THE BASIC PHYSICAL LIE OPERATIONS

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Abstract

Quantum theory can be formulated as a theory of operations, more specific, of complex represented operations from real Lie groups. Hilbert space eigenvectors of acting Lie operations are used as states or particles. The simplest simple Lie groups have three dimensions. These groups together with their contractions and their subgroups contain - in the simplest form - all physically important basic operations which come as translations for causal time, for space and for spacetime, as rotations, Lorentz transformations and as Euclidean and Poincaré transformations with scattering and particle states and also - via the Heisenberg group - as the operational structure of non-relativistic quantum mechanics. The classification of all those groups and their contractions is given together with their Hilbert spaces, constituted by energy-momentum functions. The group representation matrix elements can be written in the form of residues of energy-momentum poles - simple poles for abelian translations, e.g. in Feynman propagators, and dipoles for simple group operations, e.g. in the Schrödinger wave functions for the nonrelativistic hydrogen atom.


Contents

1 The Basic Actors .................................................. 1

2 Heisenberg Lie Algebras and Groups .......................... 3

3 Contractions .......................................................... 6
   3.1 Contractive Units and Dilations .......................... 6
   3.2 Simple Contractions ........................................ 7
   3.3 Double Contraction .......................................... 8
   3.4 Semidirect and Central Contractions .................. 9

4 Hilbert Representations .......................................... 10
   4.1 Some General Remarks ........................................ 10
   4.2 Hilbert Representations of Abelian and Compact Groups ... 12
   4.3 Induced Representations ................................... 14
   4.4 Residual Representations .................................. 15

5 Simple Poles for Translations .................................. 16
   5.1 The Affine Group in one Dimension - Causal Time ....... 16
   5.2 The Flat Euclidean Group - Scattering in the Plane ...... 17
   5.3 The Flat Poincaré Group - Spinless Free Particles ....... 19
   5.4 The Heisenberg Group - Nonrelativistic Quantum Mechanics ... 20

6 Dipoles for Simple Groups ...................................... 21
   6.1 Multipoles for the Nonrelativistic Hydrogen Atom ....... 22
   6.2 3-Space and (1,2)-Spacetime as Lie Algebras ............ 23
   6.3 Hilbert Representations of SU(2) and SU(1,1) ............ 24
   6.4 Hilbert Spaces of SU(2) ................................... 27
   6.5 Hilbert Spaces of SU(1,1) ................................ 30

7 Appendix: ℂ³-Lie Algebras ....................................... 33

8 Appendix: Residual Distributions .............................. 34
1 The Basic Actors

The abelian and simple real and complex Lie algebras are the building blocks for all real and complex Lie operations\cite{5, 9, 10}. The classification of the Lie algebras with dimensions one, two and three - 'the basic physical Lie operations (actors)' - gives also the simplest nontrivial examples for the concepts abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable and simple\cite{2}.

It makes sense to call the trivial Lie algebra $L = \{0\}$ semisimple, but not simple. Its group is the trivial group $\exp\{0\} = \{1\}$.

There is one complex 1-dimensional Lie algebra $L \cong \mathbb{C}$, it is abelian $\partial L = [L, L] = \{0\}$. It generates the linear group\cite{5, 9, 10} $\text{GL}(\mathbb{C})$.

There are two complex 2-dimensional Lie algebras\footnote{There is the covariant functor $G \mapsto \log G$ from Lie groups to their Lie algebras and, vice versa, $L \mapsto \exp L$. The linear group $\text{GL}(\mathbb{K}^n)$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ has as Lie algebra $\text{AL}(\mathbb{K}^n)$ ($(n \times n)$-matrices).} $L \cong \mathbb{C}^2$, the abelian decomposable one and - new for two dimensions - the nonabelian, solvable one. The latter one is a semidirect product\footnote{Isomorphies should be qualified, e.g. for a Lie algebra $L \cong \mathbb{K}^n$ ($\mathbb{K}$-vector space isomorphy). For a simpler notation, such qualifications are omitted - they should be obvious from the context.}, isomorphic to the Lie algebra of the 1-dimensional affine group. It is given in the 2nd column with the bracket in a basis $\{l^1, l^2\}$ and its faithful adjoint representation $\exp\{\mathbb{R}\}$.

| abelian | solvable |
|---------|----------|
| $\log \text{GL}(\mathbb{C}) \cong \mathbb{C}$ | $\log[\text{GL}(\mathbb{C}) \times \mathbb{C}] \cong \mathbb{C} \oplus \mathbb{C}$ |
| $[l^1, l^2] = 0$ | $[l^1, l^2] \neq 0$ |
| $x l \to x \in \mathbb{C}$ | $x l^1 + x l^2 \to \left(0 \begin{array}{c} 0 \end{array} \right) \in \text{AL}(\mathbb{C}^2)$ |
| $\log \text{D}(1) \cong \mathbb{R}$ | $\log[\text{D}(1) \times \mathbb{R}] \cong \mathbb{R} \oplus \mathbb{R}$ |
| $\log \text{U}(1) \cong \mathbb{R}$ | |

Complex Lie operations have real forms: $\text{D}(1) = \exp \mathbb{R}$ and $\text{U}(1) = \exp \mathbb{R}$ are the connected real 1-dimensional Lie groups. The abelian Lie algebras $x \in \mathbb{R}^n$ formalize physical translations. In the semidirect real Lie group $\left(1 \begin{array}{c} -x \end{array} \right) \in \text{D}(1) \times \mathbb{R}$ the translations are acted upon with $\text{D}(1)$-dilations $\mathbb{R} \ni x \mapsto e^{\psi x}$.

There are three nondecomposable complex 3-dimensional Lie algebras $L \cong \mathbb{C}^3$ - simple, solvable and nilpotent (proof in the appendix). They have faithful representations for a basis $\{l^1, l^2, l^3\}$ by $(3 \times 3)$-matrices $\alpha_1 l^1 + \alpha_2 l^2 + \alpha_3 l^3 \mapsto \text{AL}(\mathbb{C}^3)$.

| simple | solvable | nilpotent |
|--------|----------|-----------|
| $\log \text{SO}(\mathbb{C}^3)$ | $\log[\text{SO}(\mathbb{C}^2) \times \mathbb{C}] \cong \mathbb{C} \oplus \mathbb{C}^2$ | $\log \text{H}(\mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}^2$ |
| $[l^1, l^2] = l^3$ | $[l^1, l^2] = 0$ | $[l^1, l^2] = 0$ |
| $[l^1, l^3] = l^1$ | $[l^2, l^3] = l^1$ | $[l^2, l^3] = 0$ |
| $[l^2, l^3] = l^2$ | $[l^3, l^1] = l^2$ | $[l^3, l^1] = l^2$ |
| $\begin{pmatrix} 0 & \alpha_3 & \alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ -\alpha_2 & -\alpha_1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & \alpha_3 & \alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$ |
| $\log \text{SO}(3)$ | $\log \text{SO}(2) \oplus \mathbb{R}^2$ | $\log \text{H}(1)$ |
| $\log \text{SO}_0(1, 2)$ | $\log \text{SO}_0(1, 1) \oplus \mathbb{R}^2$ | |
| rotations | flat Euclidean | Heisenberg |
| flat Lorentz | flat Poincaré |

Their real forms (2nd last line) are - up to $\log \text{SO}(3)$ - noncompact.\footnote{denoted as $G_1 \times G_2$ for groups and $L_1 \oplus L_2$ for Lie algebras}
All nondecomposable Lie algebras with dimensions 1, 2 and 3 have rank 1. There is one generating linear invariant for the abelian case, and one quadratic Casimir invariant (inverse Killing form) for the simple case and its contracted forms for the contractions (below). The 2-dimensional Lie algebra \( \mathbb{R} \oplus \mathbb{R} \) has no nontrivial invariant.

The Lie groups for the 1-dimensional real Lie algebras act irreducibly on 1-dimensional spaces and, in the selfdual orthogonal representations, on 2-dimensional ones

\[
\text{noncompact: } \begin{cases} 
\mathbf{D}(1) = \exp \mathbb{R} \ni e^\psi, & \mathbb{R} \cong \log \mathbf{D}(1) \cong \log \mathbf{SO}(1, 1) \\
\mathbf{SO}_0(1, 1) = \exp \sigma_3 \mathbb{R} \ni \begin{pmatrix} e^\psi & 0 \\
0 & e^{-\psi} \end{pmatrix} \cong \begin{pmatrix} \cosh \psi & \sinh \psi \\
\sinh \psi & \cosh \psi \end{pmatrix} \in \exp \sigma_1 \mathbb{R}
\end{cases}
\]

\[
\text{compact: } \begin{cases} 
\mathbf{U}(1) = \exp i\mathbb{R} \ni e^{i\varphi}, & i\mathbb{R} \cong \log \mathbf{U}(1) \cong \log \mathbf{SO}(2) \\
\mathbf{SO}(2) = \exp i\sigma_3 \mathbb{R} \ni \begin{pmatrix} e^{i\varphi} & 0 \\
0 & e^{-i\varphi} \end{pmatrix} \cong \begin{pmatrix} \cos \varphi & i\sin \varphi \\
i\sin \varphi & \cos \varphi \end{pmatrix} \in \exp i\sigma_1 \mathbb{R}
\end{cases}
\]

The simply connected totally ordered group \( \mathbf{D}(1) \cong \mathbb{R} \) covers \( \mathbf{U}(1) \cong \mathbb{R}/\mathbb{Z} \) infinitely often and is a real form of the complex 1-dimensional full Lie group \( \mathbf{GL}(\mathbb{C}) \cong \mathbf{SO}(\mathbb{C}^2) \) with Lie algebra \( D_1 \).

The simple real Lie structures with dimension 3 add spherical⁵ and hyperbolic degrees of freedom to the abelian \( \mathbb{R} \)-structures

\[
\begin{align*}
\mathbf{SO}(2) & \cong \Omega^1, \\
\mathbf{SO}_0(1, 1) & \cong \mathcal{Y}^1 \\
\mathbf{SO}(3) & \text{ rotation group,} \\
\mathbf{SO}(3)/\mathbf{SO}(2) & \cong \Omega^2 \\
\mathbf{SO}_0(1, 2) & \text{ flat Lorentz group,} \\
\mathbf{SO}_0(1, 2)/\mathbf{SO}_0(1, 1) & \cong \mathcal{Y}^1 \times \Omega^1
\end{align*}
\]

The twofold covering groups are (isospin) \( \mathbf{SU}(2) \) (simply connected) and \( \mathbf{SU}(1, 1) \) (\( \mathbb{Z} \)-connected) as real forms of the complex 3-dimensional special Lie group \( \mathbf{SL}(\mathbb{C}^2) \) (considered as 3-dimensional complex Lie group).

The simple complex Lie operations, i.e. nonabelian without proper ideal, have been classified by Cartan with four main series \( \{A, B, C, D\} \) and five exceptional Lie algebras. Three main series - with the invariance operations for volumes \( A_r \cong \log \mathbf{SL}(\mathbb{C}^{1+r}) \), for odd dimensional orthogonal structures \( B_r \cong \log \mathbf{SO}(\mathbb{C}^{1+2r}) \) and for symplectic structures \( C_r \cong \log \mathbf{Sp}(\mathbb{C}^{2r}) \) - start with the same simplest simple Lie algebra which has three dimensions

\[
A_1 = B_1 = C_1 \cong \mathbb{C}^3
\]

The fourth series - for even dimensional orthogonal structures - starts with \( D_3 \cong \log \mathbf{SO}(\mathbb{C}^6) \cong \log \mathbf{SL}(\mathbb{C}^4) \) after the abelian \( D_1 \cong \log \mathbf{SO}(\mathbb{C}^2) \) and the semisimple \( D_2 \cong A_1 \oplus A_1 \cong \log \mathbf{SO}(\mathbb{C}^4) \). All simple Lie algebras are ‘fused collectives’ of several \( A_1 \)-isomorphic building blocks. The simplest example, used in physics, is the 8-dimensional Lie structure \( \mathbf{SU}(3) \) (flavor or color) where three 3-dimensional \( A_1 \)-building blocks, called \( I, U \) and \( V \)-spin, are ‘fused’ in their Cartan subalgebras by the linear dependence \( I^3 + U^3 + V^3 = 0 \) to yield the Lie algebra \( A_2 = \log \mathbf{SL}(\mathbb{C}^3) \).
The simplest real simple structures are descendants of $A_1$, compact and noncompact (definite and indefinite unitary)

$$A_1^c = B_1^c = C_1^c \cong \log SU(2) \cong \log SO(3) \cong \mathbb{R}^3$$
$$A_1^n = B_1^n = C_1^n \cong \log SU(1,1) \cong \log SO_0(1,2) \cong \log SL(\mathbb{R}^2) \cong \mathbb{R}^3$$

If each of the three compact degrees of freedom in the Lie algebra of the spin group $SU(2) \cong \exp A_1^c$ is paired with a noncompact one, there arises the rank 2 simple Lie algebra $A_1^c \oplus iA_1^c$ as ‘complexified spin’ operations for the Lie group $SL(\mathbb{C}^2) = \exp[A_1^c \oplus iA_1^c]$, considered as 6-dimensional real Lie group where three independent rotations are paired with three boosts. It is the twofold cover of the orthochronous Lorentz group for 4-dimensional Minkowski spacetime

$$A_1^c \oplus iA_1^c \cong \log SL(\mathbb{C}^2) \cong \log SO(\mathbb{C}^3) \cong \log SO_0(1,3) \cong \mathbb{R}^6$$

The not simple nondecomposable real 3-dimensional Lie operations are all semidirect groups, i.e. affine subgroups in $GL(\mathbb{R}^2) \times \mathbb{R}^2$

$$SO(2) \times \mathbb{R}^2 \quad \text{Euclidean (flat Galilei) group}$$
$$SO_0(1,1) \times \mathbb{R}^2 \quad \text{flat Poincaré group}$$
$$H(1) \cong \mathbb{R} \times \mathbb{R}^2 \quad \text{Heisenberg group}$$

They are contraction of $SO(3)$ and $SO_0(1,2)$ (below). $D(1) \times \mathbb{R}$ is a subgroup of the flat Poincaré group.

The general Euclidean, Lorentz and Poincaré groups are $SO(s) \times \mathbb{R}^s$, $SO_0(1,s)$ and $SO_0(1,s) \times \mathbb{R}^{1+s}$ respectively, for $s = 1,2,\ldots$. In the physical names of the operations ‘flat’ is meant as ‘spatially flat’, i.e. without nonabelian space rotations $s = 1,2$. The Heisenberg group $H(n) = \mathbb{R}^n \times \mathbb{R}^{1+n}$ with $n$ position-momentum pairs is looked at in more detail below.

## 2 Heisenberg Lie Algebras and Groups

The simplest nonabelian nilpotent Lie operations constitute the real 3-dimensional Heisenberg Lie algebra $\log H(1) = h(1)$ for one position-momentum pair $(x,p)$ and its bracket $I$, a basis for the centrum

$$h(1) = \{qx + yp + tI \mid q,y,t \in \mathbb{R}\} \cong \mathbb{R}^3$$

with $[x,p] = I$, $[I,x] = 0 = [I,p]$.

The central action operator $I$ is no number, e.g. not the imaginary unit $i$. $h(1)$ is a semidirect product

$$h(1) = \mathbb{R} \times [\mathbb{R}p + RI] : [\mathbb{R}x, \mathbb{R}p + RI] \subseteq RI$$

The position acts upon the ideal spanned by $\{p,I\}$, not by $\{p,x\}$. Here, and everywhere, the roles of the position $x$ and the momentum $p$ operations can be exchanged.
h(1) has a 3-dimensional faithful representation by nilpotent matrices
\[ h(1) \ni qx + yp + tI \quad \mapsto \quad \begin{pmatrix} 0 & q & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in AL(\mathbb{R}^3) \]
By exponentiation there arises the Heisenberg group H(1) with the Weyl product⁶ expressing the noncommutativity of position and momentum operators
\[ H(1) \ni e^{qx+yp+tI} \quad \mapsto \quad \begin{pmatrix} 1 & q & t+qy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in SL(\mathbb{R}^3) \]
Weyl product: \( e^{qx} e^{yp} = \begin{pmatrix} 1 & q & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{yp} e^{qx} \)
\[ \text{commutator group: } (H(1), H(1)) = e^{RI} : e^{qx} e^{yp} e^{-qx} e^{-yp} = e^{ypI} \]
H(1) leads to the chains with abelian normal subgroups
\[ H(1) = \{ e^{qx+yp+tI} \} \supset \{ e^{yp+I} \} \cong \mathbb{R}^2 \supset \text{centr } H(1) = \{ e^t \} \cong \mathbb{R} \]
Correspondingly, H(1) can be considered either as semidirect extension[2] or as central extension with the exact sequences⁷
\[ \mathbb{R}^2 \xto{\iota} H(1) \cong \mathbb{R} \times \mathbb{R}^2 \xto{\pi_1} \mathbb{R} \\
\mathbb{R} \xto{\iota} H(1) \cong \mathbb{R} \oplus \mathbb{R}^2 \xto{\pi} \mathbb{R}^2 \]
In the semidirect group product
\[ H(1) \cong \mathbb{R}^2 \circ \mathbb{R} = \mathbb{R} \times \mathbb{R}^2 \text{ with } \begin{cases} \mathbb{R} \cong \{ e^{qx} \mid q \in \mathbb{R} \} \\ \mathbb{R}^2 \cong \{ e^{yp+I} \mid y, t \in \mathbb{R} \} \end{cases} \]
illustrated in the (3 × 3)-matrix representations
\[ \begin{pmatrix} 1 & q & t+qy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t+qy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
the homogeneous group with the position x acts on the abelian normal subgroup \( \mathbb{R}^2 \) by inner automorphisms
\[ \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \begin{cases} e^{qx} \circ e^{yp+I} \circ e^{-qx} = e^{yp+(qy+t)I} \\ (\begin{pmatrix} 1 & q & t+qy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) = (\begin{pmatrix} 1 & q & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) \end{cases} \]
The adjoint representation⁸ has commuting position and momentum \([ \text{ad } x, \text{ad } p ] = 0\) - the image is the classical position-momentum Lie algebra
\[ \text{ad} : h(1) \longrightarrow AL(\mathbb{R}^3), \quad \text{Ad} : H(1) \longrightarrow SL(\mathbb{R}^3) \]
\[ \text{ad} (qx + yp + tI) = \begin{pmatrix} 0 & 0 & \frac{-y}{2q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{Ad } e^{qx+yp+tI} = \begin{pmatrix} 1 & 0 & \frac{-y}{2q} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
\[ \text{ad } h(1) \cong \{ \mathbb{R} x + \mathbb{R} p \}, \quad \text{ad } I = 0, \quad \text{Ad } I = 1_3 \]

⁶ In analogy to the commutator ideal of a Lie algebra, there is the normal commutator subgroup \((G, G) = \{ gk(kg)^{-1} \mid g, k \in G \}\).

⁷ An extended vector space or group or Lie algebra G is defined by the injection-projection structure \( N \xto{\gamma} G \longrightarrow H \) with image \( \gamma = \ker \pi \), i.e. \( H \cong G/N \).

⁸ The adjoint representation of a Lie algebra acts on itself \( \text{ad} : L \longrightarrow AL(L) \) with \( \text{ad } m(l) = [m, l] \). The adjoint representation of a Lie group on its Lie algebra \( \text{Ad} : G \longrightarrow GL(L) \) goes via \( \text{Ad } g(l) \sim glg^{-1} \). The adjoint action of a group on itself is by inner automorphisms \( \text{Int } g : G \longrightarrow G, \text{ Int } g(k) = gkg^{-1} \).
\( \mathbf{1}_n = \text{id}_V \) denotes the identity operation on a vector space \( V \cong \mathbb{K}^n \).

Also the general Heisenberg Lie algebra with \( n \) position momentum pairs, \( a = 1, \ldots, n \)

\[
\mathbf{h}(n) \cong \mathbb{R}^{1+2n} : \quad [x_a, p^b] = \delta^b_a \mathbf{I}, \quad [x_a, \mathbf{I}] = 0 = [\mathbf{I}, p^b]
\]
is nilcubic with its centrum as commutator ideal

\[
[h(n), h(n)] = \mathbb{R} \mathbf{I} = \text{centr } h(n), \quad [[h(n), h(n)], h(n)] = \{0\}
\]

It has a faithful representation by \( (2 + n) \times (2 + n) \)-matrices

\[
\mathbf{h}(n) \ni q^a x_a + y^b p^b + t \mathbf{I} \mapsto \begin{pmatrix} 0 & q^a & t \\ 0 & 0 & y^b \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{2+n} \subset \text{AL}(\mathbb{R}^{2+n})
\]

It is a semidirect product of the \( n \)-dimensional abelian positions acting upon the \( n \)-dimensional momenta and the central operator

\[
\mathbf{h}(n) = \mathbb{R}^n \bigoplus [\mathbb{R}^n \bigoplus \mathbb{R}] : \quad [\mathbb{R} x_a, \mathbb{R} p^b + \mathbb{R} \mathbf{I}] \subseteq \mathbb{R} \mathbf{I}
\]

The affine Heisenberg Lie algebra \( a\mathbf{h}(n) \) includes, in addition, all linear transformations \( f \in \text{AL}(\mathbb{R}^n) \) of the position-momentum pairs - in the matrix representation

\[
a\mathbf{h}(n) \ni \begin{pmatrix} 0 & q^a & t \\ 0 & 0 & y^b \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & \mathbb{R}^n & \mathbb{R} \\ 0 & \text{AL}(\mathbb{R}^n) & \mathbb{R} \\ 0 & 0 & 0 \end{pmatrix} \subset \text{AL}(\mathbb{R}^{2+n})
\]

with the Lie-bracket involving the dual product \( \langle q, y \rangle \) of position with momentum and the linear transformations acting on position and momentum

\[
\begin{pmatrix} 0 & q_1 & t_1 \\ 0 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q_2 & t_2 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & f^T(q_1) - f^T(q_2) & \langle q_1, y_2 \rangle - \langle q_2, y_1 \rangle \\ 0 & 0 & [f_1, f_2] \\ 0 & 0 & 0 \end{pmatrix}
\]

with \( \langle q, y \rangle = q^a y_a, \quad f(y)_b = f^b_a y_a, \quad f^T(q)^a = f^a_b q^b \)

The Heisenberg Lie algebra is the nilradical of the affine Heisenberg Lie algebra

\[
[a\mathbf{h}(n), h(n)] \subseteq h(n)
\]

Therewith the extended Heisenberg Lie algebra is a double semidirect product

\[
a\mathbf{h}(n) = \text{AL}(\mathbb{R}^n) \bigoplus h(n) = \text{AL}(\mathbb{R}^n) \bigoplus [\mathbb{R}^n \bigoplus [\mathbb{R}^n \bigoplus \mathbb{R}]]
\]


### 3 Contractions

In general, contractions of simple Lie operations go with a ‘flattening’ of degrees of freedom. In physics, this is related to a trivialization of units. The prototype Wigner-Inönü contraction from the Lorentz to the Galilei group\([20]\) ‘flattens’ the boosts by trivializing the speed of light \(\frac{1}{c} \to 0\). In the opposite procedure, a contracted group is expanded by ‘flexing’ flat degrees of freedom. An expansion resuscitates a mute unit. Constructions work with units, operationally formalized with dilations.

The not simple nondecomposable real 3-dimensional Lie operations are all contractions of the simplest simple Lie operations. Their Lie algebras are given with a basis, a defining representation in the endomorphism algebra \(AL(\mathbb{Q}^3)\) (complex \((3 \times 3)\)-matrices) - for the simple groups \(SO(3)\) and \(SO_6(1,2)\) the adjoint representations - and the invariant Casimir element \(C\).

From the two simple groups (left), 3-rotations and \((1,2)\)-Lorentz group, related to each other by the spherical-hyperbolic, i.e. compact-noncompact, i.e. imaginary-real exchange of two operations \(i\varphi_{1,2} \leftrightarrow \psi_{1,2}\), there leads one contraction to the 2-Euclidean and (or) to the \((1,1)\)-Poincaré group. From these contracted groups there leads a 2nd contraction to the double contracted Heisenberg group which can also be reached directly by a central contraction.

#### 3.1 Contractive Units and Dilations

The simple Lie algebra parametrizations above with faithful adjoint \((3 \times 3)\)-matrix representations are for a diagonal invariant metric \(1_3\) and \(\eta\), i.e. for the Killing forms in an orthonormal basis. Dilation transformations from the triad manifold \(d \in D(1)^3 \in GL(\mathbb{R}^3)/SO(3)\) and \(d \in D(1)^3 \in GL(\mathbb{R}^3)/SO_6(1,2)\) introduce three units \(\alpha_{1,2,3} > 0\) for the three operations. The units can be visualized as lengths for the three axes of a metrical 2-ellipsoid \(d 1_3 d^T\) for
\( \text{SO}(3) \) and of a metrical 2-hyperboloid \( d\eta^T \) for \( \text{SO}_0(1, 2) \)

\[
\text{SO}(3) : d1_3d^T = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\]

\[
\text{SO}_0(1, 2) : d\eta^T = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\]

The related inner automorphisms \( L \ni l \mapsto dld^{-1} \) give renormalized Lie algebra representations which leave invariant the dilation transformed metric - for \( \text{SO}_0(1, 2) \)

\[
dld^{-1} = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}
\begin{pmatrix}
0 & \psi_1 & \psi_2 \\
\psi_1 & 0 & \psi_3 \\
\psi_2 & -\psi_3 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\alpha_1} & 0 & 0 \\
0 & \frac{1}{\alpha_2} & 0 \\
0 & 0 & \frac{1}{\alpha_3}
\end{pmatrix}
\]

and - with \( \psi_{1,2} \mapsto i\varphi_{1,2} \) - for \( \text{SO}(3) \).

### 3.2 Simple Contractions

By renormalization of the Lie algebra basis and a renaming of the parameters, e.g. \( \frac{\alpha_1}{\alpha_3} \psi_2 = y_0 \), one obtains the contraction of the flat Lorentz group \( \text{SO}_0(1, 2) \) to the flat Poincaré group

\[
\begin{pmatrix}
0 & \frac{\alpha_1}{\alpha_3} \psi_1 & \frac{\alpha_1}{\alpha_3} \psi_2 \\
\frac{\alpha_1}{\alpha_3} \psi_1 & 0 & \frac{\alpha_1}{\alpha_3} \psi_2 \\
\frac{\alpha_1}{\alpha_3} \psi_2 & -\frac{\alpha_1}{\alpha_3} \psi_3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & \psi_1 & \psi_2 & \psi_3 \\
\alpha_1 y_0 & 0 & 0 & 0 \\
0 & y_0 & y_1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( \alpha_3^2 \rightarrow 0, \ \alpha_1 = \alpha_2 \) with finite \( y_{0,1} \)

Therewith the Casimir invariant (inverse Killing form) is contracted to

\[
C = (B^1)^2 + (B^2)^2 - (L^3)^2 \rightarrow (Q^0)^2 - (Q^1)^2
\]

i.e. the metrical 2-hyperboloid degenerates to a metrical 1-hyperbola. Anticipation of the contraction leads to the parametrization of metrical hyperboloid and invariance group involving one contractive unit \( \ell \) as one main axis

\[
\log \text{SO}_0(1, 2) \ni \begin{pmatrix}
\alpha_1^2 \\
0 \\
\frac{\alpha_1}{\alpha_3} \psi_1 \\
\frac{\alpha_1}{\alpha_3} \psi_2 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -\ell^2
\end{pmatrix}
\]

with \( \psi_1 \mapsto t \psi_1 \)

\[
\ell^2 \rightarrow 0: \text{SO}_0(1, 2) \ni \text{SO}_0(1, 1) \times \mathbb{R}^2
\]

\[
\begin{pmatrix}
\cosh \psi_1 & \sinh \psi_1 \\
\sinh \psi_1 & \cosh \psi_1
\end{pmatrix}
\in \text{SO}_0(1,1) \subset \text{GL}(\mathbb{R}^2)
\]
The Wigner-Inönü contraction from the flat Lorentz group to the Euclidean (flat Galilei) group is analogous

\[
\begin{bmatrix}
0 & \frac{\alpha_2}{\alpha_1} \psi_1 & \frac{\alpha_1}{\alpha_2} \psi_2 \\
\frac{\alpha_1}{\alpha_2} \psi_1 & 0 & \frac{\alpha_2}{\alpha_1} \varphi_3 \\
-\frac{\alpha_2}{\alpha_1} \psi_2 & \frac{\alpha_1}{\alpha_2} \varphi_3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & \frac{\alpha_2}{\alpha_1} x_1 & \frac{\alpha_1}{\alpha_2} x_2 \\
x_1 & 0 & -\frac{\alpha_2}{\alpha_1} \varphi_3 \\
x_2 & \frac{\alpha_1}{\alpha_2} \varphi_3 & 0
\end{bmatrix}
\]

\(\alpha_1^2 \rightarrow 0, \ \alpha_2 = \alpha_3 \) with finite \( x_{1,2} \)

\[ C = (B_1)^2 + (B_2)^2 - (L_3)^2 \rightarrow (P_1)^2 + (P_2)^2 \]

The metrical 2-hyperboloid degenerates to a metrical circle (1-sphere). The contractive unit \( c \) is the speed of light

\[
\log \text{SO}_0(1,2) \ni \begin{bmatrix}
0 & \frac{\alpha_2}{\alpha_1} x_1 & \frac{\alpha_1}{\alpha_2} x_2 \\
x_1 & 0 & -\frac{\alpha_2}{\alpha_1} \varphi_3 \\
x_2 & \frac{\alpha_1}{\alpha_2} \varphi_3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & \frac{1}{c} x_1 & \frac{1}{c} x_2 \\
x_1 & 0 & -\varphi_3 \\
x_2 & -\varphi_3 & 0
\end{bmatrix}
\]

\( \frac{1}{c} \rightarrow 0: \text{SO}_0(1,2) \rightarrow \text{SO}(2) \times \mathbb{R}^2 \)

with \((\cos \varphi_3 \sin \varphi_3, -\sin \varphi_3 \cos \varphi_3) \in \text{SO}(2) \subset \text{GL}(\mathbb{R}^2)\)

### 3.3 Double Contraction

The transition from Poincaré or Euclid to Heisenberg involves a 2nd contraction from 'hyperbolic' or spherical to flat

\[ \uparrow \text{SO}_0(1,1) \hexrightarrow{\mathbb{R}^2} \text{SO}_0(1,1) \rightarrow \mathbb{R} \]

\[ \downarrow \quad \text{SO}_0(1,2) \quad \begin{cases} \uparrow \ \text{SO}_0(1,1) \hexrightarrow{\mathbb{R}^2} \text{SO}_0(1,1) \rightarrow \mathbb{R} \\
\downarrow \quad \text{SO}(2) \hexrightarrow{\mathbb{R}^2} \text{SO}(2) \rightarrow \mathbb{R} \end{cases} \]

E.g., the double contraction from simple \( \text{SO}_0(1,2) \) to Heisenberg \( \textbf{H}(1) \) (Segal contraction[24])

\[
\begin{bmatrix}
0 & \frac{\alpha_1}{\alpha_2} \psi_1 & \frac{\alpha_2}{\alpha_1} \psi_2 \\
\frac{\alpha_2}{\alpha_1} \psi_1 & 0 & \frac{\alpha_1}{\alpha_2} \varphi_3 \\
-\frac{\alpha_1}{\alpha_2} \psi_2 & \frac{\alpha_2}{\alpha_1} \varphi_3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & q & t \\
q & 0 & y \\
t & y & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & q & t \\
q & 0 & y \\
t & y & 0
\end{bmatrix}
\]

\(\alpha_3 \rightarrow 0, \ \frac{1}{\alpha_1} \rightarrow 0, \ \alpha_2 = 1\) with finite \( q, y, t \)

\[ C = (B_1)^2 + (B_2)^2 - (L_3)^2 \rightarrow \mathbb{I}^2 \]
can be performed with two contractive units $\ell^2, \frac{1}{\ell^2}$

\[
\log \text{SO}_0(1, 2) \ni \begin{pmatrix}
\alpha_1^2 & 0 & 0 \\
0 & -\alpha_2^2 & 0 \\
0 & 0 & -\alpha_3^2
\end{pmatrix} = \begin{pmatrix}
\epsilon^2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -\epsilon^2
\end{pmatrix}
\]

$\ell^2, \frac{1}{\ell^2} \rightarrow 0 : \text{SO}_0(1, 2) \rightarrow H(1) \cong \mathbb{R} \times \mathbb{R}^2$

A 1-step contraction uses one contractive unit $\mu^2 \rightarrow 0$

\[
\log \text{SO}_0(1, 2) \ni \begin{pmatrix}
\alpha_1^2 & 0 & 0 \\
0 & -\alpha_2^2 & 0 \\
0 & 0 & -\alpha_3^2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -\mu^2 & 0 \\
0 & 0 & -\mu^4
\end{pmatrix}
\]

3.4 Semidirect and Central Contractions

Any subgroup $H$ in a finite dimensional Lie group $G$ allows vector space decompositions of the Lie algebra

\[
\log G = \log H \oplus K, \quad K \cong \log G / \log H \quad \text{with} \quad \begin{cases}
[h_1, h_2] = h_3 \\
h_1, k_1 = h_2 + k_2 \\
k_1, k_2 = h + k_3
\end{cases}
\]

A dilation transformation of the complementary vector subspaces

\[
\mu \nu \neq 0, \quad h = \frac{1}{\nu} h, \quad k = \frac{1}{\mu} k
\]

\[
\log G = \log H(\nu) \oplus K(\mu) \quad \begin{cases}
[h_1, h_2] = \nu h_3 \\
h_1, k_1 = \mu h_2 + \nu k_2 \\
k_1, k_2 = \frac{\mu^2}{\nu} h + \mu k_3
\end{cases}
\]

can be used for the semidirect contraction with the Lie subalgebra $\log H$ acting on a vector space $K$

\[
\nu = 1, \quad \mu \rightarrow 0 : \log G \rightarrow \log H \oplus K, \quad \begin{cases}
[\log H, \log H] \subseteq \log H \\
[\log H, K] \subseteq K \\
[K, K] = \{0\}
\end{cases}
\]

\[
e.g. \quad \log \text{SO}(3) \rightarrow \log \text{SO}(2) \oplus \mathbb{R}^2
\]

\[
\log \text{SO}_0(1, 2) \rightarrow \log \text{SO}_0(1, 1) \oplus \mathbb{R}^2
\]

With related dilations the central contraction gives a central Lie subalgebra $\log H$. The complementary space $K$ has a Lie bracket in $\log H$

\[
\nu = \mu^2 : \begin{cases}
[h_1, h_2] = \mu^2 h_3 \\
h_1, k_1 = \mu h_2 + \mu^2 k_2 \\
k_1, k_2 = h + \mu k_3
\end{cases}
\]

\[
\mu \rightarrow 0 : \quad \log G \rightarrow \log H \overset{\text{centr}}{\cong} K, \quad \begin{cases}
[\log H, \log H] = \{0\} \\
[\log H, K] = \{0\} \\
[K, K] \subseteq \log H
\end{cases}
\]

\[
e.g. \quad \log \text{SO}(3), \log \text{SO}_0(1, 2) \rightarrow \mathbb{R} \overset{\text{centr}}{\cong} \mathbb{R}^2
\]
In the Heisenberg example $\mathbb{R}^2 \subset \log H(1)$ is spanned by $\{p, x\}$ for the central contraction in contrast to $\{x, I\}$ for the semidirect contraction.

The diagonalizable quadratic Casimir element for a semisimple Lie algebra (Killing metric of the adjoint representation) is contracted to a bilinear vector space form in the semidirect contraction and to a central subalgebra form in the central one

$$\log G : C = \hbar \otimes \hbar + \kappa \otimes \kappa = \frac{\hbar \otimes \hbar}{\mu^2} + \frac{\kappa \otimes \kappa}{\mu^2}$$

$$\log G \rightarrow \log H \, \otimes \, K : \mu^2 C \rightarrow \kappa \otimes \kappa$$

$$\log G \rightarrow \log H \, \otimes \, K : \mu^4 C \rightarrow \hbar \otimes \hbar$$

### 4 Hilbert Representations

Groups carry ‘in themselves’ the structure of ‘their’ representation spaces: Any set $S$ where a group $G$ acts on, is a disjoint union of $G$-orbits $G \cdot v$, $v \in S$, which are irreducible $G$-sets. An orbit is isomorphic to a subgroup class $G \cdot v \cong G/H$, i.e. to the $G$-operations up to the fixgroup (Wigner’s ‘little’ groups) $H = G_v$ for the $G$-action. Therefore: The coset spaces $\{G/H \mid H \subseteq G\}$ constitute - up to isomorphy - the irreducible sets with $G$-action.

For group representations on vector spaces, linearity has to be taken into account (more below). All real Lie groups (locally compact) define ‘their’ Hilbert spaces with complex representations, compact Lie groups have only Hilbert representations. In physics, with Born, the scalar product of the Hilbert spaces acted upon is interpreted in terms of ‘probability amplitudes’. Therewith, physical Lie operations carry their probability interpretation in their own structure.

#### 4.1 Some General Remarks

First some facts about representations which, in a more detailed and exact formulation, can be found in the literature[2, 4, 11]: Group (Lie algebra) representations have a normal subgroup (an ideal) as kernel, i.e. a group (Lie algebra) without normal subgroup (ideal) has faithful, i.e. injective, or trivial representations. For the abelian and simple groups with the ‘basic physical Lie operations’ the nontrivial unfaithful representations are characterized by the discrete normal subgroups $N$ - they are faithful for the quotient groups $G/N$

| $G$  | $\dim_{\mathbb{R}} G$ | $N$  | $G/N$            |
|------|---------------------|------|-----------------|
| $D(1) \cong \mathbb{R}$ | 1     | $\mathbb{R}^* \cong \mathbb{Z}$ | $U(1)$ |
| $SU(2)$ | 3     | $\{\pm 1, 2\}$ | $SO(3)$ |
| $SU(1,1)$ | 3    | $\{\pm 1\}$ | $SO_0(1,2)$ |

For the affine group in one dimension and for the 3-dimensional contracted groups the translations are continuous normal subgroups
\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
$G$ & dim$_{\mathbb{R}}G$ & $N$ & $G/N$ \\
\hline
$D(1) \times \mathbb{R}$ & 2 & $\mathbb{R}$ & $D(1)$ \\
$SO(2) \times \mathbb{R}^2$ & 3 & $\mathbb{R}^2$ & $SO(2)$ \\
$SO_0(1, 1) \times \mathbb{R}^2$ & 3 & $\mathbb{R}^2$ & $SO_0(1, 1)$ \\
$H(1) = \mathbb{R} \times \mathbb{R}^2$ & 3 & $\mathbb{R}$ & $\mathbb{R}^2$ \\
\hline
\end{tabular}
\end{table}

In addition, there are normal subgroups with discrete factors $\mathbb{Z}$ as used for $D(1) \cong \mathbb{R}$.

There is Ado’s theorem\cite{2}: A finite dimensional Lie algebra has a faithful finite dimensional matrix representations (the nilradical becomes strictly triangular), e.g. the Heisenberg Lie algebras above.

Any vector $v \in V$ of a group representation space generates - by the closure of its group orbit span (finite linear combinations) $\overline{\mathcal{C}(G \cdot v)}$ - a $G$-action invariant cyclic subspace. A vector $v$ is called cyclic for the representation if $\overline{\mathcal{C}(G \cdot v)} = V$. A cyclic representation has a cyclic vector $v$. Cyclic representations have not to be irreducible (simple), e.g. the reducible representation\cite{1} $\mathbb{R} \ni t \mapsto \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix} \in SU(1, 1)$ with invariant subspace $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and cyclic vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This representation describes the time development of a free Newtonian mass point\cite{27} $\begin{pmatrix} x(t) \\ i \phi(t) \end{pmatrix} = \begin{pmatrix} 1 & -it \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ i \phi(0) \end{pmatrix}$. According to Maschke and Weyl\cite{22, 16}, ‘irreducible’ and ‘cyclic’ coincide for compact groups.

Physical examples for cyclic vectors are ground states where a nontrivial fixgroup characterizes a degenerate ground state (‘spontaneous symmetry breakdown’). E.g., a ground state for the electroweak standard model of quark and lepton fields with their interactions is characterized by an electromagnetic $U(1)$ as fixgroup (‘little group’) in the represented interaction inducing hypercharge-isopin group $U(2)$.

To define ‘realness’ in a complex representation of a real Lie group, the representation vector space has to come with a conjugation, i.e. the represented Lie group has to be a unitary group - definite or indefinite unitary. E.g. the complex four dimensional Dirac representation of the real Lorentz group $SL(\mathbb{C}^2)$ is a subgroup of the indefinite unitary group $SU(2, 2)$.

Group functions $\mathcal{C}^G = \{ f : G \rightarrow \mathbb{C} \}$ are a ‘huge’ representation space of the ‘doubled’ group $G \times G$ with the both sided (left and right) regular action $f \mapsto g f_k$ where $g f_k(h) = f(g^{-1} h k)$. Complex group functions come with the number induced conjugation $f \leftrightarrow \hat{f}$, $\hat{f}(g) = \overline{f(g^{-1})}$ (definite unitary $U(1)$).

Of importance are the Banach spaces with the Lebesque function classes $L^p(G)$, $1 \leq p \leq \infty$, on a locally compact group\footnote{For the function (classes) $L^p_{d\mu}(\mathcal{S})$ where the $\mathcal{S}$-measure $d\mu$ is unique up to a scalar factor, e.g. Haar measure $dg$ for a locally compact group, the measure is omitted in the notation $L^p(\mathcal{S})$. Finite groups with discrete topology are compact and have counting measure.} with, especially, the Hilbert space with the square integrable functions $L^2(G)$, the convolution group algebra $L^1(G)$ and, as its topological dual, the essentially bounded functions $L^\infty(G)$. All Lebesque spaces are $L^1(G)$-convolution modules $L^1(G) \ast L^p(G) \rightarrow L^p(G)$. For compact groups the group algebra is maximal $L^p(G) \supseteq L^q(G)$ if $p \leq q$.\footnote{For the function (classes) $L^p_{d\mu}(\mathcal{S})$ where the $\mathcal{S}$-measure $d\mu$ is unique up to a scalar factor, e.g. Haar measure $dg$ for a locally compact group, the measure is omitted in the notation $L^p(\mathcal{S})$. Finite groups with discrete topology are compact and have counting measure.}
Representations in definite unitary groups are called Hilbert (representations). All compact group representations are Hilbert. Reducible representations of compact groups are decomposable into orthogonal direct sums of irreducible ones, the irreducible (cyclic, simple) representations are - according to Weyl - finite dimensional.

In the twofold dichotomy abelian-nonabelian and compact-noncompact, exemplified for finite dimensional real Lie groups with \( r \geq 1 \)

|          | abelian | nonabelian |
|----------|---------|------------|
| compact  | \( \text{U}(1) \) | \( \text{SU}(1+r) \) |
| noncompact| \( \text{D}(1) \) | \( \text{SL}(\mathbb{H}^{r+r}) \) |

the Hilbert representation structure of nonabelian noncompact Lie operations, is much more complicated than that for compact and abelian ones.

According to Gelfand and Raikov[19], a Hilbert representation of a locally compact group \( G \) is an orthogonal direct sum of cyclic ones and - relevant for noncompact groups - an orthogonal direct integral of irreducible ones. Faithful Hilbert representations of noncompact locally compact groups are infinite dimensional.

Only for compact groups, all representations act upon Hilbert spaces with square integrable functions as group algebra subspaces \( L^2(G) \subseteq L^1(G) \). In general, the cyclic Hilbert representations of a locally compact group \( G \) are - up to equivalence - bijectively related to positive type group functions. Such a function is defined as an essentially bounded function \( L^\infty(G) \) which endows the group algebra with a definite product - \( L^1(G) \) becomes a pre-Hilbert space

\[
\omega \in L^\infty(G) \quad \text{with} \quad \langle f | f \rangle_\omega = (\hat{f} * \omega * f)(e) = \int_{G \times G} d\mu(g)d\mu(g')\overline{f(\overline{g^{-1}})}\omega(gg')f(g') \geq 0
\]

Any vector of a Hilbert representation space gives, by its diagonal matrix elements, a positive type function \( G \ni g \mapsto \langle v | g \cdot v \rangle = \omega(g) \). The positive type functions are diagonal matrix elements of cyclic vectors.

Irreducible representations of locally compact groups are characterized by invariants, constituting the dual group space, for Hilbert representations the definite dual group space. The definite dual group space (invariants) comes with a Plancherel measure, uniquely associated to a Haar measure of the group. Locally compact noncompact nonabelian groups have also Hilbert representations with trivial Plancherel measure, e.g. the supplementary representations of the nonabelian Lorentz groups \( \text{SO}_0(1, s), s \geq 2 \).

### 4.2 Hilbert Representations of Abelian and Compact Groups

The Hilbert representations of the abelian Lie subgroups \( \text{D}(1) \cong \mathbb{R} \) and \( \text{U}(1) \) are basic for Hilbert representations of all real Lie groups.

The irreducible Hilbert representations of the abelian noncompact groups (‘translations’ of rank \( r \)) are \( \mathbb{R}^r \ni x \mapsto e^{ipx} \in \text{U}(1) \) (unfaithful). The 1-dimensional Hilbert spaces \( \mathbb{C}[p] \) are spanned by one normalized eigenvector
\[ \langle p|p \rangle = 1 \] with the translation behavior \( |p(x)\rangle = e^{i px} |p\rangle \) and the matrix element \( \langle p(y)|p(x)\rangle = e^{ip(x-y)} \). In physics, the eigenvalues \( p \in \mathbb{R}^r \) as linear invariants are used as energy-momenta. The combinations \( \{\cos px, \sin px\} \) are matrix elements of selfdual representations. There, the Hilbert space \( \mathbb{C}|p\rangle \oplus \mathbb{C}\langle p| \) is spanned by the dual basic vectors of the irreducible representations.

In physics, translation representations characterize free states, e.g. free scattering states in position space \( \mathbb{R}^3 \) or free particles in spacetime \( \mathbb{R}^4 \). This structure is familiar from the simplest example, the harmonic oscillator with frequency (energy) \( E \in \mathbb{R} \) where the creation operator gives the eigenvector \( u(E)|0\rangle = |E\rangle \) with the time translation action \( |E(t)\rangle = e^{iEt}|E\rangle \). The position and momentum operator \( x = \frac{u(E) + u^*(E)}{\sqrt{2}} \) and \( ip = \frac{u(E) - u^*(E)}{\sqrt{2}} \) are linear combinations of creation and annihilation operators and span the selfdual representation space with the time development of position-momentum \( \mathbb{R} \ni t \mapsto \left( \begin{array}{cc} \cos Et & i \sin Et \\ i \sin Et & \cos Et \end{array} \right) \in \text{SO}(2) \) involving \( \langle x(s)|x(t)\rangle = i \sin E(t-s) \) etc.

The Plancherel measure, associated to the Haar-Lebesque measure \( d^r x \) of the translations \( \mathbb{R}^r \), is the Haar-Lebesque measure \( d^r \frac{E^r}{\sqrt{\pi}} \) of the (energy-)momenta \( \mathbb{R}^r \) (dual group with linear invariants). It comes with Schur’s orthogonality[4] for the matrix elements of inequivalent representations by integrating with a Haar measure over the group

\[ \int d^r x \ e^{i px} e^{-ip'y} = \delta(p-p') \]

and is used for the harmonic analysis of the translation functions \( L^2(\mathbb{R}^r) \) (Fourier integrals).

If functions, acted upon with a representation of a space and time translation group \( x \in \mathbb{R}^r \), are Fourier transformable \( (f, \omega)(x) = \int \frac{d^r p}{(2\pi)^r} (f, \omega)(p) e^{-ipx}, \) a positive type function \( \omega \) for the scalar product

\[ \langle f_1|f_2\rangle_\omega = \int d^r x \ d^r x' \overline{f_1(x)} \omega(x' - x) f_2(x) = \int \frac{d^r p}{(2\pi)^r} \overline{f_1(p)} \omega(p) f_2(p) \]

is expressed with a positive distribution \( \omega \) for the dual group with energies and momenta \( p \in \mathbb{R}^r \).

The irreducible Hilbert representations \( e^{iz\varphi} \mapsto e^{iz\varphi} \) of the compact quotient \( \mathbb{R}/\mathbb{Z} \cong U(1) \cong \text{SO}(2) \), faithful for \( z \neq 0 \), are given with the winding numbers (linear invariants) constituting the discrete dual group \( z \in \mathbb{Z} \cong \mathbb{R}/U(1) \). A physical example are the electromagnetic charge numbers, integer multiples of one basic charge. Again, the 1-dimensional Hilbert spaces \( \mathbb{C}|z\rangle \) are spanned by one normalized eigenvector \( \langle z|z\rangle = 1 \). The Plancherel measure associated to the normalized Haar measure \( d\frac{z}{2\pi} \) of \( U(1) \) is the counting measure (dimension) \( d(z) = 1 \) of the winding numbers (dual group space). Schur’s orthogonality reads

\[ \int_0^{2\pi} d\frac{\varphi}{2\pi} e^{iz\varphi} e^{-iz'\varphi} = \delta_{zz'} \]

It is used for the harmonic analysis of the group functions \( L^2(U(1)) \) (Fourier series).

Compact groups of rank \( r \) have discrete eigenvalues (weights) in \( \mathbb{Z}^r \), the dual groups of \( U(1)^r \), e.g. (iso)spin \( SU(2) \) or color \( SU(3) \) or flavor groups \( SU(1+r) \). Extending \( U(1) \) by spherical degrees of freedom, the irreducible Hilbert representations of (iso)spin \( SU(2) \cong u \mapsto 2J(u) \) and its quotient
SO(3), e.g. for $J = 1$ with Euler angles

$$\text{SU}(2) \ni u = \begin{pmatrix} e^{i\frac{\varphi + \chi}{2} \cos \frac{\theta}{2}} & e^{-i\frac{\varphi - \chi}{2} \sin \frac{\theta}{2}} \\ ie^{i\frac{\varphi - \chi}{2} \sin \frac{\theta}{2}} & e^{-i\frac{\varphi + \chi}{2} \cos \frac{\theta}{2}} \end{pmatrix}$$

$$\mapsto 2(u) = \begin{pmatrix} e^{i(\varphi + \chi) \cos^2 \frac{\theta}{2}} & ie^{-i(\varphi - \chi) \sin^2 \frac{\theta}{2}} \\ ie^{i(\varphi - \chi) \sin^2 \frac{\theta}{2}} & e^{-i(\varphi + \chi) \cos^2 \frac{\theta}{2}} \end{pmatrix} \in \text{SO}(3) \subset \text{SU}(3)$$

are given with the invariants $2J \in \mathbb{N}$ (spins as dual group space) with scalar product $\langle J; a'| J; a \rangle = \delta_{a'a}$ (for spherical bases). The Plancherel counting measure $d(J) = 1 + 2J$ (dimension of the irreducible Hilbert spaces) associated to the normalized Haar measure $d^3u$ of $\text{SU}(2)$ can be read off Schur's orthogonality for the representation matrix elements

$$f_{\text{SU}(2)} d^3u 2J(u) \sum_b 2J(b) = \frac{1}{1+2J} \delta_{J',J} \delta_{a',a} \delta_{b'}^b$$

e.g. for $J = 1$: $f_{-2\pi} d\phi f_{2\pi} d\phi f_{0} d\theta \int_{-1}^{1} d\cos \theta |e^{i(\varphi + \chi)} \sin \frac{\theta}{2}|^2 = \frac{1}{3}$

The Plancherel measure $\sum_{2J=0}^{\infty} (1+2J)$ is used in the harmonic expansion of spin group functions $L^2(\text{SU}(2))$ as an example for the Peter-Weyl theorem[23] for compact groups.

### 4.3 Induced Representations

All Hilbert representations of a locally compact Lie group $G$ are inducible\(^\text{10}\) from those of its closed subgroups $\{ H \subseteq G \}$. The induced representations[3, 26, 21, 12, 4] act upon the vector space with the mappings $W^{G/H}$ from the $H$-classes of the group into a vector space $W$ with a Hilbert representation of the subgroup $H$

$$w : G/H \longrightarrow W, \ gH \longmapsto w(gH)$$

As discussed in the literature, the mappings $W^{G/H}$ have to be ‘appropriately’ defined with respect to ‘smoothness’ and measurability.

For finite dimension $W \cong \mathbb{C}^d$ the mappings can be expanded (decomposed) into a direct integral with a $G$-invariant measure $dgH$ for the classes, a $W$-basis $\{ e^a(gH) \}_{a=1, \ldots, d}$ for each coset $gH \in G/H$ and the function values as coefficients

$$W^{G/H} \ni w = \int dgH w(gH) e^{(gH)a}, \quad \text{dim} W^{G/H} = d \text{ card } G/H$$

also in bra-ket-notation, e.g. $|w\rangle$ and $|gH, a\rangle$, with the identity decomposition (sum over $a$)

$$\text{id}_{W^{G/H}} \cong \int dgH |gH, a\rangle \langle gH, a|$$

A positive measure has an associated Dirac distribution which, for an orthonormal $W$-basis, defines a scalar product distribution

$$\langle g'H, a'|gH, a\rangle = \delta_{a'\sigma} \delta(gH, g'H) \text{ where } \int dgH \delta(gH, g'H) f(g'H) = f(gH)$$

---

\(^\text{10}\)The basic structure for induced group actions is the group left action on subgroup right classes $G \times G/H \rightarrow G/H, (k, gH) \mapsto kgH$.\)
In general, the induced $G$-representations

$$G \times W^{G/H} \longrightarrow W^{G/H}, \quad (k, w) \mapsto k \cdot |w\rangle = \int dg \ w(kgH) |kgH, a\rangle$$

are highly decomposable, e.g. with Frobenius’ reciprocity theorem\[3\] for compact groups.

For practical purposes, convenient subgroups $H \subseteq G$ have to be chosen. E.g., the trivial subgroup $\{e\} \subseteq G$ induces the left regular $G$-representation on the complex group functions $\Phi^G$ which contains all $G$-representations. A representation of the full group ‘induces itself’ with $W^{G/G} \cong W$.

### 4.4 Residual Representations

The eigenvalues (weights) for eigenvectors in a Lie group representation are from a discrete or continuous spectrum. They are linear Lie algebra forms, e.g. the winding numbers $z \in \mathbb{Z} \subset \mathbb{R}$ for $U(1)$, the spin directions $2J_3 \in \mathbb{Z} \subset \mathbb{R}$ for $SU(2)$ or the energies $E \in \mathbb{R}$ for time translations $\mathbb{R}$ and the momenta $\vec{p} \in \mathbb{R}^3$ for position translations $\mathbb{R}^3$. The invariants arise from linear Lie algebra forms in the abelian case and from at least bilinear form in the nonabelian case, e.g. the spin Casimir $\vec{J}^2$ from the Killing form or the Euclidean invariant $\vec{p}^2$. A group acts on its Lie algebra and its forms in the adjoint and coadjoint representation. There exist formulations for representation matrix elements as residues of functions on complex Lie algebra forms\[28, 29, 30, 31\], e.g. for the abelian groups

$$U(1) \ni e^{imx} = \oint dp \frac{1}{2\pi} e^{ipx} = \int dp \ \delta(p - m) e^{ipx}, \ m \in \mathbb{R}$$

A not so trivial example is the residual representation of the matrix elements of the simple group $SU(2)$, involving the derived Dirac distribution, i.e. the derived 2-sphere measure, supported by the value for the invariant $\vec{p}^2 = n^2$

$$SU(2) \ni e^{in\vec{x}} = \oint dp \left( n \mathbf{1}_2 + \vec{p} \right) \delta'(n^2 - \vec{p}^2) e^{ip\vec{x}} = \mathbf{1}_2 \cos n |\vec{x}| + i \pi \sin n |\vec{x}| / |\vec{x}|$$

Euclidean $\mathbb{R}^n$-vectors are written with arrows, e.g. here with Pauli matrices

$$\vec{x} = x_a \sigma_a = \left( \begin{array}{c} x_3 \\ x_1 + ix_2 \\ -x_3 \end{array} \right) \in \mathbb{R}^3$$

The Dirac and principal value distributions are imaginary and real part in the associated complex distribution

$$a \in \mathbb{R}: \frac{\Gamma(1+N)}{(a+io)^{1+N}} = \frac{\Gamma(1+N)}{a^{1+N}} \pm i \pi \delta^{(N)}(-a), \ N = 0, 1, \ldots$$

(Derived) Dirac distributions will be also called (multi)pole distributions.

The distributions of Lie algebra forms show, on the one side, the representation characteristic invariants as complex singularities, e.g. as poles or as support of distributions, and, on the other side, the structure of the Hilbert space where the representation acts upon. The measures and distributions lead to distributions of Hilbert bases, to distributions of the scalar product and to functions on the Lie algebra forms as Hilbert space vectors (more below).
5 Simple Poles for Translations

Hilbert representations of affine subgroups $G \rtimes \mathbb{R}^n$ with a homogeneous group $G \subseteq \text{GL}(\mathbb{R}^n)$ acting on translations $\mathbb{R}^n$ - in the following for the semidirect 'basic physical Lie operations' $D(1) \rtimes \mathbb{R}$ and $\text{SO}(2) \rtimes \mathbb{R}^2$, $\text{SO}_0(1, 1) \rtimes \mathbb{R}^2$, $H(1) \cong \mathbb{R} \rtimes \mathbb{R}^2$

are inducible\cite{26, 21} from Hilbert representations of direct product subgroups $H \times \mathbb{R}^n$ which involve a fixgroup $H \subseteq G$ of the (energy-)momenta from the dual group $\mathbb{R}^n$. An appropriate measure for the homogeneous space $G/H$ is constructed from $G$-orbits of the (energy-)momenta. The homogeneous group restricts and collects the translation representations $x \mapsto e^{ipx}$ with the invariants. E.g. for the nonrelativistic scattering group $\text{SO}(3) \rtimes \mathbb{R}^3$, all translation characters $e^{ipx}$ are collected with the momentum square $\{ \vec{p} \in \mathbb{R}^3 \mid \vec{p}^2 = P^2 \}$.

5.1 The Affine Group in one Dimension - Causal Time

In the group $\left( \begin{array}{cc} 1 & -t \\ 0 & e^\psi \end{array} \right) \in D(1) \rtimes \mathbb{R}$, interpreted as dilations $D(1)$ acting upon time translations $\mathbb{R} \ni t \mapsto e^{\psi t}$, the energies are the eigenvalues for the time translations with the dual dilation action $\mathbb{R} \ni E \mapsto e^{-\psi E}$.

A Cartan subalgebra is spanned by the dilation generator with adjoint representation $\text{ad} l^1 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$ - for the translations $\text{ad} l^2 = \left( \begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array} \right)$. The Killing form of the Lie algebra $\text{tr ad} l^a \circ \text{ad} l^b \cong \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ is degenerate, there is no nontrivial invariant.

The Hilbert representations are inducable from the $D(1)$-action on the energies. For trivial energy $E = 0$ with full fixgroup $D(1)$ and, therefore, trivial $D(1)$-orbit there are the 1-dimensional representations $D(1) \ni e^\psi \mapsto e^{im\psi} \in U(1)$ given above. These representations are faithful only for $D(1) \rtimes \mathbb{R}/\mathbb{R} \cong D(1)$.

Nontrivial energies $E = \pm |E| \neq 0$, have trivial fixgroup (little group) and, therefore, $D(1)$-isomorphic orbits $\mathbb{R}_\pm = (0, \pm \infty)$ (either positive or negative energies). They lead to the two equivalence classes of faithful Hilbert representations, induced from the $U(1)$-representation of time translations $\mathbb{R} \ni t \mapsto e^{iEt}$. These representations are orbit-integrated with invariant energy-measure and the characteristic functions for positive and negative energies to give as matrix elements for the two irreducible representations

$$t \mapsto \pm i \int dE \, \vartheta(\pm E)e^{iEt} = \frac{1}{\pm i\pi} = \frac{1}{t\nu} \mp i\pi\delta(t) \text{ where } \left\{ \begin{array}{l} \vartheta(\pm E) = \frac{1\pm\epsilon(E)}{\epsilon(E)} \\ \epsilon(E) = \frac{|E|^2}{|E|} \end{array} \right.$$ 

Now, the Hilbert spaces for causal time representations: In contrast to the 1-dimensional irreducible representations $\mathbb{R} \ni t \mapsto e^{iEt} \in U(1)$ with only one eigenvalue and eigenvector $|E\rangle$ with $\langle E|E\rangle = 1$, the $D(1) \rtimes \mathbb{R}$ representations use either all positive or all negative energies. Here, the irreducible Hilbert spaces have infinite dimensions. Basis distributions (no Hilbert vectors) on
the $D(1)$-orbit for the two Hilbert spaces are given with an orthogonal and positive scalar product distribution

$$\{|E_\pm\} \mid E = \pm|E| \in \mathbb{R} \} \text{ for }$$

$$\left\{ \begin{array}{l}
\int \frac{dE}{2\pi} \vartheta(\pm E) |E_\pm\rangle \langle E_\pm| = \text{id}_{L^2(\mathbb{R}_\pm)} \\
|E_\pm\rangle \mapsto e^{iE_\pm t} |E_\pm\rangle \\
\langle E'_\pm| E_\pm\rangle = \vartheta(\pm E) \delta(E-E') \\
\langle E'_+| E_-\rangle = 0
\end{array} \right.$$ 

Vectors $|f_\pm\rangle$ from the Hilbert spaces with their scalar product use energy packets

$$|f_\pm\rangle = \int \frac{dE}{2\pi} f(E) |E_\pm\rangle \Rightarrow \langle f'_+| f_+\rangle = \int_0^\infty \frac{dE}{2\pi} \overline{f'_+(E)} f_+(E)$$

The two types of representations act upon functions $f_\pm \in L^2(\mathbb{R}_\pm)$, square integrable on the $D(1)$-energy orbits, i.e. on the positive and on the negative energies. The Hilbert product, written with time dependent functions, employs the advanced and retarded distribution

$$f_\pm(E) = \vartheta(\pm E) \int dt \bar{f}(t) e^{iEt}$$

$$\Rightarrow \langle f'_+| f_+\rangle = \int dt dt' \overline{f'(t')} \delta(t-t') \bar{f}(t)$$

The translation function scalar product $\int dt dt' \overline{f'(t')} \delta(t-t') \bar{f}(t)$ is restricted corresponding to the action of the homogeneous dilation group $D(1)$.

The dilation invariance of the scalar product distribution determines the dilation behavior of the basis distribution

$$E \mapsto e^{-\psi} E \Rightarrow \delta(E-E') \mapsto e^{\psi} \delta(E-E')$$

$$\Rightarrow \{E_\pm\} \mapsto e^{\pm \psi} \{E_\pm\}$$

5.2 The Flat Euclidean Group - Scattering in the Plane

In the flat Euclidean group nontrivial momenta have trivial fixgroup

$$SO(2) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow H_{p\neq \theta} = \{1\}$$

Their orbits are isomorphic to the homogeneous group $SO(2)$, i.e. to circles in the momentum plane. Therewith, the Hilbert spaces with faithful representations are orthogonal direct integrals over the 1-sphere with the normalized Haar measure

$$\text{for } SO(2) \quad \text{ with } P > 0 : \left\{ \begin{array}{l}
\int \frac{dp_1}{\pi} \delta(p^2 - P^2) = \int P \frac{dp}{\sqrt{P^2-p_1^2}} = \int_0^{2\pi} \frac{d\theta}{2\pi} \\
p^2 = p_1^2 + p_2^2, \quad \frac{p_2}{p_1} = \tan \theta
\end{array} \right.$$ 

The translation representations $\mathbb{R}^2 \ni \vec{x} \mapsto e^{i\vec{p} \cdot \vec{x}}$ have the momenta as eigenvalues with a positive invariant momentum squared $P^2$ for the Casimir
element characterizing an irreducible representation. The matrix elements for the irreducible representations

\[ P^2 > 0 : \vec{x} \mapsto \int \frac{d^2 p}{\pi} \delta(\vec{p}^2 - P^2) e^{i\vec{p}\cdot\vec{x}} = \mathcal{J}_0(P|\vec{x}|) \]

contain the integer index Bessel function [15] (appendix ‘Residual Distributions’)

\[ \mathbb{R} \ni \xi \mapsto \mathcal{J}_0(\xi) = \int_0^\pi \frac{d\theta}{\pi} \cos(\xi \cos \theta) = \sum_{k=0}^{\infty} (-\frac{\pi}{\xi})^k, \quad \mathcal{J}_0(0) = 1 \]

By derivatives \( \frac{\partial}{\partial \vec{p}} \) one obtains matrix elements for nontrivial representations of axial rotations \( \text{SO}(2) \).

The infinite dimensional Hilbert spaces with the irreducible representations for \( P^2 > 0 \) have a basis distribution of scattering ‘states’ in the plane \( \mathbb{R}^2 \) with the momenta on the \( \text{SO}(2) \)-orbit, i.e. on the circle with fixed momentum radius \( P \). They have the orthogonal and positive distribution of the scalar product

\[ \{|P^2; \theta\} | 0 \leq \theta < 2\pi \} \]

\[ \{ |P^2; \theta\} | 0 \leq \theta < 2\pi \} \quad \left\{ \begin{array}{l}
\int \frac{d\theta}{2\pi} |P^2; \theta\rangle \langle P^2; \theta| \approx \text{id}_{L^2(\text{SO}(2))} \\
|P^2; \theta\rangle \mapsto_{\mathbb{R}^2} e^{i\vec{p}\cdot\vec{x}} |P^2; \theta\rangle \\
\text{with } \vec{p} = P(\cos \theta, \sin \theta) \\
\langle P^2; \theta| |P^2; \theta\rangle = \delta\left(\frac{d^2\vec{p}}{2\pi}\right) \]

The Hilbert space vectors \( |P^2; f\rangle \) are square integrable wave packets \( f \in L^2(\Omega^1) \) on the momentum sphere

\[ |P^2; f\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta) |P^2; \theta\rangle = \int \frac{d^2 p}{\pi} \delta(\vec{p}^2 - P^2) f(\vec{p}) |\vec{p}\rangle \]

\[ \langle P^2; f'|P^2; f\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \overline{f'(\theta)} f(\theta) = \int \frac{d^2 p}{\pi} \overline{f'(\vec{p})} \delta(\vec{p}^2 - P^2) f(\vec{p}) \]

The Hilbert product in translation dependent functions \( f(\vec{p}) = \int d^2x \tilde{f}(\vec{x}) e^{i\vec{p}\cdot\vec{x}} \) employs the Bessel function which modifies the scalar product for translation functions \( \int \frac{d^2 x d^2 x'}{2} \overline{f(\vec{x})} \delta(\vec{x} - \vec{x'}) \tilde{f}(\vec{x}) \) according to the action of the homogeneous group \( \text{SO}(2) \)

\[ \langle P^2; f'|P^2; f\rangle = \int \frac{d^2 x d^2 x'}{2} \overline{f(\vec{x})} \mathcal{J}_0(P|\vec{x} - \vec{x'}|) \tilde{f}(\vec{x}) \]

The angle, momentum und translation dependent functions could be written with the same symbol \( (f(\theta), f(\vec{p}), f(\vec{x})) \) - the same function expanded in \( \theta, \vec{p} \) or in \( \vec{x} \). Somewhat inconsequently, the notation is \( (f(\theta), f(\vec{p}), \tilde{f}(\vec{x})) \).

Schur’s orthogonality for the square integrable representation matrix elements involves the integration over the translations, e.g.

\[ \int d^2x \mathcal{J}_0(P|\vec{x}|) \mathcal{J}_0(P'|\vec{x}|) = 4\pi \delta(P^2 - P'^2) \]

This replaces, in 2-dimensional position space (in general in even dimensional position), the Huygen-Fresnel principle with spherical Bessel functions, e.g. \( j_0(P|\vec{x}|) = \frac{\sin(P|x|)}{P|x|} = \int \frac{d^2\vec{p}}{2\pi} \frac{1}{|\vec{p}|} \delta(\vec{p}^2 - P^2) e^{i\vec{p}\cdot\vec{x}} \) of 3-dimensional (in general odd dimensional) position \( \text{SO}(3) \times \mathbb{R}^3 \).
5.3 The Flat Poincaré Group
- Spinless Free Particles

Also in the flat Poincaré group nontrivial energy-momenta have trivial fixgroup

\[ \text{SO}_0(1, 1) \cong \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \begin{pmatrix} \cosh \psi & -\sinh \psi \\ -\sinh \psi & \cosh \psi \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \Rightarrow H_{q \neq 0} = \{1\} \]

The orbits are - up to for the lightcone energy-momenta \( q^2 = 0 \) - isomorphic to hyperbolas \( \text{SO}_0(1, 1) \cong \mathcal{Y}^1 \) in the energy-momentum plane. Therewith, the Hilbert spaces with faithful representations are orthogonal direct integrals with Haar measure

\[ \int \mathbb{R}^2 q_0 \mathbb{R}_2 q_1 \frac{2}{\pi} = \int_0^\infty \frac{dq_0}{2\pi \sqrt{q_1^2 + m^2}} = \int_0^\infty \frac{d\psi}{2\pi} \]

for \( \text{SO}_0(1, 1) \):

\[ \begin{aligned}
q^2 &= q_0^2 - q_1^2, \\
\frac{q_1}{q_0} &= \tanh \psi
\end{aligned} \]

The measure can be parametrized with momenta as familiar from particle quantum fields in 4-dimensional Minkowski spacetime or with a hyperbolic ‘angle’.

For 2-dimensional Minkowski spacetime, there is an obvious isomorphy between spacelike \( y^2 < 0 \) and timelike \( y^2 > 0 \) with a timelike and a spacelike order structure.

The translation representation matrix elements have the energy-momenta as eigenvalues with positive and negative invariant (‘mass’) for the Casimir element \( Q^2 \). The matrix elements for the two types of inequivalent irreducible representations \( \{ \pm m^2 \mid m^2 > 0 \} \) with nontrivial invariant

\[ m^2 > 0 : \ y \mapsto \int \frac{d^2 q}{\pi} \delta(q^2 \mp m^2)e^{i\varphi} = \begin{aligned}
\vartheta(\mp y^2)^2 \frac{2}{\pi} &\text{K}_0(|my|) - \vartheta(\mp y^2)\mathcal{N}_0(|my|) \\
\text{with } |y| = \sqrt{y^2} \geq 0
\end{aligned} \]

come with the order 0 Macdonald function

\[ \text{Re } \xi \mapsto 2\mathcal{K}_0(\xi) = \int d\psi \ e^{-|\xi|\cosh \psi} = -\sum_{n=0}^\infty \frac{e^{\xi^2/n}}{(n!)^2} \left[ \log \frac{\xi}{\sqrt{\xi^2}} - 2\Gamma'(1) - 2\varphi(n) \right] \]

\[ \varphi(0) = 0, \ \varphi(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \ n = 1, 2, \ldots \]
\[ -\Gamma'(1) = \lim_{n \rightarrow \infty} [\varphi(n) - \log n] = 0.5772 \ldots \]

which - for imaginary argument - is the Neumann function \( \mathcal{N}_0 \) with the Bessel function \( \mathcal{J}_0 \)

\[ 2\mathcal{K}_0(i\xi) = \int d\psi \ e^{i|\xi|\cosh \psi} = -\pi \mathcal{N}_0(\xi) + i\pi \mathcal{J}_0(\xi) \]

By derivatives \( \frac{\partial}{\partial x} \) one obtains representation matrix elements with nontrivial boost (dilation) \( \text{SO}_0(1, 1) \)-properties.

The representation matrix elements for positive translation invariant are familiar as the on-shell part of the Feynman propagator for massive particles

\[ \frac{1}{\pi} \frac{1}{q^2 - m^2} = \delta(q^2 - m^2) + \frac{1}{\pi} \frac{1}{q_0^2 - m^2} \]
Only the on-shell part gives a Hilbert representation matrix element of the Poincaré group \( \text{SO}_0(1, s) \times \mathbb{R}^{1+s} \).

The Hilbert spaces with the irreducible representations - have an orthogonal and positive basis distribution of free particles with energy-momenta \( q = (\sqrt{m^2 + q_1^2}, q_1) \) on the two-shell hyperbola of mass \( \pm m^2 \neq 0 \)

\[
\begin{align*}
\{ |m^2; q_1 \rangle \mid q_1 \in \mathbb{R} \} & \quad \text{with} \quad \langle m^2; q'_1 | m^2; q_1 \rangle = \sqrt{m^2 + q_1^2} \delta \left( \frac{q_1 - q'_1}{2\pi} \right) \\
\{ |-m^2; q_0 \rangle \mid q_0 \in \mathbb{R} \} & \quad \text{with} \quad \langle -m^2; q'_0 | -m^2; q_0 \rangle = \sqrt{m^2 + q_0^2} \delta \left( \frac{q_0 - q'_0}{2\pi} \right)
\end{align*}
\]

From now on, explicitly only for

\[
\begin{align*}
\{ |m^2; \psi \rangle \mid \psi \in \mathbb{R} \} & \quad \text{with} \quad \int \frac{d\psi}{2\pi} |m^2; \psi \rangle \langle m^2; \psi| = \text{id}_{L^2(\text{SO}(1,1))} \\
|m^2; \psi \rangle & \quad \text{with} \quad \int \frac{d\psi}{2\pi} e^{iqy} |m^2; \psi \rangle = \delta(q - m)\psi_0\sqrt{m} \\
\langle m^2; \psi' | m^2; \psi \rangle & = \delta(\psi - \psi')
\end{align*}
\]

The square integrable Hilbert space vectors are wave packets \( f \in L^2(\text{SO}(1,1)) \) of momenta on the mass hyperboloid \( \mathcal{H} = \mathcal{H}_+ \cup \mathcal{H}_- \cong \text{SO}(1,1) \)

\[
\begin{align*}
|m^2; f \rangle &= \int \frac{d\psi}{2\pi} f(\psi) |m^2; \psi \rangle = \int \frac{d\psi}{2\pi} \delta(q^2 - m^2) f(q) |q\rangle \\
\langle m^2; f' | m^2; f \rangle &= \int \frac{d\psi}{2\pi} f'(\psi) f(\psi) = \int \frac{d\psi}{2\pi} \delta(q^2 - m^2) f(q) \langle q | q \rangle
\end{align*}
\]

For spacetime translation dependent functions \( f(q) = \int d^2y \tilde{f}(y)e^{iqy} \), the Hilbert product is restricted according to the homogeneous action with the orthochronous Lorentz group by the change of \( \delta(y - y') \) into the combination of Macdonald and Neumann function

\[
\langle m^2; f' | m^2; f \rangle = \int d^2y \int \frac{d^2q}{\pi} \int \frac{d^2q'}{\pi} \delta(q^2 - m^2) \delta(q'^2 - m'^2) \tilde{f}(y) \left( \vartheta(z^2) \mathcal{K}_0(|mz|) - \vartheta(z^2) \mathcal{N}_0(|mz|) \right) \]

If, for Schur’s orthogonality with different invariants, the integration is performed over the translations there remains the infinite measure of the hyperboloid

\[
\int d^2y \int \frac{d^2q}{\pi} \delta(q^2 - m^2) \int d^2q' \delta(q'^2 - m'^2) e^{iq'y} = \delta(m^2 - m'^2)4 \int d^2q \delta(q^2 - 1)
\]

The representation matrix elements are not square integrable.

### 5.4 The Heisenberg Group

- Nonrelativistic Quantum Mechanics

In the Heisenberg group as semidirect product \( \mathbf{H}(1) = \mathbb{R} \ltimes \mathbb{R}^2 \) the homogeneous group \( e^{i\alpha} \in \mathbb{R} \) with the position \( x \cong \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) acts on the abelian normal subgroup \( \mathbb{R}^2 \) with momentum \( p \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and the central action operator

\[
\mathbf{I} = [x, p] \cong \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\mathbb{R} \ltimes \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ y \end{pmatrix} = \begin{pmatrix} t + qy \\ y \end{pmatrix}
\]
I generates the invariants.

The position action upon the dual space \((h, p) \in \mathbb{R}^2\) with \((ih, ip)\) the eigenvalues for the action of \((I, p)\)

\[
\langle (h, p), \begin{pmatrix} t & y \\ y & 1 \end{pmatrix} \rangle = th + yp, \quad (h, p) \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} = (h, p - hq)
\]

has two types of fixgroups with corresponding orbits: The fixgroups are characterized either by trivial or by nontrivial eigenvalue \(h \in \mathbb{R}\)

\(h = 0, p\) has full fixgroup \(\mathbb{R}\) and point orbit \(\{(0, p)\} \cong \{1\}\)

\(h \neq 0, p\) has trivial fixgroup \(\{1\}\) and line orbit \((h, \mathbb{R}) \cong \mathbb{R}\)

Correspondingly, there are two types of representations (Stone-von Neumann theorem[13]): The 1st type with trivial representations of the central action operator \(I \in \text{centr} H(1)\), i.e. invariant \(h = 0\), leads to classical unfaithful representations of the Heisenberg group with commuting position and momentum, i.e. of the abelian adjoint Heisenberg group \(\text{Int} H(1) = H(1)/\text{centr} H(1) \cong \mathbb{R}^2\). The Hilbert representations of \(\mathbb{R}^2\) are given above.

The 2nd type with trivial fixgroup and a nontrivial \(I\)-eigenvalue \(ih\) (there is a spectrum of \(0 \neq h \in \mathbb{R}\)) induces the quantum representations of the Heisenberg group. Different action quanta \(h \neq h'\) define inequivalent representations with \(I \mapsto ihI\). These irreducible faithful representations integrate the irreducible momentum \(p\)-representations \(\mathbb{R} \ni y \mapsto e^{ipy} \in U(1)\) for all momenta eigenvalues on the orbit line \(p \in \mathbb{R}\) with orthogonal and positive scalar product distribution

\[
|\langle h; p \rangle| \in \mathbb{R}\) with 

\[
\begin{align*}
\int_{2\pi} dp |\langle h; p \rangle| & \cong \text{id}_{L^2(\mathbb{R})} \\
|\langle h; p \rangle| & \mapsto e^{ipy}|\langle h; p \rangle| \\
|\langle h; p'|\langle h; p \rangle| & = \delta_{r(\frac{p-p'}{2\pi})}
\end{align*}
\]

The Hilbert spaces consist of the square integrable momentum functions \(f \in L^2(\mathbb{R})\) which are isomorphic to the square integrable position functions \(\tilde{f}(y) = \int dp \ f(p)e^{ipy}\)

\[
|\langle h; f \rangle| = \int_{2\pi} dp \ f(p)|\langle h; p \rangle| \\
\Rightarrow |\langle h; f'|\langle h; f \rangle| = \int_{2\pi} dp \ f'(p)f(p) = \int dy \ \tilde{f}'(y)\tilde{f}(y)
\]

The action of the Lie algebra momentum operator is given by the derivative \(p \mapsto -ih\frac{d}{dy}\).

A harmonic analysis of functions on the Heisenberg group \(H(n)\) uses the classical Fourier components \(|0; f\rangle\) with trivial Plancherel measure and the quantum components \(|h; f\rangle\) with Plancherel measure[4] \(|h|^n dh\) for the invariant values of \(I\).

### 6 Dipoles for Simple Groups

In contrast to the representations of abelian groups (translations) \(\mathbb{R}^n\) with pole singularities (Dirac distributions), simple groups use higher order poles...
(derived Dirac distributions). This will be exemplified for the simplest simple compact and noncompact Lie groups, the ‘basic Lie operations’ SU(2) and SU(1, 1) with rank 1, twofold covering SO(3) and SO_0(1, 2), the smallest nonabelian Lorentz group for an odd dimensional spacetime SO_0(1, 2R), R = 1, 2, ... Here everything is explicitly known. For the noncompact group SL(R^2) ∼ SO_0(1, 2), the definite group dual has been given by Bargman[17, 7].

In the usual procedure (more detailed formulations in the literature) the group SL(R^2) ∼ SU(1, 1) is used in the defining representation on real 2-vectors and their component ratio \((\xi_1 \xi_2) \mapsto \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)(\xi_1 \xi_2) \Rightarrow \xi = \frac{\xi_1}{\xi_2} \mapsto \frac{a \xi_1 + b \xi_2}{c \xi_1 + d \xi_2}\)

The SL(R^2)-transformation behavior of complex functions of the ratio \(\xi \mapsto F(\xi)\), induced by a representation of a Cartan subgroup SO(2) or SO_0(1, 1) is given by \((\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \cdot F)(\xi) = \eta_{\mu}(c \xi + d)^{-1+\mu}F\left(\frac{a \xi + b}{c \xi + d}\right)\)

where the power \(\mu \in \mathbb{C}\) in the overall factor is the Cartan subgroup representation characterizing invariant and \(\eta_{\mu}\) a representation dependent sign factor. The Cartan subgroup representation determines the homogeneity property of the functions in one irreducible representation.

In the following, an orientation for SU(2)-Hilbert representations - as a warm up - and for SU(1, 1)-Hilbert representations is given by applying the inducing procedure and residual representations in a spacetime and energy-momentum oriented language. For the noncompact simple group there arise both square integrable Hilbert spaces and Hilbert spaces defined with positive type functions.

6.1 Multipoles for the Nonrelativistic Hydrogen Atom

A familiar example of higher order poles for simple Lie operations, discussed in more detail in[32], are the Schrödinger bound states functions for the nonrelativistic hydrogen atom with Hamiltonian \(\frac{\mathbf{p}^2}{2} - \frac{1}{|\mathbf{x}|}\). The related Lenz-Runge invariance (perihel conservation)[18] leads to the action group SO(4) with the rotation group classes the 3-sphere \(\Omega^3 \cong SO(4)/SO(3)\). The \(\Omega^3\)-measure has a momentum parametrization by dipoles

\[
\mathbb{R}^4 \ni (\frac{\cos \varphi}{\sin \varphi}) = \frac{1}{\sqrt{p^2 + 1}}(\begin{array}{c} 1 \\ \varphi \end{array}) \quad \int d\Omega^3 = \int d^3p \frac{2P}{(p^2 + P^2)^{\frac{3}{2}}} = 2\pi^2
\]

with the imaginary ‘momentum’ invariant \(\vec{p}^2 = -P^2\). Its Fourier transform is the hydrogen ground state \(\int d\Omega^3 e^{-i\vec{p}\vec{x}} = e^{-P^2}\) with binding energy \(-2E = P^2 = 1\) for principal quantum number \(k = 1\). It is a representation matrix ele

ent of nonrelativistic position as symmetric space \(Y^1 \times \Omega^2 \cong \mathbb{R}^4\). The \(\Omega^3\)-spherical harmonics lead to higher order poles for the SO(4)-multiplets. E.g., the position representation matrix elements in the quartet \(k = 2\) (s- and p-states) come with normalized SO(4)-vectors

\[
e(\vec{p}) = \frac{1}{p^2 + 1}(\begin{array}{c} \vec{p}^2 + 1 \\ 2i\varphi \end{array}) \in \mathbb{R}^4, \quad \langle e(\vec{p})|e(\vec{p})\rangle = 1
\]
leading to 3rd order poles
\[ \int \frac{d\Omega^3}{2\pi^2} e^{i\vec{p} \cdot \vec{x}} = \int \frac{d^d \vec{p}}{(2\pi)^d} \frac{p}{(p^2 + P^2)^3} \left( \frac{p^2 - P^2}{2i P \vec{p}} \right) e^{-i\vec{p} \cdot \vec{x}} = \frac{1}{2} \left( \frac{1 - P^2}{p^2} \right) e^{-P \vec{x}} \]

In general, there are order \((1 + k)\)-poles with power \((k - 1)\)-tensors \(e(\vec{p})^{k-1}\) for bound states with energy \(-2E = P^2 = \frac{1}{\xi^2}\) and \(k = 1, 2, \ldots\)

### 6.2 3-Space and (1,2)-Spacetime as Lie Algebras

The groups \(SU(2)\) and \(SU(1, 1)\) in the defining representation by \((2 \times 2)\)-matrices \(W \otimes W^T \cong \mathbb{C}^2 \otimes \mathbb{C}^2\) have a Lie algebra parametrization almost everywhere

\[
\begin{align*}
SU(2) & \ni e^{i\vec{x}} = \mathbf{1}_2 \cos |\vec{x}| + i\vec{x} \sin |\vec{x}| \\
SU(1, 1) & \ni e^{i\vec{y}} = \vartheta(y^2) \left[\mathbf{1}_2 \cos |y| + i\vec{y} \sin |y|\right] + \vartheta(-y^2) \left[\mathbf{1}_2 \cosh |y| + \frac{i\vec{y}}{|\vec{y}|} \sinh |y|\right]
\end{align*}
\]

using a 3-dimensional Euclidean space \(\vec{x} \in i \log SU(2)\) and \((1, 2)\)-spacetime \(y \in i \log SU(1, 1)\)

\[
\begin{align*}
i\vec{x} & = ix_3\sigma_3 + ix_1\sigma_1 + ix_2\sigma_2 = \begin{pmatrix} ix_3 \\ ix_1 - x_2 \\ -ix_3 \end{pmatrix} \in \log SU(2) \cong \mathbb{R}^3 \\
\text{invariant } \vec{x}^2 & = \frac{1}{2} \text{tr} \vec{x} \circ \vec{x} = \det i\vec{x} = x_3^2 + x_1^2 + x_2^2 \\
i\vec{y} & = iy_0\sigma_3 + iy_1\sigma_1 + iy_2\sigma_2 = \begin{pmatrix} iy_0 \\ iy_1 + iy_2 \\ -iy_0 \end{pmatrix} \in \log SU(1, 1) \cong \mathbb{R}^{1,2} \\
\text{invariant } y^2 & = \frac{1}{2} \text{tr} \vec{y} \circ \vec{y} = \det iy = y_0^2 - y_1^2 - y_2^2 \\
|y| & = \sqrt{y^2}
\end{align*}
\]

The invariants are generated by the quadratic Killing invariant which gives the Euclidean and Lorentz metric. The compact parameter space is restricted to \(\vec{x}^2 < (2\pi)^2\) for \(SU(2)\) and to the timelike ‘subhyperboloid’ \(\vartheta(y^2)y^2 < (2\pi)^2\) for \(SU(1, 1)\).

The group acts adjointly on its Lie algebra \(\vec{x} \mapsto u \circ \vec{x} \circ u^{-1}\) for \(u \in SU(2)\) and \(y \mapsto v \circ y \circ v^{-1}\) for \(v \in SU(1, 1)\). Obviously in this case, the group transformations \(u\) and \(v\) cannot be parametrized with \(\vec{x}\) and \(y\). E.g. group parametrizations with independent Euler ‘angles’ can be used

\[
\begin{align*}
u(\varphi, \psi, \chi) & = \begin{pmatrix} e^{i\varphi/2} \cos \psi/2 & e^{-i\varphi/2} \sin \psi/2 \\ e^{i\varphi/2} \sin \psi/2 & e^{-i\varphi/2} \cos \psi/2 \end{pmatrix} = e^{i\varphi/2} e^{i\psi/2} e^{i\chi/2} \in SU(2) \\
u(\varphi, \psi, \chi) & = \begin{pmatrix} e^{i\varphi/2} \cosh \psi/2 & e^{-i\varphi/2} \sinh \psi/2 \\ e^{-i\varphi/2} \sinh \psi/2 & e^{i\varphi/2} \cosh \psi/2 \end{pmatrix} = e^{i\varphi/2} e^{i\psi/2} e^{i\chi/2} \in SU(1, 1)
\end{align*}
\]

The dual vector spaces, i.e. the linear forms of the Lie algebras, are parametrized by momenta and by energy-momenta

\[
\vec{p} \in \mathbb{R}^3 \cong (\log SU(2))^T, \quad q \in \mathbb{R}^3 \cong (\log SU(1, 1))^T
\]

both with the corresponding coadjoint action. The nontrivial \(SU(2)\)-momenta, e.g. \(\sigma_3\), have a spherical orbit parametrization (polar coordinates) where
the fixgroup $e^{i\hat{\sigma}_3}\in SO(2)$ parameter drops out

$$SU(2) : \quad u(\varphi, \theta, \chi) \circ \sigma_3 \circ u(\varphi, \theta, \chi)^{-1} = \left( \begin{array}{cc}
\cos \theta & -ie^{i\varphi} \sin \theta \\
 ie^{-i\varphi} \sin \theta & -\cos \theta \end{array} \right)$$

$$p^2 > 0 \Rightarrow \left( \begin{array}{c} p_3 \\
p_1 \\
p_2 \end{array} \right) = \sqrt{p^2} \left( \begin{array}{c} \sin \theta \sin \varphi \\
\sin \theta \cos \varphi \end{array} \right)$$

For the energylike $SU(1, 1)$-energy-momenta, e.g. $\sigma_3$ with fixgroup $e^{i\hat{\sigma}_3}\in SO(2)$, there are the corresponding hyperbolic orbit (‘polar’) coordinates

$$SU(1, 1) : \quad u(\varphi, \psi, \chi) \circ \sigma_3 \circ u(\varphi, \psi, \chi)^{-1} = \left( \begin{array}{cc}
cosh \psi & -e^{i\varphi} \sinh \psi \\
e^{-i\varphi} \sinh \psi & \cosh \psi \end{array} \right)$$

$$q^2 > 0 \Rightarrow \left( \begin{array}{c} q_0 \\
q_1 \\
q_2 \end{array} \right) = \sqrt{q^2} \left( \begin{array}{c} \cosh \psi \\
\sinh \psi \sin \varphi \\
\sinh \psi \cos \varphi \end{array} \right)$$

For the momentumlike energy-momenta, e.g. $i\sigma_1$ with fixgroup $e^{i\hat{\sigma}_1}\in SO_0(1, 1)$, another parametrization with two noncompact parameters is appropriate

$$w(\varphi, \psi, \xi) = e^{i\hat{\sigma}_3}e^{i\hat{\sigma}_2}e^{i\hat{\sigma}_1}\in SU(1, 1)$$

with the hyperbolic orbit (‘polar’) coordinates

$$SU(1, 1) : \quad w(\varphi, \psi, \xi) \circ i\sigma_1 \circ w(\varphi, \psi, \xi)^{-1} = \left( \begin{array}{cc}
sinh \psi & ie^{i\varphi} \cosh \psi \\
 e^{-i\varphi} \cosh \psi & -\sinh \psi \end{array} \right)$$

$$q^2 < 0 \Rightarrow \left( \begin{array}{c} q_0 \\
q_1 \\
q_2 \end{array} \right) = \sqrt{-q^2} \left( \begin{array}{c} \cosh \psi \\
\sinh \psi \cos \varphi \\
\sinh \psi \sin \varphi \end{array} \right)$$

The semidirect groups $SO(1+s) \times \mathbb{R}^{1+s}$ and $SO_0(1, s) \times \mathbb{R}^{1+s}$ describe the adjoint action of the homogeneous groups on their Lie algebras for $(\frac{1+s}{2}) = 1+s$, i.e. only for $1+s = 3$. For $1+s = 3$, the Pauli spinor representation is equivalent to its conjugated one, i.e. for group $g \cong \hat{g} = g^{-1*}$ and for the Lie algebra $I \cong \hat{l} = -l^*$

for $\log SU(2) : \quad -(ix)^* = ix$

for $\log SU(1, 1) : \quad -(iy)^* = iy_0\sigma_3 - y_1\sigma_1 - y_2\sigma_2 = \sigma_3 \circ iy \circ \sigma_3$

The orthogonal structures for $1+s = 3$ are embedded into, but not trivially generalizable to higher dimensions. E.g., even dimensional nonabelian orthogonal structures, starting with the rank 2 proper Lorentz group $SL(\mathbb{Q}^2) \sim SO_0(1, 3)$, have two inequivalent left and right handed Weyl spinor representations, for the Lorentz group $g = e^{i\bar{\alpha} + \bar{\beta}}$ and $\hat{g} = e^{i\bar{\alpha} - \bar{\beta}}$. The Hilbert representations of $SL(\mathbb{Q}^2) \sim SO_0(1, 3)$ have been given by Gel’fand and Naimark[8, 14, 7].

### 6.3 Hilbert Representations of $SU(2)$ and $SU(1, 1)$

The two types of maximal abelian subgroups (Cartan subgroups) in $SU(1, 1)$ and the corresponding classes are parametrizable by time and position trans-
The quadratic Killing invariants $\mu^2 \in \mathbb{R}$ are called polarization $n^2 = z^2$ and momentumlike invariant $P^2 = -q^2$.

The measures for the compact classes $\text{SO}(1+s)/\text{SO}(s) \cong \Omega^s$ (spheres) and for the noncompact ones $\text{SO}_0(1,s)/\text{SO}(s) \cong \Upsilon^s$ (energylike hyperboloids) and $\text{SO}_0(1,s)/\text{SO}_0(1,s-1) \cong \Upsilon^{1,1} \times \Omega^{-1} = \Upsilon^{(1,s-1)}$ (momentumlike hyperboloids) are

\[
\begin{align*}
\int d\Omega^s &= \int d^{1+s}p \, \delta(p^2 - 1) = \int_0^{\frac{\pi}{2}} (\sin \theta)^{s-1}d\theta \int d\Omega^{s-1} \\
\int d\Upsilon^s &= \int d^{1+s}q \, \delta(\pm q_0)\delta(q^2 - 1) = \int_0^{\frac{\pi}{2}} (\sinh \psi)^{s-1}d\psi \int d\Omega^{s-1} \\
\int d\Upsilon^{(1,s-1)} &= \int d^{1+s}q \, \delta(q^2 + 1) = \int_0^{\frac{\pi}{2}} (\cosh \psi)^{s-1}d\psi \int d\Omega^{s-1}
\end{align*}
\]

The distributions and residues for one dimension

\[
\int \frac{dq}{i\pi} \frac{1}{q^2 - io - 1} e^{iqt} = e^{|t|}, \quad \int \frac{dq}{\pi} \frac{1}{q^2 + 1} e^{iqz} = e^{-|z|}
\]

are both embedded in the two 3-dimensional distributions with energylike and momentumlike invariant as poles whose Dirac contributions are the measures of $\Omega^2$, $\mathcal{Y}^2_2$ and $\mathcal{Y}^{(1,1)}$

\[
\begin{align*}
\text{SU}(2) : & \quad \int \frac{d^3p}{2\pi^2} \frac{1}{p^2 - io - 1} e^{ip\vec{x}} = \frac{e^{i|\vec{x}|}}{|\vec{x}|} \\
\text{SU}(1,1) : & \quad \begin{cases} \\
\int \frac{d^3q}{2\pi^2} \frac{1}{q^2 - io - 1} e^{iqy} = \frac{\vartheta(-y^2)e^{-|y|}}{|y|} - \vartheta(y^2)e^{-|y|}e^{i|y|} \\
\int \frac{d^3q}{2\pi^2} \frac{1}{q^2 + io + 1} e^{iqy} = \frac{\vartheta(-y^2)e^{-|y|}}{|y|} + i\vartheta(y^2)e^{-|y|}e^{i|y|}
\end{cases}
\end{align*}
\]

The order structure of spacetime $\mathbb{R}^{1+2}$ is encoded in the two causal distributions, the advanced and retarded one

\[
\begin{align*}
\frac{1}{(q \pm io)^2 - 1} &= \frac{1}{q_s^2 - 1} \pm i\pi \varepsilon(q_0)\delta(q^2 - 1) \\
\text{with} \quad (q \pm io)^2 &= (q_0 \mp io)^2 - q_s^2 \\
\frac{1}{q^2 + io - 1} &= \frac{1}{q_s^2 - 1} \pm i\pi \delta(q^2 - 1)
\end{align*}
\]
The sum of advanced and retarded distribution coincides with the sum of the two conjugated Feynman distributions. The Fourier transformation of advanced and retarded distributions are supported by future and past

\[
\int \frac{d^{1+q}}{\pi} \frac{1}{q_{\pm}^{q-1}} e^{iqy} = i\epsilon(y_0) \int d^{1+q} q \epsilon(q_0) \delta(1 - q^2) e^{iqy}
\]

\[
\int \frac{d^{1+q}}{\pi} \frac{1}{(q_{\pm}^{q-1})^2} e^{iqy} = \pm i \vartheta(\pm y_0) \int d^{1+q} q \epsilon(q_0) \delta(1 - q^2) e^{iqy}
\]

\[
= \vartheta(\pm y_0) \int \frac{d^{1+q}}{\pi} \frac{1}{q_{\pm}^{q-1}} e^{iqy}
\]

The Feynman distributions are compatible with the action of any orthogonal group \(O(p, q)\) with bilinear form \(q^2\), the causal distributions only with the action of the orthochronous Lorentz groups \(SO_0(1, s)\).

One obtains for time and \((1, 2)\)-spacetime

\[
\ref{su11} \int \frac{dq}{2\pi} \frac{1}{(q_{\pm}^{q-1})^2} e^{iqy} = -\vartheta(t) \sin |t|
\]

\[
SU(1, 1) : \int \frac{d^2q}{4\pi^2} \frac{1}{(q_{\pm}^{q-1})^2} e^{iqy} = -\vartheta(y^2) \vartheta(y_0) \cos |y|
\]

The \(SU(2)\)-representations are inducible from Cartan subgroup representations with winding numbers (integer powers) \(SO(2) \ni e^{i\pi \sigma_3} \sim (e^{i\pi \sigma_3})^{\pm n} \in SO(2)\) and the decomposition \(SU(2)/\{\pm 1\} \cong SO(3) \cong SO(2) \times \Omega^2\). The matrix elements for spin \(J\) arise from dipole distributions, supported by the invariant

\[
\text{SU}(2) \ni e^{i\vec{x}} \longrightarrow \pm \int \frac{d^2y}{2\pi} \frac{n}{(r^{2}+|n-2|^2)} e^{i\vec{p} \vec{x}} = e^{\pm |n| \vec{x}}
\]

\[
\int \frac{d^2y}{2\pi} n \delta'(n^2 - \vec{p}^2) e^{i\vec{p} \vec{x}} = \cos n |\vec{x}|
\]

\[
\int \frac{d^2q}{2\pi} \vec{p} \delta'(n^2 - \vec{p}^2) e^{i\vec{p} \vec{x}} = \frac{\vec{x}}{|\vec{x}|} \sin n |\vec{x}|
\]

Matrix elements with nontrivial properties with respect to the rotation classes \(SU(2)/SO(2)\) use derivations \(\frac{\partial}{\partial \vec{x}} = \frac{\vec{x}}{|\vec{x}|} \frac{\partial}{\partial |\vec{x}|}\).

For the \(SU(1, 1)\)-representations, there are six dipole distributions (three pairs with \(\pm i\sigma\)). They involve the Cartan subgroup representation matrix elements \([e^{i|\vec{y}|}, e^{-|\vec{y}|}]\) for the invariant \(|\vec{y}| = \sqrt{|y^2|}\). There are two pairs of \(SU(1, 1)\)-representation types: The discrete ‘energylike’ polarization invariants give the two causally supported series, induced by compact Cartan subgroup representations \(SO(2) \ni e^{i\pi \sigma_3} \sim (e^{i\pi \sigma_3})^{\pm n} \in SO(2)\) and the decomposition \(SU(1, 1)/\{\pm 1\} \cong SO_0(1, 2) \cong SO(2) \times \mathbb{Y}_1^\pm\)

\[
\text{SU}(1, 1) \ni e^{i\vec{y}} \longrightarrow \pm \int \frac{d^2y}{2\pi} \frac{n}{(r^{2}+|n-2|^2)} e^{i\vec{p} \vec{x}} = e^{\pm |n| \vec{x}}
\]

\[
\int \frac{d^2y}{2\pi} n \delta'(n^2 - \vec{p}^2) e^{i\vec{p} \vec{x}} = \cos n |\vec{x}|
\]

\[
\int \frac{d^2q}{2\pi} \vec{p} \delta'(n^2 - \vec{p}^2) e^{i\vec{p} \vec{x}} = \frac{\vec{x}}{|\vec{x}|} \sin n |\vec{x}|
\]

The lightlike invariant lead to the ‘mock’ series \(0_\pm\). The two principal\(^{11}\) series are induced by noncompact Cartan subgroup representations \(SO_0(1, 1) \ni e^{i\pi \sigma_3} \sim (e^{i\pi \sigma_3})^{\pm i [P]} \in SO(2)\) with imaginary eigenvalues and a momentumlike

\(^{11}\)Principal’ in principal series and principal value are not related to each other
invariant and the decomposition $\text{SU}(1, 1)/\{\pm 1_2\} \cong \text{SO}_0(1, 2) \cong \text{SO}_0(1, 1) \times \mathcal{Y}^{(1,1)}$

\[
\text{SU}(1, 1) \ni e^{iy} \mapsto \begin{cases} P^2(y) = -\int \frac{d^3q}{\pi^2} \frac{\omega}{|q^2 + i\omega + p^2|} e^{iqy} \\
\text{principal: } P^2 > 0 \end{cases}
\]

In addition, there is the causally supported supplementary series with principal value integration

\[
\text{SU}(1, 1) \ni e^{iy} \mapsto \begin{cases} m^2_p(y) = \int \frac{d^3q}{\pi^2} \frac{\omega}{|q^2 - m^2|} e^{iqy} \\
\text{supplementary: } 0 < m^2 < 1 \end{cases}
\]

These representations can be understood to be induced from a noncompact Cartan subgroup representation by a rotation of a spacelike direction into a ‘timelike’ one $\sigma_1 \mapsto \sigma_3$ (not $i\sigma_3$) and exponentiating with a continuous imaginary eigenvalue

\[
\text{SO}_0(1, 1) \ni e^{iy_{\sigma_1}} \mapsto (w \circ e^{iy_{\sigma_1}} \circ w^*)^{\pm \, |\omega|} = (e^{iy_{\sigma_1}})^{\pm \, |\omega|} \in \text{SO}(2)
\]

With the compact $\text{SU}(1, 1)$-parameter space, the continuous energylke invariant is restricted by the lowest nontrivial polarization $n^2 = 1$.

Matrix elements with nontrivial properties with respect to the axial rotation classes $\text{SU}(1, 1)/\text{SO}(2)$ and boost classes $\text{SU}(1, 1)/\text{SO}_0(1, 1)$ use derivations $\frac{\partial}{\partial y} = \frac{y}{|y|} \frac{\partial}{\partial |y|}$.

The Plancherel measure\cite{25, 4} for the supplementary and the trivial representations is trivial, the discrete representations, induced from $\text{SO}(2)$, have counting measure $\mu(n_1^2) = n$ and the principal ones, induced from $\text{SO}_0(1, 1)$, a hyperbolic measure $d\mu(P^2) = (\tanh \frac{\pi^2}{2}, \coth \frac{\pi^2}{2}) dP^2$ (always with fixed Haar measure).

### 6.4 Hilbert Spaces of $\text{SU}(2)$

The following derivation of the familiar $\text{SU}(2)$-Hilbert spaces - finite dimensional subspaces of the 2-sphere square integrable functions $L^2(\Omega^2)$ - by starting from $\text{SU}(2)$-matrix elements in the form of Fourier transformed momentum distributions serves as a preparation for the $\text{SU}(1, 1)$-case below. It is instructive to consider, side by side, the representations of the Euclidean group $\text{SU}(2) \times \mathbb{R}^3$ (rank 2), relevant for nonrelativistic scattering structures $\text{SO}(3) \times \mathbb{R}^3$ in 3-position, and of the spin (rotation) group $\text{SU}(2)$ (rank 1).

The irreducible scalar matrix elements (spherical Bessel functions) with Dirac measure for the momentum 2-sphere

\[
\text{SU}(2) \times \mathbb{R}^3/\text{SU}(2) \ni \vec{x} \mapsto \frac{\sin \mu |\vec{x}|}{\mu |\vec{x}|} = \int \frac{d^3\omega}{4\pi} \frac{\mu}{\mu^2 - \vec{p}^2} e^{i\vec{p} \cdot \vec{x}} = \int \frac{d\mu^2}{4\pi} e^{i\mu^2 \vec{x}} = \int \frac{d^3\omega}{4\pi} e^{-i\mu^2 \vec{x}}, \mu = P > 0
\]
leads also to matrix elements with the action of the \( \text{SO}_\infty \) in addition to the action of the subgroup \( \text{P} \) and derived Dirac distribution. The normalized momentum-sphere measure \( J \) on the 2-sphere.

The relevant integral sums over the 2-sphere momentum directions

\[
\bar{\Omega} = \frac{\rho}{\sqrt{p^2}} = \left( \cos \theta - i e^{i\varphi} \sin \theta, -ie^{i\varphi} \sin \theta, -\cos \theta \right) \in \Omega^2, \quad \rho = |\rho| \bar{\Omega}
\]

\[
\int \frac{d^2\omega}{4\pi} = \int_{-1}^{1} \frac{d\cos \theta}{2} \int_{0}^{2\pi} \frac{d\varphi}{2\pi}, \quad \delta(\bar{\Omega} - \bar{\Omega}') = \delta(\frac{\cos \theta - \cos \theta'}{2}) \delta(\frac{\varphi - \varphi'}{2\pi})
\]

For the neutral group element \( \xi = 0 \), there remains, for both underived and derived Dirac distribution, the normalized momentum-sphere measure

\[
\int \frac{d^2\rho}{2\pi} \frac{1}{\mu} \delta(\mu^2 - \rho^2) = \int \frac{d^2\rho}{\pi} \mu \delta(\mu^2 - \rho^2) = \int \frac{d^2\omega}{4\pi} = 1
\]

The Hilbert space relevant restriction - in both cases - is the Dirac distribution on the 2-sphere.

The \( \text{SU}(2) \times \mathbb{R}^3 \)-representations are induced by translation representations with fixgroup \( \text{SO}(2) \), those for \( \text{SU}(2) \) by \( \text{SO}(2) \)-representations

\[
\text{for} \quad \text{SU}(2) \times \mathbb{R}^3 : \quad \mathbb{R}^3 \ni \bar{x} \mapsto e^{iP\bar{x}} \in \mathcal{U}(1)
\]

\[
\text{for} \quad \text{SU}(2) : \quad \text{SO}(2) \ni e^{i2J\bar{\varphi}\sigma_3} \mapsto e^{i2J\bar{\varphi}\sigma_3} \in \text{SO}(2)
\]

The basis distribution for the irreducible Hilbert space of the Euclidean group contain scattering ‘states’ of momentum value \( P \) in the direction \( \bar{\omega} \) and \( \text{SO}(2) \)-polarization \( \pm J = 0, \pm \frac{1}{2}, \pm 1, \ldots \) with \( \text{SO}(2) \times \mathbb{R}^3 \)-action

\[
(P^2, J) \in \text{irrep \_SU}(2) \times \mathbb{R}^3 : \quad \left\{ \left| P^2, J; \bar{\omega}, \epsilon \right> \mid \bar{\omega} \in \Omega^2, \epsilon = \pm 1 \right\} \quad \int \frac{d^2\omega}{4\pi} \left| P^2, J; \bar{\omega}, \epsilon \right> \left< P^2, J; \bar{\omega}, \epsilon \right> \cong \text{id}_{\mathbb{R}^2(\Omega^2)}
\]

\[
\left| P^2, J; \bar{\omega}, \epsilon \right> \overset{\text{SO}(2) \times \mathbb{R}^3}{\longrightarrow} e^{i2J\bar{\varphi}\sigma_3} \left| P^2, J; \bar{\omega}, \epsilon \right>
\]

\( J = 0 \) has ‘states’ with trivial polarization \( \left\{ \left| P^2, \bar{\omega} \right> \right\} \). The ‘states’ in the basis distributions for \( \text{SU}(2) \) with one invariant \( J \) (spin) have momentum direction \( \bar{\omega} \) and \( \text{SO}(2) \)-eigenvalues \( \epsilon J = \pm J \)

\[
J \in \text{irrep \_SU}(2) : \quad \left\{ \left| J; \bar{\omega}, \epsilon \right> \mid \bar{\omega} \in \Omega^2, \epsilon = \pm 1 \right\} \quad \int \frac{d^2\omega}{4\pi} \left| J; \bar{\omega}, \epsilon \right> \left< J; \bar{\omega}, \epsilon \right> \cong \text{id}_{\mathbb{R}^2(\Omega^2)}
\]

\[
\left| J; \bar{\omega}, \epsilon \right> \overset{\text{SO}(2)}{\longrightarrow} e^{i2J\bar{\varphi}\sigma_3} \left| J; \bar{\omega}, \epsilon \right>
\]

In addition to the action of the subgroup \( \text{SO}(2) \), the derived Dirac distribution leads also to matrix elements with the action of the \( \text{SO}(2) \)-Lie algebra

\[
\left| J; \bar{\omega}, \epsilon \right> \overset{\log_{\text{SO}(2)}^{\text{SU}(2)}}{\longrightarrow} e^{i2J\bar{\varphi}\sigma_3} \left| J; \bar{\omega}, \epsilon \right>
\]

The induced action for the full rotation group \( r \in \text{SU}(2) \)

\[
\left| P^2, J; \bar{\omega}, \epsilon \right> \overset{\text{SU}(2)}{\longrightarrow} o(r, \bar{\omega})^* \left| P^2, J; \bar{\omega}, \epsilon \right>, \quad \left| J; \bar{\omega}, \epsilon \right> \overset{\text{SU}(2)}{\longrightarrow} o(r, \bar{\omega})^* \left| J; \bar{\omega}, \epsilon \right>
\]
comes as Wigner axial rotation \( o(r, \vec{\omega}) \in SO(2) \) with momentum-direction dependent parameters which is determined by the rotation action on the representative \( u(\varphi, \theta, 0) \in SU(2)/SO(2) \) for the momentum direction orbit

\[
\begin{align*}
  r \in SU(2) : & \quad r \circ u(\vec{\omega}) = u(r \bullet \vec{\omega}) \circ o(r, \vec{\omega}) \\
  \Rightarrow & \quad o(r, \vec{\omega}) = u(r \bullet \vec{\omega})^* \circ o(r, \vec{\omega}) \in SO(2) \\
  \text{with } & \quad u(\vec{\omega}) = u(\varphi, \theta, 0) \in SU(2)/SO(2)
\end{align*}
\]

\( r \bullet \vec{\omega} = r \circ \vec{\omega} \circ r^{-1} \) is the \( SU(2) \)-rotated momentum direction.

This is in analogy to the familiar momentum dependent Wigner rotation for an induced Lorentz transformation: There, a boost, parametrized by an energy-momentum vector \( q \in \mathbb{R}^4 \) on the hyperboloid \( \mathcal{Y} \), e.g. in the Weyl representation

\[
s(\frac{\vec{q}}{m}) = e^{i\vec{\omega} \cdot \vec{p}} \in SL(\mathfrak{q}^2)/SU(2), \quad \frac{\vec{q}^2}{2m} = 1, \quad \sinh 2|\vec{\beta}| = \frac{|\vec{q}|}{m}
\]

is transformed by a Lorentz group action \( SL(\mathfrak{q}^2) \ni \lambda \sim \Lambda \in SO_0(1, 3) \) into a boost for the transformed momentum up to a Wigner rotation

\[
\lambda \circ s(\frac{\vec{q}}{m}) = s(\Lambda \frac{\vec{q}}{m}) \circ r(\lambda, \frac{\vec{q}}{m}), \quad r(\lambda, \frac{\vec{q}}{m}) \in SU(2)
\]

With orthogonal basis distributions on the momentum 2-sphere

\[
\langle P^2, J; \vec{\omega}', e|P^2, J; \vec{\omega}, \epsilon \rangle = \delta_{e, e'} \delta(\frac{\vec{\omega}' - \vec{\omega}}{4\pi})
\]

and analogously for the Euclidean group where also position functions \( w(\vec{p}) = \int d^3x \, \tilde{w}(\vec{x}) e^{i\vec{p} \cdot \vec{x}} \) can be used with a spherical Bessel function

\[
\langle P^2, J; w'|P^2, J; w \rangle = \int d^3\omega \, \frac{\tilde{w}'(\vec{\omega}), e}{\tilde{w}(\vec{\omega}), \epsilon} \w(\vec{\omega}, \epsilon) = \frac{d^3p}{4\pi} \delta(p^2 - 4J^2)w(\vec{p}), \epsilon\delta(p^2 - 4J^2)w(\vec{p}), \epsilon
\]

Only for \( SU(2) \) with discrete basis invariant \( \mu = 2J \), a basis for the irreducible \((1 + 2J)\)-dimensional Hilbert space is given with the totally symmetrized \( 2J \)-powers of the basis representation \( SU(2)/SO(2) \ni u(\varphi, \theta, 0) \) (spherical harmonics for integer \( J \) - starting with the two functions in the first and the three functions in the middle column of the matrices for \( J = \frac{1}{2} \) and \( J = 1 \)

\[
\begin{pmatrix}
  e^{i\varphi \cos \theta \frac{\pi}{2}} & e^{i\varphi \sin \theta \frac{\pi}{2}} \\
  e^{-i\varphi \cos \theta \frac{\pi}{2}} & e^{-i\varphi \sin \theta \frac{\pi}{2}}
\end{pmatrix} \in SU(2), \quad \begin{pmatrix}
  e^{i\varphi \cos^2 \theta \frac{\pi}{2}} & e^{i\varphi \sin \theta \cos \theta \frac{\pi}{2}} & -e^{i\varphi \sin^2 \theta \frac{\pi}{2}} \\
  e^{i\varphi \sin \theta \cos \theta \frac{\pi}{2}} & e^{-i\varphi \sin \theta \cos \theta \frac{\pi}{2}} & e^{i\varphi \cos^2 \theta \frac{\pi}{2}}
\end{pmatrix} \in SU(3)
\]

The \( SU(1+2J) \)-orthonormality of the columns \( \langle J; a'|J; a \rangle = \delta_{a-a'} \), which holds for any group element, i.e. any \((\varphi, \theta)\), has to be distinguished from Schur’s orthogonality for the matrix elements which integrates over the full group. The Euler angle parametrized components in the vectors \( |J; a \rangle \) are Schur-orthogonal.
6.5 Hilbert Spaces of SU(1, 1)

In the representation matrix elements for (1, 2)-spacetime with an invariant \( \mu > 0 \)

\[
y \mapsto \int \frac{d^3 q}{2\pi} \frac{1}{\mu} \delta(\mu^2 - q^2) e^{iyq} = \int \frac{d^2 s}{4\pi} e^{i\mu c}
\]

with \( c = \frac{q}{\sqrt{a^2}} = \left( \pm \cos \psi \frac{e^{-i\varphi}}{\cosh \psi}, -e^{i\varphi} \sinh \psi \right) \in \mathcal{Y}_\pm^2 \), \( q = |q| c \)

\[
y \mapsto \int \frac{d^3 q}{2\pi} \frac{1}{\mu} \delta(\mu^2 + q^2) e^{iyq} = \int \frac{d^2 s}{4\pi} e^{i\mu s}
\]

with \( s = \frac{q}{\sqrt{-a^2}} = \left( \sinh \psi \frac{e^{im\varphi}}{ie^{m\varphi} \cosh \psi}, -\sin \psi \right) \in \mathcal{Y}^{(1,1)}, q = |q| s \)

the unit vectors \((c, s) \in (\mathcal{Y}_\pm^2, \mathcal{Y}^{(1,1)})\) on the hyperboloids for the noncompact group SU(1, 1) are the analogue to the sphere unit vectors \(\vec{\omega} \in \Omega^2\) for the compact group SU(2).

Energy-momentum functions are expanded with basis distributions

\[
\mu^2 \in \text{irrep}_{(1,1)} SU(1, 1) : \begin{cases} 
\{ |\mu^2; q, \epsilon \rangle \mid q \in \mathbb{R}^3, \epsilon = \pm 1 \} \\
\int \frac{d^3 q}{(2\pi)^3}|\mu^2; q, \epsilon \rangle \langle \mu^2; q, \epsilon | \cong \text{id}_{\mathcal{H}^3(\mathbb{R})} \\
|\mu^2; w \rangle = \int d^3 q w(q, \epsilon) |\mu^2; q, \epsilon \rangle
\end{cases}
\]

The inducing Hilbert representations of compact and noncompact Cartan subgroup are powers with the invariant \( \mu \) and \( i\mu \)

\[
SO(2) \ni e^{i\gamma_3 s} \mapsto (e^{i\gamma_3 s})^\mu \in SO(2) \\
SO_0(1, 1) \ni e^{i\gamma_1 s} \mapsto (e^{i\gamma_1 s})^\mu \in SO(2)
\]

They act upon the basis distributions \(|\mu^2; q, \epsilon \rangle\) corresponding to the energy-momenta - either energylike \(q = c|q|\) or momentumlike \(q = s|q|\).

The induced actions for the full group \(s \in SU(1, 1)\)

\[
|c| q, \epsilon \xrightarrow{SU^{(1,1)}} o(s, c)^\epsilon |c| q, \epsilon', \quad |s| q, \epsilon \xrightarrow{SU^{(1,1)}} o(s, s)^\epsilon |s| q, \epsilon'
\]

come in the form of Wigner axial rotations \(o(s, c), o(s, s) \in SO(2)\) with parameters dependent on the hyperboloid points. They can be constructed with the coset representatives above for the orbit parametrizations of the energy-momenta

\[
s \in SU(1, 1) \Rightarrow \begin{cases} 
SO(2) \ni v(s \cdot c)^{-1} o s v(c) = o(s, c) \\
SO_0(1, 1) \ni w(s \cdot s)^{-1} o s w(s) \mapsto o(s, s) \in SO(2)
\end{cases}
\]

with \( v(c) = v(\varphi, \psi, 0) \in SU(1, 1)/SO(2) \)

\( w(s) = w(\varphi, \psi, 0) \in SU(1, 1)/SO_0(1, 1) \)

\( s \cdot c \) and \( s \cdot s \) are the SU(1, 1) transformed directions on the hyperboloids.

For the principal SU(1, 1)-representations, supported by a nontrivial momentumlike invariant \( \mu = P \), the definition of - now - infinite dimensional
irreducible Hilbert spaces with square integrable functions \(L^2(\mathcal{Y}^{(1,1)})\) on the momentumlike hyperboloid - will be given in analogy to the \(SU(2)\)-functions \(L^2(\Omega^2)\). The support of the Dirac distribution \(\delta(q^2 + P^2)\) restricts the basis distributions from all energy-momenta \(\mathbb{R}^3\) to the hyperboloid \(\mathcal{Y}^{(1,1)}\)

\[
P^2_\pm \in \text{irrep}_+SU(1,1):
\begin{align*}
\{ |P^2_\pm; \mathbf{s}, \epsilon \rangle | \mathbf{s} \in \mathcal{Y}^{(1,1)}, \epsilon = \pm 1 \} \\
\int \frac{d^2q}{4\pi} |P^2_\pm; \mathbf{s}, \epsilon \rangle \langle P^2_\pm; \mathbf{s}, \epsilon | \cong \text{id}L^2(\mathcal{Y}^{(1,1)}) \\
|P^2_\pm; \mathbf{s}, \epsilon \rangle \quad \epsilon = \pm i P \omega \mathcal{Y} |P^2_\pm; \mathbf{s}, \epsilon \rangle
\end{align*}
\]

The derived Dirac distributions give also matrix elements with the action of the \(SO_0(1,1)\)-Lie algebra. With the orthogonal scalar product distributions on the hyperboloid one obtains the product for the Hilbert vectors

\[
\langle P^2; \mathbf{s}', \epsilon' | P^2; \mathbf{s}, \epsilon \rangle = \delta_{\epsilon', \epsilon} \delta\left(\frac{\mathbf{s}' - \mathbf{s}}{4\pi}\right) \\
|P^2; \mathbf{w} \rangle = \int \frac{d^2s}{4\pi} w(\mathbf{s}, \epsilon) |P^2; \mathbf{s}, \epsilon \rangle = \int \frac{d^2q}{4\pi} \frac{1}{P} \delta(q^2 + P^2) w(q, \epsilon) |q, \epsilon \rangle \\
\langle P^2; \mathbf{w}' | P^2; \mathbf{w} \rangle = \int \frac{d^2s}{4\pi} w'(\mathbf{s}, \epsilon) w(\mathbf{s}, \epsilon) = \int \frac{d^2q}{4\pi} \frac{1}{P} \delta(q^2 + P^2) w(q, \epsilon) \frac{1}{P} \delta(q^2 + P^2) w(q, \epsilon)
\]

For \(SU(1,1)\), there is no finite dimensional definite unitarity (no hyperbolic harmonics) as seen also in the Euler 'angle' group parametrization above.

The representations for the discrete and supplementary series with energy-like invariant \(\mu = n, m\) do not involve a Dirac measure. The Hilbert spaces are not characterized by square integrable functions. The Hilbert space functions \(\{ \mu^2, \mathbf{w} \} \) use all energy-momenta from a basis distribution \(\{ |\mu^2; q, \epsilon \rangle | q \in \mathbb{R}^3 \} \) with the inducing and induced representations above. The energy-momentum dipoles in the representation matrix elements, valued in complex \((2 \times 2)\)-matrices, i.e. in the \(SO_0(1,2)\) Lie algebra and for the energy-momenta in the Lie algebra forms

\[
SU(1,1) \ni e^{iy} \longmapsto \left\{ \begin{array}{l}
\int \frac{d^3q}{2\pi^2} \frac{q}{[(q+i0)^2-n^2]^2} e^{iqy} = -\vartheta(y^2) \vartheta(\pm y_0) \frac{i}{\vartheta(y)} \cos n|y| \\
\int \frac{d^3q}{2\pi^2} \frac{q}{(q^2-m^2)^2} e^{iqy} = -\vartheta(y^2) \frac{i}{\vartheta(y)} \cos m|y|
\end{array} \right.
\]

\[
iy = \left( \frac{iy_0}{y_1 + iy_2}, \frac{y_1 - iy_2}{-y_0} \right) \in \log SU(1,1) \cong \mathbb{R}^{1+2}
\]

lead to scalar matrix elements with the invariant \(y^2 = \det iy = \frac{1}{2} \text{tr} y \circ y = y_0^2 - y_1^2 - y_2^2\)

\[
\det \left[ \vartheta(y^2) \vartheta(\pm y_0) \frac{i}{\vartheta(y)} \cos \mu |y| \right] = \vartheta(y^2) \vartheta(\pm y_0)(\cos \mu |y|^2)
\]

The corresponding Hilbert product distributions involve positive type functions

\[
\langle \mu^2; q', \epsilon' | \mu^2; q, \epsilon \rangle = q_{\epsilon', \epsilon} \omega_{\mu^2}(q^2) \delta(q - q') \quad \text{with} \quad \omega_{\mu^2}(q^2) = \left\{ \begin{array}{l}
\frac{1}{[(q+i0)^2-n^2]^2} \\
\frac{1}{(q^2-m^2)^2}
\end{array} \right.
\]

\[
\langle \mu^2; w' | \mu^2; w \rangle = \int d^3q w'(q, \epsilon) q_{\epsilon', \epsilon} \omega_{\mu^2}(q^2) w(q, \epsilon) = \int d^3q \omega_{\mu^2}(q^2) \text{tr} q \circ (w \otimes w')(q)
\]

The Hilbert space function pairs \(\{q \longmapsto w(q, \epsilon) | \epsilon = \pm 1\}\) are defined by \(\langle \mu^2; w | \mu^2; w \rangle \geq 0\) with the \((2 \times 2)\)-matrix valued function \(q \longmapsto (w \otimes w')(q)\), in analogy to the energylike energy-momenta by \(\frac{1}{2} \text{tr} q \circ q = q^2 \geq 0\).
It remains to establish explicitly how the Hilbert spaces with energy-momentum functions, of $L^2$-type and with positive type function, are related to the Hilbert spaces as constructed with homogeneous functions of one variable $\xi \mapsto F(\xi)$ as mentioned above.
7 Appendix: $\mathfrak{g}^3$-Lie Algebras

For a real or complex finite dimensional Lie algebra $L$, the classes with respect to the commutator ideal $[L, L]$, the radical $R$ (maximal solvable ideal in $L$) and the nilradical $N \subseteq R$ (maximal nilpotent ideal in $L$ and, also, in $R$) are

| up to noncommutativity | $[L, L]$ ⇒ $L/[L, L]$ | abelian |
|------------------------|------------------------|---------|
| up to solvability      | $R$ ⇒ $L/R \cong S \subseteq L$ | semisimple |
| up to nilpotency       | $N$ ⇒ $L \cong L/N \oplus N$ | semidirect product |

Since there are no semisimple Lie algebras with dimension 1 and 2, the complex 3-dimensional Lie algebras have radical $R \in \{\{0\}, \mathfrak{q}^3\}$, i.e. they are either simple or solvable.

They can be classified and constructed with the commutator ideal

$$L \cong \mathfrak{q}^3 \Rightarrow \partial L = [L, L] \in \{\mathfrak{q}^3, \mathfrak{q}^2, \mathfrak{q}, \{0\}\}$$

The perfect case is the simple Lie algebra $A_1$ with a basis $\{l^1, l^2, l^3\}$ for totally antisymmetric structure constants

$$\mathfrak{q}^3 \cong \partial L = A_1 \text{ with } \begin{cases} [l^1, l^2] = l^3 \\ [l^2, l^3] = l^1 \\ [l^3, l^1] = l^2 \end{cases}$$

The ideals $\partial L \cong \mathfrak{q}^2$ belong to semidirect product Lie algebras. An abelian $\partial L$ gives a solvable, not nilpotent Lie algebra with possible basis

$$\partial L = [L, \partial L] = \mathfrak{q}^1 \oplus \mathfrak{q}^2, \quad \partial^2 L = \{0\} \text{ with } \begin{cases} [l^1, l^2] = 0 \\ [l^2, l^3] = l^1 \\ [l^3, l^1] = l^2 \end{cases}$$

A semidirect $\mathfrak{q}^2$-ideal is not possible, since

$$\partial L = \mathfrak{q}^1 \oplus \mathfrak{q}^2 : \begin{cases} [l^1, l^2] = l^2 \\ [l^2, l^3] = \gamma l^1 + \delta l^2 \\ [l^3, l^1] = \alpha l^1 + \beta l^2 \end{cases}$$

Jacobi identity:

$$\Rightarrow (\alpha, \gamma) = (0, 0) \Rightarrow \begin{cases} [l^1, l^2] = l^2 \\ [l^2, l^3] = \delta l^2 \\ [l^3, l^1] = \beta l^2 \end{cases} \Rightarrow [L, L] = \mathfrak{q}^2$$

For $\partial L \cong \mathfrak{q}$ there is a basis with

$$\partial L = \mathfrak{q}^2 : \begin{cases} [l^1, l^2] = \beta_1 l^2 \\ [l^2, l^3] = \beta_3 l^2 \\ [l^3, l^1] = \alpha l^2 \end{cases}$$
$l^1$ and $l^3$ can be exchanged (1st and 2nd line). By renormalizations, a basis with three nontrivial brackets leads to a basis with two nontrivial brackets, and even to a basis with one nontrivial bracket

$$\begin{align*}
[l^1, l^2] &= l^2 \\
[l^2, l^3] &= l^2 \\
[l^3, l^1] &= l^2
\end{align*}$$

which arises also from a basis with the following two nontrivial brackets

$$\begin{align*}
[l^1, l^2] &= l^2 \\
[l^2, l^3] &= l^2 \\
[l^3, l^1] &= 0
\end{align*}$$

This characterizes the decomposable Lie algebra $\mathfrak{C} \oplus [\mathfrak{C} \oplus \mathfrak{C}]$.

Therefore, the only nondecomposable $\mathfrak{C}^3$-Lie algebra with $\partial L \cong \mathfrak{C}$ is the nilcubic Heisenberg Lie algebra - in a basis with two trivial brackets

$$L = \mathfrak{C}^3 \oplus [\mathfrak{C}l^1 \oplus \mathfrak{C}l^2], \quad \partial L = \mathfrak{C}l^2, \quad [L, \partial L] = \{0\} \quad \text{with} \quad \begin{align*}
[l^1, l^2] &= 0 \\
[l^2, l^3] &= 0 \\
[l^3, l^1] &= l^2
\end{align*}$$

\section{Appendix: Residual Distributions}

All residual representations, considered above, arise from the Fourier transformed generalized scalar functions\cite{6} (where the $\Gamma$-functions are defined with $\nu \in \mathbb{R}$) - for linear invariants

$$\mathbb{R} : \quad \int \frac{dq}{2\pi} \frac{\Gamma(1-\nu)}{(q-i\omega-m)^{1-\nu}} e^{iqx} = \vartheta(x) \mathcal{J}^{\nu}_{\nu(m)}(\sqrt{x}), \quad m \in \mathbb{R}$$

and for quadratic invariants in the scalar distributions for the definite orthogonal groups

$$\mathcal{O}(d)$$

$$\begin{align*}
\int \frac{d^dq}{\pi^d} \frac{\Gamma(\frac{d}{2} - \nu)}{(q^2)^{\frac{d}{2} - \nu}} e^{iq\xi} &= \frac{\Gamma(\nu)}{(\xi^2)^{\nu}}, \\
\int \frac{d^dq}{\pi^d} \frac{\Gamma(\frac{d}{2} - \nu)}{(q^2+1)^{\frac{d}{2} - \nu}} e^{iq\xi} &= \frac{2\mathcal{K}_{\nu}(r)}{(\xi^2)^{\nu}}, \\
\int \frac{d^dq}{\pi^d} \frac{\Gamma(\frac{d}{2} - \nu)}{(q^2-i\omega-1)^{\frac{d}{2} - \nu}} e^{iq\xi} &= \frac{i\pi\mathcal{H}^{(1)}_{\nu}(r)}{(\xi^2)^{\nu}} = -\pi |\mathcal{N}_\nu - i\mathcal{J}_{\nu}|(r) \quad \text{with} \quad \mathcal{K}_{\nu}, \ \mathcal{H}^{(1)}_{\nu}, \ \mathcal{H}^{(1)}_{\nu} = e^{i\pi \nu} \mathcal{H}^{(1)}_{\nu}
\end{align*}$$

As seen in the power $\frac{d}{2}$, there is a distinction between even and odd dimensions $d$ - in parallel with the Cartan series $D_R \cong \log \mathbb{SO}(\mathfrak{q}^2R)$ and $B_R \cong \log \mathbb{SO}(\mathfrak{q}^{1+2R})$.

The real-imaginary transition relates to each other Macdonald and Hankel (with Neumann and Bessel) functions

$$\xi \in \mathbb{R} : \quad 2\mathcal{K}_{\nu}(i\xi) = i\pi \mathcal{H}^{(2)}_{\nu}(\xi) = e^{i\pi \nu} i\pi \mathcal{H}^{(1)}_{\nu}(-\xi)$$

$$\pm i\mathcal{H}^{(1,2)}_{\nu} = -\mathcal{N}_{\nu} \pm i\mathcal{J}_{\nu}, \quad \mathcal{K}_{-\nu} = \mathcal{K}_{\nu}, \quad \mathcal{H}^{(1)}_{-\nu} = e^{i\pi \nu} \mathcal{H}^{(1)}_{\nu}$$
There are the special functions for $N = 0, 1, 2, \ldots$ for halfinteger index

\[
\frac{\mathcal{K}_{N-\frac{1}{2}}(\xi)}{\xi^{N-\frac{1}{2}}} = \frac{\mathcal{J}_{N-\frac{1}{2}}}{\xi^{N-\frac{1}{2}}} = \frac{\mathcal{N}_{N-\frac{1}{2}}}{\xi^{N-\frac{1}{2}}} = \frac{(-\partial}{\partial \xi^2} e^{-\xi},}{\sqrt{\xi}} = \frac{e^{\pm i\xi}}{\sqrt{\xi}}, \quad \frac{\mathcal{N}_{N-\frac{1}{2}}}{\xi^{N-\frac{1}{2}}} = \frac{\mathcal{J}_{N-\frac{1}{2}}}{\xi^{N-\frac{1}{2}}} = \frac{-i e^{\pm i\xi}}{\sqrt{\xi}}
\]

and for integer index

\[
\frac{\mathcal{K}_N(\xi)}{\xi^N} = \frac{\mathcal{J}_N(\xi)}{\xi^N} = \frac{\mathcal{N}_N(\xi)}{\xi^N} = \left( -\frac{\partial}{\partial \xi^2} \right)^N \mathcal{K}_0(\xi), \quad \mathcal{J}_0(\xi), \quad \mathcal{N}_0(\xi) \right) \]

\[
2\mathcal{K}_0(\xi) = \int d\psi e^{-|\psi|} \cosh \psi = - \sum_{n=0}^{\infty} \left( \frac{\xi}{(n+1)^2} \right)^n \log \frac{\xi^2}{4} - 2\Gamma(1) - 2\nu(n) \] 

\[
\lim_{\xi \to 0} \mathcal{J}_N(\xi) = \frac{1}{\Gamma(1+N)}, \quad \lim_{\xi \to 0} \mathcal{N}_N(\xi) = -\Gamma(N)
\]

$\mathcal{J}_N$ has no singularities. The integer index functions give a quadratic dependence, e.g. in $\frac{\mathcal{J}_N}{\xi^N} = \mathcal{E}_N(\xi^2)$.

By analytic continuation one obtains for indefinite orthogonal groups

\[
\mathcal{O}(n, m) \times \mathbb{R}^d:
\begin{align*}
\mathcal{O}(n, m) & : \\
& \begin{cases}
0 > n, m > 0 \\
d = 2, 3, \ldots \\
|x| = \sqrt{|x|^2}
\end{cases}
\end{align*}
\begin{align*}
\int \frac{d^d q}{i^n \pi^{\frac{d}{2}}} & \frac{\Gamma(\frac{d}{2}-\nu)}{(q^2-iq)^{\frac{d}{2}-\nu}} e^{i q x} = \frac{\Gamma(\nu)}{\left( \frac{x^2+i\nu}{4} \right)^{\nu}} \\
\int \frac{d^d q}{i^n \pi^{\frac{d}{2}}} & \frac{\Gamma(\frac{d}{2}-\nu)}{(q^2-iq+1)^{\frac{d}{2}-\nu}} e^{i q x} = \frac{\theta(x^2) 2\mathcal{K} \nu(|x|) - \theta(-x^2) i \pi \mathcal{H}^{(2)} \nu(|x|)}{\left( \frac{|x|^2}{4} \right)^{\nu}} \\
-\delta^N i \pi & \sum_{k=1}^{N} \frac{1}{(N-k)!} \delta(k-1)(-\frac{x^2}{4})
\end{align*}
\]

For integer $N = 1, 2, \ldots$, there arise $x^2 = 0$ supported Dirac distributions.

Orthogonally invariant distributions are embedded in hyperbolically invariant ones, e.g. for $(1, s)$-spacetime with the general Lorentz groups

\[
\mathcal{O}(1, s) \begin{cases}
\mathcal{O}(1, s) & : \\
s = 1, 2, \ldots \\
N = 1, 2, \ldots \\
|x| = \sqrt{|x|^2}
\end{cases}
\end{align*}
\begin{align*}
\int \frac{d^d q}{i^n \pi^{\frac{d}{2}}} & \frac{\Gamma(\frac{d}{2}-\nu)}{(q^2-iq+1)^{\frac{d}{2}-\nu}} e^{i q x} = \frac{\theta(x^2) 2\mathcal{K} \nu(|x|) - \theta(-x^2) i \pi \mathcal{H}^{(2)} \nu(|x|)}{\left( \frac{|x|^2}{4} \right)^{\nu}} \\
-\delta^N i \pi & \sum_{k=1}^{N} \frac{1}{(N-k)!} \delta(k-1)(-\frac{x^2}{4})
\end{align*}
\]

For $\nu = -\frac{1}{2}$ there are no singularities
These distributions are relevant for the representations of orthogonal groups in odd dimensions \( \mathbf{O}(1 + 2R) \) and \( \mathbf{O}(1, 2R) \) with rank \( R \) and poles of order \( 1 + R = 1, 2, \ldots \).

For \( \nu = 0 \) there is a logarithmic singularity in \( \mathcal{K}_0 \) and \( \mathcal{N}_0 \):

\[
\mathbf{O}(1 + s) : \begin{cases}
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{\Gamma(\frac{1+s}{2})}{(q^2 - io)^{\frac{1+s}{2}}} e^{iqx} = -|x| \left[ \vartheta(x^2) + i \vartheta(-x^2) \right] \\
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{\Gamma(\frac{1+s}{2})}{(q^2 - io+1)^{\frac{1+s}{2}}} e^{iqx} = \vartheta(x^2)e^{-|x|} + \vartheta(-x^2)e^{i|x|} \\
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{\Gamma(\frac{1+s}{2})}{(q^2 - io-1)^{\frac{1+s}{2}}} e^{iqx} = \vartheta(-x^2)e^{-|x|} + \vartheta(x^2)e^{i|x|}
\end{cases}
\]

These distributions are relevant for the representations of orthogonal groups in even dimensions \( \mathbf{O}(2R) \) and \( \mathbf{O}(1, 2R - 1) \) with rank \( R \) and order \( R = 1, 2, \ldots \) poles.

The Fourier transformed simple poles are used for representations of the Euclidean groups \( \mathbf{SO}(1 + s) \times \mathbb{R}^{1+s} \) and Poincaré groups \( \mathbf{SO}_0(1, s) \times \mathbb{R}^{1+s} \):

\[
\mathbf{O}(1 + s) : \begin{cases}
\int \frac{d^{1+s} q}{\pi^{1+s}} \frac{1}{q^2} e^{i\xi q^2} = \frac{\Gamma(\frac{1+s}{2})}{(\xi)^{s+1}} \\
\int \frac{d^{1+s} q}{\pi^{1+s}} \frac{1}{q^2+1} e^{\xi q^2} = \frac{2\mathcal{K}_{s+1}(r)}{(\xi)^{\frac{1+s}{2}}} \\
\int \frac{d^{1+s} q}{\pi^{1+s}} \frac{1}{q^2 - io-1} e^{i\xi q^2} = -\frac{\pi[N_{s+1} - i\mathcal{J}_{s+1}](r)}{(\xi)^{\frac{1+s}{2}}}
\end{cases}
\]

\[
\mathbf{O}(1, s) : \begin{cases}
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{1}{q^2 - io} e^{i\xi q^2} = \frac{\Gamma(\frac{1+s}{2})}{(\xi^2 + io)^{\frac{1+s}{2}}} \\
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{1}{q^2 - io+1} e^{i\xi q^2} = \vartheta(x^2)2\mathcal{K}_{s+1}(|x|) - \vartheta(-x^2)\pi[N_{s+1} - i\mathcal{J}_{s+1}](|x|) \\
\int \frac{d^{1+s} q}{i^n \pi^{1+s}} \frac{1}{q^2 - io-1} e^{i\xi q^2} = \vartheta(-x^2)2\mathcal{K}_{s+1}(|x|) - \vartheta(x^2)\pi[N_{s+1} - i\mathcal{J}_{s+1}](|x|)
\end{cases}
\]

The lightcone supported Dirac distributions arise for even dimensional spacetime with nonflat position, i.e. for \( (1, s) = (1, 3), (1, 5), \ldots \).
The one dimensional pole integrals

\[
O(1) = O(1, 0) : \begin{cases} 
\frac{dq}{\pi} \frac{1}{q^{s+1}} e^{iqx} = e^{-|x|} \\
\int \frac{dq}{\pi} \frac{1}{q^{s-1}} e^{iqx} = i e^{i|x|}
\end{cases}
\]

are spread to odd dimensions starting with \(1 + s = 3\) and a singularity at \(|x| = 0\)

\[
O(3) : \begin{cases} 
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s+1}} e^{iq\vec{x}} = \frac{2}{r} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s+1}+1} e^{iq\vec{x}} = - \frac{\partial}{\partial r} e^{-r} = 2 e^{-r} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-1}} e^{iq\vec{x}} = - \frac{\partial}{\partial r} i e^{ir} = 2 e^{ir}
\end{cases}
\]

\[
O(1, 2) : \begin{cases} 
- \int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \frac{2 \partial(x^2)}{|x|} i e^{i|x|} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}+1} e^{iq\vec{x}} = \frac{2 \partial(x^2) e^{-|x|} \partial(x^2)e^{-i|x|}}{|x|} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \frac{2 \partial(x^2) e^{-|x|} + \partial(x^2)e^{i|x|}}{|x|}
\end{cases}
\]

The dipoles in three dimensions are without singularity

\[
O(3) : \begin{cases} 
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s+2}} e^{iq\vec{x}} = - r \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s+2}+1} e^{iq\vec{x}} = e^{-r} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}+2} e^{iq\vec{x}} = i e^{ir}
\end{cases}
\]

\[
O(1, 2) : \begin{cases} 
- \int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = |x|[\partial(x^2) - i \partial(-x^2)] \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}+1} e^{iq\vec{x}} = \partial(x^2) e^{-|x|} + \partial(-x^2)e^{-i|x|} \\
\int \frac{d^3q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \partial(-x^2)e^{-|x|} + \partial(x^2)e^{i|x|}
\end{cases}
\]

The 2-dimensional integrals integrate over the 1-dimensional functions

\[
O(2) : \begin{cases} 
\int \frac{d^2q}{\pi} \frac{1}{q^{s+1}} e^{iq\vec{x}} = \int d\psi e^{-r \cosh \psi} = 2 K_0(r) \\
\int \frac{d^2q}{\pi} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \int d\psi e^{i r \cosh \psi} = - \pi [N_0 - i J_0](r)
\end{cases}
\]

\[
O(1, 1) : \begin{cases} 
\int \frac{d^2q}{\pi} \frac{1}{q^{s-10}+1} e^{iq\vec{x}} = \partial(x^2) 2 K_0(|x|) - \partial(-x^2) \pi [N_0 + i J_0](|x|) \\
\int \frac{d^2q}{\pi} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \partial(-x^2) 2 K_0(|x|) - \partial(x^2) \pi [N_0 - i J_0](|x|)
\end{cases}
\]

They are spread to even dimensions - starting with \(1 + s = 4\)

\[
O(4) : \begin{cases} 
\int \frac{d^4q}{\pi^2} \frac{1}{q^{s+1}} e^{iq\vec{x}} = \frac{4}{r^2} \\
\int \frac{d^4q}{\pi^2} \frac{1}{q^{s+1}+1} e^{iq\vec{x}} = - \frac{\partial}{\partial r} 2 K_0(r) = \frac{2 K_1(r)}{r} \\
\int \frac{d^4q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \frac{\partial}{\partial r} \pi [N_0 - i J_0](r) = - \frac{\pi [N_0 - i J_0](r)}{r}
\end{cases}
\]

\[
O(1, 3) : \begin{cases} 
- \int \frac{d^4q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \frac{4}{r^2} - i \pi \delta(\frac{x^2}{4}) \\
- \int \frac{d^4q}{\pi^2} \frac{1}{q^{s-10}+1} e^{iq\vec{x}} = \frac{\pi (x^2) 2 K_1(|x|) - \partial(-x^2) \pi [N_0 + i J_0](|x|)}{|x|^2} - i \pi \delta(\frac{x^2}{4}) \\
\int \frac{d^4q}{\pi^2} \frac{1}{q^{s-10}} e^{iq\vec{x}} = \frac{\partial(-x^2) 2 K_1(|x|) - \partial(x^2) \pi [N_0 - i J_0](|x|)}{|x|^2} + i \pi \delta(\frac{x^2}{4})
\end{cases}
\]

Dipoles for four dimensions lead to maximally logarithmic singularities.
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