Complete Form of Fermion Self-energy in NJL Model with External Magnetic Field

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In this paper, we aim to study the complete form of self-energy in the fermion propagator within two-flavor NJL model in the case of finite temperature, chemical potential and external magnetic field. Through Fierz transformation we prove that the self-energy is not simply proportional to dynamical mass in the presence of chemical potential, moreover, it could be more complicated after introducing external magnetic field. We find out the appropriate and complete form of self-energy and establish the new gap equations. The numerical results show that not only the dynamical mass get quantitative modification, but also the properties of Nambu phase and Wigner phase are significantly different from the previous one. Furthermore, we find that the new self-energy does generate split in the dispersion relation with fixed momentum and Landau level. Especially, through analysis of the numerical results, we find out that at some specific temperature and magnetic field, along with the increasing of chemical potential, the dynamic mass does not directly jumps from Nambu solution to Wigner solution, instead it could jump to a nonzero and small solution, which has not been found in the previous literatures

Key-words: NJL model, magnetic field, dynamical mass, gap equations, self-energy

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I. INTRODUCTION

The phase structure of QCD matter has always been an important and attractive topic in theoretical physics\textsuperscript{1–7}. In relativistic heavy-ion collisions, the produced QCD matter will go through a phase transition or a crossover as time goes by. Either way, the state of QCD matter is believed to change from quark-gluon plasma to hadronic matter in this process. Its physical properties and dynamical behaviors such as chiral symmetry and confinement are altered along with the change of the state.

At the early stage of noncentral collision, the QCD matter produces extremely strong magnetic field\textsuperscript{8}, which brings about obvious magnetic effects. Therefore studying QCD matter’s properties under the influence of magnetic field becomes a meaningful and important subject. So far, many relevant theories and models have been proposed and it is shown that the quark condensate are strengthened by magnetic field, which is known as ‘Magnetic Catalysis’\textsuperscript{9–13}. Consequently, the QCD phase diagram is related to magnetic field\textsuperscript{14, 15}.

NJL model is quite a useful and convenient tool to qualitatively study QCD matter states\textsuperscript{1, 2, 16–23}. For a NJL model we usually apply mean field approximation to deal with the four fermion interaction terms, namely, 

\begin{equation}
\langle \bar{\psi} \psi \rangle^2 \rightarrow 2 \langle \bar{\psi} \psi \rangle \langle \bar{\psi} \psi \rangle - \langle \bar{\psi} \psi \rangle^2,

\langle i \bar{\psi} \gamma^5 \tau \psi \rangle^2 \rightarrow 2 \langle i \bar{\psi} \gamma^5 \tau \psi \rangle \cdot \langle i \bar{\psi} \gamma^5 \tau \psi \rangle - \langle i \bar{\psi} \gamma^5 \tau \psi \rangle^2.
\end{equation}

It is believed that this approximation is equivalent to Dyson-Schwinger equations with contact interaction treatment, hence the gap equation can be written as

\begin{equation}
\frac{\Sigma}{G} \int d^4x = i \int d^4x \langle x | \gamma^\mu \hat{S} \gamma^\mu | x \rangle,
\end{equation}

\begin{equation}
\hat{S}^{-1} = \not{p} - m - \Sigma,
\end{equation}

\begin{equation}
\Sigma = \sigma + i \gamma^5 \vec{\pi} \cdot \vec{\tau}, \quad \sigma = -\frac{G}{N_c} \langle \bar{\psi} \psi \rangle, \quad \vec{\pi} = -\frac{G}{N_c} \langle i \bar{\psi} \gamma^5 \tau \psi \rangle.
\end{equation}

In the above equations, $\Sigma$ represents self-energy of fermion propagator, it contains dynamical mass $\sigma$, which is generated by non-perturbative effect, more specifically, dynamical chiral symmetry breaking. Generally speaking,
there is \( \vec{\pi} = 0 \) in Eq. (1), which leads to \( \Sigma = \sigma \). Therefore it is usually more convenient to study dynamical mass directly rather than discuss a general form of self-energy. But in M. Asakawa and K. Yazaki’s work [24], they have pointed that, in a self-consistent mean-field approximation, the self-energy does not simply equal dynamical mass, which reveals with the help of the Fierz transformation. When chemical potential \( \mu \) is not zero, the actual self-energy should be written as \( \Sigma = \sigma + a \gamma^0 \) to guarantee the self-consistency of gap equation (1). In this new \( \Sigma \), we can combine \( a \) with chemical potential as a renormalized chemical potential \( \mu_r = \mu - a \).

In this paper, we are about to study the self-energy problem in two flavor NJL model with temperature, chemical potential and external magnetic field, the self-energy must not simply equal dynamical mass. In order to find out the appropriate and complete self-energy, we start from the most general form, a \((4 \times 4) \otimes (2 \times 2)\) matrix (spinor space and flavor space), and rule out its inappropriate parts. The detail is discussed in the beginning of section II. In section II, we also give detailed deduction of gap equation and analyse numerical results. Section III is our conclusion.

II. THE GAP EQUATIONS AND NUMERICAL RESULTS

A. NJL Gap Equations

A two-flavor NJL model Lagrangian with external magnetic field in Minkowski space is

\[
\mathcal{L} = \bar{\psi} (i \partial + eA \otimes Q) \psi + G[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5 \vec{\tau}\psi)^2],
\]

(4)

\[
(A_0, A_1, A_2, A_3) = (0, \frac{B}{2} x^2, -\frac{B}{2} x^1, 0),
\]

(5)

\[
Q = \begin{pmatrix}
q_u & 0 \\
0 & q_d
\end{pmatrix},
\quad q_u = \frac{2}{3},
\quad q_d = -\frac{1}{3}.
\]

we define a flavor index \( 'f' \) for convenience,

\[
f = u, d, \quad q_l = q_u, q_d.
\]

According to Ref. [24], if one wants to apply mean field approximation to Eq. (4), only making \( (\bar{\psi}\psi) \) and \((i\bar{\psi}\gamma^5 \vec{\tau}\psi)\) to be mean fields is not enough. Instead, one should make Fierz transformation firstly, this will provide us more four fermion interaction terms.

Let \( \mathcal{L}_1 \) represent the four fermion interaction terms in Eq. (4),

\[
\mathcal{L}_1 = G[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5 \vec{\tau}\psi)^2],
\]

(8)

the Fierz transformation of \( \mathcal{L}_1 \) yields [25]

\[
\mathcal{F}(\mathcal{L}_1) = \frac{G}{4N_c}[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5 \vec{\tau}\psi)^2 - (\bar{\psi}\vec{\tau}\psi)^2 - (i\bar{\psi}\gamma^5 \psi)^2 - 2(\bar{\psi}\gamma^\mu \psi)^2 - 2(\bar{\psi}\gamma^5 \gamma^\mu \psi)^2 + (\bar{\psi}\sigma^{\mu\nu} \psi)^2 - (\bar{\psi}\sigma^{\mu\nu} \vec{\tau}\psi)^2].
\]

(9)

In Klevansky’s article [25], he had mentioned that with or without Fierz transformation there are three equivalent four fermion interaction terms (see Eq. (2.57) in his article, ), they are

\[
\mathcal{L}_1 = G[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5 \vec{\tau}\psi)^2],
\]

(10)

\[
\mathcal{F}(\mathcal{L}_1),
\]

(11)

\[
\frac{1}{2}[\mathcal{L}_1 + \mathcal{F}(\mathcal{L}_1)].
\]

(12)

The dynamic properties of these three interaction terms should be equivalent in original NJL model, but with external magnetic field, their equivalence might be broken. In this article, we choose Eq. (12) as our four fermion interaction terms for three reasons, first of all, Eq. (12) can provide us more structures than in Eq. (10), secondly, Eq. (12) is
the only term that has obvious Fierz transformation invariance among Eqs. (10), (12), thirdly, from Eqs. (8) and (9), we know that Eq. (12) has \(O(G)\) terms and \(O(\frac{1}{N_c})\) terms, comparing to \(O(\frac{1}{N_c})\) terms, it seems \(O(G)\) terms might be dominant in dynamic process, that’s what we would like to find out. Therefore, the new four fermion interaction lagrangian is

\[
\mathcal{L}_1 = \frac{G}{2} (1 + \frac{1}{4N_c})[(\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5\bar{\psi}\psi)^2] - \frac{G}{8N_c}[(\bar{\psi}\gamma^\mu\bar{\psi}\psi)^2 + (i\bar{\psi}\gamma^5\psi)^2 + 2(\bar{\psi}\gamma^\mu\psi)^2 + 2(\bar{\psi}\gamma^5\gamma^\mu\psi)^2 - (\bar{\psi}\gamma^\mu\gamma^\nu\psi)^2 + (\bar{\psi}\sigma^{\mu\nu}\psi)^2].
\]

Now we can apply mean field approximation to each four fermion term in Eq. (9), and we are led to a complex self-energy \(\Sigma_{sf}\), which should still preserve the same Lorentz invariance as \(\Sigma\), means field square terms such as \(\langle \bar{\psi}\gamma^5\psi \rangle^2\) and so on, the detailed content of this lagrangian will be discussed later, but firstly we need to simplify \(\Sigma_{sf}\).

Comparing Eq. (18) with Eq. (16), \(\sigma, a, b\) separately correspond to

\[
\sigma = -G(1 + \frac{1}{4N_c})\langle \bar{\psi}\psi \rangle, \quad a = \frac{G}{2N_c}\langle \bar{\psi}\gamma^0\psi \rangle, \quad b = -\frac{G}{2N_c}\langle \bar{\psi}\gamma^5\gamma^3\psi \rangle,
\]

and now we are able to write down the explicit expression of \(\mathcal{L}_M\),

\[
\mathcal{L}_M = -\frac{2N_c}{4N_c + 1}G\sigma^2 + \frac{N_c}{G}a^2 - \frac{N_c}{G}b^2.
\]

The partition function of \(\mathcal{L}'\) is

\[
Z = \int D\bar{\psi}D\psi e^{i\mathcal{L}'d^4x} = e^{i\mathcal{W}[\sigma,a,b]},
\]

\[
\mathcal{W} = \mathcal{L}_M \int d^4x - iN_c \text{Tr}_{sf} \ln(\hat{\Pi} - \Sigma_{sf}),
\]

here the trace operator ‘\(\text{Tr}_{sf}\)’ implies that besides summing up all expectation values of \(\ln(\hat{\Pi} - \Sigma_{sf})\) at every quantum state, one need to trace the matrices in flavor and spinor spaces.
Eq. (22) leads to the gap equations through the variations below

$$
\frac{\delta W}{\delta \sigma} = 0, \quad \frac{\delta W}{\delta a} = 0, \quad \frac{\delta W}{\delta b} = 0,
$$

(23)
because \(\sigma, a, b\) are constant fields, their variations are equivalent to corresponding partial differentiations. The gap equations have more explicit forms

$$
\frac{4N_c}{4N_c + 1} G \int d^4 x = i \sum_f \text{tr} \int d^4 x \langle x | \hat{S}_f | x \rangle,
$$

(24)

$$
2 \frac{\sigma}{G} \int d^4 x = -i \sum_f \text{tr} \int d^4 x \langle x | \hat{S}_f \gamma^0 | x \rangle,
$$

(25)

$$
2 \frac{b}{G} \int d^4 x = i \sum_f \text{tr} \int d^4 x \langle x | \hat{S}_f \gamma^5 \gamma^3 | x \rangle,
$$

(26)

$$
\hat{S}_f = (\bar{\Pi}^f - \Sigma)^{-1}, \quad \bar{\Pi}^f_\mu = p_\mu + q_i e A_\mu.
$$

(27)

Here the ‘tr’ operator means tracing the gamma matrices in spinor space.

Referring to our previous work [13], one can quantize the quadratic sum of \(\hat{\Pi}^f_1\) and \(\hat{\Pi}^f_2\), \([ (\Pi^f_1)^2 + (\Pi^f_2)^2 ] \rightarrow (2n + 1) | q_i e B | (n = 0, 1, 2, \ldots)\). We employ the method proposed in Ref. [13] to calculate Eqs. (24), (25), (26) and (28), summing up all expectation values of \(\hat{S}\), this gives us

$$
\int d^4 x \langle x | \hat{S} | x \rangle = \frac{| q_i e B |}{\pi} \int \frac{dp_0 dp_3}{(2\pi)^2} S \int d^4 x,
$$

(28)

S can be treated as a sum of two parts, effective part and noneffective part, effective part \(S_{\text{eff}}\) is useful to gap equations,

$$
S_{\text{eff}} = f_1 I_4 + f_2 \gamma^0 + f_3 \gamma^3,
$$

(29)

while noneffective part is useless to gap equations, because the terms noneffective part contains naturally disappear after the trace and integral operations have been made in Eqs. (24), (25), (26) and (28). By introducing Eq. (29) into Eqs. (24), (25), (26), we get the new gap equations that are qualified for numerical calculation,

$$
\frac{2N_c}{4N_c + 1} G = i \sum_f \frac{| q_i e B |}{\pi^2} \int \frac{dp_0 dp_3}{2\pi} f_1, \quad \frac{a}{G} = -i \sum_f \frac{| q_i e B |}{\pi^2} \int \frac{dp_0 dp_3}{2\pi} f_2, \quad \frac{b}{G} = i \sum_f \frac{| q_i e B |}{\pi^2} \int \frac{dp_0 dp_3}{2\pi} f_3,
$$

(30)

where \(f_1, f_2, f_3\) are functions of \(| eB, \sigma, a, b, p_0, p_3, n | n \in \mathbb{N}^0 \}). In a thermal system, \(f_1, f_2, f_3\) are also functions of \(T, \mu\). The explicit expressions of \(f_1, f_2, f_3\) will be discussed in next subsection.

Of cause we can also employ Eq. (11) as the interaction terms and make the mean field approximation, this will cause some coefficients differences in gap equations. We will discuss this kind of interaction terms and discuss the rationality of simply using Eq. (8) as interaction terms in the Appendix.

### B. Gap Equations at Finite \(T\) and \(\mu\)

In a thermal system described by NJL model, the existence of temperature \(T\) and chemical potential \(\mu\) does not change the conclusion above that \(\Sigma_{\text{eff}}\) can be simplified to Eq. (23), but in this case, the gap equations are not exactly the same as Eq. (30), a few modifications are needed. At finite temperature, imaginary number ‘\(i\)’ in the RHS of gap equations of Eq. (30) turns into \(-1\), meanwhile, \(p_0\) is quantized,

$$
\int \frac{dp_0}{2\pi} \rightarrow T \sum_m, \quad p_0 \rightarrow i \omega_m + \mu, \quad \omega_m = (2m + 1)\pi T, \quad m \in \mathbb{Z},
$$

(31)
applying these modifications to Eq. (30), we get the gap equations for a thermal NJL model with external magnetic field,

\[
\frac{2N_c - \sigma}{4N_c + 1} G = -T \frac{|q|eB}{\pi^2} \sum_i \sum_m \int f_1 \, dp_3, \tag{32}
\]

\[
\frac{a}{G} = T \frac{|q|eB}{\pi^2} \sum_i \sum_m \int f_2 \, dp_3, \tag{33}
\]

\[
\frac{b}{G} = -T \frac{|q|eB}{\pi^2} \sum_i \sum_m \int f_3 \, dp_3. \tag{34}
\]

in order to express \(f_1, f_2\) and \(f_3\) concisely, we define a few dispersion parameters firstly

\[
\omega = \sqrt{p_0^2 + \sigma^2}, \quad \omega_{\pm} = \omega \pm b, \quad \omega_{\pm,n} = \sqrt{\omega_{\pm}^2 + 2n|q|eB}, \quad n = 1, 2, 3, \ldots
\]

then the expressions for \(f_1, f_2\) and \(f_3\) are

\[
f_1 = \frac{\sigma}{4\omega} \left( \frac{1}{p_0 - \omega_-} - \frac{1}{p_0 + \omega_+} \right) + \frac{\sigma}{2\omega} \sum_{\pm} \sum_{n=1}^{+\infty} \frac{\omega_{\pm,n}}{p_0^2 - \omega_{\pm,n}^2}, \tag{36}
\]

\[
f_2 = \frac{1}{4} \left( \frac{1}{p_0 - \omega_-} + \frac{1}{p_0 + \omega_+} \right) + \frac{1}{2} \sum_{\pm} \sum_{n=1}^{+\infty} \frac{\hat{p}_0}{p_0^2 - \omega_{\pm,n}^2}, \tag{37}
\]

\[
f_3 = -\frac{1}{4} \left( \frac{1}{p_0 - \omega_-} + \frac{1}{p_0 + \omega_+} \right) + \frac{1}{2} \sum_{\pm} \sum_{n=1}^{+\infty} \left( \frac{\omega_{\pm,n}}{p_0^2 - \omega_{\pm,n}^2} - \frac{\omega_-}{p_0^2 - \omega_-^2} \right). \tag{38}
\]

\[
\hat{p}_0 = i\omega_{m} + \mu_{\tau}, \quad \mu_{\tau} = \mu - a, \tag{39}
\]

\(\mu_{\tau}\) is named as renormalized chemical potential.

Putting Eqs. (30)-(38) into Eqs. (32)-(34) separately, one can transform sum of all the polynomials about \(\omega_{m}\) at finite chemical potential to an integral of a few hyperbolic functions. Base on the equation (looking up detailed deduction in Refs. 17, 27)

\[
\sum_m \ln \{ \beta^2 [(\omega_m + i\mu)^2 + x^2] \} = \beta x + \ln[1 + e^{-\beta(x+\mu)}] + \ln[1 + e^{-\beta(x+\mu)}], \quad \beta = \frac{1}{T}, \quad x \in \mathbb{R}, \tag{40}
\]

we have the new gap equations described as below:

\[
\frac{2N_c - 4\pi^2}{4N_c + 1} G = \sum_i \frac{|q|eB}{\sqrt{\pi}} \int_0^{+\infty} \coth(|q|eBs) \, ds \int \frac{1}{2} \left( \frac{\omega e^{-\omega^2 s} + \omega e^{-\omega^2 s}}{s} \right) \, dp_3
\]

\[
- eB \int \frac{1}{\omega} \left[ \frac{1}{1 + e^{\beta(\omega + \mu)}} + \frac{1}{1 + e^{\beta(\omega - \mu)}} \right] \, dp_3
\]

\[
- \sum_n 2|q|eB \int \frac{1}{\omega} \sum_{n=1}^{+\infty} (F_{+n} + F_{-n}) \, dp_3 + 2eB \theta(|b| - \sigma) \ln \left( \frac{|b| + \sqrt{b^2 - \sigma^2}}{\sigma} \right), \tag{41}
\]

\[
\frac{4\pi^2}{G} b = \sum_i \frac{|q|eB}{2\sqrt{\pi}} \int_0^{+\infty} \coth(|q|eBs) \, ds \int (\omega e^{-\omega^2 s} - \omega e^{-\omega^2 s}) \, dp_3
\]

\[
+ eB \int \frac{1}{1 + e^{\beta(\omega - \mu)}} - \frac{1}{1 + e^{\beta(\omega + \mu)}} \, dp_3
\]

\[
+ \sum_n |q|eB \int \sum_{n=1}^{+\infty} (F_{-n} - F_{+n}) \, dp_3 + 4eB \theta(-b - \sigma) - \theta(b - \sigma) \sqrt{b^2 - \sigma^2}, \tag{42}
\]
that after substituting \( \sigma \) therefore it is reasonable to simplify Eq. (41) and Eq. (42) by Taylor expanding

we choose

because of the subtractions in RHS of Eq. (42). Eq. (43) is the equation for parameter \( \mu \) magnetic field if we set \( \sigma = 0, \mu_r = 0 \). Eq. (43) is the equation for parameter \( \mu \)

eq 0, \mu_r has replaced \( \mu \) in gap equations, it could degenerate to the usual gap equation with external magnetic field if we set \( \mu_r = 0, b = 0 \). Eq. (42) is the equation for parameter \( b \), we expect \( b \) would be a small value because of the subtractions in RHS of Eq. (42). Eq. (43) is the equation for parameter \( a \), as mentioned before, \( a \) is a dynamic modification of chemical potential, it actually means pure particle number (the difference between particle number and anti-particle number) in whole quantum states.

C. Regularization and Approximation

Because \( \mu_r \) has replaced \( \mu \) in gap equations, it is not so important to calculate the value of parameter \( a \) in Eq. (43), we can treat \( \mu_r \) as a free variable and then focus on solving Eq. (41) and Eq. (42). One can also prove that when \( T \rightarrow 0, \mu_r = 0 \) and \( b = 0 \), Eq. (41) will degenerate into the normal gap equation with external magnetic field

which means we can apply the same regularization scheme [19, 26] to proper time \( s \) in Eq. (41) and Eq. (42),

we choose

\( \Lambda = 0.991 \text{GeV}, \quad G = (25.4 \times \frac{2N_c}{4N_c + 1}) \text{GeV}^{-2} = 11.723 \text{GeV}^{-2} \).

Through numerical analysis, we find that when the solutions of \( \sigma \) are nonzero, \( b \)'s values are always small but nonzero, therefore it is reasonable to simplify Eq. (41) and Eq. (42) by Taylor expanding \( b \) to the first order. Also, we find that after substituting \( \sigma \)'s and \( b \)'s solutions back to Eq. (42), the values of \( \frac{G}{4\pi^2} \sum_f |q_f|eB \cdot \int [F_{-n} - F_{+n}] dp_3 \) are always much smaller than \( b \)'s, this term contributes little to gap equations, we can safely remove this term to facilitate the numerical calculation. Now the gap equations are modified to

\[
\frac{2N_c}{4N_c + 1} \frac{4\pi^2}{G} = \sum_f |q_f|eB \int_{1/\Lambda^2}^{+\infty} \frac{e^{-\sigma^2 s}}{s} \coth(|q_f|eBs) ds \]  

\[
- \sum_f 2|q_f|eB \int_{1/\Lambda^2}^{+\infty} \frac{1}{\omega_n \left[ 1 + e^{\beta(\omega_n - \mu_r)} \right]} + \frac{1}{1 + e^{\beta(\omega_n + \mu_r)}} dp_3 ,
\]

\[
- eB \int \frac{1}{\omega_n \left[ 1 + e^{\beta(\omega_n - \mu_r)} \right]} \coth \left( \frac{1}{1 + e^{\beta(\omega_n + \mu_r)}} \right) dp_3 + 2eB \theta(b - \sigma) \ln \left( \frac{|b| + \sqrt{b^2 - \sigma^2}}{\sigma} \right),
\]

\[
\frac{4\pi^2}{G} b = 2\sigma^2 \sum_f |q_f|eB \int_{1/\Lambda^2}^{+\infty} e^{-\sigma^2 s} \coth(|q_f|eBs) ds + 2eB \int \frac{1}{1 + e^{\beta(\omega_n - \mu_r)}} - \frac{1}{1 + e^{\beta(\omega_n + \mu_r)}} dp_3 ,
\]

\[
+ 2eB \theta(-b - \sigma) - \theta(b - \sigma) \sqrt{b^2 - \sigma^2} ,
\]

\[
\omega_{nf} = \sqrt{p^2 + \sigma^2 + 2n |q_f|eB}, \quad n = 1, 2, 3, \ldots, \quad f = u, d.
\]
FIG. 1: The $\mu_t$ dependence of dynamical mass $\sigma$ with fixed temperature and different $eB$s, $T = 0.01\text{GeV}$. For the $eB = 0.01\text{GeV}^2$ curve, when $\mu_t$ reaches 0.318GeV, the dynamic mass jumps from 0.087GeV to 0.0015GeV at $\mu_t = 0.325\text{GeV}$, it is clearly a phase transition. For the $eB = 0.1\text{GeV}^2$ curve, when $\mu_t$ reaches 0.303GeV, the dynamic mass jumps from 0.087GeV to 0.019GeV at $\mu_t = 0.325\text{GeV}$, it is also a phase transition.

FIG. 2: The $\mu_t$ dependence of dynamical mass $\sigma$ with fixed temperature and different $eB$s, $T = 0.11\text{GeV}$. For the $eB = 0.01\text{GeV}^2$ curve, when $\mu_t$ reaches 0.243GeV, the dynamic mass jumps from 0.077GeV to 0GeV at $\mu_t = 0.28\text{GeV}$. For the $eB = 0.1\text{GeV}^2$ curve, $\mu_t = 0.235\text{GeV}$, the dynamic mass jumps from 0.079GeV to 0.015GeV at $\mu_t = 0.25\text{GeV}$. For $eB = 0.28\text{GeV}^2$ curve, when $\mu_t = 0.205\text{GeV}$, the dynamic mass jumps from 0.142GeV to 0.039GeV with $\mu_t = 0.265\text{GeV}$.

For some specific values of $T$, $\mu$ and $eB$, Eq. 49 and Eq. 50 do not have solutions for $\sigma$ and $b$ any more, because at these points, dynamical mass $\sigma$ jumps from Nambu solution to Wigner solution and chiral phase transition happens. In these cases, gap equations degenerate into

$$\frac{4\pi^2}{G} b = 2eB\mu_t. \quad (51)$$

This equation is strictly deduced from Eq. 42 without any approximation. The Wigner solutions always exist, we can say that when the Wigner solution $\sigma = 0$ is preferred, $b$ is proportional to both renormalized chemical potential and magnetic field (this result is also mentioned in Shovkovy’s work [28], known as ‘chiral shift parameter’).

D. Numerical Results and Discussions

By numerically calculating Eqs. 49 and 50 with a series of $T$, $\mu_t$, $eB$ as parameters, we get the values of $\sigma$ and $b$. In these values, some representative results are shown in Figs. 1-7 in the form of $\sigma$-$\mu_t$ relations, these figures demonstrate the phase transition of dynamical mass $\sigma$ when renormalized chemical potential reaches a critical point, of course this critical point depends on temperature and magnetic field. Generally speaking, the stronger magnetic field and lower temperature is, the bigger dynamical mass is generated. However in Figs. 1 and 2, when $\mu_t$ crosses a specific point, the $eB = 0.01\text{GeV}^{-1}$ curve and $eB = 0.1\text{GeV}^{-1}$ curve show us inverse magnetic catalysis effect, which means the stronger magnetic field is, the smaller dynamical mass is generated. Fig. 8 explicitly demonstrates
such effect. For NJL model, inverse magnetic catalysis does not always exist, when \( T \) is big enough, the dynamic mass will strictly obey the rule of magnetic catalysis.

Comparing with \( \sigma-\mu_r \) relations, the corresponding \( b-\mu_r \) relations are shown in Figs. (9)-(15). From these relations, one can see that, \( b-\mu_r \) and \( \sigma-\mu_r \) relations show gaps at the same critical \( \mu_r \). After crossing these critical \( \mu_r \), \( b-\mu_r \) relations are strictly linear correlative because of Eq. (51). Similar to dynamical mass, with fixed temperature, the stronger magnetic field is, the bigger value of \( b \) is, but unlike dynamical mass, with magnetic field fixed, lower temperature makes smaller value of \( b \) when \( \mu_r \) is smaller than its critical point.

Fig. (16) shows us the dynamic mass without modification of \( b \) (called the usual dynamic mass \( \sigma_c \)), comparing it with Fig. (1), one can notice the character of \( \sigma-\mu \) relation barely changes. As for the quantitative modification, Figs. (17) and (18) give us the modification ratios between normal dynamic mass \( \sigma_c \) and actual dynamic mass \( \sigma \) of \( eB = 0.01 \text{ GeV}^2 \) and \( eB = 0.11 \text{ GeV}^2 \) curves with fixed \( T = 0.01\text{GeV} \), the ratios have a common property that

FIG. 3: The \( \mu_r \) dependance of dynamical mass \( \sigma \) with fixed temperature and different \( eBs \), \( T = 0.19\text{GeV} \).

FIG. 4: The \( \mu_r \) dependance of dynamical mass \( \sigma \) with fixed temperature and different \( eBs \), \( T = 0.21\text{GeV} \).

FIG. 5: The \( \mu_r \) dependance of dynamical mass \( \sigma \) with fixed \( eB \) and different temperatures, \( eB = 0.01\text{GeV}^2 \).
The $\mu_r$ dependance of dynamical mass $\sigma$ with fixed $eB$ and different temperatures, $eB = 0.1\text{GeV}^2$.

The $\mu_r$ dependance of dynamical mass $\sigma$ with fixed $eB$ and different temperatures, $eB = 0.22\text{GeV}^2$.

Before $\mu_r$ reaching a specific point (this point varies with $T$ and $eB$), the modification by introducing $b$ is nearly zero (the ratios are much smaller than our numerical precision, therefore its exact values are meaningless), and after $\mu_r$ crossing this point, the modification ratio has prominent increase, such drastic change of ratio happens because along with $\mu_r$ crossing a specific point, the $\sigma$-$\mu_r$ curve begins to leave the straight area, its slope goes farther and farther away from zero, the difference between $\sigma$ and $\sigma_c$ is enlarged. Above all, for the Nambu solution, $b$'s existence mainly causes $\sigma$-$\mu_r$ curves to shift a little along $\mu_r$ axis. From Figs. (11) and (16), we find out that the critical points of $\mu_r$ are barely affected by $b$, therefore assuming self-energy equals dynamical mass is a reasonable approximation, and it suffices to calculate dynamical mass only.

$b$ is too small to affect qualitative properties of phase transition, but why $b$ is so small? Actually we can find the

The $eB$ dependance of dynamical mass $\sigma$ with fixed temperature and different chemical potentials, $T = 0.01\text{GeV}$.
FIG. 9: The $\mu$ dependance of $b$ with fixed temperature and different $eB$s, $T = 0.01\text{GeV}$.

FIG. 10: The $\mu$ dependance of $b$ with fixed temperature and different $eB$s, $T = 0.11\text{GeV}$.

FIG. 11: The $\mu$ dependance of $b$ with fixed temperature and different $eB$s, $T = 0.19\text{GeV}$.

FIG. 12: The $\mu$ dependance of $b$ with fixed temperature and different $eB$s, $T = 0.21\text{GeV}$.
FIG. 13: The $\mu$ dependence of $b$ with fixed $eB$ and different temperatures, $eB = 0.01\text{GeV}^2$.

FIG. 14: The $\mu$ dependence of $b$ with fixed $eB$ and different temperatures, $eB = 0.1\text{GeV}^2$.

FIG. 15: The $\mu$ dependence of $b$ with fixed $eB$ and different temperatures, $eB = 0.22\text{GeV}^2$.

FIG. 16: The $\mu$ dependence of classic dynamic mass $\sigma_c$ with fixed temperature and different $eBs$, $T = 0.01$ GeV.
FIG. 17: The modification ratio between actual dynamic mass $\sigma$ and classic dynamic mass $\sigma_c$ with $T = 0.01$ GeV, $eB = 0.01$ GeV$^2$.

FIG. 18: The modification ratio between actual dynamic mass $\sigma$ and classic dynamic mass $\sigma_c$ with $T = 0.01$ GeV, $eB = 0.13$ GeV$^2$.

solution in Eq. (49), we separate out the zeroth order term of $b$ in the RHS of Eq. (49), it gives us

$$\frac{4\pi^2}{G} b = 2eB \int \left[ \frac{1}{1 + e^{\beta(\omega - \mu_r)}} - \frac{1}{1 + e^{\beta(\omega + \mu_r)}} \right] d\mu_3,$$

(52)

this equation implies the main contribution to $b$ is the pure particle number in LLL (Lowest Landau Level), beside that, $b$'s existence depends on nonzero chemical potential and external magnetic field, these evidences support the conclusion that in a non-neutral system made of high energy particles, external magnetic field could stimulate weak axial-vector current (because $b$ is proportional to $\langle \bar{\psi} \gamma^5 \gamma^3 \psi \rangle$ from the definition in Eq. (19)), and this axial-vector current is nearly proportional to pure particle number in LLL. $b$ is small due to the tiny difference between particle number and anti-particle number in LLL. As for the contribution from higher Landau levels, referring to Eq. (49), it is

$$\int \sum_{n=1}^{+\infty} (F_{-n} - F_{+n}) d\mu_3,$$

this is not pure particle number in higher Landau levels, and its value is much smaller than the contribution from Eq. (52).

Now let’s go back to extract more information from Figs. (1)-(4). In Fig. (1), after phase transition, all curves are not simply jump to $\sigma = 0$ Wigner solution immediately, instead, they jump to smaller solutions that asymptotically approach zero. It is a new phenomenon which has not been found in the previous literatures. We name these new results as ‘intermediate solutions’, they are connecting with Wigner solution smoothly. So as to Fig. (2), the $eB = 0.1$ GeV$^2$ and $eB = 0.28$ GeV$^2$ curves have the same phenomena. This new property is quite common, as long as $T < 0.19$ GeV and with $T$ fixed, the $\sigma-\mu_r$ relations with $eB$ around 0.1 GeV$^2$ or bigger normally have intermediate solutions, they all follow the same pattern that when $eB$ is small, the intermediate solutions are quite small or nonexistent, $eB$ increases, the intermediate solutions are more and more apparent. Meanwhile, when $eB$ exceeding 0.1 GeV$^2$ or around (depending on different $T$), the Nambu solutions shrink asymptotically (which means the phase transition points of $\mu_r$ are getting smaller and smaller), the bigger $eB$ is, the shorter Nambu solution becomes, e.x. in Fig. (1), $eB = 0.16$ GeV$^2$ is apparently shorter than the other two curves, and in Fig. (2), $eB = 0.28$ GeV$^2$ is also shorter than the other two. The increase of temperature also stimulates this shrink, and when $T$ is big enough (for a rough estimation, $T > 0.19$ GeV), all Nambu solutions shrink to null, from then on, the solutions of $\sigma$ are only
intermediate solution and Wigner solution, these results are shown in Figs. 3 and 4, most of them are below 10 MeV scale.

III. CONCLUSIONS AND REMARKS

In this paper, we have studied the self-energy of NJL model in the presence of temperature, chemical potential and external magnetic field, it turns out when chemical potential is nonzero, the self-energy is no longer equivalent to dynamical mass. In order to establish correct gap equations in the frame of NJL model, one has to employ Fierz transition, and then use symmetry analysis to simplify the structure. From Eq. (15) to Eq. (18), the analysis is general, we believe this kind of methods are quite useful to deal with self-energy problems and simplify dynamic equations.

In the new self-energy, \( \sigma \) still stands for dynamical mass. Similar to dynamical chemical potential, but its existence depends on the existence of original chemical potential \( a \), it appears to be the modification of original chemical potential. \( \sigma \) is a small quantity, therefore in this paper, we avoid treating \( \sigma \) as an independent variable, instead we make \( \mu \) absorbing \( a (\mu_r = \mu - a) \), and treat \( \mu_r \) as an independent variable. \( b \) is mean field of axial-vector current, through equation analysis and numerical results, we find \( b \)'s values are always small when \( \sigma \) has non-zero solutions (including Nambu solutions and intermediate solutions), and most of the time \( \sigma \) is bigger than \( b \). At low \( \mu_r \), \( b \) is much smaller than \( \sigma \), it implies \( b \) only has small quantitative modification to \( \sigma \) comparing to classic dynamic mass \( \sigma_c \). The prominent change by introducing \( b \) is the existence of intermediate solutions, these solutions are small (10 GeV scale), but not zero, and they asymptotically approach zero (the classic Wigner solution of massless NJL model) when \( \mu_r \) is big enough.

\( b \)'s existence depends on two conditions, nonzero external magnetic field and surplus charges in the whole system. Comparing with a normal quantum system, if the system is charge neutral, its whole magnetic moments are zero, its quantum states are degenerate, else if the system has surplus charges, an external magnetic field will couple the surplus magnetic moments of this system, and the degeneration of quantum states are broke. These similarities imply that \( b \) is a magnetic moment like or magnetic moment related quantity. We would like to find out the relation between \( b \) and magnetic moment in our following work.

Nevertheless, when dynamical mass equals zero, \( b \) is simply proportional to magnetic field and renormalized chemical potential, which brings about two meaningful results: Firstly, self-energy in Wigner solution is no longer trivial at chiral limit, although dynamical mass remains zero, \( a, b \) change along with temperature, chemical potential and magnetic field. Secondly, if chemical potential or magnetic field is large enough, \( b \)'s values would bring about possible observable effects. Importantly in physics, \( b \) causes splitting of dispersion relation according to Eq. (35). Especially in Wigner phase, \( b \) is capable of acquiring bigger values, hence the dispersion relation Eq. (53) is rewritten as

\[
\omega_{\pm n} = \sqrt{(|p_n| + b)^2 + 2n|q|eB},
\]

(53)
apparently, at specific momentum and Landau level, one particle can have two different energies.

As shown in Eqs. (35) and (53), the physical effects induced by \( b \) depend on dispersion relation, the change of dispersion relation causes different particle number densities. In a series of Tatsumi’s works, they use \( b \) to study ferromagnetism in nuclear matter and quark matter, \( b \), as a mean field of axial vector current, is the parameter to describe ‘spin polarization’ in their articles, it is said that \( b \)'s existence slightly splits dispersion relation, hence the ‘spin’-up particles and ‘spin’-down particles are separated to different Fermi surfaces, known as ‘spin polarization’. Different with our study, they does not consider external magnetic field in their works, therefore \( b \) will not exist unless quarks have nonzero original masses and the quark matter is in CSC (color super conductivity) state. As soon as quark matter leaves CSC state, ‘spin polarization’ vanishes along with \( b \), the ‘spin polarization’ effect is spontaneous. However, in our article, the fermion’s original mass is zero (chiral limit), and CSC state is not included, but we have nonzero external magnetic field, the magnetic field keeps \( b \) presenting, hence ‘spin polarization’ is automatically but not spontaneously stimulated. In this paper, the strength of external magnetic field ranges from 0.01 to 0.28 GeV^2, which correspond approximately around \( 10^{19-10^{20}} \) Gauss. This is nearly the strongest magnetic field in experiments, and even so, the physical effects are still too small to be distinguished from experiments. We are looking forward to other conditions that can bring us obvious effects. Large space scale could enlarge tiny modification in dispersion relation, for example, magnetar, one kind of neutron star, which is expected to have magnetic field of \( 10^{15} \) Gauss, although this strength is much smaller than the strength we employ in this paper, magnetars have large volumes. On the other hand, we can also simply expect larger magnetic field to show obvious effects in future experiments. It is believed that in early universe, the magnetic field could reach \( 10^{23} \) Gauss, hence the results in this paper may play an important role in early universe evolution.

The inverse magnetic catalysis effect mentioned in Fig. 5 is not exactly the same as the well-known inverse magnetic catalysis studied in these articles. The lattice results show inverse magnetic catalysis happens at
\( \mu = 0, T \neq 0 \). But in NJL model, this can not happen with mean field approximation at least. We are looking forward to use other methods beyond mean field approximation to find out inverse magnetic catalysis in NJL model. Comparing Figs. 11 and 13, we believe that with or without the modification of \( b \), the inverse magnetic catalysis effect in Fig. 8 still happens, the dynamic mass’s modification induced by \( b \) is too small to affect the qualitative properties of dynamic mass. However, from Figs. 17 and 18 we know, when chemical potential is big enough, \( b \) is more or less decreases inverse magnetic catalysis effect.

Comparing Figs. 1 and 16, we believe that with or without the modification of \( b \), the inverse magnetic catalysis effect in Fig. 8 still happens, the dynamic mass’s modification induced by \( b \) is positively bigger than \( \sigma_c \) (dynamic mass without modification of \( b \)), therefore we can conclude that \( b \) more or less decreases inverse magnetic catalysis effect.

**Appendix A: The Gap Equations with Other Four Fermions interaction terms**

In this appendix, we employ Eq. (11) instead of Eq. (12) as the four fermions interaction terms in NJL model with external magnetic field, and we still make mean field approximation to these terms in Eq. (11),

\[
\mathcal{L}'_f = \mathcal{F}(\mathcal{L}_f) \rightarrow \mathcal{L}_{\text{mean}} = -\bar{\psi} \Sigma_{sf} \psi + \mathcal{L}_M, \tag{A1}
\]

\[
\Sigma_{sf} = -\frac{G}{2N_c} [i \bar{\psi} \gamma^5 \vec{\tau} \psi] \cdot [i \bar{\tau} \gamma^5 \vec{\tau} \tau] - (i \bar{\psi} \gamma^5 \vec{\tau} \tau) i \gamma^5
- 2(\bar{\psi} \gamma_\mu \gamma_5 \psi) \gamma^\mu - 2(\bar{\psi} \gamma_5 \gamma_5 \psi) (\gamma_5 \gamma_5) + (\bar{\psi} \sigma_{\mu \nu} \psi) \gamma^{\mu \nu} - (\bar{\psi} \sigma_{\mu \nu} \gamma_5 \gamma_5 \psi) \cdot (\gamma^{\mu \nu} \vec{\tau})]. \tag{A2}
\]

In this case, we still can use symmetry analysis to simplify \( \Sigma_{sf} \) to \( \Sigma \),

\[
\Sigma = \sigma + a \gamma^0 + b \gamma^5 \gamma^3, \tag{A3}
\]

\[
\sigma = -\frac{G}{2N_c} (\bar{\psi} \gamma_5 \gamma_5 \psi), \quad a = \frac{G}{N_c} (\bar{\psi} \gamma_0 \gamma_0 \psi), \quad b = -\frac{G}{N_c} (\bar{\psi} \gamma_5 \gamma_5 \gamma_3 \psi). \tag{A4}
\]

Accordingly, \( \mathcal{L}_M \) has a simple form,

\[
\mathcal{L}_M = -\frac{N_c}{G} \sigma^2 + \frac{N_c}{2G} a^2 - \frac{N_c}{2G} b^2. \tag{A5}
\]

Similar to main body of this article, we deduct the gap equations through the extremum of effective action Eq. (22), and we get

\[
\frac{2}{G} \sigma \int d^4x = i \sum_i \text{tr} \int d^4x \langle x | \hat{S}_i | x \rangle, \tag{A6}
\]

\[
\frac{a}{G} \int d^4x = -i \sum_i \text{tr} \int d^4x \langle x | \hat{S}_i \gamma^0 | x \rangle, \tag{A7}
\]

\[
\frac{b}{G} \int d^4x = i \sum_i \text{tr} \int d^4x \langle x | \hat{S}_i \gamma^5 \gamma^3 | x \rangle. \tag{A8}
\]

Comparing Eqs. (A6)-(A8) with Eqs. (23)-(26), we find that the series of gap equations in this appendix are only have different coefficients with the ones in main body. Following Eqs. (A6)-(A8), we deduct gap equations suiting direct calculation as

\[
\frac{4\pi^2}{G} = \sum_i \frac{|q_i| eB}{\sqrt{\pi}} \int_0^{\infty} \frac{\coth(|q_i| eB \omega)}{\sqrt{\omega}} d\omega \int \frac{1}{2} \left( \frac{\omega_+ e^{-\omega_+ s} + \omega_- e^{-\omega_- s}}{\omega} \right) dp_3
- eB \int \frac{1}{\omega} \left[ \frac{1}{1 + e^{\beta (\omega_+ - \mu_c)}} + \frac{1}{1 + e^{\beta (\omega_- - \mu_c)}} \right] dp_3
- \sum_i 2 |q_i| eB \int \frac{1}{\omega} \sum_{n=1}^{\infty} (F_{i+n} + F_{i-n}) dp_3 + 2eB \Theta(|b| - \sigma) \ln \left( \frac{|b| + \sqrt{b^2 - \sigma^2}}{\sigma} \right). \tag{A9}
\]
\[
\frac{4\pi^2}{G} b = \sum_{\ell} |q| eB \int_{0}^{+\infty} \frac{\coth(q_{\ell} eB s)}{\sqrt{s}} ds \int (\omega_{+} e^{-\omega_{+}^2 s} - \omega_{-} e^{-\omega_{-}^2 s}) dp_3 \\
+ 2eB \int \left[ \frac{1}{1 + e^{\beta(\omega_{-} - \mu_{-})}} - \frac{1}{1 + e^{\beta(\omega_{+} + \mu_{+})}} \right] dp_3 \\
+ \sum_{\ell} 2|q| eB \int \sum_{n=1}^{+\infty} (F_{-n} - F_{+n}) dp_3 + 4eB[\theta(-b - \sigma) - \theta(b - \sigma)] \sqrt{b^2 - \sigma^2},
\]

(A10)

\[
\frac{4\pi^2}{G^2} = 2eB \int \left[ \frac{1}{1 + e^{\beta(\omega_{+} - \mu_{+})}} - \frac{1}{1 + e^{\beta(\omega_{-} + \mu_{-})}} \right] dp_3 \\
+ \sum_{\ell} 2|q| eB \int \sum_{n=1}^{+\infty} \left[ \frac{1}{1 + e^{\beta(\omega_{+} - \mu_{+})}} - \frac{1}{1 + e^{\beta(\omega_{+} + \mu_{+})}} + \frac{1}{1 + e^{\beta(\omega_{-} - \mu_{-})}} - \frac{1}{1 + e^{\beta(\omega_{-} + \mu_{-})}} \right] dp_3,
\]

(A11)

here the definitions of \(\omega_{\pm}, \omega_{\pm n}\) and \(F_{\pm n}\) are the same with main body.

In Wigner solution \((\sigma = 0)\), the gap equations degenerate to

\[
\frac{4\pi^2}{G} b = 4eB \mu_{+}.
\]

(A12)

One more thing, we also need a cutoff to the integral \(\int_{0}^{+\infty} ds\) in Eqs. (A9) and (A10) and decide the value of coupling constant \(G\). The cutoff \(\Lambda\) is also 0.991GeV, but the coupling constant is different here,

\[
\int_{0}^{+\infty} ds \rightarrow \int_{\frac{1}{\Lambda^2}}^{+\infty} ds,
\]

(A13)

\[
\Lambda = 0.991\text{GeV}, \quad G = 25.4\text{GeV}^{-2}.
\]

(A14)

The numerical results with the gap equations in this appendix is similar to the ones in main body, there are also intermediate solutions when \(\mu_{+}\) crosses critical points at some specific temperature and magnetic field. When temperature is high enough, no Nambu solution is available, but there is always a crossover from intermediate solution to classic Wigner solution.

With the presence of external magnetic field and finite chemical potential, the original four fermion interaction terms Eq. (5) are dynamically forbidden. Because mathematically speaking, even without \(b\gamma^5 \gamma^0\) in Eq. (18), Eq. (29) will also explicitly show \(f^3\gamma^3\) term, this indicate that there must be another parameter cooperating with this term to make the gap equation self consistent. Therefore, in order to acquire sensible results, one should study Eqs. (11) and (12) instead of Eq. (10).

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