Unambiguous angular momentum of radiative spacetimes and asymptotic structure in terms of the center of mass system

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Abstract

We present a definition of angular momentum for radiative spacetimes which does not suffer from any ambiguity of supertranslations. We succeed in providing an appropriate notion of intrinsic angular momentum; and at the same time a definition of center of mass frame at future null infinity. We use the center of mass frame to present the asymptotic structure equations for vacuum spacetimes.

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1 Introduction

The difficulty associated with a sensible definition of angular momentum at future null infinity \( I^+(\text{scri}) \) is related to the complicated structure of the Bondi-Metzner-Sachs (BMS) group of asymptotic symmetries. There is a multiplicity of Poincaré subgroups, one for each supertranslation.

Because of this difficulty one can find in the literature several inequivalent definitions of angular momentum \([\text{LW}66, \text{Win}68, \text{Bra}75, \text{Pri}77, \text{Str}78, \text{Pen}82, \text{DS}84]\) which suffer from the so called supertranslation ambiguities. The only definition of angular momentum without supertranslation ambiguities is the one we presented in reference \([\text{Mor}86]\); however our construction can be criticized on the grounds that we select a unique frame based on the properties of future null infinity in the limit for the retarded time going to \(-\infty\). A global choice of this nature can hardly be thought to be able to describe the local meaning of intrinsic angular momentum of the sources at an arbitrary retarded time of a radiating system.

In order to provide with an unambiguous notion of angular momentum in radiative spacetimes, it is essential to make use of a reference frame system, which embodies the notion of rest frame\([\text{Win}80]\).

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The problem of the characterization of the notion of a rest frame for isolated systems has been the matter of concern for many years. Early attempts used the properties of an optical parameter of a congruence of null geodesics reaching future null infinity; namely, the shear. When the spacetime is stationary, it is known \cite{NP68} that one can choose a Bondi system for which the shear $\sigma_0$ is zero; where the freedom in this choice is just a translation. The sections for which the shear is zero are known as 'good cuts'. We see then that for stationary spacetimes there is a simple geometrical way to get rid off the supertranslation problem. However for radiating spacetimes things are not so easy. In presence of gravitational radiation, the good cut equation has in general no solution; but it was suggested in the literature that the solution to this can be obtained if $I^+$ is complexified. This approach has lead to the beautiful construction of $H$ spaces\cite{NT80}. Due to the fact that $H$ spaces are intrinsically complex, it remained the difficulty of the interpretation, in this construction.

This situation motivated us in the past to introduce the so called 'nice sections' \cite{Mor88} at $I^+$. These sections are defined in terms of the nice section equation; which recently has been proved \cite{MD98,DM00} to have solutions in terms of a 4-parameter family of translations. Furthermore the solutions have the expected physical properties of a rest frame \cite{DM00}; for example, having a nice section $S_0$, all other nice sections $S_f$ obtained from future timelike translations happen to be to the future of $S_0$.

The nice section construction singles out precisely, in an intrinsic way, a Poincaré structure from the infinite dimensional BMS group. In particular, given a fixed observational point $p$ at $I^+$, there is precisely a 3-degree of freedom of spacelike translations which generate all the nice sections that contain $p$. In contrast, without this constructions there is an infinite dimensional family of general sections that contain $p$, one for each supertranslation.

The success of the nice section construction allows us to define a well behaved notion of intrinsic angular momentum along with the associated notion of center of mass frame.

In section 2 we briefly review the definition of nice sections. We make use of charge integrals of the Riemann tensor to present in section 3 both the angular momentum definition and the center of mass frame. In section 4 we study the asymptotic structure equations of a general radiating spacetime in terms of the center of mass frame. We briefly compare our presentation with previous definitions of angular momentum and make final comments in section 5.

We will make use of the GHP\cite{GHP73} notation.

2 Rest frame systems at future null infinity

If a physical system can be ascribed a Poincaré structure, one should be able to define its momentum vector $P^a$ and angular momentum tensor $J^{ab}$; and then one would define a rest frame by demanding that the total momentum has no spacial components in this frame. However an isolated gravitating system has the BMS structure\cite{Mor87}; therefore it is natural to define a rest frame by transforming to a section for which the supermomentum has no spacial components. This brings us to the issue of which supermomentum to use. In the nice section construction we use \cite{Mor88} the supermomentum, that for simplicity we are going to call 'psi'. The supermomentum psi has some interesting properties, but it is important to emphasize that it does not coincide with the Geroch supermomentum \cite{Ger77}.
or with the Winicour one [W66] [W68] [W80] [GW81].

Given an arbitrary section \( S \) of \( \mathcal{I}^+ \), one can, without loss of generality, choose a Bondi coordinate system \((u, \zeta, \bar{\zeta})\), such that \( u = 0 \) determines the section \( S \). Then, the supermomentum \( \text{psi} \) on \( S \) is given by the expression

\[
P_{lm}(S) = -\frac{1}{\sqrt{4\pi}} \int_S Y_{lm}(\zeta, \bar{\zeta})\Psi(u = 0, \zeta, \bar{\zeta})dS^2,
\]

where \( dS^2 \) is the surface element of the unit sphere on \( S \), \( Y_{lm} \) are the spherical harmonics, the scalar \( \Psi \) is given by

\[
\Psi \equiv \Psi^0_2 + \sigma_0 \dot{\sigma}_0 + \bar{\sigma}^2 \bar{\sigma}_0,
\]

where \( \Psi^0_2 \) is the leading order asymptotic behavior of the second Weyl tensor component, \( \sigma_0 \) is the leading order of the Bondi shear respectively, where we are using \( \bar{\sigma} \) to denote the GHP [GHP73] operator of the unit sphere, and where a dot means partial derivative with respect to the retarded time \( u \).

The first four components of the supermomentum, namely the case \( l = 0 \) and the three cases \( l = 1 \), determine the Bondi energy-momentum vector. More specifically

\[
(P^a) = \left( P_{00}, -\frac{1}{\sqrt{6}}(P_{11} - P_{1,-1}), i\frac{1}{\sqrt{6}}(P_{11} + P_{1,-1}), \frac{1}{\sqrt{3}}P_{10} \right),
\]

where \( a = 0, 1, 2, 3 \).

All other nice sections can be obtained from \( S \) by the appropriate supertranslations \( \gamma \). Therefore one can impose the nice section condition on \( \gamma \). It was shown [Mor88] [MD98] that this condition can be cast in the following equation

\[
\bar{\sigma}^2 \bar{\sigma}^2 \gamma = \Psi(\gamma, \zeta, \bar{\zeta}) + K^3(\gamma, \zeta, \bar{\zeta})M(\gamma),
\]

where the conformal factor \( K \) can be related to the Bondi momentum by

\[
K = \frac{M}{P^a l_a},
\]

with

\[
(l^a) = \left( 1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta \bar{\zeta})}, \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}} \right)
\]

and \( P^a \) is evaluated at the section \( u = \gamma \); which is calculated through the integral (3). The rest mass \( M \) at the section is given by

\[
M = \sqrt{P^a P_a},
\]

where the indices are raised and lowered with the Lorentzian flat metric \( \eta_{ab} \) at scri as described in reference [Mor86].

As it was stated in the introduction, there exists a four parameter family of solutions of equation (4) with the expected physical properties [MD98] [DM00]. One can always express \( \gamma(\zeta, \bar{\zeta}) \) in terms of the translation part \( \gamma_I(\zeta, \bar{\zeta}) \) and its proper supertranslation part \( \gamma_{II}(\zeta, \bar{\zeta}) \); where using spherical harmonics \( \gamma_I \) is expressed in terms of \( Y_{00} \) and \( Y_{1m} \); while \( \gamma_{II} \) is expressed in terms of \( Y_{l_2m} \) with \( l_2 \geq 2 \). The result is that for each translation \( \gamma_I \) there is a proper supertranslation \( \gamma_{II} \).
We have assumed that the original section \( S \) determined by \( u = 0 \) is a nice section. In other words \( \gamma = 0 \) is already a solution of \( \Pi \), and so \( K(0, \zeta, \bar{\zeta}) = 1 \). For any other nice section \( \gamma \neq 0 \) in general the Bondi momentum will have spacelike components with respect to the original system, and so one would have \( K \neq 1 \). In that case, in order to observe that the space like components of the supermomentum are zero, one needs to make a Lorentz transformation of the BMS group, that aligns the generator of time translations with the Bondi momentum, at the new section.

The need of supertranslations and Lorentz transformations for grasping the idea of rest frames, is what makes the discussion of radiative spacetimes so complicated; and it is clear that one single Bondi system does not provide with a rest frame unless the metric is stationary.

3 Intrinsic angular momentum and center of mass

3.1 Charge integrals of the Riemann tensor

The rest frames provided by the nice sections construction permit us to define in an unambiguous way the physical quantities of an isolated system, in particular its angular momentum.

It is convenient to approach the notion of physical quantities by the method of charge integrals of the Riemann tensor, as it was used in our previous work [Mor86].

Given a 2-sphere \( S \) a charge integral of the Riemann tensor is a quantity of the form:

\[
Q_S = \int_S C
\]

where the 2-form \( C_{ab} \) is given in terms of the Riemann tensor by

\[
C_{ab} \equiv R^s_{ab} cd w_{cd},
\]

and where the 2-form \( w_{ab} \) is to be chosen appropriately.

We will assume that the sphere \( S \) is actually a section of \( I^+ \). Since the spacetime is asymptotically flat, it is possible to express the metric, in a neighborhood of scri, around a flat background, namely

\[
g_{ab} = \eta_{ab} + h_{ab},
\]

where at scri, the conformal regular metric \( \bar{g}_{ab} \) is given by

\[
\bar{g}_{ab} |_{I^+} = \Omega^2 \eta_{ab};
\]

in other words, at scri one has \( \Omega^2 h_{ab} = 0 \).

Let us consider, for a moment, the situation in which the expansion [10] can be extended to the interior of the spacetime; which, to simplify the discussion, will be assumed to contain no singularities. Let \( \Sigma \) be a spacelike hypersurface in the interior of the spacetime but that reaches asymptotically future null infinity; in such a way that in the conformally completed spacetime, \( \Sigma \) can be extended to scri with boundary \( S \). Then, it is deduced that the charge integral on \( S \) can be expressed as an integral on \( \Sigma \), by the use of Stokes’ theorem, namely

\[
Q_S = \int_S C = \int_\Sigma dC.
\]
The exterior derivative of $C$ can be expressed by

$$dC_{abc} = \frac{1}{3} \epsilon_{abcd} \ast R^{*defg} \nabla_e w_{fg};$$  \hspace{1cm} (13)$$

where it is interesting to note that a property of the double dual of the Riemann tensor is that its trace gives the Einstein tensor, namely: $\ast R^{*}_{abcd} g^{bd} = G_{ac} = R_{ac} - \frac{1}{2} g_{ac} R$. Therefore the previous equation can be written

$$dC_{abc} = \frac{1}{3} \epsilon_{abcd} \ast R^{*defg} \left( T_{efg} + \frac{1}{3} g_{ef} v_g - \frac{1}{3} g_{eg} v_f \right)$$  \hspace{1cm} (14)$$

where $T_{abc}$ is the traceless part of $\nabla_a w_{bc}$ and $v_c$ its trace; in other words

$$\nabla_a w_{bc} = T_{abc} + \frac{1}{3} g_{ab} v_c - \frac{1}{3} g_{ac} v_b;$$  \hspace{1cm} (15)$$

where it can be checked that $\nabla_a w^{ab} = v^b$. From equation (14) one observes that if the vector $v^a$ where a Killing vector of the metric $\eta_{ab}$ and $T_{abc}$ where $O(h)$, then the charge integral will give precisely the conserved quantities in the context of linearized gravity. This analysis ensures that this charge integrals admit the appropriate physical interpretations in the linearized gravity regime.

In what follows, we will only assume that the spacetime is asymptotically flat at future null infinity [Mor87].

Another interesting property of the double dual of the Riemann tensor is the one associated with the Bianchi identities, namely $\ast R^{*}_{[def]} = 0$, from which one can prove the relations [Mor86]

$$\ast R^{*}_{defg} T_{efg} = \frac{2}{3} \ast R^{*defg} \left( T_{(ef)g} - T_{(eg)f} \right);$$  \hspace{1cm} (16)$$

where it is important to note that the factor involving the tensor $T_{abc}$ have a very simple form when expressed in terms of the spinorial notation, that is

$$\frac{2}{3} \left( T_{(ef)g} - T_{(eg)f} \right) = \nabla_{E'} (E^{w}_{FG}) \epsilon_{F'G'} + \text{c.c.};$$  \hspace{1cm} (17)$$

where c.c. means complex conjugate.

Observing equation (14) and recalling the analysis in the context of linearized gravity, it is natural to study at scri the conditions on $w$ given by the equations

$$-\nabla_A^{B'} w^{AB} + \text{c.c.} = v^{BB'}$$  \hspace{1cm} (18)$$

and

$$\nabla_{E'} (E^{w}_{FG}) = 0;$$  \hspace{1cm} (19)$$

where the vector $v^{BB'}$ is a generator of asymptotic symmetries.

The generators of Lorentz rotations are a special case of asymptotic symmetries. The vector field $v^a$, defined in a neighborhood of scri is said to be an asymptotic symmetry if it satisfies

$$\nabla_{(a} v_{b)} = \frac{S_{ab}}{\Omega},$$  \hspace{1cm} (20)$$
where \( S_{ab} \) has a regular extension to \( I^+ \).

In general an asymptotic symmetry \( v^a \) can be expressed by its components, in terms of a null tetrad frame

\[
v^a = v_n \ell^a - v_m m^a - v_m \bar{m}^a + v_\ell n^a;
\]

where we are using the standard complex null tetrad that satisfies \( \ell^a n_a = -m^a \bar{m}_a = 1 \) with all other scalar products giving zero. As usual the tetrad vector \( \ell^a \) is chosen to generate a congruence of null geodesics reaching \( \text{scri} \). A regular tetrad at future null infinity \( (\ell^a, \bar{m}^a, \hat{m}^a, \hat{n}^a) \) can be constructed from the following relations

\[
\ell^a = \Omega^2 \hat{\ell}^a, \quad (22)
\]

\[
m^a = \Omega \hat{m}^a, \quad (23)
\]

\[
n^a = \hat{n}^a; \quad (24)
\]

where \( \Omega \) can be taken as the inverse of the affine parameter \( r \) of the null geodesics reaching \( \text{scri} \).

Since the asymptotic symmetries are tangent to \( I^+ \), the tetrad components have the following behavior

\[
v_n = r v_n^0 + v_n^1 + O\left(\frac{1}{r}\right), \quad (25)
\]

\[
v_m = r v_m^0 + v_m^1 + O\left(\frac{1}{r}\right), \quad (26)
\]

\[
v_\ell = v_\ell^0 + \frac{v_\ell^1}{r} + O\left(\frac{1}{r^2}\right). \quad (27)
\]

The leading order behavior of the asymptotic symmetries is given by

\[
v_m^0 = \bar{\delta}a, \quad (28)
\]

\[
v_n^0 = \frac{1}{2} \left( \bar{\delta}v_m + \bar{\delta}v_m \right) = \frac{1}{2} \bar{\delta}\bar{\delta}(a + \bar{a}), \quad (29)
\]

\[
v_\ell^0 = \chi(\zeta, \bar{\zeta}) - \frac{1}{2} \bar{\delta}\bar{\delta}(a + \bar{a}) \quad (30)
\]

where we are using a Bondi tetrad [Mor86], \( \chi \) and \( a \) are functions on the sphere with spin weight 0 and satisfying: \( \chi = \bar{\chi}, \chi = 0, \dot{a} = 0 \) and \( \bar{\delta}^2 a = 0 \). Let us note that the function “\( a \)” only appears under the action of the operator \( \text{edth} \); then, without loss of generality we can assume that it satisfies \( \bar{\delta}\bar{\delta}a = -a \); which says that \( a \) can be expressed in terms of spherical harmonics \( Y_{lm} \) with \( l = 1 \).

The relation (18) at \( \text{scri} \) can be expressed in terms of the spinorial components of the regular dyad

\[
w^{AB} = w_0 \gamma^A \gamma^B - w_1 \left( \gamma^A \gamma^B + i^A \gamma^B \right) + w_2 \gamma^A \gamma^B, \quad (31)
\]

by

\[
w_2 = -\frac{1}{3} v_m, \quad (32)
\]
\[ w_1 + \bar{w}_1 = -\frac{1}{3}v_\ell, \quad (33) \]
\[ \dot{w}_1 + \dot{\bar{w}}_1 = -\frac{1}{2}(\bar{\sigma}w_2 + \bar{\bar{\sigma}}\bar{w}_2); \quad (34) \]

while condition (19) at scri becomes
\[ \bar{\sigma}w_2 = 0, \quad (35) \]
\[ \dot{w}_2 = 0, \quad (36) \]
\[ \bar{\sigma}w_0 = -2\sigma_0 w_1, \quad (37) \]
\[ \dot{w}_1 = -\frac{1}{2}\bar{\sigma}w_2, \quad (38) \]
\[ \frac{1}{2}\dot{w}_0 + \bar{\sigma}w_1 + \sigma_0 w_2 = 0. \quad (39) \]

It is interesting to see whether the solutions of (18) at \( I^+ \), namely (32)-(34) also satisfy (35)-(39).

Before studying this for the stationary and radiating case separately, let us note that for any 2-form \( w \) defined on a sphere \( S \), one can define the charge integral obtaining
\[ Q_S(w) = 4\int \left[ -\bar{w}_2(\Psi_1 - \Phi_{10}) + 2\bar{w}_1(\Psi_2 - \Phi_{11} - \Lambda) - \bar{w}_0(\Psi_3 - \Phi_{21}) \right] dS^2 + \text{c.c.} \quad (40) \]

where \((\bar{w}_0, \bar{w}_1, \bar{w}_2)\) are the components of \( w_{AB} \) with respect to the dyad adapted to \( S \), namely for which the vectors \( m \) and \( \bar{m} \) are tangent to \( S \) and \( \ell \) is outgoing. If \( w \) has a regular extension to scri we can take the limit of \( S \) to a section of \( I^+ \) obtaining
\[ Q_S(w) = 4\int \left[ -w_2\Psi_1^0 + 2w_1\Psi_2^0 - w_0\Psi_3^0 \right] dS^2 + \text{c.c.} \quad (41) \]

where the upper-script 0 denotes the leading order behavior of the respective Weyl components. The charge integral at \( I^+ \) turns out to be independent of the Ricci tensor since as we have shown in [Mor87] for any asymptotically flat spacetime, the components of the Ricci tensor go to zero faster than the accompanying Weyl components in the above terms. Furthermore, using that \( \Psi_3^0 = -\bar{\sigma}\bar{\sigma} \) and equation (37) one can rewrite this as
\[ Q_S(w) = 4\int \left[ -w_2\Psi_1^0 + 2w_1(\Psi_2^0 + \sigma_0\bar{\sigma}_0) \right] dS^2 + \text{c.c.} \quad (42) \]

It is important to note that the solutions of equations (32)-(34) can always be made to satisfy equations (35)-(38) and so one can obtain expression (42) for the charge integral, which is independent of the validity of the equation (39).
3.2 Stationary spacetime case

For the case of stationary spacetimes one can solve the set of equations (35)-(39) with solution

\[ w_2 = -\frac{1}{3} \ddot{\alpha} \bar{a}, \]
\[ w_1 = w_1^{00}(\zeta, \bar{\zeta}, \sigma_0) + \frac{1}{6} u \dddot{\alpha} \bar{a}, \]
\[ w_0 = w_0^{00}(\zeta, \bar{\zeta}, \sigma_0) + u \left( -2 \ddot{w}_1^{00} + \frac{2}{3} \sigma_0 \dddot{\alpha} \bar{a} \right) - \frac{1}{6} u^2 \dddot{\alpha} \bar{a} \]

where \( \dddot{\alpha} \bar{a} = 0 \), \( w_1^{00} \) and \( w_0^{00} \) are spin weight 0 and 1 functions respectively that solve the equations

\[ \ddot{w}_1^{00} = \frac{1}{3} \ddot{\sigma} \sigma_0 \dddot{\alpha} \bar{a} + \frac{1}{2} \sigma_0 \dddot{\sigma} \dddot{\alpha} \bar{a} = -\ddot{\sigma} \sigma_0 w_2 - \frac{3}{2} \sigma_0 \dddot{\sigma} \bar{a} \]
and

\[ \dddot{w}_0^{00} = -2 \sigma_0 w_2^{00}. \]

Let us note that if one uses the potential \( \delta \) of the shear satisfying

\[ \sigma_0 = \dddot{\sigma} \delta, \]

then, the component \( w_1 \) can be expressed by

\[ w_1 = b + \frac{1}{3} \dddot{\sigma} \dddot{\alpha} \bar{a} + \frac{1}{6} (u - \dot{\delta}) \dddot{\sigma} \bar{a}; \]

where the spin 0 quantity \( b \) satisfies \( \dot{b} = 0 \) and \( \dddot{\sigma} b = 0 \).

This procedure provides with a two-form with the functional dependence

\[ w_{AB}^0 = w_{AB}^0 (u, \zeta, \bar{\zeta}; \sigma_0(\zeta, \bar{\zeta}), a, b). \]

Let us observe that from one of the Bianchi identities at scri, it is obtained

\[ \ddot{\Psi}_2^0 = \dot{\Psi}_1^0 - 2 \Psi_3^0 \sigma_0, \]

but for a stationary spacetime \( \dot{\Psi}_1^0 = 0 \) and also \( \Psi_3^0 = 0 \) in a Bondi system; therefore, one deduces that \( \Psi_2^0 \) is a constant. Then, since the nice section condition requires \( \Psi = \Psi_2^0 + \sigma_0 \dot{\sigma}_0 + \dddot{\sigma}_0 = \Psi_2^0 + \dddot{\sigma}_0 \) to be a constant; one concludes that \( \dddot{\sigma}_0 = 0 \) and therefore that \( \sigma_0 = 0 \). In other words the nice section condition coincides in the stationary case with the good cut equation condition. However, as we have noticed, the nice section equation has solutions in the radiating case, while the good cut equation does not.

In the stationary case we have that the charge integrals of the Riemann tensor at scri are given by

\[ Q_S(w) = \int_S C = 4 \int_S \left( -w_2 \Psi_1^0 + 2 w_1 \Psi_2^0 \right) dS^2 + \text{c.c.}; \]

where \( w_2 \) is given by (43) and \( w_1 \) is

\[ w_1 = b + \frac{1}{6} u \dddot{\sigma} \bar{a}. \]
Let us note that \(a\) involves 6 real constants associated with the Lorentz rotations, and that since \(\Psi_2^0\) is a real quantity, \(b\) contributes to the charge integral \(Q_S(w)\) with other four real constants associated with translations.

We see then that this construction provides with the appropriate rest frames for stationary spacetimes; and that the vector \(v^a(a, b + \bar{b})\) calculated from the divergence of the 2-form \(w\), equation (18), is a generator of BMS symmetries.

The first term in the integrand of equation (52) includes the Weyl component \(\Psi_1\) which is known to describe the angular momentum in the Kerr geometry for example. In the second term we recognize the component \(\Psi_2^0\) which form the supermomentum psi for this particular stationary case. Therefore equation (52) has the same structure as the expression that, in special relativity, relates the angular momentum \(J^{ab}\), the intrinsic angular momentum \(S^{ab}\) and the linear momentum \(P^a\), namely

\[
J^{ab} = S^{ab} + R^a P^b - P^a R^b, \tag{54}
\]

where \(a, b\) are numeric spacetime indices. Given a rest reference frame in Minkowski spacetime one needs to use the spacelike translation freedom \((\bar{R}^b)\) appearing in expression (54) in order to single out the center of mass reference frame. In the center of mass frame one has \(J^{ab} = S^{ab}\); that is the total momentum coincide with the intrinsic angular momentum. Since the intrinsic angular momentum satisfies \(S^{ab} P_b = 0\), one can characterize the center of mass frame as that rest frame for which \(J^{0i} = 0\); where \(i = 1, 2, 3\) is a spacelike numeric index.

In the nice section construction we have seen that given an observation point \(p\) at \(\text{scri}\), there is a 3-family of rest frames, associated with the choice of spacelike translations. We also observe that the charge integral (52) has the appropriate angular momentum behavior. Therefore we also need a condition analogous to the Minkowskian case to single out the center of mass reference frames.

It can be seen that the condition to be imposed in the charge integral case is

\[
Q_S(a) = 0 \quad \text{for all} \quad a = \bar{a}; \tag{55}
\]

where it is understood that one takes \(b = 0\) in this equation. The quantity \(a\) is in principle complex; so this condition makes use precisely of a 3-degree of freedom, which is associated with spacelike translations.

This is the appropriate condition that leaves a one-dimensional family of nice sections \(S\) that can legitimately be called center of mass frames. We can see that the center of mass frames are generated by time translations in the nice section construction.

Using these frames \(S_{cm}\), the intrinsic angular momentum \(j\) is defined through

\[
j(w) = Q_{S_{cm}}(w); \tag{56}
\]

where to determine \(w\) one chooses \(a = -\bar{a}\) and \(b = 0\).

The same charge integral can also be used to calculate the Bondi momentum, which requires to take \(a = 0\) and \(b \neq 0\).

This prescription singles out the center of mass frame for stationary spacetimes and a Poincaré subgroup of BMS generators.
3.3 Radiating spacetime case

In the presence of gravitational radiation, the shear $\sigma_0$ depends on the time $u$ and the set of equations (32)-(39) in general have no solution. But it is interesting to note that one can define a 2-form $w$ at scri, which in some sense is the closest to the stationary prescription, as we describe next.

Let $(u, \zeta, \bar{\zeta})$ be some Bondi coordinate system at future null infinity. A generator of scri can be identified with the line determined by $\zeta = \text{constant}$ ($\bar{\zeta} = \text{constant}$).

Let us consider a point along a particular generator of scri, denoted by $p(\tau)$, with $\tau$ a monotonically increasing time parameter.

Let us recall [DM00] that the set of nice sections form a four parameter $(T, \vec{R})$ family that we now label $S_{(T, \vec{R})}$; where $(T, \vec{R})$ can be identified with a translation of the BMS group.

Then, for a given fixed $\tau$, one has a 3-parameter $(\vec{R})$ family of nice sections $S_{(T, \vec{R})}$ that contain the point $p(\tau)$; where all this family share the same $T(\tau)$. Given one of this nice sections $S_{(T, \vec{R})}$, we can always identify it with the condition $u = \gamma_S$, where $\gamma_S(\zeta, \bar{\zeta})$ is the supertranslation that defines the corresponding section. On $S_{(T, \vec{R})}$ we define the 2-form $w_{S_{(T, \vec{R})}}(u = \gamma_S, \zeta, \bar{\zeta}; a, b)$ as the solution on $S_{(T, \vec{R})}$ of equations (18) and (19) for the radiation data $\sigma_{S_{(T, \vec{R})}}(u, \zeta, \bar{\zeta}) = \sigma_0(\gamma_S, \zeta, \bar{\zeta})$. Since by definition $\sigma_{S_{(T, \vec{R})}}$ does not depend on the time coordinate $u$, the 2-form $w_{S_{(T, \vec{R})}}$ is the restriction on $S_{(T, \vec{R})}$ of the solution of the stationary problem when the radiation data is $\sigma_{S_{(T, \vec{R})}}$. More concretely, one can express the 2-form $w_{S_{(T, \vec{R})}}$ in terms of its spinor components by the prescription stated in equations (43)-(47), if we chose the Bondi system such that $u = \text{constant}$.

In order to single out the center of mass section $S_{cm}$ from the 3-parameter $(\vec{R})$ family of nice sections that contain the point $p(\tau)$, we demand

$$Q_{S_{(T, \vec{R})}}(a) = 0 \quad \text{for all} \quad a = \bar{a}. \quad (57)$$

This is the analog in the Minkowskian case of the condition $J^{0i} = 0$; that we mentioned earlier. Let us note that the quantity $a$ involves six real parameters; so that the condition $a = \bar{a}$ involves three degrees of freedom which are taking care of by the corresponding freedom involved in $(\vec{R})$. In this way, for each $\tau$ we select the unique $S_{cm}(\tau)$ which contains the observation point $p(\tau)$ at scri.

Given the radiation data $\sigma_0(u, \zeta, \bar{\zeta})$, this construction provides with a smooth [DM00] one-parameter family of non-crossing center of mass sections $S_{cm}(\tau)$ along with a smooth 2-form

$$w(\tau, \zeta, \bar{\zeta}; a, b) = w_{S_{(T, \vec{R})}}(u = \gamma_S, \zeta, \bar{\zeta}; a, b) \quad (58)$$
on scri.

The 2-form $w_{AB}$ as defined by (58) will not satisfy all equations (32)-(39); more concretely, most of the equations involving time derivatives will not be satisfied in general, but the error will be $O(\dot{\sigma}_0)$. That is, if $\dot{\sigma}_0 = 0$ on a particular center of mass section $S_{cm}$, then all equations (18) and (19), or equivalently (32)-(39), are satisfied on $S_{cm}$.

Using these center of mass frames $S_{cm}$, the intrinsic angular momentum $j$ is defined through

$$j(w) = Q_{S_{cm}}(w); \quad (59)$$

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where as before, in order to pick up the intrinsic angular momentum, one takes \( a = -\bar{a} \) and \( b = 0 \).

With this definition the charge integral of the Riemann on \( S_{cm} \), can be expressed by

\[
Q_{S_{cm}}(w) = 4 \int_{S_{cm}} C = 4 \int_{S_{cm}} \left( -w_2 \left( \Psi_1^0 + 2\sigma_0 \bar{\sigma} \sigma_0 + \bar{\sigma} (\sigma_0 \bar{\sigma}_0) \right) + 2 \Psi_2^0 \right) dS^2 + \text{c.c.} \quad (60)
\]

where we have used that

\[
\int_{S_{cm}} (w_1 \bar{\sigma}^2 \bar{\sigma}_0) \ dS^2 = \int_{S_{cm}} \left( \sigma_0 \bar{\sigma} \sigma_0 + \frac{1}{2} \bar{\sigma} (\sigma_0 \bar{\sigma}_0) \right) w_2 \ dS^2. \quad (61)
\]

It should be emphasized that in this way, given an asymptotically flat spacetime, one constructs a unique regular family of non intersecting sections [DM00] at future null infinity, which are the center of mass frames, and that can be used to describe the detailed multipolar asymptotic structure of the spacetime.

In order to see whether our definition of angular momentum has the appropriate behavior in the presence of gravitational radiation, let us imagine a system in which one can distinguish three stages; starting with a stationary regime, passing through a radiating stage and ending in a stationary regime as depicted in figure 1. By construction it is clear that our definition gives the correct intrinsic angular momentum in the first and third stage; even though the center of mass frames of the first and third stage are related in general by a supertranslation.

This is the only definition of angular momentum that satisfies this expected property.

Figure 1: Behavior of the notion of intrinsic angular momentum in a spacetime with three stages; beginning with a stationary stage, continuing with a radiating stage and ending with a stationary stage.
4 Asymptotic structure of asymptotically flat spacetimes

4.1 Basic variables

Having obtained the center of mass frames, we now take the opportunity to study the asymptotic structure in terms of a null tetrad which is adapted to the center of mass frames.

We start by defining the basic variables that we will use.

One can express the asymptotic geometry in terms of a complex null tetrad \((\ell^a, m^a, \overline{m}^a, n^a)\) with the properties:

\[
g_{ab} \ell^a n^b = -g_{ab} m^a \overline{m}^b = 1
\]

and all other possible scalar products being zero, the metric can be expressed by

\[
g_{ab} = \ell_a n_b + n_a \ell_b - m_a \overline{m}_b - \overline{m}_a m_b.
\]

Newman and Penrose\[NP62\] showed that such a null tetrad is easily related to a dyad of spinors \((o^A, \iota^A)\), and that they are very useful for the discussion of the asymptotic structure.

Using the null polar coordinate system \((x^0, x^1, x^2, x^3) = (u, r, (\zeta + \bar{\zeta}), \frac{1}{i}(\zeta - \bar{\zeta}))\) one can express the null tetrad as:

\[
\ell_a = (du)_a
\]

\[
\ell^a = \left(\frac{\partial}{\partial r}\right)^a
\]

\[
m^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a
\]

\[
\overline{m}^a = \xi^i \left(\frac{\partial}{\partial x^i}\right)^a
\]

\[
n^a = \left(\frac{\partial}{\partial u}\right)^a + U \left(\frac{\partial}{\partial r}\right)^a + X^i \left(\frac{\partial}{\partial x^i}\right)^a
\]

with \(i = 2, 3\) and where the components \(\xi^i, U\) and \(X^i\) are:

\[
\xi^2 = \frac{\xi_0^2}{r} + O\left(\frac{1}{r^2}\right), \quad \xi^3 = \frac{\xi_0^3}{r} + O\left(\frac{1}{r^2}\right),
\]

with

\[
\xi_0^2 = \sqrt{2} P_0 V, \quad \xi_0^3 = -i \xi_0^2,
\]

where \(V = V(u, \zeta, \bar{\zeta})\) and the square of \(P_0 = \frac{(1 + \zeta \bar{\zeta})}{2}\) is the conformal factor of the unit sphere;

\[
U = ru_{00} + u_0 + \frac{U_1}{r} + O\left(\frac{1}{r^2}\right),
\]

where

\[
U_{00} = \frac{\dot{V}}{V}, \quad u_0 = -\frac{1}{2} K_V, \quad U_1 = -\frac{\Psi_2^0 + \overline{\Psi}_2^0}{2},
\]
where $K_V$ is the curvature of the 2-metric

$$dS^2 = \frac{1}{V^2 P_0^2} d\zeta d\bar{\zeta};$$  \hfill (73)

where the regular conformal metric restricted to scri is precisely $\tilde{g} |_{\mathcal{I}^+} = -dS^2$. In terms of the edth operator $\bar{\partial}_V$ of the sphere $dS_2^2 = \frac{1}{V^2} P_0^2 d\zeta d\bar{\zeta}$; \hfill (73)

the curvature $K_V$ is given by

$$K_V = \frac{2}{V} \bar{\partial}_V \partial_V V - \frac{2}{V^2} \partial_V V \bar{\partial}_V V + V^2. \hfill (74)$$

Finally, the other components of the vector $n^a$ have the asymptotic form

$$X^2 = O \left( \frac{1}{r^2} \right), \quad X^3 = O \left( \frac{1}{r^2} \right). \hfill (75)$$

### 4.2 Asymptotic behavior of the spin coefficients

The optical spin coefficients of the outgoing congruence of null geodesics are

$$\rho = -\frac{1}{r} - \frac{\sigma_0 \bar{\sigma}_0}{r^3} + O \left( \frac{1}{r^4} \right) \hfill (76)$$

and

$$\sigma = \frac{\sigma_0}{r^2} + \frac{\sigma_0^2 \bar{\sigma}_0 - \frac{1}{2} \Psi_0^0}{r^4} + O \left( \frac{1}{r^5} \right); \hfill (77)$$

while their primed version, associated to the vector field $n^a$, are

$$\rho' = -\frac{U_0}{r} + \frac{\Psi_0^0 - \sigma_0 \bar{\sigma}_0}{r^2} + \bar{\partial}_V \tau_0 + O \left( \frac{1}{r^3} \right) \hfill (78)$$

and

$$\sigma' = \frac{\sigma'_0}{r} + O \left( \frac{1}{r^2} \right); \hfill (79)$$

where

$$\tau_0 = \bar{\partial}_V \sigma_0 \hfill (80)$$

and

$$\sigma'_0 = \frac{\dot{V}}{V} \bar{\sigma}_0 - \dot{\bar{\sigma}}_0. \hfill (81)$$

Let us also note that

$$\epsilon' = \epsilon'_0 + \left( \frac{1}{r} \right); \hfill (82)$$

with

$$\epsilon'_0 = \frac{\dot{V}}{2V}. \hfill (83)$$
4.3 Asymptotic behavior of the Weyl components

The leading order behavior of the Weyl components $\Psi_2$, $\Psi_3$ and $\Psi_4$ satisfy the following relations

$$\Psi_2^0 + \sigma_0 \dot{\sigma}_0 + \bar{\sigma}_0^2 \sigma_0 = \Psi_2^0 + \sigma_0 \dot{\sigma}_0 + \bar{\sigma}_0^2 \sigma_0,$$  \hfill (84)

$$\Psi_3^0 = \bar{\sigma}_0 \sigma'_0 - \bar{\sigma}_0 \rho'_0$$

$$= - \bar{\sigma}_0 \dot{\sigma}_0 + \frac{\bar{V}}{V} \bar{\sigma}_0 + \left( \frac{\bar{\sigma}_0 \dot{V}}{V} - \frac{\bar{V} \bar{\sigma}_0 V}{V^2} \right) \dot{\sigma}_0 - \frac{1}{2} \bar{\sigma}_0 K_V$$  \hfill (85)

and

$$\Psi_4^0 = \dot{\sigma}_0 - 2U_0 \sigma_0' - \bar{\sigma}_0^2 U_0$$

$$= - \dot{\sigma}_0 + 3 \frac{V}{V} \bar{\sigma}_0 + \left( \frac{V}{V} - 3 \frac{V^2}{V^2} \right) \sigma_0$$

$$- \frac{\bar{\sigma}_0 \dot{V}}{V} + 2 \frac{\bar{\sigma}_0 \dot{V} \bar{\sigma}_0 V}{V^2} - 2 \frac{\dot{V} (\bar{\sigma}_0 V)^2}{V^3} + \frac{\dot{V} \bar{\sigma}_0^2 V}{V^2}.$$  \hfill (86)

4.4 The asymptotic gauge freedom

4.4.1 Coordinate and tetrad transformations

Let us consider the main gauge freedom admitted in our calculation which is of the form

$$\bar{u} = \alpha (u, \zeta, \bar{\zeta}) + \bar{u}_1 (u, \zeta, \bar{\zeta}) + O \left( \frac{1}{r^2} \right),$$  \hfill (87)

$$\bar{r} = \frac{r}{w (u, \zeta, \bar{\zeta})} + O \left( r^0 \right),$$  \hfill (88)

$$\bar{\zeta} = \zeta + O \left( \frac{1}{r} \right).$$  \hfill (89)

with $\dot{\alpha} > 0$. The possible further transformation of the coordinates of the sphere $(\zeta, \bar{\zeta})$ into itself, will complicate the discussion unnecessarily.

The condition $g^{\bar{u} \bar{r}} = 1$ imposes the relation

$$w = \dot{\alpha}.$$  \hfill (90)

This asymptotic coordinate transformation is associated to a corresponding null tetrad transformation; which in the leading orders is given by

$$\bar{\ell} = d\bar{u} = \dot{\alpha} du + \alpha \zeta \, d\zeta + \alpha \bar{\zeta} \, d\bar{\zeta} + O \left( \frac{1}{r} \right)$$

$$= \dot{\alpha} \bar{\ell} - \frac{\bar{\sigma}_0 \alpha}{r} \bar{m} - \frac{\bar{\sigma}_0 \alpha}{r} \bar{m} + O \left( \frac{1}{r} \right),$$  \hfill (91)

$$\bar{n} = \frac{\partial}{\partial \bar{u}} + O \left( \frac{1}{r} \right) = \frac{1}{\dot{\alpha}} \frac{\partial}{\partial u} + O \left( \frac{1}{r} \right)$$

$$= \frac{1}{\dot{\alpha}} n + O \left( \frac{1}{r} \right),$$  \hfill (92)
\[ \tilde{m} = \frac{\sqrt{2} \tilde{P}}{\tilde{r}} \frac{\partial}{\partial \tilde{\zeta}} + O \left( \frac{1}{r^2} \right) = \frac{\sqrt{2} P_0 \tilde{V} w}{r} \left( -\frac{\alpha}{\tilde{\alpha}} \frac{\partial}{\partial u} + \frac{\partial}{\partial \tilde{\zeta}} \right) + O \left( \frac{1}{r^2} \right) \] (93)

\[ = -\frac{\sqrt{2} P_0 \tilde{V}}{r} \frac{\alpha}{\tilde{\alpha}} n + \tilde{V} \frac{w}{V} m + O \left( \frac{1}{r^2} \right); \]

since the metric expressed in terms of the new null tetrad must coincide with the metric expressed in terms of the original null tetrad, it is deduced that

\[ \tilde{V} = V = \frac{V}{\tilde{\alpha}}; \] (94)

therefore

\[ \tilde{m} = m - \frac{\tilde{\delta} V}{r \tilde{\alpha}} n + O \left( \frac{1}{r^2} \right). \] (95)

The null tetrad transformation equations can be used to write the leading order transformation relations for the spinor dyad associated to the null tetrad\[GHP73;\] namely

\[ \tilde{\sigma}^A = \sqrt{\tilde{\alpha}} \left( \sigma^A - \frac{\tilde{\delta} V}{r \tilde{\alpha}} \iota^A \right) \] (96)

and

\[ \tilde{\iota}^A = \frac{1}{\sqrt{\alpha}} \iota^A. \] (97)

Taking into account higher order transformations would include an equation of the form

\[ \tilde{\iota}^A = \frac{1}{\sqrt{\alpha}} \left( \iota^A + h o^A \right); \] (98)

where in principle \( h \) could be of order \( O (r^0) \).

Analogous equations to (22)-(24) relate the regular dyad at future null infinity with the spacetime ones, namely

\[ \hat{\sigma}^A = \Omega^{-1} \sigma^A, \] (99)

\[ \hat{\iota}^A = \iota^A. \] (100)

Then, the regular dyad at future null infinity is given by

\[ \hat{\sigma}^A = \hat{\Omega}^{-1} \hat{\sigma}^A = \frac{r}{w} \sqrt{\hat{\alpha}} \left( \sigma^A - \frac{\hat{\delta} V}{\hat{r} \hat{\alpha}} \iota^A \right) = \frac{1}{\sqrt{\hat{\alpha}}} \left( \hat{\sigma}^A - \frac{\hat{\delta} V}{\hat{\alpha}} \hat{\iota}^A \right), \] (101)

\[ \hat{\iota}^A = \hat{\iota}^A = \frac{1}{\sqrt{\alpha}} \iota^A; \] (102)

where we are using \( \Omega = \frac{1}{r} \).
4.4.2 Transformation of $\Psi^0_2$ and $\Psi^0_1$

We can now easily calculate the component $\Psi^0_2$ of the Weyl tensor, in leading orders, with respect to the new null tetrad, obtaining

$$\tilde{\Psi}^0_2 = \tilde{\Omega}^{-1} \Psi^0_{ABCD} \tilde{z}^A \tilde{z}^B \tilde{z}^C \tilde{z}^D = \frac{1}{\tilde{\alpha}^3} \left( \Psi^0_2 - 2 \frac{\tilde{\alpha}}{\alpha} \Psi^0_3 + \left( \frac{\tilde{\alpha}}{\alpha} \right)^2 \Psi^0_4 \right). \quad (103)$$

Similarly for $\Psi^0_1$ one has

$$\tilde{\Psi}^0_1 = \frac{1}{\tilde{\alpha}^3} \left( \Psi^0_1 - 3 \frac{\tilde{\alpha}}{\alpha} \Psi^0_2 + 3 \left( \frac{\tilde{\alpha}}{\alpha} \right)^2 \Psi^0_3 - \left( \frac{\tilde{\alpha}}{\alpha} \right)^3 \Psi^0_4 \right). \quad (104)$$

4.4.3 Transformation of some spin coefficients

Recalling that the shear is defined by

$$\sigma = o^A l^{A'} o^B \nabla_{AA'} o_B = \frac{\sigma_0}{r^2} + O \left( \frac{1}{r^3} \right); \quad (105)$$

then, using

$$\Lambda = -\frac{\tilde{\alpha}}{r} \frac{\tilde{\alpha}}{\alpha} \quad (106)$$

one obtains, for the leading orders behavior of the new shear

$$\tilde{\sigma} = \sqrt{\tilde{\alpha}} \left( o^A + \Lambda l^A \right) \left( l^{A'} + h o^{A'} \right) \left( o^B + \Lambda l^B \right) \quad (107)$$

$$\nabla_{AA'} \sqrt{\tilde{\alpha}} \left( o_B + \Lambda l_B \right) + O \left( \frac{1}{r^3} \right)$$

$$= \tilde{\alpha} \left( \sigma - \tilde{\alpha} m \Lambda - \Lambda \nabla' \Lambda + \Lambda \tau - \Lambda^2 \rho' - \Lambda^3 \kappa' \right) + O \left( \frac{1}{r^3} \right)$$

where we have used $\nabla'$ to denote the thorn primed operator, $\kappa = 0$, and where $\Lambda$ is recognized as a quantity of $\{p, q\}$ type $\{2,0\}$. This equation implies that

$$\tilde{\sigma}_0 = \frac{1}{\tilde{\alpha}} \left( \sigma_0 + \frac{1}{\tilde{\alpha}} \tilde{\alpha} \left( \tilde{\alpha} \right)^2 - \frac{2}{\tilde{\alpha}^2} \tilde{\alpha} \tilde{\alpha} \tilde{\alpha} - \frac{2}{\tilde{\alpha}^2} \left( \tilde{\alpha} \right)^2 \tilde{V} \right) \quad (108)$$

In order to calculate the transformation of other spin coefficients, as for example $\rho'$, $\sigma'$ and $\tau$, one would need to determine the quantity $h$. However since the leading order behavior of these spin coefficients can be put in terms of $V$ and $\sigma_0$, one can calculate its transformations from those of the last two quantities.

4.4.4 Transformation of the edth operator

Let us recall that the edth operator can be defined in terms of the conformal factor $P = V P_0$ of the sphere, by the relations

$$\tilde{\alpha} (P^1)^{-s} \frac{\partial}{\partial \zeta} (P^s f) \quad (109)$$
\[ \partial_{V} f = \sqrt{2} P^{1+s} \frac{\partial}{\partial \zeta} (P^{-s} f); \]  

(110)

where \( s \) is the spin weight of the quantity \( f \).

Then one can prove that the transformed edth operator acts as

\[ \tilde{\partial}_{\tilde{V}} f = \frac{\tilde{V}}{V} \left( \partial_{V} f - \frac{\tilde{\partial}_{\tilde{V}} \alpha}{\alpha} f \right) + s f \left[ \partial_{V} \left( \frac{\tilde{V}}{V} \right) - \frac{\tilde{\partial}_{\tilde{V}} \alpha}{V \alpha} \right]. \]  

(111)

### 4.4.5 Transformations to a Bondi system

If the target coordinate system is a Bondi system, then one has

\[ \tilde{V} = 1 \]  

(112)

or equivalently

\[ \dot{\alpha} = V. \]  

(113)

With these relations the previous transformations are now

\[ \tilde{\Psi}_{2}^{0} = \frac{1}{V^{3}} \left( \Psi_{2}^{0} - \frac{2}{V} \partial_{V} \alpha \Psi_{3}^{0} + \frac{(\partial_{V} \alpha)^{2}}{V^{2}} \Psi_{4}^{0} \right), \]  

(114)

\[ \tilde{\Psi}_{1}^{0} = \frac{1}{V^{3}} \left( \Psi_{1}^{0} - 3 \frac{\partial_{V} \alpha}{V} \Psi_{2}^{0} + 3 \frac{(\partial_{V} \alpha)^{2}}{V^{2}} \Psi_{3}^{0} - \frac{(\partial_{V} \alpha)^{3}}{V^{3}} \Psi_{4}^{0} \right), \]  

(115)

\[ \tilde{\sigma}_{0} = \frac{1}{V} \left( \sigma_{0} + \frac{1}{V} \partial_{V}^{2} \alpha - \frac{1}{V^{2}} \dot{V} (\partial_{V} \alpha)^{2} - \frac{2}{V^{2}} \partial_{V} \alpha \partial_{V} V \right), \]  

(116)

\[ \tilde{\partial}_{\tilde{V}} f = \frac{1}{V} \partial_{V} f - \frac{s}{V^{2}} \partial_{V} V f - \frac{1}{V^{2}} \partial_{V} \alpha \dot{f}. \]  

(117)

### 4.5 Rest frame and center of mass conditions

In order to study the condition for a section to be a rest frame we need to consider the transformation properties of the supermomentum integrand \( \Psi \) defined in a Bondi system by

\[ \Psi_{B} = \Psi_{B_{2}}^{0} + \sigma_{B_{0}} \dot{\sigma}_{B_{0}} + \tilde{\sigma}_{B_{0}} \tilde{\sigma}_{B_{0}}; \]  

(118)

where the appearance of the subindex \( B \) is used to emphasize that the quantities refer to a Bondi system.

If \( S \) is a nice section then, on \( S \) one has

\[ \Psi_{B} = -K^{3} M, \]  

(119)

where \( K \) is given by equation (13). The idea is to consider the original tetrad and coordinate system at scri \((u, \zeta, \bar{\zeta})\) as coinciding with a one parameter family of nice sections, defined by the conditions \( u = \text{constant} \).
The expression of the supermomentum $\Psi_B$ in terms of the intrinsic system is rather complicated, namely

$$
\Psi_B = \Psi_0^0 V^{-3} - 2 (\bar{\partial}_V V)^2 \sigma_0 V^{-5} + \bar{\partial}_V^2 V \sigma_0 V^{-4} + \bar{\partial}_V^2 \bar{\partial}_V \sigma V^{-4} \\
- 4 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V V^{-5} - 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-5} - 2 \bar{\partial}_V V \bar{\partial}_V \sigma_0 V^{-5} \\
+ 8 \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-5} - 2 (\bar{\partial}_V V)^2 \sigma_0 V^{-5} - 2 \bar{\partial}_V V \bar{\partial}_V V V^{-5} \\
+ 8 \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-5} - 8 \bar{\partial}_V \bar{\partial}_V V (\bar{\partial}_V V)^2 V^{-7} + 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-5} \\
+ \bar{\partial}_V^2 V \sigma_0 V^{-4} + \bar{\partial}_V^2 \sigma_0 V^{-3} - \bar{\partial}_V^2 \bar{\partial}_V (\bar{\partial}_V V)^2 V^{-6} - 4 \bar{\partial}_V \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-6} \\
+ 6 \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-5} + 4 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V V V^{-7} \\
+ 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma \bar{\partial}_V V^{-7} - \bar{\partial}_V \sigma_0 \bar{\partial}_V V^{-4} - 2 \bar{\partial}_V \bar{\partial}_V \bar{\partial}_V \sigma_0 V^{-6} \\
- 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V V^{-6} - 12 \bar{\partial}_V \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-4} + 8 \bar{\partial}_V \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} \\
+ 8 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-5} - 16 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-8} \\
- \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} - 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-8} + 2 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} \\
- 8 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} + 16 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-9} \\
+ \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} + 12 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-9} \\
- 10 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-10} - \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-6} + 4 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-7} \\
- 6 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-8} - 4 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-8} \\
+ 8 \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-9} - \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-8} \\
+ \bar{\partial}_V \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 \bar{\partial}_V \sigma_0 V^{-6}.
$$

However, it is important to note that the function $\alpha$ which generates the transformation (87)-(89) is defined up to an integration constant coming from the condition (113). Therefore at each $u = u_0$ we can choose the integration constant such that $\alpha(u = u_0) = 0$. In other words, given $u_0$, one can always define

$$
\alpha_{u_0}(u, \zeta, \bar{\zeta}) = \int_{u_0}^u V(u', \zeta, \bar{\zeta}) \, du'.
$$

Using this choice, equation (120) simplifies considerably; and the nice section equation becomes

$$
\Psi_B(u = u_0, \zeta, \bar{\zeta}) = \frac{1}{V^3} \left( \Psi_2^0 + \bar{\sigma}_0 \sigma_0 - \frac{\dot{V}}{V} \sigma_0 \bar{\sigma}_0 + \bar{\partial}_V^2 \bar{\sigma}_0 \right) + \frac{1}{V^3} \left( \bar{\partial}_V^3 V \sigma_0 + \bar{\partial}_V^2 V \bar{\sigma}_0 \right) \\
- \frac{2}{V^3} \left( (\bar{\partial}_V V)^2 \sigma_0 + (\bar{\partial}_V V)^2 \bar{\sigma}_0 \right)
$$

(122)

where we are using as $\alpha$ the expression coming from (121).

This constitutes the condition that must be satisfied at each section of a system to represent a one parameter family of nice sections.
The center of mass condition is equation (57) which in the present notation means

\[ Q_S(w) = \int_S C = 4 \int_S \left( -w_{B2} \left( \Psi^0_{B1} + 2\sigma_{B0}\bar{\sigma}_{B0} + \bar{\sigma}_B (\sigma_{B0}\bar{\sigma}_{B0}) \right) 
+ 2w_{B1}\Psi_B \right) dS^2_B + c.c. = 0 \]  

(123)

for all \( a = \bar{a} \) and \( b = 0 \); and where we are already using the nice section condition.

All these quantities are evaluated with respect to the Bondi system, so that in terms of the intrinsic system one must use the transformation laws of equations (114)-(117).

## 5 Final comments

In this work we have shown how to circumvent the difficulties inherent in the structure of the BMS group by the use of the nice section construction; in particular in the task of defining the intrinsic angular momentum in radiating spacetimes. We have provided, as far as we know, with the only definition of angular momentum which satisfies the expected physical properties, as those discussed at the end of section 3.

In particular, let us note that detectors of gravitational waves can be considered as observers at future null infinity. Our construction provides for each point \( p \) at scri with a prescription that singles out the unique center of mass section containing \( p \); and also with the appropriate rest frame to calculate intrinsic quantities like angular momentum and multipole moments; which are important for the description of the gravitational waves that one wishes to detect.

It is worth to mention the relation of our work with previous presentations of the notion of angular momentum; so next we briefly comment on some (arbitrary, incomplete) selection of previous references.

Early works had the ideology suggested by the conservation laws due to Komar [Kom59]; where associated with a symmetry \( v^a \) one can define the 2-form \( A_{ab} = \nabla_{[a}v_{b]} \), and define the ‘charge’ \( K(v) \) [Win80] on a 2-surface \( S \) by

\[ K_S(v) = -\frac{1}{8\pi} \int_S A^*; \]  

(124)

where \( A^* \) is the dual of \( A \). This has the same form as the expression that gives the electric charge \( q \) of the electromagnetic field \( F_{ab} \), namely

\[ q(S) = -\frac{1}{8\pi} \int_S F^*. \]

If \( \Sigma \) is a hypersurface that has as boundary \( S \), then from Stokes’ theorem one has

\[ K_S(v) = -\frac{1}{8\pi} \int_\Sigma dA^*; \]

the fact that this holds for any such hypersurface \( \Sigma \), is sometimes referred to as a conservation law. Noting that \( dA^*_{abc} = 2\epsilon_{abcd}\nabla_c A^d \), one can see that if \( v^a \) is a Killing symmetry then \( dA^*_a = 2\epsilon_{abcd}R^d_c v^e \) and therefore, assuming Einstein equations, the Komar charge can be related to the matter content of the spacetime. If the sources are bounded by a
world tube, then for any surface $S$ surrounding the world tube, the Komar charge will give the same value. So, in the presence of a Killing symmetry, the Komar charge provides a simple useful notion of a conserved quantity; in particular for axis symmetric spacetimes, it provides with the notion of angular momentum (one component).

Although in a general spacetime one will not have a Killing symmetry, the Winicour approach uses the Komar charge idea, at future null infinity, associated to the asymptotic symmetries of an isolated system. When expressing the integral in terms of the asymptotic fields, there appears a term with the factor $[2\Psi^0_1 - 2\sigma_0 \delta \bar{\sigma}_0 - \delta(\sigma_0 \bar{\sigma}_0)]$ plus another term containing the supermomentum part. One must first be aware that the curvature spin coefficients definitions has been varying, and in some cases it is even difficult to trace the conventions; in particular the relative negative sign seems to be the consequence of difference with our conventions [GHP73]; however, one can note the non-trivial appearance of a factor of 2 in front of $\Psi^0_1$. The Winicour charges were further developed in reference [GW81] where a flux was calculated. The relation with the symplectic fluxes was worked out in reference [AS81].

Using a gauge theory approach, Bramson [Bra75], presents a notion of angular momentum in which its integral has the factor $[\Psi^0_1 - 2\sigma_0 \delta \bar{\sigma}_0 - \delta(\sigma_0 \bar{\sigma}_0)]$ and where no term associated with supermomentum appears. In fact in Bramson’s approach no use of the generators of supertranslations is done. As we have mentioned it is essential to make use of this term in order to be able to pick up the notion of intrinsic angular momentum.

Inspired by the Komar approach, Prior presented [Pri77] a definition of angular momentum that resembles the Winicour expressions but lacking the supermomentum term.

Using a completely different approach, Streubel [Str78] found the same expression for the angular momentum as Bramson’s, but now with a supermomentum term.

Penrose presented [Pen82] a notion of quasi-local mass and angular momentum associated with any sphere in the interior of the spacetime; making use of charge integrals of the Riemann tensor. His construction, when extended to a sphere at future null infinity, gives an integral expression where only curvature terms appear. However, as Dray and Streubel have shown [DS84], one can add the shear terms with some arbitrary factors, due to some identities. They also presented a charge associated to generators of an unfortunately complex BMS Lie algebra. Their choice is such that the angular momentum factor agrees with Winicour’s one.

Recent works [KL97] [WZ00] have concentrated on the relation between angular momentum and Lagrangian formulations, although they use background geometries in their constructions.

In reference [Riz98] Rizzi presented a definition that tackles the problem of supertranslation ambiguities. However, there is no mention in this work of the issue of supermomentum, and therefore it is unable to set the center of mass frame; which is needed to define the intrinsic angular momentum.

Independently from the nature of the approach all these definitions have the characteristics that they provide with a notion of angular momentum associated to a given Lorentz rotation BMS generator $\nu^a$ and a given section $S$ of future null infinity. The choice of a Lorentz rotation can be associated with a choice of a particular Bondi frame, because to each Bondi frame there is a Lorentz subgroup of rotation that leave the origin ($u = 0$) of the Bondi frame unchanged. Since among Bondi frames there is the supertranslation freedom, there is an ambiguity associated with this choice. But also any section $S$ can be thought of as a section $u = \gamma(\zeta, \bar{\zeta})$ of a given Bondi frame; therefore there is also
a supertranslation choice associated with the election of section $S$. This becomes very important when one wants to determine the intrinsic angular momentum of the sources. All the definitions mentioned previously in this section suffer from this supertranslations ambiguities.

In reference [Mor86] we presented a definition that fixed this problem; that required a preferred choice of Lorentz generators based on the properties of the spacetime in the limit for $u \to -\infty$; that is in the limit to spacelike infinity. Therefore given any $S$ we provided with a charge integral of the Riemann tensor that had the correct angular momentum behavior, with an immediate flux law and without supertranslation ambiguities. The remaining difficulty with our previous work was that it would not necessarily pick up the natural expected notion of intrinsic angular momentum of the sources for the cases of radiating spacetimes.

From the naive comparison of the mathematical expression for angular momentum factors on a given section $S$, one could distinguish mainly the following cases:

| Angular momentum factor | Supermomentum factor | appearing in references |
|-------------------------|----------------------|------------------------|
| Winicour type           | Winicour supermomentum | TW66 [Win68] [Win80] |
| Winicour type           | absent               | Pri77                   |
| Winicour type           | Geroch supermomentum | DS84 [WZ00]            |
| Winicour type           | psi supermomentum    | KL97                    |
| Bramson type            | absent               | Bra75 Riz98            |
| Bramson type            | psi supermomentum    | Str78 Mor86 [present]  |

Actually Rizzi’s integrand does not coincide with Bramson’s exactly, but it is closer to this than to Winicour type integrands. The fact that two or more references appear in the same row, does not mean that they are equivalent. This table only shows the similarities of the factors involving the curvature tensor; but the definitions are completed with the factors involving the information of the BMS generator that one is using. In particular, references [Str78] and [Mor86] both are inspired from Penrose work [Pen82]; but applying the definition of reference [Str78] to two sections $S_1$ and $S_2$, due to the supertranslation ambiguity problem, it is not clear how to relate the two quantities. Instead reference [Mor86] constitutes an extension of Penrose expression to the whole of scri, making sense for radiating spacetimes, solving the ambiguities of supertranslations by a physical condition and providing with an immediate flux law, which is zero in the absence of gravitational radiation. It should be stressed that the Penrose expressions can be made to agree to Winicour or Bramson type angular momentum factors, equivalently to agree with Geroch or psi supermomentum factors. It seems that in reference [Str78] the choice was intended to get the Bramson angular momentum type, although later the author changed his mind for the Winicour type [DS84]; instead in reference [Mor86] the stress was on the supermomentum part, due to the properties of the psi supermomentum [Mor86] [Mor88] [MD98] [DM00], and as a consequence the angular momentum factor was obtained to be of Bramson type.

Our present construction; which can also be considered a development of the Penrose approach, and a mayor improvement over reference [Mor86], provides for the first time with a global well behaved notion of intrinsic angular momentum for radiating sources, and also gives the notion of instantaneous center of mass frame; which is essential for the discussion of higher multipole moments.

For example, if one is studying a system whose gravitational radiation is appropriately described by the quadrupole radiation formula, at an instant of time; then in a long lapse of time, the corresponding center of mass frame will be supertranslated with respect to
the original one. Therefore, in order to know what is the quadrupole moment at a later
time, one needs to settle what is the center of mass frame first; which we have done here.

We next review some expected properties that one may require a charge integral of
BMS symmetries $\nu^a$ to satisfy. The first are adapted from reference [AWS2] and we
have added another one.

1. $Q_S(\nu)$ should be linear in the BMS symmetry $\nu^a$.

2. The expression $Q_S(\nu)$ should only involve local fields at $S$ on scri.

3. For the case when $\nu^a$ is a translation, $Q_S(\nu)$ should agree with the corresponding
   component of the Bondi momentum evaluated at $S$.

4. If $\nu^a$ is the natural extension to scri of a Killing symmetry of the spacetime then
   $Q_S(\nu)$ should be independent of $S$, if the vacuum Einstein equations are satisfied en
   a neighborhood of scri, and it should be proportional to the Komar charge $K_S(\nu)$
   of equation (124).

5. In Minkowski space, $Q_S(\nu)$ should vanish for all BMS symmetries $\nu^a$ and section $S$.

6. Given a symmetry $\nu^a$ there should exist a local flux $F(\nu)$, linear in $\nu^a$, such that
   $Q_S(\nu) - Q_{S'}(\nu) = \int_\Sigma F(\nu)$; where $\Sigma$ is the region on scri bounded by $S$ and $S'$.

7. The flux $F(\nu)$ should vanish in absence of gravitational radiation.

8. Having three consecutive stages at scri, a first stationary region $\Sigma_1$, a second radiat-
ing stage $\Sigma_2$ and a third stationary region $\Sigma_3$, then there is a continuous prescription
   for the center of mass sections $S_{cm}$ provided by the angular momentum calculated
   from $Q_S(\nu)$ such that $J^a_b P^b = 0$, where $J^{ab}$ is the angular momentum calculated
   from $Q_{S_{cm}}(v_{cm})$, in which $v_{cm}$ are the Lorentz rotations leaving $S_{cm}$ invariant, and $P^b$
   is the Bondi momentum.

One can readily see that our definition of charge integrals of the Riemann tensor at scri,
which uses the charge $Q_S(\nu)$ of equation (60), satisfies all condition except 4 although it
is still an open question whether condition 4 is satisfied or not. Ours is the only definition
we know to satisfy property 8.

We have seen that the rest frame and center of mass condition for the intrinsic systems
are rather complicated. However one should have in mind that for astrophysical interesting
systems, the amount of gravitational radiation is small, and therefore several quantities
can be approximated around stationary values.

This is precisely the key idea we plan to implement in future works.

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References

[AS81] A. Ashtekar and M. Streubel. Symplectic geometry of radiative modes and conserved quantities at null infinity. Proc. R. Soc. Lond. A, 376:585–607, 1981.

[AW82] Abhay Ashtekar and Jeffrey Winicour. Linkages and hamiltonians at null infinity. J. Math. Phys., 23:2410–2417, 1982.

[Bra75] B.D. Bramson. Relativistic angular momentum for asymptotically flat Einstein-Maxwell manifolds. Proc. R. Soc. Lond. A, 341:463–490, 1975.

[DM00] Sergio Dain and Osvaldo M. Moreschi. General existence proof for rest frame system in asymptotically flat space-time. Class. Quantum Grav., 17:3663–3672, 2000.

[DS84] Tevian Dray and Michael Streubel. Angular momentum at null infinity. Class. Quantum Grav., 1:15–26, 1984.

[Ger77] R. Geroch. Asymptotic structure of space-time. In F. P. Esposito and L. Witten, editors, Asymptotic structure of space-time. Plenum Publishing Corporation, 1977.

[GHP73] R. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. J. Math. Phys., 14:874–881, 1973.

[GW81] R. Geroch and J. Winicour. Linkages in general relativity. J. Math. Phys., 22:803–812, 1981.

[KL97] Joseph Katz and Dorit Lerer. On global conservation laws at null infinity. Class. Quantum Grav., 14:2249–2266, 1997.

[Kom59] Arthur Komar. Covariant conservation laws in general relativity. Phys. Rev., 113:934–936, 1959.

[MD98] Osvaldo M. Moreschi and Sergio Dain. Rest frame system for asymptotically flat space-times. J. Math. Phys., 39(12):6631–6650, 1998.

[Mor86] Osvaldo M. Moreschi. On angular momentum at future null infinity. Class. Quantum Grav., 3:503–525, 1986.

[Mor87] Osvaldo M. Moreschi. General future asymptotically flat spacetimes. Class. Quantum Grav., 4:1063–1084, 1987.

[Mor88] Osvaldo M. Moreschi. Supercenter of mass system at future null infinity. Class. Quantum Grav., 5:423–435, 1988.

[NP62] E. T. Newman and R. Penrose. An aproach to gravitational radiation by a method of spin coefficients. J. Math. Phys., 3(3):566, 1962.

[NP68] E. T. Newman and R. Penrose. New conservation laws for zero rest-mass fields in asymptotically flat space-time. Proc. Roc. Soc. A, 305:175–204, 1968.
[NT80] E. T. Newman and K. P. Tod. Asymptotically flat space-time. In A. Held, editor, General Relativity and Gravitation, volume 2, pages 1–34. Plenum, 1980.

[Pen82] R. Penrose. Quasi-local mass and angular momentum in general relativity. Proc. R. Soc. Lond. A, 381:53–63, 1982.

[Pri77] C.R. Prior. Angular momentum in general relativity. Proc. R. Soc. Lond., A(354):379, 1977.

[Riz98] Anthony Rizzi. Angular momentum in general relativity: A new definition. Phys. Rev. Lett., 81(6), 1998.

[Str78] M. Streubel. “Conserved” quantities for isolated gravitational systems. Gen. Rel. Grav., 9:551–561, 1978.

[TW66] L. A. Tamburino and J. Winicour. Gravitational fields in finite and conformal Bondi frames. Phys. Rev., 150:1039–1053, 1966.

[Win68] J. Winicour. Some total invariants of asymptotically flat space-times. J. Math. Phys, 9:861, 1968.

[Win80] J. Winicour. Angular momentum in general relativity. In A. Held, editor, General Relativity and Gravitation, volume 2, pages 71–96. Plenum, 1980.

[WZ00] Robert M. Wald and Andreas Zoupas. General definition of “conserved quantities” in general relativity. Phys. Rev. D, 61(084027):1–16, 2000.