INTEGRABLE HIERARCHIES AND WAKIMOTO MODULES

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To our teacher D.B. Fuchs on his sixtieth birthday

1. Introduction

In our papers [20, 21] we proposed a new approach to integrable hierarchies of soliton equations and their quantum deformations. We have applied this approach to the Toda field theories and the generalized KdV and modified KdV (mKdV) hierarchies. In this paper we apply our approach to the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [1] and its generalizations. In particular, we show that the free field (Wakimoto) realization of an affine algebra [36, 16] naturally appears in the context of the generalized AKNS hierarchies. This is analogous to the appearance of the free field (quantum Miura) realization of a $W$-algebra in the context of the generalized KdV equations. As an application, we give here a new proof of the existence of the Wakimoto realization.

We also conjecture that all integrals of motion of the generalized AKNS equation can be quantized. In the case of $\hat{sl}_2$ the corresponding quantum integrals of motion can be viewed as integrals of motion of a thermal perturbation of the parafermionic conformal field theory [14]. Thus we expect that this deformation, and analogous deformations for arbitrary affine algebras, are integrable, in the sense of Zamolodchikov [37].

Let us first recall the main steps of our analysis of the Toda field theories from [20, 21].

1.1. Overview of the previous work. Let $\mathfrak{g}$ be an affine algebra, twisted or untwisted, and $\mathfrak{g}$ be the finite-dimensional simple Lie algebra, whose Dynkin diagram is obtained by deleting the 0th node of the Dynkin diagram of $\mathfrak{g}$ [31]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Heisenberg algebra. For $x \in \mathfrak{h}, n \in \mathbb{Z}$, denote $x(n) = x \otimes t^n$. Then $\mathfrak{h}$ has generators $b_i(n) = \alpha_i(n), i = 1, \ldots, \ell, n \in \mathbb{Z}$ (where $\alpha_i$’s are the simple roots of $\mathfrak{g}$), and $1$, with the commutation relations

\[ [b_i(n), b_j(m)] = n\delta_{\alpha_i, \alpha_j} \delta_{n,-m} 1, \quad [1, b_i(n)] = 0. \]

There is a family of Fock representations $\pi_{\lambda}^\nu, \nu \in \mathbb{C}, \lambda \in \mathfrak{h}^*$, of $\mathfrak{h}$, which are generated by vectors $v_\lambda$, such that

\[ b_i(n)v_\lambda = 0, n > 0, \quad b_i(0)v_\lambda = (\lambda, \alpha_i)v_\lambda, \quad 1v_\lambda = \nu v_\lambda. \]

Introduce a derivation $\partial$ on $\pi_{\lambda}^\nu$, such that $[\partial, b_i(n)] = -nb_i(n-1), \partial \cdot v_\lambda = \lambda(-1)v_\lambda$, and so $\lambda(-1) \in \mathfrak{h}$.

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The module $\pi_0^\nu$ carries a vertex operator algebra (VOA) structure described in [20] (for the definition of VOA, see [3, 24], or Sect. 4 of [21]).

Consider the space $\pi_\lambda[z, z^{-1}]$, and extend the action of $\partial$ to it by the formula $\partial \otimes 1 + 1 \otimes \partial$. Denote by $\pi_0^\nu$ the quotient of $\pi_\lambda[z] \otimes \mathbb{C}[z, z^{-1}]$ by the image of $\partial$ (total derivatives) and constants, if $\lambda = 0$. The space $\pi_0^\nu$ is a Lie algebra, the local completion of the universal enveloping algebra of $\mathfrak{h}$ [18]. Elements of $\pi_0^\nu$ can be interpreted as Fourier components of currents of the VOA $\pi_0^\nu$ [20].

For any $\gamma \in \mathfrak{h}^*$ we can define the bosonic vertex operator

$$V_\gamma^\nu(z) = \sum_{n \in \mathbb{Z}} V_\gamma^\nu(n) z^{\nu(\gamma, \lambda) - n}$$

(1.2)

$$T_\gamma z^{\nu(\gamma, \lambda)} \exp \left( - \sum_{n < 0} \frac{\gamma(n) z^{-n}}{n} \right) \exp \left( - \sum_{n > 0} \frac{\gamma(n) z^{-n}}{n} \right),$$

where $\gamma \in \mathfrak{h}^* \simeq \mathfrak{h}$ and $T_\gamma : \pi_\lambda^\nu \to \pi_\lambda^{\nu+\gamma}$ is such that $T_\gamma : v_\lambda = v_{\lambda+\gamma}$ and $[T_\gamma^\nu, b_i(n)] = 0, n < 0$. Thus, $V_\gamma^\nu(n), n \in \mathbb{Z}$, are well-defined linear operators $\pi_\lambda^\nu \to \pi_\lambda^{\nu+\gamma}$.

Now introduce operators

$$\tilde{Q}_i^\nu = V_{-\alpha_i}^\nu(1) = \int V_{-\alpha_i}^\nu(z) dz : \pi_0 \to \pi_{-\alpha_i}, \quad i = 1, \ldots, \ell,$$

where $\alpha_i, i = 1, \ldots, \ell$, are the simple roots of $\mathfrak{g}$. These operators are called the screening operators. They commute with the action of $\partial$ and hence give rise to operators $\tilde{Q}_i : \pi_0 \to \pi_{-\alpha_i}, i = 1, \ldots, \ell$.

The VOA $\mathcal{W}_\nu(\mathfrak{g})$ is defined as a vertex operator subalgebra of $\pi_0^\nu$:

$$\mathcal{W}_\nu(\mathfrak{g}) = \bigcap_{i=1}^\ell \ker_{\pi_0^\nu} \tilde{Q}_i^\nu.$$

In [20] we proved that for generic $\nu$, $\mathcal{W}_\nu(\mathfrak{g})$ is finitely and freely generated in the following sense. There exist elements $W_i^\nu$ in $\mathcal{W}_\nu(\mathfrak{g})$ of degrees $d_i + 1, i = 1, \ldots, \ell$, where $d_i$ is the $i$th exponent of $\mathfrak{g}$, such that $\mathcal{W}_\nu(\mathfrak{g})$ has a linear basis of lexicographically ordered monomials in the Fourier components $W_i^\nu(n_i), 1 \leq i \leq \ell, n_i < -d_i$, of the currents $Y(W_i^\nu, z) = \sum_{n \in \mathbb{Z}} W_i^\nu(n) z^{-n - d_i - 1}$.

The Lie algebra $L_\nu(\mathfrak{g})$ of quantum integrals of motion of the Toda theory of $\mathfrak{g}$ (the $\mathcal{W}$–algebra of $\mathfrak{g}$) is defined as

$$L_\nu(\mathfrak{g}) = \bigcap_{i=1}^\ell \ker_{\pi_0^\nu} \bar{Q}_i^\nu.$$

We proved in [21] that $L_\nu(\mathfrak{g})$ is isomorphic to the quotient of $\mathcal{W}_\nu(\mathfrak{g})[z, z^{-1}]$ by the total derivatives and constants. In other words, it consists of all Fourier components of currents defined by the VOA $\mathcal{W}_\nu(\mathfrak{g})$. The Lie algebra $L_\nu(\mathfrak{g})$ is called the quantum $\mathcal{W}$–algebra associated to $\mathfrak{g}$ and is denoted by $\mathcal{W}_\nu(\mathfrak{g})$.

Now let $a_0 = -1/a_0 \sum_{i=1}^\ell a_i \alpha_i$ be the element of $\mathfrak{h}^*$, corresponding to the affine root of $\mathfrak{g}$; here $a_i, i = 1, \ldots, \ell$, are the labels of the Dynkin diagram of $\mathfrak{g}$ [31]. We can define the screening operators corresponding to the 0th root of the affine algebra $\mathfrak{g}$, $\bar{Q}_0^\nu : \pi_0^\nu \to \pi_{-a_0}^\nu$ and $\bar{Q}_0^\nu : \pi_0^\nu \to \pi_{-a_0}^\nu$, in the same way as the operators $\tilde{Q}_i^\nu$. and
\( \tilde{Q}_i \), \( i = 1, \ldots, \ell \), which correspond to the simple roots of \( \mathfrak{g} \). The space \( I_\nu(\mathfrak{g}) \) of \textit{quantum integrals of motion} of the affine Toda theory associated to \( \mathfrak{g} \) is

\[
I_\nu(\mathfrak{g}) = \bigcap_{i=0}^{\ell} \text{Ker} \tilde{Q}_i^\nu.
\]

Clearly, \( I_\nu(\mathfrak{g}) \) is a Lie subalgebra of \( \mathcal{W}_\nu(\mathfrak{g}) \). Elements of the space \( I_\nu(\mathfrak{g}) \) can be viewed as integrals of motion of a certain perturbation of the conformal field theory with the \( \mathcal{W}_\nu(\mathfrak{g}) \)-symmetry (see \([37, 10, 30, 19]\)). In \([20, 21]\) we proved that for generic \( \nu \) this is a commutative Lie algebra linearly spanned by elements of degrees equal to the exponents of \( \mathfrak{g} \) modulo the Coxeter number.

In order to prove the above mentioned results we interpreted the spaces \( \mathcal{W}_\nu(\mathfrak{g}) \) and \( I_\nu(\mathfrak{g}) \) as cohomologies of certain complexes. These complexes were constructed using the Bernstein-Gelfand-Gelfand (BGG) resolution of the trivial representation of the quantum group \( U_q(\mathfrak{g}) \) and \( U_0(\mathfrak{g}) \), respectively, where \( q = \exp(\pi i \nu) \). The construction was based on the fact that in a certain sense the operators \( \tilde{Q}_i^\nu \) satisfy the defining relations of the nilpotent subalgebra of a quantum group – the \( q \)-Serre relations \([11]\).

We were able to compute the cohomologies of our complexes for generic \( \nu \) by computing them in the \textit{classical limit} \( \nu \to 0 \) \([20]\). In this limit, the spaces \( I_\nu(\mathfrak{g}) \) and \( \mathcal{W}_\nu(\mathfrak{g}) \) become the spaces \( I_0(\mathfrak{g}) \) and \( \mathcal{W}_0(\mathfrak{g}) \) of local integrals of motion of the Toda equation associated to \( \mathfrak{g} \) and \( \mathfrak{g} \), respectively.

In the classical limit \( \nu \to 0 \), the Fock representation can be identified with the ring of differential polynomials \( \mathbb{C}[u_i^{(n)}]_{i=1, \ldots, \ell, n \geq 0} \), where \( u_i = b_i(-1) \). We have proved in \([20, 21]\) that the classical limits \( \tilde{Q}_i, i = 0, \ldots, \ell \), of the screening operators act on \( \mathbb{C}[u_i^{(n)}]_{i=1, \ldots, \ell, n \geq 0} \) and generate the pronilpotent subalgebra \( \mathfrak{n}_+ \subset \mathfrak{g} \). Moreover, \( \mathbb{C}[u_i^{(n)}]_{i=1, \ldots, \ell, n \geq 0} \simeq \mathbb{C}[N_+/A_+] \), where \( N_+ \) is the Lie group of \( \mathfrak{n}_+ \) and its \textit{principal} abelian subgroup \( A_+ \). The left infinitesimal action of the Lie algebra \( \mathfrak{n}_+ \) on \( N_+/A_+ \) coincides with the one defined by the operators \( \tilde{Q}_i, i = 0, \ldots, \ell \).

In particular, the operators \( \tilde{Q}_i, i = 1, \ldots, \ell \), generate the action of the Lie subalgebra \( \mathfrak{n}_+ \subset \mathfrak{n}_+ \) on \( \mathbb{C}[u_i^{(n)}]_{i=1, \ldots, \ell, n \geq 0} \), and \( \mathcal{W}_0(\mathfrak{g}) \) is the ring of invariants of this action. This ring coincides with the ring of \( \mathfrak{n}_+ \)-invariants of \( \mathbb{C}[N_+/A_+] \), and we have shown that it is also a ring of differential polynomials \( \mathbb{C}[v_i^{(n)}]_{i=1, \ldots, \ell, n \geq 0} \). The embedding \( \mathbb{C}[v_i^{(n)}] \to \mathbb{C}[u_i^{(n)}] \) is called the generalized Miura transformation.

The quotient of \( \mathbb{C}[u_i^{(n)}] \otimes \mathbb{C}[z, z^{-1}] \) by the total derivatives and constants is the \textit{Heisenberg-Poisson algebra} denoted by \( \mathcal{J}_0 \). The \textit{classical} \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \), \( \mathcal{W}_0(\mathfrak{g}) = \mathcal{W}_0(\mathfrak{g}) \) is a Poisson subalgebra of \( \mathcal{J}_0 \), which can be identified with the quotient of \( \mathcal{W}_0(\mathfrak{g})[z, z^{-1}] \) by the total derivatives and constants.

Next, the space \( I_0(\mathfrak{g}) \) is identified with the Lie algebra cohomology \( H^1(\mathfrak{n}_+, \mathbb{C}[u_i^{(n)}]) \).

Using the fact that \( \mathbb{C}[u_i^{(n)}] \simeq \mathbb{C}[N_+/A_+] \), one shows that \( H^*(\mathfrak{n}_+, \pi_0) \simeq \Lambda^*(\mathfrak{a}^*_+) \), where \( \mathfrak{a}^*_+ \) is the dual space to the Lie algebra \( \mathfrak{a}_+ \) of \( A_+ \). This way we proved that \( I_0(\mathfrak{g}) \simeq \mathfrak{a}^*_+ \) \([2] \).
Each element of $I_0(\mathfrak{g})$ gives rise to a hamiltonian vector field on $N_+/A_+$ and these vector fields the $\mathfrak{g}$–mKdV hierarchy (on the set of functions $u_i(t), i = 1, \ldots \ell$). On the other hand, the abelian Lie algebra $\mathfrak{a}_-$, which is the opposite of the Lie algebra $\mathfrak{a}_+$, also acts on this homogeneous space from the right by vector fields. In \cite{21} we proved that these two actions coincide.

Finally, these derivations preserve the differential subring $\mathbb{C}[v_i^{(n)}] = \mathcal{W}_0(\mathfrak{g})$. The corresponding equations on the set of functions $v_i(t), i = 1, \ldots \ell$, form the $\mathfrak{g}$–KdV hierarchy.

1.2. The present work. In this paper we apply the same approach when the principal abelian subgroup $A_+$ of $N_+$ is replaced by the homogeneous abelian subgroup $H_+$. In this case we have a structure, which is very similar to what we have in the principal case. This is illustrated by the following table.

|                                | Principal Case | Homogeneous Case |
|--------------------------------|----------------|------------------|
| Differential polynomials       | $\mathbb{C}[u_i^{(n)}]$ | $\mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}, u_i^{(n)}]$ |
| Homogeneous space              | $N_+/A_+$       | $N_+/H_+$        |
| Screening operators            | $\int e^{-\phi_i}dz$ | $\sum_{\beta \in \Delta_+} P_{\alpha \beta}(q_\beta) p_\beta e^{-\phi_i}dz$ |
| Non-local equations            | (affine) Toda equation | equations (5.5), (8.3) |
| $I_0(\mathfrak{g})$            | classical $\mathcal{W}$–algebra $\mathcal{W}_0(\mathfrak{g})$ | $\mathbb{C}[\mathfrak{g}^*]_{\text{loc}}$ |
| $I_\nu(\mathfrak{g})$         | $\mathcal{W}$–algebra $\mathcal{W}_\nu(\mathfrak{g})$ | $U_k(\mathfrak{g})_{\text{loc}}, k = -h^\vee + \nu^{-1}$ |
| Embedding $I_\nu(\mathfrak{g}) \subset \mathcal{F}_0^\vee$ | Miura transformation | Wakimoto realization |
| $I_0(\mathfrak{g})$            | mKdV hamiltonians | mAKNS hamiltonians |
| Symmetries of the non-local equation | mKdV hierarchy | mAKNS hierarchy |
| Local equations in $I_0(\mathfrak{g})$ | KdV hierarchy | AKNS hierarchy |

Thus, the starting point of our approach is always a Heisenberg–Poisson algebra and a pair of non-local equations, which possess local integrals of motion (or symmetries) in this Heisenberg–Poisson algebra. Both equations are generated by the screening operators corresponding to the simple roots of $\mathfrak{g}$ or $\mathfrak{g}$. The action of the Poisson bracket with each of these terms on the ring of differential polynomials coincides with the action of the corresponding generator of the pronilpotent Lie algebra $\mathfrak{n}$ on the homogeneous space of $N_+$. This fact allows us to interpret the local integrals of motion as cohomologies of this Lie algebra.

In the principal case, the Poisson algebra of local integrals of motion of the non-local equation associated to $\mathfrak{g}$ (the $\mathfrak{g}$–Toda equation) is the classical $\mathcal{W}$–algebra $\mathcal{W}_0(\mathfrak{g})$, and the local integrals of motion of the non-local equation corresponding to $\mathfrak{g}$ (affine Toda equation) are the hamiltonians of the $\mathfrak{g}$–mKdV hierarchy. The embedding of $\mathcal{W}_0(\mathfrak{g})$ into the Heisenberg–Poisson algebra is the generalized Miura transformation. The equations of the mKdV hierarchy, written in terms of $\mathcal{W}_0(\mathfrak{g})$, become the equations of the $\mathfrak{g}$–KdV hierarchy.

In the homogeneous case, the role of $\mathcal{W}_0(\mathfrak{g})$ is played by the classical affine algebra, i.e., the Poisson algebra $\mathbb{C}[\mathfrak{g}^*]_{\text{loc}}$ of local functionals on a hyperplane in $\mathfrak{g}^*$. The role of the $\mathfrak{g}$–mKdV hierarchy is played by a modified AKNS hierarchy for $\mathfrak{g} = \mathfrak{sl}_2$ and its
generalization for other $\mathfrak{g}$, which we call the $\mathfrak{g}$–mAKNS hierarchy. The embedding of $\mathbb{C}[\mathfrak{g}]_{\text{loc}}$ into the Heisenberg–Poisson algebra is the Poisson analogue of the Wakimoto realization. Written in terms of $\mathbb{C}[\mathfrak{g}^*]_{\text{loc}}$, the $\mathfrak{g}$–mAKNS hierarchy becomes what we call the $\mathfrak{g}$–AKNS hierarchy.

The next step is to quantize the classical integrals of motion. In the principal case, this was done in [20]. Here we prove the existence of the quantum integrals of motion of the non-local equation corresponding to $\mathfrak{g}$ in the homogeneous case. This gives us an embedding of $\mathfrak{g}$ into a Heisenberg algebra, analogous of the embedding of the $\mathcal{W}$–algebra $\mathcal{W}_{\nu}(\mathfrak{g})$ into $F_{\nu}$. Thus we obtain a new proof of the Wakimoto realization, which has been previously defined in [36] for $\mathfrak{g} = \mathfrak{sl}_2$ and in [16] for general $\mathfrak{g}$. In this paper we give another perspective on the Wakimoto realization, which comes from the theory of integrable hierarchies of soliton equations. This confirms to the general philosophy of “free field realization” outlined in [24].

Finally, we conjecture that all classical integrals of motion of the non-local equation corresponding to $\mathfrak{g}$ can be quantized.

The main results of this paper have been obtained during our visit to Kyoto University in the Summer of 1993, and the present paper is an expanded version of a draft written in the Fall of 1994. In the mean time, some aspects of the $\mathfrak{sl}_2$ case of our program have been considered in [2]. We also became aware of the interesting papers [32], in which integrable hierarchies similar to ours have been considered, from a different point of view.

The program outlined above has been further developed in [11, 12, 25]. In [4] the results of [21] have been interpreted geometrically in terms of moduli spaces of bundles, thus allowing a uniform treatment of the integrable hierarchies associated to arbitrary maximal abelian (or Heisenberg) subalgebras of $\mathfrak{g}$.

2. Wakimoto realization

Let $\mathfrak{g}$ be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h} \oplus \mathfrak{h}^-$, where $\mathfrak{h}^+$ and $\mathfrak{h}^-$ are the upper and lower nilpotent subalgebras, respectively, and $\mathfrak{h}$ is its Cartan subalgebra.

Let $\mathfrak{g}$ be the affine algebra, corresponding to $\mathfrak{g}$: the universal central extension of the loop algebra $\mathfrak{g}(t, t^{-1})$.

Introduce the Heisenberg algebra $\mathcal{H}(\mathfrak{g})$, which has generators $a_{\alpha}(n), a_{\alpha}^*(n), \alpha \in \Delta_+, n \in \mathbb{Z}$, where $\Delta_+$ is the set of positive roots of $\mathfrak{g}$, and $1$. The commutation relations read:

$[a_{\alpha}(n), a_{\beta}^*(m)] = \delta_{\alpha,\beta} \delta_{n, -m} 1, \quad [a_{\alpha}(n), a_{\beta}(m)] = 0, \quad [a_{\alpha}^*(n), a_{\beta}^*(m)] = 0,$

and $1$ commutes with everything.

For $\nu \neq 0$, denote by $M$ the Fock representation of $\mathcal{H}(\mathfrak{g})$, which is generated from the vacuum vector $v$ satisfying

$a_{\alpha}(n)v = 0, \quad n \geq 0; \quad a_{\alpha}^*(n)v = 0, \quad n > 0,$

and on which the central element $1$ acts as the identity.
The space $M$ is a vertex operator algebra (VOA) \cite{FeingRen}. We will give an explicit formula for the currents $Y(z, t)$ defined by this VOA.

Monomials $a_{\alpha_1}(m_1)\ldots a_{\alpha_k}(m_k)a_{\beta_1}^*(n_1)\ldots a_{\beta_l}^*(n_l)v, m_p < 0, n_q \leq 0$, form a linear basis in $M$. The series $Y(z, t)$ associated to this monomial is given by

$$C : \partial_z^{-m_1-1}a_{\alpha_1}(z)\ldots \partial_z^{-m_k-1}a_{\alpha_k}(z)\partial_z^{-n_1}a_{\beta_1}^*(z)\ldots \partial_z^{-n_l}a_{\beta_l}^*(z),$$

where $C = [(-m_1 - 1)!\ldots (-m_1 - 1)!(-n_1)!\ldots (-n_l)!]^{-1}$, columns stand for normal ordering, and

$$a_{\alpha}(z) = \sum_{n \in \mathbb{Z}} a_{\alpha}(n)z^{-n-1}, \quad a_{\beta}^*(z) = \sum_{n \in \mathbb{Z}} a_{\beta}^*(n)z^{-n}.$$  

The Fourier coefficients of currents of this VOA form a Lie algebra, which lies in a certain completion of the universal enveloping algebra $U(\mathfrak{h}(\mathfrak{f}))$ factored by the ideal generated by $(1 - 1)$, respectively. Following \cite{FeingRen}, we call this Lie algebra the \textit{local completion} of the universal enveloping algebra and denote it by $U(\mathfrak{h}(\mathfrak{f}))_{\text{loc}}$.

Denote by $\mathfrak{L}N_+$ the Lie subalgebra $\mathfrak{G}[[t, t^{-1}]]$ of $\mathfrak{g}$. Consider the Lie group $LN_+$ of $\mathfrak{L}N_+$. It consists of polynomial maps from the circle to the nilpotent subgroup $\mathfrak{N}_+$ of $G$.

The Lie group $\mathfrak{N}_+$ is isomorphic to its Lie algebra $\mathfrak{N}_+$ via the exponential map. This allows us to introduce coordinates $x_\alpha, \alpha \in \Delta_+$, on $\mathfrak{N}_+$, such that $x_\alpha$ has weight $\alpha$ with respect to the natural action of the Cartan subgroup $H \subset G$, which is the Lie group of $\mathfrak{h}$. Denote the vector field corresponding to the left (respectively, right) action of a generator $e_\alpha$ of $\mathfrak{N}_+$ on $\mathfrak{N}_+$ by $e_\alpha^L$ (respectively, $e_\alpha^R$). In coordinates $x_\beta$ they are given by:

$$e_\alpha^L = \sum_{\beta \in \Delta_+} P_{\alpha, \beta}^L \frac{\partial}{\partial x_{\beta}}, \quad e_\alpha^R = \sum_{\beta \in \Delta_+} P_{\alpha, \beta}^R \frac{\partial}{\partial x_{\beta}},$$

where $P_{\alpha, \beta}^L$ and $P_{\alpha, \beta}^R$ are certain polynomials in $x_\gamma, \gamma \in \Delta_+$ of degree $\beta - \alpha$.

Coordinates $x_\alpha, \alpha \in \Delta_+$, on the group $\mathfrak{N}_+$ give us coordinates $x_\alpha(n), \alpha \in \Delta_+, n \in \mathbb{Z}$, on the group $LN_+$. Denote $a_\alpha^*(n) = x_\alpha(-n), a_\alpha(n) = \partial/\partial x_\alpha(n)$. These operators generate the Heisenberg algebra $H(\mathfrak{f})$. We have two commuting infinitesimal actions of $\mathfrak{L}N_+$ on $LN_+$ by vector fields: left and right.

Explicitly, we have

$$e_\alpha^L(z) = \sum_{n \in \mathbb{Z}} e_\alpha^L(n)z^{-n-1} = \sum_{\beta \in \Delta_+} P_{\alpha, \beta}^L(z) a_\beta(z),$$

$$e_\alpha^R(z) = \sum_{n \in \mathbb{Z}} e_\alpha^R(n)z^{-n-1} = \sum_{\beta \in \Delta_+} P_{\alpha, \beta}^R(z) a_\beta(z),$$

where $P_{\alpha, \beta}^L(z)$ and $P_{\alpha, \beta}^R(z)$ are obtained from the polynomials $P_{\alpha, \beta}^L$ and $P_{\alpha, \beta}^R$, respectively, by replacing $x_\gamma, \gamma \in \Delta_+$, by $a_\gamma^*(z)$, and we use the notation $e_\alpha(n) = e_\alpha \otimes t^n$.

These formulas define two embeddings $\mathfrak{L}N_+ \to U(\mathfrak{h}(\mathfrak{f}))_{\text{loc}}$ and hence two commuting actions of the Lie algebra $\mathfrak{L}N_+$ on the space $M$. 
Remark 2.1. We can define these actions in a coordinate independent way.

Introduce the Lie algebra $D(\overline{\mathfrak{p}}_+)$. It has generators $y^R(n), y \in \overline{\mathfrak{p}}_+, n \in \mathbb{Z},$ and $P(n), P \in \mathbb{C}[\overline{\mathbb{N}}_+], n \in \mathbb{Z},$ where $\mathbb{C}[\overline{\mathbb{N}}_+]$ stands for the ring of regular functions on $\overline{\mathbb{N}}_+$. If we choose coordinates $x_\alpha$ on $\overline{\mathbb{N}}_+$, then $\mathbb{C}[\overline{\mathbb{N}}_+] \simeq \mathbb{C}[x_\alpha]_{\alpha \in \Delta_+}$. There are the following relations:

\[
[y^R_1(n_1), y^R_2(n_2)] = [y_1, y_2]^R(n_1 + n_2), \quad [y^R(n), P(m)] = [y \cdot P](n + m),
\]

where $y \cdot P$ denotes the action of $y \in \overline{\mathfrak{p}}_+$ on $P \in \mathbb{C}[\overline{\mathbb{N}}_+]$ by vector field from the right, and

\[
[P(n), Q(m)] = 0, \quad P(n) = \sum_{n_1 + n_2 = n} P_1(n_1)P_2(n_2),
\]

if $P = P_1P_2$ in $\mathbb{C}[\overline{\mathbb{N}}_+]$. If we choose coordinates on $\overline{\mathbb{N}}_+$, the Lie algebra $D(\overline{\mathfrak{p}}_+)$ becomes isomorphic to the Heisenberg algebra $\mathfrak{h}(\overline{\mathfrak{g}})$.

The linear span of $y^R(n), y \in \overline{\mathfrak{p}}_+, n \geq 0,$ and $P(n), P \in \mathbb{C}[\overline{\mathbb{N}}_+], n > 0,$ is a Lie subalgebra $D_+(\overline{\mathfrak{p}}_+)$ of $D(\overline{\mathfrak{p}}_+)$. The module $M$ over $D(\overline{\mathfrak{p}}_+)$ can be defined as the module induced from the trivial one-dimensional representation $\mathbb{C}v$ of $D_+(\overline{\mathfrak{p}}_+)$.

The correlation functions, i.e. matrix elements of the currents

\[
y^R(z) = \sum_{n \in \mathbb{Z}} y^R(n)z^{-n-1}, \quad P(z) = \sum_{n \in \mathbb{Z}} P(n)z^{-n}
\]
can also be expressed in terms of action of $\overline{\mathfrak{p}}_+$ on $\mathbb{C}[\overline{\mathbb{N}}_+]$ in a coordinate independent way. These correlation functions uniquely determine vertex algebra structure on $M$.

One obtains, e.g., the following formula (compare with $\mathbb{B}$. $\mathbb{I}$):

\[
\left\langle v^s, \prod_{s=1}^m e_i^R(w_s) \prod_{j=1}^N P_j(z_j) v \right\rangle = \sum_{p=(1^1, \ldots, 1^N)} \prod_{j=1}^N \frac{j(e_i^R \cdots e_i^R P_{X_j})}{(w_{i_1,j} - w_{i_2,j}) \cdots (w_{i_m,j} - z_j)},
\]

where the summation is taken over all ordered partitions $1^1 \cup 1^2 \cup \ldots \cup 1^N$ of the set $\{i_1, \ldots, i_m\},$ where $P = \{i_1, i_2, \ldots, i_m\},$ and $j$ is the augmentation homomorphism $\mathbb{C}[\overline{\mathbb{N}}_+] \to \mathbb{C}$.

We note that the construction described above assigns a vertex algebra to an arbitrary affine algebraic group in place of $\overline{\mathbb{N}}_+$. \hfill \Box

Now put $W_\nu^\nu = M \otimes \pi_\nu^\nu,$ where $\pi_\nu^\nu$ was defined in the Introduction. The space $W_0^\nu$ is a VOA, the tensor product of the VOAs $M$ and $\pi_0^\nu$. We can extend the action of the differential $\partial$ to $W_\nu^\nu[z, z^{-1}]$ by the formula $\partial \otimes 1 + 1 \otimes \partial_z$. Denote by $W^\nu_\lambda$ the quotient of $W^\nu_\lambda \otimes \mathbb{C}[z, z^{-1}]$ by the total derivative and constants, if $\lambda = 0$. The space $W^\nu_\lambda$ is a Lie algebra, which is isomorphic to $U_\nu(\mathfrak{h}(\overline{\mathfrak{g}}) \oplus \mathfrak{h})_{\text{loc}}$. Introduce operators $D_i^\nu(n) : W^\nu_\lambda \to W^\nu_{\lambda - \alpha_i}, i = 1, \ldots, \ell,$ by the formula

\[
D_i^\nu(z) = \sum_{n \in \mathbb{Z}} D_i^\nu(n)z^{-n-\nu(\alpha_i, \lambda)} = e_i^R(z)V^\nu_{-\alpha_i}(z),
\]

where $V^\nu_{-\alpha_i}(z)$ is given by formula $1.2$.

Put

\[
G_i^\nu = D_i^\nu(1) = \sum_{n \in \mathbb{Z}} e_i^R(n)V^\nu_{-\alpha_i}(-n), \quad G_i^\nu : W_0^\nu \to W_{-\alpha_i}^\nu.
\]
Theorem 1. For generic $k \in \mathbb{Z}$, the affine algebra $\mathfrak{g}$ has the induced operator $\mathcal{W}_0 \to \mathcal{W}_{-\alpha_i}, i = 1, \ldots, \ell$.

Denote 
$$K_\nu(\mathfrak{g}) = \bigcap_{i=1}^\ell \ker \mathcal{W}_0^\nu G_i^\nu, \quad J_\nu(\mathfrak{g}) = \bigcap_{i=1}^\ell \ker \mathcal{W}_0^\nu \mathcal{G}_i^\nu.$$ 

According to Lemma 4.2.8 from [20], $K_\nu(\mathfrak{g})$ is a vertex algebra, and $J_\nu(\mathfrak{g})$ is a Lie algebra.

Let $V_k, k \in \mathbb{C}$, be the VOA of the affine algebra $\mathfrak{g}$. Recall that as a vector space 
$$V_k = U(\mathfrak{g}) \otimes U(\mathfrak{g}[t] \otimes \mathbb{C}_{k}) \mathbb{C},$$

where $\mathbb{C}_k$ stands for the trivial one-dimensional representation of the Lie subalgebra $\mathfrak{g}[t]$ of $\mathfrak{g}$, on which $K$ acts by multiplication by $k$. Its $\mathbb{Z}$-grading is inherited from the standard $\mathbb{Z}$-grading on $\mathfrak{g}$, such that $\deg A(n) = -n, \deg K = 0$, cf. [18, 27].

The Fourier coefficients of currents of the VOA $V_k$ form a Lie algebra $U_k(\mathfrak{g})_{\text{local}}$, the local completion of the universal enveloping algebra at level $k$, $U_k(\mathfrak{g}) = U(\mathfrak{g})/(K - k)U(\mathfrak{g})$.

**Theorem 1.** For generic $\nu$, the vertex algebra $K_\nu(\mathfrak{g})$ is isomorphic to the VOA $V_k$ of the affine algebra $\mathfrak{g}$, and $J_\nu(\mathfrak{g}) \simeq U_k(\mathfrak{g})_{\text{local}}$, where $k = -h^\vee + \nu^{-1}$, $h^\vee$ being the dual Coxeter number of $\mathfrak{g}$.

The proof of this theorem is analogous to the proof of Theorem 4.5.9 from [20]. We construct a family of complexes $C_\nu^*(\mathfrak{g})$ depending on the parameter $\nu$, whose 0th cohomology is $K_\nu(\mathfrak{g})$ and which has a well-defined classical limit, when $\nu \to 0$. We then show that in this limit all higher cohomologies of the complex vanish and the 0th cohomology can be identified with the limit of the VOA $V_k$ when $k \to \infty$. This will allow us to compute the cohomology of the complex $C_\nu(\mathfrak{g})$ for generic $\nu$ and identify $K_\nu(\mathfrak{g})$ with the VOA $V_k$.

**Example.** We give here an explicit realization of the kernel of the screening operator $G_1$ in the case when $\mathfrak{g} = \mathfrak{sl}_2$ (for generic $\nu$). In the following formulas we suppress the index 1. Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}_2$. Set 
$$e(z) = a(z), \quad h(z) = -2 : a(z) a^*(z) : + \frac{1}{\nu} a(z),$$

(2.6) 
$$f(z) = - : a(z) a^*(z) a^*(z) : + \frac{1}{\nu} \partial a^*(z) + \frac{1}{\nu} a(z) a^*(z).$$

These formulas first appeared in [24]. The Fourier coefficients of the above series satisfy the relations in $\mathfrak{sl}_2$ with $k = -2 + \nu^{-1}$. Moreover, the generating vector of $W_0^\nu$ is annihilated by all non-negative Fourier coefficients. Therefore we obtain a homomorphism $V_k \to W^\nu_0$. It is known that $V_k$ is irreducible for generic $k$. Hence for generic $\nu$ the above homomorphism is injective. On the other hand, it is shown in [17] that the generating series above commute with the screening operator 
$$G_1^\nu = - \sum_{n \in \mathbb{Z}} a(n)V_{-\alpha_1}(-n) : W^\nu_0 \to W^\nu_2.$$
Therefore the image of $V_k$ in $W_0^\nu$ lies in the kernel of $G_1$. The calculation of the characters given below proves that $V_k$ is equal to the kernel of $G_1$.

In [16, 17] we generalized the above construction to the case of an arbitrary simple Lie algebra $\mathfrak{g}$. The commutativity with the screening operators was proved in [23, 22]. This gives us a proof of Theorem [1]. In what follows, we will give an alternative proof of this theorem.

3. The complex

The $j$th group $C^j_\nu(\mathfrak{g})$ of our complex is

$$C^j_\nu(\mathfrak{g}) = \oplus_{l(s)=j} W^\nu_{s(p)-\rho},$$

where $s$ belongs to the Weyl group, $l(s)$ is its length, and $\rho \in \mathfrak{h}^*$ is the half-sum of positive roots of $\mathfrak{g}$.

The construction of the differentials of the complex $C^j_\nu(\mathfrak{g})$ follows closely the construction of the differentials of the complex $F^j_\nu(\mathfrak{g})$ from [20], Sect. 4.5 (where $\beta^j$ played the role of $\nu$).

Let $p = (p_1, \ldots, p_m)$ be a permutation of the set $(1, 2, \ldots, m)$. We define a contour of integration $C_p$ in the space $(\mathbb{C}^\times)^m$ with the coordinates $z_1, \ldots, z_m$ as the product of one-dimensional contours along each of the coordinates, going counterclockwise around the origin starting and ending at the point $z_i = 1$, and such that $|z_{p_1}| > |z_{p_2}| > \ldots > |z_{p_m}|$ whenever $z_i \neq 1$.

Denote by $i = (i_1, \ldots, i_m)$ a sequence of numbers from 1 to $l$, such that $i_1 \leq i_2 \leq \ldots \leq i_m$. We can apply a permutation $p$ to this sequence to obtain another sequence $p(i) = (i_{p(1)}, \ldots, i_{p(m)})$. Put $\gamma = \sum_{j=1}^m \alpha_{i_j}$. Let us define an operator $D^\nu_{p(i)}$ from $W^\nu_\lambda$ to $W^\nu_{\lambda-\gamma}$ as the integral

$$\int_{C_p} dz_1 \ldots dz_m D^\nu_{p(i)}(z_1) \ldots D^\nu_{p(m)}(z_m) =$$

$$\int_{C_p} dz_1 \ldots dz_m \prod_{1 \leq k < \ell \leq m} (z_k - z_\ell)^{\nu(\alpha_{i_k}, \alpha_{i_\ell})} \prod_{1 \leq k \leq m} z_k^{\nu(\lambda, \alpha_{i_k})} : V^\nu_{\lambda-\alpha_{i_1}}(z_1) \ldots V^\nu_{\lambda-\alpha_{i_m}}(z_m) : \cdot$$

$$\cdot e_{i_1, \ldots, i_m}(z_1, \ldots, z_m),$$

where

$$V^\nu_{\lambda-\alpha_{i}}(z) = \sum_{n \in \mathbb{Z}} V^\nu_{\lambda-\alpha_{i}}(n) z^{-n},$$

and

$$e_{i_1, \ldots, i_m}(z_1, \ldots, z_m) = e^R_{i_1}(z_1) \ldots e^R_{i_m}(z_m).$$

The latter can be rewritten using Wick’s formula as a linear combination of normally ordered products of currents $a_\alpha(z)$ and $a^*_\alpha(z)$, $\alpha \in \Delta_+$, multiplied by rational functions in $z_1, \ldots, z_m$, which have poles only on the diagonals $z_i = z_j$. $D^\nu_{p(i)}$ is a linear operator from $W^\nu_\lambda$ to the completion $\hat{W}^\nu_{\lambda+\gamma}$ of $W^\nu_{\lambda+\gamma}$. Note that this operator is uniquely defined by the sequence $j = p(i)$, and so we can denote it by $D^\nu_{j}$. We choose the branch of a power function appearing in the integral, which takes real values for real $z_i$’s ordered so that $z_{j_1} > z_{j_2} > \ldots > z_{j_m}$. Thus, $C_p$ should be
viewed as an element of the group of relative \(m\)-chains in \((\mathbb{C}^\times)^m\) modulo the diagonals, with values in the one-dimensional local system \(E_j\), which is defined by the multi-valued function

\[
\prod_{1 \leq k < l \leq m} (z_k - z_l)^{-\nu(\alpha_{ik}, \alpha_{jl})} \prod_{1 \leq k \leq m} z_k^{-\nu(\lambda, \alpha_{ik})}.
\]

Our integral is well-defined for generic values of \(\nu\) over any such relative chain. Indeed, the integral

\[
\int_C dz_1 \ldots dz_m \prod_{1 \leq k < l \leq m} (z_k - z_l)^{\mu_{kl}} \prod_{1 \leq k \leq m} z_k^{\nu_k}
\]

over such a chain \(C\) converges in the region \(\text{Re} \mu_{kl} \geq 0\), and can be analytically continued to other values of \(\mu_{kl}\), which do not lie on hyperplanes

\[
(3.1) \quad \sum_{k,l \in S, k < l} \mu_{kl} = -s, \quad s \in \mathbb{Z}, s \geq (\#S) - 1,
\]

where \(S\) is a subset of the set \(\{1, 2, \ldots, m\}\), cf. [35], Theorem (10.7.7), for details.

For generic \(\nu\) the exponents in our integral do not lie on those hyperplanes. Indeed, an expression of the form

\[
\sum_{1 \leq k < l \leq r} \nu(\alpha_{jk}, \alpha_{ji})
\]

can take integral value for generic \(\nu\), if and only if

\[
(3.2) \quad \sum_{1 \leq k < l \leq r} (\alpha_{jk}, \alpha_{ji}) = 0.
\]

If this is the case, for our integral to converge, the rational functions appearing in \(E_{j_1, \ldots, j_r}(z_1, \ldots, z_r)\) should not have poles on the diagonals of combined order \(r - 1\) or more.

In the case when \(\alpha_{j_1} = \alpha_{j_2} = \ldots = \alpha_{j_{r-1}} = \alpha_i\), and \(\alpha_{j_r} = \alpha_j\), this fact follows from the Serre relations:

\[
(\text{ad} e_i^R)^{a_{ij}+1} \cdot e_j(m) = 0,
\]

which the operators \(e_i^R\) satisfy. Indeed, in order to satisfy (3.2), we should put \(r = -2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_j) + 2\). But the Serre relations imply that the coefficients of \(E_{i,j_1,\ldots,j_r}(z_1, \ldots, z_r)\) are rational functions in \(z_1, \ldots, z_r\), whose combined poles have combined order less than or equal to \(-a_{ij} = -2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) < r - 1\). Therefore, the equation (3.1) can not hold in this case.

In general we have to show that if (3.2) holds then any commutator of the form

\[
[e_{m_1}, [e_{m_2}, \ldots, e_{m_r}]],
\]

where \((m_1, \ldots, m_r)\) is a permutation of the set \((j_1, \ldots, j_r)\), vanishes. This is indeed the case, since \(\gamma = \sum_{k=1}^r \alpha_{j_k}, r > 1\), can not be a root of \(\text{Ad} e_i\), if (3.2) holds.

The proof, which works for an arbitrary Kac-Moody algebra with a symmetrizable Cartan matrix, was communicated to us by V. Kac (cf. [31]): the equation (3.2) can be rewritten as \(2(\rho, \gamma) = (\gamma, \gamma)\). This equation can not hold for imaginary roots, because then \((\gamma, \gamma) \leq 0\) and \((\rho, \gamma) > 0\). If \(\gamma\) is real, i.e. \(\gamma = w \cdot \alpha_i\) for a simple root \(\alpha_i\) and an element \(w\) of the Weyl group of \(\mathfrak{g}\), then we obtain \(2(\rho, w \cdot \alpha_i) = (\alpha_i, \alpha_i)\). This is true for \(w = 1\), but each reflection from the Weyl group increases the left hand side
because \( w \cdot \alpha_i > 0 \) by assumption while the right hand side remains the same. Hence this equality can not hold for \( w \neq 1 \).

This shows that our integrals are well-defined.

We can interpret the operator \( D^\nu_j \) as a composition operator \( G^\nu_{j_1} \cdots G^\nu_{j_m} \), cf. 20, Sect. 4.5.3. The operators \( G^\nu_{j} \) satisfy the \( q \)-Serre relations, where \( q = \exp(\pi iv) \), in the following sense 3 (see also 7, 20).

Consider a free algebra \( A \) with generators \( g_i, i = 1, \ldots, \ell \). We can assign to each monomial \( g_{j_1} \cdots g_{j_m} \) the contour \( C_j \), where \( j = (j_1, \ldots, j_m) \), and hence the operator \( D^\nu_j \).

This gives us a map \( \Delta \) from \( A \) to the space of linear combinations of such contours. Given such a linear combination \( C \), we define \( D^\nu_C \) as the linear combination of the corresponding operators \( D^\nu_j \).

Consider the two-sided ideal \( S_q \) in \( A \), which is generated by the \( q \)-Serre relations \( (adg_i)_q^{-a_{ij}+1} g_j, i \neq j \), where \( q = \exp(\pi iv) \).

**Lemma 1.** If \( C \) belongs to \( \Delta(S_q) \), then \( D^\nu_C = 0 \).

The proof is given in 20 (note however that in that paper the question of convergence of integrals was not addressed). It is based on rewriting the integrals over the contours \( C_j \) as integrals over other contours, where all variables are on the unit circle with some ordering of their arguments.

Lemma 1 means that the operators \( G^\nu_{j} \) “satisfy” the \( q \)-Serre relations of \( \mathfrak{g} \). Thus, we obtain a well-defined map, which assigns to each element \( P \) of the algebra \( U_q(\mathfrak{g}) \simeq A/S_q \) the operator \( D^\nu_P \).

**Lemma 2.** Let \( P \in U_q(\mathfrak{g}) \) be such that \( P \cdot 1_{\lambda} \) is a singular vector of weight \( \lambda + \gamma \) in the Verma module \( M^\lambda_q \) of highest weight \( \lambda \) over \( U_q(\mathfrak{g}) \). Then the operator \( D^\nu_P \) is a homogeneous linear operator \( W^\nu_\lambda \rightarrow W^\nu_{\lambda+\gamma} \), which commutes with the action of \( \partial \).

The proof is given in 20, Sect. 4.5.6.

We are ready now to define the differentials \( \delta^\nu_j : C^\nu_{j} (\mathfrak{g}) \rightarrow C^{j+1,\nu}_{j} (\mathfrak{g}) \) of the quantum complex \( C^\nu_{j} (\mathfrak{g}) \). Recall that for any pair \( s, s' \) of elements of the Weyl group of \( \mathfrak{g} \) there exists a singular vector \( P^q_{s',s} \cdot q^{\nu} w_{s'-s} \) of weight \( s'-s \) in the Verma module \( M^q_{s'-s} \), cf. 20, Sect. 4.4.5. We put:

\[
(3.3) \quad \delta^\nu_j = \sum_{l(s) = j, l(s') = j+1, s < s'} \epsilon_{s', s} \cdot D^{\nu q}_{P^q_{s',s}},
\]

where \( q = \exp(\pi iv) \), and \( \epsilon_{s', s} = \pm 1 \) are signs chosen in a special way. By Lemma 2, the differentials \( \delta^\nu_j \) are well-defined homogeneous linear operators. From the nilpotency of the differential of the quantum BGG resolution, cf. 20, Sect. 4.4.6, and Lemma 1 we derive that these differentials are nilpotent: \( \delta^{\nu+1} \delta^\nu_j = 0 \).

Thus, we have constructed a family of complexes \( C^\nu_{j} (\mathfrak{g}) \). We have: \( C^0_{j} (\mathfrak{g}) = W^\nu_0, C^1_{j} (\mathfrak{g}) = \bigoplus_{i=1}^\ell W^\nu_{\alpha_i}, \) and \( \delta^\nu : C^\nu_{j} (\mathfrak{g}) \rightarrow C^0_{j} (\mathfrak{g}) \) is given by the sum of the operators \( G^\nu_{j} : W^\nu_0 \rightarrow W^\nu_{-\alpha_i} \). Therefore the 0th cohomology of the complex \( C^\nu_{j} (\mathfrak{g}) \) is nothing but the VOA \( K^\nu_{j} (\mathfrak{g}) \).
Since the differentials of the complex $C^*_\nu(\mathfrak{g})$ commute with the action of $\partial$, we can form the double complex
\[
\mathbb{C} \rightarrow C^*_\nu(\mathfrak{g}) \xrightarrow{\pm \partial} C^*_\nu(\mathfrak{g}) \rightarrow \mathbb{C}.
\]
The 0th cohomology of the total complex $\tilde{C}^*_\nu(\mathfrak{g})$ of this double complex is the space $J_\nu(\mathfrak{g})$.

In order to compute the cohomologies of the complexes $C^*_\nu(\mathfrak{g})$ and $\tilde{C}^*_\nu(\mathfrak{g})$, we will study their classical limit $\nu \rightarrow 0$.

**Remark 3.1.** Let $\mathfrak{g}$ be an arbitrary symmetrizable Kac-Moody algebra, and $\alpha_1, \ldots, \alpha_\ell$ be the set of simple roots of $\mathfrak{g}$. Consider a set of homogeneous linear operators $X_i(n), i = 1, \ldots, \ell; n \in \mathbb{Z}$, acting on a $\mathbb{Z}$-graded linear space $M$ so that $\text{deg} X_i(n) = -n$. Define the series $X_i(z)V_{\nu,\alpha_i}(z), i = 1, \ldots, \ell,$ of linear operators acting from $M \otimes \pi'_\nu$ to $M \otimes \pi'_{\nu-\alpha_i}$. We can define integrals of products of these operators in the same way as above. Our proof of convergence above applies in the general case as well, and it shows that these integrals converge and Lemma 2 holds, if and only if $X_i(n)$’s satisfy the Serre relations of $\mathfrak{g}$:

\[ [X_i(n_1), [X_i(n_2), \ldots, [X_i(n_{-\alpha_j+1}), X_j(m)]\ldots]] = 0, \quad n_i, m \in \mathbb{Z}. \]

In other words, the operators $\int X_i(z)V_{\nu,\alpha_i}(z)dz$ “satisfy” the $q$-Serre relations, if and only if the operators $X_i(n)$ satisfy the Serre relations. \qed

4. **Classical limit**

Introduce new operators $a'_\alpha(n) = \nu a_\alpha(n)$. We have the commutation relations

\[ [a'_\alpha(n), a'_\beta(m)] = \nu \delta_{\alpha,\beta} \delta_{n,-m} \]

for the operators $a'_\alpha(n)$ and $a'_\beta(m)$ acting on $M$.

Consider a linear basis in $M$, which consists of monomials in $a'_\alpha(n), \alpha \in \Delta_+, n < 0$, and $a'_\alpha(n), \alpha \in \Delta_+, n \leq 0$, applied to the vacuum vector $v$. As a basis in $\pi'_\nu$ we take monomials in $b_i(n), i = 1, \ldots, \ell, n < 0$, applied to the vacuum vector $v_\lambda$. Tensor products of elements of these bases form a basis in $W^\nu_\lambda$. Using these bases, we can identify the spaces $W^\nu_\lambda$ with different values of $\nu$, so we can omit the superscript $\nu$ and write $W_\lambda$. The linear operators defined above for different $\nu$ should be considered as operators explicitly depending on the parameter $\nu$, acting spaces $W_\lambda$, which do not depend on $\nu$.

We can also identify the spaces $\overline{W}^\nu_\lambda$ with different $\nu$ and write $\overline{W}_\lambda$. However the Lie algebra structure on $\overline{W}_0$ will be $\nu$-dependent.

Consider the operator $G^\nu_1 : W_0 \rightarrow W_{-\alpha_i}$. It can be expanded in powers of $\nu$: $G^\nu_1 = G^0_1 + \nu(\ldots)$. We will call $G^0_1$ the classical screening operators. We can consider $G^0_1$ as an operator $W_\lambda \rightarrow W_{\lambda-\alpha_i}$ for any $\lambda$.

Put $G_i = T_{\alpha_i}G^0_i : W_0 \rightarrow W_0$, where $T_{\alpha_i} : W_{-\alpha_i} \rightarrow W_0$ was defined after formula (1.2).

**Lemma 3.** The operators $G_i$ generate an action of the nilpotent subalgebra $\mathfrak{g}_+$ of $\mathfrak{g}$ on $W_0$. 


The Lemma follows from Lemma 3 in the limit $q \to 1$. A different proof will be given below.

In the same way as in [20] we can show that $D^p_{P^s} = P_s(G) + \nu(\ldots)$, where $P_s \in U(\mathfrak{P}_-)\mathfrak{p}_+$ is the expression for the singular vector at $q = 1$, and $P_s(G)$ is the operator $W_{s(\rho) - \rho} \to W_{s(\rho) - \rho}$ obtained by inserting $G^0_i$ instead of the $e_i$ in $P_s$ for all $i = 1, \ldots, \ell$.

Thus, we have a well-defined limit of the complex $C^\ast_\mathfrak{g}(\mathfrak{f})$ as $\nu \to 0$. As a linear space it does not depend on $\nu$: $C^0_\mathfrak{g}(\mathfrak{f}) = \oplus_{(s) = j} W_{s(\rho) - \rho}$, and the differential is the $\nu \to 0$ limit of the differential (3.3):

$$
\delta^j_s = \sum_{l(s) = j(l'(s)) = j+1, s' < s'} \epsilon_{s', s} \cdot P_{s', s}(G).
$$

In the same way as in [20] we obtain the following result.

**Proposition 1.** The cohomology of the complex $C^\ast_\mathfrak{g}(\mathfrak{f})$ is isomorphic to the cohomology of $\overline{\mathfrak{p}}_+$ with coefficients in $W_0$, $H^\ast(\overline{\mathfrak{p}}_+, W_0)$.

The action of $\overline{\mathfrak{p}}_+$ on $W_0$ has geometric origin. The space $W_0$ can be considered as the algebra of regular functions on an infinite-dimensional linear space $\mathfrak{u}$ with coordinates $a^\alpha_n(n), \alpha \in \Delta_+, n < 0$, $a^\alpha_n(n), \alpha \in \Delta_+, n \leq 0$, and $b^i_n(n), i = 1, \ldots, \ell, n < 0$. The Lie algebra $\mathfrak{p}_+$ acts on this space by vector fields. This is the infinitesimal action of $\mathfrak{p}_+$, corresponding to an action of the Lie group $\mathfrak{p}_+$ on $\mathfrak{u}$. We will see later that this action of $\mathfrak{p}_+$ is free. Therefore the $\mathfrak{p}_+$-module $W_0$ is co-free, i.e., dual to a free module. Hence $H^i(\mathfrak{p}_+, W_0) = 0, i \neq 0$, and $H^0(\mathfrak{p}_+, W_0)$ is the algebra $\mathbb{C}[\mathfrak{u}]^{\mathfrak{p}_+}$ of $\mathfrak{p}_+$-invariant functions on $\mathfrak{u}$. The latter is a polynomial algebra of infinitely many variables.

We can find the degrees of the generators of the algebra $\mathbb{C}[\mathfrak{u}]^{\mathfrak{p}_+}$ by computing its character, which coincides with the Euler character of the complex $C^\ast_0(\mathfrak{f})$. We obtain:

$$
\text{ch } \mathbb{C}[\mathfrak{u}]^{\mathfrak{p}_+} = \prod_{n=1}^\infty (1 - q^n)^{-l} \prod_{\alpha \in \Delta} (1 - q^n u^\alpha)^{-1}.
$$

This gives us the following result.

**Theorem 2.** The 0th cohomology $K_0(\mathfrak{f})$ of the complex $C^\ast_0(\mathfrak{f})$ is isomorphic to a graded polynomial algebra of infinitely many variables, whose character is given by formula (1.1). All higher cohomologies of the complex $C^\ast_0(\mathfrak{f})$ vanish.

The 0th cohomology $J_0(\mathfrak{f})$ of the complex $\tilde{C}_0^\ast(\mathfrak{f})$ is isomorphic to the quotient of $K_0(\mathfrak{f})[t, t^{-1}]$ by the subspace of total derivatives and constants. All higher cohomologies of the complex $\tilde{C}_0^\ast(\mathfrak{f})$ vanish.

**Corollary 1.** For generic $\nu$ all higher cohomologies of the complex $C^\ast_\nu(\mathfrak{f})$ vanish, and the character of the 0th cohomology $K_\nu(\mathfrak{f})$ is given by formula (1.1).

The 0th cohomology $J_\nu(\mathfrak{f})$ of the complex $\tilde{C}_0^\ast(\mathfrak{f})$ is isomorphic to the quotient of $K_\nu(\mathfrak{f})[z, z^{-1}]$ by the subspace of total derivatives and constants. All higher cohomologies of the complex $\tilde{C}_0^\ast(\mathfrak{f})$ vanish.
In the next section we will identify the 0th cohomology of the complex $C_0^\ast(\mathfrak{g})$ with the classical limit of the VOA $V_k$ of $\mathfrak{g}$. This will complete the proof of Theorem 1.

Remark 4.1. Theorem 2 can be proved in a simpler way by considering a different classical limit of the differentials of the complex. Namely, we can take the linear basis in $W_\nu$, which consists of monomials in $a_\alpha(n), a_\alpha^\ast(n)$, and $b'(n) = \nu^{-1}b(n)$. Then in the limit $\nu \to 0$ the vertex operator $V_\nu^\ast(z) \to \text{Id}$, and so $G_\nu^\ast \to e_\nu^R(0)$. Therefore the cohomology of the complex $C_\ast^\nu(\mathfrak{g})$ in this limit become $H^\ast(\mathfrak{g}_+, \pi_0)$, where now $\mathfrak{g}_+$ acts on $\pi_0$ via the operators $e_\nu^R(0), \alpha \in \Delta_+$. This action is co-free by definition of the operators $e_\nu^R$. Hence we obtain Theorem 2.

This proof is simpler than the proof given above, but is not clear how to identify $V_k$ with the 0th cohomology of $C_\ast^\nu(\mathfrak{g})$ for $\nu \neq 0$. For that it is more convenient to use the other classical limit introduced above.

\section{Hamiltonian structure}

Consider the loop spaces $L\mathfrak{g}_+ = \mathfrak{g}_+[t, t^{-1}]$ and $L\mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$. We have the inner product $\langle \cdot, \cdot \rangle$ on the direct sum $L\mathfrak{g}_+ \oplus (L\mathfrak{g}_+)^\ast$, induced by the pairing $L\mathfrak{g}_+ \times (L\mathfrak{g}_+)^\ast \to \mathbb{C}$, and the inner product on $L\mathfrak{h}$, which are the restrictions of the invariant inner product on $\mathfrak{g}[t, t^{-1}]$:

$$\langle u(t), v(t) \rangle = \int (u(t), dv(t)),$$

where $\langle \cdot, \cdot \rangle$ is the invariant inner product on $\mathfrak{g}$ normalized so that the square of the maximal root equals 2.

Denote by $p_\alpha, \alpha \in \Delta_+$ coordinates on $\mathfrak{g}_+$, by $q_\alpha, \alpha \in \Delta_+$, the dual coordinates on $\mathfrak{g}_+$, and by $u_i, i = 1, \ldots, \ell$, be coordinates on $L\mathfrak{h}$. Let $W_0$ be the space of local functionals on $L\mathfrak{g}_+ \oplus (L\mathfrak{g}_+)^\ast \oplus L\mathfrak{h}$. It consists of functionals of the form

$$\int P(p_\alpha^{(n)}(t), q_\alpha^{(n)}(t), u_i^{(n)}(t); t)dt,$$

where $P$ is a polynomial in $(p_\alpha(t), q_\alpha(t), u_i(t)) \in L\mathfrak{g}_+ \oplus (L\mathfrak{g}_+)^\ast \oplus L\mathfrak{h}$ and their derivatives, and $t$. We use notation $f^{(n)}(t) = \partial^n f(t)$.

Local functionals can be considered as infinite sums of monomials in the Fourier coefficients $p_\alpha(n) = \int p_\alpha(t)t^n dt, q_\alpha(n) = \int q_\alpha(t)t^{-n-1} dt, u_i(n) = \int u_i(t)t^n dt$ of the polynomials $p_\alpha(t), q_\alpha(t),$ and $u_i(t)$, cf. [20], Sect. 2.1.

Put

$$W_0 = \mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}, u_i^{(n)}]|_{\alpha \in \Delta_+, i=1,\ldots,\ell, n\geq 0}.$$  

We identify it with $W_0$ defined in the previous section by identifying

$$p_\alpha^{(n)} \sim n!a_\alpha'(-n-1), \quad q_\alpha^{(n)} \sim n!a_\alpha^\ast(-n), \quad u_i^{(n)} \sim n!b_i(-n-1).$$

The operator $\partial$ acts on $W_0$ as a derivation. There is a map $W_0 \otimes \mathbb{C}[t, t^{-1}] \to \overline{W}$, which sends $P \otimes t^n \in W_0[t, t^{-1}]$ to the corresponding local functional, which we denote for simplicity by $\int P t^n$.

Using this map we can show that the space of local functionals $\overline{W}_0$ is isomorphic to the quotient of $W_0$ by the subspace of total derivatives and constants, cf. [20]. Thus,
we can identify the space of local functionals with $\overline{W}_0$, defined in the previous section, as a linear space.

There is a Lie bracket on $\overline{W}_0$. It coincides with the well-known Poisson structure, cf. [13, 28]:

$$\{ \int P, \int Q \} = - \sum_{1 \leq i,j \leq l} (\alpha_i, \alpha_j) \int \frac{\delta P}{\delta u_i} \partial \frac{\delta Q}{\delta u_j} + \sum_{a \in \Delta_+} \int \frac{\delta P}{\delta p_a} \delta q_a - \sum_{a \in \Delta_+} \int \frac{\delta P}{\delta q_a} \delta p_a,$$

where $\delta/\delta f$ denotes variational derivative with respect to $f$.

This Lie bracket is uniquely defined by formulas (5.2) and (5.3) for any $P, Q \in \overline{W}_0$, the commutator is given by $[A, B] = \nu\{A, B\} + \nu^2(\ldots)$. For any $\lambda = \sum_{i=1}^\ell \lambda_i \alpha_i$, put $W_{\lambda} = W_0 \otimes e^{\lambda}$, where $\lambda = \sum_{i=1}^\ell \lambda_i \alpha_i$. We identify it with $W_{\lambda}$ defined in the previous section by identifying $v_{\lambda}$ with $e^{\lambda}$.

We have an action of $\partial$ on $W_{\lambda}$. We denote by $\overline{W}_{\lambda}$ the quotient of $W_{\lambda}[t, t^{-1}]$ by the subspace of total derivatives. This space can be interpreted as the space of functionals of the form $\int P(p^{(n)}_\alpha(t), q^{(n)}_\alpha(t), u^{(n)}_i(t); t)e^{\lambda v(t)} dt$. We will use simpler notation $\int Pe^{X}, P \in W_{\lambda}$ for such a functional.

For any $P \in \overline{W}_0$, $\int Qe^X \in \overline{W}_{\lambda}$ define their bracket

$$\{ \int P, \int Qe^X \} = \sum_{1 \leq i,j \leq l} (\alpha_i, \alpha_j) \int \frac{\delta P}{\delta u_i} \partial \frac{\delta Q}{\delta u_j} e^X - \partial \left( \frac{\delta Q}{\delta u_j} e^X \right) + \sum_{a \in \Delta_+} \int \frac{\delta P}{\delta p_a} \frac{\delta Q e^{\lambda n}}{\delta q_a} dt - \sum_{a \in \Delta_+} \int \frac{\delta P}{\delta q_a} \frac{\delta Q e^{\lambda n}}{\delta p_a} dt.$$

This bracket is uniquely defined by formulas (5.2) and

$$\{ \int u_i t^n, \int e^{X t^m} \} = (\lambda, \alpha_i) \int e^{X t^{n+m}}.$$

The map $\{\cdot, \cdot\} : \overline{W}_0 \times \overline{W}_{-\alpha_i} \to \overline{W}_{-\alpha_i}$ satisfies the Jacobi identity for any $\int P_1, \int P_2 \in \overline{W}_0$, $\int Qe^X \in \overline{W}_{\lambda}$ [20]. Hence it defines on $\overline{W}_{\lambda}$ a structure of module over the Lie algebra $\overline{W}_0$.

For $P \in W_0$ denote by $\xi(P)$ an operator on $W_\lambda$ given by

$$\sum_{1 \leq i,j \leq l} (\alpha_i, \alpha_j) \left[ \lambda_j \frac{\delta P}{\delta u_j} + \sum_{n=1}^\infty \left( \frac{\partial}{\partial u_i} \frac{\delta P}{\delta u_i}^{(n)} \right) \frac{\partial}{\partial u_j} \right].$$
\[
\sum_{\alpha \in \Delta_+} \sum_{n=1}^{\infty} \left( \frac{\partial^n}{\partial p_{\alpha}^{(n)}} \right) \frac{\partial}{\partial q_{\alpha}^{(n)}} - \sum_{\alpha \in \Delta_+} \sum_{n=1}^{\infty} \left( \frac{\partial^n}{\partial q_{\alpha}^{(n)}} \right) \frac{\partial}{\partial p_{\alpha}}.
\]

Clearly, \( [\xi(P), \partial] = 0 \) for any \( P \). The map \( \xi : W_0 \to \text{End} W_0 \) is the classical limit of the map \( P \to \int Y(P, z)dz \) for generic \( v \), cf. [20].

The map \( \xi : W_0 \to W_0 \) can be presented in the form \( \xi' \circ d \), where \( \xi' \) is a homomorphism \( \Omega / \text{Im} \partial \to \text{Vect} \) from the quotient of the space of one-forms \( \Omega \) by the action of \( \partial \) to the space of \( \partial \)-invariant vector field on \( \text{Spec} W_0 \). The map \( \xi' \) is a quasi-Poisson structure on \( \text{Spec} W_0 \) in the sense of Gelfand-Dickey-Dorfman [28, 29] (cf. also [21]).

We can also define for any \( Pe^{\mathbf{v}} \in W_{\lambda} \) a map \( \xi(Pe^{\mathbf{v}}) : W_0 \to W_{\lambda} \) by formula (5.4), where the first term should be replaced by

\[
\sum_{1 \leq i, j \leq \ell} (\alpha_i, \alpha_j) \left[ -\lambda_j Pe^{\mathbf{v}} + \sum_{n=1}^{\infty} \left( \frac{\partial^n}{\partial u_i^{(n)}} \right) \frac{\partial}{\partial u_j^{(n)}} \right].
\]

We have:

\[
\{ \int P, \int Q e^{\mathbf{v}} \} = \int [\xi(P) \cdot Q e^{\mathbf{v}}] = -\int [\xi(Q e^{\mathbf{v}}) \cdot Q],
\]

cf. [20, 21].

Now consider the following element of \( W_{-\alpha_i} \):

\[
\tilde{G}_i = \sum_{\beta \in \Delta_+} P_{\alpha_i, \beta}(q)p_{\beta} e^{-\phi_i}, \quad i = 1, \ldots, \ell,
\]

where \( P_{\alpha_i, \beta}(q) \) is a polynomial in \( q_\gamma, \gamma \in \Delta_+ \), obtained from \( P_{\alpha_i, \beta}^R \) in formula (2.2) by replacing \( x_\gamma \) by \( q_\gamma \). In the same way as in [20, 21] we obtain the following result.

**Lemma 4.** \( G_i^0 = \xi(\tilde{G}_i) \) and \( G_i^0 = \{ \cdot, \int \tilde{G}_i \} \).

Thus, the space \( J_0(\mathfrak{g}) \) can be considered as the space of local functionals in \( p_\alpha(t), q_\alpha(t), \) and \( u_i(t) \), which commute with \( \int \tilde{G}_i, i = 1, \ldots, \ell \), with respect to the Poisson bracket \( \{ \cdot, \cdot \} \). In other words, \( J_0(\mathfrak{g}) \) is the space of local integrals of motion of the system of hamiltonian equations, defined by the hamiltonian

\[
H = \sum_{i=1}^{\ell} \int \tilde{G}_i.
\]

This system reads:

\[
\partial_\tau p_\alpha(z, \tau) = \{ p_\alpha, H \}, \quad \partial_\tau q_\alpha(z, \tau) = \{ q_\alpha(z, \tau), H \}, \quad \partial_\tau u_i(z, \tau) = \{ u_i, H \}.
\]

Here \( p_\alpha(z, \tau), q_\alpha(z, \tau), \) and \( u_i(z, \tau) \) are considered as delta-like functionals, which are equal to the value of the corresponding function at the point \( t \), depending on the time \( \tau \).

Using formulas (5.2) and (5.3), we obtain the following equations:

\[
\partial_\tau p_\alpha = \sum_{j=1}^{\ell} \sum_{\beta \in \Delta_+} \frac{\partial P_{\alpha, \beta}^R}{\partial q_\alpha} p_{\beta} e^{-\phi_j}, \quad \alpha \in \Delta_+.
\]
\[ \partial_\tau q_\alpha = -\sum_{j=1}^\ell P^R_{\alpha_j,\alpha} e^{-\phi_j}, \quad \alpha \in \Delta_+, \]

\[ \partial_\tau \partial_\tau \phi_i = -\sum_{j=1}^\ell (\alpha_j, \alpha_j) \sum_{\beta \in \Delta_+} P^R_{\alpha_j,\beta} p_\beta e^{-\phi_j}, \quad i = 1, \ldots, \ell. \]

**Example.** In the case of \( \mathfrak{sl}_2 \) these equations read:

\[ \partial_\tau p(z, \tau) = 0, \quad \partial_\tau q(z, \tau) = -e^{-\phi}, \quad \partial_\tau \partial_\tau \phi(\tau, t) = pe^{-\phi}. \]

The system (5.5) should be compared to the system of Toda equations associated to \( \mathfrak{g} \), which reads

\[ \partial_\tau u_i(z, \tau) = \sum_{j=1}^\ell (\alpha_i, \alpha_j) e^{-\phi_j}, \quad i = 1, \ldots, \ell. \]

These equations are non-local in \( u_i(t) \), but possess local integrals of motion. The corresponding algebra of integrals of motion is the classical \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \).

**Remark 5.1.** The equations (5.3) imply that

\[ \partial_\tau \left( u_i - \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) p_\alpha q_\alpha \right) = 0, \quad i = 1, \ldots, \ell. \]

Therefore we can put:

\[ u_i = U_i = \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) p_\alpha q_\alpha \]

and eliminate \( \phi_i, i = 1, \ldots, \ell \) from the system (5.5) (recall that \( u_i = \partial_\tau \phi_i \)). We then obtain the following system of equations on the functions \( p_\alpha, q_\alpha, \alpha \in \Delta_+ \):

\[ \partial_\tau p_\alpha = \sum_{j=1}^\ell \sum_{\beta \in \Delta_+} P^R_{\alpha_j,\beta} p_\beta e^{\int z U_j dz}, \quad \alpha \in \Delta_+, \]

\[ \partial_\tau q_\alpha = -\sum_{j=1}^\ell P^R_{\alpha_j,\alpha} e^{\int z U_j dz}, \quad \alpha \in \Delta_. \]

Note that if we put: \( p_{\alpha i} = 1, i = 1, \ldots, \ell, p_\alpha = 0, \alpha \neq \alpha_i, i = 1, \ldots, \ell, \) in the resulting system, we will obtain a system of equations on the functions \( q_{\alpha i}, i = 1, \ldots, \ell \), which is equivalent to the Toda system. This is a version of Drinfeld-Sokolov reduction.

We will now show that local integrals of motion of the system (5.5) form the classical limit of the affine algebra \( \mathfrak{g} \).

Recall that the Lie group \( N_+ \) can be identified with the big cell of the flag manifold \( G/B_- \). Hence the Lie algebra \( \mathfrak{g} \) acts on \( N_+ \) from the left by vector fields. We can
write down formulas for these vector fields in terms of the coordinates \( x_\alpha, \alpha \in \Delta_+ \): the action of elements \( e_\alpha \) is given by the vector fields \( e_\alpha^L \), cf. formula (2.2), and

\[
h_i = - \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) x_\alpha \frac{\partial}{\partial x_\alpha}, \quad i = 1, \ldots, \ell,\]

(5.6)

\[
f_\alpha = \sum_{\beta \in \Delta_+} Q_{\alpha, \beta} \frac{\partial}{\partial x_\beta}, \quad \alpha \in \Delta_+,\]

where \( Q_{\alpha, \beta} \) is a certain polynomial of degree \( \alpha + \beta \).

Introduce elements \( E_i, F_i \), and \( H_i, i = 1, \ldots, \ell \) of \( W_0 \) as follows:

\[
E_i = u_i - \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) p_\alpha q_\alpha, \quad F_i = \sum_{\beta \in \Delta_+} Q_{\alpha, \beta} p_\beta + u_i q_\alpha + \frac{2}{(\alpha_i, \alpha)} \partial q_\alpha.
\]

Here \( P_{\alpha, \beta}(q) \) and \( Q_{\alpha, \beta}(q) \) are obtained from the polynomials \( P_{\alpha, \beta} \) and \( Q_{\alpha, \beta} \) given by (2.2) and (5.6), respectively, by replacing \( x_\gamma, \gamma \in \Delta_+ \), by \( q_\gamma \).

Now we define elements \( E_\alpha \) and \( F_\alpha \) for all other \( \alpha \in \Delta_+ \) by induction. If \( [e_\beta, e_\gamma] = e_{\beta+\gamma} \) in \( g \), then we put \( E_{\beta+\gamma} = \xi(E_\beta) \cdot E_\gamma \), where \( \xi(\cdot) \) is defined by formula (5.4), and analogously for \( F_\alpha \).

**Theorem 3.** The space \( K_0([g]) \) is the algebra of differential polynomials in \( E_\alpha, F_\alpha, \alpha \in \Delta_+ \), and \( H_i, i = 1, \ldots, \ell \),

\[
K_0([g]) = \mathbb{C}[E_\alpha^{(n)}, H_i^{(n)}, F_\alpha^{(n)}]_{\alpha \in \Delta_+; i = 1, \ldots, \ell; n \geq 0}.
\]

The space \( J_0([g]) \) is the space of local functionals in \( E_\alpha, F_\alpha, \alpha \in \Delta_+ \), and \( H_i, i = 1, \ldots, \ell \),

\[
J_0([g]) = K_0([g])[t, t^{-1}]/(\text{Im} \partial \oplus \mathbb{C}).
\]

**Proof.** One checks directly that the elements \( E_\alpha, F_\alpha, \) and \( H_j \) defined above lie in the kernel of the operators \( G_i^0, i = 1, \ldots, \ell \). Since \( \partial \) commutes with \( G_i^0 \), the derivatives of these elements also lie in the kernel. But \( G_i^0 \cdot (PQ) = (G_i^0 \cdot P)Q + P(G_i^0 \cdot Q) \). Hence the algebra generated by these elements lies in \( K_0([g]) \). But its character equals the character of \( K_0([g]) \), cf. Theorem 2. Hence it coincides with \( K_0([g]) \).

The second part of the theorem follows from Theorem 2. □

Now consider the hyperplane \( g_1^* \) in the restricted dual space \( g^* \) to the affine algebra \( g \), which consists of those linear functionals on \( g \) which are equal to 1 on the central element \( K \in g \). This space has a canonical Kirillov-Kostant Poisson structure.

If we choose a coordinate \( t \) on the circle, we can identify \( g_1^* \) with \( \bar{\mathfrak{g}}[t, t^{-1}] \). The space \( \mathcal{L}(g) \) of local functionals on \( \bar{\mathfrak{g}}[t, t^{-1}] \) then becomes a Lie algebra with respect to the Kirillov-Kostant bracket.

This bracket is uniquely defined by the brackets of the linear functionals on \( g_1^* \). Such a functional is simply an element of \( \bar{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \). The Kirillov-Kostant bracket of two such functionals, \( A(n) = A \otimes t^n \) and \( B(m) = B \otimes t^m \) is given by

\[
\{A(n), B(m)\} = [A, B](n + m) + n(A, B)\delta_{n,-m}.
\]
Proposition 2. $J_0(\mathfrak{g}) \simeq \mathcal{L}(\mathfrak{g})$ as Lie algebras.

Proof. We have to check that Fourier components of the differential polynomials $E_{a_i}, F_{a_i}$, and $H_i$ have the same Poisson brackets as the corresponding elements of $\mathcal{L}(\mathfrak{g})$. But this automatically follows from our construction. \hfill \Box

Thus, we have shown that the space $J_0(\mathfrak{g})$ of local integrals of motion of the system (5.5) is isomorphic to the space $\mathcal{L}(\mathfrak{g})$ of local functionals on the dual space to the affine algebra $\mathfrak{g}$. The space $\mathcal{L}(\mathfrak{g})$ is therefore the classical limit of the Lie algebra $U_k(\mathfrak{g})_{\text{loc}}$ as $k \to \infty$.

The space $K_0(\mathfrak{g})$ is the classical limit of the VOA $V_k$ as $k \to \infty$. We know from Theorem 2 and Corollary 1 that $K_0(\mathfrak{g})$ is also a VOA whose classical limit coincides with $K_0(\mathfrak{g})$. Knowing the character of $K_0(\mathfrak{g})$ it is easy to show that $V_k$ is the only possible quantum deformation of $K_0(\mathfrak{g})$. Finally, $k$ can be computed from the commutator of the elements $H_i(n)$, which do not change under deformation. Therefore $k = -h^\vee + \nu^{-1}$. This completes the proof of Theorem 2.

This implies that $K_0(\mathfrak{g})$ coincides with $U_0(\mathfrak{g})_{\text{loc}}$ for generic $\nu$. In particular, for any $k \neq -h^\vee$ there exists a free field realization of the affine algebra $\mathfrak{g}$ of level $k$, i.e. a homomorphism $\mathfrak{g} \to U(\mathfrak{g}) \oplus \mathfrak{g}_{\text{loc}}$, which sends $K$ to $k$. It coincides with Wakimoto realization, which was first constructed geometrically in [10].

6. INTEGRALS OF MOTION OF THE DEFORMED CFT

We now want to add an extra operator $G_0^\nu$, corresponding to the extra simple root $\alpha_0$ of $\mathfrak{g}$, to the set $G_i^\nu$, $i = 1, \ldots, \ell$. This operator will define certain deformations of the conformal field theories associated to $\mathfrak{g}$: Wess-Zumino-Novikov-Witten model and generalized parafermions. We will determine local integrals of motion of these deformations by an analogue of the procedure from the previous section. The classical limits of these integrals of motion will be shown to coincide with local integrals of motion of the AKNS equation and its generalizations.

6.1. The case $\mathfrak{g} = \hat{\mathfrak{sl}}_2$. In this case we have one operator $G_i^\nu = \int a(z) V_{-\alpha_i}(z) dz$. It is natural to put

$$G_0^\nu = \int a^*(z) V_{\alpha_0}^\nu(z) dz.$$  

The operator $G_0^\nu : W_0^\nu \to W_0^\nu$ commutes with the action of $\partial$ and hence defines an operator $G_0^\nu : \overline{W}_0^\nu \to \overline{W}_0^\nu$. We can now define a Lie algebra $J_\nu(\hat{\mathfrak{sl}}_2)$ as

$$J_\nu(\hat{\mathfrak{sl}}_2) = \text{Ker}_{\overline{W}_0^\nu} G_0^\nu \bigcap \text{Ker}_{\overline{W}_0^\nu} G_1^\nu.$$  

For $\lambda \in \mathbb{C}$, denote by $M_{\lambda,k}^\nu$ the module contragradient to the Verma module over $\hat{\mathfrak{sl}}_2$ with highest weight $\lambda$. There is a unique primary field (or vertex operator) of weight $\lambda$ (i.e. spin $\lambda/2$) $\Phi_{\lambda}^\mu(z) : V_k \to M_{\lambda,k}^\nu; \mu = \lambda, \lambda - 2, \ldots$. Recall that, by definition, the action of $\hat{\mathfrak{sl}}_2$ on the components of this field by commutator coincides with its action on the homogeneous components of the evaluation representation corresponding to contragradient Verma module of highest weight $\lambda$ over $\mathfrak{sl}_2$, see, e.g., [13].
We know that $\text{Ker}_{\mathcal{W}_0} G_1^\nu = V_k$ with $k = -2 + \nu^{-1}$ for generic $\nu$, according to Theorem \[1\]. Also $W_\nu^\nu \simeq M_{\alpha,k}^*$ as an $\hat{sl}_2$-module for generic $\nu$. The restriction of the fields $a^*(z)^m V_\lambda^\nu(z), m = 0, 1, \ldots : W_0^\nu \to W_\nu^\nu$ to $V_k$ is the primary field of weight $\lambda$, cf. \[17\]. In particular, $a^*(z)V^\nu_{\alpha}$ is the component $\Phi_2^\nu(z)$ of the primary field corresponding to the adjoint representation of $\hat{sl}_2$ (we can identify $\alpha$ with 2).

Hence for generic $\nu$ elements of the space $J_\nu(\hat{sl}_2)$ can be interpreted as vectors $P \in V_k$, such that $\int \Phi_2^0(z) dz \cdot P = \partial Q$ for some $Q \in W_\nu^\nu$. For such a $P$, $\int P(z)dz \in U_k(\mathfrak{g})_{\text{loc}}$ commutes with $\int \Phi_2^\nu(z) dz$. Following Zamolodchikov \[18\], we can interpret elements of $J_\nu(\hat{sl}_2)$ as local integrals of motion (in the first order of perturbation theory) of the deformation of conformal field theory of $\hat{sl}_2$ by the field $\Phi_2^0(z)$. The space $J_\nu(\mathfrak{g})$ has a $\mathbb{Z}$-gradation, which is obtained by subtracting 1 from the $\mathbb{Z}$-gradation on $W_0^\nu$. We will call an element of $J_\nu(\hat{sl}_2)$ of degree $s$ an integral of motion of spin $s$.

The field $\Phi_2^0(z)$ commutes with the homogeneous Heisenberg subalgebra $\mathfrak{h}$ of $\hat{sl}_2$. Hence it defines a primary field of the coset model $\hat{sl}_2/\mathfrak{h}$ of arbitrary level $k \neq -2$, which can be viewed as the analytic continuation of the parafermionic theory \[15, 14\] corresponding to the case of integral $k$. This field is called the first thermal operator and is denoted by $\epsilon_1(z)$. The corresponding deformation of the parafermionic theory was studied by Fateev \[14\].

It is expected that for a positive integer $k$ the parafermionic theory is equivalent to the $(k + 1, k + 2)$ minimal model of the conformal field theory associated to the $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_k)$. Hence we can assume that this deformation is equivalent to the standard deformation of the latter theory by the “adjoint” field \[14, 14\], in which spins of integrals of motion are presumably all positive integers, which are not divisible by $k$. Thus we can expect the spins of integrals of motion of the deformation of the parafermionic theory to be the same, as proposed \[14\]. This allowed Fateev to predict the factorizable $S$–matrix of the deformed theory using the standard bootstrap technique, see \[14\].

Since $\Phi_2^0(z)$ commutes with $\mathfrak{h}$, we have $U(\mathfrak{h})_{\text{loc}} \subseteq J_\nu(\hat{sl}_2)$, and hence $J_\nu(\hat{sl}_2)$ decomposes into the direct sum $U(\mathfrak{h})_{\text{loc}} \oplus J'_\nu(\hat{sl}_2)$. By definition, elements of $J'_\nu(\hat{sl}_2)$ are local integrals of motion of the deformation of the WZNW theory of level $k = -2 + \nu^{-1}$ by $\int \Phi_2^\nu(z) dz$. Note that the elements of $J'_\nu(\hat{sl}_2)$ are also in one-to-one correspondence with the integrals of motion of the deformation of the parafermionic theory by $\int \epsilon_1(z) dz$.

**Conjecture 1.** For generic $\nu$ the space $J'_\nu(\hat{sl}_2)$ is linearly spanned by elements of all positive integral spins.

We expect that when $k$ is an integer, the integral of motion of spins divisible by $k$ indeed “drops out”, in agreement with the prediction of \[14\].

In the next section we will prove Conjecture \[1\] in the classical limit $\nu \to 0$, and show that these integrals of motion are in fact quantum deformations of the hamiltonians of the AKNS hierarchy. We remark that the connection between the integrals of motion in the deformations of the parafermionic theory and the non-linear Schrödinger hierarchy (which is a reduction of the AKNS hierarchy) has been previously dicussed by Schiff \[34\].
Now we want to realize $J'_\nu(\hat{\mathfrak{sl}}_2)$ as a cohomology group of a complex $C'^*_\nu(\hat{\mathfrak{sl}}_2)$, which is constructed in the same way as the complex $C'^*_\nu(\mathfrak{g})$ in Sect. 3.

We use the BGG resolution of $U_q(\hat{\mathfrak{sl}}_2)$ with $q = \exp(\pi i \nu)$, cf. [20], Sects. 3.1, 4.4. Put $C^0_{\nu}(\hat{\mathfrak{sl}}_2) = W^\nu_0$ and $C^j_{\nu}(\hat{\mathfrak{sl}}_2) = W^\nu_j \oplus W^\nu_{-2j}$, $j > 0$. The operators $a(m), m \in \mathbb{Z}$, and $a^*(m), m \in \mathbb{Z}$, satisfy the Serre relations of $\mathfrak{sl}_2$. Hence, according to Remark 3.1, the operators $G^\nu_1$ and $G^\nu_2$ satisfy the $q$–Serre relations of $\hat{\mathfrak{sl}}_2$ in the sense of Lemma 4. Thus we can define differentials of the complex $C'^*_\nu(\hat{\mathfrak{sl}}_2)$ using the differentials of the BGG resolution of $U_q(\hat{\mathfrak{sl}}_2)$ in the same way as in Sect. 3.

We have an analogue of Lemma 4: the differentials of the complex $C'^*_\nu(\hat{\mathfrak{sl}}_2)$ are homogeneous operators, which commute with the action of $\partial$. Therefore we can define the quotient complex $\hat{C}^*_\nu(\hat{\mathfrak{sl}}_2)$, such that $\hat{C}^j(\hat{\mathfrak{sl}}_2) = C^j(\hat{\mathfrak{sl}}_2)/\text{Im } \partial$, $j > 0$, and $\hat{C}^0(\hat{\mathfrak{sl}}_2) = C^0(\hat{\mathfrak{sl}}_2)/(\text{Im } \partial + \mathbb{C})$. The 0th differential of $\hat{C}^*_\nu(\hat{\mathfrak{sl}}_2)$ is equal to $\hat{G}^\nu_1 + \hat{G}^\nu_2 : \hat{W}^\nu_0 \rightarrow \hat{W}^\nu_{-\alpha} \oplus \hat{W}^\nu_{\alpha}$. Therefore the space $J'^*_\nu(\hat{\mathfrak{sl}}_2)$ is the 0th cohomology of the complex $\hat{C}^*_\nu(\hat{\mathfrak{sl}}_2)$.

In the next section we will compute the cohomology of this complex in the classical limit $\nu \rightarrow 0$. But first we will define an analogue of the space $J'_\nu(\mathfrak{g})$ for an arbitrary affine algebra $\mathfrak{g}$.

6.2. General case. As an analogue of the operator $G^\nu_0$ for $\mathfrak{g}$ we will take a weight 0 component from the vertex operator $\Phi_{\text{adj}}(z)$ corresponding to the adjoint representation of $\mathfrak{g}$. The adjoint representation is isomorphic to $\mathfrak{g}$ itself, so its highest weight is

$$\alpha_{\text{max}} = -\alpha_0 = \sum_{i=1}^{\ell} a_i \alpha_i,$$

where $\alpha_0$ is the weight corresponding to the 0th simple root of $\mathfrak{g}$. The weight 0 component is isomorphic to the Cartan subalgebra $\mathfrak{h}$. More generally, we can consider the vertex operator corresponding to the contragradient Verma module $M_{\alpha_{\text{max}}}^*$ over $\mathfrak{g}$ with highest weight $\alpha_{\text{max}}$. This module contains the adjoint representation as a submodule, and its component $M_{\alpha_{\text{max}}}^*(0)$ of weight 0 contains $\mathfrak{h}$.

To each vector $x \in M_{\alpha_{\text{max}}}^*(0)$ we can associate a field $\Phi^x_{\text{adj}}(z) : V_k \rightarrow M_{-\alpha_0,k}^*$. For generic $k = -h^\vee + \nu^{-1}$ this field has a bosonic realization $\Psi_x(z) = P_x(a_\alpha^*(z))^\nu_{-\alpha_0} : W^\nu_0 \rightarrow W^\nu_{-\alpha_0}$, where $P_x(x_\alpha)$ is a polynomial in $x_\alpha$, $\alpha \in \Delta_+$, of weight $-\alpha_{\text{max}}$ representing $x \in M_{\alpha_{\text{max}}}^*(0) \simeq \mathbb{C}[x_\alpha]$.

Introduce the operator

$$G^\nu_0 = \int \Psi_x(z) dz : W^\nu_0 \rightarrow W^\nu_{-\alpha_0}.$$

It commutes with the action of $\partial$ and hence defines an operator $\hat{G}^\nu_0 : \hat{W}^\nu_0 \rightarrow \hat{W}^\nu_{-\alpha_0}$. Define

$$J'_\nu(\mathfrak{g})_x = \bigcap_{i=0}^\ell \text{Ker } \hat{G}^\nu_0 \cap \hat{W}^\nu_0.$$
Remark 3.1, this happens if the operators for which the integrals of motion of the deformation of the WZNW model of level \( k = -h^\nu + \nu^{-1} \) by \( \int \Phi^x_{\text{adj}}(z)dz \).

By definition, the field \( \Phi^x_{\text{adj}}(z) \) commutes with the action of the Heisenberg subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). Therefore we can write \( J_\nu(\mathfrak{g})_x = U(\mathfrak{h})_{\text{loc}} \oplus J_\nu(\mathfrak{g})_x \). The space \( J_\nu(\mathfrak{g})_x \) can be considered as the space of local integrals of motion of a deformation of the generalized parafermionic theory associated to \( \mathfrak{g} \) (the \( \mathfrak{g}/\mathfrak{h} \) coset model) by the field \( \Phi^x_{\text{adj}}(z) \).

Now we want to construct a complex \( C^*_\nu(\mathfrak{g})_x \) whose cohomology equals \( J_\nu(\mathfrak{g})_x \), using the BGG resolution of \( U_q(\mathfrak{g}) \). We put \( C^*_\nu(\mathfrak{g})_x = \oplus_{n=0}^{\nu} W^\nu_{s(n) + \rho} \). In order to define the differentials of the complex, we need to make sure that the screening operators \( G^\nu_i, i = 0, \ldots, \ell \), satisfy the \( q \)-Serre relations of \( \mathfrak{g} \) in the sense of Lemma 1. According to Remark 3.1, this happens if the operators \( e_i^R(n), i = 1, \ldots, \ell \), and \( P_x(m) \) satisfy the ordinary Serre relations. Note that \( P_x(m) \) automatically satisfies the relations \( [e_i^R(n_1), [e_i^R(n_2), P_x(m)]] = 0 \) for all \( i \). Therefore the Serre relations hold for those \( i \), for which the \( i \)-th node of the Dynkin diagram is not connected to the 0th node. In addition, the relations \( [e_i^R(n), P_x(m)] = 0 \) have to hold for all other \( i \). In other words, for those \( i \)'s the vector \( x \in M^\nu_{\alpha_{\text{max}}} \) should be annihilated by the right action of \( e_i \). In that case, according to Remark 3.1, the operators \( G^\nu_i, i = 0, \ldots, \ell \), satisfy the \( q \)-Serre relations of \( \mathfrak{g} \) in the sense of Lemma 1.

Therefore we can construct differentials of the complex \( C^*_\nu(\mathfrak{g})_x \) using the differentials of the BGG resolution of \( \mathfrak{g} \) in the same way as in Sect. 3. As before, the differentials of the complex \( C^*_\nu(\mathfrak{g})_x \) are homogeneous operators, which commute with the action of \( \partial \).

Hence we can define the quotient complex \( \tilde{C}^*_\nu(\mathfrak{g})_x \), such that \( \tilde{C}^i(\hat{\mathfrak{s}}_2) = C^i(\mathfrak{g})_x / \text{Im } \partial, j > 0, \) and \( \tilde{C}^0(\hat{\mathfrak{s}}_2)_x = C^0(\mathfrak{g})_x / \text{Im } \partial \oplus \mathbb{C} \). The 0th differential of \( \tilde{C}^*_\nu(\mathfrak{g})_x \) is equal to \( \sum_{i=0}^\ell \tilde{C}^i: \tilde{W}^\nu_0 \rightarrow \oplus_{i=0}^\ell \tilde{W}^\nu_{-\alpha_i} \). Therefore the space \( J_\nu^0(\mathfrak{g})_x \) is the 0th cohomology of the complex \( \tilde{C}^*_\nu(\mathfrak{g})_x \).

Conjecture 2. Suppose that \( x \in M^*_{\alpha_{\text{max}}}(0) \) satisfies: \( e_i^R \cdot x = 0 \) for all \( i \), such that the \( i \)-th node of the Dynkin diagram is not connected to the 0th node. Then for generic \( \nu \) the space \( J_\nu^0(\mathfrak{g})_x \) has a linear basis, which consists of \( \ell \) elements of each positive integral spin.

In Sect. 8 we will compute this cohomology of the complex \( C^*_\nu(\mathfrak{g})_x \) and prove Conjecture 2 in the classical limit, when \( P_x = x_{\alpha_{\text{max}}} \) with respect to a special coordinate system.

7. Classical limit in the case of \( \hat{\mathfrak{s}}_2 \)

We consider now the new basis defined in Sect. 4, which consists of monomials in \( a'(n), a^*(n), \) and \( b(n) \). This basis enables us to identify the spaces \( W^\nu_\lambda \) and \( \tilde{W}^\nu_\lambda \), respectively, for different \( \nu \), but consider the differentials of the complexes \( C^*(\hat{\mathfrak{s}}_2) \) and \( \tilde{C}^*(\hat{\mathfrak{s}}_2) \) as \( \nu \)-dependent, as in Sect. 4.

In particular, we have for \( G^\nu_0: W_0 \rightarrow W_0 \), and \( G^\nu_0: W_0 \rightarrow W_0 \): \( G^\nu_1 = G^0_1 + \nu(\ldots), G^\nu_0 = \nu G^0_0 + \nu^2(\ldots) \). Put \( G_1 = T_0 G^0_1: W_0 \rightarrow W_0 \), and \( G_0 = T_{-\alpha} G^0_0 \).
Lemma 5. The operators $G_1$ and $G_0$ generate an action of the nilpotent subalgebra $n_+$ of $\widehat{\mathfrak{sl}_2}$ on $W_0$.

This follows from the $q$–Serre relations to which the operators $G_i^\nu$ satisfy in the limit $\nu \to 0$. In Proposition 3 we will give another proof of this fact. This implies

**Proposition 3.** The cohomology of the complex $C^*_0(\widehat{\mathfrak{sl}_2})$ is isomorphic to the cohomology of $n_+$ with coefficients in $W_0$, $H^*_0(n_+, W_0)$.

7.1. Separation of variables. It is convenient to realize $W_0$ as $\mathbb{C}[p^{(n)}, q^{(n)}, u^{(n)}]_{n \geq 0}$, where the variables $p^{(n)}, q^{(n)}$ and $u^{(n)}$ are defined by formula (5.1). In terms of these variables we can write as in Lemma 4: $G_1^0 = \xi(-pe^{-\phi}), G_0^0 = \xi(qe^{\phi})$, and $G_0^1 = \{\cdot, -\int pe^{-\phi}\}, G_0^0 = \{\cdot, \int qe^{\phi}\}$.

More explicitly, we have:

$$G_1 = 2 \sum_{n=0}^{\infty} \partial^n(pe^{-\phi}) \frac{\partial}{\partial u^{(n)}} + \sum_{n=0}^{\infty} \partial^n e^{-\phi} \frac{\partial}{\partial q^{(n)}},$$

$$G_0 = 2 \sum_{n=0}^{\infty} \partial^n(qe^{\phi}) \frac{\partial}{\partial u^{(n)}} + \sum_{n=0}^{\infty} \partial^n e^{\phi} \frac{\partial}{\partial p^{(n)}}.$$

By definition, $J_0(\widehat{\mathfrak{sl}_2})$ is the kernel of the operator $\{\cdot, \int (-pe^{-\phi} + qe^{\phi})\}$. Therefore we can interpret $J_0(\widehat{\mathfrak{sl}_2})$ as the space of local integrals of motion of the system of equations:

$$\partial_\tau p(z, \tau) = e^\phi, \quad \partial_\tau q(z, \tau) = e^{-\phi},$$

$$\partial_\tau \partial_z \phi(z, \tau) = 2qe^{\phi} + 2pe^{-\phi}.$$

**Remark 7.1.** These equations imply that $\partial_\tau (u - 2pq) = 0$. Hence we can put $u = 2pq$. Then we obtain the system

$$(7.1) \quad \partial_\tau p(z, \tau) = e^{2\int^z pqd\z), \quad \partial_\tau q(z, \tau) = e^{-2\int^z pqd\z}.$$

If we identify $q(z, \tau) = \overline{p}(z, \tau)$ and replace $z$ by $iz$, we obtain a non-local equation

$$(7.2) \quad \partial_\tau p(z, \tau) = e^{-2iz \int^z |p|^2d\z).$$

The remarkable fact, which will be proved below, is that the integrals of motion of the modified version of the AKNS hierarchy (resp., non-local Schrödinger hierarchy) are symmetries of equation (7.1) (resp., (7.2)).

Recall the formula for the field $h(z)$ (see the Example at the end of Sect. 3):

$$(7.3) \quad h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1} = \frac{1}{\nu} b(z) - 2 : a(z)a^* (z) :.$$

It is easy to check that all Fourier coefficients of this field (they span the homogeneous Heisenberg subalgebra of $\widehat{\mathfrak{sl}_2}$) commute with the screening operators $G_0^\nu$ and $G_1^\nu$ for all values of $\nu$. Let us compute the classical limits of these Fourier coefficients in terms of the variables $p^{(n)}, q^{(n)}, u^{(n)}$, defined by formula (5.1). They will certainly commute with the operators $G_0$ and $G_1$. 

Set \( v = u - 2pq \). Then the limit of \( \nu h_n, n < 0 \), as \( \nu \to 0 \) is the operator of multiplication by \( \frac{1}{(-n-1)!} v^{(n)} = \frac{1}{(-n-1)!} \partial^{-n-1} v \). On the other hand, a straightforward calculation shows that \( h_n, n > 0 \) equals \( n! \partial_{n-1} + \nu^2 (\cdots) \), where

\[
\partial_n = 2 \frac{\partial}{\partial u^{(n)}} + 2 \sum_{m \geq 0} \left( \frac{n + m + 1}{m} \right) \left[ p^{(m)} \frac{\partial}{\partial p^{(n+m+1)}} - q^{(m)} \frac{\partial}{\partial q^{(n+m+1)}} \right], \quad n \geq 0.
\]

Thus we obtain:

\[
\partial_n = 2 \frac{\partial}{\partial u^{(n)}} + 2 \sum_{m \geq 0} \left( \frac{n + m + 1}{m} \right) \left[ p^{(m)} \frac{\partial}{\partial p^{(n+m+1)}} - q^{(m)} \frac{\partial}{\partial q^{(n+m+1)}} \right], \quad n \geq 0.
\]

Finally, the leading term of \( h_0 \) as \( \nu \to 0 \) equals

\[
h_0 = 2 \sum_{m \geq 0} \left[ p^{(m)} \frac{\partial}{\partial p^{(m)}} - q^{(m)} \frac{\partial}{\partial q^{(m)}} \right] = 2 \delta_{n,m}.
\]

We find from the definition of \( \partial_n \) the following relations:

\[
\partial_n, \partial = \partial_{n-1}, \quad n > 0, \quad \partial_0, \partial = h_0.
\]

This implies that the operator \( \partial - \frac{1}{2} vh_0 \) commutes with all \( \partial_n, n \geq 0 \). Let us define the new variables \( \tilde{p}^{(n)}, \tilde{q}^{(n)}, n \geq 0 \), by the formulas

\[
\tilde{p}^{(n)} = (\partial - \frac{1}{2} vh_0)^n p, \quad \tilde{q}^{(n)} = (\partial - \frac{1}{2} vh_0)^n q.
\]

Then \( \tilde{p}^{(n)} = p^{(n)} + P_n, \tilde{q}^{(n)} = q^{(n)} + Q_n \), where \( P_n, Q_n \in W_0 \) lie in the ideal generated by \( v^{(m)}, m \geq 0 \). Since \( \partial_m \cdot p = \partial_n \cdot q = 0 \), we find that that \( \partial_m \tilde{p}^{(n)} = \partial_n \tilde{q}^{(n)} = 0 \) for all \( m \geq 0, n \geq 0 \). Moreover, formulas (7.3) imply that

\[
G_i \cdot (R(\tilde{p}^{(n)}, \tilde{q}^{(n)}))P(v^{(n)}) = (G_i \cdot R(\tilde{p}^{(n)}, \tilde{q}^{(n)}))P(v^{(n)}).
\]

Therefore we can separate the variables, that is represent \( W_0 \) as the tensor product \( \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}]_{n \geq 0} \otimes \mathbb{C}[v^{(n)}]_{n \geq 0} \), and \( G_i \)'s will act as derivations of the factor \( W_0 = \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}]_{n \geq 0} \). It is easy to find explicit expression for these derivations. Indeed, let us identify \( \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}]_{n \geq 0} \) with the quotient of \( W_0 \) by the ideal generated by \( v^{(n)}, n \geq 0 \). Then \( G_i \cdot \tilde{p}^{(n)} \) equals the projection of \( G_i \cdot p^{(n)} n \) onto this quotient, expressed as an element of \( \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}]_{n \geq 0} \). The same is true for \( G_i \cdot \tilde{q}^{(n)} \). This way we obtain the following expressions for \( G_1 \) and \( G_0 \) in the new variables:

\[
G_1 = \sum_{n=0}^{\infty} B_n^- \frac{\partial}{\partial \tilde{q}^{(n)}}, \quad G_0 = \sum_{n=0}^{\infty} B_n^+ \frac{\partial}{\partial \tilde{p}^{(n)}},
\]

where \( B_n^\pm \) is defined recursively as follows: \( B_0^\pm = 1 \), and

\[
B_n^\pm = \partial B_{n-1}^\pm + 2\tilde{p} \tilde{q} B_{n-1}^\pm.
\]

Here we denote by \( \tilde{\partial} \) the derivation of \( \tilde{W}_0 \), such that \( \partial \tilde{p}^{(n)} = \tilde{p}^{(n+1)} \), \( \partial \tilde{q}^{(n)} = \tilde{q}^{(n+1)} \).
7.2. Isomorphism with $\mathbb{C}[N_+/H_+]$. Let $N_+$ be the Lie group of $\mathfrak{n}_+$. This is a pronilpotent proalgebraic Lie group, which is isomorphic to $\mathfrak{n}_+$ via the exponential map. Denote by $\mathfrak{h}_+$ the Lie subalgebra $\mathfrak{h} \otimes t\mathbb{C}[t]$ of $\mathfrak{n}_+$ and by $H_+$ the corresponding subgroup of $N_+$. The Lie algebra $\mathfrak{h}_+$ is called the homogeneous abelian Lie subalgebra of $\mathfrak{n}_+$. The Lie algebra $\mathfrak{n}_+$ infinitesimally acts on the homogeneous space $N_+/H_+$ from the left. On the other hand, $N_+$ is isomorphic to the big cell of $B_-\backslash G$, and therefore $N_+$ acts infinitesimally on $N_+$ from the right. The Lie algebra $\mathfrak{h}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ commutes with $\mathfrak{h}_+$, and therefore it acts infinitesimally on $N_+/H_+$ from the right. Thus, $\mathbb{C}[N_+/H_+]$ is an $\mathfrak{n}_+$-module and an $\mathfrak{h}_-$-module.

Let $E_1 = e \otimes 1$ and $E_0 = f \otimes t$ be the generators of $\mathfrak{n}_+$.

**Proposition 4.** There exists an isomorphism of rings $\mathbb{C}[\overline{p}^{(n)}, \overline{q}^{(n)}]_{n \geq 0} \simeq \mathbb{C}[N_+/H_+]$, under which $E_i = -G_i, i = 0, 1$, and $\frac{1}{2}h_{-1} = \partial$.

**Proof.** We follow the same strategy as in the principal case (see [21, 23]). Introduce the functions $\overline{p}$ and $\overline{q}$ on $N_+$ by the formulas

$$(7.8) \quad \overline{p}(K) = \frac{1}{2}(E_1, Kh_{-1}K^{-1}), \quad \overline{q}(K) = -\frac{1}{2}(E_0, Kh_{-1}K^{-1}), \quad K \in N_+.$$ 

These functions are invariant with respect to the right action of $H_+$, and hence descend to $N_+/H_+$.

Next, we define the functions

$$\overline{p}^{(n)} = (\frac{1}{2}h_{-1})^n \cdot \overline{p}, \quad \overline{q}^{(n)} = (\frac{1}{2}h_{-1})^n \cdot \overline{q}, \quad n > 0,$$

on $N_+/H_+$. Define the homomorphism $\mathbb{C}[\overline{p}^{(n)}, \overline{q}^{(n)}]_{n \geq 0} \to \mathbb{C}[N_+/H_+]$, which sends $\overline{p}^{(n)}$ to $\overline{p}^{(n)}$, and $\overline{q}^{(n)}$ to $\overline{q}^{(n)}$.

To prove that this homomorphism is injective, we have to show that the functions $\overline{p}^{(n)}, \overline{q}^{(n)}, n \geq 0$ are algebraically independent. We will do that by showing that the values of their differentials at the identity coset $\bar{1} \in N_+/H_+$, are linearly independent. Those are elements of the tangent space to $N_+/H_+$ at $\bar{1}$, which is canonically isomorphic to $(\mathfrak{n}_+/h_+)^*$. Using the invariant inner product on $\mathfrak{g}$, we identify $(\mathfrak{n}_+/h_+)^*$ with $(\mathfrak{h}_+)^\bot \cap \mathfrak{n}_- = \oplus_{j \geq 0} \mathfrak{h}_+ \otimes t^{-j} \oplus \mathfrak{n}_- \otimes t^{-j+1}$.

Let us first show that the vectors $d\overline{p}^{(1)}|_1$ and $d\overline{q}^{(1)}|_1$ generate $\mathfrak{h}_+ \otimes t^{-1}$ and $\mathfrak{h}_+ \otimes t^{-1}$, respectively. For that it suffices to check that $E_1 \cdot \overline{q} \neq 0, E_1 \cdot \overline{p} = 0, E_0 \cdot \overline{q} = 0, E_0 \cdot \overline{p} \neq 0$. We find

$$(E_1 \cdot \overline{q})(K) = \frac{1}{2}(E_0, [E_1, Kh_{-1}K^{-1}]) = \frac{1}{2}([E_0, E_1], Kh_{-1}K^{-1})$$

$$= -\frac{1}{2}(h_1, Kh_{-1}K^{-1}) = -\frac{1}{2}(h_1, Kh_{-1}K^{-1}) = -1.$$ 

Likewise, we check that $E_1 \cdot \overline{p} = 0, E_0 \cdot \overline{q} = 0, E_0 \cdot \overline{p} = -1$.

Now, by definition,

$$d\overline{p}^{(m+1)}|_1 = \frac{1}{2} \text{ad } h_{-1} \cdot d\overline{p}^{(m)}|_1, \quad d\overline{q}^{(m+1)}|_1 = \frac{1}{2} \text{ad } h_{-1} \cdot d\overline{q}^{(m)}|_1.$$
Hence the vectors $d\mu_{i}^{(m)}\mid_{1}$ are all linearly independent. But $\text{ad} h_{-1} : \mathfrak{h}_{\pm} \otimes t^{-j} \rightarrow \mathfrak{h}_{\pm} \otimes t^{-j-1}$ is an isomorphism for all $j > 0$. Therefore $d\bar{p}^{(n)}\mid_{1}, d\bar{q}^{(n)}|_{1}, n \geq 0$ are linearly independent, and so the functions $\bar{p}^{(n)}, \bar{q}^{(n)}, n \geq 0$, are algebraically independent. Therefore our homomorphism $\mathbb{C}[\bar{p}^{(n)}, \bar{q}^{(n)}]_{n \geq 0} \rightarrow \mathbb{C}[N_{+}/H_{+}]$ is injective.

To prove that it is an isomorphism, we compute the characters of both spaces with respect to the bigradation by the integers and the root lattice. The result is

$$\prod_{n=1}^{\infty} (1 - q^{n}u^{-2})^{-1}(1 - q^{n-1}u^{2})^{-1}$$

for both spaces, and so the above homomorphism is indeed an isomorphism. Moreover, by construction, the operators $\frac{1}{2}h_{-1}$ and $\bar{\partial}$ get identified under this isomorphism.

It remains to show that the formula for $E_{i}$ in terms of the coordinates $\bar{p}^{(n)}, \bar{q}^{(n)}$ coincides with the formula for $G_{i}, i = 0, 1$. Let us show that for $E_{1}$. We already know that $E_{1} \cdot \bar{p} = -1, E_{1} \cdot \bar{q} = 0$. On the other hand, in the same way as in $[21, 22]$ we obtain the relation

$$(7.10) \quad [E_{1}, \frac{1}{2}h_{-1}] = -\frac{1}{2}f_{\alpha_{1}}(h_{-1}) \cdot E_{1},$$

where $f_{\alpha_{1}}(h_{-1})$ is the function on $N_{+}/H_{+}$, which equals $(\alpha_{1}, Kh_{-1}K^{-1})$ at $K \in N_{+}/H_{+}$. We claim that this function equals $4\bar{p} \bar{q}$. To see this, note that $f_{\alpha_{1}}(h_{-1})$ has to be proportional to $\bar{p} \bar{q}$ by degree considerations. To find the coefficient of proportionality, we apply to $f_{\alpha_{1}}(h_{-1})$ the operator $E_{1}E_{0}$. Our previous computations show that $E_{1}E_{0} \cdot \bar{p} \bar{q} = 1$, while we obtain in a similar fashion: $E_{1}E_{0} \cdot f_{\alpha_{1}}(h_{-1}) = 4$.

Now formula $(7.10)$ gives us the relation

$$[E_{1}, \bar{\partial}] = -2\bar{p} \bar{q}E_{1}.$$  

Writing

$$E_{1} = -\sum_{n \geq 0} \left( \bar{B}_{n} \frac{\partial}{\partial \bar{q}^{(n)}} + \bar{B}'_{n} \frac{\partial}{\partial \bar{p}^{(n)}} \right),$$

we find the recurrence relations on $\bar{B}_{n}, \bar{B}'_{n}$:

$$\bar{B}_{n+1} = -2\bar{p} \bar{q} \bar{B}_{n} + \bar{\partial} \bar{B}_{n}$$

with the initial conditions $\bar{B}_{0} = 1, \bar{B}'_{0} = 0$. Comparing this with formula $(7.7)$, we obtain that $E_{1} = -G_{1}$. In the same way we show that $E_{0} = -G_{0}$. This completes the proof. 

7.3. The cohomology. We can now compute the cohomology of the complex $C_{0}^{*}(\hat{\mathfrak{sl}}_{2})$.

**Proposition 5.** The cohomology of the complex $C_{0}^{*}(\hat{\mathfrak{sl}}_{2})$ is isomorphic to $\bigwedge^{*}(\hat{\mathfrak{h}}_{+}^{\ast}) \otimes \mathbb{C}[v^{(n)}]_{n \geq 0}$.

**Proof.** According to the results of the previous subsection, as an $\mathfrak{n}_{+}$–module, $W_{0} \simeq \mathbb{C}[N_{+}/H_{+}] \otimes \mathbb{C}[v^{(n)}]_{n \geq 0}$, where $\mathfrak{n}_{+}$ acts trivially on the second factor. Therefore the cohomology of the complex $C_{0}^{*}(\hat{\mathfrak{sl}}_{2})$, which by Proposition 4 is isomorphic to $H^{*}(\mathfrak{n}_{+}, W_{0})$, equals $H^{*}(\mathfrak{n}_{+}, N_{+}/H_{+}) \otimes \mathbb{C}[v^{(n)}]_{n \geq 0}$. By Shapiro’s lemma (see [21]), $H^{*}(\mathfrak{n}_{+}, N_{+}/H_{+}) \simeq \bigwedge^{*}(\mathfrak{b}_{+}^{\ast})$. 

□
Now we want to compute the cohomology of the double complex $\hat{C}^0_0(\hat{\mathfrak{sl}}_2)$.

Denote
\[ \partial_v = \sum_{n \geq 0} v^{(n+1)} \frac{\partial}{\partial v^{(n)}}. \]

The results of Sect. 7 imply that
\[ \partial = \frac{1}{2} h_{-1} + \partial_v + \frac{1}{2} v h_0. \]

By Proposition 6, $h_n, n \leq 0$, act trivially on the cohomology of the complex $C^*_0(\hat{\mathfrak{sl}}_2)$. Hence $\partial$ acts on a representative of a cohomology class $\omega \otimes P, \omega \in \Lambda^n(h^*_+) P \in \mathbb{C}[v^{(n)}]_{n \geq 0}$, as follows: $\partial \cdot \omega \otimes P = \omega \otimes \partial_v P$. This leads to the following result.

**Proposition 6.** The $j$th cohomology of the complex $\hat{C}^*_0(\hat{\mathfrak{sl}}_2)$ is isomorphic to $\Lambda^{j+1}(h^*_+) \oplus \Lambda^j(h^*_+) \otimes (\mathbb{C}[v^{(n)}]_{n \geq 0}/ \text{Im} \partial_v)$.

In particular, the zeroth cohomology is isomorphic to $\mathfrak{h}^* \oplus \mathbb{C}[v^{(n)}]_{n \geq 0}/ \text{Im} \partial_v$. The second summand consists of classes of the form $\int P$, where $P \in \mathbb{C}[v^{(n)}]_{n \geq 0}$. The classes corresponding to elements of the first summand are constructed as follows.

Let $X \in W_{-\alpha} \oplus W_\alpha$ be a representative of a first cohomology class of the form $\omega \otimes 1 \in h^*_+ \otimes \mathbb{C}[v^{(n)}]_{n \geq 0}$. Then $\partial X = 0$ in the cohomology, and hence there exists $\tilde{X} \in W_0$, such that $\delta^0 \tilde{X} = X$ (we recall that $\delta_0 = G^0_1 + G^0_0$). Here $\tilde{X}$ is defined only up to an element of $\mathbb{C}[v^{(n)}]_{n \geq 0}$, but there is a representative of $h_0$-weight $0$, which lies in $\mathbb{C}[^*\!(n), \hat{q}^{(n)}]_{n \geq 0}$, and is unique up to a total $\partial$-derivative. Note that $\tilde{X}$ can not lie in the image of $\partial$ (and hence $\tilde{\partial}$), because otherwise $X$ would also be in the image of $\partial$. Therefore $\int h \neq 0$ defines a zeroth cohomology class of $\hat{C}^*_0(\hat{\mathfrak{sl}}_2)$.

Now let $\tilde{X}_n, n < 0$, be the element of $\tilde{W}_0$, which corresponds to the first cohomology class $h^*_+ \otimes 1$ of $\hat{C}^*_0(\hat{\mathfrak{sl}}_2)$ via the isomorphism of Proposition 6. Let $\eta_0 = \xi(\tilde{X}_n)$. This is a derivation of $W_0$, which commutes with $\partial$. It also has the following commutation relations with $G_1$ and $G_0$:
\[ [G_1, \eta_0] = -2 \frac{\delta \tilde{X}_n}{\delta u} G_1, \quad [G_0, \eta_0] = 2 \frac{\delta \tilde{X}_n}{\delta u} G_0. \]

Following [21] it is easy to describe all derivations $\xi$ of $W_0$, which commute with $\partial$ and have commutation relations of the form
\[ [G_i, \xi] = (-1)^i f_\xi G_i, \]
for some $f_\xi \in W_0$, with $G_i, i = 0, 1$.

**Lemma 6.** The vector space of such derivations is the direct sum of $\mathfrak{h}_-$ and the space of derivations of the form
\[ \sum_{n \geq 0} (\partial^{n+1}) P \frac{\partial}{\partial v^{(n)}} + \frac{1}{2} P h_0, \quad P \in \mathbb{C}[v^{(n)}]_{n \geq 0}. \]
Corollary 2.
\[ \eta_n = \alpha_n h_{-n} + \sum_{n \geq 0} \partial^{n+1} P \frac{\partial}{\partial v(n)} + \frac{1}{2} Ph_0 \]
for some \( \alpha_n \in \mathbb{C}^\times, P \in \mathbb{C}[v(n)]_{n \geq 0} \). In particular, \( \{ \int \tilde{X}_n, \int \tilde{X}_m \} = [\eta_n, \eta_m] = 0 \) for all \( n, m > 0 \).

The Corollary means that the action of \( h_- \) on \( W_0 \) is hamiltonian.

7.4. The zero curvature formalism. The evolutionary derivations of \( \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}] \) coming from the action of \( \tilde{b}_- \) can be written down explicitly in the zero curvature form. This is explained in detail in [11, 25] in the case of the KdV and mKdV hierarchies, and the results carry over directly to our case. The zero curvature equation corresponding to the element \( \frac{1}{2} h_{-n}, n > 0 \), of \( \tilde{b}_- \) reads:

\[ (7.11) \quad [\partial_z + (K \frac{1}{2} h_{-1} K^{-1})_-, \partial_{r_n} + (K \frac{1}{2} h_{-n} K^{-1})_-] = 0, \quad K \in N_+/H_+. \]

Here for \( A \in \mathfrak{g}, A_- \) stands for the projection of \( A \) onto the \( \mathfrak{b}_- \) part of \( \mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{n}_+ \). According to the computations made in the proof of Proposition \[ \text{we have:} \]

\[ (K \frac{1}{2} h_{-1} K^{-1})_- = \left( \begin{array}{cc} pq + q t^{-1} & -q t^{-1} \\ \frac{1}{2} t^{-2} & p - q t^{-1} \end{array} \right). \]

(to simplify notation, we remove the tildes from \( p \) and \( q \)). On the other hand, for each \( n > 0 \), \( (K \frac{1}{2} h_{-n} K^{-1})_- \) is a matrix, whose entries are polynomials in \( t^{-1} \) with coefficients in \( \mathbb{C}[p^{(n)}, q^{(n)}] \). Therefore formula \([7.11]\) defines a derivation of \( \mathbb{C}[p^{(n)}, q^{(n)}] \). The first of the equations, with \( n = 1 \), tells us that \( r_1 = z \), as expected. Straightforward calculation gives:

\[ (K \frac{1}{2} h_{-1} K^{-1})_- = \left( \begin{array}{cc} -2pq' + 2p'q - 2p^2q^2 + pq t^{-1} + \frac{1}{2} t^{-2} \\ \frac{1}{2} t^{-2} & 2p'q - 2p^2q + 2p^2q^2 - pq t^{-1} - \frac{1}{2} t^{-2} \end{array} \right) \]

(here \( p' \) stands for \( \partial p \)). Substituting this into formula \([7.11]\) we obtain the equation, corresponding to \( n = 2 \):

\[ (7.12) \quad \partial_{r_2} p = p'' - 2p^3q^2 - 2p^2q', \quad \partial_{r_2} q = -q'' + 2q^3p^2 - q^2p'. \]

Now recall from Theorem \[ \text{that the kernel of the screening operator } G_1 \text{ in } W_0 \text{ is } \mathbb{C}[E^{(n)}, H^{(n)}, F^{(n)}]_{n \geq 0}, \text{ where } E = p, H = v = u - 2pq, F = -pq^2 + uq + q'. \text{ But } G_1^0 \text{ commutes with } v^{(n)} = H^{(n)} \text{ and preserves } \tilde{W}_0 = \mathbb{C}[\tilde{p}^{(n)}, \tilde{q}^{(n)}]. \text{ Therefore we obtain that the kernel of } G_1^0 \text{ in } \tilde{W}_0 \text{ equals } \mathbb{C}[\tilde{E}^{(n)}, \tilde{F}^{(n)}]_{n \geq 0}, \text{ where } \]

\[ \tilde{E} = \tilde{p}, \quad \tilde{F} = \tilde{p} \tilde{q}^2 + \tilde{q}'. \]

The derivations \( \partial_{r_n} \) commute with \( G_1^0 \) and hence define evolutionary derivations of \( \mathbb{C}[\tilde{E}^{(n)}, \tilde{F}^{(n)}]_{n \geq 0} \), which we denote by the same symbols. In particular, we find from formula \([7.12]\) the following formula for \( \partial_{r_2} \) (we again omit tildes to simplify notation):

\[ \partial_{r_2} E = E'' - 2E^2 F, \quad \partial_{r_2} F = -F'' + 2F^2 E. \]
This is the AKNS equation. Its reduction obtained by identifying $F$ with $\overline{E}$ and replacing $\tau_2$ with $i\tau_2$ is the non-linear Schrödinger (nS) equation
\[ i\partial_{\tau_2} = E'' - 2E|E|^2. \]
Therefore the derivations $\partial_{\tau_n}$, acting on $\mathbb{C}[\overline{E}^{(n)}, \overline{F}^{(n)}]_{n \geq 0}$, define the AKNS hierarchy. Because of that, it is natural to call the equation (7.12), the modified AKNS (mAKNS) equation, and the hierarchy of the derivations $\partial_{\tau_n}$ of $\mathbb{C}[\overline{E}^{(n)}, \overline{F}^{(n)}]_{n \geq 0}$, the mAKNS hierarchy.

The AKNS hierarchy also has a zero curvature representation (7.11), where now $A_{-}$ stands for the projection of $A \in \mathfrak{g}$ onto the $\overline{\mathfrak{g}}[t^{-1}]$ part of the decomposition $\mathfrak{g} = \overline{\mathfrak{g}}[t^{-1}] \oplus (\overline{\mathfrak{g}} \otimes t\mathbb{C}[t])$, and $K \in \exp(\overline{\mathfrak{g}} \otimes t\mathbb{C}[t])/H_{+}$. In particular, we then have the well-known AKNS $L$–operator
\[ \partial_z + (K - \frac{1}{2}h_{-1}K^{-1})_{-} = \partial_z + \left( \frac{1}{2}t^{-1}E - \frac{1}{2}t^{-1} \right). \]

Finally, the non-local equation (7.13) can also be written in the zero curvature form:
\[ \left[ \partial_z + \left( \frac{pq + \frac{1}{2}t^{-1}}{p} - \frac{qt^{-1}}{p} \right), \partial_{\tau} + \left( 0 \right) \right] = 0. \]

The equations of the mAKNS hierarchy are symmetries of this equation.

8. Classical limit for an arbitrary $\mathfrak{g}$

The results of this section can be generalized to the case of an arbitrary (non-twisted) affine algebra $\overline{\mathfrak{g}}$. In this case $N_{+}/H_{+}$ also carries a good system of coordinates, in which the infinitesimal action of $\mathfrak{h}_{-}$ becomes a set of evolutionary derivations. They define a completely integrable system, which is an analogue of the AKNS hierarchy. Moreover, with respect to these coordinates, the generators $E_{i}, i = 0, \ldots, \ell$, of $\mathfrak{n}_{+}$ become the classical limits of the screening operators $G_{i}^{\nu}$.

8.1. Screening operators in a special coordinate system. Let us fix generators $e_{\alpha}, f_{\alpha}$ of the one-dimensional subspaces $\overline{\mathfrak{p}}_{\alpha} \subset \overline{\mathfrak{p}}_{+}$ and $\overline{\mathfrak{p}}_{-} \subset \overline{\mathfrak{p}}_{-}$, respectively, such that $(e_{\alpha}, f_{\alpha}) = 1$. In that case $[e_{\alpha}, f_{\alpha}] = \alpha$, where we identify $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}^{\ast}$ using the inner product on $\overline{\mathfrak{g}}$, such that $(\alpha_{\max}, \alpha_{\max}) = 2$.

First we introduce a special coordinate system $\{x_{\alpha}\}_{\alpha \in \Delta_{+}}$ on the finite-dimensional unipotent group $\overline{N}_{+}$, with respect to which the right action of $\overline{\mathfrak{p}}_{+}$ becomes particularly simple. Let $\rho^{\nu}$ be the element of $\overline{\mathfrak{p}}_{-}$, such that $(\rho^{\nu}, \alpha_{i}) = 1$, $\forall i = 1, \ldots, \ell$. Denote by $x_{\alpha}$ the regular function on $\overline{N}_{+}$ defined by the formula
\[ x_{\alpha}(K) = -(f_{\alpha}, K^{-1}\rho^{\nu}K), \quad K \in \overline{N}_{+}. \]

Then
\[ (e_{i}^{R} \cdot x_{\alpha})(K) = -(f_{\alpha}, [K^{-1}\rho^{\nu}K, e_{i}]) = -([e_{\alpha}, f_{\alpha}], K^{-1}\rho^{\nu}K) = \alpha_{i}, K^{-1}\rho^{\nu}K = -1. \]

Let us write
\[ [e_{\alpha}, e_{\beta}] = -c_{\alpha, \beta} e_{\alpha + \beta}. \]
Then
\[ [e_i, f_\alpha] = c_{\alpha_i, \alpha - \alpha_i} f_{\alpha - \alpha_i}, \quad \alpha \neq \alpha_i. \]

Now we obtain in the same way as above:
\[ e^R_i \cdot x_\alpha = c_{\alpha_i, \alpha - \alpha_i} x_{\alpha - \alpha_i}, \quad \alpha \neq \alpha_i. \]
Hence
\[ e^R_i = -\frac{\partial}{\partial x_\alpha} + \sum_{\alpha \in \Delta_+} c_{\alpha_i, \alpha} x_\alpha \frac{\partial}{\partial x_{\alpha + \alpha_i}}. \]

Moreover, we find that
\[ e^R_\alpha = -(\rho^\vee, \alpha) \frac{\partial}{\partial x_\alpha} + \sum_{\beta \in \Delta_+} c_{\alpha, \beta} x_\beta \frac{\partial}{\partial x_{\alpha + \beta}}, \]
\[ h^L_i = -\sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) x_\alpha \frac{\partial}{\partial x_\alpha}. \]

We use the above coordinates to construct the screening operators, as in Sect. 2.
Their classical limits as \( \nu \to 0 \) are
\[ G^0_i = \left\{ \cdot, \int \left( -p_\alpha + \sum_{\alpha \in \Delta} c_{\alpha_i, \alpha} q_\alpha p_{\alpha + \alpha_i} \right) e^{-\phi_i} \right\}. \]

Next we define the 0th screening operator. According to Sect. 6.2 it has the form
\( P_x(a^*_\alpha(z)) V^\nu_{\alpha_0} \), where \( P_x(x_\alpha) \in \mathbb{C}[\Delta_+] \) represents an element of \( M^\alpha_{\alpha_{\text{max}}} \simeq \mathbb{C}[\Delta_+] \) of weight 0, which satisfies the conditions of Conjecture 2. It is straightforward to check that the element \( x_{\alpha_{\text{max}}} \) satisfies these conditions. In the classical limit we obtain the following formula for the 0th screening operator:
\[ \overline{G}_i^0 = \left\{ \cdot, \int q_{\alpha_{\text{max}}} e^{-\phi_0} \right\}. \]

The operators \( \sum_{i=1}^\ell \overline{G}_i^0 \) and \( \sum_{i=0}^\ell \overline{G}_i^0 \) define non-local Toda type equations. Here are the explicit formulas for the second of these equations:
\[ \partial_\tau q_{\alpha_j} = e^{-\phi_j}, \quad \partial_\tau q_\alpha = \sum_{i=1}^\ell c_{\alpha_i, \alpha} e^{-\phi_i} q_{\alpha - \alpha_i} e^{-\phi_j}, \quad \alpha \in \Delta_+ \setminus \Delta_+^q, \]
\[ \partial_\tau p_\alpha = \sum_{i=1}^\ell c_{\alpha_i, \alpha} e^{-\phi_i} + e^{-\phi_0} \delta_{\alpha, \alpha_{\text{max}}}, \quad \alpha \in \Delta_+, \]
\[ \partial_\tau \partial_\tau \phi_j = \sum_{i=1}^\ell (\alpha_i, \alpha_j) \left( p_{\alpha_i} - \sum_{\alpha \in \Delta_+} c_{\alpha_i, \alpha} q_\alpha p_{\alpha + \alpha_i} \right) e^{-\phi_i} - (\alpha_j, \alpha_0) q_{\alpha_{\text{max}}} e^{-\phi_0}, \quad j = 1, \ldots, \ell. \]
Let
\[ v_i = u_i - \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) q_\alpha p_\alpha. \]
One finds that $G_i \cdot v_j = 0, \forall i, j$. Therefore we can further reduce the system (8.3) by setting $u_i = \sum_{\alpha \in \Delta_+} (\alpha, \alpha) q_\alpha p_\alpha$.

The classical screening operators give rise to derivations of the ring of differential polynomials $W_0 = \mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}, u_i^{(n)}]$, which we denote by $G_i, i = 0, \ldots, \ell$. We know that $G_i \cdot v_j = 0, \forall i, j, n$. In the same way as in the case of $\mathfrak{sl}_2$ we can use this fact to separate variables.

Each $h \in \mathfrak{h}$ acts on $W_0$ in a natural way: $h \cdot p_\alpha^{(n)} = \alpha(h) p_\alpha^{(n)}, h \cdot q_\alpha^{(n)} = -\alpha(h) q_\alpha^{(n)}, h \cdot u_i^{(n)} = 0$. Denote by $\{h^j\}_{j=1, \ldots, \ell}$, the dual basis to $\{\alpha_i\}_{j=1, \ldots, \ell}$ with respect to the normalized inner product on $\mathfrak{h}$. Let $\tilde{\partial} = \partial - \sum_{i=1}^{\ell} v_i h^i$, and define new variables $p_\alpha^{(n)} = \tilde{\partial}^\alpha p_\alpha^{(n)}, q_\alpha^{(n)} = \tilde{\partial}^\alpha q_\alpha^{(n)}$. In the same way as in Sect. 7.1, one shows that $W_0 = \mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}] \otimes \mathbb{C}[v_i^{(n)}]$, and that the derivations $G_i$ act along the first factor of this tensor product.

Furthermore, we find, in the same way as in Sect. 7.1, the following explicit formulas for the action of $G_i, i = 1, \ldots, \ell$, on $\mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}]$:

\begin{equation}
G_i = \sum_{n=0}^{\infty} \sum_{\alpha \in \Delta_+} \left( B_{\alpha,n}^{(i)} \frac{\partial}{\partial q_\alpha^{(n)}} + B_{-\alpha,n}^{(i)} \frac{\partial}{\partial p_\alpha^{(n)}} \right),
\end{equation}

where the polynomials $B_{\pm,\alpha,n}$ are defined recursively as follows:

$B_{0,\alpha,n}^{(i)} = \delta_{i,j}; \quad B_{0,\alpha}^{(i)} = -c_{\alpha_1,\alpha-\alpha} q_{\alpha-\alpha}, \quad \alpha \in \Delta_+ \setminus \Delta^s$,

$B_{-\alpha,0}^{(i)} = c_{\alpha_1,\alpha} p_\alpha^{(\alpha)}, \quad B_{-\alpha,0}^{(i)} = \tilde{\partial} B_{\pm,\alpha,n+1} - U_{i} B_{\pm,\alpha,n+1}^\pm, \quad n > 0$,

where

\begin{equation}
U_i = \sum_{\alpha \in \Delta_+} (\alpha, \alpha) p_\alpha q_\alpha.
\end{equation}

We also find a formula for $G_0$:

\begin{equation}
G_0 = \sum_{n=0}^{\infty} B_n^{(0)} \frac{\partial}{\partial p_\alpha^{(n)}},
\end{equation}

where

$B_0^{(0)} = 1, \quad B_n^{(0)} = \tilde{\partial} B_{n-1}^{(0)} - U_{0} B_{n-1}^{(0)}$,

with $U_0$ given by formula (8.6).

8.2. Isomorphism with $\mathbb{C}[N_+/H_+]$. Now we can identify the ring of differential polynomials $\mathbb{C}[p_\alpha^{(n)}, q_\alpha^{(n)}]$ with $\mathbb{C}[N_+/H_+]$, where $H_+$ is the subgroup of $N_+$, which is the image of the Lie algebra $\mathfrak{h}$ under the exponential map. Observe that the Lie algebra $\mathfrak{h} \otimes t \mathbb{C}[t^{-1}]$ acts on $\mathbb{C}[N_+/H_+]$ from the right. Denote by $\rho_{\alpha}^{\vee}$ the element $\rho_{\alpha}^{\vee} \otimes t^{-1}$ of $\mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}]$. Let $E_i = e_i \otimes 1, i = 1, \ldots, \ell$, and $E_0 = f_{\alpha_{\text{max}}} \otimes t$ be the generators of $n_+$.

**Proposition 7.** There exists an isomorphism of rings $\mathbb{C}[\overline{p}^{(n)}, \overline{q}^{(n)}]_{n \geq 0} \simeq \mathbb{C}[N_+/H_+]$, under which $E_i = -G_i, i = 0, \ldots, \ell$, and $\rho_{-1}^{\vee} = \tilde{\partial}$. 
Proof. The proof proceeds along the lines of the proof of Proposition 4. We introduce a system of coordinates on \( N_+/H_+ \) and then find formulas for the action of \( E_i = -G_i, i = 0, \ldots, \ell \), and \( \rho^\vee \) in these coordinates.

Introduce the following regular functions \( \tilde{p}_\alpha, \tilde{q}_\alpha \) on \( N_+/H_+ \):

\[
\tilde{p}_\alpha(K) = -(e_\alpha \otimes 1, K \rho^\vee_{-1} K^{-1}), \quad K \in N_+/H_+,
\]

\[
\tilde{q}_\alpha(K) = -(f_\alpha \otimes t, K \rho^\vee_{-1} K^{-1}), \quad K \in N_+/H_+.
\]

Straightforward computation analogous to that made in the proof of Proposition 4 gives for \( i = 1, \ldots, \ell \):

\[
E_i \cdot \tilde{q}_\alpha = -\delta_{i,j}; \quad E_i \cdot \tilde{q}_\alpha = c_{\alpha, \alpha - \alpha_i} \tilde{q}_{\alpha - \alpha_i}, \quad \alpha \in \Delta_+ \setminus \Delta^*_+,
\]

\[
E_i \cdot \tilde{p}_\alpha = -c_{\alpha, \alpha} \tilde{p}_{\alpha + \alpha_i},
\]

and

\[
(8.8) \quad E_0 \cdot \tilde{q}_\alpha = 0, \quad E_0 \cdot \tilde{p}_\alpha = (\alpha_{\text{max}}, \rho^\vee) \delta_{\alpha, \alpha_{\text{max}}} - c_{\alpha, \alpha_{\text{max}}} \cdot \rho q_{\alpha_{\text{max}} - \alpha}.
\]

Now let \( \tilde{p}^{(n)}_\alpha = (\rho^\vee)^n \cdot \tilde{p}_\alpha, \tilde{q}^{(n)}_\alpha = (\rho^\vee)^n \cdot \tilde{q}_\alpha, n \geq 0 \). We show in the same way as in the proof of Proposition 4 that these functions are algebraically independent. Here we rely only on the fact that that \( \rho^\vee \) is a regular semi-simple element of \( \mathfrak{g} \).

Next we define a homomorphism from \( C[\tilde{p}^{(n)}_\alpha, \tilde{q}^{(n)}_\alpha] \) to \( C[N_+/H_+] \). It intertwines the actions of \( \tilde{\partial} \) and \( \rho^\vee \) and maps each \( \tilde{q}_\alpha \) to \( \tilde{q}_\alpha \). It also maps each \( \tilde{p}_\alpha \) to a polynomial in \( \tilde{p}_\alpha, \tilde{q}_\alpha \) (also denoted by \( \tilde{p}_\alpha \)), such that

\[
(8.9) \quad \tilde{p}_\alpha = (\alpha, \rho^\vee) \tilde{p}_\alpha - \sum_{\beta \in \Delta_+} c_{\alpha, \beta} \tilde{p}_\beta \tilde{q}_\alpha + \beta
\]

(cf. formula (8.1)). It is clear that such polynomials exist, are unique and that the above homomorphism is injective. To prove that it is an isomorphism, we use the equality of characters of the two spaces, as in the proof of Proposition 4.

Finally, we need to show that the action of \( E_i \) on \( C[N_+/A_+] \) coincides with the action of \( -G_i \) on \( C[\tilde{p}^{(n)}_\alpha, \tilde{q}^{(n)}_\alpha] \). But the actions of \( E_i \) and \( -G_i \) on \( \tilde{q}_\alpha = \tilde{q}_\alpha \) coincide. The action of \( E_i, i = 1, \ldots, \ell \), on \( \tilde{p}_\alpha \) is given by the same formula as the action on \( \tilde{p}_\alpha \). Comparing with formula (8.3) we again find agreement between the actions of \( E_i \) and \( -G_i \). The same is true for \( E_0 \) and \( -G_0 \) as formulas (8.8), (8.7) and (8.9) show.

We also have:

\[
(8.10) \quad [E_i, \rho^\vee_{-1}] = -f_{\alpha_i} \cdot E_i,
\]

where \( f_{\alpha_i}(K) = (\alpha_i, K \rho^\vee_{-1} K^{-1}), K \in N_+/H_+ \). It is easy to see that

\[
f_{\alpha_i} = \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) p_{\alpha} q_{\alpha}.
\]

Formula (8.10) gives us a recurrence relation on the coefficients of the derivations \( E_i \), which coincides with that on the coefficients of the derivations \( G_i \) given by formulas (8.3), (8.7). We have shown above that their first coefficients differ by sign, and therefore we obtain that \( E_i = -G_i \). This completes the proof. \( \blacksquare \)
Remark 8.1. In the above isomorphism the element $\rho^{\vee}$ of $\hat{\mathfrak{h}}$ can be replaced by any regular element of $\hat{\mathfrak{h}}$. In that case we need to change accordingly the formulas for the coordinates $p_\alpha, q_\alpha$. Moreover, with appropriate changes all results of this section will remain true if we replace $\rho^{\vee}$ by any regular element of $\hat{\mathfrak{h}}$. \hfill \Box

8.3. Integrable hierarchies. Recall that the Lie algebra $\mathfrak{h}_- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$ acts on $\mathbb{C}[N_+/H_+]$ by derivations. Hence we obtain an infinite hierarchy of commuting evolutionary (i.e., commuting with $\tilde{\partial} = \rho^{\vee}_{-1}$) derivations on $\mathbb{C}[p^{(n)}_\alpha, q^{(n)}_\alpha]$. We call it the modified AKNS hierarchy associated to $\mathfrak{h}$, or $\mathfrak{g}$-mAKNS hierarchy for shorthand.

The above derivations preserve the subring $\mathbb{C}[N_+/H_+]$ of $N_-$-invariants of $\mathbb{C}[N_+/H_+]$. This subring equals the intersection of kernels of the operators $G_i, i = 1, \ldots, \ell$, in $\mathbb{C}[p^{(n)}_\alpha, q^{(n)}_\alpha]$. The latter is isomorphic to the quotient of the ring $K_0(\mathfrak{g}) = \mathbb{C}[E^{(n)}_\alpha, H^{(n)}_i, F^{(n)}_\alpha]$ introduced in Theorem 3 by the ideal generated by $H^{(n)}_i$. Hence it is also isomorphic to a ring of differential polynomials $\mathbb{C}[\tilde{E}^{(n)}_\alpha, \tilde{F}^{(n)}_\alpha]$, where $\tilde{E}^{(n)}_\alpha$ and $\tilde{F}^{(n)}_\alpha$ are certain polynomials in $p^{(m)}_\beta, q^{(m)}_\beta$. The Lie algebra $\mathfrak{h}_-$ acts on $\mathbb{C}[\tilde{E}^{(n)}_\alpha, \tilde{F}^{(n)}_\alpha]$ by evolutionary derivations. They form a hierarchy, which we call the $\mathfrak{g}$-AKNS hierarchy. It has been previously studied in the literature under the name AKNS–D hierarchy (see [8]).

The equations of both hierarchies can be written in the zero curvature form as follows. The equation of the $\mathfrak{g}$-mAKNS hierarchy corresponding to an element $y \in \mathfrak{h}_-$ reads

$$[\partial_z + (K\rho^{\vee}_{-1}K^{-1})_-, \partial_r + (KyK^{-1})_-] = 0, \quad K \in N_+/H_+,$$

where $A^-$ denotes the projection of $A \in \mathfrak{g}$ onto the $\mathfrak{b}^-$ part of the decomposition $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{n}^+$. The equation of the $\mathfrak{g}$-AKNS hierarchy corresponding to $y \in \mathfrak{h}_-$ reads

$$[\partial_z + (K\rho^{\vee}_{-1}K^{-1})_-, \partial_r + (KyK^{-1})_-] = 0,$$

where $A^-$ stands for the projection of $A \in \mathfrak{g}$ onto the $\mathfrak{g}[t^{-1}]$ part of the decomposition $\mathfrak{g} = \mathfrak{g}[t^{-1}] \oplus (\mathfrak{g} \otimes t\mathbb{C}[t])$, and $K \in \exp(\mathfrak{g} \otimes t\mathbb{C}[t])/H_+$. In the next subsection we will show that both hierarchies are Hamiltonian.

Note also that the non-local equations introduced above can also be written in the zero-curvature form (in the case of $\hat{\mathfrak{sl}}_2$ see the end of Sect. [7.4]). The equations of the $\mathfrak{g}$-mAKNS hierarchy are symmetries of the non-local equation (8.3).

8.4. Cohomology computation. In the same way as in the case of $\hat{\mathfrak{sl}}_2$ we obtain the following result.

**Proposition 8.** The cohomology of the complex $C^*_0(\mathfrak{g})$ is isomorphic to $H^*(\mathfrak{n}_+, W_0)$ and hence to $\bigwedge^*(\mathfrak{n}_+^*) \otimes \mathbb{C}[v^{(n)}_i]_{i=1,\ldots,\ell, n\geq 0}$.

Now we can to compute the cohomology of the double complex $\hat{C}^*_0(\hat{\mathfrak{sl}}_2)$.

We have

$$\partial = \rho^{\vee}_{-1} + \sum_{i=1}^{\ell} v_i h^i + \partial_v,$$
where
\[ \partial_v = \sum_{i=1}^\ell \sum_{n \geq 0} v_i^{(n+1)} \frac{\partial}{\partial v_i^{(n)}}. \]

In the same way as in the case of \( \widehat{sl}_2 \) we obtain:

**Proposition 9.** The \( j \)th cohomology of the complex \( \widehat{C}_0^n(\mathfrak{g}) \) is isomorphic to \( \wedge^{j+1}(\mathfrak{h}_+^*) \oplus \wedge^j(\mathfrak{h}_+^*) \otimes (\mathbb{C}[v_i^{(n)}]/\text{Im} \partial_v) \).

In particular, the zeroth cohomology is isomorphic to \( \mathfrak{h}^* \oplus \mathbb{C}[v_i^{(n)}]/\text{Im} \partial_v \). The classes corresponding to elements of \( \mathfrak{h}^* \) are constructed as in the case of \( \widehat{sl}_2 \).

Let \( \{h_i\}_{i=1,\ldots,\ell, n<0} \) be a basis of \( \mathfrak{h} \). Then \( \{h_n^i\}_{i=1,\ldots,\ell, n>0} \), where \( h_n^i = h_t \otimes t^{-n} \) is a basis of \( \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] \). Denote by \( \widetilde{X}_n^i \in W_0 \), the representative of the zeroth cohomology class of \( \widehat{C}_0^n(\widehat{sl}_2) \), which corresponds to the first cohomology class \( (h_n^i)^* \otimes 1 \) of \( C^n_0(\mathfrak{g}) \) via the isomorphism of Proposition 9. Let \( \eta_n^i = \xi(\widetilde{X}_n^i) \). This is a derivation of \( W_0 \), which commutes with \( \partial \) and has the commutation relations with the operators \( G_i \) of the form
\[ \{G_i, \xi\} = f_{ni}^i G_i, \quad i = 0, \ldots, \ell, \]
for some \( f_{ni}^i \in W_0 \) (in the case of \( \widehat{sl}_2 \) these relations are given in Sect. 7.3).

Following [2] it is easy to describe all such derivations \( \xi \) of \( W_0 \).

**Lemma 7.** The vector space of derivations \( \xi \) of \( W_0 \) satisfying commutation relations (8.11) is the direct sum of \( \mathfrak{h}_- \) and the space of derivations of the form
\[ \xi(\{P_j\}) = \sum_{i=1}^\ell \sum_{n \geq 0} (\partial^{n+1} P_i) \frac{\partial}{\partial v_i^{(n)}} + \sum_{i=1}^\ell P_i h_i, \quad P_j \in \mathbb{C}[v_i^{(n)}]. \]

**Corollary 3.**
\[ \eta_n^i = \alpha_n^i h_n - \xi(\{P_j\}) \]
for some \( \alpha_n^i \in \mathbb{C}^\times, P_j \in \mathbb{C}[v_i^{(n)}] \). Therefore \( \{f \widetilde{X}_n^i, f \widetilde{X}_m^j\} = [\eta_n^i, \eta_m^j] = 0 \) for all \( i, j, n, m \).

The Corollary implies that the \( \mathfrak{g} \)-mAKNS and \( \mathfrak{g} \)-AKNS hierarchies introduced above are completely integrable hamiltonian systems.

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