Simulations of Nonaxisymmetric Instability in a Rotating Star: A Comparison Between Eulerian and Smooth Particle Hydrodynamics

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\textbf{ABSTRACT}

We have carried out 3-D numerical simulations of the dynamical bar instability in a rotating star and the resulting gravitational radiation using both an Eulerian code written in cylindrical coordinates and a smooth particle hydrodynamics (SPH) code. The star is modeled initially as a polytrope with index $n = 3/2$ and $T_{\text{rot}}/|W| \approx 0.30$, where $T_{\text{rot}}$ is the rotational kinetic energy and $|W|$ is the gravitational potential energy. In both codes the gravitational field is purely Newtonian, and the gravitational radiation is calculated in the quadrupole approximation.

We have run 3 simulations with the Eulerian code, varying the number of angular zones and the treatment of the boundary between the star and the vacuum. Using the SPH code we did 7 runs, varying the number of particles, the artificial viscosity, and the type of initial model. We compare the growth rate and rotation speed of the bar, the mass and angular momentum distributions, and the gravitational radiation quantities. We highlight the successes and difficulties of both methods, and make suggestions for future improvements.

\textit{Subject headings:} hydrodynamics — methods: numerical — instabilities — radiation mechanisms: gravitational

\section{1. Introduction}

Many of the most interesting astrophysical systems can be described by the equations of hydrodynamics coupled to gravity. As computers grow more powerful, new numerical

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techniques are being developed and computer simulations of astrophysical systems are gaining importance. In fact, numerical modeling may be the only means of getting detailed understanding about certain phenomena. Each numerical method has its own strengths and weaknesses, and the choice of the most suitable method depends in part on the physical system being studied. Therefore, it is important to understand the behavior of the various techniques in different situations.

One area in which numerical simulations play a key role is the modeling of astrophysical sources of gravitational radiation. With the prospect of several gravitational wave detectors becoming operational within a decade (e.g. Abramovici, et al. 1992; Bradaschia, et al. 1990), the detailed modeling of these sources has a high priority. For example, global rotational instabilities that arise in collapsing or compact stars can potentially produce detectable amounts of gravitational radiation. A rapidly rotating stellar core that has exhausted its nuclear fuel and is prevented from collapsing to neutron star size by centrifugal forces could become unstable and possibly shed enough angular momentum to allow collapse to a supernova (Thorne 1995). Also, a neutron star that is spun up by accretion of mass from a binary companion could potentially reach fast enough rotation rates to go unstable (Schutz 1989). Since these sources are time-dependent, nonlinear, and fully 3-dimensional systems, calculating the gravitational radiation they produce requires numerical simulations.

Global rotational instabilities in fluids arise from nonradial “toroidal” modes $e^{\pm im\varphi}$, where $\varphi$ is the azimuthal coordinate and $m = 2$ is known as the “bar mode”. They can be parametrized by

$$\tau = T_{\text{rot}}/|W|,$$  

where $T_{\text{rot}}$ is the rotational kinetic energy and $W$ is the gravitational potential energy (Tassoul 1978; Shapiro & Teukolsky 1983; Durisen & Tohline 1985; Schutz 1986). We concentrate on the bar instability since it is expected to be the fastest growing mode. This instability can develop by two different physical mechanisms. The *dynamical* bar instability is driven by Newtonian hydrodynamics and gravity. It occurs for fairly large values of the stability parameter $\tau > \tau_d$ and develops on a timescale of approximately one rotation period. The *secular* instability arises from dissipative processes such as gravitational radiation reaction and operates in the range $\tau_s < \tau < \tau_d$. It develops on a timescale of several rotation periods or longer (Schutz 1989). The constant density, incompressible, uniformly rotating Maclaurin spheroids have $\tau_s \approx 0.14$ and $\tau_d \approx 0.27$. For differentially rotating fluids with a polytropic equation of state,

$$P = K \rho^\Gamma = K \rho^{1+1/n},$$

where $n$ is the polytropic index and $K$ is a constant that depends on the entropy, early studies indicated that the secular and dynamical bar instabilities should occur at about
these same values of $\tau$ (Shapiro & Teukolsky 1983; Durisen & Tohline 1985; Managan 1985; Imamura, Friedman, & Durisen 1985). Recent work by Imamura, et al. (1995) shows that both the angular momentum distribution and, to a lesser degree, the polytropic index affect the value of $\tau$ at which the $m = 2$ secular instability sets in. For the dynamical bar instability Pickett, Durisen, & Davis (1996; hereafter PDD) demonstrate that, for $n = 3/2$ polytropes, the $m = 2$ dynamical stability limit $\tau_d \approx 0.27$ is valid for initial angular momentum distributions that are centrally condensed and similar to those of Maclaurin spheroids. However, for angular momentum distributions that produce somewhat extended disk-like regions, both one- and two-armed spiral instabilities appear at considerably lower values of $\tau$.

As a first step toward understanding realistic sources we are simulating the gravitational radiation emitted when a rapidly rotating star, modeled initially as a polytrope with $n = 3/2$ ($\Gamma = 5/3$), becomes dynamically unstable. Newtonian gravity is used, and the gravitational radiation produced is calculated in the quadrupole approximation. The back reaction of the gravitational radiation on the star is not included. We have chosen the case $\tau \approx 0.30$, which is just above the dynamical stability limit and so might reasonably approximate a star that spins up (due to collapse or accretion) and goes unstable. This case has also been studied numerically and analyzed using the linearized tensor virial equations (TVE; see Chandrasekhar 1969) by Tohline, Durisen, & McCollough (1985; hereafter TDM), so their results are available for comparison.

We have carried out simulations of the dynamical bar instability using two very different computer codes, each based on numerical techniques actively used in astrophysics. One of these is a 3-D Eulerian hydrodynamics code written in cylindrical coordinates with monotonic advection. The other is a smooth particle hydrodynamics (SPH) code with variable smoothing lengths and individual particle timesteps. Since the SPH code is Lagrangian, gridless, and fully adaptive, it is intrinsically very different from the Eulerian code. By running the same calculation on these two codes, we hope to gain a better understanding of the relative merits of these methods in modeling the dynamical bar instability.

An earlier comparison between the results of using Eulerian and SPH codes to model the dynamical instability was carried out by Durisen et al. (1986; hereafter DGTB). They used rapidly rotating polytropes with $n = 3/2$ and considered the cases $\tau \approx 0.33$ and $\tau \approx 0.38$. Since they were studying this instability in the context of star formation, they did not calculate the gravitational radiation generated. They used two different Eulerian codes, one with cylindrical coordinates (the same one used by TDM) and the other with spherical coordinates. Both of these used the diffusive donor cell advection and fairly low resolution.
The SPH code used a smoothing length that was the same for all particles and varied in time, and a fairly small number of particles. Our study takes advantage of more modern and accurate numerical methods, and focuses on the gravitational radiation generated by the dynamical instability in compact stars.

This paper is organized as follows. In §2, we briefly describe the numerical techniques used in the two codes, and in §3, we discuss the calculation of gravitational radiation using the quadrupole approximation. The initial conditions are presented in §4. The results of modeling the bar instability using the Eulerian code are given in §5, and the results of using the SPH code in §6. We compare the Eulerian and SPH results in §7 and present our conclusions in §8.

2. Numerical Techniques

Both of the computer codes used in this study solve the equations of hydrodynamics coupled to Newtonian gravity. The matter is taken to be a perfect fluid with equation of state

\[ P = (\Gamma - 1)\rho\epsilon, \]

where \( \epsilon \) is the specific internal energy. Each code has been subjected to a variety of tests to insure its accuracy and stability. In this section we present a brief description of each code, referring the reader to the literature for further details.

2.1. Eulerian Code

We use the 3-D Eulerian hydrodynamics code developed by Smith, Centrella, & Clancy (1994; see also Smith 1993; Clancy 1989). This code is written in cylindrical coordinates \((\varpi, z, \varphi)\) with variable spatial zoning. This is useful for representing rotating configurations, including bars, toroids, and more complicated geometries, all of which may exhibit substantial rotational flattening. The hydrodynamical equations are solved using time explicit differencing with operator splitting (Wilson 1979; Bowers & Wilson 1991). Although the code has the option of allowing the grid to move in the \(\varpi\) and \(z\) directions, for simplicity we hold both grids fixed for the models presented in this paper. We impose reflection symmetry through the equatorial plane \(z = 0\), and calculate the full range of the angular coordinate \(\varphi : 0 - 2\pi\).
In Eulerian hydrodynamics fluid is transported from one grid zone to another, and it is important to obtain an accurate value for the quantity crossing the zone face. The simplest such advection scheme is the donor cell method, which is only accurate to first order and produces large numerical diffusion (Bowers & Wilson 1991). To achieve better accuracy and less diffusion, the advection terms can be updated using an interpolation method that preserves monotonicity in the quantity being advected. This code uses a monotonic advection scheme developed by LeBlanc (Clancy 1989; Bowers & Wilson 1991), with all spatial finite differences in the advection phase being second order. The spatial differencing in the Lagrangian phase is first order except for the “PdV” term, which is second order. The code uses first order differencing in time with operator splitting; in general, this results in a scheme that is somewhat better than first order in time, but not quite second order. The method of consistent advection is used for the angular momentum transport (Norman, Wilson, & Barton 1980; Norman & Winkler 1986). Shocks are handled using a standard artificial viscosity. For the bar instability runs presented in this paper, shocks occur during the later stages of the evolution, when the spiral arms expand supersonically and merge. This shock heating and dissipation generates entropy.

The Newtonian gravitational potential is calculated by solving Poisson’s equation on the cylindrical grid, with the boundary conditions at the edge of the grid specified using a spherical multipole expansion. In finite difference form this becomes a large, sparse, banded matrix equation which we solve using a preconditioned conjugate gradient method with diagonal scaling (Press, et al. 1992; Meijerink & Van Der Vorst 1981). This is a simple and efficient method that requires a minimum of memory overhead, since it does not need to store the entire matrix being inverted and takes advantage of existing arrays already set aside for temporary storage in the code. Comparison tests with other sparse matrix solvers showed that this method produces solutions with the same accuracy using significantly less CPU time (Smith, Centrella, & Clancy 1994). Such memory and time considerations are very important for the successful implementation of a fully 3-D Eulerian code.

2.2. TREESPH

SPH is a gridless Lagrangian hydrodynamics method that models the fluid as a collection of fluid elements of finite extent described by a smoothing kernel (Lucy 1977; Gingold & Monaghan 1977; see Monaghan (1992) for a review). We have used the implementation of SPH by Hernquist & Katz (1989) known as TREESPH. In this code each particle is assigned a smoothing length which is allowed to vary in both space and
time, thereby achieving roughly the same level of accuracy in all regions of the fluid. The use of these variable smoothing lengths as well as individual particle timesteps makes the program adaptive in both space and time. TREESPH has the option of evolving either the thermal energy or a function of the entropy. Hernquist (1993) has shown that for adaptive SPH, in which smoothing lengths vary in time, certain types of errors do not show up in the total energy if the thermal energy equation is evolved. However, if the entropy equation is used, then conservation of total energy is a good indicator of the global accuracy of the calculation. We have chosen to evolve the entropy equation.

The gravitational forces in TREESPH are calculated using a hierarchical tree method (Barnes & Hut 1986) optimized for vector computers (Hernquist 1987). The particles are first organized into a nested hierarchy of cells, and the mass multipole moments of each cell up to a fixed order, usually quadrupole, are calculated. In computing the gravitational acceleration of a particle, it is allowed to interact with different levels of the hierarchy in different ways. The force due to neighboring particles is computed by directly summing the two-body interactions. The influence of more distant particles is accounted for by including the multipole expansions of the cells which satisfy the accuracy criterion at the location of each particle. In general, the number of terms in the multipole expansions is small compared to the number of particles in the corresponding cells. This leads to a significant gain in efficiency, and allows the use of larger numbers of particles than would be possible with methods that simply sum over all possible pairs of particles.

As a Lagrangian method SPH is attractive because the computational resources can be concentrated where the mass is located, rather than spread over a grid that can be mostly empty. In addition the numerical algorithms are simpler and, in general, considerably easier to implement than standard Eulerian methods. SPH has been applied to a variety of astrophysical problems and is gaining in popularity. However, it is still a relatively new method, and less is known about its behavior in various situations than the Eulerian methods, which have been developed and used by a much larger number of researchers over the decades. Comparison studies such as this one are therefore of considerable interest.

3. Calculation of Gravitational Radiation

We calculate the gravitational radiation produced in these models using the quadrupole approximation, which is valid for nearly Newtonian sources (Misner, Thorne, & Wheeler 1973). Since the gravitational field in both codes is purely Newtonian, we calculate only the production of gravitational radiation and do not include the effects of radiation reaction.
The spacetime metric can be written
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4} \]
where \( \mu, \nu = 0, 1, 2, 3 \), \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) is the metric of flat spacetime, and \( |h_{\mu\nu}| \ll 1 \).

The gravitational waveforms are given by the transverse-traceless (TT) components of the metric perturbation \( h_{ij} \),
\[ r h_{ij}^{\text{TT}} = 2 \ddot{I}_{ij}^{\text{TT}}, \tag{5} \]
where
\[ I_{ij} = \int \rho \left( x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right) d^3r \tag{6} \]
is the trace-reduced quadrupole moment of the source. Note that we use units in which \( c = G = 1 \) only when discussing the gravitational radiation quantities. Here \( r^2 = x^2 + y^2 + z^2 \) is the distance to the source, spatial indices \( i, j = 1, 2, 3 \), and a dot indicates a time derivative \( d/dt \). For an observer located on the axis at \( \theta = 0, \varphi = 0 \) in spherical coordinates centered on the source, the waveforms for the two polarization states take the simple form
\[ h_{+, \text{axis}} = \frac{1}{r} (\ddot{I}_{xx} - \ddot{I}_{yy}), \tag{7} \]
\[ h_{\times, \text{axis}} = \frac{2}{r} \ddot{I}_{xy}. \tag{8} \]

The gravitational wave luminosity is defined by
\[ L = \frac{dE}{dt} = \frac{1}{5} \left\langle I_{ij}^{(3)} I_{ij}^{(3)} \right\rangle \tag{9} \]
and the angular momentum lost through gravitational radiation is
\[ \frac{dJ_i}{dt} = \frac{2}{5} \epsilon_{ijk} \left\langle I_{jm}^{(2)} I_{km}^{(3)} \right\rangle, \tag{10} \]
where the superscript (3) indicates 3 time derivatives, there is an implied sum on repeated indices, and the angle-brackets indicate an average over several wave periods. For these burst sources such averaging is not well-defined; therefore we display the unaveraged quantities \( \frac{1}{5} \dot{I}_{ij}^{(3)} I_{ij}^{(3)} \) and \( \frac{2}{5} \epsilon_{ijk} I_{jm}^{(2)} I_{km}^{(3)} \) below. The energy emitted as gravitational radiation is
\[ \Delta E = \int L \, dt \tag{11} \]
and the angular momentum carried away by the waves is
\[ \Delta J_i = \int (dJ_i/dt) dt. \tag{12} \]
The expressions given above for calculating gravitational radiation in the quadrupole approximation are all functions of at least the second time derivative of $I_{ij}$. The standard quadrupole formula consists of using the definition \((\ref{eq:6})\) for $I_{ij}$ in these equations. In an Eulerian code, $I_{ij}$ may be calculated directly by summing over the grid and the time derivatives may be taken numerically. However, this successive application of numerical time derivatives can introduce a great deal of noise into the calculated quantities, especially when the time step varies from cycle to cycle.

To reduce this problem, Finn & Evans (1990) have developed two partially integrated versions of the standard quadrupole formula that eliminate one of the time derivatives; they call these the momentum divergence and the first moment of momentum formulae. In these expressions, $\dot{I}_{ij}$ is calculated directly by integrating fluid quantities over the grid. Since eliminating one numerical time derivative in a finite difference code greatly increases the signal-to-noise ratio, both of these methods significantly reduce the high frequency numerical noise and produce much cleaner waveforms than the standard quadrupole formula.

Both of these formulae from Finn & Evans (1990) have been implemented in the Eulerian hydrodynamics code to calculate the gravitational radiation quantities; details are given in Smith, Centrella, & Clancy (1994). Since the resulting waveforms are very similar, we only show the waveforms obtained using the first moment of momentum expression in this paper. Note that this expression gives $\dot{I}_{ij}$; the waveform requires taking another time derivative to obtain $\ddot{I}_{ij}$, and the luminosity requires still another derivative. When these derivatives are calculated numerically the signals can still be dominated by noise, especially when the time step is changing significantly from cycle to cycle. This problem was solved by passing the data through a filter to smooth it after $\dot{I}_{ij}$ was calculated, and again after each numerical derivative was taken. These techniques produce smooth profiles for both the waveforms and luminosities, as shown below.

The gravitational radiation is computed in TREESPH using the method of Centrella & McMillan (1993) to calculate $\ddot{I}_{ij}$ analytically from the SPH equations of motion. With this, the gravitational waveforms are calculated directly using the particle positions, velocities, and accelerations which are already available in the code. The resulting waveforms are very smooth functions of time and require no filtering or smoothing to remove numerical noise. However, the luminosity and angular momentum lost by gravitational radiation do contain the time derivative of the particle acceleration, which is taken numerically and therefore introduces some noise. We have chosen to smooth the luminosity data for the SPH runs using simple averaging over a fixed interval of $0.1t_D$ centered on each point. Here, $t_D$ is the dynamical time defined in equation \((\ref{eq:14})\) below. In general this procedure makes a negligible
change in the integrated luminosity (11), which gives the energy emitted as gravitational radiation, and produces very smooth profiles (Centrella & McMillan 1993). The profiles of the angular momentum lost to gravitational radiation are not smoothed.

4. Initial Models

The initial conditions for our simulations are rotating axisymmetric equilibrium models having \( \tau \approx 0.30 \) and polytropic index \( n = 3/2 \). The bar instability then grows from nonaxisymmetric perturbations of these equilibrium spheroids. In this section, we briefly describe the construction of these initial models and their representations in the Eulerian code and in TREESPH.

The self-consistent field method of Smith & Centrella (1992) is used to generate the axisymmetric equilibrium models. This is based on the earlier work of Ostriker & Mark (1968) and Hachisu (1986), and derives from an integral formulation of the equations of hydrodynamic equilibrium which automatically incorporates the boundary conditions. We use a cylindrical grid \((\varpi, z)\) with uniform zoning. An initial “guess” density distribution is given, and the gravitational potential is calculated using a Legendre polynomial expansion to solve Poisson’s equation (Hachisu 1986). A rotation law of the general form \( j(m) = j(m(\varpi)) \) is specified, where \( j(m) \) is the specific angular momentum and \( m(\varpi) \) is the mass interior to the cylinder of radius \( \varpi \) (Ostriker & Mark 1968). Following the convention of earlier work (Bodenheimer & Ostriker 1973; TDM; DGTB; Williams & Tohline 1987, 1988) we use the rotation law for the uniformly rotating, constant density Maclaurin spheroids, which can be written in the dimensionless form

\[
h(m) = \frac{M}{J} j(m) = \frac{5}{2}(1 - (1 - m)^{2/3}),
\]

where \( J \) is the total angular momentum and \( M \) is the total mass. Since polytropes do not have constant density, this produces differentially rotating models. The rotation law is used to calculate a rotational potential, which is then used with the gravitational potential to compute an improved density distribution. This process is iterated until convergence is achieved.

The freely specifiable quantities in this method are the dimensionless rotation law \( h(m) \), the polytropic index \( n \), and the axis ratio \( R_p/R_{eq} \), where \( R_p \) is the polar radius and \( R_{eq} \) is the equatorial radius of the initial model. To get a dimensional model, we also specify the maximum density and the entropy, which is given by the constant \( K \) in the polytropic
equation of state (2). Upon convergence to a solution of the equations of hydrodynamic equilibrium, this procedure gives the density $\rho(\varpi)$, the angular velocity $\Omega(\varpi)$, the total mass $M$, the total angular momentum $J$, and the stability parameter $\tau$. Since $\tau$ is not specified initially, some experimentation with the axis ratio is generally necessary to achieve a desired value of $\tau$. In constructing our model with $\tau \approx 0.30$ we were guided in our choice of input parameters by the values given in TDM. We found that using $R_p/R_{eq} = 0.205$ gives $\tau = 0.301$; note that this configuration is highly flattened due to rotation. The central rotation period for this model is $2.15 t_D$ and the rotation period for a point on the equator is $6.90 t_D$, where

$$t_D = \left( \frac{R_{eq}^3}{GM} \right)^{1/2}$$  

(14)

is the dynamical time for a sphere of radius $R_{eq}$. To construct the initial model, we used a uniform cylindrical grid of $N_\varpi = 65$ radial zones and $N_z = 23$ axial zones. The mass distribution extended out to zone 61 in the $\varpi$ direction and to zone 19 in the $z$ direction. This model required 50 iterations to converge to a solution with a tolerance of $10^{-10}$ and used 400 seconds of CPU time on the Cray C90 at the Pittsburgh Supercomputing Center (PSC). The accuracy of this initial equilibrium model can be measured using the virial relation. Let

$$VC = \frac{|2T + W + 3\Pi|}{|W|},$$  

(15)

where $\Pi = \int P \, dV$ is the volume integral of the pressure (Hachisu 1986). For our initial model, $VC = 7.4 \times 10^{-4}$.

To evolve this model with the Eulerian code, we first interpolate it onto the non-uniform grid used in that code. We trigger the bar instability by imposing a random perturbation with amplitude $10^{-3}$ on the density in each grid zone (cf. TDM). The instability then grows from this relatively low noise level, with the start of the gravitational wave burst occurring at $\sim 15 t_D$.

We have used two different methods to convert the density $\rho(\varpi, z)$ and angular velocity $\Omega(\varpi)$ produced by the self-consistent field method into a particle model to be evolved with TREESPH. Both methods use equal-mass particles.

The first technique is a simple random or “rejection” method (Press, et al. 1992; Centrella & McMillan 1993) that randomly distributes particles within the probability distribution $\rho(\varpi, z)$, and then assigns the appropriate angular velocity. Since the particles are accepted into the model randomly, and thus independently of each other, this method results in both positive and negative density fluctuations about the target $\rho(\varpi, z)$. These fluctuations are relatively large. They trigger the bar instability, with the gravitational wave burst starting immediately (see model T6 below).
Since the perturbations imposed on the Eulerian models at the initial time are significantly smaller than those produced by the random particle method, we developed a technique for producing a quieter, “cold” initial particle model. In this method a set of equipotential surfaces is calculated for the equilibrium model produced by the self-consistent field method using the gravitational and rotational potentials. The mass distribution for this model is specified by calculating the mass interior to these equipotential surfaces. To create a particle representation, we start by placing particles within the surface boundary of the star at uniform Cartesian coordinates. The mass interior to the equipotential surfaces is then used to determine how to relocate these particles to produce the desired mass distribution. Particle velocities are assigned using $\Omega(\varpi)$. The resulting models have considerably less noise, with the gravitational wave bursts beginning at $\sim 10t_D$ (Houser & Centrella 1995).

5. Evolution of the Bar Instability: Eulerian Runs

We ran three Eulerian models, labeled E1 – E3; in all these models, we use $N_\varpi = 64$ zones in the $\varpi$ direction and $N_z = 32$ zones in the $z$ direction. To maximize both resolution and efficiency we use a finer grid in the region initially occupied by the matter and a coarser grid outside. The $\varpi$ grid is uniform up to the zone $j = 30$, and the $z$ grid is uniform up to $k = 16$. The zoning is chosen such that the center of the radial zone $j = 25$ is at the equatorial radius of the initial model $R_{\text{eq}}$ and the center of the axial zone $k = 9$ is at the polar radius $R_p$. Outside of this uniformly zoned region, the zone size increases linearly with zoning ratios $\Delta\varpi_{j+1}/\Delta\varpi_j = 1.03$ and $\Delta z_{k+1}/\Delta z_k = 1.1$. The angular grid is uniform and covers the range $\varphi : 0 − 2\pi$. For simplicity, the grids are held fixed throughout the runs. The grid boundaries for the Eulerian runs are set at $\varpi = 3.85R_{\text{eq}}$ and $z = 1.60R_{\text{eq}} = 7.91R_p$. This large amount of initially empty space is necessary to provide room for the star to expand as the bar mode grows, and to specify the boundary conditions for the solution of Poisson’s equation accurately (Smith, Centrella, & Clancy 1994). We varied the number of angular zones $N_\varphi$ and the vacuum boundary conditions to test the effects of these parameters on the bar mode instability. The properties of the Eulerian models are summarized in Table 1.

To study the development of the bar mode quantitatively, we analyze the density in a ring of fixed $\varpi$ and $z$ using a complex Fourier integral

$$C_m(\varpi, z) = \frac{1}{2\pi} \int_0^{2\pi} \rho(\varpi, \varphi, z)e^{im\varphi} d\varphi,$$

(16)
where \( m = 2 \) (TDM). The normalized bar mode amplitude is

\[
|C| = |C_2|/C_0,
\]

(17)

where \( C_0(\varpi, z) = \bar{\rho}(\varpi, z) \) is the mean density in the ring. The phase angle \( \phi_m \) is defined by

\[
\phi_m(\varpi, z) = \tan^{-1} \frac{\text{Im}(C_m)}{\text{Re}(C_m)}.
\]

(18)

The phase information can be used to describe global nonaxisymmetric structure propagating in the \( \varphi \)-direction. When such a global mode develops out of the initial noise we can write

\[
\phi_m = \sigma_m t,
\]

(19)

where the pattern speed of the \( m \)th structure is (cf. Williams & Tohline 1987; PDD)

\[
W_m(\varpi, z) = \frac{1}{m} \frac{d\phi}{dt} = \frac{\sigma_m}{m}.
\]

(20)

Thus, \( \sigma_2 \) is twice the bar rotation speed, and the rotation period of the bar is \( T_{\text{bar}} = 4\pi/\sigma_2 \).

The bar mode amplitude \(|C|\) and phase angle \( \phi_2 \) have been calculated by TDM using the linearized TVE method. This technique gives exact results for small oscillations of uniform density, incompressible ellipsoids such as the Maclaurin spheroids (Chandrasekhar 1969). It was adapted to study rotating compressible fluids by Tassoul & Ostriker (1968), and applied to rotating polytropes by Ostriker & Bodenheimer (1973). For compressible fluids, the TVE method gives only approximate results; see TDM for a discussion. Nevertheless, it provides a useful point of comparison for the numerical simulations. TDM used the Ostriker–Bodenheimer TVE code to calculate the bar mode growth rate and eigenfrequency for the case \( \tau = 0.301 \). According to their analysis, the amplitude \(|C|\) should grow exponentially with time as the instability develops, with \( d \ln |C|/dt = 0.728 \pm 0.038 t_D^{-1} \) (we have converted from their units). For the eigenfrequency they find \( \sigma_2 = 1.892 \pm 0.094 t_D^{-1} \). The errors quoted by TDM are of the order \( \pm 5\% \) and only account for the expected inaccuracies in the equilibrium models.

### 5.1. Results from our Standard Eulerian Model

Our standard Eulerian model is E2, which uses \( N_\varphi = 64 \). Density contours showing the development of the bar instability in this model are presented in Figure [I]. The growth of the \( m = 2 \) mode produces a bar-shaped structure. This rotating bar develops a spiral
arm pattern as mass is shed from the ends of the bar. The bar and spiral arms exert gravitational torques, causing angular momentum to be transported. The spiral arms expand supersonically and merge together, causing shock heating and dissipation in the disk surrounding the central core. The system remains highly flattened throughout, and evolves toward a nearly axisymmetric final state.

The distribution of mass $m(\varpi)$ is shown in Figure 2 for the initial time (dot-dashed line), the intermediate time $t = 21.1t_D$ (dashed line), and the final time $t = 34t_D$ (solid line). The angular momentum $J(\varpi)$ is shown for these same three times in Figure 3. Note that $J(\varpi)$ is normalized by the total angular momentum in the system at the time, which is less than the initial value due to non-conservation; see Table 1. We define the core to be all matter contained within cylindrical radius $\varpi = R_{eq}$. In the final state, the core has 96% of the mass, and 86% of the angular momentum; see Table 2.

The growth of the bar mode for model E2 is shown quantitatively in Figure 4, where $\ln |C|$ is plotted versus time for the ring $\varpi = 0.362R_{eq}$ in zone $j = 10$ in the equatorial plane $z = 0$. We also checked the growth of $\ln |C|$ at several other values of $\varpi$ and found that the growth rate $d\ln |C|/dt$ is essentially independent of cylindrical radius within the core. This shows that the bar mode grows at a well-defined rate (TDM). To determine the bar growth rate we fit a straight line through the data points in Figure 4 in the time interval during which the function $\ln |C|$ is growing linearly with time. The endpoints of this time interval are chosen “by eye”. Then, using the definition that a line segment consists of at least 10 successive points, the slope is calculated by linear regression for all possible line segments in this time interval. The average of these slopes is used to determine the growth rate. For model E2 we find $d\ln |C|/dt = 0.58t_D^{-1}$.

To determine the eigenfrequency $\sigma_2$, we plot $\cos \phi_2$ as a function of time, and use a trigonometric fitting routine to calculate $\phi_2$. The function $\cos \phi_2$ is used for simplicity, since $\phi_2$ itself is multi-valued due to the $\tan^{-1}$ in equation (18). The fit is performed over the same interval used to calculate the bar mode growth rate. As a check on this procedure, we also use an FFT to calculate the frequency spectrum. For model E2 we obtain the eigenfrequency $\sigma_2 = 1.8t_D^{-1}$, which gives a bar rotation period $T_{bar} \sim 7t_D$. Comparison with Figure 4 confirms that the initial exponential growth of the bar mode takes place over approximately one bar rotation period.

We calculate the growth of other Fourier components of the density in this same ring using equation (16) and normalizing the resulting amplitudes by $C_0$. Figure 5 shows the growth of the amplitudes of the components (a) $m = 1$, (b) $m = 3$, and (c) $m = 4$. In each plot, the amplitude of the bar mode is shown as a solid line for comparison. As expected, the growth of the bar mode dominates the initial stage of the evolution, with the other
components becoming important at later times. In particular, Figure 5 (c) shows that the \( m = 4 \) mode starts growing after the bar mode is well into its exponential growth regime. The \( m = 4 \) mode also grows exponentially, but at a faster rate \( d \ln(|C_4|/|C_0|)/dt = 1.1t_D^{-1} \) than the bar mode. Both modes reach their peak amplitudes at about the same time, then drop to local minima and grow again. The eigenfrequency of the \( m = 4 \) mode is \( \sigma_4 = 3.4t_D^{-1} \), giving a pattern speed for this mode \( W_4 = 0.85t_D^{-1} \). Since the pattern speed of the bar mode is \( W_2 = 0.9t_D^{-1} \sim W_4 \), this suggests that the \( m = 4 \) mode is a harmonic of the bar mode, and not an independent mode. This agrees with the expectation that if the \( m = 4 \) mode were an independent mode, then it would grow at a slower rate than the bar mode. See Williams & Tohline (1987).

In addition, Figure 5 (a) shows that the \( m = 1 \) disturbance grows to nonlinear amplitude after the bar mode amplitude has reached its maximum value. Recent work by Bonnell (1994; see also Bonnell & Bate 1994) in the context of star formation shows similar behavior. PDD also find that both \( m = 1 \) and \( m = 2 \) modes arise for certain initial angular momentum distributions. These issues should be investigated more closely in future work.

Figure 4 shows that the amplitude of the bar mode peaks at \( t \sim 22t_D \) and then drops to a local minimum at \( t \sim 26t_D \). It then rises to a local maximum near \( t \sim 30t_D \) and subsequently drops again. These features can also be seen in the density contours given in Figure 3 as follows. Focus on the second highest contour starting in frame (b). This reaches a maximum bar-like shape between frames (d) and (e), and then grows more axisymmetric until around the time of frame (g), which is near the time at which the bar mode reaches its local minimum. The contour again develops a bar-like shape before becoming more axisymmetric. The behavior of the stability parameter \( \tau = T_{\text{rot}}/|W| \) is shown as a function of time by the solid line in Figure 6; for comparison, the dashed line gives \( T_{\text{total}}/|W| \). Note that \( \tau \) reaches its minimum value when the amplitude of the bar mode peaks at \( t \sim 22t_D \). It then rises to a local maximum \( \tau \sim 0.27 \) near the time \( t \sim 26t_D \) when the bar mode amplitude is at its local minimum, and falls as the bar mode amplitude grows again. This anti-coincidence of the bar amplitude and \( \tau \) results from the fact that the higher amplitude bar has a greater moment of inertia, which reduces the rotational kinetic energy.

The behavior of the gravitational wave quantities is also strongly linked to the amplitude of the bar mode. Figure 7 shows the gravitational waveforms (a) \( rh_+ \) and (b) \( rh_x \) for an observer on the axis at \( \theta = 0, \varphi = 0 \). The gravitational waveforms show a strong initial burst that peaks around the same time that the bar mode reaches its maximum amplitude, \( t \sim 22t_D \). This is followed by a weaker secondary burst that peaks around \( t \sim 30t_D \), corresponding to the secondary maximum of the bar mode amplitude. Figure 8 shows (a) the gravitational wave luminosity \( L \), (b) the energy \( \Delta E/M \) emitted as
gravitational waves, (c) the rate \( dJ_z/dt \) at which angular momentum is carried away by the waves, and (d) the angular momentum \( \Delta J_z/J_0 \) lost to gravitational radiation. The luminosity \( L \) and \( dJ_z/dt \) both show a primary peak at \( t \sim 22t_D \) and a secondary peak at \( t \sim 30t_D \), separated by a local minimum at \( t \sim 26t_D \). Some of the interesting gravitational wave properties are summarized in Table 3.

5.2. Results from Eulerian Models with Different Parameter Values

Model E1 is the same as E2 except that the resolution in the \( \varphi \) direction is reduced by a factor of two, giving \( N_\varphi = 32 \) angular zones. We chose to change \( N_\varphi \) because we are primarily interested in the growth of nonaxisymmetric modes, and thus the angular resolution is expected to be an important parameter. The bar growth rate is \( d\ln|C|/dt = 0.53t_D^{-1} \), which is \( \sim 9\% \) smaller than in E2. Since both these models have reasonably long, well-defined linear growth regions for \( d\ln|C|/dt \) we believe that these differences are real and not just the result of the method used to compute them. The eigenfrequency obtained for model E1 is \( \sigma_2 = 1.7t_D^{-1} \).

TDM (see also Norman, Wilson, & Barton 1980; Williams & Tohline 1987) showed that, for the first-order donor cell advection scheme, the difference between the true growth rate and the actual growth rate in the code is given by a numerical diffusion term. The size of this diffusive term is proportional to the size of the angular zones \( \Delta\varphi \). They also showed that, at least to first order, the eigenfrequency \( \sigma_2 \) is not affected by this numerical diffusion. Our code uses a monotonic advection scheme which is significantly less diffusive than the donor cell method (Bowers & Wilson 1991; Hawley, Smarr, & Wilson 1984), and in fact the bar mode growth rates we obtain are larger, and hence closer to the TVE values, than those found by Tohline and collaborators. Nevertheless, we expect that increasing the size of \( \Delta\varphi \) by decreasing \( N_\varphi \) will also lower the growth rate in our code, and this is the behavior that we find in comparing E1 and E2. The differences in \( \sigma_2 \) between E1 and E2 are about a factor of 2 smaller than the differences in the growth rate.

The bar mode amplitude in run E1 peaks at \( t \sim 22t_D \), then drops off and oscillates around a lower value for \( \sim 10t_D \), and begins to grow again. Both the mass \( m(\varpi) \) and angular momentum \( J(\varpi) \) distributions in run E1 are more spread out than in run E2, resulting in a core \( (\varpi \leq R_{eq}) \) with a smaller mass and angular momentum. See Tables 1 and 2.

The gravitational waveforms for E1 show a strong initial burst with maximum amplitude \( \sim 7\% \) smaller than in E2. This is followed by some additional waves, but they
are not cleanly organized into a secondary burst as in E2. The luminosity $L$ shows both a primary and a secondary peak. And, although the maximum luminosity in E1 is only $\sim 5\%$ smaller than in E2, the ratio of the amplitudes of the primary and secondary peaks in $L$ is $\sim 20$ for E1 and $\sim 9$ for E2.

The mass density within an Eulerian grid zone can never be zero, since this leads to divisions by zero in the code. Therefore, “vacuum” regions of the grid actually have a very small mass density. However, if allowed to evolve unrestricted, these low density zones can attain very high velocities and begin to dominate the timestep calculations. To prevent this, special provisions must be made to handle these “vacuum” regions (R. Bowers, private communication, 1991). We have chosen to place the following restrictions on low density zones. For a grid zone in which the density is below a certain threshold value, the velocity is set to zero. The density threshold we use to limit the velocity for our standard run E2 (as well as for E1) is $10^{-7}\rho_{\text{max},i}$, where $\rho_{\text{max},i}$ is the maximum density at the initial time. Also, if the density in a zone is $< 10^{-10}\rho_{\text{max},i}$, the internal energy is set to a fixed value that produces consistency between equations (2) and (3). Finally, the density itself is set to $10^{-15}\rho_{\text{max},i}$ if it is less than this value. These conditions lead to some loss of energy and momentum as matter flows into these cells. DGTB report a similar loss of angular momentum that they attribute to the zeroing of velocities in low density zones.

To see how the vacuum restrictions affect the evolution of model, we ran a simulation with relaxed vacuum restrictions. Model E3 is the same as E2 except that the thresholds specifying the vacuum conditions are less restrictive, and the velocity is never set to zero. For run E3, the velocity in a zone is set to the sound speed if it exceeds the sound speed and the density is below $10^{-14}\rho_{\text{max},i}$. The threshold below which the internal energy is set to a fixed low value is $10^{-13}\rho_{\text{max},i}$. The density itself is set to $10^{-15}\rho_{\text{max},i}$ if it is less than this value, as in E2.

Table I shows that the simulation time covered by E3 is $23.9t_D$, which is considerably less than the $34.0t_D$ covered by E2. Run E3 was not continued beyond this point because this would have been too expensive in terms of CPU time. Although less simulation time is covered, E3 takes almost as much computer time as E2, with the last $1.5t_D$ of simulation time for E3 using 5 CPU hours even with a relaxed Courant condition to allow $20\%$ larger timesteps. Because of the high velocities of the matter in the low density zones, the timestep continually decreases throughout the run, and we believe that it would take $\sim 100$ hours of CPU time to run the remaining $\sim 10t_D$ to complete E3. This demonstrates the need for restrictions on the vacuum zones.

The less restrictive vacuum conditions contribute to better energy and angular momentum conservation. By the end of run E3 the model has lost $1.5\%$ of its initial energy,
and 3.4% of its initial angular momentum. At this same time in E2, the model has lost 2% of its energy and 4.9% of its angular momentum.

Although E3 was not run long enough to evolve the model completely, sufficient time has elapsed for some useful comparisons. The bar mode amplitude has peaked and spiral structure has developed, but the model has not yet settled back to the nearly axisymmetric final configuration of E2. The gravitational wave amplitudes have also peaked by this time, although the initial wave burst is not yet complete. We can thus compare the bar mode properties and the peak gravitational wave signals. To provide a reasonable comparison of other properties, we evaluate them for E2 at 23.9$t_D$ (these entries in Tables 1 – 3 are labeled E′) and compare them to the final results for E3.

Examining the bar mode properties of E3 shows that they are very similar to those for E2. The growth rate for E3 is only slightly smaller, and the eigenfrequencies are the same to within the limit of our measurement accuracy. This is not surprising, since the growth of the bar mode occurs before there is significant expansion of the model. Also, these properties are measured within the central bulk of the configuration, so they should not be strongly affected by the treatment of the vacuum regions. An examination of the density contours, however, shows that the system expands significantly more when the vacuum conditions are relaxed. Figure 9 shows density contours for (a) E2 and (b) E3 at time $t = 23.9t_D$. The inner two contours are nearly identical, but the spiral arms ejected by the spinning star extend out to a larger radius than in E2.

The gravitational wave pulses are qualitatively similar but model E3 shows somewhat reduced amplitudes, with the maximum amplitude $\sim 9\%$ smaller than that of E2. The difference is more pronounced when we examine the luminosity, with the peak luminosity of model E3 $\sim 19\%$ smaller than that of E2. The peak gravitational radiation amplitudes are thus somewhat sensitive to the treatment of the boundary between the fluid and the vacuum, which affects the outer, lower density regions of the star. However, this is not expected to be a major factor in situations that are not dominated by expansion, such as rotating stellar core collapse.

After this work was completed, we learned that R. Durisen and collaborators have carried out similar calculations using an Eulerian code (PDD). They were able to avoid problems arising from the time step becoming too small without inhibiting expansion by setting the background density to a value between $10^{-10}$ and $10^{-7} \rho_{\text{max},i}$ and limiting all velocities in the background to less than twice the maximum initial sound speed (R. Durisen, private communication). We plan to incorporate their suggestions into our future simulations.
An alternative means of achieving a better treatment of the interface between the star and the vacuum might be to use the piecewise parabolic method (PPM; Colella & Woodward 1984; Davies, et al. 1993), which is known to be very good at handling discontinuities in the flow. It also has a higher resolution for a fixed number of zones than the finite difference scheme used here. Of course, PPM is also more expensive in terms of CPU usage. A comparison of this model run on a PPM code would be very interesting.

We can also compare our results with other numerical calculations. Tohline and collaborators used a 3-D Eulerian code written in cylindrical coordinates. Their code does not solve an energy equation, and therefore has no way to handle self-consistently the shocks that form as the spiral arms expand. Instead, they required that the fluid maintain the same polytropic equation of state (\[ \mathbf{2} \]), and hence the same entropy, throughout the evolution. They used donor cell advection, which is known to be very diffusive. And, their code assumes “\( \pi \)-symmetry”, which means that the flow is calculated only in the angular range \( 0 \leq \varphi < \pi \) so that only the even mode distortions are modeled. TDM used \( N_\varpi = 31 \), \( N_z = 15 \) and \( N_\varphi = 32 \), with the model extending out to zone 24 in the \( \varpi \)-direction and zone 9 in the \( z \)-direction; this is essentially the same as our resolution for E2 and E3. They found a bar mode growth rate of \( \frac{d \ln |C|}{dt} = 0.22 t_D^{-1} \) and an eigenfrequency \( \sigma_2 = 2.1 t_D^{-1} \).

They demonstrated that the substantial deviation from the TVE result for the growth rate is due to the large numerical diffusion in their code; see TDM for details. Williams and Tohline (1987) modeled the initial development of the bar instability in a polytrope with \( \tau = 0.31 \) using the same code with \( N_\varpi = 32 \), \( N_z = 32 \), and \( N_\varphi = 64 \), again employing \( \pi \)-symmetry. This doubling of the number of angular zones increased the bar mode growth rate to \( \frac{d \ln |C|}{dt} = 0.49 t_D^{-1} \); the eigenfrequency was \( \sigma_2 = 1.9 t_D^{-1} \). They found that the \( m = 4 \) mode starts growing exponentially after the bar mode does, and that it grows at a faster rate. Also, the \( m = 4 \) pattern moves together with the \( m = 2 \) pattern, with the maxima locked in phase, implying that the \( m = 4 \) pattern is a harmonic of \( m = 2 \), and not a distinct mode. Finally, PDD used a modified version of Tohline’s code which is second order in all spatial differences, including advection terms, and second order in time. They calculated the development of the bar instability in a polytrope with \( \tau = 0.304 \) using \( N_\varpi = 64 \), \( N_z = 16 \), and \( N_\varphi = 64 \) without \( \pi \)-symmetry. They obtained \( \frac{d \ln |C|}{dt} = 0.58 t_D^{-1} \) and \( \sigma_2 = 2.1 t_D^{-1} \) (PDD).

6. Evolution of the Bar Instability: SPH Runs

We ran a series of 7 models using TREESPH, labeled T1 – T7. All of these models use equal-mass particles. The initial state for each model was produced using the “cold”
method described in §4 except for T6, in which the random method was used. We vary the number of particles \(N\) and the linear and quadratic artificial viscosity coefficients \(\alpha\) and \(\beta\), respectively. In all cases, the number of neighbors that contribute to the smoothing kernel is chosen to be \(N_S = 64\). The properties of the SPH models are summarized in Table 4.

6.1. Results from our Standard SPH Model

Our standard SPH model is T7, which has \(N = 32,914\) particles, \(\alpha = 0.25\), and \(\beta = 1.0\). Figure 10 shows all the particles in this model projected onto the equatorial plane. The system rotates in the counterclockwise direction. We see the development of the bar and then the spiral arm pattern as mass is shed from the ends of the rotating bar. The gravitational torques exerted by the bar and spiral arms cause angular momentum to be transported. The spiral arms expand supersonically and merge, causing shock heating. The system then evolves toward a nearly axisymmetric final state. Throughout its evolution the system remains flattened. Figure 11 shows the particle positions projected onto the \(x - z\) plane at the final time \(t = 35t_D\). Note that the disk around the central core is somewhat puffed up due to shock heating during the expansion of the spiral arms.

Density contours in the equatorial plane for model T7 are shown in Figure 12, with the frames corresponding to the same ones displayed in Figure 10. To produce these contours we first use kernel estimation to interpolate the density of T7 onto the cylindrical grid used for the Eulerian model E2. We then calculate contours for the matter located in the equatorial plane. The contour levels are the same as those used in Figure 1 for model E2.

The mass distribution \(m(\varpi)\) is shown in Figure 13 for the initial time (dot-dashed line), the intermediate time \(t = 18.2t_D\) (dashed line), and the final time \(t = 35t_D\) (solid line) for model T7. Figure 14 shows the angular momentum distribution \(J(\varpi)\) for these same three times. As before, we define the core to be all matter contained within cylindrical radius \(\varpi = R_{eq}\). In the final state the core has 90% of the mass and 72% of the angular momentum; see Table 5.

To study the development of the Fourier components of the density, we first interpolate the particle model onto the cylindrical grid every \(0.1t_D\). We then use the same procedure developed for the Eulerian models and analyze the density in the ring \(\varpi = 0.362R_{eq}\) in zone \(j = 10\) in the equatorial plane using equations (16) – (20). The growth of the bar mode is shown quantitatively in Figure 15, where \(\ln |C|\) is plotted versus time. Comparison of Figure 15 with Figure 4 shows that the region in which \(\ln |C|\) grows linearly with time
is shorter and less clearly defined in run T7 than in run E2. Since the endpoints of this linear region are chosen “by eye,” there is a somewhat greater element of subjectivity in the resulting bar mode growth rate for T7; this applies to all the TREESPH runs. For T7 we find that the growth rate is $d \ln |C|/dt = 0.51t_D^{-1}$ and the eigenfrequency is $\sigma_2 = 1.9t_D^{-1}$.

Figure 16 shows the growth of the amplitudes of the components (a) $m = 1$, (b) $m = 3$, and (c) $m = 4$. In each plot, the amplitude of the bar mode is shown as a solid line for comparison. As we saw in § 5, the growth of the bar mode dominates the initial stage of the evolution. The $m = 1$ and $m = 3$ components do not exhibit significant growth. The $m = 4$ mode starts growing after the bar mode is well into its exponential growth regime, and grows at a faster exponential rate $d \ln(|C_4|/|C_0|)/dt = 1.1t_D^{-1}$. Both the $m = 2$ and $m = 4$ modes reach their peak amplitudes at about the same time, then drop to local minima and grow again. The eigenfrequency of the $m = 4$ mode is $\sigma_4 = 3.8t_D^{-1}$, which gives a pattern speed for this mode $W_4 = 0.95t_D^{-1}$. Since the pattern speed of the bar mode is $W_2 = 0.95t_D^{-1} = W_4$, this implies that the $m = 4$ mode is a harmonic of the bar mode, and not an independent mode.

Figure 17 shows that the amplitude of the bar mode peaks at $t \sim 18.5t_D$ and then drops to a local minimum at $t \sim 25t_D$. The amplitude then begins to rise sharply again and levels off around $t \sim 30t_D$. Compare this behavior with that seen in the contour plots in Figure 16 by focussing on the innermost contour starting in frame (b). This shows the pronounced initial development of the bar, with the maximum elongation of the contour occurring around the time of frame (d), in agreement with Figure 15. This contour then becomes more axisymmetric until around the time of frame (g), after which it elongates again and then grows more axisymmetric. Thus, in run T7, this contour becomes nearly axisymmetric before the bar mode amplitude reaches its local minimum. For comparison, the behavior of the stability parameter $\tau$ is shown as a function of time by the solid line in Figure 17; the dashed line gives $T_{\text{total}}/|W|$. Note that $\tau$ reaches a local minimum when the amplitude of the bar mode peaks at $t \sim 18t_D$. It then rises to a local maximum $\tau \sim 0.27$ at $t \sim 22t_D$ and drops off again.

The gravitational radiation quantities also exhibit features corresponding to the behavior of these modes. Figure 18 shows the gravitational waveforms (a) $r h_+$ and (b) $r h_\times$ for an observer on the axis at $\theta = 0, \phi = 0$; cf. Table 1. The gravitational waveforms show an initial burst that peaks at $t \sim 18t_D$, corresponding to the initial growth of the bar instability. After the peak, the amplitude drops off until $t \sim 22t_D$ and then stays at a nearly constant value for about one bar rotation period, during which time the bar mode amplitude reaches its local minimum. Recall that $T_{\text{bar}} = 2T_{\text{GW}}$, where $T_{\text{GW}}$ is the period of the gravitational waves. The wave amplitude then drops again at $t \sim 30t_D$ and stays
at a nearly constant amplitude for about another bar rotation period before the run ends. These low amplitude waves are produced by the rotating, slightly non-axisymmetric core in the final state; cf. Figure 12. Figure 19 shows (a) the gravitational wave luminosity $L$, (b) the energy $\Delta E/M$ emitted as gravitational waves, (c) the rate $dJ_z/dt$ at which angular momentum is carried away by the waves, and (d) the angular momentum $\Delta J_z/J_0$ lost to gravitational radiation. The luminosity $L$ shows a broad primary peak centered around $t \sim 18t_D$, followed by a drop to a local minimum around $t \sim 23t_D$ and then a secondary feature at $t \sim 25t_D$. The signal then drops again to a nearly constant level for $t \gtrsim 30t_D$.

6.2. Results from SPH Models with Different Parameter Values

To understand how changing the resolution affects the behavior of the model, compare runs T1, T2, and T3 with the standard run T7. These runs differ only in the total number of particles $N$; see Table 4. In all these cases, the bar and spiral arms develop as in T7. Plots of the particles projected onto the equatorial plane appear visually similar, except that the extent of the spiral arm pattern increases somewhat as $N$ increases due to the larger number of particles available to resolve the outer regions. Table 5 shows that the properties of the cores of these runs are all similar. The behavior of the Fourier components of the density in these runs is also similar, except that the results for T1 are much noisier due to its very low resolution. Although T1 has the largest bar mode growth rate, we attribute this to the lack of a clearly defined linear growth region for $d \ln |C|/dt$ and therefore do not consider it to be a reliable indicator of the accuracy of this model.

The gravitational wave quantities show definite trends with particle number $N$. As $N$ decreases, the amplitude of the burst goes down and the structure of the waveforms after the burst becomes less distinct. The peak amplitude of the luminosity $L$ also decreases, and the secondary peak becomes a plateau and then disappears into noise for the poorly resolved run T1. For the sequence of models T7, T3, and T2, each run has roughly half the number of particles as the previous one. Quantitatively, the amplitude of the burst goes down by $\sim 10\%$ between model T7 and T3, and another $\sim 10\%$ between T3 and T2. The peak values of $L$ and $dJ_z/dt$ decrease by $\sim 30\%$ between T7 and T3, and by another $\sim 20\%$ between T3 and T2. See Table 6.

Thus, the larger the number of particles $N$, the larger the amplitudes of the gravitational wave signals. It is clear from Table 6 that the values of these amplitudes have not yet converged. One possible means of achieving convergence is simply to increase $N$. However, doubling the number of particles increases the effective grid resolution of the
model by only \( \sim 2^{1/3} \), since we are working in 3-D. Perhaps a better method would be to use non-equal-mass particles, with the lower mass particles distributed in the lower density regions and thus increasing the resolution in the outer parts of the star (e.g. Monaghan & Lattanzio 1985).

In numerical simulations, artificial viscosity terms are typically added to the momentum and thermal energy equations to provide a dissipative mechanism that converts the energy jump across the shock into heat (Bowers & Wilson 1991). This smooths out the discontinuities that occur in shock fronts while satisfying the Rankine-Hugoniot relations, which specify the conservation of mass, momentum, and energy across the shock (Landau & Lifshitz 1959). If there is no artificial viscosity, the kinetic energy of the matter passing through the shock is not correctly converted into heat and there can be large post-shock oscillations in the fluid.

The standard SPH artificial viscosity contains two terms, one that is linear in the particle velocity differences, and another that is quadratic (Monaghan 1992). The linear term has user-specified coefficient \( \alpha \) and tends to dominate for low Mach numbers, while the quadratic term has coefficient \( \beta \) and is important for higher Mach numbers. Typically, the values \( \alpha \sim 1 \) and \( \beta \sim 2 \) are used and the shock front is spread over \( \sim 3 - 4 \) particle smoothing lengths (Hernquist & Katz 1989). This type of SPH viscosity can introduce shear into the flow, particularly through the linear term (Hernquist & Katz 1989; Monaghan 1992). Since shear can affect the bar mode instability we want to keep the artificial viscosity coefficients, particularly \( \alpha \), as small as possible while still maintaining accuracy in the presence of the shock waves that occur as the ends of the bar and the spiral arms expand supersonically. We have chosen to use \( \alpha = 0.25 \) and \( \beta = 1.0 \) as our standard values.

TREESPH contains the option to use a version of the usual SPH artificial viscosity that reduces the amount of artificial viscosity in the presence of curl (Balsara 1989; Balsara, et al. 1989; Benz 1990). Tests carried out by Centrella & McMillan (1993) using TREESPH to calculate the gravitational radiation from the head-on collision of identical polytropes showed that the best results were obtained using this modified artificial viscosity. We have therefore chosen to use it here.

In these simulations shocks occur in the outer regions of the model as the bar and spiral arms develop. The effects of changing the artificial viscosity coefficients can be seen by comparing models T3 and T4. Both of these runs have \( N = 15,648 \) and started from the same initial equilibrium model; see Table 4. Run T3 has our standard values \( \alpha = 0.25 \) and \( \beta = 1.0 \), while T4 uses the larger values \( \alpha = 1.0 \) and \( \beta = 2.0 \). Models T3 and T4 are very similar in visual appearance and in their bulk properties, having the same core values in the final state; see Table 5. The behavior of the Fourier components of the density in
both models is also similar. Although the bar mode growth rate is slightly lower in T4, this may not be significant due to the element of subjectivity involved in calculating it. As expected, the energy conservation in T4 is improved due to the greater smoothing of shocks by the larger artificial viscosity. In addition, T4 requires significantly more CPU time than T3, due to the stability requirement on the particle timesteps in the presence of artificial viscosity (Hernquist & Katz 1989).

The gravitational waveforms and luminosities for runs T3 and T4 are similar, except that the quantities in T4 have lower amplitudes than in T3. This can be compared with the results of Zhuge, Centrella, & McMillan (1994), who used TREESPH to simulate binary neutron star coalescence. In their models, the stars merge and coalesce into a rotating bar-like structure. Spiral arms form as mass is shed from the ends of the bar; the arms then expand and merge into a disk around the central object. They found that, during this spiral arm stage, the amplitude of the gravitational waveforms decreases as the amount of artificial viscosity is increased.

For comparison, run T5 also has the same number of particles and began with the same initial state as T3, but has no explicit artificial viscosity, with \( \alpha = \beta = 0 \). In this case, the kinetic energy of the particles passing through shocks is not correctly converted into heat, resulting in large post-shock oscillations. Table 4 shows that the energy conservation errors are more than twice as large as those for run T3. The growth of the Fourier components is similar, but the spiral arms spread out and merge more quickly after the bar mode peaks in T5 compared to T3. The final core \( \varpi \leq R_{\text{eq}} \) has less mass and angular momentum than in run T3. The values of the stability parameter \( \tau \) for both the core and the entire system are about the same for T5 and T3. However, T5 has a much larger kinetic energy due to the large post-shock oscillations. This gives \( T_{\text{total}}/|W| = 0.31 \) for the whole system compared with \( T_{\text{total}}/|W| = 0.26 \) for T3. The waveforms and luminosity profiles for model T5 are similar to T3 until around the time that the amplitude of the bar mode reaches its peak value; afterwards, the profiles for T5 become more noisy.

Finally, we compare models T6 and T3 to see the effects of starting with a different initial model. Run T6 uses the same values of \( \alpha \) and \( \beta \) as T3 and \( N = 16,000 \) particles; this is the model that was presented in Houser, Centrella, & Smith (1994). The initial conditions for this run were produced using the random or rejection method. As mentioned in §4, this technique produces relatively large density fluctuations about the equilibrium solution. Thus the particles in the initial model are acted on by forces that can have large deviations from their expected values, which can lead to violent motions (Lucy 1977). Although TREESPH does perform some smoothing of the initial data (Hernquist & Katz 1989), significant fluctuations remain. This results in energy conservation errors \( \sim 4.3\% \),
which are more than twice as large as those in T3. The amplitude of the bar mode in T6
starts growing from a larger initial value and reaches a much broader peak before dropping
to a local minimum again. In particular, there is a relatively short time interval during
which we can define a linear growth region for $\ln |C|$ so we must use caution in interpreting
the relatively large growth rate reported in Table 4. Run T6 also has more mass and
angular momentum in the core at the final state than T3, although the stability parameter
has the same value in both cases. Comparison of the gravitational wave data shows that the
gravitational wave burst begins immediately in T6, compared to a starting time $\sim 10t_D$ for
T3. Since T6 goes out to $26.9t_D$ whereas T3 goes out to $35t_D$, the final states are roughly
equivalent and can be compared meaningfully. The gravitational wave quantities for T6
do show the signature of a rotating nonaxisymmetric core after the burst, although there
is not as much detail as in T3. The amplitudes of the waveforms and the luminosities are
both lower for T6. See Table 6.

It is interesting to compare our results with those of DGTB, who evolved the dynamical
bar instability using an SPH code with $N = 2000$ particles and smoothing lengths that are
allowed to vary in time but are the same for all particles. Plots of the particle positions
projected onto the $x$–$y$ plane for the case $\tau \approx 0.33$ are visually similar to our Run T1.
They report that the system has $\tau = 0.247$ at the end of their run, which is essentially the
same as our result for T1.

In summary, all the models run with TREESPH show the development of the bar and
the spiral arm pattern. Models T1 - T4 and T7 conserve total energy to $\lesssim 2\%$, with T5 and
T6 conserving energy to $\sim 4\%$. In all cases, angular momentum is conserved to $\lesssim 0.1\%$.
The bar mode growth rates for models T2 - T7 are the same to within $\sim 6\%$; we attribute
these differences largely to the element of subjectivity inherent in choosing the region of
linear growth for $d \ln |C|/dt$. The anomalous growth rate found for T1 is due to the lack
of a clearly defined linear growth region. The properties of the final cores $\varpi \leq R_{eq}$ in the
models are remarkably similar, especially if runs T5 (with no artificial viscosity) and T6
(with a noisy initial state) are excluded. Finally, the amplitudes of the gravitational wave
quantities increase as $N$ increases; higher resolution runs, or models with non-equal-mass
particles are needed to achieve convergence in these quantities.

7. Comparison of Eulerian and SPH Results

We turn now to a comparison between the results of the Eulerian and SPH codes. To
accomplish this we focus on our standard models E2 and T7. There is no simple measure
for comparing the resolution of these two models since the underlying fluid descriptions are so different. In run E2, the fluid initially occupies $\sim 14,400$ zones; after flowing through the grid during the development of the bar mode, the fluid occupies $\sim 66,200$ zones at the end of the run. In contrast, the fluid in run T7 is discretized into $N = 32,914$ equal-mass particles. As the system evolves, these particles move through space, with the smoothing length of each particle continually adjusted to keep the number of nearest neighbors approximately constant. For our purposes we simply note that the number of particles used in T7 is comfortably between the initial and final number of zones occupied by the fluid in E2, and proceed with the comparison.

Examination of Tables 1 and 4 shows that T7 conserves both total energy and angular momentum better than does E2. Run T7 also uses a larger amount of CPU time. However, as demonstrated in § 5.2., the energy and angular momentum conservation for E2 both improve when relaxed vacuum conditions are used, although at the expense of a larger usage of CPU time.

In both models, the growth of the $m = 2$ mode produces a rotating bar that develops spiral arms as mass is shed from the ends of the bar. The bar and spiral arms exert gravitational torques that cause angular momentum to be transported outward. The spiral arms expand supersonically and merge together, causing shock heating and dissipation in the disk surrounding the central core. The system remains highly flattened and evolves toward a nearly axisymmetric final state in both models.

The density contours for E2 in Figure 1 and for T7 in Figure 12 can be compared directly, since the contours for T7 were made by interpolating the particle model onto the grid used for E2 and these figures use the same contour levels. The most striking visual difference lies in the fact that T7 expands much more than does E2. These differences in the amount of expansion account for the differences in the core ($\tilde{\omega} \leq R_{eq}$) masses and angular momenta given in Tables 2 and 3. When the net transport of angular momentum is considered, the behavior of the models is more similar. At the final time, for example, in model E2 90% of the mass has 73% of the angular momentum and in T7 90% of the mass has 71% of the angular momentum.

The growth of the bar mode is shown quantitatively in Figure 4 for run E2 and in Figure 15 for run T7. In both cases there is a relatively long period of linear growth for the bar mode amplitude $\ln |C|$. The measured values in this linear growth region are $d \ln |C|/dt = 0.58t_D^{-1}$ for E2 and $d \ln |C|/dt = 0.51t_D^{-1}$ for T7. The larger slope for E2 is closer to the analytic TVE value $d \ln |C|/dt = 0.728 \pm 0.038t_D^{-1}$ given by TDM. As was discussed in § 5.2., numerical diffusion can cause the growth rate given by a simulation to be lower than the expected value (TDM). This suggests that the SPH code has more
numerical diffusion than the Eulerian code; further studies of this would be very useful. The eigenfrequency is $\sigma_2 = 1.8 t_D^{-1}$ for E2 and $\sigma_2 = 1.9 t_D^{-1}$ for T7; both of these values are within the range $\sigma_2 = 1.892 \pm 0.094 t_D^{-1}$ given for the TVE result (TDM).

In both runs, the bar mode amplitudes reach their maximum values, drop off to local minima, and then grow again. The bar mode in T7 reaches a higher value than in E2, but the peak is broader. In run E2 the amplitude reaches a second peak at a lower amplitude than the first, and then drops again. In both cases, the $m = 4$ mode starts growing after the bar mode, grows at a faster exponential rate, and then peaks at about the same time as the bar mode before dropping off; cf. Figures 3(c) and 16(c). For E2 we find $d \ln(|C_4|/|C_0|)/dt = 1.1 t_D^{-1}$ and $\sigma_4 = 3.4 t_D^{-1}$ and for T7 we get $d \ln(|C_4|/|C_0|)/dt = 1.1 t_D^{-1}$ and $\sigma_4 = 3.8 t_D^{-1}$. In both runs we find that the pattern speeds for these two modes are about the same, indicating that the $m = 4$ mode is a harmonic of the $m = 2$ mode.

One interesting difference between the models can be seen by comparing the behavior of the $m = 1$ and $m = 3$ Fourier components of the density. Figure 3 shows that these disturbances grow in run E2 whereas Figure 16 shows that they do not grow in run T7. We do not have an explanation for this behavior. However, given the importance of the $m = 1$ mode in recent work (Bonnell 1994; Bonnell & Bate 1994; PDD), this question deserves further study.

The overall behavior of the stability parameter $\tau$ and $T_{\text{total}}/|W|$ is similar in both E2 and T7, as can be seen by comparing Figures 3 and 17. In both cases, the final value of the stability parameter is in the range $\tau_s < \tau < \tau_d$. We therefore expect that the system will develop a secular bar instability when the effects of gravitational radiation reaction are included in the hydrodynamical equations (cf. Lai & Shapiro 1995).

The gravitational radiation in both models is dominated by a strong feature that corresponds to the initial growth of the bar mode, with the peak amplitudes of both the waveforms and luminosity being higher in E2 than in T7. The radiation emitted after the initial burst shows a secondary feature in both models. In E2 the radiation has a double-burst structure; this behavior is less distinct in T7, although the secondary feature is present. We showed in § 5.2 that the gravitational radiation amplitudes for E2 decrease somewhat if we halve the number of angular zones or change the treatment of the vacuum zones to allow the model to expand more. Also, § 5.2 showed that the gravitational wave amplitudes for the SPH models increase and approach the E2 amplitude as the particle number increases, and that they have not yet converged for this set of runs. Further work with both codes, including higher resolution runs, is needed to resolve these issues.
8. Conclusions

We have carried out 3-D numerical simulations of the dynamical bar instability in a rapidly rotating star and the resulting gravitational radiation using both an Eulerian finite-difference code with a cylindrical grid and monotonic advection, and an SPH code with variable smoothing lengths and a hierarchical tree method for calculating the gravitational acceleration. The star is initially modeled as a polytrope with index $n = 3/2$ and $\tau \approx 0.30$. In both codes the gravitational field is purely Newtonian and the gravitational radiation is calculated using the quadrupole formula. The back reaction of the gravitational radiation on the fluid is not included.

In both codes the dynamical instability of the $m = 2$ mode produces a rotating bar-like structure. Spiral arms develop as mass is shed from the ends of the bar, and gravitational torques cause angular momentum to be transported outward. The spiral arms expand supersonically and merge, causing shock heating in the outer regions. At the end of the simulations, both codes agree that the system consists of a nearly axisymmetric central core surrounded by an extended disk, and has $\tau \sim 0.25$.

It is interesting to compare our results with those from the earlier study by DGTB that considered rapidly rotating $n = 3/2$ polytropes with $\tau \approx 0.33$ and $\tau \approx 0.38$. The Eulerian codes used in that work were quite diffusive. They also did not solve an energy equation and thus had no means of handling self-consistently the shocks that form in the outer regions. Also, a certain amount of mass was allowed to leave the grid. In our Eulerian runs, shocks are handled using an artificial viscosity and no mass is allowed to leave the grid. Our main difficulty in the Eulerian code is with the expansion of the model into the vacuum. We believe this can be alleviated with the use of better vacuum conditions.

The SPH code used by DGTB had a smoothing length that was the same for all particles. Their runs were also limited to a fairly small number of particles. The use of variable smoothing lengths and the hierarchical tree method for calculating the gravitational accelerations in TREESPH allows us to evolve more particles. All the SPH models have no grid and expand freely. However, the fluid description can break down in the low density outer regions due to lack of resolution.

The simulations of DGTB showed the development of the bar instability with spiral arms and transport of angular momentum. However, their codes differed in the final outcome of the models in the low density outer regions. In their longest Eulerian run, most of the low density material formed a fairly narrow ring around the central remnant. (This outcome was also seen by Williams & Tohline (1988) for polytropes having $n = 0.8$ and $n = 1.8$ with $\tau = 0.31$.) However, in their SPH runs the low density material formed an
extended disk.

We are not sure what causes these differences in the final outcome. Further study, perhaps including longer runs with our codes, is needed to resolve these issues.

Interestingly, DGTB concluded that their SPH runs had less numerical diffusion than their Eulerian simulations. We believe this is due to the low-order differencing in their Eulerian codes. In our study, the use of higher-order differencing and monotonic advection in the Eulerian code resulted in much less numerical diffusion than seen in the earlier studies. (See also PDD.) Comparison of the bar growth rates for our standard Eulerian and SPH runs suggests that the SPH code has more numerical diffusion.

We agree with DGTB that the SPH code is easier to implement and use. We find that the bulk properties of the model can be obtained at very low cost using $N \sim 2000$ particles, although more particles are needed for a good measure of the bar mode growth rate. We did not carry out any very low resolution studies with our Eulerian code. However, we find that the peak amplitudes of the gravitational radiation quantities in both the Eulerian and SPH codes increase as the resolution is increased. Overall, for comparable resolution, the cost of the Eulerian and SPH runs is similar.

We note that although the version of SPH used here allows the particle smoothing lengths to vary in both space and time, the terms describing these changing smoothing lengths are not explicitly incorporated into the dynamical equations (Hernquist & Katz 1989). Although this situation is typical of most SPH codes currently used in astrophysics, it does present a potentially serious deficiency in the method. Recent work to incorporate these terms into the dynamical equations (e.g. Nelson & Papaloizou 1995) may lead to substantial improvements in the SPH method.

Finally, the suitability of a particular numerical method must be determined in the context of the astrophysical system being modeled, as well as the resources available to the investigators. We hope that this study, along with related work comparing the results of Eulerian and SPH hydrodynamic codes in modeling stellar collisions (Davies, et al. 1993) and the growth of structure in cosmology (Kang, et al. 1994), will help others to find the best approach for their applications.

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Table 1: Properties of the Eulerian models. $N_\varpi$, $N_z$, and $N_\varphi$ are the number of grid zones in the $\varpi$, $z$, and $\varphi$ directions, respectively. $N_{\text{star},i}$ is the approximate number of zones occupied by the matter distribution. The subscripts “i” and “f” denote the initial and final states of the model. The stability parameter at the initial time, calculated on the Eulerian grid, is $\tau_i$. The duration of the run is measured in units of the dynamical time $t_D$ and listed in the column labeled “time”. All models were run on a Cray C90; the amount of CPU time used is given for the duration of the run. The quantities given in the row labeled $E_2'$ are the values for model $E_2$ at the intermediate time $23.9t_D$. Model $E_3$ is the same as $E_2$ except for the use of relaxed constraints in the vacuum conditions.

| Run | $N_\varpi$ | $N_z$ | $N_\varphi$ | $N_{\text{star},i}$ | $N_{\text{star},f}$ | $\tau_i$ | time | $\frac{E_i-E_f}{E_i}$ | $\frac{J_i-J_f}{J_i}$ | CPU |
|-----|------------|------|-------------|---------------------|---------------------|--------|------|---------------------|---------------------|-----|
| E1  | 64         | 32   | 32          | 7200                | 24,600              | 0.30   | $34t_D$ | 0.035               | 0.072               | 8.2 hr |
| E2  | 64         | 32   | 64          | 14,400              | 66,200              | 0.30   | $34t_D$ | 0.043               | 0.086               | 29.7 hr |
| E2' |           |      |             | 41,500              |                     |        | $23.9t_D$ | 0.020               | 0.049               | 20.5 hr |
| E3  | 64         | 32   | 64          | 14,400              | 110,600             | 0.30   | $23.9t_D$ | 0.015               | 0.034               | 29.5 hr |

Table 2: Hydrodynamical and bar mode results for the Eulerian models. The quantities in the row labeled $E_2'$ are the values for run $E_2$ at the intermediate time $23.9t_D$; cf. Table 1. The bar growth rate $d \ln |C|/dt$ and the eigenfrequency $\sigma_2$ are calculated for the ring $\varpi = 0.362R_{\text{eq}}$ in zone $j = 10$ in the equatorial plane; both are in units of $t_D^{-1}$. The core refers to matter within cylindrical radius $\varpi = R_{\text{eq}}$, where $R_{\text{eq}}$ is the equatorial radius of the initial model, and the subscript “f” denotes the final state of the model. Note that $J_{\text{core},f}$ is normalized using the value of the total angular momentum of the system at the final time, which is less than the initial value due to non-conservation; see Table 1.

| Run | $d \ln |C|/dt$ | $\sigma_2$ | $M_{\text{core},f}$ | $J_{\text{core},f}$ | $\tau_{\text{core},f}$ | $\tau_f$ |
|-----|-------------|------------|---------------------|---------------------|------------------------|---------|
| E1  | 0.53        | 1.7        | 94%                 | 78%                 | 0.18                   | 0.19    |
| E2  | 0.58        | 1.8        | 96%                 | 86%                 | 0.24                   | 0.24    |
| E2' |             |           | 96%                 | 87%                 | 0.25                   | 0.26    |
| E3  | 0.57        | 1.8        | 95%                 | 83%                 | 0.25                   | 0.26    |
| Run  | max $|r|/m$ | max $L$ | $(\Delta E/M)_{T}$ | max $dJ_z/dt$ | $(\Delta J/J_0)_{T}$ |
|------|-----------|---------|------------------|--------------|---------------------|
| E1   | 0.63      | 0.20    | 0.68             | 0.18         | 1.6                 |
| E2   | 0.68      | 0.21    | 0.93             | 0.20         | 2.3                 |
| E2'  | –         | –       | 0.77             | –            | 2.0                 |
| E3   | 0.62      | 0.17    | 0.63             | 0.16         | 1.6                 |

Table 3: Gravitational wave results for the Eulerian models. We use $c = G = 1$. The quantities in the row labeled E2' are the values for run E2 at the intermediate time $23.9t_D$; cf. Table [4]. The values of the peak gravitational wave amplitudes $|r|/m$ are in units of $M^2/R_{eq}$. The values of the maximum luminosity $L$ are in units of $(M/R_{eq})^5$, and the values of the total energy emitted during the duration of the run $(\Delta E/M)_{T}$ are in units of $(M/R_{eq})^7/2$. The values of the maximum $dJ_z/dt$ are in units of $M(M/R_{eq})^7/2$. The quantity $(\Delta J_z/J_0)_{T}$ is the total angular momentum emitted as gravitational radiation normalized by the initial total angular momentum, and has units of $(M/R_{eq})^5/2$.

| Run  | $N$   | type | $\alpha$ | $\beta$ | $\tau_1$ | time    | $|E_i-E_f|/E_i$ | $|J_i-J_f|/J_i$ | CPU   |
|------|-------|------|----------|---------|----------|---------|----------------|----------------|-------|
| T1   | 2061  | cold | 0.25     | 1.0     | 0.305    | 35$t_D$ | 0.022          | $\lesssim .001$ | 1.01 hr |
| T2   | 8728  | cold | 0.25     | 1.0     | 0.308    | 35$t_D$ | 0.018          | $\lesssim .001$ | 8.65 hr |
| T3   | 15,648| cold | 0.25     | 1.0     | 0.314    | 35$t_D$ | 0.018          | $\lesssim .001$ | 16.9 hr |
| T4   | 15,648| cold | 1.0      | 2.0     | 0.314    | 35$t_D$ | 0.011          | $\lesssim .001$ | 23.0 hr |
| T5   | 15,648| cold | 0        | 0       | 0.314    | 35$t_D$ | 0.043          | $\lesssim .001$ | 16.7 hr |
| T6   | 16,000| random | 0.25     | 1.0     | 0.299    | 26.9$t_D$| 0.043          | $\lesssim .001$ | 11.6 hr |
| T7   | 32,914| cold | 0.25     | 1.0     | 0.316    | 35$t_D$ | 0.018          | $\lesssim .001$ | 44.6 hr |

Table 4: Properties of the SPH models. All models used the “cold” initial conditions except T6; the initial conditions for this model were produced using the random method. $N$ is the number of particles in the model, and $\alpha$ and $\beta$ are, respectively, the coefficients of the linear and quadratic terms in the artificial viscosity. The other quantities are defined as in Table [4].
\[
\frac{d \ln |C|}{dt} \quad \sigma_2 \quad M_{\text{core},f} \quad J_{\text{core},f} \quad \tau_{\text{core},f} \quad \tau_f
\]

| Run | \( \frac{d \ln |C|}{dt} \) | \( \sigma_2 \) | \( M_{\text{core},f} \% \) | \( J_{\text{core},f} \% \) | \( \tau_{\text{core},f} \) | \( \tau_f \) |
|-----|-----------------|----------|----------------|----------------|----------|----------|
| T1   | 0.66            | 2.1      | 91%            | 73%            | 0.24     | 0.25     |
| T2   | 0.51            | 1.9      | 92%            | 74%            | 0.24     | 0.26     |
| T3   | 0.53            | 1.9      | 92%            | 73%            | 0.25     | 0.26     |
| T4   | 0.51            | 1.9      | 92%            | 73%            | 0.25     | 0.26     |
| T5   | 0.54            | 1.9      | 89%            | 67%            | 0.24     | 0.26     |
| T6   | 0.54            | 1.8      | 95%            | 81%            | 0.25     | 0.26     |
| T7   | 0.51            | 1.9      | 90%            | 72%            | 0.25     | 0.26     |

Table 5: Hydrodynamical and bar mode results for the SPH models. The quantities are defined as in Table 2. The large growth rate for T1 is anomalous; see the text for details.

\[
\max |r_h| \quad \max L \quad (\Delta E/M)_f \quad \max \frac{dJ_z}{dt} \quad (\Delta J_z/J_0)_f
\]

| Run | \( \max |r_h| \) | \( \max L \) | \( (\Delta E/M)_f \) | \( \max \frac{dJ_z}{dt} \) | \( (\Delta J_z/J_0)_f \) |
|-----|----------------|----------|----------------|----------------|----------|----------|
| T1  | 0.26           | 0.030    | 0.21           | 0.015          | 0.24     |
| T2  | 0.43           | 0.078    | 0.46           | 0.039          | 0.54     |
| T3  | 0.47           | 0.091    | 0.55           | 0.045          | 0.66     |
| T4  | 0.41           | 0.066    | 0.39           | 0.035          | 0.48     |
| T5  | 0.48           | 0.10     | 0.52           | 0.050          | 0.59     |
| T6  | 0.38           | 0.051    | 0.31           | 0.027          | 0.42     |
| T7  | 0.53           | 0.12     | 0.87           | 0.060          | 1.0      |

Table 6: Gravitational wave results for the SPH models. The quantities are defined as in Table 3.
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Fig. 1.— Density contours in the equatorial plane for model E2. The contour levels are the same in all frames. The contours are spaced a factor of 10 apart, and go down to 3 decades below the maximum (central) density at the initial time $t = 0$. Since the maximum density increases slightly during the run, there are 4 contours shown for the later times, with the innermost contour being at the initial maximum density. The model rotates in the counterclockwise direction. The circular boundary of the plotted region is set at $\varpi = 2.5R_{\text{eq}}$.

Fig. 2.— The mass fraction $m(\varpi)$ is shown for model E2 at the initial time (dot-dashed line), the intermediate time $t = 21.1t_D$ (dashed line), and the final time $t = 34t_D$ (solid line). The total mass is $M$.

Fig. 3.— The angular momentum $J(\varpi)$ is shown for the model E2 at the initial time (dot-dashed line), the intermediate time $t = 21.1t_D$ (dashed line), and the final time $t = 34t_D$ (solid line). Note that $J(\varpi)$ is normalized by the total angular momentum in the system at the time.

Fig. 4.— The growth of the bar mode for model E2. The bar amplitude $\ln |C|$ is shown versus time for the ring $\varpi = 0.362R_{\text{eq}}$ in zone $j = 10$ in the equatorial plane $z = 0$. The growth rate in the region where $\ln |C|$ is growing linearly with time is $d\ln |C|/dt = 0.58t_D^{-1}$.

Fig. 5.— The growth of various Fourier components of the density for model E2. The amplitudes have been calculated for the same ring used in Figure 4, and are shown versus time. In each case, the bar mode amplitude is plotted as a solid line for comparison. (a) $m = 1$ (b) $m = 3$ (c) $m = 4$

Fig. 6.— The behavior of $T/|W|$ is shown as a function of time for model E2. The solid line shows the stability parameter $\tau = T_{\text{rot}}/|W|$ and the dashed line shows $T_{\text{total}}/|W|$.

Fig. 7.— Gravitational waveforms for an observer on the axis at $\theta = 0$ and $\varphi = 0$ for model E2. We use $c = G = 1$. (a) $r h_+$ (b) $r h_x$
Fig. 8.— Various gravitational wave quantities are shown for run E2. We use $c = G = 1$. (a) Gravitational wave luminosity $L$. (b) The energy $\Delta E/M$ emitted as gravitational radiation. (c) The rate $dJ_z/dt$ at which angular momentum is carried away by the waves. (d) The angular momentum $\Delta J/J_0$ lost to gravitational radiation. Here, $J_0$ is the initial total angular momentum.

Fig. 9.— Density contours in the equatorial plane at $t = 23.9t_D$. The contour levels are the same as in Figure [1]. (a) Model E2 (b) Model E3 (with relaxed vacuum conditions)

Fig. 10.— Particle positions are shown projected onto the equatorial plane for various times in the evolution of model T7. The vertical axis is $y/R_{eq}$ and the horizontal axis is $x/R_{eq}$. The system rotates in the counterclockwise direction.

Fig. 11.— Particle positions are shown projected onto the $x-z$ plane at the final time $t = 35t_D$ of model T7. Shock heating during the spiral arm expansion has caused the disk to puff up.

Fig. 12.— Density contours in the equatorial plane for model T7. The density has been interpolated onto the cylindrical grid used for model E2, and the contour levels are the same as those used in Figure [1].

Fig. 13.— The mass fraction $m(\varpi)$ is shown for model T7 at the initial time (dot-dashed line), the intermediate time $t = 18.2t_D$ (dashed line), and the final time $t = 35t_D$ (solid line). Only the region $\varpi \leq 5R_{eq}$ is plotted. The total mass is $M$.

Fig. 14.— The angular momentum $J(\varpi)$ is shown for the model T7 at the initial time (dot-dashed line), intermediate time $t = 18.2t_D$ (dashed line), and the final time $t = 35t_D$ (solid line). Only the region $\varpi \leq 5R_{eq}$ is plotted. The total initial angular momentum is $J_0$.

Fig. 15.— The growth of the bar mode for model T7. The particle model is interpolated onto the cylindrical grid used for model E2, and the bar amplitude is calculated as in Figure [4]. The growth rate in the region where $\ln |C|$ is growing linearly with time is $d\ln |C|/dt = 0.51t_D^{-1}$.

Fig. 16.— The growth of various Fourier components of the density for model T7. In each case, the bar mode amplitude is plotted as a solid line for comparison. (a) $m = 1$ (b) $m = 3$ (c) $m = 4$
Fig. 17.— The behavior of $T/|W|$ is shown as a function of time for model T7. The solid line shows the stability parameter $\tau = T_{\text{tot}}/|W|$ and the dashed line shows $T_{\text{total}}/|W|$. The initial oscillations in $T/|W|$ are caused by adjustments in the model due to residual noise in the initial conditions.

Fig. 18.— Gravitational waveforms for an observer on the axis at $\theta = 0$ and $\varphi = 0$ for model T7. We use $c = G = 1$. (a) $\mathcal{R}h_+$ (b) $\mathcal{R}h_\times$

Fig. 19.— Various gravitational wave quantities are shown for run T7. We use $c = G = 1$. (a) Gravitational wave luminosity $L$. (b) The energy $\Delta E/M$ emitted as gravitational radiation. (c) The rate $dJ_z/dt$ at which angular momentum is carried away by the waves. (d) The angular momentum $\Delta J/J_0$ lost to gravitational radiation.