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Research Article

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Dynamical behavior for a stochastic two-species competitive model

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Abstract: This paper deals with a stochastic two-species competitive model. Some very verifiable criteria on the global stability of the positive equilibrium of the deterministic system are established. An example with its computer simulations is given to illustrate our main theoretical findings.

Keywords: Competitive model, Global stability, Stochastic perturbation

MSC: 60H30, 60H10, 92D25

1 Introduction

It is well known that Schoener’s models play an important role in ecology and mathematical biology. In recent years, dynamics of Schoener’s models has attracted much attention due to its theoretical and practical significance. Many results on various Schoener’s models are available, for example, Lu and Chen [1] studied the asymptotic behaviors of a periodic Schoener’s model, Liu et al. [2] analyzed the persistence and global stability of a Schoener’s model with time delay and feedback control, Li and Wang [3] focused on permanence, periodic solution and globally asymptotic stability of a Schoener’s model with two populations, Wu et al. [4] addressed the permanence and global attractivity for a discrete Schoener’s model with delays, Lv et al. [5] discussed the dynamical properties of a stochastic two-species Schoener’s competitive model. For more details, we refer the reader to [6–10]. In 1991, Chen [11] investigated the following competitive system

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= r_1 x_1(t) \left[ -a_1 - a_{11} x_1(t) - a_{12} x_2(t) + \frac{c_1}{x_1(t) + d_1} \right], \\
\frac{dx_2(t)}{dt} &= r_2 x_2(t) \left[ -a_2 - a_{21} x_1(t) - a_{22} x_2(t) + \frac{c_2}{x_2(t) + d_2} \right],
\end{align*}
\]

(1)

where \(x_1(t)\) and \(x_2(t)\) represents the population densities of each species and \(r_i a_i (i = 1, 2)\) stands for its death rate, respectively, \(r_1 a_{11}\) and \(r_2 a_{22}\) denote intra-specific competition rates, \(r_1 a_{12}\) and \(r_2 a_{21}\) denote inter-specific competition rates. All the parameters \(r_i, a_{ij}, c_i, d_i (i, j = 1, 2)\) are positive constants.

Many authors argue [12–23] that in the real world the populations usually live in changing circumstances which have important effect on the growth rates. So we can think that the growth rate can be expressed as an average rate plus an error term. In view of the well known central limit theorem, we know that the error term follows a normal distribution [24–27]. Therefore the error term can be denoted by a white noise \(\sigma_i(t) \dot{B}_i(t)\), where \(\sigma_i^2(t)\) denotes the intensity of the noise and \(\dot{B}_i(t)\) denotes a standard white noise, where \(B_i(t)\) represents a Brownian motion defined on a complete probability space \((\Omega, F, P)\). Motivated by the discussed above, we can let the growth rates \(r_1\) and \(r_2\)
become \( r_1 + \sigma_1 \dot{B}_1(t) \) and \( r_2 + \sigma_2 \dot{B}_2(t) \), respectively. Then we obtain the following stochastic system:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1(t) \left[-a_1 - a_{11}x_1(t) - a_{12}x_2(t) + \frac{c_1}{x_1(t) + d_1}\right] \, \left[r_1 \, dt + \sigma_1 \, dB_1(t)\right], \\
\frac{dx_2}{dt} &= x_2(t) \left[-a_2 - a_{21}x_1(t) - a_{22}x_2(t) + \frac{c_2}{x_2(t) + d_2}\right] \, \left[r_2 \, dt + \sigma_2 \, dB_2(t)\right].
\end{align*}
\]

(2)

The initial conditions of system (2) are given by

\[ x_{10} = \varphi_1(0) > 0, \quad x_{20} = \varphi_2(0) > 0. \]

(3)

The remainder of the paper is organized as follows: in Section 2, some sufficient conditions for the existence of global positive solution of system (2) with the initial conditions (3) are established, moreover, the sufficient condition is obtained to ensure that the equilibrium of system (2) with the initial conditions (3) is globally asymptotically stable. In Section 3, an example with its computer simulations is given to illustrate the feasibility and effectiveness of our results derived in this section. A brief conclusion is drawn in Section 4.

## 2 Main results

In this section, we shall present our main results.

**Theorem 2.1.** For any given initial value \((x_{10}, x_{20}) \in \mathbb{R}_+^2\), system (2) has a unique global solution \((x_1(t), x_2(t))\) almost sure (a.s.), where \(\mathbb{R}_+^2 = \{x_1 > 0, x_2 > 0\}\).

**Proof.** Consider the following system

\[
\begin{align*}
\frac{df_1}{dt} &= \left\{ r_1 \left[-a_1 - a_{11}e^{f_1} - a_{12}e^{f_2} + \frac{c_1}{e^{f_1} + d_1}\right] \\
&\quad - \frac{\sigma_1^2}{2} \left[a_1 - a_{11}e^{f_1} - a_{12}e^{f_2} + \frac{c_1}{e^{f_1} + d_1}\right]^2 \right\} \\
&\quad + \sigma_1 \left[a_1 - a_{11}e^{f_1} - a_{12}e^{f_2} + \frac{c_1}{e^{f_1} + d_1}\right] dB_1(t), \\
\frac{df_2}{dt} &= \left\{ r_2 \left[-a_2 - a_{21}e^{f_1} - a_{22}e^{f_2} + \frac{c_2}{e^{f_2} + d_2}\right] \\
&\quad - \frac{\sigma_2^2}{2} \left[a_2 - a_{21}e^{f_1} - a_{22}e^{f_2} + \frac{c_2}{e^{f_2} + d_2}\right]^2 \right\} \\
&\quad + \sigma_2 \left[a_2 - a_{21}e^{f_1} - a_{22}e^{f_2} + \frac{c_2}{e^{f_2} + d_2}\right] dB_2(t).
\end{align*}
\]

(4)

with initial value \(f_{10} = \ln x_{10}, f_{20} = \ln x_{20}\). Obviously, the coefficients of (4) satisfy the local Lipschitz condition, then there is unique local solution \((f_1(t), f_2(t))\) on \([0, \tau_e)\), where \(\tau_e\) is the explosion time. Thus, in view of Itô’s formula, \((x_1(t), x_2(t), \cdot) = (e^{f_1(t)}, e^{f_2(t)})\) is the unique positive local solution to system (2) with the initial value \(x_{10} > 0, x_{20} > 0\).

Assume that \(k_0 > 0\) is sufficiently large such that \(x_{10}\) and \(x_{20}\) lying within \([\frac{1}{k_0}, k_0]\). For integer \(k > k_0\), define the stopping times

\[ \tau_k = \inf \left\{ t \in [0, \tau_e) : x_1(t) \notin \left(\frac{1}{k}, k\right) \text{ or } x_2(t) \notin \left(\frac{1}{k}, k\right) \right\}. \]

(5)

Let \(\tau_{\infty} = \lim_{k \to \infty} \tau_k\). Then \(\tau_{\infty} \leq \infty\) a.s. To complete the proof of Lemma 2.1 we need only to prove \(\tau_{\infty} = \infty\). Assume that the statement does not hold, then there exists a constant \(T \geq 0\) and \(\varepsilon \in (0, 1)\) such that \(P\{\tau_{\infty} \leq T\} > \varepsilon\). Thus there is an integer \(k_1 \geq k_0\) such that \(P\{\tau_{k_1} \leq T\} \geq \varepsilon\). We define

\[ V(x_1, x_2) = (\sqrt{x_1} - 1 - 0.5 \ln x_1) + (\sqrt{x_2} - 1 - 0.5 \ln x_2). \]

(6)
If \((x_1(t), x_2(t)) \in \mathbb{R}_+^2\), then

\[
\begin{align*}
    dV(x_1, x_2) &= 0.5r_1(\sqrt{x_1} - 1) \left[-a_1 - a_{11}x_1 - a_{12}x_2 + \frac{c_1}{x_1 + d_1}\right] dt \\
    &\quad + \frac{\sigma_1^2}{8} (2 - \sqrt{x_1}) \left[-a_1 - a_{11}x_1 - a_{12}x_2 + \frac{c_1}{x_1 + d_1}\right]^2 dt \\
    &\quad + 0.5\sigma_1(\sqrt{x_1} - 1) \left[-a_1 - a_{11}x_1 - a_{12}x_2 + \frac{c_1}{x_1 + d_1}\right] dB_1(t) \\
    &\quad + 0.5\sigma_2(\sqrt{x_2} - 1) \left[-a_2 - a_{21}x_1 - a_{22}x_2 + \frac{c_2}{x_2 + d_2}\right] dB_2(t) \\
    &\quad + \frac{\sigma_2^2}{8} (2 - \sqrt{x_2}) \left[-a_2 - a_{21}x_1 - a_{22}x_2 + \frac{c_2}{x_2 + d_2}\right]^2 dt \\
    \leq 0.5r_1 \left[-a_1 \sqrt{x_1} - a_{11}x_1^\frac{3}{2} - a_{12}x_2 \sqrt{x_1} + \frac{c_1 \sqrt{x_1}}{x_1 + d_1}\right] dt \\
    &\quad + a_1 + a_{11}x_1 + a_{12}x_2 - \frac{c_1}{x_1 + d_1} \right] dt \\
    &\quad + 0.5r_2 \left[-a_2 \sqrt{x_2} - a_{22}x_2^\frac{3}{2}\right] dt \\
    &\quad + \frac{\sigma_1^2}{8} \left[2a_1^2 + 2a_{11}^2x_1^2 + 2a_{12}^2x_2^2 + 2\left(\frac{c_1}{x_1 + d_1}\right)^2 + 4a_1a_{11}x_1\right] \\
    &\quad + 4a_1a_{12}x_2 - \frac{4a_1c_1}{x_1 + d_1} + 4a_{11}a_{12}x_1x_2 - \frac{4a_{11}c_1x_1}{x_1 + d_1} - \frac{4a_{12}c_1x_2}{x_1 + d_1} \\
    &\quad - a_1 \sqrt{x_1} - a_{11}x_1^\frac{3}{2} - a_{12}x_2 \sqrt{x_1} - \left(\frac{c_1}{x_1 + d_1}\right)^2 \sqrt{x_1} - 2a_1a_{11}x_1^\frac{3}{2} \\
    &\quad - 2a_1a_{12}x_2 \sqrt{x_1} + \frac{2a_1c_1 \sqrt{x_1}}{x_1 + d_1} - 2a_{11}a_{12}x_1^\frac{3}{2}x_2 + \frac{2a_{11}c_1x_1^\frac{3}{2}}{x_1 + d_1} \\
    &\quad + \frac{2a_{12}c_1x_2^\frac{3}{2}}{x_1 + d_1} \right] dt \\
    &\quad + \frac{\sigma_2^2}{8} \left[2a_2^2 + 2a_{22}^2x_2^2 + 2a_{21}^2x_1^2\right] \\
    &\quad + 2\left(\frac{c_2}{x_2 + d_2}\right)^2 + 4a_2a_{22}x_2 + 4a_2a_{21}x_1 - \frac{4a_{22}c_2}{x_2 + d_2} - 4a_{21}a_{22}x_2x_1 \\
    &\quad - a_2 \sqrt{x_2} - a_{22}x_2^\frac{3}{2} - a_{21}x_1 \sqrt{x_2} - \left(\frac{c_2}{x_2 + d_2}\right)^2 \sqrt{x_2} \\
    &\quad - 2a_2a_{22}x_2^\frac{3}{2}x_1 + \frac{2a_{22}c_2x_2^\frac{3}{2}}{x_2 + d_2} + \frac{2a_{21}c_2x_1x_2^\frac{3}{2}}{x_2 + d_2} \right] dt \\
    &\quad + 0.5\sigma_1(\sqrt{x_1} - 1) \left[-a_1 - a_{11}x_1 - a_{12}x_2 + \frac{c_1}{x_1 + d_1}\right] dB_1(t) \\
    &\quad + 0.5\sigma_2(\sqrt{x_2} - 1) \left[-a_2 - a_{21}x_1 - a_{22}x_2 + \frac{c_2}{x_2 + d_2}\right] dB_2(t) \\
    \leq 0.5r_1 \left[\frac{c_1 \sqrt{x_1}}{x_1 + d_1} + a_1 + a_{11}x_1 + a_{12}x_2\right] dt
\end{align*}
\]
Now we will state our second main result of the article.

In view of (8), we have

$$
\frac{1}{8} \left[ 2a_1^2 + 2a_1^2 x_1^2 + 2a_2^2 x_2^2 + \frac{c_1}{x_1 + d_1} \right]^2 + 4a_1 a_{11} x_1
$$

$$
+ 4a_1 a_{12} x_2 + 4a_{11} a_{12} x_1 x_2 + \frac{2a_1 c_1 \sqrt{x_1}}{x_1 + d_1} + \frac{2a_{11} c_1 x_1^2}{x_1 + d_1}
$$

$$
+ 2a_{12} c_1 x_2 x_1 \frac{x_1^2}{x_1 + d_1} \right] dt + \frac{\sigma_2^2}{8} \left[ 2a_2^2 + 2a_2^2 x_2^2 + 2a_2^2 x_1^2 \right]^2 dt
$$

$$
+ 2 \left( \frac{c_2}{x_2 + d_2} \right)^2 + 4a_2 a_{22} x_2 + 4a_2 a_{21} x_1 + 4a_{22} a_{21} x_2 x_1
$$

$$
+ 2a_1 c_2 \sqrt{x_2} + 2a_2 c_2 x_2^2 + \frac{2a_{21} c_2 x_1 x_2}{x_2 + d_2} \right] dt
$$

$$
+ 0.5 \sigma_1 \left( \sqrt{x_1} - 1 \right) \left[ -a_1 - a_{11} x_1 - a_{12} x_2 + \frac{c_1}{x_1 + d_1} \right] dB_1(t)
$$

$$
+ 0.5 \sigma_2 \left( \sqrt{x_2} - 1 \right) \left[ -a_2 - a_{21} x_1 - a_{22} x_2 + \frac{c_2}{x_2 + d_2} \right] dB_2(t)
$$

$$
\leq M_1 dt + M_2 dt
$$

$$
+ 0.5 \sigma_1 \left( \sqrt{x_1} - 1 \right) \left[ -a_1 - a_{11} x_1 - a_{12} x_2 + \frac{c_1}{x_1 + d_1} \right] dB_1(t)
$$

$$
+ 0.5 \sigma_2 \left( \sqrt{x_2} - 1 \right) \left[ -a_2 - a_{21} x_1 - a_{22} x_2 + \frac{c_2}{x_2 + d_2} \right] dB_2(t).
$$

(7)

where $M_1$ and $M_2$ are positive numbers. Integrating both sides of (7) from 0 to $t_k \wedge T$, and taking the expectations, we have

$$
EV(x_1(t_k \wedge T), x_2(t_k \wedge T)) \leq V(x_{10}, x_{20}) + (M_1 + M_2) T.
$$

(8)

Set $\Phi_k = \{ t_k \leq T \}$, then it follows that $P(\Phi_k) \geq \epsilon$. Since for every $\sigma \in \Phi_k$, there exists some $i$ such that $x_i(t_k, \sigma)$ is equal to either $k$ or $\frac{1}{k}$ for $i = 1, 2$. Thus $V(x_1(t_k, \sigma), x_2(t_k, \sigma))$ is no less than

$$
\min \left\{ \left( \sqrt{k} - 1 - 0.5 \ln k \right) \cdot \left( \sqrt{\frac{1}{k}} - 1 - 0.5 \ln \frac{1}{k} \right) \right\}.
$$

(9)

In view of (8), we have

$$
V(x_{10}, x_{20}) + (M_1 + M_2) T
$$

$$
\geq E[1_{\Phi_k}(\sigma) V(x_1(t_k), x_2(t_k))]
$$

$$
\geq \epsilon \min \left\{ \left( \sqrt{k} - 1 - 0.5 \ln k \right) \cdot \left( \sqrt{\frac{1}{k}} - 1 - 0.5 \ln \frac{1}{k} \right) \right\}.
$$

(10)

where $1_{\Phi_k}$ denotes the indicator function of $\Phi_k$. Letting $k \to \infty$, then

$$
\infty > V(x_{10}, x_{20}) + (M_1 + M_2) T = \infty.
$$

(11)

which is a contradiction. The proof of Theorem 2.1 is completed.

Let $E(x_1^*, x_2^*)$ be the positive equilibrium of system (2), then $x_1^*$ and $x_2^*$ satisfy the following equations

$$
\left\{ \begin{array}{l}
-a_1 - a_{11} x_1^* - a_{12} x_2^* + \frac{c_1}{x_1^* + d_1} = 0, \\
-a_2 - a_{21} x_1^* - a_{22} x_2^* + \frac{c_2}{x_2^* + d_2} = 0.
\end{array} \right.
$$

(12)

Now we will state our second main result of the article.
Theorem 2.2. Let

\[ \begin{align*}
\varrho_1 &= -r_1 a_{11} + \frac{r_1 c_1}{(x_1^* + d_1)d_1} + \frac{\sigma_1^2 x_1^* a_{11}^2}{2} \\
&\quad + \frac{c_1^2}{(x_1^* + d_1)d_1} + \frac{2a_{11} c_1^2}{(x_1^* + d_1)d_1} + \frac{\sigma_1^2 x_1^*}{2} a_{12}^2, \\
\varrho_2 &= -r_1 a_{12} + 2a_{11} a_{12} + \frac{2a_{12} c_1^2}{(x_1^* + d_1)d_1} \\
&\quad -r_2 a_{21} + 2a_{22} a_{21} + \frac{2a_{21} c_2^2}{(x_2^* + d_2)d_2}, \\
\varrho_3 &= -r_2 a_{22} + \frac{r_2 c_2}{(x_2^* + d_2)d_2} + \frac{\sigma_2^2 x_2^* a_{22}^2}{2} \\
&\quad + \frac{c_2^2}{(x_2^* + d_2)d_2} + \frac{2a_{22} c_2^2}{(x_2^* + d_2)d_2} + \frac{\sigma_2^2 x_2^*}{2} a_{12}^2.
\end{align*} \]

If \( \varrho_1 < 0, 4\varrho_1 \varrho_3 > \varrho_2^2 \), then the equilibrium \((x_1^*, x_2^*)\) of system (2) is globally asymptotically stable, that is, for any initial value \(x_{10}(0) > 0, x_{20}(0) > 0\), the solution of system (2) satisfies

\[ \lim_{t \to +\infty} x_1(t) = x_1^*, \lim_{t \to +\infty} x_2(t) = x_2^*, \text{a.s.} \]

Proof. Define the following functions

\[ V_1(x_1) = \int_0^{x_1 - x_1^*} \frac{\theta}{\theta + x_1^*} d\theta, \quad (13) \]

\[ V_2(x_2) = \int_0^{x_2 - x_2^*} \frac{\theta}{\theta + x_2^*} d\theta. \quad (14) \]

It follows from Itô’s formula that

\[ \begin{align*}
LV_1(x_1) &= r_1(x_1 - x_1^*) \left[ -a_1 - a_{11} x_1 - a_{12} x_2 + \frac{c_1}{x_1 + d_1} \right] \\
&\quad + \frac{\sigma_1^2 x_1^*}{2} \left[ -a_1 - a_{11} x_1 - a_{12} x_2 + \frac{c_1}{x_1 + d_1} \right]^2 \\
&\quad - r_1 a_{11} (x_1 - x_1^*)^2 - r_1 a_{12} (x_1 - x_1^*) (x_2 - x_2^*) - \frac{r_1 c_1 (x_1 - x_1^*)^2}{(x_1^* + d_1)(x_1 + d_1)} \\
&\quad + \frac{\sigma_1^2 x_1^*}{2} \left[ a_{11} (x_1 - x_1^*)^2 + a_{12} (x_2 - x_2^*)^2 + \frac{c_1^2 (x_1 - x_1^*)^2}{(x_1^* + d_1)(x_1 + d_1)} \right] \\
&\quad + 2a_{11} a_{12} (x_1 - x_1^*) (x_2 - x_2^*) + \frac{2a_{11} c_1^2 (x_1 - x_1^*)^2}{(x_1^* + d_1)(x_1 + d_1)} \\
&\quad + \frac{2a_{12} c_2^2 (x_1 - x_1^*) (x_2 - x_2^*)}{(x_2^* + d_2)(x_2 + d_2)} \right] \\
&\leq \left[ -r_1 a_{11} + \frac{r_1 c_1}{(x_1^* + d_1)d_1} + \frac{\sigma_1^2 x_1^* a_{11}^2}{2} \right]
\end{align*} \]


\[
LV_2(x_2) = r_2(x_2 - x_2^*) \left[ -a_2 - a_2 x_1 - a_2 x_2 + \frac{c_2}{x_2 + d_2} \right] + \frac{\sigma_2^2 x_2^*}{2} \left[ -a_2 - a_2 x_1 - a_2 x_2 + \frac{c_2}{x_2 + d_2} \right]^2 \\
= r_2(x_2 - x_2^*) \left[ a_2 (x_1^* - x_1) + a_2 (x_2^* - x_2) - \frac{c_2}{x_2^* + d_2} + \frac{c_2}{x_2 + d_2} \right] \\
+ \frac{\sigma_2^2 x_2^*}{2} \left[ a_2 (x_1^* - x_1) + a_2 (x_2^* - x_2) - \frac{c_2}{x_2^* + d_2} + \frac{c_2}{x_2 + d_2} \right]^2 \\
= -r_2 a_2 (x_2 - x_2^*)^2 - r_2 a_2 (x_1 - x_1^*) (x_2 - x_2^*) - \frac{r_2 c_2 (x_2 - x_2^*)^2}{(x_2^* + d_2)(x_2 + d_2)} \\
+ \frac{\sigma_2^2 x_2^*}{2} \left[ a_2 (x_1^* - x_1)^2 + a_2 (x_2^* - x_2)^2 + \frac{c_2^2 (x_2 - x_2^*)^2}{(x_2^* + d_2)(x_2 + d_2)} \right] \\
+ \frac{2 a_2 c_2^2 (x_2 - x_2^*)^2}{(x_2^* + d_2)(x_2 + d_2)} \\
+ \frac{2 a_2 c_2^2 (x_2 - x_2^*) (x_1 - x_1^*)}{(x_2^* + d_2)(x_2 + d_2)} \\
\leq \left[ -r_2 a_2 + \frac{r_2 c_2}{(x_2^* + d_2)d_2} + \frac{\sigma_2^2 x_2^* a_2^2}{2} \right] \\
+ \frac{c_2^2}{(x_2^* + d_2)^2} + \frac{a_2 c_2^2}{(x_2^* + d_2)^2} \left[ (x_2 - x_2^*)^2 + \frac{c_2^2 (x_2 - x_2^*)^2}{2} \right] \\
+ \left[ -r_2 a_{22} + \frac{2 a_{22} c_2^2}{(x_2^* + d_2)d_2} \right] |x_1 - x_1^*| |x_2 - x_2^*|. 
\] (16)

Now we define

\[
V(t) = V_1(x_1) + V_2(x_2).
\] (17)

Then we have

\[
LV(t) = LV_1(x_1) + LV_2(x_2)
\]
Let \( |Z - Z^*| = (|x_1 - x_1^*|, |x_2 - x_2^*|)^T \), then it follows from (18) that

\[
LV(t) \leq \frac{1}{2} |Z - Z^*|^T \begin{bmatrix}
2\varrho_1 & \varrho_2 \\
\varrho_2 & 2\varrho_3
\end{bmatrix} |Z - Z^*|.
\]

In view of the conditions in Theorem 2.2, we can conclude that \( LV(t) < 0 \) along all trajectories in the first quadrant except \( x_1^*, x_2^* \). Then

\[
\lim_{t \to +\infty} x_1(t) = x_1^*, \quad \lim_{t \to +\infty} x_2(t) = x_2^*.
\]

The proof of Theorem 2.2 is completed. \( \square \)

### 3 Computer simulations

In this section, we give an example to illustrate our main results obtained in previous sections by the Milstein method [27]. Consider the following stochastic three-species clockwise chain predator-prey model

\[
\begin{align*}
    x_{1,k+1} &= x_{1,k} + r_1 x_{1,k} \left[ -a_1 - a_{11} x_{1,k} - a_{12} x_{2,k} + \frac{c_1}{x_{1,k} + d_1} \right] \Delta t \\
    &\quad + \sigma_1 x_{1,k} \left[ -a_1 - a_{11} x_{1,k} - a_{12} x_{2,k} + \frac{c_1}{x_{1,k} + d_1} \sqrt{\Delta t} \xi_k \right] \\
    &\quad + \frac{\sigma_1^2}{2} x_{1,k} \left[ -a_1 - a_{11} x_{1,k} - a_{12} x_{2,k} + \frac{c_1}{x_{1,k} + d_1} (\xi_k^2 - 1) \Delta t, \right] \\
    x_{2,k+1} &= x_{2,k} + r_2 x_{2,k} \left[ -a_2 - a_{21} x_{1,k} - a_{22} x_{2,k} + \frac{c_2}{x_{2,k} + d_2} \right] \Delta t \\
    &\quad + \sigma_2 x_{2,k} \left[ -a_2 - a_{21} x_{1,k} - a_{22} x_{2,k} + \frac{c_2}{x_{2,k} + d_2} \sqrt{\Delta t} \eta_k \right] \\
    &\quad + \frac{\sigma_2^2}{2} x_{2,k} \left[ -a_2 - a_{21} x_{1,k} - a_{22} x_{2,k} + \frac{c_2}{x_{2,k} + d_2} (\eta_k^2 - 1) \Delta t, \right]
\end{align*}
\]

where \( \xi_k \) and \( \eta_k \) are Gaussian random variables that follow \( N(0, 1) \). We choose \( r_1 = 0.5, r_2 = 0.2, a_1 = 0.1, a_2 = 0.2, a_{11} = 0.4, a_{22} = 0.3, a_{21} = 0.1, a_{22} = 0.4, c_1 = 0.1, c_2 = 0.2, d_1 = 0.3, d_2 = 0.2 \). Let \( \sigma_1^2 = 0.2, \sigma_2^2 = 0.15 \). We can easily check that all the conditions in Theorem 2.2 are fulfilled. Hence we can conclude that the equilibrium \( (x_1^*, x_2^*) \) is globally asymptotically stable which is shown in Figure 1. Let \( \sigma_1^2 = 0.4, \sigma_2^2 = 0.2 \) and also we can easily check that all the conditions in Theorem 2.2 are fulfilled. Hence we can also know that the equilibrium \( (x_1^*, x_2^*) \) is globally asymptotically stable which is shown in Figure 2.

### 4 Conclusions and further research

In this paper, we are concerned with the dynamical properties of a stochastic two-species competitive model. We have shown that under suitable conditions, the stochastic system has a unique global positive solutions for any initial values. Some sufficient conditions which ensure the global stability of the stochastic system are established. Form the viewpoint of biology, the results play an important role in practical applications. The fact that a positive equilibrium is globally asymptotically stable implies that all the species could exist. It is well known that two-species competitive models have attracted much attention due to their theoretical and practical significance. To the best of our knowledge, it is the first attempt to carry out such a study in a stochastic case. Since discontinuity is a common phenomenon in real natural world and fractal calculus is valid for discontinuous problems, then the fractal effect has received much attention and have been widely investigated. Motivated by the discussion, the fractal two-species competitive models will be our future work.
Fig. 1. The solutions of system (20) with $\sigma_1^2 = 0.2$ and $\sigma_2^2 = 0.15$, where the red line stands for $x_1$ and the blue line stands for $x_2$.

Fig. 2. The solutions of system (20) with $\sigma_1^2 = 0.4$ and $\sigma_2^2 = 0.2$, where the red line stands for $x_1$ and the blue line stands for $x_2$.

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