Vibration characteristics study of micro-cantilever plate with a tip mass

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Abstract. This paper presents a systematic vibration characteristics study of micro-cantilever plate with tip mass. Unlike previous research, this work proposes that the vibration mode function varies with different values of tip mass while most of previous works assumed that it’s constant, and verifies the necessity of employing the varied mode function. Based on strain gradient and inextensible plate theory, comparative analysis found that an increase of tip mass reduces the natural frequency of the plate.

1. Introduction
Since MEMS (Microelectromechanical Systems) has been widely used in civil and military fields, there has been much research on microscale energy harvester, which is convenient, long-life, cost-effective and environmentally friendly compared with traditional chemical batteries [1-5]. The cantilever architecture can effectively produce large mechanical strain within the piezoelectric layer during vibrations [6, 7]. Adding a tip mass at the free end of the cantilever beam or plate has become one of the most common structural optimization methods, as it can effectively increase vibration amplitude and obtain higher output voltage due to piezoelectric effects. Moreover, adding a tip mass has the advantage of tuning resonant frequency to match ambient vibration frequency by reducing system stiffness [8, 9].

It has been found that deformation of the structure in the micron scale is size-dependent and the theory of classical continuum mechanics is not applicable [10, 11]. Building a constitutive model based on strain gradient theory is an effective approach to link classical elastic theory with atomic simulation. The strain gradient theory has been developed as a relatively optimal theory which is widely used to predict size effect of mechanical properties for micron size structures in the past few decades [12-14]. This paper proposes the model of the micro-cantilever plate with tip mass. Nature frequency and mode shape are calculated for different ratios of tip mass to the cantilever mass and relation between mass ratios and vibration mode function is analyzed.

2. Modeling and analysis
According to classical thin plate theory and strain gradient theory, natural frequency and mode shape of the CPWTM (cantilever plate with a tip mass) are determined by Rayleigh-Ritz method, where modal function of CPWTM is a combination of CBWTM (cantilever beam with a tip mass) modal functions and free-free beam modal functions.

2.1 Basic beam functions in x and y directions
When using the Beam Function Combination method, basic function of cantilever beam is required to obtain mode function of CPWTM.

The structure of CBWTM is shown in Fig. 1, and the function of CBWTM deflection is assumed to be:

\[ w(x,t) = X(x) \sin(\nu t + \theta) \]  

(1)

The modal function \( X(x) \) is:

\[ X(x) = c_1 \sin \left( \frac{kx}{a} \right) + c_2 \cos \left( \frac{kx}{a} \right) + c_3 \sinh \left( \frac{kx}{a} \right) + c_4 \cosh \left( \frac{kx}{a} \right) \]  

(2)

where \( c_1, c_2, c_3, c_4 \) and \( k \) are undetermined coefficients.

The boundary conditions of the CBWTM are:

a) At the fixed end \( (x=0) \), the displacement and rotation angle are equal to zero:

\[ X = 0, \quad \frac{\partial X}{\partial x} = 0 \]  

(3)

b) At the cantilever end \( (x=a) \), the bending moment is equal to zero, the shear force is equal to inertia force of the vibrating tip mass \( m_t \):

\[ EI \frac{\partial^2 X}{\partial x^2} = 0, \quad \frac{\partial}{\partial x} \left( EI \frac{\partial^2 w}{\partial x^2} \right) = m_t \frac{\partial^2 w}{\partial t^2} \]  

(4)

where \( E \) is Young’s modulus, \( I \) is the moment of inertia.

The undetermined coefficients \( c_1, c_2, c_3, c_4 \) and frequency equation can be obtained by substituting Equation (1), Equation (2) into the boundary condition Equation (3), Equation (4):

\[ c_4 = -c_2, \quad c_3 = -c_1, \quad c_2 = -\frac{c_1 (\sin k + \sinh k)}{\cos k + \cosh k} \]  

(5)

\[-r_m k \sin k \cosh k + r_b k \cos k \sinh k + \cos k \cosh k + 1 = 0 \]  

(6)

where \( r_m = \frac{m_t}{m_b} = \frac{m_t}{\rho A a} \) is mass ratio; \( m_b \) is beam mass; \( \rho \) is density of beam; \( A \) is cross sectional area; \( a \) is beam length.

Frequency coefficient \( k_i (i=1\ldots n) \) which varies with the mass ratio can be obtained by the frequency equation Equation (6), and then modal functions of CBWTM in x direction can be obtained from Equation (2), Equation (6):

\[ X_i = \sin \frac{k_i x}{a} - \sinh \frac{k_i x}{a} - \alpha_i \left( \cosh \frac{k_i x}{a} - \cos \frac{k_i x}{a} \right), \quad (i=1\ldots n) \]  

(7)

where \( \alpha_i = \frac{\sin k_i + \sinh k_i}{\cos k_i + \cosh k_i} \).

Change law of the first three orders frequency coefficients with mass ratio is shown in Fig. 2.
It can be seen that frequency coefficient $k$ decreases as mass ratio increases, especially the first order. Reduction rate of reaches 52% at while decline rate of and are only 14% and 8% in the meantime. It should be noted that frequency coefficient change very little with, which indicates that increasing does not necessarily work as expected.

2.2 Vibration characteristics analysis of CPWTM

The structure of CPWTM is shown in Fig. 3. Set the middle plane of the plate to plane and $z$-axis is perpendicular to plane. Parameters of the plate are given in Table 1.

![Fig. 3. Model diagram of micro-cantilever plate with tip mass](image)

Table 1. Parameters of plate

| Description | Symbol | Value  |
|-------------|--------|--------|
| Length      | $a$    | 400μm  |
| Width       | $b$    | 200μm  |
| Thickness   | $h$    | 1μm    |

Deflection of middle surface of micro CPWTM can be assumed as:

$$w(x, y, t) = W(x, y) \sin(pt + \phi)$$

where $p$ is natural frequency and $\phi$ is initial phase.

Modal function of CPWTM is regarded as a combination of CBWTM function in $x$ direction and the free-free beam function in $y$ direction.

$$W(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} A_{ij} X_i(x) Y_j(y)$$

where $X_i(x)$ can be obtained by Equation (7).

Based on continuum mechanics theory, strain energy destiny and corresponding strain energy of CPWTM can be written as:
\[ \bar{W} = \frac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{zz}e_{zz}) \]
\[ U = \iiint \bar{W} \, dxdydz \]  \hspace{1cm} (10)

where \( \sigma \) and \( e \) are stress and strain respectively.

Considering the size effect of micro structure, strain gradient theory is used and strain gradient expression of micro plate is:
\[ \eta_{ik} = \eta_{ij} = e_{ij,j} = e_{ij,j} (i = x, y, z \quad j,k = x, y) \]  \hspace{1cm} (11)

where \( \eta \) refers to strain gradient.

General expressions of strain energy density and corresponding strain energy deduced from strain gradient are proposed by Mindlin \cite{12}:  
\[ \bar{W} = L_1 \eta_{11} \eta_{11} + L_2 \eta_{12} \eta_{12} + L_3 \eta_{13} \eta_{13} + L_4 \eta_{41} \eta_{41} + L_5 \eta_{51} \eta_{51} \]
\[ \bar{U} = \iiint \bar{W} \, dxdydz \]  \hspace{1cm} (12)

where \( L_1 \) to \( L_5 \) are material parameters according to Ramezani \cite{15}:
\[ L_2 = \frac{1}{2}l^2 \lambda \]
\[ L_4 = l^2 \mu \]
\[ L_5 = L_6 = L_7 = 0 \]  \hspace{1cm} (13)

where \( l \) is material scale parameter, \( \lambda \) and \( \mu \) are common lame constants:
\[ \lambda = E \nu / (1 - 2\nu)(1 + \nu) \]
\[ \mu = E / 2(1 + \nu) \]  \hspace{1cm} (14)

where \( \nu \) is Poisson ratio.

Potential energy of micro cantilever plate \( U \) can be expressed as:
\[ U = \bar{U} + \hat{U} \]  \hspace{1cm} (15)

Kinetic energy of the system is:
\[ T = T_0 + T_m \]  \hspace{1cm} (16)

where \( T_0 \) is kinetic energy of a micro cantilever plate, \( T_m \) is kinetic energy of the tip mass:
\[ T_0 = \iiint \frac{\rho}{2} \ddot{w}^2 \, dV \]
\[ T_m = \frac{1}{2}m \ddot{w}^2 \bigg|_{x=a} \]  \hspace{1cm} (17)

Based on Rayleigh-Ritz method, simultaneous equations are obtained by extremum conditions for functions of several variables:
\[ \frac{\partial U_{\text{max}}}{\partial A_{ij}} - \frac{\partial T_{\text{max}}}{\partial A_{ij}} = 0 \]  \hspace{1cm} (18)

It’s a system of algebraic equations where \( A_{ij} \) are treated as generalized coordinates, and natural frequency \( p \) can be determined by non-zero condition, which refers to the coefficient determinant of \( A_{ij} \) is equal to zero; \( U_{\text{max}} \) and \( T_{\text{max}} \) can be obtained by Equation (10) and Equation (16). Then nth order natural frequency \( p^{(n)} \) is substituted into the characteristic equation for the undetermined coefficient \( A_{ij}^{(n)} \), and nth order vibration mode function corresponding to nth order natural frequency is obtained.
\[ W^{(n)} = \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}^{(n)} X_i Y_j \]  \hspace{1cm} (19)

Abaqus software is used for finite element analysis to validate the nature frequency and mode shape calculated based on modal function varied with \( r_m \) which are proposed in this paper.
Table 2. Nature frequency of plate (KHz)

| Order | Method | \( r_n = 1 \) | \( r_n = 2 \) | \( r_n = 3 \) |
|-------|--------|-------------|-------------|-------------|
|       |        | \( l = 0 \) | \( l = 0.3 \) | \( l = 0 \) | \( l = 0.3 \) |
| 1st   | Present | 9.24        | 13.3        | 4.16        | 6.00        | 2.57        | 3.70        |
|       | Abaqus  | 9.08        | 4.13        |             |             |             |             |
| 2nd   | Present | 39.8        | 57.4        | 20.5        | 29.6        | 12.9        | 18.5        |
|       | Abaqus  | 39.4        | 21.8        |             |             |             |             |
| 3rd   | Present | 57.9        | 83.5        | 43.2        | 62.3        | 42.1        | 60.6        |
|       | Abaqus  | 58.0        | 43.7        |             |             |             |             |

(The unit of \( l \) is \( \mu \text{m} \).)

Fig. 4. Maximum deflection for different \( k_i \).

Since Abaqus software cannot consider size effect, only result comparison for \( l = 0 \) is performed, it can be found that the theoretical analysis results match the finite element analysis results regarding the frequency in Table 2. Moreover, the size effect could never be underestimated when comparing the data of \( l = 0 \) and \( l = 0.3 \mu \text{m} \) columns in Table 2 for micron scale structure, the nature frequency increase when considering size effect.

Based on MATLAB software, taking constant \( k_i \) and varying \( k_i \) respectively, deflection of the plate is calculated and compared with the finite element results, where the mass ratio is set to be 3. The maximum deflection for different case is shown in Fig. 4.

3. Conclusions
This paper established a model of the micro cantilever plate. Mode function of the cantilever plate that varies with the tip mass is proposed, the modal analysis is validated by Abaqus software. Based on the proposed mode function, the vibration characteristics are analyzed. The results of the comparative study show that the mode function that varies with the mass ratio is more accurate. The nature frequency is greater when considering size effect, and decreases as the mass ratio increases.
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