A, I) \in X_{\Delta}$, $\mathcal{F}(A, I)$ is a finite projective (A)-module such that for any morphism $(A, I) \rightarrow (B, J)$, there is a canonical isomorphism $\mathcal{F}(A, I) \otimes_{\ast(A)} \ast(B) \cong \mathcal{F}(B, J)$.

Moreover, the category of such $\mathcal{F}$ is denoted as $\text{Vect}(X_{\Delta}, \ast)$.

In this way, we get

- Prismatic crystal if $\ast = \mathcal{O}_{\Delta}$. 


The category of such $\phi$-modules is a pair $(\mathcal{E}, \varphi_\mathcal{E})$ such that
- $\mathcal{E}$ is a $(\ast)$-crystal;
- $\varphi_\mathcal{E}$ is an identification

$$\varphi_\mathcal{E} : (\mathcal{E}[1/\mathcal{I}]) \cong \mathcal{E}[1/\mathcal{I}].$$

The category of such $\mathcal{F}$ is denoted as $\text{Vect}^{\ast}(X_{\Delta}, \ast)$.

In this way, we get prismatic $F$-crystals if $\ast = \mathcal{O}_\Delta$ and Laurent prismatic $F$-crystals if taking $\ast = (\mathcal{O}_\Delta[1/\mathcal{I}])_p$.

Heuristically those $F$-crystals were first studied as they naturally correspond to some nice integral $p$-adic Galois representations over $\mathbb{Q}_p$ of interest by the work of Bhattach and Scholze (Frobenius is a very strong condition to guarantee that we end up with something defined over $\mathbb{Q}_p$ or a finite extension $K/\mathbb{Q}_p$). On the other hand, if we consider those prismatic crystals without Frobenius, typically we could only get something defined over $\mathbb{C}_p$, or equivalently, over some large highly ramified fields (eg. $K_{p^\infty}$ and $K_\infty$) by Faltings almost purity theorem and decomposition theory.

To concretely relate various prismatic crystals to $p$-adic Galois representations, the story starts from the following key theorem due to Bhattach and Scholze, which was also proven by Du and Liu independently:

**Theorem 1.3 ([BS21],[DL21]).**

$$\text{Vect}^{\ast}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\cong} \text{Vect}^{\ast}(X_{\Delta}, (\mathcal{O}_{\Delta}[1/\mathcal{I}])_p)$$

$$\text{Rep}_{\mathbb{Z}_p}^{\text{cryst}}(G_K) \xrightarrow{T} \text{Rep}_{\mathbb{Z}_p}(G_K)$$

Here $K$ is a complete discretely valued field with ring of integers $\mathcal{O}_K$ of mixed characteristic with perfect residue field $k$ and $X = \text{Spf} \mathcal{O}_K$. $G_K$ is the absolute Galois group of $K$. $\text{Rep}_{\mathbb{Z}_p}^{\text{cryst}}(G_K)$ is defined to be the category of finite free $\mathbb{Z}_p$-representations of $G_K$, while $\text{Rep}_{\mathbb{Z}_p}(G_K)$ is the full subcategory consisting of crystalline $\mathbb{Z}_p$-representations of $G_K$.

From now on we fix a uniformizer $\pi$ of $\mathcal{O}_K$ with Eisenstein polynomial $E(u) \in W(k)[[u]]$. $(\mathcal{E}, E) = (W(k)[[u]], E(u))$ is the Breuil-Kisin prism.

**Remark 1.4.**
- $T$ which is called étale realization functor, was first proven to be an equivalence independently by Zhiyou Wu. Concretely, it is given by taking $\varphi$-invariants of the evaluation of $\mathcal{E}$ at $A_{\text{fppf}}$. By work of Kedlaya and Liu, the category of étale $\varphi$-modules over $W(\mathbb{C}_p^\circ)$ is equivalent to $\text{Rep}_{\mathbb{Z}_p}(G_K)$). Moreover, the crystal condition can be transferred to give a $G_K$-action (key observation of Wu).
- Also, by evaluating $\mathcal{E}$ at some other prisms $A_K^+$ (which could be viewed as certain "deperfection" of $A_{\text{int}}$), Wu gives a new proof of Fontaine’s classical result that $\text{Rep}_{\mathbb{Z}_p}(G_K)$ is equivalent to $(\varphi, \Gamma)$-modules over $A_K$. We refer the readers to [Wu21] for details.
- In Bhattach-Scholze’s work, they show that the image of $T$ when restricted to $\text{Vect}^{\ast}(X_{\Delta}, \mathcal{O}_{\Delta})$ naturally enjoys crystalline property first. The genuine difficulty is to prove essential surjectivity. To do that, given a crystalline $\mathbb{Z}_p$-representation, Bhattach and Scholze first construct a crystal over some larger rings and then show the descent data is bounded to actually descend to a prismatic crystal, which relies heavily on using Beilinson fiber sequence to calculate Frobenius invariant elements in certain large rings.

- On the other hand, Du and Liu prove the equivalence in a different method. Their study is based on the Kummer tower. In particular, they prove an étale realization in this setting and then give
a descent data via bounding the matrix on which Frobenius acts via, relying on the calculation of crystalline condition.

- Later all of the results are generalized to the relative case. Min-Wang prove that $T$ is an equivalence in the relative setting in [MW21a]. Du-Liu-Moon-Shimizu and Guo-Reinecke prove the following result independently:

**Theorem 1.5** ([GR22],[DLMS22]). For $X$ a smooth formal scheme over $\mathcal{O}_K$

$$\text{Vect}^{\mathrm{an},\forall}(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\cong} \text{Rep}_{\mathbb{Z}_p}^{\text{crys}}(X)$$

Here the right handside is crystalline $\mathbb{Z}_p$-local systems on the generic fiber of $X$, while the left handside is the so called analytic prismatic $F$-crystals consisting of crystals $\mathcal{F}$ on $X_\Delta$ equipped with Frobenius structures whose evaluations at a prism $(A,I) \in X_\Delta$ are vector bundles on the analytic locus of $\text{Spec}(A)$, we refer the readers to [GR22, Remark 1.4] for details.

On the other hand, one might ask what’s the story if one drops the Frobenius structure and works with prismatic crystals.

Towards this direction, Yu Min and Yupeng Wang study Hodge-Tate crystals on $\mathcal{O}_K$ in [MW21]. Later Hui Gao relates rational Hodge-Tate crystals on $\mathcal{O}_K$ to nearly Hodge-Tate representations in [Gao22]. Let us review their work here. For simplicity we assume $p > 2$ from now on.

Utilizing the Breuil-Kisin prism as the chosen weakly final object in $(\mathcal{O}_K)_\Delta$ and by identifying $\mathfrak{S}^l/E$ with $\mathcal{O}_K\{X_1\}^{\wedge}_{pd}$, which is involved in the stratification of the Breuil-Kisin prism, Min-Wang prove the following result (see [MW21, Theorem 3.5]):

**Theorem 1.6.** To give a rank $l$ Hodge-Tate crystal $\mathcal{M}$, it is the same as specifying a pair $(M, \varepsilon)$, where $M$ is a finite free $\mathfrak{S}$-module $M$ of rank $l$ and the stratification $\varepsilon$ is an $\mathcal{O}_K\{X_1\}^{\wedge}_{pd}$-linear isomorphism

$$\varepsilon : M \otimes_{\mathcal{O}_K} \mathcal{O}_K\{X_1\}^{\wedge}_{pd} \xrightarrow{\cong} M \otimes_{\mathcal{O}_K} \mathcal{O}_K\{X_1\}^{\wedge}_{pd}.$$ 

Moreover, $\varepsilon$ is uniquely determined by a matrix $A_1 \in M_{l \times l}(\mathcal{O}_K)$ such that $\lim_{n \to +\infty} \prod_{i=0}^{n} (iE'(\pi) + A_1) = 0$ in the sense that

$$\varepsilon(x) = \varepsilon \sum_{n=0}^{\infty} A_n X_1^{[n]}$$

with $A_0 = I$, $A_n = \prod_{i=0}^{n-1} (iE'(\pi) + A_1)$ for $n \geq 1$.

Then essentially by proving that the Sen operator of the $\mathbb{C}_p$-representation of $G_K$ associated to such a (rational) Hodge-Tate crystal is $-\frac{A_1}{\beta}$ for $\beta = E'(\pi)$, which was conjectured by Min and Wang, Gao constructs the following diagram (see [Gao22, Thm 1.1.3]):

**Theorem 1.7.**

$$\begin{array}{ccc}
\text{Vect}((\mathcal{O}_K)_\Delta, \overline{\mathcal{O}}_\Delta^{[1/p]}) & \rightarrow & \text{Vect}((\mathcal{O}_K)_\Delta^{\mathrm{perf}}, \overline{\mathcal{O}}_\Delta^{[1/p]}) \\
\cong & & \cong \\
\text{Rep}_{\mathbb{C}_p}^{\mathrm{nHT}}(G_K) & \rightarrow & \text{Rep}_{\mathbb{C}_p}(G_K)
\end{array}$$

Here $\text{Rep}_{\mathbb{C}_p}^{\mathrm{nHT}}(G_K)$ is defined by Gao, which is the full subcategory of $\text{Rep}_{\mathbb{C}_p}(G_K)$ consisting of those $W \in \text{Rep}_{\mathbb{C}_p}(G_K)$ with Sen weights in the subset $\mathbb{Z} + E'(\pi)^{-1}\mathbb{A}_p$.

Then Gao asked a natural question that whether we can deform this diagram by replacing the rational Hodge-Tate crystals with de Rham crystals in this picture, see [Gao22, Remark 1.1.4].

This is one of the motivations of our study on the de Rham prismatic crystals over $(\mathcal{O}_K)_\Delta$ and we wish to give a partial answer to this question. To state our result, we need some notations:

**Notation 1.8.**

- $\text{Rep}_{\mathbb{C}_p}^{\mathrm{fp}}(G_K)$ is defined to be the category of finite projective $\mathbb{B}_{\mathrm{dR}}^+$-modules equipped with a semi-linear $G_K$ action.
• $M \in \text{Rep}^{fp}_{B_{dr}}(G_K)$ is called nearly de Rham if $M/tM \in \text{Rep}^H_{E_p}(G_K)$, i.e. all of the Sen weights of $M/tM$ are in the subset $\mathbb{Z} + E'(\pi)^{-1}m_{\text{crpr,}}$. (In this paper $t$ is identified with $E([\pi^p])$, where $[\pi^p]$ is defined in the end of this section). Denote $\text{Rep}^{fp,ndR}_{B_{dr}}(G_K)$ to be the full subcategory of $\text{Rep}^{fp}_{B_{dr}}(G_K)$ consisting of such $M$.

Now we are ready to state our main result:

**Theorem 1.9.** We have a commutative diagram

\[
\begin{array}{ccc}
\text{Vect}((\mathcal{O}_K)_\Delta, (\mathcal{O}_\Delta[\frac{1}{p}])^\wedge) & \xrightarrow{V} & \text{Vect}((\mathcal{O}_K)_{\wedge}^\text{perf}, (\mathcal{O}_\Delta[\frac{1}{p}])^\wedge) \\
\downarrow & & \downarrow T \\
\text{Rep}^{fp,ndR}_{B_{dr}}(G_K) & \longrightarrow & \text{Rep}^{fp}_{B_{dr}}(G_K)
\end{array}
\]

such that $V$ is fully faithful and $T$ is an equivalence. Here $V$ is given by restricting a de Rham prismatic crystal to the perfect prismatic site and $T$ is the evaluation at the $A_{\text{inf}}$ prism. We will call $T$ the **de Rham realization functor** later.

As a corollary, if we abuse the notation by still using $V$ to denote the composition of $T$ and $V$, then we have that:

**Corollary 1.10.** $V : \text{Vect}((\mathcal{O}_K)_\Delta, (\mathcal{O}_\Delta[\frac{1}{p}])^\wedge) \longrightarrow \text{Rep}^{fp,ndR}_{B_{dr}}(G_K)$ is fully faithful.

We now explain how to prove **Theorem 1.9**. Once the basic properties of $V$ and $T$ are established, then the image of $V$ naturally lands in $\text{Rep}^{fp,ndR}_{B_{dr}}(G_K)$ as a de Rham crystal naturally specializes to a rational Hodge-Tate crystal (see **Corollary 2.2** for details) and rational Hodge-Tate crystals are realized to nearly de Rham realizations by **Theorem 1.7**.

To study the behavior of $V$ and $T$, we need several preliminaries.

First we give an explicit description of de Rham crystals in terms of a stratification of the Čech nerve associated to the Breuil-Kisin prism by identifying $\mathbb{B}^+_{\text{dr}}(\mathcal{G})$ (resp. $\mathbb{B}^+_{\text{dr}}(\mathcal{G}^1)$) with $K[[T]]$ (resp. $K\{X_1\}_{pd}[[T]]$) (see **Lemma 2.10** for details), then we have the following:

**Theorem 1.11** (**Theorem 2.18**). Let $M$ be a finite free $\mathbb{B}^+_{dr}(\mathcal{G})$-module and $\varepsilon$ be an $\mathcal{O}_K\{X_1\}_{pd}$-linear isomorphism

\[
\varepsilon : M \otimes_{K[[T]],\delta_0} K\{X_1\}_{pd}[[T]] \cong M \otimes_{K[[T]],\delta_1} K\{X_1\}_{pd}[[T]].
\]

Fix a basis $\underline{e}$ of $M$ and write $\varepsilon(\underline{e}) = \varepsilon = \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n} X^{(n)} t^m$, then if $(M, \varepsilon)$ is induced from a de Rham prismatic crystal $M \in \text{Vect}((\mathcal{O}_K)_\Delta, (\mathcal{O}_\Delta[\frac{1}{p}])^\wedge)$, the following holds:

- $A_{0,0} = 1$ and $A_{i,0} = 0$ for $i > 0$.
- $A_{0,1} \in M_1(K)$ satisfies that $\lim_{n \to +\infty} \prod_{i=0}^{n} (iE'(\pi) + A_{0,1}) = 0$.
- $A_{m,n+1} = (\beta(n-m) + A_{0,1}) A_{m,n} + \sum_{i+j=m} (\sum_{n \geq 0} A_{m,n} X^{(n)} t^m)$ for $m,n \in \mathbb{N}^\geq$. In particular, $\{A_{m,n}\}$ is determined by $\{A_{m,1}\}_{m \geq 0}$.

where $\theta_{1,j}$ is defined as in **Example 2.7**.

Actually we conjecture that the reverse is also true:

**Conjecture 1.12** (**Conjecture 2.24**). Assume that $\{B_{m,1}\}$ is a sequence of matrices in $M_1(K)$ such that

- $\lim_{n \to +\infty} \prod_{i=0}^{n} (iE'(\pi) + B_{0,1}) = 0$.

Then if we define a sequence of matrices $\{A_{m,n}\}$ via following:

- $A_{0,0} = 1$ and $A_{i,0} = 0$ for $i > 0$, $A_{m,1} = B_{m,1}$.
- $A_{m,n+1} = (\beta(n-m) + A_{0,1}) A_{m,n} + \sum_{i+j=m} (\sum_{n \geq 0} A_{m,n} X^{(n)} t^m)$ for $m,n \in \mathbb{N}^\geq$. 

\[ \varepsilon(\underline{\xi}) = \underline{\xi} \cdot \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n}X^{[n]}t^m. \]

can serve as the stratification associated to certain prismatic de Rham crystal, i.e. the cocycle condition is satisfied.

To show that \( V \) is fully faithful, essentially it suffices to show that \( H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \cong \mathcal{M} \langle A_{i\nu} \rangle^{G_k} \). The idea is to write \( \tilde{\nu} \in \mathcal{M}(A_{i\nu}) \) as linear combinations of \( \underline{\xi} \) with coefficients in \( B^{\text{dR}}_\mathbb{C} \cong \mathbb{C}_p[[t]] \), then characterize the Galois invariant action. However, the main difficulty is that \( B^{\text{dR}}_{\mathbb{C}} \cong \mathbb{C}_p[[t]] \) is only an abstract isomorphism, but couldn’t be Galois equivariant! To solve this problem, we want to control the coefficient of \( \tilde{\nu} \) provided that \( \tilde{\nu} \) is Galois invariant. The key ingredient is the following Sen style decompletion theorem (see the end of this section for notations):

**Theorem 1.13** (Theorem 4.29). Given \( W \in \text{Rep}_{B^{\text{dR}}_k}(G_K) \) of rank \( d \), define

\[ D_{\text{Sen}, K_{\infty}[[t]]}(W) = (W^{G_k})^{G_{-1}a, \gamma=1}. \]

Then this is a finite projective \( K_{\infty}[[t]] \)-module of rank \( d \) such that

\[ D_{\text{Sen}, K_{\infty}[[t]]}(W) \otimes_{K_{\infty}[[t]]} B^{\text{dR}}_k = W. \]

Here \( (W^{G_k})^{G_{-1}a, \gamma=1} \) is the subspace of \( W \) consisting of those vectors which are \( G_L \)-invariant, \( G \)-locally analytic and \( \text{Gal}(L/K_{\infty}) \)-invariant, see Definition 4.13 for details.

For those \( W \) associated to a de Rham crystal, we can write \( D_{\text{Sen}, K_{\infty}[[t]]}(W) \) concretely:

**Theorem 1.14** (Theorem 5.11). Suppose \( \mathcal{M} \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_K\langle \frac{1}{p} \rangle)^2) \). Let \( M = \mathcal{M}(\mathcal{G}, (E(u))) \), and \( V(M) = \mathcal{M}(A_{\text{int}}, (\xi)) \), then

\[ D_{\text{Sen}, K_{\infty}[[t]]}(V(M)) = M \otimes_{K[[t]]} K_{\infty}[[t]], \]

here we have identified \( B^{\text{dR}}_k(\mathcal{G}) \) with \( K[[T]] \).

Now if \( \tilde{\nu} \in V(M) \) is \( G \)-invariant, then in particular it is a \( G \)-locally analytic, \( \text{Gal}(L/K_{\infty}) \) invariant element, hence \( \tilde{\nu} \in D_{\text{Sen}, K_{\infty}[[t]]}(V(M)) \). Thanks to Theorem 1.14, we can assume that \( \tilde{\nu} = \underline{\xi} \sum_{m=1}^\infty D_m t^m \in M \) with \( D_m \in M_{1 \times 1}(K_{\infty}) \), from which we can calculate the \( G \)-action concretely using Proposition 5.6:

\[ 0 = (g - \text{Id})(\underline{\xi} \sum_{m=1}^\infty D_m t^m) = \underline{\xi} \left( \sum_{m=0}^\infty \sum_{n=0}^\infty A_{m,n} X_1(g)^{[n]}(\sum_{m=1}^\infty g(D_m)(at)^m) \right) - \underline{\xi} \sum_{m=1}^\infty D_m t^m. \]

We want to show that this implies \( \{D_m\} \) are actually matrices in \( M_{1 \times 1}(K) \) which are exactly determined by the same condition computing \( H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \) via Čech-Alexander complex, see Proposition 3.11 for details. Then another difficulty is that \( X_1 \) is no longer a variable and we couldn’t just compare the coefficient! To overcome it, essentially using [DL21, Lemma 2.4.4], one can show that \( \theta(X_1(\tau)) \) is transcendental over \( K \), this (plus some other conditions) implies the right hand side is always 0 when viewing \( X_1(\tau) \) as a "variable". As a result, we prove that

**Theorem 1.15** (Theorem 5.12). The restriction functor

\[ V : \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_K\langle \frac{1}{p} \rangle)^2) \rightarrow \text{Vect}((\mathcal{O}_K)_{\Delta}^{\text{perf}}, (\mathcal{O}_K\langle \frac{1}{p} \rangle)^2) \]

is fully faithful.

On the other hand, we want to understand the essential image of \( V \), i.e. to determine which kind of de Rham representation of \( G_K \) (here we implicitly identify \( \text{Vect}((\mathcal{O}_K)_{\Delta}^{\text{perf}}, (\mathcal{O}_K\langle \frac{1}{p} \rangle)^2) \) with \( \text{Rep}_{G_K}(B^+_{\text{dR}}) \) via \( T \) in Theorem 1.9) can be associated to a de Rham prismatic crystal. To state our conjecture in this direction, we need some preliminaries.

For \( W \in \text{Rep}_{G_K}(B^+_{\text{dR}}) \), \( D_{\text{Sen}, K_{\infty}[[t]]}(W) \) is actually equipped with a natural monodromy operator \( N_T \):
Theorem 1.16 (Theorem 4.31). Given $W \in \text{Rep}_{G_K}(B^+_{dR})$, there is a $K\infty$-linear operator
\[ N_{\nabla} : D_{\text{Sen},K\infty[[t]]}(W) \to D_{\text{Sen},K\infty[[t]]}(W) \]
such that $N_{\nabla}$ satisfies Leibniz rule and that
\[ N_{\nabla}(tv) = N_{\nabla}(t)v + tN_{\nabla}(v) = E'(u)\lambda u \cdot v + tN_{\nabla}(v). \]

In the special case that $W = V(M)$, where $M \in \text{Vect}((\mathcal{O}_K)_{\Delta^1},(\mathcal{O}_K[\xi])_{\Delta^2})$, is a de Rham crystal corresponding to the pair $(M, \varepsilon)$ via Corollary 2.4 for a finite free $B^+_{dR}(\mathcal{S})$-module $M$ and a stratification $\varepsilon$ such that
\[ \varepsilon(\xi) = \varepsilon \cdot \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n} X^{[n]} \right)t^m, \]
then the monodromy operator $N_{\nabla}$ on $D_{\text{Sen},K\infty[[t]]}(W)$ reflects the stratification information. More precisely,

Theorem 1.17 (Theorem 5.17). On $D_{\text{Sen},K\infty[[t]]}(W)$, we have that
\[ N_{\nabla}(\xi) = -\varepsilon \lambda_1 u \left( \sum_{m=0}^{\infty} A_{m,1} t^m \right). \]

Here $\lambda_1 = \frac{1}{\varepsilon(\xi)} \in K[[t]]^\times$.

Based on this theorem, we see that the coefficients of $N_{\nabla}$ on $D_{\text{Sen},K\infty[[t]]}(V(M))$ need to be in $K[[t]]$, not just in $K\infty[[t]]$. Motivated by this observation, we make the following definition:

Definition 1.18. $W \in \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ is called strong nearly de Rham if there exists a $K[[t]]$-module $M$ inside $D_{\text{Sen},K\infty[[t]]}(W)$ such that the following holds:
- $M$ is stable under $N_{\nabla}$;
- $M \otimes_{K[[t]]} K\infty[[t]] = D_{\text{Sen},K\infty[[t]]}(W)$.

We use $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ to denote the full subcategory of $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ consisting of such $W$.

Remark 1.19. Such $M$ is unique if exists. Actually, for $W \in \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$, if both $M_1$ and $M_2$ satisfies the desired property, then without loss of generality, we can assume $M_1 \subseteq M_2$, to show the inclusion $M_1 \hookrightarrow M_2$ is an equality, it suffices to check that the monodromy operator $N_{\nabla}$ on $D_{\text{Sen},K\infty[[t]]}(W)$ reflects the stratification information. More precisely, $N_{\nabla}(\xi) = -\varepsilon \lambda_1 u \left( \sum_{m=0}^{\infty} A_{m,1} t^m \right)$.

Then we have the following result:

Proposition 1.20 (Proposition 5.19). Consider the composition of the de Rham realization functor and the restriction functor, which we still denoted as $V$ by abuse of notation, then we get a fully faithful functor
\[ V : \text{Vect}((\mathcal{O}_K)_{\Delta^1},(\mathcal{O}_K[\xi])_{\Delta^2}) \to \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K). \]

Moreover, assume that Conjecture 1.12 holds, then $V$ is essentially surjective, in particular, it induces an equivalence of categories between $\text{Vect}((\mathcal{O}_K)_{\Delta^1},(\mathcal{O}_K[\xi])_{\Delta^2})$ and $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$.

Remark 1.21. It is natural to ask whether the artificial condition in the definition of $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ can be removed. In other words, is the natural embedding $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K) \hookrightarrow \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ essentially surjective? It is unclear to the author whether this should be true.

On the other hand, in the final stage of our project, we learned that Gao-Min-Wang have also initiated a similar project and in the forthcoming paper they can actually prove a similar version of Conjecture 1.12, in other words, they can give a sufficient condition to make $\varepsilon$ serve as a stratification of a de Rham crystal. Moreover, based on such sufficient condition, they could prove the stronger result that $V : \text{Vect}((\mathcal{O}_K)_{\Delta^1},(\mathcal{O}_K[\xi])_{\Delta^2}) \to \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ is essentially surjective, which essentially implies that the natural embedding $\text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K) \hookrightarrow \text{Rep}_{B^+_{dR}}^{sp,ndR}(G_K)$ should be essentially surjective.
Outline. The paper is organized as follows.

In section 2, after introducing stratification of de Rham crystals over \((\mathcal{O}_K)_{\hat{}}\), we give some necessary conditions on a sequence of matrices \(\{A_{m,n}\}\) controlling such a stratification and we give some evidence of the conjecture that such conditions are also sufficient.

In section 3, we study absolute prismatic cohomology of de Rham crystals over \((\mathcal{O}_K)_{\hat{}}\). In particular, we show such cohomology is concentrated on degree 0 and 1.

In section 4, we introduce locally analytic vectors and use this theory to study \(\text{Rep}_{G_K}(B_{\text{dR}}^+)\). In particular, we prove a Sen style decomposition theorem for de Rham representations of \(G_K\). As a consequence, we construct a decomposition functor \(D_{\text{Sen,}\mathcal{O}_K[[t]]}^+\) from \(\text{Rep}_{G_K}(B_{\text{dR}}^+)\) to \(\text{Mod}^{\text{NC}}_{\mathcal{O}_K[[t]]}\) which loses no information on detecting isomorphism classes.

In section 5, we apply the decomposition theory developed in section 4 to study de Rham representations arising from de Rham prismatic crystals. We show that \(D_{\text{Sen,}\mathcal{O}_K[[t]]}^+(V(\mathcal{M}))\) can be written explicitly. As a consequence, we prove that the restriction functor \(V\) in Theorem 1.9 is fully faithful. Moreover, we show that we can extract stratification information of a de Rham crystal from the monodromy operator \(N_{\mathcal{V}}\) after applying the decomposition functor \(D_{\text{Sen,}\mathcal{O}_K[[t]]}^+\) to its associated de Rham representations. As a consequence, we "narrow" the essential image of \(V\) to what so called and argue that those strong nearly de Rham representations are precisely the image of \(V\) provided Conjecture 1.12 is true.

Notations and conventions In this paper \(\mathcal{O}_K\) is a complete discrete valuation ring of mixed characteristic with fraction field \(K\) and perfect residue field \(k\) of characteristic \(p > 2\). Fix a uniformizer \(\pi\) of \(\mathcal{O}_K\). Fix an algebraic closure \(\overline{K}\) of \(K\), whose \(p\)-adic completion is denoted as \(\mathbb{C}_p\). \(\{\pi_n\}\) is a chosen sequence of \(p\)-power roots of \(\pi\) in \(\overline{K}\) such that \(\pi_0 = \pi, \pi_{n+1}^p = \pi_n\) while \(\{\xi_n\}\) is a chosen sequence of \(p\)-power roots of unity such that \(\xi_0 = 1, \xi_{n+1}^p = \xi_n\). Let \(K_{\infty} = \bigcup_{n=1}^{\infty} K(\pi_n)\) and \(K_{p\infty} = \bigcup_{n=1}^{\infty} K(\xi_n)\). Finally, we use \(L = \bigcup_{n=1}^{\infty} K(\pi_n, \xi_n)\) to denote the union of \(K_{\infty}\) and \(K_{p\infty}\). \(L\) is the \(p\)-adic completion of \(L\) with ring of integers \(\mathcal{O}_L\).

\[A_{\inf} = W((\lim_{\leftarrow\to} x \rightarrow x^p \mathcal{O}_C/p) / \theta : A_{\inf} \to \mathcal{O}_C). \quad B_{\text{dR}}^+ = (A_{\inf}(\frac{1}{\pi}))^\hat{)}\] where \(I = \ker\theta\). Similarly we define \(A_L = A_{\inf}(\mathcal{O}_L) = W((\lim_{\leftarrow\to} x \rightarrow x^p \mathcal{O}_L/p)\) and \(B_{\text{dR},L}^+ = (A_L(\frac{1}{\pi}))^\hat{)}\). Denote \(\pi^h\) to be \((\pi_0, \pi_1, \cdots) \in \mathcal{O}_L^\hat{)} = \lim_{\leftarrow\to} x \rightarrow x^p \mathcal{O}_L/p\) and \(\varepsilon\) to be \((1, \xi_1, \xi_2, \cdots) \in \mathcal{O}_L^\hat{)}\). \(\xi = \frac{[\pi]}{[\pi^h]}\) is a generator of \(\ker\theta\). \(A_{\text{cris}}\) is the usual crystalline ring, i.e. the \(p\)-completion of the divided power envelope of \(A_{\inf}\) with respect to \(\ker\theta\). \(B_{\text{cris}}^+ = A_{\text{cris}}(\frac{1}{\pi})^\hat{)}\).

In this paper we will embed the Breuil-Kisin prism \((G, E) = (W(k)[[u]], E(u))\) into \(A_{\inf, \ker\theta}\) by sending \(u\) to \([\pi^h]\). Let \(t = E([\pi^h])\), where \(E(u)\) is the Eisenstein polynomial of our chosen uniformizer \(\pi\). \(\beta = E'(\pi)\) and \(\lambda = \prod_{n \geq 0} \varphi^n(E(u)) \in B_{\text{cris}}\).

Next, we fix some Galois groups. \(G_K = \text{Gal}(\overline{K}/K)\) \(G_{K_{p\infty}} = \text{Gal}(\overline{K}/K_{p\infty})\) \(G_L = \text{Gal}(\overline{K}/L)\) while \(\Gamma_K = \text{Gal}(K_{p\infty}/K)\) and \(\hat{G} = \text{Gal}(L/K) \cong \text{Gal}(L/K_{p\infty}) \times \Gamma_K \cong \mathbb{Z}_p\tau \times \Gamma_K\), here \(\tau = \text{Gal}(L/K_{p\infty})\) is a topological generator such that \(\tau^p = \xi_p^{p\pi} \tau\pi\). Moreover, fix \(\gamma\) to be a topological generator of \(\Gamma_K\). Then \(\gamma^\tau = \gamma^\tau^{-1} = \tau\chi(\gamma)\), here \(\chi\) is the \(p\)-adic cyclotomic character. For \(g \in \hat{G}\), \(c(g) \in \mathbb{Z}_p\) is defined such that \(g(\pi^h) = \varepsilon^{c(g)\pi}\).

\(K\{X_1\}_{pd}^\hat{)}\) is the \(p\)-completion of the free \(pd\) polynomial with one variable \(X_1\) over \(K\). \(X_1^{[n]} = \frac{X_1^p}{n!}\) is the \(n\)-th divided power of \(X_1\). \(A\{X_1\}_{\delta}\) means the free \(\delta\)-algebra over \(A\) with one variable \(X_1\). \(A(T)\) is the \(p\)-completion of the polynomial ring \(A[T]\).

\((\mathcal{O}_K)_{\hat{}}\) is the absolute prismatic site of \(\mathcal{O}_K\) while \((\mathcal{O}_K)_{\hat{}}^{\text{perf}}\) is the site of perfect prisms over \(\mathcal{O}_K\).

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2. de Rham prismatic crystals

2.1. Stratification of de Rham prismatic crystals. Let us restate the definition of de Rham prismatic crystals first.

Definition 2.1. A de Rham prismatic crystal on $(\mathcal{O}_{\Delta})$ is a sheaf $\mathcal{M}$ of $(\mathcal{O}_{\Delta}[\frac{1}{p}])^{\flat}$-modules such that for any $(A,I) \in (\mathcal{O}_{\Delta}), \mathcal{M}(A,I)$ is a finite projective $(\mathcal{A}[\frac{1}{p}])^{\flat}$-module and that for any morphism $(A,I) \to (B,J)$ in $(\mathcal{O}_{\Delta})$ (hence $J = IB$), the "crystal" property is satisfied, i.e. there is a canonical isomorphism

$$\mathcal{M}(A,I) \otimes_{(\mathcal{A}[\frac{1}{p}])^{\flat}} (B[\frac{1}{p}])^{\flat} \xrightarrow{\sim} \mathcal{M}(B,J).$$

We denote by $\text{Vect}((\mathcal{O}_{\Delta}), (\mathcal{O}_{\Delta}[\frac{1}{p}])^{\flat})$ the category of de Rham prismatic crystals.

Later given $(A,I) \in (\mathcal{O}_{\Delta})$, we will use $\mathbb{B}_{\text{dR}}^{+}(A)$ to denote $(A[\frac{1}{p}])^{\flat}$ when $I$ is clear in the context. Given $\mathcal{M} \in \text{Vect}((\mathcal{O}_{\Delta}), (\mathcal{O}_{\Delta}[\frac{1}{p}])^{\flat})$, we can specialize it to get a rational Hodge-Tate crystal:

Corollary 2.2. Define $\mathcal{M} := \mathcal{M}/I$, then $\mathcal{M}$ is a rational Hodge-Tate crystal, i.e. $\mathcal{M} \in \text{Vect}((\mathcal{O}_{\Delta}), \mathcal{O}_{\Delta}[\frac{1}{p}])$, where the later is essentially defined in [MW21, Definition 3.8]. Moreover, for any $(A,I) \in (\mathcal{O}_{\Delta}), \mathcal{M}(A,I) = \mathcal{M}(A,I)/I$.

Proof. We claim that the presheaf associating each $(A,I) \in (\mathcal{O}_{\Delta})$ to $\mathcal{M}(A,I)/I$ is a sheaf. Let $(A,I) \to (B,IB)$ be a cover in $(\mathcal{O}_{\Delta})$ and $(C,IC)$ be the self product of $(B,J)$ over $(A,I)$ in $(\mathcal{O}_{\Delta})$ with two structure morphisms $p_{1}, p_{2} : (B,IB) \to (C,JC)$. As $\mathcal{O}_{\Delta}[\frac{1}{p}] \in \text{Vect}((\mathcal{O}_{\Delta}), \mathcal{O}_{\Delta}[\frac{1}{p}])$ and $\mathcal{M}(A,I)/I$ is a finite projective $(\mathcal{A}/I)[\frac{1}{p}]$-module, we have a short exact sequence

$$0 \to \mathcal{M}(A,I)/I \to (B/IB)[\frac{1}{p}] \otimes (\mathcal{A}/I)[\frac{1}{p}] (\mathcal{M}(A,I)/I) \xrightarrow{p_{1} \otimes 1 - p_{2} \otimes 1} (C/IC)[\frac{1}{p}] \otimes (\mathcal{A}/I)[\frac{1}{p}] (\mathcal{M}(A,I)/I).$$

By the crystal property of $\mathcal{M}$, this can be identified with

$$0 \to \mathcal{M}(A,I)/I \to \mathcal{M}(B,IB)/IB \to \mathcal{M}(C,IC)/IC.$$

Hence we are done. \[\square\]

Remark 2.3. A similar argument shows that $\mathcal{M}/I^{n} \in \text{Vect}((\mathcal{O}_{\Delta}), \mathcal{O}_{\Delta}/I^{n}[\frac{1}{p}])$ and that for any $(A,I) \in (\mathcal{O}_{\Delta})$, $\mathcal{M}/I^{n}(A,I) = \mathcal{M}(A,I)/I^{n}$.

Corollary 2.4. The category of de Rham prismatic crystals is equivalent to the category of finite free $\mathbb{B}_{\text{dR}}^{+}(\mathcal{G})$-modules $M$ equipped with a stratification satisfying cocycle condition, i.e a $\mathbb{B}_{\text{dR}}^{+}(\mathcal{G})$-linear isomorphism

$$\varepsilon : M \otimes_{\mathbb{B}_{\text{an}}^{+}(\mathcal{G})} \delta_{1}^{1} \mathbb{B}_{\text{dR}}^{+}(\mathcal{G}^{1}) \xrightarrow{\sim} M \otimes_{\mathbb{B}_{\text{an}}^{+}(\mathcal{G})} \delta_{1}^{1} \mathbb{B}_{\text{dR}}^{+}(\mathcal{G}^{1}),$$

where $\delta_{1}^{1}$ is defined in Lemma 2.10 such that the cocycle condition is satisfied:

$$\delta_{3}^{2} \ast (\varepsilon) = \delta_{2}^{1} \ast (\varepsilon) \circ \delta_{0}^{2} \ast (\varepsilon).$$

Proof. Clearly given a de Rham prismatic crystal $\mathcal{M}$ we can get a finite free $\mathbb{B}_{\text{dR}}^{+}(\mathcal{G})$-module by evaluating $\mathcal{M}$ at the Breuil-Kisin prism $(\mathcal{G},(E))$. Moreover, the desired stratification $\varepsilon$ follows by the crystal property of $\mathcal{M}$.

On the other hand, starting with such a pair $(\mathcal{M}, \varepsilon)$, we can construct a de Rham crystal as follows. For any $(A,I) \in (\mathcal{O}_{\Delta})$, as $(\mathcal{G},(E))$ is a weakly final object in $(\mathcal{O}_{\Delta})$, there exists $(B,J) \in (\mathcal{O}_{\Delta})$ covering $(A,I)$ (i.e. $A \to B$ is $p$-completely faithfully flat) which also lies over $(\mathcal{G},(E))$. For example, one can take $(B,J)$ to be $(A,I) \times (\mathcal{O}_{\Delta}) (\mathcal{G},(E))$, which satisfies the desired property thanks to [BS19, Corollary 3.14]. Consequently we can define

$$\mathcal{M}(A,I) = \text{Eq}(M \otimes_{\mathbb{B}_{\text{an}}^{+}(\mathcal{G})} \mathbb{B}_{\text{dR}}^{+}(B) \Rightarrow M \otimes_{\mathbb{B}_{\text{an}}^{+}(\mathcal{G})} \mathbb{B}_{\text{dR}}^{+}(\tilde{B})),$$

where $(\tilde{B}, \tilde{J}) := (B,J) \times (A,I) (\tilde{B}, \tilde{J})$ is the self product of $(B,J)$ over $(A,I)$. Here to give the two arrows in the diagram we implicitly use the universal property of $\mathcal{G}^{1}$ to base change $\varepsilon$ along $\mathcal{G}^{1} \to \tilde{B}$. But by the proof
of [BS21, Proposition 2.7], de Rham crystals satisfy \((p, l)\)-completely faithfully flat descent, hence \(\mathcal{M}(A, I)\) is a finite projective \(\mathbb{B}_{\text{dR}}^+(A)\)-module, we are done.

Let \(\mathcal{G}^n\) be the \(n + 1\)-th self product of the Breuil-Kisin prism \((\mathcal{G}, E) = (W(k)[[u]], E(u))\), then essentially using [BS19, Corollary 3.14], we have that

\[
\mathcal{G}^n = W(k)[[u_0, \ldots, u_n]](\frac{u_0 - u_1}{E(u_0)}, \ldots, \frac{u_0 - u_n}{E(u_0)})^\delta_{(p, E(u_0))}.
\]

Later we will view \(\mathcal{G}^n\) as an \(\mathcal{G}\)-algebra via \(p_0 : \mathcal{G} \to \mathcal{G}^n\) defined by sending \(u\) to \(u_0\).

First we would like to give a concrete description of \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\).

**Lemma 2.5.** we can identify \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\) with \(K\{X_1, \ldots, X_n\}_\text{pd}[|t|]\) by identifying \(t\) with the uniformizer \(E(u_0)\) in the \(E(u_0)\)-adic complete ring \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\) and identifying \(X_i\) with \(\frac{u_i - u_0}{E(u_0)}\) in \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\).

**Proof.** Recall \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/(E) = \mathcal{G}^n[1/p]/(E)\), hence we can identify the former with \(K\{X_1, \ldots, X_n\}_\text{pd}\) thanks to [MW21, Lemma 2.7] or [Tian21, Corollary 4.5], where \(X_i\) corresponds to the image of \(\frac{u_i - u_0}{E(u_0)} \in \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E\).

By [SP22, Tag 0ALJ], \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n), (E))\) is a henselian pair, hence we can embed \(K\) canonically into \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\).

We then construct a ring homomorphism \(f : K[X_1, \ldots, X_n] \to \mathbb{B}_{\text{dR}}^+(\mathcal{G})\) by sending \(X_i\) to \(\frac{u_i - u_0}{E(u_0)} \in \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\). We then construct a ring homomorphism \(K[X_1, \ldots, X_n]_\text{pd} \to \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\), by writing \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\) as \(\lim E_{\text{dR}}^+(\mathcal{G}^n)/E^j\) it suffices to show that the induced ring homomorphism \(f_j : K[X_1, \ldots, X_n] \to \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E^j\) can be extended to \(\tilde{f}_j : K[X_1, \ldots, X_n]_\text{pd} \to \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E^j\) for each \(j\). In other words, we need to check that the image of \(\frac{X_i}{t^j}\) in \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E^j\) is bounded for each \(j\).

First notice \(f_1(\frac{X_i}{t^j})\) actually lies in \(\mathcal{G}^n/E\) by the proof of [MW21, Lemma 2.7], hence for a general \(j\), \(f_j(\frac{X_i}{t^j}) = z_{i,t} + Ew_{i,t}\) for some \(z_{i,t}, w_{i,t} \in \mathcal{G}^n, w_i,t \in \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\), this implies that in \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E^j\) we have that

\[
f_j(\frac{X_i}{t^j})^k = (z_{i,t} + Ew_{i,t})^k = z_{i,t}^k + \binom{k}{1} z_{i,t}^{k-1} Ew_{i,t} + \cdots + \binom{k}{j-1} z_{i,t}^{k-j} (Ew_{i,t})^{j-1}
\]

for arbitrary \(k \in \mathbb{N}\), hence \(f_j(\frac{X_i}{t^j})^k \in (\mathcal{G}^n + w_{i,t}\mathcal{G}^n + \cdots + w_{i,t}^{j-1}\mathcal{G}^n)\) is bounded in \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/E^j\) for any \(k \geq 0\).

Now we have extended \(f\) to a ring homomorphism \(K\{X_1, \ldots, X_n\}_\text{pd} \to \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\), then after choosing the uniformizer \(E(u_0)\) in \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\), we get a ring homomorphism \(K\{X_1, \ldots, X_n\}_\text{pd}[|t|] \to \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\) by sending \(t\) to \(E(u_0)\), which is actually an isomorphism by using derived Nakayama and checking that after modulo \(t\).

**Remar 2.6.** This construction is compatible with the identification \(\mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)/(E) \cong K\{X_1, \ldots, X_n\}_\text{pd}\) after modulo \(E\).

**Example 2.7.** Let us calculate the image of \(E(u_0) / E(u_0)^n \in \mathbb{B}_{\text{dR}}^+(\mathcal{G}^n)\) under this identification. Recall that

\[
E(u_1) = E(u_0) + \sum_{n=1}^{c} \frac{E^n(u_0)}{n!} (u_1 - u_0)^n,
\]

hence

\[
\frac{E(u_1)}{E(u_0)} = 1 + \sum_{n=1}^{c} \frac{E^n(u_0)}{n!} (u_1 - u_0)^n (E(u_0))^{n-1}
\]

(2.8)

\[
= 1 + \sum_{n=1}^{c} \frac{E^n(u_0)}{n!} (-1)^{n-1} (u_0 - u_1)^n (E(u_0))^{n-1}.
\]

Let \(E^n(u_0) = \sum_{i=0}^{\infty} \theta_{n,i} t^i \in \mathbb{B}_{\text{dR}}^+ \cong K[[t]]\), here \(\theta_{n,i} \in K\). Then we have that \(\theta_{n,0} = E^n(\pi)\). We will use \(\beta\) to denote \(\theta_{1,0} = E^\beta(\pi)\).
Then Eq. (2.8) turns into
\[
\frac{E(u_1)}{E(u_0)} = 1 + \sum_{n=1}^{\infty} \left( \sum_{i=0}^{e} \theta_{n,i} t^i \right) \frac{(-1)^n X_1^n t^{n-1}}{n!} \\
= 1 + \sum_{i=0}^{\infty} t^i \left( \sum_{n=1}^{\infty} \theta_{n,i-(n-1)} \frac{(-1)^n X_1^n}{n!} \right) \\
= 1 - \beta X_1 + \sum_{i=1}^{\infty} t^i \left( \sum_{n=1}^{\infty} \theta_{n,i-(n-1)} \frac{(-1)^n X_1^n}{n!} \right).
\]

We will denote $\alpha$ to be $\frac{E(u_1)}{E(u_0)}$ for later use.

**Remark 2.9.** By definition, $\theta_{n,0} = E^{(n)}(\pi)$ is non zero as $E(x)$ is the minimal polynomial of $\pi$ over $W(k)$ and the degree of $E^{(n)}(x)$ is strictly smaller than that of $E(x)$.

**Lemma 2.10.** For any $0 \leq i \leq n + 1$, let $\delta_i^{n+1} : \mathbb{B}^+_{\text{dr}}(\mathfrak{S}^n) \to \mathbb{B}^+_{\text{dr}}(\mathfrak{S}^{n+1})$ be the structure morphism induced by the order-preserving map 
\[ \{0, \ldots, n\} \to \{0, \ldots, i-1, i+1 \ldots, n+1\}. \]
Then by identifying $\mathbb{B}^+_{\text{dr}}(\mathfrak{S}^n)$ with $K\{X_1, \ldots, X_n\}_{\text{pd}}[[t]]$ under Lemma 2.5, we have that
\[
\delta_i^{n+1}(X_j) = \begin{cases} 
(X_{j+1} - X_1)\alpha^{-1}, & i = 0 \\
X_j, & j < i \\
X_{j+1}, & 0 < i \leq j 
\end{cases}
\]
\[
\delta_i^{n+1}(t) = \begin{cases} 
\alpha t, & i = 0 \\
t, & i > 0 
\end{cases}
\]

**Proof.** For the image of $t$ under $\delta_i^{n+1}$, the result follows from the definition of $\delta_i^{n+1}$ and Example 2.7. Also,
\[
\delta_i^{n+1}(X_j) = \delta_i^{n+1}(\frac{u_0 - u_j}{E(u_0)}) = \frac{u_i - u_{j+1}}{E(u_1)} = \frac{(u_0 - u_{j+1}) - (u_0 - u_1)}{E(u_1)} = (X_{j+1} - X_1)\alpha^{-1}.
\]
For $0 < i \leq j$, $\delta_i^{n+1}$ preserves $u_0$ and sends $u_j$ to $u_{j+1}$, hence $\delta_i^{n+1}(X_j) = X_{j+1}$. For $i > j$, $\delta_i^{n+1}$ preserves both $u_0$ and $u_j$, hence it also preserves $X_j$. \hfill \Box

Let $M$ be a finite free $\mathbb{B}^+_{\text{dr}}(\mathfrak{S})$-module endowed with a stratification $\varepsilon$ with respect to the cover $\mathfrak{S}^\bullet$. Fix a $\mathbb{B}^+_{\text{dr}}(\mathfrak{S})$ basis $e_1, \ldots, e_t$ of $M$ and we write
\[
\varepsilon(\varepsilon) = \varepsilon \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n} X_1^n \right) t^m,
\]
where $A_{m,n} \in M_{f}(K)$, $A_{0,0} = I \lim_{n \to \infty} A_{m,n} = 0$ for any $m$. Here we have identified $\mathbb{B}^+_{\text{dr}}(\mathfrak{S})$ (resp. $\mathbb{B}^+_{\text{dr}}(\mathfrak{S}^1)$) with $K[[T]]$ (resp. $K\{X_1\}_{\text{pd}}[[T]]$).

To calculate the cocycle condition, notice that
\[
\delta_1^{2*}(\varepsilon)(\varepsilon) = \varepsilon \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n} X_2^n \right) t^m,
\]
and by Lemma 2.10, we have that
\[
\delta_1^{2*}(\varepsilon)(\varepsilon) = \delta_2^{2*}(\varepsilon)(\varepsilon) = \varepsilon \sum_{p,q \geq 0} A_{p,q}(X_2 - X_1)^q \alpha^{-q} \alpha^p t^p
\]
\[
\varepsilon \left( \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n} X_1^n t^m \right) \left( \sum_{p,q \geq 0} A_{p,q}(X_2 - X_1)^q \alpha^{-q} \alpha^p t^p \right) \right).
\]
For $p \in \mathbb{Z}$, write
\[
\alpha^p = \sum_{s=0}^{\infty} c_{p,s} t^s.
\]
where \( c_{p,s} = c_{p,s}(X_1) \in K\{X_1\}_{pd}^\infty \). In particular, \( c_{p,0} = (1 - \beta X_1)^p \).

Then Eq. (2.15) turns into

\[
\delta_2^2(\varepsilon) \circ \delta_0^2(\varepsilon) = \varepsilon \left( \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n} X_1^{[n]} \right) t^m \right) \left( \sum_{p,q \geq 0} A_{p,q} (X_2 - X_1)^{[q]} t^p \sum_{s=0}^\infty c_{p-q,s} t^s \right)
\]

(2.16)

Then by comparing the coefficient of \( t^m \) in Eq. (2.14) and Eq. (2.16), we see that the cocycle condition is equivalent to the following:

\[
\sum_{n \geq 0} A_{m,n} X_2^{[n]} = \sum_{i+j=m} \left( \sum_{s \geq 0} A_{i,s} X_1^{[s]} \right) \left( \sum_{p=q} A_{p,q} (X_2 - X_1)^{[q]} c_{p-q,i-p} \right)
\]

\[
= \sum_{k=0}^{\infty} X_2^{[k]} \sum_{i+j=m} \left( \sum_{s \geq 0} A_{i,s} X_1^{[s]} \right) \left( \sum_{p=0}^\infty A_{p,k+v} (-1)^v X_1^{[v]} c_{p-(k+v),i-p} \right)
\]

Further compare the coefficient of \( X_2^{[n]} \), the cocycle relation can be reinterpreted as

(2.17)

\[
A_{m,k} = \sum_{i+j=m} \left( \sum_{s \geq 0} A_{i,s} X_1^{[s]} \right) \left( \sum_{p=0}^\infty A_{p,k+v} (-1)^v X_1^{[v]} c_{p-(k+v),i-p} \right).
\]

Now we can state our main result in this section.

**Theorem 2.18.** Keep notations as above. Then

- If \( (M, \varepsilon) \) is induced from a de Rham prismatic crystal \( \mathcal{M} \in \text{Vect}((\mathcal{O}_K)^{(1)})_{\Delta} \), then the following holds:
  - \( A_{0,0} = 1 \) and \( A_{i,0} = 0 \) for \( i > 0 \).
  - \( A_{0,1} \in M_1(K) \) satisfies that \( \lim_{n \to +\infty} \prod_{i=0}^{n-1} (iE'(\pi) + A_{0,1}) = 0 \).
  - \( A_{m,m+1} = (\beta(n-m) + A_{0,1}) A_{m,n} + \sum_{i+j=m} (A_{i,j+1} + (n-i)\theta_{i,j}) A_{i,n} \) for \( m, n \in \mathbb{N}_{\geq 0} \). In particular, \( \{A_{m,n}\} \) is determined by \( \{A_{m,1}\}_{m \geq 0} \).

**Proof.** Recall by Eq. (2.17), the cocycle condition is equivalent to the following:

(2.19)

\[
A_{m,k} = \sum_{i+j=m} \left( \sum_{s \geq 0} A_{i,s} X_1^{[s]} \right) \left( \sum_{p=0}^\infty A_{p,k+v} (-1)^v X_1^{[v]} c_{p-(k+v),i-p} \right).
\]

The idea is to compare the coefficients of \( X_1^{[0]} \) and \( X_1^{[1]} \) on both sides. Write

\[
c_{p,s} = \sum_{i=0}^s d_{p,s,i} X_1^{[i]},
\]

where \( d_{p,s,i} \in K \). An easy observation is that \( d_{m,0,0} = 1, d_{p,0,0} = 0 \) for \( m \in \mathbb{Z} \) and \( i > 0 \).

First consider the constant term in Eq. (2.19), then we get

(2.20)

\[
A_{m,k} = \sum_{i+j=m} A_{i,0} \left( \sum_{p=0}^i A_{p,k} d_{p-k,i-p,0} \right).
\]

Take \( m = 1, k = 0 \), this implies that \( A_{1,0} = 0 \) as \( d_{0,1,0} = 0 \) and \( A_{0,0} = I \) by assumption.
To show $A_{m,0} = 0$ for general $m > 0$, we proceed by induction on $m$. Suppose $A_{k,0} = 0$ for all positive integers no larger than $m - 1$ ($m \geq 2$). Then Eq. (2.20) turns into
\[ A_{m,0} = d_{m,0,0}A_{m,0} + A_{m,0}, \]
from which we see $A_{m,0}$ also vanishes. Hence we finish the proof of the first part in Theorem 2.18.

Next we compare the coefficients of $X_1$ on left and right side of Eq. (2.19):
\[
0 = A_{0,0}(\sum_{p=0}^{m}(A_{p,k+1} \cdot (-1) \cdot d_{p-k-1,m-p,p} + A_{p,k}d_{p-k,m-p,1})) + \sum_{i+j=m}(A_{i,1}(\sum_{p=0}^{i+j-1} d_{p-k,i-p,0}A_{p,k})) \\
= -A_{m,k+1} + (k - m)\beta A_{m,k} + \sum_{p=0}^{m-1}(k - p)\theta_{1,m-p}A_{p,k} + \sum_{i+j=m}A_{i,1}A_{i,k}.
\]
Here the first identity follows from the vanishing of $A_{r,0}$ for $r > 0$, the second equality is due to the vanishing of $d_{r,i,0}$ for $i > 0$ and Lemma 2.22.

As a result,
\[
A_{m,k+1} = (\beta(k - m) + A_{0,1})A_{m,k} + \sum_{\substack{i+j=m \\ i \leq m-1}}(A_{j,1} + (k - i)\theta_{1,j})A_{i,k}.
\]
Take $m = 0$, this implies
\[
A_{0,n+1} = (\beta n + A_{0,1})A_{0,n}.
\]
Since $\lim_{n \to \infty} A_{0,n} = 0$, $A_{0,1} \in M_l(K)$ satisfies that $\lim_{n \to \infty} \prod_{i=0}^{n}(i\beta + A_{0,1}) = 0$. We finish the proof of Theorem 2.18. \hfill $\square$

The following lemma is used in the proof:

**Lemma 2.22.** $d_{p,s,1} = -p\theta_{1,s}$ for $p \in \mathbb{Z}$, $s \geq 0$.

**Proof.** When $s = 0$, by definition, $c_{p,0} = (1 - \beta X_1)^p$, hence $d_{p,0,1} = -p\beta = -p\theta_{1,1}$. For $s \geq 1$, we first assume $p \geq 1$, then by definition, $d_{p,s,1}$ is the coefficient of $X_1$ in
\[
c_{p,s} = \sum_{(g_1,\ldots,g_{p}) \in \mathbb{N}^{p}} \prod_{i=1}^{p}a_{g_i},
\]
here $a_i = \sum_{n=1}^{\infty}\theta_{n,i-(n-1)}(-1)^{n}X_i^n$.

Then observe that for $g_1, g_2 \geq 1$, $X_1^2$ is a factor of $a_{g_1}a_{g_2}$, hence such $\prod_{i=1}^{p}a_{g_i}$ does not contribute to the coefficient of $X_1$ in $c_{p,s}$, then the desired result follows as the coefficient of $X_1$ in $a_s$ is precisely $\theta_{1,s}$. For $p \leq 1$, a similar argument leads to the result. \hfill $\square$

**Remark 2.23.**
- Given a de Rham prismatic crystal $M \in \text{Vect}((\mathcal{O}_K)_{\overline{A}},(\mathcal{O}_{\Delta}^\beta)_{\overline{A}})$, then $M/I$ defines a Hodge Tate crystal in the sense of [MW21, Definition 3.1] by Corollary 2.2, then one can see our work is compatible with Min and Wang’s results in the sense that if we let $m$ be 0, then the equation $A_{m,n+1} = (\beta(n-m)+A_{0,1})A_{m,n}+\sum_{i+j=m}A_{i+1,j+1}\theta_{1,j}A_{i,n}$ turns into $A_{0,n+1} = A_{0,n}(\beta n + A_{0,1})$, which is precisely the condition in [MW21, Theorem 3.5].
- Thanks to Theorem 2.18, by induction on $m$ and applying the inductive formula $A_{m,n+1} = (\beta(n-m)+A_{0,1})A_{m,n}+\sum_{i+j=m}A_{i+1,j+1}\theta_{1,j}A_{i,n}$ guarantees that $\lim_{n \to \infty} A_{m,n} = 0$ is satisfied automatically thanks to the convergence property that $A_{0,1}$ satisfies. In other words, there is no new restriction on $A_{1,1}, A_{2,1}, \cdots$ to make $\lim_{n \to \infty} A_{m,n} = 0$ holds for $m \geq 1$.
- One might wonder whether the three properties in the theorem are sufficient to define the stratification of a prismatic de Rham crystal. Inspired by the proof of [MW21, Lemma 3.6], we believe that the answer is positive. We have the following conjecture:

**Conjecture 2.24.** Assume that $\{A_{m,n}\}$ is a sequences of matrices in $M_l(K)$ that the following holds:
2.18

A sequence

\[ \sum_{n=0}^{\infty} A_{m,n} X^{[n]} \]

can serve as the stratification associated to a certain prismatic de Rham crystal.

Nevertheless, although we couldn’t prove Conjecture 2.24 now, we will give an explicit expression such that in the case that \( A_{0,1} \) commutes with all \( A_{j,1} \), Conjecture 2.24 can be reduced to the study of a sequence of very explicit polynomials in variables \( \{A_{0,1}, A_{1,1}\} \) showing up in the following theorem:

**Theorem 2.25.** Assume that \( \{A_{m,n}\} \) is a sequence of matrices in \( M_i(K) \) such that the following holds (for example, thanks to Theorem 2.18, this is satisfied when coming from the stratification of a prismatic de Rham crystal such that \( A_{0,1} \) commutes with all \( A_{j,1} \)):

- \( A_{0,0} = I \) and \( A_{i,0} = 0 \) for \( i > 0 \).
- \( A_{0,1} \in M_i(K) \) satisfies that \( \lim_{n \to \infty} \prod_{i=0}^{n} (iE'(\pi) + A_{0,1}) = 0 \).
- \( A_{m,n+1} = (\beta(n-m) + A_{0,1})A_{m,n} + \sum_{i+j=m} (A_{j,1} + (n-i)\theta_{1,1})A_{j,n} \) for \( m, n \in \mathbb{N}^\geq 0 \). In particular, \( \{A_{m,n}\} \) is determined by \( \{A_{m,1}\}_{m \geq 0} \).

Then

\[ \varepsilon(e) = e \cdot \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n} X^{[n]} t^m \]

can serve as the stratification associated to a certain prismatic de Rham crystal.

Moreover, \( \{h_{m,j}\} \) satisfies the following properties:

- For \( 1 \leq j \leq m \),

\[ h_{m,j} = f_{m,j} A_{m,1} + \sum_{j \leq f, r \leq m-1} (A_{r,1} - f_{1,r})(\sum_{i=1}^{f} h_{f,i} g_{m,f,i}^j + \sum_{i=f+1}^{2f} h_{f,i} g_{m,f,i}^j A_{0,i-f}) \]

\[ \quad + \sum_{j \leq f, r \leq m-1} \theta_{1,r}(\sum_{i=0}^{f} ((j-1)g_{m,f,i}^j + (1 - \frac{1}{j})g_{m,f,i}^{j-1}(-(m - (j-1))\beta + A_{0,1}))h_{f,i} \]

\[ \quad + \sum_{i=f+1}^{2f} (h_{f,i} A_{0,i-f}((j-1)g_{m,f,i}^j + (1 - \frac{1}{j})g_{m,f,i}^{j-1}(-(m - (j-1))\beta + A_{0,1}))). \]

- For \( m+1 \leq j \leq 2m \),

\[ h_{m,j} = \sum_{f+r=m \atop 1 \leq f, r \leq m-1} (A_{r,1} - f_{1,r})(\sum_{i=f+1}^{2f} h_{f,i} g_{m,f,i}^j A_{0,j-m}) \]

\[ \quad + \sum_{f+r=m \atop f \leq m-1} \theta_{1,r}(\sum_{i=f}^{2f} h_{f,i} A_{0,j-m}((j-1)g_{m,f,i}^j + (1 - \frac{1}{j})g_{m,f,i}^{j-1} A_{0,j-1-m})). \]
Here \( f_{m,1} = 1, f_{m,i} = \frac{d^{i-1}}{d} (m - (i - 1)) \) for \( 2 \leq i \leq m \) and
\[
g^j_{m,f,i} = \begin{cases} \frac{\beta^{j-1}}{j} (m - (f + 1)) \cdots (m - (f + j - i - 1)) & m \geq f + 1 \quad i + 1 \leq j \leq m - f + i \\ 0, & \text{otherwise} \end{cases}
\]

Remark 2.26. If we define
\[
\tilde{h}_{m,j} = \begin{cases} h_{m,j}, & j \leq m \\ h_{m,j}A_0_{j-m}, & j + 1 \leq m \leq m + 1 \end{cases}
\]
Then we have a much cleaner inductive formula for calculating \( \tilde{h}_{m,j} \) and Theorem 2.25 can be restated as
\[
\sum_{n=0}^{\infty} A_{m,n}X[n] = \sum_{j=1}^{m} \tilde{h}_{m,j}(1 - \beta X)^{m-j}X^j(1 - \beta X)^{-A_0_{j+1}}.
\]
Later we will freely interchange between \( \tilde{h} \) and \( h \).

In the unramified case, we have a much cleaner expression:

Corollary 2.27. Assume as in Theorem 2.25 and moreover \( e = 1 \), then
\[
\sum_{n=0}^{\infty} A_{m,n}X[n] = \sum_{j=1}^{m} h_{m,j}(1 - \beta X)^{m-j}X^j(1 - \beta X)^{-A_0_{j+1}}
\]
such that:

- \( h_{0,0} = 1 \) and for \( m \geq 1 h_{m,j} \) is a homogeneous polynomial of degree \( m \) in variables \( A_{1,1}, \ldots, A_{m,1} \), where \( A_{i,1} \) is given weighted degree \( i \).
- \( \{h_{m,j}\} \) can be calculated inductively via
\[
h_{m,j} = f_{m,j}A_{m,1} + \sum_{f+r=m} A_{r,1} \sum_{1 \leq f, r \leq m-1} h_{f,i}g^j_{m,f,i},
\]
where \( f_{m,1} = 1 \) and \( g^j_{m,f,i} \) are defined in Theorem 2.25.

Proof. In this case \( \theta_{1,r} = 0 \) for \( r > 0 \), then the result follows from induction on \( m \) using the inductive formula for \( h_{m,j} \). \( \square \)

Example 2.28. Under the assumption of Theorem 2.25,
\[
\sum_{n=0}^{\infty} A_{1,n}X[n] = (1 - \beta X)^{-A_0_{1}}(A_{1,1}X + \frac{\theta_{1,1}}{2} A_{0,1} (1 - \beta X)^{-1} X^2).
\]
\[
\sum_{n=0}^{\infty} A_{2,n}X[n] = (1 - \beta X)^{-A_0_{1}} (A_{2,1}X (1 - \beta X) + (\frac{\beta}{2} A_{2,1} + \frac{1}{2} A_{1,1}^2 + \frac{\theta_{1,2} A_{0,1}}{2})X^2 + (\frac{\theta_{1,1} A_{1,1}}{2} + \frac{\theta_{1,2}^2}{6} + \frac{\theta_{1,2}}{3}) A_{0,1} (1 - \beta X)^{-1} X^3 + \frac{\theta_{1,2}^2}{8} A_{0,2} (1 - \beta X)^{-2} X^4).
\]

2.2. Calculation in the commutative case. Our goal is to prove Theorem 2.25. In this subsection we assume that \( \{A_{m,n}\} \) is a sequence of matrices in \( M_l(K) \) satisfies assumptions in Theorem 2.25. In particular, \( A_{0,1} \) commutes with all \( A_{j,1} \).

Our strategy is to prove the following lemma:

Lemma 2.29. There exists a sequence of polynomials \( \{h_{m,j}\} \) satisfying desired properties in Theorem 2.25 such that
\[
A_{m,s} = A_{0,s-m} \sum_{j=1}^{m} (-(m - j)\beta + A_{0,1}) \cdots (-\beta + A_{0,1}) s \cdots (s - j + 1) h_{m,j} + \sum_{j=m+1}^{2m} s \cdots (s - j + 1) h_{m,j}
\]
for non negative integers \(m, s\). Here we abuse the notation by requiring that

\[
A_{0,-j}(-j\beta + A_{0,1}) \cdots (-\beta + A_{0,1}) = I
\]

for \(1 \leq j \leq m\).

**Proof of Theorem 2.25 under Lemma 2.29.** We separate the calculation of \(\sum_{s=0}^{\infty} A_{m,s}X^s\) into two parts. For the sum over \(1 \leq j \leq m\),

\[
\sum_{j=1}^{m} \sum_{s=0}^{\infty} A_{0,s-m}(- (m-j)\beta + A_{0,1}) \cdots (-\beta + A_{0,1})s(s-1) \cdots (s-j+1)X^s h_{m,j}
\]

\[
= \sum_{j=1}^{m} \sum_{s=j}^{\infty} A_{0,s-m}(- (m-j)\beta + A_{0,1}) \cdots (-\beta + A_{0,1})s(s-1) \cdots (s-j+1)X^s h_{m,j}
\]

(2.30)

\[
= \sum_{j=1}^{m} h_{m,j}^s \sum_{s=0}^{\infty} A_{0,s-m}(- (m-j)\beta + A_{0,1}) \cdots (-\beta + A_{0,1})X^s h_{m,j}
\]

\[
= \sum_{j=1}^{m} h_{m,j}^s \sum_{s=0}^{\infty} A_{0,s-m}(- (m-j)\beta + A_{0,1}) \cdots (-\beta + A_{0,1})X^s h_{m,j}
\]

here the second to last equality follows from applying Lemma 2.32, which is a key observation from [MW21] to \(A = -(m-j)\beta + A_{0,1}\) thanks to Remark 2.23.

For \(m+1 \leq j \leq 2m\) part,

\[
\sum_{j=m+1}^{2m} \sum_{s=0}^{\infty} h_{m,j} A_{0,s-m} \cdots (s-j+1)X^s = \sum_{j=m+1}^{2m} h_{m,j} A_{0,s-m} \cdots (s-j+1)X^s
\]

(2.31)

here the last equality also follows from Lemma 2.32.

Combine Eq. (2.30), Eq. (2.31), than thanks to Lemma 2.29, we have that

\[
\sum_{n=0}^{\infty} A_{m,n}X^n = \sum_{j=1}^{m} h_{m,j} (1-\beta X)^{m-j} X^j (1-\beta X)^{-\frac{A_{0,1}}{\beta}} + \sum_{j=m+1}^{2m} h_{m,j} A_{0,s-m} (1-\beta X)^{m-j} X^j (1-\beta X)^{-\frac{A_{0,1}}{\beta}}. \quad \square
\]

**Lemma 2.32.** Suppose \(A \in M_1(K)\) satisfies that \(\lim_{n \to +\infty} \prod_{i=0}^{n} (i\beta + A) = 0\). Further we define \(A_0 = I, A_{n+1} = \prod_{i=0}^{n} (i\beta + A)\) for \(n \geq 0\), then for a fixed non negative integer \(k\),

\[
\sum_{s=0}^{\infty} A_{k+s}X^s = A_k (1-\beta X)^{-\frac{1}{\beta}} - k.
\]

**Proof.** This is proven in [MW21, Lemma 3.6]. \(\square\)

Now we proceed to prove Lemma 2.29. By assumption for \(m, n \geq 1\) we have that

\[
A_{m,n+1} = (\beta(n-m) + A_{0,1})A_{m,n} + \sum_{\substack{i+j=m \\ i \leq m-1}} (A_{i,j+1} + (n-i)\theta_{1,i})A_{i,n}.
\]
Then by induction on \( n \) we see that

\[
(2.33) \quad A_{m,s} = \sum_{b=1}^{s} A_{0,s-b} \prod_{t=1}^{b-1} (\beta(s-m-t) + A_{0,1}) + \sum_{f+r=m}^{s-1} (A_{r,1} - f\theta_{1,r}) \sum_{b=1}^{s-1} A_{f,s-b} \prod_{t=1}^{b-1} (\beta(s-m-t) + A_{0,1})
\]

\[
+ \sum_{f+r=m}^{s-1} \sum_{b=1}^{s-1} A_{f,s-b} \theta_{1,r} (s-b) \prod_{t=1}^{b-1} (\beta(s-m-t) + A_{0,1}).
\]

Our strategy is to treat the three terms on the right side of Eq. (2.33) separately. First by assumption \( A_{0,t} = \prod_{i=0}^{t-1} (t\beta + A_{0,1}) \), hence that

\[
(2.34) \quad A_{m,1} \sum_{b=1}^{s} A_{0,s-b} \prod_{t=1}^{b-1} (\beta(s-m-t) + A_{0,1}) = A_{m,1} A_{0,s-m} \sum_{b=0}^{s-1} \frac{(m-1)\beta + A_{0,1} + b\beta}{} \cdot (-\beta + A_{0,1} + b\beta),
\]

here we abuse the notation in the following sense:

- When \( m = 1 \), \( \sum_{b=0}^{n-1} (-m - 1)\beta + A_{0,1} + b\beta \cdot (-\beta + A_{0,1} + b\beta) \) is defined to be \( n \).
- \( A_{0,-j}(-j\beta + A_{0,1}) \cdot (-\beta + A_{0,1}) = I \) for \( 1 \leq j \leq t \).

To see our abuse of notation makes sense, we need the following lemma:

**Lemma 2.35.** For \( m \geq 1, 1 \leq i \leq m \), there exists a constant \( f_{m,i} \) which only depends on \( m \) and \( i \) such that

\[
\sum_{c=0}^{s-1} (-m-i)\beta + A_{0,1} + c\beta \cdot (-\beta + A_{0,1} + c\beta) = \sum_{c=0}^{s-1} (-m-i)\beta + A_{0,1} \cdot (-\beta + A_{0,1}) s(s-1) \cdot \vdots \cdot (s-i+1)f_{m,i}.
\]

Moreover, \( f_{m,i} \) is given as in Corollary 2.27.

**Proof.** We prove it by induction on \( m \). When \( m = 1 \), \( f_{1,1} = 1 \). Suppose it holds for \( m \geq 1 \), then consider \( m+1 \) case:

\[
\sum_{c=0}^{s-1} (m+i)\beta + A_{0,1} \cdot (-\beta + A_{0,1}) s(s-1) \cdot \vdots \cdot (s-i+1)f_{m,i}.
\]

By induction,

\[
(-m+i)\beta + A_{0,1} \sum_{c=0}^{s-1} \prod_{i=1}^{m-i} (-t\beta + A_{0,1}) \prod_{j=0}^{i-1} (s-j)
\]

\[
= (-m+i)\beta + A_{0,1} \sum_{i=1}^{m} f_{m,i} \left( \prod_{t=1}^{m-i} (-t\beta + A_{0,1}) \prod_{j=0}^{i-1} (s-j) \right)
\]

\[
= \sum_{i=1}^{m} (-m+i-1)\beta + A_{0,1} \prod_{t=1}^{m-i} (-t\beta + A_{0,1}) \prod_{j=0}^{i-1} (s-j) f_{m,i}
\]

\[
= \sum_{i=1}^{m} (-m+i-1)\beta + A_{0,1} \prod_{t=1}^{m-i} (s-j) f_{m,i} - (i-1)\beta f_{m,i} \left( \prod_{t=1}^{m-i} (-t\beta + A_{0,1}) \prod_{j=0}^{i-1} (s-j) \right)
\]

\[
= \sum_{i=1}^{m} \prod_{t=1}^{m-i} (-t\beta + A_{0,1}) (f_{m,i} \prod_{j=0}^{i-1} (s-j) - (i-2)\beta f_{m,i-1} \prod_{j=0}^{i-2} (s-j)).
\]
By induction and Abel’s summation formula,
\[
\sum_{c=0}^{s-1} \beta \prod_{i=1}^{m-1} (-i \beta + A_{0,1} + c \beta)
= \beta(s - 1) \sum_{c=0}^{s-1} \prod_{i=1}^{m-1} (-i \beta + A_{0,1} + c \beta) - \beta \sum_{c=0}^{s-2} \prod_{i=1}^{m-1} (-i \beta + A_{0,1} + t \beta)
\]
(2.37)

Using that
\[
\sum_{c=0}^{s-1} \prod_{i=1}^{m-1} (c + 1 - j) = \frac{1}{t} \prod_{j=0}^{i-1} (s - j),
\]
and combine Eq. (2.36) and Eq. (2.37), we have that
\[
\sum_{c=0}^{s-1} \prod_{i=1}^{m-1} (-i \beta + A_{0,1} + c \beta)
= \sum_{i=1}^{m+1} \prod_{t=1}^{m+1-i} (-t \beta + A_{0,1})((f_{m,i} - \frac{\beta}{t} f_{m,i-1}) \prod_{j=0}^{i-1} (s - j) + (\beta(s - 1)f_{m,i-1} - (i - 2)\beta f_{m,i-1}) \prod_{j=0}^{i-2} (s - j))
\]

hence the lemma holds for \(m + 1\) provided that we define \(f_{m+1,i}\) to be \(f_{m,i} + \beta f_{m,i-1} - \frac{\beta}{t} f_{m,i-1}\). Finally by induction we calculate \(f_{m,i}\) explicitly:
\[
f_{m,i} = \frac{\beta^{i-1}}{i!} (m - 1) \cdots (m - (i - 1)).
\]

Now we use similar strategy to calculate the second term on the right hand side of Eq. (2.33).
For this, we need a slight generalization of Lemma 2.35.

**Lemma 2.38.** Suppose \(m \geq f + 1\), then there exist constants \(g_{m,f,i}, g_{m,f,i}^{2}, \cdots, g_{m,f,i}^{m-f+i}\) such that
\[
\sum_{c=0}^{s-1} (-m - 1) \beta + A_{0,1} + c \beta) \cdots (-f + 1) \beta + A_{0,1} + c \beta) c \cdots (c - i + 1)
\]
\[
= \sum_{j=1}^{m-f+i} (-m - j) \beta + A_{0,1}) \cdots (-f - i + 1) \beta + A_{0,1}) c \cdots (s - (j - 1)) g_{m,f,i}^{i}
\]
Moreover, \(g_{m,f,i}^{i}\) is given as in Theorem 2.25.

**Proof.** We prove it by induction on \(m\). When \(m = f + 1\),
\[
\text{LHS} = \sum_{c=0}^{s-1} c \cdots (c - i + 1) = \frac{1}{i+1} s \cdots (s - i),
\]
hence we can take \(g_{f+1,f,i}^{i} = \cdots g_{f+1,f,i}^{i+1} = 0\) and \(g_{f+1,f,i}^{i+1} = \frac{1}{i+1}\) to make the identify holds.
Suppose the lemma is proven for $m \geq f + 1$, then consider $m + 1$ case:

\[
\sum_{c=0}^{s-1} (-m\beta + A_{0,1} + c\beta) \cdots (-f + 1)\beta + A_{0,1} + c\beta)c \cdots (c - i + 1)
\]

\[
= (-m\beta + A_{0,1}) \sum_{c=0}^{s-1} m_{t=f+1}^{i-1} (i - \beta + A_{0,1}, c\beta)(\prod_{u=0}^{c-u}) + \sum_{c=0}^{s-1} c\beta(i - \beta + A_{0,1}, c\beta)(\prod_{u=0}^{c-u}).
\]

By induction,

\[
(2.39)
\]

\[
(-m\beta + A_{0,1}) \sum_{c=0}^{m-1} m_{t=f+1}^{i-1} (i - \beta + A_{0,1}) (\prod_{u=0}^{c-u})
\]

\[
= (-m\beta + A_{0,1}) \sum_{j=1}^{m-f+i} (-m - j)\beta + A_{0,1}) \cdots (-f - i + 1)\beta + A_{0,1}) s \cdots (s - j - 1) g_{j}^{m+1, f, i}
\]

\[
= \sum_{j=1}^{m-f+i} (-m + 1 - j)\beta + A_{0,1}) - (j - 1)\beta g_{m+1, f, i}^{j-1} \prod_{u=0}^{j-1} (-t\beta + A_{0,1}) \prod_{u=0}^{j-1} (s - u)
\]

\[
\sum_{j=1}^{m-f+i} \prod_{u=0}^{j-1} (-t\beta + A_{0,1}) (\prod_{u=0}^{j-1} (s - u)) - (j - 1)\beta g_{m+1, f, i}^{j-1} \prod_{u=0}^{j-1} (s - u)
\]

By induction and Abel’s summation formula,

\[
(2.40)
\]

\[
\sum_{c=0}^{m-1} c\beta(i - \beta + A_{0,1}, c\beta)(\prod_{u=0}^{c-u})
\]

\[
= \beta(s - 1) \sum_{c=0}^{s-1} m_{t=f+1}^{i-1} (i - \beta + A_{0,1}, c\beta)(\prod_{u=0}^{c-u}) - \beta \sum_{c=0}^{s-1} m_{t=f+1}^{i-1} (i - \beta + A_{0,1} + p\beta)(\prod_{u=0}^{p-u})
\]

\[
= \beta(s - 1) \sum_{j=1}^{m-f+i} \prod_{u=0}^{j-1} (-t\beta + A_{0,1}) (\prod_{u=0}^{j-1} (s - u)) - \beta \sum_{c=1}^{m-f+i} \prod_{u=0}^{j-1} (-t\beta + A_{0,1}) (\prod_{u=0}^{j-1} (c - j))
\]

\[
= \sum_{j=1}^{m-f+i} \prod_{u=0}^{j-1} (-t\beta + A_{0,1}) (\prod_{u=0}^{j-1} (s - u)) - \beta \sum_{c=1}^{m-f+i} \prod_{u=0}^{j-1} (c - u).
\]

Using that

\[
\sum_{c=1}^{j-1} \prod_{u=0}^{j-1} (c - u) = \frac{1}{j} \prod_{u=0}^{j-1} (s - u),
\]
and combine Eq. (2.39) and Eq. (2.40), we have that

\[
\begin{align*}
\sum_{c=0}^{s-1} \left( \prod_{t=f+1}^{m} (-t\beta + A_{0,1} + c\beta) \right) \left( \prod_{u=0}^{i-1} (c-u) \right) \\
= \sum_{j=1}^{m+1-f+i} \left( \prod_{t=j-i+1}^{m+1-i} \left(-t\beta + A_{0,1}\right) \left( g_{m+f,i}^{j} - \frac{\beta}{j} g_{m+f,i}^{j-1} \prod_{u=0}^{j-1} (s-u) + \beta(s-1-(j-2)) g_{m+f,i}^{j-1} \prod_{u=0}^{j-2} (s-u) \right) \right) \\
= \sum_{j=1}^{m+1-f+i} \left( g_{m,f,i}^{j} - \frac{\beta}{j} g_{m,f,i}^{j-1} + \beta g_{m,f,i}^{j-1} \right) \left( \prod_{t=j-i+1}^{m+1-i} \left(-t\beta + A_{0,1}\right) \prod_{u=0}^{j-1} (s-u) \right).
\end{align*}
\]

Hence the lemma holds for \( m+1 \) provided that we define \( g_{m+1,f,i}^{j} \) to be \( g_{m,f,i}^{j-1} - \frac{\beta}{j} g_{m,f,i}^{j-1} + \beta g_{m,f,i}^{j-1} \).

Then an easy induction shows that

\[
g_{m,f,i}^{j} = \left\{ \begin{array}{ll}
\frac{\beta^{i-j-1}}{j} \frac{1}{(j-l-1)!} (m-(f+1)) \cdots (m-(f+j-i-1)), & m \geq f+1, \quad i + 1 \leq j \leq m - f + i \\
0, & \text{otherwise}
\end{array} \right.
\]

Now we are ready to prove Lemma 2.29.

**Proof of Lemma 2.29.** We proceed by induction on \( m \). When \( m = 1 \), by induction on \( s \), one can show \( A_{1,s} = sA_{1,1}A_{0,s-1} + A_{0,s-1}(s-1) \frac{\theta_{1,1}}{2} \), hence we can take \( h_{1,1} = A_{1,1} = f_{1,1}A_{1,1} \) and \( h_{1,2} = \frac{\theta_{1,1}}{2} \). Suppose it is proven for up to \( m - 1 \). Consider the \( m \) case. We treat three terms showing up on the right hand side of Eq. (2.33) separately.

**Step 1** For a fixed pair \((f,r)\) such that \( f + r = m \), \( 1 \leq f, r \leq m - 1 \),

\[
\sum_{b=0}^{s-1} A_{f,s-b} \prod_{t=1}^{b-1} (\beta(s-m+t) + A_{0,1})
= \sum_{b=0}^{s-1} A_{0,s-b} \left( \sum_{i=0}^{f} \left( \prod_{t=1}^{i-1} (-t\beta + A_{0,1}) \right) \prod_{u=0}^{i-1} (s-b-u) h_{f,i} \right) + \sum_{i=0}^{f} \left( \prod_{t=1}^{i-1} \left( s-b-u \right) \right) h_{f,i} \left( \prod_{t=1}^{r-1} \left( s-b-f \right) + A_{0,1} - t\beta \right)
= A_{0,s-m} \left( \sum_{i=0}^{f} \left( \prod_{t=1}^{i-1} (-t\beta + A_{0,1}) \right) \prod_{u=0}^{i-1} (s-c-u) \left( \prod_{t=1}^{r-1} \left( (c-f) + A_{0,1} - t\beta \right) \right) \right)
+ \sum_{i=0}^{f} h_{f,i} \left( \prod_{c=1}^{s-1} (c-u) \right) \left( \prod_{t=1}^{r-1} \left( (c-f) + A_{0,1} - t\beta \right) \right).
\]

Here the first identity follows from induction, the second equality is due to \( A_{0,t} = \prod_{i=0}^{t-1} (t\beta + A_{0,1}) \). The last equality just change \( s-b \) to \( c \).

For \( 1 \leq i \leq f \), we have that \( m - f + i \leq m \) and that

\[
\sum_{i=0}^{f} h_{f,i} \left( \prod_{c=1}^{s-1} (c-u) \right) \left( \prod_{t=1}^{r-1} \left( (c-f) + A_{0,1} - t\beta \right) \right)
= \sum_{j=1}^{m-f+i} \left( \prod_{t=1}^{m-j} (-t\beta + A_{0,1}) \right) \left( \prod_{u=0}^{j-1} (s-u) \right)
\]

(2.42)
by applying Lemma 2.38.

For $f + 1 \leq i \leq 2f$,

\[
h_{f,i}(\sum_{c=1}^{i-1} (\prod_{u=0}^{c-1} (c-u))(\prod_{t=1}^{i-1} ((c-f)\beta + A_{0,1} - t\beta)))
\]

\[
= \sum_{j=1}^{m-f+i} h_{f,i} g_{m,f,i}^j \left( \prod_{t=f+1}^{m-j} (-t\beta + A_{0,1}) \right) (\prod_{u=0}^{j-1} (s-u))
\]

\[
= \sum_{j=1}^{m} h_{f,i} g_{m,f,i}^j (A_{0,i-f} \prod_{t=1}^{m-j} (-t\beta + A_{0,1})) (\prod_{u=0}^{j-1} (s-u)) + \sum_{j=m+1}^{m-f+i} h_{f,i} g_{m,f,i}^j \frac{A_{0,i-f}}{A_{0,j-m}} (\prod_{u=0}^{j-1} (s-u)).
\]

**Step II** Finally we treat the last term in Eq. (2.33). For this, we fix a pair $(f, r)$ such that $f + r = m$ and $f \leq m - 1$. Repeating the calculation as in Eq. (2.41) leads to the following:

\[
= A_{0,s-m}(\sum_{i=0}^{f} \left( \prod_{t=1}^{i} (-t\beta + A_{0,1}) \right) h_{f,i} (\sum_{c=1}^{i-1} (c-u)(\prod_{t=1}^{i} (c-f)\beta + A_{0,1} - t\beta)))
\]

\[
= \sum_{j=1}^{s-1} A_{f,s-b}(s-b) \prod_{t=1}^{b-1} (\beta(s-m-t) + A_{0,1})
\]

\[
= A_{0,s-m}(\sum_{i=0}^{f} \left( \prod_{t=1}^{i} (-t\beta + A_{0,1}) \right) h_{f,i} (\sum_{c=1}^{i-1} (c-u)(\prod_{t=1}^{i} (c-f)\beta + A_{0,1} - t\beta)))
\]

For $0 \leq i \leq f$, arguing as in Eq. (2.42),

\[
= \sum_{j=1}^{m-f+i} h_{f,i} g_{m,f,i}^j \left( \prod_{t=f+1}^{m-j} (-t\beta + A_{0,1}) \right) (\prod_{u=0}^{j-1} (s-u)) + \sum_{j=m+1}^{m-f+i} h_{f,i} g_{m,f,i}^j (1 - \frac{1}{j}) g_{m,f,i}^{j-1}(-m-(j-1)) A_{0,1} \delta_j (\prod_{t=1}^{m-j} (-t\beta + A_{0,1})) (\prod_{u=0}^{j-1} (s-u)).
\]

Here the first identity follows from Lemma 2.38 and Abel’s summation formula, $\delta_j = 1$ if $j \leq m$ and $\delta_j = 0$ if $j = m + 1$.

For $f + 1 \leq i \leq 2f$, arguing as in Eq. (2.43),

\[
= \sum_{j=1}^{m} h_{f,i} (j-1) g_{m,f,i}^j + (1 - \frac{1}{j}) g_{m,f,i}^{j-1}(-m-(j-1)) A_{0,1} \delta_j (\prod_{t=1}^{m-j} (-t\beta + A_{0,1})) (\prod_{u=0}^{j-1} (s-u))
\]

\[
+ \sum_{j=m+1}^{m-f+i} h_{f,i} (A_{0,i-f}) (j-1) g_{m,f,i}^j + \frac{A_{0,i-f}}{A_{0,j-m}} (1 - \frac{1}{j}) g_{m,f,i}^{j-1}(-m-(j-1)) A_{0,1} \delta_j (\prod_{u=0}^{j-1} (s-u)).
\]
Step III Combine Eq. (2.33), Eq. (2.34), Lemma 2.35, Eq. (2.41), Eq. (2.42), Eq. (2.43), Eq. (2.44), Eq. (2.45) AND Eq. (2.46), we conclude that \( h_{m,j} \) \((1 \leq j \leq 2m)\) can be defined as in Theorem 2.25 to make Lemma 2.29 holds for \( m \). Hence Lemma 2.29 holds for \( m \).

\[\square\]

2.3. Evidence of the conjecture. We give some evidence of Conjecture 2.24 and explain why Conjecture 2.24 can be reduced to the study of polynomials \( \{h_{m,j}\} \) showing up in Theorem 2.25 in the case that \( A_{0,1} \) commutes with all \( A_{m,1} \). Indeed, given a sequence of matrices \( \{A_{m,n}\} \in M_k(K) \) as in Conjecture 2.24, then

\[\varepsilon(\mathbf{a}) = \mathbf{a} \cdot \left( \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n}X_1^{[n]} \right) t^m\]

defines an isomorphism as \( \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n}X_1^{[n]} \right) \) is invertible. To verify that it can be obtained from a de Rham prismatic crystal, it suffices to check that \( \varepsilon \) satisfies the cocycle condition, which is obvious if we could show the following conjecture:

**Conjecture 2.47.** Let \( a = \frac{-A_{0,1}}{2} \) and \( \hat{T} = \sum_{i=0}^{\infty} a_i t^i \in M_k(K)[[t]] \) such that \( a_0 = 1 \) and \( a_k \) is defined inductively for \( k \geq 1 \) using that

\[a_k = \frac{1}{k!} \left( \sum_{i+s=k, s \leq \lfloor k/2 \rfloor} (d_{i+a,s,1}a_i - A_{s,1})a_i \right)\]

Then we have the following identity:

\[a \hat{T}(\alpha t) = \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n}X_1^{[n]} t^m\]

Here \( \alpha^a \) is well defined due to the convergence property of \( A_{0,1} \).

Actually once Conjecture 2.47 can be verified, then recall \( \alpha \) is defined to be \( \frac{E(u_2)}{E(u_0)} \), hence

\[\delta_{1}^{\ast}(\varepsilon(\mathbf{a})) = \mathbf{a} \cdot \frac{E(u_2)}{E(u_0)} \cdot \frac{f(E(u_2))}{f(E(u_0))}\]

while

\[\delta_{2}^{\ast}(\varepsilon(\mathbf{a})) \circ \delta_{0}^{\ast}(\varepsilon(\mathbf{a})) = \mathbf{a} \cdot \left( \frac{E(u_2)}{E(u_1)} \right)^a \cdot \frac{f(E(u_2))}{f(E(u_1))}\]

\[= \mathbf{a} \cdot \left( \frac{E(u_2)}{E(u_0)} \right)^a \cdot \frac{f(E(u_2))}{f(E(u_0))} \cdot \left( \frac{E(u_2)}{E(u_1)} \right)^a \cdot \frac{f(E(u_2))}{f(E(u_1))}\]

\[= \mathbf{a} \cdot \frac{E(u_2)}{E(u_0)} \cdot \frac{f(E(u_2))}{f(E(u_0))}\]

Hence \( \delta_{2}^{\ast}(\varepsilon(\mathbf{a})) = \delta_{2}^{\ast}(\varepsilon(\mathbf{a})) \circ \delta_{0}^{\ast}(\varepsilon(\mathbf{a})) \), i.e. the cocycle condition is satisfied.

On the other hand, we claim that the proof of Eq. (2.48) is purely an algebraic problem with respect to polynomials \( \{h_{m,j}\} \) in Theorem 2.25. Actually, Eq. (2.48) is equivalent to that

\[\alpha^a \hat{T}(\alpha t) = \left( \sum_{m \geq 0} \sum_{n \geq 0} A_{m,n}X_1^{[n]} t^m \right) \hat{T}(t),\]

then we calculate that

\[\text{LHS} = \sum_{i=0}^{\infty} a_i \alpha^{i+a} t^i = \sum_{i=0}^{\infty} a_i \sum_{s=0}^{i} c_{i+a,s} t^s t^i = \sum_{k=0}^{\infty} t^k \left( \sum_{i+s=k} c_{i+a,s} a_i \right) = \sum_{k=0}^{\infty} t^k \left( \sum_{i+s=k} \sum_{n=0}^{\infty} d_{i+a,s,n} X_1^{[n]} a_i \right),\]
and that

\[ \text{RHS} = \left( \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m,n}X_1^{[n]} \right)^t \right) \hat{f}(t) \]

\[ = \sum_{k=0}^\infty t^k \left( \sum_{m+l=k} \left( \sum_{n \geq 0} A_{m,n}X_1^{[n]} \right) a_l \right). \]

Hence Conjecture 2.47 is true if and only if for any \( k \geq 0, \)

\[ \sum_{i=s}^\infty \left( \sum_{n \geq 0} d_{i+a,s,n}X_1^{[n]} \right) a_i = \sum_{m+l=k} \left( \sum_{n \geq 0} A_{m,n}X_1^{[n]} \right) a_l. \]  

But recall \( \alpha^p \) satisfies cocycle condition (see the paragraph after Conjecture 2.47) and in our notation \( \alpha^p = \sum_{s=0}^\infty c_{p,s}t^s = \sum_{s=0}^\infty (\sum_{k=0}^\infty d_{p,s,k}X_1^{[k]}) t^s, \) hence theoretically we can use Theorem 2.25 and Lemma 2.22 to calculate both sides of Eq. (2.50) and verify it provided that we understand the polynomial \( \{h_{m,j}\} \) in Theorem 2.25 very well!

We calculate the difference between left hand side and right hand side of Eq. (2.50) for small degrees using \( d_{p,s,1} = -p\theta_{1,s} \) thanks to Lemma 2.22 (We implicitly extend this notation to allow \( p = a = -\frac{A_{0,1}}{\beta} \) thanks to convergence property of \( A_{0,1} \)).

When \( k = 0, \)

\[ \text{LHS} - \text{RHS} = \sum_{n=0}^\infty d_{a,0,n}X_1^{[n]} - \sum_{n=0}^\infty A_{0,n}X_1^{[n]} = (1 - \beta X_1)^{-\frac{A_{0,1}}{\beta}} - (1 - \beta X_1)^{-\frac{A_{0,1}}{\beta}} = 0. \]

Here the second equality holds as \( d_{a,0,1} = -a\beta = A_{0,1} \) and then we can apply Theorem 2.25.

When \( k = 1, \) one use that \( a_0 = 1, d_{a+1,0,1} = -\beta + A_{0,1}, d_{a,1,1} = \frac{\theta_{1,1}A_{0,1}X_1}{\beta} \) and then use Example 2.28 to see

\[ \text{LHS} = \left( \sum_{n=0}^\infty d_{1+a,0,n}X_1^{[n]} \right) a_1 + \sum_{n=0}^\infty d_{a,1,n}X_1^{[n]} \]

\[ = \left( (1 - \beta X_1)^{-\frac{d_{a+1,0,1}}{\beta}} a_1 \right) + (d_{a,1,1}X_1(1 - \beta X_1)^{-\frac{d_{a,0,1}}{\beta}} + \frac{\theta_{1,1}A_{0,1}X_1^2}{\beta} - \frac{d_{a+1,0,1}}{\beta}) \]

\[ = (1 - \beta X_1)^{-\frac{A_{0,1}}{\beta}} ((1 - \beta X_1)a + \frac{A_{0,1}\theta_{1,1}X_1}{\beta} + \frac{\theta_{1,1}X_1^2}{A_{0,1} - \beta X_1}), \]

and that

\[ \text{RHS} = \left( \sum_{n=0}^\infty A_{0,n}X_1^{[n]} \right) a_1 + \sum_{n=0}^\infty A_{1,n}X_1^{[n]} \]

\[ = (1 - \beta X_1)^{-\frac{A_{0,1}}{\beta}} (a_1 + A_{1,1}X_1 + \frac{\theta_{1,1}X_1^2}{2A_{0,1} - \beta X_1}). \]

By definition \( a_1 = -\frac{A_{0,1}}{\beta} + \frac{\theta_{1,1}A_{0,1}}{\beta^2}, \) then one easily verifies that \( \text{LHS} = \text{RHS} \) for \( k = 1. \)

When \( k = 2, \)

\[ \text{LHS} = \left( \sum_{n=0}^\infty d_{2+a,1,n}X_1^{[n]} \right) a_2 + \left( \sum_{n=0}^\infty d_{1+a,1,n}X_1^{[n]} \right) a_1 + \sum_{n=0}^\infty d_{a,2,n}X_1^{[n]}, \]

\[ \text{RHS} = \left( \sum_{n=0}^\infty A_{0,n}X_1^{[n]} \right) a_2 + \left( \sum_{n=0}^\infty A_{1,n}X_1^{[n]} \right) a_1 + \sum_{n=0}^\infty A_{2,n}X_1^{[n]}. \]
Using \( d_{a,0,1} = -a\beta = A_{0,1}, d_{a,1,1} = \frac{\lambda_{0,1}}{\beta}, d_{a,2,1} = \frac{\lambda_{1}}{\beta}, a_{1} = -\frac{A_{1,1}}{\beta} + \frac{\theta_{1,1}A_{0,1}}{\beta}\) and Example 2.28, we have that

\[
(2.51)
\sum_{n=0}^{\infty} A_{a,2,n}X_{i}^{[n]} + \sum_{n=0}^{\infty} A_{a,1,n}X_{i}^{[n]} - \sum_{n=0}^{\infty} d_{2+a,1,n}X_{i}^{[n]}al_{1}
\]

\[
= (1 - \beta X_{i})^{-\frac{\lambda_{0,1}}{\beta}} (\frac{\lambda_{2}}{2}) (A_{1,1} - \frac{\lambda_{0,1}}{\beta} A_{0,1}) \frac{X_{i}^{3}}{1 - \beta X_{i}} + \beta (A_{1,1} - \frac{\lambda_{0,1}}{\beta} X_{i}^{2}) + 1 - \frac{1}{2} (A_{1,1} - \frac{\lambda_{0,1}}{\beta} A_{0,1}) X_{i}^{2}
\]

\[
+ (A_{1,1} - \frac{\lambda_{0,1}}{\beta} A_{0,1}) (1 - \beta X_{i}) X_{i} + (1 - \beta X_{i}) X_{i}^{2}
\]

\[
+ \theta_{0,1} \frac{\lambda_{2}}{2} (1 - \beta X_{i}) X_{i}^{2} \frac{\lambda_{1,1} A_{0,1} - \beta A_{1,1}}{\beta^{2}}
\]

while by definition we calculate that \( a_{2} = \frac{1}{\beta} (\frac{\lambda_{2}}{2} - A_{2,1} + (-\theta_{1,1} + \frac{\lambda_{1,1} A_{0,1}}{\beta} - A_{1,1})(\frac{-A_{1,1}}{\beta} + \frac{\theta_{1,1} A_{0,1}}{\beta^{2}}))\), hence

\[
\sum_{n=0}^{\infty} A_{0,0,n}X_{i}^{[n]} a_{2} - \sum_{n=0}^{\infty} d_{2,1,n}X_{i}^{[n]} a_{2}
\]

\[
= (1 - \beta X_{i})^{-\frac{\lambda_{0,1}}{\beta}} \frac{\lambda_{2}}{2} (1 - \beta X_{i}) X_{i}^{2} \frac{\lambda_{1,1} A_{0,1} - \beta A_{1,1}}{\beta^{2}}
\]

Combine Eq. (2.51) and Eq. (2.52), we get the desired result LHS \(-\) RHS = 0 for \( k = 2 \).

Hopefully these calculations in the small degree give some evidence of Conjecture 2.24. For larger \( k \), the verification of Conjecture 2.24 is hard, even if we further assume that \( A_{0,1} \) commutes with all \( A_{m,1} \), we still need to understand \( \{ h_{m,j} \} \) better to prove Conjecture 2.24 or Conjecture 2.47.

3. Absolute prismatic cohomology of de Rham crystals

In this section we study the absolute prismatic cohomology of de Rham crystals \( M \in \text{Vect}((O_K)_{\Delta}, (O_{\Delta}^{[1]})_{\Delta}^{2}) \).

Our strategy is to study the cohomology of its restriction to prisms in \( (O_K)_{\Delta} \) first, then use Čech-Alexander complex to calculate \( R \Gamma_{\Delta}(O_K, M) \).

Lemma 3.1. Let \( M \in \text{Vect}((O_K)_{\Delta}, (O_{\Delta}^{[1]})_{\Delta}^{2}) \), then for any \( U = (A, I) \in (O_K)_{\Delta} \) and \( q > 0 \), we have that

\[ H^{q}(U, M) = 0, \]

and that

\[ H^{q}(U, M/\mathcal{I}^{n}) = 0. \]

Here \( \mathcal{I}^{n} \) is the ideal sheaf on \( (O_K)_{\Delta} \) by sending \( (A, I) \) to \( I^{n}B_{\text{prism}}^{+}(A) \).

Proof. Let \( (A, I) \rightarrow (B, IB) \) be a cover in \( (O_K)_{\Delta} \); denote \( V = (B, IB) \) we claim that the Čech complex of \( M \) for this cover \( R \Gamma(U^{*}, M) \) is concentrated in degree 0. By derived Nakayama lemma, it suffices to check that \( R \Gamma(U^{*}, M) \otimes_{B_{\text{prism}}^{+}(A)} (B_{\text{prism}}^{+}(A)/I) \) is concentrated in degree 0. However, \( B_{\text{prism}}^{+}(A)/I \) has \( I \)-complete Tor amplitude \([-1, 0]\) as a \( B_{\text{prism}}^{+}(A) \)-module, hence \( \otimes_{B_{\text{prism}}^{+}(A)} (B_{\text{prism}}^{+}(A)/I) \) commutes with totalization by [KP21, Corollary 3.1.13], this implies that \( R \Gamma(U^{*}, M) \otimes_{B_{\text{prism}}^{+}(A)} (B_{\text{prism}}^{+}(A)/I) \) is the Čech complex of the rational Hodge-Tate crystal \( M/\mathcal{I}^{n} \) thanks to Corollary 2.2, which is concentrated in degree 0 by [MW21, Lemma 3.18].

This implies that \( H^{q}(U, M) = 0 \) by [SP22, Tag 01EY].

The desired results for \( M/\mathcal{I}^{n} \) can be proven similarly thanks to Remark 2.3.
Corollary 3.2. Let $\mathcal{M} \in \text{Vect}(\langle \mathcal{O}_K \rangle_\Delta, \langle \mathcal{O}_\Delta \langle \mathcal{I} \mathcal{P} \rangle \rangle_\Delta)$, then the absolute prismatic cohomology $R\Gamma_\Delta(\mathcal{O}_K, \mathcal{M})$ can be calculated by the Čech-Alexander complex

\[
\mathcal{M}(\mathcal{E}, (E)) \xrightarrow{d_0} \mathcal{M}(\mathcal{E}^1, (E)) \xrightarrow{d_1} \mathcal{M}(\mathcal{E}^2, (E)) \to \cdots,
\]

where $d_n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$.

Similarly, $R\Gamma_\Delta(\mathcal{O}_K, \mathcal{M}/\mathcal{I}^n)$ can be calculated by the Čech-Alexander complex

\[
\mathcal{M}/\mathcal{I}^n(\mathcal{E}, (E)) \xrightarrow{d_0} \mathcal{M}/\mathcal{I}^n(\mathcal{E}^1, (E)) \xrightarrow{d_1} \mathcal{M}/\mathcal{I}^n(\mathcal{E}^2, (E)) \to \cdots.
\]

Proof. As $(\mathcal{E}^n, (E))$ is the Čech complex associated to the weakly final object $(\mathcal{E}, (E))$ in $\mathcal{O}_K$, it is a standard fact that $R\Gamma_\Delta(\mathcal{O}_K, \mathcal{M})$ (resp. $R\Gamma_\Delta(\mathcal{O}_K, \mathcal{M}/\mathcal{I}^n)$) can be computed via $\text{Tot}(\Gamma(\mathcal{E}^*, \mathcal{M}))$ (resp. $\text{Tot}(\Gamma(\mathcal{E}^*, \mathcal{M}/\mathcal{I}^n))$, which is just Eq. (3.3) (resp. Eq. (3.4)) thanks to Lemma 3.1.

Theorem 3.5. Given $\mathcal{M} \in \text{Vect}(\langle \mathcal{O}_K \rangle_\Delta, \langle \mathcal{O}_\Delta \langle \mathcal{I} \mathcal{P} \rangle \rangle_\Delta)$, then $H^i((\mathcal{O}_K)_\Delta, \mathcal{M}) = 0$ for $i > 1$.

To prove this theorem, we need to consider the restricted site $(\mathcal{O}_K)^\prime_\Delta$ introduced in [MW21, Section 3.4], whose underlying category is the full subcategory of $(\mathcal{O}_K)_\Delta$ spanned by those objects admitting maps from $(\mathcal{E}, (E))$ with coverings inherited from those in $\mathcal{O}_K$.

The reason to restrict to $(\mathcal{O}_K)^\prime_\Delta$ is that for any $((A, \mathcal{I}) \in (\mathcal{O}_K)^\prime_\Delta, \mathcal{I} = \mathcal{E} \mathcal{A}$ is oriented by rigidity of morphism of prisms (See [BS19, Prop 1.5]), hence it is more convenient to relate $\mathcal{M}/\mathcal{I}^{n+1}$ and $\mathcal{M}/\mathcal{I}^n$, for example see the proof of Lemma 3.7.

We warn the reader that $(\mathcal{O}_K)^\prime_\Delta$ is not the relative prismatic site $(\mathcal{O}_K/\mathcal{E}, (E))_\Delta$ as for example one can see that products are calculated differently.

By abuse of notation we still use $\mathcal{M}$ to define its restriction on $(\mathcal{O}_K)^\prime_\Delta$.

Lemma 3.6. For $\mathcal{M} \in \text{Vect}(\langle \mathcal{O}_K \rangle_\Delta, \langle \mathcal{O}_\Delta \langle \mathcal{I} \mathcal{P} \rangle \rangle_\Delta)$, the following holds:

- $R\Gamma((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}) = R\Gamma((\mathcal{O}_K)_\Delta, \mathcal{M}) = \mathcal{C}(\mathcal{M})$.
- $R\Gamma((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}^n) = R\Gamma((\mathcal{O}_K)_\Delta, \mathcal{M}/\mathcal{I}^n) = \mathcal{C}(\mathcal{M}/\mathcal{I}^n)$.

Here $\mathcal{C}(\mathcal{M})$ denotes the Čech-Alexander complex associated to the Čech Nerve of the weakly final object $(\mathcal{E}, (E))$.

Proof. For the first part, in Corollary 3.2, we already prove $R\Gamma((\mathcal{O}_K)_\Delta, \mathcal{M}) = \mathcal{C}(\mathcal{M})$, then a similar argument shows that $R\Gamma((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}) = \mathcal{C}(\mathcal{M})$ as $\mathcal{C}(\mathcal{M})$ is also the Čech Nerve of the weakly final object $(\mathcal{E}, (E))$ in $(\mathcal{O}_K)^\prime_\Delta$.

The second part follows from a similar argument.

Lemma 3.7. $H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}^n) = 0$ for any $i > 1, n \geq 1$.

Proof. We proceed by induction on $n$. When $n = 1$, as $\mathcal{M}/\mathcal{I}$ is a rational Hodge-Tate crystal by Corollary 2.2, hence $H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}) = 0$ for any $i > 1$ by [MW21, Thm 3.20] and [MW21, Lem 3.26].

For the induction process, consider the short exact sequence of abelian sheaves

\[
\mathcal{M}/\mathcal{I}^{n-1} \xrightarrow{\mathcal{E}} \mathcal{M}/\mathcal{I}^n \to \mathcal{M}/\mathcal{I},
\]

which induces a long exact sequence

\[
\cdots \to H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}^{n-1}) \to H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}^n) \to H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}) \to \cdots.
\]

Hence by induction $H^i((\mathcal{O}_K)^\prime_\Delta, \mathcal{M}/\mathcal{I}^n) = 0$ for any $i > 1$, we are done.

Lemma 3.8. For any $\mathcal{M} \in \text{Vect}(\langle \mathcal{O}_K \rangle_\Delta, \langle \mathcal{O}_\Delta \langle \mathcal{I} \mathcal{P} \rangle \rangle_\Delta)$, $M = R\lim \mathcal{M}/\mathcal{I}^n$.

Proof. Notice that the topos $\text{Shv}(\langle \mathcal{O}_K \rangle_\Delta)$ is replete and that $\mathcal{M}/\mathcal{I}^{n+1} \to \mathcal{M}/\mathcal{I}^n$ is surjective for each $n$, hence $R\lim \mathcal{M}/\mathcal{I}^n \cong \lim \mathcal{M}/\mathcal{I}^n$ by [BS15, Prop 3.1.10]. Then the result follows as $\mathcal{M}$ is $\mathcal{I}$-adically complete.
Proof of Theorem 3.5. Notice that
\[ R\Gamma((\mathcal{O}_K')_{\Delta}, \mathcal{M}) = R\lim_n R\Gamma((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^n) \]
as \( R\Gamma \) commutes with taking derived inverse limit. Also, the cohomology of inverse limits can be calculated via the following short exact sequence (See [SP22, Tag 07KY])
\[ 0 \rightarrow R^1 \lim_n H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^n) \rightarrow H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}) \rightarrow \lim_n H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^n) \rightarrow 0. \]
As a result, \( H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}) = 0 \) for \( j \geq 3 \) by applying Lemma 3.7. For \( j = 2 \), we still need to show \( R^1 \lim_n H^1((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^n) = 0 \) to guarantee that \( H^2((\mathcal{O}_K')_{\Delta}, \mathcal{M}) = 0 \). To do this, it suffices to show that for any \( n \),
\[ H^1((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^{n+1}) \rightarrow H^1((\mathcal{O}_K')_{\Delta}, \mathcal{M}/\mathcal{I}^n) \]
is surjective. However, this can be seen via the long exact sequence associated to the short exact sequence
\[ 0 \rightarrow \mathcal{M}/\mathcal{I} \times \mathcal{E}_M^n \rightarrow \mathcal{M}/\mathcal{I}^{n+1} \rightarrow \mathcal{M}/\mathcal{I}^n \rightarrow 0. \]
Now we conclude that \( H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}) = 0 \) for \( j \geq 2 \). Hence \( H^j((\mathcal{O}_K')_{\Delta}, \mathcal{M}) = 0 \) for \( j \geq 2 \) thanks to Lemma 3.6 again. \( \square \)

Next we would like to give a detailed study of \( H^i((\mathcal{O}_K')_{\Delta}, \mathcal{M}) \) based on our stratification data.

To give a rank \( l \) de Rham crystal \( \mathcal{M} \in \text{Vec}(\mathcal{O}_K'_{\Delta}; \mathcal{O}_K[\mathcal{H}])_2 \), by Corollary 2.4, it is equivalent to specify a finite free \( \mathbb{B}^{+}_{dt}(\mathcal{S}) \)-module \( M \) of rank \( l \) equipped with a stratification \( \varepsilon \) satisfying cocycle condition.

Notice that every element \( v \in M \) can be written as \( v = e \bar{f} \) for some \( \bar{f} \in M_{l \times 1}(\mathbb{K}[\mathcal{T}]) \), further denote \( \bar{f} = \sum_{i=0}^{\infty} B_i t^i \) with \( B_i \in M_{l \times 1}(\mathbb{K}) \), then unwinding the definition of \( \delta^0_1 \) and use the notation in Section 2, we see that
\[ \delta^0_1(v) = \delta^0_1(e \bar{f}) = e \varepsilon(\sum_{i=0}^{\infty} B_i t^i) = \varepsilon(e)(\sum_{i=0}^{\infty} B_i (at)^i) \]
\[ = e \cdot (\sum_{m \geq 0} (\sum_{n \geq 0} A_{m,n} X^{[n]} t^m)(\sum_{i=0}^{\infty} B_i (at)^i) \]
\[ = e \cdot (\sum_{m \geq 0} (\sum_{n \geq 0} A_{m,n} X^{[n]} t^m)(\sum_{i=0}^{\infty} B_i t^i(\sum_{s=0}^{\infty} c_{i,s} t^s)) \]
\[ = e \cdot (\sum_{m \geq 0} (\sum_{n \geq 0} A_{m,n} X^{[n]} t^m)(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} B_i c_{j,i} t^j) \]
\[ = e \sum_{m=0}^{\infty} t^m(\sum_{0 \leq i,j \leq m} (\sum_{s=0}^{\infty} A_{i,s} X_1^{[s]})(\sum_{j=0}^{\infty} B_{p} e_{p,j-p})). \]

Hence by Corollary 3.2, \( v \in H^0_{\Delta}(\mathcal{O}_K, \mathcal{M}) \), which is equivalent to that \( 0 = d_0(v) = \delta^0_0(v) - \delta^1_1(v) = \delta^1_0(v) - v \), if and only if the following holds:
\[ e \sum_{m=0}^{\infty} t^m(\sum_{0 \leq i,j \leq m} (\sum_{s=0}^{\infty} A_{i,s} X_1^{[s]})(\sum_{j=0}^{\infty} B_{p} e_{p,j-p})) = e \sum_{i=0}^{\infty} B_i t^i. \]

Compare the coefficients of \( B_m \), we see this is equivalent to that
\[ B_m = \sum_{0 \leq i,j \leq m} (\sum_{s=0}^{\infty} A_{i,s} X_1^{[s]})(\sum_{j=0}^{\infty} B_{p} e_{p,j-p}). \]

We explore the conditions on \( B_m \) to guarantee that Eq. (3.9) holds.
First compare the constant term on the both sides of Eq. (3.9). On the right hand side of Eq. (3.9), we get

\[ A_{0,0} \sum_{p=0}^{m} B_p d_{p,m-p,0} = B_{m,0} = B_m, \]

here we use the fact that \( A_{j,0} = 0 \) for \( j > 0 \) by Theorem 2.18 and that \( d_{p,s,0} = 0 \) for \( s > 0 \) by unwinding definition of \( d_{p,s,j} \).

Hence we obtain nothing new when considering the constant term.

Next we compare the coefficient of \( X_1 \) on the both sides of Eq. (3.9). We compute the coefficient of \( X_1 \) on the right hand side, which is

\[ (3.10) \quad A_{0,0} \left( \sum_{p=0}^{m} B_p d_{p,m-p,1} \right) + \sum_{i+j=m} A_{i,1} B_j = \sum_{p=0}^{m} (-p\theta_{1,m-p} B_p) + \sum_{i+j=m} A_{i,1} B_j. \]

Here we have used Lemma 2.22. This calculation implies the following:

**Proposition 3.11.** Given \( v = \mathbf{f} \) where \( \mathbf{f} = \sum_{i=0}^{\infty} B_i t^i \) with \( B_i \in \mathcal{M}_{l \times 1}(K) \), then \( v \in H^0_{\Delta}(\mathcal{O}_K, \mathcal{M}) \) implies that

\[ (A_{0,1} - m\beta)B_m = \sum_{p=0}^{m-1} (p\theta_{1,m-p} - A_{m-p,1})B_p \]

for any non-negative integers. In particular, by taking \( m = 0 \), we have that \( A_{0,1}B_0 = 0 \).

**Proof.** By assumption, Eq. (3.9) holds, hence by comparing the coefficient of \( X_1 \) on the both sides, we have that

\[ 0 = \sum_{p=0}^{m} (-p\theta_{1,m-p} B_p) + \sum_{i+j=m} A_{i,1} B_j, \]

which implies the desired result. \( \square \)

**Remark 3.12.** In [MW21, Theorem 3.20], they show that for \( w \) in the rational Hodge-Tate crystal \( \mathcal{M}/\mathcal{T} \), \( w \in H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}/\mathcal{T}) \) if and only if \( A_{0,1}w = 0 \), hence our results are compatible with theirs.

**Corollary 3.13.** Suppose \( \mathcal{M} \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_K)_{\Delta})^{\mathcal{O}_K} \). Further assume that that none of the Sen weights of the rational Hodge-Tate crystal \( \mathcal{M}/\mathcal{T} \) are non-positive integers, then \( H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) = 0 \).

**Proof.** Under assumption suppose \( v = \mathbf{f} \) where \( \mathbf{f} = \sum_{i=0}^{\infty} B_i t^i \) with \( B_i \in \mathcal{M}_{l \times 1}(K) \) is an element of \( H^i((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \), then by Proposition 3.11,

\[ (3.14) \quad (A_{0,1} - m\beta)B_m = \sum_{p=0}^{m-1} (p\theta_{1,m-p} - A_{m-p,1})B_p. \]

Recall that the Sen weights of \( \mathcal{M}/\mathcal{T} \) are eigenvalues of the Sen operator \( -\Delta_{\mathcal{M}} \) by [Gao22, Theorem 1.1.7]. On the other hand, by assumption none of them are non-positive integers, hence the eigenvalues of \( A_{0,1} - m\beta \) are non-zero for any \( m \in \mathbb{N}^{\geq 0} \). In other words, all \( A_{0,1} - m\beta \) are invertible. We claim that this implies that \( B_i \) all vanish. We proceed by induction on \( i \).

First take \( m = 0 \) in Eq. (3.14), hence \( A_{0,1}B_0 = 0 \), which implies that \( B_0 = 0 \) as \( A_{0,1} \) is invertible.

For the induction process, suppose we have proven that \( B_i = 0 \) for all \( i \leq m - 1 \) (\( m \geq 2 \)), then Eq. (3.14) guarantees that \( B_m \) also vanishes as \( A_{0,1} - m\beta \) is invertible.

As a result, \( v = 0 \). This implies that there are no non-zero elements in \( H^i((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \), hence \( H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) = 0 \). \( \square \)

**Corollary 3.15.** Let \( \mathcal{M} \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_K)_{\Delta})^{\mathcal{O}_K} \) be a rank \( l \) de Rham crystal. Define \( q \) to be the number of non-positive integers (with multiplicity) among Sen weights of the rational Hodge-Tate crystal \( \mathcal{M}/\mathcal{T} \), then \( H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \) is a \( K \)-vector space of dimension at most \( q \). In particular, \( \dim_K H^0((\mathcal{O}_K)_{\Delta}, \mathcal{M}) \leq l \).
Proof. Clearly $H^0((\mathcal{O}_K)_{\hat{L}}, \mathcal{M})$ is a $K$-vector space as $d_0$ is $K$-linear. Suppose $s_1, \cdots, s_k$ are non-positive integers which show up in the set of Sen weights the rational Hodge-Tate crystal $\mathcal{M}/\mathcal{I}$ with multiplicity $\lambda_1, \cdots, \lambda_k$, then $\lambda_1 + \cdots + \lambda_k = q$ by assumption. As the Sen weights of $\mathcal{M}/\mathcal{I}$ are precisely eigenvalues of the Sen operator $-\frac{A_{0,1}}{2}$ by [Gao22, Theorem 1.1.7], hence $A_{0,1} - m\beta$ are invertible unless $m \in \{s_1, \cdots, s_k\}$. On the other hand, the solution space of $(A_{0,1} - s_i)X = Y$ is of dimension at most $\lambda_i$ for any $X, Y \in M_{1 \times 1}(K)$. By Proposition 3.11, $H^0((\mathcal{O}_K)_{\hat{L}}, \mathcal{M})$ is of dimension at most $\lambda_1 + \cdots + \lambda_k = q$ as a $K$-vector space. 

4. Locally analytic vectors and Decompletion theorem

4.1. First descent step. We aim to prove Corollary 4.7 in this subsection, which could be thought as a purity for de Rham representations. This result should be familiar to experts, but we decide to write it down for completeness. Our strategy is to imitate the proof written in [BC09, Section 15.2] treating the cyclotomic tower.

Lemma 4.1. $L_{\text{dr}}^+ := (B_{\text{dr}}^+)^{G_L}$ is a closed $L$-subalgebra of $B_{\text{dr}}^+$ that is a complete discrete valuation ring with uniformizer $t$ and residue field $\hat{L}$. Moreover, the topological ring $L_{\text{dr}}^+$ is separated and complete for its subspace topology from $B_{\text{dr}}^+$ for which the multiplication map $t : L_{\text{dr}}^+ \to L_{\text{dr}}^+$ defines a closed embedding.

Proof. The proof is verbatim as in [BC09, Lemma 15.2.1] by replacing $K_{p, \infty}$ (which is denoted as $K_{\infty}$ in loc. cit.) with $L$ and $G_{K_{p, \infty}}$ (which is denoted as $H_K$ in loc. cit.) with $G_L$. □

Remark 4.2. As a consequence, from now on we can identify $L_{\text{dr}}^+$ with $B_{\text{dr}, L}^+ := \mathbb{B}_{\text{dr}}^+(A_{\text{inf}}(\mathcal{O}_L))$.

Notice that both $L_{\text{dr}}^+$ and $B_{\text{dr}}^+$ are topologized discrete valuation rings equipped with the finer topologies (finer than the $t$-adic topology) to make their residue fields acquire the valuation topology rather than the discrete topology as the quotient topology. As a result, finitely generated modules over these rings admits a functorial topological structure, we show this is compatible with inverse limits:

Lemma 4.3. Let $N$ be a finitely generated module over the topologized discrete valuation ring $* \in \{L_{\text{dr}}^+, B_{\text{dr}}^+\}$ endowed with natural topology, then the linear continuous bijection $N \to N/t^nN$ is a homeomorphism. Moreover, the quotient topology on $L_{\text{dr}}^+/t^n$ coincides with its subspace topology from $B_{\text{dr}}^+/t^n$ for any $n \in \mathbb{N}$.

Proof. The proof of [BC09, Lemma 15.2.2] still works here. □

Proposition 4.4. Given $W \in \text{Rep}_{G_K}(B_{\text{dr}}^+)$, the $L_{\text{dr}}^+$-module $W^{G_L}$ is finitely generated with a continuous $G$-action for its natural topology as a finitely generated $L_{\text{dr}}^+$-module, and the natural $B_{\text{dr}}^+$-linear map

$$\alpha_W : B_{\text{dr}}^+ \otimes_{L_{\text{dr}}^+} W^{G_L} \to W$$

is an isomorphism. In particular, the rank and invariant factors of $W^{G_L}$ over $L_{\text{dr}}^+$ coincide with those of $W$ over $B_{\text{dr}}^+$.

Proof. First we treat the case when $W$ is killed by a power of $t$ by induction on the power of $t$ killing $W$ such that $\alpha_W$ is an isomorphism and $H^1(G_L, W)$ vanishes. We start from the case that $tW = 0$, i.e. $W \in \text{Rep}_{G_K}(\mathbb{C}_p)$ as the quotient topology on $\mathbb{C}_p \simeq B_{\text{dr}}^+/(t^\infty)$ induced from that on $B_{\text{dr}}^+$ coincides with its natural topology. Then by Faltings’s almost purity theorem

$$\alpha_W : \mathbb{C}_p \otimes_{L} W^{G_L} \to W.$$ 

Also, we have that $H^1(G_L, W) = 0$ by [KL16, Theorem 3.5.8] and [Sch13, Theorem 6.5] as $\hat{L}$ is a perfectoid field.

Suppose we have proven that $\alpha_W$ is an isomorphism and $H^1(G_L, W) = 0$ provided that $W$ is killed by $t^n$, here $n \geq 1$. Now given a $W \in \text{Rep}_{G_K}(B_{\text{dr}}^+)$ which is killed by $t^{n+1}$, we proceed by utilizing the following topologically exact sequence (as $tW$ is equipped with the subspace topology induced from $W$) in $\text{Rep}_{G_K}(B_{\text{dr}}^+)$:

$$0 \to tW \to W \to W/tW \to 0.$$
Observe that both tW and W/tW are killed by t^n, then by induction H^1(G_L, tW) = 0, which implies the exactness of the following G-equivariant sequence of L^+_{dr}-modules:

\[ 0 \rightarrow (tW)^{G_L} \rightarrow W^{G_L} \rightarrow (W/tW)^{G_L} \rightarrow 0. \]

As an upshot, we see that W^{G_L} is finitely generated over L^+_{dr}. This leads to the following B^+_{dr}-linear diagram whose rows are all exact:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B^+_{dr} \otimes_{L^+_{dr}} (tW)^{G_L} & \rightarrow & B^+_{dr} \otimes_{L^+_{dr}} W^{G_L} & \rightarrow & B^+_{dr} \otimes_{L^+_{dr}} (W/tW)^{G_L} & \rightarrow & 0 \\
 \downarrow \alpha_{W} & & \downarrow \alpha_{W} & & \downarrow \alpha_{W/tW} & & \downarrow \alpha_{W/tW} & & 0.
\end{array}
\]

Here the top row is exact as the scalar extension of injective discrete valuation rings L^+_{dr} \rightarrow B^+_{dr} is flat.

Then five lemma implies that \alpha_{W} is an isomorphism as so are for \alpha_{W} and \alpha_{W/tW} by induction. Moreover, thanks to [BC09, Exercise 2.5.3] again, Eq. (4.6) induces the exact sequence

\[ H^1(G_L, tW) \rightarrow H^1(G_L, W) \rightarrow H^1(G_L, W/tW). \]

Hence H^1(G_L, W) = 0 as H^1(G_L, tW) = H^1(G_L, W/tW) = 0 by induction.

Now we have finished the proof for those W killed by a power of t. One of the byproduct is that the functor W \mapsto W^{G_L} is exact when restricted to torsion W \in Rep_G(B^+_{dr} as H^1(G_L, \cdot) vanishes on such W.

For non-torsion \text{W} \in Rep_G(B^+_{dr}), W = \varprojlim (W/t^n). For a fixed m, consider the G_K-equivalent right exact sequence

\[ W/t^n \rightarrow W/t^m \rightarrow W/t^m \rightarrow 0. \]

Thanks to the vanishing of H^1 on torsion objects, it is still right exact after taking G_L-invariants

\[ (W/t^n)^{G_L} \rightarrow (W/t^m)^{G_L} \rightarrow (W/t^m)^{G_L} \rightarrow 0. \]

Then passing to inverse limit with respect to m preserves right exactness as it forms a Mittag-Leffler system (we have shown that all the torsion terms involved are finitely generated):

\[ W^{G_L} \rightarrow W^{G_L} \rightarrow (W/t^n)^{G_L} \rightarrow 0. \]

This implies that W^{G_L}/t^n \cong (W/t^n)^{G_L} for all n \geq 1.

In particular, take n = 1, we have that W^{G_L}/t \cong (W/t)^{G_L} is a finitely generated \hat{L}-vector space based on our study of t-torsion case. But W^{G_L} is a closed L^+_{dr}-submodule of a finitely generated B^+_d_r-submodule W, hence W^{G_L} is t-adically complete and separated. Then Nakayama’s lemma implies that W^{G_L} is finitely generated over L^+_{dr}.

Finally consider the natural map

\[ \alpha_{W} : B^+_{dr} \otimes_{L^+_{dr}} W^{G_L} \rightarrow W. \]

Both sides are finitely generated over B^+_d_r, hence t-adically complete and separated, by derived Nakayama lemma, to show \alpha_{W} is an isomorphism, it suffices to show \alpha_{W}/t (derived sense) is an isomorphism. As W is non-torsion, \alpha_{W}/t is the usual quotient, then the result follows as W^{G_L}/t \cong (W/t)^{G_L} and we have already shown \alpha_{W}/t is an isomorphism.

\[
\text{Corollary 4.7. The functor}
\]

\[ \text{Rep}_G(B^+_{dr}) \rightarrow \text{Rep}_G(B^+_{dr}, L) \]

\[ W \mapsto W^{G_L} \]

\[ \text{is well defined and induces an equivalence of categories. A quasi-inverse can be chosen as } X \mapsto B^+_{dr} \otimes_{B^+_{dr}, L} X. \]

\[ \text{Proof. Let us first explain why it is well defined. By Remark 4.2 and Proposition 4.4, } W^{G_L} \text{ is a finitely generated } B^+_{dr,L}-\text{module and} \]

\[
(4.8) \quad \alpha_{W} : B^+_{dr} \otimes_{B^+_{dr}, L} W^{G_L} \rightarrow W
\]
is an isomorphism. Passing to the case of cyclic modules using Lemma 4.3 and then apply Lemma 4.1 as well, the natural continuous injective map \( M \to B_{dR}^+ \otimes B_{dR,L}^+ \) is a homeomorphism onto its image for any finitely generated \( B_{dR,L}^+ \)-module \( M \). Apply it to \( M = W^{G_L} \), we see that the natural topology on \( W^{G_L} \) as a finitely generated \( B_{dR,L}^+ \)-module coincides with the subspace topology induced from \( W \) via \( \alpha_W \). Then the continuity of the \( \hat{G} \)-action on \( W^{G_L} \) follows from the continuity of the \( G_K \)-action on \( W \), hence \( W^{G_L} \in \text{Rep}_{G_K}(B_{dR,L}^+) \).

On the other hand, given \( X \in \text{Rep}_{\hat{G}}(B_{dR,L}^+) \), then \( V := B_{dR}^+ \otimes B_{dR,L}^+ \) is equipped with a continuous \( G_K \)-action with its natural topology as a finitely generated \( B_{dR,L}^+ \)-module as \( G_K \) acts continuously on both \( B_{dR}^+ \) and \( X \). As a result, \( V \in \text{Rep}_{G_K}(B_{dR}) \). Then one can check \( V^{G_L} = X \) and \( V(W^{G_L}) \cong W \) via \( \alpha_W \) for \( W \in \text{Rep}_{G_K}(B_{dR}) \). Hence the functors are quasi-inverses. \( \square \)

### 4.2. Sen theory over \( B_{dR,L}^+ \): decompletion along Kummer tower.

**Notation 4.9.** Recall that we embed the Breuil-Kisin prism \((W(k)[[u]], E(u))\) into \((A_{inf}, \text{Ker}(\theta))\) by sending \( u \) to \([\pi^\alpha]\). Hence the induced embedding \( B_{dR,L}^+ \to B_{dR,L}^+ \) sends \( t = E(u) \) to \( E([\pi^\alpha]) \), which will be our chosen uniformizer in the complete discrete ring \( B_{dR,L}^+ \). Define \( B_{dR,k,L}^+ \) to be \( B_{dR,k,L}^+ / t^k \), which is just \( A_{inf}(O_L)[\frac{1}{p}] / t^k \).

And we use \( \theta_k \) to denote the specialization map \( B_{dR,L}^+ \to B_{dR,k,L}^+ \). We will abuse notation by also writing \( \theta_k \) for \( A_{inf}(O_L)[\frac{1}{p}] \to A_{inf}(O_L)[\frac{1}{p}] / (t^k) \). When \( k = 1 \), we will denote \( \theta_1 \) as \( \theta \).

We give a quick review of locally analytic vectors. We refer the readers to [BC16, Section 2] [Ber16, Section 2] for details. Recall given a \( p \)-adic Lie group \( G \), and let \( V \) be a \( \mathbb{Q}_p \)-Banach representation of \( G \) with norm \( || \cdot || \). Let \( H \) be an open subgroup of \( G \) such that there exist coordinates \( c_1, \ldots, c_d : H \to \mathbb{Z}_p \) giving rise to an analytic bijection \( c : H \to \mathbb{Z}_p^d \). Then

**Definition 4.10.**

- An element \( w \in V \) is \( H \)-analytic if there exists a sequence \( \{w_k\}_{k \in \mathbb{N}} \) with \( w_k \to 0 \) in \( W \), such that
  \[
g(w) = \sum_{k \in \mathbb{N}} c(g)^k w_k, \quad \forall g \in H.
\]

The space of such vectors is denoted as \( V^{H-\text{an}} \).
- A vector \( w \in V \) is locally analytic if there exists an open subgroup \( H \) as above such that \( w \in V^{H-\text{an}} \).

The space of such vectors is denoted as \( V^{\text{Gal}-\text{an}} \).

**Definition 4.11 (Monodromy operators).** Still assume \( G = \hat{G} \), \( V \) a \( \mathbb{Q}_p \)-Banach representation of \( \hat{G} \), one can define two monodromy operators \( \nabla_\gamma \) and \( \nabla_\tau \) on \( V^{\hat{G}-\text{an}} \) as follows:

- \( \nabla_\gamma := \frac{\log g}{\log \chi(g)} \) for \( g \in \text{Gal}(L/K_\infty) \) enough close to 1, where as usual \( \chi \) is the \( p \)-adic cyclotomic character and \( \log g \) is just the formal power series \( \sum_{n=1}^{\infty} (-1)^{n-1} (g-1)^n/n \).
- \( \nabla_\tau := \log(\tau^p) \) for \( n \) large enough.

**Remark 4.12.** A useful result is that for \( x \in V^{\hat{G}-\text{an}} \),

\[
\nabla_\tau(x) = \lim_{n \to \infty} \frac{\tau^p(x) - x}{p^n}.
\]

as on \( V^{\hat{G}-\text{an}} \), \( \tau = \sum_{n=0}^{\infty} \frac{\nabla_\tau^n}{p^n} \).

**Definition 4.13.** Suppose \( G = \hat{G}, W \in \text{Rep}_{B_{dR,L}^+}(\hat{G}) \), define

\[
W^{\hat{G}-\text{an}} = \lim_\leftarrow (W/t^n W)^{\hat{G}-\text{an}}, \quad W^{\hat{G}-\text{an}, \gamma=1} = \lim_\leftarrow (W/t^n W)^{\hat{G}-\text{an}, \gamma=1},
\]

\[
W^{\hat{G}-\text{an}, \tau=1} = \lim_\leftarrow (W/t^n W)^{\hat{G}-\text{an}, \tau=1}, \quad W^{\hat{G}-\text{an}, \nabla, \gamma=0} = \lim_\leftarrow (W/t^n W)^{\hat{G}-\text{an}, \nabla, \gamma=0},
\]

and equip them with subspace topology induced from that on \( W = \lim_\leftarrow X/t^n W \). Here for \( V \) a \( \mathbb{Q}_p \)-Banach representation of \( G \),

\[
V^{\hat{G}-\text{an}, \gamma=1} := V^{\hat{G}-\text{an}}, \text{Gal}(L/K_\infty) = 1, \quad V^{\hat{G}-\text{an}, \tau=1} := V^{\hat{G}-\text{an}}, \text{Gal}(L/K_p) = 1.
\]
Also, given $V$ a $\mathbb{Q}_p$-Banach representation of $G$ with norm $\| \cdot \|$, then we define $V\{\{T\}\}_n$ to be the vector space of series $\sum_{k \geq 0} v_k T^k$ such that $p^{nk} v_k \to 0$ as $k$ tends to $\infty$, here $T$ is a variable. For $h \in V$ such that $\|v\| \leq p^{-n}$, $V\{\{h\}\}_n$ is defined to be the evaluation of $V\{\{T\}\}_n$ at $T = h$.

The story starts from the $t$-torsion case.

Let $\beta_n \in L$ such that $\|\theta(t) - \beta_n\| \leq p^{-n}$, then as in [BC16, Section 4.2] and [Gao22, Construction 3.2.5], there is an increasing sequence $\{r_n\}$ such that if $m \geq \{r_n\}$, then $\|\theta(t) - \beta_n\|\hat{G}_m = \|\theta(t) - \beta_n\|$ and $\theta(t) - \beta_n \in \hat{G}_m^{\theta - \beta_n}$, where $\hat{G}_m$ is topologically generated by $\tau^n$ and $\gamma^n$, $\|w\|\hat{G}_m = \sup_{k \in \mathbb{N}} \|w_k\|$ ($w_k$ is defined as in Definition 4.10).

Then the following result is essentially calculated by Gao and Poyeton:

**Lemma 4.14.**

- $\hat{L}^{\hat{G} - \theta_n} = \bigcup_{n \geq 1} K(\mu_{r(n)}, \pi_{r(n)})\{\{\theta(t) - \beta_n\}\}_n$.
- $\hat{L}^{\hat{G} - \theta_n, \pi_n = 0} = L$.
- $\hat{L}^{\hat{G} - \theta_n, \gamma_n = 1} = K_\infty$.

Here $t = \frac{\log(\|\cdot\|)}{p^k}$.

**Proof.** The last two is [GP21, Proposition 3.3.2] while the first identification is [Gao22, Proposition 3.2.8].

**Remark 4.15.** By [BC16, Lemma 2.5], $\hat{L}^{\hat{G} - \theta_n}$ is a field. It strictly contains $L$.

**Theorem 4.16.** Suppose $k \geq 1$, then the map

$$
\bigoplus_{i=0}^{k-1} \bigcup_{n \geq 1} K(\mu_{r(n)}, \pi_{r(n)})\{\{\theta(t) - \beta_n\}\}_n \cdot t_i \to (B_{dR,k,L}^+)\hat{G}^{-\theta_n}
$$

is an isomorphism.

**Proof.** First notice that for a fixed $k$, $t \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n}$. Actually, when $k = 1$, $t$ is 0 in $B_{dR,1,L}^+ \cong \hat{L}$. To treat the general $k \geq 2$, it suffices to show

$$
t \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n, \gamma_n = 1}
$$

thanks to [GP21, Lemma 3.2.4]. By [BC16, Lemma 2.5(i)], $(B_{dR,k,L}^+)\hat{G}^{-\theta_n, \gamma_n = 1}$ is a ring, hence it suffices to show $[\pi^\gamma] \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n, \gamma_n = 1}$.

However, $\text{Gal}(L/K_\infty)$ acts trivially on $[\pi^\gamma]$ by the definition of $[\pi^\gamma]$ and $\tau$ acts on $[\pi^\gamma]$ via

$$
\tau([\pi^\gamma]) = [\pi^\gamma] = ([\pi] - 1)[\pi^\gamma]
$$

then $([\pi] - 1)[\pi^\gamma] = ([\pi] - 1)[\pi^\gamma] \in t_1 B_{dR,k,L}^+$. Then by induction we see that $(\tau - 1)^i([\pi^\gamma]) = ([\pi] - 1)^i[\pi^\gamma] \in t^i B_{dR,k,L}^+$. As a consequence,

$$
(\log \tau)^i([\pi^\gamma]) \in t^i B_{dR,k,L}^+.
$$

hence $[\pi^\gamma] \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n}$ thanks to [GP21, Lemma 3.1.7]. As a result, $[\pi^\gamma] \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n, \gamma_n = 1}$.

Moreover, this implies $t \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n, \gamma_n = 1}$ by [BC16, Lemma 2.5(i)] again.

Now we proceed to prove the result by induction on $k$. When $k = 1$, this is Lemma 4.14. Suppose the lemma is proven for up to $k - 1$, then given $y \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n}$, $\theta(y) \in \hat{L}^{\hat{G} - \theta_n}$, hence there exists $n$ large enough, such that $\theta(y) = f(\theta(t) - \beta_n)$, where $f(T) \in K(\mu_{r(n)}, \pi_{r(n)})\{\{T\}\}_n$. Thanks to Lemma 4.26, $\theta_k(\pi) \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n}$. On the other hand, by Lemma 4.17, $f(\theta_k(t) - \beta_n)$ converges in $B_{dR,k,L}^+$, hence $f(\theta_k(t) - \beta_n) \in (B_{dR,k,L}^+)\hat{G}^{-\theta_n}$. Also, $y - f(\theta_k(t) - \beta_n) \in t \cdot B_{dR,k,L}^+$, hence there exists $z \in B_{dR,k-1,L}^+$, such that $y - f(\theta_k(t) - \beta_n) = tz$. Moreover, $z \in (B_{dR,k-1,L}^+)\hat{G}^{-\theta_n}$ as we have shown $t$ is locally analytic. Then the desired result for $k$ follows from induction.

The following lemma is used in the proof:

**Lemma 4.17.** Let $E$ be a finite extension of $\mathbb{Q}_p$ and $f(T) = \sum_{k \geq 0} a_k T^k \in E[[T]]$. Suppose $x \in B_{dR,k,L}^+$ for some $k \geq 1$, then $f(x)$ converges in $B_{dR,k,L}^+$ if and only if $f(\theta(x))$ converges in $\mathbb{C}_p$. 

Proof. This is [BC16, Lemma 4.9], we quickly review its proof here. Recall that $B^+_{\text{dR},k,L}$ is a $\mathbb{Q}_p$-Banach space with unit ball $A_{\text{inf}}(\mathcal{O}_L)$. By enlarging $E$ if necessary we can assume that $E$ contains an element whose valuation is precisely that of $\theta(x)$, hence it suffices to show if $\theta(x) \in \mathcal{O}_L$, then $\{x^n\}$ is bounded in $B^+_{\text{dR},k,L}$. Pick $x_0 \in A_{\text{inf}}(\mathcal{O}_L)$ such that $\theta(x_0) = \theta(x)$, then $x = x_0 + ty + t^k z$ for some $y \in A_{\text{inf}}(\mathcal{O}_L)[[t]]$, $z \in B^+_{\text{dR},L}$. This implies

$$x^n = x_0^n + \left(\frac{n}{1}\right)x_0^{n-1} \cdot ty + \cdots + \left(\frac{n}{k-1}\right)x_0^{n-(k-1)} \cdot (ty)^{k-1} + t^k z_k$$

for some $z_k \in B^+_{\text{dR},L}$. Hence $x^n \in (A_{\text{inf}}(\mathcal{O}_L) + gyA_{\text{inf}}(\mathcal{O}_L) + \cdots + T^n A_{\text{inf}}(\mathcal{O}_L)) + t^n B^+_{\text{dR},L}$ is bounded for any $n \geq 0$. \qed

Remark 4.18. In the cyclotomic case, a similar result is [BC16, Theorem 4.11]. In our case, we use $t$ instead of $\log([\varepsilon])$ in the expression as this makes the calculation of "$\gamma$-invariant" elements quite transparent:

Corollary 4.19. The following are isomorphisms:

$$\bigoplus_{i=0}^{k-1} K^{\infty} \cdot t^i \xrightarrow{\sim} (B^+_{\text{dR},k,L})^{\tilde{G}^{-la},\gamma=1},$$

$$\bigoplus_{i=0}^{k-1} L \cdot t^i \xrightarrow{\sim} (B^+_{\text{dR},k,L})^{\tilde{G}^{-la},\nabla,\gamma=0}.$$  

Proof. We can argue by induction on $k$ similarly as in Theorem 4.16 using Lemma 4.14. \qed

By taking inverse limits, we have that:

Corollary 4.20.

$$(B^+_{\text{dR},L})^{\tilde{G}^{-la}} = (\bigcup_{n \geq 1} K \left(\mu_{r(n)}, \pi_{r(n)}\right) \left\{t - \beta_n\right\})[[t]],$$

$$(B^+_{\text{dR},L})^{\tilde{G}^{-la},\nabla,\gamma=0} = L[[t]],$$

$$(B^+_{\text{dR},L})^{\tilde{G}^{-la},\gamma=1} = K^{\infty}[[t]].$$

Corollary 4.21. The map $K^{\infty}[[t]] \to (B^+_{\text{dR},L})^{\tilde{G}^{-la}}$ is faithfully flat, here we equip $(B^+_{\text{dR},L})^{\tilde{G}^{-la}}$ with a $K^{\infty}[[t]]$-module structure by sending $t$ to $E([\pi^n])$ as usual.

Proof. Thanks to Corollary 4.20, the flatness follows using [SP22, Tag 00MK], which implies faithful flatness by [SP22, Tag 00HR]. \qed

Remark 4.22. Given $X \in \text{Rep}_G(B^+_{\text{dR},L})$, if $X$ is killed by $t^n$, then $X^{\tilde{G}^{-la},\gamma=1}$ is a $(B^+_{\text{dR},m,L})^{\tilde{G}^{-la},\gamma=1}$, module, i.e. a $K^{\infty}[[t]]/t^m$-module thanks to Corollary 4.19. Passing to inverse limit, we have that for general $X$, $X^{\tilde{G}^{-la},\gamma=1}$ is a $K^{\infty}[[t]]$-module.

Theorem 4.23. For any $W \in \text{Rep}_G(B^+_{\text{dR},L})$, $W^{\tilde{G}^{-la}} \in \text{Rep}_G((B^+_{\text{dR},L})^{\tilde{G}^{-la}})$. Moreover, the natural map

$$\beta_W : B^+_{\text{dR},L} \otimes (B^+_{\text{dR},L})^{\tilde{G}^{-la}} W^{\tilde{G}^{-la}} \to W$$

is an isomorphism.

Proof. As the map $(B^+_{\text{dR},L})^{\tilde{G}^{-la}} \to (B^+_{\text{dR},L})$ is faithfully flat, to show $W^{\tilde{G}^{-la}}$ is finitely generated over $(B^+_{\text{dR},L})^{\tilde{G}^{-la}}$, it suffices to check this after scalar extension to $B^+_{\text{dR},L}$. Hence it reduces to show that $\beta_W$ is an isomorphism. For this, we notice that by [GP21, Prop. 3.1.6]

$$W^{\tilde{G}^{-la}} = (W^{\text{Gal}(L/K_p)})^{\text{K}^{-1}a} \otimes (B^+_{\text{dR},K_p})^{\text{K}^{-1}a} (B^+_{\text{dR},L})^{\tilde{G}^{-la}}.$$  

On the other hand,

$$W = W^{\text{Gal}(L/K_p)} \otimes B^+_{\text{dR},K_p} B^+_{\text{dR},L}.$$
Hence it suffices to show that
\[(W^\Gal(L/K_{p}\infty))^\Gamma_{K} \sim= B_{\dR,K_{p}\infty} = W^\Gal(L/K_{p}\infty).\]
But as \(\Gamma_{K}\) is a \(p\)-adic Lie group of dimension 1, \(\Gamma_{K}\)-locally analytic vectors coincide with classical \(K\)-finite vectors thanks to [BC16, Thm. 3.2], hence
\[(W^\Gal(L/K_{p}\infty))^\Gamma_{K} \sim= (W^\Gal(L/K_{p}\infty))_{f},\]
where the latter is defined in [Fon04, Section 3.3], hence the desired result follows from [Fon04, Thm.3.6].

**Remark 4.24.** As a consequence, one can see that for \(W \in \Rep_{G}(B_{\dR,L}^{+}), W^{G_{\hat{\gamma}a,\gamma}=1} \sim= (W/t^{n})^{G_{\hat{\gamma}a}}.\)

Now we are ready to state our main theorem in this section, which should be viewed as a lifting of Sen’s decomposition theory for Kummer tower developed by Gao in [Gao22].

**Theorem 4.25.** For any \(W \in \Rep_{G}(B_{\dR,L}^{+}), W^{\hat{\gamma}a,\gamma}=1 \) is a finitely generated \(K_{\infty}[t] \)-module. Moreover, the natural map
\[\alpha_W : (B_{\dR,L}^{+})^{G_{\hat{\gamma}a}} \otimes_{K_{\infty}[t]} W^{G_{\hat{\gamma}a,\gamma}=1} \rightarrow W^{G_{\hat{\gamma}a}}\]
is an isomorphism.

In the following, for \(W \in \Rep_{G}(B_{\dR,L}^{+}),\) we write \(D_{\Sen,K_{\infty}[t]}(W) := W^{G_{\hat{\gamma}a,\gamma}=1}.\)

Our strategy is to treat the torsion case first, imitating Gao’s proof in [Gao22, Proposition 3.2.7] for the \(t\)-torsion case. In particular, this should be viewed as a higher level \(\theta_{k}\)-specialization of [GP21, Remark 6.1.7]. Hence we need several preliminaries.

**Lemma 4.26.** For \(k \geq 1,\) under the specialization map \(\theta_{k} : A_{\inf}(O_{\hat{L}})[\frac{1}{p}] \rightarrow A_{\inf}(O_{\hat{L}})[\frac{1}{p}]/(\xi^{k}), \theta_{k}(t) \in (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}} \) and \(\theta_{k}(t)^{-1} \in (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}\), here \(t\) is defined in Lemma 4.14.

**Proof.** \(\theta_{k}(t)\) is nonzero by [Gao22, Lemma 3.2.1].

To prove analyticity of \(\theta_{k}(t),\) start from [GP21, Lemma 5.1.1], which tells us that there exists \(n = n(t)\) such that \(t \in (A_{\inf}(O_{\hat{L}})[\frac{1}{p}, \frac{p}{[\pi^{p}]^{n}}], \frac{1}{p})^{G_{\hat{\gamma}a}},\) hence implies the analyticity of \(t\) under
\[A_{\inf}(O_{\hat{L}})[\frac{1}{p}, \frac{p}{[\pi^{p}]^{n}}] \rightarrow B_{\dR,L}^{+} \xrightarrow{\theta_{k}} B_{\dR,k,L}^{+}.\]

In other words, \(\theta_{k}(\varphi^{-n}(t)) \in (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}.\) On the other hand, recall \(s\) satisfies that \(\varphi(t) = \frac{E(u)}{E(0)} t,\) hence
\[t = \varphi^{-n}(t) \cdot \prod_{i=1}^{n} \varphi^{-i}(\frac{pE(u)}{E(0)}),\]
so it suffices to check \(\theta_{k}(\varphi^{-i}(\frac{pE(u)}{E(0)}))\) is locally analytic in \(B_{\dR,k,L}^{+}.\) For this, we use the same trick as in the proof of Theorem 4.16. Let \(x = \varphi^{-i}(u),\) then \(x\) is \(\Gal_{L/K_{\infty}}\)-invariant, and
\[\tau^{i}(x) = [\epsilon] x,\]
from which we can deduce that \((\log(\tau^{i}))^{j}(x) \in t^{j}B_{\dR,k,L}^{+}\) by induction on \(j.\) Then notice that \(\tau^{i}\) is a topological generator of \(\Gal_{L/K_{\infty}},\) we have that \(x \in (B_{\dR,k,L}^{+})^{\varphi^{-i}},\) which further implies that \(x \in (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}.\)

Now we conclude that \(\theta_{k}(t) \in (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}.\) The analyticity of \(\theta_{k}(\frac{1}{t})\) can be proven similarly.

**Proof of Theorem 4.25.** Consider the torsion case first, i.e. suppose \(W \in \Rep_{G}(B_{\dR,L}^{+})\) is killed by \(t^{k}\) for some \(k.\) We proceed in two steps:

- First we show \(W^{G_{\hat{\gamma}a}} = 0\) satisfies that
  \[W^{G_{\hat{\gamma}a}} = 0 \otimes_{(B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}} (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}} \xrightarrow{\cong} W^{G_{\hat{\gamma}a}}.\]

- Second we show \(W^{G_{\hat{\gamma}a}} \otimes_{(B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}}} (B_{\dR,k,L}^{+})^{G_{\hat{\gamma}a}} \xrightarrow{\cong} W^{G_{\hat{\gamma}a}}.\)
Combine these two identities together we see $\alpha_W$ is an isomorphism (for $W$ killed by a power of $t$).

**Step 1** Define $\partial_\gamma$ to be the $\theta_k$-specialization of that in [GP21, 5.3.4], i.e. $\partial_\gamma = \frac{1}{\theta_k(t)} \nabla_\gamma$. Then $\partial_\gamma(\theta_k(t)) = 1$ as $\nabla_\gamma(t) = t$. Fix a minimal generating set of $\{x_1, \cdots, x_l\}$, on which $\partial_\gamma$ acts via $D_\gamma \in \text{Mat}(\mathcal{B}_{\text{DR}, k, L}^+)^{G_{la}}$. Let $\xi = (x_1, \cdots, x_l)$. It suffices to show surjectivity of $M^{\nabla_\gamma = 0} \otimes (\mathcal{B}_{\text{DR}, k, L}^+)_{\nabla = 0} \to \mathcal{B}_{\text{DR}, k, L}^+$, which can be reduced to finding a matrix $H \in GL_l((\mathcal{B}_{\text{DR}, k, L}^+)_{\nabla = 0})$ such that

$$\partial_\gamma(H) + D_\gamma H = 0.$$ 

We claim that $H = \sum_{s \geq 0} (-1)^s D_s \frac{(\theta_k(t) - \beta_s)^s}{s!}$ works for some $n$ large enough, here $D_s$ is defined to be the matrix via which $\partial_\gamma^s$ acts on $\xi$. Actually,

$$\partial_\gamma(\xi H) = \partial_\gamma(\xi) H + \xi \partial_\gamma(H)$$

$$= \sum_{s \geq 0} (-1)^s D_{s+1} \frac{(\theta_k(t) - \beta_s)^s}{s!} + \xi \sum_{s \geq 1} (-1)^s D_s \frac{(\theta_k(t) - \beta_s)^{s-1}}{(s-1)!}$$

$$= 0.$$

Also, for large enough $n$, $H$ converges as this can be checked after modulo $\theta$ thanks to Lemma 4.17.

**Step 2** By step 1, $\xi H$ forms a minimal generating set $\{\eta_1, \cdots, \eta_l\}$ for $(\mathcal{B}_{\text{DR}, k, L}^+)_{\nabla = 0}$-module $W_{\nabla = 0}$. Suppose $t^{n_1}$ is the proper annihilator of $y_l$. Then if we view $W_{\nabla = 0}$ as a $L$-vector space thanks to Corollary 4.19, then $\{t^{n_1} y_1, \cdots, t^{n_1} y_l\}$ forms a $L$-basis. By étale descent as shown in [Gao22, Thm 3.3.1], we have that $W_{\nabla = 0}$ is $K_\infty$-space of dimension $\sum m_l$ and that $W_{\nabla = 0}$ is a $K_\infty$-module as $t$ is fixed by $Gal(L/K_\infty)$. Thanks to Corollary 4.19 again, this implies that

$$(4.28) W_{\nabla = 0} = (\mathcal{B}_{\text{DR}, k, L}^+)_{\nabla = 0} \to W_{\nabla = 0}.$$ 

**Step 3** For general $W$ (not necessarily $t$-power torsion), recall that by definition,

$$W_{\nabla = 0} = \lim_{\leftarrow} (W/t^n W)_{\nabla = 0}.$$ 

Moreover, by our proof in step 1 and step 2, we see that each $(W/t^n W)_{\nabla = 0}$ is a finitely generated $K_\infty[[t]]/t^n$-module such that

$$\left(\frac{W}{t^{n+1} W}\right)_{\nabla = 0} \to \left(\frac{W}{t^n W}\right)_{\nabla = 0}.$$ 

Hence $W_{\nabla = 0}$ is a finitely generated $K_\infty[[t]]$-module and that $W_{\nabla = 0} = \lim_{\leftarrow} (W/t^n W)_{\nabla = 0}$ by [SP22, Tag 09B8]. Together with Remark 4.24, we see that $\alpha_W$ mod $t^n$ is precisely $\alpha_W/t^n$, hence an isomorphism by combining Eq. (4.27) and Eq. (4.28). Iterating all $n$ we conclude that $\alpha_W$ is an isomorphism. \[\square\]

**Theorem 4.29.** Given $W \in \text{Rep}_{G_K}(B_{\text{DR}}^+)$, by abuse of notation we still define

$$D_{\text{Sen}, K_\infty[[t]]}(W) := (W_{G_L})_{\nabla = 0}.$$ 

Then this is a finitely generated $K_\infty[[t]]$-module such that

$$D_{\text{Sen}, K_\infty[[t]]}(W) \otimes_{K_\infty[[t]]} B_{\text{DR}}^+ = W.$$ 

**Proof.** This follows from combining Proposition 4.4, Theorem 4.23 and Theorem 4.25. \[\square\]

**4.3. Kummer-Sen operator.** We consider $W \in \text{Rep}_{G_K}(B_{\text{DR}}^+)$. We can define a monodromy operator on $D_{\text{Sen}, K_\infty[[t]]}(W)$. Actually, for each $k \in \mathbb{N}$, since vectors in $(W_{G_L}/t^k)^{\nabla = 0} = (W_{G_L}/t^k)_{\nabla = 0}$ by [GP21, Lemma 3.2.4] are all $\tau$-locally analytic, hence we can define

$$N_{\nabla, k} := \frac{1}{t^k \theta_k(t)} \nabla_\tau : (W_{G_L}/t^k)^{\nabla = 0} \to (W_{G_L}/t^k)^{\nabla = 0}.$$ 


Clearly $N_{\varphi,m}$ are compatible with $N_{\varphi,n}$, hence by passing to inverse limits, we get
\begin{equation}
(4.30) \quad N_{\varphi} = \frac{1}{p^t} \nabla_{\varphi} : (W^{G_{L}})^{\tau^{-1}a, \gamma=1} \rightarrow (W^{G_{L}})^{G^{-1}a}.
\end{equation}

**Theorem 4.31.** Eq. (4.30) induces a $K_{\infty}$-linear operator
\[ N_{\varphi} : D_{\text{Sen}, K_{\infty}[[t]]}(W) \rightarrow D_{\text{Sen}, K_{\infty}[[t]]}(W) \]
such that $N_{\varphi}$ satisfies Leibniz rule and that
\[ N_{\varphi}(tv) = N_{\varphi}(t)v + tN_{\varphi}(v) = E'(u)\lambda u \cdot v + tN_{\varphi}(v). \]
We will call $N_{\varphi}$ the Kummer Sen operator.

**Proof.** Clearly $N_{\varphi}$ is $K_{\infty}$-linear as $\nabla_{\varphi}$ vanishes on $K_{\infty}$, hence it suffices to show that $N_{\varphi}$ is a operator from $D_{\text{Sen}, K_{\infty}[[t]]}(W)$ to itself, i.e. $N_{\varphi}$ preserves $\text{Gal}(L/K_{\infty})$-invariant elements. For this, notice that $t\tau \gamma^{-1} = \tau (\gamma)$, hence $\gamma \nabla_{\varphi} = \chi(\gamma) \nabla_{\varphi} \gamma$. Also, as $\gamma(t) = \chi(\gamma)t$, we conclude that
\[ \gamma N_{\varphi} = \frac{1}{p^t} \gamma \nabla_{\varphi} = \frac{1}{p^t} \chi(\gamma) \nabla_{\varphi} \gamma = N_{\varphi} \gamma. \]
This implies that $N_{\varphi}$ preserves $D_{\text{Sen}, K_{\infty}[[t]]}(W)$. To show that $N_{\varphi}$ satisfies Leibniz rule, it suffices to check $N_{\varphi}(tv) = N_{\varphi}(t)v + tN_{\varphi}(v)$. But by definition,
\[ N_{\varphi}(tv) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \left( \sum_{i=1}^{n} \binom{n}{i} (\varepsilon)^{i} \right) u = \frac{1}{p^n} \left( \sum_{i=1}^{n} \binom{n}{i} (\varepsilon)^{i} \right) u = \frac{1}{p^n} \log([\varepsilon]) u = \lambda u. \]

For the former, we just need to check that
\[ N_{\varphi}(u) = \frac{1}{p^n} \lim_{n \rightarrow \infty} \frac{\tau^{p^n}(u)}{p^n} = \frac{1}{p^n} \lim_{n \rightarrow \infty} \frac{\log(\tau^{p^n}(u))}{p^n} = \frac{1}{p^n} \lim_{n \rightarrow \infty} \frac{\log(\tau^{p^n}(u))}{p^n} = \frac{1}{p^n} \log([\varepsilon]) u = \lambda u. \]

For the later, it suffices to check after modulo $t^k$ for any $k \in \mathbb{N}$. On the other hand, notice that by Example 2.7
\[ \frac{\tau^{p^n}(t)}{t} = \alpha(\tau^{p^n}) = 1 - \beta X_1(\tau^{p^n}) + \sum_{n=1}^{\infty} t^{i}(\sum_{n=1}^{\infty} \binom{n}{i} (\varepsilon)^{i}) u = \frac{1}{n!} \log([\varepsilon]) u = \lambda u. \]

For a fixed $k$, $\theta_k(\alpha(\tau^{p^n}))$ is a finite sum and that $\lim_{n \rightarrow \infty} \theta_k(X_1(\tau^{p^n})) = 0$ by Lemma 5.7 and the proof of Proposition 5.9, hence $\lim_{n \rightarrow \infty} \theta_k(\alpha(\tau^{p^n})) = 1$, which implies the desired result.

Motivated by Theorem 4.31, we can introduce the following definition:

**Notation 4.32.** Define $\text{Mod}_{K_{\infty}[[t]]}$ to be the category consisting of objects are finite generated $K_{\infty}[[t]]$-modules $M$ equipped with a $K_{\infty}$-linear self map $N_{\varphi} : M \rightarrow M$ such that $N_{\varphi}$ satisfies Leibniz rule and that $N_{\varphi}(tv) = E'(u)\lambda u \cdot v + tN_{\varphi}(v)$. Morphisms in $\text{Mod}_{K_{\infty}[[t]]}$ are $K_{\infty}[[t]]$-module homomorphisms commuting with $N_{\varphi}$.

**Remark 4.33.** By the definition of $\lambda$, we see that as an element in $K_{\infty}[[t]]$, $\lambda = E(u)\lambda_1 = t\lambda_1$ for some units $\lambda_1 \in K[[t]]$. Hence given $\text{Mod}_{K_{\infty}[[t]]}$, for $v \in M$, $N_{\varphi}(tv) = E'(u)\lambda_1 utv + tN_{\varphi}(v) \in tM$. Then by induction we also see that $N_{\varphi}(t^v) \subseteq t^v M$.

Thanks to Theorem 4.31, we have a natural functor:
\[ D : \text{Rep}_{G_K}(B_{dR}^+) \rightarrow \text{Mod}_{K_{\infty}[[t]]} \]
\[ W \mapsto (D_{\text{Sen}, K_{\infty}[[t]]}(W), N_{\varphi}) \]
Lemma 4.34. $D$ is exact and faithful. Moreover, $D$ preserves tensor product and internal Hom such that for $W_1, W_2 \in \text{Rep}_{G_K}(B_{\text{dr}}^+)$, $f \in \text{Hom}_{B_{\text{dr}}^+}(W_1, W_2)$,

$$N\nu \cdot W_1 \otimes W_2 = 1 \otimes N\nu \cdot W_2 + N\nu \cdot W_1 \otimes 1$$

$$N\nu \cdot \text{Hom}_{B_{\text{dr}}^+}(W_1, W_2)(f) = N\nu \cdot W_2 \circ f - f \circ N\nu \cdot W_1.$$  

Proof. To show it is exact, it suffices to show that given a short exact sequence $0 \to W_1 \to W_2 \to W_3 \to 0$ in $\text{Rep}_{G_K}(B_{\text{dr}}^+)$, it is still exact after applying $D_{\text{Sen}, K}(f)(\cdot)$ functor. But as $K_{\infty}[\{t\}] \to B_{\text{dr}}^+$ is faithfully flat, it suffices to check the exactness of

$$0 \to D_{\text{Sen}, K_{\infty}}[\{t\}](W_1) \otimes_{K_{\infty}[\{t\}]} B_{\text{dr}}^+ \to D_{\text{Sen}, K_{\infty}}[\{t\}](W_2) \otimes_{K_{\infty}[\{t\}]} B_{\text{dr}}^+ \to D_{\text{Sen}, K_{\infty}}[\{t\}](W_3) \otimes_{K_{\infty}[\{t\}]} B_{\text{dr}}^+ \to 0.$$  

But this is just the exact sequence of underlying $B_{\text{dr}}^+$-modules of the original exact sequence thanks to Theorem 4.29. The faithfulness can be proven similarly.

Finally we have a natural map $D_{\text{Sen}, K_{\infty}}[\{t\}](W_1) \otimes D_{\text{Sen}, K_{\infty}}[\{t\}](W_2) \to D_{\text{Sen}, K_{\infty}}[\{t\}](W_1 \otimes W_2)$, to show that it is an equality, it suffices to check that after the faithfully flat base change $K_{\infty}[\{t\}] \to B_{\text{dr}}^+$, which follows from Theorem 4.29 again. The desired property for $N\nu$ can be calculated directly using $N\nu = \frac{1}{p^n} \nabla \tau$.

For internal Hom, the result follows from the tensor product case as $\text{Hom}_{B_{\text{dr}}^+}(W_1, W_2) = W_1 \otimes_{B_{\text{dr}}^+} W_2$ in $\text{Rep}_{G_K}(B_{\text{dr}}^+)$.

\[ \square \]

Remark 4.35. However, $D$ is neither fully faithful nor essentially surjective. For the former, just notice that morphisms in $\text{Mod}_{\text{Rep}_{G_K}}^N[\{t\}]$ are $K_{\infty}$-linear, while morphisms in $\text{Rep}_{G_K}(B_{\text{dr}}^+)$ are only $K$-linear. However, we would like to show that $D$ at least preserves the information about isomorphism classes in the remaining of this section.

Let us consider the $t$-torsion case first.

Proposition 4.36. For $W \in \text{Rep}_{G_p}(G_K)$, the $K_{\infty}$-linear operator $N\nu$ on the finite dimensional $K_{\infty}$-vector space $D_{\text{Sen}, K_{\infty}}(W)$ satisfies the following:

- The kernel $\text{Ker}(N\nu)$ consists exactly of those $x \in W^{G_L}$ which are further fixed by $\text{Gal}(L/K_{\infty})$ and whose $\hat{G}$-orbit is finite.
- $\text{Ker}(N\nu)$ is equal to $W^{G_K} \otimes_K K_{\infty}$. In particular, $N\nu$ is an isomorphism if and only if $(W)^{G_K} = 0$.
- $N\nu = 0$ if and only if $W^{G_L}$ has discrete $\hat{G}$-action. Also, $\dim_K W^{G_K} \leq \dim_{K_{\infty}} D_{\text{Sen}, K_{\infty}}(W)$.

Proof. First given $x \in \text{Ker}(N\nu)$, by definition, we see that $x \in W^{G_L}$ is fixed by $\text{Gal}(L/K_{\infty})$ and that $\log(\tau)(x) = 0$, the later implies that for $n$ large enough, $\tau^{p^n}(x) = \exp(p^n \log(\tau))(x) = x$, hence the $\hat{G}$-orbit of $x$ is finite. On the other hand, if $x \in W^{G_L}$ is $G/(L/K_{\infty})$-invariant and that the $\hat{G}$-orbit of $x$ is finite, then for $n$ large enough, $\tau^{p^n}(x) = 0$, hence $x \in W^{G_K} \otimes_K K_{\infty}$ and that $\log(\tau)(x) = \lim_{n \to \infty} \frac{\tau^{p^n}(x) - x}{p^n} = 0$.

Finally we have a natural map $K_{\infty} \otimes (W)^{G_K} \to \text{Ker}(N\nu)$, to show that it is an isomorphism, it suffices to check that after the faithfully flat field extension $K_{\infty} \to \hat{L}^{\hat{G} - \text{la}}$. Notice that due to the work of Gao (see [Gao22, Theorem 3.3.1, Theorem 3.3.2]), on $D_{\text{Sen}, K_{\infty}}(W) \otimes_{K_{\infty}} \hat{L}^{\hat{G} - \text{la}} = D_{\text{Sen}, K_{\infty}}(W) \otimes_{K_{\infty}} \hat{L}^{\hat{G} - \text{la}}$, there exist two linearly dependent monodromy operators, namely the $\hat{L}^{\hat{G} - \text{la}}$-linear extension of $N\nu$ on $D_{\text{Sen}, K_{\infty}}(W)$ (which will be denoted as $N\nu \otimes \text{Id}$) and the $\hat{L}^{\hat{G} - \text{la}}$-linear extension of the classical Serre operator $\nabla_{\gamma}$ on $D_{\text{Sen}, K_{\infty}}(W)$ (which will be denoted as $\nabla_{\gamma} \otimes \text{Id}$), and the exact relation between these two monodromy operator is that $N\nu \otimes \text{Id} = \theta(u\lambda')(\nabla_{\gamma} \otimes \text{Id})$, where $\theta(u\lambda')$ is a unit in $K$. As a result,

$$\text{Ker}(N\nu) \otimes \hat{L}^{\hat{G} - \text{la}} = \text{Ker}(N\nu \otimes \text{Id}) = \text{Ker}(\nabla_{\gamma} \otimes \text{Id}) = \text{Ker}(\nabla_{\gamma}) \otimes \hat{L}^{\hat{G} - \text{la}}.$$  

Here the first and last equality holds as any field extension is faithfully flat.

But by a similar argument as in the first paragraph, we see that $\text{Ker}(\nabla_{\gamma})$ is precisely the $K_{\infty}$-subspace of $\Gamma_{K}$-discrete vectors inside $D_{\text{Sen}, K_{\infty}}(W)$, from which a classical Galois descent argument verifies that $\text{Ker}(\nabla_{\gamma}) = K_{\infty} \otimes W^{G_K}$, hence that

$$\text{Ker}(\nabla_{\gamma}) \otimes \hat{L}^{\hat{G} - \text{la}} = W^{G_K} \otimes \hat{L}^{\hat{G} - \text{la}}.$$
Combining Eq. (4.37) and Eq. (4.38), we see that $\text{Ker}(N_{\mathcal{V}}) \otimes \hat{G}^{-\la} = W^{G_K} \otimes \hat{G}^{-\la}$, hence the natural map $K_\infty \otimes (W)^{G_K} \to \text{Ker}(N_{\mathcal{V}})$ is an equality. The other statements then follow.

**Proposition 4.39.** For $W_1, W_2 \in \text{Rep}_{C_p}(G_K)$, the natural map

$$K_\infty \otimes K \text{Hom}_{\text{Rep}_{C_p}(G_K)}(W_1, W_2) \rightarrow \text{Hom}_{\text{Mod}^{N_{\mathcal{V}}}_{K_\infty[[t]]}}((D_{\text{Sen}, K_\infty}(W_1), N_{\mathcal{V}}), (D_{\text{Sen}, K_\infty}(W_2), N_{\mathcal{V}}))$$

is an isomorphism.

**Proof.** Let $W = \text{Hom}_{C_p}(W_1, W_2)$, then thanks to Lemma 4.34, the desired identity is precisely $K_\infty \otimes K W^{G_K} = \text{Ker}(N_{\mathcal{V}}, W)$, which is precisely Proposition 4.36.

**Corollary 4.40.** Two objects $W_1, W_2 \in \text{Rep}_{C_p}(G_K)$ are isomorphic if and only if $(D_{\text{Sen}, K_\infty}(W_1), N_{\mathcal{V}})$ and $(D_{\text{Sen}, K_\infty}(W_2), N_{\mathcal{V}})$ are isomorphic as objects in $\text{Mod}^{N_{\mathcal{V}}}_{K_\infty[[t]]}$.

**Proof.** We imitate the proof for a similar result in the cyclotomic tower setting given in [Fon04, Proposition 2.6]. The only direction if follows from the functoriality of the functor $D_{\text{Sen}, K_\infty}$. For the other direction, assume that $(D_{\text{Sen}, K_\infty}(W_1), N_{\mathcal{V}})$ and $(D_{\text{Sen}, K_\infty}(W_2), N_{\mathcal{V}})$ are isomorphic as objects in $\text{Mod}^{N_{\mathcal{V}}}_{K_\infty[[t]]}$. In particular, this implies that $\text{dim}_{C_p} W_1 = \text{dim}_{C_p} W_2$ thanks to Theorem 4.29. Let $W = \text{Hom}_{C_p}(W_1, W_2) \in \text{Rep}_{C_p}(G_K)$, then our assumption precisely says that $\text{Ker}(N_{\mathcal{V}}, W)$ contains an element of $W$ that is a linear isomorphism by Lemma 4.34. We wish to further prove that $D^{G_K}$ contains an element that is a linear isomorphism, which exactly means that $W_1$ and $W_2$ are isomorphic as objects in $\text{Rep}_{C_p}(G_K)$. For this, one can apply [Fon04, Lemma 2.7] directly to conclude as $K_\infty/K$ is an algebraic extension with $K$ an infinite field.

Now we are ready to treat the general case, which is motivated by the cyclotomic analogue given in [Fon04, Proposition 3.8]. We warn the reader that in the Kummer tower case we need to start with $W \in \text{Rep}_{G_K}(B_{dR}^+)$ as $D_{\text{Sen}, K_\infty[[t]]}(W)$ is not $G_K$-stable inside $W$.

**Proposition 4.41.** For any $W \in \text{Rep}_{G_K}(B_{dR}^+)$, the $K_\infty$-vector space $\text{Ker}(N_{\mathcal{V}})$ (a subspace of $D_{\text{Sen}, K_\infty[[t]]}(W)$) is finite dimensional satisfying that

$$\text{dim}_K W^{G_K} \otimes_K K_\infty = \text{Ker}(N_{\mathcal{V}}).$$

In particular, $\text{dim}_K W^{G_K}$ is finite. Moreover, the following holds:

i) If $W \in \text{Rep}_{B_{dR}^+}(G_K)$, then $\text{dim}_K W^{G_K} \leq \text{dim}_{K_\infty}(D_{\text{Sen}, K_\infty[[t]]}(W)/tD_{\text{Sen}, K_\infty[[t]]}(W))$.

ii) For $W_1, W_2 \in \text{Rep}_{G_K}(B_{dR}^+)$,

$$K_\infty \otimes K \text{Hom}_{\text{Rep}_{G_K}(B_{dR}^+)}(W_1, W_2) \rightarrow \text{Hom}_{\text{Mod}^{N_{\mathcal{V}}}_{K_\infty[[t]]}}((D_{\text{Sen}, K_\infty[[t]]}(W_1), N_{\mathcal{V}}), (D_{\text{Sen}, K_\infty[[t]]}(W_2), N_{\mathcal{V}})).$$

Also, if we further assume that $W_1, W_2 \in \text{Rep}_{B_{dR}^+}(G_K)$, then

$$\text{dim}_K (\text{Hom}_{\text{Rep}_{G_K}(B_{dR}^+)}(W_1, W_2)) \leq \text{dim}_{K_\infty}(D_{\text{Sen}, K_\infty[[t]]}(W_1)/t) \text{dim}_{K_\infty}(D_{\text{Sen}, K_\infty[[t]]}(W_2)/t).$$

To prove this proposition, we need the following lemma:

**Lemma 4.43.** For any $W \in \text{Rep}_{G_K}(B_{dR}^+)$, $(W^{G_L})^{G^{-\la}}, N_{\mathcal{V}}=0, \gamma=0$ is a finite dimensional $L$-vector space.

**Proof.** The proof of Theorem 4.25 implicitly implies that $(W^{G_L})^{G^{-\la}}, N_{\mathcal{V}}=0, \gamma=0$ is a finite generated $L[[t]]$-module. If $W$ is $t$-power torsion, then there is nothing to prove as $(W^{G_L})^{G^{-\la}}, N_{\mathcal{V}}=0, \gamma=0$ is a $L$-subspace of $(W^{G_L})^{G^{-\la}}, \gamma=0$. From now now we assume that $W \in \text{Rep}_{B_{dR}^+}(G_K)$, hence $(W^{G_L})^{G^{-\la}}, \gamma=0$ is a finite free $L[[t]]$-module with the same rank of $W$ by the proof of Theorem 4.25. We aim to show that $\text{dim}_L((W^{G_L})^{G^{-\la}}, N_{\mathcal{V}}=0, \gamma=0)$ is bounded by that rank. Argue exactly as that in Theorem 4.31, we have that $N_{\mathcal{V}}$ maps $(W^{G_L})^{G^{-\la}}, \gamma=0$ to itself, in particular, $(W^{G_L})^{G^{-\la}}, \gamma=0, N_{\mathcal{V}})$ defines an object in $\text{Mod}^{N_{\mathcal{V}}}_{L[[t]]}$. Hence it suffices to show the following:
(\*) Suppose \((M, N_\mathcal{V}) \in \text{Mod}_{L[[t]]}^\mathcal{V}\) satisfies that \(M\) is a finite free \(L[[t]]\)-module, then the natural map
\[
L[[t]] \otimes_L M^{N_\mathcal{V}=0} \rightarrow M
\]
is injective.

To prove (\*), we argue by contradiction, otherwise we can pick a nonzero element \(x\) in the kernel with a minimal length expression \(\sum f_i \otimes v_i\) in elementary tensors. In particular, all \(f_i\) are nonzero and that \(v_i\) are linearly independent over \(L\). Suppose \(f_i = t^{n_i}g_i\) such that \(g_i(0) \neq 0\), i.e. \(g(i)\) is a unit in \(L[[t]]\). Then by dividing \(t^{\min\{n_i\}}\) and multiplying a suitable unit, we can assume without of loss generality that \(f_1 = 1\). Applying \(N_\mathcal{V}\) to 0 = \(\sum f_i v_i\) and using Leibniz rule, we have that
\[
0 = \sum (N_\mathcal{V}(f_i)v_i + f_i N_\mathcal{V}(v_i)) = \sum N_\mathcal{V}(f_i)v_i
\]
as \(N_\mathcal{V}(v_i) = 0\) by assumption. Notice that \(N_\mathcal{V}(f_1) = 0\), hence by minimal length assumption we must have that \(N_\mathcal{V}(f_i) = 0\) for arbitrary \(i\). Using Remark 4.33 and that \(N_\mathcal{V}(t) = E'(u)\lambda_1 u t\) with \(E'(u)\lambda_1 u\) being a unit, one can easily verify that for \(v_i\), we proceed as in Proposition 4.36, \((W^G_L)_{G-\text{locally analytic vectors coincides with classical } K\text{-finite vectors thanks to [BC16, Thm. 3.2], hence\}}
\[
(W^G_{\mathcal{K}})_{K-\text{finite vectors}} \rightarrow \text{Ker}(N_\mathcal{V}),
\]
where the latter denotes the subspace of \(W^G_{\mathcal{K}}\) consisting of those vectors whose \(\Gamma_\mathcal{K}\) orbits are finite, see [Fon04, Section 3.3] for details. In particular, by [Fon04, Theorem 3.6], \((W^G_{\mathcal{K}})_{K-\text{finite vectors}} \rightarrow \text{Ker}(N_\mathcal{V}),\)
\[
(W^G_{\mathcal{K}})_{K-\text{finite vectors}} \rightarrow \text{Ker}(N_\mathcal{V}),\]
which is also \(\dim_{\mathbb{K}} \text{Ker}(N_\mathcal{V}) = N_\mathcal{V}(v_i)\), i.e. \(N_\mathcal{V}(v_i) = 0\), hence the proof of Lemma 4.43 implies that it is bounded by the rank of \(D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W)\), also \(\dim_{\mathbb{K}} D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W) = \dim_{\mathbb{K}} D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W)\).

For \(i\), just notice that by Eq. (4.46), \(\dim_{\mathbb{K}} W^G_{\mathcal{K}} = \dim_{\mathbb{K}} (W^G_L)_{G-\text{finite vectors}}\), hence the proof of Lemma 4.43 implies that it is bounded by the rank of \(D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W)\), which is also \(\dim_{\mathbb{K}} D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W)\).

For \(ii\), we proceed as in Proposition 4.39. Consider \(W = \text{Hom}_{B^+_\text{der}}(W_1, W_2) \in \text{Rep}_{G_{\mathcal{K}}}(B^+_\text{der})\). By the proof of Lemma 4.34, we see that
\[
\text{Hom}_{\text{Mod}^\mathcal{V}_{\mathcal{K}-\text{finite vectors}}}(D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W_1), D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W_2)) = D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W),
\]
and that
\[
N_{\mathcal{V}, W}(f) = N_{\mathcal{V}, W_2} \circ f \circ N_{\mathcal{V}, W_1}.
\]
Hence given \(f \in D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W)\), \(f\) defines a morphism in \(\text{Mod}^\mathcal{V}_{\mathcal{K}-\text{finite vectors}}\) if and only if \(f \in \text{Ker}(N_{\mathcal{V}, W})\). In other words,
\[
\text{Hom}_{\text{Mod}^\mathcal{V}_{\mathcal{K}-\text{finite vectors}}}(D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W_1), D_{\text{Sen},\mathcal{K}-\text{finite vectors}}(W_2), N_{\mathcal{V}}) = \text{Ker}(N_{\mathcal{V}, W}).
\]
Then we can apply Eq. (4.42) to conclude that
\[(4.47) \quad K_\infty \otimes_K \text{Hom}_{\text{Rep}_{G_K}(B_{\text{dr}}^+)}(W_1, W_2) \xrightarrow{\cong} \text{Hom}_{\text{Mod}_{K_{\infty}}^N}[\text{t}](D_{\text{Sen},K_\infty}(W_1), N_{\infty}), (D_{\text{Sen},K_\infty}(W_2), N_{\infty})),\]
from which we see that for $W_1, W_2 \in \text{Rep}_{B_{\text{dr}}^+}(G_K)$,
\[
\dim_K(\text{Hom}_{\text{Rep}_{G_K}(B_{\text{dr}}^+)}(W_1, W_2) \leq \dim_K(D_{\text{Sen},K_\infty}[\text{t}](W_1)) \dim_K(D_{\text{Sen},K_\infty}[\text{t}](W_2))
\]
\[
= \dim_K(D_{\text{Sen},K_\infty}[\text{t}](W_1))/t \dim_K(D_{\text{Sen},K_\infty}[\text{t}](W_2)/t).
\]

\[\square\]

**Corollary 4.48.** Two objects $W_1, W_2 \in \text{Rep}_{G_K}(B_{\text{dr}}^+)$ are isomorphic if and only if $(D_{\text{Sen},K_\infty}[\text{t}](W_1), N_{\infty})$ and $(D_{\text{Sen},K_\infty}[\text{t}](W_2), N_{\infty})$ are isomorphic as objects in $\text{Mod}_{K_{\infty}}^N[\text{t}]$.

**Proof.** The "only if" is obvious by functoriality of $D_{\text{Sen},K_\infty}[\text{t}]$. To show the other direction, we proceed as in [Fon04, Proposition 3.8]. Assume that there is an isomorphism $f_0$ in
\[
\text{Hom}_{\text{Mod}_{K_{\infty}}^N}[\text{t}](D_{\text{Sen},K_\infty}[\text{t}](W_1), N_{\infty}), (D_{\text{Sen},K_\infty}[\text{t}](W_2), N_{\infty})),
\]
and we wish to construct a $\Gamma_\kappa$-equivariant isomorphism from $W_1$ to $W_2$ (see Eq. (4.47)). Let $\{x_1, \cdots, x_d\}$ and $\{y_1, \cdots, y_{d'}\}$ be minimal $K_\infty[\text{t}]$-module generating sets for $D_{\text{Sen},K_\infty}[\text{t}](W_1)$ and $D_{\text{Sen},K_\infty}[\text{t}](W_2)$ separately. By assumption we naturally have that $D_{\text{Sen},K_\infty}[\text{t}](W_1)$ and $D_{\text{Sen},K_\infty}[\text{t}](W_2)$ are isomorphic as $K_\infty[\text{t}]$-modules via $f_0$, in particular, $d = d'$.

Notice that $\text{Hom}_{\text{Rep}_{G_K}(B_{\text{dr}}^+)}(W_1, W_2)$ is a finite dimensional $K$-vector space by Proposition 4.41, hence we can choose a $K$-basis $\{f_1, \cdots, f_n\}$ of it. Let
\[
\overline{f}_i \in \text{Hom}_{K_\infty}(D_{\text{Sen},K_\infty}[\text{t}](W_1)/tD_{\text{Sen},K_\infty}[\text{t}](W_1), D_{\text{Sen},K_\infty}[\text{t}](W_2)/tD_{\text{Sen},K_\infty}[\text{t}](W_2))
\]
be the reduction of $f_i$ modulo $t$. Concretely, each $\overline{f}_i$ can be described by a $d \times d$ matrix over $K_\infty$ utilizing the bases $\{\overline{x}_j\}$ and $\{\overline{y}_j\}$, which are just reductions of $\{x_j\}$ and $\{y_j\}$ modulo $t$. Now apply Proposition 4.41 ii), we see that $\{f_1, \cdots, f_n\}$ forms a $K_\infty$-basis for
\[
\text{Hom}_{\text{Mod}_{K_{\infty}}^N}[\text{t}](D_{\text{Sen},K_\infty}[\text{t}](W_1), N_{\infty}), (D_{\text{Sen},K_\infty}[\text{t}](W_2), N_{\infty})),
\]
which contains an isomorphism by our assumption. As a result, there exist $\lambda_1, \cdots, \lambda_n \in K_\infty$ such that $\det(\lambda_1 \overline{f}_1 + \cdots + \lambda_n \overline{f}_n) \neq 0$, which implies that the polynomial $\det(X_1 \overline{f}_1 + \cdots + X_n \overline{f}_n) \in K_\infty[X_1, \cdots, X_n]$ is non-zero. On the other hand, as $K$ is an infinite field, we can pick $\mu_1, \cdots, \mu_n \in K$ such that $\det(\overline{f}) \neq 0$ for $f = \mu_1 f_1 + \cdots + \mu_n f_n$. Then $f$ defines a $K_\infty[\text{t}]$-linear isomorphism $(D_{\text{Sen},K_\infty}[\text{t}](W_1) \rightarrow (D_{\text{Sen},K_\infty}[\text{t}](W_2)$ as its reduction modulo $t$ is an isomorphism for determinant reasons. But now $f \in \text{Hom}_{\text{Rep}_{G_K}(B_{\text{dr}}^+)}(W_1, W_2)$, hence $f$ is an isomorphism in $\text{Rep}_{G_K}(B_{\text{dr}}^+)$, we are done.

\[\square\]

5. **Representations associated to de Rham prismatic crystals**

5.1. **De Rham crystals and $B_{\text{dr}}^+$-representations.** We define $A_L$ to be $A_{\text{inf}}(\mathcal{O}_L)$ and $A_{L, \text{perf}}$ to be the $i$-th self products of $A_L$ in $(\mathcal{O}_K)_{\Delta}^{\text{perf}}$.

We denote by $\text{Vect}((\mathcal{O}_K)_{\Delta}^{\text{perf}}, (\mathcal{O}_K_{\Delta}[1])_{\Delta}^{1/2})$ the category of de Rham prismatic crystals on $(\mathcal{O}_K)_{\Delta}^{\text{perf}}$.

**Corollary 5.1.** The category of de Rham prismatic crystals on $(\mathcal{O}_K)_{\Delta}^{\text{perf}}$ is equivalent to the category of finite free $B_{\text{dr},L}$-modules $M$ on which there is stratification satisfying cocycle condition.

**Proof.** As $(A_L, E)$ is a weakly final object in $(\mathcal{O}_K)_{\Delta}^{\text{perf}}$, one can proceed as in Corollary 2.4.

**Proposition 5.2.** There is a canonical isomorphism of cosimplicial rings
\[
(\mathcal{O}_K[\frac{1}{p}]_{\Delta})_{\text{perf}}(A_{L, \text{perf}}) \cong C(G^\bullet, B_{\text{dr},L}^+).
\]
Proof. Unwinding definition, it suffices to show that
\[ \mathbb{B}_{\text{dr}}^+(A_{L, \text{perf}}^*) \cong C(\hat{G}^*, B_{\text{dr}, L}^+). \]
Recall that \( \mathbb{B}_{\text{dr}}^+(A_{L, \text{perf}}^*) \) is equipped with the inverse limit topology induced from the quotient topology on each \( \mathbb{B}_{\text{dr}}^+(A_{L, \text{perf}}^*)/(\xi') = A_{L, \text{perf}}^*/(\xi') \), hence it suffices to show that
\[ A_{L, \text{perf}}^* \cong C(\hat{G}^*, \hat{L}). \]
Here we identify \( C(\hat{G}^*, B_{\text{dr}, L}^+)/\xi \) with \( C(\hat{G}^*, \hat{L}) \) as multiplication map by \( \xi \) on \( B_{\text{dr}, L}^+ \) is a closed embedding with respect to the topology on \( B_{\text{dr}, L}^+ \).

But this is already proven in the proof of [MW21, Proposition 3.10].

As a consequence, by Galois descent, we have the following result:

**Theorem 5.3.** The category of de Rham prismatic crystals on \( (\mathcal{O}_K)_{\Delta}^\text{perf} \) is equivalent to the category of \( B_{\text{dr}, L}^+ \)-representation of \( \hat{G} \).

By either using \( A_{\inf} \) instead of \( A_L \) in the proof of the previous theorem or utilizing Corollary 4.7, we have that:

**Theorem 5.4.** We have the following commutative diagrams in which all the arrows induce equivalences of categories
\[
\begin{array}{ccc}
\text{Vect}(((\mathcal{O}_K)_{\Delta}^\text{perf}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) & \xrightarrow{T_L} & \text{Rep}_{\hat{G}}(B_{\text{dr}, L}^+) \\
& \searrow & \downarrow \cong \text{Rep}_{\hat{G}}(B_{\text{dr}, L}^+) \\
& & T \\
\text{Rep}_{\hat{G}}(B_{\text{dr}, L}^+) & \xrightarrow{T} & \text{Rep}_{\hat{G}}(B_{\text{dr}, L}^+) \\
\end{array}
\]

Here given \( \mathcal{M} \in \text{Vect}(((\mathcal{O}_K)_{\Delta}^\text{perf}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) \), \( T(\mathcal{M}) = \mathcal{M}(A_{\inf}) \) and \( T(\mathcal{M}) = \mathcal{M}(A_L) \).

**Example 5.5.** Identify \( B_{\text{dr}, L}^+ \) (resp. \( \mathbb{B}_{\text{dr}}^+(A_{L, \text{perf}}^*) \)) with \( C(\hat{G}^0, B_{\text{dr}, L}^+) \) (resp. \( C(\hat{G}^1, B_{\text{dr}, L}^+) \)), then for \( x \in B_{\text{dr}, L}^+ \), \( \delta_1(x)(g) = x \), \( \delta_0(x)(g) = g(x) \). In particular, as \( u_0(g) = \delta_1^0(u_0)(g) = [\pi^b] \), \( u_1(g) = \delta_0^1(u_0)(g) = g[\pi^b] = [e]^{1+g}[\pi^b] \), we have that
\[
X_1(g) = \frac{u_0 - u_1}{E(u_0)}(g) = \frac{1 - [e]^{g}(g)[\pi^b]}{E([\pi^b])}. 
\]

### 5.2. Full faithfulness of the restriction functor.
Consider the natural restriction functor
\[
V : \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) \to \text{Vect}((\mathcal{O}_K)_{\Delta}^\text{perf}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) \cong \text{Rep}_{\hat{G}}(B_{\text{dr}, L}^+). 
\]
by restricting a de Rham crystal to the the perfect prismatic site.

In this section, we aim to prove \( V \) is fully faithful.

To do this, we need several preliminaries. First we would like to give an explicit description of the \( \hat{G} \) representation \( V(\mathcal{M}) \) given \( \mathcal{M} \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) \).

**Proposition 5.6.** Let \( M \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_{\Delta, [1]}^1/p)]^2) \) be a de Rham prismatic crystal associated to a pair \((M, \varepsilon)\) as in Corollary 2.4. Choose a basis \( \xi \) in \( M \) such that
\[
\varepsilon(\xi) = \varepsilon \cdot \sum_{m \geq 0} \left( \sum_{n \geq 0} A_{m, n} X_1^{[n]} \right) t^m. 
\]
Then for \( \bar{v} = \varepsilon B \) a vector in \( V(\mathcal{M}) = M \otimes_{B_{\text{dr}, L}^+} B_{\text{dr}, L}^+ \) with \( B \in M_{t \times 1}(B_{\text{dr}, L}^+) \), then for \( g \in \hat{G} \),
\[
g(\bar{v}) = \varepsilon U(g)B 
\]
where \( U(g) = \sum_{n \geq 0} A_{m, n} X_1^{[n]} t^m. \)
Proof. This follows from the way that $\hat{G}$-action is constructed from stratification.

\begin{lemma}
\theta_k(X(g)) = \theta_k\left(\frac{[\pi^r](1-[\varepsilon])}{E([\pi^r])}\right)\sum_{i=1}^{k} (-1)^{i-1}c(g)^i\cdot(c(g)-i+1)(1-[\varepsilon])^{i-1}.
\end{lemma}

\begin{proof}
By definition, $X(g) = \frac{[\pi^r](1-[\varepsilon])}{E([\pi^r])}\cdot\frac{1-[\varepsilon]}{1-[\varepsilon]}$. Let $Y(g) := \frac{1-[\varepsilon]}{1-[\varepsilon]}$, hence it suffices to show
\begin{equation}
\theta_k(Y(g)) = \sum_{i=1}^{k} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^{i-1}.
\end{equation}
We prove it by induction on $k$. When $k = 1$, $\theta_1(Y(g)) = c(g)$ as $\theta_1([\varepsilon]) = 1$. Suppose Eq. (5.8) is proven for up to $k$, then
\begin{align*}
Y(g) &= (Y(g) - \sum_{i=0}^{k-1} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^i) + \sum_{i=1}^{k} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^{i-1} \\
&= (Y(g) - \sum_{i=1}^{k} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^{i-1}, (1-[\varepsilon])^k) + \sum_{i=1}^{k} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^{i-1}.
\end{align*}
Then the desired result follows as one can use L'Hôpital's rule to calculate that
\begin{equation}
\theta_1\left(\frac{Y(g) - \sum_{i=1}^{k} \frac{(-1)^{i-1}c(g)\cdots(c(g)-i+1)}{i!}(1-[\varepsilon])^{i-1}}{(1-[\varepsilon])^k}\right) = \frac{(-1)^k c(g)\cdots(c(g)-k)}{(k+1)!}.
\end{equation}
\end{proof}

\begin{proposition}
Suppose $\mathcal{M} \in \text{Vect}(\mathcal{O}_X^{(1)}(\mathcal{O}_X^{+}[2]^+(\mathcal{D}_{\mathcal{S}}))$, then $\varepsilon \subseteq V(\mathcal{M})^{G_{l_0},\gamma=1}$.
\end{proposition}

\begin{proof}
Recall that for $g \in \hat{G}$, $g \cdot \varepsilon = \varepsilon U(g)$ and $U(g) = \sum_{m \geq 0} (\sum_{n \geq 0} A_{m,n} X(g)^n) t^m$. By Definition 4.10, it suffices to show $\varepsilon \subseteq (V(\mathcal{M})/(t^k))^{G_{l_0},\gamma=1}$ for arbitrary $k$. Fix such a $k$, thanks to Lemma 5.7, it suffices to show for $p \leq k-1$,
\begin{equation}
\lim_{s \to \infty} A_{p,s} \frac{\theta_k\left(\frac{[\pi^r](1-[\varepsilon])}{E([\pi^r])}\right)^s}{s!} = 0.
\end{equation}
This can be checked after modulo $t$ by Lemma 4.17. Then notice that
\begin{equation}
\theta_1\left(\frac{[\pi^r](1-[\varepsilon])}{E([\pi^r])}\right) = \pi(1-\xi_0)\theta_1\left(\frac{\xi}{E([\pi^r])}\right),
\end{equation}
and that $v_p((1-\xi_0)^s) = \frac{s}{p-1} > \frac{s}{p-1} \geq v_p(s)$, $\lim_{s \to \infty} A_{p,s} = 0$ by Remark 2.23, hence Eq. (5.10) holds, we win.
\end{proof}

\begin{theorem}
Suppose $\mathcal{M} \in \text{Vect}(\mathcal{O}_X^{(1)}(\mathcal{O}_X^{+}[2]^+(\mathcal{D}_{\mathcal{S}}))$ such that the associated stratification $\varepsilon$ is given by a sequence of commutative matrices. Let $M = M(\mathcal{S}, (E(u)))$, and $V(\mathcal{M}) = M(A_{\text{inf}}, (\xi))$, then
\begin{align*}
D_{\text{Sen}, K_{\infty}[t]}(V(\mathcal{M})) &= M \otimes K[t] K_{\infty}[t],
\end{align*}
here we have identified $\mathbb{B}^+_{dR}(\mathcal{S})$ with $K[[T]]$.
\end{theorem}

\begin{proof}
First we check that $M \subseteq V(\mathcal{M})^{G_{l_0},\gamma=1}$. By our choice of embedding $(\mathcal{S}, (E))$ into $(\mathcal{O}_{\text{inf}}, (\xi))$ via sending $u$ to $[\pi^r]$, $M$ is invariant under Gal($L/K_{\infty}$)-action. Hence it suffices to show that $M \subseteq V(\mathcal{M})^{l_0}$, which follows from Proposition 5.9. By extending scalars along $K[t] \to (B_{dR,L}^{+})^{G_{l_0},\gamma=1}$, which is just $K_{\infty}[t]$ by Corollary 4.20, we get a natural map
\begin{align*}
M \otimes K[t] K_{\infty}[t] \rightarrow D_{\text{Sen}, K_{\infty}[t]}(V(\mathcal{M})).
\end{align*}
To show that it is an identity, it suffices to check that after the faithfully flat base change $K_{\infty}[t] \to B_{dR}$, then the desired result follows from Theorem 4.29.
\end{proof}
Theorem 5.12. The restriction functor

\[ V : \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_{\Delta}\frac{1}{p})^2) \rightarrow \text{Vect}((\mathcal{O}_K)^{\text{perf}}, (\mathcal{O}_{\Delta}\frac{1}{p})^2) \]

is fully faithful.

Proof. It suffices to show that For \( M \in \text{Vect}((\mathcal{O}_K)_{\Delta}, (\mathcal{O}_{\Delta}\frac{1}{p})^2) \), \( H^0((\mathcal{O}_K)_{\Delta}, M) \cong M(A_{\text{inf}})^{G_K} \). Suppose \( \bar{v} \in M(A_{\text{inf}})^{G_K} \), then \( \bar{v} \in M \otimes_K [[t]] K_{\infty}[t] \) thanks to Theorem 5.11. Hence it suffices to show that given \( \bar{v} = \sum_{m=0}^{\infty} D_m t^m \in M(A_{\text{inf}}) \) with \( D_m \in M_{\times 1}(K_{\infty}) \), \( \bar{v} \in M(A_{\text{inf}})^{G_K} \) if and only if \( D_m \in M_{\times 1}(K) \) and that \( D_m \) satisfies Eq. (3.9), which is exactly the condition that

\[ \bar{v} \in H^0((\mathcal{O}_K)_{\Delta}, M). \]

For this, first we calculate the \( \hat{G} \)-action on \( \bar{v} \). Let \( g \in \hat{G} \)

\[ g(\bar{v}) = g(\sum_{i=0}^{\infty} D_i t^i) = g(U(g)(\sum_{i=0}^{\infty} g(D_i)(\alpha(g)t)^i)) \]

\[ = \sum_{m=0}^{\infty} t^m (\sum_{i+j=m}^{\infty} \sum_{0 \leq i,j \leq m} A_{i,s} \theta_k(X_1(\tau)) \tau(D_p) c_{p,j-p}(X_1(\tau))). \]

This implies that \( \gamma \bar{v} = \bar{v} \) as \( X_1(1) = 0 \) and that \( D_m \in M_{\times 1}(K_{\infty}) \) is fixed by \( \gamma \). Hence \( \bar{v} \) is \( \hat{G} \)-invariant if and only if \( \bar{v} \in \gamma \)-invariant, i.e.

\[ \sum_{m=0}^{\infty} D_m t^m = \sum_{m=0}^{\infty} t^m (\sum_{i+j=m}^{\infty} \sum_{0 \leq i,j \leq m} A_{i,s} \theta_k(X_1(\tau)) \tau(D_p) c_{p,j-p}(\theta_k(X_1(\tau)))) \]

By the \( t \)-adic completeness of the finite free \( K_{\infty}[[t]] \)-module \( M \otimes_{K[[t]]} K_{\infty}[[t]] \), Eq. (5.13) is equivalent to that for \( \forall k \geq 1 \),

\[ \sum_{m=0}^{\infty} D_m t^m = \sum_{m=0}^{\infty} t^m (\sum_{i+j=m}^{\infty} \sum_{0 \leq i,j \leq m} A_{i,s} \theta_k(X_1(\tau)) \tau(D_p) c_{p,j-p}(\theta_k(X_1(\tau)))) \]

For \( k = 1 \), Eq. (5.14) turns into

\[ D_0 = (\sum_{s=0}^{\infty} A_{0,s} \theta_1(X_1(\tau)) \tau(D_0)). \]

We claim that this is equivalent to that

\[ (*) \ f(Z) := (\sum_{s=0}^{\infty} A_{0,s} Z^{\lfloor s \rfloor}) \tau(D_0) - D_0 \text{ is always 0, where } Z \text{ is a free variable. In particular, } \tau(D_0) = D_0, \text{ hence } D_0 \in M_{\times 1}(K). \]

Clearly \( * \) implies Eq. (5.15). On the other hand, given Eq. (5.15), we can always find a finite extension \( \bar{K} \) over \( K \) such that \( \{1 - \xi_p\} \in \bar{K}, D_0 \in M_{\times 1}(\bar{K}) \) and \( \tau(D_0) \in M_{\times 1}(\bar{K}). \) By Lemma 5.7,

\[ \theta_1(X(\tau)) = \theta_1(\frac{1-[e]}{E([\pi])}) = \pi \theta_1(\frac{1-[e]}{E([\pi])} \pi (1-\xi_p) \theta_1 \frac{\xi}{E([\pi])}). \]
This implies that $\omega_0 := \theta_1(\frac{t}{s})$ is a root of $f(\pi(1 - \xi_p)Z)$. Notice that $v_p((1 - \xi_p)^s) = \frac{s}{p-1} > v_p(s!)$ and that $\lim_{s \to \infty} A_{0, s} = 0$, hence $f(\pi(1 - \xi_p)Z) \in M_n(\mathcal{O}_K)(Z)$ after multiplying a scalar. By Weierstrass preparation theorem, $\omega_0$ is algebraic over $K$ (hence also algebraic over $K$) unless $f = 0$. However, [DL21, Lem 2.4.4] shows that $\omega_0$ is not algebraic over $K$, hence $f = 0$.

Suppose that we have shown that

$$D_m = \sum_{i+j=m}^{\infty} \sum_{s=0}^{n} (\sum_{0 \leq i,j \leq m} \tau(D_p)c_{p,j-p}(Z))$$

for $m \leq n - 1$. Then take $k = n + 1$, Eq. (5.14) turns into

$$\ell \sum_{m=0}^{n} D_m t^m = \ell \sum_{m=0}^{n} t^m (\sum_{i+j=m}^{\infty} (\sum_{s=0}^{\infty} A_{i,s} \theta_{n+1}(X_1(\tau))^{[s]})(\sum_{p=0}^{j} \tau(D_p)c_{p,j-p}(\theta_{n+1}(X_1(\tau))))).$$

However, by induction (Eq. (5.16)),

$$\ell \sum_{m=0}^{n-1} D_m t^m = \ell \sum_{m=0}^{n-1} t^m (\sum_{i+j=m}^{\infty} (\sum_{s=0}^{\infty} A_{i,s} \theta_{n+1}(X_1(\tau))^{[s]})(\sum_{p=0}^{j} \tau(D_p)c_{p,j-p}(\theta_{n+1}(X_1(\tau))))).$$

This implies that

$$D_n = \sum_{i+j=n}^{\infty} (\sum_{s=0}^{\infty} A_{i,s} \theta_1(X_1(\tau))^{[s]})(\sum_{p=0}^{j} \tau(D_p)c_{p,j-p}(\theta_1(X_1(\tau))))).$$

Arguing exactly as the case $k = 1$ (this makes sense as $c_{p,j-p}(Z) \in K[Z]$), we conclude that

$$(s^n) f_n(Z) := \sum_{i+j=n} (\sum_{s=0}^{\infty} A_{i,s} Z^{[s]})(\sum_{p=0}^{j} \tau(D_p)c_{p,j-p}(Z)) - D_n$$

is always 0, where $Z$ is a free variable.

In particular, $\tau(D_n) = D_n$, hence $D_n \in M_{1 \times 1}(K)$. Now by induction we conclude that for all $m$, $D_m \in M_{1 \times 1}(K)$ and satisfies that Eq. (3.9), hence $\tilde{v} \in H^0((\mathcal{O}_K)_\Delta, \mathcal{M})$, we are done.

5.3. Essential image of the restriction functor. Recall that a de Rham crystal $\mathcal{M} \in \text{Vect}(\mathcal{O}_K)_\Delta, (\mathcal{O}_K^d[p])^2_\Delta$ is uniquely determined by a pari $(M, \varepsilon)$ where $M$ is a finite free $\mathbb{B}^+_\text{dr}(\mathcal{G})$-modules and $\varepsilon$ is a stratification

$$\varepsilon(\ell) = \ell \sum_{m=0}^{\infty} (\sum_{n \geq 0} A_{m,n} X^{[n]}) t^m.$$

On the other hand, thanks to Theorem 4.31 and Theorem 5.11, we can associate to $\mathcal{M}$ a pair $(D_{\text{Sen},K[[\ell]]}(V(\mathcal{M})), N_{\mathcal{T}})$, where $D_{\text{Sen},K[[\ell]]}(V(\mathcal{M})) = \mathcal{M} \otimes_{K[[\ell]]} K[[\ell]]$ is a finite dimensional $K[[\ell]]$-space and $N_{\mathcal{T}}$ is the Kummer-Sen operator on $D_{\text{Sen},K[[\ell]]}(V(\mathcal{M}))$. Motivated by Gao’s proof that one can extract the stratification information of a rational Hodge-Tate crystal from the Sen operator, we would like to read off the stratification of a de Rham crystal from the Kummer-Sen operator $N_{\mathcal{T}}$. The main result is the following:

Theorem 5.17. Assume as before, then

$$N_{\mathcal{T}}(\ell) = -\varepsilon(1) \mu_{\lambda_1}(\sum_{m=0}^{\infty} A_{m,1} t^m),$$

where $\lambda_1 = \frac{\lambda}{E(\pi)} \in K[[\ell]]^\times$

Remark 5.18. By construction, our $N_{\mathcal{T}}$ is just the monodromy operator on $(D_{\text{Sen},K[[\ell]]}(V(\mathcal{M})))$ after modulo $t$, where $V(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{K} \otimes \mathbb{C}_p} \in \text{Rep}_{\mathbb{C}_p}(\mathcal{G}_K)$ is associated to the rational Hodge Tate crystal $\mathcal{M} = \mathcal{M}/\mathcal{L}$. Then one can check that this coincides with the calculation in [Gao22, Theorem 4.3.3].
Proof of Theorem 5.17. By Lemma 4.26, \( \theta(t) = \theta_{\log(|\epsilon|)}^{(\log(|\epsilon|))} \) is a unit, hence \( \lambda_1 \) is a unit.

It suffices to show that for \( \forall k \in \mathbb{N} \)

\[
N_{\nabla, k}(\tilde{\epsilon}) = e \theta_k(-\lambda_1 u)(\sum_{m=0}^{k-1} A_{m,t} t^m).
\]

For this, by Lemma 5.7

\[
\theta_k(X(g)) = \theta_k(\frac{[\pi^b](1 - |\epsilon|)}{E([\pi^b])})(\sum_{i=1}^{k} (-1)^{i-1} c(g) \cdot (c(g) - i + 1)(1 - |\epsilon|)^{i-1}).
\]

Hence utilizing that \( c(\tau^n) = p^n \) and by calculating the coefficient of \( c(g) \) in \( \theta_k(X(g)) \), we have that

\[
\nabla_{\tau,k}(\epsilon) = \lim_{n \to \infty} \frac{\tau^n(\epsilon) - \epsilon}{p^n} = \lim_{n \to \infty} \frac{\epsilon^{\sum_{m=0}^{k-1} \sum_{n=0}^{\infty} (A_{m,n} \theta_k(X_1(\tau^n)))[n])t^m} - \epsilon }{p^n} = \epsilon^{\sum_{m=0}^{k-1} \sum_{j=0}^{n} (1 - |\epsilon|)^{j}} \theta_k(\frac{[\pi^b](1 - |\epsilon|)}{E([\pi^b])}).
\]

The second to last equality follows as \( A_{0,1} = I \) and \( A_{m,0} = 0 \) for \( m > 0 \) by Theorem 2.18.

Hence

\[
N_{\nabla,k}(\epsilon) = \frac{1}{p \theta_k(t)} \nabla_{\tau,k}(\epsilon) = \theta_k(E(u)\lambda_1) \frac{\theta_k(\log(|\epsilon|))}{\theta_k(\log(|\epsilon|))} \nabla_{\tau,k}(\epsilon)
\]

\[
= \frac{1}{p \theta_k(t)} \nabla_{\tau,k}(\epsilon) = \epsilon^{\sum_{m=0}^{k-1} \sum_{j=0}^{n} (1 - |\epsilon|)^{j}} \theta_k(\frac{[\pi^b](1 - |\epsilon|)}{E([\pi^b])}).
\]

This is precisely the desired result.

Motivated by this calculation, we see that \( V(\mathcal{M}) \) needs to be strong nearly de Rham (see Definition 1.18). Actually, we have the following result:

Proposition 5.19. Consider the composition of the de Rham realization functor and the restriction functor, which we still denoted as \( V \) by abuse of notation, then we get a fully faithful functor

\[
V : \text{Vect}((O_K)_{\Delta}, (O_{\Delta})_{\bar{\Delta}}^{1}) \to \text{Rep}^\text{fp, SNdR}_{B_{ir}^{\text{fp}}} (G_K).
\]

Moreover, assume that Conjecture 2.24 holds, then \( V \) is essentially surjective, in particular, it induces an equivalence of categories between \( \text{Vect}((O_K)_{\Delta}, (O_{\Delta})_{\bar{\Delta}}^{1}) \) and \( \text{Rep}^\text{fp, SNdR}_{B_{ir}^{\text{fp}}} (G_K) \).

Proof. A priori is that \( V \) defines a fully faithful functor \( \text{Vect}((O_K)_{\Delta}, (O_{\Delta})_{\bar{\Delta}}^{1}) \to \text{Rep}^\text{fp}_{B_{ir}^{\text{fp}}} (G_K) \) by Theorem 5.3 and Theorem 5.12. Utilizing Corollary 2.2 and Theorem 1.7, we see the image of \( V \) lands into \( \text{Rep}^\text{fp, SNdR}_{B_{ir}^{\text{fp}}} (G_K) \). Further thanks to Theorem 5.11 and Theorem 5.17, we see the target of \( V \) is in \( \text{Rep}^\text{fp, SNdR}_{B_{ir}^{\text{fp}}} (G_K) \).

Next we show that \( V \) is essentially surjective assuming that Conjecture 2.24 is true. Given \( W \in \text{Rep}^\text{fp, SNdR}_{B_{ir}^{\text{fp}}} (G_K) \) of rank \( l \), by definition there exists a \( N_{\nabla} \) stable \( K[[t]] \)-module \( M \) inside \( D_{\text{Sen}, K, \infty}[[t]](W) \), fix a basis \( \tilde{\epsilon} \) of \( M \), as \( -\lambda_1 u \) is a unit in \( K[[t]] \), we can find \( \{ B_{m,1} \} \in M_{l \times 1} \) such that

\[
N_{\nabla}(\tilde{\epsilon}) = -\tilde{\epsilon}\lambda_1 u(\sum_{m=0}^{\infty} B_{m,1} t^m).
\]
As a result, \( N_\mathcal{V} \) on
\[
D_{\Sen,K_\infty}(W/tW) = D_{\Sen,K_\infty[[t]]}(W)/t = \tilde{M}/t\tilde{M} \otimes_K K_\infty
\]
is given by
\[
N_\mathcal{V}(\xi) = -2\frac{\theta(\lambda_1 u)}{\theta(u\lambda')} B_{0,1}.
\]
By [Gao22, Theorem 4.3.3], the (normalized) Kummer-Sen operator on \( D_{\Sen,K_\infty}(W/tW) \) is just
\[
\frac{1}{\theta(u\lambda')} N_\mathcal{V},
\]
where \( \lambda' \) is the derivative of \( \lambda \) with respect to \( u \). Hence the Kummer-Sen operator on \( D_{\Sen,K_\infty}(W/tW) \) acts on \( \xi \) via
\[
-\frac{\theta(\lambda_1 u)}{\theta(u\lambda')} B_{0,1} = \frac{1}{\theta(\lambda')/\theta(\lambda_1)} B_{0,1} = \frac{B_{0,1}}{\beta}.
\]
The last equality holds as \( \lambda = \lambda_1 E(u) \), hence \( \lambda' = \lambda_1 E'(u) + \lambda_1' E(u) \), hence \( \theta(\lambda') = \theta(\lambda_1) E'(\pi) = \theta(\lambda_1) \beta \).

On the other hand, as \( W \) is nearly de Rham, hence \( W/tW \) is nearly Hodge-Tate, i.e. the eigenvalues of the Sen operator are all in the subset \( \mathbb{Z} + \beta^{-1}m_{\mathcal{O}_{\mathcal{V}}^c} \), from which we see all the eigenvalues of \( B_{0,1} \) are in the set \( \beta \mathbb{Z} + m_{\mathcal{O}_{\mathcal{V}}^c} \), which implies that \( \lim_{n \to +\infty} \prod_{i=0}^{n} (iE'(\pi) + B_{0,1}) = 0 \) by the proof of [Gao22, Corollary 4.1.6].

Now we have that \( \{B_{m,1}\} \) satisfies the assumption in Conjecture 2.24. As we assume that Conjecture 2.24 is true, hence we can construct a de Rham crystal \( \mathcal{M} \) whose corresponding stratification \( \varepsilon \) is determined by \( \{B_{m,1}\} \) as in Conjecture 2.24. Now let \( M = \mathcal{M}(\mathfrak{S}, E) \) and \( \xi \) be a basis of \( M \) on which \( \varepsilon \) acts as in Conjecture 2.24. Then by Theorem 5.17,
\[
N_\mathcal{V}(\xi) = -\varepsilon \lambda_1 u (\sum_{m=0}^{\infty} B_{m,1} t^m).
\]

In this way, we can construct a \( K[[t]] \)-linear isomorphism \( f: \tilde{M} \to M \) by sending \( \varepsilon_i \) to \( e_i \), which is compatible with \( N_\mathcal{V} \)-structure by Eq. (5.20) and Eq. (5.21). This can be further \( K_\infty[[t]] \)-linearly extended to \( f \otimes \text{Id}: \tilde{M} \otimes_{K[[t]]} K_\infty[[t]] \to M \otimes_{K[[t]]} K_\infty[[t]] \). Then notice that
\[
\tilde{M} \otimes_{K[[t]]} K_\infty[[t]] = D_{\Sen,K_\infty[[t]]}(W),
\]
\[
M \otimes_{K[[t]]} K_\infty[[t]] = D_{\Sen,K_\infty[[t]]}(V(\mathcal{M})),
\]
where the first equality follows from the definition of strong nearly de Rham crystals, while the second identity holds thanks to Theorem 5.11. In this way, \( f \otimes \text{Id} \) defines an isomorphism in
\[
\text{Hom}_{\text{Mod}_{N_\mathcal{V}}}((D_{\Sen,K_\infty[[t]]}(W_1), N_\mathcal{V}), (D_{\Sen,K_\infty[[t]]}(W_2), N_\mathcal{V})).
\]
This implies \( W \) is isomorphic to \( V(\mathcal{M}) \) in \( \text{Rep}_{\mathcal{G}_K}(B_{\text{dR}}^+) \) thanks to Corollary 4.48. Since \( W \) is arbitrarily chosen, we see that \( V \) is essentially surjective.

\[\square\]

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