A COMPLETE CLASSIFICATION OF GROUND-STATES FOR A
COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract. In this paper, we establish the existence of nontrivial ground-state solutions for a coupled nonlinear Schrödinger system

\[-\Delta u_j + u_j = \sum_{i=1}^{m} b_{ij} u_i^2 u_j, \quad \text{in } \mathbb{R}^n,\]
\[u_j(x) \to 0 \text{ as } |x| \to \infty, \quad j = 1, 2, \ldots, m,\]

where \( n = 1, 2, 3, m \geq 2 \) and \( b_{ij} \) are positive constants satisfying \( b_{ij} = b_{ji} \).

By nontrivial we mean a solution that has all components non-zero. Due to possible systems collapsing it is important to classify ground state solutions. For \( m = 3 \), we get a complete picture that describes whether nontrivial ground-state solutions exist or not for all possible cases according to some algebraic conditions of the matrix \( B = (b_{ij}) \). In particular, there is a nontrivial ground-state solution provided that all coupling constants \( b_{ij}, i \neq j \) are sufficiently large as opposed to cases in which any ground-state solution has at least a zero component when \( b_{ij}, i \neq j \) are all sufficiently small. Moreover, we prove that any ground-state solution is synchronized when matrix \( B = (b_{ij}) \) is positive semi-definite.

1. Introduction. In this note, we are concerned with the existence of nontrivial ground-state solutions of the time-independent Schrödinger equations

\[
\begin{cases}
-\Delta u_j + \lambda_j u_j = \sum_{i=1}^{m} b_{ij} u_i^2 u_j, & \text{in } \mathbb{R}^n, \\
u_j(x) \to 0 \text{ as } |x| \to \infty, & j = 1, 2, \ldots, m,
\end{cases}
\]

where \( n = 1, 2, 3, m \geq 2 \), and \( \lambda_j, b_{ij} \) all are positive constants for \( i, j = 1, 2, \ldots, m \), which is closely related to the following time-dependent system of \( m \)-coupled nonlinear Schrödinger(CNLS) equations

\[
\begin{cases}
-i \frac{\partial}{\partial t} v_j = \Delta v_j + \sum_{i=1}^{m} b_{ij} v_i^2 v_j, & \text{for } x \in \mathbb{R}^n, \ t > 0, \\
v_j(x, t) \to 0 & \text{as } |x| \to +\infty, \ t > 0, \ j = 1, 2, \ldots, m.
\end{cases}
\]
The relation between the two equations is that \((u_1, u_2, \cdots, u_m)\) satisfies (1) if and only if \(v_j(x,t) = e^{i\lambda_j t}u_j(x), j = 1, 2, \cdots, m\) solves (2) which arises physically under conditions when there are m-wave trains moving with nearly the same group velocities [13, 17]. The CNLS system also models physical systems whose fields have more than one component; for example, in optical fibers and wave guides, the propagating electric field has two components that are transverse to the direction of propagation. This type of systems also arises from physical models in nonlinear optics and in Bose Einstein condensates for multi-species condensates (i.e., [11, 14] and references therein). Readers are referred to the works [5, 6, 4, 11, 14, 18, 19] for the derivation as well as applications of this system.

As the terminology about ground states are not quite uniformly used throughout the literature we first fix some definitions in the paper. A solution of (1) is called \textit{nontrivial} if all components are non-zero. In contrast, semi-trivial solutions are the non-zero solutions with at least one zero component. In this case the system collapses to a system with fewer number of equations. A solution \((u_1, \cdots, u_m)\) of (1) having all components \(u_j \geq 0\) for all \(j\) and \(u_j > 0\) for at least one \(j\). Solutions \(\vec{u} = (u_1, \cdots, u_m) \in (H^1(\mathbb{R}^n))^m\) correspond to critical points of the energy functional associated with (1)

\[
\Psi(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^{m} (|\nabla u_j|^2 + \lambda_j u_j^2) dx - \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j=1}^{m} b_{ij} |u_i u_j|^2 dx.
\]

Since we suppose that \(n < 4\), the Sobolev embedding implies that \(\Psi\) is well-defined and of class \(C^2\). A solution is called ground-state solution if it has the least energy among all non-zero solutions of (1). Note that a ground state solution may be a semi-trivial solution due to collapsing of the system. Lemma 3.1 in [9] implies that there exists a ground state solution which is a non-negative radial solution. We consider the Nehari manifold

\[
\mathcal{N} = \left\{ (u_1, u_2, \cdots, u_m) \in X_m \setminus \{0\} \mid \sum_{j=1}^{m} \int_{\mathbb{R}^n} (|\nabla u_j|^2 + \lambda_j u_j^2) dx = \sum_{i,j=1}^{m} \int_{\mathbb{R}^n} b_{ij} |u_i u_j|^2 dx \right\},
\]

hereafter, \(X_m = H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times \cdots \times H^1(\mathbb{R}^n)\) is the m-times Cartesian product of \(H^1(\mathbb{R}^n)\). Then a ground-state solution of (1) is equivalent to a minimizer of the following variational problem

\[
\inf \left\{ \Psi(\vec{u}) : \vec{u} = (u_1, u_2, \cdots, u_m) \in \mathcal{N} \right\},
\]

because \(\mathcal{N}\) is a natural constraint of \(\Psi\). Thus it is important to identify conditions of the matrix \(B\) so that the ground states are non-trivial or semi-trivial. For \(m = 2\), Bartsch and Wang have investigated the following linear equations

\[
\sum_{i=1}^{m} b_{ij} x_i = 1, \quad j = 1, 2, \cdots, m
\]

in [3] in order to study existence of the positive ground-state solutions of (1) which was well understood for the case of \(m = 2\) and \(\lambda_1 = \lambda_2 > 0\). See also [2, 10] for more general cases. Some interesting progress has been made partially in [9] under some conditions on the structure of the matrix \(B\) for arbitrary \(m \geq 3\). We investigate the classification of non-trivial ground-state solutions of (1) for \(m = 3\) in the present paper for all possible cases. The results established here describe a complete picture in which we know the type of ground-state solutions, non-trivial or semi-trivial, for
each case under the condition that (3) has at least a solution with each component \( x_i > 0 \). Such solutions are called positive solutions to (3). Similarly, a solution of (3) is called non-negative if each component is non-negative.

Throughout the present paper, we suppose that the symmetric real matrix \( B \) satisfies that

\[
B = (b_{ij}), b_{ij} > 0, \ i, j = 1, 2, \ldots, m.
\] (4)

\[
\lambda_1 = \lambda_2 = \cdots = \lambda_m = 1.
\] (5)

Let \( w \) be the unique positive radial solution of the equation

\[
\Delta w - w + |w|^2w = 0 \quad \text{in} \ \mathbb{R}^n, \quad w(x) \rightarrow 0 \quad \text{as} \ |x| \rightarrow \infty.
\] (6)

The main results in the present paper are the following theorems. For simplicity and matching notations used in the literatures we write

\[
b_{ii} = \mu_i, \ i = 1, 2, 3, \quad b_{12} = \beta_1, b_{13} = \beta_2, b_{23} = \beta_3.
\]

**Theorem 1.1.** Let \( m = 3 \) and \( \det(B) > 0 \). Suppose that (3) has at least one positive solution \((t_1, t_2, t_3)\). Then, the following conclusions hold.

(1) if

\[
\beta_1 > \sqrt{\mu_2 \mu_3}, \quad \beta_2 > \sqrt{\mu_1 \mu_3}, \quad \beta_3 > \sqrt{\mu_1 \mu_2},
\] (7)

then each ground-state solution of (1) is nontrivial. In particular, 

\[
(\sqrt{t_1 w}, \sqrt{t_2 w}, \sqrt{t_3 w})
\] is a positive ground state solution.

(2) If

\[
0 < \beta_1 < \sqrt{\mu_2 \mu_3}, \quad 0 < \beta_2 < \sqrt{\mu_1 \mu_3}, \quad 0 < \beta_3 < \sqrt{\mu_1 \mu_2},
\] (8)

then each ground-state solution of (1) has exactly one non-zero component.

If \( \det(B) < 0 \), the theorem below says that every ground-state solution of (1) is semi-trivial.

**Theorem 1.2.** Let \( m = 3 \) and \( \det(B) < 0 \). Suppose that (3) has at least one positive solution \((t_1, t_2, t_3)\). Then, every ground-state solution of (1) is semi-trivial.

When \( \operatorname{rank}(B) = 1 \), we have

**Theorem 1.3.** Let \( m = 3 \) and \( \operatorname{rank}(B) = 1 \). Suppose that (3) has at least one positive solution. Then, \((\sqrt{t_1 w}, \sqrt{t_2 w}, \sqrt{t_3 w})\) is a ground-state solution of (1) provided that \( t_1 + t_2 + t_3 = \frac{1}{\mu_1} \) and \( t_j \geq 0, j = 1, 2, 3 \).

The following theorem involves the cases in which \( \operatorname{rank}(B) = 2 \).

**Theorem 1.4.** Let \( m = 3 \) and \( \operatorname{rank}(B) = 2 \). Suppose that (3) has at least one positive solution. Then, the following conclusions hold.

(a) If \( \mu_1 \mu_2 - \beta_2^2 = 0 \) and

(1) if \( \beta_1 > \sqrt{\mu_2 \mu_3} \), then \((\sqrt{t_1 w}, \sqrt{t_2 w}, \sqrt{t_3 w})\) is a ground-state solution of (1)

provided that

\[
t_1 + t_2 = \frac{\beta_3 - \mu_3}{\beta_3 - \mu_1 \mu_2}, \quad t_1, t_2 \geq 0, \quad t_3 = \frac{\beta_1 - \mu_2}{\beta_1 - \mu_2 \mu_3}.
\]

(2) if \( \beta_1 < \sqrt{\mu_2 \mu_3} \), each ground-state solution of (1) is semi-trivial.

(b) If \( \beta_1 > \sqrt{\mu_2 \mu_3}, \quad \beta_2 > \sqrt{\mu_1 \mu_3}, \quad \beta_3 > \sqrt{\mu_1 \mu_2} \), then \((\sqrt{t_1 w}, \sqrt{t_2 w}, \sqrt{t_3 w})\) is a ground-state solution of (1) provided \((t_1, t_2, t_3)\) is a non-negative solution to (3). In particular, if \( \beta_1 = \beta_2 = \mu_3 > \max\{\mu_1, \mu_2\} \) and \( \beta_3 > \max\{\mu_1, \mu_2\} \), then all non-negative solutions \((t_1, t_2, t_3)\) to (3) can be formulated as follows

\[
t_1 = \frac{\beta_3 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} (1 - \mu_3 t_3), \quad t_2 = \frac{\beta_3 - \mu_1}{\beta_3^2 - \mu_1 \mu_2} (1 - \mu_3 t_3), \quad t_3 \in [0, \frac{1}{\mu_3}].
\] (9)
(c) If \( \beta_1 < \sqrt{\mu_2 \mu_3}, \beta_2 < \sqrt{\mu_1 \mu_3}, \beta_3 < \sqrt{\mu_1 \mu_2} \). Then, any ground-state solution has exactly one non-zero component.

Let \( p \) denote the number of non-zero components in any ground-state solution of (1). For instance, \( p = 3 \) if and only if the associated solution is nontrivial. The type of ground-state solutions determined in above theorems can be displayed explicitly in the following table.

| case | condition 1 | condition 2 | type |
|------|-------------|-------------|------|
| 1    | \( \det(B) > 0 \) | \( \beta_1 > \sqrt{\mu_2 \mu_3}, \beta_2 > \sqrt{\mu_1 \mu_3}, \beta_3 > \sqrt{\mu_1 \mu_2} \) | \( p = 3 \) |
| 2    | \( \det(B) > 0 \) | \( \beta_1 < \sqrt{\mu_2 \mu_3}, \beta_2 < \sqrt{\mu_1 \mu_3}, \beta_3 < \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |
| 3    | \( \det(B) < 0 \) | \( \beta_1 > \sqrt{\mu_2 \mu_3}, \beta_2 > \sqrt{\mu_1 \mu_3}, \beta_3 > \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |
| 4    | \( \text{rank}(B) = 1 \) | \( \beta_1 > \sqrt{\mu_2 \mu_3}, \beta_2 > \sqrt{\mu_1 \mu_3}, \beta_3 > \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |
| 5    | \( \text{rank}(B) = 2 \) | \( \beta_1 > \sqrt{\mu_2 \mu_3}, \beta_2 > \sqrt{\mu_1 \mu_3}, \beta_3 = \sqrt{\mu_1 \mu_2} \) | \( p = 2, 3 \) |
| 6    | \( \text{rank}(B) = 2 \) | \( \beta_1 < \sqrt{\mu_2 \mu_3}, \beta_2 > \sqrt{\mu_1 \mu_3}, \beta_3 > \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |
| 7    | \( \text{rank}(B) = 2 \) | \( \beta_1 > \sqrt{\mu_2 \mu_3}, \beta_2 < \sqrt{\mu_1 \mu_3}, \beta_3 = \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |
| 8    | \( \text{rank}(B) = 2 \) | \( \beta_1 < \sqrt{\mu_2 \mu_3}, \beta_2 < \sqrt{\mu_1 \mu_3}, \beta_3 < \sqrt{\mu_1 \mu_2} \) | \( p = 1 \) |

Next we turn to more qualitative property of ground state solutions. When the components of a solution to (1) are proportional to one another, it is termed a synchronized solution. Under attractive couplings synchronized solutions have been studied in recent years ([3, 7, 9, 12, 15, 16] and references there in). We define another minimization problem. Let

\[
\mathcal{N} = \{ (u_1, u_2, \cdots, u_m) \in X_m | u_j \neq 0, \|u_j\|_{H^1}^2 = \sum_{i=1}^m \int_{\mathbb{R}^n} b_{ij} |u_i u_j|^2 dx, j = 1, 2, \cdots, m \}. 
\]

On synchronized ground-state solutions, we have the following result.

**Theorem 1.5.** Suppose that the matrix \( B \) is positive semi-definite. Let \( (u_1, u_2, \cdots, u_m) \) be a minimizer of \( \Psi \) on \( \mathcal{N} \), then \( u_j = \pm \sqrt{t_j} v, j = 1, 2, \cdots, m \), where \( v \) is a ground-state solution of (6) and \( \{t_1, t_2, \cdots, t_m\} \) is a positive solution to (3).

**Remark 1.** The existence of the minimizers of \( \Psi \) on \( \mathcal{N} \) is guaranteed by Lemma 2.2 here when (3) has a positive solution or by the Theorem 2 in [7] for all \( b_{ij}, i \neq j \) small (see [8]) and \( B \) is positive definite.

**Remark 2.** For \( m = 2 \), the result in the above theorem is obtained in [16] in which for small coupling constant the existence and synchronization of a minimizer on \( \mathcal{N} \) is proved. Our result here shows that, when \( B \) is positive semi-definite, the solution of m-coupled system (1) which has the least energy on \( \mathcal{N} \) is unique provided the ground-state solution of the scaler equation (6) is unique (or unique up to translations) and the solution of linear equations (3) is unique.

The rest of the paper is organized as follows. Section 2 provides several lemmas that are important in the proof of the above main theorems about ground-state solutions of (1) which will be proved in Section 3. Furthermore, the final section is to deal with the synchronized ground-state solutions.
2. Preliminary.

**Lemma 2.1.** Let $A = (a_{ij})$ be a symmetric real matrix and suppose that $s_j \geq 0, j = 1, 2, \cdots, N$, satisfy $\sum_{j=1}^N a_{ij} s_j = 1, i = 1, 2, \cdots, N$. Let $\{c_{k,j}\}_{k=1}^\infty, j = 1, 2, \cdots, N$ be any sequences of real numbers satisfying the inequality system

$$\sum_{j=1}^N a_{ij} c_{k,j} = d_{k,i} \geq 0, \quad i = 1, 2, \cdots, N,$$

$$\sum_{j=1}^N c_{k,j} \leq g(k), \quad \text{where } \lim_{k \to \infty} g(k) = 0.$$

Then $\sum_{j=1}^N c_{k,j} \to 0$ as $k \to \infty$.

**Proof.** It follows from $\sum_{j=1}^N a_{ij} s_j = 1, i = 1, 2, \cdots, N$, that

$$\sum_{i=1}^N (\sum_{j=1}^N a_{ij} s_j - 1)c_{k,i} = 0.$$

It implies that

$$\sum_{i=1}^N c_{k,i} = \sum_{j=1}^N (\sum_{i=1}^N a_{ij} c_{k,i}) s_j = \sum_{j=1}^N d_{k,j} s_j.$$

Noticing that $s_j \geq 0$ and $d_{k,j} \geq 0$, then the desired result follows from $\sum_{j=1}^N c_{k,j} \leq g(k)$ and $\lim_{k \to \infty} g(k) = 0$.

**Remark 3.** A by-product of this lemma is that the sums of all components of all solutions to linear equations $\sum_{j=1}^N a_{ij} x_j = 1, i = 1, 2, \cdots, N$, are equal to each other.

**Remark 4.** The above lemma has been proved in [9] for a special case in which the matrix $A$ is invertible. Therefore the conclusion here is weaker than that of [9] there, $c_{k,j} \to 0$ as $k \to \infty$ for all $j = 1, 2, \cdots, N$. However, it is enough for the following application.

The following lemma says that the synchronized solution is a minimizer for $\alpha_J$, which has been proved in [9] for the special case when $J = \{1, 2, \cdots, m\}$ and $B = (b_{ij})$ is invertible. Following the idea in [9], one can prove it easily by Lemma 2.1. Therefore, we state only the lemma without more details.

**Lemma 2.2.** Let $J$ be any non-empty subset of $\{1, 2, \cdots, m\}$. Suppose that $t_j > 0, j \in J$ satisfy $\sum_{j \in J} b_{ij} t_j = 1, \text{ for all } i \in J$. Then

$$\Psi(\sqrt{t_1} w, \sqrt{t_2} w, \cdots, \sqrt{t_m} w) = \alpha_J,$$

where $t_i = 0, i \notin J$ and

$$\alpha_J = \inf \{ \Psi(u_1, \cdots, u_m) \|u_j\|_{H^1}^2 = \sum_{i=1}^m \int_{\mathbb{R}^n} b_{ij} |u_i u_j|^2 dx, u_j \neq 0, j \in J; u_j = 0, j \notin J \}.$$

**Remark 5.** Suppose that $t_j > 0$ for all $j = 1, 2, \cdots, m$. Then the above lemma guarantees that the existence of positive minimizers is a nontrivial result because in general the existence of minimizers of the variational problem needs some additional conditions. For instance, the results in [7] imply that if $B$ is positive definite then the
Lemma 2.4. If $\beta$ is a minimizer with one zero component for $x$ a solution ($m$ is helpful to understand solutions of the linear equations (3) for $\beta > \min\{\mu_1, \mu_2\}$). Moreover, if $0 < \beta < \sqrt{\mu_1 \mu_2}$, then any ground-state solution of (1) is semi-trivial.

(2) If $\beta > \max\{\mu_1, \mu_2\}$, then

$$\inf\{\Psi(u, v) : (u, v) \in \mathcal{N}\} = \frac{\Lambda^2}{4} \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\}. \quad (10)$$

and any ground-state solution of (1) is nontrivial.

Furthermore, if $\beta > \sqrt{\mu_1 \mu_2}$, then

$$\frac{\Lambda^2}{4} \frac{2\beta - \mu_1 - \mu_2}{\beta^2 - \mu_1 \mu_2} \leq \inf\{\Psi(u, v) : (u, v) \in \mathcal{N}\} \leq \frac{\Lambda^2}{4} \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\}. \quad (11)$$

Proof. It follows from Theorem 0.5 in [3] that the above minimum problem admits a minimizer with one zero component for $\beta \leq \max\{\mu_1, \mu_2\}$ as well as one positive minimizer for $\beta > \max\{\mu_1, \mu_2\}$. We therefore have

$$\inf\{\Psi(u, v) : (u, v) \in \mathcal{N}\} = \left\{\begin{array}{ll}
\min\{\alpha_{(1)}, \alpha_{(2)}\}, & \beta \leq \max\{\mu_1, \mu_2\}, \\
\alpha_{(1, 2)}, & \beta > \max\{\mu_1, \mu_2\}.
\end{array}\right.\quad (12)$$

Then (10) and (11) follow from Lemma 2.2. Moreover, a straightforward calculation reveals that

$$\frac{2\beta - \mu_1 - \mu_2}{\beta^2 - \mu_1 \mu_2} \leq \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\},$$

if $\beta > \sqrt{\mu_1 \mu_2}$. Then, it follows from (10) and (11) that (12). Consequently, the rest of the lemma follows from theorem 0.5 in [3].

Before discussing the classification of ground-state solutions of (1) for $m = 3$, it is helpful to understand solutions of the linear equations (3) for $m = 3$. Recall that a solution $(x_1, x_2, x_3)$ to (3) is called positive if $x_1, x_2, x_3 > 0$ and non-negative if $x_1, x_2, x_3 \geq 0$. For the case in which $m = 3$, we set $\mu_i = b_{ii}, i = 1, 2, 3$ and $\beta_1 = b_{23}, \beta_2 = b_{13}, \beta_3 = b_{12}$.

Lemma 2.4. (1) If $\det(B) > 0$, then the following three numbers

$$\beta_3^2 - \mu_1 \mu_2, \quad \beta_2^2 - \mu_1 \mu_3, \quad \beta_1^2 - \mu_2 \mu_3,$$

have the same sign, and

(2) if there is one number of (13) being zero, then $\det(B) \leq 0$.

(3) If there exist two numbers of (13) having different signs, then $\det(B) < 0$.

(4) If $\det(B) = 0$, then

(i) if $\text{rank}(B)$, the rank of matrix $B$, equals 1, and suppose that (3) has at least one solution. Then

$$b_{ij} = b_{11}$$

for all $i, j = 1, 2, 3$. 

The following lemma describes the least energy level of (1) with two equations. Let $\Lambda^2 = \int_{\mathbb{R}^n} w^4 dx$ where $w$ is the solution of (6).

Lemma 2.3. Suppose $m = 2$. Let $b_{11} = \mu_1, b_{12} = b_{21} = \beta, b_{22} = \mu_2$. Then,

(1) if $0 < \beta \leq \max\{\mu_1, \mu_2\}$, then

$$\inf\{\Psi(u, v) : (u, v) \in \mathcal{N}\} = \frac{\Lambda^2}{4} \min\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\}. \quad (10)$$

Moreover, if $0 < \beta < \sqrt{\mu_1 \mu_2}$, then any ground-state solution of (1) is semi-trivial.

(2) If $\beta > \max\{\mu_1, \mu_2\}$, then

$$\inf\{\Psi(u, v) : (u, v) \in \mathcal{N}\} = \frac{\Lambda^2}{4} \frac{2\beta - \mu_1 - \mu_2}{\beta^2 - \mu_1 \mu_2} \quad (11)$$

and any ground-state solution of (1) is nontrivial.
(ii) If \( \text{rank}(B) = 2 \), then there exists at most one number in (13) which equals zero. In this case, if more \( \beta_3^2 = \mu_1 \mu_2 \) and suppose that (3) has at least one solution, then

\[
\mu_1 = \mu_2 = \beta_3, \quad \beta_2 = \beta_1,
\]

and all solutions of (3) can be formulated as follows

\[
t_1 + t_2 = \frac{\beta_1 - \mu_3}{\beta_1^2 - \mu_2 \mu_3}, \quad t_1, t_2 \in \mathbb{R}, \quad t_3 = \frac{\beta_1 - \mu_2}{\beta_1^2 - \mu_2 \mu_3}.
\]

(5) If \( \beta_i \in [\min\{\mu_j, \mu_k\}, \max\{\mu_j, \mu_k\}] \) for all \( i, j, k \), permutation of 1, 2, 3, and suppose that (3) has at least one solution, then \( \text{rank}(B) = 1 \).

**Proof.** A straightforward calculation reveals

\[
det(B) = \frac{1}{\mu_i}[(\beta_k^2 - \mu_i \mu_j)(\beta_j^2 - \mu_i \mu_k) - (\mu_i \beta_i - \beta_j \beta_k)^2]
\]

for any \( i, j, k \), permutation of 1, 2, 3. It implies that part (1), (2) and (3). To prove (i) of part (4), notice that \( \text{rank}(B) = 1 \), then we therefore assume that there exist three positive constants \( c_1, c_2, c_3 > 0 \) such that

\[
c_1(b_{11}, b_{12}, b_{13}) = c_2(b_{21}, b_{22}, b_{23}) = c_3(b_{31}, b_{32}, b_{33}).
\]

(16)

So, it follows from (3) having a solution that \( c_1 = c_2 = c_3 \). Then, we obtain that (14) due to the symmetry of matrix \( B \).

As for (ii) of part (4), if

\[
\beta_k^2 - \mu_i \mu_j = \beta_j^2 - \mu_i \mu_k = 0
\]

for some \( i, j, k \), permutation of 1, 2, 3. Then, (15) and (17) imply that

\[
\mu_i \beta_i - \beta_j \beta_k = 0
\]

because of \( det(B) = 0 \). Therefore, combing (17) with (18), we deduce that

\[
\beta_i^2 - \mu_j \mu_k = 0.
\]

(19)

Then, we can solve \( \beta_1, \beta_2, \beta_3 \) from equations (17) and (19) as follows

\[
\beta_1 = \sqrt{\mu_2 \mu_3}, \quad \beta_2 = \sqrt{\mu_1 \mu_3}, \quad \beta_3 = \sqrt{\mu_1 \mu_2},
\]

which implies that (16) holds for \( c_1 = \frac{1}{\sqrt{\mu_1}}, c_2 = \frac{1}{\sqrt{\mu_2}}, c_3 = \frac{1}{\sqrt{\mu_3}} \). Consequently, we obtain that \( \text{rank}(B) = 1 \), which is contradictory to \( \text{rank}(B) = 2 \).

We assume now that \( \beta_3^2 = \mu_1 \mu_2 \), then the above argument implies that \( \beta_1^2 - \mu_2 \mu_3 \neq 0 \). Notice that (15) and \( det(B) = 0 \) imply that

\[
\beta_1^2 \mu_1 = \beta_2^2 \mu_2.
\]

It follows from the equalities

\[
\mu_1 t_1 + \beta_3 t_2 + \beta_2 t_3 = \beta_3 t_1 + \mu_2 t_2 + \beta_1 t_3 = 1
\]

that

\[
\mu_1 = \mu_2 = \beta_3, \quad \beta_2 = \beta_1.
\]

We therefore solve the linear equations (3) as follows

\[
t_1 + t_2 = \frac{\beta_1 - \mu_3}{\beta_1^2 - \mu_2 \mu_3}, \quad t_1, t_2 \in \mathbb{R}, \quad t_3 = \frac{\beta_1 - \mu_2}{\beta_1^2 - \mu_2 \mu_3}.
\]

To prove the last part of this lemma, we can assume that \( \mu_1 \leq \mu_2 \leq \mu_3 \) without lost of generality. If we subtract one equation in (3) from another, then

\[
(\mu_1 - \beta_3)t_1 + (\beta_3 - \mu_2)t_2 = (\beta_1 - \beta_2)t_3 \]

(20)
and

\[(\mu_2 - \beta_1)t_2 + (\beta_1 - \mu_3)t_3 = (\beta_2 - \beta_3)t_1.\]  \tag{21}

It implies that  \(\beta_1 \leq \beta_2 \leq \beta_3\). Moreover, we have

\[\mu_1 \leq \beta_3 \leq \mu_2 \leq \beta_1 \leq \mu_3\]

because of the assumption \(\beta_i \in [\min\{\mu_j, \mu_k\}, \max\{\mu_j, \mu_k\}]\) for any \(i, j, k\), permutation of 1, 2, 3. Thus, we obtain

\[
\beta_1 = \beta_2 = \beta_3,
\]

which implies that \(\mu_1 = \beta_3 = \mu_2\) and \(\mu_2 = \beta_1 = \mu_3\) by (20) and (21). Consequently, \(b_{ij} \equiv \mu_1\). Thus, \(\text{rank}(B) = 1\).

3. Proofs of the main theorems.

The proof of theorem 1.1. To prove the part (1), it is equivalent to proving that

\[
\Psi(\sqrt{t_1}w, \sqrt{t_2}w, \sqrt{t_3}w) < \inf\{\Psi(\overrightarrow{u}) : \overrightarrow{u} = (u_1, u_2, u_3) \in \mathcal{N}, u_i = 0, \}, i = 1, 2, 3
\]

by lemma 2.2. Moreover, lemma 2.3 implies that

\[
\frac{\Lambda^2}{4} \frac{2\beta_1 - \mu_2 - \mu_3}{\beta_1^2 - \mu_2\mu_3} \leq \inf\{\Psi(0, u, v) : (0, u, v) \in \mathcal{N}\},
\]

\[
\frac{\Lambda^2}{4} \frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1\mu_2} \leq \inf\{\Psi(u, v, 0) : (u, v, 0) \in \mathcal{N}\}.
\]

and

\[
\frac{\Lambda^2}{4} \frac{2\beta_2 - \mu_1 - \mu_3}{\beta_2^2 - \mu_1\mu_3} \leq \inf\{\Psi(u, 0, v) : (u, 0, v) \in \mathcal{N}\},
\]

thanks to the assumption (7). Then, (22) will follow from the following inequality

\[
\Psi(\sqrt{t_1}w, \sqrt{t_2}w, \sqrt{t_3}w) < \frac{\Lambda^2}{4} \min\{\frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1\mu_2}, \frac{2\beta_2 - \mu_1 - \mu_3}{\beta_2^2 - \mu_1\mu_3}, \frac{2\beta_1 - \mu_2 - \mu_3}{\beta_1^2 - \mu_2\mu_3}\}. \tag{23}
\]

Next, let us to prove (23). A straightforward calculation reveals

\[
det(B) = 2\beta_1\beta_2\beta_3 - \sum_{i=1}^{3} \mu_i\beta_i^2 + \mu_1\mu_2\mu_3,
\]

\[
t_i = \frac{1}{det(B)} \left[ (\sum_{j=1}^{3} \beta_j - 2\beta_i)\beta_i - \sum_{j=1}^{3} \mu_j\beta_j + \mu_i\beta_i + \frac{\mu_1\mu_2\mu_3}{\mu_i} \right], \quad i = 1, 2, 3
\]

and

\[
\Psi(\sqrt{t_1}w, \sqrt{t_2}w, \sqrt{t_3}w) = \frac{\Lambda^2}{4} (t_1 + t_2 + t_3).
\]
Then,
\[
\frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} - (t_1 + t_2 + t_3) = \frac{1}{(\beta_3^2 - \mu_1 \mu_2) \det(B)} \left[ \beta_3^2 (\beta_1 + \beta_2 - \beta_3)^2 - 2\beta_3 (\beta_1 + \beta_2 - \beta_3) (\mu_1 \beta_1 + \mu_2 \beta_2) + (\mu_1 \beta_1 + \mu_2 \beta_2)^2 \right] \\
= \frac{1}{(\beta_3^2 - \mu_1 \mu_2) \det(B)} \left[ \beta_3 (\beta_1 + \beta_2 - \beta_3) - (\mu_1 \beta_1 + \mu_2 \beta_2) + \mu_1 \mu_2 \right]^2 \\
= \frac{1}{\beta_3^2 - \mu_1 \mu_2} \det(B). 
\]
(24)

Similarly, we have
\[
\frac{2\beta_2 - \mu_1 - \mu_3}{\beta_2^2 - \mu_1 \mu_3} - (t_1 + t_2 + t_3) = \frac{t_3^2 \det(B)}{\beta_2^2 - \mu_1 \mu_3}, 
\]
and
\[
\frac{2\beta_1 - \mu_2 - \mu_3}{\beta_1^2 - \mu_2 \mu_3} - (t_1 + t_2 + t_3) = \frac{t_3^2 \det(B)}{\beta_1^2 - \mu_2 \mu_3}. 
\]

Therefore, (23) follows from the assumption \( \det(B) > 0 \), \( t_1, t_2, t_3 > 0 \) and (7).

For the part (2), we assume that \( \beta_1 - \sqrt{\mu_1 \mu_2}, \beta_2 - \sqrt{\mu_1 \mu_3} \) and \( \beta_1 - \sqrt{\mu_2 \mu_3} \) are all negative. Then (24) implies that
\[
\frac{\Lambda^2 2\beta_3 - \mu_1 - \mu_2}{4 \beta_3^2 - \mu_1 \mu_2} < \frac{\Lambda^2}{4} (t_1 + t_2 + t_3). 
\]
(25)

Moreover, by lemma 2.2
\[
\alpha_{(1)} \left( \frac{\Lambda^2}{4} \right) \frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} = \frac{\Lambda^2}{4} \frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} = \frac{\Lambda^2}{4} \frac{(\beta_3 - \mu_1)^2}{\beta_3^2 - \mu_1 \mu_2} \leq 0, 
\]
(26)
because of \( \beta_3 - \sqrt{\mu_1 \mu_2} < 0 \). Putting (25) together with (26), we deduce that
\[
\alpha_{(1)} \left( \frac{\Lambda^2}{4} \right) < \frac{\Lambda^2}{4} (t_1 + t_2 + t_3). 
\]

On the other hand, any ground-state solution of (1) with exact one zero component does not exist by lemma 2.3. It implies that any ground-state solution of (1) has exactly one non-zero component. Consequently, it completes the proof of part (2).

\[\Box\]

**Remark 6.** The part (1) of the above theorem has been proved in [9] under the assumption
\[
\beta_1 = \beta_2 > \max \{ \frac{\beta_3^2 - \mu_1 \mu_2}{2 \beta_3 - \mu_1 - \mu_2}, \mu_3 \}, \quad \beta_3 > \max \{ \mu_1, \mu_2 \}, 
\]
(27)
which implies that \( \beta_1 > \max \{ \mu_2, \mu_3 \}, \beta_2 > \max \{ \mu_1, \mu_3 \}, \beta_3 > \max \{ \mu_1, \mu_2 \} \). Therefore, the results here generalize that of [9] in the case of \( m = 3 \). The following is an example
\[
B = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 3 \\ 5 & 3 & 2 \end{pmatrix},
\]
which satisfies all the assumptions here but not for (27).
Remark 7. The inequalities (8) and det(B) > 0 in part (2) of the above theorem are equivalent to that the matrix B is positive definite. For example,

\[ B = \begin{pmatrix} 6 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 4 \end{pmatrix}. \]

Corollary 1. Let \( k_1, k_2, k_3 > 0 \) be fixed, \(|k_1 - k_2| < k_3 \) and \( \beta_i = \beta k_i, i = 1, 2, 3, \beta > 0 \). Then, for any fixed \( \mu_1, \mu_2, \mu_3 > 0 \) there exists some constant \( \beta_0 > 0 \) depending only on \( \mu_1, k_i, i = 1, 2, 3 \) such that for any \( \beta > \beta_0 \), \((\sqrt{T_1 w}, \sqrt{T_2 w}, \sqrt{T_3 w})\) is a ground-state solution of (1). Moreover, any ground-state solution of (1) is nontrivial.

Proof. To prove the corollary, it suffices to verify that all assumptions of part (1) in theorem 1.1 hold here. Indeed, we have in current case \( \det(B) \) can be rewritten as

\[ \det(B) = 2k_1k_2k_3\beta^3 - \sum_{i=1}^{3} \mu_i k_i^2 \beta^2 + \mu_1\mu_2\mu_3 \] (28)

and the unique solution of (3)

\[ t_i = \frac{1}{\det(B)} \left( \sum_{j=1}^{3} k_j - 2k_i \right) k_i \beta^2 - \left( \sum_{j=1}^{3} \mu_j k_j - \mu_i k_i \right) \beta + \frac{\mu_1\mu_2\mu_3}{\mu_i}, \quad i = 1, 2, 3. \]

It implies that \( \det(B) \) and \( \det(B) \cdot t_i \), the product of \( \det(B) \) and \( t_i \), \( i = 1, 2, 3 \) are all polynomials with respect to variable \( \beta \) with positive leading coefficient because of \(|k_1 - k_2| < k_3 \). Therefore, they are all positive as \( \beta > 0 \) sufficiently large. Moreover, it is easy to see that

\[ \beta > \frac{\max\{\mu_1, \mu_2, \mu_3\}}{\min\{k_1, k_2, k_3\}} \]

implies that (7) holds. Consequently, the corollary is proved. \( \square \)

Remark 8. It is interesting to consider three positive numbers \( k_1, k_2, k_3 \) as the lengths of the sides of a triangle due to the inequality \(|k_1 - k_2| < k_3 \). Therefore, the assumptions of the above corollary are equivalent to that three coupling constants are corresponding to the lengths of the sides of a triangle which is similar to another one with its sides having lengths \( k_1, k_2, k_3 \) and the ratio \( \beta > 0 \) is sufficiently large.

The following corollary can be proved in a similar way to that of corollary 1, so the proof is omitted.

Corollary 2. Let \( 0 < \mu_1 \leq \mu_2 \leq \mu_3 \) be fixed. Then, there exists some constant \( \beta_0 > 0 \) depending only on \( \mu_1, \mu_2 \) and \( \mu_3 \) such that \((0,0, \frac{1}{\sqrt{\mu_3}}w)\) is a ground-state solution of (1) provided that \( \max\{\beta_1, \beta_2, \beta_3\} < \beta_0 \).

The proof of theorem 1.2. It is equivalent to proving

\[ (t_1 + t_2 + t_3) \frac{A_1^2}{4} > \inf\{\Psi(\vec{u}) : \vec{u} = (u_1, u_2, u_3) \in \mathcal{N}, u_i = 0, i = 1, 2, 3. \} \] (29)

In order to show (29), we consider the following two cases: one case is that there exists some \( i, j, k \) permutation of 1, 2, 3 such that \( \beta_i > \max\{\mu_j, \mu_k\} \), and another case is that \( \beta_i \leq \max\{\mu_2, \mu_3\}, \beta_2 \leq \max\{\mu_1, \mu_3\}, \beta_3 \leq \max\{\mu_1, \mu_2\} \). For the former, a calculation similar to (24) reveals

\[ \frac{2\beta_i - \mu_j \mu_k}{\beta_i^2 - \mu_j \mu_k} - (t_1 + t_2 + t_3) = \frac{t_i^2 \det(B)}{\beta_i - \mu_j \mu_k} < 0, \]
which implies that
\[ t_1 + t_2 + t_3 > \frac{2\beta_i - \mu_j \mu_k}{\beta_i^2 - \mu_j \mu_k}. \]

Therefore, the theorem in this case follows from Lemma 2.3. On the other hand, for the latter, we have
\[ \beta_1 \leq \max\{\mu_2, \mu_3\}, \beta_2 \leq \max\{\mu_1, \mu_3\}, \beta_3 \leq \max\{\mu_1, \mu_2\}. \]

Then, it follows from (3) that
\[ (t_1 + t_2 + t_3) \max\{\mu_1, \mu_3, \mu_3\} \geq 1. \]

We therefore deduce that
\[ t_1 + t_2 + t_3 \geq \frac{1}{\max\{\mu_1, \mu_3, \mu_3\}}. \]

It is easy to see that if
\[ t_1 + t_2 + t_3 = \frac{1}{\max\{\mu_1, \mu_3, \mu_3\}} \]
were true then it would cause \( \text{rank}(B) = 1 \). Consequently, the theorem in this case follows from
\[ t_1 + t_2 + t_3 > \frac{1}{\max\{\mu_1, \mu_3, \mu_3\}}. \]

**Remark 9.** Each of the two cases considered in the proof of the above theorem may occur. For example,
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}, \quad \begin{bmatrix}
4 & 6 & 8 \\
6 & 10 & 2 \\
8 & 2 & 8 \\
\end{bmatrix}
\]

The proof of theorem 1.3. It follows from lemma 2.4 that
\[ b_{ij} \equiv \mu_1, \]
for all \( i, j = 1, 2, 3 \). It is easy to see that
\[ \inf\{\Psi(\vec{u}) : \vec{u} = (u_1, u_2, u_3) \in \mathcal{N}\} = \frac{\Lambda^2}{4\mu_1}, \]
by lemma 2.2. Then, the theorem follows from the equality
\[ \Psi(\sqrt{t_1} w, \sqrt{t_2} w, \sqrt{t_3} w) = \frac{\Lambda^2}{4\mu_1}, \]
where \( t_1 + t_2 + t_3 = \frac{1}{\mu_1} \) and \( t_j \geq 0, j = 1, 2, 3. \)

**Remark 10.** In this case \( b_{ij} = b_{ij} \) for all \( i, j = 1, 2, 3 \), all positive solutions to (1) are synchronized solutions, which has been proved in [15] in the case of \( m = 2 \) and space dimension \( n = 1 \). For a general case of \( m \geq 3 \), the result can be proved by the same argument in [15].

Next, we discuss ground-state solutions for the case in which \( \text{rank}(B) = 2 \). Taking Lemma 2.4 into account, we see that all possible cases are contained in the three parts of Theorem 1.4.
The proof of Theorem 1.4. For part (a), the matrix $B$ can be rewritten as follows

$$B = \begin{pmatrix} \mu_2 & \mu_2 & \beta_1 \\ \mu_2 & \mu_2 & \beta_1 \\ \beta_1 & \beta_1 & \mu_3 \end{pmatrix},$$

due to lemma 2.4. Thus, all non-negative solutions of (3) can be formulated by

$$t_1 + t_2 = \frac{\beta_1 - \mu_3}{\beta_1^2 - \mu_2 \mu_3}, \quad t_1, t_2 \geq 0, \quad t_3 = \frac{\beta_1 - \mu_2}{\beta_1^2 - \mu_2 \mu_3}.$$ 

Indeed, we observe that either $\beta_1 > \max\{\mu_2, \mu_3\}$ or $\beta_1 < \min\{\mu_2, \mu_3\}$ must hold because $\beta_1 \in [\min\{\mu_2, \mu_3\}, \max\{\mu_2, \mu_3\}]$ would imply $\text{rank}(B) = 1$ by part (5) of Lemma 2.4, which is contradictory to $\text{rank}(B) = 2$. Then, the part (1) of (a) follows from the following equalities

$$t_1 + t_2 + t_3 = \frac{2\beta_1 - \mu_2 - \mu_3}{\beta_1^2 - \mu_2 \mu_3} = \inf \{ \Psi(0, u, v) : (0, u, v) \in \mathcal{N} \} = \inf \{ \Psi(u, 0, v) : (u, 0, v) \in \mathcal{N} \} < \min \left\{ \frac{1}{\mu_2}, \frac{1}{\mu_3} \right\},$$

where Lemma 2.3 has been used. Similarly, part (2) of (a) follows from the following inequality

$$\max \left\{ \frac{1}{\mu_2}, \frac{1}{\mu_3} \right\} < t_1 + t_2 + t_3,$$
	hanks to $\beta_1 < \min\{\mu_2, \mu_3\}$.

For part (b), we suppose that

$$\text{rank}(B) = 2, \quad \beta_1^2 > \mu_2 \mu_3, \quad \beta_2^2 > \mu_1 \mu_3, \quad \beta_3^2 > \mu_1 \mu_2.$$  (30)

We notice that the following three vectors

$$\left( \frac{\beta_3 - \mu_2}{\beta_3^2 - \mu_1 \mu_2}, \frac{\beta_3 - \mu_1}{\beta_3^2 - \mu_1 \mu_2}, 0 \right), \left( \frac{\beta_2 - \mu_3}{\beta_2^2 - \mu_1 \mu_3}, 0, \frac{\beta_2 - \mu_1}{\beta_2^2 - \mu_1 \mu_3} \right), \left( 0, \frac{\beta_1 - \mu_3}{\beta_1^2 - \mu_2 \mu_3}, \frac{\beta_1 - \mu_2}{\beta_1^2 - \mu_2 \mu_3} \right),$$

are all solutions to the linear equations (3). Certainly, it is possible that some of their components are negative. By remark 3, the sums of all components for all solutions are equal to each other. That is,

$$\frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} = \frac{2\beta_2 - \mu_1 - \mu_3}{\beta_3^2 - \mu_2 \mu_3} = \frac{2\beta_1 - \mu_2 - \mu_3}{\beta_1^2 - \mu_2 \mu_3}.$$ 

We can assume, by part (5) of lemma 2.4, that $\beta_3 > \max\{\mu_1, \mu_2\}$ without the generality. Thus, all non-negative solutions to (3) can be expressed as follow

$$t_1 = \frac{\beta_3 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} + \frac{\mu_2 \beta_2 - \beta_1 \beta_3}{\beta_3^2 - \mu_1 \mu_2} t_3,$$

$$t_2 = \frac{\beta_3 - \mu_1}{\beta_3^2 - \mu_1 \mu_2} + \frac{\mu_1 \beta_1 - \beta_2 \beta_3}{\beta_3^2 - \mu_1 \mu_2} t_3,$$

$$t_3 = t_3 \in [0, T],$$  (31)
where \( T \) is determined by
\[
T = \begin{cases}
\beta_3 - \mu_2 & \text{if } \mu_2 \beta_2 - \beta_1 \beta_3 < 0, \\
\frac{\beta_3 - \mu_2}{\beta_3 - \mu_1} & \text{if } \mu_1 \beta_1 - \beta_2 \beta_3 < 0, \\
\min\left\{ -\frac{\beta_3 - \mu_1}{\mu_2 \beta_2 - \beta_1 \beta_3}, -\frac{\beta_3 - \mu_1}{\mu_1 \beta_1 - \beta_2 \beta_3} \right\} & \text{if } \mu_2 \beta_2 - \beta_1 \beta_3 < 0, \mu_1 \beta_1 - \beta_2 \beta_3 < 0,
\end{cases}
\]
which is well defined because of
\[
(\mu_2 \beta_2 - \beta_1 \beta_3) + (\mu_1 \beta_1 - \beta_2 \beta_3) = \mu_1 \mu_2 - \beta_3^2 < 0,
\]
where the equality follows from the assumption that (3) has at least one solution.

Notice that \( t_1 \) and \( t_2 \) in (31) must be non-negative because both \( t_1 \) and \( t_2 \) are linear functions with respect to the variable \( t_3 \). Then, part (b) follows from
\[
t_1 + t_2 + t_3 = \frac{2\beta_3 - \mu_1 - \mu_2}{\beta_3^2 - \mu_1 \mu_2} \leq \frac{1}{\mu_3},
\]
due to \( \beta_3 > \max\{\mu_1, \mu_2\} \) and \( \beta_1 > \sqrt{\mu_2 \mu_3} \).

For part (c), we assume that
\[
\beta_2^2 < \mu_2 \beta_3, \quad \beta_2^2 < \mu_1 \beta_3, \quad \beta_2^2 < \mu_1 \mu_2.
\]

Let \((t_1, t_2, t_3)\) be a non-negative solution to (3). It follows from (32) and lemma 2.3 that there is no ground state solution having exactly one trivial component. Moreover, an argument similar to that of theorem 1.2 reveals
\[
t_1 + t_2 + t_3 > \frac{1}{\max\{\mu_1, \mu_2, \mu_3\}},
\]
which implies that part (c) holds.

**Remark 11.** The following examples satisfy all assumptions in part (a).
\[
B = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.
\]

**Remark 12.** Here are some examples such as
\[
B = \begin{pmatrix} 5 & 13 & 9 \\ 13 & 5 & 9 \\ 9 & 9 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 13 & 9 \\ 13 & 7 & 9 \\ 9 & 9 & 9 \end{pmatrix}
\]
satisfying all assumptions in part (b).

**Remark 13.** There exist an example satisfying all assumptions in part (c). For instance,
\[
B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.
\]
Remark 14. It is easy to see from Table 1 that there is no nontrivial ground-state in the cases where there exists some $i, j, k$, permutation of 1, 2, 3 such that $0 < \beta_i < \sqrt{\mu_j \mu_k}$. Indeed, the Morse index of any solution of (1) in this case is at least two if the $j^{th}$ and $k^{th}$ component are both non-zero. It implies that the $j^{th}$ or $k^{th}$ component must be zero for any ground-state solution of (1). In particular, for case 2 and case 8 in Table 1, every ground-state solution has exactly one non-zero component without the assumption that linear equations (3) has at least one positive solution.

4. Synchronized ground-state solutions. Before proving Theorem 1.5, we need the following lemma.

Lemma 4.1. Suppose that $B = (b_{ij})$ is positive semi-definite. Let $(u_1, u_2, \cdots, u_m) \in X_m$ and $u_i \neq 0$ for all $i = 1, 2, \cdots, m$. Then the function $f(t_1, t_2, \cdots, t_m) = \Psi(\sqrt{t_1}u_1, \sqrt{t_2}u_2, \cdots, \sqrt{t_m}u_m)$ has a global maximum on $(0, +\infty)^m \subset \mathbb{R}^m$ and all extreme values of $f$ are equal to one another. In particular, if $(u_1, u_2, \cdots, u_m) \in \mathcal{N}$, then

$$\Psi(\sqrt{t_1}u_1, \sqrt{t_2}u_2, \cdots, \sqrt{t_m}u_m) \leq \Psi(u_1, u_2, \cdots, u_m)$$

for all $(t_1, t_2, \cdots, t_m) \in (0, +\infty)^m$.

Proof. By the definition of $\Psi$, we have

$$f(t_1, t_2, \cdots, t_m) = \frac{1}{2} \sum_{j=1}^{m} t_j \int_{\mathbb{R}^n} (|\nabla u_j| + |u_j|^2) dx - \frac{1}{4} \sum_{i,j=1}^{m} b_{ij} t_i t_j \int_{\mathbb{R}^n} |u_i u_j|^2 dx,$$

which implies that $f$ attains its global maximum at some point in $(0, +\infty)^m$. On the other hand, it is clear that

$$\frac{\partial f}{\partial t_j} = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u_j|^2 + |u_j|^2) dx - \sum_{i=1}^{m} b_{ij} t_i \int_{\mathbb{R}^n} |u_i u_j|^2 dx, \quad j = 1, 2, \cdots, m$$

and the Hessian matrix $H_f$ of $f$ is

$$H_f = (-\frac{1}{2} b_{ij} \int_{\mathbb{R}^n} |u_i u_j|^2 dx).$$

Then, $-H_f$ is positive semi-definite because $B$ is positive semi-definite. It implies that every critical point of $f$ is a local maximum point. Consequently, $f$ takes the same value on these critical points. Indeed, let $(a_1, a_2, \cdots, a_m)$ and $(b_1, b_2, \cdots, b_m)$ be any two critical points of $f$ on $(0, +\infty)^m$. Set

$$g(\theta) = f(a_1 + \theta(b_1 - a_1), a_2 + \theta(b_2 - a_2), \cdots, a_m + \theta(b_m - a_m)), \quad \theta \in [0, 1].$$

Notice that

$$g''(\theta) = (b_1 - a_1, b_2 - a_2, \cdots, b_m - a_m) H_f (b_1 - a_1, b_2 - a_2, \cdots, b_m - a_m)^T \leq 0,$$

which implies that its first derivative $g'(\theta)$ is non-increasing. We therefore obtain that $g'(\theta) \equiv 0$ on $[0, 1]$ because of $g'(0) = g'(1) = 0$. It implies that $f(a_1, a_2, \cdots, a_m) = g(0) = g(1) = f(b_1, b_2, \cdots, b_m)$. Furthermore, (33) follows the fact that $(1, \cdots, 1)$ is a critical point of $f$ when $(u_1, u_2, \cdots, u_m) \in \mathcal{N}$. \hfill $\square$

Remark 15. When $m = 1$, the function $f$ must have only one critical point on $(0, \infty)$. However, critical points of $f$ could be more than one for $m \geq 2$. For instance, when $b_{ij} = b_{11} > 0$ for all $i, j = 1, 2, \cdots, m$ and $u_i = \phi$ for all $i = 1, 2, \cdots, m$, $f$ has infinitely many critical points.
Now let us prove theorem 1.5.

The proof of theorem 1.5. Let \((u_1, u_2, \cdots, u_m)\) be a minimizer of \(\Psi\) on \(\tilde{N}\). Suppose that
\[
\int_{\mathbb{R}^n} \frac{|u_0|^4}{\|u_0\|_{H^1}^4} \, dx = \max \left\{ \int_{\mathbb{R}^n} \frac{|u_j|^4}{\|u_j\|_{H^1}^4} \, dx \mid j = 1, 2, \cdots, m \right\}.
\]

Let \(\phi = \frac{u_0}{\|u_0\|_{H^1}}\). We assume, by way of contradiction, that there exists some \(k\) such that \(u_k\) is not proportional to \(u_0\). Then the following inequalities hold
\[
\int_{\mathbb{R}^n} \frac{|u_i u_j|^2}{\|u_i\|_{H^1}^2 \|u_j\|_{H^1}^2} \, dx \leq \int_{\mathbb{R}^n} |\phi|^4 \, dx, \text{ for } i, j = 1, 2, \cdots, m,
\]

\[
\int_{\mathbb{R}^n} \frac{|u_i u_k|^2}{\|u_i\|_{H^1}^2 \|u_k\|_{H^1}^2} \, dx < \int_{\mathbb{R}^n} |\phi|^4 \, dx
\]

by Cauchy inequality. On the other hand, by lemma 4.1 there exists \((s_1, s_2, \cdots, s_m)\) \(\in (0, \infty)^m\) such that
\[
\max \{ \Psi(\sqrt{t_1} \phi, \sqrt{t_2} \phi, \cdots, \sqrt{t_m} \phi) \mid (t_1, t_2, \cdots, t_m) \in (0, \infty)^m \}
= \Psi(\sqrt{s_1} \phi, \sqrt{s_2} \phi, \cdots, \sqrt{s_m} \phi)
\]

and \((\sqrt{s_1} \phi, \sqrt{s_2} \phi, \cdots, \sqrt{s_m} \phi) \in \tilde{N}\). Putting (33) and (34) together, we have
\[
c = \inf \{ \Psi(\frac{u}{\|u\|_{H^1}}) \mid u \in \tilde{N} \}
\leq \Psi(\sqrt{s_1} \phi, \sqrt{s_2} \phi, \cdots, \sqrt{s_m} \phi)
< \Psi(\sqrt{s_1} \frac{u_1}{\|u_1\|_{H^1}}, \cdots, \sqrt{s_m} \frac{u_m}{\|u_m\|_{H^1}})
\leq \Psi(u_1, u_2, \cdots, u_m) = c.
\]

This is a contradiction. Consequently, we deduce that \(u_i = c_i \phi\) for some non-zero constants \(c_i, i = 1, 2, \cdots, m\). It follows from \((u_1, u_2, \cdots, u_m) \in \tilde{N}\) that
\[
\sum_{j=1}^{m} b_{ij} c_j^2 = \sum_{j=1}^{m} b_{1j} c_j^2 := \gamma
\]

for all \(i = 1, 2, \cdots, m\). For \(j = 1, 2, \cdots, m\), set
\[
t_j = \frac{c_j^2}{\gamma}, \quad v = \sqrt{\gamma} \phi,
\]

then \(u_j = \pm \sqrt{t_j} v\) for \(j = 1, 2, \cdots, m\). It is now easy to complete the proof. \(\square\)

As mentioned in remark 1, the existence of the minimizer of \(\Psi\) on \(\tilde{N}\) is proved in [7]. Therefore we have the following corollary

Corollary 3. There exists a small positive constant \(\beta_0 > 0\) depending on \(b_{ii}, i = 1, 2, \cdots, m\) such that if \(b_{ij} \in (0, \beta_0), i \neq j\), then linear equations (3) has a positive solution.

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